Solution Section 5.1 – Cramer's Rule

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y=-4 \\ 2x-y=-5 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}=-3-4=-7$$

$$D_x=\begin{vmatrix} -4 & 2 \\ -5 & -1 \end{vmatrix}=4-(-10)=14$$

$$D_y=\begin{vmatrix} 3 & -4 \\ 2 & -5 \end{vmatrix}=-15-(-8)=-7$$

$$x=\frac{D_x}{D}=\frac{14}{-7}=-2$$

$$y=\frac{D_y}{D}=\frac{-7}{-7}=1$$

∴ ***Solution***: $(-2,\ 1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+5y=7 \\ 5x-2y=-3 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix}=-29 \qquad D_x=\begin{vmatrix} 7 & 5 \\ -3 & -2 \end{vmatrix}=1 \qquad D_y=\begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix}=-41$$

$$x=\frac{1}{-29}=-\frac{1}{29} \qquad y=\frac{41}{29}$$

∴ ***Solution***: $\left(-\frac{1}{29},\ \frac{41}{29}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y=-4 \\ 2x-y=-5 \end{cases}$

Solution

$$D = \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = -7 \qquad D_x = \begin{vmatrix} -4 & -5 \\ 2 & -1 \end{vmatrix} = 14 \qquad D_y = \begin{vmatrix} 3 & -4 \\ 2 & -5 \end{vmatrix} = -7$$

$$x = -\frac{14}{7} = -2 \qquad x = \frac{D_x}{D}$$

$$y = \frac{7}{7} = 1 \qquad y = \frac{D_y}{D}$$

Solution: $(-2,\ 1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$

Solution

$$D = \begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix} = -29 \qquad D_x = \begin{vmatrix} 7 & 5 \\ -3 & -2 \end{vmatrix} = 1 \qquad D_y = \begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix} = -41$$

$$x = -\frac{1}{29} \qquad x = \frac{D_x}{D}$$

$$y = \frac{41}{29} \qquad y = \frac{D_y}{D}$$

∴ ***Solution***: $\left(-\frac{1}{29},\ \frac{41}{29}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$

Solution

$$D = \begin{vmatrix} 4 & -7 \\ 2 & 5 \end{vmatrix} = 34 \qquad D_x = \begin{vmatrix} -16 & -7 \\ 9 & 5 \end{vmatrix} = -17 \qquad D_y = \begin{vmatrix} 4 & -16 \\ 2 & 9 \end{vmatrix} = 68$$

$$x = -\frac{17}{34} = -\frac{1}{2} \qquad x = \frac{D_x}{D}$$

$$y = \frac{68}{34} = 2 \qquad y = \frac{D_y}{D}$$

∴ ***Solution***: $\left(-\frac{1}{2},\ 2\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y=4 \\ 2x+y=1 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}=-1 \qquad D_x=\begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix}=2 \qquad D_y=\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix}=-5$$

$\underline{x=-2}$ $\qquad x=\dfrac{D_x}{D}$

$\underline{y=5}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(-2,\ 5)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+4y=2 \\ 2x+5y=-1 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix}=7 \qquad D_x=\begin{vmatrix} 2 & 4 \\ -1 & 5 \end{vmatrix}=14 \qquad D_y=\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}=-7$$

$x=\dfrac{14}{7}\underline{=2}$ $\qquad x=\dfrac{D_x}{D}$

$y=-\dfrac{7}{7}\underline{=-1}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(2,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 5x-2y=4 \\ -10x+4y=7 \end{cases}$

Solution

$$D=\begin{vmatrix} 5 & -2 \\ -10 & 4 \end{vmatrix}=0 \qquad D_y=\begin{vmatrix} 5 & 4 \\ -10 & 7 \end{vmatrix}=75\neq 0$$

$\therefore$ ***No Solution***

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & -4 \\ 5 & -20 \end{vmatrix}=0 \qquad D_y=\begin{vmatrix} 1 & -8 \\ 5 & -40 \end{vmatrix}=0$$

$$\frac{1}{5}\times \begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$$

$$\begin{cases} x-4y=-8 \\ x-4y=-8 \end{cases}$$

$\therefore$ ***Solution***: $\underline{(4y-8,\ y)|}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}=-3 \qquad D_x=\begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix}=-6 \qquad D_y=\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix}=3$$

$$x=\frac{6}{3}=2 \qquad x=\frac{D_x}{D}$$

$$y=-\frac{3}{3}=-1 \qquad y=\frac{D_y}{D}$$

$\therefore$ ***Solution***: $\underline{(2,\ -1)|}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+10y=-14 \\ 7x-2y=-16 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & 10 \\ 7 & -2 \end{vmatrix}=-74 \qquad D_x=\begin{vmatrix} -14 & 10 \\ -16 & -2 \end{vmatrix}=188 \qquad D_y=\begin{vmatrix} 2 & -14 \\ 7 & -16 \end{vmatrix}=66$$

$$x=-\frac{188}{74}=-\frac{94}{37} \qquad x=\frac{D_x}{D}$$

$$y = -\frac{66}{74} = -\frac{33}{37} \qquad\qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\left(-\frac{94}{37},\ -\frac{33}{37}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$

Solution

$$D = \begin{vmatrix} 4 & -3 \\ -3 & 9 \end{vmatrix} = 27 \qquad D_x = \begin{vmatrix} 24 & -3 \\ -1 & 9 \end{vmatrix} = 213 \qquad D_y = \begin{vmatrix} 4 & 24 \\ -3 & -1 \end{vmatrix} = 68$$

$$x = \frac{213}{27} = \frac{71}{9} \qquad\qquad x = \frac{D_x}{D}$$

$$y = \frac{68}{27} \qquad\qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\left(\frac{71}{9},\ \frac{68}{27}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$

Solution

$$D = \begin{vmatrix} 4 & 2 \\ 3 & -2 \end{vmatrix} = -14 \qquad D_x = \begin{vmatrix} 12 & 2 \\ 16 & -2 \end{vmatrix} = -56 \qquad D_y = \begin{vmatrix} 4 & 12 \\ 3 & 16 \end{vmatrix} = 28$$

$$x = \frac{56}{14} = 4 \qquad\qquad x = \frac{D_x}{D}$$

$$y = -\frac{28}{14} = -2 \qquad\qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $(4,\ -2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+2y=-1 \\ 4x-2y=6 \end{cases}$

Solution

$D=\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix}=-10$ $\qquad D_x=\begin{vmatrix} -1 & 2 \\ 6 & -2 \end{vmatrix}=-10$ $\qquad D_y=\begin{vmatrix} 1 & -1 \\ 4 & 6 \end{vmatrix}=10$

$x=1$ $\qquad x=\dfrac{D_x}{D}$

$y=-1$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $(1,\ -1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-2y=5 \\ -10x+2y=4 \end{cases}$

Solution

$D=\begin{vmatrix} 1 & -2 \\ -10 & 2 \end{vmatrix}=-18$ $\qquad D_x=\begin{vmatrix} 5 & -2 \\ 4 & 2 \end{vmatrix}=18$ $\qquad D_y=\begin{vmatrix} 1 & 5 \\ -10 & 4 \end{vmatrix}=54$

$x=-1$ $\qquad x=\dfrac{D_x}{D}$

$y=-\dfrac{54}{18}=-3$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $(-1,\ -3)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$

Solution

$$\begin{array}{l} \frac{1}{3}\times \\ \frac{1}{15}\times \end{array} \begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$$

$$\begin{cases} 4x+5y=-9 \\ 2x-y=-1 \end{cases}$$

$$D = \begin{vmatrix} 4 & 5 \\ 2 & -1 \end{vmatrix} = -14 \qquad D_x = \begin{vmatrix} -9 & 5 \\ -1 & -1 \end{vmatrix} = 14 \qquad D_y = \begin{vmatrix} 4 & -9 \\ 2 & -1 \end{vmatrix} = 14$$

$\underline{x = -1}$ $\qquad x = \frac{D_x}{D}$

$\underline{y = -1}$ $\qquad y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(-1, \ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$

Solution

$$\begin{matrix} \frac{1}{4} \times \\ \frac{1}{4} \times \end{matrix} \begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$$

$$\begin{cases} x - y = -3 \\ x + y = -5 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \qquad D_x = \begin{vmatrix} -3 & -1 \\ -5 & 1 \end{vmatrix} = -8 \qquad D_y = \begin{vmatrix} 1 & -3 \\ 1 & -5 \end{vmatrix} = -2$$

$\underline{x = -4}$ $\qquad x = \frac{D_x}{D}$

$\underline{y = -1}$ $\qquad y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(-4, \ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + y = 7 \\ x - y = 3 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \qquad D_x = \begin{vmatrix} 7 & 1 \\ 3 & -1 \end{vmatrix} = -10 \qquad D_y = \begin{vmatrix} 1 & 7 \\ 1 & 3 \end{vmatrix} = -4$$

$\underline{x = 5}$ $\qquad x = \frac{D_x}{D}$

$\underline{y = 2}$ $\qquad y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(5,\ 2)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$

Solution

$D = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3 \qquad D_x = \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} = -6 \qquad D_y = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 3$

$\underline{x = 2}$ $\qquad x = \frac{D_x}{D}$

$\underline{y = -1}$ $\qquad y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(2,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 12x + 3y = 15 \\ 2x - 3y = 13 \end{cases}$

Solution

$D = \begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix} = -42 \qquad D_x = \begin{vmatrix} 15 & 3 \\ 13 & -3 \end{vmatrix} = -84 \qquad D_y = \begin{vmatrix} 12 & 15 \\ 2 & 13 \end{vmatrix} = 126$

$\underline{x = 2}$ $\qquad x = \frac{D_x}{D}$

$\underline{y = -3}$ $\qquad y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(2,\ -3)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x - 2y = 5 \\ 5x - y = -2 \end{cases}$

Solution

$D = \begin{vmatrix} 1 & -2 \\ 5 & -1 \end{vmatrix} = 9 \qquad D_x = \begin{vmatrix} 5 & -2 \\ -2 & -1 \end{vmatrix} = -9 \qquad D_y = \begin{vmatrix} 1 & 5 \\ 5 & -2 \end{vmatrix} = -27$

$\boxed{x=-1}$ $\qquad x=\dfrac{D_x}{D}$

$\boxed{y=-3}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\boxed{(-1,\ -3)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x-5y=17 \\ 2x+3y=3 \end{cases}$

Solution

$D=\begin{vmatrix} 4 & -5 \\ 2 & 3 \end{vmatrix}=22$ $\qquad D_x=\begin{vmatrix} 17 & -5 \\ 3 & 3 \end{vmatrix}=66$ $\qquad D_y=\begin{vmatrix} 4 & 17 \\ 2 & 3 \end{vmatrix}=-22$

$\boxed{x=3}$ $\qquad x=\dfrac{D_x}{D}$

$\boxed{y=-1}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\boxed{(3,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y=2 \\ 2x+2y=3 \end{cases}$

Solution

$D=\begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix}=2$ $\qquad D_x=\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix}=-2$ $\qquad D_y=\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}=5$

$\boxed{x=-1}$ $\qquad x=\dfrac{D_x}{D}$

$\boxed{y=\dfrac{5}{2}}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\boxed{\left(-1,\ \dfrac{5}{2}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-3y=4 \\ 3x-4y=12 \end{cases}$

Solution

$D=\begin{vmatrix} 1 & -3 \\ 3 & -4 \end{vmatrix}=5 \qquad D_x=\begin{vmatrix} 4 & -3 \\ 12 & -4 \end{vmatrix}=20 \qquad D_y=\begin{vmatrix} 1 & 4 \\ 3 & 12 \end{vmatrix}=0$

$\underline{x=4}\qquad x=\frac{D_x}{D}$

$\underline{y=0}\qquad y=\frac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(4,\ 0)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x-9y=5 \\ 3x-3y=11 \end{cases}$

Solution

$D=\begin{vmatrix} 2 & -9 \\ 3 & -3 \end{vmatrix}=21 \qquad D_x=\begin{vmatrix} 5 & -9 \\ 11 & -3 \end{vmatrix}=84 \qquad D_y=\begin{vmatrix} 2 & 5 \\ 3 & 11 \end{vmatrix}=7$

$\underline{x=4}\qquad x=\frac{D_x}{D}$

$\underline{y=\frac{1}{3}}\qquad y=\frac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{\left(4,\ \frac{1}{3}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x-4y=4 \\ x+y=6 \end{cases}$

Solution

$D=\begin{vmatrix} 3 & -4 \\ 1 & 1 \end{vmatrix}=7 \qquad D_x=\begin{vmatrix} 4 & -4 \\ 6 & 1 \end{vmatrix}=28 \qquad D_y=\begin{vmatrix} 3 & 4 \\ 1 & 6 \end{vmatrix}=14$

$\underline{x=4}\qquad x=\frac{D_x}{D}$

$\underline{y=2}\qquad y=\frac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(4,\ 2)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x = 7y + 1 \\ 2x = 3y - 1 \end{cases}$

Solution

$$\begin{cases} 3x - 7y = 1 \\ 2x - 3y = -1 \end{cases}$$

$$D = \begin{vmatrix} 3 & -7 \\ 2 & -3 \end{vmatrix} = 5 \qquad D_x = \begin{vmatrix} 1 & -7 \\ -1 & -3 \end{vmatrix} = -10 \qquad D_y = \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} = -5$$

$\underline{x = -2}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = -1}$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(-2,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x = 3y + 2 \\ 5x = 51 - 4y \end{cases}$

Solution

$$\begin{cases} 2x - 3y = 2 \\ 5x + 4y = 51 \end{cases}$$

$$D = \begin{vmatrix} 2 & -3 \\ 5 & 4 \end{vmatrix} = 23 \qquad D_x = \begin{vmatrix} 2 & -3 \\ 51 & 4 \end{vmatrix} = 161 \qquad D_y = \begin{vmatrix} 2 & 2 \\ 5 & 51 \end{vmatrix} = 92$$

$\underline{x = 7}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = 4}$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(7,\ 4)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} y = -4x + 2 \\ 2x = 3y - 1 \end{cases}$

Solution

$$\begin{cases} 4x + y = 2 \\ 2x - 3y = -1 \end{cases}$$

$$D = \begin{vmatrix} 4 & 1 \\ 2 & -3 \end{vmatrix} = -14 \qquad D_x = \begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix} = -5 \qquad D_y = \begin{vmatrix} 4 & 2 \\ 2 & -1 \end{vmatrix} = -8$$

$$\underline{x = \frac{5}{14}} \qquad x = \frac{D_x}{D}$$

$$\underline{y = \frac{4}{7}} \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\underline{\left(\frac{15}{4}, \ \frac{4}{7}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x = 2 - 3y \\ 2y = 3 - 2x \end{cases}$

Solution

$$\begin{cases} 3x + 3y = 2 \\ 2x + 2y = 3 \end{cases}$$

$$D = \begin{vmatrix} 3 & 3 \\ 2 & 2 \end{vmatrix} = 0 \qquad D_y = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5 \neq 0$$

$\therefore$ ***No Solution***

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + 2y - 3 = 0 \\ 12 = 8y + 4x \end{cases}$

Solution

$$\begin{cases} x + 2y = 3 \\ 4x + 8y = 12 \end{cases}$$

$$\begin{cases} x + 2y = 3 \\ x + 2y = 3 \end{cases}$$

$\therefore$ ***Solution:*** $\underline{(3 - 2y, \ y)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 7x-2y=3 \\ 3x+y=5 \end{cases}$

Solution

$D = \begin{vmatrix} 7 & -2 \\ 3 & 1 \end{vmatrix} = 13 \qquad D_x = \begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 13 \qquad D_y = \begin{vmatrix} 7 & 3 \\ 3 & 5 \end{vmatrix} = 26$

$\underline{x=1} \qquad x = \dfrac{D_x}{D}$

$\underline{y=2} \qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(1,\ 2)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y-z=4 \\ 3x-2y+z=5 \\ 4x-5y-z=-1 \end{cases}$

Solution

$D = \begin{vmatrix} 3 & 2 & -1 \\ 3 & -2 & 1 \\ 4 & -5 & -1 \end{vmatrix} = 42 \qquad D_x = \begin{vmatrix} 4 & 2 & -1 \\ 5 & -2 & 1 \\ -1 & -5 & -1 \end{vmatrix} = 63$

$D_y = \begin{vmatrix} 3 & 4 & -1 \\ 3 & 5 & 1 \\ 4 & -1 & -1 \end{vmatrix} = 39 \qquad D_z = \begin{vmatrix} 3 & 2 & 4 \\ 3 & -2 & 5 \\ 4 & -5 & -1 \end{vmatrix} = 99$

$x = \dfrac{63}{42} = \underline{\dfrac{3}{2}} \qquad x = \dfrac{D_x}{D}$

$y = \dfrac{39}{42} = \underline{\dfrac{13}{14}} \qquad y = \dfrac{D_y}{D}$

$z = \dfrac{99}{42} = \underline{\dfrac{33}{14}} \qquad z = \dfrac{D_z}{D}$

Solution: $\underline{\left(\dfrac{3}{2},\ \dfrac{13}{14},\ \dfrac{33}{14}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+y+z=2 \\ 2x+y-z=5 \\ x-y+z=-2 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}=-6 \qquad D_x=\begin{vmatrix} 2 & 1 & 1 \\ 5 & 1 & -1 \\ -2 & -1 & 1 \end{vmatrix}=-6$$

$$D_y=\begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & -1 \\ 1 & -2 & 1 \end{vmatrix}=-12 \qquad D_z=\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 5 \\ 1 & -2 & 2 \end{vmatrix}=6$$

$\underline{x=1}$ $\qquad x=\dfrac{D_x}{D}$

$\underline{y=2}$ $\qquad y=\dfrac{D_y}{D}$

$\underline{z=-1}$ $\qquad z=\dfrac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(1,\ 2,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+y+z=9 \\ -x-y+z=1 \\ 3x-y+z=9 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & 1 & 1 \\ -1 & -1 & 1 \\ 3 & -1 & 1 \end{vmatrix}\begin{matrix} 2 & 1 \\ -1 & -1 \\ 3 & -1 \end{matrix}=-2+3+1+3+2+1$$

$$\underline{=8}$$

$$D_x=\begin{vmatrix} 9 & 1 & 1 \\ 1 & -1 & 1 \\ 9 & -1 & 1 \end{vmatrix}\begin{matrix} 9 & 1 \\ 1 & -1 \\ 9 & -1 \end{matrix}=-9+9-1+9+9-1$$

$$\underline{=16}$$

$$D_y=\begin{vmatrix} 2 & 9 & 1 \\ -1 & 1 & 1 \\ 3 & 9 & 1 \end{vmatrix}\begin{matrix} 2 & 9 \\ -1 & 1 \\ 3 & 9 \end{matrix}=2+27-9-3-18+9$$

$$\underline{=8}$$

$$D_z = \begin{vmatrix} 2 & 1 & 9 \\ -1 & -1 & 1 \\ 3 & -1 & 9 \end{vmatrix} \begin{matrix} 2 & 1 \\ -1 & -1 \\ 3 & -1 \end{matrix} = -18+3+9+27+2+9$$

$$= 32$$

$x = 2$ $\qquad x = \dfrac{D_x}{D}$

$y = 1$ $\qquad y = \dfrac{D_y}{D}$

$z = \dfrac{32}{8} = 4$ $\qquad z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $(2,\ 1,\ 4)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

Solution

$$D = \begin{vmatrix} 0 & 3 & -1 \\ 1 & 5 & -1 \\ -3 & 6 & 2 \end{vmatrix} \begin{matrix} 0 & 3 \\ 1 & 5 \\ -3 & 6 \end{matrix} = 9-6-15-6$$

$$= -18$$

$$D_x = \begin{vmatrix} -1 & 3 & -1 \\ -4 & 5 & -1 \\ 11 & 6 & 2 \end{vmatrix} \begin{matrix} -1 & 3 \\ -4 & 5 \\ 11 & 6 \end{matrix} = -10-33+24+55-6+24$$

$$= 54$$

$$D_y = \begin{vmatrix} 0 & -1 & -1 \\ 1 & -4 & -1 \\ -3 & 11 & 2 \end{vmatrix} \begin{matrix} 0 & -1 \\ 1 & -4 \\ -3 & 11 \end{matrix} = -3-11+12+2$$

$$= 0$$

$$D_z = \begin{vmatrix} 0 & 3 & -1 \\ 1 & 5 & -4 \\ -3 & 6 & 11 \end{vmatrix} \begin{matrix} 0 & 3 \\ 1 & 5 \\ -3 & 6 \end{matrix} = 36-6-15-33$$

$$= -18$$

$x = -3$ $\qquad x = \dfrac{D_x}{D}$

$y = 0$ $\qquad\qquad y = \frac{D_y}{D}$

$z = 1$ $\qquad\qquad z = \frac{D_z}{D}$

$\therefore$ ***Solution***: $\boxed{(-3,\ 0,\ 1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 3 & 4 \\ 2 & -3 & 2 \\ 3 & -1 & 1 \end{vmatrix} \begin{matrix} 1 & 3 \\ 2 & -3 \\ 3 & -1 \end{matrix} = -3 + 18 - 8 + 36 + 2 - 6$$

$$= 39$$

$$D_x = \begin{vmatrix} 14 & 3 & 4 \\ 10 & -3 & 2 \\ 9 & -1 & 1 \end{vmatrix} \begin{matrix} 14 & 3 \\ 10 & -3 \\ 9 & -1 \end{matrix} = -42 + 54 - 40 + 108 + 28 - 30$$

$$= 78$$

$$D_y = \begin{vmatrix} 1 & 14 & 4 \\ 2 & 10 & 2 \\ 3 & 9 & 1 \end{vmatrix} \begin{matrix} 1 & 14 \\ 2 & 10 \\ 3 & 9 \end{matrix} = 10 + 84 + 72 - 120 - 18 - 28$$

$$= 0$$

$$D_z = \begin{vmatrix} 1 & 3 & 14 \\ 2 & -3 & 10 \\ 3 & -1 & 9 \end{vmatrix} \begin{matrix} 1 & 3 \\ 2 & -3 \\ 3 & -1 \end{matrix} = -27 + 90 - 28 + 126 + 10 - 54$$

$$= 117$$

$x = \frac{78}{39} = 2$ $\qquad\qquad x = \frac{D_x}{D}$

$y = 0$ $\qquad\qquad y = \frac{D_y}{D}$

$z = \frac{117}{39} = 3$ $\qquad\qquad z = \frac{D_z}{D}$

$\therefore$ ***Solution***: $\boxed{(2,\ 0,\ 3)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+4y-z=20 \\ 3x+2y+z=8 \\ 2x-3y+2z=-16 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & 4 & -1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix}\begin{matrix} 1 & 4 \\ 3 & 2 \\ 2 & -3 \end{matrix} = 4+8+9+4+3-24$$

$$\underline{=4}$$

$$D_x=\begin{vmatrix} 20 & 4 & -1 \\ 8 & 2 & 1 \\ -16 & -3 & 2 \end{vmatrix}\begin{matrix} 20 & 4 \\ 8 & 2 \\ -16 & -3 \end{matrix} = 80-64+24-32+60-64$$

$$\underline{=4}$$

$$D_y=\begin{vmatrix} 1 & 20 & -1 \\ 3 & 8 & 1 \\ 2 & -16 & 2 \end{vmatrix}\begin{matrix} 1 & 20 \\ 3 & 8 \\ 2 & -16 \end{matrix} = 16+40+48+16+16-120$$

$$\underline{=16}$$

$$D_z=\begin{vmatrix} 1 & 4 & 20 \\ 3 & 2 & 8 \\ 2 & -3 & -16 \end{vmatrix}\begin{matrix} 1 & 4 \\ 3 & 2 \\ 2 & -3 \end{matrix} = -32+64-180-80+24+192$$

$$\underline{=-12}$$

$$x=\frac{4}{4}\ \underline{=1} \qquad x=\frac{D_x}{D}$$

$$y=\frac{16}{4}\ \underline{=4} \qquad y=\frac{D_y}{D}$$

$$z=-\frac{12}{4}\ \underline{=-3} \qquad z=\frac{D_z}{D}$$

∴ ***Solution***: $\underline{(1,\ 4,\ -3)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} -2x+6y+7z=3 \\ -4x+5y+3z=7 \\ -6x+3y+5z=-4 \end{cases}$

Solution

$$D = \begin{vmatrix} -2 & 6 & 7 \\ -4 & 5 & 3 \\ -6 & 3 & 5 \end{vmatrix} \begin{matrix} -2 & 6 \\ -4 & 5 \\ -6 & 3 \end{matrix} = -50-108-84+210+18+120$$

$$= 106$$

$$D_x = \begin{vmatrix} 3 & 6 & 7 \\ 7 & 5 & 3 \\ -4 & 3 & 5 \end{vmatrix} \begin{matrix} 3 & 6 \\ 7 & 5 \\ -4 & 3 \end{matrix} = 75-72+147+140-27-210$$

$$= 53$$

$$D_y = \begin{vmatrix} -2 & 3 & 7 \\ -4 & 7 & 3 \\ -6 & -4 & 5 \end{vmatrix} \begin{matrix} -2 & 3 \\ -4 & 7 \\ -6 & -4 \end{matrix} = -70-54+112+294-24+60$$

$$= 318$$

$$D_z = \begin{vmatrix} -2 & 6 & 3 \\ -4 & 5 & 7 \\ -6 & 3 & -4 \end{vmatrix} \begin{matrix} -2 & 6 \\ -4 & 5 \\ -6 & 3 \end{matrix} = 40-252-36+90+42-96$$

$$= -212$$

$x = \dfrac{53}{106} = \dfrac{1}{2}$ $\qquad x = \dfrac{D_x}{D}$

$y = \dfrac{318}{106} = 3$ $\qquad y = \dfrac{D_y}{D}$

$z = -\dfrac{212}{106} = -2$ $\qquad z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $\left(\frac{1}{2},\ 3,\ -2\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

Solution

$$D = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -3 & 4 \\ 4 & -2 & 3 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & -3 \\ 4 & -2 \end{matrix} = -18 - 16 - 6 + 12 + 16 + 9$$

$$= -3$$

$$D_x = \begin{vmatrix} 1 & -1 & 1 \\ 5 & -3 & 4 \\ 4 & -2 & 3 \end{vmatrix} \begin{matrix} 1 & -1 \\ 5 & -3 \\ 4 & -2 \end{matrix} = -9 - 16 - 10 + 12 + 8 + 15$$

$$= 0$$

$$D_y = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 5 & 4 \\ 4 & 4 & 3 \end{vmatrix} \begin{matrix} 2 & 1 \\ 3 & 5 \\ 4 & 4 \end{matrix} = 30 + 16 + 12 - 20 - 32 - 9$$

$$= -3$$

$$D_z = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -3 & 5 \\ 4 & -2 & 4 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & -3 \\ 4 & -2 \end{matrix} = -24 - 20 - 6 + 12 + 20 + 12$$

$$= -6$$

$x = -\frac{0}{3} = 0$ $\qquad x = \frac{D_x}{D}$

$y = \frac{-3}{-3} = 1$ $\qquad y = \frac{D_y}{D}$

$z = \frac{-6}{-3} = 2$ $\qquad z = \frac{D_z}{D}$

$\therefore$ ***Solution***: $(0,\ 1,\ 2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x-4y+4z=7 \\ x-y-2z=2 \\ 2x-3y+6z=5 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & -4 & 4 \\ 1 & -1 & -2 \\ 2 & -3 & 6 \end{vmatrix}\begin{matrix} 3 & -4 \\ 1 & -1 \\ 2 & -3 \end{matrix} = -18+16-12+8-18+24$$

$$\underline{=0}$$

$$D_z=\begin{vmatrix} 3 & -4 & 7 \\ 1 & -1 & 2 \\ 2 & -3 & 5 \end{vmatrix}\begin{matrix} 3 & -4 \\ 1 & -1 \\ 2 & -3 \end{matrix} = -15-16-21+14+18+20$$

$$\underline{=0}$$

$$-3\times(2)\quad \begin{cases} -3x+3y+6z=-6 \\ 2x-3y+6z=5 \end{cases}$$

$$-x+12z=-1$$

$$\underline{x=12z+1}$$

$$(2)\rightarrow\quad y=12z+1-2z-2$$

$$\underline{=10z-1}$$

$\therefore$ ***Solution***: $\underline{(12z+1,\ 10z-1,\ z)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix}\begin{matrix} 1 & -2 \\ 2 & -1 \\ -1 & 1 \end{matrix} = -1+2-2+1-1+4$$

$$\underline{=3}$$

$$D_x=\begin{vmatrix} 2 & -2 & -1 \\ 4 & -1 & 1 \\ 4 & 1 & 1 \end{vmatrix}\begin{matrix} 2 & -2 \\ 4 & -1 \\ 4 & 1 \end{matrix} = -2-8-4-4-2+8$$

$$\underline{=-12}$$

$$D_y = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 4 & 1 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & 4 \\ -1 & 4 \end{matrix} = 4 - 2 - 8 - 4 - 4 - 4$$

$$\underline{= -18|}$$

$$D_z = \begin{vmatrix} 1 & -2 & 2 \\ 2 & -1 & 4 \\ -1 & 1 & 4 \end{vmatrix} \begin{matrix} 1 & -2 \\ 2 & -1 \\ -1 & 1 \end{matrix} = -4 + 8 + 4 - 2 - 4 + 16$$

$$\underline{= 18|}$$

$x = -\frac{12}{3}\ \underline{= -4|}$ $\qquad x = \frac{D_x}{D}$

$y = -\frac{18}{3}\ \underline{= -6|}$ $\qquad y = \frac{D_y}{D}$

$z = \frac{18}{3}\ \underline{= 6|}$ $\qquad z = \frac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(-4,\ -6,\ 6)\ |}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + y + z = 3 \\ -y + 2z = 1 \\ -x + z = 0 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & 1 \end{vmatrix} \begin{matrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{matrix}\ \underline{= -4|}$$

$$D_x = \begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \begin{matrix} 3 & 1 \\ 1 & -1 \\ 0 & 0 \end{matrix}\ \underline{= -4|}$$

$$D_y = \begin{vmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{vmatrix} \begin{matrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{matrix}\ \underline{= -4|}$$

$$D_z = \begin{vmatrix} 1 & 1 & 3 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \end{vmatrix} \begin{matrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{matrix}\ \underline{= -4|}$$

$x = \frac{4}{4}\ \underline{= 1|}$ $\qquad x = \frac{D_x}{D}$

$y = \frac{4}{4} \underline{=1}$ $\qquad y = \frac{D_y}{D}$

$z = \frac{4}{4} \underline{=1}$ $\qquad z = \frac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(1,\ 1,\ 1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x + y + 3z = 14 \\ 7x + 5y + 8z = 37 \\ x + 3y + 2z = 9 \end{cases}$

Solution

$$D = \begin{vmatrix} 3 & 1 & 3 \\ 7 & 5 & 8 \\ 1 & 3 & 2 \end{vmatrix} \begin{matrix} 3 & 1 \\ 7 & 5 \\ 1 & 3 \end{matrix} = 30 + 8 + 62 - 15 - 72 - 14$$

$$\underline{= 0}$$

$$D_z = \begin{vmatrix} 3 & 1 & 14 \\ 7 & 5 & 37 \\ 1 & 3 & 9 \end{vmatrix} \begin{matrix} 3 & 1 \\ 7 & 5 \\ 1 & 3 \end{matrix} = 135 + 37 + 294 - 70 - 333 - 63$$

$$\underline{= 0}$$

$$\begin{array}{c} -3\times(1) \\ (3) \end{array} \begin{cases} -9x - 3y - 9z = -42 \\ \quad x + 3y + 2z = 9 \end{cases}$$

$$-8x - 7z = -33$$

$$\underline{x = -\frac{7}{8}z + \frac{33}{8}}$$

$$(1) \rightarrow \quad y = 14 - 3z - 3\left(-\frac{7}{8}z + \frac{33}{8}\right)$$

$$\underline{= \frac{13}{8} - \frac{3}{8}z}$$

$\therefore$ ***Solution***: $\underline{\left(\frac{33}{8} - \frac{7}{8}z,\ \ \frac{13}{8} - \frac{3}{8}z,\ \ z\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x-2y+z=7 \\ x+y+z=-2 \\ 4x+2y+z=3 \end{cases}$

Solution

$$D=\begin{vmatrix} 4 & -2 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix}\begin{matrix} 4 & -2 \\ 1 & 1 \\ 4 & 2 \end{matrix}$$

$$\underline{=-12}$$

$$D_x=\begin{vmatrix} 7 & -2 & 1 \\ -2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix}\begin{matrix} 7 & -2 \\ -2 & 1 \\ 3 & 2 \end{matrix}$$

$$\underline{=-24}$$

$$D_y=\begin{vmatrix} 4 & 7 & 1 \\ 1 & -2 & 1 \\ 4 & 3 & 1 \end{vmatrix}\begin{matrix} 4 & 7 \\ 1 & -2 \\ 4 & 3 \end{matrix}$$

$$\underline{=12}$$

$$D_z=\begin{vmatrix} 4 & -2 & 7 \\ 1 & 1 & -2 \\ 4 & 2 & 3 \end{vmatrix}\begin{matrix} 4 & -2 \\ 1 & 1 \\ 4 & 2 \end{matrix}$$

$$\underline{=36}$$

$x=\frac{24}{12}\ \underline{=2}$ $\qquad x=\frac{D_x}{D}$

$y=-\frac{12}{12}\ \underline{=-1}$ $\qquad y=\frac{D_y}{D}$

$z=-\frac{36}{12}\ \underline{=-3}$ $\qquad z=\frac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(2,\ -1,\ -3)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

Solution

$$D = \begin{vmatrix} 0 & 2 & -1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix} \begin{matrix} 0 & 2 \\ 1 & 2 \\ 2 & 3 \end{matrix}$$

$$= 1$$

$$D_x = \begin{vmatrix} 7 & 2 & -1 \\ 17 & 2 & 1 \\ -1 & 3 & 2 \end{vmatrix} \begin{matrix} 7 & 2 \\ 17 & 2 \\ -1 & 3 \end{matrix}$$

$$= -116$$

$$D_y = \begin{vmatrix} 0 & 7 & -1 \\ 1 & 17 & 1 \\ 2 & -1 & 2 \end{vmatrix} \begin{matrix} 0 & 7 \\ 1 & 17 \\ 2 & -1 \end{matrix}$$

$$= 35$$

$$D_z = \begin{vmatrix} 0 & 2 & 7 \\ 1 & 2 & 17 \\ 2 & 3 & -1 \end{vmatrix} \begin{matrix} 0 & 2 \\ 1 & 2 \\ 2 & 3 \end{matrix}$$

$$= 63$$

$x = -116$ $\qquad x = \dfrac{D_x}{D}$

$y = 35$ $\qquad y = \dfrac{D_y}{D}$

$z = 63$ $\qquad z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $(-116,\ 35,\ 63)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x-2y+z=-4 \\ 6x+4y-3z=-24 \\ x-2y+2z=1 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & -2 & 1 \\ 6 & 4 & -3 \\ 1 & -2 & 2 \end{vmatrix}\begin{matrix} 2 & -2 \\ 6 & 4 \\ 1 & -2 \end{matrix}$$

$$=18$$

$$D_x=\begin{vmatrix} -4 & -2 & 1 \\ -24 & 4 & -3 \\ 1 & -2 & 2 \end{vmatrix}\begin{matrix} -4 & -2 \\ -24 & 4 \\ 1 & -2 \end{matrix}$$

$$=-54$$

$$D_y=\begin{vmatrix} 2 & -4 & 1 \\ 6 & -24 & -3 \\ 1 & 1 & 2 \end{vmatrix}\begin{matrix} 2 & -4 \\ 6 & -24 \\ 1 & 1 \end{matrix}$$

$$=0$$

$$D_z=\begin{vmatrix} 2 & -2 & -4 \\ 6 & 4 & -24 \\ 1 & -2 & 1 \end{vmatrix}\begin{matrix} 2 & -2 \\ 6 & 4 \\ 1 & -2 \end{matrix}$$

$$=36$$

$x=-\frac{54}{18}$ $\qquad x=\frac{D_x}{D}$

$=-3$

$y=0$ $\qquad y=\frac{D_y}{D}$

$z=2$ $\qquad z=\frac{D_z}{D}$

$\therefore$ ***Solution***: $(-3,\ 0,\ 2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

Solution

$$D=\begin{vmatrix} 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \end{vmatrix}\begin{matrix} 9 & 3 \\ 16 & 4 \\ 25 & 5 \end{matrix}$$

$$=-2$$

$$D_x=\begin{vmatrix} 4 & 3 & 1 \\ 2 & 4 & 1 \\ 2 & 5 & 1 \end{vmatrix}\begin{matrix} 4 & 3 \\ 2 & 4 \\ 2 & 5 \end{matrix}$$

$$=-2$$

$$D_y=\begin{vmatrix} 9 & 4 & 1 \\ 16 & 2 & 1 \\ 25 & 2 & 1 \end{vmatrix}\begin{matrix} 9 & 4 \\ 16 & 2 \\ 25 & 2 \end{matrix}$$

$$=18$$

$$D_z=\begin{vmatrix} 9 & 3 & 4 \\ 16 & 4 & 2 \\ 25 & 5 & 2 \end{vmatrix}\begin{matrix} 9 & 3 \\ 16 & 4 \\ 25 & 5 \end{matrix}$$

$$=-44$$

$$x=\frac{-2}{-2} \qquad x=\frac{D_x}{D}$$

$$=1$$

$$y=\frac{18}{-2} \qquad y=\frac{D_y}{D}$$

$$=-9$$

$$z=\frac{-44}{-2} \qquad z=\frac{D_z}{D}$$

$$=22$$

$\therefore$ ***Solution***: $(1,\ -9,\ 22)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$

Solution

$$D = \left|\begin{array}{ccc} 2 & -1 & 2 \\ 1 & 2 & -3 \\ 3 & -1 & -4 \end{array}\right| \begin{array}{cc} 2 & -1 \\ 1 & 2 \\ 3 & -1 \end{array}$$

$$= -31$$

$$D_x = \left|\begin{array}{ccc} -8 & -1 & 2 \\ 9 & 2 & -3 \\ 3 & -1 & -4 \end{array}\right| \begin{array}{cc} -8 & -1 \\ 9 & 2 \\ 3 & -1 \end{array}$$

$$= 31$$

$$D_y = \left|\begin{array}{ccc} 2 & -8 & 2 \\ 1 & 9 & -3 \\ 3 & 3 & -4 \end{array}\right| \begin{array}{cc} 2 & -8 \\ 1 & 9 \\ 3 & 3 \end{array}$$

$$= -62$$

$$D_z = \left|\begin{array}{ccc} 2 & -1 & -8 \\ 1 & 2 & 9 \\ 3 & -1 & 3 \end{array}\right| \begin{array}{cc} 2 & -1 \\ 1 & 2 \\ 3 & -1 \end{array}$$

$$= 62$$

$$x = -\frac{31}{31} \qquad x = \frac{D_x}{D}$$

$$= -1$$

$$y = \frac{62}{31} \qquad y = \frac{D_y}{D}$$

$$= 2$$

$$z = -\frac{62}{31} \qquad z = \frac{D_z}{D}$$

$$= -2$$

$\therefore$ ***Solution***: $(-1,\ 2,\ -2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x \qquad -3z = -5 \\ 2x - y + 2z = 16 \\ 7x - 3y - 5z = 19 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 0 & -3 \\ 2 & -1 & 2 \\ 7 & -3 & -5 \end{vmatrix} \begin{matrix} 1 & 0 \\ 2 & -1 \\ 7 & -3 \end{matrix}$$

$$= 8$$

$$D_x = \begin{vmatrix} -5 & 0 & -3 \\ 16 & -1 & 2 \\ 19 & -3 & -5 \end{vmatrix} \begin{matrix} -5 & 0 \\ 16 & -1 \\ 19 & -3 \end{matrix}$$

$$= 32$$

$$D_y = \begin{vmatrix} 1 & -5 & -3 \\ 2 & 16 & 2 \\ 7 & 19 & -5 \end{vmatrix} \begin{matrix} 1 & -5 \\ 2 & 16 \\ 7 & 19 \end{matrix}$$

$$= -16$$

$$D_z = \begin{vmatrix} 1 & 0 & -5 \\ 2 & -1 & 16 \\ 7 & -3 & 19 \end{vmatrix} \begin{matrix} 1 & 0 \\ 2 & -1 \\ 7 & -3 \end{matrix}$$

$$= 24$$

$x = \frac{32}{8}$ $\qquad x = \frac{D_x}{D}$

$= 4$

$y = -\frac{16}{8}$ $\qquad y = \frac{D_y}{D}$

$= -2$

$z = \frac{24}{8}$ $\qquad z = \frac{D_z}{D}$

$= 3$

∴ ***Solution***: $(4, \ -2, \ 3)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+2y-z=5 \\ 2x-y+3z=0 \\ 2y+z=1 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 0 & 2 & 1 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{matrix}$$

$$= -15$$

$$D_x = \begin{vmatrix} 5 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 2 & 1 \end{vmatrix} \begin{matrix} 5 & 2 \\ 0 & -1 \\ 1 & 2 \end{matrix}$$

$$= -30$$

$$D_y = \begin{vmatrix} 1 & 5 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix} \begin{matrix} 1 & 5 \\ 2 & 0 \\ 0 & 1 \end{matrix}$$

$$= -15$$

$$D_z = \begin{vmatrix} 1 & 2 & 5 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{matrix}$$

$$= 15$$

$x = \frac{30}{15}$ $\qquad x = \frac{D_x}{D}$

$= 2$

$y = \frac{15}{15}$ $\qquad y = \frac{D_y}{D}$

$= 1$

$z = -\frac{15}{15}$ $\qquad z = \frac{D_z}{D}$

$= -1$

$\therefore$ ***Solution***: $(2,\ 1,\ -1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+y+z=6 \\ 3x+4y-7z=1 \\ 2x-y+3z=5 \end{cases}$

Solution

$$D=\left|\begin{array}{ccc}1 & 1 & 1 \\ 3 & 4 & -7 \\ 2 & -1 & 3\end{array}\right|\begin{array}{cc}1 & 1 \\ 3 & 4 \\ 2 & -1\end{array}$$

$$=-29$$

$$D_x=\left|\begin{array}{ccc}6 & 1 & 1 \\ 1 & 4 & -7 \\ 5 & -1 & 3\end{array}\right|\begin{array}{cc}6 & 1 \\ 1 & 4 \\ 5 & -1\end{array}$$

$$=-29$$

$$D_y=\left|\begin{array}{ccc}1 & 6 & 1 \\ 3 & 1 & -7 \\ 2 & 5 & 3\end{array}\right|\begin{array}{cc}1 & 6 \\ 3 & 1 \\ 2 & 5\end{array}$$

$$=-87$$

$$D_z=\left|\begin{array}{ccc}1 & 2 & 6 \\ 2 & -1 & 1 \\ 0 & 2 & 5\end{array}\right|\begin{array}{cc}1 & 2 \\ 2 & -1 \\ 0 & 2\end{array}$$

$$=-58$$

$x=\frac{29}{29}$ $\qquad\qquad$ $x=\frac{D_x}{D}$

$=1$

$y=\frac{87}{29}$ $\qquad\qquad$ $y=\frac{D_y}{D}$

$=3$

$z=\frac{58}{29}$ $\qquad\qquad$ $z=\frac{D_z}{D}$

$=2$

$\therefore$ ***Solution***: $(1,\ 3,\ 2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y+3z=3 \\ 4x-5y+7z=1 \\ 2x+3y-2z=6 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 2 & 3 \\ 4 & -5 & 7 \\ 2 & 3 & -2 \end{vmatrix}\begin{matrix} 3 & 2 \\ 4 & -5 \\ 2 & 3 \end{matrix}$$

$$=77$$

$$D_x=\begin{vmatrix} 3 & 2 & 3 \\ 1 & -5 & 7 \\ 6 & 3 & -2 \end{vmatrix}\begin{matrix} 3 & 2 \\ 1 & -5 \\ 6 & 3 \end{matrix}$$

$$=154$$

$$D_y=\begin{vmatrix} 3 & 3 & 3 \\ 4 & 1 & 7 \\ 2 & 6 & -2 \end{vmatrix}\begin{matrix} 3 & 3 \\ 4 & 1 \\ 2 & 6 \end{matrix}$$

$$=0$$

$$D_z=\begin{vmatrix} 3 & 2 & 3 \\ 4 & -5 & 1 \\ 2 & 3 & 6 \end{vmatrix}\begin{matrix} 3 & 2 \\ 4 & -5 \\ 2 & 3 \end{matrix}$$

$$=-77$$

$x=\frac{154}{77}=2$ $\qquad x=\frac{D_x}{D}$

$=2$

$y=0$ $\qquad y=\frac{D_y}{D}$

$z=-\frac{77}{77}$ $\qquad z=\frac{D_z}{D}$

$=-1$

$\therefore$ ***Solution***: $(2,\ 0,\ -1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x+5y \qquad = 2 \\ 11x+y+2z=3 \\ x+5y+2z=1 \end{cases}$

Solution

$$D = \begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} \begin{matrix} 4 & 5 \\ 11 & 1 \\ 1 & 5 \end{matrix}$$

$$= -132$$

$$D_x = \begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} \begin{matrix} 2 & 5 \\ 3 & 1 \\ 1 & 5 \end{matrix}$$

$$= -36$$

$$D_y = \begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} \begin{matrix} 4 & 2 \\ 11 & 3 \\ 1 & 1 \end{matrix}$$

$$= -24$$

$$D_z = \begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} \begin{matrix} 4 & 5 \\ 11 & 1 \\ 1 & 5 \end{matrix}$$

$$= 12$$

$$x = \frac{36}{132} \qquad\qquad x = \frac{D_x}{D}$$

$$= \frac{3}{11}$$

$$y = \frac{24}{132} \qquad\qquad y = \frac{D_y}{D}$$

$$= \frac{2}{11}$$

$$z = -\frac{12}{132} \qquad\qquad z = \frac{D_z}{D}$$

$$= -\frac{1}{11}$$

$\therefore$ ***Solution***: $\left(\frac{3}{11},\ \frac{2}{11},\ -\frac{1}{11}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-4y+z=6 \\ 4x-y+2z=-1 \\ 2x+2y-3z=-20 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix}\begin{matrix} 1 & -4 \\ 4 & -1 \\ 2 & 2 \end{matrix}$$

$$= -55$$

$$D_x=\begin{vmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{vmatrix}\begin{matrix} 6 & -4 \\ -1 & -1 \\ -20 & 2 \end{matrix}$$

$$= 144$$

$$D_y=\begin{vmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{vmatrix}\begin{matrix} 1 & 6 \\ 4 & -1 \\ 2 & -20 \end{matrix}$$

$$= 61$$

$$D_z=\begin{vmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{vmatrix}\begin{matrix} 1 & -4 \\ 4 & -1 \\ 2 & 2 \end{matrix}$$

$$= -230$$

$x = -\dfrac{144}{55}$ $\qquad x = \dfrac{D_x}{D}$

$y = -\dfrac{61}{55}$ $\qquad y = \dfrac{D_y}{D}$

$z = \dfrac{230}{55}$ $\qquad z = \dfrac{D_z}{D}$

$= \dfrac{46}{11}$

$\therefore$ ***Solution***: $\left(-\dfrac{144}{55},\ -\dfrac{61}{55},\ \dfrac{46}{11}\right)$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 2x - \ \ y + \ \ z = -1 \\ 3x + 4y - \ z = -1 \\ 4x - \ \ y + 2z = -1 \end{cases}$$

Solution

$$D = \begin{array}{|ccc|cc} 2 & -1 & 1 & 2 & -1 \\ 3 & 4 & -1 & 3 & 4 \\ 4 & -1 & 2 & 4 & -1 \end{array}$$

$$= 5$$

$$D_x = \begin{array}{|ccc|cc} -1 & -1 & 1 & -1 & -1 \\ -1 & 4 & -1 & -1 & 4 \\ -1 & -1 & 2 & -1 & -1 \end{array}$$

$$= -5$$

$$D_y = \begin{array}{|ccc|cc} 2 & -1 & 1 & 2 & -1 \\ 3 & -1 & -1 & 3 & -1 \\ 4 & -1 & 2 & 4 & -1 \end{array}$$

$$= 5$$

$$D_z = \begin{array}{|ccc|cc} 2 & -1 & -1 & 2 & -1 \\ 3 & 4 & -1 & 3 & 4 \\ 4 & -1 & -1 & 4 & -1 \end{array}$$

$$= 10$$

$$x = \frac{-5}{5} \qquad\qquad x = \frac{D_x}{D}$$

$$= -1$$

$$y = \frac{5}{5} \qquad\qquad y = \frac{D_y}{D}$$

$$= 1$$

$$z = \frac{10}{5} \qquad\qquad z = \frac{D_z}{D}$$

$$= 2$$

$\therefore$ ***Solution***: $(-1,\ 1,\ 2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 = -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 = 14 \\ -x_1 + x_2 + 3x_3 + x_4 = 11 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -4 \end{cases}$

Solution

$$D = \begin{vmatrix} -1 & -4 & 2 & 1 \\ 2 & -1 & 7 & 9 \\ -1 & 1 & 3 & 1 \\ -1 & -2 & 1 & -4 \end{vmatrix}$$

$$\underline{= -243}$$

$$D_1 = \begin{vmatrix} -32 & -4 & 2 & 1 \\ 14 & -1 & 7 & 9 \\ 11 & 1 & 3 & 1 \\ -4 & -2 & 1 & -4 \end{vmatrix}$$

$$\underline{= -2115}$$

$$D_2 = \begin{vmatrix} -1 & -32 & 2 & 1 \\ 2 & 14 & 7 & 9 \\ -1 & 11 & 3 & 1 \\ -1 & -4 & 1 & -4 \end{vmatrix}$$

$$\underline{= -1834}$$

$$D_3 = \begin{vmatrix} -1 & -4 & -32 & 1 \\ 2 & -1 & 14 & 9 \\ -1 & 1 & 11 & 1 \\ -1 & -2 & -4 & -4 \end{vmatrix}$$

$$\underline{= -1279}$$

$$D_4 = \begin{vmatrix} -1 & -4 & 2 & -32 \\ 2 & -1 & 7 & 14 \\ -1 & 1 & 3 & 11 \\ -1 & -2 & 1 & -4 \end{vmatrix}$$

$$\underline{= 883}$$

$$x_1 = \frac{-2115}{-243}$$

$$\underline{= \frac{235}{27}}$$

$$x_2 = \frac{-1834}{-243}$$
$$= \boxed{\frac{1834}{243}}$$

$$x_3 = \frac{-1279}{-243}$$
$$= \boxed{\frac{1279}{243}}$$

$$\boxed{x_4 = -\frac{883}{243}}$$

$\therefore$ ***Solution***: $\boxed{\left(\frac{235}{27}, \frac{1834}{243}, \frac{1279}{243}, -\frac{883}{243}\right)}$

Exercise

Find the quadratic function $f(x) = ax^2 + bx + c$ for which $f(1) = -10, \quad f(-2) = -31, \quad f(2) = -19$. What is the function?

Solution

$$f(1) = a(1)^2 + b(1) + c \quad \Rightarrow \quad -10 = a + b + c$$
$$f(-2) = a(-2)^2 + b(-2) + c \quad \Rightarrow \quad -31 = 4a - 2b + c$$
$$f(2) = a(2)^2 + b(2) + c \quad \Rightarrow \quad -19 = 4a + 2b + c$$

$$\begin{cases} a + b + c = -10 \\ 4a - 2b + c = -31 \\ 4a + 2b + c = -19 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 4 & -2 & 1 \\ 4 & 2 & 1 \end{vmatrix} = 12 \qquad D_a = \begin{vmatrix} -10 & 1 & 1 \\ -31 & -2 & 1 \\ -19 & 2 & 1 \end{vmatrix} = -48$$

$$D_b = \begin{vmatrix} 1 & -10 & 1 \\ 4 & -31 & 1 \\ 4 & -19 & 1 \end{vmatrix} = 36 \qquad D_c = \begin{vmatrix} 1 & 1 & -10 \\ 4 & -2 & -31 \\ 4 & 2 & -19 \end{vmatrix} = -108$$

$$a = \frac{D_a}{D} = \frac{-48}{12} \boxed{= -4}$$

$$b = \frac{D_b}{D} = \frac{36}{12} \boxed{= 3}$$

$$c = \frac{D_c}{D} = \frac{-108}{12} \underline{= -9|}$$

$\therefore$ ***Solution***: $\underline{f(x) = -x^2 + 3x - 9}$

Exercise

You wish to mix candy worth \$3.44 per pound with candy worth \$9.96 per pound to form 24 pounds of a mixture worth \$8.33 per pound.

a) Write the system equations?

b) How many pounds of each candy should you use?

Solution

Let x: total pounds of \$3.44 candy

y: total pounds of \$9.96 candy

a)
$$\begin{cases} x + y = 24 \\ 3.44x + 9.96y = 8.33(24) \end{cases}$$

$$\begin{cases} x + y = 24 \\ 344x + 996y = 19,992 \end{cases}$$

$$\begin{cases} x + y = 24 \\ 86x + 249y = 4,998 \end{cases}$$

b) $D = \begin{vmatrix} 1 & 1 \\ 86 & 249 \end{vmatrix} = 163$

$$D_x = \begin{vmatrix} 24 & 1 \\ 4998 & 249 \end{vmatrix} = 978$$

$$D_y = \begin{vmatrix} 1 & 24 \\ 86 & 4998 \end{vmatrix} = 2,934$$

Total pounds of \$3.44 candy: $\frac{978}{163} \underline{= 6 \ \ lbs|}$

Total pounds of \$9.96 candy: $\frac{2,934}{163} \underline{= 18 \ \ lbs|}$

Exercise

Anne and Nancy use a metal alloy that is 17.76% copper to make jewelry. How many ounces of a 15% alloy must be mixed with a 19% alloy to form 100 ounces of the desired alloy?

Solution

Let x: total ounces 15%

y: total ounces of 19%

$$\begin{cases} x+y=100 \\ 15x+19y=17.76(100) \end{cases}$$

$$\begin{cases} x+y=100 \\ 15x+19y=1776 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 \\ 15 & 19 \end{vmatrix}=4 \qquad D_x=\begin{vmatrix} 100 & 1 \\ 1776 & 19 \end{vmatrix}=124$$

$\therefore$ Total ounces 15%: $\frac{124}{4} = 31\ \ ounces$

Exercise

A company makes 3 types of cable. Cable ***A*** requires 3 black, 3 white, and 2 red wires. ***B*** requires 1 black, 2 white, and 1 red. ***C*** requires 2 black, 1 white, and 2 red. They used 95 black, 100 white and 80 red wires.

a) Write the system equations?

b) How many of each cable were made?

Solution

Let x: Cable ***A***

y: Cable ***B***

z: Cable ***C***

a) $$\begin{cases} 3x+y+2z=95 \\ 3x+2y+z=100 \\ 2x+y+2z=80 \end{cases}$$

b) $$D=\begin{vmatrix} 3 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix}\begin{matrix} 3 & 1 \\ 3 & 2 \\ 2 & 1 \end{matrix} = 3$$

$$D_x=\begin{vmatrix} 95 & 1 & 2 \\ 100 & 2 & 1 \\ 80 & 1 & 2 \end{vmatrix}\begin{matrix} 95 & 1 \\ 100 & 2 \\ 80 & 1 \end{matrix} = 45$$

$$D_y = \begin{vmatrix} 3 & 95 & 2 \\ 3 & 100 & 1 \\ 2 & 80 & 2 \end{vmatrix} \begin{matrix} 3 & 95 \\ 3 & 100 \\ 2 & 80 \end{matrix} \underline{=60|}$$

$$D_z = \begin{vmatrix} 3 & 1 & 95 \\ 3 & 2 & 100 \\ 2 & 1 & 80 \end{vmatrix} \begin{matrix} 3 & 1 \\ 3 & 2 \\ 2 & 1 \end{matrix} \underline{=45|}$$

$x = \frac{45}{3} \underline{=15|}$ $\qquad x = \frac{D_x}{D}$

$y = \frac{60}{3} \underline{=20|}$ $\qquad y = \frac{D_y}{D}$

$z = \frac{45}{3} \underline{=15|}$ $\qquad z = \frac{D_z}{D}$

∴ ***Solution***: 15 cable ***A*** 20 cable ***B*** 15 cable ***C***

Exercise

A basketball fieldhouse seats 15,000. Courtside seats sell for \$8.00, end zone for \$6.00, and balcony for \$5.00. Total for a sell-out is \$86,000. If half the courtside and balcony and all end zone seats are sold, ticket sales total \$49,000.

a) Write the system equations?

b) How many of each type of seat are there?

Solution

Let x: Courtside seats

y: end zone

z: balcony

a)
$$\begin{cases} x+y+z=15,000 \\ 8x+6y+5z=86,000 \\ \frac{1}{2}(8x)+6y+\frac{1}{2}(5z)=49,000 \end{cases}$$

$$\begin{cases} x+y+z=15,000 \\ 8x+6y+5z=86,000 \\ 8x+12y+5z=98,000 \end{cases}$$

b)
$$D = \begin{vmatrix} 1 & 1 & 1 \\ 8 & 6 & 5 \\ 8 & 12 & 5 \end{vmatrix} \begin{matrix} 1 & 1 \\ 8 & 6 \\ 8 & 12 \end{matrix} \underline{=18|}$$

$$D_x = \begin{vmatrix} 15,000 & 1 & 1 \\ 86,000 & 6 & 5 \\ 98,000 & 12 & 5 \end{vmatrix} \begin{matrix} 15,000 & 1 \\ 86,000 & 6 \\ 98,000 & 12 \end{matrix} \underline{=54,000|}$$

$$D_y = \begin{vmatrix} 1 & 15,000 & 1 \\ 8 & 86,000 & 5 \\ 8 & 98,000 & 5 \end{vmatrix} \begin{matrix} 1 & 15,000 \\ 8 & 86,000 \\ 8 & 98,000 \end{matrix} \underline{=36,000|}$$

$$D_z = \begin{vmatrix} 1 & 1 & 15,000 \\ 8 & 6 & 86,000 \\ 8 & 12 & 98,000 \end{vmatrix} \begin{matrix} 1 & 1 \\ 8 & 6 \\ 8 & 12 \end{matrix} \underline{=180,000|}$$

$$x = \frac{54,000}{18} \underline{=3,000|} \qquad x = \frac{D_x}{D}$$

$$y = \frac{36,000}{18} \underline{=2,000|} \qquad y = \frac{D_y}{D}$$

$$z = \frac{180,000}{18} \underline{=10,000|} \qquad z = \frac{D_z}{D}$$

∴ ***Solution***: **3,000** Courtside **2,000** End zone **10,000** Balcony

Exercise

A movie theater charges \$9.00 for adults and \$7.00 for senior citizens. On a day when 325 people paid admission, the total receipts were \$2,495.

a) Write the system equations?

b) How many who paid were adults? How many were seniors?

Solution

Let x: Adults

y: Senior citizens

a) $\begin{cases} x+y=325 \\ 9x+7y=2,495 \end{cases}$

b) $D = \begin{vmatrix} 1 & 1 \\ 9 & 7 \end{vmatrix} \underline{=-2|}$

$$D_x = \begin{vmatrix} 325 & 1 \\ 2,495 & 7 \end{vmatrix} \underline{=-220|}$$

$$D_y = \begin{vmatrix} 1 & 325 \\ 9 & 2,495 \end{vmatrix} \underline{=430|}$$

$x = \frac{220}{2} = \underline{110|}$ $\qquad x = \frac{D_x}{D}$

$y = \frac{430}{2} = \underline{215|}$ $\qquad y = \frac{D_y}{D}$

∴ ***Solution***: **110** Adults **215** Senior citizens

Exercise

A Broadway theater has 500 seats, divided into orchestra, main, and balcony seating. Orchestra seats sell for \$150, main seats for \$135, and balcony seats for \$110. If all the seats are sold, the gross revenue to the theater is \$64,250. If all the main and balcony seats are sold, but only half the orchestra seats are sold, the gross revenue is \$56,750.

a) Write the system equations?

b) How many of each kind of seat are there?

Solution

Let x: Numbers of orchestra seats

y: Numbers of main seats

z: Numbers of balcony seats

a)
$$\begin{cases} x + y + z = 500 \\ 150x + 135y + 110z = 64,250 \\ \frac{1}{2}(150)x + 135y + 110z = 56,750 \end{cases}$$

$$\begin{cases} x + y + z = 500 \\ 30x + 27y + 22z = 12,850 \\ 15x + 27y + 22z = 11,350 \end{cases}$$

b)
$$D = \begin{vmatrix} 1 & 1 & 1 \\ 30 & 27 & 22 \\ 15 & 27 & 22 \end{vmatrix} = \underline{75|}$$

$$D_x = \begin{vmatrix} 500 & 1 & 1 \\ 12850 & 27 & 22 \\ 11350 & 27 & 22 \end{vmatrix} = \underline{7,500|}$$

$$D_y = \begin{vmatrix} 1 & 500 & 1 \\ 30 & 12,850 & 22 \\ 15 & 11,350 & 22 \end{vmatrix} = \underline{15,750|}$$

$$D_z = \begin{vmatrix} 1 & 1 & 500 \\ 30 & 27 & 12,850 \\ 15 & 27 & 11,350 \end{vmatrix} = 14,250$$

$$x = \frac{7,500}{75} = 100 \qquad x = \frac{D_x}{D}$$

$$y = \frac{15,750}{75} = 210 \qquad y = \frac{D_y}{D}$$

$$z = \frac{14,250}{75} = 190 \qquad z = \frac{D_z}{D}$$

∴ ***Solution***: There are **100** orchestra seats, **210** main seats, and **190** balcony seats.

Exercise

A movie theater charges \$11 for adults, \$6.50 for children, and \$9 for senior citizens. One day the theater sold 405 tickets and collected \$3,315 in receipts. Twice as many children's tickets were sold as adult tickets.

a) Write the system equations?

b) How many adults, children, and senior citizens went to the theater that day?

Solution

Let x: Numbers of adults

y: Numbers of children

z: Numbers of senior citizens

a)
$$\begin{cases} x + y + z = 405 \\ 11x + 6.5y + 9z = 3,315 \\ y = 2x \end{cases}$$

b)
$$\begin{cases} 3x + z = 405 \\ 24x + 9z = 3,315 \end{cases}$$

$$D = \begin{vmatrix} 3 & 1 \\ 24 & 9 \end{vmatrix} = 3$$

$$D_x = \begin{vmatrix} 405 & 1 \\ 3,315 & 9 \end{vmatrix} = 330$$

$$D_y = \begin{vmatrix} 3 & 405 \\ 24 & 3,315 \end{vmatrix} = 225$$

$$x=\frac{330}{3}\ \underline{=110|} \qquad x=\frac{D_x}{D}$$

$$z=\frac{225}{3}\ \underline{=75|} \qquad z=\frac{D_z}{D}$$

$$y=2(110)\ \underline{=220|}$$

∴ ***Solution***: There are **110** adults, **220** children, and **75** senior citizens.

Exercise

Emma has \$20,000 to invest. As her financial planner, you recommend that she diversify into three investements: Treasure bills that yield 5% simple interest. Treasury bonds tht yield 7% simple interest, and corporate bonds that yield 10% simple interest. Emma wishes to earn \$1,390 per year in income. Also, Emma wants her investment in Treasury bills to be \$3,000 more than her investment in corporate bonds. How much money should Emma place in each investement?

Solution

Let x: Amount in Treasure bills.

y: Amount in Treasury bonds.

z: Amount in corporate bonds.

$$\begin{cases} x+y+z=20,000 \\ .05x+.07y+.1z=1,390 \\ x=3,000+z \end{cases}$$

$$\begin{cases} x+y+z=20,000 \\ 5x+7y+10z=139,000 \\ x-z=3,000 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 & 1 \\ 5 & 7 & 10 \\ 1 & 0 & -1 \end{vmatrix}\ \underline{=1|}$$

$$D_x=\begin{vmatrix} 20,000 & 1 & 1 \\ 139,000 & 7 & 10 \\ 3,000 & 0 & -1 \end{vmatrix}\ \underline{=8,000|}$$

$$D_y=\begin{vmatrix} 1 & 20,000 & 1 \\ 5 & 139,000 & 10 \\ 1 & 3,000 & -1 \end{vmatrix}\ \underline{=7,000|}$$

$$D_z=\begin{vmatrix} 1 & 1 & 20,000 \\ 5 & 7 & 139,000 \\ 1 & 0 & 3,000 \end{vmatrix}\ \underline{=5,000|}$$

$\therefore$ ***Solution***: Emma should invest **$8,000** in Treasure bills

$7,000 in Treasury bonds

$5,000 in corporate bonds.

Exercise

A person invested $17,000 for one year, part at 10%, part at 12%, and the remainder at 15%. The total annual income from these investements was $2,110. The amount of money invested at 12% was $1,000 less than the amounts invested at 10% and 15% combined. Find the amount invested at each rate.

Solution

Let $x =$ Amount invested at 10%

Let $y =$ Amount invested at 12%

Let $z =$ Amount invested at 15%

$$\begin{cases} x+y+z=17{,}000 \\ .1x+.12y+.15z=2{,}110 \\ y=x+z-1{,}000 \end{cases}$$

$$\begin{cases} x+y+z=17{,}000 \\ 10x+12y+15z=211{,}000 \\ x-y+z=1{,}000 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 & 1 \\ 10 & 12 & 15 \\ 1 & -1 & 1 \end{vmatrix} \underline{=10|}$$

$$D_x=\begin{vmatrix} 17{,}000 & 1 & 1 \\ 211{,}000 & 12 & 15 \\ 1{,}000 & -1 & 1 \end{vmatrix} \underline{=40{,}000|}$$

$$D_y=\begin{vmatrix} 1 & 17{,}000 & 1 \\ 10 & 211{,}000 & 15 \\ 1 & 1{,}000 & 1 \end{vmatrix} \underline{=80{,}000|}$$

$$D_z=\begin{vmatrix} 1 & 1 & 17{,}000 \\ 10 & 12 & 211{,}000 \\ 1 & -1 & 1{,}000 \end{vmatrix} \underline{=50{,}000|}$$

$$x=\frac{40{,}000}{10} \underline{=4{,}000|} \qquad x=\frac{D_x}{D}$$

$$y=\frac{80{,}000}{10} \underline{=8{,}000|} \qquad y=\frac{D_y}{D}$$

$$z = \frac{50,000}{10} = \underline{5,000} \qquad\qquad z = \frac{D_z}{D}$$

∴ ***Solution***: should invest **\$4,000** invested at 10%

\$8,000 invested at 12%

\$5,000 invested at 15%.

Exercise

At a production, 400 tickets were sold. The ticket prices were \$8, \$10, and \$12, and the total income from ticket sales was \$3,700. How many tickets of each type were sold if the combined number of \$8 and \$10 tickets sold was 7 times the number of \$12 tickets sold?

Solution

Let $x =$ Numbers of tickets sold at \$8
Let $y =$ Numbers of tickets sold at \$10
Let $z =$ Numbers of tickets sold at \$12

$$\begin{cases} x + y + z = 400 \\ 8x + 10y + 12z = 3,700 \\ x + y = 7z \end{cases}$$

$$\begin{cases} x + y + z = 400 \\ 4x + 5y + 6z = 1,850 \\ x + y - 7z = 0 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 5 & 6 \\ 1 & 1 & -7 \end{vmatrix} = \underline{-8}$$

$$D_x = \begin{vmatrix} 400 & 1 & 1 \\ 1,850 & 5 & 6 \\ 0 & 1 & -7 \end{vmatrix} = \underline{-1,600}$$

$$D_y = \begin{vmatrix} 1 & 400 & 1 \\ 4 & 1850 & 6 \\ 1 & 0 & -7 \end{vmatrix} = \underline{-1,200}$$

$$D_z = \begin{vmatrix} 1 & 1 & 400 \\ 4 & 5 & 1,850 \\ 1 & 1 & 0 \end{vmatrix} = \underline{-400}$$

$$x = \frac{1600}{8} = \underline{200} \qquad\qquad x = \frac{D_x}{D}$$

$$y = \frac{1200}{8} \underline{= 150|} \qquad y = \frac{D_y}{D}$$

$$z = \frac{400}{8} \underline{= 50|} \qquad z = \frac{D_z}{D}$$

∴ ***Solution***: **200** tickets sold at \$8

150 tickets sold at \$10

50 tickets sold at \$12

Exercise

A certain brand of razor blades comes in packages if 6, 12, and 24 blades, costing \$2, \$3, and \$4 per package, respectively. A store sold 12 packages containing a total of 162 razor blades and took in \$35. How many packages of each type were sold?

Solution

Let $x =$ Numbers of packages sold at \$2
Let $y =$ Numbers of packages sold at \$3
Let $z =$ Numbers of packages sold at \$4

$$\begin{cases} x + y + z = 12 \\ 2x + 3y + 4z = 35 \\ 6x + 12y + 24z = 162 \end{cases}$$

$$\begin{cases} x + y + z = 12 \\ 2x + 3y + 4z = 35 \\ x + 2y + 4z = 27 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & 2 & 4 \end{vmatrix} \underline{= 1|}$$

$$D_x = \begin{vmatrix} 12 & 1 & 1 \\ 35 & 3 & 4 \\ 27 & 2 & 4 \end{vmatrix} \underline{= 5|}$$

$$D_y = \begin{vmatrix} 1 & 12 & 1 \\ 2 & 35 & 4 \\ 1 & 27 & 4 \end{vmatrix} \underline{= 3|}$$

$$D_z = \begin{vmatrix} 1 & 1 & 12 \\ 2 & 3 & 35 \\ 1 & 2 & 27 \end{vmatrix} \underline{= 4|}$$

$$x = \frac{5}{1} \underline{= 5|} \qquad x = \frac{D_x}{D}$$

$$y = \frac{3}{1} \underline{= 3|} \qquad y = \frac{D_y}{D}$$

$$z = \frac{4}{1} \underline{= 4|} \qquad z = \frac{D_z}{D}$$

∴ ***Solution***: **5** packages sold at \$2
3 packages sold at \$3
4 packages sold at \$4

Exercise

A store sells cashews for \$5.00 per pound and peanuts for \$1.50 per pound. The manager decides to mix 30 pounds of peanuts with some cashews and sell the mixture for \$3.00 per pound.

a) Write the system equations?

b) How many pounds of cashews should be mixed with peanuts so that the mixture will produce the same revenue as selling the nuts separately?

Solution

Let x: pounds of cashews

y: pounds of in the mixture

a) $\begin{cases} x + 30 = y \\ 5x + \frac{3}{2}(30) = 3y \end{cases}$

$$\begin{cases} x - y = -30 \\ 5x - 3y = -45 \end{cases}$$

b) $D = \begin{vmatrix} 1 & -1 \\ 5 & -3 \end{vmatrix} \underline{= 2|}$

$$D_x = \begin{vmatrix} -30 & -1 \\ -45 & -3 \end{vmatrix} \underline{= 45|}$$

$$\underline{x = \frac{90}{7}|} \qquad x = \frac{D_x}{D}$$

$$\underline{y = \frac{120}{7}|} \qquad y = \frac{D_y}{D}$$

∴ ***Solution***: $\frac{45}{2} = 22.5$ pounds of cashews

Exercise

A wireless store takes presale orders for a new smartphone and tablet. He gets 340 preorders for the smartphone and 250 preorders for the tablet. The combined value of the preorders is \$270,500.00. If the price of a smartphone and tablet together is \$965, how much does each device cost?

Solution

Let x: Cost of a smartphone

y: Cost of a tablet

$$\begin{cases} 340x + 250y = 270,500 \\ x + y = 965 \end{cases}$$

$$\begin{cases} 34x + 25y = 27,050 \\ x + y = 965 \end{cases}$$

$$D = \begin{vmatrix} 34 & 25 \\ 1 & 1 \end{vmatrix} = 9$$

$$D_x = \begin{vmatrix} 27,050 & 25 \\ 965 & 1 \end{vmatrix} = 2,925$$

$$D_y = \begin{vmatrix} 34 & 27,050 \\ 1 & 965 \end{vmatrix} = 5,760$$

$$x = \frac{2,925}{9} = \$325 \qquad x = \frac{D_x}{D}$$

$$y = \frac{5,760}{9} = \$640 \qquad y = \frac{D_y}{D}$$

∴ ***Solution***: Cost of a smartphone is \$325

Cost of a tablet is \$640

Exercise

A restaurant manager wants to purchase 200 sets of dishes. One design costs \$25 per set, and another costs \$45 per set. If she has only \$7400 to spend, how many sets of each design should be order?

Solution

Let x: Number of sets for \$25 set.

y: Number of sets for \$45 set.

$$\begin{cases} 25x + 45y = 7,400 \\ x + y = 200 \end{cases}$$

$$\begin{cases} 5x+9y=1,480 \\ x+y=200 \end{cases}$$

$$D=\begin{vmatrix} 5 & 9 \\ 1 & 1 \end{vmatrix} \underline{=-4|}$$

$$D_x=\begin{vmatrix} 1480 & 9 \\ 200 & 1 \end{vmatrix} \underline{=-320|}$$

$$D_y=\begin{vmatrix} 5 & 1480 \\ 1 & 200 \end{vmatrix} \underline{=-480|}$$

$$x=\frac{320}{4} \underline{=80|} \qquad x=\frac{D_x}{D}$$

$$y=\frac{480}{4} \underline{=120|} \qquad y=\frac{D_y}{D}$$

∴ ***Solution***: 80 sets for \$25 set.

120 sets for \$45 set.

Exercise

One group of people purchased 10 hot dogs and 5 soft drinks at a cost of \$35.00. A second bought 7 hot dogs and 4 soft drinks at a cost of \$25.25. What is the cost of a single hot dog and a single soft drink?

Solution

Let x: Cost of a hot dog.

y: Cost of a drink

$$\begin{cases} 10x+5y=35 \\ 7x+4y=25.25 \end{cases}$$

$$\begin{cases} 2x+y=7 \\ 700x+400y=2,525 \end{cases}$$

$$D=\begin{vmatrix} 2 & 1 \\ 700 & 400 \end{vmatrix} \underline{=100|}$$

$$D_x=\begin{vmatrix} 7 & 1 \\ 2,525 & 400 \end{vmatrix} =275$$

$$D_y=\begin{vmatrix} 2 & 7 \\ 700 & 2,525 \end{vmatrix} \underline{=150|}$$

$$x=\frac{275}{100} \underline{=2.75|} \qquad x=\frac{D_x}{D}$$

$y = \frac{150}{100} \underline{= 1.5|}$ $\qquad$ $y = \frac{D_y}{D}$

$\therefore$ ***Solution***: Cost of a hot dog is $2.75

Cost of a soft drink is $1.50

Exercise

The sum of three times the first number, plus the second number, and twice the third number is 5. If 3 times the second number is subtracted from the sum of the first number and 3 times the third number, the result is 2. If the third number is subtracted from the sum of 2 times the first number and 3 times the second number, the result is 1. Find the three numbers.

Solution

Let x: be the first number.

y: be the second number.

z: be the third number.

$$\begin{cases} 3x + y + 2z = 5 \\ (x + 3z) - 3y = 2 \\ 2x + 3y - z = 1 \end{cases}$$

$$\begin{cases} 3x + y + 2z = 5 \\ x - 3y + 3z = 2 \\ 2x + 3y - z = 1 \end{cases}$$

$$D = \begin{vmatrix} 3 & 1 & 2 \\ 1 & -3 & 3 \\ 2 & 3 & -1 \end{vmatrix} \underline{= 7|}$$

$$D_x = \begin{vmatrix} 5 & 1 & 2 \\ 2 & -3 & 3 \\ 1 & 3 & -1 \end{vmatrix} \underline{= -7|}$$

$$D_y = \begin{vmatrix} 3 & 5 & 2 \\ 1 & 2 & 3 \\ 2 & 1 & -1 \end{vmatrix} \underline{= 14|}$$

$$D_z = \begin{vmatrix} 3 & 1 & 5 \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} \underline{= 21|}$$

$x = -\frac{7}{7} \underline{= -1|}$ $\qquad$ $x = \frac{D_x}{D}$

$y = \frac{14}{7} \underline{= 2|}$ $\qquad y = \frac{D_y}{D}$

$z = \frac{21}{7} \underline{= 3|}$ $\qquad z = \frac{D_z}{D}$

∴ ***Solution***: The three numbers are: $-1,\ 2,\ and\ \ 3$

Exercise

The sum of three numbers is 16. The sum of twice the first number, 3 times the second number, and 4 times the third number is 46. The difference between 5 times the first number and the second number is 31. Find the three numbers.

Solution

Let x: be the first number.
y: be the second number.
z: be the third number.

$$\begin{cases} x + y + z = 16 \\ 2x + 3y + 4z = 46 \\ 5x - y = 31 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 5 & -1 & 0 \end{vmatrix} \underline{= 7|}$$

$$D_x = \begin{vmatrix} 16 & 1 & 1 \\ 46 & 3 & 4 \\ 31 & -1 & 0 \end{vmatrix} \underline{= 49|}$$

$$D_y = \begin{vmatrix} 1 & 16 & 1 \\ 2 & 46 & 4 \\ 5 & 31 & 0 \end{vmatrix} \underline{= 28|} \qquad D_z = \begin{vmatrix} 1 & 1 & 16 \\ 2 & 3 & 46 \\ 5 & -1 & 31 \end{vmatrix} \underline{= 35|}$$

$x = \frac{49}{7} \underline{= 7|}$ $\qquad x = \frac{D_x}{D}$

$y = \frac{28}{7} \underline{= 4|}$ $\qquad y = \frac{D_y}{D}$

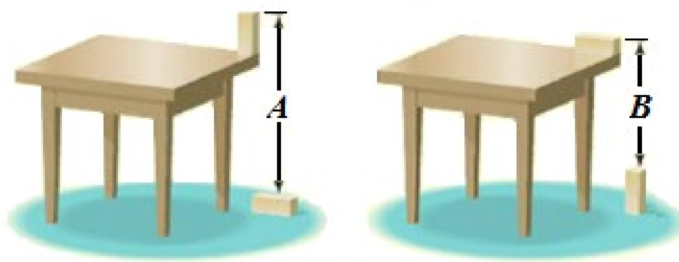
$z = \frac{35}{7} \underline{= 5|}$ $\qquad z = \frac{D_z}{D}$

∴ ***Solution***: The three numbers are: $7,\ 4,\ and\ \ 5$

Exercise

Two blocks of wood having the same length and width are placed on the top and bottom of a table. Length *A* measure 32 *cm*. The blocks are rearranged. Length *B* measures 28 *cm*. Determine the height of the table.

Solution



Let h: height of the table.
l: length of the block
w: width of the block

$$\begin{cases} (A) & h - w + l = 32 \\ (B) & h - l + w = 28 \end{cases}$$

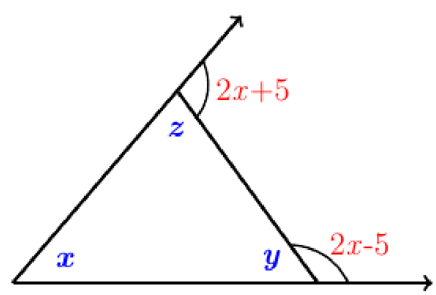
$$\overline{\quad 2h = 60 \quad}$$

$\therefore$ ***Solution***: The height of the table is **30** *cm*

Exercise

In the following triangle, the degree measures of the three interior angles and two of the exterior angles are represented with variables. Find the measure of each interior angle.

Solution



$$\begin{cases} x + y + z = 180 \\ z + 2x + 5 = 180 \\ y + 2x - 5 = 180 \end{cases}$$

$$\begin{cases} x + y + z = 180 \\ 2x \quad + z = 175 \\ 2x + y \quad = 185 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} \underline{= 3|}$$

$$D_x = \begin{vmatrix} 180 & 1 & 1 \\ 175 & 0 & 1 \\ 185 & 1 & 0 \end{vmatrix} \underline{= 180|}$$

$$D_y = \begin{vmatrix} 1 & 180 & 1 \\ 2 & 175 & 1 \\ 2 & 185 & 0 \end{vmatrix} \underline{= 195|}$$

$$D = \begin{vmatrix} 1 & 1 & 180 \\ 2 & 0 & 175 \\ 2 & 1 & 185 \end{vmatrix} \underline{= 165|}$$

$$x = \frac{180}{3} \underline{= 60^\circ|} \qquad\qquad x = \frac{D_x}{D}$$

$$y = \frac{195}{3} \underline{= 65^\circ|} \qquad\qquad y = \frac{D_y}{D}$$

$$z = \frac{165}{3} \underline{= 55^\circ|} \qquad\qquad z = \frac{D_z}{D}$$

Exercise

Three painters (Beth, Bill, and Edie), working together, can paint the exterior of a home in 10 *hours*. Bill and Edie together have painted similar house in 15 *hours*. One day, all three worked on this same kind of house for 4 *hours*, after which Edie left. Beth and Bill required 8 more *hours* to finish. Assuming no gain or loss in efficiency, how long should it take each person to complete such a job alone?

Solution

Let x: Beth's time

y: Bill's time

z: Edie's time

Let $\frac{1}{x} = a$: Beth's part of the job done in 1 *hour*.

$\frac{1}{y} = b$: Bill's part of the job done in 1 *hour*.

$\frac{1}{z} = c$: Edie's part of the job done in 1 *hour*.

All completed 1 job in 10 *hours*: $10\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 1$

Bill and Edie 1 job in 15 *hours*: $15\left(\frac{1}{y} + \frac{1}{z}\right) = 1$

All worked 1 job in 4 *hours* Beth and Bill required 8 *hours*: $4\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) + 8\left(\frac{1}{x} + \frac{1}{y}\right) = 1$

$$\begin{cases} 10a + 10b + 10c = 1 \\ 15b + 15c = 1 \\ 4a + 4b + 4c + 8a + 8b = 1 \end{cases}$$

$$\begin{cases} 10a + 10b + 10c = 1 \\ 15b + 15c = 1 \\ 12a + 12b + 4c = 1 \end{cases}$$

$$D = \begin{vmatrix} 10 & 10 & 10 \\ 0 & 15 & 15 \\ 12 & 12 & 4 \end{vmatrix} \underline{= -1200}$$

$$D_a = \begin{vmatrix} 1 & 10 & 10 \\ 1 & 15 & 15 \\ 1 & 12 & 4 \end{vmatrix} \underline{= -40}$$

$$D_b = \begin{vmatrix} 10 & 1 & 10 \\ 0 & 1 & 15 \\ 12 & 1 & 4 \end{vmatrix} \underline{= -50}$$

$$D_c = \begin{vmatrix} 10 & 10 & 1 \\ 0 & 15 & 1 \\ 12 & 12 & 1 \end{vmatrix} \underline{= -30}$$

$$a = \frac{40}{1200} = \frac{1}{30} \quad \to \quad x = \frac{1}{a} \underline{= 30}$$

$$b = \frac{50}{1200} = \frac{1}{24} \quad \to \quad y = \frac{1}{b} \underline{= 24}$$

$$c = \frac{30}{1200} = \frac{1}{40} \quad \to \quad z = \frac{1}{c} \underline{= 40}$$

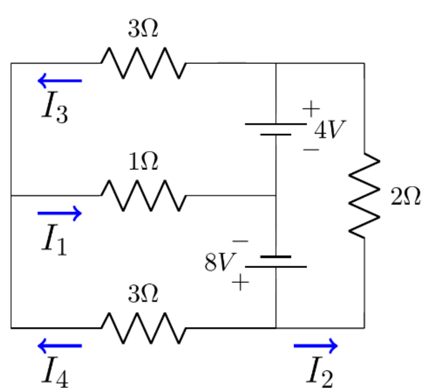
∴ ***Solution***: Took alone to complete a job: Beth **30** *hours*, Bill **24** *hours*, and Eddie **40** *hours*

Exercise

An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:

$$\begin{cases} I_1 = I_3 + I_4 \\ I_1 + 5I_4 = 8 \\ I_1 + 3I_3 = 4 \\ 8 - 4 - 2I_2 = 0 \end{cases}$$

Find the currents I_1, I_2, I_3, and I_4



Solution

$$\begin{cases} I_1 - I_3 - I_4 = 0 \\ I_1 + 5I_4 = 8 \\ I_1 + 3I_3 = 4 \\ \underline{I_2 = 2} \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & -1 \\ 1 & 0 & 5 \\ 1 & 3 & 0 \end{vmatrix} \underline{= -23|}$$

$$D_1 = \begin{vmatrix} 0 & -1 & -1 \\ 8 & 0 & 5 \\ 4 & 3 & 0 \end{vmatrix} \underline{= -44|}$$

$$D_3 = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 8 & 5 \\ 1 & 4 & 0 \end{vmatrix} \underline{= -16|}$$

$$D_4 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & 8 \\ 1 & 3 & 4 \end{vmatrix} \underline{= -28|}$$

∴ ***Solution***: $\underline{I_1 = \frac{44}{23}|}$ $\underline{I_2 = 2|}$ $\underline{I_3 = \frac{16}{23}|}$ $\underline{I_4 = \frac{28}{23}|}$

Exercise

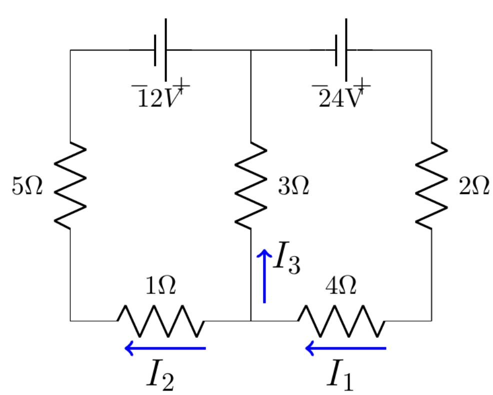
An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:

$$\begin{cases} I_1 = I_2 + I_3 \\ 24 - 6I_1 - 3I_3 = 0 \\ 12 + 24 - 6I_1 - 6I_2 = 0 \end{cases}$$

Find the currents I_1, I_2, and I_3

Solution

$$\begin{cases} I_1 - I_2 - I_3 = 0 \\ 2I_1 + I_3 = 8 \\ I_1 + I_2 = 6 \end{cases}$$



$$D = \begin{vmatrix} 1 & -1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \underline{= -4|}$$

$$D_1 = \begin{vmatrix} 0 & -1 & -1 \\ 8 & 0 & 1 \\ 6 & 1 & 0 \end{vmatrix} \underline{= -14|}$$

$$D_2 = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 8 & 1 \\ 1 & 6 & 0 \end{vmatrix} \underline{= -10|}$$

$$D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 0 & 8 \\ 1 & 1 & 6 \end{vmatrix} \underline{= -4|}$$

$\therefore$ ***Solution***: $\underline{I_1 = \frac{7}{2}|}$

$\underline{I_2 = \frac{5}{2}|}$

$\underline{I_3 = 1|}$

Exercise

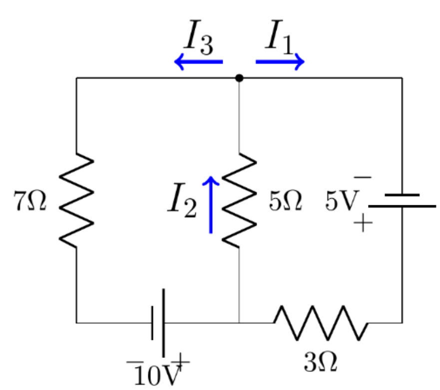
An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:

$$\begin{cases} I_2 = I_1 + I_3 \\ 5 - 3I_1 - 5I_2 = 0 \\ 10 - 5I_2 - 7I_3 = 0 \end{cases}$$

Find the currents I_1, I_2, and I_3

Solution

$$\begin{cases} -I_1 + I_2 - I_3 = 0 \\ 3I_1 + 5I_2 = 5 \\ 5I_2 + 7I_3 = 10 \end{cases}$$



$$D = \begin{vmatrix} -1 & 1 & -1 \\ 3 & 5 & 0 \\ 0 & 5 & 7 \end{vmatrix} \underline{= -71|}$$

$$D_1 = \begin{vmatrix} 0 & 1 & -1 \\ 5 & 5 & 0 \\ 10 & 5 & 7 \end{vmatrix} \underline{= -10|}$$

$$D_2 = \begin{vmatrix} -1 & 0 & -1 \\ 3 & 5 & 0 \\ 0 & 10 & 7 \end{vmatrix} \underline{= -65|}$$

$$D_3 = \begin{vmatrix} -1 & 1 & 0 \\ 3 & 5 & 5 \\ 0 & 5 & 10 \end{vmatrix} \underline{= -55|}$$

$\therefore$ ***Solution***: $\quad \underline{I_1 = \frac{10}{71}|}$

$\underline{I_2 = \frac{65}{71}|}$

$\underline{I_3 = \frac{55}{71}|}$

Exercise

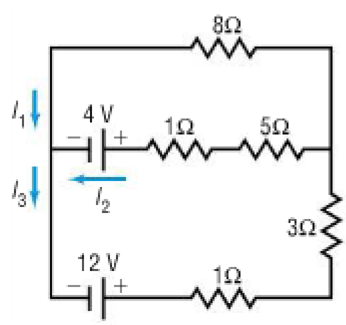
An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:

$$\begin{cases} I_3 = I_1 + I_2 \\ 6I_2 + 4I_3 = 8 \\ 8I_1 = 4 + 6I_2 \end{cases}$$

Find the currents I_1, I_2, and I_3

Solution

$$\begin{cases} I_1 + I_2 - I_3 = 0 \\ 3I_2 + 2I_3 = 4 \\ 4I_1 - 3I_2 = 2 \end{cases}$$



$$D = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 3 & 2 \\ 4 & -3 & 0 \end{vmatrix} \underline{= 26|}$$

$$D_1 = \begin{vmatrix} 0 & 1 & -1 \\ 4 & 3 & 2 \\ 2 & -3 & 0 \end{vmatrix} \underline{= 22|}$$

$$D_2 = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 4 & 2 \\ 4 & 2 & 0 \end{vmatrix} \underline{= 12|}$$

$$D_3 = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 3 & 4 \\ 4 & -3 & 2 \end{vmatrix} \underline{= 34|}$$

$\therefore$ ***Solution***: $\quad I_1 = \frac{22}{26} \underline{= \frac{11}{13}|}$

$$I_2 = \frac{12}{26} = \frac{6}{13}$$

$$I_3 = \frac{34}{26} = \frac{17}{13}$$

Solution Section 5.2 – Partial Fraction Decomposition

Exercise

Write the partial fraction decomposition of each rational expression $\frac{4}{x(x-1)}$

Solution

$$\frac{4}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$4 = A(x-1) + Bx$$

$$4 = Ax - A + Bx$$

$$4 = (A+B)x - A$$

$$\begin{cases} A+B=0 \\ -A=4 \end{cases} \rightarrow \begin{cases} B=-A=4 \\ A=-4 \end{cases}$$

$$\boxed{\frac{4}{x(x-1)} = -\frac{4}{x} + \frac{4}{x-1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{3x}{(x+2)(x-1)}$

Solution

$$\frac{3x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$3x = A(x-1) + B(x+2)$$

$$3x = Ax - A + Bx + 2B$$

$$3x = (A+B)x - A + 2B$$

$$\begin{cases} A+B=3 \\ -A+2B=0 \end{cases}$$

$$3B=3$$

$$\underline{B=1} \quad \rightarrow \quad \underline{A=2}$$

$$\boxed{\frac{3x}{(x+2)(x-1)} = \frac{2}{x+2} + \frac{1}{x-1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{1}{x\left(x^2+1\right)}$

Solution

$$\frac{1}{x\left(x^2+1\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$$

$$1=A\left(x^2+1\right)+x(Bx+C)$$

$$1=Ax^2+A+Bx^2+Cx$$

$$1=(A+B)x^2+Cx+A$$

$$\begin{cases} A+B=0 & \rightarrow \quad \underline{B=-1} \\ \underline{C=0} \\ \underline{A=1} \end{cases}$$

$$\boxed{\frac{1}{x\left(x^2+1\right)}=\frac{1}{x}-\frac{x}{x^2+1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{1}{(x+1)\left(x^2+4\right)}$

Solution

$$\frac{1}{(x+1)\left(x^2+4\right)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+4}$$

$$1=A\left(x^2+4\right)+(x+1)(Bx+C)$$

$$1=Ax^2+4A+Bx^2+Cx+Bx+C$$

$$1=(A+B)x^2+(B+C)x+4A+C$$

$$\begin{cases} A+B=0 \\ B+C=0 \\ 4A+C=1 \end{cases}$$

$$\rightarrow \begin{cases} A=-B & \rightarrow \quad \underline{A=\frac{1}{5}} \\ C=-B & \rightarrow \quad \underline{C=\frac{1}{5}} \\ -4B-B=1 & \rightarrow \quad \underline{B=-\frac{1}{5}} \end{cases}$$

$$\frac{1}{(x+1)\left(x^2+4\right)}=\frac{\frac{1}{5}}{x+1}+\frac{\frac{1}{5}x+\frac{1}{5}}{x^2+4}$$

$$=\frac{1}{5}\frac{1}{x+1}+\frac{1}{5}\frac{x+1}{x^2+4}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2}{(x-1)^2(x+1)^2}$

Solution

$$\frac{x^2}{(x-1)^2(x+1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}$$

$$x^2=A(x-1)(x+1)^2+B(x+1)^2+C(x-1)^2(x+1)+D(x-1)^2$$

$$=A(x-1)\left(x^2+2x+1\right)+B\left(x^2+2x+1\right)+C\left(x^2-2x+1\right)(x+1)+D\left(x^2-2x+1\right)$$

$$=Ax^3+2Ax^2+Ax-Ax^2-2Ax-A+Bx^2+2Bx+B$$
$$+Cx^3-2Cx^2+Cx+Cx^2-2Cx+C+Dx^2-2Dx+D$$

$$x^2=(A+C)x^3+(A+B-C+D)x^2+(-A+2B-C-2D)x-A+B+C+D$$

$$\begin{cases} A+C=0 & \rightarrow A=-C \\ A+B-C+D=1 \\ -A+2B-C-2D=0 \\ -A+B+C+D=0 \end{cases}$$

$$\begin{cases} B-2C+D=1 \\ B-D=0 \\ B+2C+D=0 \end{cases}$$

$$\Delta=\begin{vmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{vmatrix}=8 \qquad \Delta_B=\begin{vmatrix} 1 & -2 & 1 \\ 0 & 0 & -1 \\ 0 & 2 & 1 \end{vmatrix}=2$$

$$\Delta_C=\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix}=-2 \qquad \Delta_D=\begin{vmatrix} 1 & -2 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{vmatrix}=2$$

$$B=\frac{1}{4}\quad C=-\frac{1}{4}\quad D=\frac{1}{4}\quad A=\frac{1}{4}$$

$$\boxed{\frac{x^2}{(x-1)^2(x+1)^2}=\frac{1}{4}\frac{1}{x-1}+\frac{1}{4}\frac{1}{(x-1)^2}-\frac{1}{4}\frac{1}{x+1}+\frac{1}{4}\frac{1}{(x+1)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x+1}{x^2(x-2)^2}$

Solution

$$\frac{x+1}{x^2(x-2)^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}+\frac{D}{(x-2)^2}$$

$$\begin{aligned}x+1&=Ax(x-2)^2+B(x-2)^2+Cx^2(x-2)+Dx^2\\&=Ax\left(x^2-4x+4\right)+B\left(x^2-4x+4\right)+Cx^3-2Cx^2+Dx^2\\&=Ax^3-4Ax^2+4Ax+Bx^2-4Bx+4B+Cx^3-2Cx^2+Dx^2\\&=(A+C)x^3+(-4A-B-2C+D)x^2+(4A-4B)x+4B\end{aligned}$$

$$\begin{cases}A+C=0\\-4A-B-2C+D=0\\4A-4B=1\\4B=1\end{cases}$$

$$\begin{cases}C=-\frac{1}{2}\\D=2+\frac{1}{4}-1=\frac{5}{4}\\A=\frac{1}{2}\\B=\frac{1}{4}\end{cases}$$

$$\boxed{\frac{x+1}{x^2(x-2)^2}=\frac{1}{2}\frac{1}{x}+\frac{1}{4}\frac{1}{x^2}-\frac{1}{2}\frac{1}{x-2}+\frac{5}{4}\frac{1}{(x-2)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x-3}{(x+2)(x+1)^2}$

Solution

$$\frac{x-3}{(x+2)(x+1)^2}=\frac{A}{x+2}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$x-3=A(x+1)^2+B(x+1)(x+2)+C(x+2)$$

$$=Ax^2+2Ax+A+B\left(x^2+3x+2\right)+Cx+2C$$

$$=(A+B)x^2+(2A+3B+C)x+A+2B+2C$$

$$\begin{cases} A+B=0 \\ 2A+3B+C=1 \\ A+2B+2C=-3 \end{cases}$$

$$\begin{cases} A=-B \\ -2B+3B+C=1 \\ -B+2B+2C=-3 \end{cases}$$

$$\begin{cases} B+C=1 \\ B+2C=-3 \end{cases}$$

$$\Delta=\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}=1 \qquad \Delta_B=\begin{vmatrix} 1 & 1 \\ -3 & 2 \end{vmatrix}=5 \qquad \Delta_C=\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix}=-4$$

$$C=-4, \quad B=5, \quad A=-5$$

$$\frac{x-3}{(x+2)(x+1)^2}=-\frac{5}{x+2}+\frac{5}{x+1}-\frac{4}{(x+1)^2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2+x}{(x+2)(x-1)^2}$

Solution

$$\frac{x^2+x}{(x+2)(x-1)^2}=\frac{A}{x+2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$$x^2+x=A(x-1)^2+B(x-1)(x+2)+C(x+2)$$

$$=Ax^2-2Ax+A+Bx^2+Bx-2B+Cx+2C$$

$$\begin{matrix} x^2 \\ x \\ x^0 \end{matrix}\begin{cases} A+B=1 \\ -2A+B+C=1 \\ A-2B+2C=0 \end{cases}$$

$$\Delta=\begin{vmatrix} 1 & 1 & 0 \\ -2 & 1 & 1 \\ 1 & -2 & 2 \end{vmatrix}=9 \qquad \Delta_A=\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix}=2$$

$$\Delta_B = \begin{vmatrix} 1 & 1 & 0 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 7 \qquad \Delta_C = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 1 \\ 1 & -2 & 0 \end{vmatrix} = 6$$

$$A = \frac{2}{9} \quad B = \frac{7}{9} \quad C = \frac{2}{3}$$

$$\frac{x^2 + x}{(x+2)(x-1)^2} = \frac{\frac{2}{9}}{x+2} + \frac{\frac{7}{9}}{x-1} + \frac{\frac{2}{3}}{(x-1)^2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{10x^2 + 2x}{(x-1)^2(x^2+2)}$

Solution

$$\frac{10x^2 + 2x}{(x-1)^2(x^2+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+2}$$

$$\begin{aligned} 10x^2 + 2x &= A(x-1)(x^2+2) + B(x^2+2) + (Cx+D)(x-1)^2 \\ &= Ax^3 + 2Ax - Ax^2 - 2A + Bx^2 + 2B + (Cx+D)(x^2 - 2x + 1) \\ &= Ax^3 + 2Ax - Ax^2 - 2A + Bx^2 + 2B + Cx^3 - 2Cx^2 + Cx + Dx^2 - 2Dx + D \\ &= (A+C)x^3 + (B - 2A - 2C + D)x^2 + (2A + C - 2D)x - 2A + 2B + D \end{aligned}$$

$$\begin{cases} A + C = 0 & \rightarrow A = -C \\ B - 2A - 2C + D = 10 \\ 2A + C - 2D = 2 \\ -2A + 2B + D = 0 \end{cases}$$

$$\begin{cases} B + D = 10 \\ -C - 2D = 2 \\ 2B + 2C + D = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 2 & 2 & 1 \end{vmatrix} = 5 \qquad \Delta_B = \begin{vmatrix} 10 & 0 & 1 \\ 2 & -1 & -2 \\ 0 & 2 & 1 \end{vmatrix} = 34$$

$$\Delta_C = \begin{vmatrix} 1 & 10 & 1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{vmatrix} = -42 \qquad \Delta_D = \begin{vmatrix} 1 & 0 & 10 \\ 0 & -1 & 2 \\ 2 & 2 & 0 \end{vmatrix} = 16$$

$$\boxed{B=\frac{34}{5}\quad C=-\frac{42}{5}\quad D=\frac{16}{5}\quad A=\frac{42}{5}}$$

$$\boxed{\frac{10x^2+2x}{(x-1)^2\left(x^2+2\right)}=\frac{42}{5(x-1)}+\frac{34}{5(x-1)^2}+\frac{-42x+16}{5\left(x^2+2\right)}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2+2x+3}{(x+1)\left(x^2+2x+4\right)}$

Solution

$$\frac{x^2+2x+3}{(x+1)\left(x^2+2x+4\right)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+2x+4}$$

$$\begin{aligned}x^2+2x+3&=A\left(x^2+2x+4\right)+(Bx+C)(x+1)\\&=Ax^2+2Ax+4A+Bx^2+Bx+Cx+C\\&=(A+B)x^2+(2A+B+C)x+4A+C\end{aligned}$$

$$\begin{cases}A+B=1\\2A+B+C=2\\4A+C=3\end{cases}$$

$$\Delta=\begin{vmatrix}1&1&0\\2&1&1\\4&0&1\end{vmatrix}=3\qquad\Delta_A=\begin{vmatrix}1&1&0\\2&1&1\\3&0&1\end{vmatrix}=2$$

$$\Delta_B=\begin{vmatrix}1&1&0\\2&2&1\\4&3&1\end{vmatrix}=1\qquad\Delta_C=\begin{vmatrix}1&1&1\\2&1&2\\4&0&3\end{vmatrix}=1$$

$$\boxed{A=\frac{2}{3}\quad B=\frac{1}{3}\quad C=\frac{1}{3}}$$

$$\boxed{\frac{x^2+2x+3}{(x+1)\left(x^2+2x+4\right)}=\frac{\frac{2}{3}}{x+1}+\frac{\frac{1}{3}x+\frac{1}{3}}{x^2+2x+4}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2-11x-18}{x\left(x^2+3x+3\right)}$

Solution

$$\frac{x^2-11x-18}{x\left(x^2+3x+3\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+3x+3}$$

$$x^2-11x-18=Ax^2+3Ax+3A+Bx^2+Cx$$

$$=(A+B)x^2+(3A+C)x+3A$$

$$\begin{cases} A+B=1 & \rightarrow \quad \underline{B=7} \\ 3A+C=-11 & \rightarrow \quad \underline{C=7} \\ 3A=-18 \quad \rightarrow \quad \underline{A=-6} \end{cases}$$

$$\boxed{\frac{x^2-11x-18}{x\left(x^2+3x+3\right)}=-\frac{6}{x}+\frac{7x+7}{x^2+3x+3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{1}{(2x+3)(4x-1)}$

Solution

$$\frac{1}{(2x+3)(4x-1)}=\frac{A}{2x+3}+\frac{B}{4x-1}$$

$$1=4Ax-A+2Bx+3B$$

$$1=(4A+2B)x-A+3B$$

$$\begin{cases} 4A+2B=0 \\ -A+3B=1 \end{cases}$$

$$\Delta=\begin{vmatrix} 4 & 2 \\ -1 & 3 \end{vmatrix}=14 \quad \Delta_A=\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix}=-2 \quad \Delta_B=\begin{vmatrix} 4 & 0 \\ -1 & 1 \end{vmatrix}=4$$

$$\underline{A=-\tfrac{1}{7}} \quad \underline{B=\tfrac{2}{7}}$$

$$\boxed{\frac{1}{(2x+3)(4x-1)}=-\frac{\frac{1}{7}}{2x+3}+\frac{\frac{2}{7}}{4x-1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2+2x+3}{\left(x^2+4\right)^2}$

Solution

$$\frac{x^2+2x+3}{\left(x^2+4\right)^2}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{\left(x^2+4\right)^2}$$

$$x^2+2x+3=(Ax+B)\left(x^2+4\right)+Cx+D$$

$$=Ax^3+4Ax+Bx^2+4B+Cx+D$$

$$=Ax^3+Bx^2+(4A+C)x+4B+D$$

$$\begin{cases} \underline{A=0} \\ \underline{B=1} \\ 4A+C=2 \\ 4B+D=3 \end{cases} \rightarrow \quad \begin{array}{l} \\ \\ \underline{C=2} \\ D=3-4B\ \underline{=-1} \end{array}$$

$$\boxed{\frac{x^2+2x+3}{\left(x^2+4\right)^2}=\frac{1}{x^2+4}+\frac{2x-1}{\left(x^2+4\right)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^3+1}{\left(x^2+16\right)^2}$

Solution

$$\frac{x^3+1}{\left(x^2+16\right)^2}=\frac{Ax+B}{x^2+16}+\frac{Cx+D}{\left(x^2+16\right)^2}$$

$$x^3+1=(Ax+B)\left(x^2+16\right)+Cx+D$$

$$=Ax^3+16Ax+Bx^2+16B+Cx+D$$

$$\begin{cases} x^3 & \underline{A=1} \\ x^2 & \underline{B=0} \\ x & 16A+C=0 \\ x^0 & 16B+D=1 \end{cases} \rightarrow \quad \begin{array}{l} \\ \\ \underline{C=-16} \\ \underline{D=1} \end{array}$$

$$\boxed{\frac{x^3+1}{\left(x^2+16\right)^2}=\frac{x}{x^2+16}+\frac{-16x+1}{\left(x^2+16\right)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{7x+3}{x^3-2x^2-3x}$

Solution

$$\frac{7x+3}{x^3-2x^2-3x}=\frac{7x+3}{x(x+1)(x-3)}$$

$$=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-3}$$

$$7x+3=A(x+1)(x-3)+Bx(x-3)+Cx(x+1)$$

$$=Ax^2-2Ax-3A+Bx^2-3B+Cx^2+Cx$$

$$=(A+B+C)x^2+(C-2A)x-3A-3B$$

$$\begin{cases} A+B+C=0 \\ C-2A=7 \\ -3A-3B=3 \end{cases}$$

$$\Delta=\begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ -3 & -3 & 0 \end{vmatrix}=6 \qquad \Delta_A=\begin{vmatrix} 0 & 1 & 1 \\ 7 & 0 & 1 \\ 3 & -3 & 0 \end{vmatrix}=-18$$

$$\Delta_B=\begin{vmatrix} 1 & 0 & 1 \\ -2 & 7 & 1 \\ -3 & 3 & 0 \end{vmatrix}=12 \qquad \Delta_C=\begin{vmatrix} 1 & 1 & 0 \\ -2 & 0 & 7 \\ -3 & -3 & 3 \end{vmatrix}=6$$

$$\boxed{A=-3 \quad B=2 \quad C=1}$$

$$\boxed{\frac{7x+3}{x^3-2x^2-3x}=\frac{-3}{x}+\frac{2}{x+1}+\frac{1}{x-3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2}{x^3-4x^2+5x-2}$

Solution

$$\frac{x^2}{x^3-4x^2+5x-2}=\frac{x^2}{(x-2)(x-1)^2}$$

$$=\frac{A}{x-2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$$\begin{aligned}x^2&=A(x-1)^2+B(x-2)(x-1)+C(x-2)\\&=Ax^2-2Ax+A+Bx^2-3Bx+2B+Cx-2C\\&=(A+B)x^2+(-2A-3B+C)x+A+2B-2C\end{aligned}$$

$$\begin{cases}A+B=1\\-2A-3B+C=0\\A+2B-2C=0\end{cases}$$

$$D=\begin{vmatrix}1&1&0\\-2&-3&1\\1&2&-2\end{vmatrix}=1 \qquad D_A=\begin{vmatrix}1&1&0\\0&-3&1\\0&2&-2\end{vmatrix}=4$$

$$D_B=\begin{vmatrix}1&1&0\\-2&0&1\\1&0&-2\end{vmatrix}=-3 \qquad D_C=\begin{vmatrix}1&1&1\\-2&-3&0\\1&2&0\end{vmatrix}=-1$$

$$\underline{A=4 \quad B=-3 \quad C=-1}$$

$$\boxed{\frac{x^2}{x^3-4x^2+5x-2}=\frac{4}{x-2}-\frac{3}{x-1}-\frac{1}{(x-1)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^3}{\left(x^2+16\right)^3}$

Solution

$$\frac{x^3}{\left(x^2+16\right)^3}=\frac{Ax+B}{x^2+16}+\frac{Cx+D}{\left(x^2+16\right)^2}+\frac{Ex+F}{\left(x^2+16\right)^3}$$

$$\begin{aligned}x^3&=(Ax+B)\left(x^2+16\right)^2+(Cx+D)\left(x^2+16\right)+Ex+F\\&=(Ax+B)\left(x^4+32x^2+256\right)+Cx^3+16Cx+Dx^2+16D+Ex+F\\&=Ax^5+32Ax^3+256Ax+Bx^4+32Bx^2+256B+Cx^3+Dx^2+(16C+E)x+16D+F\\&=Ax^5+Bx^4+(32A+C)x^3+(32B+D)x^2+(256A+16C+E)x+256B+16D+F\end{aligned}$$

$x^5 \quad \boxed{A = 0}$

$x^4 \quad \boxed{B = 0}$

$x^3 \quad 32A + C = 1 \quad \rightarrow \quad \boxed{C = 1}$

$x^2 \quad 32B + D = 0 \quad \rightarrow \quad \boxed{D = 0}$

$x^1 \quad 256A + 16C + E = 0 \quad \rightarrow \quad \boxed{E = -16}$

$x^0 \quad 256B + 16D + F = 0 \quad \rightarrow \quad \boxed{F = 0}$

$$\boxed{\frac{x^3}{\left(x^2+16\right)^3} = \frac{x}{\left(x^2+16\right)^2} + \frac{-16x}{\left(x^2+16\right)^3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{4}{2x^2-5x-3}$

Solution

$$\frac{4}{2x^2-5x-3} = \frac{4}{(2x+1)(x-3)}$$
$$= \frac{A}{2x+1} + \frac{B}{x-3}$$

$$4 = Ax - 3A + 2Bx + B$$
$$= (A+2B)x - 3A + B$$

$$\begin{cases} A + 2B = 0 \\ -3A + B = 4 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} = 7 \qquad \Delta_A = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8 \qquad \Delta_B = \begin{vmatrix} 1 & 0 \\ -3 & 4 \end{vmatrix} = 4$$

$$\boxed{A = -\frac{8}{7} \quad B = \frac{4}{7}}$$

$$\boxed{\frac{4}{2x^2-5x-3} = \frac{-\frac{8}{7}}{2x+1} + \frac{\frac{4}{7}}{x-3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{2x+3}{x^4-9x^2}$

Solution

$$\frac{2x+3}{x^4-9x^2}=\frac{2x+3}{x^2(x-3)(x+3)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-3}+\frac{D}{x+3}$$

$$2x+3=Ax\left(x^2-9\right)+B\left(x^2-9\right)+Cx^2(x+3)+Dx^2(x-3)$$

$$=Ax^3-9Ax+Bx^2-9B+Cx^3+3Cx^2+Dx^3-3Dx^2$$

$$=(A+C+D)x^3+(B+3C-3D)x^2-9Ax-9B$$

$$\begin{cases} A+C+D=0 & & C+D=\frac{2}{9} \\ B+3C-3D=0 & & 3C-3D=\frac{1}{3} \\ -9A=2 & \rightarrow & \boxed{A=-\frac{2}{9}} \\ -9B=3 & \rightarrow & \boxed{B=-\frac{1}{3}} \end{cases}$$

$$\Delta=\begin{vmatrix}1 & 1\\ 3 & -3\end{vmatrix}=-6 \qquad \Delta_C=\begin{vmatrix}\frac{2}{9} & 1\\ \frac{1}{3} & -3\end{vmatrix}=-1 \qquad \Delta_D=\begin{vmatrix}1 & \frac{2}{9}\\ 3 & \frac{1}{3}\end{vmatrix}=-\frac{1}{3}$$

$$\boxed{C=\frac{1}{6} \quad D=\frac{1}{18}}$$

$$\boxed{\frac{2x+3}{x^4-9x^2}=-\frac{2}{9}\frac{1}{x}-\frac{1}{3}\frac{1}{x^2}+\frac{1}{6}\frac{1}{x-3}+\frac{1}{18}\frac{1}{x+3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2+9}{x^4-2x^2-8}$

Solution

$$\frac{x^2+9}{x^4-2x^2-8}=\frac{A}{x-2}+\frac{B}{x+2}+\frac{Cx+D}{x^2+2}$$

$$x^2+9=A(x+2)\left(x^2+2\right)+B(x-2)\left(x^2+2\right)+(Cx+D)\left(x^2-4\right)$$

$$=Ax^3+2Ax+2Ax^2+4A+Bx^3+2Bx-2Bx^2-4B+Cx^3-4Cx+Dx^2-4D$$

$$=(A+B+C)x^3+(2A-2B+D)x^2+(2A+2B-4C)x+4A-4B-4D$$

$$\begin{cases} A+B+C=0 \\ 2A-2B+D=1 \\ 2A+2B-4C=0 \\ 4A-4B-4D=9 \end{cases}$$

$$\boxed{A=\frac{13}{24}\quad B=-\frac{13}{24}\quad C=0\quad D=-\frac{7}{6}}$$

$$\boxed{\frac{x^2+9}{x^4-2x^2-8}=\frac{\frac{13}{24}}{x-2}-\frac{\frac{13}{24}}{x+2}-\frac{\frac{7}{6}}{x^2+2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{y}{y^2-2y-3}$

Solution

$$\frac{y}{y^2-2y-3}=\frac{A}{y-3}+\frac{B}{y+1}$$

$$y=(A+B)y+A-3B$$

$$\begin{cases} A+B=1 \\ A-3B=0 \end{cases}$$

$$\Delta=\begin{vmatrix}1 & 1\\ 1 & -3\end{vmatrix}=-4 \qquad \Delta_A=\begin{vmatrix}1 & 1\\ 0 & -3\end{vmatrix}=-3 \qquad \Delta_B=\begin{vmatrix}1 & 1\\ 1 & 0\end{vmatrix}=-1$$

$$\boxed{A=\frac{3}{4}\quad B=\frac{1}{4}}$$

$$\boxed{\frac{y}{y^2-2y-3}=\frac{\frac{3}{4}}{y-3}+\frac{\frac{1}{4}}{y+1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x+3}{2x^3-8x}$

Solution

$$\begin{aligned}\frac{x+3}{2x^3-8x}&=\frac{1}{2}\frac{x+3}{x\left(x^2-4\right)}\\ &=\frac{1}{2}\left(\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-2}\right)\\ &=\frac{1}{2}\frac{A(x+2)(x-2)+Bx(x-2)+Cx(x+2)}{x(x+2)(x-2)}\end{aligned}$$

$$(A+B+C)x^2+(2C-2B)x-4A=x+3$$

$$\begin{cases} A+B+C=0 & & B+C=\frac{3}{4} \\ 2C-2B=1 & & -B+C=\frac{1}{2} \\ -4A=3 & \rightarrow \ \underline{A=-\frac{3}{4}|} & \end{cases}$$

$$\underline{B=\frac{1}{8} \quad C=\frac{5}{8}|}$$

$$\boxed{\frac{x+3}{2x^3-8x}=\frac{1}{2}\left(-\frac{\frac{3}{4}}{x}+\frac{\frac{1}{8}}{x+2}+\frac{\frac{5}{8}}{x-2}\right)}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2}{(x-1)\left(x^2+2x+1\right)}$

Solution

$$\frac{x^2}{(x-1)\left(x^2+2x+1\right)}=\frac{x^2}{(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$\begin{aligned} x^2 &= A(x+1)^2+B(x-1)(x+1)+C(x-1) \\ &=(A+B)x^2+(2A+C)x+A-B-C \end{aligned}$$

$$\begin{cases} A+B=1 \\ 2A+C=0 \\ A-B-C=0 \end{cases}$$

$$\Delta=\begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix}=4 \qquad \Delta_A=\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{vmatrix}=1$$

$$\Delta_B=\begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & -1 \end{vmatrix}=3 \qquad \Delta_C=\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 1 & -1 & 0 \end{vmatrix}=-2$$

$$\underline{A=\frac{1}{4} \quad B=\frac{3}{4} \quad C=-\frac{1}{2}|}$$

$$\boxed{\frac{x^2}{(x-1)\left(x^2+2x+1\right)}=\frac{\frac{1}{4}}{x-1}+\frac{\frac{3}{4}}{x+1}-\frac{\frac{1}{2}}{(x+1)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{3x^2+x+4}{x^3+x}$

Solution

$$\frac{3x^2+x+4}{x^3+x}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$$

$$=\frac{(A+B)x^2+Cx+A}{x\left(x^2+1\right)}$$

$$3x^2+x+4=(A+B)x^2+Cx+A$$

$$\begin{cases} A+B=3 & \rightarrow \ \underline{B=-1} \\ \underline{C=1} \\ \underline{A=4} \end{cases}$$

$$\boxed{\frac{3x^2+x+4}{x^3+x}=\frac{4}{x}+\frac{-x+1}{x^2+1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{8x^2+8x+2}{\left(4x^2+1\right)^2}$

Solution

$$\frac{8x^2+8x+2}{\left(4x^2+1\right)^2}=\frac{Ax+B}{4x^2+1}+\frac{Cx+D}{\left(4x^2+1\right)^2}$$

$$=\frac{(Ax+B)\left(4x^2+1\right)+Cx+D}{\left(4x^2+1\right)^2}$$

$$8x^2+8x+2=(Ax+B)\left(4x^2+1\right)+Cx+D$$

$$=4Ax^3+4Bx^2+(A+C)x+B+D$$

$$\begin{cases} \underline{A=0} \\ 4B=8 & \rightarrow \ \underline{B=2} \\ A+C=8 & \rightarrow \ \underline{C=8} \\ B+D=2 & \rightarrow \ \underline{D=0} \end{cases}$$

$$\boxed{\frac{8x^2+8x+2}{\left(4x^2+1\right)^2}=\frac{2}{4x^2+1}+\frac{8x}{\left(4x^2+1\right)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{1}{x^2+2x}$

Solution

$$\frac{1}{x^2+2x}=\frac{A}{x}+\frac{B}{x+2}$$

$$1=Ax+2A+Bx$$

$$x \qquad 2A=1 \qquad \to A=\frac{1}{2}$$

$$x^0 \qquad A+B=0 \quad \to B=-\frac{1}{2}$$

$$\boxed{\frac{1}{x^2+2x}=\frac{1}{2}\frac{1}{x}-\frac{1}{2}\frac{1}{x+2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{2x+1}{x^2-7x+12}$

Solution

$$\frac{2x+1}{x^2-7x+12}=\frac{A}{x-4}+\frac{B}{x-3}$$

$$2x+1=Ax-3A+Bx-4B$$

$$x \qquad A+B=2$$

$$x^0 \qquad -3A-4B=1$$

$$A=\frac{\begin{vmatrix}2 & 1\\ 1 & -4\end{vmatrix}}{\begin{vmatrix}1 & 1\\ -3 & -4\end{vmatrix}}=\frac{-9}{-1}\ \underline{=9} \qquad B=\frac{\begin{vmatrix}1 & 2\\ -3 & 1\end{vmatrix}}{-1}=\frac{7}{-1}\ \underline{=-7}$$

$$\boxed{\frac{2x+1}{x^2-7x+12}=\frac{9}{x-4}-\frac{7}{x-3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2+x}{x^4-3x^2-4}$

Solution

$$\frac{x^2+x}{x^4-3x^2-4}=\frac{x^2+x}{\left(x^2-4\right)\left(x^2+1\right)}$$

$$=\frac{A}{x-2}+\frac{B}{x+2}+\frac{Cx+D}{x^2+1}$$

$$x^2+x=A(x+2)\left(x^2+1\right)+B(x-2)\left(x^2+1\right)+(Cx+D)\left(x^2-4\right)$$

$$=Ax^3+Ax+2Ax^2+2A+Bx^3+Bx-2Bx^2-2B+Cx^3-4Cx+Dx^2-4D$$

$$=(A+B+C)x^3+(2A-2B+D)x^2+(A+B-4C)x+2A-2B-4D$$

$$\begin{cases} x^3 & A+B+C=0 & (1) \\ x^2 & 2A-2B+D=1 & (2) \\ x & A+B-4C=1 & (3) \\ x^0 & 2A-2B-4D=0 & (4) \end{cases}$$

$$(1)-(3)\rightarrow\quad 5C=-1\quad \underline{C=-\tfrac{1}{5}}$$

$$(2)-(4)\rightarrow\quad 5D=1\quad \underline{D=\tfrac{1}{5}}$$

$$\begin{cases} A+B=\frac{1}{5} \\ 2A-2B=\frac{4}{5} \end{cases}\rightarrow\begin{cases} 2A+2B=\frac{2}{5} \\ 2A-2B=\frac{4}{5} \end{cases}$$

$$4A=\frac{6}{5}\quad\rightarrow\quad \underline{A=\frac{3}{10}}$$

$$B=\frac{1}{5}-\frac{3}{10}\quad\rightarrow\quad \underline{B=-\frac{1}{10}}$$

$$\underline{\frac{x^2+x}{x^4-3x^2-4}=\frac{3}{10}\frac{1}{x-2}-\frac{1}{10}\frac{1}{x+2}+\frac{1}{5}\frac{-x+1}{x^2+1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{\left(\theta^2+1\right)^3}$

Solution

$$\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{\left(\theta^2+1\right)^3}=\frac{A\theta+B}{\theta^2+1}+\frac{C\theta+D}{\left(\theta^2+1\right)^2}+\frac{E\theta+F}{\left(\theta^2+1\right)^3}$$

$$\begin{aligned}\theta^4-4\theta^3+2\theta^2-3\theta+1&=(A\theta+B)\left(\theta^2+1\right)^2+(C\theta+D)\left(\theta^2+1\right)+E\theta+F\\&=(A\theta+B)\left(\theta^4+2\theta^2+1\right)+C\theta^3+C\theta+D\theta^2+D+E\theta+F\\&=A\theta^5+B\theta^4+(2A+C)\theta^3+(2B+D)\theta^2+(A+C+E)\theta+B+D+F\end{aligned}$$

$$\begin{cases}\boxed{A=0}\\ \boxed{B=1}\\ 2A+C=-4\\ 2B+D=2\\ A+C+E=-3\\ B+D+F=1\end{cases}\quad\rightarrow\boxed{C=-4}\quad\boxed{D=0}\quad\boxed{E=1}\quad\boxed{F=0}$$

$$\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{\left(\theta^2+1\right)^3}=\frac{1}{\theta^2+1}-4\frac{\theta}{\left(\theta^2+1\right)^2}+\frac{\theta}{\left(\theta^2+1\right)^3}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{3x^2+7x-2}{x^3-x^2-2x}$

Solution

$$\frac{3x^2+7x-2}{x^3-x^2-2x}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-2}$$

$$\begin{aligned}3x^2+7x-2&=A(x+1)(x-2)+Bx(x-2)+Cx(x+1)\\&=Ax^2-Ax-2A\\&\quad Bx^2-2Bx\\&\quad Cx^2+Cx\end{aligned}$$

$$\begin{cases} A+B+C=3 \\ -A-2B+C=7 \\ -2A=-2 & \rightarrow \underline{A=1} \end{cases}$$

$$\begin{cases} B+C=2 \\ -2B+C=8 \end{cases} \rightarrow \underline{B=-2} \quad \underline{C=4}$$

$$\frac{3x^2+7x-2}{x^3-x^2-2x} = \frac{1}{x} - \frac{2}{x+1} + \frac{4}{x-2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)}$

Solution

$$\frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)} = \frac{A}{x-1} + \frac{B}{x-5} + \frac{C}{x+4}$$

$$3x^2+2x+5 = (A+B+C)x^2 + (-A+3B-6C)x - 20A - 4B + 5C$$

$$\begin{cases} x^2 & A+B+C=3 \\ x & -A+3B-6C=2 \\ x^0 & -20A-4B+5C=5 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 3 & -6 \\ -20 & -4 & 5 \end{vmatrix} = 180 \qquad D_A = \begin{vmatrix} 3 & 1 & 1 \\ 2 & 3 & -6 \\ 5 & -4 & 5 \end{vmatrix} = -90$$

$$D_B = \begin{vmatrix} 1 & 3 & 1 \\ -1 & 2 & -6 \\ -20 & 5 & 5 \end{vmatrix} = 450 \qquad D_C = \begin{vmatrix} 1 & 1 & 3 \\ -1 & 3 & 2 \\ -20 & -4 & 5 \end{vmatrix} = 180$$

$$A = \frac{1}{2}, \quad B = \frac{5}{2}, \quad C = 1$$

$$\frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)} = \frac{1}{2}\frac{1}{x-1} + \frac{5}{2}\frac{1}{x-5} + \frac{1}{x+4}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{5x^2-3x+2}{x^3-2x^2}$

Solution

$$\frac{5x^2-3x+2}{x^3-2x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}$$

$$5x^2-3x+2=Ax^2-2Ax+Bx-2B+Cx^2$$

$$\begin{cases} x^2 & A+C=5 & C=4 \\ x & -2A+B=-3 & A=1 \\ x^0 & -2B=2 & \rightarrow \underline{B=-1} \end{cases}$$

$$\boxed{\frac{5x^2-3x+2}{x^3-2x^2}=\frac{1}{x}-\frac{1}{x^2}+\frac{4}{x-2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}$

Solution

$$\frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}=\frac{A}{x-2}+\frac{Bx+C}{x^2-2x+3}$$

$$7x^2-13x+13=Ax^2-2Ax+3A+Bx^2-2Bx+Cx-2C$$

$$\begin{cases} x^2 & A+B=7 \\ x^1 & -2A-2B+C=-13 \\ x^0 & 3A-2C=13 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 & 0 \\ -2 & -2 & 1 \\ 3 & 0 & -2 \end{vmatrix}=3 \qquad D_A=\begin{vmatrix} 7 & 1 & 0 \\ -13 & -2 & 1 \\ 13 & 0 & -2 \end{vmatrix}=15$$

$$D_B=\begin{vmatrix} 1 & 7 & 0 \\ -2 & -13 & 1 \\ 3 & 13 & -2 \end{vmatrix}=6 \qquad D_C=\begin{vmatrix} 1 & 1 & 7 \\ -2 & -2 & -13 \\ 3 & 0 & 13 \end{vmatrix}=3$$

$$\boxed{A=5;\ \ B=2;\ \ C=1}$$

$$\boxed{\frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}=\frac{5}{x-2}+\frac{2x+1}{x^2-2x+3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{1}{x^2-5x+6}$

Solution

$$\frac{1}{x^2-5x+6}=\frac{A}{x-2}+\frac{B}{x-3}$$

$$Ax-3A+Bx-2B=1$$

$$\to\begin{cases}A+B=0\\-3A-2B=1\end{cases}\quad\to\quad A=-1\quad B=1$$

$$\boxed{\frac{1}{x^2-5x+6}=\frac{-1}{x-2}+\frac{1}{x-3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{1}{x^2-5x+5}$

Solution

$$\frac{1}{x^2-5x+5}=\frac{A}{x-\frac{5+\sqrt{5}}{2}}+\frac{B}{x-\frac{5-\sqrt{5}}{2}}$$

$$Ax-\left(\frac{5-\sqrt{5}}{2}\right)A+Bx-\left(\frac{5+\sqrt{5}}{2}\right)B=1$$

$$\begin{cases}A+B=0\\-\frac{5-\sqrt{5}}{2}A-\frac{5+\sqrt{5}}{2}B=1\end{cases}\quad\to\quad\begin{array}{l}\frac{5-\sqrt{5}}{2}A+\frac{5-\sqrt{5}}{2}B=0\\-\frac{5-\sqrt{5}}{2}A-\frac{5+\sqrt{5}}{2}B=1\end{array}$$

$$-\sqrt{5}B=1\to \boxed{B=-\frac{1}{\sqrt{5}}}\quad\Rightarrow\quad\boxed{A=\frac{1}{\sqrt{5}}}$$

$$\boxed{\frac{1}{x^2-5x+5}=\frac{\sqrt{5}}{5}\frac{2}{2x-5-\sqrt{5}}-\frac{\sqrt{5}}{5}\frac{2}{2x-5+\sqrt{5}}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{5x^2+20x+6}{x^3+2x^2+x}$

Solution

$$\frac{5x^2+20x+6}{x^3+2x^2+x}=\frac{5x^2+20x+6}{x(x+1)^2}$$

$$=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$Ax^2+2Ax+A+Bx^2+Bx+Cx=5x^2+20x+6$$

$$\begin{cases} A+B=5 \\ 2A+B+C=20 & \rightarrow \quad \underline{B=-1|} \quad \underline{C=9|} \\ \underline{A=6|} \end{cases}$$

$$\boxed{\frac{5x^2+20x+6}{x^3+2x^2+x}=\frac{6}{x}-\frac{1}{x+1}+\frac{9}{(x+1)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}$

Solution

$$\frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}=\frac{2x^3-4x-8}{x(x-1)\left(x^2+4\right)}=\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+4}$$

$$Ax^3-Ax^2+4Ax-4A+Bx^3+4Bx+Cx^3-Cx^2+Dx^2-Dx=2x^3-4x-8$$

$$\begin{cases} x^3 & A+B+C=2 \\ x^2 & -A-C+D=0 \\ x^1 & 4A+4B-D=-4 \\ x^0 & -4A=-8 \end{cases} \rightarrow \begin{cases} B+C=0 \\ -C+D=2 \\ 4B-D=-12 \\ \underline{A=2|} \end{cases}$$

$$\Rightarrow \begin{cases} B+D=2 \\ 4B-D=-12 \end{cases}$$

$$\underline{A=2 \quad B=-2 \quad C=2 \quad D=4|}$$

$$\frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}=\frac{2}{x}-\frac{2}{x-1}+\frac{2x}{x^2+4}+\frac{4}{x^2+4}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{8x^3+13x}{\left(x^2+2\right)^2}$

Solution

$$\frac{8x^3+13x}{\left(x^2+2\right)^2}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{\left(x^2+2\right)^2}$$

$$Ax^3+2Ax+Bx^2+2B+Cx+D=8x^3+13x$$

$$\begin{cases} x^3 & A=8 \\ x^2 & B=0 \\ x^1 & 2A+C=13 \\ x^0 & D=0 \end{cases} \rightarrow C=-3$$

$$\frac{8x^3+13x}{\left(x^2+2\right)^2}=\frac{8x}{x^2+2}-\frac{3x}{\left(x^2+2\right)^2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{1}{x^2-9}$

Solution

$$\frac{1}{x^2-9}=\frac{A}{x-3}+\frac{B}{x+3}$$

$$Ax+3A+Bx-3B=1$$

$$\Rightarrow\begin{cases} A+B=0 \\ 3A-3B=1 \end{cases} \rightarrow A=\frac{1}{6} \quad B=-\frac{1}{6}$$

$$\frac{1}{x^2-9}=\frac{1}{6}\frac{1}{x-3}-\frac{1}{6}\frac{1}{x+3}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{2}{9x^2-1}$

Solution

$$\frac{2}{9x^2-1}=\frac{A}{3x-1}+\frac{B}{3x+1}$$

$$3Ax+A+3Bx-B=2$$

$$\Rightarrow\begin{cases}3A+3B=0\\ A-B=2\end{cases}\rightarrow \underline{A=1\quad B=-1}$$

$$\frac{2}{9x^2-1}=\frac{1}{3x-1}-\frac{1}{3x+1}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{5}{x^2+3x-4}$

Solution

$$\frac{5}{x^2+3x-4}=\frac{A}{x-1}+\frac{B}{x+4}$$

$$Ax+4A+Bx-B=5$$

$$\Rightarrow\begin{cases}A+B=0\\ 4A-B=5\end{cases}\rightarrow \underline{A=1\quad B=-1}$$

$$\frac{5}{x^2+3x-4}=\frac{1}{x-1}-\frac{1}{x+4}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{3-x}{3x^2-2x-1}$

Solution

$$\frac{3-x}{3x^2-2x-1}=\frac{A}{x-1}+\frac{B}{3x+1}$$

$$3Ax+A+Bx-B=3-x$$

$$\Rightarrow\begin{cases}3A+B=-1\\ A-B=3\end{cases}\rightarrow \underline{A=\frac{1}{2}\quad B=-\frac{5}{2}}$$

$$\frac{3-x}{3x^2-2x-1}=\frac{1}{2}\frac{1}{x-1}-\frac{5}{2}\frac{1}{3x+1}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2+12x+12}{x^3-4x}$

Solution

$$\frac{x^2+12x+12}{x^3-4x}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2}$$

$$Ax^2-4A+Bx^2+2Bx+Cx^2-2Cx=x^2+12x+12$$

$$\begin{cases} x^2 & A+B+C=1 \\ x^1 & 2B-2C=12 \\ x^0 & -4A=12 \end{cases} \quad \rightarrow A=-3 \quad B=5 \quad C=-1$$

$$\boxed{\frac{x^2+12x+12}{x^3-4x}=-\frac{3}{x}+\frac{5}{x-2}-\frac{1}{x+2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{5x-2}{(x-2)^2}$

Solution

$$\frac{5x-2}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}$$

$$Ax-2A+B=5x-2$$

$$\Rightarrow \begin{cases} \underline{A=5|} \\ -2A+B=-2 \end{cases} \quad \rightarrow \quad \underline{B=8|}$$

$$\boxed{\frac{5x-2}{(x-2)^2}=\frac{5}{x-2}+\frac{8}{(x-2)^2}}$$

Solution   Section 5.3 – Ellipses

Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $\frac{x^2}{9}+\frac{y^2}{4}=1$

Solution

$$\begin{cases} a^2=9 \to a=3 \\ b^2=4 \to b=2 \end{cases}$$

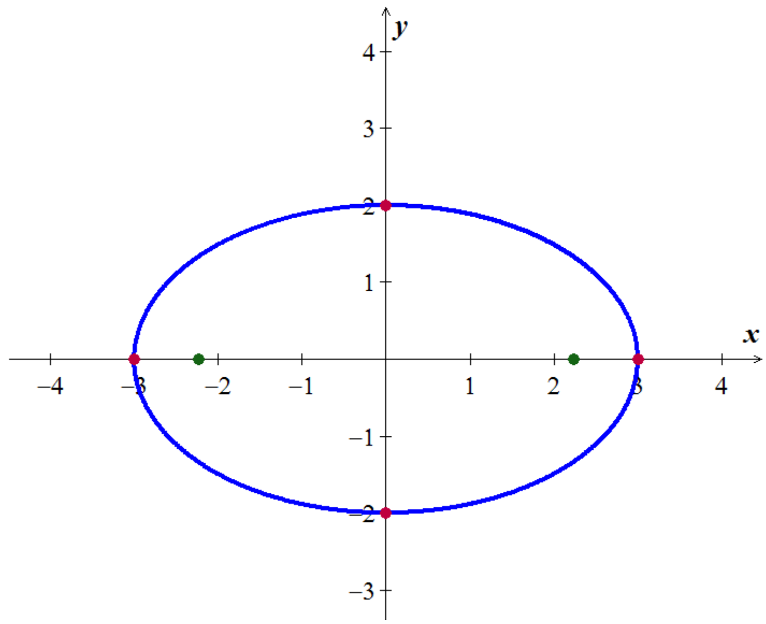
$$c=\sqrt{a^2-b^2}=\sqrt{9-4}=\sqrt{5}$$

Center: $C(0,\ 0)$

Vertices: $V(\pm 3,\ 0)$

Minor $M(0,\ \pm 2)$

Foci $F(\pm\sqrt{5},\ 0)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $\frac{x^2}{16}+\frac{y^2}{36}=1$

Solution

$$\begin{cases} a^2=36 \to a=6 \\ b^2=16 \to b=4 \end{cases}$$

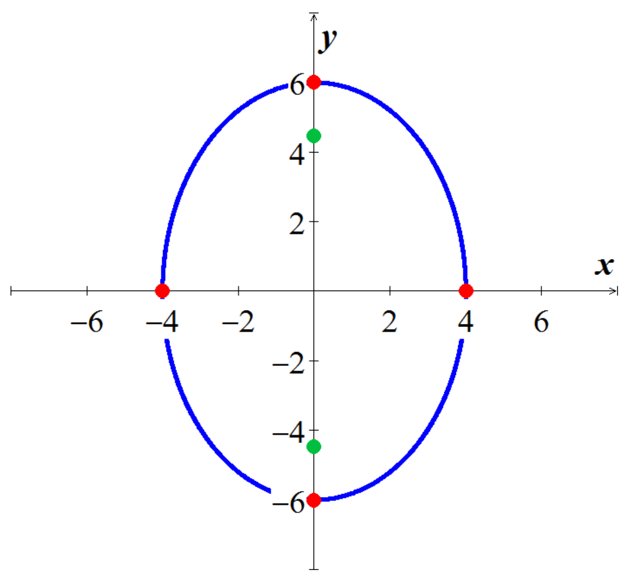
$$c=\sqrt{a^2-b^2}=\sqrt{36-16}=2\sqrt{5}$$

Center: $C(0,\ 0)$

Vertices: $V(0,\ \pm 6)$

Minors $M(\pm 4,\ 0)$

Foci $F(0,\ \pm 2\sqrt{5})$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $\frac{x^2}{15}+\frac{y^2}{16}=1$

Solution

$$\begin{cases} a^2=16 \to a=4 \\ b^2=15 \to b=\sqrt{15} \end{cases}$$

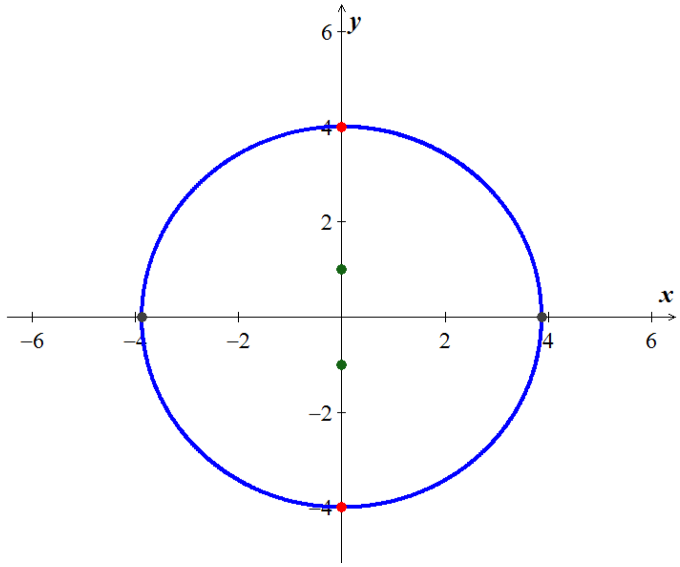
$$c=\sqrt{a^2-b^2}=\sqrt{16-15}=1$$

Center: $C(0,\ 0)$

Vertices: $V(0,\ \pm 4)$

Minors $M\left(\pm\sqrt{15},0\right)$

Foci $F(0,\ \pm 1)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $\frac{25x^2}{36}+\frac{64y^2}{9}=1$

Solution

$$\frac{x^2}{\frac{36}{25}}+\frac{y^2}{\frac{9}{64}}=1$$

$$\begin{cases} a^2=\frac{36}{25} \to a=\frac{6}{5} \\ b^2=\frac{9}{64} \to b=\frac{3}{8} \end{cases}$$

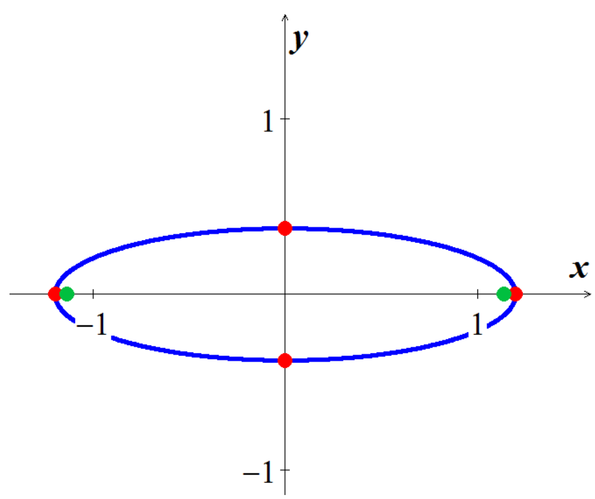
$$c=\sqrt{a^2-b^2}=\sqrt{\frac{36}{25}-\frac{9}{64}}=\sqrt{\frac{2079}{1600}}=\frac{3\sqrt{231}}{40}$$

Center: $C(0,\ 0)$

Vertices: $V\left(\pm\frac{6}{5},\ 0\right)$

Minor $M\left(0,\ \pm\frac{3}{8}\right)$

Foci $F\left(\pm\frac{3\sqrt{231}}{40},\ 0\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $12x^2 + 8y^2 = 96$

Solution

$$\frac{12}{96}x^2 + \frac{8}{96}y^2 = \frac{96}{96}$$

$$\frac{x^2}{8} + \frac{y^2}{12} = 1$$

$$\rightarrow \begin{cases} a^2 = 12 \rightarrow a = 2\sqrt{3} \\ b^2 = 8 \rightarrow b = 2\sqrt{2} \end{cases}$$

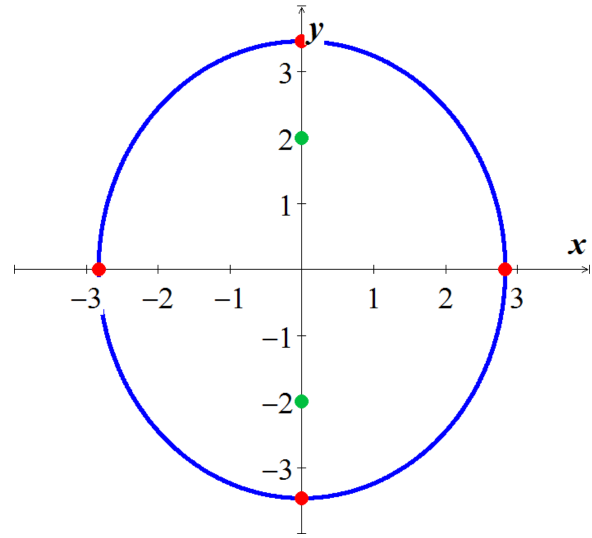
$$c = \sqrt{a^2 - b^2} = \sqrt{12 - 8} = 2$$

Center: $C(0,\ 0)$

Vertices: $V\left(0,\ \pm 2\sqrt{3}\right)$

Minors $M\left(\pm 2\sqrt{2}, 0\right)$

Foci $F(0,\ \pm 2)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $4x^2 + y^2 = 16$

Solution

$$\frac{1}{16}4x^2 + \frac{1}{16}y^2 = \frac{1}{16}16$$

$$\frac{x^2}{4} + \frac{y^2}{16} = 1$$

$$\rightarrow \begin{cases} a^2 = 16 \rightarrow a = 4 \\ b^2 = 4 \rightarrow b = 2 \end{cases}$$

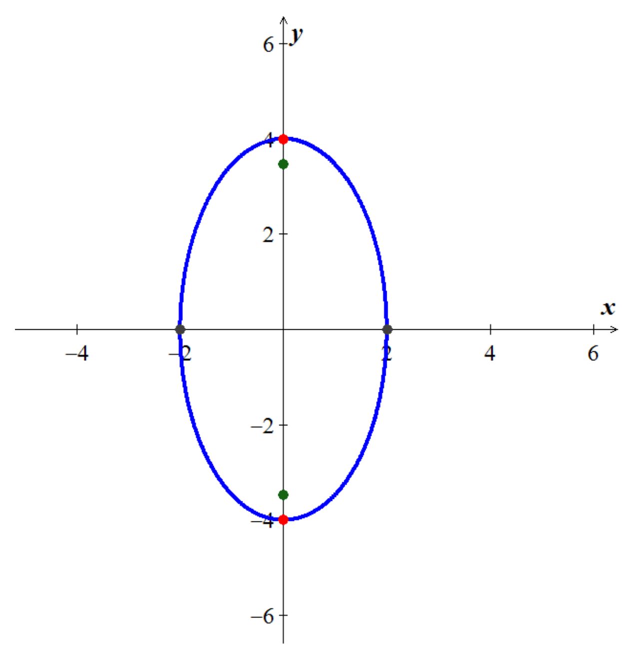
$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 4} = \sqrt{12} = 2\sqrt{3}$$

Center: $C(0,\ 0)$

Vertices: $V(0,\ \pm 4)$

Minors $M(\pm 2, 0)$

Foci $F\left(0,\ \pm 2\sqrt{3}\right)$



Exercise

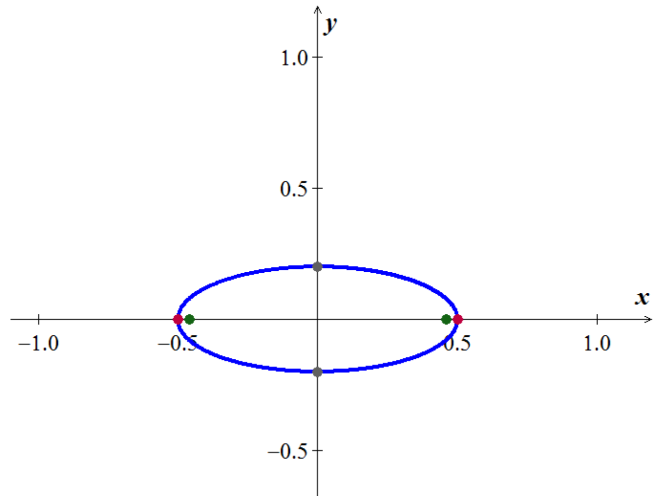
Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $4x^2 + 25y^2 = 1$

Solution

$$\frac{x^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{25}} = 1$$

$$\begin{cases} a^2 = \frac{1}{4} \to a = \frac{1}{2} \\ b^2 = \frac{1}{25} \to b = \frac{1}{5} \end{cases}$$

$$c = \sqrt{a^2 - b^2} = \sqrt{\frac{1}{4} - \frac{1}{25}} = \sqrt{\frac{21}{100}} = \frac{\sqrt{21}}{10}$$



Center: $C(0,\ 0)$

Vertices: $V\left(\pm\frac{1}{2},\ 0\right)$

Minor $M\left(0,\ \pm\frac{1}{5}\right)$

Foci $F\left(\pm\frac{\sqrt{21}}{10},\ 0\right)$

Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$$\frac{(x-3)^2}{16} + \frac{(y+4)^2}{9} = 1$$

Solution

$$\begin{cases} a^2 = 16 \to a = 4 \\ b^2 = 9 \to b = 3 \end{cases}$$

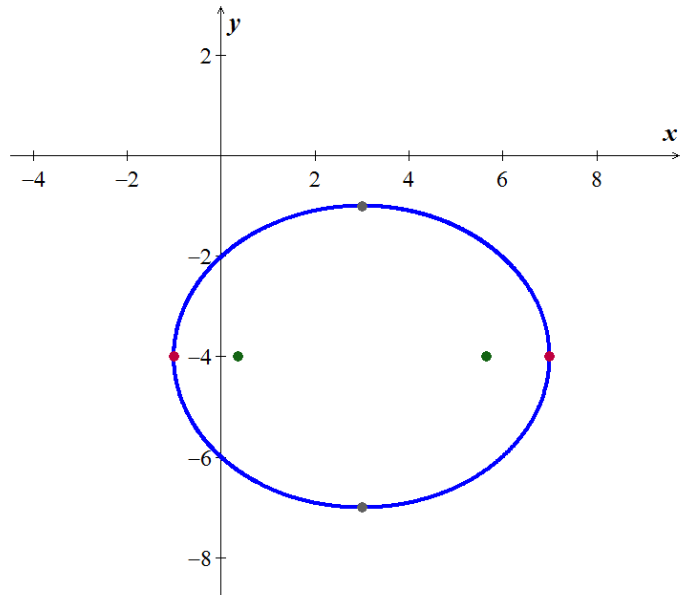
$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$$

Center: $C(3, \ -4)$

Vertices: $V(3 \pm 4, \ -4)$

Minor $M(3, \ -4 \pm 3)$

Foci $F\left(3 \pm \sqrt{7}, \ -4\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$$\frac{(x+3)^2}{16}+\frac{(y-2)^2}{36}=1$$

Solution

$$\begin{cases} a^2=36 \to a=6 \\ b^2=16 \to b=4 \end{cases}$$

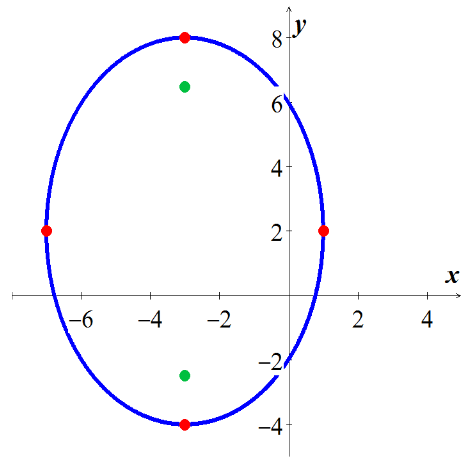
$$c=\sqrt{36-16}=2\sqrt{5}$$

Center: $C(-3,\ 2)$

Vertices: $V(-3,\ 2\pm 6)$

Minor $M(-3\pm 4,\ 2)$

Foci $F\left(-3,\ 2\pm 2\sqrt{5}\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$$\frac{(x+1)^2}{64}+\frac{(y-2)^2}{49}=1$$

Solution

$$\begin{cases} a^2=64 \to a=8 \\ b^2=49 \to b=7 \end{cases}$$

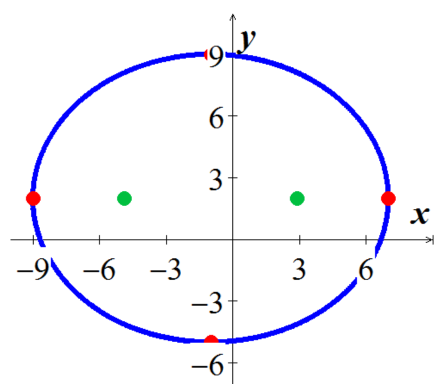
$$c=\sqrt{a^2-b^2}=\sqrt{64-49}=\sqrt{15}$$

Center: $C(-1,\ 2)$

Vertices: $V(-1\pm 8,\ 2)$

Minor $M(-1,\ 2\pm 7)$

Foci $F\left(-1\pm\sqrt{15},\ 2\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$4x^2+9y^2-32x-36y+64=0$

Solution

$$4\left(x^2-8x+\left(\frac{8}{2}\right)^2\right)+9\left(y^2-4y+\left(\frac{4}{2}\right)^2\right)=-64+4(16)+9(4)$$

$$4(x-4)^2+9(y-2)^2=36$$

$$\frac{(x-4)^2}{9}+\frac{(y-2)^2}{4}=1$$

$$\begin{cases}a^2=9\to a=3\\ b^2=4\to b=2\end{cases}$$

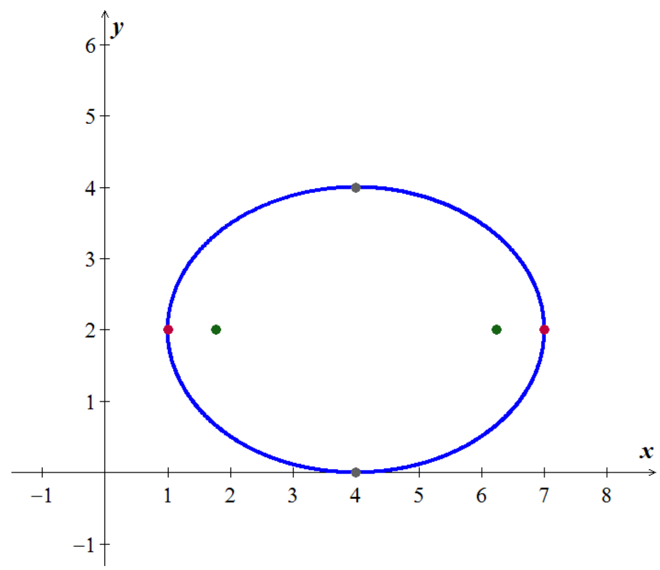
$$c=\sqrt{a^2-b^2}=\sqrt{9-4}=\sqrt{5}$$

Center: $C(4,\ 2)$

Vertices: $(4\pm3,\ 2)$ $V'(1,\ 2)$ $V(7,\ 2)$

Minor $(4,\ 2\pm2)$ $M'(4,\ 0)$ $M(4,\ 4)$

Foci $F\left(4\pm\sqrt{5},\ 2\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$x^2+2y^2+2x-20y+43=0$

Solution

$$\left(x^2+2x+\left(\frac{2}{2}\right)^2\right)+2\left(y^2-10y+\left(\frac{10}{2}\right)^2\right)=-43+1+2(100)$$

$$(x+1)^2+2(y-5)^2=8$$

$$\frac{(x+1)^2}{8}+\frac{(y-5)^2}{4}=1$$

$$\begin{cases}a^2=8\to a=2\sqrt{2}\\ b^2=4\to b=2\end{cases}$$

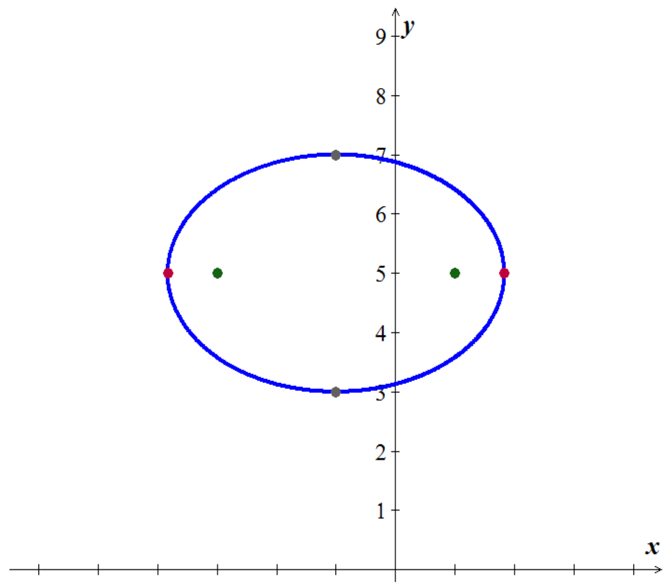
$$c=\sqrt{a^2-b^2}=\sqrt{8-4}=2$$

Center: $C(-1,\ 5)$

Vertices: $V\left(-1 \pm 2\sqrt{2},\ 5\right)$

Minor $(-1,\ 5 \pm 2) \quad \to \quad M'(-1,\ 3) \quad M(-1,\ 7)$

Foci $(-1 \pm 2,\ 5) \quad \to \quad F'(-3,\ 5) \quad F(1,\ 5)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $25x^2 + 4y^2 - 250x - 16y + 541 = 0$

Solution

$$25\left(x^2 - 10x + \left(\frac{10}{2}\right)^2\right) + 4\left(y^2 - 4y + \left(\frac{4}{2}\right)^2\right) = -541 + 25(25) + 4(4)$$

$$25(x-5)^2 + 4(y-2)^2 = 100$$

$$\frac{(x-5)^2}{4} + \frac{(y-2)^2}{25} = 1$$

$$\begin{cases} a^2 = 25 \to a = 5 \\ b^2 = 4 \to b = 2 \end{cases}$$

$$c = \sqrt{a^2 - b^2} = \sqrt{25-4}$$

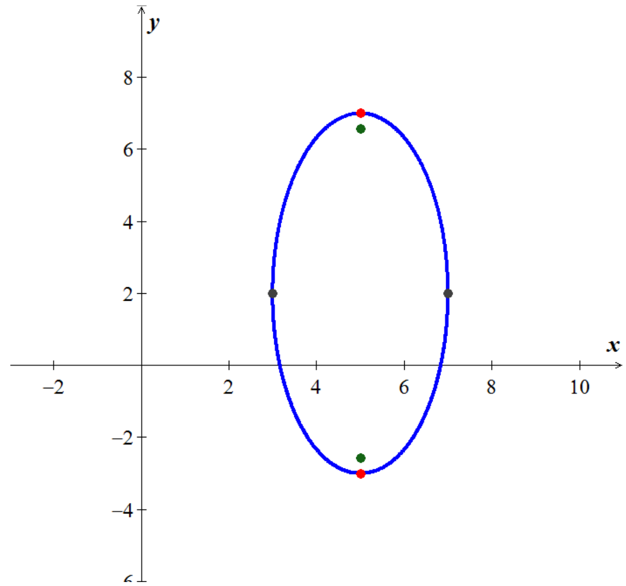
$$= \sqrt{21}$$

Center: $C(5,\ 2)$

Vertices: $(5,\ 2 \pm 5) \quad \to \quad V'(5,\ -3) \quad V(5,\ 7)$

Minor $(5 \pm 2,\ 2) \to \quad M(3,\ 2) \quad M(7,\ 2)$

Foci $F\left(5,\ 2 \pm \sqrt{21}\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $4x^2 + y^2 = 2y$

Solution

$$4x^2 + y^2 - 2y = 0$$

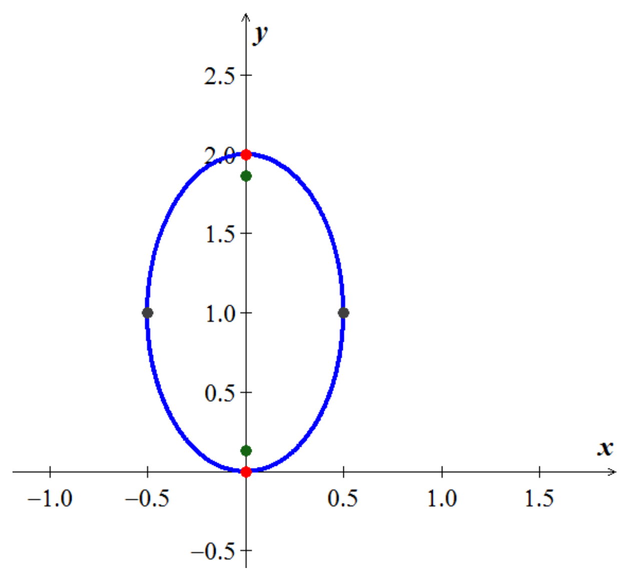
$$4x^2 + \left(y^2 - 2y + \left(\frac{2}{2}\right)^2\right) = (1)^2$$

$$4x^2 + (y-1)^2 = 1$$

$$\frac{x^2}{\frac{1}{4}} + \frac{(y-1)^2}{1} = 1$$

$$\begin{cases} a^2 = 1 \to a = 1 \\ b^2 = \frac{1}{4} \to b = \frac{1}{2} \end{cases}$$

$$c = \sqrt{a^2 - b^2} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$



Center: $C(0,\ 1)$

Vertices: $(0,\ 1 \pm 1) \to \quad V'(0,\ 0) \quad V(0,\ 2)$

Minor $\left(0 \pm \frac{1}{2},\ 1\right) \to \quad M'\left(-\frac{1}{2},\ 1\right) \quad M\left(\frac{1}{2},\ 1\right)$

Foci $F\left(0,\ 1 \pm \frac{\sqrt{3}}{2}\right)$

Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse Sketch the graph: $2x^2+3y^2-8x+6y+5=0$

Solution

$$2x^2-8x+3y^2+6y=-5$$

$$2\left(x^2-4x+\left(\frac{-4}{2}\right)^2\right)+3\left(y^2+2y+\left(\frac{2}{2}\right)^2\right)=-5+2\left(\frac{-4}{2}\right)^2+3\left(\frac{2}{2}\right)^2$$

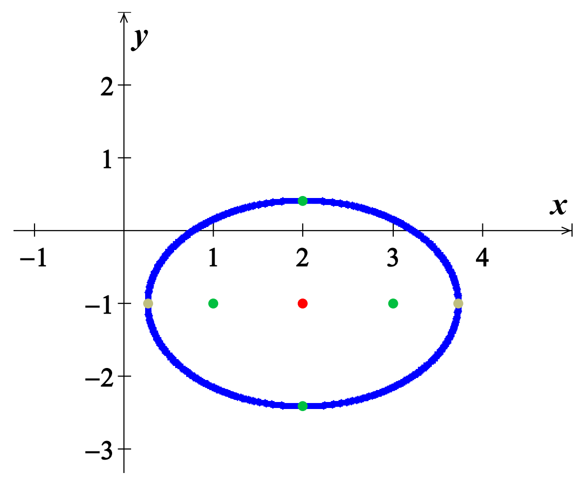
$$2(x-2)^2+3(y+1)^2=-5+8+3$$

$$2(x-2)^2+3(y+1)^2=6$$

$$\frac{2(x-2)^2}{6}+\frac{3(y+1)^2}{6}=1$$

$$\frac{(x-2)^2}{3}+\frac{(y+1)^2}{2}=1$$

$$\begin{cases} a^2=3\to a=\pm\sqrt{3} \\ b^2=2\to b=\pm\sqrt{2} \\ \hline c=\sqrt{a^2-b^2}=\sqrt{1}=1 \end{cases}$$



Center: $(2,\ -1)$

Vertices: $V\left(2\pm\sqrt{3},\ -1\right)$

Minor $M\left(2,\ -1\pm\sqrt{2}\right)$

Foci $(2\pm1,\ -1)\to\quad F'=(1,\ -1)\quad F=(3,\ -1)$

Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$4x^2+3y^2+8x-6y-5=0$

Solution

$$4x^2+8x+3y^2-6y=5$$

$$4\left(x^2+2x+\left(\frac{2}{2}\right)^2\right)+3\left(y^2-2y+\left(\frac{-2}{2}\right)^2\right)=5+4\left(\frac{2}{2}\right)^2+3\left(\frac{-2}{2}\right)^2$$

$$4(x+1)^2+3(y-1)^2=5+4+3$$

$$4(x+1)^2+3(y-1)^2=12$$

$$\frac{4(x+1)^2}{12}+\frac{3(y-1)^2}{12}=1$$

$$\frac{(x+1)^2}{3}+\frac{(y-1)^2}{4}=1$$

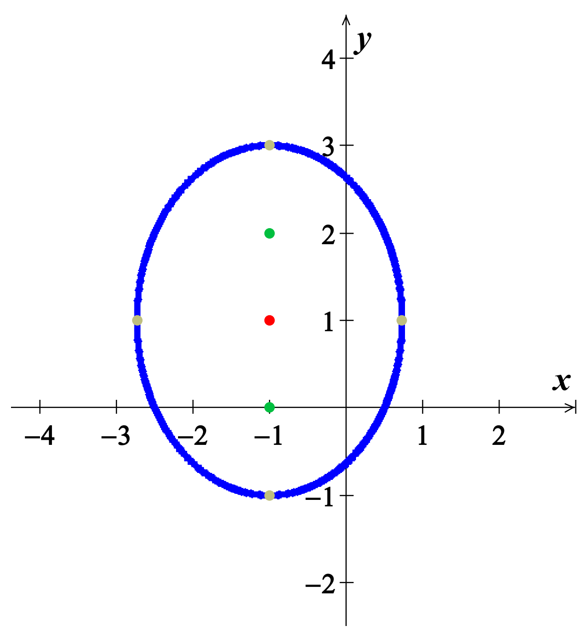
$$\begin{cases} a^2=4\to a=\pm 2 \\ b^2=3\to b=\pm\sqrt{3} \\ c=\pm\sqrt{a^2-b^2}=\pm\sqrt{4-3}=\pm 1 \end{cases}$$

Center: $(-1,\ 1)$

Vertices: $(-1,\ 1\pm 2)\to\quad V'(-1,\ -1)\quad V(-1,\ 3)$

Minor $M\left(-1\pm\sqrt{3},\ 1\right)$

Foci $(-1,\ 1\pm 1)\to\quad F'=(-1,\ 0)\quad F=(-1,\ 2)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$9x^2 + 4y^2 - 18x + 16y - 11 = 0$

Solution

$$9x^2 - 18x + 4y^2 + 16y = 11$$

$$9\left(x^2 - 2x + \left(\frac{-2}{2}\right)^2\right) + 4\left(y^2 + 4y + \left(\frac{4}{2}\right)^2\right) = 11 + 9\left(\frac{-2}{2}\right)^2 + 4\left(\frac{4}{2}\right)^2$$

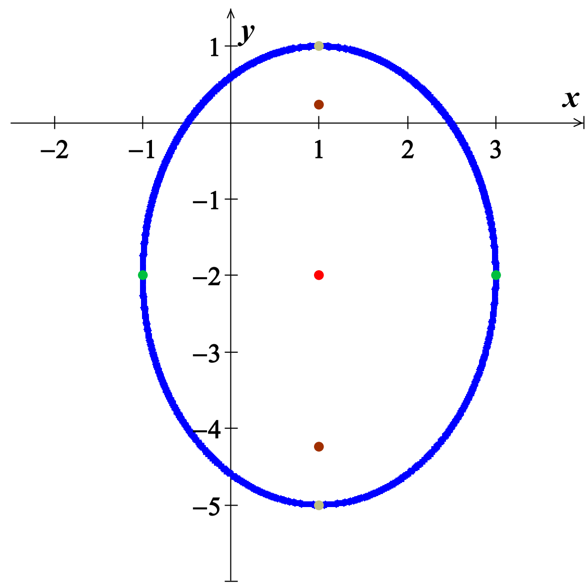
$$9(x-1)^2 + 4(y+2)^2 = 11 + 9 + 16$$

$$9(x-1)^2 + 4(y+2)^2 = 36$$

$$\frac{9(x-1)^2}{36} + \frac{4(y+2)^2}{36} = 1$$

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$$

$$\begin{cases} a^2 = 9 \to a = \pm 3 \\ b^2 = 4 \to b = \pm 2 \\ c = \pm\sqrt{a^2 - b^2} = \pm\sqrt{9-4} = \pm\sqrt{5} \end{cases}$$



Center: $(1, \ -2)$

Vertices: $(1, \ -2 \pm 3) \to \quad V'(1, \ -5) \quad V(1, \ 1)$

Minor: $(1 \pm 2, \ -2) \to \quad M'(-1, \ -2) \quad M(3, \ -2)$

Foci $\left(1, \ -2 \pm \sqrt{5}\right)$

Exercise

Find an equation for an ellipse with: $x - intercepts\!: \ \pm 4; \quad foci \ (-2, \ 0) \ and \ (2, \ 0)$

Solution

The ellipse is centered at $(0, \ 0)$

Major axis: $a = 4$

Foci: $(\pm 2, \ 0) \quad \Rightarrow \quad c = 2$

$b^2 = a^2 - c^2 = 16 - 4 = 12$

The equation is: $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$

Exercise

Find an equation for an ellipse with: *Endpoints of major axis at* $(6,\ 0)$ *and* $(-6,\ 0)$; $c = 4$

Solution

The ellipse is centered at $(0,\ 0)$ between the endpoint of the major axis

Major axis: $a = 6$

$b^2 = a^2 - c^2 = 36 - 16 = 20$

The equation is: $\frac{x^2}{36} + \frac{y^2}{20} = 1$

Exercise

Find an equation for an ellipse with: Center $(3,-2)$; $a = 5$; $c = 3$; major axis vertical

Solution

The ellipse is centered at $(3,-2)$

$b^2 = a^2 - c^2 = 25 - 9 = 16$

The equation is: $\frac{(x-3)^2}{25} + \frac{(y+2)^2}{16} = 1$

Exercise

Find an equation for an ellipse with: *major axis of length* 6; *foci* $(0,\ 2)$ *and* $(0,\ -2)$

Solution

The ellipse is centered between the foci at $(0,\ 0)$

Major axis is the vertical with $a = 3$

Foci: $(0,\ \pm 2) \ \Rightarrow \ c = 2$

$b^2 = a^2 - c^2 = 9 - 4 = 5$

The equation is: $\frac{y^2}{9} + \frac{x^2}{5} = 1$

Exercise

A patient's kidney stone is placed 12 *units* away from the source of the shock waves of a lithotripter. The lithotripter is based on an ellipse with a minor axis that measures 16 *units*. Find an equation of an ellipse that would satisfy this situation.

Solution

The patient and the emitter are 12 units apart $\Rightarrow$ these represent the foci of an ellipse, so $c = 6$.

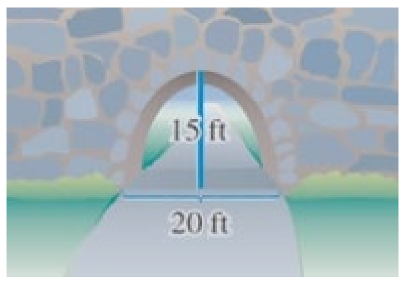
The minor axis: 16 units $\Rightarrow b = 8$.

$\therefore\ a^2 = b^2 + c^2 = 64 + 36 = 100$.

The equation is: $\dfrac{x^2}{100} + \dfrac{y^2}{64} = 1$

Exercise

A one-way road passes under an overpass in the form of half of an ellipse 15 *feet* high at the center and 20 *feet* wide. Assuming that a truck is 12 *feet* wide, what is the height of the tallest truck that can pass under the overpass?



Solution

Using a vertical major axis $\Rightarrow a = 15$.

The minor axis: 20 *ft*. $\Rightarrow b = 10$.

The equation is: $\dfrac{y^2}{225} + \dfrac{x^2}{100} = 1$

Assuming the truck drives through the middle, we want to find y when $x = 6$

$$\frac{y^2}{225} = 1 - \frac{6^2}{100} = \frac{64}{100}$$

$$\Rightarrow\quad y^2 = 225\frac{64}{100}$$

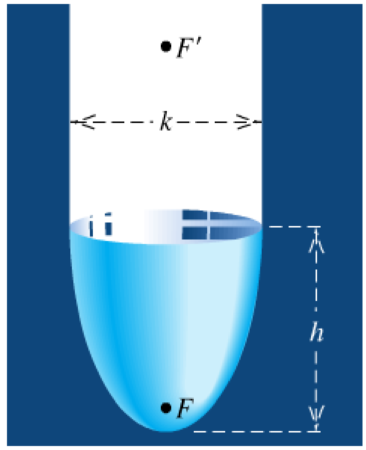
$$y = \sqrt{\frac{225(64)}{100}}$$

$$\underline{= 12}$$

The truck must be just under 12 *feet* high to pass through.

Exercise

The basic shape of an elliptical reflector is a hemi-ellipsoid of height h and diameter k. Waves emitted from focus F will reflect off the surface into focus F'



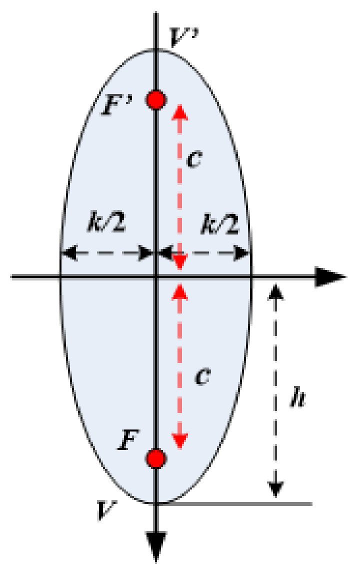
a) Express the distance $d(V,\ F)$ and $d(V,\ F')$ in terms of h and k.

b) An elliptical reflector of height 17 *cm* is to be constructed so that waves emitted from F are reflected to a point F' that is 32 *cm* from V. Find the diameter of the reflector and the location of F.

Solution

Given: $b = \frac{k}{2}, \quad a = h$

$$c^2 = a^2 - b^2 = h^2 - \left(\frac{k}{2}\right)^2$$



a) $d(V,\ F) = h - c$

$$= h - \sqrt{h^2 - \frac{1}{4}k^2}$$

$$d(V,\ F') = h + c$$

$$= h + \sqrt{h^2 - \frac{1}{4}k^2}$$

b) ***Given***: $h = 17\ cm, \quad h + c = 32\ cm$

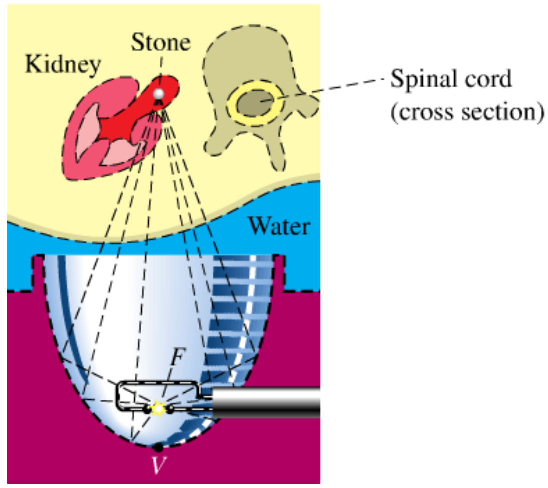
$$c = 32 - h$$
$$= 32 - 17$$
$$= 15\ cm$$

$$d(V,\ F) = h - c$$
$$= 17 - 15$$
$$= 2\ \ cm$$

The location of F is $16\ cm;\ \ 2\ cm\ \ from\ \ V'$

Exercise

A lithotripter of height 15 *cm* and diameter 18 *cm* is to be constructed. High-energy underwater shock waves will be emitted from the focus F that is closest to the vertex V.



a) Find the distance from V to F.

b) How far from V (in the vertical direction) should a kidney stone located?

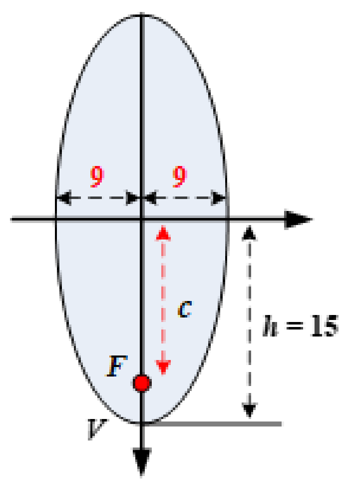
Solution

Given: $b = \frac{18}{2} = 9, \quad a = h = 15$

$$c = \sqrt{a^2 - b^2}$$

$$= \sqrt{15^2 - 9^2}$$

$$= 12 \ cm$$



a) $d(V, F) = h - c$

$$= 15 - 12$$

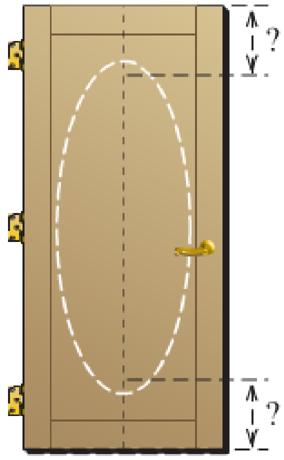
$$= 3 \ cm$$

b) $h + c = 15 + 12$

$$= 27 \ cm$$

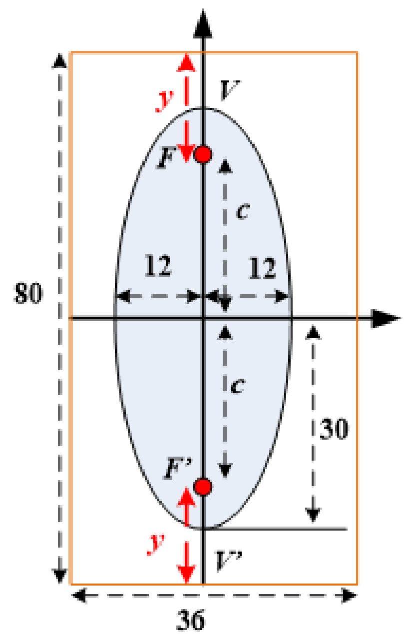
Exercise

An Artist plans to create an elliptical design with major axis 60″ and minor axis 24″, centered on a door that measures 80″ by 36″.



On a vertical line that dissects the door, approximately how far from each end of the door should the push-pins be inserted? How long should the string be?

Solution



Given: $b = \frac{24}{2} = 12''$, $\quad a = \frac{60}{2} = 30''$

$$c = \sqrt{a^2 - b^2}$$
$$= \sqrt{15^2 - 9^2}$$
$$= 27.5$$

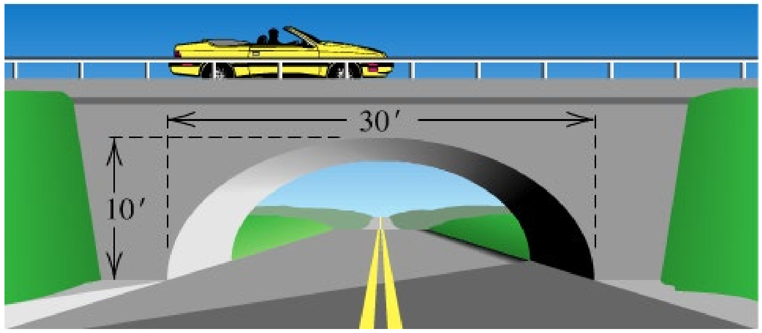
$$2y + 2c = 80$$
$$y = \frac{80 - 2c}{2}$$
$$= \frac{80 - 2\sqrt{756}}{2}$$
$$\approx 12.5''$$

Therefore, the distance from each end of the door should the push-pins be inserted, is $12.5\ in.$

The string should be $= 30 + 30 = 60\ in.$

Exercise

An arch of a bridge is semi-elliptical, with major axis horizontal. The base of the arch is 30 *feet*. across, and the highest part of the arch is 10 *feet*. above the horizontal roadway. Find the height of the arch 6 *feet*. from the center of the base.

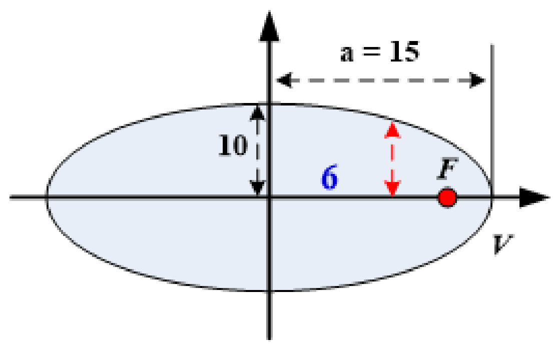


Solution

Given: $b = 10', \quad a = \frac{30}{2} = 15'$

$$c = \sqrt{a^2 - b^2}$$
$$= \sqrt{15^2 - 10^2}$$
$$= \sqrt{125}$$
$$= 5\sqrt{5}$$

$$\frac{x^2}{225} + \frac{y^2}{100} = 1 \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$$\frac{y^2}{100} = 1 - \frac{6^2}{225}$$

$$y^2 = 100\left(1 - \frac{36}{225}\right)$$

$$y = \sqrt{100\left(1 - \frac{36}{225}\right)}$$
$$= \sqrt{84} \approx 9.2 \quad ft$$

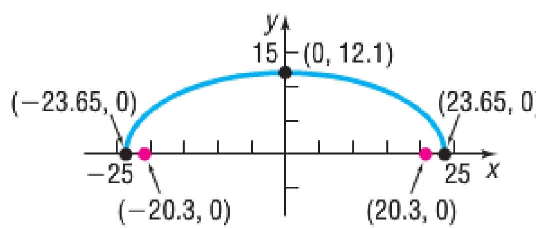
Exercise

The whispering gallery in the Museum of Science and Industry in Chicago is 47.3 *feet* long. The distance from the center of the room to the foci is 20.3 *feet*. Find an equation that describes the shape of the room. How high is the room at its center?

Solution

Set up a rectangular coordinate so that the center of the ellipse is at the origin and the major axis along the x-axis. The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



Length of the room: 47.3 *ft*.

Distance from the center of the room to each vertex:

$$a=\frac{47.3}{2}$$
$$=23.65$$

Distance from the center of the room to each focus is $c=20.3\,ft$

$$b^2=a^2-c^2$$
$$=23.65^2-20.3^2$$
$$=147.2325$$

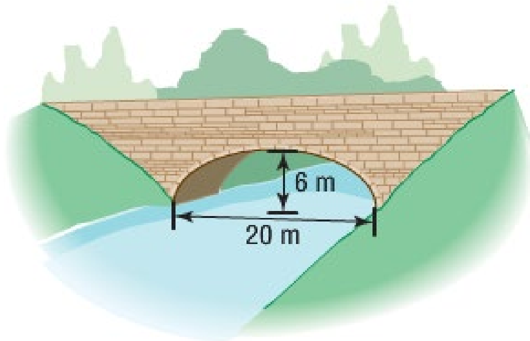
Therefore, the equation is given: $\frac{x^2}{559.3225}+\frac{y^2}{147.2325}=1$

The Height of the room: $b=\sqrt{147.2325}$

$$\approx 12.1\ \ ft$$

Exercise

An arch in the shape of the upper half of an ellipse is used to support a bridge that is to span a river 20 *meters* wide. The center of the arch is 6 *meters* above the center of the river. Write an equation for the ellipse in which the x-axis coincides with the water level and the y-axis passes through the center of the arch.



Solution

The center of the ellipse is $(0,\ 0)$. The length of the major axis is 20, so $a = 10$.

The length of the half minor axis is 6, so $b = 6$.

The ellipse is situated with its major axis on the x-axis.

The equation: $\dfrac{x^2}{10^2} + \dfrac{y^2}{6^2} = 1$

$$\dfrac{x^2}{100} + \dfrac{y^2}{36} = 1$$

Exercise

A bridge is built in the shape of a semielliptical arch. The bridge has a span of 120 *feet* and a maximum height of 25 *feet*. Choose a rectangular coordinate system and find the height of the arch at distances of 10, 30, and 50 *feet* from the center.

Solution

Since the bridge has a span of 120 feet, the length of the major axis is $120 = 2a \ \rightarrow\ a = 60$

The maximum height of the bridge is 25 feet, so $b = 25$.

The equation: $\dfrac{x^2}{60^2} + \dfrac{y^2}{25^2} = 1$

$$\dfrac{x^2}{3600} + \dfrac{y^2}{625} = 1$$

At distance 10 *feet*:

$$\dfrac{10^2}{3600} + \dfrac{y^2}{625} = 1$$

$$\dfrac{y^2}{625} = 1 - \dfrac{100}{3600}$$

$$y^2 = 625\left(1 - \frac{1}{36}\right)$$

$$y = \sqrt{625\left(\frac{35}{36}\right)}$$

The height from the center is $\underline{y \approx 24.65\ ft}$

At distance **30** *feet*:

$$\frac{30^2}{3600} + \frac{y^2}{625} = 1$$

$$\frac{y^2}{625} = 1 - \frac{900}{3600}$$

$$y^2 = 625\left(1 - \frac{9}{36}\right)$$

$$y = \sqrt{625\left(\frac{27}{36}\right)}$$

The height from the center is $\underline{y \approx 21.65\ ft}$

At distance **50** *feet*:

$$\frac{50^2}{3600} + \frac{y^2}{625} = 1$$

$$\frac{y^2}{625} = 1 - \frac{2500}{3600}$$

$$y^2 = 625\left(1 - \frac{25}{36}\right)$$

$$y = \sqrt{625\left(\frac{11}{36}\right)}$$

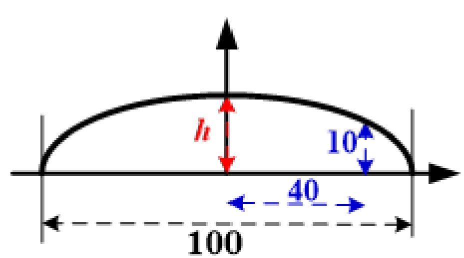
The height from the center is $\underline{y \approx 13.82\ ft}$

Exercise

A bridge is built in the shape of a semielliptical arch. The bridge has a span of 100 *feet*. The height of the arch is 10 *feet*. Find the height of the arch at its center.

Solution

Since the bridge has a span of 100 *feet*.



Length of the major axis is $100 = 2a \ \to\ a = 50$

The maximum height of the bridge is 25 *feet*, so $b = 25$.

The equation: $\frac{x^2}{2500} + \frac{y^2}{h^2} = 1$

The height of the arch 40 *feet* from the center is 10 *feet*.

So $(40, 10)$ is a point on the ellipse.

$$\frac{40^2}{2500} + \frac{10^2}{h^2} = 1$$

$$\frac{10^2}{h^2} = 1 - \frac{1600}{2500}$$

$$\frac{100}{h^2} = 1 - \frac{16}{25}$$

$$\frac{100}{h^2} = \frac{9}{25}$$

$$h^2 = \frac{100 \cdot 25}{9}$$

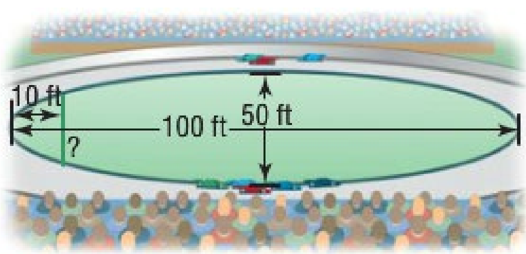
$$h = \sqrt{\frac{100 \cdot 25}{9}}$$

$$\underline{h \approx 16.67}$$

The height of the arch at its center is 16.67 *feet*.

Exercise

A racetrack is in the shape of an ellipse, 100 *feet* long and 50 *feet* wide. What is the width 10 *feet* from a vertex?



Solution

Length of the major axis is $100 = 2a \rightarrow a = 50$

The maximum height of the bridge is $50 = 2b \rightarrow b = 25$.

The equation: $\frac{x^2}{2500} + \frac{y^2}{625} = 1$

We need to find y at $x = 50 - 10 = 40$

$$\frac{40^2}{2500} + \frac{y^2}{625} = 1$$

$$\frac{y^2}{625} = 1 - \frac{1600}{2500}$$

$$y^2 = 625\frac{9}{25}$$

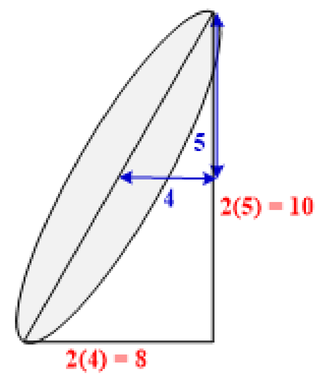
$$\underline{y = 15 \ ft}$$

The width of the ellipse at 10 *feet* from a vertex $x = 40$ is $\boxed{2\times 15 = 30\ ft}$

Exercise

A homeowner is putting in a fireplace that has a 4-*inch* radius vent pipe. He needs to cut an elliptical hole in his roof to accommodate the pipe. If the pitch of his roof is $\frac{5}{4}$ (a rise of 5, run of 4) what are the dimensions of the hole?

Solution



The length of the major axis can be determined from the pitch
By using Pythagorean Theorem:

$a = \sqrt{4^2 + 5^2} = \sqrt{41}$

The length of the major axis $2a\ \boxed{= 2\sqrt{41}\ in}$

The length of the minor axis:

$2b = 2(4) = \boxed{= 8\ \ in}$

Exercise

A football is in the shape of a ***prolate spheroid***, which is simply a solid obtained by rotating an ellipse about its major axis. An inflated NFL football averages 11.125 *inches* in length and 28.25 *inches* in center circumference. If the volume of a prolate spheroid is $\frac{4}{3}\pi ab^2$, how much air does the football contain? (Neglect material thickness)

Solution

The length of the football is $2a = 11.125 \quad \Rightarrow \quad a = 5.5625$

The center circumference is $28.25 = 2\pi b \quad \Rightarrow \quad b = \frac{28.25}{2\pi}$

The volume is:

$$V = \frac{4}{3}\pi ab^2$$

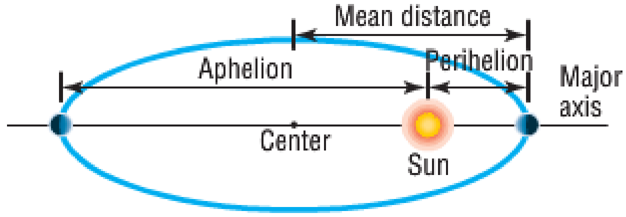
$$= \frac{4}{3}\pi(5.5625)\left(\frac{28.25}{2\pi}\right)^2$$

$$\boxed{\approx 472\ \ in^3}$$

The football contains approximately 471 *cubic inches* of air.

Exercise

The fact that the orbit of a planet about the Sun is an ellipse with the Sun at one focus. The ***aphelion*** of a planet is its greatest distance from the Sun, and the ***perihelion*** is its shortest distance. The ***mean distance*** of a planet from the Sun is the length of the semi-major axis of the elliptical orbit.



a) The mean distance of Earth from the Sun is 93 million *miles*. If the aphelion of Earth is 94.5 million *miles*, what is the perihelion? Write an equation for the orbit of Earth around the Sun.

b) The mean distance of Mars from the Sun is 142 million *miles*. If the perihelion of Mars is 128.5 million *miles*, what is the aphelion? Write an equation for the orbit of Mars about the Sun.

c) The aphelion of Jupiter is 507 million *miles*. If the distance from the center of it elliptical orbit to the Sun is 23.2 million *miles*, what is the perihelion? What is the mean distance? Write an equation for the orbit of Jupiter around the Sun.

d) The perihelion of Pluto is 4551 million *miles*, and the distance from the center of its elliptical orbit to the Sun is 897.5 million *miles*. Find the aphelion of Pluto. What is the mean distance of Pluto from the Sun? Write an equation for the orbit of Pluto about the Sun.

Solution

a) The mean distance is 93 million *miles* $\Rightarrow \underline{a = 93}$

The length of the major axis is 186 million

The perihelion is 186 − 94.5 = 91.5 million *miles*

Distance from the ellipse center to the sun is the focus: $c = 93 - 91.5 = 1.5$ million *miles*.

$$b^2 = a^2 - c^2$$
$$= 93^2 - 1.5^2$$

$$b = \sqrt{93^2 - 1.5^2}$$
$$\underline{= 92.99 \ \ million}$$

Therefore: $a = 93\times 10^6 \quad and \quad b = 92.99\times 10^6$

The equation is given by: $\dfrac{x^2}{\left(93\times 10^6\right)^2} + \dfrac{y^2}{\left(92.99\times 10^6\right)^2} = 1$

Let x and y in millions miles: $\dfrac{x^2}{93^2} + \dfrac{y^2}{92.99^2} = 1 \qquad (in\ millions\ miles)$

The equation of the orbit is: $\boxed{\dfrac{x^2}{8649} + \dfrac{y^2}{8647.14} = 1}$

b) The mean distance is 142 million *miles* $\Rightarrow \underline{a=142}|$

The length of the major axis is 284 million

The perihelion is 284 – 128.5 = 155.5 million *miles*

Distance from the ellipse center to the sun is the focus: c = 142 – 128.5 = 13.5 million *miles*.

$b^2 = a^2 - c^2 = 142^2 - 13.5^2 = 19{,}981.75$

$b = \sqrt{142^2 - 13.5^2}$ $\underline{=141.36 \ \ million}|$

Let x and y in millions miles: $\dfrac{x^2}{142^2} + \dfrac{y^2}{141.36^2} = 1$ $\quad$ $(in\ millions\ miles)$

The equation of the orbit is: $\boxed{\dfrac{x^2}{20{,}164} + \dfrac{y^2}{19{,}981.75} = 1}$

c) The mean distance is 507 – 23.2 = 483.8 million *miles* $\Rightarrow \underline{a=483.8}|$

The perihelion is 483.8 – 23.2 = 460.6 million *miles*

Distance from the ellipse center to the sun is the focus: c = 23.2 million *miles*.

$b^2 = a^2 - c^2 = 438.8^2 - 23.2^2 = 233{,}524.2$

$b = \sqrt{438.8^2 - 23.2^2}$ $\underline{=483.2 \ \ million}|$

Let x and y in millions miles: $\dfrac{x^2}{483.8^2} + \dfrac{y^2}{483.2^2} = 1$ $\quad$ $(in\ millions\ miles)$

The equation of the orbit is: $\boxed{\dfrac{x^2}{234{,}062.44} + \dfrac{y^2}{233{,}524.2} = 1}$

d) The mean distance is 4551 + 897.5 = 5448.5 million *miles* $\Rightarrow \underline{a=5448.5}|$

The aphelion is 5448.5 + 897.5 = 6346 million *miles*

Distance from the ellipse center to the sun is the focus: c = 897.5 million *miles*.

$b^2 = a^2 - c^2 = 5448.5^2 - 897.5^2 = 28{,}880{,}646$

$b = \sqrt{5448.5^2 - 897.5^2}$ $\underline{=5374.07 \ \ million}|$

Let x and y in millions miles: $\dfrac{x^2}{5448.5^2} + \dfrac{y^2}{5374.07^2} = 1$ $\quad$ $(in\ millions\ miles)$

The equation of the orbit is: $\boxed{\dfrac{x^2}{29{,}686{,}152.25} + \dfrac{y^2}{28{,}880{,}646} = 1}$

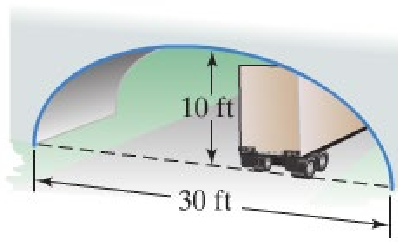
Exercise

Will a truck that is 8 *feet* wide carrying a load that reaches 7 *feet* above the ground the semielliptical arch on the one-way road that passes under the bridge?

Solution

Given: $a = 15, \quad b = 10$

$$\frac{x^2}{225} + \frac{y^2}{100} = 1 \qquad \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Given: $x = 4$

$$\frac{y^2}{100} = 1 - \frac{16}{225}$$

$$y = \pm\sqrt{100\left(1 - \frac{16}{225}\right)}$$

$$= \sqrt{\frac{836}{9}}$$

$$\approx 9.64$$

Yes, the truck will clear about $9.6 - 7 = 7.6 \ ft$.

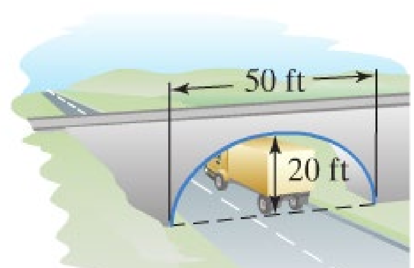
Exercise

A semielliptic archway has a height of 20 *feet* and a width of 50 *feet* and a width of 50 *feet*. Can a truck 14 *feet* high and 10 *feet* wide drive under the archway without going into the other lane?

Solution

Given: $a = 25, \quad b = 20$

$$\frac{x^2}{625} + \frac{y^2}{400} = 1 \qquad \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Given: $x = 10$

$$\frac{y^2}{400} = 1 - \frac{100}{625}$$

$$y^2 = 400\left(\frac{500}{625}\right)$$

$$y = \sqrt{400\left(\frac{500}{625}\right)}$$

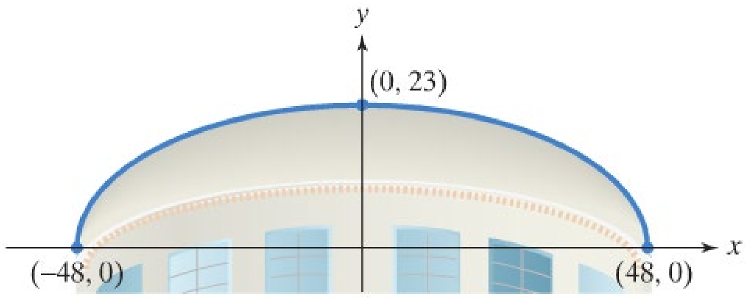
$$= \sqrt{320}$$

$$\approx 17.9$$

Yes, the truck will clear about $17.9 - 14 = 3.9 \ ft$.

Exercise

The elliptical ceiling in Statuary Hall is 96 *feet* long and 23 *feet* tall.



a) Using the rectangular coordinate system in the figure shown, write the standard form of the equation of the elliptical ceiling.

b) John Quincy Adams discovered that he could overhear the conversations of opposing party leaders near the left side of the chamber if he situated his desk at the focus at the right side of the chamber. How far from the center of the ellipse along the major axis did Adams situate his desk?

Solution

a) ***Given***: $a = 48, \quad b = 23$

$$\frac{x^2}{48^2} + \frac{y^2}{23^2} = 1 \qquad\qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\boxed{\frac{x^2}{2304} + \frac{y^2}{529} = 1}$$

b) $c^2 = a^2 - b^2$

$$= 2304 - 529$$

$$= 1775$$

$c = \sqrt{1775}$

$\quad \boxed{\approx 42.13}$

He situated desk about 42 *feet* from the center of the ellipse, along the major axis.

Solution Section 5.4 – Hyperbolas

Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $\frac{x^2}{9}-\frac{y^2}{4}=1$

Solution

$$\rightarrow \begin{cases} a^2 = 9 \rightarrow a = 3 \\ b^2 = 4 \rightarrow b = 2 \end{cases}$$

$$\Rightarrow c=\sqrt{a^2+b^2}=\sqrt{9+4}=\sqrt{13}$$

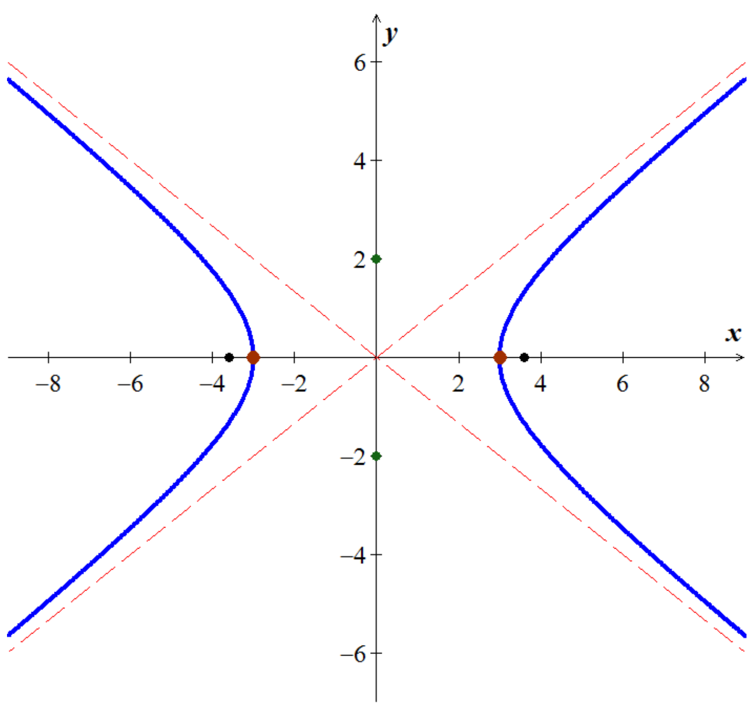
Center: $C=(0,\ 0)$

Vertices: $V=(\pm 3,\ 0)$

Endpoints: $W=(0,\ \pm 2)$

Foci: $F=\left(\pm\sqrt{13},\ 0\right)$

Equations of the ***asymptotes***: $y = \pm\frac{2}{3}x$ $\qquad$ $y=\pm\frac{b}{a}x$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $\frac{y^2}{9}-\frac{x^2}{4}=1$

Solution

$$\rightarrow \begin{cases} a^2=9 \rightarrow a=3 \\ b^2=4 \rightarrow b=2 \end{cases}$$

$$\Rightarrow c=\sqrt{a^2+b^2}=\sqrt{9+4}=\sqrt{13}$$

Center: $C=(0,\ 0)$

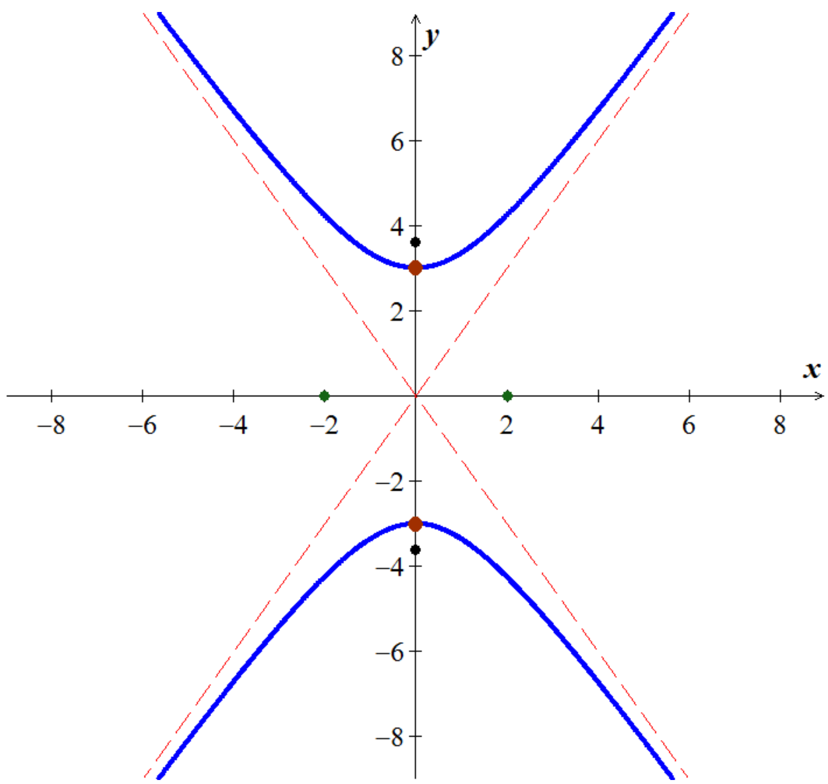
Vertices: $V=(0,\ \pm 3)$

Endpoints: $W=(\pm 2,\ 0)$

Foci: $F=\left(0,\ \pm\sqrt{13}\right)$

Equations of the ***asymptotes***:

$y=\pm\frac{3}{2}x$ $\qquad\qquad$ $y=\pm\frac{a}{b}x$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $x^2 - \frac{y^2}{24} = 1$

Solution

$$\rightarrow \begin{cases} a^2 = 1 \rightarrow a = 1 \\ b^2 = 24 \rightarrow b = 2\sqrt{6} \end{cases}$$

$$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{1+24} = 5$$

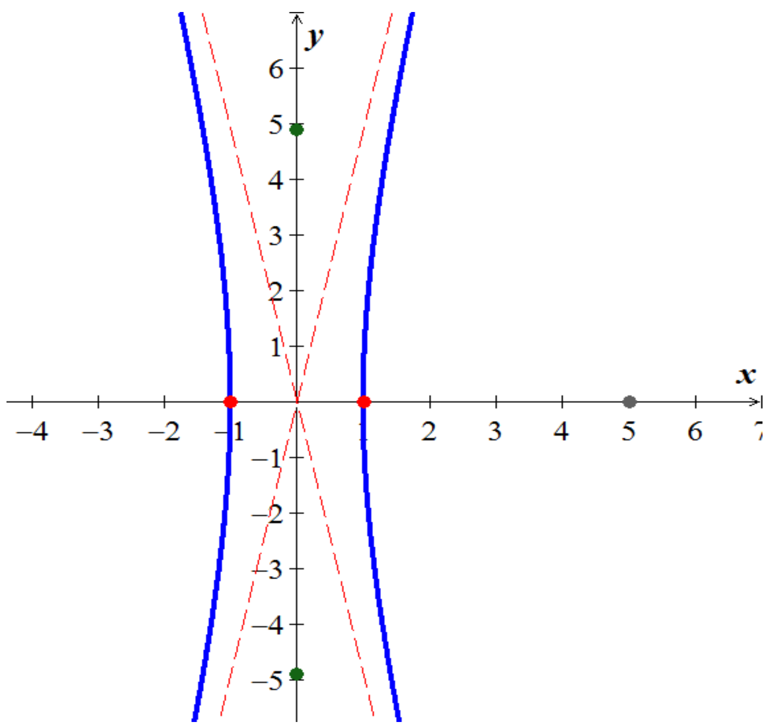
Center: $C = (0,\ 0)$

Vertices: $V = (\pm 1,\ 0)$

Endpoints: $W = \left(0,\ \pm 2\sqrt{6}\right)$

Foci: $F = (\pm 5,\ 0)$

Equations of the ***asymptotes***: $y = \pm 4\sqrt{3}x$ $\qquad$ $y = \pm \frac{b}{a}x$



Exercise

Find the ***center***, ***vertices***, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $y^2 - 4x^2 = 16$

Solution

$$\frac{y^2}{16} - \frac{x^2}{4} = 1$$

$$\begin{cases} a^2 = 16 \rightarrow a = 4 \\ b^2 = 4 \rightarrow b = 2 \end{cases}$$

$$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{16+4} = 2\sqrt{5}$$

Center: $C = (0,\ 0)$

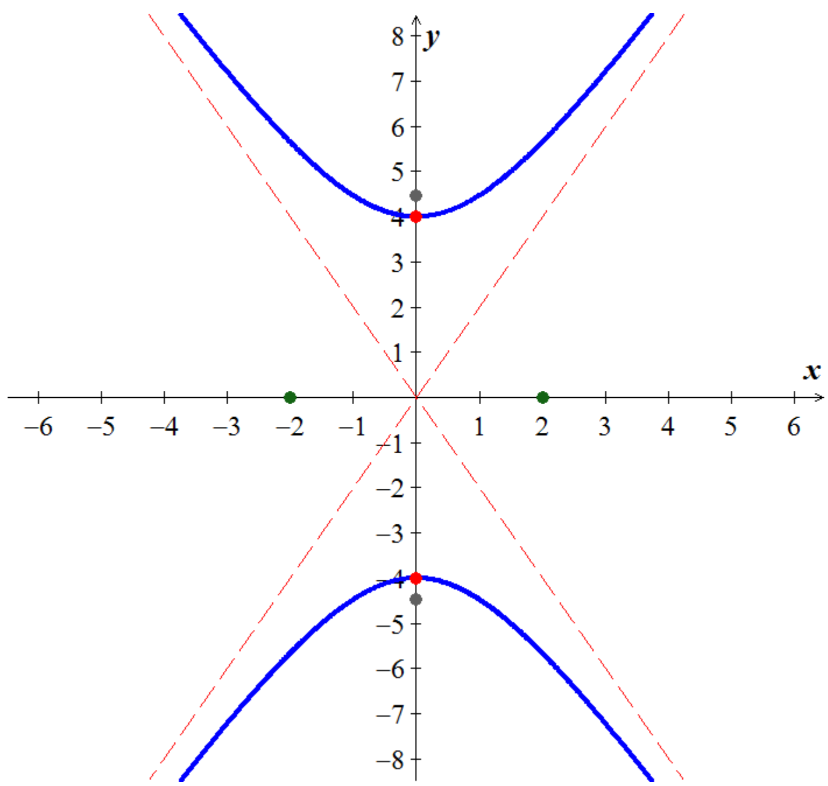
Vertices: $V = (0,\ \pm 4)$

Endpoints: $W = (\pm 2,\ 0)$

Foci: $F = \left(0,\ \pm 2\sqrt{5}\right)$

Equations of the ***asymptotes***:

$y = \pm \frac{4}{2} x \; \underline{= \pm 2x}$ $\qquad\qquad$ $y = \pm \frac{a}{b} x$



Exercise

Find the ***center***, ***vertices***, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $16x^2 - 36y^2 = 1$

Solution

$$\frac{x^2}{\frac{1}{16}} - \frac{y^2}{\frac{1}{36}} = 1$$

$$\rightarrow \begin{cases} a^2 = \frac{1}{16} \rightarrow a = \frac{1}{4} \\ b^2 = \frac{1}{36} \rightarrow b = \frac{1}{6} \end{cases}$$

$$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{\frac{1}{16} + \frac{1}{36}} = \sqrt{\frac{9+4}{144}} = \pm\frac{\sqrt{13}}{12}$$

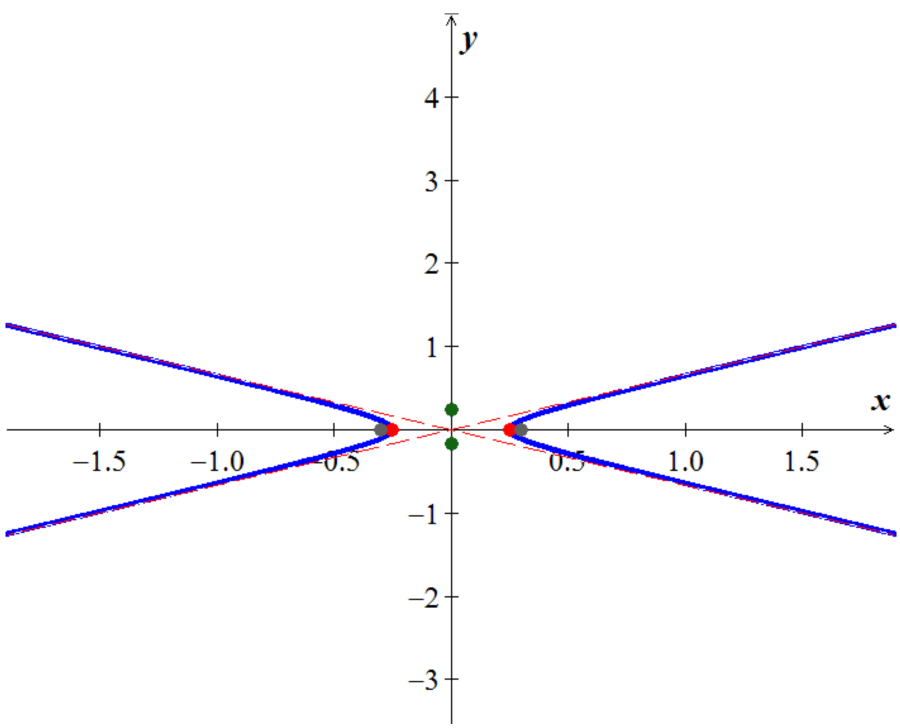
Center: $C = (0,\ 0)$

Vertices: $V = \left(\pm\frac{1}{4},\ 0\right)$

Endpoints: $W = \left(0,\ \pm\frac{1}{6}\right)$

Foci: $F = \left(\pm\frac{\sqrt{13}}{12},\ 0\right)$

Equations of the ***asymptotes***: $y = \pm\frac{\frac{1}{6}}{\frac{1}{4}}x = \pm\frac{4}{6}x \ \boxed{= \pm\frac{2}{3}x}$ $\qquad y = \pm\frac{b}{a}x$



Exercise

Find the ***center***, ***vertices***, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $\frac{(y+2)^2}{9}-\frac{(x+2)^2}{4}=1$

Solution

$$\rightarrow \begin{cases} a^2 = 9 \rightarrow a = 3 \\ b^2 = 4 \rightarrow b = 2 \end{cases}$$

$$\Rightarrow c = \sqrt{a^2+b^2} = \sqrt{9+4} = \pm\sqrt{13}$$

Center: $C = (-2,\ -2)$

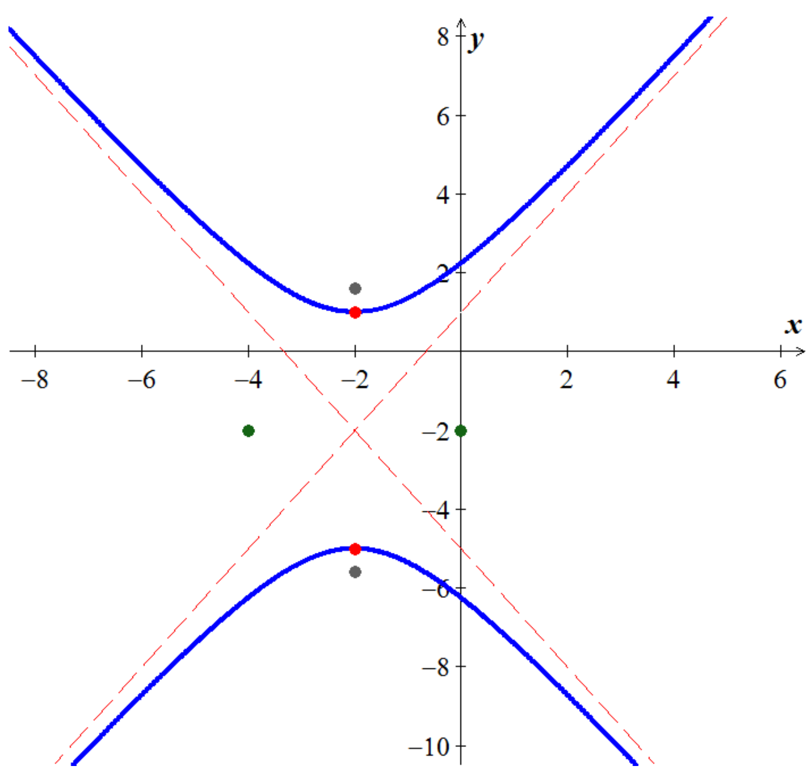
Vertices: $V = (-2,\ -2 \pm 3)$

Endpoints: $W = (-2 \pm 2,\ -2)$

Foci: $F = \left(-2,\ -2 \pm \sqrt{13}\right)$

Equations of the ***asymptotes***: $y+2 = \pm\frac{a}{b}(x+2) = \pm\frac{3}{2}(x+2)$

$y+2 = -\frac{3}{2}(x+2)$	$y+2 = \frac{3}{2}(x+2)$
$y+2 = -\frac{3}{2}x - 3$	$y+2 = \frac{3}{2}x + 3$
$y = -\frac{3}{2}x - 5$	$y = \frac{3}{2}x + 1$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $\frac{(x-2)^2}{4}-\frac{(y+3)^2}{9}=1$

Solution

$$\begin{cases} a^2=4\to a=\pm 2 \\ b^2=9\to b=\pm 3 \\ \hline c=\pm\sqrt{a^2+b^2}=\pm\sqrt{9+4}=\pm\sqrt{13} \end{cases}$$

Center: $C=(2,\ -3)$

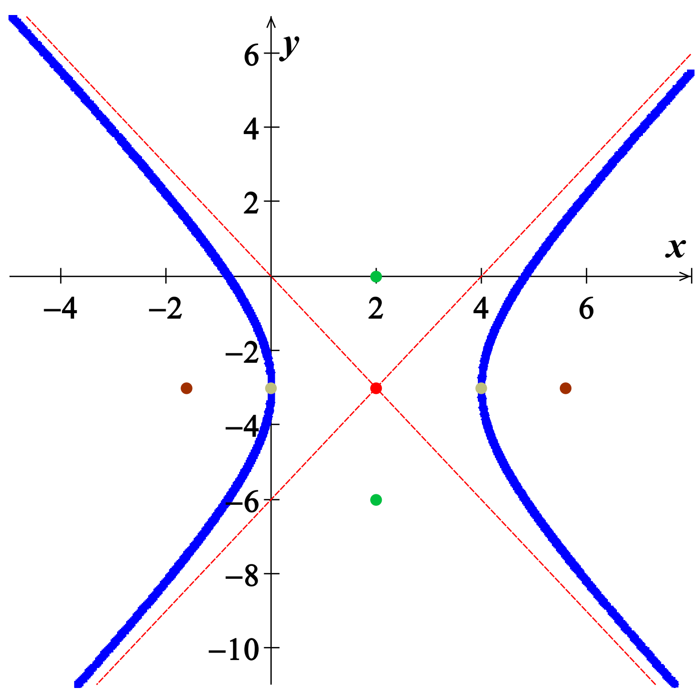
Vertices: $(2\pm 2,\ -3)\to\ V'=(0,\ -3)\quad V=(4,\ -3)$

Endpoints: $(2,\ -3\pm 3)\to\ W'(2,\ -6)\quad W=(2,\ 0)$

Foci: $F=\left(2\pm\sqrt{13},\ -3\right)$

Equations of the ***asymptotes***: $y+3=\pm\frac{b}{a}(x-2)=\pm\frac{3}{2}(x-2)$

$y+3=-\frac{3}{2}(x-2)$	$y+3=\frac{3}{2}(x-2)$
$y+3=-\frac{3}{2}x+3$	$y+3=\frac{3}{2}x-3$
$y=-\frac{3}{2}x$	$y=\frac{3}{2}x-6$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $(y-2)^2 - 4(x+2)^2 = 4$

Solution

$$\frac{(y-2)^2}{4} - \frac{4(x+2)^2}{4} = 1$$

$$\frac{(y-2)^2}{4} - \frac{(x+2)^2}{1} = 1$$

$$\begin{cases} a^2 = 4 \to a = \pm 2 \\ b^2 = 1 \to b = \pm 1 \\ c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{4+1} = \pm\sqrt{5} \end{cases}$$

Center: $C = (-2,\ 2)$

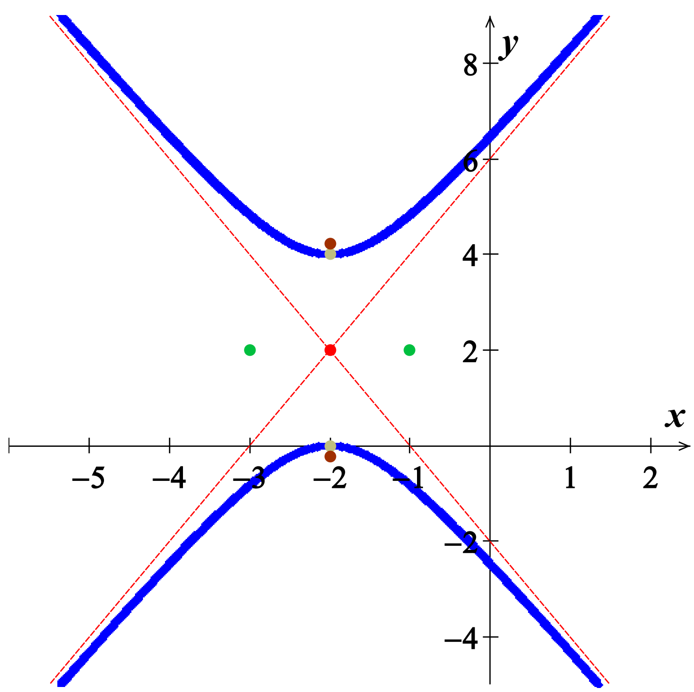
Vertices: $(-2,\ 2 \pm 2) \to \quad V' = (-2,\ 0) \quad V = (-2,\ 4)$

Endpoints: $(-2 \pm 1,\ 2) \to \quad W' = (-3,\ 2) \quad W = (-1,\ 2)$

Foci: $F = \left(-2,\ 2 \pm \sqrt{5}\right)$

*Equations of the **asymptotes***: $y - 2 = \pm \frac{a}{b}(x+2) = \pm \frac{2}{1}(x+2)$

$y-2 = -2(x+2)$	$y-2 = 2(x+2)$
$y-2 = -2x-4$	$y-2 = 2x+4$
$y = -2x-2$	$y = 2x+6$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $(x+4)^2-9(y-3)^2=9$

Solution

$$\frac{(x+4)^2}{9}-\frac{9(y-3)^2}{9}=1$$

$$\frac{(x+4)^2}{9}-\frac{(y-3)^2}{1}=1$$

$$\begin{cases} a^2=9 \rightarrow a=\pm 3 \\ b^2=1 \rightarrow b=\pm 1 \\ \hline c=\pm\sqrt{a^2+b^2}=\pm\sqrt{9+1}=\pm\sqrt{10} \end{cases}$$

Center: $C=(-4,\ 3)$

Vertices: $(-4\pm 3,\ 3)\rightarrow\ V'=(-7,\ 3)\quad V=(-1,\ 3)$

Endpoints: $(-4,\ 3\pm 1)\rightarrow\ W'(-4,\ 2)\quad W=(-4,\ 4)$

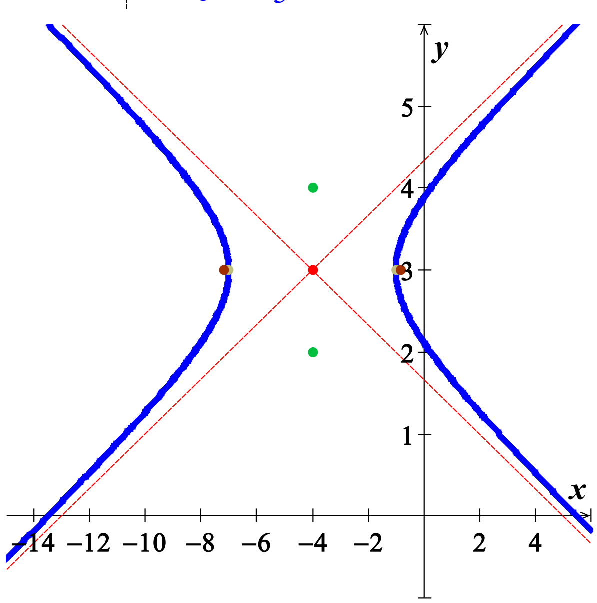
Foci: $F=\left(-4\pm\sqrt{10},\ 3\right)$

Equations of the ***asymptotes***: $y-3=\pm\frac{b}{a}(x+4)=\pm\frac{1}{3}(x+4)$

$$y-3=-\frac{1}{3}(x+4) \qquad\qquad y-3=\frac{1}{3}(x+4)$$

$$y-3=-\frac{1}{3}x-\frac{4}{3} \qquad\qquad y-3=\frac{1}{3}x+\frac{4}{3}$$

$$y=-\frac{1}{3}x+\frac{5}{3} \qquad\qquad y=\frac{1}{3}x+\frac{13}{3}$$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $144x^2 - 25y^2 + 864x - 100y - 2404 = 0$

Solution

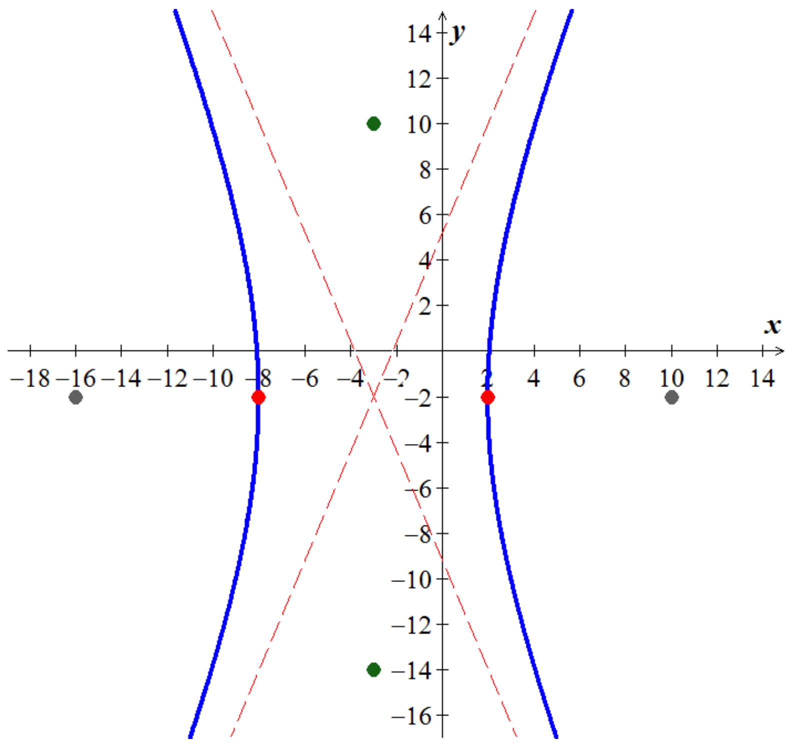
$$144\left(x^2 + 6x + \left(\frac{6}{2}\right)^2\right) - 25\left(y^2 + 4y + \left(\frac{4}{2}\right)^2\right) = 2404 + 144(4) - 25(4)$$

$$144(x+3)^2 - 25(y+2)^2 = 3600$$

$$\frac{(x+3)^2}{25} - \frac{(y+2)^2}{144} = 1$$

$$\rightarrow \begin{cases} a^2 = 25 \rightarrow a = 5 \\ b^2 = 144 \rightarrow b = 12 \end{cases}$$

$$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{25 + 144}$$
$$\underline{= \pm 13|}$$



Center: $C = (-3,\ -2)$

Vertices: $V = (-3 \pm 5,\ -2)$

Endpoints: $W = (-3,\ -2 \pm 12)$

Foci: $F = (-3 \pm 13,\ -2)$

Equations of the ***asymptotes***:

$$y + 2 = \pm\frac{b}{a}(x+3) = \pm\frac{12}{5}(x+3)$$

$y + 2 = -\frac{12}{5}(x+3)$	$y + 2 = \frac{12}{5}(x+3)$
$y + 2 = -\frac{12}{5}x - \frac{36}{5}$	$y + 2 = \frac{12}{5}x + \frac{36}{5}$
$y = -\frac{12}{5}x - \frac{46}{5}$	$y = \frac{12}{5}x + \frac{26}{5}$

Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $4y^2 - x^2 + 40y - 4x + 60 = 0$

Solution

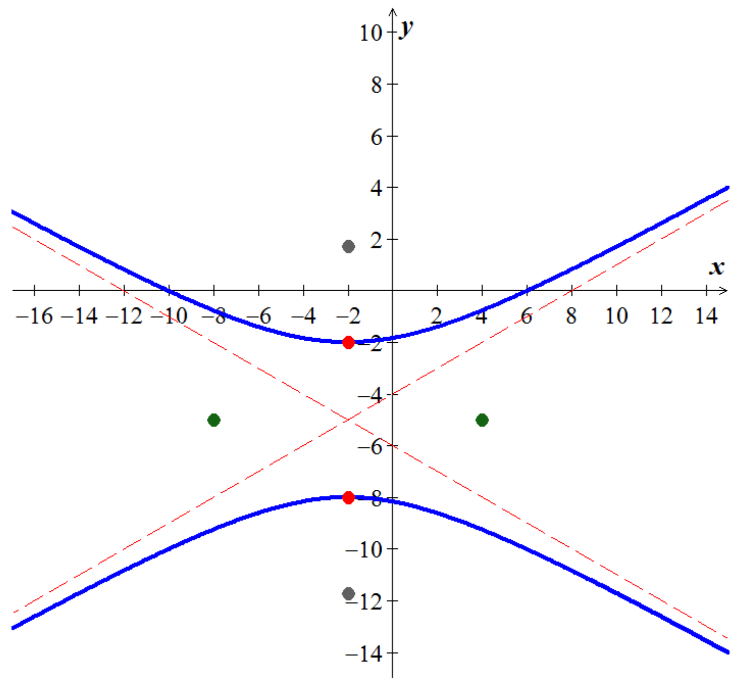
$$4\left(y^2 + 10y + \left(\frac{10}{2}\right)^2\right) - \left(x^2 + 4x + \left(\frac{4}{2}\right)^2\right) = -60 + 4(25) - (4)$$

$$4(y+5)^2 - (x+2)^2 = 36$$

$$\frac{(y+5)^2}{9} - \frac{(x+2)^2}{36} = 1$$

$$\to \begin{cases} a^2 = 9 \to a = 3 \\ b^2 = 36 \to b = 6 \end{cases}$$

$$\Rightarrow c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{9+36}$$
$$= \pm\sqrt{45}$$
$$= \pm 3\sqrt{5}$$



Center: $C = (-5,\ -2)$

Vertices: $V = (-2,\ -5 \pm 3)$

Endpoints: $W = (-2 \pm 6,\ -5)$

Foci: $F = \left(-2,\ -5 \pm 3\sqrt{5}\right)$

Equations of the ***asymptotes***:

$$y + 5 = \pm\frac{a}{b}(x+2) = \pm\frac{3}{6}(x+2) = \pm\frac{1}{2}(x+2)$$

$y + 5 = -\frac{1}{2}(x+2)$	$y + 5 = \frac{1}{2}(x+2)$
$y + 5 = -\frac{1}{2}x - 1$	$y + 5 = \frac{1}{2}x + 1$
$y = -\frac{1}{2}x - 6$	$y = \frac{1}{2}x - 4$

Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $4x^2-16x-9y^2+36y=-16$

Solution

$$4\left(x^2-4x\right)-9\left(y^2-4y\right)=-16$$

$$4\left(x^2-4x+2^2\right)-9\left(y^2-4y+2^2\right)=-16+4\left(2^2\right)-9\left(2^2\right)$$

$$4(x-2)^2-9(y-2)^2=-16+16-36$$

$$4(x-2)^2-9(y-2)^2=-36$$

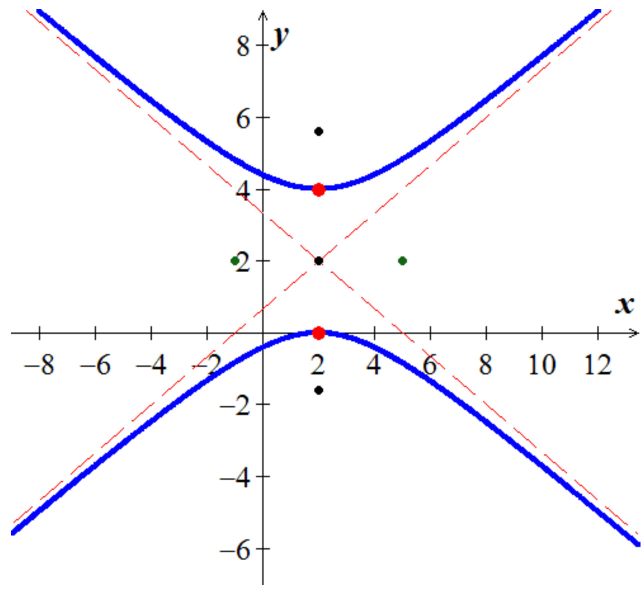
$$\frac{4(x-2)^2}{-36}-\frac{9(y-2)^2}{-36}=1$$

$$-\frac{4(x-2)^2}{36}+\frac{9(y-2)^2}{36}=1$$

$$\frac{9(y-2)^2}{36}-\frac{4(x-2)^2}{36}=1$$

$$\frac{(y-2)^2}{\frac{36}{9}}-\frac{(x-2)^2}{\frac{36}{4}}=1$$

$$\frac{(y-2)^2}{4}-\frac{(x-2)^2}{9}=1$$



$$\to\begin{cases} a^2=4\to a=\pm 2 \\ b^2=9\to b=\pm 3 \end{cases}$$

$$\Rightarrow c=\mp\sqrt{a^2+b^2}=\pm\sqrt{9+4}=\pm\sqrt{13}$$

Center: $(2,\ 2)$

The ***endpoints***: $(2\pm 3,\ -2)\Rightarrow(-1,\ 2)\quad(5,\ 2)$

The ***vertices***: $(2,\ 2\pm 2)\Rightarrow(2,\ 0)\quad(2,\ 4)$

The ***foci*** are $\left(2,\ 2\pm\sqrt{13}\right)$

The equations of the ***asymptotes*** are: $y-2=\pm\frac{a}{b}(x-2)\ \Rightarrow \underline{y=\pm\frac{2}{3}(x-2)+2}$

Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $2x^2 - y^2 + 4x + 4y = 4$

Solution

$$2\left(x^2 + 2x + \left(\frac{2}{2}\right)^2\right) - \left(y^2 - 4y + \left(\frac{-4}{2}\right)^2\right) = 4 + 2\left(\frac{2}{2}\right)^2 + (-1)\left(\frac{-4}{2}\right)^2$$

$$2(x+1)^2 - (y-2)^2 = 4 + 2 - 4$$

$$2(x+1)^2 - (y-2)^2 = 2$$

$$\frac{(x+1)^2}{1} - \frac{(y-2)^2}{2} = 1$$

$$\begin{cases} a^2 = 1 \rightarrow a = \pm 1 \\ b^2 = 2 \rightarrow b = \pm\sqrt{2} \\ c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{1+2} = \pm\sqrt{3} \end{cases}$$

Center: $C = (-1,\ 2)$

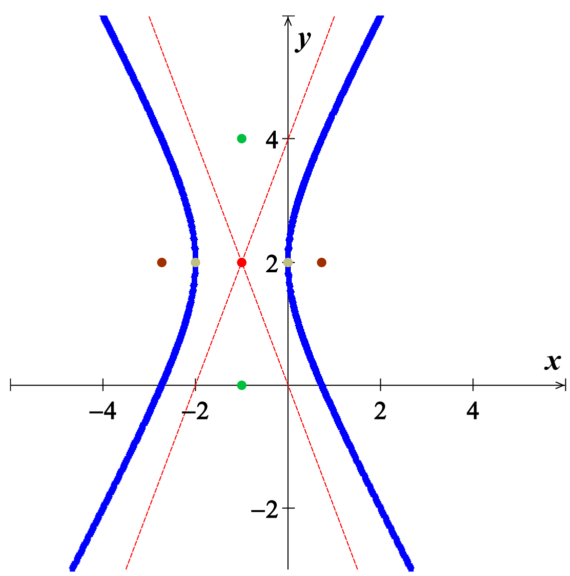
Vertices: $(-1 \pm 1,\ 2) \rightarrow \quad V' = (-2,\ 2) \quad V = (0,\ 2)$

Endpoints: $(-1,\ 2 \pm 2) \rightarrow \quad W' = (-1,\ 0) \quad W = (-1,\ 4)$

Foci: $F = \left(-1 \pm \sqrt{3},\ 2\right)$

Equations of the ***asymptotes***: $y - 2 = \pm\frac{b}{a}(x+1) = \pm\frac{2}{1}(x+1)$

$y - 2 = -2(x+1)$	$y - 2 = 2(x+1)$
$y - 2 = -2x - 2$	$y - 2 = 2x + 2$
$y = -2x$	$y = 2x + 4$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $2y^2 - x^2 + 2x + 8y + 3 = 0$

Solution

$$2y^2 + 8y - x^2 + 2x = -3$$

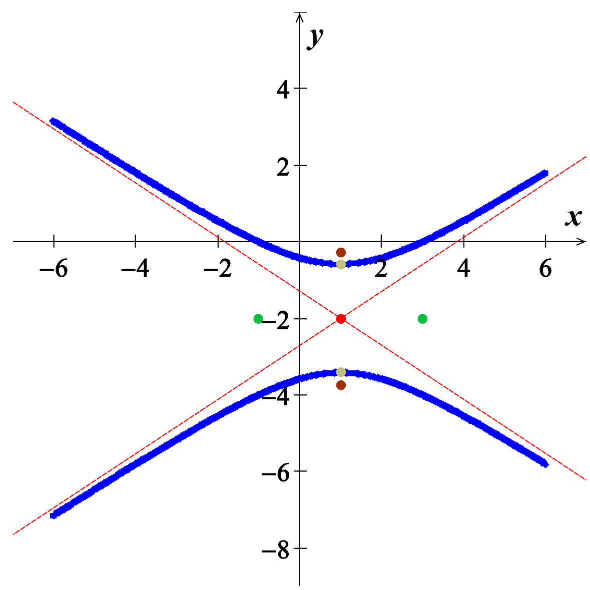
$$2\left(y^2 + 4y + \left(\frac{4}{2}\right)^2\right) - \left(x^2 - 2x + \left(\frac{-2}{2}\right)^2\right) = -3 + 2\left(\frac{4}{2}\right)^2 + (-1)\left(\frac{-2}{2}\right)^2$$

$$2(y+2)^2 - (x-1)^2 = -3 + 8 - 1$$

$$2(y+2)^2 - (x-1)^2 = 4$$

$$\frac{(y+2)^2}{2} - \frac{(x-1)^2}{4} = 1$$

$$\begin{cases} a^2 = 2 \rightarrow a = \pm\sqrt{2} \\ b^2 = 4 \rightarrow b = \pm 2 \\ c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{2+4} = \pm\sqrt{6} \end{cases}$$



Center: $C = (1,\ -2)$

Vertices: $V = \left(1,\ -2 \pm \sqrt{2}\right)$

Endpoints: $(1 \pm 2,\ -2) \rightarrow\ W' = (-1,\ -2)\quad W = (3,\ -2)$

Foci: $F = \left(1,\ -2 \pm \sqrt{3}\right)$

Equations of the ***asymptotes***: $y + 2 = \pm\frac{a}{b}(x-1) = \pm\frac{\sqrt{2}}{2}(x-1)$

$$y + 2 = -\frac{\sqrt{2}}{2}(x-1) \qquad\qquad y + 2 = \frac{\sqrt{2}}{2}(x-1)$$

$$y = -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2} - 2 \qquad\qquad y = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2} - 2$$

Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $2y^2 - 4x^2 - 16x - 2y - 19 = 0$

Solution

$$2y^2 - 2y - 4x^2 - 16x = 19$$

$$2\left(y^2 - y + \left(\frac{-1}{2}\right)^2\right) - 4\left(x^2 + 4x + \left(\frac{4}{2}\right)^2\right) = 19 + 2\left(\frac{-1}{2}\right)^2 - 4\left(\frac{4}{2}\right)^2$$

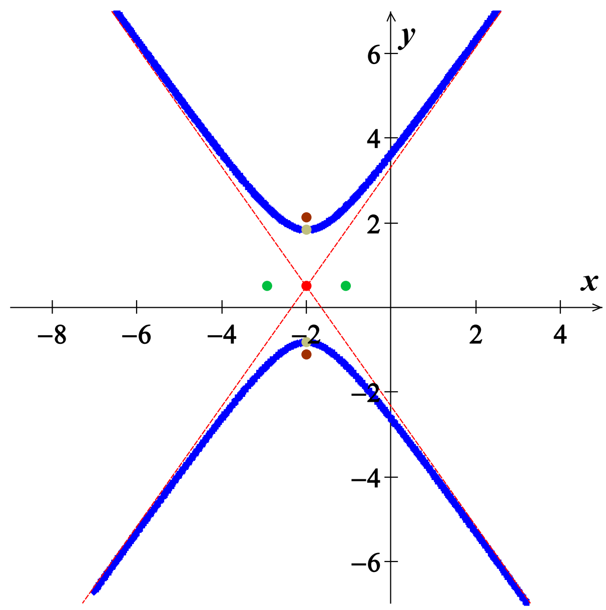
$$2\left(y - \frac{1}{2}\right)^2 - 4(x+2)^2 = 19 + \frac{1}{2} - 16$$

$$2\left(y - \frac{1}{2}\right)^2 - 4(x+2)^2 = \frac{7}{2}$$

$$\frac{2\left(y - \frac{1}{2}\right)^2}{\frac{7}{2}} - \frac{4(x+2)^2}{\frac{7}{2}} = 1$$

$$\frac{\left(y - \frac{1}{2}\right)^2}{\frac{7}{4}} - \frac{4(x+2)^2}{\frac{7}{8}} = 1$$

$$\begin{cases} a^2 = \frac{7}{4} \to a = \pm\frac{\sqrt{7}}{2} \\ b^2 = \frac{7}{8} \to b = \pm\frac{\sqrt{7}}{2\sqrt{2}} = \pm\frac{\sqrt{14}}{4} \\ \hline c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{\frac{7}{4} + \frac{7}{8}} = \pm\sqrt{\frac{21}{8}} \end{cases}$$



Center: $C = \left(-2, \ \frac{1}{2}\right)$

Vertices: $V = \left(1, \ -2 \pm \frac{\sqrt{7}}{2}\right)$

Endpoints: $W\left(-2 \pm \frac{\sqrt{14}}{4}, \ \frac{1}{2}\right)$

Foci: $F = \left(-2, \ \frac{1}{2} \pm \sqrt{\frac{21}{8}}\right)$

Equations of the ***asymptotes***: $y - \frac{1}{2} = \pm\frac{a}{b}(x+2) = \pm\frac{\frac{\sqrt{7}}{2}}{\frac{\sqrt{7}}{2\sqrt{2}}}(x+2) = \pm\sqrt{2}(x+2)$

$y - \frac{1}{2} = -\sqrt{2}(x+2)$	$y - \frac{1}{2} = \sqrt{2}(x+2)$
$y = -\sqrt{2}x - 2\sqrt{2} + \frac{1}{2}$	$y = \sqrt{2}x + 2\sqrt{2} + \frac{1}{2}$

Exercise

Suppose a hyperbola has center at the origin, foci at $F'(-c,\ 0)$ and $F(c,\ 0)$, and equation $d(P,\ F') - d(P,\ F) = 2a$. Let $b^2 = c^2 - a^2$, and show that an equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Solution

$$d(P,\ F') - d(P,\ F) = 2a \qquad\qquad d = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$$\sqrt{(x+c)^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2} = 2a$$

$$\sqrt{x^2 + 2cx + c^2 + y^2} - \sqrt{x^2 - 2cx + c^2 + y^2} = 2a$$

$$\sqrt{x^2 + 2cx + c^2 + y^2} = 2a + \sqrt{x^2 - 2cx + c^2 + y^2}$$

$$\left(\sqrt{x^2 + 2cx + c^2 + y^2}\right)^2 = \left(2a + \sqrt{x^2 - 2cx + c^2 + y^2}\right)^2$$ ***Square both sides***

$$x^2 + 2cx + c^2 + y^2 = 4a^2 + x^2 - 2cx + c^2 + y^2 + 4a\sqrt{x^2 - 2cx + c^2 + y^2}$$

$$4cx - 4a^2 = 4a\sqrt{x^2 - 2cx + c^2 + y^2}$$ ***Divide by 4***

$$cx - a^2 = a\sqrt{x^2 - 2cx + c^2 + y^2}$$

$$\left(cx - a^2\right)^2 = \left(a\sqrt{x^2 - 2cx + c^2 + y^2}\right)^2$$ ***Square both sides***

$$c^2x^2 - 2a^2cx + a^4 = a^2\left(x^2 - 2cx + c^2 + y^2\right)$$

$$c^2x^2 - 2a^2cx + a^4 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$$

$$c^2x^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4$$

$$\left(c^2 - a^2\right)x^2 - a^2y^2 = a^2\left(c^2 - a^2\right)$$

$$\frac{\left(c^2 - a^2\right)x^2}{a^2\left(c^2 - a^2\right)} - \frac{a^2y^2}{a^2\left(c^2 - a^2\right)} = 1$$

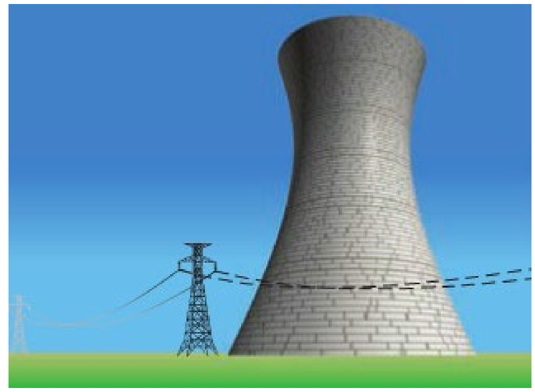
$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1 \qquad\qquad b^2 = c^2 - a^2$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Exercise

A cooling tower is a hydraulic structure. Suppose its base diameter is 100 *meters* and its smallest diameter of 48 *meters* occurs 84 *meters* from the base. If the tower is 120 *meters* high approximate its diameter at the top.

Solution



$$\textbf{\textit{Given}}: \ a = \frac{48}{2} = 24$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \rightarrow \quad \frac{x^2}{24^2} - \frac{y^2}{b^2} = 1$$

At the point $(50, \ -84)$:

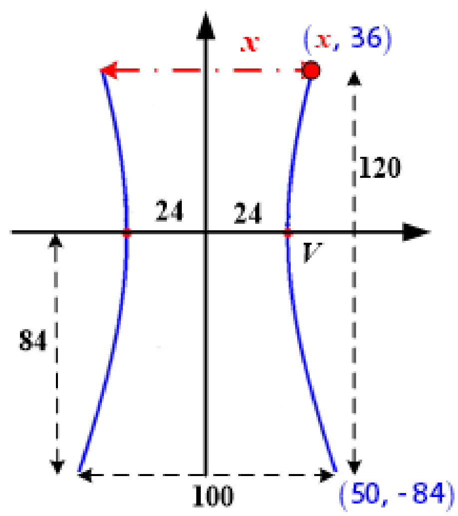
$$\frac{50^2}{24^2} - \frac{(-84)^2}{b^2} = 1$$

$$\frac{50^2}{24^2} - 1 = \frac{84^2}{b^2}$$

$$\frac{50^2 - 24^2}{24^2} = \frac{84^2}{b^2}$$

$$b^2 = \frac{84^2 \cdot 24^2}{50^2 - 24^2} = 2112.4$$

$$\Rightarrow \ \frac{x^2}{576} - \frac{y^2}{2112.4} = 1$$



At the point $(x, \ 36)$:

$$\frac{x^2}{576} - \frac{36^2}{2112.4} = 1$$

$$\frac{x^2}{576} = 1 + \frac{1296}{2112.4}$$

$$\frac{x^2}{576} = 1.61$$

$$x^2 = 929.45$$

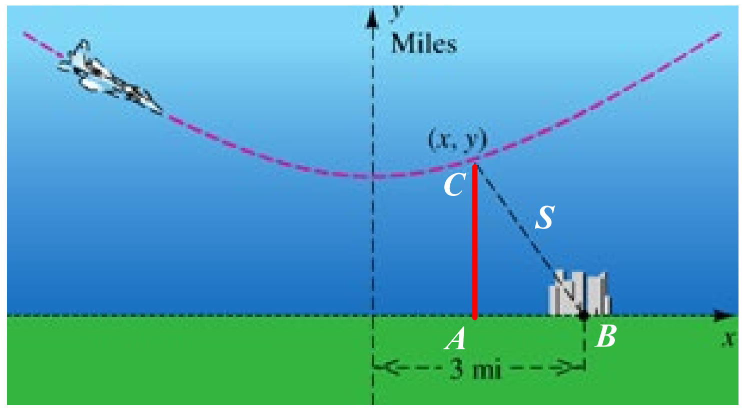
$$x = \sqrt{929.45} \approx 30.49$$

The diameter at the top: $= 2x = \underline{60.97 \ m.}$

Exercise

An airplane is flyting along the hyperbolic path. If an equation of the path is $2y^2 - x^2 = 8$, determine how close the airplane comes to town located at (3, 0). (Hunt: Let S denote the square of the distance from a point (x, y) on the path to (3, 0), and find the minimum value of S.)

Solution



$$2y^2 - x^2 = 8$$

$$y^2 = \frac{1}{2}x^2 + 4$$

$$S^2 = (3-x)^2 + y^2$$

$$= 9 - 6x + x^2 + \frac{1}{2}x^2 + 4$$

$$= \frac{3}{2}x^2 - 6x + 13$$

The vertex point of S^2

$$x = -\frac{b}{2a}$$

$$= -\frac{-6}{2\left(\frac{3}{2}\right)}$$

$$\underline{= 2|}$$

$$S^2 = \frac{3}{2}(2)^2 - 6(2) + 13$$

$$\underline{= 7|}$$

Therefore the close the town to the airplane is $S = \sqrt{7}\ miles$

Exercise

A ship is traveling a course that is 100 *miles* from, and parallel tom a straight shoreline. The ship sends out a distress signal that is received by two Coast Guard stations A and B, located 200 *miles* apart. By measuring the difference in signal reception times, it is determined that the ship is 160 *miles* closer to B than to A. Where is the ship?

Solution

Given: $c = 100 \quad and \quad BC = AC - 160$

$$d_1 - d_2 = 160 = 2a$$

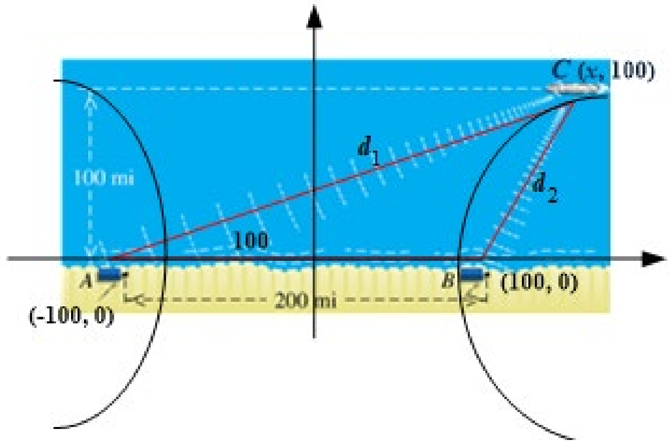
$$\underline{a = 80|}$$

$$b^2 = c^2 - a^2$$

$$= 100^2 - 80^2$$

$$= 3600$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \rightarrow \quad \frac{x^2}{6400}-\frac{y^2}{3600}=1$$



At point C(x, 100):

$$\frac{x^2}{6400}-\frac{100^2}{3600}=1$$

$$\frac{x^2}{6400}=1+\frac{100^2}{3600}$$

$$x^2=6400\left(1+\frac{100^2}{3600}\right)$$

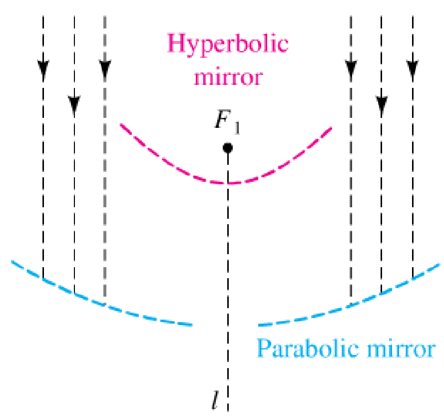
$$x=80\sqrt{\frac{13600}{3600}}$$

$$=\underline{\frac{80}{3}\sqrt{34}}$$

The ship position is $\left(\frac{80}{3}\sqrt{34},\ 100\right)=\underline{(155.5,\ 100)}$

Exercise

The Cassegrain telescope design (dating back to 1672) makes use of the reflective properties of both the parabola and the hyperbola. The figure shows a (***split***) parabolic mirror, with one focus at F_1 and axis along the line l, and a hyperbolic mirror, with one focus also at F_1 and transverse axis along l. Where do incoming light waves parallel to the common axis finally collect?



Solution

Exterior focus of hyperbolic mirror (below parabolic mirror)

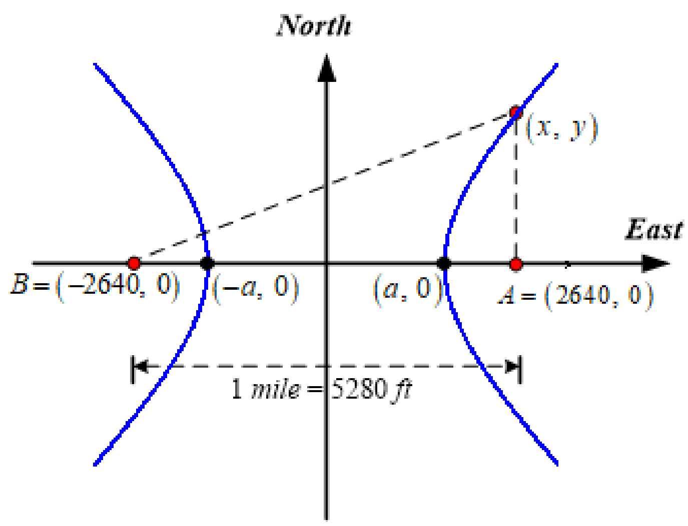
Exercise

Suppose that two people standing 1 *mile* apart both see a flash of lightning. After a period of time, the person standing at point *A* hears the thunder. One second later, the person standing at point *B* hears the thunder. If the person at *B* is due west of the person at *A* and the lightning strike is known to occur due north of the person standing at point *A*, where did the lightning strike occur?
(Sound travels at 1100 $ft\ /\ sec$ and 1 $mile = 5280\ ft$)

Solution

Person ***A*** is 1100 *feet* closer to the lightning strike than the person at point ***B***.
Distance from $(x,\ y)$ to ***B*** ***minus*** distance from $(x,\ y)$ to ***A*** is 1100.

The point $(x,\ y)$ lies on a hyperbola whose foci are at ***A*** and ***B***.



$2a = 1100 \quad \Rightarrow \quad \underline{a = 550|}$

$2c = 5280 \quad \Rightarrow \quad \underline{c = 2640|}$

$$\begin{aligned} b^2 &= c^2 - a^2 \\ &= 2640^2 - 550^2 \\ &= 6,667,100 \end{aligned}$$

An equation of the hyperbola:

$$\frac{x^2}{550^2} - \frac{y^2}{6,667,100} = 1 \qquad\qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

At point $A = (2640,\ 0)$, let $x = 2640$, and solve for *y* at that *x* value:

$$\frac{2640^2}{550^2} - \frac{y^2}{6,667,100} = 1$$

$$y^2 = 6,667,100\left(\frac{2640^2}{550^2} - 1\right)$$

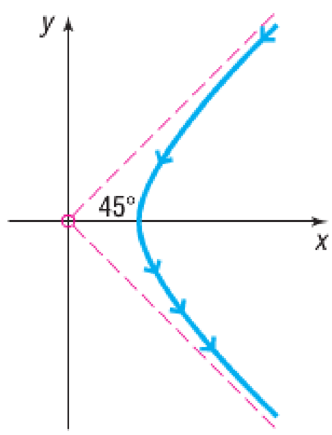
$$y = \sqrt{6,667,100\left(\frac{2640^2}{550^2} - 1\right)}$$

$$= 12,122$$

he lightning strike occurred 12,122 *feet.* north of the person standing at point A.

Exercise

Ernest Rutherford published a paper that he described the motion of alpha particles as they are shot at a piece of gold foil 0.00004 *cm* thick. Before conducting this experiment, Rutherford expected that the alpha particles would shoot through the foil just as a bullet would shoot through the foil just as a bullet would shoot through snow. Instead, a small fraction of the alpha particles bounced off the foil. This led to the conclusion that the nucleus of an atom is dense, while the remainder of the atom is sparse. Only the density of the nucleus could cause the alpha particles to deviate from their path. The figure shows a diagram from Rutherford's paper that indicates that the deflected alpha particles follow the path of one branch of a hyperbola.



a) Find an equation of the asymptotes under this scenario.

b) If the vertex of the path of the alpha particles is 10 *cm* from the center of the hyperbola, find a model that describes the path of the particle.

Solution

a) Since the particles are deflected at a 45° angle, the asymptotes will be $y = \pm x$

b) Since the vertex is 10 *cm* from the center of the hyperbola, so $a = 10$

The slope of the asymptotes is given by $\pm\frac{b}{a}$

Therfore: $\frac{b}{a} = 1 \quad \rightarrow \quad b = a = 10$

The equation of the particle path is: $\frac{x^2}{100} - \frac{y^2}{100} = 1 \quad (x \ge 0)$

Exercise

Hyperbolas have interesting reflective properties that make them useful for lenses and mirrors. For example, if a ray of light strikes a convex hyperbolic mirror on a line that would (theoretically) pass through its rear focus, it is reflected through its rear focus, it is reflected through the front focus. This property and that of the parabola were used to develop the *Cassegrain* telescope in 1672. The focus of the parabolic mirror and the rear focus of the hyperbolic mirror are the same point. The rays are collected by the parabolic mirror, reflected toward the common focus, and thus are reflected by the hyperbolic mirror through the opening to its front focus, where the eyepiece is located. If the equation of the hyperbola is $\frac{y^2}{9}-\frac{x^2}{16}=1$ and the focal length (distance from the vertex to the focus) of the parabola is 6, find the equation of the parabola.

Solution

Assume the origin lies at the center of the hyperbola. The foci of the hyperbola are located on *y*-axis at $(0,\ \pm c)$, since the hyperbola has a transverse axis that is parallel to the *y*-axis.

Given: $a^2=9 \quad and \quad b^2=16$

$c^2=a^2+b^2=9+6=25$

$c=\sqrt{25}\ \underline{=5|}$

Therefore, the foci of the foci of the hyperbola are at $(0,\ -5)$ & $(0,\ 5)$

Assume that he parabola opens up, the common focus is at $(0,\ 5)$.

The equation of the parabola: $x^2=4a(y-k)$

The focal length of the parabola is given as $a=6$

The distance focus of the parabola is located at $(0,\ k+a)=(0,\ 5)$

$$k+6=5 \quad \Rightarrow \quad k=1$$

The equation of the parabola becomes $x^2=4(8)(y-(-1))$

$\underline{x^2=24(y+1)|}$ *or* $y=\frac{1}{24}x^2-1$

Exercise

The ***eccentricity e*** of a hyperbola is defined as the number $\frac{c}{a}$, where *a* is the distance of a vertex from the center and *c* is the distance of a focus from the center. Because *c* > *a*, it follows that $e>1$. Describe the general shape of a hyperbola whose eccentricity is close to 1. What is the shape if *e* is very large?

Solution

Assume $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

If the eccentricity is close to 1, then $c\approx a$ and $b\approx 0$.

When b is close to 0, the hyperbola is very narrow, because the slopes of asymptotes are close to 0.

If the eccentricity is very large, then c is much larger than a and b. The result is a hyperbola is very wide, because the slopes of the asymptotes are very large.

For $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the opposite is true.

When the eccentricity is close to 1, the hyperbola is very wide because the slopes of the asymptotes are close to 0.

When the eccentricity is very large, the hyperbola is very narrow because the slopes of asymptotes are very large.

Exercise

An explosive is recorded by two microphone that are 1 *mile* apart. Microphone M_1 received the sound 2 *seconds* before microphone M_2. Assuming sound travels at 1,100 *feet* per *second*, determine the possible locations of the explosion relative to the location of the microphones.

Solution

$$\begin{aligned}\left|d_2-d_1\right| &= 2a \\ &= 2(1100) \\ &= 2,200\ ft\end{aligned}$$

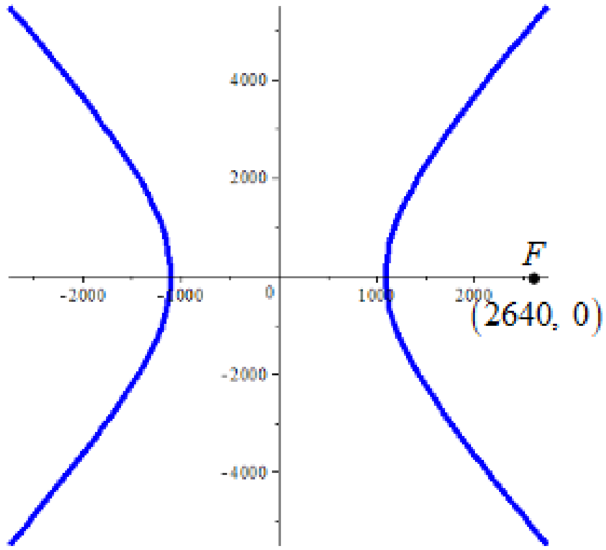
$$a = 1,100\ ft$$

$$2c = 1\ mi = 5,280\ ft$$

$$c = 2,640\ ft$$

$$\begin{aligned}b^2 &= c^2-a^2 \\ &= 2640^2-1100^2 \\ &= 5,759,600\end{aligned}$$

$$\frac{x^2}{1,210,000}-\frac{y^2}{5,759,600}=1 \qquad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$



If M_1 is located 2640 *feet* to the right of the origin on the *x-axis*, the explosive is located on the right branch of the hyperbola given by the driven equation.

Exercise

Radio towers *A* and *B*, 200 *km* apart, are situated along the coast, with *A* located due west of *B*. Simultaneous radio signals are sent from each tower to a ship, with the signal from *B* received $500\ \mu\text{sec}$ before the signal from *A*.

a) Assuming that the radio signals travel $300\ m/\mu\text{sec}$, determine the equation of the hyperbola on which the ship is located.

b) If the ship lies due north of tower *B*, how far out at sea is it?

Solution

a) $2c = 200\ km \quad \to \quad c = 100\ km$

$$\left|d_2 - d_1\right| = (500\ \mu\text{sec})\left(300\ \frac{m}{\mu\text{sec}}\right)$$

$$2a = 150{,}000\ m$$

$$a = \underline{75{,}000\ m = 75\ km}$$

$$b^2 = c^2 - a^2$$

$$= 100^2 - 75^2$$

$$= \underline{4{,}375}$$

$$\frac{x^2}{(75)^2} - \frac{y^2}{4375} = 1 \qquad\qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\boxed{\frac{x^2}{5625} - \frac{y^2}{4375} = 1}$$

b) ***Given***: $x = 100\ km$

$$\frac{100^2}{5625} - \frac{y^2}{4375} = 1$$

$$y^2 = 4375\left(\frac{10,000}{5625} - 1\right)$$

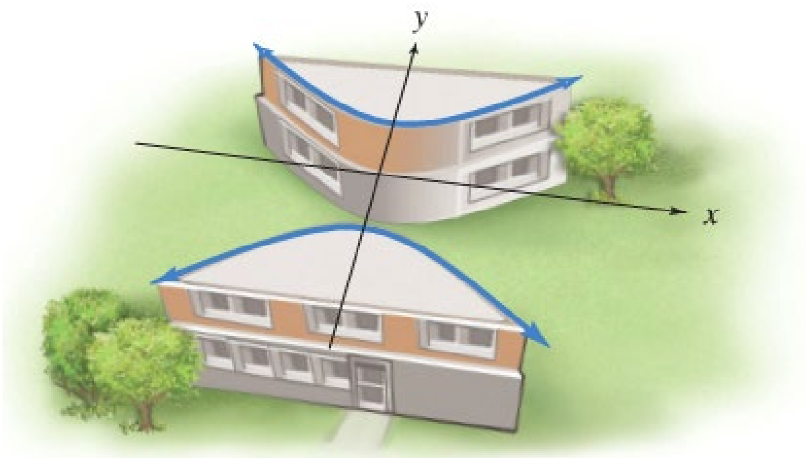
$$y = \pm\sqrt{4375\left(\frac{10,000}{5625} - 1\right)}$$

$$\approx \pm 58.3$$

The ship is about 58.3 *km* from the coast.

Exercise

An architect designs two houses that are shaped and positioned like a part of the branches of the hyperbola whose equation is $625y^2 - 400x^2 = 250,000$, where *x* and *y* are in yards. How far apart are the houses at their closest point?



Solution

$$625y^2 - 400x^2 = 250,000$$

$$\frac{625}{250,000}y^2 - \frac{400}{250,000}x^2 = 1$$

$$\frac{y^2}{400} - \frac{x^2}{625} = 1$$

$$a^2 = 400 \rightarrow a = 20$$

$$2a = 40$$

The houses are 40 *yards* apart at their closest point.

Solution Section 5.5 – Infinite Sequences and Summation Notation

Exercise

Find the first four terms and the eight term of the sequence: $\{12-3n\}$

Solution

$$a_n = 12-3n$$

$$a_1 = 12-3(1) \underline{=9}$$

$$a_2 = 12-3(2) = 6$$

$$a_3 = 12-3(3) \underline{=3}$$

$$a_4 = 12-3(4) \underline{=0}$$

$$a_8 = 12-3(8) \underline{=-12}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\dfrac{3n-2}{n^2+1}\right\}$

Solution

$$a_n = \frac{3n-2}{n^2+1}$$

$$a_1 = \frac{3-2}{1^2+1} = \frac{1}{2}$$

$$a_2 = \frac{3(2)-2}{2^2+1} = \frac{4}{5}$$

$$a_3 = \frac{3(3)-2}{3^2+1} = \frac{7}{10}$$

$$a_4 = \frac{3(4)-2}{4^2+1} = \frac{10}{17}$$

$$a_8 = \frac{3(8)-2}{8^2+1} = \frac{22}{65}$$

Exercise

Find the first four terms and the eight term of the sequence: $\{9\}$

Solution

$a_1 = 9$

$a_2 = 9$

$a_3 = 9$

$a_4 = 9$

$a_8 = 9$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{(-1)^{n-1}\frac{n+7}{2n}\right\}$

Solution

$a_1 = (-1)^{1-1}\frac{1+7}{2(1)} = 4$

$a_2 = (-1)^{2-1}\frac{2+7}{2(2)} = -\frac{9}{4}$

$a_3 = (-1)^{3-1}\frac{3+7}{2(3)} = \frac{5}{3}$

$a_4 = (-1)^{4-1}\frac{4+7}{2(4)} = -\frac{11}{8}$

$a_8 = (-1)^{8-1}\frac{8+7}{2(8)} = -\frac{15}{16}$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\frac{2^n}{n^2+2}\right\}$

Solution

$a_1 = \frac{2^1}{1^2+2} = \frac{2}{3}$

$a_2 = \frac{2^2}{2^2+2} = \frac{2}{3}$

$$a_3 = \frac{2^3}{3^2+2} = \frac{8}{11}$$

$$a_4 = \frac{2^4}{4^2+2} = \frac{8}{9}$$

$$a_8 = \frac{2^8}{8^2+2} = \frac{128}{33}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{(-1)^{n-1}\frac{n}{2n-1}\right\}$

Solution

$$a_1 = (-1)^0\frac{1}{2-1} = 1$$

$$a_2 = (-1)^1\frac{2}{4-1} = -\frac{2}{3}$$

$$a_3 = (-1)^2\frac{3}{6-1} = \frac{3}{5}$$

$$a_4 = (-1)^3\frac{4}{8-1} = -\frac{4}{7}$$

$$a_8 = (-1)^7\frac{8}{16-1} = -\frac{8}{15}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\frac{2^n}{3^n+1}\right\}$

Solution

$$a_1 = \frac{2^1}{3^1+1} = \frac{2}{4} = \frac{1}{2}$$

$$a_2 = \frac{2^2}{3^2+1} = \frac{4}{10} = \frac{2}{5}$$

$$a_3 = \frac{2^3}{3^3+1} = \frac{8}{28} = \frac{2}{7}$$

$$a_4 = \frac{2^4}{3^4+1} = \frac{16}{82} = \frac{8}{41}$$

$$a_8 = \frac{2^8}{3^8+1} = \frac{256}{6562} = \frac{128}{3281}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\frac{n^2}{2^n}\right\}$

Solution

$$a_1 = \frac{1^2}{2^1} = \frac{1}{2}$$

$$a_2 = \frac{2^2}{2^2} = 1$$

$$a_3 = \frac{3^2}{2^3} = \frac{9}{8}$$

$$a_4 = \frac{4^2}{2^4} = \frac{16}{16} = 1$$

$$a_8 = \frac{8^2}{2^8} = \frac{64}{256} = \frac{1}{4}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\frac{n}{e^n}\right\}$

Solution

$$a_1 = \frac{1}{e^1} = \frac{1}{e}$$

$$a_2 = \frac{2}{e^2}$$

$$a_3 = \frac{3}{e^3}$$

$$a_4 = \frac{4}{e^4}$$

$$a_8 = \frac{8}{e^8}$$

Exercise

Find the first four terms and the eight term of the sequence: $\{c_n\} = \left\{(-1)^{n+1} n^2\right\}$

Solution

$$c_1 = (-1)^2 1^2 \ \underline{= 1}$$

$$c_2 = (-1)^3 2^2 \ \underline{= -4}$$

$$c_3 = (-1)^4 3^2 \ \underline{= 9}$$

$$c_4 = (-1)^5 4^2 \ \underline{= -16}$$

$$c_8 = (-1)^9 8^2 \ \underline{= -64}$$

Exercise

Find the first four terms and the eight term of the sequence: $\{c_n\} = \left\{\dfrac{(-1)^n}{(n+1)(n+2)}\right\}$

Solution

$$c_1 = \frac{(-1)^1}{2\cdot 3} \ \underline{= -\frac{1}{6}}$$

$$c_2 = \frac{(-1)^2}{3\cdot 4} \ \underline{= \frac{1}{12}}$$

$$c_3 = \frac{(-1)^3}{4\cdot 5} \ \underline{= -\frac{1}{20}}$$

$$c_4 = \frac{(-1)^4}{5\cdot 6} \ \underline{= \frac{1}{30}}$$

$$c_8 = \frac{(-1)^8}{9\cdot 10} \ \underline{= \frac{1}{90}}$$

Exercise

Find the first four terms and the eight term of the sequence: $\{c_n\} = \left\{\left(\frac{4}{3}\right)^n\right\}$

Solution

$$c_1 = \left(\frac{4}{3}\right)^1 = \frac{4}{3}$$

$$c_2 = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$$

$$c_3 = \left(\frac{4}{3}\right)^3 = \frac{64}{27}$$

$$c_4 = \left(\frac{4}{3}\right)^4 = \frac{256}{81}$$

$$c_8 = \left(\frac{4}{3}\right)^8 = \frac{65,536}{6,561}$$

Exercise

Find the first four terms and the eight term of the sequence: $\{b_n\} = \left\{\frac{3^n}{n}\right\}$

Solution

$$b_1 = \frac{3^1}{1} = 3$$

$$b_2 = \frac{3^2}{2} = \frac{9}{2}$$

$$b_3 = \frac{3^3}{3} = 9$$

$$b_4 = \frac{3^4}{4} = \frac{81}{4}$$

$$c_8 = \frac{3^8}{8} = \frac{6,561}{8}$$

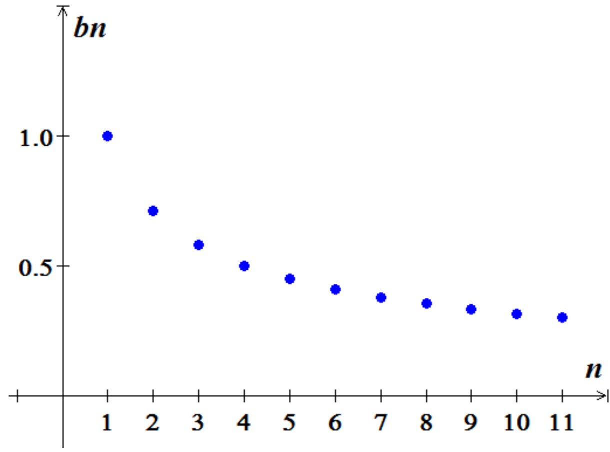
Exercise

Graph the sequence $\left\{\frac{1}{\sqrt{n}}\right\}$

Solution

$$\left\{\frac{1}{\sqrt{n}}\right\}=\left\{\frac{1}{\sqrt{1}},\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{4}},\ \dots\right\}$$

$$\approx\{1,\ 0.71,\ 0.58,\ 0.5,\ 0.45\}$$



Exercise

Find the first four terms of the sequence of partial sums for the given sequence. $\left\{3+\frac{1}{2}n\right\}$

Solution

$$S_1=a_1$$

$$=3+\frac{1}{2}(1)$$

$$=\frac{7}{2}$$

$$S_2=S_1+a_2$$

$$=\frac{7}{2}+3+\frac{1}{2}(2)$$

$$=\frac{15}{2}$$

$$S_3=S_2+a_3$$

$$=\frac{15}{2}+3+\frac{1}{2}(3)$$

$$=12$$

$$S_4=S_3+a_4$$

$$=12+3+\frac{1}{2}(4)$$

$$=17$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_{k+1} = 3a_k - 5$

Solution

$$k = 1 \rightarrow \quad a_2 = 3a_1 - 5$$
$$= 3(2) - 5$$
$$= 1$$

$$k = 2 \rightarrow \quad a_3 = 3a_2 - 5$$
$$= 3(1) - 5$$
$$= -2$$

$$k = 3 \rightarrow \quad a_4 = 3a_3 - 5$$
$$= 3(-2) - 5$$
$$= -11$$

$$k = 4 \rightarrow \quad a_5 = 3a_4 - 5$$
$$= 3(-11) - 5$$
$$= -38$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = -3, \quad a_{k+1} = a_k^2$

Solution

$$k = 1 \rightarrow \quad a_2 = a_1^2$$
$$= (-3)^2$$
$$= 9$$

$$k = 2 \rightarrow \quad a_3 = a_2^2$$
$$= (9)^2$$
$$= 81$$

$$k = 3 \rightarrow \quad a_4 = a_3^2$$
$$= \left(3^4\right)^2$$
$$= 3^8$$

$$k = 4 \rightarrow \quad a_5 = a_4^2$$
$$= \left(3^8\right)^2$$
$$\underline{= 3^{16}}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 5, \quad a_{k+1} = ka_k$

Solution

$$k = 1 \rightarrow \quad a_2 = 1a_1$$
$$\underline{= 5}$$

$$k = 2 \rightarrow \quad a_3 = 2a_2$$
$$= 2(5)$$
$$\underline{= 10}$$

$$k = 3 \rightarrow \quad a_4 = 3a_3$$
$$= 3(10)$$
$$\underline{= 30}$$

$$k = 4 \rightarrow \quad a_5 = 4a_4$$
$$= 4(30)$$
$$\underline{= 120}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_n = 3 + a_{n-1}$

Solution

$$a_2 = 3 + a_1 = 3 + 2 \; \underline{= 5}$$

$$a_3 = 3 + a_2 = 3 + 5 \; \underline{= 8}$$

$$a_4 = 3 + a_3 = 3 + 8 \; \underline{= 11}$$

$$a_5 = 3 + a_4 = 3 + 11 \; \underline{= 14}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 5, \quad a_n = 2a_{n-1}$

Solution

$$a_2 = 2a_1$$
$$= 2(5)$$
$$\boxed{= 10}$$

$$a_3 = 2a_2$$
$$= 2(10)$$
$$\boxed{= 20}$$

$$a_4 = 2a_3$$
$$= 2(20)$$
$$\boxed{= 40}$$

$$a_5 = 2a_4$$
$$= 2(40)$$
$$\boxed{= 80}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = \sqrt{2}, \quad a_n = \sqrt{2 + a_{n-1}}$

Solution

$$a_2 = \sqrt{2 + a_1}$$
$$\boxed{= \sqrt{2 + \sqrt{2}}}$$

$$a_3 = \sqrt{2 + a_2}$$
$$\boxed{= \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$

$$a_4 = \sqrt{2 + a_3}$$
$$\boxed{= \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}$$

$$a_5 = \sqrt{2 + a_4}$$
$$\boxed{= \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_{n+1} = 7 - 2a_n$

Solution

$$a_2 = 7 - 2a_1$$
$$= 7 - 4$$
$$\underline{= 3}$$

$$a_3 = 7 - 2a_2$$
$$= 7 - 6$$
$$\underline{= 1}$$

$$a_4 = 7 - 2a_3$$
$$= 7 - 2$$
$$\underline{= 5}$$

$$a_5 = 7 - 2a_4$$
$$= 7 - 10$$
$$\underline{= -5}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 128, \quad a_{n+1} = \frac{1}{4}a_n$

Solution

$$a_2 = \frac{1}{4}a_1$$
$$= \frac{1}{4}128$$
$$\underline{= 32}$$

$$a_3 = \frac{1}{4}a_2$$
$$= \frac{32}{4}$$
$$\underline{= 8}$$

$$a_4 = \frac{1}{4}a_3$$
$$\underline{= 2}$$

$$a_5 = \frac{1}{4}a_4$$
$$\underline{= \frac{1}{2}}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_{n+1} = \left(a_n\right)^n$

Solution

$$a_2 = \left(a_1\right)^1$$
$$\underline{= 2}$$

$$a_3 = \left(a_2\right)^2$$
$$= 2^2$$
$$\underline{= 4}$$

$$a_4 = \left(a_3\right)^3$$
$$= 4^3$$
$$\underline{= 64}$$

$$a_5 = \left(a_4\right)^4$$
$$\underline{= 64^4}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = A, \quad a_n = a_{n-1} + d$

Solution

$$a_2 = a_1 + d$$
$$\underline{= A + d}$$

$$a_3 = a_2 + d$$
$$= A + d + d$$
$$\underline{= A + 2d}$$

$$a_4 = a_3 + d$$
$$\underline{= A + 3d}$$

$$a_5 = a_4 + d$$
$$\underline{= A + 4d}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = A, \quad a_n = ra_{n-1}, \quad r \neq 0$

Solution

$$a_2 = ra_1$$
$$\underline{= rA}$$

$$a_3 = ra_2$$
$$\underline{= Ar^2}$$

$$a_4 = ra_3$$
$$\underline{= Ar^3}$$

$$a_5 = ra_4$$
$$\underline{= Ar^4}$$

Exercise

Find the first 5 terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_2 = 2; \quad a_n = a_{n-1} \bullet a_{n-2}$

Solution

$$a_3 = a_2 \cdot a_1$$
$$= 2 \cdot 2$$
$$\underline{= 4}$$

$$a_4 = a_3 \cdot a_2$$
$$= 4 \cdot 2$$
$$\underline{= 8}$$

$$a_5 = a_4 \cdot a_3$$
$$= 8 \cdot 4$$
$$\underline{= 32}$$

$$a_6 = a_5 \cdot a_4$$
$$= 32 \cdot 8$$
$$\underline{= 256}$$

Exercise

Express each sum using summation notation $1+2+3+\ldots+20$

Solution

$$1+2+3+4+\cdots+20=\sum_{k=1}^{20} k$$

Exercise

Express each sum using summation notation $1+2+3+\ldots+40$

Solution

$$1+2+3+\ldots+40=\sum_{k=1}^{40} k$$

Exercise

Express each sum using summation notation $1^3+2^3+3^3+\ldots+8^3$

Solution

$$1^3+2^3+3^3+\ldots+8^3=\sum_{k=1}^{8} k^3$$

Exercise

Express each sum using summation notation $1^2+2^2+3^2+\ldots+15^2$

Solution

$$1^2+2^2+3^2+\ldots+15^2=\sum_{k=1}^{15} k^2$$

Exercise

Express each sum using summation notation $2^2+2^3+2^4+\ldots+2^{11}$

Solution

$$2^2+2^3+2^4+\ldots+2^{11}=\sum_{k=2}^{11} 2^k$$

Exercise

Express each sum using summation notation $\quad \frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{13}{14}$

Solution

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{13}{14}=\sum_{k=1}^{13}\frac{k}{k+1}$$

Exercise

Express each sum using summation notation $\quad 1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\cdots+(-1)^6\frac{1}{3^6}$

Solution

$$1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\cdots+(-1)^6\frac{1}{3^6}=\sum_{k=0}^{6}(-1)^k\frac{1}{3^k}$$

Exercise

Express each sum using summation notation $\quad \frac{2}{3}-\frac{4}{9}+\frac{8}{27}-\cdots+(-1)^{12}\left(\frac{2}{3}\right)^{11}$

Solution

$$\frac{2}{3}-\frac{4}{9}+\frac{8}{27}-\cdots+(-1)^{12}\left(\frac{2}{3}\right)^{11}=\sum_{k=1}^{11}(-1)^{k+1}\left(\frac{2}{3}\right)^k$$

Exercise

Express each sum using summation notation $\quad \frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{14}{14+1}$

Solution

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{14}{14+1}=\sum_{k=1}^{14}\frac{k}{k+1}$$

Exercise

Express each sum using summation notation $\quad \frac{1}{e}+\frac{2}{e^2}+\frac{3}{e^3}+\cdots+\frac{n}{e^n}$

Solution

$$\frac{1}{e}+\frac{2}{e^2}+\frac{3}{e^3}+\cdots+\frac{n}{e^n}=\sum_{k=1}^{n}\frac{k}{e^k}$$

Exercise

Find the sum: $\sum_{k=1}^{5}(2k-7)$

Solution

$$\sum_{k=1}^{5}(2k-7)=(-5)+(-3)+(-1)+1+3$$
$$\underline{=-5\,|}$$

Exercise

Find the sum: $\sum_{k=0}^{5}k(k-2)$

Solution

$$\sum_{k=0}^{5}k(k-2)=0+(-1)+0+3+8+15$$
$$\underline{=25\,|}$$

Exercise

Find the sum: $\sum_{k=1}^{5}(-3)^{k-1}$

Solution

$$\sum_{k=1}^{5}(-3)^{k-1}=1+(-3)+9+(-27)+81$$
$$\underline{=61\,|}$$

Exercise

Find the sum: $\sum_{k=253}^{571}\left(\frac{1}{3}\right)$

Solution

$$\sum_{k=253}^{571}\left(\frac{1}{3}\right)=(571-253+1)\left(\frac{1}{3}\right) \qquad \sum_{k=m}^{n} c=(n-m+1)c$$

$$=\frac{319}{3}$$

Exercise

Find the sum: $\sum_{k=1}^{50} 8$

Solution

$$\sum_{k=1}^{50} 8=(50-1+1)8 \qquad \sum_{k=m}^{n} c=(n-m+1)c$$

$$=400$$

Exercise

Find the sum: $\sum_{k=1}^{40} k$

Solution

$$\sum_{k=1}^{40} k=\frac{40(41)}{2} \qquad \sum_{k=1}^{n} k=\frac{n(n+1)}{2}$$

$$=820$$

Exercise

Find the sum: $\sum_{k=1}^{5} (3k)$

Solution

$$\sum_{k=1}^{5} 3k=3(1)+3(2)+3(3)+3(4)+3(5)$$

$$=45$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{10}\left(k^3+1\right)$

Solution

$$\sum_{k=1}^{10}\left(k^3+1\right)=\sum_{k=1}^{10}k^3+\sum_{k=1}^{10}1$$

$$\sum_{k=1}^{n}k^3=\frac{n^2(n+1)^2}{4}$$

$$=\frac{10^2(10+1)^2}{4}+10(1)$$

$$=\frac{12100}{4}+10$$

$$=3025+10$$

$$=3035$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{24}\left(k^2-7k+2\right)$

Solution

$$\sum_{k=1}^{24}\left(k^2-7k+2\right)=\frac{24(24+1)(2\cdot 24+1)}{6}-7\frac{24(24+1)}{2}+2(24)$$

$$\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

$$=2848$$

Exercise

Find the sum: $\displaystyle\sum_{k=6}^{20}\left(4k^2\right)$

Solution

$$\sum_{k=6}^{20}\left(4k^2\right)=4\left(\sum_{k=1}^{20}k^2-\sum_{k=1}^{5}k^2\right)$$

$$\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

$$=4\left(\frac{20(20+1)(2\cdot 20+1)}{6}-\frac{5(5+1)(2\cdot 5+1)}{6}\right)$$

$$= 4\left(\frac{20(21)(41)}{6} - \frac{5(6)(11)}{6}\right)$$

$$= 4(2870 - 55)$$

$$\underline{= 11,260}$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{16}\left(k^2 - 4\right)$

Solution

$$\sum_{k=1}^{16}\left(k^2 - 4\right) = \sum_{k=1}^{16} k^2 - \sum_{k=1}^{16} 4$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{16(16+1)(2\cdot 16+1)}{6} - 4(16)$$

$$= 1496 - 64$$

$$\underline{= 1432}$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{6}(10 - 3k)$

Solution

$$\sum_{k=1}^{6}(10 - 3k) = 7 + 4 + 1 - 2 - 5 - 8$$

$$\underline{= -3}$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{10}\left[1 + (-1)^k\right]$

Solution

$$\sum_{k=1}^{10}\left[1+(-1)^k\right]=0+2+0+2+0+2+0+2+0+2$$

$$\boxed{=10}$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{6}\frac{3}{k+1}$

Solution

$$\sum_{k=1}^{6}\frac{3}{k+1}=\frac{3}{2}+1+\frac{3}{4}+\frac{3}{5}+2+\frac{3}{7}$$

$$\boxed{=\frac{879}{140}}$$

Exercise

Find the sum: $\displaystyle\sum_{k=137}^{428}2.1$

Solution

$$\sum_{k=137}^{428}2.1=(428-137+1)2.1=(292)2.1$$

$$\sum_{k=m}^{n}c=(n-m+1)c$$

$$\boxed{=613.2}$$

Exercise

Write out each sum $\displaystyle\sum_{k=1}^{n}(k+2)$

Solution

$$\sum_{k=1}^{n}(k+2)\ \boxed{=3+5+7+9+\cdots+(n+2)}$$

Exercise

Write out each sum $\sum\limits_{k=1}^{n} k^2$

Solution

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2$$

$$\boxed{= 1 + 4 + 9 + 16 + \cdots + n^2}$$

Exercise

Write out each sum $\sum\limits_{k=2}^{n} (-1)^k \ln k$

Solution

$$\sum_{k=2}^{n} (-1)^k \ln k = (-1)^2 \ln 2 + (-1)^3 \ln 3 + (-1)^4 \ln 4 + (-1)^5 \ln 5 + \cdots + (-1)^n \ln n$$

$$\boxed{= \ln 2 - \ln 3 + \ln 4 - \ln 5 + \cdots + (-1)^n \ln n}$$

Exercise

Write out each sum $\sum\limits_{k=3}^{n} (-1)^{k+1} 2^k$

Solution

$$\sum_{k=3}^{n} (-1)^{k+1}\ 2^k = (-1)^4 2^3 + (-1)^5 2^4 + (-1)^6 2^5 + (-1)^7 2^6 + \cdots + (-1)^{n+1} 2^n$$

$$\boxed{= 8 - 16 + 32 - 64 + \cdots + (-1)^{n+1} 2^n}$$

Exercise

Write out each sum $\sum\limits_{k=0}^{n} \frac{1}{3^k}$

Solution

$$\sum_{k=0}^{n} \frac{1}{3^k} \underline{= 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^n}}$$

Exercise

Fred has a balance of \$3,000 on his card which charges 1% interest per month on any unpaid balance. Fred can afford to pay \$100 toward the balance each *month*. His balance each month after making a \$100 payment is given by the recursively defined sequence

$$B_0 = \$3,000 \quad B_n = 1.01 B_{n-1} - 100$$

Determine Fred's balance after making the first payment. That is, determine B_1

Solution

$$\begin{aligned} B_1 &= 1.01 B_0 - 100 \\ &= 1.01(3,000) - 100 \\ &\underline{= \$2,930} \end{aligned}$$

Fred's balance is \$2,930 after making the first payment.

Exercise

A pond currently has 2,000 trout in it. A fish hatchery decides to add an additional 20 trout each month. Is it also known that the trout population is grwoing at a rate of 3% per *month*. The size of the population after *n* months is given bu the recursively defined sequence

$$P_0 = 2,000 \quad P_n = 1.03 P_{n-1} + 20$$

How many trout are in the pond after 2 months? That is, what is P_2?

Solution

$$\begin{aligned} P_1 &= 1.03 P_0 + 20 \\ &= 1.03(2,000) + 20 \\ &\underline{= 2,080} \end{aligned}$$

$$\begin{aligned} P_2 &= 1.03 P_1 + 20 \\ &= 1.03(2,080) + 20 \\ &\underline{= 2,162.4} \end{aligned}$$

There are approximately 2162 ***trout*** in the pond after 2 *months*.

Exercise

Fred bought a car by taking out a loan for \$18,500 at 0.5% interest per month. Fred's normal monthly payment is \$434.47 per month, but he decides that he can afford to pay \$100 extra toward the balance each month. His balance each month is given by the recursively defined sequence

$$B_0 = \$18,500 \quad B_n = 1.005B_{n-1} - 534.47$$

Determine Fred's balance after making the first payment. That is, determine B_1

Solution

$$\begin{aligned} B_1 &= 1.005B_0 - 534.47 \\ &= 1.005(18,500) - 534.47 \\ &= \$18,058.03 \end{aligned}$$

Fred's balance is \$18.058.03 after making the first payment.

Exercise

The Environmental Protection Agency (EPA) determines that Maple Lake has 250 *tons* of pollutant as a result of industrial waste and that 10% of the pollutant present is neuttralized by solar oxidation every year. The EPA imposes new pollution control laws that result in 15 *tons* of new pollutant entering the lake each year. The amount of pollutant in the lake after *n* years is given by the recursively defined sequence

$$P_0 = 250 \quad P_n = 0.9P_{n-1} + 15$$

Determine the amount of pollutant in the lake after 2 years? That is, what is P_2?

Solution

$$\begin{aligned} P_1 &= 0.9P_0 + 15 \\ &= 0.9(250) + 15 \\ &= 240 \end{aligned}$$

$$\begin{aligned} P_2 &= 0.9P_1 + 15 \\ &= 0.9(240) + 15 \\ &= 231 \end{aligned}$$

There are 231 ***tons*** of pollutants after 2 *years*.

Exercise

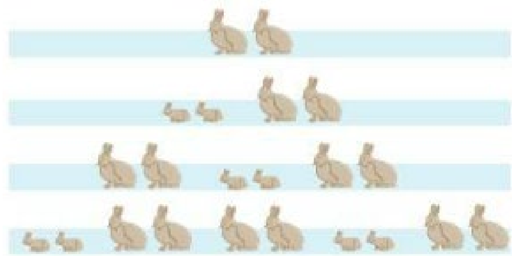
A colony of rabbits begins with one pair of mature rabbits, which will produce a pair of offspring (one male, one female) each month. Assume that all rabbits mature in 1 month and produce a pair of offspring (one male, one female) after 2 months. If no rabbits ever die, how many pairs of mature rabbits are there after 7 months?

1 mature pair

1 mature pair

2 mature pairs

3 mature pairs



Solution

$a_1 = 1$

$a_2 = 1$

$a_3 = 2$

$a_4 = 3$

$a_5 = 5$

$a_6 = 8$

$a_7 = 13$

$a_8 = 21$

$\vdots \quad \vdots$

$a_n = a_{n-1} + a_{n-2}$

After 7 *months* there are 21 *mature* pairs of rabbits.

Exercise

Let $\quad u_n = \dfrac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}$

Define the *n*th term of a sequence

a) Show that $u_1 = 1$ and $u_2 = 1$

b) Show that $u_{n+2} = u_{n+1} + u_n$

c) Draw the conclusion that $\{u_n\}$ is a Fibonacci sequence

d) Find the first ten terms of the sequence from part (*c*)

Solution

a)
$$u_1 = \frac{\left(1+\sqrt{5}\right)^1 - \left(1-\sqrt{5}\right)^1}{2^1\ \sqrt{5}}$$
$$= \frac{2\ \sqrt{5}}{2\ \sqrt{5}}$$
$$\underline{=1}\,\rfloor$$

$$u_2 = \frac{\left(1+\sqrt{5}\right)^2 - \left(1-\sqrt{5}\right)^2}{2^2\ \sqrt{5}}$$

$$a^2 - b^2 = (a-b)(a+b)$$

$$= \frac{\left(1+\sqrt{5}-1+\sqrt{5}\right)-\left(1-\sqrt{5}+1-\sqrt{5}\right)}{2^2\ \sqrt{5}}$$
$$= \frac{4\ \sqrt{5}}{4\ \sqrt{5}}$$
$$\underline{=1}\,\rfloor$$

b)
$$u_{n+1} + u_n = \frac{\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}}{2^{n+1}\ \sqrt{5}} + \frac{\left(1+\sqrt{5}\right)^{n} - \left(1-\sqrt{5}\right)^{n}}{2^{n}\ \sqrt{5}}$$
$$= \frac{\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1} + 2\left(1+\sqrt{5}\right)^{n} - 2\left(1-\sqrt{5}\right)^{n}}{2^{n+1}\ \sqrt{5}}$$
$$= \frac{\left(1+\sqrt{5}\right)^{n}\left(1+\sqrt{5}+2\right) - \left(1-\sqrt{5}\right)^{n}\left(1-\sqrt{5}+2\right)}{2^{n+1}\ \sqrt{5}}$$
$$= \frac{\left(1+\sqrt{5}\right)^{n}\left(3+\sqrt{5}\right) - \left(1-\sqrt{5}\right)^{n}\left(3-\sqrt{5}\right)}{2^{n+1}\ \sqrt{5}}$$
$$= \frac{\left(1+\sqrt{5}\right)^{n+2}\dfrac{3+\sqrt{5}}{\left(1+\sqrt{5}\right)^2} - \left(1-\sqrt{5}\right)^{n+2}\dfrac{3-\sqrt{5}}{\left(1-\sqrt{5}\right)^2}}{2^{n+1}\ \sqrt{5}}$$
$$= \frac{\left(1+\sqrt{5}\right)^{n+2}\dfrac{3+\sqrt{5}}{1+2\sqrt{5}+5} - \left(1-\sqrt{5}\right)^{n+2}\dfrac{3-\sqrt{5}}{1-2\sqrt{5}+5}}{2^{n+1}\ \sqrt{5}}$$
$$= \frac{\frac{1}{2}\left(1+\sqrt{5}\right)^{n+2}\dfrac{3+\sqrt{5}}{3+\sqrt{5}} - \frac{1}{2}\left(1-\sqrt{5}\right)^{n+2}\dfrac{3-\sqrt{5}}{3-\sqrt{5}}}{2^{n+1}\ \sqrt{5}}$$

$$=\frac{1}{2}\frac{\left(1+\sqrt{5}\right)^{n+2}-\left(1-\sqrt{5}\right)^{n+2}}{2^{n+1}\sqrt{5}}$$

$$=\frac{\left(1+\sqrt{5}\right)^{n+2}-\left(1-\sqrt{5}\right)^{n+2}}{2^{n+2}\sqrt{5}}$$

$$=u_{n+2}\ \checkmark$$

c) Since $u_1=1$ and $u_2=1$ and $u_{n+2}=u_{n+1}+u_n$

$\therefore \{u_n\}$ is a Fibonacci sequence

e) $u_1=1$

$u_2=1$

$u_3=u_2+u_1=1+1=2$

$u_4=u_3+u_2=2+1=3$

$u_5=u_4+u_3=3+2=5$

$u_6=u_5+u_4=5+3=8$

$u_7=u_6+u_5=8+5=13$

$u_8=u_7+u_5=13+8=21$

$u_9=u_8+u_7=21+13=34$

$u_{10}=u_9+u_8=34+21=55$

Solution Section 5.6 – Arithmetic and Geometric Sequences

Exercise

Show that the sequence $-6,\ -2,\ 2,\ \ldots,\ 4n-10,\ \ldots$ is arithmetic, and find the common difference.

Solution

We to show that $a_{k+1}-a_k$ equals to a constant.

$$\begin{aligned} a_{k+1}-a_k &= 4(k+1)-10-(4k-10) \\ &= 4k+4-10-4k+10 \\ &= 4 \end{aligned}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $2,\ 6,\ 10,\ 14,\ \ldots$

Solution

$$\begin{aligned} d &= 6-2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} a_n &= 2+(n-1)4 \\ &= 2+4n-4 \\ &= 4n-2 \end{aligned}$$

$a_n = a_1 + (n-1)d$

$$\begin{aligned} a_{10} &= 4(10)-2 \\ &= 38 \end{aligned}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $3,\ 2.7,\ 2.4,\ 2.1,\ \ldots$

Solution

$$\begin{aligned} d &= 2.7-3=-0.3 \\ &= -0.3 \end{aligned}$$

$$\begin{aligned} a_n &= 3+(n-1)(-0.3) \\ &= 3-0.3n+0.3 \\ &= 3.3-0.3n \end{aligned}$$

$a_n = a_1 + (n-1)d$

$$\begin{aligned} a_{10} &= 3.3-0.3(10) \\ &= 0.3 \end{aligned}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $-6,\ -4.5,\ -3,\ -1.5,\ \ldots$

Solution

$$d = -4.5 - (-6)$$
$$= 1.5$$

$$a_n = -6 + (n-1)(1.5) \qquad a_n = a_1 + (n-1)d$$
$$= -6 + 1.5n - 1.5$$
$$= 1.5n - 7.5$$

$$a_{10} = 1.5(10) - 7.5$$
$$= 7.5$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $\ln 3,\ \ln 9,\ \ln 27,\ \ln 81,\ \ldots$

Solution

$$\ln 3,\ \ln 3^2,\ \ln 3^3,\ \ln 3^4,\ \ldots$$
$$\ln 3,\ 2\ln 3,\ 3\ln 3,\ 4\ln 3,\ \ldots$$

$$d = 2\ln 3 - \ln 3$$
$$= \ln 3$$

$$a_n = \ln 3 + (n-1)\ln 3 \qquad a_n = a_1 + (n-1)d$$
$$= \ln 3 + n\ln 3 - \ln 3$$
$$= n\ln 3$$

$$a_{10} = 10\ln 3$$
$$= \ln 3^{10}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = 2, \quad d = 3$

Solution

$$a_n = 2 + 3(n-1) \qquad a_n = a_1 + (n-1)d$$
$$= 2 + 3n - 3$$
$$= 3n - 1$$

$$a_{10} = 29$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = 5, \quad d = -3$

Solution

$$a_n = 5 + (n-1)(-3) \qquad\qquad a_n = a_1 + (n-1)d$$

$$= 5 - 3n + 3$$

$$\underline{= 8 - 3n}$$

$$a_{10} = 8 - 30$$

$$\underline{= -22}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = 1, \quad d = -\frac{1}{2}$

Solution

$$a_n = 1 + (n-1)\left(-\frac{1}{2}\right) \qquad\qquad a_n = a_1 + (n-1)d$$

$$= 1 - \frac{1}{2}n + \frac{1}{2}$$

$$\underline{= \frac{3}{2} - \frac{1}{2}n}$$

$$a_{10} = \frac{3}{2} - \frac{1}{2}(10)$$

$$\underline{= -\frac{7}{2}}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = -2, \quad d = 4$

Solution

$$a_n = -2 + (n-1)(4) \qquad\qquad a_n = a_1 + (n-1)d$$

$$= -2 + 4n - 4$$

$$\underline{= 4n - 6}$$

$$a_{10} = 4(10) - 6$$

$$\underline{= 36}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = \sqrt{2}, \quad d = \sqrt{2}$

Solution

$$a_n = \sqrt{2} + (n-1)\sqrt{2} \qquad\qquad a_n = a_1 + (n-1)d$$

$$= \sqrt{2} + \sqrt{2}n - \sqrt{2}$$

$$= \sqrt{2}\,n$$

$$a_{10} = 10\sqrt{2}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = 0, \quad d = \pi$

Solution

$$a_n = 0 + (n-1)(\pi) \qquad\qquad a_n = a_1 + (n-1)d$$

$$= \pi n - \pi$$

$$a_{10} = 10\pi - \pi$$

$$= 9\pi$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = 13, \quad d = 4$

Solution

$$a_n = 13 + (n-1)(4) \qquad\qquad a_n = a_1 + (n-1)d$$

$$= 4n + 9$$

$$a_{10} = 4(10) + 9$$

$$= 49$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = -40, \quad d = 5$

Solution

$$a_n = -40 + (n-1)(5) \qquad\qquad a_n = a_1 + (n-1)d$$

$$= 5n - 45$$

$$a_{10} = 4(10) - 45$$
$$= -5$$

Exercise

Find the *n*th term, and the tenth term of the arithmetic sequence: $a_1 = -32, \quad d = 4$

Solution

$$a_n = -32 + (n-1)(4) \qquad\qquad a_n = a_1 + (n-1)d$$
$$= 4n - 36$$
$$a_{10} = 4(10) - 36$$
$$= 4$$

Exercise

Find the common difference for the arithmetic sequence with the specified terms: $a_4 = 14, \quad a_{11} = 35$

Solution

$$a_n = a_1 + (n-1)d$$
$$a_{11} = a_1 + 10d \rightarrow 35 = a_1 + 10d$$
$$a_4 = a_1 + 3d \rightarrow 14 = a_1 + 3d$$
$$21 = 7d$$
$$d = 3$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: $a_{12}; \quad a_1 = 9.1, \quad a_2 = 7.5$

Solution

$$d = a_2 - a_1$$
$$= 7.5 - 9.1$$
$$= -1.6$$
$$a_n = a_1 + (n-1)d$$
$$a_{12} = 9.1 + (11)(-1.6)$$
$$= -8.5$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_1; $a_8 = 47$, $a_9 = 53$

Solution

$$d = a_9 - a_8$$
$$= 53 - 4$$
$$= 6$$

$$a_8 = a_1 + (7)(6) \qquad\qquad a_n = a_1 + (n-1)d$$

$$a_1 = 47 - 42$$
$$= 5$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_{10}; $a_2 = 1$, $a_{18} = 49$

Solution

$$a_2 = a_1 + d$$
$$a_1 = a_2 - d$$

$$a_{18} = a_1 + (17)d$$
$$= a_2 - d + 17d$$
$$= a_2 + 16d$$

$$49 = 1 + 16d$$
$$16d = 48$$
$$d = \frac{48}{16} = 3$$

$$a_1 = a_2 - d$$
$$= 1 - 3$$
$$= -2$$

$$a_{10} = -2 + 9(3)$$
$$= 25$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_{10}; $a_8 = 8$, $a_{20} = 44$

Solution

$$d = \frac{44-8}{20-8} \qquad\qquad d = \frac{a_y - a_x}{y - x}$$

$$= \frac{36}{12}$$

$$\underline{= 3}$$

$$a_8 = a_1 + (8-1)(3) \qquad\qquad a_n = a_1 + (n-1)d$$

$$8 = a_1 + 21$$

$$\underline{a_1 = -13}$$

$$a_{10} = -13 + 9(3)$$

$$\underline{= 14}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_{12}; $a_8 = 4$, $a_{18} = -96$

Solution

$$d = \frac{-96-4}{18-8} \qquad\qquad d = \frac{a_y - a_x}{y - x}$$

$$= \frac{-100}{10}$$

$$\underline{= -10}$$

$$a_8 = a_1 + (8-1)(-10) \qquad\qquad a_n = a_1 + (n-1)d$$

$$4 = a_1 - 70$$

$$\underline{a_1 = 74}$$

$$a_{12} = 74 + (11)(-10)$$

$$\underline{= -36}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_8; $a_{15} = 0$, $a_{40} = -50$

Solution

$$d = \frac{-50-0}{40-15} \qquad\qquad d = \frac{a_y - a_x}{y - x}$$

$$= \frac{-50}{25}$$

$$\underline{= -2}$$

$a_{15} = a_1 + (15-1)(-2)$ $\qquad a_n = a_1 + (n-1)d$

$0 = a_1 - 28$

$\underline{a_1 = 28|}$

$a_8 = 28 + (7)(-2)$

$\quad \underline{= 14\ |}$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: $a_{20};\quad a_9 = -5,\quad a_{15} = 31$

Solution

$d = \dfrac{31+5}{15-9}$ $\qquad d = \dfrac{a_y - a_x}{y - x}$

$\quad = \dfrac{36}{6}$

$\quad \underline{= 6|}$

$a_9 = a_1 + (9-1)(6)$ $\qquad a_n = a_1 + (n-1)d$

$-5 = a_1 + 42$

$\underline{a_1 = -47|}$

$a_{20} = -47 + (19)(6)$

$\quad \underline{= 67\ |}$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: $a_n;\quad a_8 = 8,\quad a_{20} = 44$

Solution

$d = \dfrac{44-8}{20-8}$ $\qquad d = \dfrac{a_y - a_x}{y - x}$

$\quad = \dfrac{36}{12}$

$\quad \underline{= 3|}$

$a_8 = a_1 + 3(8-1)$ $\qquad a_n = a_1 + (n-1)d$

$8 = a_1 + 21$

$\underline{a_1 = -13|}$

$$a_n = -13 + 3(n-1)$$

$$= 3n - 16$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_n; $a_8 = 4$, $a_{18} = -96$

Solution

$$d = \frac{-96-4}{18-8} \qquad d = \frac{a_y - a_x}{y - x}$$

$$= -10$$

$$a_8 = a_1 - 10(8-1) \qquad a_n = a_1 + (n-1)d$$

$$4 = a_1 - 70$$

$$a_1 = 74$$

$$a_n = 74 - 10(n-1)$$

$$= -10n + 84$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_n; $a_{14} = -1$, $a_{15} = 31$

Solution

$$d = \frac{31+1}{15-14} \qquad d = \frac{a_y - a_x}{y - x}$$

$$= 32$$

$$a_{14} = a_1 + 32(14-1) \qquad a_n = a_1 + (n-1)d$$

$$-1 = a_1 + 416$$

$$a_1 = -417$$

$$a_n = -417 + 32(n-1)$$

$$= 32n + 449$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_n; $a_9 = -5$, $a_{15} = 31$

Solution

$$d = \frac{31+5}{15-9} \qquad\qquad d = \frac{a_y - a_x}{y - x}$$

$$= 6$$

$$a_9 = a_1 + 6(9-1) \qquad\qquad a_n = a_1 + (n-1)d$$

$$-5 = a_1 + 48$$

$$a_1 = -53$$

$$a_n = -53 + 6(n-1)$$

$$= 6n - 59$$

Exercise

Find the sum S_n of the arithmetic sequence that satisfies the conditions: $a_1 = 40$, $d = -3$, $n = 30$

Solution

$$S_n = \frac{30}{2}\left[2(40) + (30-1)(-3)\right] \qquad\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$= -105$$

Exercise

Find the sum S_n of the arithmetic sequence that satisfies the conditions: $a_7 = \frac{7}{3}$, $d = -\frac{2}{3}$, $n = 15$

Solution

$$a_7 = a_1 + (6)\left(-\frac{2}{3}\right) = \frac{7}{3}$$

$$\frac{7}{3} = a_1 - 4$$

$$a_1 = \frac{7}{3} + 4$$

$$= \frac{19}{3}$$

$$S_n = \frac{15}{2}\left[2\left(\frac{19}{3}\right) + (15-1)\left(-\frac{2}{3}\right)\right] \qquad\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$= 25$$

Exercise

Find the number of integers between 32 and 390 that are divisible by 6, find their sum

Solution

Number of terms: $n = \dfrac{390-36}{6} + 1 = 60$

$$S_n = \frac{60}{2}(36+390) \qquad\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$\underline{= 12780}$$

Exercise

Find the number of terms in the arithmetic sequence with the given conditions:

$a_1 = -2, \quad d = \frac{1}{4}, \quad S = 21$

Solution

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$21 = \frac{n}{2}\left[2(-2) + (n-1)\frac{1}{4}\right]$$

$$21 = -2n + \frac{1}{8}n(n-1)$$

$$(8)21 = -2n(8) + \frac{1}{8}n(n-1)(8)$$

$$168 = -16n + \left(n^2 - n\right)$$

$$0 = n^2 - 17n - 168$$

$$\boxed{n = 24} \qquad n = -7$$

Exercise

Express the sum in terms of summation notation and find the sum $2 + 11 + 20 + \ldots + 16,058$.

Solution

Difference in terms:

$$d = 11 - 2 = 9$$

Number of terms:

$$n = \frac{16058-2}{9} + 1 \; \underline{= 1785}$$

$$a_n = 2 + (n-1)(9) \qquad\qquad a_n = a_1 + (n-1)d$$

$$= 2 + 9n - 9$$

$$\underline{= 9n - 7}$$

Hence the nth term is: $\sum_{n=1}^{1785}(9n-7)$

$S_{1785} = \frac{1789}{2}(2+16058)$ $\qquad\qquad S_n = \frac{n}{2}(a_1 + a_n)$

$\underline{= 14,333,550}$

Exercise

Express the sum in terms of summation notation and find the sum $60+64+68+72+\cdots+120$.

Solution

Difference in terms:

$d = 64 - 60 = 4$

Number of terms:

$n = \frac{120-60}{4} + 1 = \underline{16}$ $\qquad\qquad n = \frac{a_n - a_1}{d} + 1$

$a_n = 60 + (n-1)(4)$

$\underline{= 4n - 54}$

Hence the nth term is: $\sum_{n=1}^{16}(4n-54)$

$S = \frac{16}{2}(60+120)$ $\qquad\qquad S_n = \frac{n}{2}(a_1 + a_n)$

$\underline{= 1440}$

Exercise

Find each arithmetic sum $\quad 1+3+5+\cdots+(2n-1)$

Solution

Difference in terms:

$d = 3 - 1 = 2$

Number of terms:

$n = \frac{(2n-1)-1}{2} + 1$ $\qquad\qquad n = \frac{a_n - a_1}{d} + 1$

$= \frac{2n-2}{2} + 1$

$= n - 1 + 1$

$\underline{= n}$

$$S = \frac{n}{2}(1+(2n-1)) \qquad S_n = \frac{n}{2}(a_1 + a_n)$$

$$= \frac{n}{2}(2n)$$

$$\underline{= n^2}$$

Exercise

Find each arithmetic sum $\quad 2+4+6+\cdots+2n$

Solution

Difference in terms: $\quad d = 4-2 = 2$

Number of terms:

$$n = \frac{2n-2}{2}+1 \qquad n = \frac{a_n - a_1}{d}+1$$

$$= n-1+1$$

$$\underline{= n}$$

$$S = \frac{n}{2}(2+2n) \qquad S_n = \frac{n}{2}(a_1 + a_n)$$

$$= n(n+1)$$

$$\underline{= n^2 + n}$$

Exercise

Find each arithmetic sum $\quad 2+5+8+\cdots+41$

Solution

Difference in terms:

$$d = 5-2 = 3$$

Number of terms:

$$n = \frac{41-2}{3}+1 \qquad n = \frac{a_n - a_1}{d}+1$$

$$\underline{= 14}$$

$$S = \frac{14}{2}(2+41) \qquad S_n = \frac{n}{2}(a_1 + a_n)$$

$$\underline{= 301}$$

Exercise

Find each arithmetic sum $\quad 7+12+17+\cdots+(2+5n)$

Solution

Difference in terms:

$$d = 12 - 7 = 5$$

Number of terms:

$$n = \frac{2+5n-7}{5}+1 \qquad\qquad n = \frac{a_n - a_1}{d}+1$$

$$= \frac{5n-5}{5}+1$$

$$= \frac{5n}{5}-\frac{5}{5}+1$$

$$= n$$

$$S = \frac{n}{2}(7+2+5n) \qquad\qquad S_n = \frac{n}{2}(a_1 + a_n)$$

$$= \frac{n}{2}(9+5n)$$

Exercise

Find each arithmetic sum $\quad 73+78+83+88+\cdots+558$

Solution

Difference in terms:

$$d = 78 - 73 = 5$$

Number of terms:

$$n = \frac{558-73}{5}+1 \qquad\qquad n = \frac{a_n - a_1}{d}+1$$

$$= 98$$

$$S = \frac{98}{2}(73+558) \qquad\qquad S_n = \frac{n}{2}(a_1 + a_n)$$

$$= 30,919$$

Exercise

Find each arithmetic sum $\quad 7+1-5-11-\cdots-299$

Solution

Difference in terms:

$$d = 1 - 7 = -6$$

Number of terms:

$$n = \frac{-299-7}{-6} + 1 \qquad\qquad n = \frac{a_n - a_1}{d} + 1$$

$$\underline{= 52|}$$

$$S = \frac{52}{2}(7-299) \qquad\qquad S_n = \frac{n}{2}(a_1 + a_n)$$

$$\underline{= -7592\ |}$$

Exercise

Find each arithmetic sum $\quad -1+2+7+\cdots+(4n-5)$

Solution

$$S = \frac{n}{2}(-1+4n-5) \qquad\qquad S_n = \frac{n}{2}(a_1 + a_n)$$

$$\underline{= n(2n-3)\ |}$$

Exercise

Find each arithmetic sum $\quad 5+9+13+\cdots+49$

Solution

Difference in terms: $d = 9-5 = 4$

Number of terms: $n = \frac{49-5+4}{4} \ \underline{=12|} \qquad\qquad n = \frac{a_n - a_1 + d}{d}$

$$S = \frac{12}{2}(5+49) \qquad\qquad S_n = \frac{n}{2}(a_1 + a_n)$$

$$\underline{= 324\ |}$$

Exercise

Find each arithmetic sum $\quad 2+4+6+\cdots+70$

Solution

Difference in terms: $\quad d = 4-2 = 2$

Number of terms:

$$n = \frac{70-2+2}{2} \qquad\qquad n = \frac{a_n - a_1 + d}{d}$$

$$\underline{= 35|}$$

$$S = \frac{35}{2}(70+2) \qquad\qquad S_n = \frac{n}{2}(a_1 + a_n)$$

$= 1,260$

Exercise

Find each arithmetic sum $\quad 1+3+5+\cdots+59$

Solution

Difference in terms: $\quad d=3-1=2$

Number of terms:

$$n=\frac{59-1+2}{2} \qquad\qquad n=\frac{a_n-a_1+d}{d}$$

$$\underline{=30|}$$

$$S=\frac{30}{2}(59+1) \qquad\qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$$

$$\underline{=900|}$$

Exercise

Find each arithmetic sum $\quad 4+4.5+5+5.5+\cdots+100$

Solution

Difference in terms: $d=4.5-4=0.5$

Number of terms:

$$n=\frac{100-4+0.5}{0.5} \qquad\qquad n=\frac{a_n-a_1+d}{d}$$

$$\underline{=193|}$$

$$S=\frac{193}{2}(4+100) \qquad\qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$$

$$\underline{=10,036|}$$

Exercise

Find each arithmetic sum $\quad 8+8\frac{1}{4}+8\frac{1}{2}+8\frac{3}{4}+9+\cdots+50$

Solution

Difference in terms: $\quad d=8\frac{1}{4}-8=\frac{1}{4}$

Number of terms:

$$n=\frac{50-8+0.25}{0.25} \qquad\qquad n=\frac{a_n-a_1+d}{d}$$

$$\underline{=169|}$$

$$S=\frac{169}{2}(8+50) \qquad\qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$$

$$\underline{=4,901|}$$

Exercise

Show that the given sequence is geometric, and find the common ratio

$$5,\ -\frac{5}{4},\ \frac{5}{16},\ \dots,\ 5\left(-\frac{1}{4}\right)^{n-1},\ \dots$$

Solution

To be geometric, we must show that $\frac{a_{k+1}}{a_k} = r$ is equal to some constant, which is the common ratio.

The common ratio:

$$r = \frac{a_2}{a_1} \qquad\qquad r = \frac{a_{k+1}}{a_k}$$

$$= \frac{-\frac{5}{4}}{5}$$

$$\boxed{= -\frac{1}{4}}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $8,\ 4,\ 2,\ 1,\ \dots$

Solution

Given: $a_1 = 8, \quad r = \frac{4}{8} = \frac{1}{2}$

$$a_n = a_1 r^{n-1} = 8\left(\frac{1}{2}\right)^{n-1}$$

$$= 2^3\left(2^{-1}\right)^{n-1}$$

$$= 2^3 2^{-n+1}$$

$$\boxed{= 2^{4-n}}$$

$$a_5 = 2^{4-5}$$

$$= 2^{-1}$$

$$\boxed{= \frac{1}{2}}$$

$$a_8 = 2^{4-8}$$

$$= 2^{-4}$$

$$\boxed{= \frac{1}{16}}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence

$$300, \ -30, \ 3, \ -0.3, \ \dots$$

Solution

Given: $a_1 = 300, \quad r = \frac{-30}{300} = -0.1$

$a_n = a_1 r^{n-1} = 300(-0.1)^{n-1}$

$= 3\left(10^2\right)\left(-10^{-1}\right)^{n-1} = 3(10)^2(-10)^{-n+1} = 3(-10)^{-n+3}$

$a_5 = 300(-0.1)^{5-1}$

$= 300\left(-10^{-1}\right)^4$

$= 0.03$

$a_8 = 3(-10)^{-8+3}$

$= 3(-10)^{-5}$

$= -.00003$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $1, \ -\sqrt{3}, \ 3, \ -3\sqrt{3}, \ \dots$

Solution

Given: $a_1 = 1, \quad r = \frac{a_2}{a_1} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$

$a_n = 1\left(-\sqrt{3}\right)^{n-1}$ $\qquad a_n = a_1 r^{n-1}$

$= a_1 r^{n-1}$

$a_5 = 1\left(-\sqrt{3}\right)^{5-1}$

$= 9$

$a_8 = 1\left(-\sqrt{3}\right)^{8-1}$

$= \left(-\sqrt{3}\right)^7$

$= -27\sqrt{3}$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $4,\ -6,\ 9,\ -13.5,\ \dots$

Solution

Given: $a_1 = 4, \quad r = \dfrac{a_2}{a_1} = \dfrac{-6}{4} = -\dfrac{3}{2}$

$a_n = 4\left(-\dfrac{3}{2}\right)^{n-1}$ $\qquad a_n = a_1 r^{n-1}$

$$\begin{aligned} a_5 &= 4\left(-\frac{3}{2}\right)^{5-1} \\ &= 4\left(-\frac{3}{2}\right)^{4} \\ &= 4\left(\frac{3^4}{2^4}\right) \\ &= \frac{81}{4} \end{aligned}$$

$$\begin{aligned} a_8 &= 4\left(-\frac{3}{2}\right)^{7} \\ &= -4\left(\frac{3^7}{2^7}\right) \\ &= -\frac{2187}{32} \end{aligned}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $1,\ -x^2,\ x^4,\ -x^6,\ \dots$

Solution

Given: $a_1 = 1, \quad r = \dfrac{a_2}{a_1} = \dfrac{-x^2}{1} = -x^2$

$a_n = \left(-x^2\right)^{n-1}$ $\qquad a_n = a_1 r^{n-1}$

$$\begin{aligned} a_5 &= \left(-x^2\right)^{4} \\ &= x^8 \end{aligned}$$

$a_8 = \left(-x^2\right)^{7}$

$\boxed{= -x^{14}}$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence

$$10,\ 10^{2x-1},\ 10^{4x-3},\ 10^{6x-5},\ \dots$$

Solution

Given: $a_1 = 10$

$$r = \frac{a_2}{a_1} = \frac{10^{2x-1}}{10} = 10^{2x-1-1} = 10^{2x-2}$$

$$a_n = 10\left(10^{2x-2}\right)^{n-1} \qquad\qquad a_n = a_1 r^{n-1}$$

$$= 10\left(10^{(2x-2)(n-1)}\right)$$

$$= 10\left(10^{(2x-2)n-2x+2}\right)$$

$$= 10^{2nx-2n-2x+2+1}$$

$$\boxed{= 10^{2(n-1)x-2n+3}}$$

$$a_5 = 10^{2(5-1)x-2(5)+3}$$

$$\boxed{= 10^{8x-7}}$$

$$a_8 = 10^{2(8-1)x-2(8)+3}$$

$$\boxed{= 10^{14x-13}}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $a_1 = 2, \quad r = 3$

Solution

Given: $a_1 = 10, \quad r = 3$

$$\boxed{a_n = 2 \cdot 3^{n-1}} \qquad\qquad a_n = a_1 r^{n-1}$$

$$a_5 = 2 \cdot 3^4$$

$$\boxed{= 162}$$

$$a_8 = 2 \cdot 3^7$$

$$\boxed{= 4374}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $a_1 = 1, \quad r = -\frac{1}{2}$

Solution

Given: $\quad a_1 = 1, \quad r = -\frac{1}{2}$

$$\boxed{a_n = \left(-\frac{1}{2}\right)^{n-1}} \qquad a_n = a_1 r^{n-1}$$

$$a_5 = \left(-\frac{1}{2}\right)^4$$

$$\boxed{= \frac{1}{16}}$$

$$a_8 = \left(-\frac{1}{2}\right)^7$$

$$\boxed{= -\frac{1}{128}}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $a_1 = -2, \quad r = 4$

Solution

Given: $\quad a_1 = -2, \quad r = 4$

$$\boxed{a_n = -2 \cdot (4)^{n-1}} \qquad a_n = a_1 r^{n-1}$$

$$a_5 = \left(-\frac{1}{2}\right)^4$$

$$\boxed{= \frac{1}{16}}$$

$$a_8 = \left(-\frac{1}{2}\right)^7$$

$$\boxed{= -\frac{1}{128}}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $a_1 = \sqrt{2}, \quad r = \sqrt{2}$

Solution

Given: $\quad a_1 = \sqrt{2}, \quad r = \sqrt{2}$

$$a_n = \sqrt{2}\left(\sqrt{2}\right)^{n-1} \qquad a_n = a_1 r^{n-1}$$

$$\underline{= \left(\sqrt{2}\right)^n}$$

$$a_5 = \left(\sqrt{2}\right)^5$$

$$\underline{= 4\sqrt{2}}$$

$$a_8 = \left(\sqrt{2}\right)^8$$

$$\underline{= 16}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $a_1 = 0, \quad r = \pi$

Solution

Given: $a_1 = 0, \quad r = \pi$

$$a_n = 0(\pi)^{n-1} \qquad a_n = a_1 r^{n-1}$$

$$\underline{= 0}$$

$$a_5 = 0^5$$

$$\underline{= 0}$$

$$a_8 = 0^8$$

$$\underline{= 0}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $\left\{s_n\right\} = \left\{3^n\right\}$

Solution

$$\underline{a_n = 3^n}$$

$$\underline{a_5 = 3^5}$$

$$\underline{a_8 = 3^8}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $\{s_n\} = \{(-5)^n\}$

Solution

$$\boxed{a_n = 3^n}$$

$$a_5 = (-5)^5$$
$$\boxed{= -5^5}$$

$$a_8 = (-5)^8$$
$$\boxed{= 5^8}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $\{s_n\} = \left\{-3\left(\frac{1}{2}\right)^n\right\}$

Solution

$$\boxed{a_n = -3\left(\frac{1}{2}\right)^n}$$

$$a_5 = -3\left(\frac{1}{2}\right)^5$$
$$\boxed{= -\frac{3}{32}}$$

$$a_8 = -3\left(\frac{1}{2}\right)^8$$
$$\boxed{= -\frac{3}{256}}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $\{u_n\} = \left\{\frac{3^{n-1}}{2^n}\right\}$

Solution

$$\boxed{a_n = \frac{3^{n-1}}{2^n}}$$

$$a_5 = \frac{3^4}{2^5}$$

$$\boxed{= \frac{81}{32}}$$

$$a_8 = \frac{3^7}{2^8}$$

$$\boxed{= \frac{3^7}{256}}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $\{u_n\} = \left\{\dfrac{2^n}{3^{n-1}}\right\}$

Solution

$$\boxed{a_n = \frac{2^n}{3^{n-1}}}$$

$$a_5 = \frac{2^5}{3^4}$$

$$\boxed{= \frac{32}{81}}$$

$$a_8 = \frac{2^8}{3^7}$$

$$\boxed{= \frac{256}{3^7}}$$

Exercise

Find all possible values of ***r*** for a geometric sequence with the two given terms $a_4 = 3, \quad a_6 = 9$

Solution

$$\frac{a_6}{a_4} = \frac{a_1 r^5}{a_1 r^3}$$

$$\frac{9}{3} = r^2$$

$$r^2 = 3$$

$$\boxed{r = \pm\sqrt{3}}$$

Exercise

Find the *sixth* term of the geometric sequence whose first two terms are 4 and 6

Solution

Given: $a_1 = 4, \quad a_2 = 6$

$$r = \frac{a_2}{a_1}$$

$$= \frac{6}{4}$$

$$= \frac{3}{2}$$

$$a_6 = a_1 r^{n-1}$$

$$= 4\left(\frac{3}{2}\right)^5$$

$$= \frac{243}{8}$$

Exercise

Given a geometric sequence with $a_4 = 4, \quad a_7 = 12$, find r and a_{10}

Solution

$$r = \left(\frac{12}{4}\right)^{1/(7-4)} \qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$= 3^{1/3}$$

$$= \sqrt[3]{3}$$

$$a_1 = \frac{a_4}{r^3} \qquad a_4 = a_1 r^{n-1}$$

$$= \frac{4}{3}$$

$$a_{10} = \frac{4}{3}\left(\sqrt[3]{3}\right)^9 \qquad a_{10} = a_1 r^{n-1}$$

$$= 36$$

Exercise

Find the specified term of the geometric sequence $a_6;\ a_1 = 4,\ a_2 = 6$

Solution

$$r = \left(\frac{6}{4}\right)^{1/(2-1)} \qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$= \frac{3}{2}$$

$$a_6 = 4\left(\frac{3}{2}\right)^5 \qquad a_n = a_1 r^{n-1}$$

$$= \frac{3^5}{8}$$

Exercise

Find the specified term of the geometric sequence $a_7;\ a_2 = 3,\ a_3 = -\sqrt{3}$

Solution

$$r = \left(\frac{-\sqrt{3}}{3}\right)^{1/(3-2)} \qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$= -\frac{\sqrt{3}}{3}$$

$$a_2 = a_1\left(-\frac{\sqrt{3}}{3}\right)^1 \qquad a_n = a_1 r^{n-1}$$

$$3 = -\frac{\sqrt{3}}{3}a_1$$

$$a_1 = -\frac{9}{\sqrt{3}}$$

$$= -3\sqrt{3}$$

$$a_7 = -3\sqrt{3}\left(-\frac{\sqrt{3}}{3}\right)^6$$

$$= -3\sqrt{3}\,\frac{3^3}{3^6}$$

$$= -\frac{\sqrt{3}}{9}$$

Exercise

Find the specified term of the geometric sequence $\quad a_6; \quad a_2 = 3, \quad a_3 = -\sqrt{2}$

Solution

$$r = \left(\frac{-\sqrt{2}}{3}\right)^{1/(3-2)} \qquad\qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$= \boxed{-\frac{\sqrt{2}}{3}}$$

$$a_2 = a_1\left(-\frac{\sqrt{2}}{3}\right)^1 \qquad\qquad a_n = a_1 r^{n-1}$$

$$3 = -\frac{\sqrt{2}}{3}a_1$$

$$\boxed{a_1 = -\frac{9}{\sqrt{2}}}$$

$$a_6 = -\frac{9}{\sqrt{2}}\left(-\frac{\sqrt{2}}{3}\right)^5$$

$$= 9\frac{\sqrt{2}^4}{3^5}$$

$$\boxed{= \frac{4}{27}}$$

Exercise

Find the specified term of the geometric sequence $\quad a_5; \quad a_1 = 4, \quad a_2 = 7$

Solution

$$\boxed{r = \frac{7}{4}} \qquad\qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$a_5 = 4\left(\frac{7}{4}\right)^4 \qquad\qquad a_n = a_1 r^{n-1}$$

$$\boxed{= \frac{7^4}{64}}$$

Exercise

Find the specified term of the geometric sequence $a_9;\quad a_2 = 3,\quad a_5 = -81$

Solution

$$r = \left(\frac{-81}{3}\right)^{1/(5-2)} \qquad\qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$= (-27)^{1/3}$$

$$= -3$$

$$a_2 = a_1(-3)^3 \qquad\qquad a_n = a_1 r^{n-1}$$

$$3 = -27a_1$$

$$a_1 = -\frac{1}{9}$$

$$a_9 = -\frac{1}{9}(-3)^8$$

$$= -3^6$$

Exercise

Find the specified term of the geometric sequence $a_7;\quad a_1 = -4,\quad a_3 = -1$

Solution

$$r = \left(\frac{-1}{-4}\right)^{1/(3-1)} \qquad\qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$= \left(\frac{1}{4}\right)^{1/2}$$

$$= \frac{1}{2}$$

$$a_7 = -4\left(\frac{1}{2}\right)^6 \qquad\qquad a_n = a_1 r^{n-1}$$

$$= -\frac{1}{16}$$

Exercise

Find the specified term of the geometric sequence $\quad a_8; \quad a_2 = 3, \quad a_4 = 6$

Solution

$$r = \left(\frac{-81}{3}\right)^{1/(5-2)} \qquad\qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$= (-27)^{1/3}$$

$$= -3$$

$$a_2 = a_1(-3)^3 \qquad\qquad a_n = a_1 r^{n-1}$$

$$3 = -81a_1$$

$$a_1 = -\frac{1}{9}$$

$$a_8 = -\frac{1}{9}(-3)^7$$

$$= 3^5$$

Exercise

Express the sum in terms of summation notation: $4+11+18+25+32$. (Answers are not unique)

Solution

$$n = 5$$

$$d = 11 - 4 = 7$$

$$a_n = 4 + (n-1)7 \qquad\qquad a_n = a_1 + (n-1)d$$

$$= 4 + 7n - 7$$

$$= 7n - 3$$

$$4+11+18+25+32 = \sum_{n=1}^{5}(7n-3)$$

Exercise

Express the sum in terms of summation notation: $4+11+18+\ldots+466$. (Answers are not unique)

Solution

Difference in terms: $d = 11 - 4 = 7$

Number of terms: $n = \dfrac{466-4}{7} + 1 \underline{= 67|}$

$a_n = 4 + (n-1)7$ $\qquad\qquad a_n = a_1 + (n-1)d$

$= 4 + 7n - 7$

$\underline{= 7n - 3|}$

$$4 + 11 + 18 + \ldots + 466 = \sum_{n=1}^{67} (7n-3)$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*) $2+4+8+16+32+64+128$

Solution

$$2+4+8+16+32+64+128 = 2^1+2^2+2^3+2^4+2^5+2^6+2^7$$

$$= \sum_{n=1}^{7} 2^n$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*) $2-4+8-16+32-64$

Solution

$r = \dfrac{-4}{2} = -2$ $\qquad\qquad r = \dfrac{a_2}{a_1}$

$a_n = 2(-2)^{n-1}$ $\qquad\qquad a_n = a_1 r^{n-1}$

$= (-1)^{n-1} 2^n$

$$2-4+8-16+32-64 = \sum_{n=1}^{6} (-1)^{n-1} 2^n$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*) $3+8+13+18+23$

Solution

$d = 8 - 3$ $\qquad\qquad d = a_2 - a_1$

$\underline{= 5|}$

$a_n = 3 + 5(n-1)$ $\qquad\qquad a_n = a_1 + (n-1)d$

$$\underline{= 5n - 2}$$

$$3 + 8 + 13 + 18 + 23 = \sum_{n=1}^{5} (5n - 2)$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*) $256 + 192 + 144 + 108 + \cdots$

Solution

$$r = \frac{192}{256} \qquad\qquad r = \frac{a_2}{a_1}$$

$$\underline{= \frac{3}{4}}$$

$$\underline{a_n = 256\left(\frac{3}{4}\right)^{n-1}} \qquad\qquad a_n = a_1 r^{n-1}$$

$$256 + 192 + 144 + 108 + \cdots = \sum_{n=1}^{\infty} 256\left(\frac{3}{4}\right)^{n-1}$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*): $\frac{5}{13} + \frac{10}{11} + \frac{15}{9} + \frac{20}{7}$

Solution

Number of terms: $n = 4$

Numerators : $5, 10, 15, 20$ *common difference* 5

Denominators : $13, 11, 9, 7$ *common difference* -2

Numerator:

$$a_n = 5 + (n-1)5 \qquad\qquad a_n = a_1 + (n-1)d$$

$$= 5 + 5n - 5$$

$$\underline{= 5n}$$

Denominator:

$$a_n = 13 + (n-1)(-2)$$

$$= 13 - 2n + 2$$

$$\underline{= 15 - 2n}$$

Hence the nth term is: $\frac{5}{13} + \frac{10}{11} + \frac{15}{9} + \frac{20}{7} = \sum_{n=1}^{4} \frac{5n}{15 - 2n}$

Exercise

Express the sum in terms of summation notation (Answers are not unique.) $\frac{1}{4}-\frac{1}{12}+\frac{1}{36}-\frac{1}{108}$

Solution

$$\frac{1}{4}-\frac{1}{12}+\frac{1}{36}-\frac{1}{108}=\frac{1}{4}-\frac{1}{4}\frac{1}{3^1}+\frac{1}{4}\frac{1}{3^2}-\frac{1}{4}\frac{1}{3^3}$$

$$=\sum_{n=1}^{4}(-1)^{n+1}\frac{1}{4}\left(\frac{1}{3}\right)^{n-1}$$

Exercise

Express the sum in terms of summation notation (Answers are not unique.) $3+\frac{3}{5}+\frac{3}{25}+\frac{3}{125}+\frac{3}{625}$

Solution

$$3+\frac{3}{5}+\frac{3}{25}+\frac{3}{125}+\frac{3}{625}=\frac{3}{5^0}+\frac{3}{5^1}+\frac{3}{5^2}+\frac{3}{5^3}+\frac{3}{5^4}$$

$$=\sum_{n=0}^{4}\frac{3}{5^n}$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*): $\frac{3}{7}+\frac{6}{11}+\frac{9}{15}+\frac{12}{19}+\frac{15}{23}+\frac{18}{27}$

Solution

Numerators : 3, 6, 9, 12, 15,18 *common difference* 3

Denominators : 7, 11,15, 19, 13, 27 *common difference* 4

Numerator:

$$a_n = 3+3(n-1) \qquad a_n = a_1 + (n-1)d$$

$$\underline{= 3n}$$

***Denominator*:**

$$a_n = 7+4(n-1)$$

$$\underline{= 4n+3}$$

$$\frac{3}{7}+\frac{6}{11}+\frac{9}{15}+\frac{12}{19}+\frac{15}{23}+\frac{18}{27}=\sum_{n=1}^{6}\frac{3n}{4n+3}$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique.*) $\frac{x}{3}+\frac{x^2}{9}+\frac{x^3}{27}+\cdots, \quad |x|<3$

Solution

$$\frac{x}{3}+\frac{x^2}{9}+\frac{x^3}{27}+\cdots = \frac{x}{3}+\frac{x^2}{3^2}+\frac{x^3}{3^3}+\cdots$$

$$= \sum_{n=1}^{\infty}\left(\frac{x}{3}\right)^n$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique.*) $2x+4x^2+8x^3+\cdots, \quad |x|<\frac{1}{2}$

Solution

$$2x+4x^2+8x^3+\cdots = 2x+(2x)^2+(2x)^3+\cdots$$

$$= \sum_{n=1}^{\infty}(2x)^n$$

Exercise

Find the sum of the infinite geometric series if it exists: $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$

Solution

$$a_1=1, \quad r=-\frac{1}{2}$$

$$S=\frac{1}{1+\frac{1}{2}} \qquad\qquad S=\frac{a_1}{1-r}$$

$$=\frac{1}{\frac{3}{2}}$$

$$=\frac{2}{3}$$

Exercise

Find the sum of the infinite geometric series if it exists: $1.5+0.015+0.00015+\ldots$

Solution

$$a_1=0.015$$

$$a_2 = .00015$$

$$r = \frac{.00015}{.015}$$
$$= .01$$

$$S = 1.5 + \frac{a_1}{1-r}$$
$$= 1.5 + \frac{.015}{1-.01}$$
$$= \frac{15}{10} + \frac{.015}{.99}$$
$$= \frac{15}{10} + \frac{15}{990}$$
$$= \frac{15}{10} + \frac{15}{990}$$
$$= \frac{1500}{990}$$
$$= \frac{50}{33}$$

Exercise

Find the sum of the infinite geometric series if it exists: $\sqrt{2} - 2 + \sqrt{8} - 4 + \dots$

Solution

$$r = \frac{-2}{\sqrt{2}} \qquad r = \frac{a_2}{a_1}$$
$$= -\sqrt{2}$$

$|r| = \sqrt{2} > 1 \Rightarrow$ The sum ***doesn't exist***.

Exercise

Find the sum of the infinite geometric series if it exists: $256 + 192 + 144 + 108 + \dots$

Solution

$$r = \frac{192}{256} \qquad r = \frac{a_2}{a_1}$$
$$= \frac{3}{4}$$

$$S = \frac{256}{1-\frac{3}{4}} \qquad S = \frac{a_1}{1-r}$$
$$= 1024$$

Exercise

Find the sum of the infinite geometric series if it exists: $\frac{1}{4}+\frac{2}{4}+\frac{2^2}{4}+\frac{2^3}{4}+\ldots+\frac{2^{n-1}}{4}$

Solution

$$r=\frac{\frac{2}{4}}{\frac{1}{4}} \qquad\qquad r=\frac{a_2}{a_1}$$

$$=2$$

$$S_n=\frac{1}{4}\left(\frac{1-2^n}{1-2}\right) \qquad\qquad S_n=a_1\frac{1-r^n}{1-r}$$

$$=-\frac{1}{4}\left(1-2^n\right)$$

Exercise

Find the sum of the infinite geometric series if it exists: $\frac{3}{9}+\frac{3^2}{9}+\frac{3^3}{9}+\ldots+\frac{3^n}{9}$

Solution

$$r=\frac{\frac{3^2}{9}}{\frac{3}{9}} \qquad\qquad r=\frac{a_2}{a_1}$$

$$=3$$

$$S_n=\frac{3}{9}\left(\frac{1-3^n}{1-3}\right) \qquad\qquad S_n=a_1\frac{1-r^n}{1-r}$$

$$=-\frac{1}{6}\left(1-3^n\right)$$

Exercise

Find the sum of the infinite geometric series if it exists: $-1-2-4-8-\cdots-2^{n-1}$

Solution

$$r=\frac{-2}{-1} \qquad\qquad r=\frac{a_2}{a_1}$$

$$=2$$

$$S_n=-1\left(\frac{1-2^n}{1-2}\right) \qquad\qquad S_n=a_1\frac{1-r^n}{1-r}$$

$$=1-2^n$$

Exercise

Find the sum of the infinite geometric series if it exists: $2+\frac{6}{5}+\frac{18}{25}+\ldots+2\left(\frac{3}{5}\right)^{n-1}$

Solution

$$r=\frac{\frac{6}{5}}{2} \qquad r=\frac{a_2}{a_1}$$

$$=\frac{3}{5}<1$$

$$S_n=2\cdot\frac{1-\left(\frac{3}{5}\right)^n}{1-\frac{3}{5}} \qquad S_n=a_1\frac{1-r^n}{1-r}$$

$$=2\cdot\frac{1-\left(\frac{3}{5}\right)^n}{\frac{2}{5}}$$

$$=5\left(1-\left(\frac{3}{5}\right)^n\right)$$

Exercise

Find the sum of the infinite geometric series if it exists: $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$

Solution

$$r=\frac{1}{3}<1 \qquad r=\frac{a_2}{a_1}$$

$$S=\frac{1}{1-\frac{1}{3}} \qquad S=\frac{a_1}{1-r}$$

$$=\frac{3}{2}$$ The series ***converges***

Exercise

Find the sum of the infinite geometric series if it exists: $2+\frac{4}{3}+\frac{8}{9}+\frac{16}{27}+\cdots$

Solution

$$r=\frac{\frac{4}{3}}{2} \qquad r=\frac{a_2}{a_1}$$

$=\frac{2}{3}<1$

$S=\dfrac{2}{1-\frac{2}{3}}$ $\qquad S=\dfrac{a_1}{1-r}$

$\underline{=6}$ The series ***converges***

Exercise

Find the sum of the infinite geometric series if it exists: $\quad 2-\frac{1}{2}+\frac{1}{8}-\frac{1}{32}+\cdots$

Solution

$a_1=2$

$|r|=\left|-\frac{1}{4}\right|<1$ $\qquad r=\dfrac{a_2}{a_1}$

$S=\dfrac{2}{1+\frac{1}{4}}$ $\qquad S=\dfrac{a_1}{1-r}$

$\underline{=\frac{8}{5}}$ The series ***converges***

Exercise

Find the sum of the infinite geometric series if it exists: $\quad 1-\frac{3}{4}+\frac{9}{16}-\frac{27}{64}+\cdots$

Solution

$a_1=1$

$|r|=\left|-\frac{3}{4}\right|<1$ $\qquad r=\dfrac{a_2}{a_1}$

$S=\dfrac{1}{1+\frac{3}{4}}$ $\qquad S=\dfrac{a_1}{1-r}$

$\underline{=\frac{4}{7}}$ The series ***converges***

Exercise

Find the sum of the infinite geometric series if it exists: $9+12+16+\frac{64}{3}+\cdots$

Solution

$a_1 = 9$

$|r| = \left|\frac{4}{3}\right| > 1$ The series ***diverges***

Exercise

Find the sum of the infinite geometric series if it exists: $8+12+18+27+\cdots$

Solution

$a_1 = 8$

$r = \frac{12}{8}$ $\quad r = \frac{a_2}{a_1}$

$= \frac{3}{2} > 1$ The series ***diverges***

Exercise

Find the sum of the infinite geometric series if it exists: $6+2+\frac{2}{3}+\frac{2}{9}+\cdots$

Solution

$a_1 = 6$

$|r| = \frac{2}{6}$ $\quad r = \frac{a_2}{a_1}$

$= \frac{1}{3} < 1$

$S = \frac{6}{1-\frac{1}{3}}$ $\quad S = \frac{a_1}{1-r}$

$= \frac{6}{\frac{2}{3}}$

$\underline{= 9}$ The series ***converges***

Exercise

Find the sum: $\sum_{k=1}^{20}(3k-5)$

Solution

$a_1 = 3(1) - 5 \underline{= -2|}$

$a_{20} = 3(20) - 5 \underline{= 55|}$

$\sum_{k=1}^{20}(3k-5) = \frac{20}{2}(-2+55)$ $\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$

$\underline{= 530\ |}$

Exercise

Find the sum: $\sum_{k=1}^{18}\left(\frac{1}{2}k+7\right)$

Solution

$a_1 = \frac{1}{2}(1) + 7 \underline{= \frac{15}{2}|}$

$a_{18} = \frac{1}{2}(18) + 7 \underline{= 16|}$

$\sum_{k=1}^{18}\left(\frac{1}{2}k+7\right) = \frac{18}{2}\left(\frac{15}{2}+16\right)$ $\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$

$\underline{= \frac{423}{2}\ |}$

Exercise

Find the sum: $\sum_{k=1}^{80}(2k-5)$

Solution

$a_1 = 2(1) - 5 \underline{= -3|}$

$a_{80} = 2(80) - 5 \underline{= 155|}$

$$\sum_{k=1}^{80}(2k-5)=\frac{80}{2}(-3+155) \qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$$

$$=40(152)$$

$$\underline{=6080}$$

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{90}(3-2n)$

Solution

$$a_1=3-2(1)\ \underline{=1}$$

$$a_{90}=3-2(90)\ \underline{=-177}$$

$$\sum_{n=1}^{80}(3-2n)=\frac{90}{2}(1-177) \qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$$

$$=45(-176)$$

$$\underline{=-7920}$$

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{100}\left(6-\frac{1}{2}n\right)$

Solution

$$a_1=6-\frac{1}{2}(1)\ \underline{=\frac{11}{2}}$$

$$a_{100}=6-\frac{1}{2}(100)\ \underline{=-44}$$

$$\sum_{n=1}^{100}\left(6-\frac{1}{2}n\right)=\frac{100}{2}\left(\frac{11}{2}-44\right) \qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$$

$$=50\left(-\frac{77}{2}\right)$$

$$\underline{=-1925}$$

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{80}\left(\frac{1}{3}n+\frac{1}{2}\right)$

Solution

$$a_1=\frac{1}{3}(1)+\frac{1}{2}=\frac{5}{6}$$

$$a_{80}=\frac{1}{3}(80)+\frac{1}{2}=\frac{163}{6}$$

$$\sum_{n=1}^{80}\left(\frac{1}{3}n+\frac{1}{2}\right)=\frac{80}{2}\left(\frac{5}{6}+\frac{163}{6}\right) \qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$$

$$=40\left(\frac{168}{6}\right)$$

$$=1,120$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{10}3^k$

Solution

$$\sum_{k=1}^{10}3^k=3\frac{1-3^{10}}{1-3} \qquad S_n=a_1\frac{1-r^n}{1-r}$$

$$=3\frac{-59048}{-2}$$

$$=88,572$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{9}\left(-\sqrt{5}\right)^k$

Solution

$$\begin{cases} a_1=-\sqrt{5} \\ a_2=\left(-\sqrt{5}\right)^2=5 \end{cases}$$

$$r = \frac{5}{-\sqrt{5}} \qquad\qquad r = \frac{a_2}{a_1}$$

$$= -\sqrt{5}$$

$$\sum_{k=1}^{9} \left(-\sqrt{5}\right)^{k} = \left(-\sqrt{5}\right)\frac{1-\left(-\sqrt{5}\right)^{9}}{1-\left(-\sqrt{5}\right)} \qquad\qquad S_n = a_1\frac{1-r^n}{1-r}$$

$$= \frac{\left(-\sqrt{5}\right)\left(1+625\sqrt{5}\right)}{1+\sqrt{5}} \bullet \frac{1-\sqrt{5}}{1-\sqrt{5}}$$

$$= \frac{3124\sqrt{5}-3120}{-4}$$

$$= 780-781\sqrt{5}$$

Exercise

Find the sum: $\displaystyle\sum_{k=0}^{9} \left(-\frac{1}{2}\right)^{k+1}$

Solution

$$\sum_{k=0}^{9} \left(-\frac{1}{2}\right)^{k+1} = \left(-\frac{1}{2}\right)\frac{1-\left(-\frac{1}{2}\right)^{10}}{1+\frac{1}{2}} \qquad\qquad S_n = a_1\frac{1-r^n}{1-r}$$

$$= -\frac{1}{2}\,\frac{1-\frac{1}{2^{10}}}{\frac{3}{2}}$$

$$= -\frac{\frac{1024-1}{1024}}{3}$$

$$= -\frac{1023}{3072}$$

$$= -\frac{341}{1024}$$

Exercise

Find the sum : $\displaystyle\sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1}$

Solution

$|r| = \frac{2}{3} \ < 1$

$$\sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1} = \frac{2}{1-\frac{2}{3}} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= \frac{2}{\frac{1}{3}}$$

$$\underline{= 6 \,|}, \qquad \text{the series } \textbf{\textit{converges}}$$

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n}$

Solution

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n} = \sum_{n=1}^{\infty} \frac{2}{3}\left(\frac{2}{3}\right)^{n-1}$$

$|r| = \frac{2}{3} \ < 1$

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n} = \frac{2}{3}\frac{1}{1-\frac{2}{3}} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= \frac{2}{3}(3)$$

$$\underline{= 2 \,|} \qquad \text{The series } \textbf{\textit{converges}}$$

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{\infty} 3\left(\frac{3}{2}\right)^{n}$

Solution

Since $|r| = \frac{3}{2} > 1$, the series ***diverges***

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{\infty} 5\left(\frac{1}{4}\right)^{n-1}$

Solution

$|r| = \frac{1}{4} < 1$

$a_1 = 5$

$$\sum_{n=1}^{\infty} 5\left(\frac{1}{4}\right)^{n-1} = \frac{5}{1-\frac{1}{4}} \qquad S = \frac{a_1}{1-r}$$

$$= \frac{20}{3}$$

The series ***converges***

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{\infty} 8\left(\frac{1}{3}\right)^{n-1}$

Solution

$|r| = \frac{1}{3} < 1$

$a_1 = 8$

$$\sum_{n=1}^{\infty} 8\left(\frac{1}{3}\right)^{n-1} = \frac{8}{1-\frac{1}{3}} \qquad S = \frac{a_1}{1-r}$$

$$= 12$$

The series ***converges***

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2}\cdot 3^{k-1}$

Solution

Since $|r| = 3 > 1$, the series ***diverges***

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{\infty} 6\left(-\frac{2}{3}\right)^{k-1}$

Solution

$|r| = \frac{2}{3} < 1$

$a_1 = 6$

$$\sum_{k=1}^{\infty} 6\left(-\frac{2}{3}\right)^{k-1} = \frac{6}{1+\frac{2}{3}} \qquad S = \frac{a_1}{1-r}$$

$$= \frac{18}{5}$$

The series ***converges***

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{\infty} 4\left(-\frac{1}{2}\right)^{k-1}$

Solution

$|r| = \frac{1}{2} < 1$

$a_1 = 4$

$$\sum_{k=1}^{\infty} 4\left(-\frac{1}{2}\right)^{k-1} = \frac{4}{1+\frac{1}{2}} \qquad S = \frac{a_1}{1-r}$$

$$= \frac{8}{3}$$

The series ***converges***

Exercise

Find the sum: $\displaystyle\sum_{k=8}^{14} \left(3^{k-7} + 2j^2\right)$

Solution

$a_n = 3^{n-7} \rightarrow a_1 = 3^{-6}$

$r = 3$

$n = 14 - 8 + 1 = 7$

$$\sum_{k=8}^{14}\left(3^{k-7}+2j^2\right)=\sum_{k=8}^{14}3^{k-7}+2\sum_{k=8}^{14}j^2$$

$$=3^{-6}\cdot\frac{1-3^7}{1-3}+2(7)j^2$$

$$=-\frac{1}{2}\left(\frac{1-3^7}{3^6}\right)+14j^2$$

$$=-\frac{1}{2}\left(\frac{-2,186}{729}\right)+14j^2$$

$$\underline{=\frac{1,093}{729}+14j^2}$$

Exercise

Find the sum of the first 120 terms of: $\quad 14,\ 16,\ 18,\ 20,\ \cdots$

Solution

$n=120$

$a_1=14$

$d=16-14=2$

$$S_{120}=\frac{120}{2}\left[2(14)+2(120-1)\right] \qquad S_n=\frac{n}{2}\left[2a_1+(n-1)d\right]$$

$$=60(48+238)$$

$$\underline{=17,160}$$

Exercise

Find the sum of the first 46 terms of $\quad 2,\ -1,\ -4,\ -7,\ \cdots$

Solution

$n=46$

$a_1=2$

$d=-1-2=-3$

$$S_{46}=\frac{46}{2}\left[2(2)-3(46-1)\right] \qquad S_n=\frac{n}{2}\left[2a_1+(n-1)d\right]$$

$$=23(4-135)$$

$$\underline{=-3,013}$$

Exercise

Find the rational number represented by the repeating decimal $0.\overline{23}$

Solution

$$0.\overline{23} = 0.23 + 0.0023 + .000023 + \dots$$

$$a_1 = 0.23$$

$$r = \frac{.0023}{.23} = 0.01$$

$$S = \frac{0.23}{1-0.01} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= \frac{0.23}{0.99}$$

$$= \frac{23}{99}$$

Exercise

Find the rational number represented by the repeating decimal $0.0\overline{71}$

Solution

$$0.0\overline{71} = 0.071 + 0.00071 + .0000071 + \dots$$

$$a_1 = 0.071$$

$$r = \frac{.00071}{.071} = 0.01$$

$$S = \frac{0.071}{1-0.01} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= \frac{0.071}{0.990}$$

$$= \frac{71}{990}$$

Exercise

Find the rational number represented by the repeating decimal $2.4\overline{17}$

Solution

$$2.4\overline{17} = 2.4 + 0.017 + 0.00017 + .0000017 + \dots$$

$$a_1 = 0.017$$

$$r = \frac{.00017}{.017} = 0.01$$

$$S = 2.4 + \frac{0.017}{1-0.01} \qquad S = \frac{a_1}{1-r}$$

$$= \frac{24}{10} + \frac{0.017}{0.990}$$

$$= \frac{24}{10} + \frac{17}{990}$$

$$= \frac{240+17}{990}$$

$$= \frac{2,393}{990}$$

Exercise

Find the rational number represented by the repeating decimal $10.\overline{5}$

Solution

$$10.\overline{5} = 10 + 0.5 + 0.05 + .005 + \dots$$

$$a_1 = 0.5$$

$$r = \frac{0.05}{0.5} = 0.1$$

$$S = 10 + \frac{0.5}{1-0.1} \qquad S = \frac{a_1}{1-r}$$

$$= 10 + \frac{0.5}{0.9}$$

$$= 10 + \frac{5}{9}$$

$$= \frac{95}{9}$$

Exercise

Find the rational number represented by the repeating decimal $5.\overline{146}$

Solution

$$5.\overline{146} = 5 + 0.146 + 0.000146 + .000000146 + \dots$$

$$a_1 = 0.146$$

$$r = \frac{0.000146}{0.146} = 0.001$$

$$S = 5 + \frac{0.146}{1-0.001} \qquad S = \frac{a_1}{1-r}$$

$$= 5 + \frac{0.146}{0.999}$$

$$= 5 + \frac{146}{999}$$

$$\boxed{= \frac{5{,}141}{999}}$$

Exercise

Find the rational number represented by the repeating decimal $3.2\overline{394}$

Solution

$$3.2\overline{394} = 3.2 + 0.0394 + 0.0000394 + \dots$$

$$a_1 = 0.0394$$

$$r = \frac{0.0000394}{0.0394} = 0.001$$

$$S = 3.2 + \frac{0.0394}{1 - 0.001} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= \frac{32}{10} + \frac{0.0394}{0.9990}$$

$$= \frac{32}{10} + \frac{394}{9990}$$

$$= \frac{31968 + 394}{9990}$$

$$= \frac{32{,}362}{9{,}990}$$

$$\boxed{= \frac{16{,}181}{4{,}995}}$$

Exercise

Find the rational number represented by the repeating decimal $1.\overline{6124}$

Solution

$$1.\overline{6124} = 1 + 0.6124 + 0.00006124 + \dots$$

$$a_1 = 0.6124$$

$$r = \frac{0.00006124}{0.6124} = 0.0001$$

$$S = 1 + \frac{0.6124}{1 - 0.0001} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= 1 + \frac{0.6124}{0.9999}$$

$$= 1 + \frac{6124}{9999}$$

$$= \boxed{\frac{16,123}{9,999}}$$

Exercise

Find x so that $x+3$, $2x+1$, and $5x+2$ are consecutive terms of an arithmetic sequence.

Solution

$$d = 2x+1-(x+3)$$

$$= x-2$$

$$d = 5x+2-(2x+1)$$

$$= 3x+1$$

$$d = 3x+1 = x-2$$

$$2x = -3$$

$$\boxed{x = -\frac{3}{2}}$$

Exercise

Find x so that $2x$, $3x+2$, and $5x+3$ are consecutive terms of an arithmetic sequence.

Solution

$$d = 3x+2-2x$$

$$= x+2$$

$$d = 5x+3-(3x+2)$$

$$= 2x+1$$

$$d = 2x+1 = x+2$$

$$\boxed{x = 1}$$

Exercise

Find x so that x, $x+2$, and $x+3$ are consecutive terms of a geometric sequence.

Solution

$$r = \frac{x+2}{x}$$

$$r = \frac{x+3}{x+2}$$

$$r=\frac{x+2}{x}=\frac{x+3}{x+2}$$

$$(x+2)^2=x^2+3x$$

$$x^2+4x+4-x^2-3x=0$$

$$x+4=0$$

$$\underline{x=-4}$$

Exercise

Find x so that $x-1$, x and $x+2$ are consecutive terms of a geometric sequence.

Solution

$$r=\frac{x}{x-1}=\frac{x+2}{x}$$

$$x^2=x^2+x-2$$

$$x-2=0$$

$$\underline{x=2}$$

Exercise

How many terms must be added in an arithmetic sequence whose first term is 11 and whose common difference is 3 to obtain a sum of 1092?

Solution

Given: $a_1=11;\quad d=3;\quad S=1092$

$$1092=\frac{n}{2}\left(22+3(n-1)\right) \qquad S_n=\frac{n}{2}\left[2a_1+(n-1)d\right]$$

$$n(3n+19)=2184$$

$$3n^2+19n-2184=0$$

$$n=\frac{-19\pm\sqrt{361+26208}}{6}$$

$$=\frac{-19\pm163}{6}$$

$\underline{n=24}$ & $\cancel{n=-\frac{91}{3}}$

Exercise

How many terms must be added in an arithmetic sequence whose first term is 78 and whose common difference is –4 to obtain a sum of 702?

Solution

Given: $a_1 = 78; \quad d = -4; \quad S = 702$

$702 = \frac{n}{2}\left(2(78) - 4(n-1)\right) \qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

$n(160 - 4n) = 1404$

$-4n^2 + 160n - 1404 = 0$

$n = \dfrac{-160 \pm \sqrt{25,600 - 22464}}{-8}$

$= \dfrac{160 \pm 56}{8}$

$\underline{n = 13}$ & $\underline{n = 27}$

Exercise

The first ten rows of seating in a certain section of a stadium have 30 seats, 32 seats, 34 seats, and so on. The eleventh through the twentieth rows each contain 50 seats. Find the total number of seats in the section.

Solution

Given: $a_1 = 30; \quad d = 2$

$S = S_{10} + 50(20 - 11 + 1) \qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

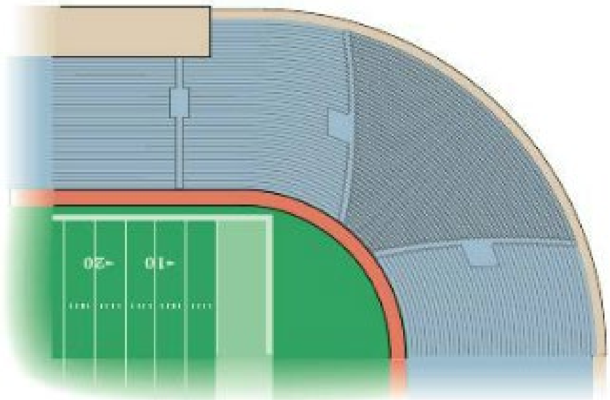
$= \frac{10}{2}\left(2(30) + 2(9)\right) + 50(10)$

$= 5(78) + 500$

$\underline{= 890 \ \ seats}$

Exercise

The corner section of a football stadium has 15 seats in the first row and 40 rows in all. Each successive row contains two additional seats. How many seats are in this section?



Solution

Given: $a_1 = 15; \quad d = 2; \quad n = 40$

$$S_{40} = \frac{40}{2}\left(30 + 2(40-1)\right) \qquad\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$= 20(30 + 78)$$

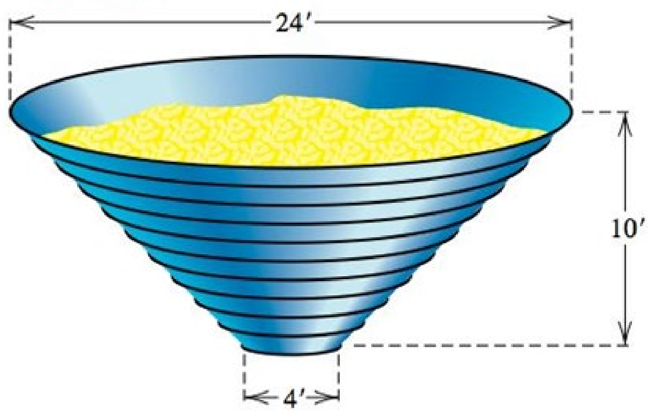
$$= 20(108)$$

$$\underline{= 2,160\,|}$$

The corner section has 2,160 seats.

Exercise

A gain bin is to be constructed in the shape of a frustum of a cone.



The bin is to be 10 *feet* tall with 11 metal rings positioned uniformly around it, from the 4-foot opening at the bottom to the 24-foot opening at the top. Find the total length of metal needed to make the rings.

Solution

The circumference of each ring is πD.

$a_1 = 4\pi; \quad a_{11} = 24\pi$

$$24 = 4 + (11-1)d \qquad\qquad a_n = a_1 + (n-1)d$$

$$10d = 20$$

$$\underline{d = 2}$$

$$S_{11} = \frac{11}{2}(4\pi + 24\pi) \qquad\qquad S_n = \frac{n}{2}(a_1 + a_n)$$

$$\underline{= 154\pi \ \ ft}$$

Exercise

A bicycle rider coasts downhill, traveling 4 *feet* the first second. In each succeeding second, the rider travels 5 *feet* farther than in the preceding second. If the rider reaches the bottom of the hill in 11 *seconds*, find the total distance traveled.

Solution

Given: $a_1 = 4 \ ft \quad \& \quad d = 5 \ ft$

$$S_{11} = \frac{11}{2}(8 + 5(10)) \qquad\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$\underline{= 319 \ \ ft}$$

∴ the total distance traveled 319 *feet*.

Exercise

A contest will have five each prizes totaling \$5,000, and there will be a \$100 difference between successive prices. Find the first prize.

Solution

Given: $n = 5 \quad S_5 = 5000 \quad d = -100$

$$5{,}000 = \frac{5}{2}\left[2a_1 + 4(-100)\right] \qquad\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$2{,}000 = 2a_1 - 400$$

$$\underline{a_1 = \$1{,}200}$$

Exercise

A Company is to distribute \$46,000 in bonuses to its top ten salespeople. The tenth salesperson on the list will receive \$1,000, and the difference in bonus money between successively ranked salesperson is to be constant. Find the bonus for each salesperson.

Solution

Given: $n=10 \quad S_{10}=46{,}000 \quad a_{10}=1{,}000$

$46{,}000=\frac{10}{2}\left(a_1+1000\right)$ $\qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$

$9{,}200=a_1+1000$

$a_1=8{,}200$

$d=\dfrac{1{,}000-8{,}200}{9}$ $\qquad a_n=a_1+(n-1)d$

$\quad \underline{=-800}$

\$8,200 \$7,400 \$6,600 \$5,800 \$5,000 \$4,200 \$3,400 \$2,600 \$1,800 \$1,000

Exercise

Assuming air resistance is negligible, a small object that is dropped from a hot air balloon falls 16 *feet* during the first second, 48 *feet* during the second second, 80 *feet* during the third second, 112 *feet* during the fourth second, and so on. Find an expression for the distance the object falls in *n* seconds.

Solution

Given the sequence: 16, 48, 80, 112, ⋯

This is an arithmetic sequence with:

$a_1=16 \quad \& \quad d=48-16=32$

$S_n=\frac{n}{2}\left(32+32(n-1)\right)$ $\qquad S_n=\frac{n}{2}\left[2a_1+(n-1)d\right]$

$\quad =\frac{n}{2}(32n)$

$\quad \underline{=16n^2}$

Exercise

A brick staircase has a total of 30 steps. The bottom step requires 100 bricks. Each successive step requires two fewer bricks than the prior step.

a) How many bricks are required for the top step?

b) How many bricks are required to build the staircase?

Solution

a) ***Given***: $n=30 \quad a_1=100 \quad d=-2$

$a_n=100-2(n-1)$ $\qquad a_n=a_1+(n-1)d$

$\quad \underline{=-2n+102}$

$a_{30}=102-60$

$\quad \underline{=42}$

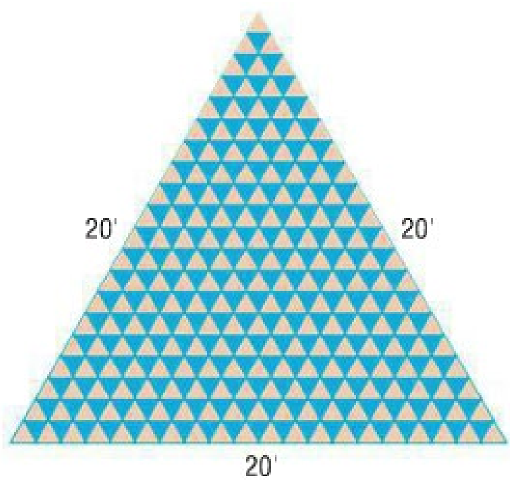
b) $S_{30} = 15(100+42)$ $\qquad\qquad$ $S_n = \frac{n}{2}\left(a_1 + a_n\right)$

$\underline{= 2,130}$

It required 2130 *bricks* to build the staircase.

Exercise

A mosaic is designed in the shape of an equilateral triangle, 20 *feet* on each side. Each tile in the mosaic is in the shape of an equilateral triangle, 12 *inches* to a side. The tiles are to alternate in color as shown below.



How many tiles of each color will be required?

Solution

Bottom row has 20 lighter colored tiles.
Top row has 1 lighter colored tile.
The number decreases by 1 as we move up the triangle.
∴ This is an arithmetic sequence with: $a_1 = 20;\quad d = -1;\quad n = 20$

$S_{20} = \frac{20}{2}\left(40 + (-1)(20-1)\right)$ $\qquad\qquad$ $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

$= 10(40-19)$

$= 10(21)$

$\underline{= 210}$

∴ There are 210 *lighter colored* tiles.

Bottom row has 19 darker colored tiles.
Top row has 1 darker colored tile.

∴ This is an arithmetic sequence with: $a_1 = 1;\quad d = -1;\quad n = 19$

$S_{19} = \frac{19}{2}\left(2(19) + (-1)(19-1)\right)$ $\qquad\qquad$ $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

$$= \frac{19}{2}(38 - 18)$$

$$\underline{= 190}$$

∴ There are 190 *darker colored* tiles.

Solution Section 5.7 – Mathematical Induction

Exercise

Find all positive integers n for which the given statement is not true

$a)\quad 3^n > 6n \qquad b)\quad 3^n > 2n+1 \qquad c)\quad 2^n > n^2 \qquad d)\quad n! > 2n$

Solution

a) $n=1 \quad 3<6$

$n=2 \quad 3^2 < 18$

$n=3, \quad 27 > 18$

The statement is true for all $n \ge 3 \quad 3^n > 6n$

The statement is not true for $\boxed{n=1,\ 2}$

b) $n=1; \quad 3=3$

$n=2; \quad 9>5$

The statement is true for all $n \ge 2 \quad 3^n > 2n+1$

The statement is not true for $\boxed{n=1}$

c) $n=1; \quad 2<4$

$n=2; \quad 4=4$

$n=3; \quad 8<9$

$n=4; \quad 16=16$

$n=5; \quad 32>25$

The statement is true for all $n \ge 5; \quad 2^n > n^2$

The statement is not true for $\boxed{n=1,\ 2,\ 3,\ 4}$

d) $n=1; \quad 1<2$

$n=2; \quad 2<4$

$n=3; \quad 6=6$

$n=4; \quad 12>8$

The statement is true for all $n \ge 4; \quad n! > 2n$

The statement is not true for $\boxed{n=1,\ 2,\ 3}$

Exercise

Prove that the statement is true for every positive integer n. $2+4+6+\ldots+2n=n(n+1)$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 2=1(1+1)=2$; hence P_1 is true.

(2) Assume $2+4+6+\ldots+2k=k(k+1)$ is true

$$\Rightarrow 2+4+6+\ldots+2k+2(k+1)=(k+1)(k+1+1)?$$

$$\begin{aligned} 2+4+6+\ldots+2k+2(k+1) &= 2+4+6+\ldots+2k+2(k+1) \\ &= k(k+1)+2(k+1) \\ &= (k+1)(k+2) \\ &= (k+1)(k+1+1)\ \checkmark \end{aligned}$$

***Factor* (k + 1)**

Hence P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $1+3+5+\ldots+(2n-1)=n^2$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 1\overset{?}{=}1^2=1$; hence P_1 is true.

(2) Assume $1+3+5+\ldots+(2k-1)=k^2$ is true

$$\Rightarrow 1+3+5+\ldots+(2(k+1)-1)=(k+1)^2?$$

$$\begin{aligned} 1+3+5+\ldots+(2k-1)+(2(k+1)-1) &= 1+3+5+\ldots+(2k-1)+(2k+2-1) \\ &= k^2+(2k+1) \\ &= k^2+2k+1 \\ &= (k+1)^2\ \checkmark \end{aligned}$$

Hence P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $2+7+12+\ldots+(5n-3)=\frac{1}{2}n(5n-1)$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 2\overset{?}{=}\frac{1}{2}(1)(5(1)-1)=\frac{1}{2}(4)=2$; hence P_1 is true.

(2) Assume $2+7+12+\ldots+(5k-3)=\frac{1}{2}k(5k-1)$ is true

$$2+7+12+\ldots+(5(k+1)-3)=\frac{1}{2}(k+1)(5(k+1)-1)?$$

$$\begin{aligned}
2+7+12+\ldots+(5k-3)+(5(k+1)-3) &= 2+7+12+\ldots+(5k-3)+(5k+5-3)\\
&=\frac{1}{2}k(5k-1)+(5k+2)\frac{2}{2}\\
&=\frac{1}{2}\left[5k^2-k+10k+4\right]\\
&=\frac{1}{2}\left[5k^2-k+5k+5k+5-1\right]\\
&=\frac{1}{2}\left[k(5k-1+5)+5k+5-1\right]\\
&=\frac{1}{2}\left[(k+1)(5k+5-1)\right]\\
&=\frac{1}{2}\left[(k+1)(5(k+1)-1)\right] \quad \checkmark \quad P_{k+1} \text{ is also true.}
\end{aligned}$$

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $1+2.2+3.2^2+\ldots+n.2^{n-1}=1+(n-1).2^n$

Solution

For $\boldsymbol{n=1}$

$1\overset{?}{=}1+(1-1)2^1=1-0=1$

Hence P_1 is true.

$1+2.2+3.2^2+\ldots+k.2^{k-1}=1+(k-1).2^k$ is true

$$1+2.2+3.2^2+\ldots+k.2^{k-1}+(k+1).2^{(k+1)-1}=1+((k+1)-1).2^{k+1}?$$

$$\begin{aligned}
1+2.2+3.2^2+\ldots+k.2^{k-1}+(k+1).2^{(k+1)-1} &=1+(k-1).2^k+(k+1).2^{k+1-1}\\
&=1+k.2^k-1.2^k+(k+1).2^k\\
&=1+k.2^k-1.2^k+k.2^k+1.2^k\\
&=1+2^1k.2^k
\end{aligned}$$

$$=1+(k+0).2^{k+1}$$

$$=1+((k+1)-1).2^{k+1} \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. $1^2+2^2+3^2+\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6}$

Solution

For $\boldsymbol{n} = \mathbf{1}$

$$1^2 \overset{?}{=} \frac{1(1+1)(2(1)+1)}{6}=\frac{1(2)(3)}{6}=\frac{6}{6}=1 \quad \surd$$

Hence P_1 is true.

$1^2+2^2+3^2+\ldots+k^2=\dfrac{k(k+1)(2k+1)}{6}$ is true

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}?$$

$$\begin{aligned}
1^2+2^2+3^2+\ldots+k^2+(k+1)^2 &= \frac{k(k+1)(2k+1)}{6}+(k+1)^2 \\
&= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} \\
&= \frac{(k+1)\left[k(2k+1)+6(k+1)\right]}{6} \\
&= \frac{(k+1)\left[2k^2+k+6k+6\right]}{6} \\
&= \frac{(k+1)\left[2k^2+7k+6\right]}{6} \\
&= \frac{(k+1)((k+2)(2k+3))}{6} \\
&= \frac{(k+1)((k+1+1)(2k+2+1))}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \quad \surd
\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. $\frac{1}{1\bullet 2}+\frac{1}{2\bullet 3}+\frac{1}{3\bullet 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

Solution

For $\boldsymbol{n=1}$

$$\frac{1}{1.2}\stackrel{?}{=}\frac{1}{1+1}=\frac{1}{2}=\frac{1}{1.2}\ \checkmark$$

Hence P_1 is true.

$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$ is true

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{(k+1)+1}?$$

$$\begin{aligned}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}&=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}\\&=\frac{k(k+2)+1}{(k+1)(k+2)}\\&=\frac{k^2+2k+1}{(k+1)(k+2)}\\&=\frac{(k+1)(k+1)}{(k+1)(k+2)}\\&=\frac{k+1}{(k+1+1)}\\&=\frac{k+1}{(k+1)+1}\ \checkmark\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$

Solution

For $\boldsymbol{n=1}$

$$\frac{1}{2}\stackrel{?}{=}1-\frac{1}{2}=\frac{1}{2}\ \checkmark$$

Hence, P_1 is true.

$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}=1-\frac{1}{2^k}$ is true

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^{k+1}}?$$

$$\begin{aligned}\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}&=1-\frac{1}{2^k}+\frac{1}{2^{k+1}}\\&=1-\frac{1}{2^k}+\frac{1}{2^k\cdot 2}\\&=\frac{2^{k+1}-2+1}{2^{k+1}}\\&=\frac{2^{k+1}-1}{2^{k+1}}\\&=\frac{2^{k+1}}{2^{k+1}}-\frac{1}{2^{k+1}}\\&=1-\frac{1}{2^{k+1}}\quad\checkmark\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{(3n-2)\cdot(3n+1)}=\frac{n}{3n+1}$

Solution

For ***n* = 1**

$$\frac{1}{1\cdot 4}\overset{?}{=}\frac{1}{3(1)+1}=\frac{1}{4}\quad\checkmark$$

Hence, P_1 is true.

$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}$ is true

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3(k+1)-2)(3(k+1)+1)}\overset{?}{=}\frac{k+1}{3(k+1)+1}$$

$$\begin{aligned}\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3(k+1)-2)(3(k+1)+1)}&=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}\\&=\frac{3k^2+4k+1}{(3k+1)(3k+4)}\end{aligned}$$

$$= \frac{(3k+1)(k+1)}{(3k+1)(3k+3+1)}$$

$$= \frac{k+1}{3(k+1)+1} \quad \checkmark$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\frac{4}{5}+\frac{4}{5^2}+\frac{4}{5^3}+\cdots+\frac{4}{5^n}=1-\frac{1}{5^n}$

Solution

For $\boldsymbol{n=1}$

$$\frac{4}{5} \stackrel{?}{=} 1-\frac{1}{5}=\frac{4}{5} \quad \checkmark$$

Hence, P_1 is true.

$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}=1-\frac{1}{5^k}$ is true

$$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}+\frac{4}{5^{k+1}} \stackrel{?}{=} 1-\frac{1}{5^{k+1}}$$

$$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}+\frac{4}{5^{k+1}}=1-\frac{1}{5^k}+\frac{4}{5^{k+1}}$$

$$=1-\left(\frac{1}{5^k}-\frac{4}{5^{k+1}}\right)$$

$$=1-\frac{5-4}{5^{k+1}}$$

$$=1-\frac{1}{5^{k+1}} \quad \checkmark$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$

Solution

For $\boldsymbol{n=1}$

$$1^3 \overset{?}{=} \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1 \quad \checkmark$$

Hence, P_1 is true.

$\frac{4}{5} + \frac{4}{5^2} + \cdots + \frac{4}{5^k} = 1 - \frac{1}{5^k}$ is true

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 \overset{?}{=} \frac{(k+1)^2\left((k+1)+1\right)^2}{4}$$

$$\begin{aligned}
1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2\left[k^2 + 4(k+1)\right]}{4} \\
&= \frac{(k+1)^2\left(k^2 + 4k + 4\right)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k+1)^2\left((k+1)+1\right)^2}{4} \quad \checkmark
\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. $3 + 3^2 + 3^3 + \ldots + 3^n = \frac{3}{2}\left(3^n - 1\right)$

Solution

For $\boldsymbol{n} = \mathbf{1}$

$$3 \overset{?}{=} \frac{3}{2}\left(3^1 - 1\right) = \frac{3}{2}2 = 3 \quad \checkmark$$

Hence, P_1 is true.

$3 + 3^2 + \cdots + 3^k = \frac{3}{2}\left(3^k - 1\right)$ is true

$$3 + 3^2 + \cdots + 3^k + 3^{k+1} \overset{?}{=} \frac{3}{2}\left(3^{k+1} - 1\right)$$

$$3+3^2+\cdots+3^k+3^{k+1}=\frac{3}{2}\left(3^k-1\right)+3^{k+1}$$
$$=\frac{1}{2}3^{k+1}-\frac{3}{2}+3^{k+1}$$
$$=\frac{3}{2}3^{k+1}-\frac{3}{2}$$
$$=\frac{3}{2}\left(3^{k+1}-1\right) \quad \checkmark$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $$x^{2n}+x^{2n-1}y+\cdots+xy^{2n-1}+y^{2n}=\frac{x^{2n+1}-y^{2n+1}}{x-y}$$

Solution

For $\boldsymbol{n}=\mathbf{1}$

$$x^2+xy+y^2\stackrel{?}{=}\frac{x^3-y^3}{x-y}$$
$$=\frac{(x-y)\left(x^2+xy+y^2\right)}{x-y}$$
$$=x^2+xy+y^2 \quad \checkmark$$

Hence, P_1 is true.

$$x^{2k}+x^{2k-1}y+\cdots+xy^{2k-1}+y^{2k}=\frac{x^{2k+1}-y^{2k+1}}{x-y} \text{ is true}$$

$$x^{2(k+1)}+x^{2(k+1)-1}y+\cdots+xy^{2(k+1)-1}+y^{2(k+1)}\stackrel{?}{=}\frac{x^{2(k+1)+1}-y^{2(k+1)+1}}{x-y}$$

$$x^{2k+2}+x^{2k+1}y+\cdots+xy^{2k+1}+y^{2k+2}=x^2\left(x^{2k}+x^{2k-1}y+\cdots+y^{2k}\right)+xy^{2k+1}+y^{2k+2}$$
$$=x^2\left(\frac{x^{2k+1}-y^{2k+1}}{x-y}\right)+xy^{2k+1}+y^{2k+2}$$
$$=\frac{x^{2k+3}-x^2y^{2k+1}+x^2y^{2k+1}+xy^{2k+2}-xy^{2k+2}-y^{2(k+1)+1}}{x-y}$$
$$=\frac{x^{2(k+1)+1}-y^{2(k+1)+1}}{x-y} \quad \checkmark$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $5\cdot6+5\cdot6^2+5\cdot6^3+\cdots+5\cdot6^n=6\left(6^n-1\right)$

Solution

For $\boldsymbol{n}$ **= 1**

$$5\cdot6\overset{?}{=}6\left(6^1-1\right)=6(5)\ \checkmark$$

Hence, P_1 is true.

$5\cdot6+5\cdot6^2+\cdots+5\cdot6^k=6\left(6^k-1\right)$ is true

$$5\cdot6+5\cdot6^2+\cdots+5\cdot6^k+5\cdot6^{k+1}\overset{?}{=}6\left(6^{k+1}-1\right)$$

$$\begin{aligned}5\cdot6+5\cdot6^2+\cdots+5\cdot6^k+5\cdot6^{k+1}&=6\left(6^k-1\right)+5\cdot6^{k+1}\\&=6^{k+1}-6+5\cdot6^{k+1}\\&=6^{k+1}(1+5)-6\\&=6\cdot6^{k+1}-6\\&=6\left(6^{k+1}-1\right)\ \checkmark\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $7\cdot8+7\cdot8^2+7\cdot8^3+\cdots+7\cdot8^n=8\left(8^n-1\right)$

Solution

For $\boldsymbol{n}$ **= 1**

$$7\cdot8\overset{?}{=}8\left(8^1-1\right)=8(7)\ \checkmark$$

Hence, P_1 is true.

$7\cdot8+7\cdot8^2+\cdots+7\cdot8^k=8\left(8^k-1\right)$ is true

$$7\cdot8+7\cdot8^2+\cdots+7\cdot8^k+7\cdot8^{k+1}\overset{?}{=}8\left(8^{k+1}-1\right)$$

$$\begin{aligned}7\cdot8+7\cdot8^2+\cdots+7\cdot8^k+7\cdot8^{k+1}&=8\left(8^k-1\right)+7\cdot8^{k+1}\\&=8^{k+1}-8+7\cdot8^{k+1}\end{aligned}$$

$$= 8^{k+1}(1+7) - 8$$

$$= 8\left(8^{k+1} - 1\right) \quad \checkmark$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\quad 3+6+9+\cdots+3n = \dfrac{3n(n+1)}{2}$

Solution

For $\boldsymbol{n = 1}$

$$3 \stackrel{?}{=} \frac{3(1)(1+1)}{2} = 3 \quad \checkmark$$

Hence, P_1 is true.

$3+6+9+\cdots+3k = \dfrac{3k(k+1)}{2}$ is true

$$3+6+9+\cdots+3k+3(k+1) \stackrel{?}{=} \frac{3(k+1)(k+2)}{2}$$

$$\begin{aligned} 3+6+9+\cdots+3k+3(k+1) &= \frac{3k(k+1)}{2} + 3(k+1) \\ &= \frac{3k(k+1)+6(k+1)}{2} \\ &= \frac{(k+1)(3k+6)}{2} \\ &= \frac{3(k+1)(k+2)}{2} \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\quad 5+10+15+\cdots+5n = \dfrac{5n(n+1)}{2}$

Solution

For $\boldsymbol{n = 1}$

$$5 \overset{?}{=} \frac{5(1)(1+1)}{2} = 5 \quad \checkmark$$

Hence, P_1 is true.

$$5+10+15+\cdots+5k = \frac{5k(k+1)}{2} \text{ is true}$$

$$5+10+15+\cdots+5k+5(k+1) \overset{?}{=} \frac{5(k+1)(k+2)}{2}$$

$$\begin{aligned} 5+10+15+\cdots+5k+5(k+1) &= \frac{5k(k+1)}{2}+5(k+1) \\ &= \frac{5k(k+1)+10(k+1)}{2} \\ &= \frac{(k+1)(5k+10)}{2} \\ &= \frac{5(k+1)(k+2)}{2} \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $1+3+5+\cdots+(2n-1) = n^2$

Solution

For $\boldsymbol{n = 1}$

$$1 \overset{?}{=} 1^2 = 1 \quad \checkmark$$

Hence, P_1 is true.

$$1+3+5+\cdots+(2k-1) = k^2 \text{ is true}$$

$$1+3+5+\cdots+(2k-1)+(2(k+1)-1) \overset{?}{=} (k+1)^2$$

$$\begin{aligned} 1+3+5+\cdots+(2k-1)+(2(k+1)-1) &= k^2+2k+2-1 \\ &= k^2+2k+1 \\ &= (k+1)^2 \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\qquad 4+7+10+\cdots+(3n+1)=\dfrac{n(3n+5)}{2}$

Solution

For $\boldsymbol{n=1}$

$$4\overset{?}{=}\frac{1(3+5)}{2}=4 \quad \checkmark$$

Hence, P_1 is true.

$$4+7+10+\cdots+(3k+1)=\frac{k(3k+5)}{2} \text{ is true}$$

$$4+7+10+\cdots+(3k+1)+(3(k+1)+1)\overset{?}{=}\frac{(k+1)(3(k+1)+5)}{2}=\frac{(k+1)(3k+8)}{2}$$

$$\begin{aligned}4+7+10+\cdots+(3k+1)+(3k+4)&=\frac{k(3k+5)}{2}+3k+4\\&=\frac{3k^2+5k+6k+8}{2}\\&=\frac{3k^2+5k+3k+3k+8}{2}\\&=\frac{k(3k+8)+(3k+8)}{2}\\&=\frac{(3k+8)(k+1)}{2} \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction:

$$2+4+6+\cdots+2(n-1)+2n=n(n+1)$$

Solution

For $\boldsymbol{n=1}$

$$2\overset{?}{=}1(1+1)$$

$$2=2 \quad \checkmark$$

Hence, P_1 is true.

For $\boldsymbol{k}$: $\quad 2+4+6+\cdots+2(k-1)+2k=k(k+1)$

$$2+4+\cdots+2k+2(k+1)=(k+1)(k+2)$$

$$\begin{aligned}2+4+\cdots+2k+2(k+1)&=k(k+1)+2(k+1)\\&=(k+1)(k+2)\quad\checkmark\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction:

$$1+(1+2)+(1+2+3)+\ \cdots\ +(1+2+\cdots+n)=\frac{n(n+1)(n+2)}{6}$$

Solution

For ***n* = 1**

$$1\overset{?}{=}\frac{1(1+1)(1+2)}{6}$$

$$1\overset{?}{=}\frac{(2)(3)}{6}$$

$$1=1\quad\checkmark$$

Hence, P_1 is true.

For ***k***: $\quad 1+(1+2)+\ \cdots\ +(1+2+\cdots+k)=\dfrac{k(k+1)(k+2)}{6}$

Is P_{k+1}: $\quad 1+(1+2)+\ \cdots\ +(1+2+\cdots+k)+(1+2+\cdots+k+(k+1))\overset{?}{=}\dfrac{(k+1)(k+2)(k+3)}{6}$

$$1+(1+2)+\ \cdots\ +(1+2+\cdots+k)+(1+2+\cdots+k+(k+1))=\frac{k(k+1)(k+2)}{6}+(1+2+\cdots+k+(k+1))$$

$$1+2+\cdots+n=\frac{1}{2}n(n+1)$$

$$\begin{aligned}1+2+\cdots+k+(k+1)&=\frac{1}{2}k(k+1)+(k+1)\\&=(k+1)\left(\frac{1}{2}k+1\right)\\&=\frac{1}{2}(k+1)(k+2)\end{aligned}$$

$$\begin{aligned}1+(1+2)+\ \cdots\ +(1+2+\cdots+k)+(1+2+\cdots+k+(k+1))&=\frac{k(k+1)(k+2)}{6}+\frac{1}{2}(k+1)(k+2)\\&=(k+1)(k+2)\left(\frac{k}{6}+\frac{1}{2}\right)\end{aligned}$$

$$=\frac{(k+1)(k+2)(k+3)}{6} \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction: $1+2+3+\cdots+n<\frac{(2n+3)^2}{7}$

Solution

For ***n*** **= 1**

$$1 \overset{?}{<} \frac{(2+3)^2}{7}$$

$$1<\frac{25}{7} \quad >1 \quad \surd$$

Hence, P_1 is true.

For ***k***: $\quad 1+2+\cdots+k<\frac{(2k+3)^2}{7}$

Is P_{k+1}: $\quad 1+2+\cdots+k+(k+1)<\frac{(2(k+1)+3)^2}{7}$

$$<\frac{(2k+5)^2}{7} \; ? \qquad \frac{4k^2+20k+25}{7}$$

$$1+2+\cdots+k+(k+1)<\frac{(2k+3)^2}{7}+(k+1)$$

$$=\frac{4k^2+12k+9+7k+7}{7}$$

$$=\frac{1}{7}\left(4k^2+19k+16+k+9-k-9\right)$$

$$=\frac{1}{7}\left(4k^2+20k+25-(k+9)\right)$$

$$=\frac{(2k+5)^2}{7}-\frac{k+9}{7}$$

$$<\frac{(2k+5)^2}{7} \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction:

$$\frac{1}{2n} \le \frac{1\cdot 3\cdot 5\cdots(2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot(2n)}$$

Solution

For $n = 1$

$$\frac{1}{2} \le \frac{1}{2} \ \checkmark$$

Hence, P_1 is true.

For k: $\quad \dfrac{1}{2k} \le \dfrac{1\cdots(2k-3)\cdot(2k-1)}{2\cdot\cdots(2k-2)\cdot(2k)}$

Is P_{k+1}: $\quad \dfrac{1}{2(k+1)} \le \dfrac{1\cdots(2k-1)\cdot(2k+1)}{2\cdot\cdots(2k)\cdot(2k+2)}$

$$\frac{1\cdots(2k-1)\cdot(2k+1)}{2\cdot\cdots(2k)\cdot(2k+2)} \ge \frac{1}{2(k+1)} \quad ?$$

$$\begin{aligned}
\frac{1\cdots(2k-1)\cdot(2k+1)}{2\cdot\cdots(2k)\cdot(2k+2)} &= \frac{1\cdots(2k-1)}{2\cdot\cdots(2k)}\frac{2k+1}{2k+2} \\
&\ge \frac{1}{2k}\cdot\frac{2k+1}{2k+2} \\
&= \frac{2k+1}{2k}\cdot\frac{1}{2(k+1)} \\
&= \left(1+\frac{1}{2k}\right)\cdot\frac{1}{2(k+1)} \\
&\ge \frac{1}{2(k+1)} \quad \checkmark
\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction: $\quad \dfrac{2n+1}{2n+2} \le \dfrac{\sqrt{n+1}}{\sqrt{n+2}}$

Solution

For $n = 1$

$$\frac{2+1}{2+2} \overset{?}{\le} \frac{\sqrt{1+1}}{\sqrt{1+2}}$$

$$\frac{3}{4} \overset{?}{\le} \frac{\sqrt{2}}{\sqrt{3}}$$

$3\sqrt{3} \quad ? \quad 4\sqrt{2}$ *Square both sides*

$27 \quad \le \quad 32$ √

Hence, P_1 is true.

For ***k***: $\quad \frac{2k+1}{2k+2} \le \frac{\sqrt{k+1}}{\sqrt{k+2}}$

$$(2k+1)\sqrt{k+2} \le (2k+2)\sqrt{k+1}$$

Is P_{k+1}: $\quad \frac{2k+3}{2k+4} \overset{?}{\le} \frac{\sqrt{k+2}}{\sqrt{k+3}}$

$$(2k+3)\sqrt{k+3} \overset{?}{\le} (2k+4)\sqrt{k+2}$$

$$(2k+3) \le (2k+4)$$

$$\underline{\sqrt{k+3} \le \sqrt{k+2}}$$

$$(2k+3)\sqrt{k+3} \le (2k+4)\sqrt{k+2} \quad √$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction: $\quad n! < n^n \quad for \quad n > 1$

Solution

For ***n* = 2**

$$2! \overset{?}{<} 2^2$$

$2 < 4$ √

Hence, P_1 is true.

For ***k***: $\quad k! < k^k$

Is P_{k+1}: $\quad (k+1)! < (k+1)^{k+1}$

$$(k+1)! = k!\,(k+1)$$

$$< k^k\,(k+1)$$

$$k < k+1$$

$$k^k < (k+1)^k$$

$$< (k+1)^k \ (k+1)$$

$$= (k+1)^{k+1} \quad \checkmark$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction: $\left(a^m\right)^n = a^{mn}$ (a and m are constant)

Solution

For $\boldsymbol{n} = \mathbf{1}$

$$\left(a^m\right)^1 \stackrel{?}{=} a^{m(1)}$$

$$a^m = a^m \quad \checkmark$$

Hence, P_1 is true.

$\left(a^m\right)^k = a^{mk}$ is true

$$\left(a^m\right)^{(k+1)} \stackrel{?}{=} a^{m(k+1)}$$

$$\left(a^m\right)^{(k+1)} = \left(a^m\right)^k a^m$$

$$= a^{km} a^m$$

$$= a^{km+m}$$

$$= a^{m(k+1)} \quad \checkmark$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. $n < 2^n$

Solution

For $\boldsymbol{n} = \mathbf{1}$

$$1 \stackrel{?}{<} 2^1 \quad \checkmark$$

Hence, P_1 is true.

Assume that P_k is true $k < 2^k$

We need to prove that P_{k+1} is true, that is $k+1 < 2^{k+1}$

$$\begin{aligned} k+1 < k+k &= 2k \\ &< 2 \cdot 2^k \\ &= 2^{k+1} \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. 3 is a factor of $n^3 - n + 3$

Solution

For $\boldsymbol{n} = \mathbf{1}$

$1^3 - 1 + 3 = 3 = 3(1)$ $\checkmark$

Hence, P_1 is true.

Assume that P_k is true 3 is a factor of $k^3 - k + 3$

We need to prove that P_{k+1} is true, that is $(k+1)^3 - (k+1) + 3$

$$\begin{aligned} (k+1)^3 - (k+1) + 3 &= k^3 + 3k^2 + 3k + 1 - k - 1 + 3 \\ &= \left(k^3 - k + 3\right) + 3k^2 + 3k \qquad\qquad k^3 - k + 3 = 3K \\ &= 3K + 3k^2 + 3k \\ &= 3\left(K + k^2 + k\right) \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. 4 is a factor of $5^n - 1$

Solution

For $\boldsymbol{n} = \mathbf{1}$

$5^{1} - 1 = 4 = 4(1)$ √

Hence, P_1 is true.

Assume that P_k is true 4 is a factor of $5^k - 1$

We need to prove that P_{k+1} is true, that is $5^{k+1} - 1$

$$
\begin{aligned}
5^{k+1} - 1 &= 5^k 5^1 - 5 + 4 \\
&= 5\left(5^k - 1\right) + 4 \\
&= 5\left(5^k - 1\right) + 4
\end{aligned}
$$

By the induction hypothesis, 4 is a factor of $5^k - 1$ and 4 is a factor of 4, so 4 is a factor of the $(k+1)$ term. √

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $2^n > 2n \quad if \quad n \geq 3$

Solution

For ***n* = 3**

$2^3 \geq 2(3)$

$8 \geq 6$ √

Hence, P_3 is true.

Assume that P_k is true: $2^k > 2k$

We need to prove that P_{k+1}: $2^{k+1} > 2(k+1)$ is true

$$
\begin{aligned}
& 2^k > 2k \\
& 2^k \cdot 2 > 2k \cdot 2 \\
2^{k+1} &> 4k = 2k + 2k \qquad k \geq 3 \\
&> 2k + 2 \\
&= 2(k+1) \quad √
\end{aligned}
$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: If $0 < a < 1$, then $a^n < a^{n-1}$

Solution

For $\boldsymbol{n} = \mathbf{1}$

$a^1 < a^{1-1}$

$a < 1$ √

since $0 < a < 1 \Rightarrow P_1$ is true.

Assume that P_k is true: $a^k < a^{k-1}$

We need to prove that $P_{k+1}: a^{k+1} < a^k$ is true

$$a^k < a^{k-1} \quad \rightarrow \quad a^k \cdot a < a^{k-1} \cdot a$$

$$a^{k+1} < a^k \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: If $n \ge 4$, then $n! > 2^n$

Solution

For $\boldsymbol{n} = \mathbf{4}$

$4! > 2^4$

$24 > 16$ √

Hence, P_4 is true.

Assume that P_k is true: $k! > 2^k$

We need to prove that $P_{k+1}: (k+1)! > 2^{k+1}$ is true

$$\begin{aligned}(k+1)! &= k! \cdot (k+1) \\ &> 2^k (k+1) \qquad \textit{Since} \quad k \ge 4 \Rightarrow k+1 > 2 \\ &> 2^k \cdot 2 \\ &= 2^{k+1} \quad \surd\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $3^n > 2n+1 \quad if \quad n \ge 2$

Solution

For $\boldsymbol{n} = \mathbf{2}$

$3^2 > 2(2)+1$

$9 > 5$ √

Hence, P_2 is true.

Assume that P_k is true: $3^k > 2k+1$;

We need to prove that P_{k+1}: $3^{k+1} > 2(k+1)+1$ is true

$$3^k > 2k+1 \quad \Rightarrow \quad 3^k \cdot 3 > (2k+1)\cdot 3$$

$$3^{k+1} > 6k+3$$

$$> 2k+2+1$$

$$= 2(k+1)+1 \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $2^n > n^2 \quad for \quad n > 4$

Solution

For $\boldsymbol{n} = \mathbf{5}$

$2^5 > 5^2$

$32 > 25$ √

Hence, P_5 is true.

Assume that P_k is true: $2^k > k^2$

We need to prove that P_{k+1}: $2^{k+1} > (k+1)^2$ is true

$$2^k > k^2$$

$$2^k \cdot 2 > k^2 \cdot 2$$

$$2^{k+1} > 2k^2$$

$$= k^2 + k^2 \qquad\qquad k < k+1 \ \Rightarrow \ k \cdot k > k+k+1 \Rightarrow \ k^2 > 2k+1$$

$$> k^2 + 2k + 1$$

$$= (k+1)^2 \quad \checkmark$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $4^n > n^4 \quad for \quad n \ge 5$

Solution

For $\boldsymbol{n = 5}$

$4^5 > 5^4$

$1024 > 625 \quad \checkmark$

Hence, P_5 is true.

Assume that P_k is true: $4^k > k^4$

We need to prove that P_{k+1} : $4^{k+1} > (k+1)^4$ is true

$4^k > k^4$

$4^k \cdot 4 > k^4 \cdot 4$

$4^{k+1} > 4k^4$

$k < k+1$

$4k > k+1$

$4k^4 > (k+1)^4$

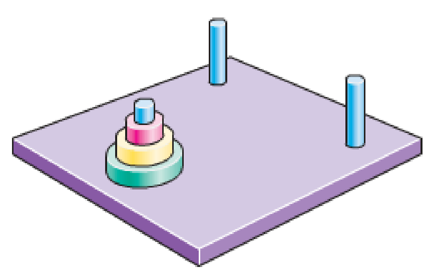
$> (k+1)^4 \quad \checkmark$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

A pile of *n* rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must be moved, one at a time, to a different peg with no ring ever placed on top of a smaller ring.
Find the least number of moves that would be required.
Prove your result by mathematical induction.



Solution

With 1 ring, 1 move is required.
With 2 rings, 3 moves are required $\Rightarrow \quad 3 = 2 + 1$

With 3 rings, 7 moves are required $\Rightarrow \quad 7 = 2^2 + 2 + 1$

With n rings, $2^{n-1} + \cdots + 2^2 + 2^1 + 2^0 = 2^n - 1$ moves are required

For $\boldsymbol{n = 1}$

$$2^0 = 2^1 - 1 = 1 \checkmark$$

Hence, P_1 is true.

Assume that P_k is true: $2^{k-1} + \cdots + 2^2 + 2^1 + 2^0 = 2^k - 1$

$$2^k + 2^{k-1} + \cdots + 2^2 + 2^1 + 1 \stackrel{?}{=} 2^{k+1} - 1$$

$$\begin{aligned} 2^k + 2^{k-1} + \cdots + 2^2 + 2^1 + 1 &= 2^k + 2^k - 1 \\ &= 2 \cdot 2^k - 1 \\ &= 2^{k+1} - 1 \quad \checkmark \end{aligned}$$

∴ By the mathematical induction, the given statement is true.