Solution   Section 2.2 – Future Value of an Annuity

Exercise

Recently, Guaranty Income Life offered an annuity that pays 6.65% compounded monthly. If $500 is deposited into this annuity every month, how much is in the account after 10 years? How much of this is interest?

Solution

Given: $PMT = 500 \quad r = 6.65\% = .0665 \quad m = 12 \quad t = 10$

$$i = \frac{r}{m} = \frac{.0665}{12} \qquad n = mt = 12(10) = 120$$

$$FV = PMT\,\frac{(1+i)^n - 1}{i}$$

$$= 500\,\frac{\left(1+\frac{.0665}{12}\right)^{120} - 1}{\frac{.0665}{12}}$$

$$= \$84,895.10$$

Total deposits: $500(120) = \$60,000.00$

$$Interest = FV - Deposits$$

$$= 84,895.40 - 60,000$$

$$\underline{= \$24,895.40}$$

Exercise

Recently, USG Annuity Life offered an annuity that pays 4.25% compounded monthly. If $1,000 is deposited into this annuity every month, how much is in the account after 15 years? How much of this is interest?

Solution

Given: $PMT = 1,000 \quad r = 4.25\% = .0425 \quad m = 12 \quad t = 15$

$$i = \frac{r}{m} = \frac{.0425}{12} \qquad n = mt = 12(15) = 180$$

$$FV = PMT\,\frac{(1+i)^n - 1}{i}$$

$$= 1000\,\frac{\left(1+\frac{.0425}{12}\right)^{180} - 1}{\frac{.0425}{12}}$$

$$= \$251,185.76$$

$$\text{Total deposits: } 1,000(180) = \$180,000.00$$

$$\begin{aligned} Interest &= FV - Deposits \\ &= 251,185.76 - 180,000 \\ &= \underline{\$71,185.76} \end{aligned}$$

Exercise

In order to accumulate enough money for a down payment on a house, a couple deposits $300 per month into an account paying 6% compounded monthly. If payments are made at the end of each period, how much money will be in the account in 5 years?

Solution

Given: $PMT = 300 \quad r = 6\% = .06 \quad m = 12 \quad t = 5$

$$i = \frac{r}{m} = \frac{.06}{12} = 0.005 \quad n = mt = 12(5) = 60$$

$$\begin{aligned} FV &= PMT\,\frac{(1+i)^n - 1}{i} \\ &= 300\,\frac{(1+.005)^{60} - 1}{.005} \\ &= \underline{\$20,931.01} \end{aligned}$$

Exercise

A self-employed person has a Keogh retirement plan. (This type of plan is free of taxes until money is withdrawn.) If deposits of $7,500 are made each year into an account paying 8% compounded annually, how much will be in the account after 20 years?

Solution

Given: $PMT = 7,500 \quad r = 8\% = .08 \quad m = 1 \quad t = 20$

$$i = \frac{r}{m} = \frac{.08}{1} = 0.08 \quad n = mt = 1(20) = 20$$

$$\begin{aligned} FV &= PMT\,\frac{(1+i)^n - 1}{i} \\ &= 7,500\,\frac{(1+.08)^{20} - 1}{.08} \\ &= \underline{\$343,214.73} \end{aligned}$$

Exercise

Sun America recently offered an annuity that pays 6.35% compounded monthly. What equal monthly deposit should be made into this annuity in order to have \$200,000 in 15 years?

Solution

Given: $FV = 200,000 \quad r = 6.35\% = .0635, \quad m = 12, \quad t = 15$

$$i = \frac{r}{m} = \frac{.0635}{12} \quad n = mt = 12(15) = 180$$

$$PMT = FV\frac{i}{(1+i)^n - 1}$$

$$= 200,000\frac{\frac{.0635}{12}}{\left(1+\frac{.0635}{12}\right)^{180} - 1} \qquad 200000(.0635/12)/((1+.0635/12)\wedge 180-1)$$

$$\underline{= \$667.43|} \quad \textit{per month}$$

Exercise

Recently, The Hartford offered an annuity that pays 5.5% compounded monthly. What equal monthly deposit should be made into this annuity in order to have \$100,000 in 10 years?

Solution

Given: $FV = 100,000 \quad r = 5.5\% = .055, \quad m = 12, \quad t = 10$

$$i = \frac{r}{m} = \frac{.055}{12} \quad n = mt = 12(10) = 120$$

$$PMT = FV\frac{i}{(1+i)^n - 1}$$

$$= 100,000\frac{\frac{.055}{12}}{\left(1+\frac{.055}{12}\right)^{120} - 1} \qquad 100000(.055/12)/((1+.055/12)\wedge 120-1)$$

$$\underline{= \$626.93|} \quad \textit{per month}$$

Exercise

Compu-bank, an online banking service, offered a money market account with an APY of 4.86%.

a) If interest is compounded monthly, what is the equivalent annual nominal rate?

b) If you wish to have $10,000 in the account after 4 years, what equal deposit should you make each month?

Solution

Given: $APY = 4.86\% = .0486$

$$APY = \left(1+\frac{r}{m}\right)^m - 1$$

a) $m = 12$

$$.0486 = \left(1+\frac{r}{12}\right)^{12} - 1$$

Add1 on both sides

$$1.0486 = \left(1+\frac{r}{12}\right)^{12}$$

$$(1.0486)^{1/12} = 1+\frac{r}{12}$$

$$\frac{r}{12} = (1.0486)^{1/12} - 1$$

$$r = 12\left[(1.0486)^{1/12} - 1\right]$$

$12\left(1.0486 \wedge (1/12) - 1\right)$

$$\approx 0.0475$$

The equivalent annual nominal rate $\boxed{r = 4.75\%}$

b) ***Given***: $FV = \$10,000 \quad r = .0475, \quad m = 12, \quad t = 4$

$$i = \frac{r}{m} = \frac{.0475}{12} \qquad n = mt = 12(4) = 48$$

$$PMT = FV\frac{i}{(1+i)^n - 1}$$

$$= 10,000\frac{\frac{.0475}{12}}{\left(1+\frac{.0475}{12}\right)^{48} - 1}$$

$10000\left(.0475/12\right)/\left(\left(1+.0475/12\right) \wedge 60 - 1\right)$

$\boxed{= \$189.58}$ *per month*

Exercise

American Express's online banking division offered a money market account with an APY of 5.65%.

a) If interest is compounded monthly, what is the equivalent annual nominal rate?

b) If you wish to have $1,000,000 in the account after 8 years, what equal deposit should you make each month?

Solution

Given: $APY = 5.65\% = .0565$

$$APY = \left(1+\frac{r}{m}\right)^m - 1$$

a) $m = 12$

$$.0565 = \left(1+\frac{r}{12}\right)^{12} - 1$$

Add1 on both sides

$$1.0565 = \left(1+\frac{r}{12}\right)^{12}$$

$$(1.0565)^{1/12} = 1+\frac{r}{12}$$

$$\frac{r}{12} = (1.0565)^{1/12} - 1$$

$$r = 12\left[(1.0565)^{1/12} - 1\right]$$

$12\left(1.0565 \wedge (1/12) - 1\right)$

$$\approx 0.0551$$

The equivalent annual nominal rate $\boxed{r = 5.51\%}$

b) ***Given***: $FV = \$1,000,000 \quad r = .0551, \quad m = 12, \quad t = 8$

$$i = \frac{r}{m} = \frac{.0551}{12} \qquad n = mt = 12(8) = 96$$

$$PMT = FV\,\frac{i}{(1+i)^n - 1}$$

$$= 1,000,000\,\frac{\frac{.0551}{12}}{\left(1+\frac{.0551}{12}\right)^{96} - 1}$$

$1000000(.0551/12)/\left((1+.051/12) \wedge 96 - 1\right)$

$\boxed{= \$8,312.47}$ *per month*

Exercise

Find the future value of an annuity due if payments of $500 are made at the beginning of each quarter for 7 years, in an account paying 6% compounded quarterly.

Solution

Given: $PMT = 500 \quad r = 6\% = .06 \quad m = 4 \quad t = 7$

$$i = \frac{r}{m} = \frac{.06}{4} = 0.015 \qquad \underline{n = mt + 1 = 4(7) + 1 = 29}$$

Since you put money at the beginning of each month, we need to add the first payment.

$$FV = PMT\,\frac{(1+i)^n - 1}{i}$$

$$= 500\,\frac{(1+.015)^{29} - 1}{.015} \qquad 500\left((1+.015)^\wedge 29 - 1\right)/.015$$

$$\underline{= \$17,499.35}$$

Exercise

A 45 year-old man puts $2500 in a retirement account at the end of each quarter until he reaches the age of 60, then makes no further deposits. If the account pays 6% interest compounded quarterly, how much will be in the account when the man retires at age 65?

Solution

For the 15 years $(60 - 45 = 15)$:

$PMT = 2,500 \quad r = 6\% = .06 \quad m = 4 \quad t = 15$

$$i = \frac{r}{m} = \frac{.06}{4} = 0.015 \qquad n = mt + 1 = 4(15) = 60$$

$$FV = PMT\,\frac{(1+i)^n - 1}{i}$$

$$= 2,500\,\frac{(1+.015)^{60} - 1}{.015} \qquad 2500\left((1+.015)^\wedge 60 - 1\right)/.015$$

$$\underline{= \$240,536.63}$$

For the remaining 5 years, the *FV* amount is the present amount (P) at 6% compounded quarterly.

$$A = P(1+i)^n$$

$$= 240,536.63(1+.015)^{4(5)} \qquad 240536.63(1+.015)^\wedge(5*4)$$

$$\underline{= \$323,967.96}$$

Exercise

A father opened a savings account for his daughter on the day she was born, depositing \$1000. Each year on her birthday he deposits another \$1000, making the last deposit on her 21^{st} birthday. If the account pays 5.25% interest compounded annually, how much is in the account at the end of the day on his daughter's 21^{st} birthday? How much interest has been earned?

Solution

Given: $PMT = 1,000 \quad r = 5.25\% = .0525 \quad m = 1 \quad t = 21$

$$i = \frac{r}{m} = \frac{.0525}{1} = 0.0525 \qquad \left\lfloor n = mt + 1 = 1(21) + 1 \underline{= 22} \right|$$

Since you put money at the beginning of each year, we need to add the first payment.

$$FV = PMT\,\frac{(1+i)^n - 1}{i}$$

$$= 1,000\,\frac{(1+.0525)^{22} - 1}{.0525} \qquad 1000\left(\left(1+.0525\right)^\wedge 22 - 1\right)/.0525$$

$$\underline{= \$39,664.40}$$

The Total contribution: $1000\left(22\right) = \$22,000.00$

The interest earned: $39,664.40 - 22,000 \ \underline{= \$17,664.40}$

Exercise

You deposits \$10,000 at the beginning of each year for 12 years in an account paying 5% compounded annually. Then you put the total amount on deposit in another account paying 6% compounded semi-annually for another 9 years. Find the final amount on deposit after the entire 21-year period.

Solution

Given: $PMT = 10,000 \quad r = 5.\% = .05 \quad m = 1 \quad t = 12$

$$i = \frac{r}{m} = \frac{.05}{1} = 0.05 \qquad \left\lfloor n = mt + 1 = 12 + 1 \underline{= 13} \right|$$

$$FV_{12} = PMT\,\frac{(1+i)^n - 1}{i}$$

$$= 10,000\,\frac{(1+.05)^{13} - 1}{.05} \qquad 10000\left(\left(1+.05\right)^\wedge 13 - 1\right)/.05$$

$$\underline{= \$177,129.83}$$

Since the last deposit did mature yet when roll over, then:

$$P = 177,129.83 - 10,000 = \$167,129.83$$

$$i = \frac{r}{m} = \frac{.06}{2} = 0.03 \qquad \left\lfloor n = 9(2) \underline{= 18} \right|$$

$$A = P(1+i)^n$$

$$= 167{,}129.83(1+.03)^{18} \qquad 167129.83(1.03)\text{^}18$$

$$\underline{= \$284{,}527.35}$$

Exercise

You need $10,000 in 8 years.

a) What amount should be deposit at the end of each quarter at 8% compounded quarterly so that he will have his $10,000?

b) Find your quarterly deposit if the money is deposited at 6% compounded quarterly.

Solution

a) ***Given***: $FV = 10{,}000 \quad r = 8\% = .08, \quad m = 4, \quad t = 8$

$$i = \frac{r}{m} = \frac{.08}{4} = .02 \qquad n = mt = 4(8) = 32$$

$$PMT = FV\frac{i}{(1+i)^n - 1}$$

$$= 10{,}000\frac{.02}{(1+.02)^{32} - 1} \qquad 10000(.02)/(1.02\text{^}32-1)$$

$$\underline{= \$226.11} \quad \textit{each quarter}$$

b) ***Given***: $FV = 10{,}000 \quad r = 6\% = .06, \quad m = 4, \quad t = 8$

$$i = \frac{r}{m} = \frac{.06}{4} = .015 \qquad n = 4(8) = 32$$

$$PMT = FV\frac{i}{(1+i)^n - 1}$$

$$= 10{,}000\frac{.015}{(1+.015)^{32} - 1} \qquad 10000(.015)/(1.015\text{^}32-1)$$

$$\underline{= \$245.77} \quad \textit{each quarter}$$

Exercise

You want to have a \$20,000 down payment when you buy a car in 6 years. How much money must you deposit at the end of each quarter in an account paying 3.2% compounded quarterly so that you will have the down payment you desire?

Solution

Given: $FV = 20,000 \quad r = 3.2\% = .032, \quad m = 4, \quad t = 6$

$$i = \frac{r}{m} = \frac{.032}{4} = .008 \quad n = 4(6) = 24$$

$$PMT = FV \frac{i}{(1+i)^n - 1}$$

$$= 20,000 \frac{.008}{(1+.008)^{24} - 1} \qquad 100000(.055/12)/((1+.055/12)\wedge 120 - 1)$$

$$\underline{= \$759.21|} \quad \textit{quarterly}$$

Exercise

You sell a land and then you will be paid a lump sum of \$60,000 in 7 years. Until then, the buyer pays 8% simple interest quarterly.

a) Find the amount of each quarterly interest payment on the \$60,000

b) The buyer sets up a sinking fund so that enough money will be present to pay off the \$60,000. The buyer will make semiannual payments into the sinking fund; the account pays 6% compounded semiannually. Find the amount of each payment into the fund.

Solution

Given: $P = 60,000 \quad r = 8\% = .08, \quad m = 4, \quad t = 7$

a) $I = Prt$

$$= 60,000(.08)\left(\frac{1}{4}\right)$$

$$= \$1,200.00$$

b) **Given**: $FV = 60,000 \quad r = 6\% = .06, \quad m = 2, \quad t = 7$

$$i = \frac{r}{m} = \frac{.06}{2} = .03 \quad n = 2(7) = 14$$

$$PMT = FV \frac{i}{(1+i)^n - 1}$$

$$= 60,000 \frac{.03}{(1+.03)^{14} - 1} \qquad 60000(.03)/((1.03)\wedge 14 - 1)$$

$$\underline{= \$3511.58|}$$

$$i = \frac{.06}{2} = .03$$

$$(Balance)i = .03(Balance)$$

Pmt #	***Deposit Amount***	$I = .03 * Balance$	***Interest Earned***		***Balance***
1	$3,511.58	----	$0		$3,511.58
2	$3,511.58	$.03 * 3,511.58$	$105.35	$2(3{,}511.58) + 105.35$	$7,128.51
3	$3,511.58	$.03 * 7128.51$	$213.86	$7128.51 + 3,511.58 + 213.86$	$10,853.95
4	$3,511.58	$.03 * 10853.95$	$325.62	$10853.95 + 3511.58 + 325.62$	$14,691.15
5	$3,511.58	$.03 * 14691.15$	$440.73	$14691.15 + 3511.58 + 440.73$	$18,643.46
6	$3,511.58	$.03 * 18643.46$	$559.30	$18643.46 + 3511.58 + 559.30$	$22,714.34
7	$3,511.58	$.03 * 22714.34$	$681.43	$22714.34 + 3511.58 + 681.43$	$26,907.35
8	$3,511.58	$.03 * 26907.35$	$807.22	$26907.35 + 3511.58 + 807.22$	$31,226.15
9	$3,511.58	$.03 * 31226.15$	$936.78	$31226.15 + 3511.58 + 936.78$	$35,674.51
10	$3,511.58	$.03 * 35674.51$	$1,070.24	$35674.51 + 3511.58 + 1070.24$	$40,256.33
11	$3,511.58	$.03 * 40256.33$	$1,207.69	$40256.33 + 3511.58 + 1207.69$	$44,975.60
12	$3,511.58	$.03 * 44975.60$	$1,349.57	$44975.60 + 3511.58 + 1349.57$	$49,843.13
13	$3,511.58	$.03 * 49.843.13$	$1,495.09	$49843.13 + 3511.58 + 1495.09$	$54,843.13
14	$3,511.58	$.03 * 54843.13$	$1,645.29	$54843.13 + 3511.58 + 1645.29$	$60,000.00