

## ***Solution***    **Section 3.6 – Solving Linear Recurrence Relations**

### ***Exercise***

Determine which of these are linear and homogeneous recurrence relations with constant coefficients. Also find the degree of those that are

a)  $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$

b)  $a_n = 2na_{n-1} + a_{n-2}$

c)  $a_n = a_{n-1} + a_{n-4}$

d)  $a_n = a_{n-1} + 2$

e)  $a_n = a_{n-1}^2 + a_{n-2}$

f)  $a_n = a_{n-2}$

g)  $a_n = a_{n-1} + n$

h)  $a_n = 3a_{n-2}$

i)  $a_n = 3$

j)  $a_n = a_{n-1}^2$

k)  $a_n = a_{n-1} + 2a_{n-3}$

l)  $a_n = \frac{a_{n-1}}{n}$

### **Solution**

- a) Linear (terms  $a_i$  all to the first power), has constant coefficients (3, 4 and 5), and is homogeneous (no terms are functions of just  $n$ ); has degree 3
- b) Linear (terms  $a_i$  all to the first power), doesn't have constant coefficients ( $2n$ ), and is homogeneous
- c) Linear, homogeneous, with constant coefficients; degree 4
- d) Linear with constant coefficients, not homogeneous because of 2
- e) Not linear since  $a_{n-1}^2$
- f) Linear, homogeneous, with constant coefficients; degree 2
- g) Linear but not homogeneous because of the  $n$ .
- h) Linear, homogeneous, with constant coefficients; degree 2
- i) Linear with constant coefficients, not homogeneous because of 3
- j) Not linear since  $a_{n-1}^2$
- k) Linear, homogeneous, with constant coefficients; degree 3
- l) Linear with constant coefficients, not homogeneous

### Exercise

Solve these recurrence relations together with the initial conditions given

- a)  $a_n = 2a_{n-1}$  for  $n \geq 1$ ,  $a_0 = 3$
- b)  $a_n = 5a_{n-1} - 6a_{n-2}$  for  $n \geq 2$ ,  $a_0 = 1$ ,  $a_1 = 0$
- c)  $a_n = 4a_{n-1} - 4a_{n-2}$  for  $n \geq 2$ ,  $a_0 = 6$ ,  $a_1 = 8$
- d)  $a_n = 4a_{n-2}$  for  $n \geq 2$ ,  $a_0 = 0$ ,  $a_1 = 4$
- e)  $a_n = \frac{a_{n-2}}{4}$  for  $n \geq 2$ ,  $a_0 = 1$ ,  $a_1 = 0$
- f)  $a_n = a_{n-1} + 6a_{n-2}$  for  $n \geq 2$ ,  $a_0 = 3$ ,  $a_1 = 6$
- g)  $a_n = 7a_{n-1} - 10a_{n-2}$  for  $n \geq 2$ ,  $a_0 = 2$ ,  $a_1 = 1$
- h)  $a_n = -6a_{n-1} - 9a_{n-2}$  for  $n \geq 2$ ,  $a_0 = 3$ ,  $a_1 = -3$
- i)  $a_{n+2} = -4a_{n+1} + 5a_n$  for  $n \geq 0$ ,  $a_0 = 2$ ,  $a_1 = 8$

### Solution

- a) The characteristic polynomial is  $r - 2 = 0 \Rightarrow r = 2$

The general solution:  $a_n = \alpha_1 2^n$

$$3 = \alpha_1 2^0 \rightarrow \alpha_1 = 3$$

Therefore, the solution is  $a_n = 3 \cdot 2^n$

- b) The characteristic polynomial is  $r^2 - 5r + 6 = 0 \Rightarrow r = 2, 3$

The general solution:  $a_n = \alpha_1 2^n + \alpha_2 3^n$

$$\begin{aligned} 1 &= \alpha_1 2^0 + \alpha_2 3^0 \rightarrow 1 = \alpha_1 + \alpha_2 \\ 0 &= \alpha_1 2^1 + \alpha_2 3^1 \rightarrow 0 = 2\alpha_1 + 3\alpha_2 \end{aligned} \Rightarrow \alpha_1 = 3, \alpha_2 = -2$$

Therefore, the solution is  $a_n = 3 \cdot 2^n - 2 \cdot 3^n$

- c) The characteristic polynomial is  $r^2 - 4r + 4 = 0 \Rightarrow r = 2, 2$

The general solution:  $a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n$

$$\begin{aligned} 6 &= \alpha_1 2^0 + \alpha_2 (0) 2^0 \rightarrow 6 = \alpha_1 \\ 8 &= \alpha_1 2^1 + \alpha_2 (1) 2^1 \rightarrow 8 = 2\alpha_1 + 2\alpha_2 \end{aligned} \Rightarrow \alpha_1 = 6, \alpha_2 = -2$$

Therefore, the solution is  $a_n = 6 \cdot 2^n - 2n \cdot 2^n = (6 - 2n) 2^n$

- d) The characteristic polynomial is  $r^2 - 4 = 0 \Rightarrow r = \pm 2$

The general solution:  $a_n = \alpha_1 (-2)^n + \alpha_2 2^n$

$$\begin{aligned} 0 &= \alpha_1 (-2)^0 + \alpha_2 2^0 \rightarrow 0 = \alpha_1 + \alpha_2 \\ 4 &= \alpha_1 (-2)^1 + \alpha_2 2^1 \rightarrow 4 = -2\alpha_1 + 2\alpha_2 \end{aligned} \Rightarrow \alpha_1 = -1, \alpha_2 = 1$$

Therefore, the solution is  $\underline{a_n = 2^n - (-2)^n}$

e) The characteristic polynomial is  $r^2 - \frac{1}{4} = 0 \Rightarrow r = \pm \frac{1}{2}$

The general solution:  $a_n = \alpha_1 \left(-\frac{1}{2}\right)^n + \alpha_2 \left(\frac{1}{2}\right)^n = \alpha_1 (-2)^{-n} + \alpha_2 (2)^{-n}$

$$\begin{aligned} 1 &= \alpha_1 \left(-\frac{1}{2}\right)^0 + \alpha_2 \left(\frac{1}{2}\right)^0 \rightarrow 1 = \alpha_1 + \alpha_2 \\ 0 &= \alpha_1 \left(-\frac{1}{2}\right)^1 + \alpha_2 \left(\frac{1}{2}\right)^1 \rightarrow 0 = -\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 \end{aligned} \Rightarrow \alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{2}$$

Therefore, the solution is  $\underline{a_n = \frac{1}{2} \left(-\frac{1}{2}\right)^n + \frac{1}{2} \left(\frac{1}{2}\right)^n}$   
 $\underline{= \left(\frac{1}{2}\right)^{n+1} - \left(-\frac{1}{2}\right)^{n+1}}$

f) The characteristic polynomial is  $r^2 - r - 6 = 0 \Rightarrow r = -2, 3$

The general solution:  $a_n = \alpha_1 (-2)^n + \alpha_2 3^n$

$$\begin{aligned} 3 &= \alpha_1 (-2)^0 + \alpha_2 3^0 \rightarrow 3 = \alpha_1 + \alpha_2 \\ 6 &= \alpha_1 (-2)^1 + \alpha_2 3^1 \rightarrow 6 = -2\alpha_1 + 3\alpha_2 \end{aligned} \Rightarrow \alpha_1 = \frac{3}{5}, \alpha_2 = \frac{12}{5}$$

Therefore, the solution is  $\underline{a_n = \frac{3}{5}(-2)^n + \frac{12}{5}3^n}$

g) The characteristic polynomial is  $r^2 - 7r + 10 = 0 \Rightarrow r = 2, 5$

The general solution:  $a_n = \alpha_1 2^n + \alpha_2 5^n$

$$\begin{aligned} 2 &= \alpha_1 2^0 + \alpha_2 5^0 \rightarrow 2 = \alpha_1 + \alpha_2 \\ 1 &= \alpha_1 2^1 + \alpha_2 5^1 \rightarrow 1 = 2\alpha_1 + 5\alpha_2 \end{aligned} \Rightarrow \alpha_1 = 3, \alpha_2 = -1$$

Therefore, the solution is  $\underline{a_n = 3 \cdot 2^n - 5^n}$

h) The characteristic polynomial is  $r^2 + 6r + 9 = 0 \Rightarrow r = -3, -3$

The general solution:  $a_n = \alpha_1 (-3)^n + \alpha_2 n(-3)^n$

$$\begin{aligned} 3 &= \alpha_1 (-3)^0 + \alpha_2 (0)(-3)^0 \rightarrow 3 = \alpha_1 \\ -3 &= \alpha_1 (-3)^1 + \alpha_2 (1)(-3)^1 \rightarrow -3 = -3\alpha_1 - 3\alpha_2 \end{aligned} \Rightarrow \alpha_1 = 3, \alpha_2 = -2$$

Therefore, the solution is  $\underline{a_n = 3 \cdot (-3)^n - 2n(-3)^n = (3 - 2n)(-3)^n}$

i) The characteristic polynomial is  $r^2 + 4r - 5 = 0 \Rightarrow r = -5, 1$

The general solution:  $a_n = \alpha_1 (-5)^n + \alpha_2 1^n = \alpha_1 (-5)^n + \alpha_2$

$$2 = \alpha_1 (-5)^0 + \alpha_2 \rightarrow 2 = \alpha_1 + \alpha_2 \Rightarrow \alpha_1 = -1, \alpha_2 = 3$$

$$8 = \alpha_1 (-5)^1 + \alpha_2 \rightarrow 8 = -5\alpha_1 + \alpha_2$$

Therefore, the solution is  $a_n = -(-5)^n + 3$

### Exercise

How many different messages can be transmitted in  $n$  microseconds using three different signals if one signal requires 1 microsecond for transmittal, the other two signals require 2 microseconds each for transmittal, and a signal in a message is followed immediately by the next signal?

### Solution

The model is the recurrence relation  $a_n = a_{n-1} + a_{n-2} + a_{n-2} = a_{n-1} + 2a_{n-2}$  with  $a_0 = a_1 = 1$

The characteristic polynomial is  $r^2 - r - 2 = 0$

So the roots are  $-1$ , and  $2$

The general solution:  $a_n = \alpha_1 (-1)^n + \alpha_2 2^n$

Plugging in initial conditions gives

$$1 = \alpha_1 (-1)^0 + \alpha_2 2^0 \rightarrow 1 = \alpha_1 + \alpha_2$$

$$1 = \alpha_1 (-1)^1 + \alpha_2 2^1 \rightarrow 1 = -\alpha_1 + 2\alpha_2 \Rightarrow \alpha_1 = \frac{1}{3}, \alpha_2 = \frac{2}{3}$$

Therefore, the solution is in  $n$  microseconds  $a_n = \frac{1}{3}(-1)^n + \frac{2}{3}2^n$  messages can be transmitted.

### Exercise

In how many ways can a  $2 \times n$  rectangular checkerboard be tiled using  $1 \times 2$  and  $2 \times 2$  pieces?

### Solution

Let  $t_n$  be the number of ways like to tile a  $2 \times n$  board with  $1 \times 2$  and  $2 \times 2$  pieces. To obtain the recurrence relation, imagine what tiles are placed at the left-hand end of the board. We can place a  $2 \times 2$  tile there, leaving a  $2 \times (n-2)$  board to be tiled, which of course can be done in  $t_{n-2}$  ways.

We can place a  $1 \times 2$  tile at the edge, oriented vertically, leaving  $2 \times (n-1)$  board to be tiled, which of course can be done in  $t_{n-1}$  ways. Finally, we can place two  $1 \times 2$  tiles horizontally, one above the other, leaving a  $2 \times (n-2)$  board to be tiled, which of course can be done in  $t_{n-2}$  ways. These 3 possibilities are disjoint.

Therefore our recurrence relation is  $t_n = t_{n-1} + 2t_{n-2}$

The initial conditions are  $t_0 = t_1 = 1$ , since there is only one way to tile as  $2 \times 0$  board and  $2 \times 1$  board. This recurrence relation has characteristic roots  $-1$  and  $2$ .

So, the general solution is  $t_n = \alpha_1 (-1)^n + \alpha_2 2^n$

Plugging in initial conditions gives

$$\begin{aligned} 1 &= \alpha_1 (-1)^0 + \alpha_2 2^0 \rightarrow 1 = \alpha_1 + \alpha_2 \\ 1 &= \alpha_1 (-1)^1 + \alpha_2 2^1 \rightarrow 1 = -\alpha_1 + 2\alpha_2 \end{aligned} \Rightarrow \alpha_1 = \frac{1}{3}, \quad \alpha_2 = \frac{2}{3}$$

Therefore, the solution is  $a_n = \frac{1}{3}(-1)^n + \frac{2}{3} \cdot 2^n$

$$= \frac{(-1)^n}{3} + \frac{2^{n+1}}{3}$$

### Exercise

Find the solution to  $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$  for  $n \geq 3$ ,  $a_0 = 3$ ,  $a_1 = 6$  and  $a_2 = 0$

### Solution

$$a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$$

The characteristic polynomial is  $r^3 - 2r^2 - r + 2 = 0$

That implies to:  $r^2(r-2) - (r-2) = (r-2)(r^2-1) = 0$

So the roots are  $1$ ,  $-1$ , and  $2$

The general solution:  $a_n = \alpha_1 1^n + \alpha_2 (-1)^n + \alpha_3 2^n = \alpha_1 + \alpha_2 (-1)^n + \alpha_3 2^n$

Plugging in initial conditions gives

$$\begin{aligned} 3 &= \alpha_1 + \alpha_2 (-1)^0 + \alpha_3 2^0 \rightarrow 3 = \alpha_1 + \alpha_2 + \alpha_3 \\ 6 &= \alpha_1 + \alpha_2 (-1)^1 + \alpha_3 2^1 \rightarrow 6 = \alpha_1 - \alpha_2 + 2\alpha_3 \\ 0 &= \alpha_1 + \alpha_2 (-1)^2 + \alpha_3 2^2 \rightarrow 0 = \alpha_1 + \alpha_2 + 4\alpha_3 \end{aligned} \Rightarrow \alpha_1 = 6, \quad \alpha_2 = -2, \quad \alpha_3 = -1$$

Therefore, the solution is  $a_n = 6 - 2(-1)^n - 2^n$

### Exercise

Find the solution to  $a_n = 7a_{n-2} + 6a_{n-3}$  with  $a_0 = 9, a_1 = 10$  and  $a_2 = 32$

### Solution

This is a third degree recurrence relation.

The characteristic polynomial is  $r^3 - 7r - 6 = 0$

By the rational root test, the possible rational roots are  $\pm \left\{ \frac{6}{1} \right\} = \pm \{1, 2, 3, 6\}$

We find that  $r = -1$  (using calculator).

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array} \quad Q(x) = r^2 - r - 6 = (r+2)(r-3)$$

$$r^3 - 6r^2 + 12r - 8 = (r+1)(r+2)(r-3) = 0$$

So the roots are  $-2, -1$ , and  $3$ .

The general solution:  $a_n = \alpha_1 (-2)^n + \alpha_2 (-1)^n + \alpha_3 3^n$

Plugging in initial conditions gives

$$a_0 = 9 = \alpha_1 (-2)^0 + \alpha_2 (-1)^0 + \alpha_3 3^0 \rightarrow 9 = \alpha_1 + \alpha_2 + \alpha_3$$

$$a_1 = 10 = \alpha_1 (-2)^1 + \alpha_2 (-1)^1 + \alpha_3 3^1 \rightarrow 10 = -2\alpha_1 - \alpha_2 + 3\alpha_3$$

$$a_2 = 32 = \alpha_1 (-2)^2 + \alpha_2 (-1)^2 + \alpha_3 3^2 \rightarrow 32 = 4\alpha_1 + \alpha_2 + 9\alpha_3$$

The solution to the system of equations is  $\alpha_1 = -3, \alpha_2 = 8$  and  $\alpha_3 = 4$

Therefore the specific solution is  $\boxed{a_n = -3(-2)^n + 8(-1)^n + 4 \cdot 3^n}$

### Exercise

Find the solution to  $a_n = 5a_{n-2} - 4a_{n-4}$  with  $a_0 = 3, a_1 = 2, a_2 = 6$  and  $a_3 = 8$

### Solution

This is a fourth degree recurrence relation.

The characteristic polynomial is  $r^4 - 5r^2 - 4 = 0$

That implies to:  $(r^2 - 1)(r^2 - 4) = (r-1)(r+1)(r-2)(r+2) = 0$

So the roots are  $1, -1, 2, -2$

The general solution:  $a_n = \alpha_1 + \alpha_2 (-1)^n + \alpha_3 2^n + \alpha_4 (-2)^n$

Plugging in initial conditions gives

$$3 = \alpha_1 + \alpha_2 (-1)^0 + \alpha_3 2^0 + \alpha_4 (-2)^0 \rightarrow 3 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

$$10 = \alpha_1 + \alpha_2 (-1)^1 + \alpha_3 2^1 + \alpha_4 (-2)^1 \rightarrow 10 = \alpha_1 - \alpha_2 + 2\alpha_3 - 2\alpha_4$$

$$6 = \alpha_1 + \alpha_2 (-1)^2 + \alpha_3 2^2 + \alpha_4 (-2)^2 \rightarrow 6 = \alpha_1 + \alpha_2 + 4\alpha_3 + 4\alpha_4$$

$$8 = \alpha_1 + \alpha_2 (-1)^3 + \alpha_3 2^3 + \alpha_4 (-2)^3 \rightarrow 8 = \alpha_1 - \alpha_2 + 8\alpha_3 - 8\alpha_4$$

The solution to the system of equations is  $\alpha_1 = \alpha_2 = \alpha_3 = 1$  and  $\alpha_4 = 0$

Therefore, the solution is  $\underline{a_n = 1 + (-1)^n + 2^n}$

### Exercise

Find the recurrence relation  $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$  with  $a_0 = -5$ ,  $a_1 = 4$  and  $a_2 = 88$

### Solution

This is a third degree recurrence relation.

The characteristic polynomial is  $r^3 - 6r^2 + 12r - 8 = 0$

By the rational root test, the possible rational roots are  $\pm 1, \pm 2, \pm 4, \pm 8$

We find that  $r = 2$  (using calculator).

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 12 & -8 \\ & & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array} \quad Q(x) = r^2 - 4r + 4 = (r - 2)^2$$

$$r^3 - 6r^2 + 12r - 8 = (r - 2)^3 = 0$$

Hence the only root is 2, with multiplicity 3.

The general solution:  $\underline{a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n + \alpha_3 n^2 \cdot (-2)^n}$

Plugging in initial conditions gives

$$\underline{-5 = a_0 = \alpha_1}$$

$$4 = a_1 = 2\alpha_1 + 2\alpha_2 + 2\alpha_3$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 2 \rightarrow \alpha_2 + \alpha_3 = 7$$

$$88 = a_2 = 4\alpha_1 + 8\alpha_2 + 16\alpha_3$$

$$\alpha_1 + 2\alpha_2 + 4\alpha_3 = 22 \rightarrow 2\alpha_2 + 4\alpha_3 = 27$$

$$\rightarrow \begin{cases} \alpha_2 + \alpha_3 = 7 \\ 2\alpha_2 + 4\alpha_3 = 27 \end{cases} \Rightarrow \underline{\begin{array}{l} \alpha_2 = \frac{1}{2} \\ \alpha_3 = \frac{13}{2} \end{array}}$$

Therefore the solution:  $a_n = -5 \cdot 2^n + \frac{1}{2} n \cdot 2^n + \frac{13}{2} n^2 \cdot (-2)^n$

$$\underline{= -5 \cdot 2^n + n \cdot 2^{n-1} + 13n^2 \cdot (-2)^{n-1}}$$

### Exercise

Find the recurrence relation  $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$  with  $a_0 = 5$ ,  $a_1 = -9$  and  $a_2 = 15$

### Solution

This is a third degree recurrence relation.

The characteristic polynomial is  $r^3 + 3r^2 + 3r + 1 = 0$

$$r^3 + 3r^2 + 3r + 1 = 0 = (r+1)^3 = 0$$

Hence the only root is  $-1$ , with multiplicity 3.

The general solution: 
$$a_n = \alpha_1 (-1)^n + \alpha_2 n \cdot (-1)^n + \alpha_3 n^2 \cdot (-1)^n$$

Plugging in initial conditions gives

$$5 = a_0 = \alpha_1$$

$$a_1 = -9 = -\alpha_1 - \alpha_2 - \alpha_3 \rightarrow \alpha_2 + \alpha_3 = 9 - \alpha_1 = 4$$

$$a_2 = 15 = \alpha_1 + 2\alpha_2 + 4\alpha_3 \rightarrow 2\alpha_2 + 4\alpha_3 = 15 - \alpha_1 = 10$$

$$\rightarrow \begin{cases} \alpha_2 + \alpha_3 = 4 \\ 2\alpha_2 + 4\alpha_3 = 10 \end{cases} \Rightarrow \begin{cases} \alpha_2 = 3 \\ \alpha_3 = 1 \end{cases}$$

Therefore the specific solution is 
$$a_n = 5(-1)^n + 3n \cdot (-1)^n + n^2 \cdot (-1)^n = (n^2 + 3n + 5)(-1)^n$$

### Exercise

Find the general form of the solutions of the recurrence relation  $a_n = 8a_{n-2} - 16a_{n-4}$

### Solution

This is a fourth degree recurrence relation.

The characteristic polynomial is  $r^4 - 8r^2 + 16 = (r^2 - 4)^2$

$$(r^2 - 4)^2 = (r-2)^2 (r+2)^2 = 0$$

The roots are  $-2$  and  $2$ , each with multiplicity 2.

The general solution: 
$$a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n + \alpha_3 (-2)^n + \alpha_4 n \cdot (-2)^n$$



### Exercise

What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots 1, 1, 1, 1, -2, -2, -2, 3, 3, -4?

### Solution

There are 4 distinct roots, so  $t = 4$ . The multiplicities are 4, 3, 2, and 1.

The general solution:

$$a_n = \left( \alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2 + \alpha_{1,3}n^3 \right) + \left( \alpha_{2,0} + \alpha_{2,1}n + \alpha_{2,2}n^2 \right) (-2)^n + \left( \alpha_{3,0} + \alpha_{3,1}n \right) 3^n + \alpha_{4,0} (-4)^n$$

### Exercise

What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots -1, -1, -1, 2, 2, 5, 5, 7?

### Solution

There are 4 distinct roots, so  $t = 4$ . The multiplicities are 3, 2, 2, and 1.

The general solution:

$$a_n = \left( \alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2 \right) (-1)^n + \left( \alpha_{2,0} + \alpha_{2,1}n \right) 2^n + \left( \alpha_{3,0} + \alpha_{3,1}n \right) 5^n + \alpha_{4,0} 7^n$$