Solution Section 1.7 - Direction Fields; Existence and Uniqueness of Solutions

Exercise

Which of the initial value problems are guaranteed a unique solution. $y' = 4 + y^2, \quad y(0) = 1$

Solution

$f(t, y) = 4 + y^2 \rightarrow f$ is continuous

$\frac{\partial f}{\partial y} = 2y$ is also continuous on the whole plane.

Hence the hypotheses are satisfied and guarantee a unique solution.

Exercise

Which of the initial value problems are guaranteed a unique solution? $y' = \sqrt{y}, \quad y(4) = 0$

Solution

$f(t, y) = \sqrt{y} \quad \Rightarrow \ y \ge 0$

$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}} \quad \rightarrow \ y > 0 \ \ (only)$

Initial condition: $y(4) = 0 \Rightarrow y_0 = 0$ *and* $t_0 = 4$

Both f *and* $\frac{\partial f}{\partial y}$ are not continuous in the rectangle containing $\left(t_0, y_0\right)$

Hence the hypotheses are not satisfied.

Exercise

Which of the initial value problems are guaranteed a unique solution?

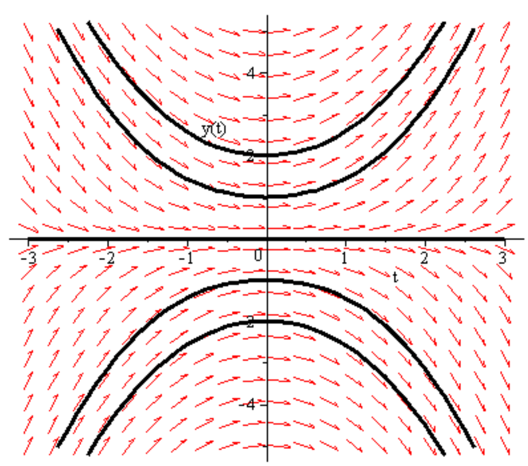
$y' = t\tan^{-1} y, \quad y(0) = 2$

Solution

The right hand side of the equation is $f\left(t, y\right) = t\tan^{-1} y$,

which is continuous in the whole plane. $\frac{\partial f}{\partial y} = \frac{t}{t + y^2}$ is also continuous in the whole plane.

Hence the hypotheses are satisfied and the theorem guarantees a unique solution.



Exercise

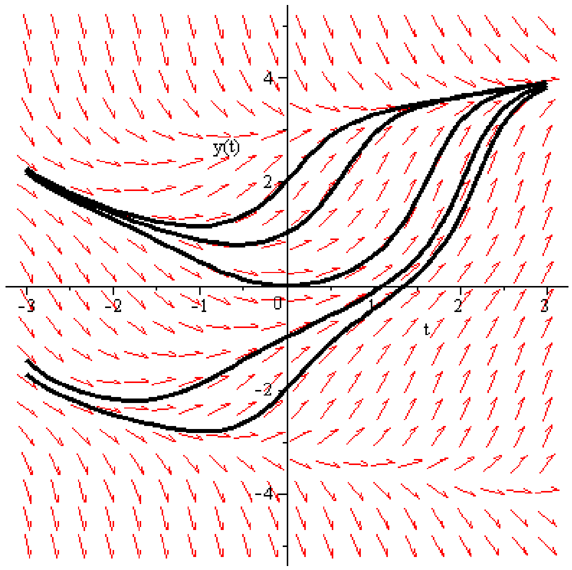
Which of the initial value problems are guaranteed a unique solution? $\omega' = \omega \sin \omega + s, \quad \omega(0) = -1$

Solution

The right hand side of the equation is $f(s, \omega) = \omega \sin \omega + s$, which is continuous in the whole plane.

$\dfrac{\partial f}{\partial \omega} = \sin \omega + \omega \cos \omega$ is also continuous in the whole plane.

Hence the hypotheses are satisfied and the theorem guarantees a unique solution.



Exercise

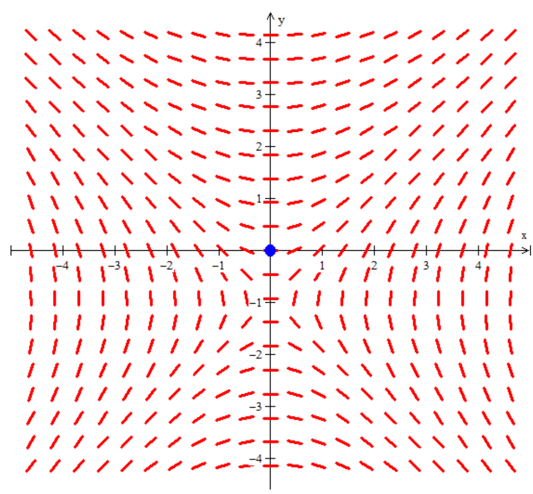
Which of the initial value problems are guaranteed a unique solution? $x' = \dfrac{t}{x+1}, \quad x(0) = 0$

Solution

The right hand side of the equation is $f(t, x) = \dfrac{t}{x+1}$, which is continuous in the whole plane, except where $x = -1$.

$\dfrac{\partial f}{\partial x} = -\dfrac{t}{(x+1)^2}$ is also continuous in the whole plane, except where $x = -1$.

Hence the hypotheses are satisfied in a rectangle containing the initial point (0, 0), so the theorem guarantees a unique solution.



Exercise

Which of the initial value problems are guaranteed a unique solution? $y' = \frac{1}{x}y + 2, \quad y(0) = 1$

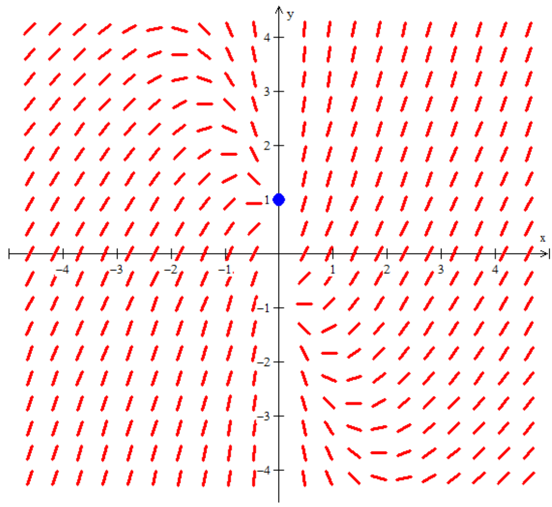
Solution

The right hand side of the equation is $f(x, y) = \frac{1}{x}y + 2$, which is continuous in the whole plane, except where $x = 0$.

Since the initial point is (0, 1), *f* is discontinuous there.

Consequently, there is no rectangle containing this point in which *f* is continuous.

The hypotheses are not satisfied, so the theorem doesn't guarantee a unique solution.



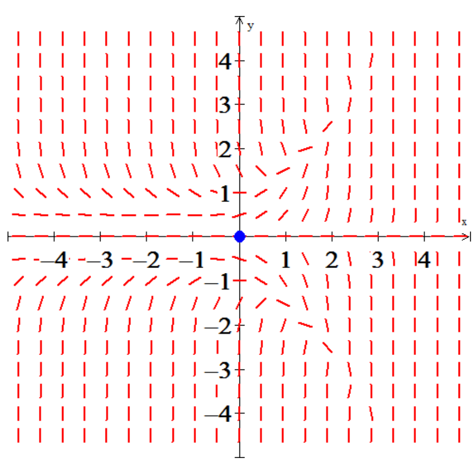
Exercise

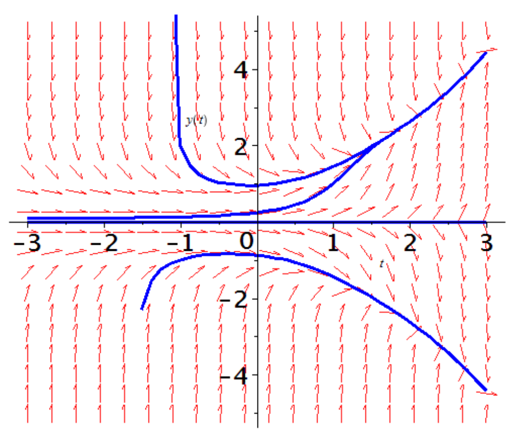
Which of the initial value problems are guaranteed a unique solution? $y' = e^t y - y^3, \quad y(0) = 0$

Solution

The right hand side of the equation is $f(t, y) = e^t y - y^3$, which is continuous in the whole plane.

$\frac{\partial f}{\partial y} = e^t - 3y^2$ is also continuous in the whole plane.





Hence the hypotheses are satisfied in a rectangle containing the initial point (0, 0), so the theorem guarantees a unique solution.

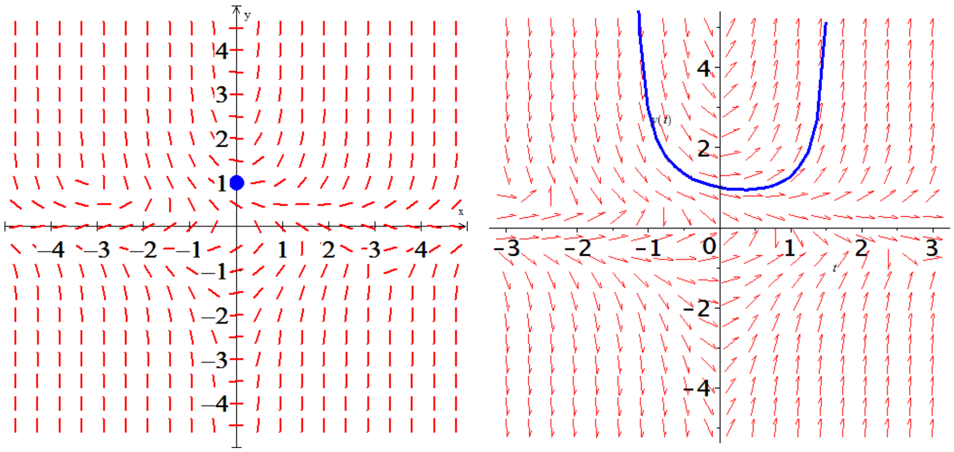
Exercise

Which of the initial value problems are guaranteed a unique solution? $\quad y' = ty^2 - \dfrac{1}{3y+t}, \quad y(0) = 1$

Solution

The right hand side of the equation is $f(t, y) = ty^2 - \dfrac{1}{3y+t}$, which is continuous in the whole plane.

$\dfrac{\partial f}{\partial y} = 2ty + \dfrac{3}{(3y+t)^2}$ is also continuous in the whole plane.



Hence the hypotheses are satisfied in a rectangle containing the initial point $(0,\ 1)$, so the theorem guarantees a unique solution.

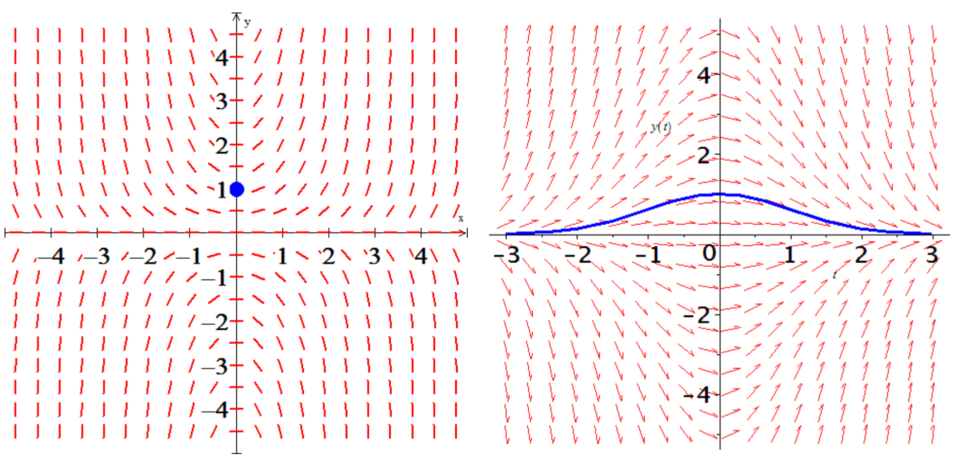
Exercise

Which of the initial value problems are guaranteed a unique solution? $\quad y' = x\,y, \quad y(0) = 1$

Solution

The right hand side of the equation is $f(x, y) = xy$, which is continuous in the whole plane.

$\dfrac{\partial f}{\partial y} = x$ is also continuous in the whole plane.



Hence the hypotheses are satisfied in a rectangle containing the initial point $(0,\ 1)$, so the theorem guarantees a unique solution.

Exercise

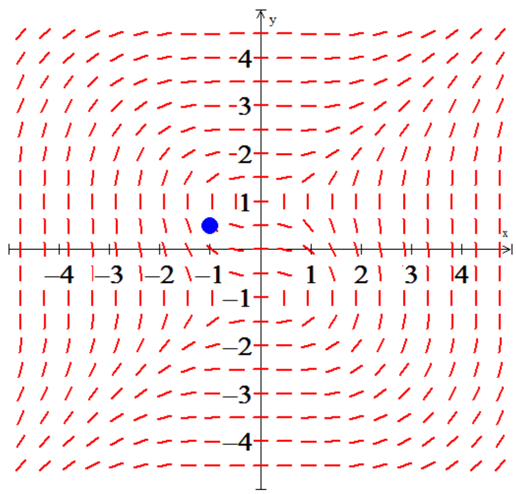
Which of the initial value problems are guaranteed a unique solution? $\qquad y' = -\dfrac{t^2}{1-y^2}, \quad y(-1) = \frac{1}{2}$

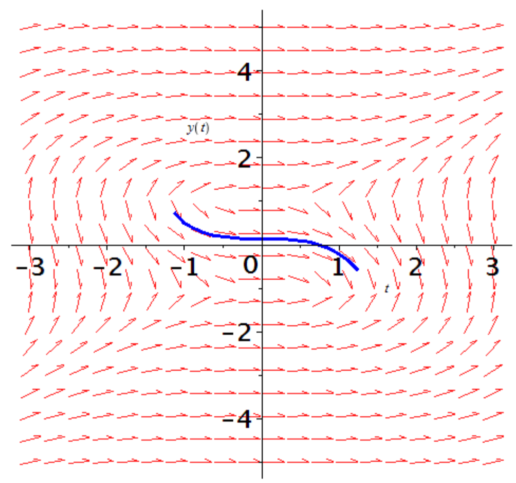
Solution

The right hand side of the equation is $f(t,y) = -\dfrac{t^2}{1-y^2}$, which is continuous in the whole plane, except where $y = \pm 1$.

$\dfrac{\partial f}{\partial y} = -\dfrac{2t^2 y}{\left(1-y^2\right)^2}$ is also continuous in the whole plane, except where $y = \pm 1$.

Since $t = -1 \rightarrow y_0 = \frac{1}{2} \quad (\neq \pm 1)$





Hence the hypotheses are satisfied in a rectangle containing the initial point $\left(-1, \frac{1}{2}\right)$, so the theorem guarantees a unique solution.

Exercise

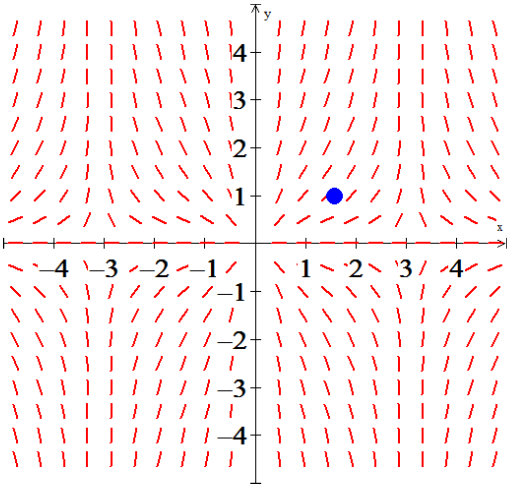
Which of the initial value problems are guaranteed a unique solution? $\qquad y' = \dfrac{y}{\sin t}, \quad y\left(\frac{\pi}{2}\right) = 1$

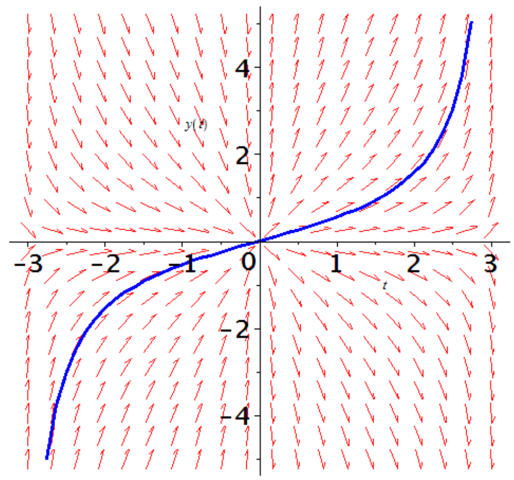
Solution

The right hand side of the equation is $f(t,y) = \dfrac{y}{\sin t}$, which is continuous in the whole plane, except where $t = n\pi$.

$\dfrac{\partial f}{\partial y} = \dfrac{1}{\sin t}$ is also continuous in the whole plane, except where $t = n\pi$

Since $t = \frac{\pi}{2} \rightarrow y_0 = 1 \quad (\neq n\pi)$





Hence the hypotheses are satisfied in a rectangle containing the initial point $\left(\frac{\pi}{2},\ 1\right)$, so the theorem guarantees a unique solution.

Exercise

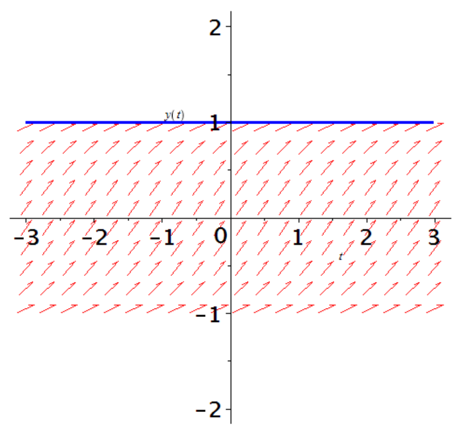
Which of the initial value problems are guaranteed a unique solution? $y' = \sqrt{1-y^2}, \quad y(0)=1$

Solution

The right hand side of the equation is $f(t,y) = \sqrt{1-y^2}$, which is continuous in the whole plane except where $y < -1\ \ \&\ \ y > 1$.

$\frac{\partial f}{\partial y} = -\frac{y}{\sqrt{1-y^2}}\Big|_{y=1} = \infty$ undefined.

So, the uniqueness theorem doesn't apply



$\frac{dy}{dt} = \sqrt{1-y^2}$

$\int \frac{dy}{\sqrt{1-y^2}} = \int dt$

$\arccos y = t + C$

$y(t) = \cos(t+C)$

$y(0)=1 \quad \rightarrow 1 = \cos C \quad \Rightarrow C = 0(=2n\pi)$

$\underline{y(t) = \cos t \quad (2n-1)\pi \le t \le 2\pi}|$

Exercise

Show that $y(t)=0$ and $y(t)=t^3$ are both solutions of the initial value problem $y'=3y^{2/3}$, where $y(0)=0$. Explain why this fact doesn't contradict Theorem

<u>**Solution**</u>

$$f(t,y)=3y^{2/3}$$

$$f'=2y^{-1/3} \quad \text{which is not continuous at } y=0$$

Exercise

Use a numerical solver to sketch the solution of the given initial value problem

$$\frac{dy}{dt}=\frac{t}{y+1}, \qquad y(2)=0$$

a) Where does your solver experience difficulty? why? Use the image of your solution to estimate the interval of existence.

b) Find an explicit solution; then use your formula to determine the interval of existence. How does it compare with the approximation found in part (*a*).

<u>**Solution**</u>

a) $(y+1)dy=tdt$

$$\int(y+1)dy=\int tdt$$

$$\tfrac{1}{2}y^2+y=\tfrac{1}{2}t^2+C$$

$$y^2+2y=t^2+C$$

$$(0)^2+2(0)=2^2+C$$

$$0=4+C$$

$$C=-4$$

$$y^2+2y=t^2-4$$

$$y^2+2y-t^2+4=0$$

Solve for *y*:

$$y=\frac{-2\pm\sqrt{2^2-4(1)(-t^2+4)}}{2(1)}=\frac{-2\pm\sqrt{4t^2-12}}{2}=\frac{-2\pm2\sqrt{t^2-3}}{2}$$

$$\underline{=-1\pm\sqrt{t^2-3}}$$

b) The only solution is: $y=-1+\sqrt{t^2-3}$ and $t^2-3>0\Rightarrow t>\sqrt{3}$

The interval of the solution $\left(\sqrt{3},\ \infty\right)$