Solution Section 3.7 – Linear Dependence and Independence

Exercise

Given three independent vectors w_1, w_2, w_3. Take combinations of those vectors to produce v_1, v_2, v_3. Write the combinations in a matrix form as $V = WM$.

$$\begin{aligned} v_1 &= w_1 + w_2 \\ v_2 &= w_1 + 2w_2 + w_3 \\ v_1 &= w_2 + cw_3 \end{aligned} \quad \text{which is } \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix}$$

What is the test on a matrix **V** to see if its columns are linearly independent?

If $c \neq 1$ show that v_1, v_2, v_3 are linearly independent.

If $c = 1$ show that v's are linearly *dependent*.

Solution

The nullspace of **V** must contain only the *zero* vector. Then $x = (0,\ 0,\ 0)$ is the only combination of the columns that gives **V**x = zero vector.

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & c \end{bmatrix} \xrightarrow[R_3 - R_2]{R_1 - R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & \boxed{c-1} \end{bmatrix}$$

If $c \neq 1$, then the matrix M is invertible. So if x is any nonzero vector we know that Mx is nonzero. Since $\boldsymbol{w}$'s are given as independent and WMx is nonzero. Since $V = WM$, this says that x is not in the nullspace of **V**. therefore; v_1, v_2, v_3 are independent.

If $c = 1$, that implies
$$\begin{aligned} v_1 &= w_1 + w_2 \\ v_2 &= w_1 + w_2 + w_2 + w_3 \\ v_3 &= w_2 + w_3 \end{aligned} \Rightarrow \begin{aligned} v_1 &= w_1 + w_2 \\ &\boxed{v_2 = v_1 + v_3} \\ v_3 &= w_2 + w_3 \end{aligned}$$

$-v_1 + v_2 - v_3 = 0$, which means that $v's$ are linearly *dependent*.

The other way, the vector $x = (1,\ -1,\ 1)$ is in that nullspace, and $Mx = 0$. Then certainly $WMx = 0$ which is the same as $Vx = 0$. So the $v's$ are dependent.

Exercise

Find the largest possible number of independent vectors among

$$v_1=\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}\quad v_2=\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}\quad v_3=\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\quad v_4=\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\quad v_5=\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\quad v_6=\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$$

Solution

Since $v_4=v_2-v_1,\ v_5=v_3-v_1,\ and\ \ v_6=v_3-v_2$, there are at most three independent vectors among these: furthermore, applying row reduction to the matrix $\begin{bmatrix}v_1 & v_2 & v_3\end{bmatrix}$ gives three pivots, showing that v_1, v_2, v_3 are independent.

Exercise

Show that $v_1,\ v_2,\ v_3$ are independent but $v_1,\ v_2,\ v_3,\ v_4$ are dependent:

$$v_1=\begin{pmatrix}1\\0\\0\end{pmatrix}\quad v_2=\begin{pmatrix}1\\1\\0\end{pmatrix}\quad v_3=\begin{pmatrix}1\\1\\1\end{pmatrix}\quad v_4=\begin{pmatrix}2\\3\\4\end{pmatrix}$$

Solve either $c_1v_1+c_2v_2+c_3v_3=0$ *or* $Ax=0$. The v's go in the columns of ***A***.

Solution

$$\begin{pmatrix}v_1 & v_2 & v_3\end{pmatrix}=\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$$

This matrix has 3 pivots with rank of 3 equals to rows that implies the $v_1,\ v_2,\ v_3$ are independent.

$v_4=v_1+v_2-4v_3$ *or* $v_1+v_2-4v_3-v_4=0$ that shows that $v_1,\ v_2,\ v_3,\ v_4$ are dependent.

Exercise

Decide the dependence or independence of

a) The vectors (1, 3, 2) and (2, 1, 3) and (3, 2, 1).

b) The vectors $(1,\ -3,\ 2)$ and $(2,\ 1,\ -3)$ and $(-3,\ 2,\ 1)$.

Solution

a) These are linearly independent. $x_1(1,\ 3,\ 2)+x_2(2,\ 1,\ 3)+x_3(3,\ 2,\ 1)=(0,\ 0,\ 0)$ only if $x_1=x_2=x_3=0$

b) These are linearly dependent: $1(1,\ -3,\ 2)+1(2,\ 1,\ -3)+1(-3,\ 2,\ 1)=(0,\ 0,\ 0)$

Exercise

Find two independent vectors on the plane $x+2y-3z-t=0$ in $\boldsymbol{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

Solution

This plane is the nullspace of the matrix

$$A=\begin{pmatrix}1 & 2 & -3 & -1\end{pmatrix}$$

$x_1+2x_2-3x_3-x_4=0$

The pivot is 1st column, and the rest are 3 variables.

If $x_2=-1 \quad x_3=x_4=0 \Rightarrow x_1=2$. The vector is $(2,\ -1,\ 0,\ 0)$

If $x_3=1 \quad x_1=x_4=0 \Rightarrow x_1=3$. The vector is $(3,\ 0,\ 1,\ 0)$

If $x_4=1 \quad x_1=x_3=0 \Rightarrow x_1=1$. The vector is $(1,\ 0,\ 0,\ 1)$

The 3 vectors $(2,\ -1,\ 0,\ 0)$, $(3,\ 0,\ 1,\ 0)$, $(1,\ 0,\ 0,\ 1)$ are linearly independent.

We can't find 4 independent vectors because the nullspace only has dimension 3 (have 3 variables).

Exercise

Determine whether the vectors are linearly dependent or linearly independent in $\boldsymbol{R}^3$

a) (4, –1, 2), (–4, 10, 2)

b) (8, –1, 3), (4, 0, 1)

c) (–3, 0, 4), (5, –1, 2), (1, 1, 3)

d) (–2, 0, 1), (3, 2, 5), (6, –1, 1), (7, 0, –2)

Solution

a) The vector equation $a(4, -1, 2) + b(-4, 10, 2) = (0, 0, 0)$

$$\left[\begin{array}{cc|c} 4 & -4 & 0 \\ -1 & 10 & 0 \\ 2 & 2 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system has only the trivial solution $a = b = 0$.
We conclude that the given set of vectors is linearly independent.

b) $a(8, -1, 3) + b(4, 0, 1) = (0, 0, 0)$

$$\left[\begin{array}{cc|c} 8 & 4 & 0 \\ -1 & 0 & 0 \\ 3 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, the system has only one trivial solution $a = b = 0$.
We conclude that the given set of vectors is linearly independent

c) The vector equation $a(-3, 0, 4) + b(5, -1, 2) + c(1, 1, 3) = (0, 0, 0)$

$$\left[\begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore the system has only the trivial solution $a = b = c = 0$.
We conclude that the given set of vectors is linearly independent.

d) The vector equation $a(-2, 0, 1) + b(3, 2, 5) + c(6, -1, 1) + d(7, 0, -2) = (0, 0, 0)$

$$\left[\begin{array}{cccc|c} -2 & 3 & 6 & 7 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 1 & 5 & 1 & -2 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{79}{29} & 0 \\ 0 & 1 & 0 & \frac{3}{29} & 0 \\ 0 & 0 & 1 & \frac{6}{29} & 0 \end{array}\right]$$

Therefore the system has nontrivial solutions $a = \frac{79}{29}t, \quad b = -\frac{3}{29}t, \quad c = -\frac{6}{29}t, \quad d = t$

We conclude that the given set of vectors is linearly dependent.

Exercise

Determine whether the vectors are linearly dependent or linearly independent in $\boldsymbol{R}^4$

a) (3, 8, 7, –3), (1, 5, 3, –1), (2, –1, 2, 6), (1, 4, 0, 3)

b) (0, 0, 2, 2), (3, 3, 0, 0), (1, 1, 0, –1)

c) (0, 3, –3, –6), (–2, 0, 0, –6), (0, –4, –2, –2), (0, –8, 4, –4)

d) (3, 0, –3, 6), (0, 2, 3, 1), (0, –2, –2, 0), (–2, 1, 2, 1)

Solution

a) $\det\begin{pmatrix} 3 & 1 & 2 & 1 \\ 8 & 5 & -1 & 4 \\ 7 & 3 & 2 & 0 \\ -3 & -1 & 6 & 3 \end{pmatrix} \underline{= 128 \neq 0}$

The system has only the trivial solution and the vectors are linearly independent.

b) $k_1(0,0,2,2) + k_2(3,3,0,0) + k_3(1,1,0,-1) = (0,0,0,0)$

$$\left[\begin{array}{ccc|c} 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 = k_2 = k_3 = 0$

The system has only the trivial solution and the vectors are linearly independent.

c) $\det\begin{pmatrix} 0 & -2 & 0 & 0 \\ 3 & 0 & -4 & -8 \\ -3 & 0 & -2 & 4 \\ -6 & -6 & -2 & -4 \end{pmatrix} \underline{= 480 \neq 0}$

The system has only the trivial solution and the vectors are linearly independent.

d) $a(3,\ 0,\ -3,\ 6) + b(0,\ 2,\ 3,\ 1) + c(0,\ -2,\ -2,\ 0) + d(-2,\ 1,\ 2,\ 1) = (0,\ 0,\ 0,\ 0)$

$$\left[\begin{array}{cccc|c} 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & 1 & 0 \\ -3 & 3 & -2 & 2 & 0 \\ 6 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore, the system has only one trivial solution $a = b = c = d = 0$.

The given set of vectors is linearly independent

Exercise

a) Show that the three vectors $v_1 = (1,2,3,4)$ $v_2 = (0,1,0,-1)$ $v_3 = (1,3,3,3)$ form a linearly dependent set in $\boldsymbol{R}^4$.

b) Express each vector in part (*a*) as a linear combination of the other two.

Solution

a) The vector equation $k_1(1,2,3,4) + k_2 = (0,1,0,-1) + k_3(1,3,3,3) = (0,0,0,0)$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ 3 & 0 & 3 & 0 \\ 4 & -1 & 3 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The solution: $k_1 = -t,\ k_2 = -t,\ k_3 = t$

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

b) Since $k_1 = -t,\ k_2 = -t,\ k_3 = t$ and if we let $t = 1$, then $-v_1 - v_2 + v_3 = 0$

$$v_1 = -v_2 + v_3, \quad v_2 = -v_1 + v_3, \quad v_3 = v_1 + v_2$$

Exercise

For which real values of λ do the following vectors form a linearly dependent set in $\boldsymbol{R}^3$

$$v_1 = \left(\lambda, -\tfrac{1}{2}, -\tfrac{1}{2}\right) \quad v_2 = \left(-\tfrac{1}{2}, \lambda, -\tfrac{1}{2}\right) \quad v_3 = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, \lambda\right)$$

Solution

$$k_1\left(\lambda, -\tfrac{1}{2}, -\tfrac{1}{2}\right) + k_2 = \left(-\tfrac{1}{2}, \lambda, -\tfrac{1}{2}\right) + k_3\left(-\tfrac{1}{2}, -\tfrac{1}{2}, \lambda\right) = (0,0,0,0)$$

$$\det\begin{pmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{pmatrix} = \frac{1}{4}\left(4\lambda^3 - 3\lambda - 1\right)$$

For $\lambda = 1$ $\lambda = -\frac{1}{2}$, the determinant is zero and the vectors form a linearly dependent set.

Exercise

Show that if $S = \{v_1, v_2, \ldots, v_n\}$ is a linearly independent set of vectors, then so is every nonempty subset of S.

Solution

Let $\{v_a, v_b, \ldots, v_r\}$ be a nonempty subset of *S*.

If this set is linearly dependent, then there would be a nonzero solution $(k_a, k_b, \ldots, k_r)$ to $k_a v_a + k_b v_b + \ldots + k_r v_r = 0$. This can be expanded to a nonzero solution of $k_1 v_1 + k_2 v_2 + \ldots + k_n v_n = 0$ by taking all other coefficients as 0. This contradicts the linear independence of S, so the subset must be linearly independent.

Exercise

Show that if $S = \{v_1, v_2, \ldots, v_r\}$ is a linearly dependent set of vectors in a vector space *V*, and if $v_{r+1}, \ldots, v_n$ are vectors in *V* that are not in *S*, then $\{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ is also linearly dependent.

Solution

If *S* is linearly dependent, then there is a nonzero solution $(k_1, k_2, \ldots, k_r)$ to $k_1 v_1 + k_2 v_2 + \ldots + k_r v_r = 0$. Thus $(k_1, k_2, \ldots, k_r, 0, 0, \ldots, 0)$ is a nonzero solution to $k_1 v_1 + k_2 v_2 + \ldots + k_r v_r + k_{r+1} v_{r+1} \ldots + k_n v_n = 0$ so the set $\{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ is linearly dependent.

Exercise

Show that $\{v_1, v_2\}$ is linearly independent and v_3 does not lie in span $\{v_1, v_2\}$, then $\{v_1, v_2, v_3\}$ is a linearly independent.

Solution

If $\{v_1, v_2, v_3\}$ are linearly dependent, there exist a nonzero solution to $k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$ with $k_3 \neq 0$ (since v_1 and v_2 are linearly independent).

$k_3 v_3 = -k_1 v_1 - k_2 v_2 \quad \Rightarrow \quad v_3 = -\frac{k_1}{k_3} v_1 - \frac{k_2}{k_3} v_2$ which contradicts that v_3 is not in span $\{v_1, v_2\}$.

Thus $\{v_1, v_2, v_3\}$ is a linearly independent.

Exercise

By using the appropriate identities, where required, determine $F(-\infty,\ \infty)$ are linearly dependent.

a) $6,\ 3\sin^2 x,\ 2\cos^2 x$

b) $x,\ \cos x$

c) $1,\ \sin x,\ \sin 2x$

d) $(3-x)^2,\ x^2-6x,\ 5$

e) $\cos 2x,\ \sin^2 x,\ \cos^2 x$

Solution

a) From the identity $\sin^2 x+\cos^2 x=1$

$$(-1)(6)+(2)\left(3\sin^2 x\right)+(3)\left(2\cos^2 x\right)=-6+6\left(\sin^2 x+\cos^2 x\right)\ \underline{=0}$$

Therefore, the set is linearly dependent.

b) $ax+b\cos x=0$

$$x=0\quad\Rightarrow\quad b=0$$

$$x=\frac{\pi}{2}\quad\Rightarrow\quad a=0$$

Therefore, the set is linearly independent.

c) $a(1)+b\sin x+c\sin 2x=0$

$$x=0\quad\Rightarrow\quad a=0$$

$$x=\frac{\pi}{2}\quad\Rightarrow\quad b=0$$

$$x=\frac{\pi}{4}\quad\Rightarrow\quad c=0$$

Therefore, the set is linearly independent.

d) $(3-x)^2=9-6x+x^2$

$$(3-x)^2-\left(9-6x+x^2\right)=0$$

$$(3-x)^2-\left(x^2-6x\right)-9=0$$

$$(1)(3-x)^2+(-1)\left(x^2-6x\right)+\left(-\frac{9}{5}\right)5=0$$

Therefore, the set is linearly dependent.

e) By using the double angle:

$\cos 2x=\cos^2 x-\sin^2 x$ are linearly dependent.

Exercise

$f_1(x)=\sin x, \quad f_2(x)=\cos x$ are linearly independent in $F(-\infty, \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using Wroński's test.

Solution

The Wronskian: $W(x)=\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$

$$=-\sin^2 x-\cos^2 x$$

$$=-\left(\sin^2 x+\cos^2 x\right)$$

$$\boxed{=-1\neq 0}$$

$\sin x$ *and* $\cos x$ are linearly independent

Exercise

Use the Wronskian to show that $f_1(x)=\sin x, \quad f_2(x)=\cos x, \quad f_3(x)=x\cos x$ span a three-dimensional subspace of $F(-\infty, \infty)$

Solution

The Wronskian: $W(x)=\begin{vmatrix} \sin x & \cos x & x\cos x \\ \cos x & -\sin x & \cos x-x\sin x \\ -\sin x & -\cos x & -2\sin x-x\cos x \end{vmatrix}$

$$=2\sin^3 x+x\sin^2 x\cos x-\sin x\cos^2 x+x\sin^2 x\cos x-x\cos^3 x$$
$$-x\sin^2 x\cos x+\sin x\cos^2 x-x\sin^2 x\cos x+2\sin x\cos^2 x+x\cos^3 x$$

$$=2\sin^3 x+2\sin x\cos^2 x$$

$$=2\sin x\left(\sin^2 x+\cos^2 x\right)$$

$$\boxed{=2\sin x}$$

Since $\sin x\neq 0$ for all real x values, the vectors are linearly independent.

Exercise

Show by inspection that the vectors are linearly dependent.

$\mathbf{v}_1(4, -1, 3), \quad \mathbf{v}_2(2, 3, -1), \quad \mathbf{v}_3(-1, 2, -1), \quad \mathbf{v}_4(5, 2, 3), \quad$ *in* $\mathbb{R}^3$

Solution

$$\begin{bmatrix} 4 & 2 & -1 & 5 \\ -1 & 3 & 2 & 2 \\ 3 & -1 & -1 & 3 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & \frac{11}{7} \\ 0 & 1 & 0 & \frac{1}{7} \\ 0 & 0 & 1 & \frac{11}{7} \end{bmatrix}$$

$$7\boldsymbol{v}_4 = 11\boldsymbol{v}_1 + \boldsymbol{v}_2 + 11\boldsymbol{v}_2$$

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

a) $(2,\ -1,\ 3),\ \ (3,\ 4,\ 1),\ \ (2,\ -3,\ 4),\ \ in\ \mathbb{R}^3.$

b) $(1,\ 0,\ 0,\ 0),\ \ (1,\ 1,\ 0,\ 0),\ \ (1,\ 1,\ 1,\ 0),\ \ (1,\ 1,\ 1,\ 1),\ \ in\ \mathbb{R}^4.$

c) $A_1\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix},\ \ A_2\begin{bmatrix} -2 & 4 \\ 1 & 0 \end{bmatrix},\ \ A_3\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix},\ \ in\ M_{22}$

Solution

a) $a(2,\ -1,\ 3) + b(3,\ 4,\ 1) + c(2,\ -3,\ 4) = (0,\ 0,\ 0)$

$$\begin{bmatrix} 2 & 3 & 2 & 0 \\ -1 & 4 & -3 & 0 \\ 3 & 1 & 4 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The system has only he trivial solution $a = b = c = 0$.

$$\begin{vmatrix} 2 & 3 & 2 \\ -1 & 4 & -3 \\ 3 & 1 & 4 \end{vmatrix} = 32 - 27 - 2 - 24 + 6 + 12 \neq 0$$

The system has only the trivial solution and the vectors are linearly independent

b) $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

The system has only the trivial solution and the vectors are linearly independent

c) $$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 2 & 4 & -1 & 0 \\ 3 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ The vectors are linearly independent

Exercise

Suppose that the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \textit{ and } \boldsymbol{u}_3$ are linearly dependent. Are the vectors $\boldsymbol{v}_1 = \boldsymbol{u}_1 + \boldsymbol{u}_2$, $\boldsymbol{v}_2 = \boldsymbol{u}_1 + \boldsymbol{u}_3$, and $\boldsymbol{v}_3 = \boldsymbol{u}_2 + \boldsymbol{u}_3$ also linearly dependent?

(***Hint***: Assume that $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 = 0$, and see what the $a_i\text{'}s$ can be.)

Solution

Given: $\boldsymbol{u}_1, \boldsymbol{u}_2, \textit{ and } \boldsymbol{u}_3$ are linearly dependent, then there are scalar $b_1, b_2, \textit{ and } b_3$ such that $b_1\boldsymbol{u}_1 + b_2\boldsymbol{u}_2 + b_3\boldsymbol{u}_3 = 0$.

Assume that $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 = 0$

$$a_1\left(\boldsymbol{u}_1 + \boldsymbol{u}_2\right) + a_2\left(\boldsymbol{u}_1 + \boldsymbol{u}_3\right) + a_3\left(\boldsymbol{u}_2 + \boldsymbol{u}_3\right) = 0$$

$$a_1\boldsymbol{u}_1 + a_1\boldsymbol{u}_2 + a_2\boldsymbol{u}_1 + a_2\boldsymbol{u}_3 + a_3\boldsymbol{u}_2 + a_3\boldsymbol{u}_3 = 0$$

$$\left(a_1 + a_2\right)\boldsymbol{u}_1 + \left(a_1 + a_3\right)\boldsymbol{u}_2 + \left(a_2 + a_3\right)\boldsymbol{u}_3 = 0$$

If $a_1 + a_2 = b_1 \quad a_1 + a_3 = b_2 \quad a_2 + a_3 = b_3$ and since $\boldsymbol{u}_1, \boldsymbol{u}_2, \textit{ and } \boldsymbol{u}_3$ are linearly dependent, therefore, $\boldsymbol{v}_1, \boldsymbol{v}_2, \textit{ and } \boldsymbol{v}_3$ are linearly dependent