Lecture Three

Section 3.1 – Inner Products

Definition

An ***inner product*** on a real vector space V is a function that associates a real number $\langle \boldsymbol{u},\ \boldsymbol{v} \rangle$ with each pair of vectors in V in such a way that the following axioms are satisfies for all vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ in V and all scalars k.

1. $\langle \boldsymbol{u},\ \boldsymbol{v} \rangle = \langle \boldsymbol{v},\ \boldsymbol{u} \rangle$ — ***Symmetry axiom***
2. $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w} \rangle = \langle \boldsymbol{u},\ \boldsymbol{w} \rangle + \langle \boldsymbol{v},\ \boldsymbol{w} \rangle$ — ***Additivity axiom***
3. $\langle k\boldsymbol{u},\ \boldsymbol{v} \rangle = k\langle \boldsymbol{u},\ \boldsymbol{v} \rangle$ — ***Homogeneity axiom***
4. $\langle \boldsymbol{v},\ \boldsymbol{v} \rangle \ge 0$ and $\langle \boldsymbol{v},\ \boldsymbol{v} \rangle = 0$ iff $\boldsymbol{v} = 0$ — ***Positivity axiom***

A real vector space with an inner product is called a ***real inner product space***.

$$\boxed{\langle \boldsymbol{u},\ \boldsymbol{v} \rangle = \boldsymbol{u}\cdot\boldsymbol{v} = u_1v_1 + u_2v_2 + \cdots u_nv_n}$$

This is called the ***Euclidean inner product*** (or the ***standard inner product***)

Definition

If V is a real inner product space, then the norm (or length) of a vector $\vec{v}$ in V is denoted by $\|\vec{v}\|$ and is defined by

$$\|\vec{v}\| = \sqrt{\langle \vec{v},\ \vec{v} \rangle}$$

And the ***distance*** between two vectors is denoted by $d(\boldsymbol{u},\ \boldsymbol{v})$ and is defined by

$$d(\boldsymbol{u},\ \boldsymbol{v}) = \|\boldsymbol{u}-\boldsymbol{v}\| = \sqrt{\langle \boldsymbol{u}-\boldsymbol{v},\ \boldsymbol{u}-\boldsymbol{v} \rangle}$$

A vector of norm 1 is called a ***unit vector***.

Theorem

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in a real inner product space V, and if k is a scalar, then:

a) $\|\boldsymbol{v}\| \ge 0$ with equality *iff* $\boldsymbol{v} = 0$

b) $\|k\boldsymbol{v}\| = |k|\|\boldsymbol{v}\|$

c) $d(\boldsymbol{u},\ \boldsymbol{v}) = d(\boldsymbol{v},\ \boldsymbol{u})$

d) $d(\boldsymbol{u},\ \boldsymbol{v}) \ge 0$ with equality *iff* $\boldsymbol{u} = \boldsymbol{v}$

Although the Euclidean inner product is the most important inner product on R^n, there are various applications in which is desirable to modify it by weighing each term differently. More precisely, if $w_1, w_2, \ldots, w_n$ are positive real numbers, which we will call weighs, and if $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and are vectors in R^n, then it can be shown that the formula

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \ldots + w_n u_n v_n$$

Defines an inner product on R^n that we call the ***weighted Euclidean inner product*** with weights $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n$

Example

Let $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$ be vectors in R^2, verify that the weighted Euclidean inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 3u_1 v_1 + 2u_2 v_2$ satisfies the four inner product axioms.

Solution

Axiom 1: $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 3u_1 v_1 + 2u_2 v_2 = 3v_1 u_1 + 2v_2 u_2 = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$

Axiom 2:
$$\begin{aligned}
\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle &= 3(u_1 + v_1)w_1 + 2(u_2 + v_2)w_2 \\
&= 3(u_1 w_1 + v_1 w_1) + 2(u_2 w_2 + v_2 w_2) \\
&= 3u_1 w_1 + 3v_1 w_1 + 2u_2 w_2 + 2v_2 w_2 \\
&= (3u_1 w_1 + 2u_2 w_2) + (3v_1 w_1 + 2v_2 w_2) \\
&= \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle
\end{aligned}$$

Axiom 3:
$$\begin{aligned}
\langle k\boldsymbol{u}, \boldsymbol{v} \rangle &= 3(ku_1)v_1 + 2(ku_2)v_2 \\
&= k(3u_1 v_1 + 2u_2 v_2) \\
&= k\langle \boldsymbol{u}, \boldsymbol{v} \rangle
\end{aligned}$$

Axiom 4:
$$\begin{aligned}
\langle \boldsymbol{v}, \boldsymbol{v} \rangle &= 3v_1 v_1 + 2v_2 v_2 \\
&= 3v_1^2 + 2v_2^2 \ge 0
\end{aligned}$$

$$v_1 = v_2 = 0 \quad \textit{iff} \quad \boldsymbol{v} = \boldsymbol{0}$$

Exercises *Section* 3.1 – Inner Products

1. Let $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ be the Euclidean inner product on R^2, and let $\boldsymbol{u} = (1,\ 1)$, $\boldsymbol{v} = (3,\ 2)$, $\boldsymbol{w} = (0,\ -1)$, and $k = 3$. Compute the following.

a) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ *c)* $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle$ *e)* $d(\boldsymbol{u}, \boldsymbol{v})$

b) $\langle k\boldsymbol{v}, \boldsymbol{w} \rangle$ *d)* $\|\boldsymbol{v}\|$ *f)* $\|\boldsymbol{u} - k\boldsymbol{v}\|$

2. Let $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ be the Euclidean inner product on R^2, and let $\boldsymbol{u} = (1,\ 1)$, $\boldsymbol{v} = (3,\ 2)$, $\boldsymbol{w} = (0,\ -1)$ and $k = 3$. Compute the following for the weighted Euclidean inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 2u_1v_1 + 3u_2v_2$.

a) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ *c)* $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle$ *e)* $d(\boldsymbol{u}, \boldsymbol{v})$

b) $\langle k\boldsymbol{v}, \boldsymbol{w} \rangle$ *d)* $\|\boldsymbol{v}\|$ *f)* $\|\boldsymbol{u} - k\boldsymbol{v}\|$

3. Let $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ be the Euclidean inner product on R^2, and let $\boldsymbol{u} = (3,\ -2)$, $\boldsymbol{v} = (4,\ 5)$, $\boldsymbol{w} = (-1,\ 6)$, and $k = -4$. Verify the following.

a) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$ *d)* $\langle k\boldsymbol{u}, \boldsymbol{v} \rangle = k\langle \boldsymbol{u}, \boldsymbol{v} \rangle$

b) $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ *e)* $\langle \boldsymbol{0}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{0} \rangle = 0$

c) $\langle \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{u}, \boldsymbol{w} \rangle$

4. Let $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ be the Euclidean inner product on R^2, and let $\boldsymbol{u} = (3,\ -2)$, $\boldsymbol{v} = (4,\ 5)$, $\boldsymbol{w} = (-1,\ 6)$, and $k = -4$. Verify the following for the weighted Euclidean inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 4u_1v_1 + 5u_2v_2$

a) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$ *d)* $\langle k\boldsymbol{u}, \boldsymbol{v} \rangle = k\langle \boldsymbol{u}, \boldsymbol{v} \rangle$

b) $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ *e)* $\langle \boldsymbol{0}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{0} \rangle = 0$

c) $\langle \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{u}, \boldsymbol{w} \rangle$

5. Let $\boldsymbol{u} = (u_1,\ u_2)$ and $\boldsymbol{v} = (v_1,\ v_2)$. Show that the following are inner product on R^3 by verifying that the inner product axioms hold. $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 3u_1v_1 + 5u_2v_2$

6. Show that the following identity holds for the vectors in any inner product space

$$\|\boldsymbol{u} + \boldsymbol{v}\|^2 + \|\boldsymbol{u} - \boldsymbol{v}\|^2 = 2\|\boldsymbol{u}\|^2 + 2\|\boldsymbol{v}\|^2$$

7. Show that the following identity holds for the vectors in any inner product space

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \frac{1}{4}\left(\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \|\boldsymbol{u}-\boldsymbol{v}\|^2 \right)$$

8. Prove that $\|k\vec{v}\| = |k| \ \|\vec{v}\|$

Section 3.2 – Angle and Orthogonality in Inner Product Spaces

Cosine Formula

If u and v are nonzero vectors that implies $\Rightarrow$ $\cos\theta = \dfrac{\vec{u}\bullet\vec{v}}{\|\vec{u}\|.\|\vec{v}\|}$ $\rightarrow$ $\boxed{\theta = \cos^{-1}\left(\dfrac{\langle \boldsymbol{u},\boldsymbol{v}\rangle}{\|\boldsymbol{u}\|.\|\boldsymbol{v}\|}\right)}$

$$-1 \le \frac{\boldsymbol{u}.\boldsymbol{v}}{\|\boldsymbol{u}\|.\|\boldsymbol{v}\|} \le 1$$

Example

Let R^4 have the Euclidean inner product. Find the cosine angle θ between the vectors $\boldsymbol{u} = (4,\ 3,\ 1,\ -2)$ and $\boldsymbol{v} = (-2,\ 1,\ 2,\ 3)$.

Solution

$$\|\boldsymbol{u}\| = \sqrt{4^2+3^2+1^2+(-2)^2} = \underline{\sqrt{30}}$$

$$\|\boldsymbol{v}\| = \sqrt{(-2)^2+1^2+2^2+3^2} = \sqrt{18} = \underline{3\sqrt{2}}$$

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle = 4(-2)+3(1)+1(2)-2(3) = \underline{-9}$$

$$\begin{aligned}\cos\theta &= \frac{\langle \boldsymbol{u},\boldsymbol{v}\rangle}{\|\boldsymbol{u}\|.\|\boldsymbol{v}\|}\\ &= -\frac{9}{3\sqrt{30}\sqrt{2}}\\ &= -\frac{3}{\sqrt{60}}\\ &= -\frac{3}{2\sqrt{15}}\end{aligned}$$

Theorem – Cauchy-Schwarz Inequality

If $\vec{v}$ and $\vec{w}$ are vectors in a real inner product space V, then

$$\|\langle \boldsymbol{u},\boldsymbol{v}\rangle\| \le \|\boldsymbol{u}\|.\|\boldsymbol{v}\|$$

The following two alternative forms of the Cauchy-Schwarz inequality are useful to know:

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle^2 \le \langle \boldsymbol{u},\boldsymbol{u}\rangle\langle \boldsymbol{v},\boldsymbol{v}\rangle$$

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle^2 \le \|\boldsymbol{u}\|^2.\|\boldsymbol{v}\|^2$$

Theorem

If $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors in a real inner product space *V*, and if *k* is any scalar, then

a) $\|\boldsymbol{u}+\boldsymbol{v}\| \le \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ (***Triangle inequality for vectors***)

b) $d(\boldsymbol{u},\boldsymbol{v}) \le d(\boldsymbol{u},\boldsymbol{w}) + d(\boldsymbol{w},\boldsymbol{v})$ (***Triangle inequality for distances***)

Proof **(*a*)**

$$\begin{aligned}\|\boldsymbol{u}+\boldsymbol{v}\|^2 &= \langle \boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}+\boldsymbol{v}\rangle \\ &= \langle \boldsymbol{u},\boldsymbol{u}\rangle + 2\langle \boldsymbol{u},\boldsymbol{v}\rangle + \langle \boldsymbol{v},\boldsymbol{v}\rangle \\ &\le \langle \boldsymbol{u},\boldsymbol{u}\rangle + 2\left|\langle \boldsymbol{u},\boldsymbol{v}\rangle\right| + \langle \boldsymbol{v},\boldsymbol{v}\rangle \\ &\le \langle \boldsymbol{u},\boldsymbol{u}\rangle + 2\|\boldsymbol{u}\|\|\boldsymbol{v}\| + \langle \boldsymbol{v},\boldsymbol{v}\rangle \\ &= \|\boldsymbol{u}\|^2 + 2\|\boldsymbol{u}\|\|\boldsymbol{v}\| + \|\boldsymbol{v}\|^2 \\ &= \left(\|\boldsymbol{u}\| + \|\boldsymbol{v}\|\right)^2\end{aligned}$$

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 \le \left(\|\boldsymbol{u}\| + \|\boldsymbol{v}\|\right)^2$$

$$\|\boldsymbol{u}+\boldsymbol{v}\| \le \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$$

Definition

Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in an inner product space are called orthogonal if $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = 0$

Example

The vectors $\boldsymbol{u} = (1,\ 1)$ and $\boldsymbol{v} = (1,\ -1)$ are orthogonal with respect to the Euclidean inner product on R^2, since

$$\boldsymbol{u} \bullet \boldsymbol{v} = 1(1) + 1(-1) = 0$$

They are not orthogonal with the respect to the weighted Euclidean inner product $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = 3u_1v_1 + 2u_2v_2$, since

$$\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = 3(1)(1) + 2(1)(-1) = 1 \ne 0$$

Example

$U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ *and* $V = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ are orthogonal, since

$$U \cdot V = 1(0) + 0(2) + 1(0) + 1(0) = \underline{0|}$$

Definition

If W is a subspace of an inner product space V, then the set of all vectors are orthogonal to every vector in W is called the ***orthogonal complement*** of W and is denoted by the symbol $W^{\perp}$

Theorem

If W is a subspace of an inner product space V, then:

a) $W^{\perp}$ is a subspace of V.

b) $W \cap W^{\perp} = \{0\}$

Proof

a) Let set $W^{\perp}$ contains at least the zero vector, since $\langle \mathbf{0}, \boldsymbol{w} \rangle = 0$ for every vector $\boldsymbol{w}$ in W. We need to show that $W^{\perp}$ is closed under addition and scalar multiplication.

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $W^{\perp}$, so every vector $\boldsymbol{w}$ in W we have $\langle \boldsymbol{u}, \boldsymbol{w} \rangle = 0$ and $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0$

$\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0 + 0 = 0$ ***Closed under addition***

$\langle k\boldsymbol{u}, \boldsymbol{w} \rangle = k\langle \boldsymbol{u}, \boldsymbol{w} \rangle = k(0) = 0$ ***Closed under scalar multiplication***

Which proves that $\boldsymbol{u} + \boldsymbol{w}$ and $k\boldsymbol{u}$ are in $W^{\perp}$

b) If $\boldsymbol{v}$ is any vector in both W and $W^{\perp}$, then $\boldsymbol{v}$ is orthogonal to itself; that is, $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$. It follows from the positivity axiom for inner products that $\boldsymbol{v} = 0$

Theorem

If W is a subspace of a finite-dimensional inner product space V, then the orthogonal complement of $W^{\perp}$ is W; that is

$$\left(W^{\perp}\right)^{\perp} = W$$

Example

Let W be the subspace of R^6 spanned by the vectors

$$\boldsymbol{w}_1 = (1,\ 3,\ -2,\ 0,\ 2,\ 0), \qquad \boldsymbol{w}_2 = (2,\ 6,\ -5,\ -2,\ 4,\ -3)$$
$$\boldsymbol{w}_3 = (0,\ 0,\ 5,\ 10,\ 0,\ 15), \qquad \boldsymbol{w}_4 = (2,\ 6,\ 0,\ 8,\ 4,\ 18)$$

Find a basis for the orthogonal complement of W.

Solution

The Space W is the same as the row space of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution

$$\begin{aligned}(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6) &= (-3x_2 - 4x_4 - 2x_5,\ x_2,\ -2x_4,\ x_4,\ x_5,\ 0) \\ &= x_2(-3,\ 1,\ 0,\ 0,\ 0,\ 0) + x_4(-4,0,-2,\ 1,\ 0,\ 0) + x_5(-2,\ 0,\ 0,\ 0,\ 1,\ 0)\end{aligned}$$

$$\vec{v}_1 = (-3,\ 1,\ 0,\ 0,\ 0,\ 0), \quad \vec{v}_2 = (-4,\ 0,\ -2,\ 1,\ 0,\ 0), \quad \vec{v}_3 = (-2,\ 0,\ 0,\ 0,\ 1,\ 0)$$

Definition

A collection of vectors in $\mathbb{R}^n$ (or inner space) is called orthogonal if any 2 are perpendicular.

$$v_i . v_j = v^T v = \begin{cases} 0 & for\ i \neq j \quad (orthogonal\ vectors) \\ 1 & for\ i = j \quad (unit\ vectors) \end{cases}$$

Theorem

If $v_1,\ \ldots,\ v_m$ are nonzero orthogonal vectors, then they are linearly independent.

Definition

A vector $\boldsymbol{v}$ is called normal if $\|v\| = 1$

A collection of vectors $v_1,\ \ldots,\ v_m$ is called orthonormal if they are orthogonal and each $\|v_i\| = 1$.

An orthonormal basis is a basis made up of orthonormal vectors.

Example

Q rotates every vector in the plane through the angle θ.

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$Q^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$\cos^2\theta + \sin^2\theta = 1$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\boxed{= Q^T}$$

The dot product $(\cos\theta\sin\theta - \sin\theta\cos\theta = 0)$, the columns are orthogonal.

They are unit vectors because $\cos^2\theta + \sin^2\theta = 1$. Those columns give an orthonormal basis for the plane $\boldsymbol{R}^2$.

We have: $QQ^T = I = Q^T Q$ (This type is called ***rotation***)

Exercises *Section* 3.2 – Angle and Orthogonality in Inner Product Spaces

1. Which of the following form orthonormal sets?

a) $(1, 0), (0, 2)$ in $\boldsymbol{R}^2$

b) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\boldsymbol{R}^2$

c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\boldsymbol{R}^2$

d) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ in $\boldsymbol{R}^3$

e) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ in $\boldsymbol{R}^3$

f) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ in $\boldsymbol{R}^3$

2. Find the cosine of the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

a) $\boldsymbol{u} = (1, -3), \quad \boldsymbol{v} = (2, 4)$

b) $\boldsymbol{u} = (-1, 0), \quad \boldsymbol{v} = (3, 8)$

c) $\boldsymbol{u} = (-1, 5, 2), \quad \boldsymbol{v} = (2, 4, -9)$

d) $\boldsymbol{u} = (4, 1, 8), \quad \boldsymbol{v} = (1, 0, -3)$

e) $\boldsymbol{u} = (1, 0, 1, 0), \quad \boldsymbol{v} = (-3, -3, -3, -3)$

f) $\boldsymbol{u} = (2, 1, 7, -1), \quad \boldsymbol{v} = (4, 0, 0, 0)$

g) $\boldsymbol{u} = (1, 3, -5, 4), \quad \boldsymbol{v} = (2, -4, 4, 1)$

h) $\boldsymbol{u} = (1, 2, 3, 4), \quad \boldsymbol{v} = (-1, -2, -3, -4)$

3. Find the cosine of the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$.

a) $A = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$

b) $A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$

c) $A = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

d) $A = \begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$

4. Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

a) $\boldsymbol{u} = (-1, 3, 2), \quad \boldsymbol{v} = (4, 2, -1)$

b) $\boldsymbol{u} = (a, b), \quad \boldsymbol{v} = (-b, a)$

c) $\boldsymbol{u} = (-2, -2, -2), \quad \boldsymbol{v} = (1, 1, 1)$

d) $\boldsymbol{u} = (-4, 6, -10, 1), \quad \boldsymbol{v} = (2, 1, -2, 9)$

e) $\boldsymbol{u} = (-4, 6, -10, 1), \quad \boldsymbol{v} = (2, 1, -2, 9)$

5. Do there exist scalars k and l such that the vectors
$\boldsymbol{u} = (2, k, 6), \quad \boldsymbol{v} = (l, 5, 3), \quad \textit{and} \quad \boldsymbol{w} = (1, 2, 3)$ are mutually orthogonal with respect to the Euclidean inner product?

6. Let $\boldsymbol{R}^3$ have the Euclidean inner product. For which values of k are $\boldsymbol{u}$ and $\boldsymbol{v}$ orthogonal?

a) $\boldsymbol{u} = (2, 1, 3), \quad \boldsymbol{v} = (1, 7, k)$ *b)* $\boldsymbol{u} = (k, k, 1), \quad \boldsymbol{v} = (k, 5, 6)$

7. Let V be an inner product space. Show that if $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal unit vectors in V, then $\|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{2}$

8. Let $\mathbf{S}$ be a subspace of $\mathbb{R}^n$. Explain what $\left(\mathbf{S}^{\perp}\right)^{\perp} = \mathbf{S}$ means and why it is true.

9. The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0,\ 0,\ 0)$, $(1,\ 1,\ 0)$, $(1,\ 0,\ 1)$ and $(0,\ 1,\ 1)$ – (***note*** that all six edges have length $\sqrt{2}$, so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$ to the vertices?

10. Determine if the given vectors are orthogonal.

$$\boldsymbol{x}_1 = (1,\ 0,\ 1,\ 0), \quad \boldsymbol{x}_2 = (0,\ 1,\ 0,\ 1), \quad \boldsymbol{x}_3 = (1,\ 0,\ -1,\ 0), \quad \boldsymbol{x}_4 = (1,\ 1,\ -1,\ -1)$$

11. Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

a) $\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\ \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)\ \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$

b) $\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right)\ \left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)\ \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)$

12. Consider vectors $\vec{u} = (2,\ 3,\ 5)\quad \vec{v} = (1,\ -4,\ 3)\quad in\quad R^3$

a) $\langle \vec{u}, \vec{v} \rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine between $\vec{u}$ *and* $\vec{v}$

13. Consider vectors $\vec{u} = (1,\ 1,\ 1)\quad \vec{v} = (1,\ 2,\ -3)\quad in\quad R^3$

a) $\langle \vec{u}, \vec{v} \rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine θ between $\vec{u}$ *and* $\vec{v}$

14. Consider vectors $\vec{u} = (1,\ 2,\ 5)\quad \vec{v} = (2,\ -3,\ 5)\quad \vec{w} = (4,\ 2,\ -3)\quad in\quad R^3$

a) $\langle \vec{u}, \vec{v} \rangle$ *d)* $\|\vec{u}\|$ *g)* Cosine α between $\vec{u}$ *and* $\vec{v}$

b) $\langle \vec{u}, \vec{w} \rangle$ *e)* $\|\vec{v}\|$ *h)* Cosine β between $\vec{u}$ *and* $\vec{w}$

i) Cosine θ between $\vec{v}$ *and* $\vec{w}$

c) $\langle \vec{v}, \vec{w} \rangle$ *f)* $\|\vec{w}\|$ *j)* $(\vec{u} + \vec{v}) \bullet \vec{w}$

15. Consider polynomial $f(t) = 3t - 5;\quad g(t) = t^2 \quad in \quad \boldsymbol{P}(t)$

a) $\langle f, g \rangle$ *b)* $\|f\|$ *c)* $\|g\|$ *d)* Cosine between f *and* g

16. Consider polynomial $f(t) = t + 2\ ;\quad g(t) = 3t - 2\ ;\quad h(t) = t^2 - 2t - 3 \quad in \quad \boldsymbol{P}(t)$

a) $\langle f, g \rangle$ *d)* $\|f\|$ *g)* Cosine α between f *and* g

b) $\langle f, h \rangle$ *e)* $\|g\|$ *h)* Cosine β between f *and* h

c) $\langle g, h \rangle$ *f)* $\|h\|$ *i)* Cosine θ between g *and* h

17. Suppose $\langle \vec{u}, \vec{v} \rangle = 3 + 2\,i$ in a complex inner product space V. Find:

a) $\langle (2 - 4\,i)\vec{u}, \vec{v} \rangle$ *b)* $\langle \vec{u}, (4 + 3\,i)\vec{v} \rangle$ *c)* $\langle (3 - 6\,i)\vec{u}, (5 - 2\,i)\vec{v} \rangle$ *d)* $\|\vec{u}, \vec{v}\|$

18. Find the Fourier coefficient c and the projection $c\vec{v}$ of $\vec{u} = (3 + 4\,i,\ 2 - 3\,i)$ along $\vec{v} = (5 + i,\ 2i)$ in $\boldsymbol{C}^2$

19. Suppose $\vec{v} = (1,\ 3,\ 5,\ 7)$. Find the projection of $\vec{v}$ onto W or find $\vec{w} \in W$ that minimizes $\|\vec{v} - \vec{w}\|$, where W is the subspace of R^4 spanned by:

a) $\vec{u}_1 = (1,\ 1,\ 1,\ 1)$ *and* $\vec{u}_2 = (1,\ -3,\ 4,\ -2)$

b) $\vec{v}_1 = (1,\ 1,\ 1,\ 1)$ *and* $\vec{v}_2 = (1,\ 2,\ 3,\ 2)$

20. Suppose $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ is an orthogonal set of vectors. Prove that (*Pythagoras*)

$$\|\vec{u}_1 + \vec{u}_2 + \ldots + \vec{u}_n\|^2 = \|\vec{u}_1\|^2 + \|\vec{u}_2\|^2 + \ldots + \|\vec{u}_n\|^2$$

21. Suppose A is an orthogonal matrix. Show that $\langle \vec{u}A, \vec{v}A \rangle = \langle \vec{u}, \vec{v} \rangle$ for any $\vec{u}, \vec{v} \in V$

22. Suppose A is an orthogonal matrix. Show that $\|\vec{u}A\| = \|\vec{u}\|$ for every $\vec{u} \in V$

Section 3.3 – Gram-Schmidt Process

Definition

A set of two or more vectors in a real inner product space is said to be ***orthogonal*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be ***orthonormal***.

Theorem

1. If $S=\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is an orthogonal basis for an inner product space *V*, and if $\boldsymbol{u}$ is any vector in *V*, then

$$\boldsymbol{u}=\frac{\langle \boldsymbol{u}, \boldsymbol{v}_1\rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1+\frac{\langle \boldsymbol{u}, \boldsymbol{v}_2\rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2+\cdots+\frac{\langle \boldsymbol{u}, \boldsymbol{v}_n\rangle}{\|\boldsymbol{v}_n\|^2}\boldsymbol{v}_n$$

2. If $S=\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is an orthonormal basis for an inner product space *V*, and if $\boldsymbol{u}$ is any vector in *V*, then

$$\boldsymbol{u}=\langle \boldsymbol{u}, \boldsymbol{v}_1\rangle\boldsymbol{v}_1+\langle \boldsymbol{u}, \boldsymbol{v}_2\rangle\boldsymbol{v}_2+\cdots+\langle \boldsymbol{u}, \boldsymbol{v}_n\rangle\boldsymbol{v}_n$$

Proof

1. Since $S=\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is a basis for *V*, every vector $\boldsymbol{u}$ in *V* can be expressed in the form

$$\boldsymbol{u}=c_1\boldsymbol{v}_1+c_2\boldsymbol{v}_2+\cdots+c_n\boldsymbol{v}_n$$

Let show that $c_i=\dfrac{\langle \boldsymbol{u}, \boldsymbol{v}_i\rangle}{\|\boldsymbol{v}_i\|^2}$ *for* $i=1,2,\ldots n$

$$\begin{aligned}\langle \boldsymbol{u}, \boldsymbol{v}_i\rangle &= \langle c_1\boldsymbol{v}_1+c_2\boldsymbol{v}_2+\cdots+c_n\boldsymbol{v}_n, \boldsymbol{v}_i\rangle \\ &= c_1\langle \boldsymbol{v}_1, \boldsymbol{v}_i\rangle+c_2\langle \boldsymbol{v}_2, \boldsymbol{v}_i\rangle+\cdots+c_n\langle \boldsymbol{v}_n, \boldsymbol{v}_i\rangle\end{aligned}$$

Since *S* is an orthogonal set, all of the inner products in the last equality are zero except the i^{th}, so we have

$$\langle \boldsymbol{u}, \boldsymbol{v}_i\rangle=c_i\langle \boldsymbol{v}_i, \boldsymbol{v}_i\rangle=c_i\|\boldsymbol{v}_i\|^2$$

The Gram-Schmidt Process

To convert a basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_r\}$ into an orthogonal basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$, perform the following computations:

***Step* 1**: $\boldsymbol{v}_1 = \boldsymbol{u}_1$

***Step* 2**: $\boldsymbol{v}_2 = \boldsymbol{u}_2 - \dfrac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$

***Step* 3**: $\boldsymbol{v}_3 = \boldsymbol{u}_3 - \dfrac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \dfrac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2$

***Step* 4**: $\boldsymbol{v}_4 = \boldsymbol{u}_4 - \dfrac{\langle \boldsymbol{u}_4, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \dfrac{\langle \boldsymbol{u}_4, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2 - \dfrac{\langle \boldsymbol{u}_4, \boldsymbol{v}_3 \rangle}{\|\boldsymbol{v}_3\|^2}\boldsymbol{v}_3$

To convert the orthogonal basis into an orthonormal basis $\{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3\}$, normalize the orthogonal basis vectors. $\boxed{\boldsymbol{q}_i = \dfrac{\boldsymbol{v}_i}{\|\boldsymbol{v}_i\|}}$

Example

Assume that the vector space R^3 has the Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vectors

$$\vec{u}_1 = (1,\ 1,\ 1) \quad \vec{u}_2 = (0,\ 1,\ 1) \quad \vec{u}_3 = (0,\ 0,\ 1)$$

Into the orthogonal basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, and then normalize the ***orthogonal*** basis vectors to obtain an orthonormal basis $\{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3\}$

Solution

$$\boldsymbol{v}_1 = \boldsymbol{u}_1 = (1,\ 1,\ 1)$$

$$\begin{aligned}\boldsymbol{v}_2 &= \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 \\ &= (0,\ 1,\ 1) - \tfrac{0+1+1}{1^2+1^2+1^2}(1,\ 1,\ 1) \\ &= (0,\ 1,\ 1) - \tfrac{2}{3}(1,\ 1,\ 1)\end{aligned}$$

$$= \left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$\boldsymbol{v}_3 = \boldsymbol{u}_3 - proj_{\boldsymbol{v}_2}\boldsymbol{u}_3$$

$$= \boldsymbol{u}_3 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2$$

$$= (0,\ 0,\ 1) - \frac{0+0+1}{1^2+1^2+1^2}(1,\ 1,\ 1) - \frac{0+0+\frac{1}{3}}{\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$= (0,\ 0,\ 1) - \frac{1}{3}(1,\ 1,\ 1) - \frac{\frac{1}{3}}{\frac{2}{3}}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$= (0,\ 0,\ 1) - \frac{1}{3}(1,\ 1,\ 1) - \frac{1}{2}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$= \left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)$$

$$\boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \frac{(1,\ 1,\ 1)}{\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\boldsymbol{q}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} = \frac{\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}{\sqrt{\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}}$$

$$= \frac{\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}{\frac{\sqrt{6}}{3}}$$

$$= \left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$\boldsymbol{q}_3 = \frac{\boldsymbol{v}_3}{\|\boldsymbol{v}_3\|} = \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\sqrt{0^2+\left(-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}}$$

$$= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\frac{\sqrt{2}}{2}}$$

$$= \left(0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

Gram-Schmidt Process (*Orthonormal*)

Suppose $\vec{v}_1,\ \ldots,\ \vec{v}_n$ linearly independent in $\mathbb{R}^n$, construct n ***orthonormal*** $\vec{u}_1,\ \ldots,\ \vec{u}_n$ that span the same space: span $\left\{\vec{u}_1,\ \ldots,\ \vec{u}_k\right\}$ = span $\left\{\vec{v}_1,\ \ldots,\ \vec{v}_k\right\}$

***Step* 1**: Since $\vec{v}_i$ are linearly independent ($\neq 0$), so $\left\|\vec{v}_1\right\| \neq 0$ (to create a normal vector)

Let $\boxed{\vec{u}_1 = \dfrac{\vec{v}_1}{\left\|\vec{v}_1\right\|} = \vec{q}_1}$, then $\left\|\vec{u}_1\right\| = 1$ since $\vec{u}_1$ is orthonormal and span $\left\{\vec{u}_1\right\} = span\left\{\vec{v}_1\right\}$

$$\vec{w}_1 = \vec{v}_1 \Rightarrow \vec{v}_1 = \left\|\vec{w}_1\right\|\vec{u}_1$$

***Step* 2**: $\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \cdot \vec{q}_1\right)\vec{q}_1$

$$\Rightarrow \vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{u}_1}{\left\|\vec{v}_1\right\|}\vec{v}_1 \qquad \left(\vec{w}_2 \perp \vec{u}_1\right)$$

$$\vec{v}_2 = \left\|\vec{w}_2\right\|\vec{u}_2 + \left(\vec{v}_2 \cdot \vec{u}_1\right)\vec{u}_1 \qquad \vec{w}_2 = \left\|\vec{w}_2\right\|\vec{u}_2$$

$$\boxed{\vec{q}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}}$$

***Step* 3**: $\boxed{\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \cdot \vec{q}_1\right)\vec{q}_1 - \left(\vec{v}_3 \cdot \vec{q}_2\right)\vec{q}_2}$

$$\boxed{\vec{q}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}}$$

	$\vec{q}_1 = \dfrac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$
$\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \cdot \vec{q}_1\right)\vec{q}_1$	$\vec{q}_2 = \dfrac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$
$\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \cdot \vec{q}_1\right)\vec{q}_1 - \left(\vec{v} \cdot \vec{q}_2\right)\vec{q}_2$	$\vec{q}_3 = \dfrac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$
$\vec{w}_n = \vec{v}_n - \left(\vec{v}_n \cdot \vec{q}_1\right)\vec{q}_1 - \left(\vec{v}_n \cdot \vec{q}_2\right)\vec{q}_2 - \ \ldots\ - \left(\vec{v}_n \cdot \vec{q}_{n-1}\right)\vec{q}_{n-1}$	$\vec{q}_n = \dfrac{\vec{w}_n}{\left\|\vec{w}_n\right\|}$

Example

Use the Gram-Schmidt process to find an ***orthonormal*** basis for the subspaces of

$$\vec{v}_1 = (1,\ 1,\ 0,\ 0) \quad \vec{v}_2 = (0,\ 1,\ 1,\ 0) \quad \vec{v}_3 = (1,\ 0,\ 1,\ 1)$$

<u>Solution</u>

Step 1: $\vec{q}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \dfrac{(1,\ 1,\ 0,\ 0)}{\sqrt{1^2+1^2+0+0}}$

$$= \frac{(1,\ 1,\ 0,\ 0)}{\sqrt{2}}$$

$$\boxed{= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)}$$

Step 2: $\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \bullet \vec{q}_1\right)\vec{q}_1$

$$= (0,\ 1,\ 1,\ 0) - \left[(0,\ 1,\ 1,\ 0).\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)\right]\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$$

$$= (0,\ 1,\ 1,\ 0) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$$

$$= (0,\ 1,\ 1,\ 0) - \left(\tfrac{1}{2},\ \tfrac{1}{2},\ 0,\ 0\right)$$

$$= \left(-\tfrac{1}{2},\ \tfrac{1}{2},\ 1,\ 0\right)$$

$$\|\vec{w}_2\| = \sqrt{\left(-\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 + 1} = \sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{2}\vec{v}$$

$$\vec{q}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|} = \frac{\left(-\tfrac{1}{2},\ \tfrac{1}{2},\ 1,\ 0\right)}{\frac{\sqrt{6}}{2}}$$

$$= \frac{2}{\sqrt{6}}\left(-\tfrac{1}{2},\ \tfrac{1}{2},\ 1,\ 0\right)$$

$$\boxed{= \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right)}$$

Step 3: $\vec{v}_3 \bullet \vec{q}_1 = (1,\ 0,\ 1,\ 1).\left(\dfrac{1}{\sqrt{2}},\ \dfrac{1}{\sqrt{2}},\ 0,\ 0\right) = \boxed{\dfrac{1}{\sqrt{2}}}$

$$\vec{v}_3 \bullet \vec{q}_2 = (1,\ 0,\ 1,\ 1).\left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right) = -\frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} = \boxed{\frac{1}{\sqrt{6}}}$$

$$\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \bullet \vec{q}_1\right)\vec{q}_1 - \left(\vec{v} \bullet \vec{q}_2\right)\vec{q}_2$$

$$= (1,\ 0,\ 1,\ 1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right) - \frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right)$$

$$= (1,\ 0,\ 1,\ 1) - \left(\frac{1}{2},\ \frac{1}{2},\ 0,\ 0\right) - \left(-\frac{1}{6},\ \frac{1}{6},\ \frac{2}{6},\ 0\right)$$

$$= \left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$\vec{q}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|} = \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + 1^2}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \frac{1}{\sqrt{\frac{21}{9}}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \frac{3}{\sqrt{21}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \left(\frac{2}{\sqrt{21}},\ -\frac{2}{\sqrt{21}},\ \frac{2}{\sqrt{21}},\ \frac{3}{\sqrt{21}}\right)$$

QR–Decomposition

Problem

If A is an $m \times n$ matrix with linearly independent column vectors, and if Q is the matrix that results by applying the Gram-Schmidt process to the column vectors of A, what relationship, if any, exists between A and Q?

To solve this problem, suppose that the column vectors of A are $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ and the orthonormal column vectors of Q are $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_n$.

$$\begin{aligned}
\boldsymbol{u}_1 &= \langle \boldsymbol{u}_1, \boldsymbol{q}_1 \rangle \boldsymbol{q}_1 + \langle \boldsymbol{u}_1, \boldsymbol{q}_2 \rangle \boldsymbol{q}_2 + \cdots + \langle \boldsymbol{u}_1, \boldsymbol{q}_n \rangle \boldsymbol{q}_n \\
\boldsymbol{u}_2 &= \langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle \boldsymbol{q}_1 + \langle \boldsymbol{u}_2, \boldsymbol{q}_2 \rangle \boldsymbol{q}_2 + \cdots + \langle \boldsymbol{u}_2, \boldsymbol{q}_n \rangle \boldsymbol{q}_n \\
\vdots \quad & \quad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
\boldsymbol{u}_n &= \langle \boldsymbol{u}_n, \boldsymbol{q}_1 \rangle \boldsymbol{q}_1 + \langle \boldsymbol{u}_n, \boldsymbol{q}_2 \rangle \boldsymbol{q}_2 + \cdots + \langle \boldsymbol{u}_n, \boldsymbol{q}_n \rangle \boldsymbol{q}_n
\end{aligned}$$

$$R = \begin{bmatrix} \langle \boldsymbol{u}_1, \boldsymbol{q}_1 \rangle & \langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle & \cdots & \langle \boldsymbol{u}_n, \boldsymbol{q}_1 \rangle \\ 0 & \langle \boldsymbol{u}_2, \boldsymbol{q}_2 \rangle & \cdots & \langle \boldsymbol{u}_n, \boldsymbol{q}_2 \rangle \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \langle \boldsymbol{u}_n, \boldsymbol{q}_n \rangle \end{bmatrix}$$

The equation $A = QR$ is a factorization of A into the product of a matrix Q with orthonormal column vectors and an invertible upper triangular matrix R. We call it the ***QR-decomposition of A***.

Theorem

If A is an $m \times n$ matrix with linearly independent column vectors, then A can be factored as

$$A = QR$$

Where Q is an $m \times n$ matrix with orthonormal column vectors, and R is an $n \times n$ invertible upper triangular matrix.

Example

Find the QR-decomposition of

$$A=\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}$$

Solution

The column vectors of are

$$\boldsymbol{u}_1=\begin{pmatrix}1\\1\\1\end{pmatrix},\quad \boldsymbol{u}_2=\begin{pmatrix}0\\1\\1\end{pmatrix},\quad \boldsymbol{u}_3=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

From the previous example

$$\boldsymbol{q}_1=\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\qquad \boldsymbol{q}_2=\left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)\ \boldsymbol{q}_3=\left(0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$R=\begin{bmatrix}\langle\boldsymbol{u}_1,\boldsymbol{q}_1\rangle & \langle\boldsymbol{u}_2,\boldsymbol{q}_1\rangle & \langle\boldsymbol{u}_3,\boldsymbol{q}_1\rangle\\ 0 & \langle\boldsymbol{u}_2,\boldsymbol{q}_2\rangle & \langle\boldsymbol{u}_3,\boldsymbol{q}_2\rangle\\ 0 & 0 & \langle\boldsymbol{u}_3,\boldsymbol{q}_3\rangle\end{bmatrix}$$

$$=\begin{bmatrix}(1)\frac{1}{\sqrt{3}}+(1)\frac{1}{\sqrt{3}}+(1)\frac{1}{\sqrt{3}} & 0+(1)\frac{1}{\sqrt{3}}+(1)\frac{1}{\sqrt{3}} & 0+0+(1)\frac{1}{\sqrt{3}}\\ 0 & 0\left(\frac{-2}{\sqrt{6}}\right)+(1)\frac{1}{\sqrt{6}}+(1)\frac{1}{\sqrt{6}} & 0\left(\frac{-2}{\sqrt{6}}\right)+0\frac{1}{\sqrt{6}}+(1)\frac{1}{\sqrt{6}}\\ 0 & 0 & 0+(0)\frac{-1}{\sqrt{2}}+(1)\frac{1}{\sqrt{2}}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\ 0 & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}$$

$$\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}}\end{pmatrix}\cdot\begin{pmatrix}\frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\ 0 & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$\boldsymbol{A}\quad = \quad \boldsymbol{Q} \qquad\qquad \boldsymbol{R}$$

Calculus: Applying the Gram-Schmidt Process

We can apply the Gram-Schmidt orthogonalization procedure to generate some polynomials that are orthonormal on the interval $x \in [-1, 1]$ with inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

Example

Apply the Gram-Schmidt orthonormalization process to the basis $B = \{1, x, x^2\}$ in P_2 using the inner product

Solution

$$B = \{1, x, x^2\} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$$

$$\vec{v}_1 = \vec{u}_1 \; \underline{=1}$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^{1} dx$$

$$= x \Big|_{-1}^{1}$$

$$\underline{= 2}$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^{1} x\,dx$$

$$= \frac{1}{2}x^2 \Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\vec{v}_2 = x - \frac{0}{2}(1)$$

$$\boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$$

$$\underline{= x}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} x^2\,dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$\underline{= \frac{2}{3}}$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = \int_{-1}^{1} x^2 \, dx$$

$$= \frac{1}{3} x^3 \Big|_{-1}^{1}$$

$$= \frac{2}{3}$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \int_{-1}^{1} x^3 \, dx$$

$$= \frac{1}{4} x^4 \Big|_{-1}^{1}$$

$$= 0$$

$$\vec{v}_3 = x^2 - \frac{3}{2}(x)(0) - \frac{1}{2}\frac{2}{3}$$

$$\boldsymbol{v}_3 = \boldsymbol{u}_3 - \frac{\boldsymbol{v}_2}{\left\|\boldsymbol{v}_2\right\|^2}\left\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \right\rangle - \frac{\boldsymbol{v}_1}{\left\|\boldsymbol{v}_1\right\|^2}\left\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \right\rangle$$

$$= x^2 - \frac{1}{3}$$

$$\left\langle \vec{v}_3, \vec{v}_3 \right\rangle = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx$$

$$= \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx$$

$$= \left(\frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{1}{9}x\right)\Big|_{-1}^{1}$$

$$= \frac{1}{5} - \frac{2}{9} + \frac{1}{9} + \frac{1}{5} - \frac{2}{9} + \frac{1}{9}$$

$$= \frac{8}{45}$$

$$\vec{q}_1 = \frac{1}{\sqrt{2}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$\vec{q}_2 = \frac{x}{\sqrt{2/3}} \qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{\sqrt{3}}{\sqrt{2}} x$$

$$\vec{q}_3 = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \frac{3\sqrt{5}}{\sqrt{8}}\left(x^2 - \frac{1}{3}\right)$$

Exercises Section 3.3 – Gram-Schmidt Process

Use the Gram-Schmidt process to find an ***orthonormal*** basis for the subspaces of $\boldsymbol{R}^m$.

1. $\boldsymbol{u}_1 = (1,\ -3),\quad \boldsymbol{u}_2 = (2,\ 2)$

2. $\boldsymbol{u}_1 = (1,\ 0),\quad \boldsymbol{u}_2 = (3,\ -5)$

3. $\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ (1,\ 2,\ 1)\}$

4. $\{(1,\ 1,\ 1),\ (0,\ 1,\ 1),\ (0,\ 0,\ 1)\}$

5. $\{(1,\ 1,\ 1),\ (0,\ 2,\ 1),\ (1,\ 0,\ 3)\}$

6. $\{(2,\ 2,\ 2),\ (1,\ 0,\ -1),\ (0,\ 3,\ 1)\}$

7. $\{(1,\ -1,\ 0),\ (0,\ 1,\ 0),\ (2,\ 3,\ 1)\}$

8. $\{(3,\ 0,\ 4),\ (-1,\ 0,\ 7),\ (2,\ 9,\ 11)\}$

9. $\{(1,\ 1,\ 1,\ 1),\ (1,\ 2,\ 1,\ 0),\ (1,\ 3,\ 0,\ 0)\}$

10. $\{(0,\ 2,\ -1,\ 1),\ (0,\ 0,\ 1,\ 1),\ (-2,\ 1,\ 1,\ -1)\}$

11. $\boldsymbol{u}_1 = (1,\ 0,\ 0),\quad \boldsymbol{u}_2 = (3,\ 7,\ -2),\quad \boldsymbol{u}_3 = (0,\ 4,\ 1)$

12. $\vec{u}_1 = (1, 1, 1, 1),\quad \vec{u}_2 = (1,\ 2,\ 4,\ 5),\quad \vec{u}_3 = (1,\ -3,\ -4,\ -2)$

13. $\vec{u}_1 = (1, 1, 1, 1),\quad \vec{u}_2 = (1, 1,\ 2,\ 4),\quad \vec{u}_3 = (1,\ 2,\ -4,\ -3)$

14. $\boldsymbol{u}_1 = (0,\ 2,\ 1,\ 0),\quad \boldsymbol{u}_2 = (1,\ -1,\ 0,\ 0),\quad \boldsymbol{u}_3 = (1,\ 2,\ 0,\ -1),\quad \boldsymbol{u}_4 = (1,\ 0,\ 0,\ 1)$

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\boldsymbol{R}^m$.

15. $\{(1,\ 1,\ 0),\ (1,\ 2,\ 0),\ (0,\ 1,\ 2)\}$

16. $\{(1,\ -2,\ 2),\ (2,\ 2,\ 1),\ (2,\ -1,\ -2)\}$

17. $\{(1,\ 0,\ 0),\ (1,\ 1,\ 1),\ (1,\ 1,\ -1)\}$

18. $\{(4,\ -3,\ 0),\ (1,\ 2,\ 0),\ (0,\ 0,\ 4)\}$

19. $\{(0,\ 1,\ 2),\ (2,\ 0,\ 0),\ (1,\ 1,\ 1)\}$

20. $\{(0,\ 1,\ 1),\ (1,\ 1,\ 0),\ (1,\ 0,\ 1)\}$

21. $\{(1,\ 2,\ -2),\ (1,\ 0,\ -4),\ (5,\ 2,\ 0),\ (1,\ 1,\ -1)\}$

22. $\{(-3,\ 1,\ 2),\ (1,\ 1,\ 1),\ (2,\ 0,\ -1),\ (1,\ -3,\ 2)\}$

23. $\{(2,\ 1,\ 1),\ (0,\ 3,\ -1),\ (3,\ -4,\ -2),\ (-1,\ -1,\ 3)\}$

24. $\vec{u}_1 = (1, 1, 0,\ -1),\quad \vec{u}_2 = (1, 3, 0, 1),\quad \vec{u}_3 = (4,\ 2,\ 2,\ 0)$

25. $\vec{u}_1 = (1, 1, 1, 1),\quad \vec{u}_2 = (1, 1,\ 2,\ 4),\quad \vec{u}_3 = (1,\ 2,\ -4,\ -3)$

26. $\{(3,\ 4,\ 0,\ 0),\ (-1,\ 1,\ 0,\ 0),\ (2,\ 1,\ 0,\ -1),\ (0,\ 1,\ 1,\ 0)\}$

27. Find the ***QR***-decomposition of

a) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

28. Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product

$\boldsymbol{u} = (0,\ -2,\ 2,\ 1), \quad \boldsymbol{v} = (-1,\ -1,\ 1,\ 1)$

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product

29. $f_1(x) = x + 2, \quad f_2(x) = x^2 - 3x + 4$

30. $f_1(x) = x, \quad f_2(x) = x^3, \quad f_3(x) = x^5$

31. $f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = \frac{1}{2}\left(3x^2 - 1\right)$

32. $f_1(x) = 1, \quad f_2(x) = \sin \pi x, \quad f_3(x) = \cos \pi x$

33. $f_1(x) = \sin \pi x, \quad f_2(x) = \sin 2\pi x, \quad f_3(x) = \sin 3\pi x$

Section 3.4 – Orthogonal Matrices

Definition

A square matrix *A* is said to be orthogonal if its transpose is the same as its inverse, that is, if

$$A^{-1} = A^T$$

or, equivalently, if

$$AA^T = A^T A = I$$

Example

The matrix $A = \begin{pmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{pmatrix}$

Solution

$$A^T A = \begin{pmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{pmatrix} \begin{pmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example

The matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Solution

$$A^T A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem

The following are equivalent for $n \times n$ matrix *A*.

a) *A* is orthogonal.

b) The row vectors of *A* form an orthonormal set in R^n with the Euclidean inner product.

c) The column vectors of *A* form an orthonormal set in R^n with the Euclidean inner product.

Theorem

a) The inverse of an orthogonal matrix is orthogonal

b) A product of orthogonal matrices is orthogonal

c) If A is orthogonal, then $\det(A)=1$ or $\det(A)=-1$

Theorem

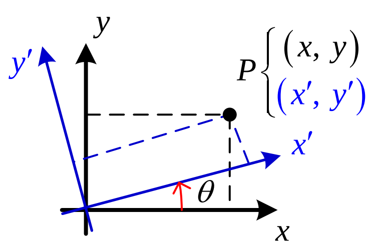
If A is an $n\times n$ matrix, then the following are equivalent

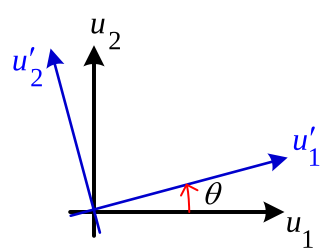
a) A is orthogonal.

b) $\|Ax\|=\|x\|$ for all $\boldsymbol{x}$ in R^n.

c) $Ax\bullet Ay=x\bullet y$ for all $\boldsymbol{x}$ and $\boldsymbol{y}$ in R^n.

Let $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ be the unit vectors along the x- and y-axes and unit vectors $\boldsymbol{u}'_1$ and $\boldsymbol{u}'_2$ along the x'- and y'-axes.





The new coordinates (x', y') and the old coordinates (x, y) of a point P will be related by

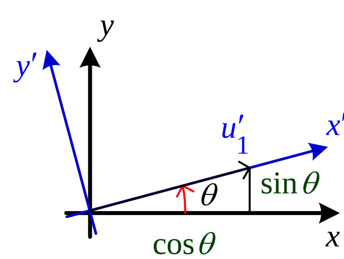
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = P^{-1}\begin{bmatrix} x \\ y \end{bmatrix} \qquad P=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

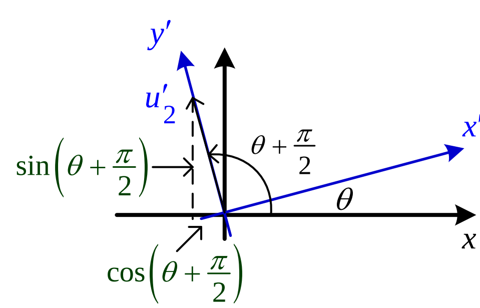
$$P^{-1}=P^T=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\rightarrow \begin{cases} x' = x\cos\theta + y\sin\theta \\ y' = -x\sin\theta + y\cos\theta \end{cases}$$

These are sometimes called the ***rotation equations***.





Example

Use the form $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ to find the new coordinates of the point $Q(2,\ 1)$ if the coordinate axes of a rectangular coordinate system are rotated through an angle of $\theta = \frac{\pi}{4}$.

Solution

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

The new coordinates of Q are $(x',\ y') = \left(\frac{3}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)$

Exercises *Section* 3.4 – Orthogonal Matrices

Show that the matrix is orthogonal

1. $A=\begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$

2. $A=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse.

3. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

5. $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

6. $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

7. $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$

8. $\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

9. $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

10. $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$

11. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$

13. Find a last column so that the resulting matrix is orthogonal

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & \cdots \end{bmatrix}$$

14. Determine if the given matrix is orthogonal. If it is, find its inverse

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{5} & \frac{3}{7} \\ \frac{4}{9} & \frac{3}{5} & -\frac{2}{7} \\ \frac{8}{9} & -\frac{2}{5} & \frac{3}{7} \end{bmatrix}$$

15. Prove that if A is orthogonal, then A^T is orthogonal.

16. Prove that if A is orthogonal, then A^{-1} is orthogonal.

17. Prove that if A and B are orthogonal, then AB is orthogonal.

18. Let Q be an $n \times n$ orthogonal matrix, and let A be an $n \times n$ matrix.
Show that $\det\left(QAQ^T\right) = \det(A)$

19. Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$

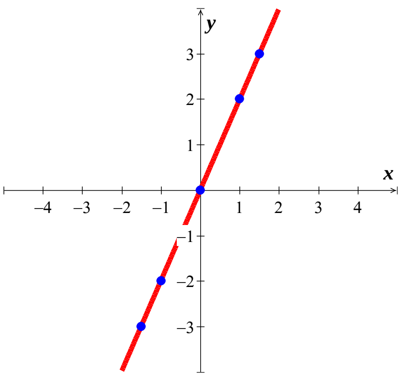
a) Is matrix A an orthogonal matrix?
b) Let B be the matrix obtained by normalizing each row of A, find B.
c) Is B an orthogonal matrix?
d) Are the columns of B orthogonal?

Section 3.5 – Least Squares Analysis

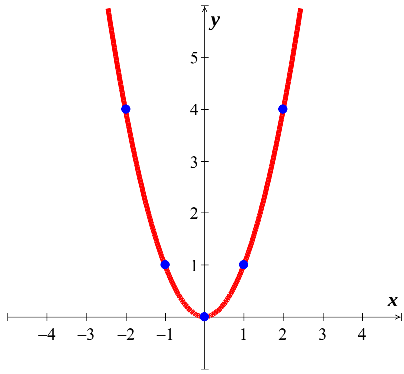
The use to ***best*** fit data, we will use results about orthogonal projections in inner product spaces to obtain a technique for fitting a line or other polynomial.

Fitting a Curve to Data

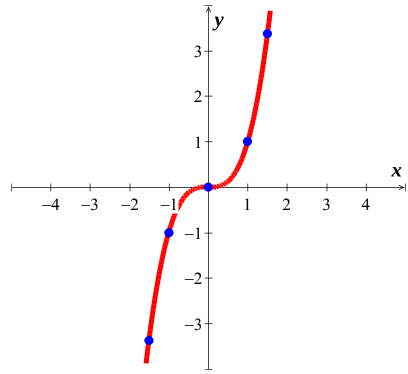
The common problem is to obtain a mathematical relationship between 2 variables x and y by ***fitting*** a curve to points in the xy-plane.
Some possibility of fitting the data



$y = ax + b$



$y = ax^2 + bx + c$



$y = ax^3 + bx^2 + cx + d$

Least Squares Fit of a Straight Line

Recall that a system of equations $A\boldsymbol{x} = \boldsymbol{y}$ is called inconsistent if it does not have a solution. Suppose we want to fit a straight line $y = mx + b$ to the determined points $\left(x_1, y_1\right), \ \ldots, \ \left(x_n, y_n\right)$

If the data points were collinear, the line would pass through all n points and the unknown coefficients m and b would satisfy the equations

$$\begin{aligned} y_1 &= mx_1 + b \\ y_2 &= mx_2 + b \\ \vdots \quad & \quad \vdots \qquad \vdots \\ y_n &= mx_n + b \end{aligned} \quad \Rightarrow \quad \underbrace{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ b \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\boldsymbol{y}}$$

The problem is to find m and b that minimize the errors is some sense.

Least Square Problem

Given a linear system $A\boldsymbol{x} = \boldsymbol{y}$ of m equations in n unknowns, find a vector $\boldsymbol{x}$ that minimizes $\|\boldsymbol{y} - A\boldsymbol{x}\|$ with respect to the Euclidean inner product on $\boldsymbol{R}^m$. We call such as $\boldsymbol{x}$ a least squares solution of the system, we call $\boldsymbol{y} - A\boldsymbol{x}$ the least squares error vectors, and we call $\|\boldsymbol{y} - A\boldsymbol{x}\|$ the least squares error.

$$A\boldsymbol{x} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix}$$

The term "***least square solution***" results from the fact the minimizing $\|\boldsymbol{y} - A\boldsymbol{x}\| = e_1^2 + e_2^2 + \ldots + e_m^2$

Example

Find the sums of squares of the errors of (2, 4), (4, 8), (6, 6)

Solution

$$4 = 2m + b \quad \Rightarrow \quad 4 - 2m - b = e_1$$
$$8 = 4m + b \quad \Rightarrow \quad 8 - 4m - b = e_2$$
$$6 = 6m + b \quad \Rightarrow \quad 6 - 6m - b = e_3$$
$$e_1^2 + e_2^2 + \ldots + e_m^2 = (4 - 2m - b)^2 + (8 - 4m - b)^2 + (6 - 6m - b)^2$$

The least squares problem for this example to find the values m and b for which $e_1^2 + e_2^2 + \ldots + e_m^2$ is a minimum.

Theorem

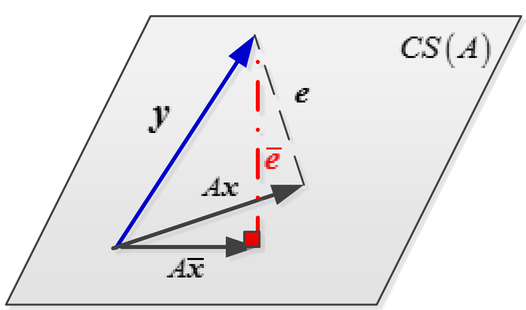
If A is an $m \times n$ matrix, the equation $A\boldsymbol{x} = \boldsymbol{y}$ has a solution if and only if y is in the column space of A.

$$\boldsymbol{y} - A\boldsymbol{x} = \boldsymbol{e}$$

$A\boldsymbol{x}$ is a vector that is in the column space of A. For this A the column space is a plane is $\boldsymbol{R}^m$

$\boldsymbol{y}$ is a vector, not in the column space of A (otherwise $A\boldsymbol{x} = \boldsymbol{y}$ has an exact solution)

$\boldsymbol{e}$ is the error vector, the difference between $\boldsymbol{y}$ and $A\boldsymbol{x}$



The length $\|\boldsymbol{e}\|$ is a minimum exactly when $\boldsymbol{e} \perp CS(A)$

Best Approximation *Theorem*

If $CS(A)$ is a finite dimensional subspace of an inner product space, and if $\boldsymbol{y}$ is a vector in $\boldsymbol{V}$, then $proj_{CS(A)}\boldsymbol{y}$ is the best approximation to $\boldsymbol{y}$ from $CS(A)$ is the sense that

$$\left\| \boldsymbol{y} - proj_{CS(A)}\boldsymbol{y} \right\| < \left\| \boldsymbol{y} - CS(A) \right\|$$

For every vector $\boldsymbol{w}$ in $CS(A)$ that is different from $proj_{CS(A)}\boldsymbol{y}$

Theorem

For every linear system $A\boldsymbol{x} = \boldsymbol{y}$, the associated normal system

$$A^T A\boldsymbol{x} = A^T\boldsymbol{y}$$

Is consistent, and all solutions are least squares solutions of $A\boldsymbol{x} = \boldsymbol{y}$

If the columns of A are linearly independent, then $A^T A$ is invertible so has a unique solution $\overline{\boldsymbol{x}}$. This solution is often expressed theoretically as

$$\left(A^T A\right)^{-1} A^T A\overline{\boldsymbol{x}} = \left(A^T A\right)^{-1} A^T\boldsymbol{y}$$

$$\overline{\boldsymbol{x}} = \left(A^T A\right)^{-1} A^T\boldsymbol{y}$$

Proof

Let the vector $\overline{\boldsymbol{x}}$ is a least squares solution to $A\boldsymbol{x} = \boldsymbol{y} \quad \Leftrightarrow \quad (\boldsymbol{y} - A\overline{\boldsymbol{x}}) \perp CS(A)$

$$(\boldsymbol{y} - A\overline{\boldsymbol{x}}) \cdot \boldsymbol{z} = 0 \qquad \boldsymbol{z} \text{ in } CS(A) \quad \& \quad \boldsymbol{z} = A\boldsymbol{w}$$

$$(\boldsymbol{y} - A\overline{\boldsymbol{x}}) \cdot A\boldsymbol{w} = 0 \qquad \boldsymbol{w} \text{ in } \boldsymbol{R}^n$$

$$A^T(\boldsymbol{y} - A\overline{\boldsymbol{x}}) \cdot \boldsymbol{w} = 0$$

$$A^T(\boldsymbol{y} - A\overline{\boldsymbol{x}}) = 0$$

$$A^T\boldsymbol{y} - A^T A\overline{\boldsymbol{x}} = 0$$

$$A^T\boldsymbol{y} = A^T A\overline{\boldsymbol{x}}$$

Theorem

If A is an $m \times n$ matrix, then the following are equivalent

a) A has linearly independent column vectors.

b) $A^T A$ is invertible.

Example

Find the equation of the line that best fits the given points in the least-squares sense.

(40, 482), (45, 467), (50, 452), (55, 432), (60, 421)

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$$

Where $A = \begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix}$ $\quad \boldsymbol{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\quad \boldsymbol{y} = \begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$

Using the normal equation formula: $A^T Ax = A^T y$

$$\begin{pmatrix} 40 & 45 & 50 & 55 & 60 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 40 & 45 & 50 & 55 & 60 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$$

$$\begin{pmatrix} 12,750 & 250 \\ 250 & 5 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 111,970 \\ 2,255 \end{pmatrix}$$

$X = A^{-1}B$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{1250}\begin{pmatrix} 5 & -250 \\ -250 & 12,750 \end{pmatrix}\begin{pmatrix} 111,970 \\ 2,255 \end{pmatrix}$$

$$= \begin{pmatrix} -3.12 \\ 607 \end{pmatrix}$$

Or

$$\begin{pmatrix} 12,750 & 250 & 111,970 \\ 250 & 5 & 2,225 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & -3.12 \\ 0 & 1 & 607 \end{pmatrix}$$

Thus $\boxed{y = -3.12x + 607}$

Example

Given the system equation: $\begin{cases} x_1 - x_2 = 4 \\ 3x_1 + 2x_2 = 1 \\ -2x_1 + 4x_2 = 3 \end{cases}$

a) Find the least-squares solution of the linear system $A\boldsymbol{x} = \boldsymbol{y}$

b) Find the orthogonal projection of $\boldsymbol{y}$ on the column space of A

c) Find the error vector and the error

Solution

a) $A = \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix} \quad \boldsymbol{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \boldsymbol{y} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$

$$A^T A\boldsymbol{x} = A^T \boldsymbol{y}$$

$$\begin{pmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 14 & -3 \\ -3 & 21 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{285}\begin{pmatrix} 21 & 3 \\ 3 & 14 \end{pmatrix}\begin{pmatrix} 1 \\ 10 \end{pmatrix} = \begin{pmatrix} \frac{51}{285} \\ \frac{143}{285} \end{pmatrix} \qquad X = A^{-1}B$$

$$= \begin{pmatrix} \frac{17}{95} \\ \frac{143}{285} \end{pmatrix}$$

Thus $\underline{y = 0.1789x + 0.5018}$

b) The orthogonal projection of $\boldsymbol{y}$ on the column space of A

$$A\boldsymbol{x} = \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} \frac{17}{95} \\ \frac{143}{285} \end{pmatrix} = \begin{pmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{pmatrix}$$

c) $\boldsymbol{y} - A\boldsymbol{x} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{pmatrix} = \begin{pmatrix} \frac{1232}{285} \\ -\frac{154}{285} \\ \frac{4}{3} \end{pmatrix}$

The error: $\|\boldsymbol{y} - A\boldsymbol{x}\| = \sqrt{\left(\frac{1232}{285}\right)^2 + \left(-\frac{154}{285}\right)^2 + \left(\frac{4}{3}\right)^2} \;\underline{\approx 4.556}$

Exercises *Section* 3.5 – Least Squares Analysis

1. Find the equation of the line that best fits the given points in the least-squares sense.

a) $\{(0,\ 2),\ (1,\ 2),\ (2,\ 0)\}$

b) $\{(1,\ 5),\ (2,\ 4),\ (3,\ 1),\ (4,\ 1),\ (5,\ -1)\}$

c) $\{(0,\ 1),\ (1,\ 3),\ (2,\ 4),\ (3,\ 4)\}$

d) $\{(-2,\ 0),\ (-1,\ 0),\ (0,\ 1),\ (1,\ 3),\ (2,\ 5)\}$

2. Find the orthogonal projection of the vector $\boldsymbol{u}$ on the subspace of $\boldsymbol{R}^4$ spanned by the vectors

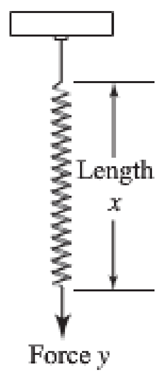
a) $\boldsymbol{u}=(-3,\ -3,\ 8,\ 9);\quad \boldsymbol{v}_1=(3,\ 1,\ 0,\ 1),\quad \boldsymbol{v}_2=(1,\ 2,\ 1,\ 1),\quad \boldsymbol{v}_3=(-1,\ 0,\ 2,\ -1)$

b) $\boldsymbol{u}=(6,\ 3,\ 9,\ 6);\quad \boldsymbol{v}_1=(2,\ 1,\ 1,\ 1),\quad \boldsymbol{v}_2=(1,\ 0,\ 1,\ 1),\quad \boldsymbol{v}_3=(-2,\ -1,\ 0,\ -1)$

c) $\boldsymbol{u}=(-2,\ 0,\ 2,\ 4);\quad \boldsymbol{v}_1=(1,\ 1,\ 3,\ 0),\quad \boldsymbol{v}_2=(-2,\ -1,\ -2,\ 1),\quad \boldsymbol{v}_3=(-3,\ -1,\ 1,\ 3)$

3. Find the standard matrix for the orthogonal projection P of $\boldsymbol{R}^2$ on the line passes through the origin and makes an angle θ with the positive x-axis.

4. Hooke's law in physics states that the length x of a uniform spring is a linear function of the force y applied to it. If we express the relationship as $y=mx+b$, then the coefficient m is called the spring constant.



Suppose a particular unstretched spring has a measured length of 6.1 *inches*.(i.e., x = 6.1 when y = 0). Forces of 2 pounds, 4 pounds, and 6 pounds are then applied to the spring, and the corresponding lengths are found to be 7.6 inches, 8.7 inches, and 10.4 inches. Find the spring constant.

5. Prove: If A has a linearly independent column vectors, and if $\boldsymbol{b}$ is orthogonal to the column space of A, then the least squares solution of $A\boldsymbol{x}=\boldsymbol{b}$ is $\boldsymbol{x}=0$.

6. Let A be an $m\times n$ matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of R^n onto the row space of A.

7. Let *W* be the line with parametric equations $x = 2t, \quad t = -t, \quad z = 4t$

a) Find a basis for *W*.

b) Find the standard matrix for the orthogonal projection on *W*.

c) Use the matrix in part (*b*) to find the orthogonal projection of a point $P_0\left(x_0, y_0, z_0\right)$ on *W*.

d) Find the distance between the point $P_0(2, 1, -3)$ and the line *W*.

8. In R^3, consider the line *l* given by the equations $x = t, \quad t = t, \quad z = t$

And the line *m* given by the equations $x = s, \quad t = 2s - 1, \quad z = 1$

Let *P* be the point on *l*, and let *Q* be a point on *m*. Find the values of t and s that minimize the distance between the lines by minimizing the squared distance $\|P - Q\|^2$

9. Determine whether the statement is true or false,

a) If *A* is an $m \times n$ matrix, then $A^T A$ is a square matrix.

b) If $A^T A$ is invertible, then *A* is invertible.

c) If *A* is invertible, then $A^T A$ is invertible.

d) If $A\boldsymbol{x} = \boldsymbol{b}$ is a consistent linear system, then $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$ is also consistent.

e) If $A\boldsymbol{x} = \boldsymbol{b}$ is an inconsistent linear system, then $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$ is also inconsistent.

f) Every linear system has a least squares solution.

g) Every linear system has a unique least squares solution.

h) If *A* is an $m \times n$ matrix with linearly independent columns and $\boldsymbol{b}$ is in R^m, then $A\boldsymbol{x} = \boldsymbol{b}$ has a unique least squares solution.