Section 4.5 – Exponential and logarithmic Equations

Exponential Equations

$$b^M = b^N \leftrightarrow M = N \quad \text{for any } b > 0, \neq 1$$

Example

Solve $5^{3x-6} = 125$

Solution

$5^{3x-6} = 5^3$

$3x - 6 = 3$

$3x = 9$

$\Rightarrow \boxed{x = 3}$

Example

Solve $8^{x+2} = 4^{x-3}$

Solution

$\left(2^3\right)^{x+2} = \left(2^2\right)^{x-3}$

$2^{3(x+2)} = 2^{2(x-3)}$

$3(x+2) = 2(x-3)$

$3x + 6 = 2x - 6$

$3x - 2x = -6 - 6$

$\boxed{x = -12}$

Using *Natural Logarithms*

1. Isolate the exponential expression
2. Take the natural logarithm on both sides of the equation
3. Simplify using one of the following properties: $\ln b^x = x\ln b$ *or* $\ln e^x = x$
4. Solve for the variable

Example

Solve: $7e^{2x} - 5 = 58$

Solution

$7e^{2x} - 5 = 58$ — ***Isolate the exponential expression***

$7e^{2x} = 63$ — ***Divide by 7 both sides***

$e^{2x} = 9$ — ***Natural logarithm on both sides***

$\ln e^{2x} = \ln 9$ — ***Use inverse Property***

$2x = \ln 9$

$\Rightarrow \boxed{x = \frac{\ln 9}{2} \approx 1.0986}$

Example

Solve: $3^{2x-1} = 7^{x+1}$

Solution

$\ln 3^{2x-1} = \ln 7^{x+1}$ — ***Natural logarithm on both sides***

$(2x-1)\ln 3 = (x+1)\ln 7$ — ***Power Rule***

$2x\ln 3 - \ln 3 = x\ln 7 + \ln 7$

$2x\ln 3 - x\ln 7 = \ln 3 + \ln 7$

$x(2\ln 3 - \ln 7) = \ln 3 + \ln 7$

$\boxed{x = \frac{\ln 3 + \ln 7}{2\ln 3 - \ln 7}} \approx 12.1143$

Logarithmic Equations

1. Express the equation in the form $\log_b M = c$
2. Use the definition of a logarithm to rewrite the equation in exponential form:

$$\log_b M = c \quad \Rightarrow \quad b^c = M$$

3. Solve for the variable
4. Check proposed solution in the original equation. Include only the set for M > 0

Example

Solve: $\log(x) + \log(x-3) = 1$

Solution

$log[x(x-3)] = 1$ ***Product Rule***

$x(x-3) = 10^1$ ***Convert to exponential form***

$x^2 - 3x = 10$

$x^2 - 3x - 10 = 0$ ***Solve for x***

$\Rightarrow x = -2, 5$

Check: $x = -2 \Rightarrow \log(-2) + \log(-2-3) = 1$

$x = 5 \Rightarrow log(5) + log(5-3) = 1$

Example

Solve: $\log_6(3x+2) + \log_6(x-1) = 1$

Solution

$\log_6[(3x+2)(x-1)] = 1$ ***Product Rule***

$(3x+2)(x-1) = 6^1$ ***Convert to exponential form***

$3x^2 - x - 2 = 6$

$3x^2 - x - 8 = 0$ ***Solve for x***

$x = \frac{1-\sqrt{97}}{6} < 0$ $\quad x = \frac{1+\sqrt{97}}{6} > 1$

Solution: $\boxed{x = \frac{1+\sqrt{97}}{6}}$

Property of Logarithmic Equality

For any $M > 0$, $N > 0$, $b > 0, \neq 1$

$$\log_b M = \log_b N \quad \Rightarrow \quad M = N$$

Example

Solve: $\ln(x-3) = \ln(7x-23) - \ln(x+1)$

Solution

$$\ln(x-3) = \ln\left(\frac{7x-23}{x+1}\right) \qquad \textit{Quotient Rule}$$

$$x - 3 = \frac{7x-23}{x+1}$$

$$(x - 3)(x + 1) = 7x - 23$$

$$x^2 - 2x - 3 = 7x - 23$$

$$x^2 - 9x + 20 = 0 \qquad \Rightarrow x = 4, 5$$

Check: $x = 4 \Rightarrow \ln(4-3) = \ln(7(4)-23) - \ln(4+1)$

$x = 5 \Rightarrow \ln(5-3) = \ln(7(5)-23) - \ln(5+1)$

Example

Solve: $\log(x+6) - \log(x+2) = \log x$

Solution

$$\log\frac{x+6}{x+2} = \log x \qquad \textit{Quotient Rule}$$

$$\frac{x+6}{x+2} = x \qquad \textit{Multiply by } x + 2$$

$$x + 6 = x(x+2)$$

$$x + 6 = x^2 + 2x$$

$$0 = x^2 + 2x - x - 6$$

$$x^2 + x - 6 = 0 \qquad \textit{Solve for } x$$

$$x = -3,\ 2$$

Check: $x = -3 \rightarrow \log(-3+6) - \log(-3+2) = \log(-3)$ *Or* ***Domain***

$x = 2 \rightarrow \log(2+6) - \log(2+2) = \log(2)$

Solution: $\boxed{x = 2}$

Exercises *Section* 4.5 – Exponential and logarithmic Equations

Solve

1. $2^{3x-7} = 32$

2. $4^{2x-1} = 64$

3. $3^{1-x} = \frac{1}{27}$

4. $\left(\frac{1}{3}\right)^{x} = 81$

5. $5^{x} = 134$

6. $7^{x} = 12$

7. $9^{x} = \frac{1}{\sqrt[3]{3}}$

8. $9e^{x} = 107$

9. $7^{2x+1} = 3^{x+2}$

10. $4^{x+3} = 3^{-x}$

11. $2^{x+4} = 8^{x-6}$

12. $8^{x+2} = 4^{x-3}$

13. $7^{x} = 12$

14. $5^{x+4} = 4^{x+5}$

15. $5^{x+2} = 4^{1-x}$

16. $27 = 3^{5x}9^{x^2}$

17. $3^{2x-1} = 0.4^{x+2}$

18. $4^{3x-5} = 16$

19. $4^{x+3} = 3^{-x}$

20. $3^{x-1} = 7^{2x+5}$

21. $4^{x-2} = 2^{3x+3}$

22. $2^{3x-7} = 32$

23. $3^{2x-1} = 0.4^{x+2}$

24. $e^{2x} - 2e^{x} - 3 = 0$

25. $e^{0.08t} = 2500$

26. $e^{x^2} = 200$

27. $e^{2x+1} \cdot e^{-4x} = 3e$

28. $e^{2x} - 8e^{x} + 7 = 0$

29. $e^{x} + e^{-x} - 6 = 0$

30. $e^{1-3x} \cdot e^{5x} = 2e$

31. $6\ln(2x) = 30$

32. $\log_5(x-7) = 2$

33. $\log_5 x + \log_5(4x-1) = 1$

34. $\log x + \log(x-3) = 1$

35. $\log x - \log(x+3) = 1$

36. $\log_3 x = -2$

37. $\log(3x+2) + \log(x-1) = 1$

38. $\log_5(x+2) + \log_5(x-2) = 1$

39. $\log x + \log(x-9) = 1$

40. $\log_2(x+1) + \log_2(x-1) = 3$

41. $\log_8(x+1) - \log_8 x = 2$

42. $\log(x+6) - \log(x+2) = \log x$

43. $\ln(x+8) + \ln(x-1) = 2\ln x$

44. $\ln(4x+6) - \ln(x+5) = \ln x$

45. $\ln(5+4x) - \ln(x+3) = \ln 3$

46. $\ln(x-5) - \ln(x+4) = \ln(x-1) - \ln(x+2)$

47. $\ln(x-3) = \ln(7x-23) - \ln(x+1)$

48. $27 = 3^{5x}9^{x^2}$

49. $\ln \sqrt[4]{x} = \sqrt{\ln x}$

50. $\sqrt{\ln x} = \ln \sqrt{x}$

51. $7^{x+6} = 7^{3x-4}$

52. $2^{-100x} = (0.5)^{x-4}$

53. $4^x \left(\frac{1}{2}\right)^{3-2x} = 8.\left(2^x\right)^2$

54. $5^{3x-6} = 125$

55. $e^{x^2} = e^{7x-12}$

56. $f(x) = xe^x + e^x$

57. $f(x) = x^3\left(4e^{4x}\right) + 3x^2e^{4x}$

58. $\log_4 x = \log_4 (8-x)$

59. $\log_7 (x-5) = \log_7 (6x)$

60. $\ln x^2 = \ln(12-x)$

61. $e^{x \ln 3} = 27$

62. $\log_6 (2x-3) = \log_6 12 - \log_6 3$

63. $\ln(-4-x) + \ln 3 = \ln(2-x)$

64. $\log_2 (x+7) + \log_2 x = 3$

65. $\log_3 (x+3) + \log_3 (x+5) = 1$

66. $\ln x = 1 - \ln(x+2)$

67. $\ln x = 1 + \ln(x+1)$

68. $\log_3 (x-2) = \log_3 27 - \log_3 (x-4) - 5^{\log_5 1}$

69. $\log_2 (x+3) = \log_2 (x-3) + \log_3 9 + 4^{\log_4 3}$

70. $\log_3 x - \log_9 (x+42) = 0$

71. Solve for *t* using logarithms with base *a*: $2a^{t/3} = 5$

72. Solve for *t* using logarithms with base *a*: $K = H - Ca^t$

Find the exact solution (2-decimal place approximation)

73. $3^{x+4} = 2^{1-3x}$

74. $3^{2-3x} = 4^{2x+1}$

75. $2^{-x^2} = 5$

76. $2^{-x} = 8$

77. $\log\left(x^2+4\right) - \log(x+2) = 2 + \log(x-2)$

78. $5^x + 125\left(5^{-x}\right) = 30$

79. $4^x - 3\left(4^{-x}\right) = 8$

Solve the equation without using the calculator

80. $\log x^2 = (\log x)^2$

81. $\log(\log x) = 2$

82. $\log\sqrt{x^3 - 9} = 2$

83. $e^{2x} + 2e^x - 15 = 0$

84. How long, to the nearest tenth of a year, will it take \$1000 to grow to \$3600 at 8% annual interest compounded quarterly?