Solution Section 4.1 – Antiderivatives

Exercise

Find each indefinite integral $\int v^2 dv$

Solution

$$\int v^2 dv = \boxed{\frac{v^3}{3} + C}$$

Exercise

Find each indefinite integral $\int x^{1/2} dx$

Solution

$$\int x^{1/2} dx = \boxed{\frac{2}{3} x^{3/2} + C}$$

Exercise

Find each indefinite integral $\int 4y^{-3} dy$

Solution

$$\int 4y^{-3} dy = 4\frac{y^{-2}}{-2} + C$$

$$\boxed{= -\frac{2}{y^2} + C}$$

Exercise

Find each indefinite integral $\int (x^3 - 4x + 2) dx$

Solution

$$\int (x^3 - 4x + 2) dx = \frac{x^4}{4} - 4\frac{x^2}{2} + 2x + C$$

$$\boxed{= \frac{1}{4} x^4 - 2x^2 + 2x + C}$$

Exercise

Find each indefinite integral $\int \left(3z^2 - 4z + 5\right) dz$

Solution

$$\int \left(3z^2 - 4z + 5\right) dz = 3\frac{z^3}{3} - 4\frac{z^2}{2} + 5z + C$$

$$= z^3 - 2z^2 + 5z + C$$

Exercise

Find each indefinite integral $\int \left(x^2 - 1\right)^2 dx$

Solution

$$\int \left(x^2 - 1\right)^2 dx = \int \left(x^4 - 2x^2 + 1\right) dx \qquad (a-b)^2 = a^2 - 2ab + b^2$$

$$= \frac{1}{5}x^5 - \frac{2}{3}x^3 + x + C$$

Exercise

Find each indefinite integral $\int \frac{x^2 + 1}{\sqrt{x}} dx$

Solution

$$\int \frac{x^2 + 1}{\sqrt{x}} dx = \int \frac{x^2}{x^{1/2}} + \frac{1}{x^{1/2}} dx$$

$$= \int \left(x^{3/2} + x^{-1/2}\right) dx$$

$$= \frac{x^{5/2}}{5/2} - \frac{x^{1/2}}{1/2} + C$$

$$= \frac{2}{5}x^{5/2} - 2x^{1/2} + C$$

Exercise

Find each indefinite integral $\int \left(\sqrt[4]{x^3}+1\right)dx$

Solution

$$\int \left(\sqrt[4]{x^3}+1\right)dx = \int \left(x^{3/4}+1\right)dx$$

$$= \frac{4}{7}x^{7/4}+x+C$$

Exercise

Find each indefinite integral $\int \sqrt{x}(x+1)dx$

Solution

$$\int \sqrt{x}(x+1)dx = \int x^{1/2}(x+1)dx$$

$$= \int \left(x^{3/2}+x^{1/2}\right)dx$$

$$= \frac{2}{5}x^{5/2}+\frac{2}{3}x^{3/2}+C$$

Exercise

Find each indefinite integral $\int (1+3t)t^2dt$

Solution

$$\int (1+3t)t^2dt = \int \left(t^2+3t^3\right)dt$$

$$= \frac{1}{3}t^3+\frac{3}{4}t^4+C$$

Exercise

Find each indefinite integral $\int \frac{x^2 - 5}{x^2} dx$

Solution

$$\int \frac{x^2 - 5}{x^2} dx = \int \left(1 - \frac{5}{x^2}\right) dx$$

$$= \int \left(1 - 5x^{-2}\right) dx$$

$$= x + 5x^{-1} + C$$

$$= x + \frac{5}{x} + C$$

Exercise

Find each indefinite integral $\int (-40x + 250) dx$

Solution

$$\int (-40x + 250) dx = -20x^2 + 250x + C$$

Exercise

Find each indefinite integral $\int \frac{x+2}{\sqrt{x}} dx$

Solution

$$\int \frac{x+2}{\sqrt{x}} dx = \int \left[\frac{x}{x^{1/2}} + \frac{2}{x^{1/2}}\right] dx$$

$$= \int \frac{x}{x^{1/2}} dx + \int \frac{2}{x^{1/2}} dx$$

$$= \int x^{1/2} dx + 2\int x^{-1/2} dx$$

$$= \frac{x^{3/2}}{3/2} + 2\frac{x^{1/2}}{1/2} + C$$

$$= \frac{2}{3}x^{3/2} + 4x^{1/2} + C$$

Exercise

Find each indefinite integral $\int\left(\frac{1}{5}-\frac{2}{x^3}+2x\right)dx$

Solution

$$\int\left(\frac{1}{5}-\frac{2}{x^3}+2x\right)dx=\int\frac{1}{5}dx-\int 2x^{-3}dx+\int 2xdx$$

$$=\frac{x}{5}-2\frac{x^{-2}}{-2}+x^2+C$$

$$=\frac{x}{5}+\frac{1}{x^2}+x^2+C$$

Exercise

Find each indefinite integral $\int\left(\sqrt{x}+\sqrt[3]{x}\right)dx$

Solution

$$\int\left(\sqrt{x}+\sqrt[3]{x}\right)dx=\int\left(x^{1/2}+x^{1/3}\right)dx$$

$$=\frac{x^{3/2}}{3/2}+\frac{x^{4/3}}{4/3}+C$$

$$=\frac{2}{3}x^{3/2}+\frac{3}{4}x^{4/3}+C$$

Exercise

Find each indefinite integral $\int 2x\left(1-x^{-3}\right)dx$

Solution

$$\int 2x\left(1-x^{-3}\right)dx=\int\left(2x-2x^{-2}\right)dx$$

$$=x^2-2\frac{x^{-1}}{-1}+C$$

$$=x^2+\frac{2}{x}+C$$

Exercise

Find each indefinite integral $\int\left(\frac{4+\sqrt{t}}{t^3}\right)dt$

Solution

$$\int\left(\frac{4+\sqrt{t}}{t^3}\right)dt=\int\left(\frac{4}{t^3}+\frac{t^{1/2}}{t^3}\right)dt$$

$$=\int\left(4t^{-3}+t^{-5/2}\right)dt$$

$$=4\frac{t^{-2}}{-2}+\frac{t^{-3/2}}{-3/2}+C$$

$$=-\frac{2}{t^2}-\frac{2}{3t^{3/2}}+C$$

Exercise

Find each indefinite integral $\int(-2\cos t)dt$

Solution

$$\int(-2\cos t)dt=-2\sin t+C$$

Exercise

Find each indefinite integral $\int 7\sin\frac{\theta}{3}d\theta$

Solution

$$\int 7\sin\frac{\theta}{3}d\theta=7\frac{-\cos\left(\frac{\theta}{3}\right)}{\frac{1}{3}}+C=-21\cos\left(\frac{\theta}{3}\right)+C$$

Exercise

Find each indefinite integral $\int\frac{2}{5}\sec\theta\tan\theta d\theta$

Solution

$$\int\frac{2}{5}\sec\theta\tan\theta d\theta=\frac{2}{5}\sec\theta+C$$

Exercise

Find each indefinite integral $\int \left(4\sec x \tan x - 2\sec^2 x\right)dx$

Solution

$$\int \left(4\sec x \tan x - 2\sec^2 x\right)dx = 4\int (\sec x \tan x)dx - 2\int \left(\sec^2 x\right)dx$$

$$= 4\sec x - 2\tan x + C$$

Exercise

Find each indefinite integral $\int (2\cos 2x - 3\sin 3x)dx$

Solution

$$\int (2\cos 2x - 3\sin 3x)dx = \sin 2x + \cos 3x + C$$

Exercise

Find each indefinite integral $\int \left(1 + \tan^2 \theta\right)d\theta$

Solution

$$\int \left(1 + \tan^2 \theta\right)d\theta = \int \left(\sec^2 \theta\right)d\theta$$

$$= \tan\theta + C$$

Exercise

Find each indefinite integral $\int \frac{\csc\theta}{\csc\theta - \sin\theta}d\theta$

Solution

$$\int \frac{\csc\theta}{\csc\theta - \sin\theta}d\theta = \int \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta} - \sin\theta}d\theta$$

$$= \int \frac{\frac{1}{\sin\theta}}{\frac{1-\sin^2\theta}{\sin\theta}}d\theta \qquad \sin^2\theta + \cos^2\theta = 1 \Rightarrow 1 - \sin^2\theta = \cos^2\theta$$

$$= \int \frac{1}{\cos^2\theta}d\theta$$

$$= \int \sec^2\theta d\theta$$

$$\boxed{= \tan\theta + C}$$

Exercise

Solve the initial value problem: $\frac{dy}{dx} = 2x - 7, \quad y(2) = 0$

Solution

$$\frac{dy}{dx} = 2x - 7 \quad \Rightarrow \quad dy = (2x-7)dx$$

$$\int dy = \int (2x-7)dx$$

$$y = x^2 - 7x + C$$

At point (2, 0): $0 = 2^2 - 7(2) + C$

$$0 = 4 - 14 + C$$

$$0 = -10 + C$$

$$\boxed{C = 10}$$

$$\boxed{y = x^2 - 7x + 10}$$

Exercise

Solve the initial value problem: $\frac{dy}{dx} = \frac{1}{x^2} + x, \quad y(2) = 1; \ \ x > 0$

Solution

$$\frac{dy}{dx} = \frac{1}{x^2} + x \quad \rightarrow dy = \left(x^{-2} + x\right)dx$$

$$\int dy = \int \left(x^{-2} + x\right)dx$$

$$y = -x^{-1} + \frac{1}{2}x^2 + C$$

$$1 = -(2)^{-1} + \frac{1}{2}(2)^2 + C$$

$$1 + \frac{1}{2} - 2 = C \quad \Rightarrow \quad \boxed{C = -\frac{1}{2}}$$

$$y = -x^{-1} + \frac{1}{2}x^2 - \frac{1}{2} \quad or \quad y = -\frac{1}{x} + \frac{1}{2}x^2 - \frac{1}{2}$$

Exercise

Solve the initial value problem: $\frac{ds}{dt} = 1 + \cos t, \quad s(0) = 4$

Solution

$$\frac{ds}{dt} = 1 + \cos t \quad \Rightarrow ds = (1 + \cos t)\,dt$$

$$\int ds = \int (1 + \cos t)\,dt$$

$$s = t + \sin t + C$$

$$4 = 0 + \sin(0) + C \quad \Rightarrow \boxed{C = 4}$$

$$\boxed{s = t + \sin t + 4}$$

Exercise

Solve the initial value problem: $\frac{ds}{dt} = \cos t + \sin t, \quad s(\pi) = 1$

Solution

$$\frac{ds}{dt} = \cos t + \sin t \quad \rightarrow ds = (\cos t + \sin t)\,dt$$

$$\int ds = \int (\cos t + \sin t)\,dt$$

$$s = \sin t - \cos t + C$$

$$1 = \sin \pi - \cos \pi + C$$

$$1 = 0 - (-1) + C$$

$$1 = 1 + C$$

$$\boxed{C = 0}$$

$$\boxed{s = \sin t - \cos t}$$

Exercise

Derive the position function if a ball is thrown upward with initial velocity of 32 ft per second from an initial height of 48 ft. When does the ball hit the ground? With what velocity does the ball hit the ground?

Solution

$$s(t) = -16t^2 + 32t + 48$$

$$s(0) = 48$$

$$s'(0) = 32$$

$$s''(t) = -32$$

$$s'(t) = \int -32dt$$

$$= -32t + C_1$$

$$s'(0) = -32(0) + C_1 = 32$$

$$\Rightarrow C_1 = 32$$

$$s'(t) = -32t + 32$$

$$s(t) = \int (-32t + 32)dt$$

$$= -32\frac{t^2}{2} + 32t + C_2$$

$$s(0) = -32\frac{0^2}{2} + 32(0) + C_2 = 48 \qquad \Rightarrow C_2 = 48$$

$$s(t) = -16t^2 + 32t + 48$$

$$s(t) = -16t^2 + 32t + 48 = 0$$

$$-t^2 + 2t + 3 = 0 \ \Rightarrow t = -1,\ t = 3$$

The ball hits the ground in 3 seconds

The velocity: $v(t) = s'(t) = -32t + 32$

$$v(t = 3) = -32(3) + 32 = -64\ ft / \sec^2$$

Exercise

Suppose a publishing company has found that the marginal cost at a level of production of x thousand books is given by

$$\frac{dC}{dx} = \frac{50}{\sqrt{x}}$$

And that the fixed cost (the cost before the first book can be produced) is a $25,000. Find the cost function $C(x)$.

Solution

$$\frac{dC}{dx} = \frac{50}{\sqrt{x}} = 50x^{-1/2}$$

$$dC = 50x^{-1/2}dx$$

$$\int dC = \int 50x^{-1/2}dx$$

$$\begin{aligned} C(x) &= 50\frac{x^{1/2}}{1/2} + C \\ &= 50(2)x^{1/2} + C \\ &= 100\sqrt{x} + C \end{aligned}$$

$25000 = 100\sqrt{0} + C$ Before the first ($x = 0$) costs 25,000

$$\boxed{25000 = C}$$

$$C(x) = 100\sqrt{x} + 25{,}000$$

Exercise

Find the general solution of $F'(x) = 4x + 2$, and find the particular solution that satisfies the initial condition $F(1) = 8$.

Solution

$$\begin{aligned} F(x) &= \int (4x+2)dx \\ &= 4\frac{x^2}{2} + 2x + C \\ &= 2x^2 + 2x + C \end{aligned}$$

$$\begin{aligned} F(x) = 2(1)^2 + 2(1) + C &= 8 \\ 2 + 2 + C &= 8 \\ 4 + C &= 8 \end{aligned}$$

$$\boxed{C = 4} \qquad \Rightarrow F(x) = 2x^2 + 2x + 4$$

Exercise

The marginal cost function for producing x units of a product is modeled by

$$\frac{dC}{dx} = 28 - 0.02x$$

It costs \$40 to produce one unit. Find the cost of producing 200 units.

Solution

$$C = \int (28 - 0.02x)dx$$

$$= 28x - 0.02\frac{x^2}{2} + K$$

Cost \$40 for one unit $\Rightarrow$ C(x=1) = 40

$$C(x=1) = 28(1) - 0.01(1)^2 + K = 40$$

K = 12.01

$$C(x) = -0.01x^2 + 28x + 12.01$$

$$C(200) = -0.01(200)^2 + 28(200) + 12.01$$

$$\underline{= \$5212.01}$$

Solution *Section* 4.2 – Area and Estimating with Finite Sums

Exercise

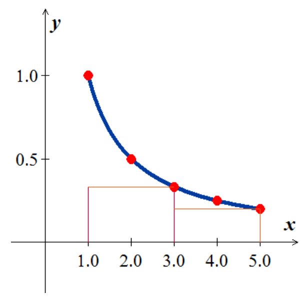
Use finite approximations to estimate the area under the graph of the function using

$$f(x)=\frac{1}{x} \quad between \quad x=1 \quad and \quad x=5$$

a) A lower sum with two rectangles of equal width
b) A lower sum with four rectangles of equal width
c) A upper sum with two rectangles of equal width
d) A upper sum with four rectangles of equal width

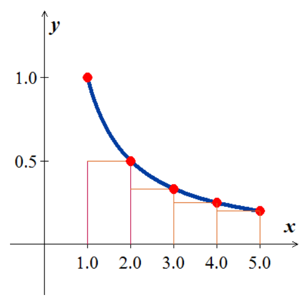
Solution

a) Using 2 lower rectangles: $\Delta x=\frac{5-1}{2}=2$



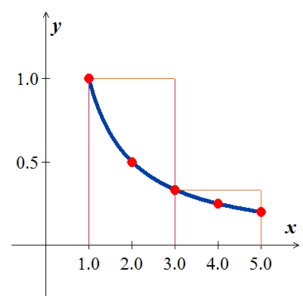
$$\begin{aligned} A &\approx \Delta x\left(f\left(x_1\right)+f\left(x_2\right)\right) \\ &\approx 2\cdot\left(f(3)+f(5)\right) \\ &\approx 2\cdot\left(\frac{1}{3}+\frac{1}{5}\right) \\ &\approx \boxed{\frac{16}{15}} \end{aligned}$$

b) Using 4 lower rectangles: $\Delta x=\frac{5-1}{4}=1$



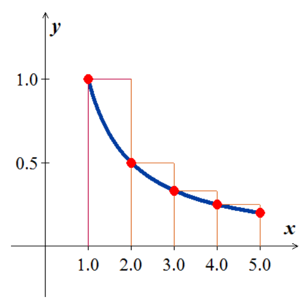
$$\begin{aligned} A &\approx 1\cdot\left(f(2)+f(3)+f(4)+f(5)\right) \\ &\approx 1\cdot\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right) \\ &\approx \boxed{\frac{77}{60}} \end{aligned}$$

c) Using 2 upper rectangles: $\Delta x=\frac{5-1}{2}=2$



$$\begin{aligned} A &\approx 2\cdot\left(f(1)+f(3)\right) \\ &\approx 2\cdot\left(1+\frac{1}{3}\right) \\ &\approx \boxed{\frac{8}{3}} \end{aligned}$$

d) Using 4 lower rectangles: $\Delta x=\frac{5-1}{4}=1$



$$\begin{aligned} A &\approx 1\cdot\left(f(1)+f(2)+f(3)+f(4)\right) \\ &\approx 1\cdot\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) \\ &\approx \boxed{\frac{25}{12}} \end{aligned}$$

Exercise

Use finite approximations to estimate the area under the graph of the function using

$$f(x) = 4 - x^2 \quad between \quad x = -2 \quad and \quad x = 2$$

a) A lower sum with two rectangles of equal width

b) A lower sum with four rectangles of equal width

c) A upper sum with two rectangles of equal width

d) A upper sum with four rectangles of equal width

Solution

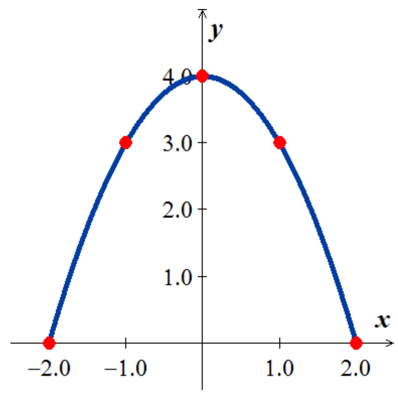
a) Using 2 lower rectangles: $\Delta x = \dfrac{2-(-2)}{2} = 2$

$$A \approx \Delta x\left(f\left(x_1\right) + f\left(x_2\right)\right)$$

$$\approx 2 \cdot \left(f(-2) + f(2)\right)$$

$$\approx 2 \cdot \left[\left(4-(-2)^2\right) + \left(4-2^2\right)\right]$$

$$\underline{= 0}$$



b) Using 4 lower rectangles: $\Delta x = \dfrac{2-(-2)}{4} = 1$

$$A \approx 1 \cdot \left(f(-2) + f(-1) + f(1) + f(2)\right)$$

$$\approx 1 \cdot (0 + 3 + 3 + 0)$$

$$\underline{= 6}$$

c) Using 2 upper rectangles: $\Delta x = \dfrac{2-(-2)}{2} = 2$

$$A \approx 2 \cdot \left(f(0) + f(0)\right)$$

$$\approx 2 \cdot (4 + 4)$$

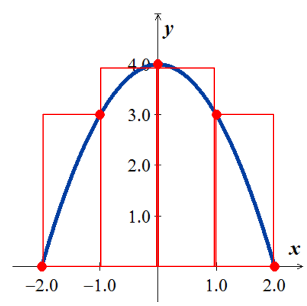
$$\underline{= 16}$$

d) Using 4 lower rectangles: $\Delta x = \dfrac{2-(-2)}{4} = 1$

$$A \approx 1 \cdot \left(f(-1) + f(0) + f(1) + f(2)\right)$$

$$\approx 1 \cdot (3 + 4 + 4 + 3)$$

$$\underline{= 14}$$

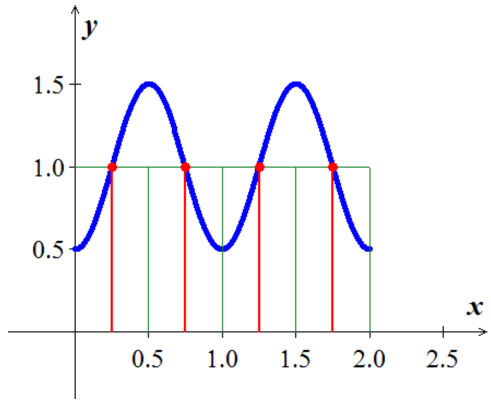


Exercise

Use finite approximations to estimate the average value of *f* on the given interval by partitioning the interval into four subintervals of equal length and evaluating *f* at the subinterval midpoints.

$$f(t)=\frac{1}{2}+\sin^2 \pi t \quad on \quad [0,\ 2]$$

Solution



$$\Delta x=\frac{2-0}{4}=0.5$$

$$f(t=.25)=\frac{1}{2}+\sin^2(.25\pi)=1$$

$$f(t=.75)=\frac{1}{2}+\sin^2(.75\pi)=1$$

$$f(t=1.25)=\frac{1}{2}+\sin^2(1.25\pi)=1$$

$$f(t=1.75)=\frac{1}{2}+\sin^2(1.75\pi)=1$$

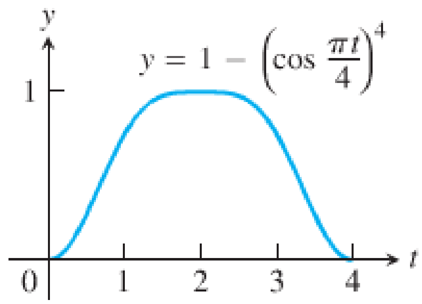
$$\begin{aligned}A &\approx .5\cdot\big(f(.25)+f(.75)+f(1.25)+f(1.75)\big)\\ &=.5(1+1+1+1)\\ &=2\end{aligned}$$

$$\begin{aligned}\text{Average value } &\approx \frac{Area}{Length\ [0,\ 2]}\\ &=\frac{2}{2}\\ &=1\end{aligned}$$

Exercise

Use finite approximations to estimate the average value of *f* on the given interval by partitioning the interval into four subintervals of equal length and evaluating *f* at the subinterval midpoints.

$$f(t) = 1 - \left(\cos\frac{\pi t}{4}\right)^4 \quad on \quad [0,\ 4]$$



Solution

$$\Delta x = \frac{4-0}{4} = 1$$

$$f(t = 0.5) = 1 - \left(\cos\frac{0.5\pi}{4}\right)^4 = 0.27145$$

$$f(t = 1.5) = 1 - \left(\cos\frac{1.5\pi}{4}\right)^4 = 0.97855$$

$$f(t = 2.5) = 1 - \left(\cos\frac{2.5\pi}{4}\right)^4 = 0.97855$$

$$f(t = 3.5) = 1 - \left(\cos\frac{3.5\pi}{4}\right)^4 = 0.27145$$

$$\begin{aligned} A &\approx 1\cdot\left(f(.5) + f(1.5) + f(2.5) + f(3.5)\right) \\ &= 1(0.27145 + 0.97855 + 0.97855 + 0.27145) \\ &= 2.5 \end{aligned}$$

$$\begin{aligned} \text{Average value } &\approx \frac{Area}{Length\ [0,\ 2]} \\ &= \frac{2.5}{4} \\ &= 0.625 \end{aligned}$$

Solution Section 4.3 – Sigma Notation and Limits of Finite Sums

Exercise

Write the sums without sigma notation. Then evaluate: $\sum_{k=1}^{2} \frac{6k}{k+1}$

Solution

$$\sum_{k=1}^{2} \frac{6k}{k+1} = \frac{6}{2} + \frac{12}{3} \underline{=7}$$

Exercise

Write the sums without sigma notation. Then evaluate: $\sum_{k=1}^{3} \frac{k-1}{k}$

Solution

$$\sum_{k=1}^{3} \frac{k-1}{k} = \frac{0}{1} + \frac{1}{2} + \frac{2}{3} \underline{= \frac{7}{6}}$$

Exercise

Write the sums without sigma notation. Then evaluate: $\sum_{k=1}^{5} \sin k\pi$

Solution

$$\sum_{k=1}^{5} \sin k\pi = \sin \pi + \sin 2\pi + \sin 3\pi + \sin 4\pi + \sin 5\pi = 0 + 0 + 0 + 0 \underline{=0}$$

Exercise

Write the sums without sigma notation. Then evaluate: $\sum_{k=1}^{4} (-1)^k \cos k\pi$

Solution

$$\sum_{k=1}^{4} (-1)^k \cos k\pi = -\cos \pi + \cos 2\pi - \cos 3\pi + \cos 4\pi = -(-1) + 1 - (-1) + 1 \underline{=4}$$

Exercise

Write the following expression 1 + 2 + 4 + 8 + 16 + 32 in sigma notation

Solution

$$1+2+4+8+16+32=\sum_{k=1}^{6}2^{k-1}$$

$$1+2+4+8+16+32=\sum_{k=0}^{5}2^{k}$$

Exercise

Write the following expression 1 – 2 + 4 – 8 + 16 – 32 in sigma notation

Solution

$$1-2+4-8+16-32=\sum_{k=1}^{6}(-2)^{k-1}$$

$$1-2+4-8+16-32=\sum_{k=0}^{2}(-1)^{k}\,2^{k}$$

Exercise

Write the following expression $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ in sigma notation

Solution

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}=\sum_{k=1}^{4}\frac{1}{2^k}$$

Exercise

Write the following expression $-\frac{1}{5}+\frac{2}{5}-\frac{3}{5}+\frac{4}{5}-\frac{5}{5}$ in sigma notation

Solution

$$-\frac{1}{5}+\frac{2}{5}-\frac{3}{5}+\frac{4}{5}-\frac{5}{5}=\sum_{k=1}^{5}(-1)^{k}\,\frac{k}{5}$$

Exercise

Suppose that $\sum_{k=1}^{n} a_k = -5$ *and* $\sum_{k=1}^{n} b_k = 6$. Find the value of $\sum_{k=1}^{n} \left(b_k - 2a_k\right)$

Solution

$$\sum_{k=1}^{n} \left(b_k - 2a_k\right) = \sum_{k=1}^{n} b_k - 2\sum_{k=1}^{n} a_k$$

$$= 6 - 2(-5)$$

$$\underline{= 16}$$

Exercise

Evaluate the sums $\sum_{k=1}^{10} k^3$

Solution

$$\sum_{k=1}^{10} k^3 = \left(\frac{10(10+1)}{2}\right)^2$$

$$= 55^2$$

$$\underline{= 3025}$$

Exercise

Evaluate the sums $\sum_{k=1}^{7} (-2k)$

Solution

$$\sum_{k=1}^{7} (-2k) = -2\sum_{k=1}^{7} k$$

$$= -2\left(\frac{7(7+1)}{2}\right)$$

$$\underline{= -56}$$

Exercise

Evaluate the sums $\displaystyle\sum_{k=1}^{5} \frac{\pi k}{15}$

Solution

$$\sum_{k=1}^{5} \frac{\pi k}{15} = \frac{\pi}{15} \sum_{k=1}^{5} k$$

$$= \frac{\pi}{15}\left(\frac{5(5+1)}{2}\right)$$

$$= \pi$$

Exercise

Evaluate the sums $\displaystyle\sum_{k=1}^{5} k(3k+5)$

Solution

$$\sum_{k=1}^{5} k(3k+5) = \sum_{k=1}^{5}\left(3k^2 + 5k\right)$$

$$= 3\sum_{k=1}^{5} k^2 + 5\sum_{k=1}^{5} k$$

$$= 3\left(\frac{5(5+1)(2(5)+1)}{6}\right) + 5\frac{5(5+1)}{2}$$

$$= 240$$

Exercise

Evaluate the sums $\displaystyle\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$

Solution

$$\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3 = \frac{1}{225}\sum_{k=1}^{5} k^3 + \left(\sum_{k=1}^{5} k\right)^3$$

$$= \frac{1}{225}\left(\frac{5(5+1)}{2}\right)^2 + \left(\frac{5(5+1)}{2}\right)^3$$

$$\underline{= 3376}$$

Exercise

Evaluate the sums $\displaystyle\sum_{k=1}^{500} 7$

Solution

$$\sum_{k=1}^{500} 7 = 7(500) \ \underline{= 3500}$$

Exercise

Evaluate the sums $\displaystyle\sum_{k=18}^{71} k(k-1)$

Solution

Let $n = (k-18)+1 = k-17 \begin{cases} k=18 & \to n=1 \\ k=71 & \to n=54 \end{cases} \Rightarrow \quad k = n+17$

$$\sum_{k=18}^{71} k(k-1) = \sum_{n=1}^{54} (n+17)(n+17-1)$$

$$= \sum_{n=1}^{54} (n+17)(n+16)$$

$$= \sum_{n=1}^{54} \left(n^2 + 33n + 272\right)$$

$$= \sum_{n=1}^{54} n^2 + 33\sum_{n=1}^{54} n + \sum_{n=1}^{54} 272$$

$$= \frac{54(54+1)(54(2)+1)}{6} + 33 \cdot \frac{54(54+1)}{6} + 272(54)$$

$$\underline{= 117648}$$

Exercise

Evaluate the sums $\displaystyle\sum_{k=1}^{n}\left(\frac{1}{n}+2n\right)$

Solution

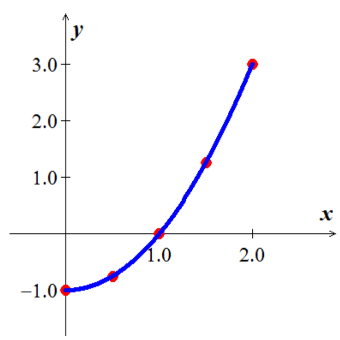
$$\sum_{k=1}^{n}\left(\frac{1}{n}+2n\right)=n\cdot\left(\frac{1}{n}+2n\right)\ \underline{=1+2n^2}$$

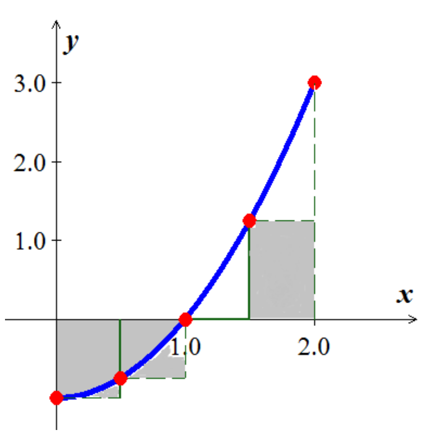
Exercise

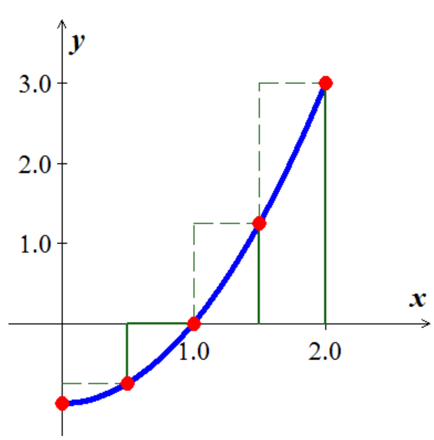
Graph the function $f(x)=x^2-1$ over the given interval [0, 2]. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\displaystyle\sum_{k=1}^{4}f\left(c_k\right)\Delta x_k$, given c_k is the

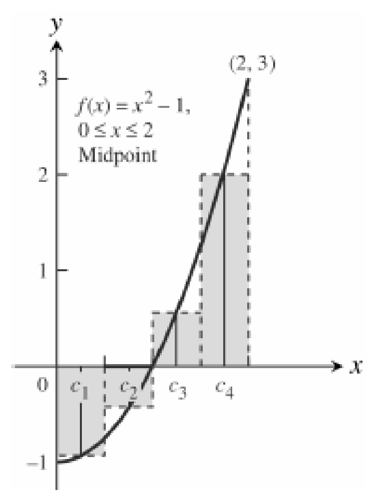
a) Left-hand endpoint
b) Right-hand endpoint
c) Midpoint of kth subinterval.

Solution









Solution *Section* 4.5 – The Fundamental Theorem of Calculus

Exercise

Evaluate the integrals $\displaystyle\int_0^2 x(x-3)\,dx$

Solution

$$\int_0^2 x(x-3)\,dx = \int_0^2 \left(x^2-3x\right)dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^2$$

$$= \left(\frac{2^3}{3} - \frac{3(2)^2}{2}\right) - \left(\frac{0^3}{3} - \frac{3(2)^2}{2}\right)$$

$$\underline{= -\frac{10}{3}}$$

Exercise

Evaluate the integrals $\displaystyle\int_0^4 \left(3x - \frac{x^3}{4}\right)dx$

Solution

$$\int_0^4 \left(3x - \frac{x^3}{4}\right)dx = \left[3\frac{x^2}{2} - \frac{x^4}{16}\right]_0^4$$

$$= \left(3\frac{(4)^2}{2} - \frac{(4)^4}{16}\right) - 0$$

$$\underline{= 8}$$

Exercise

Evaluate the integrals $\displaystyle\int_{-2}^{2}\left(x^3-2x+3\right)dx$

Solution

$$\begin{aligned}\int_{-2}^{2}\left(x^3-2x+3\right)dx &= \left[\frac{x^4}{4}-x^2+3x\right]_{-2}^{2}\\ &= \left(\frac{(2)^4}{4}-(2)^2+3(2)\right)-\left(\frac{(-2)^4}{4}-(-2)^2+3(-2)\right)\\ &= 12\end{aligned}$$

Exercise

Evaluate the integrals $\displaystyle\int_{0}^{1}\left(x^2+\sqrt{x}\right)dx$

Solution

$$\begin{aligned}\int_{0}^{1}\left(x^2+\sqrt{x}\right)dx &= \left[\frac{x^3}{3}+\frac{2}{3}x^{3/2}\right]_{0}^{1}\\ &= \left(\frac{(1)^3}{3}+\frac{2}{3}(1)^{3/2}\right)-0\\ &= 1\end{aligned}$$

Exercise

Evaluate the integrals $\displaystyle\int_{0}^{\pi/3}4\sec u\tan u\ du$

Solution

$$\begin{aligned}\int_{0}^{\pi/3}4\sec u\tan u\ du &= 4\sec u\Big|_{0}^{\pi/3}\\ &= 4\left(\sec\frac{\pi}{3}-\sec 0\right)\\ &= 4(2-1)\\ &= 4\end{aligned}$$

Exercise

Evaluate the integrals $\displaystyle\int_{\pi/4}^{3\pi/4} \csc\theta \cot\theta \, d\theta$

Solution

$$\int_{\pi/4}^{3\pi/4} \csc\theta \cot\theta \, d\theta = -\csc\theta \Big|_{\pi/4}^{3\pi/4}$$

$$= -\left(\csc\frac{3\pi}{4} - \csc\frac{\pi}{4}\right)$$

$$= -\left(\sqrt{2} - \sqrt{2}\right)$$

$$\underline{= 0}$$

Exercise

Evaluate the integrals $\displaystyle\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt$

Solution

$$\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt = \int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \pi t^{-2}\right) dt$$

$$= \left[4\tan t - \pi t^{-1}\right]_{-\pi/3}^{-\pi/4}$$

$$= \left(4\tan\left(-\frac{\pi}{4}\right) - \pi\left(-\frac{4}{\pi}\right)\right) - \left(4\tan\left(-\frac{\pi}{3}\right) - \pi\left(-\frac{3}{\pi}\right)\right)$$

$$= \left(4(-1) + 4\right) - \left(4\left(-\sqrt{3}\right) + 3\right)$$

$$= -\left(-4\sqrt{3} + 3\right)$$

$$\underline{= 4\sqrt{3} - 3}$$

Exercise

Evaluate the integrals $\displaystyle\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} \, dy$

Solution

$$\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} \, dy = \int_{-3}^{-1} \left(\frac{y^5}{y^3} - \frac{2y}{y^3}\right) dy$$

$$= \int_{-3}^{-1} \left(y^2 - 2y^{-2} \right) dy$$

$$= \left[\frac{1}{3} y^3 + 2y^{-1} \right]_{-3}^{-1}$$

$$= \left(\frac{1}{3}(-1)^3 + \frac{2}{-1} \right) - \left(\frac{1}{3}(-3)^3 + \frac{2}{-3} \right)$$

$$= \frac{22}{3}$$

Exercise

Evaluate the integrals $\displaystyle\int_1^8 \frac{\left(x^{1/3}+1\right)\left(2-x^{2/3}\right)}{x^{1/3}} dx$

Solution

$$\int_1^8 \frac{\left(x^{1/3}+1\right)\left(2-x^{2/3}\right)}{x^{1/3}} dx = \int_1^8 \frac{2x^{1/3} - x + 2 - x^{2/3}}{x^{1/3}} dx$$

$$= \int_1^8 \left(2 - x^{2/3} + 2x^{-1/3} - x^{1/3}\right) dx$$

$$= \left[2x - \frac{3}{5} x^{5/3} + 3x^{2/3} - \frac{3}{4} x^{4/3} \right]_1^8$$

$$= \left(2(8) - \frac{3}{5}(8)^{5/3} + 3(8)^{2/3} - \frac{3}{4}(8)^{4/3} \right) - \left(2(1) - \frac{3}{5}(1)^{5/3} + 3(1)^{2/3} - \frac{3}{4}(1)^{4/3} \right)$$

$$= \left(-\frac{16}{5} \right) - \left(\frac{73}{20} \right)$$

$$= -\frac{137}{20}$$

Exercise

Evaluate the integrals $\displaystyle\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x} dx$

Solution

$$\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x} dx = \int_{\pi/2}^{\pi} \frac{2\sin x \cos x}{2\sin x} dx$$

$$= \int_{\pi/2}^{\pi} \cos x dx$$

$$= \sin x \Big|_{\pi/2}^{\pi}$$

$$= \sin \pi - \sin \frac{\pi}{2}$$

$$\underline{= -1}$$

Exercise

Evaluate the integrals $\displaystyle\int_0^{\pi/3} (\cos x + \sec x)^2 \, dx$

Solution

$$\int_0^{\pi/3} (\cos x + \sec x)^2 \, dx = \int_0^{\pi/3} \left(\cos^2 x + 2 + \sec^2 x\right) dx$$

$$= \int_0^{\pi/3} \left(\frac{1}{2} + \frac{1}{2}\cos 2x + 2 + \sec^2 x\right) dx$$

$$= \int_0^{\pi/3} \left(\frac{5}{2} + \frac{1}{2}\cos 2x + \sec^2 x\right) dx$$

$$= \left[\frac{5}{2}x + \frac{1}{4}\sin 2x + \tan x\right]_0^{\pi/3}$$

$$= \left(\frac{5}{2}\frac{\pi}{3} + \frac{1}{4}\sin\frac{2\pi}{3} + \tan\frac{\pi}{3}\right) - \left(\frac{5}{2}(0) + \frac{1}{4}\sin(2\cdot 0) + \tan(0)\right)$$

$$= \frac{5\pi}{6} + \frac{1}{4}\frac{\sqrt{3}}{2} + \sqrt{3}$$

$$\underline{= \frac{5\pi}{6} + \frac{9\sqrt{3}}{8}}$$

Exercise

Evaluate the integrals $\displaystyle\int_0^{\pi} \frac{1}{2}\left(\cos x + |\cos x|\right) dx$

Solution

$$\int_0^{\pi} \frac{1}{2}\left(\cos x + |\cos x|\right) dx = \int_0^{\pi/2} \frac{1}{2}(\cos x + \cos x)\, dx + \int_{\pi/2}^{\pi} \frac{1}{2}(\cos x - \cos x)\, dx$$

$$= \int_0^{\pi/2} \cos x dx$$

$$= \sin x \Big|_{0}^{\pi/2}$$

$$\underline{\boxed{= 1}}$$

Exercise

Find the total area between the region and the x-axis $\qquad y = -x^2 - 2x, \quad -3 \le x \le 2$

Solution

$$-x^2 - 2x = 0 \rightarrow -x(x+2) = 0 \quad \Rightarrow \boxed{x = -2, 0}$$

$$A = -\int_{-3}^{-2} \left(-x^2 - 2x\right) dx + \int_{-2}^{0} \left(-x^2 - 2x\right) dx - \int_{0}^{2} \left(-x^2 - 2x\right) dx$$

$$= -\left[-\frac{1}{3}x^3 - x^2\right]_{-3}^{-2} + \left[-\frac{1}{3}x^3 - x^2\right]_{-2}^{0} - \left[-\frac{1}{3}x^3 - x^2\right]_{0}^{2}$$

$$= -\left[\left(-\tfrac{1}{3}(-2)^3 - (-2)^2\right) - \left(-\tfrac{1}{3}(-3)^3 - (-3)^2\right)\right] + \left[0 - \left(-\tfrac{1}{3}(-2)^3 - (-2)^2\right)\right] - \left[\left(-\tfrac{1}{3}(2)^3 - (2)^2\right) - 0\right]$$

$$= -\frac{4}{3} + \frac{4}{3} - \frac{20}{3}$$

$$\underline{\boxed{= \frac{28}{3}}}$$

Exercise

Find the total area between the region and the x-axis $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$

Solution

$$x^3 - 3x^2 + 2x = 0 \quad \boxed{x = 0,\ 1,\ 2}$$

$$A = \int_{0}^{1} \left(x^3 - 3x^2 + 2x\right) dx - \int_{1}^{2} \left(x^3 - 3x^2 + 2x\right) dx$$

$$= \left[\frac{1}{4}x^4 - x^3 + x^2\right]_{0}^{1} - \left[\frac{1}{4}x^4 - x^3 + x^2\right]_{1}^{2}$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$\underline{\boxed{= \frac{1}{2}}}$$

Exercise

Find the total area between the region and the x-axis $y = x^{1/3} - x, \quad -1 \le x \le 8$

Solution

$$x^{1/3} - x = 0 \to x^{1/3}\left(1 - x^{2/3}\right) = 0 \quad \boxed{x = 0, \ \pm 1}$$

$$\begin{aligned}
A &= -\int_{-1}^{0}\left(x^{1/3} - x\right)dx + \int_{0}^{1}\left(x^{1/3} - x\right)dx - \int_{1}^{8}\left(x^{1/3} - x\right)dx \\
&= -\left[\frac{3}{4}x^{4/3} - \frac{1}{2}x^2\right]_{-1}^{0} + \left[\frac{3}{4}x^{4/3} - \frac{1}{2}x^2\right]_{0}^{1} - \left[\frac{3}{4}x^{4/3} - \frac{1}{2}x^2\right]_{1}^{8} \\
&= \left(\frac{3}{4} - \frac{1}{2}\right) + \left(\frac{3}{4} - \frac{1}{2}\right) - \left[(12 - 32) - \left(\frac{3}{4} - \frac{1}{2}\right)\right] \\
&= \frac{1}{4} + \frac{1}{4} + \frac{81}{4} \\
&= \frac{83}{4}
\end{aligned}$$

Exercise

Find the total area between the region and the x-axis $f(x) = x^2 + 1, \qquad 2 \le x \le 3$

Solution

$$\begin{aligned}
\int_2^3 (x^2 + 1)dx &= \left[\frac{1}{3}x^3 + x\right]_2^3 \\
&= \left(\frac{1}{3}3^3 + 3\right) - \left(\frac{1}{3}2^3 + 2\right) \\
&= (9 + 3) - \left(\frac{8}{3} + 2\right) \\
&= 12 - \left(\frac{14}{3}\right) \\
&= \frac{22}{3} \\
&= 7.3
\end{aligned}$$

Exercise

Archimedes, inventor, military engineer, physicist, and the greatest mathematician of classical times in the Western world, discovered that the area under a parabolic arch is two-thirds the base times the height. Sketch the parabolic arch $y = h - \left(\frac{4h}{b^2}\right)x^2 \quad -\frac{b}{2} \le x \le \frac{b}{2}$, assuming that h and b are positive. Then use calculus to find the area of the region enclosed between the arch and the x-axis

Solution

$$A = \int_{-b/2}^{b/2} \left(h - \left(\frac{4h}{b^2}\right)x^2\right)dx$$

$$= \left[hx - \frac{4h}{b^2}\frac{x^3}{3}\right]_{-b/2}^{b/2}$$

$$= \left(\frac{hb}{2} - \frac{4h}{3b^2}\frac{b^3}{8}\right) - \left(-\frac{hb}{2} + \frac{4h}{3b^2}\frac{b^3}{8}\right)$$

$$= \left(\frac{hb}{2} - \frac{hb}{6}\right) - \left(-\frac{hb}{2} + \frac{hb}{6}\right)$$

$$= \frac{hb}{3} + \frac{hb}{3}$$

$$= \frac{2}{3}bh$$

Exercise

Suppose that a company's marginal revenue from the manufacture and sale of eggbeaters is

$$\frac{dr}{dx} = 2 - \frac{2}{(x+1)^2}$$

Where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand eggbeaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

Solution

$$r = \int_0^3 \left(2 - 2(x+1)^{-2}\right)dx \qquad\qquad d(x+1) = dx$$

$$= \int_0^3 2dx - \int_0^3 2(x+1)^{-2}\, d(x+1)$$

$$= 2x\Big|_0^3 + 2(x+1)^{-1}\Big|_0^3$$

$$= 6 + 2(4)^{-1} - 2$$

$$= 4.5 \quad or \quad \$4500.00$$

Exercise

The height H (ft) of a palm tree after growing for t years is given by

$$H = \sqrt{t+1} + 5t^{1/3} \quad for \quad 0 \le t \le 8$$

a) Find the tree's height when $t = 0$, $t = 4$, and $t = 8$.

b) Find the tree's average height for $0 \le t \le 8$

Solution

a) $t = 0 \Rightarrow H = 1\,ft$

$t = 4 \Rightarrow H = 10.17\,ft$

$t = 8 \Rightarrow H = 13\,ft$

b) Average height $= \frac{1}{8-0}\int_0^8 \left(\sqrt{t+1} + 5t^{1/3}\right)dt$ $\qquad d(t+1) = dt$

$$= \frac{1}{8}\int_0^8 (t+1)^{1/2}\,d(t+1) + \frac{5}{8}\int_0^8 t^{1/3}dt$$

$$= \left[\frac{1}{12}(t+1)^{3/2} + \frac{15}{32}t^{4/3}\right]_0^8$$

$$= \frac{1}{12}(9)^{3/2} + \frac{15}{32}(8)^{4/3} - \frac{1}{12}$$

$$\approx 9.67\ ft$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int 2(2x+4)^5\,dx, \quad u=2x+4$$

Solution

Let $u=2x+4 \quad \Rightarrow \quad du=2xdx$

$$\int 2(2x+4)^5\,dx = \int u^5 du$$

$$= \frac{1}{6}u^6 + C$$

$$\boxed{= \frac{1}{6}(2x+4)^6 + C}$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int \frac{4x^3}{(x4+1)^2}\,dx, \quad u=x^4+1$$

Solution

Let $u=x^4+1 \quad \Rightarrow \quad du=4x^3dx$

$$\int \frac{4x^3}{(x4+1)^2}\,dx = \int \frac{du}{u^2}$$

$$= \int u^{-2}du$$

$$= \frac{u^{-1}}{-1} + C$$

$$\boxed{= \frac{-1}{x^2+1} + C}$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int x\sin\left(2x^2\right)dx, \quad u = 2x^2$$

Solution

Let $u = 2x^2 \quad \Rightarrow \quad du = 4xdx \to \ \frac{1}{4}du = xdx$

$$\int x\sin\left(2x^2\right)dx = \int \frac{1}{4}\sin u du$$

$$= -\frac{1}{4}\cos u + C$$

$$= -\frac{1}{4}\cos\left(2x^2\right) + C$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int 12\left(y^4 + 4y^2 + 1\right)^2\left(y^3 + 2y\right)dx, \quad u = y^4 + 4y^2 + 1$$

Solution

Let $u = y^4 + 4y^2 + 1 \quad \Rightarrow \quad du = \left(4y^3 + 8y\right)dx \to \quad du = 4\left(y^3 + 2y\right)dx$

$$\int 12\left(y^4 + 4y^2 + 1\right)^2\left(y^3 + 2y\right)dx = \int 12u^2\left(\tfrac{1}{4}du\right)$$

$$= 3\int u^2\ du$$

$$= 3\frac{u^3}{3} + C$$

$$= \left(y^4 + 4y^2 + 1\right)^3 + C$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int \csc^2 2\theta \cot 2\theta \, d\theta \quad \rightarrow \begin{cases} a) \text{ Using } \ u = \cot 2\theta \\ b) \text{ Using } \ u = \csc 2\theta \end{cases}$$

Solution

Let $u = \cot 2\theta \quad \Rightarrow \quad du = -2\csc^2 2\theta d\theta \rightarrow \quad -\frac{1}{2}du = \csc^2 2\theta dx$

$$\int \csc^2 2\theta \cot 2\theta \, d\theta = -\int \frac{1}{2} u du$$

$$= -\frac{1}{2}\frac{u^2}{2} + C$$

$$\boxed{= -\frac{1}{4}\cot^2 2\theta + C}$$

Let $u = \csc 2\theta \quad \Rightarrow \quad du = -2\csc 2\theta \cot 2\theta d\theta \rightarrow \quad -\frac{1}{2}du = \csc 2\theta \cot 2\theta dx$

$$\int \csc^2 2\theta \cot 2\theta \, d\theta = \int \csc 2\theta \left(\csc 2\theta \cot 2\theta \, d\theta\right)$$

$$= -\int \frac{1}{2} u du$$

$$= -\frac{1}{2}\frac{u^2}{2} + C$$

$$\boxed{= -\frac{1}{4}\csc^2 2\theta + C}$$

Exercise

Evaluate the integrals $\quad \displaystyle\int \frac{1}{\sqrt{5s+4}} ds$

Solution

Let $u = 5s + 4 \quad \Rightarrow \quad du = 5ds \quad \rightarrow \quad \frac{1}{5}du = ds$

$$\int \frac{1}{\sqrt{5s+4}} ds = \frac{1}{5}\int u^{-1/2} du$$

$$= \frac{1}{5}\frac{u^{1/2}}{1/2} + C$$

$$\boxed{= \frac{2}{5}\sqrt{5s+4} + C}$$

Exercise

Evaluate the integrals $\int \theta\sqrt[4]{1-\theta^2}\,d\theta$

Solution

Let $u=1-\theta^2 \quad \Rightarrow \quad du=-2\theta d\theta \quad \rightarrow \quad -\frac{1}{2}du=\theta d\theta$

$$\int \theta\sqrt[4]{1-\theta^2}\,d\theta=-\frac{1}{2}\int u^{1/4}du$$

$$=-\frac{1}{2}\frac{u^{5/4}}{5/4}+C$$

$$\boxed{=-\frac{2}{5}\left(1-\theta^2\right)^{5/4}+C}$$

Exercise

Evaluate the integrals $\int \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}dx$

Solution

Let $u=1+\sqrt{x} \quad \Rightarrow \quad du=\frac{1}{2\sqrt{x}}dx \quad \rightarrow \quad 2du=\frac{1}{\sqrt{x}}dx$

$$\int \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}dx=\int \frac{2}{u^2}du$$

$$=2\int u^{-2}du$$

$$=2\frac{u^{-1}}{-1}+C$$

$$\boxed{=\frac{-2}{1+\sqrt{x}}+C}$$

Exercise

Evaluate the integrals $\int \tan^2 x\sec^2 x\,dx$

Solution

Let $u=\tan x \quad \Rightarrow \quad du=\sec^2 x dx$

$$\int \tan^2 x \sec^2 x \, dx = \int u^2 du$$
$$= \frac{1}{3}u^3 + C$$
$$\boxed{= \frac{1}{3}\tan^3 x + C}$$

Exercise

Evaluate the integrals $\int \sin^5 \frac{x}{3} \cos \frac{x}{3} \, dx$

Solution

Let $u = \sin\left(\frac{x}{3}\right) \quad \Rightarrow \quad du = \frac{1}{3}\cos\left(\frac{x}{3}\right)dx \to 3du = \cos\left(\frac{x}{3}\right)dx$

$$\int \sin^5 \frac{x}{3} \cos \frac{x}{3} \, dx = \int u^5 (3du)$$
$$= 3\frac{u^6}{6} + C$$
$$\boxed{= \frac{1}{2}\sin^6\left(\frac{x}{3}\right) + C}$$

Exercise

Evaluate the integrals $\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2} \, dx$

Solution

Let $u = \tan\left(\frac{x}{2}\right)$

$$du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx \to 2du = \sec^2\left(\frac{x}{2}\right)dx$$

$$\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2} \, dx = 2\int u^7 du$$
$$= 2\frac{1}{8}u^8 + C$$
$$\boxed{= \frac{1}{4}\tan^8 \frac{x}{2} + C}$$

Exercise

Evaluate the integrals $\displaystyle\int r^4\left(7-\frac{r^5}{10}\right)^3 dr$

Solution

Let $u = 7-\dfrac{r^5}{10}$

$$du = -\frac{1}{10}5r^4 dr \rightarrow \ -2du = r^4 dr$$

$$\int r^4\left(7-\frac{r^5}{10}\right)^3 dr = \int u^3(-2du)$$

$$= -2\int u^3 du$$

$$= -2\frac{u^4}{4}+C$$

$$\boxed{= -\frac{1}{2}\left(7-\frac{r^5}{10}\right)^4 + C}$$

Exercise

Evaluate the integrals $\displaystyle\int x^{1/2}\sin\left(x^{3/2}+1\right)dx$

Solution

Let $u = x^{3/2}+1 \ \Rightarrow \ du = \dfrac{3}{2}x^{1/2}dx \rightarrow \ \dfrac{2}{3}du = x^{1/2}dx$

$$\int x^{1/2}\sin\left(x^{3/2}+1\right)dx = \int \sin u \ \left(\tfrac{2}{3}du\right)$$

$$= \frac{2}{3}\int \sin u \ \ du$$

$$= \frac{2}{3}(-\cos u)+C$$

$$\boxed{= -\frac{2}{3}\cos\left(x^{3/2}+1\right)+C}$$

Exercise

Evaluate the integrals $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$

Solution

Let $u = \csc\left(\frac{v-\pi}{2}\right) \quad \Rightarrow \quad du = -\frac{1}{2}\csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv \quad \to \quad -2du = \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$

$$\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv = \int -2du$$

$$= -2u + C$$

$$\boxed{= -2\csc\left(\frac{v-\pi}{2}\right) + C}$$

Exercise

Evaluate the integrals $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}dt$

Solution

Let $u = \cos(2t+1) \quad \Rightarrow \quad du = -2\sin(2t+1)dt \quad \to \quad -\frac{1}{2}du = \sin(2t+1)dt$

$$\int \frac{\sin(2t+1)}{\cos^2(2t+1)}dt = -\frac{1}{2}\int \frac{du}{u^2}$$

$$= \frac{1}{2u} + C$$

$$\boxed{= \frac{1}{2\cos(2t+1)} + C}$$

Exercise

Evaluate the integrals $\int \frac{\sec z\ \tan z}{\sqrt{\sec z}}dz$

Solution

Let $u = \sec z \quad \Rightarrow \quad du = \sec z \tan z dz$

$$\int \frac{\sec z\ \tan z}{\sqrt{\sec z}}dz = \int \frac{du}{u^{1/2}}$$

$$= \int u^{-1/2}du$$

$$= 2u^{1/2} + C$$

$$\boxed{= 2\sqrt{\sec z} + C}$$

Exercise

Evaluate the integrals $\int \frac{1}{\sqrt{t}}\cos\left(\sqrt{t}+3\right)dt$

Solution

Let $u=\sqrt{t}+3 \quad \Rightarrow \quad du=\frac{1}{2\sqrt{t}}dt \quad \to \quad 2du=\frac{1}{\sqrt{t}}dt$

$$\int \frac{1}{\sqrt{t}}\cos\left(\sqrt{t}+3\right)dt=\int (\cos u)(2du)$$

$$=2\int \cos u\ du$$

$$=2\sin u+C$$

$$\boxed{=2\sin\left(\sqrt{t}+3\right)+C}$$

Exercise

Evaluate the integrals $\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\ d\theta$

Solution

Let $u=\sin\frac{1}{\theta} \quad \Rightarrow \quad du=\left(\cos\frac{1}{\theta}\right)\left(\frac{1}{\theta}\right)'=\left(\cos\frac{1}{\theta}\right)\left(-\frac{1}{\theta^2}\right)d\theta$

$$\to \quad -du=\frac{1}{\theta^2}\cos\frac{1}{\theta}d\theta$$

$$\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\ d\theta=-\int udu$$

$$=-\frac{1}{2}u^2+C$$

$$\boxed{=-\frac{1}{2}\sin^2\frac{1}{\theta}+C}$$

Exercise

Evaluate the integrals $\int t^3\left(1+t^4\right)^3 dt$

Solution

Let $u=1+t^4 \quad \Rightarrow \quad du=4t^3dt \quad \rightarrow \quad \frac{1}{4}du=t^3dt$

$$\int t^3\left(1+t^4\right)^3 dt=\frac{1}{4}\int u^3du$$

$$=\frac{1}{4}\left(\frac{u^4}{4}\right)+C$$

$$\boxed{=\frac{1}{16}\left(1+t^4\right)^3+C}$$

Exercise

Evaluate the integrals $\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}dx$

Solution

Let $u=\frac{x^2-1}{x^2}=1-\frac{1}{x^2}=1-x^{-2}$

$$du=2x^{-3}dx \quad \rightarrow \quad \frac{1}{2}du=\frac{1}{x^3}dx$$

$$\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}dx=\int u^{1/2}\left(\frac{1}{2}du\right)$$

$$=\frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)+C$$

$$\boxed{=\frac{1}{3}\left(1-\frac{1}{x^2}\right)^{3/2}+C}$$

Exercise

Evaluate the integrals $\int x^3\sqrt{x^2+1}\,dx$

Solution

Let $u = x^2 + 1 \quad \Rightarrow \quad du = 2xdx \quad \to \quad \frac{1}{2}du = xdx$

$x^2 = u - 1$

$$\int x^3\sqrt{x^2+1}\,dx = \int x^2\sqrt{x^2+1}\,xdx$$

$$= \int (u-1)u^{1/2}\left(\tfrac{1}{2}du\right)$$

$$= \tfrac{1}{2}\int \left(u^{3/2} - u^{1/2}\right)du$$

$$= \tfrac{1}{2}\left(\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right) + C$$

$$\boxed{= \tfrac{1}{5}\left(x^2+1\right)^{5/2} - \tfrac{1}{3}\left(x^2+1\right)^{3/2} + C}$$

Exercise

Evaluate the integrals $\int \frac{x}{\left(x^2-4\right)^3}dx$

Solution

Let $u = x^2 - 4 \quad \Rightarrow \quad du = 2xdx \quad \to \quad \frac{1}{2}du = xdx$

$$\int \frac{x}{\left(x^2-4\right)^3}dx = \int u^{-3}du$$

$$= \frac{u^{-2}}{-2} + C$$

$$\boxed{= -\frac{1}{2\left(x^2-4\right)^2} + C}$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\,dr$

Solution

Let $u=\sqrt{3(2r-1)^2+6} \;\Rightarrow\; du=\frac{1}{2}\left(3(2r-1)^2+6\right)^{-1/2}\left(6(2r-1)(2)\right)dr$

$$=\frac{6(2r-1)}{\left(3(2r-1)^2+6\right)^{1/2}}dr$$

$$\to\; \frac{1}{6}du=\frac{2r-1}{\sqrt{3(2r-1)^2+6}}dr$$

$$\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\,dr=\int \cos u\left(\tfrac{1}{6}du\right)$$

$$=\frac{1}{6}\sin u+C$$

$$\boxed{=\frac{1}{6}\sin\sqrt{3(2r-1)^2+6}+C}$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\,d\theta$

Solution

Let $u=\cos\sqrt{\theta} \;\Rightarrow\; du=\left(-\sin\sqrt{\theta}\right)\left(\frac{1}{2\sqrt{\theta}}\right)d\theta \;\to\; -2du=\frac{1}{\sqrt{\theta}}\sin\sqrt{\theta}\,d\theta$

$$\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\,d\theta=\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\sqrt{\cos^3\sqrt{\theta}}}\,d\theta$$

$$=\int \frac{1}{u^{3/2}}(-2du)$$

$$=-2\int u^{-3/2}du$$

$$=-2\frac{u^{-1/2}}{-1/2}+C$$

$$\boxed{=\frac{4}{\sqrt{\cos\sqrt{\theta}}}+C}$$

Exercise

Evaluate the integrals. $\int 2x\sqrt{x^2-2}\,dx$

Solution

$$u = x^2 - 2 \Rightarrow du = 2xdx$$

$$\int 2x\sqrt{x^2-2}\,dx = \int \sqrt{u}\,du$$

$$= \int u^{1/2}\,du$$

$$= \frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{3}u^{3/2} + C$$

$$\boxed{= \frac{2}{3}(x^2-2)^{3/2} + C}$$

Exercise

Evaluate the integrals $\int x^3(3x^4+1)^2 dx$

Solution

$$u = 3x^4 + 1 \Rightarrow du = 12x^3dx$$

$$\Rightarrow \frac{1}{12}du = x^3dx$$

$$\int x^3(3x^4+1)^2 dx = \int \frac{1}{12}u^2du$$

$$= \frac{1}{12}\frac{(3x^4+1)^3}{3} + C$$

$$\boxed{= \frac{1}{36}(3x^4+1)^3 + C}$$

Exercise

Evaluate the integrals $\int 2(3x^4+1)^2\,dx$

Solution

$$\int 2(3x^4+1)^2\,dx = \int 2\left((3x^4)^2+2(3x^4)(1)+1^2\right)dx \qquad (a+b)^2 = a^2+2ab+b^2$$

$$= \int 2\left(9x^8+6x^4+1\right)dx$$

$$= \int \left(18x^8+12x^4+2\right)dx$$

$$= 18\frac{x^9}{9}+12\frac{x^5}{5}+2x+C$$

$$\boxed{= 2x^9+\frac{12}{5}x^5+2x+C}$$

Exercise

Evaluate the integrals $\int 5x\sqrt{x^2-1}\,dx$

Solution

$$u = x^2-1 \quad \Rightarrow du = 2xdx$$

$$\Rightarrow \tfrac{1}{2}du = xdx$$

$$\int 5x\left(x^2-1\right)^{1/2}\,dx = 5\int u^{1/2}\ \tfrac{1}{2}du \qquad \textit{Substitute for x and dx}$$

$$= 5\int u^{1/2}\ \tfrac{1}{2}du$$

$$= \tfrac{5}{2}\int u^{1/2}du$$

$$= \tfrac{5}{2}\frac{u^{3/2}}{3/2}+C$$

$$= \tfrac{5}{3}u^{3/2}+C$$

$$\boxed{= \tfrac{5}{3}\left(x^2-1\right)^{3/2}+C}$$

Exercise

Find the integral $\int (x^2-1)^3(2x)dx$

Solution

$$u = x^2 - 1 \Rightarrow du = 2xdx$$

$$\int (x^2-1)^3(2x)dx = \int u^3 du$$

$$= \frac{1}{4}u^4 + C$$

$$\boxed{= \frac{1}{4}\left(x^2-1\right)^4 + C}$$

Exercise

Find the integral $\int \frac{6x}{(1+x^2)^3}dx$

Solution

$$u = 1 + x^2 \Rightarrow du = 2xdx \quad \Rightarrow \frac{1}{2x}du = dx$$

$$\int \frac{6x}{u^3}\frac{1}{2x}du = \int 3\frac{1}{u^3}du$$

$$= 3\int u^{-3}du$$

$$= 3\frac{u^{-2}}{-2} + C$$

$$= -\frac{3}{2}\left(1+x^2\right)^{-2} + C$$

$$\boxed{= -\frac{3}{2}\frac{1}{\left(1+x^2\right)^2} + C}$$

Exercise

Find the integral $\int u^3\sqrt{u^4+2}\ du$

Solution

$$Let\ \ x = u^4 + 2 \quad \Rightarrow \quad dx = 4u^3 du \quad \to \frac{1}{4u^3}dx = du$$

$$\int u^3\sqrt{u^4+2}\ du = \int u^3 x^{1/2}\ \frac{1}{4u^3}dx$$

$$= \frac{1}{4}\int x^{1/2}\ dx$$

$$= \frac{1}{4}\frac{2}{3}x^{3/2} + C$$

$$\boxed{= \frac{1}{6}\left(u^4+2\right)^{3/2} + C}$$

Exercise

Find the integral $\int \frac{t+2t^2}{\sqrt{t}}\ dt$

Solution

$$\int \frac{t+2t^2}{\sqrt{t}}\ dt = \int \left(\frac{t}{t^{1/2}} + 2\frac{t^2}{t^{1/2}}\right) dt$$

$$= \int \left(t^{1/2} + 2t^{3/2}\right) dt$$

$$= \frac{2}{3}t^{3/2} + 2\frac{2}{5}t^{5/2} + C$$

$$\boxed{= \frac{2}{3}t^{3/2} + \frac{4}{5}t^{5/2} + C}$$

Exercise

Find the integral $\int \left(1+\frac{1}{t}\right)^3 \frac{1}{t^2}\ dt$

Solution

$$u = 1 + t^{-1} \Rightarrow du = -t^{-2}dt \quad \rightarrow \quad -t^2 du = dt$$

$$\begin{aligned}
\int \left(1+\frac{1}{t}\right)^3 \frac{1}{t^2}\ dt &= \int u^3 \frac{1}{t^2}\ (-t^2 du) \\
&= -\int u^3 du \\
&= -\frac{1}{4}u^4 + C \\
&= \boxed{-\frac{1}{4}\left(1+\frac{1}{t}\right)^4 + C}
\end{aligned}$$

Exercise

Find the integral $\int (7-3x-3x^2)(2x+1)\ dx$

Solution

$$\begin{aligned}
u = 7-3x-3x^2 \Rightarrow du &= (-3-6x^2)dx \\
&= -3(2x^2+1)dx \\
\Rightarrow -\frac{1}{3}du &= (2x^2+1)dx
\end{aligned}$$

$$\begin{aligned}
\int (7-3x-3x^2)(2x+1)\ dx &= \int u(-\tfrac{1}{3})\ du \\
&= -\frac{1}{3}\int u\ du \\
&= -\frac{1}{6}u^2 + C \\
&= \boxed{-\frac{1}{6}\left(7-3x-3x^2\right)^2 + C}
\end{aligned}$$

Exercise

Find the integral $\int \sqrt{x}\left(4-x^{3/2}\right)^2 \ dx$

Solution

$$u = 4 - x^{3/2} \Rightarrow du = -\frac{3}{2}x^{1/2}dx \quad \to \quad -\frac{2}{3}du = \sqrt{x}dx$$

$$\begin{aligned}\int \sqrt{x}\left(4-x^{3/2}\right)^2 \ dx &= \int u^2\left(-\frac{2}{3}\right)du \\ &= -\frac{2}{3}\int u^2 du \\ &= -\frac{2}{9}u^3 + C \\ &= -\frac{2}{9}\left(4-x^{3/2}\right)^3 + C\end{aligned}$$

Exercise

Find the integral $\int \frac{1}{\sqrt{x}+\sqrt{x+1}}dx$

Solution

$$\begin{aligned}\int \frac{1}{\sqrt{x}+\sqrt{x+1}}dx &= \int \frac{1}{\sqrt{x}+\sqrt{x+1}}\frac{\sqrt{x}-\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}dx \\ &= \int \frac{\sqrt{x}-\sqrt{x+1}}{x-(x+1)}dx \\ &= \int \frac{\sqrt{x}-\sqrt{x+1}}{-1}dx \\ &= -\int \left(x^{1/2}-(x+1)^{1/2}\right)dx \\ &= -\left(\frac{2}{3}x^{3/2}-\frac{2}{3}(x+1)^{3/2}\right)+C \\ &= \frac{2}{3}(x+1)^{3/2}-\frac{2}{3}x^{3/2}+C\end{aligned}$$

Exercise

Find the integral $\int \sqrt{1-x}\ dx$

Solution

$$u = 1-x \quad \Rightarrow \quad du = -dx \quad \rightarrow \quad -du = dx$$

$$\int \sqrt{1-x}\ dx = \int \sqrt{u}\ (-du) \qquad\qquad \textit{Substitute for x and dx}$$

$$= -\int u^{1/2}\ du$$

$$= -\frac{u^{3/2}}{3/2} + C$$

$$\boxed{= -\frac{2}{3}(1-x)^{3/2} + C}$$

Exercise

Find the integral $\int x\sqrt{x^2+4}\ dx$

Solution

$$u = x^2+4 \quad \Rightarrow \quad du = 2xdx \quad \rightarrow \quad \frac{1}{2}du = xdx$$

$$\int \sqrt{x^2+4}\ xdx = \int u^{1/2}\ \frac{1}{2}du$$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{3}u^{3/2} + C$$

$$\boxed{= \frac{1}{3}\left(x^2+4\right)^{3/2} + C}$$

Exercise

Find the integral $\int \sin^2\left(\theta + \frac{\pi}{6}\right) d\theta$

Solution

$$\int \sin^2\left(\theta + \frac{\pi}{6}\right) d\theta = \frac{1}{2}\int \left(1 - \cos\left(2\theta + \frac{\pi}{3}\right)\right) d\theta$$

$$= \frac{1}{2}\left(\theta - \frac{1}{2}\sin\left(2\theta + \frac{\pi}{3}\right)\right) + C$$

$$= \frac{\theta}{2} - \frac{1}{4}\sin\left(2\theta + \frac{\pi}{3}\right) + C$$

Exercise

Find the integral $\int \cos^2(8\theta)\, d\theta$

Solution

$$\int \cos^2(8\theta)\, d\theta = \frac{1}{2}\int (1 + \cos(16\theta))\, d\theta$$

$$= \frac{1}{2}\left(1 + \frac{1}{16}\sin(16\theta)\right) + C$$

$$= \frac{1}{2} + \frac{1}{32}\sin(16\theta) + C$$

Exercise

Find the integral $\int \sin^2(2\theta)\, d\theta$

Solution

$$\int \sin^2(2\theta)\, d\theta = \frac{1}{2}\int (1 - \cos(4\theta))\, d\theta$$

$$= \frac{1}{2}\left(1 - \frac{1}{4}\sin(4\theta)\right) + C$$

$$= \frac{1}{2} - \frac{1}{8}\sin(4\theta) + C$$

Exercise

Evaluate the integrals $\int 8\cos^4 2\pi x\, dx$

Solution

$$\int 8\cos^4 2\pi x\, dx = 8\int (\cos 2\pi x)^4\, dx$$

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$= 8\int \left(\frac{1+\cos 4\pi x}{2}\right)^2 dx$$

$$= 2\int (1+\cos 4\pi x)^2\, dx$$

$$= 2\int \left(1+2\cos 4\pi x+\cos^2 4\pi x\right)dx$$

$$= 2\int dx + 4\int \cos 4\pi x dx + 2\int \cos^2 4\pi x dx$$

$$= 2x + 4\frac{1}{4\pi}\cos 4\pi x + 2\int \frac{1+\cos 8\pi x}{2}\, dx$$

$$= 2x + \frac{1}{\pi}\cos 4\pi x + \int (1+\cos 8\pi x)\, dx$$

$$= 2x + \frac{1}{\pi}\sin 4\pi x + x + \frac{1}{8\pi}\sin 8\pi x + C$$

$$= 3x + \frac{1}{\pi}\sin 4\pi x + \frac{1}{8\pi}\sin 8\pi x + C$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{18\tan^2 x\sec^2 x}{\left(2+\tan^3 x\right)^2}dx$

a) $u=\tan x$, followed by $v=u^3$ *then by* $w=2+v$

b) $u=\tan^3 x$, followed by $v=2+u$

c) $u=2+\tan^3 x$

Solution

a) Let $u=\tan x \quad \Rightarrow \quad du=\sec^2 x dx$

$v=u^3 \quad \Rightarrow \quad dv=3u^2 du$

$w=2+v \quad \Rightarrow \quad dw=dv$

$$\int \frac{18\tan^2 x\sec^2 x}{\left(2+\tan^3 x\right)^2}dx=\int \frac{18u^2 du}{\left(2+u^3\right)^2}$$

$$=\int \frac{6dv}{(2+v)^2}$$

$$=\int \frac{6dw}{w^2}$$

$$=6\int w^{-2}dw$$

$$=6\frac{w^{-1}}{-1}+C$$

$$=-\frac{6}{w}+C$$

$$=-\frac{6}{2+v}+C$$

$$=-\frac{6}{2+u^3}+C$$

$$=-\frac{6}{2+\tan^3 x}+C$$

b) Let $u=\tan^3 x \quad \Rightarrow \quad du=3\tan^2 x\sec^2 x dx$

$v=2+u \quad \Rightarrow \quad dv=du$

$$\int \frac{18\tan^2 x\sec^2 x}{\left(2+\tan^3 x\right)^2}dx=\int \frac{6du}{(2+u)^2}$$

$$= \int \frac{6dv}{v^2}$$

$$= \int 6v^{-2}dv$$

$$= -6v^{-1} + C$$

$$= -\frac{6}{v} + C$$

$$= -\frac{6}{2+u} + C$$

$$\boxed{= -\frac{6}{2+\tan^3 x} + C}$$

c) Let $u = 2 + \tan^3 x \quad \Rightarrow \quad du = 3\tan^2 x\sec^2 xdx \rightarrow \; \frac{1}{3}du = \tan^2 x\sec^2 xdx$

$du = 3\tan^2 x\sec^2 xdx \; \rightarrow \; \frac{1}{3}du = \tan^2 x\sec^2 xdx$

$$\int \frac{18\tan^2 x\sec^2 x}{\left(2+\tan^3 x\right)^2}dx = \int \frac{18}{u^2}\left(\frac{1}{3}du\right)$$

$$= 6\int u^{-2}du$$

$$= -6u^{-1} + C$$

$$= -\frac{6}{u} + C$$

$$\boxed{= -\frac{6}{2+\tan^3 x} + C}$$

Exercise

Evaluate: $\int_0^1 (2t+3)^3 dt$

Solution

$u = 2t+3 \Rightarrow du = 2dt \rightarrow \frac{du}{2} = dt$

$$\begin{aligned}\int_0^1 (2t+3)^3 dt &= \int_0^1 u^3 \tfrac{1}{2} du \\ &= \tfrac{1}{2}\int_0^1 u^3 du \\ &= \frac{1}{2}\frac{u^4}{4}\Big|_0^1 \\ &= \tfrac{1}{8}(2t+3)^4\Big|_0^1 \\ &= \tfrac{1}{8}\left[(2(1)+3)^4 - (2(0)+3)^4\right] \\ &= \tfrac{1}{8}\left[5^4 - 3^4\right] \\ &= 68\end{aligned}$$

Exercise

Evaluate the integral $\int_0^3 (2x+1)dx$

Solution

$$\begin{aligned}\int_0^3 (2x+1)dx &= x^2 + x\Big|_0^3 \\ &= 3^2 + 3 - (0+0) \\ &= 12\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$

Solution

$$\int_0^2 \sqrt{4-x^2}\,dx = \left[\tfrac{1}{2}x\sqrt{4-x^2} + \tfrac{4}{2}\sin^{-1}\tfrac{x}{2}\right]_0^2$$

$\sqrt{4-x^2}$ is a semi-circle with center (0, 0) and radius = 2

Since x from 0 to 2

$\Rightarrow$ Area = $\frac{1}{4}$(Area of this circle)

$$= \frac{1}{4}2\pi 2^2$$

$$\underline{= 2\pi}$$

Exercise

Evaluate the integral $\displaystyle\int_0^3 \sqrt{y+1}\;dy$

Solution

$u = y + 1 \Rightarrow du = dy$

$$\int_0^3 \sqrt{y+1}\;dy = \int_0^3 u^{1/2}\;du$$

$$= \frac{2}{3}u^{3/2}\Big|_0^3$$

$$= \frac{2}{3}(y+1)^{3/2}\Big|_0^3$$

$$= \frac{2}{3}\left[(3+1)^{3/2} - (0+1)^{3/2}\right]$$

$$= \frac{2}{3}[8-1]$$

$$\underline{= \frac{14}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{1} r\sqrt{1-r^2}\,dr$

Solution

Let $u = 1-r^2 \quad \Rightarrow \quad du = -2rdr \quad \to \quad -\frac{1}{2}du = rdr$

$$\int_{-1}^{1} r\sqrt{1-r^2}\,dr = \int_{-1}^{1} u^{1/2}\left(-\tfrac{1}{2}du\right)$$

$$= -\frac{1}{2}\frac{2}{3}u^{3/2}\Big|_{-1}^{1}$$

$$= -\frac{1}{3}\left[\left(1-r^2\right)^{3/2}\right]_{-1}^{1}$$

$$= -\frac{1}{3}\left[\left(1-(1)^2\right)^{3/2} - \left(1-(-1)^2\right)^{3/2}\right]$$

$$= -\frac{1}{3}[0-0]$$

$$\underline{= 0}$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{\pi/4} \tan x \sec^2 x\,dx$

Solution

Let $u = \tan x \quad \Rightarrow \quad du = \sec^2 x dx \quad \to \quad \begin{cases} x = \frac{\pi}{4} & \to u = 1 \\ x = 0 & \to u = 0 \end{cases}$

$$\int_{0}^{\pi/4} \tan x \sec^2 x\,dx = \int_{0}^{1} udu$$

$$= \left[\frac{u^2}{2}\right]_{0}^{1}$$

$$= \frac{1}{2}\left[1^2 - 0^2\right]$$

$$\underline{= \frac{1}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_{2\pi}^{3\pi} 3\cos^2 x \sin x \, dx$

Solution

Let $u = \cos x \quad \Rightarrow \quad du = -\sin x dx \quad \rightarrow \quad -du = \sin x dx \quad \begin{cases} x = 3\pi & \rightarrow u = -1 \\ x = 2\pi & \rightarrow u = 1 \end{cases}$

$$\int_{2\pi}^{3\pi} 3\cos^2 x \sin x \, dx = \int_1^{-1} 3u^2(-du)$$

$$= -3\int_1^{-1} u^2 du$$

$$= -3\left[\frac{u^3}{3}\right]_1^{-1}$$

$$= -\left[(-1)^3 - 1^3\right]$$

$$= 2$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 \mathrm{t}^3\left(1+t^4\right)^3 \, dt$

Solution

Let $u = 1 + t^4 \quad \Rightarrow \quad du = 4t^3 dt \quad \rightarrow \quad \frac{1}{4}du = t^3 dt \quad \begin{cases} t = 1 & \rightarrow u = 2 \\ t = 0 & \rightarrow u = 1 \end{cases}$

$$\int_0^1 \mathrm{t}^3\left(1+t^4\right)^3 \, dt = \int_1^2 \frac{1}{4}u^3 du$$

$$= \frac{1}{4}\left(\frac{u^4}{4}\right)_1^2$$

$$= \frac{1}{16}\left(u^4\right)_1^2$$

$$= \frac{1}{16}\left(2^4 - 1^4\right)$$

$$= \frac{15}{16}$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 \frac{r}{\left(4+r^2\right)^2}\,dr$

Solution

Let $u = 4+r^2 \quad \Rightarrow \quad du = 2rdr \quad \to \quad \frac{1}{2}du = rdr \quad \begin{cases} r=1 & \to u=5 \\ r=0 & \to u=4 \end{cases}$

$$\int_0^1 \frac{r}{\left(4+r^2\right)^2}\,dr = \int_4^5 u^{-2}\left(\frac{1}{2}du\right)$$

$$= -\frac{1}{2}\left[u^{-1}\right]_4^5$$

$$= -\frac{1}{2}\left(\frac{1}{5}-\frac{1}{4}\right)$$

$$= -\frac{1}{40}$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 \frac{10\sqrt{v}}{\left(1+v^{3/2}\right)^2}\,dv$

Solution

Let $u = 1+v^{3/2} \quad \Rightarrow \quad du = \frac{3}{2}v^{1/2}dv \quad \to \quad \frac{2}{3}du = \sqrt{v}dv \quad \begin{cases} v=1 & \to u=2 \\ v=0 & \to u=1 \end{cases}$

$$\int_0^1 \frac{10\sqrt{v}}{\left(1+v^{3/2}\right)^2}\,dv = \int_1^2 10u^{-2}\left(\frac{2}{3}du\right)$$

$$= \frac{20}{3}\int_1^2 u^{-2}du$$

$$= \frac{20}{3}\left[-u^{-1}\right]_1^2$$

$$= -\frac{20}{3}\left(\frac{1}{2}-\frac{1}{1}\right)$$

$$= \frac{10}{3}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$

Solution

Let $u = x^2 + 1$

$$du = 2xdx \quad \to \quad \frac{1}{2}du = xdx \qquad \begin{cases} x = \sqrt{3} & \to u = 4 \\ x = -\sqrt{3} & \to u = 4 \end{cases}$$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx = \int_4^4 4u^{-1}\left(\tfrac{1}{2}du\right)$$

$$\boxed{=0}$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$

Solution

Let $u = x^4 + 9$

$$du = 4x^3dx \quad \to \quad \frac{1}{4}du = x^3dx \qquad \begin{cases} x = 1 & \to u = 10 \\ x = 0 & \to u = 9 \end{cases}$$

$$\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx = \frac{1}{4}\int_3^{10} u^{-1/2}du$$

$$= \frac{1}{4}\left[2u^{1/2}\right]_9^{10}$$

$$= \frac{1}{2}\left[10^{1/2} - 9^{1/2}\right]$$

$$\boxed{= \frac{\sqrt{10}-3}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/6} (1-\cos 3t)\sin 3t\ dt$

Solution

Let $u = 1-\cos 3t$

$$du = 3\sin 3t dt \quad \rightarrow \quad \frac{1}{3}du = \sin 3t dt \quad \begin{cases} t = \frac{\pi}{6} & \rightarrow u = 1 \\ t = 0 & \rightarrow u = 0 \end{cases}$$

$$\int_0^{\pi/6} (1-\cos 3t)\sin 3t\ dt = \frac{1}{3}\int_0^1 u\ du$$

$$= \frac{1}{3}\left[\frac{u2}{2}\right]_0^1$$

$$= \frac{1}{6}\left(1^2 - 0^2\right)$$

$$\underline{= \frac{1}{6}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\ dt$

Solution

Let $u = 2+\tan\frac{t}{2}$

$$du = \frac{1}{2}\sec^2\frac{t}{2}\ dt \quad \rightarrow \quad 2du = \sec^2\frac{t}{2}\ dt \quad \begin{cases} t = \frac{\pi}{2} & \rightarrow u = 3 \\ t = -\frac{\pi}{2} & \rightarrow u = 1 \end{cases}$$

$$\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\ dt = \int_1^3 u(2du)$$

$$= 2\left[\frac{u^2}{2}\right]_1^3$$

$$= \left[3^2 - 1^2\right]$$

$$\underline{= 8}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$

Solution

Let $u = 4+3\sin z$

$du = 3\cos z\ dz \quad \to \quad \frac{1}{3}du = \cos z\ dz \qquad \begin{cases} z=\pi & \to u=4 \\ z=-\pi & \to u=4 \end{cases}$

$$\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz = \int_{4}^{4} \frac{1}{\sqrt{u}}\frac{1}{3}du$$

$$= 0$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi/2}^{0} \frac{\sin w}{(3+2\cos w)^2}\,dw$

Solution

Let $u = 3+2\cos w$

$du = -2\sin w\ dw \quad \to \quad -\frac{1}{2}du = \sin w\ dw \qquad \begin{cases} w=0 & \to u=5 \\ w=-\frac{\pi}{2} & \to u=3 \end{cases}$

$$\int_{-\pi/2}^{0} \frac{\sin w}{(3+2\cos w)^2}\,dw = \int_{3}^{5} \frac{1}{u^2}\left(-\frac{1}{2}du\right)$$

$$= -\frac{1}{2}\int_{3}^{5} u^{-2}du$$

$$= \frac{1}{2}\left[u^{-1}\right]_3^5$$

$$= \frac{1}{2}\left(\frac{1}{5}-\frac{1}{3}\right)$$

$$= -\frac{1}{15}$$

Exercise

Evaluate the integral $\int_0^1 \sqrt{t^5+2t}\,\left(5t^4+2\right)dt$

Solution

Let $u=t^5+2t \quad \Rightarrow \quad du=\left(5t^4+2\right)dt \quad \rightarrow \quad \begin{cases} t=1 & \rightarrow u=3 \\ t=0 & \rightarrow u=0 \end{cases}$

$$\int_0^1 \sqrt{t^5+2t}\,\left(5t^4+2\right)dt=\int_0^3 u^{1/2}du$$

$$=\frac{2}{3}\left[u^{3/2}\right]_0^3$$

$$=\frac{2}{3}\left(3^{3/2}-0\right)$$

$$=\frac{2}{3}\left(3\sqrt{3}\right)$$

$$=2\sqrt{3}$$

Exercise

Evaluate the integral $\int_1^4 \frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2}$

Solution

Let $u=1+y^{1/2} \quad \Rightarrow \quad du=\frac{1}{2}y^{-1/2}dy \quad \rightarrow du=\frac{1}{2\sqrt{y}}dy \quad \begin{cases} y=4 & \rightarrow u=3 \\ y=1 & \rightarrow u=2 \end{cases}$

$$\int_1^4 \frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2}=\int_2^3 \frac{du}{u^2}$$

$$=\int_2^3 u^{-2}du$$

$$=-u^{-1}\Big|_2^3$$

$$=-\left(\frac{1}{3}-\frac{1}{2}\right)$$

$$=\frac{1}{6}$$

Exercise

Evaluate the integral $\int_0^1 \left(4y - y^2 + 4y^3 + 1\right)^{-2/3} \left(12y^2 - 2y + 4\right) dy$

Solution

Let $u = 4y - y^2 + 4y^3 + 1 \quad \Rightarrow \quad du = \left(4 - 2y + 12y^2\right) dy \quad \rightarrow \quad \begin{cases} y = 1 & \rightarrow u = 8 \\ y = 0 & \rightarrow u = 1 \end{cases}$

$$\begin{aligned}
\int_0^1 \left(4y - y^2 + 4y^3 + 1\right)^{-2/3} \left(12y^2 - 2y + 4\right) dy &= \int_1^8 u^{-2/3} du \\
&= 3u^{1/3} \Big|_1^8 \\
&= 3\left(8^{1/3} - 1^{1/3}\right) \\
&= 3
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^5 |x - 2| dx$

Solution

$$|x - 2| = \begin{cases} x - 2 & x > 2 \\ -(x - 2) & x < 2 \end{cases}$$

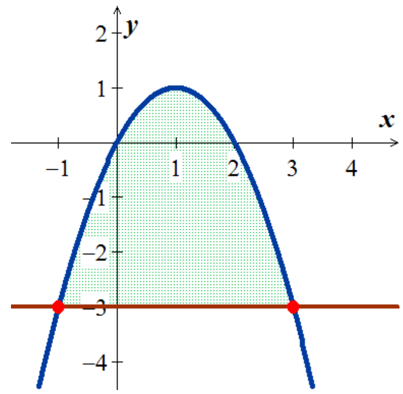
$$\begin{aligned}
\int_0^5 |x - 2| dx &= \int_0^2 -(x - 2) dx + \int_2^5 (x - 2) dx \\
&= -\frac{x^2}{2} + 2x \Big|_0^2 + \left(\frac{x^2}{2} - 2x\right)\Big|_2^5 \\
&= -\frac{4}{2} + 4 - 0 + \left(\frac{25}{2} - 10 - \left(\frac{4}{2} - 4\right)\right) \\
&= -2 + 4 + \frac{25}{2} - 10 - 2 + 4 \\
&= \frac{25}{2} - 6 \\
&= \frac{13}{2}
\end{aligned}$$

Exercise

Find the area of the region bounded by the graphs of $y = 2x - x^2$ and $y = -3$

Solution

$y = -3 \quad \to \quad 2x - x^2 = -3 \Rightarrow x^2 - 2x - 3 = 0 \quad \boxed{x = -1,\ 3}$



$$A = \int_{-1}^{3} \left[2x - x^2 - (-3) \right] dx$$

$$= \left[x^2 - \frac{x^3}{3} + 3x \right]_{-1}^{3}$$

$$= \left((3)^2 - \frac{(3)^3}{3} + 3(3) \right) - \left((-1)^2 - \frac{(-1)^3}{3} + 3(-1) \right)$$

$$= (9 - 9 + 9) - \left(1 + \frac{1}{3} - 3 \right)$$

$$= \frac{32}{3}$$

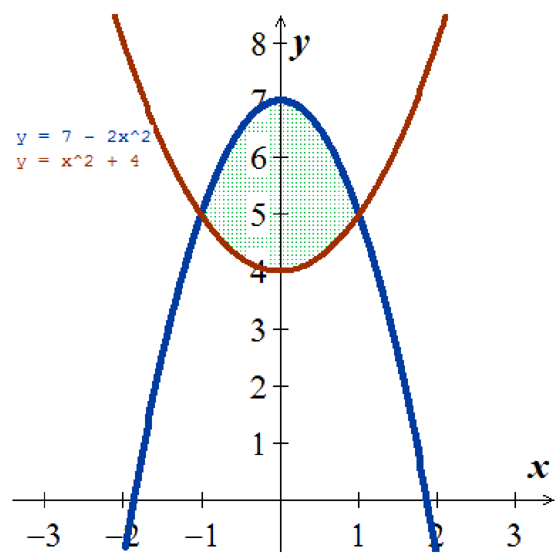
Exercise

Find the area of the region bounded by the graphs of $y = 7 - 2x^2$ and $y = x^2 + 4$

Solution

$7 - 2x^2 = x^2 + 4$

$-3x^2 = -3 \quad \to \quad x^2 = 1 \Rightarrow \boxed{x = \pm 1}$



$$A = \int_{-1}^{1} \left[\left(7 - 2x^2 \right) - \left(x^2 + 4 \right) \right] dx$$

$$= \int_{-1}^{1} \left(3 - 3x^2 \right) dx$$

$$= \left[3x - 3\frac{x^3}{3} \right]_{-1}^{1}$$

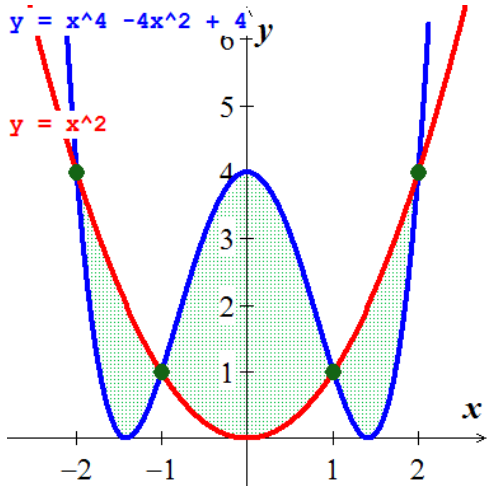
$$= \left(3(1) - (1)^3 \right) - \left(3(-1) - (-1)^3 \right)$$

$$= 4$$

Exercise

Find the area of the region bounded by the graphs of $y = x^4 - 4x^2 + 4$ and $y = x^2$

Solution



$$x^4 - 4x^2 + 4 = x^2$$

$$x^4 - 5x^2 + 4 = 0 \quad \rightarrow \quad \boxed{x = \pm 1, \ \pm 2}$$

$$A = \int_{-2}^{-1} \left(x^2 - \left(x^4 - 4x^2 + 4\right)\right) dx + \int_{-1}^{1} \left(x^4 - 4x^2 + 4 - \left(x^2\right)\right) dx + \int_{1}^{2} \left(x^2 - \left(x^4 - 4x^2 + 4\right)\right) dx$$

$$= \int_{-2}^{-1} \left(-x^4 + 5x^2 - 4\right) dx + \int_{-1}^{1} \left(x^4 - 5x^2 + 4\right) dx + \int_{1}^{2} \left(-x^4 + 5x^2 - 4\right) dx$$

$$= \left[-\frac{x^5}{5} + \frac{5}{3}x^3 - 4x \right]_{-2}^{-1} + \left[\frac{x^5}{5} - \frac{5}{3}x^3 + 4x \right]_{-1}^{1} + \left[-\frac{x^5}{5} + \frac{5}{3}x^3 - 4x \right]_{1}^{2}$$

$$= \left[\left(-\frac{(-1)^5}{5} + \frac{5}{3}(-1)^3 - 4(-1) \right) - \left(-\frac{(-2)^5}{5} + \frac{5}{3}(-2)^3 - 4(-2) \right) \right]$$

$$+ \left[\left(\frac{(1)^5}{5} - \frac{5}{3}(1)^3 + 4(1) \right) - \left(\frac{(-1)^5}{5} - \frac{5}{3}(-1)^3 + 4(-1) \right) \right]$$

$$+ \left[\left(-\frac{(2)^5}{5} + \frac{5}{3}(2)^3 - 4(2) \right) - \left(-\frac{(1)^5}{5} + \frac{5}{3}(1)^3 - 4(1) \right) \right]$$

$$= \left(\tfrac{1}{5} - \tfrac{5}{3} + 4 \right) - \left(\tfrac{32}{5} - \tfrac{40}{3} + 8 \right) + \left(\tfrac{1}{5} - \tfrac{5}{3} + 4 \right) - \left(-\tfrac{1}{5} + \tfrac{5}{3} - 4 \right) + \left(-\tfrac{32}{5} + \tfrac{40}{3} - 8 \right) - \left(-\tfrac{1}{5} + \tfrac{5}{3} - 4 \right)$$

$$\underline{= 8}$$

Exercise

Find the area of the region bounded by the graphs of $x = 2y^2, \quad x = 0, \quad \textit{and} \quad y = 3$

Solution

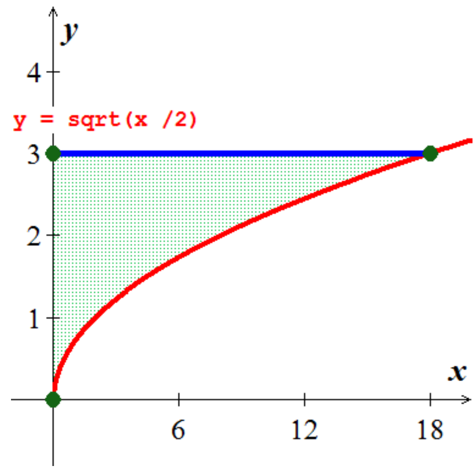
$y = 3 \quad \to \quad \underline{x = 2y^2 = 18}$

$$A = \int_0^3 2y^2 dy$$

$$= \frac{2}{3}\left[y^3\right]_0^3$$

$$= \frac{2}{3}\left(3^3 - 0\right)$$

$$= 18$$



Exercise

Find the area of the region bounded by the graphs of $x = y^3 - y^2 \quad \textit{and} \quad x = 2y$

Solution

$$y^3 - y^2 = 2y$$

$$y^3 - y^2 - 2y = 0$$

$$y\left(y^2 - y - 2\right) = 0 \quad \to \quad \boxed{y = 0, \ -1, \ 2}$$

$$A = \int_{-1}^0 \left[y^3 - y^2 - (2y)\right] dy + \int_0^2 \left[2y - \left(y^3 - y^2\right)\right] dy$$

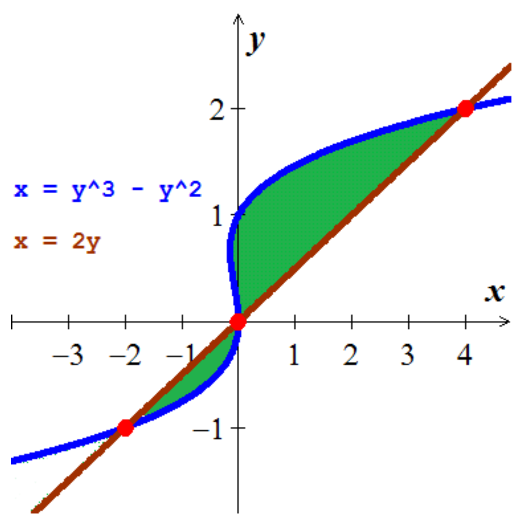
$$= \int_{-1}^0 \left(y^3 - y^2 - 2y\right) dy + \int_0^2 \left(2y - y^3 + y^2\right) dy$$

$$= \left[\frac{y^4}{4} - \frac{y^3}{3} - y^2\right]_{-1}^0 + \left[y^2 - \frac{y^4}{4} + \frac{y^3}{3}\right]_0^2$$

$$= \left[0 - \left(\frac{1}{4} + \frac{1}{3} - 1\right)\right] + \left[\left(4 - 4 + \frac{8}{3}\right) - 0\right]$$

$$= \frac{5}{12} + \frac{8}{3}$$

$$= \frac{37}{12}$$



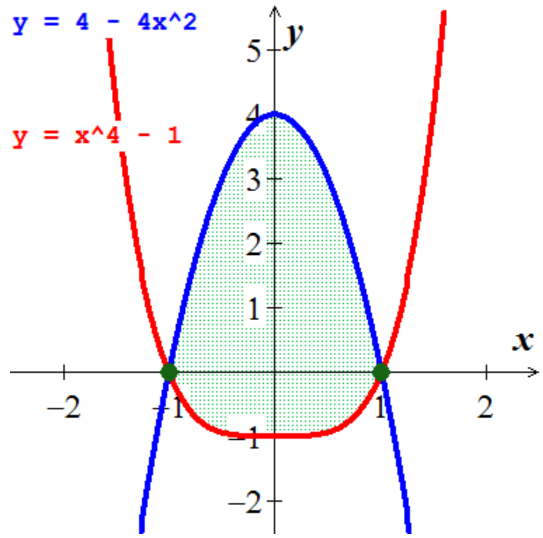
Exercise

Find the area of the region bounded by the graphs of $4x^2 + y = 4$ *and* $x^4 - y = 1$

Solution

$4x^2 + y = 4 \quad \to \quad y = 4 - 4x^2$

$x^4 - y = 1 \quad and \quad y = x^4 - 1$



$$A = \int_{-1}^{1} \left[4 - 4x^2 - \left(x^4 - 1\right) \right] dx$$

$$= \int_{-1}^{1} \left(x^4 - 4x^2 + 5 \right) dx$$

$$= \left[\frac{x^5}{5} - 4\frac{x^3}{3} + 5x \right]_{-1}^{1}$$

$$= \left(\frac{1}{5} - \frac{4}{3} + 5 \right) - \left(-\frac{1}{5} + \frac{4}{3} - 5 \right)$$

$$= \underline{\frac{105}{15}}$$

Exercise

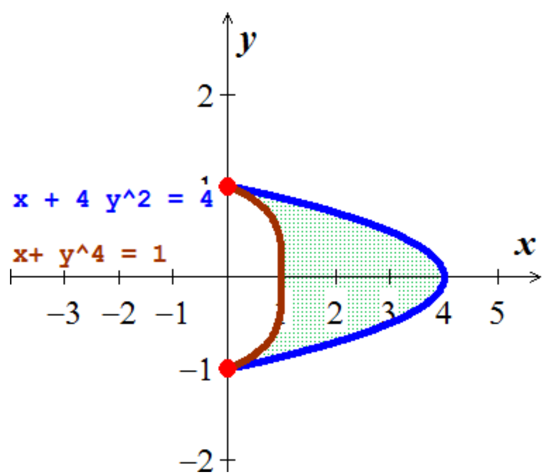
Find the area of the region bounded by the graphs of $x + 4y^2 = 4$ *and* $x + y^4 = 1,$ *for* $x \ge 0$

Solution

$x = 4 - 4y^2 \quad x = 1 - y^4 \quad \to 4 - 4y^2 = 1 - y^4$

$y^4 - 4y^2 + 3 = 0 \quad \to \quad y^2 = 1,\ 3 \Rightarrow y = \pm 1,\ \pm\sqrt{3}$

$$\begin{cases} y = \pm 1 & \to \underline{x = 1 - (\pm 1)^4 = 0} \\ y = \pm\sqrt{3} & \to x = 1 - \left(\pm\sqrt{3}\right)^4 = -8 < 0 \times \end{cases}$$



$$A = \int_{-1}^{1} \left[4 - 4y^2 - \left(1 - y^4\right) \right] dy$$

$$= \int_{-1}^{1} \left(3 - 4y^2 + y^4 \right) dy$$

$$= \left[3y - 4\frac{y^3}{3} + \frac{y^5}{5} \right]_{-1}^{1}$$

$$=\left(3-\frac{4}{3}+\frac{1}{5}\right)-\left(-3+\frac{4}{3}-\frac{1}{5}\right)$$

$$=\frac{56}{15}$$

Exercise

Find the area of the region bounded by the graphs of $y=2\sin x, \quad and \quad y=\sin 2x, \quad 0\le x\le \pi$

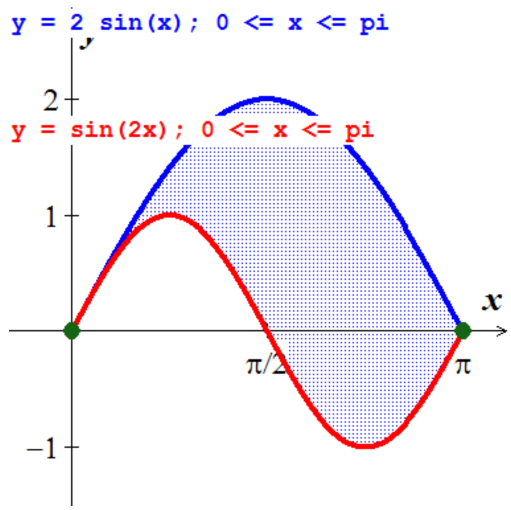
Solution

$$y=2\sin x=\sin 2x$$

$$2\sin x=2\sin x\cos x$$

$$2\sin x-2\sin x\cos x=0$$

$$2\sin x(1-\cos x)=0 \quad \rightarrow \quad \begin{cases}\sin x=0 & x=0,\pi \\ \cos x=1 & x=0\end{cases}$$



$$A=\int_0^{\pi}(2\sin x-\sin 2x)\,dx$$

$$=\left[-2\cos x+\frac{1}{2}\cos 2x\right]_0^{\pi}$$

$$=\left(-2(-1)+\frac{1}{2}(1)\right)-\left(-2+\frac{1}{2}\right)$$

$$=4$$

Exercise

Find the area of the region bounded by the graphs of $y=\sin\frac{\pi x}{2}$ *and* $y=x$

Solution

$$y=\sin\frac{\pi x}{2}=x \quad\rightarrow\quad \boxed{x=-1,\ 1}$$

$$A=\int_{-1}^{0}\left(\sin\frac{\pi x}{2}-x\right)dx+\int_{0}^{1}\left(\sin\frac{\pi x}{2}-x\right)dx$$

$$=2\int_{0}^{1}\left(\sin\frac{\pi x}{2}-x\right)dx$$

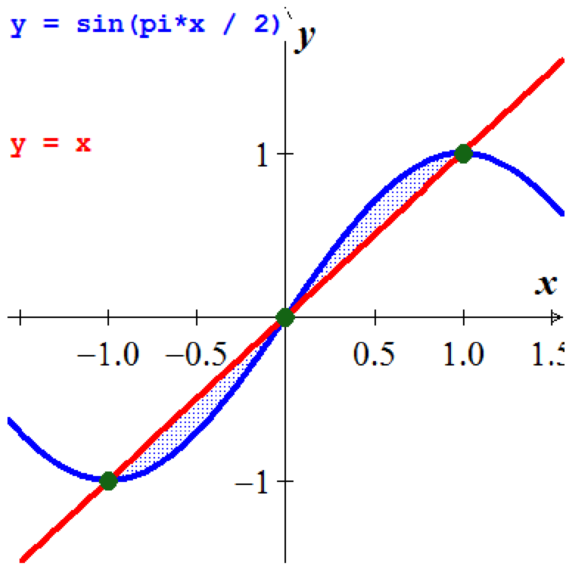
$$=2\left[-\frac{2}{\pi}\cos\frac{\pi x}{2}-\frac{x^2}{2}\right]_0^1$$

$$=2\left[\left(0-\frac{1}{2}\right)-\left(-\frac{2}{\pi}-0\right)\right]$$

$$=2\left(-\frac{1}{2}+\frac{2}{\pi}\right)$$

$$=2\left(\frac{-\pi+4}{2\pi}\right)$$

$$=\frac{4-\pi}{\pi}$$



Exercise

Find the area of the region bounded by the graphs of $y = \sec^2 x, \quad y = \tan^2 x, \quad x = -\frac{\pi}{4}, \quad and \quad x = \frac{\pi}{4}$

Solution

$$A = \int_{-\pi/4}^{\pi/4} \left(\sec^2 x - \tan^2 x\right) dx$$

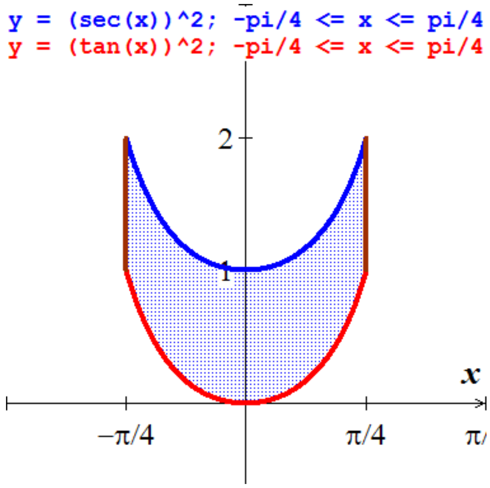
$$= \int_{-\pi/4}^{\pi/4} \left(\sec^2 x - \left(\sec^2 x - 1\right)\right) dx$$

$$= \int_{-\pi/4}^{\pi/4} dx$$

$$= x \Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right)$$

$$= \frac{\pi}{2}$$



Exercise

Find the area of the region bounded by the graphs of $x = 3\sin y\sqrt{\cos y}, \quad and \quad x = 0, \quad 0 \le y \le \frac{\pi}{2}$

Solution

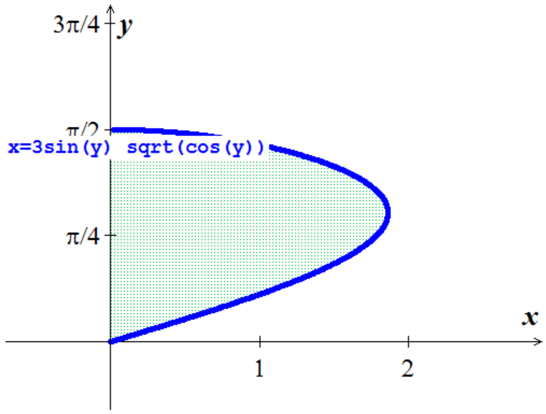
$$A = \int_0^{\pi/2} \left(3\sin y\sqrt{\cos y} - 0\right) dx \qquad\qquad d(\cos y) = -\sin y\, dy$$

$$= -3\int_0^{\pi/2} \cos^{1/2} y \ \ d(\cos y)$$

$$= -3\left[\frac{2}{3}\cos^{3/2} y\right]_0^{\pi/2}$$

$$= -2(0 - 1)$$

$$= 2$$



Exercise

Find the area of the region bounded by the graphs of $y = x^2 + 1$ and $y = x$ for $0 \le x \le 2$

Solution

$$A = \int_0^2 [(x^2 + 1) - x]dx$$

$$A = \int_0^2 [x^2 - x + 1]dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + 1x \Big|_0^2$$

$$= \frac{2^3}{3} - \frac{2^2}{2} + 1(2) - 0$$

$$= \frac{8}{3}$$

Exercise

Find the area of the region bounded by the graphs of $y = 3 - x^2$ and $y = 2x$

Solution

Determine the intersection between two functions: $y = 3 - x^2 = 2x$

$$x^2 + 2x - 3 = 0 \quad \rightarrow \boxed{x = 1, -3}$$

$$A = \int_{-3}^1 [(3 - x^2) - 2x]dx$$

$$A = \int_{-3}^1 [-x^2 - 2x + 3]dx$$

$$= -\frac{x^3}{3} - 2\frac{x^2}{2} + 3x \Big|_{-3}^1$$

$$= -\frac{1^3}{3} - 1^2 + 3(1) - \left[-\frac{(-3)^3}{3} - (-3)^2 + 3(-3) \right]$$

$$= -\frac{1}{3} - 1 + 3 - [9 - 9 - 9]$$

$$= 11 - \frac{1}{3}$$

$$= \frac{32}{3}$$

Exercise

Find the area of the region bounded by the graphs of $y = x^2 - x - 2$ and x-axis

Solution

Determine the intersection points: $x^2 - x - 2 = 0 \Rightarrow \boxed{x = -1, 2}$

$$A = \int_{-1}^{2} [0 - (x^2 - x - 2)]dx$$

$$= -\frac{x^3}{3} + \frac{x^2}{2} + 2x \Big|_{-1}^{2}$$

$$= -\frac{2^3}{3} + \frac{2^2}{2} + 2(2) - \left[-\frac{(-1)^3}{3} + \frac{(-1)^2}{2} + 2(-1)\right]$$

$$= -\frac{8}{3} + 2 + 4 - \left[\frac{1}{3} + \frac{1}{2} - 2\right]$$

$$= 4.5 \text{ unit}^2$$

Exercise

Find the area of the region bounded by the graphs of $f(x) = x^3 + 2x^2 - 3x$ and $g(x) = x^2 + 3x$

Solution

$$x^3 + 2x^2 - 3x = x^2 + 3x$$

$$x^3 + x^2 - 6x = 0$$

$$x(x^2 + x - 6) = 0 \Rightarrow \begin{cases} x = 0 \\ x^2 + x - 6 = 0 \end{cases}$$

$$x^2 + x - 6 = 0 \rightarrow x = -3, 2 \qquad \rightarrow \boxed{x = -3,\ 0,\ 2}$$

$$A = \int_{-3}^{0} (f - g)dx + \int_{0}^{2} (g - f)dx$$

$$= \int_{-3}^{0} (x^3 + 2x^2 - 3x - (x^2 + 3x))dx + \int_{0}^{2} (x^2 + 3x - (x^3 + 2x^2 - 3x))dx$$

$$= \int_{-3}^{0} (x^3 + x^2 - 6x)dx + \int_{0}^{2} (-x^3 - x^2 + 6x)dx$$

$$= \frac{x^4}{4} + \frac{x^3}{3} - 3x^2 \Big|_{-3}^{0} + \left[-\frac{x^4}{4} - \frac{x^3}{3} + 3x^2\right]_{0}^{2}$$

$$= 0 - \left(\frac{(-3)^4}{4} + \frac{(-3)^3}{3} - 3(-3)^2\right) + \left[\left(-\frac{2^4}{4} - \frac{2^3}{3} + 32^2\right) - 0\right]$$

$$= \frac{253}{12}$$

$$\approx 21.083$$

Exercise

Find the area bounded by $f(x) = -x^2 + 1$, $g(x) = 2x + 4$, $x = -1$, and $x = 2$

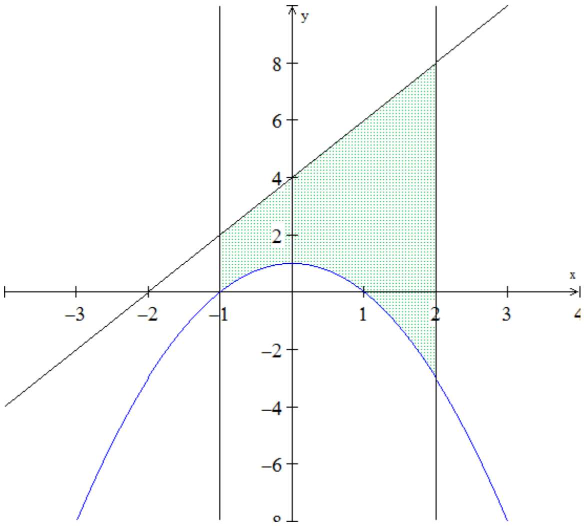
Solution

$$f \cap g \Rightarrow \quad -x^2 + 1 = 2x + 4$$

$$-x^2 - 2x - 3 = 0$$

$$x^2 + 2x + 3 = 0$$

$$\Rightarrow x = -1 \pm i\sqrt{2}$$



$$A = \int_{-1}^{2} (g - f)dx$$

$$= \int_{-1}^{2} \left(2x + 4 - \left(-x^2 + 1\right)\right)dx$$

$$= \int_{-1}^{2} \left(x^2 + 2x + 3\right)dx$$

$$= \frac{1}{3}x^3 + x^2 + 3x \Big|_{-1}^{2}$$

$$= \left(\frac{1}{3}(2)^3 + (2)^2 + 3(2)\right) - \left(\frac{1}{3}(-1)^3 + (-1)^2 + 3(-1)\right)$$

$$= \left(\frac{8}{3} + 4 + 6\right) - \left(-\frac{1}{3} + 1 - 3\right)$$

$$= \frac{8}{3} + 10 + \frac{1}{3} + 2$$

$$= 15$$

Exercise

Find the area between the curves $y = x^{1/2}$ *and* $y = x^3$

Solution

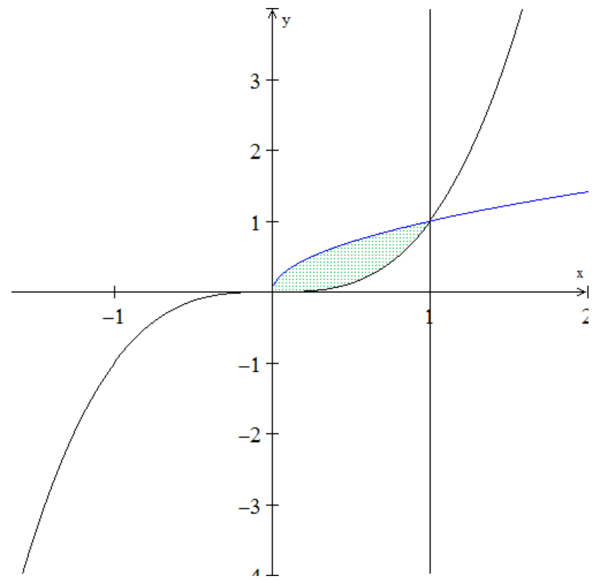
$x^3 = x^{1/2}$ ***Square both sides***

$x^6 = x$

$x^6 - x = 0$

$x\left(x^5 - 1\right) = 0$

$x = 0 \quad x^5 - 1 = 0 \Rightarrow x = 1$



$$A = \int_0^1 \left(x^{1/2} - x^3\right) dx$$

$$= \frac{2}{3}x^{3/2} - \frac{1}{4}x^4 \Big|_0^1$$

$$= \frac{2}{3}1^{3/2} - \frac{1}{4}1^4 - 0$$

$$= \frac{2}{3} - \frac{1}{4}$$

$$= \frac{8-3}{12}$$

$$= \frac{5}{12}$$

Exercise

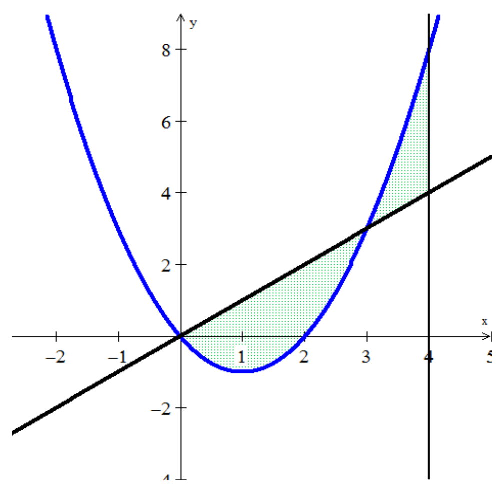
Find the area of the region bounded by the graphs of $y = x^2 - 2x$ and $y = x$ on [0, 4].

Solution

$x^2 - 2x = x$

$x^2 - 3x = 0$

$x(x-3) = 0 \Rightarrow \boxed{x = 0, 3}$



$$A = \int_0^3 \left(x - \left(x^2 - 2x\right)\right) dx + \int_3^4 \left(x^2 - 2x - x\right) dx$$

$$= \int_0^3 \left(-x^2 + 3x\right) dx + \int_3^4 \left(x^2 - 3x\right) dx$$

$$= \left(-\frac{1}{3}x^3 + \frac{3}{2}x^2\right)\Big|_0^3 + \left(\frac{1}{3}x^3 - \frac{3}{2}x^2\right)\Big|_3^4$$

$$= \left(-\frac{1}{3}3^3 + \frac{3}{2}3^2\right) + \left[\left(\frac{1}{3}4^3 - \frac{3}{2}4^2\right) - \left(\frac{1}{3}3^3 - \frac{3}{2}3^2\right)\right]$$

$$= \left(\frac{9}{2}\right) + \left[\left(-\frac{8}{3}\right) - \left(-\frac{9}{2}\right)\right]$$

$$= \frac{9}{2} - \frac{8}{3} + \frac{9}{2}$$

$$= \frac{19}{3}$$

Exercise

Find the area between the curves $x = 1, \quad x = 2, \quad y = x^3 + 2, \quad y = 0$

Solution

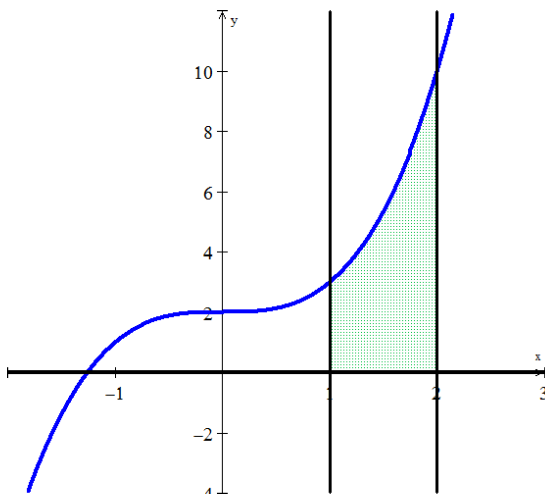
$$A = \int_1^2 \left(x^3 + 2 - 0\right) dx$$

$$= \frac{1}{4}x^4 + 2x \Big|_1^2$$

$$= \left(\frac{1}{4}2^4 + 2(2)\right) - \left(\frac{1}{4}1^4 + 2(1)\right)$$

$$= (8) - \left(\frac{9}{4}\right)$$

$$= \frac{23}{4}$$



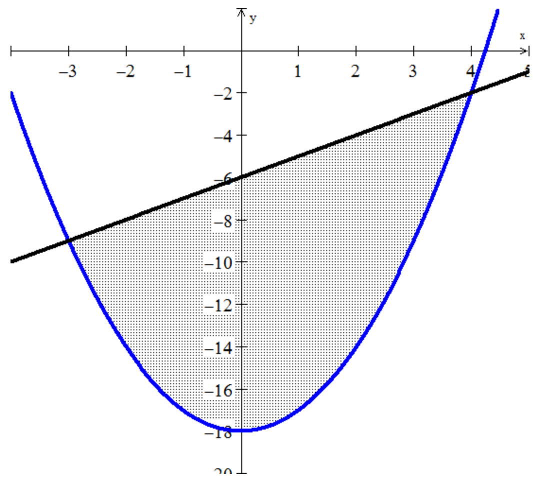
Exercise

Find the area between the curves $y = x^2 - 18, \quad y = x - 6$

Solution

$$x^2 - 18 = x - 6$$

$$x^2 - x - 12 = 0 \to \boxed{x = -3, 4}$$



$$A = \int_{-3}^{4} \left(x^2 - 18 - (x - 6)\right) dx$$

$$= \int_{-3}^{4} \left(x^2 - x - 12\right) dx$$

$$= \frac{1}{3}x^3 - \frac{1}{2}x^2 - 12x \Big|_{-3}^{4}$$

$$= \left(\frac{1}{3}4^3 - \frac{1}{2}4^2 - 12(4)\right) - \left(\frac{1}{3}(-3)^3 - \frac{1}{2}(-3)^2 - 12(-3)\right)$$

$$= \left(-\frac{104}{3}\right) - \left(\frac{45}{2}\right)$$

$$= \frac{343}{6}$$

Exercise

Find the area between the curves $y = \sqrt{x}, \quad y = x\sqrt{x}$

Solution

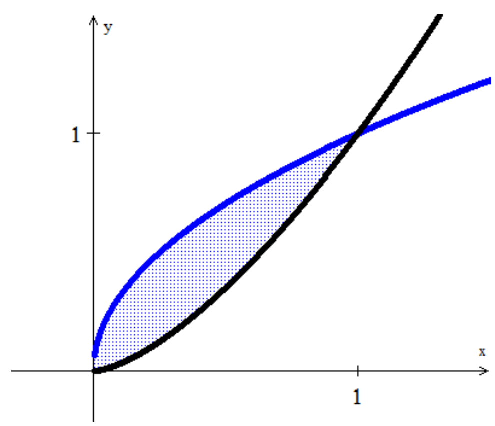
$$x\sqrt{x} = \sqrt{x}$$

$$\left(x\sqrt{x}\right)^2 = \left(\sqrt{x}\right)^2$$

$$x^2 x = x$$

$$x\left(x^2 - 1\right) = 0$$

$$\boxed{x = 0} \quad x^2 - 1 = 0 \Rightarrow x = \pm 1 (\textit{no negative}) \quad \boxed{x = 1}$$



$$A = \int_{0}^{1} \left(\sqrt{x} - x\sqrt{x}\right) dx$$

$$= \int_{0}^{1} \left(x^{1/2} - x^{3/2}\right) dx$$

$$= \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\Big|_0^1$$

$$= \left(\frac{2}{3}1^{3/2} - \frac{2}{5}1^{5/2}\right) - \left(\frac{2}{3}0^{3/2} - \frac{2}{5}0^{5/2}\right)$$

$$= \left(\frac{2}{3} - \frac{2}{5}\right) - 0$$

$$\boxed{= \frac{4}{15}}$$

Exercise

A company is considering a new manufacturing process in one of its plants. The new process provides substantial initial savings, with the savings declining with time *t* (in years) according to the rate-of-savings function

$$S'(t) = 100 - t^2$$

where $S'(t)$ is in thousands of dollars per year. At the same time, the cost of operating the new process increases with time *t* (in years), according to the rate-of-cost function (in thousands of dollars per year)

$$C'(t) = t^2 + \frac{14}{3}t$$

a) For how many years will the company realize savings?

b) What will be the net total savings during this period?

Solution

a) For how many years will the company realize savings?

$$C'(t) = S'(t)$$

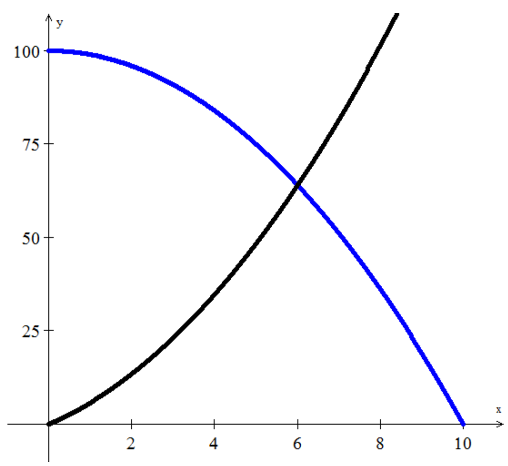
$$t^2 + \frac{14}{3}t = 100 - t^2$$

$$2t^2 + \frac{14}{3}t - 100 = 0$$

$$\to t = -\frac{25}{3} \; or \;\; 6$$

$$\boxed{t = 6}$$

The company should use this type for 6 years.



b) What will be the net total savings during this period?

$$\text{Total savings } = \int_0^6 \left[\left(100 - t^2\right) - \left(t^2 + \frac{14}{3}t\right)\right]dt$$

$$= \int_0^6 \left[100 - 2t^2 - \frac{14}{3}t\right]dt$$

$$=100t-\frac{2}{3}t^3-\frac{7}{3}t^2\Big|_0^6$$

$$=100(6)-\frac{2}{3}(6)^3-\frac{7}{3}(6)^2-\left(100(0)-\frac{2}{3}(0)^3-\frac{7}{3}(0)^2\right)$$

$$\underline{=372}$$

The company will save a total of \$372,000. Over the 6-year period

Exercise

Find the producers' surplus if the supply function for pork bellies is given by

$$S(x)=x^{5/2}+2x^{3/2}+50$$

Assume supply and demand are in equilibrium at $x=16$.

Solution

The equilibrium price: $\underline{p_0}=S(x=16)=16^{5/2}+2(16)^{3/2}+50\underline{=1202}$

Producer's surplus $=\displaystyle\int_0^{x_0}\left[p_0-S(x)\right]dx$

$$=\int_0^{16}\left[1202-\left(x^{5/2}+2x^{3/2}+50\right)\right]dx$$

$$=\int_0^{16}\left[1152-x^{5/2}-2x^{3/2}\right]dx$$

$$=1152x-\frac{2}{7}x^{7/2}-\frac{4}{5}x^{5/2}\Big|_0^{16}$$

$$=\left(1152(16)-\frac{2}{7}(16)^{7/2}-\frac{4}{5}(16)^{5/2}\right)-\left(1152(0)-\frac{2}{7}(0)^{7/2}-\frac{4}{5}(0)^{5/2}\right)$$

$$\underline{=\$12,931.66}$$

The producers' surplus is \$12,931.66