# *Lecture Two* – Trigonometric

## *Section* 2.1 – Angles

### Basic Terminology

Two distinct points determine line $AB$.

***Line segment AB:*** portion of the line between $A$ and $B$.

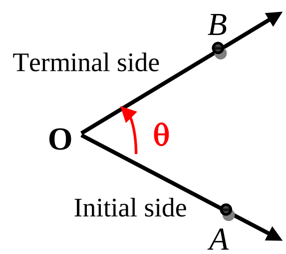

***Ray AB***: portion of the line $AB$ starts at $A$ and continues through $B$, and past $B$.


### *Angles* in General

An angle is formed by 2 rays with the same end point.

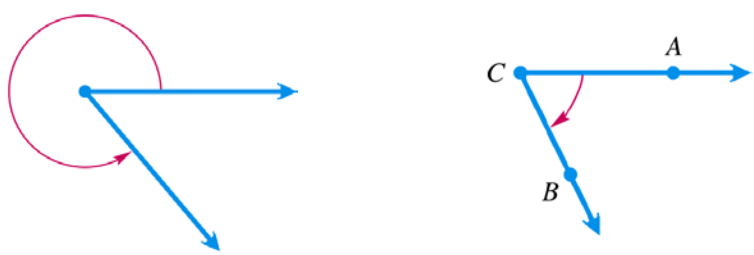
The two rays are the sides of the angle.
Angle $\theta = AOB$

**O** is the common endpoint and it is called ***vertex*** of the angle




An angle is in a Counterclockwise (***CCW***) direction: positive angle

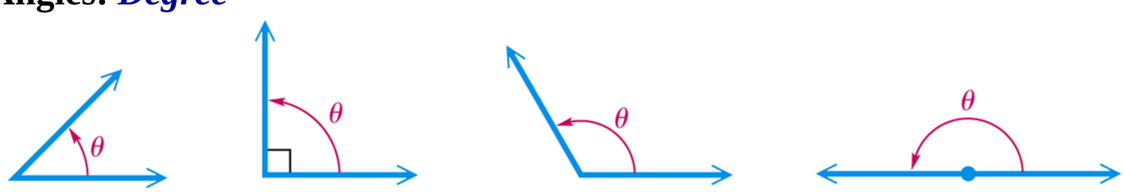
An angle is in a Clockwise (***CW***) direction: negative angle




Positive angle

Negative angle

### Type of Angles: *Degree*




Acute angle
$0° < \theta < 90°$

Right angle
$\theta = 90°$

Obtuse angle
$90° < \theta < 180°$

Straight angle
$\theta = 180°$

***Complementary angles***: $\alpha + \beta = 90°$

***Supplementary angles***: $\alpha + \beta = 180°$

### *Example*

Give the complement and the supplement of each angle: 40° 110°

***Solution***

***a.*** 40° Complement: 90° - 40° = 50° Supplement: 180° - 40° = 140°

***b.*** 110° Complement: 90° - 110° = -20° Supplement: 180° - 110° = 70°

## *Degrees, Minutes, Seconds*

$1°$: 1 *degree*
$1'$: 1 *minute*
$1''$: 1 *second*

**1** ***full Rotation*** or ***Revolution*** **= 360°** $1° = 60' = 3600''$ $1'' = \left(\frac{1}{60}\right)' = \left(\frac{1}{3600}\right)^{\circ}$

### *Example*

Change 27.25° to degrees and minutes

***Solution***

$$\begin{aligned} 27.25° &= 27° + .25° \\ &= 27° + .25(\mathbf{60'}) \\ &= 27° + 15' \\ &= 27°\ 15' \end{aligned}$$

| | |
|---|---|
| ***Example*** | ***Example*** |
| Add 48° 49′ and 72° 26′ | Subtract 24° 14′ and 90° |
| ***Solution*** | ***Solution*** |
| $\begin{array}{r} 48°\quad 49' \\ +\ 72°\quad 26' \\ \hline 120°\quad 75' \end{array}$ | $\begin{array}{r} 90° \\ -24°\quad 14' \\ \hline \end{array} = \begin{array}{r} 89°\quad 60' \\ -24°\quad 14' \\ \hline 65°\quad 46' \end{array}$ |
| $120°\ 75' = 120°\ 60' + 15'$ <br> $= 121°\ 15'$ | |

## Angles in Standard Position

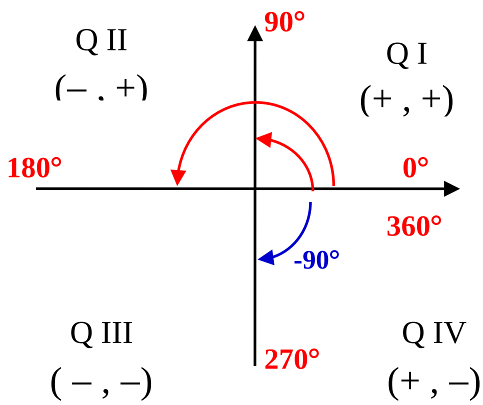
An angle is said to be in standard position if its initial side is along the positive *x*-axis and its vertex is at the origin.

If angle $\theta$ is in standard position and the terminal side of $\theta$ lies in quadrant I, then we say $\theta$ lies in QI

$$\theta \in QI$$

If the terminal side of an angle in standard position lies along one of the axes (*x*-axis or *y*-axis), such as angles with measures 90°, 180°, 270°, then that called a ***quadrantal*** *angle*.

Two angles in standard position with the same terminal side are called ***coterminal*** *angles*.




### *Example*

Find all angles that are coterminal with 120°.

***Solution***

120° + 360°k

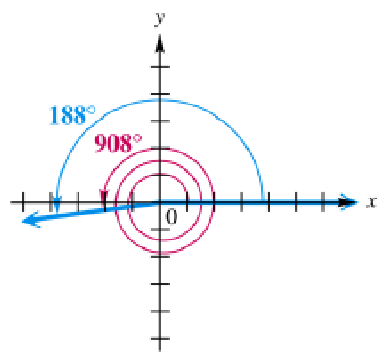
### *Example*

Find the angle of least possible positive measure coterminal with an angle of 908°.

***Solution***

$908° - 2.360° = 188°$

An angle of 908° is coterminal with an angle of 188°



### *Example*

CD players always spin at the same speed. Suppose a Constant Angular Velocity player makes 480 revolutions per minute. What degrees will a point on the edge of a CD spins for 2 *seconds*?

**Solution**

The player revolves 480 times in one minute $= \frac{480}{1'} = \frac{480}{60} = 8$ *times per* sec.

In 2 sec, the CD will spin: 2.8 = 16 times

Therefore; CD will revolve $16.360° = 5760°$

## *Triangles*

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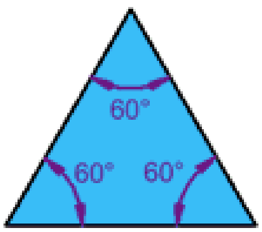
***Equilateral*** – All angles always equal to 60°& all sides are equals

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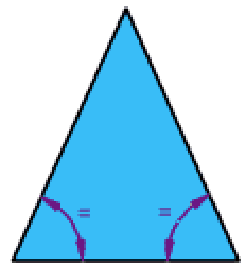
***Isosceles***: 2 sides and angles are equals

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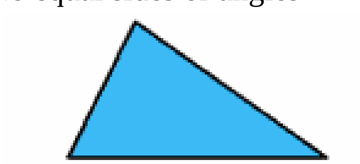
***Scalene***: No equal sides or angles



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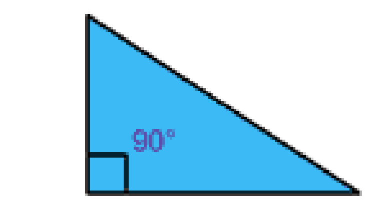
***Right***: Has a right angle 90°.

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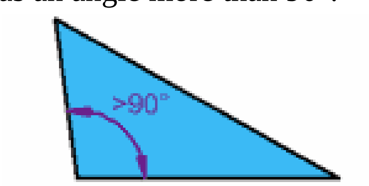
***Obtuse***: Has an angle more than 90°.

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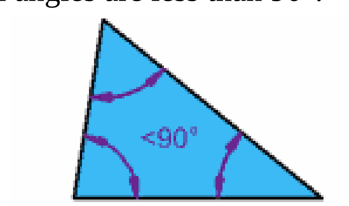
***Acute***: All angles are less than 90°.

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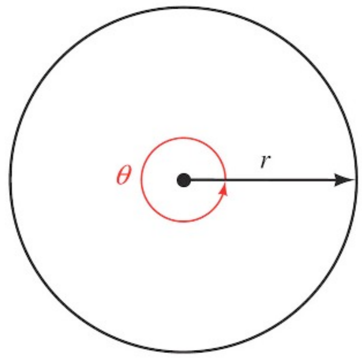
# *Radians*

## Degrees - Radians




$\theta$ *measures one full rotation* $\qquad \theta = 2\pi \qquad$ *The measure of* $\theta$ *in radians is* $2\pi$

$$\boxed{1 = 1\ rad}$$

$$1^\circ = 1 \text{ degree}$$

***If no unit of angle measure is specified, then the angle is to be measured in radians.***

$$\text{Full Rotation: } 360^\circ = 2\pi\ rad$$

$$180^\circ = \pi\ rad$$

### *Converting from Degrees to Radians*

$$\frac{180^\circ}{180} = \frac{\pi}{180}\ rad \qquad \Rightarrow 1^\circ = \frac{\pi}{180}\ rad$$

Multiply a degree measure by $\frac{\pi}{180}\ rad$ and simplify to convert to radians.

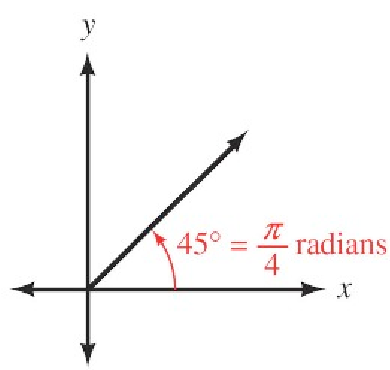
### *Example*

Convert 45° to radians

***Solution***

$$45^\circ = 45\left(\frac{\pi}{180}\right) rad$$

$$= \frac{\pi}{4}\ rad$$



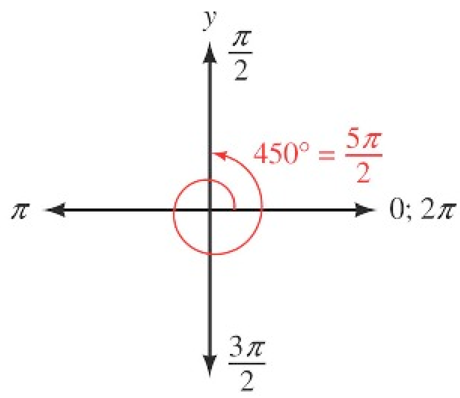
### *Example*

Convert -450° to radians

***Solution***

$$-450° = -450\left(\frac{\pi}{180}\right)rad$$

$$= -\frac{5\pi}{2}\ rad$$




### *Example*

Convert 249.8° to radians

***Solution***

$$249.8° = 249.8\left(\frac{\pi}{180}\right)rad$$

$$\approx 4.360\ rad$$

### *Converting from Radians to Degrees*

Multiply a radian measure by $\frac{180°}{\pi}$ radian and simplify to convert to degrees.

$$\frac{180°}{\pi} = \frac{\pi}{\pi}\ rad \qquad \boxed{\left(\frac{180}{\pi}\right)^{\circ} = 1\ rad}$$

### *Example*

Convert 1 to degrees

***Solution***

$$1\ rad = 1\left(\frac{180}{\pi}\right)^{\circ} = 1\left(\frac{180}{3.14}\right)^{\circ} = 57.3^{\circ}$$

### *Example*

Convert $\frac{4\pi}{3}$ to degrees

***Solution***

$$\frac{4\pi}{3} = \frac{4\pi}{3}\left(\frac{180}{\pi}\right)^{\circ} = 240^{\circ}$$

### *Example*

Convert –4.5 to degrees

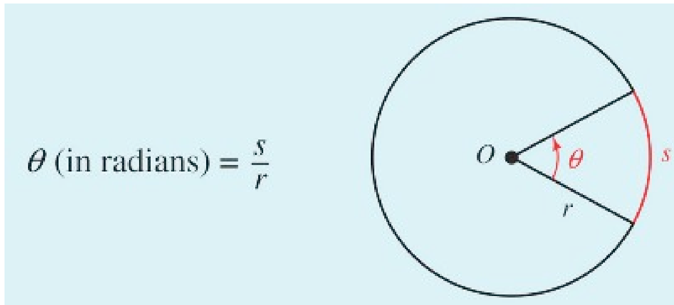
***Solution***

$$-4.5 = -4.5\left(\frac{180}{\pi}\right)^{\circ} \approx -257.8^{\circ}$$

# *Arc Length*

## *Definition*

If a central angle $\theta$, in a circle of a radius $r$, cuts off an arc of length $s$, then the measure of $\theta$, in radians is:

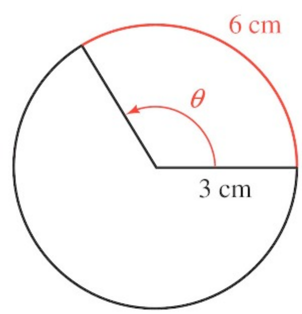



$\theta r = \frac{s}{r} r$ $\qquad\qquad\qquad\qquad s = r\theta \qquad (\theta \text{ in radians})$

> ***Note***: When applying the formula $s = r\theta$, the value of ***must*** be in ***radian***.

### *Example*

A central angle $\theta$ in a circle of radius 3 *cm* cuts off an arc of length 6 *cm*.
What is the radian measure of $\theta$.




***Solution***

$$\theta = \frac{s}{r} = \frac{6\ cm}{3\ cm} = 2\ rad$$

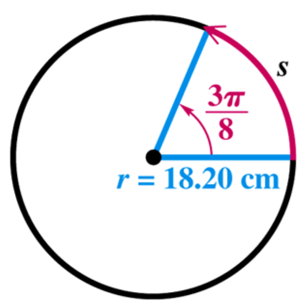
### *Example*

A circle has radius 18.20 *cm*. Find the length of the arc intercepted by a central angle with measure $\frac{3\pi}{8}$ radians.

***Solution***

***Given***: $\theta = \frac{3\pi}{8}\ rad, \quad r = 18.20\ cm$

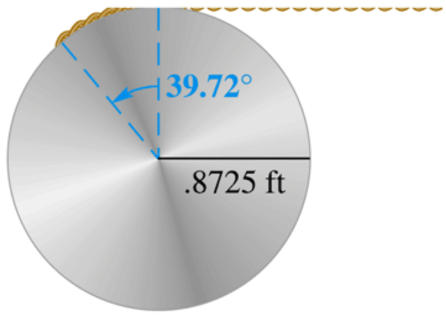
$$s = r\theta$$
$$= 18.20\left(\frac{3\pi}{8}\right) cm$$
$$\approx 21.44\ cm$$



### *Example*

A rope is being wound around a drum with radius 0.8725 *ft*. How much rope will be wound around the drum if the drum is rotated through an angle of39.72°?

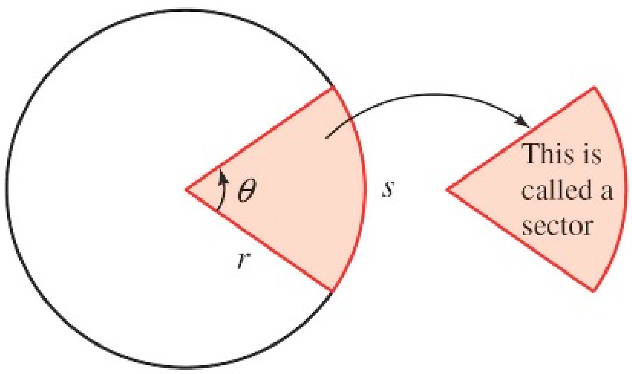
**Solution**




$$s = r\theta$$
$$= 0.8725\left(39.72\frac{\pi}{180}\right)$$
$$\approx 0.6049\ feet$$

## *Area of a Sector*

A sector of a circle is a portion of the interior of a circle intercepted by a central angle.




$$\begin{array}{lcccl} \textit{Area of sector} & \rightarrow & \dfrac{A}{\pi r^2} = \dfrac{\theta}{2\pi} & \leftarrow & \textit{Central angle } \theta \\ \textit{Area of circle} & \rightarrow & & \leftarrow & \textit{One full rotation} \end{array}$$

$$\frac{A}{\pi r^2}\pi r^2 = \frac{\theta}{2\pi}\pi r^2$$
$$A = \frac{1}{2}r^2\theta$$

### *Definition*

If $\theta$ (in *radians*) is a central angle in a circle with radius *r*, then the area of the sector formed by an angle $\theta$ is given by

$$A = \frac{1}{2}r^2\theta \qquad (\boldsymbol{\theta}\ \textbf{in radians})$$

### *Example*

Find the area of the sector formed by a central angle of 1.4 *radians* in a circle of radius 2.1 *meters*

***Solution***

**Given**: $r = 2.1\ m,\ \ \theta = 1.4$

$$A = \frac{1}{2}r^2\theta \ = \frac{1}{2}(2.1)^2(1.4)$$

$$\underline{= 3.1\ m^2}$$

### *Example*

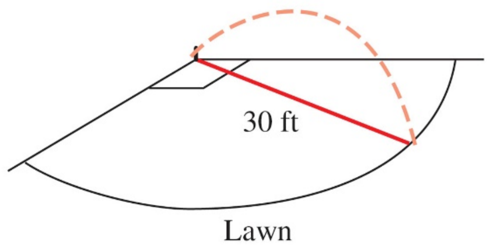
A lawn sprinkler located at the corner of a yard is set to rotate 90° and project water out 30.0 ft. To three significant digits, what area of lawn is watered by the sprinkler?

***Solution***

**Given**: $\theta = 90° = \frac{\pi}{2};\ \ r = 30\ ft$

$$A = \frac{1}{2}r^2\theta \ = \frac{1}{2}(30)^2\frac{\pi}{2}$$

$$\underline{\approx 707\ ft^2}$$



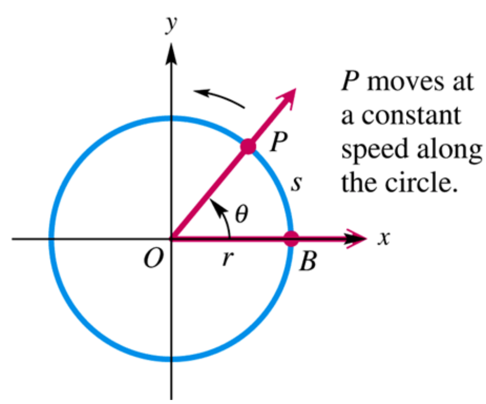

## *Linear Speed*

### *Definition*

If $P$ is a point on a circle of radius $r$, and $P$ moves a distance $s$ on the circumference of the circle in an amount of time $t$, then the ***linear velocity***, $v$, of $P$ is given by the formula

$$speed = \frac{distance}{time}$$

$$v = \frac{s}{t}$$




### *Example*

A point on a circle travels 5 cm in 2 sec. Find the linear velocity of the point.

***Solution***

**Given**: $s = 5cm;\ \ t = 2\ sec$

$$v = \frac{s}{t} = \frac{5\ cm}{2\ sec}$$

$$\underline{= 2.5\ cm / sec}$$

The most intuitive measure of the rate at which the rider is traveling around the wheel is what we call ***linear velocity***.

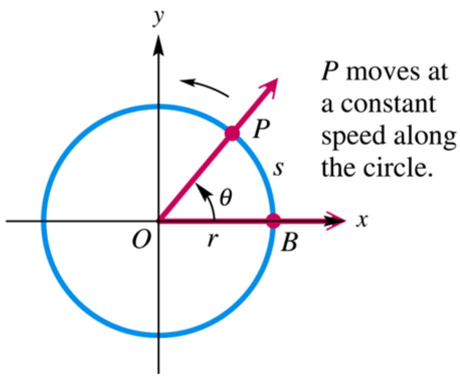
Another way to specify how fast the rider is traveling around the wheel is with what we call ***angular velocity***.

## *Angular Speed*

### *Definition*

If $P$ is a point moving with uniform circular motion on a circle of radius $r$, and the line from the center of the circle through $P$ sweeps out a central angle $\theta$ in an amount of time $t$, then the *angular velocity*, $\omega$ (omega), of $P$ is given by the formula

$$\omega = \frac{\theta}{t} \qquad \text{where } \theta \text{ is measured in radians}$$




### *Example*

A point on a circle rotates through $\frac{3\pi}{4}$ radians in 3 sec. Give the angular velocity of the point.

***Solution***

**Given**: $\theta = \frac{3\pi}{4}\, rad;\ t = 3sec$

$$\omega = \frac{\frac{3\pi}{4}\ rad}{3\ sec}$$

$$\boxed{= \frac{\pi}{4}\ rad\,/\,sec}$$

### *Example*

A bicycle wheel with a radius of 13.0 *in*. turns with an angular velocity of 3 *radians per seconds*. Find the distance traveled by a point on the bicycle tire in 1 *minute*.

***Solution***

**Given**: $r = 13.0\ in.;\ \omega = 3\text{rad}/sec;\ t = 1\ min = 60\ sec.$

$$\omega = \frac{\theta}{t} \Rightarrow \omega t = \theta \qquad s = r\theta \Rightarrow \theta = \frac{s}{r}$$

$$\omega t = \frac{s}{r}$$

$$s = \omega tr$$

$$= 3 \times 60 \times 13$$

$$\boxed{= 2{,}340\ inches} \qquad or\ \frac{2{,}340}{12} = 195\,ft$$

## *Relationship between the Two Velocities*

If $s = r\theta$

$$\frac{s}{t} = \frac{r\theta}{t}$$

$$\frac{s}{t} = r\frac{\theta}{t}$$

$$v = r\omega$$

$$v = r\frac{\theta}{t}$$

### *Linear and Angular Velocity*

If a point is moving with uniform circular motion on a circle of radius *r*, then the linear velocity *v* and angular velocity $\omega$ of the point are related by the formula

$$v = r\omega$$

### *Example*

A phonograph record is turning at 45 revolutions per minute (*rpm*). If the distance from the center of the record to a point on the edge of the record is 3 *inches*, find the angular velocity and the linear velocity of the point in feet per minute.

***Solution***

$$\begin{aligned}\omega &= 45\ rpm \\ &= 45\ \frac{rev}{min} \qquad\qquad 1\ revolution = 2\pi rad \\ &= 45\ \frac{rev}{min}\frac{2\pi\ rad}{1\ rev} \\ &= 90\pi\ rad\,/\,min\end{aligned}$$

$$\begin{aligned}v &= r\omega \\ &= (3\ in.)\left(90\pi\frac{rad}{min}\right) \\ &= 270\pi\frac{in}{min} \\ &\approx 848\ \ in\,/\,min\end{aligned}$$

$$v = 848\ \frac{in}{min}\frac{1\,ft}{12in}$$

$$v \approx 70.7\ ft\,/\,\text{min}$$

# *Exercises* *Section* 2.1– Angles

**1.** Indicate the angle if it is an acute or obtuse. Then give the complement and the supplement of each angle.

*a*) $10°$ *b*) $52°$ *c*) $90°$ *d*) $120°$ *e*) $150°$

**2.** Change to decimal degrees.

| | | | |
|---|---|---|---|
| *a)* $10°\ 45'$ | *c)* $274°\ 18'\ 59''$ | *e)* $98°\ 22'\ 45''$ | *g)* $1°\ 2'\ 3''$ |
| *b)* $34°\ 51'\ 35''$ | *d)* $74°\ 8'\ 14''$ | *f)* $9°\ 9'\ 9''$ | *h)* $73°\ 40'\ 40''$ |

**3.** Convert to degrees, minutes, and seconds.

| | | | |
|---|---|---|---|
| *a)* $89.9004°$ | *c)* $122.6853°$ | *e)* $44.01°$ | *g)* $29.411°$ |
| *b)* $34.817°$ | *d)* $178.5994°$ | *f)* $19.99°$ | *h)* $18.255°$ |

**4.** Perform each calculation

*a)* $51°\ 29' + 32°\ 46'$ *b)* $90° - 73°12'$ *c)* $90° - 36°\ 18'\ 47''$ *d)* $75°\ 15' + 83°\ 32'$

**5.** Find the angle of least possible positive measure coterminal with an angle of

*a)* $-75°$ *b)* $-800°$ *c)* $270°$

**6.** A vertical rise of the Forest Double chair lift 1,170 *feet* and the length of the chair lift as 5,570 *feet*. To the nearest foot, find the horizontal distance covered by a person riding this lift.

**7.** A tire is rotating 600 times per minute. Through how many degrees does a point of the edge of the tire move in $\frac{1}{2}$ second?

**8.** A windmill makes 90 revolutions per minute. How many revolutions does it make per second?

**9.** Convert to radians

*a)* $256°\ 20'$ *b)* $-78.4°$ *c)* $330°$ *d)* $-60°$ *e)* $-225°$

**10.** Convert to degrees

| | | | |
|---|---|---|---|
| *a)* $\frac{11\pi}{6}$ | *c)* $\frac{\pi}{6}$ | *e)* $\frac{\pi}{3}$ | *g)* $-4\pi$ |
| *b)* $-\frac{5\pi}{3}$ | *d)* $2.4$ | *f)* $-\frac{5\pi}{12}$ | *h)* $\frac{7\pi}{13}$ |

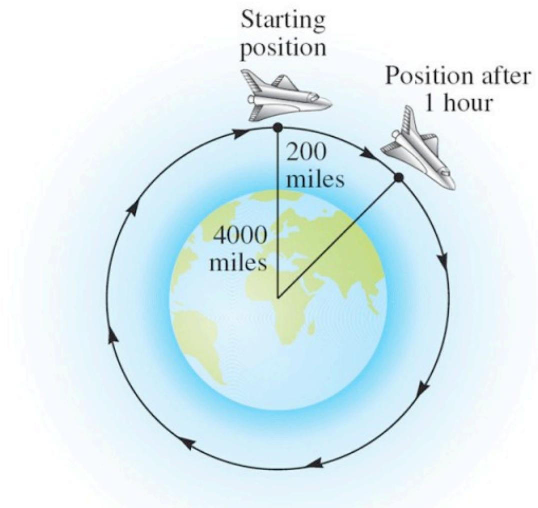
**11.** The minute hand of a clock is 1.2 *cm* long. How far does the tip of the minute hand travel in 40 *minutes*?

**12.** Find the radian measure if angle $\theta$, if $\theta$ is a central angle in a circle of radius $r = 4$ *inches*, and $\theta$ cuts off an arc of length $s = 12\pi$ *inches*.

**13.** Give the length of the arc cut off by a central angle of 2 *radians* in a circle of radius 4.3 *inches*.

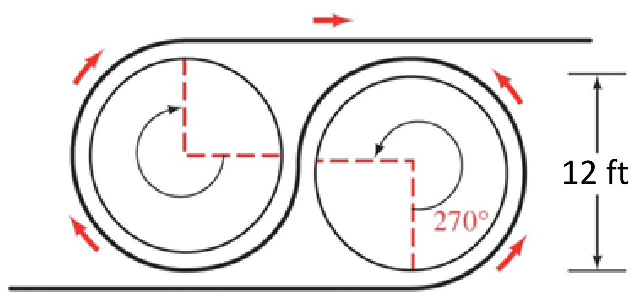
**14.** Reno, Nevada is due north of Los Angeles. The latitude of Reno is 40°, while that of Los Angeles is 34° N. The radius of Earth is about 4000 *mi*. Find the north-south distance between the two cities.

**15.** A space shuttle 200 *miles* above the earth is orbiting the earth once every 6 *hours*. How long, in hours, does it take the space shuttle to travel 8,400 *miles*? (Assume the radius of the earth is 4,000 *miles*.) Give both the exact value and an approximate value for your answer.




**16.** The pendulum on a grandfather clock swings from side to side once every second. If the length of the pendulum is 4 *feet* and the angle through which it swings is 20°. Find the total distance traveled in 1 *minute* by the tip of the pendulum on the grandfather clock.

**17.** The first cable railway to make use of the figure-eight drive system was a Sutter Street Railway. Each drive sheave was 12 *feet* in diameter. Find the length of cable riding on one of the drive sheaves.

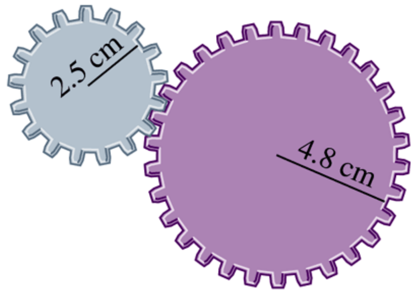



**18.** The diameter of a model of George Ferris's Ferris wheel is 250 *feet*, and $\theta$ is the central angle formed as a rider travels from his or her initial position $P_0$ to position $P_1$. Find the distance traveled by the rider if $\theta = 45°$ and if $\theta = 105°$.

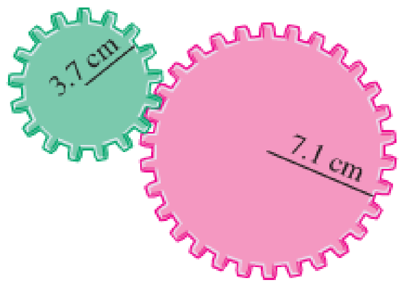
**19.** The rotation of the smaller wheel causes the larger wheel to rotate. Through how many degrees will the larger wheel rotate if the smaller one rotates through 60.0°?

**20.** Find the number of regular (statute) miles in 1 *nautical mile* to the nearest hundredth of a mile. (Use 4,000 miles for the radius of the earth).

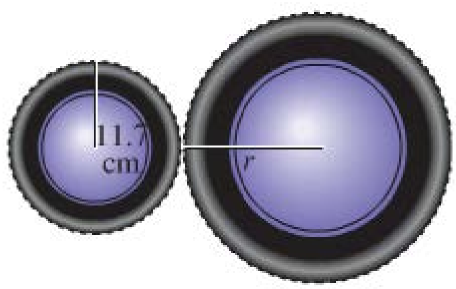
**21.** Two gears are adjusted so that the smaller gear drives the larger one. If the smaller gear rotates through an angle of 225°, through how many degrees will the larger gear rotate?




**22.** Two gears are adjusted so that the smaller gear drives the larger one. If the smaller gear rotates through an angle of 300°, through how many degrees will the larger rotate?



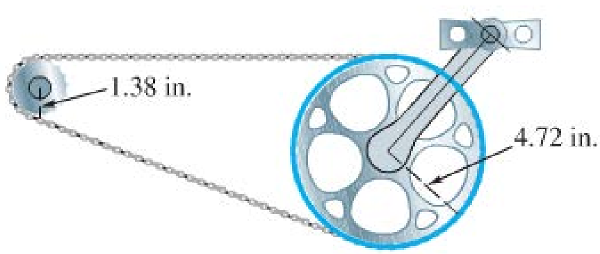

**23.** Find the radius of the larger wheel if the smaller wheel rotates 80° when the larger wheel rotates 50°.



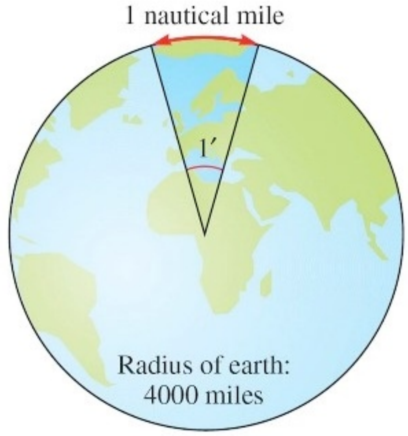

**24.** Los Angeles and New York City are approximately 2,500 *miles* apart on the surface of the earth. Assuming that the radius of the earth is 4,000 *miles*, find the radian measure of the central angle with its vertex at the center of the earth that has Los Angeles on one side and New York City in the other side.

**25.** If two ships are 20 *nautical miles* apart on the ocean, how many statute miles apart are they?

**26.** The figure shows the chain drive of a bicycle. How far will the bicycle move if the pedals are rotated through 180°? Assume the radius of the bicycle wheel is 13.6 *in*.



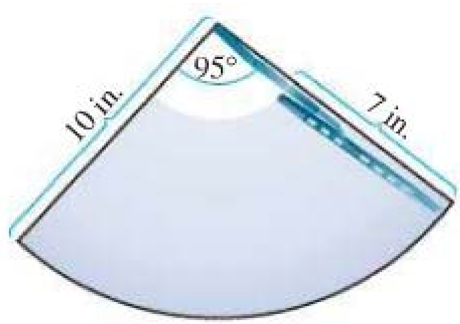
**27.** If a central angle with its vertex at the center of the earth has a measure of 1′, then the arc on the surface of the earth that is cut off by this angle (knows as the great circle distance) has a measure of 1 nautical mile.



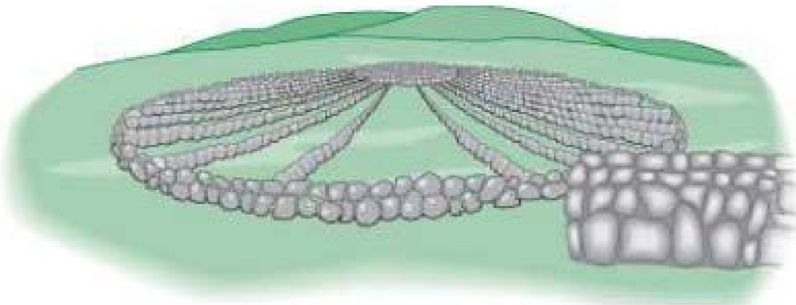

**28.** How many inches will the weight rise if the pulley is rotated through an angle of 71° 50′? Through what angle, to the nearest minute, must the pulley be rotated to raise the weight 6 *in*?


**29.** Find the radius of the pulley if a rotation of 51.6° raises the weight 11.4 *cm*.


**30.** The total arm and blade of a single windshield wiper was 10 *in*. long and rotated back and forth through an angle of 95°. The shaded region in the figure is the portion of the windshield cleaned by the 7-*in*. wiper blade. What is the area of the region cleaned?

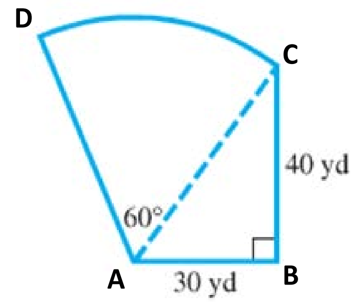


**31.** The circular of a Medicine Wheel is 2500 *yrs* old. There are 27 aboriginal spokes in the wheel, all equally spaced.

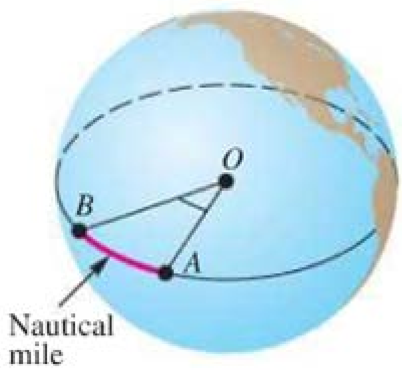


*a)* Find the measure of each central angle in degrees and in radians.
*b)* The radius measure of each of the wheel is 76.0 *ft*, find the circumference.
*c)* Find the length of each arc intercepted by consecutive pairs of spokes.
*d)* Find the area of each sector formed by consecutive spokes,

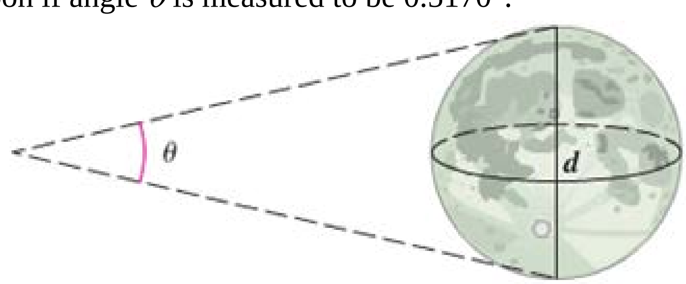
**32.** A frequent problem in surveying city lots and rural lands adjacent to curves of highways and railways is that of finding the area when one or more of the boundary lines is the arc of the circle. Find the area of the lot.




**33.** Nautical miles are used by ships and airplanes. They are different from statue miles, which equal 5280 *ft*. A nautical mile is defined to be the arc length along the equator intercepted by a central angle AOB of 1 *min*. If the equatorial radius is 3963 *mi*, use the arc length formula to approximate the number of statute miles in 1 *nautical mile*.




**34.** The distance to the moon is approximately 238,900 *mi*. Use the arc length formula to estimate the diameter $d$ of the moon if angle $\theta$ is measured to be 0.5170°.



**35.** The minute hand of a clock is 1.2 *cm* long. To two significant digits, how far does the tip of the minute hand move in 20 *minutes*?

**36.** If the sector formed by a central angle of 15° has an area of $\frac{\pi}{3}\text{cm}^2$, find the radius of a circle.

**37.** Suppose that *P* is on a circle with radius 10 *cm*, and ray O*P* is rotating with angular speed $\frac{\pi}{18}\ rad/\text{sec}$.

*a)* Find the angle generated by *P* in 6 *seconds*

*b)* Find the distance traveled by *P* along the circle in 6 seconds.

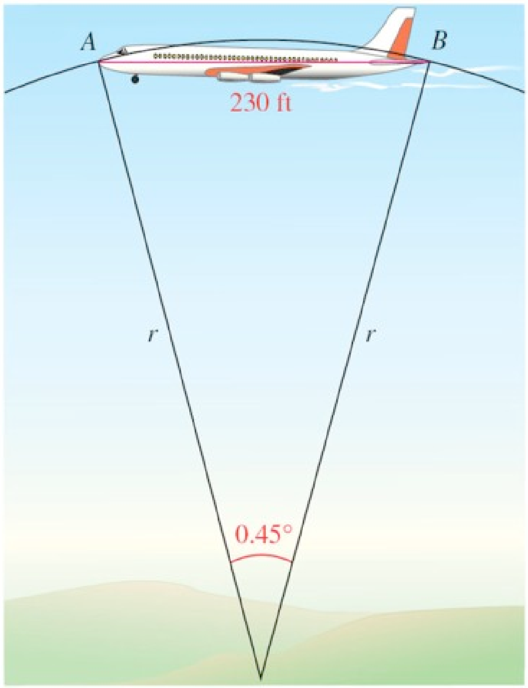
*c)* Find the linear speed of *P* in cm per sec.

**38.** A belt runs a pulley of radius 6 cm at $80\ rev/\text{min}$.

*a)* Find the angular speed of the pulley in radians per sec.

*b)* Find the linear speed of the belt in cm per *sec*.

**39.** A person standing on the earth notices that a 747 jet flying overhead subtends an angle 0.45°. If the length of the jet is 230 *ft*., find its altitude to the nearest thousand feet.




**40.** Find the linear velocity of a point moving with uniform circular motion, if *s* = 12 cm and *t* = 2 sec.

**41.** Find the distance s covered by a point moving with linear velocity *v* = 55 mi/hr and *t* = 0.5 hr.

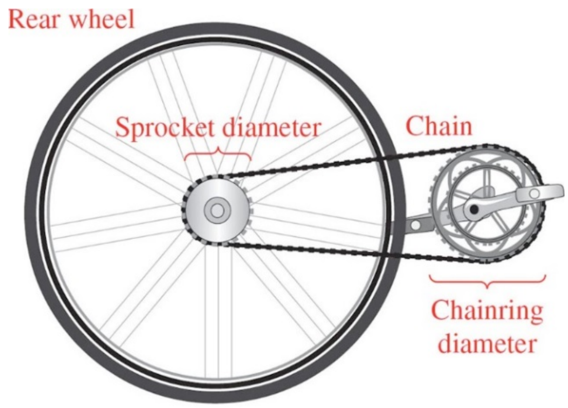
**42.** Point ***P*** sweeps out central angle θ = 12π as it rotates on a circle of radius r with t = 5π sec. Find the angular velocity of point ***P***.

**43.** Find the angular velocity, in radians per minute, associated with given 7.2 *rpm*.

**44.** Suppose that point $P$ is on a circle with radius 60 *cm*, and ray $OP$ is rotating with angular speed $\frac{\pi}{12}$ radian per sec.

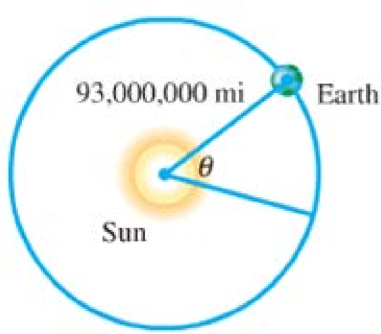
*a)* Find the angle generated by $P$ in 8 *sec*.
*b)* Find the distance traveled by $P$ along the circle in 8 *sec*.
*c)* Find the linear speed of $P$.

**45.** When Lance Armstrong blazed up Mount Ventoux in the 2002 tour, he was equipped with a 150-millimeter-diameter chainring and a 95-*millimeter*-diameter sprocket. Lance is known for maintaining a very high cadence, or pedal rate. If he was pedaling at a rate of 90 revolutions per minute, find his speed in kilometers per hour. (1 *km* = 1,000,000 *mm* or $10^6$ *mm*)




**46.** Tires of a bicycle have radius 13 *in*. and are turning at the rate of 215 revolutions per min. How fast is the bicycle traveling in miles per hour? (*Hint*: 1 *mi* = 5280 *ft*.)


**47.** Earth travels about the sun in an orbit that is almost circular. Assume that the orbit is a circle with radius 93,000,000 *mi*. Its angular and linear speeds are used in designing solar-power facilities.




*a)* Assume that a year is 365 days, and find the angle formed by Earth's movement in one day.
*b)* Give the angular speed in radians per hour.
*c)* Find the linear speed of Earth in miles per hour.

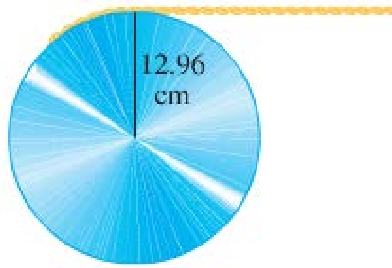
**48.** Earth revolves on its axis once every 24 *hr*. Assuming that earth's radius is 6400 *km*, find the following.

*a)* Angular speed of Earth in radians per day and radians per hour.

*b)* Linear speed at the North Pole or South Pole

*c)* Linear speed ar a city on the equator

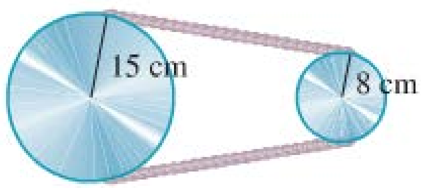
**49.** The pulley has a radius of 12.96 *cm*. Suppose it takes 18 *sec* for 56 *cm* of belt to go around the pulley.




*a)* Find the linear speed of the belt in cm per sec.

*b)* Find the angular speed of the pulley in rad per sec.

**50.** The two pulleys have radii of 15 *cm* and 8 *cm*, respectively. The larger pulley rotates 25 times in 36 sec. Find the angular speed of each pulley in rad per sec.



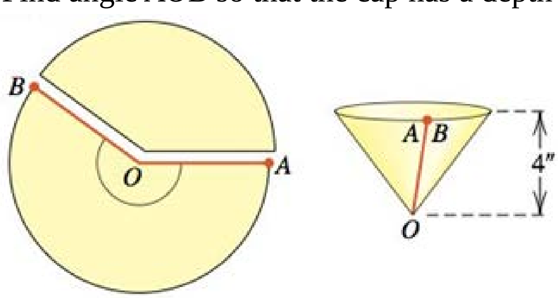

**51.** A thread is being pulled off a spool at the rate of 59.4 *cm* per *sec*. Find the radius of the spool if it makes 152 revolutions per min.

**52.** A railroad track is laid along the arc of a circle of radius 1800 *ft*. The circular part of the track subtends a central angle of 40°. How long (in seconds) will it take a point on the front of a train traveling 30 *mph* to go around this portion of the track?

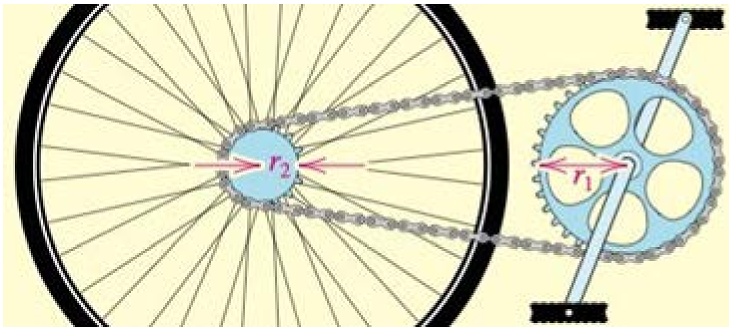
**53.** A 90-horsepower outboard motor at full throttle will rotate it propeller at exactly 5000 revolutions per min. Find the angular speed of the propeller in radians per second.

**54.** The shoulder joint can rotate at 25 *rad/min*. If a golfer's arm is straight and the distance from the shoulder to the club head is 5.00 *ft*., find the linear speed of the club head from the shoulder rotation

**55.** A conical paper cup is constructed by removing a sector from a circle of radius 5 *inches* and attaching edge *OA* to *OB*. Find angle *AOB* so that the cap has a depth of 4 *inches*.



**56.** The sprocket assembly for a bicycle is show in the figure. If the sprocket of radius $r_1$ rotates through an angle of $\theta_1$ radians, find the corresponding angle of rotation for the sprocket of radius $r_2$




**57.** A vendor sells two sizes of pizza by the slice. The small slice is $\frac{1}{6}$ of a circular 18–inch–diameter pizza, and it sells for \$2.00. The large slice is $\frac{1}{8}$ of a circular 26–inch–diameter pizza, and it sells for \$3.00. Which slice provides more pizza per dollar?

**58.** A cone–shaped tent is made from a circular piece of canvas 24 *feet* in diameter by removing a sector with central angle $100°$ and connecting the ends. What is the surface area of the tent?

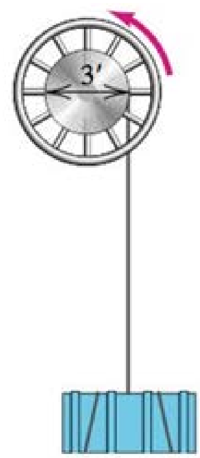
**59.** A simple model of the core of a tornado is a right circular cylinder that rotates about its axis. If a tornado has a core diameter of 200 *feet* and maximum wind speed of 180 *mi/hr*. (or 264 *ft/sec*) at the perimeter of the core, approximate the number of revolutions the core makes each minute.

$\approx 25.2\ rev/\min$

**60.** Earth rotates about its axis once every 23 *hours*, 56 *minutes*, and 4 *seconds*. Approximate the number of radians Earth rotates in one second $\approx 7.31 \times 10^{-5}\ rad/\sec$

**61.** A typical tire for a compact car is 22 *inches* in diameter. If the car is traveling at a speed of 60 *mi/hr*., find the number of revolutions the tire makes per minute. $\approx 916.73\ rev/\min$

**62.** A large winch of diameter 3 *feet* is used to hoist cargo.




*a)* Find the distance the cargo is lifted if the winch rotates through an angle measure $\frac{7\pi}{4}$. $\approx 8.25\ ft$

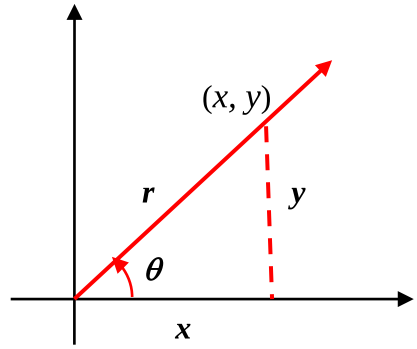
*b)* Find the angle (in *radians*) through which the winch must rotate in order to lift the cargo *d feet*. $\frac{2}{3}d$

**63.** A pendulum in a grandfather clock is 4 *feet* long and swings back and forth along a 6–*inch* arc. Approximate the angle (in *degrees*) through which the pendulum passes during one swing. $\approx 7.162°$

# *Section* 2.2 – Trigonometric Functions

Let $(x, y)$ be a point on the terminal side of an angle $\theta$ in standard position

The distance from the point to the origin is given by: $r = \sqrt{x^2 + y^2}$




## *Six* Trigonometry Functions

$$\sin\theta = \frac{Opposite}{Hypotenuse} = \frac{opp}{hyp} = \frac{y}{r} = \frac{a}{c} = \cos B$$

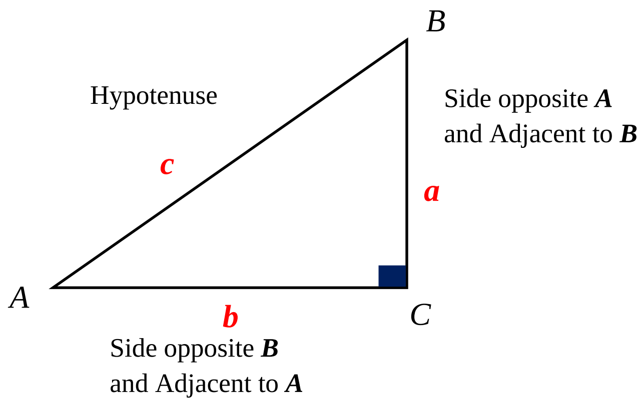
$$\cos\theta = \frac{Adjacent}{Hypotenuse} = \frac{adj}{hyp} = \frac{x}{r} = \frac{b}{c} = \sin B$$

$$\tan\theta = \frac{opp}{adj} = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} = \frac{a}{b} = \cot B$$

$$\cot\theta = \frac{adj}{opp} = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} = \frac{x}{y} = \frac{b}{a} = \tan B$$

$$\sec\theta = \frac{hyp}{adj} = \frac{1}{\cos\theta} = \frac{r}{x} = \frac{c}{b} = \csc B$$

$$\csc\theta = \frac{hyp}{opp} = \frac{1}{\sin\theta} = \frac{r}{y} = \frac{c}{a} = \sec B$$




### *Example*

Find the six trigonometry functions of $\theta$ if $\theta$ is in the standard position and the point $(8, 15)$ is on the terminal side of $\theta$.

***Solution***

$$\Rightarrow r = \sqrt{x^2 + y^2} = \sqrt{8^2 + 15^2} = 17$$

$$\sin\theta = \frac{y}{r} = \frac{15}{17} \qquad \cos\theta = \frac{x}{r} = \frac{8}{17} \qquad \tan\theta = \frac{y}{x} = \frac{15}{8}$$

$$\csc\theta = \frac{r}{y} = \frac{17}{15} \qquad \sec\theta = \frac{r}{x} = \frac{17}{8} \qquad \cot\theta = \frac{x}{y} = \frac{8}{15}$$

### Example

Which will be greater, tan 30° or tan 40°? How large could $\tan\theta$ be?

**Solution**

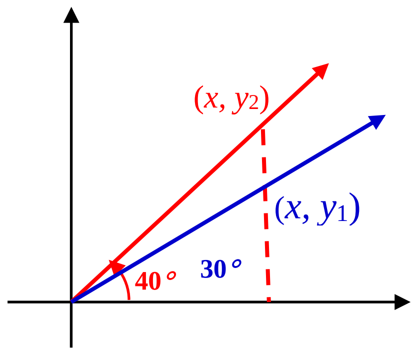
$$\tan 30^\circ = \frac{y_1}{x}$$

$$\tan 40^\circ = \frac{y_2}{x}$$

$$Ratio: \frac{y_2}{x} > \frac{y_1}{x}$$

$\to$ tan 40°> tan 30°

No limit as to how large $\tan\theta$ can be.




| *Function* | *I* | *II* | *III* | *IV* |
|---|---|---|---|---|
| *y = sin x* | + | + | – | – |
| *y = cosx* | + | – | – | + |
| *y = tan x* | + | – | + | – |
| *y = cot x* | + | – | + | – |
| *y = cscx* | + | + | – | – |
| *y = sec x* | + | – | – | + |

### Example

If $\cos\theta = \frac{\sqrt{3}}{2}$, and $\theta$ is QIV, find $\sin\theta$ and $\tan\theta$.

**Solution**

$$\cos\theta = \frac{\sqrt{3}}{2} = \frac{x}{r} \quad \to x = \sqrt{3}, \quad r = 2$$

$$y = \sqrt{r^2 - x^2}$$

$$y = \sqrt{2^2 - \left(\sqrt{3}\right)^2} = \sqrt{4-3} = 1 \qquad \text{Since } \theta \text{ is Q IV} \Rightarrow \boxed{y = -1}$$

$$\sin\theta = \frac{y}{r} = -\frac{1}{2}$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

## *Reciprocal Identities*

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

## *Ratio Identities*

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

## *Pythagorean Identities*

$$x^2 + y^2 = r^2$$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1 \quad \Rightarrow \quad \boxed{\cos^2\theta + \sin^2\theta = 1}$$

Solving for $\cos\theta$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$$

Solving for $\sin\theta$

$$\sin^2\theta = 1 - \cos^2\theta \quad \Rightarrow \quad \boxed{\sin\theta = \pm\sqrt{1 - \cos^2\theta}}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\left(\frac{\cos\theta}{\cos\theta}\right)^2 + \left(\frac{\sin\theta}{\cos\theta}\right)^2 = \left(\frac{1}{\cos\theta}\right)^2$$

$$\boxed{1 + \tan^2\theta = \sec^2\theta}$$

| |
|---|
| $\cos^2\theta + \sin^2\theta = 1$ |
| $\cos\theta = \pm\sqrt{1 - \sin^2\theta}$ |
| $\sin\theta = \pm\sqrt{1 - \cos^2\theta}$ |
| $1 + \tan^2\theta = \sec^2\theta$ |
| $1 + \cot^2\theta = \csc^2\theta$ |

### *Example*

Write $\tan\theta$ in terms of $\sin\theta$.

***Solution***

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$= \frac{\sin\theta}{\pm\sqrt{1-\sin^2\theta}}$$
$$= \pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$$

### *Example*

If $\cos\theta = \frac{1}{2}$ and $\theta$ terminated in QIV, find the remaining trigonometric ratios for $\theta$.

***Solution***

| | |
|---|---|
| $\sin\theta = -\sqrt{1-\cos^2\theta}$ $= -\sqrt{1-\left(\frac{1}{2}\right)^2}$ $= -\sqrt{1-\frac{1}{4}}$ $= -\sqrt{\frac{3}{4}}$ $\underline{= -\frac{\sqrt{3}}{2}}$ | $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{1/2} = 2$ |
| | $\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\sqrt{3}/2} = -\frac{2}{\sqrt{3}}$ |
| | $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$ |
| | $\cot\theta = -\frac{1}{\sqrt{3}}$ |

### *Example*

Simplify the expression $\sqrt{x^2+9}$ as much as possible after substituting $3\tan\theta$ for $x$

***Solution***

$$x = 3\tan\theta$$
$$\sqrt{x^2+9} = \sqrt{(3\tan\theta)^2+9}$$
$$= \sqrt{9\tan^2\theta+9}$$
$$= \sqrt{9\left(\tan^2\theta+1\right)}$$
$$= 3\sqrt{\sec^2\theta}$$
$$\underline{= 3\sec\theta}$$

### *Example*

Triangle ABC is a right triangle with $C = 90°$. If $a = 6$ and $c = 10$, find the six trigonometric functions of $A$.

**Solution**

$$b = \sqrt{c^2 - a^2} = \sqrt{10^2 - 6^2} = 8$$

| | | |
|---|---|---|
| $\sin A = \frac{a}{c} = \frac{6}{10} = \frac{3}{5}$ | $\cos A = \frac{b}{c} = \frac{8}{10} = \frac{4}{5}$ | $\tan A = \frac{a}{b} = \frac{6}{8} = \frac{3}{4}$ |
| $\csc A = \frac{c}{a} = \frac{10}{6} = \frac{5}{3}$ | $\sec A = \frac{c}{b} = \frac{10}{8} = \frac{5}{4}$ | $\tan 71°\ \cot A = \frac{b}{a} = \frac{8}{6} = \frac{4}{3}$ |

$$\boxed{if\ \ A + B = 90° \Rightarrow \begin{cases} \sin A = \cos B \\ \sec A = \csc B \\ \tan A = \cot B \end{cases}}$$

### *Cofunction Theorem*

A trigonometric function of an angle is always equal to the cofunction of the complement of the angle.

***Example***

Write each function in terms of its cofunction

*a)* $\cos 52°$

**Solution**

$$\cos 52° = \sin(90° - 52°) = \sin 38°$$

*b)* $\tan 71°$

**Solution**

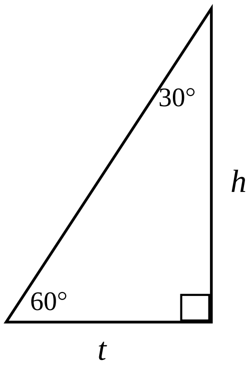
$$\tan 71° = \cot(90° - 71°) = \cot 19°$$

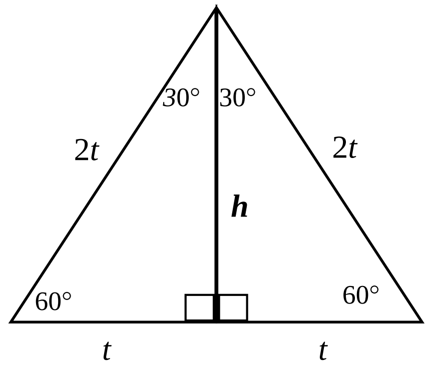
*c)* $\sec 24°$

**Solution**

$$\sec 24° = \csc(90° - 24°) = \csc 66°$$

## 30° - 60° - 90° *Triangle*



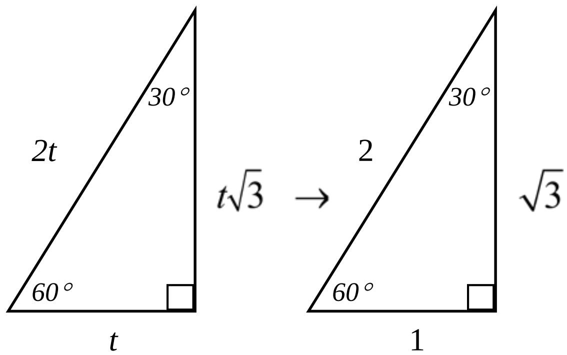

$$t^2 + h^2 = (2t)^2$$

$$t^2 + h^2 = 4t^2$$

$$h^2 = 4t^2 - t^2$$

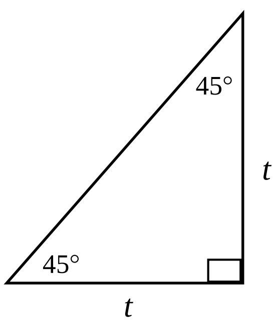
$$h^2 = 3t^2$$

$$h = t\sqrt{3}$$




$$\Rightarrow \boxed{\sin 60° = \frac{\sqrt{3}}{2}}$$

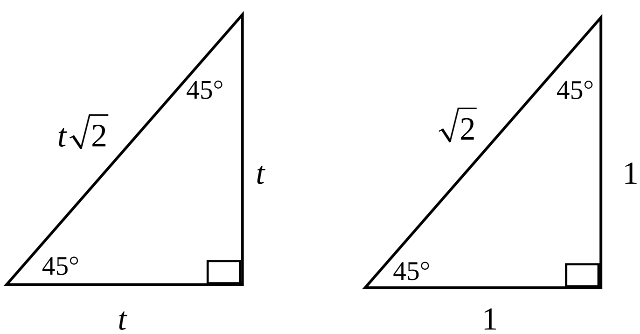
## 45° - 45° - 90° *Triangle*




$$hypotenuse^2 = t^2 + t^2$$

$$hypotenuse = \sqrt{2t^2}$$

$$hypotenuse = t\sqrt{2}$$

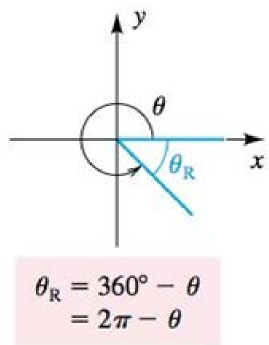



$$\Rightarrow \boxed{\cos 45° = \frac{1}{\sqrt{2}}}$$

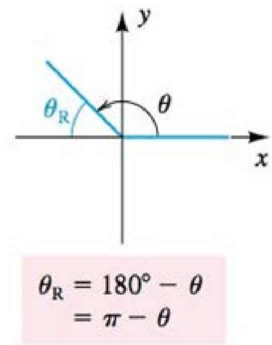
# Reference Angle

## *Definition*

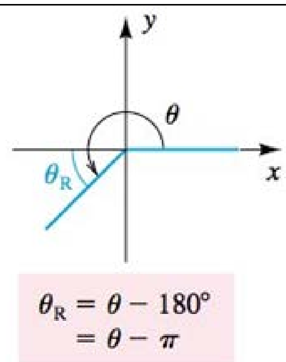
The reference angle or related angle for any angle $\theta$ in standard position ifs the positive acute angle between the terminal side of $\theta$ and the *x*-axis, and it is denoted $\hat{\theta}$



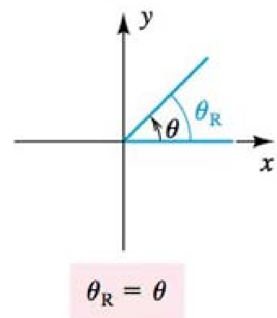

Quadrant **I**




Quadrant **II**




Quadrant **III**




$\hat{\theta} = 360° - \theta \quad \leftrightarrow \quad \theta = 360° - \hat{\theta}$

Quadrant **IV**

***Example***

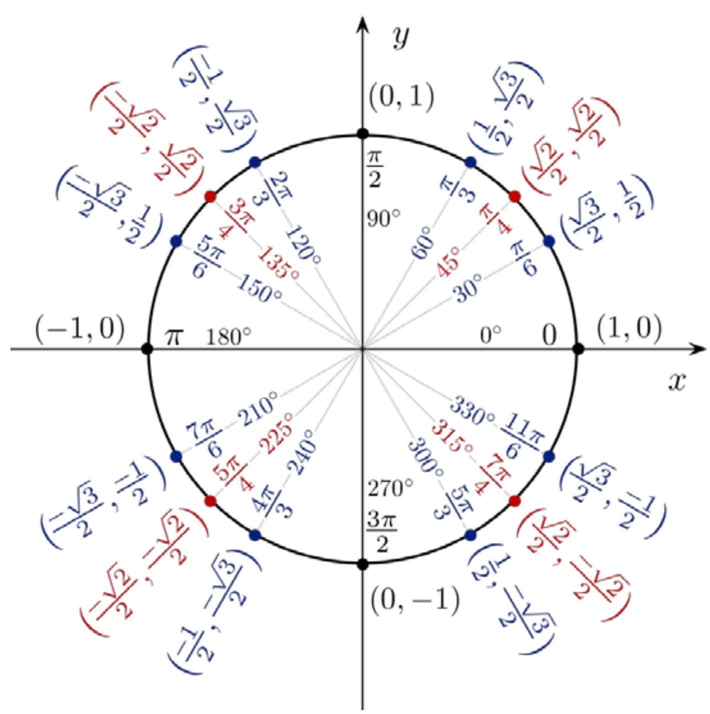
Find the exact value of $\sin 240°$

***Solution***

$\hat{\theta} = 240° - 180° = 60° \quad \to 240° \in \text{QIII}$

$\sin 240° = -\sin 60°$

$$= -\frac{\sqrt{3}}{2}$$



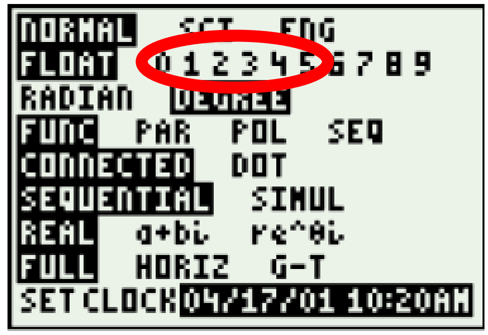
***Approximation***- Simply using calculator

$\sin 250° \approx -0.9397$

$\cos 250° \approx -0.3420$

$\tan 250° \approx 2.7475$

$\csc 250° = \dfrac{1}{\sin 250°} \approx -1.0642$




*To find the angle by using the inverse trigonometry functions, always enter a* ***positive*** *value.*

### Example

Find $\theta$ if $\sin\theta = -0.5592$ and $\theta$ terminates in QIII with $0° \le \theta < 360°$.

***Solution***

$\hat{\theta} = \sin^{-1} 0.5592 \approx 34°$

$\theta \in \text{QIII}$

$\Rightarrow \theta = 180° + 34° = 214°$

### Example

Find $\theta$ to the nearest degree if $\cot\theta = -1.6003$ and $\theta$ terminates in QII with $0° \le \theta < 360°$.

***Solution***

$\cot\theta = -1.6003 = \dfrac{1}{\tan\theta}$

$\tan\theta = \dfrac{1}{-1.6003}$

$\hat{\theta} = \tan^{-1}\dfrac{1}{1.6003} = 32°$

$\theta \in \text{QII} \quad \Rightarrow \theta = 180° - 32° = 148°$

# *Exercise* *Section* 2.2 – Trigonometric Functions

Find the six trigonometry functions of $\theta$ if $\theta$ is in the standard position and the given point is on the terminal side of $\theta$.

**1.** $(-2,\ 3)$ **2.** $(-3,\ -4)$ **3.** $(-3,\ 0)$ **4.** $(12,\ -5)$

**5.** Find the values of the six trigonometric functions for an angle of 90°.

**6.** Indicate the two quadrants $\theta$ could terminate in if $\cos\theta = \frac{1}{2}$

**7.** Indicate the two quadrants $\theta$ could terminate in if $\csc\theta = -2.45$

**8.** Find the remaining trigonometric function of $\theta$ if $\sin\theta = \frac{12}{13}$ and $\theta$ terminates in QI.

**9.** Find the remaining trigonometric function of $\theta$ if $\cot\theta = -2$ and $\theta$ terminates in QII.

**10.** Find the remaining trigonometric function of $\theta$ if $\tan\theta = \frac{3}{4}$ and $\theta$ terminates in QIII.

**11.** Find the remaining trigonometric function of $\theta$ if $\cos\theta = \frac{24}{25}$ and $\theta$ terminates in QIV.

**12.** Find the remaining trigonometric functions of $\theta$ if $\cos\theta = \frac{\sqrt{3}}{2}$ and $\theta$ is terminates in QIV.

**13.** Find the remaining trigonometric functions of $\theta$ if $\tan\theta = -\frac{1}{2}$ and $\cos\theta > 0$.

**14.** If $\sin\theta = -\frac{5}{13}$, and $\theta$ is QIII, find $\cos\theta$ and $\tan\theta$.

**15.** If $\cos\theta = \frac{3}{5}$, and $\theta$ is QIV, find $\sin\theta$ and $\tan\theta$.

**16.** Use the reciprocal identities if $\cos\theta = \frac{\sqrt{3}}{2}$ find $\sec\theta$

**17.** Find $\cos\theta$, given that $\sec\theta = \frac{5}{3}$

**18.** Find $\sin\theta$, given that $\csc\theta = -\frac{\sqrt{12}}{2}$

**19.** Use a ratio identity to find $\tan\theta$ if $\sin\theta = \frac{3}{5}$ and $\cos\theta = -\frac{4}{5}$

**20.** If $\cos\theta = -\frac{1}{2}$ and $\theta$ terminates in QII, find $\sin\theta$

**21.** If $\sin\theta = \frac{3}{5}$ and $\theta$ terminated in QII, find $\cos\theta$ and $\tan\theta$.

**22.** Find $\tan\theta$ if $\sin\theta = \frac{1}{3}$ and $\theta$ terminates in QI

**23.** Find the remaining trigonometric ratios of $\theta$, if $\sec\theta = -3$ and $\theta \in QIII$

**24.** Using the calculator and rounding your answer to the nearest hundredth, find the remaining trigonometric ratios of $\theta$ if $\csc\theta = -2.45$ and $\theta \in QIII$

**25.** Write $\dfrac{\sec\theta}{\csc\theta}$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

**26.** Write $\cot\theta - \csc\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

**27.** Write $\dfrac{\sin\theta}{\cos\theta} + \dfrac{1}{\sin\theta}$ in terms of $\sin\theta$ and/or $\cos\theta$, and then simplify if possible.

**28.** Write $\sin\theta\cot\theta + \cos\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

**29.** Multiply $(1-\cos\theta)(1+\cos\theta)$

**30.** Multiply $(\sin\theta + 2)(\sin\theta - 5)$

**31.** Simplify the expression $\sqrt{25 - x^2}$ as much as possible after substituting $5\sin\theta$ for $x$.

**32.** Simplify the expression $\sqrt{4x^2 + 16}$ as much as possible after substituting $2\tan\theta$ for $x$

Simplify by using the table

**33.** $5\sin^2 30°$ **34.** $\sin^2 60° + \cos^2 60°$ **35.** $\left(\tan 45° + \tan 60°\right)^2$

**36.** Find $\theta$ if $\sin\theta = -\frac{1}{2}$ and $\theta$ terminates in QIII with $0° \le \theta \le 360°$.

**37.** Find $\theta$ to the nearest degree if $\sec\theta = 3.8637$ and $\theta$ terminates in QIV with $0° \le \theta < 360°$.

Find the exact value of

**38.** $\cos 225°$ **39.** $\csc 300°$ **40.** $\tan 315°$ **41.** $\cos 420°$ **42.** $\cot 480°$

Use the calculator to find the value of

**43.** $\csc 166.7°$ **44.** $\sec 590.9°$ **45.** $\tan 195°\, 10'$

**46.** Use the calculator to find $\theta$ to the nearest degree if $\sin\theta = -0.3090$ with $\theta \in$ QIV with $0° \le \theta < 360°$

**47.** Use the calculator to find $\theta$ to the nearest degree if $\cos\theta = -0.7660$ with $\theta \in$ QIII with $0° \le \theta < 360°$

**48.** Use the calculator to find $\theta$ to the nearest degree if $\sec\theta = -3.4159$ with $\theta \in$ QII with $0° \le \theta < 360°$

**49.** Find $\theta$ to the nearest tenth of a degree if $\tan\theta = -0.8541$ and $\theta$ terminates in QIV with $0° \le \theta < 360°$

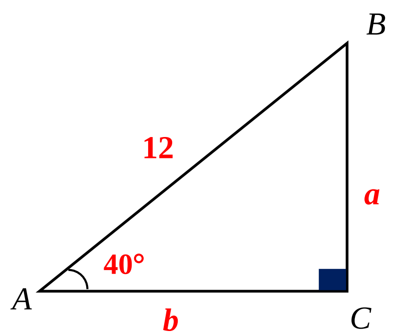
**50.** Use the calculator to find $\theta$ to the nearest degree if $\sin\theta = 0.49368329$ with $\theta \in$ QII with $0° \le \theta < 360°$

# *Section* 2.3 – Solving Right Triangle Trigonometry

### *Example*

In the right triangle *ABC*, *A* = 40° and c = 12 *cm*. Find *a*, *b*, and *B*.

***Solution***




$$\sin 40^\circ = \frac{a}{c} = \frac{a}{12}$$

$$a = 12 \sin 40^\circ$$

$$\approx 7.7\ cm$$

$$\cos 40^\circ = \frac{b}{c} = \frac{b}{12}$$

$$b = 12 \cos 40^\circ$$

$$\approx 9.2\ cm$$

$$B = 90^\circ - A$$

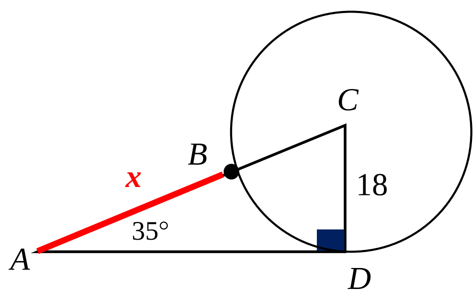
$$= 90^\circ - 40^\circ$$

$$\approx 50^\circ$$

### *Example*

A circle has its center at *C* and a radius of 18 *inches*. If triangle *ADC* is a right triangle and *A* = 35°. Find *x*, the distance from *A* to *B*.

***Solution***




$$\sin 35^\circ = \frac{18}{x+18}$$

$$(x+18)\sin 35^\circ = 18$$

$$x + 18 = \frac{18}{\sin 35^\circ}$$

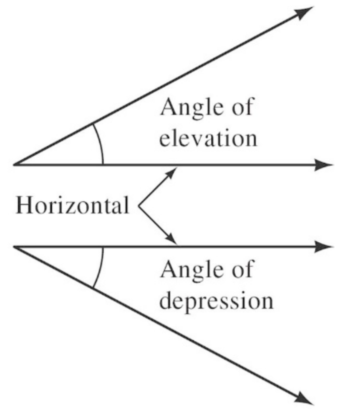
$$x = \frac{18}{\sin 35^\circ} - 18$$

$$\approx 13\ in$$

## Definitions

An angle measured from the horizontal up is called an ***angle of elevation***.

An angle measured from the horizontal down is called an ***angle of depression***.




### Example

The two equal sides of an isosceles triangle are each 24 *cm*. If each of the two equal angles measures 52°, find the length of the base and the altitude.

**Solution**

$$\sin 52° = \frac{x}{24}$$

$$x = 24\sin 52°$$

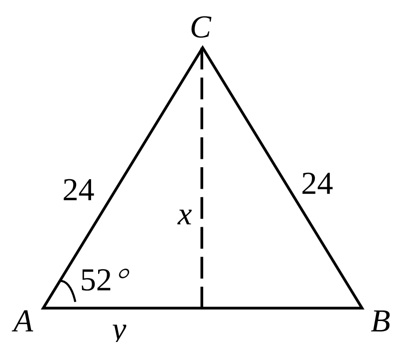
$$x \approx 19\ cm$$

$$\cos 52° = \frac{y}{24}$$

$$y = 24\cos 52°$$

$$y \approx 15\ cm$$

$$\Rightarrow \boxed{AB = 2y \approx 30\ cm}$$




### Example

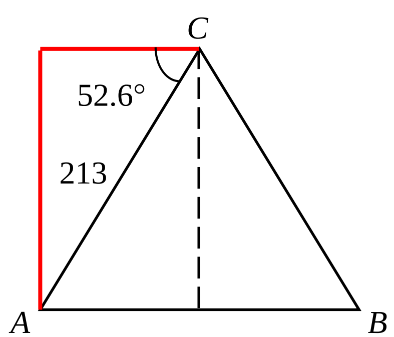
A man climbs 213 *meters* up the side of a pyramid. Find that the angle of depression to his starting point is 52.6°. How high off of the ground is he?

**Solution**

$$\sin 52.6° = \frac{h}{213}$$

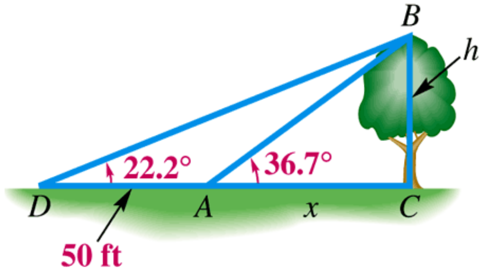
$$h = 213\sin 52.6°$$

$$h \approx 169\ m$$



### *Example*

From a given point on the ground, the angle of elevation to the top of a tree is 36.7°. From a second point, 50 *feet* back, the angle of elevation to the top of the tree is 22.2°. Find the height of the tree to the nearest foot.




***Solution***

Triangle *DCB*

$$\Rightarrow \tan 22.2° = \frac{h}{50+x}$$

$$h = (50+x)\tan 22.2°$$

Triangle *ACB*

$$\Rightarrow \tan 36.7° = \frac{h}{x}$$

$$h = x\tan 36.7°$$

$$x\tan 36.7° = (50+x)\tan 22.2°$$

$$x\tan 36.7° = 50\tan 22.2° + x\tan 22.2°$$

$$x\tan 36.7° - x\tan 22.2° = 50\tan 22.2°$$

$$x(\tan 36.7° - \tan 22.2°) = 50\tan 22.2°$$

$$x = \frac{50\tan 22.2°}{\tan 36.7° - \tan 22.2°}$$

$$h = x\tan 36.7°$$

$$= \left(\frac{50\tan 22.2°}{\tan 36.7° - \tan 22.2°}\right)\tan 36.7°$$

$$\underline{\approx 45\ ft}$$

The tree is about 45 *feet* tall.

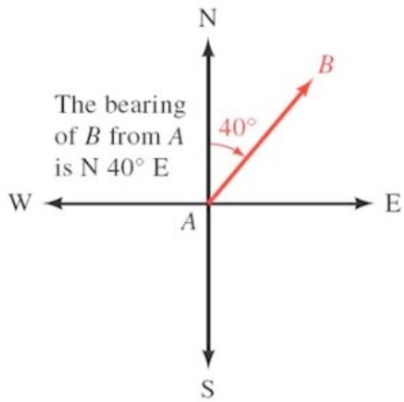
OR $$h = \frac{x\tan\alpha\tan\beta}{\tan\beta - \tan\alpha} = \frac{50\tan 22.2°\tan 36.7°}{\tan 36.7° - \tan 22.2°}$$

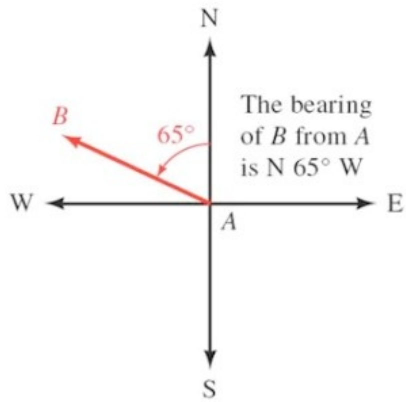
## Bearing

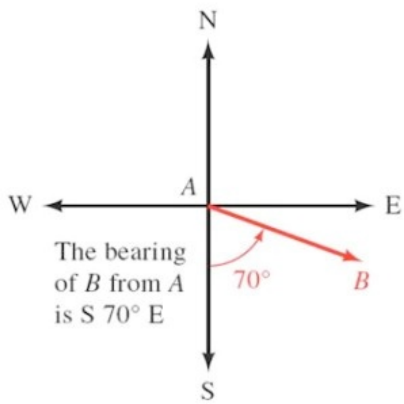
### Definition

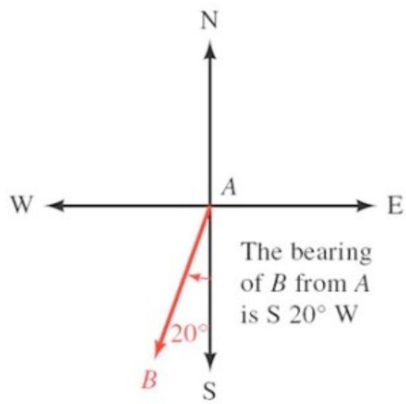
The *bearing of a line* $\ell$ is the acute angle formed by the *north-south* line and the line $\ell$.

The notation used to designate the bearing of a line begins with *N* (for ***north***) or *S* (for ***south***), followed by the number of degrees in the angle, and ends with *E* (for ***east***) or **W** (for ***west***).




### Example

A boat travels on a course of bearing N 52° 40′ E for distance of 238 *miles*. How many miles north and how many miles east have the boat traveled?

***Solution***

$$52°40' = 52° + 40'\frac{1°}{60'}$$

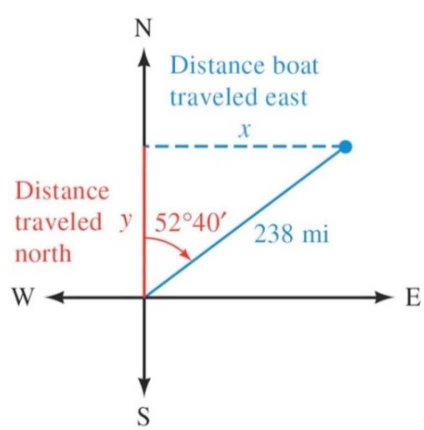
$$\approx 52.6667°$$

$$\sin 52.6667° = \frac{x}{238}$$

$$x = 238 \sin 52.6667° \approx 189\ mi$$

$$\cos 52.6667° = \frac{y}{238}$$

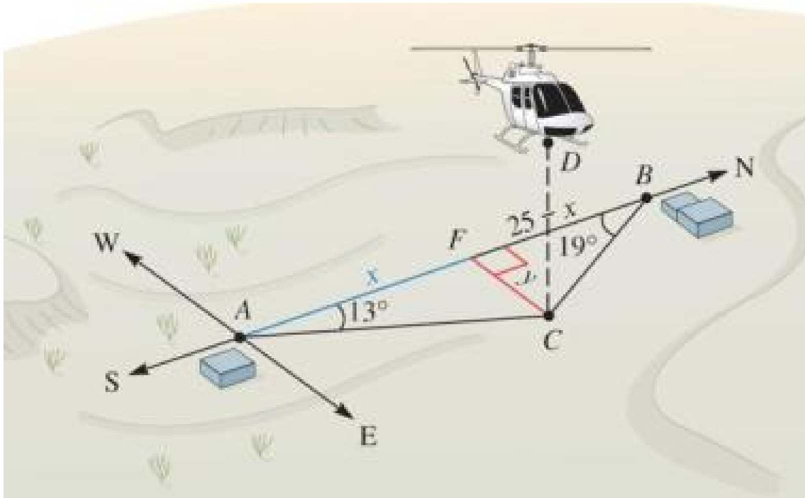
$$y = 238 \cos 52.6667° \approx 144\ mi$$



### *Example*

A helicopter is hovering over the desert when it develops mechanical problems and is forced to land. After landing, the pilot radios his position to a pair of radar station located 25 *miles* apart along a straight road running north and south. The bearing of the helicopter from one station is N 13° E, and from the other it is S 19° E. After doing a few trigonometric calculations, one of the stations instructs the pilot to walk due west for 3.5 *miles* to reach the road. Is this information correct?

**Solution**




In triangle AFC

$$\tan 13° = \frac{y}{x}$$

$$y = x\tan 13°$$

In triangle BFC

$$\tan 19° = \frac{y}{25 - x}$$

$$y = (25 - x)\tan 19°$$

$$y = y$$

$$(25 - x)\tan 19° = x\tan 13°$$

$$25\tan 19° - x\tan 19° = x\tan 13°$$

$$25\tan 19° = x\tan 13° + x\tan 19°$$

$$25\tan 19° = x(\tan 13° + \tan 19°)$$

$$\frac{25\tan 19°}{\tan 13° + \tan 19°} = x$$

$$x = 14.966$$

$$y = x\tan 13°$$

$$= 14.966\tan 13°$$

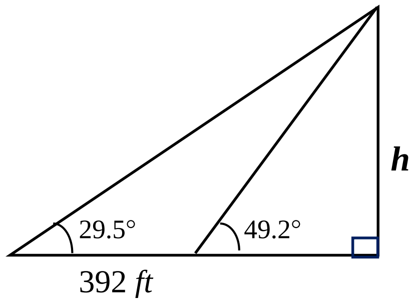
$$\approx 3.5 \ mi$$

# *Exercises* *Section* 2.3 – Solving Right Triangle Trigonometry

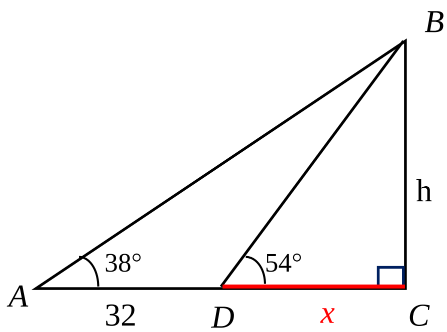
**1.** In the right triangle *ABC*, $a = 29.43$ and $c = 53.58$. Find the remaining side and angles.

**2.** In the right triangle *ABC*, $a = 2.73$ and $b = 3.41$. Find the remaining side and angles.

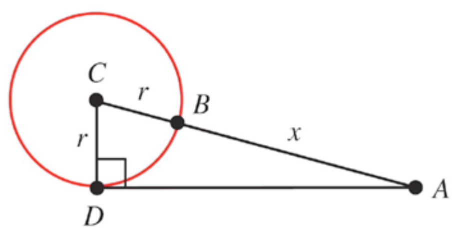
**3.** Find $h$ as indicated in the figure.



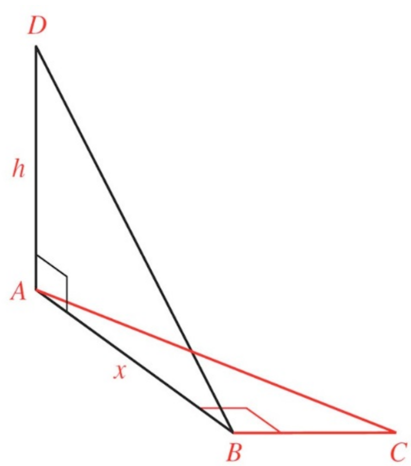

**4.** The distance from *A* to *D* is 32 feet. Use the information in figure to solve *x*, the distance between *D* and *C*.



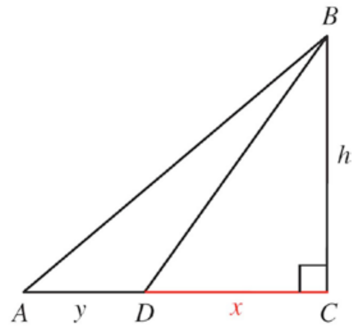

**5.** If C = 26° and $r = 19$, find $x$.



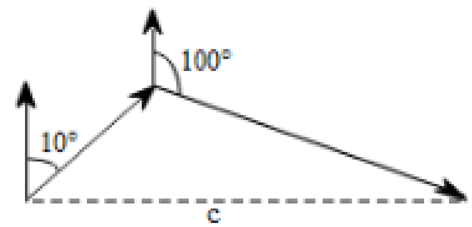

**6.** If ∠ABD = 53°, C = 48°, and BC = 42, find $\boldsymbol{x}$ and then find $\boldsymbol{h}$.



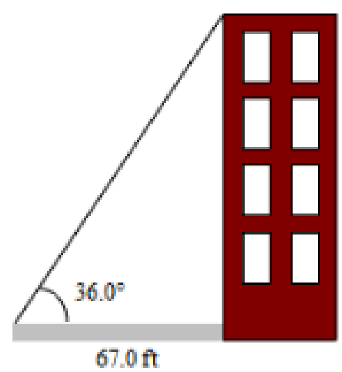
**7.** If $A = 41°$, $\angle BDC = 58°$, and $AB = 28$, find $\boldsymbol{h}$, then $\boldsymbol{x}$.



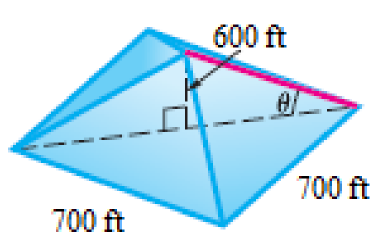

**8.** A plane flies 1.7 *hours* at 120 *mph* on a bearing of 10°. It then turns and flies 9.6 *hours* at the same speed on a bearing of 100°. How far is the plane from its starting point?




**9.** The shadow of a vertical tower is 67.0 *ft* long when the angle of elevation of the sun is 36.0°. Find the height of the tower.

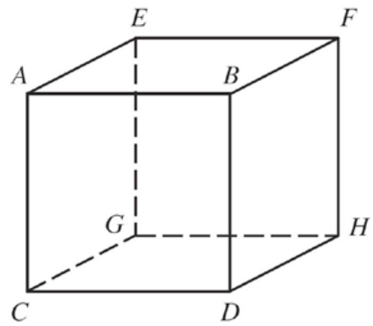



**10.** The base of a pyramid is square with sides 700 *ft* long, and the height of the pyramid is 600 *ft*. Find the angle of elevation of the edge indicated in the figure to two significant digits. (Hint: The base of the triangle in the figure is half the diagonal of the square base of the pyramid.)

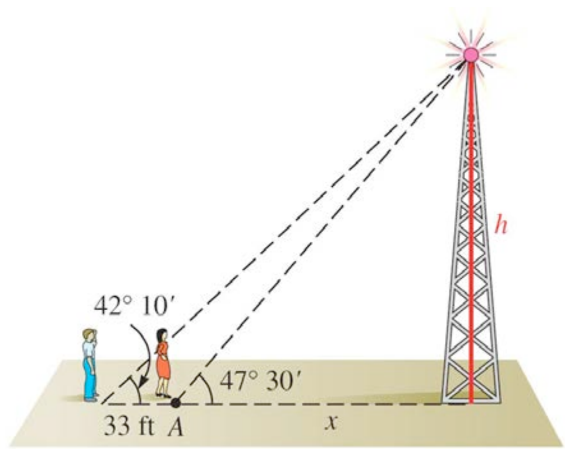



**11.** If a 73-*foot* flagpole casts a shadow 51 *feet* long, what is the angle of elevation of the sun (to the nearest tenth of a degree)?

**12.** Suppose each edge of the cube is 3.00 *inches* long. Find the measure of the angle formed by diagonals DE and DG. *Round your answer to the nearest tenth of a degree.*

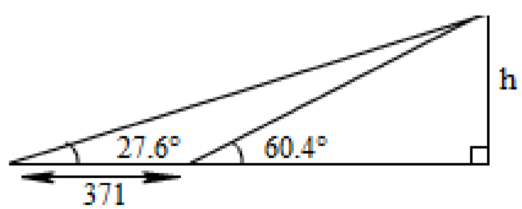



**13.** A person standing at point A notices that the angle of elevation to the top of the antenna is 47° 30′. A second person standing 33.0 *feet* farther from the antenna than the person at A finds the angle of elevation to the top of the antenna to be 42° 10′. How far is the person at A from the base of the antenna?

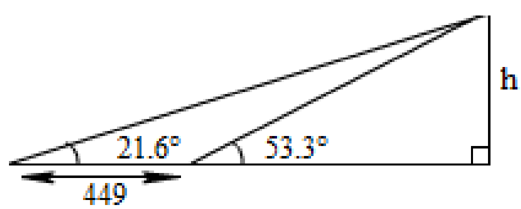



Find ***h*** as indicated in the figure.

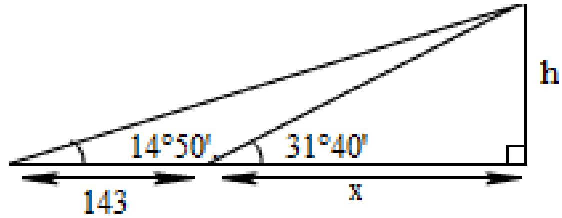
**14.**



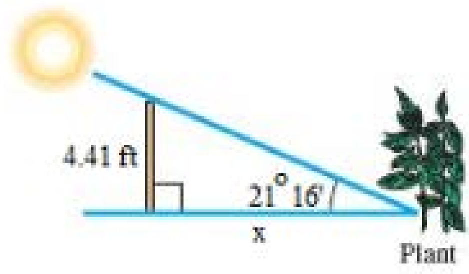

**15.**



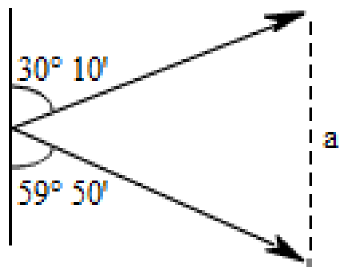

**16.** The angle of elevation from a point on the ground to the top of a pyramid is 31° 40′. The angle of elevation from a point 143 *ft* farther back to the top of the pyramid is 14° 50′. Find the height of the pyramid.



**17.** In one area, the lowest angle of elevation of the sun in winter is 21° 16′. Find the minimum distance, ***x***, that a plant needing full sun can be placed from a fence 4.41 *ft* high.

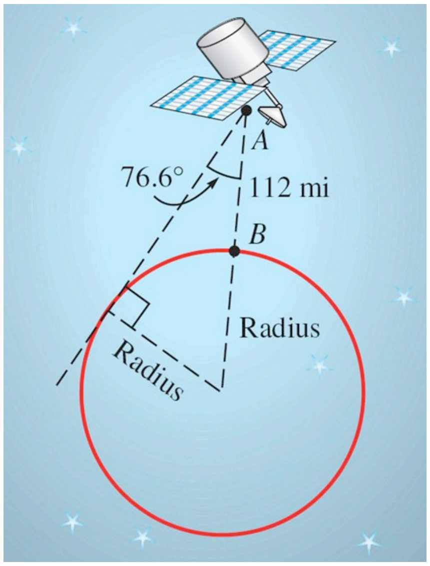



**18.** A ship leaves its port and sails on a bearing of N 30° 10′ E, at speed 29.4 *mph*. Another ship leaves the same port at the same time and sails on a bearing of S 59° 50′ E, at speed 17.1 *mph*. Find the distance between the two ships after 2 *hrs*.

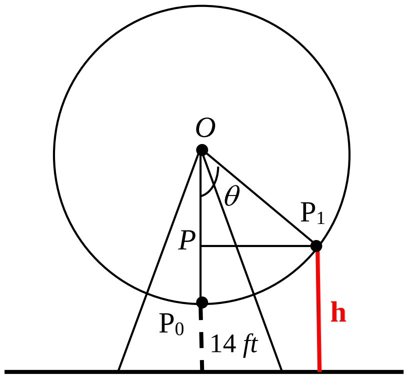



**19.** Radar stations $A$ and $B$ are on the east-west line, 3.7 *km* apart. Station $A$ detects a place at $C$, on a bearing of 61°. Station $B$ simultaneously detects the same plane, on a bearing of 331°. Find the distance from $A$ to $C$.

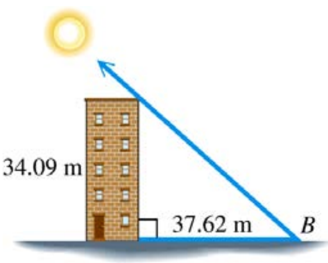
**20.** Suppose the figure below is exaggerated diagram of a plane flying above the earth. If the plane is 4.55 *miles* above the earth and the radius of the earth is 3,960 *miles*, how far is it from the plane to the horizon? What is the measure of angle $A$?



**21.** The Ferry wheel has a 250 *feet* diameter and 14 *feet* above the ground. If $\theta$ is the central angle formed as a rider moves from position $P_0$ to position $P_1$, find the rider's height above the ground $h$ when $\theta$ is 45°.



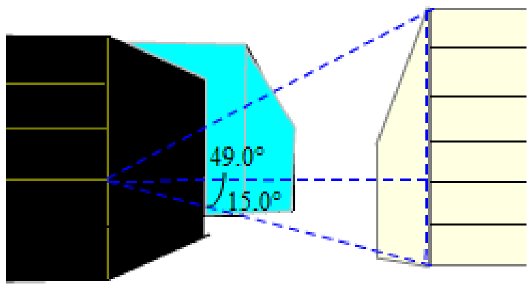

**22.** The length of the shadow of a building 34.09 *m* tall is 37.62 *m*. Find the angle of the elevation of the sun.




**23.** San Luis Obispo, California is 12 *miles* due north of Grover Beach. If Arroyo Grande is 4.6 *miles* due east of Grover Beach, what is the bearing of San Luis Obispo from Arroyo Grande?

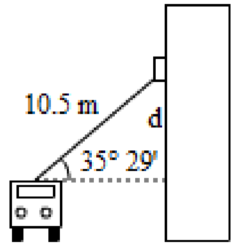
**24.** The bearing from $A$ to $C$ is S 52° E. The bearing from $A$ to $B$ is N 84° E. The bearing from $B$ to $C$ is S 38° W. A plane flying at 250 *mph* takes 2.4 hours to go from $A$ to $B$. Find the distance from $A$ to $C$.

**25.** From a window 31.0 *ft.* above the street, the angle of elevation to the top of the building across the street is 49.0° and the angle of depression to the base of this building is 15.0°. Find the height of the building across the street.

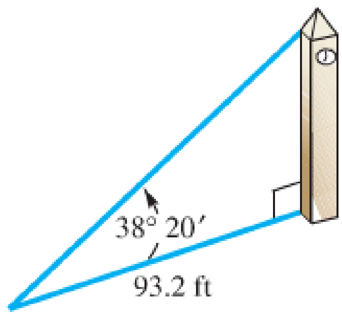



**26.** A man wondering in the desert walks 2.3 *miles* in the direction S 31° W. He then turns 90° and walks 3.5 *miles* in the direction N 59° W. At that time, how far is he from his starting point, and what is his bearing from his starting point?

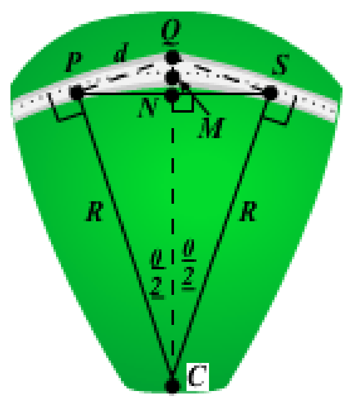
**27.** A 10.5-*m* fire truck ladder is leaning against a wall. Find the distance ***d*** the ladder goes up the wall (above the fire truck) if the ladder makes an angle of 35° 29′ with the horizontal.




**28.** The angle of elevation from a point 93.2 *ft* from the base of a tower to the top of the tower is 38° 20′. Find the height of the tower.



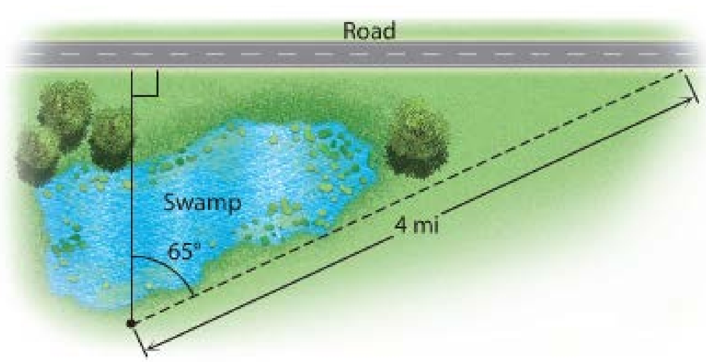

**29.** A basic curve connecting two straight sections of road is often circular. In the figure, the points ***P*** and ***S*** mark the beginning and end of the curve. Let ***Q*** be the point of intersection where the two straight sections of highway leading into the curve would meet if extended. The radius of the curve is ***R***, and the central angle denotes how many degrees the curve turns.




***a)*** If ***R*** = 965 ft. and θ = 37°, find the distance **d** between ***P*** and ***Q***.

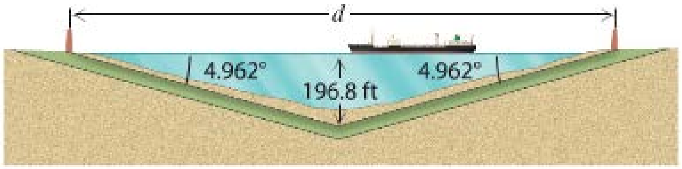
***b)*** Find an expression in terms of ***R*** and θ for the distance between points ***M*** and ***N***.

**30.** Jane was hiking directly toward a long straight road when she encountered a swamp. She turned 65° to the right and hiked 4 *mi* in that direction to reach the road. How far was she form the road when she encountered the swamp?

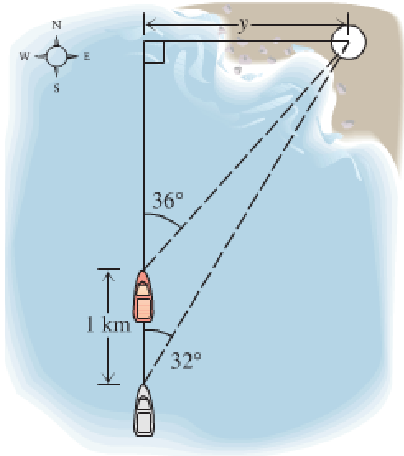


**31.** From a highway overpass, 14.3 *m* above the road, the angle of depression of an oncoming car is measured at 18.3°. How far is the car from a point on the highway directly below the observer?

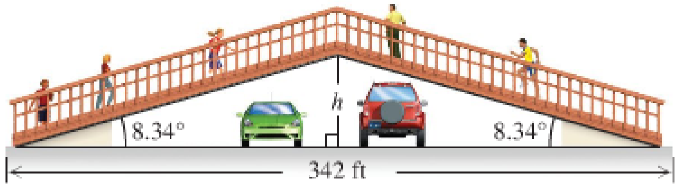
**32.** A tunnel under a river is 196.8 ft. below the surface at its lowest point. If the angle of depression of the tunnel is 4.962° , then how far apart on the surface are the entrances to the tunnel? How long is the tunnel?



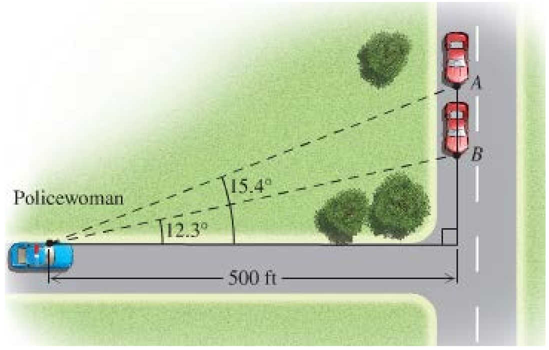

**33.** A boat sailing north sights a lighthouse to the east at an angle of 32° from the north. After the boat travels one more kilometer, the angle of the lighthouse from the north is 36°. If the boat continues to sail north, then how close will the boat come to the lighthouse?



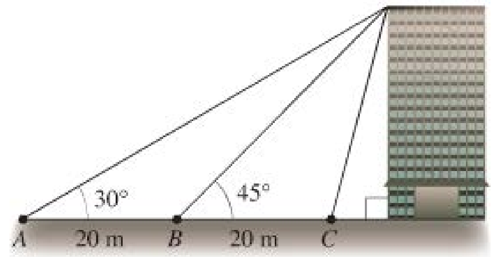

**34.** The angle of elevation of a pedestrian crosswalk over a busy highway is 8.34°, as shown in the drawing. If the distance between the ends of the crosswalk measured on the ground is 342 *ft*., then what is the height h of the crosswalk at the center?



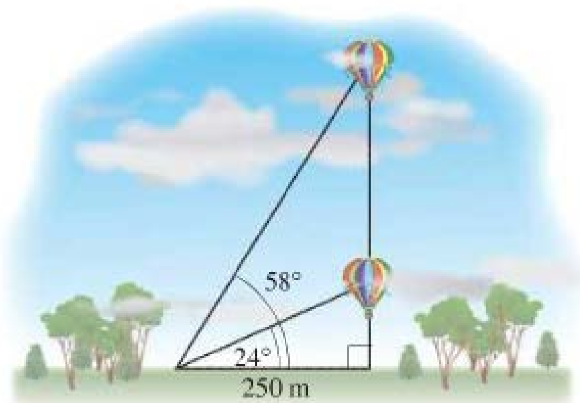

**35.** A policewoman has positioned herself 500 *ft*. from the intersection of two roads. She has carefully measured the angles of the lines of sight to points *A* and *B*. If a car passes from *A* to *B* is 1.75 *sec* and the speed limit is 55 *mph*, is the car speeding? (Hint: Find the distance from *B* to *A* and use *R* = D/T)



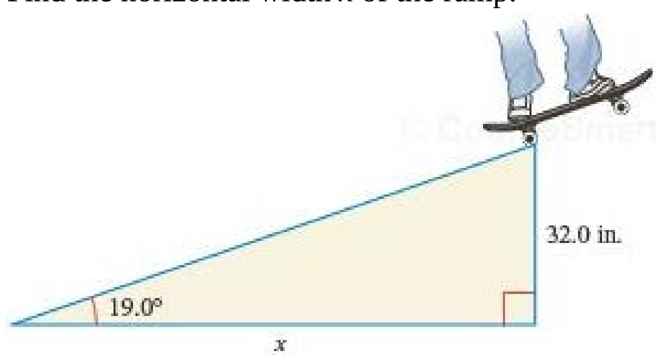
**36.** From point $A$ the angle of elevation to the top of the building is 30°. From point $B$, 20 *meters* closer to the building, the angle of elevation is 45°. Find the angle of elevation of the building from point $C$, which is another 20 *meters* closer to the building.



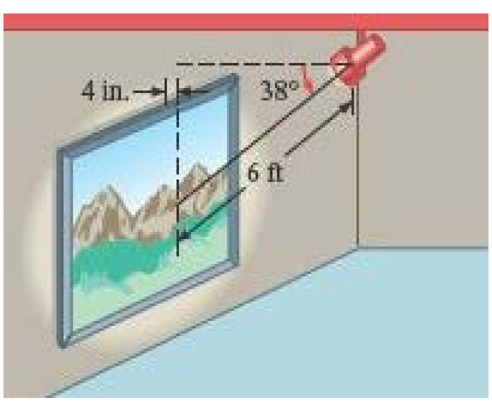

**37.** A hot air balloon is rising upward from the earth at a constant rate. An observer 250 *m* away spots the balloon at an angle of elevation of 24°. Two minutes later the angle of elevation of the balloon is 58°. At what rate is the balloon ascending?



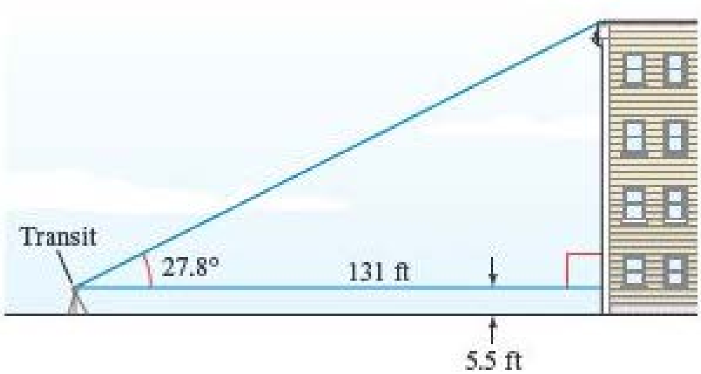

**38.** A skateboarder wishes to build a jump ramp that is inclined at a 19° angle and that has a maximum height of 32.0 *inches*. Find the horizontal width $x$ of the ramp.



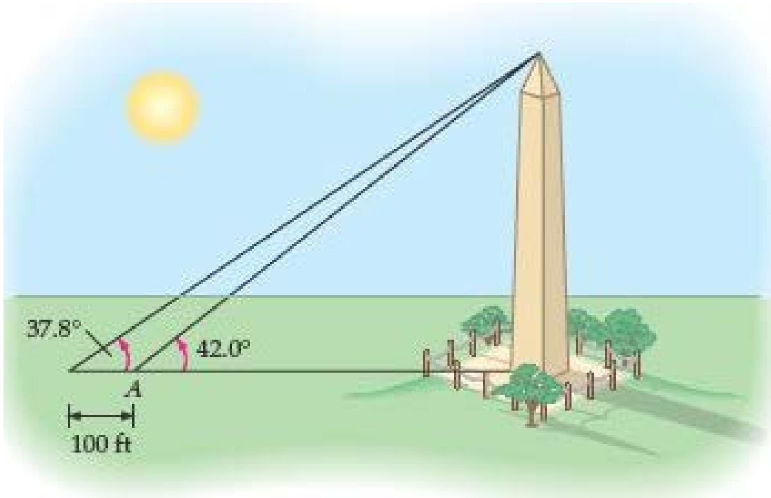

**39.** For best illumination of a piece of art, a lighting specialist for an art gallery recommends that a ceiling-mounted light be 6 *ft* from the piece of art and that the angle of depression of the light be 38°. How far from a wall should the light be placed so that the recommendations of the specialist are met? Notice that the art extends outward 4 inches from the wall.



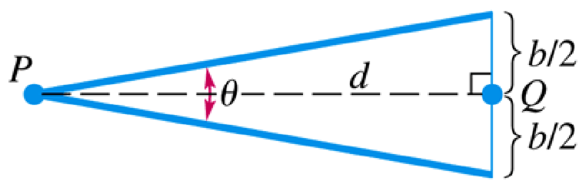
**40.** A surveyor determines that the angle of elevation from a transit to the top of a building is 27.8°. The transit is positioned 5.5 *feet* above ground level and 131 *feet* from the building. Find the height of the building to the nearest tenth of a foot.




**41.** From a point $A$ on a line from the base of the Washington Monument, the angle of elevation to the top of the monument is 42.0°. From a point 100 *ft* away from $A$ and on the same line, the angle to the top is 37.8°. Find the height, to the nearest foot, of the Monument.



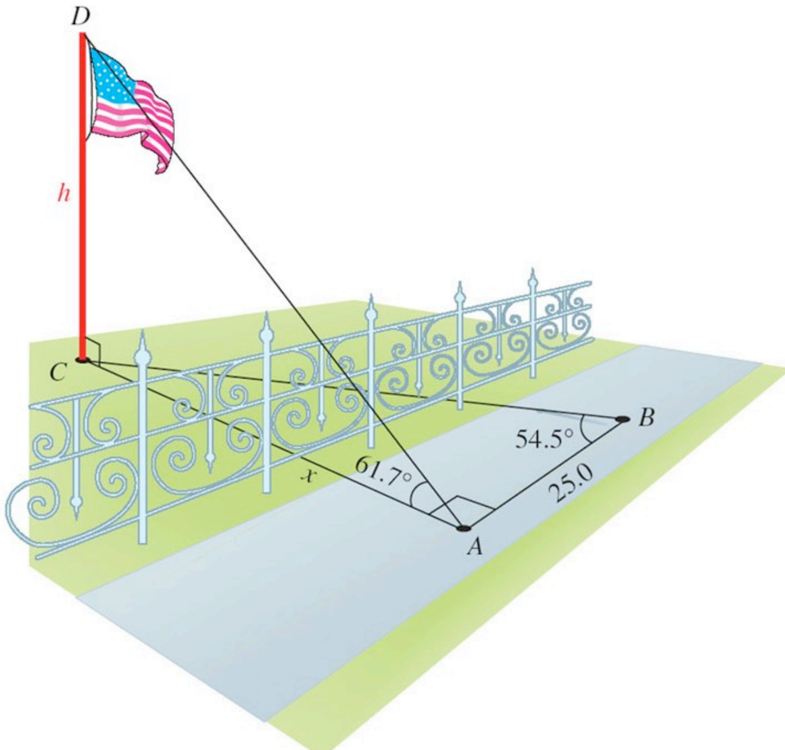

**42.** A method that surveyors use to determine a small distance $d$ between two points $P$ and $Q$ is called the ***subtense bar method***. The subtense bar with length $b$ is centered at $Q$ and situated perpendicular to the line of sight between $P$ and $Q$. Angle $\theta$ is measured, then the distance d can be determined.



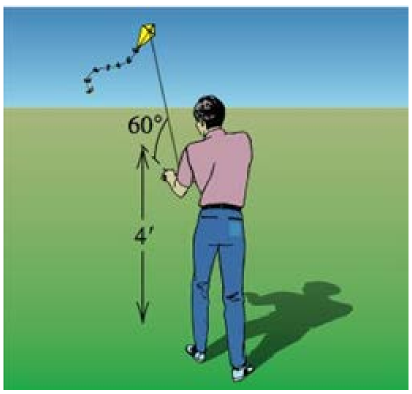

***a)*** Find $d$ with $\theta = 1° \ 23' \ 12''$ and $b = 2.000 \ cm$

***b)*** Angle $\theta$ usually cannot be measured more accurately than to the nearest 1″. How much change would there be in the value of $d$ if $\theta$ were measured 1″ larger?

**43.** A diagram that shows how Diane estimates the height of a flagpole. She can't measure the distance between herself and the flagpole directly because there is a fence in the way. So she stands at point *A* facing the pole and finds the angle of elevation from point *A* to the top of the pole to be 61.7°. Then she turns 90° and walks 25.0 *ft* to point *B*, where she measures the angle between her path and a line from *B* to the base of the pole. She finds that angle is 54.5°. Use this information to find the height of the pole.



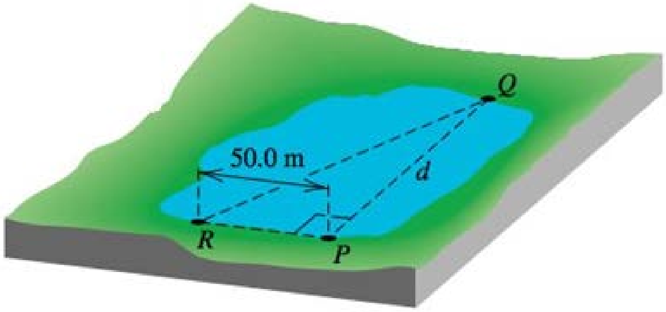

**44.** A person flying a kite holds the string 4 *feet* above ground level. The string of the kite is taut and make an angle of 60° with the horizontal. Approximate the height of the kite above level ground if 500 *feet* of sting is paved out.



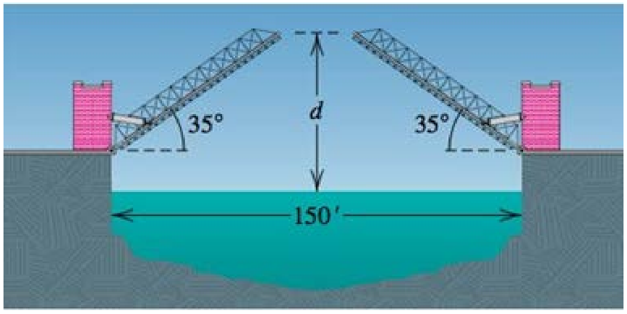

**45.** From a point 15 feet above level ground, a surveyor measures the angle of depression of an object on the ground at 68°. Approximate the distance from the object to the point on the ground directly beneath the surveyor.

**46.** A pilot, flying at an altitude of 5,000 feet wishes to approach the numbers on a runway at an angle of 10°. Approximate, to the nearest 100 feet, the distance from the airplane to the numbers at the beginning of the descent.

**47.** To find the distance d between two points *P* and *Q* on opposite shores of a lake, a surveyor locates a point *R* that is 50.0 meters from *P* such that *RP* is perpendicular to *PQ*. Nest, using a transit, the surveyor measures angle *PRQ* as $72^\circ\ 40'$. Find $d$.



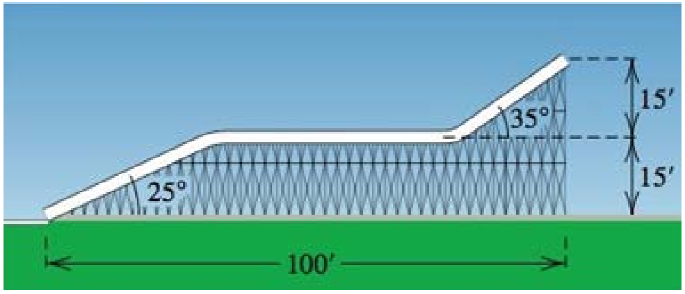

**48.** A drawbridge is 150 *feet* long when stretched across a river. The two sections of the bridge can be rotated upward through an angle of 35°.




*a)* If the water level is 15 *feet* below the closed bridge, find the distance d between the end of a section and the water level when the bridge is fully open.

*b)* Approximately how far apart are the ends of the two sections when the bridge is fully opened?

**49.** Find the total length of a design for a water slide to the nearest foot.

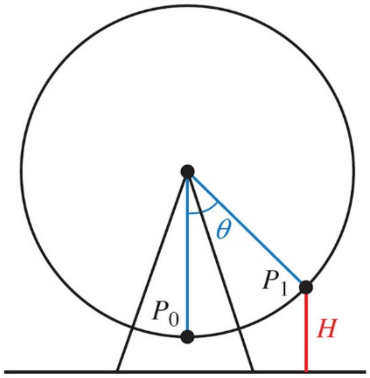



**50.** A Ferris wheel has radius 50.0 *ft*. A person takes a seat and then the wheel turns $\frac{2\pi}{3}$ *rad* .

*a)* How far is the person above the ground?

*b)* If it takes 30 sec for the wheel to turn $\frac{2\pi}{3}$ *rad* , what is the angular speed of the wheel?

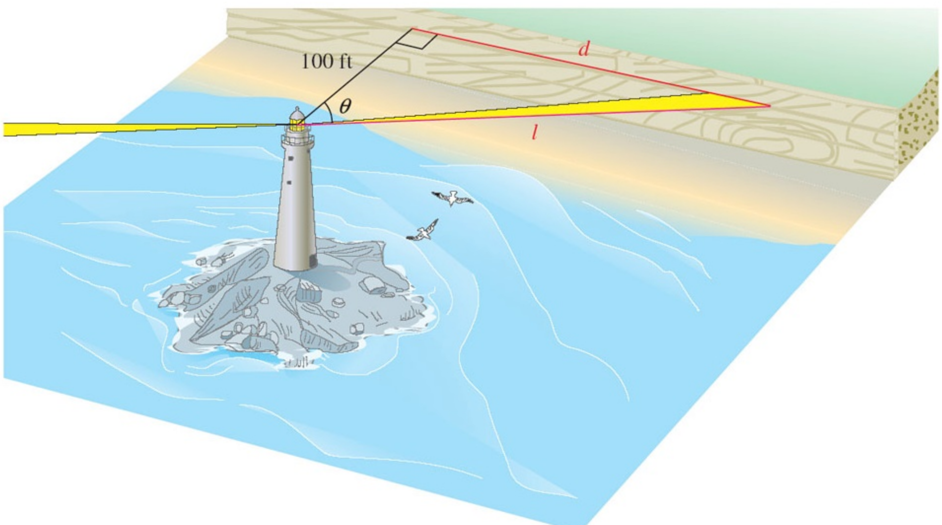
**51.** The diameter of the Ferris wheel is 250 *ft*, the distance from the ground to the bottom of the wheel is 14 *ft*, and one complete revolution takes 20 *minutes*, find



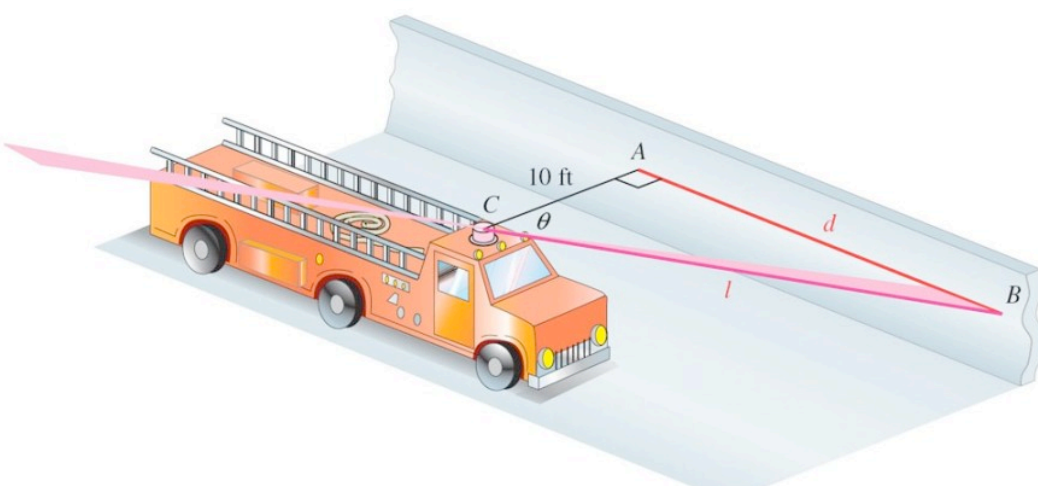

*a)* The linear velocity, in miles per hour, of a person riding on the wheel.

*b)* The height of the rider in terms of the time *t*, where *t* is measured in minutes.

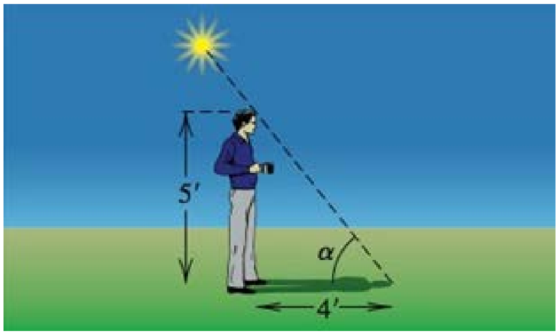
**52.** Find an equation that expresses ***l*** in terms of time *t*. Find ***l*** when *t* is 0.5 *sec*, 1.0 *sec*, and 1.5 *sec*. (assume the light goes through one rotation every 4 seconds.)




**53.** A fire truck parked on the shoulder of a freeway next to a long block wall. The red light on the top of the truck is 10 *feet* from the wall and rotates through a complete revolution every 2 *seconds*. Find the equations that give the lengths *d* and ***ℓ*** in terms of time.

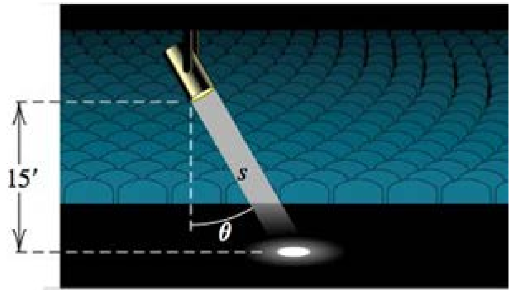


**54.** Approximate the angle of elevation $\alpha$ of the sun if a person 5.0 *feet* tall casts a shadow 4.0 *feet* long on level ground. $\approx 51.34°$




**55.** A spotlight with intensity 5000 candles is located 15 *feet* above a stage. If the spotlight is rotated through an angle $\theta$, the illuminance $E$ (in foot-candles) in the lighted area of the stage is given by

$$E = \frac{5{,}000\cos\theta}{s^2}$$

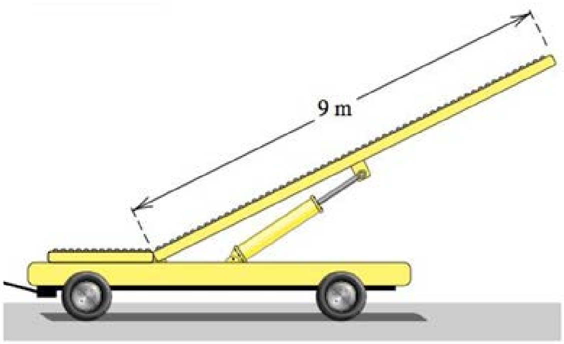



Where $s$ is the distance (in *feet*) that the light must travel.

*a)* Find the illuminance if the spotlight is rotated through an angle of 30°.

*b)* The maximum illuminance occurs when $\theta = 0°$. For what value of $\theta$ is the illuminance one-half the maximum value.

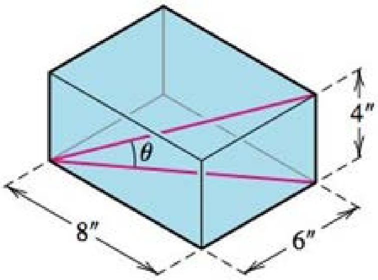
**56.** A conveyor belt 9 *meters* long can be hydraulically rotated up to an angle of 40° to unload cargo from airplanes.



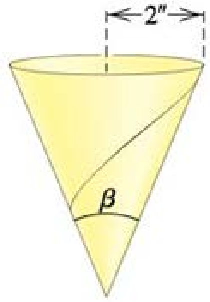

*a)* Find, to the nearest degree, the angle through which the conveyor belt should be rotated up to reach a door that is 4 *meters* above the platform supporting the belt. $\approx 26.4°$

*b)* Approximate the maximum height above the platform that the belt can reach. $\approx 5.785\ m$

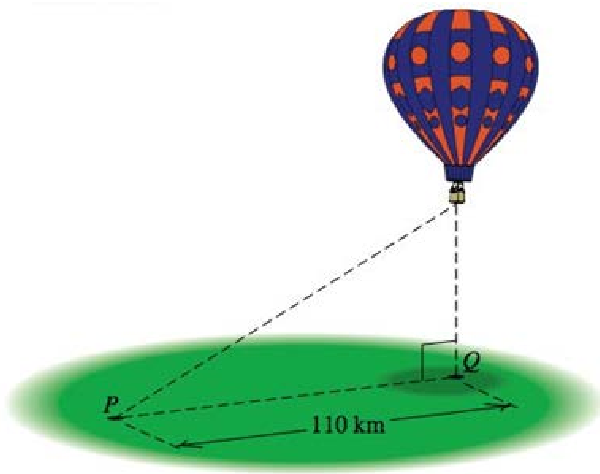
**57.** A rectangular box has dimensions $8'' \times 6'' \times 4''$. Approximate, to the nearest tenth of a degree, the angle $\theta$ formed by a diagonal of the base and the diagonal of the box. $\approx 21.8°$



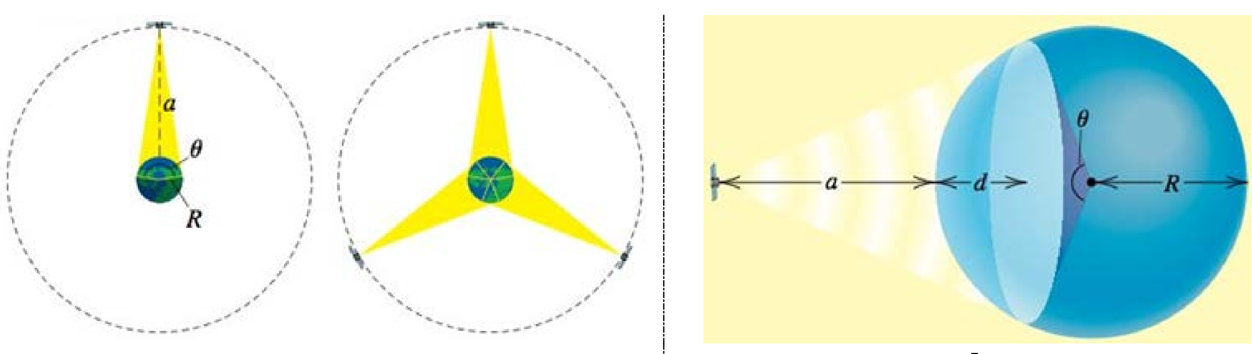
**58.** A conical paper cup has a radius of 2 *inches*, approximate, to the nearest degree, the angle $\beta$ so that the cone will have a volume of $20\ in^3$. $\approx 45.5°$




**59.** As a hot-air balloon rises vertically, its angle of elevation from a point $P$ on level ground 100 *km* from the point $Q$ directly underneath the balloon changes from $19°\ 20'$ to $31°\ 50'$. Approximately how far does the balloon rise during this period? $\approx 29.7\ km$




**60.** Shown in the left part of the figure is a communications satellite with an equatorial orbit–that is, a nearly circular orbit in the plane determined by Earth's equator. If the satellite circles Earth at an altitude of $a = 22{,}300\ mi$, its speed is the same as the rotational speed of Earth; to an observer on the equator, the satellite appears to be stationary–that is, its orbit is synchronous.




***a*** ***b***

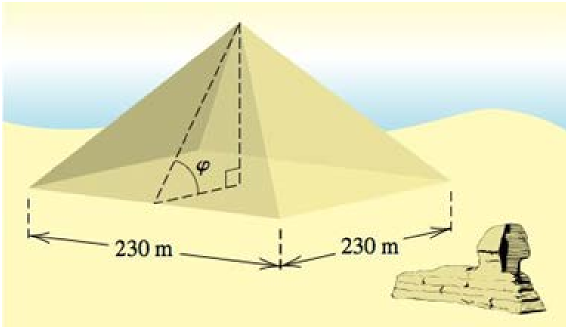
*a)* Using $R = 4{,}000\ mi$ for the radius of Earth, determine the percentage of the equator that is within signal range of such a satellite. $\theta \approx 162.5°$ 45%

*b)* As shown in the right part of the figure (***a***), three satellites are equally spaced in equatorial synchronous orbits. Use the value of $\theta$ obtained in part (*a*) to explain why all points on the equator are within signal range of at least one of the three satellites.

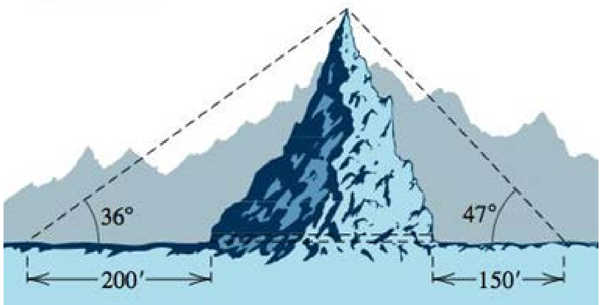
*c)* The figure (***b***) shows the area served by a communication satellite circling a planet of radius $R$ at an altitude $a$. The portion of the planet's surface within range of the satellite is a spherical cap of depth $d$ and surface area $A = 2\pi Rd$. Express $d$ in terms of $R$ and $\theta$. $R\left(1-\cos\frac{\theta}{2}\right)$

*d)* Estimate the percentage of the planet's surface that is within signal range of a single satellite in equatorial synchronous orbit. 42.39%

**61.** The great Pyramid of Egypt is 147 *meters* high, with a square base of side 230 *meters*. Approximate, to the nearest degree, the angle $\varphi$ formed when an observer stands at the midpoint of one the sides and views the apex of the pyramid. $\approx 52°$



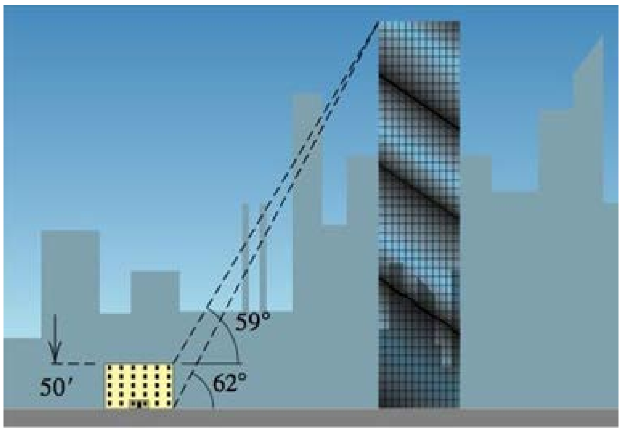

**62.** A tunnel for a new highway is to be cut through a mountain that is 260 *feet* high. At a distance of 200 *feet* from the base of the mountain, the angle of elevation is 36°. From a distance of 150 *feet* on the other side, the angle of elevation is 47°. Approximate the length of the tunnel to the nearest foot. 250 *ft*



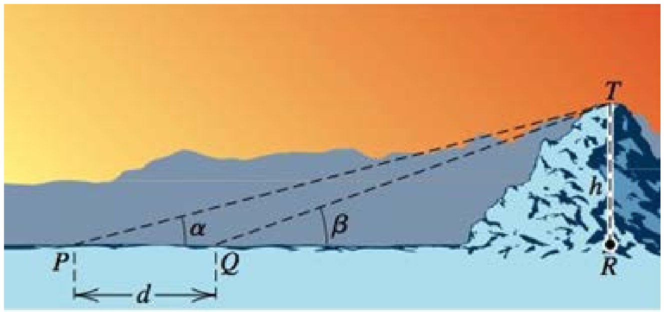

**63.** When a certain skyscraper is viewed from the top of a building 50 *feet* tall, the angle of elevation is 59°. When viewed from the street next to the shorter building, the angle of elevation is 62°.

*a)* Approximately how far apart are the two structures? 231 *ft*

*b)* Approximate the height of the skyscraper to the nearest tenth of a *foot*. 434.5 *ft*



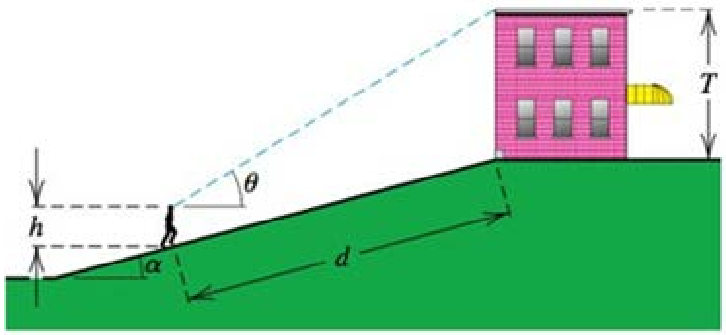
**64.** When a mountaintop is viewed from the point $P$, the angle of elevation is $\alpha$. From a point $Q$, which is $d$ *miles* closer to the mountain, the angle of elevation increases to $\beta$.




*a)* Show that the height $h$ of the mountain is given by: $h = \dfrac{d}{\cot\alpha - \cot\beta}$.

*b)* If $d = 2mi, \quad \alpha = 15°, \quad and \quad \beta = 20°$, approximate the height of the mountain. $\approx 2.03\ mi$

**65.** An observer of height $h$ stands on an incline at a distance $d$ from the base of a building of height $T$. The angle of elevation from the observer to the top of the building is $\theta$, and the incline makes an angle of $\alpha$ with the horizontal.




*a)* Express $T$ in terms of $h$, $d$, $\alpha$, and $\theta$. $T = d\left(\cos\alpha \tan\theta - \sin\alpha\right) + h$

*b)* If $d = 50\ ft, \quad h = 6\ ft, \quad \alpha = 15°, \quad and \quad \theta = 31.4°$, estimate the height of the building.

$\approx 22.54$

# *Section* 2.4 – Law of Sines and Cosines

## Oblique Triangle

A triangle, that is not a right triangle, is either acute or obtuse.

The measures of the three sides and the three angles of a triangle can be found if at least one side and any other two measures are known.

## The Law of *Sines*

There are many relationships that exist between the sides and angles in a triangle.

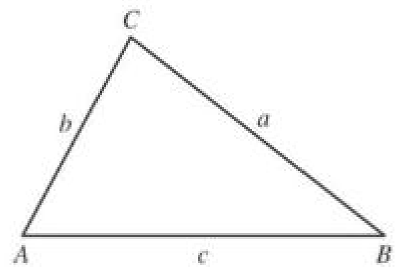
One such relation is called the law of sines.

Given triangle ABC

$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$$

*or, equivalently*

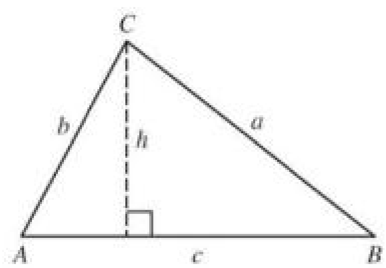
$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$$




### *Proof*

$$\sin A = \frac{h}{b} \Rightarrow h = b\sin A \quad (1)$$

$$\sin B = \frac{h}{a} \Rightarrow h = a\sin B \quad (2)$$




From (1) & (2)

$$h = h$$

$$b\sin A = a\sin B$$

$$\frac{b\sin A}{ab} = \frac{a\sin B}{ab}$$

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

## Angle – Side - Angle (*ASA* or *AAS*)

If two angles and the included side of one triangle are equal, respectively, to two angles and the included side of a second triangle, then the triangles are congruent.

### *Example*

In triangle *ABC*, $A = 30°$, $B = 70°$, and $a = 8.0\ cm$. Find the length of side *c*.

***Solution***

$$\begin{aligned} C &= 180° - (A + B) \\ &= 180° - (30° + 70°) \\ &= 180° - 100° \\ &= 80° \end{aligned}$$

$$\begin{aligned} \frac{c}{\sin C} &= \frac{a}{\sin A} \\ c &= \frac{a}{\sin A}\sin C \\ &= \frac{8}{\sin 30°}\sin 80° \\ &\approx 16\ cm \end{aligned}$$

### *Example*

Find the missing parts of triangle *ABC* if $A = 32°$, $C = 81.8°$, , and $a = 42.9\ cm$.

***Solution***

$$\begin{aligned} B &= 180° - (B + C) \\ &= 180° - (32° + 81.8°) \\ &= 66.2° \end{aligned}$$

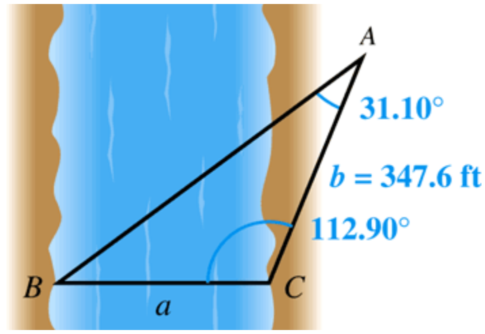
| | |
|---|---|
| $\frac{a}{\sin A} = \frac{b}{\sin B}$<br>$b = \frac{a \sin B}{\sin A}$<br>$= \frac{42.9 \sin 66.2°}{\sin 32°}$<br>$\approx 74.1\ cm$ | $\frac{c}{\sin C} = \frac{a}{\sin A}$<br>$c = \frac{a \sin C}{\sin A}$<br>$= \frac{42.9 \sin 81.8°}{\sin 32°}$<br>$\approx 80.1\ cm$ |

### *Example*

You wish to measure the distance across a River. You determine that $C = 112.90°$, $A = 31.10°$, and $b = 347.6\ ft$. Find the distance $a$ across the river.

***Solution***

$$B = 180° - A - C$$
$$= 180° - 31.10° - 112.90°$$
$$\boxed{= 36°}$$



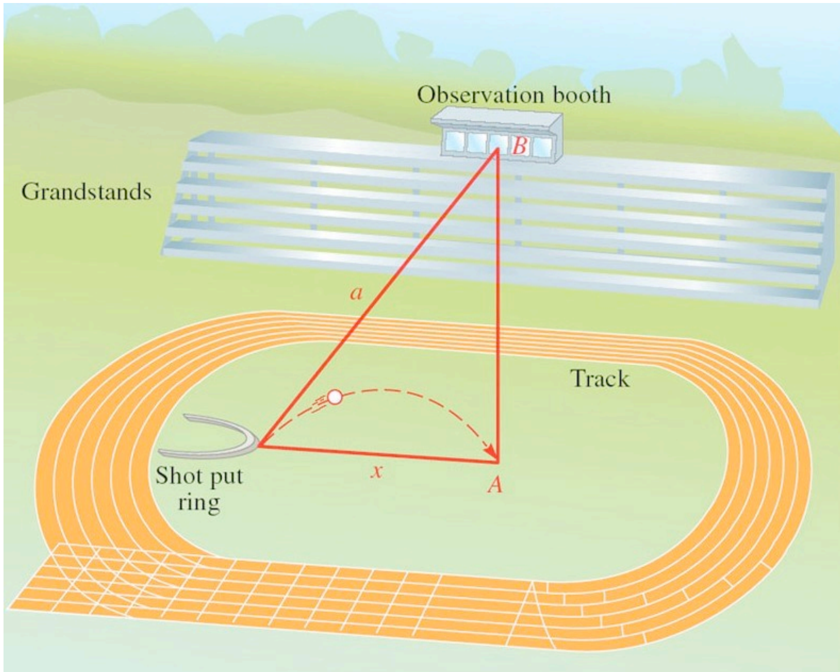

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{a}{\sin 31.1°} = \frac{347.6}{\sin 36°}$$
$$a = \frac{347.6}{\sin 36°}\sin 31.1°$$
$$\boxed{a \approx 305.5\ ft}$$

### *Example*

Find distance $x$ if $a = 562\ ft.$, $B = 5.7°$ and $A = 85.3°$

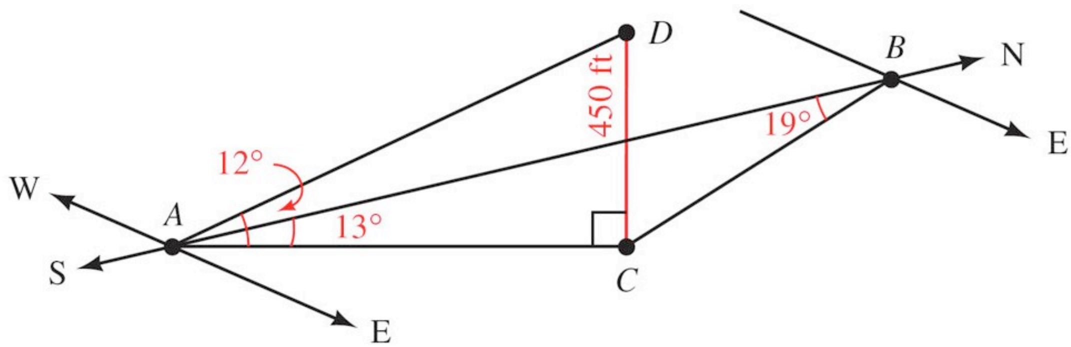
***Solution***

$$\frac{x}{\sin B} = \frac{a}{\sin A}$$
$$x = \frac{a \sin B}{\sin A}$$
$$= \frac{562 \sin 5.7°}{\sin 85.3°}$$
$$\boxed{\approx 56.0\ ft}$$



### *Example*

A hot-air balloon is flying over a dry lake when the wind stops blowing. The balloon comes to a stop 450 *feet* above the ground at point *D*. A jeep following the balloon runs out of gas at point *A*. The nearest service station is due north of the jeep at point *B*. The bearing of the balloon from the jeep at *A* is N 13° E, while the bearing of the balloon from the service station at *B* is S 19° E. If the angle of elevation of the balloon from *A* is 12°, how far will the people in the jeep have to walk to reach the service station at point *B*?




***Solution***

$$\tan 12° = \frac{DC}{AC}$$

$$AC = \frac{DC}{\tan 12°}$$

$$= \frac{450}{\tan 12°}$$

$$\approx 2{,}117 \ ft$$

$$\angle ACB = 180° - (13° + 19°)$$

$$= 148°$$

Using triangle *ABC*

$$\frac{AB}{\sin 148°} = \frac{2117}{\sin 19°}$$

$$AB = \frac{2117 \sin 148°}{\sin 19°}$$

$$\approx 3{,}446 \ ft$$

## *Ambiguous Case*

### Side – Angle – Side (*SAS*)

If two sides and the included angle of one triangle are equal, respectively, to two sides and the included angle of a second triangle, then the triangles are congruent.

### *Example*

Find angle $B$ in triangle ABC if $a = 2$, $b = 6$, and $A = 30°$

***Solution***

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\sin B = \frac{b \sin A}{a}$$

$$= \frac{6 \sin 30°}{2}$$

$$\underline{= 1.5} \quad > 1 \qquad\qquad -1 \le \sin \alpha \le 1$$

Since $sinB > 1$ is impossible, no such triangle exists.

### *Example*

Find the missing parts in triangle ABC if $C = 35.4°$, $a = 205$ *ft*., and $c = 314$ *ft*.

***Solution***

$$\sin A = \frac{a \sin C}{c}$$

$$= \frac{205 \sin 35.4°}{314}$$

$$A = \sin^{-1}\left(\frac{205 \sin 35.4°}{314}\right)$$

$$\underline{A \approx 22.2°}$$

$$A' = 180° - 22.2° = 157.8°$$

$$C + A' = 35.4° + 157.8°$$

$$= 193.2° \quad > 180°$$

$$\underline{B} = 180° - (22.2° + 35.4°) \ \underline{\approx 122.4°}$$

$$b = \frac{c \sin B}{\sin C}$$

$$= \frac{314 \sin 122.4°}{\sin 35.4°}$$

$$\underline{\approx 458 \ ft}$$

### *Example*

Find the missing parts in triangle ABC if $a = 54\ cm$, $b = 62\ cm$, and $A = 40°$.

***Solution***

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\sin B = \frac{b \sin A}{a}$$

$$= \frac{62 \sin 40°}{54}$$

$$\underline{|B} = \sin^{-1}\left(\frac{62 \sin 40°}{54}\right) \underline{\approx 48°|} \qquad B = 180° - 48° = 132°$$

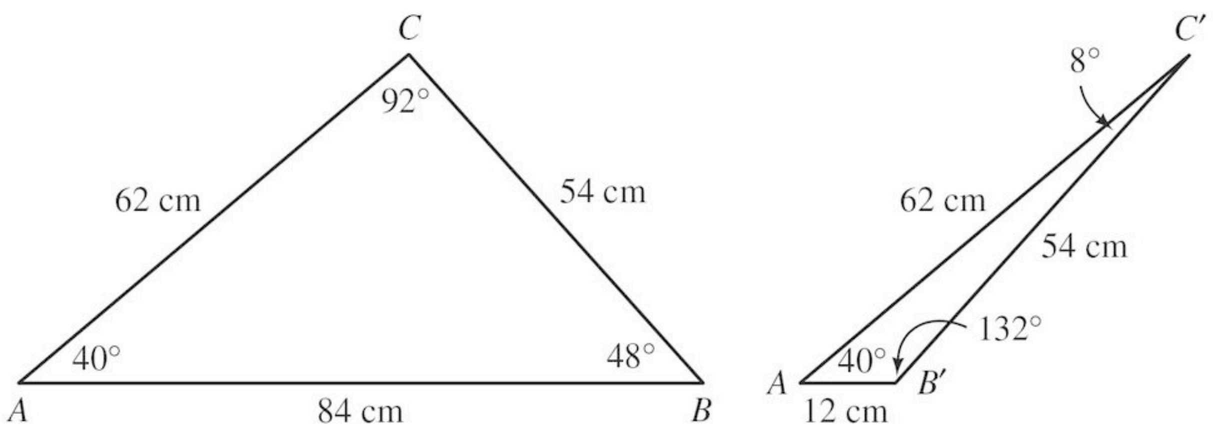
$$C = 180° - (40° + 48°) \qquad C' = 180° - (40° + 132°)$$

$$\underline{\approx 92°|} \qquad \underline{\approx 8°|}$$

$$c = \frac{a \sin C}{\sin A} \qquad c' = \frac{a \sin C'}{\sin A}$$

$$= \frac{54 \sin 92°}{\sin 40°} \qquad = \frac{54 \sin 8°}{\sin 40°}$$

$$\underline{\approx 84\ cm|} \qquad \underline{\approx 12\ cm|}$$



## *Area of a Triangle (SAS)*

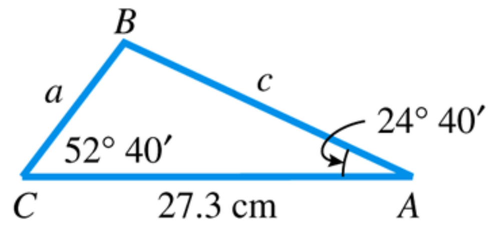
In any triangle $ABC$, the area $K$ is given by the following formulas:

$$K = \frac{1}{2}bc\sin A \qquad K = \frac{1}{2}ac\sin B \qquad K = \frac{1}{2}ab\sin C$$

### *Example*

Find the area of triangle $ABC$ if $A = 24°40',\ b = 27.3\ cm,\ \ and\ \ C = 52°40'$

***Solution***




$$B = 180° - 24°40' - 52°40'$$

$$= 180° - \left(24° + \frac{40°}{60}\right) - \left(52° + \frac{40°}{60}\right)$$

$$\approx 102.667°$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin(24°40')} = \frac{27.3}{\sin(102°40')}$$

$$a = \frac{27.3\sin(24°40')}{\sin(102°40')}$$

$$\approx 11.7\ cm$$

$$K = \frac{1}{2}ac\sin B$$

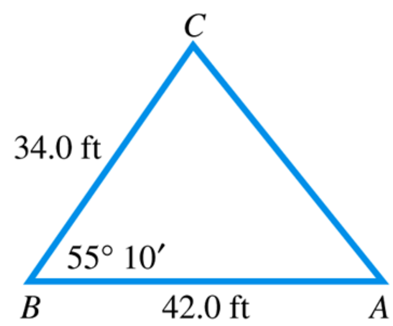
$$= \frac{1}{2}(11.7)(27.3)\sin(52°40')$$

$$\approx 127\ cm^2$$

### *Example*

Find the area of triangle $ABC$.

***Solution***




$$K = \frac{1}{2}ac\sin B$$

$$= \frac{1}{2}(34.0)(42.0)\sin(55°10')$$

$$\approx 586\ ft^2$$

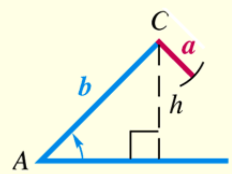
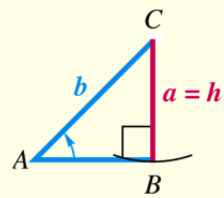
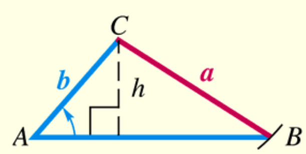
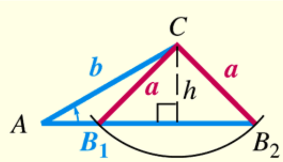
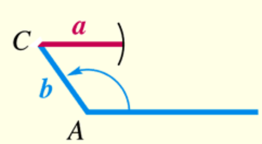
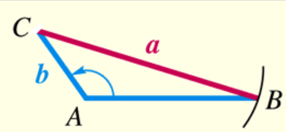
## *Number of Triangles Satisfying the Ambiguous Case* (SSA)

Let sides $a$ and $b$ and angle $A$ be given in triangle $ABC$. (The law of sines can be used to calculate the value of $\sin B$.)

1. If applying the law of sines results in an equation having $\sin B > 1$, then *no triangle* satisfies the given conditions.
2. If $\sin B = 1$, then *one triangle* satisfies the given conditions and $B = 90°$.
3. If $0 < \sin B < 1$, then either *one or two triangles* satisfy the given conditions.

   ***a)*** If $\sin B = k$, then let $B_1 = \sin^{-1} k$ and use $B_1$ for $B$ in the first triangle.

   ***b)*** Let $B_2 = 180° - B_1$. If $A + B_2 < 180°$, then a second triangle exists. In this case, use $B_2$ for $B$ in the second triangle.

| Number of Triangles | Sketch | Applying Law of Sines Leads to |
|---|---|---|
| 0 |  | $\sin B > 1$,<br>$a < h < b$ |
| 1 |  | $\sin B = 1$,<br>$a = h$ and $h < b$ |
| 1 |  | $0 < \sin B < 1$,<br>$a \geq b$ |
| 2 |  | $0 < \sin B_2 < 1$,<br>$h < a < b$ |
| 0 |  | $\sin B \geq 1$,<br>$a \leq b$ |
| 1 |  | $0 < \sin B < 1$,<br>$a > b$ |

## Law of Cosines (*SAS*)

$$a^2 = b^2 + c^2 - 2bc\cos A \quad \rightarrow \quad a = \sqrt{b^2 + c^2 - 2bc\cos A}$$

$$b^2 = a^2 + c^2 - 2ac\cos B \quad \rightarrow \quad b = \sqrt{a^2 + c^2 - 2ac\cos B}$$

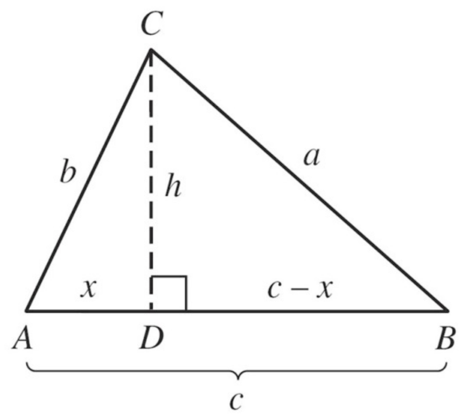
$$c^2 = a^2 + b^2 - 2ab\cos C \quad \rightarrow \quad c = \sqrt{a^2 + b^2 - 2ab\cos C}$$

### *Derivation*

$$a^2 = (c-x)^2 + h^2$$

$$= c^2 - 2cx + x^2 + h^2 \qquad \textbf{(1)}$$

$$b^2 = x^2 + h^2 \qquad \textbf{(2)}$$




From **(2)**:

**(1)** $a^2 = c^2 - 2cx + b^2$

$$a^2 = c^2 + b^2 - 2cx \qquad \textbf{(3)}$$

$$\cos A = \frac{x}{b}$$

$$b\cos A = x$$

**(3)** $\Rightarrow a^2 = c^2 + b^2 - 2cb\cos A$

### *Example*

Find the missing parts in triangle *ABC* if $A = 60°$, $b = 20$ *in*, and $c = 30$ *in*.

***Solution***

$$a = \sqrt{20^2 + 30^2 - 2(20)(30)\cos 60°} \qquad a = \sqrt{b^2 + c^2 - 2bc\cos A}$$

$$\approx 26 \ in$$

$$B = \sin^{-1}\left(\frac{20\sin 60°}{26}\right) \qquad B = \sin^{-1}\left(\frac{b\sin A}{a}\right)$$

$$\approx 42°$$

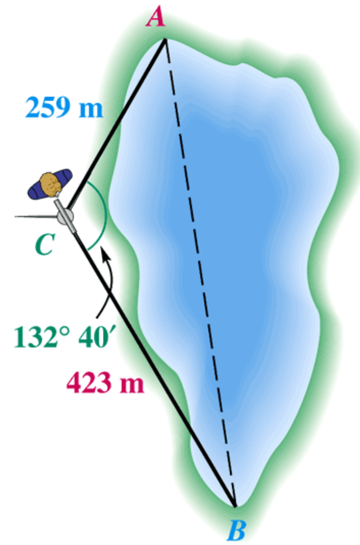
$$C = 180° - A - B$$

$$= 180° - 60° - 42°$$

$$\approx 78°$$

### *Example*

A surveyor wishes to find the distance between two inaccessible points $A$ and $B$ on opposite sides of a lake. While standing at point $C$, she finds that $AC = 259\ m$, $BC = 423\ m$, and angle $ACB = 132°40'$. Find the distance $AB$.




**Solution**

$$\begin{aligned} AB &= \sqrt{AC^2 + BC^2 - 2(AC)(BC)\cos C} \\ &= \sqrt{259^2 + 423^2 - 2(259)(423)\cos(132°40')} \\ &\approx 628\ m \end{aligned}$$

## **Law of Cosines (*SSS*)** - Three Sides

$$\cos A = \frac{b^2+c^2-a^2}{2bc} \quad \to \quad A = \cos^{-1}\frac{b^2+c^2-a^2}{2bc}$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac} \quad \to \quad B = \cos^{-1}\frac{a^2+c^2-b^2}{2ac}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab} \quad \to \quad C = \cos^{-1}\frac{a^2+b^2-c^2}{2ab}$$

### *Example*

Solve triangle *ABC* if $a = 34\ km$, $b = 20\ km$, and $c = 18\ km$

***Solution***

$$A = \cos^{-1}\left(\frac{b^2+c^2-a^2}{2bc}\right)$$

$$= \cos^{-1}\frac{20^2+18^2-34^2}{2(20)(18)}$$

$$\approx 127°$$

$$C = \cos^{-1}\frac{a^2+b^2-c^2}{2ab}$$

$$= \cos^{-1}\frac{34^2+20^2-18^2}{2(34)(20)}$$

$$\approx 25°$$

*OR*

$$\sin C = \frac{c\sin A}{a}$$

$$= \frac{18\sin 127°}{34}$$

$$C = \sin^{-1}\left(\frac{18\sin 127°}{34}\right)$$

$$\approx 25°$$

$$B = 180° - A - C$$

$$= 180° - 127° - 25°$$

$$\approx 28°$$

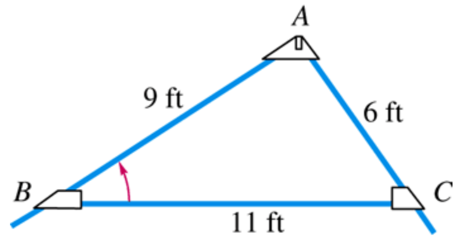
### *Example*

Find the measure of angle *B* in the figure of a roof truss.

***Solution***

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{11^2+9^2-6^2}{2(11)(9)}$$

$$B = \cos^{-1}\left(\frac{11^2+9^2-6^2}{2(11)(9)}\right) \approx 33°$$



## ***Heron's Area Formula (SSS)***

If a triangle has sides of lengths $a$, $b$, and $c$, with semi-perimeter

$$s = \frac{1}{2}(a+b+c)$$

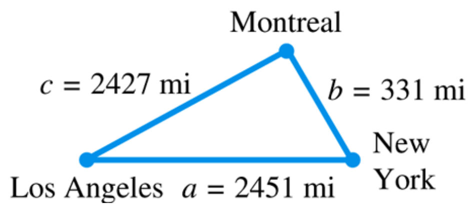
Then the area of the triangle is:

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

### ***Example***

The distance "as the crow flies" from Los Angeles to New York is 2451 *miles*, from New York to Montreal is 331 *miles*, and from Montreal to Los Angeles is 2427 *miles*. What is the area of the triangular region having these three cities as vertices? (Ignore the curvature of Earth.)

***Solution***




The semi-perimeter $s$ is:

$$\begin{aligned} s &= \frac{1}{2}(a+b+c) \\ &= \frac{1}{2}(2451+331+2427) \\ &= 2,604.5 \end{aligned}$$

$$\begin{aligned} K &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{2604.5(2604.5-2451)(2604.5-331)(2604.5-2427)} \\ &\approx 401,700\ mi^2 \end{aligned}$$

# *Exercises* Section 2.4 – Law of Sines and Cosines

**1.** In triangle *ABC*, $B = 110°$, $C = 40°$ and $b = 18\ in$. Find the length of side *c*.

Find all the missing parts

**2.** $A = 110.4°$, $C = 21.8°$ and $c = 246\ in$

**3.** $B = 34°$, $C = 82°$, and $a = 5.6\ cm$

**4.** $B = 55°40'$, $b = 8.94\ m$, and $a = 25.1\ m$.

**5.** $A = 55.3°$, $a = 22.8\ ft.$, and $b = 24.9$ ft.

**6.** $A = 43.5°$, $a = 10.7\ in.$, and $c = 7.2\ in.$

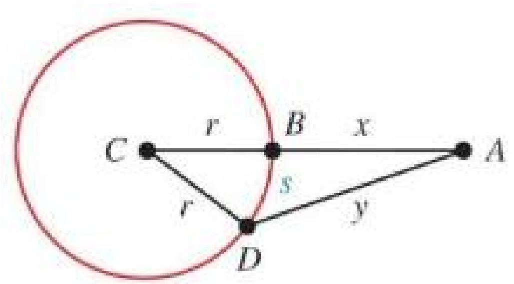
**7.** $b = 63.4\ km$, and $c = 75.2\ km$, $A = 124°\ 40'$

**8.** $A = 42.3°$, $b = 12.9m$, *and* $c = 15.4m$

**9.** $a = 832\ ft.$, $b = 623\ ft.$, and $c = 345\ ft.$

**10.** $a = 9.47\ ft$, $b = 15.9\ ft$, *and* $c = 21.1\ ft$

**11.** If $A = 26°$, $s = 22$, *and* $r = 19$, find $x$

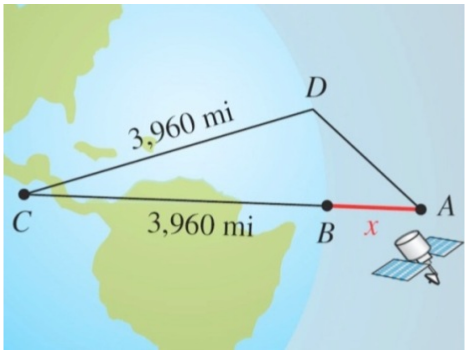



**12.** If $a = 13\ yd.$, $b = 14\ yd.$, and $c = 15\ yd.$, find the largest angle.

**13.** The diagonals of a parallelogram are 24.2 *cm* and 35.4 *cm* and intersect at an angle of 65.5°. Find the length of the shorter side of the parallelogram

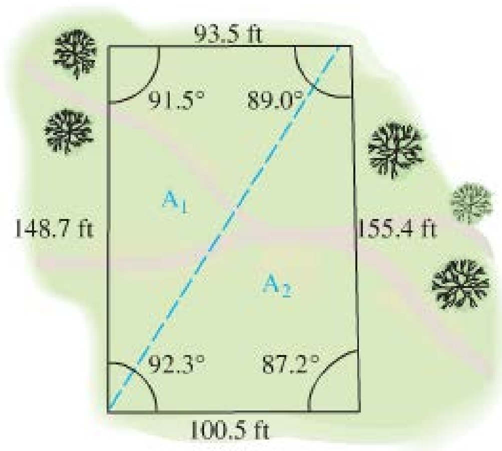
**14.** A man flying in a hot-air balloon in a straight line at a constant rate of 5 *feet per second*, while keeping it at a constant altitude. As he approaches the parking lot of a market, he notices that the angle of depression from his balloon to a friend's car in the parking lot is 35°. A minute and a half later, after flying directly over this friend's car, he looks back to see his friend getting into the car and observes the angle of depression to be 36°. At that time, what is the distance between him and his friend?

**15.** A satellite is circling above the earth. When the satellite is directly above point *B*, angle *A* is 75.4°. If the distance between points *B* and *D* on the circumference of the earth is 910 *miles* and the radius of the earth is 3,960 *miles*, how far above the earth is the satellite?

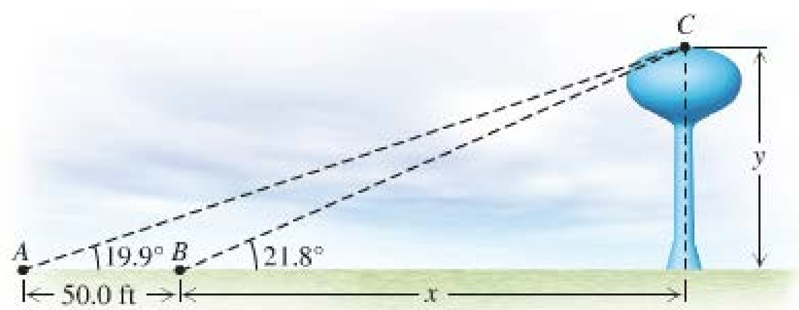


**16.** A pilot left Fairbanks in a light plane and flew 100 *miles* toward Fort in still air on a course with bearing of 18°. She then flew due east (bearing 90°) for some time drop supplies to a snowbound family. After the drop, her course to return to Fairbanks had bearing of 225°. What was her maximum distance from Fairbanks?

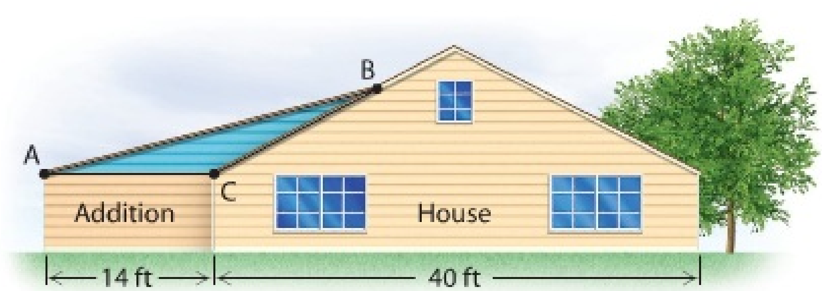
**17.** The dimensions of a land are given in the figure. Find the area of the property in square feet.



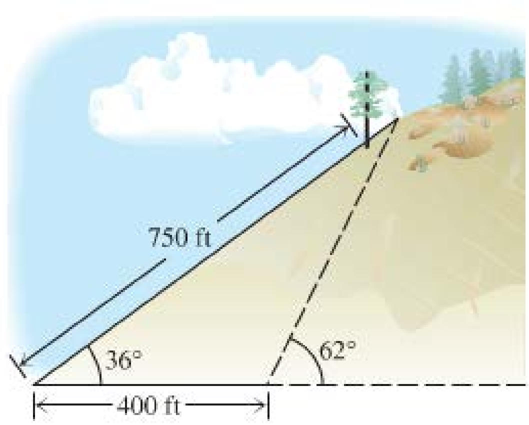

**18.** The angle of elevation of the top of a water tower from point A on the ground is 19.9°. From point B, 50.0 *feet* closer to the tower, the angle of elevation is 21.8°. What is the height of the tower?



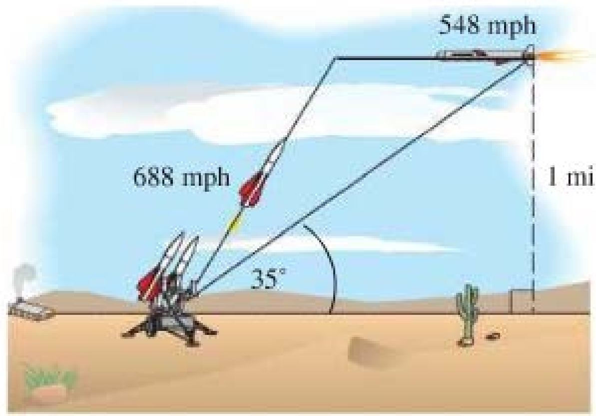

**19.** A 40-*ft* wide house has a roof with a 6-12 pitch (the roof rises 6 *ft* for a run of 12 *ft*). The owner plans a 14-*ft* wide addition that will have a 3-12 pitch to its roof. Find the lengths of $\overline{AB}$ and $\overline{BC}$



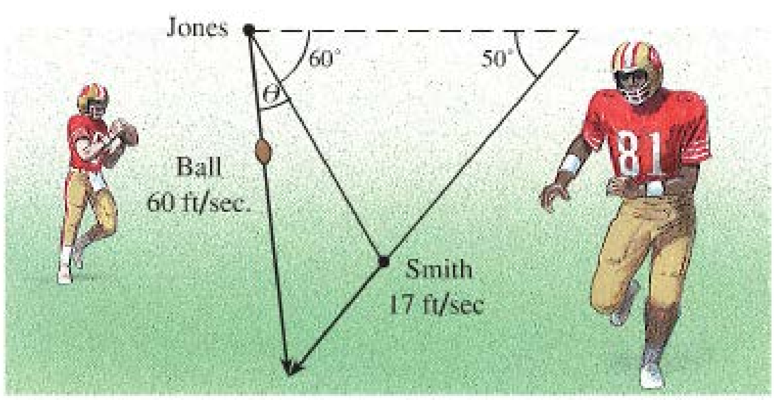

**20.** A hill has an angle of inclination of 36°. A study completed by a state's highway commission showed that the placement of a highway requires that 400 *ft* of the hill, measured horizontally, be removed. The engineers plan to leave a slope alongside the highway with an angle of inclination of 62°. Located 750 *ft* up the hill measured from the base is a tree containing the nest of an endangered hawk. Will this tree be removed in the excavation?



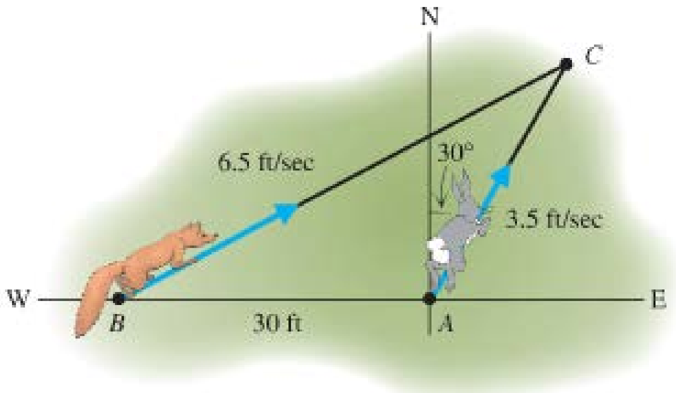

**21.** A cruise missile is traveling straight across the desert at 548 *mph* at an altitude of 1 *mile*. A gunner spots the missile coming in his direction and fires a projectile at the missile when the angle of elevation of the missile is 35°. If the speed of the projectile is 688 mph, then for what angle of elevation of the gun will the projectile hit the missile?



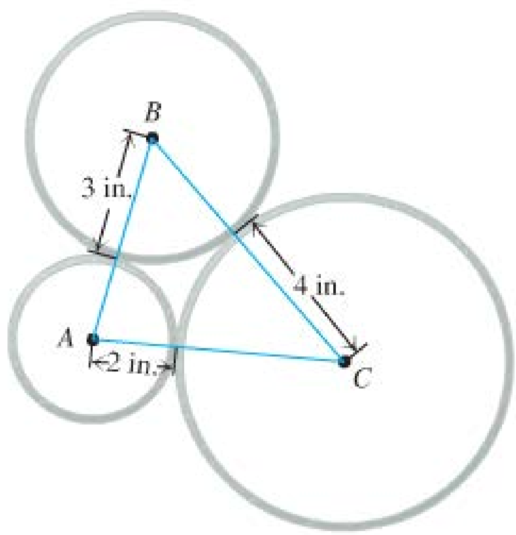

**22.** When the ball is snapped, Smith starts running at a 50° angle to the line of scrimmage. At the moment when Smith is at a 60° angle from Jones, Smith is running at 17 *ft/sec* and Jones passes the ball at 60 *ft/sec* to Smith. However, to complete the pass, Jones must lead Smith by the angle $\theta$. Find $\theta$ (find $\theta$ in his head. Note that $\theta$ can be found without knowing any distances.)



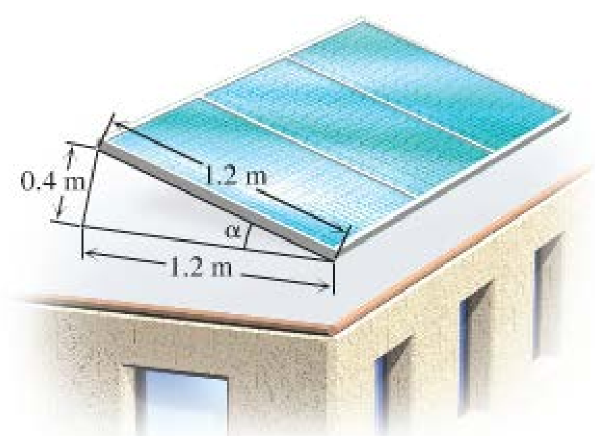
**23.** A rabbit starts running from point *A* in a straight line in the direction 30° from the north at 3.5 *ft/sec*. At the same time a fox starts running in a straight line from a position 30 *ft* to the west of the rabbit 6.5 *ft/sec*. The fox chooses his path so that he will catch the rabbit at point C. In how many seconds will the fox catch the rabbit?




**24.** An engineer wants to position three pipes at the vertices of a triangle. If the pipes *A*, *B*, and *C* have radii 2 *in*, 3 *in*, and 4 *in*, respectively, then what are the measures of the angles of the triangle *ABC*?

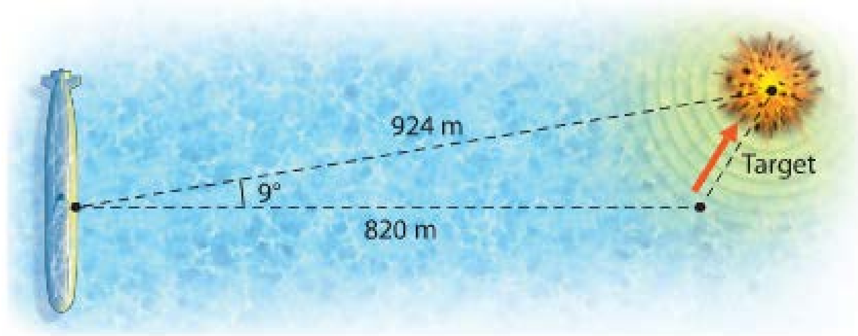



**25.** A solar panel with a width of 1.2 *m* is positioned on a flat roof. What is the angle of elevation $\alpha$ of the solar panel?

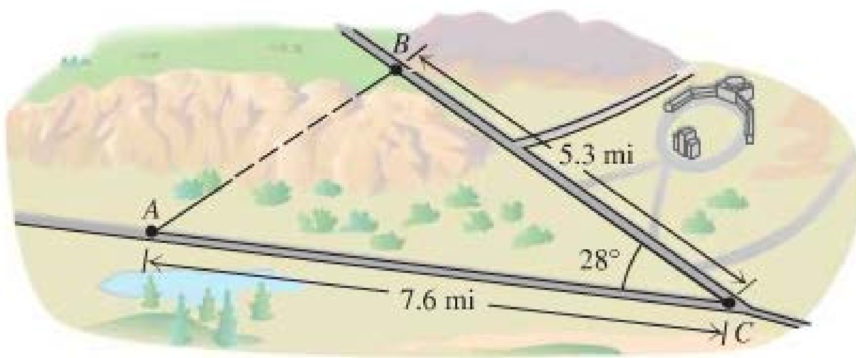


**26.** Andrea and Steve left the airport at the same time. Andrea flew at 180 *mph* on a course with bearing 80°, and Steve flew at 240 *mph* on a course with bearing 210°. How far apart were they after 3 *hr*.?

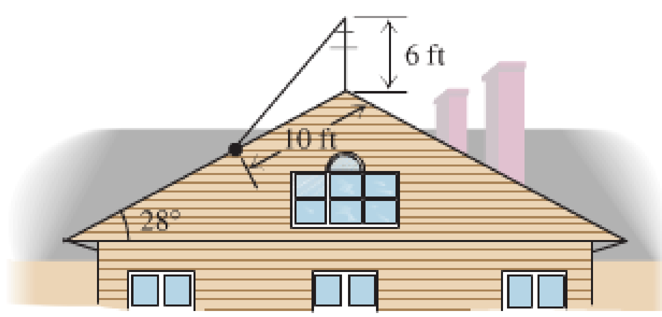
**27.** A submarine sights a moving target at a distance of 820 *m*. A torpedo is fired 9° ahead of the target, and travels 924 *m* in a straight line to hit the target. How far has the target moved from the time the torpedo is fired to the time of the hit?



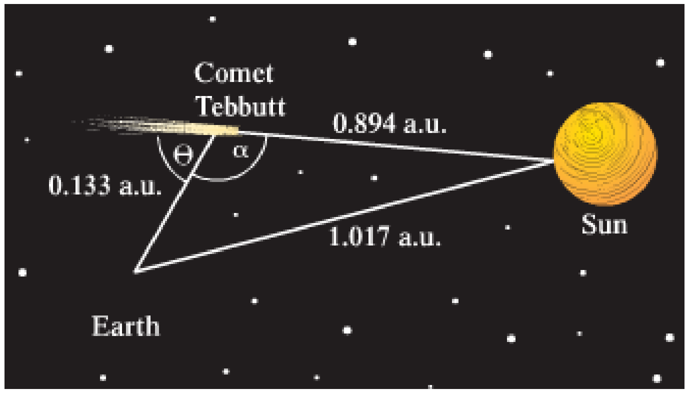

**28.** A tunnel is planned through a mountain to connect points $A$ and $B$ on two existing roads. If the angle between the roads at point $C$ is 28°, what is the distance from point $A$ to $B$? Find $\angle CBA$ and $\angle CAB$ to the nearest tenth of a degree.



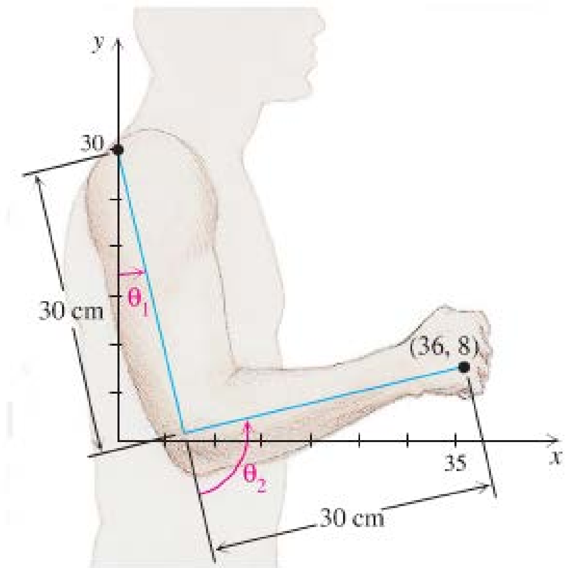

**29.** A 6- *ft* antenna is installed at the top of a roof. A guy wire is to be attached to the top of the antenna and to a point 10 *ft* down the roof. If the angle of elevation of the roof is 28°, then what length guy wire is needed?



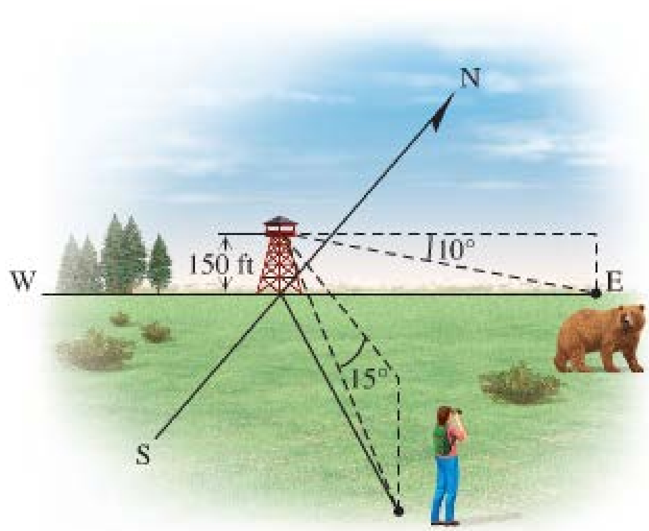

**30.** On June 30, 1861, Comet Tebutt, one of the greatest comets, was visible even before sunset. One of the factors that causes a comet to be extra bright is a small scattering angle $\theta$. When Comet Tebutt was at its brightest, it was 0.133 *a.u*. from the earth, 0.894 *a.u*. from the sun, and the earth was 1.017 *a.u.* from the sun. Find the phase angle $\alpha$ and the scattering angle $\theta$ for Comet Tebutt on June 30, 1861. (One astronomical unit (*a.u*) is the average distance between the earth and the sub.)



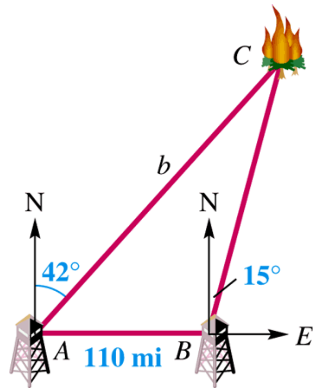

**31.** A human arm consists of an upper arm of 30 *cm* and a lower arm of 30 *cm*. To move the hand to the point (36, 8), the human brain chooses angle $\theta_1$ *and* $\theta_2$ to the nearest tenth of a degree.



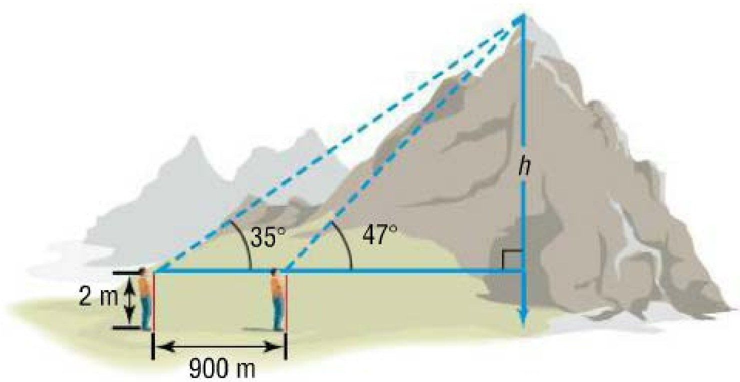

**32.** A forest ranger is 150 *ft* above the ground in a fire tower when she spots an angry grizzly bear east of the tower with an angle of depression of 10°. Southeast of the tower she spots a hiker with an angle of depression of 15°. Find the distance between the hiker and the angry bear.



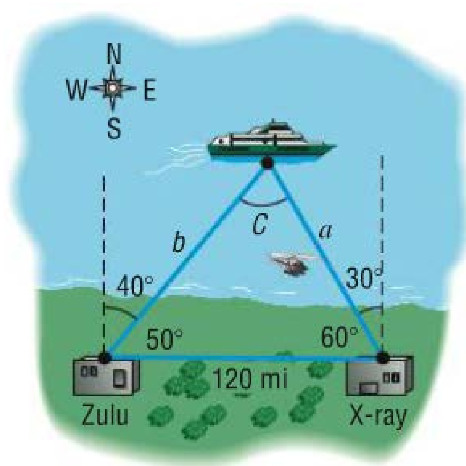
**33.** Two ranger stations are on an east-west line 110 *mi* apart. A forest fire is located on a bearing N 42° E from the western station at $A$ and a bearing of N 15° E from the eastern station at $B$. How far is the fire from the western station?




**34.** To measure the height of a mountain, a surveyor takes two sightings of the peak at a distance 900 meters apart on a direct line to the mountain. The first observation results in an angle of elevation of 47°, and the second results in an angle of elevation of 35°. If the transit is 2 *meters* high, what is the height $h$ of the mountain?



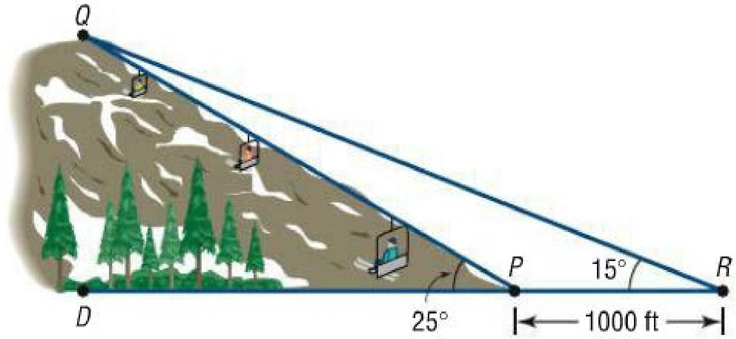

**35.** A Station Zulu is located 120 *miles* due west of Station X-ray. A ship at sea sends an SOS call that is received by each station. The call to Station Zulu indicates that the bearing of the ship from Zulu is $N\ 40°\ E$. The call to Station X-ray indicates that the bearing of the ship from X-ray is $N\ 30°\ W$.



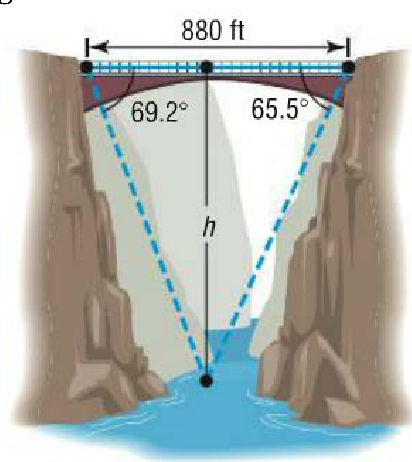

*a)* How far is each station from the ship?

*b)* If a helicopter capable of flying 200 *miles* per *hour* is dispatched from the nearest station to the ship, how long will it take to reach the ship?

**36.** To find the length of the span of a proposed ski lift from $P$ to $Q$, a surveyor measures $\angle DPQ$ to be $25°$ and then walks back a distance of 1000 *feet* to $R$ and measures $\angle DRQ$ to be $15°$. What is the distance from $P$ to $Q$.




**37.** The highest bridge in the world is the bridge over the Royal Gorge of the Arkansas River in Colorado, sightings to the same point at water level directly under the bridge are taken from each side of the 880–foot–long bridge.




Find the area of the triangle

**38.** $b=1, \quad c=3, \quad A=80°$

**39.** $b=4, \quad c=1, \quad A=120°$

**40.** $a=2, \quad c=1, \quad B=10°$

**41.** $a=3, \quad c=2, \quad B=110°$

**42.** $a=8, \quad b=6, \quad C=30°$

**43.** $a=3, \quad b=4, \quad C=60°$

**44.** $a=6, \quad b=4, \quad C=60°$

**45.** $a=4, \quad b=5, \quad c=7$

**46.** $a=12, \quad b=13, \quad c=5$

**47.** $a=3, \quad b=3, \quad c=2$

**48.** $a=4, \quad b=5, \quad c=3$

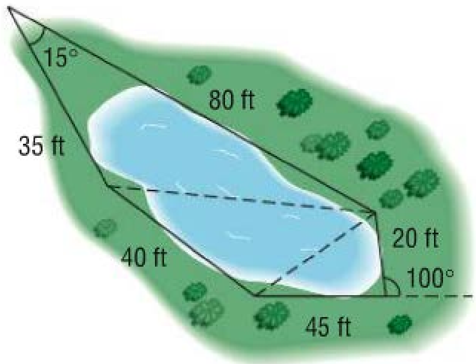
**49.** $a=5, \quad b=8, \quad c=9$

**50.** $a=2, \quad b=2, \quad c=2$

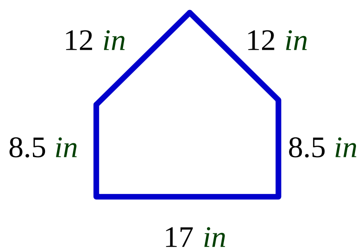
**51.** $a=4, \quad b=3, \quad c=6$

**52.** The dimensions of a triangular lot are 100 *feet* by 50 *feet* by 75 *feet*. If the price of such land is \$3 per square foot, how much does the lot cost?

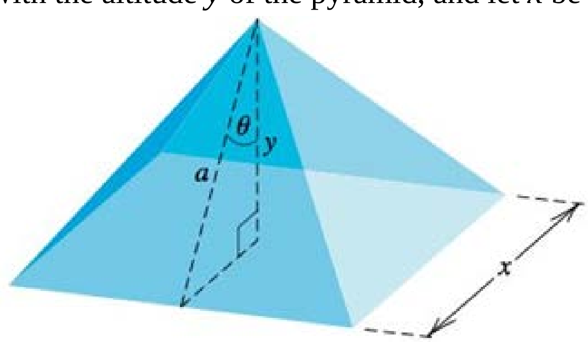
**53.** To approximate the area of a lake, a surveyor walks around the perimeter of the lake. What is the approximate area of the lake?




**54.** The dimensions of home plate at any major league baseball stadium are shown. Find the area of home plate



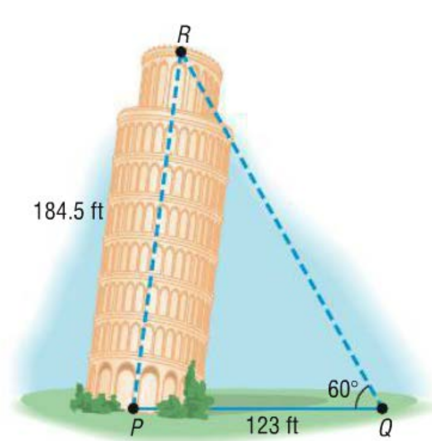

**55.** A pyramid has a square base and congruent triangular faces. Let $\theta$ be the angle that the altitude $a$ of a triangular face makes with the altitude $y$ of the pyramid, and let $x$ be the length of a side.



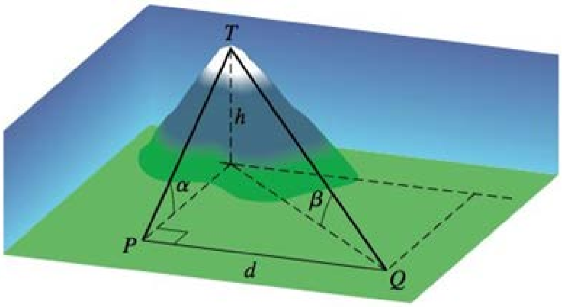

*a)* Express the total surface area $S$ of the four faces in terms of $a$ and $\theta$.

*b)* The volume $V$ of the pyramid equals one-third the area of the base times the altitude. Express $V$ in terms of $a$ and $\theta$.

**56.** The famous Leaning Tower of Pisa was originally 184.5 *feet* high. At a distance of 123 *feet* from the base if the tower, the angle of elevation to the top of the tower is found to be 60°. Find the $\angle RPQ$ indicated in the figure. Also, find the perpendicular distance from $R$ to $PQ$.



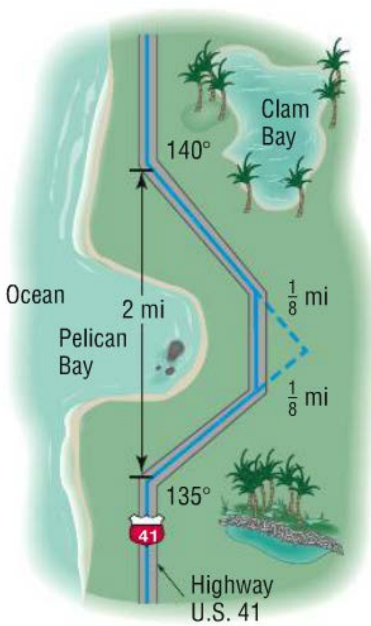
**57.** If a mountaintop is viewed from a point $P$ due south of the mountain, the angle of elevation is $\alpha$. If viewed from a point $Q$ that is $d$ miles cast of $P$, the angle of elevation is $\beta$.




*a)* Show that the height $h$ of the mountain is given by $h = \dfrac{d \sin\alpha \sin\beta}{\sqrt{\sin^2\alpha - \sin^2\beta}}$

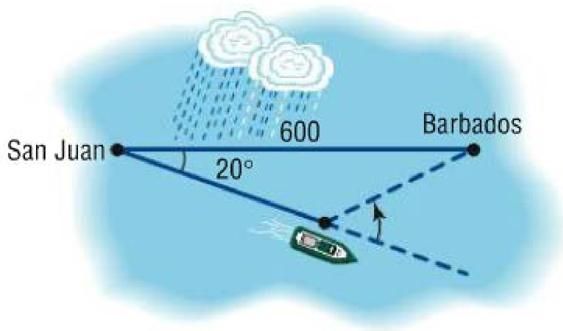
*b)* If $\alpha = 30°, \quad \beta = 20°, \quad and \quad d = 10\ mi$, approximate $h$.

**58.** A highway whose primary directions are north–south, is being constructed along the west coast of Florida. Near Naples, a bay obstructs the straight path of the road. Since the cost of a bridge is prohibitive, engineers decide to go around the bay. The path that they decide on and the measurements taken as shown in the picture. What is the length of highway needed to go around the bay?




**59.** Derive the Mollweide's formula: $\dfrac{a-b}{c} = \dfrac{\sin\left[\frac{1}{2}(A-B)\right]}{\cos\left(\frac{1}{2}C\right)}$

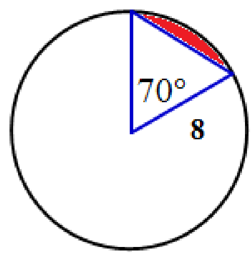
**60.** A cruise ship maintains an average speed of 15 knots in going from San Juan, Puerto Rico, to Barbados, West Indies, a distance of 600 nautical miles. To avoid a tropical storm, the captain heads out to San Juan in a direction of 20° off a direct heading to Barbados. The captain maintains the 15–knots speed for 10 *hours*, after which time the path to Barbados becomes clear of storms.



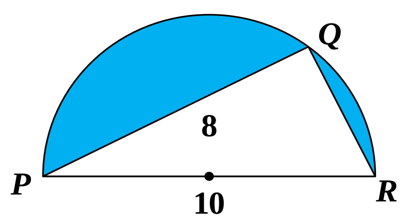

*a)* Through what angle should the captain turn to head directly to Barbados?

*b)* Once the turn is made, how long will it be before the ship reaches Barbados if the same 15–knot speed is maintained?

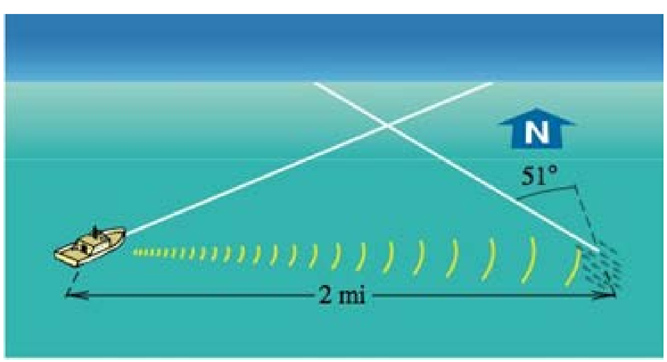
**61.** Find the area of the segment (shaded in blue in the figure) of a circle whose radius is 8 *feet*, formed by a cebtral angle of $70°$




**62.** Find the area of the shaded region enclosed in a semicircle of diameter 10 *inches*. The length of the chord *PQ* is 8 *inches*.



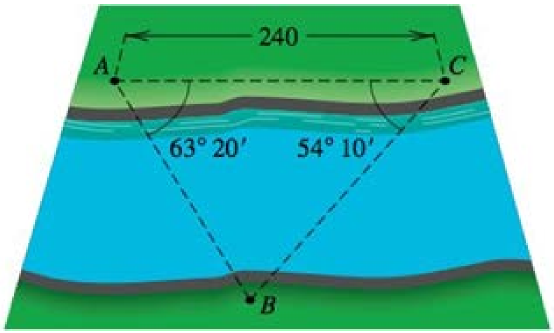

**63.** A commercial fishing boat uses sonar equipment to detect a school of fish 2 *miles* east of the boat and traveling in the direction of $N\ 51°\ W$ at a rate of 8 $mi\ /\ hr$




*a)* The boat travels at 20 $mi\ /\ hr$, approximate the direction it should head to intercept the school of fish.

*b)* Find, to the nearest minute, the time it will take the boat to reach the fish.

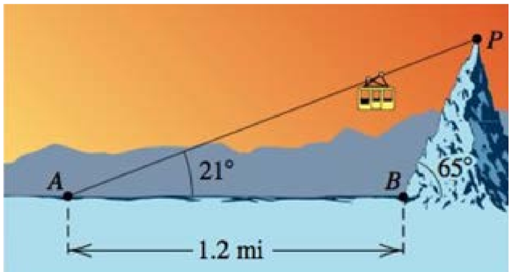
**64.** To find the distance between two points *A* and *B* that lie on opposite banks of a river, a surveyor lays off a line segment *AC* of length 240 *yards* along one bank and determines that the measures of $\angle BAC$ and $\angle ACB$ are $63° \ 20'$ and $54° \ 10'$, respectively.



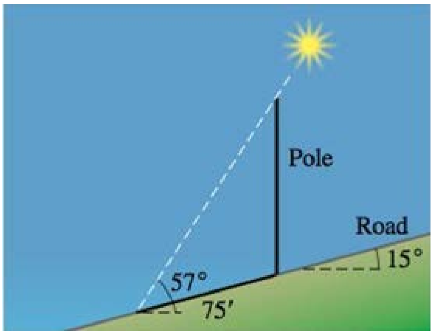

**65.** A cable car carries passengers from a point *A*, which is 1.2 *miles* from a point *B* at the base of a mountain, to a point *P* at the top of the mountain. The angle of elevation of *P* from *A* and *B* are $21°$ and $65°$, respectively.

*a)* Approximate the distance between *A* and *P*.

*b)* Approximate the height of the mountain.

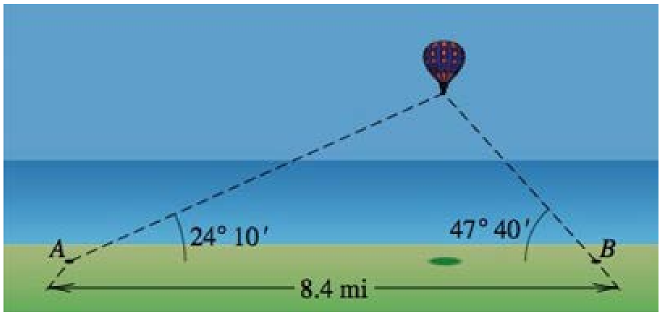



**66.** A straight road makes an angle of $15°$ with the horizontal. When the angle of elevation of the sun is $57°$, a vertical pole at the side of the road casts a shadow 75 *feet* long directly down the road.

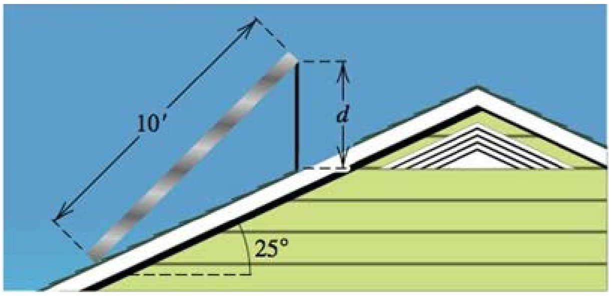



Approximate the length of the pole.

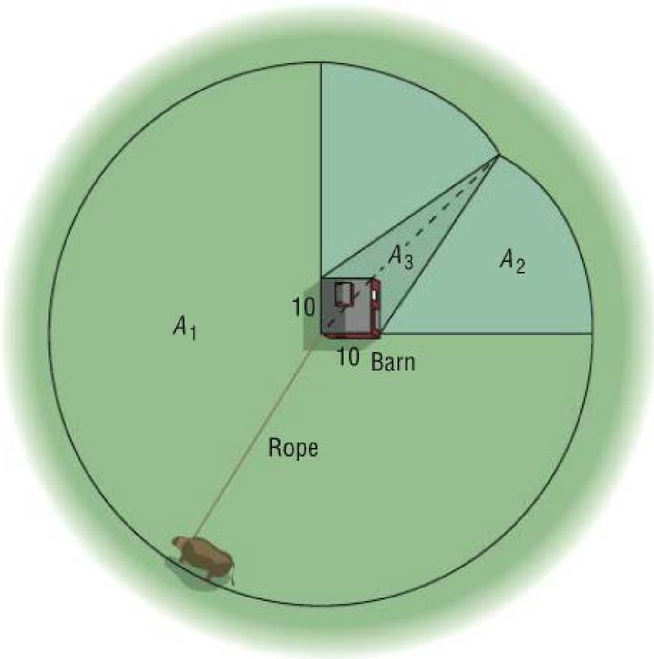
**67.** The angles of elevation of a balloon from two points *A* and *B* on level ground are $24° \ 10'$ and $47° \ 40'$, respectively. Points *A* and *B* are 8.4 *miles* apart, and the balloon is between the points, in the same vertical plane. Approximate the height of the balloon above the ground.



**68.** A solar panel 10 *feet* in width, which is to be attached to a roof that makes an angle of $25°$ with the horizontal. Approximate the length $d$ of the brace that is needed for the panel to make an angle of $45°$ with the horizontal.




**69.** A cow is tethered to one corner of a square barn, 10 *feet* by 10 *feet*, with a rope 100 *feet* long.

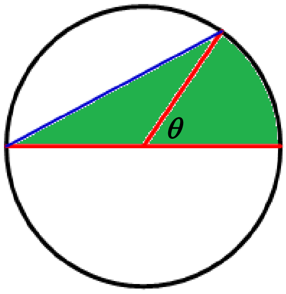



*a)* What is the maximum grazing area for the cow?

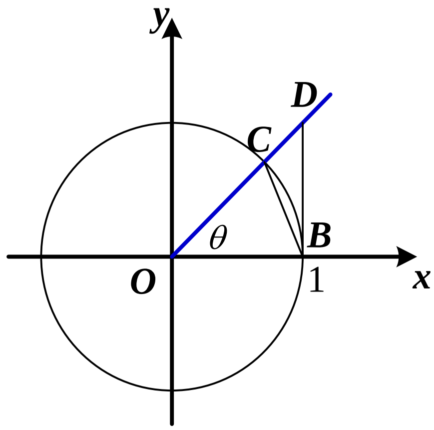
*b)* If the barn is rectangular, 10 *feet* by 20 *feet*, what is the maximum grazing area for the cow?

**70.** For any triangle, show that $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$ *where* $s=\frac{1}{2}(a+b+c)$

**71.** The figure shows a circle of radius $r$ with center at $O$. find the area $K$ of the shaded region as a function of the central angle $\theta$.



**72.** Refer to the figure, in which a unit circle is drawn. The line segment *DB* is tangent to the circle and $\theta$ is acute.




*a)* Express the area of $\Delta OBC$ in terms of $\sin\theta$ and $\cos\theta$.

*b)* Express the area of $\Delta OBD$ in terms of $\sin\theta$ and $\cos\theta$.

*c)* The area of the sector $\overset{\frown}{OBC}$ if the circle is $\frac{1}{2}\theta$, where $\theta$ is measured in *radians*. Use the results of part (*a*) and (*b*) and the fact that

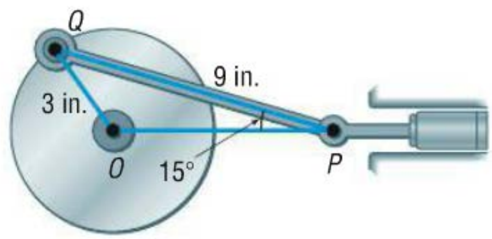
$$Area\ \Delta OBC < Area\ \overset{\frown}{OBC} < Area\ \Delta OBD$$

To show that $\quad 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$

**73.** Find the area of the segment of a circle whose radius is 5 *inches*, formed by a central angle of $40°$.

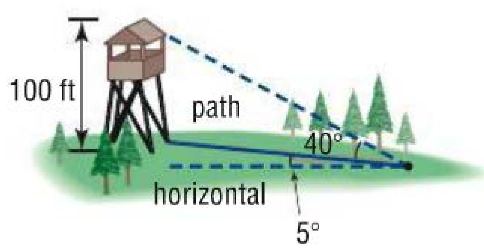
$\approx 0.69\ in^2$

**74.** On a certain automobile, the crankshaft is 3 *inches* long and the connecting rod is 9 *inches* long. At the time when $\angle OPQ$ is 15°, how far is the piston *P* from the center *O* of the crankshaft?

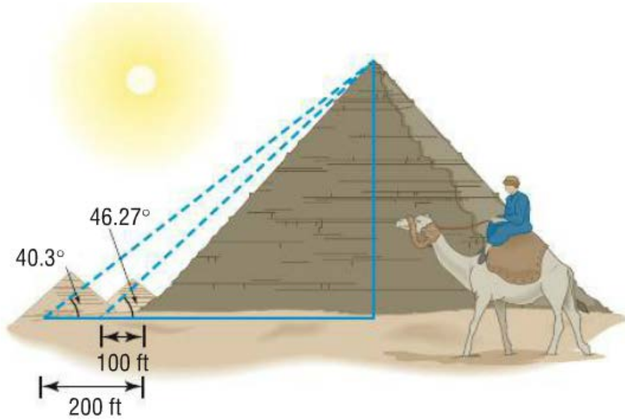



$\approx 10.58\ \ \&\ 6.80\ in$

**75.** A forest ranger is walking on a path inclined at 5° to the horizontal directly toward a 100–foot–tall fire observation tower. The angle of elevation from the path to the top of the tower is 40°. How far is the ranger from the tower at this time? $\approx 110.01\ \ ft$

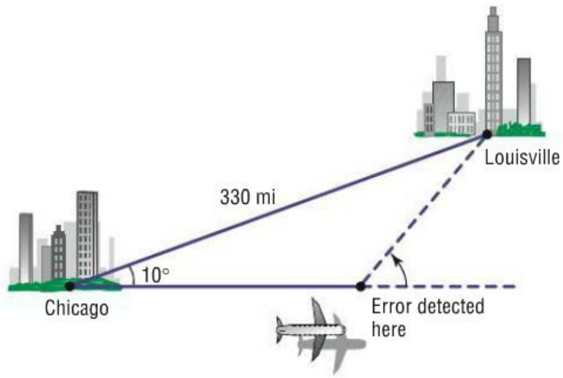


**76.** One of the original Seven Wonders of the world, the Great Pyramid of Cheops was built about 2580 BC. Its original height was 480 *feet* 11 *inches*, but owing to the loss of its topmost stones, it is now shorter. Find the current height of the Great Pyramid using the information shown in the picture.




≈ 449.36 *ft*

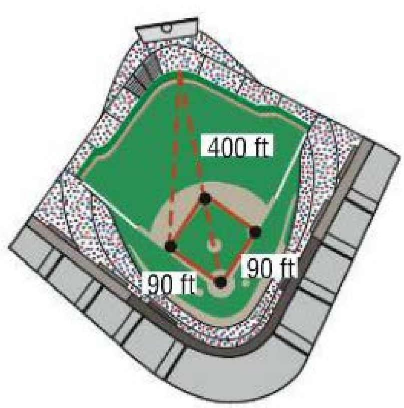
**77.** In attempting to fly from Chicago to Louisville, a distance of 330 *miles*, a pilot inadvertently took a course that was 10° in error.



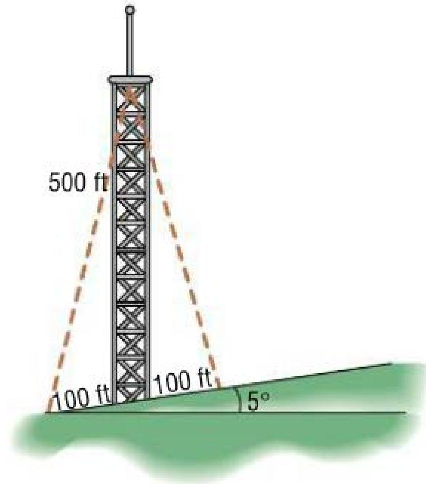

*a)* If the aircraft maintains an average speed of 220 *miles* per *hours*, and if the error in direction is discovered after 15 *minutes*, through what angle should the pilot turn to head toward Louisville? ≈ 12°

*b)* What new average speed should the pilot maintain so that the total time of the trip is 90 *minutes*? ≈ 220.8 *mi* / *hr*

**78.** The distance from home plate to the fence in dead center is 400 feet. How far is it for the fence in dead center to third base? ≈ 342.33 *ft*

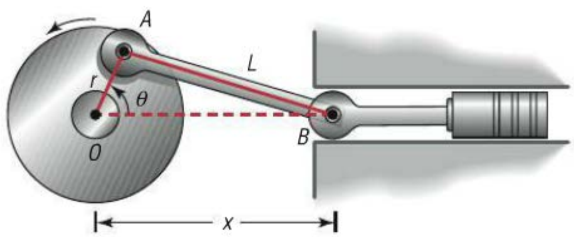


**79.** A radio tower 500 *feet* high is located on the side of a hill with an inclination to the horizontal of 5°. How long should two guy wires be if they are to connect to the top of the tower and be secured at two points 100 *feet* directly above and directly below the base of the tower? $\approx 501.28$ & $518.38$ *ft*



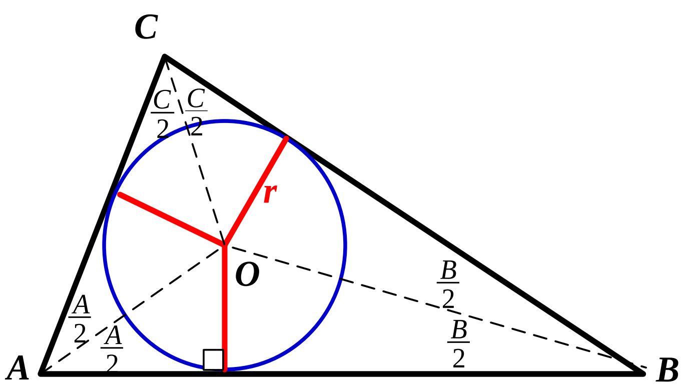

**80.** Rod *OA* rotates about the fixed point *O* so that point *A* travels on a circle of radius *r*. Connected to point *A* is another rod *AB* of length $L > 2r$, and point *B* is connected to a piston. Show that the distance *x* between point *O* and point *B* is given by

$$x = r\cos\theta + \sqrt{r^2\cos^2\theta + L^2 - r^2}$$




Where $\theta$ is the angle of rotation of rod *OA*.

**81.** The lines that bisect each angle of a triangle meet in a single point *O*, and perpendicular distance *r* from *O* to each side of the triangle is the same. The circle with center at *O* and radius *r* is called the inscribed circle of the triangle.




*a)* Show that $r = \dfrac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}$

*b)* Show that $\cot\frac{C}{2} = \frac{s-c}{r}$ where $s = \frac{1}{2}(a+b+c)$

*c)* Show that $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \frac{s}{r}$

*d)* Show that the area $K$ of triangle $ABC$ is $K = rs$, where $s = \frac{1}{2}(a+b+c)$.

*e)* Show that $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$

**82.** Derive the formula: $\frac{a-b}{a+b} = \frac{\tan\left[\frac{1}{2}(A-B)\right]}{\tan\left[\frac{1}{2}(A+B)\right]}$

**83.** For any triangle, show that $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$ *where* $s = \frac{1}{2}(a+b+c)$

**84.** Prove the identity $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}$

# *Section* 2.5 – Trigonometric Graphs

We consider graphs of the equation: $y = A\sin(Bx + C) + D \quad y = A\cos(Bx + C) + D$

## *Amplitude*

If the greatest value of $y$ is $M$ and the least value of $\boldsymbol{y}$ is $m$, then the amplitude of the graph of $y$ is defined to be

$$A = \frac{1}{2}|M - m|$$

The amplitude is $|A|$.

***Note:*** If $A > 0$, then the graph of $y = A\sin x$ and $y = A\cos x$ will have amplitude $A$ and range $[-A, A]$.

## *Period*

$$\to Period = \frac{2\pi}{|B|}$$

Many things in daily life repeat with a predictable pattern, such as weather, tides, and hours of daylight.

This periodic graph represents a normal heartbeat.

### *Example*

Find the amplitude and the period of $y = 3\sin 2x$

**Solution**

***Amplitude***: $|A| = |3| = 3$

***Period***: $P = \frac{2\pi}{|B|} = \frac{2\pi}{2} = \pi$

## *Example*

Find the amplitude and the period of $y = 2\sin\frac{1}{2}x$

***Solution***

***Amplitude***: $|A| = |2| = 2$

***Period***: $P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{1}{2}} = 4\pi$

## *Example*

Find the amplitude and the period of $y = -4\sin(-\pi x)$

***Solution***

***Amplitude***: $|A| = |-4| = 4$

***Period***: $P = \frac{2\pi}{|B|} = \frac{2\pi}{\pi} = 2$

## *Even* and *Odd* Functions

### *Definition*

An *even function* is a function for which $f(-x) = f(x)$

An *odd function* is a function for which $f(-x) = -f(x)$

| ***Even Functions*** | ***Odd Functions*** |
|---|---|
| $y = \cos(\theta)$, $y = \sec(\theta)$ | $y = \sin(\theta)$, $y = \csc(\theta)$<br>$y = \tan(\theta)$, $y = \cot(\theta)$ |
| *Graphs are symmetric about the y-axis* | *Graphs are symmetric about the origin* |

## *Phase shift*

If we add a term to the argument of the function, the graph will be translated in a ***horizontal direction***.

In the function $y = f(x - c)$, the expression $x - c$ is called the ***argument***.

$$\boxed{\textit{Phase Shift}: \quad \phi = -\frac{C}{B}}$$

### *Example*

Find the amplitude, the period, and the phase shift of $y = 3\sin\left(2x + \frac{\pi}{2}\right)$

***Solution***

***Amplitude***: $|A| = 3$

***Period***: $P = \frac{2\pi}{|B|} = \frac{2\pi}{2} = \pi$

***Phase shift***: $\phi = -\frac{C}{B} = -\frac{\pi}{2}\frac{1}{2} = -\frac{\pi}{4}$

## *Vertical* **Translations**

For $d > 0$, $y = f(x) + d \Rightarrow$ The graph shifted up $d$ units

$y = f(x) - d \Rightarrow$ The graph shifted down $d$ units

### *Example*

Find the amplitude, the period, and the vertical shift of $y = -3 - 2\sin \pi x$

***Solution***

***Amplitude***: $A = 2$

***Period***: $P = \frac{2\pi}{\pi} = 2$

***Vertical Shifting:*** $y = -3$ *Down 3 units*

## Graphing the *Sine* and *Cosine* Functions

The graphs of $y = A\sin(Bx + C) + D$ and $y = A\cos(Bx + C) + D$, will have the following characteristics:

Amplitude $= |A|$ Period: $P = \dfrac{2\pi}{|B|}$

Phase Shift: $\phi = -\dfrac{C}{B}$ Vertical translation: $y = D$

If $A < 0$ the graph will be reflected about the $x$-axis

### *Example*

Graph $y = \sin\left(x + \frac{\pi}{2}\right)$, *if* $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

***Amplitude***: $A = 1$

***Period***: $P = \dfrac{2\pi}{1} = 2\pi$

$x + \frac{\pi}{2} = 0 \rightarrow x = -\frac{\pi}{2}$

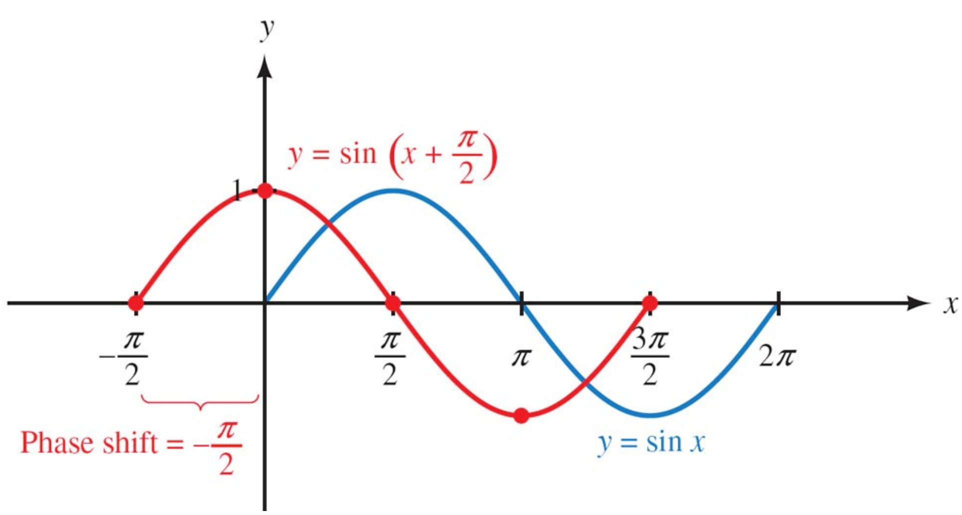
***Phase Shift***: $\phi = -\dfrac{\pi}{2}$

$0 \le argument \le 2\pi$

$0 \le x + \frac{\pi}{2} \le 2\pi$

$-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

| $x$ | | $x$ | $y = \sin\left(x + \frac{\pi}{2}\right)$ |
|---|---|---|---|
| $\phi + 0$ | $-\frac{\pi}{2} + 0$ | $-\frac{\pi}{2}$ | 0 |
| $\phi + \frac{1}{4}P$ | $-\frac{\pi}{2} + \frac{1}{2}\pi$ | 0 | 1 |
| $\phi + \frac{1}{2}P$ | $-\frac{\pi}{2} + \frac{3}{2}\pi$ | $\frac{\pi}{2}$ | 0 |
| $\phi + \frac{3}{4}P$ | $-\frac{\pi}{2} + \frac{3}{4}\pi$ | $\pi$ | –1 |
| $\phi + P$ | $-\frac{\pi}{2} + 2\pi$ | $\frac{3\pi}{2}$ | 0 |



### *Example*

Graph $y = 4\cos\left(2x - \frac{3\pi}{2}\right)$ *for* $0 \le x \le 2\pi$

***Solution***

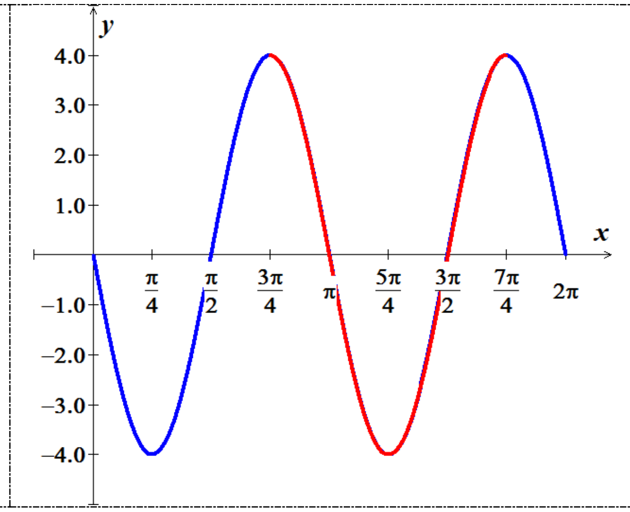
***Amplitude***: $|A| = 4$

***Period***: $P = \frac{2\pi}{2} = \pi$

***Phase Shift***: $\phi = \frac{\frac{3\pi}{2}}{2} = \frac{3\pi}{4}$

***VT:*** $y = 0$

| | $x$ | $y$ |
|---|---|---|
| $0 + \frac{3\pi}{4}$ | $\frac{3\pi}{4}$ | 4 |
| $\frac{\pi}{4} + \frac{3\pi}{4}$ | $\pi$ | 0 |
| $\frac{\pi}{2} + \frac{3\pi}{4}$ | $\frac{5\pi}{4}$ | $-4$ |
| $\frac{3\pi}{4} + \frac{3\pi}{4}$ | $\frac{3\pi}{2}$ | 0 |
| $\pi + \frac{3\pi}{4}$ | $\frac{7\pi}{4}$ | 4 |




### *Example*

Graph one complete cycle $y = 3 - 5\sin\left(\pi x + \frac{\pi}{4}\right)$

***Solution***

***Amplitude***: $|A| = 5$

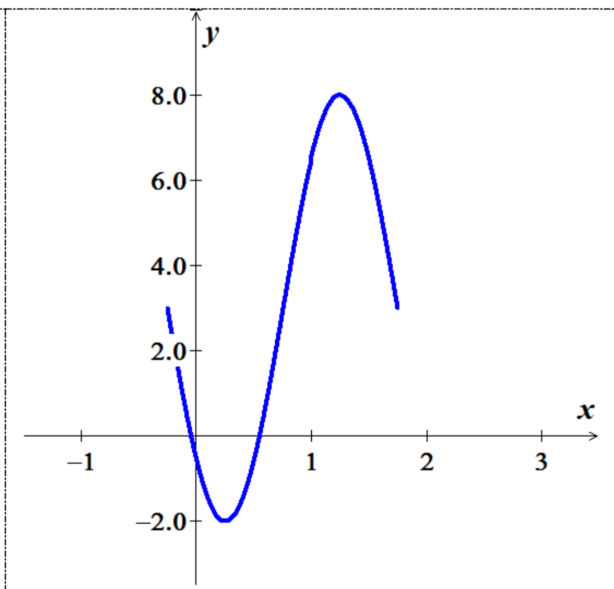
***Period***: $P = \frac{2\pi}{\pi} = 2$

***Phase Shift***:

$$\phi = -\frac{\frac{\pi}{4}}{\pi} = -\frac{1}{4}$$

***VT:*** $y = 3$

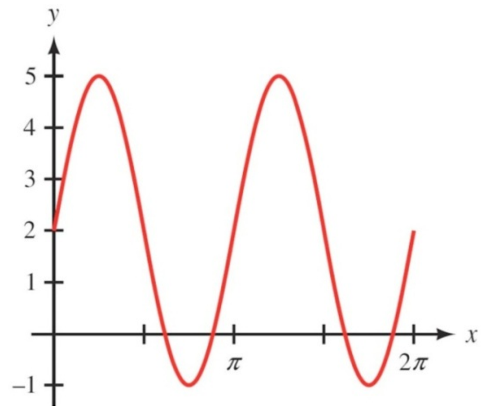
| | $x$ | $y$ |
|---|---|---|
| $0 - \frac{1}{4}$ | $-\frac{1}{4}$ | 3 |
| $\frac{1}{2} - \frac{1}{4}$ | $\frac{1}{4}$ | $-2$ |
| $1 - \frac{1}{4}$ | $\frac{3}{4}$ | 3 |
| $\frac{3}{2} - \frac{1}{4}$ | $\frac{5}{4}$ | 8 |
| $2 - \frac{1}{4}$ | $\frac{7\pi}{4}$ | 3 |



## Finding the *Sine* and *Cosine* Functions from the Graph

### *Example*

Find an equation $y = A\sin(Bx + C) + D$ or $y = A\cos(Bx + C) + D$ to match the graph

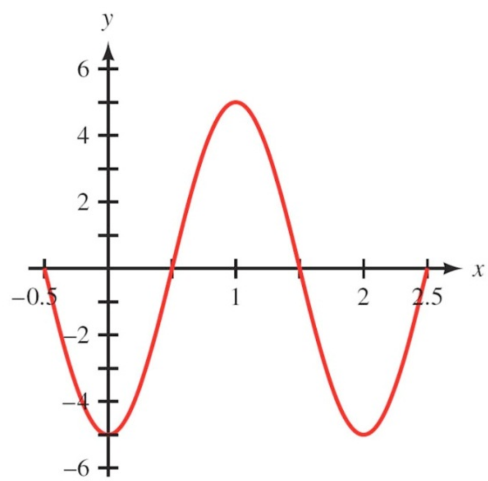



B = $\frac{2\pi}{\pi} = 2$

***Amplitude*** = 3 $\quad \Rightarrow \boxed{y = 2 + 3\sin 2x} \quad 0 \le x \le 2\pi$

### *Example*

Find an equation $y = A\sin(Bx + C) + D$ or $y = A\cos(Bx + C) + D$ to match the graph




B = $\frac{2\pi}{2} = \pi$

***Amplitude*** = 5

$y = -5\cos \pi x \quad -0.5 \le x \le 2.5$

*Or*

Phase shift = $-0.5 = -\frac{C}{B}$

$0.5 = \frac{C}{\pi}$

$0.5\pi = C$

$y = -5\sin(\pi x + \frac{\pi}{2}) \qquad -0.5 \le x \le 2.5$

# *Exercises* Section 2.5 – Trigonometric Graphs

Find the amplitude, the period, the phase shift, and the vertical translation and sketch the graph of the equation

**1.** $y = 2\sin(x - \pi)$

**2.** $y = \frac{2}{3}\sin\left(x + \frac{\pi}{2}\right)$

**3.** $y = 4\cos\left(\frac{1}{2}x + \frac{\pi}{2}\right)$

**4.** $y = \frac{1}{2}\sin\left(\frac{1}{2}x + \pi\right)$

**5.** $y = 3\cos\left(\frac{\pi}{2}\left(x - \frac{1}{2}\right)\right)$

**6.** $y = -\cos\pi\left(x - \frac{1}{3}\right)$

**7.** $y = 2 - \sin\left(3x - \frac{\pi}{5}\right)$

**8.** $y = -\frac{2}{3}\sin\left(3x - \frac{\pi}{2}\right)$

**9.** $y = -1 + \frac{1}{2}\cos(2x - 3\pi)$

**10.** $y = 2 - \frac{1}{3}\cos\left(\pi x + \frac{3\pi}{2}\right)$

**11.** $y = \frac{5}{2} - 3\cos\left(\pi x - \frac{\pi}{4}\right)$

**12.** $y = \cos\frac{1}{2}x$

**13.** $y = -3 + \sin\left(\pi x + \frac{\pi}{2}\right)$

**14.** $y = \frac{2}{3} - \frac{4}{3}\cos(3x - \pi)$

**15.** $y = 2\sin\left(x - \frac{\pi}{3}\right)$

**16.** $y = 4\cos\left(x - \frac{\pi}{4}\right)$

**17.** $y = -\sin(3x + \pi) - 1$

**18.** $y = \cos(2x - \pi) + 2$

**19.** $y = \cos\frac{1}{2}x$

**20.** $y = \sin\left(\frac{1}{2}x - \frac{\pi}{3}\right)$

**21.** $y = 5\sin\left(3x - \frac{\pi}{2}\right)$

**22.** $y = 3\cos\left(\frac{1}{2}x - \frac{\pi}{4}\right)$

**23.** $y = -5\cos\left(\frac{1}{3}x + \frac{\pi}{6}\right)$

**24.** $y = -2\sin(2\pi x + \pi)$

**25.** $y = -2\sin(2x - \pi) + 3$

**26.** $y = 3\cos(x + 3\pi) - 2$

**27.** $y = 5\cos(2x + 2\pi) + 2$

**28.** $y = -4\sin(3x - \pi) - 3$

Graph a ***one complete*** cycle

**29.** $y = \cos\left(x - \frac{\pi}{6}\right)$

**30.** $y = \frac{2}{3} - \frac{4}{3}\cos(3x - \pi)$

**31.** $y = -3 + \sin\left(\pi x + \frac{\pi}{2}\right)$

**32.** $y = 2\sin(-\pi x)$ *for* $-3 \le x \le 3$

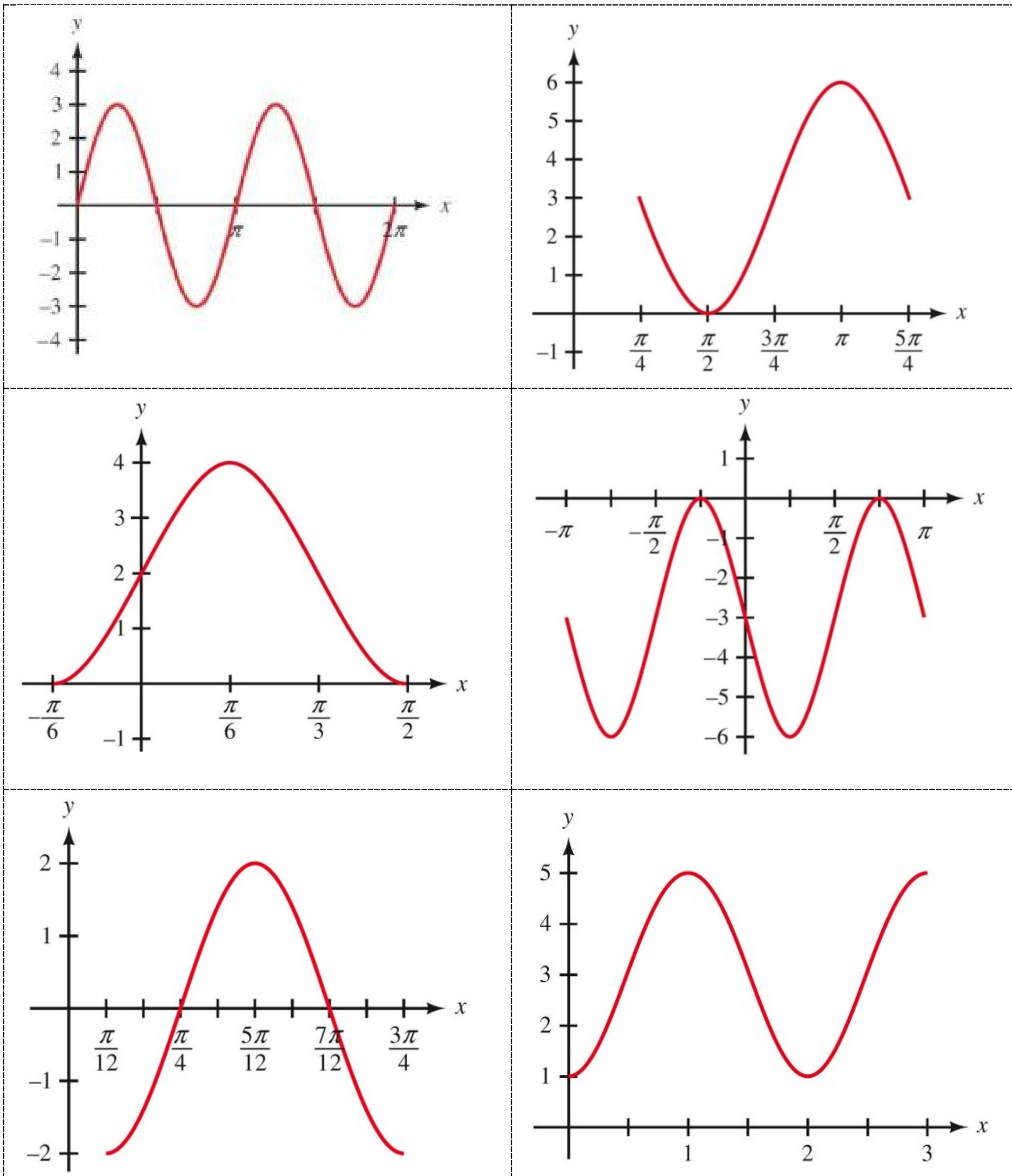
**33.** $y = 4\cos\left(-\frac{2}{3}x\right)$ *for* $-\frac{15\pi}{4} \le x \le \frac{15\pi}{4}$

**34.** $y = -1 + 2\sin(4x + \pi)$ over two periods.

**35.** The maximum afternoon temperature in a given city might be modeled by $t = 60 - 30\cos\frac{\pi x}{6}$
Where *t* represents the maximum afternoon temperature in month *x*, with *x* = 0 representing January, *x* = 1 representing February, and so on. Find the maximum afternoon temperature to the nearest degree for each month.

*a*) Jan. *b*) Apr. *c*) May. *d*) Jun. *e*) Oct.

**36.** Find an equation $y = A\sin(Bx+C)+D$ or $y = A\cos(Bx+C)+D$ to match the graph

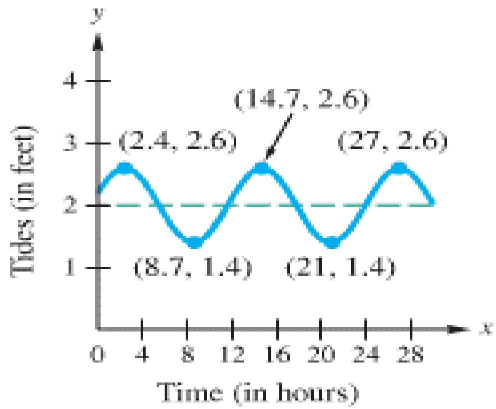



**37.** The diameter of the Ferris wheel is 250 *ft*, the distance from the ground to the bottom of the wheel is 14 *ft*. We found the height of a rider on that Ferris wheel was given by the function:

$$H = 139 - 125\cos\left(\frac{\pi}{10}t\right)$$

Where $t$ is the number of minutes from the beginning of a ride. Graph a complete cycle of this function.

**38.** The figure shows a function *f* that models the tides in feet at Clearwater Beach, *x* hours after midnight starting on Aug. 26,

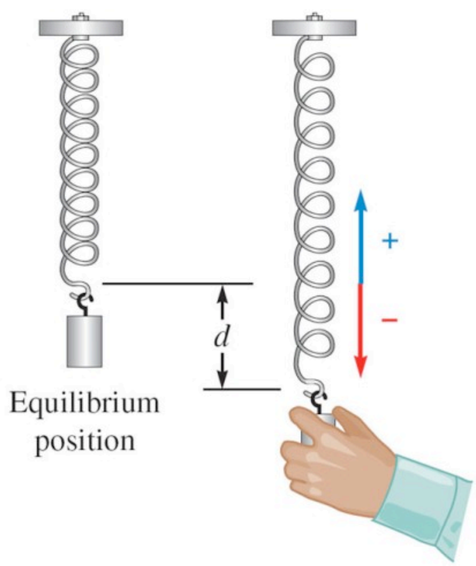



*a)* Find the time between high tides.

*b)* What is the difference in water levels between high tide and low tide?

*c)* The tides can be modeled by $f(x) = 0.6\cos[0.511x - 2.4] + 2$ Estimate the tides when $x = 10$.

**39.** A mass attached to a spring oscillates upward and downward. The length $L$ of the spring after $t$ seconds is given by the function $L = 15 - 3.5\cos(2\pi t)$, where $L$ is measured in cm.




*a)* Sketch the graph of this function for $0 \le t \le 5$

*b)* What is the length the spring when it is at equilibrium?

*c)* What is the length the spring when it is shortest?

*d)* What is the length the spring when it is longest?

**40.** Based on years of weather data, the expected low temperature $T$ (in °F) in Fairbanks, Alaska, can be approximated by

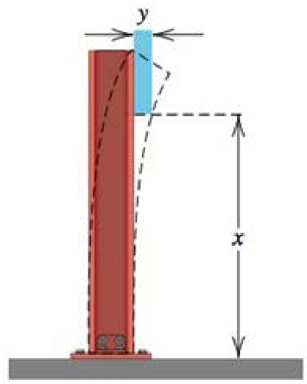
$$T = 36\sin\left(\frac{2\pi}{365}(t - 101)\right) + 14$$

*a)* Sketch the graph T for $0 \le t \le 365$

*b)* Predict when the coldest day of the year will occur.

**41.** To simulate the response of a structure to an earthquake, an engineer must choose a shape for the initial displacement of the beams in the building. When the beam has length $L$ *feet* and the maximum displacement is $a$ *feet*, the equation

$$y = a - a\cos\frac{\pi}{2L}x$$




Has been used by engineers to estimate the displacement $y$. if $a = 1$ and $L = 10$, sketch the graph of the equation for $0 \le x \le 10$.

# *Section* 2.6 – Additional Trigonometric Graphs

## *Vertical Asymptote*

A ***vertical asymptote*** is a vertical line that the graph approaches but does not intersect, while function values increase or decrease without bound as *x*-values get closer and closer to the line.

## Graphing the *Tangent* Functions

The graphs of $y = A\tan(Bx + C) + D$ will have the following characteristics:

***Domain***: $\left\{x \mid x \neq (2n+1)\frac{\pi}{2},\ \textit{where } n \in \mathbb{Z}\right\}$

***Range***: $(-\infty,\ \infty)$

- The graph is discontinuous at values of *x* of the form $\boxed{x = (2n+1)\frac{\pi}{2}}$ and has ***vertical asymptotes*** at these values.
- Its ***x-intercepts*** are of the form $x = n\pi$ .
- Its period is π.
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all *x* in the domain, $\tan(-x) = -\tan(x)$.

***No*** *Amplitude*

***Period*** $= \frac{\pi}{|B|}$

***Phase Shift*** $= -\frac{C}{B}$

***Vertical translation*** $= D$

**Vertical Asymptote (*VA*):** $bx + c = (2n+1)\frac{\pi}{2}$

One cycle: $0 \le \arg ument \le \pi$ $\boxed{or}$ $-\frac{\pi}{2} < \arg ument \le \frac{\pi}{2}$

### Example

Find the period, and the phase shift and sketch the graph of $y=\frac{1}{2}\tan\left(x+\frac{\pi}{4}\right)$

**Solution**

***Period***: $P=\frac{\pi}{|B|}=\pi$

***Phase shift***: $\phi=-\frac{C}{B}=-\frac{\frac{\pi}{4}}{1}=-\frac{\pi}{4}$

***Vertical translation***: $y = 0$

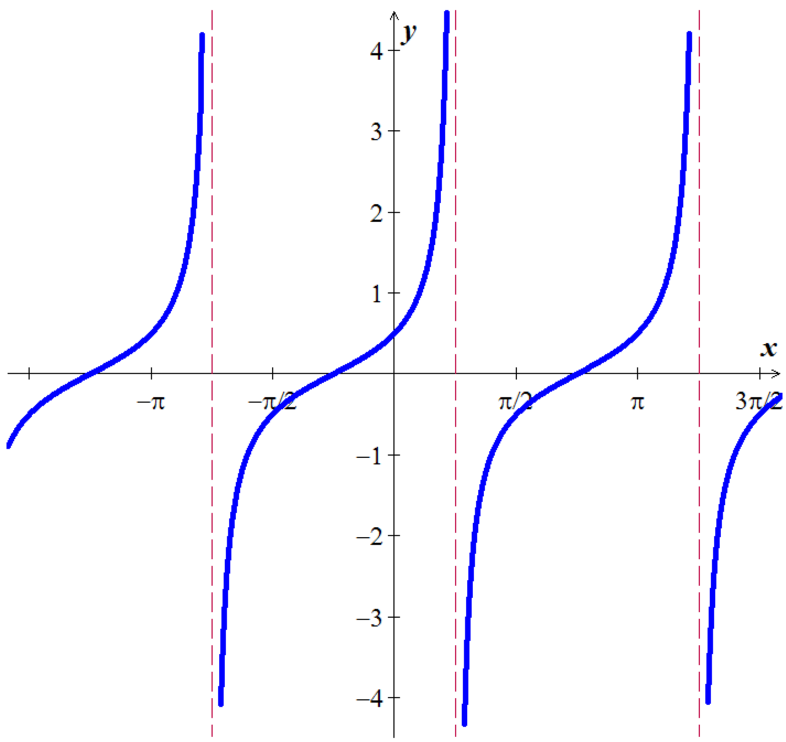
***Vertical Asymptote***: $x+\frac{\pi}{4}=(2n+1)\frac{\pi}{2}$

$$x+\frac{\pi}{4}=\pi n+\frac{\pi}{2}$$

$$x+\frac{\pi}{4}-\frac{\pi}{4}=\pi n+\frac{\pi}{2}-\frac{\pi}{4}$$

$$\boxed{x=\pi n+\frac{\pi}{4}}$$

| | $x$ | $y=\frac{1}{2}\tan\left(x+\frac{\pi}{4}\right)$ |
|---|---|---|
| $-\frac{\pi}{4}+0$ | $-\frac{\pi}{4}$ | 0 |
| $-\frac{\pi}{4}+\frac{1}{4}\pi$ | 0 | 0.5 |
| $-\frac{\pi}{4}+\frac{1}{2}\pi$ | $\frac{\pi}{4}$ | $\infty$ |
| $-\frac{\pi}{4}+\frac{3}{4}\pi$ | $\frac{\pi}{2}$ | –0.5 |
| $-\frac{\pi}{4}+\pi$ | $\frac{3\pi}{4}$ | 0 |




One Complete cycle can be determined by:

$$-\frac{\pi}{2}\le x+\frac{\pi}{4}\le\frac{\pi}{2}\Rightarrow-\frac{\pi}{2}-\frac{\pi}{4}\le x+\frac{\pi}{4}-\frac{\pi}{4}\le\frac{\pi}{2}-\frac{\pi}{4}$$

$$-\frac{3\pi}{4}\le x\le\frac{\pi}{4}$$

## *Cotangent Functions*

***Domain***: $\{x \mid x \neq n\pi,\ where\ n \in \mathbb{Z}\}$

***Range***: $(-\infty,\ \infty)$

- The graph is discontinuous at values of $x$ of the form $\boxed{x = n\pi}$ and has ***vertical asymptotes*** at these values.
- Its ***x-intercepts*** are of the form $x = (2n+1)\frac{\pi}{2}$.
- Its period is π.
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all $x$ in the domain, $\cot(-x) = -\cot(x)$.

## *Example*

Find the period, and the phase shift and sketch the graph of $y = \cot\left(2x - \frac{\pi}{2}\right)$

***Solution***

Period: $P = \frac{\pi}{|B|} = \frac{\pi}{2}$

Phase shift: $\phi = -\frac{C}{B} = -\frac{-\frac{\pi}{2}}{2} = \frac{\pi}{4}$

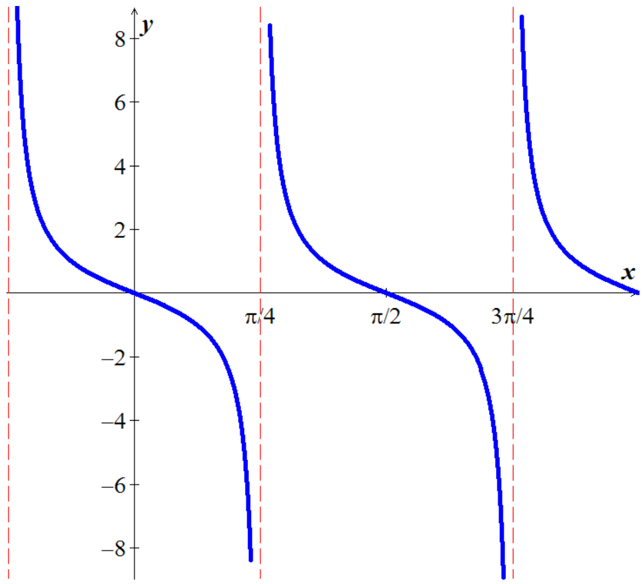
One cycle: $0 \leq 2x - \frac{\pi}{2} \leq \pi$

$$\frac{\pi}{2} \leq 2x \leq \frac{3\pi}{2}$$

$$\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$$

V.A: $2x - \frac{\pi}{2} = n\pi \Rightarrow 2x = n\pi + \frac{\pi}{2} \Rightarrow \boxed{x = n\frac{\pi}{2} + \frac{\pi}{4}}$

| | $x$ | $y = \cot\left(2x - \frac{\pi}{2}\right)$ |
|---|---|---|
| $\frac{\pi}{4} + 0$ | $\frac{\pi}{4}$ | $\infty$ |
| $\frac{\pi}{4} + \frac{\pi}{8}$ | $\frac{3\pi}{8}$ | 1 |
| $\frac{\pi}{4} + \frac{\pi}{4}$ | $\frac{\pi}{2}$ | 0 |
| $\frac{\pi}{4} + \frac{3\pi}{8}$ | $\frac{5\pi}{8}$ | $-1$ |
| $\frac{\pi}{4} + \frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\infty$ |



## Graphing the *Secant* Function

***Domain***: $\left\{ x \mid x \neq (2n+1)\frac{\pi}{2},\ \text{where } n \in \mathbb{Z} \right\}$

***Range***: $(-\infty,\ -1] \cup [1,\ \infty)$

- The graph is discontinuous at values of $x$ of the form $x = (2n+1)\frac{\pi}{2}$ and has ***vertical asymptotes*** at these values.
- There are ***no x-intercepts***.
- Its period is $2\pi$.
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the $y$-axis, so the function is an even function. For all $x$ in the domain, $\sec(-x) = \sec(x)$.

### *Example*

Sketch the graph of $y = 2\sec\left(x - \frac{\pi}{4}\right)$

***Solution***

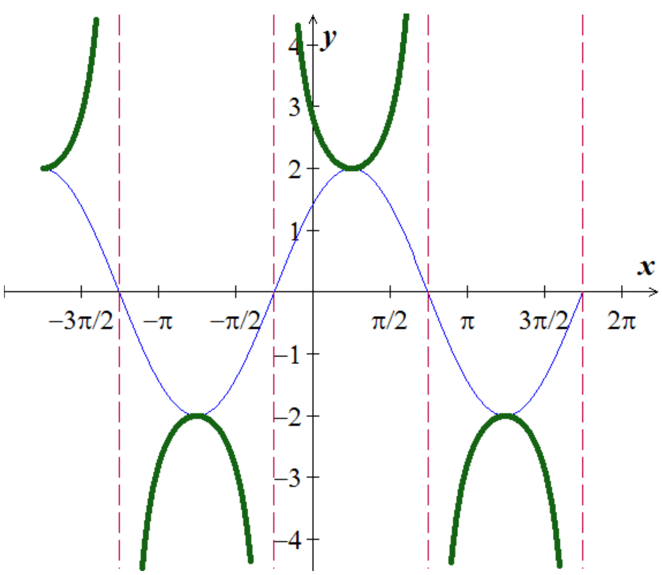
*Period* $= \frac{2\pi}{1} = 2\pi$

***First***, graph $y = 2\cos\left(x - \frac{\pi}{4}\right)$

*Phase shift*: $\phi = -\frac{C}{B} = -\frac{-\frac{\pi}{4}}{1} = \frac{\pi}{4}$

*Vertical Asymptote*: $x = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \ldots$

| $x$ | $y = 2\cos\left(x - \frac{\pi}{4}\right)$ |
|---|---|
| $0 + \frac{\pi}{4} = \frac{\pi}{4}$ | 2 |
| $\frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$ | 0 |
| $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$ | –2 |
| $\frac{3\pi}{2} + \frac{\pi}{4} = \frac{7\pi}{4}$ | 0 |
| $2\pi + \frac{\pi}{4} = \frac{9\pi}{4}$ | 2 |



## Graphing the *Cosecant* Function

***Domain***: $\{x \mid x \neq n\pi, \text{ where } n \in \mathbb{Z}\}$

***Range***: $(-\infty, -1] \cup [1, \infty)$

- The graph is discontinuous at values of *x* of the form $x = n\pi$ and has ***vertical asymptotes*** at these values.
- There are no *x*-intercepts.
- Its period is $2\pi$.
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the *origin*, so the function is an odd function. For all *x* in the domain $\csc(-x) = -\csc(x)$.

### *Example*

Find the period and sketch the graph of $y = \csc(2x + \pi)$

***Solution***

$$y = \csc(2x+\pi) = \frac{1}{\sin(2x+\pi)}$$

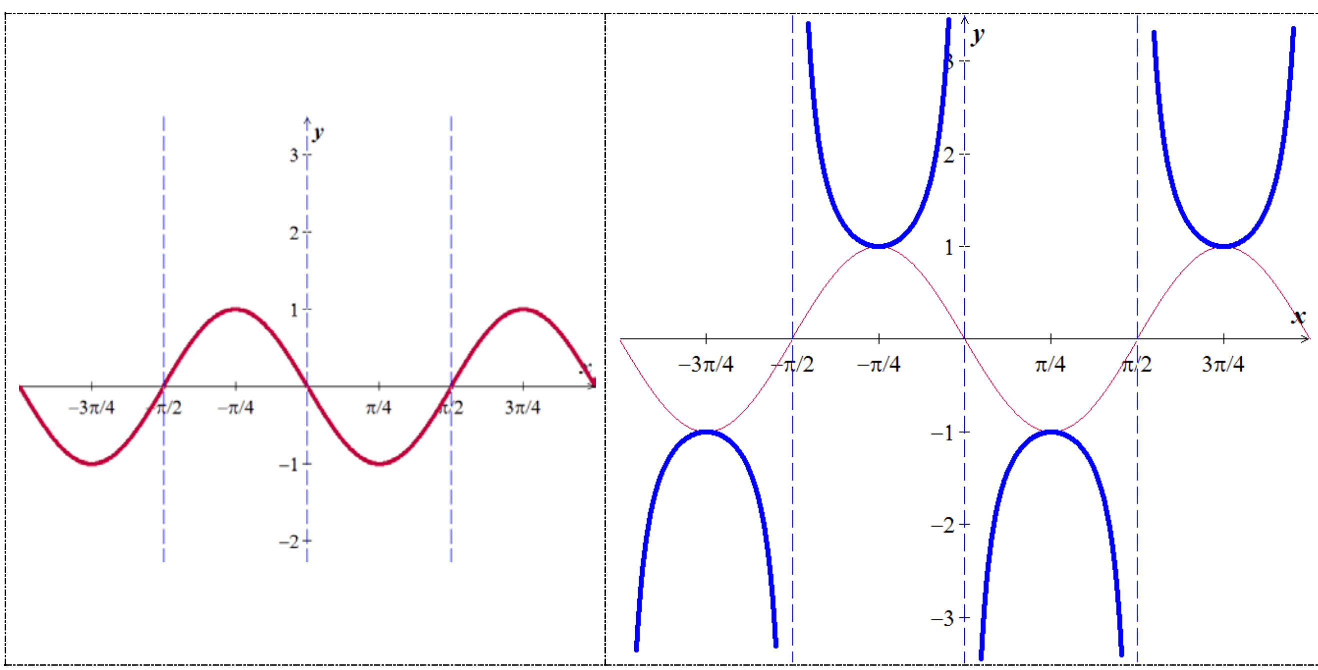
*Period* $= \frac{2\pi}{2} = \pi$

***First***, graph $y = \sin(2x + \pi)$

*Phase shift*: $\phi = -\frac{C}{B} = -\frac{\pi}{2}$

*Vertical Asymptote*: $x = 0, \pm\frac{\pi}{2}, \pm\pi, \ldots$

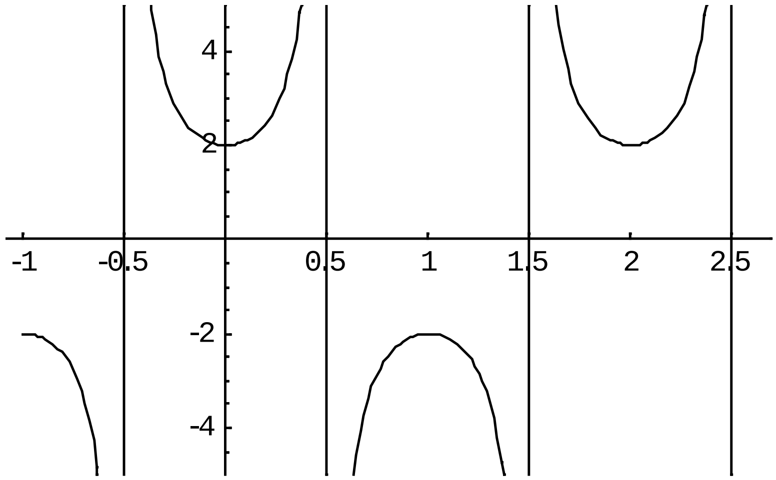
| $x$ | $y = \sin(2x+\pi)$ |
|---|---|
| $0 - \frac{\pi}{2} = -\frac{\pi}{2}$ | 0 |
| $\frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$ | 1 |
| $\frac{\pi}{2} - \frac{\pi}{2} = 0$ | 0 |
| $\frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4}$ | –1 |
| $\pi - \frac{\pi}{2} = \frac{\pi}{2}$ | 0 |



## Finding the *Secant* and *Cosecant* Functions from the Graph

### *Example*

Find an equation $y = k + A\sec(Bx + C)$ or $y = k + A\csc(Bx + C)$ to match the graph




**Solution**

For cosine:

$\boldsymbol{A} = 2$ $\quad P = 2 = \frac{2\pi}{B} \Rightarrow \boxed{B = \frac{2\pi}{2} = \pi}$ $\quad Phase\ shift = -\frac{C}{B} = 0 \Rightarrow \boxed{C = 0}$

$y = 2\sec(\pi x)$ from –1 to 2.5.

# *Exercises* Section 2.6 – Additional Trigonometric Graphs

Find the period, show the asymptotes, and sketch the graph of

**1.** $y = \tan\left(x - \frac{\pi}{4}\right)$

**2.** $y = 2\tan\left(2x + \frac{\pi}{2}\right)$

**3.** $y = -\frac{1}{4}\tan\left(\frac{1}{2}x + \frac{\pi}{3}\right)$

**4.** $y = \cot\left(x + \frac{\pi}{4}\right)$

**5.** $y = 2\cot\left(2x + \frac{\pi}{2}\right)$

**6.** $y = -\frac{1}{2}\cot\left(\frac{1}{2}x + \frac{\pi}{4}\right)$

**7.** $y = \sec\left(x - \frac{\pi}{2}\right)$

**8.** $y = 2\sec\left(2x - \frac{\pi}{2}\right)$

**9.** $y = -3\sec\left(\frac{1}{3}x + \frac{\pi}{3}\right)$

**10.** $y = \csc\left(x - \frac{\pi}{2}\right)$

**11.** $y = 2\csc\left(2x + \frac{\pi}{2}\right)$

**12.** $y = 4\csc\left(\frac{1}{2}x - \frac{\pi}{4}\right)$

Graph over a **2**-*period* interval

**13.** $y = 1 - 2\cot 2\left(x + \frac{\pi}{2}\right)$

**14.** $y = \frac{2}{3}\tan\left(\frac{3}{4}x - \pi\right) - 2$

**15.** $y = -2 - \cot\left(x - \frac{\pi}{4}\right)$

Graph over a one-period interval

**16.** $y = 1 - \frac{1}{2}\csc\left(x - \frac{3\pi}{4}\right)$

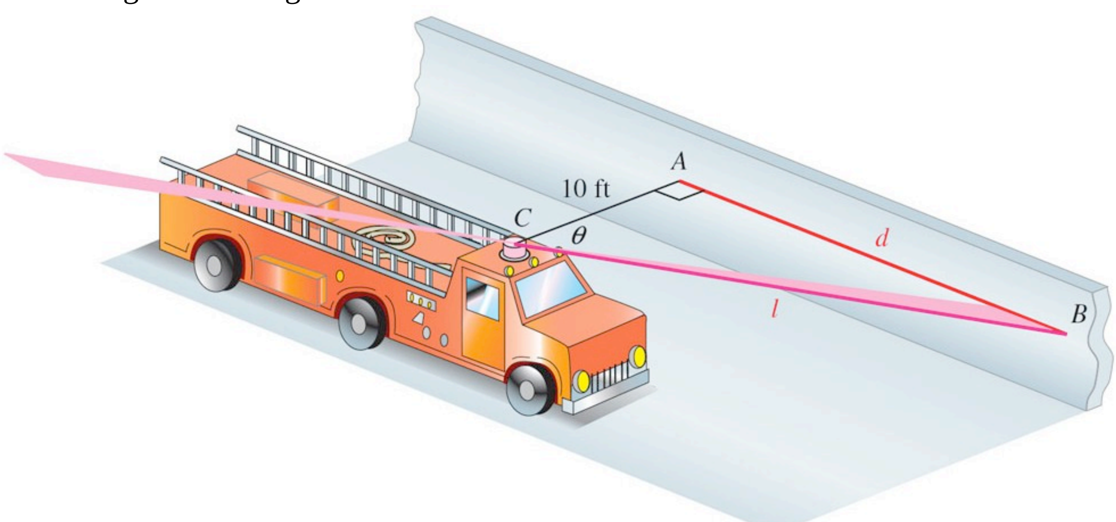
**18.** $y = 3 + 2\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)$

**19.** $y = -1 - 3\csc\left(\frac{\pi x}{2} + \frac{3\pi}{4}\right)$

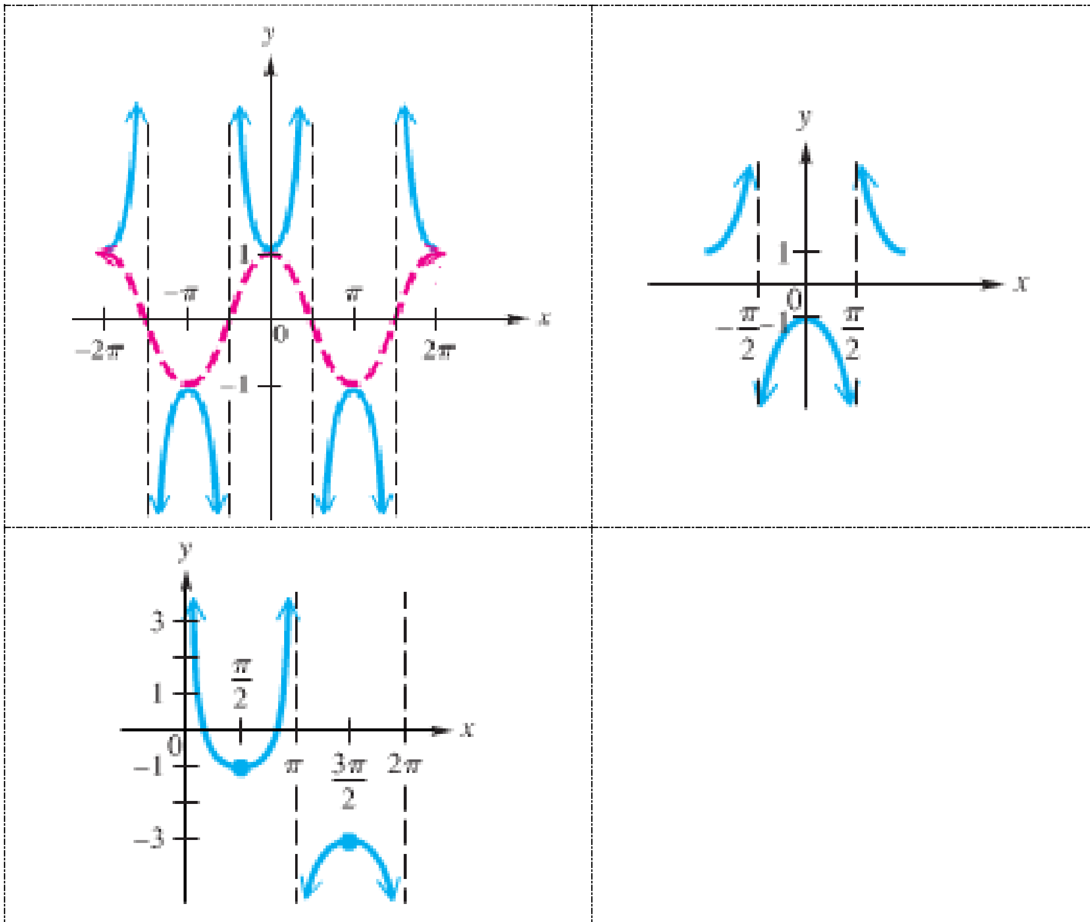
**17.** $y = 2 + \frac{1}{4}\sec\left(\frac{1}{2}x - \pi\right)$

**20.** Graph $y = \frac{1}{3}\sec 2x$ *for* $-\frac{3\pi}{2} \le x \le \frac{3\pi}{2}$

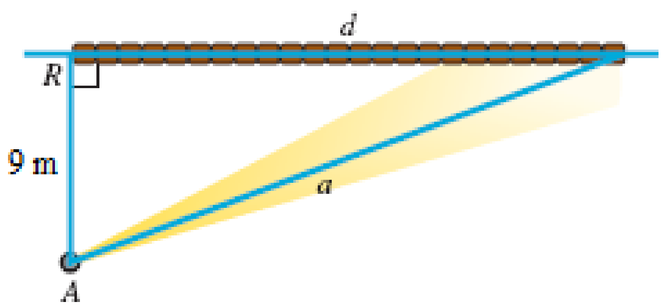
**21.** A fire truck parked on the shoulder of a freeway next to a long block wall. The red light on the top is 10 feet from the wall and rotates through one complete revolution every 2 seconds. Graph the function that gives the length ***d*** in terms of time ***t*** from ***t*** = 0 to ***t*** = 2.



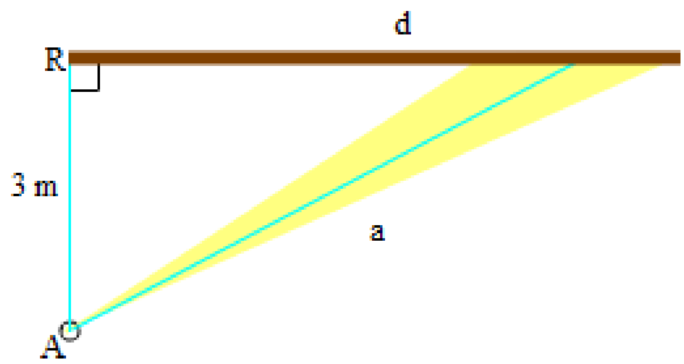
**22.** Find an equation to match the graph



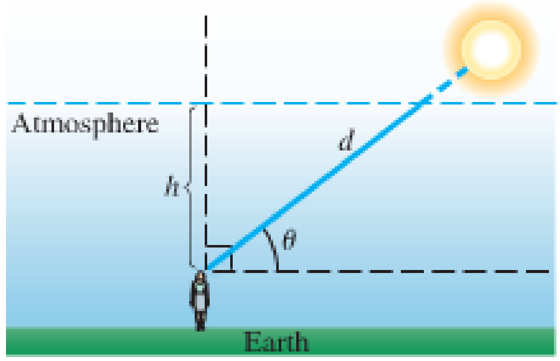

**23.** A rotating beacon is located at point $A$ next to a long wall. The beacon is 9 *m* from the wall. The distance ***a*** is given by $a = 9\left|\sec 2\pi t\right|$, where *t* is time measured in seconds since the beacon started rotating. (When *t* = 0, the beacon is aimed at point *R*.) Find ***a*** for *t* = 0.45




**24.** A rotating beacon is located 3 *m* south of point *R* on an east-west wall. *d*, the length of the light display along the wall from *R*, is given by $d = 3\tan 2\pi t$, where *t* is time measured in seconds since the beacon started rotating. (When *t* = 0, the beacon is aimed at point *R*. When the beacon is aimed to the right of *R*, the value of *d* is positive; *d* is negative if the beacon is aimed to the left of *R*.) Find a for *t* = 0.8




**25.** The shortest path for the sun's rays through Earth's atmosphere occurs when the sun is directly overhead. Disregarding the curvature of Earth, as the sun moves lower on the horizon, the distance that sunlight passes through the atmosphere increases by a factor of $\csc\theta$, where $\theta$ is the angle of elevation of the sun. This increased distance reduces both the intensity of the sun and the amount of ultraviolet light that reached Earth's surface.

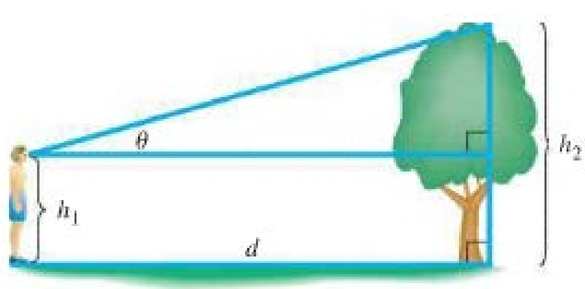



*a)* Verify that $d = h\csc\theta$

*b)* Determine $\theta$ when $d = 2h$

*c)* The atmosphere filters out the ultraviolet light that causes skin to burn, Compare the difference between sunbathing when $\theta = \frac{\pi}{2}$ and when $\theta = \frac{\pi}{3}$. Which measure gives less ultraviolet light?

**26.** Let a person whose eyes are $h_1$ feet from the ground stand ***d*** feet from an object $h_1$ feet tall, where $h_2 > h_1$ feet. Let θ be the angle of elevation to the top of the object.




*a)* Show that $d = \left(h_2 - h_1\right)\cot\theta$

*b)* Let $h_2 = 55$ and $h_1 = 5$. Graph ***d*** for the interval $0 < \theta \le \frac{\pi}{2}$