Solution Section 3.4 – Triple Integrals

Exercise

Evaluate the integral $\int_0^1\int_0^1\int_0^1\left(x^2+y^2+z^2\right)dzdydx$

Solution

$$\begin{aligned}
\int_0^1\int_0^1\int_0^1\left(x^2+y^2+z^2\right)dzdydx &= \int_0^1\int_0^1 \left(x^2z+y^2z+\frac{1}{3}z^3 \ \Big|_0^1\right) dydx \\
&= \int_0^1\int_0^1 \left(x^2+y^2+\frac{1}{3}\right) dydx \\
&= \int_0^1 \left(x^2y+\frac{1}{3}y^3+\frac{1}{3}y \ \Big|_0^1\right) dx \\
&= \int_0^1 \left(x^2+\frac{1}{3}+\frac{1}{3}\right) dx \\
&= \frac{1}{3}x^3+\frac{2}{3}x \ \Big|_0^1 \\
&= \frac{1}{3}+\frac{2}{3} \\
&= 1
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dzdxdy$

Solution

$$\begin{aligned}
\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dzdxdy &= \int_0^{\sqrt{2}}\int_0^{3y} \left[8-x^2-y^2-\left(x^2+3y^2\right)\right] dxdy \\
&= \int_0^{\sqrt{2}}\int_0^{3y}\left(8-2x^2-4y^2\right) dxdy \\
&= \int_0^{\sqrt{2}}\left(8x-\frac{2}{3}x^3-4y^2x \ \Big|_0^{3y}\right) dy
\end{aligned}$$

$$= \int_0^{\sqrt{2}} \left(24y - 18y^3 - 12y^3\right) dy$$

$$= \int_0^{\sqrt{2}} \left(24y - 30y^3\right) dy$$

$$= 12y^2 - \frac{15}{2}y^4 \Big|_0^{\sqrt{2}}$$

$$= 24 - 30$$

$$\underline{= -6}$$

Exercise

Evaluate the integral $\int_0^{\pi/6} \int_0^1 \int_{-2}^3 y \sin z \; dxdydz$

Solution

$$\int_0^{\pi/6} \int_0^1 \int_{-2}^3 y \sin z \; dxdydz = \int_0^{\pi/6} \int_0^1 y \sin z \; \left(x \Big|_{-2}^3 \right. dydz$$

$$= 5\int_0^{\pi/6} \int_0^1 y \sin z \; dydz$$

$$= 5\int_0^{\pi/6} \sin z \; \left(\frac{1}{2}y^2 \Big|_0^1 \right. dz$$

$$= \frac{5}{2}\int_0^{\pi/6} \sin z \; dz$$

$$= -\frac{5}{2} \cos z \Big|_0^{\pi/6}$$

$$= -\frac{5}{2}\left(\frac{\sqrt{3}}{2} - 1\right)$$

$$\underline{= \frac{5}{4}\left(2 - \sqrt{3}\right)}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{1}\int_{0}^{1}\int_{0}^{2}(x+y+z)\,dydxdz$

Solution

$$\int_{-1}^{1}\int_{0}^{1}\int_{0}^{2}(x+y+z)\,dydxdz=\int_{-1}^{1}\int_{0}^{1}\left(xy+\frac{1}{2}y^2+zy\ \Big|_0^2\right)dxdz$$

$$=\int_{-1}^{1}\int_{0}^{1}(2x+2+2z)\,dxdz$$

$$=\int_{-1}^{1}\left(x^2+(2+2z)x\ \Big|_0^1\right)dz$$

$$=\int_{-1}^{1}(1+2+2z)\,dz$$

$$=\int_{-1}^{1}(3+2z)\,dz$$

$$=3z+z^2\ \Big|_{-1}^{1}$$

$$=(3+1)-(-3+1)$$

$$=6$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2}}dzdydx$

Solution

$$\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2}}dzdydx=\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}}\sqrt{9-x^2}\,dydx$$

$$=\int_{0}^{3}\sqrt{9-x^2}\left(y\ \Big|_0^{\sqrt{9-x^2}}\right)dx$$

$$=\int_{0}^{3}\left(9-x^2\right)dx$$

$$= 9x - \frac{1}{3}x^3 \Big|_0^3$$

$$\underline{= 18}$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} xdzdydx$

Solution

$$\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} xdzdydx = \int_0^1 \int_0^{1-x^2} x\,z \Big|_3^{4-x^2-y} dydx$$

$$= \int_0^1 \int_0^{1-x^2} x\left(4 - x^2 - y - 3\right)dydx$$

$$= \int_0^1 \int_0^{1-x^2} \left(x - x^3 - xy\right)dydx$$

$$= \int_0^1 \left(\left(x - x^3\right)y - \frac{1}{2}xy^2 \Big|_0^{1-x^2} dx\right.$$

$$= \int_0^1 \left[x\left(1-x^2\right)\left(1-x^2\right) - \frac{1}{2}x\left(1-x^2\right)^2\right]dx$$

$$= \int_0^1 \left(1-x^2\right)^2\left(\frac{1}{2}x\right)dx \qquad d\left(1-x^2\right) = -2xdx$$

$$= -\frac{1}{4}\int_0^1 \left(1-x^2\right)^2 d\left(1-x^2\right)$$

$$= -\frac{1}{12}\left(1-x^2\right)^3 \Big|_0^1$$

$$= -\frac{1}{12}(0-1)$$

$$\underline{= \frac{1}{12}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^\pi \int_0^\pi \int_0^\pi \cos(u+v+w)\,dudvdw$

Solution

$$\begin{aligned}
\int_0^\pi \int_0^\pi \int_0^\pi \cos(u+v+w)\,dudvdw &= \int_0^\pi \int_0^\pi \sin(u+v+w)\Big|_0^\pi \,dvdw \\
&= \int_0^\pi \int_0^\pi \left[\sin(v+w+\pi)-\sin(v+w)\right]dvdw \\
&= \int_0^\pi \left[-\cos(v+w+\pi)+\cos(v+w)\right]\Big|_0^\pi \,dw \\
&= \int_0^\pi \left[-\cos(w+2\pi)+\cos(w+\pi)+\cos(w+\pi)-\cos(w)\right]dw \\
&= \int_0^\pi \left[-\cos(w+2\pi)+2\cos(w+\pi)-\cos(w)\right]dw \\
&= -\sin(w+2\pi)+2\sin(w+\pi)-\sin(w)\Big|_0^\pi \\
&= -\sin(3\pi)+2\sin(2\pi)-\sin\pi-\left(-\sin(2\pi)+2\sin(\pi)-\sin 0\right) \\
&= 0
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/4} \int_0^{\ln\sec v} \int_{-\infty}^{2t} e^x\,dxdtdv$

Solution

$$\begin{aligned}
\int_0^{\pi/4} \int_0^{\ln\sec v} \int_{-\infty}^{2t} e^x\,dxdtdv &= \int_0^{\pi/4} \int_0^{\ln\sec v} \left(e^x \Big|_{-\infty}^{2t}\right. dtdv \\
&= \int_0^{\pi/4} \int_0^{\ln\sec v} \left(e^{2t}-e^{-\infty}\right) dtdv \\
&= \int_0^{\pi/4} \int_0^{\ln\sec v} \left(e^{2t}-e^{-\infty}\right) dtdv \qquad e^{-\infty}=0
\end{aligned}$$

$$= \int_0^{\pi/4} \int_0^{\ln \sec v} e^{2t}\, dt dv$$

$$= \frac{1}{2} \int_0^{\pi/4} e^{2t} \Big|_0^{\ln \sec v} dv$$

$$= \frac{1}{2} \int_0^{\pi/4} \left(e^{2 \ln \sec v} - 1 \right) dv \qquad e^{2 \ln \sec v} = e^{\ln \sec^2 v} = \sec^2 v$$

$$= \frac{1}{2} \int_0^{\pi/4} \left(\sec^2 v - 1 \right) dv$$

$$= \frac{1}{2} \left(\tan v - v \right. \Big|_0^{\pi/4}$$

$$= \frac{1}{2} \left(1 - \frac{\pi}{4} \right)$$

$$= \frac{1}{2} - \frac{\pi}{8}$$

Exercise

Evaluate the integral $\displaystyle \int_0^1 \int_{-z}^{z} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dydxdz$

Solution

$$\int_0^1 \int_{-z}^{z} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dydxdz = \int_0^1 \int_{-z}^{z} y \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dxdz$$

$$= 2 \int_0^1 \int_{-z}^{z} \sqrt{1-x^2}\, dxdz \qquad \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$$

$$= 2 \int_0^1 \left(\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x \right. \Big|_{-z}^{z} dz$$

$$= 2 \int_0^1 \left(\frac{z}{2}\sqrt{1-z^2} + \frac{1}{2} \sin^{-1} z + \frac{z}{2}\sqrt{1-z^2} + \frac{1}{2} \sin^{-1} z \right) dz$$

$$= 2 \int_0^1 \left(z\sqrt{1-z^2} + \sin^{-1} z \right) dz \qquad \int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{1-x^2}$$

$$= -\int_0^1 \left(1-z^2\right)^{1/2} d\left(1-z^2\right) + 2\int_0^1 \left(\sin^{-1} z\right) dz$$

$$= -\frac{2}{3}\left(1-z^2\right)^{3/2} + 2\left(z\sin^{-1} z + \sqrt{1-z^2}\right) \Big|_0^1$$

$$= 2\sin^{-1} 1 + \frac{2}{3} - 2$$

$$\underline{= \pi - \frac{4}{3}}$$

Exercise

Evaluate the integral $\int_0^{\pi}\int_0^{y}\int_0^{\sin x} dzdxdy$

Solution

$$\int_0^{\pi}\int_0^{y}\int_0^{\sin x} dzdxdy = \int_0^{\pi}\int_0^{y} z \Big|_0^{\sin x} dxdy$$

$$= \int_0^{\pi}\int_0^{y} \sin x \, dxdy$$

$$= -\int_0^{\pi} \cos x \Big|_0^{y} dy$$

$$= -\int_0^{\pi} (\cos y - 1) dy$$

$$= -(\sin y - y) \Big|_0^{\pi}$$

$$\underline{= \pi}$$

Exercise

Evaluate the integral $\int_0^{9}\int_0^{1}\int_{2y}^{2} \frac{4\sin x^2}{\sqrt{z}} dxdydz$

Solution

$$\begin{cases} 2y \le x \le y & \rightarrow \quad 0 \le x \le 2 \\ 0 \le y \le 1 & \rightarrow \quad 0 \le y \le \frac{x}{2} \end{cases}$$

$$\int_0^9 \int_0^1 \int_{2y}^2 \frac{4\sin x^2}{\sqrt{z}} dxdydz = \int_0^9 z^{-1/2} dz \int_0^2 \int_0^{x/2} 4\sin\left(x^2\right) dydx$$

$$= 8z^{1/2} \Big|_0^9 \int_0^2 \sin x^2 \left(y \Big|_0^{x/2} \right. dx$$

$$= 4(3) \int_0^2 x\sin x^2 \; dx$$

$$= 6\int_0^2 \sin x^2 \; d\left(x^2\right)$$

$$= -6\cos x^2 \Big|_0^2$$

$$= -6(\cos 4 - 1)$$

$$\underline{= 6 - 6\cos 4}$$

Exercise

Evaluate the integral $\displaystyle\int_0^\pi \int_0^\pi \int_0^\pi \cos(x+y+z)\,dxdydz$

Solution

$$\int_0^\pi \int_0^\pi \int_0^\pi \cos(x+y+z)\,dxdydz = \int_0^\pi \int_0^\pi \sin(x+y+z) \Big|_0^\pi dydz$$

$$= \int_0^\pi \int_0^\pi \left(\sin(\pi+y+z) - \sin(y+z)\right) dydz$$

$$= \int_0^\pi \left(-\cos(2\pi+z) + \cos(\pi+z) + \cos(\pi+z) - \cos(z)\right) dz$$

$$\cos(2\pi+z) = \cos z \qquad \cos(\pi+z) = -\cos z$$

$$= -4\int_0^\pi \cos z \; dz$$

$$= -4\sin z \Big|_0^\pi$$

$$\underline{= 0}$$

Exercise

Evaluate the integral $\displaystyle\int_1^e \int_1^x \int_0^z \frac{2y}{z^3}\,dydzdx$

Solution

$$\int_1^e \int_1^x \int_0^z \frac{2y}{z^3}\,dydzdx = \int_1^e \int_1^x \frac{1}{z^3}y^2 \Big|_0^z \,dzdx$$

$$= \int_1^e \int_1^x \frac{1}{z}\,dzdx$$

$$= \int_1^e \ln z \Big|_1^x \,dx$$

$$= \int_1^e \ln x\,dx \qquad u = \ln x \to du = \frac{dx}{x} \quad v = \int dx = x$$

$$\int \ln x\,dx = x\ln x - \int dx$$

$$= x\ln x - x \Big|_1^e$$

$$= e - e + 1$$

$$= 1$$

Exercise

Evaluate the integral $\displaystyle\int_{\ln 6}^{\ln 7} \int_0^{\ln 2} \int_{\ln 4}^{\ln 5} e^{(x+y+z)}\,dzdydx$

Solution

$$\int_{\ln 6}^{\ln 7} \int_0^{\ln 2} \int_{\ln 4}^{\ln 5} e^{(x+y+z)}\,dzdydx = \int_{\ln 6}^{\ln 7} \int_0^{\ln 2} \int_{\ln 4}^{\ln 5} e^x\,e^y\,e^z\,dzdydx$$

$$= \int_{\ln 6}^{\ln 7} e^x dx \int_0^{\ln 2} e^y dy \int_{\ln 4}^{\ln 5} e^z dz$$

$$= e^x \Big|_{\ln 6}^{\ln 7} \; e^y \Big|_0^{\ln 2} \; e^z \Big|_{\ln 4}^{\ln 5}$$

$$= (7-6)(2-1)(5-4) \qquad e^{\ln u} = u$$

$$= 1$$

Exercise

Evaluate the integral $\displaystyle\int_0^1\int_0^{x^2}\int_0^{x+y}(2x-y-z)\,dzdydx$

Solution

$$\begin{aligned}
\int_0^1\int_0^{x^2}\int_0^{x+y}(2x-y-z)\,dzdydx &= \int_0^1\int_0^{x^2}\left((2x-y)z-\frac{1}{2}z^2\ \Big|_0^{x+y}\right)dydx \\
&= \int_0^1\int_0^{x^2}\left((2x-y)(x+y)-\frac{1}{2}(x+y)^2\right)dydx \\
&= \int_0^1\int_0^{x^2}\left(2x^2+xy-y^2-\frac{1}{2}x^2-xy-\frac{1}{2}y^2\right)dydx \\
&= \int_0^1\int_0^{x^2}\left(\frac{3}{2}x^2-\frac{3}{2}y^2\right)dydx \\
&= \int_0^1\left(\frac{3}{2}x^2y-\frac{1}{2}y^3\ \Big|_0^{x^2}\right)dx \\
&= \int_0^1\left(\frac{3}{2}x^4-\frac{1}{2}x^6\right)dx \\
&= \frac{3}{10}x^5-\frac{1}{14}x^7\ \Big|_0^1 \\
&= \frac{3}{10}-\frac{1}{14} \\
&= \frac{32}{140} \\
&= \frac{8}{35}
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int_{-2}^2\int_3^6\int_0^2 dxdydz$

Solution

$$\int_{-2}^2\int_3^6\int_0^2 dxdydz = \int_{-2}^2 dz\int_3^6 dy\int_0^2 dx$$

$$= z\Big|_{-2}^{2}\ \ y\Big|_{3}^{6}\ \ x\Big|_{0}^{2}$$

$$= (2+2)(6-3)(2-0)$$

$$\underline{= 24}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{1}\int_{-1}^{2}\int_{0}^{1} 6xyz\ dydxdz$

Solution

$$\int_{-1}^{1}\int_{-1}^{2}\int_{0}^{1} 6xyz\ dydxdz = 6\int_{-1}^{1} z\ dz\ \int_{-1}^{2} x\ dx\ \int_{0}^{1} y\ dy$$

$$= 6\left(\frac{1}{2}z^2\right)\Big|_{-1}^{1}\ \ \frac{1}{2}x^2\Big|_{-1}^{2}\ \ \frac{1}{2}y^2\Big|_{0}^{1}$$

$$= \frac{3}{4}(1-1)(4-1)(1-0)$$

$$\underline{= 0}$$

Exercise

Evaluate the integral $\displaystyle\int_{-2}^{2}\int_{1}^{2}\int_{1}^{e} \frac{xy^2}{z} dzdxdy$

Solution

$$\int_{-2}^{2}\int_{1}^{2}\int_{1}^{e} \frac{xy^2}{z} dzdxdy = \int_{-2}^{2} y^2 dy\ \int_{1}^{2} xdx\ \int_{1}^{e} \frac{dz}{z}$$

$$= \frac{1}{3}y^3\Big|_{-2}^{2}\ \ \frac{1}{2}x^2\Big|_{1}^{2}\ \ \ln z\Big|_{1}^{e}$$

$$= \frac{1}{6}(8+8)(4-1)(1-0)$$

$$\underline{= 8}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\ln 4}\int_0^{\ln 3}\int_0^{\ln 2} e^{-x+y+z}\, dxdydz$

Solution

$$\int_0^{\ln 4}\int_0^{\ln 3}\int_0^{\ln 2} e^{-x+y+z} dxdydz = \int_0^{\ln 4} e^z dz \int_0^{\ln 3} e^y dy \int_0^{\ln 2} e^{-x} dx$$

$$= e^z \Big|_0^{\ln 4} \quad e^y \Big|_0^{\ln 3} \quad \left(-e^{-x} \Big|_0^{\ln 2}\right.$$

$$= -\left(e^{\ln 4} - e^0\right)\left(e^{\ln 3} - e^0\right)\left(e^{\ln 2} - e^0\right)$$

$$= -(4-1)(3-1)\left(\frac{1}{2}-1\right)$$

$$= 3$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^1\int_0^{\frac{\pi}{2}} \sin \pi x \cos y \sin 2z \; dydxdz$

Solution

$$\int_0^{\frac{\pi}{2}}\int_0^1\int_0^{\frac{\pi}{2}} \sin \pi x \cos y \sin 2z \; dydxdz = \int_0^{\frac{\pi}{2}} \sin 2z \; dz \int_0^1 \sin \pi x \;\; cdx \int_0^{\frac{\pi}{2}} \cos y \; dy$$

$$= -\frac{1}{2}\cos 2z \Big|_0^{\frac{\pi}{2}} \quad \left(-\frac{1}{\pi}\cos \pi x \Big|_0^1\right. \quad \sin y \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2\pi}(-1-1)(-1-1)(1-0)$$

$$= \frac{2}{\pi}$$

Exercise

Evaluate the integral $\displaystyle\int_0^2\int_1^2\int_0^1 yze^x dxdzdy$

Solution

$$\int_0^2\int_1^2\int_0^1 yze^x dxdzdy = \int_0^2 ydy \int_1^2 zdz \int_0^1 e^x dx$$

$$= \frac{1}{2} y^2 \Big|_0^2 \quad \frac{1}{2} z^2 \Big|_1^2 \quad e^x \Big|_0^1$$

$$= \frac{1}{4}(4)(4-1)(e-1)$$

$$\boxed{= 3(e-1)}$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dzdydx$

Solution

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dzdydx = \int_0^1 \int_0^{\sqrt{1-x^2}} z \Big|_0^{\sqrt{1-x^2}} dydx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\, dydx$$

$$= \int_0^1 \sqrt{1-x^2}\; y \Big|_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \left(1-x^2\right) dx$$

$$= x - \frac{1}{3} x^3 \Big|_0^1$$

$$\boxed{= \frac{2}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 2xz\; dzdydx$

Solution

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 2xz\; dzdydx = \int_0^1 \int_0^{\sqrt{1-x^2}} x z^2 \Big|_0^{\sqrt{1-x^2-y^2}} dydx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} x\left(1-x^2-y^2\right) dydx$$

$$= \int_0^1 \left(xy - x^3y - \frac{1}{3}xy^3 \right|_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \left(x\left(1-x^2\right)^{1/2} - x^3\left(1-x^2\right)^{1/2} - \frac{1}{3}x\left(1-x^2\right)^{3/2}\right) dx$$

Switching dydx to dxdy

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} \left(x - x^3 - xy^2\right) dxdy$$

$$= \int_0^1 \left(\frac{1}{2}x^2 - \frac{1}{4}x^4 - \frac{1}{2}x^2y^2 \right|_0^{\sqrt{1-y^2}} dy$$

$$= \frac{1}{4}\int_0^1 \left(2\left(1-y^2\right) - \left(1-y^2\right)^2 - 2\left(1-y^2\right)y^2\right) dy$$

$$= \frac{1}{4}\int_0^1 \left(2 - 2y^2 - 1 + 2y^2 - y^4 - 2y^2 + 2y^4\right) dy$$

$$= \frac{1}{4}\int_0^1 \left(y^4 - 2y^2 + 1\right) dy$$

$$= \frac{1}{4}\left(\frac{1}{5}y^5 - \frac{2}{3}y^3 + y \right|_0^1$$

$$= \frac{1}{4}\left(\frac{1}{5} - \frac{2}{3} + 1\right)$$

$$= \frac{1}{4}\left(\frac{8}{15}\right)$$

$$= \frac{2}{15}$$

Exercise

Evaluate the integral $\displaystyle\int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} \int_0^{16-\frac{1}{4}x^2-y^2} dzdxdy$

Solution

$$\int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} \int_0^{16-\frac{1}{4}x^2-y^2} dzdxdy = \int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} z \Big|_0^{16-\frac{1}{4}x^2-y^2} dxdy$$

$$= \int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} \left(16-\frac{1}{4}x^2-y^2\right)dxdy$$

$$= \int_0^4 \left(16x-\frac{1}{12}x^3-xy^2\right)\Big|_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} dy$$

$$= 2\int_0^4 \left(16x-\frac{1}{12}x^3-xy^2\right)\Big|_0^{2\sqrt{16-y^2}} dy$$

$$= 2\int_0^4 \left(32\sqrt{16-y^2}-\frac{2}{3}\left(16-y^2\right)^{3/2}-2y^2\sqrt{16-y^2}\right)dy$$

$y = 4\sin\theta \quad \rightarrow \quad dy = 4\cos\theta d\theta \qquad \sqrt{16-y^2} = 4\cos\theta$

$$\int \sqrt{16-y^2}\ dy = 16\int \cos^2\theta\ d\theta$$

$$= 8\int (1+\cos 2\theta)\ d\theta$$

$$= 8\left(\theta+\frac{1}{2}\sin 2\theta\right)$$

$$\boxed{= 8\sin^{-1}\left(\frac{y}{4}\right)+\frac{1}{2}y\sqrt{16-y^2}}$$

$$\int \left(16-y^2\right)^{3/2} dy = \int \left(16\cos^2\theta\right)^{3/2}\ 4\cos\theta\ d\theta$$

$$= 256\int \cos^4\theta\ d\theta$$

$$= 64\int (1+\cos 2\theta)^2\ d\theta$$

$$= 64\int\left(1+2\cos 2\theta+\cos^2 2\theta\right)\,d\theta$$

$$= 64\int\left(\frac{3}{2}+2\cos 2\theta+\frac{1}{2}\cos 4\theta\right)\,d\theta$$

$$= 64\left(\frac{3}{2}\theta+\sin 2\theta+\frac{1}{8}\sin 4\theta\right)$$

$$= 64\left(\frac{3}{2}\theta+2\sin\theta\cos\theta+\frac{1}{4}\sin 2\theta\cos 2\theta\right)$$

$$= 64\left(\frac{3}{2}\theta+2\sin\theta\cos\theta+\frac{1}{2}\sin\theta\cos\theta\left(1-2\sin^2\theta\right)\right)$$

$$= 64\left(\frac{3}{2}\sin^{-1}\left(\frac{y}{4}\right)+2\frac{y}{4}\frac{\sqrt{16-y^2}}{4}+\frac{1}{2}\frac{y}{4}\frac{\sqrt{16-y^2}}{4}\left(1-\frac{y^2}{8}\right)\right)$$

$$= 64\left(\frac{3}{2}\sin^{-1}\left(\frac{y}{4}\right)+\frac{1}{8}y\sqrt{16-y^2}+\frac{1}{256}y\left(8-y^2\right)\sqrt{16-y^2}\right)$$

$$= 64\left(\frac{3}{2}\sin^{-1}\left(\frac{y}{4}\right)+\frac{1}{8}y\sqrt{16-y^2}+\frac{1}{32}y\sqrt{16-y^2}-\frac{1}{256}y^3\sqrt{16-y^2}\right)$$

$$\boxed{=\left(96\sin^{-1}\left(\frac{y}{4}\right)+10y\sqrt{16-y^2}-\frac{1}{4}y^3\sqrt{16-y^2}\right)}$$

$$\int y^2\sqrt{16-y^2}\,dy = 4^4\int\sin^2\theta\cos^2\theta\,d\theta$$

$$= 64\int\left(1-\cos 2\theta\right)\left(1+\cos 2\theta\right)\,d\theta$$

$$= 64\int\left(1-\cos^2 2\theta\right)\,d\theta$$

$$= 64\int\left(\frac{1}{2}-\frac{1}{2}\cos 4\theta\right)\,d\theta$$

$$= 32\left(\theta-\frac{1}{4}\sin 4\theta\right)$$

$$= 32\left(\theta-\frac{1}{2}\sin 2\theta\cos 2\theta\right)$$

$$= 32\left(\theta-\sin\theta\cos\theta\left(1-2\sin^2\theta\right)\right)$$

$$= 32\left(\sin^{-1}\left(\frac{y}{4}\right)-\frac{y}{4}\frac{\sqrt{16-y^2}}{4}\left(1-\frac{y^2}{8}\right)\right)$$

$$= 32\left(\sin^{-1}\left(\frac{y}{4}\right)-\frac{1}{128}y\left(8-y^2\right)\sqrt{16-y^2}\right)$$

$$= 32\sin^{-1}\left(\frac{y}{4}\right) - 2y\sqrt{16-y^2} + \frac{1}{4}y^3\sqrt{16-y^2}$$

$$\int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} \int_0^{16-\frac{1}{4}x^2-y^2} dzdxdy = \int_0^4 \left(64\sqrt{16-y^2} - \frac{4}{3}\left(16-y^2\right)^{3/2} - 4y^2\sqrt{16-y^2}\right)dy$$

$$= 64\left(8\sin^{-1}\left(\frac{y}{4}\right) + \frac{1}{2}y\sqrt{16-y^2}\right)$$

$$- \frac{4}{3}\left(96\sin^{-1}\left(\frac{y}{4}\right) + 10y\sqrt{16-y^2} - \frac{1}{4}y^3\sqrt{16-y^2}\right)$$

$$- 4\left(32\sin^{-1}\left(\frac{y}{4}\right) - 2y\sqrt{16-y^2} + \frac{1}{4}y^3\sqrt{16-y^2}\right)$$

$$= 512\sin^{-1}\left(\frac{y}{4}\right) + 32y\sqrt{16-y^2}$$

$$- 128\sin^{-1}\left(\frac{y}{4}\right) - \frac{40}{3}y\sqrt{16-y^2} + \frac{1}{3}y^3\sqrt{16-y^2}$$

$$- 128\sin^{-1}\left(\frac{y}{4}\right) + 8y\sqrt{16-y^2} - y^3\sqrt{16-y^2}$$

$$= 256\sin^{-1}\left(\frac{y}{4}\right) + \frac{80}{3}y\sqrt{16-y^2} - \frac{2}{3}y^3\sqrt{16-y^2}\ \Big|_0^4$$

$$= 256\left(\frac{\pi}{2}\right)$$

$$= 128\pi$$

Exercise

Evaluate the integral $\displaystyle\int_1^6 \int_0^{4-\frac{2}{3}y} \int_0^{12-2y-3z} \frac{1}{y}\, dxdzdy$

Solution

$$\int_1^6 \int_0^{4-\frac{2}{3}y} \int_0^{12-2y-3z} \frac{1}{y}\, dxdzdy = \int_1^6 \int_0^{4-\frac{2}{3}y} \frac{1}{y}x\ \Big|_0^{12-2y-3z} dzdy$$

$$= \int_1^6 \int_0^{4-\frac{2}{3}y} \frac{1}{y}(12-2y-3z)\,dzdy$$

$$= \int_1^6 \left(12\frac{1}{y}z - 2z - \frac{3}{2}\frac{1}{y}z^2\ \Big|_0^{4-\frac{2}{3}y}\right) dy$$

$$= \int_1^6 \left(12\frac{1}{y}\left(4-\frac{2}{3}y\right) - 2\left(4-\frac{2}{3}y\right) - \frac{3}{2}\frac{1}{y}\left(4-\frac{2}{3}y\right)^2 \right) dy$$

$$= \int_1^6 \left(\frac{48}{y} - 16 + \frac{4}{3}y - \frac{3}{2}\frac{1}{y}\left(16 - \frac{16}{3}y + \frac{4}{9}y^2\right) \right) dy$$

$$= \int_1^6 \left(\frac{48}{y} - 16 + \frac{4}{3}y - \frac{24}{y} + 8 - \frac{2}{3}y \right) dy$$

$$= \int_1^6 \left(\frac{24}{y} - 8 + \frac{2}{3}y \right) dy$$

$$= 24\ln y - 8y + \frac{1}{3}y^2 \Big|_1^6$$

$$= 24\ln 6 - 48 + 12 + 8 - \frac{1}{3}$$

$$= 24\ln 6 - \frac{85}{3}$$

Exercise

Evaluate the integral $\displaystyle\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\sqrt{1+x^2+z^2}} dydxdz$

Solution

$$\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\sqrt{1+x^2+z^2}} dydxdz = \int_0^3 \int_0^{\sqrt{9-z^2}} y \Big|_0^{\sqrt{1+x^2+z^2}} dxdz$$

$$= \int_0^3 \int_0^{\sqrt{9-z^2}} \sqrt{1+x^2+z^2}\ dxdz \qquad \text{Let } x^2+z^2=r^2$$

$$= \int_0^{\frac{\pi}{2}} \int_0^3 \sqrt{1+r^2}\ rdrd\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \int_0^3 \left(1+r^2\right)^{1/2} d\left(1+r^2\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right)\frac{2}{3}\left(1+r^2\right)^{3/2} \Big|_0^3$$

$$= \frac{\pi}{6}\left(10\sqrt{10}-1\right)$$

Exercise

Evaluate the integral $\int_0^{\pi}\int_0^{\pi}\int_0^{\sin x} \sin y \; dzdxdy$

Solution

$$\int_0^{\pi}\int_0^{\pi}\int_0^{\sin x} \sin y \; dzdxdy = \int_0^{\pi}\int_0^{\pi} (\sin y)\, z \Big|_0^{\sin x} dxdy$$

$$= \int_0^{\pi}\int_0^{\pi} (\sin y \sin x)\, dxdy$$

$$= -\int_0^{\pi} \sin y \cos x \Big|_0^{\pi} dy$$

$$= 2\int_0^{\pi} \sin y \; dy$$

$$= -2\cos y \Big|_0^{\pi}$$

$$\underline{= 4}$$

Exercise

Evaluate the integral $\int_0^{\ln 8}\int_1^{\sqrt{z}}\int_{\ln y}^{\ln 2y} e^{x+y^2-z} dxdydz$

Solution

$$\int_0^{\ln 8}\int_1^{\sqrt{z}}\int_{\ln y}^{\ln 2y} e^{x+y^2-z} \; dxdydz = \int_0^{\ln 8}\int_1^{\sqrt{z}} e^{y^2} e^{-z} \; e^{x} \Big|_{\ln y}^{\ln 2y} dydz$$

$$= \int_0^{\ln 8}\int_1^{\sqrt{z}} e^{y^2} e^{-z} (y) \; dydz$$

$$= \frac{1}{2}\int_0^{\ln 8}\int_1^{\sqrt{z}} e^{-z} e^{y^2} d\left(y^2\right) dz$$

$$= \frac{1}{2}\int_0^{\ln 8} e^{-z} \; e^{y^2} \Big|_1^{\sqrt{z}} dz$$

$$= \frac{1}{2}\int_0^{\ln 8} e^{-z}\left(e^{z} - e\right) dz$$

$$=\frac{1}{2}\int_0^{\ln 8}\left(1-e^{1-z}\right)dz$$

$$=\frac{1}{2}\left(z+e^{1-z}\ \Big|_0^{\ln 8}\right.$$

$$=\frac{1}{2}\left(\ln 8+e^{1-\ln 8}-e\right)$$

$$=\frac{1}{2}\left(\ln 8+e\left(e^{\ln 8^{-1}}\right)-e\right)$$

$$=\frac{1}{2}\left(\ln 8+\frac{1}{8}e-e\right)$$

$$=\frac{1}{2}\ln 8-\frac{7}{16}e$$

Exercise

Evaluate the integral $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{2-x} 4yz\ dzdydx$

Solution

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{2-x} 4yz\ dzdydx=\int_0^1\int_0^{\sqrt{1-x^2}} 2yz^2\ \Big|_0^{2-x}\ dydx$$

$$=\int_0^1\int_0^{\sqrt{1-x^2}} 2y(2-x)^2\ dydx$$

$$=\int_0^1\left(4-4x+x^2\right)y^2\ \Big|_0^{\sqrt{1-x^2}}\ dx$$

$$=\int_0^1\left(4-4x+x^2\right)\left(1-x^2\right)dx$$

$$=\int_0^1\left(4-4x-3x^2+4x^3-x^4\right)dx$$

$$=4x-2x^2-x^3+x^4-\frac{1}{5}x^5\ \Big|_0^1$$

$$=\frac{9}{5}$$

Exercise

Evaluate the integral $\displaystyle\int_0^2\int_0^4\int_{y^2}^4 \sqrt{x}\ dzdxdy$

Solution

$$\int_0^2\int_0^4\int_{y^2}^4 \sqrt{x}\ dzdxdy = \int_0^2\int_0^4 \sqrt{x}\ z\ \Big|_{y^2}^{4}\ dxdy$$

$$= \int_0^2 \left(4-y^2\right)dy\ \ \int_0^4 x^{1/2}\ dx$$

$$= \left(4y-\frac{1}{3}y^3\ \Big|_0^2\ \ \frac{2}{3}\left(x^{3/2}\ \Big|_0^4\right.\right.$$

$$= \frac{2}{3}\left(8-\frac{8}{3}\right)(8)$$

$$= \frac{256}{9}$$

Exercise

Evaluate the integral $\displaystyle\int_0^1\int_y^{2-y}\int_0^{2-x-y} xy\ dzdxdy$

Solution

$$\int_0^1\int_y^{2-y}\int_0^{2-x-y} xy\ dzdxdy = \int_0^1\int_y^{2-y} xyz\ \Big|_0^{2-x-y}\ dxdy$$

$$= \int_0^1\int_y^{2-y} \left(2xy-x^2y-xy^2\right)dxdy$$

$$= \int_0^1 \left(x^2y-\frac{1}{3}x^3y-\frac{1}{2}x^2y^2\ \Big|_y^{2-y}\ dy\right.$$

$$= \int_0^1 \left((2-y)^2y-\frac{1}{3}(2-y)^3y-\frac{1}{2}(2-y)^2y^2-y^3+\frac{5}{6}y^4\right)dy$$

$$= \int_0^1 \left(\left(4-4y+y^2\right)\left(y-\frac{2}{3}y+\frac{1}{3}y^2-\frac{1}{2}y^2\right)-y^3+\frac{5}{6}y^4\right)dy$$

$$= \int_0^1 \left(\left(4-4y+y^2\right)\left(\frac{1}{3}y-\frac{1}{6}y^2\right)-y^3+\frac{5}{6}y^4\right)dy$$

$$= \int_0^1 \left(\frac{4}{3}y - 2y^2 + y^3 - \frac{1}{6}y^4 - y^3 + \frac{5}{6}y^4\right)dy$$

$$= \int_0^1 \left(\frac{4}{3}y - 2y^2 + \frac{2}{3}y^4\right)dy$$

$$= \frac{2}{3}y^2 - \frac{2}{3}y^3 + \frac{2}{15}y^5 \Big|_0^1$$

$$= \frac{2}{15}$$

Exercise

Here is the region of integration of the integral

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dzdydx$$

a) *dydzdx* ***b***) *dydxdz* ***c***) *dxdydz* ***d***) *dxdzdy* ***e***) *dzdxdy*

Solution

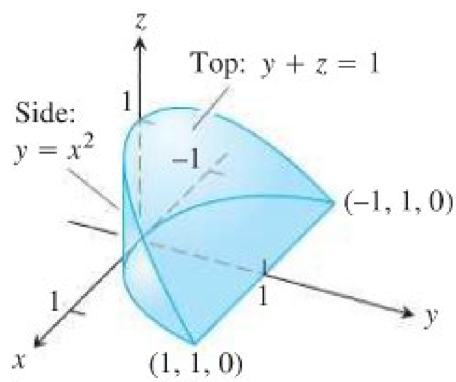
a) $\displaystyle\int_{-1}^{1}\int_{0}^{1-x^2}\int_{x^2}^{1-x} dydzdx$

b) $\displaystyle\int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-x} dydxdz$

c) $\displaystyle\int_{0}^{1}\int_{0}^{1-x}\int_{-\sqrt{y}}^{\sqrt{y}} dxdydz$

d) $\displaystyle\int_{0}^{1}\int_{0}^{1-y}\int_{-\sqrt{y}}^{\sqrt{y}} dxdzdy$

e) $\displaystyle\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}}\int_{0}^{1-y} dzdxdy$



Exercise

Use another order to evaluate $\displaystyle\int_0^5\int_{-1}^0\int_0^{4x+4} dydxdz$

Solution

$$-1\le x\le 0 \quad 0\le y\le 4x+4 \quad 0\le z\le 5$$

$$\begin{cases} x=-1 & \to y=0 \\ x=0 & \to y=4 \end{cases}$$

$$y=4x+4 \quad\to\quad x=\frac{y-4}{4}$$

$$\int_0^5\int_{-1}^0\int_0^{4x+4} dydxdz = \int_0^4\int_{\frac{y-4}{4}}^0\int_0^5 dzdxdy$$

$$= \int_0^4 x\Big|_{\frac{y-4}{4}}^{0} dy \quad z\Big|_0^5$$

$$= \frac{5}{4}\int_0^4 (4-y)dy$$

$$= \frac{5}{4}\left(4y-\frac{1}{2}y^2\right)\Big|_0^4$$

$$\underline{=10}$$

Exercise

Use another order to evaluate $\displaystyle\int_0^1\int_{-2}^2\int_0^{\sqrt{4-y^2}} dzdydx$

Solution

$$0\le x\le 1 \quad -2\le y\le 2 \quad 0\le z\le \sqrt{4-y^2}$$

$$0\le z\le 2$$

$$z=\sqrt{4-y^2} \quad\to\quad y=\pm\sqrt{4-z^2}$$

$$\int_0^1\int_{-2}^2\int_0^{\sqrt{4-y^2}} dzdydx = \int_0^1\int_0^2\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dydzdx$$

$$= \int_0^1 dx\int_0^2 y\Big|_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dz$$

$$= 2\int_0^2 \sqrt{4 - z^2}\, dz$$

$$= 2\left(\frac{z}{2}\sqrt{4 - z^2} + 2\sin^{-1}\frac{z}{2}\right|_0^2$$

$$= 2\left(2\frac{\pi}{2}\right)$$

$$\underline{= 2\pi}$$

Exercise

Use another order to evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dydzdx$

Solution

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dydzdx = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dzdydx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} z \Big|_0^{\sqrt{1-x^2}} dydx$$

$$= \int_0^1 \sqrt{1 - x^2}\ y \Big|_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \left(1 - x^2\right) dx$$

$$= x - \frac{1}{3}x^3 \Big|_0^1$$

$$\underline{= \frac{2}{3}}$$

Exercise

Use another order to evaluate $\displaystyle\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-z^2}} dydzdx$

Solution

$$0 \le x \le 4 \quad 0 \le y \le \sqrt{16 - x^2 - z^2} \quad 0 \le z \le \sqrt{16 - x^2}$$

$$x^2 + y^2 + z^2 = 16$$

$0 \le x \le \sqrt{16-y^2-z^2} \quad 0 \le y \le \sqrt{16-z^2} \quad 0 \le z \le 4$

$$\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-z^2}} dydzdx = \int_0^4 \int_0^{\sqrt{16-z^2}} \int_0^{\sqrt{16-y^2-z^2}} dxdydz$$

$$= \int_0^4 \int_0^{\sqrt{16-z^2}} x \Big|_0^{\sqrt{16-y^2-z^2}} dydz$$

$$= \int_0^4 \int_0^{\sqrt{16-z^2}} \sqrt{16-y^2-z^2}\ dydz \qquad \text{Let } y^2+z^2=r^2$$

$$= \int_0^{\frac{\pi}{2}} \int_0^4 \sqrt{16-r^2}\ r\ drd\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \int_0^4 \left(16-r^2\right)^{1/2} d\left(16-r^2\right)$$

$$= -\frac{1}{2}\left(\frac{\pi}{2}\right)\left(\frac{2}{3}\right)\left(16-r^2\right)^{3/2} \Big|_0^4$$

$$= -\frac{\pi}{6}(-64)$$

$$= \boxed{\frac{32\pi}{3}}$$

Exercise

Use another order to evaluate $\displaystyle\int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin\sqrt{yz}}{x^{3/2}} dydxdz$

Solution

$$\int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin\sqrt{yz}}{x^{3/2}} dydxdz = \int_0^{\pi^2} \int_1^4 \int_z^{4z} x^{-3/2} \sin\sqrt{yz}\ dxdzdy$$

$$= -2\int_0^{\pi^2} \int_1^4 \sin\sqrt{yz}\ \left(x^{-1/2} \Big|_z^{4z} dzdy\right.$$

$$= -2\int_0^{\pi^2} \int_1^4 \sin\sqrt{yz}\left(\frac{1}{2\sqrt{z}} - \frac{1}{\sqrt{z}}\right)dzdy$$

$$= \int_0^{\pi^2} \int_1^4 \frac{\sin\sqrt{yz}}{\sqrt{z}}\, dzdy$$

$$= 2\int_0^{\pi^2} \int_1^4 \frac{1}{\sqrt{y}} \sin\sqrt{yz}\; d\left(\sqrt{yz}\right) dy \qquad d\left(\sqrt{yz}\right) = \frac{1}{2}\frac{y}{\sqrt{yz}}dz$$

$$= -2\int_0^{\pi^2} \frac{1}{\sqrt{y}} \cos\sqrt{yz}\, \Big|_1^4 \, dy$$

$$= -2\int_0^{\pi^2} \frac{1}{\sqrt{y}}\left(\cos\left(2\sqrt{y}\right) - \cos\sqrt{y}\right)dy$$

$$= -4\int_0^{\pi^2} \left(\cos\left(2\sqrt{y}\right) - \cos\sqrt{y}\right) d\left(\sqrt{y}\right)$$

$$= -4\left(\frac{1}{2}\sin\left(2\sqrt{y}\right) - \sin\sqrt{y}\right. \Big|_0^{\pi^2}$$

$$\underline{= 0}$$

Exercise

Evaluate $\iiint\limits_D (xy + xz + yz)\, dV$; $D = \left\{(x,\ y,\ z):\ \ -1 \le x \le 1,\quad -2 \le y \le 2,\quad -3 \le z \le 3\right\}$

Solution

$$\iiint\limits_D (xy + xz + yz)\, dV = \int_{-3}^{3}\int_{-2}^{2}\int_{-1}^{1} (xy + xz + yz)\, dxdydz$$

$$= \int_{-3}^{3}\int_{-2}^{2} \left(\frac{1}{2}x^2 y + \frac{1}{2}x^2 z + xyz\, \Big|_{-1}^{1}\right. dydz$$

$$= \int_{-3}^{3}\int_{-2}^{2} \left(\frac{1}{2}y + \frac{1}{2}z + yz - \frac{1}{2}y - \frac{1}{2}z + yz\right) dydz$$

$$= \int_{-3}^{3}\int_{-2}^{2} 2yz\; dydz$$

$$= \int_{-3}^{3} zy^2 \Big|_{-2}^{2} dz$$

$$\underline{= 0}$$

Exercise

Evaluate $\iiint\limits_D xyze^{-x^2-y^2}\, dV$; $\quad D = \left\{(x, y, z):\ 0 \le x \le \sqrt{\ln 2},\ \ 0 \le y \le \sqrt{\ln 4},\ \ 0 \le z \le 1\right\}$

Solution

$$\iiint\limits_D xyze^{-x^2-y^2}\, dV = \int_0^1 \int_0^{\sqrt{\ln 4}} \int_0^{\sqrt{\ln 2}} xyze^{-x^2-y^2}\, dxdydz$$

$$= \int_0^1 z\, dz \int_0^{\sqrt{\ln 4}} ye^{-y^2} dy \int_0^{\sqrt{\ln 2}} xe^{-x^2} dx$$

$$= \frac{1}{2}z^2 \Big|_0^1 \left(-\frac{1}{2}\right) \int_0^{\sqrt{\ln 4}} e^{-y^2} d\left(-y^2\right) \quad \left(-\frac{1}{2}\right) \int_0^{\sqrt{\ln 2}} e^{-x^2} d\left(-x^2\right)$$

$$= \frac{1}{8}\ e^{-y^2} \Big|_0^{\sqrt{\ln 4}} \quad e^{-x^2} \Big|_0^{\sqrt{\ln 2}}$$

$$= \frac{1}{8}\left(\frac{1}{4} - 1\right)\left(\frac{1}{2} - 1\right)$$

$$\underline{= \frac{3}{64}}$$

Exercise

Let $D = \left\{(x, y, z):\ 0 \le x \le y^2,\ 0 \le y \le z^3,\ 0 \le z \le 2\right\}$

a) Use a triple integral to find the volume of *D*.

b) In theory, how many other possible orderings of the variables (besides the one used in part (*a*)) can be used to find the volume of *D*? Verify the result of part (*a*) using one of these other ordering.

c) What is the volume of the region $D = \left\{(x, y, z):\ 0 \le x \le y^p,\ 0 \le y \le z^q,\ 0 \le z \le 2\right\}$, where *p* and *q* are positive real numbers?

Solution

a) $V = \int_0^2 \int_0^{z^3} \int_0^{y^2} dxdydz$

$$= \int_0^2 \int_0^{z^3} x \Big|_0^{y^2} dydz$$

$$= \int_0^2 \int_0^{z^3} y^2 \, dydz$$

$$= \frac{1}{3}\int_0^2 y^3 \Big|_0^{z^3} dz$$

$$= \frac{1}{3}\int_0^2 z^9 \, dz$$

$$= \frac{1}{30} z^{10} \Big|_0^2$$

$$= \boxed{\frac{512}{15} \ unit^3}$$

b) There are total of 6: $dxdydz,\ dxdzdy,\ dydxdz,\ dydzdx,\ dzdxdy,\ dzdydx$

$0 \le x \le y^2$

$z = 2 \to y = 2^3 = 8 \qquad 0 \le y \le 8$

$y = z^3 \to z = \sqrt[3]{y} \qquad \sqrt[3]{y} \le z \le 2$

$$\int_0^8 \int_{\sqrt[3]{y}}^2 \int_0^{y^2} dxdzdy = \int_0^8 \int_{\sqrt[3]{y}}^2 x \Big|_0^{y^2} dzdy$$

$$= \int_0^8 \int_{\sqrt[3]{y}}^2 y^2 \, dzdy$$

$$= \int_0^8 y^2 z \Big|_{\sqrt[3]{y}}^2 dy$$

$$= \int_0^8 y^2\left(2 - y^{1/3}\right)dy$$

$$= \int_0^8 \left(2y^2 - y^{7/3}\right)dy$$

$$= \frac{2}{3}y^3 - \frac{3}{10}y^{10/3} \Big|_0^{2^3}$$

$$= \frac{2^{10}}{3} - \frac{3}{5}2^9$$

$$= 2^9\left(\frac{2}{3} - \frac{3}{5}\right)$$

$$= \frac{2^9}{15} \qquad = \frac{512}{15}$$

c) $D = \left\{(x, y, z):\ 0 \le x \le y^p,\ 0 \le y \le z^q,\ 0 \le z \le 2\right\},\ (p,\ q \in \mathbb{R})$

$$V = \int_0^2 \int_0^{z^q} \int_0^{y^p} dxdydz$$

$$= \int_0^2 \int_0^{z^q} x \Big|_0^{y^p} dydz$$

$$= \int_0^2 \int_0^{z^q} y^p\ dydz$$

$$= \frac{1}{p+1}\int_0^2 y^{p+1} \Big|_0^{z^q} dz$$

$$= \frac{1}{p+1}\int_0^2 z^{q(p+1)}\ dz$$

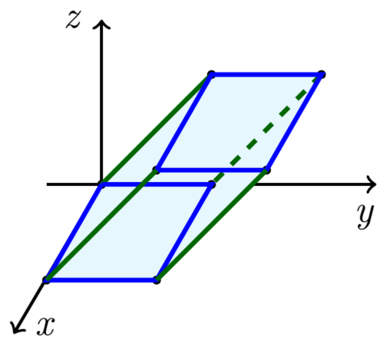
$$= \frac{1}{(p+1)(q(p+1)+1)}\ z^{q(p+1)+1} \Big|_0^2$$

$$= \frac{2^{q(p+1)+1}}{(p+1)(q(p+1)+1)}\ \ unit^3$$

Exercise

Find the volume the parallelepiped (slanted box) with vertices $(0,\ 0,\ 0)$, $(1,\ 0,\ 0)$, $(0,\ 1,\ 0)$, $(1,\ 1,\ 0)$, $(0,\ 1,\ 1)$, $(1,\ 1,\ 1)$, $(0,\ 2,\ 1)$, $(1,\ 2,\ 1)$

Solution



$$V = \int_0^1 \int_z^{z+1} \int_0^1 dxdydz$$

$$= \int_0^1 \int_z^{z+1} x \Big|_0^1 dydz$$

$$= \int_0^1 \int_z^{z+1} dydz$$

$$= \int_0^1 y \Big|_z^{z+1} dz$$

$$= \int_0^1 dz$$

$$= 1$$

Exercise

Find the volume the larger of two solids formed when the parallelepiped with vertices $(0,\ 0,\ 0)$, $(2,\ 0,\ 0)$, $(0,\ 2,\ 0)$, $(2,\ 2,\ 0)$, $(0,\ 1,\ 1)$, $(2,\ 1,\ 1)$, $(0,\ 3,\ 1)$, $(2,\ 3,\ 1)$ is sliced by the plane $y = 2$.

Solution

$$V = \int_0^1 \int_z^2 \int_0^2 dxdydz$$

$$= \int_0^1 \int_z^2 x \Big|_0^2 dydz$$

$$= 2\int_0^1 \int_z^2 dydz$$

$$= 2\int_0^1 y \Big|_z^2 dz$$

$$= 2\int_0^1 (2 - z)\, dz$$

$$= 2\left(2z - \frac{1}{2}z^2\right.\Big|_0^1$$

$$= 2\left(2 - \frac{1}{2}\right)$$

$$\underline{\underline{= 3}}$$

Exercise

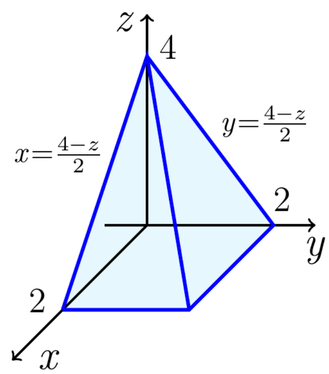
Find the volume of the pyramid with vertices $(0,\ 0,\ 0)$, $(2,\ 0,\ 0)$, $(2,\ 2,\ 0)$, $(0,\ 2,\ 0)$, $(0,\ 0,\ 4)$

Solution

$$(2,\ 0)\ \ \&\ \ (0,\ 4) \to z = \frac{4}{-2}(x-2) = -2x+4$$

$$\boxed{x = \frac{4-z}{2}}$$

$$(2,\ 0)\ \ \&\ \ (0,\ 4)\ \to\ \boxed{y = \frac{4-z}{2}}$$



$$V = \int_0^4 \int_0^{\frac{4-z}{2}} \int_0^{\frac{4-z}{2}} dxdydz$$

$$= \int_0^4 \int_0^{\frac{4-z}{2}} x \Big|_0^{\frac{4-z}{2}} dydz$$

$$= \frac{1}{2}\int_0^4 \int_0^{\frac{4-z}{2}} (4-z)\, dydz$$

$$= \frac{1}{2}\int_0^4 (4-z)\,y \Big|_0^{\frac{4-z}{2}} dz$$

$$= -\frac{1}{4}\int_0^4 (4-z)^2\, d(4-z)$$

$$= -\frac{1}{12}(4-z)^3 \Big|_0^4$$

$$= -\frac{1}{12}(-64)$$

$$\underline{\underline{= \frac{16}{3}}}$$

Exercise

Two different tetrahedrons fill the region in the first octant bounded by the coordinate planes and the plane $x+y+z=4$. Both solids have densities that vary in the z–direction between $\rho=4$ and $\rho=8$, according to the functions $\rho_1=8-z$ and $\rho_2=4+z$. Find the mass of each solid

Solution

$$m_1=\int_0^4\int_0^{4-x}\int_0^{4-x-y}(8-z)\,dzdydx$$

$$=\int_0^4\int_0^{4-x}\left(8z-\frac{1}{2}z^2\,\Big|_0^{4-x-y}\,dydx\right.$$

$$=\int_0^4\int_0^{4-x}\left(32-8x-8y-\frac{1}{2}(4-x-y)^2\right)dydx$$

$$=\int_0^4\int_0^{4-x}\left(24-4x-\frac{1}{2}x^2-4y-xy-\frac{1}{2}y^2\right)dydx$$

$$=\int_0^4\left(24y-4xy-\frac{1}{2}x^2y-2y^2-\frac{1}{2}xy^2-\frac{1}{6}y^3\,\Big|_0^{4-x}\,dx\right.$$

$$=\int_0^4\left(\frac{160}{3}-24x+2x^2+\frac{1}{6}x^3\right)dx$$

$$=\frac{160}{3}x-12x^2+\frac{2}{3}x^3+\frac{1}{24}x^4\,\Big|_0^4$$

$$\boxed{=\frac{224}{3}}$$

$$m_2=\int_0^4\int_0^{4-x}\int_0^{4-x-y}(4+z)\,dzdydx$$

$$=\int_0^4\int_0^{4-x}\left(4z+\frac{1}{2}z^2\,\Big|_0^{4-x-y}\,dydx\right.$$

$$=\int_0^4\int_0^{4-x}\left(24-8x-\frac{1}{2}x^2-8y+xy-\frac{1}{2}y^2\right)dydx$$

$$=\int_0^4\left(24y-8xy-\frac{1}{2}x^2y-4y^2+\frac{1}{2}xy^2-\frac{1}{6}y^3\,\Big|_0^{4-x}\,dx\right.$$

$$= \int_0^4 \left(\frac{128}{3} - 24x + 4x^2 - \frac{1}{6}x^3\right)dx$$

$$= \frac{128}{3}x - 12x^2 + \frac{4}{3}x^3 - \frac{1}{24}x^4 \Big|_0^4$$

$$\boxed{= \frac{160}{3}}$$

Solid 1 has ***greater mass***.

Exercise

Suppose a wedge of cheese fills the region in the first octant bounded by the planes $y = z$, $y = 4$ and $x = 4$. You could divide the wedge into two equal pieces (by volume) if you sliced the wedge with the plane $x = 2$. Instead find *a* with $0 < a < 1$ such that slicing the wedge with the plane $y = a$ divides the wedge into two pieces of equal volume

Solution

$$V = \int_0^4 \int_0^4 \int_0^y dzdydx$$

$$= \int_0^4 dx \int_0^4 z \Big|_0^y dy$$

$$= x\Big|_0^4 \int_0^4 y\, dy$$

$$= 2y^2 \Big|_0^4$$

$$\boxed{= 32}$$

$$V = \int_0^4 \int_0^a \int_0^y dzdydx$$

$$= \frac{1}{2}(32)$$

$$\boxed{= 16\ unit^3}$$

$$\int_0^4 \int_0^a \int_0^y dzdydx = \int_0^4 dx \int_0^a z \Big|_0^y dy$$

$$= x\Big|_0^4 \int_0^a y\, dy$$

$$= 2y^2 \Big|_0^a$$

$$= 2a^2 = 16$$

$$\underline{a = 2\sqrt{2}}$$

Exercise

Find the volumes of the region between the cylinder $z = y^2$ and the xy-plane that is bounded by the planes $x = 0, \quad x = 1, \quad y = -1, \quad y = 1$

Solution

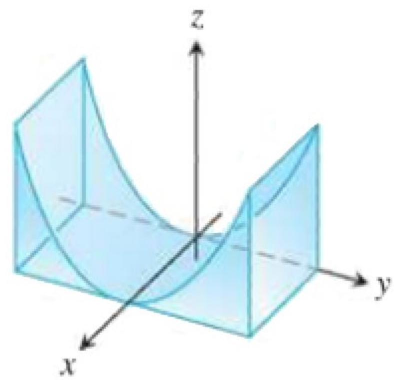
$$V = \int_0^1 \int_{-1}^1 \int_0^{y^2} dzdydx$$

$$= \int_0^1 dx \int_{-1}^1 \left(z \Big|_0^{y^2} dy\right.$$

$$= \int_{-1}^1 y^2\, dy$$

$$= \frac{1}{3} y^3 \Big|_{-1}^1$$

$$= \frac{2}{3}\ unit^3$$



Exercise

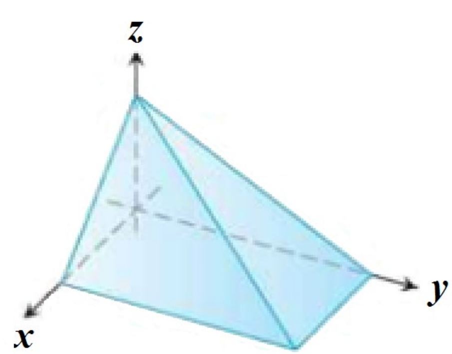
Find the volumes of the region in the first octant bounded by the coordinate planes and the planes $x + z = 1, \quad y + 2z = 2$

Solution

$$V = \int_0^1 \int_0^{1-x} \int_0^{2-2z} dydzdx$$

$$= \int_0^1 \int_0^{1-x} (2 - 2z)\, dzdx$$

$$= \int_0^1 \left(2z - z^2 \Big|_0^{1-x} dx\right.$$



$$=\int_0^1 \left[2(1-x)-(1-x)^2\right] dx$$

$$=\int_0^1 (1-x)(2-1+x)\,dx$$

$$=\int_0^1 (1-x)(1+x)\,dx$$

$$=\int_0^1 \left(1-x^2\right) dx$$

$$=x-\frac{1}{3}x^3 \Big|_0^1$$

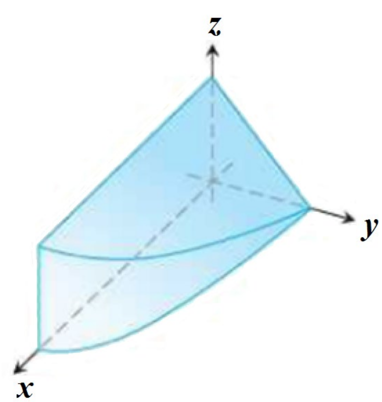
$$=1-\frac{1}{3}$$

$$\boxed{=\frac{2}{3}\ unit^3}$$

Exercise

Find the volumes of the region in the first octant bounded by the coordinate planes and the plane $y+z=2$, and the cylinder $x=4-y^2$

Solution



$$V=\int_0^4 \int_0^{\sqrt{4-x}} \int_0^{2-y} dzdydx$$

$$=\int_0^4 \int_0^{\sqrt{4-x}} (2-y)\,dydx$$

$$=\int_0^4 \left(2y-\frac{1}{2}y^2 \Big|_0^{\sqrt{4-x}} dydx\right.$$

$$=\int_0^4 \left[2\sqrt{4-x}-\frac{1}{2}(4-x)\right] dx$$

$$=-\int_0^4 \left[2(4-x)^{1/2}-\frac{1}{2}(4-x)\right] d(4-x)$$

$$=-\left(\frac{4}{3}(4-x)^{3/2}-\frac{1}{4}(4-x)^2 \Big|_0^4\right.$$

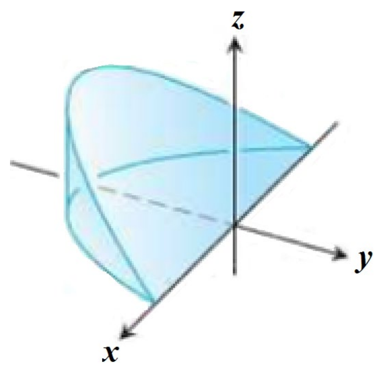
$$= -\left[0 - \left(\frac{4}{3}4^{3/2} - \frac{1}{4}4^2\right)\right]$$

$$= \frac{20}{3}\ unit^3$$

Exercise

Find the volumes of the wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y, \quad z = 0$

Solution



$$V = 2\int_0^1 \int_{-\sqrt{1-x^2}}^0 \int_0^{-y} dzdydx$$

$$= -2\int_0^1 \int_{-\sqrt{1-x^2}}^0 y\, dydx$$

$$= -2\int_0^1 \left(\frac{1}{2}y^2 \Big|_{-\sqrt{1-x^2}}^0\right) dx$$

$$= \int_0^1 \left(1 - x^2\right) dx$$

$$= x - \frac{1}{3}x^3 \Big|_0^1$$

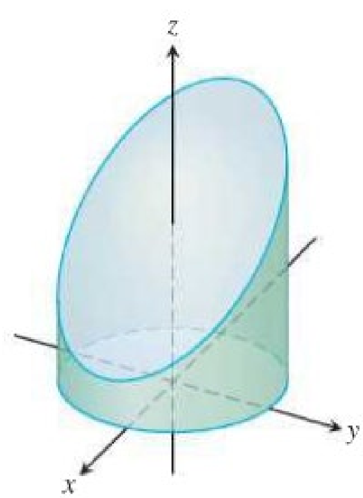
$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3}\ unit^3$$

Exercise

Find the volumes of the region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$

Solution



$$V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{3-x} dzdydx$$

$$= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3 - x)\, dydx$$

$$= 2\int_{-2}^{2} (3-x)\sqrt{4-x^2}\, dx$$

$$= 6\int_{-2}^{2} \sqrt{4-x^2}\, dx \, - \, 2\int_{-2}^{2} x\sqrt{4-x^2}\, dx \qquad\qquad d\left(4-x^2\right) = -2xdx$$

$$= 6\int_{-2}^{2} \sqrt{4-x^2}\, dx \, + \, \int_{-2}^{2} \left(4-x^2\right)^{1/2} d\left(4-x^2\right)$$

$$= 3\left(x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2} \,\Big|_{-2}^{2} \, + \, \frac{2}{3}\left(\left(4-x^2\right)^{3/2} \,\Big|_{-2}^{2}\right.\right.$$

$$= 3\left[4\sin^{-1}1 - 4\sin^{-1}(-1)\right] + \frac{2}{3}(0)$$

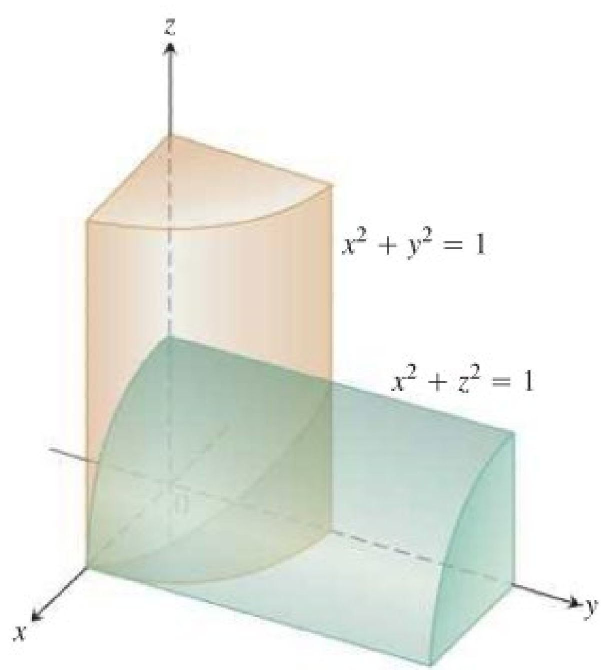
$$= 12\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$\boxed{= 12\pi \ \ unit^3}$$

Exercise

Find the volumes of the region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is shown below

Solution



$$V = 8\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dzdydx$$

$$= 8\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\, dydx$$

$$= 8\int_0^1 \sqrt{1-x^2} \left(y \,\Big|_0^{\sqrt{1-x^2}}\right. dx$$

$$= 8\int_0^1 \left(1-x^2\right) dx$$

$$= 8\left(x - \frac{1}{3}x^3 \,\Big|_0^1\right.$$

$$\boxed{= \frac{16}{3} \ \ unit^3}$$

Exercise

Find the volume of the solid in the first octant bounded by the plane $2x+3y+6z=12$ and the coordinate planes

Solution

$$z=\frac{12-2x-3y}{6}$$

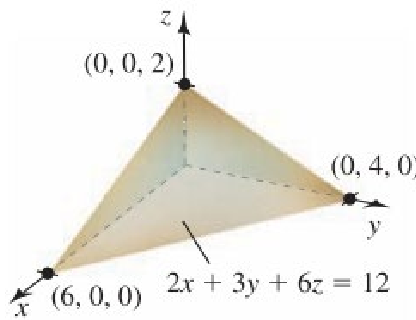
$$=2-\frac{x}{3}-\frac{y}{2}$$

$$z=0\to \quad 2x+3y=12$$

$$y=4-\frac{2x}{3}$$

$$0\le z\le 2-\frac{x}{3}-\frac{y}{2};\quad 0\le y\le 4-\frac{2x}{3};\quad 0\le x\le 6$$

$$V=\int_0^6\int_0^{4-\frac{2x}{3}}\int_0^{2-\frac{x}{3}-\frac{y}{2}} 1\, dzdydx$$



$$=\int_0^6\int_0^{4-\frac{2x}{3}} z\Big|_0^{2-\frac{x}{3}-\frac{y}{2}} dydx$$

$$=\int_0^6\int_0^{4-\frac{2x}{3}} \left(2-\frac{x}{3}-\frac{y}{2}\right) dydx$$

$$=\int_0^6 \left(2y-\frac{x}{3}y-\frac{1}{4}y^2\right)\Big|_0^{4-\frac{2x}{3}} dx$$

$$=\int_0^6 \left(8-\frac{4}{3}x-\frac{4}{3}x+\frac{2}{9}x^2-\frac{1}{4}\left(16-\frac{16}{3}x+\frac{4}{9}x^2\right)\right)dx$$

$$=\int_0^6 \left(4-\frac{4}{3}x+\frac{1}{9}x^2\right)dx$$

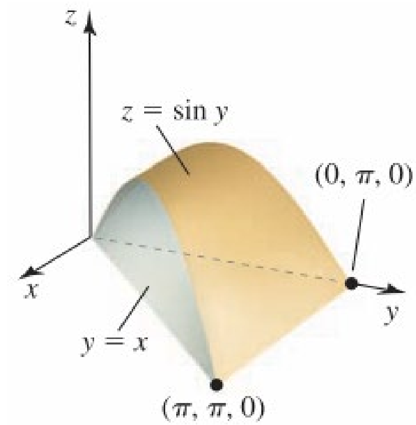
$$=4x-\frac{2}{3}x^2+\frac{1}{27}x^3\Big|_0^6$$

$$=8\ \ unit^3$$

Exercise

Find the volume of the solid in the first octant formed when the cylinder $z = \sin y$, for $0 \le y \le \pi$, is sliced by the planes $y = x$ and $x = 0$

Solution



$$V = \int_0^{\pi}\int_x^{\pi}\int_0^{\sin y} 1\, dzdydx$$

$$= \int_0^{\pi}\int_x^{\pi} z\Big|_0^{\sin y}\, dydx$$

$$= \int_0^{\pi}\int_x^{\pi} \sin y\, dydx$$

$$= -\int_0^{\pi} \cos y\Big|_x^{\pi}\, dx$$

$$= -\int_0^{\pi} (-1-\cos x)\, dx$$

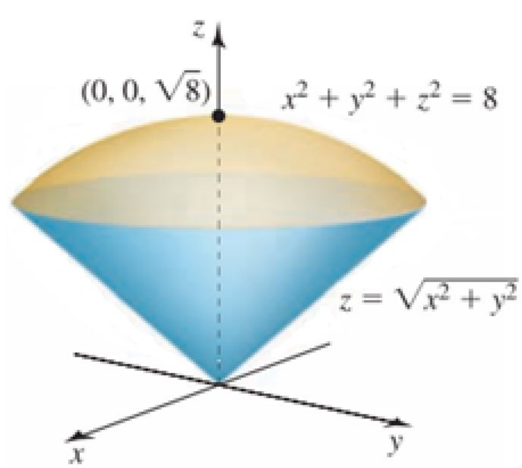
$$= x + \sin x\Big|_0^{\pi}$$

$$= \pi \ \ unit^3$$

Exercise

Find the volume of the solid bounded below by the cone $z = \sqrt{x^2+y^2}$ and bounded above the sphere $x^2+y^2+z^2 = 8$

Solution



$$z = \sqrt{x^2+y^2} \quad z = \sqrt{8-x^2-y^2}$$

$$x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2 = 8$$

$$x^2+y^2 = 4 \ \rightarrow\ y = \pm\sqrt{4-x^2}$$

$$(y=0) \rightarrow x^2 = 4 \quad x = \pm 2$$

$$V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} 1\, dzdydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} z \Big|_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} dydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(\sqrt{8-x^2-y^2} - \sqrt{x^2+y^2} \right) dydx$$ *Convert to* ***Polar coordinates***

$$= \int_{0}^{2\pi} \int_{0}^{2} \left(\sqrt{8-r^2} - r \right) r\, drd\theta$$

$$= \int_{0}^{2\pi} d\theta \int_{0}^{2} \left(r\sqrt{8-r^2} - r^2 \right) dr$$

$$= 2\pi \left(\int_{0}^{2} \frac{-1}{2} \left(8-r^2\right)^{1/2} d\left(8-r^2\right) - \left(\frac{1}{3}r^3\right) \Big|_{0}^{2} \right)$$

$$= 2\pi \left(-\frac{1}{3} \left(8-r^2\right)^{3/2} \Big|_{0}^{2} - \frac{8}{3} \right)$$

$$= 2\pi \left(-\frac{1}{3}\left(8-16\sqrt{2}\right) - \frac{8}{3} \right)$$

$$= 2\pi \left(\frac{16\sqrt{2}}{3} - \frac{16}{3} \right)$$

$$= \frac{32\pi}{3}\left(\sqrt{2}-1\right) \ unit^3$$

Exercise

The solid between the sphere $x^2+y^2+z^2=19$ and the hyperboloid $z^2-x^2-y^2=1$, for $z>0$

Solution

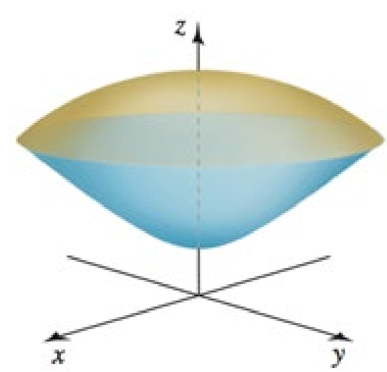
$z^2 = 1+x^2+y^2$

The intersection of the sphere and hyperboloid:

$x^2+y^2+z^2=19$

$x^2+y^2+1+x^2+y^2=19$

$2x^2+2y^2=18 \quad \rightarrow \quad x^2+y^2=9$



$$V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{1+x^2+y^2}}^{\sqrt{19-x^2-y^2}} dzdydx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z \Big|_{\sqrt{1+x^2+y^2}}^{\sqrt{19-x^2-y^2}} dydx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left(\sqrt{19-x^2-y^2} - \sqrt{1+x^2+y^2}\right) dydx$$

$$= \int_{0}^{2\pi}\int_{0}^{3} \left(\sqrt{19-r^2} - \sqrt{1+r^2}\right) r\, drd\theta$$

$$= \int_{0}^{2\pi} d\theta \left(-\frac{1}{2}\int_{0}^{3}\left(19-r^2\right)^{1/2} d\left(19-r^2\right) - \frac{1}{2}\int_{0}^{3}\left(1+r^2\right)^{1/2} d\left(1+r^2\right)\right)$$

$$= -\frac{2\pi}{3}\left(\left(19-r^2\right)^{3/2} + \left(1+r^2\right)^{3/2}\right)\Big|_{0}^{3}$$

$$= -\frac{2\pi}{3}\left(10\sqrt{10} + 10\sqrt{10} - 19\sqrt{19} - 1\right)$$

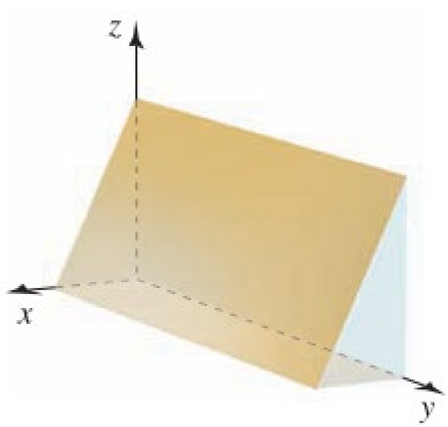
$$\boxed{= \frac{2\pi}{3}\left(1 - 20\sqrt{10} + 19\sqrt{19}\right) \;\; unit^3}$$

Exercise

Find the volume of the prism in the first octant bounded below by $z = 2 - 4x$ and $y = 8$

Solution

$$z = 2 - 4x = 0 \quad \Rightarrow \quad \boxed{x = \frac{1}{2}}$$



$$V = \int_{0}^{1/2}\int_{0}^{8}\int_{0}^{2-4x} 1\, dzdydx$$

$$= \int_{0}^{1/2}\int_{0}^{8} (2-4x)\, dydx$$

$$= \int_{0}^{1/2} (2-4x)\, y \Big|_{0}^{8} dx$$

$$= 16\int_{0}^{1/2} (1-2x)\, dx$$

$$= 16\left(x - x^2\right. \Big|_{0}^{1/2}$$

$$= 16\left(\frac{1}{2} - \frac{1}{4}\right)$$

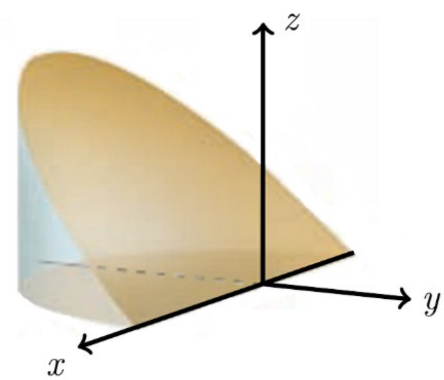
$$= 4 \quad unit^3$$

Exercise

Find the volume of the wedge above the *xy*-plane formed when the cylinder $x^2 + y^2 = 4$ is cut by the planes $z = 0$ and $y = -z$

Solution

$0 \le z \le -y \ (y<0); \quad -\sqrt{4-x^2} \le y \le 0; \quad y = 0 \to x^2 = 4 \ \Rightarrow \ -2 \le x \le 2$



$$V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{0}\int_{0}^{-y} 1 \; dzdydx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{0} (-y) \; dydx$$

$$= -\frac{1}{2}\int_{-2}^{2} \left(y^2\right)\Big|_{-\sqrt{4-x^2}}^{0} \; dx$$

$$= \frac{1}{2}\int_{-2}^{2} \left(4 - x^2\right) dx$$

$$= \frac{1}{2}\left(4x - \frac{1}{3}x^3\right)\Big|_{-2}^{2}$$

$$= 8 - \frac{8}{3}$$

$$= \frac{16}{3} \quad unit^3$$

Exercise

The wedge bounded by the parabolic cylinder $y = x^2$ and the planes $z = 3 - y$ and $z = 0$

Solution

$z = 3 - y = 0 \ \to \ y = 3$

$y = x^2 = 3 \ \to \ x = \pm\sqrt{3}$

$$V = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{x^2}^{3}\int_{0}^{3-y} dzdydx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2}^{3} z \Big|_0^{3-y} dydx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2}^{3} (3-y)\,dydx$$

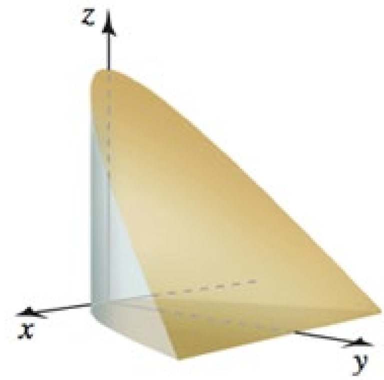
$$= \int_{-\sqrt{3}}^{\sqrt{3}} \left(3y - \frac{1}{2}y^2 \Big|_{x^2}^{3}\right) dx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{9}{2} - 3x^2 + \frac{1}{2}x^4\right) dx$$

$$= \frac{9}{2}x - x^3 + \frac{1}{10}x^5 \Big|_{-\sqrt{3}}^{\sqrt{3}}$$

$$= 2\left(\frac{9}{2}\sqrt{3} - 3\sqrt{3} + \frac{9}{10}\sqrt{3}\right)$$

$$= \frac{24\sqrt{3}}{5} \quad unit^3$$



Exercise

Find the volume of the solid bounded by the surfaces $z = e^y$ and $z = 1$ over the rectangle $\{(x,y):\ 0 \le x \le 1,\ 0 \le y \le \ln 2\}$

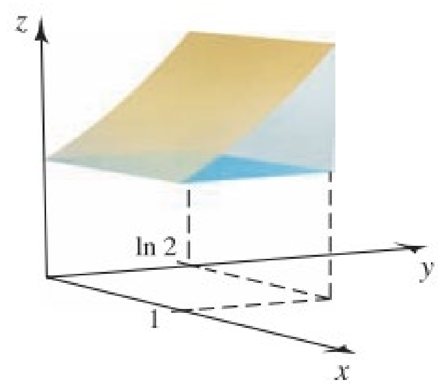
Solution

$$V = \int_0^1 \int_0^{\ln 2} \int_1^{e^y} 1\,dzdydx$$

$$= \int_0^1 dx \int_0^{\ln 2} \left(e^y - 1\right) dy$$

$$= x \Big|_0^1 \left(e^y - y \Big|_0^{\ln 2}\right)$$

$$= 1 - \ln 2 \quad unit^3$$



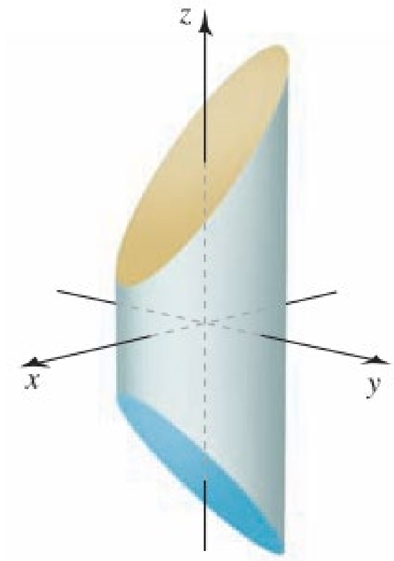
Exercise

Find the volume of the wedge of the cylinder $x^2 + 4y^2 = 4$ created by the planes $z = 3 - x$ and $z = x - 3$

Solution

$$y^2 = \frac{1}{4}\left(4 - x^2\right) \quad \to \quad y = \pm\frac{1}{2}\sqrt{4 - x^2} \text{ v}$$

$$x^2 = 4 \quad \to \quad -2 \le x \le 2$$



$$V = \int_{-2}^{2}\int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}}\int_{x-3}^{3-x} 1\, dzdydx$$

$$= \int_{-2}^{2}\int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} (6 - 2x)\, dydx$$

$$= \int_{-2}^{2} (6 - 2x)\, y \Big|_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} (6 - 2x)\sqrt{4 - x^2}\, dx$$

$$= \int_{-2}^{2} 6\sqrt{4 - x^2}\, dx + \int_{-2}^{2} \sqrt{4 - x^2}\, d\left(4 - x^2\right)$$

$$x = 2\sin\theta \qquad \sqrt{4 - x^2} = 2\cos\theta$$

$$dx = 2\cos\theta\, d\theta$$

$$\int \sqrt{4 - x^2}\, dx = \int 2\cos\theta\left(2\cos\theta\, d\theta\right)$$

$$= 4\int \cos^2\theta\, d\theta$$

$$= 2\int (1 + \cos 2\theta)\, d\theta$$

$$= 2\left(\theta + \frac{1}{2}\sin 2\theta\right)$$

$$= 2\left(\arcsin\frac{x}{2} + \sin\theta\cos\theta\right)$$

$$= 2\left(\arcsin\frac{x}{2} + \frac{x}{2}\frac{\sqrt{2 - x^2}}{2}\right)$$

$$= 2\arcsin\frac{x}{2} + \frac{1}{2}x\sqrt{4-x^2}$$

$$= 12\sin^{-1}\frac{x}{2} + 3x\sqrt{4-x^2} + \frac{2}{3}\sqrt{4-x^2}\ \Big|_{-2}^{2}$$

$$= 12\frac{\pi}{2} + 12\frac{\pi}{2}$$

$$= 12\pi \quad unit^3$$

Exercise

Find the volume of the solid in the first octant bounded by the cone $z = 1 - \sqrt{x^2 + y^2}$ and the plane $x + y + z = 1$

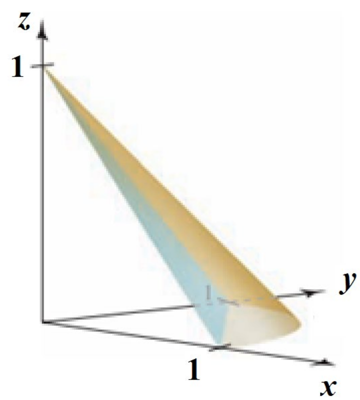
Solution

$$0 \le z \le 1$$

$$z = 1 - \sqrt{x^2 + y^2} \quad \to x^2 + y^2 = (1-z)^2 \Rightarrow x = \sqrt{(1-z)^2 - y^2}$$

$$1 - y - z \le x \le \sqrt{(1-z)^2 - y^2}$$

$$0 \le y \le 1 - z$$



$$V = \int_0^1 \int_1^{1-z} \int_{1-y-z}^{\sqrt{(1-z)^2-y^2}} 1\, dxdydz$$

$$= \int_0^1 \int_1^{1-z} x \Big|_{1-y-z}^{\sqrt{(1-z)^2-y^2}} dydz$$

$$= \int_0^1 \int_1^{1-z} \left(\sqrt{(1-z)^2 - y^2} - 1 + y + z \right) dydz$$

$$x = a\sin\theta \qquad \sqrt{a^2 - x^2} = a\cos\theta$$

$$dx = a\cos\theta\ d\theta$$

$$\int \sqrt{a^2 - x^2}\ dx = \int a\cos\theta\left(a\cos\theta\ d\theta\right)$$

$$= a^2 \int \cos^2\theta\ d\theta$$

$$= \frac{a^2}{2} \int \left(1 + \cos 2\theta\right)\ d\theta$$

$$=\frac{a^2}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)$$

$$=\frac{a^2}{2}\left(\arcsin\frac{x}{a}+\sin\theta\cos\theta\right)$$

$$=\frac{a^2}{2}\left(\arcsin\frac{x}{a}+\frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}\right)$$

$$\boxed{=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}}$$

$a=1-z$ & $x=y$

$$=\int_0^1\left(\frac{y}{2}\sqrt{(1-z)^2-y^2}+\frac{1}{2}(1-z)^2\sin^{-1}\left(\frac{y}{1-z}\right)-y+\frac{1}{2}y^2+zy\ \Big|_0^{1-z}\right)dz$$

$$=\int_0^1\left(\frac{1}{2}(1-z)^2\sin^{-1}(1)+\frac{1}{2}(1-z)^2-(1-z)^2\right)dz$$

$$=\int_0^1\left(\frac{\pi}{4}(1-z)^2-\frac{1}{2}(1-z)^2\right)dz$$

$$=\frac{\pi-2}{4}\int_0^1(1-z)^2d(1-z)$$

$$=\frac{\pi-2}{12}(1-z)^3\Big|_0^1$$

$$\boxed{=\frac{\pi-2}{12}\quad unit^3}$$

Exercise

Find the volume of the solid bounded by $x=0$, $x=1-z^2$, $y=0$, $z=0$, and $z=1-y$

Solution

$$V=\int_0^1\int_0^{1-z^2}\int_0^{1-z}1\,dydxdz$$

$$=\int_0^1\int_0^{1-z^2}(1-z)\,dxdz$$

$$= \int_0^1 (1-z)x \Big|_0^{1-z^2} dz$$

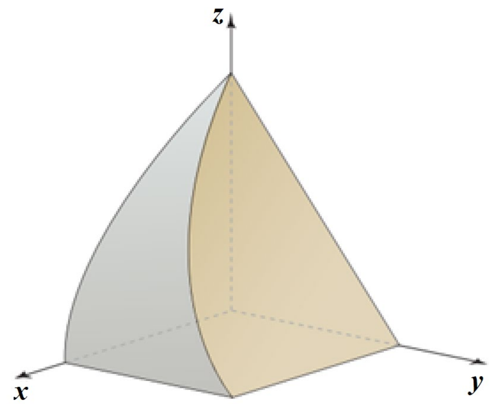
$$= \int_0^1 (1-z)\left(1-z^2\right) dz$$

$$= \int_0^1 \left(1-z^2-z+z^3\right) dz$$

$$= z - \frac{1}{3}z^3 - \frac{1}{2}z^2 + \frac{1}{4}z^4 \Big|_0^1$$

$$= 1 - \frac{1}{3} - \frac{1}{2} + \frac{1}{4}$$

$$= \frac{5}{12} \quad unit^3$$



Exercise

Find the volume of the solid bounded by $x=0$, $x=2$, $y=0$, $y=e^{-z}$, $z=0$, and $z=1$

Solution

$$V = \int_0^2 \int_0^1 \int_0^{e^{-z}} 1\, dydzdx$$

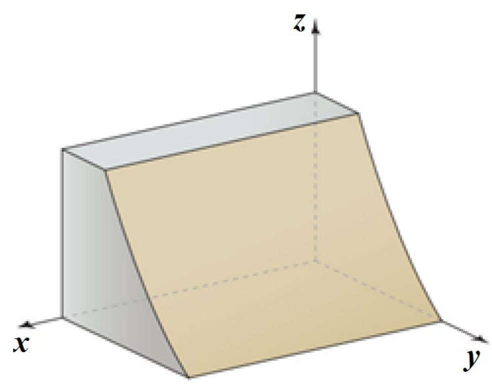
$$= \int_0^2 dx \int_0^1 y \Big|_0^{e^{-z}} dz$$

$$= 2\int_0^1 e^{-z}\, dz$$

$$= -2e^{-z} \Big|_0^1$$

$$= -2\left(e^{-1}-1\right)$$

$$= 2 - \frac{2}{e} \quad unit^3$$



Exercise

Find the volume of the solid bounded by $x=0$, $x=2$, $y=z$, $y=z+1$, $z=0$, and $z=4$

Solution

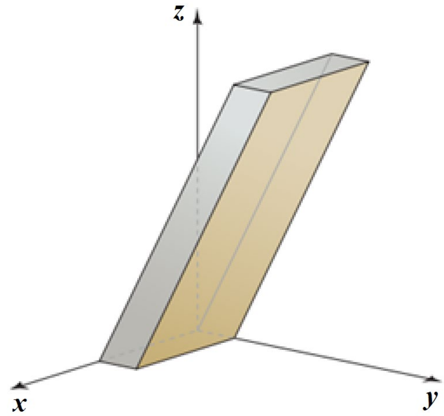
$$V=\int_0^2\int_0^4\int_z^{z+1} 1\, dydzdx$$

$$=\int_0^2\int_0^4 y\Big|_z^{z+1} dz\, dx$$

$$=\int_0^2 dx\int_0^4 dz$$

$$=(2)(4)$$

$$=8\quad unit^3$$



Exercise

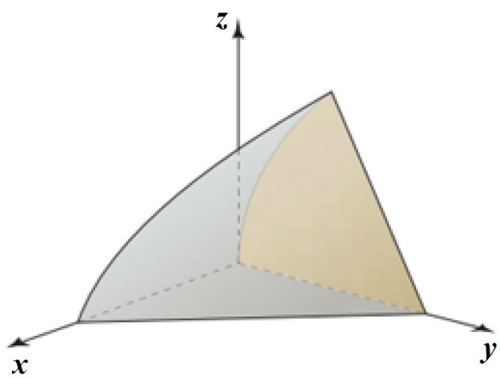
Find the volume of the solid bounded by $x=0$, $y=z^2$, $z=0$, and $z=2-x-y$

Solution

$$y=2-x-z$$

$$x=2-z-y$$

$$=2-z-z^2$$



$$V=\int_0^1\int_0^{2-z-z^2}\int_{z^2}^{2-x-z} 1\, dydxdz$$

$$=\int_0^1\int_0^{2-z-z^2}\left(2-x-z-z^2\right)dxdz$$

$$=\int_0^1\left(\left(2-z-z^2\right)x-\frac{1}{2}x^2\,\Big|_0^{2-z-z^2}\right. dz$$

$$=\frac{1}{2}\int_0^1\left(2-z-z^2\right)^2 dz$$

$$=\frac{1}{2}\int_0^1\left(4-4z-3z^2+2z^3+z^4\right)dz$$

$$=\frac{1}{2}\left(4z-2z^2-z^3+\frac{1}{2}z^4+\frac{1}{5}z^5\right)\Big|_0^1$$

$$=\frac{1}{2}\left(4-2-1+\frac{1}{2}+\frac{1}{5}\right)$$

$$=\frac{17}{20}\quad unit^3$$

Exercise

Find the volume of the solid common to the cylinders $z=\sin x$ and $z=\sin y$ over the square

$$R=\{(x,y):\ 0\le x\le \pi,\ 0\le y\le \pi\}$$

Solution

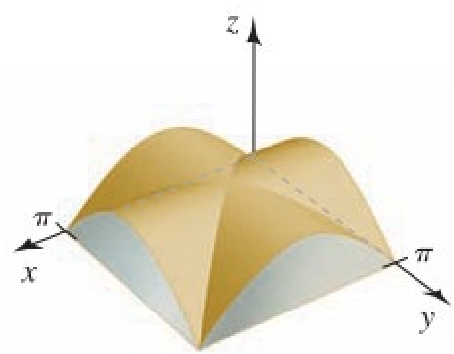
$$z=\sin x=\sin y$$

$$\Rightarrow\ x=y\ \ or\ y=\pi-x$$

$$V=4\int_0^{\pi/2}\int_x^{\pi-x}\int_0^{\sin y}1\,dzdydx$$

4: *by symmetry, volume –* **4** *times*

$$=4\int_0^{\pi/2}\int_x^{\pi-x}\sin y\ dydx$$



$$=-4\int_0^{\pi/2}\cos y\Big|_x^{\pi-x}\ dx$$

$$=-4\int_0^{\pi/2}\left(\cos(\pi-x)-\cos x\right)\ dx$$

$$=-4\int_0^{\pi/2}\left(-2\cos x\right)\ dx$$

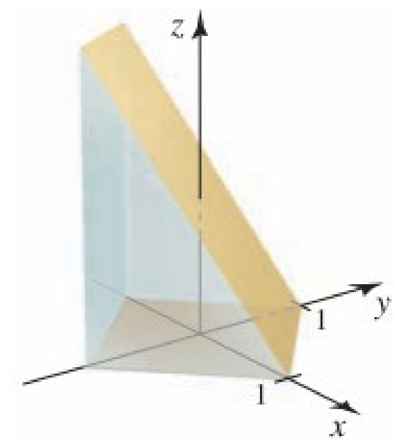
$$=8\sin x\Big|_0^{\pi/2}$$

$$=8\quad unit^3$$

Exercise

Find the volume of the wedge of the square column $|x|+|y|=1$ created by the planes $z=0$ and $x+y+z=1$

Solution



$$0 \le z \le 1-x-y$$

$$|x|+|y|=1 \quad \to \quad \begin{cases} x+y=1 \Rightarrow y=1-x \\ -x+y=1 \Rightarrow y=1+x \\ x-y=1 \Rightarrow y=x-1 \\ -x-y=1 \Rightarrow y=-x-1 \end{cases}$$

$$\begin{cases} y=-x-1 \\ y=x+1 \end{cases} \Rightarrow -1 \le x \le 0$$

$$\begin{cases} y=x-1 \\ y=-x+1 \end{cases} \Rightarrow 0 \le x \le 1$$

$$V = \int_{-1}^{0}\int_{-x-1}^{x+1}\int_{0}^{1-x-y} 1\, dzdydx \ + \int_{0}^{1}\int_{x-1}^{-x+1}\int_{0}^{1-x-y} 1\, dzdydx$$

$$= \int_{-1}^{0}\int_{-x-1}^{x+1} (1-x-y)\ dydx \ + \int_{0}^{1}\int_{x-1}^{-x+1} (1-x-y)\ dydx$$

$$= \int_{-1}^{0} \left((1-x)y - \frac{1}{2}y^2 \ \Big|_{-x-1}^{x+1}\ dx \ + \int_{0}^{1} \left((1-x)y - \frac{1}{2}y^2 \ \Big|_{x-1}^{-x+1}\ dx$$

$$= \int_{-1}^{0} 2(1-x)(x+1)\ dx \ + \int_{0}^{1} 2(1-x)^2\ dx$$

$$= \int_{-1}^{0} 2\left(1-x^2\right)\ dx \ + \ 2\int_{0}^{1} \left(1-2x+x^2\right)\ dx$$

$$= 2\left(x - \frac{1}{3}x^3 \ \Big|_{-1}^{0} + 2\left(x - x^2 + \frac{1}{3}x^3 \ \Big|_{0}^{1}$$

$$= 2\left(1-\frac{1}{3}\right) + \frac{2}{3}$$

$$= 2 \quad unit^3$$

Exercise

Find the volume of a right circular cone with height h and base radius r.

Solution

The equation of a circle is centered at the origin with radius r: $x^2+y^2=r^2$

$$-\sqrt{r^2-x^2} \le y \le \sqrt{r^2-x^2} \quad \& \quad -r \le x \le r$$

$$z = a - b\sqrt{x^2+y^2}$$

$$\begin{cases} z = h & \underline{h = a} \\ z = 0 & 0 = a - br = h - br \ \Rightarrow b = \frac{h}{r} \end{cases}$$

The equation of a cone with height h: $z = h - \frac{h}{r}\sqrt{x^2+y^2}$

$$V = \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}}\int_{0}^{h-\frac{h}{r}\sqrt{x^2+y^2}} 1\, dzdydx$$

$$= \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \left(h - \frac{h}{r}\sqrt{x^2+y^2}\right) dydx \qquad \text{Let } x^2+y^2=R^2 \ (\textit{Polar Coordinates})$$

$$= \int_{0}^{2\pi}\int_{0}^{r} \left(h - \frac{h}{r}R\right) R\, dRd\theta$$

$$= \int_{0}^{2\pi} d\theta \int_{0}^{r} \left(hR - \frac{h}{r}R^2\right) dR$$

$$= 2\pi\left(\frac{1}{2}hR^2 - \frac{h}{3r}R^3\right)\Big|_{0}^{r}$$

$$= 2\pi\left(\frac{1}{2}hr^2 - \frac{1}{3}hr^2\right)$$

$$\boxed{= \frac{1}{3}\pi r^2 h \quad unit^3}$$

Exercise

Find the volume of a tetrahedron whose vertices are located at $(0,\ 0,\ 0)$, $(a,\ 0,\ 0)$, $(0,\ b,\ 0)$, and $(0,\ 0,\ c)$

Solution

The equation of the plane through the vertices: $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$$0 \le z \le c\left(1-\frac{x}{a}-\frac{y}{b}\right) \quad 0 \le y \le b\left(1-\frac{x}{a}\right) \quad 0 \le x \le a$$

$$V = \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} 1\, dzdydx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dydx$$

$$= c\int_0^a \left(\left(1-\frac{x}{a}\right)y - \frac{1}{2b}y^2\right) \Big|_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= c\int_0^a \left(b\left(1-\frac{x}{a}\right)^2 - \frac{1}{2}b\left(1-\frac{x}{a}\right)^2\right) dx$$

$$= \frac{1}{2}bc\int_0^a \left(1-\frac{2}{a}x+\frac{1}{a^2}x^2\right) dx$$

$$= \frac{1}{2}bc\left(x-\frac{1}{a}x^2+\frac{1}{3a^2}x^3\right) \Big|_0^a$$

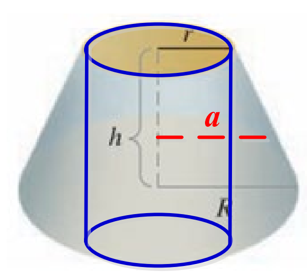
$$= \frac{1}{2}bc\left(a-a+\frac{1}{3}a\right)$$

$$\boxed{= \frac{abc}{6} \quad unit^3}$$

Exercise

Find the volume of a truncated cone of height h whose ends have radii r and R.

Solution



There are 2 volumes to consider:

1. Volume of the cylinder: $V_1 = \pi r^2 h$
2. Volume V_2 that remains when cylinder is removed.

   V_2 is the annulus centered at the origin with inner radius r and outer radius R.

   Using Polar Coordinates: the equation of the frustum is: $z = \frac{h}{R-r}(R-a)$

$$V_2 = \int_0^{2\pi} \int_r^R \int_0^{\frac{h}{R-r}(R-a)} a\, dz\, da\, d\theta$$

$$= \int_0^{2\pi} \int_r^R \frac{h}{R-r}(R-a)\, a\; da\, d\theta$$

$$= \frac{h}{R-r}\int_0^{2\pi} d\theta \int_r^R \left(Ra - a^2\right) da$$

$$= \frac{2\pi h}{R-r}\left(\frac{1}{2}Ra^2 - \frac{1}{3}a^3 \right|_r^R$$

$$= \frac{2\pi h}{R-r}\left(\frac{1}{2}R^3 - \frac{1}{3}R^3 - \frac{1}{2}Rr^2 + \frac{1}{3}r^3\right)$$

$$= \frac{2\pi h}{R-r}\left(\frac{1}{6}R^3 - \frac{1}{2}Rr^2 + \frac{1}{3}r^3\right)$$

$$= \frac{1}{3}\frac{\pi h}{R-r}\left(R^3 - 3Rr^2 + 2r^3\right)$$

$$V_1 + V_1 = \pi r^2 h + \frac{1}{3}\frac{\pi h}{R-r}\left(R^3 - 3Rr^2 + 2r^3\right)$$

$$= \frac{1}{3}\frac{\pi h}{R-r}\left(3r^2(R-r) + R^3 - 3Rr^2 + 2r^3\right)$$

$$= \frac{1}{3}\frac{\pi h}{R-r}\left(R^3 - r^3\right)$$

$$= \frac{1}{3}\frac{\pi h}{R-r}(R-r)\left(R^3 + rR + r^2\right)$$

$$= \frac{1}{3}\pi h\left(R^3 + rR + r^2\right) \quad unit^3$$

Exercise

Find the volume of the solid that is bounded above by the cylinder $z = 4 - x^2$, on the sides by the cylinder $x^2 + y^2 = 4$, and below by the *xy*-plane.

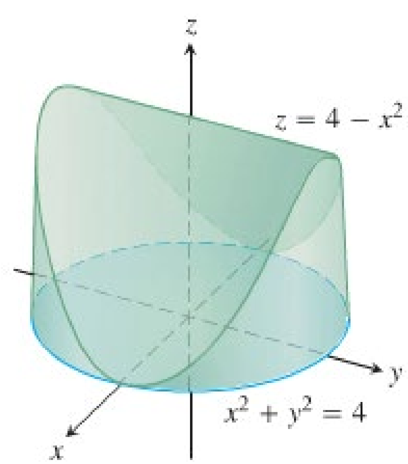
Solution

$z = 4 - x^2 \quad \to \quad 0 \le z \le 4 - x^2$

$x^2 + y^2 = 4 \quad \to \quad -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

Since it is symmetric, then $0 \le y \le \sqrt{4-x^2}$

$y = 0 \quad \to \quad x = \pm 2 \qquad 0 \le x \le 2$



$$V = 4\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{4-x^2} dzdydx$$

$$= 4\int_0^2 \int_0^{\sqrt{4-x^2}} z \Big|_0^{4-x^2} dydx$$

$$= 4\int_0^2 \int_0^{\sqrt{4-x^2}} \left(4 - x^2\right) dydx$$

$$= 4\int_0^2 \left(4-x^2\right) y \Big|_0^{\sqrt{4-x^2}} \, dx$$

$$= 4\int_0^2 \left(4-x^2\right)^{3/2} dx$$

$$x = 2\sin\alpha \quad \rightarrow \quad 4 - x^2 = 4\cos^2\alpha$$

$$dx = 2\cos\alpha \, d\alpha$$

$$\begin{cases} x = 2 & \rightarrow \alpha = \sin^{-1} 1 = \frac{\pi}{2} \\ x = 0 & \rightarrow \alpha = \sin^{-1} 0 = 0 \end{cases}$$

$$= 4\int_0^{\pi/2} 16\cos^4\alpha \, d\alpha$$

$$= 64\int_0^{\pi/2} \left(\frac{1+\cos 2\alpha}{2}\right)^2 d\alpha$$

$$= 16\int_0^{\pi/2} \left(1 + 2\cos 2\alpha + \cos^2 2\alpha\right) d\alpha$$

$$= 16\int_0^{\pi/2} \left(1 + 2\cos 2\alpha + \frac{1}{2} + \frac{1}{2}\cos 4\alpha\right) d\alpha$$

$$= 16\left(\frac{3}{2}\alpha + \sin 2\alpha + \frac{1}{8}\sin 4\alpha \,\Big|_0^{\pi/2}\right.$$

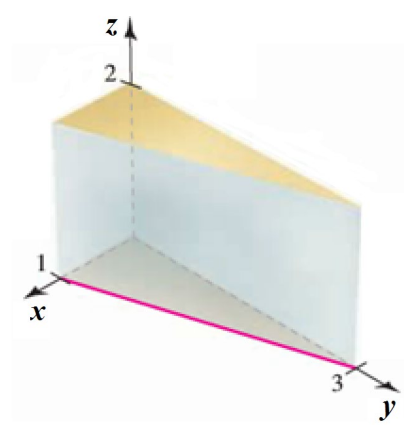
$$= 16\left(\frac{3\pi}{4}\right)$$

$$= 12\pi \quad unit^3$$

Exercise

Find the volume of the prism in the first octant bounded by the planes $y = 3 - 3x$ *and* $z = 2$

Solution



$$V = \int_0^1 \int_0^{3-3x} \int_0^2 dz\,dy\,dx$$

$$= \int_0^1 \int_0^{3-3x} z \Big|_0^2 \, dy\,dx$$

$$= 2\int_0^1 \int_0^{3-3x} dy\,dx$$

$$= 2\int_0^1 y \Big|_0^{3-3x} dx$$

$$= 2\int_0^1 (3-3x)\,dx$$

$$= 2\left(3x - \frac{3}{2}x^2 \Big|_0^1\right.$$

$$= 2\left(3 - \frac{3}{2}\right)$$

$$\underline{= 3 \quad unit^3}$$

Exercise

Find the volume of the prism in the first octant bounded by the planes $x^2 + y^2 = 4$ *and* $x^2 + z^2 = 4$

Solution

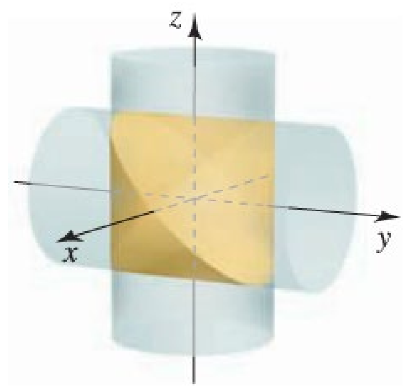
$$x^2 + y^2 = 4 \quad \to \quad y^2 \le 4 - x^2$$

$$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$$

$$x^2 + z^2 = 4 \quad \to \quad z^2 \le 4 - x^2$$

$$-\sqrt{4-x^2} \le z \le \sqrt{4-x^2}$$

$$x^2 \le 4 \quad \to \quad -2 \le x \le 2$$



$$V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dzdydx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} z \Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dydx$$

$$= 2\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4-x^2}\; dydx$$

$$= 2\int_{-2}^{2} \sqrt{4-x^2}\; y \Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= 4\int_{-2}^{2}\left(4 - x^2\right)dx$$

$$= 4\left(4x - \frac{1}{3}x^3 \right|_{-2}^{2}$$

$$= 8\left(8 - \frac{8}{3}\right)$$

$$= \frac{128}{3} \ \ unit^3$$