

## ***Solution***      **Section 3.1 – Inner Products**

### ***Exercise***

Let  $\langle \vec{u}, \vec{v} \rangle$  be the Euclidean inner product on  $\mathbb{R}^2$ , and let  $\vec{u} = (1, 1)$ ,  $\vec{v} = (3, 2)$ ,  $\vec{w} = (0, -1)$ , and  $k = 3$ . Compute the following.

- |  |   |                             |
|--|---|-----------------------------|
| a) $\langle \vec{u}, \vec{v} \rangle$  | c) $\langle \vec{u} + \vec{v}, \vec{w} \rangle$ | e) $d(\vec{u}, \vec{v})$    |
| b) $\langle k\vec{v}, \vec{w} \rangle$ | d) $\ \vec{v}\ $                                | f) $\ \vec{u} - k\vec{v}\ $ |

### ***Solution***

$$\begin{aligned} \text{a) } \langle \vec{u}, \vec{v} \rangle &= 1(3) + 1(2) \\ &= 5 \end{aligned}$$

$$\begin{aligned} \text{b) } \langle k\vec{v}, \vec{w} \rangle &= \langle 3\vec{v}, \vec{w} \rangle \\ &= 9 \cdot 0 + 6 \cdot (-1) \\ &= -6 \end{aligned}$$

$$\begin{aligned} \text{c) } \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \\ &= 1 \cdot 0 + 1 \cdot (-1) + 3 \cdot 0 + 2 \cdot (-1) \\ &= -3 \end{aligned}$$

$$\begin{aligned} \text{d) } \|\vec{v}\| &= \sqrt{\langle \vec{v}, \vec{v} \rangle} \\ &= \sqrt{3^2 + 2^2} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} \text{e) } d(\vec{u}, \vec{v}) &= \|\vec{u} - \vec{v}\| \\ &= \|(-2, -1)\| \\ &= \sqrt{(-2)^2 + (-1)^2} \\ &= \sqrt{5} \end{aligned}$$

$$\begin{aligned} \text{f) } \|\vec{u} - k\vec{v}\| &= \|(1, 1) - 3(3, 2)\| \\ &= \|(-8, -5)\| \\ &= \sqrt{(-8)^2 + (-5)^2} \\ &= \sqrt{89} \end{aligned}$$

**Exercise**

Let  $\langle \vec{u}, \vec{v} \rangle$  be the Euclidean inner product on  $\mathbb{R}^2$ , and let  $\vec{u} = (3, -2)$ ,  $\vec{v} = (4, 5)$ ,  $\vec{w} = (-1, 6)$ , and  $k = -4$ . Verify the following.

$$a) \quad \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$$

$$d) \quad \langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle$$

$$b) \quad \langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$$

$$e) \quad \langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$$

$$c) \quad \langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$$

**Solution**

$$a) \quad \langle \vec{u}, \vec{v} \rangle = 3 \cdot 4 + (-2) \cdot (5)$$

$$= 2$$

$$\langle \vec{v}, \vec{u} \rangle = 4 \cdot 3 + (5) \cdot (-2)$$

$$= 2$$

$$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \quad \checkmark$$

$$b) \quad \langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle (7, 3), (-1, 6) \rangle$$

$$= 7(-1) + 3(6)$$

$$= 11$$

$$\langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle = (3)(-1) + (-2)(6) + (4)(-1) + (5)(6)$$

$$= 11$$

$$\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \quad \checkmark$$

$$c) \quad \langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle (3, -2), (3, 11) \rangle$$

$$= 3(3) + (-2)(11)$$

$$= -13$$

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle = (3)(4) + (-2)(5) + (3)(-1) + (-2)(6)$$

$$= -13$$

$$\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle \quad \checkmark$$

$$d) \quad \langle k\vec{u}, \vec{v} \rangle = (-4 \cdot 3) \cdot 4 + ((-4)(-2)) \cdot (5)$$

$$= -8$$

$$k \langle \vec{u}, \vec{v} \rangle = (-4)(3 \cdot 4 + (-2) \cdot (5))$$

$$= -8$$

$$\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle \quad \checkmark$$

$$\begin{aligned}
 e) \quad \langle \vec{0}, \vec{v} \rangle &= 0 \cdot 4 + 0 \cdot (5) \\
 &= 0 \\
 \langle \vec{v}, \vec{0} \rangle &= 4 \cdot 0 + (5) \cdot (0) \\
 &= 0 \\
 \langle \vec{0}, \vec{v} \rangle &= \langle \vec{v}, \vec{0} \rangle = 0 \quad \checkmark
 \end{aligned}$$

### Exercise

Let  $\langle \vec{u}, \vec{v} \rangle$  be the Euclidean inner product on  $\mathbb{R}^3$ , and let  $\vec{u} = (1, 0, -2)$ ,  $\vec{v} = (5, 1, 2)$ ,  $\vec{w} = (5, 2, -1)$ , and  $k = 3$ . Compute the following.

$$\begin{array}{lll}
 a) \quad \langle \vec{u}, \vec{v} \rangle & c) \quad \langle \vec{u} + \vec{v}, \vec{w} \rangle & e) \quad d(\vec{u}, \vec{v}) \\
 b) \quad \langle k\vec{v}, \vec{w} \rangle & d) \quad \|\vec{v}\| & f) \quad \|\vec{u} - k\vec{v}\|
 \end{array}$$

### Solution

$$\begin{aligned}
 a) \quad \langle \vec{u}, \vec{v} \rangle &= (1, 0, -2) \cdot (5, 1, 2) \\
 &= 1(5) + 0(1) - 2(2) \\
 &= 1 \\
 b) \quad \langle k\vec{v}, \vec{w} \rangle &= \langle 3\vec{v}, \vec{w} \rangle \\
 &= (3(5, 1, 2)) \cdot (5, 2, -1) \\
 &= (15, 3, 6) \cdot (5, 2, -1) \\
 &= 15(5) + 3(2) + 6(-1) \\
 &= 75 \\
 c) \quad \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \\
 &= (1, 0, -2) \cdot (5, 2, -1) + (5, 1, 2) \cdot (5, 2, -1) \\
 &= (5 + 0 + 2) + (25 + 2 - 2) \\
 &= 32 \\
 d) \quad \|\vec{v}\| &= \sqrt{\langle \vec{v}, \vec{v} \rangle} \\
 &= \sqrt{(5, 1, 2) \cdot (5, 1, 2)} \\
 &= \sqrt{25 + 1 + 4} \\
 &= \sqrt{30} \\
 e) \quad d(\vec{u}, \vec{v}) &= \|\vec{u} - \vec{v}\|
 \end{aligned}$$

$$\begin{aligned} &= \|(1, 0, -2) - (5, 1, 2)\| \\ &= \|(-4, -1, -4)\| \\ &= \sqrt{16 + 1 + 16} \\ &= \sqrt{33} \end{aligned}$$

$$\begin{aligned} f) \quad \|\vec{u} - k\vec{v}\| &= \|(1, 0, -2) - 3(5, 1, 2)\| \\ &= \|(-14, -3, -8)\| \\ &= \sqrt{196 + 9 + 64} \\ &= \sqrt{269} \end{aligned}$$

### Exercise

Let  $\langle \vec{u}, \vec{v} \rangle$  be the Euclidean inner product on  $\mathbb{R}^3$ , and let  $\vec{u} = (2, 4, 3)$ ,  $\vec{v} = (0, -3, -4)$ ,  $\vec{w} = (6, 3, 1)$ , and  $k = 2$ . Verify the following.

- |   |  |
|---|--|
| a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$  | d) $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$   |
| b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ | e) $\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$ |
| c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$ |  |

### Solution

$$\begin{aligned} a) \quad \langle \vec{u}, \vec{v} \rangle &= (2, 4, 3) \cdot (0, -3, -4) \\ &= -12 - 12 \\ &= -24 \end{aligned}$$

$$\begin{aligned} \langle \vec{v}, \vec{u} \rangle &= (0, -3, -4) \cdot (2, 4, 3) \\ &= -12 - 12 \\ &= -24 \end{aligned}$$

$$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \quad \checkmark$$

$$\begin{aligned} b) \quad \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle (2, 4, 3) + (0, -3, -4), (6, 3, 1) \rangle \\ &= \langle (2, 1, -1), (6, 3, 1) \rangle \\ &= 12 + 3 - 1 \\ &= 14 \end{aligned}$$

$$\begin{aligned} \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle &= (2, 4, 3) \cdot (6, 3, 1) + (0, -3, -4) \cdot (6, 3, 1) \\ &= (12 + 12 + 3) + (0 - 9 - 4) \end{aligned}$$

$$\underline{=14}$$

$$\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \quad \checkmark$$

$$\begin{aligned} c) \quad \langle \vec{u}, \vec{v} + \vec{w} \rangle &= \langle (2, 4, 3), (0, -3, -4) + (6, 3, 1) \rangle \\ &= \langle (2, 4, 3), (6, 0, -3) \rangle \\ &= 12 + 0 - 9 \\ &\underline{=3} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle &= (2, 4, 3) \cdot (0, -3, -4) + (2, 4, 3) \cdot (6, 3, 1) \\ &= (0 - 12 - 12) + (12 + 12 + 3) \\ &\underline{=3} \end{aligned}$$

$$\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle \quad \checkmark$$

$$\begin{aligned} d) \quad \langle k\vec{u}, \vec{v} \rangle &= 2(2, 4, 3) \cdot (0, -3, -4) \\ &= (4, 8, 6) \cdot (0, -3, -4) \\ &= 0 - 24 - 24 \\ &\underline{=-48} \end{aligned}$$

$$\begin{aligned} k\langle \vec{u}, \vec{v} \rangle &= (2)((2, 4, 3) \cdot (0, -3, -4)) \\ &= (2)(-12 - 12) \\ &\underline{=-48} \end{aligned}$$

$$\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle \quad \checkmark$$

$$\begin{aligned} e) \quad \langle \vec{0}, \vec{v} \rangle &= (0, 0, 0) \cdot (0, -3, -4) \\ &= 0 + 0 + 0 \\ &\underline{=0} \end{aligned}$$

$$\begin{aligned} \langle \vec{v}, \vec{0} \rangle &= (0, -3, -4) \cdot (0, 0, 0) \\ &= 0 + 0 + 0 \\ &\underline{=0} \end{aligned}$$

$$\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0 \quad \checkmark$$

**Exercise**

Let  $\langle \vec{u}, \vec{v} \rangle$  be the Euclidean inner product on  $\mathbb{R}^2$ , and let  $\vec{u} = (1, 1)$ ,  $\vec{v} = (3, 2)$ ,  $\vec{w} = (0, -1)$  and  $k = 3$ . Compute the following for the weighted Euclidean inner product  $\langle \vec{u}, \vec{v} \rangle = 2u_1v_1 + 3u_2v_2$ .

a)  $\langle \vec{u}, \vec{v} \rangle$

c)  $\langle \vec{u} + \vec{v}, \vec{w} \rangle$

e)  $d(\vec{u}, \vec{v})$

b)  $\langle k\vec{v}, \vec{w} \rangle$

d)  $\|\vec{v}\|$

f)  $\|\vec{u} - k\vec{v}\|$

**Solution**

$$\begin{aligned} \text{a) } \langle \vec{u}, \vec{v} \rangle &= 2(1)(3) + 3(1)(2) \\ &= 12 \end{aligned}$$

$$\begin{aligned} \text{b) } \langle k\vec{v}, \vec{w} \rangle &= 2(3 \cdot 3)(0) + 3(3 \cdot 2)(-1) \\ &= -18 \end{aligned}$$

$$\begin{aligned} \text{c) } \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \\ &= 1 \cdot 0 + 1 \cdot (-1) + 3 \cdot 0 + 2 \cdot (-1) \\ &= -3 \end{aligned}$$

$$\begin{aligned} \text{d) } \|\vec{v}\| &= \sqrt{\langle \vec{v}, \vec{v} \rangle} \\ &= \sqrt{2(3)(3) + 3(2)(2)} \\ &= \sqrt{30} \end{aligned}$$

$$\begin{aligned} \text{e) } d(\vec{u}, \vec{v}) &= \|\vec{u} - \vec{v}\| \\ &= \|\langle (-2, -1) \rangle\| \\ &= \sqrt{2(-2)(-2) + 3(-1)(-1)} \\ &= \sqrt{11} \end{aligned}$$

$$\begin{aligned} \text{f) } \|\vec{u} - k\vec{v}\| &= \|(1, 1) - 3(3, 2)\| \\ &= \|\langle (-8, -5) \rangle\| \\ &= \sqrt{2(-8)^2 + 3(-5)^2} \\ &= \sqrt{203} \end{aligned}$$

**Exercise**

Let  $\langle \vec{u}, \vec{v} \rangle$  be the Euclidean inner product on  $\mathbb{R}^2$ , and let  $\vec{u} = (3, -2)$ ,  $\vec{v} = (4, 5)$ ,  $\vec{w} = (-1, 6)$ , and  $k = -4$ . Verify the following for the weighted Euclidean inner product  $\langle \vec{u}, \vec{v} \rangle = 4u_1v_1 + 5u_2v_2$ .

$$a) \quad \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$$

$$d) \quad \langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$$

$$b) \quad \langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$$

$$e) \quad \langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$$

$$c) \quad \langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$$

**Solution**

$$a) \quad \langle \vec{u}, \vec{v} \rangle = 4 \cdot 3 \cdot 4 + 5 \cdot (-2) \cdot (5) \\ = -2$$

$$\langle \vec{v}, \vec{u} \rangle = 4 \cdot 4 \cdot 3 + 5 \cdot (5) \cdot (-2) \\ = -2$$

$$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \quad \checkmark$$

$$b) \quad \langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle (7, 3), (-1, 6) \rangle \\ = 4 \cdot 7 \cdot (-1) + 5 \cdot 3 \cdot (6) \\ = 62$$

$$\langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle = (4 \cdot (3)(-1) + 5 \cdot (-2)(6)) + (4 \cdot (4)(-1) + 5 \cdot (5)(6)) \\ = (-12 - 60) + (-16 + 150) \\ = 62$$

$$\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \quad \checkmark$$

$$c) \quad \langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle (3, -2), (4, 5) + (-1, 6) \rangle \\ = \langle (3, -2), (3, 11) \rangle \\ = 4 \cdot (3)(3) + 5 \cdot (-2)(11) \\ = -74$$

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle = (4 \cdot (3)(4) + 5 \cdot (-2)(5)) + (4 \cdot (3)(-1) + 5 \cdot (-2)(6)) \\ = (48 - 50) + (-12 - 60) \\ = -74$$

$$\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle \quad \checkmark$$

$$d) \quad \langle k\vec{u}, \vec{v} \rangle = \langle -4(3, -2), (4, 5) \rangle \\ = \langle (-12, 8), (4, 5) \rangle$$

$$= 4 \cdot (-12)(4) + 5 \cdot (8)(5)$$

$$= 8$$

$$k \langle \vec{u}, \vec{v} \rangle = (-4)(4 \cdot 3 \cdot 4 + 5 \cdot (-2) \cdot (5))$$

$$= 8$$

$$\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle \quad \checkmark$$

$$e) \quad \langle \vec{0}, \vec{v} \rangle = 4 \cdot (0)(4) + 5 \cdot (0)(5)$$

$$= 0$$

$$\langle \vec{v}, \vec{0} \rangle = 4 \cdot (4)(0) + 5 \cdot (5)(0)$$

$$= 0$$

$$\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0 \quad \checkmark$$

### Exercise

Let  $\langle \vec{u}, \vec{v} \rangle$  be the Euclidean inner product on  $\mathbb{R}^2$ , and let  $\vec{u} = (-3, 2)$ ,  $\vec{v} = (5, 4)$ ,  $\vec{w} = (1, -6)$ , and  $k = 2$ . Verify the following for the weighted Euclidean inner product  $\langle \vec{u}, \vec{v} \rangle = 4u_1v_1 + 5u_2v_2$

$$a) \quad \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$$

$$d) \quad \langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle$$

$$b) \quad \langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$$

$$e) \quad \langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$$

$$c) \quad \langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$$

### Solution

$$a) \quad \langle \vec{u}, \vec{v} \rangle = 4(-3)(5) + 5(2)(4)$$

$$= -60 + 40$$

$$= -20$$

$$\langle \vec{v}, \vec{u} \rangle = 4(5)(-3) + 5(4)(2)$$

$$= -60 + 40$$

$$= -20$$

$$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \quad \checkmark$$

$$b) \quad \langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle (-3, 2) + (5, 4), (1, -6) \rangle$$

$$= \langle (2, 6), (1, -6) \rangle$$

$$= 4 \cdot (2)(1) + 5 \cdot (6)(-6)$$

$$= 8 - 180$$



$$\underline{= -172}$$

$$\begin{aligned}\langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle &= (4 \cdot (-3)(1) + 5 \cdot (2)(-6)) + (4 \cdot (5)(1) + 5 \cdot (4)(-6)) \\ &= (-12 - 60) + (20 - 120)\end{aligned}$$

$$\underline{= -172}$$

$$\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \quad \checkmark$$

$$\begin{aligned}c) \quad \langle \vec{u}, \vec{v} + \vec{w} \rangle &= \langle (-3, 2), (5, 4) + (1, -6) \rangle \\ &= \langle (-3, 2), (6, -2) \rangle \\ &= 4 \cdot (-3)(6) + 5 \cdot (2)(-2) \\ &= -72 - 20 \\ &\underline{= -92}\end{aligned}$$

$$\begin{aligned}\langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle &= (4 \cdot (-3)(5) + 5 \cdot (2)(4)) + (4 \cdot (-3)(1) + 5 \cdot (2)(-6)) \\ &= (-60 + 40) + (-12 - 60) \\ &\underline{= -92}\end{aligned}$$

$$\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle \quad \checkmark$$

$$\begin{aligned}d) \quad \langle k\vec{u}, \vec{v} \rangle &= \langle 2(-3, 2), (5, 4) \rangle \\ &= \langle (-6, 4), (5, 4) \rangle \\ &= 4 \cdot (-6)(5) + 5 \cdot (4)(4) \\ &= -120 + 80 \\ &\underline{= -40}\end{aligned}$$

$$\begin{aligned}k\langle \vec{u}, \vec{v} \rangle &= 2 \cdot (4 \cdot (-3)(5) + 5 \cdot (2)(4)) \\ &= 2 \cdot (-60 + 40) \\ &\underline{= -40}\end{aligned}$$

$$\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle \quad \checkmark$$

$$\begin{aligned}e) \quad \langle \vec{0}, \vec{v} \rangle &= 4 \cdot (0)(5) + 5 \cdot (0)(4) \\ &\underline{= 0}\end{aligned}$$

$$\begin{aligned}\langle \vec{v}, \vec{0} \rangle &= 4 \cdot (5)(0) + 5 \cdot (4)(0) \\ &\underline{= 0}\end{aligned}$$

$$\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0 \quad \checkmark$$

**Exercise**

Let  $\langle \vec{u}, \vec{v} \rangle$  be the Euclidean inner product on  $\mathbb{R}^3$ , and let  $\vec{u} = (2, 1, -2)$ ,  $\vec{v} = (-1, 3, 2)$ ,  $\vec{w} = (2, 1, 0)$  and  $k = 2$ .

Compute the following for the weighted Euclidean inner product  $\langle \vec{u}, \vec{v} \rangle = 2u_1v_1 + 3u_2v_2 + u_3v_3$ .

- |  |   |                             |
|--|---|-----------------------------|
| a) $\langle \vec{u}, \vec{v} \rangle$  | c) $\langle \vec{u} + \vec{v}, \vec{w} \rangle$ | e) $d(\vec{u}, \vec{v})$    |
| b) $\langle k\vec{v}, \vec{w} \rangle$ | d) $\ \vec{v}\ $                                | f) $\ \vec{u} - k\vec{v}\ $ |

**Solution**

$$\begin{aligned} \text{a) } \langle \vec{u}, \vec{v} \rangle &= 2 \cdot (2)(-1) + 3(1)(3) + (-2)(2) \\ &= -4 + 9 - 4 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{b) } \langle k\vec{v}, \vec{w} \rangle &= \langle 2(-1, 3, 2), (2, 1, 0) \rangle \\ &= \langle (-2, 6, 4), (2, 1, 0) \rangle \\ &= 2 \cdot (-2)(2) + 3(6)(1) + (4)(0) \\ &= -8 + 18 \\ &= 10 \end{aligned}$$

$$\begin{aligned} \text{c) } \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle (2, 1, -2) + (-1, 3, 2), (2, 1, 0) \rangle \\ &= \langle (1, 4, 0), (2, 1, 0) \rangle \\ &= 2 \cdot (1)(2) + 3(4)(1) + (0)(0) \\ &= 14 \end{aligned}$$

$$\begin{aligned} \text{d) } \|\vec{v}\| &= \sqrt{\langle \vec{v}, \vec{v} \rangle} \\ &= \sqrt{2 \cdot (-1)(-1) + 3(3)(3) + (2)(2)} \\ &= \sqrt{2 + 27 + 4} \\ &= \sqrt{33} \end{aligned}$$

$$\begin{aligned} \text{e) } d(\vec{u}, \vec{v}) &= \|\vec{u} - \vec{v}\| \\ &= \|(2, 1, -2) - (-1, 3, 2)\| \\ &= \|(3, -2, -4)\| \\ &= \sqrt{2 \cdot (3)(3) + 3(-2)(-2) + (-4)(-4)} \\ &= \sqrt{18 + 12 + 16} \\ &= \sqrt{29} \end{aligned}$$

$$\begin{aligned}
 \text{f) } \|\vec{u} - k\vec{v}\| &= \|(2, 1, -2) - 2(-1, 3, 2)\| \\
 &= \|(2, 1, -2) - (-2, 6, 4)\| \\
 &= \|(4, -5, -6)\| \\
 &= \sqrt{2(16) + 3(25) + 36} \\
 &= \sqrt{32 + 75 + 36} \\
 &= \sqrt{143}
 \end{aligned}$$

### Exercise

Let  $\vec{u} = (u_1, u_2)$  and  $\vec{v} = (v_1, v_2)$ . Show that the following are inner product on  $\mathbb{R}^2$  by verifying that the inner product axioms hold.  $\langle \vec{u}, \vec{v} \rangle = 3u_1v_1 + 5u_2v_2$

### Solution

$$\text{Let } \vec{w} = (w_1, w_2)$$

$$\begin{aligned}
 \text{Axiom 1: } \langle \vec{u}, \vec{v} \rangle &= 3u_1v_1 + 5u_2v_2 \\
 &= 3v_1u_1 + 5v_2u_2 \\
 &= \langle \vec{v}, \vec{u} \rangle \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{Axiom 2: } \langle \vec{u} + \vec{v}, \vec{w} \rangle &= 3(u_1 + v_1)w_1 + 5(u_2 + v_2)w_2 \\
 &= 3(u_1w_1 + v_1w_1) + 5(u_2w_2 + v_2w_2) \\
 &= 3u_1w_1 + 3v_1w_1 + 5u_2w_2 + 5v_2w_2 \\
 &= (3u_1w_1 + 5u_2w_2) + (3v_1w_1 + 5v_2w_2) \\
 &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{Axiom 3: } \langle k\vec{u}, \vec{v} \rangle &= 3(ku_1)v_1 + 5(ku_2)v_2 \\
 &= k(3u_1v_1 + 5u_2v_2) \\
 &= k\langle \vec{u}, \vec{v} \rangle \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{Axiom 4: } \langle \vec{v}, \vec{v} \rangle &= 3v_1v_1 + 5v_2v_2 \\
 &= 3v_1^2 + 5v_2^2 \geq 0 \\
 v_1 = v_2 = 0 &\text{ iff } \vec{v} = \vec{0} \quad \checkmark
 \end{aligned}$$

**Exercise**

Let  $\vec{u} = (u_1, u_2, u_3)$  and  $\vec{v} = (v_1, v_2, v_3)$ . Show that the following are inner product on  $\mathbb{R}^3$  by verifying that the inner product axioms hold.  $\langle \vec{u}, \vec{v} \rangle = 2u_1v_1 + 3u_2v_2 + 5u_3v_3$

**Solution**

$$\text{Let } \vec{w} = (w_1, w_2, w_3)$$

$$\begin{aligned} \text{Axiom 1: } \langle \vec{u}, \vec{v} \rangle &= 2u_1v_1 + 3u_2v_2 + 5u_3v_3 \\ &= 2v_1u_1 + 3v_2u_2 + 5v_3u_3 \\ &= \langle \vec{v}, \vec{u} \rangle \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{Axiom 2: } \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle (u_1, u_2, u_3) + (v_1, v_2, v_3), (w_1, w_2, w_3) \rangle \\ &= \langle (u_1 + v_1, u_2 + v_2, u_3 + v_3), (w_1, w_2, w_3) \rangle \\ &= 2(u_1 + v_1)w_1 + 3(u_2 + v_2)w_2 + 5(u_3 + v_3)w_3 \\ &= 2u_1w_1 + 2v_1w_1 + 3u_2w_2 + 3v_2w_2 + 5u_3w_3 + 5v_3w_3 \\ &= (2u_1w_1 + 3u_2w_2 + 5u_3w_3) + (2v_1w_1 + 3v_2w_2 + 5v_3w_3) \\ &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{Axiom 3: } \langle k\vec{u}, \vec{v} \rangle &= \langle k(u_1, u_2, u_3), (v_1, v_2, v_3) \rangle \\ &= \langle (ku_1, ku_2, ku_3), (v_1, v_2, v_3) \rangle \\ &= 2(ku_1)v_1 + 3(ku_2)v_2 + 5(ku_3)v_3 \\ &= k(2u_1v_1 + 3u_2v_2 + 5u_3v_3) \\ &= k\langle \vec{u}, \vec{v} \rangle \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{Axiom 4: } \langle \vec{v}, \vec{v} \rangle &= 2v_1v_1 + 3v_2v_2 + 5v_3v_3 \\ &= 2v_1^2 + 3v_2^2 + 5v_3^2 \geq 0 \\ v_1 = v_2 = v_3 = 0 &\text{ iff } \vec{v} = \vec{0} \quad \checkmark \end{aligned}$$

**Exercise**

Show that the following identity holds for the vectors in any inner product space

$$\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

**Solution**

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 &= \langle \vec{u} + \vec{v}, \vec{u} + \vec{v} \rangle + \langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle \\ &= \langle \vec{u}, \vec{u} + \vec{v} \rangle + \langle \vec{v}, \vec{u} + \vec{v} \rangle + \langle \vec{u}, \vec{u} - \vec{v} \rangle - \langle \vec{v}, \vec{u} - \vec{v} \rangle \\ &= \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{u} \rangle + \langle \vec{v}, \vec{v} \rangle + \langle \vec{u}, \vec{u} \rangle - \langle \vec{u}, \vec{v} \rangle - \langle \vec{v}, \vec{u} \rangle + \langle \vec{v}, \vec{v} \rangle \\ &= 2\langle \vec{u}, \vec{u} \rangle + 2\langle \vec{v}, \vec{v} \rangle \\ &= 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2 \quad \checkmark \end{aligned}$$

**Exercise**

Show that the following identity holds for the vectors in any inner product space

$$\langle \vec{u}, \vec{v} \rangle = \frac{1}{4} \left( \|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 \right)$$

**Solution**

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= \langle \vec{u} + \vec{v}, \vec{u} + \vec{v} \rangle \\ &= \|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2 \\ \|\vec{u} - \vec{v}\|^2 &= \langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle \\ &= \|\vec{u}\|^2 - 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2 \\ \|\vec{u} + \vec{v}\|^2 &= \|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2 \\ - \|\vec{u} - \vec{v}\|^2 &= \|\vec{u}\|^2 - 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2 \\ \hline \|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 &= 4\langle \vec{u}, \vec{v} \rangle \\ \langle \vec{u}, \vec{v} \rangle &= \frac{1}{4} \left( \|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 \right) \quad \checkmark \end{aligned}$$

**Exercise**

Prove that  $\|k\vec{v}\| = |k| \|\vec{v}\|$

**Solution**

$$\begin{aligned}\|k\vec{v}\|^2 &= \langle k\vec{v}, \vec{v} \rangle \\ &= k^2 \langle \vec{v}, \vec{v} \rangle \\ &= k^2 \|\vec{v}\|^2\end{aligned}$$

$$\|k\vec{v}\| = k \|\vec{v}\| \quad \checkmark$$

## ***Solution***      **Section 3.2 – Angle and Orthogonality in Inner Product Spaces**

### ***Exercise***

Which of the following form orthonormal set?

$$\{(1, 0), (0, 2)\} \text{ in } \mathbb{R}^2$$

### **Solution**

$$\begin{aligned} (1, 0) \cdot (0, 2) &= 1(0) + 0(2) \\ &= 0 \end{aligned}$$

Therefore, they are ***orthonormal*** set.

### ***Exercise***

Which of the following form orthonormal set?

$$\{(2, -4), (2, 1)\} \text{ in } \mathbb{R}^2$$

### **Solution**

$$\begin{aligned} (2, -4) \cdot (2, 1) &= 4 - 4 \\ &= 0 \end{aligned}$$

Therefore, they are ***orthonormal*** set.

### ***Exercise***

Which of the following form orthonormal set?

$$\left\{ \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right), \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right\} \text{ in } \mathbb{R}^2$$

### **Solution**

$$\begin{aligned} \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \cdot \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) &= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \left( -\frac{1}{\sqrt{2}} \right) \frac{1}{\sqrt{2}} \\ &= \frac{1}{2} - \frac{1}{2} \\ &= 0 \end{aligned}$$

Therefore, they are ***orthonormal*** set.

**Exercise**

Which of the following form orthonormal set?

$$\left\{ \left( -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right), \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right\} \text{ in } \mathbb{R}^2$$

**Solution**

$$\begin{aligned} \left( -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \cdot \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) &= \left( -\frac{1}{\sqrt{2}} \right) \frac{1}{\sqrt{2}} + \left( -\frac{1}{\sqrt{2}} \right) \frac{1}{\sqrt{2}} \\ &= -\frac{1}{2} - \frac{1}{2} \\ &= \underline{-1 \neq 0} \end{aligned}$$

Therefore, they are **not orthonormal** set

**Exercise**

Which of the following form orthonormal set?

$$\left\{ \left( \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right), \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) \right\} \text{ in } \mathbb{R}^3$$

**Solution**

$$\begin{aligned} \left( \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right) \cdot \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) &= \frac{1}{\sqrt{6}} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}} \left( -\frac{1}{\sqrt{2}} \right) + 0 \\ &= \underline{0} \end{aligned}$$

Therefore, they are **orthonormal** set.

**Exercise**

Which of the following form orthonormal set?

$$\{(4, -1, 1), (-1, 0, 4), (-4, -17, -1)\} \text{ in } \mathbb{R}^3$$

**Solution**

$$\begin{aligned} (4, -1, 1) \cdot (-1, 0, 4) \cdot (-4, -17, -1) &= 16 + 0 - 4 \\ &= \underline{12 \neq 0} \end{aligned}$$

Therefore, they are **not orthonormal** set.



**Exercise**

Which of the following form orthonormal set?

$$\left\{ \left( \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right), \left( \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right), \left( \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right) \right\} \text{ in } \mathbb{R}^3$$

**Solution**

$$\begin{aligned} \left( \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right) \cdot \left( \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right) &= \frac{2}{3} \frac{2}{3} - \frac{2}{3} \frac{1}{3} + \frac{1}{3} \left( -\frac{2}{3} \right) \\ &= \frac{4}{9} - \frac{2}{9} - \frac{2}{9} \\ &= \frac{0}{9} = 0 \end{aligned}$$

Therefore, they are **not orthonormal** set.

**Exercise**

Which of the following form orthonormal set?

$$\left\{ \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right), \left( -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \right\} \text{ in } \mathbb{R}^3$$

**Solution**

$$\begin{aligned} \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \cdot \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right) &= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{3}} + 0 - \frac{1}{\sqrt{2}} \frac{1}{\sqrt{3}} \\ &= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}} \frac{1}{\sqrt{3}} \\ &= 0 \end{aligned}$$

Therefore, they are **not orthonormal** set.

**Exercise**

Which of the following form orthonormal set?

$$\left\{ \left( \frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}} \right), \left( 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right), \left( -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right) \right\} \text{ in } \mathbb{R}^4$$

**Solution**

$$\left( \frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}} \right) \cdot \left( 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) = 0 + 0 + 0 + 0 = 0$$

$$= 0 \mid$$

Therefore, they are *orthonormal* set.

### Exercise

Which of the following form orthonormal set?

$$\left\{ \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right), \left( 0, 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right), \left( -\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{3} \right) \right\} \text{ in } \mathbb{R}^4$$

### Solution

$$\left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right) \cdot \left( 0, 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \cdot \left( -\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{3} \right) = 0 + 0 + 0 + 0$$

$$= 0 \mid$$

Therefore, they are *orthonormal* set.

### Exercise

Find the cosine of the angle between  $\vec{u}$  and  $\vec{v}$ .

$$\vec{u} = (1, -3), \quad \vec{v} = (2, 4)$$

### Solution

$$\begin{aligned} \|\vec{u}\| &= \sqrt{1^2 + (-3)^2} \\ &= \sqrt{10} \mid \end{aligned}$$

$$\begin{aligned} \|\vec{v}\| &= \sqrt{2^2 + 4^2} \\ &= \sqrt{20} \mid \end{aligned}$$

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= 1(2) + (-3)(4) \\ &= -10 \mid \end{aligned}$$

$$\begin{aligned} \cos \theta &= \frac{-10}{\sqrt{10} \sqrt{20}} \\ &= -\frac{10}{\sqrt{200}} \\ &= -\frac{1}{\sqrt{2}} \mid \end{aligned}$$

$$\cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

**Exercise**

Find the cosine of the angle between  $\vec{u}$  and  $\vec{v}$ .

$$\vec{u} = (-1, 0); \quad \vec{v} = (3, 8)$$

**Solution**

$$\begin{aligned} \|\vec{u}\| &= \sqrt{(-1)^2 + 0^2} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \|\vec{v}\| &= \sqrt{3^2 + 8^2} \\ &= \sqrt{73} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= (-1)(3) + (0)(8) \\ &= -3 \end{aligned}$$

$$\begin{aligned} \cos \theta &= \frac{-3}{1\sqrt{73}} \\ &= -\frac{3}{\sqrt{73}} \end{aligned}$$

$$\cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

**Exercise**

Find the cosine of the angle between  $\vec{u}$  and  $\vec{v}$ .

$$\vec{u} = (-1, 5, 2); \quad \vec{v} = (2, 4, -9)$$

**Solution**

$$\begin{aligned} \|\vec{u}\| &= \sqrt{(-1)^2 + 5^2 + 2^2} \\ &= \sqrt{30} \end{aligned}$$

$$\begin{aligned} \|\vec{v}\| &= \sqrt{2^2 + 4^2 + (-9)^2} \\ &= \sqrt{101} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= (-1)(2) + (5)(4) + (2)(-9) \\ &= 0 \end{aligned}$$

$$\cos \theta = 0$$

$$\cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

**Exercise**

Find the cosine of the angle between  $\vec{u}$  and  $\vec{v}$ .

$$\vec{u} = (4, 1, 8); \quad \vec{v} = (1, 0, -3)$$

**Solution**

$$\begin{aligned}\|\vec{u}\| &= \sqrt{4^2 + 1^2 + 8^2} \\ &= 9\end{aligned}$$

$$\begin{aligned}\|\vec{v}\| &= \sqrt{1 + 0 + 9} \\ &= \sqrt{10}\end{aligned}$$

$$\begin{aligned}\langle \vec{u}, \vec{v} \rangle &= (4)(1) + (1)(0) + (8)(-3) \\ &= -20\end{aligned}$$

$$\cos \theta = \frac{-20}{9\sqrt{10}} \qquad \cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

**Exercise**

Find the cosine of the angle between  $\vec{u}$  and  $\vec{v}$ .

$$\vec{u} = (1, 0, 1, 0); \quad \vec{v} = (-3, -3, -3, -3)$$

**Solution**

$$\|\vec{u}\| = \sqrt{2}$$

$$\begin{aligned}\|\vec{v}\| &= \sqrt{9 + 9 + 9 + 9} \\ &= 12\end{aligned}$$

$$\begin{aligned}\langle \vec{u}, \vec{v} \rangle &= -3 + 0 - 3 + 0 \\ &= -6\end{aligned}$$

$$\cos \theta = \frac{-6}{12\sqrt{2}} \qquad \cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$
$$= -\frac{1}{2\sqrt{2}}$$

**Exercise**

Find the cosine of the angle between  $\vec{u}$  and  $\vec{v}$ .

$$\vec{u} = (2, 1, 7, -1); \quad \vec{v} = (4, 0, 0, 0)$$

**Solution**

$$\begin{aligned} \|\vec{u}\| &= \sqrt{2^2 + 1^2 + 7^2 + (-1)^2} \\ &= \sqrt{55} \end{aligned}$$

$$\begin{aligned} \|\vec{v}\| &= \sqrt{4^2 + 0} \\ &= 4 \end{aligned}$$

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= (2)(4) + (1)(0) + (7)(0) + (-1)(0) \\ &= 8 \end{aligned}$$

$$\begin{aligned} \cos \theta &= \frac{8}{4\sqrt{55}} \\ &= \frac{2}{\sqrt{55}} \end{aligned}$$

$$\cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

**Exercise**

Find the cosine of the angle between  $\vec{u}$  and  $\vec{v}$ .

$$\vec{u} = (1, 3, -5, 4), \quad \vec{v} = (2, -4, 4, 1)$$

**Solution**

$$\begin{aligned} \|\vec{u}\| &= \sqrt{1 + 9 + 25 + 16} \\ &= \sqrt{51} \end{aligned}$$

$$\begin{aligned} \|\vec{v}\| &= \sqrt{4 + 16 + 16 + 1} \\ &= \sqrt{37} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= 2 - 12 - 20 + 4 \\ &= -26 \end{aligned}$$

$$\cos \theta = \frac{-26}{\sqrt{51}\sqrt{37}}$$

$$\cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

**Exercise**

Find the cosine of the angle between  $\vec{u}$  and  $\vec{v}$ .

$$\vec{u} = (1, 2, 3, 4), \quad \vec{v} = (-1, -2, -3, -4)$$

**Solution**

$$\begin{aligned}\|\vec{u}\| &= \sqrt{1+4+9+16} \\ &= \sqrt{30} \quad | \end{aligned}$$

$$\begin{aligned}\|\vec{v}\| &= \sqrt{1+4+9+16} \\ &= \sqrt{30} \quad | \end{aligned}$$

$$\begin{aligned}\langle \vec{u}, \vec{v} \rangle &= -1 - 4 - 9 - 16 \\ &= -30 \quad | \end{aligned}$$

$$\begin{aligned}\cos \theta &= \frac{-30}{\sqrt{30}\sqrt{30}} \\ &= -1 \quad | \end{aligned}$$

$$\cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

**Exercise**

Find the cosine of the angle between  $A$  and  $B$ .

$$A = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$$

**Solution**

$$\begin{aligned}\|A\| &= \sqrt{\langle A, A \rangle} \\ &= \sqrt{2^2 + 6^2 + 1^2 + (-3)^2} \\ &= \sqrt{50} \\ &= 5\sqrt{2} \quad | \end{aligned}$$

$$\begin{aligned}\|B\| &= \sqrt{\langle B, B \rangle} \\ &= \sqrt{9 + 4 + 1 + 0} \\ &= \sqrt{14} \quad | \end{aligned}$$

$$\begin{aligned}\langle A, B \rangle &= 2(3) + 6(2) + 1(1) + (-3)(0) \\ &= 19 \quad | \end{aligned}$$

$$\cos \theta = \frac{19}{5\sqrt{2}\sqrt{14}}$$

$$\cos \theta = \frac{\langle A, B \rangle}{\|A\| \|B\|}$$

$$= \frac{19}{10\sqrt{7}} \quad |$$

### Exercise

Find the cosine of the angle between  $A$  and  $B$ .

$$A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$$

### Solution

$$\begin{aligned} \|A\| &= \sqrt{\langle A, A \rangle} \\ &= \sqrt{2^2 + 4^2 + (-1)^2 + 3^2} \\ &= \sqrt{30} \quad | \end{aligned}$$

$$\begin{aligned} \|B\| &= \sqrt{\langle B, B \rangle} \\ &= \sqrt{(-3)^2 + 1^2 + 4^2 + 2^2} \\ &= \sqrt{30} \quad | \end{aligned}$$

$$\begin{aligned} \langle A, B \rangle &= 2(-3) + 4(1) + (-1)(4) + 3(2) \\ &= 0 \quad | \end{aligned}$$

$$\begin{aligned} \cos \theta &= \frac{0}{30} & \cos \theta &= \frac{\langle A, B \rangle}{\|A\| \|B\|} \\ &= 0 \quad | \end{aligned}$$

### Exercise

Find the cosine of the angle between  $A$  and  $B$ .

$$A = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

### Solution

$$\begin{aligned} \|A\| &= \sqrt{81 + 64 + 49 + 36 + 25 + 16} & \|A\| &= \sqrt{\langle A, A \rangle} \\ &= \sqrt{271} \quad | \end{aligned}$$

$$\begin{aligned} \|B\| &= \sqrt{1 + 4 + 9 + 16 + 25 + 36} & \|B\| &= \sqrt{\langle B, B \rangle} \\ &= \sqrt{91} \quad | \end{aligned}$$

$$\begin{aligned}\langle A, B \rangle &= 9 + 16 + 21 + 24 + 25 + 24 \\ &= 119\end{aligned}$$

$$\cos \theta = \frac{119}{\sqrt{271}\sqrt{91}} \qquad \cos \theta = \frac{\langle A, B \rangle}{\|A\| \|B\|}$$

### Exercise

Find the cosine of the angle between  $A$  and  $B$ .

$$A = \begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$$

### Solution

$$\begin{aligned}\|A\| &= \sqrt{1^2 + (-2)^2 + 7^2 + 6^2 + (-3)^2 + 4^2} & \|A\| &= \sqrt{\langle A, A \rangle} \\ &= \sqrt{115}\end{aligned}$$

$$\begin{aligned}\|B\| &= \sqrt{1^2 + (-2)^2 + 3^2 + 6^2 + 5^2 + (-4)^2} & \|B\| &= \sqrt{\langle B, B \rangle} \\ &= \sqrt{91}\end{aligned}$$

$$\begin{aligned}\langle A, B \rangle &= 1 + 4 + 21 + 36 - 15 - 16 \\ &= 31\end{aligned}$$

$$\cos \theta = \frac{31}{\sqrt{115}\sqrt{91}} \qquad \cos \theta = \frac{\langle A, B \rangle}{\|A\| \|B\|}$$

### Exercise

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

$$\vec{u} = (-1, 3, 2), \quad \vec{v} = (4, 2, -1)$$

### Solution

$$\begin{aligned}\langle \vec{u}, \vec{v} \rangle &= (-1)(4) + 3(2) + 2(-1) \\ &= 0\end{aligned}$$

Therefore, the given vectors are orthogonal.



**Exercise**

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

$$\vec{u} = (a, b), \quad \vec{v} = (-b, a)$$

**Solution**

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= a(-b) + b(a) \\ &= 0 \end{aligned}$$

Therefore, the given vectors are orthogonal.

**Exercise**

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

$$\vec{u} = (-2, -2, -2), \quad \vec{v} = (1, 1, 1)$$

**Solution**

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= (-2)(1) + (-2)(1) + (-2)(1) \\ &= -6 \end{aligned}$$

Therefore, the given vectors are **not** orthogonal.

**Exercise**

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

$$\vec{u} = (-4, 6, -10, 1), \quad \vec{v} = (2, 1, -2, 9)$$

**Solution**

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= (-4)(2) + (6)(1) + (-10)(-2) + (1)(9) \\ &= 27 \neq 0 \end{aligned}$$

Therefore, the given vectors are **not** orthogonal.

**Exercise**

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

$$\vec{u} = (-4, 6, -10, 1), \quad \vec{v} = (2, 1, -2, 9)$$

**Solution**

$$\|\vec{u}\| = \sqrt{(-4)^2 + 6^2 + (-10)^2 + 1^2}$$

$$\begin{aligned} &= \sqrt{153} \\ &= \underline{3\sqrt{17}} \end{aligned}$$

$$\begin{aligned} \|\vec{v}\| &= \sqrt{2^2 + 1^2 + (-2)^2 + 9^2} \\ &= \sqrt{90} \\ &= \underline{3\sqrt{10}} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= (-4)(2) + 6(1) - 10(-2) + 1(9) \\ &= \underline{27} \end{aligned}$$

$$\begin{aligned} \cos \theta &= \frac{27}{3\sqrt{17}(3\sqrt{10})} \\ &= \underline{\frac{3}{\sqrt{170}}} \end{aligned} \qquad \cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

The vectors  $\vec{u}$  and  $\vec{v}$  are **not** orthogonal with respect to the Euclidean

### Exercise

Do there exist scalars  $k$  and  $l$  such that the vectors  $\vec{u} = (2, k, 6)$ ,  $\vec{v} = (l, 5, 3)$ , and  $\vec{w} = (1, 2, 3)$  are mutually orthogonal with respect to the Euclidean inner product?

### Solution

$$\begin{aligned} \langle \vec{u}, \vec{w} \rangle &= (2)(1) + (k)(2) + (6)(3) \\ &= 20 + 2k = 0 \\ \Rightarrow \underline{k = -10} \end{aligned}$$

$$\begin{aligned} \langle \vec{v}, \vec{w} \rangle &= (l)(1) + (5)(2) + (3)(3) \\ &= l + 19 = 0 \\ \Rightarrow \underline{l = -19} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= (2)(l) + (k)(5) + (6)(3) \\ &= 2l + 5k + 18 = 0 \\ 2(-19) + 5(-10) + 18 &= -70 \neq 0 \end{aligned}$$

Thus, there are no scalars such that the vectors are mutually orthogonal.

**Exercise**

Let  $\mathbb{R}^3$  have the Euclidean inner product. For which values of  $k$  are  $\vec{u}$  and  $\vec{v}$  orthogonal?

a)  $\vec{u} = (2, 1, 3), \quad \vec{v} = (1, 7, k)$

b)  $\vec{u} = (k, k, 1), \quad \vec{v} = (k, 5, 6)$

**Solution**

$$\begin{aligned} \text{a) } \langle \vec{u}, \vec{v} \rangle &= (2)(1) + (1)(7) + (3)(k) \\ &= 9 + 3k = 0 \end{aligned}$$

$\vec{u}$  and  $\vec{v}$  are orthogonal for  $k = -3$

$$\begin{aligned} \text{b) } \langle \vec{u}, \vec{v} \rangle &= (k)(k) + (k)(5) + (1)(6) \\ &= k^2 + 5k + 6 = 0 \end{aligned}$$

$\vec{u}$  and  $\vec{v}$  are orthogonal for  $k = -2, -3$

**Exercise**

Let  $V$  be an inner product space. Show that if  $\vec{u}$  and  $\vec{v}$  are orthogonal unit vectors in  $V$ , then

$$\|\vec{u} - \vec{v}\| = \sqrt{2}$$

**Solution**

$$\begin{aligned} \|\vec{u} - \vec{v}\|^2 &= \langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle \\ &= \langle \vec{u}, \vec{u} - \vec{v} \rangle - \langle \vec{v}, \vec{u} - \vec{v} \rangle \\ &= \langle \vec{u}, \vec{u} \rangle - \langle \vec{u}, \vec{v} \rangle - \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{v} \rangle \\ &= \|\vec{u}\|^2 - 0 - 0 + \|\vec{v}\|^2 && \text{since } \vec{u} \text{ and } \vec{v} \text{ are orthogonal unit vectors} \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

Thus  $\|\vec{u} - \vec{v}\| = \sqrt{2}$

**Exercise**

Let  $\mathcal{S}$  be a subspace of  $\mathbb{R}^n$ . Explain what  $(\mathcal{S}^\perp)^\perp = \mathcal{S}$  means and why it is true.

**Solution**

$(\mathcal{S}^\perp)^\perp$  is the orthogonal complement of  $\mathcal{S}^\perp$ , which is itself the orthogonal complement of  $\mathcal{S}$ , so  $(\mathcal{S}^\perp)^\perp = \mathcal{S}$  means that  $\mathcal{S}$  is the orthogonal of its orthogonal complement.

We need to show that  $\mathcal{S}$  is contained in  $(\mathcal{S}^\perp)^\perp$  and, conversely, that  $(\mathcal{S}^\perp)^\perp$  is contained in  $\mathcal{S}$  to be true.

*i.* Suppose  $\vec{v} \in \mathcal{S}^\perp$  and  $\vec{w} \in \mathcal{S}^\perp$ . Then  $\langle \vec{v}, \vec{w} \rangle = 0$  by definition of  $\mathcal{S}^\perp$ .

Thus,  $\mathcal{S}$  is certainly contained in  $(\mathcal{S}^\perp)^\perp$  (which consists of all vectors in  $\mathbb{R}^n$  which are orthogonal to  $\mathcal{S}^\perp$ ).

*ii.* Suppose  $\vec{v} \in (\mathcal{S}^\perp)^\perp$  (means  $\vec{v}$  is orthogonal to all vectors in  $\mathcal{S}^\perp$ ); then we need to show that  $\vec{v} \in \mathcal{S}$ .

Let assume  $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p\}$  be a basis for  $\mathcal{S}$  and let  $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_q\}$  be a basis for  $\mathcal{S}^\perp$ . If

$\vec{v} \notin \mathcal{S}$ , then  $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p, \vec{v}\}$  is linearly independent set. Since each vector in that set is orthogonal to all of  $\mathcal{S}^\perp$ , the set  $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p, \vec{v}, \vec{w}_1, \vec{w}_2, \dots, \vec{w}_q\}$  is linearly independent.

Since there are  $p + q + 1$  vectors in this set, this means that  $p + q + 1 \leq n \Leftrightarrow p + q \leq n - 1$ .

On the other hand, If  $A$  is the matrix whose  $i^{th}$  row is  $\vec{u}_i^T$ , then the row space of  $A$  is  $\mathcal{S}$  and the nullspace of  $A$  is  $\mathcal{S}^\perp$ .

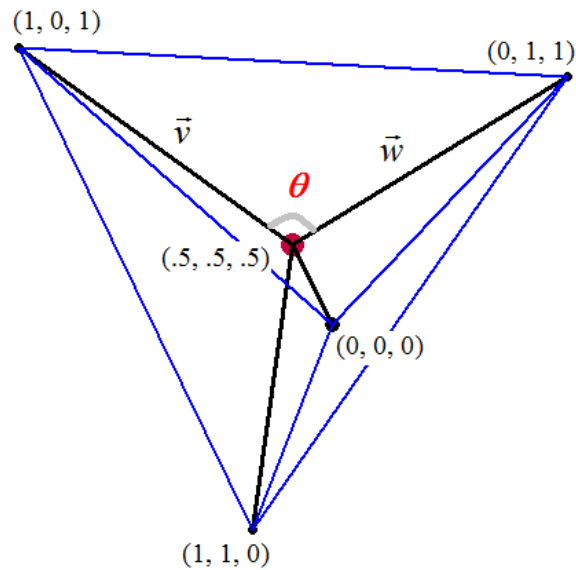
Since  $\mathcal{S}$  is  $p$ -dimensional, the rank of  $A$  is  $p$ , meaning that the dimension of  $\text{nul}(A) = \mathcal{S}^\perp$  is  $q = n - p$ . Therefore,

$$p + q = p + (n - p) = n$$

Which contradict the fact that  $p + q \leq n - 1$ . From this, we see that, if  $\vec{v} \in (\mathcal{S}^\perp)^\perp$ , it must be the case that  $\vec{v} \in \mathcal{S}$ .

**Exercise**

The methane molecule  $CH_4$  is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at  $(0, 0, 0)$ ,  $(1, 1, 0)$ ,  $(1, 0, 1)$  and  $(0, 1, 1)$  – (**note** that all six edges have length  $\sqrt{2}$ , so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center  $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$  to the vertices?

**Solution**

Let  $\vec{v}$  be the vector of the segment  $(1, 0, 1)$  and  $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$$\begin{aligned}\vec{v} &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\end{aligned}$$

Let be the vector of the segment  $(0, 1, 1)$  and  $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$$\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

We have:

$$\begin{aligned} \cos \theta &= \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \|\vec{w}\|} \\ &= \frac{\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \cdot \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}{\sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}}} \\ &= \frac{-\frac{1}{4}}{\frac{3}{4}} \\ &= -\frac{1}{3} \\ \theta &\approx 109.47^\circ \end{aligned}$$

### ***Exercise***

Determine if the given vectors are orthogonal.

$$\vec{x}_1 = (1, 0, 1, 0), \quad \vec{x}_2 = (0, 1, 0, 1), \quad \vec{x}_3 = (1, 0, -1, 0), \quad \vec{x}_4 = (1, 1, -1, -1)$$

### ***Solution***

$$\begin{aligned} \vec{x}_1 \cdot \vec{x}_2 &= (1, 0, 1, 0) \cdot (0, 1, 0, 1) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{x}_1 \cdot \vec{x}_3 &= (1, 0, 1, 0) \cdot (1, 0, -1, 0) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{x}_1 \cdot \vec{x}_4 &= (1, 0, 1, 0) \cdot (1, 1, -1, -1) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{x}_2 \cdot \vec{x}_3 &= (0, 1, 0, 1) \cdot (1, 0, -1, 0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{x}_2 \cdot \vec{x}_4 &= (0, 1, 0, 1) \cdot (1, 1, -1, -1) \\ &= 1 - 1 \end{aligned}$$

$$\begin{aligned}
 &= 0 \\
 \vec{x}_3 \cdot \vec{x}_4 &= (1, 0, -1, 0) \cdot (1, 1, -1, -1) \\
 &= 1 - 1 \\
 &= 0
 \end{aligned}$$

The given vectors are **orthogonal**.

### Exercise

Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

$$\left\{ \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right), \left( -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \right\}$$

### Solution

$$\begin{aligned}
 \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \cdot \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right) &= \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \cdot \left( -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) &= -\frac{1}{2} + 0 + \frac{1}{2} \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right) \cdot \left( -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) &= -\frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} \\
 &= -\frac{2}{\sqrt{6}} \neq 0
 \end{aligned}$$

Therefore, the given vectors are **not orthogonal**.

$$\begin{aligned}
 \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{vmatrix} &= \frac{1}{2\sqrt{3}} + \frac{1}{2\sqrt{3}} \\
 &= \frac{1}{\sqrt{3}} \neq 0
 \end{aligned}$$

**Exercise**

Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

$$\left\{ \left( \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right), \left( \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right), \left( \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right) \right\}$$

**Solution**

$$\left( \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right) \cdot \left( \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right) = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} \\ \underline{= 0}$$

$$\left( \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right) \cdot \left( \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right) = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} \\ \underline{= 0}$$

$$\left( \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right) \cdot \left( \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right) = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} \\ \underline{= 0}$$

Therefore, the given vectors are *orthogonal*.

$$\begin{vmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{vmatrix} = \frac{4}{27} + \frac{4}{27} + \frac{4}{27} - \frac{1}{27} + \frac{8}{27} + \frac{8}{27} \\ \underline{= 0}$$

**Exercise**

Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

$$\left\{ \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right), \left( -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \right\}$$

**Solution**

$$\begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{2\sqrt{3}} + \frac{1}{2\sqrt{3}} \\ \underline{= \frac{1}{\sqrt{3}} \neq 0}$$

Therefore, the given vectors are **not** orthogonal.



$$\begin{aligned} \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \cdot \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) &= -\frac{1}{2\sqrt{3}} - \frac{1}{2\sqrt{3}} \\ &= -\frac{1}{\sqrt{3}} \neq 0 \end{aligned}$$

**Exercise**

Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

$$\left\{ \left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \right\}$$

**Solution**

$$\begin{aligned} \left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right) \cdot \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \cdot \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) &= 0 + 0 + 0 + 0 \\ &= 0 \end{aligned}$$

Therefore, the given vectors are *orthogonal*.

**Exercise**

Consider vectors  $\vec{u} = (2, 3, 5)$   $\vec{v} = (1, -4, 3)$  in  $\mathbb{R}^3$

- a)  $\langle \vec{u}, \vec{v} \rangle$       b)  $\|\vec{u}\|$       c)  $\|\vec{v}\|$       d) Cosine between  $\vec{u}$  and  $\vec{v}$

**Solution**

$$\begin{aligned} a) \quad \langle \vec{u}, \vec{v} \rangle &= (2, 3, 5) \cdot (1, -4, 3) \\ &= 2 - 12 + 15 \\ &= 5 \end{aligned}$$

$$\begin{aligned} b) \quad \|\vec{u}\| &= \sqrt{4 + 9 + 25} \\ &= \sqrt{38} \end{aligned}$$

$$\begin{aligned} c) \quad \|\vec{v}\| &= \sqrt{1 + 16 + 9} \\ &= \sqrt{26} \end{aligned}$$

$$d) \quad \cos \theta = \frac{5}{\sqrt{38}\sqrt{26}} \quad \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

**Exercise**

Consider vectors  $\vec{u} = (1, 1, 1)$   $\vec{v} = (1, 2, -3)$  in  $\mathbb{R}^3$

- a)  $\langle \vec{u}, \vec{v} \rangle$       b)  $\|\vec{u}\|$       c)  $\|\vec{v}\|$       d) Cosine  $\theta$  between  $\vec{u}$  and  $\vec{v}$

**Solution**

$$\begin{aligned} \text{a) } \langle \vec{u}, \vec{v} \rangle &= (1, 1, 1) \cdot (1, 2, -3) \\ &= 1 + 2 - 3 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{b) } \|\vec{u}\| &= \sqrt{1+1+1} \\ &= \sqrt{3} \end{aligned}$$

$$\begin{aligned} \text{c) } \|\vec{v}\| &= \sqrt{1+4+9} \\ &= \sqrt{14} \end{aligned}$$

$$\text{d) } \cos \theta = 0 \qquad \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$\vec{u}$  and  $\vec{v}$  are orthogonal vectors.

**Exercise**

Consider vectors  $\vec{u} = (1, 2, 5)$   $\vec{v} = (2, -3, 5)$   $\vec{w} = (4, 2, -3)$  in  $\mathbb{R}^3$

- a)  $\langle \vec{u}, \vec{v} \rangle$       d)  $\|\vec{u}\|$       g) Cosine  $\alpha$  between  $\vec{u}$  and  $\vec{v}$   
b)  $\langle \vec{u}, \vec{w} \rangle$       e)  $\|\vec{v}\|$       h) Cosine  $\beta$  between  $\vec{u}$  and  $\vec{w}$   
c)  $\langle \vec{v}, \vec{w} \rangle$       f)  $\|\vec{w}\|$       i) Cosine  $\theta$  between  $\vec{v}$  and  $\vec{w}$   
j)  $(\vec{u} + \vec{v}) \cdot \vec{w}$

**Solution**

$$\begin{aligned} \text{a) } \langle \vec{u}, \vec{v} \rangle &= (1, 2, 5) \cdot (2, -3, 5) \\ &= 2 - 6 + 25 \\ &= 21 \end{aligned}$$

$$\begin{aligned} \text{b) } \langle \vec{u}, \vec{w} \rangle &= (1, 2, 5) \cdot (4, 2, -3) \\ &= 4 + 4 - 15 \\ &= -7 \end{aligned}$$

$$\begin{aligned} \text{c) } \langle \vec{v}, \vec{w} \rangle &= (2, -3, 5) \cdot (4, 2, -3) \\ &= 8 - 6 - 15 \\ &= -13 \end{aligned}$$

$$\text{d) } \|\vec{u}\| = \sqrt{1+4+25}$$

$$= \sqrt{30} \mid$$

$$e) \quad \|\vec{v}\| = \sqrt{4 + 9 + 25}$$

$$= \sqrt{38} \mid$$

$$f) \quad \|\vec{w}\| = \sqrt{16 + 4 + 9}$$

$$= \sqrt{29} \mid$$

$$g) \quad \cos \alpha = \frac{21}{\sqrt{30}\sqrt{38}}$$

$$= \frac{21}{2\sqrt{114}} \mid$$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$h) \quad \cos \beta = \frac{-7}{\sqrt{30}\sqrt{29}}$$

$$= \frac{-7}{\sqrt{870}} \mid$$

$$\cos \theta = \frac{\vec{u} \cdot \vec{w}}{\|\vec{u}\| \|\vec{w}\|}$$

$$i) \quad \cos \theta = \frac{-13}{\sqrt{38}\sqrt{29}}$$

$$= \frac{-13}{\sqrt{1,102}} \mid$$

$$\cos \theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}$$

$$j) \quad (\vec{u} + \vec{v}) \cdot \vec{w} = [(1, 2, 5) + (2, -3, 5)] \cdot (4, 2, -3)$$

$$= (3, -1, 10) \cdot (4, 2, -3)$$

$$= 12 - 2 - 30$$

$$= -20 \mid$$

### Exercise

Consider vectors  $\vec{u} = (-1, -1, -1)$   $\vec{v} = (2, 2, 2)$   $\vec{w} = (4, 2, -3)$  in  $\mathbb{R}^3$

$$a) \quad \langle \vec{u}, \vec{v} \rangle$$

$$d) \quad \|\vec{u}\|$$

$$g) \quad \text{Cosine } \alpha \text{ between } \vec{u} \text{ and } \vec{v}$$

$$b) \quad \langle \vec{u}, \vec{w} \rangle$$

$$e) \quad \|\vec{v}\|$$

$$h) \quad \text{Cosine } \beta \text{ between } \vec{u} \text{ and } \vec{w}$$

$$c) \quad \langle \vec{v}, \vec{w} \rangle$$

$$f) \quad \|\vec{w}\|$$

$$i) \quad \text{Cosine } \theta \text{ between } \vec{v} \text{ and } \vec{w}$$

$$j) \quad (\vec{u} + \vec{v}) \cdot \vec{w}$$

### Solution

$$a) \quad \langle \vec{u}, \vec{v} \rangle = (-1, -1, -1) \cdot (2, 2, 2)$$

$$= -2 - 2 - 2$$

$$= -6 \mid$$

$$b) \quad \langle \vec{u}, \vec{w} \rangle = (-1, -1, -1) \cdot (4, 2, -3)$$

$$= -4 - 2 + 3$$

$$\underline{= -3}$$

$$c) \quad \langle \vec{v}, \vec{w} \rangle = (2, 2, 2) \cdot (4, 2, -3)$$

$$= 8 + 4 - 6$$

$$\underline{= 6}$$

$$d) \quad \|\vec{u}\| = \sqrt{1+1+1}$$

$$\underline{= \sqrt{3}}$$

$$e) \quad \|\vec{v}\| = \sqrt{4+4+4}$$

$$\underline{= 2\sqrt{3}}$$

$$f) \quad \|\vec{w}\| = \sqrt{16+4+9}$$

$$\underline{= \sqrt{29}}$$

$$g) \quad \cos \alpha = \frac{-6}{6}$$

$$\underline{= -1}$$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$h) \quad \cos \beta = \frac{-3}{\sqrt{3}\sqrt{29}}$$

$$\underline{= -\frac{3}{\sqrt{87}}}$$

$$\cos \theta = \frac{\vec{u} \cdot \vec{w}}{\|\vec{u}\| \|\vec{w}\|}$$

$$i) \quad \cos \theta = \frac{6}{2\sqrt{3}\sqrt{29}}$$

$$\underline{= \frac{3}{\sqrt{87}}}$$

$$\cos \theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}$$

$$j) \quad (\vec{u} + \vec{v}) \cdot \vec{w} = [(-1, -1, -1) + (2, 2, 2)] \cdot (4, 2, -3)$$

$$= (1, 1, 1) \cdot (4, 2, -3)$$

$$= 4 + 2 - 3$$

$$\underline{= 3}$$

**Exercise**

Consider vectors  $\vec{u} = (-2, 0, 1, 3)$   $\vec{v} = (1, 1, 1, 1)$   $\vec{w} = (3, -1, 5, 2)$  in  $\mathbb{R}^4$

- |                                       |  |  |
|---------------------------------------|--|--|
| a) $\langle \vec{u}, \vec{v} \rangle$ | d) $\ \vec{u}\ $                       | g) Cosine $\alpha$ between $\vec{u}$ and $\vec{v}$ |
| b) $\langle \vec{u}, \vec{w} \rangle$ | e) $\ \vec{v}\ $                       | h) Cosine $\beta$ between $\vec{u}$ and $\vec{w}$  |
| c) $\langle \vec{v}, \vec{w} \rangle$ | f) $\ \vec{w}\ $                       | i) Cosine $\theta$ between $\vec{v}$ and $\vec{w}$ |
|                                       | j) $(\vec{u} + \vec{v}) \cdot \vec{w}$ |  |

**Solution**

$$\begin{aligned} \text{a) } \langle \vec{u}, \vec{v} \rangle &= (-2, 0, 1, 3) \cdot (1, 1, 1, 1) \\ &= -2 + 0 + 1 + 3 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{b) } \langle \vec{u}, \vec{w} \rangle &= (-2, 0, 1, 3) \cdot (3, -1, 5, 2) \\ &= -6 + 0 + 5 + 6 \\ &= 5 \end{aligned}$$

$$\begin{aligned} \text{c) } \langle \vec{v}, \vec{w} \rangle &= (1, 1, 1, 1) \cdot (3, -1, 5, 2) \\ &= 3 - 1 + 5 + 2 \\ &= 9 \end{aligned}$$

$$\begin{aligned} \text{d) } \|\vec{u}\| &= \sqrt{4 + 1 + 9} \\ &= \sqrt{14} \end{aligned}$$

$$\begin{aligned} \text{e) } \|\vec{v}\| &= \sqrt{1 + 1 + 1 + 1} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{f) } \|\vec{w}\| &= \sqrt{9 + 1 + 25 + 4} \\ &= \sqrt{39} \end{aligned}$$

$$\begin{aligned} \text{g) } \cos \alpha &= \frac{2}{2\sqrt{14}} \\ &= \frac{1}{\sqrt{14}} \end{aligned} \qquad \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$\begin{aligned} \text{h) } \cos \beta &= \frac{5}{\sqrt{14}\sqrt{39}} \\ &= \frac{5}{\sqrt{546}} \end{aligned} \qquad \cos \theta = \frac{\vec{u} \cdot \vec{w}}{\|\vec{u}\| \|\vec{w}\|}$$

$$\begin{aligned} \text{i) } \cos \theta &= \frac{9}{2\sqrt{39}} \end{aligned} \qquad \cos \theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}$$

$$\text{j) } (\vec{u} + \vec{v}) \cdot \vec{w} = [(-2, 0, 1, 3) + (1, 1, 1, 1)] \cdot (3, -1, 5, 2)$$

$$\begin{aligned} &= (-1, 1, 1, 3) \cdot (3, -1, 5, 2) \\ &= -3 - 1 + 5 + 6 \\ &= \underline{7} \end{aligned}$$

**Exercise**

Consider polynomial  $f(t) = 3t - 5$ ;  $g(t) = t^2$  in  $\mathbb{P}(t)$

- a)  $\langle f, g \rangle$       b)  $\|f\|$       c)  $\|g\|$       d) Cosine between  $f$  and  $g$

**Solution**

$$\begin{aligned} a) \quad \langle f, g \rangle &= \int_0^1 f(t)g(t)dt \\ &= \int_0^1 (3t-5)t^2dt \\ &= \int_0^1 (3t^3 - 5t^2)dt \\ &= \left. \frac{3}{4}t^4 - \frac{5}{3}t^3 \right|_0^1 \\ &= \frac{3}{4} - \frac{5}{3} \\ &= \underline{-\frac{11}{12}} \end{aligned}$$

$$\begin{aligned} b) \quad \langle f, f \rangle &= \int_0^1 f(t)f(t)dt \\ &= \int_0^1 (3t-5)^2dt \\ &= \frac{1}{3} \int_0^1 (3t-5)^2 d(3t-5) \\ &= \left. \frac{1}{9}(3t-5)^3 \right|_0^1 \\ &= \frac{1}{9}(8-125) \\ &= \underline{13} \end{aligned}$$

$$\|f\| = \sqrt{|\langle f, f \rangle|}$$

$$= \sqrt{13} \mid$$

$$c) \quad \langle g, g \rangle = \int_0^1 g(t)g(t)dt$$

$$= \int_0^1 t^4 dt$$

$$= \frac{1}{5} t^5 \Big|_0^1$$

$$= \frac{1}{5} \mid$$

$$\|g\| = \sqrt{\langle g, g \rangle}$$

$$= \frac{1}{\sqrt{5}} \mid$$

$$d) \quad \cos \theta = \frac{-\frac{11}{12}}{\sqrt{13} \frac{\sqrt{5}}{5}}$$

$$= \frac{-55}{12\sqrt{65}} \mid$$

$$\cos \theta = \frac{f \cdot g}{\|f\| \|g\|}$$

### Exercise

Consider polynomial  $f(t) = t + 2$ ;  $g(t) = 3t - 2$ ;  $h(t) = t^2 - 2t - 3$  in  $\mathbb{P}(t)$

$$a) \quad \langle f, g \rangle$$

$$d) \quad \|f\|$$

$$g) \quad \text{Cosine } \alpha \text{ between } f \text{ and } g$$

$$b) \quad \langle f, h \rangle$$

$$e) \quad \|g\|$$

$$h) \quad \text{Cosine } \beta \text{ between } f \text{ and } h$$

$$c) \quad \langle g, h \rangle$$

$$f) \quad \|h\|$$

$$i) \quad \text{Cosine } \theta \text{ between } g \text{ and } h$$

### Solution

$$a) \quad \langle f, g \rangle = \int_0^1 (t+2)(3t-2)dt$$

$$= \int_0^1 (3t^2 + 4t - 4)dt$$

$$= t^3 + 2t^2 - 4t \Big|_0^1$$

$$= 1 + 2 - 4$$

$$= -1 \mid$$

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt$$

$$\begin{aligned}b) \quad \langle f, h \rangle &= \int_0^1 (t+2)(t^2-2t-3) dt \\&= \int_0^1 (t^3-7t-6) dt \\&= \frac{1}{4}t^4 - \frac{7}{2}t^2 - 6t \Big|_0^1 \\&= \frac{1}{4} - \frac{7}{2} - 6 \\&= -\frac{37}{4}\end{aligned}$$

$$\langle f, h \rangle = \int_0^1 f(t)h(t)dt$$

$$\begin{aligned}c) \quad \langle g, h \rangle &= \int_0^1 (3t-2)(t^2-2t-3) dt \\&= \int_0^1 (3t^3-8t^2-5t+6) dt \\&= \frac{3}{4}t^4 - \frac{8}{3}t^3 - \frac{5}{2}t^2 + 6t \Big|_0^1 \\&= \frac{3}{4} - \frac{8}{3} - \frac{5}{2} + 6 \\&= \frac{9}{4}\end{aligned}$$

$$\langle f, h \rangle = \int_0^1 g(t)h(t)dt$$

$$\begin{aligned}d) \quad \langle f, f \rangle &= \int_0^1 (t+2)^2 dt \\&= \frac{1}{3}(t+2)^3 \Big|_0^1 \\&= \frac{1}{3}(27-8) \\&= \frac{19}{3}\end{aligned}$$

$$\langle f, f \rangle = \int_0^1 f(t)f(t)dt$$

$$\begin{aligned}\|f\| &= \sqrt{|\langle f, f \rangle|} \\&= \sqrt{\frac{19}{3}}\end{aligned}$$

$$e) \quad \langle g, g \rangle = \int_0^1 (3t-2)^2 dt$$

$$\langle g, g \rangle = \int_0^1 g(t)g(t)dt$$



$$\begin{aligned}
 &= \frac{1}{3} \int_0^1 (3t-2)^2 d(3t-2) \\
 &= \frac{1}{9} (3t-2)^3 \Big|_0^1 \\
 &= \frac{1}{9} (1+8) \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \|g\| &= \sqrt{\langle g, g \rangle} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 f) \quad \langle h, h \rangle &= \int_0^1 (t^2 - 2t - 3)^2 dt & \langle h, h \rangle &= \int_0^1 h(t)h(t)dt \\
 &= \int_0^1 (t^4 - 4t^3 - 2t^2 + 12t + 9) dt \\
 &= \left( \frac{1}{5}t^5 - t^4 - \frac{2}{3}t^3 + 6t^2 + 9t \right) \Big|_0^1 \\
 &= \frac{1}{5} - 1 - \frac{2}{3} + 6 + 9 \\
 &= \frac{203}{15}
 \end{aligned}$$

$$\begin{aligned}
 \|h\| &= \sqrt{\langle h, h \rangle} \\
 &= \sqrt{\frac{203}{15}}
 \end{aligned}$$

$$\begin{aligned}
 g) \quad \cos \alpha &= \frac{-1}{\sqrt{\frac{19}{3}}} & \cos \alpha &= \frac{f \cdot g}{\|f\| \|g\|} \\
 &= -\sqrt{\frac{3}{19}}
 \end{aligned}$$

$$\begin{aligned}
 h) \quad \cos \beta &= -\frac{37}{4} \frac{1}{\sqrt{\frac{19}{3}} \sqrt{\frac{203}{15}}} & \cos \beta &= \frac{f \cdot g}{\|f\| \|g\|} \\
 &= -\frac{111}{4} \sqrt{\frac{5}{3,857}}
 \end{aligned}$$

$$\begin{aligned}
 i) \quad \cos \theta &= \frac{9}{4} \frac{1}{\sqrt{\frac{203}{15}}} \\
 &= \frac{9}{4} \sqrt{\frac{15}{203}}
 \end{aligned}$$

$$\cos \theta = \frac{f \cdot g}{\|f\| \|g\|}$$

### Exercise

Consider polynomial  $f(x) = x^2 - 2x + 3$ ;  $g(x) = 2x + 3$ ;  $h(x) = x - 2$  in  $\mathbb{P}(x)$

- |                           |            |  |
|---------------------------|------------|--|
| a) $\langle f, g \rangle$ | d) $\ f\ $ | g) Cosine $\alpha$ between $f$ and $g$ |
| b) $\langle f, h \rangle$ | e) $\ g\ $ | h) Cosine $\beta$ between $f$ and $h$  |
| c) $\langle g, h \rangle$ | f) $\ h\ $ | i) Cosine $\theta$ between $g$ and $h$ |

### Solution

$$\begin{aligned}
 a) \quad \langle f, g \rangle &= \int_0^1 (x^2 - 2x + 3)(2x + 3) dx & \langle f, g \rangle &= \int_0^1 f(x)g(x) dx \\
 &= \int_0^1 (2x^3 - x^2 + 9) dx \\
 &= \frac{1}{2}x^4 - \frac{1}{3}x^3 + 9x \Big|_0^1 \\
 &= \frac{1}{2} - \frac{1}{3} + 9 \\
 &= \frac{55}{6}
 \end{aligned}$$

$$\begin{aligned}
 b) \quad \langle f, h \rangle &= \int_0^1 (x^2 - 2x + 3)(x - 2) dx & \langle f, h \rangle &= \int_0^1 f(x)h(x) dx \\
 &= \int_0^1 (x^3 - 4x^2 + 7x - 6) dx \\
 &= \frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{7}{2}x^2 - 6x \Big|_0^1 \\
 &= \frac{1}{4} - \frac{4}{3} + \frac{7}{2} - 6 \\
 &= \frac{3 - 16 - 42 + 72}{12} \\
 &= \frac{17}{12}
 \end{aligned}$$

$$\begin{aligned}
 c) \quad \langle g, h \rangle &= \int_0^1 (2x+3)(x-2) \, dx \\
 &= \int_0^1 (2x^2 - x - 6) \, dx \\
 &= \left. \frac{2}{3}x^3 - \frac{1}{2}x^2 - 6x \right|_0^1 \\
 &= \frac{2}{3} - \frac{1}{2} - 6 \\
 &= \underline{-\frac{37}{6}}
 \end{aligned}$$

$$\langle f, h \rangle = \int_0^1 g(x)h(x) \, dx$$

$$\begin{aligned}
 d) \quad \langle f, f \rangle &= \int_0^1 (x^2 - 2x + 3)^2 \, dx \\
 &= \int_0^1 (x^4 - 8x^3 + 6x^2 - 12x + 3) \, dx \\
 &= \left. \frac{1}{5}x^5 - 2x^4 + 2x^3 - 6x^2 + 3x \right|_0^1 \\
 &= \frac{1}{5} - 2 + 2 - 6 + 3 \\
 &= \underline{-\frac{14}{5}}
 \end{aligned}$$

$$\langle f, f \rangle = \int_0^1 f(x)f(x) \, dx$$

$$\begin{aligned}
 \|f\| &= \sqrt{\langle f, f \rangle} \\
 &= \underline{\sqrt{\frac{14}{5}}}
 \end{aligned}$$

$$\begin{aligned}
 e) \quad \langle g, g \rangle &= \int_0^1 (2x+3)^2 \, dx \\
 &= \frac{1}{2} \int_0^1 (2x+3)^2 \, d(2x+3) \\
 &= \left. \frac{1}{6}(2x+3)^3 \right|_0^1 \\
 &= \frac{1}{6}(125 + 27) \\
 &= \underline{\frac{152}{6}}
 \end{aligned}$$

$$\langle g, g \rangle = \int_0^1 g(x)g(x) \, dx$$

$$\begin{aligned}
 &= \frac{76}{3} \Big| \\
 \|g\| &= \sqrt{\langle g, g \rangle} \\
 &= \sqrt{\frac{76}{3}} \Big|
 \end{aligned}$$

$$\begin{aligned}
 f) \quad \langle h, h \rangle &= \int_0^1 (x-2)^2 dx \\
 &= \frac{1}{3}(x-2)^3 \Big|_0^1 \\
 &= \frac{1}{3}(-1+8) \\
 &= \frac{7}{3} \Big|
 \end{aligned}$$

$$\begin{aligned}
 \|h\| &= \sqrt{\langle h, h \rangle} \\
 &= \sqrt{\frac{7}{3}} \Big|
 \end{aligned}$$

$$\begin{aligned}
 g) \quad \cos \alpha &= \frac{\frac{55}{6}}{\sqrt{\frac{76}{3}} \sqrt{\frac{14}{5}}} \\
 &= \frac{55}{6} \sqrt{\frac{5}{14} \cdot \frac{3}{76}} \\
 &= \frac{55}{12} \sqrt{\frac{15}{266}} \Big|
 \end{aligned}$$

$$\begin{aligned}
 h) \quad \cos \beta &= \frac{17}{12} \sqrt{\frac{3}{7} \cdot \frac{5}{14}} \\
 &= \frac{17}{84} \sqrt{\frac{15}{2}} \Big|
 \end{aligned}$$

$$\begin{aligned}
 i) \quad \cos \theta &= -\frac{37}{6} \sqrt{\frac{3}{76} \cdot \frac{3}{7}} \\
 &= -\frac{37}{4} \sqrt{\frac{1}{133}} \Big|
 \end{aligned}$$

$$\langle h, h \rangle = \int_0^1 h(x)h(x)dx$$

$$\cos \alpha = \frac{f \bullet g}{\|f\| \|g\|}$$

$$\cos \beta = \frac{f \bullet h}{\|f\| \|h\|}$$

$$\cos \theta = \frac{h \bullet g}{\|h\| \|g\|}$$

**Exercise**

Suppose  $\langle \vec{u}, \vec{v} \rangle = 3 + 2i$  in a complex inner product space  $V$ . Find:

$$a) \langle (2 - 4i)\vec{u}, \vec{v} \rangle \quad b) \langle \vec{u}, (4 + 3i)\vec{v} \rangle \quad c) \langle (3 - 6i)\vec{u}, (5 - 2i)\vec{v} \rangle \quad d) \|\vec{u}, \vec{v}\|$$

**Solution**

$$\begin{aligned} a) \quad \langle (2 - 4i)\vec{u}, \vec{v} \rangle &= (2 - 4i)\langle \vec{u}, \vec{v} \rangle \\ &= (2 - 4i)(3 + 2i) \\ &= 6 + 4i - 12i + 8 \quad i^2 = -1 \\ &= \underline{14 - 8i} \end{aligned}$$

$$\begin{aligned} b) \quad \langle \vec{u}, (4 + 3i)\vec{v} \rangle &= (4 + 3i)\langle \vec{u}, \vec{v} \rangle \\ &= (4 + 3i)(3 + 2i) \\ &= 12 + 8i + 9i - 6 \quad i^2 = -1 \\ &= \underline{14 - 8i} \end{aligned}$$

$$\begin{aligned} c) \quad \langle (3 - 6i)\vec{u}, (5 - 2i)\vec{v} \rangle &= (3 - 6i)(5 - 2i)\langle \vec{u}, \vec{v} \rangle \\ &= (15 - 36i - 12)(3 + 2i) \\ &= (3 - 36i)(3 + 2i) \\ &= 9 - 102i + 72 \\ &= \underline{81 - 102i} \end{aligned}$$

$$\begin{aligned} d) \quad \|\vec{u}, \vec{v}\| &= \sqrt{\langle \vec{u}, \vec{v} \rangle} \\ &= \sqrt{9 + 4} \\ &= \underline{\sqrt{13}} \end{aligned}$$

**Exercise**

Suppose  $\langle \vec{u}, \vec{v} \rangle = 2 - 3i$  in a complex inner product space  $V$ . Find:

$$a) \langle (2 + 2i)\vec{u}, \vec{v} \rangle \quad b) \langle \vec{u}, (3 - 4i)\vec{v} \rangle \quad c) \langle (1 + 3i)\vec{u}, (5 - 2i)\vec{v} \rangle \quad d) \|\vec{u}, \vec{v}\|$$

**Solution**

$$\begin{aligned} a) \quad \langle (2 + 2i)\vec{u}, \vec{v} \rangle &= (2 + 2i)\langle \vec{u}, \vec{v} \rangle \\ &= (2 + 2i)(2 - 3i) \\ &= 4 - 6i + 4i + 6 \\ &= \underline{10 - 2i} \end{aligned}$$

$$b) \quad \langle \vec{u}, (3 - 4i)\vec{v} \rangle = (3 - 4i)\langle \vec{u}, \vec{v} \rangle$$

$$\begin{aligned}
 &= (3 - 4i)(2 - 3i) \\
 &= 4 - 9i - 8i - 12 \\
 &= \underline{-8 - 17i}
 \end{aligned}$$

$$\begin{aligned}
 c) \quad \langle (1 + 3i)\vec{u}, (5 - 2i)\vec{v} \rangle &= (1 + 3i)(5 - 2i)\langle \vec{u}, \vec{v} \rangle \\
 &= (5 - 2i + 15i + 6)(2 - 3i) \\
 &= (11 + 13i)(2 - 3i) \\
 &= 22 - 33i + 26i + 39 \\
 &= \underline{61 - 7i}
 \end{aligned}$$

$$\begin{aligned}
 d) \quad \|\vec{u}, \vec{v}\| &= \sqrt{\langle \vec{u}, \vec{v} \rangle} \\
 &= \sqrt{4 + 9} \\
 &= \underline{\sqrt{13}}
 \end{aligned}$$

### Exercise

Find the Fourier coefficient  $c$  and the projection  $c\vec{v}$  of  $\vec{u} = (3 + 4i, 2 - 3i)$  along  $\vec{v} = (5 + i, 2i)$  in  $\mathbb{C}^2$

### Solution

$$\begin{aligned}
 c &= \frac{(3 + 4i)(\overline{5 + i}) + (2 - 3i)(\overline{2i})}{5^2 + 1^2 + 2^2} & c &= \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \\
 &= \frac{(3 + 4i)(5 - i) + (2 - 3i)(-2i)}{25 + 1 + 4} \\
 &= \frac{15 + 17i + 4 - 4i - 6}{30} \\
 &= \frac{13 + 13i}{30} \\
 &= \underline{\frac{13}{30} + \frac{13}{30}i}
 \end{aligned}$$

$$\begin{aligned}
 \text{proj}(\vec{u}, \vec{v}) &= c\vec{v} \\
 &= \left(\frac{13}{30} + \frac{13}{30}i\right)(5 + i, 2i) \\
 &= \left(\frac{13}{6} + \frac{13}{6}i + \frac{13}{30}i - \frac{13}{30}, \frac{13}{15}i - \frac{13}{15}\right) \\
 &= \left(\frac{52}{30} + \frac{78}{30}i, -\frac{13}{15} + \frac{13}{15}i\right) \\
 &= \underline{\left(\frac{26}{15} + \frac{39}{30}i, -\frac{13}{15} + \frac{13}{15}i\right)}
 \end{aligned}$$

**Exercise**

Suppose  $\vec{v} = (1, 3, 5, 7)$ . Find the projection of  $\vec{v}$  onto  $W$  or find  $\vec{w} \in W$  that minimizes  $\|\vec{v} - \vec{w}\|$ , where  $W$  is the subspace of  $\mathbb{R}^4$  spanned by:

$$a) \quad \vec{u}_1 = (1, 1, 1, 1) \quad \text{and} \quad \vec{u}_2 = (1, -3, 4, -2)$$

$$b) \quad \vec{v}_1 = (1, 1, 1, 1) \quad \text{and} \quad \vec{v}_2 = (1, 2, 3, 2)$$

**Solution**

$$\begin{aligned} a) \quad \vec{u}_1 \cdot \vec{u}_2 &= (1, 1, 1, 1) \cdot (1, -3, 4, -2) \\ &= 1 - 3 + 4 - 2 \\ &= 0 \end{aligned}$$

Therefore,  $\vec{u}_1$  and  $\vec{u}_2$  are orthogonal.

$$\begin{aligned} c_1 &= \frac{\langle \vec{v}, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \\ &= \frac{(1, 3, 5, 7) \cdot (1, 1, 1, 1)}{\|(1, 1, 1, 1)\|^2} \\ &= \frac{1+3+5+7}{1+1+1+1} \\ &= \frac{16}{4} \\ &= 4 \end{aligned}$$

$$\begin{aligned} c_2 &= \frac{\langle \vec{v}, \vec{u}_2 \rangle}{\langle \vec{u}_2, \vec{u}_2 \rangle} \\ &= \frac{(1, 3, 5, 7) \cdot (1, -3, 4, -2)}{\|(1, -3, 4, -2)\|^2} \\ &= \frac{1-9+20-14}{1+9+16+4} \\ &= \frac{-2}{30} \\ &= -\frac{1}{15} \end{aligned}$$

$$\begin{aligned} w &= \text{proj}(\vec{v}, W) \\ &= c_1 \vec{u}_1 + c_2 \vec{u}_2 \\ &= 4(1, 1, 1, 1) + \frac{1}{15}(1, -3, 4, -2) \end{aligned}$$

$$= \left( \frac{59}{15}, \frac{63}{5}, \frac{56}{15}, \frac{62}{15} \right) \mid$$

$$\begin{aligned} b) \quad \vec{v}_1 \cdot \vec{v}_2 &= (1, 1, 1, 1) \cdot (1, 2, 3, 2) \\ &= 1 + 2 + 3 + 2 \\ &= 8 \neq 0 \end{aligned}$$

Therefore,  $\vec{v}_1$  and  $\vec{v}_2$  are *not* orthogonal.

Applying Gram-Schmidt algorithm

$$\vec{w}_1 = \vec{v}_1 = (1, 1, 1, 1) \mid$$

$$\vec{w}_2 = (1, 2, 3, 2) - \frac{(1, 2, 3, 2) \cdot (1, 1, 1, 1)}{4} (1, 1, 1, 1)$$

$$\begin{aligned} &= (1, 2, 3, 2) - 2(1, 1, 1, 1) \\ &= (-1, 0, 1, 0) \mid \end{aligned}$$

$$\begin{aligned} c_1 &= \frac{(1, 3, 5, 7) \cdot (1, 1, 1, 1)}{\|(1, 1, 1, 1)\|^2} \\ &= \frac{1+3+5+7}{1+1+1+1} \\ &= \frac{16}{4} \\ &= 4 \mid \end{aligned}$$

$$\begin{aligned} c_2 &= \frac{(1, 3, 5, 7) \cdot (-1, 0, 1, 0)}{\|(-1, 0, 1, 0)\|^2} \\ &= \frac{-1+0+5+0}{2} \\ &= -3 \mid \end{aligned}$$

$$\begin{aligned} w &= \text{proj}(\vec{v}, W) \\ &= c_1 \vec{w}_1 + c_2 \vec{w}_2 \\ &= 4(1, 1, 1, 1) - 3(-1, 0, 1, 0) \\ &= (7, 4, 1, 4) \mid \end{aligned}$$

$$\vec{w}_2 = \vec{v}_2 - \frac{\langle \vec{v}, \vec{w}_1 \rangle}{\|\vec{w}_1\|^2} \vec{w}_1$$

$$c_1 = \frac{\langle \vec{v}, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle}$$

$$c_2 = \frac{\langle \vec{v}, \vec{w}_2 \rangle}{\langle \vec{w}_2, \vec{w}_2 \rangle}$$



**Exercise**

Suppose  $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$  is an orthogonal set of vectors. Prove that (*Pythagoras*)

$$\|\vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_n\|^2 = \|\vec{u}_1\|^2 + \|\vec{u}_2\|^2 + \dots + \|\vec{u}_n\|^2$$

**Solution**

$$\begin{aligned} \|\vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_n\|^2 &= \langle \vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_n, \vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_n \rangle \\ &= \langle \vec{u}_1, \vec{u}_1 \rangle + \langle \vec{u}_2, \vec{u}_2 \rangle + \dots + \langle \vec{u}_n, \vec{u}_n \rangle \\ &= \|\vec{u}_1\|^2 + \|\vec{u}_2\|^2 + \dots + \|\vec{u}_n\|^2 \end{aligned}$$

**Exercise**

Suppose  $A$  is an orthogonal matrix. Show that  $\langle \vec{u}A, \vec{v}A \rangle = \langle \vec{u}, \vec{v} \rangle$  for any  $\vec{u}, \vec{v} \in V$

**Solution**

$$A \text{ is an orthogonal matrix} \Rightarrow AA^T = I$$

$$\text{And } \langle \vec{u}, \vec{v} \rangle = \vec{u}^T \vec{v}$$

$$\begin{aligned} \langle \vec{u}A, \vec{v}A \rangle &= (A\vec{u})^T (A\vec{v}) \\ &= \vec{u}^T (A^T A) \vec{v} \\ &= \vec{u}^T I \vec{v} \\ &= \vec{u}^T \vec{v} \\ &= \langle \vec{u}, \vec{v} \rangle \quad \checkmark \end{aligned}$$

**Exercise**

Suppose  $A$  is an orthogonal matrix. Show that  $\|\vec{u}A\| = \|\vec{u}\|$  for every  $\vec{u} \in V$

**Solution**

$$A \text{ is an orthogonal matrix}$$

$$\Rightarrow AA^T = I \text{ and } \langle \vec{u}, \vec{u} \rangle = \vec{u}^T \vec{u}$$

$$\begin{aligned} \|\vec{u}A\|^2 &= \langle \vec{u}A, \vec{u}A \rangle \\ &= (A\vec{u})^T (A\vec{u}) \end{aligned}$$

$$\begin{aligned}
&= \vec{u}^T \begin{pmatrix} A^T & A \end{pmatrix} \vec{u} \\
&= \vec{u}^T I \vec{u} \\
&= \vec{u}^T \vec{u} \\
&= \langle \vec{u}, \vec{u} \rangle \quad \checkmark
\end{aligned}$$

### Exercise

Let  $V$  be an inner product space over  $\mathbb{R}$  or  $\mathbb{C}$ . Show that

$$\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$$

If and only if

$$\|s\vec{u} + t\vec{v}\| = s\|\vec{u}\| + t\|\vec{v}\| \quad \text{for all } s, t \geq 0$$

### Solution

Suppose that  $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$ . For  $s, t \geq 0$

$$\begin{aligned}
\|s\vec{u} + t\vec{v}\|^2 &= s^2 \|\vec{u}\|^2 + t^2 \|\vec{v}\|^2 + 2st \vec{u}\vec{v} \\
&\leq s^2 \|\vec{u}\|^2 + t^2 \|\vec{v}\|^2 \\
&\leq s\|\vec{u}\| + t\|\vec{v}\|
\end{aligned}$$

$$\|s\vec{u} + t\vec{v}\| \leq s\|\vec{u}\| + t\|\vec{v}\|$$

$$\begin{aligned}
\|s\vec{u} + t\vec{v}\| &= \|s\vec{u} + t\vec{u} - t\vec{u} + t\vec{v}\| \\
&= \|t\vec{u} + t\vec{v} - t\vec{u} + s\vec{u}\| \\
&= \|t(\vec{u} + \vec{v}) - (t-s)\vec{u}\| \\
&\geq |t\|\vec{u} + \vec{v}\| - (t-s)\|\vec{u}\|| \qquad \|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\| \\
&= t\|\vec{u}\| + \|\vec{v}\| - t\|\vec{u}\| + s\|\vec{u}\| \\
&= t\|\vec{v}\| + s\|\vec{u}\|
\end{aligned}$$

$$\begin{cases} \|s\vec{u} + t\vec{v}\| \leq s\|\vec{u}\| + t\|\vec{v}\| \\ \text{and} \\ \|s\vec{u} + t\vec{v}\| \geq s\|\vec{u}\| + t\|\vec{v}\| \end{cases} \Rightarrow \|s\vec{u} + t\vec{v}\| = s\|\vec{u}\| + t\|\vec{v}\|$$

**Exercise**

Let  $V$  be an inner product vector space over  $\mathbb{R}$ .

- a) If  $e_1, e_2, e_3$  are three vectors in  $V$  with pairwise product negative, that is,

$$\langle e_i, e_j \rangle < 0, \quad i, j = 1, 2, 3, \quad i \neq j$$

Show that  $e_1, e_2, e_3$  are linearly independent.

- b) Is it possible for three vectors on the  $xy$ -plane to have pairwise negative products?  
 c) Does part (a) remain valid when the word “negative” is replaced with positive?  
 d) Suppose  $\vec{u}, \vec{v}$ , and  $\vec{w}$  are three-unit vectors in the  $xy$ -plane. What are the maximum and minimum values that

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle$$

Can attain? And when?

**Solution**

- a) Suppose that  $e_1, e_2, e_3$  are linearly dependent.

Then, assume that  $e_1, e_2, e_3$  are unit vectors and that

$$e_3 = c_1 e_1 + c_2 e_2$$

Then

$$\langle e_1, e_3 \rangle = c_1 \langle e_1, e_1 \rangle + c_2 \langle e_1, e_2 \rangle \quad \langle e_1, e_1 \rangle = 1$$

$$= c_1 + c_2 \langle e_1, e_2 \rangle < 0$$

$$c_1 < -c_2 \langle e_1, e_2 \rangle$$

$$\langle e_2, e_3 \rangle = c_1 \langle e_2, e_1 \rangle + c_2 \langle e_2, e_2 \rangle \quad \langle e_2, e_2 \rangle = 1$$

$$= c_1 \langle e_2, e_1 \rangle + c_2 < 0$$

$$c_2 < -c_1 \langle e_2, e_1 \rangle$$

$$c_2 < -c_1 \langle e_2, e_1 \rangle$$

$$< -(-c_2 \langle e_1, e_2 \rangle) \langle e_2, e_1 \rangle$$

$$= c_2 \langle e_1, e_2 \rangle^2 \quad \langle e_1, e_2 \rangle^2 > 1$$

$$c_2 < c_2 \quad \text{Contradiction}$$

Therefore,  $e_1, e_2, e_3$  are linearly independent.

- b) To have all three vectors on the  $xy$ -plane which is in 2 dimensional.  
 Therefore, it is **impossible** for three to have pairwise negative products.

c) No

d) Given:  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$  are three-unit vectors in the  $xy$ -plane and

$$|\langle \vec{u}, \vec{u} \rangle| = |\langle \vec{v}, \vec{v} \rangle| = |\langle \vec{w}, \vec{w} \rangle| = 1$$

$$\cos \alpha_1 = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|} \rightarrow \cos \alpha_1 = \langle \vec{u}, \vec{v} \rangle$$

$$\cos \alpha_2 = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \|\vec{w}\|} \rightarrow \cos \alpha_2 = \langle \vec{v}, \vec{w} \rangle$$

$$\cos \alpha_3 = \frac{\langle \vec{u}, \vec{w} \rangle}{\|\vec{u}\| \|\vec{w}\|} \rightarrow \cos \alpha_3 = \langle \vec{u}, \vec{w} \rangle$$

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = \cos \alpha_1 + \cos \alpha_2 + \cos \alpha_3$$

Since  $-1 \leq \cos \theta \leq 1$

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = 1 + 1 + 1$$

$$\underline{= 3}$$

Since the 3 vectors are unit vectors in the  $xy$ -plane and which it will divide the plane into a three equal angles  $\alpha = \frac{2\pi}{3}$

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle &= \cos \frac{2\pi}{3} + \cos \frac{2\pi}{3} + \cos \frac{2\pi}{3} \\ &= 3 \cos \frac{2\pi}{3} \\ &= 3 \left( -\frac{1}{2} \right) \\ &\underline{= -\frac{3}{2}} \end{aligned}$$

Therefore, the minimum  $\underline{\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = -\frac{3}{2}}$

The maximum:  $\underline{\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = 3}$

## ***Solution***      **Section 3.3 – Gram-Schmidt Process**

### ***Exercise***

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces  $\mathbb{R}^m$ .

$$\vec{u}_1 = (1, -3), \quad \vec{u}_2 = (2, 2)$$

### ***Solution***

$$\vec{v}_1 = \vec{u}_1 = (1, -3)$$

$$\vec{q}_1 = \frac{(1, -3)}{\sqrt{1+9}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left( \frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (2, 2) - \frac{1}{10} [(2, 2) \cdot (1, -3)] (1, -3)$$

$$= (2, 2) + \frac{2}{5} (1, -3)$$

$$= (2, 2) + \left( \frac{2}{5}, -\frac{6}{5} \right)$$

$$= \left( \frac{12}{5}, \frac{4}{5} \right)$$

$$\|\vec{v}_2\| = \sqrt{\left( \frac{12}{5} \right)^2 + \left( \frac{4}{5} \right)^2}$$

$$= \sqrt{\frac{144}{25} + \frac{16}{25}}$$

$$= \sqrt{\frac{160}{25}}$$

$$= \frac{4\sqrt{10}}{5}$$

$$\vec{q}_2 = \frac{5}{4\sqrt{10}} \left( \frac{12}{5}, \frac{4}{5} \right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left( \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right)$$

The *orthogonal* basis:  $\left\{(1, -3), \left(\frac{12}{5}, \frac{4}{5}\right)\right\}$

The *orthonormal* basis:  $\left\{\left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right), \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)\right\}$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\vec{u}_1 = (1, 0), \quad \vec{u}_2 = (3, -5)$$

### Solution

$$\vec{v}_1 = \vec{u}_1 = (1, 0) \quad |$$

$$\vec{q}_1 = \frac{(1, 0)}{\sqrt{1+0}}$$

$$= (1, 0) \quad |$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (3, -5) - [(3, -5) \cdot (1, 0)](1, 0)$$

$$= (3, -5) - (3)(1, 0)$$

$$= (3, -5) - (3, 0)$$

$$= (0, -5) \quad |$$

$$\vec{q}_2 = \frac{1}{5}(0, -5)$$

$$= (0, -1) \quad |$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

The *orthogonal* basis:  $\{(1, 0), (0, -5)\}$

The *orthonormal* basis:  $\{(1, 0), (0, -1)\}$

**Exercise**

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces  $\mathbb{R}^m$ .

$$\vec{u}_1 = (1, 0, 0), \quad \vec{u}_2 = (3, 7, -2), \quad \vec{u}_3 = (0, 4, 1)$$

**Solution**

$$\vec{v}_1 = \vec{u}_1 = (1, 0, 0) \quad |$$

$$\vec{q}_1 = \frac{(1, 0, 0)}{\sqrt{1+0+0}}$$

$$= (1, 0, 0) \quad |$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (3, 7, -2) - [(3, 7, -2) \cdot (1, 0, 0)](1, 0, 0)$$

$$= (3, 7, -2) - 3(1, 0, 0)$$

$$= (0, 7, -2) \quad |$$

$$\vec{q}_2 = \frac{1}{\sqrt{53}}(0, 7, -2)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left( 0, \frac{7}{\sqrt{53}}, -\frac{2}{\sqrt{53}} \right) \quad |$$

$$\vec{u}_3 \cdot \vec{v}_1 = (0, 4, 1) \cdot (1, 0, 0)$$

$$= 0 \quad |$$

$$\vec{u}_3 \cdot \vec{v}_2 = (0, 4, 1) \cdot (0, 7, -2)$$

$$= 26 \quad |$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2,$$

$$= (0, 4, 1) - 0 - \left( \frac{26}{53} \right) (0, 7, -2)$$

$$= (0, 4, 1) - \left( 0, \frac{182}{53}, -\frac{52}{53} \right)$$

$$= \left( 0, \frac{30}{53}, \frac{105}{53} \right) \quad |$$

$$\begin{aligned}
 \vec{q}_3 &= \frac{1}{\sqrt{\left(\frac{30}{53}\right)^2 + \left(\frac{105}{53}\right)^2}} \left(0, \frac{30}{53}, \frac{105}{53}\right) & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
 &= \frac{53}{\sqrt{11925}} \left(0, \frac{30}{53}, \frac{105}{53}\right) \\
 &= \frac{53}{15\sqrt{53}} \left(0, \frac{30}{53}, \frac{105}{53}\right) \\
 &= \left(0, \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}}\right)
 \end{aligned}$$

The *orthogonal* basis:  $\left\{(1, 0, 0), (0, 7, -2), \left(0, \frac{30}{53}, \frac{105}{53}\right)\right\}$

The *orthonormal* basis:  $\left\{(1, 0, 0), \left(0, \frac{7}{\sqrt{53}}, -\frac{2}{\sqrt{53}}\right), \left(0, \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}}\right)\right\}$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\vec{u}_1 = (1, 1, 0, -1), \quad \vec{u}_2 = (1, 3, 0, 1), \quad \vec{u}_3 = (4, 2, 2, 0)$$

### Solution

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 0, -1)$$

$$\begin{aligned}
 \vec{q}_1 &= \frac{1}{\sqrt{3}}(1, 1, 0, -1) & \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
 &= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}}\right)
 \end{aligned}$$

$$\begin{aligned}
 \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
 &= (1, 3, 0, 1) - \frac{(1, 3, 0, 1) \cdot (1, 1, 0, -1)}{3} (1, 1, 0, -1) \\
 &= (1, 3, 0, 1) - (1, 1, 0, -1) \\
 &= (0, 2, 0, 2)
 \end{aligned}$$

$$\begin{aligned}
 \vec{q}_2 &= \frac{1}{2\sqrt{2}}(0, 2, 0, 2) & \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|}
 \end{aligned}$$



$$= \left( 0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \Big|$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(4, 2, 2, 0) \cdot (1, 1, 0, -1)}{3} (1, 1, 0, -1)$$

$$= (2, 2, 0, -2) \Big|$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{(4, 2, 2, 0) \cdot (0, 2, 0, 2)}{8} (0, 2, 0, 2)$$

$$= \frac{1}{2} (0, 2, 0, 2)$$

$$= (0, 1, 0, 1) \Big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (4, 2, 2, 0) - (2, 2, 0, -2) - (0, 1, 0, 1)$$

$$= (2, -1, 2, 1) \Big|$$

$$\vec{q}_3 = \frac{1}{\sqrt{10}} (2, -1, 2, 1) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \left( \frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right) \Big|$$

The *orthogonal* basis:  $\{(1, 1, 0, -1), (0, 2, 0, 2), (2, -1, 2, 1)\}$

The *orthogonal* basis:

$$\left\{ \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}} \right), \left( 0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \left( \frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right) \right\}$$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\vec{u}_1 = (1, 1, 1, 1), \quad \vec{u}_2 = (1, 2, 4, 5), \quad \vec{u}_3 = (1, -3, -4, -2)$$

### Solution

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 1, 1) \Big|$$

$$\vec{q}_1 = \frac{(1, 1, 1, 1)}{\sqrt{4}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (1, 2, 4, 5) - \left( \frac{1}{4} \right) [(1, 2, 4, 5) \cdot (1, 1, 1, 1)] (1, 1, 1, 1)$$

$$= (1, 2, 4, 5) - \left( \frac{12}{4} \right) (1, 1, 1, 1)$$

$$= (1, 2, 4, 5) - (3, 3, 3, 3)$$

$$= (-2, -1, 1, 2)$$

$$\vec{q}_2 = \frac{1}{\sqrt{10}} (-2, -1, 1, 2)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left( -\frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}} \right)$$

$$\vec{u}_3 \cdot \vec{v}_1 = (1, -3, -4, -2) \cdot (1, 1, 1, 1)$$

$$= -8$$

$$\vec{u}_3 \cdot \vec{v}_2 = (1, -3, -4, -2) \cdot (-2, -1, 1, 2)$$

$$= -7$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (1, -3, -4, -2) + \frac{8}{4} (1, 1, 1, 1) + \frac{7}{10} (-2, -1, 1, 2)$$

$$= (1, -3, -4, -2) + (2, 2, 2, 2) + \left( -\frac{7}{5}, -\frac{7}{10}, \frac{7}{10}, \frac{7}{5} \right)$$

$$= \left( \frac{8}{5}, -\frac{17}{10}, -\frac{13}{10}, \frac{7}{5} \right)$$

$$\vec{q}_3 = \frac{1}{\sqrt{\frac{64}{25} + \frac{289}{100} + \frac{169}{100} + \frac{49}{25}}} \left( \frac{8}{5}, -\frac{17}{10}, -\frac{13}{10}, \frac{7}{5} \right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{10}{\sqrt{910}} \left( \frac{8}{5}, -\frac{17}{10}, -\frac{13}{10}, \frac{7}{5} \right)$$

$$= \left( \frac{16}{\sqrt{910}}, -\frac{17}{\sqrt{910}}, -\frac{13}{\sqrt{910}}, \frac{14}{\sqrt{910}} \right)$$

The *orthogonal* basis:  $\left\{ (1, 1, 1, 1), (-2, -1, 1, 2), \left( \frac{8}{5}, -\frac{17}{10}, -\frac{13}{10}, \frac{7}{5} \right) \right\}$

The *orthonormal* basis:

$$\left\{ \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \left( -\frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}} \right), \left( \frac{16}{\sqrt{910}}, -\frac{17}{\sqrt{910}}, -\frac{13}{\sqrt{910}}, \frac{14}{\sqrt{910}} \right) \right\}$$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\vec{u}_1 = (1, 1, 1, 1), \quad \vec{u}_2 = (1, 1, 2, 4), \quad \vec{u}_3 = (1, 2, -4, -3)$$

### Solution

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 1, 1)$$

$$\vec{q}_1 = \frac{(1, 1, 1, 1)}{\sqrt{4}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (1, 1, 2, 4) - \frac{1}{4} [(1, 1, 2, 4) \cdot (1, 1, 1, 1)] (1, 1, 1, 1)$$

$$= (1, 1, 2, 4) - (2, 2, 2, 2)$$

$$= (-1, -1, 0, 2)$$

$$\vec{q}_2 = \frac{1}{\sqrt{1+1+0+4}} (-1, -1, 0, 2)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left( -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}} \right)$$

$$\vec{u}_3 \cdot \vec{v}_1 = (1, 2, -4, -3) \cdot (1, 1, 1, 1)$$

$$= -4$$

$$\vec{u}_3 \cdot \vec{v}_2 = (1, 2, -4, -3) \cdot (-1, -1, 0, 2)$$

$$\begin{aligned}
&= -9 \mid \\
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2, \\
&= (1, 2, -4, -3) + \frac{4}{4}(1, 1, 1, 1) + \frac{9}{6}(-1, -1, 0, 2) \\
&= (1, 2, -4, -3) + (1, 1, 1, 1) + \left(-\frac{3}{2}, -\frac{3}{2}, 0, 3\right) \\
&= \left(\frac{1}{2}, \frac{3}{2}, -3, 1\right) \mid
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{1}{\sqrt{\frac{1}{4} + \frac{9}{4} + 9 + 1}} \left(\frac{1}{2}, \frac{3}{2}, -3, 1\right) & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \frac{2}{5\sqrt{2}} \left(\frac{1}{2}, \frac{3}{2}, -3, 1\right) \\
&= \left(\frac{1}{5\sqrt{2}}, \frac{3}{5\sqrt{2}}, -\frac{6}{5\sqrt{2}}, \frac{2}{5\sqrt{2}}\right) \mid
\end{aligned}$$

The *orthogonal* basis:  $\left\{(1, 1, 1, 1), (-1, -1, 0, 2), \left(\frac{1}{2}, \frac{3}{2}, -3, 1\right)\right\}$

The *orthonormal* basis:

$$\left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}\right), \left(\frac{1}{5\sqrt{2}}, \frac{3}{5\sqrt{2}}, -\frac{6}{5\sqrt{2}}, \frac{2}{5\sqrt{2}}\right)\right\}$$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\vec{u}_1 = (0, 2, 1, 0); \quad \vec{u}_2 = (1, -1, 0, 0); \quad \vec{u}_3 = (1, 2, 0, -1); \quad \vec{u}_4 = (1, 0, 0, 1)$$

### Solution

$$\begin{aligned}
\vec{v}_1 &= \vec{u}_1 = (0, 2, 1, 0) \mid \\
\vec{q}_1 &= \frac{(0, 2, 1, 0)}{\sqrt{4+1}} & \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) \mid
\end{aligned}$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
&= (1, -1, 0, 0) - \frac{1}{5} [(1, -1, 0, 0) \cdot (0, 2, 1, 0)] (0, 2, 1, 0) \\
&= (1, -1, 0, 0) + \frac{2}{5} (0, 2, 1, 0) \\
&= (1, -1, 0, 0) + \left(0, \frac{4}{5}, \frac{2}{5}, 0\right) \\
&= \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{1}{\sqrt{1 + \frac{1}{25} + \frac{4}{25}}} \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) & \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{5}{\sqrt{30}} \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) \\
&= \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right) \Big|
\end{aligned}$$

$$\begin{aligned}
u_3 \cdot v_1 &= (1, 2, 0, -1) \cdot (0, 2, 1, 0) \\
&= 4 \Big|
\end{aligned}$$

$$\begin{aligned}
u_3 \cdot v_2 &= (1, 2, 0, -1) \cdot \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) \\
&= 1 - \frac{2}{5} \\
&= \frac{3}{5} \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2, \\
&= (1, 2, 0, -1) - \left(\frac{4}{5}\right) (0, 2, 1, 0) - \frac{1}{1 + \frac{1}{25} + \frac{4}{25}} \left(\frac{3}{5}\right) \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) \\
&= (1, 2, 0, -1) - \left(0, \frac{8}{5}, \frac{4}{5}, 0\right) - \left(\frac{3}{5}\right) \left(\frac{25}{30}\right) \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) \\
&= (1, 2, 0, -1) - \left(0, \frac{8}{5}, \frac{4}{5}, 0\right) - \left(\frac{1}{2}, -\frac{1}{10}, \frac{1}{5}, 0\right) \\
&= \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (-1)^2 + (-1)^2}} \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right) & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{5}} \left( \frac{1}{2}, \frac{1}{2}, -1, -1 \right) \\
&= \frac{\sqrt{2}}{\sqrt{5}} \left( \frac{1}{2}, \frac{1}{2}, -1, -1 \right) \\
&= \frac{2}{\sqrt{10}} \left( \frac{1}{2}, \frac{1}{2}, -1, -1 \right) \\
&= \left( \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}, -\frac{2}{\sqrt{10}} \right) \Big|
\end{aligned}$$

$$\begin{aligned}
u_4 \cdot v_1 &= (1, 0, 0, 1) \cdot (0, 2, 1, 0) \\
&= 0 \Big|
\end{aligned}$$

$$\begin{aligned}
u_4 \cdot v_2 &= (1, 0, 0, 1) \cdot \left( 1, -\frac{1}{5}, \frac{2}{5}, 0 \right) \\
&= 1 \Big|
\end{aligned}$$

$$\begin{aligned}
u_4 \cdot v_3 &= (1, 0, 0, 1) \cdot \left( \frac{1}{2}, \frac{1}{2}, -1, -1 \right) \\
&= -\frac{1}{2} \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{v}_4 &= \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 \\
&= (1, 2, 0, -1) - (0) - \left( \frac{25}{5} \right) \left( 1, -\frac{1}{5}, \frac{2}{5}, 0 \right) + \left( -\frac{1}{2} \right) \left( \frac{2}{5} \right) \left( \frac{1}{2}, \frac{1}{2}, -1, -1 \right) \\
&= (1, 2, 0, -1) - \left( \frac{5}{6}, -\frac{1}{6}, \frac{1}{3}, 0 \right) + \left( \frac{1}{10}, \frac{1}{10}, -\frac{1}{5}, -\frac{1}{5} \right) \\
&= \left( \frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5} \right) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{q}_4 &= \frac{1}{\sqrt{\left(\frac{4}{15}\right)^2 + \left(\frac{4}{15}\right)^2 + \left(-\frac{8}{15}\right)^2 + \left(\frac{4}{5}\right)^2}} \left( \frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5} \right) & \vec{q}_4 &= \frac{\vec{v}_4}{\|\vec{v}_4\|} \\
&= \frac{1}{\sqrt{\frac{240}{225}}} \left( \frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5} \right) \\
&= \frac{1}{\frac{4}{\sqrt{15}}} \left( \frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5} \right) \\
&= \frac{15}{4\sqrt{15}} \left( \frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5} \right) \\
&= \left( \frac{1}{\sqrt{15}}, \frac{1}{\sqrt{15}}, -\frac{2}{\sqrt{15}}, \frac{3}{\sqrt{15}} \right) \Big|
\end{aligned}$$

The *orthogonal* basis:

$$\left\{ (0, 2, 1, 0), \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right), \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right), \left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right) \right\}$$

The *orthonormal* basis:

$$\left\{ \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right), \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right), \left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right), \left(\frac{1}{\sqrt{15}}, \frac{1}{\sqrt{15}}, -\frac{2}{\sqrt{15}}, \frac{3}{\sqrt{15}}\right) \right\}$$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(3, 4), (1, 0)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (3, 4) \quad \vec{u}_2 = (1, 0)$$

$$\vec{v}_1 = \vec{u}_1 = (3, 4)$$

$$\vec{q}_1 = \frac{(3, 4)}{\sqrt{9+16}}$$

$$= \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (1, 0) - \frac{1}{25} [(1, 0) \cdot (3, 4)] (3, 4)$$

$$= (1, 0) - \left(\frac{9}{25}, \frac{12}{25}\right)$$

$$= \left(\frac{16}{25}, -\frac{12}{25}\right)$$

$$\vec{q}_2 = \frac{25}{20} \left(\frac{16}{25}, -\frac{12}{25}\right)$$

$$= \left(\frac{4}{5}, -\frac{3}{5}\right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\text{The } \textit{orthogonal} \text{ basis: } \left\{ (3, 4), \left(\frac{16}{25}, -\frac{12}{25}\right) \right\}$$

The *orthonormal* basis:  $\left\{\left(\frac{3}{5}, \frac{4}{5}\right), \left(\frac{4}{5}, -\frac{3}{5}\right)\right\}$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(3, 0, -1), (8, 5, -6)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (3, 0, -1) \quad \vec{u}_2 = (8, 5, -6)$$

$$\vec{v}_1 = \vec{u}_1 = (3, 0, -1)$$

$$\vec{q}_1 = \frac{(3, 0, -1)}{\sqrt{9+1}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left( \frac{3}{\sqrt{10}}, 0, -\frac{1}{\sqrt{10}} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (8, 5, -6) - \frac{1}{10} [(8, 5, -6) \cdot (3, 0, -1)] (3, 0, -1)$$

$$= (8, 5, -6) - (3) (3, 0, -1)$$

$$= (8, 5, -6) - (9, 0, -3)$$

$$= (-1, 5, -3)$$

$$\vec{q}_2 = \frac{1}{\sqrt{1+25+9}} (-1, 5, -3)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{35}} (-1, 5, -3)$$

$$= \left( -\frac{1}{\sqrt{35}}, \frac{5}{\sqrt{35}}, -\frac{3}{\sqrt{35}} \right)$$

The *orthogonal* basis:  $\{(3, 0, -1), (-1, 5, -3)\}$

The *orthonormal* basis:  $\left\{ \left( \frac{3}{\sqrt{10}}, 0, -\frac{1}{\sqrt{10}} \right), \left( -\frac{1}{\sqrt{35}}, \frac{5}{\sqrt{35}}, -\frac{3}{\sqrt{35}} \right) \right\}$



**Exercise**

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(0, 4, 2), (5, 6, -7)\}$$

**Solution**

$$\text{Let } \vec{u}_1 = (0, 4, 2) \quad \vec{u}_2 = (5, 6, -7)$$

$$\vec{v}_1 = \vec{u}_1 = (0, 4, 2) \quad |$$

$$\vec{q}_1 = \frac{(0, 4, 2)}{\sqrt{16+4}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left( 0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right) \quad |$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (5, 6, -7) - \frac{1}{20} [(5, 6, -7) \cdot (0, 4, 2)] (0, 4, 2)$$

$$= (5, 6, -7) - \left( \frac{10}{20} \right) (0, 4, 2)$$

$$= (5, 6, -7) - (0, 2, 1)$$

$$= (5, 4, -8) \quad |$$

$$\vec{q}_2 = \frac{1}{\sqrt{25+16+64}} (5, 4, -8)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{105}} (5, 4, -8)$$

$$= \left( \frac{5}{\sqrt{105}}, \frac{4}{\sqrt{105}}, -\frac{8}{\sqrt{105}} \right) \quad |$$

$$\text{The orthogonal basis: } \{(0, 4, 2), (5, 4, -8)\}$$

$$\text{The orthonormal basis: } \left\{ \left( 0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right), \left( \frac{5}{\sqrt{105}}, \frac{4}{\sqrt{105}}, -\frac{8}{\sqrt{105}} \right) \right\}$$

**Exercise**

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(1, 1, 1), (-1, 1, 0), (1, 2, 1)\}$$

**Solution**

$$\text{Let } \vec{u}_1 = (1, 1, 1) \quad \vec{u}_2 = (-1, 1, 0) \quad \vec{u}_3 = (1, 2, 1)$$

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 1) \quad |$$

$$\vec{q}_1 = \frac{(1, 1, 1)}{\sqrt{1+1+1}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1, 1, 1)}{\sqrt{3}}$$

$$= \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \quad |$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (-1, 1, 0) - \frac{1}{3} [(-1, 1, 0) \cdot (1, 1, 1)] (1, 1, 1)$$

$$= (-1, 1, 0) - (0)(1, 1, 1)$$

$$= (-1, 1, 0) \quad |$$

$$\vec{q}_2 = \frac{1}{\sqrt{2}} (-1, 1, 0)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \quad |$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (1, 2, 1) \cdot (1, 1, 1)$$

$$= 4 \quad |$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (1, 2, 1) \cdot (-1, 1, 0)$$

$$= 1 \quad |$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$\begin{aligned}
&= (1, 2, 1) - \frac{4}{3}(1, 1, 1) - \frac{1}{2}(-1, 1, 0) \\
&= (1, 2, 1) - \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) - \left(-\frac{1}{2}, \frac{1}{2}, 0\right) \\
&= \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{1}{\sqrt{\frac{1}{36} + \frac{1}{36} + \frac{1}{9}}} \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right) & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \frac{6}{\sqrt{6}} \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right) \\
&= \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)
\end{aligned}$$

The *orthogonal* basis:  $\left\{(1, 1, 1), (-1, 1, 0), \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)\right\}$

The *orthonormal* basis:  $\left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)\right\}$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (1, 1, 1) \quad \vec{u}_2 = (0, 1, 1) \quad \vec{u}_3 = (0, 0, 1)$$

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 1)$$

$$\begin{aligned}
\vec{q}_1 &= \frac{(1, 1, 1)}{\sqrt{1+1+1}} & \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \frac{(1, 1, 1)}{\sqrt{3}} \\
&= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)
\end{aligned}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\&= (0, 1, 1) - \frac{1}{3}[(0, 1, 1) \cdot (1, 1, 1)](1, 1, 1) \\&= (0, 1, 1) - \left(\frac{2}{3}\right)(1, 1, 1) \\&= (0, 1, 1) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) \\&= \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \Big| \end{aligned}$$

$$\begin{aligned}\|\vec{v}_2\| &= \sqrt{\left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} \\&= \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}} \\&= \sqrt{\frac{6}{9}} \\&= \frac{\sqrt{6}}{3} \Big| \end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{3}{\sqrt{6}}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \\&= \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \Big| \end{aligned}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\begin{aligned}\langle \vec{u}_3, \vec{v}_1 \rangle &= (0, 0, 1) \cdot (1, 1, 1) \\&= 1 \Big| \end{aligned}$$

$$\begin{aligned}\langle \vec{u}_3, \vec{v}_2 \rangle &= (0, 0, 1) \cdot \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \\&= \frac{1}{3} \Big| \end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\&= (0, 0, 1) - \frac{1}{3}(1, 1, 1) - \frac{1}{3}\left(\frac{9}{6}\right)\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \\&= (0, 0, 1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) - \left(-\frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right) \\&= \left(0, -\frac{1}{2}, \frac{1}{2}\right) \Big| \end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}} \left(0, -\frac{1}{2}, \frac{1}{2}\right) & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= (\sqrt{2}) \left(0, -\frac{1}{2}, \frac{1}{2}\right) \\ &= \left(0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\end{aligned}$$

The *orthogonal* basis:  $\left\{(1, 1, 1), \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(0, -\frac{1}{2}, \frac{1}{2}\right)\right\}$

The *orthonormal* basis:  $\left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(1, 1, 1), (0, 2, 1), (1, 0, 3)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (1, 1, 1) \quad \vec{u}_2 = (0, 2, 1) \quad \vec{u}_3 = (1, 0, 3)$$

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 1)$$

$$\begin{aligned}\vec{q}_1 &= \frac{(1, 1, 1)}{\sqrt{1+1+1}} & \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(1, 1, 1)}{\sqrt{3}} \\ &= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\end{aligned}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (0, 2, 1) - \frac{1}{3} [(0, 2, 1) \cdot (1, 1, 1)] (1, 1, 1) \\ &= (0, 2, 1) - \frac{3}{3} (1, 1, 1) \\ &= (0, 2, 1) - (1, 1, 1) \\ &= (-1, 1, 0)\end{aligned}$$

$$\vec{q}_2 = \frac{1}{\sqrt{2}}(-1, 1, 0)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)$$

$$\begin{aligned} \langle \vec{u}_3, \vec{v}_1 \rangle &= (1, 0, 3) \cdot (1, 1, 1) \\ &= 4 \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_3, \vec{v}_2 \rangle &= (1, 0, 3) \cdot (-1, 1, 0) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (1, 0, 3) - \frac{4}{3}(1, 1, 1) + \frac{1}{2}(-1, 1, 0) \\ &= (1, 0, 3) - \left( \frac{4}{3}, \frac{4}{3}, \frac{4}{3} \right) + \left( -\frac{1}{2}, \frac{1}{2}, 0 \right) \\ &= \left( -\frac{5}{6}, -\frac{5}{6}, \frac{5}{3} \right) \end{aligned}$$

$$\begin{aligned} \vec{q}_3 &= \frac{1}{\sqrt{\frac{25}{36} + \frac{25}{36} + \frac{25}{9}}} \left( -\frac{5}{6}, -\frac{5}{6}, \frac{5}{3} \right) \\ &= \frac{1}{\frac{5}{6}\sqrt{6}} \left( -\frac{5}{6}, -\frac{5}{6}, \frac{5}{3} \right) \\ &= \left( -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right) \end{aligned}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

The *orthogonal* basis:  $\left\{ (1, 1, 1), (-1, 1, 0), \left( -\frac{5}{6}, -\frac{5}{6}, \frac{5}{3} \right) \right\}$

The *orthonormal* basis:  $\left\{ \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right), \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right), \left( -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right) \right\}$

**Exercise**

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces  $\mathbb{R}^m$ .

$$\{(2, 2, 2), (1, 0, -1), (0, 3, 1)\}$$

**Solution**

$$\text{Let } \vec{u}_1 = (2, 2, 2) \quad \vec{u}_2 = (1, 0, -1) \quad \vec{u}_3 = (0, 3, 1)$$

$$\vec{v}_1 = \vec{u}_1 = (2, 2, 2) \quad |$$

$$\vec{q}_1 = \frac{(2, 2, 2)}{\sqrt{4+4+4}}$$

$$= \frac{(2, 2, 2)}{2\sqrt{3}}$$

$$= \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \quad |$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (1, 0, -1) - \frac{1}{12} [(1, 0, -1) \cdot (2, 2, 2)] (2, 2, 2)$$

$$= (1, 0, -1) - \frac{0}{12} (2, 2, 2)$$

$$= (1, 0, -1) \quad |$$

$$\vec{q}_2 = \frac{(1, 0, -1)}{\sqrt{2}}$$

$$= \left( \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right) \quad |$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (0, 3, 1) \cdot (2, 2, 2)$$

$$= 8 \quad |$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (0, 3, 1) \cdot (1, 0, -1)$$

$$= -1 \quad |$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$\begin{aligned}
 &= (0, 3, 1) - \frac{8}{12}(2, 2, 2) + \frac{1}{2}(1, 0, -1) \\
 &= (0, 3, 1) - \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) + \left(\frac{1}{2}, 0, -\frac{1}{2}\right) \\
 &= \left(-\frac{5}{6}, \frac{5}{3}, -\frac{5}{6}\right)
 \end{aligned}$$

$$\begin{aligned}
 \vec{q}_3 &= \frac{1}{\sqrt{\left(-\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(-\frac{5}{6}\right)^2}} \left(-\frac{5}{6}, \frac{5}{3}, -\frac{5}{6}\right) & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
 &= \frac{1}{\sqrt{\left(-\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(-\frac{5}{6}\right)^2}} \left(-\frac{5}{6}, \frac{5}{3}, -\frac{5}{6}\right) \\
 &= \frac{1}{\sqrt{\frac{25}{36} + \frac{25}{9} + \frac{25}{36}}} \left(-\frac{5}{6}, \frac{5}{3}, -\frac{5}{6}\right) \\
 &= \frac{1}{\frac{5}{6}\sqrt{6}} \left(-\frac{5}{6}, \frac{5}{3}, -\frac{5}{6}\right) \\
 &= \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)
 \end{aligned}$$

The *orthogonal* basis:  $\left\{(2, 2, 2), (1, 0, -1), \left(-\frac{5}{6}, \frac{5}{3}, -\frac{5}{6}\right)\right\}$

The *orthonormal* basis:  $\left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)\right\}$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(1, -1, 0), (0, 1, 0), (2, 3, 1)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (1, -1, 0) \quad \vec{u}_2 = (0, 1, 0) \quad \vec{u}_3 = (2, 3, 1)$$

$$\vec{v}_1 = \vec{u}_1 = (1, -1, 0)$$

$$\vec{q}_1 = \frac{(1, -1, 0)}{\sqrt{2}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$



$$= \frac{(1, -1, 0)}{\sqrt{2}}$$

$$= \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) \Big|$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (0, 1, 0) - \frac{1}{2} [(0, 1, 0) \cdot (1, -1, 0)] (1, -1, 0)$$

$$= (0, 1, 0) + \frac{1}{2} (1, -1, 0)$$

$$= (0, 1, 0) + \left( \frac{1}{2}, -\frac{1}{2}, 0 \right)$$

$$= \left( \frac{1}{2}, \frac{1}{2}, 0 \right) \Big|$$

$$\vec{q}_2 = \frac{1}{\sqrt{\frac{1}{4} + \frac{1}{4}}} \left( \frac{1}{2}, \frac{1}{2}, 0 \right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \Big|$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (2, 3, 1) \cdot (1, -1, 0)$$

$$= -1 \Big|$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (2, 3, 1) \cdot \left( \frac{1}{2}, \frac{1}{2}, 0 \right)$$

$$= \frac{5}{2} \Big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (2, 3, 1) + \frac{1}{2} (1, -1, 0) - \frac{5}{2} (2) \left( \frac{1}{2}, \frac{1}{2}, 0 \right)$$

$$= (2, 3, 1) + \left( \frac{1}{2}, -\frac{1}{2}, 0 \right) - \left( \frac{5}{2}, \frac{5}{2}, 0 \right)$$

$$= (0, 0, 1) \Big|$$

$$\vec{q}_3 = \frac{1}{1} (0, 0, 1)$$

$$= (0, 0, 1) \Big|$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

The *orthogonal* basis:  $\left\{ (1, -1, 0), \left(\frac{1}{2}, \frac{1}{2}, 0\right), (0, 0, 1) \right\}$

The *orthonormal* basis:  $\left\{ \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), (0, 0, 1) \right\}$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(3, 0, 4), (-1, 0, 7), (2, 9, 11)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (3, 0, 4) \quad \vec{u}_2 = (-1, 0, 7) \quad \vec{u}_3 = (2, 9, 11)$$

$$\vec{v}_1 = \vec{u}_1 = (3, 0, 4)$$

$$\begin{aligned} \vec{q}_1 &= \frac{(3, 0, 4)}{\sqrt{9+16}} \\ &= \left(\frac{3}{5}, 0, \frac{4}{5}\right) \end{aligned} \quad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (-1, 0, 7) - \frac{1}{25} [(-1, 0, 7) \cdot (3, 0, 4)] (3, 0, 4) \\ &= (-1, 0, 7) - \frac{25}{25} (3, 0, 4) \\ &= (-1, 0, 7) - (3, 0, 4) \\ &= (-4, 0, 3) \end{aligned}$$

$$\begin{aligned} \vec{q}_2 &= \frac{1}{\sqrt{16+9}} (-4, 0, 3) \\ &= \left(-\frac{4}{5}, 0, \frac{3}{5}\right) \end{aligned} \quad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\begin{aligned} \langle \vec{u}_3, \vec{v}_1 \rangle &= (2, 9, 11) \cdot (3, 0, 4) \\ &= 50 \end{aligned}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (2, 9, 11) \cdot (-4, 0, 3)$$

$$\begin{aligned}
 &= 25 \Big| \\
 \vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
 &= (2, 9, 11) - \frac{50}{25}(3, 0, 4) - \frac{25}{25}(-4, 0, 3) \\
 &= (2, 9, 11) - (6, 0, 8) - (-4, 0, 3) \\
 &= (0, 9, 0) \Big|
 \end{aligned}$$

$$\begin{aligned}
 \vec{q}_3 &= \frac{1}{9}(0, 9, 0) & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
 &= (0, 1, 0) \Big|
 \end{aligned}$$

The *orthogonal* basis:  $\{(3, 0, 4), (-4, 0, 3), (0, 9, 0)\}$

The *orthonormal* basis:  $\left\{\left(\frac{3}{5}, 0, \frac{4}{5}\right), \left(-\frac{4}{5}, 0, \frac{3}{5}\right), (0, 1, 0)\right\}$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(2, 1, -1), (1, 2, 2), (2, -2, 1)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (2, 1, -1) \quad \vec{u}_2 = (1, 2, 2) \quad \vec{u}_3 = (2, -2, 1)$$

$$\vec{v}_1 = \vec{u}_1 = (2, 1, -1)$$

$$\begin{aligned}
 \vec{q}_1 &= \frac{(2, 1, -1)}{\sqrt{4+1+1}} & \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
 &= \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) \Big|
 \end{aligned}$$

$$\begin{aligned}
 \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|} \vec{v}_1 \\
 &= (1, 2, 2) - \frac{1}{4+1+1}[(1, 2, 2) \cdot (2, 1, -1)](2, 1, -1) \\
 &= (1, 2, 2) - \frac{1}{6}(2)(2, 1, -1)
 \end{aligned}$$

$$= (1, 2, 2) - \left(\frac{2}{3}, \frac{1}{3}, -\frac{1}{3}\right)$$

$$= \left(\frac{1}{3}, \frac{5}{3}, \frac{7}{3}\right)$$

$$\bar{q}_2 = \frac{1}{\sqrt{\frac{1}{9} + \frac{25}{9} + \frac{49}{9}}} \left(\frac{1}{3}, \frac{5}{3}, \frac{7}{3}\right) \quad \bar{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{3}{5\sqrt{3}} \left(\frac{1}{3}, \frac{5}{3}, \frac{7}{3}\right)$$

$$= \left(\frac{1}{5\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{7}{5\sqrt{3}}\right)$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (2, -2, 1) \cdot (2, 1, -1)$$

$$= 4 - 2 - 1$$

$$= 1$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (2, -2, 1) \cdot \left(\frac{1}{3}, \frac{5}{3}, \frac{7}{3}\right)$$

$$= \frac{2}{3} - \frac{10}{3} + \frac{7}{3}$$

$$= -\frac{1}{3}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (2, -2, 1) - \frac{1}{6}(2, 1, -1) + \left(\frac{1}{3}\right)\left(\frac{9}{75}\right)\left(\frac{1}{3}, \frac{5}{3}, \frac{7}{3}\right)$$

$$= (2, -2, 1) - \left(\frac{1}{3}, \frac{1}{6}, -\frac{1}{6}\right) + \left(\frac{1}{75}, \frac{1}{15}, \frac{7}{75}\right)$$

$$= \left(\frac{42}{25}, -\frac{63}{30}, \frac{63}{50}\right)$$

$$\bar{q}_2 = \frac{1}{\sqrt{\frac{1,764}{625} + \frac{3,969}{900} + \frac{3,969}{2,500}}} \left(\frac{42}{25}, -\frac{63}{30}, \frac{63}{50}\right) \quad \bar{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{\frac{63,504 + 99,225 + 35,721}{22,500}}} \left(\frac{42}{25}, -\frac{63}{30}, \frac{63}{50}\right)$$

$$= \frac{150}{\sqrt{198,450}} \left(\frac{42}{25}, -\frac{63}{30}, \frac{63}{50}\right)$$

$$= \frac{150}{315\sqrt{2}} \left(\frac{42}{25}, -\frac{63}{30}, \frac{63}{50}\right)$$

$$= \frac{10}{21\sqrt{2}} \left( \frac{42}{25}, -\frac{63}{30}, \frac{63}{50} \right)$$

$$= \left( \frac{4}{5\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{3}{5\sqrt{2}} \right)$$

The *orthogonal* basis:  $\left\{ (2, 1, -1), \left( \frac{1}{3}, \frac{5}{3}, \frac{7}{3} \right), \left( \frac{42}{25}, -\frac{63}{30}, \frac{63}{50} \right) \right\}$

The *orthonormal* basis:

$$\left\{ \left( \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right), \left( \frac{1}{5\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{7}{5\sqrt{3}} \right), \left( \frac{4}{5\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{3}{5\sqrt{2}} \right) \right\}$$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(1, 1, 0), (1, 2, 0), (0, 1, 2)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (1, 1, 0) \quad \vec{u}_2 = (1, 2, 0) \quad \vec{u}_3 = (0, 1, 2)$$

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 0)$$

$$\vec{q}_1 = \frac{(1, 1, 0)}{\sqrt{2}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (1, 2, 0) - \frac{(1, 2, 0) \cdot (1, 1, 0)}{1+1+0} (1, 1, 0)$$

$$= (1, 2, 0) - \frac{3}{2} (1, 1, 0)$$

$$= \left( -\frac{1}{2}, \frac{1}{2}, 0 \right)$$

$$\vec{q}_2 = \frac{1}{\sqrt{\frac{1}{4} + \frac{1}{4}}} \left( -\frac{1}{2}, \frac{1}{2}, 0 \right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{2}{\sqrt{2}} \left( -\frac{1}{2}, \frac{1}{2}, 0 \right)$$

$$= \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \Big|$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(0, 1, 2) \cdot (1, 1, 0)}{2} (1, 1, 0)$$

$$= \left( \frac{1}{2}, \frac{1}{2}, 0 \right) \Big|$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{1}{\frac{1}{4} + \frac{1}{4}} \left[ (0, 1, 2) \cdot \left( -\frac{1}{2}, \frac{1}{2}, 0 \right) \right] \left( -\frac{1}{2}, \frac{1}{2}, 0 \right)$$

$$= 2 \left( \frac{1}{2} \right) \left( -\frac{1}{2}, \frac{1}{2}, 0 \right)$$

$$= \left( -\frac{1}{2}, \frac{1}{2}, 0 \right) \Big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (0, 1, 2) - \left( \frac{1}{2}, \frac{1}{2}, 0 \right) - \left( -\frac{1}{2}, \frac{1}{2}, 0 \right)$$

$$= (0, 0, 1) \Big|$$

$$\vec{q}_3 = (0, 0, 1) \Big|$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

The *orthogonal* basis:  $\left\{ (1, 1, 0), \left( -\frac{1}{2}, \frac{1}{2}, 0 \right), (0, 0, 1) \right\}$

The *orthogonal* basis:  $\left\{ \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right), \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right), (0, 0, 1) \right\}$

**Exercise**

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces  $\mathbb{R}^m$ .

$$\{(1, -2, 2), (2, 2, 1), (2, -1, -2)\}$$

**Solution**

$$\text{Let } \vec{u}_1 = (1, -2, 2) \quad \vec{u}_2 = (2, 2, 1) \quad \vec{u}_3 = (2, -1, -2)$$

$$\vec{v}_1 = \vec{u}_1 = (1, -2, 2) \quad |$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (2, 2, 1) - \frac{(2, 2, 1) \cdot (1, -2, 2)}{9} (1, -2, 2) \\ &= (2, 2, 1) - \frac{0}{9} (1, -2, 2) \\ &= (2, 2, 1) \quad | \end{aligned}$$

$$\begin{aligned} \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(2, -1, -2) \cdot (1, -2, 2)}{9} (1, -2, 2) \\ &= \frac{0}{9} (1, -2, 2) \\ &= (0, 0, 0) \quad | \end{aligned}$$

$$\begin{aligned} \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{1}{9} [(2, -1, -2) \cdot (2, 2, 1)] (2, 2, 1) \\ &= (0, 0, 0) \quad | \end{aligned}$$

$$\begin{aligned} \vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (2, -1, -2) - (0, 0, 0) - (0, 0, 0) \\ &= (2, -1, -2) \quad | \end{aligned}$$

$$\begin{aligned} \vec{q}_1 &= \frac{1}{3} (1, -2, 2) & \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \left( \frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \right) \quad | \end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{1}{3}(2, 2, 1) & \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \underline{\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)}\end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{1}{3}(2, -1, -2) & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \underline{\left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)}\end{aligned}$$

The *orthogonal* basis:  $\{(1, -2, 2), (2, 2, 1), (2, -1, -2)\}$

The *orthogonal* basis:

$$\left\{\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)\right\}$$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(1, 0, 0), (1, 1, 1), (1, 1, -1)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (1, 0, 0) \quad \vec{u}_2 = (1, 1, 1) \quad \vec{u}_3 = (1, 1, -1)$$

$$\vec{v}_1 = \underline{\vec{u}_1 = (1, 0, 0)}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (1, 1, 1) - \frac{(1, 1, 1) \cdot (1, 0, 0)}{1} (1, 0, 0) \\ &= (1, 1, 1) - (1, 0, 0) \\ &= \underline{(0, 1, 1)}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, 1, -1) \cdot (1, 0, 0)}{1} (1, 0, 0) \\ &= \underline{(1, 0, 0)}\end{aligned}$$



$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{1}{1}[(1, 1, -1) \cdot (0, 1, 1)](0, 1, 1) \\ &= 0(0, 1, 1) \\ &= \underline{(0, 0, 0)}\end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (1, 1, -1) - (1, 0, 0) - (0, 0, 0) \\ &= \underline{(0, 1, -1)}\end{aligned}$$

$$\begin{aligned}\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{1}{1}(1, 0, 0) \\ &= \underline{(1, 0, 0)}\end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{1}{\sqrt{2}}(0, 1, 1) \\ &= \underline{\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}\end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \frac{1}{\sqrt{2}}(0, 1, -1) \\ &= \underline{\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)}\end{aligned}$$

The *orthogonal* basis:  $\{(1, 0, 0), (0, 1, 1), (0, 1, -1)\}$

The *orthogonal* basis:

$$\left\{ (1, 0, 0), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \right\}$$

**Exercise**

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(4, -3, 0), (1, 2, 0), (0, 0, 4)\}$$

**Solution**

$$\text{Let } \vec{u}_1 = (4, -3, 0) \quad \vec{u}_2 = (1, 2, 0) \quad \vec{u}_3 = (0, 0, 4)$$

$$\vec{v}_1 = \vec{u}_1 = (4, -3, 0) \quad |$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (1, 2, 0) - \frac{(1, 2, 0) \cdot (4, -3, 0)}{25} (4, -3, 0) \\ &= (1, 2, 0) + \frac{2}{25} (4, -3, 0) \\ &= \left( \frac{33}{25}, \frac{44}{25}, 0 \right) \quad | \end{aligned}$$

$$\begin{aligned} \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(0, 0, 4) \cdot (4, -3, 0)}{25} (4, -3, 0) \\ &= (0, 0, 0) \quad | \end{aligned}$$

$$\begin{aligned} \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{225}{3,025} \left[ (0, 0, 4) \cdot \left( \frac{33}{25}, \frac{44}{25}, 0 \right) \right] \left( \frac{33}{25}, \frac{44}{25}, 0 \right) \\ &= (0, 0, 0) \quad | \end{aligned}$$

$$\begin{aligned} \vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (0, 0, 4) - (0, 0, 0) - (0, 0, 0) \\ &= (0, 0, 4) \quad | \end{aligned}$$

$$\begin{aligned} \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{1}{\sqrt{16+9}} (4, -3, 0) \\ &= \left( \frac{4}{5}, -\frac{3}{5}, 0 \right) \quad | \end{aligned}$$

$$\begin{aligned}
 \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
 &= \frac{25}{\sqrt{3,025}} \left( \frac{33}{25}, \frac{44}{25}, 0 \right) \\
 &= \frac{25}{55} \left( \frac{33}{25}, \frac{44}{25}, 0 \right) \\
 &= \left( \frac{3}{5}, \frac{4}{5}, 0 \right)
 \end{aligned}$$

$$\begin{aligned}
 \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
 &= \frac{1}{4} (0, 0, 4) \\
 &= (0, 0, 1)
 \end{aligned}$$

The *orthogonal* basis:  $\left\{ (4, -3, 0), \left( \frac{33}{25}, \frac{44}{25}, 0 \right), (0, 0, 4) \right\}$

The *orthogonal* basis:  $\left\{ \left( \frac{4}{5}, -\frac{3}{5}, 0 \right), \left( \frac{3}{5}, \frac{4}{5}, 0 \right), (0, 0, 1) \right\}$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(0, 1, 2), (2, 0, 0), (1, 1, 1)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (0, 1, 2) \quad \vec{u}_2 = (2, 0, 0) \quad \vec{u}_3 = (1, 1, 1)$$

$$\vec{v}_1 = \vec{u}_1 = (0, 1, 2)$$

$$\begin{aligned}
 \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
 &= (2, 0, 0) - \frac{(2, 0, 0) \cdot (0, 1, 2)}{5} (0, 1, 2) \\
 &= (2, 0, 0) + \frac{0}{5} (0, 1, 2) \\
 &= (2, 0, 0)
 \end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, 1, 1) \cdot (0, 1, 2)}{5} (0, 1, 2) \\ &= \frac{3}{5} (0, 1, 2) \\ &= \left( 0, \frac{3}{5}, \frac{6}{5} \right) \quad | \end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{1}{4} [(1, 1, 1) \cdot (2, 0, 0)] (2, 0, 0) \\ &= \frac{1}{2} (2, 0, 0) \\ &= (1, 0, 0) \quad | \end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (1, 1, 1) - (1, 0, 0) - \left( 0, \frac{3}{5}, \frac{6}{5} \right) \\ &= \left( 0, \frac{2}{5}, -\frac{1}{5} \right) \quad | \end{aligned}$$

$$\begin{aligned}\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{1}{\sqrt{5}} (0, 1, 2) \\ &= \left( 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \quad | \end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{1}{2} (2, 0, 0) \\ &= (1, 0, 0) \quad | \end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \frac{5}{\sqrt{5}} \left( 0, \frac{2}{5}, -\frac{1}{5} \right) \\ &= \left( 0, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right) \quad | \end{aligned}$$

The *orthogonal* basis:  $\left\{ (0, 1, 2), (2, 0, 0), \left(0, \frac{2}{5}, -\frac{1}{5}\right) \right\}$

The *orthogonal* basis:  $\left\{ \left(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), (1, 0, 0), \left(0, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right) \right\}$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(0, 1, 1), (1, 1, 0), (1, 0, 1)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (0, 1, 1) \quad \vec{u}_2 = (1, 1, 0) \quad \vec{u}_3 = (1, 0, 1)$$

$$\vec{v}_1 = \vec{u}_1 = (0, 1, 1) \quad |$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (1, 1, 0) - \frac{(1, 1, 0) \cdot (0, 1, 1)}{2} (0, 1, 1) \\ &= (1, 1, 0) - \frac{1}{2} (0, 1, 1) \\ &= \left(1, \frac{1}{2}, -\frac{1}{2}\right) \quad | \end{aligned}$$

$$\begin{aligned} \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, 0, 1) \cdot (0, 1, 1)}{2} (0, 1, 1) \\ &= \left(0, \frac{1}{2}, \frac{1}{2}\right) \quad | \end{aligned}$$

$$\begin{aligned} \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{4}{6} \left[ (1, 0, 1) \cdot \left(1, \frac{1}{2}, -\frac{1}{2}\right) \right] \left(1, \frac{1}{2}, -\frac{1}{2}\right) \\ &= \frac{1}{3} \left(1, \frac{1}{2}, -\frac{1}{2}\right) \\ &= \left(\frac{1}{3}, \frac{1}{6}, -\frac{1}{6}\right) \quad | \end{aligned}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (1, 0, 1) - \left(0, \frac{1}{2}, \frac{1}{2}\right) - \left(\frac{1}{3}, \frac{1}{6}, -\frac{1}{6}\right)$$

$$= \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\right)$$

$$\bar{q}_1 = \frac{1}{\sqrt{2}}(0, 1, 1)$$

$$\bar{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\bar{q}_2 = \frac{2}{\sqrt{6}}\left(1, \frac{1}{2}, -\frac{1}{2}\right)$$

$$\bar{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$$

$$\bar{q}_3 = \frac{3}{\sqrt{12}}\left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\right)$$

$$\bar{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

The *orthogonal* basis:  $\left\{(0, 1, 1), \left(1, \frac{1}{2}, -\frac{1}{2}\right), \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\right)\right\}$

The *orthogonal* basis:  $\left\{\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right\}$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(2, -1, 3), (3, 4, 1), (2, -3, 4)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (2, -1, 3) \quad \vec{u}_2 = (3, 4, 1) \quad \vec{u}_3 = (2, -3, 4)$$

$$\vec{v}_1 = \vec{u}_1 = (2, -1, 3)$$

$$\bar{q}_1 = \frac{1}{\sqrt{4+1+9}}(2, -1, 3)$$

$$\bar{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
&= (3, 4, 1) - \frac{(3, 4, 1) \cdot (2, -1, 3)}{14} (2, -1, 3) \\
&= (3, 4, 1) - \frac{5}{14} (2, -1, 3) \\
&= (3, 4, 1) - \left( \frac{5}{7}, -\frac{5}{14}, \frac{15}{14} \right) \\
&= \left( \frac{16}{7}, \frac{61}{14}, -\frac{1}{14} \right)
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{14}{\sqrt{32^2 + 61^2 + 1}} \left( \frac{16}{7}, \frac{61}{14}, -\frac{1}{14} \right) & \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{14}{\sqrt{4,746}} \left( \frac{16}{7}, \frac{61}{14}, -\frac{1}{14} \right) \\
&= \left( \frac{32}{\sqrt{4,746}}, \frac{61}{\sqrt{4,746}}, -\frac{1}{\sqrt{4,746}} \right)
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(2, -3, 4) \cdot (2, -1, 3)}{14} (2, -1, 3) \\
&= \frac{19}{14} (2, -1, 3) \\
&= \left( \frac{19}{7}, -\frac{19}{14}, \frac{57}{14} \right)
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{196}{4,746} \left( (2, -3, 4) \cdot \left( \frac{16}{7}, \frac{61}{14}, -\frac{1}{14} \right) \right) \left( \frac{16}{7}, \frac{61}{14}, -\frac{1}{14} \right) \\
&= \frac{14}{339} \left( \frac{32}{7} - \frac{183}{14} - \frac{2}{7} \right) \left( \frac{16}{7}, \frac{61}{14}, -\frac{1}{14} \right) \\
&= \frac{14}{339} \left( -\frac{123}{14} \right) \left( \frac{16}{7}, \frac{61}{14}, -\frac{1}{14} \right) \\
&= -\frac{41}{113} \left( \frac{16}{7}, \frac{61}{14}, -\frac{1}{14} \right) \\
&= \left( -\frac{656}{791}, -\frac{2,501}{1,582}, \frac{41}{1,582} \right)
\end{aligned}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$\begin{aligned}
&= (2, -3, 4) - \left(\frac{19}{7}, -\frac{19}{14}, \frac{57}{14}\right) - \left(-\frac{656}{791}, -\frac{2,501}{1,582}, \frac{41}{1,582}\right) \\
&= \left(-\frac{5}{7}, -\frac{23}{14}, -\frac{1}{14}\right) - \left(-\frac{656}{791}, -\frac{2,501}{1,582}, \frac{41}{1,582}\right) \\
&= \left(\frac{13}{113}, -\frac{7}{113}, -\frac{11}{113}\right)
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{113}{\sqrt{169+49+121}} \left(\frac{13}{113}, -\frac{7}{113}, -\frac{11}{113}\right) & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \frac{1}{\sqrt{339}} (13, -7, -11) \\
&= \left(\frac{13}{\sqrt{339}}, -\frac{7}{\sqrt{339}}, -\frac{11}{\sqrt{339}}\right)
\end{aligned}$$

The *orthogonal* basis:  $\left\{(2, -1, 3), \left(\frac{16}{7}, \frac{61}{14}, -\frac{1}{14}\right), \left(\frac{13}{113}, -\frac{7}{113}, -\frac{11}{113}\right)\right\}$

The *orthogonal* basis:

$$\left\{\left(\frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right), \left(\frac{32}{\sqrt{4,746}}, \frac{61}{\sqrt{4,746}}, -\frac{1}{\sqrt{4,746}}\right), \left(\frac{13}{\sqrt{339}}, -\frac{7}{\sqrt{339}}, -\frac{11}{\sqrt{339}}\right)\right\}$$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(-3, 0, 4), (5, -1, 2), (1, 1, 3)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (-3, 0, 4) \quad \vec{u}_2 = (5, -1, 2) \quad \vec{u}_3 = (1, 1, 3)$$

$$\vec{v}_1 = \vec{u}_1 = (-3, 0, 4)$$

$$\begin{aligned}
\vec{q}_1 &= \frac{(-3, 0, 4)}{\sqrt{9+16}} & \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \left(-\frac{3}{5}, 0, \frac{4}{5}\right)
\end{aligned}$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
&= (5, -1, 2) - \frac{(5, -1, 2) \cdot (-3, 0, 4)}{25} (-3, 0, 4)
\end{aligned}$$



$$\begin{aligned}
&= (5, -1, 2) + \frac{7}{25}(-3, 0, 4) \\
&= (5, -1, 2) + \left(-\frac{21}{25}, 0, \frac{28}{25}\right) \\
&= \left(\frac{104}{25}, -1, \frac{78}{25}\right)
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{25}{\sqrt{104^2 + 25^2 + 78^2}} \left(\frac{104}{25}, -1, \frac{78}{25}\right) & \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{1}{\sqrt{17,525}} (104, -25, 78) \\
&= \frac{1}{5\sqrt{701}} (104, -25, 78) \\
&= \left(\frac{104}{5\sqrt{701}}, -\frac{5}{\sqrt{701}}, \frac{78}{5\sqrt{701}}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, 1, 3) \cdot (-3, 0, 4)}{25} (-3, 0, 4) \\
&= \frac{9}{25} (-3, 0, 4) \\
&= \left(-\frac{27}{25}, 0, \frac{36}{25}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{25^2}{17,525} \left( (1, 1, 3) \cdot \left(\frac{104}{25}, -1, \frac{78}{25}\right) \right) \left(\frac{104}{25}, -1, \frac{78}{25}\right) \\
&= \frac{25}{701} \left( \frac{104}{25} - 1 + \frac{234}{25} \right) \left(\frac{104}{25}, -1, \frac{78}{25}\right) \\
&= \frac{1}{701} \left( \frac{313}{25} \right) (104, -25, 78) \\
&= \frac{313}{17,525} (104, -25, 78) \\
&= \left( \frac{32,552}{17,525}, -\frac{313}{701}, \frac{24,414}{17,525} \right)
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
&= (1, 1, 3) - \left(-\frac{27}{25}, 0, \frac{36}{25}\right) - \left(\frac{32,552}{17,525}, -\frac{313}{701}, \frac{24,414}{17,525}\right) \\
&= \left(\frac{52}{25}, 1, \frac{39}{25}\right) - \left(\frac{32,552}{17,525}, -\frac{313}{701}, \frac{24,414}{17,525}\right)
\end{aligned}$$

$$= \left( \frac{156}{701}, \frac{1,014}{701}, \frac{117}{701} \right) \Big|$$

$$\begin{aligned} \vec{q}_3 &= \frac{701}{\sqrt{156^2 + 1014^2 + 117^2}} \left( \frac{156}{701}, \frac{1,014}{701}, \frac{117}{701} \right) & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \frac{701}{\sqrt{1,066,221}} \left( \frac{156}{701}, \frac{1,014}{701}, \frac{117}{701} \right) \\ &= \frac{701}{39\sqrt{701}} \left( \frac{156}{701}, \frac{1,014}{701}, \frac{117}{701} \right) \\ &= \left( \frac{4}{\sqrt{701}}, \frac{26}{\sqrt{701}}, \frac{3}{\sqrt{701}} \right) \Big| \end{aligned}$$

The *orthogonal* basis:  $\left\{ (-3, 0, 4), \left( \frac{104}{25}, -1, \frac{78}{25} \right), \left( \frac{156}{701}, \frac{1,014}{701}, \frac{117}{701} \right) \right\}$

The *orthogonal* basis:

$$\left\{ \left( -\frac{3}{5}, 0, \frac{4}{5} \right), \left( \frac{104}{5\sqrt{701}}, -\frac{5}{\sqrt{701}}, \frac{78}{5\sqrt{701}} \right), \left( \frac{4}{\sqrt{701}}, \frac{26}{\sqrt{701}}, \frac{3}{\sqrt{701}} \right) \right\}$$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(-2, 0, 1), (3, 2, 5), (6, -1, 1), (7, 0, -2)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (-2, 0, 1) \quad \vec{u}_2 = (3, 2, 5) \quad \vec{u}_3 = (6, -1, 1) \quad \vec{u}_4 = (7, 0, -2)$$

$$\vec{v}_1 = \vec{u}_1 = (-2, 0, 1) \Big|$$

$$\vec{q}_1 = \frac{(-2, 0, 1)}{\sqrt{5}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left( -\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}} \right) \Big|$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (3, 2, 5) - \frac{(3, 2, 5) \cdot (-2, 0, 1)}{5} (-2, 0, 1) \end{aligned}$$

$$\begin{aligned}
&= (3, 2, 5) + \frac{1}{5}(-2, 0, 1) \\
&= (3, 2, 5) + \left(-\frac{2}{5}, 0, \frac{1}{5}\right) \\
&= \left(\frac{13}{5}, 2, \frac{26}{5}\right)
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{5}{\sqrt{13^2 + 10^2 + 26^2}} \left(\frac{13}{5}, 2, \frac{26}{5}\right) & \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{1}{\sqrt{945}} (13, 10, 26) \\
&= \frac{1}{3\sqrt{105}} (13, 10, 26) \\
&= \left(\frac{13}{3\sqrt{105}}, -\frac{10}{3\sqrt{105}}, \frac{26}{3\sqrt{105}}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(6, -1, 1) \cdot (-2, 0, 1)}{5} (-2, 0, 1) \\
&= -\frac{11}{5} (-2, 0, 1) \\
&= \left(\frac{22}{5}, 0, -\frac{11}{5}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{25}{945} \left( (6, -1, 1) \cdot \left(\frac{13}{5}, 2, \frac{26}{5}\right) \right) \left(\frac{13}{5}, 2, \frac{26}{5}\right) \\
&= \frac{5}{189} \left( \frac{78}{5} - 2 + \frac{26}{5} \right) \left(\frac{13}{5}, 2, \frac{26}{5}\right) \\
&= \frac{94}{189} \left(\frac{13}{5}, 2, \frac{26}{5}\right) \\
&= \left(\frac{1,222}{945}, -\frac{188}{189}, \frac{2,444}{945}\right)
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
&= (6, -1, 1) - \left(\frac{22}{5}, 0, -\frac{11}{5}\right) - \left(\frac{1,222}{945}, -\frac{188}{189}, \frac{2,444}{945}\right) \\
&= \left(\frac{8}{5}, -1, \frac{16}{5}\right) - \left(\frac{1,222}{945}, -\frac{188}{189}, \frac{2,444}{945}\right) \\
&= \left(\frac{290}{945}, -\frac{377}{189}, \frac{580}{945}\right)
\end{aligned}$$

$$= \left( \frac{58}{189}, -\frac{377}{189}, \frac{116}{189} \right) \Big|$$

$$\vec{q}_3 = \frac{189}{\sqrt{58^2 + 377^2 + 116^2}} \left( \frac{58}{189}, -\frac{377}{189}, \frac{116}{189} \right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{158,949}} (58, -377, 116)$$

$$= \frac{1}{87\sqrt{21}} (58, -377, 116)$$

$$= \left( \frac{2}{3\sqrt{21}}, -\frac{13}{3\sqrt{21}}, \frac{4}{3\sqrt{21}} \right) \Big|$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(7, 0, -2) \cdot (-2, 0, 1)}{5} (-2, 0, 1)$$

$$= -\frac{16}{5} (-2, 0, 1)$$

$$= \left( \frac{32}{5}, 0, -\frac{16}{5} \right) \Big|$$

$$\begin{aligned} \|\vec{v}_2\|^2 &= \frac{13^2 + 10^2 + 26^2}{25} \\ &= \frac{945}{25} \Big| \end{aligned}$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{25}{945} \left[ (7, 0, -2) \cdot \left( \frac{13}{5}, 2, \frac{26}{5} \right) \right] \left( \frac{13}{5}, 2, \frac{26}{5} \right)$$

$$= \frac{25}{945} \left( \frac{39}{5} \right) \left( \frac{13}{5}, \frac{10}{5}, \frac{26}{5} \right)$$

$$= \frac{13}{315} (13, 10, 26)$$

$$= \left( \frac{169}{315}, \frac{2}{63}, \frac{338}{315} \right) \Big|$$

$$\begin{aligned} \|\vec{v}_3\|^2 &= \frac{58^2 + 377^2 + 116^2}{189^2} \\ &= \frac{158,949}{35,721} \\ &= \frac{841}{189} \Big| \end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 &= \frac{189}{841} \left[ (7, 0, -2) \cdot \left( \frac{58}{189}, -\frac{377}{189}, \frac{116}{189} \right) \right] \left( \frac{58}{189}, -\frac{377}{189}, \frac{116}{189} \right) \\
&= \frac{189}{841} \left( \frac{174}{189} \right) \left( \frac{58}{189}, -\frac{377}{189}, \frac{116}{189} \right) \\
&= \frac{174}{841} \left( \frac{58}{189}, -\frac{377}{189}, \frac{116}{189} \right) \\
&= \frac{58}{841} \left( \frac{58}{63}, -\frac{377}{63}, \frac{116}{63} \right) \\
&= \left( \frac{3,364}{52,983}, -\frac{21,866}{52,983}, \frac{6,728}{52,983} \right)
\end{aligned}$$

$$\begin{aligned}
\vec{v}_4 &= \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 \\
&= (7, 0, -2) - \left( \frac{32}{5}, 0, -\frac{16}{5} \right) - \left( \frac{169}{315}, \frac{2}{63}, \frac{338}{315} \right) - \left( \frac{3,364}{52,983}, -\frac{21,866}{52,983}, \frac{6,728}{52,983} \right) \\
&= \left( \frac{3}{5}, 0, \frac{6}{5} \right) - \left( \frac{3}{5}, 0, \frac{6}{5} \right) \\
&= \underline{(0, 0, 0)}
\end{aligned}$$

$$\vec{q}_4 = \underline{(0, 0, 0)} \qquad \vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

The *orthogonal* basis:

$$\left\{ (-2, 0, 1), \left( \frac{13}{5}, 2, \frac{26}{5} \right), \left( \frac{58}{189}, -\frac{377}{189}, \frac{116}{189} \right), (0, 0, 0) \right\}$$

The *orthogonal* basis:

$$\left\{ \left( -\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}} \right), \left( \frac{13}{3\sqrt{105}}, -\frac{10}{3\sqrt{105}}, \frac{26}{3\sqrt{105}} \right), \left( \frac{2}{3\sqrt{21}}, -\frac{13}{3\sqrt{21}}, \frac{4}{3\sqrt{21}} \right), (0, 0, 0) \right\}$$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(1, 2, -2), (1, 0, -4), (5, 2, 0), (1, 1, -1)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (1, 2, -2) \quad \vec{u}_2 = (1, 0, -4) \quad \vec{u}_3 = (5, 2, 0) \quad \vec{u}_4 = (1, 1, -1)$$

$$\underline{\vec{v}_1 = \vec{u}_1 = (1, 2, -2) \mid}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (1, 0, -4) - \frac{(1, 0, -4) \cdot (1, 2, -2)}{9} (1, 2, -2) \\ &= (1, 0, -4) - (1, 2, -2) \\ &= \underline{(0, -2, -2) \mid}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(5, 2, 0) \cdot (1, 2, -2)}{9} (1, 2, -2) \\ &= \underline{(1, 2, -2) \mid}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{1}{8} [(5, 2, 0) \cdot (0, -2, -2)] (0, -2, -2) \\ &= -\frac{1}{2} (0, -2, -2) \\ &= \underline{(0, 1, 1) \mid}\end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (5, 2, 0) - (1, 2, -2) - (0, 1, 1) \\ &= \underline{(4, -1, 1) \mid}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, 1, -1) \cdot (1, 2, -2)}{9} (1, 2, -2) \\ &= \frac{5}{9} (1, 2, -2) \\ &= \underline{\left(\frac{5}{9}, \frac{10}{9}, -\frac{10}{9}\right) \mid}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{1}{8} [(1, 1, -1) \cdot (0, -2, -2)] (0, -2, -2) \\ &= \underline{(0, 0, 0) \mid}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 &= \frac{1}{18} [(1, 1, -1) \cdot (4, -1, 1)] (4, -1, 1) \\ &= \frac{1}{9} (4, -1, 1) \\ &= \left( \frac{4}{9}, -\frac{1}{9}, \frac{1}{9} \right) \end{aligned}$$

$$\begin{aligned}\vec{v}_4 &= \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 \\ &= (1, 1, -1) - \left( \frac{5}{9}, \frac{10}{9}, -\frac{10}{9} \right) - \left( \frac{4}{9}, -\frac{1}{9}, \frac{1}{9} \right) \\ &= \underline{(0, 0, 0)}\end{aligned}$$

$$\begin{aligned}\vec{q}_1 &= \frac{1}{3} (1, 2, -2) & \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \underline{\left( \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right)}\end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{1}{2\sqrt{2}} (0, -2, -2) & \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \underline{\left( 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)}\end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{1}{3\sqrt{2}} (4, -1, 1) & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \underline{\left( \frac{4}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}} \right)}\end{aligned}$$

$$\begin{aligned}\vec{q}_4 &= \frac{\vec{v}_4}{\|\vec{v}_4\|} \\ &= \underline{(0, 0, 0)}\end{aligned}$$

The *orthogonal* basis:  $\{(1, 2, -2), (0, -2, -2), (4, -1, 1), (0, 0, 0)\}$

The *orthogonal* basis:

$$\left\{ \left( \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right), \left( 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right), \left( \frac{4}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}} \right), (0, 0, 0) \right\}$$

**Exercise**

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(-3, 1, 2), (1, 1, 1), (2, 0, -1), (1, -3, 2)\}$$

**Solution**

$$\text{Let } \vec{u}_1 = (-3, 1, 2) \quad \vec{u}_2 = (1, 1, 1) \quad \vec{u}_3 = (2, 0, -1) \quad \vec{u}_4 = (1, -3, 2)$$

$$\vec{v}_1 = \vec{u}_1 = \underline{(-3, 1, 2)}$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (1, 1, 1) - \frac{(1, 1, 1) \cdot (-3, 1, 2)}{14} (-3, 1, 2) \\ &= (1, 1, 1) - \frac{0}{14} (1, 2, -2) \\ &= \underline{(1, 1, 1)} \end{aligned}$$

$$\begin{aligned} \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(2, 0, -1) \cdot (-3, 1, 2)}{14} (-3, 1, 2) \\ &= -\frac{4}{7} (-3, 1, 2) \\ &= \underline{\left(\frac{12}{7}, -\frac{4}{7}, -\frac{8}{7}\right)} \end{aligned}$$

$$\begin{aligned} \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{1}{3} [(2, 0, -1) \cdot (1, 1, 1)] (1, 1, 1) \\ &= \frac{1}{3} (1, 1, 1) \\ &= \underline{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)} \end{aligned}$$

$$\begin{aligned} \vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (2, 0, -1) - \left(\frac{12}{7}, -\frac{4}{7}, -\frac{8}{7}\right) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \\ &= \underline{\left(-\frac{1}{21}, \frac{5}{21}, -\frac{4}{21}\right)} \end{aligned}$$



$$\begin{aligned}\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, -3, 2) \cdot (-3, 1, 2)}{14} (-3, 1, 2) \\ &= -\frac{1}{7} (-3, 1, 2) \\ &= \left( \frac{3}{7}, -\frac{1}{7}, -\frac{2}{7} \right)\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{(1, -3, 2) \cdot (1, 1, 1)}{3} (1, 1, 1) \\ &= (0, 0, 0)\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 &= \frac{441}{42} \left[ (1, -3, 2) \cdot \left( -\frac{1}{21}, \frac{5}{21}, -\frac{4}{21} \right) \right] \left( -\frac{1}{21}, \frac{5}{21}, -\frac{4}{21} \right) \\ &= \frac{441}{42} \left( -\frac{24}{21} \right) \left( -\frac{1}{21}, \frac{5}{21}, -\frac{4}{21} \right) \\ &= \left( \frac{4}{7}, -\frac{20}{7}, \frac{16}{7} \right)\end{aligned}$$

$$\begin{aligned}\vec{v}_4 &= \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 \\ &= (1, -3, 2) - \left( \frac{3}{7}, -\frac{1}{7}, -\frac{2}{7} \right) - (0, 0, 0) - \left( \frac{4}{7}, -\frac{20}{7}, \frac{16}{7} \right) \\ &= (0, 0, 0)\end{aligned}$$

$$\begin{aligned}\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{1}{\sqrt{14}} (-3, 1, 2) \\ &= \left( -\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}} \right)\end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{1}{\sqrt{3}} (1, 1, 1) \\ &= \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)\end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \frac{21}{\sqrt{42}} \left( -\frac{1}{21}, \frac{5}{21}, -\frac{4}{21} \right) \\ &= \left( -\frac{1}{\sqrt{42}}, \frac{5}{\sqrt{42}}, -\frac{4}{\sqrt{42}} \right)\end{aligned}$$

$$\begin{aligned}\vec{q}_4 &= \frac{\vec{v}_4}{\|\vec{v}_4\|} \\ &= (0, 0, 0)\end{aligned}$$

The *orthogonal* basis:  $\left\{ (-3, 1, 2), (1, 1, 1), \left( -\frac{1}{21}, \frac{5}{21}, -\frac{4}{21} \right), (0, 0, 0) \right\}$

The *orthogonal* basis:

$$\left\{ \left( -\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}} \right), \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right), \left( -\frac{1}{\sqrt{42}}, \frac{5}{\sqrt{42}}, -\frac{4}{\sqrt{42}} \right), (0, 0, 0) \right\}$$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(2, 1, 1), (0, 3, -1), (3, -4, -2), (-1, -1, 3)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (2, 1, 1) \quad \vec{u}_2 = (0, 3, -1) \quad \vec{u}_3 = (3, -4, -2) \quad \vec{u}_4 = (-1, -1, 3)$$

$$\vec{v}_1 = \vec{u}_1 = (2, 1, 1)$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (0, 3, -1) - \frac{(0, 3, -1) \cdot (2, 1, 1)}{6} (2, 1, 1) \\ &= (0, 3, -1) - \frac{1}{3} (2, 1, 1) \\ &= \left( -\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right)\end{aligned}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(3, -4, -2) \cdot (2, 1, 1)}{6} (2, 1, 1)$$

$$= \underline{(0, 0, 0)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{9}{84} \left[ (3, -4, -2) \cdot \left( -\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right) \right] \left( -\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right)$$

$$= \frac{3}{28} (-10) \left( -\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right)$$

$$= \underline{\left( \frac{5}{7}, -\frac{20}{7}, \frac{10}{7} \right)}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (3, -4, -2) - (0, 0, 0) - \left( \frac{5}{7}, -\frac{20}{7}, \frac{10}{7} \right)$$

$$= \underline{\left( \frac{16}{7}, -\frac{8}{7}, -\frac{24}{7} \right)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(-1, -1, 3) \cdot (2, 1, 1)}{6} (2, 1, 1)$$

$$= \underline{(0, 0, 0)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{9}{84} \left[ (-1, -1, 3) \cdot \left( -\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right) \right] \left( -\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right)$$

$$= \frac{3}{28} \left( -\frac{18}{3} \right) \left( -\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right)$$

$$= \underline{\left( \frac{3}{7}, -\frac{12}{7}, \frac{6}{7} \right)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 = \frac{49}{896} \left[ (-1, -1, 3) \cdot \left( \frac{16}{7}, -\frac{8}{7}, -\frac{24}{7} \right) \right] \left( \frac{16}{7}, -\frac{8}{7}, -\frac{24}{7} \right)$$

$$= \frac{7}{128} \left( -\frac{80}{7} \right) \left( \frac{16}{7}, -\frac{8}{7}, -\frac{24}{7} \right)$$

$$= \underline{\left( -\frac{10}{7}, \frac{5}{7}, \frac{15}{7} \right)}$$

$$\begin{aligned}
\vec{v}_4 &= \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 \\
&= (-1, -1, 3) - (0, 0, 0) - \left(\frac{3}{7}, -\frac{12}{7}, \frac{6}{7}\right) - \left(-\frac{10}{7}, \frac{5}{7}, \frac{15}{7}\right) \\
&= \underline{(0, 0, 0)}
\end{aligned}$$

$$\begin{aligned}
\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \frac{1}{\sqrt{6}}(2, 1, 1) \\
&= \underline{\left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)}
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{3}{2\sqrt{21}}\left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3}\right) \\
&= \underline{\left(-\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, -\frac{2}{\sqrt{21}}\right)}
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \frac{7}{8\sqrt{14}}\left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7}\right) \\
&= \underline{\left(\frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\right)}
\end{aligned}$$

$$\begin{aligned}
\vec{q}_4 &= \frac{\vec{v}_4}{\|\vec{v}_4\|} \\
&= \underline{(0, 0, 0)}
\end{aligned}$$

The *orthogonal* basis:  $\left\{(2, 1, 1), \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3}\right), \left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7}\right), (0, 0, 0)\right\}$

The *orthogonal* basis:

$$\left\{\left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(-\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, -\frac{2}{\sqrt{21}}\right), \left(\frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\right), (0, 0, 0)\right\}$$

**Exercise**

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces  $\mathbb{R}^m$ .

$$\{(1, 1, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1)\}$$

**Solution**

$$\text{Let } \vec{u}_1 = (1, 1, 1, 1) \quad \vec{u}_2 = (0, 1, 1, 1) \quad \vec{u}_3 = (0, 0, 1, 1)$$

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 1, 1)$$

$$\vec{q}_1 = \frac{(1, 1, 1, 1)}{2}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (0, 1, 1, 1) - \frac{1}{4} [(0, 1, 1, 1) \cdot (1, 1, 1, 1)] (1, 1, 1, 1)$$

$$= (0, 1, 1, 1) - \frac{3}{4} (1, 1, 1, 1)$$

$$= \left( -\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right)$$

$$\vec{q}_2 = \frac{4}{\sqrt{9+1+1+1}} \left( -\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{2}{\sqrt{3}} \left( -\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right)$$

$$= \left( -\frac{3}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}} \right)$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (0, 0, 1, 1) \cdot (1, 1, 1, 1)$$

$$= 2$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (0, 0, 1, 1) \cdot \left( -\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right)$$

$$= \frac{1}{2}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
&= (0, 0, 1, 1) - \frac{2}{4}(1, 1, 1, 1) - \left(\frac{1}{2}\right)\left(\frac{16}{12}\right)\left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \\
&= (0, 0, 1, 1) - \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) - \left(-\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right) \\
&= \left(0, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{3}{\sqrt{4+1+1}} \left(0, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \frac{3}{\sqrt{6}} \left(0, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \\
&= \left(0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)
\end{aligned}$$

The *orthogonal* basis:  $\left\{(1, 1, 1, 1), \left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \left(0, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$

The *orthonormal* basis:

$$\left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(-\frac{3}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\right), \left(0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)\right\}$$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(1, 2, -1, 0), (2, 2, 0, 1), (1, 1, -1, 0)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (1, 2, -1, 0) \quad \vec{u}_2 = (2, 2, 0, 1) \quad \vec{u}_3 = (1, 1, -1, 0)$$

$$\vec{v}_1 = \vec{u}_1 = (1, 2, -1, 0)$$

$$\vec{q}_1 = \frac{(1, 2, -1, 0)}{\sqrt{1+4+1}} \quad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0\right)$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
&= (2, 2, 0, 1) - \frac{1}{6}[(2, 2, 0, 1) \cdot (1, 2, -1, 0)](1, 2, -1, 0) \\
&= (2, 2, 0, 1) - (1, 2, -1, 0) \\
&= \underline{(1, 0, 1, 1)}
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{1}{\sqrt{3}}(1, 0, 1, 1) & \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \underline{\left(\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_1 \rangle &= (1, 1, -1, 0) \cdot (1, 2, -1, 0) \\
&= \underline{4}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_2 \rangle &= (1, 1, -1, 0) \cdot (1, 0, 1, 1) \\
&= \underline{0}
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2, \\
&= (1, 1, -1, 0) - \frac{4}{6}(1, 2, -1, 0) - (0)(1, 0, 1, 1) \\
&= (1, 1, -1, 0) - \left(\frac{2}{3}, \frac{4}{3}, -\frac{2}{3}, 0\right) \\
&= \underline{\left(\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 0\right)}
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{3}{\sqrt{3}}\left(\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 0\right) & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \underline{\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)}
\end{aligned}$$

The *orthogonal* basis:  $\left\{(1, 2, -1, 0), (1, 0, 1, 1), \left(\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 0\right)\right\}$

The *orthonormal* basis:

$$\left\{\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0\right), \left(\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)\right\}$$

**Exercise**

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(1, 1, 1, 1), (1, 2, 1, 0), (1, 3, 0, 0)\}$$

**Solution**

$$\text{Let } \vec{u}_1 = (1, 1, 1, 1) \quad \vec{u}_2 = (1, 2, 1, 0) \quad \vec{u}_3 = (1, 3, 0, 0)$$

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 1, 1) \quad |$$

$$\begin{aligned} \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(1, 1, 1, 1)}{\sqrt{4}} \\ &= \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \quad | \end{aligned}$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (1, 2, 1, 0) - \frac{(1, 2, 1, 0) \cdot (1, 1, 1, 1)}{\|(1, 1, 1, 1)\|^2} (1, 1, 1, 1) \\ &= (1, 2, 1, 0) - \frac{1+2+1}{4} (1, 1, 1, 1) \\ &= (1, 2, 1, 0) - \frac{4}{4} (1, 1, 1, 1) \\ &= (1, 2, 1, 0) - (1, 1, 1, 1) \\ &= (0, 1, 0, -1) \quad | \end{aligned}$$

$$\begin{aligned} \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{(0, 1, 0, -1)}{\sqrt{1+1}} \\ &= \left( 0, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right) \quad | \end{aligned}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$



$$\begin{aligned}
&= (1, 3, 0, 0) - \frac{(1, 3, 0, 0) \cdot (1, 1, 1, 1)}{\|(1, 1, 1, 1)\|^2} (1, 1, 1, 1) - \frac{(1, 3, 0, 0) \cdot (0, 1, 0, -1)}{\|(0, 1, 0, -1)\|^2} (0, 1, 0, -1) \\
&= (1, 3, 0, 0) - \frac{4}{4} (1, 1, 1, 1) - \frac{3}{2} (0, 1, 0, -1) \\
&= (1, 3, 0, 0) - (1, 1, 1, 1) - \left(0, \frac{3}{2}, 0, -\frac{3}{2}\right) \\
&= \left(0, \frac{1}{2}, -1, \frac{1}{2}\right)
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{1}{\sqrt{\frac{1}{4} + 1 + \frac{1}{4}}} \left(0, \frac{1}{2}, -1, \frac{1}{2}\right) & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \frac{2}{\sqrt{6}} \left(0, \frac{1}{2}, -1, \frac{1}{2}\right) \\
&= \left(0, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)
\end{aligned}$$

The *orthogonal* basis:  $\left\{(1, 1, 1, 1), (0, 1, 0, -1), \left(0, \frac{1}{2}, -1, \frac{1}{2}\right)\right\}$

The *orthonormal* basis:  $\left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(0, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), \left(0, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)\right\}$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(0, 2, -1, 1), (0, 0, 1, 1), (-2, 1, 1, -1)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (0, 2, -1, 1) \quad \vec{u}_2 = (0, 0, 1, 1) \quad \vec{u}_3 = (-2, 1, 1, -1)$$

$$\vec{v}_1 = \vec{u}_1 = (0, 2, -1, 1)$$

$$\vec{q}_1 = \frac{(0, 2, -1, 1)}{\sqrt{6}} \quad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(0, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$\begin{aligned}
 \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
 &= (0, 0, 1, 1) - \frac{1}{6}[(0, 0, 1, 1) \cdot (0, 2, -1, 1)](0, 2, -1, 1) \\
 &= (0, 0, 1, 1) - \frac{1}{6}(\textcolor{red}{0})(0, 2, -1, 1) \\
 &= \underline{\textcolor{blue}{(0, 0, 1, 1)}}
 \end{aligned}$$

$$\begin{aligned}
 \vec{q}_2 &= \frac{(0, 0, 1, 1)}{\sqrt{2}} & \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
 &= \underline{\textcolor{blue}{\left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}}
 \end{aligned}$$

$$\begin{aligned}
 \langle \vec{u}_3, \vec{v}_1 \rangle &= (-2, 1, 1, -1) \cdot (0, 2, -1, 1) \\
 &= \underline{0}
 \end{aligned}$$

$$\begin{aligned}
 \langle \vec{u}_3, \vec{v}_2 \rangle &= (-2, 1, 1, -1) \cdot (0, 0, 1, 1) \\
 &= \underline{0}
 \end{aligned}$$

$$\begin{aligned}
 \vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
 &= (-2, 1, 1, -1) - 0 - 0 \\
 &= \underline{\textcolor{blue}{(-2, 1, 1, -1)}}
 \end{aligned}$$

$$\begin{aligned}
 \vec{q}_3 &= \frac{(-2, 1, 1, -1)}{\sqrt{(-2)^2 + 1^2 + 1^2 + (-1)^2}} & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
 &= \frac{(-2, 1, 1, -1)}{\sqrt{7}} \\
 &= \underline{\textcolor{blue}{\left(-\frac{2}{\sqrt{7}}, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}}\right)}}
 \end{aligned}$$

The *orthogonal* basis:  $\{ \textcolor{blue}{(0, 2, -1, 1)}, \textcolor{blue}{(0, 0, 1, 1)}, \textcolor{blue}{(-2, 1, 1, -1)} \}$

The *orthonormal* basis:

$$\left\{ \textcolor{blue}{\left(0, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)}, \textcolor{blue}{\left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}, \textcolor{blue}{\left(-\frac{2}{\sqrt{7}}, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}}\right)}} \right\}$$

**Exercise**

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(-1, 3, 1, 1), (6, -8, -2, -4), (6, 3, 6, -3)\}$$

**Solution**

$$\text{Let } \vec{u}_1 = (-1, 3, 1, 1) \quad \vec{u}_2 = (6, -8, -2, -4) \quad \vec{u}_3 = (6, 3, 6, -3)$$

$$\vec{v}_1 = \vec{u}_1 = (-1, 3, 1, 1) \quad |$$

$$\vec{q}_1 = \frac{(-1, 3, 1, 1)}{\sqrt{1+9+1+1}} \quad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left( -\frac{1}{2\sqrt{3}}, \frac{3}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}} \right) \quad |$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (6, -8, -2, -4) - \frac{1}{12} [(6, -8, -2, -4) \cdot (-1, 3, 1, 1)] (-1, 3, 1, 1)$$

$$= (6, -8, -2, -4) - \frac{1}{12} (-36) (-1, 3, 1, 1)$$

$$= (6, -8, -2, -4) + (-3, 9, 3, 3)$$

$$= (3, 1, 1, -1) \quad |$$

$$\vec{q}_2 = \frac{1}{\sqrt{12}} (3, 1, 1, -1)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left( \frac{3}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}} \right) \quad |$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (6, 3, 6, -3) \cdot (-1, 3, 1, 1)$$

$$= 6 \quad |$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (6, 3, 6, -3) \cdot (3, 1, 1, -1)$$

$$= 30 \quad |$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2,$$

$$\begin{aligned}
&= (6, 3, 6, -3) - \frac{6}{12}(-1, 3, 1, 1) - \frac{30}{12}(3, 1, 1, -1) \\
&= (6, 3, 6, -3) - \left(-\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right) - \left(\frac{15}{2}, \frac{5}{2}, \frac{5}{2}, -\frac{5}{2}\right) \\
&= \underline{(-1, -1, 3, -1)}
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{1}{2\sqrt{3}}(-1, -1, 3, -1) & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \underline{\left(-\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, \frac{3}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}\right)}
\end{aligned}$$

The *orthogonal* basis:  $\{(-1, 3, 1, 1), (3, 1, 1, -1), (-1, -1, 3, -1)\}$

The *orthonormal* basis:

$$\left\{ \left(-\frac{1}{2\sqrt{3}}, \frac{3}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\right), \left(\frac{3}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}\right), \left(-\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, \frac{3}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}\right) \right\}$$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(0, 0, 2, 2), (3, 3, 0, 0), (1, 1, 0, -1)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (0, 0, 2, 2) \quad \vec{u}_2 = (3, 3, 0, 0) \quad \vec{u}_3 = (1, 1, 0, -1)$$

$$\vec{v}_1 = \vec{u}_1 = \underline{(0, 0, 2, 2)}$$

$$\vec{q}_1 = \frac{(0, 0, 2, 2)}{\sqrt{4+4}} \quad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(0, 0, 2, 2)}{2\sqrt{2}}$$

$$= \underline{\left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
&= (3, 3, 0, 0) - \frac{1}{8}[(3, 3, 0, 0) \cdot (0, 0, 2, 2)] (0, 0, 2, 2) \\
&= (3, 3, 0, 0) - \frac{1}{8}(0) (0, 0, 2, 2) \\
&= \underline{(3, 3, 0, 0)}
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{1}{3\sqrt{2}}(3, 3, 0, 0) & \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \underline{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_1 \rangle &= (1, 1, 0, -1) \cdot (0, 0, 2, 2) \\
&= \underline{-2}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_2 \rangle &= (1, 1, 0, -1) \cdot (3, 3, 0, 0) \\
&= \underline{6}
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2, \\
&= (1, 1, 0, -1) + \frac{2}{8}(0, 0, 2, 2) - \frac{6}{18}(3, 3, 0, 0) \\
&= (1, 1, 0, -1) + \left(0, 0, \frac{1}{2}, \frac{1}{2}\right) - (1, 1, 0, 0) \\
&= \underline{\left(0, 0, \frac{1}{2}, -\frac{1}{2}\right)}
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{1}{\sqrt{\frac{1}{4} + \frac{1}{4}}}\left(0, 0, \frac{1}{2}, -\frac{1}{2}\right) & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \frac{2}{\sqrt{2}}\left(0, 0, \frac{1}{2}, -\frac{1}{2}\right) \\
&= \underline{\left(0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)}
\end{aligned}$$

The *orthogonal* basis:  $\left\{(0, 0, 2, 2), (3, 3, 0, 0), \left(0, 0, \frac{1}{2}, -\frac{1}{2}\right)\right\}$

The *orthonormal* basis:  $\left\{\left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right), \left(0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\}$

**Exercise**

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)\}$$

**Solution**

$$\text{Let } \vec{u}_1 = (1, 0, 0, 0) \quad \vec{u}_2 = (1, 1, 0, 0) \quad \vec{u}_3 = (1, 1, 1, 0) \quad \vec{u}_4 = (1, 1, 1, 1)$$

$$\vec{v}_1 = \vec{u}_1 = (1, 0, 0, 0) \quad |$$

$$\vec{q}_1 = \frac{(1, 0, 0, 0)}{1}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= (1, 0, 0, 0) \quad |$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (1, 1, 0, 0) - [(1, 1, 0, 0) \cdot (1, 0, 0, 0)] (1, 0, 0, 0)$$

$$= (1, 1, 0, 0) - (1, 0, 0, 0)$$

$$= (0, 1, 0, 0) \quad |$$

$$\vec{q}_2 = \frac{1}{1}(0, 1, 0, 0)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= (0, 1, 0, 0) \quad |$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (1, 1, 1, 0) \cdot (1, 0, 0, 0)$$

$$= 1 \quad |$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (1, 1, 1, 0) \cdot (0, 1, 0, 0)$$

$$= 1 \quad |$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2,$$

$$= (1, 1, 1, 0) - \frac{1}{1}(1, 0, 0, 0) - \frac{1}{1}(0, 1, 0, 0)$$

$$= (1, 1, 1, 0) - (1, 0, 0, 0) - (0, 1, 0, 0)$$

$$= (0, 0, 1, 0) \quad |$$

$$\begin{aligned}\vec{q}_3 &= \frac{1}{1}(0, 0, 1, 0) & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \underline{(0, 0, 1, 0)}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, 1, 1, 1) \cdot (1, 0, 0, 0)}{1} (1, 0, 0, 0) \\ &= \underline{(1, 0, 0, 0)}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= [(1, 1, 1, 1) \cdot (0, 1, 0, 0)](0, 1, 0, 0) \\ &= \underline{(0, 1, 0, 0)}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 &= [(1, 1, 1, 1) \cdot (0, 0, 1, 0)](0, 0, 1, 0) \\ &= \underline{(0, 0, 1, 0)}\end{aligned}$$

$$\begin{aligned}\vec{v}_4 &= \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 \\ &= (1, 1, 1, 1) - (1, 0, 0, 0) - (0, 1, 0, 0) - (0, 0, 1, 0) \\ &= \underline{(0, 0, 0, 1)}\end{aligned}$$

$$\begin{aligned}\vec{q}_4 &= \underline{(0, 0, 0, 1)} & \vec{q}_4 &= \frac{\vec{v}_4}{\|\vec{v}_4\|}\end{aligned}$$

The *orthogonal* basis:  $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$

The *orthonormal* basis:  $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$

**Exercise**

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(3, 8, 7, -3), (1, 5, 3, -1), (2, -1, 2, 6), (1, 4, 0, 3)\}$$

**Solution**

$$\text{Let } \vec{u}_1 = (3, 8, 7, -3) \quad \vec{u}_2 = (1, 5, 3, -1) \quad \vec{u}_3 = (2, -1, 2, 6) \quad \vec{u}_4 = (1, 4, 0, 3)$$

$$\vec{v}_1 = \vec{u}_1 = (3, 8, 7, -3)$$

$$\vec{q}_1 = \frac{(3, 8, 7, -3)}{\sqrt{9+64+49+9}} \quad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(3, 8, 7, -3)}{\sqrt{131}}$$

$$= \left( \frac{3}{\sqrt{131}}, \frac{8}{\sqrt{131}}, \frac{7}{\sqrt{131}}, -\frac{3}{\sqrt{131}} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (1, 5, 3, -1) - \frac{1}{131} [(1, 5, 3, -1) \cdot (3, 8, 7, -3)] (3, 8, 7, -3)$$

$$= (1, 5, 3, -1) - \frac{67}{131} (3, 8, 7, -3)$$

$$= (1, 5, 3, -1) - \left( \frac{201}{131}, \frac{536}{131}, \frac{469}{131}, -\frac{201}{131} \right)$$

$$= \left( -\frac{70}{131}, \frac{119}{131}, -\frac{76}{131}, \frac{70}{131} \right)$$

$$\vec{q}_2 = \frac{131}{\sqrt{70^2 + 119^2 + 76^2 + 70^2}} \left( -\frac{70}{131}, \frac{119}{131}, -\frac{76}{131}, \frac{70}{131} \right) \quad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{131}{\sqrt{29,737}} \left( -\frac{70}{131}, \frac{119}{131}, -\frac{76}{131}, \frac{70}{131} \right)$$

$$= \left( -\frac{70}{\sqrt{29,737}}, \frac{119}{\sqrt{29,737}}, -\frac{76}{\sqrt{29,737}}, \frac{70}{\sqrt{29,737}} \right)$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (2, -1, 2, 6) \cdot (3, 8, 7, -3)$$

$$= -6$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (2, -1, 2, 6) \cdot \left( -\frac{70}{131}, \frac{119}{131}, -\frac{76}{131}, \frac{70}{131} \right)$$



$$\begin{aligned}
&= \frac{-140 - 119 - 152 + 420}{131} \\
&= \frac{9}{131} \Big| \\
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2, \\
&= (2, -1, 2, 6) + \frac{6}{131}(3, 8, 7, -3) - \frac{9}{131} \left( \frac{131^2}{29,737} \right) \left( -\frac{70}{131}, \frac{119}{131}, -\frac{76}{131}, \frac{70}{131} \right) \\
&= (2, -1, 2, 6) + \left( \frac{18}{131}, \frac{48}{131}, \frac{42}{131}, -\frac{18}{131} \right) - \frac{9}{131} \left( \frac{131}{227} \right) \left( -\frac{70}{131}, \frac{119}{131}, -\frac{76}{131}, \frac{70}{131} \right) \\
&= \left( \frac{280}{131}, -\frac{83}{131}, \frac{304}{131}, \frac{768}{131} \right) - \left( -\frac{630}{29,737}, \frac{1,071}{29,737}, -\frac{684}{29,737}, \frac{630}{29,737} \right) \\
&= \left( \frac{64,190}{29,737}, -\frac{19,912}{29,737}, \frac{69,692}{29,737}, \frac{173,706}{29,737} \right) \\
&= \left( \frac{490}{227}, -\frac{152}{227}, \frac{532}{227}, \frac{1,326}{227} \right) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{227}{\sqrt{490^2 + 152^2 + 532^2 + 1,326^2}} \left( \frac{490}{227}, -\frac{152}{227}, \frac{532}{227}, \frac{1,326}{227} \right) & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \frac{227}{\sqrt{2,304,504}} \left( \frac{490}{227}, -\frac{152}{227}, \frac{532}{227}, \frac{1,326}{227} \right) \\
&= \frac{1}{6\sqrt{64,014}} (490, -152, 532, 1,326) \\
&= \left( \frac{245}{3\sqrt{64,014}}, -\frac{76}{3\sqrt{64,014}}, \frac{266}{3\sqrt{64,014}}, \frac{221}{\sqrt{64,014}} \right) \Big|
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, 4, 0, 3) \cdot (3, 8, 7, -3)}{131} (3, 8, 7, -3) \\
&= \frac{26}{131} (3, 8, 7, -3) \\
&= \left( \frac{78}{131}, \frac{208}{131}, \frac{182}{131}, -\frac{78}{131} \right) \Big|
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{131^2}{29,737} \left[ (1, 4, 0, 3) \cdot \left( -\frac{70}{131}, \frac{119}{131}, -\frac{76}{131}, \frac{70}{131} \right) \right] \left( -\frac{70}{131}, \frac{119}{131}, -\frac{76}{131}, \frac{70}{131} \right) \\
&= \frac{131}{29,737} \left( \frac{-70 + 476 + 210}{131} \right) (-70, 119, -76, 70)
\end{aligned}$$

$$\begin{aligned}
&= \frac{616}{29,737} (-70, 119, -76, 70) \\
&= \left( -\frac{43,120}{29,737}, \frac{73,304}{29,737}, -\frac{46,816}{29,737}, \frac{43,120}{29,737} \right) \\
\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 &= \frac{227^2}{2,304,504} \left[ (1, 4, 0, 3) \cdot \left( \frac{490}{227}, -\frac{152}{227}, \frac{532}{227}, \frac{1,326}{227} \right) \right] \left( \frac{490}{227}, -\frac{152}{227}, \frac{532}{227}, \frac{1,326}{227} \right) \\
&= \frac{227}{2,304,504} \left( \frac{490 - 608 + 3,978}{227} \right) (490, -152, 532, 1,326) \\
&= \frac{3,860}{2,304,504} (490, -152, 532, 1,326) \\
&= \frac{965}{576,126} (490, -152, 532, 1,326) \\
&= \left( \frac{472,850}{576,126}, -\frac{146,680}{576,126}, \frac{513,380}{576,126}, \frac{1,279,590}{576,126} \right) \\
&= \left( \frac{236,425}{288,063}, -\frac{73,340}{288,063}, \frac{256,690}{288,063}, \frac{213,265}{96,021} \right)
\end{aligned}$$

$$\begin{aligned}
\vec{v}_4 &= \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 \\
&= (1, 4, 0, 3) - \left( \frac{78}{131}, \frac{208}{131}, \frac{182}{131}, -\frac{78}{131} \right) - \left( -\frac{43,120}{29,737}, \frac{73,304}{29,737}, -\frac{46,816}{29,737}, \frac{43,120}{29,737} \right) \\
&\quad - \left( \frac{236,425}{288,063}, -\frac{73,340}{288,063}, \frac{256,690}{288,063}, \frac{213,265}{96,021} \right) \\
&= \left( \frac{53}{131}, \frac{316}{131}, -\frac{182}{131}, \frac{471}{131} \right) - \left( -\frac{43,120}{29,737}, \frac{73,304}{29,737}, -\frac{46,816}{29,737}, \frac{43,120}{29,737} \right) \\
&\quad - \left( \frac{236,425}{288,063}, -\frac{73,340}{288,063}, \frac{256,690}{288,063}, \frac{213,265}{96,021} \right) \\
&= \left( \frac{55,151}{29,737}, -\frac{1,572}{29,737}, \frac{5,502}{29,737}, \frac{63,797}{29,737} \right) - \left( \frac{236,425}{288,063}, -\frac{73,340}{288,063}, \frac{256,690}{288,063}, \frac{213,265}{96,021} \right) \\
&= \left( \frac{421}{227}, -\frac{12}{227}, \frac{42}{227}, \frac{487}{227} \right) - \left( \frac{236,425}{288,063}, -\frac{73,340}{288,063}, \frac{256,690}{288,063}, \frac{213,265}{96,021} \right) \\
&= \left( \frac{297,824}{288,063}, \frac{58,112}{288,063}, -\frac{203,392}{288,063}, -\frac{7,264}{96,021} \right) \\
&= \left( \frac{1,312}{1,269}, \frac{256}{1,269}, -\frac{896}{1,269}, -\frac{32}{423} \right)
\end{aligned}$$

$$\begin{aligned}
\|\vec{v}_4\| &= \frac{1}{1269} \sqrt{(1,312)^2 + (256)^2 + (896)^2 + (96)^2} \\
&= \frac{\sqrt{2,598,912}}{1269} \\
&= \frac{96\sqrt{282}}{1269} \\
\vec{q}_4 &= \frac{1269}{96\sqrt{282}} \left( \frac{1,312}{1,269}, \frac{256}{1,269}, -\frac{896}{1,269}, -\frac{32}{423} \right) \quad \vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|} \\
&= \left( \frac{41}{3\sqrt{282}}, \frac{8}{3\sqrt{282}}, -\frac{28}{3\sqrt{282}}, -\frac{1}{\sqrt{282}} \right)
\end{aligned}$$

The *orthogonal* basis:

$$\left\{ (3, 8, 7, -3), \left( -\frac{70}{131}, \frac{119}{131}, -\frac{76}{131}, \frac{70}{131} \right), \left( \frac{490}{227}, -\frac{152}{227}, \frac{532}{227}, \frac{1,326}{227} \right), \right. \\
\left. \left( \frac{1,312}{1,269}, \frac{256}{1,269}, -\frac{896}{1,269}, -\frac{32}{423} \right) \right\}$$

The *orthonormal* basis:

$$\left\{ \left( \frac{3}{\sqrt{131}}, \frac{8}{\sqrt{131}}, \frac{7}{\sqrt{131}}, -\frac{3}{\sqrt{131}} \right), \left( -\frac{70}{\sqrt{29,737}}, \frac{119}{\sqrt{29,737}}, -\frac{76}{\sqrt{29,737}}, \frac{70}{\sqrt{29,737}} \right), \right. \\
\left( \frac{245}{3\sqrt{64,014}}, -\frac{76}{3\sqrt{64,014}}, \frac{266}{3\sqrt{64,014}}, \frac{221}{\sqrt{64,014}} \right), \\
\left. \left( \frac{41}{3\sqrt{282}}, \frac{8}{3\sqrt{282}}, -\frac{28}{3\sqrt{282}}, -\frac{1}{\sqrt{282}} \right) \right\}$$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(0, 3, -3, -6), (-2, 0, 0, -6), (0, -4, -2, -2), (0, -8, 4, -4)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (0, 3, -3, -6) \quad \vec{u}_2 = (-2, 0, 0, -6) \quad \vec{u}_3 = (0, -4, -2, -2) \quad \vec{u}_4 = (0, -8, 4, -4)$$

$$\vec{v}_1 = \vec{u}_1 = (0, 3, -3, -6)$$

$$\vec{q}_1 = \frac{(0, 3, -3, -6)}{\sqrt{9+9+36}} \quad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(0, 3, -3, -6)}{3\sqrt{6}}$$

$$= \left( 0, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right) \Big|$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (-2, 0, 0, -6) - \frac{1}{54} [(-2, 0, 0, -6) \cdot (0, 3, -3, -6)] (0, 3, -3, -6)$$

$$= (-2, 0, 0, -6) - \frac{2}{3} (0, 3, -3, -6)$$

$$= (-2, 0, 0, -6) - (0, 2, -2, -4)$$

$$= (-2, -2, 2, -2) \Big|$$

$$\vec{q}_2 = \frac{1}{\sqrt{4+4+4+4}} (-2, -2, 2, -2) \quad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{4} (-2, -2, 2, -2)$$

$$= \left( -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right) \Big|$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (0, -4, -2, -2) \cdot (0, 3, -3, -6)$$

$$= 6 \Big|$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (0, -4, -2, -2) \cdot (-2, -2, 2, -2)$$

$$= 8 \Big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2,$$

$$= (0, -4, -2, -2) - \frac{6}{54} (0, 3, -3, -6) - \frac{8}{16} (-2, -2, 2, -2)$$

$$= (0, -4, -2, -2) - \left( 0, \frac{1}{3}, -\frac{1}{3}, -\frac{2}{3} \right) - (-1, -1, 1, -1)$$

$$= \left( 1, -\frac{10}{3}, -\frac{8}{3}, -\frac{1}{3} \right) \Big|$$

$$\vec{q}_3 = \frac{3}{\sqrt{9+100+64+1}} \left( 1, -\frac{10}{3}, -\frac{8}{3}, -\frac{1}{3} \right) \quad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{3}{\sqrt{174}} \left( 1, -\frac{10}{3}, -\frac{8}{3}, -\frac{1}{3} \right)$$

$$= \left( \frac{3}{\sqrt{174}}, -\frac{10}{\sqrt{174}}, -\frac{8}{\sqrt{174}}, -\frac{1}{\sqrt{174}} \right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(0, -8, 4, -4) \cdot (0, 3, -3, -6)}{54} (0, 3, -3, -6)$$

$$= -\frac{12}{54} (0, 3, -3, -6)$$

$$= \left( 0, -\frac{2}{3}, \frac{2}{3}, \frac{4}{3} \right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{1}{16} [(0, -8, 4, -4) \cdot (-2, -2, 2, -2)] (-2, -2, 2, -2)$$

$$= \frac{32}{16} (-2, -2, 2, -2)$$

$$= (-4, -4, 4, -4)$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 = \frac{9}{174} \left[ (0, -8, 4, -4) \cdot \left( 1, -\frac{10}{3}, -\frac{8}{3}, -\frac{1}{3} \right) \right] \left( 1, -\frac{10}{3}, -\frac{8}{3}, -\frac{1}{3} \right)$$

$$= \frac{9}{174} \left( \frac{80-32+4}{3} \right) \left( 1, -\frac{10}{3}, -\frac{8}{3}, -\frac{1}{3} \right)$$

$$= \frac{9}{174} \left( \frac{52}{3} \right) \left( 1, -\frac{10}{3}, -\frac{8}{3}, -\frac{1}{3} \right)$$

$$= \frac{78}{87} \left( 1, -\frac{10}{3}, -\frac{8}{3}, -\frac{1}{3} \right)$$

$$= \left( \frac{26}{29}, -\frac{260}{87}, -\frac{208}{87}, -\frac{26}{87} \right)$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3$$

$$= (0, -8, 4, -4) - \left( 0, -\frac{2}{3}, \frac{2}{3}, \frac{4}{3} \right) - (-4, -4, 4, -4) - \left( \frac{26}{29}, -\frac{260}{87}, -\frac{208}{87}, -\frac{26}{87} \right)$$

$$= (4, -4, 0, 0) + \left( -\frac{26}{29}, \frac{318}{87}, \frac{150}{87}, -\frac{30}{29} \right)$$

$$= \left( \frac{90}{29}, -\frac{10}{29}, \frac{50}{29}, -\frac{30}{29} \right)$$

$$\begin{aligned}
\|\vec{v}_4\| &= \sqrt{\left(\frac{90}{29}\right)^2 + \left(\frac{10}{29}\right)^2 + \left(\frac{50}{29}\right)^2 + \left(\frac{90}{87}\right)^2} \\
&= \sqrt{\frac{8,100}{841} + \frac{100}{841} + \frac{2,500}{841} + \frac{8,100}{7,569}} \\
&= 10 \sqrt{\frac{729 + 9 + 225 + 81}{7,569}} \\
&= \frac{10}{87} \sqrt{1,044} \\
&= \frac{20}{29} \sqrt{29}
\end{aligned}$$

$$\begin{aligned}
\vec{q}_4 &= \frac{29}{20} \frac{1}{\sqrt{29}} \left( \frac{90}{29}, -\frac{10}{29}, \frac{50}{29}, -\frac{90}{87} \right) & \vec{q}_4 &= \frac{\vec{v}_4}{\|\vec{v}_4\|} \\
&= \left( \frac{9}{2\sqrt{29}}, -\frac{1}{2\sqrt{29}}, \frac{5}{2\sqrt{29}}, -\frac{3}{2\sqrt{29}} \right)
\end{aligned}$$

The *orthogonal* basis:

$$\left\{ (0, 3, -3, -6), (-2, -2, 2, -2), \left(1, -\frac{10}{3}, -\frac{8}{3}, -\frac{1}{3}\right), \left(\frac{90}{29}, -\frac{10}{29}, \frac{50}{29}, -\frac{30}{29}\right) \right\}$$

The *orthonormal* basis:

$$\left\{ \left(0, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right), \left(\frac{3}{\sqrt{174}}, -\frac{10}{\sqrt{174}}, -\frac{8}{\sqrt{174}}, -\frac{1}{\sqrt{174}}\right), \left(\frac{9}{2\sqrt{29}}, -\frac{1}{2\sqrt{29}}, \frac{5}{2\sqrt{29}}, -\frac{3}{2\sqrt{29}}\right) \right\}$$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(3, 0, -3, 6), (0, 2, 3, 1), (0, -2, -2, 0), (-2, 1, 2, 1)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (3, 0, -3, 6) \quad \vec{u}_2 = (0, 2, 3, 1) \quad \vec{u}_3 = (0, -2, -2, 0) \quad \vec{u}_4 = (-2, 1, 2, 1)$$

$$\vec{v}_1 = \vec{u}_1 = (3, 0, -3, 6)$$

$$\begin{aligned}\vec{q}_1 &= \frac{(3, 0, -3, 6)}{\sqrt{9+9+36}} & \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(3, 0, -3, 6)}{3\sqrt{6}} \\ &= \left( \frac{1}{\sqrt{6}}, 0, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right)\end{aligned}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (0, 2, 3, 1) - \frac{1}{54} [(0, 2, 3, 1) \cdot (3, 0, -3, 6)] (3, 0, -3, 6) \\ &= (0, 2, 3, 1) + \frac{1}{18} (3, 0, -3, 6) \\ &= (0, 2, 3, 1) + \left( \frac{1}{6}, 0, -\frac{1}{6}, \frac{1}{3} \right) \\ &= \left( \frac{1}{6}, 2, \frac{17}{6}, \frac{4}{3} \right)\end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{1}{\sqrt{\frac{1}{36} + 4 + \frac{289}{36} + \frac{16}{9}}} \left( \frac{1}{6}, 2, \frac{17}{6}, \frac{4}{3} \right) & \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{1}{\sqrt{\frac{498}{36}}} \left( \frac{1}{6}, 2, \frac{17}{6}, \frac{4}{3} \right) \\ &= \frac{6}{\sqrt{498}} \left( \frac{1}{6}, 2, \frac{17}{6}, \frac{4}{3} \right) \\ &= \left( \frac{1}{\sqrt{498}}, \frac{12}{\sqrt{498}}, \frac{17}{\sqrt{498}}, \frac{8}{\sqrt{498}} \right)\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{1}{54} [(0, -2, -2, 0) \cdot (3, 0, -3, 6)] (3, 0, -3, 6) \\ &= \frac{6}{54} (3, 0, -3, 6) \\ &= \left( \frac{1}{3}, 0, -\frac{1}{3}, \frac{2}{3} \right)\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{36}{498} \left[ (0, -2, -2, 0) \cdot \left( \frac{1}{6}, 2, \frac{17}{6}, \frac{4}{3} \right) \right] \left( \frac{1}{6}, 2, \frac{17}{6}, \frac{4}{3} \right) \\ &= \frac{6}{83} \left( -\frac{29}{3} \right) \left( \frac{1}{6}, 2, \frac{17}{6}, \frac{4}{3} \right)\end{aligned}$$

$$\begin{aligned}
&= -\frac{58}{83} \left( \frac{1}{6}, 2, \frac{17}{6}, \frac{4}{3} \right) \\
&= \left( -\frac{29}{249}, -\frac{116}{83}, -\frac{493}{249}, -\frac{232}{249} \right) \\
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2, \\
&= (0, -2, -2, 0) - \left( \frac{1}{3}, 0, -\frac{1}{3}, \frac{2}{3} \right) - \left( -\frac{29}{249}, -\frac{116}{83}, -\frac{493}{249}, -\frac{232}{249} \right) \\
&= \left( -\frac{1}{3}, -2, -\frac{5}{3}, -\frac{2}{3} \right) - \left( -\frac{29}{249}, -\frac{116}{83}, -\frac{493}{249}, -\frac{232}{249} \right) \\
&= \left( -\frac{18}{83}, -\frac{50}{83}, \frac{26}{83}, \frac{22}{83} \right)
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{83}{\sqrt{324 + 2500 + 676 + 484}} \left( -\frac{18}{83}, -\frac{50}{83}, \frac{26}{83}, \frac{22}{83} \right) \\
&= \frac{1}{\sqrt{3,984}} (-18, -50, 26, 22) \\
&= \frac{1}{4\sqrt{249}} (-18, -50, 26, 22) \\
&= \left( -\frac{9}{2\sqrt{249}}, -\frac{25}{2\sqrt{249}}, \frac{13}{2\sqrt{249}}, \frac{11}{2\sqrt{249}} \right)
\end{aligned}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$\begin{aligned}
\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(-2, 1, 2, 1) \cdot (3, 0, -3, 6)}{54} (3, 0, -3, 6) \\
&= -\frac{6}{54} (3, 0, -3, 6) \\
&= -\frac{1}{9} (3, 0, -3, 6) \\
&= \left( -\frac{1}{3}, 0, \frac{1}{3}, -\frac{2}{3} \right)
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{36}{498} \left[ (-2, 1, 2, 1) \cdot \left( \frac{1}{6}, 2, \frac{17}{6}, \frac{4}{3} \right) \right] \left( \frac{1}{6}, 2, \frac{17}{6}, \frac{4}{3} \right) \\
&= \frac{6}{83} \left( -\frac{1}{3} + 2 + \frac{17}{3} + \frac{4}{3} \right) \left( \frac{1}{6}, 2, \frac{17}{6}, \frac{4}{3} \right) \\
&= \frac{6}{83} \left( \frac{26}{3} \right) \left( \frac{1}{6}, 2, \frac{17}{6}, \frac{4}{3} \right)
\end{aligned}$$



$$= \frac{52}{83} \left( \frac{1}{6}, 2, \frac{17}{6}, \frac{4}{3} \right)$$

$$= \left( \frac{26}{249}, \frac{104}{83}, \frac{442}{249}, \frac{208}{249} \right) \Bigg|$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 = \frac{83^2}{3,984} \left[ (-2, 1, 2, 1) \cdot \left( -\frac{18}{83}, -\frac{50}{83}, \frac{26}{83}, \frac{22}{83} \right) \right] \left( -\frac{18}{83}, -\frac{50}{83}, \frac{26}{83}, \frac{22}{83} \right)$$

$$= \frac{83}{3,984} \frac{36 - 50 + 52 + 22}{83} (-18, -50, 26, 22)$$

$$= \frac{60}{3,984} (-18, -50, 26, 22)$$

$$= \frac{15}{996} (-18, -50, 26, 22)$$

$$= \frac{5}{332} (-18, -50, 26, 22)$$

$$= \left( -\frac{45}{166}, -\frac{125}{166}, \frac{65}{166}, \frac{55}{166} \right) \Bigg|$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3$$

$$= (-2, 1, 2, 1) - \left( -\frac{1}{3}, 0, \frac{1}{3}, -\frac{2}{3} \right) - \left( \frac{26}{249}, \frac{104}{83}, \frac{442}{249}, \frac{208}{249} \right) - \left( -\frac{45}{166}, -\frac{125}{166}, \frac{65}{166}, \frac{55}{166} \right)$$

$$= \left( -\frac{5}{3}, 1, \frac{5}{3}, \frac{5}{3} \right) - \left( \frac{26}{249}, \frac{104}{83}, \frac{442}{249}, \frac{208}{249} \right) + \left( \frac{45}{166}, \frac{125}{166}, -\frac{65}{166}, -\frac{55}{166} \right)$$

$$= \left( -\frac{441}{249}, -\frac{21}{83}, -\frac{27}{249}, \frac{207}{249} \right) + \left( \frac{45}{166}, \frac{125}{166}, -\frac{65}{166}, -\frac{55}{166} \right)$$

$$= \left( -\frac{147}{83}, -\frac{21}{83}, -\frac{9}{83}, \frac{69}{83} \right) + \left( \frac{45}{166}, \frac{125}{166}, -\frac{65}{166}, -\frac{55}{166} \right)$$

$$= \left( -\frac{249}{166}, \frac{83}{166}, -\frac{83}{166}, \frac{83}{166} \right)$$

$$= \left( -\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right) \Bigg|$$

$$\vec{q}_4 = \frac{1}{\sqrt{\frac{9}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}} \left( -\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right)$$

$$= \frac{2}{\sqrt{12}} \left( -\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right)$$

$$= \frac{1}{\sqrt{3}} \left( -\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right)$$

$$= \left( -\frac{3}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}} \right) \Bigg|$$

$$\vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

The *orthogonal* basis:

$$\left\{ (3, 0, -3, 6), \left(\frac{1}{6}, 2, \frac{17}{6}, \frac{4}{3}\right), \left(-\frac{18}{83}, -\frac{50}{83}, \frac{26}{83}, \frac{22}{83}\right), \right. \\ \left. \left(-\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \right\}$$

The *orthonormal* basis:

$$\left\{ \left(\frac{1}{\sqrt{6}}, 0, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{498}}, \frac{12}{\sqrt{498}}, \frac{17}{\sqrt{498}}, \frac{8}{\sqrt{498}}\right), \right. \\ \left. \left(-\frac{9}{2\sqrt{249}}, -\frac{25}{2\sqrt{249}}, \frac{13}{2\sqrt{249}}, \frac{11}{2\sqrt{249}}\right), \left(-\frac{3}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\right) \right\}$$

### Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces  $\mathbb{R}^m$ .

$$\{(3, 4, 0, 0), (-1, 1, 0, 0), (2, 1, 0, -1), (0, 1, 1, 0)\}$$

### Solution

$$\text{Let } \vec{u}_1 = (3, 4, 0, 0) \quad \vec{u}_2 = (-1, 1, 0, 0) \quad \vec{u}_3 = (2, 1, 0, -1) \quad \vec{u}_4 = (0, 1, 1, 0)$$

$$\vec{v}_1 = \vec{u}_1 = (3, 4, 0, 0) \quad |$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ = (-1, 1, 0, 0) - \frac{(-1, 1, 0, 0) \cdot (3, 4, 0, 0)}{25} (3, 4, 0, 0) \\ = (-1, 1, 0, 0) - \frac{1}{25} (3, 4, 0, 0) \\ = \left(-\frac{28}{25}, \frac{21}{25}, 0, 0\right) \quad |$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(2, 1, 0, -1) \cdot (3, 4, 0, 0)}{25} (3, 4, 0, 0) \\ = \frac{10}{25} (3, 4, 0, 0) \\ = \left(\frac{6}{5}, \frac{8}{5}, 0, 0\right) \quad |$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{625}{1,225} \left[ (2, 1, 0, -1) \cdot \left( -\frac{28}{25}, \frac{21}{25}, 0, 0 \right) \right] \left( -\frac{28}{25}, \frac{21}{25}, 0, 0 \right) \\
&= \frac{25}{49} \left( -\frac{35}{25} \right) \left( -\frac{28}{25}, \frac{21}{25}, 0, 0 \right) \\
&= \left( \frac{4}{5}, -\frac{3}{5}, 0, 0 \right) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
&= (2, 1, 0, -1) - \left( \frac{6}{5}, \frac{8}{5}, 0, 0 \right) - \left( \frac{4}{5}, -\frac{3}{5}, 0, 0 \right) \\
&= \underline{(0, 0, 0, -1)} \Big|
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(0, 1, 1, 0) \cdot (3, 4, 0, 0)}{25} (3, 4, 0, 0) \\
&= \frac{4}{25} (3, 4, 0, 0) \\
&= \left( \frac{12}{25}, \frac{16}{25}, 0, 0 \right) \Big|
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{25}{49} \left[ (0, 1, 1, 0) \cdot \left( -\frac{28}{25}, \frac{21}{25}, 0, 0 \right) \right] \left( -\frac{28}{25}, \frac{21}{25}, 0, 0 \right) \\
&= \frac{25}{49} \left( \frac{21}{25} \right) \left( -\frac{28}{25}, \frac{21}{25}, 0, 0 \right) \\
&= \left( -\frac{12}{25}, \frac{9}{25}, 0, 0 \right) \Big|
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 &= [(0, 1, 1, 0) \cdot (0, 0, 0, -1)] (0, 0, 0, -1) \\
&= \underline{(0, 0, 0, 0)} \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{v}_4 &= \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 \\
&= (0, 1, 1, 0) - \left( \frac{12}{25}, \frac{16}{25}, 0, 0 \right) - \left( -\frac{12}{25}, \frac{9}{25}, 0, 0 \right) - (0, 0, 0, 0) \\
&= \underline{(0, 0, 1, 0)} \Big|
\end{aligned}$$

$$\begin{aligned}\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{1}{5}(3, 4, 0, 0) \\ &= \left(\frac{3}{5}, \frac{4}{5}, 0, 0\right)\end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{25}{35}\left(-\frac{28}{25}, \frac{21}{25}, 0, 0\right) \\ &= \left(-\frac{4}{5}, \frac{3}{5}, 0, 0\right)\end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= (0, 0, 0, -1)\end{aligned}$$

$$\begin{aligned}\vec{q}_4 &= \frac{\vec{v}_4}{\|\vec{v}_4\|} \\ &= (0, 0, 1, 0)\end{aligned}$$

The *orthogonal* basis:

$$\left\{(3, 4, 0, 0), \left(-\frac{28}{25}, \frac{21}{25}, 0, 0\right), (0, 0, 0, -1), (0, 0, 1, 0)\right\}$$

The *orthonormal* basis:

$$\left\{\left(\frac{3}{5}, \frac{4}{5}, 0, 0\right), \left(-\frac{4}{5}, \frac{3}{5}, 0, 0\right), (0, 0, 0, -1), (0, 0, 1, 0)\right\}$$

### Exercise

Find the *QR*-decomposition of  $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

### Solution

Since  $\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5 \neq 0$ , The matrix is invertible

$$\vec{u}_1(1, 2), \quad \vec{u}_2 = (-1, 3)$$

$$\vec{v}_1 = \vec{u}_1 = (1, 2)$$

$$\begin{aligned}
 \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
 &= \frac{(1, 2)}{\sqrt{1^2 + 2^2}} \\
 &= \frac{(1, 2)}{\sqrt{5}} \\
 &= \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)
 \end{aligned}$$

$$\begin{aligned}
 \vec{v}_2 &= \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1) \vec{v}_1 \\
 &= (-1, 3) - \left[ (-1, 3) \cdot \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \right] \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \\
 &= (-1, 3) - \left( \frac{5}{\sqrt{5}} \right) \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \\
 &= (-1, 3) - (1, 2) \\
 &= (-2, 1)
 \end{aligned}$$

$$\begin{aligned}
 \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
 &= \frac{(-2, 1)}{\sqrt{(-2)^2 + 1^2}} \\
 &= \left( -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right)
 \end{aligned}$$

$$\begin{aligned}
 \langle \vec{u}_1, \vec{q}_1 \rangle &= (1, 2) \cdot \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \\
 &= \frac{5}{\sqrt{5}} \\
 &= \sqrt{5}
 \end{aligned}$$

$$\begin{aligned}
 \langle \vec{u}_2, \vec{q}_1 \rangle &= (-1, 3) \cdot \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \\
 &= \frac{5}{\sqrt{5}} \\
 &= \sqrt{5}
 \end{aligned}$$

$$\langle \vec{u}_2, \vec{q}_2 \rangle = (-1, 3) \cdot \left( -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right)$$

$$\begin{aligned}
 &= \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{5}} \\
 &= \sqrt{5} \mid \\
 R &= \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix} \\
 &= \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}
 \end{aligned}$$

The  $QR$ -decomposition of the matrix is

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}$$

### Exercise

Find the  $QR$ -decomposition of  $\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}$

### Solution

The column vectors of are:  $\vec{u}_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ ,  $\vec{u}_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$$\vec{v}_1 = \vec{u}_1 = (3, -4) \mid$$

$$\vec{q}_1 = \frac{(3, -4)}{\sqrt{9+16}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left( \frac{3}{5}, -\frac{4}{5} \right) \mid$$

$$\begin{aligned}
 \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
 &= (5, 0) - \frac{(5, 0) \cdot (3, -4)}{25} (3, -4) \\
 &= (5, 0) - \frac{15}{25} (3, -4) \\
 &= (5, 0) - \frac{3}{5} (3, -4) \\
 &= (5, 0) - \left( \frac{9}{5}, -\frac{12}{5} \right)
 \end{aligned}$$

$$\begin{aligned}
&= \left( \frac{16}{5}, \frac{12}{5} \right) \\
\vec{q}_2 &= \frac{1}{\sqrt{\frac{256}{25} + \frac{144}{25}}} \left( \frac{16}{5}, \frac{12}{5} \right) & \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{1}{\sqrt{\frac{400}{25}}} \left( \frac{16}{5}, \frac{12}{5} \right) \\
&= \frac{1}{\sqrt{16}} \left( \frac{16}{5}, \frac{12}{5} \right) \\
&= \frac{1}{4} \left( \frac{16}{5}, \frac{12}{5} \right) \\
&= \left( \frac{4}{5}, \frac{3}{5} \right) \\
R &= \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix} \\
&= \begin{bmatrix} 3\left(\frac{3}{5}\right) - 4\left(-\frac{4}{5}\right) & 5\left(\frac{3}{5}\right) + 0\left(-\frac{4}{5}\right) \\ 0 & 5\left(\frac{4}{5}\right) - 0\left(\frac{3}{5}\right) \end{bmatrix} \\
&= \begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix} \\
\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix} &= \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \cdot \begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix} \\
\mathbf{A} &= \mathbf{Q} \mathbf{R}
\end{aligned}$$

### Exercise

Find the  $QR$ -decomposition of  $\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$

### Solution

The column vectors of are:  $\vec{u}_1 = (1, -2)$   $\vec{u}_2 = (3, 4)$

$$\vec{v}_1 = \vec{u}_1 = (1, -2)$$

$$\vec{q}_1 = \frac{(1, -2)}{\sqrt{1+4}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left( \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right)$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (3, 4) - \frac{(3, 4) \cdot (1, -2)}{5} (1, -2) \\ &= (3, 4) + \frac{5}{5} (1, -2) \\ &= (3, 4) + (1, -2) \\ &= (4, 2)\end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{1}{\sqrt{16+4}} (4, 2) & \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{1}{2\sqrt{5}} (4, 2) \\ &= \left( \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right)\end{aligned}$$

$$\begin{aligned}R &= \begin{bmatrix} 1\left(\frac{1}{\sqrt{5}}\right) - 2\left(-\frac{2}{\sqrt{5}}\right) & 3\left(\frac{1}{\sqrt{5}}\right) + 4\left(-\frac{2}{\sqrt{5}}\right) \\ 0 & 3\left(\frac{2}{\sqrt{5}}\right) + 4\left(\frac{1}{\sqrt{5}}\right) \end{bmatrix} \\ &= \begin{pmatrix} \frac{5}{\sqrt{5}} & -\frac{5}{\sqrt{5}} \\ 0 & \frac{10}{\sqrt{5}} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{5} & -\sqrt{5} \\ 0 & 2\sqrt{5} \end{pmatrix}\end{aligned}$$

$$\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{5} & -\sqrt{5} \\ 0 & 2\sqrt{5} \end{pmatrix}$$

$$\mathbf{A} = \mathbf{Q} \mathbf{R}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$



**Exercise**

Find the  $QR$ -decomposition of  $\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}$

**Solution**

The column vectors of are:  $\vec{u}_1 = (1, -1)$   $\vec{u}_2 = (-2, 4)$

$$\vec{v}_1 = \vec{u}_1 = (1, -1)$$

$$\vec{q}_1 = \frac{(1, -1)}{\sqrt{2}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (-2, 4) - \frac{(-2, 4) \cdot (1, -1)}{2} (1, -1)$$

$$= (-2, 4) + \frac{6}{2} (1, -1)$$

$$= (-2, 4) + (3, -3)$$

$$= (1, 1)$$

$$\vec{q}_2 = \frac{1}{\sqrt{2}} (1, 1) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$R = \begin{bmatrix} \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) & -2\left(\frac{1}{\sqrt{2}}\right) + 4\left(-\frac{1}{\sqrt{2}}\right) \\ 0 & -2\left(\frac{1}{\sqrt{2}}\right) + 4\left(\frac{1}{\sqrt{2}}\right) \end{bmatrix}$$

$$= \begin{pmatrix} \frac{2}{\sqrt{2}} & -\frac{6}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2} & -3\sqrt{2} \\ 0 & \sqrt{2} \end{pmatrix}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & -3\sqrt{2} \\ 0 & \sqrt{2} \end{pmatrix}$$

$\mathbf{A} = \mathbf{Q} \mathbf{R}$

**Exercise**

Find the  $QR$ -decomposition of  $\begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}$

**Solution**

The column vectors of are:  $\vec{u}_1 = (-1, 2)$   $\vec{u}_2 = (1, -4)$

$$\vec{v}_1 = \vec{u}_1 = (-1, 2)$$

$$\vec{q}_1 = \frac{(-1, 2)}{\sqrt{5}} \quad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left( -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (1, -4) - \frac{(1, -4) \cdot (-1, 2)}{5} (-1, 2)$$

$$= (1, -4) + \frac{9}{5} (-1, 2)$$

$$= (1, -4) + \left( -\frac{9}{5}, \frac{18}{5} \right)$$

$$= \left( -\frac{4}{5}, -\frac{2}{5} \right)$$

$$\vec{q}_2 = \frac{5}{\sqrt{16+4}} \left( -\frac{4}{5}, -\frac{2}{5} \right) \quad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{5}{2\sqrt{5}} \left( -\frac{4}{5}, -\frac{2}{5} \right)$$

$$= \left( -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right)$$

$$R = \begin{bmatrix} \frac{1}{\sqrt{5}} + 2\left(\frac{2}{\sqrt{5}}\right) & \left(-\frac{1}{\sqrt{5}}\right) - 4\left(\frac{2}{\sqrt{5}}\right) \\ 0 & \left(-\frac{2}{\sqrt{5}}\right) - 4\left(-\frac{1}{\sqrt{5}}\right) \end{bmatrix}$$

$$= \begin{pmatrix} \frac{5}{\sqrt{5}} & -\frac{9}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \cdot \begin{pmatrix} \frac{5}{\sqrt{5}} & -\frac{9}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$\mathbf{A} = \mathbf{Q} \mathbf{R}$

### Exercise

Find the  $QR$ -decomposition of  $\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}$

### Solution

The column vectors of are:  $\vec{u}_1 = (1, -2)$   $\vec{u}_2 = (-3, 4)$

$$\vec{v}_1 = \vec{u}_1 = (1, -2)$$

$$\vec{q}_1 = \frac{(1, -2)}{\sqrt{5}} \quad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left( \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (-3, 4) - \frac{(-3, 4) \cdot (1, -2)}{5} (1, -2)$$

$$= (-3, 4) + \frac{11}{5} (1, -2)$$

$$= \left( -\frac{4}{5}, -\frac{2}{5} \right)$$

$$\vec{q}_2 = \frac{5}{\sqrt{16+4}} \left( -\frac{4}{5}, -\frac{2}{5} \right) \quad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{5}{2\sqrt{5}} \left( -\frac{4}{5}, -\frac{2}{5} \right)$$

$$= \left( -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right)$$

$$R = \begin{bmatrix} \frac{1}{\sqrt{5}} - 2 \left( -\frac{2}{\sqrt{5}} \right) & -3 \left( \frac{1}{\sqrt{5}} \right) + 4 \left( -\frac{2}{\sqrt{5}} \right) \\ 0 & -3 \left( -\frac{2}{\sqrt{5}} \right) + 4 \left( -\frac{1}{\sqrt{5}} \right) \end{bmatrix}$$

$$= \begin{pmatrix} \frac{5}{\sqrt{5}} & -\frac{11}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \cdot \begin{pmatrix} \frac{5}{\sqrt{5}} & -\frac{11}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$\mathbf{A} = \mathbf{Q} \mathbf{R}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

### Exercise

Find the  $QR$ -decomposition of  $\begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix}$

### Solution

The column vectors of are:  $\vec{u}_1 = (-2, 4)$   $\vec{u}_2 = (6, -8)$

$$\vec{v}_1 = \vec{u}_1 = (-2, 4)$$

$$\vec{q}_1 = \frac{(-2, 4)}{\sqrt{20}} \quad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left( -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (6, -8) - \frac{(6, -8) \cdot (-2, 4)}{20} (-2, 4)$$

$$= (6, -8) + \frac{44}{20} (-2, 4)$$

$$= (6, -8) + \frac{11}{5}(-2, 4)$$

$$= \left( \frac{8}{5}, \frac{4}{5} \right)$$

$$\vec{q}_2 = \frac{5}{\sqrt{64+16}} \left( \frac{8}{5}, \frac{4}{5} \right) \quad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{5}{4\sqrt{5}} \left( \frac{8}{5}, \frac{4}{5} \right)$$

$$= \left( \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right)$$

$$R = \begin{bmatrix} -2\left(-\frac{1}{\sqrt{5}}\right) + 4\left(\frac{2}{\sqrt{5}}\right) & 6\left(-\frac{1}{\sqrt{5}}\right) - 8\left(\frac{2}{\sqrt{5}}\right) \\ 0 & 6\left(\frac{2}{\sqrt{5}}\right) - 8\left(\frac{1}{\sqrt{5}}\right) \end{bmatrix}$$

$$= \begin{pmatrix} \frac{10}{\sqrt{5}} & -\frac{22}{\sqrt{5}} \\ 0 & \frac{4}{\sqrt{5}} \end{pmatrix}$$

$$\begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \cdot \begin{pmatrix} \frac{10}{\sqrt{5}} & -\frac{22}{\sqrt{5}} \\ 0 & \frac{4}{\sqrt{5}} \end{pmatrix}$$

$$\mathbf{A} = \mathbf{Q} \mathbf{R}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

### Exercise

Find the  $QR$ -decomposition of  $\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$

### Solution

The column vectors of are:  $\vec{u}_1 = (-1, 1)$   $\vec{u}_2 = (2, 0)$

$$\vec{v}_1 = \vec{u}_1 = (-1, 1)$$

$$\vec{q}_1 = \frac{(-1, 1)}{\sqrt{2}} \quad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$\begin{aligned}
 \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
 &= (2, 0) - \frac{(2, 0) \cdot (-1, 1)}{2} (-1, 1) \\
 &= (2, 0) + (-1, 1) \\
 &= (1, 1) \quad |
 \end{aligned}$$

$$\begin{aligned}
 \vec{q}_2 &= \frac{1}{\sqrt{2}}(1, 1) & \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
 &= \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \quad |
 \end{aligned}$$

$$\begin{aligned}
 R &= \begin{bmatrix} -\left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right) & 2\left(-\frac{1}{\sqrt{2}}\right) + 0 \\ 0 & 2\left(\frac{1}{\sqrt{2}}\right) + 0 \end{bmatrix} \\
 &= \begin{pmatrix} \frac{2}{\sqrt{2}} & -\frac{2}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} \end{pmatrix} \\
 \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{\sqrt{2}} & -\frac{2}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} \end{pmatrix} \\
 \mathbf{A} &= \mathbf{Q} \mathbf{R}
 \end{aligned}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

### Exercise

Find the  $QR$ -decomposition of  $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$

### Solution

Since the column vectors  $\vec{u}_1 = (1, 0, 1)$ ,  $\vec{u}_2 = (2, 1, 4)$  are linearly independent, so has a  $QR$ -decomposition.

$$\underline{\vec{v}_1 = \vec{u}_1 = (1, 0, 1) \quad |}$$

$$\begin{aligned}\vec{q}_1 &= \frac{(1, 0, 1)}{\sqrt{1^2 + 0 + 1^2}} & \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(1, 0, 1)}{\sqrt{2}} \\ &= \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)\end{aligned}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \langle \vec{u}_2, \vec{v}_1 \rangle \vec{v}_1 \\ &= (2, 1, 4) - \left[ (2, 1, 4) \cdot \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \right] \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \\ &= (2, 1, 4) - \left( \frac{6}{\sqrt{2}} \right) \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \\ &= (2, 1, 4) - (3, 0, 3) \\ &= (-1, 1, 1)\end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{(-1, 1, 1)}{\sqrt{(-1)^2 + 1^2 + 1^2}} & \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \left( -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)\end{aligned}$$

$$\begin{aligned}\langle \vec{u}_1, \vec{q}_1 \rangle &= (1, 0, 1) \cdot \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \\ &= \frac{2}{\sqrt{2}} \\ &= \sqrt{2}\end{aligned}$$

$$\begin{aligned}\langle \vec{u}_2, \vec{q}_1 \rangle &= (2, 1, 4) \cdot \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \\ &= \frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}} \\ &= 3\sqrt{2}\end{aligned}$$

$$\begin{aligned}\langle \vec{u}_2, \vec{q}_2 \rangle &= (2, 1, 4) \cdot \left( -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= -\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{4}{\sqrt{3}} \\ &= \frac{3}{\sqrt{3}} \\ &= \sqrt{3}\end{aligned}$$

$$\begin{aligned}
 R &= \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix} \\
 &= \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}
 \end{aligned}$$

The  $QR$ -decomposition of the matrix is

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

$A = Q R$

### Exercise

Find the  $QR$ -decomposition of  $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{pmatrix}$

### Solution

Since  $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{vmatrix} = -4 \neq 0$ ,

The matrix is invertible, so it has a  $QR$ -decomposition.

$$\vec{u}_1 = (1, 1, 0), \quad \vec{u}_2 = (2, 1, 3), \quad \vec{u}_3 = (1, 1, 1)$$

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 0)$$

$$\vec{q}_1 = \frac{(1, 1, 0)}{\sqrt{1+1+0}}$$

$$= \frac{(1, 1, 0)}{\sqrt{2}}$$

$$= \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$



$$\begin{aligned}
 \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
 &= (2, 1, 3) - \frac{1}{2} [(2, 1, 3) \cdot (1, 1, 0)] (1, 1, 0) \\
 &= (2, 1, 3) - \frac{3}{2} (1, 1, 0) \\
 &= (2, 1, 3) - \left(\frac{3}{2}, \frac{3}{2}, 0\right) \\
 &= \left(\frac{1}{2}, -\frac{1}{2}, 3\right) \Big|
 \end{aligned}$$

$$\begin{aligned}
 \vec{q}_2 &= \frac{\left(\frac{1}{2}, -\frac{1}{2}, 3\right)}{\sqrt{\frac{1}{4} + \frac{1}{4} + 9}} & \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
 &= \frac{2}{\sqrt{38}} \left(\frac{1}{2}, -\frac{1}{2}, 3\right) \\
 &= \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}}\right) \Big|
 \end{aligned}$$

$$\begin{aligned}
 \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, 1, 1) \cdot (1, 1, 0)}{2} (1, 1, 0) \\
 &= (1, 1, 0) \Big|
 \end{aligned}$$

$$\begin{aligned}
 \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{4}{38} \left( (1, 1, 1) \cdot \left(\frac{1}{2}, -\frac{1}{2}, 3\right) \right) \left(\frac{1}{2}, -\frac{1}{2}, 3\right) \\
 &= \frac{6}{19} \left(\frac{1}{2}, -\frac{1}{2}, 3\right) \\
 &= \left(\frac{3}{19}, -\frac{3}{19}, \frac{18}{19}\right) \Big|
 \end{aligned}$$

$$\begin{aligned}
 \vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
 &= (1, 1, 1) - (1, 1, 0) - \left(\frac{3}{19}, -\frac{3}{19}, \frac{18}{19}\right) \\
 &= (0, 0, 1) - \left(\frac{3}{19}, -\frac{3}{19}, \frac{18}{19}\right) \\
 &= \left(-\frac{3}{19}, \frac{3}{19}, \frac{1}{19}\right) \Big|
 \end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{1}{\sqrt{\left(-\frac{3}{19}\right)^2 + \left(\frac{3}{19}\right)^2 + \left(\frac{1}{19}\right)^2}} \left(-\frac{3}{19}, \frac{3}{19}, \frac{1}{19}\right) & \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \frac{19}{\sqrt{19}} \left(-\frac{3}{19}, \frac{3}{19}, \frac{1}{19}\right) \\ &= \left(-\frac{3}{\sqrt{19}}, \frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}}\right)\end{aligned}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix}$$

$$\begin{aligned}\langle \vec{u}_1, \vec{q}_1 \rangle &= (1, 1, 0) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \\ &= \frac{2}{\sqrt{2}} \\ &= \sqrt{2}\end{aligned}$$

$$\begin{aligned}\langle \vec{u}_2, \vec{q}_1 \rangle &= (2, 1, 3) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \\ &= \frac{3}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\langle \vec{u}_2, \vec{q}_2 \rangle &= (2, 1, 3) \cdot \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}}\right) \\ &= \frac{2-1+18}{\sqrt{38}} \\ &= \frac{19}{\sqrt{38}} \\ &= \frac{19}{\sqrt{2}\sqrt{19}} \\ &= \frac{\sqrt{19}}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\langle \vec{u}_3, \vec{q}_1 \rangle &= (1, 1, 1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \\ &= \frac{2}{\sqrt{2}} \\ &= \sqrt{2}\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{q}_2 \rangle &= (1, 1, 1) \cdot \left( \frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right) \\
&= \frac{1-1+6}{\sqrt{38}} \\
&= \frac{6}{\sqrt{2}\sqrt{19}} \frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{3\sqrt{2}}{\sqrt{19}}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{q}_3 \rangle &= (1, 1, 1) \cdot \left( -\frac{3}{\sqrt{19}}, \frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}} \right) \\
&= \frac{-3+3+1}{\sqrt{19}} \\
&= \frac{1}{\sqrt{19}}
\end{aligned}$$

$$\begin{aligned}
R &= \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \langle \vec{u}_3, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \langle \vec{u}_3, \vec{q}_2 \rangle \\ 0 & 0 & \langle \vec{u}_3, \vec{q}_3 \rangle \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}
\end{aligned}$$

The  $QR$ -decomposition of the matrix is

$$\begin{aligned}
\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix} \\
\mathbf{A} &= \mathbf{Q} \mathbf{R}
\end{aligned}$$

**Exercise**

Find the  $QR$ -decomposition of  $\begin{pmatrix} 7 & 1 & 2 \\ 2 & -1 & 3 \\ -3 & 0 & 4 \end{pmatrix}$

**Solution**

$$\text{Since } \begin{vmatrix} 7 & 1 & 2 \\ 2 & -1 & 3 \\ -3 & 0 & 4 \end{vmatrix} = -51 \neq 0,$$

The matrix is invertible, so it has a  $QR$ -decomposition.

$$\vec{u}_1 = (7, 2, -3), \quad \vec{u}_2 = (1, -1, 0), \quad \vec{u}_3 = (2, 3, 4)$$

$$\vec{v}_1 = \vec{u}_1 = (7, 2, -3)$$

$$\begin{aligned} \vec{q}_1 &= \frac{(7, 2, -3)}{\sqrt{49+4+9}} & \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(7, 2, -3)}{\sqrt{62}} \\ &= \left( \frac{7}{\sqrt{62}}, \frac{2}{\sqrt{62}}, -\frac{3}{\sqrt{62}} \right) \end{aligned}$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (1, -1, 0) - \frac{1}{62} [(1, -1, 0) \cdot (7, 2, -3)] (7, 2, -3) \\ &= (1, -1, 0) - \frac{5}{62} (7, 2, -3) \\ &= (1, -1, 0) - \left( \frac{35}{62}, \frac{5}{31}, -\frac{15}{62} \right) \\ &= \left( \frac{27}{62}, -\frac{36}{31}, \frac{15}{62} \right) \end{aligned}$$

$$\begin{aligned} \vec{q}_2 &= \frac{62}{\sqrt{27^2 + 72^2 + 15^2}} \left( \frac{27}{62}, -\frac{36}{31}, \frac{15}{62} \right) & \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{62}{\sqrt{6,138}} \left( \frac{27}{62}, -\frac{36}{31}, \frac{15}{62} \right) \\ &= \frac{62}{3\sqrt{682}} \left( \frac{27}{62}, -\frac{36}{31}, \frac{15}{62} \right) \\ &= \left( \frac{9}{\sqrt{682}}, -\frac{24}{\sqrt{682}}, \frac{5}{\sqrt{682}} \right) \end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(2, 3, 4) \cdot (7, 2, -3)}{62} (7, 2, -3) \\
&= \frac{8}{62} (7, 2, -3) \\
&= \frac{4}{31} (7, 2, -3) \\
&= \left( \frac{28}{31}, \frac{8}{31}, -\frac{12}{31} \right)
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{3,844}{6,138} \left( (2, 3, 4) \cdot \left( \frac{27}{62}, -\frac{36}{31}, \frac{15}{62} \right) \right) \left( \frac{27}{62}, -\frac{36}{31}, \frac{15}{62} \right) \\
&= \frac{62}{6,138} \left( \frac{54}{62} - \frac{108}{31} + \frac{60}{62} \right) (27, -72, 15) \\
&= -\frac{102}{6,138} (27, -72, 15) \\
&= -\frac{17}{1,023} (27, -72, 15) \\
&= \left( -\frac{153}{341}, \frac{408}{341}, -\frac{85}{341} \right)
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
&= (2, 3, 4) - \left( \frac{28}{31}, \frac{8}{31}, -\frac{12}{31} \right) - \left( -\frac{153}{341}, \frac{408}{341}, -\frac{85}{341} \right) \\
&= \left( \frac{34}{31}, \frac{85}{31}, \frac{136}{31} \right) - \left( -\frac{153}{341}, \frac{408}{341}, -\frac{85}{341} \right) \\
&= \left( \frac{17}{11}, \frac{17}{11}, \frac{51}{11} \right)
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{11}{\sqrt{17^2 + 17^2 + 51^2}} \left( \frac{17}{11}, \frac{17}{11}, \frac{51}{11} \right) \\
&= \frac{1}{\sqrt{3,179}} (17, 17, 51) \\
&= \frac{1}{17\sqrt{11}} (17, 17, 51) \\
&= \left( \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}} \right)
\end{aligned}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$Q = \begin{bmatrix} \frac{7}{\sqrt{62}} & \frac{3}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{62}} & -\frac{24}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ -\frac{3}{\sqrt{62}} & \frac{5}{\sqrt{19}} & \frac{3}{\sqrt{11}} \end{bmatrix}$$

$$\begin{aligned} \langle \vec{u}_1, \vec{q}_1 \rangle &= (7, 2, -3) \cdot \left( \frac{7}{\sqrt{62}}, \frac{2}{\sqrt{62}}, -\frac{3}{\sqrt{62}} \right) \\ &= \frac{49 + 4 + 9}{\sqrt{62}} \\ &= \frac{62}{\sqrt{62}} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{q}_1 \rangle &= (1, -1, 0) \cdot \left( \frac{7}{\sqrt{62}}, \frac{2}{\sqrt{62}}, -\frac{3}{\sqrt{62}} \right) \\ &= \frac{5}{\sqrt{62}} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{q}_2 \rangle &= (1, -1, 0) \cdot \left( \frac{9}{\sqrt{682}}, -\frac{24}{\sqrt{682}}, \frac{5}{\sqrt{682}} \right) \\ &= \frac{33}{\sqrt{682}} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_3, \vec{q}_1 \rangle &= (2, 3, 4) \cdot \left( \frac{7}{\sqrt{62}}, \frac{2}{\sqrt{62}}, -\frac{3}{\sqrt{62}} \right) \\ &= \frac{14 + 6 - 12}{\sqrt{62}} \\ &= \frac{8}{\sqrt{62}} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_3, \vec{q}_2 \rangle &= (2, 3, 4) \cdot \left( \frac{9}{\sqrt{682}}, -\frac{24}{\sqrt{682}}, \frac{5}{\sqrt{682}} \right) \\ &= \frac{18 - 72 + 20}{\sqrt{682}} \\ &= -\frac{34}{\sqrt{682}} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_3, \vec{q}_3 \rangle &= (2, 3, 4) \cdot \left( \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}} \right) \\ &= \frac{2 + 3 + 12}{\sqrt{11}} \\ &= \frac{17}{\sqrt{11}} \end{aligned}$$

$$R = \begin{bmatrix} \frac{62}{\sqrt{62}} & \frac{5}{\sqrt{62}} & \frac{8}{\sqrt{62}} \\ 0 & \frac{33}{\sqrt{682}} & -\frac{34}{\sqrt{682}} \\ 0 & 0 & \frac{17}{\sqrt{11}} \end{bmatrix}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \langle \vec{u}_3, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \langle \vec{u}_3, \vec{q}_2 \rangle \\ 0 & 0 & \langle \vec{u}_3, \vec{q}_3 \rangle \end{bmatrix}$$

The  $QR$ -decomposition of the matrix is

$$\begin{pmatrix} 7 & 1 & 2 \\ 2 & -1 & 3 \\ -3 & 0 & 4 \end{pmatrix} = \begin{pmatrix} \frac{7}{\sqrt{62}} & \frac{9}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{62}} & -\frac{24}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ -\frac{3}{\sqrt{62}} & \frac{5}{\sqrt{682}} & \frac{3}{\sqrt{11}} \end{pmatrix} \begin{pmatrix} \frac{62}{\sqrt{62}} & \frac{5}{\sqrt{62}} & \frac{8}{\sqrt{62}} \\ 0 & \frac{33}{\sqrt{682}} & -\frac{34}{\sqrt{682}} \\ 0 & 0 & \frac{17}{\sqrt{11}} \end{pmatrix}$$

$A \quad = \quad Q \quad R$

### Exercise

Find the  $QR$ -decomposition of

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

### Solution

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \\ R_4 + R_1 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{array}{l} \\ \\ \\ R_4 - R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is linearly dependent, so *doesn't* have a  $QR$ -decomposition.

**Exercise**Find the  $QR$ -decomposition of

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

**Solution**

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{array}{l} \\ R_2 - R_1 \\ \\ R_4 - R_3 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \\ \\ R_3 - R_1 \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \\ \\ R_3 - R_2 \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The matrix has a  $QR$ -decomposition.

$$\vec{u}_1 = (1, 1, 1, 1), \quad \vec{u}_2 = (0, 1, 1, 1), \quad \vec{u}_3 = (0, 0, 1, 1)$$

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 1, 1) \quad |$$

$$\vec{q}_1 = \frac{(1, 1, 1, 1)}{\sqrt{4}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \quad |$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (0, 1, 1, 1) - \frac{1}{4} [(0, 1, 1, 1) \cdot (1, 1, 1, 1)] (1, 1, 1, 1)$$

$$= (0, 1, 1, 1) - \left( \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4} \right)$$



$$= \left( -\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right) \Big|$$

$$\vec{q}_2 = \frac{4}{\sqrt{9+1+1+1}} \left( -\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right) \quad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{2}{\sqrt{3}} \left( -\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right)$$

$$= \left( -\frac{3}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}} \right) \Big|$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(0, 0, 1, 1) \cdot (1, 1, 1, 1)}{4} (1, 1, 1, 1)$$

$$= \frac{1}{2} (1, 1, 1, 1)$$

$$= \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \Big|$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{16}{12} \left( (0, 0, 1, 1) \cdot \left( -\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right) \right) \left( -\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right)$$

$$= \frac{4}{3} \left( \frac{1}{2} \right) \left( -\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right)$$

$$= \left( -\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right) \Big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (0, 0, 1, 1) - \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) - \left( -\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right)$$

$$= \left( 0, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right) \Big|$$

$$\vec{q}_3 = \frac{3}{\sqrt{6}} \left( 0, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right)$$

$$= \left( 0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) \Big|$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$Q = \begin{pmatrix} \frac{1}{2} & -\frac{3}{2\sqrt{3}} & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$\begin{aligned} \langle \vec{u}_1, \vec{q}_1 \rangle &= (1, 1, 1, 1) \cdot \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \\ &= 2 \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{q}_1 \rangle &= (0, 1, 1, 1) \cdot \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \\ &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{q}_2 \rangle &= (0, 1, 1, 1) \cdot \left( -\frac{3}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}} \right) \\ &= \frac{3}{2\sqrt{3}} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_3, \vec{q}_1 \rangle &= (0, 0, 1, 1) \cdot \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \\ &= 1 \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_3, \vec{q}_2 \rangle &= (0, 0, 1, 1) \cdot \left( -\frac{3}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}} \right) \\ &= \frac{1}{\sqrt{3}} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_3, \vec{q}_3 \rangle &= (0, 0, 1, 1) \cdot \left( 0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) \\ &= \frac{2}{\sqrt{6}} \end{aligned}$$

$$R = \begin{pmatrix} 2 & \frac{3}{2} & 1 \\ 0 & \frac{3}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \langle \vec{u}_3, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \langle \vec{u}_3, \vec{q}_2 \rangle \\ 0 & 0 & \langle \vec{u}_3, \vec{q}_3 \rangle \end{bmatrix}$$

The  $QR$ -decomposition of the matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{3}{2\sqrt{3}} & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 2 & \frac{3}{2} & 1 \\ 0 & \frac{3}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$$

$A = Q R$

**Exercise**

Find the  $QR$ -decomposition of

$$\begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix}$$
**Solution**

The matrix has a  $QR$ -decomposition.

$$\vec{u}_1 = (3, 1, -1, 3), \quad \vec{u}_2 = (-5, 1, 5, -7), \quad \vec{u}_3 = (1, 1, -2, 8)$$

$$\vec{v}_1 = \vec{u}_1 = (3, 1, -1, 3) \quad |$$

$$\vec{q}_1 = \frac{(3, 1, -1, 3)}{\sqrt{20}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left( \frac{3}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, -\frac{1}{2\sqrt{5}}, \frac{3}{2\sqrt{5}} \right) \quad |$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (-5, 1, 5, -7) - \frac{1}{20} [(-5, 1, 5, -7) \cdot (3, 1, -1, 3)] (3, 1, -1, 3)$$

$$= (-5, 1, 5, -7) + 2(3, 1, -1, 3)$$

$$= (1, 3, 3, -1) \quad |$$

$$\vec{q}_2 = \frac{1}{\sqrt{20}}(1, 3, 3, -1) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left( \frac{1}{2\sqrt{5}}, \frac{3}{2\sqrt{5}}, \frac{3}{2\sqrt{5}}, -\frac{1}{2\sqrt{5}} \right) \quad |$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, 1, -2, 8) \cdot (3, 1, -1, 3)}{20} (3, 1, -1, 3) \\ &= \frac{3}{2} (3, 1, -1, 3) \\ &= \left( \frac{9}{2}, \frac{3}{2}, -\frac{3}{2}, \frac{9}{2} \right) \mid\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{(1, 1, -2, 8) \cdot (1, 3, 3, -1)}{20} (1, 3, 3, -1) \\ &= -\frac{1}{2} (1, 3, 3, -1) \\ &= \left( -\frac{1}{2}, -\frac{3}{2}, -\frac{3}{2}, \frac{1}{2} \right) \mid\end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (1, 1, -2, 8) - \left( \frac{9}{2}, \frac{3}{2}, -\frac{3}{2}, \frac{9}{2} \right) - \left( -\frac{1}{2}, -\frac{3}{2}, -\frac{3}{2}, \frac{1}{2} \right) \\ &= (-3, 1, 1, 3) \mid\end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{1}{2\sqrt{5}} (-3, 1, 1, 3) \\ &= \left( -\frac{3}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, \frac{3}{2\sqrt{5}} \right) \mid\end{aligned}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$Q = \begin{pmatrix} \frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} & -\frac{3}{2\sqrt{5}} \\ \frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} \\ \frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} \\ \frac{3}{2\sqrt{5}} & -\frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} \end{pmatrix}$$

$$\begin{aligned}\langle \vec{u}_1, \vec{q}_1 \rangle &= (3, 1, -1, 3) \cdot \left( \frac{3}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, -\frac{1}{2\sqrt{5}}, \frac{3}{2\sqrt{5}} \right) \\ &= \frac{10}{\sqrt{5}} \mid\end{aligned}$$

$$\langle \vec{u}_2, \vec{q}_1 \rangle = (-5, 1, 5, -7) \cdot \left( \frac{3}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, -\frac{1}{2\sqrt{5}}, \frac{3}{2\sqrt{5}} \right)$$

$$= -\frac{20}{\sqrt{5}} \Bigg|$$

$$\langle \vec{u}_2, \vec{q}_2 \rangle = (-5, 1, 5, -7) \cdot \left( \frac{1}{2\sqrt{5}}, \frac{3}{2\sqrt{5}}, \frac{3}{2\sqrt{5}}, -\frac{1}{2\sqrt{5}} \right)$$

$$= \frac{10}{\sqrt{5}} \Bigg|$$

$$\langle \vec{u}_3, \vec{q}_1 \rangle = (1, 1, -2, 8) \cdot \left( \frac{3}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, -\frac{1}{2\sqrt{5}}, \frac{3}{2\sqrt{5}} \right)$$

$$= \frac{15}{\sqrt{5}} \Bigg|$$

$$\langle \vec{u}_3, \vec{q}_2 \rangle = (1, 1, -2, 8) \cdot \left( \frac{1}{2\sqrt{5}}, \frac{3}{2\sqrt{5}}, \frac{3}{2\sqrt{5}}, -\frac{1}{2\sqrt{5}} \right)$$

$$= -\frac{5}{\sqrt{5}} \Bigg|$$

$$\langle \vec{u}_3, \vec{q}_3 \rangle = (1, 1, -2, 8) \cdot \left( -\frac{3}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, \frac{3}{2\sqrt{5}} \right)$$

$$= \frac{10}{\sqrt{5}} \Bigg|$$

$$R = \begin{pmatrix} \frac{10}{\sqrt{5}} & -\frac{20}{\sqrt{5}} & \frac{15}{\sqrt{5}} \\ 0 & \frac{10}{\sqrt{5}} & -\frac{5}{\sqrt{5}} \\ 0 & 0 & \frac{10}{\sqrt{5}} \end{pmatrix}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \langle \vec{u}_3, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \langle \vec{u}_3, \vec{q}_2 \rangle \\ 0 & 0 & \langle \vec{u}_3, \vec{q}_3 \rangle \end{bmatrix}$$

The  $QR$ -decomposition of the matrix is

$$\begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix} = \begin{pmatrix} \frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} & -\frac{3}{2\sqrt{5}} \\ \frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} \\ \frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} \\ \frac{3}{2\sqrt{5}} & -\frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{10}{\sqrt{5}} & -\frac{20}{\sqrt{5}} & \frac{15}{\sqrt{5}} \\ 0 & \frac{10}{\sqrt{5}} & -\frac{5}{\sqrt{5}} \\ 0 & 0 & \frac{10}{\sqrt{5}} \end{pmatrix}$$

$\mathbf{A} \quad = \quad \mathbf{Q} \quad \mathbf{R}$

**Exercise**Find the  $QR$ -decomposition of

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & 5 & -1 \\ 2 & 8 & 0 \\ -3 & 3 & 5 \end{pmatrix}$$

**Solution**

$$\begin{pmatrix} 1 & -2 & -2 \\ 2 & 5 & -1 \\ 2 & 8 & 0 \\ -3 & 3 & 5 \end{pmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \\ R_4 + 3R_1 \end{array}$$

$$\begin{pmatrix} 1 & -2 & -2 \\ 0 & 9 & 3 \\ 0 & 12 & 4 \\ 0 & -3 & -1 \end{pmatrix} \begin{array}{l} \\ \frac{1}{3}R_2 \\ \frac{1}{4}R_3 \\ \end{array}$$

$$\begin{pmatrix} 1 & -2 & -2 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \\ 0 & -3 & -1 \end{pmatrix} \begin{array}{l} \\ \\ R_3 - R_2 \\ R_4 + R_2 \end{array}$$

$$\begin{pmatrix} 1 & -2 & -2 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The matrix is linearly dependent, so *doesn't* have a  $QR$ -decomposition.

**Exercise**

Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product

$$\vec{u} = (0, -2, 2, 1), \quad \vec{v} = (-1, -1, 1, 1)$$

**Solution**

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= 0 - 2(-1) + 2(1) + 1(1) \\ &= 5 \end{aligned}$$

$$\|\langle \vec{u}, \vec{v} \rangle\| = \sqrt{5}$$

$$\begin{aligned} \|\vec{u}\| \|\vec{v}\| &= \sqrt{0+4+4+1} \sqrt{1+1+1+1} \\ &= \sqrt{9}\sqrt{4} \end{aligned}$$

$$\begin{aligned} &= 6 \\ \sqrt{5} < 6 &\Rightarrow \|\langle \vec{u}, \vec{v} \rangle\| \leq \|\vec{u}\| \|\vec{v}\| \end{aligned}$$

### Exercise

Apply the Gram-Schmidt **orthonormalization** process in  $C^0[-1, 1]$  spanned by the functions, using the inner product  $f_1(x) = x + 2$ ,  $f_2(x) = x^2 - 3x + 4$

### Solution

$$\text{Let } \vec{u}_1 = f_1 = x + 2, \quad \vec{u}_2 = f_2 = x^2 - 3x + 4$$

$$\vec{v}_1 = \vec{u}_1 = x + 2$$

$$\begin{aligned} \langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 (x+2)^2 dx \\ &= \frac{1}{3}(x+2)^3 \Big|_{-1}^1 \\ &= \frac{1}{3}(27-1) \\ &= \frac{26}{3} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 (x^2 - 3x + 4)(x+2) dx \\ &= \int_{-1}^1 (x^3 - x^2 - 2x + 8) dx \\ &= \left( \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 + 8x \right) \Big|_{-1}^1 \\ &= \frac{1}{4} - \frac{1}{3} - 1 + 8 - \frac{1}{4} - \frac{1}{3} + 1 + 8 \\ &= \frac{46}{3} \end{aligned}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$\vec{v}_2 = x^2 - 3x + 4 - \frac{46}{3} \left( \frac{3}{26} \right) (x+2)$$

$$\begin{aligned}
 &= x^2 - 3x + 4 - \frac{23}{13}x - \frac{46}{13} \\
 &= \underline{x^2 - \frac{62}{13}x + \frac{6}{13}}
 \end{aligned}$$

The orthogonal basis is  $\left\{ x+2, \quad x^2 - \frac{62}{13}x + \frac{6}{13} \right\}$

$$\begin{aligned}
 \langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 \left( x^2 - \frac{62}{13}x + \frac{6}{13} \right)^2 dx \\
 &= \frac{1}{169} \int_{-1}^1 \left( 13x^2 - 62x + 6 \right)^2 dx \\
 &= \frac{1}{169} \int_{-1}^1 \left( 169x^4 + 3,844x^2 + 36 - 1,612x^3 + 156x^2 - 744x \right) dx \\
 &= \frac{1}{169} \left( \frac{169}{5}x^5 + \frac{4,000}{3}x^3 + 36x - 403x^4 - 372x^2 \right) \Big|_{-1}^1 \\
 &= \frac{1}{169} \left( \frac{169}{5} + \frac{4,000}{3} + 36 - 403 - 372 + \frac{169}{5} + \frac{4,000}{3} + 36 + 403 + 372 \right) \\
 &= \frac{1}{169} \left( \frac{338}{5} + \frac{8,000}{3} + 72 \right) \\
 &= \underline{\frac{3,238}{195}}
 \end{aligned}$$

$$\begin{aligned}
 \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
 &= \underline{\frac{\sqrt{3}}{\sqrt{26}}(x+2)}
 \end{aligned}$$

$$\begin{aligned}
 \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
 &= \underline{\sqrt{\frac{195}{3238}} \left( x^2 - \frac{62}{13}x + \frac{6}{13} \right)}
 \end{aligned}$$

The *orthonormal* basis is  $\left\{ \frac{\sqrt{3}}{\sqrt{26}}(x+2), \quad \sqrt{\frac{195}{3238}} \left( x^2 - \frac{62}{13}x + \frac{6}{13} \right) \right\}$



**Exercise**

Apply the Gram-Schmidt **orthonormalization** process in  $C^0[-1, 1]$  spanned by the functions, using the inner product  $f_1(x) = 1 + 3x^2$ ,  $f_2(x) = x - x^2$

**Solution**

$$\text{Let } \vec{u}_1 = 1 + 3x^2, \quad \vec{u}_2 = x - x^2$$

$$\vec{v}_1 = \vec{u}_1 = 1 + 3x^2$$

$$\begin{aligned} \langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 (1 + 3x^2)^2 dx \\ &= 2 \int_0^1 (1 + 6x^2 + 9x^4) dx \\ &= 2 \left( x + 2x^3 + \frac{9}{5}x^5 \right) \Big|_0^1 \\ &= 2 \left( 1 + 2 + \frac{9}{5} \right) \\ &= \frac{48}{5} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 (1 + 3x^2)(x - x^2) dx \\ &= \int_{-1}^1 (x - x^2 + 3x^3 - 3x^4) dx \\ &= \left( \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{3}{4}x^4 - \frac{3}{5}x^5 \right) \Big|_{-1}^1 \\ &= \frac{1}{2} - \frac{1}{3} + \frac{3}{4} - \frac{3}{5} - \left( \frac{1}{2} + \frac{1}{3} + \frac{3}{4} + \frac{3}{5} \right) \\ &= -\frac{28}{15} \end{aligned}$$

$$\vec{v}_2 = x - x^2 - \frac{5}{48} \left( -\frac{28}{15} \right) (1 + 3x^2)$$

$$= x - x^2 + \frac{7}{36} + \frac{7}{12}x^2$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$\left. = -\frac{5}{12}x^2 + x + \frac{7}{36} \right|$$

The *orthogonal* basis is  $\left\{ 1+3x^2, \quad -\frac{5}{12}x^2 + x + \frac{7}{36} \right\}$

$$\begin{aligned} \langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 \left( -\frac{5}{12}x^2 + x + \frac{7}{36} \right)^2 dx \\ &= \frac{1}{1,296} \int_{-1}^1 \left( -15x^2 + 36x + 7 \right)^2 dx \\ &= \frac{1}{1,296} \int_{-1}^1 \left( 225x^4 - 1,080x^3 - 210x^2 + 504x + 49 \right) dx \\ &= \frac{1}{1,296} \left( 45x^5 - 270x^4 - 70x^3 + 252x^2 + 49x \right) \Big|_{-1}^1 \\ &= \frac{1}{648} \left( -270x^4 + 252x^2 \right) \Big|_0^1 \\ &= \frac{1}{648} (-270 + 252) \\ &= \frac{12}{648} \\ &= \frac{1}{54} \Big| \end{aligned}$$

$$\begin{aligned} \vec{q}_1 &= \frac{\sqrt{5}}{\sqrt{48}} (1+3x^2) \\ &= \frac{\sqrt{15}}{12} (1+3x^2) \Big| \end{aligned}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\begin{aligned} \vec{q}_2 &= 54 \left( -\frac{4}{12}x^2 + x + \frac{7}{36} \right) \\ &= -18x^2 + 54x + \frac{63}{2} \Big| \end{aligned}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

The *orthonormal* basis is  $\left\{ \frac{\sqrt{15}}{12} (1+3x^2), \quad -18x^2 + 54x + \frac{63}{2} \right\}$

**Exercise**

Apply the Gram-Schmidt **orthonormalization** process in  $C^0[-1, 1]$  spanned by the functions, using the inner product  $f_1(x) = 5x - 3$ ,  $f_2(x) = x^3 - x^2$

**Solution**

$$\text{Let } \vec{u}_1 = 5x - 3, \quad \vec{u}_2 = x^3 - x^2$$

$$\underline{\vec{v}_1 = \vec{u}_1 = 5x - 3}$$

$$\begin{aligned} \langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 (5x - 3)^2 dx \\ &= \int_{-1}^1 (25x^2 - 30x + 9) dx \\ &= \left( \frac{25}{3}x^3 - 15x^2 + 9x \right) \Big|_{-1}^1 \\ &= \frac{25}{3} - 15 + 9 + \frac{25}{3} + 15 + 9 \\ &= \frac{104}{3} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 (x^3 - x^2)(5x - 3) dx \\ &= \int_{-1}^1 (5x^4 - 8x^3 - 3x^2) dx \\ &= \left( x^5 - 2x^4 + x^3 \right) \Big|_{-1}^1 \\ &= 4 \end{aligned}$$

$$\vec{v}_2 = x^3 - x^2 - \frac{3}{104}(4)(1 + 3x^2)$$

$$= x^3 - x^2 - \frac{3}{26}(1 + 3x^2)$$

$$\underline{= x^3 - \frac{35}{26}x^2 - \frac{3}{26}}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

The *orthogonal* basis is  $\left\{ 1 + 3x^2, \quad x^3 - \frac{35}{26}x^2 - \frac{3}{26} \right\}$

$$\begin{aligned}
\langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 \left( x^3 - \frac{35}{26}x^2 - \frac{3}{26} \right)^2 dx \\
&= \frac{1}{676} \int_{-1}^1 \left( 26x^3 - 35x^2 - 3 \right)^2 dx \\
&= \frac{1}{676} \int_{-1}^1 \left( 676x^6 - 1,820x^5 + 1,225x^4 - 156x^3 + 210x^2 + 9 \right) dx \\
&= \frac{1}{676} \left( \frac{676}{7}x^7 - \frac{910}{3}x^6 + 245x^5 - 39x^4 + 70x^3 + 9x \right) \Big|_{-1}^1 \\
&= \frac{2}{676} \left( \frac{676}{7} + 245 + 70 + 9 \right) \\
&= \frac{1}{338} \left( \frac{2,944}{7} \right) \\
&= \frac{1,472}{1,183}
\end{aligned}$$

$$\vec{q}_1 = \frac{3}{104} (1 + 3x^2)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{q}_2 = \frac{1,472}{1,183} \left( x^3 - \frac{35}{26}x^2 - \frac{3}{26} \right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

The *orthonormal* basis is  $\left\{ \frac{3}{104}(1 + 3x^2), \frac{1,472}{1,183} \left( x^3 - \frac{35}{26}x^2 - \frac{3}{26} \right) \right\}$

### Exercise

Apply the Gram-Schmidt *orthonormalization* process in  $C^0[-1, 1]$  spanned by the functions, using the inner product  $f_1(x) = 1$ ,  $f_2(x) = 2x - 1$

### Solution

Let  $\vec{u}_1 = 1$ ,  $\vec{u}_2 = 2x - 1$

$$\vec{v}_1 = \vec{u}_1 = 1$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^1 (1)^2 dx$$

$$= x \Big|_{-1}^1$$

$$= 2 \Big|$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^1 (2x-1)(1) \, dx$$

$$= x^2 - x \Big|_{-1}^1$$

$$= -2 \Big|$$

$$\vec{v}_2 = 2x - 1 - \frac{1}{2}(-2)(1)$$

$$= 2x \Big|$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

The *orthogonal* basis is  $\{1, 2x\}$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^1 (2x)^2 \, dx$$

$$= \int_{-1}^1 4x^2 \, dx$$

$$= \frac{4}{3}x^3 \Big|_{-1}^1$$

$$= \frac{8}{3} \Big|$$

$$\vec{q}_1 = \frac{1}{\sqrt{2}} \Big|$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{q}_2 = \frac{3}{\sqrt{2}}x \Big|$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

The *orthonormal* basis is  $\left\{ \frac{1}{2}, \frac{3}{4}x \right\}$

**Exercise**

Apply the Gram-Schmidt **orthonormalization** process in  $C^0[-1, 1]$  spanned by the functions, using the inner product  $f_1(x) = e^x$ ,  $f_2(x) = x$

**Solution**

$$\text{Let } \vec{u}_1 = e^x, \quad \vec{u}_2 = x$$

$$\vec{v}_1 = \vec{u}_1 = e^x$$

$$\begin{aligned} \langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 e^{2x} dx \\ &= \frac{1}{2} e^x \Big|_{-1}^1 \\ &= \frac{1}{2} (e - e^{-1}) \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 x e^x dx \\ &= e^x (x-1) \Big|_{-1}^1 \\ &= (0)e + 2e \\ &= 2e \end{aligned}$$

$$\begin{aligned} \vec{v}_2 &= x - \frac{2}{e - e^{-1}} (2e) (e^x) \\ &= x - \frac{4e^2}{e^2 - 1} e^x \end{aligned}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

The orthogonal basis is  $\left\{ e^x, x - \frac{4e^2}{e^2 - 1} e^x \right\}$

$$\begin{aligned} \langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 \left( x - \frac{4e^2}{e^2 - 1} e^x \right)^2 dx \\ &= \int_{-1}^1 \left( x^2 - \frac{8e^2}{e^2 - 1} x e^x + \left( \frac{4e^2}{e^2 - 1} \right)^2 e^{2x} \right) dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3}x^3 - \frac{8e^2}{e^2-1}(x-1)e^x + \frac{8e^4}{(e^2-1)^2}e^{2x} \Big|_{-1}^1 \\
&= \frac{1}{3} + \frac{8e^4}{(e^2-1)^2}e^2 + \frac{1}{3} + \frac{16e^2}{e^2-1}e^{-1} + \frac{8e^4}{(e^2-1)^2}e^{-2} \\
&= \frac{2}{3} + \frac{8e^6}{(e^2-1)^2} + \frac{16e}{e^2-1} + \frac{8e^2}{(e^2-1)^2} \\
&= \frac{1}{(e^2-1)^2} \left( \frac{2}{3}e^4 - \frac{4}{3}e^2 + \frac{4}{3} + 8e^6 + 16e^3 - 16e + 8e^2 \right) \\
&= \frac{1}{(e^2-1)^2} \left( 8e^6 + \frac{2}{3}e^4 + 16e^3 + \frac{20}{3}e^2 - 16e + \frac{4}{3} \right)
\end{aligned}$$

$$\vec{q}_1 = \sqrt{\frac{2}{e-e^{-1}}} e^x$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{q}_2 = \frac{e^2-1}{8e^6 + \frac{2}{3}e^4 + 16e^3 + \frac{20}{3}e^2 - 16e + \frac{4}{3}} \left( x - \frac{4e^2}{e^2-1} e^x \right) \quad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

The *orthonormal* basis is

$$\left\{ \sqrt{\frac{2}{e-e^{-1}}} e^x, \frac{e^2-1}{8e^6 + \frac{2}{3}e^4 + 16e^3 + \frac{20}{3}e^2 - 16e + \frac{4}{3}} \left( x - \frac{4e^2}{e^2-1} e^x \right) \right\}$$

### Exercise

Apply the Gram-Schmidt **orthonormalization** process in  $C^0[-1, 1]$  spanned by the functions, using the inner product  $f_1(x) = x$ ,  $f_2(x) = x^3$ ,  $f_3(x) = x^5$

### Solution

Let  $\vec{u}_1 = f_1 = x$ ,  $\vec{u}_2 = f_2 = x^3$ ,  $\vec{u}_3 = f_3 = x^5$

$$\vec{v}_1 = \vec{u}_1 = x$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^1 x^2 dx$$

$$\begin{aligned} &= \frac{1}{3}x^3 \Big|_{-1}^1 \\ &= \frac{2}{3} \Big| \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 x^4 \, dx \\ &= \frac{1}{5}x^5 \Big|_{-1}^1 \\ &= \frac{2}{5} \Big| \end{aligned}$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= x^3 - \frac{2}{5} \left( \frac{3}{2} \right) (x) \\ &= x^3 - \frac{3}{5}x \Big| \end{aligned}$$

$$\begin{aligned} \langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 \left( x^3 - \frac{3}{5}x \right)^2 \, dx \\ &= \int_{-1}^1 \left( x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2 \right) \, dx \\ &= \left( \frac{1}{7}x^7 - \frac{6}{25}x^5 + \frac{3}{25}x^3 \right) \Big|_{-1}^1 \\ &= 2 \left( \frac{1}{7} - \frac{6}{25} + \frac{3}{25} \right) \\ &= \frac{8}{175} \Big| \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_3, \vec{v}_1 \rangle &= \int_{-1}^1 x^6 \, dx \\ &= \frac{1}{7}x^7 \Big|_{-1}^1 \\ &= \frac{2}{7} \Big| \end{aligned}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^1 x^5 \left( x^3 - \frac{3}{5}x \right) \, dx$$



$$\begin{aligned}
&= \int_{-1}^1 \left( x^8 - \frac{3}{5}x^6 \right) dx \\
&= \left( \frac{1}{9}x^9 - \frac{3}{35}x^7 \right) \Big|_{-1}^1 \\
&= 2 \left( \frac{1}{9} - \frac{3}{35} \right) \\
&= \frac{16}{315} \quad |
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
&= x^5 - \frac{16}{315} \left( \frac{175}{8} \right) \left( x^3 - \frac{3}{5}x \right) - \frac{2}{7} \left( \frac{3}{2} \right) x \\
&= x^5 - \frac{70}{63} \left( x^3 - \frac{3}{5}x \right) - \frac{3}{7}x \\
&= x^5 - \frac{70}{63}x^3 + \frac{14}{21}x - \frac{3}{7}x \\
&= x^5 - \frac{70}{63}x^3 + \frac{5}{21}x \quad |
\end{aligned}$$

The orthogonal basis is  $\left\{ x, \quad x^3 - \frac{3}{5}x, \quad x^5 - \frac{70}{63}x^3 + \frac{5}{21}x \right\}$

$$\begin{aligned}
\langle \vec{v}_3, \vec{v}_3 \rangle &= \int_{-1}^1 \left( x^5 - \frac{70}{63}x^3 + \frac{5}{21}x \right)^2 dx \\
&= \int_{-1}^1 \frac{1}{3,969} \left( 63x^5 - 70x^3 + 15x \right)^2 dx \\
&= \frac{1}{3,969} \int_{-1}^1 \left( 3,969x^{10} - 8,820x^8 + 1,890x^6 - 2,100x^4 + 4,900x^2 + 225x^0 \right) dx \\
&= \frac{1}{3,969} \left( \frac{3,969}{11}x^{11} - 980x^9 + 970x^7 - 420x^5 + 75x^3 \right) \Big|_{-1}^1 \\
&= \frac{2}{3,969} \left( \frac{3,969}{11} - 980 + 970 - 420 + 75 \right) \\
&= \frac{2}{3,969} \left( \frac{3,969}{11} - 355 \right) \\
&= \frac{2}{3,969} \left( \frac{64}{11} \right) \\
&= \frac{128}{43,659} \quad |
\end{aligned}$$

$$\begin{aligned}\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{x}{\sqrt{2/3}} \\ &= \frac{\sqrt{3}}{\sqrt{2}}x \Big| \end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \sqrt{\frac{175}{8}} \left( x^3 - \frac{3}{5}x \right) \\ &= \frac{5\sqrt{7}}{2\sqrt{2}} \left( x^3 - \frac{3}{5}x \right) \Big| \end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \sqrt{\frac{43,659}{128}} \left( x^5 - \frac{70}{63}x^3 + \frac{5}{21}x \right) \\ &= \frac{63\sqrt{11}}{8\sqrt{2}} \left( x^5 - \frac{70}{63}x^3 + \frac{5}{21}x \right) \Big| \end{aligned}$$

The *orthonormal* basis is  $\left\{ \frac{\sqrt{3}}{\sqrt{2}}x, \frac{5}{2}\sqrt{\frac{7}{2}}\left(x^3 - \frac{3}{5}x\right), \frac{63}{8}\sqrt{\frac{11}{2}}\left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x\right) \right\}$

### Exercise

Apply the Gram-Schmidt *orthonormalization* process in  $C^0[-1, 1]$  spanned by the functions, using the inner product  $f_1(x) = 1, f_2(x) = x, f_3(x) = \frac{1}{2}(3x^2 - 1)$

### Solution

Let  $\vec{u}_1 = f_1 = 1, \vec{u}_2 = f_2 = x, \vec{u}_3 = f_3 = \frac{3}{2}x^2 - \frac{1}{2}$

$$\vec{v}_1 = \vec{u}_1 = 1 \Big|$$

$$\begin{aligned}\langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 1 \, dx \\ &= x \Big|_{-1}^1\end{aligned}$$

$$\underline{= 2}$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^1 x \, dx$$

$$= \frac{1}{2} x^2 \Big|_{-1}^1$$

$$\underline{= 0}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= x - \frac{0}{2}(1)$$

$$\underline{= x}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^1 x^2 \, dx$$

$$= \frac{1}{3} x^3 \Big|_{-1}^1$$

$$\underline{= \frac{2}{3}}$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \frac{1}{2} \int_{-1}^1 (3x^2 - 1) \, dx$$

$$= \frac{1}{2} (x^3 - x) \Big|_{-1}^1$$

$$\underline{= 0}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \frac{1}{2} \int_{-1}^1 x(3x^2 - 1) \, dx$$

$$= \frac{1}{2} \int_{-1}^1 (3x^3 - x) \, dx$$

$$= \frac{1}{2} \left( \frac{3}{4} x^4 - \frac{1}{2} x^2 \right) \Big|_{-1}^1$$

$$\underline{= 0}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= \frac{3}{2}x^2 - \frac{1}{2} - \frac{0}{1}(1) - \frac{0}{\frac{2}{3}}(x) \\ &= \frac{1}{2}(3x^2 - 1) \quad \Big| \end{aligned}$$

The *orthogonal* basis is  $\left\{ 1, \ x, \ \frac{1}{2}(3x^2 - 1) \right\}$

$$\begin{aligned}\langle \vec{v}_3, \vec{v}_3 \rangle &= \frac{1}{4} \int_{-1}^1 (3x^2 - 1)^2 \, dx \\ &= \frac{1}{4} \int_{-1}^1 (9x^4 - 6x^2 + 1) \, dx \\ &= \frac{1}{4} \left( \frac{9}{5}x^5 - 2x^3 + x \right) \Big|_{-1}^1 \\ &= \frac{1}{2} \left( \frac{9}{5} - 2 + 1 \right) \\ &= \frac{2}{5} \quad \Big| \end{aligned}$$

$$\begin{aligned}\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{1}{\sqrt{2}} \quad \Big| \end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \sqrt{\frac{3}{2}} x \quad \Big| \end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \frac{1}{\sqrt{\frac{2}{5}}} \frac{1}{2} (3x^2 - 1) \\ &= \frac{1}{2} \sqrt{\frac{5}{2}} (3x^2 - 1) \quad \Big| \end{aligned}$$

The *orthonormal* basis is  $\left\{ \frac{1}{\sqrt{2}}, \ \sqrt{\frac{3}{2}}x, \ \frac{1}{2}\sqrt{\frac{5}{2}}(3x^2 - 1) \right\}$

**Exercise**

Apply the Gram-Schmidt **orthonormalization** process in  $C^0[-1, 1]$  spanned by the functions, using the inner product  $f_1(x) = 5$ ,  $f_2(x) = x^2 - 6x$ ,  $f_3(x) = (3 - x)^2$

**Solution**

$$\text{Let } \vec{u}_1 = 5, \quad \vec{u}_2 = x^2 - 6x, \quad \vec{u}_3 = x^2 - 6x + 9$$

$$\underline{\vec{v}_1 = \vec{u}_1 = 5}$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^1 5 \, dx$$

$$= 5x \Big|_{-1}^1$$

$$= 10$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = 5 \int_{-1}^1 (x^2 - 6x) \, dx$$

$$= 5 \left( \frac{1}{3}x^3 - 3x^2 \right) \Big|_{-1}^1$$

$$= \frac{10}{3}$$

$$\vec{v}_2 = x^2 - 6x + 9 - \frac{1}{100} \left( \frac{10}{3} \right) (5) \qquad \vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= x^2 - 6x + 9 - \frac{1}{6}$$

$$\underline{= x^2 - 6x + \frac{53}{6}}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \frac{1}{36} \int_{-1}^1 (6x^2 - 36x + 53)^2 \, dx$$

$$= \frac{1}{36} \int_{-1}^1 (36x^4 - 432x^3 + 1,932x^2 - 3,816x + 2,809) \, dx$$

$$= \frac{1}{36} \left( \frac{36}{5}x^5 - 108x^4 + 644x^3 - 1,908x^2 + 2,809x \right) \Big|_{-1}^1$$

$$\begin{aligned}&= \frac{1}{18} \left( \frac{36}{5} x^5 + 644x^3 + 2,809x \right) \Big|_0^1 \\&= \frac{1}{18} \left( \frac{36}{5} + 644 + 2,809 \right) \\&= \frac{17,301}{90} \\&= \frac{5,767}{30}\end{aligned}$$

$$\begin{aligned}\langle \vec{u}_3, \vec{v}_1 \rangle &= 5 \int_{-1}^1 (x^2 - 6x + 9) dx \\&= 5 \left( \frac{1}{3} x^3 - 3x^2 + 9x \right) \Big|_{-1}^1 \\&= 10 \left( \frac{1}{3} x^3 + 9x \right) \Big|_0^1 \\&= 10 \left( \frac{1}{3} + 9 \right) \\&= \frac{280}{3}\end{aligned}$$

$$\begin{aligned}\langle \vec{u}_3, \vec{v}_2 \rangle &= \int_{-1}^1 (x^2 - 6x + 9) \left( x^2 - 6x + \frac{53}{6} \right) dx \\&= \int_{-1}^1 \left( x^4 - 12x^3 + \frac{323}{6} x^2 - 107x + \frac{159}{2} \right) dx \\&= \left( \frac{1}{5} x^5 - 3x^4 + \frac{323}{18} x^3 - \frac{107}{2} x^2 + \frac{159}{2} x \right) \Big|_{-1}^1 \\&= 2 \left( \frac{1}{5} x^5 + \frac{323}{18} x^3 + \frac{159}{2} x \right) \Big|_0^1 \\&= 2 \left( \frac{1}{5} + \frac{323}{18} + \frac{159}{2} \right) \\&= 2 \left( \frac{18 + 1,615 + 7,155}{90} \right) \\&= \frac{8,788}{45}\end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\&= x^2 - 6x + 9 - \left( \frac{280}{3} \right) \left( \frac{1}{10} \right) (5) - \left( \frac{8,788}{45} \right) \left( \frac{30}{5,767} \right) \left( x^2 - 6x + \frac{53}{6} \right)\end{aligned}$$

$$\begin{aligned}
&= x^2 - 6x + 9 - \frac{140}{3} - \frac{17,576}{17,301} \left( x^2 - 6x + \frac{53}{6} \right) \\
&= x^2 - 6x - \frac{113}{3} - \frac{17,576}{17,301} x^2 - \frac{35,152}{5,767} x + \frac{465,764}{51,903} \\
&= \left| -\frac{275}{17,301} x^2 - \frac{69,754}{5,767} x - \frac{1,489,249}{51,903} \right|
\end{aligned}$$

The *orthogonal* basis is  $\left\{ 5, x^2 - 6x + \frac{53}{6}, -\frac{275}{17,301} x^2 - \frac{69,754}{5,767} x - \frac{1,489,249}{51,903} \right\}$

$$\begin{aligned}
\langle \vec{v}_3, \vec{v}_3 \rangle &= \int_{-1}^1 \left( -\frac{275}{17,301} x^2 - \frac{69,754}{5,767} x - \frac{1,489,249}{51,903} \right)^2 dx \\
&= \left| \frac{4,700,107,122,016}{2,693,921,409} \right|
\end{aligned}$$

$$\vec{q}_1 = \left| \frac{5}{\sqrt{10}} \right|$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{q}_2 = \left| \sqrt{\frac{60}{5,767}} \left( x^2 - 6x + \frac{53}{6} \right) \right|$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\vec{q}_3 = \left| \frac{2,693,921,409}{4,700,107,122,016} \left( -\frac{275}{17,301} x^2 - \frac{69,754}{5,767} x - \frac{1,489,249}{51,903} \right) \right|$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{2} (3x^2 - 1)$$

$$= \left| \frac{1}{2} \sqrt{\frac{5}{2}} (3x^2 - 1) \right|$$

The *orthonormal* basis is

$$\left\{ \frac{5}{\sqrt{10}}, \sqrt{\frac{60}{5,767}} \left( x^2 - 6x + \frac{53}{6} \right), \frac{2,693,921,409}{4,700,107,122,016} \left( -\frac{275}{17,301} x^2 - \frac{69,754}{5,767} x - \frac{1,489,249}{51,903} \right) \right\}$$

### Exercise

Apply the Gram-Schmidt **orthonormalization** process in  $C^0[-1, 1]$  spanned by the functions, using the inner product  $f_1(x) = 1$ ,  $f_2(x) = \sin \pi x$ ,  $f_3(x) = \cos \pi x$

### Solution

Let  $\vec{u}_1 = 1$ ,  $\vec{u}_2 = \sin \pi x$ ,  $\vec{u}_3 = \cos \pi x$

$$\vec{v}_1 = \vec{u}_1 = 1$$

$$\begin{aligned}\langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 1 \, dx \\ &= x \Big|_{-1}^1 \\ &= 2 \end{aligned}$$

$$\begin{aligned}\langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 \sin \pi x \, dx \\ &= -\frac{1}{\pi} \cos \pi x \Big|_{-1}^1 \\ &= 0 \end{aligned}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= \sin \pi x \end{aligned}$$

$$\begin{aligned}\langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 \sin^2 \pi x \, dx \\ &= \frac{1}{2} \int_{-1}^1 (1 - \cos 2\pi x) \, dx \\ &= \frac{1}{2} \left( x - \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^1 \\ &= 1 \end{aligned}$$

$$\begin{aligned}\langle \vec{u}_3, \vec{v}_1 \rangle &= \int_{-1}^1 \cos \pi x \, dx \\ &= \frac{1}{\pi} \sin \pi x \Big|_{-1}^1 \\ &= 0 \end{aligned}$$

$$\begin{aligned}\langle \vec{u}_3, \vec{v}_2 \rangle &= \int_{-1}^1 \cos \pi x \sin \pi x \, dx \\ &= \frac{1}{2} \int_{-1}^1 \sin 2\pi x \, dx \end{aligned}$$



$$= -\frac{1}{4\pi} \cos 2\pi x \Big|_{-1}^1$$

$$= 0$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= \cos \pi x - \frac{0}{0} - \frac{0}{0}$$

$$= \cos \pi x$$

The *orthogonal* basis is  $\left\{ 1, \sin \pi x - \frac{1}{\pi}, \cos \pi x \right\}$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^1 \cos^2 \pi x \, dx$$

$$= \frac{1}{2} \int_{-1}^1 (1 + \cos 2\pi x) \, dx$$

$$= \frac{1}{2} \left( x + \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^1$$

$$= 1$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{2}}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \sin \pi x$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \cos \pi x$$

The *orthonormal* basis is  $\left\{ \frac{1}{\sqrt{2}}, \sin \pi x, \cos \pi x \right\}$

**Exercise**

Apply the Gram-Schmidt **orthonormalization** process in  $C^0[-1, 1]$  spanned by the functions, using the inner product  $f_1(x) = 1$ ,  $f_2(x) = \sin x$ ,  $f_3(x) = \sin 2x$

**Solution**

$$\text{Let } \vec{u}_1 = 1, \quad \vec{u}_2 = \sin x, \quad \vec{u}_3 = \sin 2x$$

$$\vec{v}_1 = \vec{u}_1 = 1$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^1 1 \, dx$$

$$= x \Big|_{-1}^1$$

$$= 2$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^1 \sin x \, dx$$

$$= -\cos x \Big|_{-1}^1$$

$$= -(\cos 1 - \cos(-1))$$

$$= -(\cos 1 - \cos 1)$$

$$= 0$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= \sin x - (0)$$

$$= \sin x$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^1 \sin^2 x \, dx$$

$$= \frac{1}{2} \int_{-1}^1 (1 - \cos 2x) \, dx$$

$$= \frac{1}{2} \left( x - \frac{1}{2} \sin 2x \right) \Big|_{-1}^1$$

$$= \frac{1}{2} \left( 1 - \frac{1}{2} \sin 2 + 1 - \frac{1}{2} \sin 2 \right)$$

$$\begin{aligned}
 &= 1 - \frac{1}{2} \sin 2 \\
 &= 1 - \sin(1) \cos(1) \quad |
 \end{aligned}$$

$$\begin{aligned}
 \langle \vec{u}_3, \vec{v}_1 \rangle &= \int_{-1}^1 \sin 2x \, dx \\
 &= -\frac{1}{2} \cos 2x \Big|_{-1}^1 \\
 &= -\frac{1}{2} (\cos 2 - \cos 2) \\
 &= 0 \quad |
 \end{aligned}$$

$$\begin{aligned}
 \langle \vec{u}_3, \vec{v}_2 \rangle &= \int_{-1}^1 \sin 2x \sin x \, dx \\
 &= 2 \int_{-1}^1 \sin x \cos x \sin x \, dx \\
 &= 2 \int_{-1}^1 \sin^2 x \, d(\sin x) \\
 &= \frac{2}{3} \sin^3 x \Big|_{-1}^1 \\
 &= \frac{2}{3} (\sin^3 1 + \sin^3 1) \\
 &= \frac{4}{3} \sin^3 1 \quad |
 \end{aligned}$$

$$\begin{aligned}
 \vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
 &= \sin 2x - (0) - \frac{4}{3} \sin^3(1) \frac{1}{1 - \sin(1) \cos(1)} \sin x \\
 &= \sin 2x - \frac{4}{3} \frac{\sin^3(1)}{1 - \sin(1) \cos(1)} \sin x \quad |
 \end{aligned}$$

The *orthogonal* basis is  $\left\{ 1, \sin x, \sin 2x - \frac{4}{3} \frac{\sin^3(1)}{1 - \sin(1) \cos(1)} \sin x \right\}$

$$\begin{aligned}
\langle \vec{v}_3, \vec{v}_3 \rangle &= \int_{-1}^1 \left( \sin 2x - \frac{4 \sin^3(1)}{3 - 3 \sin(1) \cos(1)} \sin x \right)^2 dx \\
&= \int_{-1}^1 \left( \sin^2 2x - \frac{8 \sin^3(1)}{3 - 3 \sin(1) \cos(1)} \sin 2x \sin x + \frac{16 \sin^6(1)}{(3 - 3 \sin(1) \cos(1))^2} \sin^2 x \right) dx \\
&= \int_{-1}^1 \left( \frac{1}{2} - \frac{1}{2} \cos 4x \right) dx - \frac{16 \sin^3(1)}{3 - 3 \sin(1) \cos(1)} \int_{-1}^1 \sin^2 x d(\sin x) \\
&\quad + \frac{8 \sin^6(1)}{(3 - 3 \sin(1) \cos(1))^2} \int_{-1}^1 (1 - \cos 2x) dx \\
&= \frac{1}{2}x - \frac{1}{8} \sin 4x - \frac{16 \sin^3(1)}{9 - 9 \sin(1) \cos(1)} \sin^3 x + \frac{8 \sin^6(1)}{(3 - 3 \sin(1) \cos(1))^2} \left( x - \frac{1}{2} \sin 2x \right) \Big|_{-1}^1 \\
&= \frac{1}{2} - \frac{1}{8} \sin 4 - \frac{16 \sin^6 1}{9 - 9 \sin 1 \cos 1} + \frac{8 \sin^6 1}{(3 - 3 \sin 1 \cos 1)^2} \left( 1 - \frac{1}{2} \sin 2 \right) \\
&\quad - \left( -\frac{1}{2} + \frac{1}{8} \sin 4 + \frac{16 \sin^6 1}{9 - 9 \sin 1 \cos 1} + \frac{8 \sin^6 1}{(3 - 3 \sin 1 \cos 1)^2} \left( -1 + \frac{1}{2} \sin 2 \right) \right) \\
&= \frac{1}{2} - \frac{1}{8} \sin 4 - \frac{16 \sin^6 1}{9 - 9 \sin 1 \cos 1} + \frac{8 \sin^6 1}{(3 - 3 \sin 1 \cos 1)^2} - \frac{4 \sin^6 1 \sin 2}{(3 - 3 \sin 1 \cos 1)^2} \\
&\quad + \frac{1}{2} - \frac{1}{8} \sin 4 - \frac{16 \sin^6 1}{9 - 9 \sin 1 \cos 1} + \frac{8 \sin^6 1}{(3 - 3 \sin 1 \cos 1)^2} - \frac{4 \sin^6 1 \sin 2}{(3 - 3 \sin 1 \cos 1)^2} \\
&= 1 - \frac{1}{4} \sin 4 - \frac{32 \sin^6 1}{9 - 9 \sin 1 \cos 1} + \frac{16 \sin^6 1}{(3 - 3 \sin 1 \cos 1)^2} - \frac{8 \sin^6 1 \sin 2}{(3 - 3 \sin 1 \cos 1)^2} \Big|
\end{aligned}$$

$$\vec{q}_1 = \frac{1}{\sqrt{2}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{q}_2 = \frac{\sin x}{\sqrt{1 - \sin(1) \cos(1)}}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\vec{q}_3 = \frac{\sin 2x - \frac{4}{3} \frac{\sin^3(1)}{1 - \sin(1) \cos(1)} \sin x}{\sqrt{1 - \frac{1}{4} \sin 4 - \frac{32 \sin^6 1}{9 - 9 \sin 1 \cos 1} + \frac{16 \sin^6 1}{(3 - 3 \sin 1 \cos 1)^2} - \frac{8 \sin^6 1 \sin 2}{(3 - 3 \sin 1 \cos 1)^2}}}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

The *orthonormal* basis is

$$\left\{ \frac{1}{\sqrt{2}}, \frac{\sin x}{\sqrt{1 - \sin(1)\cos(1)}}, \frac{\sin 2x - \frac{4}{3} \frac{\sin^3(1)}{1 - \sin(1)\cos(1)} \sin x}{\sqrt{1 - \frac{1}{4}\sin 4 - \frac{32\sin^6 1}{9 - 9\sin 1 \cos 1} + \frac{16\sin^6 1}{(3 - 3\sin 1 \cos 1)^2} - \frac{8 \sin^6 1 \sin 2}{(3 - 3\sin 1 \cos 1)^2}}} \right\}$$

### Exercise

Apply the Gram-Schmidt *orthonormalization* process in  $C^0[-1, 1]$  spanned by the functions, using the inner product  $f_1(x) = 6$ ,  $f_2(x) = 3\sin^2 x$ ,  $f_3(x) = 2\cos^2 x$

### Solution

$$\text{Let } \vec{u}_1 = 6, \quad \vec{u}_2 = 3\sin^2 x, \quad \vec{u}_3 = 2\cos^2 x$$

$$\vec{v}_1 = \vec{u}_1 = 6$$

$$\begin{aligned} \langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 6 \, dx \\ &= 12 \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{v}_1 \rangle &= 18 \int_{-1}^1 \sin^2 x \, dx \\ &= 9 \int_{-1}^1 (1 - \cos 2x) \, dx \\ &= 9 \left( x - \frac{1}{2} \sin 2x \right) \Big|_{-1}^1 \\ &= 9 \left( 1 - \frac{1}{2} \sin 2 + 1 - \frac{1}{2} \sin 2 \right) \\ &= 18 - 9 \sin 2 \end{aligned}$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= 3\sin^2 x - \frac{1}{12}(18 - 9\sin 2)(6) \end{aligned}$$

$$\left. = 3\sin^2 x + \frac{9}{2}\sin 2 - 9 \right|$$

$$\begin{aligned}
 \langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 \left( 3\sin^2 x + \frac{9}{2}\sin 2 - 9 \right)^2 dx \\
 &= \int_{-1}^1 \left( 9\sin^4 x + 6\left(\frac{9}{2}\sin 2 - 9\right)\sin^2 x + \left(\frac{9}{2}\sin 2 - 9\right)^2 \right) dx \\
 &= \int_{-1}^1 \left( 9\sin^4 x + 6\left(\frac{9}{2}\sin 2 - 9\right)\sin^2 x + \left(\frac{9}{2}\sin 2 - 9\right)^2 \right) dx \\
 9 \int_{-1}^1 \sin^4 x dx &= \frac{9}{4} \int_{-1}^1 (1 - \cos 2x)^2 dx \\
 &= \frac{9}{4} \int_{-1}^1 (1 - 2\cos 2x + \cos^2 2x) dx \\
 &= \frac{9}{4} \int_{-1}^1 \left( 1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x \right) dx \\
 &= \frac{9}{4} \int_{-1}^1 \left( \frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x \right) dx \\
 &= \frac{9}{4} \left( \frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x \right) \Big|_{-1}^1 \\
 &= \frac{9}{4} \left( 3 - 2\sin 2 + \frac{1}{4}\sin 4 \right) \\
 6\left(\frac{9}{2}\sin 2 - 9\right) \int_{-1}^1 \sin^2 x dx &= \left(\frac{27}{2}\sin 2 - 27\right) \int_{-1}^1 (1 - \cos 2x) dx \\
 &= \frac{27}{2}(\sin 2 - 2) \left( x - \frac{1}{2}\sin 2x \right) \Big|_{-1}^1 \\
 &= 27(\sin 2 - 2) \left( 1 - \frac{1}{2}\sin 2 \right) \\
 \int_{-1}^1 \left( \frac{9}{2}\sin 2 - 9 \right)^2 dx &= 2 \left( \frac{9}{2}\sin 2 - 9 \right)^2 \\
 &= \frac{27}{4} - \frac{9}{2}\sin 2 + \frac{9}{16}\sin 4 + 27(\sin 2 - 2) \left( 1 - \frac{1}{2}\sin 2 \right) + 2 \left( \frac{9}{2}\sin 2 - 9 \right)^2
 \end{aligned}$$

$$\begin{aligned}
&= \frac{27}{4} - \frac{9}{2} \sin 2 + \frac{9}{16} \sin 4 + 54 \sin 2 - \frac{27}{2} \sin^2 2 - 54 + \frac{81}{2} \sin^2 2 - 162 \sin 2 + 162 \\
&= \frac{459}{4} - \frac{225}{2} \sin 2 + \frac{9}{16} \sin 4 + \frac{54}{2} \sin^2 2 \quad |
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_1 \rangle &= 12 \int_{-1}^1 \cos^2 x \, dx \\
&= 6 \int_{-1}^1 (1 + \cos 2x) \, dx \\
&= 12 \left( x + \frac{1}{2} \sin 2x \right) \Big|_0^1 \\
&= 12 + 6 \sin 2 \quad |
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_2 \rangle &= \int_{-1}^1 \left( 3 \sin^2 x + \frac{9}{2} \sin 2 - 9 \right) (2 \cos^2 x) \, dx \\
&= 3 \int_{-1}^1 \left( 2 \sin^2 x \cos^2 x + (3 \sin 2 - 6) \cos^2 x \right) \, dx \\
&\quad \int_{-1}^1 2 \sin^2 x \cos^2 x \, dx = \frac{1}{2} \int_{-1}^1 (1 - \cos 2x)(1 + \cos 2x) \, dx \\
&\quad = \frac{1}{2} \int_{-1}^1 (1 - \cos^2 2x) \, dx \\
&\quad = \frac{1}{2} \int_{-1}^1 \left( \frac{1}{2} - \frac{1}{2} \cos 4x \right) \, dx \\
&\quad = \frac{1}{2} \left( x - \frac{1}{4} \sin 4x \right) \Big|_0^1 \\
&\quad = \frac{1}{2} (1 - \sin 4) \\
(3 \sin 2 - 6) \int_{-1}^1 \cos^2 x \, dx &= \frac{1}{2} (3 \sin 2 - 6) \int_{-1}^1 (1 + \cos 2x) \, dx \\
&= (3 \sin 2 - 6) \left( x + \frac{1}{2} \sin 2x \right) \Big|_0^1 \\
&= (3 \sin 2 - 6) \left( 1 + \frac{1}{2} \sin 2 \right) \\
&= \frac{3}{2} \sin^2 2 - 6
\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} - \frac{1}{2} \sin 4 + \frac{3}{2} \sin^2 2 - 6 \\
 &= \frac{3}{2} \sin^2 2 - \frac{1}{2} \sin 4 - \frac{11}{2} \Big|
 \end{aligned}$$

$$\begin{aligned}
 \vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
 &= 2 \cos^2 x - \left( \frac{1}{12} \right) (12 + 6 \sin 2) (6) - \frac{\left( 3 \sin^2 x + \frac{9}{2} \sin 2 - 9 \right) \left( \frac{3}{2} \sin^2 2 - \frac{1}{2} \sin 4 - \frac{11}{2} \right)}{\frac{459}{4} - \frac{225}{2} \sin 2 + \frac{9}{16} \sin 4 + \frac{54}{2} \sin^2 2} \\
 &= 2 \cos^2 x - 6 - 3 \sin 2 - \frac{\left( \frac{3}{2} \sin^2 2 - \frac{1}{2} \sin 4 - \frac{11}{2} \right)}{\frac{459}{4} - \frac{225}{2} \sin 2 + \frac{9}{16} \sin 4 + \frac{54}{2} \sin^2 2} \left( 3 \sin^2 x + \frac{9}{2} \sin 2 - 9 \right) \Big|
 \end{aligned}$$

The *orthogonal* basis is

$$\left\{ \begin{aligned} &6, \quad 3 \sin^2 x + \frac{9}{2} \sin 2 - 9, \\ &2 \cos^2 x - 6 - 3 \sin 2 - \frac{\left( \frac{3}{2} \sin^2 2 - \frac{1}{2} \sin 4 - \frac{11}{2} \right)}{\frac{459}{4} - \frac{225}{2} \sin 2 + \frac{9}{16} \sin 4 + \frac{54}{2} \sin^2 2} \left( 3 \sin^2 x + \frac{9}{2} \sin 2 - 9 \right) \end{aligned} \right\}$$

$$\begin{aligned}
 \vec{v}_3 &= 2 \cos^2 x - \frac{3 \left( \frac{3}{2} \sin^2 2 - \frac{1}{2} \sin 4 - \frac{11}{2} \right)}{\frac{459}{4} - \frac{225}{2} \sin 2 + \frac{9}{16} \sin 4 + \frac{54}{2} \sin^2 2} \sin^2 x \\
 &\quad - \left( \frac{\left( \frac{3}{2} \sin^2 2 - \frac{1}{2} \sin 4 - \frac{11}{2} \right) \left( \frac{9}{2} \sin 2 - 9 \right)}{\frac{459}{4} - \frac{225}{2} \sin 2 + \frac{9}{16} \sin 4 + \frac{54}{2} \sin^2 2} + 6 + 3 \sin 2 \right)
 \end{aligned}$$

$$\begin{aligned}
 2 \int_{-1}^1 \cos^2 x \, dx &= \int_{-1}^1 (1 + \cos 2x) \, dx \\
 &= x + \frac{1}{2} \sin 2x \Big|_{-1}^1 \\
 &= 2 + \sin 2 \Big|
 \end{aligned}$$

$$\begin{aligned}
 \int_{-1}^1 \sin^2 x \, dx &= \frac{1}{2} \int_{-1}^1 (1 - \cos 2x) \, dx \\
 &= \left( x - \frac{1}{2} \sin 2x \right) \Big|_{-1}^1
 \end{aligned}$$



$$= 1 - \frac{1}{2} \sin 2 \quad \Bigg|$$

$$\int_{-1}^1 dx = 2$$

$$\begin{aligned} \langle \vec{v}_3, \vec{v}_3 \rangle &= 2 + \sin 2 - \frac{3\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}\right)\left(1 - \frac{1}{2}\sin 2\right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2} \\ &\quad - 2\left(\frac{\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}\right)\left(\frac{9}{2}\sin 2 - 9\right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2} + 6 + 3\sin 2\right) \end{aligned}$$

$$\vec{q}_1 = \frac{6}{\sqrt{12}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{3}{\sqrt{3}} \quad \Bigg|$$

$$\vec{q}_2 = \frac{3\sin^2 x + \frac{9}{2}\sin 2 - 9}{\sqrt{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2}}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{3\sin^2 x + \frac{9}{2}\sin 2 - 9}{\frac{1}{4}\sqrt{1,836 - 1,800\sin 2 + 9\sin 4 + 432\sin^2 2}}$$

$$= \frac{12\sin^2 x + 18\sin 2 - 36}{\sqrt{1,836 - 1,800\sin 2 + 9\sin 4 + 432\sin^2 2}} \quad \Bigg|$$

$$\vec{q}_3 = \frac{2\cos^2 x - 6 - 3\sin 2 - \frac{\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}\right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2} \left(3\sin^2 x + \frac{9}{2}\sin 2 - 9\right)}{\sqrt{\langle \vec{v}_3, \vec{v}_3 \rangle}}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

The *orthonormal* basis is

$$\left\{ \begin{aligned} &\frac{3}{\sqrt{3}}, \quad \frac{12\sin^2 x + 18\sin 2 - 36}{\sqrt{1,836 - 1,800\sin 2 + 9\sin 4 + 432\sin^2 2}}, \\ &\frac{2\cos^2 x - 6 - 3\sin 2 - \frac{\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}\right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2} \left(3\sin^2 x + \frac{9}{2}\sin 2 - 9\right)}{\sqrt{\langle \vec{v}_3, \vec{v}_3 \rangle}} \end{aligned} \right\}$$

**Exercise**

Apply the Gram-Schmidt **orthonormalization** process in  $C^0[-1, 1]$  spanned by the functions, using the inner product  $f_1(x) = \cos 2x$ ,  $f_2(x) = \sin^2 x$ ,  $f_3(x) = \cos^2 x$

**Solution**

$$\text{Let } \vec{u}_1 = \cos 2x, \quad \vec{u}_2 = \sin^2 x, \quad \vec{u}_3 = \cos^2 x$$

$$\vec{v}_1 = \vec{u}_1 = \cos 2x$$

$$\begin{aligned} \langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 \cos 2x \, dx \\ &= \frac{1}{2} \sin 2x \Big|_{-1}^1 \\ &= \sin 2 \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 \sin^2 x \cos 2x \, dx \\ &= \frac{1}{2} \int_{-1}^1 (1 - \cos 2x) \cos 2x \, dx \\ &= \frac{1}{2} \int_{-1}^1 (\cos 2x - \cos^2 2x) \, dx \\ &= \frac{1}{2} \int_{-1}^1 \left( \cos 2x - \frac{1}{2} - \frac{1}{2} \cos 4x \right) \, dx \\ &= \left( \frac{1}{2} \sin 2x - \frac{1}{2} x - \frac{1}{8} \sin 8x \right) \Big|_0^1 \\ &= \frac{1}{2} \sin 2 - \frac{1}{2} - \frac{1}{8} \sin 8 \end{aligned}$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= \sin^2 x - \left( \frac{1}{\sin 2} \right) \left( \frac{1}{2} \sin 2 - \frac{1}{2} - \frac{1}{8} \sin 8 \right) (\cos 2x) \\ &= \sin^2 x - \left( \frac{4 \sin 2 - 4 - \sin 8}{8 \sin 2} \right) \cos 2x \end{aligned}$$

$$= \frac{1}{2} - \left( \frac{1}{2} + \frac{4\sin 2 - \sin 8 - 4}{8\sin 2} \right) \cos 2x$$

$$= \frac{1}{2} - \left( \frac{8\sin 2 - \sin 8 - 4}{8\sin 2} \right) \cos 2x$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^1 \left( \sin^2 x - \left( \frac{4\sin 2 - 4 - \sin 8}{8\sin 2} \right) \cos 2x \right)^2 dx$$

$$= \int_{-1}^1 \left( \sin^4 x - \left( \frac{4\sin 2 - 4 - \sin 8}{4\sin 2} \right) \cos 2x \sin^2 x + \frac{1}{64} \left( \frac{4\sin 2 - 4 - \sin 8}{\sin 2} \right)^2 \cos^2 2x \right) dx$$

$$\int_{-1}^1 \sin^4 x \, dx = \frac{1}{4} \int_{-1}^1 (1 - \cos 2x)^2 \, dx$$

$$= \frac{1}{4} \int_{-1}^1 (1 - 2\cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{4} \int_{-1}^1 \left( \frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x \right) \, dx$$

$$= \frac{1}{4} \left( \frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x \right) \Big|_{-1}^1$$

$$= \frac{1}{4} \left( 3 - 2\sin 2 + \frac{1}{4}\sin 4 \right)$$

$$\int_{-1}^1 \cos 2x \sin^2 x \, dx = \frac{1}{2} \int_{-1}^1 \cos 2x (1 - \cos 2x) \, dx$$

$$= \frac{1}{2} \int_{-1}^1 (\cos 2x - \cos^2 2x) \, dx$$

$$= \frac{1}{2} \int_{-1}^1 \left( \cos 2x - \frac{1}{2} - \frac{1}{2}\cos 4x \right) \, dx$$

$$= \left( \frac{1}{2}\sin 2x - \frac{1}{2}x - \frac{1}{8}\sin 8x \right) \Big|_0^1$$

$$= \frac{1}{2}\sin 2 - \frac{1}{2} - \frac{1}{8}\sin 8$$

$$\int_{-1}^1 \cos^2 2x \, dx = \frac{1}{2} \int_{-1}^1 (1 + \cos 4x) \, dx$$

$$\begin{aligned} &= \left( x + \frac{1}{4} \sin 4x \right) \Big|_0^1 \\ &= \underline{1 + \frac{1}{4} \sin 4} \end{aligned}$$

$$\begin{aligned} \langle \vec{v}_2, \vec{v}_2 \rangle &= \frac{1}{4} \left( 3 - 2 \sin 2 + \frac{1}{4} \sin 4 \right) - \left( \frac{4 \sin 2 - 4 - \sin 8}{4 \sin 2} \right) \left( \frac{1}{2} \sin 2 - \frac{1}{2} - \frac{1}{8} \sin 8 \right) \\ &\quad + \left( \frac{4 \sin 2 - 4 - \sin 8}{8 \sin 2} \right)^2 \left( 1 + \frac{1}{4} \sin 4 \right) \\ &= \frac{3}{4} - \frac{1}{2} \sin 2 + \frac{1}{16} \sin 4 - \frac{(4 \sin 2 - 4 - \sin 8)^2}{32 \sin 2} + \frac{(4 + \sin 4)(4 \sin 2 - 4 - \sin 8)^2}{256 \sin^2 2} \\ &= \frac{3}{4} - \frac{1}{2} \sin 2 + \frac{1}{16} \sin 4 + \frac{(4 + \sin 4)(4 \sin 2 - 4 - \sin 8)^2 - 8 \sin 2 (4 \sin 2 - 4 - \sin 8)^2}{256 \sin^2 2} \\ &= \underline{\frac{3}{4} - \frac{1}{2} \sin 2 + \frac{1}{16} \sin 4 + \frac{(4 - 7 \sin 2)(4 \sin 2 - 4 - \sin 8)^2}{256 \sin^2 2}} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_3, \vec{v}_1 \rangle &= \int_{-1}^1 \cos^2 x \cos 2x \, dx \\ &= \frac{1}{2} \int_{-1}^1 (1 + \cos 2x) \cos 2x \, dx \\ &= \frac{1}{2} \int_{-1}^1 (\cos 2x + \cos^2 2x) \, dx \\ &= \frac{1}{2} \int_{-1}^1 \left( \cos 2x + \frac{1}{2} + \frac{1}{2} \cos 4x \right) \, dx \\ &= \left( \frac{1}{2} \sin 2x + \frac{1}{2} x + \frac{1}{8} \sin 8x \right) \Big|_0^1 \\ &= \underline{\frac{1}{2} \sin 2 + \frac{1}{2} + \frac{1}{8} \sin 8} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_3, \vec{v}_2 \rangle &= \int_{-1}^1 \left( 3 \sin^2 x + \frac{9}{2} \sin 2 - 9 \right) (2 \cos^2 x) \, dx \\ &= 3 \int_{-1}^1 \left( 2 \sin^2 x \cos^2 x + (3 \sin 2 - 6) \cos^2 x \right) \, dx \end{aligned}$$

$$\begin{aligned}
6 \int_{-1}^1 \sin^2 x \cos^2 x \, dx &= \frac{3}{2} \int_{-1}^1 (1 - \cos 2x)(1 + \cos 2x) \, dx \\
&= \frac{3}{2} \int_{-1}^1 (1 - \cos^2 2x) \, dx \\
&= \frac{3}{2} \int_{-1}^1 \left( \frac{1}{2} - \frac{1}{2} \cos 4x \right) \, dx \\
&= \frac{3}{2} \left( x - \frac{1}{4} \sin 4x \right) \Big|_0^1 \\
&= \frac{3}{2} (1 - \sin 4)
\end{aligned}$$

$$\begin{aligned}
(9 \sin 2 - 18) \int_{-1}^1 \cos^2 x \, dx &= \frac{1}{2} (9 \sin 2 - 18) \int_{-1}^1 (1 + \cos 2x) \, dx \\
&= (9 \sin 2 - 18) \left( x + \frac{1}{2} \sin 2x \right) \Big|_0^1 \\
&= (9 \sin 2 - 18) \left( 1 + \frac{1}{2} \sin 2 \right) \\
&= \frac{9}{2} \sin^2 2 - 18
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_2 \rangle &= \frac{3}{2} - \frac{3}{2} \sin 4 + \frac{9}{2} \sin^2 2 - 18 \\
&= \frac{9}{2} \sin^2 2 - \frac{3}{2} \sin 4 - \frac{33}{2}
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
&= \cos^2 x - \frac{4 \sin 2 + 4 + \sin 8}{8 \sin 2} \cos 2x \\
&\quad - \frac{72 \sin^2 2 - 24 \sin 4 - 264}{6 - 8 \sin 2 + \sin 4 + \frac{(4 - 7 \sin 2)(4 \sin 2 - 4 - \sin 8)^2}{16 \sin^2 2}} \left( \sin^2 x - \left( \frac{4 \sin 2 - 4 - \sin 8}{8 \sin 2} \right) \cos 2x \right) \\
&= \frac{1}{2} + \frac{1}{2} \cos 2x - \frac{4 \sin 2 + 4 + \sin 8}{8 \sin 2} \cos 2x \\
&\quad - \frac{72 \sin^2 2 - 24 \sin 4 - 264}{6 - 8 \sin 2 + \sin 4 + \frac{(4 - 7 \sin 2)(4 \sin 2 - 4 - \sin 8)^2}{16 \sin^2 2}} \left( \frac{1}{2} - \frac{1}{2} \cos 2x - \left( \frac{4 \sin 2 - 4 - \sin 8}{8 \sin 2} \right) \cos 2x \right)
\end{aligned}$$

$$= \frac{1}{2} - \frac{4 + \sin 8}{8 \sin 2} \cos 2x$$

$$- \frac{128 \sin^2 2 \left( 9 \sin^2 2 - 3 \sin 4 - 33 \right)}{16 (6 - 8 \sin 2 + \sin 4) \sin^2 2 + (4 - 7 \sin 2)(4 \sin 2 - 4 - \sin 8)^2} \left( \frac{1}{2} - \left( \frac{8 \sin 2 - 4 - \sin 8}{8 \sin 2} \right) \cos 2x \right)$$

The *orthogonal* basis is

$$\left\{ \cos 2x, \quad \frac{1}{2} - \left( \frac{8 \sin 2 - \sin 8 - 4}{8 \sin 2} \right) \cos 2x, \right\}$$

$$\langle \vec{v}_3, \vec{v}_3 \rangle =$$

$$\vec{q}_1 = \frac{\cos 2x}{\sin 2}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{q}_2 = \frac{1}{\sqrt{\frac{3}{4} - \frac{1}{2} \sin 2 + \frac{1}{16} \sin 4 + \frac{(4 - 7 \sin 2)(4 \sin 2 - 4 - \sin 8)^2}{256 \sin^2 2}}} \left( \frac{1}{2} - \left( \frac{8 \sin 2 - \sin 8 - 4}{8 \sin 2} \right) \cos 2x \right)$$

$$\frac{1}{2} - \frac{4 + \sin 8}{8 \sin 2} \cos 2x$$

$$\vec{q}_3 = \frac{- \frac{128 \sin^2 2 \left( 9 \sin^2 2 - 3 \sin 4 - 33 \right)}{16 (6 - 8 \sin 2 + \sin 4) \sin^2 2 + (4 - 7 \sin 2)(4 \sin 2 - 4 - \sin 8)^2} \left( \frac{1}{2} - \left( \frac{8 \sin 2 - 4 - \sin 8}{8 \sin 2} \right) \cos 2x \right)}{\sqrt{\langle \vec{v}_3, \vec{v}_3 \rangle}}$$

### Exercise

Apply the Gram-Schmidt *orthonormalization* process in  $C^0[-1, 1]$  spanned by the functions, using the inner product  $f_1(x) = \sin \pi x$ ,  $f_2(x) = \sin 2\pi x$ ,  $f_3(x) = \sin 3\pi x$

### Solution

$$\text{Let } \vec{u}_1 = \sin \pi x, \quad \vec{u}_2 = \sin 2\pi x, \quad \vec{u}_3 = \sin 3\pi x$$

$$\vec{v}_1 = \vec{u}_1 = \sin \pi x$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^1 \sin^2 \pi x \, dx$$

$$\begin{aligned}
 &= \frac{1}{2} \int_{-1}^1 (1 - \cos 2\pi x) \, dx \\
 &= \frac{1}{2} \left( x - \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^1 \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 \sin \pi x \sin 2\pi x \, dx \\
 &= \frac{1}{2} \int_{-1}^1 (\cos 3\pi x - \cos(-\pi x)) \, dx \\
 &= \frac{1}{2} \int_{-1}^1 (\cos 3\pi x - \cos \pi x) \, dx \\
 &= \frac{1}{2} \left( \frac{1}{3\pi} \sin 3\pi x - \frac{1}{\pi} \sin \pi x \right) \Big|_{-1}^1 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
 &= \sin 2\pi x
 \end{aligned}$$

$$\begin{aligned}
 \langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 \sin^2 2\pi x \, dx \\
 &= \frac{1}{2} \int_{-1}^1 (1 - \cos 4\pi x) \, dx \\
 &= \frac{1}{2} \left( x - \frac{1}{4\pi} \sin 4\pi x \right) \Big|_{-1}^1 \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \langle \vec{u}_3, \vec{v}_1 \rangle &= \int_{-1}^1 \sin \pi x \sin 3\pi x \, dx \\
 &= \frac{1}{2} \int_{-1}^1 (\cos 4\pi x - \cos(-2\pi x)) \, dx
 \end{aligned}$$

$$\sin a \sin b = \frac{1}{2} [\cos(a+b) - \cos(a-b)]$$

$$\sin a \sin b = \frac{1}{2} [\cos(a+b) - \cos(a-b)]$$

$$\begin{aligned}
&= \frac{1}{2} \int_{-1}^1 (\cos 4\pi x - \cos 2\pi x) \, dx \\
&= \frac{1}{2} \left( \frac{1}{4\pi} \sin 4\pi x - \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^1 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_2 \rangle &= \int_{-1}^1 \sin 3\pi x \sin 2\pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (\cos 5\pi x - \cos \pi x) \, dx \\
&= \frac{1}{2} \left( \frac{1}{5\pi} \sin 5\pi x - \frac{1}{\pi} \sin \pi x \right) \Big|_{-1}^1 \\
&= 0
\end{aligned}$$

$$\sin a \sin b = \frac{1}{2} [\cos(a+b) - \cos(a-b)]$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
&= \sin 3\pi x
\end{aligned}$$

The *orthogonal* basis is  $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$

$$\begin{aligned}
\langle \vec{v}_3, \vec{v}_3 \rangle &= \int_{-1}^1 \sin^2 3\pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (1 - \cos 6\pi x) \, dx \\
&= \frac{1}{2} \left( x - \frac{1}{6\pi} \sin 6\pi x \right) \Big|_{-1}^1 \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \sin \pi x
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \sin 2\pi x
\end{aligned}$$



$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \sin 3\pi x$$

The *orthonormal* basis is  $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$

### Exercise

Apply the Gram-Schmidt *orthonormalization* process in  $C^0[-1, 1]$  spanned by the functions, using the inner product  $f_1(x) = \cos \pi x$ ,  $f_2(x) = \cos 2\pi x$ ,  $f_3(x) = \cos 3\pi x$

### Solution

Let  $\vec{u}_1 = f_1 = \cos \pi x$ ,  $\vec{u}_2 = f_2 = \cos 2\pi x$ ,  $\vec{u}_3 = f_3 = \cos 3\pi x$

$$\vec{v}_1 = \vec{u}_1 = \cos \pi x$$

$$\begin{aligned} \langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 \cos^2 \pi x \, dx \\ &= \frac{1}{2} \int_{-1}^1 (1 + \cos 2\pi x) \, dx \\ &= \frac{1}{2} \left( x + \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 \cos 2\pi x \cos \pi x \, dx \\ &= \frac{1}{2} \int_{-1}^1 (\cos 3\pi x + \cos \pi x) \, dx \\ &= \frac{1}{2} \left( \frac{1}{3\pi} \sin 3\pi x + \frac{1}{\pi} \sin \pi x \right) \Big|_{-1}^1 \\ &= 0 \end{aligned}$$

$$\cos a \cos b = \frac{1}{2} [\cos(a+b) + \cos(a-b)]$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= \cos 2\pi x$$

$$\begin{aligned}
\langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 \cos^2 2\pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (1 + \cos 4\pi x) \, dx \\
&= \frac{1}{2} \left( x + \frac{1}{4\pi} \sin 4\pi x \right) \Big|_{-1}^1 \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_1 \rangle &= \int_{-1}^1 \cos 3\pi x \cos \pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (\cos 4\pi x + \cos 2\pi x) \, dx \\
&= \frac{1}{2} \left( \frac{1}{4\pi} \sin 4\pi x + \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^1 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_2 \rangle &= \int_{-1}^1 \cos 3\pi x \cos 2\pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (\cos 5\pi x + \cos \pi x) \, dx \\
&= \frac{1}{2} \left( \frac{1}{5\pi} \sin 5\pi x + \frac{1}{\pi} \sin \pi x \right) \Big|_{-1}^1 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
&= \cos 3\pi x
\end{aligned}$$

The *orthogonal* basis is  $\{\cos \pi x, \cos 2\pi x, \cos 3\pi x\}$

$$\begin{aligned}
\langle \vec{v}_3, \vec{v}_3 \rangle &= \int_{-1}^1 \cos^2 3\pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (1 + \cos 6\pi x) \, dx
\end{aligned}$$

$$\cos a \cos b = \frac{1}{2} [\cos(a+b) + \cos(a-b)]$$

$$\cos a \cos b = \frac{1}{2} [\cos(a+b) + \cos(a-b)]$$

$$= \frac{1}{2} \left( x + \frac{1}{6\pi} \sin 6\pi x \right) \Big|_{-1}^1$$

$$= 1$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \cos \pi x$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \cos 2\pi x$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \cos 3\pi x$$

The *orthonormal* basis is  $\{\cos \pi x, \cos 2\pi x, \cos 3\pi x\}$

### Exercise

For  $\mathcal{P}_3[x]$ , define the inner product over  $\mathbb{R}$  as

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx$$

- If  $f(x) = 1$  is a unit vector in  $\mathcal{P}_3[x]$ ?
- Find an orthonormal basis for the subspace spanned by  $x$  and  $x^2$ .
- Complete the basis in part (b) to an orthonormal basis for  $\mathcal{P}_3[x]$  with respect to the inner product.
- Is

$$[f, g] = \int_0^1 f(x)g(x) dx$$

Also, an inner product for  $\mathcal{P}_3[x]$

- Find a pair of vectors  $\vec{v}$  and  $\vec{w}$  such that

$$\langle \vec{v}, \vec{w} \rangle = 0 \quad \text{but} \quad [\vec{v}, \vec{w}] \neq 0$$

- Is the basis found in part (c) are orthonormal basis for  $\mathcal{P}_3[x]$  with respect to the inner product in part (d)?

### Solution

$$a) \quad f(x) = 1$$

$$\begin{aligned} \langle f, f \rangle &= \int_{-1}^1 f(x) f(x) \, dx \\ &= \int_{-1}^1 dx \\ &= x \Big|_{-1}^1 \\ &= 1 + 1 \\ &= \underline{2 \neq 1} \end{aligned}$$

Therefore, when  $f(x) = 1$  is **not** a unit vector in  $\mathcal{P}_3[x]$

$$b) \quad \text{Let } \vec{u}_1 = f = x, \quad \vec{u}_2 = g = x^2$$

$$\underline{\vec{v}_1 = \vec{u}_1 = x}$$

$$\begin{aligned} \langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 x^2 \, dx \\ &= \frac{1}{3} x^3 \Big|_{-1}^1 \\ &= \frac{1}{3}(1+1) \\ &= \underline{\frac{2}{3}} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 x^2(x) \, dx \\ &= \int_{-1}^1 x^3 \, dx \\ &= \frac{1}{4} x^4 \Big|_{-1}^1 \\ &= \frac{1}{4}(1-1) \\ &= \underline{0} \end{aligned}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= x^2 - \frac{0}{*} ( )$$

$$= x^2 \Big|$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^1 x^4 dx$$

$$= \frac{1}{5} x^5 \Big|_{-1}^1$$

$$= \frac{1}{5} (1+1)$$

$$= \frac{2}{5} \Big|$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{x}{\sqrt{\frac{2}{3}}}$$

$$= \sqrt{\frac{3}{2}} x \Big|$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{x^2}{\sqrt{\frac{2}{5}}}$$

$$= \sqrt{\frac{5}{2}} x^2 \Big|$$

The orthonormal basis is  $\left\{ \sqrt{\frac{3}{2}} x, \sqrt{\frac{5}{2}} x^2 \right\}$

c) Since  $\vec{u}_1 = x$ ,  $\vec{u}_2 = x^2$  in  $\mathbb{P}_3[x]$

Then, let  $\vec{u}_3 = 1$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^1 (1)(x) dx$$

$$= \int_{-1}^1 x dx$$

$$= \frac{1}{2} x^2 \Big|_{-1}^1$$

$$= \frac{1}{2}(1-1)$$

$$= \underline{0}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^1 (1)(x^2) \, dx$$

$$= \int_{-1}^1 x^2 \, dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^1$$

$$= \frac{1}{3}(1+1)$$

$$= \underline{\frac{2}{3}}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= 1 - \frac{0}{\frac{2}{3}}(x) - \frac{\frac{2}{3}}{\frac{5}{5}}(x^2)$$

$$= \underline{1 - \frac{5}{3}x^2}$$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^1 \left(1 - \frac{5}{3}x^2\right)^2 \, dx$$

$$= \int_{-1}^1 \left(1 - \frac{10}{3}x^2 + \frac{25}{9}x^4\right) \, dx$$

$$= \left(x - \frac{10}{9}x^3 + \frac{5}{9}x^5\right) \Big|_{-1}^1$$

$$= 2\left(1 - \frac{10}{9} + \frac{5}{9}\right)$$

$$= 2\left(\frac{9-5}{9}\right)$$

$$= \underline{\frac{8}{9}}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$\begin{aligned}
&= \left( \sqrt{\frac{9}{8}} \right) \left( 1 - \frac{5}{3} x^2 \right) \\
&= \frac{3}{2\sqrt{2}} \left( 1 - \frac{5}{3} x^2 \right) \\
&= \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}} x^2 \quad |
\end{aligned}$$

The orthonormal basis is  $\left\{ \sqrt{\frac{3}{2}} x, \sqrt{\frac{5}{2}} x^2, \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}} x^2 \right\}$

$$d) [f, g] = \int_0^1 f(x) g(x) dx$$

Let  $\vec{u}_1 = 1, \vec{u}_2 = x, \vec{u}_3 = x^2$

$$\vec{v}_1 = \vec{u}_1 = 1 \quad |$$

$$\begin{aligned}
\langle \vec{v}_1, \vec{v}_1 \rangle &= \int_0^1 1 dx \\
&= x \Big|_0^1 \\
&= 1 \quad |
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_2, \vec{v}_1 \rangle &= \int_0^1 x(1) dx \\
&= \int_0^1 x dx \\
&= \frac{1}{2} x^2 \Big|_0^1 \\
&= \frac{1}{2} \quad |
\end{aligned}$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
&= x - \frac{1}{2} (1) \\
&= x - \frac{1}{2} \quad |
\end{aligned}$$

$$\begin{aligned}\langle \vec{v}_2, \vec{v}_2 \rangle &= \int_0^1 \left(x - \frac{1}{2}\right)^2 dx \\&= \int_0^1 \left(x - \frac{1}{2}\right)^2 d\left(x - \frac{1}{2}\right) \\&= \frac{1}{3} \left(x - \frac{1}{2}\right)^3 \Big|_0^1 \\&= \frac{1}{3} \left(\left(\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^3\right) \\&= \frac{1}{3} \left(\frac{1}{8} + \frac{1}{8}\right) \\&= \frac{1}{12}\end{aligned}$$

$$\begin{aligned}\langle \vec{u}_3, \vec{v}_1 \rangle &= \int_0^1 (x^2)(1) dx \\&= \int_0^1 x^2 dx \\&= \frac{1}{3} x^3 \Big|_0^1 \\&= \frac{1}{3}\end{aligned}$$

$$\begin{aligned}\langle \vec{u}_3, \vec{v}_2 \rangle &= \int_0^1 \left(x^2\right)\left(x - \frac{1}{2}\right) dx \\&= \int_0^1 \left(x^3 - \frac{1}{2}x^2\right) dx \\&= \left(\frac{1}{4}x^4 - \frac{1}{6}x^3\right) \Big|_0^1 \\&= \frac{1}{4} - \frac{1}{6} \\&= \frac{1}{12}\end{aligned}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$



$$= x^2 - \frac{\frac{1}{3}}{1}(1) - \frac{\frac{1}{12}}{\frac{1}{12}}\left(x - \frac{1}{2}\right)$$

$$= x^2 - \frac{1}{3} - x + \frac{1}{2}$$

$$= x^2 - x + \frac{1}{6} \Big|$$

$$\begin{aligned} \langle \vec{v}_3, \vec{v}_3 \rangle &= \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx \\ &= \int_0^1 \left(x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36}\right) dx \\ &= \left(\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{4}{9}x^3 - \frac{1}{6}x^2 + \frac{1}{36}x\right) \Big|_0^1 \\ &= \frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36} \\ &= \frac{2-5}{10} + \frac{16-6+1}{36} \\ &= -\frac{3}{10} + \frac{11}{36} \\ &= \frac{-108+110}{360} \\ &= \frac{1}{180} \Big| \end{aligned}$$

$$\begin{aligned} \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= 1 \Big| \end{aligned}$$

$$\begin{aligned} \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{x - \frac{1}{2}}{\sqrt{\frac{1}{12}}} \\ &= 2\sqrt{3} \left(x - \frac{1}{2}\right) \Big| \end{aligned}$$

$$\begin{aligned} \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= (\sqrt{180}) \left(x^2 - x + \frac{1}{6}\right) \end{aligned}$$

$$= (6\sqrt{5}) \left( x^2 - x + \frac{1}{6} \right)$$

$$= \underline{6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}} \quad |$$

The orthonormal basis is  $\left\{ 1, 2\sqrt{3} \left( x - \frac{1}{2} \right), \sqrt{5} \left( 6x^2 - 6x + 1 \right) \right\}$

Therefore,  $[f, g] = \int_0^1 f(x)g(x) dx$  is an inner product for  $\mathbb{P}_3[x]$

e) Let assume:  $\vec{v} = 1$  and  $\vec{w} = x$

$$\langle \vec{v}, \vec{w} \rangle = \int_{-1}^1 1(x) dx$$

$$= \int_{-1}^1 x dx$$

$$= \frac{1}{2}x^2 \Big|_{-1}^1$$

$$= \frac{1}{2}(1-1)$$

$$= \underline{0} \quad \checkmark$$

$$[\vec{v}, \vec{w}] = \int_0^1 1(x) dx$$

$$= \frac{1}{2}x^2 \Big|_0^1$$

$$= \underline{\frac{1}{2} \neq 0} \quad \checkmark$$

f) The orthonormal basis in part (c)  $\left\{ \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{2}}x^2, \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}}x^2 \right\}$  are **not** the same as

the orthonormal basis in part (d)  $\left\{ 1, 2\sqrt{3} \left( x - \frac{1}{2} \right), \sqrt{5} \left( 6x^2 - 6x + 1 \right) \right\}$

## ***Solution***      ***Section 3.4 – Orthogonal Matrices***

### ***Exercise***

Show that the matrix is orthogonal       $A = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$

### ***Solution***

$$\begin{aligned} AA^T &= \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix} \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I \quad \checkmark \end{aligned}$$

$$\begin{aligned} A^T A &= \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix} \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I \quad \checkmark \end{aligned}$$

$$AA^T = A^T A = I$$

$\therefore A$  is orthogonal

### ***Exercise***

Show that the matrix is orthogonal       $A = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}$

### ***Solution***

$$\begin{aligned} AA^T &= \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I \quad \checkmark \end{aligned}$$

$$\begin{aligned}
 A^T A &= \begin{pmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
 &= \underline{I} \quad \checkmark
 \end{aligned}$$

$$AA^T = A^T A = I$$

$\therefore A$  is orthogonal

### Exercise

Show that the matrix is orthogonal  $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$

### Solution

$$\begin{aligned}
 AA^T &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 &= \underline{I} \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 A^T A &= \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 &= \underline{I} \quad \checkmark
 \end{aligned}$$

$$AA^T = A^T A = I$$

$\therefore A$  is orthogonal

**Exercise**

Show that the matrix is orthogonal  $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{7}{\sqrt{53}} & \frac{2}{\sqrt{53}} \\ 0 & -\frac{2}{\sqrt{53}} & \frac{7}{\sqrt{53}} \end{pmatrix}$

**Solution**

$$\begin{aligned}
 AA^T &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{7}{\sqrt{53}} & \frac{2}{\sqrt{53}} \\ 0 & -\frac{2}{\sqrt{53}} & \frac{7}{\sqrt{53}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{7}{\sqrt{53}} & -\frac{2}{\sqrt{53}} \\ 0 & \frac{2}{\sqrt{53}} & \frac{7}{\sqrt{53}} \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 &= I \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 A^T A &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{7}{\sqrt{53}} & -\frac{2}{\sqrt{53}} \\ 0 & \frac{2}{\sqrt{53}} & \frac{7}{\sqrt{53}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{7}{\sqrt{53}} & \frac{2}{\sqrt{53}} \\ 0 & -\frac{2}{\sqrt{53}} & \frac{7}{\sqrt{53}} \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 &= I \quad \checkmark
 \end{aligned}$$

$$AA^T = A^T A = I$$

$\therefore A$  is orthogonal

**Exercise**

Show that the matrix is orthogonal  $A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$

**Solution**

$$\begin{aligned}
 AA^T &= \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix} \begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= I
 \end{aligned}$$

$$\begin{aligned}
 A^T A &= \begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix} \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= I
 \end{aligned}$$

$$AA^T = A^T A = I$$

$\therefore A$  is orthogonal

### Exercise

Show that the matrix is orthogonal  $A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

### Solution

$$\begin{aligned}
 AA^T &= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

$= I$  | ✓

$$A^T A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= I$  | ✓

$$AA^T = A^T A = I$$

∴  $A$  is orthogonal.

### Exercise

Show that the matrix is orthogonal

$$A = \begin{pmatrix} 0 & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & -\frac{2}{\sqrt{10}} & -\frac{2}{\sqrt{15}} \\ 0 & 0 & -\frac{2}{\sqrt{10}} & \frac{3}{\sqrt{15}} \end{pmatrix}$$

### Solution

$$AA^T = \begin{pmatrix} 0 & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & -\frac{2}{\sqrt{10}} & -\frac{2}{\sqrt{15}} \\ 0 & 0 & -\frac{2}{\sqrt{10}} & \frac{3}{\sqrt{15}} \end{pmatrix} \begin{pmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{5}{\sqrt{30}} & -\frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & 0 \\ \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & -\frac{2}{\sqrt{10}} & -\frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{15}} & \frac{1}{\sqrt{15}} & -\frac{2}{\sqrt{15}} & \frac{3}{\sqrt{15}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$= I$  | ✓

$$\begin{aligned}
A^T A &= \begin{pmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{5}{\sqrt{30}} & -\frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & 0 \\ \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & -\frac{2}{\sqrt{10}} & -\frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{15}} & \frac{1}{\sqrt{15}} & -\frac{2}{\sqrt{15}} & \frac{3}{\sqrt{15}} \end{pmatrix} \begin{pmatrix} 0 & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & -\frac{2}{\sqrt{10}} & -\frac{2}{\sqrt{15}} \\ 0 & 0 & -\frac{2}{\sqrt{10}} & \frac{3}{\sqrt{15}} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \underline{I} \quad \checkmark
\end{aligned}$$

$$AA^T = A^T A = I$$

$\therefore A$  is orthogonal.

### Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

### Solution

$$\begin{aligned}
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \underline{I} \quad \checkmark
\end{aligned}$$

$$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ is orthogonal with inverse } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



**Exercise**

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

**Solution**

$$\begin{aligned} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^T &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \underline{I} \quad \checkmark \end{aligned}$$

$$\therefore \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ is orthogonal with inverse } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

**Exercise**

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

**Solution**

$$\det \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \underline{2 \neq \pm 1}$$

$$\therefore \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{ is not orthogonal}$$

$$\text{Inverse } \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

**Exercise**

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix}$$

**Solution**

$$\begin{vmatrix} 4 & 1 \\ 3 & -1 \end{vmatrix} = -7 \neq \pm 1$$

$\therefore \begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix}$  is *not* orthogonal

$$\text{Inverse} \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{3}{7} & -\frac{4}{7} \end{pmatrix}$$

### ***Exercise***

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$$

### **Solution**

$$\begin{vmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{vmatrix} = -\frac{9}{25} - \frac{16}{25} \\ = -1 \quad \checkmark$$

$\therefore \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$  is orthogonal

$$\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$$

### ***Exercise***

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \frac{2}{\sqrt{6}} & 0 \\ 0 & \frac{3}{\sqrt{6}} \end{pmatrix}$$

### **Solution**

$$\left| \begin{array}{cc} \frac{2}{\sqrt{6}} & 0 \\ 0 & \frac{3}{\sqrt{6}} \end{array} \right| = 1$$

$$\therefore \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 \\ 0 & \frac{3}{\sqrt{6}} \end{pmatrix} \text{ is orthogonal}$$

$$\begin{pmatrix} \frac{2}{\sqrt{6}} & 0 \\ 0 & \frac{3}{\sqrt{6}} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 \\ 0 & \frac{3}{\sqrt{6}} \end{pmatrix}$$

### Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

### Solution

$$\begin{aligned} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \quad \checkmark \end{aligned}$$

$$\therefore \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ is orthogonal with inverse } \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(It is a standard matrix for a rotation of  $45^\circ$ )

**Exercise**

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

**Solution**

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}^T = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\therefore \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \text{ is orthogonal with inverse } \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

**Exercise**

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$$

**Solution**

$$\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}^T = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\therefore \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \text{ is orthogonal with an inverse } \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$$

**Exercise**

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$$

**Solution**

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 7 \\ 1 & 3 & -5 \\ -1 & 4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & & \\ & & \\ & & \end{pmatrix} \neq I$$

$$\therefore \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix} \text{ is not orthogonal}$$

### Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

### Solution

$$\begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}^T = \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2} & & \\ & & \\ & & \end{pmatrix} \neq I$$

$$\text{Or } \|r_1\| = \sqrt{0+1^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{3}{2}} \neq 1 \quad \therefore A \text{ is not orthogonal}$$

**Exercise**

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

**Solution**

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \text{ is orthogonal with inverse } \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

**Exercise**

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**Solution**

$$\begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \left( \frac{1}{2} + \frac{1}{2} \right)$$

$$=1 \quad \checkmark$$

∴ The matrix is orthogonal.

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

### Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}$$

### Solution

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}^T = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ = I \quad \checkmark$$

∴ The matrix is orthogonal.

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}$$

### Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{5\sqrt{3}} & \frac{4}{5\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{7}{5\sqrt{3}} & \frac{3}{5\sqrt{2}} \end{pmatrix}$$

### Solution

$$\begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{5\sqrt{3}} & \frac{4}{5\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{7}{5\sqrt{3}} & \frac{3}{5\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{5\sqrt{3}} & \frac{4}{5\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{7}{5\sqrt{3}} & \frac{3}{5\sqrt{2}} \end{pmatrix}^T = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{5\sqrt{3}} & \frac{4}{5\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{7}{5\sqrt{3}} & \frac{3}{5\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{5\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{7}{5\sqrt{3}} \\ \frac{4}{5\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{3}{5\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I \quad \checkmark$$

∴ The matrix is orthogonal.

$$\begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{5\sqrt{3}} & \frac{4}{5\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{7}{5\sqrt{3}} & \frac{3}{5\sqrt{2}} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{5\sqrt{3}} & \frac{4}{5\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{7}{5\sqrt{3}} & \frac{3}{5\sqrt{2}} \end{pmatrix}$$



**Exercise**

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$$

**Solution**

$$\begin{vmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{vmatrix} = \frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{1}{3}$$

$$= 1 \quad \checkmark$$

∴ The matrix is orthogonal.

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$$

**Exercise**

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix}$$

**Solution**

$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix}^T = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}$$

$\underline{=I} \quad \checkmark$

∴ The matrix is orthogonal.

$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix}$$

.....

$$\begin{vmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{vmatrix} = -\left(-\frac{1}{5} - \frac{4}{5}\right)$$

$\underline{=1} \quad \checkmark$

### Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{42}} \\ \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{4}{\sqrt{42}} \end{pmatrix}$$

### Solution

$$\begin{pmatrix} -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{42}} \\ \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{4}{\sqrt{42}} \end{pmatrix} \begin{pmatrix} -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{42}} \\ \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{4}{\sqrt{42}} \end{pmatrix}^T = \begin{pmatrix} -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{42}} \\ \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{4}{\sqrt{42}} \end{pmatrix} \begin{pmatrix} -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{42}} & \frac{5}{\sqrt{42}} & -\frac{4}{\sqrt{42}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I \quad \checkmark$$

∴ The matrix is orthogonal.

$$\begin{pmatrix} -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{42}} \\ \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{4}{\sqrt{42}} \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{42}} \\ \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{4}{\sqrt{42}} \end{pmatrix}$$

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$$\begin{vmatrix} -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{42}} \\ \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{4}{\sqrt{42}} \end{vmatrix} = \frac{12+10-1+2+15+4}{\sqrt{14} \sqrt{3} \sqrt{42}}$$

$$= \frac{42}{42}$$

$$= 1 \quad \checkmark$$

### Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

### Solution

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I \quad \checkmark$$

∴ The matrix is orthogonal &

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

### Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \frac{7}{\sqrt{62}} & \frac{9}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{62}} & -\frac{24}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ -\frac{3}{\sqrt{62}} & \frac{5}{\sqrt{682}} & \frac{3}{\sqrt{11}} \end{pmatrix}$$

### Solution

$$\begin{pmatrix} \frac{7}{\sqrt{62}} & \frac{9}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{62}} & -\frac{24}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ -\frac{3}{\sqrt{62}} & \frac{5}{\sqrt{682}} & \frac{3}{\sqrt{11}} \end{pmatrix} \begin{pmatrix} \frac{7}{\sqrt{62}} & \frac{9}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{62}} & -\frac{24}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ -\frac{3}{\sqrt{62}} & \frac{5}{\sqrt{682}} & \frac{3}{\sqrt{11}} \end{pmatrix}^T = \begin{pmatrix} \frac{7}{\sqrt{62}} & \frac{9}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{62}} & -\frac{24}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ -\frac{3}{\sqrt{62}} & \frac{5}{\sqrt{682}} & \frac{3}{\sqrt{11}} \end{pmatrix} \begin{pmatrix} \frac{7}{\sqrt{62}} & \frac{2}{\sqrt{62}} & -\frac{3}{\sqrt{62}} \\ \frac{9}{\sqrt{682}} & -\frac{24}{\sqrt{682}} & \frac{5}{\sqrt{682}} \\ \frac{1}{\sqrt{11}} & \frac{1}{\sqrt{11}} & \frac{3}{\sqrt{11}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I \quad \checkmark$$

∴ The matrix is orthogonal &

$$\begin{pmatrix} \frac{7}{\sqrt{62}} & \frac{9}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{62}} & -\frac{24}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ -\frac{3}{\sqrt{62}} & \frac{5}{\sqrt{682}} & \frac{3}{\sqrt{11}} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{7}{\sqrt{62}} & \frac{9}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{62}} & -\frac{24}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ -\frac{3}{\sqrt{62}} & \frac{5}{\sqrt{682}} & \frac{3}{\sqrt{11}} \end{pmatrix}$$

**Exercise**

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

**Solution**

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I \quad \checkmark$$

∴ The matrix is orthogonal &

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

**Exercise**

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$$

**Solution**

$$\begin{aligned} \|r_2\| &= \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{2}\right)^2} \\ &= \sqrt{\frac{1}{3} + \frac{1}{4}} \\ &= \sqrt{\frac{7}{12}} \neq 1 \end{aligned}$$

**Or**

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix}^T &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{5}{6} & & \\ & & & \\ & & & \end{pmatrix} \neq I \end{aligned}$$

∴ The matrix is **not** orthogonal

**Exercise**

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 0 & -\frac{1}{2} & \frac{3}{\sqrt{174}} & \frac{9}{2\sqrt{29}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{2} & -\frac{10}{\sqrt{174}} & -\frac{1}{2\sqrt{29}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{8}{\sqrt{174}} & \frac{5}{2\sqrt{29}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{2} & -\frac{1}{\sqrt{174}} & -\frac{3}{2\sqrt{29}} \end{pmatrix}$$

**Solution**

$$\begin{aligned} & \begin{pmatrix} 0 & -\frac{1}{2} & \frac{3}{\sqrt{174}} & \frac{9}{2\sqrt{29}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{2} & -\frac{10}{\sqrt{174}} & -\frac{1}{2\sqrt{29}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{8}{\sqrt{174}} & \frac{5}{2\sqrt{29}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{2} & -\frac{1}{\sqrt{174}} & -\frac{3}{2\sqrt{29}} \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{2} & \frac{3}{\sqrt{174}} & \frac{9}{2\sqrt{29}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{2} & -\frac{10}{\sqrt{174}} & -\frac{1}{2\sqrt{29}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{8}{\sqrt{174}} & \frac{5}{2\sqrt{29}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{2} & -\frac{1}{\sqrt{174}} & -\frac{3}{2\sqrt{29}} \end{pmatrix}^T \\ &= \begin{pmatrix} 0 & -\frac{1}{2} & \frac{3}{\sqrt{174}} & \frac{9}{2\sqrt{29}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{2} & -\frac{10}{\sqrt{174}} & -\frac{1}{2\sqrt{29}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{8}{\sqrt{174}} & \frac{5}{2\sqrt{29}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{2} & -\frac{1}{\sqrt{174}} & -\frac{3}{2\sqrt{29}} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{\sqrt{174}} & -\frac{10}{\sqrt{174}} & -\frac{8}{\sqrt{174}} & -\frac{1}{\sqrt{174}} \\ \frac{9}{2\sqrt{29}} & -\frac{1}{2\sqrt{29}} & \frac{5}{2\sqrt{29}} & -\frac{3}{2\sqrt{29}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \underline{I} \quad \checkmark \end{aligned}$$

∴ The matrix is orthogonal &

$$\begin{pmatrix} 0 & -\frac{1}{2} & \frac{3}{\sqrt{174}} & \frac{9}{2\sqrt{29}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{2} & -\frac{10}{\sqrt{174}} & -\frac{1}{2\sqrt{29}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{8}{\sqrt{174}} & \frac{5}{2\sqrt{29}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{2} & -\frac{1}{\sqrt{174}} & -\frac{3}{2\sqrt{29}} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -\frac{1}{2} & \frac{3}{\sqrt{174}} & \frac{9}{2\sqrt{29}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{2} & -\frac{10}{\sqrt{174}} & -\frac{1}{2\sqrt{29}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{8}{\sqrt{174}} & \frac{5}{2\sqrt{29}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{2} & -\frac{1}{\sqrt{174}} & -\frac{3}{2\sqrt{29}} \end{pmatrix}$$

**Exercise**

Determine if the given matrix is orthogonal. If it is, find its inverse

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{5} & \frac{3}{7} \\ \frac{4}{9} & \frac{3}{5} & -\frac{2}{7} \\ \frac{8}{9} & -\frac{2}{5} & \frac{3}{7} \end{bmatrix}$$

**Solution**

$$\vec{q}_1 = \begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{8}{9} \end{bmatrix}^T \quad \vec{q}_2 = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} & -\frac{2}{5} \end{bmatrix}^T \quad \vec{q}_3 = \begin{bmatrix} \frac{3}{7} & -\frac{2}{7} & \frac{3}{7} \end{bmatrix}^T$$

$$\begin{aligned} \vec{q}_1 \cdot \vec{q}_2 &= \frac{4}{45} + \frac{12}{45} - \frac{16}{45} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{q}_1 \cdot \vec{q}_3 &= \frac{3}{63} - \frac{8}{63} + \frac{24}{63} \\ &= \frac{19}{63} \neq 0 \end{aligned}$$

$$\begin{aligned} \vec{q}_2 \cdot \vec{q}_3 &= \frac{12}{35} - \frac{6}{35} + \frac{6}{35} \\ &= \frac{12}{35} \neq 0 \end{aligned}$$

The given matrix is **not** orthogonal

**Exercise**

Find a last column so that the resulting matrix is orthogonal

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \dots \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \dots \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & \dots \end{bmatrix}$$

**Solution**

$$\vec{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}^T$$

$$\begin{aligned} \|\vec{q}_1\| &= \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} \\ &= 1 \end{aligned}$$



$$\vec{q}_2 = \left[ \frac{1}{\sqrt{6}} \quad \frac{1}{\sqrt{6}} \quad \frac{2}{\sqrt{6}} \right]^T$$

$$\begin{aligned} \|\vec{q}_2\| &= \sqrt{\frac{1}{6} + \frac{1}{6} + \frac{4}{6}} \\ &= 1 \end{aligned}$$

$$\text{Let } \vec{q}_3 = [x \quad y \quad z]^T$$

$$\vec{q}_1 \cdot \vec{q}_3 = \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y - \frac{1}{\sqrt{3}}z = 0 \rightarrow x + y - z = 0$$

$$\vec{q}_2 \cdot \vec{q}_3 = \frac{1}{\sqrt{6}}x + \frac{1}{\sqrt{6}}y - \frac{2}{\sqrt{6}}z = 0 \rightarrow x + y - 2z = 0$$

$$\begin{cases} x + y - z = 0 \\ x + y - 2z = 0 \end{cases} \rightarrow \underline{z = 0} \quad \text{and} \quad x + y = 0 \Rightarrow \underline{x = -y}$$

$$x^2 + y^2 = 1$$

$$2x^2 = 1$$

$$x = \pm \frac{1}{\sqrt{2}}$$

$$\vec{q}_3 = \left[ -\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad 0 \right]^T$$

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \end{pmatrix}$$

### Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{pmatrix} \frac{3}{5} & 0 & \dots \\ \frac{4}{5} & 0 & \dots \\ 0 & 1 & \dots \end{pmatrix}$$

### Solution

$$\vec{q}_1 = \left[ \frac{3}{5} \quad \frac{4}{5} \quad 0 \right]^T$$

$$\begin{aligned}\|\vec{q}_1\| &= \sqrt{\frac{9}{25} + \frac{16}{25}} \\ &= 1\end{aligned}$$

$$\vec{q}_2 = [0 \ 0 \ 1]^T$$

$$\|\vec{q}_2\| = 1$$

$$\text{Let } \vec{q}_3 = [x \ y \ z]^T$$

$$\vec{q}_1 \cdot \vec{q}_3 = \frac{3}{5}x + \frac{4}{5}y - 0z = 0$$

$$\frac{3}{5}x + \frac{4}{5}y = 0$$

$$3x + 4y = 0$$

$$\vec{q}_2 \cdot \vec{q}_3 = 0 \cdot x + 0 \cdot y + 1 \cdot z = 0$$

$$z = 0$$

$$\text{From } 3x + 4y = 0$$

$$x = -\frac{4}{3}y$$

$$x^2 + y^2 + z^2 = 1$$

$$\frac{16}{9}y^2 + y^2 = 1$$

$$\frac{25}{9}y^2 = 1$$

$$y = \pm \frac{3}{5}$$

$$y = \frac{3}{5} \Rightarrow x = -\frac{4}{5}$$

$$\text{Then } \vec{q}_3 = \left[ -\frac{4}{5} \ \frac{3}{5} \ 0 \right]^T$$

$$\begin{pmatrix} \frac{3}{5} & 0 & -\frac{4}{5} \\ \frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \end{pmatrix}$$

$$\vec{q}_1 \cdot \vec{q}_3 = \frac{3}{5}\left(-\frac{4}{5}\right) + \frac{4}{5}\left(\frac{3}{5}\right) = 0 \quad \checkmark$$

**Exercise**

Find a last column so that the resulting matrix is orthogonal

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & \dots \\ 0 & 0 & \dots \\ \frac{4}{5} & \frac{3}{5} & \dots \end{pmatrix}$$

**Solution**

$$\vec{q}_1 = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}^T$$

$$\begin{aligned} \|\vec{q}_1\| &= \sqrt{\frac{9}{25} + \frac{16}{25}} \\ &= 1 \end{aligned}$$

$$\vec{q}_2 = \begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}^T$$

$$\begin{aligned} \|\vec{q}_2\| &= \sqrt{\frac{16}{25} + \frac{9}{25}} \\ &= 1 \end{aligned}$$

$$\text{Let } \vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$$

$$\vec{q}_1 \cdot \vec{q}_3 = \frac{3}{5}x - 0y + \frac{4}{5}z = 0$$

$$\frac{3}{5}x + \frac{4}{5}z = 0$$

$$\underline{3x + 4z = 0}$$

$$\vec{q}_2 \cdot \vec{q}_3 = \left(-\frac{4}{5}\right) \cdot x + 0 \cdot y + \left(\frac{3}{5}\right) \cdot z = 0$$

$$-\frac{4}{5}x + \frac{3}{5}z = 0$$

$$\underline{-4x + 3z = 0}$$

$$\begin{cases} 3x + 4z = 0 \\ -4x + 3z = 0 \end{cases}$$

$$x = \frac{\begin{vmatrix} 0 & 4 \\ 0 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ -4 & 3 \end{vmatrix}}$$

$$= \frac{0}{25}$$

$$\underline{= 0}$$

$$\Rightarrow \underline{z = 0}$$

$$x^2 + y^2 + z^2 = 1$$

$$y^2 = 1$$

$$\underline{y = \pm 1}$$

$$\text{Then } \vec{q}_3 = [0 \quad 1 \quad 0]^T$$

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}$$

$$\underline{\vec{q}_1 \cdot \vec{q}_3 = \frac{3}{5}(0) + 0 + \frac{4}{5}(0) = 0} \quad \checkmark$$

### Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \dots \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \dots \\ 0 & 0 & \dots \end{pmatrix}$$

### Solution

$$\vec{q}_1 = \left[ \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad 0 \right]^T$$

$$\begin{aligned} \|\vec{q}_1\| &= \sqrt{\frac{1}{2} + \frac{1}{2}} \\ &= 1 \end{aligned}$$

$$\vec{q}_2 = \left[ \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad 0 \right]^T$$

$$\begin{aligned} \|\vec{q}_2\| &= \sqrt{\frac{1}{2} + \frac{1}{2}} \\ &= 1 \end{aligned}$$

$$\text{Let } \vec{q}_3 = [x \quad y \quad z]^T$$

$$\vec{q}_1 \cdot \vec{q}_3 = \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y + 0 \cdot z = 0$$

$$\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y = 0$$

$$\underline{x = y}$$

$$\vec{q}_2 \cdot \vec{q}_3 = \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y + 0 \cdot z = 0$$

$$\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y = 0$$

$$\underline{x = -y}$$

$$x = \pm y \Rightarrow \underline{x = y = 0}$$

$$x^2 + y^2 + z^2 = 1$$

$$z^2 = 1$$

$$\underline{z = \pm 1}$$

$$\text{Then } \vec{q}_3 = [0 \ 0 \ 1]^T$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

### Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{pmatrix} \frac{2}{7} & \frac{3}{7} & \dots \\ \frac{6}{7} & \frac{2}{7} & \dots \\ -\frac{3}{7} & \frac{6}{7} & \dots \end{pmatrix}$$

### Solution

$$\vec{q}_1 = \left[ \frac{2}{7} \ \frac{6}{7} \ -\frac{3}{7} \right]^T$$

$$\|\vec{q}_1\| = \sqrt{\frac{4}{49} + \frac{36}{49} + \frac{9}{49}}$$

$$\underline{= 1}$$

$$\vec{q}_2 = \left[ \frac{3}{7} \ \frac{2}{7} \ \frac{6}{7} \right]^T$$

$$\begin{aligned}\|\vec{q}_2\| &= \sqrt{\frac{9}{49} + \frac{4}{49} + \frac{36}{49}} \\ &= 1\end{aligned}$$

$$\text{Let } \vec{q}_3 = [x \ y \ z]^T$$

$$\begin{aligned}\vec{q}_1 \cdot \vec{q}_3 &= \frac{2}{7}x + \frac{6}{7}y - \frac{3}{7} \cdot z = 0 \\ 2x + 6y - 3z &= 0\end{aligned}$$

$$\begin{aligned}\vec{q}_2 \cdot \vec{q}_3 &= \frac{3}{7}x + \frac{2}{7}y + \frac{6}{7} \cdot z = 0 \\ 3x + 2y + 6z &= 0\end{aligned}$$

$$\begin{array}{r} 2 \\ + \end{array} \begin{cases} 2x + 6y - 3z = 0 \\ 3x + 2y + 6z = 0 \end{cases}$$


---


$$7x + 14y = 0$$

$$x = -2y$$

$$z = \frac{2}{3}x + 2y$$

$$= -\frac{4}{3}y + 2y$$

$$= \frac{2}{3}y$$

$$x^2 + y^2 + z^2 = 1$$

$$4y^2 + y^2 + \frac{4}{9}y^2 = 1$$

$$\left(\frac{36+9+4}{9}\right)y^2 = 1$$

$$y^2 = \frac{9}{49}$$

$$y = \pm \frac{3}{7}$$

$$\text{If } y = \frac{3}{7} \Rightarrow x = -\frac{6}{7}, \ z = \frac{2}{7}$$

Or

$$y = -\frac{3}{7} \Rightarrow x = \frac{6}{7}, \ z = -\frac{2}{7}$$

$$\text{Then } \vec{q}_3 = \begin{bmatrix} -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \end{bmatrix}^T$$

$$\begin{pmatrix} \frac{2}{7} & \frac{3}{7} & -\frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & \frac{3}{7} \\ -\frac{3}{7} & \frac{6}{7} & \frac{2}{7} \end{pmatrix} \text{ or } \begin{pmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \\ -\frac{3}{7} & \frac{6}{7} & -\frac{2}{7} \end{pmatrix}$$

### Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \dots \\ \frac{2}{3} & \frac{1}{3} & \dots \\ -\frac{2}{3} & \frac{2}{3} & \dots \end{pmatrix}$$

### Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}^T$$

$$\begin{aligned} \|\vec{q}_1\| &= \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} \\ &= 1 \end{aligned}$$

$$\vec{q}_2 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}^T$$

$$\begin{aligned} \|\vec{q}_2\| &= \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} \\ &= 1 \end{aligned}$$

$$\text{Let } \vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$$

$$\begin{aligned} \vec{q}_1 \cdot \vec{q}_3 &= \frac{1}{3}x + \frac{2}{3}y - \frac{2}{3}z = 0 \\ x + 2y - 2z &= 0 \end{aligned}$$

$$\begin{aligned} \vec{q}_2 \cdot \vec{q}_3 &= \frac{2}{3}x + \frac{1}{3}y + \frac{2}{3}z = 0 \\ 2x + y + 2z &= 0 \end{aligned}$$

$$\begin{aligned} &+ \begin{cases} x + 2y - 2z = 0 \\ 2x + y + 2z = 0 \end{cases} \\ \hline &3x + 3y = 0 \end{aligned}$$

$$x = -y$$

$$\begin{aligned}
 z &= \frac{1}{2}x + y \\
 &= -\frac{1}{2}y + y \\
 &= \frac{1}{2}y
 \end{aligned}$$

$$x^2 + y^2 + z^2 = 1$$

$$y^2 + y^2 + \frac{1}{4}y^2 = 1$$

$$\left(\frac{9}{4}\right)y^2 = 1$$

$$y = \pm \frac{2}{3}$$

$$\text{If } y = \frac{2}{3} \Rightarrow x = -\frac{2}{3}, \quad z = \frac{1}{3}$$

Or

$$y = -\frac{2}{3} \Rightarrow x = \frac{2}{3}, \quad z = -\frac{1}{3}$$

$$\vec{q}_3 = \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}^T \quad \text{or} \quad \vec{q}_3 = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}^T$$

$$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

### Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \dots \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} & \dots \\ -\frac{1}{3} & 0 & \dots \end{pmatrix}$$

### Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}^T$$

$$\begin{aligned}
 \|\vec{q}_1\| &= \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} \\
 &= 1
 \end{aligned}$$



$$\vec{q}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$$

$$\begin{aligned} \|\vec{q}_2\| &= \sqrt{\frac{1}{2} + \frac{1}{2}} \\ &= 1 \end{aligned}$$

$$\text{Let } \vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$$

$$\begin{aligned} \vec{q}_1 \cdot \vec{q}_3 &= \frac{2}{3}x + \frac{2}{3}y - \frac{1}{3}z = 0 \\ \underline{2x + 2y - z = 0} \end{aligned}$$

$$\begin{aligned} \vec{q}_2 \cdot \vec{q}_3 &= \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y + 0 \cdot z = 0 \\ \underline{y = x} \end{aligned}$$

$$2x + 2y - z = 0$$

$$2x + 2x - z = 0$$

$$\underline{z = 4x}$$

$$x^2 + y^2 + z^2 = 1$$

$$x^2 + x^2 + 16x^2 = 1$$

$$x^2 = \frac{1}{18}$$

$$x = \pm \frac{1}{3\sqrt{2}}$$

$$\text{If } x = \frac{1}{3\sqrt{2}} \Rightarrow y = \frac{1}{3\sqrt{2}}, \quad z = \frac{4}{3\sqrt{2}} \quad \Big|$$

Or

$$\underline{x = -\frac{1}{3\sqrt{2}} \Rightarrow y = -\frac{1}{3\sqrt{2}}, \quad z = -\frac{4}{3\sqrt{2}} \quad \Big|}$$

$$\vec{q}_3 = \begin{bmatrix} \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} \end{bmatrix}^T \quad \text{or} \quad \vec{q}_3 = \begin{bmatrix} -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{4}{3\sqrt{2}} \end{bmatrix}^T$$

$$\text{For } \vec{q}_3 = \begin{bmatrix} \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} \end{bmatrix}^T$$

$$\vec{q}_1 \cdot \vec{q}_3 = \frac{2}{3} \cdot \frac{1}{3\sqrt{2}} + \frac{2}{3} \cdot \frac{1}{3\sqrt{2}} - \frac{1}{3} \cdot \frac{4}{3\sqrt{2}} = 0 \quad \checkmark$$

$$\vec{q}_2 \cdot \vec{q}_3 = \frac{1}{\sqrt{2}} \cdot \frac{1}{3\sqrt{2}} - \frac{1}{\sqrt{2}} \cdot \frac{1}{3\sqrt{2}} + 0 \cdot \frac{4}{3\sqrt{2}} = 0 \quad \checkmark$$

$$\text{For } \vec{q}_3 = \begin{bmatrix} -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{4}{3\sqrt{2}} \end{bmatrix}^T$$

$$\vec{q}_1 \cdot \vec{q}_3 = -\frac{2}{3} \cdot \frac{1}{3\sqrt{2}} - \frac{2}{3} \cdot \frac{1}{3\sqrt{2}} + \frac{1}{3} \cdot \frac{4}{3\sqrt{2}} = 0 \quad \checkmark$$

$$\vec{q}_2 \cdot \vec{q}_3 = \frac{1}{\sqrt{2}} \cdot \frac{1}{3\sqrt{2}} - \frac{1}{\sqrt{2}} \cdot \frac{1}{3\sqrt{2}} + 0 \cdot \frac{4}{3\sqrt{2}} = 0 \quad \checkmark$$

$$\begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ -\frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ -\frac{1}{3} & 0 & -\frac{4}{3\sqrt{2}} \end{pmatrix}$$

### Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{3}{\sqrt{70}} & \dots \\ -\frac{1}{\sqrt{5}} & \frac{6}{\sqrt{70}} & \dots \\ 0 & -\frac{5}{\sqrt{70}} & \dots \end{pmatrix}$$

### Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \end{bmatrix}^T$$

$$\begin{aligned} \|\vec{q}_1\| &= \sqrt{\frac{4}{5} + \frac{1}{5}} \\ &= 1 \end{aligned}$$

$$\vec{q}_2 = \begin{bmatrix} \frac{3}{\sqrt{70}} & \frac{6}{\sqrt{70}} & -\frac{5}{\sqrt{70}} \end{bmatrix}^T$$

$$\begin{aligned} \|\vec{q}_2\| &= \sqrt{\frac{9}{70} + \frac{36}{70} + \frac{25}{70}} \\ &= 1 \end{aligned}$$

$$\text{Let } \vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$$

$$\vec{q}_1 \cdot \vec{q}_3 = \frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}y + 0 \cdot z = 0$$

$$\underline{y = 2x}$$

$$\vec{q}_2 \cdot \vec{q}_3 = \frac{3}{\sqrt{70}}x + \frac{6}{\sqrt{70}}y - \frac{5}{\sqrt{70}}z = 0$$

$$\underline{3x + 6y - 5z = 0}$$

$$3x + 12x - 5z = 0$$

$$5z = 15x$$

$$\underline{z = 3x}$$

$$x^2 + y^2 + z^2 = 1$$

$$x^2 + 4x^2 + 9x^2 = 1$$

$$x^2 = \frac{1}{14}$$

$$x = \pm \frac{1}{\sqrt{14}}$$

$$\underline{\text{If } x = \frac{1}{\sqrt{14}} \Rightarrow y = \frac{2}{\sqrt{14}}, z = \frac{3}{\sqrt{14}}}$$

Or

$$\underline{x = -\frac{1}{\sqrt{14}} \Rightarrow y = -\frac{2}{\sqrt{14}}, z = -\frac{3}{\sqrt{14}}}$$

$$\vec{q}_3 = \begin{bmatrix} \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \end{bmatrix}^T \quad \text{or} \quad \vec{q}_3 = \begin{bmatrix} -\frac{1}{\sqrt{14}} & -\frac{2}{\sqrt{14}} & -\frac{3}{\sqrt{14}} \end{bmatrix}^T$$

$$\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{3}{\sqrt{70}} & \frac{1}{\sqrt{14}} \\ -\frac{1}{\sqrt{5}} & \frac{6}{\sqrt{70}} & \frac{2}{\sqrt{14}} \\ 0 & -\frac{5}{\sqrt{70}} & \frac{3}{\sqrt{14}} \end{pmatrix}$$

### Exercise

Prove that if  $A$  is orthogonal, then  $A^T$  is orthogonal.

### Solution

Since  $A$  is orthogonal then  $A^T = A^{-1}$  and  $A = (A^T)^T$

Then  $(A^T)^T A^T = A A^T = I \Rightarrow A^T$  is orthogonal

Another word, since  $A$  is orthogonal, then both column and row vectors of  $A$  form an orthonormal set.

$A^T$  is just  $A$  with its row and column vectors are swapped.

The column vectors of  $A^T$  (which are the row vectors of  $A$ ) and row vectors of  $A^T$  (which are the column vectors of  $A$ ) form orthonormal sets, therefore  $A^T$  is orthogonal

### Exercise

Prove that if  $A$  is orthogonal, then  $A^{-1}$  is orthogonal

#### Solution

Since  $A$  is orthogonal then  $A^T = A^{-1}$  and  $A = (A^{-1})^{-1}$

$$\begin{aligned} (A^{-1})^{-1} &= (A^T)^{-1} & A^T &= A^{-1} \\ &= (A^{-1})^T \end{aligned}$$

$\therefore A^{-1}$  is orthogonal

### Exercise

Prove that if  $A$  and  $B$  are orthogonal, then  $AB$  is orthogonal

#### Solution

Since  $A$  is orthogonal then  $A^T = A^{-1}$

and  $B$  is orthogonal then  $B^T = B^{-1}$

$$\begin{aligned} (AB)^T &= B^T A^T \\ &= B^{-1} A^{-1} \\ &= (AB)^{-1} \end{aligned}$$

$\therefore AB$  is orthogonal

### Exercise

Let  $Q$  be an  $n \times n$  orthogonal matrix, and let  $A$  be an  $n \times n$  matrix.

Show that  $\det(QAQ^T) = \det(A)$

#### Solution

$$\begin{aligned} \det(QAQ^T) &= \det(Q)\det(A)\det(Q^T) \\ &= \det(A)\det(QQ^T) \end{aligned} \quad \text{Since } Q \text{ is an orthogonal matrix } \det(QQ^T) = \det(I)$$

$$= \det(A)\det(I)$$

$$= \det(A) \quad \checkmark$$

### Exercise

$$\text{Let } A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$$

- Is matrix  $A$  an orthogonal matrix?
- Let  $B$  be the matrix obtained by normalizing each row of  $A$ , find  $B$ .
- Is  $B$  an orthogonal matrix?
- Are the columns of  $B$  orthogonal?

### Solution

$$\begin{aligned} a) \quad AA^T &= \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 7 \\ 1 & 3 & -5 \\ -1 & 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 & & \\ & & \\ & & \end{pmatrix} \neq I \end{aligned}$$

$$\therefore \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix} \text{ is } \textbf{not} \text{ orthogonal}$$

$$\begin{aligned} b) \quad \|(1, 1, -1)\| &= \sqrt{1+1+1} \\ &= \sqrt{3} \end{aligned}$$

$$\begin{aligned} \|(1, 3, 4)\| &= \sqrt{1+9+16} \\ &= \sqrt{26} \end{aligned}$$

$$\begin{aligned} \|(7, -5, 2)\| &= \sqrt{49+25+4} \\ &= \sqrt{78} \end{aligned}$$

$$B = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{26}} & \frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \\ \frac{7}{\sqrt{78}} & -\frac{5}{\sqrt{78}} & \frac{2}{\sqrt{78}} \end{pmatrix}$$

c) Yes, since the rows are orthogonal with unit vectors.

$$\begin{aligned}
 BB^T &= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{26}} & -\frac{1}{\sqrt{78}} \\ \frac{1}{\sqrt{26}} & \frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \\ \frac{7}{\sqrt{78}} & -\frac{5}{\sqrt{78}} & \frac{2}{\sqrt{78}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{26}} & \frac{7}{\sqrt{78}} \\ \frac{1}{\sqrt{3}} & \frac{3}{\sqrt{26}} & -\frac{5}{\sqrt{78}} \\ -\frac{1}{\sqrt{3}} & \frac{4}{\sqrt{26}} & \frac{2}{\sqrt{78}} \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I
 \end{aligned}$$

d) Yes, since the rows of  $B$  form an orthonormal set of vectors. Then, the column of  $B$  must form an orthonormal set.

$$\begin{aligned}
 \left\| \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{26}}, \frac{7}{\sqrt{78}} \right) \right\| &= \sqrt{\frac{1}{3} + \frac{1}{26} + \frac{49}{78}} \\
 &= \sqrt{\frac{26 + 3 + 49}{78}} \\
 &= \sqrt{\frac{78}{78}} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \left\| \left( \frac{1}{\sqrt{3}}, \frac{3}{\sqrt{26}}, -\frac{5}{\sqrt{78}} \right) \right\| &= \sqrt{\frac{1}{3} + \frac{9}{26} + \frac{25}{78}} \\
 &= \sqrt{\frac{26 + 27 + 25}{78}} \\
 &= \sqrt{\frac{78}{78}} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \left\| \left( -\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{26}}, \frac{2}{\sqrt{78}} \right) \right\| &= \sqrt{\frac{1}{3} + \frac{16}{26} + \frac{4}{78}} \\
 &= \sqrt{\frac{78}{78}} \\
 &= 1
 \end{aligned}$$

## **Solution**      **Section 3.5 – Least Squares Analysis**

### **Exercise**

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0, 2), (1, 2), (2, 0)\}$$

### **Solution**

Let  $y = mx + b$  be the equation of the line that best fits the given points.

Then

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\text{where } A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

The normal equation formula:  $A^T A \vec{x} = A^T \vec{y}$

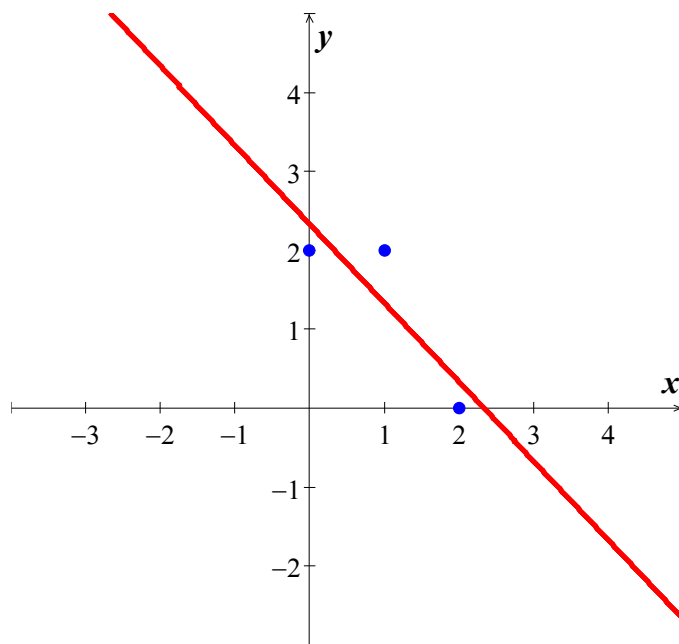
$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$m = \frac{\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 3 & 3 \end{vmatrix}} = \frac{-6}{6} = -1$$

$$b = \frac{\begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix}}{6} = \frac{7}{3}$$

$$\text{Thus, } y = -x + \frac{7}{3}$$



$$\begin{aligned} A\vec{x} &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ \frac{7}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{7}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \vec{y} - A\vec{x} &= \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{7}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \textbf{Error:} \quad \|\vec{y} - A\vec{x}\| &= \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}} \\ &= \frac{\sqrt{6}}{3} \\ &\approx 0.8164966 \end{aligned}$$

\*\*\*\*\*

The *second order* equation:  $y = -x^2 + x + 2$

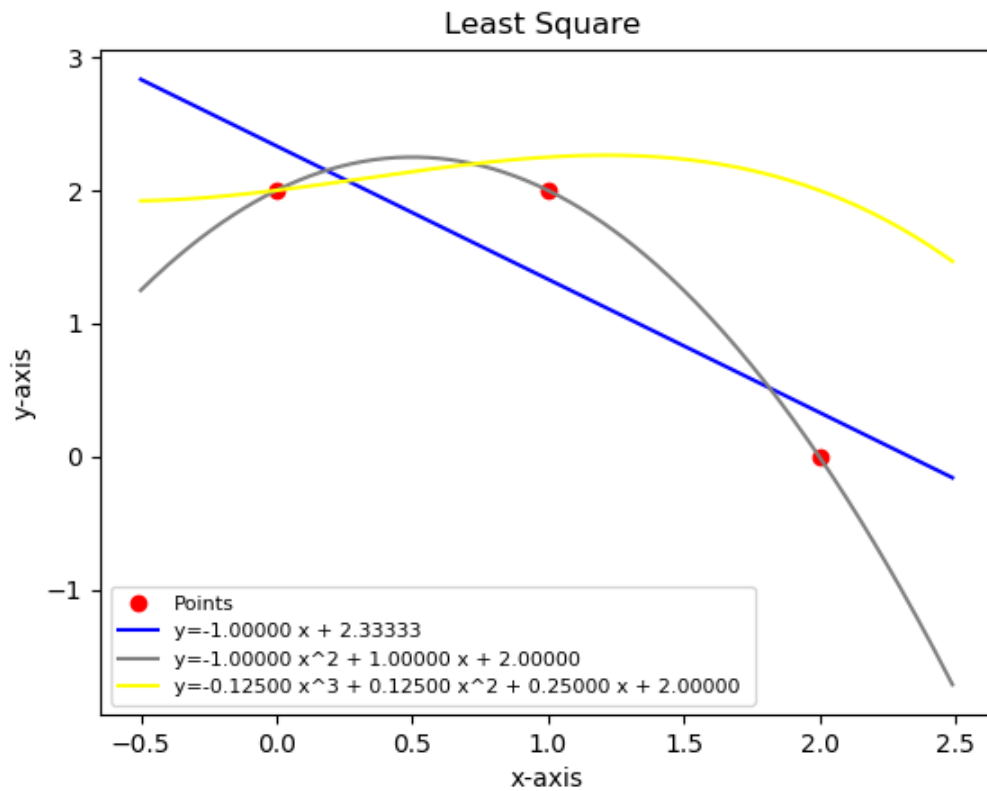
**Error** = 0.00000

\*\*\*\*\*

The *third order* equation:  $y = -.1250x^3 - 0.1250x^2 + 0.25x + 2$

**Error** = 2.01556





### Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0, 0), (1, 1), (2, 4)\}$$

### Solution

Let  $y = mx + b$  be the equation of the line that best fits the given points.

Then

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

$$\text{where } A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

The normal equation formula:  $A^T A \vec{x} = A^T \vec{y}$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

$$m = \frac{\begin{vmatrix} 9 & 3 \\ 5 & 3 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 3 & 3 \end{vmatrix}}$$

$$= \frac{12}{6}$$

$$= 2$$

$$b = \frac{\begin{vmatrix} 5 & 9 \\ 3 & 5 \end{vmatrix}}{6}$$

$$= -\frac{1}{3}$$

$$\text{Thus, } y = 2x - \frac{1}{3}$$

$$A\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -\frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} \\ \frac{5}{3} \\ \frac{11}{3} \end{pmatrix}$$

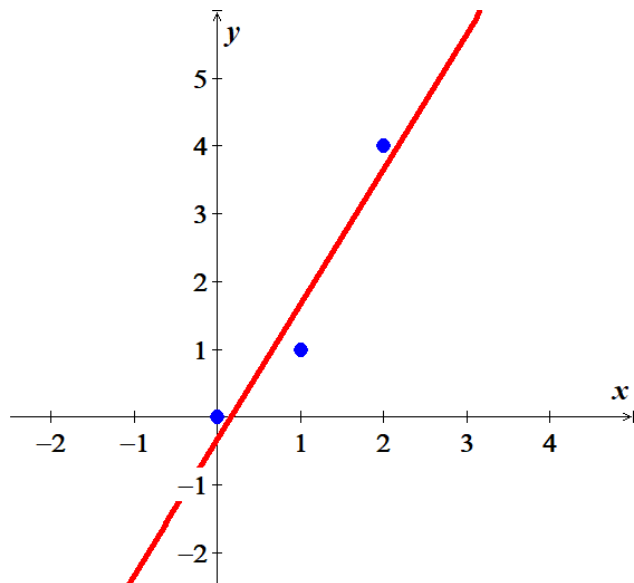
$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} -\frac{1}{3} \\ \frac{5}{3} \\ \frac{11}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$\text{Error: } \|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}}$$

$$= \frac{\sqrt{6}}{3}$$

$$\approx 0.8164966$$



\*\*\*\*\*

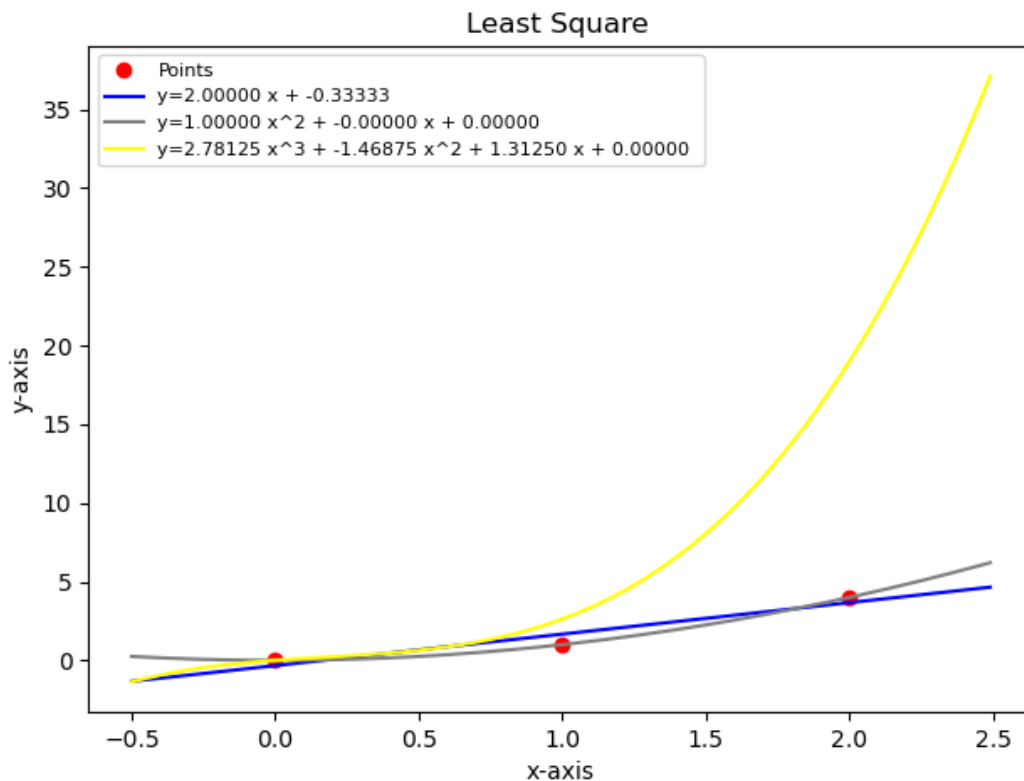
The *second order* equation:  $y = -x^2$

**Error** = 0.00000

\*\*\*\*\*

The *third order* equation:  $y = 2.78x^3 - 1.469x^2 + 1.3125x$

**Error** = 15.08776



### Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0, 0), (2, 1), (4, 1)\}$$

### Solution

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then,

$$\begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

The normal equation formula:  $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 0 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 20 & 6 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$m = \frac{\begin{vmatrix} 6 & 6 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} 20 & 6 \\ 6 & 3 \end{vmatrix}}$$

$$= \frac{6}{24}$$

$$= \frac{1}{4}$$

$$b = \frac{4}{24}$$

$$= \frac{1}{6}$$

Thus,  $y = \frac{1}{4}x + \frac{1}{6}$

$$A\vec{x} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{6} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{6} \\ \frac{2}{3} \\ \frac{7}{6} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{6} \\ \frac{2}{3} \\ \frac{7}{6} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{6} \\ \frac{1}{3} \\ -\frac{1}{6} \end{pmatrix}$$

**Error:**  $\|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{36} + \frac{1}{9} + \frac{1}{36}}$

$$= \frac{\sqrt{6}}{6}$$

$$\approx 0.40825$$

\*\*\*\*\*

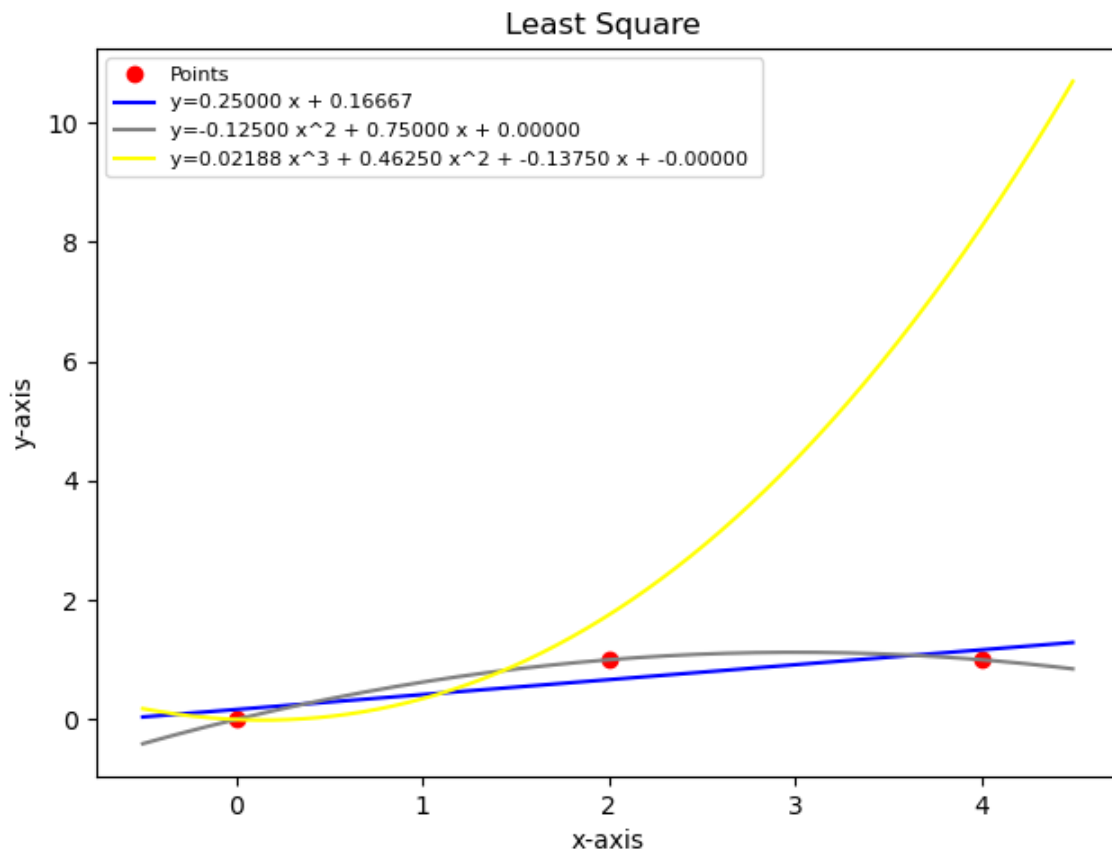
The *second* order equation:  $y = -0.125x^2 + .75x$

**Error** = 0.00000

\*\*\*\*\*

The *third* order equation:  $y = 0.01288x^3 + .4625x^2 - .1375x$

**Error** = 7.28869



**Exercise**

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-1, -1), (1, 0), (2, 4)\}$$

**Solution**

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then,

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$$

The normal equation formula:  $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \end{pmatrix}$$

$$m = \frac{21}{14}$$
$$\underline{= \frac{3}{2}} \quad |$$

$$b = \frac{0}{14}$$
$$\underline{= 0} \quad |$$

Thus,  $\underline{y = \frac{3}{2}x} \quad |$

$$A\vec{x} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ 3 \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 1 \end{pmatrix}$$

**Error:**  $\|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{4} + \frac{9}{4} + 1}$

$$= \frac{\sqrt{14}}{2}$$

$$\approx 1.87083$$

\*\*\*\*\*

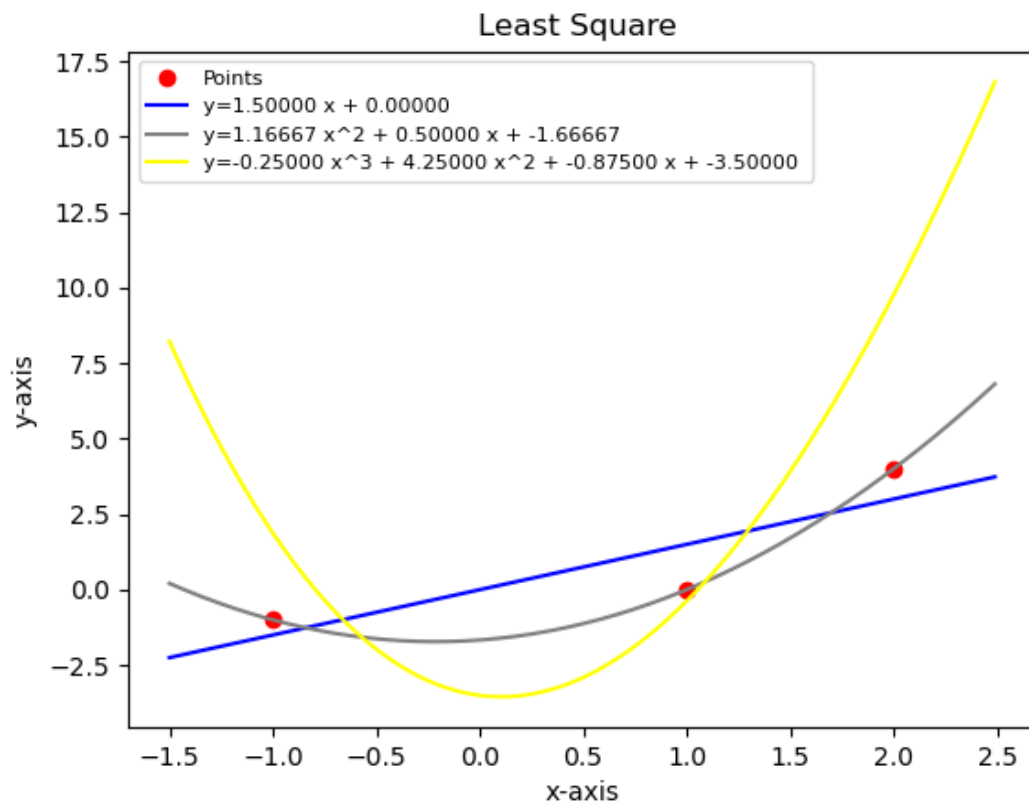
The *second order* equation:  $y = 1.1667x^2 + .5x - 1.667$

**Error** = 0.00000

\*\*\*\*\*

The *third order* equation:  $y = -.25x^3 + 4.25x^2 - .875x - 3.5$

**Error** = 6.43962



**Exercise**

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0, 1), (1, 1), (2, 2), (3, 2)\}$$

**Solution**

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then,

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

The normal equation formula:  $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}$$

$$m = \frac{8}{20}$$

$$= \frac{2}{5}$$

$$b = \frac{18}{20}$$

$$= \frac{9}{10}$$

Thus,  $y = \frac{2}{5}x + \frac{9}{10}$

$$A\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{5} \\ \frac{9}{10} \end{pmatrix}$$



$$\begin{aligned}
 &= \begin{pmatrix} \frac{9}{10} \\ \frac{13}{10} \\ \frac{17}{10} \\ \frac{21}{10} \end{pmatrix} \\
 \vec{y} - A\vec{x} &= \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} \frac{9}{10} \\ \frac{13}{10} \\ \frac{17}{10} \\ \frac{21}{10} \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{10} \\ -\frac{3}{10} \\ \frac{3}{10} \\ -\frac{1}{10} \end{pmatrix}
 \end{aligned}$$

**Error:**  $\|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{100} + \frac{9}{100} + \frac{9}{100} + \frac{1}{100}}$

$$\begin{aligned}
 &= \frac{2\sqrt{5}}{10} \\
 &\approx 0.44721
 \end{aligned}$$

\*\*\*\*\*

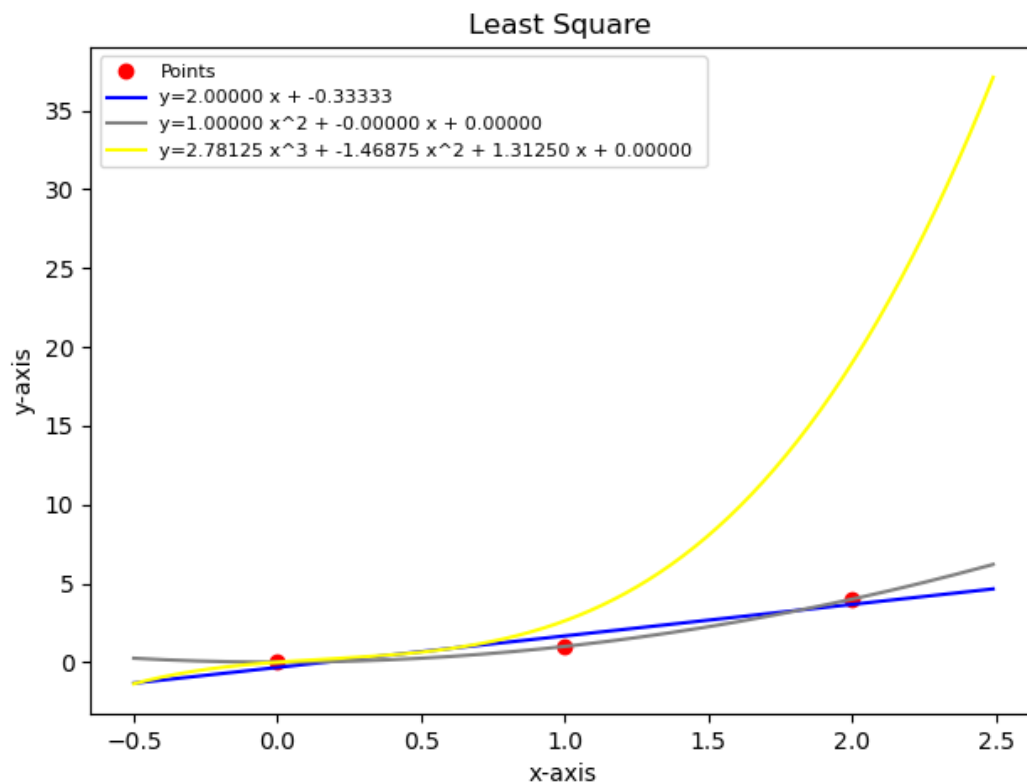
The *second order* equation:  $y = 0x^2 + .4x + .9$

**Error** = 0. 44721

\*\*\*\*\*

The *third order* equation:  $y = -.333x^3 + 1.5x^2 - 1.1667x + 1$

**Error** = 0.00000



### Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(2, 1), (5, 2), (7, 3), (8, 3)\}$$

### Solution

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then,

$$\begin{pmatrix} 2 & 1 \\ 5 & 1 \\ 7 & 1 \\ 8 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} 2 & 1 \\ 5 & 1 \\ 7 & 1 \\ 8 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}$$

The normal equation formula:  $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 2 & 5 & 7 & 8 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 5 & 1 \\ 7 & 1 \\ 8 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 2 & 5 & 7 & 8 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 142 & 22 \\ 22 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 57 \\ 9 \end{pmatrix}$$

$$m = \frac{30}{84}$$

$$= \frac{5}{14}$$

$$b = \frac{24}{84}$$

$$= \frac{4}{14}$$

$$\text{Thus, } y = \frac{5}{14}x + \frac{4}{14}$$

$$A\vec{x} = \begin{pmatrix} 2 & 1 \\ 5 & 1 \\ 7 & 1 \\ 8 & 1 \end{pmatrix} \begin{pmatrix} \frac{5}{14} \\ \frac{4}{14} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ \frac{29}{14} \\ \frac{39}{14} \\ \frac{22}{7} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ \frac{29}{14} \\ \frac{39}{14} \\ \frac{22}{7} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -\frac{1}{14} \\ \frac{3}{14} \\ -\frac{1}{7} \end{pmatrix}$$

$$\text{Error: } \|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{196} + \frac{9}{196} + \frac{1}{49}}$$

$$= \frac{\sqrt{14}}{14}$$

$$\approx 0.26726$$

\*\*\*\*\*

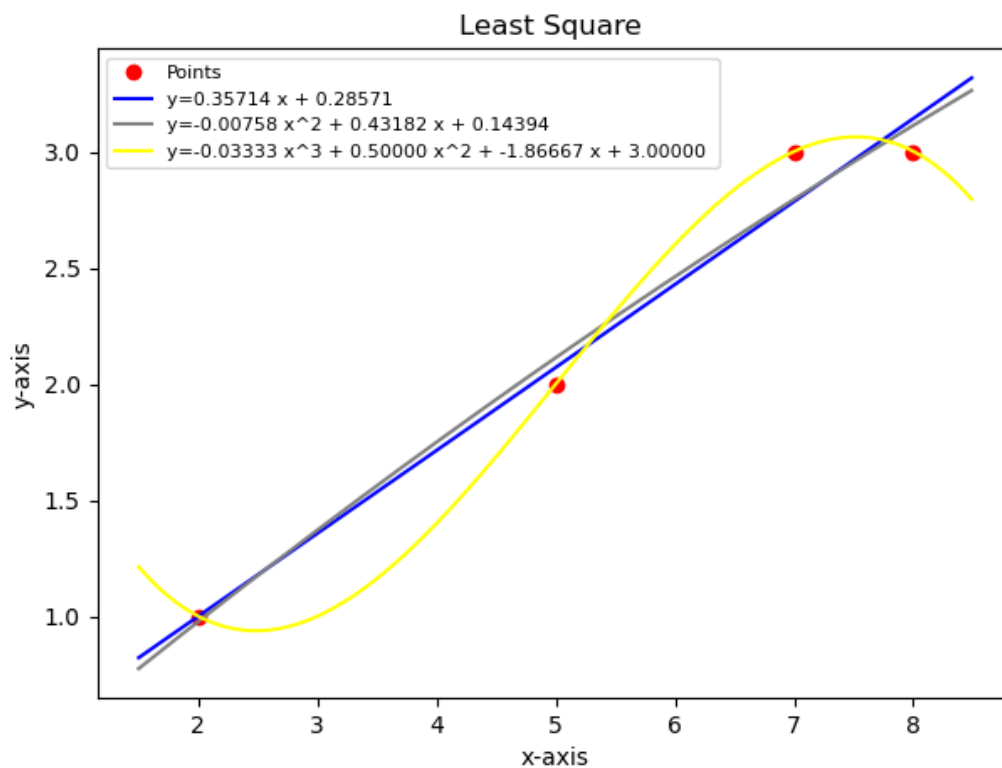
The **second order** equation:  $y = -.00758x^2 + .43182x + .14394$

**Error** = 0.26112

\*\*\*\*\*

The **third order** equation:  $y = -.0333x^3 + .5x^2 - 1.86667x + 3$

**Error** = 0.00000



### ***Exercise***

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-2, -2), (-1, 0), (0, -2), (1, 0)\}$$

### **Solution**

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then,

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

The normal equation formula:  $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -2 & -1 & 0 & 1 \\ -2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & 1 \\ -2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

$$m = \frac{8}{20}$$

$$\underline{= \frac{2}{5}}$$

$$b = -\frac{16}{20}$$

$$\underline{= -\frac{4}{5}}$$

$$\text{Thus, } \underline{y = \frac{2}{5}x - \frac{4}{5}}$$

$$A\vec{x} = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{5} \\ -\frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{8}{5} \\ -\frac{6}{5} \\ -\frac{4}{5} \\ -\frac{2}{5} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} -\frac{8}{5} \\ -\frac{6}{5} \\ -\frac{4}{5} \\ -\frac{2}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{5} \\ \frac{6}{5} \\ -\frac{6}{5} \\ \frac{2}{5} \end{pmatrix}$$

**Error:**  $\|\vec{y} - A\vec{x}\| = \sqrt{\frac{4}{25} + \frac{36}{25} + \frac{36}{25} + \frac{4}{25}}$   
 $= \frac{4\sqrt{5}}{5}$   
 $\approx 1.78885$

\*\*\*\*\*

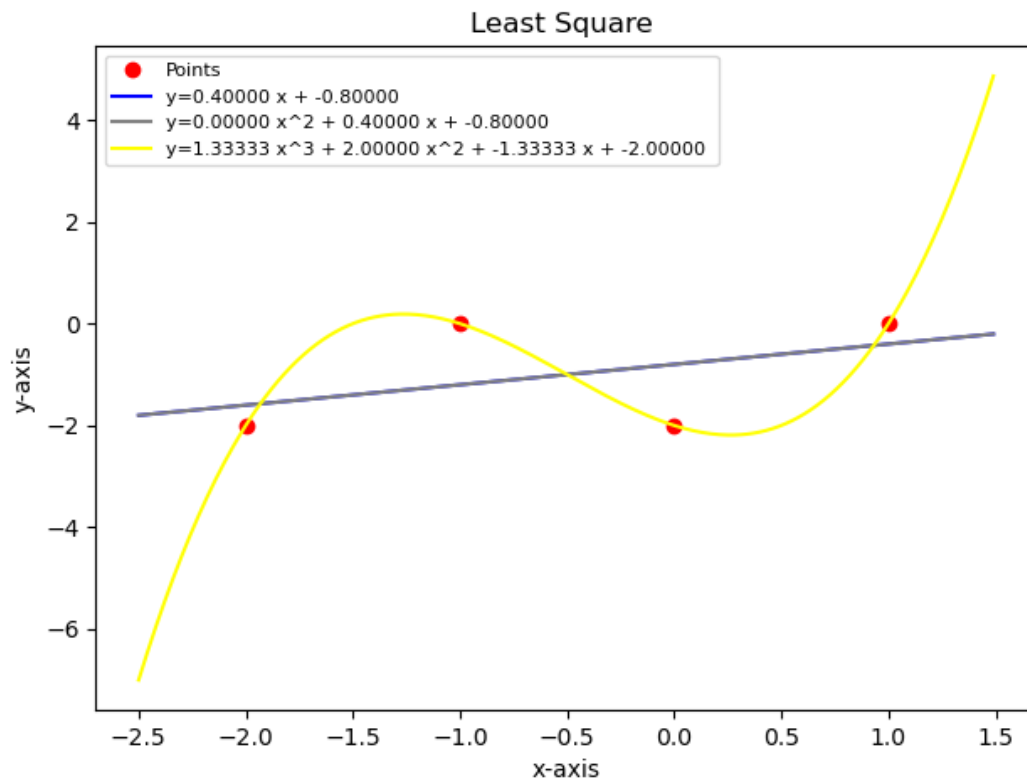
The *second order* equation:  $y = 0x^2 + .4x + .8$

**Error** = 1.78885

\*\*\*\*\*

The *third order* equation:  $y = 1.333x^3 + 2x^2 - 1.333x - 2$

**Error** = 0.00000



**Exercise**

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-2, 0), (-1, 1), (0, 1), (1, 2)\}$$

**Solution**

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then,

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

The normal equation formula:  $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$m = \frac{12}{20}$$

$$\underline{= \frac{3}{5}}$$

$$b = \frac{26}{20}$$

$$\underline{= \frac{13}{10}}$$

Thus,  $\underline{y = \frac{3}{5}x + \frac{13}{10}} \quad | \quad y = 0.60000x + 1.3000$

$$A\vec{x} = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{5} \\ \frac{13}{10} \end{pmatrix}$$

$$\begin{aligned} &= \begin{pmatrix} -\frac{1}{10} \\ \frac{7}{10} \\ \frac{13}{10} \\ \frac{19}{10} \end{pmatrix} \\ \vec{y} - A\vec{x} &= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -\frac{1}{10} \\ \frac{7}{10} \\ \frac{13}{10} \\ \frac{19}{10} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{10} \\ \frac{3}{10} \\ -\frac{3}{10} \\ \frac{1}{10} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \textbf{Error:} \quad \|\vec{y} - A\vec{x}\| &= \sqrt{\frac{1}{100} + \frac{9}{100} + \frac{9}{100} + \frac{1}{100}} \\ &= \frac{2\sqrt{5}}{10} \\ &\approx 0.44721 \end{aligned}$$

\*\*\*\*\*

The *second order* equation:  $y = 0x^2 + .6x + 1.3$

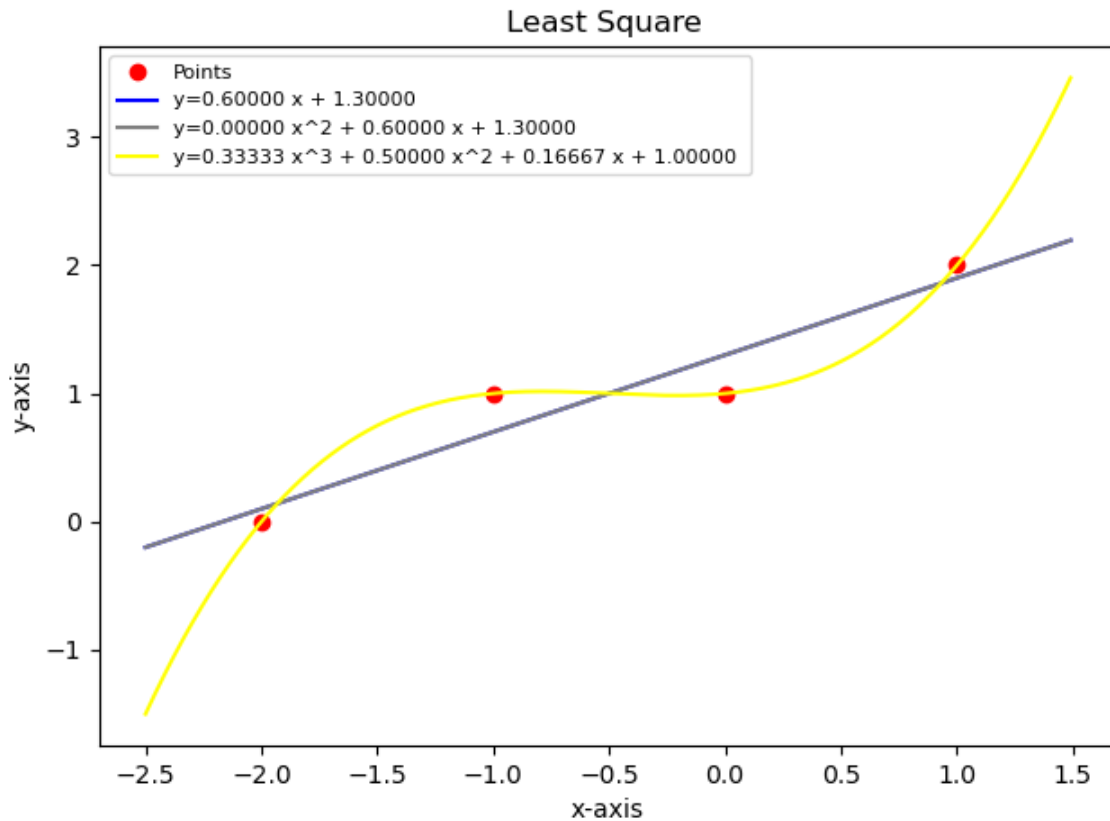
**Error** = 0.44721

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The *third order* equation:  $y = 0.333x^3 + .5x^2 + 1.6667x + 1$

**Error** = 0.00000





### Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0, 1), (1, 3), (2, 4), (3, 4)\}$$

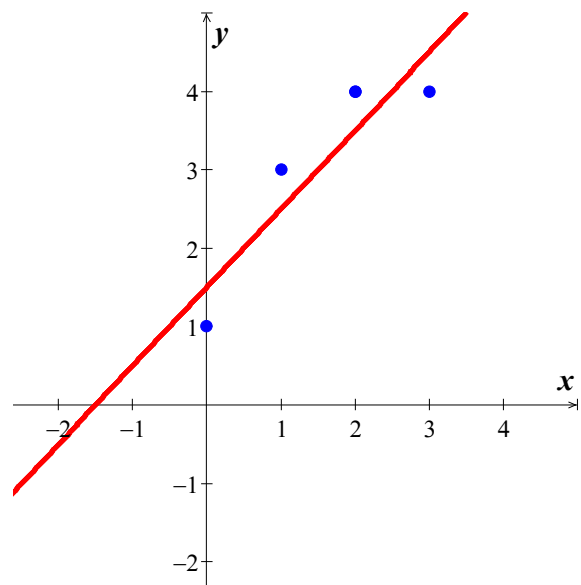
### Solution

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

The normal equation:  $A^T A \vec{x} = A^T \vec{y}$



$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 23 \\ 12 \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} m \\ b \end{pmatrix} &= \frac{1}{20} \begin{pmatrix} 4 & -6 \\ -6 & 14 \end{pmatrix} \begin{pmatrix} 23 \\ 12 \end{pmatrix} & \quad X = A^{-1}B \\ &= \frac{1}{20} \begin{pmatrix} 20 \\ 30 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix} \end{aligned}$$

We have:  $m=1$  and  $b=\frac{3}{2}$ .

Thus,  $\underline{y = x + \frac{3}{2}}$

$$\begin{aligned} A\vec{x} &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ \frac{7}{2} \\ \frac{9}{2} \end{pmatrix} \end{aligned}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ \frac{7}{2} \\ \frac{9}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

**Error:**

$$\|\vec{y} - A\vec{x}\| = \sqrt{4\left(\frac{1}{4}\right)}$$

$$= 1$$

\*\*\*\*\*

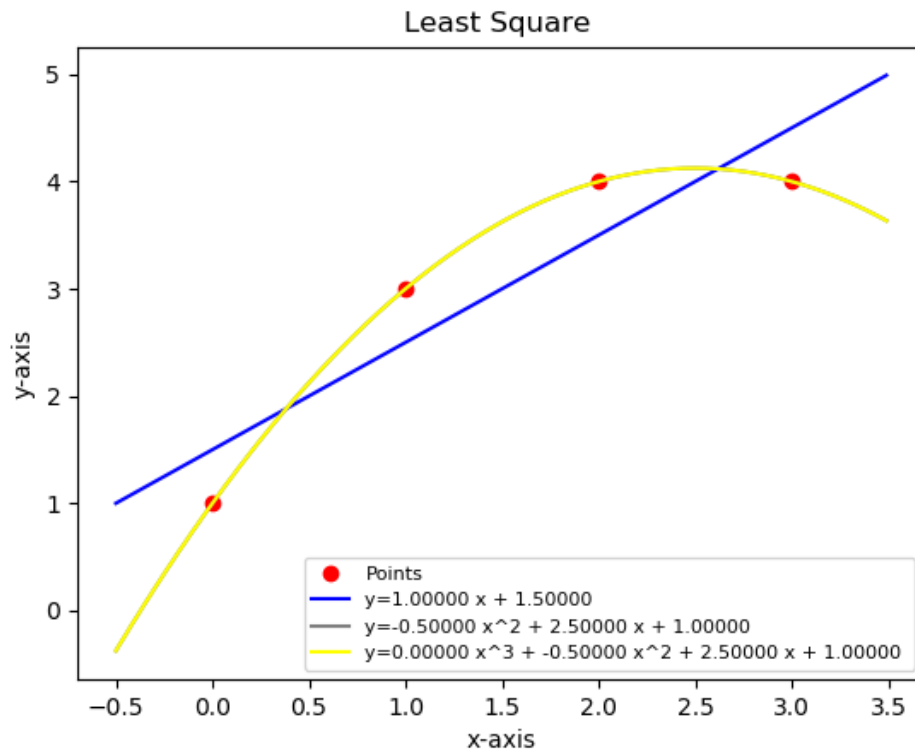
The *second* order equation:  $y = -0.50x^2 + 2.5x + 1.0$

**Error** = 0.00000

\*\*\*\*\*

The *third* order equation:  $y = 0.0x^3 - 0.5x^2 + 2.5x + 1$

**Error** = 0.00000



**Exercise**

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(2, 3), (3, 2), (5, 1), (6, 0)\}$$

**Solution**

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

The normal equation:  $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 74 & 16 \\ 16 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 17 \\ 6 \end{pmatrix}$$

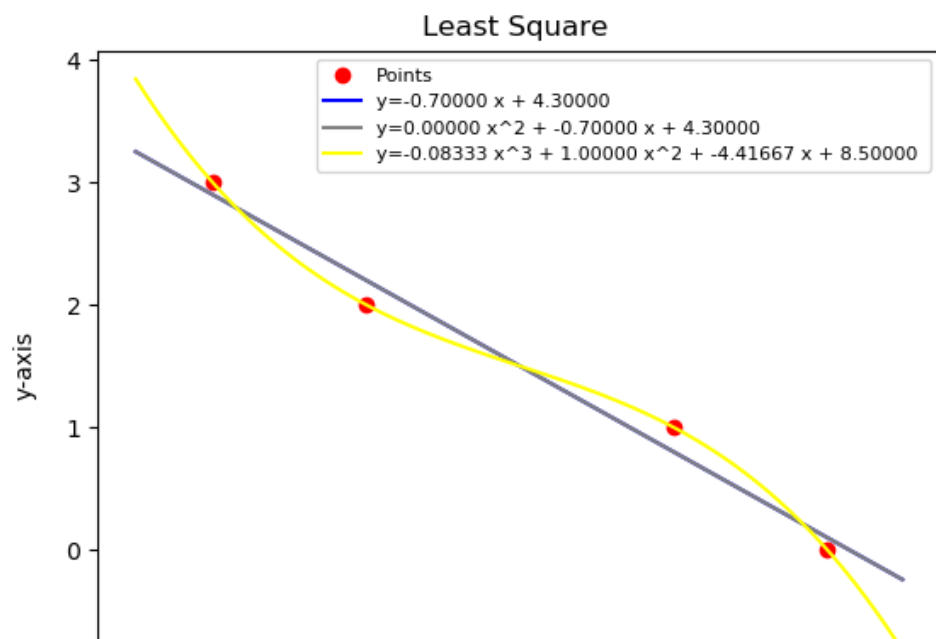
$$\Delta = \begin{vmatrix} 74 & 16 \\ 16 & 4 \end{vmatrix} = 40 \quad \Delta_m = \begin{vmatrix} 17 & 16 \\ 6 & 4 \end{vmatrix} = -28 \quad \Delta_b = \begin{vmatrix} 74 & 17 \\ 16 & 6 \end{vmatrix} = 172$$

$$m = -\frac{28}{40} = -\frac{7}{10}$$

$$b = \frac{172}{40} = \frac{43}{10}$$

$$\text{Thus, } y = -\frac{7}{10}x + \frac{43}{10}$$

$$A\vec{x} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} -\frac{7}{10} \\ \frac{43}{10} \end{pmatrix}$$



$$\begin{aligned}
 &= \begin{pmatrix} \frac{29}{10} \\ \frac{22}{10} \\ \frac{8}{10} \\ \frac{1}{10} \end{pmatrix} \\
 \vec{y} - A\vec{x} &= \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{29}{10} \\ \frac{22}{10} \\ \frac{8}{10} \\ \frac{1}{10} \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{10} \\ -\frac{1}{5} \\ \frac{1}{5} \\ -\frac{1}{10} \end{pmatrix}
 \end{aligned}$$

**Error:**

$$\begin{aligned}
 \|\vec{y} - A\vec{x}\| &= \sqrt{2\left(\frac{1}{100} + \frac{1}{25}\right)} \\
 &= \frac{\sqrt{10}}{10} \\
 &= 0.31623
 \end{aligned}$$

\*\*\*\*\*

The *second order* equation:  $y = 0.0x^2 - 0.7x + 4.3$

**Error** = 0.31623

\*\*\*\*\*

The *third order* equation:  $y = -0.08333x^3 + x^2 - 4.41667x + 8.5$

**Error** = 0.00000

**Exercise**

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-1, 0), (0, 1), (1, 2), (2, 4)\}$$

**Solution**

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

The normal equation:  $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 10 \\ 7 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 6 & 2 \\ 2 & 4 \end{vmatrix} = 20 \quad \Delta_m = \begin{vmatrix} 10 & 2 \\ 7 & 4 \end{vmatrix} = 26 \quad \Delta_b = \begin{vmatrix} 6 & 10 \\ 2 & 7 \end{vmatrix} = 22$$

$$m = \frac{26}{20} = \frac{13}{10}$$

$$b = \frac{22}{20} = \frac{11}{10}$$

$$\text{Thus, } y = \frac{13}{10}x + \frac{11}{10}$$

$$A\vec{x} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{13}{10} \\ \frac{11}{10} \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} -\frac{1}{5} \\ \frac{11}{10} \\ \frac{12}{5} \\ \frac{37}{10} \end{pmatrix} \\
\vec{y} - A\vec{x} &= \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} -\frac{1}{5} \\ \frac{11}{10} \\ \frac{12}{5} \\ \frac{37}{10} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{5} \\ -\frac{1}{10} \\ -\frac{2}{5} \\ \frac{3}{10} \end{pmatrix}
\end{aligned}$$

**Error:**

$$\begin{aligned}
\|\vec{y} - A\vec{x}\| &= \sqrt{\frac{1}{25} + \frac{1}{100} + \frac{4}{25} + \frac{9}{100}} \\
&= \sqrt{\frac{4+1+16+9}{100}} \\
&= \frac{\sqrt{30}}{10} \\
&= 0.54772
\end{aligned}$$

\*\*\*\*\*

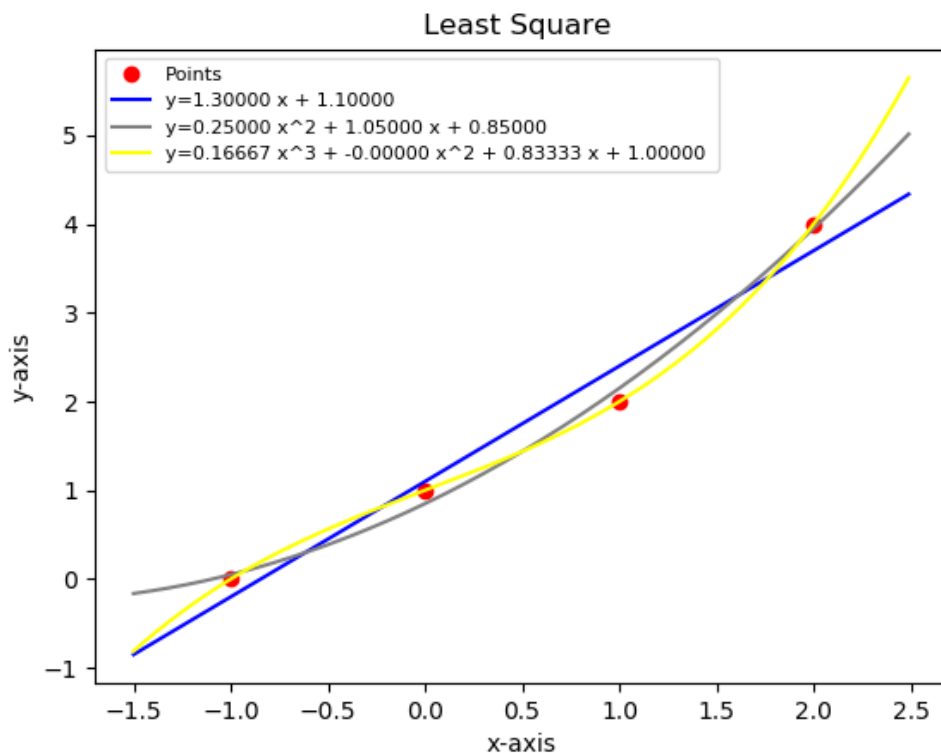
The *second order* equation:  $y = 0.25x^2 + 1.05x + 0.85$

**Error** = 0.22361

\*\*\*\*\*

The *third order* equation:  $y = 0.16667x^3 + 0.82222x + 1$

**Error** = 0.00000



### Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(1, 0), (2, 1), (4, 2), (5, 3)\}$$

### Solution

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

The normal equation:  $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$



$$\begin{pmatrix} 46 & 12 \\ 12 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 25 \\ 6 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 46 & 12 \\ 12 & 4 \end{vmatrix} = 40 \quad \Delta_m = \begin{vmatrix} 25 & 12 \\ 6 & 4 \end{vmatrix} = 28 \quad \Delta_b = \begin{vmatrix} 46 & 25 \\ 12 & 6 \end{vmatrix} = -24$$

$$m = \frac{28}{40} = \frac{7}{10}$$

$$b = -\frac{24}{40} = -\frac{3}{5}$$

$$\text{Thus, } y = \frac{7}{10}x - \frac{3}{5}$$

$$A\vec{x} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} \frac{7}{10} \\ -\frac{3}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{10} \\ \frac{4}{5} \\ \frac{11}{5} \\ \frac{29}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} \frac{1}{10} \\ \frac{4}{5} \\ \frac{11}{5} \\ \frac{29}{10} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{10} \\ \frac{1}{5} \\ -\frac{1}{5} \\ \frac{1}{10} \end{pmatrix}$$

**Error:**

$$\|\vec{y} - A\vec{x}\| = \sqrt{2\left(\frac{1}{100} + \frac{1}{25}\right)}$$

$$= \frac{\sqrt{10}}{10}$$

$$= 0.31623$$

\*\*\*\*\*

The *second order* equation:

$$y = 0.0x^2 + 0.7x - .6$$

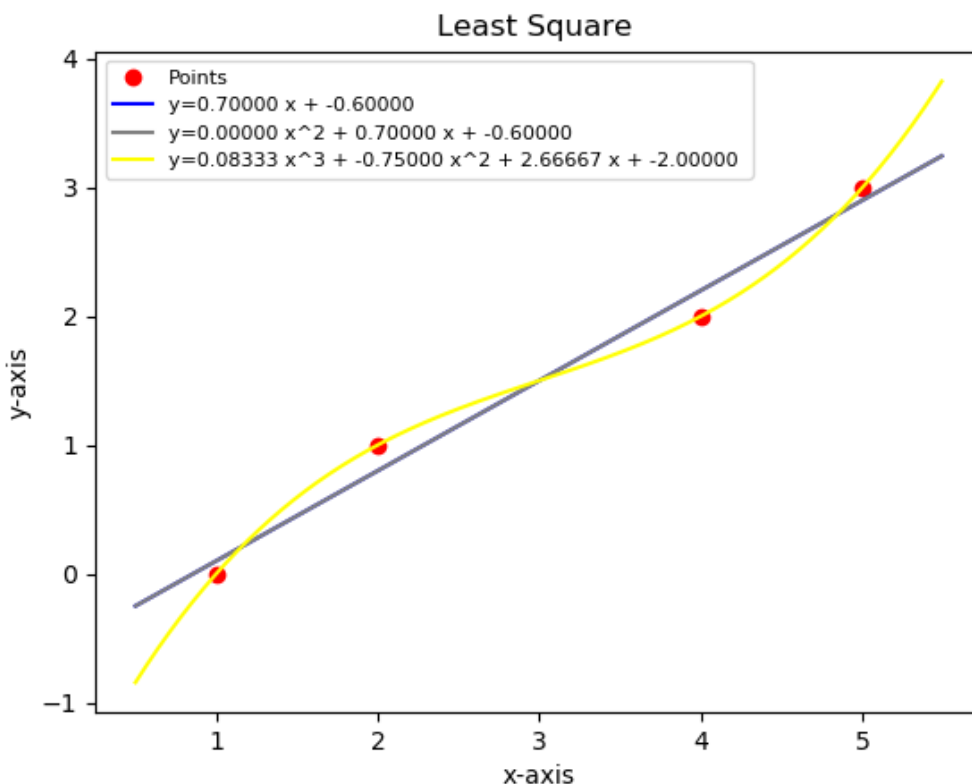
$$\text{Error} = 0.31623$$

\*\*\*\*\*

The *third order* equation:

$$y = 0.08333x^3 - 0.75x^2 + 2.66667x - 2$$

$$\text{Error} = 0.00000$$



### Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(1, 5), (2, 4), (3, 1), (4, 1), (5, -1)\}$$

### Solution

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{where } A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

The normal equation:  $A^T A \vec{x} = A^T \vec{y}$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

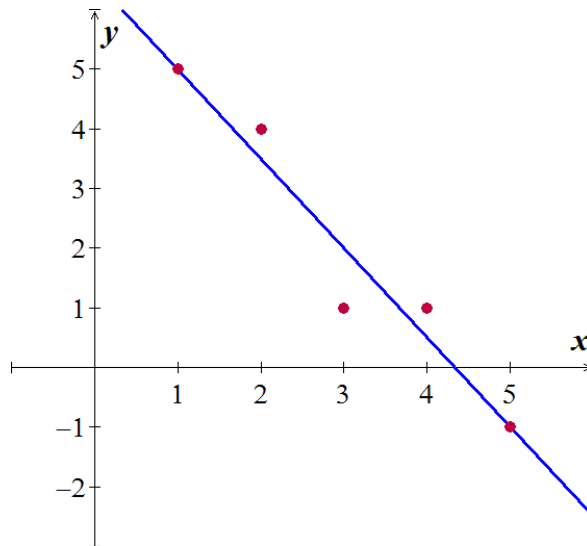
$$\begin{bmatrix} 55 & 15 \\ 15 & 5 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}$$

$$m = \frac{\begin{vmatrix} 15 & 15 \\ 10 & 5 \end{vmatrix}}{\begin{vmatrix} 55 & 15 \\ 15 & 5 \end{vmatrix}} = \frac{-75}{50} = -\frac{3}{2}$$

$$b = \frac{\begin{vmatrix} 55 & 15 \\ 15 & 10 \end{vmatrix}}{50} = \frac{325}{50} = \frac{13}{2}$$

Thus,  $y = -\frac{3}{2}x + \frac{13}{2}$  or  $y = -1.5x + 6.5$

$$A\vec{x} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ \frac{13}{2} \end{pmatrix}$$



$$= \begin{pmatrix} 5 \\ \frac{7}{2} \\ 2 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ \frac{7}{2} \\ 2 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{1}{2} \\ -1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

**Error:**

$$\begin{aligned} \|\vec{y} - A\vec{x}\| &= \sqrt{\frac{1}{4} + 1 + \frac{1}{4}} \\ &= \frac{\sqrt{6}}{2} \\ &\approx 1.224745 \end{aligned}$$

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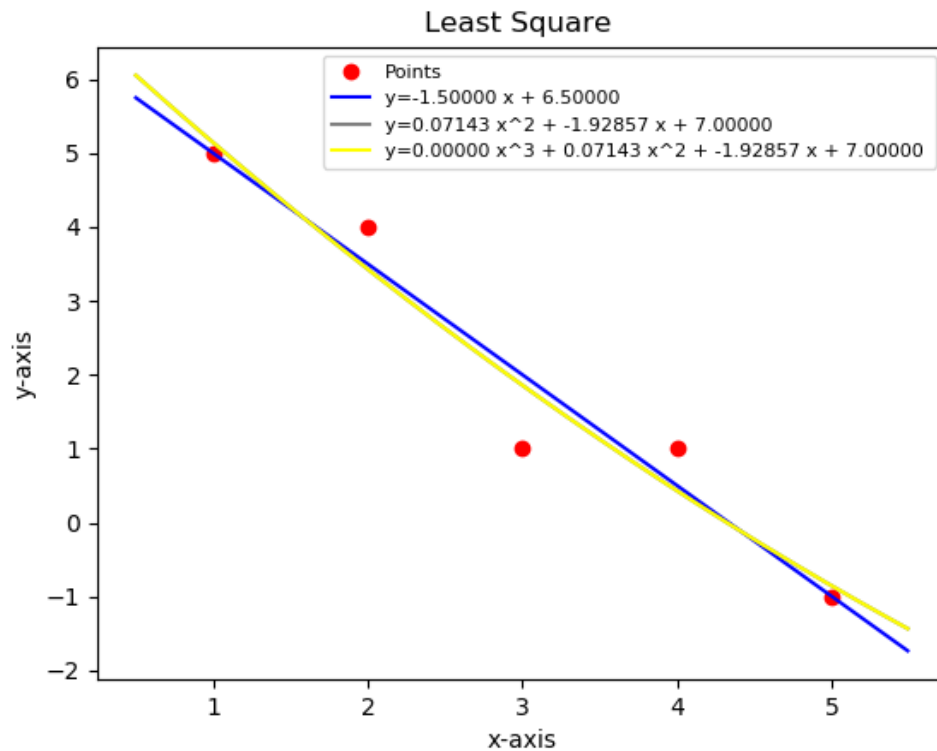
The **second order** equation:  $y = 0.07143x^2 - 1.92857x + 7$

**Error** = 1.19523

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The **third order** equation:  $y = 0.0x^3 + 0.07143x^2 - 1.92857x + 7$

**Error** = 1.19523



### Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-2, 0), (-1, 0), (0, 1), (1, 3), (2, 5)\}$$

### Solution

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

The normal equation:  $A^T A \vec{x} = A^T \vec{y}$

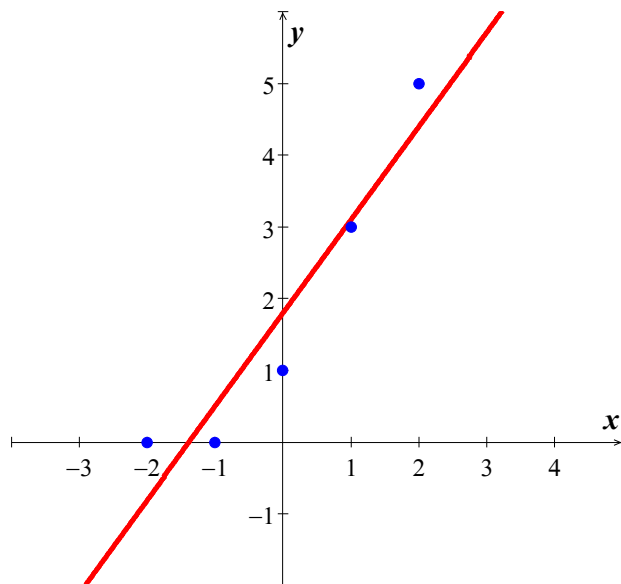
$$\begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 13 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{50} \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} 13 \\ 9 \end{pmatrix} = \begin{pmatrix} \frac{13}{10} \\ \frac{9}{5} \end{pmatrix}$$

We have:  $m = 1.3$  and  $b = 1.8$

Thus,  $y = \frac{13}{10}x + \frac{9}{5}$



$$A\vec{x} = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{13}{10} \\ \frac{9}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{5} \\ \frac{1}{2} \\ \frac{9}{5} \\ \frac{31}{10} \\ \frac{22}{5} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix} - \begin{pmatrix} -\frac{4}{5} \\ \frac{1}{2} \\ \frac{9}{5} \\ \frac{31}{10} \\ \frac{22}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{5} \\ -\frac{1}{2} \\ -\frac{4}{5} \\ -\frac{1}{10} \\ \frac{3}{5} \end{pmatrix}$$

**Error:**  $\|\vec{y} - A\vec{x}\| = \sqrt{\frac{16}{25} + \frac{1}{4} + \frac{16}{25} + \frac{1}{100} + \frac{9}{25}}$

$$= \sqrt{\frac{41}{25} + \frac{26}{100}}$$

$$= \frac{\sqrt{190}}{10}$$

$$\approx 1.37840$$

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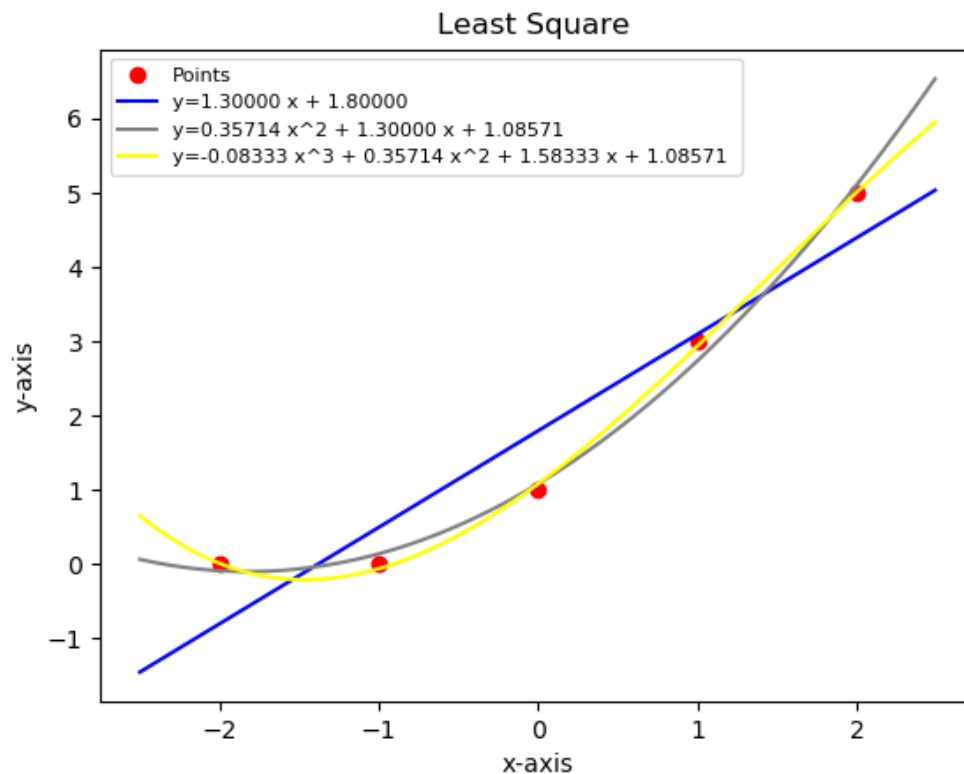
The *second order* equation:  $y = 0.35714x^2 + 1.30x + 1.08571$

**Error** = 0.33806

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The *third order* equation:  $y = -0.08333x^3 + 0.35714x^2 + 1.58333x + 1.08571$

**Error** = 0.11952



**Exercise**

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-5, 10), (-1, 8), (3, 6), (7, 4), (5, 5)\}$$

**Solution**

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -5 & 1 \\ -1 & 1 \\ 3 & 1 \\ 7 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 5 \end{pmatrix} \quad \text{where } A = \begin{pmatrix} -5 & 1 \\ -1 & 1 \\ 3 & 1 \\ 7 & 1 \\ 5 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 5 \end{pmatrix}$$

The normal equation:  $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -5 & -1 & 3 & 7 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -5 & 1 \\ -1 & 1 \\ 3 & 1 \\ 7 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -5 & -1 & 3 & 7 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 109 & 9 \\ 9 & 5 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 13 \\ 33 \end{pmatrix}$$

$$m = \frac{232}{464}$$

$$= -\frac{1}{2}$$

$$b = \frac{3,480}{464}$$

$$= \frac{15}{2}$$

$$\text{Thus, } y = -\frac{1}{2}x + \frac{15}{2}$$

$$A\vec{x} = \begin{pmatrix} -5 & 1 \\ -1 & 1 \\ 3 & 1 \\ 7 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ \frac{15}{2} \end{pmatrix}$$

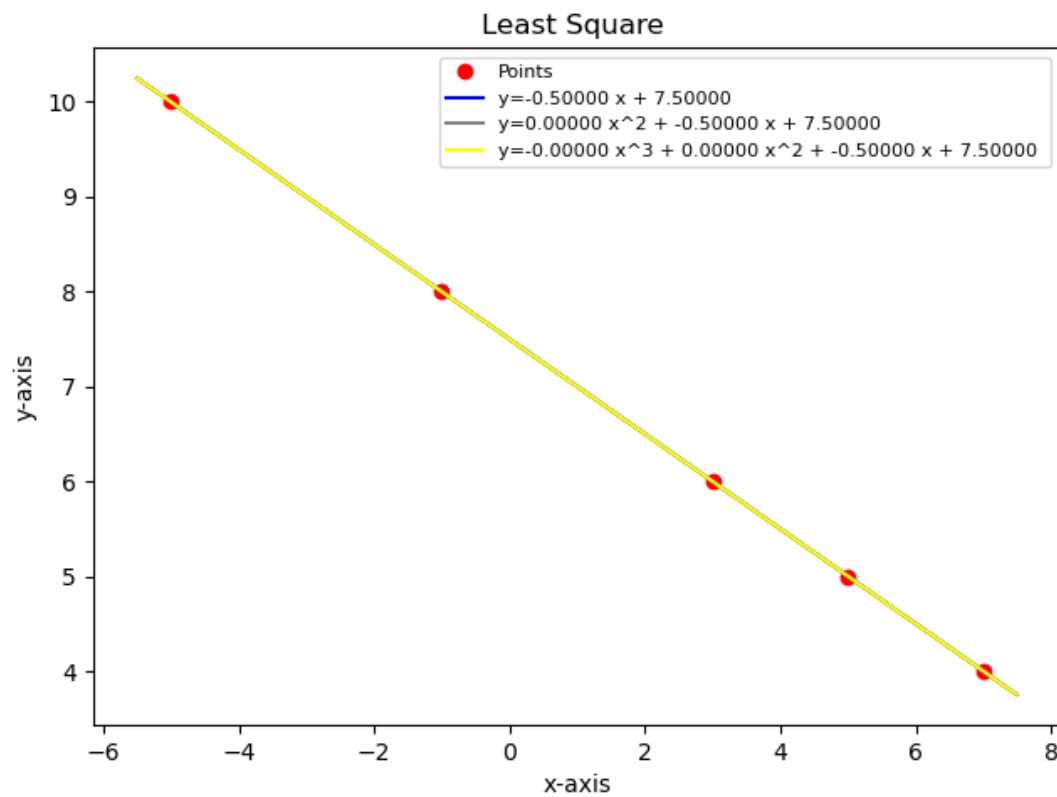


$$= \begin{pmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 5 \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

**Error:**  $\|\vec{y} - A\vec{x}\| = 0$



**Exercise**

Find the orthogonal projection of the vector  $\vec{u}$  on the subspace of  $\mathbb{R}^4$  spanned by the vectors

$$\vec{u} = (-3, -3, 8, 9); \quad \vec{v}_1 = (3, 1, 0, 1), \quad \vec{v}_2 = (1, 2, 1, 1), \quad \vec{v}_3 = (-1, 0, 2, -1)$$

**Solution**

$$\text{Let } A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{pmatrix}$$

$$A^T \vec{u} = \begin{pmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -3 \\ 8 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 8 \\ 10 \end{pmatrix}$$

The normal solution is  $A^T A \vec{x} = A^T \vec{u}$

$$\begin{pmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ 10 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{vmatrix} = 134 \quad \Delta_1 = \begin{vmatrix} -3 & 6 & -4 \\ 8 & 7 & 0 \\ 10 & 0 & 6 \end{vmatrix} = -134$$

$$\Delta_2 = \begin{vmatrix} 11 & -3 & -4 \\ 6 & 8 & 0 \\ -4 & 10 & 6 \end{vmatrix} = 268$$

$$\underline{x_1 = \frac{-134}{134} = -1} \quad \underline{x_2 = \frac{268}{134} = 2} \quad \underline{x_3 = \frac{134}{134} = 1}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

So  $\text{proj}_W \vec{u} = A\vec{x}$

$$= \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 3 \\ 4 \\ 0 \end{pmatrix}$$

$$\underline{\text{proj}_W \vec{u} = (-2, 3, 4, 0)}$$

### Exercise

Find the orthogonal projection of the vector  $\vec{u}$  on the subspace of  $\mathbb{R}^4$  spanned by the vectors

$$\vec{u} = (6, 3, 9, 6); \quad \vec{v}_1 = (2, 1, 1, 1), \quad \vec{v}_2 = (1, 0, 1, 1), \quad \vec{v}_3 = (-2, -1, 0, -1)$$

### Solution

$$\text{Let } A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix}$$

$$A^T \vec{u} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 9 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 30 \\ 21 \\ -21 \end{pmatrix}$$

The normal solution is  $A^T A \vec{x} = A^T \vec{u}$

$$\begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 30 \\ 21 \\ -21 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{vmatrix} = 3$$

$$\Delta_1 = \begin{vmatrix} 30 & 4 & -6 \\ 21 & 3 & -3 \\ -21 & -3 & 6 \end{vmatrix} = 18$$

$$\Delta_2 = \begin{vmatrix} 7 & 30 & -6 \\ 4 & 21 & -3 \\ -6 & -21 & 6 \end{vmatrix} = 9$$

$$\underline{x_1 = \frac{18}{3} = 6} \quad \underline{x_2 = \frac{9}{3} = 3} \quad \underline{x_3 = 4}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

So  $\text{proj}_W \vec{u} = A \vec{x}$

$$= \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 2 \\ 9 \\ 5 \end{pmatrix}$$

$$\underline{\text{proj}_W \vec{v} = (7, 2, 9, 5)}$$

### Exercise

Find the orthogonal projection of the vector  $\vec{u}$  on the subspace of  $\mathbb{R}^4$  spanned by the vectors

$$\vec{u} = (-2, 0, 2, 4); \quad \vec{v}_1 = (1, 1, 3, 0), \quad \vec{v}_2 = (-2, -1, -2, 1), \quad \vec{v}_3 = (-3, -1, 1, 3)$$

### Solution

$$\text{Let } A = \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{pmatrix}$$

$$A^T \vec{u} = \begin{pmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix}$$

The normal solution is  $A^T A \vec{x} = A^T \vec{u}$

$$\begin{pmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{vmatrix} = 10 \quad \Delta_1 = \begin{vmatrix} 4 & -9 & -1 \\ 4 & 10 & 8 \\ 20 & 8 & 20 \end{vmatrix} = -8 \quad \Delta_2 = \begin{vmatrix} 11 & 4 & -1 \\ -9 & 4 & 8 \\ -1 & 20 & 20 \end{vmatrix} = -16$$

$$\underline{x_1 = \frac{-8}{10} = -\frac{4}{5}} \quad \underline{x_2 = \frac{-16}{10} = -\frac{8}{5}} \quad \underline{x_3 = \frac{8}{5}}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} \\ -\frac{8}{5} \\ \frac{8}{5} \end{pmatrix}$$

$$\text{proj}_W \vec{u} = A \vec{x}$$

$$= \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} \\ -\frac{8}{5} \\ \frac{8}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{12}{5} \\ -\frac{4}{5} \\ \frac{12}{5} \\ \frac{16}{5} \end{pmatrix}$$

$$\underline{\text{proj}_W \vec{u} = \left( -\frac{12}{5}, -\frac{4}{5}, \frac{12}{5}, \frac{16}{5} \right)}$$

### Exercise

For the given matrix  $A$  and  $\vec{y}$

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 2 \\ 0 \\ 11 \end{pmatrix}$$

- Find the least-squares solution of the linear system  $A\vec{x} = \vec{y}$
- Find the orthogonal projection of  $\vec{y}$  on the column space of  $A$
- Find the **error vector** and the **error**

### Solution

a) Let  $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$

$$A^T A \vec{x} = A^T \vec{y}$$

$$\begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 11 \end{pmatrix}$$

$$\begin{pmatrix} 17 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 19 \\ 11 \end{pmatrix}$$

$$m = \frac{84}{84}$$

$$\underline{\underline{= 1}}$$

$$b = \frac{168}{84}$$

$$= 2$$

Thus  $y = x + 2$

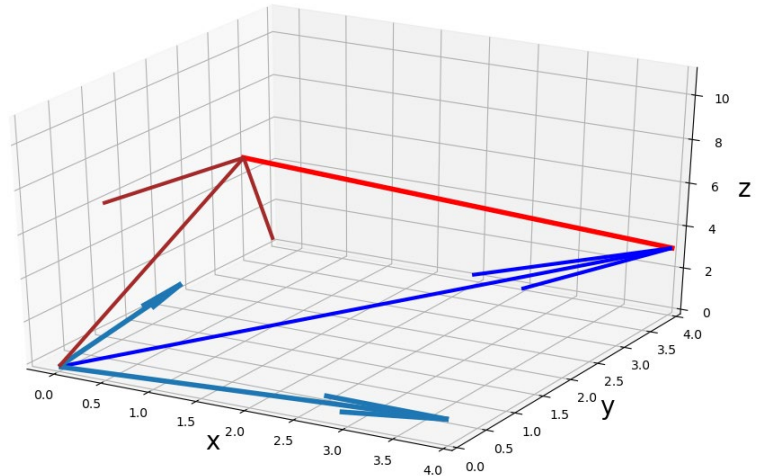
b) The orthogonal projection of  $\vec{y}$  on the column space of  $A$

$$A\vec{x} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix}$$

$$c) \quad \vec{y} - A\vec{x} = \begin{pmatrix} 2 \\ 0 \\ 11 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ -4 \\ 8 \end{pmatrix}$$



The **error**:  $\|\vec{y} - A\vec{x}\| = \sqrt{4 + 16 + 64}$

$$= 2\sqrt{21}$$

$$\approx 9.16515$$

### Exercise

For the given matrix  $A$  and  $\vec{y}$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -3 \\ -1 & 3 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

- Find the least-squares solution of the linear system  $A\vec{x} = \vec{y}$
- Find the orthogonal projection of  $\vec{y}$  on the column space of  $A$
- Find the **error vector** and the **error**

### Solution

a) Let  $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$

$$A^T A \vec{x} = A^T \vec{y}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -7 \\ -7 & 22 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \end{pmatrix}$$

$$m = \frac{165}{83}$$

$$b = \frac{94}{83}$$

$$\text{Thus } y = \frac{165}{83}x + \frac{94}{83}$$

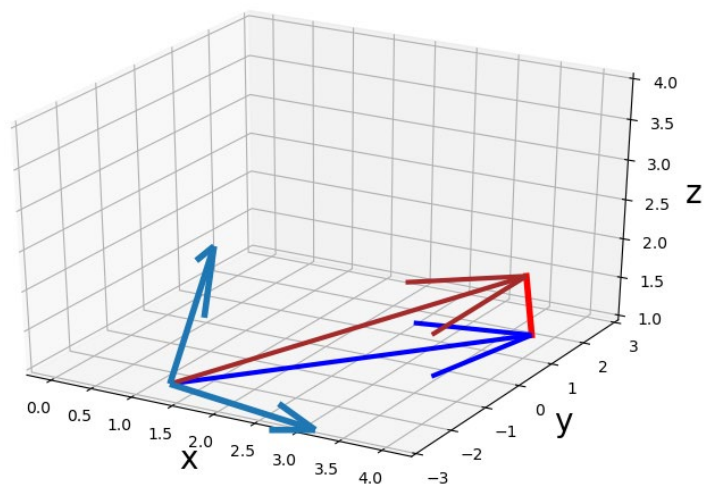
b) The orthogonal projection of  $\vec{y}$  on the column space of  $A$

$$A\vec{x} = \begin{pmatrix} 1 & 2 \\ 2 & -3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{165}{83} \\ \frac{94}{83} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{353}{83} \\ \frac{48}{83} \\ \frac{117}{83} \end{pmatrix}$$

$$c) \vec{y} - A\vec{x} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} \frac{353}{83} \\ \frac{48}{83} \\ \frac{117}{83} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{21}{83} \\ -\frac{35}{83} \\ \frac{49}{83} \end{pmatrix}$$



$$\text{The error: } \|\vec{y} - A\vec{x}\| = \frac{1}{83} \sqrt{441 + 1,225 + 2,401}$$

$$= \frac{\sqrt{4,067}}{83}$$

$$= \frac{7\sqrt{83}}{83}$$

$$\approx 0.76835$$



**Exercise**

For the given matrix  $A$  and  $\vec{y}$

$$A = \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix}$$

- Find the least-squares solution of the linear system  $A\vec{x} = \vec{y}$
- Find the orthogonal projection of  $\vec{y}$  on the column space of  $A$
- Find the **error vector** and the **error**

**Solution**

a) Let  $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$

$$A^T A \vec{x} = A^T \vec{y}$$

$$\begin{pmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 12 & 8 \\ 8 & 10 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -24 \\ -2 \end{pmatrix}$$

$$m = -\frac{224}{56}$$

$$= -4$$

$$b = \frac{168}{56}$$

$$= 3$$

Thus  $y = -4x + 3$

- b) The orthogonal projection of  $\vec{y}$  on the column space of  $A$

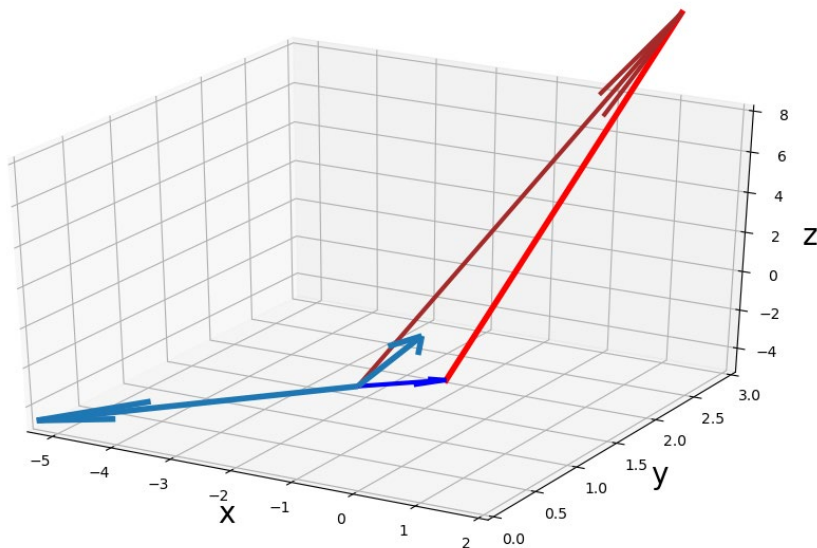
$$A\vec{x} = \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix}$$

c)  $\vec{y} - A\vec{x} = \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix} - \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The **error**:  $\|\vec{y} - A\vec{x}\| = 0$



### Exercise

For the given matrix  $A$  and  $\vec{y}$

$$A = \begin{pmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 6 \end{pmatrix}$$

- Find the least-squares solution of the linear system  $A\vec{x} = \vec{y}$
- Find the orthogonal projection of  $\vec{y}$  on the column space of  $A$
- Find the **error vector** and the **error**

### Solution

a) Let  $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$

$$A^T A \vec{x} = A^T \vec{y}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -6 & -2 & 1 & 7 \end{pmatrix} \begin{pmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -6 & -2 & 1 & 7 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 0 \\ 0 & 90 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 8 \\ 45 \end{pmatrix}$$

$$\begin{aligned} m &= \frac{720}{360} \\ &= 2 \end{aligned}$$

$$\begin{aligned} b &= \frac{180}{360} \\ &= \frac{1}{2} \end{aligned}$$

$$\text{Thus } \underline{y = 2x + \frac{1}{2}}$$

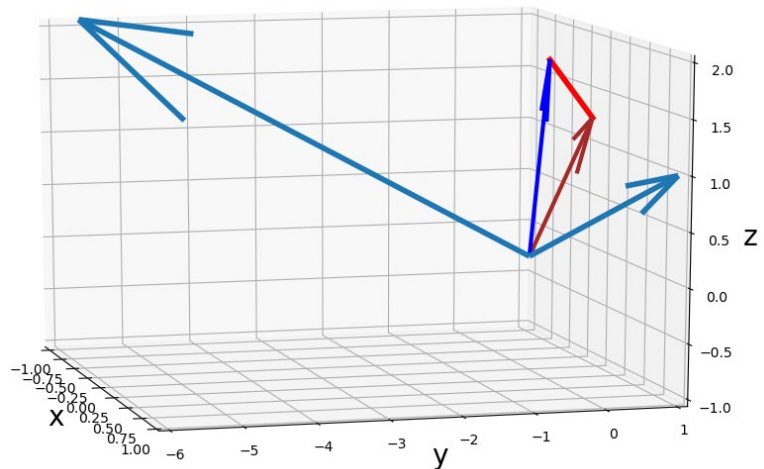
**b)** The orthogonal projection of  $\vec{y}$  on the column space of  $A$

$$\begin{aligned} A\vec{x} &= \begin{pmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 1 \\ 3 \\ \frac{11}{2} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{c) } \vec{y} - A\vec{x} &= \begin{pmatrix} -1 \\ 2 \\ 1 \\ 6 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 3 \\ \frac{11}{2} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \\ -2 \\ \frac{1}{2} \end{pmatrix} \end{aligned}$$

$$\text{The error: } \|\vec{y} - A\vec{x}\| = \sqrt{1 + 4 + \frac{1}{4}}$$

$$\begin{aligned} &= \frac{\sqrt{21}}{2} \\ &\approx 2.2913 \end{aligned}$$



**Exercise**

For the given matrix  $A$  and  $\vec{y}$

$$A = \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix}$$

- Find the least-squares solution of the linear system  $A\vec{x} = \vec{y}$
- Find the orthogonal projection of  $\vec{y}$  on the column space of  $A$
- Find the **error vector** and the **error**

**Solution**

a) Let  $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$

$$A^T A \vec{x} = A^T \vec{y}$$

$$\begin{pmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 6 \\ 6 & 42 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

$$m = \frac{288}{216}$$

$$= \frac{4}{3}$$

$$b = -\frac{72}{216}$$

$$= -\frac{1}{3}$$

Thus  $y = \frac{4}{3}x - \frac{1}{3}$

- b) The orthogonal projection of  $\vec{y}$  on the column space of  $A$

$$A\vec{x} = \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -2 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{aligned} c) \quad \vec{y} - A\vec{x} &= \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 3 \\ -3 \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{The error: } \|\vec{y} - A\vec{x}\| &= \sqrt{1+9+9+1} \\ &= 2\sqrt{5} \quad \approx 4.47214 \end{aligned}$$

### Exercise

For the given matrix  $A$  and  $\vec{y}$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}$$

- Find the least-squares solution of the linear system  $A\vec{x} = \vec{y}$
- Find the orthogonal projection of  $\vec{y}$  on the column space of  $A$
- Find the **error vector** and the **error**

### Solution

$$a) \quad \text{Let } \vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$A^T A \vec{x} = A^T \vec{y}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 14 \\ 4 \\ 10 \end{pmatrix}$$

$$\begin{cases} 4a + 2b + 2c = 14 \\ 2a + 2b = 4 \\ 2a + 2c = 10 \end{cases}$$

$$\begin{cases} 2a + b + c = 7 \\ a + b = 2 \\ a + c = 5 \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 0 \qquad \Delta_c = \begin{vmatrix} 2 & 1 & 7 \\ 1 & 1 & 2 \\ 1 & 0 & 5 \end{vmatrix} = 0$$

$$\begin{cases} b = 2 - a \\ c = 5 - a \end{cases}$$

Assume  $\underline{a = 1 \rightarrow b = 1 \quad c = 4}$

Thus  $\underline{x + y + 4z = 0}$

**b)** The orthogonal projection of  $\vec{y}$  on the column space of  $A$

$$A\vec{x} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \\ 5 \\ 5 \end{pmatrix}$$

**c)**  $\vec{y} - A\vec{x} = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 5 \\ 5 \end{pmatrix}$

$$= \begin{pmatrix} -1 \\ 1 \\ 3 \\ -3 \end{pmatrix}$$

The **error**:  $\|\vec{y} - A\vec{x}\| = \sqrt{1+1+9+9}$

$$= \underline{2\sqrt{5}}$$

$$\approx \underline{4.47214}$$

**Exercise**

For the given matrix  $A$  and  $\vec{y}$

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 3 \\ 5 \\ 7 \\ -3 \end{pmatrix}$$

- Find the least-squares solution of the linear system  $A\vec{x} = \vec{y}$
- Find the orthogonal projection of  $\vec{y}$  on the column space of  $A$
- Find the **error vector** and the **error**

**Solution**

a) Let  $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$A^T A \vec{x} = A^T \vec{y}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 7 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 8 & 10 \\ 8 & 20 & 26 \\ 10 & 26 & 38 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \\ 20 \end{pmatrix}$$

$$\begin{cases} 2a + 4b + 5c = 6 \\ 4a + 10b + 13c = 6 \\ 5a + 13b + 19c = 10 \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & 4 & 5 \\ 4 & 10 & 13 \\ 5 & 13 & 19 \end{vmatrix} = 8$$

$$\Delta_a = \begin{vmatrix} 6 & 4 & 5 \\ 6 & 10 & 13 \\ 10 & 13 & 19 \end{vmatrix} = 80$$

$$\Delta_b = \begin{vmatrix} 2 & 6 & 5 \\ 4 & 6 & 13 \\ 5 & 10 & 19 \end{vmatrix} = -48$$

$$\Delta_c = \begin{vmatrix} 2 & 4 & 6 \\ 4 & 10 & 6 \\ 5 & 13 & 10 \end{vmatrix} = 16$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 10 \\ -6 \\ 2 \end{pmatrix}$$

Thus  $\underline{10x - 6y + 2z = 0}$  |

b) The orthogonal projection of  $\vec{y}$  on the column space of  $A$

$$\begin{aligned} A\vec{x} &= \begin{pmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix} \begin{pmatrix} 10 \\ -6 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 4 \\ 8 \\ -2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} c) \quad \vec{y} - A\vec{x} &= \begin{pmatrix} 3 \\ 5 \\ 7 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 8 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \end{aligned}$$

The **error**:  $\underline{\|\vec{y} - A\vec{x}\| = 2}$  |

### Exercise

For the given matrix  $A$  and  $\vec{y}$

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{pmatrix}$$

- Find the least-squares solution of the linear system  $A\vec{x} = \vec{y}$
- Find the orthogonal projection of  $\vec{y}$  on the column space of  $A$
- Find the **error vector** and the **error**

### Solution



d) Let  $\vec{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

$$A^T A \vec{x} = A^T \vec{y}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 4 & 2 & 2 \\ 4 & 4 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 6 \end{pmatrix}$$

$$\begin{cases} 3a + 2b + c + d = 2 \\ 2a + 2b + d = 1 \\ a + c = 1 \\ a + b + d = 3 \end{cases}$$

$$\Delta = \begin{vmatrix} 3 & 2 & 1 & 1 \\ 2 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} = 0$$

$$\Delta_d = \begin{vmatrix} 3 & 2 & 1 & 2 \\ 2 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 3 \end{vmatrix} = 0$$

$$(3) \rightarrow \underline{c = 1 - a}$$

$$\begin{cases} 2a + 2b + d = 1 \\ a + b + d = 3 \end{cases}$$

$$\underline{a + b = -2}$$

$$\underline{b = -2 - a}$$

$$\underline{d = 5}$$

Assume  $\underline{a = 0 \rightarrow b = -2 \quad c = 1}$

$$-2y + z + 5 = 0$$

Therefore,  $\underline{-2y + z = -5}$

e) The orthogonal projection of  $\vec{y}$  on the column space of  $A$

$$A\vec{x} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ 1 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ -2 \\ 1 \\ 1 \\ 3 \\ 3 \end{pmatrix}$$

$$\text{f) } \vec{y} - A\vec{x} = \begin{pmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \\ 1 \\ 1 \\ 3 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 2 \\ -2 \end{pmatrix}$$

The **error**:  $\|\vec{y} - A\vec{x}\| = \sqrt{1+1+1+1+4+4}$

$$= \sqrt{12}$$

$$= 2\sqrt{3}$$

$$\approx 3.4641$$

### Exercise

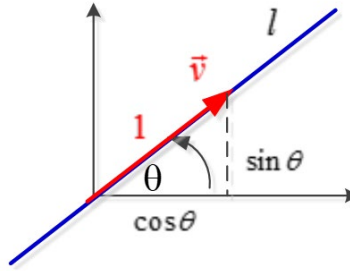
Find the standard matrix for the orthogonal projection  $P$  of  $\mathbb{R}^2$  on the line passes through the origin and makes an angle  $\theta$  with the positive  $x$ -axis.

### Solution

Since the line  $l$  in 2-dimensional, than we can take  $\vec{v} = (\cos \theta, \sin \theta)$  as a basis for this subspace

$$A = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

$$\begin{aligned} [P] &= A^T A \\ &= [\cos \theta \quad \sin \theta] \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix} \end{aligned}$$



### Exercise

Hooke's law in physics states that the length  $x$  of a uniform spring is a linear function of the force  $y$  applied to it. If we express the relationship as  $y = mx + b$ , then the coefficient  $m$  is called the spring constant.

Suppose a particular unstretched spring has a measured length of 6.1 *inches*. (i.e.,  $x = 6.1$  when  $y = 0$ ). Forces of 2 pounds, 4 pounds, and 6 pounds are then applied to the spring, and the corresponding lengths are found to be 7.6 inches, 8.7 inches, and 10.4 inches. Find the spring constant.

### Solution

$$M = \begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$

The normal equation:  $A^T A \vec{x} = A^T \vec{y}$

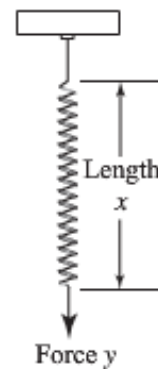
$$\begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 278.82 & 32.8 \\ 32.8 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 112.4 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{39.44} \begin{pmatrix} 4 & -32.8 \\ -32.8 & 278.2 \end{pmatrix} \begin{pmatrix} 112.4 \\ 12 \end{pmatrix}$$

$$= \frac{1}{39.44} \begin{pmatrix} 56 \\ -348.32 \end{pmatrix}$$

$$= \begin{pmatrix} 1.4 \\ -8.8 \end{pmatrix}$$



Thus, the estimated value of the spring constant is  $\approx 1.4$  pounds

**Exercise**

Prove:

If  $A$  has a linearly independent column vectors, and if  $\vec{b}$  is orthogonal to the column space of  $A$ , then the least squares solution of  $A\vec{x} = \vec{b}$  is  $\vec{x} = \vec{0}$ .

**Solution**

If  $A$  has linearly independent column vectors, then  $A^T A$  is invertible and the least squares solution of  $A\vec{x} = \vec{b}$  is the solution of  $A^T A\vec{x} = A^T \vec{b}$ , but since  $\vec{b}$  is orthogonal to the column space of  $A$ .

$A^T \vec{b} = 0$ , so  $\vec{x}$  is a solution of  $A^T A\vec{x} = 0$ .

Thus  $\vec{x} = \vec{0}$  since  $A^T A$  is invertible.

**Exercise**

Let  $A$  be an  $m \times n$  matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of  $\mathbb{R}^n$  onto the row space of  $A$ .

**Solution**

$A^T$  will have linearly independent column vectors, and the column space  $A^T$  is the row space of  $A$ .

Thus, the standard matrix for the orthogonal projection of  $\mathbb{R}^n$  onto the row space of  $A$  is

$$\begin{aligned} [P] &= A^T \left[ \left( A^T \right)^T A^T \right]^{-1} \left( A^T \right)^T \\ &= A^T \left( A A^T \right)^{-1} A \end{aligned}$$

**Exercise**

Let  $W$  be the line with parametric equations  $x = 2t$ ,  $y = -t$ ,  $z = 4t$

- Find a basis for  $W$ .
- Find the standard matrix for the orthogonal projection on  $W$ .
- Use the matrix in part (b) to find the orthogonal projection of a point  $P_0(x_0, y_0, z_0)$  on  $W$ .
- Find the distance between the point  $P_0(2, 1, -3)$  and the line  $W$ .

**Solution**

a)  $W = \text{span}\{(2, -1, 4)\}$

So that the vector  $(2, -1, 4)$  forms a basis for  $W$  (linear independence)

**b)** Let  $A = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$

$$[P] = A(A^T A)^{-1} A^T$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \left( \begin{bmatrix} 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} [21]^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \frac{1}{21} \begin{bmatrix} 4 & -2 & 8 \\ -2 & 1 & -4 \\ 8 & -4 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix}$$

**c)**  $\begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \frac{4}{21}x_0 - \frac{2}{21}y_0 + \frac{8}{21}z_0 \\ -\frac{2}{21}x_0 + y_0 - \frac{4}{21}z_0 \\ \frac{8}{21}x_0 - \frac{4}{21}y_0 + \frac{16}{21}z_0 \end{bmatrix}$

**d)**  $\begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{6}{7} \\ \frac{3}{7} \\ -\frac{12}{7} \end{bmatrix}$

The distance between  $P_0$  and  $W$  equals to the distance between  $P_0$  and its projection on  $W$ .

The distance between  $(2, 1, -3)$  and  $\left(-\frac{6}{7}, \frac{3}{7}, -\frac{12}{7}\right)$  is

$$d = \sqrt{\left(2 + \frac{6}{7}\right)^2 + \left(1 - \frac{3}{7}\right)^2 + \left(-3 + \frac{12}{7}\right)^2}$$

$$= \sqrt{\frac{400}{49} + \frac{16}{49} + \frac{81}{49}}$$

$$= \frac{\sqrt{497}}{7}$$

### Exercise

In  $R^3$ , consider the line  $l$  given by the equations  $x = t, y = t, z = t$

And the line  $m$  given by the equations  $x = s, y = 2s - 1, z = 1$

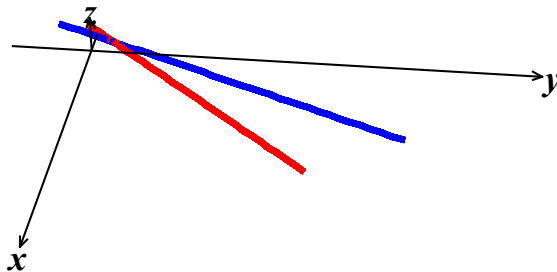
Let  $P$  be the point on  $l$ , and let  $Q$  be a point on  $m$ . Find the values of  $t$  and  $s$  that minimize the distance between the lines by minimizing the squared distance  $\|P - Q\|^2$

### Solution

When  $t = 1 \Rightarrow$  Let  $P = (1, 1, 1)$  is on line  $l$

When  $s = 1 \Rightarrow$  Let  $Q = (1, 1, 1)$  is on line  $m$

$$\|P - Q\| = \sqrt{(1-1)^2 + (1-1)^2 + (1-1)^2} = 0 \geq 0$$



Thus, these are the values  $P = (1, 1, 1)$  and  $Q = (1, 1, 1)$  are the values for  $s = t = 1$  that minimize the distance between the lines.

### Exercise

Determine whether the statement is true or false,

- If  $A$  is an  $m \times n$  matrix, then  $A^T A$  is a square matrix.
- If  $A^T A$  is invertible, then  $A$  is invertible.
- If  $A$  is invertible, then  $A^T A$  is invertible.
- If  $A\vec{x} = \vec{b}$  is a consistent linear system, then  $A^T A\vec{x} = A^T \vec{b}$  is also consistent.
- If  $A\vec{x} = \vec{b}$  is an inconsistent linear system, then  $A^T A\vec{x} = A^T \vec{b}$  is also inconsistent.
- Every linear system has a least squares solution.
- Every linear system has a unique least squares solution.

- h) If  $A$  is an  $m \times n$  matrix with linearly independent columns and  $\vec{b}$  is in  $R^m$ , then  $A\vec{x} = \vec{b}$  has a unique least squares solution.

**Solution**

- a) **True**;  $A^T A$  is an  $n \times n$  matrix
- b) **False**; only square matrix has inverses, but  $A^T A$  can be invertible when  $A$  is not square matrix.
- c) **True**; if  $A$  is invertible, so is  $A^T$ , so the product  $A^T A$  is also invertible
- d) **True**
- e) **False**; the system  $A^T A\vec{x} = A^T \vec{b}$  may be consistent
- f) **True**
- g) **False**; the least squares solution may involve a parameter
- h) **True**; if  $A$  has linearly independent column vectors; then  $A^T A$  is invertible, so  $A^T A\vec{x} = A^T \vec{b}$  has a unique solution

**Exercise**

A certain experiment produces the data  $\{(1, 1.8), (2, 2.7), (3, 3.4), (4, 3.8), (5, 3.9)\}$ .

Find the function that it will fit these data in the form of  $y = \beta_1 x + \beta_2 x^2$

**Solution**

**Given:** the equation  $y = \beta_1 x + \beta_2 x^2$  that best fits the given points. Then

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$$

The normal equation formula:  $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix} \begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$$

$$\begin{pmatrix} 55 & 225 \\ 225 & 979 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 52.1 \\ 201.5 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 55 & 225 \\ 225 & 979 \end{vmatrix} = 3,220 \quad \Delta_{\beta_1} = \begin{vmatrix} 52.1 & 225 \\ 201.5 & 979 \end{vmatrix} = 5,668.4 \quad \Delta_{\beta_2} = \begin{vmatrix} 55 & 52.1 \\ 225 & 201.5 \end{vmatrix} = -640$$

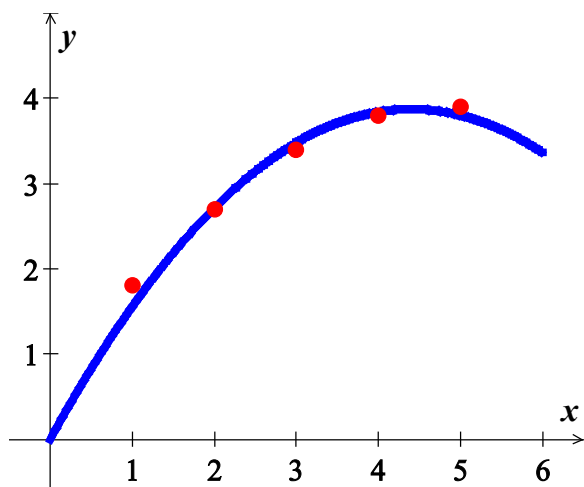
$$\beta_1 = \frac{5,668.4}{3,220}$$

$$\approx 1.76$$

$$\beta_2 = -\frac{640}{3,220}$$

$$\approx -0.199$$

$$y = 1.76x - .2x^2$$





**Exercise**

According to Kepler's first law, a comet should have an ellipse, parabolic, or hyperbolic orbit (with gravitational attractions from the planets ignored). In suitable polar coordinates, the position  $(r, \nu)$  of a comet satisfies an equation of the form

$$r = \beta + e(r \cdot \cos \nu)$$

Where  $\beta$  is a constant and  $e$  is the eccentricity of the orbit, with  $0 \leq e < 1$  for an ellipse,  $e = 1$  for a parabolic, and  $e > 1$  for a hyperbola.

Suppose observations of a newly discovered comet provide the data below.

$\nu$	.88	1.10	1.42	1.77	2.14
$r$	3.00	2.30	1.65	1.25	1.01

Determine the type of orbit, and predict where the orbit will be when  $\nu = 4.6$  (*radians*)?

**Solution**

**Given:** the equation in the form  $r = \beta + e(r \cdot \cos \nu)$

$$3 = \beta + e(3 \cdot \cos(.88)) = \beta + 1.911e$$

$$2.3 = \beta + e(2.3 \cos(1.1)) = \beta + 1.043e$$

$$1.65 = \beta + e(1.65 \cos(1.42)) = \beta + .248e$$

$$1.25 = \beta + e(1.25 \cos(1.77)) = \beta - .247e$$

$$1.01 = \beta + e(1.01 \cos(2.14)) = \beta - .544e$$

$$\begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix} \begin{pmatrix} \beta \\ e \end{pmatrix} = \begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} \beta \\ e \end{pmatrix} \quad \vec{r} = \begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$$

The normal equation formula:  $A^T A \vec{v} = A^T \vec{r}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1.911 & 1.043 & .248 & -.247 & -.544 \end{pmatrix} \begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix} \begin{pmatrix} \beta \\ e \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1.911 & 1.043 & .248 & -.247 & -.544 \end{pmatrix} \begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 2.411 \\ 2.411 & 5.158 \end{pmatrix} \begin{pmatrix} \beta \\ e \end{pmatrix} = \begin{pmatrix} 9.21 \\ 7.683 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 5 & 2.411 \\ 2.411 & 5.158 \end{vmatrix} = 19.98 \quad \Delta_{\beta} = \begin{vmatrix} 9.21 & 2.411 \\ 7.683 & 5.158 \end{vmatrix} = 28.98 \quad \Delta_e = \begin{vmatrix} 5 & 9.21 \\ 2.411 & 7.683 \end{vmatrix} = 16.21$$

$$\beta = \frac{28.98}{19.98}$$

$$\approx 1.45$$

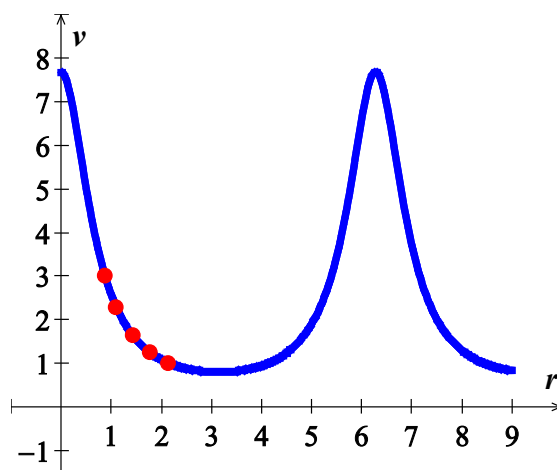
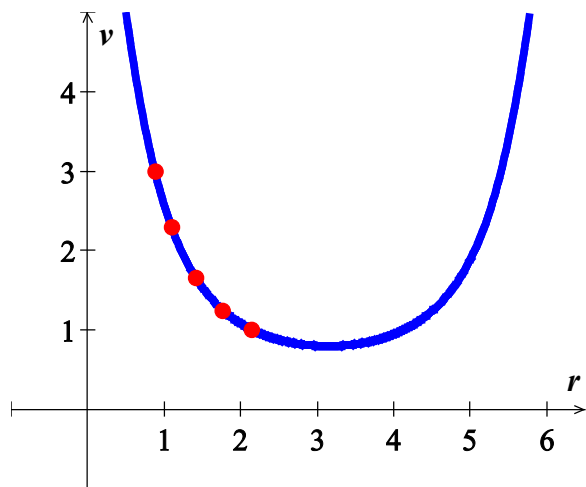
$$e = \frac{16.21}{19.98}$$

$$\approx 0.811 < 1$$

Therefore, the orbit is an *ellipse* type since  $e \approx 0.811 < 1$

Since  $r = \beta + e(r \cdot \cos v)$

$$\text{Then, } r(v) = \frac{1.45}{1 - 0.811 \cdot \cos v}$$



$$r(4.6) = \frac{1.45}{1 - 0.811 \cdot \cos 4.6}$$

$$\approx 1.329$$

**Exercise**

To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every second, from  $t = 0$  to  $t = 12$

The position (in *feet*) were:

0, 8.8, 29.9, 62.0, 104.7, 159.1, 222.0, 294.5, 380.4, 471.1, 571.7, 686.8, and 809.2

- Find the least square cubic curve  $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$  for these data.
- Estimate the velocity of the plane when  $t = 4.5$  *sec*, using the result from part (a).

**Solution**

**Given:** the equation is in form  $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \\ 1 & 6 & 36 & 216 \\ 1 & 7 & 49 & 343 \\ 1 & 8 & 64 & 512 \\ 1 & 9 & 81 & 729 \\ 1 & 10 & 100 & 1000 \\ 1 & 11 & 121 & 1331 \\ 1 & 12 & 144 & 1728 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8.8 \\ 29.9 \\ 62 \\ 104.7 \\ 159.1 \\ 222.0 \\ 294.5 \\ 380.4 \\ 471.1 \\ 571.7 \\ 686.8 \\ 809.2 \end{pmatrix}$$

$$\mathbf{A} \quad \quad \mathbf{\vec{t}} \quad = \quad \mathbf{\vec{y}}$$

The normal equation formula:  $\mathbf{A}^T \mathbf{A} \vec{t} = \mathbf{A}^T \vec{y}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 & 100 & 121 & 144 \\ 0 & 1 & 8 & 27 & 64 & 125 & 216 & 343 & 512 & 729 & 1000 & 1331 & 1728 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \\ 1 & 6 & 36 & 216 \\ 1 & 7 & 49 & 343 \\ 1 & 8 & 64 & 512 \\ 1 & 9 & 81 & 729 \\ 1 & 10 & 100 & 1000 \\ 1 & 11 & 121 & 1331 \\ 1 & 12 & 144 & 1728 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 & 100 & 121 & 144 \\ 0 & 1 & 8 & 27 & 64 & 125 & 216 & 343 & 512 & 729 & 1000 & 1331 & 1728 \end{pmatrix} \begin{pmatrix} 0 \\ 8.8 \\ 29.9 \\ 62 \\ 104.7 \\ 159.1 \\ 222.0 \\ 294.5 \\ 380.4 \\ 471.1 \\ 571.7 \\ 686.8 \\ 809.2 \end{pmatrix}$$

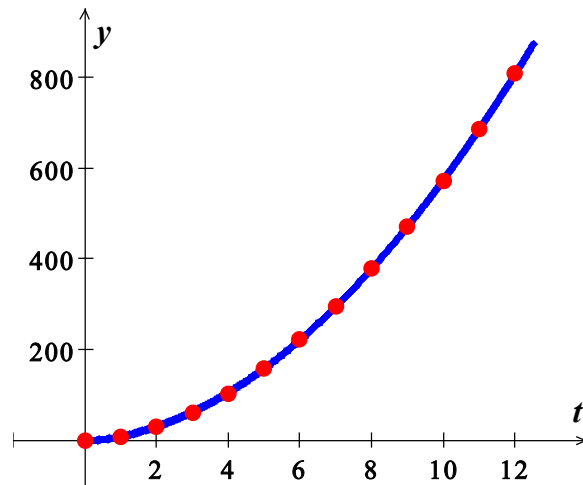
$$\begin{pmatrix} 13 & 78 & 650 & 6,084 \\ 78 & 650 & 6,084 & 60,710 \\ 650 & 6,084 & 60,710 & 630,708 \\ 6,084 & 60,710 & 630,708 & 6,735,950 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 3,800.2 \\ 35,127.7 \\ 348,063.9 \\ 3,599,800.9 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 13 & 78 & 650 & 6,084 \\ 78 & 650 & 6,084 & 60,710 \\ 650 & 6,084 & 60,710 & 630,708 \\ 6,084 & 60,710 & 630,708 & 6,735,950 \end{vmatrix} = 97,538,785,344$$

$$\Delta_0 = \begin{vmatrix} 3800.2 & 78 & 650 & 6,084 \\ 35,127.7 & 650 & 6,084 & 60,710 \\ 348,063.9 & 6,084 & 60,710 & 630,708 \\ 3,599,800.9 & 60,710 & 630,708 & 6,735,950 \end{vmatrix} = -83,470,691,303.8916$$

Or I use my program to find the values

```
rref = (Matrix([
[1, 0, 0, 0, -0.855769230765803],
[0, 1, 0, 0, 4.70248501498163],
[0, 0, 1, 0, 5.55536963037029],
[0, 0, 0, 1, -0.0273601398601744]]))
```



$$\beta_0 \approx -0.855769$$

$$\beta_1 \approx 4.702485$$

$$\beta_2 \approx 5.55537$$

$$\beta_3 \approx -0.02736$$

$$y(t) = -0.855769 + 4.702485t + 5.55537t^2 - 0.02736t^3$$

$$\text{Error} = 3.9734$$

