Divergence & Curl

Defn The divergence of a $\vec{F} = \langle f, g, h \rangle$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= \cancel{f_x + f_y + f_z}$$

If $\nabla \cdot \vec{F} = 0$, the vector field is source free

Ex $\vec{F} = \langle x, y, z \rangle$

$$\nabla \cdot \vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 1 + 1 + 1$$

$$= 3 > 0$$

divergence is positive, the flow expands outward at all points

Ex

$\vec{F} = \langle -y, x - z, y \rangle$

$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x - z) + \frac{\partial}{\partial z}(y)$

$= 0$

$\therefore$ The field is source free

Ex $\vec{F} = \langle -y, x, z \rangle$

$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z)$

$= 1$

$\nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle f, g, h \rangle$

$= \frac{\partial}{\partial x} f + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|} \qquad div?$$

$$= \frac{\langle x, y, z \rangle}{\sqrt{x^2+y^2+z^2}}$$

$$div(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)$$

$(u^n v^m)'$
$u^{n-1} v^{m-1}(nu'v + muv')$

$$= \frac{x^2+y^2+z^2 - \frac{1}{2}x(2x)}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial g}{\partial y} = \frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial h}{\partial z} = \frac{z}{(x^2+y^2+z^2)^{1/2}}$$

$$= \frac{x^2+y^2+z^2 - \frac{1}{2}z(2z)}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$$

$$\nabla \cdot \vec{F} = \frac{y^2+z^2+x^2+z^2+x^2+y^2}{|\vec{r}|^3}$$

$$= \frac{2(x^2+y^2+z^2)}{|\vec{r}|^3} = \frac{2|\vec{r}|^2}{|\vec{r}|^3}$$

$$= \boxed{\frac{2}{|\vec{r}|}}$$

Theorem

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|^p} \Rightarrow \nabla\cdot\vec{F} = \frac{3-p}{|\vec{r}|^p}$$

Ex) $\vec{F} = \langle f, g \rangle$ $\qquad$ C: $r = 2$ @ O

$= \langle x^2, y \rangle$ $\qquad$ $x^2+y^2=4$

a) divergence (+) or (−) @ $Q(1,1)$

$f_x > 0 \quad g_y > 0$

b) $\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial y}{\partial y}$

$= 2x+1 \Big|_{(1,1)}$

$= \boxed{3}$

c) $2x+1=0 \Rightarrow x = -\frac{1}{2}$ free source

$x < -\frac{1}{2} \Rightarrow$ div < 0

$x > \frac{1}{2} \Rightarrow$ div > 0

$$\text{curl } \vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \hat{i} + \left(\frac{\partial z}{\partial f} - \frac{\partial h}{\partial x} \right) \hat{j}$$

$$+ \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \hat{k}$$

$\nabla \times \vec{F} = 0 \rightarrow$ vector field is irrotational

Ex $\quad \vec{F} = \vec{a} \times \vec{r} \qquad \vec{a} = \langle a_1, a_2, a_3 \rangle$

$\qquad\qquad\qquad\qquad\quad \vec{r} = \langle x, y, z \rangle$

$$\vec{F} = \vec{a} \times \vec{r}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= (a_2 z - a_3 y)\hat{i} + (a_3 x - a_1 z)\hat{j} + (a_1 y - a_2 x)\hat{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= (a_1 + a_1)\hat{i} + (a_2 + a_2)\hat{j} + (a_3 + a_3)\hat{k}$$

$$= 2(a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k})$$

$$\boxed{= 2\vec{a}}$$

Theorem Let $\vec{F} = \nabla \Phi$

$$\nabla \times \vec{F} = \nabla \times \nabla \Phi = 0$$

$$\nabla \times \nabla \Phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \Phi_x & \Phi_y & \Phi_z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\Phi_z - \frac{\partial}{\partial z}\Phi_y\right)\hat{i} + \left(\frac{\partial}{\partial z}\Phi_x - \frac{\partial}{\partial x}\Phi_z\right)\hat{j}$$

$$+ \left(\frac{\partial}{\partial x}\Phi_y - \frac{\partial}{\partial y}\Phi_x\right)\hat{k}$$

$$= \left(\Phi_{zy} - \Phi_{yz}\right)\hat{i} + \left(\Phi_{xz} - \Phi_{zx}\right)\hat{j}$$

$$\left(\Phi_{yx} - \Phi_{xy}\right)\hat{k}$$

$\Phi_{zy} = \Phi_{yz}$

$\Phi_{zx} = \Phi_{xz}$

$\Phi_{xy} = \Phi_{yx}$

$$= 0$$

Theorem Product Rule for div.

$$\nabla \cdot (u\vec{F}) = \nabla u \cdot \vec{F} + u(\nabla \cdot \vec{F})$$

Ex $\vec{r} = \langle x, y, z \rangle$

$\phi = \frac{1}{|\vec{r}|}$

a) $\vec{F} = \nabla\left(\frac{1}{|\vec{r}|}\right) = \nabla\phi = \phi_x \hat{i} + \phi_y \hat{j} + \phi_z \hat{k}$

$= \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$ $\qquad \left(\frac{1}{u^n}\right)' = -\frac{nu'}{u^{n+1}}$

$\frac{\partial\phi}{\partial x} = -\frac{2x}{2(x^2+y^2+z^2)^{3/2}}$

$= -\frac{x}{|\vec{r}|^3}$

$\frac{\partial\phi}{\partial y} = -\frac{2y}{2(x^2+y^2+z^2)^{3/2}}$

$= -\frac{y}{|\vec{r}|^3}$

$\frac{\partial\phi}{\partial z} = -\frac{2z}{2(x^2+y^2+z^2)^{3/2}}$

$= -\frac{z}{|\vec{r}|^3}$

$\vec{F} = \nabla(\phi)$

$= -\frac{x\hat{i} + y\hat{j} + z\hat{k}}{|\vec{r}|^3}$

$= -\frac{\vec{r}}{|\vec{r}|^3}$

b) $\nabla \cdot \vec{F} = \nabla \cdot \left(-\frac{\vec{r}}{|\vec{r}|^3}\right) = -\frac{3-3}{|\vec{r}|^3}$

$= \underline{0}$

$\exists$ Φ (potential fctn) $\ni$ $\vec{F} = \nabla \Phi$

$$\int_C \vec{F} \cdot d\vec{r} = \Phi(B) - \Phi(A)$$

$\oint_C \vec{F} \cdot d\vec{r} = 0$ closed curve C.

$\nabla \times \vec{F} = 0$ @ all pts of D.

1.5 #5: $\vec{F} = \langle e^{-x+y}, e^{-y+z}, e^{-z+x} \rangle$

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(e^{-x+y}) + \frac{\partial}{y}(e^{-y+z}) + \frac{\partial}{\partial z}(e^{-z+x})$$

$$= -e^{-x+y} - e^{-y+z} - e^{-z+x}$$

#9 $\vec{F} = \frac{\vec{r}}{|\vec{r}|^2} = \frac{\langle x, y, z \rangle}{x^2+y^2+z^2}$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

$$= \frac{1}{|\vec{r}|^2} \qquad \frac{3-p}{|\vec{r}|^p}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right) = \frac{x^2+y^2+z^2-2x^2}{(x^2+y^2+z^2)^2}$$

$$= \frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}$$

$$\nabla \cdot \vec{F} = \frac{-x^2+y^2+z^2+x^2-y^2+z^2+x^2+y^2-z^2}{(|\vec{r}|^2)^2}$$

$$= \frac{x^2+y^2+z^2}{(|\vec{r}|^2)^2}$$

$$= \frac{|\vec{r}|^2}{(|\vec{r}|^2)^2}$$

$$= \frac{1}{|\vec{r}|^2}$$

#24 $\vec{F} = \langle 3xz^3e^{y^2}, 2xz^3e^{y^2}, 3xz^2e^{y^2} \rangle$

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xz^3e^{y^2} & 2xz^3e^{y^2} & 3xz^2e^{y^2} \end{vmatrix}$$

$$= (6xyz^2e^{y^2} - 6xz^2e^{y^2})\,\hat{i}$$
$$+ (9xz^2e^{y^2} - 3z^2e^{y^2})\,\hat{j}$$
$$+ (2z^3e^{y^2} - 6xyz^3e^{y^2})\,\hat{k}$$

$$= z^2e^{y^2}\left[(6xy - 6x)\hat{i} + (9x - 3)\hat{j} + (2z - 6xy)\hat{k}\right]$$

#28 $\vec{F} = \vec{r}\,|\vec{r}| = \dfrac{\vec{r}}{|\vec{r}|^{-1}}$ $\quad p = -1$

$$\operatorname{div}\vec{F} = \frac{4}{|\vec{r}|^{-1}}$$

$\langle x, y, z \rangle \sqrt{x^2+y^2+z^2}$

$$= 4\,|\vec{r}|$$

$\langle x|\vec{r}|, y|\vec{r}|, z|\vec{r}| \rangle$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x|\vec{r}| & y|\vec{r}| & z|\vec{r}| \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y} z|\vec{r}| - \frac{\partial}{\partial z}(y|\vec{r}|)\right)\hat{i}$$
$$+ \left(\frac{\partial}{\partial z} x|\vec{r}| - \frac{\partial}{\partial x} z(|\vec{r}|)\right)\hat{j}$$
$$+ \left(\frac{\partial}{\partial x} y|\vec{r}| - \frac{\partial}{\partial y} x|\vec{r}|\right)\hat{k}$$

$$|\vec{r}| = \sqrt{x^2+y^2+z^2} \quad (\quad)^{1/2}$$

$$\frac{\partial}{\partial x}|\vec{r}| = \frac{x}{|\vec{r}|}$$

$$\frac{\partial}{\partial y}|\vec{r}| = \frac{y}{|\vec{r}|}$$

$$\frac{\partial}{\partial z}|\vec{r}| = \frac{z}{|\vec{r}|}$$

$$\text{curl } \vec{F} = \left(\frac{yz}{|\vec{r}|} - \frac{yz}{|\vec{r}|}\right)\hat{i} + \left(\frac{xz}{|\vec{r}|} - \frac{xz}{|\vec{r}|}\right)\hat{j} + \left(\frac{xy}{|\vec{r}|} - \frac{xy}{|\vec{r}|}\right)\hat{k}$$

$$= 0$$

it is irrotational but not source free

4.6 Surfaces

$$\vec{r}(u,v) = f(u,v)\,\hat{i} + g(u,v)\,\hat{j} + h(u,v)\,\hat{k}$$

$$\vec{r}_u = \frac{\partial f}{\partial u}\hat{i} + \frac{\partial g}{\partial u}\hat{j} + \frac{\partial h}{\partial u}\hat{k}$$

$$\vec{r}_v = \langle f_v, g_v, h_v \rangle$$

$$\vec{r}_u \times \vec{r}_v$$

$$\text{Area} = \int_c^d \int_a^b |\vec{r}_u \times \vec{r}_v| \, du\, dv$$

Ex / Soln Surface $z = \sqrt{x^2+y^2}$ $0 \le z \le 1$

$$r = \sqrt{x^2+y^2} = z$$

$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r \rangle$$

$\vec{r}_r$ $\vec{r}_\theta$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \langle -r\cos\theta, -r\sin\theta, r(\cos^2\theta + \sin^2\theta) \rangle$$

$$= \langle -r\cos\theta, -r\sin\theta, r \rangle$$

$$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{\underbrace{r^2\cos^2\theta + r^2\sin^2\theta}_{r^2} + r^2}$$

$$= r\sqrt{2}$$

$$A = \int_0^{2\pi} d\theta \int_0^1 r\sqrt{2}\, dr$$

$$= 2\pi\sqrt{2}\left(\frac{1}{2} r^2 \Big|_0^1\right) = \boxed{\pi\sqrt{2}}\ \text{unit}^2$$

Ex Surface area sphere w/ radius a

soln $\vec{r}(\phi,\theta) = \langle a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi \rangle$

$$\begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \end{cases}$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= (a^2\sin^2\phi\cos\theta)\hat{i} + (a^2\sin^2\phi\sin\theta)\hat{j}$$
$$+ (a^2\cos\phi\sin\phi\cos^2\theta + a^2\sin\phi\cos\phi\sin^2\theta)\hat{k}$$

$$= (a^2\sin^2\phi\cos\theta)\hat{i} + (a^2\sin^2\phi\sin\theta)\hat{j}$$
$$+ (a^2\cos\phi\sin\phi)\hat{k}$$

$$|\vec{r}_\phi \times \vec{r}_\theta| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\cos^2\phi\sin^2\phi}$$

$$= a^2\sqrt{\sin^4\phi + \cos^2\phi\sin^2\phi}$$
$$= a^2\sin\phi\sqrt{\sin^2\phi + \cos^2\phi}$$
$$= a^2\sin\phi$$

$$S = a^2\int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi$$
$$= 2\pi a^2 \left(-\cos\phi\right|_0^{\pi}$$
$$= 4\pi a^2 \text{ unit}^2$$

Ex $\quad r = \cos z \qquad -\frac{\pi}{2} \le z \le \frac{\pi}{2} \qquad x^2+y^2=1$

$y = 0$

$x = r\cos\theta = \cos u \cos v \qquad x = \cos z$

$y = r\sin\theta = \cos u \sin v \qquad = \cos u$

$z = u \quad = r\cos$

$v = \theta$

$$\begin{aligned} x &= r\sin\phi\cos\theta = \cos u \cos v \\ y &= r\sin\phi\sin\theta = \cos u \sin v \\ z &= r\cos\phi = u \end{aligned}$$

$0 \le v \le 2\pi \qquad -\frac{\pi}{2} \le u \le \frac{\pi}{2}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin u \cos v & -\sin u \sin v & 1 \\ -\cos u \sin v & \cos u \cos v & 0 \end{vmatrix}$$

$$= (-\cos u \cos v)\hat{i} - (\cos u \sin v)\hat{j} + (-\sin u \cos u \cos^2 v - \sin u \cos u \sin^2 v)\hat{k}$$

$$= -(\cos u \cos v)\hat{i} - (\cos u \sin v)\hat{j} - (\sin u \cos u)\hat{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\cos^2 u \cos^2 v + \cos^2 u \sin^2 v + \sin^2 u \cos^2 u}$$

$$= \sqrt{\cos^2 u + \sin^2 u \cos^2 u}$$

$$= \cos u \sqrt{1 + \sin^2 u}$$

$$S = \int_0^{2\pi} d\nu \int_{-\pi/2}^{\pi/2} \cos u \sqrt{1+\sin^2 u}\, du\, d\nu$$

2π

$$\omega = \sin u$$
$$d\omega = \cos u\, du$$

$$= 2\pi \int_{-\pi/2}^{\pi/2} \sqrt{1+\omega^2}\, d\omega$$

$$\omega = \tan\alpha \qquad \sqrt{1+\omega^2} = \sec\alpha$$
$$d\omega = \sec^2\alpha\, d\alpha$$

$$= 2\pi \int_{-\pi/2}^{\pi/2} \sec^3\alpha\, d\alpha$$

$$= 2\pi \left(\frac{1}{2}\sec\alpha\tan\alpha + \frac{1}{2}\ln|\sec\alpha + \tan\alpha| \right|_{-\pi/2}^{\pi/2}$$

$$= \pi \left(\omega\sqrt{1+\omega^2} + \ln(\sqrt{1+\omega^2} + \omega) \right|_{-\pi/2}^{\pi/2}$$

$$= \pi \left(\sin u\sqrt{1+\sin^2 u} + \ln(\sqrt{1+\sin^2 u} + \sin u) \right|_{-\pi/2}^{\pi/2}$$

$$= \pi \left(\sqrt{2} + \ln(\sqrt{2}+1) - (-\sqrt{2} + \ln(\sqrt{2}-1)) \right)$$

$$= \pi \left(2\sqrt{2} + \ln(1+\sqrt{2}) - \ln(\sqrt{2}-1) \right)$$

$$= \pi \left(2\sqrt{2} + \ln\frac{1+\sqrt{2}}{\sqrt{2}-1} \right)$$