Solution   *Section* 4.3 – LU–Decompositions

Exercise

What matrix E puts A into triangular form $EA=U$? Multiply by $E^{-1}=L$ to factor A into LU:

$$A=\begin{bmatrix}2 & 1 & 0\\ 0 & 4 & 2\\ 6 & 3 & 5\end{bmatrix}$$

Solution

$$A=\begin{pmatrix}2 & 1 & 0\\ 0 & 4 & 2\\ 6 & 3 & 5\end{pmatrix}$$

$$\begin{pmatrix}2 & 1 & 0\\ 0 & 4 & 2\\ 6 & 3 & 5\end{pmatrix} R_3 - 3R_1 : \ell_{31}$$

$$\begin{pmatrix}2 & 1 & 0\\ 0 & 4 & 2\\ 0 & 0 & 5\end{pmatrix} = U$$

$$E_{31}A=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -3 & 0 & 1\end{pmatrix}\begin{pmatrix}2 & 1 & 0\\ 0 & 4 & 2\\ 6 & 3 & 5\end{pmatrix}$$

$$=\begin{pmatrix}2 & 1 & 0\\ 0 & 4 & 2\\ 0 & 0 & 5\end{pmatrix} = U$$

$$E=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -3 & 0 & 1\end{pmatrix}$$

$$L=E^{-1}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3 & 0 & 1\end{pmatrix}$$

$$A=LU=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3 & 0 & 1\end{pmatrix}\begin{pmatrix}2 & 1 & 0\\ 0 & 4 & 2\\ 0 & 0 & 5\end{pmatrix}$$

Exercise

Solve $Lc=b$ to find $\boldsymbol{c}$. Then solve $Ux=c$ to find $\boldsymbol{x}$. What was A?

$$L=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\qquad U=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\qquad b=\begin{bmatrix}4\\5\\6\end{bmatrix}$$

Solution

$Lc=b$

$$\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}4\\5\\6\end{pmatrix}$$

$$\to\begin{cases}\boxed{c_1=4}\\ c_1+c_2=5\Rightarrow \underline{c_2=5-4=1}\\ c_1+c_2+c_3=6\Rightarrow \underline{c_3=6-4-1=1}\end{cases}\quad\Rightarrow\ c=\begin{pmatrix}4\\1\\1\end{pmatrix}$$

$Ux=c$

$$\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}4\\1\\1\end{pmatrix}$$

$$\Rightarrow\begin{cases}x+y+z=4\\ y+z=1\\ z=1\end{cases}\to\begin{cases}x=3\\ y=0\end{cases}\qquad\Rightarrow\ x=\begin{pmatrix}3\\0\\1\end{pmatrix}$$

$Lc=b\Rightarrow LUx=b$

$$\underbrace{\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}3\\0\\1\end{pmatrix}}_{x}=\underbrace{\begin{pmatrix}4\\5\\6\end{pmatrix}}_{b}$$

Exercise

Find L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots

Solution

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

Exercise

For which c is $A = LU$ impossible – with three pivots?

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & c-6 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{matrix} \\ \\ R_3 - R_1 \end{matrix} \quad \to c-6 \neq 0 \Rightarrow \boxed{c \neq 6}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & c-6 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{matrix} \\ \frac{1}{c-6} R_1 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{c-6} \\ 0 & 1 & 1 \end{pmatrix} \begin{matrix} \\ \\ R_3 - R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{c-6} \\ 0 & 0 & \frac{c-7}{c-6} \end{pmatrix} \to c-7 \neq 0 \Rightarrow \boxed{c \neq 7}$$

LU will be impossible for $c = 6$ *and* $c = 7$

Exercise

Find an LU-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix} \quad \tfrac{1}{2}R_1 \quad \boxed{\tfrac{1}{2} : \ell_1}$$

$$\to \begin{bmatrix} 1 & 4 \\ -1 & -1 \end{bmatrix} \quad R_2 + R_1 \quad \boxed{1 : \ell_{21}}$$

$$\begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \quad \tfrac{1}{3}R_2 \quad \boxed{\tfrac{1}{3} : \ell_2}$$

$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \quad \boldsymbol{U}$$

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & \frac{1}{3} \end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} \quad \boldsymbol{L}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} 2y_1 = -2 & y_1 = -1 \\ -y_1 + 3y_2 = -2 & y_2 = -1 \end{cases}$$

$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} x_1 + 4x_2 = -1 \\ x_2 = -1 \end{cases} \quad \to x_1 = 3$$

The solution: $x_1 = 3$ *and* $x_2 = -1$

Exercise

Find an LU-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} -5 & -10 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix}$$

Solution

$$\begin{bmatrix} -5 & -10 \\ 6 & 5 \end{bmatrix} \quad -\tfrac{1}{5}R_1 \quad \boxed{-\tfrac{1}{5}:\ell_1} \qquad\qquad \begin{bmatrix} -\frac{1}{5} & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix} \quad R_2 - 6R_1 \quad \boxed{-6:\ell_{21}} \qquad\qquad \begin{bmatrix} -5 & 0 \\ -6 & 1 \end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix} -5 & 0 \\ 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & -7 \end{bmatrix} \quad -\tfrac{1}{7}R_2 \quad \boxed{-\tfrac{1}{7}:\ell_2} \qquad\qquad \begin{bmatrix} -5 & 0 \\ 6 & -\frac{1}{7} \end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \boldsymbol{U} \qquad\qquad \to \begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix} \quad \boldsymbol{L}$$

$$\begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} -5y_1 = -10 & y_1 = 2 \\ 6y_1 - 7y_2 = 19 & y_2 = -1 \end{cases}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} x_1 + 2x_2 = 2 \\ x_2 = -1 \end{cases} \quad \to x_1 = 4$$

The solution: $x_1 = 4$ *and* $x_2 = -1$

Exercise

Find an LU-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix} \quad \tfrac{1}{2}R_1 \quad \boxed{\tfrac{1}{2}:\ell_1} \qquad\qquad \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix} \quad R_3 + R_1 \quad \boxed{1 : \ell_{31}} \qquad\qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ 0 & 4 & 1 \end{bmatrix} \quad -\tfrac{1}{2} R_2 \quad \boxed{-\frac{1}{2} : \ell_{22}} \qquad\qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 4 & 1 \end{bmatrix} \quad R_3 - 4R_2 \quad \boxed{-4 : \ell_{32}} \qquad\qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & -4 & 1 \end{bmatrix} \xrightarrow{E_4^{-1}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} \quad \tfrac{1}{5} R_3 \quad \boxed{\frac{1}{5} : \ell_{32}} \qquad\qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & -4 & \frac{1}{5} \end{bmatrix} \xrightarrow{E_5^{-1}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \; U \qquad\qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix} \; L$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix} \to \begin{cases} 2y_1 = -4 \\ -2y_2 = -2 \\ y_1 + 4y_2 + 5y_3 = 6 \end{cases} \Rightarrow \begin{matrix} y_1 = -2 \\ y_2 = 1 \\ y_3 = 0 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \to \begin{cases} x_1 - x_2 - x_3 = -2 \to x_1 = -1 \\ x_2 - x_3 = 1 \to x_2 = 1 \\ x_3 = 0 \end{cases}$$

Solution: $x_1 = -1, \quad x_2 = 1, \quad x_3 = 0$

Exercise

Find an *LU*-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$$

Solution

$$\begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad -\tfrac{1}{3}R_1 \quad \boxed{-\tfrac{1}{3}:\ell_1}$$

$$\begin{bmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad R_2-R_1 \quad \boxed{-1:\ell_{21}}$$

$$\begin{bmatrix} -3 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix} -3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \tfrac{1}{2}R_2 \quad \boxed{\tfrac{1}{2}:\ell_{22}}$$

$$\begin{bmatrix} -3 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad R_3-R_2 \quad \boxed{-1:\ell_{32}}$$

$$\begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{E_4^{-1}} \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad U \qquad\qquad \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad L$$

$$\begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix} \rightarrow \begin{cases} -3y_1=-33 \Rightarrow \underline{y_1=11} \\ y_1+2y_2=7 \Rightarrow \underline{y_2=-2} \\ y_2+y_3=-1 \Rightarrow \underline{y_3=1} \end{cases}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 11 \\ -2 \\ 1 \end{bmatrix} \rightarrow \begin{cases} x_1-4x_2+2x_3=11 \Rightarrow x_1=1 \\ x_2=-2 \\ x_3=1 \end{cases}$$

Solution: $\underline{x_1=1, \quad x_2=-2, \quad x_3=1}$

Exercise

Find an *LU*-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \\ 7 \end{bmatrix}$$

Solution

$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$	$-R_1$	$\boxed{-1 : \ell_{11}}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$	$R_2 - 2R_1$	$\boxed{-2 : \ell_{21}}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 3 & 0 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$	$\frac{1}{3}R_2$	$\boxed{\frac{1}{3} : \ell_{22}}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$	$R_3 + R_2$	$\boxed{1 : \ell_{23}}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_4^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$	$\frac{1}{2}R_3$	$\boxed{\frac{1}{2} : \ell_{23}}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_4^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$	$R_4 - R_3$	$\boxed{-1 : \ell_{43}}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{E_5^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$	$\frac{1}{4}R_4$	$\boxed{\frac{1}{4} : \ell_{43}}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & \frac{1}{4} \end{bmatrix} \xrightarrow{E_6^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{U}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix} \quad \boldsymbol{L}$$

For lower triangular: $\begin{bmatrix} -1 & 0 & 0 & 0 \\ -2 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & \frac{1}{4} \end{bmatrix} \xrightarrow{E^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \\ 7 \end{bmatrix} \rightarrow \begin{cases} -y_1 = 5 \rightarrow \underline{y_1 = -5|} \\ 2y_1 + 3y_2 = -1 \rightarrow \underline{y_2 = 3|} \\ -y_2 + 2y_3 = 3 \rightarrow \underline{y_3 = 3|} \\ y_3 + 4y_4 = 7 \rightarrow \underline{y_4 = 1|} \end{cases}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 3 \\ 1 \end{bmatrix} \rightarrow \begin{cases} x_1 - x_3 = -5 \rightarrow \underline{x_1 = -3|} \\ x_2 + 2x_4 = 3 \rightarrow \underline{x_2 = 1|} \\ x_3 + x_4 = 3 \rightarrow \underline{x_3 = 2|} \\ \underline{x_4 = 1|} \end{cases}$$

Solution: $\underline{x_1 = -3, \quad x_2 = 1, \quad x_3 = 2, \quad x_4 = 3|}$