Solution *Section* 3.6 – Vectors in 2-Space, 3-Space, and *n*-Space

Exercise

Sketch the following vectors with initial points located at the origin

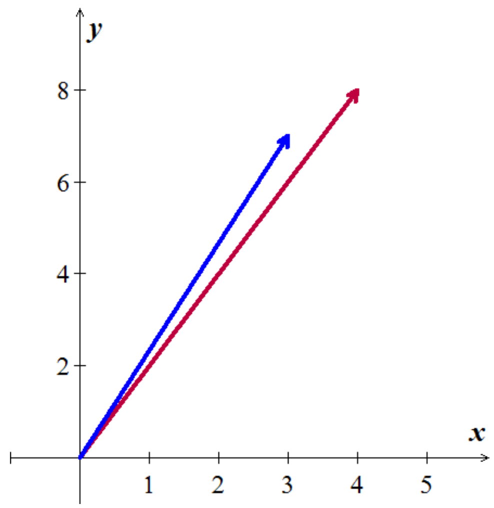
a) $P_1(4,8)\quad P_2(3,7)$

b) $P_1(-1,0,2)\quad P_2(0,-1,0)$

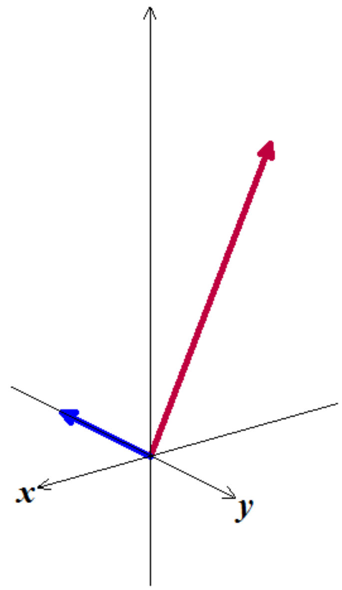
c) $P_1(3,-7,2)\quad P_2(-2,5,-4)$

Solution

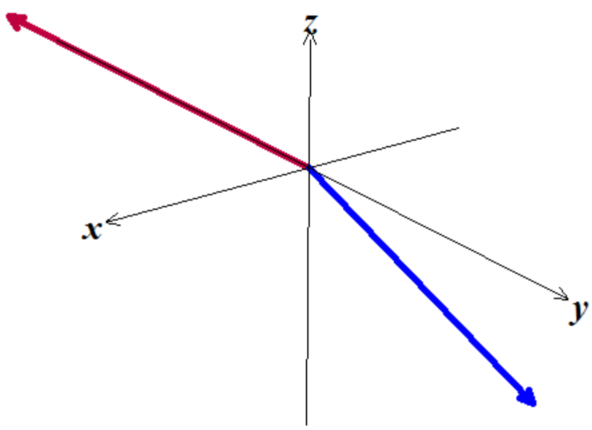
a)



b)



c)



Exercise

Find the components of the vector $\overrightarrow{P_1P_2}$

a) $P_1(3,5)\quad P_2(2,8)$

b) $P_1(5,-2,1)\quad P_2(2,4,2)$

c) $P_1(0,0,0)\quad P_2(-1,6,1)$

Solution

a) $\overrightarrow{P_1P_2}=(2-3,\ 8-5)\ \underline{=(-1,\ 3)}$

b) $\overrightarrow{P_1P_2}=(2-5,\ 4-(-2),\ 2-1)\ \underline{=(-3,\ 6,\ 1)}$

c) $\overrightarrow{P_1P_2}=(-1-0,\ 6-0,\ 1-0)\ \underline{=(-1,\ 6,\ 1)}$

Exercise

Find the terminal point of the vector that is equivalent to ***u*** = (1, 2) and whose initial point is $A(1,1)$

Solution

The terminal point: $B(b_1,\ b_2)$

$(b_1-1,\ b_2-1)=(1,2)$

$$\begin{cases} b_1-1=1 & \Rightarrow b_1=2 \\ b_2-1=2 & \Rightarrow b_2=3 \end{cases}$$

The terminal point: $\underline{B(2,\ 3)}$

Exercise

Find the initial point of the vector that is equivalent to ***u*** = (1, 1, 3) and whose terminal point is $B(-1,-1,2)$

Solution

The initial point: $A(x,y,z)$

$(-1-x,-1-y,2-z)=(1,1,3)$

$$\begin{cases} -1-x=1 & \Rightarrow x=-2 \\ -1-y=1 & \Rightarrow y=-2 \\ 2-z=3 & \Rightarrow z=-1 \end{cases}$$

The initial point: $\underline{A(-2,-2,-1)}$

Exercise

Find a nonzero vector $\boldsymbol{u}$ with initial point $P(-1, 3, -5)$ such that

a) $\boldsymbol{u}$ has the same direction as $\boldsymbol{v} = (6, 7, -3)$

b) $\boldsymbol{u}$ is oppositely directed as $\boldsymbol{v} = (6, 7, -3)$

Solution

a) $\boldsymbol{u}$ has the same direction as $\boldsymbol{v} \Rightarrow \boldsymbol{u} = \boldsymbol{v} = (6, 7, -3)$

The initial point $P(-1, 3, -5)$ then the terminal point : $(-1+6, 3+7, -5-3) = \boxed{(5, 10, -8)}$

b) $\boldsymbol{u}$ is oppositely as $\boldsymbol{v} \Rightarrow \boldsymbol{u} = -\boldsymbol{v} = (-6, -7, 3)$

The initial point $P(-1, 3, -5)$ then the terminal point : $(-1-6, 3-7, -5+3) = \boxed{(-7, -4, -2)}$

Exercise

Let $\boldsymbol{u} = (-3, 1, 2)$, $\boldsymbol{v} = (4, 0, -8)$, and $\boldsymbol{w} = (6, -1, -4)$. Find the components

a) $v - w$

b) $6u + 2v$

c) $5(v - 4u)$

d) $-3(v - 8w)$

e) $(2u - 7w) - (8v + u)$

f) $-u + (v - 4w)$

Solution

a) $v - w = (4-6,\ 0-(-1),\ -8-(-4)) = \boxed{(-2,\ 1,\ -4)}$

b) $6u + 2v = (-18, 6, 12) + (8, 0, -16) = \boxed{(-10,\ 6,\ -4)}$

c) $5(v - 4u) = 5(4-(-12), 0-4, -8-8) = 5(16, -4, -16) = \boxed{(80,\ -20,\ -80)}$

d) $-3(v - 8w) = -3(4-48, 0-(-8), -8-(-32)) = -3(-44, 8, 24) = \boxed{(32,\ -24,\ -72)}$

e)
$$\begin{aligned}(2u - 7w) - (8v + u) &= \left[(-6, 2, 4) - (42, -7, -28)\right] - \left[(32, 0, -64) + (-3, 1, 2)\right] \\ &= (-48, 9, 32) - (29, 1, -62) \\ &= \boxed{(-77,\ 8,\ 94)}\end{aligned}$$

f)
$$\begin{aligned}-u + (v - 4w) &= (3, -1, -2) + \left[(4, 0, -8) - (24, -4, -16)\right] \\ &= (3, -1, -2) + (-20, 4, 8) \\ &= \boxed{(-17,\ 3,\ 6)}\end{aligned}$$

Exercise

Let $\boldsymbol{u}$ = (2, 1, 0, 1, –1) and $\boldsymbol{v}$ = (–2, 3, 1, 0, 2). Find scalars a and b so that $a\boldsymbol{u} + b\boldsymbol{v}$ = (–8, 8, 3, –1, 7)

Solution

$$\begin{aligned} au+bv &= a(2,1,0,1,-1)+b(-2,3,1,0,2) \\ &= (a-2b,\ a+3b,\ b,\ a,\ -a+2b) \\ &= (-8,8,3,-1,7) \end{aligned}$$

$$\begin{cases} a-2b=-8 \\ a+3b=8 \\ b=3 \\ a \quad =-1 \\ -a+2b=7 \end{cases} \quad \to \quad a=-1 \quad b=3 \ \textit{Unique solution}$$

Exercise

Find all scalars $c_1,\ c_2,\ and\ c_3$ such that $c_1(1,2,0)+c_2(2,1,1,)+c_3(0,3,1)=(0,0,0)$

Solution

$$c_1(1,2,0)+c_2(2,1,1,)+c_3(0,3,1)=(c_1+2c_2, 2c_1+c_2+3c_3, c_2+c_3)=(0,0,0)$$

$$\begin{cases} c_1+2c_2 \qquad =0 \\ 2c_1+c_2+3c_3=0 \\ \qquad c_2+c_3=0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\boxed{c_1=c_2=c_3=0}$$

Exercise

Find the distance between the given points $\begin{bmatrix}5 & 1 & 8 & -1 & 2 & 9\end{bmatrix}$, $\begin{bmatrix}4 & 1 & 4 & 3 & 2 & 8\end{bmatrix}$

Solution

$$\begin{aligned} d &= \sqrt{(4-5)^2+(1-1)^2+(4-8)^2+(3+1)^2+(2-2)^2+(8-9)^2} \\ &= \sqrt{1+0+16+16+0+1} \\ &= \sqrt{34} \end{aligned}$$

Exercise

Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operation on $\boldsymbol{u}=(u_1,\ u_2)\quad \boldsymbol{v}=(v_1,\ v_2)$

$$\boldsymbol{u}+\boldsymbol{v}=(u_1+v_1+1,\ u_2+v_2+1)\qquad k\boldsymbol{u}=(ku_1,\ ku_2)$$

a) Compute $\boldsymbol{u}+\boldsymbol{v}$ and $k\boldsymbol{u}$ for $\boldsymbol{u}$ = (0, 4), $\boldsymbol{v}$ = (1, –3), and k = 2.

b) Show that (0, 0) ≠ **0**.

c) Show that (–1, –1) = 0.

d) Show that $\boldsymbol{u}+(-\boldsymbol{u})=0$ for $\boldsymbol{u}=(u_1,\ u_2)$

e) Find two vector space axioms that fail to hold.

Solution

a) $\boldsymbol{u}+\boldsymbol{v}=(0+1+1,\ 4-3+1)=\boxed{(2,\ 2)}$

$k\boldsymbol{u}=(ku_1,\ ku_2)=(2(0),\ 2(4))=\boxed{(0,\ 8)}$

b)
$$\begin{aligned}(0,0)+(u_1,u_2)&=(0+u_1+1,\ 0+u_2+1)\\&=(u_1+1,\ u_2+1)\\&\neq(u_1,\ u_2)\end{aligned}$$

Therefore (0, 0) is not the zero vector **0** required (by Axiom).

c)
$$\begin{aligned}(-1,-1)+(u_1,u_2)&=(-1+u_1+1,\ -1+u_2+1)\\&=(u_1,\ u_2)\end{aligned}$$
$$\begin{aligned}(u_1,u_2)+(-1,-1)&=(u_1-1+1,\ u_2-1+1)\\&=(u_1,\ u_2)\end{aligned}$$

Therefore (–1, –1) = **0** holds.

d) Let $\boldsymbol{u}=(u_1,\ u_2)\ -\boldsymbol{u}=(-2-u_1,\ -2-u_2)$

$$\begin{aligned}\boldsymbol{u}+(-\boldsymbol{u})&=(u_1+(-2-u_1)+1,\ u_2+(-2-u_2)+1)\\&=(-1,\ -1)\\&\boxed{=\mathbf{0}}\end{aligned}$$

$\boldsymbol{u}+(-\boldsymbol{u})=0$ holds

e) Axiom 7: $k(\boldsymbol{u}+\boldsymbol{v})\overset{?}{=}k\boldsymbol{u}+k\boldsymbol{v}$

$$k(\boldsymbol{u}+\boldsymbol{v}) = k(u_1+v_1+1,\ u_2+v_2+1) = (ku_1+kv_1+k,\ ku_2+kv_2+k)$$

$$k\boldsymbol{u}+k\boldsymbol{v} = (ku_1,\ ku_2)+(kv_1,\ kv_2) = (ku_1+kv_1+1,\ ku_2+kv_2+1)$$

Therefore, $k(\boldsymbol{u}+\boldsymbol{v}) \neq k\boldsymbol{u}+k\boldsymbol{v}$; Axiom 7 fails to hold

Axiom 8: $(k+m)\boldsymbol{u} \overset{?}{=} k\boldsymbol{u}+m\boldsymbol{u}$

$$(k+m)\boldsymbol{u} = ((k+m)u_1,\ (k+m)u_2) = (ku_1+mu_1,\ ku_2+mu_2)$$

$$k\boldsymbol{u}+m\boldsymbol{u} = (ku_1,\ ku_2)+(mu_1,\ mu_2) = (ku_1+mu_1+1,\ ku_2+mu_2+1)$$

Therefore, $(k+m)\boldsymbol{u} \neq k\boldsymbol{u}+m\boldsymbol{u}$; Axiom 8 fails to hold