Notebook 16: Multiple Integrals

Double Integrals

Double integrals are entered by nesting two single integrals. Parentheses are not required.

> $\int_0^1 \int_0^x x \cdot y \, dy \, dx$

$$\frac{1}{8}$$

The integral will be evaluated numerically if a decimal is used in one of the limits.

> $\int_0^1 \int_0^x x^x \, dy \, dx;$
> $\int_0^{1.0} \int_0^x x^x \, dy \, dx$

$$\int_0^1 x^x x \, dx$$

$$0.4030344444$$

The *evalf* command can be used to the same effect.

> $\int_0^1 \int_0^1 e^{-(x^2+y^2)} \, dy \, dx;$
> $evalf(\%)$

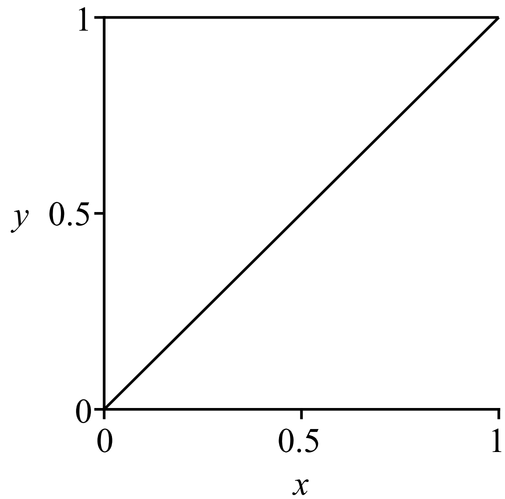
$$\frac{1}{4} \operatorname{erf}(1)^2 \pi$$

$$0.5577462855$$

Double Integrals in Polar Form

Consider the integral $\int_0^1 \int_x^1 \frac{y}{x^2+y^2} \, dy \, dx$ in Cartesian coordinates. To convert this integral to polar coordinates, the region in the xy plane must be converted to a region in the $r\theta$ plane. The region in the xy plane is defined by $0 \le x \le 1$ and $x \le y \le 1$. See the plot below.

> $plot([x, 1], x = 0..1, y = 0..1, color = black, tickmarks = [3, 3])$



The line $y = x$ converts to $\theta = \frac{\pi}{4}$ in polar coordinates, and the y-axis is $\theta = \frac{\pi}{2}$. So $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$, and r will go from 0 to the line $y = 1$ which is $r\sin(\theta) = 1$ in polar coordinates. That is,

$$0 \le r \le \frac{1}{\sin(\theta)} = \csc(\theta)$$

The integral then becomes

> $\int_{\pi/4}^{\pi/2} \int_0^{\csc(\theta)} \frac{r\sin(\theta)}{r^2} \cdot r \, dr \, d\theta$

$$\frac{1}{4}\pi$$

Triple Integrals in Rectangular and Cylindrical Coordinates

Consider the integral of $f(x, y, z) = x^2y^2z$ over the solid cylinder bounded by $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$.

In Cartesian coordinates, the region in the xy plane is defined by $-1 \le y \le 1$ and $-\sqrt{1 - y^2} \le x \le \sqrt{1 - y^2}$, so the integral would be

> $\int_0^1 \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2y^2z \, dx \, dy \, dz$

$$\frac{1}{48}\pi$$

By exploiting symmetry of this region, the above integral can be reduced to 4 times the integral of the region in the first quadrant.

> $4\int_0^1\int_0^1\int_0^{\sqrt{1-y^2}} x^2y^2z\,dx\,dy\,dz$

$$\frac{1}{48}\pi$$

In cylindrical coordinates, the solid cylinder bounded by $x^2+y^2=1$ is described by the region $0 \le \theta \le 2\pi$ and $0 \le r \le 1$. Converted to cylindrical coordinates, the integral becomes

> $\int_0^1\int_0^{2\pi}\int_0^1 r^4\cos^2(\theta)\sin^2(\theta)z\cdot r\,dr\,d\theta\,dz$

$$\frac{1}{48}\pi$$