Section 2.8 – Lagrange Multipliers

Constrained Maxima and Minima

We consider a problem where a constrained minimum can be found by eliminating a variable.

Example

Find the point $P(x,y,z)$ on the plane $2x+y-z-5=0$ that is closest to the origin.

Solution

$$\left|\overrightarrow{OP}\right| = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$$
$$= \sqrt{x^2+y^2+z^2}$$

Subject to the constraint that $2x+y-z-5=0$

Since $\left|\overrightarrow{OP}\right|$ has a minimum value wherever the function $f(x,y,z)=x^2+y^2+z^2$ has a minimum value.

$$2x+y-z-5=0 \implies z=2x+y-5$$

$$h(x,y)=f(x,\ y,\ 2x+y-5)$$
$$=x^2+y^2+(2x+y-5)^2$$

$h_x = 2x+2(2x+y-5)(2)$ $=10x+4y-20=0$	$h_y = 2y+2(2x+y-5)(1)$ $=4x+4y-10=0$

$$\rightarrow \begin{cases} 10x+4y=20 \\ 4x+4y=10 \end{cases} \implies \boxed{x=\frac{5}{3},\ y=\frac{5}{6}}$$

$$z=2\left(\frac{5}{3}\right)+\frac{5}{6}-5=-\frac{5}{6}$$

Therefore, the closest point to the origin is: $P\left(\frac{5}{3},\ \frac{5}{6},\ -\frac{5}{6}\right)$

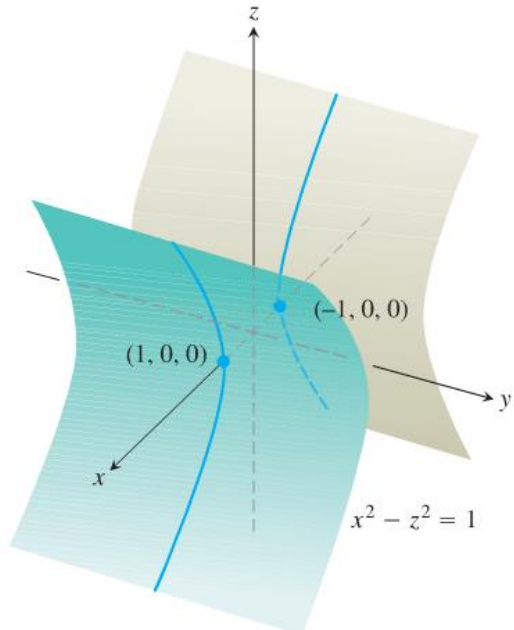
The distance from P to the origin is: $\sqrt{\left(\frac{5}{3}\right)^2+\left(\frac{5}{6}\right)^2+\left(-\frac{5}{6}\right)^2} \approx 2.04$

Example

Find the points on the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ that are closest to the origin.

Solution

The points closest to the origin are the points whose coordinates minimize the value of the function $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint that $x^2 - z^2 - 1 = 0$



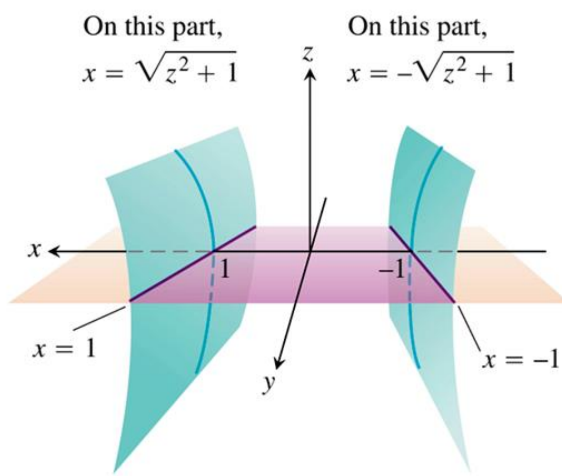
$x^2 - z^2 - 1 = 0 \quad \to \quad z^2 = x^2 - 1$

$$h(x, y) = f\left(x,\ y,\ \sqrt{x^2 - 1}\right)$$
$$= x^2 + y^2 + \left(x^2 - 1\right)$$
$$= 2x^2 + y^2 - 1$$

$h_x = 4x = 0$	$h_y = 2y = 0$

That is, at the point (0, 0) ????

The domain of h is the entire xy-plane, the domain from which we can select the first two coordinates of the points (x,y,z) on the cylinder is restricted to the shadow of the cylinder on the xy-plane; it does not include the band between the lines $x = -1$ and $x = 1$.



$x^2 - z^2 - 1 = 0 \quad \to \quad x^2 = z^2 + 1$

$$k(y,z) = f\left(z^2 + 1,\ y,\ z\right)$$
$$= z^2 + 1 + y^2 + z^2$$
$$= y^2 + 2z^2 + 1$$

$k_y = 2y = 0$	$k_z = 4z = 0$

That implies to $y = z = 0$ and which leads to

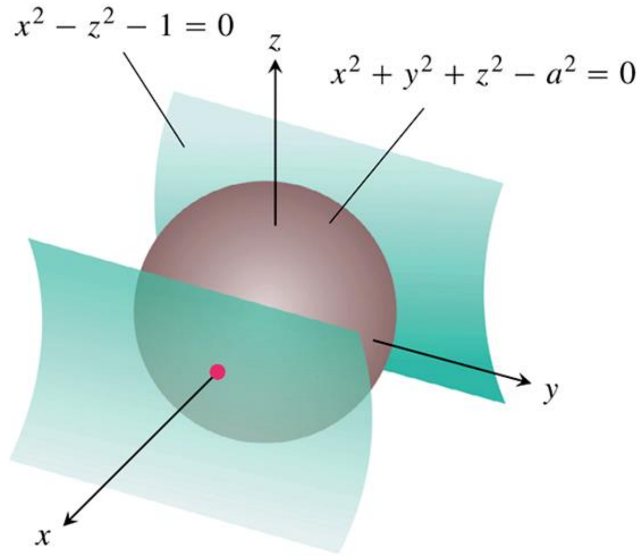
$x^2 = z^2 + 1 = 1 \quad \to \quad x = \pm 1$

The corresponding points on the cylinder are $(\pm 1,\ 0,\ 0)$.

$k(y,z) = y^2 + 2z^2 + 1 \geq 1$ gives a minimum value for k. We can also see that the minimum distance from the origin to a point on the cylinder is 1 unit.

Solution 2

Another way to find the points on the cylinder closet to the origin is to imagine a small sphere centered at the origin expanding until it touches the cylinder



$f(x,y,z) = x^2 + y^2 + z^2 - a^2 \quad and \quad g(x,y,z) = x^2 - z^2 - 1$

$\nabla f = \lambda \nabla g$

$2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k} = \lambda(2x\boldsymbol{i} - 2z\boldsymbol{k})$ $\qquad \nabla f = f_x\boldsymbol{i} + f_y\boldsymbol{j} + f_z\boldsymbol{k}$

$2x = 2\lambda x, \quad 2y = 0, \quad 2z = -2\lambda z$

Since that no point on the surface has a zero x-coordinate to conclude that $x \neq 0$.

Hence, $2x = 2\lambda x$ only if

$$2 = 2\lambda \quad \Rightarrow \quad \boxed{\lambda = 1}$$

For $\lambda = 1 \quad \rightarrow \quad 2z = -2\lambda z = -2z$, for this to satisfies, z must be zero.

Also $2y = 0 \ \Rightarrow \ y = 0$

We conclude that the points have coordinates of the form $(x,\ 0,\ 0)$

$x^2 = z^2 + 1 = 1 \quad \rightarrow \quad x = \pm 1$

The points on the cylinder closest to the origin are the points $(\pm 1,\ 0,\ 0)$.

The Method of *Lagrange* Multipliers

The method of Lagrange multipliers:

$$\nabla f = \lambda \nabla g$$

For some scalar λ (called a ***Lagrange multiplier***)

Theorem – The orthogonal Gradient Theorem

Suppose that $f(x,y,z)$ is differentiable in a region whose interior contains a smooth curve

$$C: \quad \boldsymbol{r}(t) = g(t)\boldsymbol{i} + h(t)\boldsymbol{j} + k(t)\boldsymbol{k}$$

If P_0 is a point on C where *f* has a local maximum or minimum relative to the values on C, then ∇f is orthogonal to *C* at P_0.

Corollary

At the points on a smooth curve $\boldsymbol{r}(t) = g(t)\boldsymbol{i} + h(t)\boldsymbol{j}$ where differentiable function $f(x,y)$ takes on its local maxima and minima relative to its values on the curve, $\nabla f \bullet \boldsymbol{v} = 0$ where $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$.

The Method of Lagrange Multipliers

Suppose that $f(x,y,z)$ and $g(x,y,z)$ are differentiable and $\nabla g \neq 0$ when $g(x,y,z) = 0$. To find the local maximum and minimum values of *f* subject to the constraint $g(x,y,z) = 0$ (if these exist), find the values of *x*, *y*, *z*, and λ that simultaneously satisfy the equations

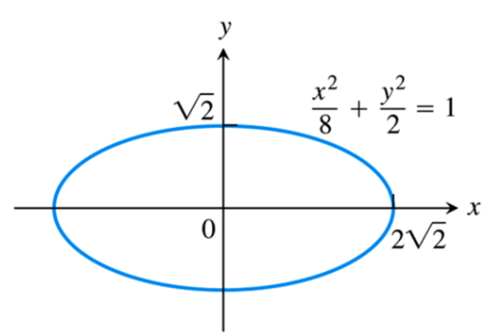
$$\nabla f = \lambda \nabla g \quad and \quad g(x,y,z) = 0$$

For functions of two independent variables, the condition is similar, but without the variables *z*.

Example

Find the greatest and smallest values that the function $f(x,y) = xy$ takes on the ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} = 1$$



Solution

$f(x,y) = xy$ subject to the constraint

$$g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$$

We need to find: $\nabla f = \lambda \nabla g \quad and \quad g(x,y,z) = 0$

$y\boldsymbol{i} + x\boldsymbol{j} = \frac{1}{4}\lambda x\boldsymbol{i} + \lambda y\boldsymbol{j}$ $\qquad\qquad \nabla f = f_x \boldsymbol{i} + f_y \boldsymbol{j}$

$y = \frac{1}{4}\lambda x, \quad x = \lambda y$

$y = \frac{1}{4}\lambda(\lambda y) = \frac{1}{4}\lambda^2 y$

$y = 0 \quad or \quad 1 = \frac{1}{4}\lambda^2$

$\lambda^2 = 4 \quad \Rightarrow \quad \lambda = \pm 2$

Consider these two cases:

***Case* 1**: If $y = 0$, then $x = y = 0$. But (0, 0) is not on the ellipse. Hence, $y \neq 0$.

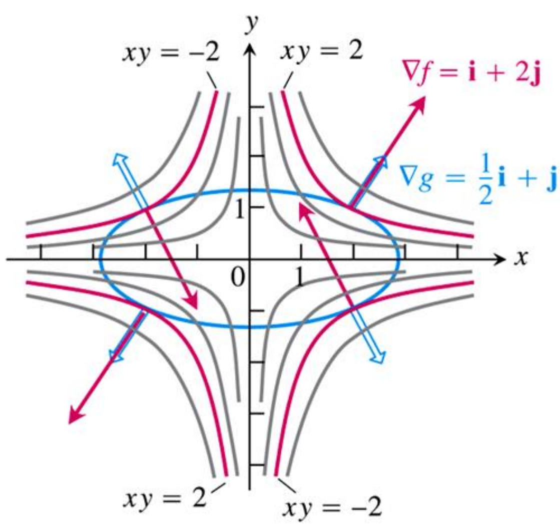
***Case* 2**: If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$.

$$g(x,y) = \frac{(\pm 2y)^2}{8} + \frac{y^2}{2} - 1 = 0$$

$$\frac{y^2}{2} + \frac{y^2}{2} = 1$$

$$y^2 = 1 \quad \Rightarrow \quad \boxed{y = \pm 1}$$

Therefore, $f(x,y) = xy$ takes on its extreme values on the ellipse at the points $(\pm 2, \ \pm 1)$. The extreme values are $xy = 2 \quad and \quad xy = -2$



The Geometry of the solution: The level curves of the function $f(x,y) = xy$ are the hyperbolas $xy = c$

At the point (2, 1): $\quad \nabla f = y\boldsymbol{i} + x\boldsymbol{j} = \boldsymbol{i} + 2\boldsymbol{j}, \quad \nabla g = \frac{1}{4}\lambda x\boldsymbol{i} + \lambda y\boldsymbol{j} = \frac{1}{2}\boldsymbol{i} + \boldsymbol{j}, \quad \nabla f = 2\nabla g$

At the point (–2, 1): $\quad \nabla f = \boldsymbol{i} - 2\boldsymbol{j}, \quad \nabla g = -\frac{1}{2}\boldsymbol{i} + \boldsymbol{j}, \quad \nabla f = -2\nabla g$

Example

Find the maximum and minimum values that the function $f(x,y)=3x+4y$ on the circle $x^2+y^2=1$

Solution

$f(x,y)=3x+4y \qquad g(x,y)=x^2+y^2-1=0$

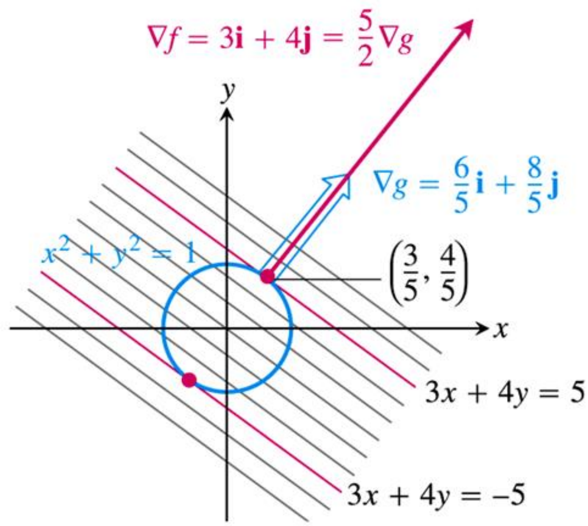
$\nabla f = f_x\boldsymbol{i}+f_y\boldsymbol{j}=3\boldsymbol{i}+4\boldsymbol{j} \qquad \nabla g = 2x\boldsymbol{i}+2y\boldsymbol{j}$

$\nabla f=\lambda\nabla g$

$3\boldsymbol{i}+4\boldsymbol{j}=2\lambda x\boldsymbol{i}+2\lambda y\boldsymbol{j}$

$2\lambda x=3, \quad 2\lambda y=4$

$x=\frac{3}{2\lambda}, \quad y=\frac{2}{\lambda}$



$\left(\frac{3}{2\lambda}\right)^2+\left(\frac{2}{\lambda}\right)^2-1=0$

$\frac{9}{4\lambda^2}+\frac{4}{\lambda^2}=1$

$9+16=4\lambda^2$

$25=4\lambda^2$

$\lambda^2=\frac{25}{4} \quad \rightarrow \quad \boxed{\lambda=\pm\frac{5}{2}}$

$x=\frac{3}{2\lambda}=\pm\frac{3}{5}, \quad y=\frac{2}{\lambda}=\pm\frac{4}{5}$

Therefore, $f(x,y)=3x+4y$ has extreme values $\left(\pm\frac{3}{5},\ \pm\frac{4}{5}\right)$

$f\left(\frac{3}{5},\ \frac{4}{5}\right)=3\left(\frac{3}{5}\right)+4\left(\frac{4}{5}\right)=5$

$f\left(-\frac{3}{5},\ -\frac{4}{5}\right)=3\left(-\frac{3}{5}\right)+4\left(-\frac{4}{5}\right)=-5$

The Geometry of the solution: The level curves of the function $f(x,y)=3x+4y$ are the lines $3x+4y=c$

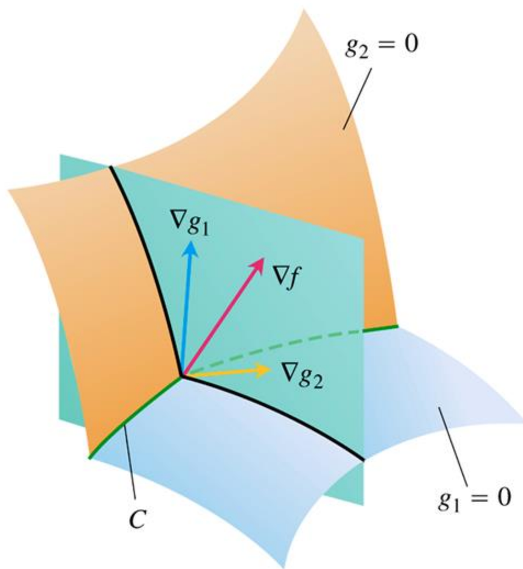
At the point $\left(\frac{3}{5},\ \frac{4}{5}\right)$: $\qquad \nabla f=3\boldsymbol{i}+4\boldsymbol{j}, \quad \nabla g=\frac{6}{5}\boldsymbol{i}+\frac{8}{5}\boldsymbol{j}, \quad \nabla f=\frac{5}{2}\nabla g$

Lagrange Multipliers with Two Constraints

To find the extreme values of a differentiable function $f(x,y,z)$ whose variables are subject to two constraints. If the constraints are

$$g_1(x,y,z)=0 \quad and \quad g_2(x,y,z)=0$$

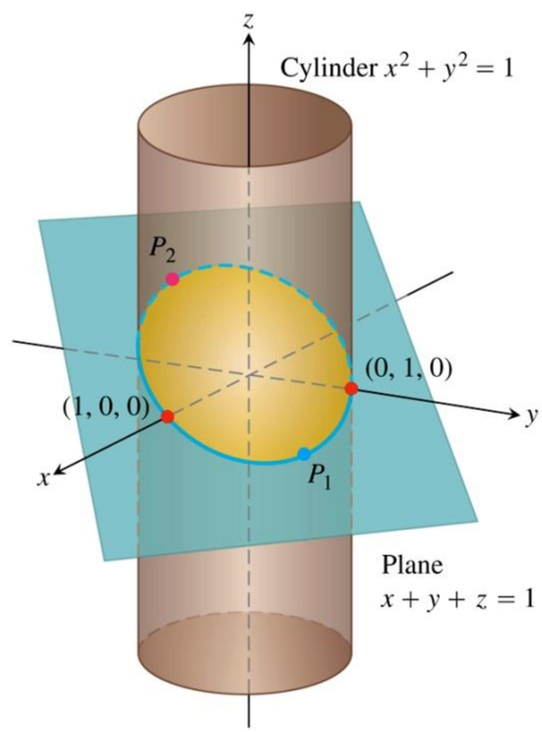
g_1 *and* g_2 are differentiable, with ∇g_1 not parallel to ∇g_2



$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \quad g_1(x,y,z)=0, \quad g_2(x,y,z)=0$$

Example

The plane $x+y+z=1$ cuts the cylinder $x^2+y^2=1$ in an ellipse. Find the points on the ellipse that lie closet to and farthest from the origin.



Solution

$$f(x,y,z)=x^2+y^2+z^2$$

$$g_1(x,y,z)=x^2+y^2-1=0$$

$$g_2(x,y,z)=x+y+z-1=0$$

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

$$2x\boldsymbol{i}+2y\boldsymbol{j}+2z\boldsymbol{k}=\lambda(2x\boldsymbol{i}+2y\boldsymbol{j})+\mu(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})$$

$$2x\boldsymbol{i}+2y\boldsymbol{j}+2z\boldsymbol{k}=(2\lambda x+\mu)\boldsymbol{i}+(2\lambda y+\mu)\boldsymbol{j}+\mu\boldsymbol{k}$$

$2z=\mu$	$2x=2\lambda x+\mu$ $2(1-\lambda)x=\mu=2z$ $(1-\lambda)x=z$	$2y=2\lambda y+\mu$ $2(1-\lambda)y=\mu=2z$ $(1-\lambda)y=z$

$(1-\lambda)x=z=(1-\lambda)y$

These satisfy if either λ = 1 and $z = 0$ or λ ≠ 1 and $x=y=\dfrac{z}{1-\lambda}$

If $z = 0$,

$$\begin{cases} x^2+y^2-1=0 \\ x+y-1=0 \end{cases} \rightarrow \quad x=1-y$$

$$(1-y)^2+y^2-1=0$$

$$1-2y+y^2+y^2-1=0$$

$$2y(y-1)=0 \quad \Rightarrow \quad \begin{cases} y=0 & \rightarrow x=1 \\ y=1 & \rightarrow x=0 \end{cases}$$

The points are: $(1,\ 0,\ 0)$ and $(0,\ 1,\ 0)$

If $x=y$,

$x^2+y^2-1=0$ $x^2+x^2-1=0$ $2x^2=1$ $x^2=\frac{1}{2}$ $x=\pm\frac{1}{\sqrt{2}}=\pm\frac{\sqrt{2}}{2}$	$x+y+z-1=0$ $2x+z=1$ $z=1-2x$ $z=1\pm\sqrt{2}$

The points are: $\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2},\ 1-\sqrt{2}\right)$ and $\left(-\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2},\ 1+\sqrt{2}\right)$

The points on the ellipse closest to the origin are $(1,\ 0,\ 0)$ and $(0,\ 1,\ 0)$. The point on the farthest from the origin is $\left(-\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2},\ 1+\sqrt{2}\right)$.

Exercises *Section* 2.8 – Lagrange Multipliers

1. Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x,y) = xy$ has its extreme values.

2. Find the extreme values of $f(x,y) = xy$ subject to the constraint $g(x,y) = x^2 + y^2 - 10 = 0$.

3. Find the extreme values of $f(x,y) = x^3 + y^2$ on the circle $x^2 + y^2 = 1$

4. Find the extreme values of $f(x,y) = x^2 + y^2 - 3x - xy$ on the circle $x^2 + y^2 \le 9$

5. Find the maximum value of $f(x,y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$.

6. Find the points on the curve $x^2 y = 2$ nearest the origin.

7. Use the method of Lagrange multipliers to find

a) The minimum value of $x + y$, subject to the constraints $xy = 16, \quad x > 0, \quad y > 0$

b) The maximum value of xy, subject to the constraints $x + y = 16$

8. Find the radius and height of the open right circular cylinder of largest surface area that can be inscribed in a sphere of radius *a*. What is the largest surface area?

9. Use the method of Lagrange multipliers to find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ with sides parallel to the coordinate axes.

10. Find the maximum and minimum values of $x^2 + y^2$ subject to the constraint $x^2 - 2x + y^2 - 4y = 0$

11. The temperature at a point (x,y) on a metal plate is $T(x,y) = 4x^2 - 4xy + y^2$. An ant on the plate walks around the circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the ant?

12. Your firm has been asked to design a storage tank for liquid petroleum gas. The customer's specifications call for a cylindrical tank with hemispherical ends, and the tank is to hold $8000\ m^3$ of gas. He customer also wants to use the smallest amount of material possible in building the tank. What radius and height do you recommend for the cylindrical portion of the tank?

13. A closed rectangular box is to have volume $V\ cm^3$. The cost of the material used in the box is a $a\ cents/cm^2$ for top and bottom, $b\ cents/cm^2$ for front and back, and $c\ cents/cm^2$ for the remaining sides. What dimensions minimize the total cost of materials?

14. Find the extreme values of $f(x,y,z) = x(y+z)$ on the curve of intersection of the right circular cylinder $x^2 + y^2 = 1$ and the hyperbolic cylinder $xz = 1$.

15. Find the point closest to the origin on the curve of intersection of the plane $x+y+z=1$ and the cone $z^2=2x^2+2y^2$

16. Find the point on the plane $x+2y+3z=13$ closest to the point $(1,\ 1,\ 1)$

17. Find the point on the sphere $x^2+y^2+z^2=4$ farthest from the point $(1,\ -1,\ 1)$

18. Find the minimum distance from the surface $x^2-y^2-z^2=1$ to the origin

19. Find the maximum and minimum values of $f(x,y,z)=x-2y+5z$ on the sphere $x^2+y^2+z^2=30$

20. Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

21. A space probe in the shape of the ellipsoid $4x^2+y^2+4z^2=16$ enters Earth's atmosphere and its surface begins to heat. After 1 *hour*, the temperature at the point (x,y,z) on the probe's surface is

$T(x,y,z)=8x^2+4yz-16z+600$.

Find the hottest point on the probe's surface.

22. What point on the plane $x+y+4z=8$ is closest to the origin? Give an argument showing you have found an absolute minimum of the distance function.

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

23. $f(x,\ y)=2x+y+10$ *subject to* $2(x-1)^2+4(y-1)^2=1$

24. $f(x,\ y)=x^2y^2$ *subject to* $2x^2+y^2=1$

25. $f(x,\ y)=x+2y$ *subject to* $x^2+y^2=4$

26. $f(x,\ y)=xy^2$ *subject to* $x^2+y^2=1$

27. $f(x,\ y)=x+y$ *subject to* $x^2-xy+y^2=1$

28. $f(x,\ y)=x^2+y^2$ *subject to* $2x^2+3xy+2y^2=7$

29. $f(x,\ y)=xy$ *subject to* $x^2+y^2-xy=9$

30. $f(x,\ y)=x-y$ *subject to* $x^2-3xy+y^2=20$

31. $f(x,\ y)=e^{2xy}$ *subject to* $x^2+y^2=16$

32. $f(x,\ y)=x^2+y^2$ *subject to* $x^6+y^6=1$

33. $f(x,\ y)=y^2-4x^2$ *subject to* $x^2+2y^2=4$

34. $f(x,\ y)=xy+x+y$ *subject to* $x^2y^2=4$

35. $f(x,\ y,\ z)=x+3y-z$ *subject to* $x^2+y^2+z^2=4$

36. $f(x, y, z) = xyz$ *subject to* $x^2 + 2y^2 + 4z^2 = 9$

37. $f(x, y, z) = x$ *subject to* $x^2 + y^2 + z^2 - z = 1$

38. $f(x, y, z) = x - z$ *subject to* $x^2 + y^2 + z^2 - y = 2$

39. $f(x, y, z) = x^2 + y^2 + z^2$ *subject to* $x^2 + y^2 + z^2 - 4xy = 1$

40. $f(x, y, z) = x + y + z$ *subject to* $x^2 + y^2 + z^2 - 2x - 2y = 1$

41. $f(x, y, z) = 2x + z^2$ *subject to* $x^2 + y^2 + 2z^2 = 25$

42. $f(x, y, z) = x^2 + y^2 - z$ *subject to* $z = 2x^2y^2 + 1$

43. $f(x, y, z) = \sqrt{xyz}$ *subject to* $x + y + z = 1$ *with* $x \geq 0,\ y \geq 0,\ z \geq 0$

44. $f(x, y, z) = x^2 + y^2 + z^2$ *subject to* $x^2 + y^2 + z^2 - 4xy = 1$

45. $f(x, y, z) = x + 2y - z$ *subject to* $x^2 + y^2 + z^2 = 1$

46. $f(x, y, z) = x^2y^2z$ *subject to* $2x^2 + y^2 + z^2 = 25$

47. Use Lagrange multipliers to find the dimensions of the rectangle with the maximum perimeter that can be inscribed with sides parallel to the coordinate axes in the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

48. Use Lagrange multipliers to find the dimensions of the right circular cylinder of minimum surface area (including the circular ends) with a volume of $32\pi\ in^3$

49. Find the point(s) on the cone $z^2 - x^2 - y^2 = 0$ that are closest to the point $(1,\ 3,\ 1)$. Give an argument showing you have found an absolute minimum of the distance function.

50. Let $P_0(a,\ b,\ c)$ be a fixed point in $\mathbb{R}^3$ and let $d(x,\ y,\ z)$ be the distance between P_0 and a variable point $P(x,\ y,\ z)$.

a) Compute $\nabla d(x,\ y,\ z)$

b) Show that $\nabla d(x,\ y,\ z)$ points in the direction from P_0 to P and has magnitude 1 for all $(x,\ y,\ z)$.

c) Describe the level surfaces of d and give the direction of $\nabla d(x,\ y,\ z)$ relative to the level surfaces of d.

d) Discuss $\lim\limits_{P \to P_0} \nabla d(x,\ y,\ z)$

51. A shipping company requires that the sum of length plus girth of rectangular boxes must not exceed 108 in. Find the dimensions of the box with maximum volume that meets this condition. (the girth is the perimeter of the smallest base of the box).

52. Find the rectangular box with a volume of $16\ ft^3$ that has minimum surface area.

53. Find the minimum and maximum distances between the ellipse $x^2 + xy + y^2 = 1$ and the origin.

54. Find the dimensions of the rectangle of maximum area with sides parallel to the coordinate axes that can be inscribed in the ellipse $4x^2 + 16y^2 = 16$

55. Find the dimensions of the rectangle of maximum perimeter with sides parallel to the coordinate axes that can be inscribed in the ellipse $2x^2 + 4y^2 = 3$

56. Find the point on the plane $2x + 3y + 6z - 10 = 0$ *closest to the point* $(-2,\ 5,\ 1)$

57. Find the point on the surface $4x + y - 1 = 0$ *closest to the point* $(1,\ 2,\ -3)$

58. Find the points on the cone $z^2 = x^2 + y^2$ *closest to the point* $(1,\ 2,\ 0)$

59. Find the minimum and maximum distances between the sphere $x^2 + y^2 + z^2 = 9$ *closest to the point* $(2,\ 3,\ 4)$

60. Find the maximum value of $x_1 + x_2 + x_3 + x_4$ *subject to* $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 16$

61. Find the maximum value of $x_1 + x_2 + \cdots + x_n$ *subject to* $x_1^2 + x_2^2 + \cdots + x_n^2 = c^2$

62. Find the maximum value of $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$ *subject to* $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ for the given positive real numbers $a_1, a_2, \cdots, a_n$

63. The planes $x + 2z = 12$ and $x + y = 6$ intersect in a line *L*. Find the point on *L* nearest the origin.

64. Find the maximum and minimum values of $f(x,\ y,\ z) = xyz$ *subject to* $x^2 + y^2 = 4$ *and* $x + y + z = 1$

65. The paraboloid $z = x^2 + 2y^2 + 1$ and the plane $x - y + 2z = 4$ intersect in a curve *C*. Find the points on *C* that have minimum and maximum distance from the origin.

66. Find the maximum and minimum values of $f(x,\ y,\ z) = x^2 + y^2 + z^2$ on the curve on which the cone $z^2 = 4x^2 + 4y^2$ and the plane $2x + 4z = 5$ intersect.

67. The temperature of points on a elliptical plate $x^2 + y^2 + xy \le 1$ is given by $T(x,\ y) = 25\left(x^2 + y^2\right)$. Find the hottest and coldest temperatures on the edge of the elliptical plate.