Solution Section 2.7 – First-Order Linear Equations

Exercise

Write an equivalent first-order differential equation and initial condition for y. $y = \int_1^x \frac{1}{t}\,dt$

Solution

$$\int_1^x \frac{1}{t}\,dt \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{x}$$

$$y(1) = \int_1^1 \frac{1}{t}\,dt = \ln t \Big|_1^1 = \ln 1 - \ln 1 = 0 \qquad\qquad \int_a^a f(x)dx = 0$$

$$\boxed{\frac{dy}{dx} = \frac{1}{x}; \quad y(1) = 0}$$

Exercise

Write an equivalent first-order differential equation and initial condition for $y = 2 - \int_0^x (1 + y(t))\sin t\,dt$

Solution

$$y = 2 - \int_0^x (1 + y(t))\sin t\ dt \quad \Rightarrow \quad \frac{dy}{dx} = -(1 + y(x))\sin x$$

$$y(0) = 2 - \int_0^0 (1 + y(t))\sin t\ dt = 2 \qquad\qquad \int_a^a f(x)dx = 0$$

$$\boxed{\frac{dy}{dx} = -(1 + y(x))\sin x; \quad y(0) = 2}$$

Exercise

Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals

$$y' = 1 - \frac{y}{x}, \quad y(2) = -1, \quad dx = 0.5$$

Solution

$$y_1 = y_0 + \left(1 - \frac{y_0}{x_0}\right)dx = -1 + \left(1 - \frac{-1}{2}\right)(0.5) = -0.25$$

$$y_2 = y_1 + \left(1 - \frac{y_1}{x_1}\right)dx = -0.25 + \left(1 - \frac{-0.25}{2.5}\right)(0.5) = 0.3$$

$$y_3 = y_2 + \left(1 - \frac{y_2}{x_2}\right)dx = 0.3 + \left(1 - \frac{0.3}{3}\right)(0.5) = 0.75$$

$$y' + \frac{1}{x}y = 1 \qquad P(x) = \frac{1}{x}, \quad Q(x) = 1$$

$$y_h = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int (1)e^{\int \frac{1}{x}dx}dx = \int xdx = \frac{1}{2}x^2$$

$$y(x) = \frac{1}{x}\left(\frac{1}{2}x^2 + C\right) = \frac{1}{2}x + \frac{C}{x}$$

$$y(2) = \frac{1}{2}(2) + \frac{C}{2} = -1$$

$$1 + \frac{C}{2} = -1$$

$$\frac{C}{2} = -2 \qquad \underline{\to C = -4|}$$

$$\underline{y(x) = \frac{x}{2} - \frac{4}{x}|}$$

$$y(3.5) = \frac{3.5}{2} - \frac{4}{3.5} \ \underline{\approx 0.6071|}$$

Exercise

Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals

$$y' = x(1-y), \quad y(1) = 0, \quad dx = 0.2$$

Solution

$$y_1 = y_0 + x_0\left(1 - y_0\right)dx = 0 + 1(1-0)(0.2) = 0.2$$

$$y_2 = y_1 + x_1\left(1 - y_1\right)dx = 0.2 + 1.2(1-0.2)(0.2) = 0.392$$

$$y_3 = y_2 + x_2\left(1 - y_2\right)dx = 0.392 + 1.4(1-0.392)(0.2) = .5622$$

$$\frac{y'}{1-y} = xdx \quad \Rightarrow \quad \int \frac{dy}{1-y} = \int xdx$$

$$\ln|1-y| = \frac{1}{2}x^2 + C$$

$$1 - y = e^{\frac{1}{2}x^2 + C}$$

$$y = 1 - e^{\frac{1}{2}x^2 + C}$$

$$y(1) = 1 - e^{\frac{1}{2}1^2 + C} = 0$$

$$e^{\frac{1}{2}+C} = 1$$

$$\frac{1}{2} + C = 0 \Rightarrow \underline{C = -\frac{1}{2}}$$

$$y(x) = 1 - e^{\frac{1}{2}\left(x^2-1\right)}$$

$$y(1.6) = 1 - e^{\frac{1}{2}\left(1.6^2-1\right)} \approx \underline{0.5416}$$

Exercise

Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals

$$y' = y^2(1+2x), \quad y(-1) = 1, \quad dx = 0.5$$

Solution

$$y_1 = y_0 + y_0^2\left(1+2x_0\right)dx = 1 + 1^2(1+2(-1))(0.5) = .5$$

$$y_2 = y_1 + y_1^2\left(1+2x_1\right)dx = 0.5 + 0.5^2(1+2(-0.5))(0.5) = .5$$

$$y_3 = y_2 + y_2^2\left(1+2x_2\right)dx = .5 + .5^2(1+2(0))(0.5) = .625$$

$$\frac{dy}{y^2} = (1+2x)\,dx \Rightarrow \int \frac{dy}{y^2} = \int (1+2x)\,dx$$

$$-\frac{1}{y} = x + x^2 + C$$

$$y = -\frac{1}{x + x^2 + C}$$

$$y(-1) = -\frac{1}{-1 + (-1)^2 + C}$$

$$1 = -\frac{1}{C} \Rightarrow \underline{C = -1}$$

$$y(x) = -\frac{1}{x + x^2 - 1} = \frac{1}{1 - x - x^2}$$

$$y(.5) = \frac{1}{1 - .5 - .5^2} = 4$$

Exercise

Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals

$y' = ye^x, \quad y(0) = 2, \quad dx = 0.5$

Solution

$$y_1 = y_0 + \left(y_0 e^{x_0}\right)dx = 2 + \left(2e^0\right)(0.5) = 3$$

$$y_2 = y_1 + \left(y_1 e^{x_1}\right)dx = 3 + \left(3e^{0.5}\right)(0.5) = 5.47308$$

$$y_3 = y_2 + \left(y_2 e^{x_2}\right)dx = 5.47308 + \left(5.47308e^1\right)(0.5) = 12.9118$$

$$\frac{dy}{dx} = ye^x \quad \Rightarrow \quad \int \frac{dy}{y} = \int e^x dx$$

$$\ln y = e^x + C$$

$$\ln 2 = e^0 + C \Rightarrow \boxed{C = \ln 2 - 1}$$

$$\ln|y| = e^x + \ln 2 - 1$$

$$y = e^{e^x + \ln 2 - 1}$$

$$= e^{\ln 2} e^{e^x - 1}$$

$$= 2e^{e^x - 1}$$

$$y(1.5) = 2e^{e^{1.5} - 1} \approx 65.0292$$

Exercise

Use the Euler method with $dx = 0.2$ to estimate $y(2)$ if $y' = \frac{y}{x}$ and $y(1) = 2$. What is the exact value of $y(2)$?

Solution

$$y_1(1) = y_0 + \left(\frac{y_0}{x_0}\right)dx = 2 + \left(\frac{2}{1}\right)(0.2) = 2.4$$

$$y_2(1.2) = y_1 + \left(\frac{y_1}{x_1}\right)dx = 2.4 + \left(\frac{2.4}{1.2}\right)(0.2) = 2.8$$

$$y_3 = y_2 + \left(\frac{y_2}{x_2}\right)dx = 2.8 + \left(\frac{2.8}{1.4}\right)(0.2) = 3.2$$

$$y_4 = y_3 + \left(\frac{y_3}{x_3}\right)dx = 3.2 + \left(\frac{3.2}{1.6}\right)(0.2) = 3.6$$

$$y_5 = y_4 + \left(\frac{y_4}{x_4}\right)dx = 3.6 + \left(\frac{3.6}{1.8}\right)(0.2) = 4$$

$$\frac{dy}{dx} = \frac{y}{x} \quad \Rightarrow \quad \int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + C$$

$$\ln 2 = \ln 1 + C \quad \rightarrow \quad \boxed{C = \ln 2}$$

$$\ln y = \ln x + \ln 2 = \ln 2x$$

$$\boxed{y = 2x} \qquad y(2) = 2(2) = 4$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y = Ce^{-5t};\quad y'(t) + 5y = 0$

Solution

$$y = Ce^{-5t} \quad \Rightarrow \quad y' = -5Ce^{-5t} = -5y$$

$$y'(t) + 5y = -5y + 5y = 0 \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y = Ct^{-3};\quad ty'(t) + 3y = 0$

Solution

$$y = Ct^{-3} \quad \Rightarrow \quad y' = -3Ct^{-4}$$

$$t\left(-3Ct^{-4}\right) + 3Ct^{-3} = -3Ct^{-3} + 3Ct^{-3} = 0 \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y = C_1 \sin 4t + C_2 \cos 4t;\quad y''(t) + 16y = 0$

Solution

$$y' = 4C_1 \cos 4t - 4C_2 \sin 4t$$

$$y'' = -16C_1 \sin 4t - 16C_2 \cos 4t$$

$$y''(t) + 16y = -16C_1 \sin 4t - 16C_2 \cos 4t + 16C_1 \sin 4t + 16C_2 \cos 4t = 0 \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y = C_1e^{-x} + C_2e^{x};\quad y''(x) - y = 0$

Solution

$$y' = -C_1e^{-x} + C_2e^{x}$$

$$y'' = C_1e^{-x} + C_2e^{x}$$

$$y''(x) - y = C_1e^{-x} + C_2e^{x} - C_1e^{-x} - C_2e^{x} \underline{= 0} \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants.

$y' + 4y = \cos t,\quad y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t + Ce^{-4t},\qquad y(0) = -1$

Solution

$$y(0) = \frac{4}{17}\cos(0) + \frac{1}{17}\sin(0) + Ce^{-4(0)}$$

$$-1 = \frac{4}{17} + C$$

$$C = -1 - \frac{4}{17} = -\frac{21}{17}$$

$$\boxed{y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t - \frac{21}{17}e^{-4t}}$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad ty' + (t+1)y = 2te^{-t},\quad y(t) = e^{-t}\left(t + \frac{C}{t}\right),\quad y(1) = \frac{1}{e}$

Solution

$$y(1) = \frac{1}{e} = e^{-1}$$

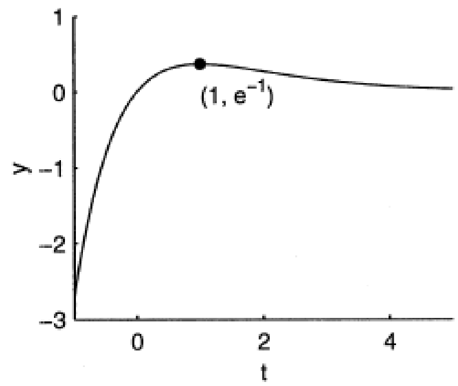
$$y(1) = e^{-1}\left(1 + \frac{C}{1}\right)$$

$$e^{-1} = e^{-1}(1 + C) \Rightarrow 1 = 1 + C$$

Hence, $C = 0$

The solution is: $y(t) = te^{-t}$

This function is defined and differentiable on the whole real line. Hence, the interval of existence is the whole real line.



Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y' = y(2+y),\quad y(t) = \dfrac{2}{-1+Ce^{-2t}},\quad y(0) = -3$

Solution

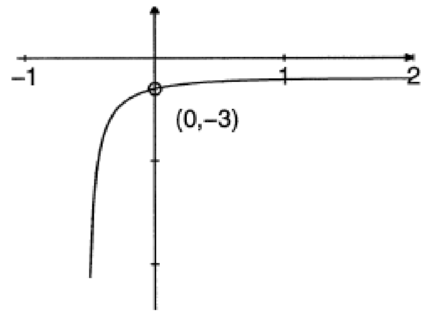
$$y(0) = \frac{2}{-1+Ce^{-2(0)}}$$

$$-3 = \frac{2}{-1+C}$$

$$3-3C = 2$$

$$-3C = -1$$

$$\boxed{C = \frac{1}{3}}$$



The solution is:

$$y(t) = \frac{2}{-1+\frac{1}{3}e^{-2t}}$$

$$= \frac{6}{-3+e^{-2t}}$$

Exercise

Verify that the given function y is a solution of the initial value problem that follows it.

$$y = 16e^{2t} - 10;\quad y' - 2y = 20,\quad y(0) = 6$$

Solution

$$y(0) = 6 \quad \to \quad y(0) = 16 - 10 = 6 \quad \checkmark$$

$$y = 16e^{2t} - 10 \quad \to \quad y' = 32e^{2t}$$

$$y' - 2y = 32e^{2t} - 32e^{2t} + 20 = 20 \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the initial value problem that follows it.

$$y = 8t^6 - 3;\quad ty' - 6y = 18,\quad y(1) = 5$$

Solution

$$y = 8t^6 - 3 \quad \to \quad y(1) = 8 - 3 = 5 \quad \checkmark$$

$$y' = 48t^5$$

$$ty' - 6y = 48t^6 - 48t^6 + 18 = 18 \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the initial value problem that follows it.

$$y = -3\cos 3t; \quad y'' + 9y = 0, \quad y(0) = -3, \ \ y'(0) = 0$$

Solution

$$y = -3\cos 3t \quad \to \quad y(0) = -3\cos 0 \ \underline{= -3} \quad \checkmark$$

$$y' = 9\sin 3t \quad \to \quad \underline{y(0) = 0} \quad \checkmark$$

$$y'' = 27\cos 3t$$

$$y'' + 9y = 27\cos 3t - 27\cos 3t \ \underline{= 0}$$

Exercise

Verify that the given function y is a solution of the initial value problem that follows it.

$$y = \frac{1}{4}\left(e^{2x} - e^{-2x}\right); \quad y'' - 4y = 0, \quad y(0) = 0, \ \ y'(0) = 1$$

Solution

$$y = \frac{1}{4}\left(e^{2x} - e^{-2x}\right) \quad \to \quad y(0) = \frac{1}{4}(1-1) \ \underline{= 0} \quad \checkmark$$

$$y' = \frac{1}{2}\left(e^{2x} + e^{-2x}\right) \quad \to \quad y'(0) = \frac{1}{2}(1+1) \ \underline{= 1} \quad \checkmark$$

$$y'' = e^{2x} - e^{-2x}$$

$$y'' - 4y = e^{2x} - e^{-2x} - e^{2x} + e^{-2x} \ \underline{= 0} \quad \checkmark$$

Exercise

Find the general solution of the differential equation $y' = xy$

Solution

$$\frac{dy}{dx} = xy$$

$$\frac{dy}{y} = xdx$$

$$\int \frac{dy}{y} = \int xdx$$

$$\ln|y| = \frac{1}{2}x^2 + C$$

$$|y| = e^{x^2/2 + C}$$

$$y(x) = \pm e^{x^2/2} e^C$$

$$\underline{= Ae^{x^2/2}} \qquad \text{Where } A = \pm e^C$$

Exercise

Find the general solution of the differential equation $xy' = 2y$

Solution

$$x\frac{dy}{dx} = 2y$$

$$\frac{dy}{y} = 2\frac{dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{2}{x}dx$$

$$\ln|y| = 2\ln|x| + C$$

$$= \ln x^2 + C$$

$$y(x) = \pm e^{\ln x^2 + C}$$

$$= \pm e^C x^2$$

$$\boxed{= Ax^2}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = e^{x-y}$

Solution

$$\frac{dy}{dx} = e^x e^{-y}$$

$$\frac{dy}{e^{-y}} = e^x dx$$

$$\int e^y dy = \int e^x dx$$

$$e^y = e^x + C$$

$$\boxed{y(x) = \ln\left(e^x + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \left(1 + y^2\right)e^x$

Solution

$$\frac{dy}{dx} = \left(1 + y^2\right)e^x$$

$$\frac{dy}{1 + y^2} = e^x dx$$

$$\int \frac{dy}{1+y^2} = \int e^x dx$$

$$\tan^{-1} y = e^x + C$$

$$\boxed{y(x) = \tan\left(e^x + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = xy + y$

Solution

$$\frac{dy}{dx} = (x+1)\,y$$

$$\frac{dy}{y} = (x+1)\,dx$$

$$\int \frac{dy}{y} = \int (x+1)\,dx$$

$$\ln y = \frac{1}{2}x^2 + x + C$$

$$\boxed{y = e^{x^2/2+x+C}}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$y' = ye^x - 2e^x + y - 2$$

Solution

$$\frac{dy}{dx} = (y-2)\,e^x + y - 2$$

$$\frac{dy}{dx} = (y-2)\left(e^x + 1\right)$$

$$\frac{dy}{y-2} = \left(e^x + 1\right)dx$$

$$\int \frac{dy}{y-2} = \int \left(e^x + 1\right)dx$$

$$\ln|y-2| = e^x + x + C$$

$$y - 2 = \pm e^{e^x + x + C}$$

$$y - 2 = \pm e^C e^{e^x + x}$$

$$\boxed{y(x) = De^{e^x + x} + 2} \quad D = \pm e^C$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \dfrac{x}{y+2}$

Solution

$$\frac{dy}{dx} = \frac{x}{y+2}$$

$$(y+2)\,dy = x\,dx$$

$$\int (y+2)\,dy = \int x\,dx$$

$$\tfrac{1}{2}y^2 + 2y = \tfrac{1}{2}x^2 + C$$

$$\boxed{y^2 + 4y = x^2 + 2C}$$

$$y^2 + 4y - x^2 - D = 0, \quad (D = 2C)$$

$$y = \frac{-4 \pm \sqrt{16 - 4\left(-x^2 - D\right)}}{2} = \frac{-4 \pm \sqrt{16 + 4x^2 + 4D}}{2}$$

$$= \frac{-4 \pm 2\sqrt{x^2 + (4+D)}}{2} \qquad E = 4 + D$$

$$= -2 \pm \sqrt{x^2 + E}$$

$$\boxed{y(x) = -2 \pm \sqrt{x^2 + E}}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \dfrac{xy}{x-1}$

Solution

$$\frac{dy}{dx} = y\left(\frac{x}{x-1}\right)$$

$$\frac{dy}{y} = \left(\frac{x}{x-1}\right)dx$$

$$\int \frac{dy}{y} = \int \left(1 + \frac{1}{x-1}\right)dx$$

$$\ln|y| = x + \ln|x-1| + C$$

$$y(x) = \pm e^{x + \ln|x-1| + C}$$

$$= \pm e^{C} e^{x} e^{\ln|x-1|}$$

$$\boxed{= De^{x}\,|x-1|}$$

Exercise

Solve the differential equations: $x\dfrac{dy}{dx}+y=e^{x}, \quad x>0$

Solution

$$\frac{dy}{dx}+\frac{1}{x}y=\frac{e^x}{x}$$

$$y_h=e^{\int \frac{1}{x}dx}=e^{\ln x}=x$$

$$\int \frac{e^x}{x}e^{\int \frac{1}{x}dx}dx=\int x\frac{e^x}{x}dx=\int e^x dx=e^x$$

$$\boxed{y(x)=\frac{1}{x}\left(e^x+C\right)}, \quad x>0$$

Exercise

Solve the differential equations: $y'+(\tan x)y=\cos^2 x, \quad -\frac{\pi}{2}<x<\frac{\pi}{2}$

Solution

$$y'+(\tan x)y=\cos^2 x$$

$$y_h=e^{\int \tan x dx}=e^{\ln(\cos x)^{-1}}=(\cos x)^{-1} \qquad \int \tan x dx=-\ln|\cos x|=\ln(\cos x)^{-1}$$

$$\int \cos^2 x(\cos x)^{-1}dx=\int \cos x dx=\sin x$$

$$y(x)=\frac{1}{(\cos x)^{-1}}(\sin x+C)$$

$$y(x)=\cos x(\sin x+C)$$

$$\boxed{y(x)=\cos x\sin x+C\cos x}$$

Exercise

Solve the differential equations: $x\dfrac{dy}{dx}+2y=1-\dfrac{1}{x}, \quad x>0$

Solution

$$y'+\frac{2}{x}y=\frac{1}{x}-\frac{1}{x^2}$$

$$y_h=e^{\int \frac{2}{x}dx}=e^{2\ln x}=e^{\ln x^2}=x^2$$

$$\int\left(\frac{1}{x}-\frac{1}{x^2}\right)x^2dx=\int(x-1)dx=\frac{1}{2}x^2-x$$

$$y(x)=\frac{1}{x^2}\left(\frac{1}{2}x^2-x+C\right)$$

$$\boxed{y(x)=\frac{1}{2}-\frac{1}{x}+\frac{C}{x^2}, \quad x>0}$$

Exercise

Solve the differential equations: $(1+x)y'+y=\sqrt{x}$

Solution

$$y'+\frac{1}{1+x}y=\frac{\sqrt{x}}{1+x}$$

$$e^{\int\frac{1}{1+x}dx}=e^{\ln(1+x)}=\underline{1+x}$$

$$\int\frac{\sqrt{x}}{1+x}(1+x)\,dx=\int x^{1/2}dx=\frac{2}{3}x^{3/2}$$

$$y(x)=\frac{1}{1+x}\left(\frac{2}{3}x^{3/2}+C\right)$$

$$\boxed{=\frac{2x^{3/2}}{3(1+x)}+\frac{C}{1+x}}$$

Exercise

Solve the differential equations: $e^{2x}y'+2e^{2x}y=2x$

Solution

$$y'+2y=2xe^{-2x}$$

$$e^{\int 2dx}=e^{2x}$$

$$\int 2xe^{-2x}\left(e^{2x}\right)dx=2\int xdx=x^2$$

$$\underline{y(x)}=\frac{1}{e^{2x}}\left(x^2+C\right)$$

$$\boxed{=x^2e^{-2x}+Ce^{-2x}}$$

Exercise

Solve the differential equations: $(t+1)\frac{ds}{dt}+2s=3(t+1)+\frac{1}{(t+1)^2}, \quad t>-1$

Solution

$$s' + \frac{2}{t+1}s = 3 + \frac{1}{(t+1)^3}$$

$$e^{\int \frac{2}{t+1}dt} = e^{2\ln(t+1)} = e^{\ln(t+1)^2} = (t+1)^2$$

$$\int \left(3 + \frac{1}{(t+1)^3}\right)(t+1)^2\, dt = \int \left(3(t+1)^2 + \frac{1}{t+1}\right)dt \qquad d(t+1) = dt$$

$$= 3\int (t+1)^2\, d(t+1) + \int \frac{1}{t+1} d(t+1)$$

$$= (t+1)^3 + \ln(t+1)$$

$$s(t) = \frac{1}{(t+1)^2}\left((t+1)^3 + \ln(t+1) + C\right)$$

$$\boxed{= t + 1 + \frac{\ln(t+1)}{(t+1)^2} + \frac{C}{(t+1)^2}, \quad t > -1}$$

Exercise

Solve the differential equations: $\tan\theta \frac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \frac{\pi}{2}$

Solution

$$\frac{dr}{d\theta} + \frac{1}{\tan\theta}r = \frac{\sin^2\theta}{\tan\theta}$$

$$\frac{dr}{d\theta} + \frac{1}{\tan\theta}r = \sin^2\theta \frac{\cos\theta}{\sin\theta}$$

$$\frac{dr}{d\theta} + \frac{1}{\tan\theta}r = \sin\theta\cos\theta$$

$$e^{\int \cot\theta d\theta} = e^{\ln|\sin\theta|} = \sin\theta, \quad 0 < \theta < \frac{\pi}{2}$$

$$\int (\sin\theta\cos\theta)(\sin\theta)\,d\theta = \int \left(\sin^2\theta\cos\theta\right)d\theta \qquad d(\sin\theta) = \cos\theta d\theta$$

$$= \int \sin^2\theta d(\sin\theta)$$

$$= \frac{1}{3}\sin^3\theta$$

$$r(\theta) = \frac{1}{\sin\theta}\left(\frac{1}{3}\sin^3\theta + C\right)$$

$$\boxed{= \frac{1}{3}\sin^2\theta + \frac{C}{\sin\theta}}$$

Exercise

Find the general solution of $y' = \cos x - y\sec x$

Solution

$$y' + (\sec x)\, y = \cos x$$

$$e^{\int \sec x dx} = e^{\ln|\sec x + \tan x|} = \sec x + \tan x$$

$$\int \cos x(\sec x + \tan x)\,dx = \int \cos x\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)dx$$

$$= \int (1 + \sin x)\,dx$$

$$= x - \cos x$$

$$\boxed{y(x) = \frac{1}{\sec x + \tan x}(x - \cos x + C)}$$

Exercise

Find the general solution of $\left(1 + x^3\right)y' = 3x^2 y + x^2 + x^5$

Solution

$$y' - \frac{3x^2}{1 + x^3}\, y = \frac{x^2\left(1 + x^3\right)}{1 + x^3} = x^2$$

$$e^{\int -\frac{3x^2}{1+x^3}dx} = e^{-\int \frac{d\left(1+x^3\right)}{1+x^3}} = e^{-\ln\left(1+x^3\right)} = e^{\ln\left(1+x^3\right)^{-1}} = \frac{1}{1 + x^3}$$

$$\int \frac{1}{1 + x^3} \cdot x^2 dx = \frac{1}{3}\int \frac{d\left(1 + x^3\right)}{1 + x^3} = \frac{1}{3}\ln\left(1 + x^3\right)$$

$$y(x) = \left(1 + x^3\right)\left(\frac{1}{3}\ln\left(1 + x^3\right) + C\right)$$

$$\boxed{= \frac{1}{3}\left(1 + x^3\right)\ln\left(1 + x^3\right) + C\left(1 + x^3\right)}$$

Exercise

Find the general solution of $\frac{dy}{dt} - 2y = 4 - t$

Solution

$$e^{\int -2dt} = e^{-2t}$$

$$\int (4-t)e^{-2t}\, dt = \int \left(4e^{-2t} - te^{-2t}\right) dt$$

$$= -2e^{-2t} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t}$$

$$= -\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t}$$

		$\int e^{-2t}$
+	t	$-\frac{1}{2}e^{-2t}$
−	1	$\frac{1}{4}e^{-2t}$

$$y(t) = \frac{1}{e^{-2t}}\left(-\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t} + C\right)$$

$$\boxed{y(t) = \frac{1}{2}t - \frac{7}{4} + Ce^{2t}}$$

Exercise

Find the general solution of $y' + y = \frac{1}{1+e^t}$

Solution

$$e^{\int dt} = e^t$$

$$\int \frac{1}{1+e^t}e^t\, dt = \int \frac{1}{1+e^t}d\left(1+e^t\right) = \ln\left(1+e^t\right)$$

$$y(t) = \frac{1}{e^t}\left(\ln\left(1+e^t\right) + C\right)$$

$$\boxed{y(t) = e^{-t}\ln\left(1+e^t\right) + Ce^{-t}}$$

Exercise

Solve the differential equation $y' = 3y - 4$

Solution

$$y' - 3y = -4$$

$$e^{\int -3dx} = e^{-3x}$$

$$\int -4e^{-3x}dx = \frac{4}{3}e^{-3x}$$

$$y(x) = \frac{1}{e^{-3x}}\left(\frac{4}{3}e^{-3x} + C\right)$$

$$\boxed{= \frac{4}{3} + Ce^{3x}}$$

Exercise

Solve the differential equation $\quad y' = -2y - 4$

Solution

$$y' + 2y = -4$$

$$e^{\int 2dx} = e^{2x}$$

$$\int -4e^{2x}dx = 2e^{2x}$$

$$y(x) = \frac{1}{e^{2x}}\left(2e^{2x} + C\right)$$

$$\underline{= 2 + Ce^{-2x}|}$$

Exercise

Solve the differential equation $\quad y' = -y + 2$

Solution

$$y' + y = 2$$

$$e^{\int dx} = e^{x}$$

$$\int 2e^{x}dx = 2e^{x}$$

$$y(x) = \frac{1}{e^{x}}\left(2e^{x} + C\right)$$

$$\underline{= 2 + Ce^{-x}|}$$

Exercise

Solve the differential equation $\quad y' = 2y + 6$

Solution

$$y' - 2y = 6$$

$$e^{\int -2dx} = e^{-2x}$$

$$\int 6e^{-2x}dx = -3e^{-2x}$$

$$y(x) = e^{2x}\left(-3e^{-2x} + C\right)$$

$$\underline{= -3 + Ce^{2x}|}$$

Exercise

Solve the initial value problem: $t\dfrac{dy}{dt}+2y=t^3, \quad t>0, \quad y(2)=1$

Solution

$$y'+\frac{2}{t}y=t^2$$

$$e^{\int \frac{2}{t}dt}=e^{2\ln t}=e^{\ln t^2}=t^2$$

$$\int t^2t^2dt=\int t^4dt=\frac{1}{5}t^5$$

$$y(t)=\frac{1}{t^2}\left(\frac{1}{5}t^5+C\right)=\frac{1}{5}t^3+\frac{C}{t^2}$$

$$y(2)=\frac{1}{5}2^3+\frac{C}{2^2}$$

$$1=\frac{8}{5}+\frac{C}{4}$$

$$\frac{C}{4}=1-\frac{8}{5}=-\frac{3}{5}\Rightarrow \boxed{C=-\frac{12}{5}}$$

$$y(t)=\frac{1}{5}t^3-\frac{12}{5t^2}$$

Exercise

Solve the initial value problem: $\theta\dfrac{dy}{d\theta}+y=\sin\theta, \quad \theta>0, \quad y\left(\frac{\pi}{2}\right)=1$

Solution

$$y'+\frac{1}{\theta}y=\frac{\sin\theta}{\theta}$$

$$e^{\int \frac{1}{\theta}d\theta}=e^{\ln|\theta|}=\theta \quad (>0)$$

$$\int \frac{\sin\theta}{\theta}\theta d\theta=\int \sin\theta d\theta=-\cos\theta$$

$$y(\theta)=\frac{1}{\theta}(-\cos\theta+C)$$

$$y\left(\frac{\pi}{2}\right)=\frac{2}{\pi}\left(-\cos\frac{\pi}{2}+C\right)$$

$$1=\frac{2}{\pi}(0+C)$$

$$1=\frac{2}{\pi}C \qquad C=\frac{\pi}{2}$$

$$y(\theta)=-\frac{\cos\theta}{\theta}+\frac{\pi}{2\theta}$$

Exercise

Solve the initial value problem: $\dfrac{dy}{dx} + xy = x, \quad y(0) = -6$

Solution

$$y' + xy = x$$

$$e^{\int x\,dx} = e^{x^2/2}$$

$$\int x e^{x^2/2}\,dx = \int e^{x^2/2}\,d\left(\frac{x^2}{2}\right) = e^{x^2/2} \qquad d\left(\frac{x^2}{2}\right) = x\,dx$$

$$y(x) = \frac{1}{e^{x^2/2}}\left(e^{x^2/2} + C\right)$$

$$y(0) = \frac{1}{e^{0^2/2}}\left(e^{0^2/2} + C\right)$$

$$-6 = 1(1 + C)$$

$$-6 = 1 + C \;\rightarrow\; \underline{C = -7}$$

$$y(x) = \frac{1}{e^{x^2/2}}\left(e^{x^2/2} - 7\right)$$

$$= 1 - \frac{7}{e^{x^2/2}}$$

Exercise

Solve the initial value problem $y' = \dfrac{y}{x}, \quad y(1) = -2$

Solution

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln|y| = \ln|x| + C$$

$$y = \pm e^{\ln|x| + C}$$

$$= \pm e^{C} e^{\ln|x|}$$

$$= Dx$$

$$y = Dx \;\Rightarrow\; D = \frac{y}{x} = \frac{-2}{1} = -2$$

$$y = -2x$$

Exercise

Solve the initial value problem $y' = \dfrac{\sin x}{y}, \quad y\left(\frac{\pi}{2}\right) = 1$

Solution

$$\frac{dy}{dx} = \frac{\sin x}{y}$$

$$ydy = \sin x dx$$

$$\int ydy = \int \sin x dx$$

$$\frac{1}{2}y^2 = -\cos x + C_1$$

$$y^2 = -2\cos x + C \quad \left(C = 2C_1\right)$$

$$y(x) = \pm\sqrt{-2\cos x + C}$$

$$y\left(\frac{\pi}{2}\right) = \sqrt{-2\cos\frac{\pi}{2} + C} \qquad 1 = \sqrt{C} \quad \Rightarrow \quad \boxed{C = 1}$$

$$\underline{y(x) = \sqrt{1 - 2\cos x}}$$

The interval of existence will be the interval containing $\frac{\pi}{2}$ and $1 - 2\cos x > 0$

$$\cos x < \frac{1}{2} \quad \Rightarrow \quad \boxed{\frac{\pi}{3} < x < \frac{5\pi}{3}}$$

Exercise

Find the general solution of $y' = y + 2xe^{2x}; \quad y(0) = 3$

Solution

$$y' - y = 2xe^{2x}$$

$$e^{\int -1dx} = e^{-x}$$

$$\int 2xe^{2x}\left(e^{-x}\right)dx = 2\int xe^{x}dx = 2\left(xe^{x} - e^{x}\right)$$

$$y(x) = \frac{1}{e^{-x}}\left(2xe^{x} - 2e^{x} + C\right)$$

$$= e^{x}\left(2xe^{x} - 2e^{x} + C\right)$$

$$\underline{= 2xe^{2x} - 2e^{2x} + Ce^{x}}$$

$$y(x = 0) = 2(0)e^{2(0)} - 2e^{2(0)} + Ce^{(0)}$$

$$3 = -2 + C \quad \rightarrow \quad \boxed{C = 5}$$

$$\underline{y(x) = 2xe^{2x} - 2e^{2x} + 5e^{x}}$$

Exercise

Find the general solution of $\left(x^2+1\right)y'+3xy=6x; \quad y(0)=-1$

Solution

$$y'+\frac{3x}{x^2+1}y=\frac{6x}{x^2+1}$$

$$e^{\int\frac{3x}{x^2+1}dx}=e^{\frac{3}{2}\ln\left(x^2+1\right)}=e^{\ln\left(x^2+1\right)^{\frac{3}{2}}}\underline{=\left(x^2+1\right)^{\frac{3}{2}}}$$

$$\int\left(x^2+1\right)^{\frac{3}{2}}\frac{6x}{x^2+1}dx=3\int\left(x^2+1\right)^{\frac{1}{2}}d\left(x^2+1\right)$$

$$=2\left(x^2+1\right)^{\frac{3}{2}}$$

$$y(x)=2+C\left(x^2+1\right)^{-\frac{3}{2}} \qquad y(0)=2+C\left((0)^2+1\right)^{-\frac{3}{2}}$$

$$-1=2+C(1)^{-\frac{3}{2}} \qquad \to \underline{C=-3}$$

$$\underline{y(x)=2-3\left(x^2+1\right)^{-\frac{3}{2}}}$$

Exercise

Solve the initial value problem $\quad y'=\left(4t^3+1\right)y, \quad y(0)=4$

Solution

$$\frac{dy}{dt}=\left(4t^3+1\right)y$$

$$\int\frac{dy}{y}=\int\left(4t^3+1\right)dt$$

$$\ln y=t^4+t+C$$

$$y(t)=e^{t^4+t+C}$$

$$\underline{=Ae^{t^4+t}}$$

$$y(0)=4 \quad \underline{\to 4=A}$$

$$y(t)\ \underline{=4e^{t^4+t}}$$

Exercise

Solve the initial value problem $y' = \dfrac{e^t}{2y}, \quad y(\ln 2) = 1$

Solution

$$\int 2y\,dy = \int e^t\,dt$$

$$y^2 = e^t + C$$

$$y(\ln 2) = 1, \quad \to 1 = 2 + C \Rightarrow \underline{C = -1}$$

$$\boxed{y^2 = e^t - 1}$$

Exercise

Solve the initial value problem $(\sec x)\,y' = y^3, \quad y(0) = 3$

Solution

$$\int y^{-3}dy = \int \frac{dx}{\sec x} = \int \cos x\,dx$$

$$-\frac{1}{2}\frac{1}{y^2} = \sin x + C_1$$

$$y^2 = \frac{1}{-2\sin x + C}$$

$$y = \pm\sqrt{\frac{1}{-2\sin x + C}} \qquad \textit{Since the initial value is positive}$$

$$y = \frac{1}{\sqrt{-2\sin x + C}}$$

$$3 = \sqrt{\frac{1}{C}} \Rightarrow \underline{C = \frac{1}{9}}$$

$$y = \frac{1}{\sqrt{-2\sin x + \frac{1}{9}}}$$

$$\boxed{= \frac{3}{\sqrt{-2\sin x + 1}}}$$

Exercise

Solve the initial value problem $\dfrac{dy}{dx} = e^{x-y}, \quad y(0) = \ln 3$

Solution

$$dy = \left(e^x e^{-y}\right)dx$$

$$\int e^y\,dy = \int e^x\,dx$$

$$e^{y} = e^{x} + C$$

$$y = \ln\left(e^{x} + C\right)$$

$$y(0) = \ln 3 \rightarrow \quad \ln 3 = \ln(1+C)$$

$$1 + C = 3 \quad \Rightarrow \quad \underline{C = 2}$$

$$\underline{y(x) = \ln\left(e^{x} + 2\right)}$$

Exercise

Solve the initial value problem $\quad y' = 2e^{3y-t}, \quad y(0) = 0$

Solution

$$\frac{dy}{dt} = 2e^{3y}e^{-t}$$

$$\int e^{-3y}dy = \int 2e^{-t}dt$$

$$-\frac{1}{3}e^{-3y} = -2e^{-t} + C_1$$

$$e^{-3y} = 6e^{-t} + C$$

$$y(0) = 0 \ \rightarrow \ 1 = 6 + C \ \Rightarrow \ \underline{C = -5}$$

$$e^{-3y} = 6e^{-t} - 5$$

$$-3y = \ln\left(6e^{-t} - 5\right)$$

$$\underline{y(t) = -\frac{1}{3}\ln\left(6e^{-t} - 5\right)}$$

Exercise

Solve the initial value problem $\quad y' = 3y - 6, \quad y(0) = 9$

Solution

$$y' - 3y = -6$$

$$e^{\int -3dx} = e^{-3x}$$

$$\int -6e^{-3x}dx = 2e^{-3x}$$

$$y = \frac{1}{e^{-3x}}\left(2e^{-3x} + C\right)$$

$$= 2 + Ce^{3x}$$

$$y(0) = 9 \qquad 9 = 2 + C \rightarrow \underline{C = 7}$$

$$\underline{y = 7e^{3x} + 2}$$

Exercise

Solve the initial value problem $\quad y' = -y + 2, \quad y(0) = -2$

Solution

$$y' + y = 2$$

$$e^{\int dx} = e^x$$

$$\int 2e^x dx = 2e^x$$

$$y = \frac{1}{e^x}\left(2e^x + C\right) \underline{= 2 + Ce^{-x}|}$$

$$y(0) = -2 \qquad -2 = 2 + C \to \underline{C = -4|}$$

$$\underline{y(x) = 2 - 4e^{-x}|}$$

Exercise

Solve the initial value problem $\quad y' = -2y - 4, \quad y(0) = 0$

Solution

$$y' + 2y = -4$$

$$e^{\int 2dx} = e^{2x}$$

$$\int -4e^{2x} dx = -2e^{2x}$$

$$y = \frac{1}{e^{2x}}\left(-2e^{2x} + C\right) \underline{= -2 + Ce^{-2x}|}$$

$$y(0) = 0 \qquad 0 = -2 + C \to \underline{C = 2|}$$

$$\underline{y(x) = 2e^{-2x} - 2|}$$