

## ***Solution***      **Section 2.1 – Vectors in 2-Space, 3-Space, and $n$ -Space**

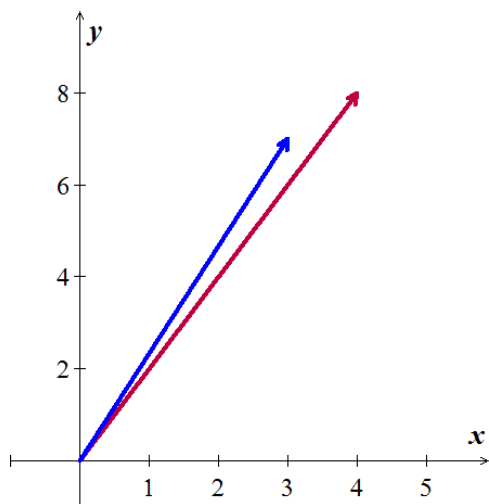
### ***Exercise***

Sketch the following vectors with initial points located at the origin

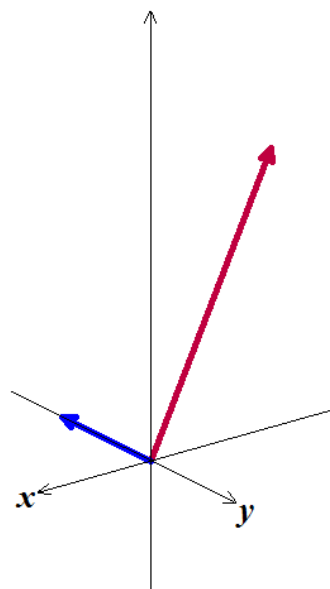
- a)  $P_1(4, 8)$     $P_2(3, 7)$
- b)  $P_1(-1, 0, 2)$     $P_2(0, -1, 0)$
- c)  $P_1(3, -7, 2)$     $P_2(-2, 5, -4)$

### **Solution**

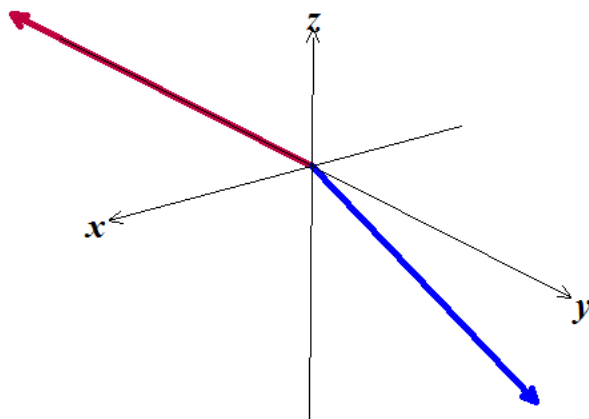
a)



b)



c)



### Exercise

Find the components of the vector  $\overrightarrow{P_1 P_2}$

a)  $P_1(3, 5) \quad P_2(2, 8)$

b)  $P_1(5, -2, 1) \quad P_2(2, 4, 2)$

c)  $P_1(0, 0, 0) \quad P_2(-1, 6, 1)$

### Solution

a)  $\overrightarrow{P_1 P_2} = (2-3, 8-5)$   
 $\quad \quad \quad = (-1, 3)$

b)  $\overrightarrow{P_1 P_2} = (2-5, 4-(-2), 2-1)$   
 $\quad \quad \quad = (-3, 6, 1)$

c)  $\overrightarrow{P_1 P_2} = (-1-0, 6-0, 1-0)$   
 $\quad \quad \quad = (-1, 6, 1)$

### Exercise

Find the terminal point of the vector that is equivalent to  $\vec{u} = (1, 2)$  and whose initial point is  $A(1, 1)$

### Solution

The terminal point:  $B(b_1, b_2)$

$$(b_1 - 1, b_2 - 1) = (1, 2)$$

$$\begin{cases} b_1 - 1 = 1 & \Rightarrow b_1 = 2 \\ b_2 - 1 = 2 & \Rightarrow b_2 = 3 \end{cases}$$

The terminal point:  $B(2, 3)$

### Exercise

Find the initial point of the vector that is equivalent to  $\vec{u} = (1, 1, 3)$  and whose terminal point is  $B(-1, -1, 2)$

### Solution

The initial point:  $A(x, y, z)$

$$(-1-x, -1-y, 2-z) = (1, 1, 3)$$

$$\begin{cases} -1 - x = 1 & \Rightarrow x = -2 \\ -1 - y = 1 & \Rightarrow y = -2 \\ 2 - z = 3 & \Rightarrow z = -1 \end{cases}$$

The initial point:  $A(-2, -2, -1)$

### Exercise

Find a nonzero vector  $\vec{u}$  with initial point  $P(-1, 3, -5)$  such that

- a)  $\vec{u}$  has the same direction as  $\vec{v} = (6, 7, -3)$
- b)  $\vec{u}$  is oppositely directed as  $\vec{v} = (6, 7, -3)$

### Solution

- a)  $\vec{u}$  has the same direction as  $\vec{v}$

$$\vec{u} = \vec{v} = (6, 7, -3)$$

The initial point  $P(-1, 3, -5)$  then the terminal point:

$$(-1+6, 3+7, -5-3) = \underline{(5, 10, -8)}$$

- b)  $\vec{u}$  is oppositely directed as  $\vec{v} = (6, 7, -3)$

$$\vec{u} = -\vec{v} = (-6, -7, 3)$$

The initial point  $P(-1, 3, -5)$  then the terminal point:

$$(-1-6, 3-7, -5+3) = \underline{(-7, -4, -2)}$$

### Exercise

Let  $\vec{u} = (-3, 1, 2)$ ,  $\vec{v} = (4, 0, -8)$ , and  $\vec{w} = (6, -1, -4)$ . Find the components

- a)  $\vec{v} - \vec{w}$
- b)  $6\vec{u} + 2\vec{v}$
- c)  $5(\vec{v} - 4\vec{u})$
- d)  $-3(\vec{v} - 8\vec{w})$
- e)  $(2\vec{u} - 7\vec{w}) - (8\vec{v} + \vec{u})$
- f)  $-\vec{u} + (\vec{v} - 4\vec{w})$

### Solution

$$\begin{aligned} \text{a) } \vec{v} - \vec{w} &= (4-6, 0-(-1), -8-(-4)) \\ &= \underline{(-2, 1, -4)} \end{aligned}$$

$$\begin{aligned} \text{b) } 6\vec{u} + 2\vec{v} &= (-18, 6, 12) + (8, 0, -16) \\ &= \underline{(-10, 6, -4)} \end{aligned}$$

$$\begin{aligned} \text{c) } 5(\vec{v} - 4\vec{u}) &= 5(4-(-12), 0-4, -8-8) \\ &= 5(16, -4, -16) \end{aligned}$$

$$= \underline{(80, -20, -80)}$$

$$\begin{aligned} d) \quad -3(\vec{v} - 8\vec{w}) &= -3(4 - 48, 0 - (-8), -8 - (-32)) \\ &= -3(-44, 8, 24) \\ &= \underline{(32, -24, -72)} \end{aligned}$$

$$\begin{aligned} e) \quad (2\vec{u} - 7\vec{w}) - (8\vec{v} + \vec{u}) &= [(-6, 2, 4) - (42, -7, -28)] - [(32, 0, -64) + (-3, 1, 2)] \\ &= (-48, 9, 32) - (29, 1, -62) \\ &= \underline{(-77, 8, 94)} \end{aligned}$$

$$\begin{aligned} f) \quad -u + (v - 4w) &= (3, -1, -2) + [(4, 0, -8) - (24, -4, -16)] \\ &= (3, -1, -2) + (-20, 4, 8) \\ &= \underline{(-17, 3, 6)} \end{aligned}$$

### Exercise

Let  $\vec{u} = (2, 1, 0, 1, -1)$  and  $\vec{v} = (-2, 3, 1, 0, 2)$ . Find scalars  $a$  and  $b$  so that  $a\vec{u} + b\vec{v} = (-8, 8, 3, -1, 7)$

### Solution

$$\begin{aligned} a\vec{u} + b\vec{v} &= a(2, 1, 0, 1, -1) + b(-2, 3, 1, 0, 2) \\ &= (a - 2b, a + 3b, b, a, -a + 2b) \\ &= \underline{(-8, 8, 3, -1, 7)} \end{aligned}$$

$$\begin{cases} a - 2b = -8 \\ a + 3b = 8 \\ b = 3 \\ a = -1 \\ -a + 2b = 7 \end{cases}$$

$$\rightarrow a = -1 \quad b = 3 \quad \text{Unique solution}$$

### Exercise

Find all scalars  $c_1$ ,  $c_2$ , and  $c_3$  such that  $c_1(1, 2, 0) + c_2(2, 1, 1) + c_3(0, 3, 1) = (0, 0, 0)$

### Solution

$$\begin{aligned} c_1(1, 2, 0) + c_2(2, 1, 1) + c_3(0, 3, 1) &= (c_1 + 2c_2, 2c_1 + c_2 + 3c_3, c_2 + c_3) \\ &= (0, 0, 0) \end{aligned}$$

$$\begin{cases} c_1 + 2c_2 = 0 \\ 2c_1 + c_2 + 3c_3 = 0 \\ c_2 + c_3 = 0 \end{cases}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right] \quad R_2 - 2R_1$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right] \quad -\frac{1}{3}R_2$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right] \quad \begin{array}{l} R_1 - 2R_2 \\ R_3 - R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right] \quad \frac{1}{2}R_3$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \quad R_2 + R_3$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$\underline{c_1 = c_2 = c_3 = 0}$$

### ***Exercise***

Find the distance between the given points  $[5 \ 1 \ 8 \ -1 \ 2 \ 9]$ ,  $[4 \ 1 \ 4 \ 3 \ 2 \ 8]$

### **Solution**

$$\begin{aligned} d &= \sqrt{(4-5)^2 + (1-1)^2 + (4-8)^2 + (3+1)^2 + (2-2)^2 + (8-9)^2} \\ &= \sqrt{1+0+16+16+0+1} \\ &= \underline{\sqrt{34}} \end{aligned}$$

### Exercise

Let  $V$  be the set of all ordered pairs of real numbers, and consider the following addition and scalar

multiplication operation on  $\vec{u} = (u_1, u_2)$   $\vec{v} = (v_1, v_2)$

$$\vec{u} + \vec{v} = (u_1 + v_1 + 1, u_2 + v_2 + 1) \quad k\vec{u} = (ku_1, ku_2)$$

- a) Compute  $\vec{u} + \vec{v}$  and  $k\vec{u}$  for  $\vec{u} = (0, 4)$ ,  $\vec{v} = (1, -3)$ , and  $k = 2$ .
- b) Show that  $(0, 0) \neq \vec{0}$ .
- c) Show that  $(-1, -1) = \vec{0}$ .
- d) Show that  $\vec{u} + (-\vec{u}) = \vec{0}$  for  $\vec{u} = (u_1, u_2)$
- e) Find two vector space axioms that fail to hold.

### Solution

$$\begin{aligned} \text{a)} \quad \vec{u} + \vec{v} &= (0+1+1, 4-3+1) \\ &= (2, 2) \end{aligned}$$

$$\begin{aligned} k\vec{u} &= (ku_1, ku_2) \\ &= (2(0), 2(4)) \\ &= (0, 8) \end{aligned}$$

$$\begin{aligned} \text{b)} \quad (0, 0) + (u_1, u_2) &= (0+u_1+1, 0+u_2+1) \\ &= (u_1+1, u_2+1) \\ &\neq (u_1, u_2) \end{aligned}$$

Therefore  $(0, 0)$  is not the zero vector  $\vec{0}$  required (by Axiom).

$$\begin{aligned} \text{c)} \quad (-1, -1) + (u_1, u_2) &= (-1+u_1+1, -1+u_2+1) \\ &= (u_1, u_2) \\ (u_1, u_2) + (-1, -1) &= (u_1-1+1, u_2-1+1) \\ &= (u_1, u_2) \end{aligned}$$

Therefore  $(-1, -1) = \vec{0}$  holds.

$$\begin{aligned} \text{d)} \quad \text{Let } \vec{u} &= (u_1, u_2) \text{ \&} \\ -\vec{u} &= (-2-u_1, -2-u_2) \\ \vec{u} + (-\vec{u}) &= (u_1 + (-2-u_1)+1, u_2 + (-2-u_2)+1) \end{aligned}$$

$$= (-1, -1)$$

$$= \underline{\vec{0}}$$

$$\vec{u} + (-\vec{u}) = 0 \text{ holds}$$

e) Axiom 7:  $k(\vec{u} + \vec{v}) \stackrel{?}{=} k\vec{u} + k\vec{v}$

$$\begin{aligned} k(\vec{u} + \vec{v}) &= k(u_1 + v_1 + 1, u_2 + v_2 + 1) \\ &= (ku_1 + kv_1 + k, ku_2 + kv_2 + k) \end{aligned}$$

$$\begin{aligned} k\vec{u} + k\vec{v} &= (ku_1, ku_2) + (kv_1, kv_2) \\ &= (ku_1 + kv_1 + 1, ku_2 + kv_2 + 1) \end{aligned}$$

Therefore,  $k(\vec{u} + \vec{v}) \neq k\vec{u} + k\vec{v}$ ; Axiom 7 fails to hold

Axiom 8:  $(k + m)\vec{u} \stackrel{?}{=} k\vec{u} + m\vec{u}$

$$\begin{aligned} (k + m)\vec{u} &= ((k + m)u_1, (k + m)u_2) \\ &= (ku_1 + mu_1, ku_2 + mu_2) \end{aligned}$$

$$\begin{aligned} k\vec{u} + m\vec{u} &= (ku_1, ku_2) + (mu_1, mu_2) \\ &= (ku_1 + mu_1 + 1, ku_2 + mu_2 + 1) \end{aligned}$$

Therefore,  $(k + m)\vec{u} \neq k\vec{u} + m\vec{u}$ ; Axiom 8 fails to hold

### Exercise

Find  $\vec{w}$  given that  $10\vec{u} + 3\vec{w} = 4\vec{v} - 2\vec{w}$ ,  $\vec{u} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$  and  $\vec{v} = \begin{pmatrix} -20 \\ 5 \end{pmatrix}$ .

### Solution

$$-10\vec{u} + 10\vec{u} + 3\vec{w} + 2\vec{w} = -10\vec{u} + 4\vec{v} - 2\vec{w} + 2\vec{w}$$

$$5\vec{w} = -10\vec{u} + 4\vec{v}$$

$$\vec{w} = -2\vec{u} + \frac{4}{5}\vec{v}$$

$$= -2\begin{pmatrix} 1 \\ -6 \end{pmatrix} + \frac{4}{5}\begin{pmatrix} -20 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 12 \end{pmatrix} + \begin{pmatrix} -16 \\ 4 \end{pmatrix}$$

$$\underline{= \begin{pmatrix} -18 \\ 16 \end{pmatrix}}$$

### ***Exercise***

Find  $\vec{w}$  given that  $\vec{u} + 3\vec{v} - 2\vec{w} = 5\vec{u} + \vec{v} - 4\vec{w}$ ,  $\vec{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$  and  $\vec{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

### **Solution**

$$\vec{u} - \vec{u} + 3\vec{v} - 3\vec{v} - 2\vec{w} + 4\vec{w} = 5\vec{u} - \vec{u} + \vec{v} - 3\vec{v} - 4\vec{w} + 4\vec{w}$$

$$2\vec{w} = 4\vec{u} - 2\vec{v}$$

$$\vec{w} = 2\vec{u} - \vec{v}$$

$$= 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -2 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\underline{= \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

### ***Exercise***

Find  $\vec{w}$  given that  $2\vec{u} + \vec{v} - 3\vec{w} = 5\vec{u} + 7\vec{v} + 3\vec{w}$ ,  $\vec{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$  and  $\vec{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

### **Solution**

$$2\vec{u} - 2\vec{u} + \vec{v} - \vec{v} - 3\vec{w} - 3\vec{w} = 5\vec{u} - 2\vec{u} + 7\vec{v} - \vec{v} + 3\vec{w} - 3\vec{w}$$

$$-6\vec{w} = 3\vec{u} + 6\vec{v}$$

$$\vec{w} = -\frac{1}{2}\vec{u} - \vec{v}$$

$$= -\frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\underline{= \begin{pmatrix} \frac{3}{2} \\ -\frac{5}{2} \end{pmatrix}}$$



### Exercise

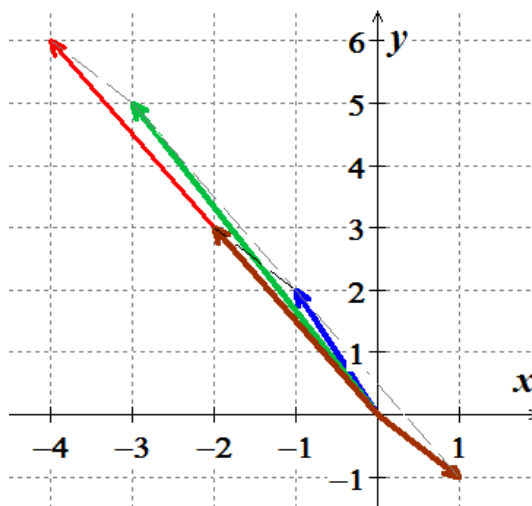
Draw  $\vec{u}$ ,  $\vec{v}$ ,  $\vec{u} + \vec{v}$ , and  $\vec{u} + 2\vec{v}$

$$\vec{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

### Solution

$$\begin{aligned} \vec{u} + \vec{v} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \vec{u} + 2\vec{v} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} -2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -4 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 5 \end{pmatrix} \end{aligned}$$



### Exercise

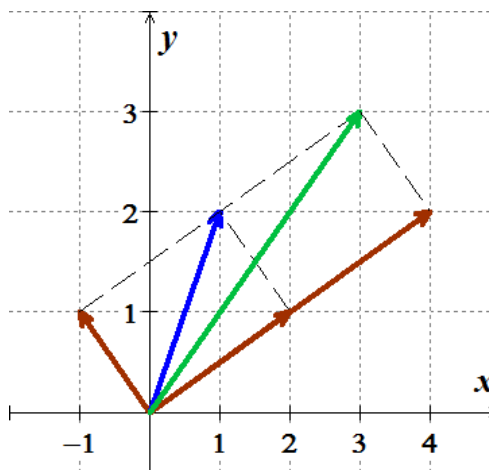
Draw  $\vec{u}$ ,  $\vec{v}$ ,  $\vec{u} + \vec{v}$ , and  $\vec{u} + 2\vec{v}$

$$\vec{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

### Solution

$$\begin{aligned} \vec{u} + \vec{v} &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \vec{u} + 2\vec{v} &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 3 \end{pmatrix} \end{aligned}$$



### Exercise

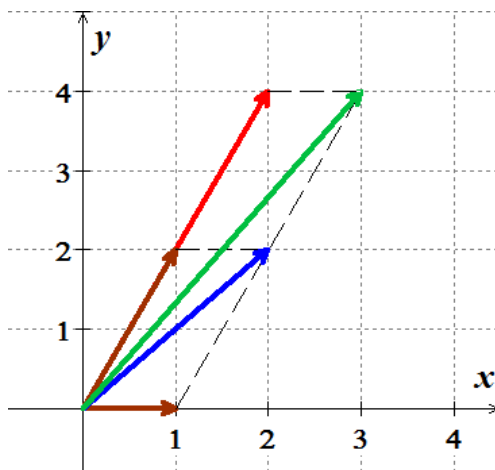
Draw  $\vec{u}$ ,  $\vec{v}$ ,  $\vec{u} + \vec{v}$ , and  $\vec{u} + 2\vec{v}$

$$\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

### Solution

$$\begin{aligned} \vec{u} + \vec{v} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \vec{u} + 2\vec{v} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 4 \end{pmatrix} \end{aligned}$$



### Exercise

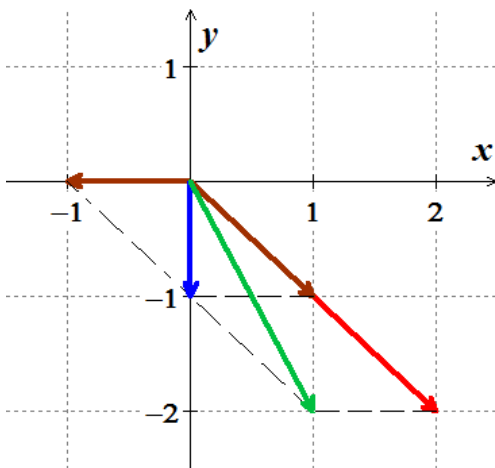
Draw  $\vec{u}$ ,  $\vec{v}$ ,  $\vec{u} + \vec{v}$ , and  $\vec{u} + 2\vec{v}$

$$\vec{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

### Solution

$$\begin{aligned} \vec{u} + \vec{v} &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \vec{u} + 2\vec{v} &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \end{aligned}$$



## ***Solution***      **Section 2.2 – Norm, Dot product, and distance in $R^n$**

### ***Exercise***

If  $\|\vec{v}\| = 5$  and  $\|\vec{w}\| = 3$ , what are the smallest and largest possible values of  $\|\vec{v} - \vec{w}\|$  and  $\vec{v} \cdot \vec{w}$ ?

### **Solution**

$$\|\vec{v} - \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\| = 5 + 3 = 8$$

$$\|\vec{v} - \vec{w}\| \geq \|\vec{v}\| - \|\vec{w}\| = 5 - 3 = 2$$

$$|\vec{v} \cdot \vec{w}| = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos \theta \leq \|\vec{v}\| \cdot \|\vec{w}\|$$

$$-\|\vec{v}\| \cdot \|\vec{w}\| \leq \vec{v} \cdot \vec{w} \leq \|\vec{v}\| \cdot \|\vec{w}\|$$

$$-(3)(5) \leq \vec{v} \cdot \vec{w} \leq (3)(5)$$

$$-15 \leq \vec{v} \cdot \vec{w} \leq 15$$

The minimum value occurs when the dot product is as small as possible,  $\vec{v}$  and  $\vec{w}$  are parallel, but point in opposite directions. Thus, the smallest value is  $-15$ .

The maximum value occurs when the dot product is as large as possible,  $\vec{v}$  and  $\vec{w}$  are parallel and point in same direction. Thus, the largest value is  $15$ .

### ***Exercise***

If  $\|\vec{v}\| = 7$  and  $\|\vec{w}\| = 3$ , what are the smallest and largest possible values of  $\|\vec{v} + \vec{w}\|$  and  $\vec{v} \cdot \vec{w}$ ?

### **Solution**

$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\| = 7 + 3 = 10$$

$$\|\vec{v} + \vec{w}\| \geq \|\vec{v}\| - \|\vec{w}\| = 7 - 3 = 4$$

$$|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\| \cdot \|\vec{w}\|$$

$$-\|\vec{v}\| \cdot \|\vec{w}\| \leq \vec{v} \cdot \vec{w} \leq \|\vec{v}\| \cdot \|\vec{w}\|$$

$$-(7)(3) \leq \vec{v} \cdot \vec{w} \leq (7)(3)$$

$$-21 \leq \vec{v} \cdot \vec{w} \leq 21$$

The minimum value occurs when the dot product is as small as possible,  $\vec{v}$  and  $\vec{w}$  are parallel, but point in opposite directions. Thus, the smallest value is  $-21$ .  $\vec{v} = (7, 0, 0, \dots)$  and  $\vec{w} = (-3, 0, 0, \dots)$

The maximum value occurs when the dot product is as large as possible,  $\vec{v}$  and  $\vec{w}$  are parallel and point in same direction. Thus, the largest value is  $21$ .  $\vec{v} = (7, 0, 0, \dots)$  and  $\vec{w} = (3, 0, 0, \dots)$

### Exercise

Given that  $\cos(\alpha) = \frac{\vec{v}_1}{\|\vec{v}\|}$  and  $\sin(\alpha) = \frac{\vec{v}_2}{\|\vec{v}\|}$ . Similarly,  $\cos(\beta) = \frac{\vec{w}_1}{\|\vec{w}\|}$  and  $\sin(\beta) = \frac{\vec{w}_2}{\|\vec{w}\|}$ . The angle  $\theta$  is  $\beta - \alpha$ . Substitute into the trigonometry formula  $\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$  for  $\cos(\beta - \alpha)$  to find  $\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}$

### Solution

$$\cos\beta = \frac{\vec{w}_1}{\|\vec{w}\|}$$

$$\sin\beta = \frac{\vec{w}_2}{\|\vec{w}\|}$$

$$\cos(\beta - \alpha) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

$$\begin{aligned} &= \frac{\vec{v}_1}{\|\vec{v}\|} \frac{\vec{w}_1}{\|\vec{w}\|} + \frac{\vec{v}_2}{\|\vec{v}\|} \frac{\vec{w}_2}{\|\vec{w}\|} \\ &= \frac{\vec{v}_1 \vec{w}_1 + \vec{v}_2 \vec{w}_2}{\|\vec{v}\| \cdot \|\vec{w}\|} \\ &= \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|} \end{aligned}$$

### Exercise

Can three vectors in the xy plane have  $\vec{u} \cdot \vec{v} < 0$  and  $\vec{v} \cdot \vec{w} < 0$  and  $\vec{u} \cdot \vec{w} < 0$ ?

### Solution

$$\text{Let consider: } \vec{u} = (1, 0), \vec{v} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \vec{w} = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

$$\begin{aligned} \vec{u} \cdot \vec{v} &= (1)\left(-\frac{1}{2}\right) + 0 \\ &= -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \vec{v} \cdot \vec{w} &= \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) \\ &= \frac{1}{4} - \frac{3}{4} \\ &= -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \vec{u} \cdot \vec{w} &= (1)\left(-\frac{1}{2}\right) + (0)\left(-\frac{\sqrt{3}}{2}\right) \\ &= -\frac{1}{2} < 0 \end{aligned}$$

Yes, it is.

### Exercise

Find the norm of  $\vec{v}$ , a unit vector that has the same direction as  $\vec{v}$ , and a unit vector that is oppositely directed.

a)  $\vec{v} = (4, -3)$

b)  $\vec{v} = (1, -1, 2)$

c)  $\vec{v} = (-2, 3, 3, -1)$

### Solution

a)  $\|\vec{v}\| = \sqrt{4^2 + (-3)^2}$   
 $= 5$

**Same direction unit vector:**

$$\begin{aligned}\vec{u}_1 &= \frac{\vec{v}}{\|\vec{v}\|} \\ &= \frac{1}{5}(4, -3) \\ &= \left(\frac{4}{5}, -\frac{3}{5}\right)\end{aligned}$$

**Opposite direction unit vector:**

$$\begin{aligned}\vec{u}_2 &= -\frac{1}{5}(4, -3) \\ &= \left(-\frac{4}{5}, \frac{3}{5}\right)\end{aligned}\quad \vec{u}_2 = -\frac{\vec{v}}{\|\vec{v}\|}$$

b)  $\|\vec{v}\| = \sqrt{1^2 + (-1)^2 + 2^2}$   
 $= \sqrt{6}$

**Same direction unit vector:**

$$\begin{aligned}\vec{u}_1 &= \frac{1}{\sqrt{6}}(1, -1, 2) \\ &= \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)\end{aligned}$$

$$\vec{u}_1 = \frac{\vec{v}}{\|\vec{v}\|}$$

**Opposite direction unit vector:**

$$\begin{aligned}\vec{u}_2 &= -\frac{1}{\sqrt{6}}(1, -1, 2) \\ &= \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)\end{aligned}\quad \vec{u}_2 = -\frac{\vec{v}}{\|\vec{v}\|}$$

c)  $\|\vec{v}\| = \sqrt{(-2)^2 + (3)^2 + (3)^2 + (-1)^2}$   
 $= \sqrt{23}$



$$\begin{aligned}
 d) \quad \|3\vec{v}\| - 3\|\vec{v}\| &= \|3(1, -3, 4)\| - 3\|(1, -3, 4)\| \\
 &= \|(3, -9, 12)\| - 3\|(1, -3, 4)\| \\
 &= \sqrt{3^2 + (-9)^2 + 12^2} - 3\sqrt{1^2 + (-3)^2 + 4^2} \\
 &= \sqrt{234} - 3\sqrt{26} \\
 &= 3\sqrt{26} - 3\sqrt{26} \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 e) \quad \|\vec{u}\| + \|-2\vec{v}\| + \|-3\vec{w}\| &= \|\vec{u}\| - 2\|\vec{v}\| - 3\|\vec{w}\| \\
 &= \sqrt{2^2 + (-2)^2 + 3^2} - 2\sqrt{1^2 + (-3)^2 + 4^2} - 3\sqrt{3^2 + 6^2 + (-4)^2} \\
 &= \sqrt{17} - 2\sqrt{26} - 3\sqrt{61}
 \end{aligned}$$

### Exercise

Let  $\vec{v} = (1, 1, 2, -3, 1)$ . Find all scalars  $k$  such that  $\|k\vec{v}\| = 5$

### Solution

$$\begin{aligned}
 \|k\vec{v}\| &= |k|\|\vec{v}\| \\
 &= |k| \|(1, 1, 2, -3, 1)\| \\
 &= |k| \sqrt{1^2 + 1^2 + 2^2 + (-3)^2 + 1^2} \\
 &= |k|\sqrt{49} \\
 &= 7|k| \\
 7|k| &= 5 \\
 |k| &= \frac{5}{7} \\
 k &= \pm \frac{5}{7}
 \end{aligned}$$

### Exercise

Find  $\vec{u} \cdot \vec{v}$ ,  $\vec{u} \cdot \vec{u}$ , and  $\vec{v} \cdot \vec{v}$

$$a) \quad \vec{u} = (3, 1, 4), \vec{v} = (2, 2, -4)$$

$$b) \quad \vec{u} = (1, 1, 4, 6), \vec{v} = (2, -2, 3, -2)$$

$$c) \quad \vec{u} = (2, -1, 1, 0, -2), \vec{v} = (1, 2, 2, 2, 1)$$

### Solution

$$\begin{aligned}
 a) \quad \vec{u} \cdot \vec{v} &= (3, 1, 4) \cdot (2, 2, -4) \\
 &= 3(2) + 1(2) + 4(-4) \\
 &= -8 \quad |
 \end{aligned}$$

$$\begin{aligned}
 \vec{u} \cdot \vec{u} &= \|\vec{u}\|^2 \\
 &= 3^2 + 1^2 + 4^2 \\
 &= 26 \quad |
 \end{aligned}$$

$$\begin{aligned}
 \vec{v} \cdot \vec{v} &= \|\vec{v}\|^2 \\
 &= 2^2 + 2^2 + (-4)^2 \\
 &= 24 \quad |
 \end{aligned}$$

$$\begin{aligned}
 b) \quad \vec{u} \cdot \vec{v} &= (1, 1, 4, 6) \cdot (2, -2, 3, -2) \\
 &= 1(2) + 1(-2) + 4(3) + 6(-2) \\
 &= 0 \quad |
 \end{aligned}$$

$$\begin{aligned}
 \vec{u} \cdot \vec{u} &= \|\vec{u}\|^2 \\
 &= 1^2 + 1^2 + 4^2 + 6^2 \\
 &= 54 \quad |
 \end{aligned}$$

$$\begin{aligned}
 \vec{v} \cdot \vec{v} &= \|\vec{v}\|^2 \\
 &= 2^2 + (-2)^2 + 3^2 + (-2)^2 \\
 &= 21 \quad |
 \end{aligned}$$

$$\begin{aligned}
 c) \quad \vec{u} \cdot \vec{v} &= (2, -1, 1, 0, -2) \cdot (1, 2, 2, 2, 1) \\
 &= 2(1) - 1(2) + 1(2) + 0(2) - 2(1) \\
 &= 0 \quad |
 \end{aligned}$$

$$\begin{aligned}
 \vec{u} \cdot \vec{u} &= \|\vec{u}\|^2 \\
 &= 2^2 + (-1)^2 + 1^2 + 0 + (-2)^2 \\
 &= 10 \quad |
 \end{aligned}$$

$$\begin{aligned}
 \vec{v} \cdot \vec{v} &= \|\vec{v}\|^2 \\
 &= 1^2 + 2^2 + 2^2 + 2^2 + 1^2 \\
 &= 14 \quad |
 \end{aligned}$$



### Exercise

Find the Euclidean distance between  $\vec{u}$  and  $\vec{v}$ , then find the angle between them

a)  $\vec{u} = (3, 3, 3)$ ,  $\vec{v} = (1, 0, 4)$

b)  $\vec{u} = (1, 2, -3, 0)$ ,  $\vec{v} = (5, 1, 2, -2)$

c)  $\vec{u} = (0, 1, 1, 1, 2)$ ,  $\vec{v} = (2, 1, 0, -1, 3)$

### Solution

$$\begin{aligned} \text{a) } d = \|\vec{u} - \vec{v}\| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(-2)^2 + (-3)^2 + (1)^2} \\ &= \sqrt{14} \end{aligned}$$

$$\begin{aligned} \cos \theta &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \\ &= \frac{3(1) + 3(0) + 3(4)}{\sqrt{3^2 + 3^2 + 3^2} \sqrt{1^2 + 0^2 + 4^2}} \\ &= \frac{15}{\sqrt{27} \sqrt{17}} \end{aligned}$$

$$\begin{aligned} \theta &= \cos^{-1} \left( \frac{15}{\sqrt{27} \sqrt{17}} \right) \\ &\approx 45.56^\circ \end{aligned}$$

$$\begin{aligned} \text{b) } d = \|\vec{u} - \vec{v}\| &= \sqrt{(1-5)^2 + (-2-1)^2 + (-3-2)^2 + (-2-0)^2} \\ &= \sqrt{(-4)^2 + (-3)^2 + (-5)^2 + (-2)^2} \\ &= \sqrt{46} \end{aligned}$$

$$\begin{aligned} \cos \theta &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \\ &= \frac{1(5) + 2(1) - 3(2) + 0(-2)}{\sqrt{1^2 + 2^2 + (-3)^2 + 0} \sqrt{5^2 + 1^2 + 2^2 + (-2)^2}} \\ &= \frac{1}{\sqrt{14} \sqrt{34}} \end{aligned}$$

$$\begin{aligned} \theta &= \cos^{-1} \left( \frac{1}{\sqrt{14} \sqrt{34}} \right) \\ &\approx 87.37^\circ \end{aligned}$$

$$\text{c) } d = \|\vec{u} - \vec{v}\| = \sqrt{(0-2)^2 + (1-1)^2 + (1-0)^2 + (1-(-1))^2 + (2-3)^2}$$

$$\begin{aligned}
& \underline{= \sqrt{10} \mid} \\
\cos \theta &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \\
&= \frac{0(2) + 1(1) + 1(0) + 1(-1) + 2(3)}{\sqrt{0^2 + 1^2 + 1^2 + 1^2 + 2^2} \sqrt{2^2 + 1^2 + 0 + (-1)^2 + (3)^2}} \\
&= \frac{6}{\sqrt{7} \sqrt{15}} \\
\theta &= \cos^{-1} \left( \frac{6}{\sqrt{7} \sqrt{15}} \right) \\
&\underline{\approx 54.16^\circ \mid}
\end{aligned}$$

### Exercise

Find a unit vector that has the same direction as the given vector

a)  $(-4, -3)$       b)  $(-3, 2, \sqrt{3})$       c)  $(1, 2, 3, 4, 5)$

### Solution

$$\begin{aligned}
a) \quad \vec{u} &= \frac{\vec{v}}{\|\vec{v}\|} = \frac{(-4, -3)}{\sqrt{(-4)^2 + (-3)^2}} \\
&= \frac{(-4, -3)}{\sqrt{25}} \\
&\underline{= \left( -\frac{4}{5}, -\frac{3}{5} \right) \mid}
\end{aligned}$$

$$\begin{aligned}
b) \quad \vec{u} &= \frac{1}{\sqrt{(-3)^2 + (2)^2 + (\sqrt{3})^2}} (-3, 2, \sqrt{3}) \\
&= \frac{1}{\sqrt{17}} (-3, 2, \sqrt{3}) \\
&\underline{= \left( -\frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{\sqrt{3}}{\sqrt{17}} \right) \mid}
\end{aligned}$$

$$\begin{aligned}
c) \quad \vec{u} &= \frac{1}{\sqrt{1^2 + 2^2 + 3^2 + 4^2 + 5^2}} (1, 2, 3, 4, 5) \\
&= \frac{1}{\sqrt{55}} (1, 2, 3, 4, 5) \\
&\underline{= \left( \frac{1}{\sqrt{55}}, \frac{2}{\sqrt{55}}, \frac{3}{\sqrt{55}}, \frac{4}{\sqrt{55}}, \frac{5}{\sqrt{55}} \right) \mid}
\end{aligned}$$

### Exercise

Find a unit vector that is oppositely to the given vector

- a)  $(-12, -5)$
- b)  $(3, -3, 3)$
- c)  $(-3, 1, \sqrt{6}, 3)$

### Solution

$$\begin{aligned} \text{a) } \vec{u} &= -\frac{1}{\sqrt{(-12)^2 + (-5)^2}}(-12, -5) \\ &= -\frac{1}{\sqrt{169}}(-12, -5) \\ &= \left(\frac{12}{13}, \frac{5}{13}\right) \end{aligned}$$

$$\begin{aligned} \text{b) } \vec{u} &= -\frac{1}{\sqrt{(3)^2 + (-3)^2 + (3)^2}}(3, -3, 3) \\ &= -\frac{1}{\sqrt{27}}(3, -3, 3) \\ &= -\frac{1}{3\sqrt{3}}(3, -3, 3) \\ &= \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \end{aligned}$$

$$\begin{aligned} \text{c) } \vec{u} &= -\frac{1}{\sqrt{(-3)^2 + 1^2 + (\sqrt{6})^2 + 3^2}}(-3, 1, \sqrt{6}, 3) \\ &= -\frac{1}{\sqrt{25}}(-3, 1, \sqrt{6}, 3) \\ &= \left(\frac{3}{5}, -\frac{1}{5}, -\frac{\sqrt{6}}{5}, -\frac{3}{5}\right) \end{aligned}$$

### Exercise

Verify that the Cauchy-Schwarz inequality holds

- a)  $\vec{u} = (-3, 1, 0), \vec{v} = (2, -1, 3)$
- b)  $\vec{u} = (0, 2, 2, 1), \vec{v} = (1, 1, 1, 1)$
- c)  $\vec{u} = (1, 3, 5, 2, 0, 1), \vec{v} = (0, 2, 4, 1, 3, 5)$

### Solution

$$\begin{aligned}
 a) \quad |\vec{u} \cdot \vec{v}| &= |(-3, 1, 0) \cdot (2, -1, 3)| \\
 &= |-3(2) + 1(-1) + 0(3)| \\
 &= |-7| \\
 &= \underline{7}
 \end{aligned}$$

$$\begin{aligned}
 \|\vec{u}\| \|\vec{v}\| &= \sqrt{(-3)^2 + 1^2 + 0} \sqrt{(2)^2 + (-1)^2 + 3^2} \\
 &= \sqrt{10} \sqrt{14} \\
 &= \underline{2\sqrt{35}} \\
 &\approx \underline{11.83}
 \end{aligned}$$

$$7 < 2\sqrt{35} \quad \checkmark$$

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\| \quad \text{Cauchy-Schwarz inequality holds}$$

$$\begin{aligned}
 b) \quad |\vec{u} \cdot \vec{v}| &= |(0, 2, 2, 1) \cdot (1, 1, 1, 1)| \\
 &= |0 + 2 + 2 + 1| \\
 &= \underline{5}
 \end{aligned}$$

$$\begin{aligned}
 \|\vec{u}\| \|\vec{v}\| &= \sqrt{0^2 + 2^2 + 2^2 + 1^2} \sqrt{1^2 + 1^2 + 1^2 + 1^2} \\
 &= \sqrt{9} \sqrt{4} \\
 &= \underline{6}
 \end{aligned}$$

$$5 < 6 \quad \checkmark$$

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\| \quad \text{Cauchy-Schwarz inequality holds}$$

$$\begin{aligned}
 c) \quad |\vec{u} \cdot \vec{v}| &= |(1, 3, 5, 2, 0, 1) \cdot (0, 2, 4, 1, 3, 5)| \\
 &= |0 + 6 + 20 + 2 + 0 + 5| \\
 &= \underline{23}
 \end{aligned}$$

$$\begin{aligned}
 \|\vec{u}\| \|\vec{v}\| &= \sqrt{1^2 + 3^2 + 5^2 + 2^2 + 0 + 1^2} \sqrt{0 + 2^2 + 4^2 + 1^2 + 3^2 + 5^2} \\
 &= \sqrt{40} \sqrt{55} \\
 &= 10\sqrt{22} \\
 &\approx \underline{46}
 \end{aligned}$$

$$23 < 10\sqrt{22} \quad \checkmark$$

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\| \quad \text{Cauchy-Schwarz inequality holds}$$

### Exercise

Find  $\vec{u} \cdot \vec{v}$  and then the angle  $\theta$  between  $\vec{u}$  and  $\vec{v}$   $\vec{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$   $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$

### Solution

$$\vec{u} \cdot \vec{v} = 3 + 0 - 2 - 1$$

$$= 0$$

$$\|\vec{u}\| = \sqrt{9 + 1 + 4 + 1}$$

$$= \sqrt{15}$$

$$\|\vec{v}\| = \sqrt{1 + 1 + 1}$$

$$= \sqrt{3}$$

$$\theta = \cos^{-1} \frac{0}{\sqrt{15}\sqrt{3}}$$

$$= \cos^{-1}(0)$$

$$= 90^\circ$$

### Exercise

Find the norm:  $\|\vec{u}\| + \|\vec{v}\|$ ,  $\|\vec{u} + \vec{v}\|$  for  $\vec{u} = (3, -1, -2, 1, 4)$   $\vec{v} = (1, 1, 1, 1, 1)$

### Solution

$$\|\vec{u}\| + \|\vec{v}\| = \sqrt{3^2 + (-1)^2 + (-2)^2 + 1^2 + 4^2} + \sqrt{1 + 1 + 1 + 1 + 1}$$

$$= \sqrt{31} + \sqrt{5}$$

$$\|\vec{u} + \vec{v}\| = \|(3, -1, -2, 1, 4) + (1, 1, 1, 1, 1)\|$$

$$= \|(4, 0, -1, 2, 5)\|$$

$$= \sqrt{16 + 0 + 1 + 4 + 25}$$

$$= \sqrt{46}$$

### Exercise

Find all numbers  $r$  such that:  $\|r(1, 0, -3, -1, 4, 1)\| = 1$

#### Solution

$$r\sqrt{1+9+1+16+1} = \pm 1$$

$$r\sqrt{28} = \pm 1$$

$$r = \pm \frac{1}{2\sqrt{7}}$$

$$= \pm \frac{\sqrt{7}}{14}$$

### Exercise

Find the distance between  $P_1(7, -5, 1)$  and  $P_2(-7, -2, -1)$

#### Solution

$$\begin{aligned}\|P_1P_2\| &= \sqrt{(-7-7)^2 + (-2+5)^2 + (-1-1)^2} \\ &= \sqrt{14^2 + 3^2 + (-2)^2} \\ &= \sqrt{196 + 9 + 4} \\ &= \sqrt{209}\end{aligned}$$

### Exercise

Given  $\vec{u} = (1, -5, 4)$ ,  $\vec{v} = (3, 3, 3)$

- a) Find  $\vec{u} \cdot \vec{v}$
- b) Find the cosine of the angle  $\theta$  between  $\vec{u}$  and  $\vec{v}$ .

#### Solution

$$\begin{aligned}a) \quad \vec{u} \cdot \vec{v} &= 3 - 15 + 12 \\ &= 0\end{aligned}$$

$$\begin{aligned}b) \quad \cos \theta &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \\ &= 0\end{aligned}$$

### Exercise

Let  $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$  and  $\vec{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ . Find  $\left\| \frac{1}{\|2\vec{u} + \vec{v}\|} (2\vec{u} + \vec{v}) \right\|$

### Solution

Since, the unit vector equals to a vector  $(2\vec{u} + \vec{v})$  divided by its magnitude.

Therefore,

$$\left\| \frac{1}{\|2\vec{u} + \vec{v}\|} (2\vec{u} + \vec{v}) \right\| = \frac{1}{\|2\vec{u} + \vec{v}\|} \|2\vec{u} + \vec{v}\|$$
$$= 1$$

**Or**

$$\begin{aligned} \left\| \frac{1}{\|2\vec{u} + \vec{v}\|} (2\vec{u} + \vec{v}) \right\| &= \left\| \frac{1}{\|2(2, 1) + (1, 4)\|} (2(2, 1) + (1, 4)) \right\| \\ &= \left\| \frac{1}{\|(5, 6)\|} (5, 6) \right\| \\ &= \left\| \frac{1}{\sqrt{25 + 36}} (5, 6) \right\| \\ &= \left\| \frac{1}{\sqrt{61}} (5, 6) \right\| \\ &= \frac{1}{\sqrt{61}} \|(5, 6)\| \\ &= \frac{1}{\sqrt{61}} \sqrt{25 + 36} \\ &= \frac{\sqrt{61}}{\sqrt{61}} \\ &= 1 \end{aligned}$$

### Exercise

Let  $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$  and  $\vec{v} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ . Find  $\left\| \frac{1}{\|\vec{u} - \vec{v}\|} (\vec{u} - \vec{v}) \right\|$

### Solution

$$\left\| \frac{1}{\|\vec{u} - \vec{v}\|} (\vec{u} - \vec{v}) \right\| = \frac{1}{\|\vec{u} - \vec{v}\|} \|\vec{u} - \vec{v}\|$$
$$= 1$$

**Or**

$$\begin{aligned}
\left\| \frac{1}{\|\vec{u} - \vec{v}\|} (\vec{u} - \vec{v}) \right\| &= \left\| \frac{1}{\|(2, 1) - (3, 3)\|} ((2, 1) - (3, 3)) \right\| \\
&= \left\| \frac{1}{\|(-1, -2)\|} (-1, -2) \right\| \\
&= \left\| \frac{1}{\sqrt{1+4}} (-1, -2) \right\| \\
&= \left\| \frac{1}{\sqrt{5}} (-1, -2) \right\| \\
&= \frac{1}{\sqrt{5}} \|(-1, -2)\| \\
&= \frac{1}{\sqrt{5}} \sqrt{1+4} \\
&= \frac{\sqrt{5}}{\sqrt{5}} \\
&= \underline{1}
\end{aligned}$$

### ***Exercise***

Let  $\vec{u} = \begin{pmatrix} 18 \\ 6 \end{pmatrix}$  and  $\vec{v} = \begin{pmatrix} -11 \\ 12 \end{pmatrix}$ . Find  $\left\| \frac{1}{\|5\vec{u} + 3\vec{v}\|} (5\vec{u} + 3\vec{v}) \right\|$

### **Solution**

$$\begin{aligned}
\left\| \frac{1}{\|5\vec{u} + 3\vec{v}\|} (5\vec{u} + 3\vec{v}) \right\| &= \frac{1}{\|5\vec{u} + 3\vec{v}\|} \|5\vec{u} + 3\vec{v}\| \\
&= \underline{1}
\end{aligned}$$

$$\begin{aligned}
\left\| \frac{1}{\|5\vec{u} + 3\vec{v}\|} (5\vec{u} + 3\vec{v}) \right\| &= \left\| \frac{1}{\|5(18, 6) + 3(-11, 12)\|} (5(18, 6) + 3(-11, 12)) \right\| \\
&= \left\| \frac{1}{\|(90, 30) + (-33, 36)\|} ((90, 30) + (-33, 36)) \right\| \\
&= \left\| \frac{1}{\|(57, 66)\|} (57, 66) \right\| \\
&= \left\| \frac{1}{\|3(19, 22)\|} (57, 66) \right\| \\
&= \left\| \frac{3(19, 22)}{3\sqrt{361 + 484}} \right\|
\end{aligned}$$



$$\begin{aligned}
&= \frac{1}{\sqrt{845}} \|(19, 22)\| \\
&= \frac{1}{\sqrt{845}} \sqrt{845} \\
&= 1
\end{aligned}$$

### Exercise

Let  $\vec{u} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$  and  $\vec{v} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ . Calculate the following:

- a)  $\vec{u} + \vec{v}$       b)  $2\vec{u} + 3\vec{v}$       c)  $\vec{v} + (2\vec{u} - 3\vec{v})$       d)  $\|\vec{u}\|$       e)  $\|\vec{v}\|$       f) unit vector of  $\vec{v}$

### Solution

$$\begin{aligned}
a) \quad \vec{u} + \vec{v} &= \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
b) \quad 2\vec{u} + 3\vec{v} &= 2 \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} 6 \\ 2 \\ -2 \end{pmatrix} + \begin{pmatrix} -6 \\ 3 \\ 6 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 5 \\ 4 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
c) \quad \vec{v} + (2\vec{u} - 3\vec{v}) &= \vec{v} + 2\vec{u} - 3\vec{v} \\
&= 2\vec{u} - 2\vec{v} \\
&= 2 \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} - 2 \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} 6 \\ 2 \\ -2 \end{pmatrix} - \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} 10 \\ 0 \\ -6 \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} 10 \\ 0 \\ -6 \end{pmatrix}$$

$$\begin{aligned} d) \quad \|\vec{u}\| &= \sqrt{3^2 + 1^2 + (-1)^2} \\ &= \sqrt{9 + 1 + 1} \\ &= \sqrt{11} \end{aligned}$$

$$\begin{aligned} e) \quad \|\vec{v}\| &= \sqrt{(-2)^2 + 1^2 + 2^2} \\ &= \sqrt{4 + 1 + 4} \\ &= 3 \end{aligned}$$

$$\begin{aligned} f) \quad \text{unit vector of } \vec{v} &= \frac{\vec{v}}{\|\vec{v}\|} \\ &= \frac{(-2, 1, 2)}{3} \\ &= \left(-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) \end{aligned}$$

### ***Exercise***

Let  $\vec{u} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$  and  $\vec{v} = \begin{pmatrix} 1 \\ 4 \\ 3 \\ 1 \end{pmatrix}$ . Calculate the following:

$$a) \quad \vec{u} - \vec{v} \qquad b) \quad 3\vec{u} - 2\vec{v} \qquad c) \quad 2(\vec{u} - \vec{v}) + 3\vec{u} \qquad d) \quad \|\vec{u}\| \qquad e) \quad \text{unit vector of } \vec{v}$$

### **Solution**

$$\begin{aligned} a) \quad \vec{u} - \vec{v} &= \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 3 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -5 \\ -3 \\ 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
 b) \quad 3\vec{u} - 2\vec{v} &= 3 \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 4 \\ 3 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} 6 \\ -3 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 8 \\ 6 \\ 2 \end{pmatrix} \\
 &= \begin{pmatrix} 4 \\ -11 \\ -6 \\ 1 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 c) \quad 2(\vec{u} - \vec{v}) + 3\vec{u} &= 2\vec{u} - 2\vec{v} + 3\vec{u} \\
 &= 5\vec{u} - 2\vec{v} \\
 &= 5 \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 4 \\ 3 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} 10 \\ -5 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 8 \\ 6 \\ 2 \end{pmatrix} \\
 &= \begin{pmatrix} 8 \\ -13 \\ -6 \\ 3 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 d) \quad \|\vec{u}\| &= \sqrt{2^2 + (-1)^2 + 0 + 1^2} \\
 &= \sqrt{4 + 1 + 1} \\
 &= \sqrt{6}
 \end{aligned}$$

$$\begin{aligned}
 e) \quad \text{unit vector of } \vec{v} &= \frac{\vec{v}}{\|\vec{v}\|} \\
 &= \frac{(1, 4, 3, 1)}{\sqrt{1 + 16 + 9 + 1}} \\
 &= \frac{(1, 4, 3, 1)}{\sqrt{27}}
 \end{aligned}$$

$$= \left( \frac{1}{3\sqrt{3}}, \frac{4}{3\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{3\sqrt{3}} \right) \Big|$$

### Exercise

Let  $\vec{u} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \\ -1 \end{pmatrix}$  and  $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ -1 \end{pmatrix}$ . Calculate the following:

- a)  $\vec{v} - \vec{u}$       b)  $\vec{u} + 3\vec{v}$       c)  $3(\vec{u} + \vec{v}) - 3\vec{u}$       d)  $\|\vec{v}\|$       e) unit vector of  $\vec{v}$

### Solution

$$\begin{aligned} a) \quad \vec{v} - \vec{u} &= \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} b) \quad \vec{u} + 3\vec{v} &= \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 6 \\ 3 \\ -3 \end{pmatrix} \\ &= (5, 1, 9, 3, -4) \Big| \end{aligned}$$

$$\begin{aligned} c) \quad 3(\vec{u} + \vec{v}) - 3\vec{u} &= 3\vec{u} + 3\vec{v} - 3\vec{u} \\ &= 3\vec{v} \\ &= 3(1, 0, 2, 1, -1) \end{aligned}$$

$$= \underline{(3, 0, 6, 3, -3)}$$

$$\begin{aligned} d) \quad \|\vec{v}\| &= \sqrt{1^2 + 0 + 2^2 + 1^2 + (-1)^2} \\ &= \sqrt{1 + 4 + 1 + 1} \\ &= \underline{\sqrt{7}} \end{aligned}$$

$$\begin{aligned} e) \quad \text{unit vector of } \vec{v} &= \frac{\vec{v}}{\|\vec{v}\|} \\ &= \frac{(1, 0, 2, 1, -1)}{\sqrt{7}} \\ &= \underline{\left(\frac{1}{\sqrt{7}}, 0, \frac{2}{\sqrt{7}}, \frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}}\right)} \end{aligned}$$

### Exercise

Let  $\vec{u} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ ,  $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$  and  $\vec{w} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ . Calculate the following:

$$a) \quad \vec{u} \cdot \vec{v} \qquad b) \quad \vec{u} \cdot (\vec{v} + \vec{w}) \qquad c) \quad (\vec{u} + 2\vec{v}) \cdot \vec{w} \qquad d) \quad \|(\vec{w} \cdot \vec{v})\vec{u}\|$$

### Solution

$$\begin{aligned} a) \quad \vec{u} \cdot \vec{v} &= (2, -1, 1) \cdot (1, 2, -2) \\ &= 2 - 2 - 2 \\ &= \underline{-2} \end{aligned}$$

$$\begin{aligned} b) \quad \vec{u} \cdot (\vec{v} + \vec{w}) &= (2, -1, 1) \cdot [(1, 2, -2) + (3, 2, 1)] \\ &= (2, -1, 1) \cdot (4, 4, -1) \\ &= 8 - 4 - 1 \\ &= \underline{3} \end{aligned}$$

$$\begin{aligned} c) \quad (\vec{u} + 2\vec{v}) \cdot \vec{w} &= [(2, -1, 1) + 2(1, 2, -2)] \cdot (3, 2, 1) \\ &= (4, 3, -3) \cdot (3, 2, 1) \\ &= 12 + 6 - 3 \\ &= \underline{15} \end{aligned}$$

$$\begin{aligned} d) \quad \|(\vec{w} \cdot \vec{v})\vec{u}\| &= |\vec{w} \cdot \vec{v}| \|\vec{u}\| \\ &= |(3, 2, 1) \cdot (1, 2, -2)| \sqrt{2^2 + (-1)^2 + 1^2} \\ &= |3 + 4 - 2| \sqrt{4 + 1 + 1} \\ &= \underline{5\sqrt{6}} \end{aligned}$$

### Exercise

Let  $\vec{u} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \end{pmatrix}$ ,  $\vec{v} = \begin{pmatrix} -2 \\ 5 \\ 2 \\ -6 \end{pmatrix}$  and  $\vec{w} = \begin{pmatrix} 4 \\ -1 \\ 0 \\ -2 \end{pmatrix}$ . Calculate the following:

a)  $\vec{u} \cdot \vec{v}$       b)  $\vec{u} \cdot (\vec{v} + \vec{w})$       c)  $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v})$       d)  $\|(\vec{w} \cdot \vec{v})\vec{u}\|$

### Solution

a)  $\vec{u} \cdot \vec{v} = (1, 3, 2, 1) \cdot (-2, 5, 2, -6)$   
 $= -2 + 15 + 4 - 6$   
 $= 11$

b)  $\vec{u} \cdot (\vec{v} + \vec{w}) = (1, 3, 2, 1) \cdot [(-2, 5, 2, -6) + (4, -1, 0, -2)]$   
 $= (1, 3, 2, 1) \cdot (2, 4, 2, -8)$   
 $= 2 + 12 + 4 - 8$   
 $= 10$

c)  $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = [(1, 3, 2, 1) + (-2, 5, 2, -6)] \cdot [(1, 3, 2, 1) - (-2, 5, 2, -6)]$   
 $= (-1, 8, 4, -5) \cdot (3, -2, 0, 7)$   
 $= -3 - 16 + 0 - 35$   
 $= -54$

**Or**  $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} - \vec{v} \cdot \vec{v}$   
 $= \vec{u} \cdot \vec{u} - \vec{v} \cdot \vec{v}$   
 $= (1 + 9 + 4 + 1) - (4 + 25 + 4 + 36)$   
 $= 15 - 69$   
 $= -54$

d)  $\|(\vec{w} \cdot \vec{v})\vec{u}\| = |\vec{w} \cdot \vec{v}| \|\vec{u}\|$   
 $= |(4, -1, 0, -2) \cdot (-2, 5, 2, -6)| \sqrt{1 + 9 + 4 + 1}$   
 $= |-8 - 5 + 12| \sqrt{15}$   
 $= \sqrt{15}$

### Exercise

Let  $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}$ ,  $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$  and  $\vec{w} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$ . Calculate the following:

a)  $\vec{u} \cdot (\vec{v} + \vec{w})$     b)  $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v})$     c)  $(\vec{u} \cdot \vec{w})\vec{v} + (\vec{v} \cdot \vec{w})\vec{u}$     d)  $(\vec{u} + 2\vec{v}) \cdot (\vec{u} - \vec{v})$

### Solution

a)  $\vec{u} \cdot (\vec{v} + \vec{w}) = (1, 0, -2, 1) \cdot [(0, 1, 1, 0) + (1, -1, -1, 1)]$   
 $= (1, 0, -2, 1) \cdot (1, 0, 0, 1)$   
 $= 2$

b)  $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = [(1, 0, -2, 1) + (0, 1, 1, 0)] \cdot [(1, 0, -2, 1) - (0, 1, 1, 0)]$   
 $= (1, 1, -1, 1) \cdot (1, -1, -3, 1)$   
 $= 1 - 1 + 3 + 1$   
 $= 4$

c)  $(\vec{u} \cdot \vec{w})\vec{v} + (\vec{v} \cdot \vec{w})\vec{u} = [(1, 0, -2, 1) \cdot (1, -1, -1, 1)](0, 1, 1, 0)$   
 $+ [(0, 1, 1, 0) \cdot (1, -1, -1, 1)](1, 0, -2, 1)$   
 $= 4(0, 1, 1, 0) - 2(1, 0, -2, 1)$   
 $= (-2, 4, 8, 2)$

d)  $(\vec{u} + 2\vec{v}) \cdot (\vec{u} - \vec{v}) = [(1, 0, -2, 1) + 2(0, 1, 1, 0)] \cdot [(1, 0, -2, 1) - (0, 1, 1, 0)]$   
 $= (1, 2, 0, 1) \cdot (1, -1, -3, 1)$   
 $= 1 - 2 + 1$   
 $= 0$

### Exercise

Suppose  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$  are vectors in  $\mathbb{R}^n$  such that  $\vec{u} \cdot \vec{v} = 2$ ,  $\vec{u} \cdot \vec{w} = -3$ , and  $\vec{v} \cdot \vec{w} = 5$ . If possible, calculate the following values:

a)  $\vec{u} \cdot (\vec{v} + \vec{w})$     d)  $\vec{w} \cdot (2\vec{v} - 4\vec{u})$     g)  $\vec{w} \cdot ((\vec{u} \cdot \vec{w})\vec{u})$   
b)  $(\vec{u} + \vec{v}) \cdot \vec{w}$     e)  $(\vec{u} + \vec{v}) \cdot (\vec{v} + \vec{w})$     h)  $\vec{u} \cdot ((\vec{u} \cdot \vec{v})\vec{v} + (\vec{u} \cdot \vec{w})\vec{w})$   
c)  $\vec{u} \cdot (2\vec{v} - \vec{w})$     f)  $\vec{w} \cdot (5\vec{v} + \pi\vec{u})$

### Solution

a)  $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

$$= 2 + (-3)$$

$$\underline{= -1}$$

$$b) \quad (\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$$

$$= -3 + 5$$

$$\underline{= 2}$$

$$c) \quad \vec{u} \cdot (2\vec{v} - \vec{w}) = \vec{u} \cdot (2\vec{v}) - \vec{u} \cdot \vec{w}$$

$$= 2\vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{w}$$

$$= 2(2) - (-3)$$

$$\underline{= 7}$$

$$d) \quad \vec{w} \cdot (2\vec{v} - 4\vec{u}) = \vec{w} \cdot (2\vec{v}) - \vec{w} \cdot (4\vec{u})$$

$$= 2\vec{w} \cdot \vec{v} - 4\vec{w} \cdot \vec{u}$$

$$= 2(\vec{v} \cdot \vec{w}) - 4(\vec{u} \cdot \vec{w})$$

$$= 2(5) - 4(-3)$$

$$\underline{= 22}$$

$$e) \quad (\vec{u} + \vec{v}) \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{v} + \vec{v} \cdot \vec{w}$$

$$= 2 + (-3) + \vec{v} \cdot \vec{v} + 5$$

$$\underline{= 4 + \vec{v}^2}$$

$$f) \quad \vec{w} \cdot (5\vec{v} + \pi\vec{u}) = \vec{w} \cdot (5\vec{v}) + \vec{w} \cdot (\pi\vec{u})$$

$$= 5(\vec{w} \cdot \vec{v}) + \pi(\vec{w} \cdot \vec{u})$$

$$= 5(\vec{v} \cdot \vec{w}) + \pi(\vec{u} \cdot \vec{w})$$

$$= 5(5) + \pi(-3)$$

$$\underline{= 25 - 3\pi}$$

$$g) \quad \vec{w} \cdot ((\vec{u} \cdot \vec{w})\vec{u}) = \vec{w} \cdot ((-3)\vec{u})$$

$$= -3(\vec{w} \cdot \vec{u})$$

$$= -3(\vec{u} \cdot \vec{w})$$

$$= -3(-3)$$

$$\underline{= 9}$$

$$h) \quad \vec{u} \cdot ((\vec{u} \cdot \vec{v})\vec{v} + (\vec{u} \cdot \vec{w})\vec{w}) = \vec{u} \cdot (2\vec{v} + 5\vec{w})$$

$$= \vec{u} \cdot (2\vec{v}) + \vec{u} \cdot (5\vec{w})$$

$$= 2\vec{u} \cdot \vec{v} + 5\vec{u} \cdot \vec{w}$$

$$= 2(2) + 5(-3)$$

$$\underline{= -11}$$



### Exercise

You are in an airplane flying from Chicago to Boston for a job interview. The compass in the cockpit of the plane shows that your plane is pointed due East, and the airspeed indicator on the plane shows that the plane is traveling through the air at 400 *mph*. there is a crosswind that affects your plane however, and the crosswind is blowing due South at 40 *mph*.

Given the crosswind you wonder; relative to the ground, in what direction are you really flying and how fast are you really traveling?

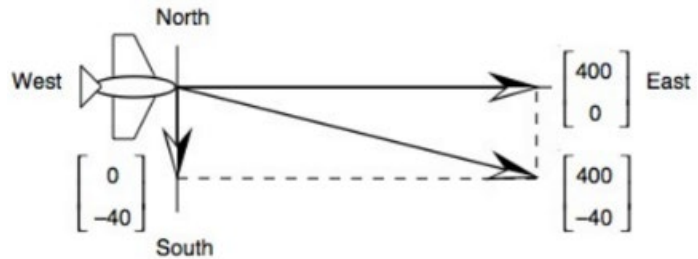
### Solution

Let the air velocity of the plane be:  $\vec{a} = \begin{pmatrix} 400 \\ 0 \end{pmatrix}$

The wind velocity be:  $\vec{w} = \begin{pmatrix} 0 \\ -40 \end{pmatrix}$

The ground the velocity is:

$$\begin{aligned} \vec{g} &= \vec{a} + \vec{w} \\ &= \begin{pmatrix} 400 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -40 \end{pmatrix} \\ &= \begin{pmatrix} 400 \\ -40 \end{pmatrix} \end{aligned}$$



The *magnitude*:  $\sqrt{400^2 + (-40)^2} = \underline{402 \text{ } mph}$

The *direction*:  $\theta = \tan^{-1} \frac{-40}{400}$   
 $= \tan^{-1} \left( -\frac{1}{10} \right)$   
 $\approx \underline{5.71^\circ}$

### Exercise

A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What speed and direction should the jetliner have in order for the resultant vector to be 500 *mph* due east?

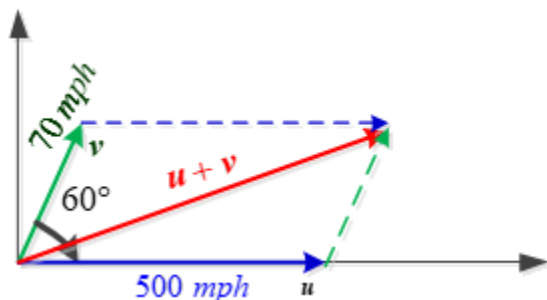
### Solution

$\vec{u} = (x, y)$ : the velocity of the airplane

$\vec{v}$ : the velocity of the tailwind

$$\begin{aligned} \vec{v} &= (70 \cos 60^\circ, 70 \sin 60^\circ) \\ &= (35, 35\sqrt{3}) \end{aligned}$$

$$\vec{u} + \vec{v} = (500, 0)$$



$$(x, y) + (35, 35\sqrt{3}) = (500, 0)$$

$$(x, y) = (500, 0) - (35, 35\sqrt{3})$$

$$= (765, -35\sqrt{3})$$

$$\vec{u} = (765, -35\sqrt{3})$$

$$|\vec{u}| = \sqrt{465^2 + (-35\sqrt{3})^2}$$

$$\approx 468.9 \text{ mph}$$

$$\theta = \tan^{-1} \frac{-35\sqrt{3}}{465}$$

$$\approx -7.4^\circ$$

$\therefore$  The direction is  $7.4^\circ$  south of east.

### Example

A jet airliner, flying due east at 500 mph in still air, encounters a 70-mph tailwind blowing in the direction  $60^\circ$  north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?

### Solution

$\vec{u}$ : the velocity of the airplane

$\vec{v}$ : the velocity of the tailwind

**Given:**  $|\vec{u}| = 500$   $|\vec{v}| = 70$

$$\vec{u} = \langle 500, 0 \rangle$$

$$\vec{v} = \langle 70 \cos 60^\circ, 70 \sin 60^\circ \rangle$$

$$= \langle 35, 35\sqrt{3} \rangle$$

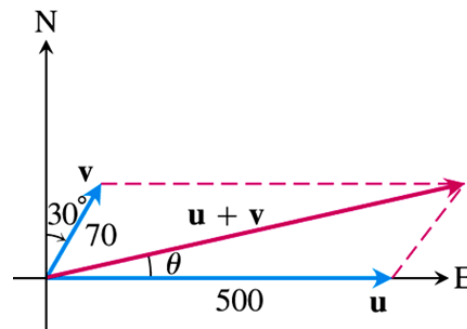
$$\vec{u} + \vec{v} = \langle 535, 35\sqrt{3} \rangle$$

$$= 535\hat{i} + 35\sqrt{3}\hat{j}$$

$$|\vec{u} + \vec{v}| = \sqrt{535^2 + (35\sqrt{3})^2}$$

$$\approx 538.4 \text{ mph}$$

$$\theta = \tan^{-1} \frac{35\sqrt{3}}{535}$$



$$\approx 6.5^\circ$$

The ground speed of the airplane is about  $538.4 \text{ mph}$ , and its direction is about  $6.5^\circ$  north of east.

### Exercise

A bird flies from its nest  $5 \text{ km}$  in the direction  $60^\circ$  north east, where it stops to rest on a tree. It then flies  $10 \text{ km}$  in the direction due southeast and lands atop a telephone pole. Place an  $xy$ -coordinate system so that the origin is the bird's nest, the  $x$ -axis points east, and the  $y$ -axis points north.

- At what point is the tree located?
- At what point is the telephone pole?

### Solution

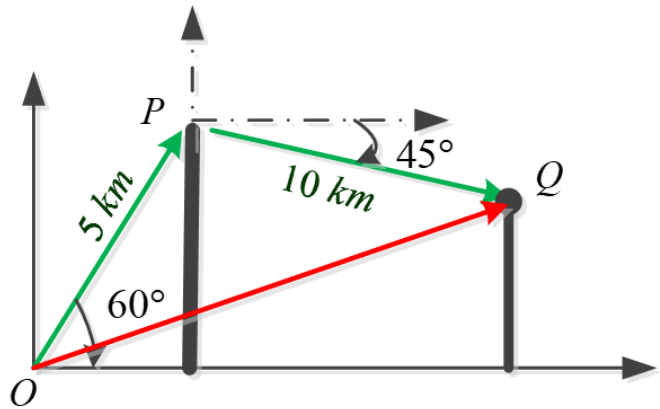
$$\begin{aligned} \text{a) } \overrightarrow{OP} &= (5 \cos 60^\circ) \hat{i} + (5 \sin 60^\circ) \hat{j} \\ &= \frac{5}{2} \hat{i} + \frac{5\sqrt{3}}{2} \hat{j} \end{aligned}$$

The tree is located at the point

$$P = \left( \frac{5}{2}, \frac{5\sqrt{3}}{2} \right)$$

$$\begin{aligned} \text{b) } \overrightarrow{OQ} &= \overrightarrow{OP} + \overrightarrow{PQ} \\ &= \frac{5}{2} \hat{i} + \frac{5\sqrt{3}}{2} \hat{j} + (10 \cos 315^\circ) \hat{i} + (10 \sin 315^\circ) \hat{j} \\ &= \frac{5}{2} \hat{i} + \frac{5\sqrt{3}}{2} \hat{j} + \left( 10 \frac{\sqrt{2}}{2} \right) \hat{i} + \left( 10 \left( -\frac{\sqrt{2}}{2} \right) \right) \hat{j} \\ &= \left( \frac{5}{2} + 5\sqrt{2} \right) \hat{i} + \left( \frac{5\sqrt{3}}{2} - \frac{10\sqrt{2}}{2} \right) \hat{j} \\ &= \left( \frac{5+10\sqrt{2}}{2} \right) \hat{i} + \left( \frac{5\sqrt{3}-10\sqrt{2}}{2} \right) \hat{j} \end{aligned}$$

The pole is located at the point  $Q = \left( \frac{5+10\sqrt{2}}{2}, \frac{5\sqrt{3}-10\sqrt{2}}{2} \right)$



### Exercise

Prove  $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2 \geq 0$

#### Solution

Let  $\vec{u} = (u_1, u_2, \dots, u_n)$

$$\begin{aligned}\vec{u} \cdot \vec{u} &= (u_1, u_2, \dots, u_n) \cdot (u_1, u_2, \dots, u_n) \\ &= u_1 u_1 + u_2 u_2 + \dots + u_n u_n \\ &= u_1^2 + u_2^2 + \dots + u_n^2\end{aligned}$$

$$\begin{aligned}\|\vec{u}\|^2 &= \left( \sqrt{u_1^2 + u_2^2 + \dots + u_n^2} \right)^2 \\ &= u_1^2 + u_2^2 + \dots + u_n^2\end{aligned}$$

Thus,  $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$

Each  $u_i \in \mathbb{R}$ , then  $u_i^2 \geq 0$  for each  $1 \leq i \leq n$ , thus  $u_1^2 + u_2^2 + \dots + u_n^2 \geq 0$ .

Hence,  $\|\vec{u}\|^2 \geq 0$

### Exercise

Prove, for any vectors and  $\vec{v}$  in  $\mathbb{R}^2$  and any scalars  $c$  and  $d$ ,

$$(c\vec{u} + d\vec{v}) \cdot (c\vec{u} + d\vec{v}) = c^2 \|\vec{u}\|^2 + 2cd\vec{u} \cdot \vec{v} + d^2 \|\vec{v}\|^2$$

#### Solution

$$\begin{aligned}(c\vec{u} + d\vec{v}) \cdot (c\vec{u} + d\vec{v}) &= (c\vec{u} + d\vec{v}) \cdot c\vec{u} + (c\vec{u} + d\vec{v}) \cdot d\vec{v} \\ &= c\vec{u} \cdot c\vec{u} + d\vec{v} \cdot c\vec{u} + c\vec{u} \cdot d\vec{v} + d\vec{v} \cdot d\vec{v} \\ &= c^2 (\vec{u} \cdot \vec{u}) + cd (\vec{u} \cdot \vec{v}) + cd (\vec{u} \cdot \vec{v}) + d^2 (\vec{v} \cdot \vec{v}) \\ &= c^2 \|\vec{u}\|^2 + 2cd (\vec{u} \cdot \vec{v}) + d^2 \|\vec{v}\|^2 \quad \checkmark\end{aligned}$$

### Exercise

Prove  $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

#### Solution

Let  $\vec{u} = (u_1, u_2, \dots, u_n)$ ,  $\vec{v} = (v_1, v_2, \dots, v_n)$ , and  $\vec{w} = (w_1, w_2, \dots, w_n)$

$$\begin{aligned}
\vec{u} \cdot (\vec{v} + \vec{w}) &= (u_1, u_2, \dots, u_n) \cdot \left( (v_1, v_2, \dots, v_n) + (w_1, w_2, \dots, w_n) \right) \\
&= (u_1, u_2, \dots, u_n) \cdot (v_1 + w_1, v_2 + w_2, \dots, v_n + w_n) \\
&= u_1(v_1 + w_1) + u_2(v_2 + w_2) + \dots + u_n(v_n + w_n) \\
&= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + \dots + u_nv_n + u_nw_n \\
&= (u_1v_1 + u_2v_2 + \dots + u_nv_n) + (u_1w_1 + u_2w_2 + \dots + u_nw_n) \\
&= (u_1, u_2, \dots, u_n) \cdot (v_1, v_2, \dots, v_n) + (u_1, u_2, \dots, u_n) \cdot (w_1, w_2, \dots, w_n) \\
&= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \quad \checkmark
\end{aligned}$$

### Exercise

Prove Minkowski theorem:  $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

### Solution

$$\begin{aligned}
\|\vec{u} + \vec{v}\|^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \\
&= \vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} \\
&\leq \|\vec{u}\|^2 + 2\|\vec{u}\| \|\vec{v}\| + \|\vec{v}\|^2 \\
&= (\|\vec{u}\| + \|\vec{v}\|)^2
\end{aligned}$$

$$\sqrt{\|\vec{u} + \vec{v}\|^2} \leq \sqrt{(\|\vec{u}\| + \|\vec{v}\|)^2}$$

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\| \quad \checkmark$$

## ***Solution***      ***Section 2.3 – Orthogonality***

### ***Exercise***

Determine whether  $\vec{u}$  and  $\vec{v}$  are orthogonal

a)  $\vec{u} = (-6, -2), \quad \vec{v} = (5, -7)$

b)  $\vec{u} = (6, 1, 4), \quad \vec{v} = (2, 0, -3)$

c)  $\vec{u} = (1, -5, 4), \quad \vec{v} = (3, 3, 3)$

d)  $\vec{u} = (-2, 2, 3), \quad \vec{v} = (1, 7, -4)$

### **Solution**

$$\begin{aligned} \text{a) } \vec{u} \cdot \vec{v} &= (-6)(5) + (-2)(-7) \\ &= -30 + 14 \\ &= \underline{-16 \neq 0} \end{aligned}$$

$\therefore \vec{u}$  and  $\vec{v}$  are not orthogonal

$$\begin{aligned} \text{b) } \vec{u} \cdot \vec{v} &= 6(2) + 1(0) + 4(-3) \\ &= \underline{0} \end{aligned}$$

$\therefore \vec{u}$  and  $\vec{v}$  are orthogonal

$$\begin{aligned} \text{c) } \vec{u} \cdot \vec{v} &= 1(3) - 5(3) + 4(3) \\ &= \underline{0} \end{aligned}$$

$\therefore \vec{u}$  and  $\vec{v}$  are orthogonal

$$\begin{aligned} \text{d) } \vec{u} \cdot \vec{v} &= -2(1) + 2(7) + 3(-4) \\ &= \underline{0} \end{aligned}$$

$\therefore \vec{u}$  and  $\vec{v}$  are orthogonal

### ***Exercise***

Determine whether the vectors form an orthogonal set

a)  $\vec{v}_1 = (2, 3), \quad \vec{v}_2 = (3, 2)$

b)  $\vec{v}_1 = (1, -2), \quad \vec{v}_2 = (-2, 1)$

c)  $\vec{u} = (-4, 6, -10, 1) \quad \vec{v} = (2, 1, -2, 9)$

d)  $\vec{u} = (a, b) \quad \vec{v} = (-b, a)$

e)  $\vec{v}_1 = (-2, 1, 1), \quad \vec{v}_2 = (1, 0, 2), \quad \vec{v}_3 = (-2, -5, 1)$

$$f) \quad \vec{v}_1 = (1, 0, 1), \quad \vec{v}_2 = (1, 1, 1), \quad \vec{v}_3 = (-1, 0, 1)$$

$$g) \quad \vec{v}_1 = (2, -2, 1), \quad \vec{v}_2 = (2, 1, -2), \quad \vec{v}_3 = (1, 2, 2)$$

### **Solution**

$$a) \quad \vec{v}_1 \cdot \vec{v}_2 = 2(3) + 3(2)$$

$$= 12 \neq 0$$

$\therefore$  Vectors don't form an orthogonal set

$$b) \quad \vec{v}_1 \cdot \vec{v}_2 = 1(-2) - 2(1)$$

$$= -4 \neq 0$$

$\therefore$  Vectors don't form an orthogonal set

$$c) \quad \vec{u} \cdot \vec{v} = -8 + 6 + 20 + 9$$

$$= 27 \neq 0$$

$\therefore$  These vectors are not orthogonal

$$d) \quad \vec{u} \cdot \vec{v} = -ab + ab$$

$$= 0$$

$\therefore$  These vectors are orthogonal

$$e) \quad \vec{v}_1 \cdot \vec{v}_2 = -2(1) + 1(0) + 1(2)$$

$$= 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = -2(-2) + 1(-5) + 1(1)$$

$$= 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = 1(-2) + 0(-5) + 2(1)$$

$$= 0$$

$\therefore$  Vectors form an orthogonal set

$$f) \quad \vec{v}_1 \cdot \vec{v}_2 = 1(1) + 0(1) + 1(1)$$

$$= 2 \neq 0$$

$\therefore$  Vectors don't form an orthogonal set

$$g) \quad \vec{v}_1 \cdot \vec{v}_2 = 2(2) - 2(1) + 1(-2)$$

$$= 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = 2(1) - 2(2) + 1(2)$$

$$= 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = 2(1) + 1(2) - 2(2) \\ = 0$$

$\therefore$  Vectors form an orthogonal set

### Exercise

Find a unit vector that is orthogonal to both  $\vec{u} = (1, 0, 1)$  and  $\vec{v} = (0, 1, 1)$

### Solution

Let  $\vec{w} = (w_1, w_2, w_3)$  be the unit vector that is orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

$$\vec{u} \cdot \vec{w} = 1(w_1) + 0(w_2) + 1(w_3) \\ = w_1 + w_3 = 0 \\ w_3 = -w_1$$

$$\vec{v} \cdot \vec{w} = 0(w_1) + 1(w_2) + 1(w_3) \\ = w_2 + w_3 = 0 \\ w_3 = -w_2$$

$$w_1 = w_2 = -w_3$$

The orthogonal vector to both  $\vec{u}$  and  $\vec{v}$  is  $\vec{w} = (1, 1, -1)$ , therefore the unit vector is

$$\frac{\vec{w}}{\|\vec{w}\|} = \frac{1}{\sqrt{1^2 + 1^2 + (-1)^2}}(1, 1, -1) \\ = \frac{1}{\sqrt{3}}(1, 1, -1) \\ = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right)$$

The possible vectors are:  $\pm \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right)$

### Exercise

a) Show that  $\vec{v} = (a, b)$  and  $\vec{w} = (-b, a)$  are orthogonal vectors.

b) Use the result to find two vectors that are orthogonal to  $\vec{v} = (2, -3)$ .

c) Find two unit vectors that are orthogonal to  $(-3, 4)$

### Solution



$$\begin{aligned}
 a) \quad \vec{v} \cdot \vec{w} &= a(-b) + b(a) \\
 &= -ab + ab \\
 &= \underline{0}
 \end{aligned}$$

$\vec{v}$  and  $\vec{w}$  are orthogonal vectors.

$$b) \quad (2, 3) \text{ and } (-2, 3).$$

$$\begin{aligned}
 c) \quad \vec{u}_1 &= \frac{1}{\sqrt{4^2 + 3^2}}(4, 3) \\
 &= \underline{\left(\frac{4}{5}, \frac{3}{5}\right)}
 \end{aligned}$$

$$\begin{aligned}
 \vec{u}_2 &= -\frac{1}{\sqrt{4^2 + 3^2}}(4, 3) \\
 &= \underline{\left(-\frac{4}{5}, -\frac{3}{5}\right)}
 \end{aligned}$$

### Exercise

Find the vector component of  $\vec{u}$  along  $\vec{a}$  and the vector component of  $\vec{u}$  orthogonal to

$$a) \quad \vec{u} = (6, 2), \quad \vec{a} = (3, -9)$$

$$d) \quad \vec{u} = (1, 1, 1), \quad \vec{a} = (0, 2, -1)$$

$$b) \quad \vec{u} = (3, 1, -7), \quad \vec{a} = (1, 0, 5)$$

$$e) \quad \vec{u} = (2, 1, 1, 2), \quad \vec{a} = (4, -4, 2, -2)$$

$$c) \quad \vec{u} = (1, 0, 0), \quad \vec{a} = (4, 3, 8)$$

$$f) \quad \vec{u} = (5, 0, -3, 7), \quad \vec{a} = (2, 1, -1, -1)$$

### Solution

$$\begin{aligned}
 a) \quad \text{proj}_{\vec{a}} \vec{u} &= \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} \\
 &= \frac{6(3) + 2(-9)}{3^2 + (-9)^2} (3, -9) \\
 &= \frac{0}{90} (3, -9) \\
 &= \underline{(0, 0)}
 \end{aligned}$$

$$\begin{aligned}
 \vec{u} - \text{proj}_{\vec{a}} \vec{u} &= (6, 2) - (0, 0) \\
 &= \underline{(6, 2)}
 \end{aligned}$$

$$\begin{aligned}
 b) \quad \text{proj}_{\vec{a}} \vec{u} &= \frac{3(1) + 0 - 7(5)}{1^2 + 0 + 5^2} (1, 0, 5) \\
 &= \frac{-32}{26} (1, 0, 5)
 \end{aligned}$$

$$\text{proj}_{\vec{a}} \vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$$

$$= \left( -\frac{16}{13}, 0, -\frac{80}{13} \right) \Big|$$

$$\begin{aligned} \vec{u} - \text{proj}_{\vec{a}} \vec{u} &= (1, 0, 5) - \left( -\frac{16}{13}, 0, -\frac{80}{13} \right) \\ &= \left( \frac{55}{13}, 1, -\frac{11}{13} \right) \Big| \end{aligned}$$

$$\begin{aligned} c) \quad \text{proj}_{\vec{a}} \vec{u} &= \frac{1(4) + 0 + 0}{4^2 + 3^2 + 8^2} (4, 3, 8) \\ &= \frac{4}{89} (4, 3, 8) \\ &= \left( \frac{16}{89}, \frac{12}{89}, \frac{32}{89} \right) \Big| \end{aligned}$$

$$\text{proj}_{\vec{a}} \vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$$

$$\begin{aligned} \vec{u} - \text{proj}_{\vec{a}} \vec{u} &= (1, 0, 0) - \left( \frac{16}{89}, \frac{12}{89}, \frac{32}{89} \right) \\ &= \left( \frac{73}{89}, -\frac{12}{89}, -\frac{32}{89} \right) \Big| \end{aligned}$$

$$\begin{aligned} d) \quad \text{proj}_{\vec{a}} \vec{u} &= \frac{1(0) + 1(2) + 1(-1)}{0^2 + 2^2 + (-1)^2} (0, 2, -1) \\ &= \frac{1}{5} (0, 2, -1) \\ &= \left( 0, \frac{2}{5}, -\frac{1}{5} \right) \Big| \end{aligned}$$

$$\text{proj}_{\vec{a}} \vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$$

$$\begin{aligned} \vec{u} - \text{proj}_{\vec{a}} \vec{u} &= (1, 1, 1) - \left( 0, \frac{2}{5}, -\frac{1}{5} \right) \\ &= \left( 1, \frac{3}{5}, \frac{6}{5} \right) \Big| \end{aligned}$$

$$\begin{aligned} e) \quad \text{proj}_{\vec{a}} \vec{u} &= \frac{2(4) + 1(-4) + 1(2) + 2(-2)}{4^2 + (-4)^2 + 2^2 + (-2)^2} (4, -4, 2, -2) \\ &= \frac{2}{40} (4, -4, 2, -2) \\ &= \left( \frac{1}{5}, -\frac{1}{5}, \frac{1}{10}, -\frac{1}{10} \right) \Big| \end{aligned}$$

$$\text{proj}_{\vec{a}} \vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$$

$$\begin{aligned} \vec{u} - \text{proj}_{\vec{a}} \vec{u} &= (2, 1, 1, 2) - \left( \frac{1}{5}, -\frac{1}{5}, \frac{1}{10}, -\frac{1}{10} \right) \\ &= \left( \frac{9}{5}, \frac{6}{5}, \frac{9}{10}, \frac{21}{10} \right) \Big| \end{aligned}$$

$$\begin{aligned}
 f) \quad \text{proj}_{\vec{a}} \vec{u} &= \frac{5(2)+0(1)-3(-1)+7(-1)}{2^2+1^2+(-1)^2+(-1)^2} (2, 1, -1, -1) & \text{proj}_{\vec{a}} \vec{u} &= \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} \\
 &= \frac{6}{7} (2, 1, -1, -1) \\
 &= \left( \frac{12}{7}, \frac{6}{7}, -\frac{6}{7}, -\frac{6}{7} \right) \\
 \vec{u} - \text{proj}_{\vec{a}} \vec{u} &= (5, 0, -3, 7) - \left( \frac{12}{7}, \frac{6}{7}, -\frac{6}{7}, -\frac{6}{7} \right) \\
 &= \left( \frac{23}{7}, -\frac{6}{7}, -\frac{15}{7}, \frac{55}{7} \right)
 \end{aligned}$$

### Exercise

Project the vector  $\vec{v}$  onto the line through  $\vec{a}$ , check that  $\vec{e} = \vec{u} - \text{proj}_{\vec{a}} \vec{u}$  is perpendicular to  $\vec{a}$ :

$$a) \quad \vec{v} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \text{ and } \vec{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$b) \quad \vec{v} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \text{ and } \vec{a} = \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}$$

$$c) \quad \vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } \vec{a} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

### Solution

$$\begin{aligned}
 a) \quad \text{proj}_{\vec{a}} \vec{v} &= \frac{1(1)+2(1)+2(1)}{1^2+1^2+1^2} (1, 1, 1) & \text{proj}_{\vec{a}} \vec{v} &= \frac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} \\
 &= \frac{5}{3} (1, 1, 1) \\
 &= \left( \frac{5}{3}, \frac{5}{3}, \frac{5}{3} \right)
 \end{aligned}$$

$$\begin{aligned}
 \vec{e} &= \vec{v} - \text{proj}_{\vec{a}} \vec{v} \\
 &= (1, 2, 2) - \left( \frac{5}{3}, \frac{5}{3}, \frac{5}{3} \right) \\
 &= \left( -\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right)
 \end{aligned}$$

$$\vec{e} \cdot \vec{a} = \left( -\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right) \cdot (1, 1, 1)$$

$$= -\frac{2}{3} + \frac{1}{3} + \frac{1}{3}$$

$$\underline{=0}$$

$\vec{e}$  is perpendicular to  $\vec{a}$

$$b) \quad \text{proj}_{\vec{a}} \vec{v} = \frac{1(-1) + 3(-3) + 1(-1)}{(-1)^2 + (-3)^2 + (-1)^2} (-1, -3, -1)$$

$$\text{proj}_{\vec{a}} \vec{v} = \frac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$$

$$= \frac{-11}{11} (-1, -3, -1)$$

$$\underline{= (1, 3, 1)}$$

$$\vec{e} = \vec{v} - \text{proj}_{\vec{a}} \vec{v}$$

$$= (1, 3, 1) - (1, 3, 1)$$

$$\underline{= (0, 0, 0)}$$

$$\vec{e} \cdot \vec{a} = (0, 0, 0) \cdot (-1, -3, -1)$$

$$\underline{=0}$$

$\vec{e}$  is perpendicular to  $\vec{a}$

$$c) \quad \text{proj}_{\vec{a}} \vec{v} = \frac{1(1) + 1(2) + 1(2)}{(1)^2 + (2)^2 + (2)^2} (1, 2, 2)$$

$$\text{proj}_{\vec{a}} \vec{v} = \frac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$$

$$= \frac{5}{9} (1, 2, 2)$$

$$\underline{= \left( \frac{5}{9}, \frac{10}{9}, \frac{10}{9} \right)}$$

$$\vec{e} = \vec{v} - \text{proj}_{\vec{a}} \vec{v}$$

$$= (1, 1, 1) - \left( \frac{5}{9}, \frac{10}{9}, \frac{10}{9} \right)$$

$$\underline{= \left( \frac{4}{9}, -\frac{1}{9}, -\frac{1}{9} \right)}$$

$$\vec{e} \cdot \vec{a} = \left( \frac{4}{9}, -\frac{1}{9}, -\frac{1}{9} \right) \cdot (1, 2, 2)$$

$$= \frac{4}{9} - \frac{2}{9} - \frac{2}{9}$$

$$\underline{=0}$$

$\vec{e}$  is perpendicular to  $\vec{a}$

### Exercise

Find the projection matrix  $proj_{\vec{a}} \vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$  onto the line through  $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

### Solution

$$\vec{a}^T \vec{a} = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 9$$

$$\begin{aligned} P &= \frac{1}{\vec{a}^T \vec{a}} \vec{a} \vec{a}^T \\ &= \frac{1}{9} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \end{pmatrix} \\ &= \frac{1}{9} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix} \end{aligned}$$

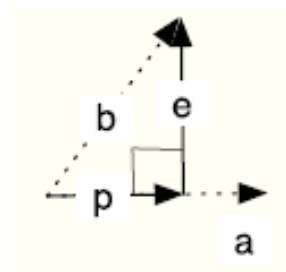
### Exercise

Draw the projection of  $\vec{b}$  onto  $\vec{a}$  and also compute it from  $proj_{\vec{a}} \vec{b} = \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$

$$\vec{b} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \quad \text{and} \quad \vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

### Solution

$$\begin{aligned} proj_{\vec{a}} \vec{b} &= \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} \\ &= \frac{\cos \theta(1) + \sin \theta(0)}{(1)^2 + 0} (1, 0) \\ &= \cos \theta (1, 0) \\ &= (\cos \theta, 0) \end{aligned}$$



$$\begin{aligned} \vec{e} &= \vec{b} - proj_{\vec{a}} \vec{b} \\ &= (\cos \theta, \sin \theta) - (\cos \theta, 0) \\ &= (0, \sin \theta) \end{aligned}$$

### Exercise

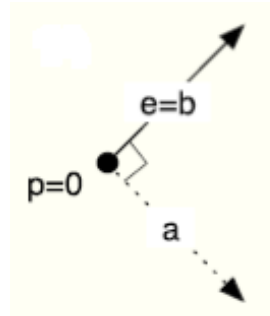
Draw the projection of  $\vec{b}$  onto  $\vec{a}$  and also compute it from  $proj_{\vec{a}} \vec{b} = \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

### Solution

$$\begin{aligned} proj_{\vec{a}} \vec{b} &= \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} \\ &= \frac{1(1) + 1(-1)}{1^2 + (-1)^2} (1, -1) \\ &= \frac{0}{2} (1, -1) \\ &= \underline{(0, 0)} \end{aligned}$$

$$\begin{aligned} \vec{e} &= \vec{b} - proj_{\vec{a}} \vec{b} \\ &= (1, 1) - (0, 0) \\ &= \underline{(1, 1)} \end{aligned}$$



### Exercise

Show that if  $\vec{v}$  is orthogonal to both  $\vec{w}_1$  and  $\vec{w}_2$ , then  $\vec{v}$  is orthogonal to  $k_1 \vec{w}_1 + k_2 \vec{w}_2$  for all scalars  $k_1$  and  $k_2$ .

### Solution

$$\begin{aligned} \vec{v} \cdot (k_1 \vec{w}_1 + k_2 \vec{w}_2) &= \vec{v} \cdot (k_1 \vec{w}_1) + \vec{v} \cdot (k_2 \vec{w}_2) \\ &= k_1 (\vec{v} \cdot \vec{w}_1) + k_2 (\vec{v} \cdot \vec{w}_2) \\ &= k_1 (0) + k_2 (0) \\ &= \underline{0} \end{aligned}$$

*If  $\vec{v}$  is orthogonal to  $\vec{w}_1$  &  $\vec{w}_2$*

$$\rightarrow \vec{v} \cdot \vec{w}_1 = \vec{v} \cdot \vec{w}_2 = 0$$

### Exercise

- a) Project the vector  $\vec{v} = (3, 4, 4)$  onto the line through  $\vec{a} = (2, 2, 1)$  and then onto the plane that also contains  $\vec{a}^* = (1, 0, 0)$ .
- b) Check that the first error vector  $\vec{v} - \vec{p}$  is perpendicular to  $\vec{a}$ , and the second error vector  $\vec{v} - \vec{p}^*$  is also perpendicular to  $\vec{a}^*$ .

### Solution

$$\begin{aligned} \text{a) } \text{proj}_{\vec{a}} \vec{v} &= \frac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} \\ &= \frac{3(2) + 4(2) + 4(1)}{(2)^2 + (2)^2 + (1)^2} (2, 2, 1) \\ &= \frac{18}{9} (2, 2, 1) \\ &= (4, 4, 2) \end{aligned}$$

The plane contains the vectors  $\vec{a}$  and  $\vec{a}^*$  is the column space of  $A$ .

$$\begin{aligned} A &= \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} \\ A^T A &= \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix} \\ (A^T A)^{-1} &= \begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix}^{-1} \\ &= \frac{1}{5} \begin{bmatrix} 1 & -2 \\ -2 & 9 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} P &= A (A^T A)^{-1} A^T \\ &= \frac{1}{5} \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 9 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & .8 & .4 \\ 0 & .4 & .2 \end{bmatrix} \end{aligned}$$

**b)** The error vector:

$$\vec{e} = \vec{v} - \vec{p}$$

$$= \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$$\begin{aligned} \vec{a} \vec{e} &= \begin{pmatrix} 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \\ &= 2(-1) + 2(0) + 1(2) \\ &= 0 \end{aligned}$$

Therefore,  $\vec{e}$  is perpendicular to  $\vec{a}$

$$p^* = P\vec{v}$$

$$\begin{aligned} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & .8 & .4 \\ 0 & .4 & .2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 4.8 \\ 2.4 \end{pmatrix} \end{aligned}$$

The error vector:

$$\vec{e}^* = \vec{v} - \vec{p}^*$$

$$= \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 4.8 \\ 2.4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -.8 \\ 1.6 \end{pmatrix}$$

$$\begin{aligned} \vec{a}^* \vec{e}^* &= \begin{pmatrix} 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -.8 \\ 1.6 \end{pmatrix} \\ &= 2(0) + 2(-.8) + 1(1.6) \\ &= 0 \end{aligned}$$

Therefore,  $\vec{e}^*$  is perpendicular to  $\vec{a}^*$



### Exercise

Compute the projection matrices  $\vec{a}\vec{a}^T / \vec{a}^T\vec{a}$  onto the lines through  $\vec{a}_1 = (-1, 2, 2)$  and  $\vec{a}_2 = (2, 2, -1)$ . Multiply those projection matrices and explain why their product  $P_1 P_2$  is what it is. Project  $\vec{v} = (1, 0, 0)$  onto the lines  $\vec{a}_1$ ,  $\vec{a}_2$ , and also onto  $\vec{a}_3 = (2, -1, 2)$ . Add up the three projections  $P_1 + P_2 + P_3$ .

### Solution

For  $\vec{a}_1 = (-1, 2, 2)$

$$\begin{aligned}\vec{a}_1 \vec{a}_1^T &= \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} -1 & 2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\vec{a}_1^T \vec{a}_1 &= \begin{pmatrix} -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \\ &= 9 \end{aligned}$$

$$\begin{aligned}P_1 &= \frac{\vec{a}_1 \vec{a}_1^T}{\vec{a}_1^T \vec{a}_1} \\ &= \frac{1}{9} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}\end{aligned}$$

For  $\vec{a}_2 = (2, 2, -1)$

$$\begin{aligned}\vec{a}_2 \vec{a}_2^T &= \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 2 & 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\vec{a}_2^T \vec{a}_2 &= \begin{pmatrix} 2 & 2 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \\ &= 9 \end{aligned}$$

$$\begin{aligned}
P_2 &= \frac{\vec{a} \vec{a}^T}{\vec{a}^T \vec{a}} \\
&= \frac{1}{9} \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix} \\
P_1 P_2 &= \frac{1}{9} \left( \frac{1}{9} \right) \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix} \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix} \\
&= \frac{1}{81} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \underline{0}
\end{aligned}$$

This because  $\vec{a}_1$  and  $\vec{a}_2$  are perpendicular.

For  $\vec{a}_3 = (2, -1, 2)$

$$\begin{aligned}
\vec{a}_3 \vec{a}_3^T &= \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
a_3^T a_3 &= \begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \\
&= \underline{9}
\end{aligned}$$

$$\begin{aligned}
P_3 &= \frac{\vec{a}_3 \vec{a}_3^T}{\vec{a}_3^T \vec{a}_3} \\
&= \frac{1}{9} \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
p_3 &= P_3 \vec{v} \\
&= \frac{1}{9} \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

$$= \frac{1}{9} \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ \frac{4}{9} \end{pmatrix}$$

$$p_1 = P_1 \vec{v}$$

$$= \frac{1}{9} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{9} \\ -\frac{2}{9} \\ -\frac{2}{9} \end{pmatrix}$$

$$p_2 = P_2 \vec{v}$$

$$= \frac{1}{9} \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{2}{9} \end{pmatrix}$$

$$p_1 + p_2 + p_3 = \begin{pmatrix} \frac{1}{9} \\ -\frac{2}{9} \\ -\frac{2}{9} \end{pmatrix} + \begin{pmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{2}{9} \end{pmatrix} + \begin{pmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ \frac{4}{9} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \vec{v} \quad |$$

The reason is that  $\vec{a}_3$  is perpendicular to  $\vec{a}_1$  and  $\vec{a}_2$ .

Hence, when you compute the three projections of a vector and add them up you get back to the vector you start with.

### Exercise

If  $P^2 = P$  show that  $(I - P)^2 = I - P$ . When  $P$  projects onto the column space of  $A$ ,  $I - P$  projects onto the \_\_\_\_.

### Solution

$$\begin{aligned} (I - P)^2 \vec{v} &= (I - P)(I - P)\vec{v} \\ &= (I - P)(I\vec{v} - P\vec{v}) \\ &= I^2\vec{v} - IP\vec{v} - PI\vec{v} + P^2\vec{v} \\ &= \vec{v} - P\vec{v} - P\vec{v} + P^2\vec{v} & P^2\vec{v} = P\vec{v} \quad \text{By given definition} \\ &= \vec{v} - P\vec{v} - P\vec{v} + P\vec{v} \\ &= \vec{v} - P\vec{v} \end{aligned}$$

$$(I - P)^2 \vec{v} = (I - P)\vec{v}$$

$$(I - P)^2 = (I - P) \quad |$$

When  $P$  projects onto the column space of  $A$ , then  $I - P$  projects onto the left nullspace.

Because  $(I - P)^2 \vec{v} = (I - P)\vec{v}$ ; if  $P\vec{v}$  is in the column space of  $A$ , then  $\vec{v} - P\vec{v}$  is a vector perpendicular to  $C(A)$ .

### Exercise

What linear combination of  $(1, 2, -1)$  and  $(1, 0, 1)$  is closest to  $\vec{v} = (2, 1, 1)$ ?

### Solution

$$\frac{1}{2}(1, 2, -1) + \frac{3}{2}(1, 0, 1) = (2, 1, 1)$$

So, this  $\vec{v}$  is actually in the span of the two given vectors.

### Exercise

Show that  $\vec{u} - \vec{v}$  is orthogonal to  $\vec{u} + \vec{v}$  if and only if  $\|\vec{u}\| = \|\vec{v}\|$

### Solution

Suppose that  $\vec{u} - \vec{v}$  is orthogonal to  $\vec{u} + \vec{v}$ . Then

$$\begin{aligned} 0 &= \langle \vec{u} - \vec{v}, \vec{u} + \vec{v} \rangle \\ &= (\vec{u} - \vec{v})^T (\vec{u} + \vec{v}) \\ &= (\vec{u}^T - \vec{v}^T)(\vec{u} + \vec{v}) \\ &= \vec{u}^T \vec{u} + \vec{u}^T \vec{v} - \vec{v}^T \vec{u} - \vec{v}^T \vec{v} \\ &= \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle - \langle \vec{v}, \vec{u} \rangle - \langle \vec{v}, \vec{v} \rangle & \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \\ &= \langle \vec{u}, \vec{u} \rangle - \langle \vec{v}, \vec{v} \rangle \end{aligned}$$

So  $\langle \vec{u}, \vec{u} \rangle = \langle \vec{v}, \vec{v} \rangle$ .

Therefore,  $\|\vec{u}\|^2 = \|\vec{v}\|^2 \Rightarrow \|\vec{u}\| = \|\vec{v}\|$ .

Suppose  $\|\vec{u}\| = \|\vec{v}\|$ . Then

$$\begin{aligned} \langle \vec{u} - \vec{v}, \vec{u} + \vec{v} \rangle &= (\vec{u} - \vec{v})^T (\vec{u} + \vec{v}) \\ &= (\vec{u}^T - \vec{v}^T)(\vec{u} + \vec{v}) \\ &= \vec{u}^T \vec{u} + \vec{u}^T \vec{v} - \vec{v}^T \vec{u} - \vec{v}^T \vec{v} \\ &= \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle - \langle \vec{v}, \vec{u} \rangle - \langle \vec{v}, \vec{v} \rangle & \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \\ &= \langle \vec{u}, \vec{u} \rangle - \langle \vec{v}, \vec{v} \rangle \\ &= \|\vec{u}\|^2 - \|\vec{v}\|^2 \\ &= 0 \end{aligned}$$

So, we can see that  $\vec{u} - \vec{v}$  is orthogonal to  $\vec{u} + \vec{v}$

We conclude that  $\vec{u} - \vec{v}$  is orthogonal to  $\vec{u} + \vec{v}$  if and only if  $\|\vec{u}\| = \|\vec{v}\|$ , as desired.

### Exercise

Given  $\vec{u} = (3, -1, 2)$   $\vec{v} = (4, -1, 5)$  and  $\vec{w} = (8, -7, -6)$

- Find  $3\vec{v} - 4(5\vec{u} - 6\vec{w})$
- Find  $\vec{u} \cdot \vec{v}$  and then the angle  $\theta$  between  $\vec{u}$  and  $\vec{v}$ .

### Solution

$$a) \quad 3\vec{v} - 4(5\vec{u} - 6\vec{w}) = 3(4, -1, 5) - 4(5(3, -1, 2) - 6(8, -7, -6))$$

$$\begin{aligned}
&= (12, -3, 15) - 4((15, -5, 10) - (48, -42, -36)) \\
&= (12, -3, 15) - 4(-33, 37, 46) \\
&= (12, -3, 15) - (-132, 148, 184) \\
&= \underline{(144, -151, -169)}
\end{aligned}$$

$$\begin{aligned}
b) \quad \vec{u} \cdot \vec{v} &= (3, -1, 2) \cdot (1, 1, -1) \\
&= 3 - 1 - 2 \\
&= \underline{0} \\
\theta &= \underline{90^\circ}
\end{aligned}$$

### Exercise

Given:  $\vec{u} = (3, 1, 3)$   $\vec{v} = (4, 1, -2)$

- Compute the projection  $\vec{w}$  of  $\vec{u}$  on  $\vec{v}$
- Find  $\vec{p} = \vec{u} - \vec{v}$  and show that  $\vec{p}$  is perpendicular to  $\vec{v}$ .

### Solution

$$\begin{aligned}
a) \quad \vec{w} &= \text{proj}_{\vec{v}} \vec{u} \\
&= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} \\
&= \frac{(3, 1, 3) \cdot (4, 1, -2)}{4^2 + 1^2 + (-2)^2} (4, 1, -2) \\
&= \frac{12 + 1 - 6}{21} (4, 1, -2) \\
&= \frac{7}{21} (4, 1, -2) \\
&= \frac{1}{3} (4, 1, -2) \\
&= \underline{\left( \frac{4}{3}, \frac{1}{3}, -\frac{2}{3} \right)}
\end{aligned}$$

$$\begin{aligned}
b) \quad \vec{p} &= (3, 1, 3) - \left( \frac{4}{3}, \frac{1}{3}, -\frac{2}{3} \right) \\
&= \left( \frac{5}{3}, \frac{2}{3}, \frac{11}{3} \right) \\
\vec{p} \cdot \vec{u} &= \left( \frac{5}{3}, \frac{2}{3}, \frac{11}{3} \right) \cdot (4, 1, -2) \\
&= \frac{20}{3} + \frac{2}{3} - \frac{22}{3} \\
&= \underline{0}
\end{aligned}$$

$\vec{p}$  is perpendicular to  $\vec{v}$ .

### Exercise

- a) Show that  $\vec{v} = (a, b)$  and  $\vec{w} = (-b, a)$  are orthogonal vectors
- b) Use the result in part (a) to find two vectors that are orthogonal to  $\vec{v} = (2, -3)$
- c) Find two unit vectors that are orthogonal to  $(-3, 4)$

### Solution

$$\begin{aligned} \text{a) } \vec{u} \cdot \vec{v} &= -ab + ba \\ &= 0 \end{aligned}$$

The 2 vectors are orthogonal vectors.

$$\text{b) } \vec{v} = (2, -3)$$

$$\vec{w} = (-3, -2) \text{ and } \vec{w} = (3, 2)$$

$$\text{c) } (-3, 4)$$

$$\begin{aligned} \vec{u} &= \frac{(-3, 4)}{\sqrt{9+16}} \\ &= \left(-\frac{3}{5}, \frac{4}{5}\right) \end{aligned}$$

$$\vec{u}_1 = \left(\frac{4}{5}, \frac{3}{5}\right) \text{ and } \vec{u}_2 = \left(-\frac{4}{5}, -\frac{3}{5}\right)$$

### Exercise

Show that  $A(3, 0, 2)$ ,  $B(4, 3, 0)$ , and  $C(8, 1, -1)$  are vertices of a right triangle. At which vertex is the right angle?

### Solution

$$AB = (4-3, 3-0, 0-2) = (1, 3, -2)$$

$$AC = (5, 1, -3)$$

$$BC = (4, -2, -1)$$

$$AB \bullet AC = 5 + 3 + 6 = 14$$

$$AB \bullet BC = 4 - 6 + 2 = 0$$

$$AC \bullet BC = 20 - 2 + 3 = 21$$

The right triangle at point  $B$

### Exercise

Establish the identity:  $\vec{u} \cdot \vec{v} = \frac{1}{4} \|\vec{u} + \vec{v}\|^2 - \frac{1}{4} \|\vec{u} - \vec{v}\|^2$

### Solution

Let  $\vec{u} = (u_1, u_2, \dots, u_n)$  and  $\vec{v} = (v_1, v_2, \dots, v_n)$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

$$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= (u_1 + v_1)^2 + (u_2 + v_2)^2 + \dots + (u_n + v_n)^2 \\ &= u_1^2 + v_1^2 + 2u_1 v_1 + u_2^2 + v_2^2 + 2u_2 v_2 + \dots + u_n^2 + v_n^2 + 2u_n v_n \end{aligned}$$

$$\vec{u} - \vec{v} = (u_1 - v_1, u_2 - v_2, \dots, u_n - v_n)$$

$$\begin{aligned} \|\vec{u} - \vec{v}\|^2 &= (u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots + (u_n - v_n)^2 \\ &= u_1^2 + v_1^2 - 2u_1 v_1 + u_2^2 + v_2^2 - 2u_2 v_2 + \dots + u_n^2 + v_n^2 - 2u_n v_n \end{aligned}$$

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 &= u_1^2 + v_1^2 + 2u_1 v_1 + u_2^2 + v_2^2 + 2u_2 v_2 + \dots + u_n^2 + v_n^2 + 2u_n v_n \\ &\quad - (u_1^2 + v_1^2 - 2u_1 v_1 + u_2^2 + v_2^2 - 2u_2 v_2 + \dots + u_n^2 + v_n^2 - 2u_n v_n) \\ &= u_1^2 + v_1^2 + 2u_1 v_1 + u_2^2 + v_2^2 + 2u_2 v_2 + \dots + u_n^2 + v_n^2 + 2u_n v_n \\ &\quad - u_1^2 - v_1^2 + 2u_1 v_1 - u_2^2 - v_2^2 + 2u_2 v_2 - \dots - u_n^2 - v_n^2 + 2u_n v_n \\ &= 4u_1 v_1 + 4u_2 v_2 + \dots + 4u_n v_n \end{aligned}$$

$$\frac{1}{4} (\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2) = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

Therefore;  $\vec{u} \cdot \vec{v} = \frac{1}{4} \|\vec{u} + \vec{v}\|^2 - \frac{1}{4} \|\vec{u} - \vec{v}\|^2$  is true.

### **2<sup>nd</sup> method:**

$$\begin{aligned} \frac{1}{4} \|\vec{u} + \vec{v}\|^2 - \frac{1}{4} \|\vec{u} - \vec{v}\|^2 &= \frac{1}{4} [(\vec{u} + \vec{v})(\vec{u} + \vec{v}) - (\vec{u} - \vec{v})(\vec{u} - \vec{v})] \\ &= \frac{1}{4} [\vec{u}\vec{u} + 2\vec{u}\vec{v} + \vec{v}\vec{v} - (\vec{u}\vec{u} - 2\vec{u}\vec{v} + \vec{v}\vec{v})] \\ &= \frac{1}{4} [\vec{u}\vec{u} + 2\vec{u}\vec{v} + \vec{v}\vec{v} - \vec{u}\vec{u} + 2\vec{u}\vec{v} - \vec{v}\vec{v}] \end{aligned}$$



$$\begin{aligned}
&= \frac{1}{4}(4\vec{u}\vec{v}) \\
&= \vec{u} \cdot \vec{v}
\end{aligned}$$

### Exercise

Find the Euclidean inner product  $\vec{u} \cdot \vec{v}$  :  $\vec{u} = (-1, 1, 0, 4, -3)$   $\vec{v} = (-2, -2, 0, 2, -1)$

### Solution

$$\begin{aligned}
\vec{u} \cdot \vec{v} &= 2 - 2 + 0 + 8 + 3 \\
&= 11
\end{aligned}$$

### Exercise

Find the Euclidean distance between  $\vec{u}$  and  $\vec{v}$  :  $\vec{u} = (3, -3, -2, 0, -3)$   $\vec{v} = (-4, 1, -1, 5, 0)$

### Solution

$$\begin{aligned}
d(\vec{u}, \vec{v}) &= \|\vec{u} - \vec{v}\| \\
&= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots + (u_n - v_n)^2} \\
&= \sqrt{(3 + 4)^2 + (-3 - 1)^2 + (-2 + 1)^2 + (0 - 5)^2 + (-3 - 0)^2} \\
&= \sqrt{49 + 16 + 1 + 25 + 9} \\
&= \sqrt{100} \\
&= 10
\end{aligned}$$

### Exercise

Find for  $\vec{v} = 2\hat{i} - 4\hat{j} + \sqrt{5}\hat{k}$ ,  $\vec{u} = -2\hat{i} + 4\hat{j} - \sqrt{5}\hat{k}$

- $\vec{v} \cdot \vec{u}$ ,  $|\vec{v}|$ ,  $|\vec{u}|$
- The cosine of the angle between  $\vec{v}$  and  $\vec{u}$
- The scalar component of  $\vec{u}$  in the direction of  $\vec{v}$
- The vector  $\text{proj}_{\vec{v}} \vec{u}$

### Solution

$$\begin{aligned}
a) \quad \vec{v} \cdot \vec{u} &= (2\hat{i} - 4\hat{j} + \sqrt{5}\hat{k}) \cdot (-2\hat{i} + 4\hat{j} - \sqrt{5}\hat{k}) \\
&= -4 - 16 - 5 \\
&= -25
\end{aligned}$$

$$|\vec{v}| = \sqrt{2^2 + (-4)^2 + (\sqrt{5})^2}$$

$$\begin{aligned}
 &= \sqrt{4+16+5} \\
 &= \sqrt{25} \\
 &= 5
 \end{aligned}$$

$$\begin{aligned}
 |\vec{u}| &= \sqrt{(-2)^2 + 4^2 + (-\sqrt{5})^2} \\
 &= \sqrt{25} \\
 &= 5
 \end{aligned}$$

$$\begin{aligned}
 b) \quad \cos \theta &= \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} \\
 &= \frac{-25}{(5)(5)} \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 c) \quad |\vec{u}| \cos \theta &= (5)(-1) \\
 &= -5
 \end{aligned}$$

$$\begin{aligned}
 d) \quad \text{proj}_{\vec{v}} \vec{u} &= \left( \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v} \\
 &= \left( \frac{-25}{5^2} \right) (2\hat{i} - 4\hat{j} + \sqrt{5}\hat{k}) \\
 &= -(2\hat{i} - 4\hat{j} + \sqrt{5}\hat{k}) \\
 &= -2\hat{i} + 4\hat{j} - \sqrt{5}\hat{k}
 \end{aligned}$$

### Exercise

Find for  $\vec{v} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{k}$ ,  $\vec{u} = 5\hat{i} + 12\hat{j}$

- $\vec{v} \cdot \vec{u}$ ,  $|\vec{v}|$ ,  $|\vec{u}|$
- The cosine of the angle between  $\vec{v}$  and  $\vec{u}$
- The scalar component of  $\vec{u}$  in the direction of  $\vec{v}$
- The vector  $\text{proj}_{\vec{v}} \vec{u}$

### Solution

$$\begin{aligned}
 a) \quad \vec{v} \cdot \vec{u} &= \left( \frac{3}{5}\hat{i} + \frac{4}{5}\hat{k} \right) \cdot (5\hat{i} + 12\hat{j}) \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 |\vec{v}| &= \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} \\
 &= \sqrt{\frac{9}{25} + \frac{16}{25}} \\
 &= \sqrt{\frac{25}{25}} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 |\vec{u}| &= \sqrt{5^2 + 12^2} \\
 &= 13
 \end{aligned}$$

$$\begin{aligned}
 b) \quad \cos \theta &= \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} \\
 &= \frac{3}{(1)(13)} \\
 &= \frac{3}{13}
 \end{aligned}$$

$$\begin{aligned}
 c) \quad |\vec{u}| \cos \theta &= (13) \left( \frac{3}{13} \right) \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 d) \quad \text{proj}_{\vec{v}} \vec{u} &= \left( \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v} \\
 &= \left( \frac{3}{1^2} \right) \left( \frac{3}{5} \hat{i} + \frac{4}{5} \hat{k} \right) \\
 &= \frac{9}{5} \hat{i} + \frac{12}{5} \hat{k}
 \end{aligned}$$

### Exercise

Find for  $\vec{v} = 2\hat{i} + 10\hat{j} - 11\hat{k}$ ,  $\vec{u} = 2\hat{i} + 2\hat{j} + \hat{k}$

- $\vec{v} \cdot \vec{u}$ ,  $|\vec{v}|$ ,  $|\vec{u}|$
- The cosine of the angle between  $\vec{v}$  and  $\vec{u}$
- The scalar component of  $\vec{u}$  in the direction of  $\vec{v}$
- The vector  $\text{proj}_{\vec{v}} \vec{u}$

### Solution

$$\begin{aligned}
 a) \quad \vec{v} \cdot \vec{u} &= (2\hat{i} + 10\hat{j} - 11\hat{k}) \cdot (2\hat{i} + 2\hat{j} + \hat{k}) \\
 &= 4 + 20 - 11
 \end{aligned}$$

$$\underline{=13}$$

$$\begin{aligned} |\vec{v}| &= \sqrt{2^2 + 10^2 + (-11)^2} \\ &= \sqrt{4 + 100 + 121} \\ &= \sqrt{225} \\ &\underline{=15} \end{aligned}$$

$$\begin{aligned} |\vec{u}| &= \sqrt{2^2 + 2^2 + 1^2} \\ &\underline{=3} \end{aligned}$$

$$\begin{aligned} b) \quad \cos \theta &= \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} \\ &= \frac{13}{(3)(15)} \\ &\underline{= \frac{13}{45}} \end{aligned}$$

$$\begin{aligned} c) \quad |\vec{u}| \cos \theta &= (3) \left( \frac{13}{45} \right) \\ &\underline{= \frac{13}{15}} \end{aligned}$$

$$\begin{aligned} d) \quad \text{proj}_{\vec{v}} \vec{u} &= \left( \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v} \\ &= \left( \frac{13}{15^2} \right) (2\hat{i} + 10\hat{j} - 11\hat{k}) \hat{j} \\ &\underline{= \frac{13}{225} (2\hat{i} + 10\hat{j} - 11\hat{k})} \end{aligned}$$

### Exercise

Find for  $\vec{v} = 5\hat{i} + \hat{j}$ ,  $\vec{u} = 2\hat{i} + \sqrt{17}\hat{j}$

$$a) \quad \vec{v} \cdot \vec{u}, \quad |\vec{v}|, \quad |\vec{u}|$$

b) The cosine of the angle between  $\vec{v}$  and  $\vec{u}$

c) The scalar component of  $\vec{u}$  in the direction of  $\vec{v}$

d) The vector  $\text{proj}_{\vec{v}} \vec{u}$

### Solution

$$\begin{aligned} a) \quad \vec{v} \cdot \vec{u} &= (5\hat{i} + \hat{j}) \cdot (2\hat{i} + \sqrt{17}\hat{j}) \\ &\underline{= 10 + \sqrt{17}} \end{aligned}$$

$$|\vec{v}| = \sqrt{25+1}$$

$$= \sqrt{26}$$

$$|\vec{u}| = \sqrt{4+17}$$

$$= \sqrt{21}$$

$$b) \quad \cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}$$

$$= \frac{10 + \sqrt{17}}{\sqrt{21}\sqrt{26}}$$

$$= \frac{10 + \sqrt{17}}{\sqrt{546}}$$

$$c) \quad |\vec{u}| \cos \theta = (\sqrt{21}) \left( \frac{10 + \sqrt{17}}{\sqrt{546}} \right)$$

$$= \frac{10 + \sqrt{17}}{\sqrt{26}}$$

$$d) \quad \text{proj}_{\vec{v}} \vec{u} = \left( \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v}$$

$$= \left( \frac{10 + \sqrt{17}}{26} \right) (5\hat{i} + \hat{j})$$

### Exercise

Find for  $\vec{v} = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right)$ ,  $\vec{u} = \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}} \right)$

- $\vec{v} \cdot \vec{u}$ ,  $|\vec{v}|$ ,  $|\vec{u}|$
- The cosine of the angle between  $\vec{v}$  and  $\vec{u}$
- The scalar component of  $\vec{u}$  in the direction of  $\vec{v}$
- The vector  $\text{proj}_{\vec{v}} \vec{u}$

### Solution

$$a) \quad \vec{v} \cdot \vec{u} = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right) \cdot \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}} \right)$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}$$

$$\begin{aligned}
 |\vec{v}| &= \sqrt{\frac{1}{2} + \frac{1}{3}} \\
 &= \frac{\sqrt{5}}{\sqrt{6}} \frac{\sqrt{6}}{\sqrt{6}} \\
 &= \frac{\sqrt{30}}{6}
 \end{aligned}$$

$$\begin{aligned}
 |\vec{u}| &= \sqrt{\frac{1}{2} + \frac{1}{3}} \\
 &= \frac{\sqrt{5}}{\sqrt{6}} \frac{\sqrt{6}}{\sqrt{6}} \\
 &= \frac{\sqrt{30}}{6}
 \end{aligned}$$

$$\begin{aligned}
 b) \quad \cos \theta &= \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} \\
 &= \frac{\frac{1}{6}}{\frac{\sqrt{30}}{6} \frac{\sqrt{30}}{6}} \\
 &= \frac{1}{6} \left( \frac{36}{30} \right) \\
 &= \frac{1}{5}
 \end{aligned}$$

$$\begin{aligned}
 c) \quad |\vec{u}| \cos \theta &= \left( \frac{\sqrt{30}}{6} \right) \left( \frac{1}{5} \right) \\
 &= \frac{\sqrt{30}}{30} \\
 &= \frac{1}{\sqrt{30}}
 \end{aligned}$$

$$\begin{aligned}
 d) \quad \text{proj}_{\vec{v}} \vec{u} &= \left( \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v} \\
 &= \frac{1}{6} \left( \frac{36}{30} \right) \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right) \\
 &= \frac{1}{5} \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right)
 \end{aligned}$$

### Exercise

Suppose Ted weighs 180 *lb.* and he is sitting on an inclined plane that drops 3 *units* for every 4 horizontal units. The gravitational force vector is  $\vec{F}_g = \begin{pmatrix} 0 \\ -180 \end{pmatrix}$ .

- a) Find the force pushing Ted down the slope.
- b) Find the force acting to hold Ted against the slope

### Solution

A vector parallel to the slope of the inclined plane is  $\vec{v} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ .

- a) The vector of the force acting to push Ted down the slope is

$$\begin{aligned}\vec{F}_s &= \frac{\vec{v} \cdot \vec{F}_g}{|\vec{v}|^2} \vec{v} \\ &= \frac{(4, -3) \cdot (0, -180)}{16+9} (4, -3) \\ &= \frac{540}{25} (4, -3) \\ &= \left( \frac{432}{5}, -\frac{324}{5} \right)\end{aligned}$$

The magnitude of the force pushing Ted down the slope is

$$\begin{aligned}\|\vec{F}_s\| &= \sqrt{\left(\frac{432}{5}\right)^2 + \left(\frac{324}{5}\right)^2} \\ &= \frac{540}{5} \\ &= \underline{108 \text{ lb}}\end{aligned}$$

- b) The vector of the force acting to hold Ted against the slope is

$$\begin{aligned}\vec{F}_p &= \vec{F}_g - \vec{F}_s \\ &= \begin{pmatrix} 0 \\ -180 \end{pmatrix} - \begin{pmatrix} \frac{432}{5} \\ -\frac{324}{5} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{432}{5} \\ -\frac{576}{5} \end{pmatrix} \\ \|\vec{F}_p\| &= \sqrt{\left(\frac{432}{5}\right)^2 + \left(\frac{576}{5}\right)^2} \\ &= \frac{720}{5} \\ &= \underline{144 \text{ lb}}\end{aligned}$$

### ***Exercise***

Prove that if two vectors  $\vec{u}$  and  $\vec{v}$  in  $\mathbb{R}^2$  are orthogonal to nonzero vector  $\vec{w}$  in  $\mathbb{R}^2$ , then  $\vec{u}$  and  $\vec{v}$  are scalar multiples of each other.

### **Solution**

Since  $\vec{u}$  is orthogonal to  $\vec{w} \rightarrow \vec{u} \cdot \vec{w} = 0$

$\vec{v}$  is orthogonal to  $\vec{w} \rightarrow \vec{v} \cdot \vec{w} = 0$

$$\Rightarrow \vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w} = 0$$

There exist  $a \in \mathbb{R}$  such that  $(a\vec{v}) \cdot \vec{w} = a(\vec{v} \cdot \vec{w}) = 0$

$$\vec{u} = a\vec{v}$$

$$\vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w} = 0 = (a\vec{v}) \cdot \vec{w}$$

Therefore,  $\vec{u}$  and  $\vec{v}$  are scalar multiples of each other



## ***Solution***      **Section 2.4 – Cross Product**

### ***Exercise***

Prove when the cross product  $\vec{u} \times \vec{v}$  is perpendicular to  $\vec{u}$ , then  $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$

### **Solution**

Let  $\vec{u} = (u_1, u_2, u_3)$  and  $\vec{v} = (v_1, v_2, v_3)$

$$\begin{aligned}\vec{u} \cdot (\vec{u} \times \vec{v}) &= (u_1, u_2, u_3) \cdot (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1) \\ &= u_1(u_2 v_3 - u_3 v_2) + u_2(u_3 v_1 - u_1 v_3) + u_3(u_1 v_2 - u_2 v_1) \\ &= u_1 u_2 v_3 - u_1 u_3 v_2 + u_2 u_3 v_1 - u_2 u_1 v_3 + u_3 u_1 v_2 - u_3 u_2 v_1 \\ &= \underline{0}\end{aligned}$$

### ***Exercise***

Find  $\vec{u} \times \vec{v}$ , where  $\vec{u} = (1, 2, -2)$  and  $\vec{v} = (3, 0, 1)$  and show that  $\vec{u} \times \vec{v}$  is perpendicular to  $\vec{u}$  and to  $\vec{v}$ .

### **Solution**

$$\begin{aligned}\vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix} \\ &= \left( \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \right) \\ &= \underline{(2, -7, -6)}\end{aligned}$$

$$\begin{aligned}\vec{u} \cdot (\vec{u} \times \vec{v}) &= (1, 2, -2) \cdot (2, -7, -6) \\ &= 2 - 14 + 12 \\ &= \underline{0}\end{aligned}$$

$$\begin{aligned}\vec{v} \cdot (\vec{u} \times \vec{v}) &= (3, 0, 1) \cdot (2, -7, -6) \\ &= 6 - 0 - 6 \\ &= \underline{0}\end{aligned}$$

$\vec{u} \times \vec{v}$  is orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

### Exercise

Given  $\vec{u} = (3, 2, -1)$ ,  $\vec{v} = (0, 2, -3)$ , and  $\vec{w} = (2, 6, 7)$  Compute the vectors

a)  $\vec{u} \times \vec{v}$

b)  $\vec{v} \times \vec{w}$

c)  $\vec{u} \times (\vec{v} \times \vec{w})$

d)  $(\vec{u} \times \vec{v}) \times \vec{w}$

e)  $\vec{u} \times (\vec{v} - 2\vec{w})$

### Solution

$$\begin{aligned} \text{a) } \vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ 0 & 2 & -3 \end{vmatrix} \\ &= \left( \begin{vmatrix} 2 & -1 \\ 2 & -3 \end{vmatrix}, -\begin{vmatrix} 3 & -1 \\ 0 & -3 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} \right) \\ &= \underline{(-4, 9, 6)} \end{aligned}$$

$$\begin{aligned} \text{b) } \vec{v} \times \vec{w} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & -3 \\ 2 & 6 & 7 \end{vmatrix} \\ &= \left( \begin{vmatrix} 2 & -3 \\ 6 & 7 \end{vmatrix}, -\begin{vmatrix} 0 & -3 \\ 2 & 7 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 2 & 6 \end{vmatrix} \right) \\ &= \underline{(32, -6, -4)} \end{aligned}$$

$$\begin{aligned} \text{c) } \vec{u} \times (\vec{v} \times \vec{w}) &= (3, 2, -1) \times (32, -6, -4) \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ 32 & -6 & -4 \end{vmatrix} \\ &= \left( \begin{vmatrix} 2 & -1 \\ -6 & -4 \end{vmatrix}, -\begin{vmatrix} 3 & -1 \\ 32 & -4 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ 32 & -6 \end{vmatrix} \right) \\ &= \underline{(-14, -20, -82)} \end{aligned}$$

$$\begin{aligned} \text{d) } (\vec{u} \times \vec{v}) \times \vec{w} &= (-4, 9, 6) \times (2, 6, 7) \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 9 & 6 \\ 2 & 6 & 7 \end{vmatrix} \end{aligned}$$

$$= \left( \begin{vmatrix} 9 & 6 \\ 6 & 7 \end{vmatrix}, -\begin{vmatrix} -4 & 6 \\ 2 & 7 \end{vmatrix}, \begin{vmatrix} -4 & 9 \\ 2 & 6 \end{vmatrix} \right) \\ = \underline{(27, 40, -42)}$$

$$\begin{aligned} e) \quad u \times (v - 2w) &= (3, 2, -1) \times [(0, 2, -3) - 2(2, 6, 7)] \\ &= (3, 2, -1) \times (-4, -10, -17) \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ -4 & -1 & -17 \end{vmatrix} \\ &= \left( \begin{vmatrix} 2 & -1 \\ -10 & -17 \end{vmatrix}, -\begin{vmatrix} 3 & -1 \\ -4 & -17 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ -4 & -10 \end{vmatrix} \right) \\ &= \underline{(-44, 47, -22)} \end{aligned}$$

### ***Exercise***

Use the cross product to find a vector that is orthogonal to both

$$a) \quad \vec{u} = (-6, 4, 2), \quad \vec{v} = (3, 1, 5)$$

$$b) \quad \vec{u} = (1, 1, -2), \quad \vec{v} = (2, -1, 2)$$

$$c) \quad \vec{u} = (-2, 1, 5), \quad \vec{v} = (3, 0, -3)$$

### ***Solution***

$$\begin{aligned} a) \quad \vec{u} \times \vec{v} &= (-6, 4, 2) \times (3, 1, 5) \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6 & 4 & 2 \\ 3 & 1 & 5 \end{vmatrix} \\ &= \left( \begin{vmatrix} 4 & 2 \\ 1 & 5 \end{vmatrix}, -\begin{vmatrix} -6 & 2 \\ 3 & 5 \end{vmatrix}, \begin{vmatrix} -6 & 4 \\ 3 & 1 \end{vmatrix} \right) \\ &= \underline{(18, 36, -18)} \end{aligned}$$

$$\begin{aligned} b) \quad \vec{u} \times \vec{v} &= (1, 1, -2) \times (2, -1, 2) \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ 2 & -1 & 2 \end{vmatrix} \\ &= \left( \begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix}, -\begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \right) \end{aligned}$$

$$= \underline{(0, -6, -3)}$$

$$c) \vec{u} \times \vec{v} = (-2, 1, 5) \times (3, 0, -3)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 5 \\ 3 & 0 & -3 \end{vmatrix}$$

$$= \left( \begin{vmatrix} 1 & 5 \\ 0 & -3 \end{vmatrix}, -\begin{vmatrix} -2 & 5 \\ 3 & -3 \end{vmatrix}, \begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} \right)$$

$$= \underline{(-3, 9, -3)}$$

### ***Exercise***

Find the area of the parallelogram determined by the given vectors

$$a) \vec{u} = (1, -1, 2) \text{ and } \vec{v} = (0, 3, 1)$$

$$b) \vec{u} = (3, -1, 4) \text{ and } \vec{v} = (6, -2, 8)$$

$$c) \vec{u} = (2, 3, 0) \text{ and } \vec{v} = (-1, 2, -2)$$

### **Solution**

$$a) \text{ Area} = \|\vec{u} \times \vec{v}\|$$

$$= \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 0 & 3 & 1 \end{vmatrix} \right\|$$

$$= \left| \left( \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} \right) \right|$$

$$= |(-7, -1, 3)|$$

$$= \sqrt{7^2 + 1^2 + 3^2}$$

$$= \underline{\sqrt{59}} \quad (\text{unit}^2)$$

$$b) \text{ Area} = \|\vec{u} \times \vec{v}\|$$

$$= \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 4 \\ 6 & -2 & 8 \end{vmatrix} \right\|$$

$$= \left| \left( \begin{vmatrix} -1 & 4 \\ -2 & 8 \end{vmatrix}, -\begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix}, \begin{vmatrix} 3 & -1 \\ 6 & -2 \end{vmatrix} \right) \right|$$

$$= |(0, 0, 0)|$$

$$= 0$$

$$\begin{aligned} \text{c) Area} &= \|\vec{u} \times \vec{v}\| \\ &= \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 0 \\ -1 & 2 & -2 \end{vmatrix} \right\| \\ &= \left| \left( \begin{vmatrix} 3 & 0 \\ 2 & -2 \end{vmatrix}, -\begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} \right) \right| \\ &= |(-6, 4, 7)| \\ &= \sqrt{(-6)^2 + 4^2 + 7^2} \\ &= \sqrt{101} \quad (\text{unit}^2) \end{aligned}$$

### Exercise

Find the area of the parallelogram with the given vertices  $P_1(3,2)$ ,  $P_2(5,4)$ ,  $P_3(9,4)$ ,  $P_4(7,2)$

### Solution

$$\overrightarrow{P_1P_2} = (5-3, 4-2) = (2, 2)$$

$$\overrightarrow{P_4P_3} = (9-7, 4-2) = (2, 2)$$

$$\overrightarrow{P_1P_4} = (7-3, 2-2) = (4, 0)$$

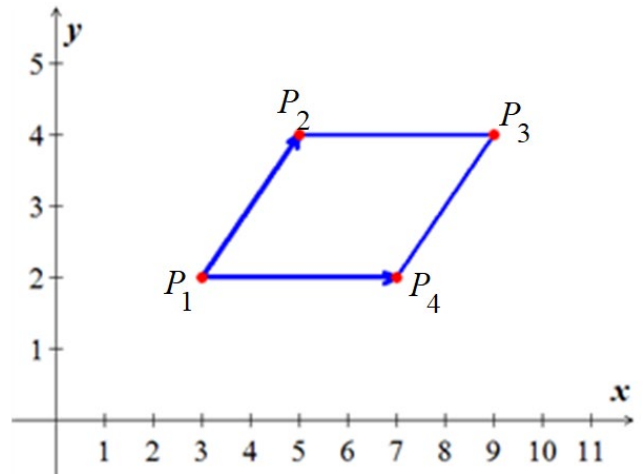
$$\overrightarrow{P_2P_3} = (9-5, 4-4) = (4, 0)$$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_4} = (2, 2) \times (4, 0)$$

$$\begin{aligned} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 0 \\ 4 & 0 & 0 \end{vmatrix} \\ &= \left( \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix}, -\begin{vmatrix} 2 & 0 \\ 4 & 0 \end{vmatrix}, \begin{vmatrix} 2 & 2 \\ 4 & 0 \end{vmatrix} \right) \\ &= (0, 0, -8) \end{aligned}$$

The area of the parallelogram is

$$\begin{aligned} \|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_4}\| &= \sqrt{0^2 + 0^2 + (-8)^2} \\ &= 8 \end{aligned}$$



### Exercise

Find the area of the triangle with the given vertices:

- a)  $A(2, 0)$   $B(3, 4)$   $C(-1, 2)$
- b)  $A(1, 1)$   $B(2, 2)$   $C(3, -3)$
- c)  $P(2, 6, -1)$   $Q(1, 1, 1)$   $R(4, 6, 2)$

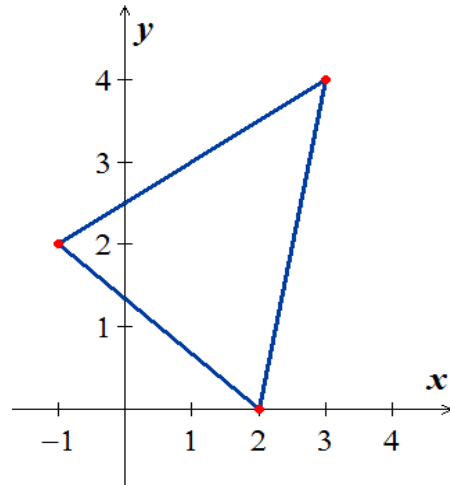
### Solution

$$\begin{aligned} \text{a) } \overrightarrow{AB} &= (1, 4) & \overrightarrow{AC} &= (-3, 2) \\ \overrightarrow{AB} \times \overrightarrow{AC} &= (1, 4, 0) \times (-3, 2, 0) \\ &= \left( \begin{vmatrix} 4 & 0 \\ 2 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ -3 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 4 \\ -3 & 2 \end{vmatrix} \right) \\ &= (0, 0, 14) \end{aligned}$$

$$\begin{aligned} \|\overrightarrow{AB} \times \overrightarrow{AC}\| &= \sqrt{0+0+14^2} \\ &= 14 \end{aligned}$$

The area of the triangle is

$$\begin{aligned} \frac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AC}\| &= \frac{1}{2} 14 \\ &= 7 \end{aligned}$$



$$\begin{aligned} \text{b) } \overrightarrow{AB} &= (1, 1) & \overrightarrow{AC} &= (2, -4) \\ \overrightarrow{AB} \times \overrightarrow{AC} &= (1, 1, 0) \times (2, -4, 0) \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 2 & -4 & 0 \end{vmatrix} \\ &= \left( \begin{vmatrix} 1 & 0 \\ -4 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} \right) \\ &= (0, 0, -6) \end{aligned}$$

$$\begin{aligned} \|\overrightarrow{AB} \times \overrightarrow{AC}\| &= \sqrt{0+0+(-6)^2} \\ &= 6 \end{aligned}$$

The area of the triangle is

$$\begin{aligned} \frac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AC}\| &= \frac{1}{2} (6) \\ &= 3 \end{aligned}$$

$$c) \quad \overrightarrow{PQ} = (-1, -5, 2) \quad \overrightarrow{PR} = (2, 0, 3)$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (-1, -5, 2) \times (2, 0, 3)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -5 & 2 \\ 2 & 0 & 3 \end{vmatrix} \quad \begin{array}{ccccc} -1 & -5 & 2 & -1 & -5 \\ 2 & 0 & 3 & 2 & 0 \end{array}$$

$$= (-15, 7, 10)$$

$$\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \sqrt{(-15)^2 + 7^2 + 10^2}$$

$$= \sqrt{374}$$

The area of the triangle is

$$\frac{1}{2} \|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \frac{1}{2} \sqrt{374} \quad \text{unit}^2$$

### Exercise

a) Find the area of the parallelogram with edges  $v = (3, 2)$  and  $w = (1, 4)$

b) Find the area of the triangle with sides  $\vec{v}$ ,  $\vec{w}$ , and  $\vec{v} + \vec{w}$ . Draw it.

c) Find the area of the triangle with sides  $\vec{v}$ ,  $\vec{w}$ , and  $\vec{v} - \vec{w}$ . Draw it.

### Solution

$$a) \quad \text{Area} = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} \\ = 10$$

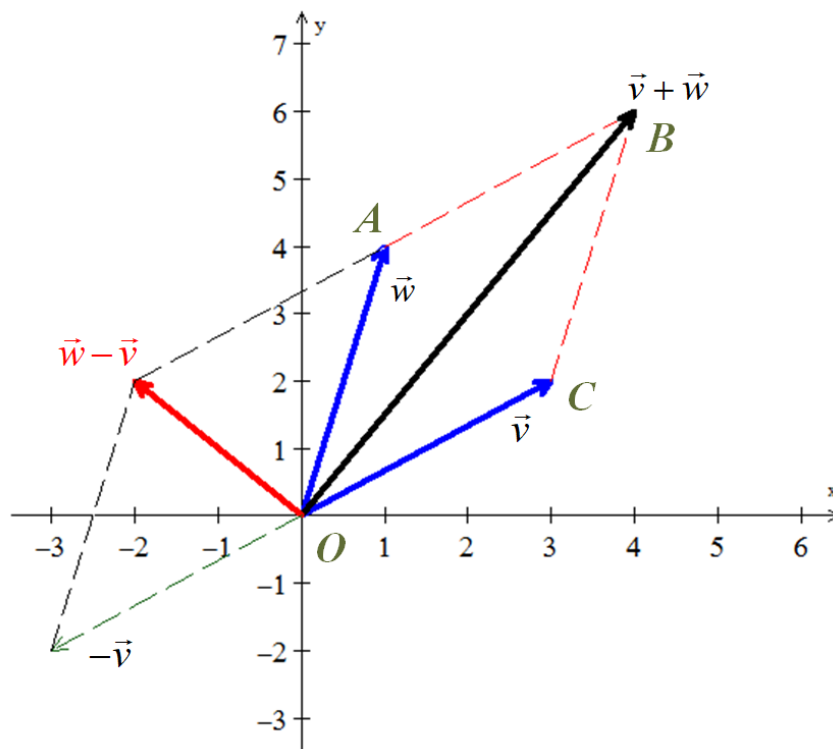
which is the parallelogram  $OABC$

b) The area of the triangle with sides  $\vec{v}$ ,  $\vec{w}$ , and  $\vec{v} + \vec{w}$  is the triangle  $OCB$  or  $OAB$  which it is half the parallelogram (by definition).

$$\text{Area} = 5$$

$$\vec{v} + \vec{w} = (3, 2) + (1, 4) \\ = (4, 6)$$

$$\text{Area} = \frac{1}{2} \begin{vmatrix} 3 & 4 \\ 2 & 6 \end{vmatrix} \\ = \frac{1}{2}(10) \\ = 5$$



- c) The area of the triangle with sides  $\vec{v}$ ,  $\vec{w}$ , and  $\vec{v} - \vec{w}$  is equivalent to the triangle  $OAC$  which it is half the parallelogram (by definition).

$$\begin{aligned}
 \text{Area} &= 5 \\
 \text{Area} &= \frac{1}{2} \begin{vmatrix} 2 & -2 \\ -3 & -2 \end{vmatrix} \\
 &= \frac{1}{2} |-10| \\
 &= \underline{5}
 \end{aligned}$$

### Exercise

Find the volume of the parallelepiped with sides  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$ .

a)  $\vec{u} = (2, -6, 2)$ ,  $\vec{v} = (0, 4, -2)$ ,  $\vec{w} = (2, 2, -4)$

b)  $\vec{u} = (3, 1, 2)$ ,  $\vec{v} = (4, 5, 1)$ ,  $\vec{w} = (1, 2, 4)$

### Solution

$$\begin{aligned}
 \text{a) } \vec{u} \cdot (\vec{v} \times \vec{w}) &= \begin{vmatrix} 2 & -6 & 2 \\ 0 & 4 & -2 \\ 2 & 2 & -4 \end{vmatrix} \\
 &= \underline{-16}
 \end{aligned}$$

The volume of the parallelepiped is  $|-16| = \underline{16} \text{ unit}^3$



$$b) \quad \vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 3 & 1 & 2 \\ 4 & 5 & 1 \\ 1 & 2 & 4 \end{vmatrix} \\ = \underline{45}$$

The volume of the parallelepiped is  $45 \text{ unit}^3$

### ***Exercise***

Compute the scalar triple product  $\vec{u} \cdot (\vec{v} \times \vec{w})$

$$a) \quad \vec{u} = (-2, 0, 6), \quad \vec{v} = (1, -3, 1), \quad \vec{w} = (-5, -1, 1)$$

$$b) \quad \vec{u} = (-1, 2, 4), \quad \vec{v} = (3, 4, -2), \quad \vec{w} = (-1, 2, 5)$$

$$c) \quad \vec{u} = (a, 0, 0), \quad \vec{v} = (0, b, 0), \quad \vec{w} = (0, 0, c)$$

$$d) \quad \vec{u} = 3\hat{i} - 2\hat{j} - 5\hat{k}, \quad \vec{v} = \hat{i} + 4\hat{j} - 4\hat{k}, \quad \vec{w} = 3\hat{j} + 2\hat{k}$$

$$e) \quad \vec{u} = (3, -1, 6) \quad \vec{v} = (2, 4, 3) \quad \vec{w} = (5, -1, 2)$$

### ***Solution***

$$a) \quad \vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} -2 & 0 & 6 \\ 1 & -3 & 1 \\ -5 & -1 & 1 \end{vmatrix} \\ = \underline{-92}$$

$$b) \quad \vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} -1 & 2 & 4 \\ 3 & 4 & -2 \\ -1 & 2 & 5 \end{vmatrix} \\ = \underline{-10}$$

$$c) \quad \vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \\ = \underline{abc}$$

$$d) \quad \vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} \\ = \underline{49}$$

$$e) \quad \vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 3 & -1 & 6 \\ 2 & 4 & 3 \\ 5 & -1 & 2 \end{vmatrix} \\ = -110$$

### Exercise

Use the cross product to find the sine of the angle between the vectors  $\vec{u} = (2, 3, -6)$ ,  $\vec{v} = (2, 3, 6)$

### Solution

$$\vec{u} \times \vec{v} = (2, 3, -6) \times (2, 3, 6) \\ = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -6 \\ 2 & 3 & 6 \end{vmatrix} \\ = \left( \begin{vmatrix} 3 & -6 \\ 3 & 6 \end{vmatrix}, - \begin{vmatrix} 2 & -6 \\ 2 & 6 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} \right) \\ = (36, -24, 0)$$

$$\|\vec{u} \times \vec{v}\| = \sqrt{36^2 + (-24)^2 + 0} \\ = \sqrt{1872} \\ = 12\sqrt{13}$$

$$\sin \theta = \left( \frac{\|\vec{u} \times \vec{v}\|}{\|\vec{u}\| \|\vec{v}\|} \right) \\ = \frac{12\sqrt{13}}{\sqrt{2^2 + 3^2 + (-6)^2} \sqrt{2^2 + 3^2 + 6^2}} \\ = \frac{12\sqrt{13}}{(7)(7)} \\ = \frac{12}{49} \sqrt{13}$$

### Exercise

Simplify  $(\vec{u} + \vec{v}) \times (\vec{u} - \vec{v})$

### Solution

$$\begin{aligned}(\vec{u} + \vec{v}) \times (\vec{u} - \vec{v}) &= (\vec{u} + \vec{v}) \times \vec{u} - (\vec{u} + \vec{v}) \times \vec{v} \\&= (\vec{u} \times \vec{u}) + (\vec{v} \times \vec{u}) - [(\vec{u} \times \vec{v}) + (\vec{v} \times \vec{v})] \\&= 0 + (\vec{v} \times \vec{u}) - [(\vec{u} \times \vec{v}) + 0] \\&= (\vec{v} \times \vec{u}) - (\vec{u} \times \vec{v}) \\&= (\vec{v} \times \vec{u}) - (-\vec{v} \times \vec{u}) \\&= (\vec{v} \times \vec{u}) + (\vec{v} \times \vec{u}) \\&= \underline{2(\vec{v} \times \vec{u})} \quad | \end{aligned}$$

### Exercise

Prove Lagrange's identity:  $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$

### Solution

Let  $\vec{u} = (u_1, u_2, u_3)$  and  $\vec{v} = (v_1, v_2, v_3)$

$$\|\vec{u}\|^2 = u_1^2 + u_2^2 + u_3^2$$

$$\|\vec{v}\|^2 = v_1^2 + v_2^2 + v_3^2$$

$$(\vec{u} \cdot \vec{v})^2 = (u_1 v_1 + u_2 v_2 + u_3 v_3)^2$$

$$\|\vec{u} \times \vec{v}\|^2 = (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2$$

$$= u_2^2 v_3^2 - 2u_2 v_3 u_3 v_2 + u_3^2 v_2^2 + u_3^2 v_1^2 - 2u_3 v_1 u_1 v_3 + u_1^2 v_3^2 + u_1^2 v_2^2 - 2u_2 v_1 u_2 v_1 + u_2^2 v_1^2$$

$$\|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2 = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2$$

$$\begin{aligned}
&= u_1^2 v_1^2 + u_1^2 v_2^2 + u_1^2 v_3^2 + u_2^2 v_1^2 + u_2^2 v_2^2 + u_2^2 v_3^2 + u_3^2 v_1^2 + u_3^2 v_2^2 + u_3^2 v_3^2 \\
&\quad - u_1^2 v_1^2 - u_1 v_1 u_2 v_2 - u_1 v_1 u_3 v_3 \\
&\quad - u_2 v_2 u_1 v_1 - u_2^2 v_2^2 - u_2 v_2 u_3 v_3 \\
&\quad - u_1 v_1 u_3 v_3 - u_2 v_2 u_3 v_3 - u_3^2 v_3^2 \\
&= u_2^2 v_3^2 - 2u_2 v_2 u_3 v_3 + u_3^2 v_2^2 \\
&\quad + u_3^2 v_1^2 - 2u_1 v_1 u_3 v_3 + u_1^2 v_3^2 \\
&\quad + u_1^2 v_2^2 - 2u_1 v_1 u_2 v_2 + u_2^2 v_1^2
\end{aligned}$$

$$\Rightarrow \underline{\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2}$$

### Exercise

Polar coordinates satisfy  $x = r \cos \theta$  and  $y = r \sin \theta$ . Polar area  $J dr d\theta$  includes  $J$ :

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$$

The two columns are orthogonal. Their lengths are \_\_\_\_\_. Thus  $J =$  \_\_\_\_\_.

### Solution

The length of the first column is:

$$\ell_1 = \sqrt{\cos^2 \theta + \sin^2 \theta}$$

$$\underline{= 1}$$

The length of the second column is:

$$\begin{aligned} \ell_2 &= \sqrt{r^2 \sin^2 \theta + r^2 \cos^2 \theta} \\ &= \sqrt{r^2 (\sin^2 \theta + \cos^2 \theta)} \\ &= \sqrt{r^2} \\ &\underline{= r} \end{aligned}$$

So,  $J$  is the product 1.  $r = r$ .

$$\begin{aligned}
 \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} &= r \cos^2 \theta + r \sin^2 \theta \\
 &= r (\cos^2 \theta + \sin^2 \theta) \\
 &= \underline{r}
 \end{aligned}$$

### Exercise

Prove that  $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$  if and only if  $\vec{u}$  and  $\vec{v}$  are parallel vectors.

### Solution

If  $\vec{u}$  and  $\vec{v}$  are parallel vectors, then  $\vec{u} \times \vec{v} = 0$

Which the two vectors are collinear, which implies that  $\vec{v} = a\vec{u}$

$$\begin{aligned}
 \|\vec{u} + \vec{v}\| &= \|\vec{u} + a\vec{u}\| \\
 &= \|(1+a)\vec{u}\| \\
 &= (1+a)\|\vec{u}\| \\
 &= \|\vec{u}\| + a\|\vec{u}\| \\
 &= \|\vec{u}\| + \|a\vec{u}\| \\
 &= \|\vec{u}\| + \|\vec{v}\| \quad \checkmark
 \end{aligned}$$

### Exercise

State the following statements as True or False

- The cross product of two nonzero vectors  $\vec{u}$  and  $\vec{v}$  is a nonzero vector if and only if  $\vec{u}$  and  $\vec{v}$  are not parallel.
- A normal vector to a plane can be obtained by taking the cross product of two nonzero and noncollinear vectors lying in the plane.
- The scalar triple product of  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$  determines a vector whose length is equal to the volume of the parallelepiped determined by  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$ .
- If  $\vec{u}$  and  $\vec{v}$  are vectors in 3-space, then  $\|\vec{u} \times \vec{v}\|$  is equal to the area of the parallelogram determined by  $\vec{u}$  and  $\vec{v}$ .
- For all vectors  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$  in  $R^3$ , the vectors  $(\vec{u} \times \vec{v}) \times \vec{w}$  and  $\vec{u} \times (\vec{v} \times \vec{w})$  are the same.
- If  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$  are vectors in  $R^3$ , where  $\vec{u}$  is nonzero and  $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$ , then  $\vec{v} = \vec{w}$

### Solution

- True,  
 $\vec{u} \times \vec{v} = \|\vec{u}\| \|\vec{v}\| \sin \theta = 0$  if  $\theta = 0$  which the two vectors are parallel.
- True;

The cross product of two nonzero and non collinear vectors will be perpendicular to both vectors, hence normal to the plane containing the vectors.

c) False;

The scalar triple product is a scalar, not a vector.

d) True;

e) False;

$$\text{Let } \vec{u} = \hat{i} \quad \vec{v} = \vec{w} = \hat{j}$$

$$\begin{aligned} (\vec{u} \times \vec{v}) \times \vec{w} &= (\hat{i} \times \hat{j}) \times \hat{j} \\ &= \hat{k} \times \hat{j} \\ &= -\hat{i} \end{aligned}$$

$$\begin{aligned} \vec{u} \times (\vec{v} \times \vec{w}) &= \hat{i} \times (\hat{j} \times \hat{j}) \\ &= \hat{i} \times \vec{0} \\ &= \vec{0} \end{aligned}$$

$$\text{Hence, } (\vec{u} \times \vec{v}) \times \vec{w} \neq (\vec{u} \times \vec{v}) \times \vec{w}$$

f) False;

$$\text{Let } \vec{u} = \hat{i} + \hat{j} \quad \vec{v} = \hat{i} + \hat{j} + \hat{k} \quad \vec{w} = -\hat{i} - \hat{j} + \hat{k}$$

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} \\ &= \hat{i} - \hat{j} \end{aligned}$$

$$\begin{aligned} \vec{u} \times \vec{w} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{vmatrix} \\ &= \hat{i} - \hat{j} \end{aligned}$$

$$\vec{u} \times \vec{v} = \vec{u} \times \vec{w}, \text{ but } \vec{v} \neq \vec{w}$$

## ***Solution***

## **Section 2.5 – Subspaces, Span and Null Space**

### ***Exercise***

Suppose  $S$  and  $T$  are two subspaces of a vector space  $V$ .

- a) The sum  $S + T$  contains all sums  $s + t$  of a vector  $s$  in  $S$  and a vector  $t$  in  $T$ . Show that  $S + T$  satisfies the requirements (addition and scalar multiplication) for a vector space.
- b) If  $S$  and  $T$  are lines in  $\mathbb{R}^m$ , what is the difference between  $S + T$  and  $S \cup T$ ? That union contains all vectors from  $S$  and  $T$  or both. Explain this statement: The span of  $S \cup T$  is  $S + T$ .

### **Solution**

- a) Let  $s, s'$  be vectors in  $S$ , Let  $t, t'$  be vectors in  $T$ , and let  $c$  be a scalar. Then

$$(s + t) + (s' + t') = (s + s') + (t + t') \text{ and}$$

$$c(s + t) = cs + ct$$

Thus  $S + T$  is closed under addition and scalar multiplication, it satisfies the two requirements for a vector space.

- b) If  $S$  and  $T$  are distinct lines, then  $S$  and  $T$  is a plane, whereas  $S \cup T$  is not even closed under addition. The span of  $S \cup T$  is the set of all combinations of vectors in this union. In particular, it contains all sums  $s + t$  of a vector  $s$  in  $S$  and a vector  $t$  in  $T$ , and these sums form  $S + T$ .  $S + T$  contains both  $S$  and  $T$ ; so, it contains  $S \cup T$ .  $S + T$  is a vector space.
- c) So, it contains all combinations of vectors in itself; in particular, it contains the span of  $S \cup T$ . Thus, the span of  $S \cup T$  is  $S + T$ .

### ***Exercise***

Determine which of the following are subspaces of  $\mathbb{R}^3$ ?

- a) All vectors of the form  $(a, 0, 0)$
- b) All vectors of the form  $(a, 1, 1)$
- c) All vectors of the form  $(a, b, c)$ , where  $b = a + c$
- d) All vectors of the form  $(a, b, c)$ , where  $b = a + c + 1$
- e) All vectors of the form  $(a, b, 0)$

### **Solution**

- a)  $(a_1, 0, 0) + (a_2, 0, 0) = (a_1 + a_2, 0, 0)$

$$k(a, 0, 0) = (ka, 0, 0)$$

This is a subspace of  $\mathbb{R}^3$

b)  $(a_1, 1, 1) + (a_2, 1, 1) = (a_1 + a_2, 2, 2)$  which is not in the set.

Therefore, this is not a subspace of  $\mathbb{R}^3$

$$\begin{aligned} c) \quad (a_1, b_1, c_1) + (a_2, b_2, c_2) &= (a_1 + a_2, b_1 + b_2, c_1 + c_2) \\ &= (a_1 + a_2, a_1 + c_1 + a_2 + c_2, c_1 + c_2) \\ &= (a_1 + a_2, (a_1 + a_2) + (c_1 + c_2), c_1 + c_2) \\ &= (a_1, a_1 + c_1, c_1) + (a_2, a_2 + c_2, c_2) \end{aligned}$$

$$\begin{aligned} k(a, b, c) &= (ka, kb, kc) \\ &= (ka, k(a + c), kc) \\ &= k(a, (a + c), c) \end{aligned}$$

This is a subspace of  $\mathbb{R}^3$

d)  $k(a + c + 1) \neq ka + kc + 1$  so  $k(a, b, c)$  is not in the set.

Therefore, this is not a subspace of  $\mathbb{R}^3$

$$\begin{aligned} e) \quad (a_1, b_1, 0) + (a_2, b_2, 0) &= (a_1 + a_2, b_1 + b_2, 0) \\ k(a, b, 0) &= (ka, kb, 0) \end{aligned}$$

This is a subspace of  $\mathbb{R}^3$

## Exercise

Determine which of the following are subspaces of  $\mathbb{R}^\infty$ ?

- a) All sequences  $\vec{v}$  in  $\mathbb{R}^\infty$  of the form  $\vec{v} = (v, 0, v, 0, \dots)$
- b) All sequences  $\vec{v}$  in  $\mathbb{R}^\infty$  of the form  $\vec{v} = (v, 1, v, 1, \dots)$
- c) All sequences  $\vec{v}$  in  $\mathbb{R}^\infty$  of the form  $\vec{v} = (v, 2v, 4v, 8v, 16v, \dots)$

## Solution

$$\begin{aligned} a) \quad \text{Let } \vec{v}_1 &= (v_1, 0, v_1, 0, \dots) \quad \vec{v}_2 = (v_2, 0, v_2, 0, \dots) \\ \vec{v}_1 + \vec{v}_2 &= (v_1, 0, v_1, 0, \dots) + (v_2, 0, v_2, 0, \dots) \\ &= (v_1 + v_2, 0, v_1 + v_2, 0, \dots) \quad w = v_1 + v_2 \\ &= (w, 0, w, 0, \dots) \end{aligned}$$



$$\begin{aligned}
k\vec{v} &= k(v, 0, v, 0, \dots) \\
&= (kv, 0, kv, 0, \dots) & w = kv \\
&= (w, 0, w, 0, \dots)
\end{aligned}$$

This is a **subspace** of  $\mathbb{R}^\infty$

**b)** Let  $\vec{v} = (v, 1, v, 1, \dots)$

$$\begin{aligned}
k\vec{v} &= k(v, 1, v, 1, \dots) \\
&= (kv, \textcolor{red}{k}, kv, \textcolor{red}{k}, \dots) \\
&\neq (kv, 1, kv, 1, \dots)
\end{aligned}$$

$k\vec{v}$  is not in the set

Since  $k \neq 1$ , then is **not** a **subspace** of  $\mathbb{R}^\infty$

**c)** Let  $\vec{v}_1 = (v_1, 2v_1, 4v_1, 8v_1, \dots)$   $\vec{v}_2 = (v_2, 2v_2, 4v_2, 8v_2, \dots)$

$$\begin{aligned}
\vec{v}_1 + \vec{v}_2 &= (v_1, 2v_1, 4v_1, 8v_1, \dots) + (v_2, 2v_2, 4v_2, 8v_2, \dots) \\
&= (v_1 + v_2, 2v_1 + 2v_2, 4v_1 + 4v_2, 8v_1 + 8v_2, \dots) \\
&= (v_1 + v_2, 2(v_1 + v_2), 4(v_1 + v_2), 8(v_1 + v_2), \dots) & w = v_1 + v_2 \\
&= (w, 2w, 4w, 8w, \dots)
\end{aligned}$$

$$\begin{aligned}
k\vec{v} &= k(v, 2v, 4v, 8v, 16v, \dots) \\
&= (kv, 2kv, 4kv, 8kv, 16kv, \dots) \\
&= (w, 2w, 4w, 8w, 16w, \dots) & w = kv
\end{aligned}$$

This is a **subspace** of  $\mathbb{R}^\infty$

## Exercise

Which of the following are linear combinations of  $\vec{u} = (0, -2, 2)$  and  $\vec{v} = (1, 3, -1)$ ?

- a)  $(2, 2, 2)$       b)  $(3, 1, 5)$       c)  $(0, 4, 5)$       d)  $(0, 0, 0)$

## Solution

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \\ 2 & -1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$a) \quad b = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\left[ \begin{array}{cc|c} 0 & 1 & 2 \\ -2 & 3 & 2 \\ 2 & -1 & 2 \end{array} \right] \quad \text{Switch } R_1 \text{ \& } R_2$$

$$\left[ \begin{array}{cc|c} -2 & 3 & 2 \\ 0 & 1 & 2 \\ 2 & -1 & 2 \end{array} \right] \quad R_3 + R_1$$

$$\left[ \begin{array}{cc|c} -2 & 3 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{array} \right] \quad -\frac{1}{2}R_1$$

$$\left[ \begin{array}{cc|c} 1 & -\frac{3}{2} & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{array} \right] \quad \begin{array}{l} R_1 + \frac{3}{2}R_2 \\ R_3 - 2R_2 \end{array}$$

$$\left[ \begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right]$$

$(2, 2, 2) = 2\vec{u} + 2\vec{v}$  is a linear combination of  $\vec{u}$  and  $\vec{v}$ .

$$b) \quad b = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

$$\left[ \begin{array}{cc|c} 0 & 1 & 3 \\ -2 & 3 & 1 \\ 2 & -1 & 5 \end{array} \right] \quad \text{Switch } R_1 \text{ \& } R_2$$

$$\left[ \begin{array}{cc|c} -2 & 3 & 1 \\ 0 & 1 & 3 \\ 2 & -1 & 5 \end{array} \right] \quad R_3 + R_1$$

$$\left[ \begin{array}{cc|c} -2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{array} \right] \quad \begin{array}{l} R_1 - 3R_2 \\ R_3 - 2R_2 \end{array}$$

$$\left[ \begin{array}{cc|c} -2 & 0 & -8 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array} \right] \quad -\frac{1}{2}R_1$$

$$\left[ \begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array} \right]$$

$(3, 1, 5) = 4\vec{u} + 3\vec{v}$  is a linear combination of  $\vec{u}$  and  $\vec{v}$ .

c)  $b = \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$

$$\left[ \begin{array}{cc|c} 0 & 1 & 0 \\ -2 & 3 & 4 \\ 2 & -1 & 5 \end{array} \right] \quad \text{Switch } R_1 \text{ \& } R_2$$

$$\left[ \begin{array}{cc|c} -2 & 3 & 4 \\ 0 & 1 & 0 \\ 2 & -1 & 5 \end{array} \right] \quad R_3 + R_1$$

$$\left[ \begin{array}{cc|c} -2 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 2 & 9 \end{array} \right] \quad \begin{array}{l} R_1 - 3R_2 \\ R_3 - 2R_2 \end{array}$$

$$\left[ \begin{array}{cc|c} -2 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{array} \right] \quad \begin{array}{l} -\frac{1}{2}R_1 \\ \frac{1}{9}R_3 \end{array}$$

$$\left[ \begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \quad R_1 + 2R_3$$

$$\left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

$(0, 4, 5)$  is not a linear combination of  $\vec{u}$  and  $\vec{v}$ .

d)  $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[ \begin{array}{cc|c} 0 & 1 & 0 \\ -2 & 3 & 0 \\ 2 & -1 & 0 \end{array} \right] \quad \text{Switch } R_1 \text{ \& } R_2$$

$$\left[ \begin{array}{cc|c} -2 & 3 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 0 \end{array} \right] \quad R_3 + R_1$$

$$\left[ \begin{array}{cc|c} -2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{array} \right] \quad \begin{array}{l} R_1 - 3R_2 \\ R_3 - 2R_2 \end{array}$$

$$\left[ \begin{array}{cc|c} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad -\frac{1}{2}R_1$$

$$\left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$(0, 0, 0) = 0\vec{u} + 0\vec{v}$  is a linear combination of  $\vec{u}$  and  $\vec{v}$ .

### Exercise

Which of the following are linear combinations of  $\vec{u} = (2, 1, 4)$ ,  $\vec{v} = (1, -1, 3)$  and  $\vec{w} = (3, 2, 5)$ ?

a)  $(-9, -7, -15)$

b)  $(6, 11, 6)$

c)  $(0, 0, 0)$

### Solution

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{pmatrix}$$

$$a) \left[ \begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array} \right] \quad \begin{array}{l} 2R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 0 & -3 & 1 & -5 \\ 0 & 1 & -1 & 3 \end{array} \right] \quad \begin{array}{l} 3R_1 + R_2 \\ 3R_3 + R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 6 & 0 & 10 & -32 \\ 0 & -3 & 1 & -5 \\ 0 & 0 & -2 & 4 \end{array} \right] \quad \begin{array}{l} R_1 + 5R_3 \\ 2R_2 + R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 6 & 0 & 0 & -12 \\ 0 & -6 & 0 & -6 \\ 0 & 0 & -2 & 4 \end{array} \right] \quad \begin{array}{l} \frac{1}{6}R_1 \\ -\frac{1}{6}R_2 \\ -\frac{1}{2}R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

Therefore,  $(-9, -7, -15) = -2\vec{u} + 1\vec{v} - 2\vec{w}$

$$b) \left[ \begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array} \right] \quad \begin{array}{l} 2R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 0 & -3 & 1 & 16 \\ 0 & 1 & -1 & -6 \end{array} \right] \quad \begin{array}{l} 3R_1 + R_2 \\ 3R_3 + R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 6 & 0 & 10 & 34 \\ 0 & -3 & 1 & 16 \\ 0 & 0 & -2 & -2 \end{array} \right] \quad \begin{array}{l} R_1 + 5R_3 \\ 2R_2 + R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 6 & 0 & 0 & 24 \\ 0 & -6 & 0 & 30 \\ 0 & 0 & -2 & -2 \end{array} \right] \quad \begin{array}{l} \frac{1}{6}R_1 \\ -\frac{1}{6}R_2 \\ -\frac{1}{2}R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

Therefore,  $(6, 11, 6) = 4\vec{u} - 5\vec{v} + 1\vec{w}$

$$c) \left[ \begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array} \right] \quad \begin{array}{l} 2R_2 - R_1 \\ R_2 - 2R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right] \quad \begin{array}{l} 2R_2 - R_1 \\ R_2 - 2R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 6 & 0 & 10 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array} \right] \quad \begin{array}{l} R_1 + 5R_3 \\ 2R_2 + R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 6 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array} \right] \quad \begin{array}{l} \frac{1}{6}R_1 \\ -\frac{1}{6}R_2 \\ -\frac{1}{2}R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Therefore,  $(0, 0, 0) = 0\vec{u} + 0\vec{v} + 0\vec{w}$

### ***Exercise***

Determine whether the given vectors span  $\mathbb{R}^3$

a)  $\vec{v}_1 = (2, 2, 2), \vec{v}_2 = (0, 0, 3), \vec{v}_3 = (0, 1, 1)$

b)  $\vec{v}_1 = (2, -1, 3), \vec{v}_2 = (4, 1, 2), \vec{v}_3 = (8, -1, 8)$

c)  $\vec{v}_1 = (3, 1, 4), \vec{v}_2 = (2, -3, 5), \vec{v}_3 = (5, -2, 9), \vec{v}_4 = (1, 4, -1)$

### ***Solution***

a)  $\det \begin{pmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{pmatrix} = -6 \neq 0$

The system is consistent for all values so the given vectors span  $\mathbb{R}^3$ .

b)  $\det \begin{pmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{pmatrix} = 0$

The system is not consistent for all values so the given vectors do not span  $\mathbb{R}^3$ .

$$\begin{aligned}
c) \quad & \left[ \begin{array}{cccc|c} 3 & 2 & 5 & 1 & b_1 \\ 1 & -3 & -2 & 4 & b_2 \\ 4 & 5 & 9 & -1 & b_3 \end{array} \right] \quad \begin{array}{l} 3R_2 - R_1 \\ 3R_3 - 4R_1 \end{array} \\
& \left[ \begin{array}{cccc|c} 3 & 2 & 5 & 1 & b_1 \\ 0 & -11 & -11 & 11 & 3b_2 - b_1 \\ 0 & 7 & 7 & -7 & 3b_3 - 4b_1 \end{array} \right] \quad \begin{array}{l} 11R_1 + 2R_2 \\ 11R_3 + 7R_2 \end{array} \\
& \left[ \begin{array}{cccc|c} 33 & 0 & 33 & 33 & 9b_1 + 6b_2 \\ 0 & -11 & -11 & 11 & 3b_2 - b_1 \\ 0 & 0 & 0 & 0 & 33b_3 - 51b_1 + 21b_2 \end{array} \right] \quad \begin{array}{l} \frac{1}{33}R_1 \\ -\frac{1}{11}R_2 \\ \frac{1}{33}R_2 \end{array} \\
& \left[ \begin{array}{cccc|c} 1 & 0 & 1 & 1 & \frac{3}{11}b_1 + \frac{2}{11}b_2 \\ 0 & 1 & 1 & -1 & \frac{1}{11}b_1 - \frac{3}{11}b_2 \\ 0 & 0 & 0 & 0 & -\frac{17}{11}b_1 + \frac{7}{11}b_2 + b_3 \end{array} \right]
\end{aligned}$$

The system has a solution only if  $-\frac{17}{11}b_1 + \frac{7}{11}b_2 + b_3 = 0$ . But since this is a restriction that the given vectors don't span on all of  $\mathbb{R}^3$ . So the given vectors do not span  $\mathbb{R}^3$ .

### Exercise

Which of the following are linear combinations of  $A = \begin{pmatrix} 4 & 0 \\ -2 & -2 \end{pmatrix}$ ,  $B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ ,  $C = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}$

$$a) \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} \quad b) \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad c) \begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$$

### Solution

$$\begin{aligned}
& \begin{pmatrix} 4 & 1 & 0 \\ 0 & -1 & 2 \\ -2 & 2 & 1 \\ -2 & 3 & 4 \end{pmatrix} \\
a) \quad & \left[ \begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -8 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -8 \end{array} \right] \quad \begin{array}{l} 2R_3 + R_1 \\ 2R_4 + R_1 \end{array}
\end{aligned}$$

$$\left[ \begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -8 \\ 0 & 5 & 2 & 4 \\ 0 & 7 & 8 & -10 \end{array} \right] \quad \begin{array}{l} R_1 + R_2 \\ \\ R_3 + 5R_2 \\ R_4 + 7R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 4 & 0 & 2 & -2 \\ 0 & -1 & 2 & -8 \\ 0 & 0 & 12 & -36 \\ 0 & 0 & 22 & -66 \end{array} \right] \quad -R_2$$

$$\left[ \begin{array}{ccc|c} 4 & 0 & 2 & -2 \\ 0 & 1 & -2 & 8 \\ 0 & 0 & 12 & -36 \\ 0 & 0 & 22 & -66 \end{array} \right] \quad \frac{1}{12}R_3$$

$$\left[ \begin{array}{ccc|c} 4 & 0 & 2 & -2 \\ 0 & 1 & -2 & 8 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 22 & -66 \end{array} \right] \quad \begin{array}{l} R_1 - 2R_3 \\ R_3 + 2R_3 \\ \\ R_4 - 22R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 4 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \frac{1}{4}R_1$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[ \begin{array}{cc} 6 & -8 \\ -1 & -8 \end{array} \right] = 1A + 2B - 3C \text{ is a linear combinations of } A, B, \text{ and } C.$$

$$b) \left[ \begin{array}{ccc|c} 4 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ -2 & 2 & 1 & 0 \\ -2 & 3 & 4 & 0 \end{array} \right] \quad \begin{array}{l} \\ 2R_3 + R_1 \\ 2R_4 + R_1 \\ \end{array}$$

$$\left[ \begin{array}{ccc|c} 4 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 7 & 8 & 0 \end{array} \right] \quad \begin{array}{l} R_1 + R_2 \\ \\ R_3 + 5R_2 \\ R_4 + 7R_2 \end{array}$$



$$\left[ \begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 22 & 0 \end{array} \right] \quad -R_2$$

$$\left[ \begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 22 & 0 \end{array} \right] \quad \begin{array}{l} \frac{1}{12}R_3 \\ \frac{1}{22}R_4 \end{array}$$

$$\left[ \begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \quad \begin{array}{l} R_1 - 2R_3 \\ R_3 + 2R_3 \\ R_4 - R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \frac{1}{4}R_1$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] = 0A + 0B + 0C \text{ is a linear combinations of } A, B, \text{ and } C.$$

$$c) \left[ \begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & 0 \\ -2 & 2 & 1 & 3 \\ -2 & 3 & 4 & 8 \end{array} \right] \quad \begin{array}{l} 2R_3 + R_1 \\ 2R_4 + R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 5 & 2 & 12 \\ 0 & 7 & 8 & 22 \end{array} \right] \quad \begin{array}{l} R_1 + R_2 \\ R_3 + 5R_2 \\ R_4 + 7R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 4 & 0 & 2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 12 & 12 \\ 0 & 0 & 22 & 22 \end{array} \right] \quad \begin{array}{l} -R_2 \\ \frac{1}{12}R_3 \\ \frac{1}{22}R_4 \end{array}$$

$$\left[ \begin{array}{ccc|c} 4 & 0 & 2 & 6 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right] \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 + 2R_3 \\ R_4 - R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 4 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \frac{1}{4}R_1$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix} = 1A + 2B + 1C \text{ is a linear combination of } A, B, \text{ and } C.$$

### ***Exercise***

Suppose that  $\vec{v}_1 = (2, 1, 0, 3)$ ,  $\vec{v}_2 = (3, -1, 5, 2)$ ,  $\vec{v}_3 = (-1, 0, 2, 1)$ . Which of the following vectors are in  $\text{span} \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$

- a)  $(2, 3, -7, 3)$       b)  $(0, 0, 0, 0)$       c)  $(1, 1, 1, 1)$       d)  $(-4, 6, -13, 4)$

### **Solution**

In order to be  $\text{span} \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$ , there must exist scalars  $a, b, c$  that  $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{w}$

$$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

**a)**  $(2, 3, -7, 3)$

$$\left[ \begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 1 & -1 & 0 & 3 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{array} \right] \quad \begin{array}{l} 2R_2 - R_1 \\ 2R_4 - 3R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 0 & -5 & 1 & 4 \\ 0 & 5 & 2 & -7 \\ 0 & -5 & 5 & 0 \end{array} \right] \quad \begin{array}{l} 5R_1 + 3R_2 \\ R_3 + R_2 \\ R_4 - R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 10 & 0 & -2 & 22 \\ 0 & -5 & 1 & 4 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 4 & -4 \end{array} \right] \quad \begin{array}{l} \frac{1}{3}R_3 \\ \frac{1}{4}R_4 \end{array}$$

$$\left[ \begin{array}{ccc|c} 10 & 0 & -2 & 22 \\ 0 & -5 & 1 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{array} \right] \quad \begin{array}{l} R_1 + 2R_3 \\ R_2 - R_3 \\ R_4 - R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 10 & 0 & 0 & 20 \\ 0 & -5 & 0 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} \frac{1}{10}R_1 \\ -\frac{1}{5}R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This system is consistent, it has only solution is  $a = 2, b = -1, c = -1$

$$2\vec{v}_1 - 1\vec{v}_2 - 1\vec{v}_3 = (2, 3, -7, 3)$$

Therefore,  $(2, 3, -7, 3)$  is in  $\text{span} \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$

**b)** The vector  $(0, 0, 0, 0)$  is obviously in  $\text{span} \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$

$$\text{Since } 0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = (0, 0, 0, 0)$$

c) For the vector  $b = (1, 1, 1, 1)$

$$\left[ \begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 5 & 2 & 1 \\ 3 & 2 & 1 & 1 \end{array} \right] \quad \begin{array}{l} 2R_2 - R_1 \\ \\ 2R_4 - 3R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 0 & -5 & 1 & 1 \\ 0 & 5 & 2 & 1 \\ 0 & -5 & 5 & -1 \end{array} \right] \quad \begin{array}{l} 5R_1 + 3R_2 \\ \\ R_3 + R_2 \\ R_4 - R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 10 & 0 & -2 & 8 \\ 0 & -5 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 4 & -2 \end{array} \right] \quad \begin{array}{l} 3R_1 + 2R_3 \\ 3R_2 - R_3 \\ \\ 3R_4 - 4R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 10 & 0 & 0 & 28 \\ 0 & -5 & 0 & 1 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & -14 \end{array} \right] \quad \begin{array}{l} \\ \frac{1}{3}R_3 \\ -\frac{1}{14}R_4 \end{array}$$

$$\left[ \begin{array}{ccc|c} 10 & 0 & 0 & 28 \\ 0 & -5 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad \begin{array}{l} R_1 - 28R_4 \\ R_2 - R_4 \\ R_3 + R_4 \end{array}$$

$$\left[ \begin{array}{ccc|c} 10 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad \begin{array}{l} \frac{1}{10}R_1 \\ -\frac{1}{5}R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

This system is inconsistent, therefore  $(1, 1, 1, 1)$  is *not* in  $\text{span} \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$

d) For the vector  $b = (-4, 6, -13, 4)$

$$\left[ \begin{array}{ccc|c} 2 & 3 & -1 & -4 \\ 1 & -1 & 0 & 6 \\ 0 & 5 & 2 & -13 \\ 3 & 2 & 1 & 4 \end{array} \right] \quad \begin{array}{l} 2R_2 - R_1 \\ 2R_4 - 3R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 2 & 3 & -1 & -4 \\ 0 & -5 & 1 & 16 \\ 0 & 5 & 2 & -13 \\ 0 & -5 & 5 & 20 \end{array} \right] \quad \begin{array}{l} 5R_1 + 3R_2 \\ R_3 + R_2 \\ R_4 - R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 10 & 0 & -2 & 28 \\ 0 & -5 & 1 & 16 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 4 & 4 \end{array} \right] \quad \begin{array}{l} \frac{1}{3}R_3 \\ \frac{1}{4}R_4 \end{array}$$

$$\left[ \begin{array}{ccc|c} 10 & 0 & -2 & 28 \\ 0 & -5 & 1 & 16 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right] \quad \begin{array}{l} R_1 + 2R_3 \\ R_2 - R_3 \\ R_4 - R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 10 & 0 & 0 & 30 \\ 0 & -5 & 0 & 15 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} \frac{1}{10}R_1 \\ -\frac{1}{5}R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This system is consistent, it has only solution is  $a = 3, b = -3, c = 1$

$$3\vec{v}_1 - 3\vec{v}_2 + 1\vec{v}_3 = (-4, 6, -13, 4)$$

Therefore,  $(-4, 6, -13, 4)$  is in  $\text{span} \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$

### Exercise

Let  $f = \cos^2 x$  and  $g = \sin^2 x$ . Which of the following lie in the space spanned by  $f$  and  $g$

- a)  $\cos 2x$       b)  $3 + x^2$       c)  $\sin x$       d)  $0$

### Solution

a)  $\cos 2x = \cos^2 x - \sin^2 x$ , therefore  $\cos 2x$  is in  $\text{span} \{f, g\}$

b) In order for  $3 + x^2$  to be in  $\text{span} \{f, g\}$ , there must exist scalars  $a$  and  $b$  such that

$$a \cos^2 x + b \sin^2 x = 3 + x^2$$

$$\text{When } \left. \begin{array}{l} x = 0 \Rightarrow a = 3 \\ x = \pi \Rightarrow a = 3 + \pi^2 \end{array} \right\} \rightarrow \text{contradiction}$$

Therefore  $3 + x^2$  is *not* in  $\text{span} \{f, g\}$

c) In order for  $\sin x$  to be in  $\text{span} \{f, g\}$ , there must exist scalars  $a$  and  $b$  such that

$$a \cos^2 x + b \sin^2 x = \sin x$$

$$\text{When } \left. \begin{array}{l} x = \frac{\pi}{2} \Rightarrow b = 1 \\ x = -\frac{\pi}{2} \Rightarrow b = -1 \end{array} \right\} \rightarrow \text{contradiction}$$

Therefore  $\sin x$  is *not* in  $\text{span} \{f, g\}$

d) In order for  $0$  to be in  $\text{span} \{f, g\}$ , there must exist scalars  $a$  and  $b$  such that

$$0 \cos^2 x + 0 \sin^2 x = 0$$

Therefore  $0$  is in  $\text{span} \{f, g\}$

### Exercise

Let  $S = \left\{ (x, y) \mid x^2 + y^2 = 0; x, y \in \mathbb{R} \right\}$ , Determine:

- a) Is  $S$  closed under addition?  
b) Is  $S$  closed under scalar multiplication?  
c) Is  $S$  a subspace of  $\mathbb{R}^2$ ?

### Solution

$$x^2 + y^2 = 0 \rightarrow x = y = 0 \quad (x, y \in \mathbb{R})$$

a) Let  $\vec{u} = (x_1, y_1) \Rightarrow x_1^2 + y_1^2 = 0$  &  $x_1 = y_1 = 0$ , and

$$\vec{v} = (x_2, y_2) \quad \ni \quad x_2^2 + y_2^2 = 0 \quad \& \quad x_2 = y_2 = 0$$

$$\vec{u} + \vec{v} = (x_1 + x_2, y_1 + y_2)$$

$$\begin{aligned} (x_1 + x_2)^2 + (y_1 + y_2)^2 &= x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2 \\ &= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2(x_1x_2 + y_1y_2) \\ &= (0) + (0) + 2(0+0) \quad x_i = y_i = 0 \\ &= 0 \end{aligned}$$

$S$  is closed under addition

$$\begin{aligned} b) \quad k\vec{u} &= k(x_1, y_1) \\ &= (kx_1, ky_1) \end{aligned}$$

$$\begin{aligned} (kx_1)^2 + (ky_1)^2 &= k^2x_1^2 + k^2y_1^2 \\ &= k^2(x_1^2 + y_1^2) \\ &= k^2(0) \\ &= 0 \end{aligned}$$

$S$  is closed under scalar multiplication

c) Since  $S$  is closed under addition and scalar multiplication, then  $S$  is a subspace of  $\mathbb{R}^2$ .

### Exercise

Let  $S = \{(x, y) \mid x^2 + y^2 = 0; \ x, y \in \mathbb{C}\}$ , Determine:

- Is  $S$  closed under addition?
- Is  $S$  closed under scalar multiplication?
- Is  $S$  a subspace of  $\mathbb{C}^2$ ?

### Solution

$$x^2 + y^2 = 0 \rightarrow x = \pm iy \quad (x, y \in \mathbb{C})$$

$$a) \quad \text{Let } \vec{u} = (x_1, y_1) \quad \ni \quad x_1^2 + y_1^2 = 0 \rightarrow x_1 = i y_1, \text{ and}$$

$$\vec{v} = (x_2, y_2) \quad \ni \quad x_2^2 + y_2^2 = 0 \rightarrow x_2 = -i y_2$$

$$\vec{u} + \vec{v} = (x_1 + x_2, y_1 + y_2)$$

$$\begin{aligned}
(x_1 + x_2)^2 + (y_1 + y_2)^2 &= x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2 \\
&= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2(x_1x_2 + y_1y_2) \\
&= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2(iy_1(-iy_2) + y_1y_2) \\
&= 0 + 0 + 2(-i^2y_1y_2 + y_1y_2) \\
&= 2(y_1y_2 + y_1y_2) \\
&= 4y_1y_2 \\
&\neq 0
\end{aligned}$$

$S$  is not closed under addition

$$\begin{aligned}
b) \quad k\vec{u} &= k(x_1, y_1) \\
&= (kx_1, ky_1)
\end{aligned}$$

$$\begin{aligned}
(kx_1)^2 + (ky_1)^2 &= k^2x_1^2 + k^2y_1^2 \\
&= k^2(x_1^2 + y_1^2) \\
&= k^2(0) \\
&= 0
\end{aligned}$$

$S$  is closed under scalar multiplication

c) Since  $S$  is *not* closed under addition, then  $S$  is **not** a subspace of  $\mathbb{C}^2$ .

### Exercise

Let  $S = \{(x, y) \mid x^2 - y^2 = 0; x, y \in \mathbb{R}\}$ , Determine:

- a) Is  $S$  closed under addition?
- b) Is  $S$  closed under scalar multiplication?
- c) Is  $S$  a subspace of  $\mathbb{R}^2$ ?

### Solution

$$x^2 - y^2 = 0 \rightarrow x = \pm y \quad (x, y \in \mathbb{R})$$

$$\begin{aligned}
a) \quad \text{Let } \vec{u} &= (x_1, y_1) \quad \ni \quad x_1^2 - y_1^2 = 0 \rightarrow x_1 = y_1, \text{ and} \\
\vec{v} &= (x_2, y_2) \quad \ni \quad x_2^2 - y_2^2 = 0 \rightarrow x_2 = -y_2
\end{aligned}$$



$$\begin{aligned}
\vec{u} + \vec{v} &= (x_1 + x_2, y_1 + y_2) \\
(x_1 + x_2)^2 - (y_1 + y_2)^2 &= x_1^2 + x_2^2 + 2x_1x_2 - y_1^2 - y_2^2 - 2y_1y_2 \\
&= (x_1^2 - y_1^2) + (x_2^2 - y_2^2) + 2(x_1x_2 - y_1y_2) \\
&= (0) + (0) + 2(y_1(-y_2) - y_1y_2) \\
&= 2(-y_1y_2 - y_1y_2) \\
&= -4y_1y_2 \\
&\neq 0
\end{aligned}$$

$S$  is not closed under addition

$$\begin{aligned}
b) \quad k\vec{u} &= k(x_1, y_1) \\
&= (kx_1, ky_1) \\
(kx_1)^2 - (ky_1)^2 &= k^2x_1^2 - k^2y_1^2 \\
&= k^2(x_1^2 - y_1^2) \\
&= k^2(0) \\
&= 0
\end{aligned}$$

$S$  is closed under scalar multiplication

c) Since  $S$  is *not* closed under addition, then  $S$  is **not** a subspace of  $\mathbb{R}^2$ .

### Exercise

Let  $S = \{(x, y) \mid x - y = 0; x, y \in \mathbb{R}\}$ , Determine:

- Is  $S$  closed under addition?
- Is  $S$  closed under scalar multiplication?
- Is  $S$  a subspace of  $\mathbb{R}^2$ ?

### Solution

$$x - y = 0 \rightarrow x = y \quad (x, y \in \mathbb{R})$$

a) Let  $\vec{u} = (x_1, y_1) \ni x_1 - y_1 = 0$ , and

$$\vec{v} = (x_2, y_2) \ni x_2 - y_2 = 0$$

$$\begin{aligned}
\vec{u} + \vec{v} &= (x_1 + x_2, y_1 + y_2) \\
(x_1 + x_2) - (y_1 + y_2) &= x_1 + x_2 - y_1 - y_2 \\
&= (x_1 - y_1) + (x_2 - y_2) \\
&= 0
\end{aligned}$$

$S$  is closed under addition

$$\begin{aligned}
b) \quad k\vec{u} &= k(x_1, y_1) \\
&= (kx_1, ky_1)
\end{aligned}$$

$$\begin{aligned}
kx_1 - ky_1 &= k(x_1 - y_1) \\
&= k(0) \\
&= 0
\end{aligned}$$

$S$  is closed under scalar multiplication

c) Since  $S$  is closed under addition and scalar multiplication, then  $S$  is a subspace of  $\mathbb{R}^2$ .

### Exercise

Let  $S = \{(x, y) \mid x - y = 1; x, y \in \mathbb{R}\}$ , Determine:

- a) Is  $S$  closed under addition?
- b) Is  $S$  closed under scalar multiplication?
- c) Is  $S$  a subspace of  $\mathbb{R}^2$ ?

### Solution

$$x - y = 0 \rightarrow x = y \quad (x, y \in \mathbb{R})$$

a) Let  $\vec{u} = (x_1, y_1) \ni x_1 - y_1 = 1$ , and

$$\vec{v} = (x_2, y_2) \ni x_2 - y_2 = 1$$

$$\begin{aligned}
\vec{u} + \vec{v} &= (x_1 + x_2, y_1 + y_2) \\
(x_1 + x_2) - (y_1 + y_2) &= x_1 + x_2 - y_1 - y_2 \\
&= (x_1 - y_1) + (x_2 - y_2) \\
&= 1 + 1 \\
&= 2 \neq 1
\end{aligned}$$

$S$  is not closed under addition

$$\begin{aligned} b) \quad k\vec{u} &= k(x_1, y_1) \\ &= (kx_1, ky_1) \end{aligned}$$

$$\begin{aligned} kx_1 - ky_1 &= k(x_1 - y_1) \\ &= k(1) \\ &= \underline{k \neq 1} \end{aligned}$$

$S$  is not closed under scalar multiplication

- c) Since  $S$  is not closed under addition and not closed scalar multiplication, then  $S$  is **not** a subspace of  $\mathbb{R}^2$ .

### Exercise

$V = \mathbb{R}^3$ ,  $S = \{(0, s, t) \mid s, t \text{ are real numbers}\}$  where  $V$  is a vector space and  $S$  is subset of  $V$

- a) Is  $S$  closed under addition?
- b) Is  $S$  closed under scalar multiplication?
- c) Is  $S$  a subspace of  $V$ ?

### Solution

$$\begin{aligned} d) \quad \text{Let } \vec{u} &= (0, s_1, t_1) \quad \text{and} \quad \vec{v} = (0, s_2, t_2) \\ \vec{u} + \vec{v} &= (0, s_1 + s_2, t_1 + t_2) \\ &= (0, s, t) \end{aligned}$$

Yes,  $S$  is closed under addition

$$\begin{aligned} e) \quad k\vec{u} &= (0, ks_1, kt_1) \\ &= (0, s, t) \end{aligned}$$

Yes,  $S$  is closed under scalar multiplication

- f) Since  $S$  is closed under addition and scalar multiplication, then  $S$  is a subspace of  $V$ .

### Exercise

$V = \mathbb{R}^3$ ,  $S = \{(x, y, z) \mid x, y, z \geq 0\}$  where  $V$  is a vector space and  $S$  is subset of  $V$

- a) Is  $S$  closed under addition?
- b) Is  $S$  closed under scalar multiplication?
- c) Is  $S$  a subspace of  $V$ ?

### Solution

a) Let  $\vec{u} = (x_1, y_1, z_1)$  and  $\vec{v} = (x_2, y_2, z_2)$

$$\begin{aligned}\vec{u} + \vec{v} &= (x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (x, y, z)\end{aligned}$$

where  $x = x_1 + x_2, y = y_1 + y_2, z = z_1 + z_2$

Yes,  $S$  is closed under addition

b)  $(-1)\vec{u} = (-x_1, -y_1, -z_1)$

$S$  is **not** closed under scalar multiplication since  $x_1 \geq 0 \Rightarrow -x_1 \leq 0$

c) Since  $S$  is closed under addition but it is not closed scalar multiplication, then  $S$  is **not** a subspace of  $V$ .

### Exercise

$V = \mathbb{R}^3$ ,  $S = \{(x, y, z) \mid z = x + y + 1\}$  where  $V$  is a vector space and  $S$  is subset of  $V$

a) Is  $S$  closed under addition?

b) Is  $S$  closed under scalar multiplication?

c) Is  $S$  a subspace of  $V$ ?

### Solution

a) Let  $\vec{u} = (0, 1, 2)$  and  $\vec{v} = (1, 2, 4)$

$$\vec{u} + \vec{v} = (1, 3, 6)$$

$$\neq (1, 3, 1+3+1)$$

$S$  is **not** closed under addition

b)  $k\vec{u} = (kx_1, ky_1, kz_1)$

$$= (kx_1, ky_1, k(x_1 + y_1 + 1))$$

$$= (kx_1, ky_1, kx_1 + ky_1 + k)$$

$$= (x, y, z)$$

Where  $x = kx_1, y = ky_1, z = k(x_1 + y_1 + 1)$

$S$  is closed under scalar multiplication

c) Since  $S$  is not closed under addition and closed scalar multiplication, then  $S$  is **not** a subspace of  $V$ .

### Exercise

Let  $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2 \right\}$ , Determine:

- a) Is  $S$  closed under addition?
- b) Is  $S$  closed under scalar multiplication?
- c) Is  $S$  a subspace of  $\mathbb{R}^3$ ?

### Solution

a) Let  $\vec{u} = (a_1, a_2, a_3)$  and  $\vec{v} = (b_1, b_2, b_3)$

$$\begin{aligned}\vec{u} + \vec{v} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) \\ &= (3a_2, a_2, -a_2) + (3b_2, b_2, -b_2) \\ &= (3a_2 + 3b_2, a_2 + b_2, -a_2 - b_2) \\ &= (3(a_2 + b_2), a_2 + b_2, -(a_2 + b_2)) \quad c_2 = a_2 + b_2 \\ &= (3c_2, c_2, -c_2) \\ &= (c_1, c_2, c_3) : c_1 = 3c_2 \quad c_3 = -c_2\end{aligned}$$

$S$  is closed under addition

b)  $k\vec{u} = k(a_1, a_2, a_3)$

$$\begin{aligned}&= k(3a_2, a_2, -a_2) \\ &= (3ka_2, ka_2, -ka_2) \quad c_2 = ka_2 \\ &= (3c_2, c_2, -c_2) \\ &= (c_1, c_2, c_3) : c_1 = 3c_2 \quad c_3 = -c_2\end{aligned}$$

$S$  is closed under scalar multiplication.

- c) Since  $S$  is closed under addition and scalar multiplication, then  $S$  is a subspace of  $V$ .

### Exercise

Let  $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2 \right\}$ , Determine:

- a) Is  $S$  closed under addition?
- b) Is  $S$  closed under scalar multiplication?
- c) Is  $S$  a subspace of  $\mathbb{R}^3$ ?

### Solution

a) Let  $\vec{u} = (2, 1, 0)$  and  $\vec{v} = (3, 0, 1)$   $a_1 = a_3 + 2$

$$\vec{u} + \vec{v} = (2, 1, 0) + (3, 0, 1)$$

$$= (5, 1, 1) \quad \begin{matrix} ? \\ 5 = 1 + 2 \end{matrix}$$

$$\neq (3, 1, 1)$$

$S$  is *not* closed under addition

b)  $k\vec{u} = k(a_1, a_2, a_3)$

$$= k(a_3 + 2, a_2, a_3)$$

$$= (ka_3 + 2k, ka_2, ka_3)$$

$$a_1 = a_3 + 2 \rightarrow ka_3 + 2k = a_3 + 2$$

$$2k \neq 2 \quad (\forall k)$$

$S$  is *not* closed under scalar multiplication.

- c) Since  $S$  is not closed under addition and not closed scalar multiplication, then  $S$  is **not** a subspace of  $V$ .

### Exercise

Let  $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0 \right\}$ , Determine:

- a) Is  $S$  closed under addition?
- b) Is  $S$  closed under scalar multiplication?
- c) Is  $S$  a subspace of  $\mathbb{R}^3$ ?

### Solution

a) Let  $\vec{u} = (a_1, a_2, a_3)$  and  $\vec{v} = (b_1, b_2, b_3)$

$$2a_1 - 7a_2 + a_3 = 0 \rightarrow a_3 = 7a_2 - 2a_1$$

$$\begin{aligned}
\vec{u} + \vec{v} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) \\
&= (a_1, a_2, 7a_2 - 2a_1) + (b_1, b_2, 7b_2 - 2b_1) \\
&= (a_1 + b_1, a_2 + b_2, 7a_2 - 2a_1 + 7b_2 - 2b_1) \\
&= (a_1 + b_1, a_2 + b_2, 7(a_2 + b_2) - 2(a_1 + b_1)) \quad \text{Let } c_1 = a_1 + b_1 \quad c_2 = a_2 + b_2 \\
&= (c_1, c_2, 7c_2 - 2c_1) \quad c_3 = 7c_2 - 2c_1 \rightarrow 2c_1 - 7c_2 + c_3 = 0 \\
&= (c_1, c_2, c_3)
\end{aligned}$$

$S$  is closed under addition

$$\begin{aligned}
b) \quad k\vec{u} &= k(a_1, a_2, a_3) \\
&= k(a_1, a_2, 7a_2 - 2a_1) \\
&= (ka_1, ka_2, 7ka_2 - 2ka_1) \quad \text{Let } c_1 = ka_1 \quad c_2 = ka_2 \\
&= (c_1, c_2, 7c_2 - 2c_1) \quad c_3 = 7c_2 - 2c_1 \rightarrow 2c_1 - 7c_2 + c_3 = 0 \\
&= (c_1, c_2, c_3)
\end{aligned}$$

$S$  is closed under scalar multiplication.

c) Since  $S$  is closed under addition and scalar multiplication, then  $S$  is a subspace of  $V$ .

### Exercise

Let  $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0 \right\}$ , Determine:

- Is  $S$  closed under addition?
- Is  $S$  closed under scalar multiplication?
- Is  $S$  a subspace of  $\mathbb{R}^3$ ?

### Solution

$$a_1 - 4a_2 - a_3 = 0 \rightarrow a_1 = 4a_2 + a_3$$

$$\begin{aligned}
a) \quad \text{Let } \vec{u} &= (a_1, a_2, a_3) \quad \text{and} \quad \vec{v} = (b_1, b_2, b_3) \\
\vec{u} + \vec{v} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) \\
&= (4a_2 + a_3, a_2, a_3) + (4b_2 + b_3, b_2, b_3) \\
&= (4a_2 + a_3 + 4b_2 + b_3, a_2 + b_2, a_3 + b_3)
\end{aligned}$$

$$\begin{aligned}
&= \left( 4(a_2 + b_2) + (a_3 + b_3), a_2 + b_2, a_3 + b_3 \right) \quad \text{Let } c_2 = a_2 + b_2 \quad c_3 = a_3 + b_3 \\
&= \left( 4c_2 + c_3, c_2, c_3 \right) \quad c_1 - 4c_2 - c_3 = 0 \rightarrow c_1 = 4c_2 + c_3 \\
&= \left( c_1, c_2, c_3 \right)
\end{aligned}$$

$S$  is closed under addition

$$\begin{aligned}
b) \quad k\vec{u} &= k(a_1, a_2, a_3) \\
&= k(4a_2 + a_3, a_2, a_3) \\
&= (4ka_2 + ka_3, ka_2, ka_3) \quad \text{Let } c_2 = ka_2 \quad c_3 = ka_3 \\
&= (4c_2 + c_3, c_2, c_3) \quad c_1 = 4c_2 + c_3 \rightarrow c_1 - 4c_2 - c_3 = 0 \\
&= (c_1, c_2, c_3)
\end{aligned}$$

$S$  is closed under scalar multiplication.

c) Since  $S$  is closed under addition and scalar multiplication, then  $S$  is a subspace of  $V$ .

### Exercise

Let  $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 0 \right\}$ , Determine:

- a) Is  $S$  closed under addition?
- b) Is  $S$  closed under scalar multiplication?
- c) Is  $S$  a subspace of  $\mathbb{R}^3$ ?

### Solution

$$a_1 + 2a_2 - 3a_3 = 0 \rightarrow a_1 = -2a_2 + 3a_3$$

$$\begin{aligned}
a) \quad \text{Let } \vec{u} &= (a_1, a_2, a_3) \quad \text{and} \quad \vec{v} = (b_1, b_2, b_3) \\
\vec{u} + \vec{v} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) \\
&= (-2a_2 + 3a_3, a_2, a_3) + (-2b_2 + 3b_3, b_2, b_3) \\
&= (-2a_2 + 3a_3 - 2b_2 + 3b_3, a_2 + b_2, a_3 + b_3) \\
&= \left( -2(a_2 + b_2) + 3(a_3 + b_3), a_2 + b_2, a_3 + b_3 \right) \quad \text{Let } c_2 = a_2 + b_2 \quad c_3 = a_3 + b_3 \\
&= (-2c_2 + 3c_3, c_2, c_3) \quad c_1 + 2c_2 - 3c_3 = 0 \rightarrow c_1 = -2c_2 + 3c_3 \\
&= (c_1, c_2, c_3)
\end{aligned}$$



$S$  is closed under addition

$$\begin{aligned} b) \quad k\vec{u} &= k(a_1, a_2, a_3) \\ &= k(4a_2 + a_3, a_2, a_3) \\ &= (-2ka_2 + 3ka_3, ka_2, ka_3) \quad \text{Let } c_2 = ka_2 \quad c_3 = ka_3 \\ &= (-2c_2 + 3c_3, c_2, c_3) \quad c_1 = -2c_2 + 3c_3 \rightarrow c_1 - 2c_2 + 3c_3 = 0 \\ &= (c_1, c_2, c_3) \end{aligned}$$

$S$  is closed under scalar multiplication.

c) Since  $S$  is closed under addition and scalar multiplication, then  $S$  is a subspace of  $V$ .

### Exercise

Let  $S = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1\}$ , Determine:

- a) Is  $S$  closed under addition?
- b) Is  $S$  closed under scalar multiplication?
- c) Is  $S$  a subspace of  $\mathbb{R}^3$ ?

### Solution

$$a_1 + 2a_2 - 3a_3 = 1 \rightarrow a_1 = 1 - 2a_2 + 3a_3$$

$$a) \quad \text{Let } \vec{u} = (a_1, a_2, a_3) \quad \text{and} \quad \vec{v} = (b_1, b_2, b_3)$$

$$\begin{aligned} \vec{u} + \vec{v} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) \\ &= (1 - 2a_2 + 3a_3, a_2, a_3) + (1 - 2b_2 + 3b_3, b_2, b_3) \\ &= (1 - 2a_2 + 3a_3 + 1 - 2b_2 + 3b_3, a_2 + b_2, a_3 + b_3) \\ &= (2 - 2(a_2 + b_2) + 3(a_3 + b_3), a_2 + b_2, a_3 + b_3) \quad \text{Let } c_2 = a_2 + b_2 \quad c_3 = a_3 + b_3 \\ &= (2 - 2c_2 + 3c_3, c_2, c_3) \quad c_1 + 2c_2 - 3c_3 = 1 \rightarrow c_1 = 1 - 2c_2 + 3c_3 \\ &\neq (1 - 2c_2 + 3c_3, c_2, c_3) \end{aligned}$$

$S$  is *not* closed under addition

$$b) \quad \vec{u} = (2, 1, 1)$$

$$k\vec{u} = k(2, 1, 1)$$

$$= (2k, k, k) \quad a_1 + 2a_2 - 3a_3 = 1 \xrightarrow{?} 2k + 2k - 3k = 1$$

$$k \neq 1 \quad (\forall k)$$

$S$  is *not* closed under scalar multiplication.

- c) Since  $S$  is not closed under addition and not closed scalar multiplication, then  $S$  is **not** a subspace of  $V$ .

### Exercise

Let  $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0 \right\}$ , Determine:

- a) Is  $S$  closed under addition?  
b) Is  $S$  closed under scalar multiplication?  
c) Is  $S$  a subspace of  $\mathbb{R}^3$ ?

### Solution

$$5a_1^2 - 3a_2^2 + 6a_3^2 = 0 \rightarrow a_2^2 = \frac{5}{3}a_1^2 + 2a_3^2$$

a) Let  $\vec{u} = (0, \sqrt{2}, 1)$  and  $\vec{v} = (3, \sqrt{17}, 1)$

$$\begin{aligned} \vec{u} + \vec{v} &= (0, \sqrt{2}, 1) + (3, \sqrt{17}, 1) \\ &= (3, \sqrt{2} + \sqrt{17}, 2) \end{aligned}$$

$$a_2^2 = \frac{5}{3}a_1^2 + 2a_3^2 \rightarrow (\sqrt{2} + \sqrt{17})^2 \neq 15 + 8$$

$S$  is *not* closed under addition

b)  $k\vec{u} = k(a_1, a_2, a_3)$

$$= (ka_1, ka_2, ka_3)$$

$$5(ka_1)^2 - 3(ka_2)^2 + 6(ka_3)^2 = 0$$

$$5k^2a_1^2 - 3k^2a_2^2 + 6k^2a_3^2 = 0$$

$$5a_1^2 - 3a_2^2 + 6a_3^2 = 0$$

$S$  is closed under scalar multiplication.

- c) Since  $S$  is *not* closed under addition and is closed scalar multiplication, then  $S$  is **not** a subspace of  $V$ .

### Exercise

Let  $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_3 = a_1 + a_2 \right\}$ , Determine:

- a) Is  $S$  closed under addition?
- b) Is  $S$  closed under scalar multiplication?
- c) Is  $S$  a subspace of  $\mathbb{R}^3$ ?

### Solution

a) Let  $\vec{u} = (a_1, a_2, a_3)$  and  $\vec{v} = (b_1, b_2, b_3)$

$$\begin{aligned}\vec{u} + \vec{v} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) \\ &= (a_1, a_2, a_1 + a_2) + (b_1, b_2, b_1 + b_2) \\ &= (a_1 + b_1, a_2 + b_2, a_1 + a_2 + b_1 + b_2) \quad \text{Let } c_1 = a_1 + b_1 \quad c_2 = a_2 + b_2 \\ &= (c_1, c_2, c_1 + c_2) \quad \text{Then, } c_3 = c_1 + c_2 \\ &= (c_1, c_2, c_3)\end{aligned}$$

$S$  is closed under addition

b)  $k\vec{u} = k(a_1, a_2, a_3)$

$$\begin{aligned}&= k(a_1, a_2, a_1 + a_2) \\ &= (ka_1, ka_2, k(a_1 + a_2)) \\ &= (ka_1, ka_2, ka_1 + ka_2) \quad \text{Where } c_1 = ka_1, \quad c_2 = ka_2, \quad c_3 = ka_1 + ka_2 \\ &= (c_1, c_2, c_3) \quad c_3 = ka_1 + ka_2 = c_1 + c_2\end{aligned}$$

$S$  is closed under scalar multiplication.

- c) Since  $S$  is closed under addition and scalar multiplication, then  $S$  is a subspace of  $V$ .

### Exercise

Let  $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + a_2 + a_3 = 0 \right\}$ , Determine:

- a) Is  $S$  closed under addition?
- b) Is  $S$  closed under scalar multiplication?
- c) Is  $S$  a subspace of  $\mathbb{R}^3$ ?

### Solution

a) Let  $\vec{u} = (a_1, a_2, a_3) \ni a_1 + a_2 + a_3 = 0$

$\vec{v} = (b_1, b_2, b_3) \ni b_1 + b_2 + b_3 = 0$

$$\begin{aligned}\vec{u} + \vec{v} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) \\ &= (a_1 + b_1, a_2 + b_2, a_3 + b_3)\end{aligned}$$

Since  $a_1 + a_2 + a_3 = 0$  &  $b_1 + b_2 + b_3 = 0$

Then,  $\rightarrow (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) = 0$

$S$  is closed under addition

b)  $k\vec{u} = k(a_1, a_2, a_3)$   
 $= (ka_1, ka_2, ka_3)$

$ka_1 + ka_2 + ka_3 = k(a_1 + a_2 + a_3) = k(0) = 0$

$S$  is closed under scalar multiplication.

c) Since  $S$  is closed under addition and scalar multiplication, then  $S$  is a subspace of  $V$ .

### Exercise

Let  $S = \{(x_1, x_2, 1) : x_1 \text{ and } x_2 \text{ are real numbers}\}$ , Determine:

- a) Is  $S$  closed under addition?
- b) Is  $S$  closed under scalar multiplication?
- c) Is  $S$  a subspace of  $\mathbb{R}^3$ ?

### Solution

a) Let  $\vec{u} = (x_1, x_2, 1)$  &  $\vec{v} = (y_1, y_2, 1)$

$$\begin{aligned}\vec{u} + \vec{v} &= (x_1, x_2, 1) + (y_1, y_2, 1) \\ &= (x_1 + y_1, x_2 + y_2, 2) \quad \text{If we let } z_1 = x_1 + y_1 \quad z_2 = x_2 + y_2 \\ &= (z_1, z_2, 2) \\ &\neq (z_1, z_2, 1)\end{aligned}$$

$S$  is **not** closed under addition

b)  $k\vec{u} = k(x_1, x_2, 1)$

$$= (kx_1, kx_2, k) \quad \text{If we let } z_1 = kx_1 \quad z_2 = kx_2$$

$$\neq (z_1, z_2, 1) \quad k \neq 1 \quad (\forall k)$$

$S$  is **not** closed under scalar multiplication.

- c) Since  $S$  is *not* closed under addition and is *not* closed scalar multiplication, then  $S$  is **not** a subspace of  $V$ .

### ***Exercise***

Let  $S = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 = x_1 + 2x_3 \right\}$ , Determine:

- a) Is  $S$  closed under addition?  
b) Is  $S$  closed under scalar multiplication?  
c) Is  $S$  a subspace of  $\mathbb{R}^3$ ?

### **Solution**

a) Let  $\vec{u} = (x_1, x_2, x_3)$  &  $\vec{v} = (y_1, y_2, y_3)$

$$\vec{u} + \vec{v} = (x_1, x_2, x_3) + (y_1, y_2, y_3)$$

$$= (x_1 + y_1, x_2 + y_2, x_3 + y_3) \quad S = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 = x_1 + 2x_3 \right\}$$

$$x_2 + y_2 = x_1 + 2x_3 + y_1 + 2y_3$$

$$= x_1 + y_1 + 2(x_3 + y_3)$$

$S$  is closed under addition

b)  $k\vec{u} = k(x_1, x_2, x_3)$

$$= (kx_1, kx_2, kx_3) \quad \text{If we let } z_1 = kx_1 \quad z_2 = kx_2$$

$$kx_2 = kx_1 + 2kx_3$$

$$kx_2 = k(x_1 + 2x_3)$$

$$x_2 = x_1 + 2x_3$$

$S$  is closed under scalar multiplication.

- c) Since  $S$  is closed under addition and scalar multiplication, then  $S$  is a subspace of  $V$ .

### Exercise

Let  $S = \left\{ \begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \in M_{2 \times 2} \mid a, c, d \in \mathbb{R} \right\}$  and  $V = M_{2,2}$ , Determine:

- a) Is  $S$  closed under addition?
- b) Is  $S$  closed under scalar multiplication?
- c) Is  $S$  a subspace of  $V$ ?

### Solution

a) Let  $A = \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$  &  $B = \begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix}$

$$\begin{aligned} A + B &= \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + a_2 & 2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix} && \text{If we let } a = a_1 + a_2 \quad c = c_1 + c_2 \quad d = d_1 + d_2 \\ &= \begin{pmatrix} a & 2 \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \end{aligned}$$

$S$  is **not** closed under addition

b)  $kA = k \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$

$$\begin{aligned} &= \begin{pmatrix} ka_1 & k \\ kc_1 & kd_1 \end{pmatrix} && \text{If we let } a = ka_1 \quad c = kc_1 \quad d = kd_1 \\ &= \begin{pmatrix} a & k \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix} && k \neq 1 \quad (\forall k) \end{aligned}$$

$S$  is **not** closed under scalar multiplication.

- c) Since  $S$  is **not** closed under addition and is **not** closed scalar multiplication, then  $S$  is **not** a subspace of  $V$ .

### Exercise

Let  $S = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in M_{2 \times 2} \mid a, c, d \in \mathbb{R} \right\}$  and  $V = M_{2,2}$ , Determine:

- a) Is  $S$  closed under addition?
- b) Is  $S$  closed under scalar multiplication?
- c) Is  $S$  a subspace of  $V$ ?

### Solution

a) Let  $A = \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$  &  $B = \begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix}$

$$A + B = \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + a_2 & 2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$

$$= \begin{pmatrix} a & 2 \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$$

If we let  $a = a_1 + a_2$   $c = c_1 + c_2$   $d = d_1 + d_2$

$S$  is **not** closed under addition

b)  $kA = k \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$

$$= \begin{pmatrix} ka_1 & k \\ kc_1 & kd_1 \end{pmatrix}$$

If we let  $a = ka_1$   $c = kc_1$   $d = kd_1$

$$= \begin{pmatrix} a & k \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$$

$k \neq 1 \quad (\forall k)$

$S$  is **not** closed under scalar multiplication.

- c) Since  $S$  is **not** closed under addition and is **not** closed scalar multiplication, then  $S$  is **not** a subspace of  $V$ .

### Exercise

Let  $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in M_{2 \times 2} \mid a, d \in \mathbb{R} \text{ \& } ad \geq 0 \right\}$  and  $V = M_{2,2}$ , Determine:

- a) Is  $S$  closed under addition?
- b) Is  $S$  closed under scalar multiplication?
- c) Is  $S$  a subspace of  $V$ ?

### Solution

a) Let  $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \rightarrow 1(2) > 0$  &  $B = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow (-2)(-1) > 0$

$$A + B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \\ = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$ad \geq 0 \rightarrow (-1)(1) = -1 < 0$$

$S$  is **not** closed under addition

b)  $kA = k \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \\ = \begin{pmatrix} ka & 0 \\ 0 & kd \end{pmatrix}$

$$(ka)(kd) = k^2(ad)$$

$$\text{Since, } ad \geq 0 \text{ \& } k^2 \geq 0$$

$$k^2(ad) \geq 0$$

$S$  is closed under scalar multiplication.

c) Since  $S$  is **not** closed under addition and is closed scalar multiplication, then  $S$  is **not** a subspace of  $V$ .

### Exercise

$V = M_{33}$ ,  $S = \{A \mid A \text{ is invertible}\}$  where  $V$  is a vector space and  $S$  is subset of  $V$

- a) Is  $S$  closed under addition?
- b) Is  $S$  closed under scalar multiplication?
- c) Is  $S$  a subspace of  $V$ ?

### Solution

a) Let assume:  $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$  are invertible

$$\text{But } A + B = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \text{ is not invertible.}$$

$S$  is **not** closed under addition

b)  $S$  is **not** closed under scalar multiplication if  $k = 0$

c) Since  $S$  is **not** closed under addition and is **not** closed scalar multiplication, then  $S$  is **not** a subspace of  $V$ .



### Exercise

Let  $S = \{p(t) = a + 2at + 3at^3 \mid a \in \mathbb{R} \text{ \& } p(t) \in P_2\}$  and  $V = P_2$ , Determine:

- a) Is  $S$  closed under addition?
- b) Is  $S$  closed under scalar multiplication?
- c) Is  $S$  a subspace of  $V$ ?

### Solution

a) Let  $p_1(t) = a + 2at + 3at^3$  &  $p_2(t) = b + 2bt + 3bt^3$

$$\begin{aligned} p_1(t) + p_2(t) &= a + 2at + 3at^3 + b + 2bt + 3bt^3 \\ &= (a+b) + 2(a+b)t + 3(a+b)t^3 && \text{Let } c = a+b \in \mathbb{R} \\ &= c + 2ct + 3ct^3 \end{aligned}$$

$S$  is closed under addition

b)  $kp_1(t) = k(a + 2at + 3at^3)$

$$\begin{aligned} &= ka + 2kat + 3kat^3 && \text{Let } c = ka \in \mathbb{R} \\ &= c + 2ct + 3ct^3 \end{aligned}$$

$S$  is closed under scalar multiplication.

- c) Since  $S$  is closed under addition and is closed scalar multiplication, then  $S$  is a subspace of  $V$ .

### Exercise

Let  $S = \{p(t) \mid p(t) \in P[t] \text{ has degree } 3\}$ , Determine:

- a) Is  $S$  closed under addition?
- b) Is  $S$  closed under scalar multiplication?
- c) Is  $S$  a subspace of  $P[t]$ ?

### Solution

a) Let  $p_1(t) = at^3 + b_1t^2 + c_1t + d_1$  &  $p_2(t) = -at^3 + b_2t^2 + c_2t + d_2$

$$\begin{aligned} p_1(t) + p_2(t) &= at^3 + b_1t^2 + c_1t + d_1 - at^3 + b_2t^2 + c_2t + d_2 \\ &= (b_1 + b_2)t^2 + (c_1 + c_2)t + (d_1 + d_2) \\ &= bt^2 + ct + d \end{aligned}$$

Has no 3<sup>rd</sup> degree polynomial.

$S$  is *not* closed under addition

$$\begin{aligned} b) \quad kp_1(t) &= k(at^3 + b_1t^2 + c_1t + d_1) \\ &= kat^3 + kb_1t^2 + kc_1t + kd_1 \\ &= k_1t^3 + k_2t^2 + k_3t + k_4 \end{aligned}$$

It is 3<sup>rd</sup> degree polynomial.

$S$  is closed under scalar multiplication.

c) Since  $S$  is *not* closed under addition and is closed scalar multiplication, then  $S$  is **not** a subspace of  $V$ .

### Exercise

Let  $S = \{p(t) \mid p(0) = 0, p(t) \in \mathbf{P}[t]\}$ , Determine:

- d) Is  $S$  closed under addition?
- e) Is  $S$  closed under scalar multiplication?
- f) Is  $S$  a subspace of  $\mathbf{P}[t]$ ?

### Solution

$$\begin{aligned} a) \quad \text{Let } p_1(t) &= a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t \Rightarrow p_1(0) = 0 \\ p_2(t) &= b_n t^n + b_{n-1} t^{n-1} + \dots + b_1 t \Rightarrow p_2(0) = 0 \\ p_1(t) + p_2(t) &= a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + b_n t^n + b_{n-1} t^{n-1} + \dots + b_1 t \\ &= (a_n + b_n) t^n + (a_{n-1} + b_{n-1}) t^{n-1} + \dots + (a_1 + b_1) t \end{aligned}$$

$$p_1(0) + p_2(0) = 0$$

$S$  is closed under addition

$$\begin{aligned} b) \quad kp_1(t) &= k(a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t) \\ &= ka_n t^n + ka_{n-1} t^{n-1} + \dots + ka_1 t \\ &= k_1 t^3 + k_2 t^2 + k_3 t + k_4 \end{aligned}$$

$$kp_1(0) = 0$$

$S$  is closed under scalar multiplication.

c) Since  $S$  is closed under addition and is closed scalar multiplication, then  $S$  is a subspace of  $V$ .

### Exercise

Given:  $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix}$

- a) Find  $NS(A)$
- b) For which  $n$  is  $NS(A)$  a subspace of  $\mathbb{R}^n$
- c) Sketch  $NS(A)$  in  $\mathbb{R}^2$  or  $\mathbb{R}^3$

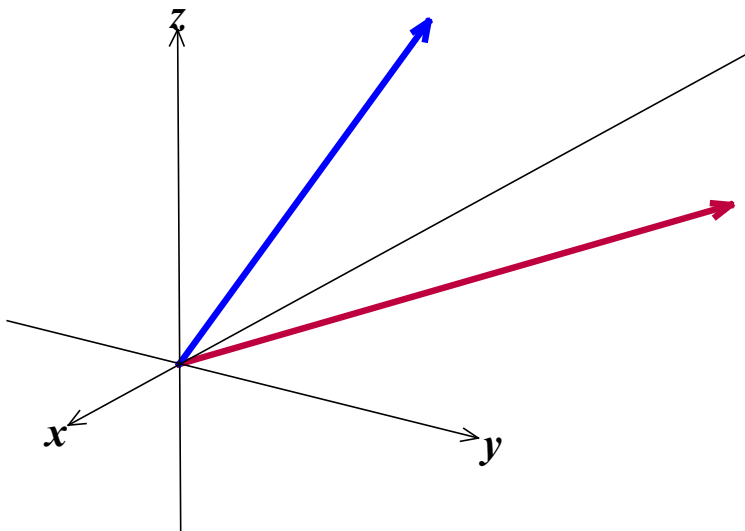
### Solution

a)  $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad x = -3y - 2z$

$$\left\{ y \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \mid y, z \in \mathbb{R} \right\}$$

b)  $n = 3$

c)



### Exercise

Determine which of the following are subspaces of  $M_{22}$

- a) All  $2 \times 2$  matrices with integer entries  
b) All matrices  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  where  $a + b + c + d = 0$

### Solution

a) Let  $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$  and  $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

where  $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$  are integers.

$$A + B = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix}$$

where  $a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4$  are integers too.

Then, it is closed under addition.

$$\begin{aligned} \frac{1}{2}A &= \frac{1}{2} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix} \end{aligned}$$

It is not closed under multiplication if the scalar is a real number.

Therefore; it is **not** a subspace of  $M_{22}$

b) Let  $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$   $a_1 + a_2 + a_3 + a_4 = 0$

and  $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$   $b_1 + b_2 + b_3 + b_4 = 0$

$$A + B = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix}$$

$$a_1 + a_2 + a_3 + a_4 + b_1 + b_2 + b_3 + b_4 = 0$$

$$(a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + (a_4 + b_4) = 0$$

Then, it is closed under addition.

$$kA = \begin{bmatrix} ka_1 & ka_2 \\ ka_3 & ka_4 \end{bmatrix}$$

$$ka_1 + ka_2 + ka_3 + ka_4 = k(a_1 + a_2 + a_3 + a_4) = k(0) = 0$$

It is closed under multiplication

Therefore; it is a subspace of  $M_{22}$

### ***Exercise***

Let  $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1 \right\}$ . Is  $V$  a vector space?

### **Solution**

$$k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

$$\begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = k^2 ad - k^2 bc$$

$$= k^2 (ad - bc)$$

$$= k^2 \neq k$$

$\therefore V$  is *not* a vector space

### ***Exercise***

Let  $V = \{(x, 0, y) : x \text{ \& } y \text{ are arbitrary } \mathbb{R}\}$ . Define addition and scalar multiplication as follows:

$$\begin{cases} (x_1, 0, y_1) + (x_2, 0, y_2) = (x_1 + x_2, y_1 + y_2) \\ c(x, 0, y) = (cx, cy) \end{cases}$$

Is  $V$  a vector space?

### **Solution**

$$\text{Let } \vec{v}_1(x_1, 0, y_1) \text{ \& } \vec{v}_2(x_2, 0, y_2)$$

$$\vec{v}_1 + \vec{v}_2 = (x_1, 0, y_1) + (x_2, 0, y_2)$$

$$= (x_1 + x_2, y_1 + y_2)$$

$$\neq (x_1 + x_2, 0, y_1 + y_2)$$

$$= \vec{V}_1 + \vec{V}_2$$

$\therefore V$  is *not* a vector space

### Exercise

Construct a matrix whose column space contains  $(1, 1, 0)$ ,  $(0, 1, 1)$ , and whose nullspace contains  $(1, 0, 1)$  and  $(0, 0, 1)$

### Solution

It is *not* possible.

Since a matrix  $(A)$  must be  $3 \times 3$ .

Since the nullspace contains 2 independent vectors, then  $A$  can have at most  $3 - 2 = 1$  pivot.

But the column space contains 2 independent vectors,  $A$  must have at least 2 pivots.

These 2 conditions can't both be met.

### Exercise

How is the nullspace  $N(C)$  related to the spaces  $N(A)$  and  $N(B)$ , is  $C = \begin{bmatrix} A \\ B \end{bmatrix}$ ?

### Solution

$$N(C) = N(A) \cap N(B)$$

$$Cx = \begin{bmatrix} Ax \\ Bx \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Iff } Ax = 0 \quad \& \quad Bx = 0$$

### Exercise

True or False (check addition or give a counterexample)

- If  $V$  is a vector space and  $W$  is a subset of  $V$  that is a vector space, then  $W$  is a subspace of  $V$ .
- The empty set is a subspace of every vector space.
- If  $V$  is a vector space other than the zero vector space, then  $V$  contains a subspace  $W$  such that  $W \neq V$ .
- The intersection of any two subsets of  $V$  is a subspace of  $V$ .
- Let  $W$  be the  $xy$ -plane in  $\mathbb{R}^3$ ; that is,  $W = \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\}$ . Then  $W = \mathbb{R}^2$

### Solution

a) False

$W$  is a subset of  $V$ , but not necessary that the scalar of a vector in  $W$  is in  $V$ .

Therefore,  $W$  is *not* a subspace of  $V$

**b)** False

Since not every subspace has an empty space, example  $\mathbb{R}$

**c)** True

If  $V$  is a vector space in  $\mathbb{R}^n$  and  $W$  is a vector space in  $\mathbb{Z}^n$ . Then  $V$  contains a subspace  $W$  and  $W \neq V$

**d)** False

**e)** False

### ***Exercise***

Let  $A\vec{x} = \vec{0}$  be a homogeneous system of  $n$  linear equations in  $n$  unknowns that has only the trivial solution. Show that if  $k$  is any positive integer, then the system  $A^k \vec{x} = \vec{0}$  also has only trivial solution.

#### **Solution**

Since  $A$  is a square matrix, thus  $A$  has only the trivial solution that implies  $A$  is invertible.

But  $A^k$  is also invertible so  $A^k \vec{x} = \vec{0}$  has only trivial solution.

### ***Exercise***

Let  $A\vec{x} = \vec{0}$  be a homogeneous system of  $n$  linear equations in  $n$  unknowns and let  $Q$  be an invertible  $n \times n$  matrix. Show that  $A\vec{x} = \vec{0}$  has just trivial solution if and only if  $(QA)\vec{x} = \vec{0}$  has just trivial solution.

#### **Solution**

Since  $A$  is a square matrix  $n \times n$ . If  $A\vec{x} = \vec{0}$  has just trivial solution, then  $A$  is invertible. Since  $Q$  is an invertible  $n \times n$  matrix that implies  $QA$  is also invertible. Thus,  $(QA)\vec{x} = \vec{0}$  has trivial solution.

On the other hand, if  $(QA)\vec{x} = \vec{0}$  has trivial solution then  $QA$  is invertible.

Since  $Q$  is invertible that implies  $Q^{-1}$  is also invertible.

Thus,  $A = Q^{-1}QA$  is invertible i.e.  $A\vec{x} = \vec{0}$  has just trivial solution.

$A\vec{x} = \vec{0}$  has just trivial solution *iff*  $(QA)\vec{x} = \vec{0}$  has just trivial solution.

### Exercise

Let  $A\vec{x} = \vec{b}$  be a consistent system of linear equations and let  $\vec{x}_1$  be a fixed solution. Show that every solution to the system can be written in the form  $\vec{x} = \vec{x}_1 + \vec{x}_0$  where  $\vec{x}_0$  is a solution to  $A\vec{x} = \vec{0}$ . Show also that every matrix of this form is a solution.

### Solution

Since  $\vec{x}_0$  is a solution to  $A\vec{x} = \vec{0}$ , we have  $A\vec{x}_0 = \vec{0}$ .

The sum of  $A\vec{x}_0 = \vec{0}$  and  $A\vec{x} = \vec{b}$

$$\begin{array}{r} A\vec{x}_0 = \vec{0} \\ + \quad A\vec{x} = \vec{b} \\ \hline A\vec{x}_0 + A\vec{x} = \vec{0} + \vec{b} \end{array}$$

$$A(\vec{x} + \vec{x}_0) = \vec{b}$$

As adding an equation to the original equation does not affect the solution.

If we let  $\vec{x}_1$  be a fixed solution, then every solution to  $A\vec{x} = \vec{b}$  is  $\vec{x} = \vec{x}_1 + \vec{x}_0$

Besides that

$$\begin{aligned} A(\vec{x}_1 + \vec{x}_0) &= A\vec{x}_1 + A\vec{x}_0 \\ &= \vec{b} + \vec{0} \\ &= \vec{b} \end{aligned}$$

So, every matrix (vector) in the form  $\vec{x}_1 + \vec{x}_0$  is a solution to  $A\vec{x} = \vec{b}$ .



## ***Solution***      **Section 2.6 – Linear Independence**

### ***Exercise***

State the following statements as true or false

- a) If  $S$  is a linearly dependent set, then each vector in  $S$  is a linear combination of other vectors in  $S$ .
- b) Any set containing the zero vector is linearly dependent.
- c) The empty set is linearly dependent.
- d) Subsets of linearly dependent sets are linearly dependent.
- e) Subsets of linearly independent sets are linearly independent.
- f) If  $a_1 \vec{x}_1 + a_2 \vec{x}_2 + \dots + a_n \vec{x}_n = \vec{0}$  and  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$  are linearly independent, then the scalars  $a_i$  are zero

### **Solution**

- a) False
- b) True
- c) False
- d) False
- e) True
- f) True

### ***Exercise***

Given three independent vectors  $\vec{w}_1, \vec{w}_2, \vec{w}_3$ . Take combinations of those vectors to produce  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ . Write the combinations in a matrix form as  $V = WM$ .

$$\vec{v}_1 = \vec{w}_1 + \vec{w}_2$$

$$\vec{v}_2 = \vec{w}_1 + 2\vec{w}_2 + \vec{w}_3$$

$$\vec{v}_3 = \vec{w}_2 + c\vec{w}_3$$

$$\text{which is } \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix}$$

What is the test on a matrix  $V$  to see if its columns are linearly independent?

If  $c \neq 1$  show that  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are linearly independent.

If  $c = 1$  show that  $\vec{v}$ 's are linearly *dependent*.

### **Solution**

The nullspace of  $\mathbf{V}$  must contain only the *zero* vector. Then  $\vec{x} = (0, 0, 0)$  is the only combination of the columns that gives  $\mathbf{V}\vec{x} = \text{zero vector}$ .

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix} \quad \begin{array}{l} R_2 - R_1 \\ R_1 - R_2 \\ R_3 - R_2 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & c \end{bmatrix} \quad \begin{array}{l} R_1 - R_2 \\ R_3 - R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & c-1 \end{bmatrix}$$

If  $c \neq 1$ , then the matrix  $M$  is invertible. So if  $x$  is any nonzero vector we know that  $Mx$  is nonzero. Since  $\vec{w}$ 's are given as independent and  $WM\vec{x}$  is nonzero. Since  $V = WM$ , this says that  $x$  is not in the nullspace of  $\mathbf{V}$ . therefore;  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are independent.

If  $c = 1$ , that implies

$$\begin{cases} \vec{v}_1 = \vec{w}_1 + \vec{w}_2 \\ \vec{v}_2 = \vec{w}_1 + \vec{w}_2 + \vec{w}_2 + \vec{w}_3 \\ \vec{v}_3 = \vec{w}_2 + \vec{w}_3 \end{cases} \Rightarrow \begin{cases} \vec{v}_1 = \vec{w}_1 + \vec{w}_2 \\ \boxed{\vec{v}_2 = \vec{v}_1 + \vec{v}_3} \\ \vec{v}_3 = \vec{w}_2 + \vec{w}_3 \end{cases}$$

$-\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = 0$ , which means that  $\vec{v}$ 's are linearly *dependent*.

The other way, the vector  $x = (1, -1, 1)$  is in that nullspace, and  $M\vec{x} = 0$ . Then certainly  $WM\vec{x} = 0$  which is the same as  $V\vec{x} = 0$ . So, the  $\vec{v}$ 's are dependent.

## Exercise

Find the largest possible number of independent vectors among

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \vec{v}_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad \vec{v}_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad \vec{v}_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

## Solution

Since  $\vec{v}_4 = \vec{v}_2 - \vec{v}_1$ ,  $\vec{v}_5 = \vec{v}_3 - \vec{v}_1$ , and  $\vec{v}_6 = \vec{v}_3 - \vec{v}_2$ , there are at most three

independent vectors among these: furthermore, applying row reduction to the matrix

$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$  gives three pivots, showing that  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are independent.

### Exercise

Show that  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are independent but  $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$  are dependent:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \vec{v}_4 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

Solve either  $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$  *or*  $A\vec{x} = \vec{0}$ . The  $v$ 's go in the columns of  $A$ .

### Solution

$$\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

This matrix has 3 pivots with rank of 3 equals to rows that implies the  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are independent.

$$\vec{v}_4 = \vec{v}_1 + \vec{v}_2 - 4\vec{v}_3 \quad \text{or} \quad \vec{v}_1 + \vec{v}_2 - 4\vec{v}_3 - \vec{v}_4 = \vec{0}$$

That shows that  $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$  are dependent.

### Exercise

Decide the dependence or independence of

- a) The vectors  $(1, 3, 2)$ ,  $(2, 1, 3)$ , and  $(3, 2, 1)$ .
- b) The vectors  $(1, -3, 2)$ ,  $(2, 1, -3)$ , and  $(-3, 2, 1)$ .

### Solution

- a) These are *linearly independent*.

$$x_1(1, 3, 2) + x_2(2, 1, 3) + x_3(3, 2, 1) = (0, 0, 0) \text{ only if } x_1 = x_2 = x_3 = 0$$

- b) These are *linearly dependent*:

$$1(1, -3, 2) + 1(2, 1, -3) + 1(-3, 2, 1) = (0, 0, 0)$$

### Exercise

Find two independent vectors on the plane  $x + 2y - 3z - t = 0$  in  $\mathbb{R}^4$ . Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

### Solution

This plane is the nullspace of the matrix  $A = \begin{pmatrix} 1 & 2 & -3 & -1 \end{pmatrix}$

$$x_1 + 2x_2 - 3x_3 - x_4 = 0$$

The pivot is 1<sup>st</sup> column, and the rest are 3 variables.

If  $x_2 = -1$   $x_3 = x_4 = 0 \Rightarrow x_1 = 2$ . The vector is  $(2, -1, 0, 0)$

If  $x_3 = 1$   $x_1 = x_4 = 0 \Rightarrow x_1 = 3$ . The vector is  $(3, 0, 1, 0)$

If  $x_4 = 1$   $x_1 = x_3 = 0 \Rightarrow x_1 = 1$ . The vector is  $(1, 0, 0, 1)$

The 3 vectors  $(2, -1, 0, 0)$ ,  $(3, 0, 1, 0)$ ,  $(1, 0, 0, 1)$  are linearly independent.

We can't find 4 independent vectors because the nullspace only has dimension 3 (have 3 variables).

### Exercise

Determine whether the vectors are linearly dependent or linearly independent in  $\mathbb{R}^3$

a)  $(4, -1, 2), (-4, 10, 2)$

c)  $(-3, 0, 4), (5, -1, 2), (1, 1, 3)$

b)  $(8, -1, 3), (4, 0, 1)$

d)  $(-2, 0, 1), (3, 2, 5), (6, -1, 1), (7, 0, -2)$

### Solution

a) The vector equation  $a(4, -1, 2) + b(-4, 10, 2) = (0, 0, 0)$

$$\left[ \begin{array}{cc|c} 4 & -4 & 0 \\ -1 & 10 & 0 \\ 2 & 2 & 0 \end{array} \right] \quad \frac{1}{4}R_1$$

$$\left[ \begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 10 & 0 \\ 2 & 2 & 0 \end{array} \right] \quad \begin{array}{l} R_2 + R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[ \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad 9R_1 + R_2$$

$$\left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad \frac{1}{9}R_2$$

$$\left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore, the system has only the trivial solution  $a = b = 0$ .

We conclude that the given set of vectors is linearly independent.

**b)**  $A(8, -1, 3) + b(4, 0, 1) = (0, 0, 0)$

$$\left[ \begin{array}{cc|c} 8 & 4 & 0 \\ -1 & 0 & 0 \\ 3 & 1 & 0 \end{array} \right] \quad \frac{1}{8}R_1$$

$$\left[ \begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ -1 & 0 & 0 \\ 3 & 1 & 0 \end{array} \right] \quad \begin{array}{l} R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[ \begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 \end{array} \right] \quad \begin{array}{l} R_1 - R_2 \\ R_3 + R_2 \end{array}$$

$$\left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array} \right] \quad 2R_2$$

$$\left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore, the system has only one trivial solution  $a = b = 0$ .

We conclude that the given set of vectors is linearly independent

**c)** The vector equation:

$$a(-3, 0, 4) + b(5, -1, 2) + c(1, 1, 3) = (0, 0, 0)$$

$$\left[ \begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{array} \right] \quad 3R_3 + 4R_1$$

$$\left[ \begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 26 & 13 & 0 \end{array} \right] \quad -R_2$$

$$\left[ \begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 26 & 13 & 0 \end{array} \right] \quad \begin{array}{l} R_1 - 5R_2 \\ R_3 - 26R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} -3 & 0 & 6 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 39 & 0 \end{array} \right] \quad \frac{1}{39}R_3$$

$$\left[ \begin{array}{ccc|c} -3 & 0 & 6 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \quad \begin{array}{l} R_1 - 6R_3 \\ R_2 + R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \quad -\frac{1}{3}R_1$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Therefore, the system has only the trivial solution  $a = b = c = 0$ .

We conclude that the given set of vectors is linearly independent.

**d)** The vector equation:

$$a(-2, 0, 1) + b(3, 2, 5) + c(6, -1, 1) + d(7, 0, -2) = (0, 0, 0)$$

$$\left[ \begin{array}{cccc|c} -2 & 3 & 6 & 7 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 1 & 5 & 1 & -2 & 0 \end{array} \right] \quad 2R_3 + R_1$$

$$\left[ \begin{array}{cccc|c} -2 & 3 & 6 & 7 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 13 & 8 & 3 & 0 \end{array} \right] \quad \begin{array}{l} 2R_1 - 3R_2 \\ 2R_3 - 13R_2 \end{array}$$

$$\left[ \begin{array}{cccc|c} -4 & 0 & 15 & 14 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 29 & 6 & 0 \end{array} \right] \quad \begin{array}{l} 29R_1 - 15R_3 \\ 29R_2 + R_3 \end{array}$$

$$\left[ \begin{array}{cccc|c} -116 & 0 & 0 & 316 & 0 \\ 0 & 58 & 0 & 6 & 0 \\ 0 & 0 & 29 & 6 & 0 \end{array} \right] \begin{array}{l} -\frac{1}{4}R_1 \\ \frac{1}{58}R_2 \\ \frac{1}{29}R_3 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & -\frac{79}{29} & 0 \\ 0 & 1 & 0 & \frac{3}{29} & 0 \\ 0 & 0 & 1 & \frac{6}{29} & 0 \end{array} \right]$$

Therefore, the system has nontrivial solutions  $a = \frac{79}{29}t$ ,  $b = -\frac{3}{29}t$ ,  $c = -\frac{6}{29}t$ ,  $d = t$

We conclude that the given set of vectors is linearly dependent.

### Exercise

Determine whether the vectors are linearly dependent or linearly independent in  $\mathbb{R}^4$

- a)  $\{(3, 8, 7, -3), (1, 5, 3, -1), (2, -1, 2, 6), (1, 4, 0, 3)\}$
- b)  $\{(0, 0, 2, 2), (3, 3, 0, 0), (1, 1, 0, -1)\}$
- c)  $\{(0, 3, -3, -6), (-2, 0, 0, -6), (0, -4, -2, -2), (0, -8, 4, -4)\}$
- d)  $\{(3, 0, -3, 6), (0, 2, 3, 1), (0, -2, -2, 0), (-2, 1, 2, 1)\}$
- e)  $\{(1, 3, -4, 2), (2, 2, -4, 0), (2, 3, 2, -4), (-1, 0, 1, 0)\}$
- f)  $\{(1, 3, -4, 2), (2, 2, -4, 0), (1, -3, 2, -4), (-1, 0, 1, 0)\}$
- g)  $\{(1, 0, 0, -1), (0, 1, 0, -1), (0, 1, 0, -1), (0, 0, 0, 1)\}$

### Solution

$$a) \det \begin{pmatrix} 3 & 1 & 2 & 1 \\ 8 & 5 & -1 & 4 \\ 7 & 3 & 2 & 0 \\ -3 & -1 & 6 & 3 \end{pmatrix} = 128 \neq 0$$

The system has only the trivial solution and the vectors are *linearly independent*.

$$b) k_1(0,0,2,2) + k_2(3,3,0,0) + k_3(1,1,0,-1) = (0,0,0,0)$$

$$\left[ \begin{array}{ccc|c} 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \end{array} \right]$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 - R_1 \\ R_4 - R_3 \end{array}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 + R_3$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 - R_2$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \frac{1}{2}R_3 \\ \frac{1}{3}R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$k_1 = k_2 = k_3 = 0$$

The system has only the trivial solution and the vectors are *linearly independent*.

$$c) \det \begin{pmatrix} 0 & -2 & 0 & 0 \\ 3 & 0 & -4 & -8 \\ -3 & 0 & -2 & 4 \\ -6 & -6 & -2 & -4 \end{pmatrix} = \underline{480 \neq 0}$$

The system has only the trivial solution and the vectors are *linearly independent*.

$$d) a(3, 0, -3, 6) + b(0, 2, 3, 1) + c(0, -2, -2, 0) + d(-2, 1, 2, 1) = (0, 0, 0, 0)$$

$$\left[ \begin{array}{cccc|c} 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & 1 & 0 \\ -3 & 3 & -2 & 2 & 0 \\ 6 & 1 & 0 & 1 & 0 \end{array} \right] \quad \begin{array}{l} R_3 + R_1 \\ R_4 - 2R_1 \end{array}$$



$$\left[ \begin{array}{cccc|c} 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & 1 & 0 \\ 0 & 3 & -2 & 0 & 0 \\ 0 & 1 & 0 & 5 & 0 \end{array} \right] \quad \begin{array}{l} 2R_3 - 3R_2 \\ 2R_4 - R_2 \end{array}$$

$$\left[ \begin{array}{cccc|c} 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & 1 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 2 & 12 & 0 \end{array} \right] \quad \begin{array}{l} R_2 + R_3 \\ R_4 - R_3 \end{array}$$

$$\left[ \begin{array}{cccc|c} 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 9 & 0 \end{array} \right] \quad \frac{1}{9}R_4$$

$$\left[ \begin{array}{cccc|c} 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right] \quad \begin{array}{l} R_1 + 2R_4 \\ R_2 + 2R_4 \\ R_3 + 3R_4 \end{array}$$

$$\left[ \begin{array}{cccc|c} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right] \quad \begin{array}{l} \frac{1}{3}R_1 \\ \frac{1}{2}R_2 \\ \frac{1}{2}R_3 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

Therefore, the system has only one trivial solution  $a = b = c = d = 0$ .

The given set of vectors is *linearly independent*

e)  $\{(1, 3, -4, 2), (2, 2, -4, 0), (2, 3, 2, -4), (-1, 0, 1, 0)\}$

$$\begin{vmatrix} 1 & 2 & 2 & -1 \\ 3 & 2 & 3 & 0 \\ -4 & -4 & 2 & 1 \\ 2 & 0 & -4 & 0 \end{vmatrix} = 28 \neq 0$$

The system has only the trivial solution and the vectors are *linearly independent*.

$$\mathcal{J} \quad \{(1, 3, -4, 2), (2, 2, -4, 0), (1, -3, 2, -4), (-1, 0, 1, 0)\}$$

$$\begin{vmatrix} 1 & 2 & 1 & -1 \\ 3 & 2 & -3 & 0 \\ -4 & -4 & 2 & 1 \\ 2 & 0 & -4 & 0 \end{vmatrix} \quad \underline{=0}$$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & 2 & -3 & 0 \\ -4 & -4 & 2 & 1 \\ 2 & 0 & -4 & 0 \end{pmatrix} \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 + 4R_1 \\ R_4 - 2R_1 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -4 & -6 & 3 \\ 0 & 4 & 6 & -3 \\ 0 & -4 & -6 & 2 \end{pmatrix} \quad \begin{array}{l} 2R_1 + R_2 \\ R_3 + R_2 \\ R_4 - R_2 \end{array}$$

$$\begin{pmatrix} 2 & 0 & -4 & 1 \\ 0 & -4 & -6 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & -4 & 1 \\ 0 & -4 & -6 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} R_1 - R_3 \\ R_2 - 3R_3 \end{array}$$

$$\begin{pmatrix} 2 & 0 & -4 & 0 \\ 0 & -4 & -6 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} \frac{1}{2}R_1 \\ -\frac{1}{4}R_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underline{x_4 = 0, \quad x_2 = -\frac{3}{2}x_3, \quad x_1 = 2x_3}$$

$\therefore$  The set is *linearly independent*.

$$g) \{(1, 0, 0, -1), (0, 1, 0, -1), (0, 1, 0, -1), (0, 0, 0, 1)\}$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 \end{vmatrix} \stackrel{=0}{=}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_3 + R_1$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_3 + R_2$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} x_1 = 0 \\ x_2 = -x_3 \\ x_4 = 0 \end{array}$$

$\therefore$  The set is *linearly independent*.

### ***Exercise***

a) Show that the three vectors  $\vec{v}_1 = (1, 2, 3, 4)$   $\vec{v}_2 = (0, 1, 0, -1)$   $\vec{v}_3 = (1, 3, 3, 3)$  form a linearly dependent set in  $\mathbb{R}^4$ .

b) Express each vector in part (a) as a linear combination of the other two.

### **Solution**

a) The vector equation:

$$k_1(1, 2, 3, 4) + k_2(0, 1, 0, -1) + k_3(1, 3, 3, 3) = (0, 0, 0, 0)$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ 3 & 0 & 3 & 0 \\ 4 & -1 & 3 & 0 \end{array} \right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 4R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{array} \right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 4R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{array} \right] \quad \begin{array}{l} \\ \\ \\ R_4 + R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The solution:  $k_1 = -t$ ,  $k_2 = -t$ ,  $k_3 = t$

Since the system has nontrivial solutions, the given set of vectors is *linearly dependent*.

**b)** Since  $k_1 = -t$ ,  $k_2 = -t$ ,  $k_3 = t$  and if we let  $t = 1$ , then  $-\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = 0$

$$\vec{v}_1 = -\vec{v}_2 + \vec{v}_3, \quad \vec{v}_2 = -\vec{v}_1 + \vec{v}_3, \quad \vec{v}_3 = \vec{v}_1 + \vec{v}_2$$

## Exercise

For which real values of  $\lambda$  do the following vectors form a linearly dependent set in  $\mathbf{R}^3$

$$\vec{v}_1 = \left( \lambda, -\frac{1}{2}, -\frac{1}{2} \right) \quad \vec{v}_2 = \left( -\frac{1}{2}, \lambda, -\frac{1}{2} \right) \quad \vec{v}_3 = \left( -\frac{1}{2}, -\frac{1}{2}, \lambda \right)$$

## Solution

$$k_1 \left( \lambda, -\frac{1}{2}, -\frac{1}{2} \right) + k_2 \left( -\frac{1}{2}, \lambda, -\frac{1}{2} \right) + k_3 \left( -\frac{1}{2}, -\frac{1}{2}, \lambda \right) = (0, 0, 0, 0)$$

$$\det \begin{pmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{pmatrix} = \frac{1}{4} (4\lambda^3 - 3\lambda - 1)$$

$$4\lambda^3 - 3\lambda - 1 = 0$$

For  $\lambda = 1$   $\lambda = -\frac{1}{2}$ , the determinant is zero and the vectors form a *linearly dependent* set.

### ***Exercise***

Show that if  $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$  is a linearly independent set of vectors, then so is every nonempty subset of  $S$ .

### **Solution**

Let  $\{\vec{v}_a, \vec{v}_b, \dots, \vec{v}_r\}$  be a nonempty subset of  $S$ .

If this set is linearly dependent, then there would be a nonzero solution  $(k_a, k_b, \dots, k_r)$  to

$$k_a \vec{v}_a + k_b \vec{v}_b + \dots + k_r \vec{v}_r = 0.$$

This can be expanded to a nonzero solution of

$k_1 \vec{v}_1 + k_2 \vec{v}_2 + \dots + k_n \vec{v}_n = 0$  by taking all other coefficients as 0. This contradicts the linear independence of  $S$ , so the subset must be *linearly independent*.

### ***Exercise***

Show that if  $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$  is a linearly independent set of vectors in a vector space  $V$ , and if  $\vec{v}_{r+1}, \dots, \vec{v}_n$  are vectors in  $V$  that are not in  $S$ , then  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r, \vec{v}_{r+1}, \dots, \vec{v}_n\}$  is also linearly independent.

### **Solution**

If  $S$  is linearly dependent, then there is a nonzero solution  $(k_1, k_2, \dots, k_r)$  to

$$k_1 \vec{v}_1 + k_2 \vec{v}_2 + \dots + k_r \vec{v}_r = 0.$$

Thus  $(k_1, k_2, \dots, k_r, 0, 0, \dots, 0)$  is a nonzero solution to

$$k_1 \vec{v}_1 + k_2 \vec{v}_2 + \dots + k_r \vec{v}_r + k_{r+1} \vec{v}_{r+1} + \dots + k_n \vec{v}_n = 0$$

So, the set  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r, \vec{v}_{r+1}, \dots, \vec{v}_n\}$  is *linearly dependent*.

### Exercise

Show that  $\{\vec{v}_1, \vec{v}_2\}$  is linearly independent and  $\vec{v}_3$  does not lie in  $\text{span}\{\vec{v}_1, \vec{v}_2\}$ , then  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  is a linearly independent.

### Solution

If  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  are linearly dependent, there exist a nonzero solution to  $k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = 0$  with  $k_3 \neq 0$  (since  $\vec{v}_1$  and  $\vec{v}_2$  are linearly independent).

$$k_3\vec{v}_3 = -k_1\vec{v}_1 - k_2\vec{v}_2 \Rightarrow \vec{v}_3 = -\frac{k_1}{k_3}\vec{v}_1 - \frac{k_2}{k_3}\vec{v}_2 \text{ which contradicts that } \vec{v}_3 \text{ is not in span } \{\vec{v}_1, \vec{v}_2\}. \text{ Thus } \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \text{ is a linearly independent.}$$

### Exercise

By using the appropriate identities, where required, determine  $F(-\infty, \infty)$  are linearly dependent.

- a)  $6, 3\sin^2 x, 2\cos^2 x$       c)  $1, \sin x, \sin 2x$       e)  $\cos 2x, \sin^2 x, \cos^2 x$   
b)  $x, \cos x$       d)  $(3-x)^2, x^2 - 6x, 5$

### Solution

a) From the identity  $\sin^2 x + \cos^2 x = 1$

$$\begin{aligned} (-1)(6) + (2)(3\sin^2 x) + (3)(2\cos^2 x) &= -6 + 6(\sin^2 x + \cos^2 x) \\ &= 0 \end{aligned}$$

Therefore, the set is *linearly dependent*.

b)  $ax + b\cos x = 0$

$$\begin{aligned} x = 0 &\Rightarrow b = 0 \\ x = \frac{\pi}{2} &\Rightarrow a = 0 \end{aligned}$$

Therefore, the set is *linearly independent*.

c)  $a(1) + b\sin x + c\sin 2x = 0$

$$\begin{aligned} x = 0 &\Rightarrow a = 0 \\ x = \frac{\pi}{2} &\Rightarrow b = 0 \\ x = \frac{\pi}{4} &\Rightarrow c = 0 \end{aligned}$$

Therefore, the set is *linearly independent*.

$$d) (3-x)^2 = 9 - 6x + x^2$$

$$(3-x)^2 - (9 - 6x + x^2) = 0$$

$$(3-x)^2 - (x^2 - 6x) - 9 = 0$$

$$(1)(3-x)^2 + (-1)(x^2 - 6x) + \left(-\frac{9}{5}\right)5 = 0$$

Therefore, the set is linearly dependent.

e) By using the double angle:

$$\cos 2x = \cos^2 x - \sin^2 x \text{ are linearly dependent.}$$

### Exercise

$f_1(x) = \sin x$ ,  $f_2(x) = \cos x$  are linearly independent in  $F(-\infty, \infty)$  because neither function is a scalar multiple of the other. Confirm the linear independence using Wronski's test.

### Solution

$$\begin{aligned} \text{The Wronskian: } W(x) &= \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} \\ &= -\sin^2 x - \cos^2 x \\ &= -(\sin^2 x + \cos^2 x) \\ &= -1 \neq 0 \end{aligned}$$

$\sin x$  and  $\cos x$  are linearly independent

### Exercise

Show  $f_1(x) = e^x$ ,  $f_2(x) = xe^x$ ,  $f_3(x) = x^2e^x$  are linearly independent in  $F(-\infty, \infty)$

### Solution

$$\begin{aligned} W &= \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & e^x + xe^x & 2xe^x + x^2e^x \\ e^x & 2e^x + xe^x & 2e^x + 4xe^x + x^2e^x \end{vmatrix} \\ &= e^{3x} \begin{vmatrix} 1 & x & x^2 \\ 1 & 1+x & 2x+x^2 \\ 1 & 2+x & 2+4x+x^2 \end{vmatrix} \end{aligned}$$

$e^x$   
factor  $e^x$   
 $e^x$

$$\begin{aligned}
&= e^{3x} \left[ (1+x)(2+4x+x^2) + 2x^2 + x^3 + 2x^2 + x^3 - x^2 - x^3 - (2x+x^2)(2+x) - 2x - 4x^2 - x^3 \right] \\
&= e^{3x} \left[ 2+4x+x^2+2x+4x^2+x^3-4x-2x^2-2x^2-x^3-2x-x^2 \right] \\
&= \underline{2e^{3x} \neq 0}
\end{aligned}$$

$\{e^x, xe^x, x^2e^x\}$  are linearly independent

### Exercise

Use the Wronskian to show that  $f_1(x) = \sin x$ ,  $f_2(x) = \cos x$ ,  $f_3(x) = x \cos x$  span a three-dimensional subspace of  $F(-\infty, \infty)$

### Solution

$$\begin{aligned}
\text{The Wronskian: } W(x) &= \begin{vmatrix} \sin x & \cos x & x \cos x \\ \cos x & -\sin x & \cos x - x \sin x \\ -\sin x & -\cos x & -2 \sin x - x \cos x \end{vmatrix} \\
&= 2 \sin^3 x + x \sin^2 x \cos x - \sin x \cos^2 x + x \sin^2 x \cos x - x \cos^3 x \\
&\quad - x \sin^2 x \cos x + \sin x \cos^2 x - x \sin^2 x \cos x + 2 \sin x \cos^2 x + x \cos^3 x \\
&= 2 \sin^3 x + 2 \sin x \cos^2 x \\
&= 2 \sin x (\sin^2 x + \cos^2 x) \\
&= \underline{2 \sin x}
\end{aligned}$$

Since  $\sin x \neq 0$  for all real  $x$  values, the vectors are *linearly independent*.

### Exercise

Show by inspection that the vectors are linearly dependent.

$$\vec{v}_1(4, -1, 3), \quad \vec{v}_2(2, 3, -1), \quad \vec{v}_3(-1, 2, -1), \quad \vec{v}_4(5, 2, 3), \quad \text{in } \mathbb{R}^3$$

### Solution

$$\begin{bmatrix} 4 & 2 & -1 & 5 \\ -1 & 3 & 2 & 2 \\ 3 & -1 & -1 & 3 \end{bmatrix} \quad \begin{array}{l} \\ 4R_2 + R_1 \\ 4R_3 - 3R_1 \end{array}$$



$$\begin{bmatrix} 4 & 2 & -1 & 5 \\ 0 & 14 & 7 & 13 \\ 0 & -10 & -1 & -3 \end{bmatrix} \quad \begin{array}{l} 7R_1 - R_2 \\ 14R_3 + 10R_2 \end{array}$$

$$\begin{bmatrix} 28 & 0 & -14 & 22 \\ 0 & 14 & 7 & 13 \\ 0 & 0 & 56 & 88 \end{bmatrix} \quad \begin{array}{l} 4R_1 + R_3 \\ 8R_2 - R_3 \end{array}$$

$$\begin{bmatrix} 112 & 0 & 0 & 176 \\ 0 & 112 & 0 & 16 \\ 0 & 0 & 56 & 88 \end{bmatrix} \quad \begin{array}{l} \frac{1}{112}R_1 \\ \frac{1}{112}R_2 \\ \frac{1}{56}R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{11}{7} \\ 0 & 1 & 0 & \frac{1}{7} \\ 0 & 0 & 1 & \frac{11}{7} \end{bmatrix} \quad \begin{array}{l} \vec{v}_1 = -\frac{11}{7}\vec{v}_4 \\ \vec{v}_2 = -\frac{1}{7}\vec{v}_4 \\ \vec{v}_3 = -\frac{11}{7}\vec{v}_4 \end{array}$$

$$-\frac{11}{7}\vec{v}_1 - \frac{1}{7}\vec{v}_2 - \frac{11}{7}\vec{v}_3 + \vec{v}_4 = 0$$

$$\underline{7\vec{v}_4 = 11\vec{v}_1 + \vec{v}_2 + 11\vec{v}_3}$$

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$(2, -1, 3), (3, 4, 1), (2, -3, 4), \text{ in } \mathbb{R}^3$$

### Solution

$$a(2, -1, 3) + b(3, 4, 1) + c(2, -3, 4) = (0, 0, 0)$$

$$\begin{bmatrix} 2 & 3 & 2 & 0 \\ -1 & 4 & -3 & 0 \\ 3 & 1 & 4 & 0 \end{bmatrix} \quad \begin{array}{l} 2R_2 + R_1 \\ 2R_3 - 3R_1 \end{array}$$

$$\begin{bmatrix} 2 & 3 & 2 & 0 \\ 0 & 11 & -4 & 0 \\ 0 & -7 & 2 & 0 \end{bmatrix} \quad \begin{array}{l} 11R_1 - 3R_2 \\ 11R_3 + 7R_2 \end{array}$$

$$\begin{bmatrix} 22 & 0 & 34 & 0 \\ 0 & 11 & -4 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix} \quad -\frac{1}{6}R_3$$

$$\begin{bmatrix} 22 & 0 & 34 & 0 \\ 0 & 11 & -4 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} R_1 - 34R_3 \\ R_2 + 4R_3 \end{array}$$

$$\begin{bmatrix} 22 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} \frac{1}{22}R_1 \\ \frac{1}{11}R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The system has only the trivial solution  $a = b = c = 0$ .

$$\begin{vmatrix} 2 & 3 & 2 \\ -1 & 4 & -3 \\ 3 & 1 & 4 \end{vmatrix} = 32 - 27 - 2 - 24 + 6 + 12 \neq 0$$

The system has only the trivial solution and the vectors are *linearly independent*

### ***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)

$$(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1), \text{ in } \mathbb{R}^4$$

### **Solution**

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

The system has only the trivial solution and the vectors are *linearly independent*

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$A_1 \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \quad A_2 \begin{bmatrix} -2 & 4 \\ 1 & 0 \end{bmatrix}, \quad A_3 \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}, \quad \text{in } M_{22}$$

### Solution

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 2 & 4 & -1 & 0 \\ 3 & 1 & 2 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 8 & -7 & 0 \\ 0 & 7 & -7 & 0 \end{bmatrix} \quad \begin{array}{l} 4R_1 + R_2 \\ 8R_3 - 7R_2 \end{array}$$

$$\begin{bmatrix} 4 & 0 & 5 & 0 \\ 0 & 8 & -7 & 0 \\ 0 & 0 & -7 & 0 \end{bmatrix} \quad -\frac{1}{7}R_3$$

$$\begin{bmatrix} 4 & 0 & 5 & 0 \\ 0 & 8 & -7 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} R_1 - 5R_3 \\ R_2 + 7R_3 \end{array}$$

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} \frac{1}{4}R_1 \\ \frac{1}{8}R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The vectors are *linearly independent*

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix}, \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix}, \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} \right\} \text{ in } M_{2 \times 3}(\mathbb{R})$$

### Solution

$$\left[ \begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -3 & 7 & 3 & 0 \\ 2 & 4 & 11 & 0 \\ -4 & 6 & -1 & 0 \\ 0 & -2 & -3 & 0 \\ 5 & -7 & 2 & 0 \end{array} \right] \begin{array}{l} R_2 + 3R_1 \\ R_3 - 2R_1 \\ R_4 + 4R_1 \\ \\ R_6 - 5R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 10 & 15 & 0 \\ 0 & -6 & -9 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 8 & 12 & 0 \end{array} \right] \begin{array}{l} \\ R_3 + 5R_2 \\ R_4 - 3R_2 \\ R_5 - R_2 \\ R_6 + 4R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} \rightarrow a_1 = 3a_2 + 2a_3 \\ \rightarrow 2a_2 = -3a_3 \\ \\ \\ \\ \end{array}$$

$$\underline{a_2 = -\frac{3}{2}a_3}$$

$$\begin{aligned} a_1 &= -\frac{9}{2}a_3 + 2a_3 \\ &= -\frac{5}{2}a_3 \end{aligned} \underline{\hspace{1cm}}$$

It is *linearly dependent*.

$$\text{if } a_3 = -2 \quad a_2 = 3 \quad a_1 = 5$$

$$5 \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} + 3 \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix} - 2 \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

### ***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} \right\} \text{ in } M_{2 \times 2}(\mathbb{R})$$

### **Solution**

$$\begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} = -2 \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}$$

$$2 \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\therefore$  Linearly dependent

### ***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} \right\} \text{ in } M_{2 \times 2}(\mathbb{R})$$

### **Solution**

$$\begin{bmatrix} 1 & -1 \\ -2 & 1 \\ -1 & 2 \\ 4 & -4 \end{bmatrix} \quad \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 + R_1 \\ R_4 - 4R_1 \end{array}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$a_2 = 0 \rightarrow a_1 = a_2 = 0$$

$\therefore$  Linearly independent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix} \right\} \text{ in } M_{2 \times 2}(\mathbb{R})$$

### Solution

$$\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ -2 & 1 & 1 & -4 \\ 1 & 1 & 0 & 4 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 1 \\ 1 & 1 & -4 \\ 1 & 0 & 4 \end{vmatrix} - \begin{vmatrix} 0 & -1 & 1 \\ -2 & 1 & -4 \\ 1 & 1 & 4 \end{vmatrix} - 2 \begin{vmatrix} 0 & -1 & 2 \\ -2 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} \\ = -21 + 7 + 14 \\ = 0$$

$\therefore$  Linearly dependent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & -2 \end{pmatrix} \right\} \text{ in } M_{2 \times 2}(\mathbb{R})$$

### Solution

$$W = \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ -2 & 1 & 1 & 2 \\ 1 & 1 & 0 & -2 \end{vmatrix} = 24 \neq 0$$

$\therefore$  Linearly independent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\{e^x, \ln x\} \text{ in } \mathbb{R}$$

### Solution

$$W = \begin{vmatrix} e^x & \ln x \\ e^x & \frac{1}{x} \end{vmatrix}$$

$$\underline{= e^x \left( \frac{1}{x} - \ln x \right) \neq 0}$$

$\therefore$  Linearly independent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ x, \frac{1}{x} \right\} \text{ in } \mathbb{R}$$

### Solution

$$\begin{aligned} W &= \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix} \\ &= -\frac{1}{x} - \frac{1}{x} \\ &= -\frac{2}{x} \neq 0 \end{aligned}$$

$\therefore$  Linearly independent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\{1+x, 1-x\} \text{ in } P_2(\mathbb{R})$$

### Solution

$$\begin{aligned} W &= \begin{vmatrix} 1+x & 1-x \\ 1 & -1 \end{vmatrix} \\ &= -1 - x - 1 + x \\ &= -2 \neq 0 \end{aligned}$$

$\therefore$  Linearly independent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\{9x^2 - x + 3, 3x^2 - 6x + 5, -5x^2 + x + 1\} \text{ in } P_3(\mathbb{R})$$

### Solution

$$W = \begin{vmatrix} 9 & -1 & 3 \\ 3 & -6 & 5 \\ -5 & 1 & 1 \end{vmatrix} = \underline{-152 \neq 0}$$

$\therefore$  Linearly independent

$$\begin{aligned}
 W &= \begin{vmatrix} 9x^2 - x + 3 & 3x^2 - 6x + 5 & -5x^2 + x + 1 \\ 18x - 1 & 6x - 6 & -10x + 1 \\ 18 & 6 & -10 \end{vmatrix} \\
 &= (-60x - 60)(9x^2 - x + 3) + (-180x + 18)(3x^2 - 6x + 5) + (108x - 6)(-5x^2 + x + 1) \\
 &\quad - (108x - 108)(-5x^2 + x + 1) - (-60x + 6)(9x^2 - x + 3) - (-180x + 10)(3x^2 - 6x + 5) \\
 &\quad \begin{matrix} x^3 & -540 - 540 - 540 + 540 + 540 + 540 \\ x^2 & -540 + 60 + 54 + 1080 + 30 + 108 + 540 - 108 - 54 - 60 - 30 - 1080 \\ x^1 & 60 - 180 - 900 - 6 + 108 + 108 - 108 + 6 + 180 + 60 + 900 \\ x^0 & -180 + 90 - 6 + 108 - 18 - 50 \end{matrix} \\
 &= \underline{228x - 56 \neq 0}
 \end{aligned}$$

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\{-x^2, 1 + 4x^2\} \text{ in } P_3(\mathbb{R})$$

#### Solution

$$\begin{aligned}
 W &= \begin{vmatrix} -x^2 & 4x^2 + 1 \\ -2x & 8x \end{vmatrix} \\
 &= \underline{2x \neq 0}
 \end{aligned}$$

$\therefore$  Linearly independent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\{7x^2 + x + 2, 2x^2 - x + 3, -3x^2 + 4\} \text{ in } P_3(\mathbb{R})$$

#### Solution



$$W = \begin{vmatrix} 7 & 1 & 2 \\ 2 & -1 & 3 \\ -3 & 0 & 4 \end{vmatrix}$$

$$= -51 \neq 0$$

$\therefore$  Linearly independent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\{3x^2 + 3x + 8, 2x^2 + x, 2x^2 + 2x + 2, 5x^2 - 2x + 8\} \text{ in } P_3(\mathbb{R})$$

### Solution

$$\begin{bmatrix} 3 & 2 & 2 & 5 \\ 3 & 1 & 2 & -2 \\ 8 & 0 & 2 & 8 \end{bmatrix} \quad \begin{array}{l} R_2 - R_1 \\ 3R_3 - 8R_1 \end{array}$$

$$\begin{bmatrix} 3 & 2 & 2 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & -16 & -10 & -16 \end{bmatrix} \quad R_3 + 16R_2$$

$$\begin{bmatrix} 3 & 2 & 2 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & -10 & 96 \end{bmatrix} \quad \begin{array}{l} 3a_1 = -2a_2 - 2a_3 - 5a_4 \\ a_2 = -7a_4 \\ a_3 = \frac{48}{5}a_4 \end{array}$$

$$3a_1 = 14a_4 - \frac{96}{5}a_4 - 5a_4$$

$$a_1 = \frac{3}{5}a_4$$

$\therefore$  Linearly dependent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\{x^3 + 2x^2, -x^2 + 3x + 1, x^3 - x^2 + 2x - 1\} \text{ in } P_3(\mathbb{R})$$

### Solution

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & -1 \\ 0 & 3 & 2 \\ 0 & 1 & -1 \end{bmatrix} \quad R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -3 \\ 0 & 3 & 2 \\ 0 & 1 & -1 \end{bmatrix} \quad \begin{array}{l} R_3 + 3R_2 \\ R_4 + R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & -7 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\begin{cases} a_3 = 0 \\ a_1 = -3a_3 = 0 \\ a_1 = -a_3 = 0 \end{cases}$$

$\therefore$  Linearly independent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\{x^3 - x, 2x^2 - 4, -2x^3 + 3x^2 + 2x + 6\} \text{ in } P_3(\mathbb{R})$$

### Solution

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & 4 & 6 \end{bmatrix} \quad R_3 + R_1$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 4 & 6 \end{bmatrix} \quad R_4 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} 2a_2 = -3a_3 \\ a_1 = 2a_3 \end{cases}$$

$$\text{If } a_3 = 2 \rightarrow a_1 = 4 \quad a_2 = -3$$

$$\rightarrow 4(x^3 - x) - 3(2x^2 + 4) + 2(-2x^3 + 3x^2 + 2x + 6) = 0$$

$\therefore$  Linearly dependent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ \begin{array}{l} x^4 - x^3 + 5x^2 - 8x + 6, \quad -x^4 + x^3 - 5x^2 + 5x - 3, \quad x^4 + 3x^2 - 3x + 5, \\ 2x^4 + 3x^3 + 4x^2 - x + 1, \quad x^3 - x + 2 \end{array} \right\} \text{ in } P_4(\mathbb{R})$$

### Solution

$$\begin{vmatrix} 1 & -1 & 1 & 2 & 0 \\ -1 & 1 & 0 & 3 & 1 \\ 5 & -5 & 3 & 4 & 0 \\ -8 & 5 & -3 & -1 & -1 \\ 6 & -3 & 5 & 1 & 2 \end{vmatrix} = -60 \neq 0$$

$\therefore$  Linearly dependent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ \begin{array}{l} x^4 - x^3 + 5x^2 - 8x + 6, \quad -x^4 + x^3 - 5x^2 + 5x - 3, \\ x^4 + 3x^2 - 3x + 5, \quad 2x^4 + x^3 + 4x^2 + 8x \end{array} \right\} \text{ in } P_4(\mathbb{R})$$

### Solution

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ -1 & 1 & 0 & 1 \\ 5 & -5 & 3 & 4 \\ -8 & 5 & -3 & 8 \\ 6 & -3 & 5 & 0 \end{bmatrix} \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 5R_1 \\ R_4 + 8R_1 \\ R_5 - 6R_1 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -6 \\ 0 & -3 & 5 & 24 \\ 0 & 3 & -1 & -12 \end{bmatrix} \quad R_5 + R_4$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -6 \\ 0 & -3 & 5 & 24 \\ 0 & 0 & 4 & 12 \end{bmatrix} \quad \begin{array}{l} R_3 + 2R_2 \\ R_5 - 4R_2 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 5 & 24 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \rightarrow a_1 = a_2 - a_3 - 2a_4 \\ \rightarrow a_3 = -3a_4 \\ \rightarrow 3a_2 = 5a_3 + 24a_4 \end{array}$$

$$\text{If } a_4 = -1 \rightarrow a_3 = 3 \quad a_2 = -3 \quad a_1 = -4$$

$$\begin{aligned} & -4(x^4 - x^3 + 5x^2 - 8x + 6) - 3(-x^4 + x^3 - 5x^2 + 5x - 3) \\ & + 3(x^4 + 3x^2 - 3x + 5) - (2x^4 + x^3 + 4x^2 + 8x) = 0 \end{aligned}$$

$\therefore$  Linearly dependent

### Exercise

Suppose that the vectors  $\vec{u}_1$ ,  $\vec{u}_2$ , and  $\vec{u}_3$  are linearly dependent. Are the vectors  $\vec{v}_1 = \vec{u}_1 + \vec{u}_2$ ,  $\vec{v}_2 = \vec{u}_1 + \vec{u}_3$ , and  $\vec{v}_3 = \vec{u}_2 + \vec{u}_3$  also linearly dependent?

(**Hint:** Assume that  $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}$ , and see what the  $a_i$ 's can be.)

### Solution

**Given:**  $\vec{u}_1$ ,  $\vec{u}_2$ , and  $\vec{u}_3$  are linearly dependent, then there are scalar  $b_1$ ,  $b_2$ , and  $b_3$  such that

$$b_1\vec{u}_1 + b_2\vec{u}_2 + b_3\vec{u}_3 = \vec{0}.$$

Assume that  $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}$

$$a_1(\vec{u}_1 + \vec{u}_2) + a_2(\vec{u}_1 + \vec{u}_3) + a_3(\vec{u}_2 + \vec{u}_3) = \vec{0}$$

$$a_1\vec{u}_1 + a_1\vec{u}_2 + a_2\vec{u}_1 + a_2\vec{u}_3 + a_3\vec{u}_2 + a_3\vec{u}_3 = \vec{0}$$

$$(a_1 + a_2)\vec{u}_1 + (a_1 + a_3)\vec{u}_2 + (a_2 + a_3)\vec{u}_3 = 0$$

If  $a_1 + a_2 = b_1$   $a_1 + a_3 = b_2$   $a_2 + a_3 = b_3$  and since  $\vec{u}_1$ ,  $\vec{u}_2$ , and  $\vec{u}_3$  are linearly dependent, therefore,  $\vec{v}_1$ ,  $\vec{v}_2$ , and  $\vec{v}_3$  are linearly dependent.

### Exercise

Show that the set  $F = \{1+t, t^2, t-2\}$  is a linearly independent subset of  $P_2$ .

### Solution

$$\begin{aligned} W &= \begin{vmatrix} 1+t & t^2 & t-2 \\ 1 & 2t & 1 \\ 0 & 2 & 0 \end{vmatrix} \\ &= 2t - 4 - 2 - 2t \\ &= \underline{-6 \neq 0} \quad \therefore \text{Linearly Independent.} \end{aligned}$$

$$\exists c_1, c_2, c_3 \text{ constants } \ni 0 = c_1(1+t) + c_2 t^2 + c_3(t-2)$$

$$\Rightarrow \begin{cases} t^0 & c_1 - 2c_3 = 0 \\ t & c_1 + c_3 = 0 \\ t^2 & c_2 = 0 \end{cases} \rightarrow \underline{c_1 = c_3 = 0}$$

Since the only solution to this system is the trivial one.  $F$  is Linearly Independent subset of  $P_2$

### Exercise

Suppose that  $A$  is linearly dependent set of vectors and  $B$  is any set containing  $A$ . Show that  $B$  must be linearly dependent.

### Solution

If  $A$  is linearly dependent, then there are vectors  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$  in  $A$  and  $\mathbb{R}, c_1, c_2, \dots, c_n$  with all not  $c_i = 0$  and  $c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n = \vec{0}$

If  $B$  any set that contains  $A$ , then this same relation holds in  $B$  set.  
 $B$  is also dependent.

### Exercise

Show that  $\{\sin t, \sin 2t, \cos t\}$  is a linearly independent, subset of  $C[0, 1]$ . Does it span  $C[0, 1]$

### Solution

$$\begin{aligned} W &= \begin{vmatrix} \sin t & \sin 2t & \cos t \\ \cos t & 2\cos 2t & -\sin t \\ -\sin t & -4\sin 2t & -\cos t \end{vmatrix} \\ &= -2\sin t \cos t \cos 2t + \sin^2 t \sin 2t - 4\cos^2 t \sin 2t + 2\sin t \cos t \cos 2t - 4\sin^2 t \sin 2t + \cos^2 t \sin 2t \\ &= \sin 2t - 4\sin 2t \\ &= -3\sin 2t \neq 0 \end{aligned}$$

$\therefore$  Linearly Independent.

$$a \sin t + b \sin 2t + d \cos t = 0$$

$$\text{If } \begin{cases} t = 0 & \rightarrow d = 0 \\ t = \frac{\pi}{2} & \rightarrow a = 0 \\ t = \frac{\pi}{4} & \rightarrow b = 0 \end{cases}$$

Since all the polynomials are in  $C[0, 1]$  and there is no other way that we can write them as linear combinations of  $\sin t$ ,  $\sin 2t$ , and  $\cos t$ .

The set can't possible span  $C[0, 1]$

### Exercise

Show that the set  $\{\sin(t+a), \sin(t+b), \sin(t+c)\}$  is linearly dependent on  $C[0, 1]$ .

### Solution

$$\begin{aligned} W &= \begin{vmatrix} \sin(t+a) & \sin(t+b) & \sin(t+c) \\ \cos(t+a) & \cos(t+b) & \cos(t+c) \\ -\sin(t+a) & -\sin(t+b) & -\sin(t+c) \end{vmatrix} \\ &= -\sin(t+a)\cos(t+b)\sin(t+c) - \sin(t+a)\cos(t+c)\sin(t+b) - \sin(t+b)\cos(t+a)\sin(t+c) \\ &\quad + \sin(t+a)\cos(t+b)\sin(t+c) + \sin(t+a)\cos(t+c)\sin(t+b) + \sin(t+b)\cos(t+a)\sin(t+c) \\ &= 0 \end{aligned}$$

$\therefore$  The set is linearly dependent on  $C[0, 1]$

$$k_1 \sin(t+a) + k_2 \sin(t+b) + k_3 \sin(t+c) = 0$$

$$\text{If } \begin{cases} t = -a & \rightarrow k_2 + k_3 = 0 \\ t = -b & \rightarrow k_1 + k_3 = 0 \\ t = -c & \rightarrow k_1 + k_2 = 0 \end{cases}$$

$$t = 0 \rightarrow k_1 \sin a + k_2 \sin b + k_3 \sin c = 0$$

$$t = \frac{\pi}{2} \rightarrow k_1 \cos a + k_2 \cos b + k_3 \cos c = 0$$

$$t = \pi \rightarrow -(k_1 \sin a + k_2 \sin b + k_3 \sin c) = 0$$

### ***Exercise***

Show that if  $\alpha_1, \alpha_2, \dots, \alpha_n$  are linearly independent and  $\alpha_1, \alpha_2, \dots, \alpha_n, \beta$  are linearly dependent, then  $\beta$  can be uniquely expressed as a linear combination of  $\alpha_1, \alpha_2, \dots, \alpha_n$

### **Solution**

Since,  $\alpha_1, \alpha_2, \dots, \alpha_n$  are linearly independent, then

$$a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_n \alpha_n = 0 \text{ when all } a_i = 0.$$

Let assume that:

$$a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_n \alpha_n = b\beta$$

$$a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_n \alpha_n - b\beta = 0$$

If  $b = 0$ , then  $a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_n \alpha_n = 0$  and  $\beta$  doesn't exist.

If  $b \neq 0$ , and  $\alpha_1, \alpha_2, \dots, \alpha_n, \beta$  are linearly dependent, then

$$\begin{aligned} \beta &= \frac{a_1}{b} \alpha_1 + \frac{a_2}{b} \alpha_2 + \dots + \frac{a_n}{b} \alpha_n \\ &= c_1 \alpha_1 + c_2 \alpha_2 + \dots + c_n \alpha_n \end{aligned}$$

$$\text{If } \frac{a_1}{b} = c_1 \Rightarrow \frac{a_1}{b} - c_1 = 0$$

$$\left( \frac{a_1}{b} - c_1 \right) \alpha_1 + \left( \frac{a_2}{b} - c_2 \right) \alpha_2 + \dots + \left( \frac{a_n}{b} - c_n \right) \alpha_n = 0$$

Then  $\frac{a_i}{b} - c_i = 0$  ( $1 \leq i \leq n$ ) since  $\alpha_1, \alpha_2, \dots, \alpha_n$  are linearly independent and contradict that  $\alpha_1, \alpha_2, \dots, \alpha_n, \beta$  are linearly dependent.

Therefore,  $\beta$  can be uniquely expressed as a linear combination of  $\alpha_1, \alpha_2, \dots, \alpha_n$  in the form

$$\beta = c_1 \alpha_1 + c_2 \alpha_2 + \dots + c_n \alpha_n$$

### ***Exercise***

Show that if  $\alpha_1, \alpha_2, \dots, \alpha_n$  are linearly dependent with  $(\alpha_1 \neq 0)$  if and only if there exists an integer  $k$  ( $1 < k \leq n$ ), such that  $\alpha_k$  is a linear combination of  $\alpha_1, \alpha_2, \dots, \alpha_{k-1}$

### **Solution**

Since,  $\alpha_1, \alpha_2, \dots, \alpha_n$  are linearly dependent, then if

$$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0$$

then there exists an  $\alpha_k \neq 0$  ( $1 < k \leq n$ )

If we let  $i > k$  where  $a_i = 0$ , then  $a_1\alpha_1 + a_2\alpha_2 + \dots + a_{k-1}\alpha_{k-1} - a_k\alpha_k = 0$

$$a_k\alpha_k = a_1\alpha_1 + a_2\alpha_2 + \dots + a_{k-1}\alpha_{k-1}$$



## ***Solution***

## **Section 2.7 – Coordinates, Basis and Dimension**

### ***Exercise***

Suppose  $\vec{v}_1, \dots, \vec{v}_n$  is a basis for  $R^n$  and the  $n$  by  $n$  matrix  $A$  is invertible. Show that  $A\vec{v}_1, \dots, A\vec{v}_n$  is also a basis for  $R^n$ .

### **Solution**

Put the basis vectors  $\vec{v}_1, \dots, \vec{v}_n$  in the columns of an invertible matrix  $V$ . then  $A\vec{v}_1, \dots, A\vec{v}_n$  are the columns of  $AV$ . Since  $A$  is invertible, so is  $AV$  and its column give a basis.

Suppose  $c_1 A\vec{v}_1 + \dots + c_n A\vec{v}_n = 0$ . This is  $A\vec{v} = 0$  with  $\vec{v} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$ . Multiply by  $A^{-1}$  to get  $\vec{v} = 0$ . By linear independence of  $\vec{v}$ 's, all  $c_i = 0$ . So, the  $A\vec{v}$ 's are independent.

### ***Exercise***

Consider the matrix  $A = \begin{pmatrix} 1 & 0 & a \\ 2 & -1 & b \\ 1 & 1 & c \\ -2 & 1 & d \end{pmatrix}$

a) Which vectors  $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$  will make the columns of  $A$  linearly dependent?

b) Which vectors  $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$  will make the columns of  $A$  a basis for  $\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : y + w = 0 \right\}$ ?

c) For  $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix}$ , compute a basis for the four subspaces.

### **Solution**

a) All linear combination of  $\begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

**b)** To satisfy  $b + d = 0$ . For example,  $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$A \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + B \begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix} + C \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}; \quad A \neq 0$$

**c)**  $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & 5 \\ -2 & 1 & 2 \end{pmatrix} \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -4 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{pmatrix} \quad \begin{array}{l} R_3 + R_2 \\ R_4 + R_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} x_1 + x_3 = 0 \\ -x_2 - 4x_3 = 0 \end{array}$$

The first 2 columns span the column space  $C(A)$ .

If  $x_3 = 1$  that implies that the nullspace

$$N(A): \left\{ \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} \right\}$$

$\text{Rank}(A) = 2$  and  $[-1 \ -4 \ 1]^T$  is a basis for the one-dimensional  $N(A)$ .

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

### ***Exercise***

Find a basis for  $x - 2y + 3z = 0$  in  $\mathbb{R}^3$ .

Find a basis for the intersection of that plane with  $xy$  plane. Then find a basis for all vectors perpendicular to the plane.

### **Solution**

This plane is the nullspace of the matrix

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The special solutions:  $s_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$   $s_2 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$  give a basis for the nullspace, and for the plane.

The intersection of this plane with the  $xy$ -plane is a line  $(x, -2x, 3x)$  and the vector  $(1, -2, 3)^T$  lies in the  $xy$ -plane.

The vector  $v_3 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$  is perpendicular to both vectors  $s_1$  and  $s_2$ : the space vectors perpendicular to a plane  $\mathbb{R}^3$  is one-dimensional, it gives a basis.

### Exercise

$\mathbf{U}$  comes from  $\mathbf{A}$  by subtracting row 1 from row 3:

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Find the bases for the two column spaces. Find the bases for the two row spaces. Find bases for the two nullspaces.

### Solution

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 - 3R_2$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \end{array}$$

a) The pivots are in the first two columns, so one possible basis for  $C(\mathbf{A})$  is  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\}$  and for

$$C(\mathbf{U}) \text{ is } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

b) Both  $\mathbf{A}$  and  $\mathbf{U}$  have the same nullspace  $N(\mathbf{A}) = N(\mathbf{U})$ ,

$$\text{with basis } \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

c) Both  $\mathbf{A}$  and  $\mathbf{U}$  have the same row space

$$C(\mathbf{A}^T) = C(\mathbf{U}^T), \quad \text{with basis } \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

### Exercise

Write a 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives  $c_1 P_1 + \dots + c_5 P_5 = 0$ , and check entries to prove  $c_i$  is zero.) The five permutation matrices are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

### Solution

Assume:

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad P_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad P_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad P_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$P_1 + P_2 + P_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\text{and } P_4 + P_5 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$P_1 + P_2 + P_3 - P_4 - P_5 = I$$

$$c_1 P_1 + c_2 P_2 + c_3 P_3 + c_4 P_4 + c_5 P_5 = \begin{pmatrix} c_3 & c_1 + c_4 & c_2 + c_5 \\ c_1 + c_5 & c_2 & c_3 + c_4 \\ c_2 + c_4 & c_3 + c_5 & c_1 \end{pmatrix} = 0$$

$$c_1 = c_2 = c_3 = 0 \quad (\text{diagonal})$$

$$\begin{pmatrix} 0 & 0 + c_4 & 0 + c_5 \\ 0 + c_5 & 0 & 0 + c_4 \\ 0 + c_4 & 0 + c_5 & 0 \end{pmatrix} = 0$$

$$\underline{c_4 = c_5 = 0}$$

### Exercise

Choose three independent columns of  $A = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 12 & 15 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix}$ . Then choose a different three independent

columns. Explain whether either of these choices forms a basis for  $C(A)$ .

### Solution

$$\begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 12 & 15 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix} \quad R_2 - 2R_1$$

$$\begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix} \quad \begin{matrix} 2R_2 - R_2 \\ R_4 - R_2 \end{matrix}$$

$$\begin{pmatrix} 4 & 0 & 1 & 2 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 0 & 1 & 2 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \frac{1}{9}R_3$$

$$\begin{pmatrix} 4 & 0 & 1 & 2 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_1 - 2R_3$$

$$\begin{pmatrix} 4 & 0 & 1 & 0 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} \frac{1}{4}R_1 \\ \frac{1}{6}R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{7}{6} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\text{Rank}(A) = 3$ , the columns space is 3 which form a basis of  $C(A)$ . The variable is  $x_3$

If  $x_3 = 1$

$$\begin{pmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{7}{6} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{aligned} x_1 + \frac{1}{4}x_3 &= 0 \\ x_2 + \frac{7}{6}x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

$$\underline{x_1 = -\frac{1}{4} \quad x_2 = -\frac{7}{6} \quad x_4 = 0}$$

$$N(A) \text{ is spanned by } x_n = \begin{pmatrix} -\frac{1}{4} \\ -\frac{7}{6} \\ 1 \\ 0 \end{pmatrix}, \text{ which gives the relation of the columns.}$$

The special solution  $x_n$  gives a relation  $-\frac{1}{4}\vec{v}_1 - \frac{7}{6}\vec{v}_2 + \vec{v}_3 = 0$ . If we take any two columns from the first three columns and the column 4, they will span a three-dimensional space since there will be no relation among them. Hence, they form a basis of  $C(A)$ .

### ***Exercise***

Which of the following sets of vectors are bases for  $\mathbb{R}^2$ ?

- a)  $\{(2, 1), (3, 0)\}$
- b)  $\{(0, 0), (1, 3)\}$

### **Solution**

$$a) \quad k_1(2, 1) + k_2(3, 0) = (0, 0)$$

$$k_1(2, 1) + k_2(3, 0) = (b_1, b_2)$$

$$\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} = -3 \neq 0$$

Therefore, the vectors  $\{(2, 1), (3, 0)\}$  are linearly independent and span  $\mathbb{R}^2$ , so they form a basis for  $\mathbb{R}^2$ .

$$b) \quad k_1(0, 0) + k_2(1, 3) = (0, 0)$$

$$k_1(0, 0) + k_2(1, 3) = (b_1, b_2)$$

$$\begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} = 0$$

Therefore; the vectors  $\{(0, 0), (1, 3)\}$  are linearly dependent, so they don't form a basis for  $\mathbb{R}^2$ .

### ***Exercise***

Which of the following sets of vectors are bases for  $\mathbb{R}^3$ ?

$$a) \quad \{(1, 0, 0), (2, 2, 0), (3, 3, 3)\}$$

$$c) \quad \{(2, -3, 1), (4, 1, 1), (0, -7, 1)\}$$

$$b) \quad \{(3, 1, -4), (2, 5, 6), (1, 4, 8)\}$$

$$d) \quad \{(1, 6, 4), (2, 4, -1), (-1, 2, 5)\}$$

### **Solution**

$$a) \quad \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{vmatrix} = 6 \neq 0$$

Therefore, the set of vectors are linearly independent.

The set form a basis for  $\mathbb{R}^3$ .

$$b) \quad \begin{vmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{vmatrix} = 26 \neq 0$$

Therefore, the set of vectors are linearly independent.

The set form a basis for  $\mathbb{R}^3$ .

$$c) \quad \begin{vmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Therefore, the set of vectors are linearly dependent.

The set don't form a basis for  $\mathbb{R}^3$ .



$$d) \begin{vmatrix} 1 & 2 & -1 \\ 6 & 4 & 2 \\ 4 & -1 & 5 \end{vmatrix} = \underline{\underline{0}}$$

Therefore; the set of vectors are linearly dependent.

The set don't form a basis for  $\mathbb{R}^3$ .

### ***Exercise***

Let  $V$  be the space spanned by  $\vec{v}_1 = \cos^2 x$ ,  $\vec{v}_2 = \sin^2 x$ ,  $\vec{v}_3 = \cos 2x$

- Show that  $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  is not a basis for  $V$ .
- Find a basis for  $V$ .

### ***Solution***

$$\begin{aligned} a) \quad \cos 2x &= \cos^2 x - \sin^2 x \\ k_1 \cos^2 x + k_2 \sin^2 x + k_3 \cos 2x &= 0 \\ k_1 \cos^2 x + k_2 \sin^2 x + k_3 (\cos^2 x - \sin^2 x) &= 0 \\ (k_1 + k_3) \cos^2 x + (k_2 - k_3) \sin^2 x &= 0 \\ \begin{cases} k_1 + k_3 = 0 & \rightarrow k_1 = -k_3 \\ k_2 - k_3 = 0 & \rightarrow k_2 = k_3 \end{cases} \end{aligned}$$

$$\text{If } k_3 = -1 \Rightarrow k_1 = 1, \quad k_2 = -1$$

$$(1)\cos^2 x + (-1)\sin^2 x + (-1)\cos 2x = 0$$

This shows that  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  is linearly dependent, therefore it is ***not*** a basis for  $V$ .

- For  $c_1 \cos^2 x + c_2 \sin^2 x = 0$  to hold for all real  $x$  values, we must have  $c_1 = 0$  ( $x = 0$ ) and  $c_2 = 0$  ( $x = \frac{\pi}{2}$ ).

Therefore, the vectors  $\vec{v}_1 = \cos^2 x$   $\vec{v}_2 = \sin^2 x$  are linearly independent.

$$\begin{aligned} v &= k_1 \cos^2 x + k_2 \sin^2 x + k_3 \cos 2x \\ &= (k_1 + k_3) \cos^2 x + (k_2 - k_3) \sin^2 x \end{aligned}$$

This proves that the vectors  $\vec{v}_1 = \cos^2 x$  and  $\vec{v}_2 = \sin^2 x$  span  $V$ .

We can conclude that  $\vec{v}_1 = \cos^2 x$  and  $\vec{v}_2 = \sin^2 x$  can form a basis for  $V$ .

### Exercise

Find the coordinate vector of  $\vec{w}$  relative to the basis  $S = \{\vec{u}_1, \vec{u}_2\}$  for  $\mathbb{R}^2$

a)  $\vec{u}_1 = (1, 0), \vec{u}_2 = (0, 1), \vec{w} = (3, -7)$       d)  $\vec{u}_1 = (1, -1), \vec{u}_2 = (1, 1), \vec{w} = (0, 1)$

b)  $\vec{u}_1 = (2, -4), \vec{u}_2 = (3, 8), \vec{w} = (1, 1)$       e)  $\vec{u}_1 = (1, -1), \vec{u}_2 = (1, 1), \vec{w} = (1, 1)$

c)  $\vec{u}_1 = (1, 1), \vec{u}_2 = (0, 2), \vec{w} = (a, b)$

### Solution

a)  $\vec{u}_1 = (1, 0), \vec{u}_2 = (0, 1), \vec{w} = (3, -7)$

We must first express  $\vec{w}$  as a linear combination of the vectors in  $S$ :  $\vec{w} = c_1 \vec{u}_1 + c_2 \vec{u}_2$

$$\left( \begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -7 \end{array} \right) \quad \begin{array}{l} c_1 = 3 \\ c_2 = -7 \end{array}$$

$$\begin{aligned} (3, -7) &= 3(1, 0) - 7(0, 1) \\ &= 3\vec{u}_1 - 7\vec{u}_2 \end{aligned}$$

Therefore,  $\underline{(\vec{w})_S = (3, -7)}$

b)  $\vec{u}_1 = (2, -4), \vec{u}_2 = (3, 8), \vec{w} = (1, 1)$

Solve:  $\vec{w} = c_1 \vec{u}_1 + c_2 \vec{u}_2$

$$c_1 (2, -4) + c_2 (3, 8) = (1, 1)$$

$$\begin{cases} 2c_1 + 3c_2 = 1 \\ -4c_1 + 8c_2 = 1 \end{cases}$$

$$\left[ \begin{array}{cc|c} 2 & 3 & 1 \\ -4 & 8 & 1 \end{array} \right] \quad R_2 + 2R_1$$

$$\left[ \begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 14 & 3 \end{array} \right] \quad \frac{1}{14}R_2$$

$$\left[ \begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 1 & \frac{3}{14} \end{array} \right] \quad R_1 - 3R_2$$

$$\left[ \begin{array}{cc|c} 2 & 0 & \frac{5}{14} \\ 0 & 1 & \frac{3}{14} \end{array} \right] \quad \frac{1}{2}R_1$$

$$\left[ \begin{array}{cc|c} 1 & 0 & \frac{5}{28} \\ 0 & 1 & \frac{3}{14} \end{array} \right]$$

$$\frac{5}{28}(2, -4) + \frac{3}{14}(3, 8) = (1, 1)$$

$$\text{Therefore, } (\vec{w})_S = \left( \frac{5}{28}, \frac{3}{14} \right)$$

$$c) \quad \vec{u}_1 = (1, 1), \quad \vec{u}_2 = (0, 2), \quad \vec{w} = (a, b)$$

$$\text{Solve: } \vec{w} = c_1 \vec{u}_1 + c_2 \vec{u}_2$$

$$c_1(1, 1) + c_2(0, 2) = (a, b)$$

$$\left\{ \begin{array}{l} c_1 = a \\ c_1 + 2c_2 = b \end{array} \right. \Rightarrow \underline{c_2 = \frac{b-a}{2}}$$

$$a(1, 1) + \frac{b-a}{2}(0, 2) = (a, b)$$

$$\text{Therefore, } (\vec{w})_S = \left( a, \frac{b-a}{2} \right)$$

$$d) \quad \vec{u}_1 = (1, -1), \quad \vec{u}_2 = (1, 1), \quad \vec{w} = (0, 1)$$

$$\text{Solve: } \vec{w} = c_1 \vec{u}_1 + c_2 \vec{u}_2$$

$$c_1(1, -1) + c_2(1, 1) = (0, 1)$$

$$\left\{ \begin{array}{l} c_1 + c_2 = 0 \\ -c_1 + c_2 = 1 \end{array} \right.$$

$$c_1 = \frac{\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}} = -\frac{1}{2}$$

$$c_2 = \frac{\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}}{2} = \frac{1}{2}$$

$$-\frac{1}{2}(1, -1) + \frac{1}{2}(1, 1) = (0, 1)$$

$$\text{Therefore, } (\vec{w})_S = \left( -\frac{1}{2}, \frac{1}{2} \right)$$

$$e) \quad \vec{u}_1 = (1, -1), \quad \vec{u}_2 = (1, 1), \quad \vec{w} = (1, 1)$$

$$\text{Solve: } \vec{w} = c_1 \vec{u}_1 + c_2 \vec{u}_2$$

$$c_1(1, -1) + c_2(1, 1) = (1, 1)$$

$$\begin{cases} c_1 + c_2 = 1 \\ -c_1 + c_2 = 1 \end{cases}$$

$$c_1 = \frac{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}} \equiv 0$$

$$c_2 = \frac{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}}{2} \equiv 1$$

$$0(1, -1) + 1(1, 1) = (1, 1)$$

$$\text{Therefore, } \underline{(\vec{w})_S = (0, 1)}$$

### Exercise

Find the coordinate vector of  $\vec{v}$  relative to the basis  $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$a) \vec{v} = (2, -1, 3), \quad \vec{v}_1 = (1, 0, 0), \quad \vec{v}_2 = (2, 2, 0), \quad \vec{v}_3 = (3, 3, 3)$$

$$b) \vec{v} = (5, -12, 3), \quad \vec{v}_1 = (1, 2, 3), \quad \vec{v}_2 = (-4, 5, 6), \quad \vec{v}_3 = (7, -8, 9)$$

### Solution

$$a) \vec{v} = (2, -1, 3), \quad \vec{v}_1 = (1, 0, 0), \quad \vec{v}_2 = (2, 2, 0), \quad \vec{v}_3 = (3, 3, 3)$$

$$\text{Solve: } c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{v}$$

$$c_1(1, 0, 0) + c_2(2, 2, 0) + c_3(3, 3, 3) = (2, -1, 3)$$

$$\begin{cases} c_1 + 2c_2 + 3c_3 = 2 & \rightarrow c_1 = 2 - 2c_2 - 3c_3 \equiv 3 \\ 2c_2 + 3c_3 = -1 & \rightarrow c_2 = \frac{-3c_3 - 1}{2} \equiv -2 \\ 3c_3 = 3 & \rightarrow c_3 = 1 \end{cases}$$

$$3(1, 0, 0) - 2(2, 2, 0) + 1(3, 3, 3) = (2, -1, 3)$$

$$\text{Therefore, } \underline{(\vec{v})_S = (3, -2, 1)}$$

$$b) \vec{v} = (5, -12, 3), \quad \vec{v}_1 = (1, 2, 3), \quad \vec{v}_2 = (-4, 5, 6), \quad \vec{v}_3 = (7, -8, 9)$$

$$\text{Solve: } c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{v}$$

$$c_1(1, 2, 3) + c_2(-4, 5, 6) + c_3(7, -8, 9) = (5, -12, 3)$$

$$\begin{cases} c_1 - 4c_2 + 7c_3 = 5 \\ 2c_1 + 5c_2 - 8c_3 = -12 \\ 3c_1 + 6c_2 + 9c_3 = 3 \end{cases}$$

$$c_1 = \frac{\begin{vmatrix} 5 & -4 & 7 \\ -12 & 5 & -8 \\ 3 & 6 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & -4 & 7 \\ 2 & 5 & -8 \\ 3 & 6 & 9 \end{vmatrix}} = \frac{-480}{240} = -2$$

$$c_2 = \frac{\begin{vmatrix} 1 & 5 & 7 \\ 2 & -12 & -8 \\ 3 & 3 & 9 \end{vmatrix}}{240} = \frac{0}{240} = 0$$

$$c_3 = \frac{\begin{vmatrix} 1 & -4 & 5 \\ 2 & 5 & -12 \\ 3 & 6 & 3 \end{vmatrix}}{240} = \frac{240}{240} = 1$$

$$-2(1, 2, 3) + 0(-4, 5, 6) + 1(7, -8, 9) = (5, -12, 3)$$

$$\text{Therefore, } (\vec{v})_S = (-2, 0, 1)$$

### Exercise

Show that  $\{A_1, A_2, A_3, A_4\}$  is a basis for  $M_{22}$ , and express  $A$  as a linear combination of the basis vectors

$$a) \quad A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$$

$$b) \quad A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$c) \quad A = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

### Solution

a) Matrices  $\{A_1, A_2, A_3, A_4\}$  are linearly independent if the equation

$$k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4 = \mathbf{0}$$

$$k_1 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + k_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k_4 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \mathbf{0}$$

Has only the trivial solution.

For these matrices to span  $M_{22}$ , it must be expressed every matrix  $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$  as

$$k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4 = \mathbf{A}$$

$$k_1 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + k_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k_4 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$$

The 2 equations can be written as linear systems

$$\begin{array}{rclcl} k_1 + k_2 + k_3 & = & 0 & k_1 + k_2 + k_3 & = a_1 \\ k_2 & = & 0 & k_2 & = a_2 \\ k_1 & + & k_4 = 0 & \text{and} & k_1 & + & k_4 = a_3 \\ k_3 & = & 0 & k_3 & = a_4 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} \right] = -1 \neq 0,$$

That the homogeneous system has only the trivial solution.

$$\{A_1, A_2, A_3, A_4\} \text{ span } M_{22}$$

$$\rightarrow \begin{cases} k_1 + k_2 + k_3 = 6 \\ k_2 = 2 \\ k_1 + k_4 = 5 \\ k_3 = 3 \end{cases}$$

$$\left[ \begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 0 & 3 \end{array} \right] \quad R_3 - R_1$$

$$\left[ \begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & -1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 3 \end{array} \right] \quad \begin{array}{l} R_1 - R_2 \\ R_3 + R_2 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 3 \end{array} \right] \quad \begin{array}{l} R_1 + R_3 \\ \\ R_4 + R_3 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{array} \right] \quad -R_3$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 4 \end{array} \right] \quad \begin{array}{l} R_1 - R_4 \\ \\ R_3 + R_4 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array} \right] \quad \begin{array}{l} k_1 = 1 \\ k_2 = 2 \\ k_3 = 3 \\ k_4 = 4 \end{array}$$

$$\mathbf{A} = \underline{A_1 + 2A_2 + 3A_3 + 4A_4}$$

**b)** Matrices  $\{A_1, A_2, A_3, A_4\}$  are linearly independent if the equation

$$k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4 = \mathbf{0}$$

$$k_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + k_2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + k_3 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + k_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{0}$$

Has only the trivial solution.

For these matrices to span  $M_{22}$ , it must be expressed every matrix  $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$  as

$$k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4 = \mathbf{A}$$

$$k_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + k_2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + k_3 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + k_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

The 2 equations can be written as linear systems

$$\begin{array}{llll}
k_1 & = 0 & k_1 & = a_1 \\
k_1 + k_2 & = 0 & k_1 + k_2 & = a_2 \\
k_1 + k_2 + k_3 & = 0 & k_1 + k_2 + k_3 & = a_3 \\
k_1 + k_2 + k_3 + k_4 & = 0 & k_1 + k_2 + k_3 + k_4 & = a_4
\end{array}$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1 \neq 0, \text{ that the homogeneous system has only the trivial solution.}$$

$$\{A_1, A_2, A_3, A_4\} \text{ span } M_{22}$$

$$\begin{array}{l}
\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array} \right] \\
\begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}
\end{array}$$

$$\begin{array}{l}
\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 \end{array} \right] \\
\begin{array}{l} R_3 - R_2 \\ R_4 - R_2 \end{array}
\end{array}$$

$$\begin{array}{l}
\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array} \right] \\
R_4 - R_3
\end{array}$$

$$\begin{array}{l}
\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right] \\
\begin{array}{l} k_1 = 1 \\ k_2 = -1 \\ k_3 = 1 \\ k_4 = -1 \end{array}
\end{array}$$

$$\mathbf{A} = A_1 - A_2 + A_3 - A_4$$

$$c) \quad k_1 \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + k_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + k_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$$

$$\begin{array}{l}
\left[ \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array} \right] \\
R_2 + R_1
\end{array}$$



$$\left[ \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array} \right] \quad \frac{1}{2}R_2$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array} \right] \quad \begin{array}{l} k_1 = 1 \\ k_2 = 1 \\ k_3 = -1 \\ k_4 = 3 \end{array}$$

$$\underline{\mathbf{A} = A_1 + A_2 - A_3 + 3A_4}$$

### Exercise

Consider the eight vectors

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

- List all of the one-element, linearly dependent sets formed from these.
- What are the two-element, linearly dependent sets?
- Find a three-element set spanning a subspace of dimension three, and dimension of two? One? Zero?
- Which four-element sets are linearly dependent? Explain why.

### Solution

$$a) \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ zero vector is the only linearly dependent.}$$

b) The set that contains zero vector and any other vector.

c) 2-dimension:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{1-dimensional subspace if we allow duplicates (zero vector)} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

d) All four-element sets are linearly dependent in three-dimensional space.

### Exercise

Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space

$$a) \begin{cases} x_1 + x_2 - x_3 = 0 \\ -2x_1 - x_2 + 2x_3 = 0 \\ -x_1 \quad \quad + x_3 = 0 \end{cases}$$

$$d) \begin{cases} x + y + z = 0 \\ 3x + 2y - 2z = 0 \\ 4x + 3y - z = 0 \\ 6x + 5y + z = 0 \end{cases}$$

$$b) \begin{cases} 3x_1 + x_2 + x_3 + x_4 = 0 \\ 5x_1 - x_2 + x_3 - x_4 = 0 \end{cases}$$

$$e) \begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 \quad \quad + 5x_3 = 0 \\ \quad \quad x_2 + x_3 = 0 \end{cases}$$

$$c) \begin{cases} x_1 - 3x_2 + x_3 = 0 \\ 2x_1 - 6x_2 + 2x_3 = 0 \\ 3x_1 - 9x_2 + 3x_3 = 0 \end{cases}$$

### Solution

$$a) \left[ \begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{array} \right] \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \begin{array}{l} R_2 - R_2 \\ \\ R_3 - R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} x_1 - x_3 = 0 \rightarrow \underline{x_1 = x_3} \\ \underline{x_2 = 0} \\ \end{array}$$

The solution:  $(x_1, 0, x_1) = x_1 (1, 0, 1)$

The solution space has dimension 1 and a basis  $\underline{(1, 0, 1)}$

$$b) \left[ \begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array} \right] \quad 3R_2 - 5R_1$$

$$\left[ \begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 0 & -8 & -2 & -8 & 0 \end{array} \right] \quad 8R_1 + R_2$$

$$\left[ \begin{array}{cccc|c} 24 & 0 & 6 & 0 & 0 \\ 0 & -8 & -2 & -8 & 0 \end{array} \right] \begin{array}{l} \frac{1}{24}R_1 \\ -\frac{1}{8}R_2 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{array} \right] \quad \begin{array}{l} x_3 = s \\ x_4 = t \end{array} \quad \begin{array}{l} \underline{x_1} = -\frac{1}{4}x_3 = \underline{-s} \\ \underline{x_2} = -\frac{1}{4}x_3 - x_4 = \underline{-\frac{1}{4}s - t} \end{array}$$

The solution:

$$\begin{aligned} (x_1, x_2, x_3, x_4) &= \left(-\frac{1}{4}s, -\frac{1}{4}s - t, s, t\right) \\ &= s\left(-\frac{1}{4}, -\frac{1}{4}, 1, 0\right) + t(0, -1, 0, 1) \end{aligned}$$

The solution space has dimension 2 and a basis  $\underline{\left(-\frac{1}{4}, -\frac{1}{4}, 1, 0\right), (0, -1, 0, 1)}$

$$\begin{array}{l} c) \left[ \begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \\ \left[ \begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad x_1 - 3x_2 + x_3 = 0 \rightarrow x_1 = 3x_2 - x_3 \end{array}$$

The solution:

$$\begin{aligned} (x_1, x_2, x_3) &= (3x_2 - x_3, x_2, x_3) \\ &= x_2(3, 1, 0) + x_3(-1, 0, 1) \end{aligned}$$

The solution space has dimension 2 and a basis  $\underline{(3, 1, 0) \text{ and } (-1, 0, 1)}$

$$\begin{array}{l} d) \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 2 & -2 & 0 \\ 4 & 3 & -1 & 0 \\ 6 & 5 & 1 & 0 \end{array} \right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - 4R_1 \\ R_4 - 6R_1 \end{array} \\ \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -1 & -5 & 0 \end{array} \right] \quad \begin{array}{l} R_1 + R_2 \\ R_3 - R_2 \\ R_4 - R_2 \end{array} \\ \left[ \begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad -R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} x = 4z \\ y = -5z \end{array}$$

The solution:  $(x, y, z) = (4z, -5z, z) = z(4, -5, 1)$

The solution space has dimension 1 and a basis  $(4, -5, 1)$

$$e) \left[ \begin{array}{ccc} 2 & 1 & 3 \\ 1 & 0 & 5 \\ 0 & 1 & 1 \end{array} \right] \quad 2R_2 - R_1$$

$$\left[ \begin{array}{ccc} 2 & 1 & 3 \\ 0 & -1 & 7 \\ 0 & 1 & 1 \end{array} \right] \quad \begin{array}{l} R_1 + R_2 \\ R_3 + R_2 \end{array}$$

$$\left[ \begin{array}{ccc} 2 & 0 & 10 \\ 0 & -1 & 7 \\ 0 & 0 & 8 \end{array} \right] \quad \begin{array}{l} \frac{1}{2}R_1 \\ -R_2 \\ \frac{1}{8}R_3 \end{array}$$

$$\left[ \begin{array}{ccc} 1 & 0 & 5 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{array} \right] \quad \begin{array}{l} R_1 - 5R_3 \\ R_2 + 7R_3 \end{array}$$

$$\left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

No basis and dimension = 0

### Exercise

If  $AS = SA$  for the shift matrix  $S$ . Show that  $A$  must have this special form:

$$\text{If } \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\text{then } A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$$

“The subspace of matrices that commute with the shift  $S$  has dimension \_\_\_\_\_.”

**Solution**

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow d = g = h = 0 \quad b = f \quad a = e = i$$

$$A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$$

The subspace of matrices that commute with the shift  $S$  has dimension 3, because the matrix has only three variables.

### Exercise

Find bases for the following subspaces of  $\mathbb{R}^3$

- a) All vectors of the form  $(a, b, c, 0)$
- b) All vectors of the form  $(a, b, c, d)$ , where  $d = a + b$  and  $c = a - b$ .
- c) All vectors of the form  $(a, b, c, d)$ , where  $a = b = c = d$ .

### Solution

- a) The subspace can be expressed as  $\text{span } S = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)\}$  is a set of linearly independent vectors. Therefore;  $S$  forms a basis for the subspace, so its dimension is 3.
- b) The subspace contains all vectors  $(a, b, a+b, a-b) = a(1, 0, 1, 1) + b(0, 1, 1, -1)$ , the set  $S = \{(1, 0, 1, 1), (0, 1, 1, -1)\}$  is linearly independent vectors. Therefore;  $S$  forms a basis for the subspace, so its dimension is 2.
- c) The subspace contains all vectors  $(a, a, a, a) = a(1, 1, 1, 1)$ , we can express the set  $S = \{(1, 1, 1, 1)\}$  as  $\text{span } S$  and it is linearly independent. Therefore,  $S$  forms a basis for the subspace, so its dimension is 1.

### Exercise

Find a basis for the null space of  $A$ .

$$a) \quad A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$$

$$b) \quad A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

$$c) \quad A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$$

### Solution

$$a) \quad A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \quad \begin{array}{l} R_2 - 5R_1 \\ R_3 - 7R_1 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix} \quad \begin{array}{l} R_1 + R_2 \\ R_3 - R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \rightarrow x_1 = 16x_3 = 16t \\ \rightarrow x_2 = 19x_3 = 19t \end{array}$$

$$\text{The general form of the solution of } A\vec{x} = \vec{0} \text{ is } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$$

Therefore, the vector  $\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$  forms a basis for the null space of  $A$ .

$$b) \quad A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{bmatrix} \quad R_3 + R_2$$

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad -\frac{1}{7}R_2$$

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 - 4R_2$$

$$\begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{l} x_1 = -x_3 + \frac{2}{7}x_4 = -t + \frac{2}{7}s \\ x_2 = -x_3 - \frac{4}{7}x_4 = -t - \frac{4}{7}s \end{array}$$

The general form of the solution of  $A\vec{x} = \vec{0}$  is 
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the vectors  $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$  form a basis for the null space of  $A$ .

c) 
$$\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix} \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 + R_1 \\ R_4 - 2R_1 \end{array}$$

$$\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & -14 & -14 & -14 & -28 \\ 0 & 4 & 4 & 4 & 8 \\ 0 & -5 & -5 & -5 & -10 \end{bmatrix} \quad \begin{array}{l} -\frac{1}{14}R_2 \\ \frac{1}{4}R_3 \\ -\frac{1}{5}R_4 \end{array}$$

$$\begin{array}{l}
\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
\begin{array}{l}
R_1 - 4R_2 \\
R_3 - R_2 \\
R_4 - R_2 \\
\rightarrow \begin{cases} x_1 = -x_3 - 2x_4 - x_5 = -r - 2s - t \\ x_2 = -x_3 - x_4 - 2x_5 = -r - s - 2t \end{cases}
\end{array}$$

The general form of the solution of  $A\vec{x} = \vec{0}$  is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the vectors  $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ , and  $\begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$  form a basis for the null space of  $A$ .

### Exercise

Find a basis for the subspace of  $\mathbb{R}^4$  spanned by the given vectors

a)  $(1, 1, -4, -3)$ ,  $(2, 0, 2, -2)$ ,  $(2, -1, 3, 2)$

b)  $(-1, 1, -2, 0)$ ,  $(3, 3, 6, 0)$ ,  $(9, 0, 0, 3)$

### Solution

a)  $(1, 1, -4, -3)$ ,  $(2, 0, 2, -2)$ ,  $(2, -1, 3, 2)$

$$\begin{pmatrix} 1 & 1 & -4 & -3 \\ 2 & 0 & 2 & -2 \\ 2 & -1 & 3 & 2 \end{pmatrix}
\begin{array}{l}
R_2 - 2R_1 \\
R_3 - 2R_1
\end{array}$$

$$\begin{pmatrix} 1 & 1 & -4 & -3 \\ 0 & -2 & 10 & 4 \\ 0 & -3 & 11 & 8 \end{pmatrix}
\begin{array}{l}
-\frac{1}{2}R_2
\end{array}$$



$$\begin{pmatrix} 1 & 1 & -4 & -3 \\ 0 & 1 & -5 & -2 \\ 0 & -3 & 11 & 8 \end{pmatrix} \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 3R_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & -4 & 2 \end{pmatrix} \quad -\frac{1}{4}R_3$$

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & 1 & -\frac{1}{2} \end{pmatrix} \quad \begin{array}{l} R_1 - R_3 \\ R_2 + 5R_3 \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{9}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{pmatrix}$$

A basis for the subspace is  $\left(1, 0, 0, -\frac{1}{2}\right), \left(0, 1, 0, -\frac{9}{2}\right), \left(0, 0, 1, -\frac{1}{2}\right)$

**b)**  $(-1, 1, -2, 0), (3, 3, 6, 0), (9, 0, 0, 3)$

$$\begin{pmatrix} -1 & 1 & -2 & 0 \\ 3 & 3 & 6 & 0 \\ 9 & 0 & 0 & 3 \end{pmatrix} \quad -R_1$$

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 3 & 3 & 6 & 0 \\ 9 & 0 & 0 & 3 \end{pmatrix} \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - 9R_1 \end{array}$$

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 9 & -18 & 3 \end{pmatrix} \quad \frac{1}{6}R_2$$

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 9 & -18 & 3 \end{pmatrix} \quad \begin{array}{l} R_1 + R_2 \\ R_3 - 9R_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -18 & 3 \end{pmatrix} \quad -\frac{1}{18}R_3$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{6} \end{pmatrix} \quad R_1 - 2R_3$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{6} \end{pmatrix}$$

A basis for the subspace is  $(1, 0, 0, 0)$ ,  $(0, 1, 0, 0)$ ,  $(0, 0, 1, -\frac{1}{6})$

### Exercise

Determine whether the given vectors form a basis for the given vector space

a)  $\vec{v}_1(3, -2, 1)$ ,  $\vec{v}_2(2, 3, 1)$ ,  $\vec{v}_3(2, 1, -3)$ , in  $\mathbb{R}^3$

b)  $\vec{v}_1=(1, 1, 0, 0)$ ,  $\vec{v}_2=(0, 1, 1, 0)$ ,  $\vec{v}_3=(0, 0, 1, 1)$ ,  $\vec{v}_4=(1, 0, 0, 1)$ , for  $\mathbb{R}^4$

c)  $M_1=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $M_2=\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ ,  $M_3=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ ,  $M_4=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$   $M_{22}$

### Solution

a)  $\vec{v}_1(3, -2, 1)$ ,  $\vec{v}_2(2, 3, 1)$ ,  $\vec{v}_3(2, 1, -3)$ , in  $\mathbb{R}^3$

$$\begin{vmatrix} 3 & 2 & 2 \\ -2 & 3 & 1 \\ 1 & 1 & -3 \end{vmatrix} = 14 \neq 0$$

The given vectors are linearly independent and span  $\mathbb{R}^3$ , so they form a basis for  $\mathbb{R}^3$ .

b)  $\vec{v}_1=(1, 1, 0, 0)$ ,  $\vec{v}_2=(0, 1, 1, 0)$ ,  $\vec{v}_3=(0, 0, 1, 1)$ ,  $\vec{v}_4=(1, 0, 0, 1)$ , for  $\mathbb{R}^4$

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 2 \neq 0$$

The given vectors are linearly independent and span  $\mathbb{R}^4$ , so they form a basis for  $\mathbb{R}^4$ .

$$c) \quad M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad M_{22}$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

They form a basis for  $M_{22}$ .

### ***Exercise***

Find a basis for, and the dimension of, the null space of the given matrix  $\begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & -2 & -2 & 1 \end{bmatrix}$

### **Solution**

$$\begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & -2 & -2 & 1 \end{bmatrix} \quad R_2 - 2R_1$$

$$\begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & -4 & 0 & -1 \end{bmatrix} \quad 4R_1 + R_2$$

$$\begin{bmatrix} 8 & 0 & -4 & 3 \\ 0 & -4 & 0 & -1 \end{bmatrix} \quad \begin{array}{l} \frac{1}{8}R_1 \\ -\frac{1}{4}R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & \frac{3}{8} \\ 0 & 1 & 0 & \frac{1}{4} \end{bmatrix} \quad \begin{array}{l} x_1 = -\frac{1}{2}x_3 - \frac{3}{8}x_4 \\ x_2 = -\frac{1}{4}x_4 \end{array}$$

$$\mathbf{x} = x_3 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{3}{8} \\ \frac{1}{4} \\ 0 \\ 1 \end{bmatrix}$$

$$\text{The bases are: } \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\frac{3}{8} \\ \frac{1}{4} \\ 0 \\ 1 \end{bmatrix}$$

Dimension: 2

### Exercise

Let  $\mathbb{R}$  be the set of all real numbers and let  $\mathbb{R}^+$  be the set of all positive real numbers. Show that  $\mathbb{R}^+$  is a vector space over  $\mathbb{R}$  under the addition

$$\alpha \oplus \beta = \alpha\beta \quad \alpha, \beta \in \mathbb{R}^+$$

And the scalar multiplication

$$a \odot \alpha = \alpha^a \quad \alpha \in \mathbb{R}^+, a \in \mathbb{R}$$

Find the dimension of the vector space. Is  $\mathbb{R}^+$  also a vector space if the scalar multiplication is instead defined as

$$a \otimes \alpha = a^\alpha \quad \alpha \in \mathbb{R}^+, a \in \mathbb{R}?$$

### Solution

$$\begin{aligned} ab \odot \alpha &= \alpha^{ab} & \alpha \in \mathbb{R}^+, a, b \in \mathbb{R} \\ &= (\alpha^b)^a \\ &= a \odot (\alpha^b) \\ &= a \odot (b \odot \alpha) \end{aligned}$$

Since for  $\alpha \in \mathbb{R}^+$ , then

$$\alpha = (\log \alpha) \odot 10$$

Thus  $\{10\}$  is a basis, therefore the dimension of the vector space is **1**.

$\mathbb{R}^+$  is not a vector space over  $\mathbb{R}$  with respect to  $\otimes$ .

Since,

$$\begin{aligned} 2 \otimes (1 \oplus 1) &= 2 \otimes ((1)(1)) \\ &= 2 \otimes 1 \\ &= 2^1 \\ &= \underline{2} \end{aligned}$$

$$\begin{aligned} (2 \otimes 1) \oplus (2 \otimes 1) &= (2^1) \oplus (2^1) \\ &= 2 \oplus 2 \\ &= (2)(2) \\ &= \underline{4} \end{aligned}$$

$$2 \neq 4$$

$$2 \otimes (1 \oplus 1) \neq (2 \otimes 1) \oplus (2 \otimes 1)$$

## ***Solution***

### **Section 2.8 – Row and Column Spaces**

#### ***Exercise***

List the row vectors and column vectors of the matrix

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ 3 & 5 & 7 & -1 \\ 1 & 4 & 2 & 7 \end{bmatrix}$$

#### **Solution**

Row vectors:

$$r_1 = [2 \quad -1 \quad 0 \quad 1], \quad r_2 = [3 \quad 5 \quad 7 \quad -1], \quad r_3 = [1 \quad 4 \quad 2 \quad 7]$$

Column vectors:

$$c_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}, \quad c_4 = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$$

#### ***Exercise***

Express the product  $A\vec{x}$  as a linear combination of the column vectors of  $A$ .  $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

#### **Solution**

$$\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

#### ***Exercise***

Express the product  $A\vec{x}$  as a linear combination of the column vectors of  $A$ .  $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$

#### **Solution**

$$\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = -2 \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 6 \\ -1 \end{bmatrix} + 5 \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$$

### Exercise

Express the product  $A\vec{x}$  as a linear combination of the column vectors of  $A$ .

$$\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

### Solution

$$\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = -1 \begin{bmatrix} -3 \\ 5 \\ 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 6 \\ -4 \\ 3 \\ 8 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}$$

### Exercise

Determine whether  $\vec{b}$  is in the column space of  $A$ , and if so, express  $\vec{b}$  as a linear combination of the column vectors of  $A$ .

$$A = \begin{bmatrix} 1 & 3 \\ 4 & -6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$$

### Solution

$$\left[ \begin{array}{cc|c} 1 & 3 & -2 \\ 4 & -6 & 10 \end{array} \right] \quad R_2 - 4R_1$$

$$\left[ \begin{array}{cc|c} 1 & 3 & -2 \\ 0 & -18 & 18 \end{array} \right] \quad -\frac{1}{18}R_2$$

$$\left[ \begin{array}{cc|c} 1 & 3 & -2 \\ 0 & 1 & -1 \end{array} \right] \quad R_1 - 3R_2$$

$$\left[ \begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array} \right]$$

$$\begin{bmatrix} -2 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ -6 \end{bmatrix}$$

### Exercise

Determine whether  $\vec{b}$  is in the column space of  $A$ , and if so, express  $\vec{b}$  as a linear combination of the column vectors of  $A$ .

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$$

### Solution

$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 9 & 3 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array} \right] \quad \begin{array}{l} R_2 - 9R_1 \\ R_3 - R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 12 & -8 & -44 \\ 0 & 2 & 0 & -6 \end{array} \right] \quad \frac{1}{4}R_2$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 3 & -2 & -11 \\ 0 & 2 & 0 & -6 \end{array} \right] \quad \begin{array}{l} 3R_1 + R_2 \\ 3R_3 - 2R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 3 & 0 & 1 & 4 \\ 0 & 3 & -2 & -11 \\ 0 & 0 & 4 & 4 \end{array} \right] \quad \frac{1}{4}R_3$$

$$\left[ \begin{array}{ccc|c} 3 & 0 & 1 & 4 \\ 0 & 3 & -2 & -11 \\ 0 & 0 & 1 & 1 \end{array} \right] \quad \begin{array}{l} R_1 - R_3 \\ R_2 + 2R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 3 & 0 & 0 & 3 \\ 0 & 3 & 0 & -9 \\ 0 & 0 & 1 & 1 \end{array} \right] \quad \begin{array}{l} \frac{1}{3}R_1 \\ \frac{1}{3}R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 9 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The system  $A\vec{x} = \vec{b}$  is consistent and  $\vec{b}$  is in the column space of  $A$ .

### Exercise

Determine whether  $\vec{b}$  is in the column space of  $A$ , and if so, express  $\vec{b}$  as a linear combination of the column vectors of  $A$ .

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

### Solution

$$\left[ \begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 2 \end{array} \right] \quad \begin{array}{l} R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & 4 \end{array} \right] \quad \begin{array}{l} R_1 + R_2 \\ R_3 - R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{array} \right] \quad \begin{array}{l} -R_2 \\ \frac{1}{3}R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad R_2 + R_3$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad \text{~~0=1~~}$$

The system  $A\vec{x} = \vec{b}$  is inconsistent and  $\vec{b}$  is not in the column space of  $A$ .

### Exercise

Determine whether  $\vec{b}$  is in the column space of  $A$ , and if so, express  $\vec{b}$  as a linear combination of the column vectors of  $A$ .

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 7 \end{bmatrix}$$

### Solution



$$\left[ \begin{array}{cccc|c} 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 2 & 1 & 3 \\ 1 & 2 & 1 & 3 & 5 \\ 0 & 1 & 2 & 2 & 7 \end{array} \right] \quad R_3 - R_1$$

$$\left[ \begin{array}{cccc|c} 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 & 7 \end{array} \right] \quad \begin{array}{l} R_1 - 2R_2 \\ R_4 - R_2 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & -4 & -1 & -2 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{array} \right] \quad \begin{array}{l} R_1 + 4R_3 \\ R_2 - 2R_3 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 7 & 2 \\ 0 & 1 & 0 & -3 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{array} \right] \quad \begin{array}{l} R_1 - 7R_4 \\ R_2 + 3R_4 \\ R_3 - 2R_4 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & -26 \\ 0 & 1 & 0 & 0 & 13 \\ 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & 4 \end{array} \right]$$

$$\begin{bmatrix} 4 \\ 3 \\ 5 \\ 7 \end{bmatrix} = -26 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 13 \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} - 7 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

The system  $A\vec{x} = \vec{b}$  is consistent and  $\vec{b}$  is in the column space of  $A$

### Exercise

Suppose that  $x_1 = -1$ ,  $x_2 = 2$ ,  $x_3 = 4$ ,  $x_4 = -3$  is a solution of a nonhomogeneous linear system  $A\vec{x} = \vec{b}$  and that the solution set of the homogeneous system  $A\vec{x} = \vec{0}$  is given by the formulas

$$x_1 = -3r + 4s, \quad x_2 = r - s, \quad x_3 = r, \quad x_4 = s$$

- a) Find a vector form of the general solution of  $A\vec{x} = \vec{0}$
- b) Find a vector form of the general solution of  $A\vec{x} = \vec{b}$

### Solution

a)  $x_1 = -3r + 4s, \quad x_2 = r - s, \quad x_3 = r, \quad x_4 = s$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

b) Special Solution:  $x_1 = -1, \quad x_2 = 2, \quad x_3 = 4, \quad x_4 = -3$

$$x_p = \begin{pmatrix} -1 \\ 2 \\ 4 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 4 \\ -3 \end{pmatrix} + r \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 4 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

### Exercise

Find the vector form of the general solution of the given linear system  $A\vec{x} = \vec{b}$ ; then use that result to find the vector form of the general solution of  $A\vec{x} = \vec{0}$ .

$$\begin{cases} x_1 - 3x_2 = 1 \\ 2x_1 - 6x_2 = 2 \end{cases}$$

### Solution

$$\begin{bmatrix} 1 & -3 & | & 1 \\ 2 & -6 & | & 2 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & -3 & | & 1 \\ 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \underline{x_1 = 1 + 3x_2}$$

The solution of  $A\vec{x} = \vec{b}$  is

$$x_1 = 1 + 3t, \quad x_2 = t \text{ *or* }$$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

The general form of the solution  $A\vec{x} = \vec{0}$  is  $\vec{x} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

### ***Exercise***

Find the vector form of the general solution of the given linear system  $A\vec{x} = \vec{b}$ ; then use that result to find the vector form of the general solution of  $A\vec{x} = \vec{0}$ .

$$\begin{cases} x_1 + x_2 + 2x_3 = 5 \\ x_1 + x_3 = -2 \\ 2x_1 + x_2 + 3x_3 = 3 \end{cases}$$

### ***Solution***

$$\begin{bmatrix} 1 & 1 & 2 & | & 5 \\ 1 & 0 & 1 & | & -2 \\ 2 & 1 & 3 & | & 3 \end{bmatrix} \xrightarrow{\begin{matrix} R_2 - R_1 \\ R_3 - 2R_1 \end{matrix}}$$

$$\begin{bmatrix} 1 & 1 & 2 & | & 5 \\ 0 & -1 & -1 & | & -7 \\ 0 & -1 & -1 & | & -7 \end{bmatrix} \xrightarrow{\begin{matrix} R_1 + R_2 \\ R_3 - R_2 \end{matrix}}$$

$$\begin{bmatrix} 1 & 0 & 1 & | & -2 \\ 0 & -1 & -1 & | & -7 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \xrightarrow{\begin{matrix} -R_2 \\ R \end{matrix}}$$

$$\begin{bmatrix} 1 & 0 & 1 & | & -2 \\ 0 & 1 & 1 & | & 7 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \underline{\begin{matrix} x_1 = -2 - x_3 \\ x_2 = 7 - x_3 \end{matrix}}$$

The solution of  $A\vec{x} = \vec{b}$  is

$$x_1 = -2 - t, \quad x_2 = 7 - t, \quad x_3 = t \text{ or}$$

$$\vec{x} = \begin{bmatrix} -2 \\ 7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

The general form of the solution of  $A\vec{x} = \vec{0}$  is

$$\vec{x} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

### ***Exercise***

Find the vector form of the general solution of the given linear system  $A\vec{x} = \vec{b}$ ; then use that result to find the vector form of the general solution of  $A\vec{x} = \vec{0}$ .

$$\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 = 4 \\ -2x_1 + x_2 + 2x_3 + x_4 = -1 \\ -x_1 + 3x_2 - x_3 + 2x_4 = 3 \\ 4x_1 - 7x_2 - 5x_4 = -5 \end{cases}$$

### **Solution**

$$\left[ \begin{array}{cccc|c} 1 & 2 & -3 & 1 & 4 \\ -2 & 1 & 2 & 1 & -1 \\ -1 & 3 & -1 & 2 & 3 \\ 4 & -7 & 0 & -5 & -5 \end{array} \right] \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 + R_1 \\ R_4 - 4R_1 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 2 & -3 & 1 & 4 \\ 0 & 5 & -4 & 3 & 7 \\ 0 & 5 & -4 & 3 & 7 \\ 0 & -15 & 12 & -9 & 21 \end{array} \right] \begin{array}{l} 5R_1 - 2R_2 \\ \\ R_3 - R_2 \\ R_4 + 3R_2 \end{array}$$

$$\left[ \begin{array}{cccc|c} 5 & 0 & -7 & -1 & 6 \\ 0 & 5 & -4 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} \frac{1}{5}R_1 \\ \frac{1}{5}R_2 \\ \\ \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & -\frac{7}{5} & -\frac{1}{5} & \frac{6}{5} \\ 0 & 1 & -\frac{4}{5} & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{array}{l} x_1 = \frac{6}{5} + \frac{7}{5}x_3 + \frac{1}{5}x_4 \\ x_2 = \frac{7}{5} + \frac{4}{5}x_3 - \frac{3}{5}x_4 \end{array}$$

The solution of  $A\vec{x} = \vec{b}$  is

$$\vec{x} = \begin{bmatrix} \frac{6}{5} \\ \frac{7}{5} \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} \frac{7}{5} \\ \frac{4}{5} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{5} \\ -\frac{3}{5} \\ 0 \\ 1 \end{bmatrix}$$

The general form of the solution of  $A\vec{x} = \vec{0}$  is

$$\vec{x} = s \begin{bmatrix} \frac{7}{5} \\ \frac{4}{5} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{5} \\ -\frac{3}{5} \\ 0 \\ 1 \end{bmatrix}$$

### ***Exercise***

Find the vector form of the general solution of the given linear system  $A\vec{x} = \vec{b}$ ; then use that result to find the vector form of the general solution of  $A\vec{x} = \vec{0}$ .

$$\begin{cases} x_1 - 2x_2 + x_3 + 2x_4 = -1 \\ 2x_1 - 4x_2 + 2x_3 + 4x_4 = -2 \\ -x_1 + 2x_2 - x_3 - 2x_4 = 1 \\ 3x_1 - 6x_2 + 3x_3 + 6x_4 = -3 \end{cases}$$

### **Solution**

$$\left[ \begin{array}{cccc|c} 1 & -2 & 1 & 2 & -1 \\ 2 & -4 & 2 & 4 & -2 \\ -1 & 2 & -1 & -2 & 1 \\ 3 & -6 & 3 & 6 & -3 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 - 3R_1 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & -2 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \underline{x_1 = -1 + 2x_2 - x_3 - 2x_4}$$

Let  $x_2 = s$   $x_3 = t$   $x_4 = r$

The solution of  $A\vec{x} = \vec{b}$  is

$$\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The general form of the solution of  $A\vec{x} = \vec{0}$  is

$$\vec{x} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

## Exercise

Given the vectors  $\vec{v}_1 = (1, 2, 0)$  and  $\vec{v}_2 = (2, 3, 0)$

- Are they linearly independent?
- Are they a basis for any space?
- What space  $\mathbf{V}$  do they span?
- What is the dimension of that space?
- What matrices  $A$  have  $\mathbf{V}$  as their column space?
- Which matrices have  $\mathbf{V}$  as their nullspace?
- Describe all vectors  $\vec{v}_3$  that complete a basis  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  for  $\mathbb{R}^3$ .

## Solution

- $\vec{v}_1, \vec{v}_2$  are independent – the only combination to give  $\vec{0}$  is  $0\vec{v}_1 + 0\vec{v}_2$ .
- Yes, they are a basis for whatever space  $\mathbf{V}$  they span.
- That space  $\mathbf{V}$  contains all vectors  $(x, y, 0)$ . It is the  $xy$  plane in  $\mathbb{R}^3$ .
- The dimension of  $\mathbf{V}$  is 2 since the basis contains 2 vectors.
- This  $\mathbf{V}$  is the column space of any 3 by  $n$  matrix  $A$  of rank 2, if every column is a combination of  $\vec{v}_1$  and  $\vec{v}_2$ . In particular  $A$  could just have columns  $\vec{v}_1$  and  $\vec{v}_2$ .

- f)** This  $\mathbf{V}$  is the nullspace of any  $m$  by 3 matrix  $\mathbf{B}$  of rank 1, if every row is a multiple of  $(0, 0, 1)$ . In particular, take  $B = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ . Then  $B\vec{v}_1 = \vec{0}$  and  $B\vec{v}_2 = \vec{0}$ .
- g)** Any third vector  $\vec{v}_3 = (a, b, c)$  will complete a basis for  $\mathbb{R}^3$  provided  $c \neq 0$ .

### Exercise

a) Let  $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Show that relative to an  $xyz$ -coordinate system in 3-space the null space of  $A$  consists of all points on the  $z$ -axis and that the column space consists of all points in the  $xy$ -plane.

- b) Find a  $3 \times 3$  matrix whose null space is the  $x$ -axis and whose column space is the  $yz$ -plane.

### Solution

a)  $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  *Interchange  $R_1$  &  $R_2$*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x = 0 \\ y = 0 \\ z = t \end{array}$$

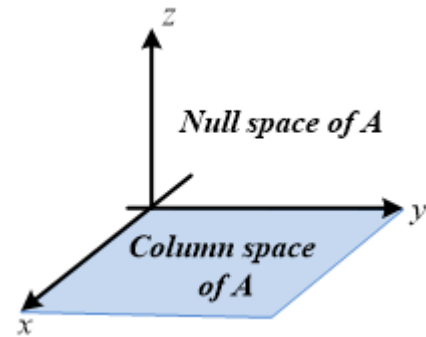
The general form of the solution of  $A\vec{x} = \vec{0}$  is,

$$t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the null space of  $A$  is the  $z$ -axis, and the column space is the span of

$$c_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ which is all linear combinations of } y \text{ and } x \text{ (} xy\text{-plane)}$$

b)  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



### Exercise

If we add an extra column  $\vec{b}$  to a matrix  $A$ , then the column space gets larger unless \_\_\_\_\_. Give an example where the column space gets larger and an example where it doesn't. Why is  $A\vec{x} = \vec{b}$  solvable exactly when the column space doesn't get larger – it is the same for  $A$  and  $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ ?

### Solution

If we add an extra column  $\vec{b}$  to a matrix  $A$ , then the column space gets larger unless **it contains  $\vec{b}$**  that is a linear combination of the columns of  $A$ .

Let  $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ; then the column space gets larger if  $\vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and it doesn't if  $\vec{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

The equation  $A\vec{x} = \vec{b}$  is solvable exactly when  $\vec{b}$  is a (nontrivial) linear combination of the column of  $A$ .

The equation  $A\vec{x} = \vec{b}$  is solvable exactly when  $\vec{b}$  lies in the column space, when the column space doesn't get larger.

### Exercise

For which right sides (find a condition on  $b_1, b_2, b_3$ ) are these solvable. (Use the column space  $C(A)$  and the equation  $A\vec{x} = \vec{b}$ )

$$a) \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$b) \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

### Solution

a) The column space consists of the vectors for

$$\begin{pmatrix} x_1 + 4x_2 + 2x_3 \\ 2x_1 + 8x_2 + 4x_3 \\ -x_1 - 4x_2 - 2x_3 \end{pmatrix} \text{ is } \begin{pmatrix} z \\ 2z \\ -z \end{pmatrix} = z \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

They are scalar multiples of  $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$



b) By substituting  $x_1 + 4x_2$  with new variable  $z$ , then the column space consists of the vectors

$$\begin{pmatrix} x_1 + 4x_2 \\ 2x_1 + 9x_2 \\ -x_1 - 4x_2 \end{pmatrix} = z \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

They are linear combinations of  $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

### Exercise

Show that the matrices  $A$  and  $[A \ AB]$  (with extra columns) have the same column space. But find a square matrix with  $C(A^2)$  smaller than  $C(A)$ . Important point: An  $n$  by  $n$  matrix has  $C(A) = \mathbb{R}^n$  exactly when  $A$  is an \_\_\_\_\_ matrix.

### Solution

Each column of  $AB$  is a combination of the columns of  $A$  (the combining coefficients are the entries in the corresponding column of  $B$ ). So, any combination of the columns of  $[A \ AB]$  is a combination of the columns of  $A$  alone. Thus,  $A$  and  $[A \ AB]$  have the same column space.

Let  $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ; then  $A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ , so  $C(A^2) = \{0\}$ .

$C(A)$  is the line through  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

Any  $n$  by  $n$  matrix has  $C(A) = \mathbb{R}^n$  exactly when  $A$  is an **invertible** matrix, because  $Ax = b$  is solvable for any given  $b$  when  $A$  is invertible.

### Exercise

The column of  $AB$  are combinations of the columns of  $A$ . This means: The column space of  $AB$  is contained in (possibly equal to) to the column space of  $A$ . Give an example where the column spaces  $A$  and  $AB$  are not equal.

### Solution

The column space of  $AB$  is contained in (possibly equal to) to the column space of  $A$ .

$B = 0$  and  $A \neq 0$  is a case when  $AB = 0$  has a smaller column space than  $A$ .

### Exercise

Find a square matrix  $A$  where  $C(A^2)$  (the column space of  $A^2$  is smaller than  $C(A)$ .

### Solution

For example,  $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ; then  $A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ .

Thus  $C(A)$  is generated by vector  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , which is of one dimensional, but  $C(A^2)$  is a zero space.

Hence,  $C(A^2)$  is strictly smaller than  $C(A)$ .

### Exercise

Suppose  $A\vec{x} = \vec{b}$  and  $C\vec{x} = \vec{b}$  have the same (complete) solutions for every  $\vec{b}$ . Is true that  $A = C$ ?

### Solution

Yes, if  $A = C$ , let  $\vec{y}$  be any vector of the correct size, and set  $\vec{b} = A\vec{y}$ . Then  $\vec{y}$  is a solution to

$A\vec{x} = \vec{b}$  and it is also a solution to  $C\vec{x} = \vec{b}$ ;

$$\vec{b} = A\vec{y} = C\vec{y}$$

### Exercise

Apply Gauss-Jordan elimination to  $U\vec{x} = 0$  and  $U\vec{x} = c$ . Reach  $R\vec{x} = 0$  and  $R\vec{x} = d$ :

$$[U \quad 0] = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \quad [U \quad c] = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

Solve  $Rx = 0$  to find  $x_n$  (its free variable is  $x_2 = 1$ ).

Solve  $Rx = d$  to find  $x_p$  (its free variable is  $x_2 = 0$ ).

### Solution

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \quad \frac{1}{4}R_2$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad R_1 - 3R_2$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The free variable is  $x_2$ , since it is the only one. We have to let  $x_2 = 1$

$$\begin{cases} x_1 + 2x_2 = 0 \\ x_3 = 0 \end{cases} \rightarrow x_1 = -2x_2$$

The special solution is  $s_1(-2, 1, 0)$

$$\bar{x}_n = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix} \quad \frac{1}{4}R_2$$

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad R_1 - 3R_2$$

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The free variable is  $x_2$  that implies to  $x_2 = 0$

$$\begin{cases} x_1 + 2x_2 = -1 \\ x_3 = 2 \end{cases} \rightarrow x_1 = -1$$

The particular solution is  $\bar{x}_p = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

## Exercise

Which of the following subsets of  $\mathbb{R}^3$  are actually subspaces?

- a) The plane of vectors  $(b_1, b_2, b_3)$  with  $b_1 = b_2$
- b) The plane of vectors with  $b_1 = 1$ .
- c) The vectors with  $b_1 b_2 b_3 = 0$ .
- d) All linear combinations of  $v = (1, 4, 0)$  and  $w = (2, 2, 2)$ .
- e) All vectors that satisfies  $b_1 + b_2 + b_3 = 0$
- f) All vectors with  $b_1 \leq b_2 \leq b_3$ .

## Solution

- a) This is subspace

- For  $\vec{v} = (b_1, b_2, b_3)$  with  $b_1 = b_2$  and  $\vec{w} = (c_1, c_2, c_3)$  with  $c_1 = c_2$  the sum  $\vec{v} + \vec{w} = (b_1 + c_1, b_2 + c_2, b_3 + c_3)$  is in the same set as  $b_1 + c_1 = b_2 + c_2$
- For an element  $\vec{v} = (b_1, b_2, b_3)$  with  $b_1 = b_2$ ,  $c\vec{v} = (cb_1, cb_2, cb_3)$  and  $cb_1 = cb_2$ , thus it is in the same set.

**b)** This is not a subspace. For example, for  $\vec{v} = (1, 0, 0)$  and  $c\vec{v} = -\vec{v} = (-1, 0, 0)$  is not in the set.

**c)** This is not a subspace. For example, for  $\vec{v} = (1, 1, 0)$  and  $\vec{w} = (1, 0, 1)$  are in the set, but their sum  $\vec{v} + \vec{w} = (2, 1, 1)$  is not in the set.

**d)** This is subspace, by definition of linear combination.

- For 2 vectors  $\vec{v}_1 = \alpha_1 \vec{v} + \beta_1 \vec{w}$  and  $\vec{v}_2 = \alpha_2 \vec{v} + \beta_2 \vec{w}$  the sum

$$\begin{aligned}\vec{v}_1 + \vec{v}_2 &= \alpha_1 \vec{v} + \beta_1 \vec{w} + \alpha_2 \vec{v} + \beta_2 \vec{w} \\ &= (\alpha_1 + \alpha_2) \vec{v} + (\beta_1 + \beta_2) \vec{w}\end{aligned}$$

is still the linear combination of  $v$  and  $w$ .

- For an element  $\vec{v}_1 = \alpha_1 \vec{v} + \beta_1 \vec{w}$ ,  $c\vec{v}_1 = c\alpha_1 \vec{v} + c\beta_1 \vec{w}$  is still the linear combination of  $\vec{v}$  and  $\vec{w}$ , thus it is the same set

**e)** This is subspace, these are the vectors orthogonal to  $(1, 1, 1)$

- For  $\vec{v} = (b_1, b_2, b_3)$  with  $b_1 + b_2 + b_3 = 0$   
and  $\vec{w} = (c_1, c_2, c_3)$  with  $c_1 + c_2 + c_3 = 0$

The sum  $\vec{v} + \vec{w} = (b_1 + c_1, b_2 + c_2, b_3 + c_3)$  is in the same set as

$$b_1 + c_1 + b_2 + c_2 + b_3 + c_3 = 0$$

- For an element  $\vec{v} = (b_1, b_2, b_3)$  with  $b_1 + b_2 + b_3 = 0$ ,  $c\vec{v} = (cb_1, cb_2, cb_3)$  and  $cb_1 + cb_2 + cb_3 = 0$ , thus it is in the same set.

**f)** This is not a subspace. For example, for  $\vec{v} = (1, 2, 3)$  and  $-\vec{v} = (-1, -2, -3)$  is not in the set.

### Exercise

We are given three different vectors  $\vec{b}_1, \vec{b}_2, \vec{b}_3$ . Construct a matrix so that the equations  $A\vec{x} = \vec{b}_1$  and  $A\vec{x} = \vec{b}_2$  are solvable, but  $A\vec{x} = \vec{b}_3$  is not solvable.

a) How can you decide if this possible?

b) How could you construct  $A$ ?

### Solution

The equations  $A\vec{x} = \vec{b}_1$  and  $A\vec{x} = \vec{b}_2$  will be solvable.

$$A\vec{x} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}_3 \text{ (solvable?)}$$

If  $A\vec{x} = \vec{b}_3$  is not solvable, we have the desired matrix  $A$ .

If  $A\vec{x} = \vec{b}_3$  is solvable, then it is not possible to construct  $A$ .

When the column space contains  $\vec{b}_1$  and  $\vec{b}_2$ , it will have to contain their linear combinations.

So  $\vec{b}_3$  would necessarily be in that column space and  $A\vec{x} = \vec{b}_3$  would necessarily be solvable.

### Exercise

For which vectors  $(b_1, b_2, b_3)$  do these systems have a solution?

$$a) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$c) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$b) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

### Solution

$$a) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \begin{aligned} &\rightarrow x_1 + x_2 + x_3 = b_1 \\ &\rightarrow x_2 + x_3 = b_2 \\ &\rightarrow x_3 = b_3 \end{aligned}$$

$$\begin{cases} x_1 = b_1 - b_2 - b_3 \\ x_2 = b_2 - b_3 \\ x_3 = b_3 \end{cases} \Rightarrow (b_1 - b_2 - b_3, b_2 - b_3, b_3)$$

Solution for every  $b$ .

$$b) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \begin{aligned} &\rightarrow x_1 + x_2 + x_3 = b_1 \\ &\rightarrow x_2 + x_3 = b_2 \\ &\rightarrow 0x_3 = b_3 \end{aligned}$$

$$\begin{cases} x_1 = b_1 - b_2 \\ x_2 = b_2 \\ 0 = b_3 \end{cases} \Rightarrow (b_1 - b_2, b_2, 0)$$

Solvable only if  $b_3 = 0$

$$c) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad R_2 - R_3$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 0 & b_3 - b_2 \\ 0 & 0 & 1 & b_3 \end{array} \right]$$

$$\Rightarrow b_3 - b_2 = 0$$

$$\underline{b_3 = b_2}$$

$$\begin{cases} x_1 = b_1 - 2b_3 \\ 0 = 0 \\ x_3 = b_3 \end{cases} \Rightarrow (b_1 - 2b_3, 0, b_3)$$

Solvable only if  $b_3 = b_2$

### Exercise

Find a basis for the null space of  $A$ .  $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

### Solution

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix} \begin{array}{l} \\ R_3 - 2R_1 \\ R_4 - 3R_1 \\ R_5 + 2R_1 \end{array}$$

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & 6 & 0 & -3 \end{bmatrix} \begin{array}{l} R_1 + R_2 \\ R_3 - R_2 \\ R_4 - R_2 \\ R_5 - R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 8 & 2 & -2 \\ 0 & 3 & 6 & 0 & -3 \\ 0 & 0 & -12 & 0 & 5 \\ 0 & 0 & -12 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \frac{1}{3}R_2 \\ R_4 - R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 8 & 2 & -2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & -12 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} 3R_1 + 2R_3 \\ 6R_3 + R_3 \end{array}$$

$$\begin{bmatrix} 3 & 0 & 0 & 6 & 4 \\ 0 & 6 & 0 & 0 & -1 \\ 0 & 0 & -12 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \frac{1}{3}R_1 \\ \frac{1}{6}R_2 \\ -\frac{1}{12}R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & \frac{4}{3} \\ 0 & 1 & 0 & 0 & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & -\frac{5}{12} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let  $x_4 = s$   $x_5 = t$

$$\begin{cases} x_1 = -2x_4 - \frac{4}{3}x_5 = 2s - \frac{4}{3}t \\ x_2 = \frac{1}{6}x_5 = \frac{1}{6}t \\ x_3 = \frac{5}{12}x_5 = \frac{5}{12}t \end{cases}$$

The general form of the solution of  $A\vec{x} = \vec{0}$  is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{4}{3} \\ \frac{1}{6} \\ \frac{5}{12} \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the vectors  $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -\frac{4}{3} \\ \frac{1}{6} \\ \frac{5}{12} \\ 0 \\ 1 \end{bmatrix}$  form a basis for the null space of  $A$ .

### ***Exercise***

Is it true that if  $m = n$  then the row space of  $A$  equals the column space.

### **Solution**

False

Counterexample, let  $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$

We have  $m = n = 2$ , but the row space of  $A$  contains multiples of  $(1, 2)$  while the column space of  $A$  contains multiples of  $(1, 3)$ .



### Exercise

If the row space equals the column space the  $A^T = A$

#### Solution

False,

Counterexample, let  $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ .

Here, the row space and column space are both equal to all of  $\mathbb{R}^2$  (since  $A$  is invertible).

But  $A \neq A^T$

### Exercise

If  $A^T = -A$ , then the row space of  $A$  equals the column space.

#### Solution

True,

The row space of  $A$  equals to the column space of  $A^T$ , which for this particular  $A$  equals the column space of  $-A$ .

Since  $A$  and  $-A$  have the same fundamental subsequences. We conclude that the row space of  $A$  equals the column space of  $A$ .

### Exercise

Does the matrices  $A$  and  $-A$  share the same 4 subspaces?

#### Solution

True.

The nullspaces are identical because  $A\vec{x} = 0 \Leftrightarrow -A\vec{x} = 0$

The column spaces are identical because any vector  $\vec{v}$  that can be expressed as  $\vec{v} = A\vec{x}$  for some  $\vec{x}$  can also be expressed as  $\vec{v} = (-A)(-\vec{x})$

### Exercise

Is  $A$  and  $B$  share the same 4 subspaces then  $A$  is multiple of  $B$ .

#### Solution

False

Any invertible  $2 \times 2$  matrix will have  $\mathbb{R}^2$  as its column space and row space and zero vector as its (left and right) nullspace.

However, it is easy to produce 2 invertible  $2 \times 2$  matrices that are not multiples of each other, as example

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

### ***Exercise***

Suppose  $A\vec{x} = \vec{b}$  &  $C\vec{x} = \vec{b}$  have the same (complete) solutions for every  $\vec{b}$ . Is it true that  $A = C$

### **Solution**

If  $A\vec{x} = C\vec{x} = \vec{b}$  for all vectors  $\vec{x}$  of the correct size.

Then, it is true that  $A = C$

### ***Exercise***

$A$  and  $A^T$  have the same left nullspace?

### **Solution**

False,

Counterexample, take any a  $1 \times 2$  matrix, such as  $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$ .

The left nullspace of  $A$  contains vectors in  $\mathbb{R}$  while the left nullspace of  $A^T$ , which is the right nullspace of  $A$ , contains vectors in  $\mathbb{R}^2$ .

So, they can't be the same.

## ***Solution***

## **Section 2.9 – Rank and the Fundamental Matrix Spaces**

### ***Exercise***

Verify that  $\text{rank}(A) = \text{rank}(A^T)$

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

### **Solution**

$$\begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix} \quad \begin{array}{l} R_2 + 3R_1 \\ R_3 + 2R_1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 7 & 17 & 2 \\ 0 & 7 & 17 & 2 \end{bmatrix} \quad R_3 - R_2$$

$$\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 7 & 17 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad 7R_1 - 2R_2$$

$$\begin{bmatrix} 7 & 0 & -6 & -4 \\ 0 & 7 & 17 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \frac{1}{7}R_1 \\ \frac{1}{7}R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -\frac{6}{7} & -\frac{4}{7} \\ 0 & 1 & \frac{17}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = 2$$

$$A^T = \begin{bmatrix} 1 & -3 & -2 \\ 2 & 1 & 3 \\ 4 & 5 & 9 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & -2 \\ 2 & 1 & 3 \\ 4 & 5 & 9 \\ 0 & 2 & 2 \end{bmatrix} \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 4R_1 \end{array}$$

$$\begin{bmatrix} 1 & -3 & -2 \\ 0 & 7 & 7 \\ 0 & 17 & 17 \\ 0 & 2 & 2 \end{bmatrix} \quad \begin{array}{l} \frac{1}{7}R_2 \\ \\ \frac{1}{2}R_4 \end{array}$$

$$\begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & 1 \\ 0 & 17 & 17 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{array}{l} R_1 + 3R_2 \\ \\ R_3 - 17R_2 \\ R_4 - R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A^T) = 2$$

$$\underline{\text{rank}(A) = \text{rank}(A^T) = 2}$$

### ***Exercise***

Find the rank and nullity of the matrix; then verify that the values obtained satisfy

$$\text{rank}(A) + N(A) = n$$

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$$

### **Solution**

$$\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \quad \begin{array}{l} \\ R_2 - 5R_1 \\ R_3 - 7R_1 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix} \quad \begin{array}{l} R_1 + R_2 \\ \\ R_3 - R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = 2$$

$$\text{nullity}(A) = 1$$

$$\text{rank}(A) + \text{nullity}(A) = 2 + 1 = 3 = \textcolor{red}{n} \leftarrow \text{number of columns}$$

### ***Exercise***

Find the rank and nullity of the matrix; then verify that the values obtained satisfy

$$\text{rank}(A) + N(A) = n$$

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

### **Solution**

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{bmatrix} \begin{array}{l} 7R_1 + 4R_2 \\ \\ R_3 + R_2 \end{array}$$

$$\begin{bmatrix} 7 & 0 & 7 & -2 \\ 0 & -7 & -7 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \frac{1}{7}R_1 \\ -\frac{1}{7}R_2 \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = 2$$

$$\text{nullity}(A) = 1$$

$$\text{rank}(A) + \text{nullity}(A) = 2 + 1 = 3 = \textcolor{red}{n}$$

### Exercise

Find the rank and nullity of the matrix; then verify that the values obtained satisfy

$$\text{rank}(A) + N(A) = n$$

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$$

### Solution

$$\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix} \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 + R_1 \\ R_4 - 2R_1 \end{array}$$

$$\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & -14 & -14 & -14 & -28 \\ 0 & 4 & 4 & 4 & 8 \\ 0 & -5 & -5 & -5 & -10 \end{bmatrix} \begin{array}{l} \\ -\frac{1}{14}R_2 \\ \frac{1}{4}R_3 \\ -\frac{1}{5}R_4 \end{array}$$

$$\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix} \begin{array}{l} R_1 - 4R_2 \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = 2$$

$$\text{nullity}(A) = 2$$

$$\text{rank}(A) + \text{nullity}(A) = 2 + 2 = 4 = n$$

### Exercise

Find the rank and nullity of the matrix; then verify that the values obtained satisfy

$$\text{rank}(A) + N(A) = n$$

$$A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$$

### Solution

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix} \begin{array}{l} \\ R_3 - 2R_1 \\ R_4 - 3R_1 \\ R_5 + 2R_1 \end{array}$$

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & 6 & 0 & -3 \end{bmatrix} \begin{array}{l} \\ R_4 - R_3 \\ R_5 - R_2 \end{array}$$

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 + R_2 \\ \\ R_3 - R_2 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 8 & 2 & -2 \\ 0 & 3 & 6 & 0 & -3 \\ 0 & 0 & -12 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \frac{1}{3}R_2 \\ -\frac{1}{12}R_3 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 8 & 2 & -2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 & -\frac{5}{12} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 - 8R_3 \\ R_2 - 2R_3 \\ \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & \frac{4}{3} \\ 0 & 1 & 0 & 0 & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & -\frac{5}{12} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = 3$$

$$NS(A) = 2$$

$$\text{Number of columns} = 5$$

$$\text{rank}(A) + NS(A) = 3 + 2 = 5 = n$$

### Exercise

If  $A$  is an  $m \times n$  matrix, what is the largest possible value for its rank and the smallest possible value of the nullity of  $A$ .

### Solution

The largest possible value for the rank of an  $m \times n$  matrix:

- $n$  if  $m \geq n$  (when every column of the  $rref(A)$  contains a leading 1)
- $m$  if  $m < n$  (when every row of the  $rref(A)$  contains a leading 1)

The smallest possible value for the nullity of an  $m \times n$  matrix:

- $0$  if  $m \geq n$  (when every column of the  $rref(A)$  contains a leading 1)
- $n - m$  if  $m < n$  (when every row of the  $rref(A)$  contains a leading 1)

### Exercise

Discuss how the rank of  $A$  varies with  $t$ .

$$a) A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix} \quad b) A = \begin{bmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{bmatrix}$$

### Solution

$$\begin{aligned} a) \begin{vmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{vmatrix} &= t + t + t - t^3 - 1 - 1 \\ &= -t^3 + 3t - 2 = 0 \end{aligned}$$



Solve for  $t$ :  $\underline{t = 1, -2, -2}$

Therefore,  $\text{rank}(A) = 3$  for  $\forall t - \{1, -2, -2\}$

$$\text{If } t = 1, A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = 1$$

$$\text{If } t = -2, A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \begin{array}{l} \\ \\ R_3 + R_2 \end{array}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ -\frac{1}{3}R_2 \\ \end{array}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 - R_2 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = 2$$

$$\begin{aligned} b) \quad \begin{vmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{vmatrix} &= 6t^2 + 6 + 9 - 6 - 6t - 9t \\ &= 6t^2 - 15t + 9 = 0 \end{aligned}$$

Solve for  $t$ :  $t=1, \frac{3}{2}$

Therefore,  $\text{rank}(A)=3$  for  $\forall t - \left\{1, \frac{3}{2}\right\}$

$$\text{If } t=1, A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & 1 \end{bmatrix} \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 + R_1 \end{array}$$

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 + R_2 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} -\frac{1}{3}R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A)=2$$

$$\text{If } t=\frac{3}{2}, A = \begin{bmatrix} \frac{3}{2} & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & \frac{3}{2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2} & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & \frac{3}{2} \end{bmatrix} \begin{array}{l} 2R_1 \\ \\ 2R_3 \end{array}$$

$$\begin{bmatrix} 3 & 6 & -2 \\ 3 & 6 & -2 \\ -2 & -6 & 3 \end{bmatrix} \begin{array}{l} \\ R_2 - R_1 \\ 3R_3 + 2R_1 \end{array}$$

$$\begin{bmatrix} 3 & 6 & -2 \\ 0 & 0 & 0 \\ 0 & -6 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & -2 \\ 0 & -6 & 5 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 + R_2$$

$$\begin{bmatrix} 3 & 0 & 3 \\ 0 & -6 & 5 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \frac{1}{3}R_1 \\ -\frac{1}{6}R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{5}{6} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = 2$$

### ***Exercise***

Are there values of  $r$  and  $s$  for which

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

Has rank 1? Has rank 2? If so, find those values.

### ***Solution***

Since the third column will always have a nonzero entry, the *rank* will never be 1. (row 1 and row 4 never have a nonzero entry).

If  $r = 2$  and  $s = 1$ , that implies to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\rightarrow \text{rank} = 2$$

### Exercise

Find the row reduced form  $\mathbf{R}$  and the rank  $r$  of  $\mathbf{A}$  (those depend on  $c$ ).

Which are the pivot columns of  $\mathbf{A}$ ? Which variables are free? What are the special solutions and the nullspace matrix  $\mathbf{N}$  (always depending on  $c$ )?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & c \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} c & c \\ c & c \end{bmatrix}$$

### Solution

$$a) \quad c \neq 4 \quad \mathbf{R} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$\text{rank}(\mathbf{A}) = 2$ , the pivot columns are 1 and 3, the second variable  $x_2$  is free.

$$\text{The special solution: } x_2 = 1 \text{ which yields to } \mathbf{N} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$c = 4 \quad \mathbf{R} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$\text{rank}(\mathbf{A}) = 1$ , the pivot column is column 1, the second and third variables  $x_2, x_3$  are free.

$$\text{The special solution goes into } \mathbf{N} = \begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$b) \quad c \neq 0 \quad \mathbf{R} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

$\text{rank}(\mathbf{A}) = 1$ , the pivot column is the first column, the second variable  $x_2$  is free.

$$\text{The special solution: } x_2 = 1 \text{ which yields to } \mathbf{N} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$c = 0 \quad \mathbf{R} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\text{rank}(\mathbf{A}) = 0$ , the matrix has no pivot column, and both variables are free.

$$\text{The special solution is: } \mathbf{N} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

### Exercise

Find the row reduced form  $R$  and the rank  $r$  of  $A$  (those depend on  $c$ ).

Which are the pivot columns of  $A$ ? Which variables are free? What are the special solutions and the nullspace matrix  $N$  (always depending on  $c$ )?

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$$

### Solution

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & c-1 & 0 & 0 \end{bmatrix} \quad \text{Interchange } R_2 \text{ \& } R_1$$

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a) If  $c = 1$ , then

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This has only one pivot (first column) and 3 free variables  $x_2, x_3, x_4$ .

$$\text{The nullspace matrix: } \begin{pmatrix} -1 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

b) If  $c \neq 1$ , then

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \frac{1}{c-1} R_2$$

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_2 - R_1$$

$$\begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are two pivots  $(C_1, C_2)$  and 2 free variables  $x_3, x_4$

The nullspace matrix:  $\begin{pmatrix} -2 & -2 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

---

$$A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$$

**a)** If  $c = 1$

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad R_1 - 2R_2$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow 0a + b = 0 \Rightarrow b = 0$$

This has a single pivot in the second column and one free variable with the nullspace matrix

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

**b)** If  $c = 2$

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \quad -R_1$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow a - 2b = 0$$

$$\text{if } b = 1 \quad a = 2$$

This has a single pivot in the first column with the nullspace matrix  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

**c)** Otherwise  $c \neq 1, 2$

$$A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix} \quad \frac{1}{1-c} R_1$$

$$\begin{bmatrix} 1 & \frac{2}{1-c} \\ 0 & 2-c \end{bmatrix} \quad \frac{1}{2-c} R_2$$

$$\begin{bmatrix} 1 & \frac{2}{1-c} \\ 0 & 1 \end{bmatrix} \quad R_1 - \frac{2}{1-c} R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The result is the identity matrix with 2 pivots, which has  $(2 - 2) = 0$  null space.

### Exercise

If  $A$  has a rank  $r$ , then it has an  $r$  by  $r$  sub-matrix  $S$  that is invertible. Remove  $m - r$  rows and  $n - r$  columns to find an invertible sub-matrix  $S$  inside each  $A$  (you could keep the pivot rows and pivot columns of  $A$ ).

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

### Solution

If a matrix  $A$  has rank  $r$ , then the

***(dimension of the column space) = (dimension of the row space) =  $r$***

For the invertible sub-matrix  $S$ , we need to find  $r$  linearly independent rows and  $r$  linearly independent columns.

For matrix  $A$ :

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} \quad R_2 - R_1$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} \quad R_1 - 3R_2$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The 1<sup>st</sup> and 3<sup>rd</sup> columns are linearly independent, and the 1<sup>st</sup> and 2<sup>nd</sup> rows are also linearly independent.

Rank ( $A$ ) = 2.

The sub matrices are:  $S_A = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$   $S_A = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$   $S_A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

For matrix **B**:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \quad R_2 - 2R_1$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Rank (**B**) = 1.

The submatrix is:  $S_A = (1)$

For matrix **C**:

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rank (**C**) = 2.

The submatrix is by disregarding (deleting) 1<sup>st</sup> column and 2<sup>nd</sup> row:  $S_A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

## ***Exercise***

Suppose that column 3 of 4 x 6 matrix is all zero. Then  $x_3$  must be a \_\_\_\_\_ variable. Give one special solution for this matrix.

## **Solution**

The  $x_3$  must be a ***free variable***.

A special solution for this variable can be taken to be.

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



### Exercise

Fill in the missing numbers to make  $A$  rank 1, rank 2, rank 3. (your solution should be 3 matrices)

$$A = \begin{pmatrix} & -3 & \\ 1 & 3 & -1 \\ & 9 & -3 \end{pmatrix}$$

### Solution

$$A = \begin{pmatrix} a & -3 & b \\ 1 & 3 & -1 \\ c & 9 & -3 \end{pmatrix}$$

If  $\text{rank}(A) = 1$ , then we need the 1<sup>st</sup> and 3<sup>rd</sup> to be multiple of the 2<sup>nd</sup> row to get zero in these rows.

$$A = \begin{pmatrix} a & -3 & b \\ 1 & 3 & -1 \\ c & 9 & -3 \end{pmatrix} \begin{array}{l} R_1 + R_2 \\ \\ R_3 - 3R_2 \end{array}$$

$$\begin{pmatrix} a+1 & 0 & b-1 \\ 1 & 3 & -1 \\ c-3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} a+1=0 \\ b-1=0 \\ c-3=0 \end{cases} \rightarrow \begin{cases} a=-1 \\ b=1 \\ c=3 \end{cases}$$

$$A = \begin{pmatrix} -1 & -3 & 1 \\ 1 & 3 & -1 \\ 3 & 9 & -3 \end{pmatrix}$$

If  $\text{rank}(A) = 2$ , then we need the 1<sup>st</sup> **or** 3<sup>rd</sup> to be multiple of the 2<sup>nd</sup> row to get zero row.

$$A = \begin{pmatrix} a & -3 & b \\ 1 & 3 & -1 \\ c & 9 & -3 \end{pmatrix} \begin{array}{l} R_1 + R_2 \\ \\ R_3 - 3R_2 \end{array}$$

$$\begin{pmatrix} a+1 & 0 & b-1 \\ 1 & 3 & -1 \\ c-3 & 0 & 0 \end{pmatrix} \quad \boxed{c \neq 3}$$

$$A = \begin{pmatrix} -1 & -3 & 1 \\ 1 & 3 & -1 \\ 2 & 9 & -3 \end{pmatrix}$$

If rank  $(A) = 3$  (full rank), then the appropriate to start using 0's or 1's to fill the blank.

$$A = \begin{pmatrix} 0 & -3 & 0 \\ 1 & 3 & -1 \\ 1 & 9 & -3 \end{pmatrix} \quad \text{Interchange } R_1 \text{ \& } R_2$$

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & -3 & 0 \\ 1 & 9 & -3 \end{pmatrix} \quad R_3 - R_1$$

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & -3 & 0 \\ 0 & 6 & -2 \end{pmatrix} \quad -\frac{1}{3}R_2$$

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 6 & -2 \end{pmatrix} \quad \begin{matrix} R_1 - 3R_2 \\ R_1 - 6R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad -\frac{1}{2}R_3$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_1 + R_3$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, it has *rank* 3.

### ***Exercise***

Fill out these matrices so that they have rank 1:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{pmatrix} \quad B = \begin{pmatrix} 2 & & \\ 1 & & \\ 2 & 6 & -3 \end{pmatrix} \quad M = \begin{pmatrix} a & b \\ c & \end{pmatrix}$$

### **Solution**

Rank = 1 means that all the rows are multiples of each other.

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & a & b \\ 4 & c & d \end{pmatrix} \xrightarrow[R_3=4R_1]{R_2=2R_1} \begin{matrix} a = 2(2) & b = 2(4) \\ c = 4(2) & d = 4(4) \end{matrix}$$

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & a & b \\ 1 & c & d \\ 2 & 6 & -3 \end{pmatrix} \xrightarrow[R_2=\frac{1}{2}R_3]{R_1=R_3} \begin{matrix} a = 6 & b = -3 \\ c = 3 & d = -\frac{3}{2} \end{matrix}$$

$$B = \begin{pmatrix} 2 & 6 & -3 \\ 1 & 3 & -\frac{3}{2} \\ 2 & 6 & -3 \end{pmatrix}$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{R_2=\frac{c}{a}R_1} d = \frac{c}{a}b$$

$$M = \begin{pmatrix} a & b \\ c & \frac{bc}{a} \end{pmatrix}$$

### ***Exercise***

Suppose  $A$  and  $B$  are  $n$  by  $n$  matrices, and  $AB = I$ . Prove from  $\text{rank}(AB) \leq \text{rank}(A)$  that the  $\text{rank}(A) = n$ . So,  $A$  is invertible and  $B$  must be its two-sided inverse. Therefore  $BA = I$  (which is not so obvious!).

### **Solution**

Since  $A$  is  $n$  by  $n \Rightarrow \text{rank}(A) \leq n$

$$n = \text{rank}(I_n) = \text{rank}(AB) \leq \text{rank}(A)$$

### Exercise

Every  $m$  by  $n$  matrix of rank  $r$  reduces to  $(m$  by  $r)$  times  $(r$  by  $n)$ :

$$A = (\text{pivot columns of } A) (\text{first } r \text{ rows of } R) = (COL)(ROW)^T$$

Write the 3 by 4 matrix  $A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{pmatrix}$  as the product of the 3 by 2 from the pivot columns and the 2 by 4 matrix from  $R$ .

### Solution

$$A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{pmatrix} \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \end{pmatrix} \begin{array}{l} \\ \\ R_3 - 2R_2 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} R_1 - R_2 \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The pivots columns are the 1<sup>st</sup> and 2<sup>nd</sup> column.

$$A = (\text{pivot columns of } A) (\text{first } r \text{ rows of } R) = (COL)(ROW)^T$$

$$\begin{aligned} A &= \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{aligned}$$

### Exercise

Suppose  $R$  is  $m$  by  $n$  matrix of rank  $r$ , with pivot columns first:  $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

- What are the shapes of those 4 blocks?
- Find the right-inverse  $B$  with  $RB = I$  if  $r = m$ .
- Find the right-inverse  $C$  with  $CR = I$  if  $r = n$ .
- What is the reduced row echelon form of  $R^T$  (with shapes)?
- What is the reduced row echelon form of  $R^T R$  (with shapes)?

Prove that  $R^T R$  has the same nullspace as  $R$ . Then show that  $A^T A$  always has the same nullspace as  $A$  (a value fact).

- Suppose you allow elementary column operations on  $A$  as well as elementary row operations (which get to  $R$ ). What is the “row-and-column reduced form” for an  $m$  by  $n$  matrix of rank  $r$ ?

### Solution

$$a) \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} r \times r & r \times (n-r) \\ (m-r) \times r & (m-r) \times (n-r) \end{bmatrix}$$

$$b) R = \begin{bmatrix} I & F \end{bmatrix}$$

$$RB = I \Rightarrow \begin{bmatrix} I & F \end{bmatrix} B = I$$

$$\begin{bmatrix} I & F \end{bmatrix} \begin{pmatrix} M \\ N \end{pmatrix} = I$$

$$IM + FN = I$$

$$\Rightarrow \begin{cases} M = I \\ N = 0 \end{cases} \rightarrow F : r \times (n-r)$$

$$B = \begin{bmatrix} I_{r \times r} \\ 0_{(n-r) \times r} \end{bmatrix}$$

$$c) R = \begin{bmatrix} I & 0 \end{bmatrix}$$

$$CR = I \Rightarrow C \begin{bmatrix} I & 0 \end{bmatrix} = I$$

$$C = \begin{bmatrix} I_{r \times r} & 0_{r \times (m-r)} \end{bmatrix}$$

$$d) R^T = \begin{bmatrix} I_{r \times r} & 0_{(m-r) \times r} \\ F_{r \times (n-r)} & 0_{(m-r) \times (n-r)} \end{bmatrix} \quad R_2 - F_{r \times (n-r)} R_1$$

$$rref(R^T) = \begin{bmatrix} I_{r \times r} & 0_{(m-r) \times r} \\ 0_{(n-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix}$$

$$\begin{aligned} e) \quad R^T R &= \begin{bmatrix} I & 0 \\ F & 0 \end{bmatrix} \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I & IF \\ FI & F^2 \end{bmatrix} \end{aligned}$$

$FI: r \times (n-r) \quad r \times r$ , the inner is not equal but to make work, we can use the  $F$  transpose.

$$(n-r) \times r \quad r \times r \Rightarrow F^T I = F^T$$

$$\begin{aligned} R^T R &= \begin{bmatrix} I & 0 \\ F & 0 \end{bmatrix} \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I & F \\ F & F^2 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} I & F \\ F & F^2 \end{bmatrix} \quad R_2 - FR_1$$

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

$$rref(R^T R) = \begin{bmatrix} I_{r \times r} & F_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix}$$

$= R$

So, that  $N(A) = N(rref(A))$  for any matrix  $A$ . So,  $N(A) = N(R^T R)$

**f)** After getting to  $R$  we can use the column operations to get rid of  $F$ .

$$\begin{bmatrix} I_{r \times r} & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix}$$

### Exercise

True or False (check addition or give a counterexample)

- a) The symmetric matrices in  $M$  (with  $A^T = A$ ) form a subspace.
- b) The skew-symmetric matrices in  $M$  (with  $A^T = -A$ ) form a subspace.
- c) The un-symmetric matrices in  $M$  (with  $A^T \neq A$ ) form a subspace.
- d) Invertible matrices
- e) Singular matrices

### Solution

a) True:  $A^T = A$  and  $B^T = B$  lead to  $(A+B)^T = A^T + B^T = A+B$

b) True:  $A^T = -A$  and  $B^T = -B$  lead to  $(A+B)^T = A^T + B^T = -A-B = -(A+B)$

c) False:  $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

d) False:  $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$  are invertible matrices but  $A+B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$  is not invertible.

$\therefore$  The zero matrix is not invertible but any linear subspace should contain the zero matrix. So, invertible matrices do not form a linear subspace.

e) False:  $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  are singular matrices

But  $A+B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$  is not singular.

### Exercise

$$\text{Let } A = \begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 2 & 4 & -3 & 7 & 0 \\ 3 & 6 & -5 & 10 & -2 \\ 5 & 10 & -9 & 16 & 0 \end{pmatrix}$$

- a) Reduce  $A$  to row-reduced echelon form.
- b) What is the rank of  $A$ ?
- c) What are the pivots?
- d) What are the free variables?
- e) Find the special solutions. What is the nullspace  $N(A)$ ?
- f) Exhibit an  $r \times r$  submatrix of  $A$  which is invertible, where  $r = \text{rank}(A)$ . (An  $r \times r$  submatrix of  $A$  is obtained by keeping  $r$  rows and  $r$  columns of  $A$ )

### Solution

$$a) \quad A = \begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 2 & 4 & -3 & 7 & 0 \\ 3 & 6 & -5 & 10 & -2 \\ 5 & 10 & -9 & 16 & 0 \end{pmatrix} \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 5R_1 \end{array}$$

$$\begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \quad \begin{array}{l} R_3 - R_2 \\ R_4 - R_2 \end{array}$$

$$\begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- b)  $\text{Rank}(A) = 3$
- c) The pivots are  $x_1, x_3, x_5$
- d) The free variables are  $x_2, x_4$

$$e) \quad \begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad -\frac{1}{2}R_3$$



$$\begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad R_1 + 2R_2$$

$$\begin{pmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let  $x = x_2 s_2 + x_4 s_4$

$$Rx = \begin{pmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} x_1 + 2x_2 + 5x_4 = 0 \\ x_3 + x_4 = 0 \\ x_5 = 0 \end{cases}$$

$$1. \text{ Set } x_2 = 1, \quad x_4 = 0 \rightarrow \begin{cases} x_1 + 2 = 0 \Rightarrow x_1 = -2 \\ x_3 = 0 \\ x_5 = 0 \end{cases}$$

The special solution:  $\vec{s}_2 = (-2, 1, 0, 0, 0)$

$$2. \text{ Set } x_2 = 0, \quad x_4 = 1 \rightarrow \begin{cases} x_1 + 5 = 0 \Rightarrow x_1 = -5 \\ x_3 + 1 = 0 \Rightarrow x_3 = -1 \\ x_5 = 0 \end{cases}$$

The special solution:  $\vec{s}_3 = (-5, 0, -1, 1, 0)$

The nullspace is the set  $\left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

- f) The pivot rows and columns must be included in a submatrix. To do that, just take the rows and columns of  $A$  containing pivots, which are columns 1, 3, 5 and rows 1, 2, 3. That will give us a 3 by 3 submatrix. Therefore, this submatrix of  $A$  will be invertible.

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & -3 & 0 \\ 3 & -5 & -2 \end{pmatrix}$$

### Exercise

Let  $A = \begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 2 & 1 & 0 & 0 & -15 \\ 6 & -1 & -8 & -1 & -47 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix}$

- Reduce  $A$  to (ordinary) echelon form.
- What the pivots?
- What are the free variables?
- Reduce  $A$  to row-reduced echelon form.
- Find the special solutions. What is the nullspace  $N(A)$ ?
- What is the rank of  $A$ ?

g) Give the complete solution to  $Ax = b$ , where  $b = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

### Solution

a)  $A = \begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 2 & 1 & 0 & 0 & -15 \\ 6 & -1 & -8 & -1 & -47 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix} \quad \begin{matrix} R_2 + 2R_1 \\ R_3 + 6R_1 \end{matrix}$

$$\begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 0 & 5 & 10 & 0 & -5 \\ 0 & 11 & 22 & -1 & -17 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix} \quad \begin{matrix} 5R_3 - 11R_2 \\ 5R_4 - 2R_2 \end{matrix}$$

$$\begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 0 & 5 & 10 & 0 & -5 \\ 0 & 0 & 0 & -5 & -30 \\ 0 & 0 & 0 & 15 & 90 \end{pmatrix} \quad R_4 + 3R_3$$

$$\begin{pmatrix} \boxed{-1} & 2 & 5 & 0 & 5 \\ 0 & \boxed{5} & 10 & 0 & -5 \\ 0 & 0 & 0 & \boxed{-5} & -30 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**b)** The pivots are  $-1$ ,  $5$ , and  $-5$  (Columns 1, 2, 4)

**c)** The free variables are 3<sup>rd</sup> and 5<sup>th</sup> ( $x_3, x_5$ )

$$d) \begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 0 & 5 & 10 & 0 & -5 \\ 0 & 0 & 0 & -5 & -30 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} -R_1 \\ \frac{1}{5}R_2 \\ -\frac{1}{5}R_3 \end{array}$$

$$\begin{pmatrix} 1 & -2 & -5 & 0 & -5 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad R_1 + 2R_2$$

$$R = \begin{pmatrix} 1 & 0 & -1 & 0 & -7 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**e)** Let  $x = x_3 s_1 + x_5 s_2$

$$R\vec{x} = \begin{pmatrix} 1 & 0 & -1 & 0 & -7 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} x_1 - x_3 - 7x_5 = 0 \\ x_2 + 2x_3 - x_5 = 0 \\ x_4 + 6x_5 = 0 \end{cases}$$

**1.** Set  $x_3 = 1, x_5 = 0$

$$\rightarrow \begin{cases} x_1 - 1 = 0 & \underline{x_1 = 1} \\ x_2 + 2 = 0 & \underline{x_2 = -2} \\ x_4 = 0 \end{cases}$$

The special solution:  $\vec{s}_1 = (1, -2, 1, 0, 0)$

2. Set  $x_3 = 0, x_5 = 1$

$$\rightarrow \begin{cases} x_1 - 7 = 0 & \underline{x_1 = 7} \\ x_2 - 1 = 0 & \underline{x_2 = 1} \\ x_4 + 6 = 0 & \underline{x_4 = -6} \end{cases}$$

The special solution:  $\vec{s}_2 = (7, 1, 0, -6, 1)$

The nullspace is the set  $\left\{ x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 7 \\ 1 \\ 0 \\ -6 \\ 1 \end{pmatrix} \right\}$

f)  $\text{Rank}(A) = 3$

g)  $A\vec{x} = \vec{b}$ , where  $\vec{b} = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

The complete solution = (the particular solution) + (special solution)

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 7 \\ 1 \\ 0 \\ -6 \\ 1 \end{pmatrix}$$

### Exercise

$$\text{Let } A = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- a) Reduce  $A$  to row-reduced echelon form.
- b) What is the rank of  $A$ ?
- c) What the pivots variables?
- d) What are the free variables?
- e) Find the special solutions.
- f) What is the nullspace  $N(A)$ ?

### Solution

$$a) \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} \\ R_2 - R_1 \\ \\ R_3 - 2R_1 \\ \end{array}$$

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & -1 & -4 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} \\ -R_2 \\ R_3 - R_2 \\ R_4 - 2R_2 \\ \end{array}$$

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} R_5 - R_4$$

$$R = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{array}{l} x_1 = -2x_2 - 3x_4 \\ x_3 = -4x_4 \\ x_5 = 0 \end{array}$$

- b)  $\text{Rank}(A) = 3$
- c) The pivots variables are:  $x_1, x_3, x_5$

d) The free variables are:  $x_2, x_4$

e) Let  $x = x_2 s_1 + x_4 s_2$

$$\begin{cases} x_1 = -2x_2 - 3x_4 \\ x_3 = -4x_4 \end{cases}$$

Set  $x_2 = 1, x_4 = 0$

The special solution:  $\vec{s}_1 = (-2, 1, 0, 0, 0)$

Set  $x_2 = 0, x_4 = 1$  ;

The special solution:  $\vec{s}_2 = (-3, 0, -4, 1, 0)$

f) The nullspace is the set  $\left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix} \right\}$

$$N(A) = \begin{pmatrix} -2 & -3 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

### Exercise

$$\text{Let } A = \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix}$$

a) Reduce  $A$  to row-reduced echelon form.

b) What is the rank of  $A$ ?

c) What the pivots?

d) What are the free variables?

e) Find the special solutions.

f) What is the nullspace  $N(A)$ ?

### Solution

$$\begin{aligned}
 a) \quad & \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix} \begin{array}{l} 3R_2 - R_1 \\ 3R_3 - 2R_1 \\ R_4 - 2R_1 \end{array} \\
 & \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 0 & 0 & -3 & -15 & -3 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & -1 & -5 & 0 \end{pmatrix} \begin{array}{l} -R_2 \\ 3R_4 + R_2 \end{array} \\
 & \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 0 & 0 & 3 & 15 & 3 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \begin{array}{l} \frac{1}{3}R_1 \\ \frac{1}{3}R_2 \\ \frac{1}{3}R_3 \\ R_4 - R_3 \end{array} \\
 & \begin{pmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} R_2 - R_3 \end{array} \\
 & \begin{pmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} x_1 = -7x_2 - 3x_4 \\ x_3 = -5x_4 \\ x_5 = 0 \end{array}
 \end{aligned}$$

**b)**  $\text{Rank}(A) = 3$

**c)** The pivots variables are:  $x_1, x_3, x_5$

**d)** The free variables are:  $x_2, x_4$

**e)** Let  $x = x_2 s_1 + x_4 s_2$

$$\begin{cases} x_1 = -7x_2 - 3x_4 \\ x_3 = -5x_4 \end{cases}$$

Set  $x_2 = 1, x_4 = 0$

The special solution:  $\vec{s}_1 = (-7, 1, 0, 0, 0)$

Set  $x_2 = 0, x_4 = 1$  ;

The special solution:  $\vec{s}_2 = (-3, 0, -5, 1, 0)$

f) The nullspace is the set  $\left\{ x_2 \begin{pmatrix} -7 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix} \right\}$

$$N(A) = \begin{pmatrix} -7 & -3 \\ 1 & 0 \\ 0 & -5 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

### Exercise

The 3 by 3 matrix  $A$  has rank 2.

$$A\vec{x} = \vec{b} \quad \text{is} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 + 5x_4 &= b_1 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 &= b_2 \\ 3x_1 + 6x_2 + 7x_3 + 13x_4 &= b_3 \end{aligned}$$

- Reduce  $\begin{bmatrix} A & \vec{b} \end{bmatrix}$  to  $\begin{bmatrix} U & \vec{c} \end{bmatrix}$ , so that  $A\vec{x} = \vec{b}$  becomes triangular system  $U\vec{x} = \vec{c}$ .
- Find the condition on  $(b_1, b_2, b_3)$  for  $A\vec{x} = \vec{b}$  to have a solution
- Describe the column space of  $A$ . Which plane in  $\mathbb{R}^3$ ?
- Describe the nullspace of  $A$ . Which special solutions in  $\mathbb{R}^4$ ?
- Find a particular solution to  $A\vec{x} = (0, 6, -6)$  and then complete solution.

### Solution

$$a) \left[ \begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & -2 & -2 & b_3 - 3b_1 \end{array} \right] \quad R_3 + R_2$$

$$\left[ \begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{array} \right]$$



**b)** The last equation  $b_3 + b_2 - 5b_1 = 0$  shows the solvability condition.

**c) (i)** The column space is the plane containing all combinations of the pivot columns: 1st (1, 2, 3) and 3<sup>rd</sup> (3, 8, 7).

**(ii)** The column space contains all vectors with  $b_3 + b_2 - 5b_1 = 0$ . That makes  $A\vec{x} = \vec{b}$

solvable, so  $\vec{b}$  is in the column space. All columns of  $A$  pass this test  $b_3 + b_2 - 5b_1 = 0$ . This is the equation for the plane in **(i)**.

**d)** The special solutions have free variables:

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 5x_4 = 0 \\ 2x_3 + 2x_4 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = -2x_2 - 2x_4 \\ x_3 = -x_4 \end{cases}$$

Let  $x_2 = 1, x_4 = 0$

$$\Rightarrow \begin{cases} x_1 = -2 \\ x_3 = 0 \end{cases}$$

$$\vec{s}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Let  $x_2 = 0, x_4 = 1$

$$\Rightarrow \begin{cases} x_1 = -2 \\ x_3 = -1 \end{cases}$$

$$\vec{s}_2 = \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

The nullspace  $N(A)$  in  $\mathbb{R}^4$  contains all

$$\vec{x}_n = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

**e)** One particular solution  $x_p$  has free variables = zero.

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 5x_4 = 0 \\ 2x_3 + 2x_4 = 6 \end{cases}$$

$$\begin{cases} x_1 = -2x_2 - 3x_3 - 5x_4 \\ x_3 = 3 - x_4 \end{cases}$$

$$\begin{cases} x_1 = -2x_2 - 9 - 2x_4 \\ x_3 = 3 - x_4 \end{cases}$$

Let  $x_2 = x_4 = 0$

$$\begin{cases} x_1 = -9 \\ x_3 = 3 \end{cases}$$

$$\vec{x}_p = \begin{pmatrix} -9 \\ 0 \\ -3 \\ 0 \end{pmatrix}$$

The complete solution to  $A\vec{x} = (0, 6, -6)$  is  $\vec{x} = \vec{x}_p + \text{all } \vec{x}_n$

$$\vec{x} = \begin{pmatrix} -9 \\ 0 \\ -3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

### ***Exercise***

Find the special solutions and describe the complete solution to  $A\vec{x} = \vec{0}$  for

$$A_1 = 3 \text{ by } 4 \text{ zero matrix} \quad A_2 = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \quad A_3 = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$$

Which are the pivot columns? Which are the free variables? What is the  $R$  (Reduced Row Echelon matrix) in each case?

### **Solution**

$A_1 \vec{x} = \vec{0}$  has 4 solutions. They are the columns  $\vec{s}_1, \vec{s}_2, \vec{s}_3, \vec{s}_4$  of the identity matrix (4 by 4).

The Nullspace is of  $\mathbb{R}^4$ .

The complete solution:  $\vec{x} = c_1 \vec{s}_1 + c_2 \vec{s}_2 + c_3 \vec{s}_3 + c_4 \vec{s}_4$  in  $\mathbb{R}^4$ .

There are no pivot columns; all variables are free; the reduced  $R$  is the same zero matrix as  $A_1$ .

$$A_2 \vec{x} = \vec{0}$$

$$A_2 \vec{x} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x_1 + 2x_2 = 0$$

The vector solution:  $\vec{s} = (-2, 1)$ , The first column of  $A_2$  is its pivot column, and  $x_2$  is the free variable.

$$\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \quad \frac{1}{3}R_1$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad R_2 - R_1$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

$$R_3 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

All variables are free. There are three special solutions to  $A_3 \vec{x} = 0$

$$\vec{s}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{s}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{s}_3 = \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The complete solution:

$$\vec{x} = c_1 \vec{s}_1 + c_2 \vec{s}_2 + c_3 \vec{s}_3$$

$$= c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

### Exercise

Create a 3 by 4 matrix whose special solutions to  $A\vec{x} = \vec{0}$  are  $\vec{s}_1$  and  $\vec{s}_2$  :

$$\vec{s}_1 = \begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{s}_2 = \begin{pmatrix} -2 \\ 0 \\ -6 \\ 1 \end{pmatrix}$$

You could create the matrix  $A$  in row reduced form  $R$ . Then describe all possible matrices  $A$  with the required Nullspace  $N(A) = \text{all combinations of } \vec{s}_1 \text{ and } \vec{s}_2$ .

### Solution

We can write the solution:

$$\vec{x} = x_2 \vec{s}_1 + x_4 \vec{s}_2$$

$$x_2 \begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -6 \\ 1 \end{pmatrix} = \begin{pmatrix} -3x_2 - 2x_4 \\ 2x_2 \\ -6x_4 \\ x_4 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3x_2 - 2x_4 \\ 2x_2 \\ -6x_4 \\ x_4 \end{pmatrix} \rightarrow \begin{cases} x_1 = -3x_2 - 2x_4 \\ x_3 = -6x_4 \end{cases}$$

$$\rightarrow \begin{cases} x_1 + 3x_2 + 2x_4 = 0 \\ x_3 + 6x_4 = 0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The entries 3, 2, 6 are the negatives of  $-3$ ,  $-2$ ,  $-6$  in the special solutions.

Every 3 by 4 matrix has at least one special solution. These  $A$ 's have two.

### Exercise

The plane  $x - 3y - z = 12$  is parallel to the plane  $x - 3y - z = 0$ . One particular point on this plane is  $(12, 0, 0)$ . All points on the plane have the form (fill the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

### Solution

$$x - 3y - z = 12$$

$$x = 3y + z + 12$$

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{pmatrix} 12 + 3y + z \\ y \\ z \end{pmatrix} \\ &= \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

### Exercise

Construct a matrix whose column space contains  $(1, 1, 5)$  and  $(0, 3, 1)$  and whose Nullspace contains  $(1, 1, 2)$ .

### Solution

$$A = \begin{pmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 + 2a \\ 1 + 3 + 2b \\ 5 + 1 + 2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 1 + 2a = 0 & \rightarrow 2a = -1 & \Rightarrow a = -\frac{1}{2} \\ 4 + 2b = 0 & \rightarrow 2b = -4 & \Rightarrow b = -2 \\ 6 + 2c = 0 & \rightarrow 2c = -6 & \Rightarrow c = -3 \end{cases}$$

$$A = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{pmatrix}$$

### Exercise

Construct a matrix whose column space contains  $(1, 1, 0)$  and  $(0, 1, 1)$  and whose Nullspace contains  $(1, 0, 1)$  and  $(0, 0, 1)$ .

### Solution

It is impossible. Matrix  $A$  must be 3 by 3.

Since the nullspace is supposed to contain two independent vectors,  $A$  can have at most  $3 - 2 = 1$  pivots.

Since the column space supposes to contain two independent vectors.  $A$  must have at least 2 pivots.

These conditions can't both be met.

### Exercise

Construct a matrix whose column space contains  $(1, 1, 1)$  and whose Nullspace contains  $(1, 1, 1, 1)$ .

### Solution

The matrix needs to be 3 by 4 matrix.

$$\begin{pmatrix} 1 & a & b & c \\ 1 & d & e & f \\ 1 & g & h & i \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 1 + a + b + c = 0 \\ 1 + d + e + f = 0 \\ 1 + g + h + i = 0 \end{cases}$$

$$\begin{cases} a + b + c = -1 \\ d + e + f = -1 \\ g + h + i = -1 \end{cases}$$

$$\begin{cases} \text{if } b = c = 0 & a = -1 \\ \text{if } d = f = 0 & e = -1 \\ \text{if } g = h = 0 & i = -1 \end{cases}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

### Exercise

How is the Nullspace  $N(C)$  related to the spaces  $N(A)$  and  $N(B)$ , if  $C = \begin{bmatrix} A \\ B \end{bmatrix}$ ?

### Solution

$$Cx = \begin{bmatrix} Ax \\ Bx \end{bmatrix} = 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If and only if  $A\vec{x} = \vec{0}$  and  $B\vec{x} = \vec{0}$

$$N(C) = N(A) \cap N(B)$$

### Exercise

Why does no 3 by 3 matrix have a nullspace that equals its column space?

### Solution

If nullspace = column space, then  $n - r = r$  (there are  $r$  pivots).

For  $n = 3 \Rightarrow 3 = 2r$  is impossible.

### Exercise

If  $AB = 0$  then the column space  $B$  is contained in the \_\_\_\_\_ of  $A$ . Give an example of  $A$  and  $B$ .

### Solution

If  $AB = 0$  then the column space  $B$  is contained in the **nullspace** of  $A$ .

$$\text{Example: } A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

### Exercise

True or false (with reason if true or example to show it is false)

- a) A square matrix has no free variables.
- b) An invertible matrix has no free variables.
- c) An  $m$  by  $n$  matrix has no more than  $n$  pivot variables.
- d) An  $m$  by  $n$  matrix has no more than  $m$  pivot variables.

### Solution

- a) False. Any matrix with fewer than full number of pivots will.  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

- b) True. Since it is invertible, we will get the full number of pivots. The nullspace has dimension, so we have 0 free variables.
- c) True, the number of pivot variables is the dimension of the nullspace, which is at most the number of columns. The nullspace dimension + column space dimension = number of columns.
- d) True, in reduced echelon matrix the pivot columns are all 0 except for a single 1, and there are only up to  $m$  vectors of this type.

### Exercise

Suppose an  $m$  by  $n$  matrix has  $r$  pivots. The number of special solutions is \_\_\_\_\_.

The Nullspace contains only  $x = 0$  when  $r =$  \_\_\_\_\_.

The column space is all of  $\mathbf{R}^m$  when  $r =$  \_\_\_\_\_.

### Solution

Suppose an  $m$  by  $n$  matrix has  $r$  pivots. The number of special solutions is  $\mathbf{n - r}$ .

The Nullspace contains only  $x = 0$  when  $r = \mathbf{n}$ .

The column space is all of  $\mathbf{R}^m$  when  $r = \mathbf{m}$ .

### Exercise

Find the complete solution in the form  $x_p + x_n$  to these full rank system:

$$\begin{array}{ll} a) & x + y + z = 4 \\ b) & \begin{array}{l} x + y + z = 4 \\ x - y + z = 4 \end{array} \end{array}$$

### Solution

a)  $x + y + z = 4$

The equivalent matrix is given by:  $\begin{cases} Ax = 4 \\ A = (1 \quad 1 \quad 1) \end{cases}$

The complete solution in the form  $\vec{x} = \vec{x}_p + \vec{x}_n$

$\vec{x}_n$  is the homogeneous solution to  $A\vec{x}_n = \vec{0}$

Size of  $A$  is  $m = 1$  and  $n = 3$ ,  $\text{rank}(A) = r = 1$

$$A\vec{x}_n = \vec{0}$$

$$(1 \quad 1 \quad 1) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + x_2 + x_3 = 0 \Rightarrow \underline{x_1 = -x_2 - x_3}$$



Set  $x_2 = 1, \quad x_3 = 0$

The special solution:  $\vec{s}_1 = (-1, 1, 0)$

Set  $x_2 = 0, \quad x_3 = 1$

The special solution:  $\vec{s}_2 = (-1, 0, 1)$

The nullspace is the set  $\left\{ x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$x = 4 - y - z$$

$$\Rightarrow x_1 = 4 - x_2 - x_3$$

Set  $x_2 = 0, \quad x_3 = 0$  that implies to the particular solution:  $\vec{x}_p = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$

The complete solution in the form  $\vec{x} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Note: that the null space of A is spanned by the two linearly independent vectors

$$(-1, 1, 0)^T \quad \text{and} \quad (-1, 0, 1)^T$$

**b)** 
$$\begin{cases} x + y + z = 4 \\ x - y + z = 4 \end{cases}$$

The equivalent matrix is given by:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \text{ and } A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & 4 \end{array} \right] \quad R_2 - R_1$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & 0 \end{array} \right] \quad -\frac{1}{2}R_2$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{array} \right] \quad R_1 - R_2$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{array} \right]$$

The pivots are  $x_1, x_2$  ; The free variable is  $x_3$

Rank  $r = 2$ ,  $n = 2$ ,  $m = 3$ .

The nullspace has dimension  $m - r = 1$ .

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1 + x_3 = 0 \rightarrow x_1 = -x_3 \\ x_2 = 0 \end{cases}$$

$$\text{If } x_3 = 1 \Rightarrow x_1 = -1$$

The special solution:  $\vec{s}_1 = (-1, 0, 1)$

The nullspace is the set  $\left\{ x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Set  $x_3 = 0$  that implies

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1 = 4 \\ x_2 = 0 \end{cases}$$

Then the particular solution:  $\vec{x}_p = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$

The complete solution in the form:

$$\vec{x} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

### Exercise

Find the complete solution in the form  $\vec{x}_p + \vec{x}_n$  to the system:

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

### Solution

$$\left( \begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{array} \right) \quad R_2 - 2R_1$$

$$\left( \begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 2 & 4 & 1 \end{array} \right) \quad \begin{array}{l} 2R_1 - R_2 \\ R_3 - R_2 \end{array}$$

$$\left( \begin{array}{cccc|c} 2 & 6 & 0 & 0 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} \frac{1}{2}R_1 \\ \frac{1}{2}R_2 \end{array}$$

$$\left( \begin{array}{cccc|c} 1 & 3 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 2 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

The pivots are  $x_1, x_3$ ; The free variables are  $x_2, x_4$

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 + 3x_2 = 0 & \underline{x_1 = -3x_2} \\ x_3 + 2x_4 = 0 & \underline{x_3 = -2x_4} \end{cases}$$

1. Set  $x_2 = 1, x_4 = 0$

The special solution:  $\vec{s}_1 = (-3, 1, 0, 0)$

2. Set  $x_2 = 0, x_4 = 1$

The special solution:  $\vec{s}_2 = (0, 0, -2, 1)$

The special solution:  $\vec{x}_n = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 + 3(0) = \frac{1}{2} & \underline{x_1 = \frac{1}{2}} \\ x_3 + 2(0) = \frac{1}{2} & \underline{x_3 = \frac{1}{2}} \end{cases}$$

Then the particular solution:

$$\vec{x}_p = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

The complete solution in the form:

$$\vec{x} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

### Exercise

If  $A$  is  $3 \times 7$  matrix, its largest possible rank is \_\_\_\_\_. In this case, there is a pivot in every \_\_\_\_\_ of  $U$  and  $R$ , the solution to  $Ax = b$  \_\_\_\_\_ (always exists or is unique), and the column space of  $A$  is \_\_\_\_\_. Construct an example of such a matrix  $A$ .

### Solution

If  $A$  is  $3 \times 7$  matrix, its largest possible rank is **3**. In this case, there is a pivot in every **row** of  $U$  and  $R$ , the solution to  $Ax = b$  **always exists**, and the column space of  $A$  is  $\mathbb{R}^3$ .

$$A = \begin{pmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{pmatrix}$$

$\text{rank}(A) \leq 3$ , that implies that you have 3 pivots (1 each row)

$$A = \begin{pmatrix} 1 & 0 & 0 & * & * & * & * \\ 0 & 1 & 0 & * & * & * & * \\ 0 & 0 & 1 & * & * & * & * \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 5 & 6 & 7 & 8 \\ 0 & 0 & 1 & 9 & 10 & 11 & 12 \end{pmatrix}$$

### Exercise

If  $A$  is  $6 \times 3$  matrix, its largest possible rank is \_\_\_\_\_. In this case, there is a pivot in every \_\_\_\_\_ of  $U$  and  $R$ , the solution to  $A\vec{x} = \vec{b}$  \_\_\_\_\_ (always exists or is unique), and the nullspace of  $A$  is \_\_\_\_\_. Construct an example of such a matrix  $A$ .

### Solution

If  $A$  is  $6 \times 3$  matrix, its largest possible rank is **3**. In this case, there is a pivot in every **column** of  $U$  and  $R$ , the solution to  $A\vec{x} = \vec{b}$  **is unique**, and the column space of  $A$  is  $\{\vec{0}\}$ .

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

### Exercise

Find the rank of  $A$ ,  $A^T A$  and  $AA^T$  for  $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix}$

### Solution

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix} \quad R_3 + R_1$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{pmatrix} \quad \frac{1}{2}R_2$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 3 \end{pmatrix} \quad R_3 - 3R_2$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\underline{\text{rank}(A) = 2}$$

$$\begin{aligned} A^T A &= \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \\ -1 & 9 \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 9 \end{pmatrix} \quad 2R_2 + R_1$$

$$\begin{pmatrix} 2 & -1 \\ 0 & 17 \end{pmatrix}$$

$$\underline{\text{rank}(A^T A) = 2}$$

$$AA^T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 4 \\ 1 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 4 \\ 1 & 4 & 5 \end{pmatrix} \begin{array}{l} \\ R_2 - R_1 \\ 2R_3 - R_1 \end{array}$$

$$\begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 6 & 9 \end{pmatrix} \begin{array}{l} \\ \\ R_3 - 3R_2 \end{array}$$

$$\begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{\text{rank}(A^T A) = 2}$$

$\therefore \text{rank}(A) = \text{rank}(A^T A) = \text{rank}(AA^T)$  for any matrix,  $A$ .

### Exercise

Explain why these are all false:

- The complete solution is any linear combination of  $\vec{x}_p$  and  $\vec{x}_n$ .
- A system  $A\vec{x} = \vec{b}$  has at most one particular solution.
- The solution  $\vec{x}_p$  with all free variables zero is the shortest solution (minimum length  $\|\vec{x}\|$ ). Find a 2 by 2 counterexample.
- If  $A$  is invertible there is no solution  $\vec{x}_n$  in the null space.

### Solution

- The coefficient of  $\vec{x}_p$  must be one.
- If  $\vec{x}_n \in N(A)$  is the nullspace of  $A$  and  $\vec{x}_p$  is one particular solution, then  $\vec{x}_p$  and  $\vec{x}_n$  is also a particular solution.
- If  $A$  is a 2 by 2 matrix of rank 1, then the solution to  $A\vec{x} = \vec{b}$  form a line parallel to the line that the nullspace. The line  $x + y = 1$  gives such an example.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \vec{x}_p = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\begin{aligned}
 \text{Then } \|\vec{x}_p\| &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \\
 &= \sqrt{2 \cdot \frac{1}{4}} \\
 &= \frac{1}{\sqrt{2}} < 1
 \end{aligned}$$

while the particular solutions having some coordinate equal to zero are (1, 0) and (0, 1) and they both have  $\|\vec{x}_p\| = 1$

*d)* There is always  $\vec{x}_n = 0$

### Exercise

Write down all known relation between  $r$  and  $m$  and  $n$  if  $A\vec{x} = \vec{b}$  has

- a)* No solution for some  $\vec{b}$ .
- b)* Infinitely many solutions for every  $\vec{b}$ .
- c)* Exactly one solution for some  $\vec{b}$ , no solution for another  $\vec{b}$ .
- d)* Exactly one solution for every  $\vec{b}$ .

### Solution

- a)* The system has less than full row rank:  $r < m$ .
- b)* The system has full row rank and less than full column rank:  $m = r < n$ .
- c)* The system has full column rank and less than full row rank:  $n = r < m$ .
- d)* The system has full row and column rank (it is invertible):  $m = r = n$ .

### Exercise

Find a basis for its row space, find a basis for its column space, and determine its rank

$$\begin{array}{ll}
 a) \begin{bmatrix} 0 & 2 & -3 & 4 & 1 & 2 & 1 & 7 \\ 0 & 0 & 3 & -2 & 0 & 4 & -5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 6 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & b) \begin{bmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{bmatrix}
 \end{array}$$

### Solution

*a) Row Space:* every row



$$\text{Column Space: } \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

$$\text{Rank} = 4$$

$$b) \begin{pmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{pmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ \end{array}$$

$$\begin{pmatrix} 3 & 2 & -1 \\ 0 & -1 & 7 \\ 0 & 1 & -7 \\ 0 & -1 & 7 \end{pmatrix} \begin{array}{l} R_1 + 2R_2 \\ \\ R_3 + R_2 \\ R_4 - R_2 \end{array}$$

$$\begin{pmatrix} 3 & 0 & 13 \\ 0 & -1 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \frac{1}{3}R_1 \\ -R_2 \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & \frac{13}{3} \\ 0 & 1 & -7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{Row Space: } [3 \ 2 \ -1], [6 \ 3 \ 5]$$

$$\text{Column Space: } \begin{bmatrix} 3 \\ 6 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \\ -1 \end{bmatrix}$$

$$\text{Rank} = 2$$

### Exercise

Find a basis for the row space, find a basis for the null space, find  $\dim RS$ , find  $\dim NS$ , and verify  $\dim RS + \dim NS = n$

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{bmatrix}$$

### Solution

$$\begin{pmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{pmatrix} \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - 5R_1 \end{array}$$

$$\begin{pmatrix} 1 & -2 & 4 & 1 \\ 0 & 7 & -15 & -4 \\ 0 & 7 & -15 & -4 \end{pmatrix} \quad \begin{array}{l} 7R_1 + 2R_2 \\ R_3 - R_2 \end{array}$$

$$\begin{pmatrix} 7 & 0 & -2 & -1 \\ 0 & 7 & -15 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} \frac{1}{7}R_1 \\ -\frac{1}{7}R_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & -\frac{2}{7} & -\frac{1}{7} \\ 0 & 1 & -\frac{15}{7} & -\frac{4}{7} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Row Space:  $\begin{bmatrix} 1 & -2 & 4 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 & -3 & -1 \end{bmatrix}$

$$\text{Column Space: } \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$$

$$\dim RS = 2$$

$$\dim NS = 2$$

$$2 + 2 = 4 \Rightarrow \dim RS + \dim NS = n$$

### Exercise

Determine if  $\vec{b}$  lies in the column space of the given matrix. If it does, express  $\vec{b}$  as linear combination of the column.

$$\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

### Solution

$$\left[ \begin{array}{cc|c} 2 & -3 & 4 \\ -4 & 6 & -6 \end{array} \right] \quad R_2 + 2R_1$$

$$\left[ \begin{array}{cc|c} 2 & -3 & 4 \\ 0 & 0 & 2 \end{array} \right] \quad \begin{array}{l} \frac{1}{2}R_1 \\ \frac{1}{2}R_2 \end{array}$$

$$\left[ \begin{array}{cc|c} 1 & -\frac{3}{2} & 2 \\ 0 & 0 & 1 \end{array} \right] \quad R_1 - 2R_2$$

$$\left[ \begin{array}{cc|c} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 1 \end{array} \right]$$

$\vec{b}$  does not lie in the column space

### Exercise

Find the transition matrix from  $B$  to  $C$  and find  $[\vec{x}]_C$

$$a) \quad B = \{(3, 1), (-1, -2)\}, \quad C = \{(1, -3), (5, 0)\}, \quad [\vec{x}]_B = [-1 \quad -2]^T$$

$$b) \quad B = \{(1, 1, 1), (-2, -1, 0), (2, 1, 2)\}, \quad C = \{(-6, -2, 1), (-1, 1, 5), (-1, -1, 1)\},$$

$$[\vec{x}]_B = [-3 \quad 2 \quad 4]^T$$

### Solution

$$a) \quad \left[ \begin{array}{cc|cc} 1 & 5 & 3 & -1 \\ -3 & 0 & 1 & -2 \end{array} \right] \quad R_2 + 3R_1$$

$$\left[ \begin{array}{cc|cc} 1 & 5 & 3 & -1 \\ 0 & 15 & 10 & -5 \end{array} \right] \quad 3R_1 - R_2$$

$$\left[ \begin{array}{cc|cc} 3 & 0 & -1 & 2 \\ 0 & 15 & 10 & -5 \end{array} \right] \quad \begin{array}{l} \frac{1}{3}R_1 \\ \frac{1}{15}R_2 \end{array}$$

$$\left[ \begin{array}{cc|cc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{array} \right]$$

$$\begin{aligned}
[\vec{x}]_c &= \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 0 \end{bmatrix}
\end{aligned}$$

$$b) \left[ \begin{array}{ccc|ccc} -6 & -1 & -1 & 1 & -2 & 2 \\ -2 & 1 & -1 & 1 & -1 & 1 \\ 1 & 5 & 1 & 1 & 0 & 2 \end{array} \right] \begin{array}{l} 3R_2 - R_1 \\ 6R_3 + R_1 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} -6 & -1 & -1 & 1 & -2 & 2 \\ 0 & 4 & -2 & 2 & -1 & 1 \\ 0 & 29 & 5 & 7 & -2 & 14 \end{array} \right] \begin{array}{l} 4R_1 + R_2 \\ 4R_3 - 29R_2 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} -24 & 0 & -6 & 6 & -9 & 9 \\ 0 & 4 & -2 & 2 & -1 & 1 \\ 0 & 0 & 78 & -30 & 21 & 27 \end{array} \right] \begin{array}{l} 13R_1 + R_3 \\ 39R_2 + R_3 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} -312 & 0 & 0 & 48 & -96 & 144 \\ 0 & 156 & 0 & 48 & -18 & 66 \\ 0 & 0 & 78 & -30 & 21 & 27 \end{array} \right] \begin{array}{l} -\frac{1}{312}R_1 \\ \frac{1}{156}R_2 \\ \frac{1}{78}R_3 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{13} & \frac{4}{13} & -\frac{6}{13} \\ 0 & 1 & 0 & \frac{4}{13} & -\frac{3}{26} & \frac{11}{26} \\ 0 & 0 & 1 & -\frac{5}{13} & \frac{7}{26} & \frac{9}{26} \end{array} \right]$$

$$\begin{aligned}
[\vec{x}]_c &= \begin{bmatrix} -\frac{2}{13} & \frac{4}{13} & -\frac{6}{13} \\ \frac{4}{13} & -\frac{3}{26} & \frac{11}{26} \\ -\frac{5}{13} & \frac{7}{26} & \frac{9}{26} \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{10}{13} \\ \frac{17}{13} \\ \frac{35}{13} \end{bmatrix}
\end{aligned}$$

### Exercise

Does  $A$  and  $A^T$  have the same number of pivots.

### Solution

True

The number of pivots of  $A$  is its column rank,  $r$ .

We know that the column rank of  $A$  equals the row rank of  $A$ , which is the column rank of  $A^T$ .

Hence,  $A^T$  must have the same number of pivots as  $A$ .

### Exercise

Let  $A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$  where  $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

- a) What is the rank of  $A$ ?
- b) What is the dimension of  $A$ ?
- c) What are the pivots variables?
- d) What are the free variables?
- e) Find the special (homogeneous) solutions.
- f) What is the nullspace  $N(A)$ ?
- g) Find the particular solution to  $A\vec{x} = \vec{b}$
- h) Give the complete solution.

### Solution

- a)  $\text{Rank}(A) = 2$
- b) Dimension of  $A = 2$
- c) The pivots variables are:  $x_1, x_3$
- d) The free variables are:  $x_2, x_4$
- e) Let  $x = x_2 s_1 + x_4 s_2$

$$A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 = -3x_2 - 2x_4 \\ x_3 = -4x_4 \end{cases}$$

Set  $x_2 = 1, x_4 = 0$

The special solution:  $\vec{s}_1 = (-3, 1, 0, 0)$

Set  $x_2 = 0, x_4 = 1$  ;

The special solution:  $\vec{s}_2 = (-2, 0, -4, 1)$

*f)* The nullspace is the set  $\left\{ x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix} \right\}$

$$N(A) = \begin{pmatrix} -3 & -2 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \end{pmatrix}$$

*g)*  $\vec{x}_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

*h)*  $\vec{x} = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

### Exercise

Let  $A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$  where  $\vec{b} = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

- What is the rank of  $A$ ?
- What is the dimension of  $A$ ?
- What are the pivots variables?
- What are the free variables?
- Find the special (homogeneous) solutions.
- What is the nullspace  $N(A)$ ?
- Find the particular solution to  $A\vec{x} = \vec{b}$
- Give the complete solution.

### Solution

- Rank( $A$ ) = 3
- Dimension of  $A = 1$
- The pivots variables are:  $x_1, x_2, x_4$

d) The free variables are:  $x_3$

e) Let  $x = x_3 s_1$

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{cases} x_1 = -2x_3 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

The special solution:  $\vec{s}_1 = (-2, 0, 1, 0)$

$$f) N(A) = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$g) \vec{x}_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$h) \vec{x} = x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

### Exercise

$$\text{Let } A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{where } \vec{b} = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

- a) What is the rank of  $A$ ?
- b) What is the dimension of  $A$ ?
- c) What are the pivots variables?
- d) What are the free variables?
- e) Find the special (homogeneous) solutions.
- f) What is the nullspace  $N(A)$ ?
- g) Find the particular solution to  $A\vec{x} = \vec{b}$
- h) Give the complete solution.

### Solution

$$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{3}R_2} \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -\frac{1}{3} & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 - 2R_2} \begin{pmatrix} 1 & 0 & \frac{2}{3} & -4 \\ 0 & 1 & -\frac{1}{3} & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 = -\frac{2}{3}x_3 + 4x_4 \\ x_2 = \frac{1}{3}x_3 - 4x_4 \end{cases}$$

**a)**  $\text{Rank}(A) = 2$

**b)** Dimension of  $A = 2$

**c)** The pivots variables are:  $x_1, x_2$

**d)** The free variables are:  $x_3, x_4$

**e)** Let  $x = x_3 \vec{s}_1 + x_4 \vec{s}_2$

Set  $x_3 = 1, x_4 = 0$

The special solution:  $\vec{s}_1 = \left(-\frac{2}{3}, \frac{1}{3}, 1, 0\right)$

Set  $x_3 = 0, x_4 = 1$  ;

The special solution:  $\vec{s}_2 = (4, -4, 0, 1)$

**f)** The nullspace is the set  $\left\{ x_3 \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ -4 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$N(A) = \begin{pmatrix} -\frac{2}{3} & 4 \\ \frac{1}{3} & -4 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$



$$g) \quad \vec{x}_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$h) \quad \vec{x} = x_3 \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ -4 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

### ***Exercise***

Let  $A = \begin{pmatrix} 1 & 0 & 0 & \frac{13}{11} \\ 0 & 1 & 0 & -\frac{17}{11} \\ 0 & 0 & 1 & \frac{6}{11} \\ 0 & 0 & 0 & 0 \end{pmatrix}$  where  $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

- What is the rank of  $A$ ?
- What is the dimension of  $A$ ?
- What are the pivots variables?
- What are the free variables?
- Find the special (homogeneous) solutions.
- What is the nullspace  $N(A)$ ?
- Find the particular solution to  $A\vec{x} = \vec{b}$
- Give the complete solution.

### **Solution**

- $\text{Rank}(A) = 3$
- Dimension of  $A = 1$
- The pivots variables are:  $x_1, x_2, x_3$
- The free variables are:  $x_4$
- Let  $x = x_4 \begin{pmatrix} s \\ 1 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 0 & 0 & \frac{13}{11} \\ 0 & 1 & 0 & -\frac{17}{11} \\ 0 & 0 & 1 & \frac{6}{11} \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 = -\frac{13}{11} \\ x_2 = \frac{17}{11} \\ x_3 = -\frac{6}{11} \end{cases}$$

Set  $x_4 = 1$

The special solution:  $\vec{s}_1 = \left(-\frac{13}{11}, \frac{17}{11}, -\frac{6}{11}, 1\right)$

$$f) \quad N(A) = \begin{pmatrix} -\frac{13}{11} \\ \frac{17}{11} \\ -\frac{6}{11} \\ 1 \end{pmatrix}$$

$$g) \quad \vec{x}_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$h) \quad \vec{x} = x_4 \begin{pmatrix} -\frac{13}{11} \\ \frac{17}{11} \\ -\frac{6}{11} \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

### Exercise

Let  $A = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$  where  $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

- What is the rank of  $A$ ?
- What is the dimension of  $A$ ?
- What are the pivots variables?
- What are the free variables?
- Find the special (homogeneous) solutions.
- What is the nullspace  $N(A)$ ?
- Find the particular solution to  $A\vec{x} = \vec{b}$
- Give the complete solution.

### Solution

- $\text{Rank}(A) = 3$
- Dimension of  $A = 1$
- The pivots variables are:  $x_1, x_2, x_3$
- The free variables are:  $x_4$
- Let  $x = x_4 s_1$

$$A = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 = \frac{1}{3} \\ x_2 = 1 \\ x_3 = \frac{1}{3} \end{cases}$$

Set  $x_4 = 1$

The special solution:  $\vec{s}_1 = \left(\frac{1}{3}, 1, \frac{1}{3}, 1\right)$

$f) N(A) = \begin{pmatrix} \frac{1}{3} \\ 1 \\ \frac{1}{3} \\ 1 \end{pmatrix}$

$$g) \quad \vec{x}_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$h) \quad \vec{x} = x_4 \begin{pmatrix} \frac{1}{3} \\ 1 \\ \frac{1}{3} \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

### Exercise

$$\text{Let } A = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ where } \vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

- What is the rank of  $A$ ?
- What is the dimension of  $A$ ?
- What are the pivots variables?
- What are the free variables?
- Find the special (homogeneous) solutions.
- What is the nullspace  $N(A)$ ?
- Find the particular solution to  $A\vec{x} = \vec{b}$
- Give the complete solution.

### Solution

- Rank( $A$ ) = 3
- Dimension of  $A = 2$
- The pivots variables are:  $x_1, x_3, x_5$
- The free variables are:  $x_2, x_4$
- Let  $x = x_2 s_1 + x_4 s_2$

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 = -2x_2 + x_4 \\ x_3 = -x_4 \\ x_5 = 0 \end{cases}$$

$$\text{Set } x_2 = 1 \quad x_4 = 0$$

The special solution:  $\vec{s}_1 = (-2, 1, 0, 0, 0)$

Set  $x_2 = 0$   $x_4 = 1$

The special solution:  $\vec{s}_2 = (1, 0, -1, 1, 0)$

$$f) \quad N(A) = \begin{pmatrix} -2 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$g) \quad \vec{x}_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$h) \quad \vec{x} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

### ***Exercise***

Find a basis for each of the four subspaces associated with the given matrix

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{pmatrix}$$

### **Solution**

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{pmatrix} \quad R_2 - 2R_1$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \end{pmatrix} \quad R_1 - 2R_2$$

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \end{pmatrix} \quad x_1 = -4x_3 \quad \leftarrow \text{Row space}$$

$$\text{Rank}(A) = 1$$

$$\text{Dimension of } A = 1$$

1. Basis for **row space**:  $\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$

The pivots variables are:  $x_1, x_2$

2. Basis of the **column spaces**:  $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

The free variable is:  $x_3$

Set  $x_3 = 1 \Rightarrow s_1 = (-4, 0, 1)$

3. Basis of the **Nullspace**:  $\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$

$$A^T = \begin{pmatrix} 1 & 2 \\ 2 & 5 \\ 4 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 5 \\ 4 & 8 \end{pmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{array}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{array}{l} R_1 - 2R_2 \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

4. Basis of the **Left Nullspace**:  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

### ***Exercise***

Find a basis for each of the four subspaces associated with the given matrix

$$B = \begin{pmatrix} 1 & 3 & 0 & 5 \\ 2 & 6 & 1 & 16 \\ 5 & 15 & 0 & 25 \end{pmatrix}$$

### **Solution**

$$\begin{pmatrix} 1 & 3 & 0 & 5 \\ 2 & 6 & 1 & 16 \\ 5 & 15 & 0 & 25 \end{pmatrix} \begin{array}{l} R_2 - 2R_1 \\ R_3 - 5R_1 \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} x_1 = -3x_2 - 5x_4 \\ x_3 = -6x_4 \end{array} \begin{array}{l} \leftarrow \text{Row space} \\ \leftarrow \text{Row space} \end{array}$$

$$\text{Rank}(A) = 2$$

$$\text{Dimension of } A = 2$$

1. Basis for **row space**:

$$\begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 6 \end{pmatrix}$$

$$\text{The pivots variables are: } x_1, x_3$$

2. Basis of the **column spaces**:

$$\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{The free variable is: } x_2, x_4$$

$$\begin{cases} x_1 = -3x_2 - 5x_4 \\ x_3 = -6x_4 \end{cases}$$

$$\text{Set } x_2 = 1 \quad x_4 = 0$$

$$\text{The special solution: } s_1 = (-3, 1, 0, 0)$$

$$\text{Set } x_2 = 0 \quad x_4 = 1$$

$$\text{The special solution: } s_2 = (-5, 0, -6, 1)$$

3. Basis of the **Nullspace**:

$$\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -5 \\ 0 \\ -6 \\ 1 \end{pmatrix}$$

$$B^T = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 6 & 15 \\ 0 & 1 & 0 \\ 5 & 16 & 25 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 5 \\ 3 & 6 & 15 \\ 0 & 1 & 0 \\ 5 & 16 & 25 \end{pmatrix} \quad \begin{array}{l} R_2 - 3R_1 \\ R_4 - 5R_1 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 0 \end{pmatrix} \quad \begin{array}{l} R_4 - 2R_3 \\ R_4 - 6R_3 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad y_1 = -5y_3$$

Let  $y_3 = 1$

4. Basis of the **Left Nullspace**:  $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$

### Exercise

Find a basis for each of the four subspaces associated with the given matrix

$$C = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

### Solution

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad R_2 - R_1$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad \begin{array}{l} R_1 - 3R_2 \\ R_3 - R_2 \end{array}$$

$$\begin{pmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{ll} x_2 = -2x_3 + 2x_5 & \leftarrow \text{Row space} \\ x_4 = -2x_5 & \leftarrow \text{Row space} \end{array}$$

Rank ( $A$ ) = 2

Dimension of  $A = 2$



1. Basis for **row space**:

$$\begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ -2 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

The pivots variables are:  $x_2, x_4$

2. Basis of the **column spaces**:

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

The free variable is:  $x_3, x_5$

$$\begin{cases} x_2 = -2x_3 + 2x_5 \\ x_4 = -2x_5 \end{cases}$$

Set  $x_3 = 1 \quad x_5 = 0$

The special solution:  $s_1 = (0, -2, 1, 0, 2)$

Set  $x_3 = 0 \quad x_5 = 1$

The special solution:  $s_2 = (0, 2, 0, -2, 1)$

3. Basis of the **Nullspace**:

$$\begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

$$C^T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \end{pmatrix} \quad \begin{matrix} \\ R_3 - 2R_2 \\ R_4 - 3R_2 \\ R_5 - 4R_2 \end{matrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 2 & 2 \end{pmatrix} \begin{array}{l} \\ 3R_2 - R_4 \\ \\ 3R_5 - 2R_4 \\ \end{array}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{array}{l} \\ \frac{1}{3}R_4 \\ \\ \frac{1}{4}R_5 \\ \end{array}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{l} \\ R_2 + R_5 \\ \\ R_4 - \frac{1}{3}R_5 \\ \end{array}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Basis of the **Left Nullspace**:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

### Exercise

Find a basis for each of the four subspaces associated with the given matrix

$$D = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix}$$

### Solution

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix} \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} R_2 + 2R_1 \\ -R_2 \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{ll} x_1 = -x_3 - x_4 & \leftarrow \text{Row space} \\ x_2 = -x_3 & \leftarrow \text{Row space} \end{array}$$

$$\text{Rank}(A) = 2$$

$$\text{Dimension of } A = 2$$

1. Basis for **row space**:  $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

The pivots variables are:  $x_1, x_2$

2. Basis of the **column spaces**:  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

The free variable is:  $x_3, x_4$

$$\begin{cases} x_1 = -x_3 - x_4 \\ x_2 = -x_3 \end{cases}$$

$$\text{Set } x_3 = 1 \quad x_4 = 0$$

$$\text{The special solution: } s_1 = (-1, 0, 1, 0)$$

$$\text{Set } x_3 = 0 \quad x_4 = 1$$

$$\text{The special solution: } s_2 = (-1, -1, 0, 1)$$

3. Basis of the **Nullspace**:

$$\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$D^T = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - R_1 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} R_1 + R_2 \\ \\ R_3 - 2R_2 \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \\ -R_2 \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad y_1 = -y_3$$

Let  $y_3 = 1$

4. Basis of the **Left Nullspace**:  $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

### Exercise

Find a basis for each of the four subspaces associated with the given matrix

$$M = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{pmatrix}$$

### Solution

$$\begin{pmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{pmatrix} \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 5R_1 \end{array}$$

$$\begin{pmatrix} 1 & -2 & 4 & 1 \\ 0 & 7 & -15 & -4 \\ 0 & 7 & -15 & -4 \end{pmatrix} \begin{array}{l} 7R_1 + 2R_2 \\ \\ R_3 - R_2 \end{array}$$

$$\begin{pmatrix} 7 & 0 & -2 & -1 \\ 0 & 7 & -15 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \frac{1}{7}R_1 \\ \frac{1}{7}R_2 \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & -\frac{2}{7} & -\frac{1}{7} \\ 0 & 1 & -\frac{15}{7} & -\frac{4}{7} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} x_1 = \frac{2}{7}x_3 + \frac{1}{7}x_4 \\ x_2 = \frac{15}{7}x_3 + \frac{4}{7}x_4 \\ \end{array} \begin{array}{l} \leftarrow \text{Row space} \\ \leftarrow \text{Row space} \end{array}$$

$$\text{Rank}(A) = 2$$

$$\text{Dimension of } A = 2$$

1. Basis for **row space**:

$$\begin{pmatrix} 1 \\ 0 \\ -\frac{2}{7} \\ -\frac{1}{7} \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ -\frac{15}{7} \\ -\frac{4}{7} \end{pmatrix}$$

The pivots variables are:  $x_1, x_2$

2. Basis of the **column spaces**:

$$\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \quad \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$$

The free variable is:  $x_3, x_4$

$$\begin{cases} x_1 = \frac{2}{7}x_3 + \frac{1}{7}x_4 \\ x_2 = \frac{15}{7}x_3 + \frac{4}{7}x_4 \end{cases}$$

$$\text{Set } x_3 = 1 \quad x_4 = 0$$

$$\text{The special solution: } s_1 = \left( \frac{2}{7}, \frac{15}{7}, 1, 0 \right)$$

$$\text{Set } x_3 = 0 \quad x_4 = 1$$

$$\text{The special solution: } s_2 = \left( \frac{1}{7}, \frac{4}{7}, 0, 1 \right)$$

3. Basis of the **Nullspace**:

$$\begin{pmatrix} \frac{2}{7} \\ \frac{15}{7} \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} \frac{1}{7} \\ \frac{4}{7} \\ 0 \\ 1 \end{pmatrix}$$

$$M^T = \begin{pmatrix} 1 & 3 & 5 \\ -2 & 1 & -3 \\ 4 & -3 & 5 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 5 \\ -2 & 1 & -3 \\ 4 & -3 & 5 \\ 1 & -1 & 1 \end{pmatrix} \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 - 4R_1 \\ R_4 - R_1 \end{array}$$

$$\begin{pmatrix} 1 & 3 & 5 \\ 0 & 7 & 7 \\ 0 & -15 & -15 \\ 0 & -4 & -4 \end{pmatrix} \begin{array}{l} \\ \frac{1}{7}R_2 \\ \frac{1}{15}R_3 \\ \frac{1}{4}R_4 \end{array}$$

$$\begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{array}{l} R_1 - 3R_2 \\ \\ R_3 + R_2 \\ R_4 + R_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} y_1 = -2y_3 \\ y_1 = -y_3 \\ \\ \end{array}$$

Let  $y_3 = 1$

4. Basis of the **Left Nullspace**:  $\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

### ***Exercise***

Find a basis for each of the four subspaces associated with the given matrix

$$N = \begin{pmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{pmatrix}$$

### **Solution**

$$\begin{pmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{pmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ \end{array}$$

$$\begin{pmatrix} 3 & 2 & -1 \\ 0 & -1 & 7 \\ 0 & 1 & -7 \\ 0 & -1 & 7 \end{pmatrix} \begin{array}{l} R_1 + 2R_2 \\ \\ R_3 + R_2 \\ R_4 - R_2 \end{array}$$

$$\begin{pmatrix} 3 & 0 & 13 \\ 0 & -1 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \frac{1}{3}R_1 \\ -R_2 \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & \frac{13}{3} \\ 0 & 1 & -7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} x_1 = -\frac{13}{3}x_3 \\ x_2 = 7x_3 \\ \leftarrow \text{Row space} \\ \leftarrow \text{Row space} \end{array}$$

$$\text{Rank}(A) = 2$$

$$\text{Dimension of } A = 2$$

1. Basis for **row space**:  $\begin{pmatrix} 1 \\ 0 \\ \frac{13}{3} \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -7 \end{pmatrix}$

The pivots variables are:  $x_1, x_2$

2. Basis of the **column spaces**:  $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$

The free variable is:  $x_3, x_4$

$$\begin{cases} x_1 = -\frac{13}{7}x_3 \\ x_2 = 7x_3 \end{cases}$$

Set  $x_3 = 1$ . The special solution:  $s_1 = \left(-\frac{13}{3}, 7, 1\right)$

3. Basis of the **Nullspace**:  $\begin{pmatrix} -\frac{13}{3} \\ 7 \\ 1 \end{pmatrix}$

$$N^T = \begin{pmatrix} 3 & 6 & -3 & 0 \\ 2 & 3 & -1 & -1 \\ -1 & 5 & -6 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 6 & -3 & 0 \\ 2 & 3 & -1 & -1 \\ -1 & 5 & -6 & 7 \end{pmatrix} \begin{array}{l} \\ 3R_2 - 3R_1 \\ 3R_3 + R_1 \end{array}$$

$$\begin{pmatrix} 3 & 6 & -3 & 0 \\ 0 & -9 & 6 & -3 \\ 0 & 21 & -21 & 21 \end{pmatrix} \begin{array}{l} \frac{1}{3}R_1 \\ -\frac{1}{9}R_2 \\ \frac{1}{21}R_3 \end{array}$$

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 1 & -1 & 1 \end{pmatrix} \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 - R_2 \end{array}$$



$$\begin{pmatrix} 1 & 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{array}{l} R_1 + R_3 \\ R_1 - 2R_3 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{4}{3} \\ 0 & 1 & 0 & \frac{5}{3} \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \end{pmatrix} -3R_3$$

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{4}{3} \\ 0 & 1 & 0 & \frac{5}{3} \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{array}{l} y_1 = \frac{4}{3}y_4 \\ y_2 = -\frac{5}{3}y_4 \\ y_3 = -2y_4 \end{array}$$

Let  $y_4 = 1$

4. Basis of the **Left Nullspace**:  $\begin{pmatrix} \frac{4}{3} \\ -\frac{5}{3} \\ -2 \\ 1 \end{pmatrix}$