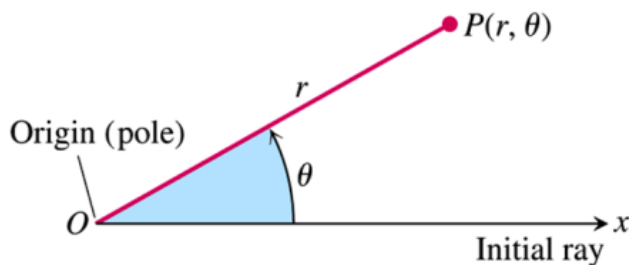


Section 6.6 – Polar Coordinates

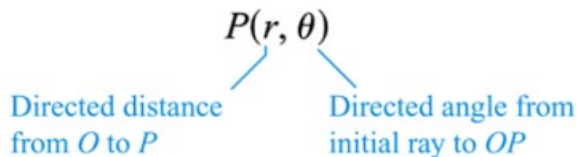
To reach the point whose address is $(2, 1)$, we start from origin and travel 2 units right and then 1 unit up. Another way to get to that point, we can travel $\sqrt{5}$ units on the terminal side of an angle in standard position and this type is called *Polar Coordinates*.

Definition of Polar Coordinates

To define polar coordinates, let an **origin** O (called the **pole**) and an **initial ray** from O . Then each point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives the directed from O to P and θ gives the directed angle from the initial ray to ray OP .



Polar Coordinates

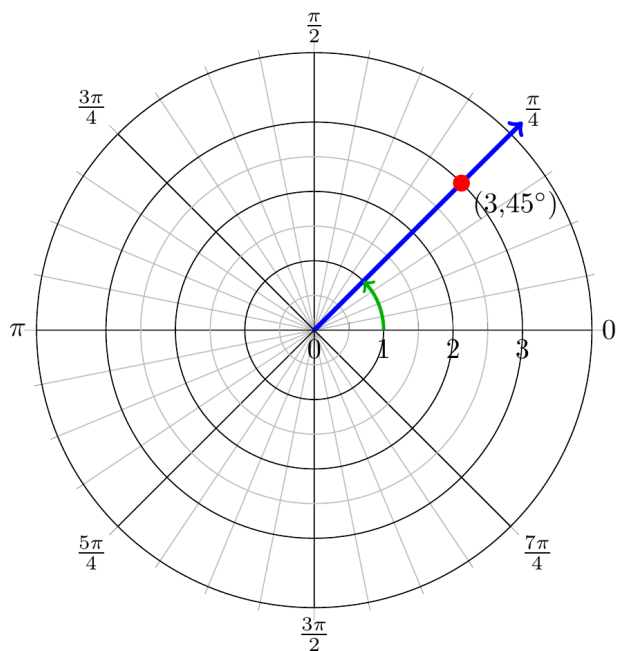


Definition – Relationships between Rectangular and Polar Coordinates

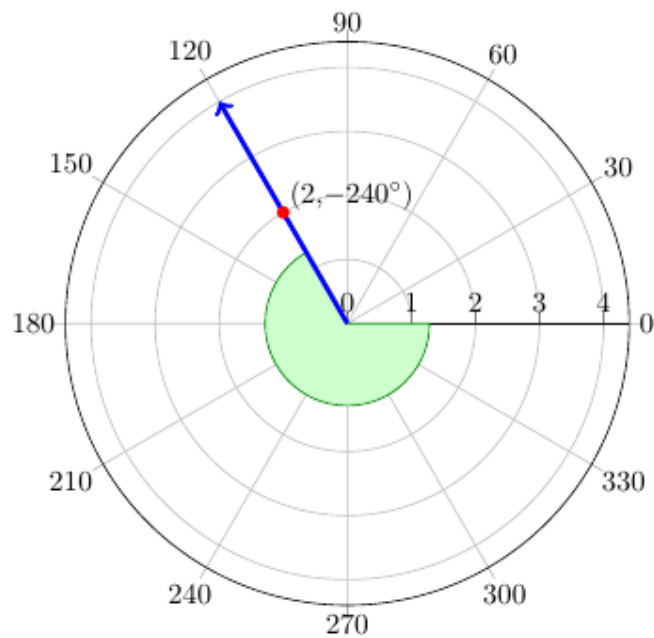
The rectangular coordinates (x, y) and polar coordinates (r, θ) of a point P are related as follows:

1. $x = r \cos \theta, \quad y = r \sin \theta$
2. $r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x} \quad \text{if } x \neq 0$

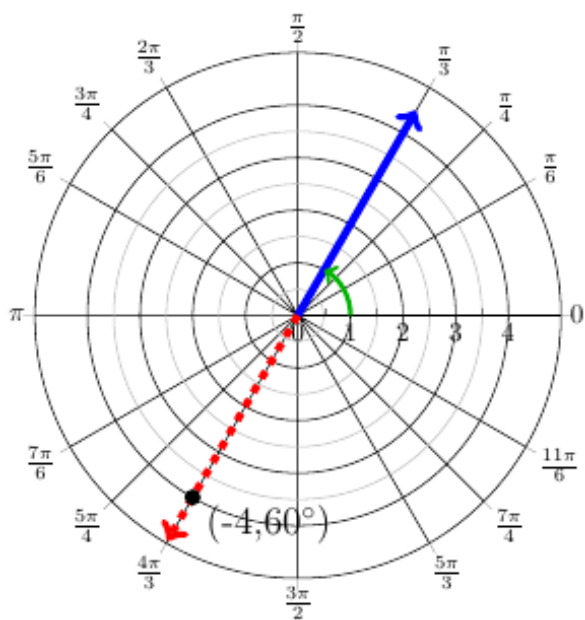
Graphing Polar Coordinates



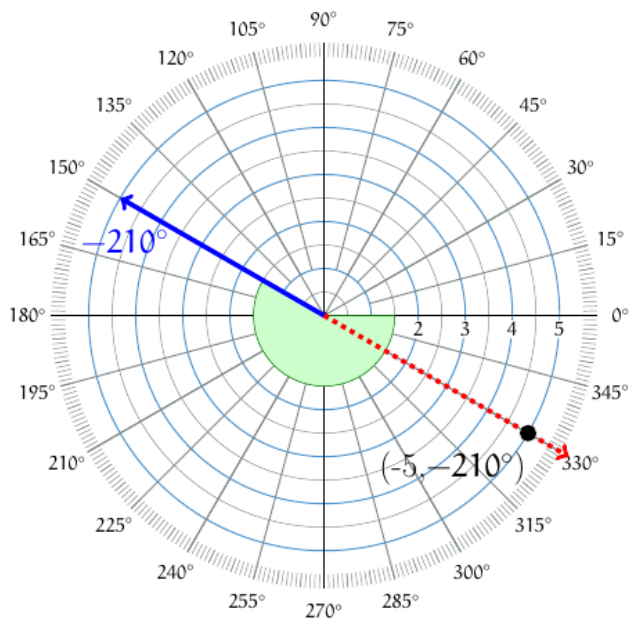
$(3, 45^\circ)$



$(2, -\frac{4\pi}{3})$



$(-4, \frac{\pi}{3})$



$(-5, -210^\circ)$

Example

If $(r, \theta) = \left(4, \frac{7\pi}{6}\right)$ are polar coordinates of a point P , find the rectangular coordinates of P .

Solution

$$\begin{aligned} x &= r \cos \theta \\ &= 4 \cos \frac{7\pi}{6} \\ &= 4 \left(-\frac{\sqrt{3}}{2} \right) \\ &= -2\sqrt{3} \end{aligned}$$

$$\begin{aligned} y &= r \sin \theta \\ &= 4 \sin \frac{7\pi}{6} \\ &= 4 \left(-\frac{1}{2} \right) \\ &= -2 \end{aligned}$$

The rectangular coordinates of P are $(x, y) = (-2\sqrt{3}, -2)$

Example

If $(x, y) = (-1, \sqrt{3})$ are rectangular coordinates of a point P , find three different pairs the polar coordinates of P .

Solution

$$\begin{aligned} r &= \pm \sqrt{x^2 + y^2} \\ &= \pm \sqrt{(-1)^2 + (\sqrt{3})^2} \\ &= \pm \sqrt{1+3} \\ &= \pm \sqrt{4} \\ &= \pm 2 \end{aligned}$$

$$\begin{aligned} \tan \theta &= \frac{y}{x} = \frac{\sqrt{3}}{-1} \\ &= -\sqrt{3} \end{aligned}$$

$$\hat{\theta} = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$$\theta_1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\theta_2 = \frac{2\pi}{3} + 2\pi = \frac{3\pi}{3}$$

$$\theta_3 = -\frac{\pi}{3}$$

The polar coordinates of P are: $\left(2, \frac{2\pi}{3}\right)$, $\left(-2, \frac{5\pi}{3}\right)$, $\left(2, -\frac{4\pi}{3}\right)$, and $\left(-2, -\frac{\pi}{3}\right)$

Example

Find a polar equation of an arbitrary line.

Solution

An equation of a line can be written in the form: $ax + by = c$.

$$ax + by = c$$

$$ar \cos \theta + br \sin \theta = c$$

$$r(a \cos \theta + b \sin \theta) = c$$

$$r = \frac{c}{a \cos \theta + b \sin \theta}$$

Example

Find a polar equation of the hyperbola $x^2 - y^2 = 16$.

Solution

$$(r \cos \theta)^2 - (r \sin \theta)^2 = 16$$

$$r^2 \cos^2 \theta - r^2 \sin^2 \theta = 16$$

$$r^2 (\cos^2 \theta - \sin^2 \theta) = 16$$

$$r^2 (\cos 2\theta) = 16$$

$$r^2 = \frac{16}{\cos 2\theta} \quad \cos 2\theta \neq 0$$

$$\text{or } r^2 = 16 \sec 2\theta$$

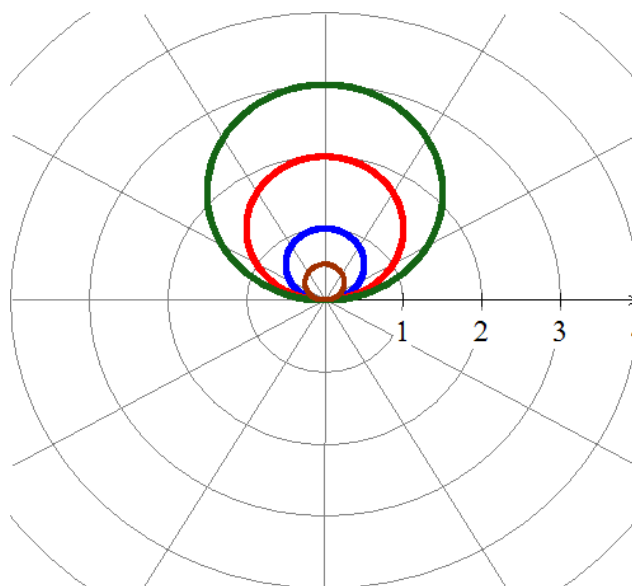
Example

Find an equation in x and y that has the same graph as the polar equation $r = a \sin \theta$, $a \neq 0$. Sketch the graph.

Solution

$$r^2 = ar \sin \theta$$

$$x^2 + y^2 = ay$$

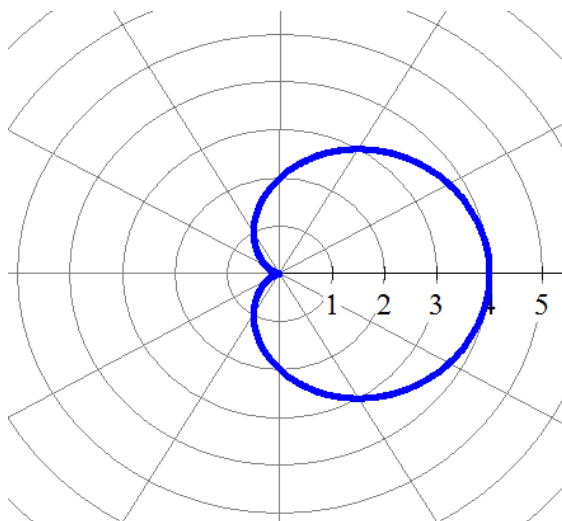


Example

Sketch the graph of the polar equation $r = 2 + 2 \cos \theta$.

Solution

θ	r
0	4
$\frac{\pi}{4}$	$2 + \sqrt{2}$
$\frac{\pi}{2}$	2
$\frac{3\pi}{4}$	$2 - \sqrt{2}$
π	0
$\frac{3\pi}{2}$	2
2π	4



Exercises Section 6.6 – Polar Coordinates

(1 – 6) Convert to rectangular coordinates

- | | | |
|----------------------------------|---------------------|----------------------------------|
| 1. $(4, 30^\circ)$ | 3. $(3, 270^\circ)$ | 5. $(\sqrt{2}, -225^\circ)$ |
| 2. $(-\sqrt{2}, \frac{3\pi}{4})$ | 4. $(2, 60^\circ)$ | 6. $(4\sqrt{3}, -\frac{\pi}{6})$ |

7. Change the polar coordinates to rectangular coordinates $(-2, \frac{7\pi}{6})$
8. Change the polar coordinates to rectangular coordinates $(6, \arctan \frac{3}{4})$
9. Change the polar coordinates to rectangular coordinates $(10, \arccos(-\frac{1}{3}))$

(10 – 16) Convert to polar coordinates

- | | |
|----------------------|--|
| 10. $(3, 3)$ | 13. $(-3, -3) \quad r \geq 0 \quad 0^\circ \leq \theta < 360^\circ$ |
| 11. $(-2, 0)$ | 14. $(2, -2\sqrt{3}) \quad r \geq 0 \quad 0^\circ \leq \theta < 360^\circ$ |
| 12. $(-1, \sqrt{3})$ | 15. $(-2, 0) \quad r \geq 0 \quad 0 \leq \theta < 2\pi$ |
| | 16. $(-1, -\sqrt{3}) \quad r \geq 0 \quad 0 \leq \theta < 2\pi$ |

17. Change the rectangular coordinates to polar coordinates $(7, -7\sqrt{3}) \quad r > 0 \quad 0 \leq \theta < 2\pi$
18. Change the rectangular coordinates to polar coordinates $(-2\sqrt{2}, -2\sqrt{2}) \quad r > 0 \quad 0 \leq \theta < 2\pi$
19. The point $(0, -3)$ in rectangular coordinates is equivalent to $(3, 270^\circ)$ in polar coordinates.
20. The point $(1, -1)$ in rectangular coordinates is equivalent to $(-\sqrt{2}, \frac{3\pi}{4})$ in polar coordinates.
21. A point lies at $(4, 4)$ on a rectangular coordinate system. Give its address in polar coordinates (r, θ)

(22 – 34) Write the equation in rectangular coordinates

- | | | |
|--|---|--|
| 22. $r^2 = 4$ | 27. $r \sin \theta = -2$ | 31. $r(\sin \theta - 2 \cos \theta) = 6$ |
| 23. $r = 6 \cos \theta$ | 28. $\theta = \frac{\pi}{4}$ | 32. $r = 8 \sin \theta - 2 \cos \theta$ |
| 24. $r^2 = 4 \cos 2\theta$ | 29. $r^2(4 \sin^2 \theta - 9 \cos^2 \theta) = 36$ | 33. $r = \tan \theta$ |
| 25. $r(\cos \theta - \sin \theta) = 2$ | 30. $r^2(\cos^2 \theta + 4 \sin^2 \theta) = 16$ | 34. $r(\sin \theta + r \cos^2 \theta) = 1$ |
| 26. $r^2 = 4 \sin 2\theta$ | | |

(35 – 38) Find a polar equation that has the same graph as the equation in x and y

35. $y^2 = 6x$

37. $(x + 2)^2 + (y - 3)^2 = 13$

36. $xy = 8$

38. $y^2 - x^2 = 4$

(39 – 42) Write the equation in polar coordinates

39. $x + y = 5$

41. $x^2 + y^2 = 4x$

43. $x + y = 4$

40. $x^2 + y^2 = 9$

42. $y = -x$

(44 – 54) Sketch the graph of the polar equation

44. $r = 5$

48. $r = 2 - \cos \theta$

52. $r = e^{2\theta} \quad \theta \geq 0$

45. $\theta = \frac{\pi}{4}$

49. $r = 4 \csc \theta$

53. $r\theta = 1 \quad \theta > 0$

46. $r = 4 \cos \theta + 2 \sin \theta$

50. $r^2 = 4 \cos 2\theta$

54. $r = 2 + 2 \sec \theta$

47. $r = 2 + 4 \sin \theta$

51. $r = 2^\theta \quad \theta \geq 0$