Solution — Section 1.1 – Velocity and Net Change

Exercise

Assume t is time measured in seconds and velocities have units of *m/s*. $v(t)=6-2t; \quad 0\le t\le 6$

a) Graph the velocity function over the given interval. Then determine when the motion is in the positive direction.

b) Find the displacement over the given interval.

c) Find the distance traveled over the given interval.

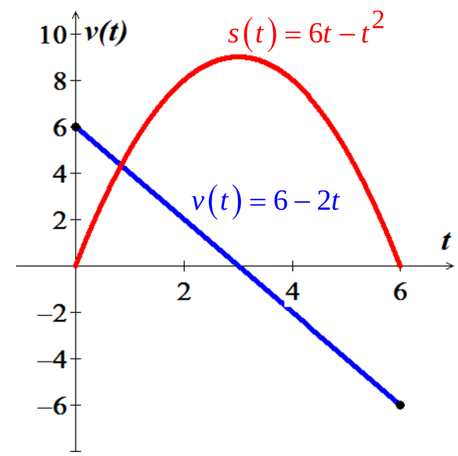
Solution

a) The motion is positive for $0\le t<3$ and negative for $3<t\le 6$

b) Displacement $=\int_0^6 (6-2t)\,dt$

$$=\left[6t-t^2\right]_0^6$$

$$\underline{=0 \ \ m}$$



c) Distance traveled is $=\int_0^3 (6-2t)\,dt+\int_3^6 (2t-6)\,dt$

$$=\left[6t-t^2\right]_0^3+\left[t^2-6t\right]_3^6$$

$$=9+9$$

$$\underline{=18 \ \ m}$$

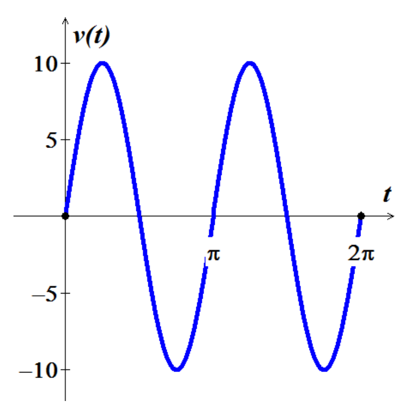
Exercise

Assume t is time measured in seconds and velocities have units of *m/s*. $v(t)=10\sin 2t; \quad 0\le t\le 2\pi$

a) Graph the velocity function over the given interval. Then determine when the motion is in the positive direction.

b) Find the displacement over the given interval.

c) Find the distance traveled over the given interval.



Solution

a) The motion is positive for $0<t<\frac{\pi}{2}$ & $\pi<t<\frac{3\pi}{2}$

and negative for $\frac{\pi}{2}<t<\pi$ & $\frac{3\pi}{2}<t<2\pi$

b) Displacement $=\int_0^{2\pi} (10\sin 2t)\,dt$

$$=-5\cos 2t\Big|_0^{2\pi}$$

$$\underline{=0}$$

c) Distance traveled is $= \int_0^{2\pi} (10\sin 2t)\,dt = 4 \cdot \int_0^{\pi/2} (10\sin 2t)\,dt$

$$= -20\cos 2t \Big|_0^{\pi/2}$$

$$= -20(-1-1)$$

$$\underline{= 40\ m}$$

Exercise

Assume *t* is time measured in seconds and velocities have units of *m/s*. $v(t) = 50e^{-2t};\quad 0 \le t \le 4$

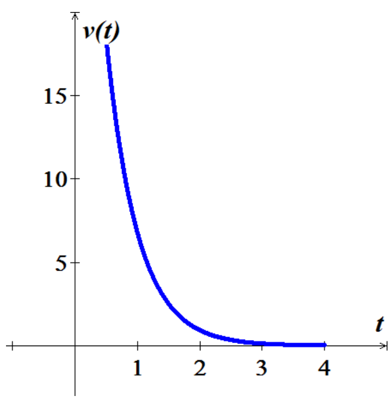
a) Graph the velocity function over the given interval. Then determine when the motion is in the positive direction.

b) Find the displacement over the given interval.

c) Find the distance traveled over the given interval.

Solution

a) The motion is positive for $0 \le t \le 4$



b) Displacement $= \int_0^4 50e^{-2t}\,dt$

$$= -25e^{-2t} \Big|_0^4$$

$$= -25\left(e^{-8} - 1\right)$$

$$\underline{= 25\left(1 - e^{-8}\right)\ m} \quad \underline{\approx 24.992\ m}$$

c) Distance traveled is the same displacement si $\approx 24.992\ m$

Exercise

Consider an object moving along a line with the following velocities and initial positions

$v(t) = 6 - 2t \quad on\ [0,\ 5] \quad s(0) = 0$

a) Graph the velocity function on the given interval. Then determine when the object is moving in the positive direction and when it is moving in the negative direction.

b) Determine the position function for $t \ge 0$ using both the antiderivative method and the Fundamental Theorem of Calculus. Check for agreement between the two methods.

c) Graph the position function on the given interval.

Solution

a) The motion is positive for $0 \le t < 3$ and negative for $3 < t \le 5$

b) $s(t)=\int(6-2t)\,dt=6t-t^2+C$

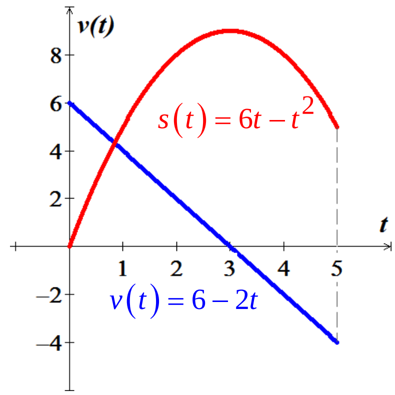
Given: $s(0)=0 \quad\rightarrow\quad \boxed{0=C}$

$\boxed{s(t)=6t-t^2}$

Also, $s(t)=s(0)+\int_0^t(6-2x)\,dx$

$=0+\left[6x-x^2\right]_0^t$

$\boxed{=6t-t^2}$



Exercise

Consider an object moving along a line with the following velocities and initial positions

$v(t)=9-t^2 \quad on\ [0,\ 4] \quad s(0)=-2$

a) Graph the velocity function on the given interval. Then determine when the object is moving in the positive direction and when it is moving in the negative direction.

b) Determine the position function for $t \ge 0$ using both the antiderivative method and the Fundamental Theorem of Calculus. Check for agreement between the two methods.

c) Graph the position function on the given interval.

Solution

a) The motion is positive for $0 \le t < 3$ and negative for $3 < t \le 4$

b) $s(t)=\int\left(9-t^2\right)dt=9t-\frac{1}{3}t^3+C$

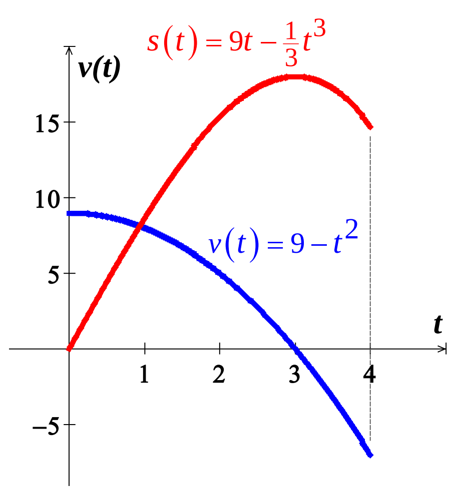
Given: $s(0)=-2 \quad\rightarrow\quad \boxed{-2=C}$

$\boxed{s(t)=9t-\frac{1}{3}t^3-2}$

Also, $s(t)=s(0)+\int_0^t\left(9-x^2\right)dx$

$=-2+\left[9x-\frac{1}{3}x^3\right]_0^t$

$\boxed{=9t-\frac{1}{3}t^3-2}$



Exercise

Find the position and velocity of an object moving along a straight line with the given acceleration, initial velocity, ansd initial position. Assume units of meters and seconds.

$a(t) = -9.8, \quad v(0) = 20, \quad s(0) = 0$

Solution

$$v(t) = \int a(t)\,dt = -\int 9.8\,dt = -9.8t + C_1$$

Since $v(0) = 20 \quad \rightarrow \quad 20 = C_1$

$$v(t) = -9.8t + 20$$

$$s(t) = \int v(t)\,dt = \int (20 - 9.8t)\,dt$$

$$= 20t - 4.9t^2 + C_2$$

Since $s(0) = 0 \quad \rightarrow \quad 0 = C_2$

$$s(t) = 20t - 4.9t^2$$

Exercise

Find the position and velocity of an object moving along a straight line with the given acceleration, initial velocity, ansd initial position. Assume units of meters and seconds.

$a(t) = e^{-t}, \quad v(0) = 60, \quad s(0) = 40$

Solution

$$v(t) = \int a(t)\,dt = \int e^{-t}\,dt = -e^{-t} + C_1$$

Since $v(0) = 60 \quad \rightarrow \quad 20 = -1 + C_1 \Rightarrow C_1 = 61$

$$v(t) = -e^{-t} + 61$$

$$s(t) = \int v(t)\,dt = \int \left(61 - e^{-t}\right)dt$$

$$= 61t + e^{-t} + C_2$$

Since $s(0) = 40 \quad \rightarrow \quad 40 = 1 + C_2 \Rightarrow C_2 = 39$

$$s(t) = 61t + e^{-t} + 39$$

Exercise

A mass hanging from a spring is set in motion and its ensuing velocity is given by $v(t) = 2\pi\cos\pi t$ for $t \ge 0$. Assume that the position direction is upward and $s(0) = 0$.

a) Determine the position function for $t \ge 0$.
b) Graph the position function on the interval [0, 3].
c) At what times does the mass reach its lowest point the first three times?
d) At what times does the mass reach its highest point the first three times?

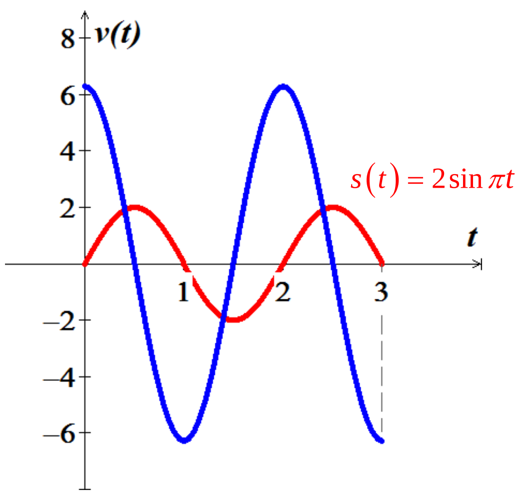
Solution

a) $s(t) = s(0) + \int_0^t (2\pi\cos\pi x)\,dx$

$$= 2\sin\pi x \Big|_0^t$$

$$= 2\sin\pi t$$

b)



c) The smallest value of Sine is -1, therefore the angle are

$$\frac{4k+3}{2}\pi = \pi t \quad \Rightarrow \quad t = \frac{4k+3}{2} \quad (k = 0,1,2)$$

The mass reaches its lowest point at $t = 1.5,\ t = 3.5,\ \textit{and}\ \ t = 5.5$

d) The Largest value of Sine is 1, therefore the angle are

$$\frac{4k+1}{2}\pi = \pi t \quad \Rightarrow \quad t = \frac{4k+1}{2} \quad (k = 0,1,2)$$

The mass reaches its highest point at $t = 0.5,\ t = 2.5,\ \textit{and}\ \ t = 4.5$

Exercise

The velocity of an airplane flying into a headwind is given by $v(t)=30\left(16-t^2\right)$ *mi/hr* for $0\le t\le 3$ *hr*.

Assume that $s(0)=0$

a) Determine and graph the position function for $0\le t\le 3$.

b) How far does the airplane travel in the first 2 *hr*.?

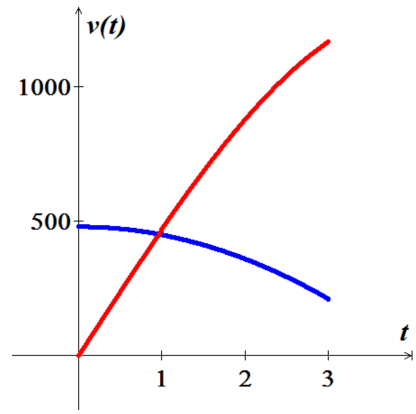
c) How far has the airplane traveled at the instant its velocity reaches 400 *mi/hr*.?

Solution

a) $s(t)=s(0)+30\int_0^t\left(16-x^2\right)dx$

$=30\left(16x-\frac{1}{3}x^3\right)\Big|_0^t$

$\boxed{=480t-10t^3}$



b) $s(2)=480(2)-10\left(2^3\right)\ \boxed{=880\ \ miles}$

c) *Given*: $v=400;\ \to 480-30t^2=400$

$t^2=\frac{8}{3}\quad\Rightarrow\quad \boxed{t=\sqrt{\frac{8}{3}}}$

$s\left(\sqrt{\frac{8}{3}}\right)=480\sqrt{\frac{8}{3}}-10\left(\sqrt{\frac{8}{3}}\right)^3\ \boxed{\approx 740.290\ \ miles}$

Exercise

A car slows down with an acceleration of $a(t)=-15\ ft/s^2$. Assume that $v(0)=60\ ft/s$ and $s(0)=0$

a) Determine and graph the position function for $t\ge 0$.

b) How far does the car travel in the time it takes to come to rest?

Solution

a) $v(t)=\int a(t)\,dt=-\int 15\ dt=-15t+C_1$

Given: $v(0)=60\quad\Rightarrow\quad \boxed{C_1=60}$

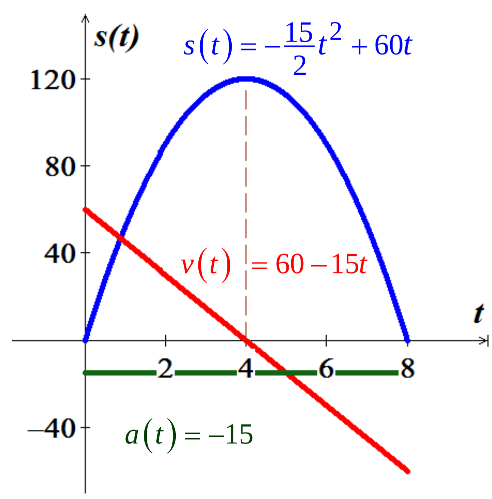
$v(t)\ \boxed{=60-15t}$

$s(t)=\int v(t)\,dt=\int(60-15t)\ dt$

$=60t-\frac{15}{2}t^2+C_2$

Given: $s(0)=0\quad\Rightarrow\quad \boxed{C_2=0}$

$\boxed{s(t)=-\frac{15}{2}t^2+60t}$



b) The car comes to rest when $v(t) = 0 = 60 - 15t \quad \rightarrow \quad \underline{t = 4}$

$$s(4) = -\frac{15}{2}(4)^2 + 60(4) \ \underline{= 120 \ ft}$$

Exercise

The owners of an oil reserve begin extracting oil at $t = 0$. Based on estimates of the reserves, suppose the projected extraction rate is given by $Q'(t) = 3t^2(40-t)^2$, where $0 \le t \le 40$, Q is measured in millions of barrels, and t is measured in years.

a) When does the peak extraction rate occur?

b) How much oil is extracted in the first 10, 20, and 30 years?

c) What is the total amount of oil extracted in 40 year?

d) Is one-fourth of the total oil extracted in the first one-fourth of the extraction period? Explain.

Solution

a)
$$\begin{aligned} Q''(t) &= 6t(40-t)^2 - 6t^2(40-t) \\ &= 6t(40-t)(40-2t) \\ &= 12t(40-t)(20-t) = 0 \end{aligned}$$

0	20	40
+	–	

From the table, Q' is maximized at $t = 20$; therefore the peak extraction rate is at $t = 20$ *yrs*

b) In the first **10 years**:

$$\begin{aligned} Q(t) &= \int Q'(t)\,dt = \int_0^{10} \left(3t^4 - 240t^3 + 4800t^2\right)dt \\ &= \left[\frac{3}{5}t^5 - 60t^4 + 1600t^3\right]_0^{10} \\ &= \frac{3}{5}10^5 - 6\left(10^5\right) + 16\left(10^5\right) \\ &= \frac{53}{5}\cdot 10^5 \\ &= \underline{106 \cdot 10^4} \end{aligned}$$

1,060,000 millions of barrels

In the first **20** years

$$\begin{aligned} Q(t) &= \int Q'(t)\,dt = \int_0^{20} \left(3t^4 - 240t^3 + 4800t^2\right)dt \\ &= \left[\frac{3}{5}t^5 - 60t^4 + 1600t^3\right]_0^{20} \\ &= \frac{3}{5}\cdot 2^5 \cdot 10^5 - 6 \cdot 2^4 \cdot 10^5 + 16 \cdot 2^3 \cdot 10^5 \\ &= 2^5 \cdot 10^5\left(\frac{3}{5} - 3 + 4\right) \\ &= 8 \cdot 2^6 \cdot 10^4 \\ &= \underline{2^9 \cdot 10^4} \end{aligned}$$

5,120,000 *millions* of barrels

In the first **30** years

$$Q(t)=\int Q'(t)\,dt=\int_0^{30}\left(3t^4-240t^3+4800t^2\right)dt$$

$$=\left[\frac{3}{5}t^5-60t^4+1600t^3\right]_0^{30}$$

$$=\frac{3}{5}\cdot 3^5\cdot 10^5-6\cdot 3^4\cdot 10^5+16\cdot 3^3\cdot 10^5$$

$$=3^3\cdot 10^5\left(\frac{27}{5}-18+16\right)$$

$$=3^3\cdot 10^5\cdot\frac{17}{5}$$

$=9{,}180{,}000$ *millions* of barrels

c) Total amount of oil extracted in 40 year

$$Q(t)=\int Q'(t)\,dt=\int_0^{40}\left(3t^4-240t^3+4800t^2\right)dt$$

$$=\left[\frac{3}{5}t^5-60t^4+1600t^3\right]_0^{40}$$

$$=\frac{3}{5}\cdot 4^5\cdot 10^5-6\cdot 4^4\cdot 10^5+16\cdot 4^3\cdot 10^5$$

$$=4^4\cdot 10^5\left(\frac{12}{5}-6+4\right)$$

$$=4^4\cdot 10^5\cdot\frac{2}{5}$$

$=10{,}240{,}000$ *millions* of barrels

d) $\frac{1}{4}(10{,}240{,}000)=2{,}560{,}000\neq 5{,}120{,}000$

No, the amount extracted in the first 10 years is not $\frac{1}{4}$ of the total amount extracted.

Exercise

Starting with an initial value of $P(0)=55$, the population of a prairie dog community grows at a rate of $P'(t)=20-\frac{t}{5}$ (in units of prairie *dogs/month*), for $0\le t\le 200$.

a) What is the population 6 months later?

b) Find the population $P(t)$ for $0\le t\le 200$.

Solution

a) $P(t)=P(0)+\int_0^{6}\left(20-\frac{t}{5}\right)dt$

$$=55+\left[20t-\frac{1}{10}t^2\right]_0^{6}$$

$$= 55 + 120 - \frac{18}{5}$$

$$\underline{= \frac{857}{5} \quad \approx 171.4}$$

b) $P(t) = 55 + \left[20t - \frac{1}{10}t^2\right]_0^{200}$

$$= 55 + 4{,}000 - 4{,}000$$

$$\underline{= 55}$$

Exercise

The population of a community of foxes is observed to fluctuate on a 10-year cycle due to variations in the availability of prey. When population measurements began (t = 0 years), the population was 35 foxes. The growth rate in units of *foxes/yr*. was observed to be

$$P'(t) = 5 + 10\sin\left(\frac{\pi t}{5}\right)$$

a) What is the population 15 years later? 35 years later?

b) Find the population $P(t)$ at any time $t \ge 0$.

Solution

a) $P(t) = P(0) + \int_0^{15}\left(5 + 10\sin\frac{\pi t}{5}\right)dt$

$$= 35 + \left[5t - \frac{50}{\pi}\cos\frac{\pi t}{5}\right]_0^{15}$$

$$= 35 + 75 + \frac{50}{\pi} + \frac{50}{\pi}$$

$$\underline{= 110 + \frac{100}{\pi} \quad \approx 142 \ \textit{foxes}}$$

$$P(t) = 35 + \left[5t - \frac{50}{\pi}\cos\frac{\pi t}{5}\right]_0^{35}$$

$$= 35 + 175 + \frac{50}{\pi} + \frac{50}{\pi}$$

$$\underline{= 210 + \frac{100}{\pi} \quad \approx 242 \ \textit{foxes}}$$

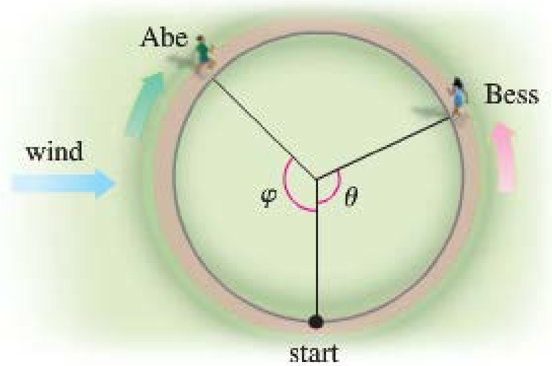
b) $P(t) = 35 + \int_0^{t}\left(5 + 10\sin\left(\frac{\pi}{5}x\right)\right)dx$

$$= 35 + \left[5x - \frac{50}{\pi}\cos\frac{\pi x}{5}\right]_0^{t}$$

$$\underline{= 35 + 5t - \frac{50}{\pi}\cos\frac{\pi t}{5} + \frac{50}{\pi} \quad \textit{foxes}}$$

Exercise

A strong west wind blows across a circular running track. Abe and Bess start at the south end of the track and at the same time, Abe starts running clockwise and Bess starts running counterclockwise. Abe runs with a speed (in units of *mi/hr*.) given by $u(\varphi)=3-2\cos\varphi$ and Bess runs with a speed given by $v(\theta)=3+2\cos\theta$, where φ and θ are the central angles of the runners



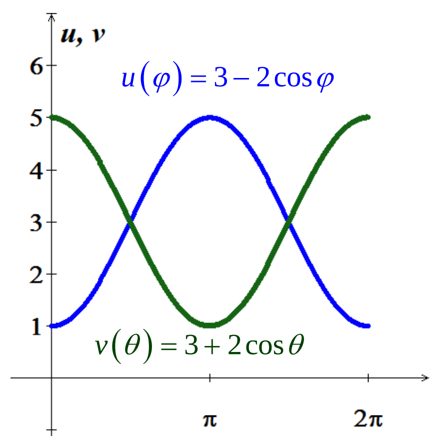
a) Graph the speed functions *u* and *v*, and explain why they describe the runners' speed (in light of the wind).

b) Which runner has the greater average speed for one lap?

c) If the track has a radius of $\frac{1}{10}$ *mi*, how long does it take each runner to complete one lap and who wins the race?

Solution

a) Abe starts out running into a headwind
Bess starts out running with a tailwind.



b) Abe's average speed $=\frac{1}{2\pi}\int_0^{2\pi}(3-2\cos\varphi)\,d\varphi$

$$=\frac{1}{2\pi}\left[3\varphi-2\sin\varphi\right]_0^{2\pi}$$

$$=3\ \ mph$$

Bess' average speed $=\frac{1}{2\pi}\int_0^{2\pi}(3+2\cos\theta)\,d\theta$

$$=\frac{1}{2\pi}\left[3\theta+2\sin\theta\right]_0^{2\pi}$$

$$=3\ \ mph$$

They have the same average speed.

c) The track is $\frac{1}{10}$ *mi* in radius $\Rightarrow s=\frac{1}{10}\varphi$ $\boxed{s=r\theta}$

We have $u=\frac{ds}{dt}=\frac{1}{10}\frac{d\varphi}{dt}=3-2\cos\varphi$

$$dt=\frac{1}{10(3-2\cos\varphi)}d\varphi$$

Abe's time:

$$T=\int_0^T dt=\int_0^{2\pi}\frac{1}{10(3-2\cos\varphi)}d\varphi$$

$$=\frac{2}{10\sqrt{5}}\tan^{-1}\left(\sqrt{5}\,\tan\frac{\varphi}{2}\right)\Bigg|_0^{2\pi}$$

$$=\frac{1}{5\sqrt{5}}[\pi-0]$$

$$\boxed{=\frac{\pi\sqrt{5}}{25}}$$

Let $\varphi=2\tan^{-1}u \quad\Rightarrow\quad u=\tan\frac{\varphi}{2}$

$$d\varphi=\frac{2}{1+u^2}du$$

$$\cos\varphi=\frac{1-\tan^2\frac{\varphi}{2}}{1+\tan^2\frac{\varphi}{2}}=\frac{1-u^2}{1+u^2}$$

$$\int\frac{d\varphi}{3-2\cos\varphi}=\int\frac{2}{1+u^2}\frac{1}{3-2\frac{1-u^2}{1+u^2}}du$$

$$=\int\frac{2}{1+5u^2}du$$

$$=\frac{2}{\sqrt{5}}\int\frac{1}{1+\left(\sqrt{5}u\right)^2}d\left(\sqrt{5}u\right)$$

$$=\frac{2}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}u\right)$$

$$=\frac{2}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\,\tan\frac{\varphi}{2}\right)$$

Bess' time:

$$T=\int_0^T dt=\int_0^{2\pi}\frac{1}{10(3+2\cos\theta)}d\theta$$

$$=\frac{1}{10}\int_0^{2\pi}\frac{2}{1+u^2}\frac{1}{3+2\frac{1-u^2}{1+u^2}}du$$

$$=\frac{1}{5}\int_0^{2\pi}\frac{du}{5+u^2}$$

$$=\frac{1}{5\sqrt{5}}\tan^{-1}\left(\frac{1}{\sqrt{5}}\,\tan u\right)\Bigg|_0^{2\pi}$$

$$=\frac{1}{5\sqrt{5}}\tan^{-1}\left(\frac{1}{\sqrt{5}}\,\tan\frac{\theta}{2}\right)\Bigg|_0^{2\pi}$$

$$\boxed{=\frac{\pi\sqrt{5}}{25}}$$

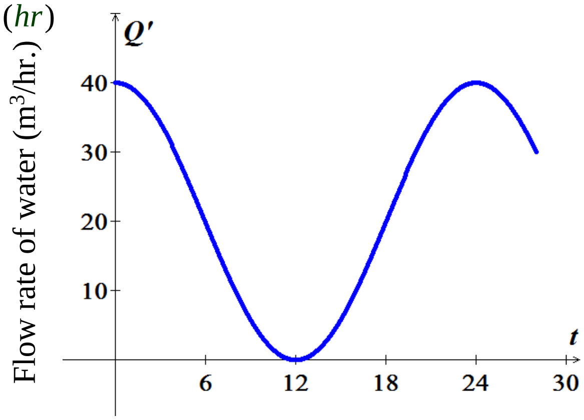
They tie the race, both have average speed $\frac{2\pi}{10T}=\frac{2\pi}{10\frac{\pi\sqrt{5}}{25}}=\frac{1}{\frac{\sqrt{5}}{5}}\ \boxed{=\sqrt{5}}$

Exercise

A reservoir with a capacity of 2500 m^3 is filled with a single inflow pipe. The reservoir is empty and the inflow pipe is opened at $t = 0$. Letting $Q(t)$ be the amount of water in the reservoir at time t, the flow rate of water into reservoir (in m^3 / hr) oscillates on 24-*hr* cycle and is given by

$$Q'(t) = 20\left[1 + \cos\frac{\pi t}{12}\right]$$

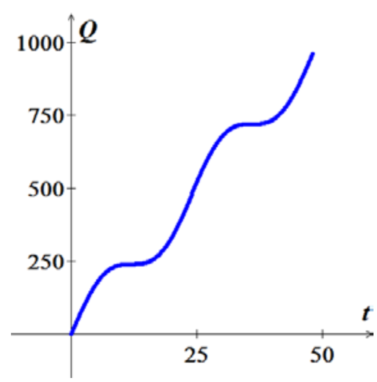
Time (*hr*)



a) How much water flows into the reservoir in the first 2 hr.?

b) Find and graph the function that gives the amount of water in the reservoir over the interval $[0,\ t]$ where $t \ge 0$.

c) When is the reservoir full?



Solution

a) $$Q(t) = \int_0^2 20\left(1 + \cos\frac{\pi t}{12}\right)dt$$

$$= \left(20t + \frac{240}{\pi}\sin\frac{\pi t}{12}\right)\Big|_0^2$$

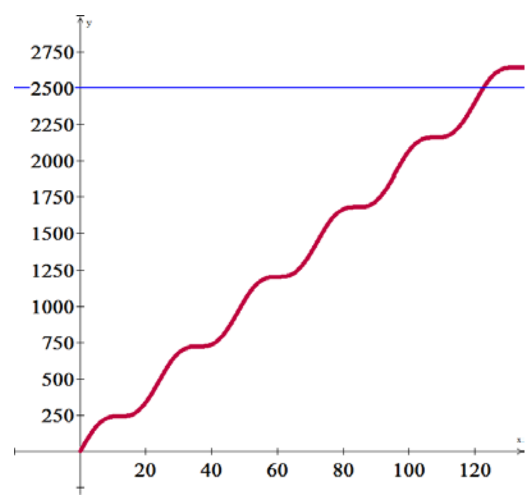
$$= 400 + \frac{240}{\pi}\cdot\frac{1}{2}$$

$$= 400 + \frac{120}{\pi} \quad \approx 78.197\ m^3$$

b) $$Q(t) = \int_0^t 20\left(1 + \cos\frac{\pi x}{12}\right)dx$$

$$= \left(20x + \frac{240}{\pi}\sin\frac{\pi x}{12}\right)\Big|_0^t$$

$$= 20t + \frac{240}{\pi}\sin\frac{\pi t}{12} \quad m^3$$



c) The reservoir is full when $20t + \frac{240}{\pi}\sin\frac{\pi t}{12} = 2500$

Using program: $T \approx 122.6\ \ hrs$