Solution   Section 2.4 – Multiplication Rule and Conditional

Exercise

Use the data below:

	No (Did Not Lie)	***Yes*** (Lied)
Positive test result	15 (*false positive*)	42 (*true positive*)
Negative test result	32 (*true negative*)	9 (*false negative*)

a) If 2 of the 98 test subjects are randomly selected without replacement find the probability that they both had false positive results. Is it unusual to randomly select 2 subjects without replacement and get 2 results that are both false positive results? Explain.

b) If 3 of the 98 test subjects are randomly selected without replacement, find the probability that all had false positive results. Is it unusual to randomly select 3 subjects without replacement and get 3 results that are all false positive results? Explain.

c) If 4 of the test subjects are randomly selected without replacement find the probability that, in each case, the polygraph indicated that the subject lied. Is such an event unusual?

d) If 4 of the test subjects are randomly selected without replacement find the probability that they all had incorrect test result (either false positive or false negative). Is such an event Likely?

e) Assume that 1 of the 98 test subjects is randomly selected. Find the probability of selecting a subject with a negative test result, given that the subject lied. What does this result suggest about the polygraph test?

f) Find $P(\text{negative test result} \mid \text{subject did not lie})$

g) Find $P(\text{subject did not lie} \mid \text{negative test result})$

Solution

	$\bar{L}$	L	
P	15	42	57
N	32	9	41
	47	51	**98**

Let F = selected person had false positive results

Let P = selected person tested positive

a) $P(F_1 \text{ and } F_2) = P(F_1) \cdot P(F_2 | F_1)$

$$= \frac{15}{98} \cdot \frac{14}{97}$$

$$= 0.0221$$

Yes; since $0.0221 \le 0.05$, getting 2 subjects who had false positives would be unusual.

b) $P(F_1 \text{ and } F_2 \text{ and } F_3) = P(F_1) \cdot P(F_2 | F_1) \cdot P(F_3 | F_1 \text{ and } F_2)$

$$= \frac{15}{98} \cdot \frac{14}{97} \cdot \frac{13}{96}$$

$$= 0.00299$$

Yes; since $0.00229 \le 0.05$, getting 3 subjects who had false positives would be unusual.

c) $P\left(P_1 \text{ and } P_2 \text{ and } P_3 \text{ and } P_4\right) = P\left(P_1\right)\cdot P\left(P_2 | P_1\right)\cdot P\left(P_3 | P_1 \text{ and } P_2\right)\cdot P\left(P_4 | P_1 \text{ and } P_2 \text{ and } P_3\right)$

$$= \frac{57}{98}\cdot\frac{56}{97}\cdot\frac{55}{96}\cdot\frac{54}{95}$$

$$\underline{= 0.109|}$$

No; since 0.109 > 0.05, getting 4 subjects who had false positives would not be unusual.

d) Let I = selected person had incorrect results.

$$P\left(I_1\right) = \frac{15}{98} + \frac{9}{98} = \frac{24}{98}$$

$$P\left(I_1 \text{ and } I_2 \text{ and } I_3 \text{ and } I_4\right) = P\left(I_1\right)\cdot P\left(I_2 | I_1\right)\cdot P\left(I_3 | I_1 \text{ and } I_2\right)\cdot P\left(I_4 | I_1 \text{ and } I_2 \text{ and } I_3\right)$$

$$= \frac{24}{98}\cdot\frac{23}{97}\cdot\frac{22}{96}\cdot\frac{21}{95}$$

$$\underline{= 0.00294|}$$

Yes; since $0.00294 \le 0.05$, getting 4 subjects who had incorrect results would be unusual.

e) $P\left(\overline{P} | Y\right) = \frac{9}{51}$ $\underline{= 0.176|}$

This result suggests that the polygraph is not very reliable because 17.6% of the time it fails to catch a person who really is lying.

f) $P\left(\text{negative test result} | \text{subject did not lie}\right) = P\left(\overline{P} | \overline{Y}\right) = \frac{32}{47}$ $\underline{= 0.681|}$

g) $P\left(\text{subject did not lie} | \text{negative test result}\right) = P\left(\overline{Y} | \overline{P}\right) = \frac{32}{41}$ $\underline{= 0.780|}$

Exercise

Use the data in the table below

	Group			
Type	***O***	***A***	***B***	***AB***
Rh^+	39	35	8	4
Rh^-	6	5	2	1

a) If 2 of the 100 subjects are randomly selected, find the probability that they are both group O and type Rh^+

i. Assume that the selections are made with replacement.

ii. Assume that the selections are made without replacement.

b) If 3 of the 100 subjects are randomly selected, find the probability that they are both group B and type Rh^-

i. Assume that the selections are made with replacement.

ii. Assume that the selections are made without replacement.

c) People with blood that is group O and type Rh^{-} are considered to be universal donors, because they can give blood to anyone. If 4 of the 100 subjects are randomly selected, find the probability that they are all universal donors.

i. Assume that the selections are made with replacement.

ii. Assume that the selections are made without replacement.

d) People with blood that is group AB and type Rh^{+} are considered to be universal donors, because they can give blood to anyone. If 3 of the 100 subjects are randomly selected, find the probability that they are all universal recipients.

i. Assume that the selections are made with replacement.

ii. Assume that the selections are made without replacement.

Solution

a) $P(O \text{ and } Rh+) = \frac{39}{100}$

i. $P(\text{With replacement}) = \frac{39}{100} \cdot \frac{39}{100} \underline{= 0.152|}$

ii. $P(\text{Without replacement}) = \frac{39}{100} \cdot \frac{38}{99} \underline{= 0.150|}$

b) $P(B \text{ and } Rh-) = \frac{2}{100}$

iii. $P(\text{With replacement}) = \frac{2}{100} \cdot \frac{2}{100} \cdot \frac{2}{100} \underline{= 0.000008|}$

iv. $P(\text{Without replacement}) = \frac{39}{100} \cdot \frac{1}{99} \cdot \frac{0}{98} \underline{= 0|}$

c) $P(O \text{ and } Rh-) = \frac{6}{100}$

v. $P(\text{With replacement}) = \frac{6}{100} \cdot \frac{6}{100} \cdot \frac{6}{100} \cdot \frac{6}{100} \underline{= 0.0000130|}$

vi. $P(\text{Without replacement}) = \frac{6}{100} \cdot \frac{5}{99} \cdot \frac{4}{98} \cdot \frac{3}{97} \underline{= 0.00000383|}$

d) $P(AB \text{ and } Rh+) = \frac{4}{100}$

vii. $P(\text{With replacement}) = \frac{4}{100} \cdot \frac{4}{100} \cdot \frac{4}{100} \underline{= 0.000064|}$

viii. $P(\text{Without replacement}) = \frac{4}{100} \cdot \frac{3}{99} \cdot \frac{2}{98} \underline{= 0.0000247|}$

Exercise

With one method of a procedure called *acceptance sampling*, a sample of items is randomly selected without replacement and the entire batch is accepted if every item in the sample is okay. The Telektronics Company manufactured a batch of 400 backup power supply units for computers, and 8 of them are defective. If 3 of the units are randomly selected for testing, what is the probability that the entire batch will be accepted?

Solution

Let P = power supply unit is OK.

$$\begin{aligned} P(\text{entire batch is accepted}) &= P\left(P_1 \text{ and } P_2 \text{ and } P_3\right) \\ &= P\left(P_1\right) \cdot P\left(P_2 | P_1\right) \cdot P\left(P_3 | P_1 \text{ and } P_2\right) \\ &= \frac{392}{400} \cdot \frac{391}{399} \cdot \frac{390}{398} \\ &= 0.941 \end{aligned}$$

Exercise

It is common for public opinion polls to have a "confidence level" of 95% meaning that there is a 095 probability that the poll results are accurate within the claimed margins of error. If each of the following organizations conducts an independent poll, find the probability that all of them are accurate within the claim margins of error: Gallup, Roper, Yankelovich, Harris, CNN, ABC, CBS, and NBC, New York Times. Does the result suggest that with a confidence level of 95%, we can expect that almost all polls will be within the claimed margin of error?

Solution

Let A = public opinion poll is accurate within its margin of error.

Each polling organization: $P(A) = 0.95$

$$\begin{aligned} P(\text{all 9 are accurate}) &= P\left(A_1 \text{ and } A_2 \text{ and } \cdots \text{ and } A_9\right) \\ &= P\left(A_1\right) \cdot P\left(A_2\right) \cdots P\left(A_9\right) \\ &= (.95)^9 \\ &= 0.630 \end{aligned}$$

No, with 9 independent polls the probability that at least one of them is not accurate within its margin of error is

$$\begin{aligned} P(\text{all accurate}) &= 1 - 0.630 \\ &= 0.370 \end{aligned}$$

Exercise

The principle of redundancy is used when system reliability is improved through redundant or back up components. Assume that your alarm clock has a 0.9 probability of working on any given morning.

a) What is the probability that your alarm clock will not work on the morning of an important final exam?

b) If you have 2 such alarm clocks, what is the probability that they both fail on the morning of an important final exam?

c) With one alarm clock, you have a 0.9 probability of being awakened. What is the probability of being awakened if you use two alarm clocks?

d) Does a second alarm clock result in greatly improved reliability?

Solution

Let A = alarm works.

For each alarm: $P(A)=0.9$

a) $P(\bar{A})=1-P(A)=1-0.9 \ \underline{=0.1}$

b) $P(\bar{A}_1 \ and \ \bar{A}_2)=P(\bar{A}_1)\cdot P(\bar{A}_2)$

$$=(0.1)(.1)$$

$$\underline{=0.01}$$

c) $P(being \ awakened)=P\left(\overline{\bar{A}_1 \ and \ \bar{A}_2}\right)$

$$=1-P(\bar{A}_1 \ and \ \bar{A}_2)$$

$$=1-.01$$

$$\underline{=0.99}$$

d) From parts (b) and (c) assume that the alarm clocks work independently of each other. This would not be true if they are both electric alarm clocks.

Exercise

The wheeling Tire Company produced a batch of 5,000 tires that includes exactly 200 that are defective.

a) If 4 tires are randomly selected for installation on a car, what is the probability that they are all good?

b) If 100 tires are randomly selected for shipment to an outlet, what is the probability that they are all good? Should this outlet plan to deal with defective tires returned by consumers?

Solution

Let G = getting a good tire.

$$P(A)=\frac{4800}{5000}=0.96$$

a) $P(G_1 and \ G_2 and \ G_3 and \ G_4)=P(G_1)\cdot P(G_2|G_1)\cdot P(G_3|G_1 \ and \ G_2)\cdot P(G_4|G_1 \ and \ G_2 and \ G_3)$

$$= \frac{4800}{5000} \cdot \frac{4799}{4999} \cdot \frac{4798}{4998} \cdot \frac{4797}{4997}$$

$$\underline{= 0.849}|$$

b) Since n = 100 represents $\frac{100}{5000} = 0.02 \le 0.05$ of the population, use the 5% guideline and treat the repeated selections as being independent.

$$P\left(G_1 \text{ and } G_2 \text{ and } \ldots \text{and } G_4\right) = P\left(G_1\right) \cdot P\left(G_2\right) \cdot \cdots \cdot P\left(G_{100}\right)$$

$$= (0.96)(0.96) \cdots (0.96)$$

$$= (0.96)^{100}$$

$$\underline{= 0.0169}|$$

Yes; since 0.0169 ≤ 0.05, getting 100 good tires would be unusual event and the outlet should plan on dealing with returns of defective tires.

Exercise

When the 15 players on the LA Lakers basketball team are tested for steroids, at least one of them tests positive. Provide a written description of the complement of this event.

Solution

If it is not true that at least one of the 15 tests positive, then all 15 of them test negative.

Exercise

If a couple plans to have 6 children, what is the probability that they will have at least one girl? Is that probability high enough for the couple to be very confident that they will get at least one girl in six children?

Solution

$$P(\text{at least one girl}) = 1 - P(\text{all boys})$$

$$= 1 - P\left(B_1 \text{ and } B_2 \text{ and } B_3 \text{ and } B_4 \text{ and } B_5 \text{ and } B_6\right)$$

$$= 1 - P\left(B_1\right) \cdot P\left(B_2\right) \cdot P\left(B_3\right) \cdot P\left(B_4\right) \cdot P\left(B_5\right) \cdot P\left(B_6\right)$$

$$= 1 - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$\underline{= 0.984}|$$

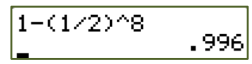
Yes, the probability is high enough for the couple to be very confident that they will get at least one girl in 6 children.

Exercise

If a couple plans to have 8 children (it could happen), what is the probability that they will have at least one girl? Is the couple eventually has 8 children and they are all boys, what can the couple conclude?

Solution

$$
\begin{aligned}
P(\text{at least one girl}) &= 1 - P(\text{all boys}) \\
&= 1 - P\left(B_1 \text{ and } B_2 \cdots \text{and } B_8\right) \\
&= 1 - P\left(B_1\right)\cdot P\left(B_2\right)\cdot P\left(B_3\right)\cdot P\left(B_4\right)\cdot P\left(B_5\right)\cdot P\left(B_6\right)\cdot P\left(B_7\right)\cdot P\left(B_8\right) \\
&= 1 - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} \\
&= 0.996
\end{aligned}
$$



If the couple has 8 boys, either a very rare event has occurred or there is some environmental or genetic factor that makes boys more likely for this couple.

Exercise

If you make guesses for 4 multiple-choice test questions (each with 5 possible answers), what is the probability of getting at least one correct? If a very lenient instructor says that passing test occurs if there is at least one correct answer, can you reasonably expect to pass by guessing?

Solution

$$
\begin{aligned}
P(\text{at least one correct}) &= 1 - P(\text{all wrong}) \\
&= 1 - P\left(W_1 \text{ and } W_2 \text{ and } W_3 \text{ and } W_4\right) \\
&= 1 - P\left(W_1\right)\cdot P\left(W_2\right)\cdot P\left(W_3\right)\cdot P\left(W_4\right) \\
&= 1 - \frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5} \\
&= 0.590
\end{aligned}
$$

Since there is a greater chance of passing than of failing, the expectation is that such a strategy would lead to passing. In that sense, one can reasonably expect to pass by guess. Nut while the expectation for a single test may be pass, such a strategy can be expected to lead to failing about 4 times every 10 times it is applied.

Exercise

Find the probability of a couple having a baby girl when their fourth child is born, given that the first 3 children were all girls. Is the result the same as the probability of getting 4 girls among 4 children?

Solution

Sample Space:

$$S=\{BBBB,\ BBBG,\ BBGB,\ BGBB,\ GBBB,\ BBGG,\ BGBG,\ BGGB,$$
$$GBBG,\ GBGB,\ GGBB,\ BGGG,\ GBGG,\ GGBG,\ GGGB,\ GGGG\}$$

Let F = first 3 children are girls.

Let G_4 = fourth child is a girl.

$P(F)=\frac{2}{16}$ $\qquad\qquad$ $P(G_4)=\frac{8}{16}=\frac{1}{2}$

$$P(F\ and\ G_4)=\frac{1}{16}$$

$$P(G_4|F)=\frac{P(G_4\ and\ F)}{P(F)}$$
$$=\frac{\frac{1}{16}}{\frac{2}{16}}$$
$$\boxed{=\frac{1}{2}}$$

$P(G_4)=\frac{1}{2}=P(G)$ independent (occurred during the first 3 births)

This probability is not true the same as $P(GGGG)=\frac{1}{16}$

Exercise

In China, the probability of a baby being a boy is 0.5845. Couples are allowed to have only one child. If relatives give birth to 5 babies, what is the probability that there is at least one girl? Can that system continue to work indefinitely?

Solution

$$P(at\ least\ one\ girl)=1-P(all\ boys)$$
$$=1-P(B_1 and\ B_2 and\ B_3 and\ B_4 and\ B_5)$$
$$=1-P(B_1)\cdot P(B_2)\cdot P(B_3)\cdot P(B_4)\cdot P(B_5)$$
$$=1-(0.5845)\cdot(0.5845)\cdot(0.5845)\cdot(0.5845)\cdot(0.5845)$$

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1-.5845^5
                .932
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$$\boxed{=0.932}$$

Probably not; the system will produce such a shortage of females that baby girls will become a valuable asset and parents will take appropriate measures to change the probabilities.

Exercise

An experiment with fruit flies involves one parent with normal wings and one parent with vestigial wings. When these parents have an offspring, there is a $\frac{3}{4}$ probability that the offspring has normal wings and a $\frac{1}{4}$ probability of vestigial wings. If the parents give birth to 10 offspring, what is the probability that at least 1 of the offspring has vestigial wings? If researchers need at least one offspring with vestigial wings, can they be reasonably confident of getting one?

Solution

$$\begin{aligned} P(at\ least\ 1\ w/vestigial\ wings) &= 1 - P(all\ have\ normal\ wings) \\ &= 1 - P\left(N_1 and\ N_2 and\ \ldots and\ N_{10}\right) \\ &= 1 - P\left(N_1\right)\cdot P\left(N_2\right)\cdot P\left(N_3\right)\cdots\cdot P\left(N_{10}\right) \\ &= 1 - \left(\frac{3}{4}\right)\cdot\left(\frac{3}{4}\right)\cdot\left(\frac{3}{4}\right)\cdots\cdot\left(\frac{3}{4}\right) \\ &= 1 - \left(\frac{3}{4}\right)^{10} \\ &= \underline{0.944} \end{aligned}$$

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1-(3/4)^10
                .944
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Yes, the researchers can be 94.4% certain of getting at least one such offspring.

Exercise

According to FBI data, 24.9% of robberies are cleared with arrests. A new detective is assigned to 10 different robberies.

a) What is the probability that at least 1 of them is cleared with an arrest?
b) What is the probability that the detective clears all 10 robberies with arrests?
c) What should we conclude if the detective clears all 10 robberies with arrests?

Solution

Let $P(C) = P(cleared\ with\ arrest) = (24.9\%) = 0.249$

$P(N) = P(not\ cleared\ with\ arrest) = 1 - .249 = 0.751$

a) $P(at\ least\ 1\ cleared) = 1 - P(all\ not\ cleared)$

$$\begin{aligned} &= 1 - P\left(N_1 and\ N_2 and\ \ldots and\ N_{10}\right) \\ &= 1 - P\left(N_1\right)\cdot P\left(N_2\right)\cdot P\left(N_3\right)\cdots\cdot P\left(N_{10}\right) \\ &= 1 - (.751)\cdot(.751)\cdots\cdot(.751) \\ &= 1 - (.751)^{10} \\ &= \underline{0.943} \end{aligned}$$

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1-.751^10
                .943
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b) $P(\textit{cleared all } 10) = P\left(C_1 \textit{ and } C_2 \textit{ and } \ldots \textit{and } C_{10}\right)$

$$= P\left(C_1\right) \cdot P\left(C_2\right) \cdot \cdots \cdot P\left(C_{10}\right)$$

$$= (.249) \cdot (.249) \cdot \cdots \cdot (.249)$$

$$= (.249)^{10}$$

$$\underline{= 0.000000916}$$

c) If the detective clears all 10 cases with arrests, we should conclude that the $P(C) = 0.249$ rate does not apply to this detective – The probability he clears a case with an arrest is much higher than 0.249.

Exercise

A statistics student wants to ensure that she is not late for an early statistics class because of a malfunctioning alarm clock. Instead of using one alarm clock, she decides to use three. What is the probability that at least one of her alarm clocks works correctly if each individual alarm clock has a 90% chance of working correctly? Does the student really gain much by using three alarm clocks instead on only one? How are the results affected if all of the alarm clocks run on electricity instead of batteries?

Solution

Let $P(F) = P(\textit{alarm clock fails}) = 1 - .9 = 0.1$

$P(\textit{at least } 1 \textit{ works}) = 1 - P(\textit{all fail})$

$$= 1 - P\left(F_1 \textit{ and } F_2 \textit{ and } F_3\right)$$

$$= 1 - P\left(F_1\right) \cdot P\left(F_2\right) \cdot P\left(F_3\right)$$

$$= 1 - (0.1) \cdot (0.1) \cdot (0.1)$$

$$\underline{= 0.999}$$

Yes, the probability of a working clock rises from 90% with just one clock to 99.9% with 3 clocks? If the alarm clocks run on electricity instead of batteries, then the clocks do not operate independently and the failure of one could be the result of a power failure or interruption and may be related to the failure of another $P\left(F_2 \middle| F_1\right)$ no longer $P(F) = 0.90$

Exercise

In a batch of 8,000 clock radios 8% are defective. A sample of 5 clock radios is randomly selected without replacement from the 8,000 and tested. The entire batch will be rejected if at least one of those tested is defective. Find the probability that the entire batch will be rejected.

Solution

Number of defective radios: $8000 \times .08 = 640$

$$P(at\ least\ 1) = 1 - P(none\ defective)$$

$$= 1 - \frac{\binom{7,360}{5}\binom{640}{0}}{\binom{8,000}{5}}$$

$$= 1 - .659$$

$$\approx 0.341$$

Exercise

In a blood testing procedure, blood samples from 3 people are combined into one mixture. The mixture will only test negative if all the individual samples are negative. If the probability that an individual sample tests positive is 0.1, find the probability that the mixture will test positive.

Solution

Probability that the mixture will test positive = 1 − (probability all negative)

$$P(all\ -) = 1 - 0.1 = 0.9$$

$$P = 1 - .9^3 \ = 0.271$$

Exercise

A sample of 4 different calculators is randomly selected from a group containing 16 that are defective and 36 that have no effects. Find the probability that at least one of the calculator is defective.

Solution

$$P(at\ least\ 1) = 1 - P(none\ defective)$$

$$= 1 - \frac{\binom{36}{4}\binom{16}{0}}{\binom{52}{4}}$$

$$= 1 - .218$$

$$\approx 0.782$$

Exercise

Among the contestants in a competition are 46 women and 29 men. If 5 winners are randomly selected, find the probability that they are all men?

Solution

$$P(\text{All men}) = \left(\frac{29}{29+46}\right)^5 \; \underline{= 0.009}$$

Exercise

A bin contains 60 lights bubs of which 7 are defective. If 4 light bulbs are randomly selected from the bin with replacement, find the probability that all the bulbs selected are good ones.

Solution

$$P(\text{All good ones}) = \left(\frac{53}{60}\right)^4 \; \underline{= 0.609}$$

Exercise

You are dealt two cards successively (without replacement) from a shuffled deck of 52 playing cards. Find the probability that both cards are black. Express your answer as a simplified fraction.

Solution

$$P(2 \text{ cards Black}) = \frac{C_{26,2}}{C_{52,2}} = \frac{325}{1326} \; \underline{= \frac{25}{102}}$$