

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = e^{2x}$, $a = 0$

Solution

$$f(x) = e^{2x} \rightarrow f(0) = e^{2(0)} = 1$$

$$f'(x) = 2e^{2x} \rightarrow f'(0) = 2$$

$$f''(x) = 4e^{2x} \rightarrow f''(0) = 4$$

$$f'''(x) = 8e^{2x} \rightarrow f'''(0) = 8$$

$$P_0(x) = f(0) = 1$$

$$P_1(x) = f(0) + f'(0)(x-0) = 1 + 2x$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 = 1 + 2x + 2x^2$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$\underline{= 1 + 2x + 2x^2 + \frac{4}{3}x^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \sin x$, $a = 0$

Solution

$$f(x) = \sin x \rightarrow f(0) = 0$$

$$f'(x) = \cos x \rightarrow f'(0) = 1$$

$$f''(x) = -\sin x \rightarrow f''(0) = 0$$

$$f'''(x) = -\cos x \rightarrow f'''(0) = -1$$

$$P_0(x) = f(0) = 0$$

$$P_1(x) = f(0) + f'(0)(x-0) = x$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 = x$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 = \underline{x - \frac{1}{6}x^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \ln(1+x)$, $a = 0$

Solution

$$f(x) = \ln(1+x) \rightarrow f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \rightarrow f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \rightarrow f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \rightarrow f'''(0) = 2$$

$$P_0(x) = f(0) = 0$$

$$P_1(x) = f(0) + f'(0)(x-0) = x$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 = x - \frac{1}{2}x^2$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 = \underline{x - \frac{1}{2}x^2 + \frac{1}{3}x^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \frac{1}{x+2}$, $a = 0$

Solution

$$f(x) = (x+2)^{-1} \rightarrow f(0) = \frac{1}{2}$$

$$f'(x) = -(x+2)^{-2} \rightarrow f'(0) = -\frac{1}{4}$$

$$f''(x) = 2(x+2)^{-3} \rightarrow f''(0) = \frac{1}{4}$$

$$f'''(x) = -6(x+2)^{-4} \rightarrow f'''(0) = -\frac{3}{8}$$

$$P_0(x) = f(0) = \frac{1}{2}$$

$$P_1(x) = f(0) + f'(0)(x-0) = \frac{1}{2} - \frac{1}{4}x$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 = \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 = \underline{\frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \sqrt{1-x}$, $a = 0$

Solution

$$f(x) = (1-x)^{1/2} \rightarrow f(0) = 1$$

$$f'(x) = -\frac{1}{2}(1-x)^{-1/2} \rightarrow f'(0) = -\frac{1}{2}$$

$$f''(x) = -\frac{1}{4}(1-x)^{-3/2} \rightarrow f''(0) = -\frac{1}{4}$$

$$f'''(x) = -\frac{3}{8}(1-x)^{-5/2} \rightarrow f'''(0) = -\frac{3}{8}$$

$$P_0(x) = f(0) = 1$$

$$P_1(x) = f(0) + f'(0)(x-0) = 1 - \frac{1}{2}x$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 = 1 - \frac{1}{2}x - \frac{1}{8}x^2$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 = \underline{1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = x^3$, $a = 1$

Solution

$$f(x) = x^3 \rightarrow f(1) = 1$$

$$f'(x) = 3x^2 \rightarrow f'(1) = 3$$

$$f''(x) = 6x \rightarrow f''(1) = 6$$

$$f'''(x) = 6 \rightarrow f'''(1) = 6$$

$$P_0(x) = \underline{1}$$

$$P_0(x) = f(a)$$

$$P_1(x) = \underline{1 + 3(x-1)}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$P_2(x) = \underline{1 + 3(x-1) + 3(x-1)^2}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x) = \underline{1 + 3(x-1) + 3(x-1)^2 + (x-1)^3}$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = 8\sqrt{x}$, $a = 1$

Solution

$$f(x) = 8x^{1/2} \rightarrow f(1) = 8$$

$$f'(x) = 4x^{-1/2} \rightarrow f'(1) = 4$$

$$f''(x) = -2x^{-3/2} \rightarrow f''(1) = -2$$

$$f'''(x) = 3x^{-5/2} \rightarrow f'''(1) = 3$$

$$P_0(x) = \underline{8}$$

$$P_1(x) = \underline{8 + 4(x-1)}$$

$$P_2(x) = \underline{8 + 4(x-1) - (x-1)^2}$$

$$P_3(x) = \underline{8 + 4(x-1) - (x-1)^2 + 3(x-1)^3}$$

$$P_0(x) = f(a)$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \sin x$, $a = \frac{\pi}{4}$

Solution

$$f(x) = \sin x \rightarrow f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = \cos x \rightarrow f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f''(x) = -\sin x \rightarrow f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f'''(x) = -\cos x \rightarrow f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$P_0(x) = \underline{\frac{\sqrt{2}}{2}}$$

$$P_1(x) = \underline{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)}$$

$$P_2(x) = \underline{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2}$$

$$P_3(x) = \underline{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3}$$

$$P_0(x) = f(a)$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \cos x$, $a = \frac{\pi}{6}$

Solution

$$f(x) = \cos x \rightarrow f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$f'(x) = -\sin x \rightarrow f'\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

$$f''(x) = -\cos x \rightarrow f''\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$f'''(x) = \sin x \rightarrow f'''\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$P_0(x) = \left[\frac{\sqrt{3}}{2} \right]$$

$$P_0(x) = f(a)$$

$$P_1(x) = \left[\frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) \right]$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$P_2(x) = \left[\frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2 \right]$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x) = \left[\frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x - \frac{\pi}{6}\right)^3 \right]$$

$$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \sqrt{x}$, $a = 9$

Solution

$$f(x) = x^{1/2} \rightarrow f(9) = 3$$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \rightarrow f'(9) = \frac{1}{6}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} = -\frac{1}{4x\sqrt{x}} \rightarrow f''(9) = -\frac{1}{4 \times 3^3}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} = \frac{3}{8x^2\sqrt{x}} \rightarrow f'''(9) = \frac{1}{2^3 \times 3^4}$$

$$P_0(x) = [3]$$

$$P_0(x) = f(a)$$

$$P_1(x) = \left[3 + \frac{1}{6}(x-9) \right]$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$P_2(x) = \left[3 + \frac{1}{2 \cdot 3}(x-9) - \frac{1}{2^3 \cdot 3^3}(x-9)^2 \right]$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x) = \left[3 + \frac{1}{2 \cdot 3}(x-9) - \frac{1}{2^2 \cdot 3^3}(x-9)^2 + \frac{1}{2^4 \cdot 3^5}(x-9)^3 \right]$$

$$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \sqrt[3]{x}$, $a = 8$

Solution

$$f(x) = x^{1/3} \rightarrow f(8) = 2$$

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} \rightarrow f'(8) = \frac{1}{2^2 \times 3}$$

$$f''(x) = -\frac{2}{9}x^{-5/3} = -\frac{2}{3^2 x^{5/3}} \rightarrow f''(8) = -\frac{1}{3^2 \times 2^4}$$

$$f'''(x) = \frac{10}{3^3}x^{-8/3} = \frac{2 \cdot 5}{3^3 x^{8/3}} \rightarrow f'''(8) = \frac{5}{2^7 \times 3^3}$$

$$P_0(x) = \underline{2}$$

$$P_0(x) = f(a)$$

$$P_1(x) = \underline{2 + \frac{1}{2^2 \cdot 3}(x-8)}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$P_2(x) = \underline{2 + \frac{1}{2^2 \cdot 3}(x-8) - \frac{1}{2^5 \cdot 3^2}(x-8)^2}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x) = \underline{2 + \frac{1}{2^2 \cdot 3}(x-8) - \frac{1}{2^5 \cdot 3^2}(x-8)^2 + \frac{1}{2^8 \cdot 3^4}(x-8)^3}$$

$$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \ln x$, $a = e$

Solution

$$f(x) = \ln x \rightarrow f(e) = 1$$

$$f'(x) = \frac{1}{x} \rightarrow f'(e) = \frac{1}{e}$$

$$f''(x) = -\frac{1}{x^2} \rightarrow f''(e) = -\frac{1}{e^2}$$

$$f'''(x) = \frac{2}{x^3} \rightarrow f'''(e) = \frac{2}{e^3}$$

$$P_0(x) = \underline{1}$$

$$P_0(x) = f(a)$$

$$P_1(x) = \underline{1 + \frac{1}{e}(x-e)}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$P_2(x) = \underline{1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x) = \underline{1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3}$$

$$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \sqrt[4]{x}$, $a = 8$

Solution

$$f(x) = x^{1/4} \rightarrow f(8) = \sqrt[4]{8}$$

$$f'(x) = \frac{1}{4}x^{-3/4} = \frac{1}{4x^{3/4}} \rightarrow f'(8 = 2^3) = \frac{1}{2^2 \times 2^{9/4}} = \frac{1}{2^4 \sqrt[4]{2}}$$

$$f''(x) = -\frac{3}{16}x^{-7/4} = -\frac{3}{2^4 x^{7/4}} \rightarrow f''(8) = -\frac{3}{2^4 2^{21/4}} = -\frac{3}{2^9 \sqrt[4]{2}}$$

$$f'''(x) = \frac{21}{2^6}x^{-11/4} = \frac{21}{2^6 x^{11/4}} \rightarrow f'''(8) = \frac{21}{2^6 \times 2^{33/4}} = \frac{21}{2^{14} \sqrt[4]{2}}$$

$$P_0(x) = \sqrt[4]{8}$$

$$P_0(x) = f(a)$$

$$P_1(x) = \sqrt[4]{8} + \frac{1}{2^4 \cdot \sqrt[4]{2}}(x-8)$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$P_2(x) = \sqrt[4]{8} + \frac{1}{2^4 \cdot \sqrt[4]{2}}(x-8) - \frac{3}{2^{10} \cdot \sqrt[4]{2}}(x-8)^2$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x) = \sqrt[4]{8} + \frac{1}{2^4 \cdot \sqrt[4]{2}}(x-8) - \frac{3}{2^{10} \cdot \sqrt[4]{2}}(x-8)^2 + \frac{7}{2^{15} \cdot \sqrt[4]{2}}(x-8)^3$$

$$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \tan^{-1}x + x^2 + 1$, $a = 1$

Solution

$$f(x) = \tan^{-1}x + x^2 + 1 \rightarrow f(1) = \frac{\pi}{4} + 2$$

$$f'(x) = \frac{1}{x^2 + 1} + 2x \rightarrow f'(1) = \frac{5}{2}$$

$$f''(x) = -\frac{2x}{(x^2 + 1)^2} + 2 \rightarrow f''(1) = -\frac{1}{2} + 2 = \frac{3}{2}$$

$$f'''(x) = -\frac{2x^2 + 2 - 8x^2}{(x^2 + 1)^3} = -\frac{2 - 2x^2}{(x^2 + 1)^3} \rightarrow f'''(1) = 0$$

$$(U^n V^m)' = U^{n-1} V^{m-1} (nU'V + mUV')$$

$$P_0(x) = \frac{\pi}{4} + 2$$

$$P_0(x) = f(a)$$

$$P_1(x) = \frac{\pi}{4} + 2 + \frac{5}{2}(x-1)$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$P_2(x) = \frac{\pi}{4} + 2 + \frac{5}{2}(x-1) - \frac{3}{4}(x-1)^2$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x) = \frac{\pi}{4} + 2 + \frac{5}{2}(x-1) - \frac{3}{4}(x-1)^2$$

$$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = e^x$, $a = \ln 2$

Solution

$$f(x) = e^x \rightarrow f(\ln 2) = 2$$

$$f'(x) = e^x \rightarrow f'(\ln 2) = 2$$

$$f''(x) = e^x \rightarrow f''(\ln 2) = 2$$

$$f'''(x) = e^x \rightarrow f'''(\ln 2) = 2$$

$$P_0(x) = \underline{2}$$

$$P_0(x) = f(a)$$

$$P_1(x) = \underline{2 + 2(x - \ln 2)}$$

$$P_1(x) = f(a) + f'(a)(x - a)$$

$$P_2(x) = \underline{2 + 2(x - \ln 2) + (x - \ln 2)^2}$$

$$P_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2$$

$$P_3(x) = \underline{2 + 2(x - \ln 2) + (x - \ln 2)^2 + \frac{1}{3}(x - \ln 2)^3}$$

$$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x - a)^3$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = e^{4x}$, $n = 4$

Solution

$$f(x) = e^{4x} \rightarrow f(0) = 1$$

$$f'(x) = 4e^{4x} \rightarrow f'(0) = 4$$

$$f''(x) = 16e^{4x} \rightarrow f''(0) = 16$$

$$f'''(x) = 64e^{4x} \rightarrow f'''(0) = 64$$

$$f^{(4)}(x) = 256e^{4x} \rightarrow f^{(4)}(0) = 256$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x) = \underline{1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = e^{-x}$, $n = 5$

Solution

$$f(x) = e^{-x} \rightarrow f(0) = 1$$

$$f'(x) = -e^{-x} \rightarrow f'(0) = -1$$

$$f''(x) = e^{-x} \rightarrow f''(0) = 1$$

$$f'''(x) = -e^{-x} \rightarrow f'''(0) = -1$$

$$f^{(4)}(x) = e^{-x} \rightarrow f^{(4)}(0) = 1$$

$$f^{(5)}(x) = -e^{-x} \rightarrow f^{(5)}(0) = -1$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$\underline{P_5(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = e^{-x/2}$, $n = 4$

Solution

$$f(x) = e^{-x/2} \rightarrow f(0) = 1$$

$$f'(x) = -\frac{1}{2}e^{-x/2} \rightarrow f'(0) = -\frac{1}{2}$$

$$f''(x) = \frac{1}{4}e^{-x/2} \rightarrow f''(0) = \frac{1}{4}$$

$$f'''(x) = -\frac{1}{8}e^{-x/2} \rightarrow f'''(0) = -\frac{1}{8}$$

$$f^{(4)}(x) = \frac{1}{16}e^{-x/2} \rightarrow f^{(4)}(0) = \frac{1}{16}$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = e^{x/3}$, $n = 4$

Solution

$$f(x) = e^{x/3} \rightarrow f(0) = 1$$

$$f'(x) = \frac{1}{3}e^{x/3} \rightarrow f'(0) = \frac{1}{3}$$

$$f''(x) = \frac{1}{9}e^{x/3} \rightarrow f''(0) = \frac{1}{9}$$

$$f'''(x) = \frac{1}{27}e^{x/3} \rightarrow f'''(0) = \frac{1}{27}$$

$$f^{(4)}(x) = \frac{1}{81}e^{x/3} \rightarrow f^{(4)}(0) = \frac{1}{81}$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = 1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \frac{1}{1944}x^4}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = \sin x$, $n = 5$

Solution

$$f(x) = \sin x \rightarrow f(0) = 0$$

$$f'(x) = \cos x \rightarrow f'(0) = 1$$

$$f''(x) = -\sin x \rightarrow f''(0) = 0$$

$$f'''(x) = -\cos x \rightarrow f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \rightarrow f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \cos x \rightarrow f^{(5)}(0) = 1$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$\underline{P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = \cos \pi x$, $n = 4$

Solution

$$f(x) = \cos \pi x \rightarrow f(0) = 1$$

$$f'(x) = -\pi \sin \pi x \rightarrow f'(0) = 0$$

$$f''(x) = -\pi^2 \cos \pi x \rightarrow f''(0) = -\pi^2$$

$$f'''(x) = \pi^3 \sin \pi x \rightarrow f'''(0) = 0$$

$$f^{(4)}(x) = \pi^4 \cos \pi x \rightarrow f^{(4)}(0) = \pi^4$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = 1 - \frac{\pi^2}{2}x^2 + \frac{\pi^4}{24}x^4}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = xe^x$, $n = 4$

Solution

$$f(x) = xe^x \rightarrow f(0) = 0$$

$$f'(x) = e^x + xe^x \rightarrow f'(0) = 1$$

$$f''(x) = 2e^x + xe^x \rightarrow f''(0) = 2$$

$$f'''(x) = 3e^x + xe^x \rightarrow f'''(0) = 3$$

$$f^{(4)}(x) = 4e^x + xe^x \rightarrow f^{(4)}(0) = 4$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = x^2e^{-x}$, $n = 4$

Solution

$$f(x) = x^2e^{-x} \rightarrow f(0) = 0$$

$$f'(x) = 2xe^{-x} - x^2e^{-x} \rightarrow f'(0) = 0$$

$$f''(x) = 2e^{-x} - 4xe^{-x} + x^2e^{-x} \rightarrow f''(0) = 2$$

$$f'''(x) = -6e^{-x} + 6xe^{-x} - x^2e^{-x} \rightarrow f'''(0) = -6$$

$$f^{(4)}(x) = 12e^{-x} - 8xe^{-x} + x^2e^{-x} \rightarrow f^{(4)}(0) = 12$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = x^2 - x^3 + \frac{1}{2}x^4}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = \frac{1}{x+1}$, $n = 5$

Solution

$$f(x) = \frac{1}{x+1} \rightarrow f(0) = 1$$

$$f'(x) = -(x+1)^{-2} \rightarrow f'(0) = -1$$

$$f''(x) = 2(x+1)^{-3} \rightarrow f''(0) = 2$$

$$f'''(x) = -6(x+1)^{-4} \rightarrow f'''(0) = -6$$

$$f^{(4)}(x) = 24(x+1)^{-5} \rightarrow f^{(4)}(0) = 24$$

$$f^{(5)}(x) = -120(x+1)^{-6} \rightarrow f^{(5)}(0) = -120$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$\underline{P_5(x) = 1 - x + x^2 - x^3 + x^4 - x^5}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = \frac{x}{x+1}$, $n = 4$

Solution

$$f(x) = \frac{x}{x+1} = 1 - \frac{1}{x+1} \rightarrow f(0) = 0$$

$$f'(x) = (x+1)^{-2} \rightarrow f'(0) = 1$$

$$f''(x) = -2(x+1)^{-3} \rightarrow f''(0) = -2$$

$$f'''(x) = 6(x+1)^{-4} \rightarrow f'''(0) = 6$$

$$f^{(4)}(x) = -24(x+1)^{-5} \rightarrow f^{(4)}(0) = -24$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = x - x^2 + x^3 - x^4}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = \sec x$, $n = 2$

Solution

$$f(x) = \sec x \rightarrow f(0) = 1$$

$$f'(x) = \sec x \tan x \rightarrow f'(0) = 0$$

$$f''(x) = \sec x \tan^2 x + \sec^3 x \rightarrow f''(0) = 1$$

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

$$\underline{P_2(x) = 1 + \frac{1}{2}x^2}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = \tan x$, $n = 3$

Solution

$$f(x) = \tan x \rightarrow f(0) = 0$$

$$f'(x) = \sec^2 x \rightarrow f'(0) = 1$$

$$f''(x) = 2 \sec^2 x \tan x \rightarrow f''(0) = 0$$

$$f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x \rightarrow f'''(0) = 2$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3$$

$$P_4(x) = x + \frac{1}{3}x^3$$

Exercise

Find the Maclaurin series for: xe^x

Solution

$$f(x) = xe^x \rightarrow f(0) = 0$$

$$f'(x) = e^x + xe^x \rightarrow f'(0) = 1$$

$$f''(x) = 2e^x + xe^x \rightarrow f''(0) = 2$$

$$f'''(x) = 3e^x + xe^x \rightarrow f'''(0) = 3$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$f^{(n)}(x) = ne^x + xe^x \rightarrow f^{(n)}(0) = n$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$xe^x = x + x^2 + \frac{1}{2}x^3 + \dots = \sum_{n=0}^{\infty} \frac{1}{(n-1)!} x^n$$

Exercise

Find the Maclaurin series for: $5 \cos \pi x$

Solution

$$f(x) = 5 \cos \pi x \rightarrow f(0) = 5$$

$$f'(x) = -5\pi \sin \pi x \rightarrow f'(0) = 0$$

$$f''(x) = -5\pi^2 \cos \pi x \rightarrow f''(0) = -5\pi^2$$

$$f'''(x) = 5\pi^3 \sin \pi x \rightarrow f'''(0) = 0$$

$$5 \cos \pi x = 5 - \frac{5\pi^2 x^2}{2!} + \frac{5\pi^4 x^4}{4!} - \frac{5\pi^6 x^6}{6!} + \dots = 5 \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!}$$

Exercise

Find the Maclaurin series for: $\frac{x^2}{x+1}$

Solution

$$f(x) = \frac{x^2}{x+1} \rightarrow f(0) = 0$$

$$f'(x) = \frac{2x(x+1) - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2} \rightarrow f'(0) = 0$$

$$f''(x) = \frac{(2x+2)(x+1)^2 - 2(x+1)(x^2 + 2x)}{(x+1)^4}$$

$$= \frac{2x^2 + 4x + 2 - 2x^2 - 4x}{(x+1)^3}$$

$$= \frac{2}{(x+1)^3} \rightarrow f''(0) = 2$$

$$f'''(x) = \frac{-6}{(x+1)^4} \rightarrow f'''(0) = -6$$

$$\frac{x^2}{x+1} = \frac{2}{2!}x^2 - \frac{6x^3}{3!} + \frac{24x^4}{4!} - \dots = x^2 - x^3 + x^4 - \dots = \sum_{n=2}^{\infty} (-1)^n x^n$$

Exercise

Find the Maclaurin series for: e^{3x+1}

Solution

$$e^{3x+1} = e \cdot e^{3x}$$

$$= e \left(\sum_{n=0}^{\infty} \frac{(3x)^n}{n!} \right)$$

$$= \sum_{n=0}^{\infty} \frac{e 3^n x^n}{n!} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\cos(2x^3)$

Solution

$$\begin{aligned}\cos(2x^3) &= 1 - \frac{(2x^3)^2}{2!} + \frac{(2x^3)^4}{4!} - \frac{(2x^3)^6}{6!} + \dots \\ &= 1 - \frac{2^2 x^3}{2!} + \frac{2^4 x^{12}}{4!} - \frac{2^6 x^{18}}{6!} + \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} x^{6n} \quad \left| \text{ (for all } x \text{)} \right.\end{aligned}$$

Exercise

Find the Maclaurin series for: $\cos(2x - \pi)$

Solution

$$\begin{aligned}\cos(2x - \pi) &= \cos(2x)\cos\pi + \sin(2x)\sin\pi \\ &= -\cos(2x) \\ &= -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (2x)^{2n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 4^n}{(2n)!} x^{2n} \quad \left| \text{ (for all } x \text{)} \right.\end{aligned}$$

Exercise

Find the Maclaurin series for: $x^2 \sin\left(\frac{x}{3}\right)$

Solution

$$\begin{aligned}x^2 \sin\left(\frac{x}{3}\right) &= x^2 \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{x}{3}\right)^{2n+1}}{(2n+1)!} \\ &= x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{3^{2n+1} (2n+1)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{3^{2n+1} (2n+1)!} \quad \left| \text{ (for all } x \text{)} \right.\end{aligned}$$

Exercise

Find the Maclaurin series for: $\cos^2\left(\frac{x}{2}\right)$

Solution

$$\begin{aligned}\cos^2\left(\frac{x}{2}\right) &= \frac{1}{2}(1 + \cos x) \\ &= \frac{1}{2}\left(1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\right) \\ &= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \\ &= \underline{\underline{1 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}}} \quad (\text{for all } x)\end{aligned}$$

Exercise

Find the Maclaurin series for: $\sin x \cos x$

Solution

$$\begin{aligned}\sin x \cos x &= \frac{1}{2} \sin(2x) \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x)^{2n+1} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 2^{2n+1} x^{2n+1} \\ &= \underline{\underline{\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{(2n+1)!} x^{2n+1}}} \quad (\text{for all } x)\end{aligned}$$

Exercise

Find the Maclaurin series for: $\tan^{-1}(5x^2)$

Solution

$$\begin{aligned}\tan^{-1}(5x^2) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (5x^2)^{2n+1} \\ &= \underline{\underline{\sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1}}{2n+1} x^{4n+2}}} \quad \left(\text{for } -\frac{1}{\sqrt{5}} \leq x \leq \frac{1}{\sqrt{5}} \right)\end{aligned}$$

$5x^2 \leq 1 \rightarrow x^2 \leq \frac{1}{5} \Rightarrow -\frac{1}{\sqrt{5}} \leq x \leq \frac{1}{\sqrt{5}}$

Exercise

Find the Maclaurin series for: $\ln(2+x^2)$

Solution

$$\begin{aligned}
\ln(2+x^2) &= \ln 2 \left(1 + \frac{x^2}{2}\right) \\
&= \ln 2 + \ln \left(1 + \frac{x^2}{2}\right) \\
&= \ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{x^2}{2}\right)^n & \frac{x^2}{2} \leq 1 \rightarrow x^2 \leq 2 \Rightarrow -\sqrt{2} \leq x \leq \sqrt{2} \\
&= \ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} \frac{x^{2n}}{2^n} \quad \left(\text{for } -\sqrt{2} \leq x \leq \sqrt{2} \right)
\end{aligned}$$

Exercise

Find the Maclaurin series for: $\frac{1+x^3}{1+x^2}$

Solution

$$\begin{aligned}
\frac{1}{1+x} &= \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \dots \\
\frac{1}{1+x^2} &= 1 - x^2 + x^4 - x^6 + \dots \\
\frac{1+x^3}{1+x^2} &= (1+x^3)(1-x^2+x^4-x^6+\dots) \\
&= 1 - x^2 + x^4 - x^6 + \dots + x^3 - x^5 + x^7 - x^9 + \dots \\
&= 1 - x^2 + x^3 + x^4 - x^5 - x^6 + x^7 + x^8 - x^9 - \dots \\
&= 1 - x^2 + \sum_{n=2}^{\infty} (-1)^n (x^{2n-1} + x^{2n}) \quad \left(\text{for } |x| < 1 \right)
\end{aligned}$$

Exercise

Find the Maclaurin series for: $\ln \frac{1+x}{1-x}$

Solution

$$\begin{aligned}
\ln \frac{1+x}{1-x} &= \ln(1+x) - \ln(1-x) & = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \right) = 2x + 2\frac{x^3}{3} + \dots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \left((-1)^n + 1 \right) \frac{x^{n+1}}{n+1} \\
&= 2 \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1} \quad (-1 < x < 1)
\end{aligned}$$

Exercise

Find the Maclaurin series for: $\frac{e^{2x^2} - 1}{x^2}$

Solution

$$\begin{aligned}
\frac{e^{2x^2} - 1}{x^2} &= \frac{1}{x^2} \left(e^{2x^2} - 1 \right) \\
&= \frac{1}{x^2} \left(1 + 2x^2 + \frac{(2x^2)^2}{2!} + \frac{(2x^2)^3}{3!} + \dots - 1 \right) \\
&= \frac{1}{x^2} \left(2x^2 + \frac{2^2 x^4}{2!} + \frac{2^3 x^6}{3!} + \dots \right) \\
&= 2 + \frac{2^2 x^2}{2!} + \frac{2^3 x^4}{3!} + \frac{2^4 x^6}{4!} + \dots \\
&= \sum_{n=0}^{\infty} \frac{2^{n+1}}{(n+1)!} x^{2n} \quad (\text{for all } x \neq 0)
\end{aligned}$$

Exercise

Find the Maclaurin series for: $\cosh x - \cos x$

Solution

$$\begin{aligned}
\cosh x - \cos x &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \\
&= \sum_{n=0}^{\infty} \left(1 - (-1)^n \right) \frac{x^{2n}}{(2n)!} \\
&= 2 \sum_{n=0}^{\infty} \frac{x^{4n+2}}{(4n+2)!} \quad (\text{for all } x)
\end{aligned}$$

Exercise

Find the Maclaurin series for: $\sinh x - \sin x$

Solution

$$\begin{aligned}\sinh x - \sin x &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \\&= \sum_{n=0}^{\infty} \left(1 - (-1)^n\right) \frac{x^{2n+1}}{(2n+1)!} \\&= 2 \sum_{n=0}^{\infty} \frac{x^{4n+3}}{(4n+3)!} \quad \left(\text{for all } x \right)\end{aligned}$$

Exercise

Finding Taylor and Maclaurin Series generated by f at $x = a$: $f(x) = x^3 - 2x + 4$, $a = 2$

Solution

$$f(x) = x^3 - 2x + 4 \rightarrow f(2) = 8$$

$$f'(x) = 3x^2 - 2 \rightarrow f'(2) = 10$$

$$f''(x) = 6x \rightarrow f''(2) = 12$$

$$f'''(x) = 6 \rightarrow f'''(2) = 6$$

$$f^{(n)}(x) = 0 \rightarrow f^{(n)}(2) = 0 \quad (n > 3)$$

$$P_n(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \dots$$

$$\underline{x^3 - 2x + 4 = 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3}$$

Exercise

Finding Taylor and Maclaurin Series generated by f at $x = a$: $f(x) = 2x^3 + x^2 + 3x - 8$, $a = 1$

Solution

$$f(x) = 2x^3 + x^2 + 3x - 8 \rightarrow f(1) = -2$$

$$f'(x) = 6x^2 + 2x + 3 \rightarrow f'(1) = 11$$

$$f''(x) = 12x + 2 \rightarrow f''(1) = 14$$

$$f'''(x) = 12 \rightarrow f'''(1) = 12$$

$$f^{(n)}(x)=0 \rightarrow f^{(n)}(1)=0 \quad (n \geq 4)$$

$$P_n(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \dots$$

$$\underline{2x^3 + x^2 + 3x - 8 = -2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3}$$

Exercise

Finding Taylor and Maclaurin Series generated by f at $x = a$:

$$f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2, \quad a = -1$$

Solution

$$f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2 \rightarrow f(-1) = -7$$

$$f'(x) = 15x^4 - 4x^3 + 6x^2 + 2x \rightarrow f'(-1) = 23$$

$$f''(x) = 60x^3 - 12x^2 + 12x + 2 \rightarrow f''(-1) = -82$$

$$f'''(x) = 180x^2 - 24x + 12 \rightarrow f'''(-1) = 216$$

$$f^{(4)}(x) = 360x - 24 \rightarrow f^{(4)}(-1) = -384$$

$$f^{(5)}(x) = 360 \rightarrow f^{(5)}(-1) = 360$$

$$f^{(n)}(x) = 0 \rightarrow f^{(n)}(-1) = 0 \quad (n \geq 6)$$

$$P_n(x) = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3 + \frac{f^{(4)}(-1)}{4!}(x+1)^4 + \frac{f^{(5)}(-1)}{5!}(x+1)^5$$

$$3x^5 - x^4 + 2x^3 + x^2 - 2 = -7 + 23(x+1) - \frac{82}{2!}(x+1)^2 + \frac{216}{3!}(x+1)^3 - \frac{384}{4!}(x+1)^4 + \frac{360}{5!}(x+1)^5$$

$$\underline{= -7 + 23(x+1) - 41(x+1)^2 + 36(x+1)^3 - 16(x+1)^4 + 3(x+1)^5}$$

Exercise

Finding Taylor and Maclaurin Series generated by f at $x = a$: $f(x) = \cos\left(2x + \frac{\pi}{2}\right), \quad a = \frac{\pi}{4}$

Solution

$$f(x) = \cos\left(2x + \frac{\pi}{2}\right) \rightarrow f\left(\frac{\pi}{4}\right) = -1$$

$$f'(x) = -2\sin\left(2x + \frac{\pi}{2}\right) \rightarrow f'\left(\frac{\pi}{4}\right) = 0$$

$$f''(x) = -4\cos\left(2x + \frac{\pi}{2}\right) \rightarrow f''\left(\frac{\pi}{4}\right) = 4$$

$$f'''(x) = 8\sin\left(2x + \frac{\pi}{2}\right) \rightarrow f'''\left(\frac{\pi}{4}\right) = 0$$

$$f^{(4)}(x) = 16 \cos\left(2x + \frac{\pi}{2}\right) \rightarrow f^{(4)}\left(\frac{\pi}{4}\right) = -16$$

$$f^{(5)}(x) = -32 \sin\left(2x + \frac{\pi}{2}\right) \rightarrow f^{(5)}\left(\frac{\pi}{4}\right) = 0$$

$$\rightarrow f^{(2n)}\left(\frac{\pi}{4}\right) = (-1)^n 2^{2n}$$

$$\begin{aligned} \cos\left(2x + \frac{\pi}{2}\right) &= -1 + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{16}{4!}\left(x - \frac{\pi}{4}\right)^4 + \dots \\ &= -1 + 2\left(x - \frac{\pi}{4}\right)^2 - \frac{2}{3}\left(x - \frac{\pi}{4}\right)^4 + \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} \left(x - \frac{\pi}{4}\right)^{2n} \end{aligned}$$

Exercise

Find the Taylor series of the functions, where is the series representation valid? $f(x) = e^{-2x}$ about -1

Solution

$$\text{Let } t = x + 1 \Rightarrow x = t - 1$$

$$\begin{aligned} f(x) &= e^{-2x} = e^{-2x-2+2} = e^{-2(x+1)+2} \\ &= e^2 \sum_{n=0}^{\infty} \frac{(-2(x+1))^n}{n!} \\ &= e^2 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} (x+1)^n \end{aligned} \quad (\text{for all } x)$$

Exercise

Find the Taylor series of the functions, where is the series representation valid? $f(x) = \sin x$ about $\frac{\pi}{2}$

Solution

$$\text{Let } y = x - \frac{\pi}{2} \Rightarrow x = y + \frac{\pi}{2}$$

$$\begin{aligned} \sin x &= \sin\left(y + \frac{\pi}{2}\right) = \cos y \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} y^{2n} \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(x - \frac{\pi}{2}\right)^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(x - \frac{\pi}{2}\right)^{2n} \quad \left(\text{for all } x \right)
\end{aligned}$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = \ln x \quad \text{in powers of } x - 3$$

Solution

$$\text{Let } y = x - 3 \Rightarrow x = y + 3$$

$$\ln x = \ln(y + 3) = \ln 3 \left(1 + \frac{y}{3}\right)$$

$$= \ln 3 + \ln \left(1 + \frac{y}{3}\right)$$

$$= \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\frac{y}{3}\right)^{n+1}$$

$$= \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \frac{(x-3)^{n+1}}{3^{n+1}}$$

$$= \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)3^{n+1}} (x-3)^{n+1} \quad (0 < x \leq 6)$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = \ln(2+x) \quad \text{in powers of } x - 2$$

Solution

$$\ln(2+x) = \ln(2+x-2+2) = \ln(4+x-2)$$

$$= \ln 4 \left(1 + \frac{x-2}{4}\right)$$

$$= \ln 4 + \ln \left(1 + \frac{x-2}{4}\right)$$

$$\begin{aligned}
&= \ln 4 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\frac{x-2}{4} \right)^{n+1} & \frac{|x-2|}{4} < 1 \Rightarrow |x-2| < 4 & -4 < x-2 < 4 & -2 < x < 6 \\
&\boxed{= \ln 4 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)4^{n+1}} (x-2)^{n+1}} & & & (-2 < x \leq 6)
\end{aligned}$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = e^{2x+3} \quad \text{in powers of } x+1$$

Solution

$$\begin{aligned}
e^{2x+3} &= e^{2x+2-2+3} = e^{2(x+1)+1} \\
&= e \cdot e^{2(x+1)} \\
&= e \sum_{n=0}^{\infty} \frac{(2(x+1))^n}{n!} \\
&\boxed{= \sum_{n=0}^{\infty} \frac{e 2^n (x+1)^n}{n!}} & (\text{for all } x)
\end{aligned}$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = \sin x - \cos x \quad \text{about } \frac{\pi}{4}$$

Solution

$$\begin{aligned}
\sin x - \cos x &= \sin \left(x - \frac{\pi}{4} + \frac{\pi}{4} \right) - \cos \left(x - \frac{\pi}{4} + \frac{\pi}{4} \right) \\
&= \cos \left(x - \frac{\pi}{4} \right) \sin \left(\frac{\pi}{4} \right) + \sin \left(x - \frac{\pi}{4} \right) \cos \left(\frac{\pi}{4} \right) - \cos \left(x - \frac{\pi}{4} \right) \cos \left(\frac{\pi}{4} \right) + \sin \left(x - \frac{\pi}{4} \right) \sin \left(\frac{\pi}{4} \right) \\
&= \cos \left(x - \frac{\pi}{4} \right) \frac{\sqrt{2}}{2} + \sin \left(x - \frac{\pi}{4} \right) \frac{\sqrt{2}}{2} - \cos \left(x - \frac{\pi}{4} \right) \frac{\sqrt{2}}{2} + \sin \left(x - \frac{\pi}{4} \right) \frac{\sqrt{2}}{2} \\
&= \sqrt{2} \sin \left(x - \frac{\pi}{4} \right) \\
&\boxed{= \sqrt{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(x - \frac{\pi}{4} \right)^{2n+1}} & (\text{for all } x)
\end{aligned}$$

Exercise

Find the Taylor series of the functions, where is the series representation valid? $f(x) = \cos^2 x$ about $\frac{\pi}{8}$

Solution

$$\begin{aligned}\cos^2(x) &= \frac{1}{2}(1 + \cos 2x) = \frac{1}{2}\left(1 + \cos\left(2x - \frac{\pi}{4} + \frac{\pi}{4}\right)\right) \\&= \frac{1}{2}\left(1 + \cos\left(2x - \frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(2x - \frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)\right) \\&= \frac{1}{2}\left(1 + \frac{\sqrt{2}}{2}\cos\left(2\left(x - \frac{\pi}{8}\right)\right) - \frac{\sqrt{2}}{2}\sin\left(2\left(x - \frac{\pi}{8}\right)\right)\right) \\&= \frac{1}{2} + \frac{\sqrt{2}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(2\left(x - \frac{\pi}{8}\right)\right)^{2n} - \frac{\sqrt{2}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(2\left(x - \frac{\pi}{8}\right)\right)^{2n+1} \\&= \frac{1}{2} + \frac{1}{2\sqrt{2}} \sum_{n=0}^{\infty} (-1)^n \left[\frac{2^{2n}}{(2n)!} \left(x - \frac{\pi}{8}\right)^{2n} - \frac{2^{2n+1}}{(2n+1)!} \left(x - \frac{\pi}{8}\right)^{2n+1} \right] \quad (\text{for all } x)\end{aligned}$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = \frac{x}{1+x} \quad \text{in powers of } x-1$$

Solution

$$\begin{aligned}\frac{x}{1+x} &= \frac{x-1+1}{1+x-1+1} = \frac{(x-1)+1}{(x-1)+2} \\&= 1 - \frac{1}{(x-1)+2} \\&= 1 - \frac{1}{2\left(1 + \frac{x-1}{2}\right)} \\&= 1 - \frac{1}{2} \left(1 - \frac{x-1}{2} + \left(\frac{x-1}{2}\right)^2 - \left(\frac{x-1}{2}\right)^3 + \dots \right) \\&= \frac{1}{2} - \frac{1}{2^2}(x-1) + \frac{1}{2^3}(x-1)^2 - \frac{1}{2^4}(x-1)^3 + \dots \quad |x-1| < 1 \quad -1 < x-1 < 1 \\&= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n+1}} (x-1)^n \quad (0 < x < 2)\end{aligned}$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = xe^x \quad \text{in powers of } x+2$$

Solution

$$\begin{aligned} xe^x &= (x+2-2)e^{x+2-2} = (x+2-2)e^{-2}e^{x+2} \\ &= (x+2)e^{-2}e^{x+2} - 2e^{-2}e^{x+2} \\ &= \frac{1}{e^2}(x+2) \sum_{n=0}^{\infty} \frac{(x+2)^n}{n!} - \frac{2}{e^2} \sum_{n=0}^{\infty} \frac{(x+2)^n}{n!} \\ &= \frac{1}{e^2} \sum_{n=0}^{\infty} \frac{(x+2)^{n+1}}{n!} - \frac{2}{e^2} \sum_{n=0}^{\infty} \frac{(x+2)^n}{n!} \\ &= \frac{1}{e^2} \sum_{n=1}^{\infty} \frac{(x+2)^n}{(n-1)!} - \frac{2}{e^2} \left(1 + \sum_{n=1}^{\infty} \frac{(x+2)^n}{n!} \right) \\ &= -\frac{2}{e^2} + \frac{1}{e^2} \sum_{n=1}^{\infty} \frac{(x+2)^n}{(n-1)!} - \frac{2}{e^2} \sum_{n=1}^{\infty} \frac{(x+2)^n}{n!} \\ &= -\frac{2}{e^2} + \frac{1}{e^2} \sum_{n=1}^{\infty} \left(\frac{1}{(n-1)!} - \frac{2}{n!} \right) (x+2)^n \end{aligned} \quad \left(\text{for all } x \right)$$

Exercise

Find the n th Taylor polynomial centered at c for the function $f(x) = \frac{2}{x}$, $n=3$, $c=1$

Solution

$$f(x) = \frac{2}{x} \rightarrow f(1) = 2$$

$$f'(x) = -\frac{2}{x^2} \rightarrow f'(1) = -2$$

$$f''(x) = \frac{4}{x^3} \rightarrow f''(1) = 4$$

$$f'''(x) = -\frac{12}{x^4} \rightarrow f'''(1) = -12$$

$$P_3(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3$$

$$P_3(x) = 2 - 2(x-1) + 2(x-1)^2 - 2(x-1)^3$$

Exercise

Find the n th Taylor polynomial centered at c for the function $f(x) = \frac{1}{x^2}$, $n = 4$, $c = 2$

Solution

$$f(x) = \frac{1}{x^2} \rightarrow f(2) = \frac{1}{4}$$

$$f'(x) = -\frac{2}{x^3} \rightarrow f'(2) = -\frac{1}{4}$$

$$f''(x) = \frac{6}{x^4} \rightarrow f''(2) = \frac{3}{8}$$

$$f'''(x) = -\frac{24}{x^5} \rightarrow f'''(2) = -\frac{3}{4}$$

$$f^{(4)}(x) = \frac{120}{x^6} \rightarrow f^{(4)}(2) = \frac{15}{8}$$

$$P_4(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4$$

$$P_4(x) = \frac{1}{4} - \frac{1}{4}(x-2) + \frac{3}{16}(x-2)^2 - \frac{1}{8}(x-2)^3 + \frac{5}{64}(x-2)^4$$

Exercise

Find the n th Taylor polynomial centered at c for the function $f(x) = \sqrt{x}$, $n = 3$, $c = 4$

Solution

$$f(x) = x^{1/2} \rightarrow f(4) = 2$$

$$f'(x) = \frac{1}{2}x^{-1/2} \rightarrow f'(4) = \frac{1}{4}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \rightarrow f''(4) = -\frac{1}{4} \frac{1}{(2^2)^{3/2}} = -\frac{1}{32}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} \rightarrow f'''(4) = \frac{3}{256}$$

$$P_3(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3$$

$$P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$$

Exercise

Find the n th Taylor polynomial centered at c for the function $f(x) = \sqrt[3]{x}$, $n = 3$, $c = 8$

Solution

$$f(x) = x^{1/3} \rightarrow f(8) = 2$$

$$f'(x) = \frac{1}{3}x^{-2/3} \rightarrow f'(8) = \frac{1}{12}$$

$$f''(x) = -\frac{2}{9}x^{-5/3} \rightarrow f''(8) = -\frac{1}{144}$$

$$f'''(x) = \frac{10}{27}x^{-8/3} \rightarrow f'''(8) = \frac{5}{3456}$$

$$P_3(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3$$

$$P_3(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20,736}(x-8)^3$$

Exercise

Find the n th Taylor polynomial centered at c for the function $f(x) = \ln x$, $n = 4$, $c = 2$

Solution

$$f(x) = \ln x \rightarrow f(2) = \ln 2$$

$$f'(x) = \frac{1}{x} \rightarrow f'(2) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{x^2} \rightarrow f''(2) = -\frac{1}{4}$$

$$f'''(x) = \frac{2}{x^3} \rightarrow f'''(2) = \frac{1}{4}$$

$$f^{(4)}(x) = -\frac{6}{x^4} \rightarrow f^{(4)}(2) = -\frac{3}{8}$$

$$P_4(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4$$

$$P_4(x) = \ln 2 + \frac{1}{2}\frac{1}{4}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4$$

Exercise

Find the n th Taylor polynomial centered at c for the function $f(x) = x^2 \cos x$, $n = 2$, $c = \pi$

Solution

$$f(x) = x^2 \cos x \rightarrow f(\pi) = -\pi^2$$

$$f'(x) = 2x \cos x - x^2 \sin x \rightarrow f'(\pi) = -2\pi$$

$$f''(x) = 2 \cos x - 4x \sin x - x^2 \cos x \rightarrow f''(\pi) = -2 + \pi^2$$

$$P_2(x) = -\pi^2 - 2\pi(x-\pi) + \frac{\pi^2-2}{2}(x-\pi)^2$$

$$P_2(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2$$

Exercise

Find the sums of the series $1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots$

Solution

$$1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots = 1 + \frac{(x^2)^1}{1!} + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!} + \dots$$

$$\underline{= e^{x^2}}$$

Exercise

Find the sums of the series $1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \frac{x^8}{9!} + \dots$

Solution

$$1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \frac{x^8}{9!} + \dots = \frac{1}{x} \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots \right)$$

$$= \frac{1}{x} \sinh x$$

$$= \begin{cases} \frac{e^x - e^{-x}}{2x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Exercise

Find the sums of the series $x^3 - \frac{x^9}{3 \times 4} + \frac{x^{15}}{5 \times 16} - \frac{x^{21}}{7 \times 64} + \frac{x^{27}}{9 \times 256} - \dots$

Solution

$$x^3 - \frac{x^9}{3 \times 4} + \frac{x^{15}}{5 \times 16} - \frac{x^{21}}{7 \times 64} + \frac{x^{27}}{9 \times 256} - \dots = 2 \left(\frac{x^3}{2} - \frac{1}{3!} \left(\frac{x^3}{2} \right)^3 + \frac{1}{5!} \left(\frac{x^3}{2} \right)^5 - \frac{1}{7!} \left(\frac{x^3}{2} \right)^7 + \dots \right)$$

$$\underline{= 2 \sin \left(\frac{x^3}{2} \right)} \quad \text{for all } x$$

Exercise

The limit $\lim_{n \rightarrow \infty} \frac{n!}{\sqrt{2\pi} n^{n+1/2} e^{-n}} = 1$ that is the relative error in the approximation $n! \approx \sqrt{2\pi} n^{n+1/2} e^{-n}$

Approaches zero as n increases. That is $n!$ grows at a rate comparable to $\sqrt{2\pi} n^{n+1/2} e^{-n}$. This result, known as Stirling's Formula, is often very useful in applied mathematics and statistics. Prove it by carrying out the following steps.

a) Use the identity $\ln(n!) = \sum_{j=1}^n \ln j$ and the increasing nature of \ln to show that if $n \geq 1$,

$$\int_0^n \ln x \, dx < \ln(n!) < \int_1^{n+1} \ln x \, dx$$

And hence that $n \ln n - n < \ln(n!) < (n+1) \ln(n+1) - n$

b) If $c_n = \ln(n!) - \left(n + \frac{1}{2}\right) \ln n + n$, show that

$$\begin{aligned} c_n - c_{n+1} &= \left(n + \frac{1}{2}\right) \ln\left(\frac{n+1}{n}\right) - 1 \\ &= \left(n + \frac{1}{2}\right) \ln\left(\frac{1 + \frac{1}{2n+1}}{1 - \frac{1}{2n+1}}\right) - 1 \end{aligned}$$

c) Use the Maclaurin series for $\ln \frac{1+t}{1-t}$ to show that

$$\begin{aligned} 0 < c_n - c_{n+1} &< \frac{1}{3} \left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^4} + \frac{1}{(2n+1)^6} + \dots \right) \\ &= \frac{1}{12} \left(\frac{1}{n} - \frac{1}{n+1} \right) \end{aligned}$$

and therefore that $\{c_n\}$ is decreasing and $\{c_n - \frac{1}{12n}\}$ is increasing. Hence conclude that

$\lim_{n \rightarrow \infty} c_n = c$ exists, and that

$$\lim_{n \rightarrow \infty} \frac{n!}{n^{n+1/2} e^{-n}} = \lim_{n \rightarrow \infty} e^{c_n} = e^c$$

Solution

$$a) \ln(k-1) < \int_{k-1}^k \ln x \, dx < \ln k, \quad k = 1, 2, 3, \dots$$

$$\begin{aligned} n \ln n - n &= \int_0^n \ln x \, dx < \ln(n!) < \int_1^{n+1} \ln x \, dx \\ &= (n+1) \ln(n+1) - n - 1 < (n+1) \ln(n+1) \end{aligned}$$

b) If $c_n = \ln(n!) - \left(n + \frac{1}{2}\right) \ln n + n$, then

$$\begin{aligned} c_n - c_{n+1} &= \ln(n!) - \left(n + \frac{1}{2}\right) \ln n + n - \left[\ln((n+1)!) - \left(n + 1 + \frac{1}{2}\right) \ln(n+1) + n + 1 \right] \\ &= \ln(n!) - \left(n + \frac{1}{2}\right) \ln n - \ln((n+1)!) + \left(n + \frac{1}{2} + 1\right) \ln(n+1) - 1 \\ &= \ln(n!) - \ln((n+1)!) - \left(n + \frac{1}{2}\right) \ln n^{n+1/2} + \left(n + \frac{1}{2}\right) \ln(n+1) + \ln(n+1) - 1 \end{aligned}$$

$$= \ln\left(\frac{n!}{(n+1)!}\right) - \left(n + \frac{1}{2}\right) \ln \frac{n}{n+1} + \ln(n+1) - 1$$

$$= \ln\left(\frac{1}{n+1}\right) - \left(n + \frac{1}{2}\right) \ln \frac{n}{n+1} + \ln(n+1) - 1$$

$$= -\ln(n+1) + \left(n + \frac{1}{2}\right) \ln\left(\frac{n}{n+1}\right)^{-1} + \ln(n+1) - 1$$

$$= \left(n + \frac{1}{2}\right) \ln\left(\frac{n+1}{n}\right) - 1$$

$$= \left(n + \frac{1}{2}\right) \ln\left(\frac{2n+2}{2n}\right) - 1$$

$$= \left(n + \frac{1}{2}\right) \ln\left(\frac{\frac{2n+2}{2n+1}}{\frac{2n}{2n+1}}\right) - 1$$

$$= \left(n + \frac{1}{2}\right) \ln\left(\frac{\frac{2n+1+1}{2n+1}}{\frac{2n+1-1}{2n+1}}\right) - 1$$

$$= \left(n + \frac{1}{2}\right) \ln\left(\frac{1 + \frac{1}{2n+1}}{1 - \frac{1}{2n+1}}\right) - 1$$

$$c) \quad \ln \frac{1+t}{1-t} = \ln(1+t) - \ln(1-t)$$

$$= t - \frac{t^2}{2} + \frac{t^3}{3} - \dots + t + \frac{t^2}{2} + \frac{t^3}{3} - \dots$$

$$= 2\left(t + \frac{t^3}{3} + \frac{t^5}{5} + \dots\right) \quad \text{for } -1 < t < 1$$

$$0 < c_n - c_{n+1} = \left(\frac{2n+1}{2}\right) \ln\left(\frac{1 + \frac{1}{2n+1}}{1 - \frac{1}{2n+1}}\right) - 1$$

$$\ln \frac{1+t}{1-t} = 2\left(t + \frac{t^3}{3} + \frac{t^5}{5} + \dots\right)$$

$$= \frac{1}{2}(2n+1)(2)\left(\frac{1}{2n+1} + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \dots\right) - 1$$

$$= 1 + \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + \dots - 1$$

$$= \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + \frac{1}{7(2n+1)^6} + \dots$$

$$< \frac{1}{3}\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^4} + \dots\right)$$

$$\text{Geometric series } S_n = a \frac{1}{1-r}$$

$$\begin{aligned}
&= \frac{1}{3} \cdot \frac{1}{(2n+1)^2} \cdot \frac{1}{1 - \frac{1}{(2n+1)^2}} \\
&= \frac{1}{3} \cdot \frac{1}{(2n+1)^2} \cdot \frac{1}{1 - \frac{1}{(2n+1)^2}} \\
&= \frac{1}{3} \cdot \frac{1}{(2n+1)^2 - 1} \\
&= \frac{1}{3} \cdot \frac{1}{4n^2 + 4n} \\
&= \frac{1}{12} \cdot \frac{1}{n(n+1)} \\
&= \frac{1}{12} \left(\frac{1}{n} - \frac{1}{n+1} \right)
\end{aligned}$$

These inequalities imply that $\{c_n\}$ is decreasing and $\left\{c_n - \frac{1}{12n}\right\}$ is increasing.

Thus $\{c_n\}$ is bounded below by $c_1 - \frac{1}{12} = 1 - \frac{1}{12} = \frac{11}{12}$ $\left(c_1 = \ln(1!) - \left(1 + \frac{1}{2}\right) \ln 1 + 1\right)$

So $\lim_{n \rightarrow \infty} c_n = c$ exists.

Since $e^{\frac{c}{n}} = \frac{n!}{n^{(n+1/2)} e^{-n}}$, we have

$\lim_{n \rightarrow \infty} \frac{n!}{n^{(n+1/2)} e^{-n}} = \lim_{n \rightarrow \infty} e^{\frac{c}{n}} = e^c$ exists.