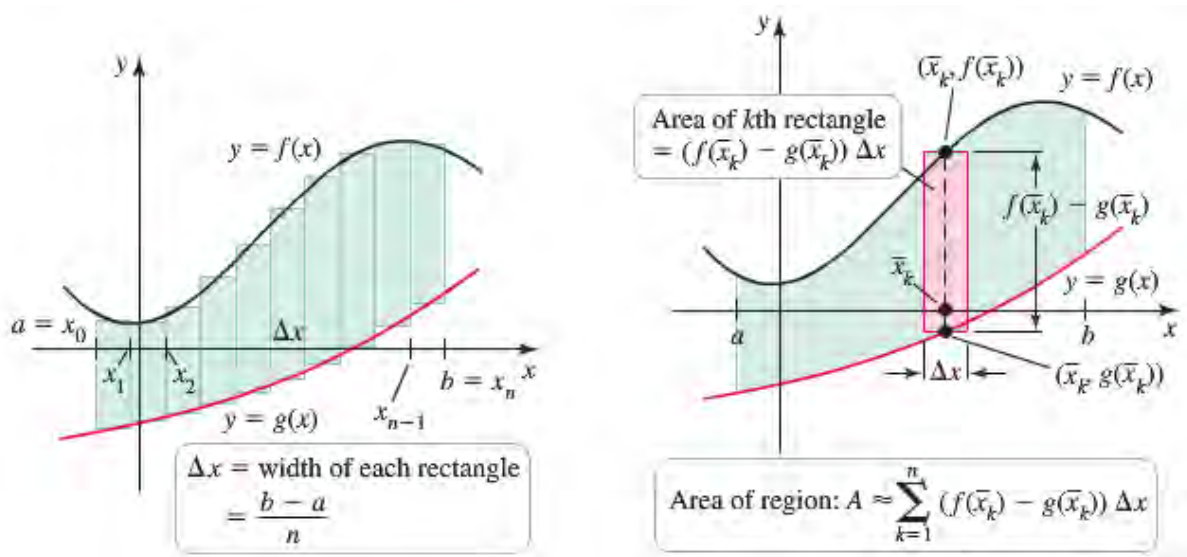


Section 1.2 – Region between Curves

Areas between Curves



Definition

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves** $y = f(x)$ and $y = g(x)$ **from a to b** is:

$$A = \int_a^b [f(x) - g(x)] dx$$

Example

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution

The limits of integrations are found by letting:

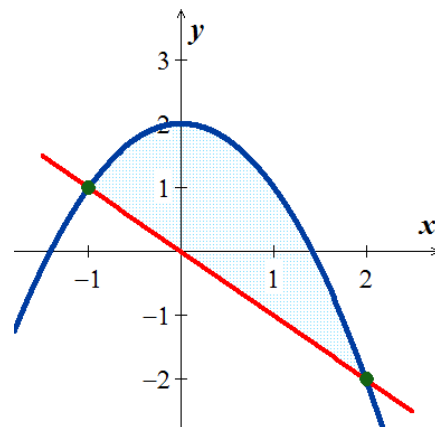
$$2 - x^2 = -x$$

$$x^2 - x - 2 = 0 \rightarrow \underline{x = -1, 2}$$

$$A = \int_{-1}^2 [f(x) - g(x)] dx$$

$$= \int_{-1}^2 (2 - x^2 - (-x)) dx$$

$$= \int_{-1}^2 (2 - x^2 + x) dx$$



$$\begin{aligned}
&= 2x - \frac{x^3}{3} + \frac{x^2}{2} \Big|_{-1}^2 \\
&= 4 - \frac{8}{3} + 2 - \left(-2 + \frac{1}{3} + \frac{1}{2}\right) \\
&= 8 - \frac{8}{3} - \frac{1}{3} - \frac{1}{2} \\
&= 5 - \frac{1}{2} \\
&= \frac{9}{2} \text{ unit}^2
\end{aligned}$$

Example

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below the x -axis and the line $y = x - 2$

Solution

$$(y = \sqrt{x}) \cap (y = 0) \rightarrow (0, 0)$$

$$(y = \sqrt{x}) \cap (y = x - 2) \rightarrow \sqrt{x} = x - 2$$

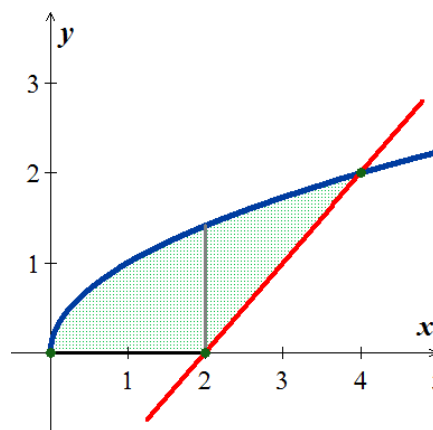
$$(\sqrt{x})^2 = (x - 2)^2$$

$$x = x^2 - 4x + 4$$

$$x^2 - 5x + 4 = 0$$

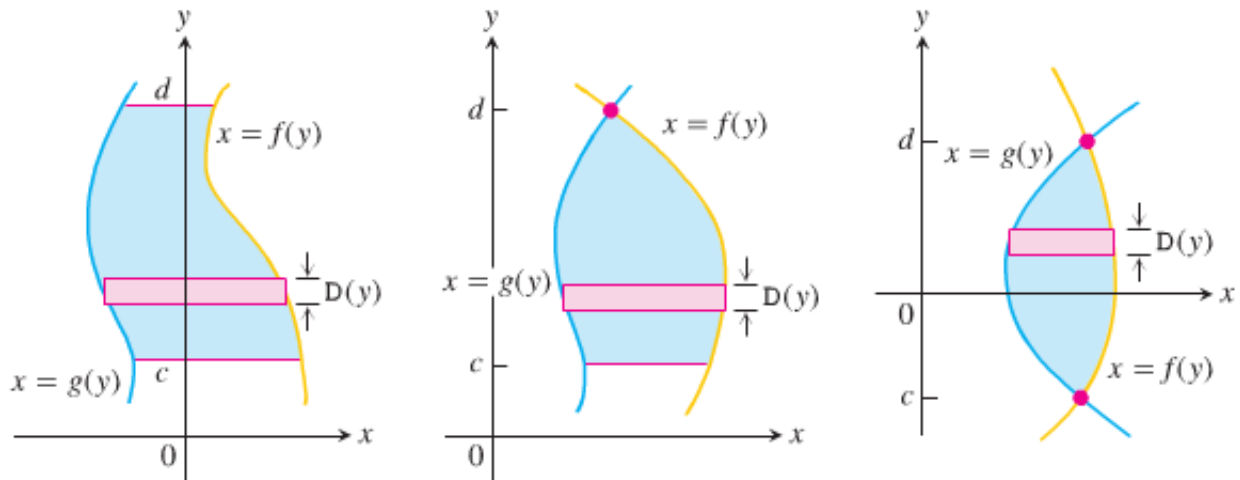
$$\rightarrow x = \text{X}, 4$$

$$(y = 0) \cap (y = x - 2) \rightarrow x = 2$$



$$\begin{aligned}
\text{Total Area} &= \int_0^2 (\sqrt{x} - 0) dx + \int_2^4 (\sqrt{x} - (-x + 2)) dx \\
&= \frac{2}{3} x^{3/2} \Big|_0^2 + \left(\frac{2}{3} x^{3/2} - \frac{x^2}{2} + 2x \right) \Big|_2^4 \\
&= \frac{2}{3} (2^{3/2}) - 0 + \left(\frac{2}{3} 4^{3/2} - \frac{4^2}{2} + 2(4) \right) - \left(\frac{2}{3} 2^{3/2} - \frac{2^2}{2} + 2(2) \right) \\
&= \frac{2}{3} (2^{3/2}) + \frac{2}{3} 4^{3/2} - \frac{16}{2} + 8 - \frac{2}{3} 2^{3/2} + \frac{4}{2} - 4 \\
&= \frac{2}{3} (8) - 2 \\
&= \frac{10}{3} \text{ unit}^2
\end{aligned}$$

Integration with Respect to y



$$A = \int_c^d (f(y) - g(y)) dy \quad (\text{From right hand to left hand})$$

Example

Find the area of the region by integrating with respect to y , in the first quadrant that is bounded above by $y = \sqrt{x}$ and below the x -axis and the line $y = x - 2$.

Solution

$$y = \sqrt{x} \rightarrow x = y^2$$

$$y = x - 2 \rightarrow x = y + 2$$

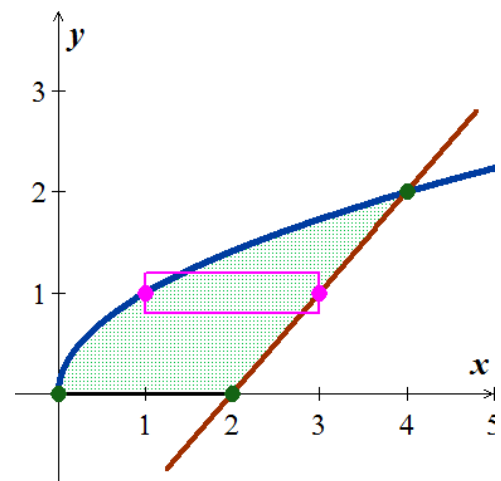
$$(x = y^2) \cap (y = 0) \rightarrow (0, 0)$$

$$(x = y^2) \cap (x = y + 2) \rightarrow y^2 = y + 2$$

$$y^2 - y - 2 = 0 \rightarrow y = -1, 2$$

$$(y = 0) \cap (x = y + 2) \rightarrow y = 0$$

$$\begin{aligned} A &= \int_0^2 (y + 2 - y^2) dy \\ &= \left. \frac{y^2}{2} + 2y - \frac{y^3}{3} \right|_0^2 \\ &= \frac{2^2}{2} + 2(2) - \frac{2^3}{3} - 0 \\ &= \frac{10}{3} \text{ unit}^2 \end{aligned}$$



Exercises Section 1.2 – Region between Curves

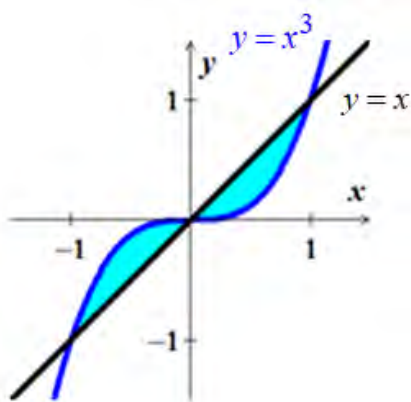
Find the area of the region bounded by the graphs of

1. $y = 2x - x^2$ and $y = -3$
2. $y = 7 - 2x^2$ and $y = x^2 + 4$
3. $y = x^4 - 4x^2 + 4$ and $y = x^2$
4. $x = 2y^2$, $x = 0$, and $y = 3$
5. $x = y^3 - y^2$ and $x = 2y$
6. $4x^2 + y = 4$ and $x^4 - y = 1$
7. $y = \sin \frac{\pi x}{2}$ and $y = x$
8. $y = 3 - x^2$ and $y = 2x$
9. $y = x^2 - x - 2$ and x -axis
10. $y = \sqrt{x}$, $y = x\sqrt{x}$
11. $y = x^{1/2}$ and $y = x^3$
12. $x + 4y^2 = 4$, $x + y^4 = 1$, $x \geq 0$
13. $y = 2\sin x$, $y = \sin 2x$, $0 \leq x \leq \pi$
14. $y = x^2 + 1$ and $y = x$ for $0 \leq x \leq 2$
15. $y = x^2 - 2x$ and $y = x$ on $[0, 4]$
16. $x = 1$, $x = 2$, $y = x^3 + 2$, $y = 0$
17. $y = x^2 - 18$, $y = x - 6$
18. $y = -x^2 + 3x + 1$, $y = -x + 1$
19. $y = x$, $y = 2 - x$, $y = 0$
20. $y = \frac{4}{x^2}$, $y = 0$, $x = 1$, $x = 4$
21. $f(y) = y^2$, $g(y) = y + 2$
22. $f(x) = 2^x$, $g(x) = \frac{3}{2}x + 1$
23. $x = \sqrt[3]{y}$ and $x = \sqrt[5]{y}$
24. $f(x) = x^3 + 2x^2 - 3x$, $g(x) = x^2 + 3x$
25. $y = \sec^2 x$, $y = \tan^2 x$, $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$
26. $f(x) = -x^2 + 1$, $g(x) = 2x + 4$, $x = -1$, $x = 2$
27. $f(x) = \sqrt{x} + 3$, $g(x) = \frac{1}{2}x + 3$
28. $f(x) = \sqrt[3]{x-1}$, $g(x) = x - 1$
29. $f(y) = y(2 - y)$, $g(y) = -y$
30. $f(y) = \frac{y}{\sqrt{16 - y^2}}$, $g(y) = 0$, $y = 3$
31. $f(y) = y^2 + 1$, $g(y) = 0$, $y = -1$, $y = 2$
32. $f(x) = \frac{10}{x}$, $x = 0$, $y = 2$, $y = 10$
33. $g(x) = \frac{4}{2 - x}$, $y = 4$, $x = 0$
34. $f(x) = \cos x$, $g(x) = 2 - \cos x$, $0 \leq x \leq 2\pi$
35. $f(x) = \sin x$, $g(x) = \cos 2x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{6}$
36. $f(x) = 2\sin x$, $g(x) = \tan x$, $-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$
37. $f(x) = \sec \frac{\pi x}{4} \tan \frac{\pi x}{4}$, $g(x) = (\sqrt{2} - 4)x + 4$, $x = 0$
38. $f(x) = xe^{-x^2}$, $y = 0$, $0 \leq x \leq 1$
39. $y = \sin x$ and $y = x$ $0 \leq x \leq 2\pi$
40. $y = x^2$, $y = 2x^2 - 4x$ and $y = 0$
41. $y = 8\cos x$, $y = \sec^2 x$, $-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$
42. $y^2 = 4x + 4$, $y = 4x - 16$
43. $x = 2y^2$, $x = 0$, $y = 3$
44. $x = y^3$ and $y = x$

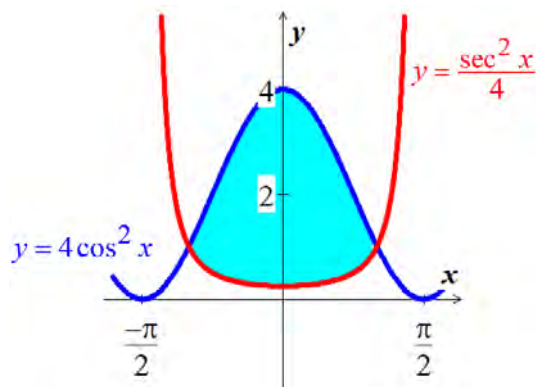
45. Find the area of the region in the first quadrant bounded by $y = 4x$ and $y = x\sqrt{25 - x^2}$
46. Find the area of the region in the first quadrant bounded by the curve $\sqrt{x} + \sqrt{y} = 1$
47. Find the area of the region in the first quadrant bounded by $y = \frac{x}{6}$ and $y = 1 - \left| \frac{x}{2} - 1 \right|$
48. Find the area of the region in the first quadrant bounded by $y = x^p$ and $y = \sqrt[p]{x}$ where $p = 100$ and $p = 1000$
49. Consider the functions $y = \frac{x^2}{a}$ and $y = \sqrt{\frac{x}{a}}$, where $a > 0$. Find $A(a)$, the area of the region between the curves.
50. Find the area between the curves $y = \ln x$ and $y = \ln 2x$ from $x = 1$ to $x = 5$.
51. Find the total area of the region enclosed by the curve $x = y^{2/3}$ and lines $x = y$ and $y = -1$.
52. Find the area of the “triangular region in the first quadrant bounded on the left by the y -axis and on the right by the curves $\sin x$ and $\cos x$.
53. Find the area of the “triangular region in the first quadrant bounded above by the curve $y = e^{2x}$, below by the curve $y = e^x$, and on the right by the line $x = \ln 3$.
54. Find the area of the triangular region bounded on the left by $x + y = 2$, on the right by $y = x^2$, and above by $y = 2$
55. Find the extreme values of $f(x) = x^3 - 3x^2$ and find the area of the region enclosed by the graph of f and the x -axis.

(56 – 59) Determine the area of the shaded region in the following

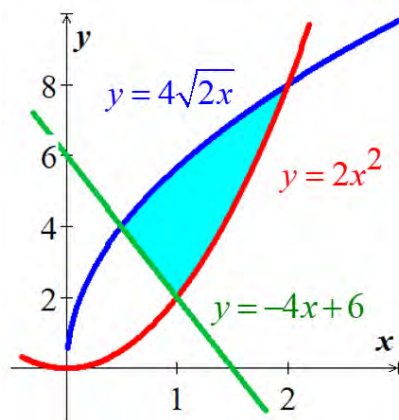
56.



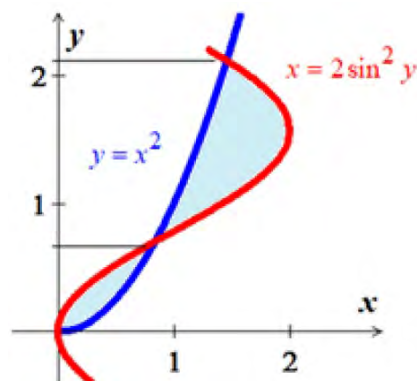
57.



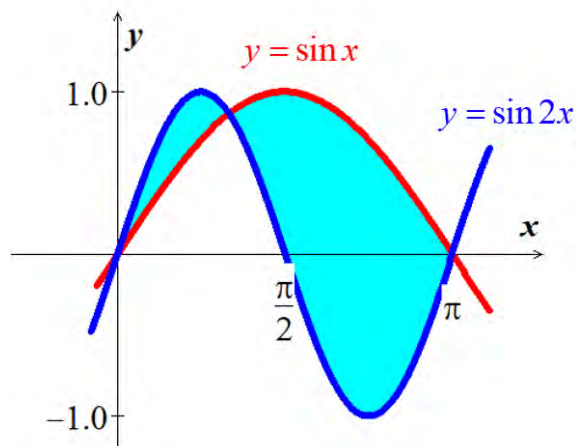
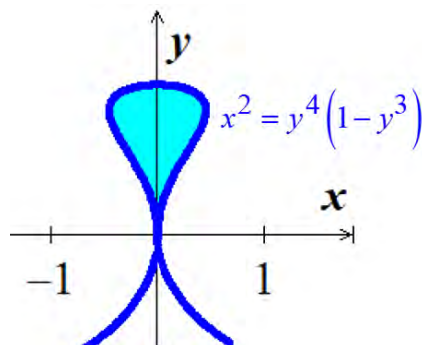
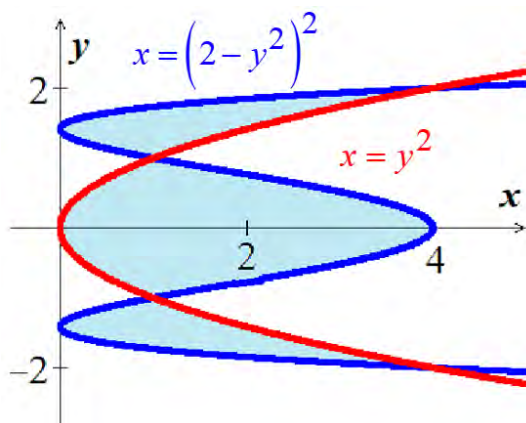
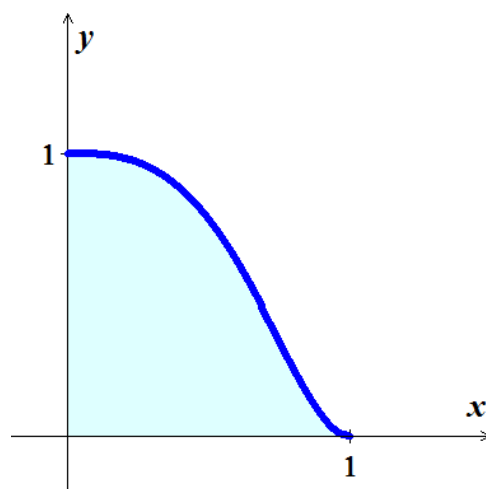
58.



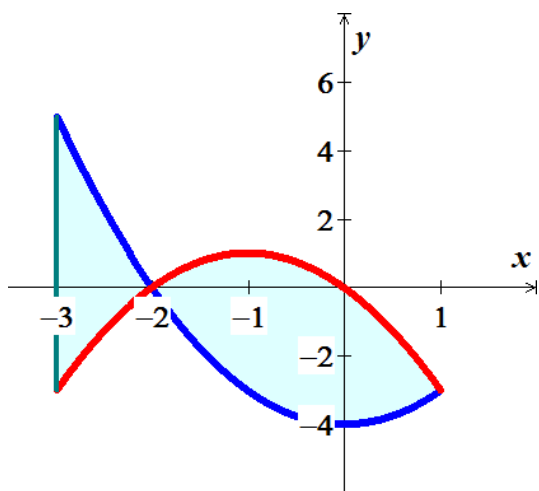
59.



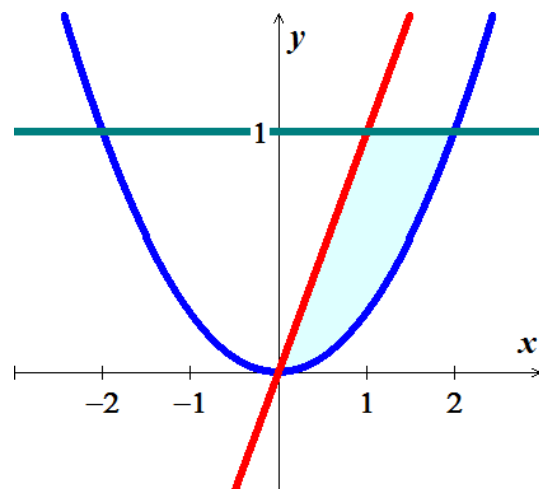
(60 – 71) Determine the area of the shaded regions

60. $y = \sin x$ and $y = \sin 2x$, for $0 \leq x \leq \pi$ 61. Bounded by $x^2 = y^4(1 - y^3)$ 62. bounded by $x = y^2$ and $x = (2 - y^2)^2$ 63. $x^3 + \sqrt{y} = 1$, $x = 0$, $y = 0$, $0 \leq x \leq 1$ 

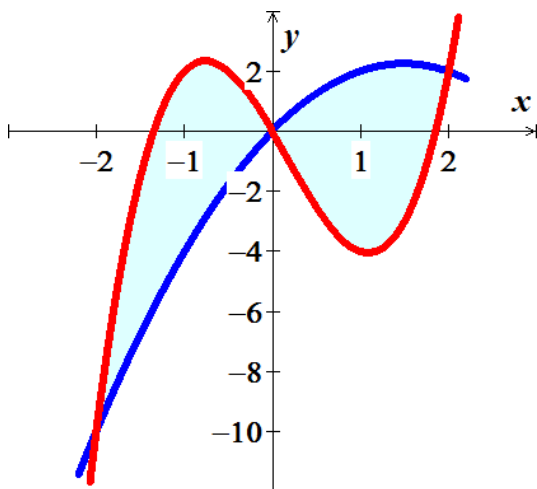
64. $y = x^2 - 4$, $y = -x^2 - 2x$, $-3 \leq x \leq 1$



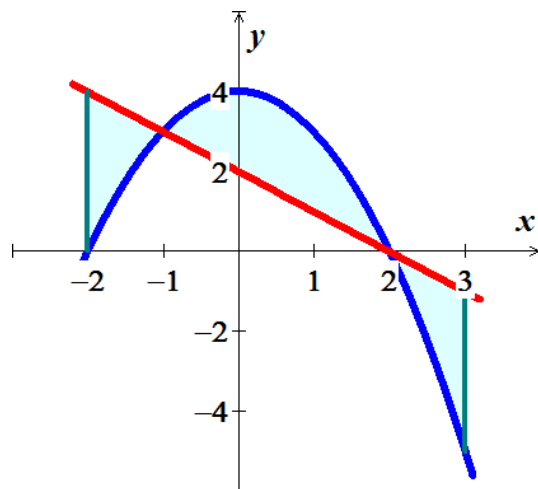
65. $y = \frac{1}{4}x^2$, $y = x$, $y = 1$



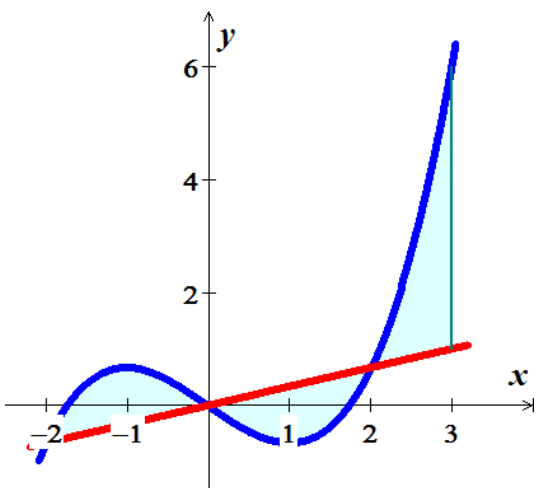
66. $y = -x^2 + 3x$, $y = 2x^3 - x^2 - 5x$, $-2 \leq x \leq 2$



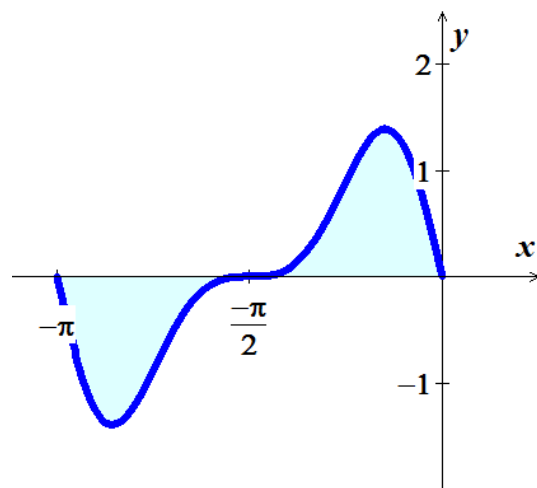
67. $y = 4 - x^2$, $y = -x + 2$, $-2 \leq x \leq 3$



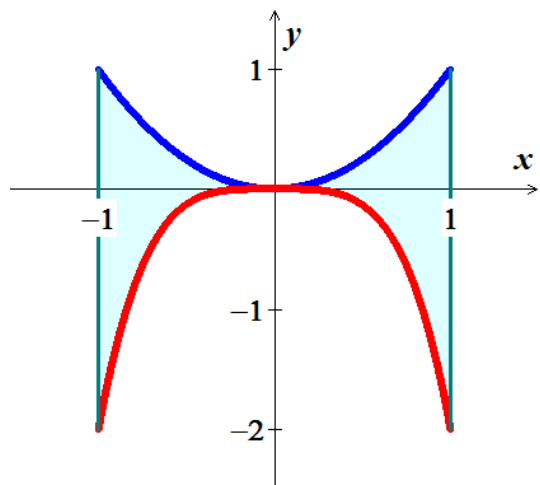
68. $y = \frac{1}{3}x^3 - x$, $y = \frac{1}{3}x$, $-2 \leq x \leq 3$



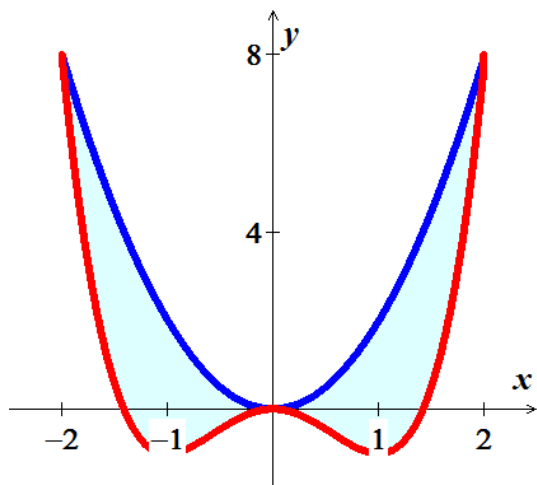
69. $y = \frac{\pi}{2} \cos x \sin(\pi + \pi \sin x)$, $-\pi \leq x \leq 0$



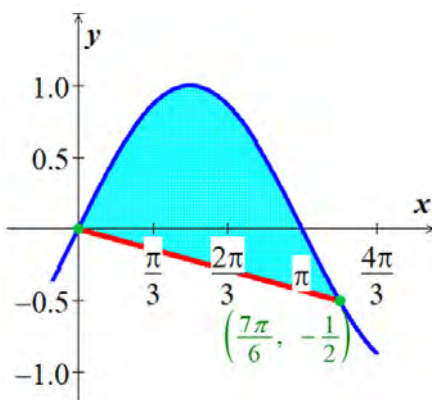
70. $y = x^2$, $y = -2x^4$, $-1 \leq x \leq 1$



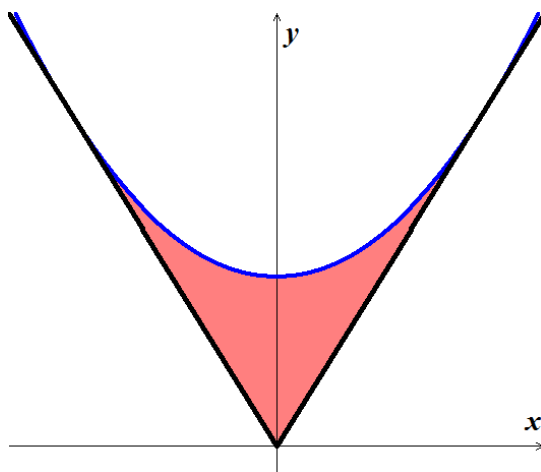
71. $y = 2x^2$, $y = x^4 - 2x^2$, $-2 \leq x \leq 2$



72. Find the area between the graph of $y = \sin x$ and the line segment joining the points $(0, 0)$ and $(\frac{7\pi}{6}, -\frac{1}{2})$.

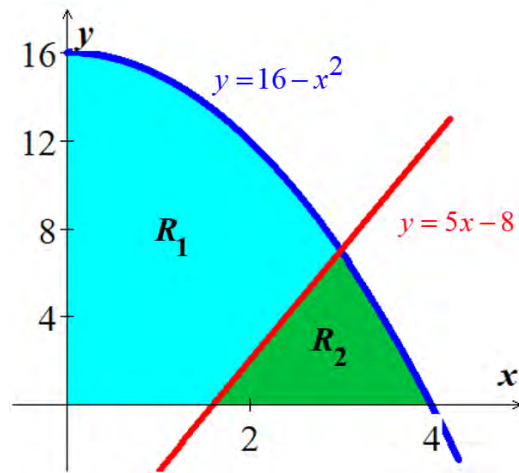


73. The surface of a machine part is the region between the graphs of $y_1 = |x|$ and $y_2 = 0.08x^2 + k$

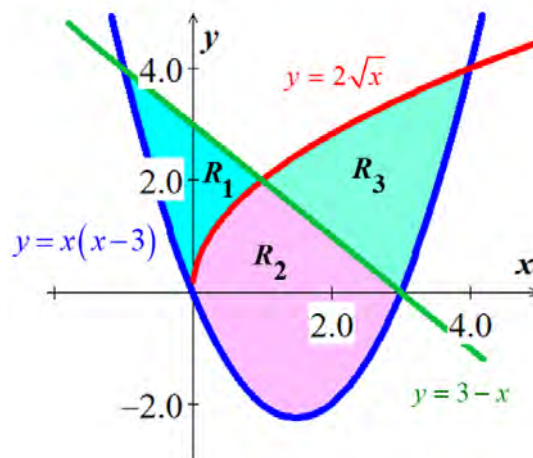


- Find k where the parabola is tangent to the graph of y_1
- Find the area of the surface of the machine part.

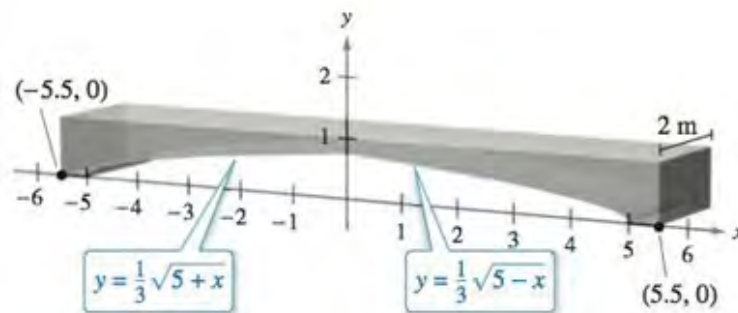
74. Find the area of the regions R_1 and R_2 (separately) shown in the figure, which are formed by the graphs of $y = 16 - x^2$ and $y = 5x - 8$



75. Find the area of the regions R_1 , R_2 and R_3 (separately) shown in the figure, which are formed by the graphs of $y = 2\sqrt{x}$, $y = 3 - x$, and $y = x(x - 3)$



76. Concrete sections for a new building have the dimensions (in meters) and shape shown in figure



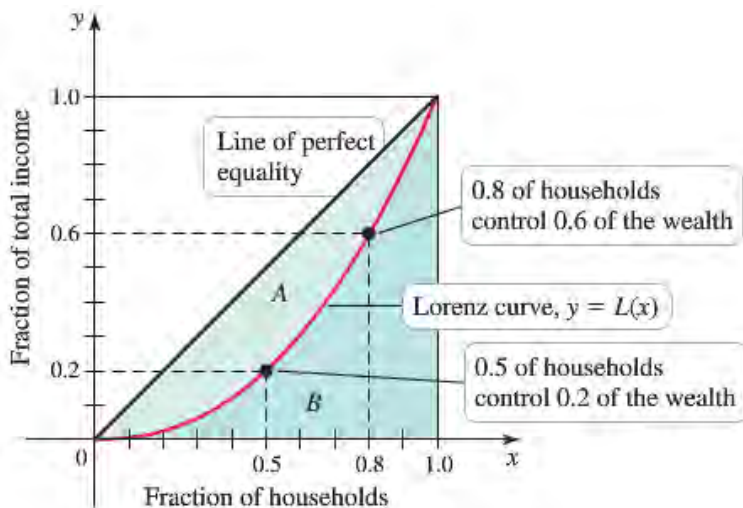
- Find the area of the face of the section superimposed on the rectangular coordinate system.
- Find the volume of concrete in one of the sections by multiplying the area in part (a) by 2 meters.
- One cubic meter of concrete weighs 5,000 pounds. Find the weight of the section.

77. A Lorenz curve is given by $y = L(x)$, where $0 \leq x \leq 1$ represents the lowest fraction of the population of a society in terms of wealth and $0 \leq y \leq 1$ represents the fraction of the total wealth that is owned by that fraction of the society. For example, the Lorenz curve in the figure shows that $L(0.5) = 0.2$, which means that the lowest 0.5 (50%) of the society owns 0.2 (20%) of the wealth.

a) A Lorenz curve $y = L(x)$ is accompanied by the line $y = x$, called the **line of perfect equality**.

Explain why this line is given the name.

b) Explain why a Lorenz curve satisfies the conditions $L(0) = 0$, $L(1) = 1$, and $L'(x) \geq 0$ on $[0, 1]$



c) Graph the Lorenz curves $L(x) = x^p$ corresponding to $p = 1.1, 1.5, 2, 3, 4$. Which value of p corresponds to the **most** equitable distribution of wealth (closest to the line of perfect equality)? Which value of p corresponds to the **least** equitable distribution of wealth? Explain.

d) The information in the Lorenz curve is often summarized in a single measure called the **Gini index**, which is defined as follows. Let A be the area of the region between $y = x$ and $y = L(x)$ and Let B be the area of the region between $y = L(x)$ and the x -axis. Then the Gini index is $G = \frac{A}{A+B}$.

Show that $G = 2A = 1 - 2 \int_0^1 L(x) dx$.

e) Compute the Gini index for the cases $L(x) = x^p$ and $p = 1.1, 1.5, 2, 3, 4$.

f) What is the smallest interval $[a, b]$ on which values of the Gini index lie, for $L(x) = x^p$ with $p \geq 1$? Which endpoints of $[a, b]$ correspond to the least and most equitable distribution of wealth?

g) Consider the Lorenz curve described by $L(x) = \frac{5x^2}{6} + \frac{x}{6}$. Show that it satisfies the conditions $L(0) = 0$, $L(1) = 1$, and $L'(x) \geq 0$ on $[0, 1]$. Find the Gini index for this function.