Chapter 3 – Vectors

Physical quantities are classified into two based on how they are represented. Physical quantities that can be represented completely by a number and a unit are called scalars. Examples are mass (2 kg), time (6h), length (7m), volume ($4cm^3$), and temperature (20°C).
On the other hand physical quantities that require specification of direction in addition to a number and unit are called vectors. Examples are displacement (2m east), velocity (5 m/s west), acceleration (5 m/s^2 north), force (6 N south) and area (4 m^2 east). Symbolically, vectors are represented by a capital letter with an arrow on top.
For example: $\vec{A} = 2m\ east$

Graphical Representation of Vectors

Graphically, a vector is represented by means of an arrow. The direction of the vector is represented by means of the direction of the arrow. The magnitude (numerical value) of the vector is represented by the length of the vector. The length of the arrow is drawn in such a way that it is proportional to the magnitude of the vector. To this end, a scale relating the magnitude of the vector and the length of the direction of a vector is stated by specifying the angle measure with respect to a certain reference line; Ex: 2m 30° west of north. The default reference line is the positive x-axis or east (i.e. horizontal line to the right). That means if no reference line is specified, the reference line is assumes to be the positive x-axis. Angles measure in a counter clockwise direction from this reference line are taken to be positive, while angles measure in a clockwise direction are taken to be negative.

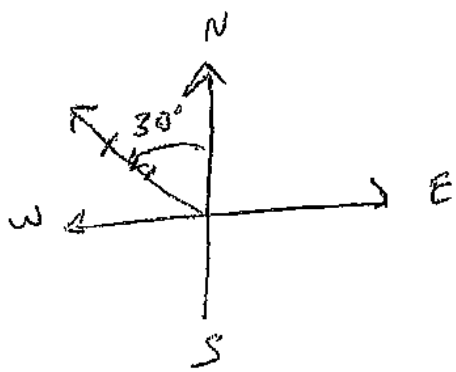
Example:

In each of the following represent the vector graphically by means of a suitable scale.

a) $\vec{A} = 300m\ 30°\ west\ of\ north$

Solution:

Scale: Let 100m be represented by 1cm.



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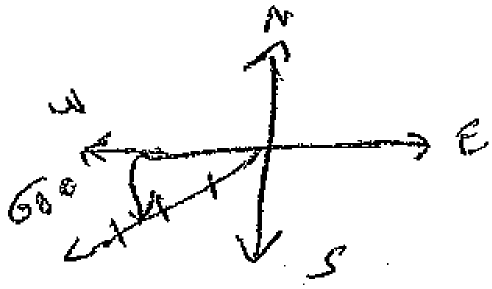
b) $\vec{A} = 40m\ 60°\ south\ of\ west$

Solution:

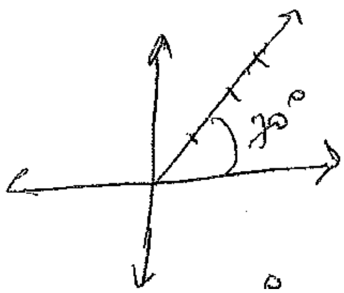
Let 20m be represented by 1cm



c) $\vec{A} = 16m\ at\ 70°$

<u>Solution:</u>

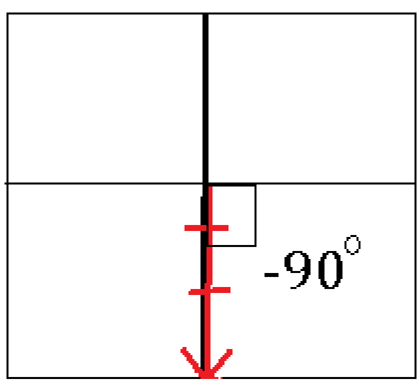
Since no reference line is specified, the reference line is assume to be the default reference line which is the positive x-axis and since it is positive, it is measure in a counter clockwise direction. Scale: 4m to 1cm.



d) $\vec{A} = 12\ m\ at\ -90°$

<u>Solution:</u>

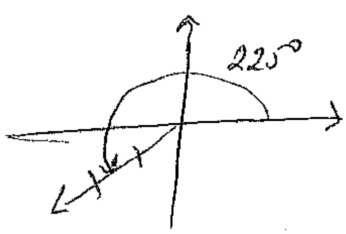
Since the angle is negative, it is measure in a clockwise direction from the positive x-axis. Scale: 3m for 1cm



e) $\vec{A} = 21\ m\ at\ 225°$

<u>Solution:</u>

Scale: 7m to 1cm



Negative of a vector: is defined to be the vector with the same magnitude but opposite direction.

$- - - \rightarrow \quad \vec{A}$

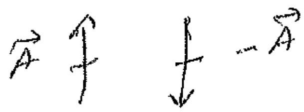
$< - - - \quad -\vec{A}$

Multiplying a vector by a constant

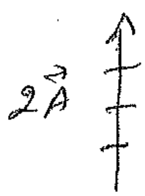
Has the effect of multiplying the magnitude of the vector by the constant. If the constant is positive the direction remains the same, while the constant is negative, the direction becomes opposite.

Example: If $\vec{A} = 2m\ north$, represent the following vectors graphically with a scale of 1m to 1cm

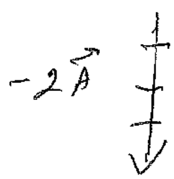
a) -$\vec{A}$



b) $2\vec{A}$



c) -2$\vec{A}$



Adding Vectors Graphically

The sum of two or more vectors is the single vector with the same effect. To add vectors graphically, first join the vectors head to tail. Then the sum vector is the vector whose tail is the tail of the first vector and whose head is the head of the last vector.

Example:

Add the following 3 vectors graphically

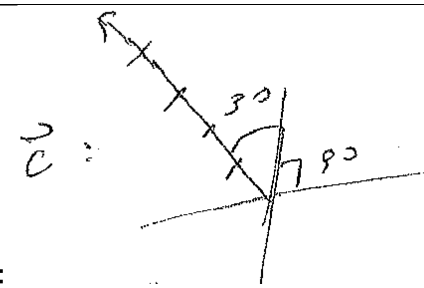
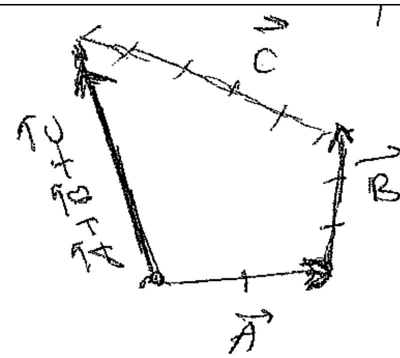
$\vec{A} = 2$ m east

$\vec{B} = 3$ m north

$\vec{C} = 5\text{m at } 120°$

Solution:

Scale: Let 1m be represented by 1cm

 $\vec{A}$:	 $\vec{C}$:
 $\vec{B}$:	 $\vec{A}+\vec{B}+\vec{C}$:

<u>Vector Subtraction: Vector</u>

Vector $\vec{B}$ can be subtracted from vector $\vec{A}$ by adding the negative of $\vec{B}$ to $\vec{A}$

$$\vec{A}-\vec{B}=\vec{A}+(-\vec{B})$$

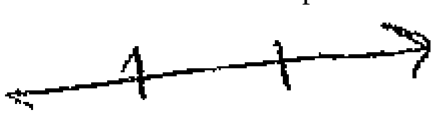
<u>Example:</u> Find $\vec{A}-\vec{B}$ of

$\vec{A}=3m\ east$

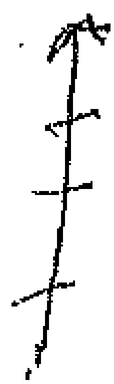
$\vec{B}=4m\ north$

Solution:

Scale: Let 1m be represented by 1cm



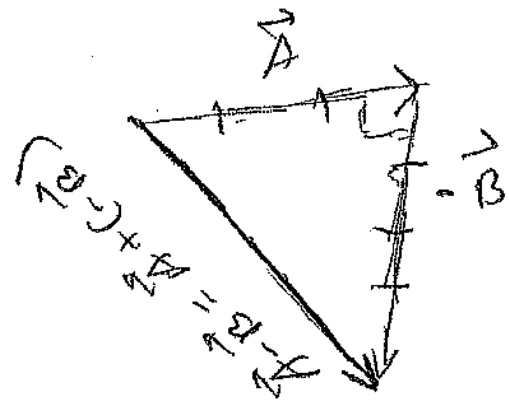
$\vec{\boldsymbol{A}}$:



$\vec{\boldsymbol{B}}$:

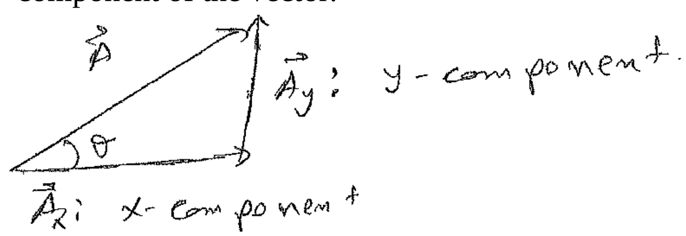
$-\vec{\boldsymbol{B}}$:

$\vec{\boldsymbol{A}}-\vec{\boldsymbol{B}}=\vec{\boldsymbol{A}}+(-\vec{\boldsymbol{B}})$



Components of a Vector:

Any vector can be written as a sum of a horizontal vector and a vertical vector. The horizontal vector is called horizontal or x-component of the vector and the vertical vector is called the vertical or y-component of the vector.



If the magnitude of vector $\vec{A}$ is A and its default angle measure (angle measured wrt. to the positive x-axix) is 0, then using the definitions of a cosine & sine in temrs of a right angles traingle. $\left[\mathrm{Cos}\ \theta = \frac{\mathrm{Ax}}{\mathrm{A}} \text{ and } \sin\theta = \frac{\mathrm{Ay}}{\mathrm{A}}\right]$

$$\boxed{\begin{aligned} A_x &= A * cos\theta \\ A_y &= A * sin\theta \end{aligned}}$$

The components A_x & A_y can be positive or negative. The x-component(A_x) is positive if $\vec{A}_x$ is to the right and negative if $\vec{A}_x$ is to the left. The y-component (A_y) positive if $\vec{A}_y$ is up and negative if $\vec{A}_y$ is down. If the default angle is used, the cosine and sine will result in the correct sign for the x & y component, respectfully.

Example: In each of the following calculate the x & y components of the vector.

a) $\vec{A} = 2m\ east$

$A = 2m \qquad \theta = 0$

$$A_x = Acos(\theta)$$
$$= (2m)\cos(\theta) = 2m$$
$$A_y = Asin(\theta)$$
$$= (2m)(\sin(0)) = 0$$

b) $\vec{A} = 4m\ west$

$A = 4m \qquad \theta = 180°$

$$A_x = Acos(\theta)$$
$$= (4m)\cos(180°) = -4m$$
$$A_y = Asin(\theta)$$

$$= (4m)(\sin(180°)) = 0$$

c) $\vec{A} = 5m\ north$

$A = 5m \qquad \theta = 90°$

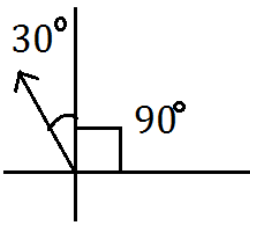
$$A_x = Acos(\theta)$$
$$= (5m)\cos(90°) = 0$$
$$A_y = Asin(\theta)$$
$$= (5m)(\sin(90°)) = 5\text{ m}$$

d) $\vec{A} = 2m\ south$

$A = 2m \qquad \theta = -90°$

$$A_x = Acos(\theta)$$
$$= (2m)\cos(-90°) = 0$$
$$A_y = Asin(\theta)$$
$$= (2m)(\sin(-90°)) = -2\text{m}$$

e) $\vec{A} = 10\ m/s\ 30°\ west\ of\ north$



$A = 10\,m/s$

$\theta = 90 + 30$

$= 120$

$$A_z = (10\,m/s)\cos 120° = -5\,m/s$$
$$A_y = (10\,m/s)\sin 120° = 8.7\,m/s$$

f) $\vec{A} = 100\ m\ 45°\ south\ of\ east$

$A = 100\ m$

$\theta = -45°$

$$A_x = (100\ m)\cos -45° = 70.7\ m$$
$$A_y = (100m)\sin -45° = -70.7\ m$$

g) $\vec{A} = 10\ m\ at\ 225°$

$A = 10\ m$

$\theta = 225\ °$

$$A_x = (10m)\cos 225° = -7.07\ m$$
$$A_y = (10m)\sin 225° = -7.07\ m$$

<u>Obtaining magnitude and direction of a vector from its components</u>

<u>Magnitude (A)</u>

From pythagorean theorem

$$A^2 = A_x^2 + A_y^2$$
$$\boxed{A = \sqrt{A_x^2 + A_y^2}}$$

<u>Direction (θ)</u>

$$\tan(\theta) = \frac{A_x}{A_y}$$
$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

To obtain the default angle, this formula has to be modified. The periodicity of tangent is 180° and not 360°. The calculator will give you angles in the range between -90° and 90° only(that is, angles in the x>0 quadrants). For angles that fall in the range between 90° and 270°(that is, angles in the x<0 quadrants) the formula has to be modified by adding 180°. For example the calculator will give you the same angle for a vector whose x & y components are 3&4 and a vector whose components are -3& -4.

$$\theta = \tan^{-1}\left(\frac{3}{4}\right) = \tan^{-1}\left(\frac{3}{4}\right) = 37°$$

But obviously the angle for the later is 37° + 180° = 217°

Further, the calculator will give you an error when $A_x = 0$.

Therefore the default angle is given by:

$$\theta = \begin{Bmatrix} \tan^{-1}\left(\frac{A_y}{A_x}\right) if\ A_x > 0 \\ \tan^{-1}\left(\frac{A_y}{A_x}\right) + 180\ if\ A_x < 0 \\ 90°\ if\ A_x = 0 \quad \& \quad A_y > 0 \\ -90°\ if\ A_x = 0 \quad \& \quad A_y < 0 \end{Bmatrix}$$

<u>Example:</u> Find the magnitude & direction of a vector whose compnents are

a) $A_x = 3 \quad A_y = 4$

$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{3^2 + 4^2} = 5$$

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) = \tan^{-1}\left(\frac{4}{3}\right) = 53.13°$$

b) $A_x = 3 \quad A_y = -4$

$$A = \sqrt{3^2 + (-4)^2} = 5$$

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) = \tan^{-1}\left(\frac{-4}{3}\right) = -53.13°$$

c) $A_x = -3 \quad A_y = 4$

$$A = \sqrt{(-3)^2 + 4^2} = 5$$

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) = \tan^{-1}\left(\frac{4}{-3}\right) + 180 = -53.13° + 180° = 126.87°$$

$$(+180°\ because\ A_x < 0)$$

d) $A_x = -3 \quad A_y = 4$

$$A = \sqrt{(-3)^2 + 4^2} = 5$$

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) = \tan^{-1}\left(\frac{-4}{-3}\right) + 180 = 53.13° + 180° = 233.13°$$

$$(+180°\ because\ A_x < 0)$$

e) $A_x = 0 \quad A_y = 4$

$$A = \sqrt{0^2 + 4^2} = 4$$

$\theta = \tan^{-1}\left(\frac{4}{0}\right) = 90°$ b/c $A_y > 0$

f) $A_x = 0 \quad A_y = -4$

$$A = \sqrt{0^2 + (-4)^2} = 4$$

$\theta = \tan^{-1}\left(\frac{4}{0}\right) = -90°$ b/c $A_y < 0$

g) $A_x = 3 \quad A_y = 0$

$A = \sqrt{3^2 + 0^2} = 3$

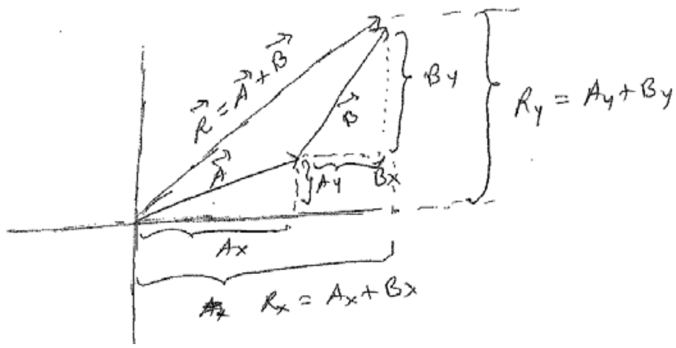
$\theta = \tan^{-1}\left(\frac{0}{3}\right) = 0°$

h) $A_x = -3 \quad A_y = 0$

$A = \sqrt{(-3)^2 + 0^2} = 3$

$\theta = \tan^{-1}\left(\frac{0}{-3}\right) = 0° + 180° = 180°$ (+180 b/c $A_x < 0$)

<u>Adding Vectors Analytically</u>



As can be seen from the diagram above

If $\vec{R} = \vec{A} + \vec{B}$, then

$$R_x = A_x + B_x$$
$$R_y = A_y + B_y$$

Therefore the magnitude and the direction of the sum vector are given

$$R = \sqrt{(A_x + B_x)\text{^}2 + (A_y + B_y)\text{^}2}$$

$$\theta = \begin{Bmatrix} \tan^{-1}(\dfrac{A_y + B_y}{A_x + B_x}) \text{ If } \quad A_x + B_x > 0 \\ \tan^{-1}(\dfrac{A_y + B_y}{A_x + B_x}) \text{ If } \quad A_x + B_x < 0 \\ 90° \text{ if } \ A_x = 0 \ \& \ A_y > 0 \\ -90° \text{ if } A_x = 0 \ \& \ A_y < 0 \end{Bmatrix}$$

<u>Example:</u> Given the vector

$$\vec{A} = 10m\ 45°\ west\ of\ North$$
$$\vec{B} = 20m\ \ 37° east\ of\ North$$

Find the magnitude and direction of the sum vector $\vec{A} + \vec{B}$

<u>Solution:</u> $\theta = ?? \quad R = ??$

Components of $\vec{A}$

$$A = 10m \quad \theta_A = 45° + 90° = 135°$$

$$A_x = A\cos\theta_A$$

$$= 10\cos 135° = -7.07$$

$$A_y = A\sin\theta_A$$

$$= 10\sin 135° = 7.07$$

Components of $\vec{B} = 20\ m \quad \theta_B = 37°$

$$B_x = B\cos\theta_B = 20\cos 37° = 16$$

$$B_y = B\sin\theta_B = 20\sin 37° = 12$$

$$R = \sqrt{(A_x + B_x)\hat{}2 + (A_y + B_y)\hat{}2}$$

$$R = \sqrt{(-7.07 + 16)^2 + (7.07 + 12)^2} = 21.06$$

$$\theta = \tan^{-1}\left(\frac{A_y + B_y}{A_x + B_x}\right) = \theta = \tan^{-1}\left(\frac{7.07 + 12}{-7.07 + 16}\right) = 64.9°$$

Unit Vectors

A unit vector is a vector whose magnitude is one (1). A unit vector in the direction of vector $\vec{A}$ ($\vec{e}_A$) is obtained by dividing the vector by its magnitude.

$$\boxed{\vec{e}_A = \frac{\vec{A}}{A}}$$

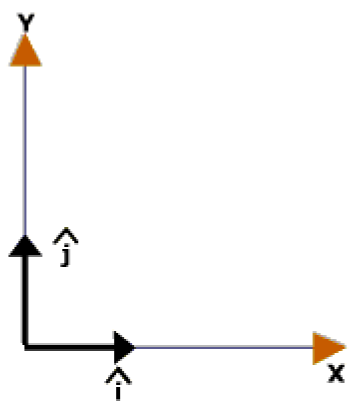
Where A is the magnitude of the vector $\vec{A}$

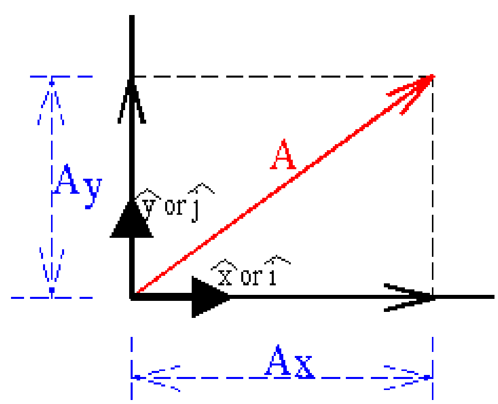
<u>The $\hat{\imath} - \hat{\jmath}$ representation of vectors</u>

Unit vectors in the direction of the positive x-axis and y-axis are customarily represented by <u>$\hat{\imath}$ *and* $\hat{\jmath}$.</u> Therefore the horizontal and vertical components of a vector

$$\vec{A}_x = A_x\hat{\imath}$$

$$\vec{A}_y = A_y\hat{\jmath}$$





where A_x & A_y are the x & y components of the vector. Also, if the magnitude of the vector is A the default angle is θ.

$$\boxed{\begin{aligned}\vec{A}_x &= A\cos\theta\,\hat{\imath}\\ \vec{A}_y &= A\sin\theta\,\hat{\jmath}\end{aligned}}$$

Any vector can be written as the sum of its x & y component vectors

$\vec{A} = \vec{A}_x + \vec{A}_y$ but $\vec{A}_x = A_x\hat{\imath}$ & $\vec{A}_y = A_y\hat{\jmath}$

therefore the vector $\vec{A}$ can be represented in terms of the unit vectors $\hat{\imath}$ & $\hat{\jmath}$ as

$$\boxed{\begin{aligned}\vec{A} &= A_x\hat{\imath} + A_y\hat{\jmath} \\ \vec{A} &= A\cos\theta\,\hat{\imath} + A\sin\theta\,\hat{\jmath}\end{aligned}}$$

Example: Express the following vectors in terms of the $\hat{\imath}$ & $\hat{\jmath}$ unit vectors

a) $\vec{A} = 10m\ \ 37°\ south\ of\ west$

Solution:

$A = 10m \quad \theta = 180 + 37 = 217°$

$$\begin{aligned}\vec{A} &= A\cos\theta\,\hat{\imath} + A\sin\theta\,\hat{\jmath} \\ &= 10\cos 217\ \hat{\imath} + 10\sin 217\,\hat{\jmath} \\ &= -8\hat{\imath} - 6\hat{\jmath}\end{aligned}$$

b) $\vec{A} = 2m\ west$

$A = 2m \quad \theta = 180°$

$$\begin{aligned}\vec{A} &= 2\cos 180\,\hat{\imath} + 2\sin 180\,\hat{\jmath} \\ &= -2m\,\hat{\imath} + 0\hat{\jmath} = \underline{-2m\,\hat{\imath}}\end{aligned}$$

Magnitude & Direction of a Vector expressed in the $\hat{\imath} - \hat{\jmath}$ notation

Since the coefficient of $\hat{\imath}$ is the x-component and the coefficient of $\hat{\jmath}$ is the y-component (i.e $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$)

$$A = \sqrt{A_x^2 + A_y^2}$$

$$\theta = \begin{cases} \boxed{\tan^{-1}(\frac{A_y}{A_x})} & \text{If } A_x > 0 \\ \boxed{\tan^{-1}(\frac{A_y}{A_x}) + 180°} & \text{If } A_x < 0 \\ 90° \text{ if } A_x = 0 & A_y > 0 \\ -90° \text{ if } A_x = 0 & A_y < 0 \end{cases}$$

Example: Find the magnitude & direction of the following vectors expressed in the $\hat{\imath} - \hat{\jmath}$ notation

a) $\vec{A} = -3\hat{\imath} + 4\hat{\jmath}$

Solution: $A_x = -3 \quad A_y = 4$

$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{(-3)^2 + 4^2} = 5$$

$$\theta = \tan^{-1}(\frac{4}{-3}) + 180° = -53 + 180 = 127°$$

b) $\vec{A} = 5\,\hat{\imath}$

Solution: $A_x = 5 \quad A_y = 0$

$A = \sqrt{A_x^2 + A_y^2} = A = \sqrt{5^2 + 0^2} = 5$

$\theta = \tan^{-1}\left(\frac{0}{5}\right) = 0°$

Operating with vectors in the $\hat{\imath} - \hat{\jmath}$ notation

The general rule is to treat vectors in the $\hat{\imath} - \hat{\jmath}$ notation like any algebraic expression while adding (subtracting) or multiplying by a constant provided the $\hat{\imath}$ & $\hat{\jmath}$ unit vectors are treated as independent algebraic entities. That is $\hat{\imath}$ terms can be combined with $\hat{\imath}$ terms & $\hat{\jmath}$ terms can be combined with $\hat{\jmath}$ terms only.

That is:

$$C[A_x\hat{\imath} + A_y\hat{\jmath}] = CA_x\hat{\imath} + CA_y\hat{\jmath}$$

$$[A_x\hat{\imath} + A_y\hat{\jmath}] + [B_x\hat{\imath} + B_y\hat{\jmath}] = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath}$$

Example: Find the magnitude & direction of the following if

$$\vec{A} = -2\hat{\imath} + 3\hat{\jmath} \quad \vec{B} = 4\,\hat{\jmath} \quad \vec{C} = 4\hat{\imath} - 2\hat{\jmath}$$

a) $2\vec{A} + \vec{B}$

Solution:

$$2\vec{A} + \vec{B} = 2(-2\hat{\imath} + 3\hat{\jmath}) + (4\hat{\jmath})$$

$$= -4\hat{\imath} + 6\hat{\jmath} + 4\hat{\jmath}$$

$$= -4\hat{\imath} + 10\hat{\jmath}$$

Magnitude $= \sqrt{(-4)^2 + 10^2} = 10.8$

Direction = $\tan^{-1}\frac{10}{-4} + 180° = 111.8°$

b) $3\vec{A} - 4\vec{B} - 2\vec{C}$

Solution

$$3\vec{A} - 4\vec{B} - 2\vec{C} = 3\,(-2\hat{\imath} + 3\hat{\jmath}) - 4(4\hat{\jmath}) - 2(4\hat{\imath} - 2\hat{\jmath})$$

$$= -6\hat{\imath} + 9\hat{\jmath} - 16\hat{\jmath} - 8\hat{\imath} + 4\hat{\jmath}$$

$$= -14\hat{\imath} - 3\hat{\jmath}$$

Magnitude = $\sqrt{(-14)^2 + (-3)^2} = 14.3$

Direction = $\tan^{-1}\frac{-3}{-14} + 180° = 192.1°$

Three Dimensional Vector

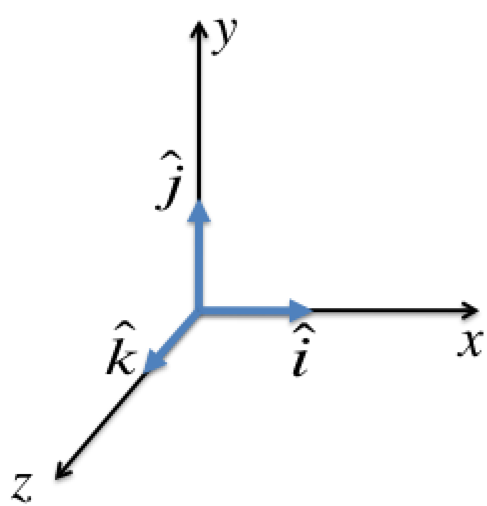
A three dimensional vector can be written as a sum of vectors along the x-axis, y-axis, and z-axis

$$\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z$$

The unit vector along the z-axis is customarily represented by $\hat{k}$. Thus the vector of $\vec{A}$ can be written in the $\hat{\imath} - \hat{\jmath} - \hat{k}$ notation as:

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

Where $A_x, A_y,$ *and* A_z are components of $\vec{A}$ along the x, y, and z axis respectively



If the projection of the 3-dimensional vector on the x-y plane is A_{xy}, then

$$A_{xy} = \sqrt{A_x^2 + A_y^2}$$

And since the projection on the x-y plane is perpendicular to the z-component vector, the magnitude of the vector A is given by

$$A = \sqrt{A_{xy}^2 + A_z^2} \quad \text{substituting} \quad A_{xy}^2 = A_x^2 + A_y^2$$

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Magnitude of a 3-dimensional vector

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

Example: Given the vectors

$\vec{A} = -2\hat{\imath} + 4\hat{\jmath} - \hat{k}$ & $\vec{B} = \hat{\imath} - 3\hat{k}$

Find the magnitude of

a) Vector $\vec{A}$

Solution

$\vec{A} = -2\hat{\imath} + 4\hat{\jmath} - \hat{k} = \sqrt{(-2)^2 + 4^2 + (-1)^2} = \sqrt{21} = 4.6$

b) $2\vec{A} - 3\vec{B}$

Solution

$$2\vec{A} - 3\vec{B} = 2(-2\hat{\imath} + 4\hat{\jmath} - \hat{k}) - 3(\hat{\imath} - 3\hat{k})$$
$$= -4\hat{\imath} + 8\hat{\jmath} - 2\hat{k} - 3\hat{\imath} + 9\hat{k}$$
$$= -7\hat{\imath} + 8\hat{\jmath} + 7\hat{k}$$

Magnitude = $\sqrt{(-7)^2 + 8^2 + 7^2} = 12.7$

Dot Product

The dot product of two vectors $\vec{A}$ & $\vec{B}$ is defined to be the product of the magnitude of vector $\vec{A}$ and the component of vector $\vec{B}$ in the direction of vector $\vec{A}$.

$$\vec{A} \cdot \vec{B} = |\vec{A}|\, B$$

and if the angle between $\vec{A}$ & $\vec{B}$ is θ, then $B_{II} = B\cos\theta$

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

Some properties of a dot product

1) $\vec{A} \cdot \vec{A} = |\vec{A}|^2$ because $\theta = 0$
2) $\vec{A} \cdot \vec{B} = 0$ if $\vec{A}$ & $\vec{B}$ are perpendicular to each other.
3) Dot product is commutative
   $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
4) Dot product is distributive over addition
   $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{B}$
5) $\vec{A} \cdot C\vec{B} = C\,\vec{A} \cdot \vec{B}$ (C is constant)

Example:

a) Calculate the dot product of $\vec{A}$ & $\vec{B}$ if :

$\vec{A} = 5\,m\ east$ & $\vec{B} = 2m\ 60°\ north\ of\ east$

Solution

$|\vec{A}| = 5m \quad |\vec{B}| = 2m \quad \theta = 60°$

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos\theta$$
$$= (5m)(2m)\cos 60° = 5m^2$$

b) If $\vec{A} = 8m\ east$ & $\vec{B} = 4m\ \ at\ \ 37°$

$|\vec{A}| = 8m \quad |\vec{B}| = 4\,m$

$$\theta = 90° + 37° = 127°$$
$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos 127° = -7.2\,m^2$$

Dot Product in terms of **x-y-z** components

$\vec{A}\cdot\vec{A} = |\vec{A}||\vec{A}|\cos\theta = |\vec{A}|^2$

since $|\hat{\imath}| = |\hat{\jmath}| = |\hat{k}| = 1$ it follows that

$$\boxed{\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1}$$

And since the unit vectors are perpendicular to each other

$$\boxed{\hat{\imath}\cdot\hat{\jmath} = \hat{\imath}\cdot\hat{k} = \hat{\jmath}\cdot\hat{k} = 0}$$

Now if $\vec{A} = \vec{A}_x\hat{\imath} + \vec{A}_y\hat{\jmath} + \vec{A}_z\hat{k}$

& $\vec{B} = \vec{B}_x\hat{\imath} + \vec{B}_y\hat{\jmath} + \vec{B}_z\hat{k}$

$$\vec{A}\cdot\vec{B} = \left(\vec{A}_x\hat{\imath} + \vec{A}_y\hat{\jmath} + \vec{A}_z\hat{k}\right)\cdot\left(\vec{B} = \vec{B}_x\hat{\imath} + \vec{B}_y\hat{\jmath} + \vec{B}_z\hat{k}\right)$$

$$= \vec{A}_x\hat{\imath}\cdot\left(\vec{B}_x\hat{\imath} + \vec{B}_y\hat{\jmath} + \vec{B}_z\hat{k}\right)$$

$$+ \vec{A}_y\hat{\jmath}\cdot(\vec{B}_x\hat{\imath} + \vec{B}_y\hat{\jmath} + \vec{B}_z\hat{k})$$

$$+ \vec{A}_z\hat{k}\cdot\left(\vec{B}_x\hat{\imath} + \vec{B}_y\hat{\jmath} + \vec{B}_z\hat{k}\right)$$

since $\hat{\imath}\cdot\hat{\imath} = 1$ & $\hat{\imath}\cdot\hat{\jmath} = \hat{\imath}\cdot\hat{k} = 0$

$$\boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z}$$

The angle between two vectors may be obtained by equating the two different forms of a dot product

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta = A_xB_x + A_yB_y + A_zB_z$$

$$\boxed{\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{|\vec{A}||\vec{B}|}}$$

<u>Example:</u>

Given $\vec{A} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$ and

$\vec{B} = -4\hat{\imath} + \hat{\jmath} - \hat{k}$

Calculate $\vec{A}\cdot\vec{B}$

<u>Solution</u>

$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z = 2(-4) + (-3)(4) + (4)(-1) = -24$

<u>Example:</u>

Given the vectors

$\vec{A} = 2\hat{\imath} - 4\hat{\jmath}$ & $\vec{B} = \hat{\imath} + 3\hat{\jmath}$

Calculate the angle formed between $\vec{A}$ & $\vec{B}$

<u>Solution</u>

$$\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{|\vec{A}||\vec{B}|}$$

$A_x = 2$; $A_y = -4$; $A_z = 0$; $B_x = 1$; $B_y = 3$; $B_z = 0$

$|\vec{A}| = \sqrt{A_x^2 + A_y^2} = \sqrt{2^2 + (-4)^2} = \sqrt{20}$

$|\vec{B}| = \sqrt{B_x^2 + B_y^2} = \sqrt{1^2 + 3^2} = \sqrt{10}$

$\cos\theta = \frac{(2)(1)+(-4)(1)}{\sqrt{20}*\sqrt{10}} = -\frac{10}{10\sqrt{2}} = -\frac{1}{\sqrt{2}}$

$\theta = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = 135°$

<u>Example:</u>

Given the vectors

$\vec{A} = 3\hat{\imath} - 6\hat{\jmath} + \hat{k}$

$\vec{B} = 2\hat{\imath} - 5\hat{\jmath}$

$\vec{C} = -\hat{\imath} + 7\hat{\jmath}$

a) Calculate $\vec{A}\cdot(\vec{B} + \vec{C})$

Solution

$A_x = 3\,; A_y = -6\,; A_z = 1$
$B_x = 2\,; B_y = -5\,; B_z = 0$
$C_x = -1; C_y = 7; C_z = 0$

$\vec{A}\cdot(\vec{B}+\vec{C})$
$= \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$
$= A_xB_x + A_yB_y + A_zB_z \quad + \quad A_xC_x + A_yC_y + A_zC_z$
$= (3)(2) + (-6)(-5) + 0 + (3)(-1) + (-6)(7) + 0$
$= 6 + 30 - 3 - 42 = \underline{-9}$

b) Calculate $\underline{\vec{B}\cdot(2\vec{A}-3\vec{C})}$

Solution

$$\vec{B}\cdot(2\vec{A}-3\vec{C}) = 2\vec{B}\vec{A} - 3\vec{B}\vec{C}$$
$$= 2(A_xB_x + A_yB_y + A_zB_z) - 3(B_xC_x + B_yC_y + B_zC_z)$$
$$= 2[(3)(2) + (-6)(-5) + 0] - 3[(2)(-1) + (-5)(7) + 0]$$
$$= 2(36) - 3(-37) = 72 + 107 = 179$$

The Cross Product

The cross product between two vectors $\vec{A}$ & $\vec{B}$, written as $\vec{A}\times\vec{B}$, is a vector whose magnitude is equal to the area of the parallelogram determined by the two vectors and whose direction is perpendicular to the plane determined by the two vectors.

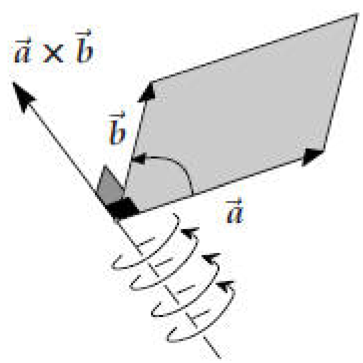
If the angle between the vectors is θ then:

$$\boxed{|\vec{A}\times\vec{B}| = AB\sin\theta}$$

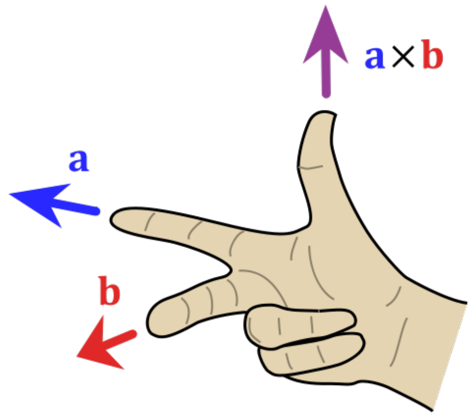
Where A & B are the magnitudes of $\vec{A}$ & $\vec{B}$, respectively

The direction can be either perpendicularly out of the plane—represented by a dot (·)—or perpendicularly into the plane—represented by a cross (×). To distinguish between these two possibilities either the screw rule or the right hand rule can be used.

The Screw Rule: First join the vectors tail to tail. Then place a screw at the intersection of the two vectors perpendicularly to the plane determined by the two vectors. To find the direction of $\vec{A}\times\vec{B}$ rotate the screw from $\vec{A}$ towards $\vec{B}$ (and to find direction of $\vec{B}\times\vec{A}$ rotate from $\vec{B}$ towards $\vec{A}$). Then the direction of movement of the screw gives the direction of $\vec{A}\times\vec{B}$)



The Right Hand Rule: First align the thumb, index finger and middle finger of the right hand perpendicular to each other. To find the direction of $\vec{A}\times\vec{B}$, $\vec{A}$ is represented by the index finger, $\vec{B}$ is represented by the middle finger. Then the thumb represents the direction of $\vec{A}\times\vec{B}$.



<u>Example:</u> If $\vec{A} = 2m\ east$ and $\vec{B} = 4m\ north$

a) Find the magnitude and direction of $\vec{A} \times \vec{B}$

$|\vec{A} \times \vec{B}| = AB \sin\theta \quad A = 2m \;\; B = 4m \;\; \theta = 90°$

$= (2)(4)\sin 90 = 8m^2$

Direction- Perpendicularly out $(\cdot)$

b) Find the magnitude and direction of $\vec{B} \times \vec{A}$

$|\vec{A} \times \vec{B}| = A * B * \sin\theta$

$= (2)(4)\sin 90 = 8m^2$

Direction. Perpendicularly in $(\times)$

<u>Some Properties of a Cross Product</u>

1) $\vec{A} \times \vec{A} = 0$ (because $\theta = 0$ & $\sin 0 = 0$)
2) $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
3) $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

<u>11.2 The cross product in terms of the $\hat{\imath} - \hat{\jmath} - \hat{k}$ Cartesian unit vectors</u>

$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$ (because $\vec{A} \times \vec{A} = 0$)

$\hat{\imath} \times \hat{\jmath} = \hat{k}$ $\hat{\jmath} \times \hat{k} = \hat{\imath}$ $\hat{k} \times \hat{\imath} = \hat{\jmath}$	$\hat{\jmath} \times \hat{\imath} = -\hat{k}$ $\hat{k} \times \hat{\jmath} = -\hat{\imath}$ $\hat{\imath} \times \hat{k} = -\hat{\jmath}$

Which can be easily shown using the basic definition of cross-product

If $\quad \vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$

$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$

then

$$\vec{A} \times \vec{B} = [A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\,] \times [B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\,]$$

$$= A_x\hat{\imath}[B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}] + \vec{A}_y\hat{\jmath}[B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}] + \vec{A}_z\hat{k}[B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}]$$

$$= A_xB_y\hat{k} - A_xB_z\hat{\jmath} - A_yB_z\hat{k} + A_yB_z\hat{\imath} + A_zB_x\hat{\jmath} - A_zB_y\hat{\imath}$$

$$\boxed{\vec{A} \times \vec{B} = \hat{\imath}(A_yB_z - A_zB_y) + \hat{\jmath}(A_zB_x - A_xB_z) + \hat{k}(A_xB_y - A_yB_x)}$$

This formula can be written also as a determinant

$$\boxed{\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}}$$

<u>Proof</u> – Expanding the determinant

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{\imath} \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{\jmath} \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$= \hat{\imath}(A_yB_z - A_zB_y) - \hat{\jmath}(A_xB_z - A_zB_x) + \hat{k}(A_xB_y - A_yB_x)$$

$$= \hat{\imath}(A_yB_z - A_zB_y) + \hat{\jmath}(A_xB_z - A_zB_x) + \hat{k}(A_xB_y - A_yB_x)$$

$$= \vec{A} \times \vec{B}$$

<u>Example</u>:

If $\begin{bmatrix} \vec{A} = \hat{\imath} - 3\hat{\jmath} + 4\hat{k} \\ \vec{B} = 2\hat{\jmath} - 5\hat{k} \\ \vec{C} = -2\hat{\imath} + 4\hat{\jmath} + \hat{k} \end{bmatrix}$

a) Find $\vec{A} \times \vec{B}$

$$\vec{A} \times \vec{B} = \hat{\imath}(A_yB_z - A_zB_y) + \hat{\jmath}(A_zB_x - A_xB_z) + \hat{k}(A_xB_y - A_yB_x)$$

$A_x = 1$	$A_y = -3$	$A_z = 4$
$B_x = 0$	$B_y = 2$	$B_z = -5$

$$\vec{A} \times \vec{B} = \hat{\imath}[(-3)(-5) - 4(2)] + j[(4)(0) - (1)(-5)] + \hat{k}[(1)(2) - (-3)(0)]$$

$$= 7\hat{\imath} + 5\hat{\jmath} + 2\hat{k}$$

b) Find $\vec{B} \times \vec{A}$

$\vec{B} \times \vec{A} = -(\vec{A} \times \vec{B}) = -[7\hat{\imath} + 5\hat{\jmath} + 2\hat{k}] = -7\hat{\imath} - 5\hat{\jmath} - 2\hat{k}$

c) $\vec{A} \times (2\vec{B} - \vec{C})$

$$= [\hat{\imath} - 3\hat{\jmath} + 4\hat{k}] \times [2\hat{\imath} - 11\hat{k}]$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -3 & 4 \\ 2 & 0 & -11 \end{vmatrix}$$

$$= \hat{\imath} \begin{vmatrix} -3 & 4 \\ 0 & -11 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 1 & 4 \\ 2 & -11 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & -3 \\ 2 & 0 \end{vmatrix}$$

$$= \hat{\imath}[(-3)(-11) - (0)(4)] - \hat{\jmath}[1(-11) - 4(2)] + \hat{k}[0(1) - (-3)(2)]$$

$$= \hat{\imath}(33) - j(-19) + \hat{k}(6)$$

$$= 33\hat{\imath} + 19\hat{\jmath} + 6\hat{k}$$