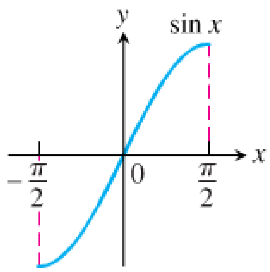
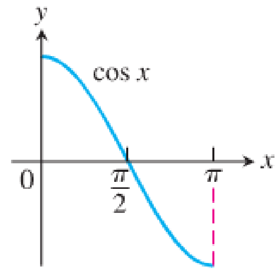
Section 2.9 – Derivatives of Inverse Trigonometric Functions

Defining the inverses

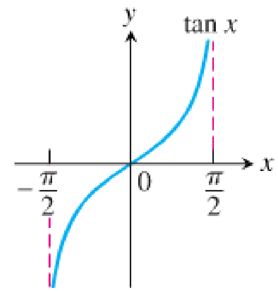
The six basic trigonometric functions are not one-to-one.



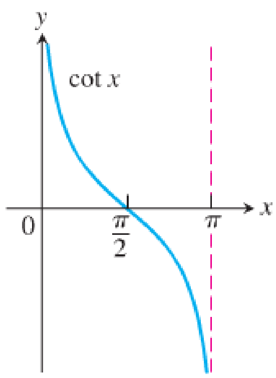
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



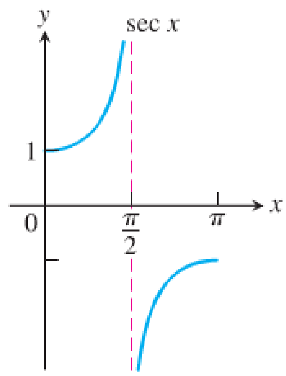
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



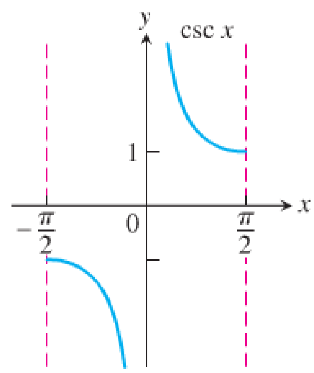
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

Since these restricted functions are now one-to-one, they have inverses, which we denoted by

$$y = \sin^{-1} x \quad or \quad y = \arcsin x$$

$$y = \cos^{-1} x \quad or \quad y = \arccos x$$

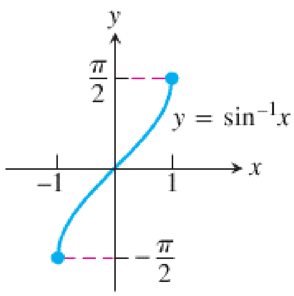
$$y = \tan^{-1} x \quad or \quad y = \arctan x$$

$$y = \cot^{-1} x \quad or \quad y = \operatorname{arccot} x$$

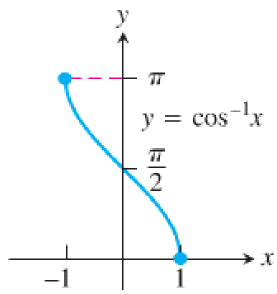
$$y = \sec^{-1} x \quad or \quad y = \operatorname{arcsec} x$$

$$y = \csc^{-1} x \quad or \quad y = \operatorname{arccsc} x$$

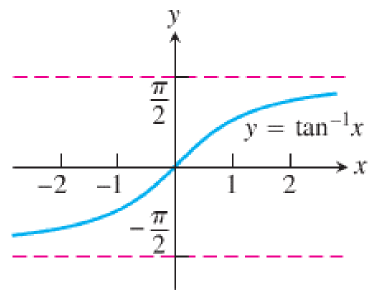
Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



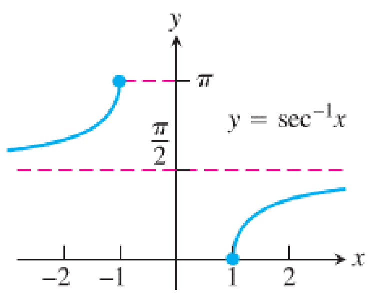
Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi$



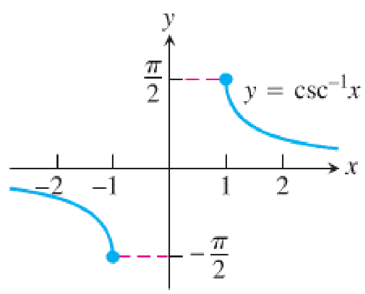
Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



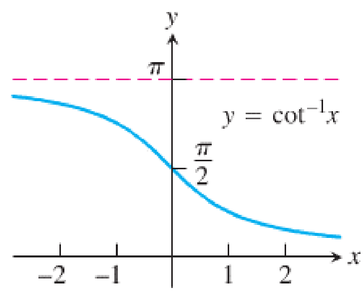
Domain: $x \le -1$ or $x \ge 1$
Range: $0 \le y \le \pi,\ y \ne \frac{\pi}{2}$



Domain: $x \le -1$ or $x \ge 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$



Domain: $-\infty < x < \infty$
Range: $0 < y < \pi$



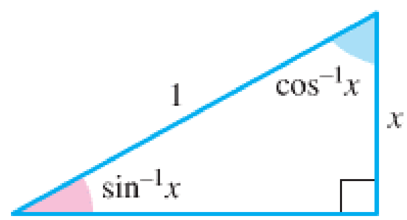
Definitions

✓ $y = \sin^{-1} x$ is the number in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for which $\sin y = x$

✓ $y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$

✓ $y = \tan^{-1} x$ is the number in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which $\tan y = x$

✓ $y = \cot^{-1} x$ is the number in $(0, \pi)$ for which $\cot y = x$



Example $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

Inverse Function – Inverse Cofunction Identities

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

$$\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x$$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

$$\cot^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{x}\right)$$

$$\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right)$$

$$\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1}(x)$$

Derivative of $y = \sin^{-1} u$

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

$$= \frac{1}{\cos\left(\sin^{-1} x\right)}$$

$$= \frac{1}{\sqrt{1-\sin^2\left(\sin^{-1} x\right)}}$$

$$\boxed{= \frac{1}{\sqrt{1-x^2}}}$$

$$\boxed{\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}}, \quad |u| < 1 \qquad \boxed{\frac{d}{dx}\sin^{-1} u = \frac{u'}{\sqrt{1-u^2}}}$$

$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad \|u\| < 1$	$\left(\sin^{-1} u\right)' = \frac{u'}{\sqrt{1-u^2}}$
$\frac{d}{dx}\cos^{-1} u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad \|u\| < 1$	$\left(\cos^{-1} u\right)' = -\frac{u'}{\sqrt{1-u^2}}$
$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}$	$\left(\tan^{-1} u\right)' = \frac{u'}{1+u^2}$
$\frac{d}{dx}\cot^{-1} u = -\frac{1}{1+u^2}\frac{du}{dx}$	$\left(\cot^{-1} u\right)' = -\frac{u'}{1+u^2}$
$\frac{d}{dx}\sec^{-1} u = \frac{1}{\|u\|\sqrt{u^2-1}}\frac{du}{dx}, \quad \|u\| > 1$	$\left(\sec^{-1} u\right)' = \frac{u'}{\|u\|\sqrt{u^2-1}}$
$\frac{d}{dx}\csc^{-1} u = -\frac{1}{\|u\|\sqrt{u^2-1}}\frac{du}{dx}, \quad \|u\| > 1$	$\left(\csc^{-1} u\right)' = -\frac{u'}{\|u\|\sqrt{u^2-1}}$

Example

Find the derivative of $\frac{d}{dx}\left(\sin^{-1} x^2\right)$

Solution

$$\frac{d}{dx}\left(\sin^{-1} x^2\right) = \frac{1}{\sqrt{1-\left(x^2\right)^2}} \cdot \frac{d}{dx}\left(x^2\right)$$

$$= \frac{2x}{\sqrt{1-x^4}}$$

Example

Find the derivative of $\frac{d}{dx}\left(\sec^{-1} 5x^4\right)$

Solution

$$\frac{d}{dx}\left(\sec^{-1} 5x^4\right) = \frac{\left(5x^4\right)'}{5x^4\sqrt{\left(5x^4\right)^2-1}} \qquad 5x^4-1>0$$

$$= \frac{20x^3}{5x^4\sqrt{25x^8-1}}$$

$$= \frac{4}{x\sqrt{25x^8-1}}$$

Exercises — *Section* 2.9 – Derivatives of Inverse Trigonometric Functions

(1 – 2) Find the value of

1. $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

2. $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

(3 – 5) Find the limit

3. $\lim\limits_{x\to -1^{+}} \cos^{-1} x$

4. $\lim\limits_{x\to -\infty} \tan^{-1} x$

5. $\lim\limits_{x\to \infty} \csc^{-1} x$

(6 – 17) Find the derivative

6. $y=\cos^{-1}\left(\frac{1}{x}\right)$

7. $y=\sin^{-1}\sqrt{2}t$

8. $y=\sec^{-1}(5s)$

9. $y=\cot^{-1}\sqrt{t-1}$

10. $y=\ln\left(\tan^{-1} x\right)$

11. $y=\tan^{-1}(\ln x)$

12. $y=\csc^{-1}\left(e^{t}\right)$

13. $y=x\sqrt{1-x^2}+\cos^{-1} x$

14. $y=\ln\left(x^2+4\right)-x\tan^{-1}\left(\frac{x}{2}\right)$

15. $f(x)=\sin^{-1}\frac{1}{x}$

16. $\frac{d}{dx}\left(x\sec^{-1} x\right)\Big|_{x=\frac{2}{\sqrt{3}}}$

17. $\frac{d}{dx}\left(\tan^{-1} e^{-x}\right)\Big|_{x=0}$

18. Find the angle α