Lecture Two

Second Order Differential Equations

Section 2.1 – Second-Order Linear Differential Equations

A second order linear differential equation is an equation which can be written in the form

$$y'' + p(x)y' + q(x)y = f(x)$$

Where *p, q*, and *f* are continuous functions on some interval *I*.

The function $f(x)$ is called the ***forcing function*** or the ***nonhomogeneous*** term.

The equation is said to be ***homogeneous*** when:

$$y'' + p(t)y' + q(t)y = 0$$

Second-Order Equations and Planar Systems

$$y'' = f(t, y, y')$$

Let's re-write in first-order system:

$$\begin{aligned} y' &= v \\ v' &= F(t, y, v) \end{aligned}$$

$$y'' + p(t)y' + q(t)y = F(t)$$

$$y'' = F(t) - p(t)y' - q(t)y$$

$$v' = F(t) - p(t)v - q(t)y$$

$$\begin{aligned} y' &= v \\ v' &= F(t) - p(t)v - q(t)y \end{aligned}$$

Example

Consider a damped unforced spring: $y'' + 0.4y' + 3y = 0$; which satisfies the initial conditions $y(0) = 2$ and $v(0) = y'(0) = -1$

Solution

$$\begin{cases} y' = v \\ v' = -0.4v - 3y \end{cases}$$

$$v' + 0.4v = -3y$$

$$ve^{\int 0.4dy} = \int -3ye^{\int 0.4dy} + C$$

$$ve^{0.4y} = -3\int ye^{0.4y} + C$$

$$\int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$= -3\frac{e^{0.4y}}{0.4^2}(0.4y-1) + C$$

$$= -18.75e^{0.4y}(0.4y-1) + C$$

$$v = -7.5y + 18.75 + Ce^{-0.4y}$$

$$v(0) = -7.5(0) + 18.75 + Ce^{-0.4(0)}$$
$$-1 = 18.75 + C$$
$$C = -19.75$$

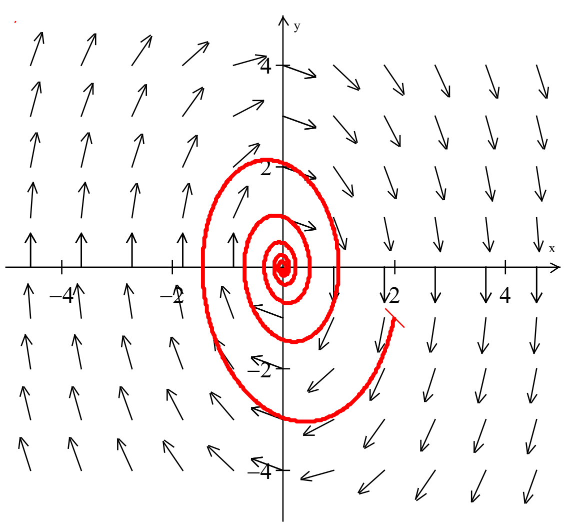
$$v(y) = -7.5y + 18.75 - 19.75e^{-0.4y}$$

$$y' = v = -7.5y + 18.75 - 19.75e^{-0.4y}$$

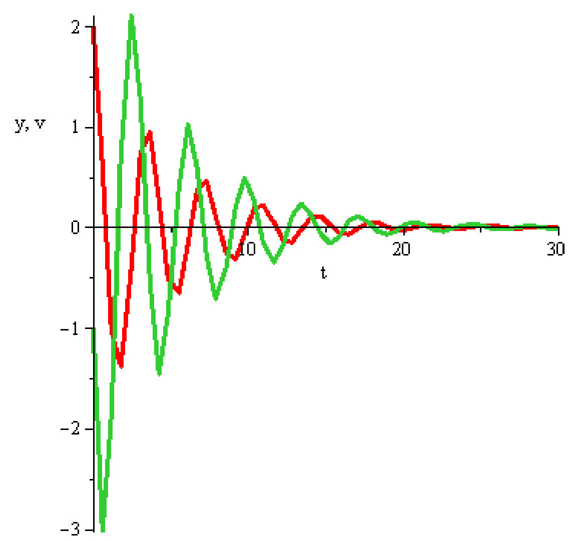
$$\frac{dy}{dt} = -7.5y + 18.75 - 19.75e^{-0.4y}$$

$$\boxed{y(t) = -\frac{3\sqrt{74}}{74}e^{-t/5}\sin\left(\frac{\sqrt{74}}{5}t\right) + 2e^{-t/5}\cos\left(\frac{\sqrt{74}}{5}t\right)}$$

The *yv*-plane is called the ***phase plane***.



Phase Plane Plot



Displacement y and the velocity v

Proposition

$$y'' + p(t)y' + q(t)y = 0$$

Solutions: $y = C_1 y_1 + C_2 y_2$

C_1, C_2 are any constant.

$y_1(t)$ & $y_2(t)$ are linearly independent solutions forming a ***fundamental set of solutions***.

Linear

Set $L[y] = y'' + p(t)y' + q(t)y$

Then, for any two twice differentiable functions $y_1(t)$ & $y_2(t)$,

- $L\lfloor y_1(t) + y_2(t) \rfloor = L\lfloor y_1(t) \rfloor + L\lfloor y_2(t) \rfloor$

  Proof

$$\begin{aligned} L\lfloor y_1(t) \rfloor + L\lfloor y_2(t) \rfloor &= y_1'' + py_1' + qy_1 + y_2'' + py_2' + qy_2 \\ &= y_1'' + y_2'' + p\left(y_1' + y_2'\right) + q\left(y_1 + y_2\right) \\ &= \left(y_1 + y_2\right)'' + p\left(y_1 + y_2\right)' + q\left(y_1 + y_2\right) \\ &= L\lfloor y_1(t) + y_2(t) \rfloor \end{aligned}$$

- $L\lfloor cy(t) \rfloor = cL\lfloor y(t) \rfloor$

  Proof

$$\begin{aligned} L\left[cy(t)\right] &= (cy)'' + p(t)(cy)' + q(t)(cy) \\ &= cy'' + cp(t)y' + cq(t)y \\ &= c\left(y'' + p(t)y' + q(t)y\right) \\ &= cL\lfloor y(t) \rfloor \end{aligned}$$

That is, L is a linear differential operator.

Definition

A linear combination of the two functions u & v is any function of the form

$$w = Au + Bv$$

Definition

Two functions u & v are said to be linearly independent on the interval (α, β), if neither is a constant multiple of the order on that interval. If one is a constant multiple of the other on (α, β), they said to be linearly dependent there.

Existence and Uniqueness for Linear Equations

Theorem

Suppose that the functions *p*, *q*, and *f* are continuous on the open interval *I* containing the point *a*. Then, given any two numbers b_1 and b_2, the equation

$$y'' + p(x)y' + q(x)y = f(x)$$

Has a unique solution on the entire interval *I* that satisfies the initial conditions

$$y(a) = b_1, \quad y'(a) = b_2$$

Example

Verify that the functions $y_1(x) = e^x$ *and* $y_2(x) = xe^x$ are solutions of the differential equation $y'' - 2y' + y = 0$ and then find a solution satisfying the initial conditions $y(0) = 3, \quad y'(0) = 1$

Solution

$$y(x) = C_1 e^x + C_2 xe^x$$

$$y'(x) = C_1 e^x + C_2 e^x + C_2 xe^x = (C_1 + C_2)e^x + C_2 xe^x$$

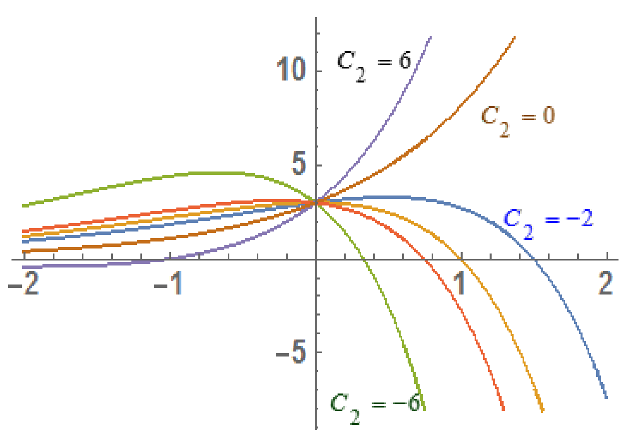
$$y(0) = \boxed{C_1 = 3}$$

$$y'(0) = C_1 + C_2 = 1 \quad \Rightarrow \quad \boxed{C_2 = -2}$$

Hence the solution of the original initial value problem is

$$\boxed{y(x) = 3e^x - 2xe^x}$$

The plot shows several addition solutions of $y'' - 2y' + y = 0$, all having the same initial value $y(0) = 3$



Linearly Independent (LI) Solutions

Definition

Two functions defined on an open interval I are said to be ***linearly independent*** on I provided that neither is a constant multiple of the other.

Example

The following pair functions are linearly independent on the entire real line.

$$f(x) = \sin x \quad and \quad g(x) = \cos x$$

$$f(x) = e^{x} \quad and \quad g(x) = e^{-2x}$$

$$f(x) = x + 1 \quad and \quad g(x) = x^2$$

Two functions are said to be ***linearly dependent*** on an open interval provided that they are not linearly independent there; that is, one of them is a constant multiplication of the other.

Example

Let $f(x) = \sin 2x \quad and \quad g(x) = \sin x \cos x$

Are linearly dependent on any interval because $f(x) = 2g(x)$.

Wronskian

The Wronskian is a function named after the Polish mathematician Józef Hoene-Wroński and it is used to determine whether a set of differentiable functions (solutions) is ***linearly independent*** on a given interval.

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_n(x) \\ f_1'(x) & f_2'(x) & f_n'(x) \\ & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & f_n^{(n-1)}(x) \end{vmatrix}$$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu'$$

If $W = 0 \Rightarrow u \ \& \ v$ are linearly dependent.
If $W \neq 0 \Rightarrow u \ \& \ v$ are linearly independent.

Theorem

Suppose that y_1 and y_2 are two solutions of the homogeneous second-order linear equation

$$y'' + p(x)y' + q(x)y = 0$$

On an open interval *I* on which *p* and *q* are continuous

1. If y_1 and y_2 are linearly dependent, then $W(y_1, y_2) \equiv 0$ on *I*.
2. If y_1 and y_2 are linearly independent, then $W(y_1, y_2) \neq 0$ at each point of *I*.

Example

Use the Wronskian to show that $\mathbf{f}_1 = x, \quad \mathbf{f}_2 = \sin x$ are linearly independence

Solution

The Wronskian is

$$W(x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x \neq 0$$

This function is not identically zero. Thus, the functions are linearly independent.

Example

Use the Wronskian to show that $\mathbf{f}_1 = 1, \quad \mathbf{f}_2 = e^x, \quad \mathbf{f}_3 = e^{2x}$ are linearly independence

Solution

The Wronskian is

$$W(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} = e^x 4e^{2x} - 2e^{2x}e^x = 2e^{3x} \neq 0$$

Thus, the functions are linearly independent.

Exercises *Section* 2.1 – Second-Order Linear Differential Equations

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation.

1. $y'' + 2y' - 3y = 0$

2. $y'' + 3y' + 4y = 2\cos 2t$

3. $y'' + 2y' + 2y = 2\sin 2\pi t$

4. $y'' + \mu\left(t^2 - 1\right)y' + y = 0$

5. $4y'' + 4y' + y = 0$

1. Show that the functions $y_1(x) = e^{-3x}$, $y_2(x) = \cos 2x$, $y_3(x) = \sin 2x$ are linearly independent.

2. Determine whether $\left\{e^x,\ xe^x,\ (x+1)e^x\right\}$ is a set of linearly independent.

Use the Wronskian to show that are linearly independence

3. $y_1(x) = e^{-3x}, \quad y_2(x) = e^{3x}$

4. $\mathbf{f}_1 = 1, \quad \mathbf{f}_2 = e^x, \quad \mathbf{f}_3 = e^{2x}$

5. $\left\{e^x,\ xe^x,\ (x+1)e^x\right\}$

6. $y_1(x) = e^{-3x}, \quad y_2(x) = \cos 2x, \quad y_3(x) = \sin 2x$

7. $y_1(x) = e^x, \quad y_2(x) = e^{2x}, \quad y_3(x) = e^{3x}$

8. $y_1(x) = \cos^2 x, \quad y_2(x) = \sin^2 x, \quad y_3(x) = \sec^2 x, \quad y_4(x) = \tan^2 x$

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0,\ 1)$

9. $y_1(t) = \cos t \sin t, \quad y_2(t) = \sin 2t$

10. $y_1(t) = e^{3t}, \quad y_2(t) = e^{-4t}$

11. $y_1(t) = te^{2t}, \quad y_2(t) = e^{2t}$

12. $y_1(t) = t^2\cos(\ln t), \quad y_2(t) = t^2\sin(\ln t)$

13. $y_1(t) = \tan^2 t - \sec^2 t, \quad y_2(t) = 3$

14. $y_1(t) \equiv 0, \quad y_2(t) = e^t$

Find a particular solution satisfying the given initial conditions

15. $y'' - 4y = 0$; $\quad y_1(t) = e^{2t},\ y_2(t) = 2e^{-2t}$; $\quad y(0) = 1,\ y'(0) = -2$

16. $y'' - y = 0$; $\quad y_1(t) = 2e^{t},\ y_2(t) = e^{-t+3}$; $\quad y(-1) = 1,\ y'(-1) = 0$

17. $y'' + y = 0$; $\quad y_1(t) = 0,\ y_2(t) = \sin t$; $\quad y\left(\frac{\pi}{2}\right) = 1,\ y'\left(\frac{\pi}{2}\right) = 1$

18. $y'' + y = 0$; $\quad y_1(t) = \cos t,\ y_2(t) = \sin t$; $\quad y\left(\frac{\pi}{2}\right) = 1,\ y'\left(\frac{\pi}{2}\right) = 1$

19. $y'' - 4y' + 4y = 0$; $y_1(t) = e^{2t}$, $y_2(t) = te^{2t}$; $y(0) = 2$, $y'(0) = 0$

20. $2y'' - y' = 0$; $y_1(t) = 1$, $y_2(t) = e^{t/2}$; $y(2) = 0$, $y'(2) = 2$

21. $y'' - 3y' + 2y = 0$; $y_1(t) = 2e^{t}$, $y_2(t) = e^{2t}$; $y(-1) = 1$, $y'(-1) = 0$

22. $ty'' + y' = 0$; $y_1(t) = \ln t$, $y_2(t) = \ln 3t$; $y(3) = 0$, $y'(3) = 3$

23. $t^2y'' - ty' - 3y = 0$; $y_1(t) = t^3$, $y_2(t) = -\frac{1}{t}$; $y(-1) = 0$, $y'(-1) = -2$ $(t < 0)$

24. $y'' + \pi^2 y = 0$; $y_1(t) = \sin \pi t + \cos \pi t$, $y_2(t) = \sin \pi t - \cos \pi t$; $y\left(\frac{1}{2}\right) = 1$, $y'\left(\frac{1}{2}\right) = 0$

25. $x^3y^{(3)} - x^2y'' + 2xy' - 2y = 0$

$y(1) = 3$, $y'(1) = 2$, $y''(1) = 1$ $\qquad$ $y_1(x) = x$, $y_2(x) = x\ln x$, $y_3(x) = x^2$

26. $y^{(3)} + 2y'' - y' - 2y = 0$

$y(0) = 1$, $y'(0) = 2$, $y''(0) = 0$ $\qquad$ $y_1(x) = e^{x}$, $y_2(x) = e^{-x}$, $y_3(x) = e^{-2x}$

27. $y^{(3)} - 6y'' + 11y' - 6y = 0$

$y(0) = 0$, $y'(0) = 0$, $y''(0) = 3$ $\qquad$ $y_1(x) = e^{x}$, $y_2(x) = e^{2x}$, $y_3(x) = e^{3x}$

28. $y^{(3)} - 3y'' + 3y' - y = 0$

$y(0) = 2$, $y'(0) = 0$, $y''(0) = 0$ $\qquad$ $y_1(x) = e^{x}$, $y_2(x) = xe^{x}$, $y_3(x) = x^2e^{x}$

29. $y^{(3)} - 5y'' + 8y' - 4y = 0$

$y(0) = 1$, $y'(0) = 4$, $y''(0) = 0$ $\qquad$ $y_1(x) = e^{x}$, $y_2(x) = e^{2x}$, $y_3(x) = xe^{2x}$

30. $y^{(3)} + 9y'' = 0$

$y(0) = 3$, $y'(0) = -1$, $y''(0) = 2$ $\qquad$ $y_1(x) = 1$, $y_2(x) = \cos 3x$, $y_3(x) = \sin 3x$

31. $y^{(3)} - 3y'' + 4y' - 2y = 0$

$y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$ $\qquad$ $y_1(x) = e^{x}$, $y_2(x) = e^{x}\cos x$, $y_3(x) = e^{x}\sin x$

32. $x^3y^{(3)} - 3x^2y'' + 6xy' - 6y = 0$

$y(1) = 6$, $y'(1) = 14$, $y''(1) = 1$ $\qquad$ $y_1(x) = x$, $y_2(x) = x^2$, $y_3(x) = x^3$

33. $x^3y^{(3)} + 6x^2y'' + 4xy' - 4y = 0$

$y(1) = 1$, $y'(1) = 5$, $y''(1) = -11$ $\qquad$ $y_1(x) = x$, $y_2(x) = x^{-2}$, $y_3(x) = x^{-2}\ln x$

34. When the values of a solution to a differential equation are specified at two different points, these conditions. (In contrast, initial conditions specify the values of a function and its derivative at the same point). The purpose of this is to show that for boundary value problems there is no existence-uniqueness theorem. Given that every solution to

$$y'' + y = 0 \qquad \text{is of the form} \qquad y(t) = c_1 \cos t + c_2 \sin t$$

Where c_1 and c_2 are arbitrary constants, show that

a) There is a unique solution to the given differential equation that satisfies the boundary conditions $y(0) = 2$ and $y\left(\frac{\pi}{2}\right) = 0$

b) There is no solution to given equation that satisfies $y(2) = 0$ and $y(\pi) = 0$

c) There are infinitely many solution to the given DE equation that satisfy $y(0) = 2$ and $y(\pi) = -2$