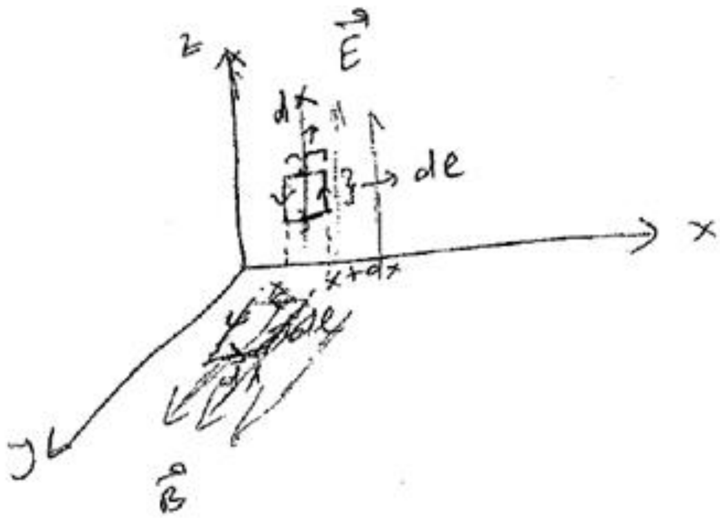
4.5 – Electromagnetic Wave

By combining his equations, Maxwell was able to show that whenever a charge is accelerated electrometric waves are propagated. In other words he showed that electric field and the magnetic field satisfy the wave equation. The full deviation of the wave equation is considering a special situation to obtain the one dimensional version of the wave equations.

At any point in space the electric field and the magnetic field are perpendicular to each other and to the direction of propagation of energy. Let's assume that the direction of the electric field is along the *z*-direction and the magnetic field is along the *y*-direction. Then the direction of the propagation of energy is along the *x*-direction



Let's apply Faraday's law over the small rectangle of width d*x* and length $d\ell$ on the *x*-*z* plane.

$$\oint \vec{E} \bullet d\vec{s} = -\frac{d\phi_B}{dt}$$

$$\phi_B = Bdt$$

But

$$A = dxd\ell = Bdxd\ell$$

$$\frac{d\phi_B}{dt} = dxd\ell \frac{\partial B}{\partial t} \qquad (eq.\ 1)$$

The line integral $\int \vec{E} \bullet d\vec{s}$ will be zero on the horizontal path's because the electric field and the path because the electric field and the perpendicular to each other. Since the rectangle is small, the electric field can be assumed to be constant on the vertical paths ($E(x)$) on the left and $E(x = dx)$ on the right one.

$$\oint \vec{E} \bullet d\vec{s} = E(x + dx)d\ell - E(x)d\ell$$

Using first term. Taylor expansion.

$$E(x + dx) = E(x) + \frac{\partial E}{\partial x}dx$$

$$\oint \vec{E} \cdot d\vec{s} = d\ell \left\{ E(x) + \frac{\partial E}{\partial x} dx - E(x) \right\}$$

$$\oint \vec{E} \cdot d\vec{s} = d\ell \cdot dx \frac{\partial E}{\partial x} \qquad (eq.\ 2)$$

Substituting equations 1 and 2 into Faraday's law.

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\phi_B}{dt}$$

$$d\ell \cdot dx \frac{\partial E}{\partial x} = -dxd\ell \frac{\partial B}{\partial t}$$

$$\frac{\partial E}{\partial x} = \frac{\partial B}{\partial t} \qquad (eq.\ 3)$$

Again applying Ampere's law without source (current) over a small rectangle of side's dx and dl another relationship between E and B.

Ampere's law

$$\oint \vec{B} \cdot d\vec{s} = \varepsilon_0 \mu_0 \frac{d\phi_E}{dt}$$

$$\phi_E = EA = Edxd\ell$$

$$\frac{d\phi_E}{dt} = dxd\ell \frac{\partial E}{\partial t} \qquad (eq.\ 4)$$

The line integrals in the paths along the *x*-axis will be zero because the magnetic field and the path are perpendicular to each other on the path's along the y-direction, the magnetic field can be taken to be approximately constant (i.e. B(*x*) on the left side and $B(x+dx)$ on the right side) because the rectangle is infinitely small.

$$\oint B \cdot d\vec{s} = B(x)d\ell - B(x+dx)d\ell$$

But

$$B(x+dx) \approx B(x) + \frac{\partial B}{\partial x} dx$$

$$\oint B \cdot d\vec{s} = -\frac{\partial B}{\partial x} dxd\ell \qquad (eq.\ 5)$$

Substituting equations 1 and 2 in Amperes law results

$$\oint \vec{B} \cdot \overrightarrow{ds} = \varepsilon_0 \mu_0 \frac{d\phi_E}{dt}$$

$$-\frac{\partial B}{\partial x} dxd\ell = \varepsilon_0 \mu_0 dxd\ell \frac{\partial E}{\partial t}$$

$$\frac{\partial B}{\partial x} = -\varepsilon_0 \mu_0 \frac{\partial E}{\partial t} \qquad (eq.\ 6)$$

Taking the derivative of the equation 6 with respect to x gives

$$\frac{\partial}{\partial x}\left(\frac{\partial B}{\partial x}\right) = -\varepsilon_0 \mu_0 \frac{\partial}{\partial x}\left(\frac{\partial E}{\partial t}\right)$$

$$\frac{\partial^2 B}{\partial x^2} = -\varepsilon_0 \mu_0 \frac{\partial E}{\partial x}\frac{\partial E}{\partial t}$$

But

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$$

$$\frac{\partial^2 B}{\partial x^2} = \varepsilon_0 \mu_0 \frac{\partial^2 B}{\partial t^2} \qquad (eq.\ 7)$$

Comparing with the wave equation $\left(\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}\right)$ the magnetic field propagates in space as a wave with speed $v = \sqrt{\frac{1}{\varepsilon_0 \mu_0}}$

Similarly taking the derivative of equation 3 with respect to x

$$\frac{\partial}{\partial x}\left(\frac{\partial E}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial B}{\partial t}\right) = -\frac{\partial}{\partial t}\frac{\partial B}{\partial x}$$

But

$$\frac{\partial B}{\partial t} = -\varepsilon_0 \mu_0 \frac{\partial B}{\partial t}$$

$$\frac{\partial^2 E}{\partial x^2} = \varepsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2} \qquad (eq.\ 8)$$

Again comparing with the wave equation, the electric field is propagated in space as a way with a speed $v = \sqrt{1/\varepsilon_0 \mu_0}$.

Equations 7 and 8 show that all electromagnetic waves travel with the speed $v = \sqrt{1/\varepsilon_0 \mu_0}$ which is constant with $\varepsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{n^2 N}$ and $\mu_0 = 4\pi \times 10^{-7} \frac{mT}{A}$

The volume of this speed turns out to be approximately $3 \times 10^8\ m/s$ which is the known speed of light denoted by C

$$\boxed{C = \sqrt{1/\varepsilon_0 \mu_0} \approx 3 \times 10^8\ m/s}$$

C is speed of all electromagnetic wave

This lead to the realization that light is actually a type of electromagnetic wave. Other examples of electromagnetic waves are ultraviolet rays, infrared waves, microwaves, x-rays, radio waves, gamma radiation and so on.

As shown before the speed of a wave is related to its wavelength and frequency (*or to its wave number and angular frequency*) as $v = \lambda f$ or $v = \frac{\omega}{k}$. For electromagnetic waves with $v = C = 3 \times 10^8\ {}^{m}/_{s}$

$$C = \lambda f = \frac{\omega}{k}$$

Example

Red light has a wavelength of $7 \times 10^{-7}\ m$. Calculate its frequency.

Solution

$$\lambda = 7 \times 10^{-7} m \qquad f = ?$$

$$C = f\lambda$$

$$f = \frac{C}{\lambda} = \frac{3 \times 10^8}{7 \times 10^{-7}} = \frac{3}{7} \times 10^{15}\ Hz$$

The most common solutions of the wave equation are harmonic waves. The harmonic wave solution for the electric field and magnetic field may be written as

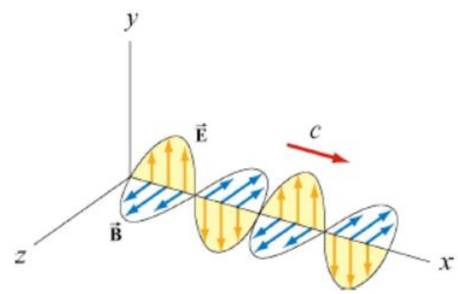
$$\boxed{E = E_{max} \cos(kx - \omega t)}$$

$$\boxed{B = B_{max} \cos(kx - \omega t)}$$

Where $\frac{\omega}{K} = C$

The ratio between these two equations shows that the ratio between the fields is equal to the ratio between amplitude

$$\frac{E}{B} = \frac{E_{max}}{B_{max}}$$



Substituting the harmonic solutions to equation 3

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$$

$$\frac{\partial}{\partial x}\{E_{max}\cos(kx - \omega t)\} = -\frac{\partial}{\partial t}\{B_{max}\cos(kx - \omega t)\}$$

$$kE_{max}\sin(kx - \omega t) = \omega B_{max}\sin(kx - \omega t)$$

$$kE_{max} = \omega B_{max}$$

$$\frac{\omega}{K} = \frac{E_{max}}{B_{max}}$$

But $\frac{E_{max}}{B_{max}} = \frac{E}{B}$ and $\frac{\omega}{K} = C \quad \Rightarrow \quad \boxed{\frac{E}{B} = C}$

The ratio between the electric field and the magnetic field is always equal; to the speed of light.

Example

An electromagnetic wave has a wavelength of $2\times10^{-6} m$. Its amplitude is $3\times10^{-3}\ N/C$.
Give the harmonic wave solution of this electromagnetic wave as a function of position and time.

Solution

Given: $\lambda = 2\times10^{-6}m,\quad E_{max} = 3\times10^{-3}\ {}^{N}/_{C}$

$$E = E_{max}\cos(kx - \omega t)$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{3\times10^{-3}}$$

$$\frac{1}{m} \approx 2\times10^{3}$$

$$\omega = KC = (2\times10^{3})(3\times10^{8}) = 6\times10^{11}\ {}^{rad}/_{s}$$

$$E = (3\times10^{-3}m)\cos\{(2\times10^{3})x - (6\times10^{11})t\}$$

Similarly, $B = B_{max}\cos(kx - \omega t)$

But $\dfrac{E_{max}}{B_{max}} = C$

$$B_{max} = \frac{E_{max}}{C} = \frac{3\times10^{-3}}{3\times10^{8}} = 10^{-11}\ T$$

$$\therefore\ B = \left(10^{-11}T\right)\cos\left\{\left(2\times10^{3}\right)x - \left(6\times10^{11}\right)t\right\}$$

Energy density of an electromagnetic wave

The energy density of an electromagnetic wave is the sum of the energy densities due to its electric field and magnetic field as shown in previous chapters, the energy densities due to electric field and magnetic field are respectively given by $u_E = \frac{1}{2}\varepsilon_0 E^2$ and $u_B = \frac{1}{2u_0}B^2$. Therefore the electromagnetic energy density u is given by

$$u = u_E + u_B = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2u_0}B^2$$

But $$B = \frac{E}{C} = \frac{E}{1/\sqrt{\varepsilon_0 \mu_0}} = \sqrt{\varepsilon_0 \mu_0}E$$

$$u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}\varepsilon_0 \mu_0 E^2$$

$$= \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\varepsilon_0 E^2$$

$$\boxed{u = \varepsilon_0 E^2}$$

Also substituting $E = \sqrt{\varepsilon_0 \mu_0}\ B$, u can be expressed in terms of B as

$$\boxed{u = \frac{B^2}{\mu_0}}$$

This is instantaneous energy density. The average of the density over one or more of its periods may be evaluated for a harmonic wave as follows. If

$$E = E_{max}\cos(kx - \omega t)$$

$$E^2 = E^2_{max} cos^2(kx - \omega t)$$

(The line on p means average)

$$cos^2(kx - \omega t) = \frac{1}{T}\int_0^T (kx - \omega t)dt = \frac{1}{2}$$

(T is its period)

$$\therefore\ E^2 = \frac{1}{2}E^2_{max}$$

$$\boxed{\bar{u} = \frac{1}{2}\varepsilon_0 E^2_{max} \quad or \quad \bar{u} = \frac{1}{2\mu_0}B^2_{max}}$$

$\bar{u}$ → Average energy density of an electromagnetic wave

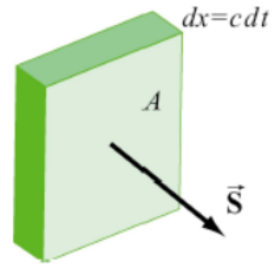
$E_{\max}\left(B_{\max}\right)$ → Amplitude of electric (magnetic) field

Poynting Vector

Poynting Vector is a vector that represents the amount of electromagnetic energy that crosses a unit ⊥ are per a unit time. Its direction is the direction of the propagation of energy which is perpendicular to both the electric and magnetic field. The pointing vector $\vec{S}$ is given in terms of the electric and magnetic field as

$$\boxed{\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}}$$

Since $\vec{E}$ and $\vec{B}$ are ⊥ to each other (remember $|\vec{A}\times\vec{B}| = AB\sin(\theta)$), the magnitude of $\vec{S}$ is given by



$$S = \frac{1}{\mu_0}EB\sin\left(90^\circ\right) = \frac{EB}{\mu_0}$$

Substituting for B from $B = \frac{E}{C}$

$$\boxed{S = \frac{E^2}{\mu_0 C}}$$

Or substituting for $E = CB$

$$\boxed{S = \frac{C}{\mu_0}B^2}$$

As shown before, assuming a harmonic wave, $\overline{E^2} = \frac{1}{2}E_{max}^2 \quad \overline{B^2} = \frac{1}{2}B_{max}^2$

$$\boxed{\bar{S} = \frac{1}{2}\frac{E_{max}^2}{\mu_0 C} = \frac{1}{2}\frac{C}{\mu_0}B_{max}^2}$$

$\bar{S} \rightarrow$ Average amount of electromagnetic energy that crosses a unit ⊥ area per unit time

$$\therefore E = \left(4\times10^{-6}\ {}^{W}\!/_{C}\right)\ \cos\left\{1.67\times10^{3}x - 4.9\times10^{11}t\right\}$$

$$B = B_{max}\cos\left(kx - \omega t\right)$$

$$B_{max} = \frac{E_{max}}{C} = \frac{4\times10^{-6}}{3\times10^{8}} = \underline{1.3\times10^{-14}\ T|}$$

$$\therefore\ B = \left(1.3\times10^{-14}T\right)\ \cos\left\{1.63\times10^{3}x - 4.9\times10^{11}t\right\}$$

Calculate the average energy density for this radiation

$$\bar{u} = \frac{\varepsilon_0 E_{max}^2}{2} = \frac{\left(8.85\times10^{-12}\right)\left(4\times10^{-6}\right)}{2}\ \underline{\approx 7.08\times10^{-23}\ {}^{J}\!/_{m^3}|}$$

Calculate the average of electromagnetic energy that crosses a unit perpendicular are per a unit time $\bar{s}$ =?

$$\bar{s} = c\bar{u} = \left(3\times10^{8}\right)\left(7.08\times10^{-23}\right)\ \underline{\approx 2.1\times10^{-15}\ {}^{w}\!/_{m^2}|}$$

Properties of Electromagnetic Waves with Light as an Example

Reflection of Light

Reflection of light is the bouncing of light from a surface. The light ray that hits the surface is called incident ray. The light ray reflected from the surface is called the reflected ray. The line perpendicular to the surface at the point of impact is called the normal line. The angle formed between the incident ray and the normal line is called the angle of incidence. The angle formed between the reflected ray and the normal line is called the angle of reflection. The ***law of reflection*** states that the angle of incidence (θ_i) and the angle of reflection (θ_r) are equal.

$$\theta_i = \theta_r$$

Example

A light ray is incident on a flat mirror. The angle formed between the surface of the mirror and the incident ray is *25°*. Calculate the angle of reflection.

Solution

Since the angle of incidence is angle formed with the normal and the incident ray makes an angle of 25° with the surface, the angle of incidence is 90° – 25° = 65°.

$\theta_i = \boxed{65° = \theta_r}$

Example

Two mirrors are connected at an angle of 120°. A light ray is incident on one of the mirrors at an angle of incidence of 70°. Calculate its angle of reflection on the second mirror.

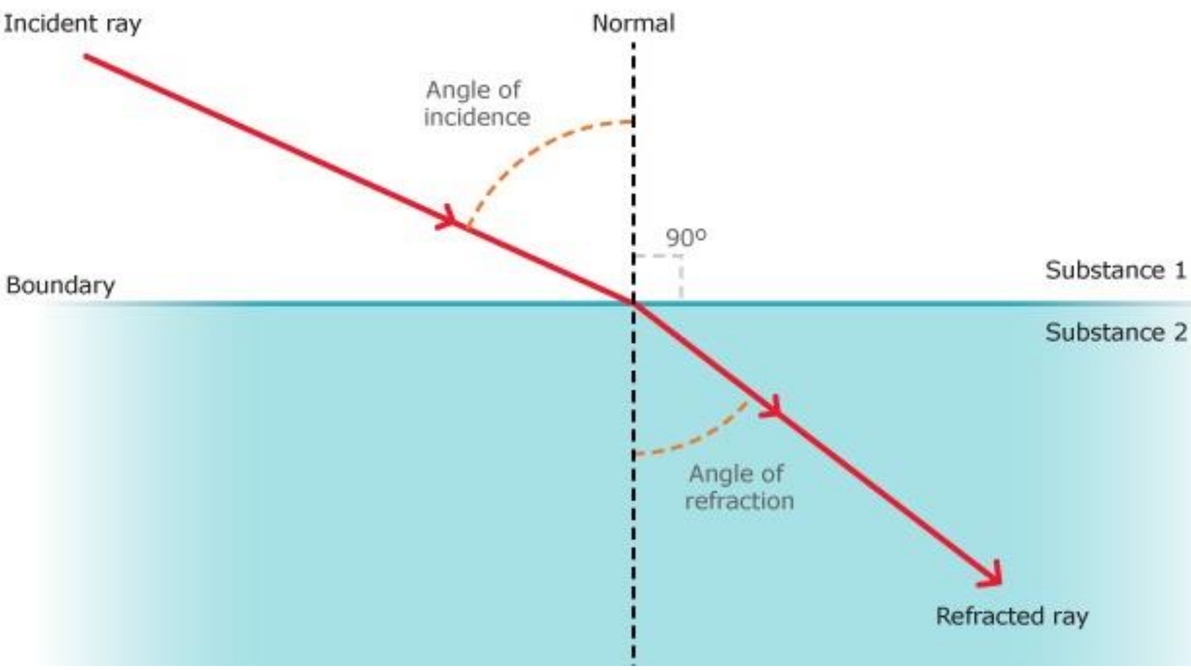
Solution

Since the angle of incidence on the first mirror is 70°, the angle of reflection on the first mirror is 70°. The angle formed between the surface of the first mirror and the reflected ray is $90° - 70° = 20°$. The reflected ray will continue to hit the second mirror and reflected. The two surfaces of the mirror and the path of the light ray from the first to the second mirror form a triangle. The angle formed between the second mirror and the light ray incident on the second mirror is $180° - (120° + 20°) = 40°$. The angle of incidence on the second mirror is $90° - 40° = 50°$

Therefore the angle of reflection on the second mirror is 50°

Refraction

Refraction is the bending of light as light crosses the boundary between two mediums. The light ray incident on the boundary is called incident ray. The ray past the boundary is called the refracted ray. The line that is perpendicular to the boundary at the point of impact is called the normal line. The angle formed between the incident ray and the normal line is called the angle of incidence. The angle formed between the refracted ray and the normal line is called the angle of refraction.



As light enters a medium from a vacuum (or approximately air), the speed and the wavelength of the light decrease while the frequency remains the same. The ratio between the speed of light (c) in vacuum and the speed of light (v) in a medium is called the ***refractive index*** (n) of the medium.

$$n = \frac{c}{v}$$

Refractive index is unit-less. From this definition of refractive index, it is clear that the refractive index $\left(n_g\right)$ of vacuum (air) is one and the refractive index of any other medium is greater than one. The refractive index of water $\left(n_w\right)$ and glass $\left(n_g\right)$ are $\frac{4}{3}$ and $\frac{3}{2}$ respectively. If the wavelength of light is λ_v in vacuum and λ_w in a medium, since frequency remains the same $n = \frac{c}{v} = \frac{f\lambda_v}{f\lambda_w}$ and

$$\boxed{n = \frac{\lambda_v}{\lambda_w}}$$

Example

Calculate the speed of light in glass.

Solution

Given: $n_g = 1.5, \quad Find\ v_g$

$$v_g = \frac{c}{n_g} = \frac{3\times10^8}{1.5} \underline{= 2\times10^8\ m/s|}$$

Example

The wavelength of violet light in vacuum is 400 nm. Calculate its wavelength in water.

Solution

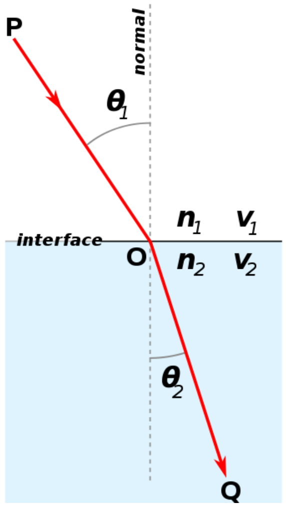
$$\lambda_v = 4\times10^{-7} m, \quad n_w = \frac{4}{3}$$

$$\lambda_w = \frac{\lambda_v}{n_w}$$

$$= \frac{4\times10^{-7}}{\frac{4}{3}}$$

$$\underline{= 3\times10^{-7}\ m|}$$

Snell's law (law of refraction) states that the ratio between the sine of the angle of incidence and the sine of the angle of refraction is equal to the ratio between the speeds of light in the respective mediums. If light enters medium **2** from medium **1** at an angle of incidence θ_1 and the angle of refraction in medium **2** is θ_2, then

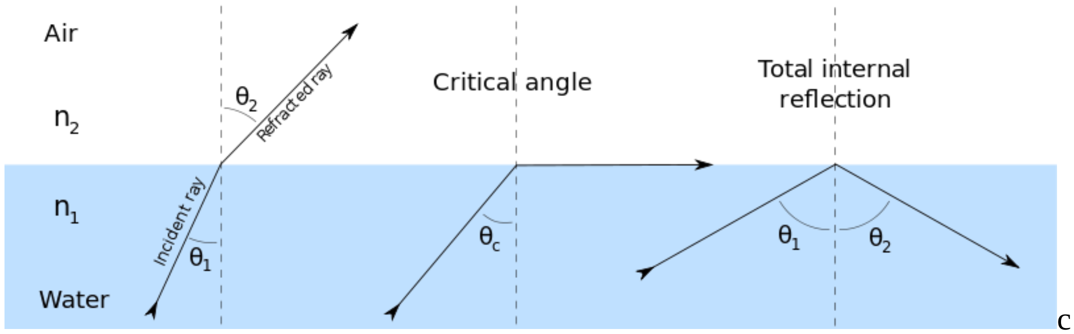


$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} = \frac{c_1/n_1}{c_2/n_2} = \frac{n_2}{n_1}$$

Therefore, Snell's can be mathematically expresses as

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$

It can be easily deduced from this equation that light bends towards the normal as it enters an optically denser (higher refractive index) medium and bends away from the normal as it enters an optically less dense medium. A light ray perpendicular to the boundary passes straight unbent.



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Example

A light ray enters water (from air) at an angle of incidence 56°. Calculate the angle of refraction in water.

Solution

Given: $n_a = 1, \quad n_w = \frac{4}{3}, \quad \theta_a = 56°$ ***Find*** $\theta_w = ?$

$$n_w \sin\theta_w = n_a \sin\theta_a$$

$$\frac{4}{3}\sin\theta_w = 1\sin 56°$$

$$\sin\theta_w = \frac{3}{4}\sin 56° \quad \Rightarrow \quad \boxed{\theta_w = \arcsin\left(\frac{3}{4}\sin 56°\right) = 38.3°}$$

Example

A light ray enters water (from air) on glass at an angle of incidence of 65°. Calculate the refraction angle in glass. (The air-water boundary and the water-glass boundary are parallel)

Solution

First the angle of refraction in water should be calculated from the air-water boundary. The angle of refraction in water and the angle of incidence on the water-glass boundary are equal because they are alternate interior angles. Then the angle of refraction in glass can be obtained from the water-glass boundary.

Given: $n_a = 1, \quad n_w = \frac{4}{3}, \quad n_g = 1.5, \quad \theta_a = 65°$

Find $\theta_g = ?$

$$n_w \sin\theta_w = n_a \sin\theta_a$$

$$\frac{4}{3}\sin\theta_w = (1)\sin 65°$$

$$\sin\theta_w = 0.7$$

$$\Rightarrow \quad \boxed{\theta_w = \arcsin(0.7) = 44.4°}$$

$$n_g \sin\theta_g = n_w \sin\theta_w$$

$$(1.5)\sin\theta_g = \frac{4}{3}\sin 44.4°$$

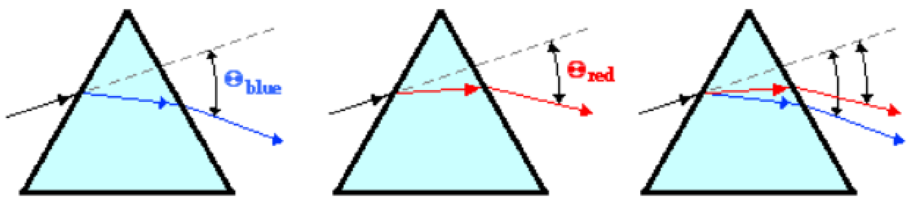
$$\sin\theta_g = \frac{4}{3}\frac{\sin 44.4°}{1.5} = 0.5$$

$$\Rightarrow \quad \boxed{\theta_g = \arcsin(0.5) = 30°}$$

Dispersion of Light

Dispersion is the separation of white light into different colors as light enters a medium from air. White light is composed of seven different colors. Arranged in increasing order of wavelength. These are violet, indigo, blue, green, yellow, orange and red (Abbreviated as VIBGYOR). Violet has the shortest wavelength and red has the longest wavelength. The reason white separates into its component colors as light enters a medium is because the refractive index of a medium depends on the wavelength of the light. The refractive index increases as the wavelength decreases. That is, violet has the largest refractive index and red has the smallest refractive index. According to Snell's law, the greater the refractive index the small the angle of refraction or the greater the deviation angle from the path of the incident light. Thus, as light enters a medium from air, violet light will be bent by the largest angle and red light will be bent by the smallest angle.

A good example of dispersion is the rainbow. Rainbow happens because the cloud has different refractive indexes for the different colors of light.



Example

White light enters glass (from air) at an angle of incidence of 65°. The refractive indices of the glass for violet and red light are respectively 1.52 and 1.48. Calculate the angle formed between red light and violet light after refraction.

Solution

Given: $n_g = 1, \quad n_{gr} = 1.48, \quad n_{gy} = 1.52, \quad \theta_a = 65°$

Find $\theta = \theta_{gr} - \theta_{gy} = ?$

$$n_{nr} \sin\theta_{gr} = n_g \sin\theta_g$$

$$1.48 \sin\theta_{gr} = (1)\sin 65° \quad \Rightarrow \quad \boxed{\theta_{gr} = \arcsin(0.612) = 37.8°}$$

$$n_{ny} \sin\theta_{gy} = n_g \sin\theta_g$$

$$1.42 \sin\theta_{gy} = (1)\sin 65° \quad \Rightarrow \quad \boxed{\theta_{gr} = \arcsin(0.596) = 36.6°}$$

$$\theta = \theta_{gr} - \theta_{gy}$$
$$= 37.8° - 36.6°$$
$$\boxed{= 1.2°}$$

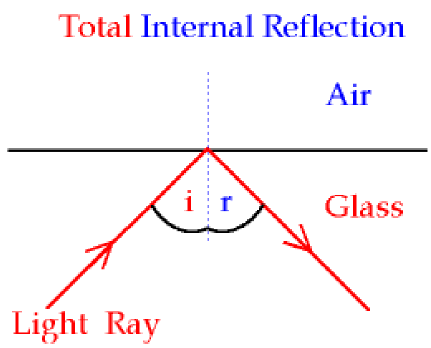
Total Internal Reflection

As light enters an optically less dense medium, it bends away from the normal. As the angle of incidence in the more dense medium is increased, for a certain angle the angle of refraction will be 90°; that is, the light ray will be refracted parallel to the boundary. The angle of incidence in the more dense medium for which the angle of refraction is 90° is called the ***critical angle*** (θ_c) of the boundary.

If the refractive indexes of the less dense and more dense medium are $n_<$ and $n_>$ respectively, then $n_> \sin\theta_c = n_< \sin 90° = n_<$; and the critical angle of the boundary is given as follows.

$$\theta_c = \arcsin\left(\frac{n_<}{n_>}\right)$$

For angles of incidence less than the critical angle, both reflection and refraction take place. The fact that we can see our face in water indicates that some of the light rays are reflected; and the fact that we can see objects inside water indicates that some of the light rays are refracted. But for angles greater than the critical angle. Only reflection takes place.



Total internal reflection is a phenomenon where only reflection takes place at the boundary between two mediums and occurs only when light enters a less dense medium at an angle of incidence greater than the critical angle.

Example

Calculate the critical angle for the boundary formed between

a) Air & water.
b) Water & glass

Solution

a) Given: $n_< = n_a = 1, \quad n_w = n_> = \frac{4}{3}$ ***Find*** $\theta_c = ?$

$$\theta_c = \arcsin\left(\frac{n_<}{n_>}\right) = \arcsin\left(\frac{1}{4/3}\right) = 48.6°$$

b) Given: $n_< = n_g = \frac{4}{3}, \quad n_> = n_g = 1.5$ ***Find*** $\theta_c = ?$

$$\theta_c = \arcsin\left(\frac{n_<}{n_>}\right) = \arcsin\left(\frac{4/3}{1.5}\right) \underline{= 62.8°}$$

Example

A light source is placed in water at a depth of 0.5 m. Calculate the radius of a circular region at the surface of the water from which light rays come out.

<u>Solution</u>

Light rays from the light source will hit the air-water boundary at different angles of incidence. Only the light rays whose angle of incidence is less than the critical angle can be refracted into air. The light rays whose angle of incidence is greater than the critical angle will be reflected back because total internal reflection takes place. Because of this, light rays will come out of a certain circular region of the surface only. The angle of incidence for the light rays that fall on the boundary of this circular region is equal to the critical angle of the boundary. The radius can be calculated from the right angled triangle formed by a line connecting the source with the center of the circle (a), the line that connect the center of the circle to a point on the boundary of the circular region (r) and the line connecting the source to the point on the boundary. The first two lines are perpendicular to each other and the angle formed between the last two lines is the critical angle.

Thus $\theta_c = \arctan\left(\frac{r}{a}\right)$.

Given: $n_< = n_g = 1, \quad n_> = n_g = \frac{4}{3}$ ***Find*** $r = ?$

$$\theta_c = \arcsin\left(\frac{n_<}{n_>}\right) = \arcsin\left(\frac{1}{4/3}\right) \underline{= 48.6°}$$

$$\tan\theta_c = \frac{r}{a}$$

$$\Rightarrow \underline{r} = a\tan\theta_c = 0.5\tan(48.6°) \underline{= 0.57\ m}$$

Example

A light ray is incident on one of the legs of a 45° right angled glass prism perpendicularly. Trace the path of the light ray.

<u>Solution</u>

Since the light ray is perpendicular to the surface (angle of incidence zero), it will enter undeflected. Then it will be incident on the glass-air boundary on the larger face (hypotenuse). Since it is a 45° prism, from simple geometry, it can be shown that the angle of incidence on this boundary is 45°. Since the light ray is incident on the denser medium (glass), what happens depends on the critical angle of the boundary. If the critical angle is greater than 45° it can be refracted. But if the critical angle is less than 45°, total internal reflection will take place and the light ray will be incident on the other leg of the prism perpendicularly (as can be shown by simple geometry) and will be refracted to air undeflected.

Given: $n_{>} = n_g = 1.5, \quad n_{<} = n_g = 1 \qquad r = ?$

$$\theta_c = \arcsin\left(\frac{n_{<}}{n_{>}}\right) = \arcsin\left(\frac{1}{1.5}\right) \underline{= 41.8°}$$

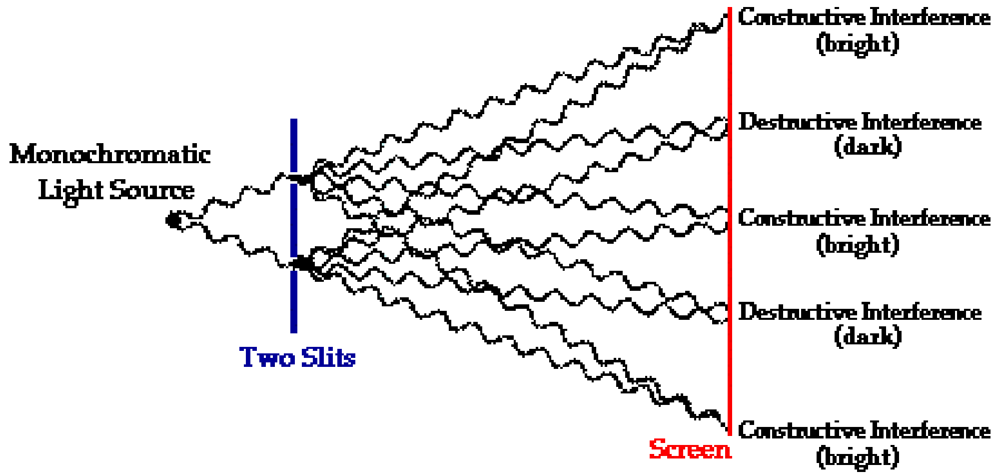
Since the critical angle is less than the angle of incidence at the hypotenuse, total internal reflection takes place and the light ray is incident on the other leg perpendicularly and is reflected to air deflected.

Interference of Light

Interference of Light is the meeting of two or more light waves at the same point at the same time. The net instantaneous effect of the interfering waves is obtained by adding the instantaneous values of the waves algebraically. The net effect of the interfering waves

$$y_1 = A_1 \cos\left(\omega t - kx_1\right) \quad and \quad y_2 = A_2 \cos\left(\omega t - kx_2\right)$$

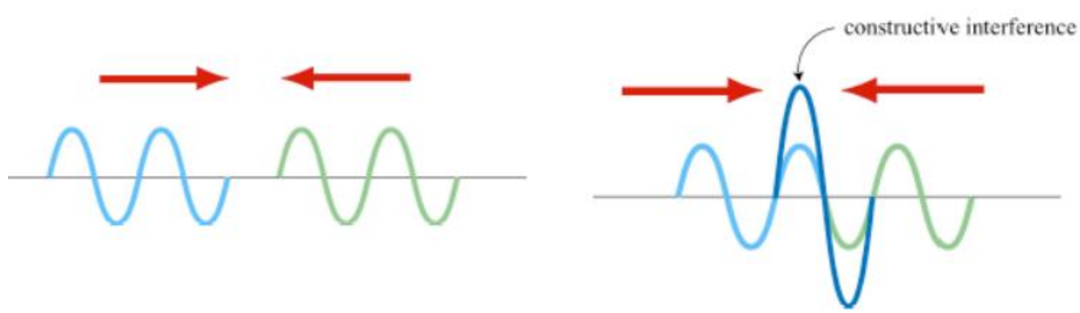
is given as $y_{net} = y_1 + y_2 = A_1 \cos\left(\omega t - kx_1\right) + A_2 \cos\left(\omega t - kx_2\right)$



Constructive interference is interference with the maximum possible effect. For light waves, constructive interference results in a bright spot. The amplitude of the net wave of two interfering waves is equal to the sum of the amplitudes of the interfering waves. It occurs when the phase shift (δ) of the interfering waves is an integral multiple of 2π. The following equation is the condition for constructive interference.

$$\delta_n = 2n\pi$$

where n is an integer; that is, n is a member of the set { . . ., –2, –1, 0, 1, 2, . . . }. and δ_n is a member of the set { . . ., -4π, -2π, 0, 2π, 4π, . . . }.



Destructive interference is interference with the minimum possible effect. For light waves, destructive interference results in a dark spot. The amplitude of the net wave of two interfering waves is equal to the difference between the amplitudes of the interfering waves. It occurs when the phase shift between the interfering waves is an odd-integral multiple of π. The following equation is the condition for destructive interference.

$$\delta_n = (2n+1)\pi$$

where n is integer; that is n is a member of the set { . . ., –2, –1, 0, 1, 2, 3, . . . } and δ_n is a member of the set { . . ., –3π, –π, π, 3π, . . . }.

Example

Determine if the following waves will interfere constructively, destructively, or neither constructively nor destructively.

a) $y_1 = 5\cos(20t+\pi)$ *and* $y_2 = 7\cos(20t+4\pi)$

b) $y_1 = 30\cos\left(40t+\frac{\pi}{2}\right)$ *and* $y_2 = 80\cos(40t+7\pi)$

c) $y_1 = 2\cos\left(50t-\frac{5\pi}{2}\right)$ *and* $y_2 = 80\cos\left(50t-\frac{\pi}{2}\right)$

<u>Solution</u>

Solution: $\beta_1 = \pi$; $\beta_2 = 4\pi$; $\delta = ?$

$$\delta = \beta_2 - \beta_1 = 4\pi - \pi = 3\pi$$

The two waves will interfere destructively because $\delta = 3\pi$ is a member of the set *{ . . . -3π, -π, π, 3π, . . . }*

Solution: $\beta_1 = \pi/2$; $\beta_2 = 7\pi$; $\delta = ?$

$$\delta = \beta_2 - \beta_1 = 7\pi - \pi/2 = 13\pi/2$$

The two waves will interfere neither constructively nor destructively because $\delta = 3\pi$ is not a member of the set *{ . . . -3π, -π, π, 3π, . . . }* or the set *{ . . . -4π, -2π, 0, 2π, 4π, . . . }*

Solution: $\beta_1 = -5\pi/2$; $\beta_2 = -\pi/2$; $\delta = ?$

$$\delta = \beta_2 - \beta_1 = -\pi/2 - -5\pi/2 = 2\pi$$

The two waves will interfere constructively because $\delta = 2\pi$ is a member of the set *{ . . . -4π, -2π, 0, 2π, 4π, . . . }*

The conditions of constructive and destructive interference can also be expressed in terms of path difference (difference between the distance travelled by the two waves) between the two waves. If the two interfering waves are given as $y_1 = A_1 \cos(\omega t - kx_1)$ and $y = A_2 \cos(\omega t - kx_2)$ (where $k = 2\pi/\lambda$), then the phase shift between the two waves is $\delta = 2\pi x_2/\lambda - 2\pi x_1/\lambda = (2\pi/\lambda)(x_2 - x_1) = (2\pi/\lambda)\Delta$ where $\Delta = x_2 - x_1$ is the path difference between the two waves. The condition of constructive interference may be writen in terms of path difference as $\delta_n = 2n\pi = (2\pi/\lambda)\Delta_n$. This implies that the path difference between two waves has to satisfy the following condition for constructive interference.

$$\Delta_n = n\lambda$$

where n is an integer; that is n is a member of the set $\{\ldots -2, -1, 0, 1, 2, \ldots\}$ and Δ_n is a member of the set $\{\ldots -2\lambda, -\lambda, 0, \lambda, 2\lambda, \ldots\}$. Two waves will interfere constructively if their path difference is an integral multiple of the wavelength of the waves.

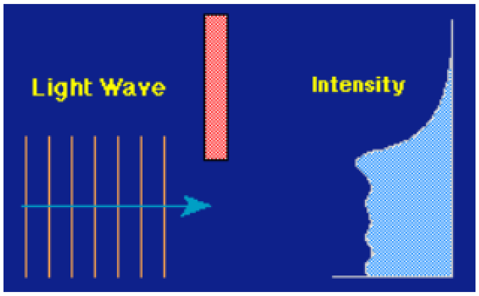
The condition of destructive interference can be written in terms of path difference as $\delta_n = (2n + 1)\pi = (2\pi/\lambda)\Delta_n$. This implies that the path difference between two waves has to satisfy the following condition if the waves are to interfere destructively.

$$\Delta_n = (n + 1/2)\lambda$$

where n is integer; that is n is a member of the set $\{\ldots -2, -1, 0, 1, 2, \ldots\}$ and Δ_n is a member of the set $\{\ldots -3\lambda/2, -\lambda/2, \lambda/2, 3\lambda/2, \ldots\}$. Two waves will interfere destructively if their path difference is half-odd-integral multiple of the wavelength.

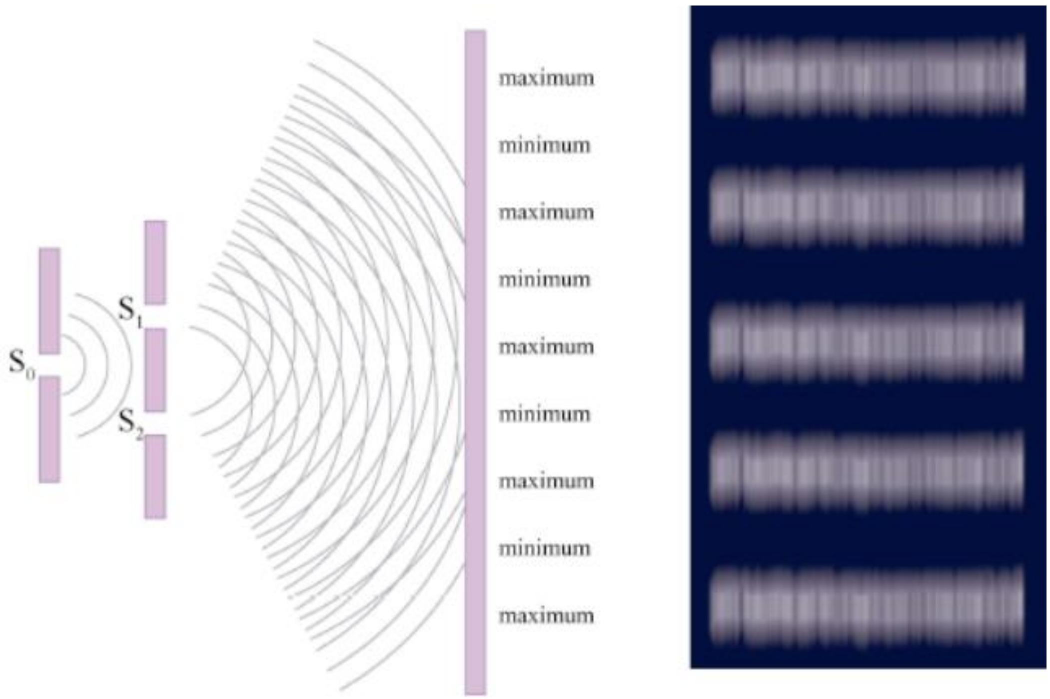
Diffraction of Light

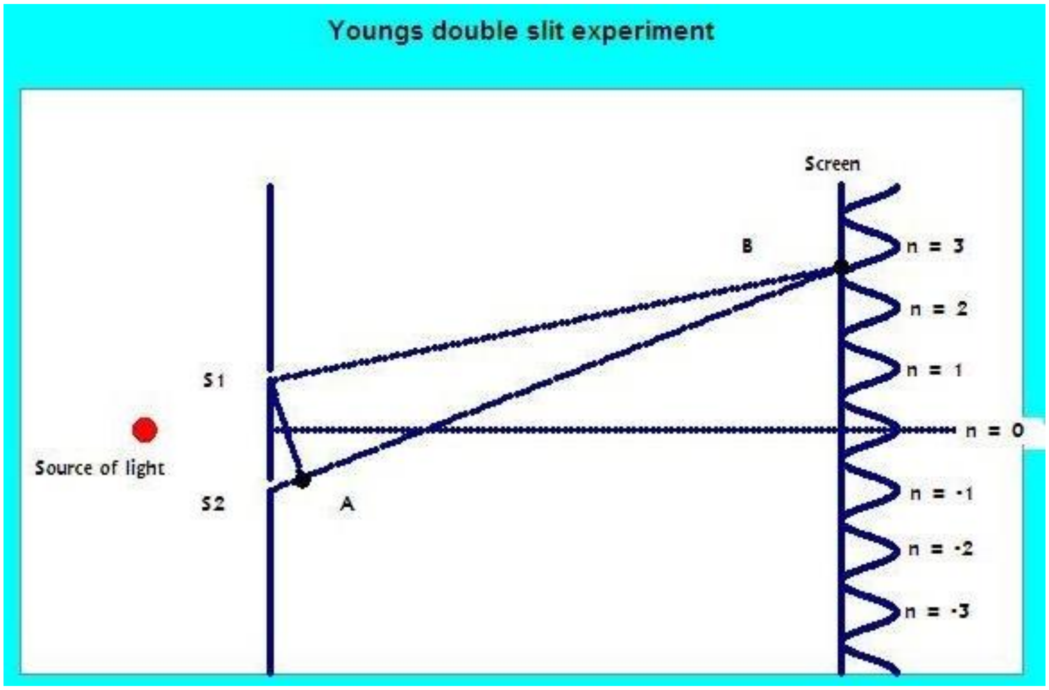
Diffraction of light is the bending of light as light encounters an obstacle. Light travels in a straight line. But when light encounters an obstacle it scatters in all directions. When light is blocked by an paque object, it can be still seen behind the opaque object because of diffraction of light at the edges of the object. A large part of a room can be seen through a key hole even though light travels in straight lines. This is because of diffraction of light at the key hole which bends the light. This property of light enables one to use a narrow slit as a source of light because as light crosses the slit it is scattered in all directions.



Young’s Double Slit Experiment

The following diagram shows the setup of Young's double slit experiment.





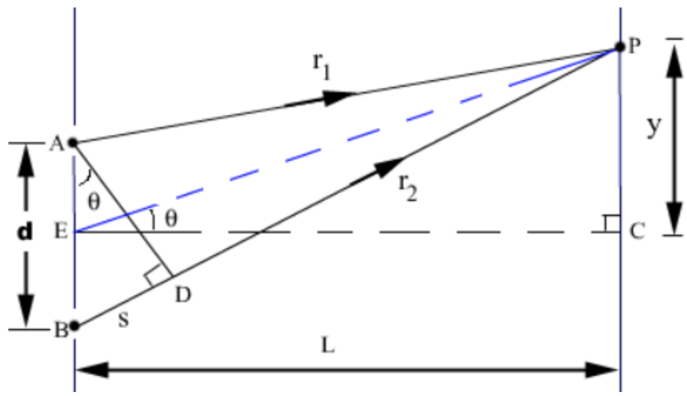
Young's double slit involves of an opaque material with two very narrow slits (s_1 and s_2 in the diagram) and a screen at some distance from this material. When the opaque material is exposed to a source of light, the two slits serve as two sources of light because light is diffracted in all directions as it passes through the slits. The light waves from the two slits interfere on the screen. The diagram shows light waves from the two slits interfering at point B. The experiment shows that bright (constructive interference) and dark (destructive interference) spots appear alternatively on the screen. The graph on the screen is a representation of the intensity of light observed on the screen. The interference pattern observed on the screen depends on the path difference between the two waves. If the path difference is an integral multiple of the wavelength of the light, the two waves interfere constructively and a bright spot is observed. If the path difference is half-odd-integral multiple of the wavelength of the light, destructive interference takes place and a dark spot is observed. At the center of the screen the two waves travel the same distance and the path difference is zero which implies constructive interference and a bright spot is observed. This corresponds to $n = 0$ and is called the zeroth order. As one goes further from the center, the path difference between the waves increases. At a point where the path difference is half of the wavelength, destructive interference takes place and a dark spot is observed. This way, as the path difference alternates between integral multiples of the wavelength and half-odd-integral multiples of the wavelength, the interference pattern alternates between bright spots and dark spots. The n^{th} bright spot is called the n^{th} order bright spot.

The path difference between the waves from slit s_1 and slit s_2 can be obtained by dropping the perpendicular from slit s_1 to the light wave from slit s_2 (the line joining slit s1 and point A in the diagram). Then the path difference (δ) is the distance between slit s_2 and point A. If the distance between the slits is d and the angle formed between the line joining the slits and the line joining slit s_1 and point A is θ (This angle is also equal to the angle formed by the line joining the mid-point of the slits to point B with the horizontal), then the path difference is given as $\delta = d \sin\theta$. Therefore the condition for constructive interference (bright spot) for Young's double slit experiment is

$$d\sin\theta = n\lambda$$

where n is an integer and λ is the wavelength of the light. Similarly, the condition for destructive interference (dark spot) is

$$d\sin\theta = \left(n + \frac{1}{2}\right)\lambda$$



Example

In Young's double slit experiment, the slits are separated by a distance of $2\times10^{-9}\,m$. The second order bright spot is observed at an angle of 26°. Calculate the wavelength of the light.

Solution

Given: $d = 2\times10^{-9}\,m, \quad \theta = 26°, \quad n = 2$ **Find**: $\lambda = ?$

$$d\sin\theta = n\lambda \quad \Rightarrow \quad \lambda = \frac{d\sin\theta}{n}$$

$$\boxed{\lambda = \frac{2\times10^{-9}\sin 26°}{2} = 4.4\times10^{-9}}$$