Section 3.5 – Linear Approximation / Mean Value Theorem

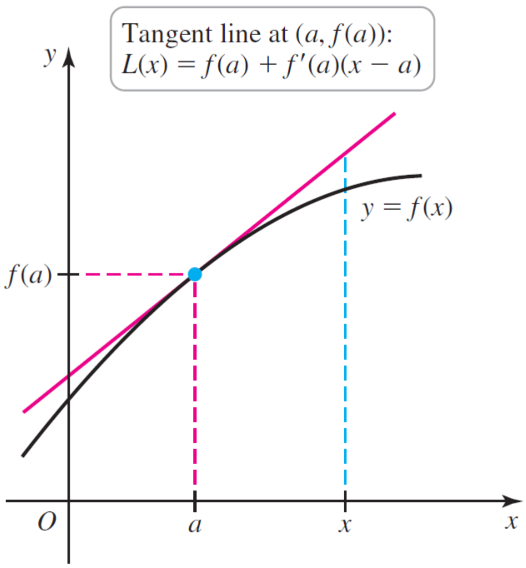
A line tangent to a graph of a function f at a point $(x, f(x))$ is used to approximate the value of f at points near x.

Linear Approximation

Definition

Suppose f is differentiable on an interval I containing the point a. The linear approximation to f at a is the linear function

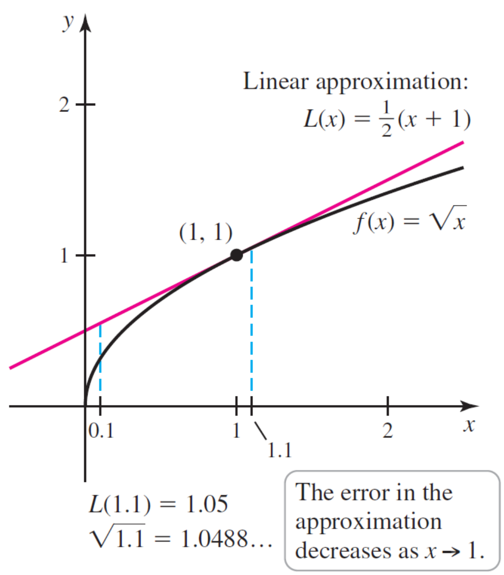
$$L(x) = f(a) + f'(a)(x-a), \quad \textit{for } x \textit{ in } I$$



Example

Find the linear approximation to $f(x) = \sqrt{x}$ at $x = 1$ and use it to approximate $\sqrt{1.1}$

Solution



$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= f(1) + f'(1)(x-1) \\ &= 1 + \frac{1}{2}(x-1) \end{aligned}$$

$$\begin{aligned} \sqrt{1.1} \approx L(1.1) &= 1 + \frac{1}{2}(1.1-1) \\ &= 1.05 \end{aligned}$$

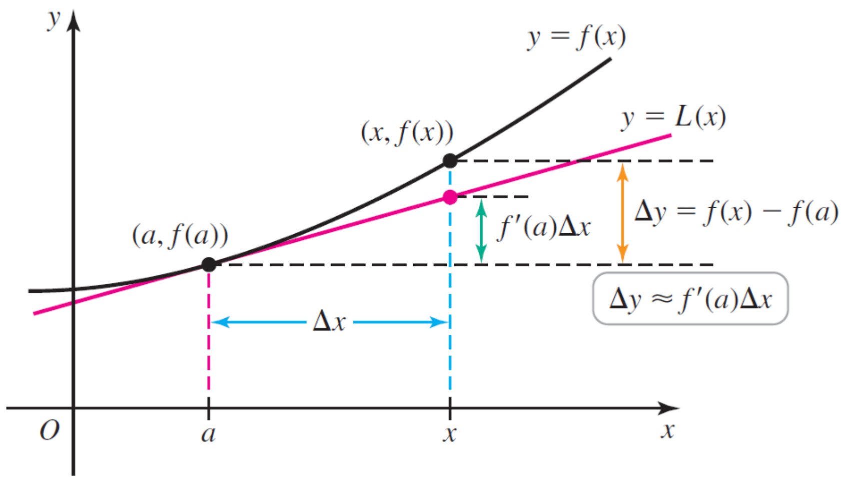
The exact value $\sqrt{1.1}$ ≈ 1.0488

$$Error = \frac{1.05 - 1.0488}{1.0488} = 0.001144 \quad (0.11\%)$$

Relationship Between Δx and Δy

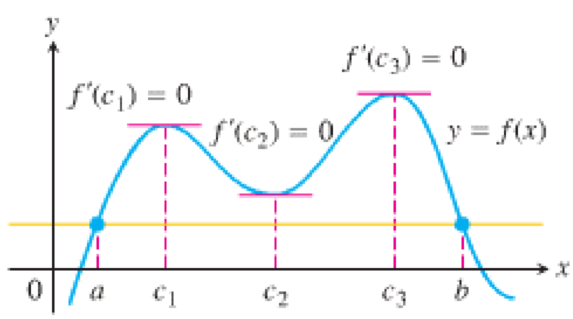
Suppose f is differentiable on an interval *I* containing the point a. The change in the value of f between two points a and $a+\Delta x$ is approximately

$$\Delta y \approx f'(a)\Delta x \qquad \textit{where } a+\Delta x \textit{ is in } I.$$



Rolle's Theorem

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least on number c in (a, b) at which $f'(c) = 0$



Proof

Being continuous, f assumes absolute maximum and minimum values on $[a, b]$. These can occur only

1. At interior points where f' is zero,
2. At interior points where f' does not exist,
3. At the endpoints of the function's domain, in this case a and b.

By hypothesis, f has a derivative at every interior point. That rules out possibility (2), leaving us with interior points where $f' = 0$ and with the two endpoints a and b.

If either maximum or the minimum occurs at a point c between a and b. then $f' = 0$.

If both the absolute maximum and the absolute minimum occur at the endpoints, then because $f(a) = f(b)$ it must be the case that f is a constant function with $f(x) = f(a) = f(b)$ for every $x \in [a, b]$. Therefore $f'(x) = 0$ and the point c can be taken anywhere in the interior (a, b).

Example

Show that the equation $x^3 + 3x + 1 = 0$ has exactly one real solution.

Solution

$f(x) = x^3 + 3x + 1$

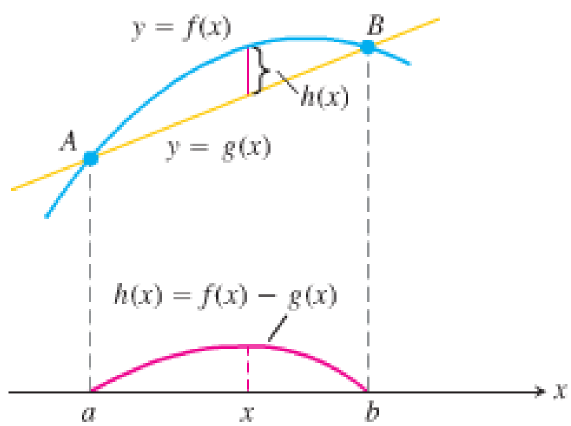
$f(-1) = -3$ and $f(0) = 1$, the Intermediate Value Theorem the equation has one real solution in the open interval $(-1, 0)$.

$f'(x) = 3x^2 + 3 > 0$ (***always positive***). Rolle's Theorem would guarantees the existence of a point $x = c$ in between them where f' was zero. Therefore, f has no more than one zero.

The Mean Value *Theorem*

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



Example

The function $f(x)=x^2$ is continuous for $0 \le x \le 2$ and differentiable for $0 < x < 2$. Since $f(0)=0$ and $f(2)=4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x)=2x$ must have the value $\frac{4-0}{2-0}=2$. In this case we can identify c by solving the equation $2c = 2$ to get $c = 1$. However, it is not always easy to find c algebraically, even though we know it always exists.

If $f'(x)=0$ at each point x of an open interval (a, b), then $f(x)=C$ for all $x \in (a, b)$, where C is a constant.

Corollary

If $f'(x)=g'(x)$ at each point x of an open interval (a, b), then there exists a constant C such that $f(x)=g(x)+C$ for all $x \in (a, b)$. That is, $f-g$ is a constant function on (a, b).