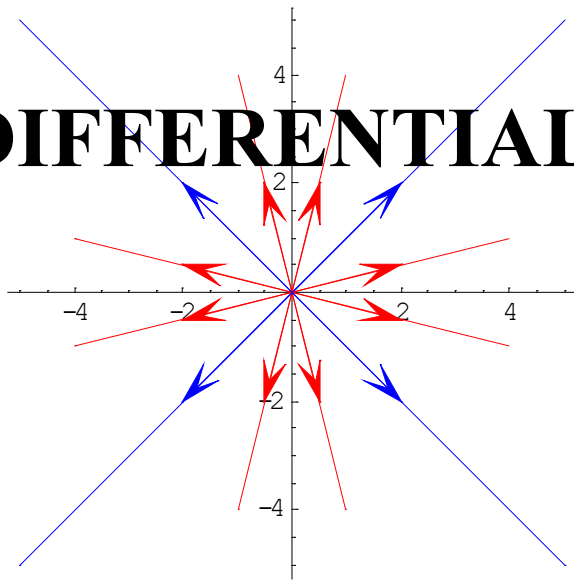
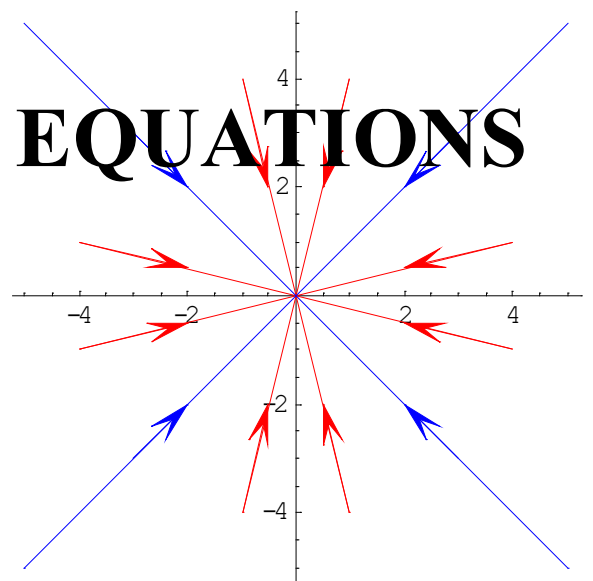


DIFFERENTIAL



EQUATIONS



Solution Manual

Volume 4

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Lecture 4 – Series

Section 4.1 – Introduction and Review of Power Series	1893
Section 4.2 – Series Solutions Near Ordinary Points	1921
Section 4.3 – Legendre’s Equation	2037
Section 4.4 – Solution about Singular Points	2051
Section 4.5 – Bessel’s Equation and Bessel Functions	2155

Solution

Section 4.1 – Introduction and Review of Power Series

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^{2n}}{\sqrt{n+1}}$

Solution

$$\begin{aligned}
 R &= \lim_{n \rightarrow \infty} \left| \frac{1}{\sqrt{n+2}} \cdot \sqrt{n+1} \right| & R &= \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n}} & & \\
 &= 1
 \end{aligned}$$

The radius of convergence is 1.

The centre of convergence is 0.

The interval of convergence is $(-1, 1)$.

The series does not converge at $x = -1$ or $x = 1$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=0}^{\infty} 3n(x+1)^n$

Solution

$$\begin{aligned}
 R &= \lim_{n \rightarrow \infty} \left| \frac{3n}{3(n+1)} \right| & R &= \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{3n}{3n} & & \\
 &= 1
 \end{aligned}$$

The radius of convergence is 1, and the centre of convergence is -1 . ($x+1=0$)

$$a - R < x < a + R \Rightarrow -1 - 1 < x < -1 + 1$$

Therefore, the given series converges absolutely on $(-2, 0)$

At $x = -2$, the series is $\sum_{n=0}^{\infty} 3n(-1)^n$ which diverges.

At $x = 0$, the series is $\sum_{n=0}^{\infty} 3n(1)^n = \sum_{n=0}^{\infty} 3n$ which diverges.

Hence, the interval of convergence is $(-2, 0)$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} x^n$

Solution

$$\begin{aligned}
 R &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^4 2^{2n+2}}{n^4 2^{2n}} \right| & R &= \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \\
 &= 4 \lim_{n \rightarrow \infty} \left| \left(\frac{n+1}{n} \right)^4 \right| \\
 &= 4
 \end{aligned}$$

The radius of convergence is 4, and the centre of convergence is 0.

$a - R < x < a + R \Rightarrow -4 < x < 4$, the given series converges absolutely on $(-4, 4)$

At $x = -4$,

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (-4)^n &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (-1)^n 2^{2n} \\
 &= \sum_{n=1}^{\infty} \frac{1}{n^4} \quad \text{which converges (p-series).}
 \end{aligned}$$

$$\text{At } x = 4, \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} 2^{2n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \text{ which also converges.}$$

Hence, the interval of convergence is $[-4, 4]$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{e^n}{n^3} (4-x)^n$

Solution

$$\begin{aligned}
 R &= \lim_{n \rightarrow \infty} \left| \frac{e^n}{n^3} \cdot \frac{(n+1)^3}{e^{n+1}} \right| & R &= \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \\
 &= \frac{1}{e} \lim_{n \rightarrow \infty} \left| \left(\frac{n+1}{n} \right)^3 \right| \\
 &= \frac{1}{e}
 \end{aligned}$$

The radius of convergence is $\frac{1}{e}$.

The centre of convergence is 4. ($4 - x = 0 \Rightarrow x = 4$)

$a - R < x < a + R \Rightarrow 4 - \frac{1}{e} < x < 4 + \frac{1}{e}$, which the given series converges absolutely

At $x = 4 - \frac{1}{e}$

The series is $\sum_{n=1}^{\infty} \frac{e^n}{n^3} \left(\frac{1}{e}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ which converges (p -series).

At $x = 4 + \frac{1}{e}$,

the series is $\sum_{n=1}^{\infty} \frac{e^n}{n^3} \left(-\frac{1}{e}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ which also converges (p -series).

Hence, the interval of convergence is $\left[4 - \frac{1}{e}, 4 + \frac{1}{e}\right]$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n}$

Solution

$$\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n} = \sum_{n=1}^{\infty} \frac{4^n \left(x - \frac{1}{4}\right)^n}{n^n}$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{4^n}{n^n} \cdot \frac{(n+1)^{n+1}}{4^{n+1}} \right|$$

$$= \frac{1}{4} \lim_{n \rightarrow \infty} \left| \left(\frac{n+1}{n}\right)^n (n+1) \right|$$

$$= \infty$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

The radius of convergence is ∞ .

The centre of convergence is $x = \frac{1}{4}$.

The interval of convergence is the real line $(-\infty, \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{1+5^n}{n!} x^n$

Solution

$$\begin{aligned}
 R &= \lim_{n \rightarrow \infty} \left| \frac{1+5^n}{n!} \cdot \frac{(n+1)!}{1+5^{n+1}} \right| & R &= \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{1+5^n}{1+5^{n+1}} (n+1) \right| \\
 &= \infty
 \end{aligned}$$

The radius of convergence is ∞ .

The centre of convergence is 0.

The interval of convergence is the real line $(-\infty, \infty)$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = e^{2x}$, $a = 0$

Solution

$$f(x) = e^{2x} \rightarrow f(0) = e^{2(0)} = 1$$

$$f'(x) = 2e^{2x} \rightarrow f'(0) = 2$$

$$f''(x) = 4e^{2x} \rightarrow f''(0) = 4$$

$$f'''(x) = 8e^{2x} \rightarrow f'''(0) = 8$$

$$P_0(x) = f(0)$$

$$= 1$$

$$P_1(x) = f(0) + f'(0)(x-0)$$

$$= 1 + 2x$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$= 1 + 2x + 2x^2$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \sin x$, $a = 0$

Solution

$$f(x) = \sin x \rightarrow f(0) = 0$$

$$f'(x) = \cos x \rightarrow f'(0) = 1$$

$$f''(x) = -\sin x \rightarrow f''(0) = 0$$

$$f'''(x) = -\cos x \rightarrow f'''(0) = -1$$

$$P_0(x) = f(0)$$

$$= 0 \mid$$

$$P_1(x) = f(0) + f'(0)(x - 0)$$

$$= x \mid$$

$$P_2(x) = f(0) + f'(0)(x - 0) + \frac{f''(0)}{2!}(x - 0)^2$$

$$= x \mid$$

$$P_3(x) = f(0) + f'(0)(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 + \frac{f'''(0)}{3!}(x - 0)^3$$

$$= x - \frac{1}{6}x^3 \mid$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \ln(1 + x)$, $a = 0$

Solution

$$f(x) = \ln(1 + x) \rightarrow f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \rightarrow f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \rightarrow f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \rightarrow f'''(0) = 2$$

$$P_0(x) = f(0)$$

$$= 0 \mid$$

$$P_1(x) = f(0) + f'(0)(x - 0)$$

$$\underline{= x} \mid$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$\underline{= x - \frac{1}{2}x^2} \mid$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$\underline{= x - \frac{1}{2}x^2 + \frac{1}{3}x^3} \mid$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \frac{1}{x+2}$, $a = 0$

Solution

$$f(x) = (x+2)^{-1} \rightarrow f(0) = \frac{1}{2}$$

$$f'(x) = -(x+2)^{-2} \rightarrow f'(0) = -\frac{1}{4}$$

$$f''(x) = 2(x+2)^{-3} \rightarrow f''(0) = \frac{1}{4}$$

$$f'''(x) = -6(x+2)^{-4} \rightarrow f'''(0) = -\frac{3}{8}$$

$$P_0(x) = f(0)$$

$$\underline{= \frac{1}{2}} \mid$$

$$P_1(x) = f(0) + f'(0)(x-0)$$

$$\underline{= \frac{1}{2} - \frac{1}{4}x} \mid$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$\underline{= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2} \mid$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$\underline{= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3} \mid$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \sqrt{1-x}$, $a = 0$

Solution

$$f(x) = (1-x)^{1/2} \rightarrow f(0) = 1$$

$$f'(x) = -\frac{1}{2}(1-x)^{-1/2} \rightarrow f'(0) = -\frac{1}{2}$$

$$f''(x) = -\frac{1}{4}(1-x)^{-3/2} \rightarrow f''(0) = -\frac{1}{4}$$

$$f'''(x) = -\frac{3}{8}(1-x)^{-5/2} \rightarrow f'''(0) = -\frac{3}{8}$$

$$P_0(x) = f(0)$$

$$= 1$$

$$P_1(x) = f(0) + f'(0)(x-0)$$

$$= 1 - \frac{1}{2}x$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = x^3$, $a = 1$

Solution

$$f(x) = x^3 \rightarrow f(1) = 1$$

$$f'(x) = 3x^2 \rightarrow f'(1) = 3$$

$$f''(x) = 6x \rightarrow f''(1) = 6$$

$$f'''(x) = 6 \rightarrow f'''(1) = 6$$

$$P_0(x) = 1$$

$$P_0(x) = f(a)$$

$$P_1(x) = 1 + 3(x-1)$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$\underline{P_2(x) = 1 + 3(x-1) + 3(x-1)^2}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\underline{P_3(x) = 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3}$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = 8\sqrt{x}$, $a = 1$

Solution

$$f(x) = 8x^{1/2} \rightarrow f(1) = 8$$

$$f'(x) = 4x^{-1/2} \rightarrow f'(1) = 4$$

$$f''(x) = -2x^{-3/2} \rightarrow f''(1) = -2$$

$$f'''(x) = 3x^{-5/2} \rightarrow f'''(1) = 3$$

$$\underline{P_0(x) = 8}$$

$$P_0(x) = f(a)$$

$$\underline{P_1(x) = 8 + 4(x-1)}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$\underline{P_2(x) = 8 + 4(x-1) - (x-1)^2}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\underline{P_3(x) = 8 + 4(x-1) - (x-1)^2 + 3(x-1)^3}$$

$$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \sin x$, $a = \frac{\pi}{4}$

Solution

$$f(x) = \sin x \rightarrow f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = \cos x \rightarrow f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f''(x) = -\sin x \rightarrow f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f'''(x) = -\cos x \rightarrow f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$P_0(x) = \frac{\sqrt{2}}{2}$$

$$P_0(x) = f(a)$$

$$P_1(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$$

$$P_1(x) = f(a) + f'(a)(x - a)$$

$$P_2(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2$$

$$P_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2$$

$$P_3(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3$$

$$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x - a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \cos x$, $a = \frac{\pi}{6}$

Solution

$$f(x) = \cos x \rightarrow f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$f'(x) = -\sin x \rightarrow f'\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

$$f''(x) = -\cos x \rightarrow f''\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$f'''(x) = \sin x \rightarrow f'''\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$P_0(x) = \frac{\sqrt{3}}{2}$$

$$P_0(x) = f(a)$$

$$P_1(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right)$$

$$P_1(x) = f(a) + f'(a)(x - a)$$

$$P_2(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2$$

$$P_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2$$

$$P_3(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x - \frac{\pi}{6}\right)^3$$

$$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x - a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \sqrt{x}$, $a = 9$

Solution

$$f(x) = x^{1/2} \rightarrow f(9) = 3$$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \rightarrow f'(9) = \frac{1}{6}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} = -\frac{1}{4x\sqrt{x}} \rightarrow f''(9) = -\frac{1}{4 \times 3^3}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} = \frac{3}{8x^2\sqrt{x}} \rightarrow f'''(9) = \frac{1}{2^3 \times 3^4}$$

$$\boxed{P_0(x) = 3} \qquad P_0(x) = f(a)$$

$$\boxed{P_1(x) = 3 + \frac{1}{6}(x-9)} \qquad P_1(x) = f(a) + f'(a)(x-a)$$

$$\boxed{P_2(x) = 3 + \frac{1}{2 \cdot 3}(x-9) - \frac{1}{2^3 \cdot 3^3}(x-9)^2} \qquad P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x) = 3 + \frac{1}{2 \cdot 3}(x-9) - \frac{1}{2^2 \cdot 3^3}(x-9)^2 + \frac{1}{2^4 \cdot 3^5}(x-9)^3} \qquad P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \sqrt[3]{x}$, $a = 8$

Solution

$$f(x) = x^{1/3} \rightarrow f(8) = 2$$

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} \rightarrow f'(8) = \frac{1}{2^2 \times 3}$$

$$f''(x) = -\frac{2}{9}x^{-5/3} = -\frac{2}{3^2 x^{5/3}} \rightarrow f''(8) = -\frac{1}{3^2 \times 2^4}$$

$$f'''(x) = \frac{10}{3^3}x^{-8/3} = \frac{2 \cdot 5}{3^3 x^{8/3}} \rightarrow f'''(8) = \frac{5}{2^7 \times 3^3}$$

$$\boxed{P_0(x) = 2} \qquad P_0(x) = f(a)$$

$$\boxed{P_1(x) = 2 + \frac{1}{2^2 \cdot 3}(x-8)} \qquad P_1(x) = f(a) + f'(a)(x-a)$$

$$\boxed{P_2(x) = 2 + \frac{1}{2^2 \cdot 3}(x-8) - \frac{1}{2^5 \cdot 3^2}(x-8)^2} \qquad P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x) = 2 + \frac{1}{2^2 \cdot 3}(x-8) - \frac{1}{2^5 \cdot 3^2}(x-8)^2 + \frac{1}{2^8 \cdot 3^4}(x-8)^3} \quad P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \ln x$, $a = e$

Solution

$$f(x) = \ln x \rightarrow f(e) = 1$$

$$f'(x) = \frac{1}{x} \rightarrow f'(e) = \frac{1}{e}$$

$$f''(x) = -\frac{1}{x^2} \rightarrow f''(e) = -\frac{1}{e^2}$$

$$f'''(x) = \frac{2}{x^3} \rightarrow f'''(e) = \frac{2}{e^3}$$

$$\boxed{P_0(x) = 1}$$

$$P_0(x) = f(a)$$

$$\boxed{P_1(x) = 1 + \frac{1}{e}(x-e)}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$\boxed{P_2(x) = 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x) = 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3}$$

$$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = \sqrt[4]{x}$, $a = 8$

Solution

$$f(x) = x^{1/4} \rightarrow f(8) = \sqrt[4]{8}$$

$$f'(x) = \frac{1}{4}x^{-3/4} = \frac{1}{4x^{3/4}} \rightarrow f'(8 = 2^3) = \frac{1}{2^2 \times 2^{9/4}} = \frac{1}{2^4 \sqrt[4]{2}}$$

$$f''(x) = -\frac{3}{16}x^{-7/4} = -\frac{3}{2^4 x^{7/4}} \rightarrow f''(8) = -\frac{3}{2^4 2^{21/4}} = -\frac{3}{2^9 \sqrt[4]{2}}$$

$$f'''(x) = \frac{21}{6}x^{-11/4} = \frac{21}{2^6 x^{11/4}} \rightarrow f'''(8) = \frac{21}{2^6 \times 2^{33/4}} = \frac{21}{2^{14} \sqrt[4]{2}}$$

$$\boxed{P_0(x) = \sqrt[4]{8}}$$

$$P_0(x) = f(a)$$

$$\underline{P_1(x) = \sqrt[4]{8} + \frac{1}{2^4 \cdot \sqrt[4]{2}}(x-8)}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\underline{P_2(x) = \sqrt[4]{8} + \frac{1}{2^4 \cdot \sqrt[4]{2}}(x-8) - \frac{3}{2^{10} \cdot \sqrt[4]{2}}(x-8)^2}$$

$$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

$$\underline{P_3(x) = \sqrt[4]{8} + \frac{1}{2^4 \cdot \sqrt[4]{2}}(x-8) - \frac{3}{2^{10} \cdot \sqrt[4]{2}}(x-8)^2 + \frac{7}{2^{15} \cdot \sqrt[4]{2}}(x-8)^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a :

$$f(x) = \tan^{-1} x + x^2 + 1, \quad a = 1$$

Solution

$$f(x) = \tan^{-1} x + x^2 + 1 \rightarrow f(1) = \frac{\pi}{4} + 2$$

$$f'(x) = \frac{1}{x^2 + 1} + 2x \rightarrow f'(1) = \frac{5}{2}$$

$$f''(x) = -\frac{2x}{(x^2 + 1)^2} + 2 \rightarrow f''(1) = -\frac{1}{2} + 2 = \frac{3}{2}$$

$$f'''(x) = -\frac{2x^2 + 2 - 8x^2}{(x^2 + 1)^3}$$

$$= -\frac{2 - 2x^2}{(x^2 + 1)^3} \rightarrow f'''(1) = 0$$

$$(U^n V^m)' = U^{n-1} V^{m-1} (n U' V + m U V')$$

$$\underline{P_0(x) = \frac{\pi}{4} + 2}$$

$$P_0(x) = f(a)$$

$$\underline{P_1(x) = \frac{\pi}{4} + 2 + \frac{5}{2}(x-1)}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$\underline{P_2(x) = \frac{\pi}{4} + 2 + \frac{5}{2}(x-1) - \frac{3}{4}(x-1)^2}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\underline{P_3(x) = \frac{\pi}{4} + 2 + \frac{5}{2}(x-1) - \frac{3}{4}(x-1)^2}$$

$$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a : $f(x) = e^x$, $a = \ln 2$

Solution

$$f(x) = e^x \rightarrow f(\ln 2) = 2$$

$$f'(x) = e^x \rightarrow f'(\ln 2) = 2$$

$$f''(x) = e^x \rightarrow f''(\ln 2) = 2$$

$$f'''(x) = e^x \rightarrow f'''(\ln 2) = 2$$

$$\underline{P_0(x) = 2}$$

$$P_0(x) = f(a)$$

$$\underline{P_1(x) = 2 + 2(x - \ln 2)}$$

$$P_1(x) = f(a) + f'(a)(x - a)$$

$$\underline{P_2(x) = 2 + 2(x - \ln 2) + (x - \ln 2)^2}$$

$$P_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2$$

$$\underline{P_3(x) = 2 + 2(x - \ln 2) + (x - \ln 2)^2 + \frac{1}{3}(x - \ln 2)^3}$$

$$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x - a)^3$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = e^{4x}$, $n = 4$

Solution

$$f(x) = e^{4x} \rightarrow f(0) = 1$$

$$f'(x) = 4e^{4x} \rightarrow f'(0) = 4$$

$$f''(x) = 16e^{4x} \rightarrow f''(0) = 16$$

$$f'''(x) = 64e^{4x} \rightarrow f'''(0) = 64$$

$$f^{(4)}(x) = 256e^{4x} \rightarrow f^{(4)}(0) = 256$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = e^{-x}$, $n = 5$

Solution

$$f(x) = e^{-x} \rightarrow f(0) = 1$$

$$f'(x) = -e^{-x} \rightarrow f'(0) = -1$$

$$f''(x) = e^{-x} \rightarrow f''(0) = 1$$

$$f'''(x) = -e^{-x} \rightarrow f'''(0) = -1$$

$$f^{(4)}(x) = e^{-x} \rightarrow f^{(4)}(0) = 1$$

$$f^{(5)}(x) = -e^{-x} \rightarrow f^{(5)}(0) = -1$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$\underline{P_5(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = e^{-x/2}$, $n = 4$

Solution

$$f(x) = e^{-x/2} \rightarrow f(0) = 1$$

$$f'(x) = -\frac{1}{2}e^{-x/2} \rightarrow f'(0) = -\frac{1}{2}$$

$$f''(x) = \frac{1}{4}e^{-x/2} \rightarrow f''(0) = \frac{1}{4}$$

$$f'''(x) = -\frac{1}{8}e^{-x/2} \rightarrow f'''(0) = -\frac{1}{8}$$

$$f^{(4)}(x) = \frac{1}{16}e^{-x/2} \rightarrow f^{(4)}(0) = \frac{1}{16}$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = e^{x/3}$, $n = 4$

Solution

$$f(x) = e^{x/3} \rightarrow f(0) = 1$$

$$f'(x) = \frac{1}{3}e^{x/3} \rightarrow f'(0) = \frac{1}{3}$$

$$f''(x) = \frac{1}{9}e^{x/3} \rightarrow f''(0) = \frac{1}{9}$$

$$f'''(x) = \frac{1}{27}e^{x/3} \rightarrow f'''(0) = \frac{1}{27}$$

$$f^{(4)}(x) = \frac{1}{81}e^{x/3} \rightarrow f^{(4)}(0) = \frac{1}{81}$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x) = 1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \frac{1}{1944}x^4$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = \sin x$, $n = 5$

Solution

$$f(x) = \sin x \rightarrow f(0) = 0$$

$$f'(x) = \cos x \rightarrow f'(0) = 1$$

$$f''(x) = -\sin x \rightarrow f''(0) = 0$$

$$f'''(x) = -\cos x \rightarrow f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \rightarrow f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \cos x \rightarrow f^{(5)}(0) = 1$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = \cos \pi x$, $n = 4$

Solution

$$f(x) = \cos \pi x \rightarrow f(0) = 1$$

$$f'(x) = -\pi \sin \pi x \rightarrow f'(0) = 0$$

$$f''(x) = -\pi^2 \cos \pi x \rightarrow f''(0) = -\pi^2$$

$$f'''(x) = \pi^3 \sin \pi x \rightarrow f'''(0) = 0$$

$$f^{(4)}(x) = \pi^4 \cos \pi x \rightarrow f^{(4)}(0) = \pi^4$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = 1 - \frac{\pi^2}{2}x^2 + \frac{\pi^4}{24}x^4}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = xe^x$, $n = 4$

Solution

$$f(x) = xe^x \rightarrow f(0) = 0$$

$$f'(x) = e^x + xe^x \rightarrow f'(0) = 1$$

$$f''(x) = 2e^x + xe^x \rightarrow f''(0) = 2$$

$$f'''(x) = 3e^x + xe^x \rightarrow f'''(0) = 3$$

$$f^{(4)}(x) = 4e^x + xe^x \rightarrow f^{(4)}(0) = 4$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = x^2e^{-x}$, $n = 4$

Solution

$$f(x) = x^2 e^{-x} \rightarrow f(0) = 0$$

$$f'(x) = 2xe^{-x} - x^2 e^{-x} \rightarrow f'(0) = 0$$

$$f''(x) = 2e^{-x} - 4xe^{-x} + x^2 e^{-x} \rightarrow f''(0) = 2$$

$$f'''(x) = -6e^{-x} + 6xe^{-x} - x^2 e^{-x} \rightarrow f'''(0) = -6$$

$$f^{(4)}(x) = 12e^{-x} - 8xe^{-x} + x^2 e^{-x} \rightarrow f^{(4)}(0) = 12$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = x^2 - x^3 + \frac{1}{2}x^4}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = \frac{1}{x+1}$, $n = 5$

Solution

$$f(x) = \frac{1}{x+1} \rightarrow f(0) = 1$$

$$f'(x) = -(x+1)^{-2} \rightarrow f'(0) = -1$$

$$f''(x) = 2(x+1)^{-3} \rightarrow f''(0) = 2$$

$$f'''(x) = -6(x+1)^{-4} \rightarrow f'''(0) = -6$$

$$f^{(4)}(x) = 24(x+1)^{-5} \rightarrow f^{(4)}(0) = 24$$

$$f^{(5)}(x) = -120(x+1)^{-6} \rightarrow f^{(5)}(0) = -120$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$\underline{P_5(x) = 1 - x + x^2 - x^3 + x^4 - x^5}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = \frac{x}{x+1}$, $n = 4$

Solution

$$f(x) = \frac{x}{x+1} = 1 - \frac{1}{x+1} \rightarrow f(0) = 0$$

$$f'(x) = (x+1)^{-2} \rightarrow f'(0) = 1$$

$$f''(x) = -2(x+1)^{-3} \rightarrow f''(0) = -2$$

$$f'''(x) = 6(x+1)^{-4} \rightarrow f'''(0) = 6$$

$$f^{(4)}(x) = -24(x+1)^{-5} \rightarrow f^{(4)}(0) = -24$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = x - x^2 + x^3 - x^4}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = \sec x$, $n = 2$

Solution

$$f(x) = \sec x \rightarrow f(0) = 1$$

$$f'(x) = \sec x \tan x \rightarrow f'(0) = 0$$

$$f''(x) = \sec x \tan^2 x + \sec^3 x \rightarrow f''(0) = 1$$

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

$$\underline{P_2(x) = 1 + \frac{1}{2}x^2}$$

Exercise

Find the n th Maclaurin polynomial for the function $f(x) = \tan x$, $n = 3$

Solution

$$f(x) = \tan x \rightarrow f(0) = 0$$

$$f'(x) = \sec^2 x \rightarrow f'(0) = 1$$

$$f''(x) = 2\sec^2 x \tan x \rightarrow f''(0) = 0$$

$$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x \rightarrow f'''(0) = 2$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3$$

$$\underline{P_4(x) = x + \frac{1}{3}x^3}$$

ExerciseFind the Maclaurin series for: xe^x **Solution**

$$f(x) = xe^x \rightarrow f(0) = 0$$

$$f'(x) = e^x + xe^x \rightarrow f'(0) = 1$$

$$f''(x) = 2e^x + xe^x \rightarrow f''(0) = 2$$

$$f'''(x) = 3e^x + xe^x \rightarrow f'''(0) = 3$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$f^{(n)}(x) = ne^x + xe^x \rightarrow f^{(n)}(0) = n$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$xe^x = x + x^2 + \frac{1}{2}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{(n-1)!} x^n$$

ExerciseFind the Maclaurin series for: $5 \cos \pi x$ **Solution**

$$f(x) = 5 \cos \pi x \rightarrow f(0) = 5$$

$$f'(x) = -5\pi \sin \pi x \rightarrow f'(0) = 0$$

$$f''(x) = -5\pi^2 \cos \pi x \rightarrow f''(0) = -5\pi^2$$

$$f'''(x) = 5\pi^3 \sin \pi x \rightarrow f'''(0) = 0$$

$$5 \cos \pi x = 5 - \frac{5\pi^2 x^2}{2!} + \frac{5\pi^4 x^4}{4!} - \frac{5\pi^6 x^6}{6!} + \cdots$$

$$= 5 \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!}$$

Exercise

Find the Maclaurin series for: $\frac{x^2}{x+1}$

Solution

$$f(x) = \frac{x^2}{x+1} \rightarrow f(0) = 0$$

$$f'(x) = \frac{2x(x+1) - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2} \rightarrow f'(0) = 0$$

$$\begin{aligned} f''(x) &= \frac{(2x+2)(x+1)^2 - 2(x+1)(x^2 + 2x)}{(x+1)^4} \\ &= \frac{2x^2 + 4x + 2 - 2x^2 - 4x}{(x+1)^3} \\ &= \frac{2}{(x+1)^3} \rightarrow f''(0) = 2 \end{aligned}$$

$$f'''(x) = \frac{-6}{(x+1)^4} \rightarrow f'''(0) = -6$$

$$\begin{aligned} \frac{x^2}{x+1} &= \frac{2}{2!}x^2 - \frac{6x^3}{3!} + \frac{24x^4}{4!} - \dots = x^2 - x^3 + x^4 - \dots \\ &= \sum_{n=2}^{\infty} (-1)^n x^n \end{aligned}$$

Exercise

Find the Maclaurin series for: e^{3x+1}

Solution

$$\begin{aligned} e^{3x+1} &= e \cdot e^{3x} \\ &= e \left(\sum_{n=0}^{\infty} \frac{(3x)^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \frac{e 3^n x^n}{n!} \quad (\text{for all } x) \end{aligned}$$

Exercise

Find the Maclaurin series for: $\cos(2x^3)$

Solution

$$\begin{aligned}
 \cos(2x^3) &= 1 - \frac{(2x^3)^2}{2!} + \frac{(2x^3)^4}{4!} - \frac{(2x^3)^6}{6!} + \dots \\
 &= 1 - \frac{2^2 x^3}{2!} + \frac{2^4 x^{12}}{4!} - \frac{2^6 x^{18}}{6!} + \dots \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} x^{6n} \quad \left| \quad (\text{for all } x) \right.
 \end{aligned}$$

Exercise

Find the Maclaurin series for: $\cos(2x - \pi)$

Solution

$$\begin{aligned}
 \cos(2x - \pi) &= \cos(2x)\cos\pi + \sin(2x)\sin\pi \\
 &= -\cos(2x) \\
 &= -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (2x)^{2n} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 4^n}{(2n)!} x^{2n} \quad \left| \quad (\text{for all } x) \right.
 \end{aligned}$$

Exercise

Find the Maclaurin series for: $x^2 \sin\left(\frac{x}{3}\right)$

Solution

$$\begin{aligned}
 x^2 \sin\left(\frac{x}{3}\right) &= x^2 \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{x}{3}\right)^{2n+1}}{(2n+1)!} \\
 &= x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{3^{2n+1} (2n+1)!}
 \end{aligned}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{3^{2n+1}(2n+1)!} \quad \left| \quad (\text{for all } x) \right.$$

Exercise

Find the Maclaurin series for: $\cos^2\left(\frac{x}{2}\right)$

Solution

$$\begin{aligned} \cos^2\left(\frac{x}{2}\right) &= \frac{1}{2}(1 + \cos x) \\ &= \frac{1}{2} \left(1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \right) \\ &= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \\ &= 1 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \left| \quad (\text{for all } x) \right. \end{aligned}$$

Exercise

Find the Maclaurin series for: $\sin x \cos x$

Solution

$$\begin{aligned} \sin x \cos x &= \frac{1}{2} \sin(2x) \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x)^{2n+1} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 2^{2n+1} x^{2n+1} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{(2n+1)!} x^{2n+1} \quad \left| \quad (\text{for all } x) \right. \end{aligned}$$

ExerciseFind the Maclaurin series for: $\tan^{-1}(5x^2)$ **Solution**

$$\begin{aligned}\tan^{-1}(5x^2) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (5x^2)^{2n+1} && 5x^2 \leq 1 \rightarrow x^2 \leq \frac{1}{5} \Rightarrow -\frac{1}{\sqrt{5}} \leq x \leq \frac{1}{\sqrt{5}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1}}{2n+1} x^{4n+2} && \left(\text{for } -\frac{1}{\sqrt{5}} \leq x \leq \frac{1}{\sqrt{5}} \right)\end{aligned}$$

ExerciseFind the Maclaurin series for: $\ln(2+x^2)$ **Solution**

$$\begin{aligned}\ln(2+x^2) &= \ln 2 \left(1 + \frac{x^2}{2} \right) \\ &= \ln 2 + \ln \left(1 + \frac{x^2}{2} \right) \\ &= \ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{x^2}{2} \right)^n && \frac{x^2}{2} \leq 1 \rightarrow x^2 \leq 2 \Rightarrow -\sqrt{2} \leq x \leq \sqrt{2} \\ &= \ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} \frac{x^{2n}}{2^n} && \left(\text{for } -\sqrt{2} \leq x \leq \sqrt{2} \right)\end{aligned}$$

ExerciseFind the Maclaurin series for: $\frac{1+x^3}{1+x^2}$ **Solution**

$$\begin{aligned}\frac{1}{1+x} &= \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \dots \\ \frac{1}{1+x^2} &= 1 - x^2 + x^4 - x^6 + \dots \\ \frac{1+x^3}{1+x^2} &= (1+x^3)(1-x^2+x^4-x^6+\dots)\end{aligned}$$

$$\begin{aligned}
&= 1 - x^2 + x^4 - x^6 + \dots + x^3 - x^5 + x^7 - x^9 + \dots \\
&= 1 - x^2 + x^3 + x^4 - x^5 - x^6 + x^7 + x^8 - x^9 - \dots \\
&= 1 - x^2 + \sum_{n=2}^{\infty} (-1)^n (x^{2n-1} + x^{2n}) \quad \left(\text{for } |x| < 1 \right)
\end{aligned}$$

Exercise

Find the Maclaurin series for: $\ln \frac{1+x}{1-x}$

Solution

$$\begin{aligned}
\ln \frac{1+x}{1-x} &= \ln(1+x) - \ln(1-x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \right) = 2x + 2\frac{x^3}{3} + \dots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \\
&= \sum_{n=0}^{\infty} \left((-1)^n + 1 \right) \frac{x^{n+1}}{n+1} \\
&= 2 \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1} \quad \left(-1 < x < 1 \right)
\end{aligned}$$

Exercise

Find the Maclaurin series for: $\frac{e^{2x^2} - 1}{x^2}$

Solution

$$\begin{aligned}
\frac{e^{2x^2} - 1}{x^2} &= \frac{1}{x^2} \left(e^{2x^2} - 1 \right) \\
&= \frac{1}{x^2} \left(1 + 2x^2 + \frac{(2x^2)^2}{2!} + \frac{(2x^2)^3}{3!} + \dots - 1 \right) \\
&= \frac{1}{x^2} \left(2x^2 + \frac{2^2 x^4}{2!} + \frac{2^3 x^6}{3!} + \dots \right) \\
&= 2 + \frac{2^2 x^2}{2!} + \frac{2^3 x^4}{3!} + \frac{2^4 x^6}{4!} + \dots
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{2^{n+1}}{(n+1)!} x^{2n} \quad \left(\text{for all } x \neq 0 \right)$$

Exercise

Find the Maclaurin series for: $\cosh x - \cos x$

Solution

$$\cosh x - \cos x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) = 2\frac{x^2}{2!} + 2\frac{x^6}{6!} + \dots$$

$$= \sum_{n=0}^{\infty} \left(1 - (-1)^n \right) \frac{x^{2n}}{(2n)!}$$

$$= 2 \sum_{n=0}^{\infty} \frac{x^{4n+2}}{(4n+2)!} \quad \left(\text{for all } x \right)$$

Exercise

Find the Maclaurin series for: $\sinh x - \sin x$

Solution

$$\sinh x - \sin x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \left(1 - (-1)^n \right) \frac{x^{2n+1}}{(2n+1)!}$$

$$= 2 \sum_{n=0}^{\infty} \frac{x^{4n+3}}{(4n+3)!} \quad \left(\text{for all } x \right)$$

Exercise

Finding Taylor and Maclaurin Series generated by f at $x = a$: $f(x) = x^3 - 2x + 4$, $a = 2$

Solution

$$f(x) = x^3 - 2x + 4 \rightarrow f(2) = 8$$

$$f'(x) = 3x^2 - 2 \rightarrow f'(2) = 10$$

$$f''(x) = 6x \rightarrow f''(2) = 12$$

$$f'''(x) = 6 \rightarrow f'''(2) = 6$$

$$f^{(n)}(x) = 0 \rightarrow f^{(n)}(2) = 0 \quad (n > 3)$$

$$P_n(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \dots$$

$$\underline{x^3 - 2x + 4 = 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3 \quad |}$$

Exercise

Finding Taylor and Maclaurin Series generated by f at $x = a$: $f(x) = 2x^3 + x^2 + 3x - 8$, $a = 1$

Solution

$$f(x) = 2x^3 + x^2 + 3x - 8 \rightarrow f(1) = -2$$

$$f'(x) = 6x^2 + 2x + 3 \rightarrow f'(1) = 11$$

$$f''(x) = 12x + 2 \rightarrow f''(1) = 14$$

$$f'''(x) = 12 \rightarrow f'''(1) = 12$$

$$f^{(n)}(x) = 0 \rightarrow f^{(n)}(1) = 0 \quad (n \geq 4)$$

$$P_n(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \dots$$

$$\underline{2x^3 + x^2 + 3x - 8 = -2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3 \quad |}$$

Exercise

Finding Taylor and Maclaurin Series generated by f at $x = a$:

$$f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2, \quad a = -1$$

Solution

$$f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2 \rightarrow f(-1) = -7$$

$$f'(x) = 15x^4 - 4x^3 + 6x^2 + 2x \rightarrow f'(-1) = 23$$

$$f''(x) = 60x^3 - 12x^2 + 12x + 2 \rightarrow f''(-1) = -82$$

$$f'''(x) = 180x^2 - 24x + 12 \rightarrow f'''(-1) = 216$$

$$f^{(4)}(x) = 360x - 24 \rightarrow f^{(4)}(-1) = -384$$

$$f^{(5)}(x) = 360 \rightarrow f^{(5)}(-1) = 360$$

$$f^{(n)}(x) = 0 \rightarrow f^{(n)}(-1) = 0 \quad (n \geq 6)$$

$$P_n(x) = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3 + \frac{f^{(4)}(-1)}{4!}(x+1)^4 + \frac{f^{(5)}(-1)}{5!}(x+1)^5$$

$$\begin{aligned} 3x^5 - x^4 + 2x^3 + x^2 - 2 &= -7 + 23(x+1) - \frac{82}{2!}(x+1)^2 + \frac{216}{3!}(x+1)^3 - \frac{384}{4!}(x+1)^4 + \frac{360}{5!}(x+1)^5 \\ &= -7 + 23(x+1) - 41(x+1)^2 + 36(x+1)^3 - 16(x+1)^4 + 3(x+1)^5 \end{aligned}$$

Exercise

Finding Taylor and Maclaurin Series generated by f at $x = a$: $f(x) = \cos\left(2x + \frac{\pi}{2}\right)$, $a = \frac{\pi}{4}$

Solution

$$f(x) = \cos\left(2x + \frac{\pi}{2}\right) \rightarrow f\left(\frac{\pi}{4}\right) = -1$$

$$f'(x) = -2\sin\left(2x + \frac{\pi}{2}\right) \rightarrow f'\left(\frac{\pi}{4}\right) = 0$$

$$f''(x) = -4\cos\left(2x + \frac{\pi}{2}\right) \rightarrow f''\left(\frac{\pi}{4}\right) = 4$$

$$f'''(x) = 8\sin\left(2x + \frac{\pi}{2}\right) \rightarrow f'''\left(\frac{\pi}{4}\right) = 0$$

$$f^{(4)}(x) = 16\cos\left(2x + \frac{\pi}{2}\right) \rightarrow f^{(4)}\left(\frac{\pi}{4}\right) = -16$$

$$f^{(5)}(x) = -32\sin\left(2x + \frac{\pi}{2}\right) \rightarrow f^{(5)}\left(\frac{\pi}{4}\right) = 0$$

$$\rightarrow f^{(2n)}\left(\frac{\pi}{4}\right) = (-1)^n 2^{2n}$$

$$\cos\left(2x + \frac{\pi}{2}\right) = -1 + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{16}{4!}\left(x - \frac{\pi}{4}\right)^4 + \dots$$

$$= -1 + 2\left(x - \frac{\pi}{4}\right)^2 - \frac{2}{3}\left(x - \frac{\pi}{4}\right)^4 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} \left(x - \frac{\pi}{4}\right)^{2n}$$

Solution

Section 4.2 – Series Solutions near Ordinary Points

Exercise

Find a power series solution. $y' = 3y$

Solution

The equation $y' = 3y$ is separable with solution

$$\frac{dy}{dx} = 3y$$

$$\int \frac{dy}{y} = \int 3 dx$$

$$y = Ce^{3x}$$

$$\ln(y) = 3x + C$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y' - 3y = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 3 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+1) a_{n+1} - 3a_n] x^n = 0$$

$$(n+1) a_{n+1} - 3a_n = 0$$

$$a_{n+1} = \frac{3a_n}{n+1}; \quad n \geq 0$$

With $y(0) = 3a_0$

$$a_1 = 3a_0$$

$$a_2 = \frac{3}{2} a_1 = \frac{3 \cdot 3}{2} a_0$$

$$a_3 = \frac{3}{3}a_2 = \frac{3 \cdot 3 \cdot 3}{2 \cdot 3}a_0$$

$$a_4 = \frac{3}{4}a_3 = \frac{3 \cdot 3 \cdot 3 \cdot 3}{2 \cdot 3 \cdot 4}a_0$$

$$\vdots \quad \quad \vdots$$

$$\boxed{a_n = \frac{3^n}{n!}a_0}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{3^n}{n!} a_0 x^n$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$$

$$\boxed{= a_0 e^{3x}} \quad \checkmark$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Exercise

Find a power series solution. $y' = 4y$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y' = 4y$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = 4 \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} 4 a_n x^n$$

$$(n+1) a_{n+1} x^n = 4 a_n x^n$$

$$(n+1) a_{n+1} = 4 a_n$$

$$\boxed{a_{n+1} = \frac{4}{n+1} a_n}$$

$$n=0 \rightarrow a_1 = 4a_0$$

$$n=1 \rightarrow a_2 = \frac{4}{2}a_1 = \frac{4^2}{2!}a_0$$

$$n=2 \rightarrow a_3 = \frac{4}{3}a_2 = \frac{4^3}{3!}a_0$$

$$n=3 \rightarrow a_4 = \frac{4}{4}a_3 = \frac{4^4}{4!}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = \frac{4^n}{n!}a_0}$$

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} \frac{4^n}{n!} a_0 x^n \\ &= a_0 \left(1 + 4x + \frac{4^2}{2!}x^2 + \frac{4^3}{3!}x^3 + \dots \right) \\ &= a_0 e^{4x} \end{aligned}$$

Exercise

Find a power series solution. $y' = x^2 y$

Solution

$$\frac{dy}{dx} = x^2 y$$

$$\int \frac{dy}{y} = \int x^2 dx$$

$$\ln y = \frac{1}{3}x^3 + C_1$$

$$y = e^{\frac{1}{3}x^3 + C_1}$$

$$y = C e^{\frac{1}{3}x^3}$$

$$y(0) = C(1) = a_0$$

$$\boxed{C = a_0}$$

$$\boxed{y = a_0 e^{x^3/3}}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y' - x^2 y = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\begin{aligned} \sum_{n=1}^{\infty} n a_n x^{n-1} &= \sum_{k=-2}^{\infty} (k+3) a_{k+3} x^{k+2} \\ &= \sum_{n=-2}^{\infty} (n+3) a_{n+3} x^{n+2} \end{aligned}$$

$$\sum_{n=-2}^{\infty} (n+3) a_{n+3} x^{n+2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$a_1 + 2a_2 x + \sum_{n=-2}^{\infty} (n+3) a_{n+3} x^{n+2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$a_1 + 2a_2 x + \sum_{n=-2}^{\infty} [(n+3) a_{n+3} - a_n] x^{n+2} = 0$$

If we set $a_1 = a_2 = 0$, then

$$(n+3) a_{n+3} - a_n = 0$$

$$\Rightarrow a_{n+3} = \frac{a_n}{n+3}, \quad n \geq 0$$

$$a_3 = \frac{1}{3} a_0$$

$$a_4 = \frac{1}{4} a_1 = 0$$

$$a_5 = \frac{1}{5}a_2 = 0$$

$$a_6 = \frac{1}{6}a_3 = \frac{1}{3 \cdot 6}a_0$$

$$a_7 = \frac{1}{7}a_4 = 0$$

$$a_9 = \frac{1}{9}a_6 = \frac{1}{3 \cdot 6 \cdot 9}a_0 = \frac{1}{3^3(1 \cdot 2 \cdot 3)}a_0$$

$$a_{12} = \frac{1}{12}a_9 = \frac{1}{3 \cdot 6 \cdot 9 \cdot 12}a_0 = \frac{1}{3^4(1 \cdot 2 \cdot 3 \cdot 4)}a_0$$

$$\left. a_{3n} = \frac{1}{3^n \cdot n!}a_0 \right|$$

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{3^n \cdot n!}a_0 x^{3n}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x^3}{3} \right)^n$$

$$\left. = a_0 e^{x^3/3} \right| \quad \checkmark$$

$$P = \lim_{n \rightarrow \infty} \left| \frac{3^k k!}{1} \right| = \infty$$

Exercise

Find a power series solution. $y' + 2xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y' + 2xy = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

$$a_1 + \sum_{n=0}^{\infty} (n+2)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty} 2a_nx^{n+1} = 0$$

$$a_1 + \sum_{n=0}^{\infty} [(n+2)a_{n+2} + 2a_n]x^{n+1} = 0$$

$$\left\{ \begin{array}{l} \underline{a_1 = 0} \\ (n+2)a_{n+2} + 2a_n = 0 \rightarrow \underline{a_{n+2} = -\frac{2}{n+2}a_0} \end{array} \right|$$

$$n=0 \rightarrow a_2 = -a_1 \qquad n=1 \rightarrow a_3 = -\frac{2}{3}a_1 = 0$$

$$n=2 \rightarrow a_4 = -\frac{1}{2}a_2 = \frac{1}{2}a_0 \qquad n=3 \rightarrow a_5 = -\frac{2}{7}a_3 = 0$$

$$n=4 \rightarrow a_6 = -\frac{1}{3}a_4 = -\frac{1}{2 \cdot 3}a_0 \qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n=6 \rightarrow a_8 = -\frac{1}{4}a_6 = \frac{1}{4!}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{a_{2k} = \frac{(-1)^k}{k!}a_0}$$

$$y(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}a_0x^{2k}$$

$$= a_0 \left(1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \dots \right)$$

$$\underline{= a_0 e^{-x^2}}$$

$$P = \lim_{n \rightarrow \infty} \left| \frac{n+2}{-2} \right| = \underline{\infty}$$

Exercise

Find a power series solution. $(x-2)y' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$(x-2) \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$x \sum_{n=1}^{\infty} n a_n x^{n-1} - 2 \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^n - 2 \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n a_n x^n - 2 \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [n a_n - 2(n+1) a_{n+1} + a_n] x^n = 0$$

$$n a_n - 2(n+1) a_{n+1} + a_n = 0$$

$$2(n+1) a_{n+1} = (n+1) a_n$$

$$\boxed{a_{n+1} = \frac{1}{2} a_n}$$

$$n=0 \rightarrow a_1 = \frac{1}{2} a_0$$

$$n=1 \rightarrow a_2 = \frac{1}{2} a_1 = \frac{1}{2^2} a_0$$

$$n=2 \rightarrow a_3 = \frac{1}{2} a_2 = \frac{1}{2^3} a_0$$

$$n=3 \rightarrow a_4 = \frac{1}{2} a_3 = \frac{1}{2^4} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = \frac{1}{2^n} a_0}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} a_0 x^n$$

$$= a_0 \left(1 + \frac{1}{2} x + \frac{1}{2^2} x^2 + \frac{1}{2^3} x^3 + \dots \right)$$

$$= a_0 \left(1 + \frac{x}{2} + \left(\frac{x}{2} \right)^2 + \left(\frac{x}{2} \right)^3 + \dots \right)$$

$$= a_0 \frac{1}{1 - \frac{x}{2}}$$

$$= \frac{2a_0}{2-x}$$

Exercise

Find a power series solution. $(2x-1)y' + 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$(2x-1) \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$0 + \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [2n a_n - (n+1) a_{n+1} + 2a_n] x^n = 0$$

$$2n a_n - (n+1) a_{n+1} + 2a_n = 0$$

$$(n+1) a_{n+1} = 2(n+1) a_n$$

$$a_{n+1} = 2a_n$$

$$n=0 \rightarrow a_1 = 2a_0$$

$$n=1 \rightarrow a_2 = 2a_1 = 2^2 a_0$$

$$n=2 \rightarrow a_3 = 2a_2 = 2^3 a_0$$

$$\begin{array}{cccc}
 n=3 & \rightarrow & a_4 = 2a_3 = 2^4 a_0 \\
 \vdots & & \vdots & \vdots \\
 \vdots & & \vdots & \vdots \\
 \hline
 a_n = 2^n a_0 & & &
 \end{array}$$

$$\begin{aligned}
 y(x) &= \sum_{n=0}^{\infty} 2^n a_0 x^n \\
 &= a_0 (1 + 2x + 2^2 x^2 + 2^3 x^3 + \dots) \\
 &= a_0 (1 + 2x + (2x)^2 + (2x)^3 + \dots) \\
 &= \frac{a_0}{1-2x}
 \end{aligned}$$

Exercise

Find a power series solution. $2(x-1)y' = 3y$

Solution

$$\begin{aligned}
 y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
 y'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
 (2x-2) \sum_{n=1}^{\infty} n a_n x^{n-1} &= \sum_{n=0}^{\infty} 3 a_n x^n \\
 2x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} 2n a_n x^{n-1} &= \sum_{n=0}^{\infty} 3 a_n x^n \\
 0 + \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^{n-1} &= \sum_{n=0}^{\infty} 3 a_n x^n \\
 \sum_{n=0}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n &= \sum_{n=0}^{\infty} 3 a_n x^n
 \end{aligned}$$

$$\sum_{n=0}^{\infty} [2na_n - 2(n+1)a_{n+1}]x^n = \sum_{n=0}^{\infty} 3a_n x^n$$

$$2na_n - 2(n+1)a_{n+1} = 3a_n$$

$$-2(n+1)a_{n+1} = (3-2n)a_n$$

$$\underline{a_{n+1} = \frac{2n-3}{2n+2}a_n}$$

$$\rho = \lim_{n \rightarrow \infty} \frac{2n-3}{2n+2} = 1$$

$$n=0 \rightarrow a_1 = -\frac{3}{2}a_0$$

$$n=1 \rightarrow a_2 = -\frac{1}{4}a_1 = \frac{3}{8}a_0$$

$$n=2 \rightarrow a_3 = \frac{1}{6}a_2 = \frac{1}{16}a_0$$

$$n=3 \rightarrow a_4 = \frac{3}{8}a_3 = \frac{3}{128}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y(x) = a_0 \left(1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \frac{3}{128}x^4 + \cdots \right)}$$

$$\frac{y'}{y} = \frac{3}{2} \frac{1}{x-1}$$

$$\int \frac{dy}{y} = \frac{3}{2} \int \frac{1}{x-1} dx$$

$$\ln y = \frac{3}{2} \ln(x-1) + \ln C$$

$$\ln y = \ln C(x-1)^{3/2}$$

$$\underline{y(x) = C(x-1)^{3/2}}$$

Exercise

Find a power series solution. $(1+x)y' - y = 0$

Solution

$$(1+x)\frac{dy}{dx} = y$$

$$\frac{dy}{y} = \frac{dx}{1+x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\ln(y) = \ln(x+1) + C \Rightarrow \underline{y = C(x+1)}$$

With $y(\mathbf{0}) = C(\mathbf{0}+1) = a_0 \rightarrow C = a_0$

$$\Rightarrow \underline{y = a_0(x+1)}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$(1+x)y' - y = 0$$

$$(1+x) \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left((n+1) a_{n+1} x^n + n a_n x^n - a_n x^n \right) = 0$$

$$(n+1) a_{n+1} + (n-1) a_n = 0$$

$$\rightarrow a_{n+1} = \frac{1-n}{n+1} a_n; \quad n \geq 0$$

$$a_1 = a_0 \quad a_2 = 0 a_1 = 0 \quad a_3 = \frac{-1}{3} a_2 = 0$$

$$a_n = 0 \quad \text{for} \quad n \geq 2$$

$$y(x) = a_0 + a_1 x$$

$$\begin{aligned}
 &= a_0 + a_0 x \\
 &= a_0 (1+x) \quad \checkmark
 \end{aligned}$$

Exercise

Find a power series solution. $(2-x)y' + 2y = 0$

Solution

$$(2-x)\frac{dy}{dx} + 2y = 0$$

$$(2-x)\frac{dy}{dx} = -2y$$

$$\int \frac{dy}{y} = \int \frac{2d(2-x)}{2-x}$$

$$\ln y = 2\ln(2-x) + C_1$$

$$\ln y = \ln(2-x)^2 + C_1$$

$$\ln y = \ln C(2-x)^2$$

$$y = C(2-x)^2$$

$$y(0) = C(2-0)^2 = a_0$$

$$C = \frac{1}{4}a_0$$

$$y = \frac{1}{4}a_0(2-x)^2$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$(2-x)y' + 2y = 0$$

$$(2-x) \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2 \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^n = 0 + \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n$$

$$2 \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [2(n+1)a_{n+1} - n a_n + 2a_n] x^n = 0$$

$$2(n+1)a_{n+1} - (n-2)a_n = 0$$

$$2(n+1)a_{n+1} = (n-2)a_n$$

$$a_{n+1} = \frac{n-2}{2(n+1)} a_n, \quad n \geq 0$$

$$a_1 = \frac{-2}{2} a_0 = -a_0$$

$$a_2 = \frac{-1}{4} a_1 = \frac{1}{4} a_0$$

$$a_3 = \frac{0}{6} a_0 = 0$$

$$\vdots$$

$$a_n = 0$$

$$\begin{aligned} y(x) &= a_0 - a_0 x + \frac{1}{4} a_0 x^2 \\ &= a_0 \left(1 - x + \frac{1}{4} x^2 \right) \\ &= \frac{1}{4} a_0 (4 - 4x + x^2) \\ &= \frac{1}{4} a_0 (2-x)^2 \end{aligned} \quad \checkmark$$

Exercise

Find a power series solution. $(x-4)y' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$(x-4)y' + y = 0$$

$$(x-4) \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} 4(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} [a_n - 4(n+1) a_{n+1}] x^n = 0$$

$$\sum_{n=0}^{\infty} n a_n x^n + a_0 - 4a_1 + \sum_{n=1}^{\infty} [a_n - 4(n+1) a_{n+1}] x^n = 0$$

$$a_0 - 4a_1 + \sum_{n=1}^{\infty} [(n+1) a_n - 4(n+1) a_{n+1}] x^n = 0$$

$$a_0 - 4a_1 = 0 \quad \rightarrow \quad \underline{a_1 = \frac{1}{4} a_0}$$

$$(n+1) a_n - 4(n+1) a_{n+1} = 0 \quad \rightarrow \quad \underline{a_{n+1} = \frac{1}{4} a_n}$$

$$a_2 = \frac{1}{4} a_1 = \frac{1}{4^2} a_0$$

$$a_3 = \frac{1}{4} a_2 = \frac{1}{4^3} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{a_n = \frac{1}{4^n} a_0}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{4^n} a_0 x^n$$

$$= a_0 \sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$$

$$\begin{aligned}
&= a_0 \left(\frac{1}{1 - \frac{x}{4}} \right) \\
&= a_0 \left(\frac{4}{4 - x} \right) \\
&= \frac{-4a_0}{x - 4} \quad \checkmark
\end{aligned}$$

Exercise

Find a power series solution. $x^2 y' = y - x - 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$x^2 y' = y - x - 1$$

$$x^2 \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = -x - 1 + \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+2} = -x - 1 + \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=2}^{\infty} (n-1) a_{n-1} x^n = -x - 1 + \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=2}^{\infty} (n-1) a_{n-1} x^n = -x - 1 + a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n$$

$$-x - 1 + a_0 + a_1 x = 0$$

$$a_0 + a_1 x = 1 + x \Rightarrow a_0 = 1; a_1 = 1$$

$$(n-1) a_{n-1} = a_n$$

$$a_2 = a_1 = 1$$

$$a_3 = 2a_2 = 2$$

$$a_4 = 3a_3 = 1 \cdot 2 \cdot 3$$

$$\vdots \quad \vdots$$

$$a_n = (n-1)!$$

$$\underline{y(x) = 1 + x + x^2 + 2!x^3 + 3!x^4 + \dots}$$

Exercise

Find a power series solution. $(x-3)y' + 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$(x-3)y' + 2y = 0$$

$$(x-3) \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} 3(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 3(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 3(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [n a_n - 3(n+1) a_{n+1} + 2a_n] x^n = 0$$

$$-3(n+1) a_{n+1} + (n+2) a_n = 0 \rightarrow \underline{a_{n+1} = \frac{n+2}{3(n+1)} a_n} \quad n = 0, 1, 2, \dots$$

$$a_1 = \frac{2}{3}a_0$$

$$a_2 = \frac{3}{3 \cdot 2}a_1 = \frac{3}{3^2}a_0$$

$$a_3 = \frac{4}{3 \cdot 3}a_2 = \frac{4}{3^3}a_0$$

$$a_4 = \frac{5}{3 \cdot 4}a_3 = \frac{5}{3^4}a_0$$

$$\vdots$$

$$a_n = \frac{n+1}{3^n}a_0 \quad n \geq 1$$

$$y(x) = \left(1 + \frac{2}{3}x + \frac{3}{3^2}x^2 + \frac{4}{3^3}x^3 + \dots \right)$$

$$= a_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n} x^n$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \rightarrow \infty} \frac{3n+3}{n+2}$$

$$= 3$$

$$y(x) = \frac{1}{(3-x)^2}$$

Exercise

Find a power series solution. $xy' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$xy' + y = 0$$

$$x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)a_n x^n = 0$$

$$(n+1)a_n = 0 \rightarrow \underline{a_n = 0}$$

$$\underline{y(x) \equiv 0}$$

\therefore The equation has no non-trivial power series.

Exercise

Find a power series solution. $x^3 y' - 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$$

$$x^3 y' - 2y = 0$$

$$x^3 \sum_{n=1}^{\infty} na_n x^{n-1} - 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} na_n x^{n+2} - 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=3}^{\infty} (n-2)a_{n-2} x^n - 2(a_0 + a_1 x + a_2 x^2) - \sum_{n=3}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=3}^{\infty} [(n-2)a_{n-2} - 2a_n] x^n - 2(a_0 + a_1 x + a_2 x^2) = 0$$

$$\begin{cases} a_0 = a_1 = a_2 = 0 \\ (n-2)a_{n-2} = 2a_n \rightarrow ka_k = 2a_{k+2} \quad (k = n-2) \end{cases}$$

$$\underline{a_{k+2} = \frac{k}{2}a_k}$$

$$a_3 = \frac{1}{2}a_1 = 0$$

$$a_4 = a_2 = 0$$

$$a_n \equiv 0$$

$$\underline{y(x) \equiv 0}$$

\therefore The equation has no non-trivial power series.

Exercise

Find a power series solution. $y'' = 4y$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' = 4y$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = 4 \sum_{n=0}^{\infty} a_n x^n$$

$$(n+2)(n+1) a_{n+2} = 4 a_n$$

$$a_{n+2} = \frac{4}{(n+2)(n+1)} a_n$$

$$n=0 \rightarrow a_2 = 2a_0 = \frac{4}{2}a_0$$

$$n=1 \rightarrow a_3 = \frac{4}{2 \cdot 3}a_1$$

$$n=2 \rightarrow a_4 = \frac{4}{3 \cdot 4}a_2 = \frac{4^2}{4!}a_0$$

$$n=3 \rightarrow a_5 = \frac{4}{4 \cdot 5}a_3 = \frac{4^2}{5!}a_1$$

$$\begin{array}{ll}
 n=4 \rightarrow a_6 = \frac{4}{5 \cdot 6} a_4 = \frac{4^3}{6!} a_0 & n=5 \rightarrow a_7 = \frac{4}{6 \cdot 7} a_5 = \frac{4^3}{7!} a_1 \\
 \vdots & \vdots
 \end{array}$$

$$\left. \begin{array}{l} a_{2k} = \frac{2^{2k}}{(2k)!} a_0 \\ a_{2k+1} = \frac{2^{2k}}{(2k+1)!} a_1 \end{array} \right|$$

$$\begin{aligned}
 y(x) &= a_0 \left(1 + \frac{2^2}{2!} x^2 + \frac{2^4}{4!} x^4 + \frac{2^6}{6!} x^6 + \dots \right) + a_1 \left(x + \frac{2^3}{3!} x^3 + \frac{2^5}{5!} x^5 + \dots \right) \\
 &= a_0 \left(1 + \frac{1}{2!} (2x)^2 + \frac{1}{4!} (2x)^4 + \frac{1}{6!} (2x)^6 + \dots \right) + a_1 \left(x + \frac{1}{3!} (2x)^3 + \frac{1}{5!} (2x)^5 + \dots \right) \\
 &= a_0 \cosh 2x + a_1 \sinh 2x
 \end{aligned}$$

Exercise

Find a power series solution. $y'' = 9y$

Solution

The equation $y'' = 9y$ has a characteristic equation:

$$\lambda^2 - 9 = 0$$

$$\lambda = \pm 3$$

\therefore The general solution: $y(x) = C_1 e^{3x} + C_2 e^{-3x}$

With $y(0) = a_0$ and $y'(0) = a_1$

$$y(0) = C_1 + C_2 = a_0$$

$$y'(x) = 3C_1 e^{3x} - 3C_2 e^{-3x}$$

$$y(0) = 3C_1 e^{3(0)} - 3C_2 e^{-3(0)} \rightarrow 3C_1 - 3C_2 = a_1$$

$$\begin{cases} C_1 + C_2 = a_0 \\ 3C_1 - 3C_2 = a_1 \end{cases} \rightarrow \begin{cases} 3C_1 + 3C_2 = 3a_0 \\ 3C_1 - 3C_2 = a_1 \end{cases}$$

$$6C_1 = 3a_0 + a_1$$

$$C_1 = \frac{3a_0 + a_1}{6}$$

$$C_2 = a_0 - C_1$$

$$C_2 = a_0 - \frac{3a_0 + a_1}{6}$$

$$= \frac{3a_0 - a_1}{6} \Bigg|$$

$$y(x) = \frac{3a_0 + a_1}{6} e^{3x} + \frac{3a_0 - a_1}{6} e^{-3x} \Bigg|$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - 9y = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 9 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 9 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - 9a_n] x^n = 0$$

$$(n+2)(n+1) a_{n+2} - 9a_n = 0$$

$$a_{n+2} = \frac{9}{(n+2)(n+1)} a_n, \quad n \geq 0$$

$$a_2 = \frac{9}{(2)(1)} a_0 = \frac{9}{2} a_0$$

$$a_4 = \frac{3^2}{(4)(3)} a_2 = \frac{9 \cdot 9}{2 \cdot 3 \cdot 4} a_0 = \frac{3^4}{2 \cdot 3 \cdot 4} a_0$$

$$a_6 = \frac{3^2}{(6)(5)} a_4 = \frac{3^6}{6!} a_0$$

$$a_{2n} = \frac{3^{2n}}{(2n)!} a_0$$

$$a_3 = \frac{9}{(3)(2)} a_1 = \frac{9}{2 \cdot 3} a_1$$

$$a_5 = \frac{9}{(5)(4)} a_3 = \frac{3^4}{2 \cdot 3 \cdot 4 \cdot 5} a_1$$

$$a_7 = \frac{9}{(7)(6)} a_5 = \frac{3^6}{7!} a_1$$

$$a_{2n+1} = \frac{3^{2n}}{(2n+1)!} a_1$$

$$\begin{aligned}
y(x) &= a_0 \left(1 + \frac{3^2}{2!}x^2 + \frac{3^4}{4!}x^4 + \frac{3^6}{6!}x^6 + \dots \right) + a_1 \left(x + \frac{3^2}{3!}x^3 + \frac{3^4}{5!}x^5 + \frac{3^6}{7!}x^7 + \dots \right) \\
y(x) &= \frac{3a_0 + a_1}{6}e^{3x} + \frac{3a_0 - a_1}{6}e^{-3x} \\
&= \frac{3a_0 + a_1}{6} \left[1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots \right] + \frac{3a_0 - a_1}{6} \left[1 - 3x + \frac{(-3x)^2}{2!} + \frac{(-3x)^3}{3!} + \dots \right] \\
&= \frac{3a_0}{6} \left[1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \dots \right] + \frac{a_1}{6} \left[1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \dots \right] \\
&\quad + \frac{3a_0}{6} \left[1 - 3x + \frac{(3x)^2}{2!} - \frac{(3x)^3}{3!} + \dots \right] - \frac{a_1}{6} \left[1 - 3x + \frac{(3x)^2}{2!} - \frac{(3x)^3}{3!} + \dots \right] \\
&= \frac{1}{2}a_0 \left[1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \dots + 1 - 3x + \frac{(3x)^2}{2!} - \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \dots \right] \\
&\quad + \frac{a_1}{6} \left[1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \dots - 1 + 3x - \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} - \frac{(3x)^4}{4!} + \dots \right] \\
&= \frac{1}{2}a_0 \left[2 + 2\frac{(3x)^2}{2!} + 2\frac{(3x)^4}{4!} + \dots \right] + \frac{a_1}{6} \left[6x + 2\frac{(3x)^3}{3!} + 2\frac{(3x)^5}{5!} + \dots \right] \\
&= a_0 \left(1 + \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} + \dots \right) + a_1 \left(x + \frac{3^2 x^3}{3!} + \frac{3^4 x^5}{5!} + \dots \right)
\end{aligned}$$

Which are identical.

Exercise

Find a power series solution. $y'' + y = 0$

Solution

The equation $y'' + y = 0$ has a characteristic equation:

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

\therefore The general solution: $y(x) = C_1 \sin x + C_2 \cos x$

With $y(0) = a_0$ and $y'(0) = a_1$

$$y(0) = C_2 = a_0$$

$$y'(x) = C_1 \cos x - C_2 \sin x$$

$$y(0) = C_1 = a_1$$

$$y(x) = a_1 \sin x + a_0 \cos x$$

$$= a_0 \cos x + a_1 \sin x$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + a_n] x^n = 0$$

$$(n+2)(n+1) a_{n+2} + a_n = 0 \rightarrow a_{n+2} = \frac{-1}{(n+2)(n+1)} a_n, \quad n \geq 0$$

$$a_2 = \frac{1}{(2)(1)} a_0 = -\frac{1}{2} a_0$$

$$a_3 = \frac{1}{(3)(2)} a_1 = -\frac{1}{2 \cdot 3} a_1$$

$$a_4 = -\frac{1}{(4)(3)} a_2 = \frac{1}{2 \cdot 3 \cdot 4} a_0$$

$$a_5 = -\frac{1}{(5)(4)} a_3 = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} a_1$$

$$a_6 = -\frac{1}{(6)(5)} a_4 = -\frac{1}{6!} a_0$$

$$a_7 = -\frac{1}{(7)(6)} a_5 = -\frac{1}{7!} a_1$$

$$a_{2n} = \frac{(-1)^n}{(2n)!} a_0$$

$$a_{2n+1} = \frac{(-1)^n}{(2n+1)!} a_1$$

$$y(x) = a_0 \left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \dots + \frac{(-1)^n}{(2n)!} x^{2n} + \dots \right) + a_1 \left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \dots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \dots \right)$$

$$= a_0 \cos x + a_1 \sin x$$



ExerciseFind a power series solution. $y'' - y = 0$ **Solution**

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - a_n] x^n = 0$$

$$a_{n+2} = \frac{1}{(n+2)(n+1)} a_n \quad \Bigg|$$

$$n=0 \rightarrow a_2 = \frac{1}{1 \cdot 2} a_0$$

$$n=2 \rightarrow a_4 = \frac{1}{3 \cdot 4} a_2 = \frac{1}{4!} a_0$$

$$n=4 \rightarrow a_6 = \frac{1}{5 \cdot 6} a_4 = \frac{1}{6!} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2k} = \frac{1}{(2k)!} a_0 \quad \Bigg|$$

$$n=1 \rightarrow a_3 = \frac{1}{2 \cdot 3} a_1$$

$$n=3 \rightarrow a_5 = \frac{1}{4 \cdot 5} a_3 = \frac{1}{5!} a_1$$

$$n=5 \rightarrow a_7 = \frac{1}{6 \cdot 7} a_5 = \frac{1}{7!} a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2k+1} = \frac{1}{(2k+1)!} a_1 \quad \Bigg|$$

$$y(x) = a_0 \left(1 + \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \frac{1}{6!} x^6 + \dots \right) + a_1 \left(x + \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \dots \right)$$

$$= a_0 \cosh x + a_1 \sinh x \quad \Bigg|$$

ExerciseFind a power series solution. $y'' + y = x$ **Solution**

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + y = x$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = x$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n + a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n - x = 0$$

$$\sum_{n=2}^{\infty} [(n+2)(n+1) a_{n+2} + a_n] x^n + a_0 + 2a_2 + (6a_3 + a_1 - 1)x = 0$$

$$\begin{cases} a_0 + 2a_2 = 0 \\ 6a_3 + a_1 - 1 = 0 \\ (n+2)(n+1) a_{n+2} + a_n = 0 \end{cases}$$

$$\begin{cases} a_2 = -\frac{1}{2} a_0 \\ a_3 = -\frac{1}{6} (a_1 - 1) \\ a_{n+2} = -\frac{1}{(n+2)(n+1)} a_n \end{cases}$$

$$n=2 \rightarrow a_4 = -\frac{1}{3 \cdot 4} a_2 = \frac{1}{4!} a_0 \quad n=3 \rightarrow a_5 = -\frac{1}{4 \cdot 5} a_3 = \frac{1}{5!} (a_1 - 1)$$

$$n=4 \rightarrow a_6 = -\frac{1}{5 \cdot 6} a_4 = -\frac{1}{6!} a_0 \quad n=5 \rightarrow a_7 = -\frac{1}{6 \cdot 7} a_5 = -\frac{1}{7!} (a_1 - 1)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\left. a_{2k} = \frac{(-1)^k}{(2k)!} a_0 \right| \qquad \left. a_{2k+1} = \frac{(-1)^k}{(2k+1)!} (a_1 - 1) \right|$$

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_0 \sum_{k=2}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + (a_1 - 1) \sum_{k=3}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\ &= a_0 + a_0 \left(-\frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \dots \right) + (a_1 - 1) \left(-\frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \dots \right) + a_1 x - \textcolor{red}{x} + \textcolor{red}{x} \\ &= a_0 \left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \dots \right) + (a_1 - 1) \left(-\frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \dots \right) + (a_1 - 1)x + x \\ &= x + a_0 \left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \dots \right) + (a_1 - 1) \left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \dots \right) \\ &= \textcolor{blue}{x + a_0 \cos x + (a_1 - 1) \sin x} \end{aligned}$$

Exercise

Find a power series solution. $y'' - xy = 0$

Solution

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \\ y''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \end{aligned}$$

$$\textcolor{red}{y'' - xy = 0}$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - a_{n-1}]x^n = 0$$

$$2a_2 = 0 \rightarrow \underline{a_2 = 0}$$

$$(n+2)(n+1)a_{n+2} - a_{n-1} = 0 \rightarrow \underline{a_{n+2} = \frac{a_{n-1}}{(n+1)(n+2)}}$$

$$a_3 = \frac{a_0}{2 \cdot 3} = \frac{1}{6}a_0 \qquad a_4 = \frac{a_1}{3 \cdot 4} = \frac{1}{12}a_1 \qquad a_5 = \frac{a_2}{4 \cdot 5} = 0$$

$$a_6 = \frac{a_3}{5 \cdot 6} = \frac{1}{180}a_0 \qquad a_7 = \frac{a_4}{6 \cdot 7} = \frac{1}{504}a_1 \qquad a_8 = \frac{a_5}{7 \cdot 8} = 0$$

$$\vdots \vdots$$

$$\vdots \vdots$$

$$\vdots \vdots$$

$$\left\{ \begin{array}{l} y_1(x) = \left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots\right)a_0 \\ y_2(x) = \left(x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots\right)a_1 \end{array} \right.$$

Exercise

Find a power series solution. $y'' + xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + xy = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} + a_{n-1}]x^n = 0$$

$$\left\{ \begin{array}{l} 2a_2 = 0 \rightarrow \underline{a_2 = 0} \\ (n+2)(n+1)a_{n+2} + a_{n-1} = 0 \end{array} \right.$$

$$\underline{a_{n+2} = -\frac{a_{n-1}}{(n+1)(n+2)}}$$

 a_0 a_1 $a_2 = 0$

$$n=1 \rightarrow a_3 = -\frac{a_0}{2 \cdot 3} = -\frac{1}{6}a_0$$

$$n=2 \rightarrow a_4 = -\frac{1}{3 \cdot 4}a_1$$

$$n=3 \rightarrow a_5 = -\frac{a_2}{20} = 0$$

$$n=4 \rightarrow a_6 = -\frac{a_3}{5 \cdot 6} = \frac{1}{180}a_0$$

$$n=5 \rightarrow a_7 = -\frac{a_4}{6 \cdot 7} = \frac{1}{504}a_1$$

$$n=6 \rightarrow a_8 = -\frac{a_5}{56} = 0$$

$$n=7 \rightarrow a_9 = -\frac{a_6}{8 \cdot 9} = -\frac{1}{12,960}a_0$$

$$\begin{array}{cccc} \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

$$\left\{ \begin{array}{l} y_1(x) = \left(1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \frac{1}{12,960}x^9 + \dots \right) a_0 \\ y_2(x) = \left(x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \dots \right) a_1 \end{array} \right.$$

$$\underline{y(x) = \left(1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \frac{1}{12,960}x^9 + \dots \right) a_0 + \left(x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \dots \right) a_1}$$

ExerciseFind a power series solution. $y'' + xy' + y = 0$ **Solution**

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + xy' + y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n+1) a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} + (n+1) a_n = 0$$

$$\boxed{a_{n+2} = -\frac{a_n}{n+2}}$$

 a_0

$$n=0 \rightarrow a_2 = -\frac{1}{2} a_0$$

$$n=2 \rightarrow a_4 = -\frac{1}{4} a_2 = \frac{1}{4 \cdot 2} a_0$$

$$n=4 \rightarrow a_6 = -\frac{a_4}{6} = -\frac{1}{6 \cdot 4 \cdot 2} a_0$$

 a_1

$$n=1 \rightarrow a_3 = -\frac{1}{3} a_1$$

$$n=3 \rightarrow a_5 = -\frac{a_3}{5} = \frac{1}{3 \cdot 5} a_1$$

$$n=5 \rightarrow a_7 = -\frac{a_5}{7} = -\frac{1}{7 \cdot 5 \cdot 3} a_1$$

$$n=6 \rightarrow a_8 = -\frac{a_6}{8} = \frac{1}{8 \cdot 6 \cdot 4 \cdot 2} a_0 \qquad n=7 \rightarrow a_9 = -\frac{a_7}{9} = \frac{1}{9 \cdot 7 \cdot 5 \cdot 3} a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n} = \frac{(-1)^n a_0}{(2n)(2n-2)\cdots 4 \cdot 2} = \frac{(-1)^n}{n! \, 2^n} a_0$$

$$a_{2n} = \frac{(-1)^n a_1}{(2n+1)(2n-1)\cdots 5 \cdot 3} = \frac{(-1)^n}{(2n+1)!!} a_1$$

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \, 2^n} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!!} x^{2n+1}$$

$$y(x) = a_0 \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \cdots \right) + a_1 \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5 - \frac{1}{105}x^7 + \cdots \right)$$

Exercise

Find a power series solution. $y'' - xy' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy' - y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} na_nx^n - \sum_{n=0}^{\infty} a_nx^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n+1)a_n \right] x^n = 0$$

$$(n+2)(n+1)a_{n+2} - (n+1)a_n = 0$$

$$\boxed{a_{n+2} = \frac{a_n}{n+2}}$$

 a_0

$$n=0 \rightarrow a_2 = \frac{1}{2}a_0$$

$$n=2 \rightarrow a_4 = \frac{1}{4}a_2 = \frac{1}{4 \cdot 2}a_0$$

$$n=4 \rightarrow a_6 = \frac{a_4}{6} = \frac{1}{6 \cdot 4 \cdot 2}a_0$$

$$n=6 \rightarrow a_8 = \frac{a_6}{8} = \frac{1}{8 \cdot 6 \cdot 4 \cdot 2}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n} = \frac{a_0}{(2n)(2n-2)\cdots 4 \cdot 2} = \frac{1}{n!} \frac{1}{2^n} a_0$$

$$a_{2n} = \frac{a_1}{(2n+1)(2n-1)\cdots 5 \cdot 3} = \frac{1}{(2n+1)!!} a_1$$

 a_1

$$n=1 \rightarrow a_3 = \frac{1}{3}a_1$$

$$n=3 \rightarrow a_5 = \frac{a_3}{5} = \frac{1}{3 \cdot 5}a_1$$

$$n=5 \rightarrow a_7 = \frac{a_5}{7} = \frac{1}{7 \cdot 5 \cdot 3}a_1$$

$$n=7 \rightarrow a_9 = \frac{a_7}{9} = \frac{1}{9 \cdot 7 \cdot 5 \cdot 3}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = a_0 \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{2^n} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!!} x^{2n+1}}$$

$$\boxed{y(x) = a_0 \left(1 + \frac{x^2}{2} + \frac{x^4}{2^2 \cdot 2} + \cdots + \frac{x^{2n}}{2^n \cdot n!} \right) + a_1 \left(x + \frac{x^3}{3} + \frac{x^5}{5 \cdot 3} + \cdots + \frac{2^n \cdot n!}{(2n+1)!!} x^{2n+1} \right)}$$

Exercise

Find a power series solution. $y'' + x^2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + x^2 y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} [(n+2)(n+1) a_{n+2} + a_{n-2}] x^n = 0$$

$$2a_2 = 0 \rightarrow \underline{a_2 = 0}$$

$$6a_3 = 0 \rightarrow \underline{a_3 = 0}$$

$$(n+2)(n+1) a_{n+2} + a_{n-2} = 0 \rightarrow a_{n+2} = \frac{-a_{n-2}}{(n+1)(n+2)}$$

$$a_4 = \frac{-a_0}{3 \cdot 4} = -\frac{1}{12} a_0 \quad a_5 = \frac{-a_1}{4 \cdot 5} = -\frac{1}{20} a_1 \quad a_6 = \frac{-a_2}{5 \cdot 6} = 0 \quad a_7 = \frac{-a_3}{6 \cdot 7} = 0$$

$$a_8 = \frac{-a_4}{7 \cdot 8} = \frac{1}{672} a_0 \quad a_9 = \frac{-a_5}{8 \cdot 9} = \frac{1}{1,440} a_1 \quad a_{10} = \frac{-a_6}{9 \cdot 10} = 0 \quad a_{11} = \frac{-a_7}{10 \cdot 11} = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\left\{ \begin{array}{l} y_1(x) = \left(1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 - \cdots\right)a_0 \\ y_2(x) = \left(x - \frac{1}{20}x^5 + \frac{1}{1,440}x^9 + \cdots\right)a_1 \end{array} \right.$$

Exercise

Find a power series solution. $y'' + k^2 x^2 y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + k^2 x^2 y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + k^2 x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} k^2 a_{n-2} x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} + k^2 a_{n-2} \right] x^n = 0$$

$$\left\{ \begin{array}{l} a_2 = 0 \\ a_3 = 0 \\ (n+2)(n+1) a_{n+2} + k^2 a_{n-2} = 0 \end{array} \right.$$

$$a_{n+2} = -\frac{k^2}{(n+1)(n+2)} a_{n-2} \quad (n \geq 2)$$

$$n = 2 \rightarrow a_4 = -\frac{k^2}{3 \cdot 4} a_0$$

$$n = 3 \rightarrow a_5 = -\frac{k^2}{4 \cdot 5} a_1$$

$$n = 6 \rightarrow a_8 = -\frac{k^2}{7 \cdot 8} a_4 = \frac{k^4}{3 \cdot 4 \cdot 7 \cdot 8} a_0 \qquad n = 7 \rightarrow a_9 = -\frac{k^2}{8 \cdot 9} a_5 = \frac{k^4}{4 \cdot 5 \cdot 8 \cdot 9} a_1$$

$$n = 10 \rightarrow a_{12} = -\frac{k^2}{11 \cdot 12} a_8 = \frac{k^6}{3 \cdot 4 \cdot 7 \cdot 8 \cdot 11 \cdot 12} a_0 \qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{4m} = -\frac{k^2}{(4m)(4m-1)} a_{4m-4}$$

$$a_{4m+1} = -\frac{k^2}{(4m)(4m+1)} a_{4m-3}$$

$$n = 4 \rightarrow a_6 = -\frac{k^2}{5 \cdot 6} a_2 = 0$$

$$n = 5 \rightarrow a_7 = -\frac{k^2}{6 \cdot 7} a_3 = 0$$

$$n = 8 \rightarrow a_{10} = -\frac{k^2}{7 \cdot 8} a_6 = 0$$

$$n = 9 \rightarrow a_{11} = -\frac{k^2}{10 \cdot 11} a_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(x) = a_0 \left(1 - \frac{k^2}{3 \cdot 4} x^4 + \frac{k^4}{3 \cdot 4 \cdot 7 \cdot 8} x^8 - \frac{k^6}{3 \cdot 4 \cdot 7 \cdot 8 \cdot 11 \cdot 12} x^{12} + \dots \right) \\ + a_1 \left(x - \frac{k^2}{4 \cdot 5} x^5 + \frac{k^4}{4 \cdot 5 \cdot 8 \cdot 9} x^9 - \frac{k^6}{4 \cdot 5 \cdot 8 \cdot 9 \cdot 12 \cdot 13} x^{13} + \dots \right)$$

Exercise

Find a power series solution. $y'' + 3xy' + 3y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + 3xy' + 3y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 3x \sum_{n=1}^{\infty} n a_n x^{n-1} + 3 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 3na_nx^n + \sum_{n=0}^{\infty} 3a_nx^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + (3n+3)a_n \right] x^n = 0$$

$$(n+2)(n+1)a_{n+2} + 3(n+1)a_n = 0$$

$$\boxed{a_{n+2} = -\frac{3}{n+2}a_n}$$

$$a_0$$

$$n=2 \rightarrow a_2 = -\frac{3}{2}a_0$$

$$n=4 \rightarrow a_4 = -\frac{3}{4}a_2 = \frac{3^2}{2^2 \cdot 2}a_0$$

$$n=6 \rightarrow a_6 = -\frac{3}{6}a_4 = \frac{3^3}{2^3 \cdot 2 \cdot 3}a_0$$

$$n=8 \rightarrow a_8 = -\frac{3}{8}a_6 = \frac{3^4}{2^4 \cdot 2 \cdot 3 \cdot 4}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2k} = \frac{(-3)^k}{2^k k!} a_0$$

$$a_1$$

$$n=3 \rightarrow a_3 = -\frac{3}{3}a_1$$

$$n=5 \rightarrow a_5 = -\frac{3}{5}a_3 = \frac{3^2}{3 \cdot 5}a_1$$

$$n=7 \rightarrow a_7 = -\frac{3}{7}a_5 = \frac{3^3}{3 \cdot 5 \cdot 7}a_1$$

$$n=9 \rightarrow a_9 = -\frac{3}{9}a_7 = \frac{3^3}{3 \cdot 5 \cdot 7 \cdot 9}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2k+1} = \frac{(-3)^k}{3 \cdot 5 \cdot 7 \cdots (2k+1)} a_1$$

$$\begin{aligned} y(x) &= a_0 \left(1 + \sum_{k=1}^{\infty} \frac{(-3)^k}{2^k k!} x^{2k} \right) + a_1 \left(x + \sum_{k=1}^{\infty} \frac{(-3)^k}{3 \cdot 5 \cdot 7 \cdots (2k+1)} x^{2k+1} \right) \\ &= a_0 \left(1 - \frac{3}{2}x^2 + \frac{9}{8}x^4 - \frac{27}{56}x^6 + \cdots \right) + a_1 \left(x - x^3 + \frac{3^2}{3 \cdot 5}x^5 - \frac{27}{3 \cdot 5 \cdot 7}x^7 + \cdots \right) \end{aligned}$$

Exercise

Find a power series solution. $y'' - 2xy' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - 2xy' + y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n = 0$$

$$2a_2 + a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - 2n a_n + a_n] x^n = 0$$

$$2a_2 + a_0 = 0 \rightarrow \underline{a_2 = -\frac{1}{2}a_0}$$

$$(n+2)(n+1) a_{n+2} - (2n-1) a_n = 0 \rightarrow \underline{a_{n+2} = \frac{(2n-1) a_n}{(n+1)(n+2)}}$$

$$a_3 = \frac{a_1}{2 \cdot 3} = \frac{1}{6} a_1$$

$$a_4 = \frac{3a_2}{3 \cdot 4} = -\frac{1}{4} \frac{1}{4} a_0 = -\frac{1}{8} a_0$$

$$a_5 = \frac{5a_3}{4 \cdot 5} = \frac{1}{2 \cdot 3 \cdot 4} a_1 = \frac{1}{4!} a_1$$

$$a_6 = \frac{7a_4}{5 \cdot 6} = -\frac{7}{240} a_0$$

$$a_7 = \frac{9a_5}{6 \cdot 7} = \frac{1}{112} a_1$$

$$\vdots \vdots$$

$$\vdots \vdots$$

$$\left\{ \begin{array}{l} y_1(x) = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6 - \dots \right) a_0 \\ y_2(x) = \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{112}x^7 + \dots \right) a_1 \end{array} \right.$$

ExerciseFind a power series solution. $y'' - xy' + 2y = 0$ **Solution**

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy' + 2y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + 2a_0 + \sum_{n=1}^{\infty} 2a_n x^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - n a_n + 2a_n] x^n = 0$$

$$2a_2 + 2a_0 = 0 \rightarrow \underline{a_2 = -a_0}$$

$$(n+2)(n+1) a_{n+2} - (n-2) a_n = 0$$

$$\rightarrow a_{n+2} = \frac{(n-2) a_n}{(n+1)(n+2)}$$

$$a_3 = \frac{-a_1}{2 \cdot 3} = -\frac{1}{6} a_1$$

$$a_5 = \frac{a_3}{4 \cdot 5} = -\frac{1}{5!} a_1$$

$$a_7 = \frac{3a_5}{6 \cdot 7} = \frac{3}{7!} a_1$$

$$a_4 = 0$$

$$a_6 = \frac{2a_4}{5 \cdot 6} = 0$$

$$a_8 = \frac{4a_6}{5 \cdot 6} = 0$$

$$\vdots$$

$$a_9 = \frac{5a_7}{8 \cdot 9} = \frac{3 \cdot 5}{9!} a_1$$

$$\vdots$$

$$\left\{ \begin{array}{l} y_1(x) = 1 - x^2 \\ y_2(x) = \left(x - \frac{1}{6}x^3 + \frac{1}{5!}x^5 + \frac{3}{7!}x^7 + \cdots \right) a_1 \end{array} \right.$$

Exercise

Find a power series solution. $y'' - xy' - x^2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy' - x^2y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=1}^{\infty} n a_n x^{n-1} - x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$2a_2 + 6a_3x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n - a_1x - \sum_{n=2}^{\infty} n a_n x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$2a_2 + 6a_3x - a_1x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n - a_{n-2} \right] x^n = 0$$

$$\left\{ \begin{array}{l} 2a_2 = 0 \rightarrow \underline{a_2 = 0} \\ (6a_3 - a_1)x = 0 \rightarrow \underline{a_3 = \frac{1}{6}a_1} \\ (n+1)(n+2)a_{n+2} - na_n - a_{n-2} = 0 \\ \underline{a_{n+2} = \frac{na_n + a_{n-2}}{(n+1)(n+2)}} \end{array} \right.$$

$$\begin{array}{c} a_0 \\ a_2 = 0 \end{array}$$

$$\begin{array}{c} a_1 \\ a_3 = \frac{1}{6}a_1 \end{array}$$

$$n = 2 \rightarrow a_4 = \frac{2a_2 + a_0}{3 \cdot 4} = \frac{1}{12}a_0$$

$$n = 3 \rightarrow a_5 = \frac{3a_3 + a_1}{20} = \frac{1}{20}\left(\frac{3}{6} + 1\right)a_1 = \frac{1}{12}a_1$$

$$n = 4 \rightarrow a_6 = \frac{4a_4 + a_2}{5 \cdot 6} = \frac{1}{90}a_0$$

$$n = 5 \rightarrow a_7 = \frac{5a_5 + a_3}{6 \cdot 7} = \frac{1}{42}\left(\frac{5}{12} + \frac{1}{6}\right)a_1 = \frac{1}{72}a_1$$

$$n = 6 \rightarrow a_8 = \frac{6a_6 + a_4}{7 \cdot 8} = \frac{1}{56}\left(\frac{1}{15} + \frac{1}{12}\right)a_0 = \frac{3}{1120}a_0$$

$$\begin{array}{cccc} \vdots & \vdots & \vdots & \vdots \end{array}$$

$$\begin{array}{cccc} \vdots & \vdots & \vdots & \vdots \end{array}$$

$$\underline{y(x) = a_0 \left(1 + \frac{1}{12}x^2 + \frac{1}{90}x^4 + \frac{3}{1120}x^6 + \cdots\right) + a_1 \left(x + \frac{1}{12}x^3 + \frac{1}{72}x^5 + \cdots\right)}$$

Exercise

Find a power series solution. $y'' + x^2y' + xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$$y'' + x^2y' + xy = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + x^2 \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} (n+1)a_{n+1}x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty} (n-1)a_{n-1}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

$$2a_2 + 6a_3x + \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty} (n-1)a_{n-1}x^n + a_0x + \sum_{n=2}^{\infty} a_{n-1}x^n = 0$$

$$2a_2 + (6a_3 + a_0)x + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} + (n-1)a_{n-1} + a_{n-1}]x^n = 0$$

$$2a_2 + (6a_3 + a_0)x = 0 \Rightarrow \begin{cases} a_2 = 0 \\ a_3 = -\frac{1}{6}a_0 \end{cases}$$

$$(n+2)(n+1)a_{n+2} + na_{n-1} = 0 \Rightarrow a_{n+2} = -\frac{n}{(n+1)(n+2)}a_{n-1}$$

$$\begin{array}{lll} a_4 = -\frac{2}{3 \cdot 4}a_1 = -\frac{1}{6}a_1 & a_5 = -\frac{3}{4 \cdot 5}a_2 = 0 & a_6 = -\frac{4}{5 \cdot 6}a_3 = \frac{1}{45}a_0 \\ a_7 = -\frac{5}{6 \cdot 7}a_4 = \frac{5}{252}a_1 & a_8 = -\frac{6}{7 \cdot 8}a_5 = 0 & a_9 = -\frac{7}{8 \cdot 9}a_3 = -\frac{7}{3,240}a_0 \\ \vdots & \vdots & \vdots \end{array}$$

$$\left\{ \begin{array}{l} y_1(x) = \left(1 - \frac{1}{6}x^3 + \frac{1}{45}x^6 - \frac{7}{3,240}x^9 + \cdots \right) a_0 \\ y_2(x) = \left(x - \frac{1}{6}x^4 + \frac{5}{252}x^7 - \cdots \right) a_1 \end{array} \right.$$

Exercise

Find a power series solution. $y'' + x^2y' + 2xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$$y'' + x^2 y' + 2xy = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + x^2 \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + 2x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1)a_{n+1} x^{n+2} + \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=2}^{\infty} (n-1)a_{n-1} x^n + \sum_{n=1}^{\infty} 2a_{n-1} x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=2}^{\infty} (n-1)a_{n-1} x^n + 2a_0 x + \sum_{n=2}^{\infty} 2a_{n-1} x^n = 0$$

$$2a_2 + (6a_3 + 2a_0)x + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} + (n-1)a_{n-1} + 2a_{n-1}] x^n = 0$$

$$2a_2 + (6a_3 + 2a_0)x = 0 \Rightarrow \begin{cases} a_2 = 0 \\ a_3 = -\frac{1}{3}a_0 \end{cases}$$

$$(n+2)(n+1)a_{n+2} + (n+1)a_{n-1} = 0$$

$$\Rightarrow a_{n+2} = -\frac{a_{n-1}}{n+2} \quad \left| \quad a_{n+3} = -\frac{a_n}{n+3} \right|$$

a_0	a_1	$a_2 = 0$
$a_3 = -\frac{1}{3}a_0$	$n=1 \rightarrow a_4 = -\frac{1}{4}a_1$	$n=2 \rightarrow a_5 = -\frac{a_2}{5} = 0$
$n=3 \rightarrow a_6 = -\frac{a_3}{6} = \frac{1}{2 \cdot 3^2}a_0$	$n=4 \rightarrow a_7 = -\frac{a_4}{7} = \frac{1}{7 \cdot 4}a_1$	$n=5 \rightarrow a_8 = -\frac{a_5}{8} = 0$
$n=6 \rightarrow a_9 = -\frac{a_6}{9} = -\frac{1}{3! \cdot 3^3}a_0$	$n=7 \rightarrow a_{10} = -\frac{a_7}{10} = -\frac{1}{10 \cdot 7 \cdot 4 \cdot 1}a_1$	$n=8 \rightarrow a_{11} = 0$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$a_{3n} = \frac{(-1)^n}{n! \cdot 3^n}$	$a_{3n+1} = \frac{(-1)^n}{1 \cdot 4 \cdot 7 \cdots (3n+1)}$	$a_{3n+2} = 0$

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n! 3^n} x^{3n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{1 \cdot 4 \cdot 7 \cdots (3n+1)} x^{3n+1}$$

$$y(x) = a_0 \left(1 - \frac{1}{3}x^3 + \frac{1}{18}x^6 - \frac{1}{162}x^9 + \cdots \right) + a_1 \left(x - \frac{1}{4}x^4 + \frac{1}{28}x^7 - \frac{1}{280}x^{10} + \cdots \right)$$

Exercise

Find a power series solution. $y'' - x^2 y' - 3xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - x^2 y' - 3xy = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x^2 \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 3x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+2} - \sum_{n=0}^{\infty} 3a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n - \sum_{n=1}^{\infty} 3a_{n-1} x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n - 3a_0 x - \sum_{n=2}^{\infty} 3a_{n-1} x^n = 0$$

$$2a_2 + 3(2a_3 - a_0)x + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} - (n+2)a_{n-1}] x^n = 0$$

$$\left\{ \begin{array}{l} 2a_2 = 0 \rightarrow \underline{a_2 = 0} \\ 2a_3 - a_0 = 0 \rightarrow \underline{a_3 = \frac{1}{2}a_0} \end{array} \right|$$

$$(n+2)(n+1)a_{n+2} = (n+2)a_{n-1}$$

$$a_{n+2} = \frac{a_{n-1}}{n+1} \rightarrow \underline{a_{n+3} = \frac{a_n}{n+2}}$$

$$a_0$$

$$a_3 = \frac{1}{2}a_0$$

$$n=3 \rightarrow a_6 = \frac{a_3}{5} = \frac{1}{2 \cdot 5}a_0$$

$$n=6 \rightarrow a_9 = \frac{a_6}{8} = -\frac{1}{2 \cdot 5 \cdot 8}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{3n} = \frac{1}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$$

$$a_1$$

$$n=1 \rightarrow a_4 = \frac{1}{3}a_1$$

$$n=4 \rightarrow a_7 = \frac{a_4}{6} = \frac{1}{2 \cdot 3^2}a_1$$

$$n=7 \rightarrow a_{10} = \frac{a_7}{9} = \frac{1}{2 \cdot 3 \cdot 3^3}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{3n+1} = \frac{1}{n! 3^n}$$

$$a_2 = 0$$

$$n=2 \rightarrow a_5 = \frac{a_2}{5} = 0$$

$$n=5 \rightarrow a_8 = 0$$

$$n=8 \rightarrow a_{11} = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{3n+2} = 0$$

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{2 \cdot 5 \cdot 8 \cdots (3n-1)} + a_1 \sum_{n=0}^{\infty} \frac{x^{3n+1}}{n! 3^n}$$

$$y(x) = a_0 \left(1 + \frac{1}{2}x^3 + \frac{1}{10}x^6 + \frac{1}{80}x^9 + \cdots \right) + a_1 \left(x + \frac{1}{3}x^4 + \frac{1}{18}x^7 + \frac{1}{162}x^{10} + \cdots \right)$$

Exercise

Find a power series solution. $y'' + 2xy' + 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + 2xy' + 2y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + 2x \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 2(n+1)a_{n+1}x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} 2na_n x^n + 2a_0 + \sum_{n=1}^{\infty} 2a_n x^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + 2na_n + 2a_n \right] x^n = 0$$

$$2a_2 + 2a_0 = 0 \rightarrow \underline{a_2 = -a_0}$$

$$(n+2)(n+1)a_{n+2} + 2(n+1)a_n = 0$$

$$\rightarrow \underline{a_{n+2} = -\frac{2}{n+2}a_n} \quad n=1, 2, \dots$$

$$a_3 = -\frac{2}{3}a_1 \qquad a_4 = -\frac{2}{4}a_2 = \frac{1}{2}a_0$$

$$a_5 = -\frac{2}{5}a_3 = \frac{4}{15}a_1 \qquad a_6 = -\frac{2}{6}a_4 = -\frac{1}{6}a_0$$

$$a_7 = -\frac{2}{7}a_5 = -\frac{8}{105}a_1 \qquad a_8 = -\frac{2}{8}a_6 = \frac{1}{24}a_0$$

$$\vdots \quad \vdots$$

$$\left\{ \begin{array}{l} y_1(x) = \left(1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 - \dots \right) a_0 \\ y_2(x) = \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7 + \dots \right) a_1 \end{array} \right.$$

Exercise

Find a power series solution. $2y'' + xy' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$2y'' + xy' + y = 0$$

$$2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$4a_2 + \sum_{n=1}^{\infty} 2(n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n = 0$$

$$4a_2 + a_0 + \sum_{n=1}^{\infty} [2(n+2)(n+1) a_{n+2} + (n+1) a_n] x^n = 0$$

$$4a_2 + a_0 = 0 \rightarrow \underline{a_2 = -\frac{1}{4}a_0}$$

$$2(n+2)(n+1) a_{n+2} + (n+1) a_n = 0$$

$$\underline{a_{n+2} = -\frac{1}{2(n+2)} a_n}$$

 a_0

$$n=0 \rightarrow a_2 = -\frac{1}{4}a_0$$

$$n=2 \rightarrow a_4 = -\frac{1}{8}a_2 = \frac{1}{2^4 \cdot 2}a_0$$

$$n=4 \rightarrow a_6 = -\frac{1}{2 \cdot 6}a_4 = -\frac{1}{2^6 \cdot 2 \cdot 3}a_0$$

 a_1

$$n=1 \rightarrow a_3 = -\frac{1}{6}a_1$$

$$n=3 \rightarrow a_5 = -\frac{1}{2 \cdot 5}a_3 = \frac{1}{2^2 \cdot 3 \cdot 5}a_1$$

$$n=5 \rightarrow a_7 = -\frac{1}{2 \cdot 7}a_5 = -\frac{1}{2^3 \cdot 3 \cdot 5 \cdot 7}a_1$$

$$n = 6 \rightarrow a_8 = -\frac{1}{2 \cdot 8} a_6 = \frac{1}{2^8 \cdot 4!} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\begin{aligned} n \geq 1 \quad a_{2n} &= \frac{(-1)^n}{2^{2n} n!} a_0 \\ n \geq 1 \quad a_{2n+1} &= \frac{(-1)^n}{2^n 1 \cdot 3 \cdot 5 \cdots (2n+1)} a_1 \\ &= \frac{(-1)^n}{2^n (2n+1)!!} a_1 \end{aligned}$$

$$y(x) = a_0 \left(1 - \frac{1}{4}x^2 + \frac{1}{32}x^4 - \dots \right) + a_1 \left(x - \frac{1}{6}x^3 + \frac{1}{60}x^5 - \dots \right)$$

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n (2n+1)!!} x^{2n+1}$$

Exercise

Find a power series solution. $3y'' + xy' - 4y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$3y'' + xy' - 4y = 0$$

$$3 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 3(n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 3(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} na_nx^n - \sum_{n=0}^{\infty} 4a_nx^n = 0$$

$$6a_2 + \sum_{n=1}^{\infty} 3(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} na_nx^n - 4a_0 - \sum_{n=1}^{\infty} 4a_nx^n = 0$$

$$6a_2 - 4a_0 + \sum_{n=1}^{\infty} [3(n+2)(n+1)a_{n+2} + (n-4)a_n]x^n = 0$$

$$6a_2 - 4a_0 \rightarrow \underline{a_2 = \frac{2}{3}a_0}$$

$$3(n+2)(n+1)a_{n+2} + (n-4)a_n = 0$$

$$\underline{a_{n+2} = -\frac{(n-4)}{3(n+1)(n+2)}a_n}$$

 a_0

$$a_2 = \frac{2}{3}a_0$$

$$n=2 \rightarrow a_4 = \frac{2}{36}a_2 = \frac{1}{27}a_0$$

$$n=4 \rightarrow a_6 = 0$$

 $\vdots \quad \vdots \quad \vdots \quad \vdots$
 a_1

$$n=1 \rightarrow a_3 = \frac{3}{2 \cdot 3 \cdot 3}a_1 = \frac{1}{2 \cdot 3}a_1$$

$$n=3 \rightarrow a_5 = \frac{1}{4 \cdot 5 \cdot 3}a_3 = \frac{1}{5! \cdot 3}a_1$$

$$n=5 \rightarrow a_7 = -\frac{1}{3 \cdot 6 \cdot 7}a_5 = -\frac{1}{7! \cdot 3^2}a_1$$

$$n=7 \rightarrow a_9 = -\frac{3}{3 \cdot 9 \cdot 8}a_7 = \frac{3}{9! \cdot 3^3}a_1$$

$$n=9 \rightarrow a_{11} = -\frac{5}{3 \cdot 11 \cdot 10}a_9 = -\frac{3 \cdot 5}{11! \cdot 3^4}a_1$$

 $\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\begin{aligned} n \geq 3 \quad a_{2n+1} &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-5)(-1)^n}{(2n+1)! \, 3^n} a_1 \\ &= \frac{(2n-5)!!(-1)^n}{(2n+1)! \, 3^n} a_1 \end{aligned}$$

$$\underline{y(x) = a_0 \left(1 + \frac{2}{3}x^2 + \frac{1}{27}x^4 \right) + a_1 \left(x + \frac{1}{6}x^3 + \frac{1}{360}x^5 + \sum_{n=3}^{\infty} \frac{(2n-5)!!(-1)^n}{(2n+1)! \, 3^n} \right)}$$

$$\underline{= a_0 \left(1 + \frac{2}{3}x^2 + \frac{1}{27}x^4 \right) + a_1 \left(x + \frac{1}{6}x^3 + \frac{1}{360}x^5 - \frac{1}{45 \cdot 360}x^7 + \cdots \right)}$$

ExerciseFind a power series solution. $5y'' - 2xy' + 10y = 0$ **Solution**

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$5y'' - 2xy' + 10y = 0$$

$$5 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 10 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 5(n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} 10a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 5(n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 10a_n x^n = 0$$

$$10a_2 + \sum_{n=1}^{\infty} 5(n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + 10a_0 + \sum_{n=1}^{\infty} 10a_n x^n = 0$$

$$10a_2 + 10a_0 + \sum_{n=1}^{\infty} [5(n+2)(n+1) a_{n+2} - 2(n-5) a_n] x^n = 0$$

$$10a_2 + 10a_0 \rightarrow \underline{a_2 = -a_0}$$

$$5(n+2)(n+1) a_{n+2} - 2(n-5) a_n = 0$$

$$\underline{a_{n+2} = \frac{2(n-5)}{5(n+1)(n+2)} a_n}$$

 a_0

$$a_2 = -a_0$$

 a_1

$$n=1 \rightarrow a_3 = -\frac{8}{30} a_1 = -\frac{4}{15} a_1$$

$$n=2 \rightarrow a_4 = -\frac{6}{60}a_2 = \frac{1}{10}a_0$$

$$n=3 \rightarrow a_5 = -\frac{4}{100}a_3 = \frac{4}{375}a_1$$

$$n=4 \rightarrow a_6 = -\frac{2}{5 \cdot 5 \cdot 6}a_4 = -\frac{1}{750}a_0$$

$$n=5 \rightarrow a_7 = 0$$

$$n=6 \rightarrow a_8 = \frac{2}{5 \cdot 7 \cdot 8}a_6 = -\frac{2}{8! \cdot 5^2}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n=8 \rightarrow a_{10} = \frac{2 \cdot 3}{5 \cdot 9 \cdot 10}a_8 = -\frac{2^2 \cdot 3}{10! \cdot 5^3}a_0$$

$$n=10 \rightarrow a_{12} = \frac{2 \cdot 5}{5 \cdot 11 \cdot 12}a_8 = -\frac{2^3 \cdot 3 \cdot 5}{12! \cdot 5^4}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n \geq 4 \quad a_{2n} = -15 \cdot \frac{2^n (2n-7)!!}{5^n (2n)!} a_0$$

$$y(x) = a_0 \left(1 - x^2 + \frac{1}{10}x^4 - \frac{1}{750}x^6 - \frac{1}{105,000}x^8 - \dots \right) + a_1 \left(x - \frac{4}{15}x^3 + \frac{4}{375}x^5 \right) \Big|$$

Exercise

Find a power series solution. $(x-1)y'' + y' = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x-1)y'' + y' = 0$$

$$(x-1) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = 0$$

$$x \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^{n+1} - \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n = 0$$

$$\sum_{n=1}^{\infty} n(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n = 0$$

$$\sum_{n=1}^{\infty} n(n+1)a_{n+1}x^n - 2a_2 - \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n + a_1 + \sum_{n=1}^{\infty} (n+1)a_{n+1}x^n = 0$$

$$a_1 - 2a_2 + \sum_{n=1}^{\infty} [n(n+1)a_{n+1} - (n+2)(n+1)a_{n+2} + (n+1)a_{n+1}]x^n = 0$$

$$a_1 - 2a_2 + \sum_{n=1}^{\infty} [-(n+2)(n+1)a_{n+2} + (n+1)^2 a_{n+1}]x^n = 0$$

$$a_1 - 2a_2 = 0 \rightarrow \underline{a_2 = \frac{1}{2}a_1}$$

$$-(n+2)(n+1)a_{n+2} + (n+1)^2 a_{n+1}$$

$$\rightarrow \underline{a_{n+2} = \frac{n+1}{n+2}a_{n+1}} \quad n = 1, 2, \dots$$

$$a_3 = \frac{2}{3}a_2 = \frac{1}{3}a_1$$

$$a_4 = \frac{3}{4}a_3 = \frac{1}{4}a_1$$

$$a_5 = \frac{4}{5}a_4 = \frac{1}{5}a_1$$

$$\vdots$$

$$\left\{ \begin{array}{l} y_1(x) = a_0 \\ y_2(x) = \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots \right) a_1 \end{array} \right.$$

ExerciseFind a power series solution. $(x+2)y'' + xy' - y = 0$ **Solution**

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x+2)y'' + xy' - y = 0$$

$$(x+2) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+1} + \sum_{n=0}^{\infty} 2(n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2(n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n(n+1) a_{n+1} x^n + 4a_2 + \sum_{n=1}^{\infty} 2(n+2)(n+1) a_{n+2} x^n + 4a_2 + \sum_{n=1}^{\infty} n a_n x^n - a_0 - \sum_{n=1}^{\infty} a_n x^n = 0$$

$$4a_2 - a_0 + \sum_{n=1}^{\infty} (n(n+1) a_{n+1} + 2(n+2)(n+1) a_{n+2} + (n-1) a_n) x^n = 0$$

$$4a_2 - a_0 = 0 \rightarrow \underline{a_2 = \frac{1}{4} a_0} \quad \underline{a_1}$$

$$n(n+1) a_{n+1} + 2(n+2)(n+1) a_{n+2} + (n-1) a_n = 0$$

$$\underline{a_{n+2} = -\frac{n-1}{2(n+2)(n+1)} a_n - \frac{n}{2(n+2)} a_{n+1}} \quad n=1, 2, \dots$$

$$a_3 = -\frac{1}{6} a_2 = -\frac{1}{24} a_0$$

$$a_4 = -\frac{1}{24} a_2 - \frac{1}{4} a_3 = -\frac{1}{96} a_0 + \frac{1}{96} a_0 = 0$$

$$a_5 = -\frac{1}{20}a_3 - \frac{3}{10}a_4 = \frac{1}{480}a_0$$

$$a_6 = -\frac{3}{60}a_4 - \frac{1}{3}a_5 = -\frac{1}{1,440}a_0$$

$$\vdots \quad \vdots$$

$$\left\{ \begin{array}{l} y_1(x) = a_1 \\ y_2(x) = \left(1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 - \dots\right)a_0 \end{array} \right.$$

Exercise

Find a power series solution. $y'' - (x+1)y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - (x+1)y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - a_n] x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 - a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - a_n] x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 - a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - a_n - a_{n-1}] x^n = 0$$

$$2a_2 - a_0 = 0 \rightarrow \underline{a_2 = \frac{1}{2}a_0}$$

$$(n+2)(n+1)a_{n+2} - a_n - a_{n-1} = 0$$

$$\underline{a_{n+2} = \frac{a_n + a_{n-1}}{(n+1)(n+2)}}$$

$$a_0 \quad a_1$$

$$a_2 = \frac{1}{2}a_0$$

$$n=1 \rightarrow a_3 = \frac{1}{6}(a_1 + a_0)$$

$$n=2 \rightarrow a_4 = \frac{1}{12}(a_2 + a_1) = \frac{1}{12}\left(\frac{1}{2}a_0 + a_1\right)$$

$$n=3 \rightarrow a_5 = \frac{1}{20}(a_3 + a_2) = \frac{1}{20}\left(\frac{2}{3}a_0 + \frac{1}{6}a_1\right)$$

$$n=4 \rightarrow a_6 = \frac{1}{30}(a_4 + a_3) = \frac{1}{30}\left(\frac{1}{2}a_0 + a_1 + \frac{1}{6}a_0 + \frac{1}{6}a_1\right) = \frac{1}{30}\left(\frac{2}{3}a_0 + \frac{7}{6}a_1\right)$$

$$a_0 \neq 0 \quad a_1 = 0$$

$$a_2 = \frac{1}{2}a_0$$

$$a_3 = \frac{1}{6}a_0$$

$$a_4 = \frac{1}{24}a_0$$

$$a_5 = \frac{1}{30}a_0$$

$$a_6 = \frac{1}{45}a_0$$

$$a_0 = 0 \quad a_1 \neq 0$$

$$a_2 = 0$$

$$a_3 = \frac{1}{6}a_1$$

$$a_4 = \frac{1}{12}a_1$$

$$a_5 = \frac{1}{120}a_1$$

$$a_6 = \frac{7}{180}a_1$$

$$\left\{ \begin{array}{l} y_1(x) = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + \frac{1}{45}x^6 + \dots\right)a_0 \\ y_2(x) = \left(x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \frac{7}{180}x^6 + \dots\right)a_1 \end{array} \right.$$

Exercise

Find a power series solution. $y'' - (x+1)y' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - (x+1)y' - y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - (x+1) \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - a_1 - \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n - a_0 - \sum_{n=1}^{\infty} a_n x^n = 0$$

$$2a_2 - a_1 - a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - n a_n - (n+1) a_{n+1} - a_n] x^n = 0$$

$$2a_2 - a_1 - a_0 = 0 \rightarrow a_2 = \frac{1}{2}a_0 + \frac{1}{2}a_1$$

$$(n+2)(n+1) a_{n+2} - (n+1) a_{n+1} - (n+1) a_n = 0$$

$$a_{n+2} = \frac{1}{n+2} a_{n+1} + \frac{1}{n+2} a_n$$

$$a_3 = \frac{1}{3}a_2 + \frac{1}{3}a_1 = \frac{1}{6}a_0 + \frac{1}{2}a_1$$

$$a_4 = \frac{1}{4}a_3 + \frac{1}{4}a_2 = \frac{1}{24}a_0 + \frac{1}{8}a_1 + \frac{1}{8}a_0 + \frac{1}{8}a_1 = \frac{1}{6}a_0 + \frac{1}{4}a_1$$

$$a_5 = \frac{1}{5}a_4 + \frac{1}{5}a_3 = \frac{1}{30}a_0 + \frac{1}{20}a_1 + \frac{1}{30}a_0 + \frac{1}{10}a_1 = \frac{1}{15}a_0 + \frac{3}{20}a_1$$

$$\vdots$$

$$\left\{ \begin{array}{l} y_1(x) = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \dots\right) a_0 \\ y_2(x) = \left(x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \dots\right) a_1 \end{array} \right.$$

Exercise

Find a power series solution. $(x^2 + 1)y'' - 6y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x^2 + 1)y'' - 6y = 0$$

$$(x^2 + 1) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 6 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+2} + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 6 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 6 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n - 6a_0 - 6a_1 x - 6 \sum_{n=2}^{\infty} a_n x^n = 0$$

$$2a_2 - 6a_0 + (6a_3 - 6a_1)x + \sum_{n=2}^{\infty} \left[(n^2 - n - 6) a_n + (n+2)(n+1) a_{n+2} \right] x^n = 0$$

$$2a_2 - 6a_0 + (6a_3 - 6a_1)x = 0 \rightarrow \begin{cases} a_2 = 3a_0 \\ a_3 = a_1 \end{cases}$$

$$(n+2)(n-3) a_n + (n+2)(n+1) a_{n+2} = 0$$

$$\Rightarrow a_{n+2} = -\frac{n-3}{n+1} a_n \quad n = 2, 3, \dots$$

$$a_4 = \frac{1}{3} a_2 = a_0 \qquad a_5 = 0$$

$$a_6 = -\frac{1}{5}a_4 = -\frac{1}{5}a_0 \qquad a_7 = -\frac{1}{3}a_5 = 0$$

$$a_8 = -\frac{3}{7}a_6 = \frac{3}{35}a_0$$

$$\vdots$$

$$\left\{ \begin{array}{l} y_1(x) = \left(1 + 3x^2 + x^4 - \frac{1}{5}x^6 + \cdots\right)a_0 \\ y_2(x) = (x + x^3)a_1 \end{array} \right.$$

Exercise

Find a power series solution. $(x^2 + 2)y'' + 3xy' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x^2 + 2)y'' + 3xy' - y = 0$$

$$(x^2 + 2) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 3x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+2} + \sum_{n=0}^{\infty} 2(n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 3(n+1) a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} [2(n+2)(n+1) a_{n+2} - a_n] x^n + \sum_{n=1}^{\infty} 3n a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + 4a_2 - a_0 + (12a_3 - a_1)x + \sum_{n=2}^{\infty} [2(n+2)(n+1)a_{n+2} - a_n] x^n$$

$$+ 3a_1 x + \sum_{n=2}^{\infty} 3na_n x^n = 0$$

$$4a_2 - a_0 + (12a_3 + 2a_1)x + \sum_{n=2}^{\infty} [n(n-1)a_n + 2(n+2)(n+1)a_{n+2} + (3n-1)a_n] x^n = 0$$

$$4a_2 - a_0 + (12a_3 + 2a_1)x$$

$$\left| \begin{array}{l} a_2 = \frac{1}{4}a_0 \\ a_3 = -\frac{1}{6}a_1 \end{array} \right|$$

$$2(n+2)(n+1)a_{n+2} + (n^2 + 2n - 1)a_n = 0$$

$$\rightarrow a_{n+2} = -\frac{n^2 + 2n - 1}{2(n+2)(n+1)} a_n \quad n = 2, 3, \dots$$

$$a_4 = -\frac{7}{24}a_2 = -\frac{7}{96}a_0$$

$$a_5 = -\frac{7}{20}a_3 = \frac{7}{120}a_1$$

$$a_6 = -\frac{23}{60}a_4 = \frac{161}{5760}a_0$$

$$a_7 = -\frac{17}{42}a_5 = -\frac{17}{720}a_1$$

$$\vdots$$

$$\left| \begin{array}{l} y_1(x) = \left(1 + \frac{1}{4}x^2 - \frac{7}{96}x^4 + \frac{161}{5760}x^6 - \dots\right)a_0 \\ y_2(x) = \left(1 - \frac{1}{6}x^3 + \frac{7}{120}x^5 - \frac{17}{720}x^7 + \dots\right)a_1 \end{array} \right|$$

Exercise

Find a power series solution. $(x^2 - 1)y'' + xy' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$$(x^2 - 1)y'' + xy' - y = 0$$

$$(x^2 - 1) \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + x \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^{n+2} - \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1)a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + a_n] x^n + \sum_{n=1}^{\infty} n a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n - (2a_2 + a_0) - (6a_3 + a_1)x - \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} + a_n] x^n$$

$$+ a_1 x + \sum_{n=2}^{\infty} n a_n x^n = 0$$

$$-2a_2 - a_0 - 6a_3 x + \sum_{n=2}^{\infty} [n(n-1)a_n - (n+2)(n+1)a_{n+2} + (n-1)a_n] x^n = 0$$

$$-2a_2 - a_0 - 6a_3 x = 0$$

$$\left| a_2 = -\frac{1}{2}a_0 \right| \quad \left| a_3 = 0 \right|$$

$$-(n+2)(n+1)a_{n+2} + (n+1)(n-1)a_n = 0$$

$$\rightarrow a_{n+2} = \frac{n-1}{n+2} a_n \quad n = 2, 3, \dots$$

$$a_4 = \frac{1}{4}a_2 = -\frac{1}{8}a_0 \quad a_5 = -\frac{2}{5}a_3 = 0$$

$$a_6 = \frac{1}{2}a_4 = -\frac{1}{16}a_0 \quad a_7 = \frac{4}{7}a_5 = 0$$

$$\vdots$$

$$\left\{ \begin{array}{l} y_1(x) = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \dots\right)a_0 \\ y_2(x) = a_1 \end{array} \right.$$

$$\begin{cases} y_1(x) = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \dots \\ y_2(x) = x \end{cases}$$

Exercise

Find a power series solution. $(x^2 + 1)y'' + xy' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x^2 + 1)y'' + xy' - y = 0$$

$$(x^2 + 1) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+2} + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - a_n] x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + a_1 x + \sum_{n=2}^{\infty} n a_n x^n + (2a_2 - a_0) + (6a_3 - a_1) x + \sum_{n=2}^{\infty} [(n+2)(n+1) a_{n+2} - a_n] x^n = 0$$

$$2a_2 - a_0 + 6a_3x + \sum_{n=2}^{\infty} \left[(n^2 - n + n - 1)a_n + (n+2)(n+1)a_{n+2} \right] x^n = 0$$

$$\begin{cases} 2a_2 - a_0 = 0 & \rightarrow a_2 = \frac{1}{2}a_0 \\ 6a_3x = 0 & \rightarrow a_3 = 0 \end{cases}$$

$$(n-1)(n+1)a_n + (n+2)(n+1)a_{n+2} = 0$$

$$a_{n+2} = -\frac{n-1}{n+2}a_n$$

a_0	a_1
$n=0 \rightarrow a_2 = \frac{1}{2}a_0$	$n=1 \rightarrow a_3 = 0$
$n=2 \rightarrow a_4 = -\frac{1}{4}a_2 = -\frac{1}{2^3}a_0$	$n=3 \rightarrow a_5 = -\frac{2}{5}a_3 = 0$
$n=4 \rightarrow a_6 = -\frac{3}{6}a_4 = \frac{1 \cdot 3}{2^3 \cdot 3!}a_0$	$n=5 \rightarrow a_7 = 0$
$n=6 \rightarrow a_8 = -\frac{5}{8}a_6 = -\frac{1 \cdot 3 \cdot 5}{2^4 \cdot 4!}a_0$	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	

$$a_{2n} = (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n \cdot n!} a_0 \quad (n \geq 2)$$

$$y(x) = a_0 \left(1 + \frac{1}{2}x^2 - \frac{1}{2^2 \cdot 2!}x^4 + \frac{1 \cdot 3}{2^3 \cdot 3!}x^6 - \frac{1 \cdot 3 \cdot 5}{2^4 \cdot 4!}x^8 + \cdots \right) + a_1x$$

$$= a_0 \left(1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{(2n-3)!!}{2^n \cdot n!} x^{2n} \right) + a_1x$$

Exercise

Find a power series solution. $(x^2 + 1)y'' - xy' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x^2 + 1)y'' - xy' + y = 0$$

$$(x^2 + 1) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+2} + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + a_n] x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n - a_1 x - \sum_{n=2}^{\infty} n a_n x^n + (2a_2 + a_0) + (6a_3 + a_1)x$$

$$+ \sum_{n=2}^{\infty} [(n+2)(n+1) a_{n+2} + a_n] x^n = 0$$

$$2a_2 + a_0 + 6a_3 x + \sum_{n=2}^{\infty} [(n^2 - 2n + 1) a_n + (n+2)(n+1) a_{n+2}] x^n = 0$$

$$\begin{cases} 2a_2 + a_0 = 0 & \rightarrow a_2 = -\frac{1}{2}a_0 \\ 6a_3 x = 0 & \rightarrow a_3 = 0 \end{cases}$$

$$(n-1)^2 a_n + (n+1)(n+2) a_{n+2} = 0$$

$$\left| a_{n+2} = -\frac{(n-1)^2}{(n+1)(n+2)} a_n \right|$$

$$\begin{array}{l} a_0 \\ n=0 \rightarrow a_2 = -\frac{1}{2}a_0 \\ n=2 \rightarrow a_4 = -\frac{1}{12}a_2 = \frac{1}{4!}a_0 \\ n=4 \rightarrow a_6 = -\frac{3^2}{5 \cdot 6}a_4 = -\frac{3^2}{6!}a_0 \\ n=6 \rightarrow a_8 = -\frac{5^2}{7 \cdot 8}a_6 = \frac{1 \cdot 3^2 \cdot 5^2}{8!}a_0 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \end{array}$$

$$\begin{array}{l} a_1 \\ n=1 \rightarrow a_3 = 0 \\ n=3 \rightarrow a_5 = -\frac{2}{5}a_3 = 0 \\ n=5 \rightarrow a_7 = 0 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \end{array}$$

$$a_{2n} = (-1)^{n-1} \frac{1 \cdot 3^2 \cdot 5^2 \cdots (2n-3)^2}{(2n)!} a_0 \quad (n \geq 3)$$

$$\left| y(x) = a_0 \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{9}{6!}x^6 - \frac{1 \cdot 3^2 \cdot 5^2}{8!}x^8 - \cdots \right) + a_1 x \right|$$

$$= a_0 \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \sum_{n=3}^{\infty} (-1)^n \frac{(2n-3)^{2!!}}{(2n)!} x^{2n} \right) + a_1 x$$

Exercise

Find a power series solution. $(1-x^2)y'' - 6xy' - 4y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(1-x^2)y'' - 6xy' - 4y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - x^2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 6x \sum_{n=1}^{\infty} na_n x^{n-1} - 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 6na_n x^n - \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 6na_n x^n - \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n(n-1) + 6n + 4)a_n \right] x^n = 0$$

$$(n+2)(n+1)a_{n+2} - (n^2 + 5n + 4)a_n = 0$$

$$(n+2)(n+1)a_{n+2} = (n+4)(n+1)a_n$$

$$\underline{a_{n+2} = \frac{n+4}{n+2}a_n}$$

$$\begin{array}{c} a_0 \\ n=2 \rightarrow a_2 = 2a_0 \end{array}$$

$$n=4 \rightarrow a_4 = \frac{6}{4}a_2 = 3a_0$$

$$n=6 \rightarrow a_6 = \frac{8}{6}a_3 = 4a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2k} = (k+1)a_0$$

$$a_1$$

$$n=3 \rightarrow a_3 = \frac{5}{3}a_1$$

$$n=5 \rightarrow a_5 = \frac{7}{5}a_3 = \frac{7}{3}a_1$$

$$n=7 \rightarrow a_7 = \frac{11}{7}a_5 = \frac{11}{3}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2k+1} = \frac{2k+3}{3}a_1$$

$$y(x) = a_0 \left(1 + 2x^2 + 3x^4 + 4x^6 + \cdots \right) + a_1 \left(x + \frac{5}{3}x^3 + \frac{7}{3}x^5 + \frac{11}{3}x^7 + \cdots \right)$$

$$\underline{= \frac{a_0}{(1-x^2)^2} + \frac{3x-x^3}{3(1-x^2)^2}a_1}$$

Exercise

Find a power series solution. $y'' + (x-1)^2 y' - 4(x-1)y = 0$

Solution

$$\text{Let } z = x-1 \rightarrow dz = dx$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + z^2 y' - 4zy = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n + z^2 \sum_{n=1}^{\infty} n a_n z^{n-1} - 4z \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n + \sum_{n=1}^{\infty} n a_n z^{n+1} - \sum_{n=0}^{\infty} 4a_n z^{n+1} = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} z^n + \sum_{n=1}^{\infty} (n-4) a_n z^{n+1} = 0$$

$$2a_2 + \sum_{n=0}^{\infty} (n+2)(n+3) a_{n+3} z^{n+1} + \sum_{n=1}^{\infty} (n-4) a_n z^{n+1} = 0$$

$$2a_2 + \sum_{n=0}^{\infty} [(n+2)(n+3) a_{n+3} + (n-4) a_n] z^{n+1} = 0$$

$$a_2 = 0$$

$$(n+2)(n+3) a_{n+3} + (n-4) a_n = 0$$

$$a_{n+3} = -\frac{n-4}{(n+2)(n+3)} a_n$$

$$a_0$$

$$a_1$$

$$a_2 = 0$$

$$n=0 \rightarrow a_3 = \frac{4}{2 \cdot 3} a_0$$

$$n=1 \rightarrow a_4 = \frac{3}{3 \cdot 4} a_1$$

$$n=2 \rightarrow a_5 = \frac{2}{20} a_2 = 0$$

$$n=3 \rightarrow a_6 = \frac{1}{5 \cdot 6} a_3 = \frac{4}{2 \cdot 3 \cdot 5 \cdot 6} a_0$$

$$n=4 \rightarrow a_7 = 0$$

$$n=5 \rightarrow a_8 = 0$$

$$n=6 \rightarrow a_9 = -\frac{2}{8 \cdot 9} a_6 = -\frac{8}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} a_0$$

$$n=7 \rightarrow a_{10} = 0$$

$$n=8 \rightarrow a_5 = 0$$

$$\begin{array}{ccccccc}
 \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
 y(x) = a_0 \left(1 + \frac{4}{2 \cdot 3} z^3 + \frac{4}{2 \cdot 3 \cdot 5 \cdot 6} z^6 - \frac{8}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 8} z^9 + \cdots \right) + a_1 \left(z + \frac{1}{4} z^4 \right) \\
 = a_0 \left(1 + \frac{2}{3} (x-1)^3 + \frac{1}{45} (x-1)^6 - \frac{1}{1,620} (x-1)^9 + \cdots \right) + a_1 \left(x-1 + \frac{1}{4} (x-1)^4 \right)
 \end{array}$$

Exercise

Find a power series solution. $(2 - x^2)y'' - xy' + 16y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(2 - x^2)y'' - xy' + 16y = 0$$

$$2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} + 16 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 16 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 16 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+2)(n+1) a_{n+2} - (n^2 - n + n - 16) a_n \right] x^n = 0$$

$$2(n+1)(n+2) a_{n+2} - (n^2 - 16) a_n = 0$$

$$2(n+1)(n+2) a_{n+2} = (n+4)(n-4) a_n$$

$$\left| a_{n+2} = \frac{(n+4)(n-4)}{2(n+1)(n+2)} a_n \right|$$

$$a_0$$

$$n = 0 \rightarrow a_2 = -4a_0$$

$$n = 2 \rightarrow a_4 = -\frac{1}{2}a_2 = 2a_0$$

$$n = 4 \rightarrow a_6 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_1$$

$$n = 1 \rightarrow a_3 = -\frac{5}{4}a_1$$

$$n = 3 \rightarrow a_5 = -\frac{7}{40}a_3 = \frac{7}{320}a_1$$

$$n = 5 \rightarrow a_7 = \frac{9}{70}a_5 = \frac{9}{320}a_1$$

$$n = 7 \rightarrow a_9 = \frac{33}{144}a_7 = \frac{33}{5,120}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n+1} = \frac{(2n-5)!! (2n+3)!!}{2^n (2n+1)!} a_1$$

$$\left| y(x) = a_0 \left(1 - 4x^2 + 2x^4 \right) + a_1 \left(x - \frac{5}{4}x^3 + \frac{7}{32}x^5 + \frac{9}{320}x^7 + \dots \right) \right|$$

$$= a_0 \left(1 - 4x^2 + 2x^4 \right) + a_1 \sum_{n=0}^{\infty} \frac{(2n-5)!! (2n+3)!!}{2^n (2n+1)!} x^{2n+1}$$

Exercise

Find a power series solution. $(x^2 + 1)y'' - y' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x^2 + 1)y'' - y' + y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} x^n - \sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{\textcolor{red}{n}=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} x^n - \sum_{\textcolor{red}{n}=0}^{\infty} (n+1)a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} + (n^2 - n + 1)a_n \right] x^n = 0$$

$$(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} + (n^2 - n + 1)a_n = 0$$

$$a_{n+2} = \frac{(n+1)a_{n+1} - (n^2 - n + 1)a_n}{(n+1)(n+2)}$$

$$\textcolor{red}{n}=0 \rightarrow a_2 = \frac{1}{2}(a_1 - a_0)$$

$$\textcolor{red}{n}=1 \rightarrow a_3 = \frac{1}{6}(2a_2 - a_1) = \frac{1}{6}(a_1 - a_0 - a_1) = -\frac{1}{6}a_0$$

$$\textcolor{red}{n}=2 \rightarrow a_4 = \frac{1}{12}(3a_3 - 3a_2) = \frac{1}{4}\left(-\frac{1}{6}a_0 - \frac{1}{2}a_1 + \frac{1}{2}a_0\right) = \frac{1}{12}a_0 - \frac{1}{8}a_1$$

$$\textcolor{red}{n}=3 \rightarrow a_5 = \frac{1}{20}(4a_4 - 7a_3) = \frac{1}{20}\left(\frac{1}{3}a_0 - \frac{1}{2}a_1 + \frac{7}{6}a_0\right) = \frac{3}{40}a_0 - \frac{1}{40}a_1$$

$$\textcolor{red}{n}=4 \rightarrow a_6 = \frac{1}{30}(5a_5 - 13a_4) = \frac{1}{30}\left(\frac{3}{8}a_0 - \frac{1}{8}a_1 - \frac{13}{12}a_0 + \frac{13}{8}a_1\right) = -\frac{17}{720}a_0 + \frac{1}{20}a_1$$

$$y(x) = a_0 + a_1 x + \left(\frac{1}{2}a_0 - \frac{1}{2}a_1\right)x^2 - \frac{1}{6}a_0 x^3 + \left(\frac{1}{12}a_0 - \frac{1}{8}a_1\right)x^4 + \left(\frac{3}{40}a_0 - \frac{1}{40}a_1\right)x^5 \\ + \left(-\frac{17}{720}a_0 + \frac{1}{20}a_1\right)x^6 + \dots$$

$$y(x) = a_0 \left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 - \frac{17}{720}x^6 + \dots\right) \\ + a_1 \left(x - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{40}x^5 + \frac{1}{20}x^6 + \dots\right)$$

Exercise

Find a power series solution. $(x^2 + 1)y'' + 6xy' + 4y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x^2 + 1)y'' + 6xy' + 4y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 6x \sum_{n=1}^{\infty} n a_n x^{n-1} + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} 6n a_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 6n a_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n^2 + 5n + 4) a_n \right] x^n = 0$$

$$(n+1)(n+2) a_{n+2} + (n+1)(n+4) a_n = 0$$

$$\boxed{a_{n+2} = -\frac{n+4}{n+2} a_n}$$

$$a_0$$

$$n=0 \rightarrow a_2 = -2a_0$$

$$n=2 \rightarrow a_4 = -\frac{3}{2} a_2 = 3a_0$$

$$n=4 \rightarrow a_6 = -\frac{8}{6} a_4 = -4a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_1$$

$$n=1 \rightarrow a_3 = -\frac{5}{3} a_1$$

$$n=3 \rightarrow a_5 = -\frac{7}{5} a_3 = \frac{7}{3} a_1$$

$$n=5 \rightarrow a_7 = -\frac{9}{7} a_5 = -\frac{9}{3} a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n} = (-1)^n (n+1) a_0$$

$$a_{2n+1} = (-1)^n (2n+3) a_1$$

$$y(x) = a_0 \sum_{n=0}^{\infty} (-1)^n (n+1) x^{2n} + \frac{1}{3} a_1 \sum_{n=0}^{\infty} (-1)^n (2n+3) x^{2n+1}$$

$$y(x) = a_0 \left(1 - 2x^2 + 3x^4 - 4x^6 + \dots \right) + \frac{1}{3} a_1 \left(x - \frac{5}{3}x^3 + \frac{7}{3}x^5 - \frac{9}{3}x^7 + \dots \right)$$

Exercise

Find a power series solution. $(x^2 - 1)y'' - 6xy' + 12y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x^2 - 1)y'' - 6xy' + 12y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 6x \sum_{n=1}^{\infty} n a_n x^{n-1} + 12 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} 6n a_n x^n + \sum_{n=0}^{\infty} 12a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} 6n a_n x^n + \sum_{n=0}^{\infty} 12a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[-(n+2)(n+1) a_{n+2} + (n^2 - 7n + 12) a_n \right] x^n = 0$$

$$-(n+1)(n+2) a_{n+2} + (n-3)(n-4) a_n = 0$$

$$\boxed{a_{n+2} = \frac{(n-3)(n-4)}{(n+1)(n+2)} a_n}$$

$$a_0$$

$$n = 0 \rightarrow a_2 = 6a_0$$

$$n = 2 \rightarrow a_4 = \frac{1}{6}a_2 = a_0$$

$$n = 4 \rightarrow a_6 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_1$$

$$n = 1 \rightarrow a_3 = a_1$$

$$n = 3 \rightarrow a_5 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = a_0(1 + 6x^2 + x^4) + a_1(x + x^3)}$$

Exercise

Find a power series solution. $(x^2 - 1)y'' + 8xy' + 12y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x^2 - 1)y'' + 8xy' + 12y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 8x \sum_{n=1}^{\infty} n a_n x^{n-1} + 12 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} 8n a_n x^n + \sum_{n=0}^{\infty} 12a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 8n a_n x^n + \sum_{n=0}^{\infty} 12a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[-(n+2)(n+1)a_{n+2} + (n^2 + 7n + 12)a_n \right] x^n = 0$$

$$-(n+1)(n+2)a_{n+2} + (n+3)(n+4)a_n = 0$$

$$a_{n+2} = \frac{(n+3)(n+4)}{(n+1)(n+2)} a_n$$

$$a_0$$

$$a_1$$

$$n=0 \rightarrow a_2 = 6a_0$$

$$n=1 \rightarrow a_3 = \frac{10}{3}a_1$$

$$n=2 \rightarrow a_4 = \frac{5}{2}a_2 = 15a_0$$

$$n=3 \rightarrow a_5 = \frac{42}{20}a_3 = \frac{21}{3}a_1$$

$$n=4 \rightarrow a_6 = \frac{5}{2}a_4 = 28a_0$$

$$n=5 \rightarrow a_7 = \frac{12}{7}a_5 = \frac{36}{3}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n} = (n+1)(2n+1)a_0$$

$$a_{2n+1} = \frac{1}{3}(n+1)(2n+3)a_1$$

$$y(x) = a_0 \left(1 + 6x^2 + 15x^4 + 28x^6 + \cdots \right) + a_1 \left(x + \frac{10}{3}x^3 + 7x^5 + 12x^7 + \cdots \right)$$

$$= a_0 \sum_{n=0}^{\infty} (n+1)(2n+1)x^{2n} + \frac{1}{3}a_1 \sum_{n=0}^{\infty} (n+1)(2n+3)x^{2n+1}$$

Exercise

Find a power series solution. $(x^2 - 1)y'' + 4xy' + 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x^2 - 1)y'' + 4xy' + 2y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + 4x \sum_{n=1}^{\infty} na_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} 4na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} 4na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[-(n+2)(n+1)a_{n+2} + (n^2 + 3n + 2)a_n \right] x^n = 0$$

$$-(n+1)(n+2)a_{n+2} + (n+1)(n+2)a_n = 0$$

$$\underline{a_{n+2} = a_n}$$

$$a_0$$

$$n=0 \rightarrow a_2 = a_0$$

$$n=2 \rightarrow a_4 = a_2 = a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_1$$

$$n=1 \rightarrow a_3 = a_1$$

$$n=3 \rightarrow a_5 = a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y(x) = a_0(1 + x^2 + x^4 + x^6 + \cdots) + a_1(x + x^3 + x^5 + x^7 + \cdots)}$$

$$= a_0 \sum_{n=0}^{\infty} x^{2n} + a_1 \sum_{n=0}^{\infty} x^{2n+1}$$

$$\underline{= \frac{a_0 + a_1 x}{1 - x^2}}$$

Exercise

Find a power series solution. $(x^2 + 1)y'' - 4xy' + 6y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x^2 + 1)y'' - 4xy' + 6y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 4x \sum_{n=1}^{\infty} n a_n x^{n-1} + 6 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} + (n^2 - 5n + 6) a_n \right] x^n = 0$$

$$(n+1)(n+2) a_{n+2} + (n-2)(n-3) a_n = 0$$

$$a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)} a_n$$

 a_0

$$n=0 \rightarrow a_2 = -3a_0$$

$$n=2 \rightarrow a_4 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(x) = a_0 \left(1 - 3x^2 \right) + a_1 \left(x - \frac{1}{3}x^3 \right)$$

 a_1

$$n=1 \rightarrow a_3 = -\frac{1}{3}a_1$$

$$n=3 \rightarrow a_5 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

Exercise

Find a power series solution. $(x^2 + 2)y'' + 4xy' + 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x^2 + 2)y'' + 4xy' + 2y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 4x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 2(n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 2(n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+2)(n+1) a_{n+2} + (n^2 + 3n + 2) a_n \right] x^n = 0$$

$$2(n+1)(n+2) a_{n+2} + (n+1)(n+2) a_n = 0$$

$$\underline{a_{n+2} = -\frac{1}{2} a_n}$$

$$a_0$$

$$n=0 \rightarrow a_2 = -\frac{1}{2} a_0$$

$$n=2 \rightarrow a_4 = -\frac{1}{2} a_2 = \frac{1}{2^2} a_0$$

$$n=4 \rightarrow a_6 = -\frac{1}{2} a_4 = -\frac{1}{2^3} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n} = (-1)^n \frac{1}{2^n} a_0$$

$$a_1$$

$$n=1 \rightarrow a_3 = -\frac{1}{2} a_1$$

$$n=3 \rightarrow a_5 = -\frac{1}{2} a_3 = \frac{1}{2^2} a_1$$

$$n=5 \rightarrow a_7 = -\frac{1}{2} a_5 = -\frac{1}{2^3} a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n+1} = (-1)^n \frac{1}{2^n} a_1$$

$$\underline{y(x) = a_0 \left(1 - \frac{1}{2} x^2 + \frac{1}{4} x^4 - \frac{1}{8} x^6 + \cdots \right) + a_1 \left(x - \frac{1}{2} x^3 + \frac{1}{4} x^5 - \frac{1}{8} x^7 + \cdots \right)}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^{2n+1} \quad \Bigg|$$

Exercise

Find a power series solution. $(x^2 - 3)y'' + 2xy' = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x^2 - 3)y'' + 2xy' = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 3 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 2x \sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} 3(n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} 2n a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} 3(n+1)(n+2) a_{n+2} x^n + \sum_{n=0}^{\infty} 2n a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[-3(n+1)(n+2) a_{n+2} + (n^2 + n) a_n \right] x^n = 0$$

$$-3(n+1)(n+2) a_{n+2} + n(n+1) a_n = 0$$

$$a_{n+2} = \frac{1}{3} \frac{n}{n+2} a_n \quad \Bigg|$$

$$a_0$$

$$n = 0 \rightarrow a_2 = 0$$

$$a_1$$

$$n = 1 \rightarrow a_3 = \frac{1}{3^2} a_1$$

$$n = 2 \rightarrow a_4 = \frac{2}{12}a_2 = 0$$

$$n = 4 \rightarrow a_6 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n = 3 \rightarrow a_5 = \frac{1}{3} \frac{3}{5} a_3 = \frac{1}{3^2 \cdot 5} a_1$$

$$n = 5 \rightarrow a_7 = \frac{1}{3} \frac{5}{7} a_5 = \frac{1}{3^3 \cdot 7} a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n+1} = (-1)^n \frac{1}{(2n+1)3^n} a_1$$

$$y(x) = a_0 + a_1 \left(x + \frac{1}{9}x^3 + \frac{1}{45}x^5 + \frac{1}{189}x^7 + \cdots \right)$$

$$= a_0 + a_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)3^n} x^{2n+1}$$

Exercise

Find a power series solution. $(x^2 + 3)y'' - 7xy' + 16y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x^2 + 3)y'' - 7xy' + 16y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 3 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 7x \sum_{n=1}^{\infty} n a_n x^{n-1} + 16 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 3(n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 7n a_n x^n + \sum_{n=0}^{\infty} 16 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 3(n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{\infty} 7n a_n x^n + \sum_{n=0}^{\infty} 16 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[3(n+1)(n+2)a_{n+2} + (n^2 - 8n + 16)a_n \right] x^n = 0$$

$$3(n+1)(n+2)a_{n+2} + (n-4)^2 a_n = 0$$

$$\boxed{a_{n+2} = -\frac{(n-4)^2}{3(n+1)(n+2)} a_n}$$

$$a_0$$

$$n=0 \rightarrow a_2 = -\frac{16}{6}a_0 = -\frac{8}{3}a_0$$

$$n=2 \rightarrow a_4 = -\frac{1}{9}a_2 = \frac{8}{27}a_0$$

$$n=4 \rightarrow a_6 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_1$$

$$n=1 \rightarrow a_3 = -\frac{9}{18}a_1 = -\frac{1}{2}a_1$$

$$n=3 \rightarrow a_5 = -\frac{1}{60}a_3 = \frac{1}{120}a_1$$

$$n=5 \rightarrow a_7 = -\frac{1}{126}a_5 = -\frac{1}{560 \cdot 3^3}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = a_0 \left(1 - \frac{8}{3}x^2 + \frac{8}{27}x^4 \right) + a_1 \left(x - \frac{1}{2}x^3 + \frac{1}{120}x^5 + \frac{1}{15,120}x^7 + \cdots \right)}$$

Exercise

Find the series solution to the initial value problem $y'' + 4y = 0$; $y(0) = 0$, $y'(0) = 3$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + 4y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + 4a_n \right] x^n = 0$$

$$(n+2)(n+1)a_{n+2} + 4a_n = 0$$

$$a_{n+2} = -\frac{4}{(n+1)(n+2)}a_n$$

Given: $y(0) = 0 = a_0$, $y'(0) = 3 = a_1$

$$a_0 = 0$$

$$n=0 \rightarrow a_2 = -2a_0 = 0$$

$$n=2 \rightarrow a_4 = -\frac{4}{12}a_2 = 0$$

$$n=4 \rightarrow a_6 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_1 = 3$$

$$n=1 \rightarrow a_3 = -\frac{4}{6}a_1 = -\frac{2^2}{3!}a_1 = -2$$

$$n=3 \rightarrow a_5 = -\frac{4}{20}a_3 = -\frac{2^4}{5!}a_1 = \frac{2}{5}$$

$$n=5 \rightarrow a_7 = -\frac{4}{42}a_5 = -\frac{2^6}{7!}a_1 = -\frac{4}{105}$$

$$a_{2k+1} = \frac{(-1)^k 2^{2k}}{(2k+1)!}a_1$$

$$y(x) = 3x - 2x^3 + \frac{2}{5}x^5 - \frac{4}{105}x^7 + \dots$$

$$= 3 \left(x - \frac{2^2}{3!}x^3 + \frac{2^4}{5!}x^5 - \frac{2^6}{7!}x^7 + \dots \right)$$

$$= \frac{3}{2} \left((2x) - \frac{1}{3!}(2x)^3 + \frac{1}{5!}(2x)^5 - \frac{1}{7!}(2x)^7 + \dots \right)$$

$$= \frac{3}{2} \sin 2x$$

Exercise

Find the series solution to the initial value problem $y'' + x^2y = 0$; $y(0) = 1$, $y'(0) = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + x^2 y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

$$2a_2 + 6a_3x + \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

$$2a_2 + 6a_3x + \sum_{n=2}^{\infty} [(n+1)(n+2)a_{n+2} + a_{n-2}]x^n = 0$$

$$(n+1)(n+2)a_{n+2} + a_{n-2} = 0$$

$$a_{n+2} = -\frac{1}{(n+1)(n+2)}a_{n-2}$$

Given: $y(0) = 1 = a_0$, $y'(0) = 0 = a_1$

$$2a_2 + 6a_3x = 0 \rightarrow a_2 = a_3 = 0$$

$$a_0 = 1$$

$$a_1 = a_2 = a_3 = 0$$

$$n=2 \rightarrow a_4 = -\frac{1}{12}a_0 = -\frac{1}{12}$$

$$n=3 \rightarrow a_5 = -\frac{1}{20}a_1 = 0$$

$$n=4 \rightarrow a_6 = *a_2 = 0$$

$$n=5 \rightarrow a_7 = *a_3 = 0$$

$$n=6 \rightarrow a_8 = -\frac{1}{56}a_4 = \frac{1}{672}$$

$$n=10 \rightarrow a_{12} = -\frac{1}{132}a_8 = \frac{1}{88,704}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(x) = 1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 - \frac{1}{88,704}x^{12} + \dots$$

Exercise

Find the series solution to the initial value problem $y'' - 2xy' + 8y = 0$; $y(0) = 3$, $y'(0) = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - 2xy' + 8y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 8 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} 8a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + 8a_n] x^n - \sum_{n=1}^{\infty} 2n a_n x^n = 0$$

$$2a_2 + 8a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} + 8a_n] x^n - \sum_{n=1}^{\infty} 2n a_n x^n = 0$$

$$2a_2 + 8a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} + (8-2n) a_n] x^n = 0$$

$$\text{Given: } y(0) = 3 = a_0, \quad y'(0) = 0 = a_1$$

$$2a_2 + 8a_0 = 0 \rightarrow a_2 = -4a_0 = -12$$

$$(n+2)(n+1) a_{n+2} + (8-2n) a_n = 0$$

$$\rightarrow a_{n+2} = \frac{2n-8}{(n+1)(n+2)} a_n \quad n=1, 2, \dots$$

$$a_3 = -a_1 = 0$$

$$a_4 = -\frac{1}{3} a_2 = 4$$

$$a_5 = -\frac{1}{10} a_3 = 0$$

$$a_6 = 0 a_4 = 0$$

$$a_7 = \frac{1}{21} a_5 = 0$$

$$a_8 = 0 a_6 = 0$$

$$y(x) = 3 - 12x^2 + 4x^4$$

Exercise

Find the series solution to the initial value problem $y'' + y' - 2y = 0$; $y(0) = 1$, $y'(0) = -2$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + y' - 2y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n+1) a_{n+1} - 2a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} + (n+1) a_{n+1} - 2a_n = 0$$

$$a_{n+2} = \frac{2a_n - (n+1)a_{n+1}}{(n+1)(n+2)}$$

Given: $y(0) = 1 = a_0$, $y'(0) = -2 = a_1$

$$a_0 = 1$$

$$a_1 = -2$$

$$n=0 \rightarrow a_2 = \frac{2a_0 - a_1}{2} = 2$$

$$n=1 \rightarrow a_3 = \frac{2a_1 - 2a_2}{6} = -\frac{8}{6}$$

$$n=2 \rightarrow a_4 = \frac{2a_2 - 3a_3}{12} = \frac{2}{3}$$

$$n=3 \rightarrow a_5 = \frac{2a_3 - 4a_4}{20} = -\frac{4}{15}$$

$$n=4 \rightarrow a_6 = \frac{2a_4 - 5a_5}{30} = \frac{4}{45}$$

$$n=5 \rightarrow a_7 = \frac{2a_5 - 6a_6}{42} = -\frac{8}{315}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \frac{4}{15}x^5 + \frac{4}{45}x^6 - \frac{8}{315}x^7 + \dots$$

$$= 1 + (-2x) + \frac{1}{2!}(-2x)^2 + \frac{1}{3!}(-2x)^3 + \frac{1}{4!}(-2x)^4 + \frac{1}{5!}(-2x)^5 + \frac{1}{6!}(-2x)^6 + \frac{1}{7!}(-2x)^7 + \dots$$

$$= e^{-2x} \Big|$$

Exercise

Find the series solution to the initial value problem $y'' - 2y' + y = 0$; $y(0) = 0$, $y'(0) = 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - 2y' + y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2 \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - 2(n+1) a_{n+1} + a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} - 2(n+1) a_{n+1} + a_n = 0$$

$$a_{n+2} = \frac{2(n+1) a_{n+1} - a_n}{(n+1)(n+2)} \Big|$$

$$\text{Given: } y(0) = 0 = a_0, \quad y'(0) = 1 = a_1$$

$$a_0 = 0$$

$$a_1 = 1$$

$$n=0 \rightarrow a_2 = \frac{2a_1 - a_0}{2} = 1$$

$$n=1 \rightarrow a_3 = \frac{4a_2 - a_1}{6} = \frac{1}{2}$$

$$n=2 \rightarrow a_4 = \frac{6a_3 - a_2}{12} = \frac{1}{6}$$

$$n=3 \rightarrow a_5 = \frac{8a_4 - a_3}{20} = \frac{1}{20} \left(\frac{4}{3} - \frac{1}{2} \right) = \frac{1}{24}$$

$$n=4 \rightarrow a_6 = \frac{10a_5 - a_4}{30} = \frac{1}{30} \left(\frac{5}{12} - \frac{1}{6} \right) = \frac{1}{120}$$

$$n=5 \rightarrow a_7 = \frac{12a_6 - a_5}{42} = \frac{1}{42} \left(\frac{1}{10} - \frac{1}{24} \right) = \frac{1}{720}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\begin{aligned}
 y(x) &= x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \frac{1}{120}x^6 + \frac{1}{720}x^7 + \cdots \\
 &= x \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots \right) \\
 &= \underline{xe^x}
 \end{aligned}$$

Exercise

Find the series solution to the initial value problem $y'' + xy' + y = 0$ $y(0) = 1$ $y'(0) = 0$

Solution

$$\begin{aligned}
 y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
 y'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \\
 y''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n
 \end{aligned}$$

$$y'' + xy' + y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + n a_n + a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} + (n+1) a_n = 0$$

$$(n+2)(n+1) a_{n+2} = -(n+1) a_n$$

$$a_{n+2} = -\frac{1}{n+2} a_n$$

$$a_0 = y(0) = 1$$

$$a_1 = y'(0) = 0$$

$$a_2 = -\frac{1}{2} a_0 = -\frac{1}{2}$$

$$a_3 = -\frac{1}{3} a_1 = 0$$

$$a_4 = -\frac{1}{4}a_2 = \frac{1}{2 \cdot 4} = \frac{1}{2^2 \cdot 1 \cdot 2} \quad a_5 = -\frac{1}{5}a_3 = 0$$

$$a_6 = -\frac{1}{6}a_4 = -\frac{1}{2^3 \cdot 1 \cdot 2 \cdot 3} \quad a_7 = -\frac{1}{7}a_7 = 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{2^2 2!}x^4 - \frac{1}{2^3 3!}x^5 + \dots$$

Exercise

Find the series solution to the initial value problem $y'' - xy' - y = 0$ $y(0) = 2$ $y'(0) = 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy' - y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - n a_n - a_n] x^n = 0$$

$$(n+2)(n+1) a_{n+2} - (n+1) a_n = 0$$

$$(n+2)(n+1) a_{n+2} = -(n+1) a_n$$

$$a_{n+2} = -\frac{1}{n+2} a_n$$

$$a_0 = y(0) = 2$$

$$a_1 = y'(0) = 1$$

$$a_2 = \frac{1}{2}a_0 = 1$$

$$a_3 = \frac{1}{3}a_1 = \frac{1}{3}$$

$$a_4 = \frac{1}{4}a_2 = \frac{1}{4}$$

$$a_5 = \frac{1}{5}a_3 = \frac{1}{3 \cdot 5}$$

$$a_6 = \frac{1}{6}a_4 = \frac{1}{4} \frac{1}{6} = \frac{1}{24}$$

$$a_7 = \frac{1}{7}a_5 = \frac{1}{3 \cdot 5 \cdot 7}$$

$$\vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots$$

$$y(x) = 2 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{15}x^5 + \frac{1}{24}x^6 + \dots$$

Exercise

Find the series solution to the initial value problem $y'' - xy' - y = 0$; $y(0) = 1$ $y'(0) = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy' - y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n - a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} - (n+1) a_n = 0$$

$$(n+2)(n+1) a_{n+2} = -(n+1) a_n$$

$$\underline{a_{n+2} = \frac{1}{n+2} a_n}$$

Given: $a_0 = y(0) = 1$

$a_1 = y'(0) = 0$

$$a_2 = \frac{1}{2} a_0 = \frac{1}{2}$$

$$a_3 = \frac{1}{3} a_1 = 0$$

$$a_4 = \frac{1}{4} a_2 = \frac{1}{2 \cdot 2^2}$$

$$a_5 = \frac{1}{5} a_3 = 0$$

$$a_6 = \frac{1}{6} a_4 = \frac{1}{2^3 \cdot 3!}$$

$$\vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots$$

$$\underline{y(x) = 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \dots}$$

Exercise

Find a power series solution. $y'' + xy' - 2y = 0$; $y(0) = 1$ $y'(0) = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$y'' + xy' - 2y = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+1)(n+2) a_{n+2} + (n-2) a_n] x^n = 0$$

$$(n+2)(n+1)a_{n+2} + (n-2)a_n = 0$$

$$\underline{a_{n+2} = -\frac{n-2}{(n+1)(n+2)}a_n}$$

$$a_0 = y(0) = 1$$

$$a_1 = y'(0) = 0$$

$$n=0 \rightarrow a_2 = \frac{2}{2}a_0 = 1$$

$$n=1 \rightarrow a_3 = \frac{1}{6}a_1 = 0$$

$$n=2 \rightarrow a_4 = 0$$

$$n=3 \rightarrow a_5 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y(x) = 1 + x^2}$$

Exercise

Find the series solution to the initial value problem $y'' + (x-1)y' + y = 0$ $y(1) = 2$ $y'(1) = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + (x-1)y' + y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n + (x-1) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n + \sum_{n=1}^{\infty} n a_n (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + n a_n + a_n \right] x^{n-1} = 0$$

$$(n+2)(n+1) a_{n+2} + (n+1) a_n = 0$$

$$\underline{a_{n+2} = -\frac{1}{n+2} a_n}$$

$$a_0 = y(1) = 2$$

$$a_1 = y'(1) = 0$$

$$a_2 = -\frac{1}{2}a_0 = -1$$

$$a_3 = -\frac{1}{3}a_1 = 0$$

$$a_4 = -\frac{1}{4}a_2 = \frac{1}{2 \cdot 4}a_0 = \frac{1}{4}$$

$$a_5 = -\frac{1}{5}a_3 = 0$$

$$a_6 = -\frac{1}{6}a_4 = -\frac{1}{24}$$

$$a_7 = -\frac{1}{7}a_5 = 0$$

$$\vdots \quad \vdots \quad \vdots$$

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n (x-1)^n \\ &= a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + \dots \\ &= 2 - (x-1)^2 + \frac{1}{4}(x-1)^4 - \frac{1}{24}(x-1)^6 + \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{2n}}{n! 2^n} \end{aligned}$$

Exercise

Find the series solution to the initial value problem $(x-1)y'' - xy' + y = 0$; $y(0) = -2$, $y'(0) = 6$

Solution

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \\ y''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \end{aligned}$$

$$(x-1)y'' - xy' + y = 0$$

$$(x-1) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^{n+1} - \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} (n+1)a_{n+1}x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n(n+1)a_{n+1}x^n - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} [a_n - (n+2)(n+1)a_{n+2}]x^n = 0$$

$$\sum_{n=1}^{\infty} [n(n+1)a_{n+1} - na_n]x^n + a_0 - 2a_2 + \sum_{n=1}^{\infty} [a_n - (n+2)(n+1)a_{n+2}]x^n = 0$$

$$a_0 - 2a_2 + \sum_{n=1}^{\infty} [n(n+1)a_{n+1} - (n-1)a_n - (n+2)(n+1)a_{n+2}]x^n = 0$$

Given: $y(0) = -2 = a_0$, $y'(0) = 6 = a_1$

$$a_0 - 2a_2 = 0 \rightarrow a_2 = \frac{1}{2}a_0 = -1$$

$$n(n+1)a_{n+1} - (n-1)a_n - (n+2)(n+1)a_{n+2} = 0$$

$$\rightarrow a_{n+2} = \frac{n}{n+2}a_{n+1} - \frac{n-1}{(n+2)(n+1)}a_n$$

$$a_3 = \frac{1}{3}a_2 - 0a_1 = -\frac{1}{3}$$

$$a_4 = \frac{1}{2}a_3 - \frac{1}{12}a_2 = -\frac{1}{6} + \frac{1}{12} = -\frac{1}{12}$$

$$a_5 = \frac{3}{5}a_4 - \frac{1}{10}a_3 = -\frac{3}{60} + \frac{1}{30} = -\frac{1}{60}$$

$$a_6 = \frac{2}{3}a_5 - \frac{1}{10}a_4 = -\frac{1}{90} + \frac{1}{120} = -\frac{1}{360}$$

$$\begin{aligned} y(x) &= \left(-2 - x^2 - \frac{1}{3}x^3 - \frac{1}{12}x^4 - \dots\right) + a_1 x \\ &= -2\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots\right) + 6x \\ &= -2\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots\right) + 6x + 2x \\ &= \underline{8x - 2e^x} \end{aligned}$$

Exercise

Find the series solution to the initial value problem

$$(x+1)y'' - (2-x)y' + y = 0; \quad y(0) = 2, \quad y'(0) = -1$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x+1)y'' - (2-x)y' + y = 0$$

$$(x+1) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - (2-x) \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\begin{aligned} & \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+1} + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n \\ & + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0 \end{aligned}$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + (n+1) a_{n+1}] x^{n+1} + \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - 2(n+1) a_{n+1} + a_n] x^n = 0$$

$$\sum_{n=1}^{\infty} [n(n+1) a_{n+1} + n a_n] x^n + \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - 2(n+1) a_{n+1} + a_n] x^n = 0$$

$$\sum_{n=1}^{\infty} [n(n+1) a_{n+1} + n a_n] x^n + 2a_2 - 2a_1 + a_0$$

$$+ \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - 2(n+1) a_{n+1} + a_n] x^n = 0$$

$$2a_2 - 2a_1 + a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} + (n-2)(n+1) a_{n+1} + (n+1) a_n] x^n = 0$$

$$\text{Given: } y(0) = 2 = a_0, \quad y'(0) = -1 = a_1$$

$$2a_2 - 2a_1 + a_0 = 0$$

$$a_2 = \frac{1}{2}(2a_1 - a_0) = -2$$

$$(n+2)(n+1) a_{n+2} + (n-2)(n+1) a_{n+1} + (n+1) a_n = 0$$

$$\rightarrow a_{n+2} = -\frac{n-2}{n+2}a_{n+1} - \frac{1}{n+2}a_n$$

$$a_3 = \frac{1}{3}a_2 - \frac{1}{3}a_1 = \frac{2}{3} + \frac{1}{3} = 1$$

$$a_4 = 0a_3 - \frac{1}{4}a_2 = -\frac{1}{2}$$

$$a_5 = -\frac{1}{5}a_4 - \frac{1}{5}a_3 = -\frac{1}{10} - \frac{1}{5} = -\frac{3}{10}$$

$$a_6 = -\frac{1}{3}a_5 - \frac{1}{6}a_4 = \frac{1}{10} - \frac{1}{12} = \frac{1}{60}$$

$$y(x) = 2 - x - 2x^2 + x^3 + \frac{1}{2}x^4 - \frac{3}{10}x^5 + \dots$$

Exercise

Find the series solution to the initial value problem

$$(1-x)y'' + xy' - 2y = 0; \quad y(0) = 0, \quad y'(0) = 1$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(1-x)y'' + xy' - 2y = 0$$

$$(1-x) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+1} + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - 2a_n] x^n - \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - (n+1) a_{n+1}] x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - 2a_n] x^n - \sum_{n=1}^{\infty} [n(n+1) a_{n+1} - n a_n] x^n = 0$$

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - 2a_n]x^n - \sum_{n=1}^{\infty} [n(n+1)a_{n+1} - na_n]x^n = 0$$

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - n(n+1)a_{n+1} + (n-2)a_n]x^n = 0$$

$$(n+2)(n+1)a_{n+2} - n(n+1)a_{n+1} + (n-2)a_n = 0$$

$$a_{n+2} = \frac{n(n+1)a_{n+1} - (n-2)a_n}{(n+1)(n+2)}$$

Given: $y(0) = 0 = a_0$, $y'(0) = 1 = a_1$

$$2a_2 - 2a_0 = 0 \rightarrow a_2 = a_0 = 0$$

$$n=1 \rightarrow a_3 = \frac{2a_2 + a_1}{6} = \frac{1}{6}$$

$$n=2 \rightarrow a_4 = \frac{6a_3}{12} = \frac{1}{12}$$

$$n=3 \rightarrow a_5 = \frac{1}{20}(12a_4 - a_3) = \frac{1}{20}\left(1 - \frac{1}{6}\right) = \frac{1}{24}$$

$$n=4 \rightarrow a_6 = \frac{1}{30}(20a_5 - 2a_4) = \frac{1}{30}\left(\frac{5}{6} - \frac{1}{6}\right) = \frac{1}{45}$$

$$n=5 \rightarrow a_7 = \frac{1}{42}(30a_6 - 3a_5) = \frac{1}{30}\left(\frac{2}{3} - \frac{1}{8}\right) = \frac{13}{1008}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{24}x^5 + \frac{1}{45}x^6 + \frac{13}{1008}x^7 + \dots$$

Exercise

Find the series solution to the initial value problem

$$(x^2 + 1)y'' + 2xy' = 0; \quad y(0) = 0, \quad y'(0) = 1$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$$(x^2 + 1)y'' + 2xy' = 0$$

$$(x^2 + 1) \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + 2x \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^{n+2} + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} 2(n+1)a_{n+1} x^{n+1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} 2na_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2} x^n + 2a_1 x + \sum_{n=2}^{\infty} 2na_n x^n = 0$$

$$2a_2 + (6a_3 + 2a_1)x + \sum_{n=2}^{\infty} [(n(n-1) + 2n)a_n + (n+2)(n+1)a_{n+2}] x^n = 0$$

$$\text{Given: } y(0) = 0 = a_0, \quad y'(0) = 1 = a_1$$

$$2a_2 + (6a_3 + 2a_1)x = 0 \rightarrow \begin{cases} a_2 = 0 \\ a_3 = -\frac{1}{3}a_1 = -\frac{1}{3} \end{cases}$$

$$n(n+1)a_n + (n+2)(n+1)a_{n+2} = 0$$

$$\rightarrow a_{n+2} = -\frac{n}{n+2}a_n \quad n = 2, 3, \dots$$

$$a_4 = -\frac{1}{2}a_2 = 0$$

$$a_5 = -\frac{3}{5}a_3 = \frac{1}{5}$$

$$a_6 = -\frac{2}{3}a_4 = 0$$

$$a_7 = -\frac{5}{7}a_5 = -\frac{1}{7}$$

$$\underline{y(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots}$$

Exercise

Find the series solution to the initial value problem

$$(x^2 - 1)y'' + 3xy' + xy = 0 ; \quad y(0) = 4, \quad y'(0) = 6$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x^2 - 1)y'' + 3xy' + xy = 0$$

$$(x^2 - 1) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 3x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\begin{aligned} \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+2} - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \\ + \sum_{n=0}^{\infty} 3(n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \end{aligned}$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^{n+2} + \sum_{n=0}^{\infty} [(3n+3) a_{n+1} + a_n] x^{n+1} - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} (3n a_n + a_{n-1}) x^n - \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n = 0$$

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1) a_n x^n + (3a_1 + a_0)x + \sum_{n=2}^{\infty} (3n a_n + a_{n-1}) x^n \\ - 2a_2 - 6a_3 x - \sum_{n=2}^{\infty} (n+1)(n+2) a_{n+2} x^n = 0 \end{aligned}$$

$$-2a_2 + (3a_1 + a_0 - 6a_3)x + \sum_{n=2}^{\infty} \left[(n^2 + 2n)a_n + a_{n-1} - (n+2)(n+1)a_{n+2} \right] x^n = 0$$

Given: $y(0) = 4 = a_0$, $y'(0) = 6 = a_1$

$$\left\{ \begin{array}{l} -2a_2 = 0 \quad \rightarrow \underline{a_2 = 0} \\ 3a_1 + a_0 - 6a_3 = 0 \quad \rightarrow \underline{a_3 = \frac{22}{6} = \frac{11}{3}} \end{array} \right.$$

$$(n^2 + 2n)a_n + a_{n-1} - (n+2)(n+1)a_{n+2} = 0$$

$$\underline{a_{n+2} = \frac{(n^2 + 2n)a_n + a_{n-1}}{(n+1)(n+2)}}$$

$$n = 2 \rightarrow a_4 = \frac{8a_2 + a_1}{12} = \frac{6}{12} = \frac{1}{2}$$

$$n = 3 \rightarrow a_5 = \frac{1}{20}(15a_3 + a_2) = \frac{1}{20}(55) = \frac{11}{4}$$

$$n = 4 \rightarrow a_6 = \frac{1}{30}(24a_4 + a_3) = \frac{1}{30}\left(12 + \frac{11}{3}\right) = \frac{47}{90}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y(x) = 4 + 6x + \frac{11}{3}x^3 + \frac{1}{2}x^4 + \frac{11}{4}x^5 + \frac{47}{90}x^6 + \cdots}$$

Exercise

Find the series solution to the initial value problem

$$(2 + x^2)y'' - xy' + 4y = 0 \quad y(0) = -1 \quad y'(0) = 3$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(2+x^2)y'' - xy' + 4y = 0$$

$$(2+x^2) \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x \sum_{n=1}^{\infty} na_n x^{n-1} + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x^2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} na_n x^{n-1} + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$2(n+2)(n+1)a_{n+2} + n(n-1)a_n - na_n + 4a_n = 0$$

$$2(n+2)(n+1)a_{n+2} + (n^2 - 2n + 4)a_n = 0$$

$$a_{n+2} = -\frac{n^2 - 2n + 4}{2(n+2)(n+1)}a_n$$

$$a_0 = y(0) = -1$$

$$a_1 = y'(0) = 3$$

$$n=0 \rightarrow a_2 = -\frac{4}{4}a_0 = 1$$

$$n=1 \rightarrow a_3 = -\frac{3}{12}a_1 = -\frac{1}{4}(3) = -\frac{3}{4}$$

$$n=2 \rightarrow a_4 = -\frac{4}{24}a_2 = -\frac{1}{6}$$

$$n=3 \rightarrow a_5 = -\frac{7}{40}a_3 = -\frac{7}{40}\left(-\frac{3}{4}\right) = \frac{21}{160}$$

$$y(x) = -1 + 3x + x^2 - \frac{3}{4}x^3 - \frac{1}{6}x^4 + \frac{21}{160}x^5 + \dots$$

Exercise

Find the series solution to the initial value problem

$$(2-x^2)y'' - xy' + 4y = 0 \quad y(0) = 1 \quad y'(0) = 0$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$$(2-x^2)y'' - xy' + 4y = 0$$

$$(2-x^2) \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x \sum_{n=1}^{\infty} na_n x^{n-1} + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x^2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} na_n x^{n-1} + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+2)(n+1)a_{n+2} - (n^2 - n + n - 4)a_n \right] x^n = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+2)(n+1)a_{n+2} - (n^2 - 4)a_n \right] x^n = 0$$

$$2(n+2)(n+1)a_{n+2} - (n^2 - 4)a_n = 0$$

$$\boxed{a_{n+2} = \frac{n-2}{2(n+1)}a_n}$$

$$a_0 = y(0) = 1$$

$$a_1 = y'(0) = 0$$

$$n=0 \rightarrow a_2 = \frac{-2}{2}a_0 = -1$$

$$n=1 \rightarrow a_3 = -\frac{1}{4}a_1 = 0$$

$$n=2 \rightarrow a_4 = 0$$

$$n=3 \rightarrow a_5 = *a_3 = 0$$

$$\vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = 1 - x^2}$$

Exercise

Find the series solution to the initial value problem

$$(4-x^2)y'' + 2y = 0 \quad y(0) = 0 \quad y'(0) = 1$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(4 - x^2) y'' + 2y = 0$$

$$(4 - x^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2} - x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 4(n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[4(n+1)(n+2) a_{n+2} - (n^2 - n - 2) a_n \right] x^n = 0$$

$$4(n+1)(n+2) a_{n+2} - (n+1)(n-2) a_n = 0$$

$$\boxed{a_{n+2} = \frac{n-2}{4(n+2)} a_n}$$

$$a_0 = y(0) = 0$$

$$a_1 = y'(0) = 1$$

$$n=0 \rightarrow a_2 = \frac{-2}{8} a_0 = 0$$

$$n=1 \rightarrow a_3 = -\frac{1}{12} a_1 = -\frac{1}{12}$$

$$n=2 \rightarrow a_4 = 0$$

$$n=3 \rightarrow a_5 = \frac{1}{20} a_3 = -\frac{1}{240}$$

$$\vdots \quad \vdots \quad \vdots$$

$$n=5 \rightarrow a_7 = \frac{3}{28} a_5 = -\frac{1}{2,240}$$

$$\vdots \quad \vdots \quad \vdots$$

$$y(x) = x - \frac{1}{12}x^3 - \frac{1}{240}x^5 - \frac{1}{2240}x^7 - \frac{1}{16,128}x^9 - \dots$$

Exercise

Find a power series solution. $(x^2 - 4)y'' + 3xy' + y = 0$; $y(0) = 4$, $y'(0) = 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x^2 - 4)y'' + 3xy' + y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 4 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 3x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} 4(n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n^2 - n + 3n + 1) a_n - 4(n+2)(n+1) a_{n+2} \right] x^n = 0$$

$$(n^2 + 2n + 1) a_n - 4(n+2)(n+1) a_{n+2} = 0$$

$$4(n+2)(n+1) a_{n+2} = (n+1)^2 a_n$$

$$a_{n+2} = \frac{n+1}{4(n+2)} a_n$$

$$a_0 = y(0) = 4$$

$$a_1 = y'(0) = 1$$

$$n=0 \rightarrow a_2 = \frac{1}{8}a_0 = \frac{1}{2}$$

$$n=1 \rightarrow a_3 = \frac{2}{4 \cdot 3}a_1 = \frac{1}{6}$$

$$n=2 \rightarrow a_4 = \frac{3}{16}a_2 = \frac{3}{32}$$

$$n=3 \rightarrow a_5 = \frac{1}{5}a_3 = \frac{1}{30}$$

$$n=4 \rightarrow a_6 = \frac{5}{24}a_4 = \frac{5}{256}$$

$$n=5 \rightarrow a_7 = \frac{6}{4 \cdot 7}a_5 = \frac{1}{140}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(x) = 4 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{3}{32}x^4 + \frac{1}{30}x^5 + \frac{5}{256}x^6 + \frac{1}{140}x^7 + \dots$$

Exercise

Find a power series solution. $(x^2 + 1)y'' + 2xy' - 2y = 0$; $y(0) = 0$, $y'(0) = 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x^2 + 1)y'' + 2xy' - 2y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2x \sum_{n=1}^{\infty} n a_n x^{n-1} - 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} + (n^2 + n - 2) a_n \right] x^n = 0$$

$$(n+1)(n+2)a_{n+2} + (n-1)(n+2)a_n = 0$$

$$\underline{a_{n+2} = -\frac{n-1}{n+1}a_n}$$

$$a_0 = y(0) = 0$$

$$a_1 = y'(0) = 1$$

$$n=0 \rightarrow a_2 = a_0$$

$$n=1 \rightarrow a_3 = 0$$

$$n=2 \rightarrow a_4 = -\frac{1}{3}a_2 = -\frac{1}{3}a_0$$

$$n=3 \rightarrow a_5 = 0$$

$$n=4 \rightarrow a_6 = -\frac{3}{5}a_4 = \frac{1}{5}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n=6 \rightarrow a_8 = -\frac{5}{7}a_6 = -\frac{1}{7}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n} = \frac{(-1)^n}{2n-1}a_0$$

$$\underline{y(x) = a_1x + a_0 \left(1 + x^2 - \frac{1}{3}x^4 + \frac{1}{5}x^6 - \frac{1}{7}x^8 + \dots\right)}$$

$$y(x) = a_1x + a_0 \left(1 + x \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots\right)\right)$$

$$\underline{= a_1x + a_0 \left(1 + x \tan^{-1}x\right)}$$

Exercise

Find a power series solution. $(2x - x^2)y'' - 6(x-1)y' - 4y = 0$; $y(1) = 0$, $y'(1) = 1$

Solution

$$\text{Let } z = x - 1 \Rightarrow \begin{cases} x = z + 1 \\ dz = dx \end{cases}$$

$$(2x - x^2)y'' - 6(x-1)y' - 4y = 0$$

$$(2z + 2 - z^2 - 2z - 1)y'' - 6zy' - 4y = 0$$

$$(1 - z^2)y'' - 6zy' - 4y = 0$$

$$y(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$y'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

$$y''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n$$

$$(1-z^2)y'' - 6zy' - 4y = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} - z^2 \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} - 6z \sum_{n=1}^{\infty} n a_n z^{n-1} - 4 \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n z^n - \sum_{n=1}^{\infty} 6n a_n z^n - 4 \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} z^n - \sum_{n=0}^{\infty} n(n-1) a_n z^n - \sum_{n=0}^{\infty} 6n a_n z^n - 4 \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} - (n^2 + 5n + 4) a_n \right] z^n = 0$$

$$(n+1)(n+2) a_{n+2} - (n+1)(n+4) a_n = 0$$

$$a_{n+2} = \frac{n+4}{n+2} a_n$$

$$\text{Given: } y(1) = 0 = a_0, \quad y'(1) = 1 = a_1$$

$$a_0 = 0$$

$$a_1 = 1$$

$$n=0 \rightarrow a_2 = 2a_0 = 0$$

$$n=1 \rightarrow a_3 = \frac{5}{3} a_1 = \frac{5}{3}$$

$$n=2 \rightarrow a_4 = \frac{3}{2} a_2 = 0$$

$$n=3 \rightarrow a_5 = \frac{7}{5} a_3 = \frac{7}{3}$$

$$n=4 \rightarrow a_6 = 0$$

$$n=5 \rightarrow a_7 = \frac{9}{7} a_5 = \frac{9}{3}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n=7 \rightarrow a_9 = \frac{11}{9} a_7 = \frac{11}{3}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n+1} = \frac{2n+3}{3}$$

$$y(z) = z + \frac{5}{3}z^3 + \frac{7}{3}z^5 + 3z^7 + \frac{11}{3}z^9 + \dots + \frac{2n+3}{3}z^{2n+1} + \dots$$

$$y(x) = (x-1) + \frac{5}{3}(x-1)^3 + \frac{7}{3}(x-1)^5 + 3(x-1)^7 + \frac{11}{3}(x-1)^9 + \dots$$

Exercise

Find a power series solution. $(x^2 - 6x + 10)y'' - 4(x-3)y' + 6y = 0$; $y(3) = 2$, $y'(3) = 0$

Solution

$$\text{Let } z = x - 3 \Rightarrow \begin{cases} x = z + 3 \\ dz = dx \end{cases}$$

$$(x^2 - 6x + 10)y'' - 4(x-3)y' + 6y = 0$$

$$(z^2 + 6z + 9 - 6z - 18 + 10)y'' - 4zy' + 6y = 0$$

$$(z^2 + 1)y'' - 4zy' + 6y = 0$$

$$y(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$y'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

$$y''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n$$

$$(z^2 + 1)y'' - 4zy' + 6y = 0$$

$$z^2 \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} - 4z \sum_{n=1}^{\infty} n a_n z^{n-1} + 6 \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n z^n + \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} - \sum_{n=1}^{\infty} 4n a_n z^n + \sum_{n=0}^{\infty} 6a_n z^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n z^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n - \sum_{n=0}^{\infty} 4n a_n z^n + \sum_{n=0}^{\infty} 6a_n z^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} + (n^2 - 5n + 6)a_n \right] z^n = 0$$

$$(n+1)(n+2)a_{n+2} + (n-2)(n-3)a_n = 0$$

$$a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)}a_n$$

Given: $y(3) = 2 = a_0$, $y'(3) = 0 = a_1$

$$a_0 = 2$$

$$a_1 = 0$$

$$n=0 \rightarrow a_2 = -3a_0 = -6$$

$$n=1 \rightarrow a_3 = -\frac{2}{4}a_1 = 0$$

$$n=2 \rightarrow a_4 = 0$$

$$n=3 \rightarrow a_5 = 0$$

$$n=4 \rightarrow a_6 = 0$$

$$n=5 \rightarrow a_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(z) = 2 - 6z^2$$

$$y(x) = 2 - 6(x-3)^2$$

Exercise

Find a power series solution. $(4x^2 + 16x + 17)y'' - 8y = 0$; $y(-2) = 1$, $y'(-2) = 0$

Solution

$$\text{Let } z = x + 2 \Rightarrow \begin{cases} x = z - 2 \\ dz = dx \end{cases}$$

$$(4z^2 - 16z + 16 + 16z - 32 + 17)y'' - 8y = 0$$

$$(4z^2 + 1)y'' - 8y = 0$$

$$y(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$y'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

$$y''(z) = \sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} z^n$$

$$(4z^2 + 1)y'' - 8y = 0$$

$$4z^2 \sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} - 8 \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=2}^{\infty} 4n(n-1)a_n z^n + \sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} - 8 \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} 4n(n-1)a_n z^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} z^n - 8 \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} + (4n^2 - 4n - 8)a_n \right] z^n = 0$$

$$(n+1)(n+2)a_{n+2} + 4(n+1)(n-2)a_n = 0$$

$$\underline{a_{n+2} = -\frac{4(n-2)}{n+2}a_n}$$

$$\text{Given: } y(-2) = 1 = a_0, \quad y'(-2) = 0 = a_1$$

$$a_0 = 1$$

$$a_1 = 0$$

$$n=0 \rightarrow a_2 = \frac{8}{2}a_0 = 4$$

$$n=1 \rightarrow a_3 = \frac{4}{3}a_1 = 0$$

$$n=2 \rightarrow a_4 = -0a_2 = 0$$

$$n=3 \rightarrow a_5 = 0$$

$$n=4 \rightarrow a_6 = 0$$

$$n=5 \rightarrow a_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(z) = 1 + 4z^2$$

$$\underline{y(x) = 1 + 4(x+2)^2}$$

Exercise

Find a power series solution. $(x^2 + 6x)y'' + (3x + 9)y' - 3y = 0$; $y(-3) = 0$, $y'(-3) = 2$

Solution

$$\text{Let } z = x + 3 \Rightarrow \begin{cases} x = z - 3 \\ dz = dx \end{cases}$$

$$(x^2 + 6x)y'' + (3x + 9)y' - 3y = 0$$

$$(z^2 - 6z + 9 + 6z - 18)y'' + 3zy' - 3y = 0$$

$$(z^2 - 9)y'' + 3zy' - 3y = 0$$

$$y(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$y'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

$$y''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n$$

$$(z^2 - 9)y'' + 3zy' - 3y = 0$$

$$z^2 \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} - 9 \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} + 3z \sum_{n=1}^{\infty} n a_n z^{n-1} - 3 \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n z^n - \sum_{n=2}^{\infty} 9n(n-1) a_n z^{n-2} + \sum_{n=1}^{\infty} 3n a_n z^n - \sum_{n=0}^{\infty} 3a_n z^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n z^n - \sum_{n=0}^{\infty} 9(n+2)(n+1) a_{n+2} z^n + \sum_{n=0}^{\infty} 3n a_n z^n - \sum_{n=0}^{\infty} 3a_n z^n = 0$$

$$\sum_{n=0}^{\infty} \left[-9(n+2)(n+1) a_{n+2} + (n^2 - n + 3n - 3) a_n \right] z^n = 0$$

$$-9(n+2)(n+1) a_{n+2} + (n+3)(n-1) a_n = 0$$

$$a_{n+2} = \frac{(n+3)(n-1)}{9(n+1)(n+2)} a_n$$

Given: $y(-3) = 0 = a_0$, $y'(-3) = 2 = a_1$

$$a_0 = 0$$

$$a_1 = 2$$

$$n=0 \rightarrow a_2 = \frac{-3}{18} a_0 = 0$$

$$n=1 \rightarrow a_3 = 0 a_1 = 0$$

$$n=2 \rightarrow a_4 = \frac{5}{108} a_2 = 0$$

$$n=3 \rightarrow a_5 = 0$$

$$n=4 \rightarrow a_6 = 0$$

$$n=5 \rightarrow a_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(z) = 2z$$

$$y(x) = 2x + 6$$

Exercise

Find the series solution near the given value $y'' - (x-2)y' + 2y = 0$; near $x = 2$

Solution

$$y = \sum_{n=0}^{\infty} a_n (x-2)^n$$

$$y' = \sum_{n=1}^{\infty} n a_n (x-2)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-2)^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-2)^n$$

$$y'' - (x-2)y' + 2y = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-2)^n - (x-2) \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-2)^n + 2 \sum_{n=0}^{\infty} a_n (x-2)^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-2)^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-2)^{n+1} + \sum_{n=0}^{\infty} 2a_n (x-2)^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} + 2a_n \right] (x-2)^n - \sum_{n=0}^{\infty} (n+1)a_{n+1} (x-2)^{n+1} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} + 2a_n \right] (x-2)^n - \sum_{n=1}^{\infty} na_n (x-2)^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left[(n+1)(n+2)a_{n+2} + 2a_n \right] (x-2)^n - \sum_{n=1}^{\infty} na_n (x-2)^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left[(n+1)(n+2)a_{n+2} - (n-2)a_n \right] (x-2)^n = 0$$

$$\text{For } n=0 \rightarrow 2a_2 + 2a_0 = 0 \Rightarrow \underline{a_2 = -a_0}$$

$$(n+1)(n+2)a_{n+2} - (n-2)a_n = 0$$

$$\underline{a_{n+2} = \frac{n-2}{(n+1)(n+2)} a_n}$$

$$a_0$$

$$n=0 \rightarrow a_2 = -a_0$$

$$n=2 \rightarrow a_4 = 0$$

$$n=4 \rightarrow a_6 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_1$$

$$n=1 \rightarrow a_3 = -\frac{1}{6}a_1$$

$$n=3 \rightarrow a_5 = \frac{1}{20}a_3 = -\frac{1}{120}a_1$$

$$n=5 \rightarrow a_7 = \frac{3}{42}a_5 = -\frac{1}{1680}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y(x) = a_0 \left(1 - (x-2)^2 \right) + a_1 \left((x-2) - \frac{1}{6}(x-2)^3 - \frac{1}{120}(x-2)^5 - \frac{1}{1680}(x-2)^7 - \dots \right)}$$

Exercise

Find the series solution near the given value $y'' + (x-1)^2 y' - 4(x-1)y = 0$; *near $x=1$*

Solution

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-1)^n$$

$$y'' + (x-1)^2 y' - 4(x-1)y = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-1)^n + (x-1)^2 \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n$$

$$- 4(x-1) \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-1)^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^{n+2} - \sum_{n=0}^{\infty} 4a_n (x-1)^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-1)^n + \sum_{n=2}^{\infty} (n-1) a_{n-1} (x-1)^n - \sum_{n=1}^{\infty} 4a_{n-1} (x-1)^n = 0$$

$$2a_2 + 6a_3(x-1) + \sum_{n=2}^{\infty} (n+1)(n+2) a_{n+2} (x-1)^n + \sum_{n=2}^{\infty} (n-1) a_{n-1} (x-1)^n$$

$$- 4a_0(x-1) - \sum_{n=2}^{\infty} 4a_{n-1} (x-1)^n = 0$$

$$2a_2 + (6a_3 - 4a_0)(x-1) + \sum_{n=2}^{\infty} [(n+1)(n+2) a_{n+2} + (n-5) a_{n-1}] (x-1)^n = 0$$

$$\begin{cases} 2a_2 = 0 & \rightarrow \underline{a_2 = 0} \\ 6a_3 - 4a_0 = 0 & \rightarrow \underline{a_3 = \frac{2}{3}a_0} \end{cases}$$

$$(n+1)(n+2)a_{n+2} + (n-5)a_{n-1} = 0$$

$$\underline{a_{n+2} = -\frac{(n-5)}{(n+1)(n+2)}a_{n-1}}$$

a_0	a_1	$a_2 = 0$
$n=1 \rightarrow a_3 = \frac{2}{3}a_0$	$n=2 \rightarrow a_4 = \frac{1}{4}a_1$	$n=3 \rightarrow a_5 = \frac{2}{20}a_2 = 0$
$n=4 \rightarrow a_6 = \frac{1}{30}a_3 = \frac{1}{45}a_0$	$n=5 \rightarrow a_7 = 0$	$n=6 \rightarrow a_8 = 0$
$n=7 \rightarrow a_9 = -\frac{2}{8 \cdot 9}a_6 = -\frac{1}{1,620}a_0$	$n=8 \rightarrow a_{10} = 0$	$n=9 \rightarrow a_{11} = 0$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\underline{y(x) = a_0 \left(1 + \frac{2}{3}(x-1)^3 + \frac{1}{45}(x-1)^6 - \frac{1}{1,620}(x-1)^9 + \dots \right) + a_1 \left((x-1) + \frac{1}{4}(x-1)^4 \right)}$$

$$\underline{y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n 4(x-1)^{3n}}{3^n (3n-1)(3n-4)n!} + a_1 \left((x-1) + \frac{1}{4}(x-1)^4 \right)}$$

Exercise

Find the series solution near the given value $y'' + (x-1)y = e^x$; *near* $x=1$

Solution

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-1)^n$$

$$y'' + (x-1)y = e^{x-1+1}$$

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}(x-1)^n + (x-1) \sum_{n=0}^{\infty} a_n(x-1)^n = e \cdot e^{x-1} \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}(x-1)^n + \sum_{n=0}^{\infty} a_n(x-1)^{n+1} = e \cdot \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$$

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}(x-1)^n + \sum_{n=1}^{\infty} a_{n-1}(x-1)^n = e \cdot \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$$

$$2a_2 + \sum_{n=1}^{\infty} (n+1)(n+2)a_{n+2}(x-1)^n + \sum_{n=1}^{\infty} a_{n-1}(x-1)^n = e + e \cdot \sum_{n=1}^{\infty} \frac{(x-1)^n}{n!}$$

$$2a_2 + \sum_{n=1}^{\infty} [(n+1)(n+2)a_{n+2} + a_{n-1}](x-1)^n = e + e \cdot \sum_{n=1}^{\infty} \frac{(x-1)^n}{n!}$$

$$2a_2 = e \rightarrow \underline{a_2 = \frac{e}{2}}$$

$$(n+1)(n+2)a_{n+2} + a_{n-1} = \frac{e}{n!}$$

$$\underline{a_{n+2} = \frac{e}{(n+1)(n+2)n!} - \frac{1}{(n+1)(n+2)}a_{n-1}}$$

$$a_0$$

$$n=1 \rightarrow a_3 = \frac{e}{6} - \frac{1}{6}a_0$$

$$n=4 \rightarrow a_6 = \frac{e}{720} - \frac{1}{30}a_3 = -\frac{11e}{720} + \frac{1}{180}a_0$$

$$a_1$$

$$n=2 \rightarrow a_4 = \frac{e}{24} - \frac{1}{12}a_1$$

$$n=5 \rightarrow a_7 = \frac{e}{5040} - \frac{1}{42}a_4 = \frac{e}{1260} + \frac{1}{504}a_0$$

$$a_2 = \frac{e}{2}$$

$$n=3 \rightarrow a_5 = \frac{e}{120} - \frac{1}{20}a_2 = \frac{e}{120} - \frac{e}{40} = -\frac{e}{60}$$

$$n=6 \rightarrow a_8 = \frac{e}{40,320} - \frac{1}{56}a_5 = \frac{e}{40,320} + \frac{e}{3360} = \frac{13e}{40,320}$$

$$\begin{array}{cccc} \vdots & \vdots & \vdots & \vdots \end{array}$$

$$\begin{aligned} y(x) &= a_0 + (x-1)a_1 + \frac{e}{2}(x-1)^2 + \left(\frac{e}{6} - \frac{1}{6}a_0\right)(x-1)^3 + \left(\frac{e}{24} - \frac{1}{12}a_1\right)(x-1)^4 - \frac{e}{60}(x-1)^5 + \dots \\ &= a_0 + (x-1)a_1 + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + -\frac{1}{6}a_0(x-1)^3 + \frac{e}{24}(x-1)^4 \\ &\quad - \frac{1}{12}a_1(x-1)^4 - \frac{e}{60}(x-1)^5 + \dots \end{aligned}$$

$$y(x) = e \left(\frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4 - \frac{1}{60}(x-1)^5 + \dots \right) + a_0 \left(1 - \frac{1}{6}(x-1)^3 + \dots \right) + a_1 \left((x-1) - \frac{1}{12}(x-1)^4 + \dots \right)$$

Exercise

Find the series solution near the given value

$$y'' + xy' + (2x-1)y = 0 ; \quad \text{near } x = -1 \quad y(-1) = 2, \quad y'(-1) = -2$$

Solution

$$t = x + 1 \rightarrow x = t - 1$$

$$y = \sum_{n=0}^{\infty} a_n t^n$$

$$y' = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} t^n$$

$$y'' + xy' + (2x-1)y = 0$$

$$y'' + (t-1)y' + (2t-3)y = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} t^n + (t-1) \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n + (2t-3) \sum_{n=0}^{\infty} a_n t^n = 0$$

$$\begin{aligned} \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}t^n + \sum_{n=0}^{\infty} (n+1)a_{n+1}t^{n+1} - \sum_{n=0}^{\infty} (n+1)a_{n+1}t^n \\ + \sum_{n=0}^{\infty} 2a_n t^{n+1} - \sum_{n=0}^{\infty} 3a_n t^n = 0 \end{aligned}$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} - 3a_n \right] t^n + \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} + 2a_n \right] t^{n+1} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} - 3a_n \right] t^n + \sum_{n=1}^{\infty} \left[na_n + 2a_{n-1} \right] t^n = 0$$

$$\begin{aligned} 2a_2 - a_1 - 3a_0 + \sum_{n=1}^{\infty} \left[(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} - 3a_n \right] t^n \\ + \sum_{n=1}^{\infty} (na_n + 2a_{n-1})t^n = 0 \end{aligned}$$

$$2a_2 - a_1 - 3a_0 + \sum_{n=1}^{\infty} \left[(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} + (n-3)a_n + 2a_{n-1} \right] t^n = 0$$

$$2a_2 - a_1 - 3a_0 = 0 \rightarrow \underline{a_2 = \frac{1}{2}(a_1 + 3a_0)}$$

$$(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} + (n-3)a_n + 2a_{n-1} = 0$$

$$\underline{a_{n+2} = \frac{1}{n+2}a_{n+1} - \frac{n-3}{(n+1)(n+2)}a_n - \frac{2}{(n+1)(n+2)}a_{n-1}}$$

Given: $t = x + 1$

$$y(x = -1) = y(t = 0) = 2 = a_0, \quad y'(x = -1) = y'(t = 0) = -2 = a_1$$

$$\underline{a_2 = \frac{1}{2}(a_1 + 3a_0) = \frac{1}{2}(-2 + 6) = 2}$$

$$n = 1 \rightarrow a_3 = \frac{1}{3}a_2 + \frac{1}{3}a_1 - \frac{1}{3}a_0$$

$$= \frac{2}{3} - \frac{2}{3} - \frac{2}{3}$$

$$\underline{= -\frac{2}{3}}$$

$$n = 2 \rightarrow a_4 = \frac{1}{4}a_3 + \frac{1}{12}a_2 - \frac{1}{6}a_1$$

$$= -\frac{1}{6} + \frac{1}{6} + \frac{1}{3}$$

$$= \frac{1}{3}$$

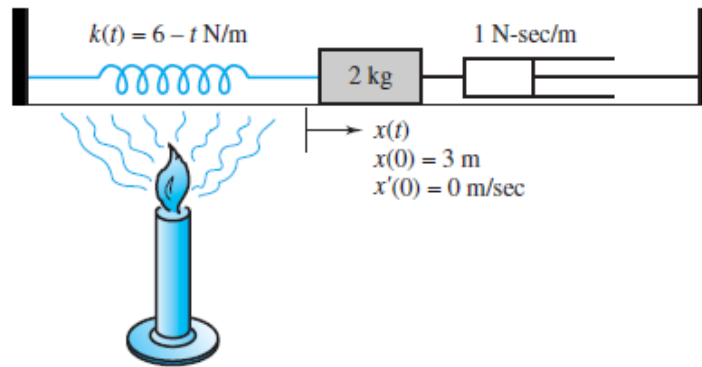
$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(t) = 2 - 2t + 3t^2 - \frac{1}{3}t^3 + \frac{1}{3}t^4 + \dots$$

$$y(x) = 2 - 2(x+1) + 3(x+1)^2 - \frac{1}{3}(x+1)^3 + \frac{1}{3}(x+1)^4 + \dots$$

Exercise

As a spring is heated, its spring “constant” decreases. Suppose the spring is heated so that the spring “constant” at time t is $k(t) = 6 - t$ N/m.



If the unforced mass-spring system has mass $m = 2$ kg and a damping constant $b = 1$ N-sec/m with initial conditions $x(0) = 3$ m and $x'(0) = 0$ m/sec, then the displacement $x(t)$ is governed by the initial value problem

$$2x''(t) + x'(t) + (6 - t)x(t) = 0 ; \quad x(0) = 3, \quad x'(0) = 0$$

Find at least the first four nonzero terms in a power series expansion about $t = 0$ for the displacement.

Solution

$$x(t) = \sum_{n=0}^{\infty} a_n t^n$$

$$x'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n$$

$$x''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n$$

$$2x'' + x' + (6-t)x = 0$$

$$2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n + \sum_{n=0}^{\infty} (n+1)a_{n+1} t^n + (6-t) \sum_{n=0}^{\infty} a_n t^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} t^n + \sum_{n=0}^{\infty} (n+1)a_{n+1} t^n + \sum_{n=0}^{\infty} 6a_n t^n - \sum_{n=0}^{\infty} a_n t^{n+1} = 0$$

$$\sum_{n=0}^{\infty} [2(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + 6a_n] t^n - \sum_{n=1}^{\infty} a_{n-1} t^n = 0$$

$$4a_2 + a_1 + 6a_0 + \sum_{n=1}^{\infty} [2(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + 6a_n] t^n - \sum_{n=1}^{\infty} a_{n-1} t^n = 0$$

$$4a_2 + a_1 + 6a_0 + \sum_{n=1}^{\infty} [2(n+1)(n+2)a_{n+2} + (n+1)a_{n+1} + 6a_n - a_{n-1}] t^n = 0$$

$$\text{Given: } x(0) = 3 = a_0, \quad x'(0) = 0 = a_1$$

$$4a_2 + a_1 + 6a_0 = 0 \rightarrow a_2 = -\frac{9}{2}$$

$$2(n+1)(n+2)a_{n+2} + (n+1)a_{n+1} + 6a_n - a_{n-1} = 0$$

$$a_{n+2} = \frac{a_{n-1} - 6a_n - (n+1)a_{n+1}}{2(n+1)(n+2)}$$

$$n=1 \rightarrow a_3 = \frac{1}{12}(a_0 - 6a_1 - 2a_2) = \frac{1}{12}(3+9) = 1$$

$$n=2 \rightarrow a_4 = \frac{1}{24}(a_1 - 6a_2 - 3a_3) = \frac{1}{24}(27-3) = 1$$

$$x(t) = 3 - \frac{9}{2}t^2 + t^3 + t^4 + \dots$$

Solution

Section 4.3 – Legendre's Equation

Exercise

Establish the recursion formula using the following two steps

a) Differentiate both sides of equation

$$g(t, x) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n \text{ with respect to } t \text{ to show that}$$

$$(x-t) \sum_{n=0}^{\infty} P_n(x)t^n = (1-2xt+t^2) \sum_{n=1}^{\infty} nP_n(x)t^{n-1}$$

b) Equate the coefficients of t^n in this equation to show that

$$P_1(x) = xP_0(x) \quad \text{and} \quad (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \quad \text{for } n \geq 1$$

Solution

a) Let: $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$

Differentiate both sides with respect to t :

$$\left((1-2xt+t^2)^{-1/2} \right)' = \left(\sum_{n=0}^{\infty} P_n(x)t^n \right)'$$

$$-\frac{1}{2}(-2x+2t)(1-2xt+t^2)^{-3/2} = \sum_{n=1}^{\infty} nP_n(x)t^{n-1}$$

$$(x-t)(1-2xt+t^2)^{-1/2} = (1-2xt+t^2) \sum_{n=1}^{\infty} nP_n(x)t^{n-1}$$

Multiply both sides by:

$$1-2xt+t^2$$

$$(x-t) \sum_{n=0}^{\infty} P_n(x)t^n = (1-2xt+t^2) \sum_{n=1}^{\infty} nP_n(x)t^{n-1}$$

b) $(x-t) \sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} xP_n(x)t^n - \underbrace{\sum_{n=0}^{\infty} P_n(x)t^{n+1}}_{n=n+1}$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} xP_n(x)t^n - \sum_{n=1}^{\infty} P_{n-1}(x)t^n \\
(1-2xt+t^2) \sum_{n=1}^{\infty} nP_n(x)t^{n-1} &= \sum_{n=1}^{\infty} nP_n(x)t^{n-1} - \sum_{n=1}^{\infty} 2xnP_n(x)t^n + \sum_{n=1}^{\infty} nP_n(x)t^{n+1} \\
&= \sum_{n=0}^{\infty} (n+1)P_{n+1}(x)t^n - \sum_{n=1}^{\infty} 2nxP_n(x)t^n + \sum_{n=2}^{\infty} (n-1)P_{n-1}(x)t^n
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{n=0}^{\infty} xP_n(x)t^n - \sum_{n=1}^{\infty} P_{n-1}(x)t^n &= \sum_{n=0}^{\infty} (n+1)P_{n+1}(x)t^n - \sum_{n=1}^{\infty} 2nxP_n(x)t^n \\
&\quad + \sum_{n=2}^{\infty} (n-1)P_{n-1}(x)t^n
\end{aligned}$$

Therefore;

$$\begin{aligned}
0 &= [xP_0(x) - P_1(x)]t^0 + [xP_1(x) - P_0(x) - 2P_2(x) + 2xP_1(x)]t^1 \\
&\quad + \sum_{n=0}^{\infty} [xP_n(x) - P_{n-1}(x) - (n+1)P_{n+1}(x) + 2nxP_n(x) - (n-1)P_{n-1}(x)]t^n \\
0 &= [xP_0(x) - P_1(x)]t^0 + [3xP_1(x) - P_0(x) - 2P_2(x)]t^1 \\
&\quad + \sum_{n=0}^{\infty} [(2n+1)xP_n(x) - nP_{n-1}(x) - (n+1)P_{n+1}(x)]t^n
\end{aligned}$$

That implies:

$$xP_0(x) - P_1(x) = 0 \Rightarrow P_1(x) = xP_0(x)$$

$$3xP_1(x) - P_0(x) - 2P_2(x) = 0 \Rightarrow 2P_2(x) = P_0(x) - 3xP_1(x)$$

$$(2n+1)xP_n(x) - nP_{n-1}(x) - (n+1)P_{n+1}(x) = 0$$

$$\Rightarrow (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

If $n = 1$ then: $2P_2(x) = 3xP_1(x) - P_0(x)$ ✓

Exercise

Show that $P_{2n+1}(0) = 0$ and $P_{2n}(0) = (-1)^n \frac{2n!}{2^{2n}(n!)^2}$

Solution

Given the formula:

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x) \quad \text{for } n \geq 2$$

By letting $x = 0$, then the formula can be:

$$nP_n(0) = -(n-1)P_{n-2}(0)$$

Replacing n with $2n$, then

$$2nP_{2n}(0) = -(2n-1)P_{2n-2}(0)$$

$$P_{2n}(0) = \frac{1-2n}{2n}P_{2n-2}(0)$$

$$P_2(0) = \frac{1-2}{2}P_0(0) = -\frac{1}{2}P_0(0)$$

$$P_4(0) = \frac{1-4}{4}P_2(0) = \frac{1-2}{2} \cdot \frac{1-4}{4}P_0(0) = \frac{1 \cdot 3}{2^2 \cdot 1 \cdot 2}P_0(0)$$

$$P_6(0) = \frac{1-6}{6}P_4(0) = \frac{1-2}{2} \cdot \frac{1-4}{4} \cdot \frac{1-6}{6}P_0(0) = -\frac{1 \cdot 3 \cdot 5}{2^3 \cdot 1 \cdot 2 \cdot 3}P_0(0)$$

$$\vdots \quad \vdots \quad \vdots$$

$$P_{2n}(0) = \frac{1-2}{2} \cdot \frac{1-4}{4} \cdot \frac{1-6}{6} \cdots \frac{1-2n}{2n}P_0(0)$$

$$= (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot 1 \cdot 2 \cdot 3 \cdots n}P_0(0)$$

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n-1)(2n)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

$$= \frac{(2n)!}{2^n n!}$$

$$= (-1)^n \frac{(2n)!}{2^n \cdot (n!)^2}P_0(0)$$

With $P_0(0) = 1$

$$\boxed{P_{2n}(0) = (-1)^n \frac{(2n)!}{2^n \cdot (n!)^2}}$$

Exercise

Show that $P'_n(0) = (-1)^{n+1} P'_n(-1) = \frac{n(n+1)}{2}$

Hint: Use Legendre's equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

Solution

Because $P_n(x)$ is a solution of Legendre's equation, then

$$(1-x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0$$

Let $x=1$, then

$$-2P'_n(1) + n(n+1)P_n(1) = 0$$

$$P'_n(1) = \frac{n(n+1)}{2} P_n(1)$$

Let $x=-1$, then

$$2P'_n(-1) + n(n+1)P_n(-1) = 0$$

$$P'_n(-1) = -\frac{n(n+1)}{2} P_n(-1)$$

However, $P_n(1) = P_n(-1) = 1$

$$\underline{(-1)^{n+1} P'_n(-1) = \frac{n(n+1)}{2}}$$

Exercise

The differential equation $y'' + xy = 0$ is called **Airy's equation**, and its solutions are called **Airy functions**. Find the series for the solutions y_1 and y_2 where $y_1(0) = 1$ and $y'_1(0) = 0$, while $y_2(0) = 0$ and $y'_2(0) = 1$. What is the radius of convergence for these two series?

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + xy = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} + a_{n-1}]x^n = 0$$

$$2a_2 = 0 \quad \text{or} \quad (n+2)(n+1)a_{n+2} + a_{n-1} = 0$$

$$a_2 = 0 \quad \text{or} \quad a_{n+2} = \frac{-a_{n-1}}{(n+2)(n+1)} \quad n \geq 1$$

$$a_3 = -\frac{a_0}{3 \cdot 2}$$

$$a_4 = -\frac{a_1}{4 \cdot 3}$$

$$a_6 = -\frac{a_3}{6 \cdot 5} = \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2}$$

$$a_7 = -\frac{a_4}{7 \cdot 6} = \frac{a_1}{7 \cdot 6 \cdot 4 \cdot 3}$$

$$a_9 = -\frac{a_6}{9 \cdot 8} = -\frac{a_0}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2}$$

$$a_{10} = -\frac{a_7}{10 \cdot 9} = -\frac{a_1}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots$$

$$a_5 = -\frac{a_2}{5 \cdot 4} = 0$$

$$a_8 = -\frac{a_5}{8 \cdot 7} = 0$$

$$a_{11} = 0$$

$$a_{3n} = \frac{(-1)^n a_0}{(2 \cdot 3) \cdot (5 \cdot 6) \cdots (3n-1)(3n)}$$

$$a_{3n+1} = \frac{(-1)^n a_1}{(3 \cdot 4) \cdot (6 \cdot 7) \cdots (3n)(3n+1)}$$

$$a_{3n+2} = 0$$

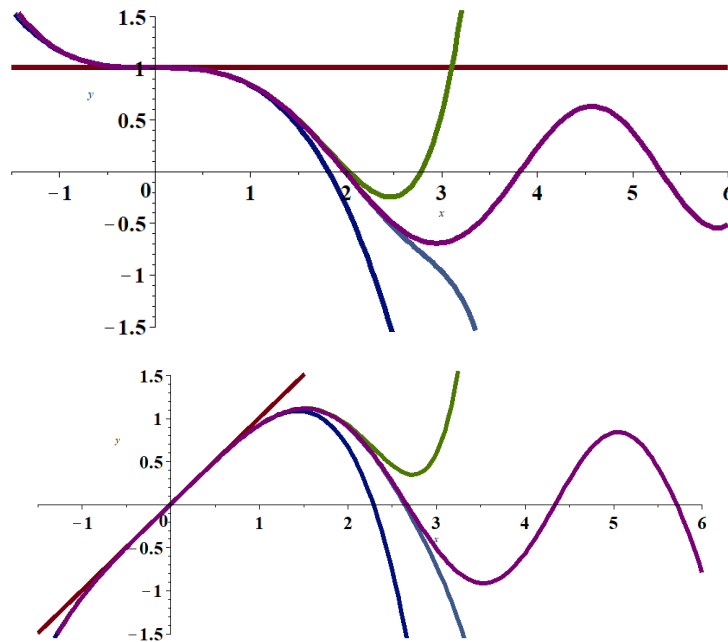
$$y(x) = a_0 \left[1 - \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 - \dots \right] + a_1 \left[x - \frac{1}{3 \cdot 4} x^4 + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 - \dots \right]$$

$$y_1(x) = 1 - \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 - \dots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{3n}}{(2 \cdot 3) \cdot (5 \cdot 6) \cdots (3n-1)(3n)}$$

$$y_2(x) = x - \frac{1}{3 \cdot 4} x^4 + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 - \dots$$

$$= x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{3n+1}}{(3 \cdot 4) \cdot (6 \cdot 7) \cdots (3n)(3n+1)}$$



Exercise

The Hermite equation of order α is $y'' - 2xy' + 2\alpha y = 0$

a) Find the general solution is $y(x) = a_0 y_1(x) + a_1 y_2(x)$

Show that $y_1(x)$ is a polynomial if α is an even integer, whereas $y_2(x)$ is a polynomial if α is an odd integer.

b) When $\alpha = n$, use $y_1(x)$ to find polynomial solutions for $n = 0$, $n = 2$, and $n = 4$, then use $y_2(x)$ to find polynomial solutions for $n = 1$, $n = 3$, and $n = 5$.

- c) The Hermite polynomial of degree n is denoted by $H_n(x)$. It is the n th-degree polynomial solution of Hermite's equation, multiplied by a suitable constant so that the coefficient of x^n is 2^n . Use part (b) to show the first six Hermite polynomials are

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

- d) A general formula for the Hermite polynomials is

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2} \right)$$

Verify that this formula does in fact give an n th-degree polynomial.

Solution

$$a) \quad y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - 2xy' + 2\alpha y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2\alpha \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} 2\alpha a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2\alpha a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} - 2(n-\alpha)a_n \right] x^n = 0$$

$$(n+1)(n+2)a_{n+2} - 2(n-\alpha)a_n = 0$$

$$a_{n+2} = \frac{2(n-\alpha)}{(n+1)(n+2)} a_n$$

$$a_0$$

$$n=0 \rightarrow a_2 = -\frac{2\alpha}{2} a_0$$

$$n=2 \rightarrow a_4 = \frac{2(2-\alpha)}{3 \cdot 4} a_2 = -\frac{2^2 \alpha (2-\alpha)}{4!} a_0$$

$$n=4 \rightarrow a_6 = \frac{2(4-\alpha)}{5 \cdot 6} a_4 = -\frac{2^3 \alpha (2-\alpha)(4-\alpha)}{6!} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x) = 1 - \frac{2\alpha}{2!} x^2 - \frac{2^2(2-\alpha)}{4!} x^4 - \frac{2^3 \alpha (2-\alpha)(4-\alpha)}{6!} x^6 - \dots$$

$$= 1 - \frac{2\alpha}{2!} x^2 + \frac{2^2(\alpha-2)}{4!} x^4 - \frac{2^3 \alpha (\alpha-2)(\alpha-4)}{6!} x^6 + \dots$$

$$a_1$$

$$n=1 \rightarrow a_3 = \frac{2(1-\alpha)}{6} a_1 = \frac{2(1-\alpha)}{3!} a_1$$

$$n=3 \rightarrow a_5 = \frac{2(3-\alpha)}{4 \cdot 5} a_3 = \frac{2^2(1-\alpha)(3-\alpha)}{5!} a_1$$

$$n=5 \rightarrow a_7 = \frac{2(5-\alpha)}{6 \cdot 7} a_5 = \frac{2^3(1-\alpha)(3-\alpha)(5-\alpha)}{7!} a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = x + \frac{2(1-\alpha)}{3!} x^3 + \frac{2^2(1-\alpha)(3-\alpha)}{5!} x^5 + \frac{2^3(1-\alpha)(3-\alpha)(5-\alpha)}{7!} x^7 + \dots$$

$$= x - \frac{2(\alpha-1)}{3!} x^3 + \frac{2^2(\alpha-1)(\alpha-3)}{5!} x^5 - \frac{2^3(\alpha-1)(\alpha-3)(\alpha-5)}{7!} x^7 + \dots$$

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

$$= a_0 \left(1 - \frac{2\alpha}{2!}x^2 + \frac{2^2\alpha(\alpha-2)}{4!}x^4 - \frac{2^3\alpha(\alpha-2)(\alpha-4)}{6!}x^6 + \dots \right) \\ + a_1 \left(x - \frac{2(\alpha-1)}{3!}x^3 + \frac{2^2(\alpha-1)(\alpha-3)}{5!}x^5 - \frac{2^3(\alpha-1)(\alpha-3)(\alpha-5)}{7!}x^7 + \dots \right)$$

$$= a_0 + a_0 \sum_{m=1}^{\infty} (-1)^m \frac{2^m \alpha(\alpha-2) \cdots (\alpha-2m+2)}{(2m)!} x^{2m} \\ + a_1 x + a_1 \sum_{m=1}^{\infty} (-1)^m \frac{2^m (\alpha-1)(\alpha-3) \cdots (\alpha-2m+1)}{(2m+1)!} x^{2m+1}$$

$$y_1(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{2^m \alpha(\alpha-2) \cdots (\alpha-2m+2)}{(2m)!} x^{2m}$$

$$y_2(x) = x + \sum_{m=1}^{\infty} (-1)^m \frac{2^m (\alpha-1)(\alpha-3) \cdots (\alpha-2m+1)}{(2m+1)!} x^{2m+1}$$

b) $n = \alpha = 0 \rightarrow y_1(x) = \underline{1}$

$$n = \alpha = 2 \rightarrow y_1(x) = 1 - \alpha x^2 = \underline{1 - 2x^2}$$

$$n = \alpha = 4 \rightarrow y_1(x) = 1 - \alpha x^2 + \frac{\alpha(\alpha-2)}{6}x^4 \\ = \underline{1 - 4x^2 + \frac{4}{3}x^4}$$

$$n = \alpha = 1 \rightarrow y_2(x) = \underline{x}$$

$$n = \alpha = 3 \rightarrow y_2(x) = x - \frac{2(\alpha-1)}{3!}x^3 = \underline{x - \frac{2}{3}x^3}$$

$$n = \alpha = 5 \rightarrow y_2(x) = x - \frac{2(\alpha-1)}{3!}x^3 + \frac{2^2(\alpha-1)(\alpha-3)}{5!}x^5$$

$$y_2(x) = \underline{x - \frac{4}{3}x^3 + \frac{4}{15}x^5}$$

c) $H_0(x) = 2^0 \cdot 1 = 1$

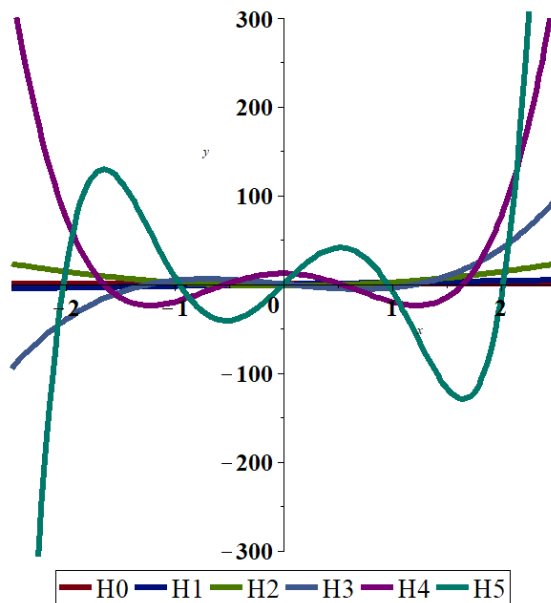
$$H_1(x) = 2^1 \cdot x = 2x$$

$$H_2(x) = -2(1 - 2x^2) = 4x^2 - 2$$

$$H_3(x) = -2^2 \cdot 3 \left(x - \frac{2}{3}x^3 \right) = 8x^3 - 12x$$

$$H_4(x) = 2^2 \cdot 3 \left(1 - 4x^2 + \frac{4}{3}x^4 \right) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 2^3 \cdot 3 \cdot 5 \left(\frac{4}{15}x^5 - \frac{4}{3}x^3 + x \right) = 32x^5 - 160x^3 + 120x$$



$$d) \quad \frac{d}{dx} \left(e^{-x^2} \right) = -2xe^{-x^2}$$

$$\frac{d^2}{dx^2} \left(e^{-x^2} \right) = -2 \frac{d}{dx} \left(xe^{-x^2} \right) = -2(1 - 2x^2)e^{-x^2}$$

$$\frac{d^3}{dx^3} \left(e^{-x^2} \right) = 2 \frac{d}{dx} \left((2x^2 - 1)e^{-x^2} \right) = 2(4x - 4x^3 + 2x)e^{-x^2} = (12x - 8x^3)e^{-x^2}$$

$$\frac{d^4}{dx^4} \left(e^{-x^2} \right) = 4 \frac{d}{dx} \left((3x - 2x^3)e^{-x^2} \right) = 4(3 - 6x^2 - 6x^2 + 4x^4)e^{-x^2} = (16x^4 - 48x^2 + 12)e^{-x^2}$$

$$H_1(x) = -e^{x^2} \frac{d}{dx} \left(e^{-x^2} \right) = 2xe^{x^2} e^{-x^2} = 2x \quad \checkmark$$

$$H_2(x) = e^{x^2} \frac{d^2}{dx^2} \left(e^{-x^2} \right) = -2e^{x^2} (1 - 2x^2)e^{-x^2} = 4x^2 - 2 \quad \checkmark$$

$$H_3(x) = e^{x^2} \frac{d^3}{dx^3} \left(e^{-x^2} \right) = e^{x^2} (12x - 8x^3)e^{-x^2} = 12x - 8x^3 \quad \checkmark$$

$$H_4(x) = e^{x^2} \frac{d^4}{dx^4} \left(e^{-x^2} \right) = e^{x^2} (16x^4 - 48x^2 + 12) e^{-x^2} = 16x^4 - 48x^2 + 12 \quad \checkmark$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2} \right)$$

Exercise

Rodrigues's Formula is given by: $P_n(x) = \frac{1}{n! 2^n} \frac{d^n}{dx^n} (x^2 - 1)^n$

For the n th-degree Legendre polynomial.

- a) Show that $v = (x^2 - 1)^n$ satisfies the differential equation $(1 - x^2)v' + 2nxv = 0$

Differentiate each side of this equation to obtain

$$(1 - x^2)v'' + 2(n-1)xv' + 2nv = 0$$

- b) Differentiate each side of the last equation n times in succession to obtain

$$(1 - x^2)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0$$

which satisfies Legendre's equation of order n .

- c) Show that the coefficient of x^n in u is $\frac{(2n)!}{n!}$; then state why this proves Rodrigues.

Formula.

Note: That the coefficient of x^n in $P_n(x)$ is $\frac{(2n)!}{2^n (n!)^2}$

$$u = v^{(n)} = D^n (x^2 - 1)^n$$

Solution

$$a) \quad v = (x^2 - 1)^n \rightarrow v' = 2nx(x^2 - 1)^{n-1}$$

$$v' = 2nx(x^2 - 1)^{n-1}$$

$$(1 - x^2)v' + 2nxv = -2nx(x^2 - 1)(x^2 - 1)^{n-1} + 2nx(x^2 - 1)^n$$

$$= -2nx(x^2 - 1)^n + 2nx(x^2 - 1)^n$$

$$= 0$$

$$\frac{d}{dx} \left((1-x^2)v' + 2nxv \right) = 0$$

$$(1-x^2)v'' - 2xv' + 2nxv' + 2nv = 0$$

$$\boxed{(1-x^2)v'' + 2(n-1)xv' + 2nv = 0}$$

$$b) \frac{d}{dx} \left((1-x^2)v'' - 2xv' + 2nxv' + 2nv \right) = 0$$

$$(1-x^2)v^{(3)} - 2xv'' - 2v' - 2xv'' + 2nxv'' + 2nv' + 2nv'$$

$$\boxed{(1-x^2)v^{(3)} + 2(n-2)xv'' + 2(2n-1)v' = 0}$$

$$n=1 \rightarrow (1-x^2)v^{(3)} - 2xv'' + 2v' = 0$$

$$\boxed{(1-x^2)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0} \quad \checkmark$$

$$\frac{d}{dx} \left((1-x^2)v^{(3)} + 2x(n-2)v'' + 2(2n-1)v' \right) = 0$$

$$(1-x^2)v^{(4)} - 2xv^{(3)} + 2x(n-2)v^{(3)} + 2(n-2)v'' + 2(2n-1)v'' = 0$$

$$(1-x^2)v^{(4)} + 2x(n-3)v^{(3)} + 6(n-1)v'' = 0$$

$$\boxed{(1-x^2)v^{(4)} + 2(n-3)xv^{(3)} + 3(2n-2)v'' = 0}$$

$$n=2 \rightarrow (1-x^2)v^{(4)} - 2xv^{(3)} + 3 \cdot 2v'' = 0$$

$$\boxed{(1-x^2)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0} \quad \checkmark$$

$$\frac{d}{dx} \left((1-x^2)v^{(4)} + 2(n-3)xv^{(3)} + 3(2n-2)v'' \right) = 0$$

$$(1-x^2)v^{(5)} - 2xv^{(4)} + 2(n-3)xv^{(4)} + 2(n-3)v^{(3)} + 3(2n-2)v^{(3)} = 0$$

$$(1-x^2)v^{(5)} + (2n-6-2)xv^{(4)} + (2n-6+6n-6)v^{(3)} = 0$$

$$(1-x^2)v^{(5)} + (2n-8)xv^{(4)} + (8n-12)v^{(3)} = 0$$

$$\boxed{(1-x^2)v^{(5)} + 2(n-4)xv^{(4)} + 4(2n-3)v^{(3)} = 0}$$

$$n=3 \rightarrow (1-x^2)v^{(5)} - 2xv^{(4)} + 4 \cdot 3v^{(3)} = 0$$

$$\boxed{(1-x^2)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0} \quad \checkmark$$

After m differentiations:

$$(1-x^2)v^{(m+2)} + 2(n-(m+1))xv^{(m+1)} + (m+1)(2n-m)v^{(m)} = 0$$

If we let $m = n$, then

$$\underline{(1-x^2)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0}$$

Let assume that $(1-x^2)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0$ is true.

We need to prove that next derivative is also true.

$$\frac{d}{dx} \left((1-x^2)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} \right) = 0$$

$$(1-x^2)v^{(n+3)} - 2xv^{(n+2)} - 2v^{(n+1)} - 2xv^{(n+2)} + (2n-n)(n+1)v^{(n+1)} = 0$$

$$(1-x^2)v^{(n+3)} - 4xv^{(n+2)} + (2n-n-2)(n+1)v^{(n+1)} = 0$$

$$(1-x^2)v^{(n+3)} - 2(2)xv^{(n+2)} + (n-1)(n+2)v^{(n+1)} = 0$$

$$(1-x^2)v^{(n+3)} + 2(n-n-2)xv^{(n+2)} + (n-1)(n+2)v^{(n+1)} = 0$$

$$(1-x^2)v^{(n+3)} + 2(n-(n+2))xv^{(n+2)} + (2n-(n+1))((n+1)+1)v^{(n+1)} = 0$$

If we let $m = n+1$, then

$$(1-x^2)v^{(m+2)} + 2(n-(m+1))xv^{(m+1)} + (2n-m)(m+1)v^{(m)} = 0$$

If we let $m = n$, then

$$(1-x^2)v^{(n+2)} + 2(n-n-1)xv^{(n+1)} + (2n-n)(n+1)v^{(n)} = 0$$

$$\underline{(1-x^2)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0} \quad \checkmark$$

$$c) \quad u = v^{(n)} = D^n(x^2 - 1)^n$$

$$= \frac{d^n}{dx^n} (x^{2n} - nx^{2n-1} + \dots - 1)$$

$$= 2n(2n-1)\dots(2n-(n-1))x^n - \frac{d^n}{dx^n} (nx^{2n-1} + \dots - 1)$$

$$= \frac{(2n)!}{n!} x^n - \frac{d^n}{dx^n} (nx^{2n-1} + \dots - 1)$$

Since $u = v^{(n)}$ satisfies Legendre's equation of order n , $\frac{u}{2^n n!}$

From the notes:

The solution of Legendre polynomial of degree n is

$$P_n(x) = \sum_{k=0}^K (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}$$

For the highest order, the result when:

$$k=0 \rightarrow P_n(x) = \frac{(2n)!}{2^n (n!)^2} x^n$$

$$\frac{u}{2^n n!} = \frac{(2n)!}{2^n (n!)^2} x^n + \dots$$

$$P_n(x) = \frac{1}{n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Solution **Section 4.4 – Solution about Singular Points**

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^2 y'' + 3y' - xy = 0$$

Solution

$$y'' + \frac{3}{x^2} y' - \frac{x}{x^2} y = 0$$

$$P(x) = \frac{3}{x^2} \quad Q(x) = -\frac{x}{x^2}$$

$$\text{For } P(x) = \frac{3}{x^2} \rightarrow \underline{x=0}$$

$$\therefore p(x) \text{ is analytic except at } \underline{x=0}$$

$$\text{For } Q(x) = -\frac{x}{x^2} = -\frac{1}{x}$$

$$\therefore q(x) \text{ is not analytic at } \underline{x=0}$$

$$\text{The singular point is: } \underline{x=0}$$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$(x^2 + x)y'' + 3y' - 6xy = 0$$

Solution

$$y'' + \frac{3}{x(x+1)} y' - \frac{6x}{x(x+1)} y = 0$$

$$P(x) = \frac{3}{x(x+1)} \quad Q(x) = -\frac{6x}{x(x+1)}$$

$$\text{For } P(x) = \frac{3}{x(x+1)} \rightarrow \underline{x=0, -1}$$

$$\therefore p(x) \text{ is analytic except at } \underline{x=0, -1}$$

$$\text{For } q(x) = -\frac{6x}{x(x+1)} \rightarrow x=0, -1$$

$$q(x) = -\frac{6x}{x(x+1)} = -\frac{6}{x+1}; \text{ is actually analytic at } x=0$$

$\therefore q(x)$ is analytic except at $\underline{x = -1}$ |

The singular points are: $\underline{x = 0, -1}$ |

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$(x^2 - 1)y'' + (1 - x)y' + (x^2 - 2x + 1)y = 0$$

Solution

$$(x - 1)(x + 1)y'' + (1 - x)y' + (x - 1)^2 y = 0$$

$$y'' - \frac{1}{x + 1}y' + \frac{x - 1}{x + 1}y = 0$$

$$P(x) = \frac{1 - x}{x^2 - 1} \quad Q(x) = \frac{(x - 1)^2}{x^2 - 1}$$

$$\text{For } p(x) = \frac{1 - x}{(x - 1)(x + 1)} \rightarrow \underline{x = -1, 1} |$$

$$p(x) = \frac{1 - x}{(x - 1)(x + 1)} = -\frac{1}{x + 1}; \text{ is actually analytic at } x = 1$$

$\therefore p(x)$ is analytic except at $\underline{x = -1}$ |

$$\text{For } q(x) = \frac{(x - 1)^2}{(x - 1)(x + 1)} \rightarrow \underline{x = -1, 1} |$$

$$q(x) = \frac{(x - 1)^2}{(x - 1)(x + 1)} = \frac{x - 1}{x + 1}; \text{ is actually analytic at } x = 1$$

$\therefore q(x)$ is analytic except at $\underline{x = -1}$ |

The singular point is: $\underline{x = -1}$ |

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$e^x y'' - (x^2 - 1)y' + 2xy = 0$$

Solution

$$y'' - \frac{x^2 - 1}{e^x}y' + \frac{2x}{e^x}y = 0$$

$$P(x) = -\frac{x^2 - 1}{e^x} \quad Q(x) = \frac{2x}{e^x}$$

Since $e^x \neq 0$, there are **no** singular points.

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\ln(x-1)y'' + (\sin 2x)y' - e^x y = 0$$

Solution

$$y'' + \frac{\sin 2x}{\ln(x-1)}y' - \frac{e^x}{\ln(x-1)}y = 0$$

$$P(x) = \frac{\sin 2x}{\ln(x-1)} \quad Q(x) = -\frac{e^x}{\ln(x-1)}$$

$$\ln(x-1) = 0$$

$$x-1 = 1$$

$$\underline{x = 2}$$

The singular point is: $\underline{x \leq 1, x = 2}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$xy'' + x(1-x)^{-1}y' + (\sin x)y = 0$$

Solution

$$y'' + \frac{x}{x(1-x)}y' + \frac{\sin x}{x}y = 0$$

$$P(x) = \frac{x}{x(1-x)} \quad Q(x) = \frac{\sin x}{x}$$

$$\text{For } p(x) = \frac{x}{x(1-x)} \rightarrow \underline{x = 0, 1}$$

$$p(x) = \frac{x}{x(1-x)} = \frac{1}{1-x}; \text{ is actually analytic at } x = 0$$

$$\therefore p(x) \text{ is analytic except at } \underline{x = 1}$$

$$\text{For } q(x) = \frac{\sin x}{x}$$

$$\begin{aligned} &= \frac{x - \frac{1}{3!}x^3 + \dots}{x} \\ &= 1 - \frac{1}{3!}x^2 + \dots \text{ is analytic everywhere } (x = 0 \text{ is removable}). \end{aligned}$$

The only singular point is $x = 1$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x(x+3)^2 y'' - y = 0$$

Solution

$$y'' - \frac{1}{x(x+3)^2} y = 0$$

$$P(x) = 0 \quad Q(x) = -\frac{1}{x(x+3)^2}$$

For $q(x) = -\frac{1}{x(x+3)^2} \rightarrow \underline{x = 0, -3}$, is analytic elsewhere

The *Regular* singular points are $\underline{x = 0, -3}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$(x^2 - 9)^2 y'' + (x+3)y' + 2y = 0$$

Solution

$$y'' + \frac{x+3}{(x^2-9)^2} y' + \frac{2}{(x^2-9)^2} y = 0$$

$$P(x) = \frac{x+3}{(x^2-9)^2} \quad Q(x) = \frac{2}{(x^2-9)^2}$$

For $P(x) = \frac{x+3}{(x^2-9)^2} \rightarrow \underline{x = \pm 3}$

$$p(x) = \frac{x+3}{((x+3)(x-3))^2}$$

$$= \frac{(x+3)(x-3)}{(x+3)(x-3)^2}; \quad \text{is analytic at } x = -3$$

$$\text{For } Q(x) = \frac{2(x^2-9)^2}{(x^2-9)^2} \rightarrow \underline{x = \pm 3}$$

$$\therefore q(x) \text{ is analytic at } \underline{x = \pm 3}$$

The *Regular* singular point: $\underline{x = -3}$, and *Irregular* singular point: $\underline{x = 3}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$y'' - \frac{1}{x}y' + \frac{1}{(x-1)^3}y = 0$$

Solution

$$y'' - \frac{1}{x}y' + \frac{1}{(x-1)^3}y = 0$$

$$P(x) = -\frac{1}{x} \quad Q(x) = \frac{1}{(x-1)^3}$$

$$\text{For } P(x) = -\frac{1}{x} \rightarrow \underline{x = 0}$$

$$p(x) = \frac{x}{x} = 1 \text{ is analytic at } \underline{x = 0}$$

$$\text{For } Q(x) = \frac{1}{(x-1)^3} \rightarrow \underline{x = 1}$$

$$q(x) = \frac{(x-1)^2}{(x-1)^3} = \frac{1}{x-1} \text{ is not an analytic at } x = 1$$

The *Regular* singular point: $\underline{x = 0}$, and *Irregular* singular point: $\underline{x = 1}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$(x^3 + 4x)y'' - 2xy' + 6y = 0$$

Solution

$$y'' - \frac{2x}{x(x^2 + 4)}y' + \frac{6}{x(x^2 + 4)}y = 0$$

$$P(x) = -\frac{2x}{x(x^2 + 4)} \quad Q(x) = \frac{6}{x(x^2 + 4)}$$

$$\text{For } P(x) = -\frac{2x}{x(x^2 + 4)} \rightarrow x = 0, \pm 2i$$

$$p(x) = -\frac{2}{x^2 + 4} \text{ is analytic at } x = \pm 2i$$

$$\text{For } Q(x) = \frac{6}{x(x^2 + 4)} \rightarrow x = 0, \pm 2i \text{ is analytic}$$

The *Regular* singular points: $x = 0, \pm 2i$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^2(x-5)^2 y'' + 4xy' + (x^2 - 25)y = 0$$

Solution

$$y'' + \frac{4x}{x^2(x-5)^2}y' + \frac{x^2 - 25}{x^2(x-5)^2}y = 0$$

$$P(x) = \frac{4x}{x^2(x-5)^2} \quad Q(x) = \frac{x^2 - 25}{x^2(x-5)^2}$$

$$\text{For } P(x) = \frac{4x}{x^2(x-5)^2} \rightarrow x = 0, 5$$

$$p(x) = \frac{4x}{x^2(x-5)^2} = x(x-5) \frac{4}{x(x-5)^2} \text{ is not an analytic at } x = 5$$

$$\text{For } Q(x) = \frac{(x-5)(x+5)}{x^2(x-5)^2} \rightarrow x = 0, 5$$

$$q(x) = x^2(x-5)^2 \frac{x+5}{x^2(x-5)} \text{ is analytic at } x = 0, 5$$

The *Regular* singular point: $x = 0$, and *Irregular* singular point: $x = 5$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$(x^2 + x - 6)y'' + (x + 3)y' + (x - 2)y = 0$$

Solution

$$y'' + \frac{x+3}{x^2+x-6}y' + \frac{x-2}{x^2+x-6}y = 0$$

$$P(x) = \frac{x+3}{(x+3)(x-2)} \quad Q(x) = \frac{x-2}{(x+3)(x-2)}$$

$$\text{For } P(x) = \frac{x+3}{(x+3)(x-2)} \rightarrow x = -3, 2$$

$$p(x) = \frac{1}{x-2} \text{ is analytic at } x = 2$$

$$\text{For } Q(x) = \frac{x-2}{(x+3)(x-2)} \rightarrow x = -3, 2$$

$$q(x) = \frac{1}{x+3} \text{ is analytic at } x = -3$$

The *Regular* singular points: $x = -3, 2$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x(x^2 + 1)^2 y'' + y = 0$$

Solution

$$y'' + \frac{1}{x(x^2+1)^2}y = 0$$

$$P(x) = 0 \quad Q(x) = \frac{1}{x(x^2+1)^2}$$

For $Q(x) = \frac{1}{x(x^2 + 1)^2} \rightarrow x = 0, \pm i$

$q(x) = x^2(x^2 + 1)^2 \frac{1}{x(x^2 + 1)^2}$ is an analytic at $x = 0, \pm i$

The *Regular* singular points: $x = 0, \pm i$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^3(x^2 - 25)(x - 2)^2 y'' + 3x(x - 2)y' + 7(x + 5)y = 0$$

Solution

$$y'' + \frac{3x(x - 2)}{x^3(x^2 - 25)(x - 2)^2} y' + \frac{7(x + 5)}{x^3(x^2 - 25)(x - 2)^2} y = 0$$

$$P(x) = \frac{3x(x - 2)}{x^3(x - 5)(x + 5)(x - 2)^2} \quad Q(x) = \frac{7(x + 5)}{x^3(x - 5)(x + 5)(x - 2)^2}$$

For $P(x) = \frac{3x(x - 2)}{x^3(x - 5)(x + 5)(x - 2)^2} \rightarrow x = 0, \pm 5, 2$

$p(x) = \frac{3x(x - 5)(x + 5)(x - 2)}{x^2(x - 5)(x + 5)(x - 2)}$ is not an analytic at $x = 0$

For $Q(x) = \frac{7(x + 5)}{x^3(x - 5)(x + 5)(x - 2)^2} \rightarrow x = 0, \pm 5, 2$

$q(x) = \frac{3x^2(x - 5)^2(x + 5)^2(x - 2)^2}{x^3(x - 5)(x - 2)^2}$ is not an analytic at $x = 0$

The *Regular* singular point: $x = 2, \pm 5$, and *Irregular* singular point: $x = 0$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^3 - 2x^2 - 3x\right)^2 y'' + x(x-3)^2 y' - (x+1)y = 0$$

Solution

$$y'' + \frac{x(x-3)^2}{\left(x^3 - 2x^2 - 3x\right)^2} y' - \frac{x+1}{\left(x^3 - 2x^2 - 3x\right)^2} y = 0$$

$$P(x) = \frac{x(x-3)^2}{x^2(x-3)^2(x+1)^2} \quad Q(x) = -\frac{x+1}{x^2(x-3)^2(x+1)^2}$$

$$\text{For } P(x) = \frac{x(x-3)^2}{x^2(x-3)^2(x+1)^2} \rightarrow x = 0, -1, 3$$

$$p(x) = \frac{1}{x(x+1)^2} \text{ is not an analytic at } x = -1$$

$$\text{For } Q(x) = \frac{x+1}{x^2(x-3)^2(x+1)^2} \rightarrow x = 0, -1, 3$$

$$q(x) = \frac{x^2(x-3)^2(x+1)^2}{x^2(x-3)^2(x+1)} \text{ is an analytic at } x = 0, -1, 3$$

The *Regular* singular point: $x = 0, 3$, and *Irregular* singular point: $x = -1$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$(1-x^2)y'' + (\tan x)y' + x^{5/3}y = 0$$

Solution

$$y'' + \frac{\tan x}{1-x^2} y' + \frac{x^{5/3}}{1-x^2} y = 0$$

$$P(x) = \frac{\tan x}{1-x^2} \quad Q(x) = \frac{x^{5/3}}{1-x^2}$$

$$\text{For } P(x) = \frac{\tan x}{1-x^2} \rightarrow x = \pm 1$$

$$\tan x = \pm\infty \rightarrow x = \pm \frac{\pi}{2} \text{ (Vertical Asymptotes).}$$

For $Q(x) = \frac{x^{5/3}}{1-x^2} \rightarrow x = \pm 1$ is not analytic

The second derivatives doesn't exist at $x = 0$

The *Regular* singular point: $x = 0, \pm 1, \pm \frac{\pi}{2}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x(x-1)^2(x+2)y'' + x^2y' - (x^3 + 2x - 1)y = 0$$

Solution

$$y'' + \frac{x^2}{x(x-1)^2(x+2)}y' - \frac{x^3 + 2x - 1}{x(x-1)^2(x+2)}y = 0$$

$$P(x) = \frac{x}{(x-1)^2(x+2)} \quad \& \quad Q(x) = -\frac{x^3 + 2x - 1}{x(x-1)^2(x+2)}$$

$$\text{For } P(x) = \frac{x}{(x-1)^2(x+2)} \rightarrow x = 1, -2$$

$$p_0 = \lim_{x \rightarrow 1} (x-1)P(x) = \lim_{x \rightarrow 1} \frac{x}{(x-1)(x+2)} = \infty \text{ is not analytic}$$

$$\begin{aligned} p_0 &= \lim_{x \rightarrow -2} (x+2)P(x) \\ &= \lim_{x \rightarrow -2} \frac{x}{(x-1)^2} \\ &= -\frac{2}{9} \end{aligned}$$

$$\text{For } Q(x) = -\frac{x^3 + 2x - 1}{x(x-1)^2(x+2)} \rightarrow x = 0, 1, -2$$

$$\begin{aligned} q_0 &= \lim_{x \rightarrow 0} x^2Q(x) \\ &= \lim_{x \rightarrow 0} \frac{x(x^3 + 2x - 1)}{(x-1)^2(x+2)} \\ &= 0 \end{aligned}$$

$$q_0 = \lim_{x \rightarrow 1} (x-1)^2Q(x)$$

$$= \lim_{x \rightarrow 1} \frac{x^3 + 2x - 1}{x(x+2)}$$

$$= \frac{2}{3} \mid$$

$$q_0 = \lim_{x \rightarrow -2} (x+2)^2 Q(x)$$

$$= - \lim_{x \rightarrow -2} \frac{(x^3 + 2x - 1)(x+2)}{x(x-1)^2}$$

$$= 0 \mid$$

The *Regular* singular point: $\underline{x = 0, -2} \mid$, and *Irregular* singular point: $\underline{x = 1} \mid$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^4(x^2 + 1)(x - 1)^2 y'' + 4x^3(x - 1)y' + (x + 1)y = 0$$

Solution

$$y'' + \frac{4x^3(x-1)}{x^4(x^2+1)(x-1)^2} y' + \frac{x+1}{x^4(x^2+1)(x-1)^2} y = 0$$

$$P(x) = \frac{4}{x(x^2+1)(x-1)} \quad \& \quad Q(x) = \frac{x+1}{x^4(x^2+1)(x-1)^2}$$

$$\text{For } P(x) = \frac{4}{x(x^2+1)(x-1)} \rightarrow \underline{x = 0, 1, \pm i} \mid$$

$$p_0 = \lim_{x \rightarrow 0} xP(x)$$

$$= \lim_{x \rightarrow 0} \frac{4}{(x^2+1)(x-1)}$$

$$= -4 \mid$$

$$p_0 = \lim_{x \rightarrow 1} (x-1)P(x)$$

$$= \lim_{x \rightarrow 1} \frac{4}{x(x^2+1)}$$

$$= 2 \mid$$

$$\begin{aligned}
 p_0 &= \lim_{x \rightarrow i} (x-i)P(x) \\
 &= \lim_{x \rightarrow i} \frac{4}{x(x-1)(x+i)} \\
 &= -\frac{2}{i-1} \\
 &= -\frac{2}{i-1} \frac{i+1}{i+1} \\
 &= \underline{i+1}
 \end{aligned}$$

$$\begin{aligned}
 p_0 &= \lim_{x \rightarrow -i} (x+i)P(x) \\
 &= \lim_{x \rightarrow -i} \frac{4}{x(x-1)(x-i)} \\
 &= \frac{2}{i-1} \\
 &= \frac{2}{i-1} \frac{i+1}{i+1} \\
 &= \underline{-i-1}
 \end{aligned}$$

$$\text{For } Q(x) = \frac{x+1}{x^4(x^2+1)(x-1)^2} \rightarrow x=0, 1, \pm i$$

$$\begin{aligned}
 q_0 &= \lim_{x \rightarrow 0} x^2 Q(x) \\
 &= \lim_{x \rightarrow 0} \frac{x+1}{x^2(x^2+1)(x-1)^2} \\
 &= \underline{\infty} \text{ is not analytic}
 \end{aligned}$$

$$\begin{aligned}
 q_0 &= \lim_{x \rightarrow 1} (x-1)^2 Q(x) \\
 &= \lim_{x \rightarrow 0} \frac{x+1}{x^4(x^2+1)} \\
 &= \underline{1}
 \end{aligned}$$

$$\begin{aligned}
 q_0 &= \lim_{x \rightarrow \pm i} (x^2+1)^2 Q(x) \\
 &= \lim_{x \rightarrow \pm i} \frac{(x+1)(x^2+1)}{x^2(x-1)^2} \\
 &= \underline{0}
 \end{aligned}$$

The *Regular* singular point: $\underline{x=0, \pm i}$, and *Irregular* singular point: $\underline{x=0}$

Exercise

Determine whether $x = 0$ is an ordinary point, singular point, or irregular singular point of the given differential equation $xy'' + (1 - \cos x)y' + x^2y = 0$

Solution

$$y'' + \frac{1 - \cos x}{x} y' + xy = 0$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\begin{aligned} \frac{1 - \cos x}{x} &= \frac{1}{x} \left(1 - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \dots \right) \\ &= \frac{x}{2} - \frac{x^3}{24} + \frac{x^5}{720} - \dots \end{aligned} \quad \left| \text{ is analytic at } x = 0 \right.$$

$\therefore x = 0$ is an ordinary point of the differential equation.

Exercise

Determine whether $x = 0$ is an ordinary point, singular point, or irregular singular point of the given differential equation $(e^x - 1 - x)y'' + xy = 0$

Solution

$$x^2 y'' + x^2 \frac{x}{e^x - 1 - x} y = 0$$

$$x^2 y'' + \frac{x^3}{e^x - 1 - x} y = 0$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\begin{aligned} e^x - 1 - x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots - 1 - x \\ &= \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots \end{aligned}$$

$$\begin{aligned} \frac{x^3}{e^x - 1 - x} &= \frac{1}{\frac{1}{x^3} \left(\frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots \right)} \\ &= \frac{1}{\frac{1}{2x} + \frac{1}{6} + \frac{x}{24} + \dots} \end{aligned}$$

$\therefore x = 0$ is a regular singular point of the differential equation

Exercise

Find the Frobenius series solutions of $2x^2y'' + 3xy' - (1+x^2)y = 0$

Solution

$$y'' + \frac{3}{2} \frac{1}{x} y' - \frac{1}{2} \frac{1+x^2}{x^2} y = 0 \quad \text{Divide each term by } 2x^2$$

Therefore, $x = 0$ is a regular singular point, and that $p_0 = \frac{3}{2}$, $q_0 = -\frac{1}{2}$

$p(x) \equiv \frac{3}{2}$, $q(x) = -\frac{1}{2} - \frac{1}{2}x^2$ are polynomials.

The Frobenius series will converge for all $x > 0$. The indicial equation is

$$r(r-1) + \frac{3}{2}r - \frac{1}{2} = r^2 + \frac{1}{2}r - \frac{1}{2} = (r+1)\left(r - \frac{1}{2}\right) = 0$$

So the roots are $r_1 = \frac{1}{2}$ and $r_2 = -1$.

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-1} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2 y'' + 3xy' - (1+x^2)y = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 3x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} - x^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$- \sum_{n=0}^{\infty} a_n x^{n+r} - x^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(2n+2r-2) a_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(2n+2r-2)+3(n+r)-1]a_n x^{n+r} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(2n+2r+1)-1]a_n x^{n+r} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$(2r^2+r-1)a_0 + (2r^2+5r+2)a_1 x +$$

$$\sum_{n=2}^{\infty} [(n+r)(2n+2r+1)-1]a_n x^{n+r} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$(2r^2+r-1)a_0 + (2r^2+5r+2)a_1 x + \sum_{n=2}^{\infty} ([(n+r)(2n+2r+1)-1]a_n - a_{n-2})x^{n+r} = 0$$

$$\text{For } n=0 \rightarrow (2r^2+r-1)a_0 = 0$$

$$r = -1 \quad \text{or} \quad r = \frac{1}{2} \quad \checkmark$$

$$\text{For } n=1 \rightarrow (2r^2+5r+2)a_1 = 0$$

$$\cancel{r = -2 \quad \text{or} \quad r = -\frac{1}{2}} \quad \text{Therefore } a_1 = 0$$

$$[(n+r)(2n+2r+1)-1]a_n - a_{n-2} = 0$$

$$a_n = \frac{1}{(n+r)(2(n+r)+1)-1} a_{n-2}$$

$$= \frac{1}{2(n+r)^2 + (n+r) - 1} a_{n-2} \quad \text{for } n \geq 2$$

$$r = \frac{1}{2}$$

$$r = -1$$

$$a_n = \frac{1}{2\left(n+\frac{1}{2}\right)^2 + \left(n+\frac{1}{2}\right) - 1} a_{n-2} = \frac{1}{2n^2 + 3n} a_{n-2}$$

$$b_n = \frac{1}{2(n-1)^2 + (n-1) - 1} b_{n-2}$$

$$a_2 = \frac{1}{14} a_0$$

$$b_2 = \frac{1}{2} b_0$$

$$a_3 = \frac{1}{24} a_1 = 0$$

$$b_3 = \frac{1}{9} b_1 = 0$$

$$a_4 = \frac{1}{44} a_2 = \frac{1}{616} a_0$$

$$b_4 = \frac{1}{20} b_2 = \frac{1}{40} b_0$$

$$a_5 = 0$$

$$b_5 = 0$$

$$a_6 = \frac{1}{90}a_4 = \frac{1}{55440}a_0$$

$$b_6 = \frac{1}{54}b_4 = \frac{1}{2160}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x) = a_0 x^{1/2} \left(1 + \frac{x^2}{14} + \frac{x^4}{616} + \frac{x^6}{55,440} + \cdots \right) \quad y_2(x) = b_0 x^{-1} \left(1 + \frac{x^2}{2} + \frac{x^4}{40} + \frac{x^6}{2160} + \cdots \right)$$

$$y(x) = C_1 \left(1 + \sum_{n=0}^{\infty} \frac{4}{n! \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)} x^n \right) + C_2 x^{1/2} \left(1 + \sum_{n=0}^{\infty} \frac{4^n}{n! \cdot 3 \cdot 5 \cdots (2n+1)} x^n \right)$$

Exercise

Find the Frobenius series solutions of $2x^2 y'' - xy' + (1+x^2)y = 0$

Solution

$$y'' - \frac{1}{2} \frac{1}{x} y' + \frac{1}{2} \frac{1+x^2}{x^2} y = 0 \quad \text{Divide each term by } 2x^2$$

Therefore, $x = 0$ is a *regular singular point*, and that $p_0 = -\frac{1}{2}$, $q_0 = \frac{1}{2}$

$p(x) = -\frac{1}{2}$, $q(x) = \frac{1}{2} + \frac{1}{2}x^2$ are polynomials.

The Frobenius series will converge for all $x > 0$. The indicial equation is

$$r(r-1) - \frac{1}{2}r + \frac{1}{2} = r^2 - \frac{3}{2}r + \frac{1}{2} = \frac{1}{2}(2r^2 - 3r + 1) = 0$$

So the roots are $\underline{r_1 = \frac{1}{2} \text{ and } r_2 = 1}$.

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^1 \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2 y'' - xy' + (1+x^2)y = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} - x \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} + x^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(2n+2r-2) - (n+r) + 1] a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(2n+2r-3) + 1] a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$(2r^2 - 3r + 1)a_0 + ((1+r)(2r-1) + 1)a_1 x +$$

$$\sum_{n=2}^{\infty} [(n+r)(2n+2r-3) + 1] a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$(2r^2 - 3r + 1)a_0 + (2r^2 + r)a_1 x + \sum_{n=2}^{\infty} [((n+r)(2n+2r-3) + 1)a_n + a_{n-2}] x^{n+r} = 0$$

$$\text{For } n=0 \rightarrow (2r^2 - 3r + 1)a_0 = 0 \Rightarrow \underline{r=1 \text{ or } r=\frac{1}{2}} \quad \checkmark$$

$$\text{For } n=1 \rightarrow (2r^2 + r)a_1 = 0 \Rightarrow r = \underline{0, -\frac{1}{2}} \quad \text{(crossed out)}$$

$$\text{Therefore } \underline{a_1 = 0}$$

$$((n+r)(2n+2r-3) + 1)a_n + a_{n-2} = 0$$

$$\underline{a_n = -\frac{1}{(n+r)(2n+2r-3) + 1} a_{n-2}} \quad \text{for } n \geq 2$$

$$r = \frac{1}{2}$$

$$a_n = -\frac{1}{\left(n + \frac{1}{2}\right)(2n-2) + 1} a_{n-2}$$

$$= -\frac{1}{2n^2 - n} a_{n-2} \Bigg|$$

$$n = 2 \rightarrow a_2 = -\frac{1}{6} a_0$$

$$n = 3 \rightarrow a_3 = -\frac{1}{15} a_1 = 0$$

$$n = 4 \rightarrow a_4 = -\frac{1}{28} a_2 = \frac{1}{168} a_0$$

$$n = 5 \rightarrow a_5 = 0$$

$$n = 6 \rightarrow a_6 = -\frac{1}{66} a_4 = -\frac{1}{11,088} a_0$$

$$\begin{aligned} y_1(x) &= a_0 x^{1/2} \left(1 - \frac{x^2}{6} + \frac{x^4}{168} - \frac{x^6}{11,088} + \dots \right) \\ &= a_0 \left(x^{1/2} - \frac{1}{6} x^{5/2} + \frac{1}{168} x^{9/2} - \frac{1}{11,088} x^{13/2} + \dots \right) \Bigg| \end{aligned}$$

$$r = 1$$

$$\begin{aligned} b_n &= -\frac{1}{(n+1)(2n-1)+1} b_{n-2} \\ &= -\frac{1}{2n^2 + n} b_{n-2} \Bigg| \end{aligned}$$

$$n = 2 \rightarrow b_2 = -\frac{1}{10} b_0$$

$$n = 3 \rightarrow b_3 = -\frac{1}{21} b_1 = 0$$

$$n = 4 \rightarrow b_4 = -\frac{1}{36} b_2 = \frac{1}{360} b_0$$

$$n = 5 \rightarrow b_5 = 0$$

$$n = 6 \rightarrow b_6 = -\frac{1}{78} b_4 = -\frac{1}{28,080} b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\begin{aligned} y_2(x) &= b_0 x \left(1 - \frac{x^2}{10} + \frac{x^4}{360} - \frac{x^6}{28,080} + \dots \right) \\ &= b_0 \left(x - \frac{1}{10} x^3 + \frac{1}{360} x^5 - \frac{1}{28,080} x^7 + \dots \right) \Bigg| \end{aligned}$$

$$y(x) = a_0 \left(x^{1/2} - \frac{1}{6} x^{5/2} + \frac{1}{168} x^{9/2} - \frac{1}{11,088} x^{13/2} + \dots \right) + b_0 \left(x - \frac{1}{10} x^3 + \frac{1}{360} x^5 - \frac{1}{28,080} x^7 + \dots \right) \Bigg|$$

Exercise

Find the general solution to the equation $2xy'' + (1+x)y' + y = 0$

Solution

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' + (1+x)y' + y = 0$$

$$2x \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} + (1+x) \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\begin{aligned} \sum_{n=0}^{\infty} 2(n+r)(n+r-1) c_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \\ + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0 \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{\infty} 2(n+r)(n+r-1) c_n x^{n-1} x^r + \sum_{n=0}^{\infty} (n+r) c_n x^{n-1} x^r \\ + \sum_{n=0}^{\infty} (n+r) c_n x^n x^r + \sum_{n=0}^{\infty} c_n x^n x^r = 0 \end{aligned}$$

$$x^r \left(\sum_{n=0}^{\infty} (n+r)(2n+2r-2+1) c_n x^{n-1} + \sum_{n=0}^{\infty} (n+r+1) c_n x^n \right) = 0$$

$$x^r \left(\sum_{n=0}^{\infty} (n+r)(2n+2r-1) c_n x^{n-1} + \sum_{n=0}^{\infty} (n+r+1) c_n x^n \right) = 0$$

$$x^r \left(c_0 r(2r-1)x^{-1} + \underbrace{\sum_{n=1}^{\infty} c_n (n+r)(2n+2r-1)x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=0}^{\infty} (n+r+1)c_n x^n}_{k=n} \right) = 0$$

$$x^r \left(c_0 r(2r-1)x^{-1} + \sum_{k=0}^{\infty} c_{k+1} (r+k+1)(2k+2r+1)x^k + \sum_{k=0}^{\infty} c_k (r+k+1)x^k \right) = 0$$

$$x^r \left(c_0 r(2r-1)x^{-1} + \sum_{k=0}^{\infty} [c_{k+1} (r+k+1)(2k+2r+1) + c_k (r+k+1)]x^k \right) = 0$$

$$\begin{cases} c_0 r(2r-1) = 0 \\ c_{k+1} (r+k+1)(2k+2r+1) + c_k (r+k) = 0 \end{cases} \Rightarrow \begin{matrix} \boxed{r=0} & \boxed{r=\frac{1}{2}} \\ \boxed{c_{k+1} = -\frac{r+k+1}{(r+k+1)(2k+2r+1)} c_k} \end{matrix}$$

$$r = 0$$

$$r = \frac{1}{2}$$

$$c_{k+1} = -\frac{1}{2k+1} c_k$$

$$c_{k+1} = -\frac{k+\frac{3}{2}}{\left(k+\frac{3}{2}\right)(2k+2)} c_k = -\frac{1}{2(k+1)} c_k$$

$$c_1 = -\frac{1}{1} c_0$$

$$c_1 = -\frac{1}{2} c_0$$

$$c_2 = -\frac{1}{3} c_1 = \frac{1}{3} c_0$$

$$c_2 = -\frac{1}{2 \cdot 2} c_1 = \frac{1}{2 \cdot 2 \cdot 2} c_0$$

$$c_3 = -\frac{1}{5} c_2 = -\frac{1}{1 \cdot 3 \cdot 5} c_0$$

$$c_3 = -\frac{1}{2 \cdot 3} c_2 = -\frac{1}{2^3 (2 \cdot 3)} c_0 = -\frac{1}{2^3 \cdot 3!} c_0$$

$$c_4 = -\frac{1}{7} c_3 = \frac{1}{1 \cdot 3 \cdot 5 \cdot 7} c_0$$

$$c_4 = -\frac{1}{2 \cdot 4} c_3 = \frac{1}{2^4 \cdot 4!} c_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$c_n = \frac{(-1)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} c_0$$

$$c_n = \frac{(-1)^n}{2^n n!} c_0$$

$$y_1(x) = c_0 \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} x^n \right)$$

$$y_2(x) = c_0 x^{1/2} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!} x^n \right)$$

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

$$= C_1 \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} x^n \right) + C_2 \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{n+1/2} \right)$$

Exercise

Find the Frobenius series solutions of $xy'' + 2y' + xy = 0$

Solution

$$\frac{1}{x}(xy'' + 2y' + xy = 0)$$

$$y'' + 2\frac{1}{x}y' + \frac{x^2}{x^2}y = 0$$

$\therefore x = 0$ is a regular singular point with $p_0 = 2$ and $q_0 = 0$

The indicial equation is: $r(r-1) + 2r = r(r+1) = 0 \rightarrow \underline{r = 0, -1}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-1} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$xy'' + 2y' + xy = 0$$

$$x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 2 \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + x \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + 2(n+r)] a_n x^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0$$

$$r(r+1)a_0x^{r-1} + (r+1)(r+2)a_1x^r + \sum_{n=2}^{\infty} (n+r)(n+r+1)a_nx^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r-1} = 0$$

For $n = 0$

$$r(r+1)a_0 = 0$$

$$\underline{r = 0 \text{ or } r = -1} \quad \checkmark$$

For $n = 1$

$$(r+1)(r+2)a_1 = 0$$

$$\underline{\cancel{r = -1, -2}}$$

$$\therefore \underline{a_1 = 0}$$

$$(n+r)(n+r+1)a_n + a_{n-2} = 0$$

$$\underline{a_n = -\frac{1}{(n+r)(n+r+1)}a_{n-2}}$$

$$\underline{r = 0 \rightarrow a_n = -\frac{1}{n(n+1)}a_{n-2}}$$

$$n = 2 \rightarrow a_2 = -\frac{1}{2 \cdot 3}a_0$$

$$n = 4 \rightarrow a_4 = -\frac{1}{4 \cdot 5}a_2 = \frac{1}{5!}a_0$$

$$n = 6 \rightarrow a_6 = -\frac{1}{6 \cdot 7}a_4 = -\frac{1}{7!}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x) = a_0 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots \right)$$

$$\underline{= \frac{a_0}{x} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right)}$$

$$\underline{r = -1 \rightarrow b_n = -\frac{1}{n(n-1)}b_{n-2}}$$

$$n = 2 \rightarrow b_2 = -\frac{1}{2 \cdot 1}b_0$$

$$n = 4 \rightarrow b_4 = -\frac{1}{4 \cdot 3}b_2 = \frac{1}{4!}b_0$$

$$n = 6 \rightarrow b_6 = -\frac{1}{6 \cdot 5}b_4 = -\frac{1}{6!}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n = 3 \rightarrow a_3 = -\frac{1}{12}a_1 = 0$$

$$n = 5 \rightarrow a_5 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n = 3 \rightarrow b_3 = -\frac{1}{6}b_1 = 0$$

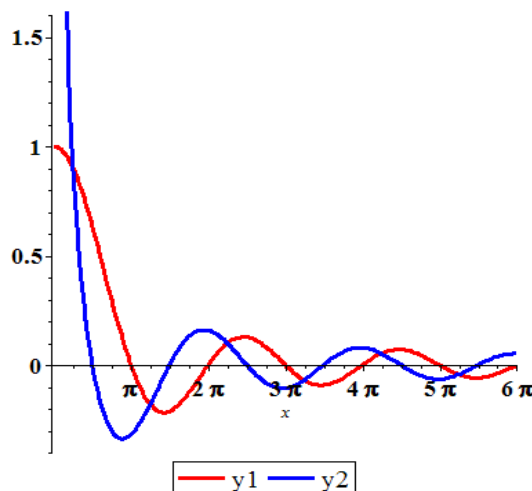
$$n = 5 \rightarrow b_5 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{-1} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right)$$

$$y(x) = \frac{a_0}{x} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right) + \frac{b_0}{x} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right)$$

$$= a_0 \frac{\sin x}{x} + b_0 \frac{\cos x}{x}$$



Exercise

Find the Frobenius series solutions of $2xy'' - y' + 2y = 0$

Solution

$$\left(\frac{x}{2}\right)2xy'' - \left(\frac{x}{2}\right)y' + \left(\frac{x}{2}\right)2y = 0$$

$$x^2 y'' - \frac{1}{2}xy' + xy = 0$$

That implies to $p(x) = -\frac{1}{2}$ and $q(x) = x$, both are analytic.

Hence, $x_0 = 0$ is a regular point

$$\text{The indicial equation is: } r(r-1) - \frac{1}{2}r = r\left(r - \frac{3}{2}\right) = 0 \rightarrow \underline{r = 0, \frac{3}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{3/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' - y' + 2y = 0$$

$$2x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + 2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [2(n+r)(n+r-1) - (n+r)] a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(2n+2r-3) a_n + 2a_{n-1}] x^{n+r-1} = 0$$

$$(n+r)(2n+2r-3) a_n + 2a_{n-1} = 0$$

$$a_n = -\frac{2}{(n+r)(2n+2r-3)} a_{n-1}$$

$$r=0 \rightarrow a_n = -\frac{2}{n(2n-3)} a_{n-1}$$

$$n=1 \rightarrow a_1 = 2a_0$$

$$n=2 \rightarrow a_2 = -a_1 = -2a_0$$

$$n=3 \rightarrow a_3 = -\frac{2}{9} a_2 = \frac{4}{9} a_0$$

$$n=4 \rightarrow a_4 = -\frac{1}{10} a_3 = -\frac{2}{45} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x) = a_0 \left(1 + 2x - 2x^2 + \frac{4}{9}x^3 - \frac{2}{45}x^4 + \dots \right)$$

$$r=\frac{3}{2} \rightarrow b_n = -\frac{1}{n\left(n+\frac{3}{2}\right)} b_{n-1}$$

$$n=1 \rightarrow b_1 = -\frac{2}{5} b_0$$

$$n=2 \rightarrow b_2 = -\frac{1}{7}b_1 = \frac{2}{35}b_0$$

$$n=3 \rightarrow b_3 = -\frac{2}{27}b_2 = -\frac{4}{945}b_0$$

$$n=4 \rightarrow b_4 = -\frac{1}{22}b_3 = \frac{2}{20,790}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\begin{aligned} y_2(x) &= b_0 x^{3/2} \left(1 - \frac{2}{5}x + \frac{2}{35}x^2 - \frac{4}{945}x^3 + \frac{2}{20,790}x^4 - \dots \right) \\ &= b_0 \sqrt{x} \left(x - \frac{2}{5}x^2 + \frac{2}{35}x^3 - \frac{4}{945}x^4 + \frac{2}{20,790}x^5 - \dots \right) \end{aligned}$$

$$y(x) = a_0 \left(1 + 2x - 2x^2 + \frac{4}{9}x^3 - \frac{2}{45}x^4 + \dots \right) + b_0 \sqrt{x} \left(x - \frac{2}{5}x^2 + \frac{2}{35}x^3 - \frac{4}{945}x^4 + \frac{2}{20,790}x^5 - \dots \right)$$

Exercise

Find the Frobenius series solutions of $2xy'' + 5y' + xy = 0$

Solution

$$\left(\frac{x}{2}\right)2xy'' + 5\left(\frac{x}{2}\right)y' + \left(\frac{x}{2}\right)xy = 0$$

$$x^2y'' + \frac{5}{2}xy' + \frac{1}{2}x^2y = 0$$

That implies to $p(x) = \frac{5}{2}$ and $q(x) = \frac{1}{2}x^2$, both are analytic.

Hence, $x_0 = 0$ is a regular point

$$\text{The indicial equation is: } r(r-1) + \frac{5}{2}r = r^2 + \frac{3}{2}r = 0 \rightarrow r = 0, -\frac{3}{2}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-3/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$2xy'' + 5y' + xy = 0$$

$$2x \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty} 5(n+r)a_n x^{n+r-1} + x \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} 5(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} [2(n+r)(n+r-1) + 5(n+r)]a_n x^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0$$

$$r(2r+3)a_0 + (r+1)(2r+5)a_1 + \sum_{n=2}^{\infty} (n+r)(2n+2r+3)a_n x^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0$$

$$r(2r+3)a_0 + (r+1)(2r+5)a_1 + \sum_{n=2}^{\infty} [(n+r)(2n+2r+3)a_n + a_{n-2}] x^{n+r-1} = 0$$

For $n = 0$

$$r(2r+3)a_0 = 0$$

$$\underline{r = 0 \text{ or } r = -\frac{3}{2}} \quad \checkmark$$

For $n = 1$

$$(r+1)(2r+5)a_1 = 0$$

$$\underline{\cancel{r = -1, -\frac{5}{2}}} \\ \rightarrow \underline{a_1 = 0}$$

$$(n+r)(2n+2r+3)a_n + a_{n-2} = 0$$

$$\underline{a_n = -\frac{1}{(n+r)(2n+2r+3)}a_{n-2}}$$

$$r = 0 \rightarrow \underline{a_n = -\frac{1}{n(2n+3)}a_{n-2}}$$

$$n = 2 \rightarrow a_2 = -\frac{1}{14}a_0$$

$$n = 3 \rightarrow a_3 = -\frac{1}{27}a_1 = 0$$

$$\begin{array}{cccc} n=4 & \rightarrow & a_4 & = -\frac{1}{88}a_2 = \frac{1}{616}a_0 \\ & & \vdots & \vdots \end{array}$$

$$\begin{array}{cccc} n=5 & \rightarrow & a_5 & = 0 \\ & & \vdots & \vdots \end{array}$$

$$\underline{y_1(x) = a_0 \left(1 - \frac{1}{14}x^2 + \frac{1}{616}x^4 - \dots \right)}$$

$$r = -\frac{3}{2}$$

$$\begin{aligned} b_n &= -\frac{1}{2n\left(n - \frac{3}{2}\right)}b_{n-2} \\ &= -\frac{1}{n(2n-3)}b_{n-2} \end{aligned}$$

$$n=2 \rightarrow b_2 = -\frac{1}{2}b_0$$

$$n=3 \rightarrow b_3 = -\frac{1}{9}b_1 = 0$$

$$n=4 \rightarrow b_4 = -\frac{1}{20}b_2 = \frac{1}{40}b_0$$

$$n=5 \rightarrow b_5 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\begin{aligned} y_2(x) &= b_0 x^{-3/2} \left(1 - \frac{1}{2}x^2 + \frac{1}{40}x^3 - \dots \right) \\ &= b_0 \left(x^{-3/2} - \frac{1}{2}x^{1/2} + \frac{1}{40}x^{3/2} - \dots \right) \end{aligned}$$

$$\underline{y(x) = a_0 \left(1 - \frac{1}{14}x^2 + \frac{1}{616}x^4 - \dots \right) + b_0 \left(x^{-3/2} - \frac{1}{2}x^{1/2} + \frac{1}{40}x^{3/2} - \dots \right)}$$

Exercise

Find the Frobenius series solutions of $4xy'' + \frac{1}{2}y' + y = 0$

Solution

$$\left(\frac{x}{4}\right)4xy'' + \frac{1}{2}\left(\frac{x}{4}\right)y' + \left(\frac{x}{4}\right)y = 0$$

$$x^2y'' + \frac{1}{8}xy' + \frac{1}{4}x^2y = 0$$

$$y'' + \frac{1}{8x}y' + \frac{1}{4}y = 0$$

That implies to $p(x) = \frac{1}{8x}$ and $q(x) = \frac{1}{4}$

$$p_0 = \lim_{x \rightarrow 0} xp(x)$$

$$= \lim_{x \rightarrow 0} x \frac{1}{8x}$$

$$\underline{= \frac{1}{8} \mid}$$

$$\begin{aligned} q_0 &= \lim_{x \rightarrow 0} x^2 q(x) \\ &= \lim_{x \rightarrow 0} x^2 \frac{1}{8} \\ &= 0 \end{aligned}$$

The indicial equation is:

$$r(r-1) + \frac{1}{8}r = r^2 - \frac{7}{8}r = 0$$

$$\underline{r = 0, \frac{7}{8} \mid}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{7/8} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$8xy'' + y' + 2y = 0$$

$$8x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + 2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 8(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [8(n+r)(n+r-1) + (n+r)] a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1} = 0$$

$$r(8r-7)a_0 + \sum_{n=0}^{\infty} (n+r)(8n+8r-7) a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1} = 0$$

$$r(8r-7)a_0 + \sum_{n=0}^{\infty} \left[(n+r)(8n+8r-7)a_n + 2a_{n-1} \right] x^{n+r-1} = 0$$

For $n=0$

$$r(8r-7)a_0 = 0$$

$$\underline{r=0, \frac{7}{8}} \quad \checkmark$$

$$(n+r)(8n+8r-7)a_n + 2a_{n-1} = 0$$

$$\underline{a_n = -\frac{2}{(n+r)(8n+8r-7)}a_{n-1}}$$

$$r=0 \rightarrow \underline{a_n = -\frac{2}{n(8n-7)}a_{n-1}}$$

$$n=1 \rightarrow a_1 = -2a_0$$

$$n=2 \rightarrow a_2 = -\frac{1}{9}a_1 = \frac{2}{9}a_0$$

$$n=3 \rightarrow a_3 = -\frac{2}{51}a_2 = -\frac{4}{459}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 \left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \dots \right)}$$

$$r = \frac{7}{8}$$

$$b_n = -\frac{2}{\left(n + \frac{7}{8}\right)(8n)}b_{n-1}$$

$$\underline{= -\frac{2}{n(8n+7)}b_{n-1}}$$

$$n=1 \rightarrow b_1 = -\frac{2}{15}b_0$$

$$n=2 \rightarrow b_2 = -\frac{1}{23}b_1 = \frac{2}{345}b_0$$

$$n=3 \rightarrow b_3 = -\frac{2}{93}b_2 = -\frac{4}{32,085}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_2(x) = b_0 x^{7/8} \left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32,085}x^3 + \dots \right)}$$

$$\underline{y(x) = a_0 \left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \dots \right) + b_0 x^{7/8} \left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32,085}x^3 + \dots \right)}$$

Exercise

Find the Frobenius series solutions of $2x^2y'' - xy' + (x^2 + 1)y = 0$

Solution

$$\frac{1}{2}2x^2y'' - \frac{1}{2}xy' + \frac{1}{2}(x^2 + 1)y = 0$$

$$x^2y'' - \frac{1}{2}xy' + \frac{1}{2}(x^2 + 1)y = 0$$

$$y'' - \frac{1}{2x}y' + \left(\frac{1}{2} + \frac{1}{2x^2}\right)y = 0$$

That implies to $p(x) = -\frac{1}{2x}$ and $q(x) = \frac{1}{2} + \frac{1}{2x^2}$.

$$p_0 = \lim_{x \rightarrow 0} xp(x)$$

$$= -\lim_{x \rightarrow 0} x \frac{1}{2x}$$

$$= -\frac{1}{2} \Big|$$

$$q_0 = \lim_{x \rightarrow 0} x^2q(x)$$

$$= \lim_{x \rightarrow 0} x^2 \left(\frac{1}{2} + \frac{1}{2x^2} \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{1}{2}x^2 + \frac{1}{2} \right)$$

$$= \frac{1}{2} \Big|$$

The indicial equation is:

$$r(r-1) - \frac{1}{2}r + \frac{1}{2} = r^2 - \frac{3}{2}r + \frac{1}{2} = 0$$

$$r = 1, \frac{1}{2} \Big|$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^1 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2 y'' - xy' + (x^2 + 1)y = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + (x^2 + 1) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (2(n+r)(n+r-1) - (n+r) + 1) a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

$$(r(2r-3)+1)a_0 + ((r+1)(2r-1)+1)a_1 + \sum_{n=2}^{\infty} ((n+r)(2n+2r-3)+1)a_n x^{n+r}$$

$$+ \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$(2r^2 - 3r + 1)a_0 + (2r^2 + r)a_1 + \sum_{n=2}^{\infty} [((n+r)(2n+2r-3)+1)a_n + a_{n-2}] x^{n+r} = 0$$

For $n=0$

$$(2r^2 - 3r + 1)a_0 = 0$$

$$\underline{r=1, \frac{1}{2}} \quad \checkmark$$

For $n=1$

$$(2r^2 + r)a_1 = 0$$

$$\cancel{\underline{r=0, -\frac{1}{2}}}$$

$$\rightarrow \underline{a_1 = 0}$$

$$((n+r)(2n+2r-3)+1)a_n + a_{n-2} = 0$$

$$a_n = -\frac{1}{(n+r)(2n+2r-3)+1}a_{n-2}$$

$$r = 1$$

$$a_n = -\frac{1}{(n+1)(2n-1)+1}a_{n-2}$$

$$= -\frac{1}{2n^2+n}a_{n-2}$$

$$n = 2 \rightarrow a_2 = -\frac{1}{10}a_0$$

$$n = 4 \rightarrow a_4 = -\frac{1}{36}a_2 = \frac{1}{360}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x) = a_0 x \left(1 - \frac{1}{10}x^2 + \frac{1}{360}x^4 - \dots \right)$$

$$= a_0 \left(x - \frac{1}{10}x^3 + \frac{1}{360}x^5 - \dots \right)$$

$$r = \frac{1}{2}$$

$$b_n = -\frac{1}{\left(n+\frac{1}{2}\right)(2n-2)+1}b_{n-2}$$

$$= -\frac{1}{2n^2-n}b_{n-2}$$

$$n = 2 \rightarrow b_2 = -\frac{1}{6}b_0$$

$$n = 4 \rightarrow b_4 = -\frac{1}{28}b_2 = \frac{1}{168}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{1/2} \left(1 - \frac{1}{6}x^2 + \frac{1}{168}x^4 - \dots \right)$$

$$y(x) = a_0 \left(x - \frac{1}{10}x^3 + \frac{1}{360}x^5 - \dots \right) + b_0 x^{1/2} \left(1 - \frac{1}{6}x^2 + \frac{1}{168}x^4 - \dots \right)$$

$$n = 3 \rightarrow a_3 = -\frac{1}{21}a_1 = 0$$

$$n = 5 \rightarrow a_5 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n = 3 \rightarrow a_3 = -\frac{1}{15}a_1 = 0$$

$$n = 5 \rightarrow a_5 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

Exercise

Find the Frobenius series solutions of $3xy'' + (2-x)y' - y = 0$

Solution

$$\frac{x}{3}3xy'' + \frac{x}{3}(2-x)y' - \frac{x}{3}y = 0$$

$$x^2y'' + \left(\frac{2}{3}x - \frac{1}{3}x^2\right)y' - \frac{x}{3}y = 0$$

$$y'' + \left(\frac{2}{3x} - \frac{1}{3}\right)y' - \frac{1}{3x}y = 0$$

That implies to $p(x) = \frac{2}{3x} - \frac{1}{3}$ and $q(x) = -\frac{1}{3x}$.

$$\begin{aligned} p_0 &= \lim_{x \rightarrow 0} xp(x) \\ &= \lim_{x \rightarrow 0} x\left(\frac{2}{3x} - \frac{1}{3}\right) \\ &= \lim_{x \rightarrow 0} \left(\frac{2}{3} - \frac{1}{3}x\right) \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x \rightarrow 0} x^2q(x) \\ &= -\lim_{x \rightarrow 0} x^2 \frac{1}{3x} \\ &= \lim_{x \rightarrow 0} \frac{x}{3} \\ &= 0 \end{aligned}$$

The indicial equation is: $r(r-1) + \frac{2}{3}r = r^2 - \frac{1}{3}r = 0$

$$\underline{r = 0, \frac{1}{3}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/3} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$3xy'' + (2-x)y' - y = 0$$

$$3x \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + (2-x) \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r)a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(3n+3r-3) + 2(n+r)]a_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r+1)a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(3n+3r-1)a_n x^{n+r-1} - \sum_{n=1}^{\infty} (n+r)a_{n-1} x^{n+r-1} = 0$$

$$r(3r-1)a_0 + \sum_{n=1}^{\infty} (n+r)(3n+3r-1)a_n x^{n+r-1} - \sum_{n=1}^{\infty} (n+r)a_{n-1} x^{n+r-1} = 0$$

$$r(3r-1)a_0 + \sum_{n=1}^{\infty} [(n+r)(3n+3r-1)a_n - (n+r)a_{n-1}] x^{n+r-1} = 0$$

For $n = 0$

$$r(3r-1)a_0 = 0$$

$$\left| r = 0, \frac{1}{3} \right| \quad \checkmark$$

$$(n+r)(3n+3r-1)a_n - (n+r)a_{n-1} = 0$$

$$\left| a_n = \frac{1}{3n+3r-1} a_{n-1} \right|$$

$$r = 0$$

$$\left| a_n = \frac{1}{3n-1} a_{n-1} \right|$$

$$n=1 \rightarrow a_1 = \frac{1}{2}a_0$$

$$n=2 \rightarrow a_2 = \frac{1}{5}a_1 = \frac{1}{10}a_0$$

$$n=3 \rightarrow a_3 = \frac{1}{8}a_2 = \frac{1}{80}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 \left(1 + \frac{1}{2}x + \frac{1}{10}x^2 + \frac{1}{80}x^3 + \dots \right) \quad |}$$

$$r = \frac{1}{3}$$

$$\underline{b_n = \frac{1}{3n}b_{n-1} \quad |}$$

$$n=1 \rightarrow b_1 = \frac{1}{3}b_0$$

$$n=2 \rightarrow b_2 = \frac{1}{6}b_1 = \frac{1}{18}b_0$$

$$n=3 \rightarrow b_3 = \frac{1}{9}b_2 = \frac{1}{162}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_2(x) = b_0 x^{1/3} \left(1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \dots \right) \quad |}$$

$$\underline{y(x) = a_0 \left(1 + \frac{1}{2}x + \frac{1}{10}x^2 + \frac{1}{80}x^3 + \dots \right) + b_0 x^{1/3} \left(1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \dots \right) \quad |}$$

Exercise

Find the Frobenius series solutions of $2xy'' - (3+2x)y' + y = 0$

Solution

$$\frac{x}{2}2xy'' - \frac{x}{2}(3+2x)y' + \frac{x}{2}y = 0$$

$$x^2y'' - \left(\frac{3}{2}x + x^2 \right)y' + \frac{1}{2}xy = 0$$

$$y'' - \left(\frac{3}{2x} + 1 \right)y' + \frac{1}{2x}y = 0$$

That implies to $p(x) = -\frac{3}{2x} - 1$ and $q(x) = \frac{1}{2x}$.

$$p_0 = \lim_{x \rightarrow 0} xp(x)$$

$$= \lim_{x \rightarrow 0} x \left(-\frac{3}{2x} - 1 \right)$$

$$= \lim_{x \rightarrow 0} \left(-\frac{3}{2} - x \right)$$

$$= -\frac{3}{2}$$

$$q_0 = \lim_{x \rightarrow 0} x^2 q(x)$$

$$= \lim_{x \rightarrow 0} x^2 \frac{1}{2x}$$

$$= \lim_{x \rightarrow 0} \frac{x}{2}$$

$$= 0$$

The indicial equation is: $r(r-1) - \frac{3}{2}r = r^2 - \frac{5}{2}r = 0$

$$\Rightarrow r = 0, \frac{5}{2}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{5/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' - (3+2x)y' + y = 0$$

$$2x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - (3+2x) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} - \sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\begin{aligned}
& \sum_{n=0}^{\infty} [2(n+r)(n+r-1) - 3(n+r)] a_n x^{n+r-1} - \sum_{n=0}^{\infty} (2n+2r-1) a_n x^{n+r} = 0 \\
& \sum_{n=0}^{\infty} (n+r)(2n+2r-5) a_n x^{n+r-1} - \sum_{n=1}^{\infty} (2n+2r-3) a_{n-1} x^{n+r-1} = 0 \\
& r(2r-5)a_0 + \sum_{n=1}^{\infty} (n+r)(2n+2r-5) a_n x^{n+r-1} - \sum_{n=1}^{\infty} (2n+2r-3) a_{n-1} x^{n+r-1} = 0 \\
& r(2r-5)a_0 + \sum_{n=1}^{\infty} [(n+r)(2n+2r-5) a_n - (2n+2r-3) a_{n-1}] x^{n+r-1} = 0
\end{aligned}$$

For $n = 0$

$$r(2r-5)a_0 = 0$$

$$\underline{r = 0, \frac{5}{2}} \quad \checkmark$$

$$(n+r)(2n+2r-5)a_n - (2n+2r-3)a_{n-1} = 0$$

$$\underline{a_n = \frac{2n+2r-3}{(n+r)(2n+2r-5)} a_{n-1}}$$

$$\underline{r = 0 \rightarrow a_n = \frac{2n-3}{n(2n-5)} a_{n-1}}$$

$$n = 1 \rightarrow a_1 = \frac{1}{3} a_0$$

$$n = 2 \rightarrow a_2 = -\frac{1}{2} a_1 = -\frac{1}{6} a_0$$

$$n = 3 \rightarrow a_3 = -a_2 = -\frac{1}{6} a_0$$

$$n = 4 \rightarrow a_4 = \frac{5}{12} a_3 = -\frac{5}{72} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 \left(1 + \frac{1}{3}x - \frac{1}{6}x^2 - \frac{1}{6}x^3 - \frac{5}{72}x^4 - \dots \right)}$$

$$r = \frac{5}{2}$$

$$b_n = \frac{2n+2}{2n\left(n+\frac{5}{2}\right)} b_{n-1}$$

$$\underline{= \frac{2n+2}{n(2n+5)} b_{n-1}}$$

$$n=1 \rightarrow b_1 = \frac{4}{7}b_0$$

$$n=2 \rightarrow b_2 = \frac{1}{3}b_1 = \frac{4}{21}b_0$$

$$n=3 \rightarrow b_3 = \frac{8}{33}b_2 = \frac{32}{693}b_0$$

$$n=4 \rightarrow b_4 = \frac{5}{26}b_3 = \frac{80}{9,009}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x) = b_0 x^{5/2} \left(1 + \frac{4}{7}x + \frac{4}{21}x^2 + \frac{32}{693}x^3 + \frac{80}{9,009}x^4 + \dots \right)$$

$$y(x) = a_0 \left(1 + \frac{1}{3}x - \frac{1}{6}x^2 - \frac{1}{6}x^3 - \frac{5}{72}x^4 - \dots \right) + b_0 x^{5/2} \left(1 + \frac{4}{7}x + \frac{4}{21}x^2 + \frac{32}{693}x^3 + \dots \right)$$

Exercise

Find the Frobenius series solutions of $xy'' + (x-6)y' - 3y = 0$

Solution

$$xxy'' + x(x-6)y' - 3xy = 0$$

$$x^2y'' + (x^2 - 6x)y' - 3xy = 0$$

$$y'' + \left(1 - \frac{6}{x}\right)y' - \frac{3}{x}y = 0$$

That implies to $p(x) = 1 - \frac{6}{x}$ and $q(x) = -\frac{3}{x}$.

$$\begin{aligned} p_0 &= \lim_{x \rightarrow 0} xp(x) \\ &= \lim_{x \rightarrow 0} x \left(1 - \frac{6}{x} \right) \\ &= \lim_{x \rightarrow 0} (x - 6) \\ &= -6 \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x \rightarrow 0} x^2 q(x) \\ &= - \lim_{x \rightarrow 0} x^2 \frac{3}{x} \\ &= - \lim_{x \rightarrow 0} 3x \\ &= 0 \end{aligned}$$

The indicial equation is: $r(r-1) - 6r = r^2 - 7r = 0$

$$\rightarrow \underline{r = 0, 7}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^7 \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$xy'' + (x-6)y' - 3y = 0$$

$$x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + (x-6) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - 3 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\begin{aligned} \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} 6(n+r) a_n x^{n+r-1} \\ - \sum_{n=0}^{\infty} 3a_n x^{n+r} = 0 \end{aligned}$$

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) - 6(n+r)] a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r-3) a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-7) a_n x^{n+r-1} + \sum_{n=1}^{\infty} (n+r-4) a_{n-1} x^{n+r-1} = 0$$

$$r(r-7)a_0 + \sum_{n=1}^{\infty} (n+r)(n+r-7) a_n x^{n+r-1} + \sum_{n=1}^{\infty} (n+r-4) a_{n-1} x^{n+r-1} = 0$$

$$r(r-7)a_0 + \sum_{n=1}^{\infty} [(n+r)(n+r-7) a_n + (n+r-4) a_{n-1}] x^{n+r-1} = 0$$

For $n = 0$

$$r(r-7)a_0 = 0$$

$$\Rightarrow \underline{r=0, 7} \quad \checkmark$$

$$(n+r)(n+r-7)a_n + (n+r-4)a_{n-1} = 0$$

$$\underline{a_n = -\frac{n+r-4}{(n+r)(n+r-7)}a_{n-1}}$$

$$\underline{r=0 \rightarrow a_n = -\frac{n-4}{n(n-7)}a_{n-1}}$$

$$n=1 \rightarrow a_1 = -\frac{1}{2}a_0$$

$$n=2 \rightarrow a_2 = -\frac{1}{5}a_1 = \frac{1}{10}a_0$$

$$n=3 \rightarrow a_3 = -\frac{1}{12}a_2 = -\frac{1}{120}a_0$$

$$n=4 \rightarrow a_4 = 0a_3 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 \left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3 \right)}$$

$$\underline{r=7 \rightarrow b_n = -\frac{n+3}{n(n+7)}b_{n-1}}$$

$$n=1 \rightarrow b_1 = -\frac{1}{2}b_0$$

$$n=2 \rightarrow b_2 = -\frac{5}{18}b_1 = \frac{5}{36}b_0$$

$$n=3 \rightarrow b_3 = -\frac{1}{5}b_2 = -\frac{1}{36}b_0$$

$$n=4 \rightarrow b_4 = -\frac{7}{44}b_3 = \frac{7}{1,584}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_2(x) = b_0 x^7 \left(1 - \frac{1}{2}x + \frac{5}{36}x^2 - \frac{1}{36}x^3 + \frac{7}{1,584}x^4 - \dots \right)}$$

$$\underline{y(x) = a_0 \left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3 \right) + b_0 x^7 \left(1 - \frac{1}{2}x + \frac{5}{36}x^2 - \frac{1}{36}x^3 + \frac{7}{1,584}x^4 - \dots \right)}$$

Exercise

Find the Frobenius series solutions of $x(x-1)y'' + 3y' - 2y = 0$

Solution

$$\frac{1}{x}x(x-1)y'' + 3\frac{1}{x}y' - 2\frac{1}{x}y = 0$$

$$(x-1)y'' + \frac{3}{x}y' - \frac{2}{x}y = 0$$

That implies to $p(x) = \frac{3}{x}$ and $q(x) = -\frac{2}{x}$.

$$\begin{aligned} p_0 &= \lim_{x \rightarrow 0} xp(x) \\ &= \lim_{x \rightarrow 0} x \frac{3}{x} \\ &= 3 \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x \rightarrow 0} x^2 q(x) \\ &= - \lim_{x \rightarrow 0} x^2 \frac{2}{x} \\ &= - \lim_{x \rightarrow 0} 2x \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{The indicial equation is: } -r(r-1) + 3r &= -r^2 + 4r = 0 \\ &\rightarrow r = 0, 4 \end{aligned}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^4 \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x(x-1)y'' + 3y' - 2y = 0$$

$$(x^2 - x) \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 3 \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - 2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\begin{aligned} & \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} \\ & + \sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0 \end{aligned}$$

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) - 2] a_n x^{n+r} - \sum_{n=0}^{\infty} [(n+r)(n+r-1) - 3(n+r)] a_n x^{n+r-1} = 0$$

$$\sum_{n=1}^{\infty} [(n-1+r)(n+r-2) - 2] a_{n-1} x^{n+r-1} - \sum_{n=0}^{\infty} (n+r)(n+r-4) a_n x^{n+r-1} = 0$$

$$\begin{aligned} & \sum_{n=1}^{\infty} [(n-1+r)(n+r-2) - 2] a_{n-1} x^{n+r-1} - r(r-4) a_0 \\ & - \sum_{n=1}^{\infty} (n+r)(n+r-4) a_n x^{n+r-1} = 0 \end{aligned}$$

$$-r(r-4) a_0 + \sum_{n=1}^{\infty} [((n+r-1)(n+r-2) - 2) a_{n-1} - (n+r)(n+r-4) a_n] x^{n+r-1} = 0$$

For $n = 0$

$$-r(r-4) a_0 = 0$$

$$\Rightarrow \underline{r = 0, 4} \quad \checkmark$$

$$((n+r-1)(n+r-2) - 2) a_{n-1} - (n+r)(n+r-4) a_n = 0$$

$$\underline{a_n = \frac{(n+r-1)(n+r-2) - 2}{(n+r)(n+r-4)} a_{n-1}}$$

$r = 0$

$$a_n = \frac{(n-1)(n-2) - 2}{n(n-4)} a_{n-1}$$

$$n=1 \rightarrow a_1 = \frac{2}{3}a_0$$

$$n=2 \rightarrow a_2 = \frac{1}{2}a_1 = \frac{1}{3}a_0$$

$$n=3 \rightarrow a_3 = \frac{0}{3}a_2 = 0$$

$$n=4 \rightarrow a_4 = 0a_3 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 \left(1 + \frac{2}{3}x + \frac{1}{3}x^2 \right) \quad |}$$

$$r=4$$

$$b_n = \frac{(n+3)(n+2)-2}{n(n+4)}b_{n-1}$$

$$n=1 \rightarrow b_1 = 2b_0$$

$$n=2 \rightarrow b_2 = \frac{3}{2}b_1 = 3b_0$$

$$n=3 \rightarrow b_3 = \frac{28}{21}b_2 = 4b_0$$

$$n=4 \rightarrow b_4 = \frac{5}{4}b_3 = 5b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_2(x) = b_0 x^4 \left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots \right) \quad |}$$

$$\underline{y(x) = a_0 \left(1 + \frac{2}{3}x + \frac{1}{3}x^2 \right) + b_0 \left(x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + \dots \right) \quad |}$$

Exercise

Find the Frobenius series solutions of $x^2 y'' - \left(x - \frac{2}{9}\right)y = 0$

Solution

$$x^2 y'' - \left(x - \frac{2}{9}\right)y = 0$$

$$y'' - \left(\frac{1}{x} - \frac{2}{9x^2}\right)y = 0$$

That implies to $p(x) = 0$ and $q(x) = \frac{2}{9x^2} - \frac{1}{x}$.

$$q_0 = \lim_{x \rightarrow 0} x^2 q(x)$$

$$\begin{aligned}
&= \lim_{x \rightarrow 0} x^2 \left(\frac{2}{9x^2} - \frac{1}{x} \right) \\
&= \lim_{x \rightarrow 0} \left(\frac{2}{9} - x \right) \\
&= \frac{2}{9}
\end{aligned}$$

The indicial equation is: $r(r-1) + \frac{2}{9} = r^2 - r + \frac{2}{9} = 0$

$$9r^2 - 9r + 2 = 0$$

$$r = \frac{9 \pm 3}{18} = \frac{1}{3}, \frac{2}{3}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/3} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{2/3} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x^2 y'' - \left(x - \frac{2}{9}\right) y = 0$$

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - \left(x - \frac{2}{9}\right) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} + \sum_{n=0}^{\infty} \frac{2}{9} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + \frac{2}{9} \right] a_n x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0$$

$$\left(r^2 - r + \frac{2}{9}\right) a_0 + \sum_{n=1}^{\infty} \left[(n+r)(n+r-1) + \frac{2}{9} \right] a_n x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0$$

$$\left(r^2 - r + \frac{2}{9}\right)a_0 + \sum_{n=1}^{\infty} \left[\left((n+r)(n+r-1) + \frac{2}{9} \right) a_n - a_{n-1} \right] x^{n+r} = 0$$

For $n = 0$

$$\left(r^2 - r + \frac{2}{9}\right)a_0 = 0$$

$$\Rightarrow \underline{r = \frac{1}{3}, \frac{2}{3}} \quad \checkmark$$

$$\left((n+r)(n+r-1) + \frac{2}{9} \right) a_n - a_{n-1} = 0$$

$$\underline{a_n = \frac{1}{(n+r)(n+r-1) + \frac{2}{9}} a_{n-1}}$$

$$r = \frac{1}{3}$$

$$a_n = \frac{1}{\left(n + \frac{1}{3}\right)\left(n - \frac{2}{3}\right) + \frac{2}{9}} a_{n-1}$$

$$= \frac{1}{n^2 - \frac{1}{3}n} a_{n-1}$$

$$= \frac{3}{3n^2 - n} a_{n-1}$$

$$n = 1 \rightarrow a_1 = \frac{3}{2} a_0$$

$$n = 2 \rightarrow a_2 = \frac{3}{10} a_1 = \frac{9}{20} a_0$$

$$n = 3 \rightarrow a_3 = \frac{1}{8} a_2 = \frac{9}{160} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 x^{1/3} \left(1 + \frac{3}{2}x + \frac{9}{20}x^2 + \frac{9}{160}x^3 + \cdots \right)}$$

$$r = \frac{2}{3}$$

$$b_n = \frac{1}{\left(n + \frac{2}{3}\right)\left(n - \frac{1}{3}\right) + \frac{2}{9}} b_{n-1}$$

$$= \frac{3}{3n^2 + n} b_{n-1}$$

$$n = 1 \rightarrow b_1 = \frac{3}{4} b_0$$

$$n = 2 \rightarrow b_2 = \frac{3}{14} b_1 = \frac{9}{56} b_0$$

$$n=3 \rightarrow b_3 = \frac{1}{10}b_2 = \frac{9}{560}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{2/3} \left(1 + \frac{3}{4}x + \frac{9}{56}x^2 + \frac{9}{560}x^3 + \dots \right)$$

$$y(x) = a_0 x^{1/3} \left(1 + \frac{3}{2}x + \frac{9}{20}x^2 + \frac{9}{160}x^3 + \dots \right) + b_0 x^{2/3} \left(1 + \frac{3}{4}x + \frac{9}{56}x^2 + \frac{9}{560}x^3 + \dots \right)$$

Exercise

Find the Frobenius series solutions of $x^2 y'' + x(3+x)y' - 3y = 0$

Solution

$$\frac{1}{x^2} x^2 y'' + \frac{1}{x^2} x(3+x)y' - 3 \frac{1}{x^2} y = 0$$

$$y'' + \left(\frac{3}{x} + 1 \right) y' - \frac{3}{x^2} y = 0$$

That implies to $p(x) = \frac{3}{x} + 1$ and $q(x) = -\frac{3}{x^2}$.

$$\begin{aligned} p_0 &= \lim_{x \rightarrow 0} x p(x) \\ &= \lim_{x \rightarrow 0} x \left(\frac{3}{x} + 1 \right) \\ &= \lim_{x \rightarrow 0} (3 + x) \\ &= 3 \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x \rightarrow 0} x^2 q(x) \\ &= - \lim_{x \rightarrow 0} x^2 \frac{3}{x^2} \\ &= -3 \end{aligned}$$

The indicial equation is: $r(r-1) + 3r - 3 = r^2 + 2r - 3 = 0$

$$\rightarrow r = 1, -3$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-3} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x^2 y'' + x(3+x)y' - 3y = 0$$

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + (3x+x^2) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - 3 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r+1} - \sum_{n=0}^{\infty} 3a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + 3(n+r) - 3] a_n x^{n+r} + \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^{n+r} = 0$$

$$(r^2 + 2r - 3) a_0 + \sum_{n=1}^{\infty} [(n+r)(n+r+2) - 3] a_n x^{n+r} + \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^{n+r} = 0$$

$$(r^2 + 2r - 3) a_0 + \sum_{n=1}^{\infty} [((n+r)(n+r+2) - 3) a_n + (n+r-1) a_{n-1}] x^{n+r} = 0$$

For $n = 0$

$$(r^2 + 2r - 3) a_0 = 0$$

$$\Rightarrow \underline{r=1, -3} \quad \checkmark$$

$$((n+r)(n+r+2) - 3) a_n + (n+r-1) a_{n-1} = 0$$

$$a_n = -\frac{n+r-1}{(n+r)(n+r+2) - 3} a_{n-1}$$

$r = 1$

$$a_n = -\frac{n}{(n+1)(n+3) - 3} a_{n-1}$$

$$= -\frac{n}{n^2 + 4n} a_{n-1} \Big|$$

$$n=1 \rightarrow a_1 = -\frac{1}{5} a_0$$

$$n=2 \rightarrow a_2 = -\frac{2}{12} a_1 = \frac{1}{30} a_0$$

$$n=3 \rightarrow a_3 = -\frac{3}{21} a_2 = -\frac{1}{210} a_0$$

$$n=4 \rightarrow a_4 = -\frac{4}{32} a_3 = \frac{1}{1,680} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x) = a_0 x \left(1 - \frac{1}{5}x + \frac{1}{30}x^2 - \frac{1}{210}x^3 + \frac{1}{1,680}x^4 - \dots \right) \Big|$$

$$r = -3$$

$$b_n = -\frac{n-4}{(n-3)(n-1)-3} b_{n-1}$$

$$= -\frac{n-4}{n^2 - 4n} b_{n-1} \Big|$$

$$n=1 \rightarrow b_1 = -b_0$$

$$n=2 \rightarrow b_2 = -\frac{2}{4} b_1 = \frac{1}{2} b_0$$

$$n=3 \rightarrow b_3 = -\frac{1}{3} b_2 = -\frac{1}{6} b_0$$

$$n=4 \rightarrow b_4 = 0 b_3 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{-3} \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 \right) \Big|$$

$$y(x) = a_0 x \left(1 - \frac{1}{5}x + \frac{1}{30}x^2 - \frac{1}{210}x^3 + \frac{1}{1,680}x^4 - \dots \right) + b_0 x^{-3} \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 \right) \Big|$$

Exercise

Find the Frobenius series solutions of $x^2 y'' + (x^2 - 2x)y' + 2y = 0$

Solution

$$\frac{1}{x^2} x^2 y'' + \frac{1}{x^2} (x^2 - 2x)y' + 2 \frac{1}{x^2} y = 0$$

$$y'' + \left(1 - \frac{2}{x}\right)y' + \frac{2}{x^2}y = 0$$

That implies to $p(x) = 1 - \frac{2}{x}$ and $q(x) = \frac{2}{x^2}$.

$$\begin{aligned} p_0 &= \lim_{x \rightarrow 0} xp(x) \\ &= \lim_{x \rightarrow 0} x \left(1 - \frac{2}{x}\right) \\ &= \lim_{x \rightarrow 0} (x - 2) \\ &= \underline{-2} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x \rightarrow 0} x^2 q(x) \\ &= \lim_{x \rightarrow 0} x^2 \frac{2}{x^2} \\ &= \underline{2} \end{aligned}$$

The indicial equation is: $r(r-1) - 2r + 2 = r^2 - 3r + 2 = 0$
 $\rightarrow \underline{r = 1, 2}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^2 \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x^2 y'' + (x^2 - 2x) y' + 2y = 0$$

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + (x^2 - 2x) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + 2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r+1} - \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) - 2(n+r) + 2]a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(n+r-3) + 2]a_n x^{n+r} + \sum_{n=1}^{\infty} (n+r-1)a_{n-1} x^{n+r} = 0$$

$$(r^2 - 3r + 2)a_0 + \sum_{n=1}^{\infty} [(n+r)(n+r-3) + 2]a_n x^{n+r} + \sum_{n=1}^{\infty} (n+r-1)a_{n-1} x^{n+r} = 0$$

$$(r^2 - 3r + 2)a_0 + \sum_{n=1}^{\infty} [((n+r)(n+r-3) + 2)a_n + (n+r-1)a_{n-1}]x^{n+r} = 0$$

For $n = 0$

$$(r^2 - 3r + 2)a_0 = 0$$

$$\Rightarrow \underline{r=1, 2} \quad \checkmark$$

$$((n+r)(n+r-3) + 2)a_n + (n+r-1)a_{n-1} = 0$$

$$a_n = -\frac{n+r-1}{(n+r-1)(n+r-2)}a_{n-1}$$

$$= -\frac{1}{n+r-2}a_{n-1}$$

$$r = 2$$

$$a_n = -\frac{1}{n}a_{n-1}$$

$$n = 1 \rightarrow a_1 = -a_0$$

$$n = 2 \rightarrow a_2 = -\frac{1}{2}a_1 = \frac{1}{2}a_0$$

$$n = 3 \rightarrow a_3 = -\frac{1}{3}a_2 = -\frac{1}{3!}a_0$$

$$n = 4 \rightarrow a_4 = -\frac{1}{4}a_3 = \frac{1}{4!}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 x \left(1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \dots \right)}$$

$$r = 1$$

$$b_n = -\frac{1}{n-1} b_{n-1}$$

Since $n \neq 1$

$$y(x) = a_0 x \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \dots \right) + b_0 x \ln x \left(-x + x^2 - \frac{x^3}{2} + \frac{x^4}{6} - \dots \right) + x \ln x \left(1 - x + \frac{x^3}{4} - \frac{5}{36} x^4 + \dots \right)$$

Exercise

Find the Frobenius series solutions of $x^2 y'' + (x^2 + 2x) y' - 2y = 0$

Solution

$$\frac{1}{x^2} x^2 y'' + \frac{1}{x^2} (x^2 + 2x) y' - 2 \frac{1}{x^2} y = 0$$

$$y'' + \left(1 + \frac{2}{x} \right) y' - \frac{2}{x^2} y = 0$$

That implies to $p(x) = 1 + \frac{2}{x}$ and $q(x) = -\frac{2}{x^2}$.

$$\begin{aligned} p_0 &= \lim_{x \rightarrow 0} x p(x) \\ &= \lim_{x \rightarrow 0} x \left(1 + \frac{2}{x} \right) \\ &= \lim_{x \rightarrow 0} (x + 2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x \rightarrow 0} x^2 q(x) \\ &= - \lim_{x \rightarrow 0} x^2 \frac{2}{x^2} \\ &= -2 \end{aligned}$$

The indicial equation is: $r(r-1) + 2r - 2 = r^2 + r - 2 = 0$

$$\rightarrow r = 1, -2$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x^2 y'' + (x^2 + 2x) y' - 2y = 0$$

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + (x^2 + 2x) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - 2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r+1} + \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r} - 2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + 2(n+r) - 2] a_n x^{n+r} + \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^{n+r} = 0$$

$$(r^2 + r - 2) a_0 + \sum_{n=1}^{\infty} [(n+r)(n+r-1) - 2] a_n x^{n+r} + \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^{n+r} = 0$$

$$(r^2 + r - 2) a_0 + \sum_{n=0}^{\infty} [((n+r)(n+r-1) - 2) a_n + (n+r-1) a_{n-1}] x^{n+r} = 0$$

For $n = 0$

$$(r^2 + r - 2) a_0 = 0$$

$$\Rightarrow \underline{r=1, 2} \quad \checkmark$$

$$((n+r)^2 + (n+r) - 2) a_n + (n+r-1) a_{n-1} = 0$$

$$a_n = -\frac{n+r-1}{(n+r-1)(n+r+2)} a_{n-1}$$

$$= -\frac{1}{n+r+2} a_{n-1}$$

$$r = 1$$

$$a_n = -\frac{1}{n+3}a_{n-1}$$

$$n=1 \rightarrow a_1 = -\frac{1}{4}a_0$$

$$n=2 \rightarrow a_2 = -\frac{1}{5}a_1 = \frac{1}{20}a_0$$

$$n=3 \rightarrow a_3 = -\frac{1}{6}a_2 = -\frac{1}{120}a_0$$

$$n=4 \rightarrow a_4 = -\frac{1}{7}a_3 = \frac{1}{840}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 x \left(1 - \frac{1}{4}x + \frac{1}{20}x^2 - \frac{1}{120}x^3 + \frac{1}{840}x^4 - \dots \right) \Big|}$$

$$r=2$$

$$b_n = -\frac{1}{n+4}b_{n-1}$$

$$n=1 \rightarrow b_1 = -\frac{1}{5}b_0$$

$$n=2 \rightarrow b_2 = -\frac{1}{6}b_1 = \frac{1}{30}b_0$$

$$n=3 \rightarrow b_3 = -\frac{1}{7}b_2 = -\frac{1}{210}b_0$$

$$n=4 \rightarrow b_4 = -\frac{1}{8}b_3 = \frac{1}{1,680}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_2(x) = b_0 x^2 \left(1 - \frac{x}{5} + \frac{x^2}{30} - \frac{x^3}{210} + \frac{x^4}{1,680} - \dots \right) \Big|}$$

$$\underline{y(x) = a_0 x \left(1 - \frac{x}{4} + \frac{x^2}{20} - \frac{x^3}{120} + \frac{x^4}{840} - \dots \right) + b_0 x^2 \left(1 - \frac{x}{5} + \frac{x^2}{30} - \frac{x^3}{210} + \frac{x^4}{1,680} - \dots \right) \Big|}$$

Exercise

Find the Frobenius series solutions of $2xy'' + 3y' - y = 0$

Solution

$$\frac{x}{2}2xy'' + 3\frac{x}{2}y' - \frac{x}{2}y = 0$$

$$x^2y'' + \frac{3}{2}xy' - \frac{1}{2}xy = 0$$

$$y'' + \frac{3}{2x}y' - \frac{1}{2x}y = 0$$

That implies to $p(x) = \frac{3}{2x}$ and $q(x) = -\frac{1}{2x}$

$$\begin{aligned} p_0 &= \lim_{x \rightarrow 0} xp(x) \\ &= \lim_{x \rightarrow 0} x \frac{3}{2x} \\ &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x \rightarrow 0} x^2 q(x) \\ &= \lim_{x \rightarrow 0} x^2 \frac{1}{2x} \\ &= 0 \end{aligned}$$

The indicial equation is: $r(r-1) + \frac{3}{2}r = r^2 + \frac{1}{2}r = 0$

$$\rightarrow r = 0, -\frac{1}{2}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' + 3y' - y = 0$$

$$2x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+r)(n+r-1)a_n + 3(n+r) \right] x^{n+r-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$r(2r+1)a_0 + \sum_{n=1}^{\infty} (n+r)(2n+2r+1)a_n x^{n+r-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$r(2r+1)a_0 + \sum_{n=1}^{\infty} \left[(n+r)(2n+2r+1)a_n - a_{n-1} \right] x^{n+r-1} = 0$$

For $n = 0$

$$r(2r+1)a_0 = 0$$

$$\Rightarrow \underline{r = 0, -\frac{1}{2}} \quad \checkmark$$

$$(n+r)(2n+2r+1)a_n - a_{n-1} = 0$$

$$\underline{a_n = \frac{1}{(n+r)(2n+2r+1)} a_{n-1}}$$

$$r = 0$$

$$a_n = \frac{1}{n(2n+1)} a_{n-1}$$

$$n = 1 \rightarrow a_1 = \frac{1}{3} a_0$$

$$n = 2 \rightarrow a_2 = \frac{1}{15} a_1 = \frac{1}{30} a_0$$

$$n = 3 \rightarrow a_3 = \frac{1}{21} a_2 = \frac{1}{630} a_0$$

$$n = 4 \rightarrow a_4 = \frac{1}{36} a_3 = \frac{1}{22,680} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 \left(1 + \frac{1}{3}x + \frac{1}{30}x^2 + \frac{1}{630}x^3 + \frac{1}{22,680}x^4 + \dots \right)}$$

$$r = -\frac{1}{2}$$

$$b_n = \frac{1}{n(2n-1)} b_{n-1}$$

$$n = 1 \rightarrow b_1 = b_0$$

$$n = 2 \rightarrow b_2 = \frac{1}{6} b_1 = \frac{1}{6} b_0$$

$$n=3 \rightarrow b_3 = \frac{1}{15}b_2 = \frac{1}{90}b_0$$

$$n=4 \rightarrow b_4 = \frac{1}{28}b_3 = \frac{1}{2,520}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{-1/2} \left(1 + x + \frac{1}{6}x^2 + \frac{1}{90}x^3 + \frac{1}{2,520}x^4 + \dots \right)$$

$$y(x) = a_0 \left(1 + \frac{1}{3}x + \frac{1}{30}x^2 + \frac{1}{630}x^3 + \frac{1}{22,680}x^4 + \dots \right) + b_0 x^{-1/2} \left(1 + x + \frac{1}{6}x^2 + \frac{1}{90}x^3 + \frac{1}{2,520}x^4 + \dots \right)$$

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!(2n+1)!!} + \frac{b_0}{\sqrt{x}} \left(1 + \sum_{n=0}^{\infty} \frac{x^n}{n!(2n-1)!!} \right)$$

Exercise

Find the Frobenius series solutions of $2xy'' - y' - y = 0$

Solution

$$\frac{1}{2x} 2xy'' - \frac{1}{2x} y' - \frac{1}{2x} y = 0$$

$$y'' - \frac{1}{2x} y' - \frac{1}{2x} y = 0$$

That implies to $p(x) = -\frac{1}{2x}$ and $q(x) = \frac{1}{2x}$

$$p_0 = \lim_{x \rightarrow 0} xp(x)$$

$$= \lim_{x \rightarrow 0} x \frac{-1}{2x}$$

$$= -\frac{1}{2}$$

$$q_0 = \lim_{x \rightarrow 0} x^2 q(x)$$

$$= \lim_{x \rightarrow 0} x^2 \frac{-1}{2x}$$

$$= -\lim_{x \rightarrow 0} \frac{x}{2}$$

$$= 0$$

The indicial equation is: $r(r-1) - \frac{1}{2}r = r^2 - \frac{3}{2}r = 0$

$$\rightarrow r = 0, \frac{3}{2}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{3/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' - y' - y = 0$$

$$2x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [2(n+r)(n+r-1) a_n - (n+r) a_n] x^{n+r-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$r(2r-3) a_0 + \sum_{n=1}^{\infty} (n+r)(2n+2r-3) a_n x^{n+r-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$r(2r-3) a_0 + \sum_{n=1}^{\infty} [(n+r)(2n+2r-3) a_n - a_{n-1}] x^{n+r-1} = 0$$

For $n = 0$

$$r(2r-3) a_0 = 0$$

$$\Rightarrow \underline{r = 0, \frac{3}{2}} \quad \checkmark$$

$$(n+r)(2n+2r-3) a_n - a_{n-1} = 0$$

$$\underline{a_n = \frac{1}{(n+r)(2n+2r-3)} a_{n-1}}$$

$$r = 0$$

$$a_n = \frac{1}{n(2n-3)} a_{n-1}$$

$$n = 1 \rightarrow a_1 = -a_0$$

$$n = 2 \rightarrow a_2 = \frac{1}{2}a_1 = -\frac{1}{2}a_0$$

$$n = 3 \rightarrow a_3 = \frac{1}{9}a_2 = -\frac{1}{18}a_0$$

$$n = 4 \rightarrow a_4 = \frac{1}{20}a_3 = -\frac{1}{360}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 \left(1 - x - \frac{1}{2}x^2 - \frac{1}{18}x^3 - \frac{1}{360}x^4 - \dots \right) \quad |}$$

$$r = \frac{3}{2}$$

$$b_n = \frac{1}{n(2n+3)} b_{n-1}$$

$$n = 1 \rightarrow b_1 = \frac{1}{5}b_0$$

$$n = 2 \rightarrow b_2 = \frac{1}{14}b_1 = \frac{1}{70}b_0$$

$$n = 3 \rightarrow b_3 = \frac{1}{27}b_2 = \frac{1}{1890}b_0$$

$$n = 4 \rightarrow b_4 = \frac{1}{44}b_3 = \frac{1}{83,160}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_2(x) = b_0 x^{3/2} \left(1 + \frac{1}{5}x + \frac{1}{70}x^2 + \frac{1}{1890}x^3 + \frac{1}{83,160}x^4 + \dots \right) \quad |}$$

$$\underline{y(x) = a_0 \left(1 - x - \frac{1}{2}x^2 - \frac{1}{18}x^3 - \frac{1}{360}x^4 + \dots \right) + b_0 x^{3/2} \left(1 + \frac{1}{5}x + \frac{1}{70}x^2 + \frac{1}{1,890}x^3 + \frac{1}{83,160}x^4 - \dots \right) \quad |}$$

Exercise

Find the Frobenius series solutions of $2xy'' + (1+x)y' + y = 0$

Solution

$$\frac{1}{2x} 2xy'' + \frac{1}{2x} (1+x)y' + \frac{1}{2x} y = 0$$

$$y'' + \left(\frac{1}{2x} + \frac{1}{2} \right) y' + \frac{1}{2x} y = 0$$

That implies to $p(x) = \frac{1}{2x} + \frac{1}{2}$ and $q(x) = \frac{1}{2x}$

$$\begin{aligned} p_0 &= \lim_{x \rightarrow 0} xp(x) \\ &= \lim_{x \rightarrow 0} x \left(\frac{1}{2x} + \frac{1}{2} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{1}{2} + \frac{1}{2}x \right) \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x \rightarrow 0} x^2 q(x) \\ &= \lim_{x \rightarrow 0} x^2 \frac{1}{2x} \\ &= \lim_{x \rightarrow 0} \frac{x}{2} \\ &= 0 \end{aligned}$$

The indicial equation is: $r(r-1) + \frac{1}{2}r = r^2 - \frac{1}{2}r = 0$

$$\rightarrow \underline{r = 0, \frac{1}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' + (1+x)y' + y = 0$$

$$2x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + (1+x) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [2(n+r)(n+r-1) + (n+r)]a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r+1)a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(2n+2r-1)]a_n x^{n+r-1} + \sum_{n=1}^{\infty} (n+r)a_{n-1} x^{n+r-1} = 0$$

$$r(2r-1)a_0 + \sum_{n=1}^{\infty} (n+r)(2n+2r-1)a_n x^{n+r-1} + \sum_{n=1}^{\infty} (n+r)a_{n-1} x^{n+r-1} = 0$$

$$r(2r-1)a_0 + \sum_{n=1}^{\infty} [(n+r)(2n+2r-1)a_n + (n+r)a_{n-1}] x^{n+r-1} = 0$$

For $n=0$

$$r(2r-1)a_0 = 0$$

$$\Rightarrow \underline{r=0, \frac{1}{2}} \quad \checkmark$$

$$(n+r)(2n+2r-1)a_n + (n+r)a_{n-1} = 0$$

$$\underline{a_n = -\frac{1}{2n+2r-1}a_{n-1}}$$

$$r=0$$

$$a_n = -\frac{1}{2n-1}a_{n-1}$$

$$n=1 \rightarrow a_1 = -a_0$$

$$n=2 \rightarrow a_2 = \frac{1}{3}a_1 = -\frac{1}{3}a_0$$

$$n=3 \rightarrow a_3 = -\frac{1}{5}a_2 = \frac{1}{15}a_0$$

$$n=4 \rightarrow a_4 = -\frac{1}{7}a_3 = -\frac{1}{105}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 \left(1 - x + \frac{1}{3}x^2 - \frac{1}{15}x^3 + \frac{1}{105}x^4 - \dots \right) \quad |}$$

$$r = \frac{1}{2}$$

$$b_n = -\frac{1}{2n}b_{n-1}$$

$$n = 1 \rightarrow b_1 = -\frac{1}{2}b_0$$

$$n = 2 \rightarrow b_2 = -\frac{1}{4}b_1 = \frac{1}{8}b_0$$

$$n = 3 \rightarrow b_3 = -\frac{1}{6}b_2 = -\frac{1}{48}b_0$$

$$n = 4 \rightarrow b_4 = -\frac{1}{8}b_3 = -\frac{1}{384}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_2(x) = b_0 x^{1/2} \left(1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4 + \dots \right) \quad |}$$

$$\underline{y(x) = a_0 \left(1 - x + \frac{1}{3}x^2 - \frac{1}{15}x^3 + \frac{1}{105}x^4 - \dots \right) + b_0 x^{1/2} \left(1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4 + \dots \right) \quad |}$$

Exercise

Find the Frobenius series solutions of $2xy'' + (1 - 2x^2)y' - 4xy = 0$

Solution

$$\frac{x}{2}2xy'' + \frac{x}{2}(1 - 2x^2)y' - \frac{x}{2}4xy = 0$$

$$x^2y'' + \left(\frac{1}{2}x - x^3\right)y' + 2x^2y = 0$$

$$y'' + \left(\frac{1}{2x} - x\right)y' + 2y = 0$$

That implies to $p(x) = \frac{1}{2x} - x$ and $q(x) = 2$

$$p_0 = \lim_{x \rightarrow 0} xp(x)$$

$$= \lim_{x \rightarrow 0} x\left(\frac{1}{2x} - x\right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{1}{2} - x^2\right)$$

$$= \frac{1}{2}$$

$$\begin{aligned}
 q_0 &= \lim_{x \rightarrow 0} x^2 q(x) \\
 &= \lim_{x \rightarrow 0} 2x^2 \\
 &= 0
 \end{aligned}$$

The indicial equation is: $r(r-1) + \frac{1}{2}r = r^2 - \frac{1}{2}r = 0$

$$\rightarrow \underline{r=0, \frac{1}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' + (1-2x^2)y' - 4xy = 0$$

$$2x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + (1-2x^2) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - 4x \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\begin{aligned}
 &\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r+1} \\
 &\quad - \sum_{n=0}^{\infty} 4a_n x^{n+r+1} = 0
 \end{aligned}$$

$$\sum_{n=0}^{\infty} [(n+r)(2n+2r-2) + (n+r)] a_n x^{n+r-1} - \sum_{n=0}^{\infty} (2n+2r+4) a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(2n+2r-2) + (n+r)] a_n x^{n+r-1} - \sum_{n=2}^{\infty} (2n+2r) a_{n-2} x^{n+r-1} = 0$$

$$\begin{aligned}
 & r(2r-1)a_0 + (r+2)(2r+1)a_1 + \sum_{n=2}^{\infty} (n+r)(2n+2r-1)a_n x^{n+r-1} \\
 & - \sum_{n=2}^{\infty} (2n+2r)a_{n-2} x^{n+r-1} = 0 \\
 & r(2r-1)a_0 + (r+2)(2r+1)a_1 + \sum_{n=2}^{\infty} \left[(n+r)(2n+2r-1)a_n - 2(n+r)a_{n-2} \right] x^{n+r-1} = 0
 \end{aligned}$$

For $n=0$

$$\begin{aligned}
 & r(2r+1)a_0 = 0 \\
 & \Rightarrow \underline{r=0, -\frac{1}{2}} \quad \checkmark
 \end{aligned}$$

For $n=1$

$$\begin{aligned}
 & (r+2)(2r+1)a_1 = 0 \\
 & \Rightarrow \underline{\cancel{r=-2, -\frac{1}{2}}} \\
 & \underline{a_1 = 0}
 \end{aligned}$$

$$(n+r)(2n+2r-1)a_n - 2(n+r)a_{n-2} = 0$$

$$\underline{a_n = \frac{2}{2n+2r-1}a_{n-2}}$$

$$r=0$$

$$a_n = \frac{2}{2n-1}a_{n-2}$$

$$n=2 \rightarrow a_2 = \frac{2}{3}a_0$$

$$n=3 \rightarrow a_3 = \frac{2}{5}a_1 = 0$$

$$n=4 \rightarrow a_4 = \frac{2}{7}a_2 = \frac{4}{21}a_0$$

$$n=5 \rightarrow a_5 = \frac{2}{9}a_3 = 0$$

$$n=6 \rightarrow a_6 = \frac{2}{11}a_4 = \frac{8}{231}a_0$$

$$n=7 \rightarrow a_7 = 0$$

$$n=8 \rightarrow a_8 = \frac{2}{15}a_6 = \frac{16}{3,465}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 \left(1 + \frac{2}{3}x^2 + \frac{4}{21}x^4 + \frac{8}{231}x^6 + \frac{16}{3465}x^8 + \dots \right)}$$

$$r = \frac{1}{2}$$

$$b_n = \frac{1}{n} b_{n-2}$$

$$n=2 \rightarrow b_2 = \frac{1}{2} b_0$$

$$n=3 \rightarrow b_3 = \frac{1}{3} b_1 = 0$$

$$n=4 \rightarrow b_4 = \frac{1}{4} b_2 = \frac{1}{8} b_0$$

$$n=5 \rightarrow b_5 = \frac{1}{5} b_3 = 0$$

$$n=6 \rightarrow b_6 = \frac{1}{6} b_4 = \frac{1}{48} b_0$$

$$n=7 \rightarrow b_7 = 0$$

$$n=8 \rightarrow b_8 = \frac{1}{8} b_6 = \frac{1}{384} b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{1/2} \left(1 + \frac{1}{2} x^2 + \frac{1}{8} x^4 + \frac{1}{48} x^6 + \frac{1}{384} x^8 + \dots \right)$$

$$y(x) = a_0 \left(1 + \frac{2}{3} x^2 + \frac{4}{21} x^4 + \frac{8}{231} x^6 + \frac{16}{3465} x^8 + \dots \right) + b_0 x^{1/2} \left(1 + \frac{1}{2} x^2 + \frac{1}{8} x^4 + \frac{1}{48} x^6 + \frac{1}{384} x^8 + \dots \right)$$

Exercise

Find the Frobenius series solutions of $2x^2 y'' + xy' - (1 + 2x^2)y = 0$

Solution

$$\frac{1}{2} 2x^2 y'' + \frac{1}{2} xy' - \frac{1}{2} (1 + 2x^2)y = 0$$

$$x^2 y'' + \frac{1}{2} xy' - \frac{1}{2} (1 + 2x^2)y = 0$$

$$y'' + \frac{1}{2x} y' - \left(\frac{1}{2x^2} + 1 \right) y = 0$$

That implies to $p(x) = \frac{1}{2x}$ and $q(x) = -\frac{1}{2x^2} - 1$

$$p_0 = \lim_{x \rightarrow 0} xp(x)$$

$$= \lim_{x \rightarrow 0} x \frac{1}{2x}$$

$$= \lim_{x \rightarrow 0} \frac{1}{2}$$

$$= \frac{1}{2}$$

$$q_0 = \lim_{x \rightarrow 0} x^2 q(x)$$

$$\begin{aligned}
&= \lim_{x \rightarrow 0} x^2 \left(-\frac{1}{2x^2} - 1 \right) \\
&= \lim_{x \rightarrow 0} x^2 \left(-\frac{1}{2} - x^2 \right) \\
&= -\frac{1}{2}
\end{aligned}$$

The indicial equation is: $r(r-1) + \frac{1}{2}r - \frac{1}{2} = r^2 - \frac{1}{2}r - \frac{1}{2} = 0$

$$\rightarrow \underline{r=1, -\frac{1}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^1 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2 y'' + xy' - (1 + 2x^2)y = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - (1 + 2x^2) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} 2a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(2n+2r-2) + (n+r) - 1] a_n x^{n+r} - \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r} = 0$$

$$\begin{aligned}
& \left(2r^2 - r - 1\right)a_0 + \left((r+1)(2r+1) - 1\right)a_1 + \sum_{n=2}^{\infty} \left[(n+r)(2n+2r-1) - 1\right]a_n x^{n+r} \\
& - \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r} = 0 \\
& \left(2r^2 - r - 1\right)a_0 + \left(2r^2 + 3r\right)a_1 + \sum_{n=2}^{\infty} \left[\left((n+r)(2n+2r-1) - 1\right)a_n - 2a_{n-2}\right]x^{n+r} = 0
\end{aligned}$$

For $n = 0$

$$\begin{aligned}
& \left(2r^2 - r - 1\right)a_0 = 0 \\
& \Rightarrow \underline{r=1, -\frac{1}{2}} \quad \checkmark
\end{aligned}$$

For $n = 1$

$$\begin{aligned}
& r(2r+3)a_1 = 0 \\
& \Rightarrow \underline{\cancel{r=0, -\frac{3}{2}}} \\
& \underline{a_1 = 0}
\end{aligned}$$

$$\left((n+r)(2n+2r-1) - 1\right)a_n - 2a_{n-2} = 0$$

$$\underline{a_n = \frac{2}{(n+r)(2n+2r-1) - 1} a_{n-2}}$$

 $r = 1$

$$\begin{aligned}
a_n &= \frac{2}{(n+1)(2n+1) - 1} a_{n-2} \\
&= \frac{2}{2n^2 + 3n} a_{n-2}
\end{aligned}$$

$$n = 2 \rightarrow a_2 = \frac{2}{14}a_0 = \frac{1}{7}a_0$$

$$n = 3 \rightarrow a_3 = \frac{2}{27}a_1 = 0$$

$$n = 4 \rightarrow a_4 = \frac{2}{44}a_2 = \frac{1}{154}a_0$$

$$n = 5 \rightarrow a_5 = \frac{2}{65}a_3 = 0$$

$$n = 6 \rightarrow a_6 = \frac{2}{90}a_4 = \frac{1}{6,390}a_0$$

$$n = 7 \rightarrow a_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 x \left(1 + \frac{1}{7}x^2 + \frac{1}{154}x^4 + \frac{1}{6,930}x^6 + \dots \right)}$$

$$r = -\frac{1}{2}$$

$$\begin{aligned} b_n &= \frac{2b_{n-2}}{\left(n - \frac{1}{2}\right)(2n-2) - 1} \\ &= \frac{2}{2n^2 - 3n} b_{n-2} \end{aligned}$$

$$n = 2 \rightarrow b_2 = \frac{2}{2} b_0 = b_0$$

$$n = 3 \rightarrow b_3 = \frac{2}{9} b_1 = 0$$

$$n = 4 \rightarrow b_4 = \frac{2}{20} b_2 = \frac{1}{10} b_0$$

$$n = 5 \rightarrow b_5 = \frac{2}{35} b_3 = 0$$

$$n = 6 \rightarrow b_6 = \frac{2}{54} b_4 = \frac{1}{270} b_0$$

$$n = 7 \rightarrow b_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{-1/2} \left(1 + x^2 + \frac{1}{10} x^4 + \frac{1}{270} x^6 + \cdots \right)$$

$$y(x) = a_0 x \left(1 + \frac{1}{7} x^2 + \frac{1}{154} x^4 + \frac{1}{6,930} x^6 + \cdots \right) + b_0 x^{-1/2} \left(1 + x^2 + \frac{1}{10} x^4 + \frac{1}{270} x^6 + \cdots \right)$$

Exercise

Find the Frobenius series solutions of $2x^2 y'' + xy' - (3 - 2x^2)y = 0$

Solution

$$\frac{1}{2x^2} 2x^2 y'' + \frac{1}{2x^2} xy' - \frac{1}{2x^2} (3 - 2x^2)y = 0$$

$$y'' + \frac{1}{2x} y' - \frac{1}{2x^2} (3 - 2x^2)y = 0$$

That implies to $p(x) = \frac{1}{2x}$ and $q(x) = -\frac{1}{2x^2} (3 - 2x^2)$

$$p_0 = \lim_{x \rightarrow 0} xp(x)$$

$$= \lim_{x \rightarrow 0} x \frac{1}{2x}$$

$$= \frac{1}{2}$$

$$q_0 = \lim_{x \rightarrow 0} x^2 q(x)$$

$$= - \lim_{x \rightarrow 0} x^2 \left(\frac{1}{2x^2} (3 - 2x^2) \right)$$

$$= -\lim_{x \rightarrow 0} \left(\frac{1}{2} (3 - 2x^2) \right)$$

$$= \frac{3}{2}$$

The indicial equation is: $r(r-1) + \frac{1}{2}r - \frac{3}{2} = r^2 - \frac{1}{2}r - \frac{3}{2} = 0$

$$\rightarrow \underline{r = -1, \frac{3}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{-1} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{3/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2 y'' + xy' - (3 - 2x^2)y = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - (3 - 2x^2) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} 3a_n x^{n+r} + \sum_{n=0}^{\infty} 2a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(2n+2r-2) + (n+r) - 3] a_n x^{n+r} + \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r} = 0$$

$$(2r^2 - r - 3)a_0 + ((r+1)(2r+1) - 3)a_1 + \sum_{n=2}^{\infty} [(n+r)(2n+2r-1) - 3] a_n x^{n+r}$$

$$+ \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r} = 0$$

$$(2r^2 - r - 3)a_0 + (2r^2 + 3r - 2)a_1 + \sum_{n=2}^{\infty} \left[((n+r)(2n+2r-1) - 3)a_n + 2a_{n-2} \right] x^{n+r} = 0$$

For $n=0$

$$(2r^2 - r - 3)a_0 = 0$$

$$\Rightarrow \underline{r = -1, \frac{3}{2}} \quad \checkmark$$

For $n=1$

$$(2r^2 + 3r - 2)a_1 = 0$$

$$\Rightarrow \underline{\cancel{r = 2, \frac{1}{2}}}$$

$$\underline{a_1 = 0}$$

$$((n+r)(2n+2r-1) - 3)a_n + 2a_{n-2} = 0$$

$$\underline{a_n = -\frac{2}{(n+r)(2n+2r-1) - 3} a_{n-2}}$$

$$r = -1$$

$$a_n = -\frac{2}{(n-1)(2n-3) - 3} a_{n-2}$$

$$= -\frac{2}{2n^2 - 5n} a_{n-2}$$

$$n=2 \rightarrow a_2 = a_0$$

$$n=4 \rightarrow a_4 = -\frac{2}{12} a_2 = -\frac{1}{6} a_0$$

$$n=6 \rightarrow a_6 = -\frac{2}{252} a_4 = \frac{1}{126} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n=3 \rightarrow a_3 = -\frac{2}{3} a_1 = 0$$

$$n=5 \rightarrow a_5 = -\frac{2}{25} a_3 = 0$$

$$n=7 \rightarrow a_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 x^{-1} \left(1 + x^2 - \frac{1}{6} x^4 + \frac{1}{126} x^6 - \dots \right)}$$

$$r = \frac{3}{2}$$

$$b_n = -\frac{2b_{n-2}}{\left(n + \frac{3}{2}\right)(2n+2) - 3}$$

$$= -\frac{2}{2n^2 + 5n} b_{n-2}$$

$$n=2 \rightarrow b_2 = -\frac{2}{18}b_0 = -\frac{1}{9}b_0$$

$$n=3 \rightarrow b_3 = -\frac{2}{33}b_1 = 0$$

$$n=4 \rightarrow b_4 = -\frac{2}{568}b_2 = \frac{1}{234}b_0$$

$$n=5 \rightarrow b_5 = 0$$

$$n=6 \rightarrow b_6 = \frac{2}{102}b_4 = \frac{1}{11,934}b_0$$

$$n=7 \rightarrow b_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{3/2} \left(1 - \frac{1}{9}x^2 + \frac{1}{234}x^4 - \frac{1}{11,934}x^6 + \dots \right)$$

$$y(x) = a_0 x^{-1} \left(1 + x^2 - \frac{1}{6}x^4 + \frac{1}{126}x^6 - \dots \right) + b_0 x^{3/2} \left(1 - \frac{1}{9}x^2 + \frac{1}{234}x^4 - \frac{1}{11,934}x^6 + \dots \right)$$

Exercise

Find the Frobenius series solutions of $3xy'' + 2y' + 2y = 0$

Solution

$$\frac{x}{3}3xy'' + 2\frac{x}{3}y' + 2\frac{x}{3}y = 0$$

$$x^2y'' + \frac{2}{3}xy' + \frac{2}{3}xy = 0$$

$$y'' + \frac{2}{3x}y' + \frac{2}{3x}y = 0$$

That implies to $p(x) = \frac{2}{3x}$ and $q(x) = \frac{2}{3x}$

$$p_0 = \lim_{x \rightarrow 0} xp(x)$$

$$= \lim_{x \rightarrow 0} x \frac{2}{3x}$$

$$= \frac{2}{3}$$

$$q_0 = \lim_{x \rightarrow 0} x^2 q(x)$$

$$= \lim_{x \rightarrow 0} x^2 \frac{2}{3x}$$

$$= \lim_{x \rightarrow 0} \frac{2}{3}x$$

$$= 0$$

The indicial equation is: $r(r-1) + \frac{2}{3}r = r^2 - \frac{1}{3}r = 0$

$$\rightarrow \underline{r = 0, \frac{1}{3}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/3} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$3xy'' + 2y' + 2y = 0$$

$$3x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 2 \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + 2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(3n+3r-3) + 2(n+r)] a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1} = 0$$

$$r(3r-1)a_n + \sum_{n=1}^{\infty} (n+r)(3n+3r-1) a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1} = 0$$

$$r(3r-1)a_n + \sum_{n=1}^{\infty} [(n+r)(3n+3r-1) a_n + 2a_{n-1}] x^{n+r-1} = 0$$

For $n=0$

$$r(3r-1)a_0 = 0$$

$$\Rightarrow \underline{r=0, \frac{1}{3}} \quad \checkmark$$

$$(n+r)(3n+3r-1)a_n + 2a_{n-1} = 0$$

$$\underline{a_n = -\frac{2}{(n+r)(3n+3r-1)} a_{n-1}}$$

$$r = 0$$

$$a_n = -\frac{2}{3n^2 - n} a_{n-1}$$

$$n = 1 \rightarrow a_1 = -a_0$$

$$n = 2 \rightarrow a_2 = -\frac{1}{5}a_1 = \frac{1}{5}a_0$$

$$n = 3 \rightarrow a_3 = -\frac{2}{24}a_2 = -\frac{1}{60}a_0$$

$$n = 4 \rightarrow a_4 = -\frac{2}{44}a_3 = \frac{1}{1320}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 x^0 \left(1 - x + \frac{1}{5}x^2 - \frac{1}{60}x^3 + \frac{1}{1320}x^4 - \dots \right) \quad |}$$

$$r = \frac{1}{3}$$

$$b_n = -\frac{2}{3n^2 + n} b_{n-1}$$

$$n = 1 \rightarrow b_1 = -\frac{1}{2}b_0$$

$$n = 2 \rightarrow b_2 = -\frac{2}{14}b_1 = \frac{1}{14}b_0$$

$$n = 3 \rightarrow b_3 = -\frac{2}{30}b_2 = -\frac{1}{210}b_0$$

$$n = 4 \rightarrow b_4 = -\frac{2}{52}b_3 = \frac{1}{5460}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_2(x) = b_0 x^{1/3} \left(1 - \frac{1}{2}x + \frac{1}{14}x^2 - \frac{1}{210}x^3 + \frac{1}{5460}x^4 - \dots \right) \quad |}$$

$$\underline{y(x) = a_0 \left(1 - x + \frac{1}{5}x^2 - \frac{1}{60}x^3 + \frac{1}{1320}x^4 - \dots \right) + b_0 x^{1/3} \left(1 - \frac{1}{2}x + \frac{1}{14}x^2 - \frac{1}{210}x^3 + \frac{1}{5460}x^4 - \dots \right) \quad |}$$

Exercise

Find the Frobenius series solutions of $3x^2 y'' + 2xy' + x^2 y = 0$

Solution

$$\frac{1}{3}3x^2 y'' + 2\frac{1}{3}xy' + \frac{1}{3}x^2 y = 0$$

$$x^2 y'' + \frac{2}{3}xy' + \frac{1}{3}x^2 y = 0$$

$$y'' + \frac{2}{3x}y' + \frac{1}{3}y = 0$$

That implies to $p(x) = \frac{2}{3x}$ and $q(x) = \frac{1}{3}$.

$$\begin{aligned} p_0 &= \lim_{x \rightarrow 0} xp(x) \\ &= \lim_{x \rightarrow 0} x \frac{2}{3x} \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x \rightarrow 0} x^2 q(x) \\ &= \lim_{x \rightarrow 0} x^2 \frac{1}{3} \\ &= 0 \end{aligned}$$

The indicial equation is: $r(r-1) + \frac{2}{3}r = r^2 - \frac{1}{3}r = 0$

$$\rightarrow \underline{r = 0, \frac{1}{3}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/3} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$3x^2 y'' + 2xy' + x^2 y = 0$$

$$3x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 2x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + x^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(3n+3r-3)+2(n+r)]a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$r(3r-1)a_0 + (r+1)(3r+2)a_1 + \sum_{n=2}^{\infty} (n+r)(3n+3r-1)a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$r(3r-1)a_0 + (r+1)(3r+2)a_1 + \sum_{n=2}^{\infty} [(n+r)(3n+3r-1)a_n + a_{n-2}] x^{n+r} = 0$$

For $n=0$

$$r(3r-1)a_0 = 0$$

$$\Rightarrow \underline{r=0, \frac{1}{3}} \quad \checkmark$$

For $n=1$

$$(r+1)(3r+2)a_1 = 0$$

$$\Rightarrow \underline{\cancel{r=-1, -\frac{2}{3}}}$$

$$\underline{a_1 = 0}$$

$$(n+r)(3n+3r-1)a_n + a_{n-2} = 0$$

$$\underline{a_n = -\frac{1}{(n+r)(3n+3r-1)}a_{n-2}}$$

$$r=0$$

$$a_n = -\frac{1}{n(3n-1)}a_{n-2}$$

$$n=2 \rightarrow a_2 = -\frac{1}{10}a_0$$

$$n=3 \rightarrow a_3 = -\frac{1}{24}a_1 = 0$$

$$n=4 \rightarrow a_4 = -\frac{1}{44}a_2 = \frac{1}{440}a_0$$

$$n=5 \rightarrow a_5 = -\frac{1}{70}a_3 = 0$$

$$n=6 \rightarrow a_6 = -\frac{1}{102}a_4 = -\frac{1}{44,880}a_0$$

$$n=7 \rightarrow a_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 x^0 \left(1 - \frac{1}{10}x^2 + \frac{1}{440}x^4 - \frac{1}{44,880}x^6 + \dots \right)}$$

$$r = \frac{1}{3}$$

$$b_n = -\frac{1}{\left(n + \frac{1}{3}\right)(3n)} b_{n-2}$$

$$= -\frac{1}{n(3n+1)} b_{n-2}$$

$$n=2 \rightarrow b_2 = -\frac{1}{14} b_0 \qquad n=3 \rightarrow b_3 = -\frac{1}{30} b_1 = 0$$

$$n=4 \rightarrow b_4 = -\frac{1}{72} b_2 = \frac{1}{728} b_0 \qquad n=5 \rightarrow b_5 = 0$$

$$n=6 \rightarrow b_6 = -\frac{1}{114} b_4 = -\frac{1}{82,992} b_0 \qquad n=7 \rightarrow b_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{1/3} \left(1 - \frac{1}{14} x^2 + \frac{1}{728} x^4 - \frac{1}{82,992} x^6 + \cdots \right)$$

$$y(x) = a_0 \left(1 - \frac{1}{10} x^2 + \frac{1}{440} x^4 - \frac{1}{44,880} x^6 + \cdots \right) + b_0 x^{1/3} \left(1 - \frac{1}{14} x^2 + \frac{1}{728} x^4 - \frac{1}{82,992} x^6 + \cdots \right)$$

Exercise

Find the Frobenius series solutions of $3x^2 y'' - xy' + y = 0$

Solution

$$\frac{1}{3} 3x^2 y'' - \frac{1}{3} xy' + \frac{1}{3} y = 0$$

$$x^2 y'' - \frac{1}{3} xy' + \frac{1}{3} y = 0$$

$$y'' - \frac{1}{3x} y' + \frac{1}{3x^2} y = 0$$

That implies to $p(x) = -\frac{1}{3x}$ and $q(x) = \frac{1}{3x^2}$.

$$p_0 = \lim_{x \rightarrow 0} xp(x)$$

$$= -\lim_{x \rightarrow 0} x \frac{1}{3x}$$

$$= -\frac{1}{3}$$

$$q_0 = \lim_{x \rightarrow 0} x^2 q(x)$$

$$= \lim_{x \rightarrow 0} x^2 \frac{1}{3x^2}$$

$$= \frac{1}{3}$$

The indicial equation is: $r(r-1) - \frac{1}{3}r + \frac{1}{3} = r^2 - \frac{4}{3}r + \frac{1}{3} = 0$

$$\rightarrow \underline{r = 1, \frac{1}{3}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/3} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$3x^2 y'' - xy' + y = 0$$

$$3x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(3n+3r-3) - (n+r) + 1] a_n x^{n+r} = 0$$

Since neither of λ , then let assume $a_n = 0, \quad n \geq 1$

$$y_1(x) = x \sum_{n=0}^{\infty} a_n x^n = \underline{a_0 x}$$

$$y_2(x) = x^{1/3} \sum_{n=0}^{\infty} b_n x^n = \underline{b_0 x^{1/3}}$$

$$\underline{y(x) = a_0 x + b_0 x^{1/3}}$$

Exercise

Find the Frobenius series solutions of $4xy'' + 2y' + y = 0$

Solution

$$\frac{x}{4}4xy'' + 2\frac{x}{4}y' + \frac{x}{4}y = 0$$

$$x^2y'' + \frac{1}{2}xy' + \frac{x}{4}y = 0$$

$$y'' + \frac{1}{2x}y' + \frac{1}{4x}y = 0$$

That implies to $p(x) = \frac{1}{2x}$ and $q(x) = \frac{1}{4x}$.

$$\begin{aligned} p_0 &= \lim_{x \rightarrow 0} xp(x) \\ &= \lim_{x \rightarrow 0} x \frac{1}{2x} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x \rightarrow 0} x^2q(x) \\ &= \lim_{x \rightarrow 0} x^2 \frac{1}{4x} \\ &= \lim_{x \rightarrow 0} \frac{x}{4} \\ &= 0 \end{aligned}$$

The indicial equation is: $r(r-1) + \frac{1}{2}r = r^2 - \frac{1}{2}r = 0$

$$\rightarrow \underline{r = 0, \frac{1}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$4xy'' + 2y' + y = 0$$

$$4x \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + 2 \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 4(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(4n+4r-4) + 2(n+r)]a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$r(4r-2)a_n + \sum_{n=1}^{\infty} (n+r)(4n+4r-2)a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$2r(2r-1)a_n + \sum_{n=1}^{\infty} [2(n+r)(2n+2r-1)a_n + a_{n-1}]x^{n+r-1} = 0$$

For $n = 0$

$$2r(2r-1)a_0 = 0$$

$$\Rightarrow \underline{r = 0, \frac{1}{2}} \quad \checkmark$$

$$2(n+r)(2n+2r-1)a_n + a_{n-1} = 0$$

$$\underline{a_n = -\frac{1}{2(n+r)(2n+2r-1)}a_{n-1}}$$

$$r = 0$$

$$a_n = -\frac{1}{2n(2n-1)}a_{n-1}$$

$$n = 1 \rightarrow a_1 = -\frac{1}{2}a_0$$

$$n = 2 \rightarrow a_2 = -\frac{1}{12}a_1 = \frac{1}{24}a_0$$

$$n = 3 \rightarrow a_3 = -\frac{1}{30}a_2 = -\frac{1}{720}a_0$$

$$n = 4 \rightarrow a_4 = -\frac{1}{42}a_3 = \frac{1}{30,240}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 x^0 \left(1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{1}{720}x^3 + \frac{1}{30,240}x^4 - \dots \right)}$$

$$r = \frac{1}{2}$$

$$b_n = -\frac{1}{2\left(n + \frac{1}{2}\right)(2n)} b_{n-1}$$

$$= -\frac{1}{4n^2 + 2n} b_{n-1}$$

$$n = 1 \rightarrow b_1 = -\frac{1}{6} b_0$$

$$n = 2 \rightarrow b_2 = -\frac{1}{20} b_1 = \frac{1}{120} b_0$$

$$n = 3 \rightarrow b_3 = -\frac{1}{42} b_2 = -\frac{1}{5040} b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{1/2} \left(1 - \frac{1}{6}x + \frac{1}{120}x^2 - \frac{1}{5,040}x^3 + \dots \right)$$

$$y(x) = a_0 \left(1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{1}{720}x^3 + \frac{1}{30,240}x^4 - \dots \right) + b_0 x^{1/2} \left(1 - \frac{1}{6}x + \frac{1}{120}x^2 - \frac{1}{5,040}x^3 + \dots \right)$$

Exercise

Find the Frobenius series solutions of $6x^2y'' + 7xy' - (x^2 + 2)y = 0$

Solution

$$\frac{1}{6}6x^2y'' + \frac{1}{6}7xy' - \frac{1}{6}(x^2 + 2)y = 0$$

$$x^2y'' + \frac{7}{6}xy' - \frac{1}{6}(x^2 + 2)y = 0$$

$$y'' + \frac{7}{6x}y' - \frac{1}{6x^2}(x^2 + 2)y = 0$$

That implies to $p(x) = \frac{7}{6x}$ and $q(x) = -\frac{1}{6x^2}(x^2 + 2)$.

$$p_0 = \lim_{x \rightarrow 0} xp(x)$$

$$= \lim_{x \rightarrow 0} x \frac{7}{6x}$$

$$= \frac{7}{6}$$

$$q_0 = \lim_{x \rightarrow 0} x^2 q(x)$$

$$\begin{aligned}
&= -\lim_{x \rightarrow 0} x^2 \frac{1}{6x^2} (x^2 + 2) \\
&= -\lim_{x \rightarrow 0} \left(\frac{1}{6}x^2 + \frac{1}{3} \right) \\
&= -\frac{1}{3}
\end{aligned}$$

The indicial equation is: $r(r-1) + \frac{7}{6}r - \frac{1}{3} = r^2 + \frac{1}{6}r - \frac{1}{3} = 0$

$$6r^2 + r - 2 = 0$$

$$\left. r = \frac{1}{2}, -\frac{2}{3} \right|$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-2/3} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$6x^2 y'' + 7xy' - (x^2 + 2)y = 0$$

$$\begin{aligned}
&6x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 7x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - x^2 \sum_{n=0}^{\infty} a_n x^{n+r} \\
&\quad - 2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0
\end{aligned}$$

$$\sum_{n=0}^{\infty} 6(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} 7(n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [6(n+r)(n+r-1) + 7(n+r) - 2] a_n x^{n+r} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$(6r^2 + r - 2)a_0 + ((r+1)(6r+7)-2)a_1 + \sum_{n=2}^{\infty} [(n+r)(6n+6r+1)-2]a_n x^{n+r}$$

$$- \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$(6r^2 + r - 2)a_0 + (6r^2 + 13r + 5)a_1 + \sum_{n=2}^{\infty} [((n+r)(6n+6r+1)-2)a_n - a_{n-2}]x^{n+r} = 0$$

For $n=0$

$$(6r^2 + r - 2)a_0 = 0$$

$$\Rightarrow \underline{r = \frac{1}{2}, -\frac{2}{3}} \quad \checkmark$$

For $n=1$

$$(6r^2 + 13r + 5)a_1 = 0$$

$$\Rightarrow r = \frac{-13 \pm 7}{12} = \cancel{-\frac{1}{2}}, \cancel{-\frac{5}{3}} \quad \left| \right.$$

$$\underline{a_1 = 0} \quad \left| \right.$$

$$((n+r)(6n+6r+1)-2)a_n - a_{n-2} = 0$$

$$\underline{a_n = \frac{1}{(n+r)(6n+6r+1)-2} a_{n-2}} \quad \left| \right.$$

$$r = \frac{1}{2}$$

$$a_n = \frac{1}{n(6n+7)} a_{n-2}$$

$$n=2 \rightarrow a_2 = \frac{1}{38} a_0$$

$$n=3 \rightarrow a_3 = \frac{1}{75} a_1 = 0$$

$$n=4 \rightarrow a_4 = \frac{1}{124} a_2 = \frac{1}{4,712} a_0$$

$$n=5 \rightarrow a_5 = \frac{1}{185} a_3 = 0$$

$$n=6 \rightarrow a_6 = \frac{1}{258} a_4 = \frac{1}{1,215,696} a_0$$

$$n=7 \rightarrow a_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 x^{1/2} \left(1 + \frac{1}{38} x^2 + \frac{1}{4,712} x^4 + \frac{1}{1,215,696} x^6 + \dots \right)} \quad \left| \right.$$

$$r = -\frac{2}{3}$$

$$b_n = \frac{1}{n(6n-7)} b_{n-2}$$

$$n=2 \rightarrow b_2 = \frac{1}{10} b_0$$

$$n=3 \rightarrow b_3 = \frac{1}{33} b_1 = 0$$

$$n=4 \rightarrow b_4 = \frac{1}{68} b_2 = \frac{1}{680} b_0$$

$$n=5 \rightarrow b_5 = 0$$

$$n=6 \rightarrow b_6 = \frac{1}{174} b_4 = \frac{1}{118,320} b_0$$

$$n=7 \rightarrow b_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{-2/3} \left(1 + \frac{1}{10} x^2 + \frac{1}{680} x^4 + \frac{1}{118,320} x^6 + \dots \right)$$

$$y(x) = a_0 x^{1/2} \left(1 + \frac{x^2}{38} + \frac{x^4}{4,712} + \frac{x^6}{1,215,696} + \dots \right) + b_0 x^{-2/3} \left(1 + \frac{x^2}{10} + \frac{x^4}{680} + \frac{x^6}{118,320} + \dots \right)$$

Exercise

Find the Frobenius series solutions of $xy'' + y' + 2y = 0$

Solution

$$x \times xy'' + y' + 2y = 0$$

$$x^2 y'' + xy' + 2xy = 0$$

$$y'' + \frac{1}{x} y' + \frac{2}{x} y = 0$$

That implies to $p(x) = \frac{1}{x}$ and $q(x) = \frac{2}{x}$.

$$p_0 = \lim_{x \rightarrow 0} xp(x)$$

$$= \lim_{x \rightarrow 0} x \frac{1}{x}$$

$$= 1$$

$$q_0 = \lim_{x \rightarrow 0} x^2 q(x)$$

$$= \lim_{x \rightarrow 0} x^2 \frac{2}{x}$$

$$= \lim_{x \rightarrow 0} 2x$$

$$= 0$$

The indicial equation is: $r^2 + (1-1)r = 0$

$$\rightarrow \underline{r_{1,2} = 0}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = y_1(x) \ln|x| + \sum_{n=0}^{\infty} c_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} a_n x^n \quad (r = r_1 = 0)$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n+r-1} = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$xy'' + y' + 2y = 0$$

$$x \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n^2 a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n^2 a_n x^{n-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n-1} = 0$$

$$\sum_{n=1}^{\infty} n^2 a_n x^{n-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n-1} = 0$$

$$\sum_{n=1}^{\infty} (n^2 a_n + 2a_{n-1}) x^{n-1} = 0$$

$$n^2 a_n + 2a_{n-1} = 0$$

$$\boxed{a_n = -\frac{2}{n^2} a_{n-1}}$$

$$n=1 \rightarrow a_1 = -2a_0$$

$$n=2 \rightarrow a_2 = -\frac{2}{2^2}a_1 = a_0 = \frac{2^2}{2^2}a_0$$

$$n=3 \rightarrow a_3 = -\frac{2}{9}a_2 = -\frac{2^3}{(2 \cdot 3)^2}a_0 = -\frac{2}{9}a_0$$

$$n=4 \rightarrow a_4 = -\frac{2}{4^2}a_3 = \frac{2^4}{(4!)^2}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = \frac{(-1)^n 2^n}{(n!)^2} a_0}$$

$$\boxed{y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(n!)^2} x^n}$$

$$y_1(x) = x^1 \sum_{n=0}^{\infty} a_n x^n \quad (a_0 = 1)$$

$$y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} c_n x^n$$

$$y_2(x) = y_1(x) \ln x + x^1 \sum_{n=0}^{\infty} c_n x^n$$

$$xy_2'' + y_2' + 2y_2 = 0$$

$$x \left(y_1(x) \ln x + \sum_{n=0}^{\infty} c_n x^n \right)'' + \left(y_1(x) \ln x + \sum_{n=0}^{\infty} c_n x^n \right)' + 2 \left(y_1(x) \ln x + \sum_{n=0}^{\infty} c_n x^n \right) = 0$$

$$x \left(y_1' \ln x + \frac{1}{x} y_1 \right)' + x \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + y_1' \ln x + \frac{1}{x} y_1 + \sum_{n=0}^{\infty} n c_n x^{n-1} + 2y_1 \ln x + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

$$x \left(y_1'' \ln x + \frac{1}{x} y_1' - \frac{1}{x^2} y_1 + \frac{1}{x} y_1' \right) + y_1' \ln x + \frac{1}{x} y_1 + 2y_1 \ln x + \sum_{n=0}^{\infty} n(n-1) c_n x^{n-1} + \sum_{n=0}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

$$xy_1'' \ln x + 2y_1' - \frac{1}{x} y_1 + \frac{1}{x} y_1 + (y_1' + 2y_1) \ln x + \sum_{n=0}^{\infty} n^2 c_n x^{n-1} + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

$$(xy_1'' + y_1' + 2y_1)\ln x + 2y_1' + \sum_{n=0}^{\infty} n^2 c_n x^{n-1} + \sum_{n=1}^{\infty} 2c_{n-1} x^{n-1} = 0$$

$$\text{Since: } xy_1'' + y_1' + 2y_1 = 0$$

$$2y_1' + \sum_{n=1}^{\infty} n^2 c_n x^{n-1} + \sum_{n=1}^{\infty} 2c_{n-1} x^{n-1} = 0$$

$$2y_1' + \sum_{n=1}^{\infty} (n^2 c_n + 2c_{n-1}) x^{n-1} = 0$$

$$2 \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{(n!)^2} n x^{n-1} + \sum_{n=1}^{\infty} (n^2 c_n + 2c_{n-1}) x^{n-1} = 0$$

$$\sum_{n=1}^{\infty} \left[(-1)^n \frac{2^{n+1}}{(n-1)!n!} + n^2 c_n + 2c_{n-1} \right] x^{n-1} = 0$$

$$n^2 c_n + 2c_{n-1} + (-1)^n \frac{2^{n+1}}{(n-1)!n!} = 0$$

$$c_n = -\frac{2}{n^2} c_{n-1} + (-1)^{n+1} \frac{2^{n+1}}{(n-1)!n^2}$$

$$n=1 \rightarrow c_1 = -2c_0 + 4$$

$$c_0 = 0 \rightarrow c_1 = 4$$

$$n=2 \rightarrow c_2 = -\frac{1}{2}c_1 - 1$$

$$c_2 = -3$$

$$n=3 \rightarrow c_3 = -\frac{2}{9}c_2 + \frac{16}{12(9)}$$

$$c_3 = \frac{22}{27}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = y_1(x) \ln|x| + 4x - 3x^2 + \frac{22}{27}x^3 - \dots$$

$$y_2(x) = y_1(x) \ln x + x^1 \sum_{n=0}^{\infty} c_n x^n$$

Exercise

Find the Frobenius series solutions of $xy'' - y = 0$

Solution

$$\text{✗ } xy'' - y = 0$$

$$x^2 y'' - xy = 0$$

$$y'' - \frac{1}{x}y = 0$$

That implies to $p(x) = 0$ and $q(x) = -\frac{1}{x}$.

$$\begin{aligned} p_0 &= \lim_{x \rightarrow 0} xp(x) \\ &= 0 \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x \rightarrow 0} x^2 q(x) \\ &= \lim_{x \rightarrow 0} x^2 \left(-\frac{1}{x} \right) \\ &= - \lim_{x \rightarrow 0} x \\ &= 0 \end{aligned}$$

The indicial equation is: $r^2 - r = 0$
 $\rightarrow \underline{r_1 = 1, r_2 = 0}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1} \quad \text{and} \quad y_2(x) = \alpha y_1(x) \ln|x| + \sum_{n=0}^{\infty} c_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} a_n x^{n+1} \quad (r = r_1 = 1)$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n+r-1} = \sum_{n=0}^{\infty} n a_n x^n$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} = \sum_{n=0}^{\infty} n(n+1) a_n x^{n-1}$$

$$xy'' - y = 0$$

$$x \sum_{n=0}^{\infty} n(n+1) a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} n(n+1) a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} n(n+1) a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\sum_{n=1}^{\infty} n(n+1)a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\sum_{n=1}^{\infty} [n(n+1)a_n - a_{n-1}] x^n = 0$$

$$n(n+1)a_n - a_{n-1} = 0$$

$$\boxed{a_n = \frac{1}{n(n+1)} a_{n-1}}$$

$$n=1 \rightarrow a_1 = \frac{1}{2} a_0$$

$$n=2 \rightarrow a_2 = \frac{1}{6} a_1 = a_0 = \frac{1}{(2 \cdot 3)!} a_0$$

$$n=3 \rightarrow a_3 = \frac{1}{3 \cdot 4} a_2 = \frac{1}{(2 \cdot 3)4!} a_0$$

$$n=4 \rightarrow a_4 = \frac{1}{4 \cdot 5} a_3 = \frac{1}{4!5!} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = \frac{1}{n!(n+1)!} a_0}$$

$$y_1(x) = x \sum_{n=1}^{\infty} \frac{1}{n!(n+1)!} x^n$$

$$\boxed{= \sum_{n=1}^{\infty} \frac{1}{n!(n+1)!} x^{n+1}}$$

$$y_1(x) = x^1 \sum_{n=0}^{\infty} a_n x^n \quad (a_0 = 1)$$

$$y_2(x) = \alpha y_1 \ln x + \sum_{n=0}^{\infty} d_n x^n$$

$$y_2(x) = \alpha y_1(x) \ln x + \sum_{n=0}^{\infty} d_n x^n \quad (d_0 = 1)$$

$$y_2' = \alpha y_1' \ln x + \frac{\alpha}{x} y_1 + \sum_{n=0}^{\infty} n d_n x^{n-1}$$

$$xy_2'' - y_2 = 0$$

$$x \left(\alpha y_1' \ln x + \frac{\alpha}{x} y_1 + \sum_{n=0}^{\infty} n d_n x^{n-1} \right)' - \left(\alpha y_1 \ln x + \sum_{n=0}^{\infty} d_n x^n \right) = 0$$

$$x \left(\alpha y_1' \ln x + \frac{\alpha}{x} y_1 \right)' + x \sum_{n=0}^{\infty} n(n-1) d_n x^{n-2} - \alpha y_1 \ln x - \sum_{n=0}^{\infty} d_n x^n = 0$$

$$x \left(\alpha y_1'' \ln x + \frac{\alpha}{x} y_1' - \frac{\alpha}{x^2} y_1 + \frac{\alpha}{x} y_1' \right) - \alpha y_1 (x) \ln x + \sum_{n=0}^{\infty} n(n-1) d_n x^{n-1} - \sum_{n=0}^{\infty} d_n x^n = 0$$

$$\alpha \left(2y_1' - \frac{1}{x} y_1 \right) + \alpha (xy_1'' - y_1) \ln x + \sum_{n=0}^{\infty} n(n+1) d_{n+1} x^n - \sum_{n=0}^{\infty} d_n x^n = 0$$

Since: $xy_1'' - y_1 = 0$

$$y_1(x) = \sum_{n=1}^{\infty} \frac{1}{n!(n+1)!} x^{n+1}$$

$$y_1' = \sum_{n=1}^{\infty} \frac{n+1}{n!(n+1)!} x^n$$

$$\alpha \left(2y_1' - \frac{1}{x} y_1 \right) + \sum_{n=0}^{\infty} [n(n+1) d_{n+1} - d_n] x^n = 0$$

$$\alpha \left(2 \sum_{n=1}^{\infty} \frac{n+1}{n!(n+1)!} x^n - \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{n!(n+1)!} x^{n+1} \right) + \sum_{n=0}^{\infty} [n(n+1) d_{n+1} - d_n] x^n = 0$$

$$\alpha \left(\sum_{n=1}^{\infty} \frac{2n+2}{n!(n+1)!} x^n - \sum_{n=1}^{\infty} \frac{1}{n!(n+1)!} x^n \right) + \sum_{n=0}^{\infty} [n(n+1) d_{n+1} - d_n] x^n = 0$$

$$\alpha \sum_{n=1}^{\infty} \frac{2n+1}{n!(n+1)!} x^n + \sum_{n=0}^{\infty} [n(n+1) d_{n+1} - d_n] x^n = 0$$

$$\sum_{n=0}^{\infty} [n(n+1) d_{n+1} - d_n] x^n = -\alpha \sum_{n=1}^{\infty} \frac{2n+1}{n!(n+1)!} x^n$$

$$\underline{n(n+1) d_{n+1} - d_n = -\alpha \frac{2n+1}{n!(n+1)!} \quad \left(d_0 = 1 \right)}$$

$$\underline{n=0 \rightarrow -d_0 = -\alpha \quad \Rightarrow \alpha = d_0 = 1}$$

$$d_{n+1} = \frac{1}{n(n+1)} \left(d_n - \frac{2n+1}{n!(n+1)!} \right)$$

$$\begin{aligned} n=1 \rightarrow d_2 &= \frac{1}{2} \left(d_1 - \frac{3}{2} \right) \\ &= \frac{1}{2} d_1 - \frac{3}{4} \end{aligned}$$

$$\begin{aligned} n=2 \rightarrow d_3 &= \frac{1}{6} \left(d_2 - \frac{5}{12} \right) \\ &= \frac{1}{6} \left(\frac{1}{2} d_1 - \frac{3}{4} - \frac{5}{12} \right) \\ &= \frac{1}{12} d_1 - \frac{7}{36} \end{aligned}$$

$$\begin{aligned} n=3 \rightarrow d_4 &= \frac{1}{12} \left(d_3 - \frac{7}{144} \right) \\ &= \frac{1}{12} \left(\frac{1}{12} d_1 - \frac{7}{36} - \frac{7}{144} \right) \\ &= \frac{1}{144} d_1 - \frac{35}{1,728} \end{aligned}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

If we let $d_1 = 0$

$$y_2(x) = y_1(x) \ln x + 1 - \frac{3}{4}x^2 - \frac{7}{36}x^3 - \dots$$

$$y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} d_n x^n$$

Exercise

Find the Frobenius series solutions of $2x(1-x)y'' + (1+x)y' - y = 0$

Solution

$$xy'' + \frac{x+1}{2(1-x)}y' - \frac{1}{2(1-x)}y = 0$$

$$x^2y'' + \frac{1}{2} \frac{x(x+1)}{1-x}y' - \frac{x}{2(1-x)}y = 0$$

$$y'' + \frac{1}{2} \frac{x+1}{x(1-x)}y' - \frac{1}{2x(1-x)}y = 0$$

That implies to $p(x) = \frac{1}{2} \frac{x+1}{x(1-x)}$ and $q(x) = -\frac{1}{2x(1-x)}$.

$$p_0 = \lim_{x \rightarrow 0} xp(x)$$

$$\begin{aligned}
&= \frac{1}{2} \lim_{x \rightarrow 0} x \frac{x+1}{x(1-x)} \\
&= \frac{1}{2} \lim_{x \rightarrow 0} \frac{x+1}{1-x} \\
&= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
q_0 &= \lim_{x \rightarrow 0} x^2 q(x) \\
&= -\frac{1}{2} \lim_{x \rightarrow 0} \frac{x}{1-x} \\
&= 0
\end{aligned}$$

The indicial equation is: $r^2 + \left(\frac{1}{2} - 1\right)r = r^2 - \frac{1}{2}r = 0$

$$\rightarrow \underline{r = 0, \frac{1}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x(1-x)y'' + (1+x)y' - y = 0$$

$$(2x - 2x^2) \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + (1+x) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\begin{aligned}
&\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} - \sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\
&\quad + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0
\end{aligned}$$

$$\sum_{n=0}^{\infty} (n+r)(2n+2r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} [-2(n+r)(n+r-1) + n+r-1]a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(2n+2r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r-1)(-2(n+r)+1)a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(2n+2r-1)a_n x^{n+r-1} + \sum_{n=1}^{\infty} (n+r-2)(-2n-2r+3)a_{n-1} x^{n+r-1} = 0$$

$$r(2r-1)a_0 x^{r-1} + \sum_{n=1}^{\infty} (n+r)(2n+2r-1)a_n x^{n+r-1} - \sum_{n=1}^{\infty} (n+r-2)(2n+2r-3)a_{n-1} x^{n+r-1} = 0$$

$$r(2r-1)a_0 x^{r-1} + \sum_{n=1}^{\infty} [(n+r)(2n+2r-1)a_n - (n+r-2)(2n+2r-3)a_{n-1}] x^{n+r-1} = 0$$

For $n = 0$

$$r(2r-1)a_0 = 0$$

$$\Rightarrow \underline{r = 0, \frac{1}{2}} \quad \checkmark$$

$$(n+r)(2n+2r-1)a_n - (n+r-2)(2n+2r-3)a_{n-1} = 0$$

$$\underline{a_n = \frac{(n+r-2)(2n+2r-3)}{(n+r)(2n+2r-1)} a_{n-1}}$$

$$r = 0$$

$$a_n = \frac{(n-2)(2n-3)}{n(2n-1)} a_{n-1}$$

$$n = 1 \rightarrow a_1 = a_0$$

$$n = 2 \rightarrow a_2 = 0a_1 = 0$$

$$n = 3 \rightarrow a_3 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0(1+x)}$$

$$r = \frac{1}{2}$$

$$\begin{aligned} b_n &= \frac{\left(n - \frac{3}{2}\right)(2n-2)}{2n\left(n + \frac{1}{2}\right)} b_{n-1} \\ &= \frac{(2n-3)(n-1)}{n(2n+1)} b_{n-1} \end{aligned}$$

$$n = 1 \rightarrow b_1 = 0, b_0 = 0$$

$$n = 2 \rightarrow b_2 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_2(x) = b_0 x^{1/2} \mid}$$

$$\underline{y(x) = a_0(1+x) + b_0\sqrt{x} \mid}$$

Exercise

Find the Frobenius series solutions of $x^2 y'' + \left(x^2 + \frac{1}{2}x\right)y' + xy = 0$

Solution

$$y'' + \left(1 + \frac{1}{2x}\right)y' + \frac{1}{x}y = 0$$

That implies to $p(x) = 1 + \frac{1}{2x}$ and $q(x) = \frac{1}{x}$.

$$\begin{aligned} p_0 &= \lim_{x \rightarrow 0} xp(x) \\ &= \lim_{x \rightarrow 0} x\left(1 + \frac{1}{2x}\right) \\ &= \lim_{x \rightarrow 0} \left(x + \frac{1}{2}\right) \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x \rightarrow 0} x^2 q(x) \\ &= \lim_{x \rightarrow 0} x \\ &= 0 \end{aligned}$$

The indicial equation is: $r^2 + \left(\frac{1}{2} - 1\right)r = r^2 - \frac{1}{2}r = 0$

$$\rightarrow \underline{r = 0, \frac{1}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x^2 y'' + \left(x^2 + \frac{1}{2}x\right) y' + xy = 0$$

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \left(x^2 + \frac{1}{2}x\right) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + x \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\begin{aligned} & \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r+1} + \sum_{n=0}^{\infty} \frac{1}{2} (n+r) a_n x^{n+r} \\ & + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0 \end{aligned}$$

$$\sum_{n=0}^{\infty} (n+r+1) a_n x^{n+r+1} + \sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + \frac{1}{2}(n+r) \right] a_n x^{n+r} = 0$$

$$\sum_{n=1}^{\infty} (n+r) a_{n-1} x^{n+r} + \sum_{n=0}^{\infty} (n+r) \left(n+r - \frac{1}{2} \right) a_n x^{n+r} = 0$$

$$\sum_{n=1}^{\infty} (n+r) a_{n-1} x^{n+r} + r \left(r - \frac{1}{2} \right) a_0 + \sum_{n=1}^{\infty} (n+r) \left(n+r - \frac{1}{2} \right) a_n x^{n+r} = 0$$

$$r\left(r-\frac{1}{2}\right)a_0 + \sum_{n=1}^{\infty} \left[(n+r)\left(n+r-\frac{1}{2}\right)a_n + (n+r)a_{n-1} \right] x^{n+r} = 0$$

For $n=0$

$$r(2r-1)a_0 = 0$$

$$\Rightarrow \underline{r=0, \frac{1}{2}} \quad \checkmark$$

$$(n+r)\left(n+r-\frac{1}{2}\right)a_n + (n+r)a_{n-1} = 0$$

$$\underline{a_n = -2 \frac{(n+r)}{(n+r)(2n+2r-1)} a_{n-1}}$$

$$r=0$$

$$a_n = -\frac{2n}{n(2n-1)} a_{n-1}$$

$$= -\frac{2}{2n-1} a_{n-1}$$

$$n=1 \rightarrow a_1 = -2a_0$$

$$n=2 \rightarrow a_2 = -\frac{2}{3}a_1 = \frac{4}{3}a_0$$

$$n=3 \rightarrow a_3 = -\frac{2}{5}a_2 = -\frac{8}{15}a_0$$

$$n=4 \rightarrow a_4 = -\frac{2}{7}a_3 = \frac{16}{105}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 \left(1 - 2x + \frac{4}{3}x^2 - \frac{8}{15}x^3 + \frac{16}{105}x^4 - \dots \right)}$$

$$r = \frac{1}{2}$$

$$b_n = -\frac{2\left(n+\frac{1}{2}\right)}{\left(n+\frac{1}{2}\right)(2n)} b_{n-1}$$

$$= -\frac{1}{n} b_{n-1}$$

$$n=1 \rightarrow b_1 = -b_0$$

$$n=2 \rightarrow b_2 = -\frac{1}{2}b_1 = \frac{1}{2}b_0$$

$$n=3 \rightarrow b_3 = -\frac{1}{3}b_2 = -\frac{1}{6}b_0$$

$$n=4 \rightarrow b_4 = -\frac{1}{4}b_3 = \frac{1}{24}b_0$$

$$n=5 \rightarrow b_5 = -\frac{1}{5}b_4 = -\frac{1}{120}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{1/2} \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \dots \right)$$

$$y(x) = a_0 \left(1 - 2x + \frac{4}{3}x^2 - \frac{8}{15}x^3 + \frac{16}{105}x^4 - \dots \right) + b_0 \sqrt{x} \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \dots \right)$$

Exercise

Find the Frobenius series solutions of $18x^2 y'' + 3x(x+5)y' - (10x+1)y = 0$

Solution

$$y'' + \frac{x+5}{6x} y' - \frac{10x+1}{18x^2} y = 0$$

That implies to $p(x) = \frac{x+5}{6x}$ and $q(x) = -\frac{10x+1}{18x^2}$.

$$\begin{aligned} p_0 &= \lim_{x \rightarrow 0} x p(x) \\ &= \lim_{x \rightarrow 0} x \left(\frac{x+5}{6x} \right) \\ &= \lim_{x \rightarrow 0} \frac{x+5}{6} \\ &= \frac{5}{6} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x \rightarrow 0} x^2 q(x) \\ &= - \lim_{x \rightarrow 0} \frac{10x+1}{18} \\ &= -\frac{1}{18} \end{aligned}$$

The indicial equation is: $r^2 + \left(\frac{5}{6} - 1\right)r - \frac{1}{18} = r^2 - \frac{1}{6}r - \frac{1}{18} = 0$

$$18r^2 - 3r - 1 = 0 \rightarrow r = -\frac{1}{6}, \frac{1}{3}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{-1/6} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/3} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$18x^2 y'' + 3x(x+5)y' - (10x+1)y = 0$$

$$18x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + (3x^2 + 15x) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\ - (10x+1) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 18(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r+1} + \sum_{n=0}^{\infty} 15(n+r) a_n x^{n+r} \\ - \sum_{n=0}^{\infty} 10a_n x^{n+r+1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [18(n+r)(n+r-1) + 15(n+r) - 1] a_n x^{n+r} + \sum_{n=0}^{\infty} (3n+3r-10) a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(18n+18r-3) - 1] a_n x^{n+r} + \sum_{n=1}^{\infty} (3n+3r-13) a_{n-1} x^{n+r} = 0$$

$$(r(18r-3)-1)a_0 x^r + \sum_{n=1}^{\infty} [(n+r)(18n+18r-3) - 1] a_n x^{n+r} + \sum_{n=1}^{\infty} (3n+3r-13) a_{n-1} x^{n+r} = 0$$

$$(r(18r-3)-1)a_0 x^r + \sum_{n=1}^{\infty} [(n+r)(18n+18r-3) - 1] a_n + (3n+3r-13) a_{n-1} x^{n+r} = 0$$

For $n = 0$

$$(18r^2 - 3r - 1)a_0 = 0$$

$$\Rightarrow \underline{r = -\frac{1}{6}, \frac{1}{3}} \quad \checkmark$$

$$((n+r)(18n+18r-3)-1)a_n + (3n+3r-13)a_{n-1} = 0$$

$$\underline{a_n = -\frac{3n+3r-13}{(n+r)(18n+18r-3)-1}a_{n-1}}$$

$$r = -\frac{1}{6}$$

$$\begin{aligned} a_n &= -\frac{3n-\frac{1}{2}-13}{\left(n-\frac{1}{6}\right)(18n-6)-1}a_{n-1} \\ &= -\frac{1}{2}\frac{6n-27}{(6n-1)(3n-1)-1}a_{n-1} \end{aligned}$$

$$n=1 \rightarrow a_1 = -\frac{1}{2}\frac{-21}{9}a_0 = \frac{7}{6}a_0$$

$$n=2 \rightarrow a_2 = -\frac{1}{2}\frac{-15}{54}a_1 = \frac{5}{36}\frac{7}{6}a_0 = \frac{35}{216}a_0$$

$$n=3 \rightarrow a_3 = -\frac{1}{2}\frac{-9}{135}a_2 = \frac{1}{30}\frac{35}{216}a_0 = \frac{7}{1,296}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 x^{-1/6} \left(1 + \frac{7}{6}x + \frac{35}{216}x^2 + \frac{7}{1,296}x^3 + \dots \right)}$$

$$r = \frac{1}{3}$$

$$\begin{aligned} b_n &= -\frac{3n-12}{\left(n+\frac{1}{3}\right)(18n+3)-1}b_{n-1} \\ &= -\frac{3(n-4)}{(3n+1)(6n+1)-1}b_{n-1} \end{aligned}$$

$$n=1 \rightarrow b_1 = -\frac{-9}{27}b_0 = \frac{1}{3}b_0$$

$$n=2 \rightarrow b_2 = \frac{6}{90}b_1 = \frac{1}{15}\frac{1}{3}b_0 = \frac{1}{45}b_0$$

$$n=3 \rightarrow b_3 = \frac{3}{189}b_2 = \frac{1}{63}\frac{1}{45}b_0 = \frac{1}{2,835}b_0$$

$$n=4 \rightarrow b_4 = 0b_3 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_2(x) = b_0 x^{1/3} \left(1 + \frac{1}{3}x + \frac{1}{45}x^2 + \frac{1}{2835}x^3 \right)}$$

$$\underline{y(x) = a_0 \frac{1}{x^{1/6}} \left(1 + \frac{7}{6}x + \frac{35}{216}x^2 + \frac{7}{1296}x^3 + \dots \right) + b_0 x^{1/3} \left(1 + \frac{1}{3}x + \frac{1}{45}x^2 + \frac{1}{2835}x^3 \right)}$$

Exercise

Find the Frobenius series solutions of $2x^2y'' + 7x(x+1)y' - 3y = 0$

Solution

$$y'' + \frac{7}{2} \frac{x+1}{x} y' - \frac{3}{2x^2} y = 0$$

That implies to $p(x) = \frac{7}{2} \frac{x+1}{x}$ and $q(x) = -\frac{3}{2x^2}$.

$$\begin{aligned} p_0 &= \lim_{x \rightarrow 0} xp(x) \\ &= \frac{7}{2} \lim_{x \rightarrow 0} x \left(\frac{x+1}{x} \right) \\ &= \frac{7}{2} \lim_{x \rightarrow 0} (x+1) \\ &= \frac{7}{2} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x \rightarrow 0} x^2 q(x) \\ &= - \lim_{x \rightarrow 0} \frac{3}{2} \\ &= -\frac{3}{2} \end{aligned}$$

The indicial equation is: $r^2 + \left(\frac{7}{2} - 1\right)r - \frac{3}{2} = r^2 + \frac{5}{2}r - \frac{3}{2} = 0$

$$2r^2 + 5r - 3 = 0 \rightarrow \underline{r = -3, \frac{1}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{-3} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2y'' + 7x(x+1)y' - 3y = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + (7x^2 + 7x) \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} - 3 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} 7(n+r)a_n x^{n+r+1} + \sum_{n=0}^{\infty} 7(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty} 3a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(2n+2r-2) + 7(n+r) - 3]a_n x^{n+r} + \sum_{n=0}^{\infty} 7(n+r)a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(2n+2r+5) - 3]a_n x^{n+r} + \sum_{n=1}^{\infty} 7(n+r-1)a_{n-1} x^{n+r} = 0$$

$$(2r^2 + 5r - 3)a_0 x^r + \sum_{n=1}^{\infty} [(n+r)(2n+2r+5) - 3]a_n x^{n+r} + \sum_{n=1}^{\infty} 7(n+r-1)a_{n-1} x^{n+r} = 0$$

$$(2r^2 + 5r - 3)a_0 x^r + \sum_{n=1}^{\infty} [((n+r)(2n+2r+5) - 3)a_n + 7(n+r-1)a_{n-1}] x^{n+r} = 0$$

For $n = 0$

$$(2r^2 + 5r - 3)a_0 = 0$$

$$\Rightarrow \underline{r = -3, \frac{1}{2}} \quad \checkmark$$

$$((n+r)(2n+2r+5) - 3)a_n + 7(n+r-1)a_{n-1} = 0$$

$$a_n = -\frac{7(n+r-1)}{(n+r)(2n+2r+5) - 3} a_{n-1}$$

$$r = -3$$

$$a_n = -\frac{7(n-4)}{(n-3)(2n-1) - 3} a_{n-1}$$

$$n = 1 \rightarrow a_1 = -\frac{21}{5} a_0$$

$$n=2 \rightarrow a_2 = -\frac{14}{6}a_1 = -\frac{7}{3}\left(-\frac{21}{5}\right)a_0 = \frac{49}{5}a_0$$

$$n=3 \rightarrow a_3 = -\frac{-7}{-3}a_2 = -\frac{7}{3}\frac{49}{5}a_0 = -\frac{343}{15}a_0$$

$$n=4 \rightarrow a_4 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 x^{-3} \left(1 - \frac{21}{5}x + \frac{49}{5}x^2 - \frac{343}{15}x^3 \right) \quad |}$$

$$r = \frac{1}{2}$$

$$\begin{aligned} b_n &= -\frac{7\left(n - \frac{1}{2}\right)}{\left(n + \frac{1}{2}\right)(2n+6) - 3} b_{n-1} \\ &= -\frac{7}{2} \frac{2n-1}{(2n+1)(n+3) - 3} b_{n-1} \end{aligned}$$

$$n=1 \rightarrow b_1 = -\frac{7}{2} \frac{1}{9} b_0 = -\frac{7}{18} b_0$$

$$n=2 \rightarrow b_2 = -\frac{7}{2} \frac{3}{22} b_1 = -\frac{21}{44} \frac{-7}{18} b_0 = \frac{49}{264} b_0$$

$$n=3 \rightarrow b_3 = -\frac{7}{2} \frac{5}{39} b_2 = -\frac{35}{78} \frac{49}{264} b_0 = -\frac{1,715}{20,592} b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_2(x) = b_0 x^{1/2} \left(1 - \frac{7}{18}x + \frac{49}{264}x^2 - \frac{1,715}{20,592}x^3 + \dots \right) \quad |}$$

$$\underline{y(x) = a_0 \frac{1}{x^3} \left(1 - \frac{21}{5}x + \frac{49}{5}x^2 - \frac{343}{15}x^3 \right) + b_0 \sqrt{x} \left(1 - \frac{7}{18}x + \frac{49}{264}x^2 - \frac{1,715}{20,592}x^3 + \dots \right) \quad |}$$

Exercise

Find the Frobenius series solutions:

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0 \quad (\text{Gauss' Hypergeometric})$$

Solution

$$y'' + \frac{c - (a+b+1)x}{x(1-x)}y' - \frac{ab}{x(1-x)}y = 0$$

$$\text{That implies to } p(x) = \frac{c - (a+b+1)x}{x(1-x)} \text{ and } q(x) = -\frac{ab}{x(1-x)}.$$

$$p_0 = \lim_{x \rightarrow 0} xp(x)$$

$$= \lim_{x \rightarrow 0} x \left(\frac{c - (a + b + 1)x}{x(1 - x)} \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{c - (a + b + 1)x}{1 - x} \right)$$

$$= \underline{c} \quad |$$

$$q_0 = \lim_{x \rightarrow 0} x^2 q(x)$$

$$= - \lim_{x \rightarrow 0} x^2 \frac{ab}{x(1 - x)}$$

$$= - \lim_{x \rightarrow 0} \frac{abx}{1 - x}$$

$$= \underline{0} \quad |$$

$$p_1 = \lim_{x \rightarrow 1} (x - 1) p(x)$$

$$= \lim_{x \rightarrow 1} (x - 1) \left(\frac{c - (a + b + 1)x}{x(1 - x)} \right)$$

$$= \lim_{x \rightarrow 1} \left(- \frac{c - (a + b + 1)x}{x} \right)$$

$$= \underline{a + b + 1 - c} \quad |$$

$$q_1 = \lim_{x \rightarrow 1} (x - 1)^2 q(x)$$

$$= - \lim_{x \rightarrow 1} (x - 1)^2 \frac{ab}{x(1 - x)}$$

$$= \lim_{x \rightarrow 1} \frac{ab}{x} (x - 1)$$

$$= \underline{0} \quad |$$

The *Regular* singular points: $x = 0, 1$ |

The indicial equation is: $r(r - 1) - cr = r^2 + (c - 1)r = 0 \rightarrow \underline{r = 0, 1 - c}$ |

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1-c} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$$

$$\begin{aligned} (x-x^2) \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + [c - (a+b+1)x] \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\ - ab \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} c(n+r) a_n x^{n+r-1} \\ - \sum_{n=0}^{\infty} (a+b+1)(n+r) a_n x^{n+r} - ab \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{\infty} [(n+r)(n+r-1) + c(n+r)] a_n x^{n+r-1} \\ - \sum_{n=0}^{\infty} [(n+r)(n+r-1) + (a+b+1)(n+r) + ab] a_n x^{n+r} = 0 \end{aligned}$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1+c) a_n x^{n+r-1} - \sum_{n=1}^{\infty} [(n+r-1)(n+r-2+a+b+1) + ab] a_{n-1} x^{n+r-1} = 0$$

$$\begin{aligned} r(r+c-1) a_0 x^{r-1} + \sum_{n=1}^{\infty} (n+r)(n+r-1+c) a_n x^{n+r-1} \\ - \sum_{n=1}^{\infty} [(n+r-1)(n+r-1+a+b) + ab] a_{n-1} x^{n+r-1} = 0 \end{aligned}$$

$$r(r+c-1)a_0x^{r-1}$$

$$+ \sum_{n=1}^{\infty} \left[(n+r)(n+r-1+c)a_n - ((n+r-1)(n+r-1+a+b)+ab)a_{n-1} \right] x^{n+r-1} = 0$$

For $n=0$

$$r(r+c-1)a_0 = 0$$

$$\Rightarrow \underline{r=0, 1-c} \quad \checkmark$$

$$(n+r)(n+r-1+c)a_n - ((n+r-1)(n+r-1+a+b)+ab)a_{n-1} = 0$$

$$(n+r)(n+r-1+c)a_n = ((n+r-1)(n+r-1+a+b)+ab)a_{n-1}$$

$$a_n = \frac{(n+r-1)(n+r-1+a+b)+ab}{(n+r)(n+r-1+c)} a_{n-1}$$

$$r=0$$

$$a_n = \frac{(n-1)(n-1+a+b)+ab}{n(n-1+c)} a_{n-1}$$

$$n=1 \rightarrow a_1 = \frac{ab}{c} a_0$$

$$n=2 \rightarrow a_2 = \frac{1+a+b+ab}{2 \cdot (c+1)} a_1 = \frac{(a+1)(b+1)}{2 \cdot (c+1)} a_1 = \frac{ab(a+1)(b+1)}{2 \cdot c \cdot (c+1)} a_0$$

$$n=3 \rightarrow a_3 = \frac{4+2a+2b+ab}{3 \cdot (c+2)} a_2 = \frac{(a+2)(b+2)}{3 \cdot (c+2)} a_2 = \frac{a(a+1)(a+2) \cdot b(b+1)(b+2)}{2 \cdot 3 \cdot c(c+1)(c+2)} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\rightarrow a_n = \frac{a(a+1)(a+2) \cdots (a+n-1) \cdot b(b+1)(b+2) \cdots (b+n-1)}{n! \cdot c(c+1)(c+2) \cdots (c+n-1)} a_0$$

$$y_1(x) = a_0 \left(1 + \frac{ab}{c} x + \frac{ab(a+1)(b+1)}{2 \cdot c \cdot (c+1)} x^2 + \frac{a(a+1)(a+2) \cdot b(b+1)(b+2)}{2 \cdot 3 \cdot c(c+1)(c+2)} x^3 + \cdots \right)$$

$$= a_0 \left(1 + \sum_{n=0}^{\infty} \frac{a(a+1) \cdots (a+n-1) \cdot b(b+1) \cdots (b+n-1)}{n! \cdot c(c+1) \cdots (c+n-1)} x^n \right)$$

$$r=1-c$$

$$b_n = \frac{(n-c)(n-c+a+b)+ab}{n(n+1-c)} b_{n-1}$$

$$n=1 \rightarrow b_1 = \frac{(1-c)(1-c+a+b)+ab}{2-c} b_0$$

$$\begin{aligned} n=2 \rightarrow b_2 &= \frac{(2-c)(2-c+a+b)+ab}{2(3-c)} b_1 = \frac{(1-c)(1-c+a+b)+ab}{2-c} b_0 \\ &= \frac{((2-c)(2-c+a+b)+ab)((1-c)(1-c+a+b)+ab)}{2(2-c)(3-c)} b_0 \end{aligned}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\rightarrow b_n = \frac{((n-c)(n-c+a+b)+ab) \cdots ((1-c)(1-c+a+b)+ab)}{2(2-c)(3-c) \cdots (n+1-c)} b_0$$

$$y_2(x) = b_0 x^{1-c} \left(1 + \frac{(1-c)(1-c+a+b)+ab}{2-c} x + \frac{((2-c)(2-c+a+b)+ab)((1-c)(1-c+a+b)+ab)}{2(2-c)(3-c)} x^2 + \cdots \right)$$

$$\begin{aligned} y(x) = a_0 & \left(1 + \sum_{n=0}^{\infty} \frac{a(a+1) \cdots (a+n-1) \cdot b(b+1) \cdots (b+n-1)}{n! \cdot c(c+1) \cdots (c+n-1)} x^n \right) \\ & + b_0 x^{1-c} \left(1 + \sum_{n=0}^{\infty} \frac{((n-c)(n-c+a+b)+ab) \cdots ((1-c)(1-c+a+b)+ab)}{2(2-c)(3-c) \cdots (n+1-c)} x^n \right) \end{aligned}$$

Solution **Section 4.5 – Bessel's Equation and Bessel Functions**

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0: \quad x^2 y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$$

Solution

$$v^2 = \frac{1}{9} \rightarrow v = \frac{1}{3}$$

$$\text{The general solution is: } \underline{y(x) = c_1 J_{1/3}(x) + c_2 J_{-1/3}(x)}$$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$$x^2 y'' + xy' + (x^2 - 1)y = 0: \quad x^2 y'' + xy' + (x^2 - 1)y = 0$$

Solution

$$v^2 = 1 \rightarrow v = 1$$

$$\text{The general solution is: } \underline{y(x) = c_1 J_1(x) + c_2 Y_1(x)}$$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$$x^2 y'' + xy' + (x^2 - 25)y = 0: \quad 4x^2 y'' + 4xy' + (4x^2 - 25)y = 0$$

Solution

$$v^2 = \frac{25}{4} \rightarrow v = \pm \frac{5}{2}$$

$$\text{The general solution is: } \underline{y(x) = c_1 J_{5/2}(x) + c_2 J_{-5/2}(x)}$$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$$x^2 y'' + xy' + (x^2 - 1)y = 0: \quad 16x^2 y'' + 16xy' + (16x^2 - 1)y = 0$$

Solution

$$v^2 = \frac{1}{15} \rightarrow v = \pm \frac{1}{4}$$

The general solution is: $\underline{y(x) = c_1 J_{1/4}(x) + c_2 J_{-1/4}(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$$x^2 y'' + xy' + (x^2 - v^2)y = 0: \quad xy'' + y' + xy = 0$$

Solution

$$v^2 = 0 \rightarrow v = 0$$

The general solution is: $\underline{y(x) = c_1 J_0(x) + c_2 Y_0(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$$x^2 y'' + xy' + \left(x^2 - v^2\right)y = 0: \quad xy'' + y' + \left(x - \frac{4}{x}\right)y = 0$$

Solution

$$v^2 = 4 \rightarrow v = 2$$

The general solution is: $\underline{y(x) = c_1 J_2(x) + c_2 Y_2(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$$x^2 y'' + xy' + (\alpha^2 x^2 - v^2)y = 0: \quad x^2 y'' + xy' + (9x^2 - 4)y = 0$$

Solution

$$\begin{cases} \alpha^2 = 9 \rightarrow \alpha = 3 \\ v^2 = 4 \rightarrow v = 2 \end{cases}$$

The general solution is: $\underline{y(x) = c_1 J_2(3x) + c_2 Y_2(3x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$$x^2 y'' + xy' + \left(\alpha^2 x^2 - \nu^2\right)y = 0 : \quad x^2 y'' + xy' + \left(36x^2 - \frac{1}{4}\right)y = 0$$

Solution

$$\begin{cases} \alpha^2 = 36 \rightarrow \alpha = 6 \\ \nu^2 = \frac{1}{4} \rightarrow \nu = \frac{1}{2} \end{cases}$$

The general solution is: $\underline{y(x) = c_1 J_{1/2}(6x) + c_2 J_{-1/2}(6x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$$x^2 y'' + xy' + \left(\alpha^2 x^2 - \nu^2\right)y = 0 : \quad x^2 y'' + xy' + \left(25x^2 - \frac{4}{9}\right)y = 0$$

Solution

$$\begin{cases} \alpha^2 = 25 \rightarrow \alpha = 5 \\ \nu^2 = \frac{4}{9} \rightarrow \nu = \frac{2}{3} \end{cases}$$

The general solution is: $\underline{y(x) = c_1 J_{2/3}(5x) + c_2 J_{-2/3}(5x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$$x^2 y'' + xy' + \left(\alpha^2 x^2 - \nu^2\right)y = 0 : \quad x^2 y'' + xy' + \left(2x^2 - 64\right)y = 0$$

Solution

$$\begin{cases} \alpha^2 = 2 \rightarrow \alpha = \sqrt{2} \\ \nu^2 = 64 \rightarrow \nu = 8 \end{cases}$$

The general solution is: $\underline{y(x) = c_1 J_8(\sqrt{2}x) + c_2 Y_8(\sqrt{2}x)}$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$4x^2y'' + 8xy' + (x^4 - 3)y = 0$$

Solution

$$\frac{1}{4} \times 4x^2y'' + 8xy' + (x^4 - 3)y = 0$$

$$x^2y'' + 2xy' + \left(-\frac{3}{4} + \frac{1}{4}x^4\right)y = 0$$

$$A = 2, \quad B = -\frac{3}{4}, \quad C = \frac{1}{4}, \quad p = 4$$

$$\begin{cases} \alpha = \frac{1-2}{2} = -\frac{1}{2} \\ \beta = \frac{4}{2} = 2 \\ k = \frac{2\sqrt{\frac{1}{4}}}{4} = \frac{1}{4} \\ \nu = \frac{\sqrt{1+3}}{4} = \frac{1}{2} \end{cases}$$

$$y(x) = x^{-1/2} \left[c_1 J_{1/2} \left(\frac{1}{4}x^2 \right) + c_2 J_{-1/2} \left(\frac{1}{4}x^2 \right) \right]$$

$$\begin{aligned} y(x) &= x^\alpha \left[c_1 J_\nu(kx^\beta) + c_2 J_{-\nu}(kx^\beta) \right] \\ &= x^{-1/2} \left(c_1 \sqrt{\frac{2}{\pi z}} \sin z + c_2 \sqrt{\frac{2}{\pi z}} \cos z \right) \\ &= c_1 \left(\frac{2}{\pi x} \right)^{1/2} \sin x + c_2 \left(\frac{2}{\pi x} \right)^{1/2} \cos x \\ &= x^{-1/2} \left(c_1 \frac{2}{x} \sqrt{\frac{2}{\pi}} \sin \frac{x^2}{4} + c_2 \frac{2}{x} \sqrt{\frac{2}{\pi}} \cos \frac{x^2}{4} \right) \\ &= x^{-3/2} \left(C_1 \sqrt{\frac{2}{\pi}} \sin \frac{x^2}{4} + C_2 \sqrt{\frac{2}{\pi}} \cos \frac{x^2}{4} \right) \end{aligned}$$

$$x^2 \frac{d^2 y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p)y = 0$$

$$z = kx^\beta = \frac{x^2}{4}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y'' + 9xy = 0$$

Solution

$$x^2 \times y'' + 9xy = 0$$

$$x^2 y'' + 9x^3 y = 0$$

$$A = 0, \quad B = 0, \quad C = 9, \quad p = 3$$

$$\alpha = \frac{1}{2}, \quad \beta = \frac{3}{2}, \quad k = 2, \quad \nu = \frac{1}{3}$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$y(x) = x^{1/2} \left[c_1 J_{1/3} \left(2x^{3/2} \right) + c_2 J_{-1/3} \left(2x^{3/2} \right) \right]$$

$$y(x) = x^\alpha \left[c_1 J_\nu \left(kx^\beta \right) + c_2 J_{-\nu} \left(kx^\beta \right) \right]$$

$$x^2 \frac{d^2 y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p) y = 0$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' + (x-3)y = 0$$

Solution

$$x \times xy'' - 3y + xy = 0$$

$$x^2 y'' - 3xy + x^2 y = 0$$

$$A = -3, \quad B = 0, \quad C = 1, \quad p = 2$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\alpha = 2, \quad \beta = 1, \quad k = 1, \quad \nu = \frac{\sqrt{16}}{2} = 2$$

$$y(x) = x^2 \left[c_1 Y_2(x) + c_2 J_2(x) \right]$$

$$x^2 \frac{d^2 y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p) y = 0$$

$$y(x) = x^\alpha \left[c_1 J_\nu \left(kx^\beta \right) + c_2 J_{-\nu} \left(kx^\beta \right) \right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' + (4x^3 - 1)y = 0$$

Solution

$$x \times xy'' - y + 4x^3 y = 0$$

$$x^2 y'' - xy + 4x^4 y = 0$$

$$A = -1, \quad B = 0, \quad C = 4, \quad p = 4$$

$$\alpha = 1, \quad \beta = 2, \quad k = 1, \quad \nu = \frac{1}{2}$$

$$\begin{aligned} y(x) &= x \left[c_1 J_{1/2}(x^2) + c_2 J_{-1/2}(x^2) \right] \\ &= x \left(c_1 \frac{1}{x} \sqrt{\frac{2}{\pi}} \sin x^2 + c_2 \frac{1}{x} \sqrt{\frac{2}{\pi}} \cos x^2 \right) \\ &= c_1 \sqrt{\frac{2}{\pi}} \sin x^2 + c_2 \sqrt{\frac{2}{\pi}} \cos x^2 \\ &= \underline{C_1 \sin x^2 + C_2 \cos x^2} \end{aligned}$$

$$x^2 \frac{d^2 y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p)y = 0$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$y(x) = x^\alpha \left[c_1 J_\nu(kx^\beta) + c_2 J_{-\nu}(kx^\beta) \right]$$

$$y(z) = x^\alpha \left(c_1 \left(\frac{2}{\pi z} \right)^{1/2} \sin z + c_2 \left(\frac{2}{\pi z} \right)^{1/2} \cos z \right)$$

$$z = kx^\beta = x^2$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2 y'' + xy' - \left(\frac{1}{4} + x^2 \right) y = 0$$

Solution

$$x^2 y'' + xy' + \left(-\frac{1}{4} - x^2 \right) y = 0$$

$$A = 1, \quad B = -\frac{1}{4}, \quad C = -1, \quad p = 2$$

$$\alpha = 0, \quad \beta = 1, \quad k = i, \quad \nu = \frac{1}{2}$$

$$y(x) = c_1 I_{1/2}(x) + c_2 I_{-1/2}(x)$$

$$y(x) = \underline{c_1 \sqrt{\frac{2}{\pi x}} \sinh x + c_2 \sqrt{\frac{2}{\pi x}} \cosh x}$$

$$x^2 \frac{d^2 y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p)y = 0$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$y(x) = x^\alpha \left[c_1 I_\nu(kx^\beta) + c_2 I_{-\nu}(kx^\beta) \right]$$

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' + (2x+1)y' + (2x+1)y = 0$$

Solution

$$\text{✗ } xy'' + (2x+1)y' + (2x+1)y = 0$$

$$x^2 y'' + x(2x+1)y' + (2x^2+x)y = 0$$

$$\text{Let } Y = ye^x \rightarrow y = Ye^{-x}$$

$$x^2 (Y'' - 2Y' + Y)e^{-x} + x(2x+1)(Y' - Y)e^{-x} + (2x^2+x)Ye^{-x} = 0$$

$$x^2 Y'' - 2x^2 Y' + x^2 Y + (2x^2+x)Y' - (2x^2+x)Y + (2x^2+x)Y = 0$$

$$x^2 Y'' + xY' + x^2 Y = 0 \qquad x^2 \frac{d^2 y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p)y = 0$$

$$A=1, \quad B=0, \quad C=1, \quad p=2$$

$$\alpha=0, \quad \beta=1, \quad k=1, \quad \nu=0$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$Y(x) = c_1 J_0(x) + c_2 Y_0(x)$$

$$y(x) = x^\alpha \left[c_1 J_0(kx^\beta) + c_2 Y_0(kx^\beta) \right]$$

$$\underline{y(x) = (c_1 J_0(x) + c_2 Y_0(x))e^{-x}}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' - y' - xy = 0$$

Solution

$$\text{✗ } xy'' - y' - xy = 0$$

$$x^2 y'' - xy' - x^2 y = 0$$

$$A=-1, \quad B=0, \quad C=-1=i, \quad p=2$$

$$\text{Let } Y = \frac{y}{x} \quad \& \quad X = ix$$

$$y = xY \quad \& \quad x = -iX$$

$$x^2 (2Y' + xY'') - x(Y + xY') - x^3 Y = 0$$

$$x^2 \frac{d^2 y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p)y = 0$$

$$x^3 Y'' + x^2 Y' - x(x^2 + 1)Y = 0$$

$$x^2 Y'' + x Y' - (x^2 + 1)Y = 0$$

$$-X^2 Y'' - iXY' - (-X^2 + 1)Y = 0$$

$$X^2 Y'' + XY' + (X^2 - 1)Y = 0$$

$$A = 1, \quad B = -1, \quad C = 1, \quad p = 2$$

$$\alpha = 0, \quad \beta = 1, \quad k = 1, \quad \nu = 1$$

$$Y = Z_1(X)$$

$$y(x) = xZ_1(ix)$$

$$= x(c_1 I_1(x) + c_2 K_1(x))$$

$$x^2 \frac{d^2 y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p)y = 0$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$y(x) = x^\alpha \left[c_1 J_\nu(kx^\beta) + c_2 Y_\nu(kx^\beta) \right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^4 y'' + a^2 y = 0$$

Solution

$$\frac{1}{x^2} \times x^4 y'' + a^2 y = 0$$

$$x^2 y'' + \frac{a^2}{x^2} y = 0$$

$$\text{Let } Y = \frac{y}{\sqrt{x}} \rightarrow y = \sqrt{x} Y$$

$$X = \frac{a}{x} \rightarrow x = \frac{a}{X}$$

$$X^2 Y'' + XY' + (X^2 - K^2)Y = 0$$

$$Y = x^{-1/2} y$$

$$Y' = -\frac{1}{2}x^{-3/2}y + x^{-1/2}y'$$

$$Y'' = \frac{3}{4}x^{-5/2}y - x^{-3/2}y' + x^{-1/2}y''$$

$$x^2 \left(x^{-1/2} y'' - x^{-3/2} y' + \frac{3}{4} x^{-5/2} y \right) + x \left(-\frac{1}{2} x^{-3/2} y + x^{-1/2} y' \right) + (x^2 - K^2) x^{-1/2} y = 0$$

$$x^{3/2}y'' - x^{1/2}y' + \frac{3}{4}x^{-1/2}y - \frac{1}{2}x^{-1/2}y + x^{1/2}y' + (x^2 - K^2)x^{-1/2}y = 0$$

$$x^{3/2}y'' + \left(x^2 - K^2 + \frac{1}{4}\right)x^{-1/2}y = 0 \quad \times x^{1/2}$$

$$x^2y'' + \left(x^2 - K^2 + \frac{1}{4}\right)y = 0$$

$$x^2 - K^2 + \frac{1}{4} = x^2$$

$$-K^2 + \frac{1}{4} = 0 \rightarrow K^2 = \frac{1}{4}$$

$$X^2Y'' + XY' + \left(X^2 - \frac{1}{4}\right)Y = 0$$

$$A=1, \quad B=-\frac{1}{4}, \quad C=1, \quad p=2$$

$$\alpha=0, \quad \beta=1, \quad k=1, \quad \nu=\frac{1}{2}$$

$$Y = Z_{1/2}(X)$$

$$y(x) = \sqrt{x}Z_{1/2}\left(\frac{a}{x}\right)$$

$$= \sqrt{x}\left(c_1J_{1/2}\left(\frac{a}{x}\right) + c_2J_{-1/2}\left(\frac{a}{x}\right)\right)$$

$$y(z) = x^\alpha \left(c_1 \left(\frac{2}{\pi z}\right)^{1/2} \sin z + c_2 \left(\frac{2}{\pi z}\right)^{1/2} \cos z \right)$$

$$= \sqrt{x} \left(c_1 \sqrt{\frac{2x}{\pi a}} \sin \frac{a}{x} + c_2 \sqrt{\frac{2x}{\pi a}} \cos \frac{a}{x} \right)$$

$$= x \left(c_1 \sqrt{\frac{2}{\pi a}} \sin \frac{a}{x} + c_2 \sqrt{\frac{2}{\pi a}} \cos \frac{a}{x} \right)$$

$$= x \left(C_1 \sin \frac{a}{x} + C_2 \cos \frac{a}{x} \right)$$

$$x^2 \frac{d^2y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p)y = 0$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$y(x) = x^\alpha \left[c_1 J_\nu \left(kx^\beta \right) + c_2 J_{-\nu} \left(kx^\beta \right) \right]$$

$$z = kX^\beta = \frac{a}{x}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y'' - x^2y = 0$$

Solution

$$x^2 \times y'' - x^2y = 0$$

$$x^2y'' - x^4y = 0$$

$$A=0, \quad B=0, \quad C=-1, \quad p=4$$

$$\alpha = \frac{1}{2}, \quad \beta = 1, \quad k = \frac{i}{2}, \quad \nu = 0$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\text{Let } Y = \frac{y}{\sqrt{x}} \rightarrow y = \sqrt{x} Y$$

$$X = \frac{1}{2}ix^2 \rightarrow x^2 = -2iX$$

$$X^2 Y'' + XY' + (X^2 - K^2)Y = 0$$

$$Y = x^{-1/2}y$$

$$Y' = -\frac{1}{2}x^{-3/2}y + x^{-1/2}y'$$

$$Y'' = \frac{3}{4}x^{-5/2}y - x^{-3/2}y' + x^{-1/2}y''$$

$$x^2 \left(x^{-1/2}y'' - x^{-3/2}y' + \frac{3}{4}x^{-5/2}y \right) + x \left(-\frac{1}{2}x^{-3/2}y + x^{-1/2}y' \right) + (x^2 - K^2)x^{-1/2}y = 0$$

$$x^{3/2}y'' - x^{1/2}y' + \frac{3}{4}x^{-1/2}y - \frac{1}{2}x^{-1/2}y + x^{1/2}y' + (x^2 - K^2)x^{-1/2}y = 0$$

$$x^{3/2}y'' + \left(x^2 - K^2 + \frac{1}{4} \right) x^{-1/2}y = 0 \quad \times x^{1/2}$$

$$x^2 y'' + \left(x^2 - K^2 + \frac{1}{4} \right) y = 0$$

$$x^2 - K^2 + \frac{1}{4} = -x^4$$

$$K = \frac{1}{4} \rightarrow K^2 = \frac{1}{16}$$

$$X^2 Y'' + XY' + \left(X^2 - \frac{1}{16} \right) Y = 0$$

$$A = 1, \quad B = -\frac{1}{16}, \quad C = 1, \quad p = 2$$

$$\alpha = 0, \quad \beta = 1, \quad k = 1, \quad \nu = \frac{1}{4}$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$Y = Z_{1/4}(X)$$

$$y(x) = \sqrt{x} Z_{\frac{1}{4}} \left(\frac{i}{2} x^2 \right)$$

$$= \sqrt{x} \left[c_1 I_{\frac{1}{4}} \left(\frac{x^2}{2} \right) + c_2 I_{-\frac{1}{4}} \left(\frac{x^2}{2} \right) \right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2 y'' - xy' + (1 + x^2)y = 0$$

Solution

$$x^2 y'' - xy' + (1 + x^2)y = 0$$

$$A = -1, \quad B = 1, \quad C = 1, \quad p = 2$$

$$\alpha = 1, \quad \beta = 1, \quad k = 1, \quad \nu = 0$$

$$y(x) = x \left[c_1 J_0(x) + c_2 Y_0(x) \right]$$

$$x^2 \frac{d^2 y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p)y = 0$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' + 3y' + xy = 0$$

Solution

$$x \times xy'' + 3y' + xy = 0$$

$$x^2 y'' + 3xy' + x^2 y = 0$$

$$A = 3, \quad B = 0, \quad C = 1, \quad p = 2$$

$$\alpha = -1, \quad \beta = 1, \quad k = 1, \quad \nu = 1$$

$$y(x) = x^{-1} \left[c_1 J_1(x) + c_2 Y_1(x) \right]$$

$$x^2 \frac{d^2 y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p)y = 0$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' - y' + 36x^3 y = 0$$

Solution

$$x \times xy'' - y' + 36x^3 y = 0$$

$$x^2 y'' - xy' + 36x^4 y = 0$$

$$A = -1, \quad B = 0, \quad C = 36, \quad p = 4$$

$$\alpha = 1, \quad \beta = 2, \quad k = 3, \quad \nu = \frac{1}{2}$$

$$y(x) = x \left[c_1 J_{1/2}(3x^2) + c_2 J_{-1/2}(3x^2) \right]$$

$$x^2 \frac{d^2 y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p)y = 0$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$y(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2 y'' - 5xy' + (8+x)y = 0$$

Solution

$$x^2 y'' - 5xy' + (8+x)y = 0$$

$$A = -5, \quad B = 8, \quad C = 1, \quad p = 1$$

$$\alpha = 3, \quad \beta = \frac{1}{2}, \quad k = 2, \quad \nu = 2$$

$$y(x) = x^3 \left[c_1 J_2(2x^{1/2}) + c_2 Y_2(2x^{1/2}) \right]$$

$$x^2 \frac{d^2 y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p)y = 0$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$y(x) = x^\alpha \left[c_1 J_\nu(kx^\beta) + c_2 Y_\nu(kx^\beta) \right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$36x^2 y'' + 60xy' + (9x^3 - 5)y = 0$$

Solution

$$x^2 y'' + \frac{5}{3}xy' + \left(\frac{1}{4}x^3 - \frac{5}{36}\right)y = 0$$

$$A = \frac{5}{3}, \quad B = -\frac{5}{36}, \quad C = \frac{1}{4}, \quad p = 3$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\alpha = -\frac{1}{3}, \quad \beta = \frac{3}{2}, \quad k = \frac{1}{3}, \quad \nu = \frac{\sqrt{\left(-\frac{2}{3}\right)^2 + \frac{5}{9}}}{3} = \frac{1}{3}$$

$$x^2 \frac{d^2 y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p)y = 0$$

$$y(x) = x^\alpha \left[c_1 J_\nu(kx^\beta) + c_2 J_{-\nu}(kx^\beta) \right]$$

$$y(x) = x^{-1/3} \left[c_1 J_{1/3} \left(\frac{1}{3} x^{3/2} \right) + c_2 J_{-1/3} \left(\frac{1}{3} x^{3/2} \right) \right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$16x^2 y'' + 24xy' + (1 + 144x^3)y = 0$$

Solution

$$x^2 y'' + \frac{3}{2} xy' + \left(\frac{1}{16} + 9x^3 \right) y = 0$$

$$A = \frac{3}{2}, \quad B = \frac{1}{16}, \quad C = 9, \quad p = 3$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\alpha = -\frac{1}{4}, \quad \beta = \frac{3}{2}, \quad k = 2, \quad \nu = \frac{\sqrt{\left(-\frac{1}{2}\right)^2 - \frac{1}{4}}}{3} = 0$$

$$y(x) = x^{-1/4} \left[c_1 J_0 \left(2x^{3/2} \right) + c_2 Y_0 \left(2x^{3/2} \right) \right]$$

$$x^2 \frac{d^2 y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p)y = 0$$

$$y(x) = x^\alpha \left[c_1 J_\nu(kx^\beta) + c_2 Y_\nu(kx^\beta) \right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2 y'' + 3xy' + (1 + x^2)y = 0$$

Solution

$$x^2 y'' + 3xy' + (1 + x^2)y = 0$$

$$A = 3, \quad B = 1, \quad C = 1, \quad p = 2$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\alpha = -1, \quad \beta = 1, \quad k = 1, \quad \nu = \frac{\sqrt{(-2)^2 - 4}}{3} = 0$$

$$x^2 \frac{d^2 y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p)y = 0$$

$$\underline{y(x) = x^{-1} \left[c_1 J_0(x) + c_2 Y_0(x) \right]}$$

$$y(x) = x^\alpha \left[c_1 J_\nu(kx^\beta) + c_2 Y_\nu(kx^\beta) \right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$4x^2 y'' - 12xy' + (15 + 16x)y = 0$$

Solution

$$x^2 y'' - 3xy' + \left(\frac{15}{4} + 4x \right) y = 0$$

$$x^2 \frac{d^2 y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p) y = 0$$

$$A = -3, \quad B = \frac{15}{4}, \quad C = 4, \quad p = 1$$

$$\alpha = 2, \quad \beta = \frac{1}{2}, \quad k = 4, \quad \nu = \frac{\sqrt{(4)^2 - 15}}{1} = 1$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\underline{y(x) = x^2 \left[c_1 J_1(4x^{1/2}) + c_2 Y_1(4x^{1/2}) \right]}$$

$$y(x) = x^\alpha \left[c_1 J_\nu(kx^\beta) + c_2 Y_\nu(kx^\beta) \right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$16x^2 y'' - (5 - 144x^3) y = 0$$

Solution

$$x^2 y'' + \left(9x^3 - \frac{5}{16} \right) y = 0$$

$$x^2 \frac{d^2 y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p) y = 0$$

$$A = 0, \quad B = -\frac{5}{16}, \quad C = 9, \quad p = 3$$

$$\alpha = \frac{1}{2}, \quad \beta = \frac{3}{2}, \quad k = 2, \quad \nu = \frac{\sqrt{1 + \frac{5}{4}}}{3} = \frac{1}{2}$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\underline{y(x) = x^{1/2} \left[c_1 J_{1/2}(2x^{3/2}) + c_2 J_{-1/2}(2x^{3/2}) \right]}$$

$$y(x) = x^\alpha \left[c_1 J_\nu(kx^\beta) + c_2 J_{-\nu}(kx^\beta) \right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$2x^2y'' + 3xy' - (28 - 2x^5)y = 0$$

Solution

$$x^2y'' + \frac{3}{2}xy' + (x^5 - 14)y = 0$$

$$x^2 \frac{d^2y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p)y = 0$$

$$A = \frac{3}{2}, \quad B = -14, \quad C = 1, \quad p = 5$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\alpha = -\frac{1}{4}, \quad \beta = \frac{5}{2}, \quad k = \frac{2}{5}, \quad \nu = \frac{\sqrt{\left(-\frac{1}{2}\right)^2 + 56}}{5} = \frac{\frac{15}{2}}{5} = \frac{3}{2}$$

$$y(x) = x^{-1/4} \left[c_1 J_{3/2} \left(\frac{2}{5} x^{5/2} \right) + c_2 J_{-3/2} \left(\frac{2}{5} x^{5/2} \right) \right] \quad \left| \quad y(x) = x^\alpha \left[c_1 J_\nu \left(kx^\beta \right) + c_2 J_{-\nu} \left(kx^\beta \right) \right] \right.$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y'' + x^4y = 0$$

Solution

$$x^2 \times y'' + x^4y = 0$$

$$x^2y'' + x^6y = 0$$

$$x^2 \frac{d^2y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p)y = 0$$

$$A = 0, \quad B = 0, \quad C = 1, \quad p = 6$$

$$\alpha = \frac{1}{2}, \quad \beta = 3, \quad k = \frac{1}{3}, \quad \nu = \frac{1}{6}$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$y(x) = x^{1/2} \left[c_1 J_{1/6} \left(\frac{1}{3} x^3 \right) + c_2 J_{-1/6} \left(\frac{1}{3} x^3 \right) \right] \quad \left| \quad y(x) = x^\alpha \left[c_1 J_\nu \left(kx^\beta \right) + c_2 J_{-\nu} \left(kx^\beta \right) \right] \right.$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y'' + 4x^3 y = 0$$

Solution

$$x^2 \times y'' + 4x^3 y = 0$$

$$x^2 y'' + 4x^5 y = 0$$

$$A = 0, \quad B = 0, \quad C = 4, \quad p = 5$$

$$\alpha = \frac{1}{2}, \quad \beta = \frac{5}{2}, \quad k = \frac{4}{5}, \quad \nu = \frac{1}{5}$$

$$y(x) = x^{1/2} \left[c_1 J_{1/5} \left(\frac{4}{5} x^{5/2} \right) + c_2 J_{-1/5} \left(\frac{4}{5} x^{5/2} \right) \right]$$

$$x^2 \frac{d^2 y}{dx^2} + Ax \frac{dy}{dx} + (B + Cx^p) y = 0$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$y(x) = x^\alpha \left[c_1 J_\nu(kx^\beta) + c_2 J_{-\nu}(kx^\beta) \right]$$

Exercise

Find a Frobenius solution of Bessel's equation of order zero $x^2 y'' + xy' + x^2 y = 0$

Solution

$$y'' + \frac{1}{x} y' + y = 0$$

Therefore, $x = 0$ is a regular singular point, and that $p_0 = 1$, $q_0 = 0$ and $p(x) \equiv 1$, $q(x) = x^2$.

The indicial equation is: $r(r-1) + r = r^2 = 0 \rightarrow \underline{r = 0}$

There is only one Frobenius series solution: $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$x^2 y'' + xy' + x^2 y = 0$$

$$x^2 \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + x \sum_{n=0}^{\infty} na_n x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} [n(n-1) + n]a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\sum_{n=0}^{\infty} n^2 a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0 \quad a_4 = -\frac{a_2}{4^2} = \frac{a_0}{2^2 \cdot 4^2}$$

$$0 + a_1 x + \sum_{n=2}^{\infty} n^2 a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$a_1 x + \sum_{n=2}^{\infty} (n^2 a_n + a_{n-2}) x^n = 0$$

$$a_1 = 0 \rightarrow a_{n(\text{odd})} = 0$$

$$n^2 a_n + a_{n-2} = 0$$

$$a_n = -\frac{a_{n-2}}{n^2} \quad (n \geq 2)$$

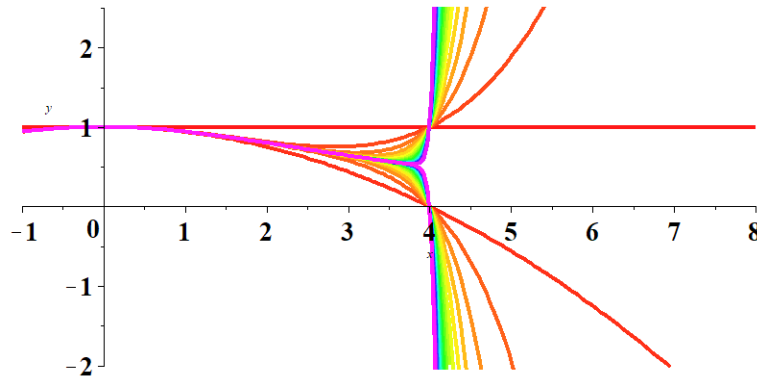
$$a_2 = -\frac{a_0}{4}$$

$$a_4 = -\frac{a_2}{6^2} = -\frac{a_0}{2^2 \cdot 4^2 \cdot 6^2}$$

$$a_{2n} = \frac{(-1)^n}{2^2 \cdot 4^2 \cdots (2n)^2} a_0 = \frac{(-1)^n}{2^{2n} \cdot (n!)^2} a_0$$

The choice $a_0 = 1$ gives us the Bessel function of order zero of the first kind.

$$J_0(x) = \frac{(-1)^n x^{2n}}{2^{2n} \cdot (n!)^2} = \underline{1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots}$$



Exercise

Derive the formula $x J'_\nu(x) = \nu J_\nu(x) - x J_{\nu+1}(x)$

Solution

$$\begin{aligned}
 x J_\nu(x) &= x \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \\
 x J'_\nu(x) &= x \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \\
 &= 2 \sum_{n=0}^{\infty} \frac{(-1)^n n}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} + \nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n n}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} + \nu J_\nu(x) \\
 &= \nu J_\nu(x) + x \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1} \\
 &= \nu J_\nu(x) + x \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n! \Gamma(2+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu+1} \\
 &= \nu J_\nu(x) - x J_{\nu+1}(x) \quad \checkmark
 \end{aligned}$$

Exercise

Derive the formula $x J'_\nu(x) = -\nu J_\nu(x) + x J_{\nu-1}(x)$

Solution

$$\begin{aligned}
 x J'_\nu(x) &= x \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1} \\
 &= 2 \sum_{n=0}^{\infty} \frac{(-1)^n n}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} + \nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \\
 -\nu J_\nu(x) + x J_{\nu-1}(x) &= -\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} + x \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1} \\
 &= -\sum_{n=0}^{\infty} \frac{(-1)^n \nu}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \\
 &= -\sum_{n=0}^{\infty} \frac{(-1)^n \nu}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} + \sum_{n=0}^{\infty} \frac{(-1)^n (\nu+n)}{n! \Gamma(1+\nu+n)} 2 \left(\frac{x}{2}\right) \left(\frac{x}{2}\right)^{2n+\nu-1} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n (-\nu + 2n + 2\nu)}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \\
 &= x \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1} \\
 &= x J'_\nu(x) \quad \checkmark
 \end{aligned}$$

Exercise

Derive the formula $2\nu J'_\nu(x) = x J_{\nu+1}(x) + x J_{\nu-1}(x)$

Solution

From previous proofs:

$$\begin{aligned}
 x J'_\nu(x) &= \nu J_\nu(x) - x J_{\nu+1}(x) \\
 - \quad x J'_\nu(x) &= -\nu J_\nu(x) + x J_{\nu-1}(x) \\
 \hline
 0 &= 2\nu J_\nu(x) - x J_{\nu+1}(x) - x J_{\nu-1}(x) \\
 \hline
 2\nu J'_\nu(x) &= x J_{\nu+1}(x) + x J_{\nu-1}(x) \quad \checkmark
 \end{aligned}$$

Exercise

Prove that $\frac{d}{dx} \left[x^{\nu+1} J_{\nu+1}(x) \right] = x^{\nu+1} J_\nu(x)$

Solution

$$\begin{aligned}
 \frac{d}{dx} \left[x^{\nu+1} J_{\nu+1}(x) \right] &= \frac{d}{dx} \left[x^{\nu+1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1+\nu+1+n)} \left(\frac{x}{2} \right)^{2n+\nu+1} \right] \\
 &= \frac{d}{dx} \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\nu+n+2)} \left(\frac{x}{2} \right)^{2n+2\nu+2} \right] \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2\nu+2)}{n! \Gamma(\nu+n+2)} \left(\frac{x}{2} \right)^{2n+2\nu+1} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n 2(n+\nu+1)}{n! 2\Gamma(\nu+n+2)} \left(\frac{x}{2} \right)^{2n+2\nu+1} \\
 &\quad 2\Gamma(\nu+n+2) = 2(\nu+n+1)\Gamma(\nu+n+1) \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n 2(n+\nu+1)}{n! 2(\nu+n+1)\Gamma(\nu+n+1)} \left(\frac{x}{2} \right)^{2n+2\nu+1} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\nu+n+1)} \left(\frac{x}{2} \right)^{2n+2\nu+1} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\nu+n+1)} \left(\frac{x}{2} \right)^{2n+\nu} \left(\frac{x}{2} \right)^{\nu+1} \\
 &= x^{\nu+1} J_\nu(x) \quad \checkmark
 \end{aligned}$$

Exercise

Show that $y = \sqrt{x} J_{3/2}(x)$ is a solution of $x^2 y'' + (x^2 - 2)y = 0$

Solution

$$x^2 y'' + (x^2 - 2)y = 0$$

$J_{3/2}(x)$ is the solution of Bessel's equation of order $\frac{3}{2}$:

$$x^2 J''_{3/2}(x) + x J'_{3/2}(x) + \left(x^2 - \frac{9}{4}\right) J_{3/2}(x) = 0$$

$$\begin{aligned} x^2 \left(\sqrt{x} J_{3/2}(x) \right)'' + (x^2 - 2) \sqrt{x} J_{3/2}(x) &= \\ &= x^2 \left[-\frac{1}{4} x^{-3/2} J_{3/2}(x) + x^{-1/2} J'_{3/2}(x) + x^{1/2} J''_{3/2}(x) \right] + (x^2 - 2) \sqrt{x} J_{3/2}(x) \\ &= -\frac{1}{4} x^{1/2} J_{3/2}(x) + x^{3/2} J'_{3/2}(x) + x^{5/2} J''_{3/2}(x) + x^{5/2} J_{3/2}(x) - 2 \sqrt{x} J_{3/2}(x) \\ &= \sqrt{x} \left[x^2 J''_{3/2}(x) + x J'_{3/2}(x) + \left(x^2 - \frac{9}{4}\right) J_{3/2}(x) \right] \\ &= 0 \end{aligned}$$

Exercise

Show that $4J''_{\nu}(x) = J_{\nu-2}(x) - 2J_{\nu}(x) + J_{\nu+2}(x)$

Solution

$$\begin{aligned} J_{\nu}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 + \nu + n)} \left(\frac{x}{2}\right)^{2n+\nu} \\ x J'_{\nu}(x) &= x \sum_{n=0}^{\infty} \frac{(-1)^n (2n + \nu)}{n! \Gamma(1 + \nu + n)} \left(\frac{x}{2}\right)^{2n+\nu-1} \\ &= 2 \sum_{n=0}^{\infty} \frac{(-1)^n n}{n! \Gamma(1 + \nu + n)} \left(\frac{x}{2}\right)^{2n+\nu} + \nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 + \nu + n)} \left(\frac{x}{2}\right)^{2n+\nu} \\ -\nu J_{\nu}(x) + x J_{\nu-1}(x) &= -\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 + \nu + n)} \left(\frac{x}{2}\right)^{2n+\nu} + x \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\nu + n)} \left(\frac{x}{2}\right)^{2n+\nu-1} \\ &= -\sum_{n=0}^{\infty} \frac{(-1)^n \nu}{n! \Gamma(1 + \nu + n)} \left(\frac{x}{2}\right)^{2n+\nu} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\nu + n)} \left(\frac{x}{2}\right)^{2n+\nu} \end{aligned}$$

$$\begin{aligned}
&= - \sum_{n=0}^{\infty} \frac{(-1)^n \nu}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} + \sum_{n=0}^{\infty} \frac{(-1)^n (\nu+n)}{n! \Gamma(1+\nu+n)} 2 \left(\frac{x}{2}\right) \left(\frac{x}{2}\right)^{2n+\nu-1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (-\nu+2n+2\nu)}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \\
&= x \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1} \\
&= \underline{x J'_{\nu}(x)} \quad |
\end{aligned}$$

$$\begin{aligned}
x J'_{\nu}(x) &= x \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \\
&= 2 \sum_{n=0}^{\infty} \frac{(-1)^n n}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} + \nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n n}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} + \nu J_{\nu}(x) \\
&= \nu J_{\nu}(x) + \sum_{\substack{n=1 \\ \text{blue}}}^{\infty} \frac{(-1)^n}{(n-1)! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1} \\
&= \nu J_{\nu}(x) + x \sum_{\substack{n=0 \\ \text{red}}}^{\infty} \frac{(-1)^{n-1}}{n! \Gamma(2+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu+1} \\
&= \underline{\nu J_{\nu}(x) - x J_{\nu+1}(x)} \quad | \\
x J'_{\nu}(x) &= -\nu J_{\nu}(x) + x J_{\nu-1}(x)
\end{aligned}$$

$$\begin{aligned}
 x J'_\nu(x) &= \nu J_\nu(x) - x J_{\nu+1}(x) \\
 + \quad x J'_\nu(x) &= -\nu J_\nu(x) + x J_{\nu-1}(x) \\
 \hline
 2x J'_\nu(x) &= -x J_{\nu+1}(x) + x J_{\nu-1}(x)
 \end{aligned}$$

$$J'_\nu(x) = \frac{1}{2} (J_{\nu-1}(x) - J_{\nu+1}(x))$$

$$J''_\nu(x) = \frac{1}{2} (J'_{\nu-1}(x) - J'_{\nu+1}(x))$$

$$J'_\nu(x) = \frac{1}{2} (J_{\nu-1}(x) - J_{\nu+1}(x)) \rightarrow (\nu = \nu - 1) \quad J'_{\nu-1}(x) = \frac{1}{2} (J_{\nu-2}(x) - J_\nu(x))$$

$$J'_\nu(x) = \frac{1}{2} (J_{\nu-1}(x) - J_{\nu+1}(x)) \rightarrow (\nu = \nu + 1) \quad J'_{\nu+1}(x) = \frac{1}{2} (J_\nu(x) - J_{\nu+2}(x))$$

$$\begin{aligned}
 J''_\nu(x) &= \frac{1}{2} (J'_{\nu-1}(x) - J'_{\nu+1}(x)) \\
 &= \frac{1}{2} \left(\frac{1}{2} J_{\nu-2}(x) - \frac{1}{2} J_\nu(x) - \frac{1}{2} J_\nu(x) + \frac{1}{2} J_{\nu+2}(x) \right) \\
 &= \frac{1}{4} (J_{\nu-2}(x) - 2J_\nu(x) + J_{\nu+2}(x))
 \end{aligned}$$

$$\underline{4J''_\nu(x) = J_{\nu-2}(x) - 2J_\nu(x) + J_{\nu+2}(x)} \quad \checkmark$$

Exercise

Show that $y = x^{1/2} w\left(\frac{2}{3}\alpha x^{3/2}\right)$ is a solution of Airy's differential equation $y'' + \alpha^2 xy = 0$, $x > 0$,

whenever w is a solution of Bessel's equation of order $\frac{2}{3}$, that is, $t^2 w'' + tw' + \left(t^2 - \frac{1}{9}\right)w = 0$, $t > 0$.

[Hint: After differentiating, substituting, and simplifying, then let $t = \frac{2}{3}\alpha x^{3/2}$].

Solution

$$y = x^{1/2} w\left(\frac{2}{3}\alpha x^{3/2}\right)$$

$$\begin{aligned}
 y' &= \frac{1}{2} x^{-1/2} w\left(\frac{2}{3}\alpha x^{3/2}\right) + x^{1/2} \left(\alpha x^{1/2}\right) w'\left(\frac{2}{3}\alpha x^{3/2}\right) \\
 &= \alpha x w'\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{1}{2} x^{-1/2} w\left(\frac{2}{3}\alpha x^{3/2}\right)
 \end{aligned}$$

$$\begin{aligned}
 y'' &= \alpha x \left(\alpha x^{1/2}\right) w''\left(\frac{2}{3}\alpha x^{3/2}\right) + \alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{1}{2} x^{-1/2} \left(\alpha x^{1/2}\right) w'\left(\frac{2}{3}\alpha x^{3/2}\right) - \frac{1}{4} x^{-3/2} w\left(\frac{2}{3}\alpha x^{3/2}\right) \\
 &= \alpha^2 x^{3/2} w''\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{3}{2} \alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right) - \frac{1}{4} x^{-3/2} w\left(\frac{2}{3}\alpha x^{3/2}\right)
 \end{aligned}$$

$$y'' + \alpha^2 xy = 0$$

$$\alpha^2 x^{3/2} w'' \left(\frac{2}{3} \alpha x^{3/2} \right) + \frac{3}{2} \alpha w' \left(\frac{2}{3} \alpha x^{3/2} \right) - \frac{1}{4} x^{-3/2} w \left(\frac{2}{3} \alpha x^{3/2} \right) + \alpha^2 x^{3/2} w \left(\frac{2}{3} \alpha x^{3/2} \right) = 0$$

$$\alpha^2 x^{3/2} w'' \left(\frac{2}{3} \alpha x^{3/2} \right) + \frac{3}{2} \alpha w' \left(\frac{2}{3} \alpha x^{3/2} \right) + \left(\alpha^2 x^{3/2} - \frac{1}{4x^{3/2}} \right) w \left(\frac{2}{3} \alpha x^{3/2} \right) = 0$$

$$t = \frac{2}{3} \alpha x^{3/2} \rightarrow \alpha x^{3/2} = \frac{3}{2} t$$

$$\frac{3}{2} \frac{\alpha}{t} \left[t^2 w''(t) + t w'(t) + \left(t^2 - \frac{1}{9} \right) w(t) \right] = 0$$

$$\underline{t^2 w'' + t w' + \left(t^2 - \frac{1}{9} \right) w = 0} \quad \checkmark$$

Exercise

Use the relation $\Gamma(x+1) = x\Gamma(x)$ and if p is nonnegative integer, then show that

$$J_{\nu}(x) = \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2} \right)^{\nu} \left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\nu+1)(\nu+2)\cdots(\nu+n)} \left(\frac{x}{2} \right)^{2n} \right]$$

Solution

$$J_{\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\nu+n+1)} \left(\frac{x}{2} \right)^{2n+\nu}$$

Given: $\Gamma(x+1) = x\Gamma(x)$

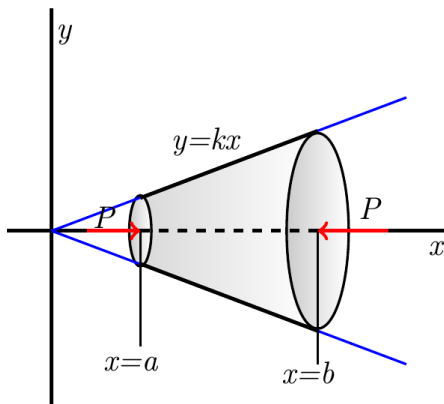
$$\Gamma(\nu+n+1) = (\nu+1)(\nu+2)\cdots(\nu+n)\Gamma(\nu+n)$$

$$\begin{aligned} J_{\nu}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n! (\nu+1)(\nu+2)\cdots(\nu+n)\Gamma(\nu+n)} \left(\frac{x}{2} \right)^{2n} \left(\frac{x}{2} \right)^{\nu} \\ &= \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2} \right)^{\nu} \left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\nu+1)(\nu+2)\cdots(\nu+n)} \left(\frac{x}{2} \right)^{2n} \right] \quad \checkmark \end{aligned}$$

Exercise

A linearly tapered rod with circular cross section, subject to an axial force P of compression. Its deflection curve $y = y(x)$ satisfies the endpoint value problem

$$EIy'' + Py = 0 ; \quad y(a) = y(b) = 0 \quad (1)$$



Here, however, the moment of inertia $I = I(x)$ of the cross section at x is given by

$$I(x) = \frac{1}{4}\pi(kx)^4 = I_0 \left(\frac{x}{b}\right)^4 \quad (2)$$

Where $I_0 = I(b)$, the value of I at $x = b$. Substitution of $I(x)$ in the differential equation (1) yields to the eigenvalue problem

$$x^4 y'' + \lambda y = 0 ; \quad y(a) = y(b) = 0 \quad (3)$$

Where $\lambda = \mu^2 = \frac{Pb^4}{EI_0}$

a) Show that the general solution of $x^4 y'' + \mu^2 y = 0$ is $y(x) = x \left(A \cos \frac{\mu}{x} + B \sin \frac{\mu}{x} \right)$

b) Conclude that the n th eigenvalue is given by $\mu_n = n\pi \frac{ab}{L}$, where $L = b - a$ is the length of the rod, and hence that the n th buckling force is

$$P_n = \frac{n^2 \pi^2}{L^2} \left(\frac{a}{b}\right)^2 EI_0$$

Solution

a) $x^{-2} \times x^4 y'' + \mu^2 y = 0$

$$x^2 y'' + \mu^2 x^{-2} y = 0$$

$$A = 0, \quad B = 0, \quad C = \mu^2, \quad p = -2$$

$$\alpha = \frac{1}{2}, \quad \beta = -1, \quad k = \mu, \quad \nu = \frac{1}{2}$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$y(x) = x^{1/2} \left[c_1 J_{1/2}(\mu x^{-1}) + c_2 J_{-1/2}(\mu x^{-1}) \right]$$

$$y(x) = x^\alpha \left(c_1 J_{1/2}(kx^\beta) + c_2 J_{-1/2}(kx^\beta) \right)$$

$$\begin{aligned}
&= \sqrt{x} \left(c_1 \sqrt{\frac{2x}{\pi\mu}} \cos\left(\frac{\mu}{x}\right) + c_2 \sqrt{\frac{2x}{\pi\mu}} \sin\left(\frac{\mu}{x}\right) \right) \\
&= x^\alpha \left(c_1 \left(\frac{2}{\pi k x^\beta}\right)^{1/2} \sin(kx^\beta) + c_2 \left(\frac{2}{\pi k x^\beta}\right)^{1/2} \cos(kx^\beta) \right) \\
&= x \left(c_1 \sqrt{\frac{2}{\pi\mu}} \cos\left(\frac{\mu}{x}\right) + c_2 \sqrt{\frac{2}{\pi\mu}} \sin\left(\frac{\mu}{x}\right) \right) \qquad A = c_1 \sqrt{\frac{2}{\pi\mu}}, \quad B = c_2 \sqrt{\frac{2}{\pi\mu}} \\
&= x \left(A \cos\left(\frac{\mu}{x}\right) + B \sin\left(\frac{\mu}{x}\right) \right)
\end{aligned}$$

b) Given: $\mu_n = n\pi \frac{ab}{L}$; $y(a) = y(b) = 0$, $L = b - a$

$$\begin{cases} y(a) = a \left(A \cos\left(\frac{\mu}{a}\right) + B \sin\left(\frac{\mu}{a}\right) \right) = 0 \\ y(b) = b \left(A \cos\left(\frac{\mu}{b}\right) + B \sin\left(\frac{\mu}{b}\right) \right) = 0 \end{cases}$$

$$\begin{cases} A \cos\left(\frac{\mu}{a}\right) + B \sin\left(\frac{\mu}{a}\right) = 0 \\ A \cos\left(\frac{\mu}{b}\right) + B \sin\left(\frac{\mu}{b}\right) = 0 \end{cases} \quad (a, b \neq 0)$$

$$\begin{aligned}
\Delta &= \begin{vmatrix} \cos \frac{\mu}{a} & \sin \frac{\mu}{a} \\ \cos \frac{\mu}{b} & \sin \frac{\mu}{b} \end{vmatrix} \\
&= \cos \frac{\mu}{a} \sin \frac{\mu}{b} - \sin \frac{\mu}{a} \cos \frac{\mu}{b} \\
&= \sin \left(\frac{\mu}{b} - \frac{\mu}{a} \right) \\
&= \sin \left(\frac{b-a}{ab} \mu \right) \\
&= \sin \left(\frac{L}{ab} \mu \right)
\end{aligned}$$

$$\lambda = \mu^2 = \frac{Pb^4}{EI_0}$$

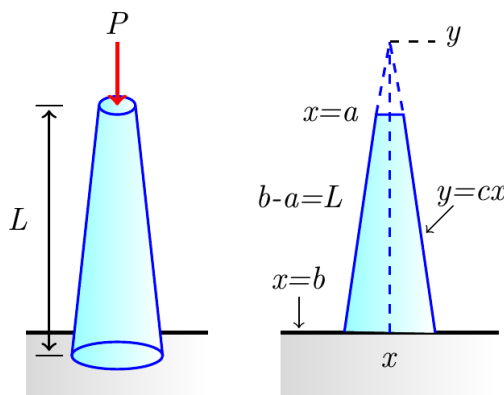
$$\begin{aligned}
P &= \frac{EI_0}{b^4} \mu^2 \\
&= \frac{EI_0}{b^4} \left(n\pi \frac{ab}{L} \right)^2 \\
&= \frac{n^2 \pi^2}{L^2} (EI_0) \left(\frac{a}{b} \right)^2
\end{aligned}$$

Exercise

When a constant vertical compressive force or load P was applied to a thin column of uniform cross section, the deflection $y(x)$ was a solution of the boundary-value problem

$$EI \frac{d^2 y}{dx^2} + Py = 0 ; \quad y(0) = 0, \quad y(L) = 0$$

The assumption here is that the column is hinged at both ends. The column will buckle or deflect only when the compression force is a critical load P_n



- a) Let assume that the column is of length L , is hinged at both ends, has circular cross sections, and is tapered. If the column, a truncated cone, has a linear taper $y = cx$ in cross section, the moment of inertia of a cross section with respect to an axis perpendicular to the xy - plane is

$$I = \frac{1}{4} \pi r^4, \text{ where } r = y \text{ and } y = cx. \text{ Hence we can write } I(x) = I_0 (x/b)^4, \text{ where}$$

$I_0 = I(b) = \frac{1}{4} \pi (cb)^4$. Substituting $I(x)$ into the differential equation, we see that the deflection in this case is determine from the BVP?

$$x^4 \frac{d^2 y}{dx^2} + \lambda y = 0 ; \quad y(a) = 0, \quad y(b) = 0$$

$$\text{Where } \lambda = Pb^4 EI_0$$

Find the critical loads P_n for the tapered column. Use an appropriate identity to express the buckling modes $y_n(x)$ as a single function.

- b) Plot the graph of the first buckling mode $y_1(x)$ corresponding to the Euler load P_1 when $b = 11$ and $a = 1$

Solution

$$a) \quad x^{-2} \times x^4 y'' + \lambda y = 0$$

$$x^2 y'' + \lambda x^{-2} y = 0$$

$$A = 0, \quad B = 0, \quad C = \lambda, \quad p = -2$$

$$\alpha = \frac{1}{2}, \quad \beta = -1, \quad k = \sqrt{\lambda}, \quad \nu = \frac{1}{2}$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$y(x) = x^{1/2} \left[c_1 J_{1/2} \left(\sqrt{\lambda} x^{-1} \right) + c_2 J_{-1/2} \left(\sqrt{\lambda} x^{-1} \right) \right]$$

$$\begin{aligned} y(x) &= x^\alpha \left(c_1 J_{1/2} \left(kx^\beta \right) + c_2 J_{-1/2} \left(kx^\beta \right) \right) \\ &= \sqrt{x} \left(c_1 \sqrt{\frac{2x}{\pi\sqrt{\lambda}}} \cos \left(\frac{\sqrt{\lambda}}{x} \right) + c_2 \sqrt{\frac{2x}{\pi\sqrt{\lambda}}} \sin \left(\frac{\sqrt{\lambda}}{x} \right) \right) \\ &= x^\alpha \left(c_1 \sqrt{\frac{2}{\pi k x^\beta}} \sin \left(kx^\beta \right) + c_2 \sqrt{\frac{2}{\pi k x^\beta}} \cos \left(kx^\beta \right) \right) \\ &= x \left(c_1 \sqrt{\frac{2x}{\pi\sqrt{\lambda}}} \cos \left(\frac{\sqrt{\lambda}}{x} \right) + c_2 \sqrt{\frac{2x}{\pi\sqrt{\lambda}}} \sin \left(\frac{\sqrt{\lambda}}{x} \right) \right) \quad A = c_1 \sqrt{\frac{2}{\pi\sqrt{\lambda}}}, \quad B = c_2 \sqrt{\frac{2}{\pi\sqrt{\lambda}}} \\ &= x \left(A \cos \left(\frac{\sqrt{\lambda}}{x} \right) + B \sin \left(\frac{\sqrt{\lambda}}{x} \right) \right) \end{aligned}$$

Given: $\lambda = Pb^4 EI_0$; $y(a) = y(b) = 0$, $L = b - a$

$$\begin{cases} y(a) = a \left(A \cos \left(\frac{\sqrt{\lambda}}{a} \right) + B \sin \left(\frac{\sqrt{\lambda}}{a} \right) \right) = 0 \\ y(b) = b \left(A \cos \left(\frac{\sqrt{\lambda}}{b} \right) + B \sin \left(\frac{\sqrt{\lambda}}{b} \right) \right) = 0 \end{cases}$$

$$\begin{cases} A \cos \left(\frac{\sqrt{\lambda}}{a} \right) + B \sin \left(\frac{\sqrt{\lambda}}{a} \right) = 0 \\ A \cos \left(\frac{\sqrt{\lambda}}{b} \right) + B \sin \left(\frac{\sqrt{\lambda}}{b} \right) = 0 \end{cases} \quad (a, b \neq 0)$$

$$\begin{aligned} \Delta &= \begin{vmatrix} \cos \frac{\sqrt{\lambda}}{a} & \sin \frac{\sqrt{\lambda}}{a} \\ \cos \frac{\sqrt{\lambda}}{b} & \sin \frac{\sqrt{\lambda}}{b} \end{vmatrix} \\ &= \cos \frac{\sqrt{\lambda}}{a} \sin \frac{\sqrt{\lambda}}{b} - \sin \frac{\sqrt{\lambda}}{a} \cos \frac{\sqrt{\lambda}}{b} \\ &= \sin \left(\frac{\sqrt{\lambda}}{b} - \frac{\sqrt{\lambda}}{a} \right) \end{aligned}$$

$$= \sin\left(\frac{b-a}{ab}\sqrt{\lambda}\right)$$

$$= \sin\left(\frac{L}{ab}\sqrt{\lambda}\right) = 0$$

$$\frac{L}{ab}\sqrt{\lambda} = n\pi \rightarrow \sqrt{\lambda} = \frac{n\pi ab}{L} \quad (n \in \mathbb{N})$$

$$\lambda = \frac{n^2 \pi^2 a^2 b^2}{L^2} = P b^4 E I_0$$

$$P_n = \frac{n^2 \pi^2}{L^2} (E I_0) \left(\frac{a}{b}\right)^2$$

If we let $B = -A \frac{\sin \frac{\sqrt{\lambda}}{a}}{\cos \frac{\sqrt{\lambda}}{a}}$

$$y(x) = x \left(A \cos\left(\frac{\sqrt{\lambda}}{x}\right) + B \sin\left(\frac{\sqrt{\lambda}}{x}\right) \right)$$

$$= x \left(A \cos\left(\frac{\sqrt{\lambda}}{x}\right) - A \frac{\sin \frac{\sqrt{\lambda}}{a}}{\cos \frac{\sqrt{\lambda}}{a}} \sin\left(\frac{\sqrt{\lambda}}{x}\right) \right)$$

$$= \frac{A}{\cos \frac{\sqrt{\lambda}}{a}} x \left(\cos \frac{\sqrt{\lambda}}{a} \cos\left(\frac{\sqrt{\lambda}}{x}\right) - \sin \frac{\sqrt{\lambda}}{a} \sin\left(\frac{\sqrt{\lambda}}{x}\right) \right)$$

$$\frac{A}{\cos \frac{\sqrt{\lambda}}{a}} = C$$

$$= Cx \sin\left(\frac{\sqrt{\lambda}}{x} - \frac{\sqrt{\lambda}}{a}\right)$$

$$= Cx \sin \sqrt{\lambda} \left(\frac{1}{x} - \frac{1}{a} \right)$$

$$y_n(x) = Cx \sin \sqrt{\lambda} \left(\frac{1}{x} - \frac{1}{a} \right) \quad \left(\sqrt{\lambda} = \frac{n\pi ab}{L} \right)$$

$$= Cx \sin \frac{n\pi ab}{L} \left(\frac{1}{x} - \frac{1}{a} \right)$$

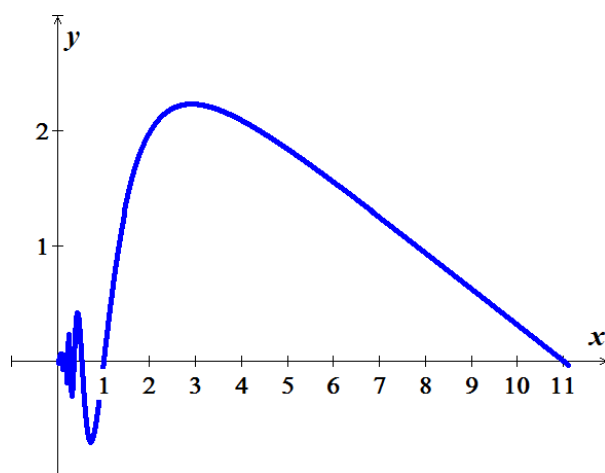
$$= Cx \sin \frac{n\pi b}{L} \left(\frac{a}{x} - 1 \right)$$

$$= C_1 x \sin \frac{n\pi b}{L} \left(1 - \frac{a}{x} \right)$$

b) Given: $n=1, \quad a=1, \quad b=11$

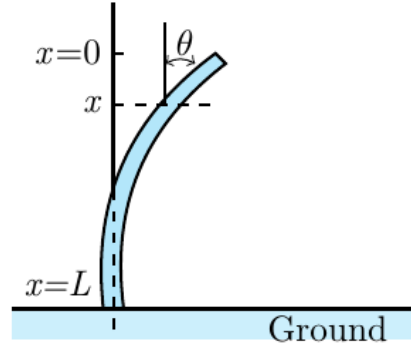
Let $C_1 = 1$

$$y_1(x) = x \sin \frac{11\pi}{10} \left(1 - \frac{1}{x} \right)$$



Exercise

For a practical application, we now consider the problem of determining when a uniform vertical column will buckle under its own weight (after, perhaps, being nudged laterally just a bit by a passing breeze). We take $x = 0$ at the free top end of the column and $x = L > 0$ at its bottom; we assume that the bottom is rigidly imbedded in the ground, perhaps in concrete.



Denote the angular deflection of the column at the point x by $\theta(x)$. From the theory of elasticity it follows that

$$EI \frac{d^2\theta}{dx^2} + g\rho x\theta = 0$$

Where E is the Young's modulus of the material of the column,

I is its cross-sectional moment of inertia

ρ is the linear density of the column

g is gravitational acceleration.

For physical reasons – no bending at the free top of the column and no deflection at its imbedded bottom – the boundary conditions are $\theta'(0) = 0$, $\theta(L) = 0$

Determine the general equation of the length L .

Solution

$$EI\theta'' + g\rho x\theta = 0$$

$$\theta'' + \frac{g\rho}{EI} x\theta = 0$$

$$\text{Let } \lambda = \frac{g\rho}{EI} = \gamma^2$$

$$x^2 \times \theta'' + \gamma^2 x\theta = 0$$

$$x^2\theta'' + \gamma^2 x^3\theta = 0; \quad \theta'(0) = 0, \quad \theta(L) = 0$$

$$A = 0, \quad B = 0, \quad C = \gamma^2, \quad p = 3$$

$$\alpha = \frac{1}{2}, \quad \beta = \frac{3}{2}, \quad k = \frac{2\gamma}{3}, \quad \nu = \frac{1}{3}$$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \nu = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\theta(x) = x^{1/2} \left[c_1 J_{1/3} \left(\frac{2}{3} \gamma x^{3/2} \right) + c_2 J_{-1/3} \left(\frac{2}{3} \gamma x^{3/2} \right) \right]$$

$$y(x) = x^\alpha \left(c_1 J_\nu \left(kx^\beta \right) + c_2 J_{-\nu} \left(kx^\beta \right) \right)$$

$$J_{1/3}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma\left(1 + \frac{1}{3} + n\right)} \left(\frac{x}{2}\right)^{2n + \frac{1}{3}}$$

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 + \nu + n)} \left(\frac{x}{2}\right)^{2n + \nu}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma\left(n + \frac{4}{3}\right)} \left(\frac{x}{2}\right)^{2n + \frac{1}{3}}$$

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \left\{ 1 - \frac{x^2}{2(2\nu+2)} + \frac{x^4}{2 \cdot 4 \cdot (2\nu+2)(2\nu+4)} - \dots \right\}$$

$$= \frac{x^{1/3}}{2^{1/3} \Gamma\left(\frac{4}{3}\right)} \left\{ 1 - \frac{x^2}{2\left(\frac{2}{3} + 2\right)} + \frac{x^4}{2 \cdot 4 \cdot \left(\frac{2}{3} + 2\right)\left(\frac{2}{3} + 4\right)} - \dots \right\}$$

$$= \frac{x^{1/3}}{2^{1/3} \Gamma\left(\frac{4}{3}\right)} \left\{ 1 - \frac{3x^2}{2^3} + \frac{3^2 x^4}{112 \times 2^3} - \dots \right\}$$

$$J_{1/3}\left(\frac{2}{3} \gamma x^{3/2}\right) = \frac{1}{2^{1/3} \Gamma\left(\frac{4}{3}\right)} \left(\frac{2}{3} \gamma x^{3/2}\right)^{1/3} \left\{ 1 - \frac{3}{2^3} \left(\frac{2}{3} \gamma x^{3/2}\right)^2 + \frac{3^2}{896} \left(\frac{2}{3} \gamma x^{3/2}\right)^4 - \dots \right\}$$

$$= \frac{\gamma^{1/3}}{3^{1/3} \Gamma\left(\frac{4}{3}\right)} x^{1/2} \left\{ 1 - \frac{1}{12} \gamma^2 x^3 + \frac{1}{504} \gamma^4 x^6 - \dots \right\}$$

$$J_{-1/3}\left(\frac{2}{3} \gamma x^{3/2}\right) = \frac{1}{2^{-1/3} \Gamma\left(\frac{4}{3}\right)} \left(\frac{2}{3} \gamma x^{3/2}\right)^{-1/3} \left\{ 1 - \frac{1}{2\left(2 - \frac{1}{3}\right)} \left(\frac{2}{3} \gamma x^{3/2}\right)^2 + \frac{1}{8\left(2 - \frac{1}{3}\right)\left(4 - \frac{1}{3}\right)} \left(\frac{2}{3} \gamma x^{3/2}\right)^4 - \dots \right\}$$

$$= \frac{3^{1/3}}{\gamma^{1/3} \Gamma\left(\frac{2}{3}\right)} x^{-1/2} \left\{ 1 - \frac{1}{6} \gamma^2 x^3 + \frac{1}{180} \gamma^4 x^6 - \dots \right\}$$

$$\theta(x) = x^{1/2} \left[c_1 J_{1/3} \left(\frac{2}{3} \gamma x^{3/2} \right) + c_2 J_{-1/3} \left(\frac{2}{3} \gamma x^{3/2} \right) \right]$$

$$= x^{1/2} \left[c_1 \frac{\gamma^{1/3}}{3^{1/3} \Gamma\left(\frac{4}{3}\right)} x^{1/2} \left\{ 1 - \frac{1}{12} \gamma^2 x^3 + \frac{1}{504} \gamma^4 x^6 - \dots \right\} + c_2 \frac{3^{1/3}}{\gamma^{1/3} \Gamma\left(\frac{2}{3}\right)} x^{-1/2} \left\{ 1 - \frac{1}{6} \gamma^2 x^3 + \frac{1}{180} \gamma^4 x^6 - \dots \right\} \right]$$

$$= c_1 \frac{\gamma^{1/3}}{3^{1/3} \Gamma\left(\frac{4}{3}\right)} \left\{ x - \frac{1}{12} \gamma^2 x^4 + \frac{1}{504} \gamma^4 x^7 - \dots \right\} + c_2 \frac{3^{1/3}}{\gamma^{1/3} \Gamma\left(\frac{2}{3}\right)} \left\{ 1 - \frac{1}{6} \gamma^2 x^3 + \frac{1}{180} \gamma^4 x^6 - \dots \right\}$$

Given: $\theta(L) = 0$, $\theta'(0) = 0$

$$\theta'(x) = c_1 \frac{\gamma^{1/3}}{3^{1/3} \Gamma\left(\frac{4}{3}\right)} \left\{ 1 - \frac{1}{3} \gamma^2 x^3 + \frac{1}{72} \gamma^4 x^6 - \dots \right\} + \frac{3^{1/3}}{\gamma^{1/3} \Gamma\left(\frac{2}{3}\right)} \left\{ \frac{1}{2} \gamma^2 x^2 + \frac{1}{30} \gamma^4 x^5 - \dots \right\}$$

$$\theta'(0) = c_1 \frac{\gamma^{1/3}}{3^{1/3} \Gamma\left(\frac{4}{3}\right)} = 0 \rightarrow \underline{c_1 = 0}$$

$$\frac{3^{1/3} c_2}{\gamma^{1/3} \Gamma\left(\frac{2}{3}\right)} \left\{ 1 - \frac{1}{6} \gamma^2 L^3 + \frac{1}{180} \gamma^4 L^6 - \dots \right\} = 0$$

$$c_2 J_{-1/3} \left(\frac{2}{3} \gamma L^{3/2} \right) = 0 \rightarrow J_{-1/3} \left(\frac{2}{3} \gamma L^{3/2} \right) = 0$$

$$J_{-1/3} \left(z = \frac{2}{3} \gamma L^{3/2} \right) = 0$$

Using MatLab:

```
fzero('besselj(-1/3,z)',2)
syms z y
fplot(besselj(-1/3, z))
axis([0 10 -0.5 1.1])
grid on
ylabel('J_{-1/3}(z)')
legend('J_{-1/3}')
```

$$\underline{z = 1.8664}$$

$$z = \frac{2}{3} \gamma L^{3/2} \rightarrow L = \left(\frac{3z}{2\gamma} \right)^{2/3}$$

$$L = \left(\frac{3(1.86635)}{2\sqrt{\frac{g\rho}{EI}}} \right)^{2/3}$$

$$\underline{\approx 1.986352 \left(\frac{EI}{g\rho} \right)^{1/3}}$$

