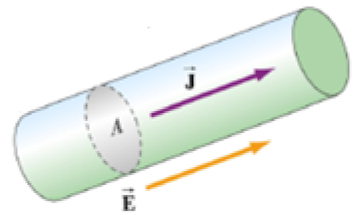
3.1 – Current and Resistance

Current: is defined to be amount of charge that crosses a cross-sectional per a unit time.

$$\boxed{I_{av} = \frac{Q}{\Delta t}}$$



I_{av} : Current (*average*)

Q: Amount of charge that crosses a cross sectional area in time interval Δt

Instantaneous current is defined to be amount of charge dQ that crosses a cross-sectional area in infinitesimal time interval dt .

$$\boxed{I = \frac{d\theta}{dt}}$$

Unit of current is coulomb/second which is defined to be the Ampere abbreviated as A.

Example

If 10^{10} electrons cross a cross-sectional area in 1*m/s*, calculate the average current.

Solution

$\Delta t = 10^{-3} s$

$Q = N|e|$ where N is number of electrons and e is the charge of one electron

$N = 10^{10}$ $|e| = 1.6 \times 10^{-19} C$

$$I_{av} = \frac{Q}{\Delta t} = \frac{\left(10^{10}\right)\left(1.6 \times 10^{-19}\right)}{10^{-3}} \underline{= 1.6 \times 10^{-6} A|}$$

Current Density $(\vec{j})$: Crossing a cross-sectional area is defined to be current per a unit perpendicular area.

$$\boxed{J = \frac{I}{A_{\perp}}}$$

J = current density

I = current

$A_{\perp}$ = perpendicular area

Unit of current density is A / m^2

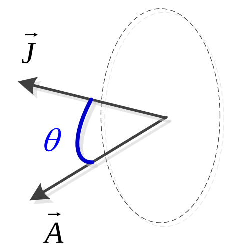
Current density is a vector quantity.

Current may be written in terms of current density and area as

$$I = \vec{J} \cdot \vec{A}$$

(If $\vec{J}$ is a constant if $\vec{J}$ varies then the integral form $I = \int \vec{J} \cdot d\vec{A}$ should be used)

$$I = \vec{J} \bullet \vec{A} = JA\cos(\theta)$$



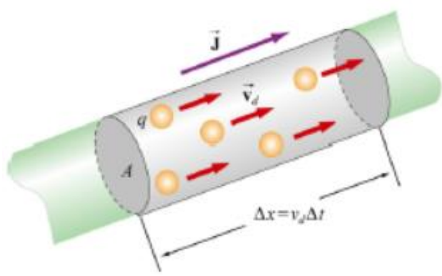
But

$$A\cos(\theta) = A_{\perp}$$

$$\boxed{I = JA_{\perp}}$$

Drift Velocity of Electrons

Normally the electrons are normally colliding with each other. If there is no electric field, the net velocity of the electrons with zero. But if there is a net electric field, the electrons will have a net velocity opposite to the directions of the electrons. This velocity is called the drift velocity of electrons. Even though the carriers of charges are negative charges (electrons), conventionally it is assumed that the carriers of charge are positive charges. Thus the direction of current density is taken to be the direction of movement of positive charges which is the same as the directions of the electric field.



Consider carriers of charge of charge q crossing a cross-sectional area with a drift velocity V_d .

Suppose the charges travel a distance Δx in a time interval Δt. $\left(i.e.\ V_d = \frac{\Delta x}{\Delta t}\right)$.

let n represent the number of charges per unit volume. Therefore the total amount of charge that crosses the cross-sectional is in time interval Δt is $n\Delta x\, A_{\perp}\, q$ (where $\Delta x\, A_{\perp}$ is the volume of the cylinder of base $A_{\perp}$ and height Δx)

$$Q = n\Delta x\, A_{\perp}\, q$$

$$I = \frac{Q}{\Delta t} = \frac{n\Delta x\, A_{\perp}\, q}{\Delta t} \qquad but \quad \frac{\Delta x}{\Delta t} = V_d$$

$$I = nV_d\, A_{\perp}\, q$$

$$J = \frac{I}{A_{\perp}} = nV_d\, q$$

n: Number of charges per a unit volume
q: Charge of one charges carrier
V_d : Drift velocity
$A_\perp$: Cross-sectional area

Example

Aluminum has a density of $2.7\times10^3\ kg/m^3$. Aluminum has 3 free (valence) electrons per atom. If there is a current of 2*A* in an aluminum wire of cross-sectional radius of 4*mm*, calculate the drift velocity of the electrons.

Solution

Given: $\rho_{Al} = 2.7\times10^3\ kg/m^3,\quad I = 2A,\quad r = 4\times10^{-3}m$

$$A_\perp = \pi r^2 = \pi\left(4\times10^{-3}\right)^2 \approx 5\times10^{-5}\ m^2$$

Find: V_d ?

The atomic mass of *Al* is 27μ. This means the gram molecular weight of aluminum is 27 *g*(*Mg*). In one gram molecular weight of any substance the Avogadro number of atoms $\left(6.02\times10^{23}\right)$.

Therefore the number of atoms per unit volume is equal to the ratio *b/n* Avogadro number (N_a) and the volume of gram molecular weight $\left(V_g\right)$

$$\rho = \frac{M}{V_g} \Rightarrow V_g = \frac{M}{\rho} = \frac{0.27}{2.7\times10^3} = 10^{-5}\ m^3$$

$$\text{Number of atoms per a unit volume} = \frac{6.02\times10^{23}}{10^{-5}m^3} = 6.02\times10^{28}\ atom/m^3$$

And since there are 3 free electrons per atom the number of electrons per unit volume is 3 times the number of atoms per unit volume

$$n = 3\left(6.02\times10^{28}\right)/m^3 = 18.06\times10^{28}/m^3$$

$q = |e| = 1.6\times10^{-19}$ (since the carriers of charge are electrons)

$$I = nV_d A_\perp q$$

$$\Rightarrow V_d = \frac{I}{nA_\perp q}$$

$$= \frac{2}{\left(18.26\times10^{28}\right)\left(1.6\times10^{-19}\right)\left(5\times10^{-5}\right)}$$

$$= 1.4\times10^{-6}\ m/s$$

Ohm's Law

States that the current density and the electric field in metals are directly proportional.

$$\boxed{J = \sigma E}$$

J: Current density

E: Electric field

σ: A material constant called the conductivity of the materials

Resistance (R)

Resistance of a material is defined to be the ratio between the potential difference across its terminals and the current flowing through it.

$$\boxed{R = \frac{\Delta V}{I}}$$

Unit if resistance is ***volt/ampere*** which is defined to be the ***Ohm*** abbreviated as Ω.

Circuit symbol for resistance is

Example

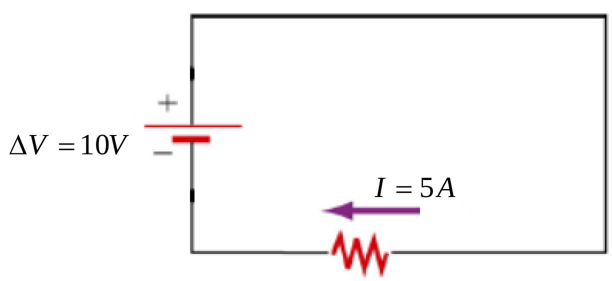
Calculate the resistance of a resistor if a current of 5A flows through it when it is connected to a 10V battery.

Solution

Given: $\Delta V = 10V, \quad I = 5A$

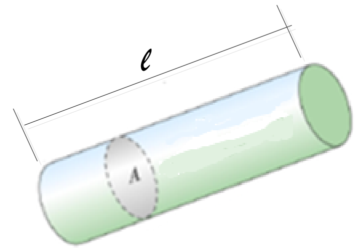
Find: R?

$R = \frac{\Delta V}{I} = \frac{10}{5} = 2 \;\Omega$



Resistance of a Wire of Length ℓ and Cross Sectional Area A

$$R = \frac{\Delta V}{I}$$



If the electric field in the wire is E, then

$$\Delta V = E\ell$$

(Assuming the electric field is constant.)

$$I = JA \quad but \quad J = \sigma E$$

$$\therefore\ I = \sigma EA$$

$$R = \frac{E\ell}{\sigma EA}$$

$$\Rightarrow\ R = \frac{1}{\sigma}\frac{\ell}{A}$$

$\frac{1}{\sigma}$ (the inverse of the conductivity of the material) is defined to be the resistivity (ρ) of the material

$$\boxed{\rho = \frac{1}{\sigma}}$$

$$R = \frac{1}{\sigma}\frac{\ell}{A}$$

$$\boxed{R = \frac{\rho\,\ell}{A}}$$

R: resistance of a wire
ℓ: length of the wire
A: cross-sectional area

Therefore it follows that the resistance of a wire is directly proportional to its length and inversely proportional to its cross-sectional area unit measurement for resistivity is Ωm.

Example

Silver has a resistivity of $1.59\times10^{-8}\,\Omega m$. Calculate the resistance of a silver wire of length 10m and cross-sectional radius 2mm.

Solution

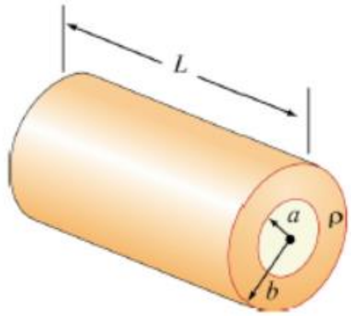
Given: $\rho_{av} = 1.59\times10^{-8}\,\Omega m, \quad \ell = 10m, \quad r = 2\times10^{-3}m$

$$A = \pi r^2 = \pi\left(2\times10^{-3}\right)^2 \ \underline{\approx 12.56\times10^{-6}\ m^2}$$

$$R = \frac{\rho\,\ell}{A} = \frac{\left(1.59\times10^{-8}\right)(10)}{12.56\times10^{-6}} \ \underline{= 1.2\times10^{-2}\ \Omega}$$

Example

Find an expression for the resistance of a coaxial cable of inner radius *a* and outer radius *b*. The resistance of the material is ρ. The length of the wire is *L*. Assume the current is flowing radially outwards.



Consider a cylindrical shell of radius *r* ($a < r < b$) of thickness *dr*. Then for a radial current the cross-sectional area is the surface area of the shell– $2\pi Lr$ and the length of the shell in the radial directions is *dr*. The resistance of the shell of thickness *dr* (*dR*) is given by

$$dR = \frac{\rho dr}{2\pi Lr}$$

and the total resistance is obtained by integrating from *a* to *b*.

$$\begin{aligned} R &= \int dR \\ &= \int_{r=a}^{r=b} \frac{\rho dr}{2\pi Lr} \\ &= \frac{\rho}{2\pi L}\int_a^b \frac{dr}{r} \\ &= \frac{\rho}{2\pi L}\left[\ln r\right]_a^b \\ &= \frac{\rho}{2\pi L}\left(\ln b - \ln a\right) \\ &= \frac{\rho}{2\pi L}\ln\left(\frac{b}{a}\right) \end{aligned}$$

Dependence of Resistivity on Temperature

The change in resistivity of a material due to the change in temperature is directly proportional to the change in temperature and to the initial resistivity.

$$\Delta\rho \propto \rho_0 \Delta T$$

$$\frac{\Delta\rho}{\rho_0 \Delta T} = constant$$

This constant is a material constant called temperature coefficient of resistivity and denoted by α

$$\therefore \frac{\Delta\rho}{\rho_0 \Delta T} = \alpha$$

$$\boxed{\Delta\rho = \alpha\rho_0 \Delta T}$$

$\Delta\rho = \rho - \rho_0 \rightarrow$ Change in resistivity

$\rho_0 \rightarrow$ Initial resistivity

$\Delta T = T - T_0 \rightarrow$ Change in temperature

$\alpha \rightarrow$ Temperature coefficient of resistivity

Unit measurement for temperature coefficient of resistivity is 1/°C.

If the value of the resistivity at a particular temperature is desired the above equation can be simplified as follows

$$\Delta\rho = \rho - \rho_0 = \alpha\ \rho_0 \left(T - T_0\right)$$

$$\rho = \rho_0 + \alpha\ \rho_0 \left(T - T_0\right)$$

$$\boxed{\rho = \rho_0 \left[1 + \alpha\ \left(T - T_0\right)\right]}$$

Example

Silver has a resistivity of $1.59 \times 10^{-8}\,\Omega m$ at a temperature of 20°C. Its temperature coefficient is $3.8 \times 10^{-3}\ /\ °C$. Calculate its resistivity at a temperature of 120°C.

Solution

Given: $\rho_0 = 1.59 \times 10^{-8}\Omega m, \quad T_0 = 20°C, \quad T = 120°C, \quad \alpha = 3.8 \times 10^{-3}\ /\ °C$

$$\rho = \rho_0 \left[1 + \alpha\ \left(T - T_0\right)\right]$$

$$= \left(1.59 \times 10^{-8}\right)\left[1 + 3.8 \times 10^{-3}\left(120 - 20\right)\right]$$

$$= 2.862 \times 10^{-8}\ \ \Omega m$$

Example

By how much would the resistivity of a silver wire change when its temperature changes by 200℃ (Assume initial temp to be 20℃)

Solution

Given: $\rho_0 = 1.59\times10^{-8}\Omega m, \quad \Delta T = 200°C, \quad \alpha = 3.8\times10^{-3} / °C$

$$\Delta\rho = \alpha\rho_0\Delta T$$

$$= \left(3.8\times10^{-3}\right)\left(1.59\times10^{-8}\right)(200)$$

$$\boxed{= 1.2\times10^{-8} \ \ \Omega m}$$

Dependence of Resistance on Temperature

Since resistance is proportional to resistivity $\left(R = \frac{\rho \ell}{A} \ \text{for a given } \ell \ \& \ A\right)$, the same proportional as resistivity apply. That is resistance of a resistor is directly proportional to change in temperature and to the initial resistance

$$\Delta R = R_0 \alpha \ \Delta T$$

ΔR: Change in resistance $\left(R - R_0\right)$

ΔT: Change in temperature $\left(T - T_0\right)$

$\propto$: Change in temperature coefficient of resistivity

R_0 : Initial resistance

Also

$$\Delta R = R - R_0 = R_0 \alpha \left(T - T_0\right)$$

$$R = R_0 \left[1 + \alpha \left(T - T_0\right)\right]$$

Example

A silver wire (resistivity $1.59 \times 10^{-8} \Omega m$ and a temperature coefficient 3.8×10^{-3}) has a length of $10m$ and a cross-sectional radius $2mm$. Calculate the current flowing through it when it is connected to a $0.5V$ battery.

a) At temperature 20℃

b) At temperature 120℃

<u>Solution</u>

Given: $\rho_{20} = 1.59 \times 10^{-8} \Omega m, \quad \ell = 10m, \quad \alpha = 3.8 \times 10^{-3} / {}^\circ C, \quad r = 2 \times 10^{-3} m$

a) $A = \pi r^2 = \pi \left(2 \times 10^{-3}\right)^2 \approx 12.56 \times 10^{-6} \ m^2$

$$R_{20} = \frac{\rho_{20} \ell}{A} = \frac{\left(1.59 \times 10^{-8}\right)(10)}{12.56 \times 10^{-6}} = 0.012 \ \Omega$$

$$I_{20} = \frac{\Delta V}{R_{20}} = \frac{0.5}{0.012} = 41.7 \ A$$

b) $R_{120} = R_{20} \left[1 + \alpha \left(T_{120} - T_{20}\right)\right]$

$$= 0.12 \left[1 + 3.8 \times 10^{-3} (120 - 20)\right]$$

$$= 0.0166 \ \Omega$$

$$I_{120} = \frac{\Delta V}{R_{120}} = \frac{0.5}{0.0166} = 30.17 \ A$$

Electrical Power and Energy

Electrical power dissipated in a resistor defined to be equal to the rate of conversion of electrical energy to non-electrical energy in the resistor

$$\boxed{P = \frac{\Delta u}{\Delta t}}$$

P: Power
Δu: Change in electrical energy in a time interval Δt
Unit of power is ***Joule/second*** which is defined to be ***Watt*** abbreviated as ***W***.

If a charge q is displaced through a potential difference ΔV, then

$$\Delta u = q\Delta V$$

and

$$P = \frac{q\Delta V}{\Delta t} \qquad but \quad I = \frac{q}{\Delta t}$$

$$\boxed{P = I\ \Delta V}$$

P: Power
I: Current
ΔV: Potential difference

Power dissipated in a resistor is equal to the product of the current through the resistor and the potential difference across the resistor.

Also since $\quad \Delta V = RI$

$$P = I\ \Delta V = I\left(RI\right) = I^2R$$

$$\boxed{P = I^2R}$$

Or

$$I = \frac{\Delta V}{R} \quad \Rightarrow \quad P = \frac{\Delta V}{R}\cdot\Delta V$$

$$\boxed{P = \frac{\Delta V^2}{R}}$$

Example

A 10Ω resistor is connected to a $6V$ battery.

a) Calculate the power dissipated in the resistor
b) Calculate the energy dissipated in the resistor in 0.2 seconds
c) Calculate the current flowing through it

Solution

a) $P = \dfrac{\Delta V^2}{R} = \dfrac{6^2}{10} \; \underline{= 3.6 \; W|}$

b) $\Delta u = P\Delta t = (3.6)(0.2) \; \underline{= 0.72 \; J|}$

c) $I = \dfrac{\Delta V}{R} = \dfrac{6}{10} \; \underline{= 0.6 \; A|}$

Example

When a 100 *watt* lamp is connected to a certain potential difference, a current of 2*A* flows across it.

a) Calculate the potential difference
b) Calculate its resistance

Solution

Given: $P = 100W, \quad I = 2A$

a) $P = I\Delta V$

$\Delta V = \dfrac{P}{I} = \dfrac{100}{2} \; \underline{= 50 \; V|}$

b) $R = \dfrac{\Delta V}{I} = \dfrac{50}{2} \; \underline{= 25 \; \Omega|}$

The Kilo Watt hour (*kWh*)

The kilo-Watt is a unit of energy defined to be equal to the amount of energy dissipated by $1kW$ device when used for one hour.

$$kWh = (1{,}000W)(3600s) = 3.6 \times 10^6\, J$$

$$\boxed{1\ kWh = 3.6 \times 10^6\ J}$$

Example

a) Convert $1.8 \times 10^3 J$ to kWh

$$1.8 \times 10^3 J\, \frac{1\ kWh}{3.6 \times 10^6 J} = 5 \times 10^{-4}\ kW$$

b) Convert 10 kWh to Joules

$$10\ kWh \cdot \frac{3.6 \times 10^6 J}{1\ kWh} = 3.6 \times 10^7\ J$$

Example

If electrical energy costs 10 *cents* per kilo Watt hour, how much would it cost to run a $100W$ lamp for 30 days at $5hrs$ per day

Solution

$$P = 100\ W = 0.1\ kW$$

$$\Delta t = (30)(5) = 150\ hrs$$

$$\begin{aligned}\Delta u &= P\Delta t \\ &= (0.1\ kW)(150\ hrs) \\ &= 15\ kWh\end{aligned}$$

$$\begin{aligned}cost &= (energy)(price) \\ &= (15\ kWh)(10\ \text{¢}/kWh) \\ &= 150\ \text{¢} \\ &= \$1.5\end{aligned}$$