Lecture Three - Identities

Section 3.1 – Proving Identities

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Ratio Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos\theta = \pm\sqrt{1-\sin^2\theta}$$

$$\sin\theta = \pm\sqrt{1-\cos^2\theta}$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Example

Write $\sec\theta\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify.

Solution

$$\sec\theta\tan\theta = \frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta}$$

$$= \frac{\sin\theta}{\cos^2\theta}$$

Example

Add $\frac{1}{\sin\theta}+\frac{1}{\cos\theta}$

Solution

$$\frac{1}{\sin\theta}+\frac{1}{\cos\theta}=\frac{\cos\theta+\sin\theta}{\sin\theta\cos\theta}$$

$$\frac{1}{\sin\theta}\frac{\cos\theta}{\cos\theta}+\frac{1}{\cos\theta}\frac{\sin\theta}{\sin\theta}$$

Example

Write: $\tan\alpha+\cot\alpha$ in terms of $\sin\alpha$ *and* $\cos\alpha$

Solution

$$\tan\alpha+\cot\alpha=\frac{\sin\alpha}{\cos\alpha}+\frac{\cos\alpha}{\sin\alpha}$$

$$=\frac{\sin\alpha}{\cos\alpha}\frac{\sin\alpha}{\sin\alpha}+\frac{\cos\alpha}{\sin\alpha}\frac{\cos\alpha}{\cos\alpha}$$

$$=\frac{\sin^2\alpha+\cos^2\alpha}{\cos\alpha\sin\alpha}$$

$$=\frac{1}{\cos\alpha\sin\alpha}$$

Example

Prove: $\tan x+\cos x=\sin x(\sec x+\cot x)$

Solution

$$\tan x+\cos x=\frac{\sin x}{\cos x}+\cos x$$

$$=\sin x\frac{1}{\cos x}+\cos x\frac{\sin x}{\sin x}$$

$$=\sin x\sec x+\sin x\frac{\cos x}{\sin x}$$

$$=\sin x(\sec x+\cot x)$$

or

$$\sin x(\sec x+\cot x)=\sin x\left(\frac{1}{\cos x}+\frac{\cos x}{\sin x}\right)$$

$$=\frac{\sin x}{\cos x}+\sin x\frac{\cos x}{\sin x}$$

$$=\tan x+\cos x$$

Example

Prove: $\cot\alpha + 1 = \csc\alpha\left(\cos\alpha + \sin\alpha\right)$

Solution

$$\csc\alpha\left(\cos\alpha + \sin\alpha\right) = \frac{1}{\sin\alpha}\left(\cos\alpha + \sin\alpha\right)$$

$$= \frac{1}{\sin\alpha}\cos\alpha + \frac{1}{\sin\alpha}\sin\alpha$$

$$= \cot\alpha + 1$$

Guidelines for Proving Identities

1. Work on the complicated side first (more trigonometry functions)
2. Look for trigonometry substitutions.
3. Look for algebraic operations
4. If not always change everything to sines and cosines
5. Keep an eye on the side you are not working.

Example

Prove $\dfrac{\cos^4 t - \sin^4 t}{\cos^2 t} = 1 - \tan^2 t$

Solution

$$\frac{\cos^4 t - \sin^4 t}{\cos^2 t} = \frac{\left(\cos^2 t - \sin^2 t\right)\left(\cos^2 t + \sin^2 t\right)}{\cos^2 t} \qquad a^2 - b^2 = (a-b)(a+b)$$

$$= \frac{\left(\cos^2 t - \sin^2 t\right)(1)}{\cos^2 t} \qquad \cos^2 t + \sin^2 t = 1$$

$$= \frac{\cos^2 t - \sin^2 t}{\cos^2 t}$$

$$= \frac{\cos^2 t}{\cos^2 t} - \frac{\sin^2 t}{\cos^2 t}$$

$$= 1 - \tan^2 t$$

Example

Prove: $1+\cos\theta = \dfrac{\sin^2\theta}{1-\cos\theta}$

Solution

$$\frac{\sin^2\theta}{1-\cos\theta} = \frac{1-\cos^2\theta}{1-\cos\theta} \qquad \sin^2\theta = 1-\cos^2\theta$$

$$= \frac{(1-\cos\theta)(1+\cos\theta)}{1-\cos\theta} \qquad a^2-b^2 = (a-b)(a+b)$$

$$= 1+\cos\theta$$

Example

Prove: $\tan^2\alpha\left(1+\cot^2\alpha\right) = \dfrac{1}{1-\sin^2\alpha}$

Solution

$$\tan^2\alpha\left(1+\cot^2\alpha\right) = \tan^2\alpha + \tan^2\alpha\cot^2\alpha$$

$$= \tan^2\alpha + \tan^2\alpha\frac{1}{\tan^2\alpha}$$

$$= \tan^2\alpha + 1 \qquad \tan^2\alpha+1 = \sec^2\alpha$$

$$= \sec^2\alpha$$

$$= \frac{1}{\cos^2\alpha} \qquad \cos^2\alpha = 1-\sin^2\alpha$$

$$= \frac{1}{1-\sin^2\alpha}$$

Example

Prove : $\dfrac{\sin\alpha}{1+\cos\alpha} + \dfrac{1+\cos\alpha}{\sin\alpha} = 2\csc\alpha$

Solution

$$\frac{\sin\alpha}{1+\cos\alpha} + \frac{1+\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{\sin\alpha}\cdot\frac{\sin\alpha}{1+\cos\alpha} + \frac{1+\cos\alpha}{\sin\alpha}\cdot\frac{1+\cos\alpha}{1+\cos\alpha}$$

$$= \frac{\sin^2\alpha + (1+\cos\alpha)^2}{\sin\alpha(1+\cos\alpha)}$$

$$= \frac{\sin^2\alpha + 1 + \cos^2\alpha + 2\cos\alpha}{\sin\alpha(1+\cos\alpha)}$$

$$= \frac{2 + 2\cos\alpha}{\sin\alpha(1+\cos\alpha)}$$

$$= \frac{2(1+\cos\alpha)}{\sin\alpha(1+\cos\alpha)}$$

$$= \frac{2}{\sin\alpha}$$

$$= 2\csc\alpha$$

Example

Prove $\frac{1+\sin t}{\cos t} = \frac{\cos t}{1-\sin t}$

Solution

$$\frac{1+\sin t}{\cos t} = \frac{1+\sin t}{\cos t} \cdot \frac{1-\sin t}{1-\sin t}$$

$$= \frac{1-\sin^2 t}{\cos t(1-\sin t)}$$

$$= \frac{\cos^2 t}{\cos t(1-\sin t)}$$

$$= \frac{\cos t}{1-\sin t}$$

Example

Show that $\cot^2\theta + \cos^2\theta = \cot^2\theta\cos^2\theta$ is not an identity by finding a counterexample

Solution

$$\cot^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} = \cot^2\frac{\pi}{4}\cos^2\frac{\pi}{4}$$

$$1^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1^2\left(\frac{1}{\sqrt{2}}\right)^2$$

$$1 + \frac{1}{2} = \frac{1}{2}$$

$$\frac{3}{2} \neq \frac{1}{2}$$

Exercises *Section* 3.1 – Proving Identities

1. Prove the identity: $\cos\theta\cot\theta+\sin\theta=\csc\theta$

2. Prove the identity: $\sec\theta\cot\theta-\sin\theta=\dfrac{\cos^2\theta}{\sin\theta}$

3. Prove the identity: $\dfrac{\csc\theta\tan\theta}{\sec\theta}=1$

4. Prove the identity: $(\sin\theta+\cos\theta)^2=1+2\sin\theta\cos\theta$

5. Prove the identity: $\sin\theta(\sec\theta+\cot\theta)=\tan\theta+\cos\theta$

6. Prove the identity: $\cos\theta(\csc\theta+\tan\theta)=\cot\theta+\sin\theta$

7. Prove the identity: $\cot\theta+\tan\theta=\csc\theta\sec\theta$

8. Prove the identity: $\tan x(\cos x+\cot x)=\sin x+1$

9. Prove the identity: $\dfrac{1-\cos^4\theta}{1+\cos^2\theta}=\sin^2\theta$

10. Prove the identity: $\dfrac{1-\sec x}{1+\sec x}=\dfrac{\cos x-1}{\cos x+1}$

11. Prove the identity: $\dfrac{\cos x}{1-\sin x}-\dfrac{1-\sin x}{\cos x}=0$

12. Prove the identity: $\dfrac{1+\cot^3 t}{1+\cot t}=\csc^2 t-\cot t$

13. Prove the identity: $\tan x+\cot x=\sec x\csc x$

14. Prove the identity: $\dfrac{\tan x-\cot x}{\sin x\cos x}=\sec^2 x-\csc^2 x$

15. Prove the identity: $\dfrac{\sec x+\tan x}{\sec x-\tan x}=\dfrac{1+2\sin x+\sin^2 x}{\cos^2 x}$

16. Prove the identity: $\sin^2 x-\cos^2 x=2\sin^2 x-1$

17. Prove the identity: $\sin^4 x-\cos^4 x=\sin^2 x-\cos^2 x$

18. Prove the identity: $\dfrac{\cos\alpha}{1+\sin\alpha}=\sec\alpha-\tan\alpha$

19. Prove the identity: $\dfrac{\sin\alpha}{1-\sin\alpha}-\dfrac{\cos\alpha}{1-\sin\alpha}=\dfrac{1-\cot\alpha}{\csc\alpha-1}$

20. Prove the identity: $\dfrac{\dfrac{1}{\tan x}+\cot x}{\dfrac{1}{\tan x}+\tan x}=\dfrac{2}{\sec^2 x}$

21. Prove the following equation is an identity: $\dfrac{\cot^2\theta+3\cot\theta-4}{\cot\theta+4}=\cot\theta-1$

22. Prove the following equation is an identity: $\dfrac{\sin\theta}{1+\cos\theta}=\dfrac{1-\cos\theta}{\sin\theta}$

23. Prove the following equation is an identity: $\tan x\left(\csc x-\sin x\right)=\cos x$

24. Prove the following equation is an identity: $\sin x\left(\tan x\cos x-\cot x\cos x\right)=1-2\cos^2 x$

25. Prove the following equation is an identity: $\left(1+\tan x\right)^2+\left(\tan x-1\right)^2=2\sec^2 x$

26. Prove the following equation is an identity: $\sec x+\tan x=\dfrac{\cos x}{1-\sin x}$

27. Prove the following equation is an identity: $\dfrac{\tan x-1}{\tan x+1}=\dfrac{1-\cot x}{1+\cot x}$

28. Prove the following equation is an identity: $7\csc^2 x-5\cot^2 x=2\csc^2 x+5$

29. Prove the following equation is an identity: $1-\dfrac{\cos^2 x}{1-\sin x}=-\sin x$

30. Prove the following equation is an identity: $\dfrac{1-\cos x}{1+\cos x}=\dfrac{\sec x-1}{\sec x+1}$

31. Prove the following equation is an identity: $\dfrac{\sec x-1}{\tan x}=\dfrac{\tan x}{\sec x+1}$

32. Prove the following equation is an identity: $\dfrac{\cos x}{\cos x-\sin x}=\dfrac{1}{1-\tan x}$

33. Prove the following equation is an identity: $\left(\sec x+\tan x\right)^2=\dfrac{1+\sin x}{1-\sin x}$

34. Prove the following equation is an identity: $\dfrac{\cos x}{1+\tan x}-\dfrac{\sin x}{1+\cot x}=\cos x-\sin x$

35. Prove the following equation is an identity: $\dfrac{\cot x+\csc x-1}{\cot x-\csc x+1}=\csc x+\cot x$

36. Prove the following equation is an identity: $\dfrac{\tan x+\cot x}{\tan x-\cot x}=\dfrac{1}{\sin^2 x-\cos^2 x}$

37. Prove the following equation is an identity: $\dfrac{1-\cot^2 x}{1+\cot^2 x}+1=2\sin^2 x$

38. Prove the following equation is an identity: $\dfrac{1+\cos x}{1-\cos x}-\dfrac{1-\cos x}{1+\cos x}=4\cot x\csc x$

39. Prove the following equation is an identity: $\dfrac{\sin^3 x-\cos^3 x}{\sin x-\cos x}=1+\sin x\cos x$

40. Prove the following equation is an identity: $1+\sec^2 x\sin^2 x=\sec^2 x$

41. Prove the following equation is an identity: $\dfrac{1+\csc x}{\sec x}=\cos x+\cot x$

42. Prove the following equation is an identity: $\tan^2 x = \sec^2 x - \sin^2 x - \cos^2 x$

43. Prove the following equation is an identity: $\frac{\sin x}{1-\cos x} + \frac{\sin x}{1+\cos x} = 2\csc x$

44. Prove the following equation is an identity: $\cos^2(\alpha-\beta) - \cos^2(\alpha+\beta) = \sin^2(\alpha+\beta) - \sin^2(\alpha-\beta)$

45. Prove the following equation is an identity: $\tan x \csc x - \sec^2 x \cos x = 0$

46. Prove the following equation is an identity: $(1+\tan x)^2 - 2\tan x = \frac{1}{(1-\sin x)(1+\sin x)}$

47. Prove the following equation is an identity: $\frac{3\csc^2 x - 5\csc x - 28}{\csc x - 4} = \frac{3}{\sin x} + 7$

48. Prove the following equation is an identity: $\left(\sec^2 x - 1\right)\left(\sec^2 x + 1\right) = \tan^4 x + 2\tan^2 x$

49. Prove the following equation is an identity: $\frac{\csc x}{\cot x} - \frac{\cot x}{\csc x} = \frac{\sin x}{\cot x}$

50. Prove the following equation is an identity: $\frac{1-\cos^2 x}{1+\cos x} = \frac{\sec x - 1}{\sec x}$

51. Prove the following equation is an identity: $\frac{\cos x}{1+\cos x} = \frac{\sec x - 1}{\tan^2 x}$

52. Prove the following equation is an identity: $\frac{1-2\sin^2 x}{1+2\sin x\cos x} = \frac{\cos x - \sin x}{\cos x + \sin x}$

53. Prove the following equation is an identity: $(\cos x - \sin x)^2 + (\cos x + \sin x)^2 = 2$

54. Prove the following equation is an identity: $\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x} = 2\csc x$

55. Prove the following equation is an identity: $\frac{\sin x + \tan x}{\cot x + \csc x} = \sin x \tan x$

56. Prove the following equation is an identity: $\csc^2 x \sec^2 x = \sec^2 x + \csc^2 x$

57. Prove the following equation is an identity: $\cos^2 x + 1 = 2\cos^2 x + \sin^2 x$

58. Prove the following equation is an identity: $1 - \frac{\cos^2 x}{1+\sin x} = \sin x$

59. Prove the following equation is an identity: $\cot^2 x = (\csc x - 1)(\csc x + 1)$

60. Prove the following equation is an identity: $\frac{\sec x - 1}{\tan x} = \frac{\tan x}{\sec x + 1}$

61. Prove the following equation is an identity: $10\csc^2 x - 6\cot^2 x = 4\csc^2 x + 6$

62. Prove the following equation is an identity: $\frac{\csc x + \cot x}{\tan x + \sin x} = \csc x \cot x$

63. Prove the following equation is an identity: $\frac{1-\sec x}{\tan x}+\frac{\tan x}{1-\sec x}=-2\csc x$

64. Prove the following equation is an identity: $\csc x-\sin x=\cos x\cot x$

65. Prove the following equation is an identity: $\frac{\tan x+\sec x}{\sec x}-\frac{\tan x+\sec x}{\tan x}=-\cos x\cot x$

66. Prove the following equation is an identity: $\cot^3 x=\cot x\left(\csc^2 x-1\right)$

67. Prove the following equation is an identity: $\frac{\cot^2 x}{\csc x-1}=\frac{1+\sin x}{\sin x}$

68. Prove the following equation is an identity: $\cot^2 x+\csc^2 x=2\csc^2 x-1$

69. Prove the following equation is an identity: $\frac{\cot^2 x}{1+\csc x}=\csc x-1$

70. Prove the following equation is an identity: $\sec^4 x-\tan^4 x=\sec^2 x+\tan^2 x$

71. Prove the following equation is an identity: $\frac{\cos x}{1+\sin x}+\frac{1+\sin x}{\cos x}=2\sec x$

72. Prove the following equation is an identity: $\frac{\sin x+\cos x}{\sin x-\cos x}=\frac{1+2\sin x\cos x}{2\sin^2 x-1}$

73. Prove the following equation is an identity: $\frac{\csc x-1}{\csc x+1}=\frac{\cot^2 x}{\csc^2 x+2\csc x+1}$

74. Prove the following equation is an identity: $\csc^4 x-\cot^4 x=\csc^2 x+\cot^2 x$

75. Prove the following equation is an identity: $\tan\left(\frac{\pi}{4}+x\right)=\cot\left(\frac{\pi}{4}-x\right)$

76. Prove the identity: $\frac{\sin\theta}{1+\sin\theta}-\frac{\sin\theta}{1-\sin\theta}=-2\tan^2\theta$

77. Prove the identity: $\csc^2 x-\cos^2 x\csc^2 x=1$

78. Prove the identity: $1-2\sin^2 x=2\cos^2 x-1$

79. Prove the identity: $\csc^2 x-\cos x\sec x=\cot^2 x$

80. Prove the identity: $\left(\sec x-\tan x\right)\left(\sec x+\tan x\right)=1$

81. Prove the identity: $\left(1+\tan^2 x\right)\left(1-\sin^2 x\right)=1$