Solution *Section* 3.1 – Maxima and Minima

Exercise

Find the absolute maximum and minimum values of the function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

$f(x)=\frac{2}{3}x-5 \qquad -2\le x\le 3$

Solution

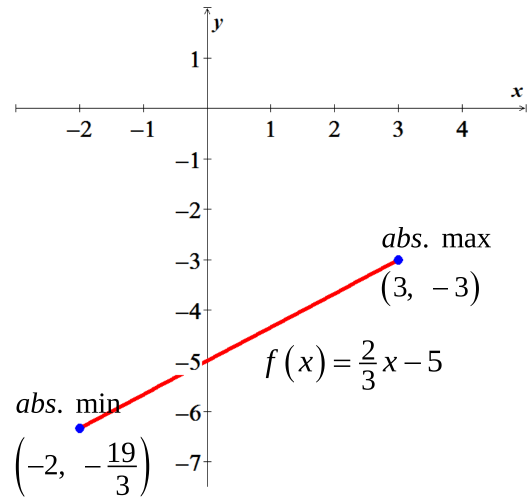
$f'(x)=\frac{2}{3}$ No Critical Points (*CP*) or (*CN*).

$f(-2)=\frac{2}{3}(-2)-5=-\frac{19}{3}$

$f(3)=\frac{2}{3}(3)-5=-3$

Absolute Maximum: $(3,\ -3)$

Absolute Minimum: $\left(-2,\ -\frac{19}{3}\right)$



Exercise

Find the absolute maximum and minimum values of the function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

$f(x)=x^2-1 \qquad -1\le x\le 2$

Solution

$f'(x)=2x=0 \quad \Rightarrow \quad \boxed{x=0} \qquad (CN)$

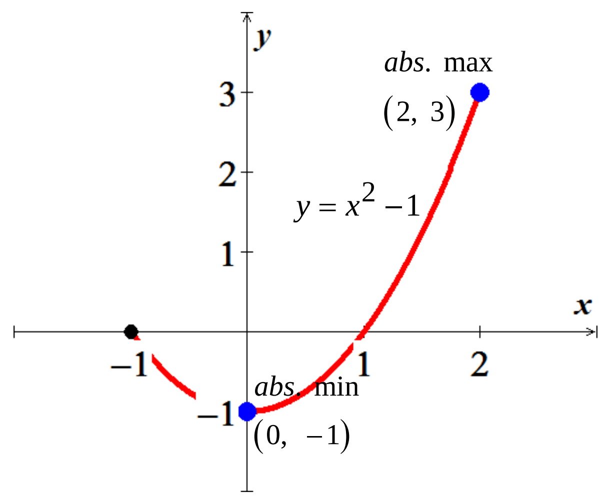
$f(-1)=(-1)^2-1=0$

$f(0)=(0)^2-1=-1$

$f(2)=(2)^2-1=3$

Abs. Maximum: $(2,\ 3)$

Abs. Minimum: $(0,\ -1)$



Exercise

Find the absolute maximum and minimum values of the function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

$f(x) = -\dfrac{1}{x^2} \qquad 0.5 \le x \le 2$

Solution

$f'(x) = \dfrac{1}{2x^3} \qquad \boxed{x=0}$ Which it is not in the domain

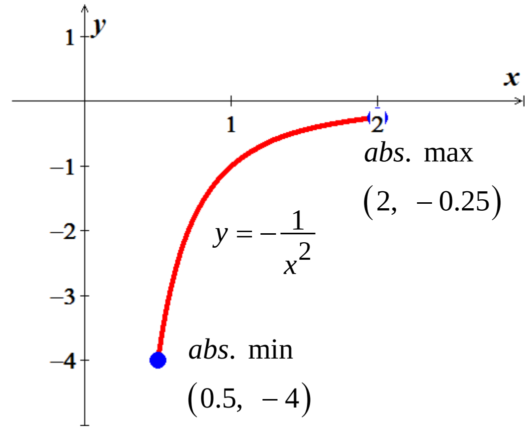
No critical point.

$f(0.5) = -\dfrac{1}{(0.5)^2} = -4$

$f(2) = -\dfrac{1}{(2)^2} = -0.25$

Abs. Max: $(2,\ -0.25)$

Abs. Min: $(0.5,\ -4)$



Exercise

Find the absolute maximum and minimum values of the function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

$f(x) = \sqrt{4-x^2} \qquad -2 \le x \le 1$

Solution

$f(x) = \left(4-x^2\right)^{1/2}$

$f'(x) = \frac{1}{2}\left(4-x^2\right)^{-1/2}(-2x)$

$= \dfrac{-x}{\sqrt{4-x^2}} = 0 \quad \rightarrow \begin{cases} \boxed{x=0} \\ 4-x^2=0 \Rightarrow \boxed{x=\pm 2} \end{cases}$

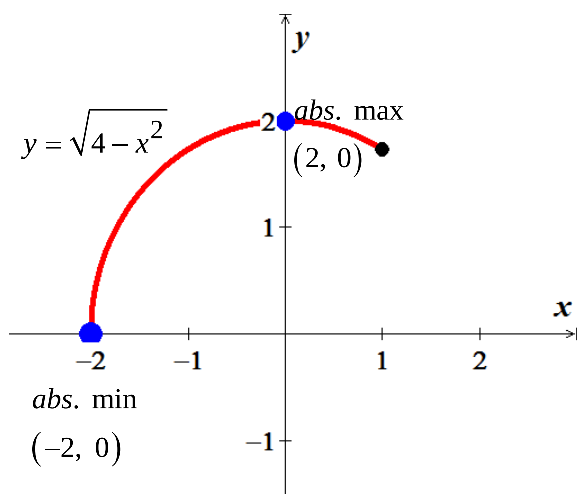
Critical points: $x = 0,\ -2$

$f(-2) = \sqrt{4-(-2)^2} = 0$

$f(0) = \sqrt{4-(0)^2} = 2$

$f(1) = \sqrt{4-(1)^2} = \sqrt{3}$

Abs. Max: $(0,\ 2)$ ***Abs. Min***: $(-2,\ 0)$



Exercise

Find the absolute maximum and minimum values of the function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

$$f(\theta)=\sin\theta \qquad -\frac{\pi}{2}\le\theta\le\frac{5\pi}{6}$$

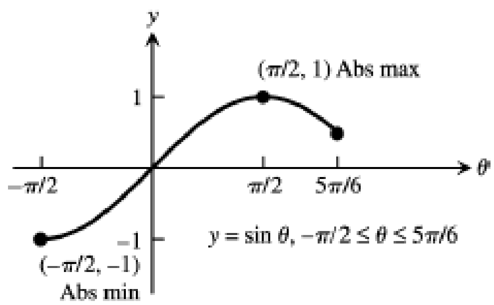
Solution

$$f'(\theta)=\cos\theta=0 \quad\Rightarrow\quad \theta=\frac{\pi}{2}\ (CN)$$

$$f\left(-\frac{\pi}{2}\right)=\sin\left(-\frac{\pi}{2}\right)=-1$$

$$f\left(\frac{\pi}{2}\right)=\sin\left(\frac{\pi}{2}\right)=1$$

$$f\left(\frac{5\pi}{6}\right)=\sin\left(\frac{5\pi}{6}\right)=\frac{1}{2}$$



Abs. Min: $\left(-\frac{\pi}{2},\ -1\right)$

Abs. Max: $\left(\frac{\pi}{2},\ 1\right)$

Exercise

Find the absolute maximum and minimum values of the function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

$$g(x)=\sec x \qquad -\frac{\pi}{3}\le x\le\frac{\pi}{6}$$

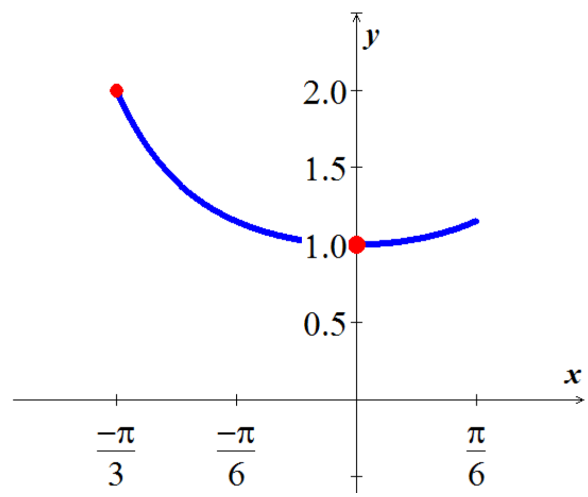
Solution

$$g'(x)=\sec x\tan x=0 \quad\Rightarrow\quad \boxed{x=0}\quad (CN)$$

$$g\left(-\frac{\pi}{3}\right)=\sec\left(-\frac{\pi}{3}\right)=2$$

$$g(0)=\sec(0)=1$$

$$g\left(\frac{\pi}{6}\right)=\sec\left(\frac{\pi}{6}\right)=\frac{2}{\sqrt{3}}$$



Abs. Max: $\left(-\frac{\pi}{3},\ 2\right)$

Abs. Min: $(0,\ 1)$

Exercise

Find the absolute maximum and minimum values of $f(x) = x^{4/3}, \quad -1 \le x \le 8$

Solution

$$f'(x) = \frac{4}{3}x^{1/3} = 0 \quad \Rightarrow \quad \boxed{x = 0} \qquad (CN)$$

$$f(-1) = 1$$

$$f(0) = 0$$

$$f(8) = 16$$

Abs. Max: $\underline{(8,\ 16)|}$ ***Abs. Min***: $\underline{(0,\ 0)|}$

Exercise

Find the absolute maximum and minimum values of $f(\theta) = \theta^{3/5}, \quad -32 \le \theta \le 1$

Solution

$$f'(\theta) = \frac{3}{5}\theta^{-2/5} = 0 \quad \Rightarrow \quad \boxed{\theta = 0} \qquad (CN)$$

$$f(-32) = -8$$

$$f(0) = 0$$

$$f(1) = 1$$

Abs. Max: $\underline{(1,\ 1)|}$ ***Abs. Min***: $\underline{(-32,\ -8)|}$

Exercise

Find the absolute maximum and minimum values of $f(x) = 2^x \sin x \quad [-2,\ 6]$

Solution

$$f'(x) = 2^x(\ln 2)\sin x + 2^x \cos x$$

$$= 2^x(\ln 2\ \sin x + \cos x) = 0$$

$$\ln 2\ \sin x + \cos x = 0; \quad 2^x \ne 0$$

$$\cos x = -\ln 2\ \sin x$$

$$\frac{\sin x}{\cos x} = -\frac{1}{\ln 2} = \tan x$$

$$x = \tan^{-1}(-\ln 2) \approx \underline{-.96468|}$$

$$x = -.96468 + \pi \ \approx \underline{2.1769|}$$

$$x = -.96468 + 2\pi \ \approx \underline{5.3185|}$$

(since it is ***periodic***)

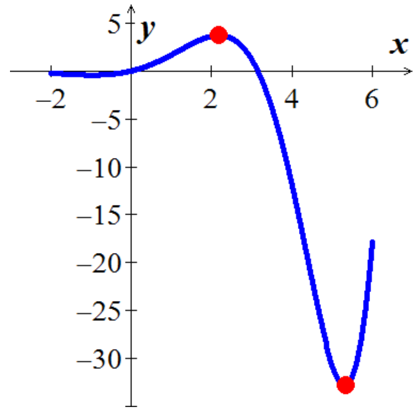
$$f(-2) = -0.227$$

$$f(-.96468) \approx -0.4211$$

$f(2.1769) \approx 3.7164$

$f(5.3185) \approx -32.7968$

$f(6) = -17.88$



Abs. Max: $\underline{(2.1769,\ 3.7164)|}$ ***Abs. Min***: $\underline{(5.3185,\ -32.7968)|}$

Exercise

Find the absolute maximum and minimum values of $f(x) = \sec x \quad \left[-\frac{\pi}{4},\ \frac{\pi}{4}\right]$

Solution

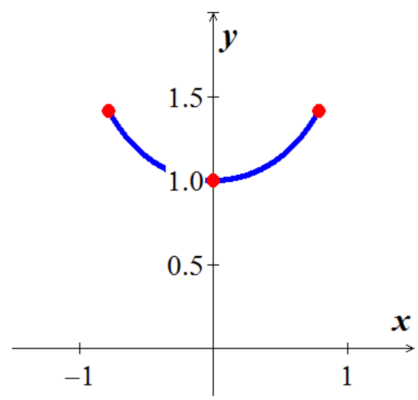
$f'(x) = \sec x \tan x = 0 \quad \Rightarrow \quad \boxed{x = 0} \qquad (CN)$

$f\left(-\frac{\pi}{4}\right) = \sqrt{2}$

$f(0) = 1$

$f\left(\frac{\pi}{4}\right) = \sqrt{2}$

Abs. Max: $\underline{\left(-\frac{\pi}{4},\ \sqrt{2}\right) \& \left(\frac{\pi}{4},\ \sqrt{2}\right)|}$ ***Abs. Min***: $\underline{(0,\ 1)|}$



Exercise

Find the absolute maximum and minimum values of $f(x) = x^3 e^{-x} \quad [-1,\ 5]$

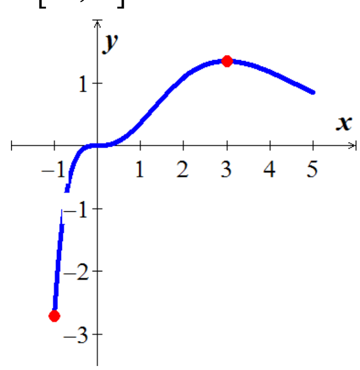
Solution

$f'(x) = 3x^2 e^{-x} - x^3 e^{-x}$

$\quad = x^2 e^{-x}(3 - x) = 0 \quad \rightarrow \quad \underline{x = 0,\ 3|} \quad (CN)$

$f(-1) = -e \ \approx -2.718$

$f(0) = 0$



$f(3) = 27e^{-3} \approx 1.344$

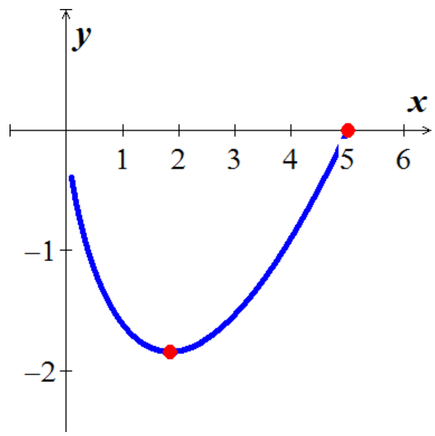
$f(5) = 125e^{-5} \approx 0.842$

Abs. Max: $\boxed{(3,\ 1.344)}$ ***Abs. Min***: $\boxed{(-1,\ -2.718)}$

Exercise

Find the absolute maximum and minimum values of $f(x) = x\ln\left(\frac{x}{5}\right)$ $[0.1,\ 5]$

Solution



$$f'(x) = \ln\left(\frac{x}{5}\right) + x\left(\frac{1}{5} \div \frac{x}{5}\right)$$

$$= \ln\left(\frac{x}{5}\right) + 1 = 0$$

$$\ln\left(\frac{x}{5}\right) = -1 \quad \rightarrow \quad \boxed{x = 5e^{-1}} \quad (CN)$$

$f(0.1) \approx -.3912$

$f(1.8394) \approx -1.8394$

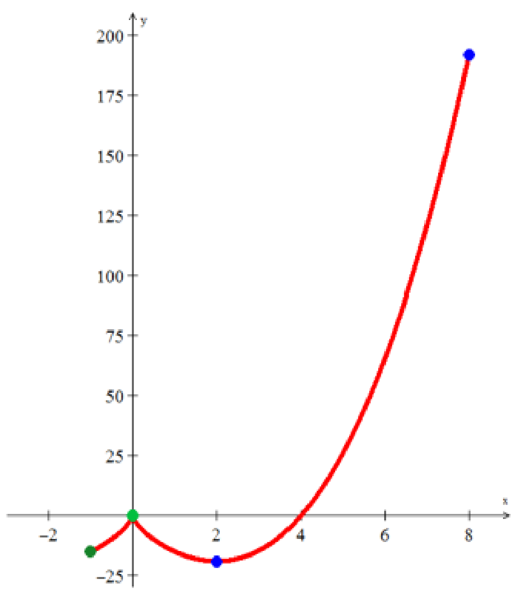
$f(5) \approx 0$

Abs. Max: $\boxed{(5,\ 0)}$ ***Abs. Min***: $\boxed{(1.8394,\ -1.8394)}$

Exercise

Find the absolute extrema of $f(x) = x^{8/3} - 16x^{2/3}$ $[-1,\ 8]$

Solution



$$f'(x) = \frac{8}{3}x^{5/3} - \frac{32}{3}x^{-1/3}$$

$$= \frac{8}{3}\left(x^{5/3} - \frac{4}{x^{1/3}}\right)$$

$$= \frac{8}{3}\left(\frac{x^2 - 4}{x^{1/3}}\right) = 0$$

$$CN: \boxed{x = \pm 2}$$

$x \neq -2 \notin [-1,\ 8]$

The derivative is undefined at $\boxed{x = 0}$

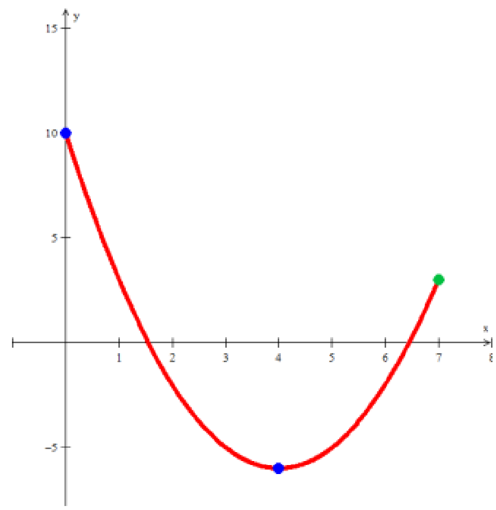
x	$f(x)$
-1	-15
0	0
2	-19.05
8	192

Abs. max: $(8,\ 192)$ ***Abs. Min*** $(2,\ -19.05)$

Exercise

Find the minimum and maximum values of $f(x)=x^2-8x+10\quad [0,\ 7]$

Solution



$$f'(x)=2x-8=0$$

$$\Rightarrow x=4\ \ (CN)$$

$$\rightarrow y = 16 - 32 + 10 = -6$$

$$\begin{cases} x=0 \rightarrow y=10 \\ x=7 \rightarrow y=3 \end{cases}$$

Abs. Maximum $(0, 10)$

Abs. Minimum $(4, -6)$

Exercise

Find the absolute extrema of the function on the closed interval $f(x)=2(3-x),\ \ [-1,\ 2]$

Solution

$$f'=-2$$

$$f(-1)=2(3-(-1))=8$$

$$f(2)=2(3-2)=2$$

Abs. Max: $\underline{(-1,\ 8)|}$ ***abs Min***: $\underline{(2,\ 2)|}$

Exercise

Find the absolute extrema of the function on the closed interval $f(x) = x^3 - 3x^2, \ [0, \ 4]$

Solution

$$f'(x) = 3x^2 - 6x = 0$$

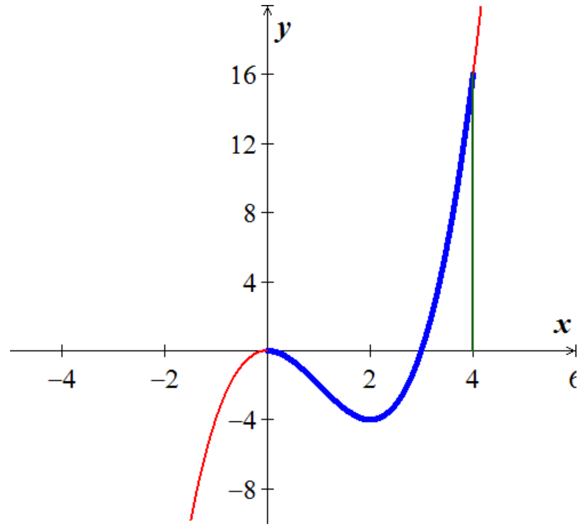
$$3x(x-2) = 0 \quad \rightarrow \begin{cases} \boxed{x = 0} \\ x - 2 = 0 \Rightarrow \boxed{x = 2} \end{cases}$$

$$f(0) = 0^3 - 3(0)^2 = 0$$

$$f(2) = 2^3 - 3(2)^2 = -4$$

$$f(4) = 4^3 - 3(4)^2 = 16$$

Abs. Max: $(4, \ 16)$ ***LMIN***: $(2, \ -4)$



Exercise

Find the absolute extrema of the function on the closed interval $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x - 4, \ [-2, \ 5]$

Solution

$$f'(x) = x^2 - 4x + 3 = 0 \quad \rightarrow \begin{cases} \boxed{x = 1} \\ \boxed{x = 3} \end{cases}$$

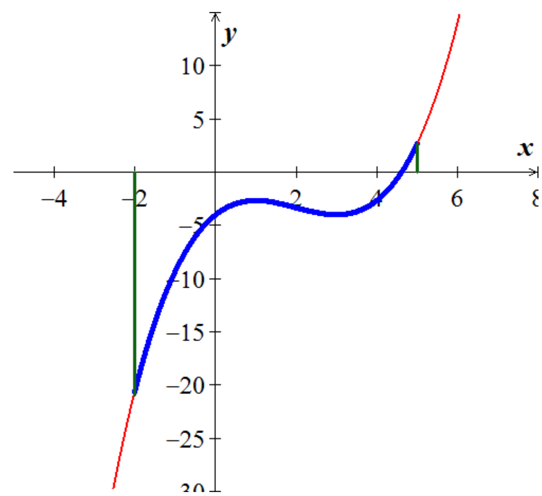
$$f(-2) = \frac{1}{3}(-2)^3 - 2(-2)^2 + 3(-2) - 4 = -20.66$$

$$f(1) = \frac{1}{3}(1)^3 - 2(1)^2 + 3(1) - 4 = -2.66$$

$$f(3) = \frac{1}{3}(3)^3 - 2(3)^2 + 3(3) - 4 = -4$$

$$f(5) = \frac{1}{3}(5)^3 - 2(5)^2 + 3(5) - 4 = 2.66$$

Abs. max: $(5, \ 2.66)$ ***abs. min***: $(-2, \ -20.66)$



Exercise

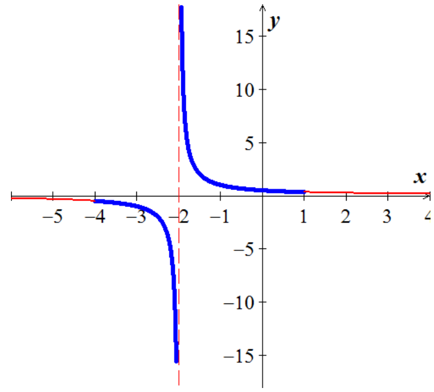
Find the absolute extrema of the function on the closed interval $f(x)=\frac{1}{x+2}, \quad [-4,\ 1]$

Solution

$$x+2\neq 0 \to x\neq -2 \quad (Asymptote)$$

$$f'(x)=-\frac{1}{(x+2)^2}\neq 0$$

There is ***no*** Relative Extrema.



Exercise

Find the absolute extrema of the function on the closed interval $f(x)=\left(x^2+4\right)^{2/3}, \quad [-2,\ 2]$

Solution

$$f'(x)=\frac{2}{3}(2x)\left(x^2+4\right)^{2/3-1}$$

$$=\frac{4x}{3}\left(x^2+4\right)^{-1/3}$$

$$f'=\frac{4x}{3}\left(x^2+4\right)^{-1/3}=0; \quad x^2+4\neq 0$$

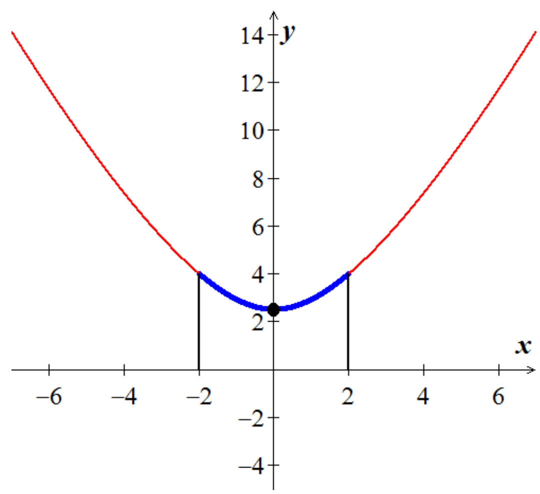
$$\boxed{x=0}$$

$$f(x=-2)=\left((-2)^2+4\right)^{2/3}=4$$

$$f(x=0)=\left((0)^2+4\right)^{2/3}=2.52$$

$$f(x=2)=\left((2)^2+4\right)^{2/3}=4$$

RMAX: $(-2,\ 4)\cup(2,\ 4)$ ***RMIN***: $(0,\ 2.52)$



Exercise

Find the absolute maximum and minimum values of each function (if they exist).

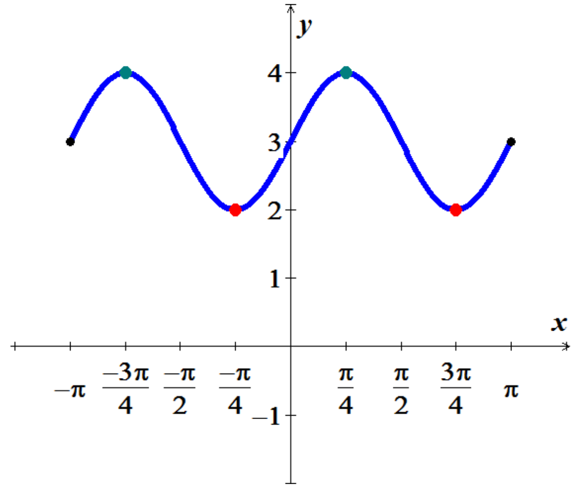
$$f(x)=\sin 2x+3 \quad on \quad [-\pi,\ \pi]$$

Solution

$$f'(x)=2\cos 2x=0 \quad \to \quad 2x=\pm\frac{\pi}{2}, \quad \pm\frac{3\pi}{2}$$

$\boxed{x = \pm\frac{\pi}{4}, \quad \pm\frac{3\pi}{4}}$ (CN)

x	$f(x)$
$-\pi$	3
$-\frac{3\pi}{4}$	4
$-\frac{\pi}{4}$	2
$\frac{\pi}{4}$	4
$\frac{3\pi}{4}$	2
π	3



Abs. Min: $\boxed{\left(-\frac{\pi}{4},\ 2\right) \quad \left(\frac{3\pi}{4},\ 2\right)}$

Abs. Max: $\boxed{\left(-\frac{3\pi}{4},\ 4\right) \quad \left(\frac{\pi}{4},\ 4\right)}$

Exercise

Find the absolute maximum and minimum values of each function (if they exist).

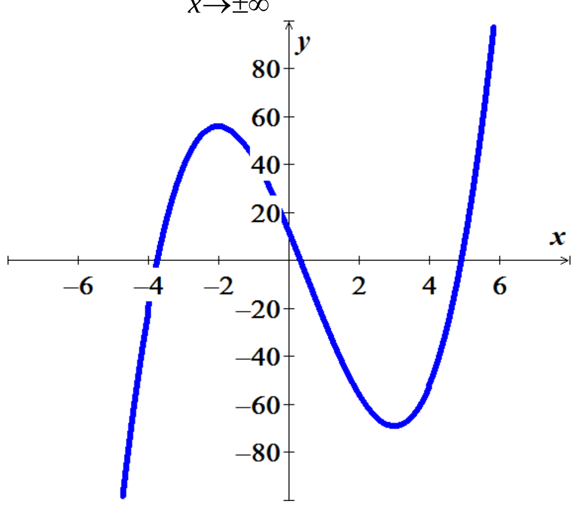
$$f(x) = 2x^3 - 3x^2 - 36x + 12 \quad on \quad (-\infty,\ \infty)$$

Solution

$$f'(x) = 6x^2 - 6x - 36 = 0$$

$$x^2 - x - 3 = 0 \quad \rightarrow \quad CN: \ \boxed{x = -2,\ 3}$$

There is no ***absolute Max.*** or ***Min***. since $\lim\limits_{x \to \pm\infty} f(x) = \pm\infty$



Exercise

Find the absolute maximum and minimum values of each function (if they exist).

$$f(x) = 4x^{1/2} - x^{5/2} \quad on \quad [0,\ 4]$$

Solution

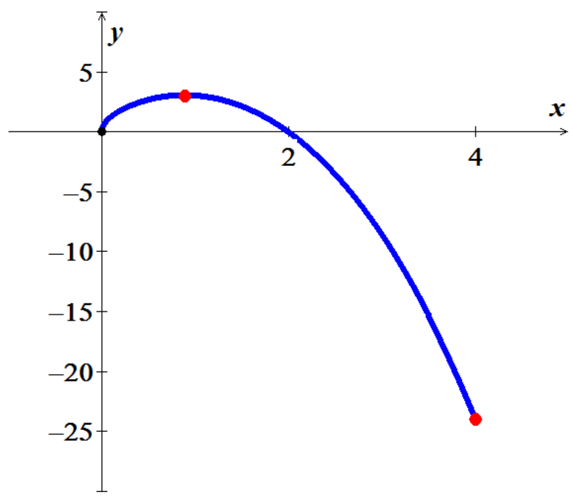
$$f'(x) = 2x^{-1/2} - \frac{5}{2}x^{3/2} = 0$$

$$\left(2x^{1/2} \times\right) \quad 2x^{-1/2} - \frac{5}{2}x^{3/2} = 0 \quad (x \neq 0)$$

$$4 - 5x^2 = 0$$

$$CN: \ x = \pm\frac{2}{\sqrt{5}},\ 0$$

x	$f(x)$
0	0
$\frac{2}{\sqrt{5}}$	$4\left(\frac{2}{\sqrt{5}}\right)^{1/2} - \left(\frac{2}{\sqrt{5}}\right)^{5/2} = 4\left(\frac{2}{\sqrt{5}}\right)^{1/2} - \frac{4}{5}\left(\frac{2}{\sqrt{5}}\right)^{1/2} = \frac{16}{5}\left(\frac{2}{\sqrt{5}}\right)^{1/2}$
4	$4(4)^{1/2} - (4)^{5/2} = 8 - 32 = -24$



Abs. Min: $(4,\ -24)$ ***Abs. Max***: $\left(\frac{2}{\sqrt{5}},\ \frac{16}{5}\left(\frac{2}{\sqrt{5}}\right)^{1/2}\right)$

Exercise

Find the absolute maximum and minimum values of each function (if they exist).

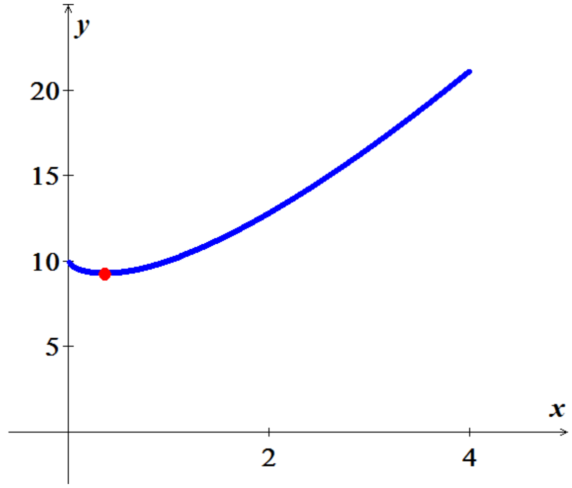
$$f(x) = 2x\ln x + 10 \quad on \quad (0,\ 4)$$

Solution

$$f'(x) = 2\ln x + 2 = 0 \quad \rightarrow \quad \ln x = -1$$

$CN: \ x = e^{-1}$

$$f\left(\frac{1}{e}\right) = \frac{2}{e}\ln e^{-1} + 10 = 10 - \frac{2}{e}$$



Abs. Min: $\left(\frac{1}{e}, \ 10 - \frac{2}{e}\right)$

Exercise

Find the absolute maximum and minimum values of each function (if they exist).

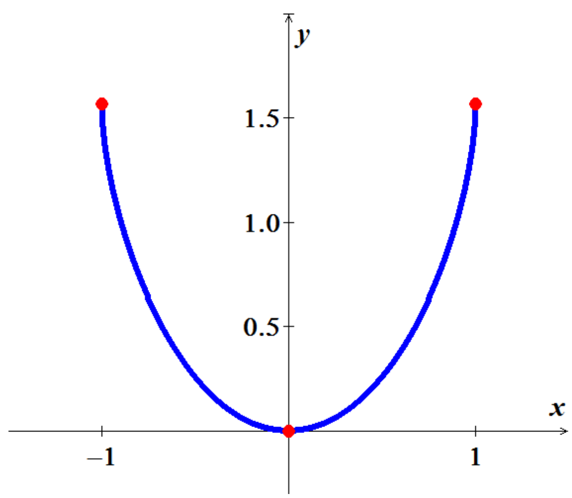
$f(x) = x\sin^{-1}x \quad on \quad [-1, 1]$

Solution

$$f'(x) = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} = 0$$

$CN: \ x = 0$

x	$f(x)$
-1	$-\sin^{-1}(-1) = \frac{\pi}{2}$
0	0
1	$\sin^{-1}(1) = \frac{\pi}{2}$



Abs. Min: $(0, \ 0)$

Abs. Max: $\left(\pm 1, \ \frac{\pi}{2}\right)$

Exercise

Determine all critical points of $y = x^2 - 6x + 7$

Solution

$$y' = 2x - 6 = 0 \quad \Rightarrow \quad \boxed{x = 3} \qquad (CN)$$

$$y\Big|_{x=3} = 3^2 - 6(3) + 7 = -2$$

Critical point: $(3,\ -2)$

Exercise

Determine all critical points of $g(x) = (x-1)^2 (x-3)^2$

Solution

$$g'(x) = 2(x-1)(x-3)^2 + 2(x-1)^2(x-3) \qquad (uv)' = u'v + v'u$$

$$= 2(x-1)(x-3)(x-3+x-1)$$

$$= 2(x-1)(x-3)(2x-4)$$

The ***critical numbers*** are: x = 1, 2, 3

$$g(1) = 0 \qquad g(2) = 1 \qquad g(3) = 0$$

Critical points: (1, 0), (2, 1) and (3, 0)

Exercise

Determine all critical points of $f(x) = \dfrac{x^2}{x-2}$

Solution

$$f'(x) = \frac{2x(x-2) - x^2}{(x-2)^2} = \frac{x^2 - 4x}{(x-2)^2} = 0 \qquad \left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

x = 2 is not in the domain

The critical numbers are: x = 0, 4

$$f(0) = 0 \qquad f(4) = 8$$

Critical points: $(0,\ 0),\ (4,\ 8)$

Exercise

Determine all critical points of $g(x) = x^2 - 32\sqrt{x}$

Solution

$$g'(x) = 2x - \frac{16}{\sqrt{x}} = \frac{2x^{3/2} - 16}{\sqrt{x}} = 0$$

$$\begin{cases} 2x^{3/2} - 16 = 0 \Rightarrow x^{3/2} = 8 \to \boxed{x = 4} \\ \sqrt{x} = 0 \Rightarrow \boxed{x = 0} \end{cases}$$

The critical numbers are: $x = 0,\ 4$

$g(0) = 0 \qquad\qquad g(4) = 16 - 32\sqrt{4} = 48$

Critical points: $(0,0),\ (4,48)$

Exercise

Find the extreme values (absolute and local) of the function and where they occur $y = x^3 - 2x + 4$

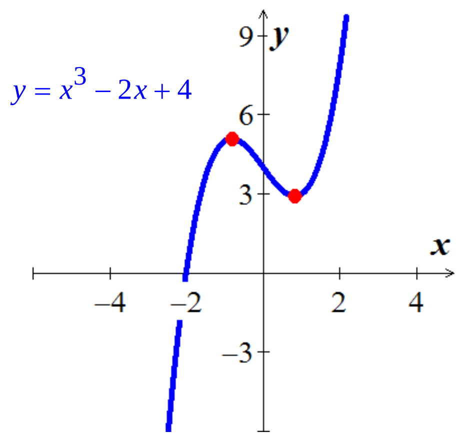
Solution

$$y' = 3x^2 - 2 = 0 \quad \Rightarrow \quad \boxed{x = \pm\sqrt{\frac{2}{3}}}$$

$$x = -\sqrt{\frac{2}{3}} \quad \Rightarrow \quad y = \left(-\sqrt{\frac{2}{3}}\right)^3 - 2\left(-\sqrt{\frac{2}{3}}\right) + 4 = 5.089$$

$$x = \sqrt{\frac{2}{3}} \quad \Rightarrow \quad y = \left(\sqrt{\frac{2}{3}}\right)^3 - 2\left(\sqrt{\frac{2}{3}}\right) + 4 = 2.911$$

LMAX: $(-.816,\ 5.089)$ ***LMIN***: $(.816,\ 2.911)$



Exercise

Find the extreme values (absolute and local) of the function and where they occur $y = \sqrt{x^2 - 1}$

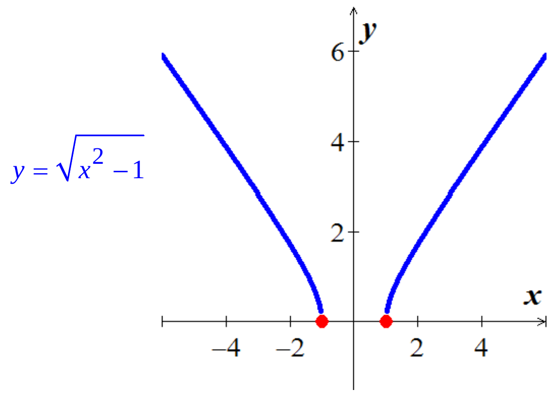
Solution

Domain: $x \le -1 \quad x \ge 1$

$$y' = \frac{x}{\sqrt{x^2 - 1}} = 0 \quad \Rightarrow \quad x = \cancel{0},\ \pm 1 \ \ (CN)$$

$$y = \sqrt{(\pm 1)^2 - 1} = 0$$

LMIN: $(-1,\ 0)\ \&\ (1,\ 0)$



Exercise

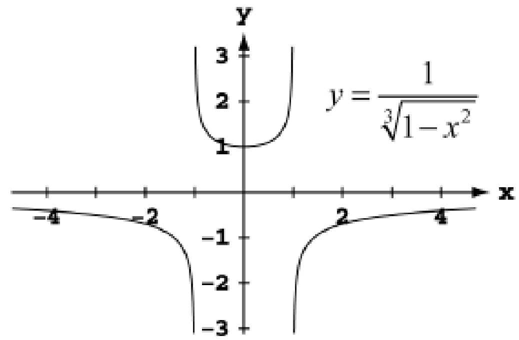
Find the extreme values (absolute and local) of the function and where they occur $y = \dfrac{1}{\sqrt[3]{1-x^2}}$

Solution

VA: $x = \pm 1$

$$y = \left(1-x^2\right)^{-1/3}$$

$$y' = -\frac{1}{3}\left(1-x^2\right)^{-4/3}(-2x) = \frac{2}{3}\frac{x}{\left(1-x^2\right)^{4/3}} = 0$$



$\boxed{x = 0,\ \cancel{\pm 1}}$ (*CN*)

LMIN $(0,\ 1)$

Exercise

Find the extreme values (absolute and local) of the function and where they occur $y = x^2\sqrt{3-x}$

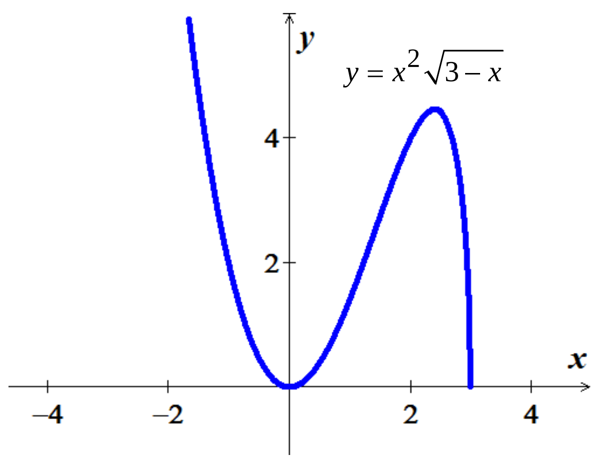
Solution

$$y' = 2x\sqrt{3-x} + \frac{1}{2}\left(\frac{-1}{\sqrt{3-x}}\right)x^2 \qquad (uv)' = u'v + v'u$$

$$= \frac{4x(3-x) - x^2}{2\sqrt{3-x}}$$

$$= \frac{4x(3-x) - x^2}{2\sqrt{3-x}}$$

$$= \frac{12x - 5x^2}{2\sqrt{3-x}} = 0 \qquad \boxed{x = \frac{5}{12},\ 0,\ 3} \quad (CN)$$



$$y\Big|_{x=\frac{5}{12}} = 0.279$$

$$y\Big|_{x=0} = 0$$

$$y\Big|_{x=3} = 0$$

LMAX: $\left(\frac{5}{12},\ 0.279\right)$ ***LMIN***: $(0,\ 0) \cup (3,\ 0)$

Exercise

Find the extreme values (absolute and local) of the function and where they occur $y = \dfrac{x+1}{x^2+2x+2}$

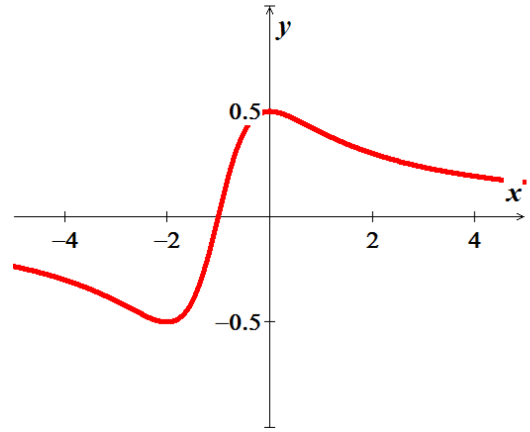
Solution

$$y' = \frac{x^2+2x+2-(2x+2)(x+1)}{\left(x^2+2x+2\right)^2} \qquad \left(\frac{u}{v}\right)' = \frac{u'v-v'u}{v^2}$$

$$= \frac{x^2+2x+2-2x^2-2x-2x-2}{\left(x^2+2x+2\right)^2}$$

$$= \frac{-x^2-2x}{\left(x^2+2x+2\right)^2} = 0$$

$\boxed{x = 0, \ -2}$ (*CN*)



$$y\Big|_{x=0} = \frac{1}{2}$$

$$y\Big|_{x=-2} = -2$$

LMAX: $\left(0, \ \frac{1}{2}\right)$ ***LMIN***: $\left(-2, \ -\frac{1}{2}\right)$

Exercise

Find the extreme values (absolute and local) of the function and where they occur $y = x^{2/3}(x+2)$

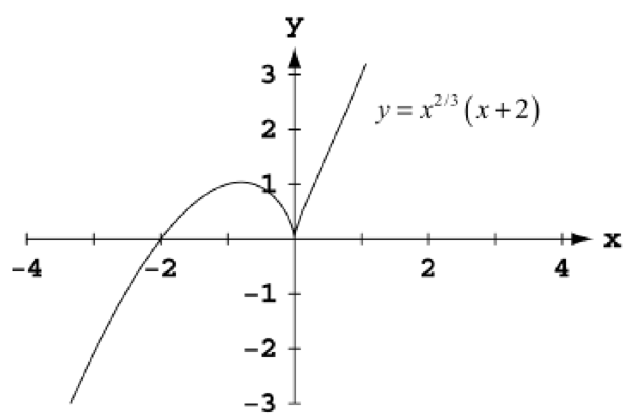
Solution

$$y' = \frac{2}{3}x^{-1/3}(x+2) + x^{2/3} \qquad (uv)' = u'v + v'u$$

$$= \frac{2}{3}\frac{x+2}{x^{1/3}} + x^{2/3}$$

$$= \frac{2x+4+3x}{x^{1/3}}$$

$$= \frac{5x+4}{\sqrt[3]{x}} = 0 \qquad \boxed{x = -\frac{4}{5}, \ 0} \quad (CN)$$



$$y\Big|_{x=-\frac{4}{5}} = \left(-\frac{4}{5}\right)^{2/3}\left(-\frac{4}{5}+2\right) = 1.034$$

$$y\Big|_{x=0} = 0$$

LMAX: $\left(-\frac{4}{5}, \ 1.034\right)$ ***LMIN***: $(0, \ 0)$

Exercise

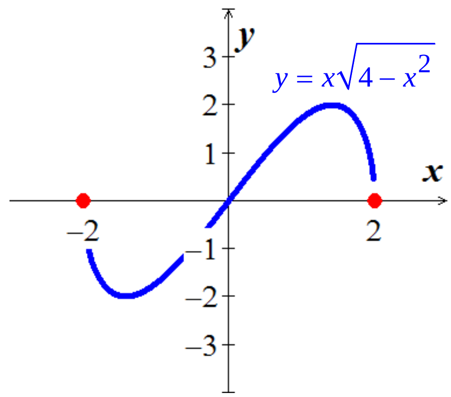
Find the extreme values (absolute and local) of the function and where they occur $y = x\sqrt{4-x^2}$

Solution

$$y' = \sqrt{4-x^2} + \left(\frac{1}{2}\frac{-2x}{\sqrt{4-x^2}}\right)(x) \qquad (uv)' = u'v + v'u$$

$$= \frac{4-x^2-x^2}{\sqrt{4-x^2}}$$

$$= \frac{4-2x^2}{\sqrt{4-x^2}} = 0$$



$$\begin{cases} 4-2x^2 = 0 \\ 4-x^2 = 0 \end{cases} \rightarrow \boxed{x = \pm\sqrt{2},\ \pm 2} \quad (CN)$$

$$y\Big|_{x=-\sqrt{2}} = -\sqrt{2}\sqrt{2} = -2$$

$$y\Big|_{x=\sqrt{2}} = 2$$

$$y\Big|_{x=\pm 2} = 0$$

LMAX: $\left(\sqrt{2},\ 2\right)$ *LMIN*: $\left(-\sqrt{2},\ -2\right)$

Exercise

Find the extreme values (absolute and local) of the function and where they occur $f(x) = \dfrac{e^x + e^{-x}}{2}$

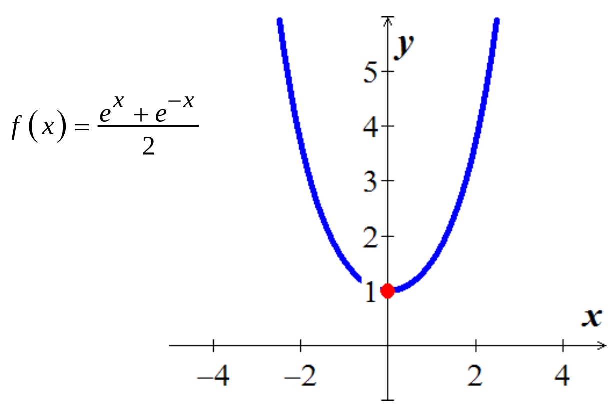
Solution

$$f'(x) = \frac{1}{2}\left(e^x - e^{-x}\right) = 0$$

$$e^x = e^{-x} \quad \rightarrow \quad x = 0 \quad (CN)$$

$$f(0) = \frac{1+1}{2} = 1$$

LMIN: $(0,\ 1)$

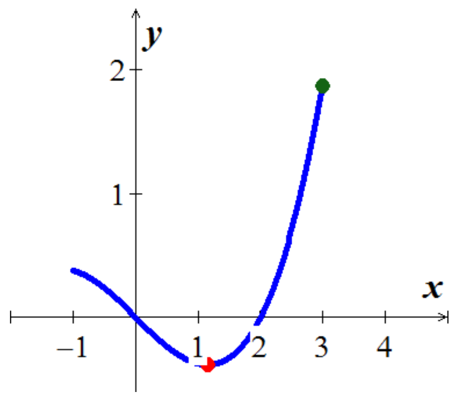


Exercise

Find the extreme values (absolute and local) of the function and where they occur

$$f(x)=\frac{1}{8}x^3-\frac{1}{2}x \quad [-1,\ 3]$$

Solution



$$f'(x)=\frac{3}{8}x^2-\frac{1}{2}=0$$

$$x^2=\frac{4}{3} \quad \Rightarrow \quad x=\cancel{-\frac{2}{\sqrt{3}}}\ (<-1),\ \ \frac{2}{\sqrt{3}}$$

$$(CN): \quad x=\frac{2}{\sqrt{3}}$$

$$f(-1)=-\frac{1}{8}+\frac{1}{2}=\frac{3}{8}$$

$$f(3)=\frac{27}{8}-\frac{3}{2}=\frac{15}{8}$$

$$f\left(\frac{2}{\sqrt{3}}\right)=\frac{1}{3\sqrt{3}}-\frac{1}{\sqrt{3}}=-\frac{2}{3\sqrt{3}}$$

LMIN: $\left(\frac{2}{\sqrt{3}},\ -\frac{2}{3\sqrt{3}}\right)$ ***abs. Max***: $\left(3,\ \frac{15}{8}\right)$

Exercise

Find the extreme values (absolute and local) of the function and where they occur $f(x)=\frac{1}{x}-\ln x$

Solution

$$f(x)=-\frac{1}{x^2}-\frac{1}{x}=-\frac{1+x}{x^2}=0$$

$x=0,\ -1$ (*CN*)

Since the critical number are not within the domain; inside the log has to be positive.
No abs or local extreme

Exercise

Find the extreme values (absolute and local) of the function and where they occur

$$f(x)=\sin x\cos x \quad [0,\ 2\pi]$$

Solution

$$f'(x)=\cos^2 x-\sin^2 x=\cos 2x \ =0$$

$$2x = \frac{\pi}{2} + k\pi \quad \to \quad \underline{x = \frac{\pi}{4}, \ \frac{3\pi}{4}, \ \frac{5\pi}{4}, \ \frac{7\pi}{4}|} \quad (CN)$$

$f(0) = 0$

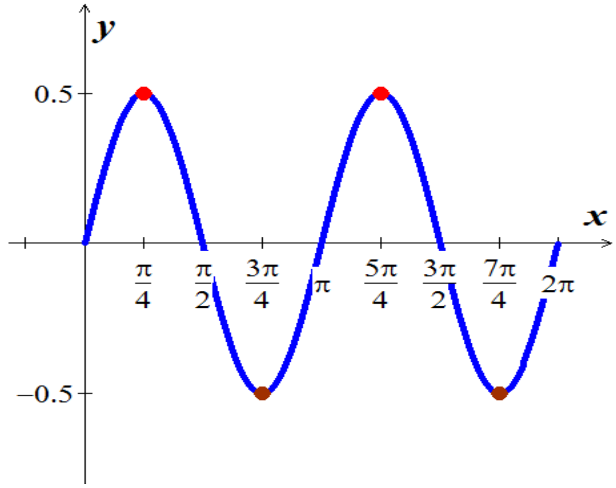
$f\left(\frac{\pi}{4}\right) = \frac{1}{2}$

$f\left(\frac{3\pi}{4}\right) = -\frac{1}{2}$

$f\left(\frac{5\pi}{4}\right) = \frac{1}{2}$

$f\left(\frac{7\pi}{4}\right) = -\frac{1}{2}$

$f(2\pi) = 0$



LMIN: $\underline{\left(\frac{3\pi}{4}, \ -\frac{1}{2}\right) \& \left(\frac{7\pi}{4}, \ -\frac{1}{2}\right)|}$ ***LMAX***: $\underline{\left(\frac{\pi}{4}, \ \frac{1}{2}\right) \& \left(\frac{5\pi}{4}, \ \frac{1}{2}\right)|}$

Exercise

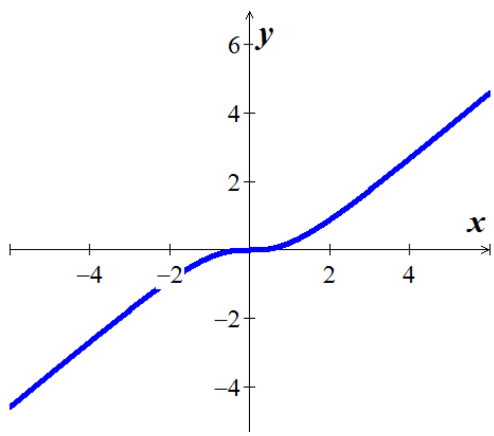
Find the extreme values (absolute and local) of the function and where they occur $f(x) = x - \tan^{-1} x$

Solution

$$f'(x) = 1 - \frac{1}{1+x^2} = 0$$

$$1 + x^2 = 1 \quad \to \quad \underline{x = 0, 0|} \quad (CN)$$

No extreme values



Exercise

Let $f(x) = (x-2)^{2/3}$

a) Does $f'(2)$ exist?

b) Show the only local extreme value of f occurs at $x = 2$.

c) Does the result in part (*b*) contradict the Extreme Value Theorem?

Solution

a) $f'(x) = \frac{2}{3}(x-2)^{-1/3}$ is undefined at $x = 2$

b) $f(x=2)=(2-2)^{2/3}=0$ and $f(x)>0 \quad \forall x \neq 2$

c) No, $f(x)$ domain is all real numbers and doesn't need to have a global maximum. Any restriction of *f* to a closed interval of the form $[a, b]$ would have a maximum and minimum value on the interval.

Exercise

When a telephone wire is hung between two poles, the wire forms a U-shape curve called a Catenary. For instance, the function $y=30\left(e^{x/60}+e^{-x/60}\right) \quad -30 \le x \le 30$ models the shape of the telephone wire strung between two poles that are 60 *ft*. apart (*x* & *y* are measured in *ft*.). Show that the lowest point on the wire is midway between two poles. How much does the wire sag between the two poles?

Solution

$$y' = 30\left(\frac{1}{60}e^{x/60} - \frac{1}{60}e^{-x/60}\right)$$

$$= \frac{1}{2}\left(e^{x/60} - e^{-x/60}\right)$$

Critical number(s)

$$y' = 0$$

$$\frac{1}{2}\left(e^{x/60} - e^{-x/60}\right) = 0$$

$$e^{x/60} - e^{-x/60} = 0$$

$$e^{x/60} = e^{-x/60}$$

$$\frac{x}{60} = -\frac{x}{60}$$

$$\Rightarrow x = 0$$

$$y(x=-30) = 30\left(e^{-30/60} + e^{-(-30)/60}\right) \approx 67.7\ ft$$

$$y(x=0) = 30\left(e^{0} + e^{0}\right) = 30(2) = 60\ ft$$

$$y(x=30) = 30\left(e^{30/60} + e^{-(30)/60}\right) \approx 67.7\ ft$$

$$\boxed{Sag:\ 67.7\ ft}$$

Exercise

You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 12 *feet* long and starts 3 *feet* from the wall you are sitting next to.

a) Show that your viewing angle is $\alpha = \cot^{-1}\frac{x}{15} - \cot^{-1}\frac{x}{3}$ If you are *x feet* from the front wall

b) Find *x* so that α is as large as possible

Solution

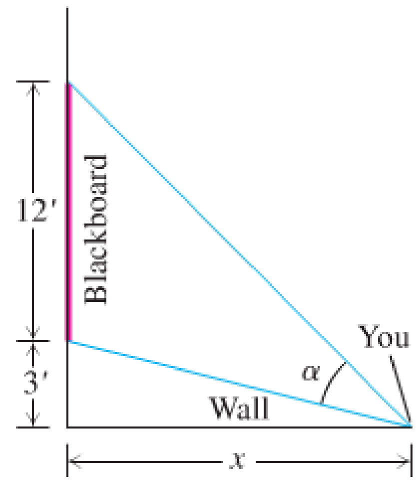
a) $\cot(wall) = \frac{x}{3} \quad \Rightarrow \angle wall = \cot^{-1}\left(\frac{x}{3}\right)$

$\cot(\triangle) = \frac{x}{15} \quad \Rightarrow \angle\triangle = \cot^{-1}\left(\frac{x}{15}\right)$

α = Angle of the large triangle – Wall triangle angle

$\alpha = \cot^{-1}\frac{x}{15} - \cot^{-1}\frac{x}{3}$

b) $\frac{d\alpha}{dx} = -\frac{\frac{1}{15}}{1+\left(\frac{x}{15}\right)^2} + \frac{\frac{1}{3}}{1+\left(\frac{x}{3}\right)^2}$



$= -\frac{1}{15}\frac{1}{1+\frac{x^2}{225}} + \frac{1}{3}\frac{1}{1+\frac{x^2}{9}}$

$= -\frac{1}{15}\frac{225}{225+x^2} + \frac{1}{3}\frac{9}{9+x^2}$

$= -\frac{15}{225+x^2} + \frac{3}{9+x^2}$

$= \frac{-15\left(9+x^2\right)+3\left(225+x^2\right)}{\left(225+x^2\right)\left(9+x^2\right)}$

$= \frac{-135-15x^2+675+3x^2}{\left(225+x^2\right)\left(9+x^2\right)}$

$= \frac{-12x^2+540}{\left(225+x^2\right)\left(9+x^2\right)} = 0$

$-12x^2+540=0 \quad \Rightarrow x^2 = \frac{540}{12} = 45 \to \boxed{x = \pm 3\sqrt{5}}$

$x = 3\sqrt{5} \approx 6.7082$

$\alpha\left(x=3\sqrt{5}\right) = \cot^{-1}\frac{3\sqrt{5}}{15} - \cot^{-1}\frac{3\sqrt{5}}{3} \approx 0.729728 \quad \approx 41.8103°$

Local maximum of 41.8103° when $x \approx 6.7082$ *ft*.