

Solution **Section 2.2 – Trigonometric Integrals**

Exercise

Evaluate the integral $\int \sin^5 \frac{x}{2} dx$

Solution

$$\begin{aligned}\sin^5 \frac{x}{2} &= \left(\sin^2 \frac{x}{2} \right)^2 \sin \frac{x}{2} \\&= \left(1 - \cos^2 \frac{x}{2} \right)^2 \sin \frac{x}{2} \\&= \left(1 - 2 \cos^2 \frac{x}{2} + \cos^4 \frac{x}{2} \right) \sin \frac{x}{2} \\d\left(\cos \frac{x}{2} \right) &= -\frac{1}{2} \sin \frac{x}{2} dx \quad \rightarrow \quad -2d\left(\cos \frac{x}{2} \right) = \sin \frac{x}{2} dx \\ \int \sin^5 \frac{x}{2} dx &= -2 \int \left(1 - 2 \cos^2 \frac{x}{2} + \cos^4 \frac{x}{2} \right) d\left(\cos \frac{x}{2} \right) \\&= -2 \left(\cos \frac{x}{2} - \frac{2}{3} \cos^3 \frac{x}{2} + \frac{1}{5} \cos^5 \frac{x}{2} \right) + C \\&= \underline{-2 \cos \frac{x}{2} + \frac{4}{3} \cos^3 \frac{x}{2} - \frac{2}{5} \cos^5 \frac{x}{2} + C}\end{aligned}$$

Exercise

Evaluate $\int \sin^4 6\theta \, d\theta$

Solution

$$\begin{aligned}\int \sin^4 6\theta \, d\theta &= \int \left(\frac{1 - \cos 12\theta}{2} \right)^2 d\theta & \sin^2 \alpha &= \frac{1}{2}(1 - \cos 2\alpha) \\&= \frac{1}{4} \int (1 - 2 \cos 12\theta + \cos^2 12\theta) d\theta & \cos^2 \alpha &= \frac{1}{2}(1 + \cos 2\alpha) \\&= \frac{1}{4} \int \left(1 - 2 \cos 12\theta + \frac{1}{2} + \frac{1}{2} \cos 24\theta \right) d\theta \\&= \underline{\frac{1}{4} \left(\frac{3}{2} \theta - \frac{1}{6} \sin 12\theta + \frac{1}{48} \sin 24\theta \right) + C}\end{aligned}$$

Exercise

Evaluate $\int x^2 \sin^2 x \, dx$

Solution

		$\int \cos 2x \, dx$
+	x^2	$\frac{1}{2} \sin 2x$
-	$2x$	$-\frac{1}{4} \cos 2x$
+	2	$-\frac{1}{8} \sin 2x$

$$\begin{aligned}
 \int x^2 \sin^2 x \, dx &= \frac{1}{2} \int x^2 (1 - \cos 2x) \, dx \\
 &= \frac{1}{2} \int (x^2 - x^2 \cos 2x) \, dx \\
 &= \frac{1}{2} \left(\frac{1}{3} x^3 + \frac{1}{2} x^2 \sin 2x + \frac{1}{2} x \cos 2x - \frac{1}{4} \sin 2x \right) + C
 \end{aligned}$$

Exercise

Evaluate $\int \sin^3 3x \, dx$

Solution

$$\begin{aligned}
 \int \sin^3 3x \, dx &= \int \sin^2 3x (\sin 3x) \, dx & d(\cos 3x) &= -3 \sin 3x \, dx & \cos^2 \alpha + \sin^2 \alpha &= 1 \\
 &= -\frac{1}{3} \int (1 - \cos^2 3x) \, d(\cos 3x) \\
 &= -\frac{1}{3} \left(\cos 3x - \frac{1}{3} \cos^3 3x \right) + C \\
 &= \frac{1}{9} \cos^3 3x - \frac{1}{3} \cos 3x + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \sin^5 x \, dx$

Solution

$$\begin{aligned}
 \int \sin^5 x \, dx &= \int \sin^4 x \sin x \, dx \\
 &= - \int (1 - \cos^2 x)^2 \, d(\cos x)
 \end{aligned}$$

$$\begin{aligned}
 &= - \int \left(1 - 2 \cos^2 x + \cos^4 x \right) d(\cos x) \\
 &= - \cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int 8 \cos^4 2\pi x \, dx$

Solution

$$\begin{aligned}
 \int 8 \cos^4 2\pi x \, dx &= 8 \int (\cos 2\pi x)^4 \, dx \\
 &= 8 \int \left(\frac{1 + \cos 4\pi x}{2} \right)^2 \, dx \\
 &= 2 \int (1 + \cos 4\pi x)^2 \, dx \\
 &= 2 \int (1 + 2 \cos 4\pi x + \cos^2 4\pi x) \, dx \\
 &= 2 \int dx + 4 \int \cos 4\pi x \, dx + 2 \int \cos^2 4\pi x \, dx \\
 &= 2x + 4 \frac{1}{4\pi} \sin 4\pi x + 2 \int \frac{1 + \cos 8\pi x}{2} \, dx \\
 &= 2x + \frac{1}{\pi} \sin 4\pi x + \int (1 + \cos 8\pi x) \, dx \\
 &= 2x + \frac{1}{\pi} \sin 4\pi x + x + \frac{1}{8\pi} \sin 8\pi x + C \\
 &= 3x + \frac{1}{\pi} \sin 4\pi x + \frac{1}{8\pi} \sin 8\pi x + C
 \end{aligned}$$

$$\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}$$

Exercise

Evaluate the integral $\int x \cos^3 x \, dx$

Solution

$$\begin{aligned}
 \int x \cos^3 x \, dx &= \int x \cos^2 x \cos x \, dx \\
 &= \int x (1 - \sin^2 x) \cos x \, dx
 \end{aligned}$$

$$\cos^2 \alpha + \sin^2 \alpha = 1$$

$$\begin{aligned}
&= \int x \cos x \, dx - \int x \sin^2 x \cos x \, dx \\
&\quad \begin{array}{ll} u = x & dv = \cos x \, dx \\ du = dx & v = \sin x \end{array} \quad \begin{array}{ll} u = x & dv = \sin^2 x \cos x \, dx \\ du = dx & v = \frac{1}{3} \sin^3 x \end{array} \\
&= x \sin x - \int \sin x \, dx - \left(\frac{1}{3} x \sin^3 x - \frac{1}{3} \int \sin^3 x \, dx \right) \\
&= x \sin x + \cos x - \frac{1}{3} x \sin^3 x + \frac{1}{3} \int \sin^2 x \sin x \, dx \\
&= x \sin x + \cos x - \frac{1}{3} x \sin^3 x - \frac{1}{3} \int (1 - \cos^2 x) d(\cos x) \\
&= x \sin x + \cos x - \frac{1}{3} x \sin^3 x - \frac{1}{3} \left(\cos x - \frac{1}{3} \cos^3 x \right) + C \\
&= x \sin x + \cos x - \frac{1}{3} x \sin^3 x - \frac{1}{3} \cos x + \frac{1}{9} \cos^3 x + C \\
&= \underline{x \sin x + \frac{2}{3} \cos x - \frac{1}{3} x \sin^3 x + \frac{1}{9} \cos^3 x + C}
\end{aligned}$$

Exercise

Evaluate the integral $\int \cos^4 x \, dx$

Solution

$$\begin{aligned}
\int \cos^4 x \, dx &= \frac{1}{4} \int (1 + \cos 2x)^2 \, dx & \cos^2 x &= \frac{1}{2}(1 + \cos 2x) \\
&= \frac{1}{4} \int (1 + 2 \cos 2x + \cos^2 2x) \, dx \\
&= \frac{1}{4} \int \left(1 + 2 \cos 2x + \frac{1}{2} + \frac{1}{2} \cos 4x \right) \, dx \\
&= \frac{1}{4} \int \left(\frac{3}{2} + 2 \cos 2x + \frac{1}{2} \cos 4x \right) \, dx \\
&= \underline{\frac{1}{4} \left(\frac{3}{2} x + \sin 2x + \frac{1}{8} \sin 4x \right) + C}
\end{aligned}$$

Exercise

Evaluate the integral $\int \cos^4 5x \, dx$

Solution

$$\begin{aligned}\int \cos^4 5x \, dx &= \frac{1}{4} \int (1 + \cos 10x)^2 \, dx & \cos^2 x &= \frac{1}{2}(1 + \cos 2x) \\&= \frac{1}{4} \int (1 + 2 \cos 10x + \cos^2 10x) \, dx \\&= \frac{1}{4} \int \left(1 + 2 \cos 10x + \frac{1}{2} + \frac{1}{2} \cos 20x\right) \, dx \\&= \frac{1}{4} \int \left(\frac{3}{2} + 2 \cos 10x + \frac{1}{2} \cos 20x\right) \, dx \\&= \frac{1}{4} \left(\frac{3}{2}x + \frac{1}{5} \sin 10x + \frac{1}{40} \sin 20x\right) + C \\&= \underline{\underline{\frac{3}{8}x + \frac{1}{20} \sin 10x + \frac{1}{160} \sin 20x + C}}\end{aligned}$$

Exercise

Evaluate $\int \cos^2 3x \, dx$

Solution

$$\begin{aligned}\int \cos^2 3x \, dx &= \frac{1}{2} \int (1 + \cos 6x) \, dx & \cos^2 \alpha &= \frac{1}{2}(1 + \cos 2\alpha) \\&= \underline{\underline{\frac{1}{2} \left(x + \frac{1}{6} \sin 6x\right) + C}}\end{aligned}$$

Exercise

Evaluate $\int \cos^3 \frac{x}{3} \, dx$

Solution

$$\begin{aligned}\int \cos^3 \frac{x}{3} \, dx &= \int \cos^2 \frac{x}{3} \cos \frac{x}{3} \, dx \\&= 3 \int \left(1 - \sin^2 \frac{x}{3}\right) d\left(\sin \frac{x}{3}\right) \\&= \underline{\underline{3 \sin \frac{x}{3} - \sin^3 \frac{x}{3} + C}}\end{aligned}$$

Exercise

Evaluate the integral $\int \cos^2 4x \, dx$

Solution

$$\begin{aligned}\int \cos^2 4x \, dx &= \frac{1}{2} \int (1 + \cos 8x) \, dx \\ &= \frac{1}{2} \left(x + \frac{1}{8} \sin 8x \right) + C\end{aligned}$$

$$\cos^2 \alpha = \frac{1}{2}(1 + \cos 2\alpha)$$

Exercise

Evaluate the integral $\int \sqrt{1 + \cos \frac{x}{2}} \, dx$

Solution

$$2 \cos^2 \alpha = 1 + \cos 2\alpha$$

$$2\alpha = \frac{x}{2} \rightarrow \alpha = \frac{x}{4}$$

$$1 + \cos \frac{x}{2} = 2 \cos^2 \frac{x}{4}$$

$$\begin{aligned}\int \sqrt{1 + \cos \frac{x}{2}} \, dx &= \int \sqrt{2 \cos^2 \frac{x}{4}} \, dx \\ &= \sqrt{2} \int \cos \frac{x}{4} \, dx \\ &= 4\sqrt{2} \sin \frac{x}{4} + C\end{aligned}$$

Exercise

Evaluate $\int \sec^4 2x \, dx$

Solution

$$\begin{aligned}\int \sec^4 2x \, dx &= \int (1 + \tan^2 2x) \sec^2 2x \, dx \\ &= \frac{1}{2} \int (1 + \tan^2 2x) \, d(\tan 2x) \\ &= \frac{1}{2} \tan 2x + \frac{1}{6} \tan^3 2x + C\end{aligned}$$

Exercise

Evaluate the integral $\int 6 \sec^4 x \, dx$

Solution

$$\begin{aligned}\int 6 \sec^4 x \, dx &= 6 \int \sec^2 x \sec^2 x \, dx \\&= 6 \int \sec^2 x (\tan^2 x + 1) \, dx \\&= 6 \int (\tan^2 x + 1) d(\tan x) \\&= 6 \left(\frac{1}{3} \tan^3 x + \tan x \right) + C \\&= \underline{2 \tan^3 x + 6 \tan x + C}\end{aligned}$$

Exercise

Evaluate $\int \sec^3 \pi x \, dx$

Solution

$$\begin{aligned}u &= \sec \pi x & dv &= \sec^2 \pi x \, dx \\du &= \pi \sec \pi x \tan \pi x & v &= \frac{1}{\pi} \tan \pi x \\ \int \sec^3 \pi x \, dx &= \frac{1}{\pi} \sec \pi x \tan \pi x - \int \sec \pi x \tan^2 \pi x \, dx \\&= \frac{1}{\pi} \sec \pi x \tan \pi x - \int \sec \pi x (\sec^2 \pi x - 1) \, dx \\ \int \sec^3 \pi x \, dx &= \frac{1}{\pi} \sec \pi x \tan \pi x - \int \sec^3 \pi x \, dx + \int \sec \pi x \, dx \\ 2 \int \sec^3 \pi x \, dx &= \frac{1}{\pi} \sec \pi x \tan \pi x + \ln |\sec \pi x + \tan \pi x| + C_1 \\ \int \sec^3 \pi x \, dx &= \underline{\frac{1}{2\pi} \sec \pi x \tan \pi x + \frac{1}{2} \ln |\sec \pi x + \tan \pi x| + C}\end{aligned}$$

Exercise

Evaluate the integral $\int \sec 4x \, dx$

Solution

$$\begin{aligned}\int \sec 4x \, dx &= \frac{1}{4} \int \sec 4x \, d(4x) \\&= \frac{1}{4} \int \sec 4x \frac{\sec 4x + \tan 4x}{\sec 4x + \tan 4x} d(4x) \\&= \frac{1}{4} \int \frac{\sec^2 4x + \sec 4x \tan 4x}{\sec 4x + \tan 4x} d(4x) \\&= \frac{1}{4} \int \frac{1}{\sec 4x + \tan 4x} d(\sec 4x + \tan 4x) \\&= \frac{1}{4} \ln |\sec 4x + \tan 4x| + C\end{aligned}$$

Exercise

Evaluate the integral $\int \csc^6 x \, dx$

Solution

$$\begin{aligned}\int \csc^6 x \, dx &= \int \csc^4 x \csc^2 x \, dx \\&= - \int (\cot^2 x + 1)^2 d(\cot x) \\&= - \int (\cot^4 x + 2 \cot^2 x + 1) d(\cot x) \\&= -\frac{1}{5} \cot^5 x - \frac{2}{3} \cot^3 x - \cot x + C\end{aligned}$$

Exercise

Evaluate the integral $\int \tan^5 \frac{x}{2} \, dx$

Solution

$$\int \tan^5 \frac{x}{2} \, dx = 2 \int \tan^2 \frac{x}{2} \tan^3 \frac{x}{2} d\left(\frac{x}{2}\right)$$

$$\begin{aligned}
&= 2 \int \left(\sec^2 \frac{x}{2} - 1 \right) \tan^3 \frac{x}{2} d\left(\frac{x}{2}\right) \\
&= 2 \int \sec^2 \frac{x}{2} \tan^3 \frac{x}{2} d\left(\frac{x}{2}\right) - 2 \int \tan^2 \frac{x}{2} \tan \frac{x}{2} d\left(\frac{x}{2}\right) \\
&= 2 \int \tan^3 \frac{x}{2} d\left(\tan \frac{x}{2}\right) - 2 \int \left(\sec^2 \frac{x}{2} - 1 \right) \tan \frac{x}{2} d\left(\frac{x}{2}\right) \\
&= \frac{1}{2} \tan^4 \frac{x}{2} - 2 \int \sec^2 \frac{x}{2} \tan \frac{x}{2} d\left(\frac{x}{2}\right) + 2 \int \tan \frac{x}{2} d\left(\frac{x}{2}\right) \\
&= \frac{1}{2} \tan^4 \frac{x}{2} - 2 \int \tan \frac{x}{2} d\left(\tan \frac{x}{2}\right) + 2 \int \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} d\left(\frac{x}{2}\right) \\
&= \frac{1}{2} \tan^4 \frac{x}{2} - \tan^2 \frac{x}{2} - 2 \int \frac{1}{\cos \frac{x}{2}} d\left(\cos \frac{x}{2}\right) \\
&= \frac{1}{2} \tan^4 \frac{x}{2} - \tan^2 \frac{x}{2} - 2 \ln \left| \cos \frac{x}{2} \right| + C \\
&= \frac{1}{2} \tan^4 \frac{x}{2} - \tan^2 \frac{x}{2} + 2 \ln \left| \sec \frac{x}{2} \right| + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \tan^5 x \, dx$

Solution

$$\begin{aligned}
\int \tan^5 x \, dx &= \int \tan^2 x \tan^3 x \, dx \\
&= \int \left(\sec^2 x - 1 \right) \tan^3 x \, dx \\
&= \int \sec^2 x \tan^3 x \, dx - \int \tan^2 x \tan x \, dx \\
&= \int \tan^3 x \, d(\tan x) - \int \left(\sec^2 x - 1 \right) \tan x \, dx \\
&= \frac{1}{4} \tan^4 x - \int \sec^2 x \tan x \, dx + \int \tan x \, dx \\
&= \frac{1}{4} \tan^4 x - \int \tan x \, d(\tan x) + \int \frac{\sin x}{\cos x} \, dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \int \frac{1}{\cos x} d(\cos x) \\
&= \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \ln |\cos x| + C \\
&= \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + \ln |\sec x| + C
\end{aligned}$$

Exercise

Evaluate $\int \tan^6 3x \, dx$

Solution

$$\begin{aligned}
\int \tan^6 3x \, dx &= \int (\sec^2 3x - 1) \tan^4 3x \, dx \\
&= \frac{1}{3} \int \tan^4 3x \, d(\tan 3x) - \int (\sec^2 3x - 1) \tan^2 3x \, dx \\
&= \frac{1}{15} \tan^5 3x - \int \sec^2 3x \tan^2 3x \, dx + \int \tan^2 3x \, dx \\
&= \frac{1}{15} \tan^5 3x - \frac{1}{3} \int \tan^2 3x \, d(\tan 3x) + \int (\sec^2 3x - 1) \, dx \\
&= \frac{1}{15} \tan^5 3x - \frac{1}{9} \tan^3 3x + \frac{1}{3} \int d(\tan 3x) - \int dx \\
&= \frac{1}{15} \tan^5 3x - \frac{1}{9} \tan^3 3x + \frac{1}{3} \tan 3x - x + C
\end{aligned}$$

Exercise

Evaluate the integral $\int 20 \tan^6 x \, dx$

Solution

$$\begin{aligned}
\int 20 \tan^6 x \, dx &= 20 \int (\sec^2 x - 1) \tan^4 x \, dx \\
&= 20 \int \tan^4 x \, d(\tan x) - 20 \int (\sec^2 x - 1) \tan^2 x \, dx \\
&= 4 \tan^5 x - 20 \int \sec^2 x \tan^2 x \, dx + 20 \int \tan^2 x \, dx \\
&= 4 \tan^5 x - 20 \int \tan^2 x \, d(\tan x) + 20 \int (\sec^2 x - 1) \, dx
\end{aligned}$$

$$= 4 \tan^5 x - \frac{20}{3} \tan^3 x + 20 \tan x - 20x + C \quad |$$

Exercise

Evaluate the integral $\int \tan^4 x \, dx$

Solution

$$\begin{aligned} \int \tan^4 x \, dx &= \int (\tan^2 x)(\tan^2 x) \, dx \\ &= \int \tan^2 x (\sec^2 x - 1) \, dx \\ &= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\ &= \int \tan^2 x \, d(\tan x) - \int (\sec^2 x - 1) \, dx \\ &= \frac{1}{3} \tan^3 x - \tan x + x + C \quad | \end{aligned}$$

Exercise

Evaluate the integral $\int \tan^3 \theta \, d\theta$

Solution

$$\begin{aligned} \int \tan^3 \theta \, d\theta &= \int \tan^2 \theta \tan \theta \, d\theta \\ &= \int (\sec^2 \theta - 1) \tan \theta \, d\theta \\ &= \int \sec^2 \theta \tan \theta \, d\theta - \int \tan \theta \, d\theta \\ &= \int \tan \theta \, d(\tan \theta) - \ln |\sec \theta| \\ &= \frac{1}{2} \tan^2 \theta - \ln |\sec \theta| + C \quad | \end{aligned}$$

Exercise

Evaluate the integral $\int \tan^3 4x \, dx$

Solution

$$\begin{aligned}\int \tan^3 4x \, dx &= \int \tan^2 4x \tan 4x \, dx \\&= \int (\sec^2 4x - 1) \tan 4x \, dx \\&= \int \sec^2 4x \tan 4x \, dx - \int \tan 4x \, dx \\&= \frac{1}{4} \int \tan 4x \, d(\tan 4x) - \int \frac{\sin 4x}{\cos 4x} \, dx \\&= \frac{1}{8} \tan^2 4x + \frac{1}{4} \int \frac{1}{\cos 4x} \, d(\cos 4x) \\&= \frac{1}{8} \tan^2 4x + \frac{1}{4} \ln |\cos 4x| + C\end{aligned}$$

Exercise

Evaluate the integral $\int \cot^4 x \, dx$

Solution

$$\begin{aligned}\int \cot^4 x \, dx &= \int \cot^2 x (\csc^2 x - 1) \, dx \\&= \int \cot^2 x \csc^2 x \, dx - \int \cot^2 x \, dx \\&= - \int \cot^2 x \, d(\cot x) - \int (\csc^2 x - 1) \, dx \\&= -\frac{1}{3} \cot^3 x - \cot x + x + C\end{aligned}$$

Exercise

Evaluate the integral $\int \cot^5 3x \, dx$

Solution

$$\begin{aligned}
\int \cot^5 3x \, dx &= \int \cot^3 3x (\csc^2 3x - 1) \, dx \\
&= \int \cot^3 3x \csc^2 3x \, dx - \int \cot^3 3x \, dx \\
&= -\frac{1}{3} \int \cot^3 3x \, d(\cot 3x) - \int \cot 3x (\csc^2 3x - 1) \, dx \\
&= -\frac{1}{12} \cot^4 3x - \int \cot 3x \csc^2 3x \, dx + \int \cot 3x \, dx \\
&= -\frac{1}{12} \cot^4 3x + \frac{1}{3} \int \cot 3x \, d(\cot 3x) + \int \frac{\cos 3x}{\sin 3x} \, dx \\
&= -\frac{1}{12} \cot^4 3x + \frac{1}{6} \cot^2 3x + \frac{1}{3} \int \frac{1}{\sin 3x} \, d(\sin 3x) \\
&= \underline{-\frac{1}{12} \cot^4 x + \frac{1}{6} \cot^2 3x + \frac{1}{3} \ln |\sin 3x| + C}
\end{aligned}$$

Exercise

Evaluate the integral $\int \sin^2 x \cos^2 x \, dx$

Solution

$$\begin{aligned}
\int \sin^2 x \cos^2 x \, dx &= \frac{1}{4} \int (1 - \cos 2x)(1 + \cos 2x) \, dx \\
&= \frac{1}{4} \int (1 - \cos^2 2x) \, dx \\
&= \frac{1}{4} \int \left(1 - \frac{1}{2} - \frac{1}{2} \cos 4x\right) \, dx \\
&= \frac{1}{4} \int \left(\frac{1}{2} - \frac{1}{2} \cos 4x\right) \, dx \\
&= \frac{1}{4} \left(\frac{1}{2} x - \frac{1}{8} \sin 4x\right) + C \\
&= \underline{\frac{1}{8} x - \frac{1}{32} \sin 4x + C}
\end{aligned}$$

Exercise

Evaluate the integral $\int \sin^2 x \cos^3 x \, dx$

Solution

$$\begin{aligned}\int \sin^2 x \cos^3 x \, dx &= \int \sin^2 x \cos^2 x \cos x \, dx \\&= \int \sin^2 x (1 - \sin^2 x) \, d(\sin x) \\&= \int (\sin^2 x - \sin^4 x) \, d(\sin x) \\&= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C\end{aligned}$$

Exercise

Evaluate the integral $\int \sin^2 x \cos^5 x \, dx$

Solution

$$\begin{aligned}\int \sin^2 x \cos^5 x \, dx &= \int \sin^2 x \cos^4 x \cos x \, dx \\&= \int \sin^2 x (1 - \sin^2 x)^2 \, d(\sin x) \\&= \int \sin^2 x (1 - 2\sin^2 x + \sin^4 x) \, d(\sin x) \\&= \int (\sin^2 x - 2\sin^4 x + \sin^6 x) \, d(\sin x) \\&= \frac{1}{3} \sin^3 x - \frac{2}{5} \sin^5 x + \frac{1}{7} \sin^7 x + C\end{aligned}$$

Exercise

Evaluate the integral $\int \sin^3 x \cos^5 x \, dx$

Solution

$$\int \sin^3 x \cos^5 x \, dx = \int \sin^2 x \cos^5 x \sin x \, dx$$

$$\begin{aligned}
&= - \int (1 - \cos^2 x) \cos^5 x \, d(\cos x) \\
&= - \int (\cos^5 x - \cos^7 x) \, d(\cos x) \\
&= -\frac{1}{6} \cos^6 x + \frac{1}{8} \cos^8 x + C \quad |
\end{aligned}$$

Exercise

Evaluate the integral $\int \sin^3 x \cos^4 x \, dx$

Solution

$$\begin{aligned}
\int \sin^3 x \cos^4 x \, dx &= \int \sin^2 x \sin x \cos^4 x \, dx \\
&= - \int (1 - \cos^2 x) \cos^4 x \, d(\cos x) \\
&= - \int (\cos^4 x - \cos^6 x) \, d(\cos x) \\
&= - \left(\frac{1}{5} \cos^5 x - \frac{1}{7} \cos^7 x \right) + C \\
&= \frac{1}{7} \cos^7 x - \frac{1}{5} \cos^5 x + C \quad |
\end{aligned}$$

Exercise

Evaluate the integral $\int \sin^3 2x \cos^4 x \, dx$

Solution

$$\begin{aligned}
\int \sin^3 2x \cos^4 x \, dx &= \int (2 \sin x \cos x)^3 \cos^4 x \, dx \\
&= 8 \int \sin^3 x \cos^7 x \, dx \\
&= -8 \int \sin^2 x \cos^7 x \, d(\cos x) \\
&= -8 \int (1 - \cos^2 x) \cos^7 x \, d(\cos x)
\end{aligned}$$

$$\begin{aligned}
&= -8 \int (\cos^7 x - \cos^9 x) d(\cos x) \\
&= -8 \left(\frac{1}{8} \cos^8 x - \frac{1}{10} \cos^{10} x \right) + C \\
&= -\cos^8 x + \frac{4}{5} \cos^{10} x + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \sin^3 2x \cos^3 2x dx$

Solution

$$\begin{aligned}
\int \sin^3 2x \cos^3 2x dx &= \frac{1}{2} \int \sin^3 2x \cos^2 2x d(\sin 2x) \\
&= \frac{1}{2} \int \sin^3 2x (1 - \sin^2 2x) d(\sin 2x) \\
&= \frac{1}{2} \int (\sin^3 2x - \sin^5 2x) d(\sin 2x) \\
&= \frac{1}{2} \left(\frac{1}{4} \sin^4 2x - \frac{1}{6} \sin^6 2x \right) + C \\
&= \frac{1}{8} \sin^4 2x - \frac{1}{12} \sin^6 2x + C
\end{aligned}$$

Exercise

Evaluate $\int \sin^4 x \cos^2 x dx$

Solution

$$\begin{aligned}
\int \sin^4 x \cos^2 x dx &= \int \left(\frac{1 - \cos 2x}{2} \right)^2 \left(\frac{1 + \cos 2x}{2} \right) dx \\
&= \frac{1}{8} \int (1 - 2 \cos 2x + \cos^2 2x)(1 + \cos 2x) dx \\
&= \frac{1}{8} \int (1 - \cos 2x - \cos^2 2x + \cos^3 2x) dx \\
&= \frac{1}{8} \int \left(1 - \cos 2x - \frac{1}{2} - \frac{1}{2} \cos 4x \right) dx + \frac{1}{8} \int \cos^2 2x \cos 2x dx \\
&= \frac{1}{8} \int \left(\frac{1}{2} - \cos 2x - \frac{1}{2} \cos 4x \right) dx + \frac{1}{16} \int (1 - \sin^2 2x) d(\sin 2x)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{8} \left(\frac{1}{2} x - \frac{1}{2} \sin 2x - \frac{1}{4} \sin 4x \right) + \frac{1}{16} \sin 2x - \frac{1}{48} \sin^3 2x + C \\
&= \frac{1}{16} x - \frac{1}{64} \sin 4x - \frac{1}{48} \sin^3 2x + C \quad |
\end{aligned}$$

Exercise

Evaluate the integral $\int \sin^4 x \cos^3 x \, dx$

Solution

$$\begin{aligned}
\int \sin^4 x \cos^3 x \, dx &= \int \sin^4 x (1 - \sin^2 x) \, d(\sin x) \\
&= \int (\sin^4 x - \sin^6 x) \, d(\sin x) \\
&= \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C \quad |
\end{aligned}$$

Exercise

Evaluate the integral $\int \sin^4 x \cos^4 x \, dx$

Solution

$$\begin{aligned}
\int \sin^4 x \cos^4 x \, dx &= \int (\sin^2 x)^2 (\cos^2 x)^2 \, dx \\
&= \int \left(\frac{1 - \cos 2x}{2} \right)^2 \left(\frac{1 + \cos 2x}{2} \right)^2 \, dx \\
&= \frac{1}{16} \int (1 - \cos^2 2x)^2 \, dx \\
&= \frac{1}{16} \int (1 - 2 \cos^2 2x + \cos^4 2x) \, dx \\
&= \frac{1}{16} \int \left(1 - 1 - \cos 4x + \left(\frac{1 + \cos 4x}{2} \right)^2 \right) \, dx \\
&= \frac{1}{16} \int \left(-\cos 4x + \frac{1}{4} (1 + 2 \cos 4x + \cos^2 4x) \right) \, dx \\
&= \frac{1}{64} \int (-4 \cos 4x + 1 + 2 \cos 4x + \cos^2 4x) \, dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{64} \int \left(1 - 2 \cos 4x + \frac{1}{2} + \frac{1}{2} \cos 8x \right) dx \\
&= \frac{1}{128} \int (3 - 4 \cos 4x + \cos 8x) dx \\
&= \frac{1}{128} \left(3x - \sin 4x + \frac{1}{8} \sin 8x \right) + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \sin^4 x \cos^5 x dx$

Solution

$$\begin{aligned}
\int \sin^4 x \cos^5 x dx &= \int \sin^4 x \cos^4 x \cos x dx \\
&= \int \sin^4 x (1 - \sin^2 x)^2 d(\sin x) \\
&= \int \sin^4 x (1 - 2 \sin^2 x + \sin^4 x) d(\sin x) \\
&= \int (\sin^4 x - 2 \sin^6 x + \sin^8 x) d(\sin x) \\
&= \frac{1}{5} \sin^5 x - \frac{2}{7} \sin^7 x + \frac{1}{9} \sin^9 x + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \sin^5 x \cos^5 x dx$

Solution

$$\begin{aligned}
\int \sin^5 x \cos^5 x dx &= \int \sin^5 x \cos^4 x (\cos x) dx \\
&= \int \sin^5 x (1 - \sin^2 x)^2 d(\sin x) \\
&= \int \sin^5 x (1 - 2 \sin^2 x + \sin^4 x) d(\sin x) \\
&= \int (\sin^5 x - 2 \sin^7 x + \sin^9 x) d(\sin x) \\
&= \frac{1}{6} \sin^6 x - \frac{1}{4} \sin^8 x + \frac{1}{10} \sin^{10} x + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \sin^5 x \cos^{-2} x \, dx$

Solution

$$\begin{aligned}\int \sin^5 x \cos^{-2} x \, dx &= \int \sin^4 x \cos^{-2} x \sin x \, dx \\&= -\int \left(1 - \cos^2 x\right)^2 \cos^{-2} x \, d(\cos x) \\&= -\int \left(1 - 2\cos^2 x + \cos^4 x\right) \cos^{-2} x \, d(\cos x) \\&= -\int \left(\cos^{-2} x - 2 + \cos^2 x\right) \, d(\cos x) \\&= \cos^{-1} x + 2\cos x - \frac{1}{3}\cos^3 x + C \\&= \sec x + 2\cos x - \frac{1}{3}\cos^3 x + C\end{aligned}$$

Exercise

Evaluate the integral $\int \sin 3x \cos^6 3x \, dx$

Solution

$$\begin{aligned}\int \sin 3x \cos^6 3x \, dx &= -\frac{1}{3} \int \cos^6 3x \, d(\cos 3x) \\&= -\frac{1}{21} \cos^7 3x + C\end{aligned}$$

Exercise

Evaluate the integral $\int \sin^4 2x \cos 2x \, dx$

Solution

$$\begin{aligned}d(\sin 2x) &= 2 \cos 2x \, dx \\ \int \sin^4 2x \cos 2x \, dx &= \frac{1}{2} \int \sin^4 2x \, d(\sin 2x) \\&= \frac{1}{10} \sin^5 2x + C\end{aligned}$$

Exercise

Evaluate the integral $\int \cos^3 2x \sin^5 2x \, dx$

Solution

$$\begin{aligned}\int \cos^3 2x \sin^5 2x \, dx &= \int (\cos^2 2x) \cos 2x \sin^5 2x \, dx & d(\sin 2x) &= 2 \cos 2x \, dx \\&= \int (1 - \sin^2 2x) \sin^5 2x \left(\frac{1}{2} d \sin 2x\right) \\&= \frac{1}{2} \int (\sin^5 2x - \sin^7 2x) (d \sin 2x) \\&= \frac{1}{2} \left(\frac{1}{6} \sin^6 2x - \frac{1}{8} \sin^8 2x \right) + C \\&= \underline{\underline{\frac{1}{12} \sin^6 2x - \frac{1}{16} \sin^8 2x + C}}\end{aligned}$$

Exercise

Evaluate the integral $\int 16 \sin^2 x \cos^2 x \, dx$

Solution

$$\begin{aligned}\int 16 \sin^2 x \cos^2 x \, dx &= 16 \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right) dx & \cos^2 \alpha &= \frac{1 + \cos 2\alpha}{2} \quad \sin^2 \alpha = \frac{1 - \cos 2\alpha}{2} \\&= 4 \int (1 - \cos^2 2x) \, dx \\&= 4 \int \left(1 - \frac{1 + \cos 4x}{2} \right) dx \\&= 4 \int \frac{1 - \cos 4x}{2} \, dx \\&= 2 \left(x - \frac{1}{4} \sin 4x \right) + C \\&= 2x - \frac{1}{2} (2 \sin 2x \cos 2x) + C \\&= 2x - (2 \sin x \cos x) (2 \cos^2 x - 1) + C \\&= \underline{\underline{2x - 4 \sin x \cos^3 x + 2 \sin x \cos x + C}}\end{aligned}$$

Exercise

Evaluate the integral $\int \sin 2x \cos 3x \, dx$

Solution

$$\begin{aligned}\int \sin 2x \cos 3x \, dx &= \frac{1}{2} \int (\sin 5x + \sin(-x)) \, dx \\ &= \frac{1}{2} \int (\sin 5x - \sin x) \, dx \\ &= \frac{1}{2} \left(-\frac{1}{5} \cos 5x + \cos x \right) + C \\ &= \frac{1}{2} \cos x - \frac{1}{10} \cos 5x + C\end{aligned}$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

Exercise

Evaluate the integral $\int \sin^2 \theta \cos 3\theta \, d\theta$

Solution

$$\begin{aligned}\int \sin^2 \theta \cos 3\theta \, d\theta &= \int \frac{1 - \cos 2\theta}{2} \cos 3\theta \, d\theta \\ &= \frac{1}{2} \int (\cos 3\theta - \cos 2\theta \cos 3\theta) \, d\theta \\ &= \frac{1}{2} \int \cos 3\theta \, d\theta - \frac{1}{2} \int \cos 2\theta \cos 3\theta \, d\theta \\ &= \frac{1}{6} \sin 3\theta - \frac{1}{2} \int \frac{1}{2} (\cos(5\theta) + \cos(-\theta)) \, d\theta \\ &= \frac{1}{6} \sin 3\theta - \frac{1}{4} \left(\frac{1}{5} \sin 5\theta + \sin \theta \right) + C \\ &= \frac{1}{6} \sin 3\theta - \frac{1}{20} \sin 5\theta - \frac{1}{4} \sin \theta + C\end{aligned}$$

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

Exercise

Evaluate the integral $\int \cos^3 \theta \sin 2\theta \, d\theta$

Solution

$$\int \cos^3 \theta \sin 2\theta \, d\theta = \int \cos^3 \theta (2 \sin \theta \cos \theta) \, d\theta$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\begin{aligned}
 &= -2 \int \cos^4 \theta \, d(\cos \theta) \\
 &= -\frac{2}{5} \cos^5 \theta + C
 \end{aligned}$$

$$d(\cos \theta) = -\sin \theta d\theta$$

Exercise

Evaluate the integral $\int \sin^{-3/2} x \cos^3 x \, dx$

Solution

$$\begin{aligned}
 \int \sin^{-3/2} x \cos^3 x \, dx &= \int \sin^{-3/2} x \cos^2 x \cos x \, dx \\
 &= \int \sin^{-3/2} x (1 - \sin^2 x) \, d(\sin x) \\
 &= \int (\sin^{-3/2} x - \sin^{1/2} x) \, d(\sin x) \\
 &= -2 \sin^{-1/2} x - \frac{2}{3} \sin^{3/2} x + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \sin^3 x \cos^{3/2} x \, dx$

Solution

$$\begin{aligned}
 \int \sin^3 x \cos^{3/2} x \, dx &= \int \sin^2 x \cos^{3/2} x \sin x \, dx \\
 &= - \int (1 - \cos^2 x) \cos^{3/2} x \, d(\cos x) \\
 &= \int (-\cos^{3/2} x + \cos^{7/2} x) \, d(\cos x) \\
 &= \frac{2}{9} \cos^{9/2} x - \frac{2}{5} \cos^{5/2} x + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \sin \theta \sin 2\theta \sin 3\theta \, d\theta$

Solution

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\begin{aligned} \int \sin \theta \sin 2\theta \sin 3\theta \, d\theta &= \int \frac{1}{2} (\cos(1-2)\theta - \cos(1+2)\theta) \sin 3\theta \, d\theta \\ &= \frac{1}{2} \int (\cos(-\theta) - \cos(3\theta)) \sin 3\theta \, d\theta \\ &= \frac{1}{2} \int \cos \theta \sin 3\theta \, d\theta - \frac{1}{2} \int \cos 3\theta \sin 3\theta \, d\theta \end{aligned}$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\begin{aligned} &= \frac{1}{4} \int (\sin 4\theta + \sin 2\theta) \, d\theta - \frac{1}{4} \int (\sin 6\theta + \sin(0)) \, d\theta \\ &= \frac{1}{4} \left(-\frac{1}{4} \cos 4\theta - \frac{1}{2} \cos 2\theta \right) + \frac{1}{24} \cos 6\theta + C \\ &= \underline{-\frac{1}{16} \cos 4\theta - \frac{1}{8} \cos 2\theta + \frac{1}{24} \cos 6\theta + C} \end{aligned}$$

Exercise

Evaluate the integral $\int \sin 3x \cos 6x \, dx$

Solution

$$\begin{aligned} \int \sin 3x \cos 6x \, dx &= \frac{1}{2} \int (\sin 9x + \sin(-3x)) \, dx \\ &= \frac{1}{2} \int (\sin 9x - \sin 3x) \, dx \\ &= \frac{1}{2} \left(-\frac{1}{9} \cos 9x + \frac{1}{3} \cos 3x \right) + C \\ &= \underline{\frac{1}{6} \cos 3x - \frac{1}{18} \cos 9x + C} \end{aligned}$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

Exercise

Evaluate the integral $\int \sin 3x \cos 7x \, dx$

Solution

$$\int \sin 3x \cos 7x \, dx = \frac{1}{2} \int (\sin 10x + \sin(-4x)) \, dx$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\begin{aligned}
&= \frac{1}{2} \int (\sin 10x - \sin 4x) dx \\
&= \frac{1}{2} \left(-\frac{1}{10} \cos 10x + \frac{1}{4} \cos 4x \right) + C \\
&= \frac{1}{8} \cos 4x - \frac{1}{20} \cos 10x + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \sin 5x \cos 4x \, dx$

Solution

$$\begin{aligned}
\int \sin 5x \cos 4x \, dx &= \frac{1}{2} \int (\sin x + \sin 9x) dx \\
&= \frac{1}{2} \left(-\cos x - \frac{1}{9} \cos 9x \right) + C \\
&= \frac{1}{2} - \cos x - \frac{1}{18} \cos 9x + C
\end{aligned}$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

Exercise

Evaluate the integral $\int \cos 2\theta \cos 6\theta \, d\theta$

Solution

$$\begin{aligned}
\int \cos 2\theta \cos 6\theta \, d\theta &= \frac{1}{2} \int (\cos 8\theta + \cos(-4\theta)) \, d\theta \\
&= \frac{1}{2} \int (\cos 8\theta + \cos 4\theta) \, d\theta \\
&= \frac{1}{2} \left(\frac{1}{8} \sin 8\theta + \frac{1}{4} \sin 4\theta \right) + C \\
&= \frac{1}{16} \sin 8\theta + \frac{1}{8} \sin 4\theta + C
\end{aligned}$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

Exercise

Evaluate the integral $\int \cos 5\theta \cos 3\theta \, d\theta$

Solution

$$\begin{aligned}
 \int \cos 5\theta \cos 3\theta \, d\theta &= \frac{1}{2} \int (\cos 8\theta + \cos 2\theta) \, d\theta \\
 &= \frac{1}{2} \left(\frac{1}{8} \sin 8\theta + \frac{1}{2} \sin 2\theta \right) + C \\
 &= \underline{\underline{\frac{1}{16} \sin 8\theta + \frac{1}{4} \sin 2\theta + C}}
 \end{aligned}$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

Exercise

Evaluate the integral $\int \sin 2\theta \cos 4\theta \, d\theta$

Solution

$$\begin{aligned}
 \int \sin 2\theta \cos 4\theta \, d\theta &= \frac{1}{2} \int (\sin 6\theta + \sin(-2\theta)) \, d\theta \\
 &= \frac{1}{2} \int (\sin 6\theta - \sin 2\theta) \, d\theta \\
 &= \frac{1}{2} \left(-\frac{1}{6} \cos 6\theta + \frac{1}{2} \cos 2\theta \right) + C \\
 &= \underline{\underline{\frac{1}{4} \cos 4\theta - \frac{1}{12} \cos 6\theta + C}}
 \end{aligned}$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

Exercise

Evaluate the integral $\int \sin(-7\theta) \cos 6\theta \, d\theta$

Solution

$$\begin{aligned}
 \int \sin(-7\theta) \cos 6\theta \, d\theta &= - \int \sin 7\theta \cos 6\theta \, d\theta \\
 &= -\frac{1}{2} \int (\sin 13\theta + \sin \theta) \, d\theta \\
 &= -\frac{1}{2} \left(-\frac{1}{13} \cos 13\theta - \cos \theta \right) + C \\
 &= \underline{\underline{\frac{1}{26} \cos 13\theta + \frac{1}{2} \cos \theta + C}}
 \end{aligned}$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

Exercise

Evaluate the integral $\int \sin \theta \sin 3\theta \, d\theta$

Solution

$$\begin{aligned}\int \sin \theta \sin 3\theta \, d\theta &= \frac{1}{2} \int (\cos(-2\theta) - \cos 4\theta) \, d\theta \\ &= \frac{1}{2} \int (\cos 2\theta - \cos 4\theta) \, d\theta \\ &= \frac{1}{2} \left(\frac{1}{2} \sin 2\theta - \frac{1}{4} \sin 4\theta \right) + C \\ &= \frac{1}{4} \sin 2\theta - \frac{1}{8} \sin 4\theta + C\end{aligned}$$

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

Exercise

Evaluate the integral $\int \sin 5\theta \sin 4\theta \, d\theta$

Solution

$$\begin{aligned}\int \sin 5\theta \sin 4\theta \, d\theta &= \frac{1}{2} \int (\cos \theta - \cos 9\theta) \, d\theta \\ &= \frac{1}{2} \left(\sin \theta - \frac{1}{9} \sin 9\theta \right) + C \\ &= \frac{1}{2} \sin \theta - \frac{1}{18} \sin 9\theta + C\end{aligned}$$

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

Exercise

Evaluate the integral $\int \sin x \cos^5 x \, dx$

Solution

$$\begin{aligned}\int \sin x \cos^5 x \, dx &= - \int \cos^5 x \, d(\cos x) \\ &= -\frac{1}{6} \cos^6 x + C\end{aligned}$$

Exercise

Evaluate the integral $\int \sin^7 2x \cos 2x \, dx$

Solution

$$\begin{aligned} \int \sin^7 2x \cos 2x \, dx &= \frac{1}{2} \int \sin^7 2x \, d(\sin 2x) \\ &= \frac{1}{16} \sin^8 2x + C \end{aligned}$$

Exercise

Evaluate the integral $\int \sin^3 2x \sqrt{\cos 2x} \, dx$

Solution

$$\begin{aligned} \int \sin^3 2x \sqrt{\cos 2x} \, dx &= -\frac{1}{2} \int (1 - \cos^2 2x) (\cos 2x)^{1/2} \, d(\cos 2x) \\ &= -\frac{1}{2} \int ((\cos 2x)^{1/2} - (\cos 2x)^{5/2}) \, d(\cos 2x) \\ &= -\frac{1}{2} \left(\frac{2}{3} (\cos 2x)^{3/2} - \frac{2}{7} (\cos 2x)^{7/2} \right) + C \\ &= \frac{1}{7} (\cos 2x)^{7/2} - \frac{1}{3} (\cos 2x)^{3/2} + C \end{aligned}$$

Exercise

Evaluate $\int \sin^3 x \cos^2 x \, dx$

Solution

$$\begin{aligned} \int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx & d(\cos x) &= -\sin x \, dx \\ &= -\int (1 - \cos^2 x) \cos^2 x \, d(\cos x) \\ &= -\int (\cos^4 x - \cos^2 x) \, d(\cos x) \\ &= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\cos^3 \theta}{\sqrt{\sin \theta}} d\theta$

Solution

$$\begin{aligned}
 \int \frac{\cos^3 \theta}{\sqrt{\sin \theta}} d\theta &= \int (\sin \theta)^{-1/2} \cos^2 \theta \cos \theta d\theta \\
 &= \int (\sin \theta)^{-1/2} (1 - \sin^2 \theta) d(\sin \theta) \\
 &= \int (\sin^{-1/2} \theta - \sin^{3/2} \theta) d(\sin \theta) \\
 &= 2 \sqrt{\sin \theta} - \frac{2}{5} \sin^{5/2} \theta + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\cos^5 \theta}{\sqrt{\sin \theta}} d\theta$

Solution

$$\begin{aligned}
 \int \frac{\cos^5 \theta}{\sqrt{\sin \theta}} d\theta &= \int (\sin \theta)^{-1/2} (1 - \sin^2 \theta)^2 d(\sin \theta) \\
 &= \int (\sin \theta)^{-1/2} (1 - 2\sin^2 \theta + \sin^4 \theta) d(\sin \theta) \\
 &= \int ((\sin \theta)^{-1/2} - 2(\sin \theta)^{3/2} + (\sin \theta)^{7/2}) d(\sin \theta) \\
 &= 2(\sin \theta)^{1/2} - \frac{1}{5}(\sin \theta)^{5/2} + \frac{2}{9}(\sin \theta)^{9/2} + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\cos^2 x}{\sin^5 x} dx$

Solution

$$\begin{aligned}
 \int \frac{\cos^2 x}{\sin^5 x} dx &= \int \frac{\cos^2 x}{\sin^2 x} \frac{1}{\sin^3 x} dx \\
 &= \int \cot^2 x \csc^3 x dx
 \end{aligned}$$

$$\begin{aligned}
&= \int (\csc^2 x - 1) \csc^3 x \, dx \\
&= \int \csc^5 x \, dx - \int \csc^3 x \, dx
\end{aligned}$$

$$\begin{aligned}
u &= \csc^3 x & dv &= \csc^2 x \, dx \\
du &= -3 \csc^3 x \cot x \, dx & v &= -\cot x
\end{aligned}$$

$$\begin{aligned}
\int \csc^5 x \, dx &= -\csc^3 x \cot x - 3 \int \csc^3 x \cot^2 x \, dx \\
&= -\csc^3 x \cot x - 3 \int \csc^3 x (\csc^2 x - 1) \, dx \\
&= -\csc^3 x \cot x - 3 \int \csc^5 x \, dx + 3 \int \csc^3 x \, dx
\end{aligned}$$

$$\begin{aligned}
5 \int \csc^5 x \, dx &= -\csc^3 x \cot x + 3 \int \csc^3 x \, dx \\
\int \csc^5 x \, dx &= -\frac{1}{5} \csc^3 x \cot x + \frac{3}{5} \int \csc^3 x \, dx
\end{aligned}$$

$$\begin{aligned}
u &= \csc x & dv &= \csc^2 x \, dx \\
du &= -\csc x \cot x \, dx & v &= -\cot x
\end{aligned}$$

$$\begin{aligned}
\int \csc^3 x \, dx &= -\csc x \cot x - \int \csc x \cot^2 x \, dx \\
&= -\csc x \cot x - \int \csc x (\csc^2 x - 1) \, dx \\
&= -\csc x \cot x - \int \csc^3 x \, dx + \int \csc x \frac{\csc x + \cot x}{\csc x + \cot x} \, dx
\end{aligned}$$

$$\begin{aligned}
2 \int \csc^3 x \, dx &= -\csc x \cot x + \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \, dx \\
2 \int \csc^3 x \, dx &= -\csc x \cot x - \int \frac{1}{\csc x + \cot x} d(\csc x + \cot x) \\
\int \csc^3 x \, dx &= -\frac{1}{2} \csc x \cot x - \ln |\csc x + \cot x|
\end{aligned}$$

$$\int \frac{\cos^2 x}{\sin^5 x} \, dx = -\frac{1}{5} \csc^3 x \cot x + \frac{3}{5} \left(-\frac{1}{2} \csc x \cot x - \ln |\csc x + \cot x| \right) + \frac{1}{2} \csc x \cot x + \ln |\csc x + \cot x|$$

$$\begin{aligned}
&= -\frac{1}{5} \csc^3 x \cot x - \frac{3}{10} \csc x \cot x - \frac{3}{5} \ln |\csc x + \cot x| + \frac{1}{2} \csc x \cot x + \ln |\csc x + \cot x| \\
&= \frac{1}{5} \csc x \cot x \left(1 - \csc^2 x\right) + \frac{2}{5} \ln |\csc x + \cot x| + C \\
&= \frac{1}{5} \csc x \cot^3 x + \frac{2}{5} \ln |\csc x + \cot x| + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\sin^3 x}{\cos^4 x} dx$

Solution

$$\begin{aligned}
\int \frac{\sin^3 x}{\cos^4 x} dx &= \int \frac{\sin^2 x \sin x}{\cos^4 x} dx & \cos^2 \alpha + \sin^2 \alpha &= 1 \\
&= \int \frac{(1 - \cos^2 x) \sin x}{\cos^4 x} dx \\
&= - \int \left(\frac{1}{\cos^4 x} - \frac{\cos^2 x}{\cos^4 x} \right) d(\cos x) \\
&= - \int (\cos^{-4} x - \cos^{-2} x) d(\cos x) \\
&= - \left(-\frac{1}{3} \cos^{-3} x + \cos^{-1} x \right) + C \\
&= \frac{1}{3} \frac{1}{\cos^3 x} - \frac{1}{\cos x} + C \\
&= \frac{1}{3} \csc^3 x - \csc x + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\sin^4 x}{\cos^6 x} dx$

Solution

$$\begin{aligned}
\int \frac{\sin^4 x}{\cos^6 x} dx &= \int \frac{\sin^4 x}{\cos^4 x} \cdot \frac{1}{\cos^2 x} dx \\
&= \int \tan^4 x \sec^2 x dx
\end{aligned}$$

$$= \int \tan^4 x \, d(\tan x)$$

$$= \frac{1}{5} \tan^5 x + C \quad |$$

Exercise

Evaluate the integral $\int \frac{2 \cos x + 3 \sin x}{\sin^3 x} \, dx$

Solution

$$\int \frac{2 \cos x + 3 \sin x}{\sin^3 x} \, dx = 2 \int \frac{\cos x}{\sin^3 x} \, dx + 3 \int \frac{\sin x}{\sin^3 x} \, dx$$

$$= 2 \int \sin^{-3} x \, d(\sin x) + 3 \int \frac{1}{\sin^2 x} \, dx$$

$$= -\sin^{-2} x + 3 \int \csc^2 x \, dx$$

$$= -\csc^2 x - 3 \cot x + C \quad |$$

Exercise

Evaluate the integral $\int \frac{2 + \sin x + 2 \cos x}{1 + \cos x} \, dx$

Solution

$$\int \frac{2 + \sin x + 2 \cos x}{1 + \cos x} \, dx = \int \frac{2(1 + \cos x)}{1 + \cos x} \, dx + \int \frac{\sin x}{1 + \cos x} \, dx$$

$$= 2 \int dx - \int \frac{1}{1 + \cos x} \, d(1 + \cos x)$$

$$= 2x - \ln|1 + \cos x| + C \quad |$$

Exercise

Evaluate the integral $\int \frac{dx}{1 - \cos x}$

Solution

$$\begin{aligned}
\int \frac{dx}{1 - \cos x} &= \int \frac{1}{1 - \cos x} \cdot \frac{1 + \cos x}{1 + \cos x} dx \\
&= \int \frac{1 + \cos x}{1 - \cos^2 x} dx \\
&= \int \frac{1 + \cos x}{\sin^2 x} dx \\
&= \int \frac{1}{\sin^2 x} dx + \int \frac{\cos x}{\sin^2 x} dx \\
&= \int \csc^2 x dx + \int \frac{1}{\sin^2 x} d(\sin x) \\
&= -\cot x - \frac{1}{\sin x} + C \\
&= \underline{-\cot x - \csc x + C}
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{1 - \sin x}$

Solution

$$\begin{aligned}
\int \frac{dx}{1 - \sin x} &= \int \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} dx \\
&= \int \frac{1 + \sin x}{1 - \sin^2 x} dx \\
&= \int \frac{1 + \sin x}{\cos^2 x} dx \\
&= \int \frac{1}{\cos^2 x} dx + \int \frac{\sin x}{\cos^2 x} dx \\
&= \int \sec^2 x dx - \int \frac{1}{\cos^2 x} d(\cos x) \\
&= \tan x + \frac{1}{\cos x} + C \\
&= \underline{\tan x + \sec x + C}
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\sin \theta \cos \theta}{2 - \cos \theta} d\theta$

Solution

$$\begin{aligned}\int \frac{\sin \theta \cos \theta}{2 - \cos \theta} d\theta &= \int \frac{\sin \theta (\cos \theta + 2 - 2)}{2 - \cos \theta} d\theta \\&= \int \frac{\sin \theta (\cos \theta - 2) + 2 \sin \theta}{2 - \cos \theta} d\theta \\&= - \int \frac{\sin \theta (2 - \cos \theta)}{2 - \cos \theta} d\theta + \int \frac{2 \sin \theta}{2 - \cos \theta} d\theta \\&= - \int \sin \theta d\theta + \int \frac{2}{2 - \cos \theta} d(2 - \cos \theta) \\&= \cos \theta + \ln |2 - \cos \theta| + C\end{aligned}$$

Exercise

Evaluate the integral $\int \tan^3 x \sec^3 x dx$

Solution

$$\begin{aligned}\int \tan^3 x \sec^3 x dx &= \int \tan^2 x \sec^2 x (\tan x \sec x) dx \\&= \int (\sec^2 x - 1) \sec^2 x d(\sec x) \\&= \int (\sec^4 x - \sec^2 x) d(\sec x) \\&= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C\end{aligned}$$

Exercise

Evaluate the integral $\int \sec x \tan^2 x dx$

Solution

$$\int \sec x \tan^2 x dx = \int \sec x \tan x \tan x dx$$

$$u = \tan x \quad dv = \sec x \tan x dx$$

$$du = \sec^2 x dx \quad v = \sec x$$

$$\begin{aligned} \int \sec x \tan^2 x dx &= \tan x \sec x - \int \sec x \sec^2 x dx \\ &= \tan x \sec x - \int \sec x (1 + \tan^2 x) dx \\ &= \tan x \sec x - \left[\int \sec x dx + \int \sec x \tan^2 x dx \right] \\ &= \tan x \sec x - \ln |\sec x + \tan x| - \int \sec x \tan^2 x dx \end{aligned}$$

$$\int \sec x \tan^2 x dx + \int \sec x \tan^2 x dx = \tan x \sec x - \ln |\sec x + \tan x|$$

$$2 \int \sec x \tan^2 x dx = \tan x \sec x - \ln |\sec x + \tan x|$$

$$\int \sec x \tan^2 x dx = \underline{\underline{\frac{1}{2} \tan x \sec x - \frac{1}{2} \ln |\sec x + \tan x| + C}}$$

Exercise

Evaluate the integral $\int \sec^2 x \tan^2 x dx$

Solution

$$\begin{aligned} \int \sec^2 x \tan^2 x dx &= \int \tan^2 x d(\tan x) & d(\tan x) &= \sec^2 x dx \\ &= \underline{\underline{\frac{1}{3} \tan^3 x + C}} \end{aligned}$$

Exercise

Evaluate the integral $\int \sec^4 x \tan^2 x dx$

Solution

$$\begin{aligned} \int \sec^4 x \tan^2 x dx &= \int \sec^2 x \sec^2 x \tan^2 x dx & d(\tan x) &= \sec^2 x dx \quad \sec^2 x = 1 + \tan^2 x \\ &= \int (1 + \tan^2 x) \tan^2 x d(\tan x) \end{aligned}$$

$$\begin{aligned}
 &= \int (\tan^2 x + \tan^4 x) d(\tan x) \\
 &= \frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \sec^6 4x \tan 4x \, dx$

Solution

$$\begin{aligned}
 \int \sec^6 4x \tan 4x \, dx &= \frac{1}{4} \int \sec^5 4x (\sec 4x \tan 4x) d(4x) \\
 &= \frac{1}{4} \int \sec^5 4x d(\sec 4x) \\
 &= \frac{1}{24} \sec^6 4x + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \sec^2 \frac{x}{2} \tan \frac{x}{2} \, dx$

Solution

$$\begin{aligned}
 \int \sec^2 \frac{x}{2} \tan \frac{x}{2} \, dx &= 2 \int \sec \frac{x}{2} \left(\sec \frac{x}{2} \tan \frac{x}{2} \right) d\left(\frac{x}{2}\right) \\
 &= 2 \int \sec \frac{x}{2} d\left(\sec \frac{x}{2}\right) \\
 &= \sec^2 \frac{x}{2} + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \tan^3 2x \sec^3 2x \, dx$

Solution

$$\begin{aligned}
 \int \tan^3 2x \sec^3 2x \, dx &= \frac{1}{2} \int (\tan^2 2x) \sec^2 2x (\sec 2x \tan 2x) d(2x) \\
 &= \frac{1}{2} \int (\sec^2 2x - 1) \sec^2 2x d(\sec 2x)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int (\sec^4 2x - \sec^2 2x) d(\sec 2x) \\
&= \frac{1}{2} \left(\frac{1}{5} \sec^5 2x - \frac{1}{3} \sec^3 2x \right) + C \\
&= \frac{1}{10} \sec^5 2x - \frac{1}{6} \sec^3 2x + C \quad |
\end{aligned}$$

Exercise

Evaluate the integral $\int \tan^5 2x \sec^4 2x dx$

Solution

$$\begin{aligned}
\int \tan^5 2x \sec^4 2x dx &= \frac{1}{2} \int \tan^5 2x \sec^2 2x \sec^2 2x d(2x) \\
&= \frac{1}{2} \int \tan^5 2x (\tan^2 2x + 1) d(\tan 2x) \\
&= \frac{1}{2} \int (\tan^7 2x + \tan^5 2x) d(\tan 2x) \\
&= \frac{1}{2} \left(\frac{1}{8} \tan^8 2x + \frac{1}{6} \tan^6 2x \right) + C \\
&= \frac{1}{16} \tan^8 2x + \frac{1}{12} \tan^6 2x + C \quad |
\end{aligned}$$

Exercise

Evaluate the integral $\int \tan^3 x \sec^5 x dx$

Solution

$$\begin{aligned}
\int \tan^3 x \sec^5 x dx &= \int \tan^4 x \sec^4 x (\sec x \tan x) dx \\
&= \int (\sec^2 x - 1)^2 \sec^4 x d(\sec x) \\
&= \int (\sec^4 x - 2\sec^2 x + 1) \sec^4 x d(\sec x) \\
&= \int (\sec^8 x - 2\sec^6 x + \sec^4 x) d(\sec x) \\
&= \frac{1}{9} \sec^9 x - \frac{2}{7} \sec^7 x + \frac{1}{5} \sec^5 x + C \quad |
\end{aligned}$$

Exercise

Evaluate $\int \tan^3 x \sec^4 x \, dx$

Solution

$$\begin{aligned}\int \tan^3 x \sec^4 x \, dx &= \int \tan^3 x (1 + \tan^2 x) \sec^2 x \, dx & \sec^2 x &= 1 + \tan^2 x \\ &= \int (\tan^3 x + \tan^5 x) \, d(\tan x) & d(\tan x) &= \sec^2 x dx \\ &= \frac{1}{4} \tan^4 x + \frac{1}{6} \tan^6 x + C\end{aligned}$$

Exercise

Evaluate the integral $\int \tan^5 \theta \sec^4 \theta \, d\theta$

Solution

$$\begin{aligned}\int \tan^5 \theta \sec^4 \theta \, d\theta &= \int \tan^5 \theta \sec^2 \theta \sec^2 \theta \, d\theta \\ &= \int \tan^5 \theta (1 + \tan^2 \theta) \, d(\tan \theta) \\ &= \int (\tan^5 \theta + \tan^7 \theta) \, d(\tan \theta) \\ &= \frac{1}{6} \tan^6 \theta + \frac{1}{8} \tan^8 \theta + C\end{aligned}$$

Exercise

Evaluate the integral $\int \tan^5 \theta \sec^7 \theta \, d\theta$

Solution

$$\begin{aligned}\int \tan^5 \theta \sec^7 \theta \, d\theta &= \int \tan^4 \theta \sec^6 \theta (\tan \theta \sec \theta) \, d\theta \\ &= \int (\sec^2 \theta - 1)^2 \sec^6 \theta \, d(\sec \theta) \\ &= \int (\sec^4 \theta - 2 \sec^2 \theta + 1) \sec^6 \theta \, d(\sec \theta)\end{aligned}$$

$$\begin{aligned}
&= \int \left(\sec^{10} \theta - 2 \sec^8 \theta + \sec^6 \theta \right) d(\sec \theta) \\
&= \frac{1}{11} \sec^{11} \theta - \frac{2}{9} \sec^9 \theta + \frac{1}{7} \sec^7 \theta + C \quad |
\end{aligned}$$

Exercise

Evaluate the integral $\int \tan^7 \theta \sec^5 \theta d\theta$

Solution

$$\begin{aligned}
\int \tan^7 \theta \sec^5 \theta d\theta &= \int \tan^6 \theta \sec^4 \theta (\tan \theta \sec \theta) d\theta \\
&= \int \left(\sec^2 \theta - 1 \right)^3 \sec^4 \theta d(\sec \theta) \\
&= \int \left(\sec^6 \theta - 3 \sec^4 \theta + 3 \sec^2 \theta - 1 \right) \sec^4 \theta d(\sec \theta) \\
&= \int \left(\sec^{10} \theta - 3 \sec^8 \theta + 3 \sec^6 \theta - \sec^4 \theta \right) d(\sec \theta) \\
&= \frac{1}{11} \sec^{11} \theta - \frac{3}{9} \sec^9 \theta + \frac{3}{7} \sec^7 \theta - \frac{1}{5} \sec^5 \theta + C \quad |
\end{aligned}$$

Exercise

Evaluate the integral $\int \sec^4 3x \tan^3 3x dx$

Solution

$$\begin{aligned}
\int \sec^4 3x \tan^3 3x dx &= \int \sec^2 3x \tan^3 3x \sec^2 3x dx \\
&= \frac{1}{3} \int \left(1 + \tan^2 3x \right) \tan^3 3x d(\tan 3x) \\
&= \frac{1}{3} \int \left(\tan^3 3x + \tan^5 3x \right) d(\tan 3x) \\
&= \frac{1}{3} \left(\frac{1}{4} \tan^4 3x + \frac{1}{6} \tan^6 3x \right) + C \\
&= \frac{1}{12} \tan^4 3x + \frac{1}{18} \tan^6 3x + C \quad |
\end{aligned}$$

Exercise

Evaluate $\int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2} dx$

Solution

$$\begin{aligned} \int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2} dx &= \frac{2}{\pi} \int \tan^3 \frac{\pi x}{2} d\left(\tan \frac{\pi x}{2}\right) \\ &= \frac{1}{2\pi} \tan^4 \frac{\pi x}{2} + C \end{aligned}$$

Exercise

Evaluate the integral $\int \sec^{-2} x \tan^3 x dx$

Solution

$$\begin{aligned} \int \sec^{-2} x \tan^3 x dx &= \int \sec^{-2} x \tan^2 x \tan x dx \\ &= \int \sec^{-2} x (\sec^2 x - 1) \tan x dx \\ &= \int (1 - \sec^{-2} x) \tan x dx \\ &= \int \tan x dx - \int \sec^{-2} x \tan x dx \\ &= \int \frac{\sin x}{\cos x} dx - \int \cos^2 x \cdot \frac{\sin x}{\cos x} dx \\ &= -\int \frac{1}{\cos x} d(\cos x) + \int \cos x d(\cos x) \\ &= -\ln|\cos x| + \frac{1}{2} \cos^2 x + C \end{aligned}$$

Exercise

Evaluate the integral $\int \sqrt{\tan x} \sec^4 x dx$

Solution

$$\int \sqrt{\tan x} \sec^4 x dx = \int (\tan x)^{1/2} (\tan^2 x + 1) \sec^2 x dx$$

$$\begin{aligned}
 &= \int \left(\tan^{5/2} x + \tan^{1/2} x \right) d(\tan x) \\
 &= \frac{2}{7} \tan^{7/2} x + \frac{2}{3} \tan^{3/2} x + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \tan^5 \theta \csc^2 \theta d\theta$

Solution

$$\begin{aligned}
 \int \tan^5 \theta \csc^2 \theta d\theta &= \int \frac{\sin^5 \theta}{\cos^5 \theta} \cdot \frac{1}{\sin^2 \theta} d\theta \\
 &= \int \frac{\sin^3 \theta}{\cos^5 \theta} d\theta \\
 &= \int \frac{\sin^3 \theta}{\cos^3 \theta} \cdot \frac{1}{\cos^2 \theta} d\theta \\
 &= \int \tan^3 \theta \sec^2 \theta d\theta \\
 &= \int \tan^3 \theta d(\tan \theta) \\
 &= \frac{1}{4} \tan^4 \theta + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \csc^2 x \cot x dx$

Solution

$$\begin{aligned}
 \int \csc^2 x \cot x dx &= - \int \cot x d(\cot x) \\
 &= -\frac{1}{2} \cot^2 x + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \csc^{10} x \cot x dx$

Solution

$$\int \csc^{10} x \cot x \, dx = - \int \csc^9 x \, d(\csc x)$$

$$\underline{= -\frac{1}{10} \csc^{10} x + C}$$

Exercise

Evaluate the integral $\int (\cot 2x - \csc 2x)^2 \, dx$

Solution

$$\begin{aligned} \int (\cot 2x - \csc 2x)^2 \, dx &= \int (\cot^2 2x - 2 \cot 2x \csc 2x + \csc^2 2x) \, dx \\ &= \int \cot^2 2x \, dx - \int (2 \cot 2x \csc 2x) \, dx + \int \csc^2 2x \, dx \\ &= \int (\csc^2 2x - 1) \, dx - \int (\cot 2x \csc 2x) \, d(2x) - \frac{1}{2} \cot 2x \\ &= -\frac{1}{2} \cot 2x - x + \csc 2x - \frac{1}{2} \cot 2x + C \\ &\underline{= \csc 2x - \cot 2x - x + C} \end{aligned}$$

Exercise

Evaluate the integral $\int \operatorname{sech}^4 x \, dx$

Solution

$$\begin{aligned} \int \operatorname{sech}^4 x \, dx &= \int \operatorname{sech}^2 x \operatorname{sech}^2 x \, dx \\ &= \int \operatorname{sech}^2 x (1 - \tanh^2 x) \, dx \\ &= \int \operatorname{sech}^2 x \, dx - \int \operatorname{sech}^2 x \tanh^2 x \, dx \\ &= \tanh x - \int \tanh^2 x \, d(\tanh x) \\ &\underline{= \tanh x - \frac{1}{3} \tanh^3 x + C} \end{aligned}$$

Exercise

Evaluate the integral $\int \sinh^3 x \cosh^2 x \, dx$

Solution

$$\begin{aligned}\int \sinh^3 x \cosh^2 x \, dx &= \int \sinh^2 x \cosh^2 x (\sinh x \, dx) \\&= \int (\cosh^2 x - 1) \cosh^2 x \, d(\cosh x) \\&= \int (\cosh^4 x - \cosh^2 x) \, d(\cosh x) \\&= \frac{1}{5} \cosh^5 x - \frac{1}{3} \cosh^3 x + C \quad | \end{aligned}$$

Exercise

Evaluate the integral $\int \operatorname{sech}^2 x \sinh x \, dx$

Solution

$$\begin{aligned}\int \operatorname{sech}^2 x \sinh x \, dx &= \int \frac{d(\cosh x)}{\cosh^2 x} \\&= -\frac{1}{\cosh x} + C \\&= -\operatorname{sech} x + C \quad | \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\tan^3 x}{\sqrt{\sec x}} \, dx$

Solution

$$\begin{aligned}\int \frac{\tan^3 x}{\sqrt{\sec x}} \, dx &= \int \frac{\tan^2 x \tan x \frac{\sec x}{\sec x}}{(\sec x)^{1/2}} \, dx \\&= \int (\sec x)^{-3/2} (\sec^2 x - 1) \, d(\sec x) \\&= \int \left((\sec x)^{1/2} - (\sec x)^{-3/2} \right) \, d(\sec x) \\&= \frac{2}{3} (\sec x)^{3/2} + 2 (\sec x)^{-1/2} + C \quad | \end{aligned}$$

$$1 + \tan^2 \alpha = \sec^2 \alpha$$

Exercise

Evaluate the integral $\int \frac{\tan^2 x}{\sec x} dx$

Solution

$$\begin{aligned}\int \frac{\tan^2 x}{\sec x} dx &= \int \frac{\sec^2 x - 1}{\sec x} dx \\&= \int \left(\sec x - \frac{1}{\sec x} \right) dx \\&= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx - \int \cos x dx \\&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx - \sin x \\&= \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} - \sin x \\&= \ln |\sec x + \tan x| - \sin x + C\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\sec x}{\tan^2 x} dx$

Solution

$$\begin{aligned}\int \frac{\sec x}{\tan^2 x} dx &= \int \frac{1}{\cos x} \frac{\cos^2 x}{\sin^2 x} dx \\&= \int \frac{\cos x}{\sin^2 x} dx \\&= \int \frac{1}{\sin^2 x} d(\sin x) \\&= -\frac{1}{\sin x} + C \\&= -\csc x + C\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\sec^2 x}{\tan^5 x} dx$

Solution

$$\int \frac{\sec^2 x}{\tan^5 x} dx = \int \tan^{-5} x d(\tan x)$$

$$\underline{= -\frac{1}{4} \tan^{-4} x + C}$$

Exercise

Evaluate the integral $\int \frac{\csc^4 x}{\cot^2 x} dx$

Solution

$$\int \frac{\csc^4 x}{\cot^2 x} dx = \int \frac{\csc^2 x (\cot^2 x + 1)}{\cot^2 x} dx$$

$$= - \int \frac{\cot^2 x + 1}{\cot^2 x} d(\cot x)$$

$$= - \int \left(1 + \frac{1}{\cot^2 x} \right) d(\cot x)$$

$$= -\cot x + \frac{1}{\cot x} + C$$

$$\underline{= -\cot x + \tan x + C}$$

Exercise

Evaluate the integral $\int \frac{\sec^4(\ln x)}{x} dx$

Solution

$$\int \frac{\sec^4(\ln x)}{x} dx = \int \sec^4(\ln x) d(\ln x)$$

$$= \int \sec^2(\ln x) (\tan^2(\ln x) + 1) d(\ln x)$$

$$= \int \sec^2(\ln x) \tan^2(\ln x) d(\ln x) + \int \sec^2(\ln x) d(\ln x)$$

$$= \int \tan^2(\ln x) d(\tan(\ln x)) + \tan(\ln x)$$

$$\underline{= \frac{1}{3} \tan^3(\ln x) + \tan(\ln x) + C}$$

Exercise

Evaluate the integral $\int e^x \sec(e^x + 1) dx$

Solution

$$\begin{aligned}\int e^x \sec(e^x + 1) dx &= \int \sec(e^x + 1) d(e^x + 1) \\&= \int \sec(e^x + 1) \frac{\sec(e^x + 1) + \tan(e^x + 1)}{\sec(e^x + 1) + \tan(e^x + 1)} d(e^x + 1) \\&= \int \frac{\sec^2(e^x + 1) + \sec(e^x + 1) \tan(e^x + 1)}{\sec(e^x + 1) + \tan(e^x + 1)} d(e^x + 1) \\&= \int \frac{1}{\sec(e^x + 1) + \tan(e^x + 1)} d(\sec(e^x + 1) + \tan(e^x + 1)) \\&= \ln |\sec(e^x + 1) + \tan(e^x + 1)| + C\end{aligned}$$

Exercise

Evaluate the integral $\int e^x \sec^3(e^x) dx$

Solution

$$\begin{aligned}u &= \sec(e^x) & dv &= \sec(e^x) e^x dx \\du &= \sec(e^x) \tan(e^x) e^x dx & v &= \int \sec(e^x) d(e^x) = \tan(e^x) \\ \int e^x \sec^3(e^x) dx &= \sec(e^x) \tan(e^x) - \int \sec(e^x) \tan^2(e^x) e^x dx \\&= \sec(e^x) \tan(e^x) - \int \sec(e^x) (\sec^2(e^x) - 1) e^x dx \\&= \sec(e^x) \tan(e^x) - \int (\sec^3(e^x) - \sec(e^x)) e^x dx \\&= \sec(e^x) \tan(e^x) - \int \sec^3(e^x) e^x dx + \int \sec(e^x) e^x dx & d(e^x) &= e^x dx \\&= \sec(e^x) \tan(e^x) - \int \sec^3(e^x) e^x dx + \int \sec(e^x) d(e^x)\end{aligned}$$

$$\int \sec^3(e^x) e^x dx = \sec(e^x) \tan(e^x) - \int \sec^3(e^x) e^x dx + \ln |\sec(e^x) + \tan(e^x)|$$

$$2 \int \sec^3(e^x) e^x dx = \sec(e^x) \tan(e^x) + \ln |\sec(e^x) + \tan(e^x)| + C$$

$$\int \sec^3(e^x) e^x dx = \underline{\frac{1}{2} \sec(e^x) \tan(e^x) + \frac{1}{2} \ln |\sec(e^x) + \tan(e^x)| + C}$$

Exercise

Evaluate the integral $\int e^x \sqrt{\tan^2 e^x + 1} dx$

Solution

$$\begin{aligned} \int e^x \sqrt{\tan^2(e^x) + 1} dx &= \int \sqrt{\tan^2(e^x) + 1} d(e^x) \\ &= \int \sqrt{\sec^2(e^x)} d(e^x) \\ &= \int \sec e^x \cdot \frac{\sec e^x + \tan e^x}{\sec e^x + \tan e^x} d(e^x) \\ &= \int \frac{\sec^2 e^x + \sec e^x \tan e^x}{\sec e^x + \tan e^x} d(e^x) \\ &= \int \frac{1}{\sec e^x + \tan e^x} d(\sec e^x + \tan e^x) \\ &= \underline{\ln |\sec e^x + \tan e^x| + C} \end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\sqrt{\frac{\pi}{2}}} x \sin^3(x^2) dx$

Solution

$$\begin{aligned} \int_0^{\sqrt{\frac{\pi}{2}}} x \sin^3(x^2) dx &= \frac{1}{2} \int_0^{\sqrt{\frac{\pi}{2}}} \sin^3(x^2) d(x^2) \\ &= \frac{1}{2} \int_0^{\sqrt{\frac{\pi}{2}}} \sin^2(x^2) \sin(x^2) d(x^2) \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2} \int_0^{\sqrt{\frac{\pi}{2}}} \left(1 - \cos^2(x^2)\right) d(\cos x^2) \\
&= -\frac{1}{2} \left(\cos x^2 - \frac{1}{3} \cos^3(x^2) \right) \bigg|_0^{\sqrt{\frac{\pi}{2}}} \\
&= -\frac{1}{2} \left(\cos \frac{\pi}{2} - \frac{1}{3} \cos^3 \frac{\pi}{2} - 1 + \frac{1}{3} \right) \\
&= -\frac{1}{2} \left(-\frac{2}{3} \right) \\
&= \frac{1}{3}
\end{aligned}$$

Exercise

Evaluate the integral $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}} dx$

Solution

$$\begin{aligned}
\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}} dx &= \int_{\pi/6}^{\pi/3} (\sin x)^{-1/2} (1 - \sin^2 x) d(\sin x) \\
&= \int_{\pi/6}^{\pi/3} \left((\sin x)^{-1/2} - (\sin x)^{3/2} \right) d(\sin x) \\
&= 2(\sin x)^{1/2} - \frac{2}{5}(\sin x)^{5/2} \bigg|_{\pi/6}^{\pi/3} \\
&= 2 \left(\frac{\sqrt{3}}{2} \right)^{1/2} - \frac{2}{5} \left(\frac{\sqrt{3}}{2} \right)^{5/2} - 2 \left(\frac{1}{2} \right)^{1/2} + \frac{2}{5} \left(\frac{1}{2} \right)^{5/2} \\
&= \sqrt[4]{3} \sqrt{2} - \frac{3}{10} \frac{\sqrt[4]{3}}{\sqrt{2}} - \sqrt{2} + \frac{\sqrt{2}}{20} \\
&= \frac{\sqrt{2}}{20} (17 \sqrt[4]{3} - 19)
\end{aligned}$$

Exercise

Evaluate the integral $\int_{\pi/6}^{\pi/2} \frac{dx}{\sin x}$

Solution

$$\begin{aligned}
\int_{\pi/6}^{\pi/2} \frac{dx}{\sin x} &= \int_{\pi/6}^{\pi/2} \csc x \frac{\csc x + \cot x}{\csc x + \cot x} dx \\
&= \int_{\pi/6}^{\pi/2} \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx \\
&= - \int_{\pi/6}^{\pi/2} \frac{1}{\csc x + \cot x} d(\csc x + \cot x) \\
&= - \ln |\csc x + \cot x| \Big|_{\pi/6}^{\pi/2} \\
&= - \ln |1 + 0| + \ln |2 + \sqrt{3}| \\
&= \ln(2 + \sqrt{3})
\end{aligned}$$

Exercise

Evaluate the integral $\int_{\pi/6}^{\pi/3} \cot^3 \theta d\theta$

Solution

$$\begin{aligned}
\int_{\pi/6}^{\pi/3} \cot^3 \theta d\theta &= \int_{\pi/6}^{\pi/3} \cot \theta (\csc^2 \theta - 1) d\theta \\
&= \int_{\pi/6}^{\pi/3} \cot \theta \csc^2 \theta d\theta - \int_{\pi/6}^{\pi/3} \cot \theta d\theta \\
&= - \int_{\pi/6}^{\pi/3} \cot \theta d(\cot \theta) - \int_{\pi/6}^{\pi/3} \frac{\cos \theta}{\sin \theta} d\theta \\
&= -\frac{1}{2} \cot^2 \theta - \int_{\pi/6}^{\pi/3} \frac{1}{\sin \theta} d(\sin \theta) \\
&= -\frac{1}{2} \cot^2 \theta - \ln |\sin \theta| \Big|_{\pi/6}^{\pi/3} \\
&= \frac{3}{2} - \frac{1}{6} + \ln \frac{1}{2} - \ln \frac{\sqrt{3}}{2} \\
&= \frac{4}{3} + \ln \frac{1}{\sqrt{3}} \\
&= \frac{4}{3} - \ln \sqrt{3}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/3} \tan^2 x \, dx$

Solution

$$\begin{aligned}\int_0^{\pi/3} \tan^2 x \, dx &= \int_0^{\pi/3} (\sec^2 x - 1) \, dx \\ &= \tan x - x \Big|_0^{\pi/3} \\ &= \sqrt{3} - \frac{\pi}{3}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/4} 6 \tan^3 x \, dx$

Solution

$$\begin{aligned}\int_0^{\pi/4} 6 \tan^3 x \, dx &= 6 \int_0^{\pi/4} \tan x \tan^2 x \, dx \\ &= 6 \int_0^{\pi/4} \tan x (\sec^2 x - 1) \, dx \\ &= 6 \int_0^{\pi/4} \tan x \sec^2 x \, dx - 6 \int_0^{\pi/4} \tan x \, dx \\ &= 6 \int_0^{\pi/4} \tan x \, d(\tan x) - 6 \int_0^{\pi/4} \frac{\sin x}{\cos x} \, dx \\ &= 3 \tan^2 x \Big|_0^{\pi/4} + 6 \int_0^{\pi/4} \frac{1}{\cos x} \, d(\cos x) \\ &= 3 + 6 \ln(\cos x) \Big|_0^{\pi/4} \\ &= 3 + 6 \ln \frac{\sqrt{2}}{2} - 6 \ln 1 \\ &= 3 + 6 \ln \frac{\sqrt{2}}{2}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/4} \tan^4 x \, dx$

Solution

$$\begin{aligned}\int_0^{\pi/4} \tan^4 x \, dx &= \int_0^{\pi/4} \tan^2 x (\sec^2 x - 1) \, dx \\&= \int_0^{\pi/4} \tan^2 x (\sec^2 x - 1) \, dx \\&= \int_0^{\pi/4} \tan^2 x \sec^2 x \, dx - \int_0^{\pi/4} \tan^2 x \, dx \\&= \int_0^{\pi/4} \tan^2 x \, d(\tan x) - \int_0^{\pi/4} (\sec^2 x - 1) \, dx \\&= \frac{1}{3} \tan^3 x - \tan x + x \Big|_0^{\pi/4} \\&= \frac{1}{3} - 1 + \frac{\pi}{4} \\&= \frac{\pi}{4} - \frac{2}{3}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi} 8 \sin^4 y \cos^2 y \, dy$

Solution

$$\begin{aligned}\int_0^{\pi} 8 \sin^4 y \cos^2 y \, dy &= 8 \int_0^{\pi} \left(\frac{1 - \cos 2y}{2} \right)^2 \left(\frac{1 + \cos 2y}{2} \right) dy \\&= \int_0^{\pi} (1 - 2 \cos 2y + \cos^2 2y)(1 + \cos 2y) \, dy \\&= \int_0^{\pi} (1 - 2 \cos 2y + \cos^2 2y + \cos 2y - 2 \cos^2 2y + \cos^3 2y) \, dy \\&= \int_0^{\pi} (1 - \cos 2y - \cos^2 2y + \cos^3 2y) \, dy\end{aligned}$$

$$\begin{aligned}
&= \int_0^{\pi} \left(1 - \cos 2y - \frac{1}{2} - \frac{1}{2} \cos 4y\right) dy + \int_0^{\pi} \cos^2 2y \cos 2y dy \\
&= \int_0^{\pi} \left(\frac{1}{2} - \cos 2y - \frac{1}{2} \cos 4y\right) dy + \frac{1}{2} \int_0^{\pi} (1 - \sin^2 2y) d(\sin 2y) \\
&= \frac{1}{2} y - \frac{1}{2} \sin 2y - \frac{1}{8} \sin 4y + \frac{1}{2} \left(\sin 2y - \frac{1}{3} \sin^3 2y \right) \Big|_0^{\pi} \\
&= \frac{\pi}{2}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/6} 3 \cos^5 3x dx$

Solution

$$\begin{aligned}
\int_0^{\pi/6} 3 \cos^5 3x dx &= \int_0^{\pi/6} 3 (\cos^2 3x)^2 \cos 3x dx \\
&= \int_0^{\pi/6} (1 - \sin^2 3x)^2 d(\sin 3x) \\
&= \int_0^{\pi/6} (1 - 2 \sin^2 3x + \sin^4 3x) d(\sin 3x) \\
&= \sin 3x - \frac{2}{3} \sin^3 3x + \frac{1}{5} \sin^5 3x \Big|_0^{\pi/6} \\
&= \sin \frac{\pi}{2} - \frac{2}{3} \sin^3 \frac{\pi}{2} + \frac{1}{5} \sin^5 \frac{\pi}{2} - 0 \\
&= 1 - \frac{2}{3} + \frac{1}{5} \\
&= \frac{8}{15}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta d\theta$

Solution

$$\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta d\theta = \int_0^{\pi/2} \sin^2 2\theta (\cos^2 2\theta) \cos 2\theta d\theta \qquad d(\sin 2\theta) = 2 \cos 2\theta d\theta$$

$$\begin{aligned}
&= \frac{1}{2} \int_0^{\pi/2} \sin^2 2\theta (1 - \sin^2 2\theta) d(\sin 2\theta) \\
&= \frac{1}{2} \int_0^{\pi/2} (\sin^2 2\theta - \sin^4 2\theta) d(\sin 2\theta) \\
&= \frac{1}{2} \left(\frac{1}{3} \sin^3 2\theta - \frac{1}{5} \sin^5 2\theta \right) \Big|_0^{\pi/2} \\
&= \frac{1}{2} \left(\frac{1}{3} \sin^3 \pi - \frac{1}{5} \sin^5 \pi - 0 \right) \\
&= 0
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{2\pi} \sqrt{\frac{1 - \cos x}{2}} dx$

Solution

$$\begin{aligned}
\int_0^{2\pi} \sqrt{\frac{1 - \cos x}{2}} dx &= \int_0^{2\pi} \sin \frac{x}{2} dx \\
&= -2 \cos \frac{x}{2} \Big|_0^{2\pi} \\
&= -2(\cos \pi - \cos 0) \\
&= 2
\end{aligned}$$

$$\left| \sin \left(\frac{\alpha}{2} \right) \right| = \sqrt{\frac{1 - \cos \alpha}{2}}$$

Exercise

Evaluate the integral $\int_0^{\pi} \sqrt{1 - \cos^2 \theta} d\theta$

Solution

$$\begin{aligned}
\int_0^{\pi} \sqrt{1 - \cos^2 \theta} d\theta &= \int_0^{\pi} |\sin \theta| d\theta \\
&= -\cos \theta \Big|_0^{\pi} \\
&= -\cos \pi + \cos 0 \\
&= 2
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/6} \sqrt{1 + \sin x} \, dx$

Solution

$$\begin{aligned} \int_0^{\pi/6} \sqrt{1 + \sin x} \, dx &= \int_0^{\pi/6} \sqrt{1 + \sin x} \frac{\sqrt{1 - \sin x}}{\sqrt{1 - \sin x}} \, dx \\ &= \int_0^{\pi/6} \frac{\sqrt{1 - \sin^2 x}}{\sqrt{1 - \sin x}} \, dx \\ &= \int_0^{\pi/6} \frac{\cos x}{\sqrt{1 - \sin x}} \, dx \\ &= - \int_0^{\pi/6} (1 - \sin x)^{-1/2} d(1 - \sin x) \\ &= -2 (1 - \sin x)^{1/2} \Big|_0^{\pi/6} \\ &= -2 \left(\sqrt{1 - \sin \frac{\pi}{6}} - 1 \right) \\ &= -2 \left(\sqrt{1 - \frac{1}{2}} - 1 \right) \\ &= -2 \left(\frac{1}{\sqrt{2}} - 1 \right) \\ &= -2 \left(\frac{\sqrt{2}}{2} - 1 \right) \\ &= \underline{2 - \sqrt{2}} \end{aligned}$$

$$\cos x = \sqrt{1 - \sin^2 x}$$

$$d(1 - \sin x) = -\cos x \, dx$$

Exercise

Evaluate the integral $\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2 \theta - 1} \, d\theta$

Solution

$$\begin{aligned} \int_{-\pi/3}^{\pi/3} \sqrt{\sec^2 \theta - 1} \, d\theta &= \int_{-\pi/3}^{\pi/3} \tan \theta \, d\theta \\ &= \int_{-\pi/3}^{\pi/3} \frac{\sin \theta}{\cos \theta} \, d\theta \end{aligned}$$

$$\begin{aligned}
&= - \int_{-\pi/3}^{\pi/3} \frac{1}{\cos \theta} d(\cos \theta) \\
&= - \ln |\cos \theta| \Big|_{-\pi/3}^{\pi/3} \\
&= - \left(\ln \left(\cos \frac{\pi}{3} \right) - \ln \left(\cos \frac{\pi}{3} \right) \right) \\
&= \underline{0}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi} (1 - \cos 2x)^{3/2} dx$

Solution

$$\begin{aligned}
\int_0^{\pi} (1 - \cos 2x)^{3/2} dx &= \int_0^{\pi} (2 \sin^2 x)^{3/2} dx \\
&= 2\sqrt{2} \int_0^{\pi} \sin^3 x dx \\
&= 2\sqrt{2} \int_0^{\pi} \sin^2 x \sin x dx \\
&= -2\sqrt{2} \int_0^{\pi} (1 - \cos^2 x) d(\cos x) \\
&= -2\sqrt{2} \left(\cos x - \frac{1}{3} \cos^3 x \right) \Big|_0^{\pi} \\
&= -2\sqrt{2} \left(-1 + \frac{1}{3} - 1 + \frac{1}{3} \right) \\
&= -2\sqrt{2} \left(-\frac{4}{3} \right) \\
&= \underline{\frac{8\sqrt{2}}{3}}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi} (1 - \cos^2 x)^{3/2} dx$

Solution

$$\begin{aligned}
\int_0^{\pi} (1 - \cos^2 x)^{3/2} dx &= \int_0^{\pi} (\sin^2 x)^{3/2} dx \\
&= \int_0^{\pi} \sin^3 x dx \\
&= \int_0^{\pi} \sin^2 x \sin x dx & \sin^2 x = 1 - \cos^2 x \\
&= \int_0^{\pi} (1 - \cos^2 x) \sin x dx & d(\cos x) = -\sin x dx \\
&= - \int_0^{\pi} (1 - \cos^2 x) d(\cos x) \\
&= - \left(\cos x - \frac{1}{3} \cos^3 x \right) \Big|_0^{\pi} \\
&= - \left(-1 + \frac{1}{3} - \left(1 - \frac{1}{3} \right) \right) \\
&= 1 - \frac{1}{3} + 1 - \frac{1}{3} \\
&= 2 - \frac{2}{3} \\
&= \frac{4}{3}
\end{aligned}$$

Exercise

Evaluate the integral $\int_{-\pi}^{\pi} (1 - \cos^2 x)^{3/2} dx$

Solution

$$\begin{aligned}
\int_{-\pi}^{\pi} (1 - \cos^2 x)^{3/2} dx &= \int_{-\pi}^{\pi} (\sin^2 x)^{3/2} dx \\
&= \int_{-\pi}^{\pi} |\sin^3 x| dx \\
&= - \int_{-\pi}^0 \sin^3 x dx + \int_0^{\pi} \sin^3 x dx & \sin^2 x = 1 - \cos^2 x \\
&= - \int_{-\pi}^0 (1 - \cos^2 x) \sin x dx + \int_0^{\pi} (1 - \cos^2 x) \sin x dx & d(\cos x) = -\sin x dx
\end{aligned}$$

$$\begin{aligned}
&= \int_{-\pi}^0 (1 - \cos^2 x) d(\cos x) - \int_0^{\pi} (1 - \cos^2 x) d(\cos x) \\
&= \left(\cos x - \frac{1}{3} \cos^3 x \right) \Big|_{-\pi}^0 - \left(\cos x - \frac{1}{3} \cos^3 x \right) \Big|_0^{\pi} \\
&= \left(1 - \frac{1}{3} - \left(-1 + \frac{1}{3} \right) \right) - \left(-1 + \frac{1}{3} - \left(1 - \frac{1}{3} \right) \right) \\
&= 1 - \frac{1}{3} + 1 - \frac{1}{3} + 1 - \frac{1}{3} + 1 - \frac{1}{3} \\
&= 4 - \frac{4}{3} \\
&= \frac{8}{3}
\end{aligned}$$

Exercise

Evaluate the integral $\int_{\pi/4}^{\pi/2} \csc^4 \theta d\theta$

Solution

$$\begin{aligned}
\int_{\pi/4}^{\pi/2} \csc^4 \theta d\theta &= \int_{\pi/4}^{\pi/2} (1 + \cot^2 \theta) \csc^2 \theta d\theta & \csc^2 \theta &= 1 + \cot^2 \theta \\
&= \int_{\pi/4}^{\pi/2} \csc^2 \theta d\theta + \int_{\pi/4}^{\pi/2} \cot^2 \theta \csc^2 \theta d\theta & d(\cot \theta) &= -\csc^2 \theta d\theta \\
&= \int_{\pi/4}^{\pi/2} \csc^2 \theta d\theta - \int_{\pi/4}^{\pi/2} \cot^2 \theta d(\cot \theta) \\
&= \left(-\cot \theta - \frac{1}{3} \cot^3 \theta \right) \Big|_{\pi/4}^{\pi/2} \\
&= -\left(\cot \frac{\pi}{2} + \frac{1}{3} \cot^3 \frac{\pi}{2} - \cot \frac{\pi}{4} - \frac{1}{3} \cot^3 \frac{\pi}{4} \right) \\
&= -\left(0 + \frac{1}{3}(0) - 1 - \frac{1}{3} \right) \\
&= \frac{4}{3}
\end{aligned}$$

Exercise

Evaluate the integral $\int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx$

Solution

$$\begin{aligned}\int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} (\cos 0 - \cos 6x) \, dx \\&= \frac{1}{2} \int_{-\pi}^{\pi} (1 - \cos 6x) \, dx \\&= \frac{1}{2} \left(x - \frac{1}{6} \sin 6x \right) \Big|_{-\pi}^{\pi} \\&= \frac{1}{2} \left(\pi - \frac{1}{6} \sin 6\pi - \left(-\pi - \frac{1}{6} \sin(-6\pi) \right) \right) \\&= \frac{1}{2} (\pi + \pi) \\&= \pi\end{aligned}$$

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

Exercise

Evaluate the integral $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x \, dx$

Solution

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \cos x \cos 7x \, dx &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} (\cos 8x + \cos(-6x)) \, dx \\&= \frac{1}{2} \int_{-\pi/2}^{\pi/2} (\cos 8x + \cos 6x) \, dx \\&= \frac{1}{2} \left(\frac{1}{8} \sin 8x + \frac{1}{6} \sin 6x \right) \Big|_{-\pi/2}^{\pi/2} \\&= \frac{1}{2} \left(\frac{1}{8} \sin(4\pi) + \frac{1}{6} \sin(3\pi) - \frac{1}{8} \sin(-4\pi) - \frac{1}{6} \sin(-3\pi) \right) \\&= 0\end{aligned}$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

Exercise

Evaluate the integral $\int_0^{\pi/4} \cos^5 2x \sin^2 2x \, dx$

Solution

$$\begin{aligned}\int_0^{\pi/4} \cos^5 2x \sin^2 2x \, dx &= \int_0^{\pi/4} \cos^4 2x \cos 2x \sin^2 2x \, dx \\&= \frac{1}{2} \int_0^{\pi/4} (1 - \sin^2 2x)^2 \sin^2 2x \, d(\sin 2x) \\&= \frac{1}{2} \int_0^{\pi/4} (1 - 2\sin^2 2x + \sin^4 2x) \sin^2 2x \, d(\sin 2x) \\&= \frac{1}{2} \int_0^{\pi/4} (\sin^2 2x - 2\sin^4 2x + \sin^6 2x) \, d(\sin 2x) \\&= \frac{1}{2} \left(\frac{1}{3} \sin^3 2x - \frac{2}{5} \sin^5 2x + \frac{1}{7} \sin^7 2x \right) \Big|_0^{\pi/4} \\&= \frac{1}{2} \left(\frac{1}{3} \sin^3 \frac{\pi}{2} - \frac{2}{5} \sin^5 \frac{\pi}{2} + \frac{1}{7} \sin^7 \frac{\pi}{2} - 0 \right) \\&= \frac{1}{2} \left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7} \right) \\&= \frac{1}{2} \left(\frac{35 - 42 + 25}{105} \right) \\&= \underline{\underline{\frac{4}{105}}}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/6} \sin^5 x \, dx$

Solution

$$\begin{aligned}\int_0^{\pi/6} \sin^5 x \, dx &= \int_0^{\pi/6} \sin^4 x \sin x \, dx \\&= - \int_0^{\pi/6} (1 - \cos^2 x)^2 \, d(\cos x) \\&= - \int_0^{\pi/6} (1 - 2\cos^2 x + \cos^4 x) \, d(\cos x)\end{aligned}$$

$$\begin{aligned}
&= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x \Big|_0^{\pi/6} \\
&= -\frac{\sqrt{3}}{2} + \frac{2}{3}\frac{3\sqrt{3}}{8} - \frac{1}{5}\frac{9\sqrt{3}}{32} + 1 - \frac{2}{3} + \frac{1}{5} \\
&= -\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4} - \frac{9\sqrt{3}}{160} + \frac{15-10+3}{15} \\
&= \left(\frac{-80+40-9}{160}\right)\sqrt{3} + \frac{8}{15} \\
&= -\frac{49\sqrt{3}}{160} + \frac{8}{15} \\
&= \frac{256-147\sqrt{3}}{480}
\end{aligned}$$

Exercise

Evaluate the integral $\int_{-\pi}^{\pi} \sin^2 x \, dx$

Solution

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin^2 x \, dx &= 2 \int_0^{\pi} \sin^2 x \, dx \\
&= 2 \int_0^{\pi} \frac{1}{2}(1 - \cos 2x) \, dx \\
&= x - \frac{1}{2}\sin 2x \Big|_0^{\pi} \\
&= \pi
\end{aligned}$$

Exercise

Evaluate the integral $\int_{-\pi/2}^{\pi/2} (\sin^2 x + 1) \, dx$

Solution

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} (\sin^2 x + 1) \, dx &= \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2} - \frac{1}{2}\cos 2x + 1\right) \, dx \\
&= \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} - \frac{1}{2}\cos 2x\right) \, dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{3}{2}x - \frac{1}{4}\sin 2x \Big|_{-\pi/2}^{\pi/2} \\
&= \frac{3\pi}{4} + \frac{3\pi}{4} \\
&= \frac{3\pi}{2}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/3} \sec^{3/2} x \tan x \, dx$

Solution

$$\begin{aligned}
\int_0^{\pi/3} \sec^{3/2} x \tan x \, dx &= \int_0^{\pi/3} \sec^{1/2} x (\sec x \tan x) \, dx \\
&= \int_0^{\pi/3} \sec^{1/2} x \, d(\sec x) \\
&= \frac{2}{3} \sec^{3/2} x \Big|_0^{\pi/3} \\
&= \frac{2}{3} \left((2)^{3/2} - 1 \right) \\
&= \frac{2}{3} (2\sqrt{2} - 1)
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/2} \frac{\cos x}{1 + \sin x} \, dx$

Solution

$$\begin{aligned}
\int_0^{\pi/2} \frac{\cos x}{1 + \sin x} \, dx &= \int_0^{\pi/2} \frac{1}{1 + \sin x} \, d(1 + \sin x) \\
&= \ln |1 + \sin x| \Big|_0^{\pi/2} \\
&= \ln 2 - \ln 1 \\
&= \ln 2
\end{aligned}$$

Exercise

Evaluate the integral $\int_{\pi/6}^{\pi/3} \sin 6x \cos 4x \, dx$

Solution

$$\begin{aligned} \int_{\pi/6}^{\pi/3} \sin 6x \cos 4x \, dx &= \frac{1}{2} \int_{\pi/6}^{\pi/3} (\sin 10x + \sin 2x) \, dx & \sin \alpha \cos \beta &= \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)] \\ &= \frac{1}{2} \left(-\frac{1}{10} \cos 10x - \frac{1}{2} \cos 2x \right) \Big|_{\pi/6}^{\pi/3} \\ &= -\frac{1}{2} \left(\frac{1}{10} \cos \frac{10\pi}{3} + \frac{1}{2} \cos \frac{2\pi}{3} - \frac{1}{10} \cos \frac{5\pi}{3} - \frac{1}{2} \cos \frac{\pi}{3} \right) \\ &= -\frac{1}{2} \left(-\frac{1}{10} - \frac{1}{4} - \frac{1}{10} - \frac{1}{4} \right) \\ &= \frac{1}{2} \left(\frac{1}{10} + \frac{1}{2} \right) \\ &= \frac{6}{20} \\ &= \frac{3}{10} \end{aligned}$$

Exercise

Evaluate the integral $\int_{-\pi/2}^{\pi/2} 3 \cos^3 x \, dx$

Solution

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} 3 \cos^3 x \, dx &= 3 \int_{-\pi/2}^{\pi/2} \cos^2 x \cos x \, dx \\ &= 3 \int_{-\pi/2}^{\pi/2} (1 - \sin^2 x) \, d(\sin x) \\ &= 3 \left(\sin x - \frac{1}{3} \sin^3 x \right) \Big|_{-\pi/2}^{\pi/2} \\ &= 3 \left(1 - \frac{1}{3} + 1 - \frac{1}{3} \right) \\ &= 3 \left(2 - \frac{2}{3} \right) \\ &= 4 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi} \sec^2 x \, dx$

Solution

$$\begin{aligned}\int_0^{\pi} \sec^2 x \, dx &= \tan x \Big|_0^{\pi} \\ &= \tan \pi - \tan 0 \\ &= 0\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\ln(\sqrt{3}+2)} \frac{\cosh x}{\sqrt{4 - \sinh^2 x}} \, dx$

Solution

$$\begin{aligned}\int_0^{\ln(\sqrt{3}+2)} \frac{\cosh x}{\sqrt{4 - \sinh^2 x}} \, dx &= \int_0^{\ln(\sqrt{3}+2)} \frac{1}{\sqrt{4 - \sinh^2 x}} \, d(\sinh x) \\ &= \sin^{-1} \left(\frac{1}{2} \sinh x \right) \Big|_0^{\ln(\sqrt{3}+2)} \\ &= \sin^{-1} \left(\frac{1}{2} \sinh \left(\ln(\sqrt{3} + 2) \right) \right) - \sin^{-1} \left(\frac{1}{2} \sinh 0 \right) \\ &= \sin^{-1} \left(\frac{1}{4} \left(e^{\ln(\sqrt{3}+2)} - e^{-\ln(\sqrt{3}+2)} \right) \right) \\ &= \sin^{-1} \left(\frac{1}{4} \left(\sqrt{3} + 2 - \frac{1}{\sqrt{3} + 2} \right) \right) \\ &= \sin^{-1} \left(\frac{1}{4} \left(\frac{3 + 4\sqrt{3} + 4 - 1}{\sqrt{3} + 2} \right) \right) \\ &= \sin^{-1} \left(\frac{1}{4} \left(\frac{6 + 4\sqrt{3}}{\sqrt{3} + 2} \cdot \frac{\sqrt{3} - 2}{\sqrt{3} - 2} \right) \right) \\ &= \sin^{-1} \left(\frac{1}{2} \left(\frac{-\sqrt{3}}{3 - 4} \right) \right) \\ &= \sin^{-1} \left(\frac{\sqrt{3}}{2} \right) \\ &= \frac{\pi}{3}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/2} \cos^4 x \, dx$

Solution

$$\int_0^{\pi/2} \cos^4 x \, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{\pi}{2}\right) \\ = \frac{3\pi}{16} \Big|$$

$$\int_0^{\pi/2} \cos^n x \, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\dots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

Exercise

Evaluate $\int_0^{\pi/2} \cos^{10} x \, dx$

Solution

$$\begin{aligned} \int_0^{\pi/2} \cos^{10} \theta \, d\theta &= \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^5 d\theta \\ &= \frac{1}{32} \int_0^{\pi/2} \left(1 + 5\cos 2\theta + 10\cos^2 2\theta + 10\cos^3 2\theta + 5\cos^4 2\theta + \cos^5 2\theta\right) d\theta \\ &= \frac{1}{32} \int_0^{\pi/2} \left(1 + 5\cos 2\theta + 5 + 5\cos 4\theta + \frac{5}{4}(1+\cos 4\theta)^2\right. \\ &\quad \left.+ (10 + \cos^2 2\theta)\cos^3 2\theta\right) d\theta \\ &= \frac{1}{32} \int_0^{\pi/2} \left(6 + 5\cos 2\theta + 5\cos 4\theta + \frac{5}{4}(1 + 2\cos 4\theta + \cos^2 4\theta)\right) d\theta \\ &\quad + \frac{5}{16} \int_0^{\pi/2} \cos^3 2\theta \, d\theta + \frac{1}{32} \int_0^{\pi/2} \cos^5 2\theta \, d\theta \\ &= \frac{1}{32} \int_0^{\pi/2} \left(\frac{63}{8} + 5\cos 2\theta + \frac{11}{2}\cos 4\theta + \frac{5}{8}\cos 8\theta\right) d\theta \\ &\quad + \frac{5}{32} \int_0^{\pi/2} (1 - \sin^2 2\theta) d(\sin 2\theta) + \frac{1}{64} \int_0^{\pi/2} (1 - \sin^2 2\theta)^2 d(\sin 2\theta) \\ &= \frac{1}{32} \left(\frac{63}{8}\theta + \frac{5}{2}\sin 2\theta + \frac{11}{8}\sin 4\theta + \frac{5}{64}\sin 8\theta\right) + \frac{5}{32} \left(\sin 2\theta - \frac{1}{3}\sin^3 2\theta\right) \Big|_0^{\pi/2} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{64} \int_0^{\pi/2} \left(1 - 2 \sin^2 2\theta + \sin^4 2\theta \right) d(\sin 2\theta) \\
& = \frac{1}{32} \left(\frac{63\pi}{16} \right) + \frac{1}{64} \left(\sin 2\theta - \frac{2}{3} \sin^3 2\theta + \frac{1}{5} \sin^5 2\theta \right) \Big|_0^{\pi/2} \\
& \quad = \frac{63\pi}{512}
\end{aligned}$$

Or

$$\begin{aligned}
\int_0^{\pi/2} \cos^{10} x \, dx &= \left(\frac{1}{2} \right) \left(\frac{3}{4} \right) \left(\frac{5}{6} \right) \left(\frac{7}{8} \right) \left(\frac{9}{10} \right) \left(\frac{\pi}{2} \right) \\
& \quad = \frac{63\pi}{512}
\end{aligned}
\qquad
\int_0^{\pi/2} \cos^n x \, dx = \left(\frac{1}{2} \right) \left(\frac{3}{4} \right) \left(\frac{5}{6} \right) \dots \left(\frac{n-1}{n} \right) \left(\frac{\pi}{2} \right)$$

Exercise

Evaluate $\int_0^{\pi/2} \cos^7 x \, dx$

Solution

$$\begin{aligned}
\int_0^{\pi/2} \cos^7 x \, dx &= \left(\frac{2}{3} \right) \left(\frac{4}{5} \right) \left(\frac{6}{7} \right) \\
& \quad = \frac{16}{35}
\end{aligned}
\qquad
\int_0^{\pi/2} \cos^n \theta \, d\theta = \left(\frac{2}{3} \right) \left(\frac{4}{5} \right) \left(\frac{6}{7} \right) \dots \left(\frac{n-1}{n} \right)$$

Or

$$\begin{aligned}
\int_0^{\pi/2} \cos^7 x \, dx &= \int_0^{\pi/2} \left(\cos^2 x \right)^3 d(\sin x) \\
&= \int_0^{\pi/2} \left(1 - \sin^2 x \right)^3 d(\sin x) \\
&= \int_0^{\pi/2} \left(1 - 3 \sin^2 x + 3 \sin^4 x - \sin^6 x \right) d(\sin x) \\
&= \left(\sin x - \sin^3 x + \frac{3}{5} \sin^5 x - \frac{1}{7} \sin^7 x \right) \Big|_0^{\pi/2} \\
&= \frac{3}{5} - \frac{1}{7} \\
& \quad = \frac{16}{35}
\end{aligned}$$

Exercise

Evaluate $\int_0^{\pi/2} \cos^9 x \, dx$

Solution

$$\int_0^{\pi/2} \cos^9 x \, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\left(\frac{8}{9}\right)$$
$$= \frac{128}{315} \Big|$$

$$\int_0^{\pi/2} \cos^n \theta \, d\theta = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\dots\left(\frac{n-1}{n}\right)$$

Or

$$\int_0^{\pi/2} \cos^9 \theta \, d\theta = \int_0^{\pi/2} (1 - \sin^2 x)^4 \, d(\sin x)$$
$$= \int_0^{\pi/2} (1 - 4\sin^2 x + 6\sin^4 x - 4\sin^6 x + \sin^8 x) \, d(\sin x)$$
$$= \sin x - \frac{4}{3}\sin^3 x + \frac{6}{5}\sin^5 x - \frac{4}{7}\sin^7 x + \frac{1}{9}\sin^9 x \Big|_0^{\pi/2}$$
$$= 1 - \frac{4}{3} + \frac{6}{5} - \frac{4}{7} + \frac{1}{9}$$
$$= \frac{128}{315} \Big|$$

Exercise

Evaluate $\int_0^{\pi/2} \sin^5 x \, dx$

Solution

$$\int_0^{\pi/2} \sin^5 x \, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)$$
$$= \frac{8}{15} \Big|$$

$$\int_0^{\pi/2} \cos^n \theta \, d\theta = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\dots\left(\frac{n-1}{n}\right)$$

Or

$$\int_0^{\pi/2} \sin^5 x \, dx = \int_0^{\pi/2} (1 - \cos^2 x)^2 \, d(\cos x)$$
$$= \int_0^{\pi/2} (1 - 2\cos^2 x + \cos^4 x) \, d(\cos x)$$

$$\begin{aligned}
&= \cos x - \frac{2}{3} \cos^3 x + \frac{1}{5} \cos^5 x \Big|_0^{\pi/2} \\
&= -1 + \frac{2}{3} - \frac{1}{5} \\
&= -\frac{8}{15}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/2} \sin^6 x \, dx$

Solution

$$\begin{aligned}
\int_0^{\pi/2} \sin^6 x \, dx &= \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{\pi}{2}\right) \\
&= \frac{5\pi}{32}
\end{aligned}$$

$$\int_0^{\pi/2} \sin^n x \, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) \cdots \left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

Exercise

Evaluate the integral $\int_0^{\pi/2} \sin^8 x \, dx$

Solution

$$\begin{aligned}
\int_0^{\pi/2} \sin^8 x \, dx &= \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\left(\frac{\pi}{2}\right) \\
&= \frac{35\pi}{256}
\end{aligned}$$

$$\int_0^{\pi/2} \sin^n x \, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) \cdots \left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

Exercise

Evaluate the integral $\int_0^{\pi/2} \tan^2 \frac{x}{2} \, dx$

Solution

$$\begin{aligned}
\int_0^{\pi/2} \tan^2 \frac{x}{2} \, dx &= \int_0^{\pi/2} \left(\sec^2 \frac{x}{2} - 1 \right) dx \\
&= 2 \tan \frac{x}{2} - x \Big|_0^{\pi/2}
\end{aligned}$$

$$= 2 \tan \frac{\pi}{4} - \frac{\pi}{2}$$

$$\underline{= 2 - \frac{\pi}{2}}$$

Exercise

Find the area of the region bounded by the graphs of $y = \tan x$ and $y = \sec x$ on the interval $\left[0, \frac{\pi}{4}\right]$

Solution

$$A = \int_0^{\pi/4} (\sec x - \tan x) dx$$

$$= \ln |\sec x + \tan x| + \ln |\cos x| \Big|_0^{\pi/4}$$

$$= \ln(\sqrt{2} + 1) + \ln \frac{\sqrt{2}}{2} - 0$$

$$= \ln\left(\frac{\sqrt{2}}{2}(\sqrt{2} + 1)\right)$$

$$\underline{= \ln\left(1 + \frac{\sqrt{2}}{2}\right) \text{ unit}^2}$$

Exercise

Find the area of the region bounded by the graphs of the equations $y = \sin x$, $y = \sin^3 x$, $x = 0$, $x = \frac{\pi}{2}$

Solution

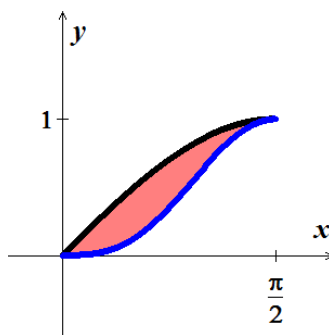
$$A = \int_0^{\pi/2} (\sin x - \sin^3 x) dx$$

$$= \int_0^{\pi/2} \sin x dx - \int_0^{\pi/2} \sin^3 x dx$$

$$= -\cos x \Big|_0^{\pi/2} - \frac{2}{3}$$

$$= 1 - \frac{2}{3}$$

$$\underline{= \frac{1}{3} \text{ unit}^2}$$

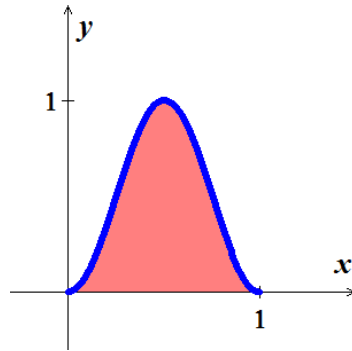


Exercise

Find the area of the region bounded by the graphs of the equations $y = \sin^2 \pi x$, $y = 0$, $x = 0$, $x = 1$

Solution

$$\begin{aligned} A &= \int_0^1 \sin^2 \pi x \, dx \\ &= \frac{1}{2} \int_0^1 (1 + \cos 2\pi x) \, dx \\ &= \frac{1}{2} \left(x + \frac{1}{2\pi} \sin 2\pi x \right) \Big|_0^1 \\ &= \frac{1}{2} \text{ unit}^2 \end{aligned}$$



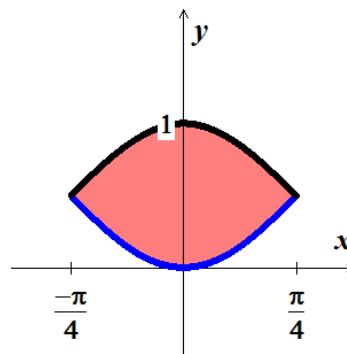
Exercise

Find the area of the region bounded by the graphs of the equations

$$y = \cos^2 x, \quad y = \sin^2 x, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$$

Solution

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} (\cos^2 x - \sin^2 x) \, dx \\ &= \int_{-\pi/4}^{\pi/4} \cos 2x \, dx \\ &= \frac{1}{2} \sin 2x \Big|_{-\pi/4}^{\pi/4} \\ &= \frac{1}{2} (1 + 1) \\ &= 1 \text{ unit}^2 \end{aligned}$$



Exercise

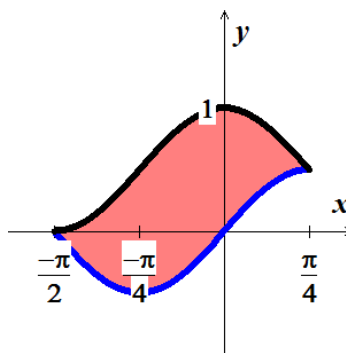
Find the area of the region bounded by the graphs of the equations

$$y = \cos^2 x, \quad y = \sin x \cos x, \quad x = -\frac{\pi}{2}, \quad x = \frac{\pi}{4}$$

Solution

$$A = \int_{-\pi/2}^{\pi/4} (\cos^2 x - \sin x \cos x) \, dx$$

$$\begin{aligned}
 &= \int_{-\pi/2}^{\pi/4} \left(\frac{1}{2} + \frac{1}{2} \cos 2x - \frac{1}{2} \sin 2x \right) dx \\
 &= \frac{1}{2} \left(x + \frac{1}{2} \sin 2x + \frac{1}{2} \cos 2x \right) \Big|_{-\pi/2}^{\pi/4} \\
 &= \frac{1}{2} \left(\frac{\pi}{4} + \frac{1}{2} + \frac{\pi}{2} + \frac{1}{2} \right) \\
 &= \frac{1}{2} \left(\frac{3\pi}{4} + 1 \right) \\
 &= \frac{3\pi}{8} + \frac{1}{2} \text{ unit}^2
 \end{aligned}$$



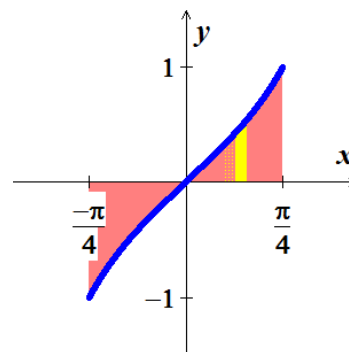
Exercise

Find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the x -axis $y = \tan x$, $y = 0$, $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$

Solution

Disks Method:

$$\begin{aligned}
 V &= 2\pi \int_0^{\pi/4} \tan^2 x \, dx \\
 &= 2\pi \int_0^{\pi/4} (\sec^2 x - 1) \, dx \\
 &= 2\pi \left(\tan x - x \right) \Big|_0^{\pi/4} \\
 &= 2\pi \left(1 - \frac{\pi}{4} \right) \\
 &= 2\pi - \frac{\pi^2}{2} \text{ unit}^3
 \end{aligned}$$



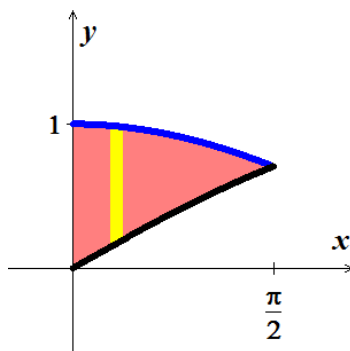
Exercise

Find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the x -axis $y = \cos \frac{x}{2}$, $y = \sin \frac{x}{2}$, $x = 0$, $x = \frac{\pi}{2}$

Solution

$$V = \pi \int_0^{\pi/2} \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right) dx \qquad \cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\begin{aligned}
 &= \pi \int_0^{\pi/2} \cos x \, dx \\
 &= \pi \sin x \Big|_0^{\pi/2} \\
 &= \pi \text{ unit}^3
 \end{aligned}$$



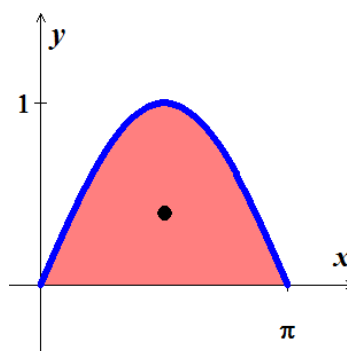
Exercise

Find the **volume** of the solid generated by revolving the region bounded by the graphs of the equations about the x -axis, then find the **centroid** of the region

$$y = \sin x, \quad y = 0, \quad x = 0, \quad x = \pi$$

Solution

$$\begin{aligned}
 V &= \pi \int_0^{\pi} \sin^2 x \, dx \\
 &= \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x) \, dx \\
 &= \frac{\pi}{2} \left(x - \frac{1}{2} \sin 2x \right) \Big|_0^{\pi} \\
 &= \frac{\pi^2}{2} \text{ unit}^3
 \end{aligned}$$



$$\begin{aligned}
 A &= \int_0^{\pi} \sin x \, dx \\
 &= -\cos x \Big|_0^{\pi} \\
 &= -(-1 - 1) \\
 &= 2 \text{ unit}^2
 \end{aligned}$$

$$\bar{x} = \frac{1}{A} \int_0^{\pi} x \sin x \, dx$$

		$\int \sin x$
+	x	$-\cos x$
-	1	$-\sin x$

$$= \frac{1}{2} \left(-x \cos x + \sin x \right) \Big|_0^{\pi}$$

$$= \frac{\pi}{2}$$

$$\bar{y} = \frac{1}{2A} \int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{1}{8} \int_0^{\pi} (1 - \cos 2x) \, dx$$

$$= \frac{1}{8} \left(x - \frac{1}{2} \sin 2x \right) \Big|_0^{\pi}$$

$$= \frac{\pi}{8}$$

$$(\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{\pi}{8} \right)$$

Exercise

Find the **volume** of the solid generated by revolving the region bounded by the graphs of the equations about the x -axis, then find the **centroid** of the region

$$y = \cos x, \quad y = 0, \quad x = 0, \quad x = \frac{\pi}{2}$$

Solution

$$V = \pi \int_0^{\pi/2} \cos^2 x \, dx$$

$$= \frac{\pi}{2} \int_0^{\pi/2} (1 + \cos 2x) \, dx$$

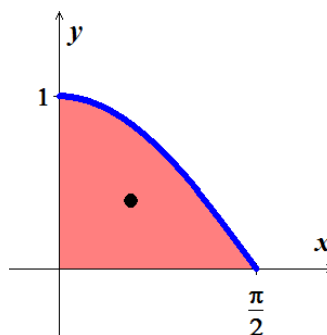
$$= \frac{\pi}{2} \left(x + \frac{1}{2} \sin 2x \right) \Big|_0^{\pi/2}$$

$$= \frac{\pi^2}{4} \text{ unit}^3$$

$$A = \int_0^{\pi/2} \cos x \, dx$$

$$= \sin x \Big|_0^{\pi/2}$$

$$= 1 \text{ unit}^2$$



$$\bar{x} = \frac{1}{A} \int_0^{\pi/2} x \cos x \, dx$$

		$\int \cos x \, dx$
+	x	$\sin x$
-	1	$-\cos x$

$$= x \sin x + \cos x \Big|_0^{\pi/2}$$

$$= \frac{\pi}{2} - 1 \Big|$$

$$\bar{y} = \frac{1}{2A} \int_0^{\pi/2} \cos^2 x \, dx$$

$$= \frac{1}{4} \int_0^{\pi/2} (1 + \cos 2x) \, dx$$

$$= \frac{1}{4} \left(x + \frac{1}{2} \sin 2x \right) \Big|_0^{\pi/2}$$

$$= \frac{\pi}{8} \Big|$$

$$(\bar{x}, \bar{y}) = \left(\frac{\pi - 2}{2}, \frac{\pi}{8} \right) \Big|$$