

## ***Solution***

## **Section 4.3 – Mathematical Induction**

### ***Exercise***

Find all positive integers  $n$  for which the given statement is not true

- a)  $3^n > 6n$       b)  $3^n > 2n + 1$       c)  $2^n > n^2$       d)  $n! > 2n$

### **Solution**

a)  $n = 1$     $3 < 6$

$n = 2$     $3^2 < 18$

$n = 3$ ,    $27 > 18$

The statement is true for all  $n \geq 3$     $3^n > 6n$

The statement is not true for  $n = 1, 2$

b)  $n = 1$ ;    $3 = 3$

$n = 2$ ;    $9 > 5$

The statement is true for all  $n \geq 2$     $3^n > 2n + 1$

The statement is not true for  $n = 1$

c)  $n = 1$ ;    $2 < 4$

$n = 2$ ;    $4 = 4$

$n = 3$ ;    $8 < 9$

$n = 4$ ;    $16 = 16$

$n = 5$ ;    $32 > 25$

The statement is true for all  $n \geq 5$ ;    $2^n > n^2$

The statement is not true for  $n = 1, 2, 3, 4$

d)  $n = 1$ ;    $1 < 2$

$n = 2$ ;    $2 < 4$

$n = 3$ ;    $6 = 6$

$n = 4$ ;    $12 > 8$

The statement is true for all  $n \geq 4$ ;    $n! > 2n$

The statement is not true for  $n = 1, 2, 3$

### Exercise

Prove that the statement is true for every positive integer  $n$ .  $2 + 4 + 6 + \dots + 2n = n(n + 1)$

### Solution

(1) For  $n = 1 \Rightarrow 2 = 1(1 + 1) = 2$ ; hence  $P_1$  is true.

(2) Assume  $2 + 4 + 6 + \dots + 2k = k(k + 1)$  is true

$$\Rightarrow 2 + 4 + 6 + \dots + 2k + 2(k + 1) = (k + 1)(k + 1 + 1) \text{ ?}$$

$$2 + 4 + 6 + \dots + 2k + 2(k + 1) = 2 + 4 + 6 + \dots + 2k + 2(k + 1)$$

$$= k(k + 1) + 2(k + 1)$$

*Factor (k + 1)*

$$= (k + 1)(k + 2)$$

$$= (k + 1)(k + 1 + 1) \quad \checkmark$$

Hence  $P_{k+1}$  is also true.

$\therefore$  By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true for every positive integer  $n$ .  $1 + 3 + 5 + \dots + (2n - 1) = n^2$

### Solution

(1) For  $n = 1 \Rightarrow 1 = 1^2 = 1$ ; hence  $P_1$  is true.

(2) Assume  $1 + 3 + 5 + \dots + (2k - 1) = k^2$  is true

$$\Rightarrow 1 + 3 + 5 + \dots + (2(k + 1) - 1) = (k + 1)^2 \text{ ?}$$

$$1 + 3 + 5 + \dots + (2k - 1) + (2(k + 1) - 1) = 1 + 3 + 5 + \dots + (2k - 1) + (2k + 2 - 1)$$

$$= k^2 + (2k + 1)$$

$$= k^2 + 2k + 1$$

$$= (k + 1)^2 \quad \checkmark$$

Hence  $P_{k+1}$  is also true.

$\therefore$  By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true for every positive integer  $n$ .  $2 + 7 + 12 + \dots + (5n - 3) = \frac{1}{2}n(5n - 1)$

#### Solution

(1) For  $n = 1 \Rightarrow 2 = \frac{1}{2}(\overset{?}{1})(5(\overset{?}{1}) - 1) = \frac{1}{2}(4) = \overset{?}{2}$ ; hence  $P_1$  is true.

(2) Assume  $2 + 7 + 12 + \dots + (5k - 3) = \frac{1}{2}k(5k - 1)$  is true

$$2 + 7 + 12 + \dots + (5(k + 1) - 3) = \frac{1}{2}(k + 1)(5(k + 1) - 1) \quad ?$$

$$\begin{aligned} 2 + 7 + 12 + \dots + (5k - 3) + (5(k + 1) - 3) &= 2 + 7 + 12 + \dots + (5k - 3) + (5k + 5 - 3) \\ &= \frac{1}{2}k(5k - 1) + (5k + 2) \quad \overset{?}{2} \\ &= \frac{1}{2}[5k^2 - k + 10k + 4] \\ &= \frac{1}{2}[5k^2 - k + 5k + 5k + 5 - 1] \\ &= \frac{1}{2}[k(5k - 1 + 5) + 5k + 5 - 1] \\ &= \frac{1}{2}[(k + 1)(5k + 5 - 1)] \\ &= \frac{1}{2}[(k + 1)(5(k + 1) - 1)] \quad \checkmark \quad P_{k+1} \text{ is also true.} \end{aligned}$$

$\therefore$  By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true for every positive integer  $n$ .  $1 + 2.2 + 3.2^2 + \dots + n.2^{n-1} = 1 + (n - 1).2^n$

#### Solution

(1) For  $n = 1 \Rightarrow 1 = 1 + (\overset{?}{1} - 1)2^{\overset{?}{1}} = 1 - 0 = \overset{?}{1}$ ; hence  $P_1$  is true.

(2)  $1 + 2.2 + 3.2^2 + \dots + k.2^{k-1} = 1 + (k - 1).2^k$  is true

$$1 + 2.2 + 3.2^2 + \dots + k.2^{k-1} + (k + 1).2^{(k+1)-1} = 1 + ((k + 1) - 1).2^{k+1} \quad ?$$

$$\begin{aligned} 1 + 2.2 + 3.2^2 + \dots + k.2^{k-1} + (k + 1).2^{(k+1)-1} &= 1 + (k - 1).2^k + (k + 1).2^{k+1-1} \\ &= 1 + k.2^k - 1.2^k + (k + 1).2^k \\ &= 1 + k.2^k - 1.2^k + k.2^k + 1.2^k \\ &= 1 + 2^1 k.2^k \\ &= 1 + (k + 0).2^{k+1} \\ &= 1 + ((k + 1) - 1).2^{k+1} \quad \checkmark \quad P_{k+1} \text{ is also true.} \end{aligned}$$

$\therefore$  By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true for every positive integer  $n$ .  $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

### Solution

(1) For  $n = 1 \Rightarrow 1^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = \frac{6}{6} = 1$  ✓; hence  $P_1$  is true.

(2)  $1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$  is true

$$1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} ?$$

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\ &= \frac{(k+1)[2k^2 + k + 6k + 6]}{6} \\ &= \frac{(k+1)[2k^2 + 7k + 6]}{6} \\ &= \frac{(k+1)((k+2)(2k+3))}{6} \\ &= \frac{(k+1)((k+1+1)(2k+2+1))}{6} \\ &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \quad \checkmark \quad P_{k+1} \text{ is also true.} \end{aligned}$$

∴ By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true for every positive integer  $n$ .  $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

### Solution

(1) For  $n = 1 \Rightarrow \frac{1}{1 \cdot 2} = \frac{1}{1+1} = \frac{1}{2} = \frac{1}{1 \cdot 2}$  ✓; hence  $P_1$  is true.

(2)  $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$  is true

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{(k+1)+1} ?$$

$$\begin{aligned}
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)(k+1)}{(k+1)(k+2)} \\
&= \frac{k+1}{(k+1)+1} \\
&= \frac{k+1}{(k+1)+1} \quad \checkmark \quad P_{k+1} \text{ is also true.}
\end{aligned}$$

∴ By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true:  $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

### Solution

(1) For  $n = 1 \Rightarrow \frac{1}{2} \stackrel{?}{=} 1 - \frac{1}{2} = \frac{1}{2} \quad \checkmark$ ;  $P_1$  is true.

(2)  $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^k} = 1 - \frac{1}{2^k}$  is true

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}} \quad ?$$

$$\begin{aligned}
\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^k} + \frac{1}{2^{k+1}} &= 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} \\
&= 1 - \frac{1}{2^k} + \frac{1}{2^k \cdot 2} \\
&= \frac{2^{k+1} - 2 + 1}{2^{k+1}} \\
&= \frac{2^{k+1} - 1}{2^{k+1}} \\
&= \frac{2^{k+1}}{2^{k+1}} - \frac{1}{2^{k+1}} \\
&= 1 - \frac{1}{2^{k+1}} \quad \checkmark \quad P_{k+1} \text{ is also true.}
\end{aligned}$$

∴ By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true:  $\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + \dots + \frac{1}{(3n-2) \cdot (3n+1)} = \frac{n}{3n+1}$

### Solution

(1) For  $n = 1 \Rightarrow \frac{1}{1 \cdot 4} \stackrel{?}{=} \frac{1}{3(1)+1} = \frac{1}{4}$  ✓ ;  $P_1$  is true.

(2)  $\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \dots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1}$  is true

$$\begin{aligned} & \frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \dots + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3(k+1)-2)(3(k+1)+1)} \stackrel{?}{=} \frac{k+1}{3(k+1)+1} \\ & \frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \dots + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3(k+1)-2)(3(k+1)+1)} = \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)} \\ & = \frac{3k^2 + 4k + 1}{(3k+1)(3k+4)} \\ & = \frac{(3k+1)(k+1)}{(3k+1)(3k+3+1)} \\ & = \frac{k+1}{3(k+1)+1} \quad \checkmark \quad P_{k+1} \text{ is also true} \end{aligned}$$

∴ By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true:  $\frac{4}{5} + \frac{4}{5^2} + \frac{4}{5^3} + \dots + \frac{4}{5^n} = 1 - \frac{1}{5^n}$

### Solution

(1) For  $n = 1 \Rightarrow \frac{4}{5} \stackrel{?}{=} 1 - \frac{1}{5} = \frac{4}{5}$  ✓ ;  $P_1$  is true.

(2)  $\frac{4}{5} + \frac{4}{5^2} + \dots + \frac{4}{5^k} = 1 - \frac{1}{5^k}$  is true

$$\begin{aligned} & \frac{4}{5} + \frac{4}{5^2} + \dots + \frac{4}{5^k} + \frac{4}{5^{k+1}} \stackrel{?}{=} 1 - \frac{1}{5^{k+1}} \\ & \frac{4}{5} + \frac{4}{5^2} + \dots + \frac{4}{5^k} + \frac{4}{5^{k+1}} = 1 - \frac{1}{5^k} + \frac{4}{5^{k+1}} \\ & = 1 - \left( \frac{1}{5^k} - \frac{4}{5^{k+1}} \right) \\ & = 1 - \frac{5-4}{5^{k+1}} \\ & = 1 - \frac{1}{5^{k+1}} \quad \checkmark \quad P_{k+1} \text{ is also true.} \end{aligned}$$

∴ By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true:  $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$

### Solution

(1) For  $n = 1 \Rightarrow 1^3 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$  ✓ ;  $P_1$  is true.

(2)  $\frac{4}{5} + \frac{4}{5^2} + \dots + \frac{4}{5^k} = 1 - \frac{1}{5^k}$  is true

$$1^3 + 2^3 + \dots + k^3 + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$$

$$\begin{aligned} 1^3 + 2^3 + \dots + k^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} \\ &= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4} \\ &= \frac{(k+1)^2((k+1)+1)^2}{4} \quad \checkmark \quad P_{k+1} \text{ is also true.} \end{aligned}$$

∴ By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true for every positive integer  $n$ .  $3 + 3^2 + 3^3 + \dots + 3^n = \frac{3}{2}(3^n - 1)$

### Solution

(1) For  $n = 1 \Rightarrow 3 = \frac{3}{2}(3^1 - 1) = \frac{3}{2} \cdot 2 = 3$  ✓ ;  $P_1$  is true.

(2)  $3 + 3^2 + \dots + 3^k = \frac{3}{2}(3^k - 1)$  is true  $\rightarrow$  Is  $3 + 3^2 + \dots + 3^k + 3^{k+1} = \frac{3}{2}(3^{k+1} - 1)$

$$\begin{aligned} 3 + 3^2 + \dots + 3^k + 3^{k+1} &= \frac{3}{2}(3^k - 1) + 3^{k+1} \\ &= \frac{1}{2}3^{k+1} - \frac{3}{2} + 3^{k+1} \\ &= \frac{3}{2}3^{k+1} - \frac{3}{2} \\ &= \frac{3}{2}(3^{k+1} - 1) \quad \checkmark \quad P_{k+1} \text{ is also true.} \end{aligned}$$

∴ By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true:  $x^{2n} + x^{2n-1}y + \dots + xy^{2n-1} + y^{2n} = \frac{x^{2n+1} - y^{2n+1}}{x - y}$

### Solution

$$\begin{aligned} (1) \text{ For } n = 1 \Rightarrow x^2 + xy + y^2 &= \frac{x^3 - y^3}{x - y} \\ &= \frac{(x - y)(x^2 + xy + y^2)}{x - y} \\ &= x^2 + xy + y^2 \quad \checkmark; P_1 \text{ is true.} \end{aligned}$$

$$(2) x^{2k} + x^{2k-1}y + \dots + xy^{2k-1} + y^{2k} = \frac{x^{2k+1} - y^{2k+1}}{x - y} \text{ is true}$$

$$x^{2(k+1)} + x^{2(k+1)-1}y + \dots + xy^{2(k+1)-1} + y^{2(k+1)} = \frac{x^{2(k+1)+1} - y^{2(k+1)+1}}{x - y}$$

$$\begin{aligned} x^{2k+2} + x^{2k+1}y + \dots + xy^{2k+1} + y^{2k+2} &= x^2 \left( x^{2k} + x^{2k-1}y + \dots + y^{2k} \right) + xy^{2k+1} + y^{2k+2} \\ &= x^2 \left( \frac{x^{2k+1} - y^{2k+1}}{x - y} \right) + xy^{2k+1} + y^{2k+2} \\ &= \frac{x^{2k+3} - x^2 y^{2k+1} + x^2 y^{2k+1} + xy^{2k+2} - xy^{2k+2} - y^{2(k+1)+1}}{x - y} \\ &= \frac{x^{2(k+1)+1} - y^{2(k+1)+1}}{x - y} \quad \checkmark \quad P_{k+1} \text{ is also true.} \end{aligned}$$

∴ By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true:  $5 \cdot 6 + 5 \cdot 6^2 + 5 \cdot 6^3 + \dots + 5 \cdot 6^n = 6(6^n - 1)$

### Solution

$$(1) \text{ For } n = 1 \Rightarrow 5 \cdot 6 = 6(6^1 - 1) = 6(5) \quad \checkmark; P_1 \text{ is true.}$$

$$(2) 5 \cdot 6 + 5 \cdot 6^2 + \dots + 5 \cdot 6^k = 6(6^k - 1) \text{ is true}$$

$$\begin{aligned} 5 \cdot 6 + 5 \cdot 6^2 + \dots + 5 \cdot 6^k + 5 \cdot 6^{k+1} &= 6(6^k - 1) + 5 \cdot 6^{k+1} \\ 5 \cdot 6 + 5 \cdot 6^2 + \dots + 5 \cdot 6^k + 5 \cdot 6^{k+1} &= 6(6^k - 1) + 5 \cdot 6^{k+1} \\ &= 6^{k+1} - 6 + 5 \cdot 6^{k+1} \\ &= 6^{k+1}(1 + 5) - 6 \\ &= 6 \cdot 6^{k+1} - 6 \end{aligned}$$



$$= 6(6^{k+1} - 1) \quad \checkmark \quad P_{k+1} \text{ is also true.}$$

∴ By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true:  $7 \cdot 8 + 7 \cdot 8^2 + 7 \cdot 8^3 + \dots + 7 \cdot 8^n = 8(8^n - 1)$

#### Solution

(1) For  $n = 1 \Rightarrow 7 \cdot 8 = 8(8^1 - 1) = 8(7) \quad \checkmark$ ;  $P_1$  is true.

(2)  $7 \cdot 8 + 7 \cdot 8^2 + \dots + 7 \cdot 8^k = 8(8^k - 1)$  is true

$$\begin{aligned} 7 \cdot 8 + 7 \cdot 8^2 + \dots + 7 \cdot 8^k + 7 \cdot 8^{k+1} &= 8(8^k - 1) + 7 \cdot 8^{k+1} \\ 7 \cdot 8 + 7 \cdot 8^2 + \dots + 7 \cdot 8^k + 7 \cdot 8^{k+1} &= 8(8^k - 1) + 7 \cdot 8^{k+1} \\ &= 8^{k+1} - 8 + 7 \cdot 8^{k+1} \\ &= 8^{k+1}(1 + 7) - 8 \\ &= 8 \cdot 8^{k+1} - 8 \\ &= 8(8^{k+1} - 1) \quad \checkmark \quad P_{k+1} \text{ is also true.} \end{aligned}$$

∴ By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true:  $3 + 6 + 9 + \dots + 3n = \frac{3n(n+1)}{2}$

#### Solution

(1) For  $n = 1 \Rightarrow 3 = \frac{3(1)(1+1)}{2} = 3 \quad \checkmark$ ;  $P_1$  is true.

(2)  $3 + 6 + 9 + \dots + 3k = \frac{3k(k+1)}{2}$  is true

$$\begin{aligned} 3 + 6 + 9 + \dots + 3k + 3(k+1) &= \frac{3(k+1)(k+2)}{2} \\ 3 + 6 + 9 + \dots + 3k + 3(k+1) &= \frac{3k(k+1)}{2} + 3(k+1) \\ &= \frac{3k(k+1) + 6(k+1)}{2} \\ &= \frac{(k+1)(3k+6)}{2} \\ &= \frac{3(k+1)(k+2)}{2} \quad \checkmark \quad P_{k+1} \text{ is also true.} \end{aligned}$$

∴ By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true:  $5 + 10 + 15 + \dots + 5n = \frac{5n(n+1)}{2}$

### Solution

(1) For  $n = 1 \Rightarrow 5 = \frac{5(1)(1+1)}{2} = 5 \checkmark$ ;  $P_1$  is true.

(2)  $5 + 10 + 15 + \dots + 5k = \frac{5k(k+1)}{2}$  is true

$$5 + 10 + 15 + \dots + 5k + 5(k+1) = \frac{5(k+1)(k+2)}{2}$$

$$5 + 10 + 15 + \dots + 5k + 5(k+1) = \frac{5k(k+1)}{2} + 5(k+1)$$

$$= \frac{5k(k+1) + 10(k+1)}{2}$$

$$= \frac{(k+1)(5k+10)}{2}$$

$$= \frac{5(k+1)(k+2)}{2} \checkmark$$

$P_{k+1}$  is also true.

$\therefore$  By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true:  $1 + 3 + 5 + \dots + (2n-1) = n^2$

### Solution

(1) For  $n = 1 \Rightarrow 1 = 1^2 = 1 \checkmark$ ;  $P_1$  is true.

(2)  $1 + 3 + 5 + \dots + (2k-1) = k^2$  is true

$$1 + 3 + 5 + \dots + (2k-1) + (2(k+1)-1) = (k+1)^2$$

$$1 + 3 + 5 + \dots + (2k-1) + (2(k+1)-1) = k^2 + 2k + 2 - 1$$

$$= k^2 + 2k + 1$$

$$= (k+1)^2 \checkmark$$

$P_{k+1}$  is also true.

$\therefore$  By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true:  $4 + 7 + 10 + \dots + (3n + 1) = \frac{n(3n + 5)}{2}$

### Solution

(1) For  $n = 1 \Rightarrow 4 = \frac{1(3+5)}{2} = 4 \checkmark$ ;  $P_1$  is true.

(2)  $4 + 7 + 10 + \dots + (3k + 1) = \frac{k(3k + 5)}{2}$  is true

$$4 + 7 + 10 + \dots + (3k + 1) + (3(k + 1) + 1) = \frac{(k + 1)(3(k + 1) + 5)}{2} = \frac{(k + 1)(3k + 8)}{2}$$

$$4 + 7 + 10 + \dots + (3k + 1) + (3k + 4) = \frac{k(3k + 5)}{2} + 3k + 4$$

$$= \frac{3k^2 + 5k + 6k + 8}{2}$$

$$= \frac{3k^2 + 5k + 3k + 3k + 8}{2}$$

$$= \frac{k(3k + 8) + (3k + 8)}{2}$$

$$= \frac{(3k + 8)(k + 1)}{2} \checkmark$$

$P_{k+1}$  is also true.

$\therefore$  By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement by mathematical induction:  $(a^m)^n = a^{mn}$  ( $a$  and  $m$  are constant)

### Solution

➤ For  $n = 1 \Rightarrow (a^m)^1 = a^{m(1)} \rightarrow a^m = a^m \checkmark$ ;  $P_1$  is true.

➤  $(a^m)^k = a^{mk}$  is true

$$(a^m)^{(k+1)} = a^{m(k+1)}$$

$$(a^m)^{(k+1)} = (a^m)^k a^m$$

$$= a^{km} a^m$$

$$= a^{km+m}$$

$$= a^{m(k+1)} \checkmark \quad P_{k+1} \text{ is also true.}$$

$\therefore$  By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true for every positive integer  $n$ .  $n < 2^n$

#### Solution

**Step 1.** For  $n = 1 \Rightarrow 1 < 2^1 \checkmark \Rightarrow P_1$  is true.

**Step 2.** Assume that  $P_k$  is true  $k < 2^k$

We need to prove that  $P_{k+1}$  is true, that is  $k+1 < 2^{k+1}$

$$\begin{aligned} k+1 &< k+k = 2k \\ &< 2 \cdot 2^k \\ &= 2^{k+1} \checkmark \quad P_{k+1} \text{ is also true.} \end{aligned}$$

$\therefore$  By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true for every positive integer  $n$ . 3 is a factor of  $n^3 - n + 3$

#### Solution

➤ For  $n = 1 \Rightarrow 1^3 - 1 + 3 = 3 = 3(1) \checkmark \Rightarrow P_1$  is true.

➤ Assume that  $P_k$  is true 3 is a factor of  $k^3 - k + 3$

We need to prove that  $P_{k+1}$  is true, that is  $(k+1)^3 - (k+1) + 3$

$$\begin{aligned} (k+1)^3 - (k+1) + 3 &= k^3 + 3k^2 + 3k + 1 - k - 1 + 3 \\ &= (k^3 - k + 3) + 3k^2 + 3k \\ &= 3K + 3k^2 + 3k \\ &= 3(K + k^2 + k) \checkmark \quad P_{k+1} \text{ is also true.} \end{aligned}$$

$\therefore$  By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement is true for every positive integer  $n$ . 4 is a factor of  $5^n - 1$

#### Solution

➤ For  $n = 1 \Rightarrow 5^1 - 1 = 4 = 4(1) \checkmark \Rightarrow P_1$  is true.

➤ Assume that  $P_k$  is true 4 is a factor of  $5^k - 1$

We need to prove that  $P_{k+1}$  is true, that is  $5^{k+1} - 1$

$$5^{k+1} - 1 = 5^k 5^1 - 5 + 4$$

$$= 5(5^k - 1) + 4$$

$$= 5(5^k - 1) + 4$$

By the induction hypothesis, 4 is a factor of  $5^k - 1$  and 4 is a factor of 4, so 4 is a factor of the  $(k+1)$  term. ✓

Thus,  $P_{k+1}$  is also true.

∴ By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement by mathematical induction:  $2^n > 2n$  if  $n \geq 3$

#### Solution

- For  $n = 3 \Rightarrow 2^3 \geq 2(3) \Rightarrow 8 \geq 6$  ✓  $\Rightarrow P_1$  is true.
- Assume that  $P_k$  is true:  $2^k > 2k$ ; we need to prove that  $P_{k+1} : 2^{k+1} > 2(k+1)$  is true

$$2^k > 2k$$

$$2^k \cdot 2 > 2k \cdot 2$$

$$2^{k+1} > 4k = 2k + 2k \quad k \geq 3$$

$$> 2k + 2$$

$$= 2(k+1) \quad \checkmark \quad P_{k+1} \text{ is also true.}$$

∴ By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement by mathematical induction: If  $0 < a < 1$ , then  $a^n < a^{n-1}$

#### Solution

- For  $n = 1 \Rightarrow a^1 < a^{1-1} \Rightarrow a < 1$  ✓ since  $0 < a < 1 \Rightarrow P_1$  is true.
- Assume that  $P_k$  is true:  $a^k < a^{k-1}$ ; we need to prove that  $P_{k+1} : a^{k+1} < a^k$  is true

$$a^k < a^{k-1} \rightarrow a^k \cdot a < a^{k-1} \cdot a$$

$$a^{k+1} < a^k \quad \checkmark \quad P_{k+1} \text{ is also true.}$$

∴ By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement by mathematical induction: If  $n \geq 4$ , then  $n! > 2^n$

#### Solution

- For  $n = 4 \Rightarrow 4! > 2^4 \Rightarrow 24 > 16 \checkmark \Rightarrow P_1$  is true.
  - Assume that  $P_k$  is true:  $k! > 2^k$ ; we need to prove that  $P_{k+1} : (k+1)! > 2^{k+1}$  is true
$$\begin{aligned}(k+1)! &= k! \cdot (k+1) \\ &> 2^k (k+1) && k \geq 4 \Rightarrow k+1 > 2 \\ &> 2^k \cdot 2 \\ &= 2^{k+1} \checkmark\end{aligned}$$
Thus,  $P_{k+1}$  is also true.
- $\therefore$  By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement by mathematical induction:  $3^n > 2n+1$  if  $n \geq 2$

#### Solution

- For  $n = 2 \Rightarrow 3^2 > 2(2)+1 \Rightarrow 9 > 5 \checkmark \Rightarrow P_1$  is true.
  - Assume that  $P_k$  is true:  $3^k > 2k+1$ ; we need to prove that  $P_{k+1} : 3^{k+1} > 2(k+1)+1$  is true
$$\begin{aligned}3^k > 2k+1 &\Rightarrow 3^k \cdot 3 > (2k+1) \cdot 3 \\ 3^{k+1} &> 6k+3 \\ &> 2k+2+1 \\ &= 2(k+1)+1 \checkmark\end{aligned}$$
Thus,  $P_{k+1}$  is also true.
- $\therefore$  By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement by mathematical induction:  $2^n > n^2$  for  $n > 4$

#### Solution

- For  $n = 5 \Rightarrow 2^5 > 5^2 \Rightarrow 32 > 25 \checkmark \Rightarrow P_1$  is true.
  - Assume that  $P_k$  is true:  $2^k > k^2$ ; we need to prove that  $P_{k+1} : 2^{k+1} > (k+1)^2$  is true
$$\begin{aligned}2^k > k^2 &\Rightarrow 2^k \cdot 2 > k^2 \cdot 2 \\ 2^{k+1} &> 2k^2 && k < k+1 \Rightarrow k^2 > 2k+1 \\ &> (k+1)^2 \checkmark\end{aligned}$$
Thus,  $P_{k+1}$  is also true.
- $\therefore$  By the mathematical induction, the proof is completed.

### Exercise

Prove that the statement by mathematical induction:  $4^n > n^4$  for  $n \geq 5$

#### Solution

- For  $n = 5 \Rightarrow 4^5 > 5^4 \Rightarrow 1024 > 625 \checkmark \Rightarrow P_1$  is true.
- Assume that  $P_k$  is true:  $4^k > k^4$ ; we need to prove that  $P_{k+1} : 4^{k+1} > (k+1)^4$  is true
 
$$\begin{aligned}
 4^k > k^4 &\Rightarrow 4^k \cdot 4 > k^4 \cdot 4 \\
 4^{k+1} &> 4k^4 & k < k+1 &\Rightarrow k^2 > 2k+1 \\
 &> (k+1)^4 \checkmark & \text{Thus, } P_{k+1} &\text{ is also true}
 \end{aligned}$$

$\therefore$  By the mathematical induction, the proof is completed.

### Exercise

A pile of  $n$  rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must moved, one at a time, to a different peg with no ring ever placed on top of a smaller ring. Find the least number of moves that would be required. Prove your result by mathematical induction.

#### Solution

- With 1 ring, 1 move is required.
- With 2 rings, 3 moves are required  $\Rightarrow 3 = 2 + 1$
- With 3 rings, 7 moves are required  $\Rightarrow 7 = 2^2 + 2 + 1$
- With  $n$  rings,  $2^{n-1} + \dots + 2^2 + 2^1 + 2^0 = 2^n - 1$  moves are required
- For  $n = 1 \Rightarrow 2^0 = 2^1 - 1 = 1 \checkmark \Rightarrow P_1$  is true.
- Assume that  $P_k$  is true:  $2^{k-1} + \dots + 2^2 + 2^1 + 2^0 = 2^k - 1$

$$\begin{aligned}
 2^k + 2^{k-1} + \dots + 2^2 + 2^1 + 1 & \stackrel{?}{=} 2^{k+1} - 1 \\
 2^k + 2^{k-1} + \dots + 2^2 + 2^1 + 1 &= 2^k + 2^k - 1 \\
 &= 2 \cdot 2^k - 1 \\
 &= 2^{k+1} - 1 \checkmark
 \end{aligned}$$

