Solution   *Section* 4.1 – Introduction and Review of Power Series

Exercise

Determine the centre, radius, and interval of convergencae of the power series $\sum_{n=0}^{\infty} \frac{x^{2n}}{\sqrt{n+1}}$

Solution

$$R = \lim_{n\to\infty} \left| \frac{1}{\sqrt{n+2}} \cdot \sqrt{n+1} \right| = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}} = 1$$

The radius of convergence is 1.
The centre of convergence is 0.
The interval of convergence is (–1, 1).
The series does not converge at $x = -1$ or $x = 1$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=0}^{\infty} 3n(x+1)^n$

Solution

$$R = \lim_{n\to\infty} \left| \frac{3n}{3(n+1)} \right| \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{3n}{3n}$$

$$\underline{=1|}$$

The radius of convergence is 1, and the centre of convergence is –1. $(x+1=0)$

$a - R < x < a + R \quad \Rightarrow \quad -1-1 < x < -1+1$

Therefore, the given series convergences absolutely on $(-2, \ 0)$

At $x = -2$, the series is $\sum_{n=0}^{\infty} 3n(-1)^n$ which diverges.

At $x = 0$, the series is $\sum_{n=0}^{\infty} 3n(1)^n = \sum_{n=0}^{\infty} 3n$ which diverges.

Hence, the interval of convergence is $(-2, \ 0)$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} x^n$

Solution

$$R = \lim_{n\to\infty} \left| \frac{(n+1)^4 2^{2n+2}}{n^4 2^{2n}} \right| \qquad\qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= 4 \lim_{n\to\infty} \left| \left(\frac{n+1}{n} \right)^4 \right|$$

$$\underline{= 4}$$

The radius of convergence is 4, and the centre of convergence is 0.

$a - R < x < a + R \quad \Rightarrow \quad -4 < x < 4$, the given series convergences absolutely on $(-4,\ 4)$

At $x = -4$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (-1)^n 2^{2n} = \sum_{n=1}^{\infty} \frac{1}{n^4}$ which converges (p-series).

At $x = 4$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} 2^{2n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ which also converges.

Hence, the interval of convergence is $[-4,\ 4]$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{e^n}{n^3} (4 - x)^n$

Solution

$$R = \lim_{n\to\infty} \left| \frac{e^n}{n^3} \cdot \frac{(n+1)^3}{e^{n+1}} \right| \qquad\qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \frac{1}{e} \lim_{n\to\infty} \left| \left(\frac{n+1}{n} \right)^3 \right|$$

$$\underline{= \frac{1}{e}}$$

The radius of convergence is $\frac{1}{e}$.

The centre of convergence is 4. $(4 - x = 0 \ \Rightarrow \ x = 4)$

$a - R < x < a + R \quad \Rightarrow \quad 4 - \frac{1}{e} < x < 4 + \frac{1}{e}$, which the given series convergences absolutely

At $x = 4 - \frac{1}{e}$, the series is $\sum_{n=1}^{\infty} \frac{e^n}{n^3} \left(\frac{1}{e} \right)^n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ which converges (p-series).

At $x = 4 + \frac{1}{e}$, the series is $\sum_{n=1}^{\infty} \frac{e^n}{n^3}\left(-\frac{1}{e}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ which also converges (p-series).

Hence, the interval of convergence is $\left[4 - \frac{1}{e},\ 4 + \frac{1}{e}\right]$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n}$

Solution

$$\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n} = \sum_{n=1}^{\infty} \frac{4^n\left(x - \frac{1}{4}\right)^n}{n^n}$$

$$R = \lim_{n\to\infty}\left|\frac{4^n}{n^n} \cdot \frac{(n+1)^{n+1}}{4^{n+1}}\right| \qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \frac{1}{4}\lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^n (n+1)\right|$$

$$\underline{= \infty|}$$

The radius of convergence is ∞ .

The centre of convergence is $\underline{x = \frac{1}{4}|}$.

The interval of convergence is the real line $(-\infty,\ \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{1+5^n}{n!}x^n$

Solution

$$R = \lim_{n\to\infty}\left|\frac{1+5^n}{n!} \cdot \frac{(n+1)!}{1+5^{n+1}}\right| \qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1+5^n}{1+5^{n+1}}(n+1)\right|$$

$$\underline{= \infty|}$$

The radius of convergence is ∞ .
The centre of convergence is 0.
The interval of convergence is the real line $(-\infty,\ \infty)$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=e^{2x}, \quad a=0$

Solution

$$f(x)=e^{2x} \quad \to \quad f(0)=e^{2(0)}=1$$

$$f'(x)=2e^{2x} \quad \to \quad f'(0)=2$$

$$f''(x)=4e^{2x} \quad \to \quad f''(0)=4$$

$$f'''(x)=8e^{2x} \quad \to \quad f'''(0)=8$$

$$P_0(x)=f(0)=1$$

$$P_1(x)=f(0)+f'(0)(x-0)=1+2x$$

$$P_2(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2=1+2x+2x^2$$

$$P_3(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3$$

$$=1+2x+2x^2+\frac{4}{3}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\sin x, \quad a=0$

Solution

$$f(x)=\sin x \quad \to \quad f(0)=0$$

$$f'(x)=\cos x \quad \to \quad f'(0)=1$$

$$f''(x)=-\sin x \quad \to \quad f''(0)=0$$

$$f'''(x)=-\cos x \quad \to \quad f'''(0)=-1$$

$$P_0(x)=f(0)=0$$

$$P_1(x)=f(0)+f'(0)(x-0)=x$$

$$P_2(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2=x$$

$$P_3(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3=x-\frac{1}{6}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\ln(1+x), \quad a=0$

Solution

$$f(x)=\ln(1+x) \quad \rightarrow \quad f(0)=0$$

$$f'(x)=\frac{1}{1+x} \quad \rightarrow \quad f'(0)=1$$

$$f''(x)=-\frac{1}{(1+x)^2} \quad \rightarrow \quad f''(0)=-1$$

$$f'''(x)=\frac{2}{(1+x)^3} \quad \rightarrow \quad f'''(0)=2$$

$$P_0(x)=f(0)=0$$

$$P_1(x)=f(0)+f'(0)(x-0)=x$$

$$P_2(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2=x-\frac{1}{2}x^2$$

$$P_3(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3=\underline{x-\frac{1}{2}x^2+\frac{1}{3}x^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\frac{1}{x+2}, \quad a=0$

Solution

$$f(x)=(x+2)^{-1} \quad \rightarrow \quad f(0)=\frac{1}{2}$$

$$f'(x)=-(x+2)^{-2} \quad \rightarrow \quad f'(0)=-\frac{1}{4}$$

$$f''(x)=2(x+2)^{-3} \quad \rightarrow \quad f''(0)=\frac{1}{4}$$

$$f'''(x)=-6(x+2)^{-4} \quad \rightarrow \quad f'''(0)=-\frac{3}{8}$$

$$P_0(x)=f(0)=\frac{1}{2}$$

$$P_1(x)=f(0)+f'(0)(x-0)=\frac{1}{2}-\frac{1}{4}x$$

$$P_2(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2=\frac{1}{2}-\frac{1}{4}x+\frac{1}{8}x^2$$

$$P_3(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3=\underline{\frac{1}{2}-\frac{1}{4}x+\frac{1}{8}x^2-\frac{1}{16}x^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\sqrt{1-x}, \quad a=0$

Solution

$$f(x)=(1-x)^{1/2} \quad \to \quad f(0)=1$$

$$f'(x)=-\tfrac{1}{2}(1-x)^{-1/2} \quad \to \quad f(0)=-\tfrac{1}{2}$$

$$f''(x)=-\tfrac{1}{4}(1-x)^{-3/2} \quad \to \quad f(0)=-\tfrac{1}{4}$$

$$f'''(x)=-\tfrac{3}{8}(1-x)^{-5/2} \quad \to \quad f(0)=-\tfrac{3}{8}$$

$$P_0(x)=f(0)=1$$

$$P_1(x)=f(0)+f'(0)(x-0)=1-\tfrac{1}{2}x$$

$$P_2(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2=1-\tfrac{1}{2}x-\tfrac{1}{8}x^2$$

$$P_3(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3=\boxed{1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=x^3, \quad a=1$

Solution

$$f(x)=x^3 \quad \to \quad f(1)=1$$

$$f'(x)=3x^2 \quad \to \quad f'(1)=3$$

$$f''(x)=6x \quad \to \quad f''(1)=6$$

$$f'''(x)=6 \quad \to \quad f'''(1)=6$$

$$P_0(x)\ \boxed{=1} \qquad\qquad P_0(x)=f(a)$$

$$P_1(x)\ \boxed{=1+3(x-1)} \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)\ \boxed{=1+3(x-1)+3(x-1)^2} \qquad\qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)\ \boxed{=1+3(x-1)+3(x-1)^2+(x-1)^3} \qquad P_3(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=8\sqrt{x}, \quad a=1$

Solution

$$f(x)=8x^{1/2} \quad \to \quad f(1)=8$$

$$f'(x)=4x^{-1/2} \quad \to \quad f'(1)=4$$

$$f''(x)=-2x^{-3/2} \quad \to \quad f''(1)=-2$$

$$f'''(x)=3x^{-5/2} \quad \to \quad f'''(1)=3$$

$$P_0(x)=8 \qquad\qquad P_0(x)=f(a)$$

$$P_1(x)=8+4(x-1) \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)=8+4(x-1)-(x-1)^2 \qquad\qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)=8+4(x-1)-(x-1)^2+3(x-1)^3 \qquad\qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\sin x, \quad a=\frac{\pi}{4}$

Solution

$$f(x)=\sin x \quad \to \quad f\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

$$f'(x)=\cos x \quad \to \quad f'\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

$$f''(x)=-\sin x \quad \to \quad f''\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$

$$f'''(x)=-\cos x \quad \to \quad f'''\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$

$$P_0(x)=\frac{\sqrt{2}}{2} \qquad\qquad P_0(x)=f(a)$$

$$P_1(x)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 \qquad\qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2-\frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3 \qquad\qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\cos x, \quad a=\frac{\pi}{6}$

Solution

$$f(x)=\cos x \quad \to \quad f\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$

$$f'(x)=-\sin x \quad \to \quad f'\left(\frac{\pi}{6}\right)=-\frac{1}{2}$$

$$f''(x)=-\cos x \quad \to \quad f''\left(\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{2}$$

$$f'''(x)=\sin x \quad \to \quad f'''\left(\frac{\pi}{6}\right)=\frac{1}{2}$$

$$P_0(x)\ \boxed{=\frac{\sqrt{3}}{2}} \qquad\qquad P_0(x)=f(a)$$

$$P_1(x)\ \boxed{=\frac{\sqrt{2}}{2}-\frac{1}{2}\left(x-\frac{\pi}{6}\right)} \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)\ \boxed{=\frac{\sqrt{2}}{2}-\frac{1}{2}\left(x-\frac{\pi}{6}\right)-\frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)^2} \qquad\qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)\ \boxed{=\frac{\sqrt{3}}{2}-\frac{1}{2}\left(x-\frac{\pi}{6}\right)-\frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)^2+\frac{1}{12}\left(x-\frac{\pi}{6}\right)^3} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\sqrt{x}, \quad a=9$

Solution

$$f(x)=x^{1/2} \quad \to \quad f(9)=3$$

$$f'(x)=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}} \quad \to \quad f'(9)=\frac{1}{6}$$

$$f''(x)=-\frac{1}{4}x^{-3/2}=-\frac{1}{4x\sqrt{x}} \quad \to \quad f''(9)=-\frac{1}{4\times 3^3}$$

$$f'''(x)=\frac{3}{8}x^{-5/2}=\frac{3}{8x^2\sqrt{x}} \quad \to \quad f'''(9)=\frac{1}{2^3\times 3^4}$$

$$P_0(x)\ \boxed{=3} \qquad\qquad P_0(x)=f(a)$$

$$P_1(x)\ \boxed{=3+\frac{1}{6}(x-9)} \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)\ \boxed{=3+\frac{1}{2\cdot 3}(x-9)-\frac{1}{2^3\cdot 3^3}(x-9)^2} \qquad\qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)\ \boxed{=3+\frac{1}{2\cdot 3}(x-9)-\frac{1}{2^2\cdot 3^3}(x-9)^2+\frac{1}{2^4\cdot 3^5}(x-9)^3} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\sqrt[3]{x}, \quad a=8$

Solution

$$f(x)=x^{1/3} \quad \to \quad f(8)=2$$

$$f'(x)=\frac{1}{3}x^{-2/3}=\frac{1}{3x^{2/3}} \quad \to \quad f'(8)=\frac{1}{2^2\times 3}$$

$$f''(x)=-\frac{2}{9}x^{-5/3}=-\frac{2}{3^2 x^{5/3}} \quad \to \quad f''(8)=-\frac{1}{3^2\times 2^4}$$

$$f'''(x)=\frac{10}{3^3}x^{-8/3}=\frac{2\cdot 5}{3^3 x^{8/3}} \quad \to \quad f'''(8)=\frac{5}{2^7\times 3^3}$$

$$P_0(x)\ \underline{\left.=2\right|} \qquad\qquad P_0(x)=f(a)$$

$$P_1(x)\ \underline{\left.=2+\frac{1}{2^2\cdot 3}(x-8)\right|} \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)\ \underline{\left.=2+\frac{1}{2^2\cdot 3}(x-8)-\frac{1}{2^5\cdot 3^2}(x-8)^2\right|} \qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)\ \underline{\left.=2+\frac{1}{2^2\cdot 3}(x-8)-\frac{1}{2^5\cdot 3^2}(x-8)^2+\frac{1}{2^8\cdot 3^4}(x-8)^3\right|} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\ln x, \quad a=e$

Solution

$$f(x)=\ln x \quad \to \quad f(e)=1$$

$$f'(x)=\frac{1}{x} \quad \to \quad f'(e)=\frac{1}{e}$$

$$f''(x)=-\frac{1}{x^2} \quad \to \quad f''(e)=-\frac{1}{e^2}$$

$$f'''(x)=\frac{2}{x^3} \quad \to \quad f'''(e)=\frac{2}{e^3}$$

$$P_0(x)\ \underline{\left.=1\right|} \qquad\qquad P_0(x)=f(a)$$

$$P_1(x)\ \underline{\left.=1+\frac{1}{e}(x-e)\right|} \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)\ \underline{\left.=1+\frac{1}{e}(x-e)-\frac{1}{2e^2}(x-e)^2\right|} \qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)\ \underline{\left.=1+\frac{1}{e}(x-e)-\frac{1}{2e^2}(x-e)^2+\frac{1}{3e^3}(x-e)^3\right|} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\sqrt[4]{x}, \quad a=8$

Solution

$$f(x)=x^{1/4} \quad \to \quad f(8)=\sqrt[4]{8}$$

$$f'(x)=\frac{1}{4}x^{-3/4}=\frac{1}{4x^{3/4}} \quad \to \quad f'\left(8=2^3\right)=\frac{1}{2^2\times 2^{9/4}}=\frac{1}{2^4\sqrt[4]{2}}$$

$$f''(x)=-\frac{3}{16}x^{-7/4}=-\frac{3}{2^4x^{7/4}} \quad \to \quad f''(8)=-\frac{3}{2^4\,2^{21/4}}=-\frac{3}{2^9\sqrt[4]{2}}$$

$$f'''(x)=\frac{21}{2^6}x^{-11/4}=\frac{21}{2^6x^{11/4}} \quad \to \quad f'''(8)=\frac{21}{2^6\times 2^{33/4}}=\frac{21}{2^{14}\sqrt[4]{2}}$$

$$P_0(x)\ \boxed{=\sqrt[4]{8}} \qquad P_0(x)=f(a)$$

$$P_1(x)\ \boxed{=\sqrt[4]{8}+\frac{1}{2^4\cdot\sqrt[4]{2}}(x-8)} \qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)\ \boxed{=\sqrt[4]{8}+\frac{1}{2^4\cdot\sqrt[4]{2}}(x-8)-\frac{3}{2^{10}\cdot\sqrt[4]{2}}(x-8)^2} \qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)\ \boxed{=\sqrt[4]{8}+\frac{1}{2^4\cdot\sqrt[4]{2}}(x-8)-\frac{3}{2^{10}\cdot\sqrt[4]{2}}(x-8)^2+\frac{7}{2^{15}\cdot\sqrt[4]{2}}(x-8)^3} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\tan^{-1}x+x^2+1, \quad a=1$

Solution

$$f(x)=\tan^{-1}x+x^2+1 \quad \to \quad f(1)=\frac{\pi}{4}+2$$

$$f'(x)=\frac{1}{x^2+1}+2x \quad \to \quad f'(1)=\frac{5}{2}$$

$$f''(x)=-\frac{2x}{\left(x^2+1\right)^2}+2 \quad \to \quad f''(1)=-\frac{1}{2}+2=\frac{3}{2}$$

$$f'''(x)=-\frac{2x^2+2-8x^2}{\left(x^2+1\right)^3}=-\frac{2-2x^2}{\left(x^2+1\right)^3} \quad \to \quad f'''(1)=0 \qquad \left(U^nV^m\right)'=U^{n-1}V^{m-1}\left(nU'V+mUV'\right)$$

$$P_0(x)\ \boxed{=\frac{\pi}{4}+2} \qquad P_0(x)=f(a)$$

$$P_1(x)\ \boxed{=\frac{\pi}{4}+2+\frac{5}{2}(x-1)} \qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)\ \boxed{=\frac{\pi}{4}+2+\frac{5}{2}(x-1)-\frac{3}{4}(x-1)^2} \qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)\ \boxed{=\frac{\pi}{4}+2+\frac{5}{2}(x-1)-\frac{3}{4}(x-1)^2} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=e^x, \quad a=\ln 2$

Solution

$$f(x)=e^x \quad \rightarrow \quad f(\ln 2)=2$$

$$f'(x)=e^x \quad \rightarrow \quad f'(\ln 2)=2$$

$$f''(x)=e^x \quad \rightarrow \quad f''(\ln 2)=2$$

$$f'''(x)=e^x \quad \rightarrow \quad f'''(\ln 2)=2$$

$$P_0(x)=2 \qquad\qquad P_0(x)=f(a)$$

$$P_1(x)=2+2(x-\ln 2) \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)=2+2(x-\ln 2)+(x-\ln 2)^2 \qquad\qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)=2+2(x-\ln 2)+(x-\ln 2)^2+\frac{1}{3}(x-\ln 2)^3 \qquad\qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x)=e^{4x}, \quad n=4$

Solution

$$f(x)=e^{4x} \quad \rightarrow \quad f(0)=1$$

$$f'(x)=4e^{4x} \quad \rightarrow \quad f'(0)=4$$

$$f''(x)=16e^{4x} \quad \rightarrow \quad f''(0)=16$$

$$f'''(x)=64e^{4x} \quad \rightarrow \quad f'''(0)=64$$

$$f^{(4)}(x)=256e^{4x} \quad \rightarrow \quad f^{(4)}(0)=256$$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x)=1+4x+8x^2+\frac{32}{3}x^3+\frac{32}{3}x^4$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x)=e^{-x}, \quad n=5$

Solution

$$f(x)=e^{-x} \quad \rightarrow \quad f(0)=1$$

$$f'(x)=-e^{-x} \quad \rightarrow \quad f'(0)=-1$$

$$f''(x) = e^{-x} \quad \rightarrow \quad f''(0) = 1$$

$$f'''(x) = -e^{-x} \quad \rightarrow \quad f'''(0) = -1$$

$$f^{(4)}(x) = e^{-x} \quad \rightarrow \quad f^{(4)}(0) = 1$$

$$f^{(5)}(x) = -e^{-x} \quad \rightarrow \quad f^{(5)}(0) = -1$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$P_5(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x) = e^{-x/2}, \quad n = 4$

Solution

$$f(x) = e^{-x/2} \quad \rightarrow \quad f(0) = 1$$

$$f'(x) = -\frac{1}{2}e^{-x/2} \quad \rightarrow \quad f'(0) = -\frac{1}{2}$$

$$f''(x) = \frac{1}{4}e^{-x/2} \quad \rightarrow \quad f''(0) = \frac{1}{4}$$

$$f'''(x) = -\frac{1}{8}e^{-x/2} \quad \rightarrow \quad f'''(0) = -\frac{1}{8}$$

$$f^{(4)}(x) = \frac{1}{16}e^{-x/2} \quad \rightarrow \quad f^{(4)}(0) = \frac{1}{16}$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x) = 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x) = e^{x/3}, \quad n = 4$

Solution

$$f(x) = e^{x/3} \quad \rightarrow \quad f(0) = 1$$

$$f'(x) = \frac{1}{3}e^{x/3} \quad \rightarrow \quad f'(0) = \frac{1}{3}$$

$$f''(x) = \frac{1}{9}e^{x/3} \quad \rightarrow \quad f''(0) = \frac{1}{9}$$

$$f'''(x) = \frac{1}{27}e^{x/3} \quad \rightarrow \quad f'''(0) = \frac{1}{27}$$

$$f^{(4)}(x) = \frac{1}{81}e^{x/3} \quad \rightarrow \quad f^{(4)}(0) = \frac{1}{81}$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x) = 1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \frac{1}{1944}x^4}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x) = \sin x, \quad n = 5$

Solution

$$f(x) = \sin x \quad \to \quad f(0) = 0$$

$$f'(x) = \cos x \quad \to \quad f'(0) = 1$$

$$f''(x) = -\sin x \quad \to \quad f''(0) = 0$$

$$f'''(x) = -\cos x \quad \to \quad f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \quad \to \quad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \cos x \quad \to \quad f^{(5)}(0) = 1$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$\boxed{P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x) = \cos \pi x, \quad n = 4$

Solution

$$f(x) = \cos \pi x \quad \to \quad f(0) = 1$$

$$f'(x) = -\pi \sin \pi x \quad \to \quad f'(0) = 0$$

$$f''(x) = -\pi^2 \cos \pi x \quad \to \quad f''(0) = -\pi^2$$

$$f'''(x) = \pi^3 \sin \pi x \quad \to \quad f'''(0) = 0$$

$$f^{(4)}(x) = \pi^4 \cos \pi x \quad \to \quad f^{(4)}(0) = \pi^4$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x) = 1 - \frac{\pi^2}{2}x^2 + \frac{\pi^4}{24}x^4}$$

Exercise

Find the nth Maclaurin polynomial for the function $f(x) = xe^x, \quad n = 4$

Solution

$$f(x) = xe^x \quad \to \quad f(0) = 0$$

$$f'(x) = e^x + xe^x \quad \to \quad f'(0) = 1$$

$$f''(x) = 2e^x + xe^x \quad \to \quad f''(0) = 2$$

$$f'''(x) = 3e^x + xe^x \quad \to \quad f'''(0) = 3$$

$$f^{(4)}(x) = 4e^x + xe^x \quad \to \quad f^{(4)}(0) = 4$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$$

Exercise

Find the nth Maclaurin polynomial for the function $f(x) = x^2e^{-x}, \quad n = 4$

Solution

$$f(x) = x^2e^{-x} \quad \to \quad f(0) = 0$$

$$f'(x) = 2xe^{-x} - x^2e^{-x} \quad \to \quad f'(0) = 0$$

$$f''(x) = 2e^{-x} - 4xe^x + x^2e^{-x} \quad \to \quad f''(0) = 2$$

$$f'''(x) = -6e^{-x} + 6xe^{-x} - x^2e^{-x} \quad \to \quad f'''(0) = -6$$

$$f^{(4)}(x) = 12e^{-x} - 8xe^{-x} + x^2e^{-x} \quad \to \quad f^{(4)}(0) = 12$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x) = x^2 - x^3 + \frac{1}{2}x^4$$

Exercise

Find the nth Maclaurin polynomial for the function $f(x) = \frac{1}{x+1}, \quad n = 5$

Solution

$$f(x) = \frac{1}{x+1} \quad \to \quad f(0) = 1$$

$$f'(x) = -(x+1)^{-2} \quad \to \quad f'(0) = -1$$

$$f''(x) = 2(x+1)^{-3} \quad \to \quad f''(0) = 2$$

$$f'''(x) = -6(x+1)^{-4} \quad \to \quad f'''(0) = -6$$

$$f^{(4)}(x) = 24(x+1)^{-5} \quad \to \quad f^{(4)}(0) = 24$$

$$f^{(5)}(x) = -120(x+1)^{-6} \quad \to \quad f^{(5)}(0) = -120$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$P_5(x) = 1 - x + x^2 - x^3 + x^4 - x^5$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x) = \frac{x}{x+1}, \quad n = 4$

Solution

$$f(x) = \frac{x}{x+1} = 1 - \frac{1}{x+1} \quad \to \quad f(0) = 0$$

$$f'(x) = (x+1)^{-2} \quad \to \quad f'(0) = 1$$

$$f''(x) = -2(x+1)^{-3} \quad \to \quad f''(0) = -2$$

$$f'''(x) = 6(x+1)^{-4} \quad \to \quad f'''(0) = 6$$

$$f^{(4)}(x) = -24(x+1)^{-5} \quad \to \quad f^{(4)}(0) = -24$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x) = x - x^2 + x^3 - x^4$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x) = \sec x, \quad n = 2$

Solution

$$f(x) = \sec x \quad \to \quad f(0) = 1$$

$$f'(x) = \sec x \tan x \quad \to \quad f'(0) = 0$$

$$f''(x) = \sec x \tan^2 x + \sec^3 x \quad \to \quad f''(0) = 1$$

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

$$P_2(x) = 1 + \frac{1}{2}x^2$$

Exercise

Find the nth Maclaurin polynomial for the function $f(x) = \tan x, \quad n = 3$

Solution

$$f(x) = \tan x \quad \to \quad f(0) = 0$$

$$f'(x) = \sec^2 x \quad \to \quad f'(0) = 1$$

$$f''(x) = 2\sec^2 x \tan x \quad \to \quad f''(0) = 0$$

$$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x \quad \to \quad f'''(0) = 2$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3$$

$$\underline{P_4(x) = x + \frac{1}{3}x^3}$$

Exercise

Find the Maclaurin series for: xe^x

Solution

$$f(x) = xe^x \quad \to \quad f(0) = 0$$

$$f'(x) = e^x + xe^x \quad \to \quad f'(0) = 1$$

$$f''(x) = 2e^x + xe^x \quad \to \quad f''(0) = 2$$

$$f'''(x) = 3e^x + xe^x \quad \to \quad f'''(0) = 3$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$f^{(n)}(x) = ne^x + xe^x \quad \to \quad f^{(n)}(0) = n$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$xe^x = x + x^2 + \frac{1}{2}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{(n-1)!}x^n$$

Exercise

Find the Maclaurin series for: $5\cos \pi x$

Solution

$$f(x) = 5\cos \pi x \quad \to \quad f(0) = 5$$

$$f'(x) = -5\pi \sin \pi x \quad \to \quad f'(0) = 0$$

$$f''(x) = -5\pi^2 \cos \pi x \quad \to \quad f''(0) = -5\pi^2$$

$$f'''(x) = 5\pi^3 \sin \pi x \quad \to \quad f'''(0) = 0$$

$$5\cos \pi x = 5 - \frac{5\pi^2 x^2}{2!} + \frac{5\pi^4 x^4}{4!} - \frac{5\pi^6 x^6}{6!} + \cdots = 5\sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!}$$

Exercise

Find the Maclaurin series for: $\dfrac{x^2}{x+1}$

Solution

$$f(x) = \frac{x^2}{x+1} \quad \to \quad f(0) = 0$$

$$f'(x) = \frac{2x(x+1) - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2} \quad \to \quad f'(0) = 0$$

$$f''(x) = \frac{(2x+2)(x+1)^2 - 2(x+1)\left(x^2 + 2x\right)}{(x+1)^4}$$

$$= \frac{2x^2 + 4x + 2 - 2x^2 - 4x}{(x+1)^3}$$

$$= \frac{2}{(x+1)^3} \quad \to \quad f''(0) = 2$$

$$f'''(x) = \frac{-6}{(x+1)^4} \quad \to \quad f'''(0) = -6$$

$$\frac{x^2}{x+1} = \frac{2}{2!}x^2 - \frac{6x^3}{3!} + \frac{24x^4}{4!} - \cdots = x^2 - x^3 + x^4 - \cdots = \sum_{n=2}^{\infty} (-1)^n x^n$$

Exercise

Find the Maclaurin series for: e^{3x+1}

Solution

$$e^{3x+1} = e \cdot e^{3x}$$

$$= e\left(\sum_{n=0}^{\infty} \frac{(3x)^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty} \frac{e3^n x^n}{n!} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\cos\left(2x^3\right)$

Solution

$$\begin{aligned}
\cos\left(2x^3\right) &= 1 - \frac{\left(2x^3\right)^2}{2!} + \frac{\left(2x^3\right)^4}{4!} - \frac{\left(2x^3\right)^6}{6!} + \cdots \\
&= 1 - \frac{2^2 x^3}{2!} + \frac{2^4 x^{12}}{4!} - \frac{2^6 x^{18}}{6!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} x^{6n} \quad (\text{for all } x)
\end{aligned}$$

Exercise

Find the Maclaurin series for: $\cos(2x - \pi)$

Solution

$$\begin{aligned}
\cos(2x - \pi) &= \cos(2x)\cos\pi + \sin(2x)\sin\pi \\
&= -\cos(2x) \\
&= -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 4^n}{(2n)!} x^{2n} \quad (\text{for all } x)
\end{aligned}$$

Exercise

Find the Maclaurin series for: $x^2 \sin\left(\frac{x}{3}\right)$

Solution

$$\begin{aligned}
x^2 \sin\left(\frac{x}{3}\right) &= x^2 \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{x}{3}\right)^{2n+1}}{(2n+1)!} \\
&= x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{3^{2n+1}(2n+1)!}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{3^{2n+1}(2n+1)!} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\cos^2\left(\frac{x}{2}\right)$

Solution

$$\cos^2\left(\frac{x}{2}\right) = \frac{1}{2}(1+\cos x)$$

$$= \frac{1}{2}\left(1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\right)$$

$$= \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$= 1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\sin x \cos x$

Solution

$$\sin x \cos x = \frac{1}{2}\sin(2x)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2x)^{2n+1}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 2^{2n+1} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{(2n+1)!} x^{2n+1} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\tan^{-1}\left(5x^2\right)$

Solution

$$\tan^{-1}\left(5x^2\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(5x^2\right)^{2n+1} \qquad 5x^2\le 1 \rightarrow x^2\le\frac{1}{5} \;\Rightarrow\; -\frac{1}{\sqrt{5}}\le x\le\frac{1}{\sqrt{5}}$$

$$=\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n 5^{2n+1}}{2n+1}x^{4n+2}} \quad \left(for\quad -\frac{1}{\sqrt{5}}\le x\le\frac{1}{\sqrt{5}}\right)$$

Exercise

Find the Maclaurin series for: $\ln\left(2+x^2\right)$

Solution

$$\ln\left(2+x^2\right)=\ln 2\left(1+\frac{x^2}{2}\right)$$

$$=\ln 2+\ln\left(1+\frac{x^2}{2}\right)$$

$$=\ln 2+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{x^2}{2}\right)^n \qquad \frac{x^2}{2}\le 1 \rightarrow x^2\le 2 \;\Rightarrow\; -\sqrt{2}\le x\le\sqrt{2}$$

$$\boxed{=\ln 2+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}\frac{x^{2n}}{2^n}} \quad \left(for\quad -\sqrt{2}\le x\le\sqrt{2}\right)$$

Exercise

Find the Maclaurin series for: $\dfrac{1+x^3}{1+x^2}$

Solution

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n=1-x+x^2-x^3+\cdots$$

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots$$

$$\frac{1+x^3}{1+x^2}=\left(1+x^3\right)\left(1-x^2+x^4-x^6+\cdots\right)$$

$$=1-x^2+x^4-x^6+\cdots+x^3-x^5+x^7-x^9+\cdots$$

$$=1-x^2+x^3+x^4-x^5-x^6+x^7+x^8-x^9-\cdots$$

$$\boxed{=1-x^2+\sum_{n=2}^{\infty}(-1)^n\left(x^{2n-1}+x^{2n}\right)} \quad \left(for\quad |x|<1\right)$$

Exercise

Find the Maclaurin series for: $\ln\frac{1+x}{1-x}$

Solution

$$\ln\frac{1+x}{1-x}=\ln(1+x)-\ln(1-x)$$

$$= x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots-\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots\right)=2x+2\frac{x^3}{3}+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}+\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$$

$$=\sum_{n=0}^{\infty}\left((-1)^n+1\right)\frac{x^{n+1}}{n+1}$$

$$=2\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2n-1} \qquad (-1<x<1)$$

Exercise

Find the Maclaurin series for: $\frac{e^{2x^2}-1}{x^2}$

Solution

$$\frac{e^{2x^2}-1}{x^2}=\frac{1}{x^2}\left(e^{2x^2}-1\right)$$

$$=\frac{1}{x^2}\left(1+2x^2+\frac{\left(2x^2\right)^2}{2!}+\frac{\left(2x^2\right)^3}{3!}+\cdots-1\right)$$

$$=\frac{1}{x^2}\left(2x^2+\frac{2^2x^4}{2!}+\frac{2^3x^6}{3!}+\cdots\right)$$

$$=2+\frac{2^2x^2}{2!}+\frac{2^3x^4}{3!}+\frac{2^4x^6}{4!}+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{2^{n+1}}{(n+1)!}x^{2n} \qquad (\text{for all } x\neq 0)$$

Exercise

Find the Maclaurin series for: $\cosh x-\cos x$

Solution

$$\cosh x - \cos x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = 2\frac{x^2}{2!} + 2\frac{x^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} \left(1 - (-1)^n\right) \frac{x^{2n}}{(2n)!}$$

$$\boxed{= 2\sum_{n=0}^{\infty} \frac{x^{4n+2}}{(4n+2)!}} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\sinh x - \sin x$

Solution

$$\sinh x - \sin x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \left(1 - (-1)^n\right) \frac{x^{2n+1}}{(2n+1)!}$$

$$\boxed{= 2\sum_{n=0}^{\infty} \frac{x^{4n+3}}{(4n+3)!}} \quad (\text{for all } x)$$

Exercise

Finding Taylor and Maclaurin Series generated by f *at* $x = a$: $f(x) = x^3 - 2x + 4, \quad a = 2$

Solution

$f(x) = x^3 - 2x + 4 \quad \rightarrow \quad f(2) = 8$

$f'(x) = 3x^2 - 2 \quad \rightarrow \quad f'(2) = 10$

$f''(x) = 6x \quad \rightarrow \quad f''(2) = 12$

$f'''(x) = 6 \quad \rightarrow \quad f'''(2) = 6$

$f^{(n)}(x) = 0 \quad \rightarrow \quad f^{(n)}(2) = 0 \quad (n > 3)$

$$P_n(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \cdots$$

$$\boxed{x^3 - 2x + 4 = 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3}$$

Exercise

Finding Taylor and Maclaurin Series generated by f *at* $x = a$: $f(x) = 2x^3 + x^2 + 3x - 8, \quad a = 1$

Solution

$$f(x) = 2x^3 + x^2 + 3x - 8 \quad \to \quad f(1) = -2$$

$$f'(x) = 6x^2 + 2x + 3 \quad \to \quad f'(1) = 11$$

$$f''(x) = 12x + 2 \quad \to \quad f''(1) = 14$$

$$f'''(x) = 12 \quad \to \quad f'''(1) = 12$$

$$f^{(n)}(x) = 0 \quad \to \quad f^{(n)}(1) = 0 \quad (n \ge 4)$$

$$P_n(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots$$

$$\underline{2x^3 + x^2 + 3x - 8 = -2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3}$$

Exercise

Finding Taylor and Maclaurin Series generated by f *at* $x = a$:

$$f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2, \quad a = -1$$

Solution

$$f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2 \quad \to \quad f(-1) = -7$$

$$f'(x) = 15x^4 - 4x^3 + 6x^2 + 2x \quad \to \quad f'(-1) = 23$$

$$f''(x) = 60x^3 - 12x^2 + 12x + 2 \quad \to \quad f''(-1) = -82$$

$$f'''(x) = 180x^2 - 24x + 12 \quad \to \quad f'''(-1) = 216$$

$$f^{(4)}(x) = 360x - 24 \quad \to \quad f^{(4)}(-1) = -384$$

$$f^{(5)}(x) = 360 \quad \to \quad f^{(5)}(-1) = 360$$

$$f^{(n)}(x) = 0 \quad \to \quad f^{(n)}(-1) = 0 \quad (n \ge 6)$$

$$P_n(x) = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3 + \frac{f^{(4)}(-1)}{4!}(x+1)^2 + \frac{f^{(4)}(-1)}{5!}(x+1)^3$$

$$3x^5 - x^4 + 2x^3 + x^2 - 2 = -7 + 23(x+1) - \frac{82}{2!}(x+1)^2 + \frac{216}{3!}(x+1)^3 - \frac{384}{4!}(x+1)^4 + \frac{360}{5!}(x+1)^3$$

$$\underline{= -7 + 23(x+1) - 41(x+1)^2 + 36(x+1)^3 - 16(x+1)^4 + 3(x+1)^3}$$

Exercise

Finding Taylor and Maclaurin Series generated by f *at* $x = a$: $\quad f(x)=\cos\left(2x+\frac{\pi}{2}\right), \quad a=\frac{\pi}{4}$

Solution

$$f(x)=\cos\left(2x+\frac{\pi}{2}\right) \quad\rightarrow\quad f\left(\frac{\pi}{4}\right)=-1$$

$$f'(x)=-2\sin\left(2x+\frac{\pi}{2}\right) \quad\rightarrow\quad f'\left(\frac{\pi}{4}\right)=0$$

$$f''(x)=-4\cos\left(2x+\frac{\pi}{2}\right) \quad\rightarrow\quad f''\left(\frac{\pi}{4}\right)=4$$

$$f'''(x)=8\sin\left(2x+\frac{\pi}{2}\right) \quad\rightarrow\quad f'''\left(\frac{\pi}{4}\right)=0$$

$$f^{(4)}(x)=16\cos\left(2x+\frac{\pi}{2}\right) \quad\rightarrow\quad f^{(4)}\left(\frac{\pi}{4}\right)=-16$$

$$f^{(5)}(x)=-32\sin\left(2x+\frac{\pi}{2}\right) \quad\rightarrow\quad f^{(5)}\left(\frac{\pi}{4}\right)=0$$

$$\rightarrow\quad f^{(2n)}\left(\frac{\pi}{4}\right)=(-1)^n 2^{2n}$$

$$\begin{aligned}\cos\left(2x+\frac{\pi}{2}\right)&=-1+\frac{4}{2!}\left(x-\frac{\pi}{4}\right)^2-\frac{16}{4!}\left(x-\frac{\pi}{4}\right)^4+\cdots\\&=-1+2\left(x-\frac{\pi}{4}\right)^2-\frac{2}{3}\left(x-\frac{\pi}{4}\right)^4+\cdots\\&=\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(2n)!}\left(x-\frac{\pi}{4}\right)^{2n}\end{aligned}$$

Solution *Section* 4.2 – Series Solutions near Ordinary Points

Exercise

Find a power series solution. $y' = 3y$

Solution

The equation $y' = 3y$ is separable with solution

$$\frac{dy}{dx} = 3y \;\Rightarrow\; \frac{dy}{y} = 3dx \qquad \to \qquad \boxed{y = Ce^{3x}}$$

$$\boxed{\ln(y) = 3x + C}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y' - 3y = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 3\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1) a_{n+1} - 3a_n\right] x^n = 0$$

$$(n+1) a_{n+1} - 3a_n = 0 \;\Rightarrow\; a_{n+1} = \frac{3a_n}{n+1}; \quad n \ge 0$$

With $y(0) = 3a_0$

$$a_1 = 3a_0$$

$$a_2 = \frac{3}{2} a_1 = \frac{3\cdot 3}{2} a_0$$

$$a_3 = \frac{3}{3} a_2 = \frac{3\cdot 3\cdot 3}{2\cdot 3} a_0$$

$$a_4 = \frac{3}{4} a_3 = \frac{3\cdot 3\cdot 3\cdot 3}{2\cdot 3\cdot 4} a_0$$

$$\vdots \qquad \vdots$$

$$\boxed{a_n = \frac{3^n}{n!} a_0}$$

$$y(x)=\sum_{n=0}^{\infty}\frac{3^n}{n!}a_0x^n$$

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$=a_0\sum_{n=0}^{\infty}\frac{(3x)^n}{n!}$$

$$\boxed{=a_0e^{3x}} \quad \checkmark$$

Exercise

Find a power series solution. $y'=4y$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}$$

$$y'=4y$$

$$\sum_{n=1}^{\infty}na_nx^{n-1}=4\sum_{n=0}^{\infty}a_nx^n$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=\sum_{n=0}^{\infty}4a_nx^n$$

$$(n+1)a_{n+1}x^n=4a_nx^n$$

$$(n+1)a_{n+1}=4a_n$$

$$\boxed{a_{n+1}=\frac{4}{n+1}a_n}$$

$$n=0 \quad \to \quad a_1=4a_0$$

$$n=1 \quad \to \quad a_2=\frac{4}{2}a_1=\frac{4^2}{2!}a_0$$

$$n=2 \quad \to \quad a_3=\frac{4}{3}a_2=\frac{4^3}{3!}a_0$$

$$n=3 \quad \to \quad a_4=\frac{4}{4}a_3=\frac{4^4}{4!}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = \frac{4^n}{n!}a_0}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{4^n}{n!}a_0 x$$

$$= a_0\left(1 + 4x + \frac{4^2}{2!}x^2 + \frac{4^3}{3!}x^3 + \ \dots\right)$$

$$\boxed{= a_0 e^{4x}}$$

Exercise

Find a power series solution. $y' = x^2 y$

Solution

$$\frac{dy}{dx} = x^2 y$$

$$\int \frac{dy}{y} = \int x^2 dx$$

$$\ln y = \frac{1}{3}x^3 + C_1$$

$$y = e^{\frac{1}{3}x^3 + C_1}$$

$$y = Ce^{\frac{1}{3}x^3}$$

$$y(0) = C(1) = a_0 \quad \rightarrow \quad C = a_0$$

$$\boxed{y = a_0 e^{x^3/3}}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$$

$$y' - x^2 y = 0$$

$$\sum_{n=1}^{\infty} na_n x^{n-1} - x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{k=-2}^{\infty} (k+3) a_{k+3} x^{k+2}$$

$$= \sum_{n=-2}^{\infty} (n+3) a_{n+3} x^{n+2}$$

$$\sum_{n=-2}^{\infty} (n+3) a_{n+3} x^{n+2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$a_1 + 2a_2 x + \sum_{n=-2}^{\infty} (n+3) a_{n+3} x^{n+2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$a_1 + 2a_2 x + \sum_{n=-2}^{\infty} \left[(n+3) a_{n+3} - a_n \right] x^{n+2} = 0$$

If we set $a_1 = a_2 = 0$, then

$$(n+3) a_{n+3} - a_n = 0 \Rightarrow \quad a_{n+3} = \frac{a_n}{n+3}, \quad n \ge 0$$

$$a_3 = \frac{1}{3} a_0$$

$$a_4 = \frac{1}{4} a_1 = 0$$

$$a_5 = \frac{1}{5} a_2 = 0$$

$$a_6 = \frac{1}{6} a_3 = \frac{1}{3 \cdot 6} a_0$$

$$a_7 = \frac{1}{7} a_4 = 0$$

$$a_9 = \frac{1}{9} a_6 = \frac{1}{3 \cdot 6 \cdot 9} a_0 = \frac{1}{3^3 (1 \cdot 2 \cdot 3)} a_0$$

$$a_{12} = \frac{1}{12} a_9 = \frac{1}{3 \cdot 6 \cdot 9 \cdot 12} a_0 = \frac{1}{3^4 (1 \cdot 2 \cdot 3 \cdot 4)} a_0$$

$$\boxed{a_{3n} = \frac{1}{3^n \cdot n!} a_0}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{3^n \cdot n!} a_0 x^{3n}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x^3}{3} \right)^n$$

$$\boxed{= a_0 e^{x^3/3}} \quad \checkmark \qquad P = \lim_{n \to \infty} \left| \frac{3^k k!}{1} \right| \boxed{= \infty}$$

Exercise

Find a power series solution. $y' + 2xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y' + 2xy = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

$$a_1 + \sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

$$a_1 + \sum_{n=0}^{\infty} \left[(n+2) a_{n+2} + 2a_n \right] x^{n+1} = 0$$

$$\begin{cases} \boxed{a_1 = 0} \\ (n+2) a_{n+2} + 2a_n = 0 \quad \rightarrow \quad \boxed{a_{n+2} = -\dfrac{2}{n+2} a_0} \end{cases}$$

$$n = 0 \quad \rightarrow \quad a_2 = -a_1 \qquad\qquad n = 1 \quad \rightarrow \quad a_3 = -\tfrac{2}{3} a_1 = 0$$

$$n = 2 \quad \rightarrow \quad a_4 = -\tfrac{1}{2} a_2 = \tfrac{1}{2} a_0 \qquad\qquad n = 3 \quad \rightarrow \quad a_5 = -\tfrac{2}{7} a_3 = 0$$

$$n = 4 \quad \rightarrow \quad a_6 = -\tfrac{1}{3} a_4 = -\tfrac{1}{2 \cdot 3} a_0 \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n = 6 \quad \rightarrow \quad a_8 = -\tfrac{1}{4} a_6 = \tfrac{1}{4!} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_{2k} = \frac{(-1)^k}{k!} a_0}$$

$$y(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} a_0 x^{2k}$$

$$= a_0 \left(1 - x^2 + \tfrac{1}{2!} x^4 - \tfrac{1}{3!} x^6 + \cdots \right)$$

$$\boxed{= a_0 e^{-x^2}} \qquad\qquad P = \lim_{n \to \infty} \left| \frac{n+2}{-2} \right| \boxed{= \infty}$$

Exercise

Find a power series solution. $(x-2)y'+y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n \quad\Rightarrow\quad y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}$$

$$(x-2)\sum_{n=1}^{\infty}na_n x^{n-1}+\sum_{n=0}^{\infty}a_n x^n=0$$

$$x\sum_{n=1}^{\infty}na_n x^{n-1}-2\sum_{n=1}^{\infty}na_n x^{n-1}+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=1}^{\infty}na_n x^n-2\sum_{n=1}^{\infty}na_n x^{n-1}+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}na_n x^n-2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[na_n-2(n+1)a_{n+1}+a_n\right]x^n=0$$

$$na_n-2(n+1)a_{n+1}+a_n=0$$

$$2(n+1)a_{n+1}=(n+1)a_n$$

$$\boxed{a_{n+1}=\frac{1}{2}a_n}$$

$$n=0 \quad\rightarrow\quad a_1=\frac{1}{2}a_0$$

$$n=1 \quad\rightarrow\quad a_2=\frac{1}{2}a_1=\frac{1}{2^2}a_0$$

$$n=2 \quad\rightarrow\quad a_3=\frac{1}{2}a_2=\frac{1}{2^3}a_0$$

$$n=3 \quad\rightarrow\quad a_4=\frac{1}{2}a_3=\frac{1}{2^4}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n=\frac{1}{2^n}a_0}$$

$$y(x)=\sum_{n=0}^{\infty}\frac{1}{2^n}a_0 x^n$$

$$= a_0\left(1+\frac{1}{2}x+\frac{1}{2^2}x^2+\frac{1}{2^3}x^3+\ \cdots\right)$$

$$= a_0\left(1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^3+\ \cdots\right)$$

$$= a_0\frac{1}{1-\frac{x}{2}}$$

$$\boxed{= \frac{2a_0}{2-x}}$$

Exercise

Find a power series solution. $(2x-1)y'+2y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n \quad\Rightarrow\quad y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}$$

$$(2x-1)\sum_{n=1}^{\infty}na_n x^{n-1}+2\sum_{n=0}^{\infty}a_n x^n=0$$

$$0+\sum_{n=1}^{\infty}2na_n x^n-\sum_{n=1}^{\infty}na_n x^{n-1}+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=0}^{\infty}2na_n x^n-\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[2na_n-(n+1)a_{n+1}+2a_n\right]x^n=0$$

$$2na_n-(n+1)a_{n+1}+2a_n=0$$

$$(n+1)a_{n+1}=2(n+1)a_n$$

$$\boxed{a_{n+1}=2a_n}$$

$$n=0 \quad\to\quad a_1=2a_0$$

$$n=1 \quad\to\quad a_2=2a_1=2^2a_0$$

$$n=2 \quad\to\quad a_3=2a_2=2^3a_0$$

$$n=3 \quad\to\quad a_4=2a_3=2^4a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = 2^n a_0}$$

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} 2^n a_0 x^n \\
&= a_0\left(1 + 2x + 2^2x^2 + 2^3x^3 + \cdots\right) \\
&= a_0\left(1 + 2x + (2x)^2 + (2x)^3 + \cdots\right) \\
&= \boxed{\frac{a_0}{1-2x}}
\end{aligned}$$

Exercise

Find a power series solution. $2(x-1)y' = 3y$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$(2x-2)\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} 3a_n x^n$$

$$2x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 3a_n x^n$$

$$0 + \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 3a_n x^n$$

$$\sum_{n=0}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} 3a_n x^n$$

$$\sum_{n=0}^{\infty}\left[2n a_n - 2(n+1)a_{n+1}\right]x^n = \sum_{n=0}^{\infty} 3a_n x^n$$

$$2n a_n - 2(n+1)a_{n+1} = 3a_n$$

$$-2(n+1)a_{n+1} = (3-2n)a_n$$

$$\boxed{a_{n+1} = \frac{2n-3}{2n+2}a_n} \qquad\qquad \rho = \lim_{n\to\infty} \frac{2n-3}{2n+2}\ \boxed{=1}$$

$$n=0 \quad \to \quad a_1 = -\frac{3}{2}a_0$$

$$n=1 \quad \to \quad a_2 = -\frac{1}{4}a_1 = \frac{3}{8}a_0$$

$$n=2 \quad \to \quad a_3 = \frac{1}{6}a_2 = \frac{1}{16}a_0$$

$$n=3 \quad \to \quad a_4 = \frac{3}{8}a_3 = \frac{3}{128}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = a_0\left(1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \frac{3}{128}x^4 + \cdots\right)}$$

$$\frac{y'}{y} = \frac{3}{2}\frac{1}{x-1}$$

$$\int \frac{dy}{y} = \frac{3}{2}\int \frac{1}{x-1}dx$$

$$\ln y = \frac{3}{2}\ln(x-1) + \ln C$$

$$\ln y = \ln C(x-1)^{3/2}$$

$$\boxed{y(x) = C(x-1)^{3/2}}$$

Exercise

Find a power series solution. $(1+x)y' - y = 0$

Solution

$$(1+x)\frac{dy}{dx} = y$$

$$\frac{dy}{y} = \frac{dx}{1+x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\ln(y) = \ln(x+1) + C \quad \Rightarrow \quad \boxed{y = C(x+1)}$$

With $y(0) = C(0+1) = a_0 \quad \to \quad C = a_0$

$$\Rightarrow \quad \boxed{y = a_0(x+1)}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$(1+x)y' - y = 0$$

$$(1+x)\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=1}^{\infty}na_n x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}na_n x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left((n+1)a_{n+1}x^n + na_n x^n - a_n x^n\right) = 0$$

$$(n+1)a_{n+1} + (n-1)a_n = 0 \quad \to a_{n+1} = \frac{1-n}{n+1}a_n; \quad n \ge 0$$

$$a_1 = a_0 \qquad a_2 = 0a_1 = 0 \qquad a_3 = \frac{-1}{3}a_2 = 0$$

$$a_n = 0 \quad \text{for} \quad n \ge 2$$

$$\begin{aligned} y(x) &= a_0 + a_1 x \\ &= a_0 + a_0 x \\ &= \underline{a_0(1+x)} \end{aligned}$$

Exercise

Find a power series solution. $(2-x)y' + 2y = 0$

Solution

$$(2-x)\frac{dy}{dx} + 2y = 0$$

$$(2-x)\frac{dy}{dx} = -2y$$

$$\int \frac{dy}{y} = \int \frac{2d(2-x)}{2-x}$$

$$\ln y = 2\ln(2-x) + C_1$$

$$\ln y = \ln(2-x)^2 + C_1$$

$$\ln y = \ln C(2-x)^2$$

$$y = C(2-x)^2$$

$$y(0) = C(2-0)^2 = a_0 \quad \rightarrow \quad \boxed{C = \tfrac{1}{4}a_0}$$

$$\boxed{y = \tfrac{1}{4}a_0(2-x)^2}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$(2-x)y' + 2y = 0$$

$$(2-x)\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$2\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0 \qquad \sum_{n=1}^{\infty} n a_n x^n = 0 + \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n$$

$$2\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+1) a_{n+1} - n a_n + 2a_n \right] x^n = 0$$

$$2(n+1)a_{n+1} - (n-2)a_n = 0$$

$$2(n+1)a_{n+1} = (n-2)a_n$$

$$a_{n+1} = \frac{n-2}{2(n+1)} a_n, \qquad n \ge 0$$

$$a_1 = \frac{-2}{2} a_0 = -a_0$$

$$a_2 = \frac{-1}{4} a_1 = \frac{1}{4} a_0$$

$$a_3 = \frac{0}{6} a_0 = 0$$

$$\vdots$$

$$a_n = 0$$

$$y(x) = a_0 - a_0 x + \frac{1}{4} a_0 x^2$$

$$= a_0\left(1 - x + \frac{1}{4}x^2\right)$$

$$= \frac{1}{4} a_0 \left(4 - 4x + x^2\right)$$

$$\boxed{= \frac{1}{4} a_0 (2-x)^2} \quad \checkmark$$

Exercise

Find a power series solution. $(x-4)y' + y = 0$

Solution

$$y(x)=\sum_{n=0}^{\infty} a_n x^n \qquad y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$(x-4)y' + y = 0$$

$$(x-4)\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}4(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=1}^{\infty}na_n x^n+\sum_{n=0}^{\infty}\left[a_n-4(n+1)a_{n+1}\right]x^n=0$$

$$\sum_{n=0}^{\infty}na_n x^n+a_0-4a_1+\sum_{n=1}^{\infty}\left[a_n-4(n+1)a_{n+1}\right]x^n=0$$

$$a_0-4a_1+\sum_{n=1}^{\infty}\left[(n+1)a_n-4(n+1)a_{n+1}\right]x^n=0$$

$$a_0-4a_1=0 \quad\to\quad \boxed{a_1=\frac{1}{4}a_0}$$

$$(n+1)a_n-4(n+1)a_{n+1}=0 \quad\to\quad a_{n+1}=\frac{1}{4}a_n$$

$$a_2=\frac{1}{4}a_1=\frac{1}{4^2}a_0$$

$$a_3=\frac{1}{4}a_2=\frac{1}{4^3}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n=\frac{1}{4^n}a_0}$$

$$y(x)=\sum_{n=0}^{\infty}\frac{1}{4^n}a_0x^n$$

$$=a_0\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n$$

$$= a_0\left(\frac{1}{1-\frac{x}{4}}\right)$$

$$= a_0\left(\frac{4}{4-x}\right)$$

$$\boxed{= \frac{-4a_0}{x-4}} \quad \checkmark$$

Exercise

Find a power series solution. $x^2 y' = y - x - 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$x^2 y' = y - x - 1$$

$$x^2 \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = -x - 1 + \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+2} = -x - 1 + \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=2}^{\infty} (n-1) a_{n-1} x^n = -x - 1 + \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=2}^{\infty} (n-1) a_{n-1} x^n = -x - 1 + a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n$$

$$-x - 1 + a_0 + a_1 x = 0$$

$$a_0 + a_1 x = 1 + x \quad \Rightarrow \quad a_0 = 1; \;\; a_1 = 1$$

$$(n-1) a_{n-1} = a_n$$

$$a_2 = a_1 = 1$$

$$a_3 = 2a_2 = 2$$

$$a_4 = 3a_3 = 1 \cdot 2 \cdot 3$$

$$\vdots \qquad \vdots$$

$$a_n = (n-1)!$$

$$\boxed{y(x) = 1 + x + x^2 + 2!x^3 + 3!x^4 + \cdots}$$

Exercise

Find a power series solution. $(x-3)y' + 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$(x-3)y' + 2y = 0$$

$$(x-3)\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty} 3(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 3(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 3(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[n a_n - 3(n+1)a_{n+1} + 2a_n\right]x^n = 0$$

$$-3(n+1)a_{n+1} + (n+2)a_n = 0 \quad \rightarrow \quad \boxed{a_{n+1} = \frac{n+2}{3(n+1)}a_n} \qquad n = 0,\ 1,\ 2,\ \cdots$$

$$a_1 = \frac{2}{3}a_0 \qquad\qquad a_2 = \frac{3}{3\cdot 2}a_1 = \frac{3}{3^2}a_0$$

$$a_3 = \frac{4}{3\cdot 3}a_2 = \frac{4}{3^3}a_0 \qquad\qquad a_4 = \frac{5}{3\cdot 4}a_3 = \frac{5}{3^4}a_0$$

$$\vdots \quad \vdots$$

$$a_n = \frac{n+1}{3^n}a_0 \qquad n \ge 1$$

$$y(x) = \left(1 + \frac{2}{3}x + \frac{3}{3^2}x^2 + \frac{4}{3^3}x^3 + \ldots\right) = a_0\sum_{n=0}^{\infty}\frac{n+1}{3^n}x^n$$

$$\rho = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{3n+3}{n+2} = 3$$

$$\boxed{y(x) = \frac{1}{(3-x)^2}}$$

Exercise

Find a power series solution. $xy' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$xy' + y = 0$

$$x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_n x^n = 0$$

$$(n+1) a_n = 0 \quad \rightarrow \quad \underline{a_n = 0}$$

$$\underline{y(x) \equiv 0}$$

$\therefore$ The equation has no non-trivial power series.

Exercise

Find a power series solution. $x^3 y' - 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$x^3 y' - 2y = 0$

$$x^3 \sum_{n=1}^{\infty} n a_n x^{n-1} - 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} na_n x^{n+2} - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=3}^{\infty} (n-2)a_{n-2}x^n - 2\left(a_0 + a_1 x + a_2 x^2\right) - \sum_{n=3}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=3}^{\infty} \left[(n-2)a_{n-2} - 2a_n\right]x^n - 2\left(a_0 + a_1 x + a_2 x^2\right) = 0$$

$$\begin{cases} a_0 = a_1 = a_2 = 0 \\ (n-2)a_{n-2} = 2a_n \end{cases} \rightarrow ka_k = 2a_{k+2} \quad (k = n-2)$$

$$\boxed{a_{k+2} = \frac{k}{2}a_k}$$

$$a_3 = \frac{1}{2}a_1 = 0$$

$$a_4 = a_2 = 0$$

$$a_n \equiv 0$$

$$\boxed{y(x) \equiv 0}$$

∴ The equation has no non-trivial power series.

Exercise

Find a power series solution. $y'' = 4y$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$y'' = 4y$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = 4\sum_{n=0}^{\infty} a_n x^n$$

$$(n+2)(n+1)a_{n+2} = 4a_n$$

$$a_{n+2} = \frac{4}{(n+2)(n+1)}a_n$$

$$n=0 \quad \to \quad a_2 = 2a_0 = \frac{4}{2}a_0 \qquad\qquad n=1 \quad \to \quad a_3 = \frac{4}{2\cdot 3}a_1$$

$$n=2 \quad \to \quad a_4 = \frac{4}{3\cdot 4}a_2 = \frac{4^2}{4!}a_0 \qquad\qquad n=3 \quad \to \quad a_5 = \frac{4}{4\cdot 5}a_3 = \frac{4^2}{5!}a_1$$

$$n=4 \quad \to \quad a_6 = \frac{4}{5\cdot 6}a_4 = \frac{4^3}{6!}a_0 \qquad\qquad n=5 \quad \to \quad a_7 = \frac{4}{6\cdot 7}a_5 = \frac{4^3}{7!}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_{2k} = \frac{2^{2k}}{(2k)!}a_0} \qquad\qquad \boxed{a_{2k+1} = \frac{2^{2k}}{(2k+1)!}a_1}$$

$$y(x) = a_0\left(1+\frac{2^2}{2!}x^2+\frac{2^4}{4!}x^4+\frac{2^6}{6!}x^6+\cdots\right)+a_1\left(x+\frac{2^3}{3!}x^3+\frac{2^5}{5!}x^5+\cdots\right)$$

$$= a_0\left(1+\frac{1}{2!}(2x)^2+\frac{1}{4!}(2x)^4+\frac{1}{6!}(2x)^6+\cdots\right)+a_1\left(x+\frac{1}{3!}(2x)^3+\frac{1}{5!}(2x)^5+\cdots\right)$$

$$\boxed{= a_0 \cosh 2x + a_1 \sinh 2x}$$

Exercise

Find a power series solution. $y'' = 9y$

Solution

The equation $y'' = 9y$ has a characteristic equation $\lambda^2 - 9 = 0 \quad \Rightarrow \quad \boxed{\lambda = \pm 3}$

$\therefore$ The general solution: $y(x) = C_1e^{3x} + C_2e^{-3x}$

With $y(0) = a_0$ *and* $y'(0) = a_1$

$$y(0) = C_1e^{3(0)} + C_2e^{-3(0)} \quad \to \quad C_1 + C_2 = a_0$$

$$y'(x) = 3C_1e^{3x} - 3C_2e^{-3x}$$

$$y(0) = 3C_1e^{3(0)} - 3C_2e^{-3(0)} \quad \to \quad 3C_1 - 3C_2 = a_1$$

$$\begin{cases} C_1 + C_2 = a_0 \\ 3C_1 - 3C_2 = a_1 \end{cases} \quad \to \quad \begin{cases} 3C_1 + 3C_2 = 3a_0 \\ 3C_1 - 3C_2 = a_1 \end{cases}$$

$$6C_1 = 3a_0 + a_1 \quad \to \quad \boxed{C_1 = \frac{3a_0 + a_1}{6}}$$

$$C_2 = a_0 - C_1 \quad \to \quad \boxed{C_2 = a_0 - \frac{3a_0 + a_1}{6} = \frac{3a_0 - a_1}{6}}$$

$$\boxed{y(x)=\frac{3a_0+a_1}{6}e^{3x}+\frac{3a_0-a_1}{6}e^{-3x}}$$

$$y(x)=\sum_{n=0}^{\infty}a_nx^n \qquad y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''-9y=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-9\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-9\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-9a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}-9a_n=0$$

$$a_{n+2}=\frac{9}{(n+2)(n+1)}a_n, \quad n\ge 0$$

$$a_2=\frac{9}{(2)(1)}a_0=\frac{9}{2}a_0 \qquad a_3=\frac{9}{(3)(2)}a_1=\frac{9}{2\cdot 3}a_1$$

$$a_4=\frac{3^2}{(4)(3)}a_2=\frac{9\cdot 9}{2\cdot 3\cdot 4}a_0=\frac{3^4}{2\cdot 3\cdot 4}a_0 \qquad a_5=\frac{9}{(5)(4)}a_3=\frac{3^4}{2\cdot 3\cdot 4\cdot 5}a_1$$

$$a_6=\frac{3^2}{(6)(5)}a_4=\frac{3^6}{6!}a_0 \qquad a_7=\frac{9}{(7)(6)}a_5=\frac{3^6}{7!}a_1$$

$$a_{2n}=\frac{3^{2n}}{(2n)!}a_0 \qquad a_{2n+1}=\frac{3^{2n}}{(2n+1)!}a_1$$

$$y(x)=a_0\left(1+\frac{3^2}{2!}x^2+\frac{3^4}{4!}x^4+\frac{3^6}{6!}x^6+\cdots\right)+a_1\left(x+\frac{3^2}{3!}x^3+\frac{3^4}{5!}x^5+\frac{3^6}{7!}x^7+\cdots\right)$$

$$y(x)=\frac{3a_0+a_1}{6}e^{3x}+\frac{3a_0-a_1}{6}e^{-3x}$$

$$=\frac{3a_0+a_1}{6}\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\cdots\right]+\frac{3a_0-a_1}{6}\left[1-3x+\frac{(-3x)^2}{2!}+\frac{(-3x)^3}{3!}+\cdots\right]$$

$$=\frac{3a_0}{6}\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots\right]+\frac{a_1}{6}\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots\right]$$

$$+\frac{3a_0}{6}\left[1-3x+\frac{(3x)^2}{2!}-\frac{(3x)^3}{3!}+\cdots\right]-\frac{a_1}{6}\left[1-3x+\frac{(3x)^2}{2!}-\frac{(3x)^3}{3!}+\cdots\right]$$

$$=\frac{1}{2}a_0\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots+1-3x+\frac{(3x)^2}{2!}-\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots\right]$$

$$+\frac{a_1}{6}\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots-1+3x-\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}-\frac{(3x)^4}{4!}+\cdots\right]$$

$$=\frac{1}{2}a_0\left[2+2\frac{(3x)^2}{2!}+2\frac{(3x)^4}{4!}+\cdots\right]+\frac{a_1}{6}\left[6x+2\frac{(3x)^3}{3!}+2\frac{(3x)^5}{5!}+\cdots\right]$$

$$=a_0\left[1+\frac{(3x)^2}{2!}+\frac{(3x)^4}{4!}+\cdots\right]+a_1\left[x+\frac{3^2x^3}{3!}+\frac{3^4x^5}{5!}+\cdots\right]$$

Which are identical.

Exercise

Find a power series solution. $y''+y=0$

Solution

The equation $y''+y=0$ has a characteristic equation $\lambda^2+1=0 \Rightarrow \boxed{\lambda=\pm i}$

$\therefore$ The general solution: $y(x)=C_1\sin x+C_2\cos x$

With $y(0)=a_0$ *and* $y'(0)=a_1$

$$y(0)=C_1\sin(0)+C_2\cos(0) \;\to\; C_2=a_0$$

$$y'(x)=C_1\cos x-C_2\sin x$$

$$y(0)=C_1\cos(0)-C_2\sin(0) \;\to\; C_1=a_1$$

$$y(x)=a_1\sin x+a_0\cos x$$

$$=a_0\cos x+a_1\sin x$$

$$y(x)=\sum_{n=0}^{\infty}a_nx^n \qquad y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y'' + y = 0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_n\right]x^n = 0$$

$$(n+2)(n+1)a_{n+2} + a_n = 0 \quad \rightarrow \quad a_{n+2} = \frac{-1}{(n+2)(n+1)}a_n, \quad n \ge 0$$

$$a_2 = \frac{1}{(2)(1)}a_0 = -\frac{1}{2}a_0 \qquad\qquad a_3 = \frac{1}{(3)(2)}a_1 = -\frac{1}{2\cdot 3}a_1$$

$$a_4 = -\frac{1}{(4)(3)}a_2 = \frac{1}{2\cdot 3\cdot 4}a_0 \qquad\qquad a_5 = -\frac{1}{(5)(4)}a_3 = \frac{1}{2\cdot 3\cdot 4\cdot 5}a_1$$

$$a_6 = -\frac{1}{(6)(5)}a_4 = -\frac{1}{6!}a_0 \qquad\qquad a_7 = -\frac{1}{(7)(6)}a_5 = -\frac{1}{7!}a_1$$

$$a_{2n} = \frac{(-1)^n}{(2n)!}a_0 \qquad\qquad a_{2n+1} = \frac{(-1)^n}{(2n+1)!}a_1$$

$$y(x) = a_0\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots\right) + a_1\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots\right)$$

$$\boxed{= a_0 \cos x + a_1 \sin x}$$

Exercise

Find a power series solution. $y'' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty}a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty}na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty}n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y'' - y = 0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-a_n\right]x^n=0$$

$$\boxed{a_{n+2}=\frac{1}{(n+2)(n+1)}a_n}$$

$$n=0 \quad\rightarrow\quad a_2=\frac{1}{1\cdot 2}a_0 \qquad\qquad n=1 \quad\rightarrow\quad a_3=\frac{1}{2\cdot 3}a_1$$

$$n=2 \quad\rightarrow\quad a_4=\frac{1}{3\cdot 4}a_2=\frac{1}{4!}a_0 \qquad\qquad n=3 \quad\rightarrow\quad a_5=\frac{1}{4\cdot 5}a_3=\frac{1}{5!}a_1$$

$$n=4 \quad\rightarrow\quad a_6=\frac{1}{5\cdot 6}a_4=\frac{1}{6!}a_0 \qquad\qquad n=5 \quad\rightarrow\quad a_7=\frac{1}{6\cdot 7}a_5=\frac{1}{7!}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_{2k}=\frac{1}{(2k)!}a_0} \qquad\qquad \boxed{a_{2k+1}=\frac{1}{(2k+1)!}a_1}$$

$$y(x)=a_0\left(1+\frac{1}{2!}x^2+\frac{1}{4!}x^4+\frac{1}{6!}x^6+\cdots\right)+a_1\left(x+\frac{1}{3!}x^3+\frac{1}{5!}x^5+\cdots\right)$$

$$\boxed{=a_0\cosh x+a_1\sinh x}$$

Exercise

Find a power series solution. $y''+y=x$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''+y=x$$

$$\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+\sum_{n=0}^{\infty}a_nx^n=x$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n+a_0+a_1x+\sum_{n=2}^{\infty}a_nx^n-x=0$$

$$\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_n\right]x^n+a_0+2a_2+\left(6a_3+a_1-1\right)x=0$$

$$\begin{cases} a_0+2a_2=0 \\ 6a_3+a_1-1=0 \\ (n+2)(n+1)a_{n+2}+a_n=0 \end{cases}$$

$$\begin{cases} a_2=-\frac{1}{2}a_0 \\ a_3=-\frac{1}{6}\left(a_1-1\right) \\ a_{n+2}=-\frac{1}{(n+2)(n+1)}a_n \end{cases}$$

$$n=2 \quad \to \quad a_4=-\frac{1}{3\cdot 4}a_2=\frac{1}{4!}a_0 \qquad n=3 \quad \to \quad a_5=-\frac{1}{4\cdot 5}a_3=\frac{1}{5!}\left(a_1-1\right)$$

$$n=4 \quad \to \quad a_6=-\frac{1}{5\cdot 6}a_4=-\frac{1}{6!}a_0 \qquad n=5 \quad \to \quad a_7=-\frac{1}{6\cdot 7}a_5=-\frac{1}{7!}\left(a_1-1\right)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_{2k}=\frac{(-1)^k}{(2k)!}a_0} \qquad \boxed{a_{2k+1}=\frac{(-1)^k}{(2k+1)!}\left(a_1-1\right)}$$

$$\begin{aligned} y(x)&=a_0+a_1x+a_0\sum_{k=2}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}+\left(a_1-1\right)\sum_{k=3}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1} \\ &=a_0+a_0\left(-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots\right)+\left(a_1-1\right)\left(-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots\right)+a_1x-x+x \\ &=a_0\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots\right)+\left(a_1-1\right)\left(-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots\right)+\left(a_1-1\right)x+x \\ &=x+a_0\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots\right)+\left(a_1-1\right)\left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots\right) \\ &\boxed{=x+a_0\cos x+\left(a_1-1\right)\sin x} \end{aligned}$$

Exercise

Find a power series solution. $y''-xy=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty} na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y''-xy=0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}a_n x^{n+1}=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}a_{n-1} x^{n}=0$$

$$2a_2+\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}a_{n-1} x^{n}=0$$

$$2a_2+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}-a_{n-1}\right]x^n=0$$

$$2a_2=0\to \boxed{a_2=0}$$

$$(n+2)(n+1)a_{n+2}-a_{n-1}=0 \quad\to\quad a_{n+2}=\frac{a_{n-1}}{(n+1)(n+2)}$$

$$a_3=\frac{a_0}{2\cdot 3}=\frac{1}{6}a_0 \qquad a_4=\frac{a_1}{3\cdot 4}=\frac{1}{12}a_1 \qquad a_5=\frac{a_2}{4\cdot 5}=0$$

$$a_6=\frac{a_3}{5\cdot 6}=\frac{1}{180}a_0 \qquad a_7=\frac{a_3}{6\cdot 7}=\frac{1}{504}a_1 \qquad a_8=\frac{a_5}{7\cdot 8}=0$$

$$\vdots\quad\vdots \qquad\qquad \vdots\quad\vdots \qquad\qquad \vdots\quad\vdots$$

$$\boxed{\begin{cases} y_1(x)=\left(1+\frac{1}{6}x^3+\frac{1}{180}x^6+\cdots\right)a_0 \\ y_2(x)=\left(x+\frac{1}{12}x^4+\frac{1}{504}x^7+\cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $y''+xy=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''+xy=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+x\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}a_n x^{n+1}=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}a_{n-1}x^n=0$$

$$2a_2+\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}a_{n-1}x^n=0$$

$$2a_2+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_{n-1}\right]x^n=0$$

$$\begin{cases} 2a_2=0 \rightarrow \boxed{a_2=0} \\ (n+2)(n+1)a_{n+2}+a_{n-1}=0 \end{cases}$$

$$\boxed{a_{n+2}=-\frac{a_{n-1}}{(n+1)(n+2)}}$$

a_0	a_1	$a_2=0$
$n=1 \rightarrow a_3=-\frac{a_0}{2\cdot 3}=-\frac{1}{6}a_0$	$n=2 \rightarrow a_4=-\frac{1}{3\cdot 4}a_1$	$n=3 \rightarrow a_5=-\frac{a_2}{20}=0$
$n=4 \rightarrow a_6=-\frac{a_3}{5\cdot 6}=\frac{1}{180}a_0$	$n=5 \rightarrow a_7=-\frac{a_3}{6\cdot 7}=\frac{1}{504}a_1$	$n=6 \rightarrow a_8=-\frac{a_5}{56}=0$
$n=7 \rightarrow a_9=-\frac{a_6}{8\cdot 9}=-\frac{1}{12,960}a_0$		
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{\begin{cases} y_1(x) = \left(1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \frac{1}{12,960}x^9 + \cdots\right)a_0 \\ y_2(x) = \left(x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \cdots\right)a_1 \end{cases}}$$

$$\boxed{y(x) = \left(1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \frac{1}{12,960}x^9 + \cdots\right)a_0 + \left(x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \cdots\right)a_1}$$

Exercise

Find a power series solution. $y'' + xy' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + xy' + y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n+1) a_n\right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} + (n+1) a_n = 0$$

$$\boxed{a_{n+2} = -\frac{a_n}{n+2}}$$

a_0 | a_1

$n = 0 \rightarrow a_2 = -\frac{1}{2} a_0$ | $n = 1 \rightarrow a_3 = -\frac{1}{3} a_1$

$$n = 2 \;\to\; a_4 = -\tfrac{1}{4}a_2 = \frac{1}{4\cdot 2}a_0 \qquad\qquad n = 3 \;\to\; a_5 = -\frac{a_3}{5} = \frac{1}{3\cdot 5}a_1$$

$$n = 4 \;\to\; a_6 = -\frac{a_4}{6} = -\frac{1}{6\cdot 4\cdot 2}a_0 \qquad\qquad n = 5 \;\to\; a_7 = -\frac{a_5}{7} = -\frac{1}{7\cdot 5\cdot 3}a_1$$

$$n = 6 \;\to\; a_8 = -\frac{a_6}{8} = \frac{1}{8\cdot 6\cdot 4\cdot 2}a_0 \qquad\qquad n = 7 \;\to\; a_9 = -\frac{a_7}{9} = \frac{1}{9\cdot 7\cdot 5\cdot 3}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n} = \frac{(-1)^n a_0}{(2n)(2n-2)\cdots 4\cdot 2} = \frac{(-1)^n}{n!\ 2^n}a_0 \qquad\qquad a_{2n} = \frac{(-1)^n a_1}{(2n+1)(2n-1)\cdots 5\cdot 3} = \frac{(-1)^n}{(2n+1)!!}a_1$$

$$\boxed{y(x) = a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\ 2^n}x^{2n} + a_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!!}x^{2n+1}}$$

$$\boxed{y(x) = a_0\left(1 - \tfrac{1}{2}x^2 + \tfrac{1}{8}x^4 - \tfrac{1}{48}x^6 + \cdots\right) + a_1\left(x - \tfrac{1}{3}x^3 + \tfrac{1}{15}x^5 - \tfrac{1}{105}x^7 + \cdots\right)}$$

Exercise

Find a power series solution. $y'' - xy' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y'' - xy' - y = 0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=1}^{\infty}na_n x^{n-1} - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}na_n x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}na_n x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-(n+1)a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}-(n+1)a_n=0$$

$$\boxed{a_{n+2}=\frac{a_n}{n+2}}$$

a_0 $\qquad\qquad$ a_1

$$n=0 \rightarrow a_2=\frac{1}{2}a_0 \qquad\qquad n=1 \rightarrow a_3=\frac{1}{3}a_1$$

$$n=2 \rightarrow a_4=\frac{1}{4}a_2=\frac{1}{4\cdot 2}a_0 \qquad\qquad n=3 \rightarrow a_5=\frac{a_3}{5}=\frac{1}{3\cdot 5}a_1$$

$$n=4 \rightarrow a_6=\frac{a_4}{6}=\frac{1}{6\cdot 4\cdot 2}a_0 \qquad\qquad n=5 \rightarrow a_7=\frac{a_5}{7}=\frac{1}{7\cdot 5\cdot 3}a_1$$

$$n=6 \rightarrow a_8=\frac{a_6}{8}=\frac{1}{8\cdot 6\cdot 4\cdot 2}a_0 \qquad\qquad n=7 \rightarrow a_9=\frac{a_7}{9}=\frac{1}{9\cdot 7\cdot 5\cdot 3}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n}=\frac{a_0}{(2n)(2n-2)\cdots 4\cdot 2}=\frac{1}{n!\ 2^n}a_0 \qquad\qquad a_{2n}=\frac{a_1}{(2n+1)(2n-1)\cdots 5\cdot 3}=\frac{1}{(2n+1)!!}a_1$$

$$\boxed{y(x)=a_0\sum_{n=0}^{\infty}\frac{1}{n!\ 2^n}x^{2n}+a_1\sum_{n=0}^{\infty}\frac{1}{(2n+1)!!}x^{2n+1}}$$

$$\boxed{y(x)=a_0\left(1+\frac{x^2}{2}+\frac{x^4}{2^2\cdot 2}+\cdots+\frac{x^{2n}}{2^n\cdot n!}\right)+a_1\left(x+\frac{x^3}{3}+\frac{x^5}{5\cdot 3}+\cdots+\frac{2^n\cdot n!}{(2n+1)!!}x^{2n+1}\right)}$$

Exercise

Find a power series solution. $y''+x^2y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y'' + x^2 y = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x^2\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}a_{n-2} x^{n} = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}a_{n-2} x^{n} = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_{n-2}\right]x^n = 0$$

$$2a_2 = 0 \rightarrow \underline{a_2 = 0} \qquad 6a_3 = 0 \rightarrow \underline{a_3 = 0}$$

$$(n+2)(n+1)a_{n+2} + a_{n-2} = 0 \quad \rightarrow \quad a_{n+2} = \frac{-a_{n-2}}{(n+1)(n+2)}$$

$$a_4 = \frac{-a_0}{3\cdot 4} = -\frac{1}{12}a_0 \qquad a_5 = \frac{-a_1}{4\cdot 5} = -\frac{1}{20}a_1 \qquad a_6 = \frac{-a_2}{5\cdot 6} = 0 \qquad a_7 = \frac{-a_3}{6\cdot 7} = 0$$

$$a_8 = \frac{-a_4}{7\cdot 8} = \frac{1}{672}a_0 \qquad a_9 = \frac{-a_5}{8\cdot 9} = \frac{1}{1,440}a_1 \qquad a_{10} = \frac{-a_6}{9\cdot 10} = 0 \qquad a_{11} = \frac{a_7}{10\cdot 11} = 0$$

$$\vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots$$

$$\begin{cases} y_1(x) = \left(1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 - \cdots\right)a_0 \\ y_2(x) = \left(x - \frac{1}{20}x^5 + \frac{1}{1,440}x^9 + \cdots\right)a_1 \end{cases}$$

Exercise

Find a power series solution. $y'' + k^2x^2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty}a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty}na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''+k^2x^2y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+k^2x^2\sum_{n=0}^{\infty}a_n x^n=0$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}k^2a_{n-2}x^n=0$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}+k^2a_{n-2}\right]x^n=0$$

$$\begin{cases} a_2=0 \\ a_3=0 \\ (n+2)(n+1)a_{n+2}+k^2a_{n-2}=0 \end{cases}$$

$$\boxed{a_{n+2}=-\frac{k^2}{(n+1)(n+2)}a_{n-2}} \quad (n\geq 2)$$

$$n=2 \;\rightarrow\; a_4=-\frac{k^2}{3\cdot 4}a_0 \qquad\qquad n=3\rightarrow\; a_5=-\frac{k^2}{4\cdot 5}a_1$$

$$n=6 \;\rightarrow\; a_8=-\frac{k^2}{7\cdot 8}a_4=\frac{k^4}{3\cdot 4\cdot 7\cdot 8}a_0 \qquad\qquad n=7\rightarrow\; a_9=-\frac{k^2}{8\cdot 9}a_5=\frac{k^4}{4\cdot 5\cdot 8\cdot 9}a_1$$

$$n=10 \;\rightarrow\; a_{12}=-\frac{k^2}{11\cdot 12}a_8=\frac{k^6}{3\cdot 4\cdot 7\cdot 8\cdot 11\cdot 12}a_0 \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{4m}=-\frac{k^2}{(4m)(4m-1)}a_{4m-4} \qquad\qquad a_{4m+1}=-\frac{k^2}{(4m)(4m+1)}a_{4m-3}$$

$$n=4\rightarrow\; a_6=-\frac{k^2}{5\cdot 6}a_2=0 \qquad\qquad n=5\rightarrow\; a_7=-\frac{k^2}{6\cdot 7}a_3=0$$

$$n=8 \;\rightarrow\; a_{10}=-\frac{k^2}{7\cdot 8}a_6=0 \qquad\qquad n=9\rightarrow\; a_{11}=-\frac{k^2}{10\cdot 11}a_7=0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{\begin{aligned} y(x)&=a_0\left(1-\frac{k^2}{3\cdot 4}x^4+-\frac{k^4}{3\cdot 4\cdot 7\cdot 8}x^8-\frac{k^6}{3\cdot 4\cdot 7\cdot 8\cdot 11\cdot 12}x^{12}+\cdots\right)\\ &\quad+a_1\left(x-\frac{k^2}{4\cdot 5}x^5+-\frac{k^4}{4\cdot 5\cdot 8\cdot 9}x^9-\frac{k^6}{4\cdot 5\cdot 8\cdot 9\cdot 12\cdot 13}x^{13}+\cdots\right)\end{aligned}}$$

Exercise

Find a power series solution. $y'' + 3xy' + 3y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$y'' + 3xy' + 3y = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 3x \sum_{n=1}^{\infty} n a_n x^{n-1} + 3 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (3n+3) a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} + 3(n+1) a_n = 0$$

$$\boxed{a_{n+2} = -\frac{3}{n+2} a_n}$$

a_0	a_1
$n = 2 \rightarrow a_2 = -\frac{3}{2} a_0$	$n = 3 \rightarrow a_3 = -\frac{3}{3} a_1$
$n = 4 \rightarrow a_4 = -\frac{3}{4} a_2 = \frac{3^2}{2^2 \cdot 2} a_0$	$n = 5 \rightarrow a_5 = -\frac{3}{5} a_3 = \frac{3^2}{3 \cdot 5} a_1$
$n = 6 \rightarrow a_6 = -\frac{3}{6} a_3 = \frac{3^3}{2^3 \cdot 2 \cdot 3} a_0$	$n = 7 \rightarrow a_7 = -\frac{3}{7} a_5 = \frac{3^3}{3 \cdot 5 \cdot 7} a_1$
$n = 8 \rightarrow a_8 = -\frac{3}{8} a_6 = \frac{3^4}{2^4 \cdot 2 \cdot 3 \cdot 4} a_0$	$n = 9 \rightarrow a_9 = -\frac{3}{9} a_7 = \frac{3^3}{3 \cdot 5 \cdot 7 \cdot 9} a_1$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$a_{2k} = \frac{(-3)^k}{2^k k!} a_0$	$a_{2k+1} = \frac{(-3)^k}{3 \cdot 5 \cdot 7 \cdots (2k+1)} a_1$

$$y(x)=a_0\left(1+\sum_{k=1}^{\infty}\frac{(-3)^k}{2^k k!}x^{2k}\right)+a_1\left(x+\sum_{k=1}^{\infty}\frac{(-3)^k}{3\cdot 5\cdot 7\cdots(2k+1)}x^{2k+1}\right)$$

$$\boxed{=a_0\left(1-\frac{3}{2}x^2+\frac{9}{8}x^4-\frac{27}{56}x^6+\cdots\right)+a_1\left(x-x^3+\frac{3^2}{3\cdot 5}x^5-\frac{27}{3\cdot 5\cdot 7}x^7+\cdots\right)}$$

Exercise

Find a power series solution. $y''-2xy'+y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''-2xy'+y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=1}^{\infty}2na_n x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$2a_2+\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=1}^{\infty}2na_n x^n+a_0+\sum_{n=1}^{\infty}a_n x^n=0$$

$$2a_2+a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}-2na_n+a_n\right]x^n=0$$

$$2a_2+a_0=0\ \rightarrow\ \boxed{a_2=-\frac{1}{2}a_0}$$

$$(n+2)(n+1)a_{n+2}-(2n-1)a_n=0\ \rightarrow\ a_{n+2}=\frac{(2n-1)a_n}{(n+1)(n+2)}$$

$$a_3=\frac{a_1}{2\cdot 3}=\frac{1}{6}a_1 \qquad\qquad a_4=\frac{3a_2}{3\cdot 4}=-\frac{1}{4}\frac{1}{4}a_0=-\frac{1}{8}a_0$$

$$a_5 = \frac{5a_3}{4\cdot 5} = \frac{1}{2\cdot 3\cdot 4}a_1 = \frac{1}{4!}a_1 \qquad\qquad a_6 = \frac{7a_4}{5\cdot 6} = -\frac{7}{240}a_0$$

$$a_7 = \frac{9a_5}{6\cdot 7} = \frac{1}{112}a_1 \qquad\qquad \vdots \ \vdots$$

$$\vdots \ \vdots$$

$$\boxed{\begin{cases} y_1(x) = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6 - \cdots\right)a_0 \\ y_2(x) = \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{112}x^7 + \cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $y'' - xy' + 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy' + 2y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + 2a_0 + \sum_{n=1}^{\infty} 2a_n x^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n + 2a_n\right] x^n = 0$$

$$2a_2 + 2a_0 = 0 \ \rightarrow \ \boxed{a_2 = -a_0}$$

$$(n+2)(n+1) a_{n+2} - (n-2) a_n = 0 \ \rightarrow \ a_{n+2} = \frac{(n-2)a_n}{(n+1)(n+2)}$$

$$a_3 = \frac{-a_1}{2\cdot 3} = -\frac{1}{6}a_1$$

$$a_5 = \frac{a_3}{4\cdot 5} = -\frac{1}{5!}a_1$$

$$a_7 = \frac{3a_5}{6\cdot 7} = \frac{3}{7!}a_1$$

$$a_9 = \frac{5a_7}{8\cdot 9} = \frac{3\cdot 5}{9!}a_1$$

$$\vdots \quad \vdots$$

$$a_4 = 0$$

$$a_6 = \frac{2a_4}{5\cdot 6} = 0$$

$$a_8 = \frac{4a_6}{5\cdot 6} = 0$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = 1 - x^2 \\ y_2(x) = \left(x - \frac{1}{6}x^3 + \frac{1}{5!}x^5 + \frac{3}{7!}x^7 + \cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $y'' - xy' - x^2 y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy' - x^2 y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x\sum_{n=1}^{\infty} n a_n x^{n-1} - x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n - a_1 x - \sum_{n=2}^{\infty} n a_n x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$2a_2 + 6a_3 x - a_1 x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} - na_n - a_{n-2}\right]x^n = 0$$

$$\begin{cases} 2a_2 = 0 \ \rightarrow \ \boxed{a_2 = 0} \\ (6a_3 - a_1)x = 0 \ \rightarrow \ \boxed{a_3 = \frac{1}{6}a_1} \\ (n+1)(n+2)a_{n+2} - na_n - a_{n-2} = 0 \end{cases}$$

$$\boxed{a_{n+2} = \frac{na_n + a_{n-2}}{(n+1)(n+2)}}$$

$$a_0 \qquad\qquad a_1$$

$$a_2 = 0 \qquad\qquad a_3 = \frac{1}{6}a_1$$

$$n = 2 \ \rightarrow \ a_4 = \frac{2a_2 + a_0}{3\cdot 4} = \frac{1}{12}a_0$$

$$n = 4 \ \rightarrow \ a_6 = \frac{4a_4 + a_2}{5\cdot 6} = \frac{1}{90}a_0$$

$$n = 6 \ \rightarrow \ a_8 = \frac{6a_6 + a_4}{7\cdot 8} = \frac{1}{56}\left(\frac{1}{15} + \frac{1}{12}\right)a_0 = \frac{3}{1120}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n = 3 \ \rightarrow \ a_5 = \frac{3a_3 + a_1}{20} = \frac{1}{20}\left(\frac{3}{6} + 1\right)a_1 = \frac{1}{12}a_1$$

$$n = 5 \ \rightarrow \ a_7 = \frac{5a_5 + a_3}{6\cdot 7} = \frac{1}{42}\left(\frac{5}{12} + \frac{1}{6}\right)a_1 = \frac{1}{72}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = a_0\left(1 + \frac{1}{12}x^2 + \frac{1}{90}x^4 + \frac{3}{1120}x^6 + \cdots\right) + a_1\left(x + \frac{1}{12}x^3 + \frac{1}{72}x^5 + \cdots\right)}$$

Exercise

Find a power series solution. $y'' + x^2 y' + xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y'' + x^2 y' + xy = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x^2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + x\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+2}+\sum_{n=0}^{\infty}a_n x^{n+1}=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n+\sum_{n=1}^{\infty}a_{n-1}x^n=0$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n+a_0x+\sum_{n=2}^{\infty}a_{n-1}x^n=0$$

$$2a_2+\left(6a_3+a_0\right)x+\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}+(n-1)a_{n-1}+a_{n-1}\right]x^n=0$$

$$2a_2+\left(6a_3+a_0\right)x=0 \Rightarrow \begin{cases} a_2=0 \\ a_3=-\frac{1}{6}a_0 \end{cases}$$

$$(n+2)(n+1)a_{n+2}+na_{n-1}=0 \Rightarrow a_{n+2}=-\frac{n}{(n+1)(n+2)}a_{n-1}$$

$$a_4=-\frac{2}{3\cdot 4}a_1=-\frac{1}{6}a_1 \qquad a_5=-\frac{3}{4\cdot 5}a_2=0 \qquad a_6=-\frac{4}{5\cdot 6}a_3=\frac{1}{45}a_0$$

$$a_7=-\frac{5}{6\cdot 7}a_4=\frac{5}{252}a_1 \qquad a_8=-\frac{6}{7\cdot 8}a_5=0 \qquad a_9=-\frac{7}{8\cdot 9}a_3=-\frac{7}{3,240}a_0$$

$$\vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x)=\left(1-\frac{1}{6}x^3+\frac{1}{45}x^6-\frac{7}{3,240}x^9+\cdots\right)a_0 \\ y_2(x)=\left(x-\frac{1}{6}x^4+\frac{5}{252}x^7-\cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $y''+x^2y'+2xy=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y'' + x^2 y' + 2xy = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x^2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 2x\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+2} + \sum_{n=0}^{\infty}2a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}(n-1)a_{n-1}x^n + \sum_{n=1}^{\infty}2a_{n-1}x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}(n-1)a_{n-1}x^n + 2a_0 x + \sum_{n=2}^{\infty}2a_{n-1}x^n = 0$$

$$2a_2 + \left(6a_3 + 2a_0\right)x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + (n-1)a_{n-1} + 2a_{n-1}\right]x^n = 0$$

$$2a_2 + \left(6a_3 + 2a_0\right)x = 0 \Rightarrow \begin{cases} a_2 = 0 \\ a_3 = -\frac{1}{3}a_0 \end{cases}$$

$$(n+2)(n+1)a_{n+2} + (n+1)a_{n-1} = 0 \Rightarrow a_{n+2} = -\frac{a_{n-1}}{n+2}$$

$$\boxed{a_{n+3} = -\frac{a_n}{n+3}}$$

a_0	a_1	$a_2 = 0$
$a_3 = -\frac{1}{3}a_0$	$n=1 \to a_4 = -\frac{1}{4}a_1$	$n=2 \to a_5 = -\frac{a_2}{5} = 0$
$n=3 \to a_6 = -\frac{a_3}{6} = \frac{1}{2\cdot 3^2}a_0$	$n=4 \to a_7 = -\frac{a_4}{7} = \frac{1}{7\cdot 4}a_1$	$n=5 \to a_8 = -\frac{a_5}{8} = 0$
$n=6 \to a_9 = -\frac{a_6}{9} = -\frac{1}{3!\,3^3}a_0$	$n=7 \to a_{10} = -\frac{a_7}{10} = -\frac{1}{10\cdot 7\cdot 4\cdot 1}a_1$	$n=8 \to a_{11} = 0$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$a_{3n} = \frac{(-1)^n}{n!\,3^n}$	$a_{3n+1} = \frac{(-1)^n}{1\cdot 4\cdot 7\cdots(3n+1)}$	$a_{3n+2} = 0$

$$\boxed{y(x) = a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,3^n}x^{3n} + a_1\sum_{n=0}^{\infty}\frac{(-1)^n}{1\cdot 4\cdot 7\cdots(3n+1)}x^{3n+1}}$$

$$\boxed{y(x) = a_0\left(1 - \frac{1}{3}x^3 + \frac{1}{18}x^6 - \frac{1}{162}x^9 + \cdots\right) + a_1\left(x - \frac{1}{4}x^4 + \frac{1}{28}x^7 - \frac{1}{280}x^{10} + \cdots\right)}$$

Exercise

Find a power series solution. $y'' - x^2 y' - 3xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - x^2 y' - 3xy = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x^2 \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 3x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+2} - \sum_{n=0}^{\infty} 3a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n - \sum_{n=1}^{\infty} 3a_{n-1} x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n - 3a_0 x - \sum_{n=2}^{\infty} 3a_{n-1} x^n = 0$$

$$2a_2 + 3(2a_3 - a_0)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n+2) a_{n-1} \right] x^n = 0$$

$$\begin{cases} 2a_2 = 0 & \rightarrow a_2 = 0 \\ 2a_3 - a_0 = 0 & \rightarrow a_3 = \frac{1}{2} a_0 \end{cases}$$

$$(n+2)(n+1) a_{n+2} = (n+2) a_{n-1}$$

$$a_{n+2} = \frac{a_{n-1}}{n+1} \quad \rightarrow \quad a_{n+3} = \frac{a_n}{n+2}$$

a_0	a_1	$a_2 = 0$
$a_3 = \frac{1}{2} a_0$	$n = 1 \rightarrow a_4 = \frac{1}{3} a_1$	$n = 2 \rightarrow a_5 = \frac{a_2}{5} = 0$
$n = 3 \rightarrow a_6 = \frac{a_3}{5} = \frac{1}{2 \cdot 5} a_0$	$n = 4 \rightarrow a_7 = \frac{a_4}{6} = \frac{1}{2 \cdot 3^2} a_1$	$n = 5 \rightarrow a_8 = 0$

$n=6 \rightarrow a_9 = \frac{a_6}{8} = -\frac{1}{2\cdot5\cdot8}a_0$ $\quad n=7 \rightarrow a_{10} = \frac{a_7}{9} = \frac{1}{2\cdot3\cdot3^3}a_1$ $\quad n=8 \rightarrow a_{11} = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad \vdots \quad \vdots \quad \vdots \quad \vdots \qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$a_{3n} = \frac{1}{2\cdot5\cdot8\cdots(3n-1)}$ $\qquad a_{3n+1} = \frac{1}{n!\,3^n}$ $\qquad a_{3n+2} = 0$

$$\boxed{y(x) = a_0\sum_{n=0}^{\infty}\frac{x^{3n}}{2\cdot5\cdot8\cdots(3n-1)} + a_1\sum_{n=0}^{\infty}\frac{x^{3n+1}}{n!\,3^n}}$$

$$\boxed{y(x) = a_0\left(1+\frac{1}{2}x^3+\frac{1}{10}x^6+\frac{1}{80}x^9+\cdots\right) + a_1\left(x+\frac{1}{3}x^4+\frac{1}{18}x^7+\frac{1}{162}x^{10}+\cdots\right)}$$

Exercise

Find a power series solution. $y'' + 2xy' + 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty}a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty}na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty}n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y'' + 2xy' + 2y = 0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 2\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}2(n+1)a_{n+1}x^{n+1} + \sum_{n=0}^{\infty}2a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}2na_n x^n + \sum_{n=0}^{\infty}2a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}2na_n x^n + 2a_0 + \sum_{n=1}^{\infty}2a_n x^n = 0$$

$$2a_2+2a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}+2na_n+2a_n\right]x^n=0$$

$$2a_2+2a_0=0 \quad\rightarrow\quad \boxed{a_2=-a_0}$$

$$(n+2)(n+1)a_{n+2}+2(n+1)a_n=0 \quad\rightarrow\quad a_{n+2}=-\frac{2}{n+2}a_n \qquad n=1,\ 2,\ \cdots$$

$$a_3=-\frac{2}{3}a_1 \qquad\qquad a_4=-\frac{2}{4}a_2=\frac{1}{2}a_0$$

$$a_5=-\frac{2}{5}a_3=\frac{4}{15}a_1 \qquad\qquad a_6=-\frac{2}{6}a_4=-\frac{1}{6}a_0$$

$$a_7=-\frac{2}{7}a_3=-\frac{8}{105}a_1 \qquad\qquad a_8=-\frac{2}{8}a_6=\frac{1}{24}a_0$$

$$\vdots\ \ \vdots \qquad\qquad \vdots\ \ \vdots$$

$$\boxed{\begin{cases} y_1(x)=\left(1-x^2+\frac{1}{2!}x^4-\frac{1}{3!}x^6+\frac{1}{4!}x^8-\cdots\right)a_0 \\ y_2(x)=\left(x-\frac{2}{3}x^3+\frac{4}{15}x^5-\frac{8}{105}x^7+\cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $2y''+xy'+y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$2y''+xy'+y=0$$

$$2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}na_nx^n+\sum_{n=0}^{\infty}a_nx^n=0$$

$$4a_2 + \sum_{n=1}^{\infty} 2(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} na_n x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n = 0$$

$$4a_2 + a_0 + \sum_{n=1}^{\infty}\left[2(n+2)(n+1)a_{n+2} + (n+1)a_n\right]x^n = 0$$

$$4a_2 + a_0 = 0 \quad \rightarrow \quad \boxed{a_2 = -\frac{1}{4}a_0}$$

$$2(n+2)(n+1)a_{n+2} + (n+1)a_n = 0$$

$$\boxed{a_{n+2} = -\frac{1}{2(n+2)}a_n}$$

a_0	a_1
$n=0 \rightarrow a_2 = -\frac{1}{4}a_0$	$n=1 \rightarrow a_3 = -\frac{1}{6}a_1$
$n=2 \rightarrow a_4 = -\frac{1}{8}a_2 = \frac{1}{2^4 \cdot 2}a_0$	$n=3 \rightarrow a_5 = -\frac{1}{2\cdot 5}a_3 = \frac{1}{2^2 \cdot 3\cdot 5}a_1$
$n=4 \rightarrow a_6 = -\frac{1}{2\cdot 6}a_4 = -\frac{1}{2^6 \cdot 2\cdot 3}a_0$	$n=5 \rightarrow a_7 = -\frac{1}{2\cdot 7}a_5 = -\frac{1}{2^3 \cdot 3\cdot 5\cdot 7}a_1$
$n=6 \rightarrow a_8 = -\frac{1}{2\cdot 8}a_6 = \frac{1}{2^8 \cdot 4!}a_0$	
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$n \ge 1 \quad a_{2n} = \frac{(-1)^n}{2^{2n}\ n!}a_0$$

$$n \ge 1 \quad a_{2n+1} = \frac{(-1)^n}{2^n\ 1\cdot 3\cdot 5\cdots(2n+1)}a_1 = \frac{(-1)^n}{2^n\ (2n+1)!!}a_1$$

$$\boxed{y(x) = a_0\left(1 - \frac{1}{4}x^2 + \frac{1}{32}x^4 - \ldots\right) + a_1\left(x - \frac{1}{6}x^3 + \frac{1}{60}x^5 - \cdots\right)}$$

$$\boxed{y(x) = a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}\ n!}x^{2n} + a_1\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n\ (2n+1)!!}x^{2n+1}}$$

Exercise

Find a power series solution. $3y'' + xy' - 4y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$3y''+xy'-4y=0$

$$3\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-4\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}3(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}4a_n x^n=0$$

$$\sum_{n=0}^{\infty}3(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}na_n x^n-\sum_{n=0}^{\infty}4a_n x^n=0$$

$$6a_2+\sum_{n=1}^{\infty}3(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}na_n x^n-4a_0-\sum_{n=1}^{\infty}4a_n x^n=0$$

$$6a_2-4a_0+\sum_{n=1}^{\infty}\left[3(n+2)(n+1)a_{n+2}+(n-4)a_n\right]x^n=0$$

$$6a_2-4a_0 \quad\rightarrow\quad a_2=\frac{2}{3}a_0$$

$$3(n+2)(n+1)a_{n+2}+(n-4)a_n=0$$

$$a_{n+2}=-\frac{(n-4)}{3(n+1)(n+2)}a_n$$

a_0

$$a_2=\frac{2}{3}a_0$$

$$n=2\ \rightarrow\ a_4=\frac{2}{36}a_2=\frac{1}{27}a_0$$

$$n=4\ \rightarrow\ a_6=0$$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

a_1

$$n=1\ \rightarrow\ a_3=\frac{3}{2\cdot3\cdot3}a_1=\frac{1}{2\cdot3}a_1$$

$$n=3\ \rightarrow\ a_5=\frac{1}{4\cdot5\cdot3}a_3=\frac{1}{5!\cdot3}a_1$$

$$n=5\ \rightarrow\ a_7=-\frac{1}{3\cdot6\cdot7}a_5=-\frac{1}{7!\cdot3^2}a_1$$

$$n=7\ \rightarrow\ a_9=-\frac{3}{3\cdot9\cdot8}a_7=\frac{3}{9!\cdot3^3}a_1$$

$$n=9\ \rightarrow\ a_{11}=-\frac{5}{3\cdot11\cdot10}a_9=-\frac{3\cdot5}{11!\cdot3^4}a_1$$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$n\ge3 \quad a_{2n+1}=\frac{1\cdot3\cdot5\cdots(2n-5)(-1)^n}{(2n+1)!\ 3^n}a_1$$

$$= \frac{(2n-5)!!(-1)^n}{(2n+1)!\ 3^n} a_1$$

$$\boxed{y(x) = a_0\left(1+\frac{2}{3}x^2+\frac{1}{27}x^4\right) + a_1\left(x+\frac{1}{6}x^3+\frac{1}{360}x^5+\sum_{n=3}^{\infty}\frac{(2n-5)!!(-1)^n}{(2n+1)!\ 3^n}\right)}$$

$$\boxed{= a_0\left(1+\frac{2}{3}x^2+\frac{1}{27}x^4\right) + a_1\left(x+\frac{1}{6}x^3+\frac{1}{360}x^5-\frac{1}{45,360}x^7+\cdots\right)}$$

Exercise

Find a power series solution. $5y'' - 2xy' + 10y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$5y'' - 2xy' + 10y = 0$$

$$5\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - 2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 10\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}5(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}2(n+1)a_{n+1}x^{n+1} + \sum_{n=0}^{\infty}10a_n x^n = 0$$

$$\sum_{n=0}^{\infty}5(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}2na_n x^n + \sum_{n=0}^{\infty}10a_n x^n = 0$$

$$10a_2 + \sum_{n=1}^{\infty}5(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}2na_n x^n + 10a_0 + \sum_{n=1}^{\infty}10a_n x^n = 0$$

$$10a_2 + 10a_0 + \sum_{n=1}^{\infty}\left[5(n+2)(n+1)a_{n+2} - 2(n-5)a_n\right]x^n = 0$$

$$10a_2 + 10a_0 \quad \rightarrow \quad \boxed{a_2 = -a_0}$$

$$5(n+2)(n+1)a_{n+2}-2(n-5)a_n=0$$

$$\boxed{a_{n+2}=\frac{2(n-5)}{5(n+1)(n+2)}a_n}$$

a_0

$a_2=-a_0$

$n=2 \rightarrow a_4=-\frac{6}{60}a_2=\frac{1}{10}a_0$

$n=4 \rightarrow a_6=-\frac{2}{5\cdot 5\cdot 6}a_4=-\frac{1}{750}a_0$

$n=6 \rightarrow a_8=\frac{2}{5\cdot 7\cdot 8}a_6=-\frac{2}{8!\cdot 5^2}a_0$

$n=8 \rightarrow a_{10}=\frac{2\cdot 3}{5\cdot 9\cdot 10}a_8=-\frac{2^2\cdot 3}{10!\cdot 5^3}a_0$

$n=10 \rightarrow a_{12}=\frac{2\cdot 5}{5\cdot 11\cdot 12}a_8=-\frac{2^3\cdot 3\cdot 5}{12!\cdot 5^4}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$n\geq 4 \quad a_{2n}=-15\cdot\frac{2^n(2n-7)!!}{5^n\,(2n)!}a_0$

a_1

$n=1 \rightarrow a_3=-\frac{8}{30}a_1=-\frac{4}{15}a_1$

$n=3 \rightarrow a_5=-\frac{4}{100}a_3=\frac{4}{375}a_1$

$n=5 \rightarrow a_7=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x)=a_0\left(1-x^2+\frac{1}{10}x^4-\frac{1}{750}x^6-\frac{1}{105{,}000}x^8-\cdots\right)+a_1\left(x-\frac{4}{15}x^3+\frac{4}{375}x^5\right)}$$

Exercise

Find a power series solution. $(x-1)y''+y'=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$(x-1)y''+y'=0$$

$$(x-1)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=0$$

$$x\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=0$$

$$\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=0$$

$$\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^n-2a_2-\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n+a_1+\sum_{n=1}^{\infty}(n+1)a_{n+1}x^n=0$$

$$a_1-2a_2+\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}-(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}\right]x^n=0$$

$$a_1-2a_2+\sum_{n=1}^{\infty}\left[-(n+2)(n+1)a_{n+2}+(n+1)^2a_{n+1}\right]x^n=0$$

$$a_1-2a_2=0 \quad\rightarrow\quad \boxed{a_2=\tfrac{1}{2}a_1}$$

$$-(n+2)(n+1)a_{n+2}+(n+1)^2a_{n+1}$$

$$\rightarrow\quad \boxed{a_{n+2}=\frac{n+1}{n+2}a_{n+1}} \qquad n=1,\ 2,\ \dots$$

$$a_3=\frac{2}{3}a_2=\frac{1}{3}a_1$$

$$a_4=\frac{3}{4}a_3=\frac{1}{4}a_1$$

$$a_5=\frac{4}{5}a_4=\frac{1}{5}a_1$$

$$\vdots\quad\vdots$$

$$\boxed{\begin{cases} y_1(x)=a_0 \\ y_2(x)=\left(x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $(x+2)y''+xy'-y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$(x+2)y''+xy'-y=0$$

$$(x+2)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1}+\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}na_nx^n-\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^n+4a_2+\sum_{n=1}^{\infty}2(n+2)(n+1)a_{n+2}x^n+4a_2+\sum_{n=1}^{\infty}na_nx^n-a_0-\sum_{n=1}^{\infty}a_nx^n=0$$

$$4a_2-a_0+\sum_{n=1}^{\infty}\left(n(n+1)a_{n+1}+2(n+2)(n+1)a_{n+2}+(n-1)a_n\right)x^n=0$$

$$4a_2-a_0=0 \quad\rightarrow\quad \underline{a_2=\tfrac{1}{4}a_0}\big| \quad \underline{a_1}\big|$$

$$n(n+1)a_{n+1}+2(n+2)(n+1)a_{n+2}+(n-1)a_n=0$$

$$\underline{a_{n+2}=-\frac{n-1}{2(n+2)(n+1)}a_n-\frac{n}{2(n+2)}a_{n+1}}\bigg| \quad n=1,\ 2,\ \ldots$$

$$a_3=-\frac{1}{6}a_2=-\frac{1}{24}a_0$$

$$a_4=-\frac{1}{24}a_2-\frac{1}{4}a_3=-\frac{1}{96}a_0+\frac{1}{96}a_0=0$$

$$a_5=-\frac{1}{20}a_3-\frac{3}{10}a_4=\frac{1}{480}a_0$$

$$a_6 = -\frac{3}{60}a_4 - \frac{1}{3}a_5 = -\frac{1}{1,440}a_0$$

$$\vdots \quad \vdots \qquad\qquad \vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = a_1 \\ y_2(x) = \left(1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 - \cdots\right)a_0 \end{cases}}$$

Exercise

Find a power series solution. $y'' - (x+1)y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$y'' - (x+1)y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - x\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_n\right]x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

$$2a_2 - a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_n\right]x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

$$2a_2 - a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_n - a_{n-1}\right]x^n = 0$$

$$2a_2 - a_0 = 0 \quad \rightarrow \quad \boxed{a_2 = \frac{1}{2}a_0}$$

$$(n+2)(n+1)a_{n+2} - a_n - a_{n-1} = 0$$

$$\boxed{a_{n+2}=\frac{a_n+a_{n-1}}{(n+1)(n+2)}}$$

$$a_0 \qquad a_1$$

$$a_2=\frac{1}{2}a_0$$

$$n=1 \ \to\ a_3=\frac{1}{6}\left(a_1+a_0\right)$$

$$n=2 \ \to\ a_4=\frac{1}{12}\left(a_2+a_1\right)=\frac{1}{12}\left(\frac{1}{2}a_0+a_1\right)$$

$$n=3 \ \to\ a_5=\frac{1}{20}\left(a_3+a_2\right)=\frac{1}{20}\left(\frac{2}{3}a_0+\frac{1}{6}a_1\right)$$

$$n=4 \ \to\ a_6=\frac{1}{30}\left(a_4+a_3\right)=\frac{1}{30}\left(\frac{1}{2}a_0+a_1+\frac{1}{6}a_0+\frac{1}{6}a_1\right)=\frac{1}{30}\left(\frac{2}{3}a_0+\frac{7}{6}a_1\right)$$

$a_0 \neq 0 \quad a_1=0$	$a_0=0 \quad a_1 \neq 0$
$a_2=\frac{1}{2}a_0$	$a_2=0$
$a_3=\frac{1}{6}a_0$	$a_3=\frac{1}{6}a_1$
$a_4=\frac{1}{24}a_0$	$a_4=\frac{1}{12}a_1$
$a_5=\frac{1}{30}a_0$	$a_5=\frac{1}{120}a_1$
$a_6=\frac{1}{45}a_0$	$a_6=\frac{7}{180}a_1$

$$\boxed{\begin{cases} y_1(x)=\left(1+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{30}x^5+\frac{1}{45}x^6+\cdots\right)a_0 \\ y_2(x)=\left(x+\frac{1}{6}x^3+\frac{1}{12}x^4+\frac{1}{120}x^5+\frac{7}{180}x^6+\cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $y''-(x+1)y'-y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y'' - (x+1)y' - y = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - (x+1)\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}na_n x^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}na_n x^n - a_1 - \sum_{n=1}^{\infty}(n+1)a_{n+1}x^n - a_0 - \sum_{n=1}^{\infty}a_n x^n = 0$$

$$2a_2 - a_1 - a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - na_n - (n+1)a_{n+1} - a_n\right]x^n = 0$$

$$2a_2 - a_1 - a_0 = 0 \quad \rightarrow \quad a_2 = \frac{1}{2}a_0 + \frac{1}{2}a_1$$

$$(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} - (n+1)a_n = 0$$

$$\boxed{a_{n+2} = \frac{1}{n+2}a_{n+1} + \frac{1}{n+2}a_n}$$

$$a_3 = \frac{1}{3}a_2 + \frac{1}{3}a_1 = \frac{1}{6}a_0 + \frac{1}{2}a_1$$

$$a_4 = \frac{1}{4}a_3 + \frac{1}{4}a_2 = \frac{1}{24}a_0 + \frac{1}{8}a_1 + \frac{1}{8}a_0 + \frac{1}{8}a_1 = \frac{1}{6}a_0 + \frac{1}{4}a_1$$

$$a_5 = \frac{1}{5}a_4 + \frac{1}{5}a_3 = \frac{1}{30}a_0 + \frac{1}{20}a_1 + \frac{1}{30}a_0 + \frac{1}{10}a_1 = \frac{1}{15}a_0 + \frac{3}{20}a_1$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \cdots\right)a_0 \\ y_2(x) = \left(x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $(x^2+1)y'' - 6y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y''-6y=0$$

$$\left(x^2+1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-6\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-6\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-6\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n-6a_0-6a_1x-6\sum_{n=2}^{\infty}a_n x^n=0$$

$$2a_2-6a_0+\left(6a_3-6a_1\right)x+\sum_{n=2}^{\infty}\left[\left(n^2-n-6\right)a_n+(n+2)(n+1)a_{n+2}\right]x^n=0$$

$$2a_2-6a_0+\left(6a_3-6a_1\right)x=0 \quad \to \quad \begin{cases} a_2=3a_0 \\ a_3=a_1 \end{cases}$$

$$(n+2)(n-3)a_n+(n+2)(n+1)a_{n+2}=0$$

$$\Rightarrow \quad a_{n+2}=-\frac{n-3}{n+1}a_n \qquad n=2,3,\ldots$$

$$a_4=\frac{1}{3}a_2=a_0 \qquad\qquad a_5=0$$

$$a_6=-\frac{1}{5}a_4=-\frac{1}{5}a_0 \qquad\qquad a_7=-\frac{1}{3}a_5=0$$

$$a_8=-\frac{3}{7}a_6=\frac{3}{35}a_0$$

$$\vdots \quad \vdots$$

$$\begin{cases} y_1(x)=\left(1+3x^2+x^4-\frac{1}{5}x^6+\cdots\right)a_0 \\ y_2(x)=\left(x+x^3\right)a_1 \end{cases}$$

Exercise

Find a power series solution. $\left(x^2+2\right)y''+3xy'-y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+2\right)y''+3xy'-y=0$$

$$\left(x^2+2\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+3x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}3(n+1)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_nx^n+\sum_{n=0}^{\infty}\left[2(n+2)(n+1)a_{n+2}-a_n\right]x^n+\sum_{n=1}^{\infty}3na_nx^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_nx^n+4a_2-a_0+\left(12a_3-a_1\right)x+\sum_{n=2}^{\infty}\left[2(n+2)(n+1)a_{n+2}-a_n\right]x^n$$

$$+3a_1x+\sum_{n=2}^{\infty}3na_nx^n=0$$

$$4a_2-a_0+\left(12a_3+2a_1\right)x+\sum_{n=2}^{\infty}\left[n(n-1)a_n+2(n+2)(n+1)a_{n+2}+(3n-1)a_n\right]x^n=0$$

$$4a_2-a_0+\left(12a_3+2a_1\right)x\quad\Rightarrow\quad \boxed{a_2=\frac{1}{4}a_0}\quad \boxed{a_3=-\frac{1}{6}a_1}$$

$$2(n+2)(n+1)a_{n+2}+\left(n^2+2n-1\right)a_n=0$$

$$\to\quad a_{n+2}=-\frac{n^2+2n-1}{2(n+2)(n+1)}a_n\qquad n=2,3,\ldots$$

$$a_4=-\frac{7}{24}a_2=-\frac{7}{96}a_0\qquad\qquad a_5=-\frac{7}{20}a_3=\frac{7}{120}a_1$$

$$a_6 = -\frac{23}{60}a_4 = \frac{161}{5760}a_0 \qquad\qquad a_7 = -\frac{17}{42}a_5 = -\frac{17}{720}a_1$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = \left(1 + \frac{1}{4}x^2 - \frac{7}{96}x^4 + \frac{161}{5760}x^6 - \cdots\right)a_0 \\ y_2(x) = \left(1 - \frac{1}{6}x^3 + \frac{7}{120}x^5 - \frac{17}{720}x^7 + \cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $\left(x^2 - 1\right)y'' + xy' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2 - 1\right)y'' + xy' - y = 0$$

$$\left(x^2 - 1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2} - \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n - \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_n\right]x^n + \sum_{n=1}^{\infty}na_n x^n = 0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n - \left(2a_2 + a_0\right) - \left(6a_3 + a_1\right)x - \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_n\right]x^n$$

$$+ a_1 x + \sum_{n=2}^{\infty}na_n x^n = 0$$

$$-2a_2 - a_0 - 6a_3 x + \sum_{n=2}^{\infty}\left[n(n-1)a_n - (n+2)(n+1)a_{n+2} + (n-1)a_n\right]x^n = 0$$

$$-2a_2 - a_0 - 6a_3 x = 0 \quad \underline{\rightarrow a_2 = -\tfrac{1}{2}a_0} \quad \underline{a_3 = 0}$$

$$-(n+2)(n+1)a_{n+2} + (n+1)(n-1)a_n = 0$$

$$\rightarrow \quad a_{n+2} = \frac{n-1}{n+2}a_n \qquad n = 2,3,\ldots$$

$$a_4 = \tfrac{1}{4}a_2 = -\tfrac{1}{8}a_0 \qquad\qquad a_5 = -\tfrac{2}{5}a_3 = 0$$

$$a_6 = \tfrac{1}{2}a_4 = -\tfrac{1}{16}a_0 \qquad\qquad a_7 = \tfrac{4}{7}a_5 = 0$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = \left(1 - \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 - \tfrac{1}{16}x^6 - \cdots\right)a_0 \\ y_2(x) = a_1 \end{cases}}$$

$$\begin{cases} y_1(x) = 1 - \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 - \tfrac{1}{16}x^6 - \cdots \\ y_2(x) = x \end{cases}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y'' + xy' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y'' + xy' - y = 0$$

$$\left(x^2+1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$x^2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n}+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}a_{n}x^{n}=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_{n}x^{n}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n}+\sum_{n=1}^{\infty}na_{n}x^{n}-\sum_{n=0}^{\infty}a_{n}x^{n}=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_{n}x^{n}+\sum_{n=1}^{\infty}na_{n}x^{n}+\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-a_{n}\right]x^{n}=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_{n}x^{n}+a_{1}x+\sum_{n=2}^{\infty}na_{n}x^{n}+(2a_{2}-a_{0})+(6a_{3}-a_{1})x+\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}-a_{n}\right]x^{n}=0$$

$$2a_{2}-a_{0}+6a_{3}x+\sum_{n=2}^{\infty}\left[\left(n^{2}-n+n-1\right)a_{n}+(n+2)(n+1)a_{n+2}\right]x^{n}=0$$

$$\begin{cases}2a_{2}-a_{0}=0 & \rightarrow \boxed{a_{2}=\frac{1}{2}a_{0}}\\ 6a_{3}x=0 & \rightarrow \boxed{a_{3}=0}\end{cases}$$

$$(n-1)(n+1)a_{n}+(n+2)(n+1)a_{n+2}=0$$

$$\boxed{a_{n+2}=-\frac{n-1}{n+2}a_{n}}$$

a_0	a_1
$n=0\rightarrow a_{2}=\frac{1}{2}a_{0}$	$n=1\rightarrow a_{3}=0$
$n=2\rightarrow a_{4}=-\frac{1}{4}a_{2}=-\frac{1}{2^{3}}a_{0}$	$n=3\rightarrow a_{5}=-\frac{2}{5}a_{3}=0$
$n=4\rightarrow a_{6}=-\frac{3}{6}a_{4}=\frac{1\cdot 3}{2^{3}\cdot 3!}a_{0}$	$n=5\rightarrow a_{7}=0$
$n=6\rightarrow a_{8}=-\frac{5}{8}a_{6}=-\frac{1\cdot 3\cdot 5}{2^{4}\cdot 4!}a_{0}$	$\vdots\quad\vdots\quad\vdots\quad\vdots$
$\vdots\quad\vdots\quad\vdots\quad\vdots$	

$$a_{2n}=(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^{n}\cdot n!}a_{0}\quad (n\geq 2)$$

$$\boxed{y(x)=a_{0}\left(1+\frac{1}{2}x^{2}-\frac{1}{2^{2}\cdot 2!}x^{4}+\frac{1\cdot 3}{2^{3}\cdot 3!}x^{6}-\frac{1\cdot 3\cdot 5}{2^{4}\cdot 4!}x^{8}+\cdots\right)+a_{1}x}$$

$$\boxed{=a_{0}\left(1+\frac{1}{2}x^{2}+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(2n-3)!!}{2^{n}\cdot n!}x^{2n}\right)+a_{1}x}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y''-xy'+y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y''-xy'+y=0$$

$$\left(x^2+1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$x^2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=1}^{\infty}na_n x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n-\sum_{n=1}^{\infty}na_n x^n+\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_n\right]x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n-a_1x-\sum_{n=2}^{\infty}na_n x^n+\left(2a_2+a_0\right)+\left(6a_3+a_1\right)x+\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_n\right]x^n=0$$

$$2a_2+a_0+6a_3x+\sum_{n=2}^{\infty}\left[\left(n^2-2n+1\right)a_n+(n+2)(n+1)a_{n+2}\right]x^n=0$$

$$\begin{cases}2a_2+a_0=0 & \rightarrow \boxed{a_2=-\frac{1}{2}a_0}\\ 6a_3x=0 & \rightarrow \boxed{a_3=0}\end{cases}$$

$$(n-1)^2a_n+(n+1)(n+2)a_{n+2}=0$$

$$\boxed{a_{n+2} = -\frac{(n-1)^2}{(n+1)(n+2)}a_n}$$

a_0

$n=0 \to a_2 = -\frac{1}{2}a_0$

$n=2 \to a_4 = -\frac{1}{12}a_2 = \frac{1}{4!}a_0$

$n=4 \to a_6 = -\frac{3^2}{5\cdot 6}a_4 = -\frac{3^2}{6!}a_0$

$n=6 \to a_8 = -\frac{5^2}{7\cdot 8}a_6 = \frac{1\cdot 3^2\cdot 5^2}{8!}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

a_1

$n=1 \to a_3 = 0$

$n=3 \to a_5 = -\frac{2}{5}a_3 = 0$

$n=5 \to a_7 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{2n} = (-1)^{n-1}\frac{1\cdot 3^2\cdot 5^2\cdots(2n-3)^2}{(2n)!}a_0 \quad (n \ge 3)$$

$$\boxed{y(x) = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{9}{6!}x^6 - \frac{1\cdot 3^2\cdot 5^2}{8!}x^8 - \cdots\right) + a_1 x}$$

$$\boxed{= a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \sum_{n=3}^{\infty}(-1)^n\frac{(2n-3)^2!!}{(2n)!}x^{2n}\right) + a_1 x}$$

Exercise

Find a power series solution. $\left(1-x^2\right)y'' - 6xy' - 4y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(1-x^2\right)y'' - 6xy' - 4y = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2} - 6x\sum_{n=1}^{\infty}na_n x^{n-1} - 4\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty}n(n-1)a_n x^n - \sum_{n=1}^{\infty}6na_n x^n - \sum_{n=0}^{\infty}4a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}n(n-1)a_n x^n - \sum_{n=0}^{\infty}6na_n x^n - \sum_{n=0}^{\infty}4a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - (n(n-1)+6n+4)a_n\right]x^n = 0$$

$$(n+2)(n+1)a_{n+2} - \left(n^2+5n+4\right)a_n = 0$$

$$(n+2)(n+1)a_{n+2} = (n+4)(n+1)a_n$$

$$\boxed{a_{n+2} = \frac{n+4}{n+2}a_n}$$

a_0	a_1
$n=2 \rightarrow a_2 = 2a_0$	$n=3 \rightarrow a_3 = \frac{5}{3}a_1$
$n=4 \rightarrow a_4 = \frac{6}{4}a_2 = 3a_0$	$n=5 \rightarrow a_5 = \frac{7}{5}a_3 = \frac{7}{3}a_1$
$n=6 \rightarrow a_6 = \frac{8}{6}a_3 = 4a_0$	$n=7 \rightarrow a_7 = \frac{11}{7}a_5 = \frac{11}{3}a_1$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$a_{2k} = (k+1)a_0$	$a_{2k+1} = \frac{2k+3}{3}a_1$

$$y(x) = a_0\left(1+2x^2+3x^4+4x^6+\cdots\right) + a_1\left(x+\frac{5}{3}x^3+\frac{7}{3}x^5+\frac{11}{3}x^7+\cdots\right)$$

$$\boxed{= \frac{a_0}{\left(1-x^2\right)^2} + \frac{3x-x^3}{3\left(1-x^2\right)^2}a_1}$$

Exercise

Find a power series solution. $y'' + (x-1)^2 y' - 4(x-1)y = 0$

Solution

Let $z = x-1 \rightarrow dz = dx$

$$y(x) = \sum_{n=0}^{\infty}a_n z^n$$

$$y'(x) = \sum_{n=1}^{\infty}na_n z^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n$$

$y''+z^2y'-4zy=0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n + z^2\sum_{n=1}^{\infty} na_n z^{n-1} - 4z\sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n + \sum_{n=1}^{\infty} na_n z^{n+1} - \sum_{n=0}^{\infty} 4a_n z^{n+1} = 0$$

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}z^n + \sum_{n=1}^{\infty}(n-4)a_n z^{n+1} = 0$$

$$2a_2 + \sum_{n=0}^{\infty}(n+2)(n+3)a_{n+3}z^{n+1} + \sum_{n=1}^{\infty}(n-4)a_n z^{n+1} = 0$$

$$2a_2 + \sum_{n=0}^{\infty}\left[(n+2)(n+3)a_{n+3} + (n-4)a_n\right]z^{n+1} = 0$$

$\boxed{a_2 = 0}$

$(n+2)(n+3)a_{n+3} + (n-4)a_n = 0$

$$\boxed{a_{n+3} = -\frac{n-4}{(n+2)(n+3)}a_n}$$

a_0	a_1	$a_2 = 0$
$n=0 \rightarrow\ a_3 = \frac{4}{2\cdot 3}a_0$	$n=1 \rightarrow\ a_4 = \frac{3}{3\cdot 4}a_1$	$n=2\ \rightarrow\ a_5 = \frac{2}{20}a_2 = 0$
$n=3 \rightarrow\ a_6 = \frac{1}{5\cdot 6}a_3 = \frac{4}{2\cdot 3\cdot 5\cdot 6}a_0$	$n=4 \rightarrow\ a_7 = 0$	$n=5\ \rightarrow\ a_8 = 0$
$n=6 \rightarrow\ a_9 = -\frac{2}{8\cdot 9}a_6 = -\frac{8}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}a_0$	$n=7 \rightarrow\ a_{10} = 0$	$n=8\ \rightarrow\ a_5 = 0$
$\vdots\quad \vdots\quad \vdots\quad \vdots$		$\vdots\quad \vdots\quad \vdots\quad \vdots$

$$y(x) = a_0\left(1+\frac{4}{2\cdot 3}z^3 + \frac{4}{2\cdot 3\cdot 5\cdot 6}z^6 - \frac{8}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 8}z^9 + \cdots\right) + a_1\left(z+\frac{1}{4}z^4\right)$$

$$\boxed{= a_0\left(1+\tfrac{2}{3}(x-1)^3 + \tfrac{1}{45}(x-1)^6 - \tfrac{1}{1,620}(x-1)^9 + \cdots\right) + a_1\left(x-1+\tfrac{1}{4}(x-1)^4\right)}$$

Exercise

Find a power series solution. $\left(2-x^2\right)y''-xy'+16y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(2-x^2\right)y''-xy'+16y=0$$

$$2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-x^2\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-x\sum_{n=1}^{\infty}na_nx^{n-1}+16\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n-\sum_{n=2}^{\infty}n(n-1)a_nx^n-\sum_{n=1}^{\infty}na_nx^n+\sum_{n=0}^{\infty}16a_nx^n=0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}n(n-1)a_nx^n-\sum_{n=0}^{\infty}na_nx^n+\sum_{n=0}^{\infty}16a_nx^n=0$$

$$\sum_{n=0}^{\infty}\left[2(n+2)(n+1)a_{n+2}-\left(n^2-n+n-16\right)a_n\right]x^n=0$$

$$2(n+1)(n+2)a_{n+2}-\left(n^2-16\right)a_n=0$$

$$2(n+1)(n+2)a_{n+2}=(n+4)(n-4)a_n$$

$$\boxed{a_{n+2}=\frac{(n+4)(n-4)}{2(n+1)(n+2)}a_n}$$

a_0	a_1
$n=0\to a_2=-4a_0$	$n=1\to a_3=-\frac{5}{4}a_1$
$n=2\to a_4=-\frac{1}{2}a_2=2a_0$	$n=3\to a_5=-\frac{7}{40}a_3=\frac{7}{32}a_1$
$n=4\to a_6=0$	$n=5\to a_7=\frac{9}{70}a_5=\frac{9}{320}a_1$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$n=7\to a_9=\frac{33}{144}a_7=\frac{33}{5{,}120}a_1$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n+1} = \frac{(2n-5)!!\ (2n+3)!!}{2^n\ (2n+1)!} a_1$$

$$\boxed{y(x) = a_0\left(1-4x^2+2x^4\right) + a_1\left(x - \frac{5}{4}x^3 + \frac{7}{32}x^5 + \frac{9}{320}x^7 + \cdots\right)}$$

$$\boxed{= a_0\left(1-4x^2+2x^4\right) + a_1 \sum_{n=0}^{\infty} \frac{(2n-5)!!\ (2n+3)!!}{2^n\ (2n+1)!} x^{2n+1}}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y'' - y' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2+1\right)y'' - y' + y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} - (n+1) a_{n+1} + \left(n^2 - n + 1\right) a_n\right] x^n = 0$$

$$(n+1)(n+2) a_{n+2} - (n+1) a_{n+1} + \left(n^2 - n + 1\right) a_n = 0$$

$$\boxed{a_{n+2} = \frac{(n+1)a_{n+1} - \left(n^2 - n + 1\right)a_n}{(n+1)(n+2)}}$$

$$n = 0 \to \ a_2 = \frac{1}{2}\left(a_1 - a_0\right)$$

$$n = 1 \to \ a_3 = \frac{1}{6}\left(2a_2 - a_1\right) = \frac{1}{6}\left(a_1 - a_0 - a_1\right) = -\frac{1}{6}a_0$$

$$n = 2 \to \ a_4 = \frac{1}{12}\left(3a_3 - 3a_2\right) = \frac{1}{4}\left(-\frac{1}{6}a_0 - \frac{1}{2}a_1 + \frac{1}{2}a_0\right) = \frac{1}{12}a_0 - \frac{1}{8}a_1$$

$$n = 3 \to \ a_5 = \frac{1}{20}\left(4a_4 - 7a_3\right) = \frac{1}{20}\left(\frac{1}{3}a_0 - \frac{1}{2}a_1 + \frac{7}{6}a_0\right) = \frac{3}{40}a_0 - \frac{1}{40}a_1$$

$$n = 4 \to \ a_6 = \frac{1}{30}\left(5a_5 - 13a_4\right) = \frac{1}{30}\left(\frac{3}{8}a_0 - \frac{1}{8}a_1 - \frac{13}{12}a_0 + \frac{13}{8}a_1\right) = -\frac{17}{720}a_0 + \frac{1}{20}a_1$$

$$y(x) = a_0 + a_1 x + \left(\frac{1}{2}a_0 - \frac{1}{2}a_1\right)x^2 - \frac{1}{6}a_0 x^3 + \left(\frac{1}{12}a_0 - \frac{1}{8}a_1\right)x^4 + \left(\frac{3}{40}a_0 - \frac{1}{40}a_1\right)x^5$$
$$+\left(-\frac{17}{720}a_0 + \frac{1}{20}a_1\right)x^6 + \cdots$$

$$\boxed{\begin{aligned} y(x) = {} & a_0\left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 - \frac{17}{720}x^6 + \cdots\right) \\ & + a_1\left(x - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{40}x^5 + \frac{1}{20}x^6 + \cdots\right) \end{aligned}}$$

Exercise

Find a power series solution. $\left(x^2 + 1\right)y'' + 6xy' + 4y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2 + 1\right)y'' + 6xy' + 4y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 6x\sum_{n=1}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} 6na_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 6na_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + \left(n^2+5n+4\right)a_n\right]x^n = 0$$

$$(n+1)(n+2)a_{n+2} + (n+1)(n+4)a_n = 0$$

$$\boxed{a_{n+2} = -\frac{n+4}{n+2}a_n}$$

a_0

$n=0 \rightarrow a_2 = -2a_0$

$n=2 \rightarrow a_4 = -\frac{3}{2}a_2 = 3a_0$

$n=4 \rightarrow a_6 = -\frac{8}{6}a_4 = -4a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

a_1

$n=1 \rightarrow a_3 = -\frac{5}{3}a_1$

$n=3 \rightarrow a_5 = -\frac{7}{5}a_3 = \frac{7}{3}a_1$

$n=5 \rightarrow a_7 = -\frac{9}{7}a_5 = -\frac{9}{3}a_1$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{2n} = (-1)^n(n+1)a_0 \qquad\qquad a_{2n+1} = (-1)^n(2n+3)a_1$$

$$\boxed{y(x) = a_0\sum_{n=0}^{\infty}(-1)^n(n+1)x^{2n} + \frac{1}{3}a_1\sum_{n=0}^{\infty}(-1)^n(2n+3)x^{2n+1}}$$

$$\boxed{y(x) = a_0\left(1-2x^2+3x^4-4x^6+\cdots\right) + \frac{1}{3}a_1\left(x-\frac{5}{3}x^3+\frac{7}{3}x^5-\frac{9}{3}x^7+\cdots\right)}$$

Exercise

Find a power series solution. $\left(x^2-1\right)y'' - 6xy' + 12y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2-1\right)y''-6xy'+12y=0$$

$$x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-6x\sum_{n=1}^{\infty}na_n x^{n-1}+12\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=1}^{\infty}6na_n x^n+\sum_{n=0}^{\infty}12a_n x^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}6na_n x^n+\sum_{n=0}^{\infty}12a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[-(n+2)(n+1)a_{n+2}+\left(n^2-7n+12\right)a_n\right]x^n=0$$

$$-(n+1)(n+2)a_{n+2}+(n-3)(n-4)a_n=0$$

$$\boxed{a_{n+2}=\frac{(n-3)(n-4)}{(n+1)(n+2)}a_n}$$

a_0	a_1
$n=0\to\ a_2=6a_0$	$n=1\ \to\ a_3=a_1$
$n=2\ \to\ a_4=\frac{1}{6}a_2=a_0$	$n=3\ \to\ a_5=0$
$n=4\ \to\ a_6=0$	$\vdots\quad\vdots\quad\vdots\quad\vdots$
$\vdots\quad\vdots\quad\vdots\quad\vdots$	

$$\boxed{y(x)=a_0\left(1+6x^2+x^4\right)+a_1\left(x+x^3\right)}$$

Exercise

Find a power series solution. $\left(x^2-1\right)y''+8xy'+12y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2-1\right)y''+8xy'+12y=0$$

$$x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+8x\sum_{n=1}^{\infty}na_n x^{n-1}+12\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}8na_n x^n+\sum_{n=0}^{\infty}12a_n x^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}8na_n x^n+\sum_{n=0}^{\infty}12a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[-(n+2)(n+1)a_{n+2}+\left(n^2+7n+12\right)a_n\right]x^n=0$$

$$-(n+1)(n+2)a_{n+2}+(n+3)(n+4)a_n=0$$

$$a_{n+2}=\frac{(n+3)(n+4)}{(n+1)(n+2)}a_n$$

$$a_0$$

$$n=0\rightarrow\ a_2=6a_0$$

$$n=2\ \rightarrow\ a_4=\frac{5}{2}a_2=15a_0$$

$$n=4\ \rightarrow\ a_6=\frac{5}{2}a_4=28a_0$$

$$\vdots\quad\vdots\quad\vdots\quad\vdots$$

$$a_{2n}=(n+1)(2n+1)a_0$$

$$a_1$$

$$n=1\ \rightarrow\ a_3=\frac{10}{3}a_1$$

$$n=3\ \rightarrow\ a_5=\frac{42}{20}a_3=\frac{21}{3}a_1$$

$$n=5\ \rightarrow\ a_7=\frac{12}{7}a_5=\frac{36}{3}a_1$$

$$\vdots\quad\vdots\quad\vdots\quad\vdots$$

$$a_{2n+1}=\frac{1}{3}(n+1)(2n+3)a_0$$

$$y(x)=a_0\left(1+6x^2+15x^4+28x^6+\cdots\right)+a_1\left(x+\frac{10}{3}x^3+7x^5+12x^7+\cdots\right)$$

$$=a_0\sum_{n=0}^{\infty}(n+1)(2n+1)x^{2n}+\frac{1}{3}a_1\sum_{n=0}^{\infty}(n+1)(2n+3)x^{2n+1}$$

Exercise

Find a power series solution. $\left(x^2-1\right)y''+4xy'+2y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2-1\right)y''+4xy'+2y=0$$

$$x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+4x\sum_{n=1}^{\infty}na_n x^{n-1}+2\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}4na_n x^n+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}4na_n x^n+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[-(n+2)(n+1)a_{n+2}+\left(n^2+3n+2\right)a_n\right]x^n=0$$

$$-(n+1)(n+2)a_{n+2}+(n+1)(n+2)a_n=0$$

$$\boxed{a_{n+2}=a_n}$$

a_0	a_1
$n=0\to\ a_2=a_0$	$n=1\ \to\ a_3=a_1$
$n=2\ \to\ a_4=a_2=a_0$	$n=3\ \to\ a_5=a_1$
$\vdots\quad\vdots\quad\vdots\quad\vdots$	$\vdots\quad\vdots\quad\vdots\quad\vdots$

$$\boxed{y(x)=a_0\left(1+x^2+x^4+x^6+\cdots\right)+a_1\left(x+x^3+x^5+x^7+\cdots\right)}$$

$$=a_0\sum_{n=0}^{\infty}x^{2n}+a_1\sum_{n=0}^{\infty}x^{2n+1}$$

$$= \boxed{\frac{a_0 + a_1 x}{1 - x^2}}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y'' - 4xy' + 6y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2+1\right)y'' - 4xy' + 6y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 4x \sum_{n=1}^{\infty} n a_n x^{n-1} + 6 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} + \left(n^2 - 5n + 6\right) a_n \right] x^n = 0$$

$$(n+1)(n+2) a_{n+2} + (n-2)(n-3) a_n = 0$$

$$\boxed{a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)} a_n}$$

a_0	a_1
$n=0 \rightarrow a_2 = -3a_0$	$n=1 \rightarrow a_3 = -\frac{1}{3} a_1$
$n=2 \rightarrow a_4 = 0$	$n=3 \rightarrow a_5 = 0$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = a_0\left(1 - 3x^2\right) + a_1\left(x - \frac{1}{3}x^3\right)}$$

Exercise

Find a power series solution. $\left(x^2+2\right)y''+4xy'+2y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+2\right)y''+4xy'+2y=0$$

$$x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}+2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+4x\sum_{n=1}^{\infty}na_n x^{n-1}+2\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}4na_n x^n+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}4na_n x^n+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[2(n+2)(n+1)a_{n+2}+\left(n^2+3n+2\right)a_n\right]x^n=0$$

$$2(n+1)(n+2)a_{n+2}+(n+1)(n+2)a_n=0$$

$$\boxed{a_{n+2}=-\frac{1}{2}a_n}$$

a_0

$n=0\to\ a_2=-\frac{1}{2}a_0$

$n=2\ \to\ a_4=-\frac{1}{2}a_2=\frac{1}{2^2}a_0$

$n=4\ \to\ a_6=-\frac{1}{2}a_4=-\frac{1}{2^3}a_0$

$\vdots\quad\vdots\quad\vdots\quad\vdots$

$$a_{2n}=(-1)^n\frac{1}{2^n}a_0$$

a_1

$n=1\ \to\ a_3=-\frac{1}{2}a_1$

$n=3\ \to\ a_5=-\frac{1}{2}a_3=\frac{1}{2^2}a_1$

$n=5\ \to\ a_7=-\frac{1}{2}a_5=-\frac{1}{2^3}a_1$

$\vdots\quad\vdots\quad\vdots\quad\vdots$

$$a_{2n+1}=(-1)^n\frac{1}{2^n}a_1$$

$$y(x) = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{4}x^4 - \frac{1}{8}x^6 + \cdots\right) + a_1\left(x - \frac{1}{2}x^3 + \frac{1}{4}x^5 - \frac{1}{8}x^7 + \cdots\right)$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^{2n+1}$$

Exercise

Find a power series solution. $\left(x^2 - 3\right)y'' + 2xy' = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2 - 3\right)y'' + 2xy' = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 3\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 2x \sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} 3(n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} 2n a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} 3(n+1)(n+2) a_{n+2} x^n + \sum_{n=0}^{\infty} 2n a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[-3(n+1)(n+2) a_{n+2} + \left(n^2 + n\right) a_n\right] x^n = 0$$

$$-3(n+1)(n+2) a_{n+2} + n(n+1) a_n = 0$$

$$a_{n+2} = \frac{1}{3}\frac{n}{n+2} a_n$$

a_0

$n = 0 \rightarrow a_2 = 0$

a_1

$n = 1 \rightarrow a_3 = \frac{1}{3^2} a_1$

$n=2 \;\rightarrow\; a_4 = \frac{2}{12}a_2 = 0$

$n=4 \;\rightarrow\; a_6 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$n=3 \;\rightarrow\; a_5 = \frac{1}{3}\frac{3}{5}a_3 = \frac{1}{3^2 \cdot 5}a_1$

$n=5 \;\rightarrow\; a_7 = \frac{1}{3}\frac{5}{7}a_5 = \frac{1}{3^3 \cdot 7}a_1$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{2n+1} = (-1)^n \frac{1}{(2n+1)3^n}a_1$$

$$y(x) = a_0 + a_1\left(x + \frac{1}{9}x^3 + \frac{1}{45}x^5 + \frac{1}{189}x^7 + \cdots\right)$$

$$= a_0 + a_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)3^n} x^{2n+1}$$

Exercise

Find a power series solution. $\left(x^2+3\right)y'' - 7xy' + 16y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+3\right)y'' - 7xy' + 16y = 0$$

$$x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 3\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - 7x\sum_{n=1}^{\infty} n a_n x^{n-1} + 16\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} 3(n+1)(n+2)a_{n+2}x^n - \sum_{n=1}^{\infty} 7n a_n x^n + \sum_{n=0}^{\infty} 16 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} 3(n+1)(n+2)a_{n+2}x^n - \sum_{n=0}^{\infty} 7n a_n x^n + \sum_{n=0}^{\infty} 16 a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[3(n+1)(n+2)a_{n+2}+\left(n^2-8n+16\right)a_n\right]x^n=0$$

$$3(n+1)(n+2)a_{n+2}+(n-4)^2a_n=0$$

$$\boxed{a_{n+2}=-\frac{(n-4)^2}{3(n+1)(n+2)}a_n}$$

a_0	a_1
$n=0\to\ a_2=-\frac{16}{6}a_0=-\frac{8}{3}a_0$	$n=1\ \to\ a_3=-\frac{9}{18}a_1=-\frac{1}{2}a_1$
$n=2\ \to\ a_4=-\frac{1}{9}a_2=\frac{8}{27}a_0$	$n=3\ \to\ a_5=-\frac{1}{60}a_3=\frac{1}{120}a_1$
$n=4\ \to\ a_6=0$	$n=5\ \to\ a_7=-\frac{1}{126}a_5=-\frac{1}{560\cdot 3^3}a_1$
$\vdots\quad\vdots\quad\vdots\quad\vdots$	$\vdots\quad\vdots\quad\vdots\quad\vdots$

$$\boxed{y(x)=a_0\left(1-\frac{8}{3}x^2+\frac{8}{27}x^4\right)+a_1\left(x-\frac{1}{2}x^3+\frac{1}{120}x^5+\frac{1}{15,120}x^7+\cdots\right)}$$

Exercise

Find the series solution to the initial value problem $y''+4y=0$; $y(0)=0,\quad y'(0)=3$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''+4y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+4\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+4a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}+4a_n=0$$

$$\boxed{a_{n+2} = -\frac{4}{(n+1)(n+2)}a_n}$$

Given: $y(0) = 0 = a_0,\quad y'(0) = 3 = a_1$

$$a_0 = 0$$

$$n = 0 \;\rightarrow\; a_2 = -2a_0 = 0$$

$$n = 2 \;\rightarrow\; a_4 = -\frac{4}{12}a_2 = 0$$

$$n = 4 \;\rightarrow\; a_6 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_1 = 3$$

$$n = 1 \;\rightarrow\; a_3 = -\frac{4}{6}a_1 = -\frac{2^2}{3!}a_1 = -2$$

$$n = 3 \;\rightarrow\; a_5 = -\frac{4}{20}a_3 = -\frac{2^4}{5!}a_1 = \frac{2}{5}$$

$$n = 5 \;\rightarrow\; a_7 = -\frac{4}{42}a_5 = -\frac{2^6}{7!}a_1 = -\frac{4}{105}$$

$$a_{2k+1} = \frac{(-1)^k 2^{2k}}{(2k+1)!}a_1$$

$$\begin{aligned} y(x) &= 3x - 2x^3 + \frac{2}{5}x^5 - \frac{4}{105}x^7 + \cdots \\ &= 3\left(x - \frac{2^2}{3!}x^3 + \frac{2^4}{5!}x^5 - \frac{2^6}{7!}x^7 + \cdots \right) \\ &= \frac{3}{2}\left((2x) - \frac{1}{3!}(2x)^3 + \frac{1}{5!}(2x)^5 - \frac{1}{7!}(2x)^7 + \cdots \right) \\ &\boxed{= \frac{3}{2}\sin 2x} \end{aligned}$$

Exercise

Find the series solution to the initial value problem $y'' + x^2 y = 0$; $y(0) = 1,\quad y'(0) = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + x^2 y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}a_n x^{n+2}=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}a_{n-2}x^n=0$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}a_{n-2}x^n=0$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2}+a_{n-2}\right]x^n=0$$

$$(n+1)(n+2)a_{n+2}+a_{n-2}=0$$

$$a_{n+2}=-\frac{1}{(n+1)(n+2)}a_{n-2}$$

Given: $\quad y(0)=1=a_0, \quad y'(0)=0=a_1$

$2a_2+6a_3x=0 \quad \rightarrow \quad a_2=a_3=0$

$a_0=1$

$n=2 \quad \rightarrow \quad a_4=-\frac{1}{12}a_0=-\frac{1}{12}$

$a_1=a_2=a_3=0$

$n=3 \rightarrow a_5=-\frac{1}{20}a_1=0$

$n=4 \rightarrow a_6={}^*a_2=0$

$n=5 \rightarrow a_7={}^*a_3=0$

$n=6 \rightarrow a_8=-\frac{1}{56}a_4=\frac{1}{672}$

$n=10 \rightarrow a_{12}=-\frac{1}{132}a_8=\frac{1}{88{,}704}$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y(x)=1-\frac{1}{12}x^4+\frac{1}{672}x^8-\frac{1}{88{,}704}x^{12}+\cdots$$

Exercise

Find the series solution to the initial value problem $y''-2xy'+8y=0; \quad y(0)=3, \quad y'(0)=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$y'' - 2xy' + 8y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - 2x\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n + 8\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} 2(n+1)a_{n+1}x^{n+1} + \sum_{n=0}^{\infty} 8a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + 8a_n\right]x^n - \sum_{n=1}^{\infty} 2na_n x^n = 0$$

$$2a_2 + 8a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + 8a_n\right]x^n - \sum_{n=1}^{\infty} 2na_n x^n = 0$$

$$2a_2 + 8a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + (8-2n)a_n\right]x^n = 0$$

Given: $y(0) = 3 = a_0, \quad y'(0) = 0 = a_1$

$2a_2 + 8a_0 = 0 \quad \rightarrow \quad \boxed{a_2 = -4a_0 = -12}$

$(n+2)(n+1)a_{n+2} + (8-2n)a_n = 0$

$\rightarrow a_{n+2} = \dfrac{2n-8}{(n+1)(n+2)}a_n \qquad n = 1, 2, \ldots$

$a_3 = -a_1 = 0 \qquad\qquad a_4 = -\frac{1}{3}a_2 = 4$

$a_5 = -\frac{1}{10}a_3 = 0 \qquad\qquad a_6 = 0a_4 = 0$

$a_7 = \frac{1}{21}a_5 = 0 \qquad\qquad a_6 = 0a_4 = 0$

$\boxed{y(x) = 3 - 12x^2 + 4x^4}$

Exercise

Find the series solution to the initial value problem $y'' + y' - 2y = 0$; $\quad y(0) = 1, \quad y'(0) = -2$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''+y'-2y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-2\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}-2a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}-2a_n=0$$

$$\boxed{a_{n+2}=\frac{2a_n-(n+1)a_{n+1}}{(n+1)(n+2)}}$$

Given: $y(0)=1=a_0,\quad y'(0)=-2=a_1$

$a_0=1 \qquad\qquad a_1=-2$

$n=0 \;\to\; a_2=\frac{2a_0-a_1}{2}=2 \qquad\qquad n=1 \;\to\; a_3=\frac{2a_1-2a_2}{6}=-\frac{8}{6}$

$n=2 \;\to\; a_4=\frac{2a_2-3a_3}{12}=\frac{2}{3} \qquad\qquad n=3 \;\to\; a_5=\frac{2a_3-4a_4}{20}=-\frac{4}{15}$

$n=4 \;\to\; a_6=\frac{2a_4-5a_5}{30}=\frac{4}{45} \qquad\qquad n=5 \;\to\; a_7=\frac{2a_5-6a_6}{42}=\frac{8}{315}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x)=1-2x+2x^2-\frac{4}{3}x^3+\frac{2}{3}x^4-\frac{4}{15}x^5+\frac{4}{45}x^6-\frac{8}{315}x^7+\cdots}$$

$$=1+(-2x)+\frac{1}{2!}(-2x)^2+\frac{1}{3!}(-2x)^3+\frac{1}{4!}(-2x)^4+\frac{1}{5!}(-2x)^5+\frac{1}{6!}(-2x)^6+\frac{1}{7!}(-2x)^7+\cdots$$

$$\boxed{=e^{-2x}}$$

Exercise

Find the series solution to the initial value problem $y''-2y'+y=0$; $\quad y(0)=0, \quad y'(0)=1$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$y'' - 2y' + y = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - 2(n+1) a_{n+1} + a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} - 2(n+1) a_{n+1} + a_n = 0$$

$$\boxed{a_{n+2} = \frac{2(n+1) a_{n+1} - a_n}{(n+1)(n+2)}}$$

Given: $y(0) = 0 = a_0, \quad y'(0) = 1 = a_1$

$a_0 = 0$ $\qquad\qquad$ $a_1 = 1$

$n = 0 \ \rightarrow \ a_2 = \dfrac{2a_1 - a_0}{2} = 1$ $\qquad$ $n = 1 \ \rightarrow \ a_3 = \dfrac{4a_2 - a_1}{6} = \dfrac{1}{2}$

$n = 2 \ \rightarrow \ a_4 = \dfrac{6a_3 - a_2}{12} = \dfrac{1}{6}$ $\qquad$ $n = 3 \ \rightarrow \ a_5 = \dfrac{8a_4 - a_3}{20} = \dfrac{1}{20}\left(\dfrac{4}{3} - \dfrac{1}{2}\right) = \dfrac{1}{24}$

$n = 4 \ \rightarrow \ a_6 = \dfrac{10a_5 - a_4}{30} = \dfrac{1}{30}\left(\dfrac{5}{12} - \dfrac{1}{6}\right) = \dfrac{1}{120}$ $\qquad$ $n = 5 \ \rightarrow \ a_7 = \dfrac{12a_6 - a_5}{42} = \dfrac{1}{42}\left(\dfrac{1}{10} - \dfrac{1}{24}\right) = \dfrac{1}{720}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\begin{aligned} y(x) &= x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \frac{1}{120}x^6 + \frac{1}{720}x^7 + \cdots \\ &= x\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots\right) \\ &\boxed{= xe^x} \end{aligned}$$

Exercise

Find the series solution to the initial value problem $y'' + xy' + y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$

Solution

$$y(x)=\sum_{n=0}^{\infty} a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''+xy'+y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+x\sum_{n=1}^{\infty} n a_n x^{n-1}+\sum_{n=0}^{\infty} a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty} n a_n x^n+\sum_{n=0}^{\infty} a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+na_n+a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}+(n+1)a_n=0$$

$$(n+2)(n+1)a_{n+2}=-(n+1)a_n$$

$$a_{n+2}=-\frac{1}{n+2}a_n$$

$$a_0=y(0)=1 \qquad a_1=y'(0)=0$$

$$a_2=-\frac{1}{2}a_0=-\frac{1}{2} \qquad a_3=-\frac{1}{3}a_1=0$$

$$a_4=-\frac{1}{4}a_2=\frac{1}{2\cdot 4}=\frac{1}{2^2\cdot 1\cdot 2} \qquad a_5=-\frac{1}{5}a_3=0$$

$$a_6=-\frac{1}{6}a_4=-\frac{1}{2^3\cdot 1\cdot 2\cdot 3} \qquad a_7=-\frac{1}{7}a_7=0$$

$$y(x)=\sum_{n=0}^{\infty} a_n x^n=a_0+a_1x+a_2x^2++a_3x^3+a_4x^4+a_5x^5+\cdots$$

$$=1-\frac{1}{2}x^2+\frac{1}{2^2 2!}x^4-\frac{1}{2^3 3!}x^5+\cdots$$

Exercise

Find the series solution to the initial value problem $y'' - xy' - y = 0 \quad y(0) = 2 \quad y'(0) = 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy' - y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n - a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} - (n+1) a_n = 0$$

$$(n+2)(n+1) a_{n+2} = -(n+1) a_n$$

$$\boxed{a_{n+2} = \frac{1}{n+2} a_n}$$

$$a_0 = y(0) = 2 \qquad\qquad a_1 = y'(0) = 1$$

$$a_2 = \frac{1}{2} a_0 = 1 \qquad\qquad a_3 = \frac{1}{3} a_1 = \frac{1}{3}$$

$$a_4 = \frac{1}{4} a_2 = \frac{1}{4} \qquad\qquad a_5 = \frac{1}{5} a_3 = \frac{1}{3\cdot 5}$$

$$a_6 = \frac{1}{6} a_4 = \frac{1}{4}\frac{1}{6} = \frac{1}{24} \qquad\qquad a_7 = \frac{1}{7} a_7 = \frac{1}{3\cdot 5\cdot 7}$$

$$\vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = 2 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{15}x^5 + \frac{1}{24}x^6 + \cdots}$$

Exercise

Find the series solution to the initial value problem $y'' - xy' - y = 0$; $y(0) = 1$ $y'(0) = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy' - y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n - a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} - (n+1) a_n = 0$$

$$(n+2)(n+1) a_{n+2} = -(n+1) a_n$$

$$\boxed{a_{n+2} = \frac{1}{n+2} a_n}$$

Given: $a_0 = y(0) = 1$ $\qquad$ $a_1 = y'(0) = 0$

$a_2 = \frac{1}{2} a_0 = \frac{1}{2}$ $\qquad$ $a_3 = \frac{1}{3} a_1 = 0$

$a_4 = \frac{1}{4} a_2 = \frac{1}{2 \cdot 2^2}$ $\qquad$ $a_5 = \frac{1}{5} a_3 = 0$

$a_6 = \frac{1}{6} a_4 = \frac{1}{2^3 \cdot 3!}$ $\qquad$ $\vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = 1 + \frac{1}{2} x^2 + \frac{1}{8} x^4 + \frac{1}{48} x^6 + \cdots}$$

Exercise

Find a power series solution. $y'' + xy' - 2y = 0$; $y(0) = 1$ $y'(0) = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$y'' + xy' - 2y = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} + (n-2) a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} + (n-2) a_n = 0$$

$$\boxed{a_{n+2} = -\frac{n-2}{(n+1)(n+2)} a_n}$$

$a_0 = y(0) = 1$ $\qquad$ $a_1 = y'(0) = 0$

$n = 0 \to a_2 = \frac{2}{2} a_0 = 1$ $\qquad$ $n = 1 \to a_3 = \frac{1}{6} a_1 = 0$

$n = 2 \to a_4 = 0$ $\qquad$ $n = 3 \to a_5 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$ $\qquad$ $\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = 1 + x^2}$$

Exercise

Find the series solution to the initial value problem $y'' + (x-1)y' + y = 0 \quad y(1) = 2 \quad y'(1) = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$y'' + (x-1)y' + y = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} (x-1)^n + (x-1)\sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} (x-1)^n + \sum_{n=1}^{\infty} n a_n (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + n a_n + a_n\right] x^{n-1} = 0$$

$$(n+2)(n+1)a_{n+2} + (n+1)a_n = 0$$

$$\boxed{a_{n+2} = -\frac{1}{n+2} a_n}$$

$$a_0 = y(1) = 2 \qquad\qquad a_1 = y'(1) = 0$$

$$a_2 = -\frac{1}{2} a_0 = -1 \qquad\qquad a_3 = -\frac{1}{3} a_1 = 0$$

$$a_4 = -\frac{1}{4} a_2 = \frac{1}{2 \cdot 4} a_0 = \frac{1}{4} \qquad\qquad a_5 = -\frac{1}{5} a_3 = 0$$

$$a_6 = -\frac{1}{6} a_4 = -\frac{1}{24} \qquad\qquad a_7 = -\frac{1}{7} a_5 = 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n = a_0 + a_1(x-1) + a_2(x-1) + a_3(x-1)^3 + a_4(x-1)^4 + \cdots$$

$$\boxed{= 2 - (x-1)^2 + \frac{1}{4}(x-1)^4 - \frac{1}{24}(x-1)^6 + \cdots}$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{2n}}{n!\ 2^n}}$$

Exercise

Find the series solution to the initial value problem $(x-1)y''-xy'+y=0; \quad y(0)=-2, \quad y'(0)=6$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$(x-1)y''-xy'+y=0$$

$$(x-1)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^n-\sum_{n=1}^{\infty}na_n x^n+\sum_{n=0}^{\infty}\left[a_n-(n+2)(n+1)a_{n+2}\right]x^n=0$$

$$\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}-na_n\right]x^n+a_0-2a_2+\sum_{n=1}^{\infty}\left[a_n-(n+2)(n+1)a_{n+2}\right]x^n=0$$

$$a_0-2a_2+\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}-(n-1)a_n-(n+2)(n+1)a_{n+2}\right]x^n=0$$

Given: $y(0)=-2=a_0, \quad y'(0)=6=a_1$

$a_0-2a_2=0 \quad\rightarrow\quad a_2=\frac{1}{2}a_0=-1$

$n(n+1)a_{n+1}-(n-1)a_n-(n+2)(n+1)a_{n+2}=0$

$\rightarrow\ a_{n+2}=\frac{n}{n+2}a_{n+1}-\frac{n-1}{(n+2)(n+1)}a_n$

$a_3=\frac{1}{3}a_2-0a_1=-\frac{1}{3}$

$a_4=\frac{1}{2}a_3-\frac{1}{12}a_2=-\frac{1}{6}+\frac{1}{12}=-\frac{1}{12}$

$a_5=\frac{3}{5}a_4-\frac{1}{10}a_3=-\frac{3}{60}+\frac{1}{30}=-\frac{1}{60}$

$a_6=\frac{2}{3}a_5-\frac{1}{10}a_4=-\frac{1}{90}+\frac{1}{120}=-\frac{1}{360}$

$$y(x)=\left(-2-x^2-\frac{1}{3}x^3-\frac{1}{12}x^4-\ldots\right)+a_1x$$

$$= -2\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots\right) + 6x$$

$$= -2\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots\right) + 6x + 2x$$

$$\underline{= 8x - 2e^x}$$

Exercise

Find the series solution to the initial value problem

$$(x+1)y'' - (2-x)y' + y = 0; \quad y(0) = 2, \quad y'(0) = -1$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x+1)y'' - (2-x)y' + y = 0$$

$$(x+1)\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - (2-x)\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+1} + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n$$

$$+ \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n+1) a_{n+1}\right] x^{n+1} + \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - 2(n+1) a_{n+1} + a_n\right] x^n = 0$$

$$\sum_{n=1}^{\infty} \left[n(n+1) a_{n+1} + n a_n\right] x^n + \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - 2(n+1) a_{n+1} + a_n\right] x^n = 0$$

$$\sum_{n=1}^{\infty} \left[n(n+1) a_{n+1} + n a_n\right] x^n + 2a_2 - 2a_1 + a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - 2(n+1) a_{n+1} + a_n\right] x^n = 0$$

$$2a_2 - 2a_1 + a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + (n-2)(n+1)a_{n+1} + (n+1)a_n\right]x^n = 0$$

Given: $y(0) = 2 = a_0, \quad y'(0) = -1 = a_1$

$$2a_2 - 2a_1 + a_0 = 0 \quad \to \quad \boxed{a_2 = \frac{1}{2}(2a_1 - a_0) = -2}$$

$$(n+2)(n+1)a_{n+2} + (n-2)(n+1)a_{n+1} + (n+1)a_n = 0$$

$$\to a_{n+2} = -\frac{n-2}{n+2}a_{n+1} - \frac{1}{n+2}a_n$$

$$a_3 = \frac{1}{3}a_2 - \frac{1}{3}a_1 = \frac{2}{3} + \frac{1}{3} = 1 \qquad a_4 = 0a_3 - \frac{1}{4}a_2 = \frac{1}{2}$$

$$a_5 = -\frac{1}{5}a_4 - \frac{1}{5}a_3 = -\frac{1}{10} - \frac{1}{5} = -\frac{3}{10} \qquad a_6 = -\frac{1}{3}a_5 - \frac{1}{6}a_4 = \frac{1}{10} - \frac{1}{12} = \frac{1}{60}$$

$$\boxed{y(x) = 2 - x - 2x^2 + x^3 + \frac{1}{2}x^4 - \frac{3}{10}x^5 + \dots}$$

Exercise

Find the series solution to the initial value problem

$$(1-x)y'' + xy' - 2y = 0\ ; \quad y(0) = 0, \quad y'(0) = 1$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$(1-x)y'' + xy' - 2y = 0$$

$$(1-x)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - 2\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty}2a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - 2a_n\right]x^n - \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - (n+1)a_{n+1}\right]x^{n+1} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-2a_n\right]x^n-\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}-na_n\right]x^n=0$$

$$2a_2-2a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}-2a_n\right]x^n-\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}-na_n\right]x^n=0$$

$$2a_2-2a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}-n(n+1)a_{n+1}+(n-2)a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}-n(n+1)a_{n+1}+(n-2)a_n=0$$

$$\boxed{a_{n+2}=\frac{n(n+1)a_{n+1}-(n-2)a_n}{(n+1)(n+2)}}$$

Given: $y(0)=0=a_0, \quad y'(0)=1=a_1$

$$2a_2-2a_0=0 \quad\rightarrow\quad \boxed{a_2=a_0=0}$$

$$n=1\rightarrow\ a_3=\frac{2a_2+a_1}{6}=\frac{1}{6}$$

$$n=2\ \rightarrow\ a_4=\frac{6a_3}{12}=\frac{1}{12}$$

$$n=3\rightarrow\ a_5=\frac{1}{20}\left(12a_4-a_3\right)=\frac{1}{20}\left(1-\frac{1}{6}\right)=\frac{1}{24}$$

$$n=4\rightarrow\ a_6=\frac{1}{30}\left(20a_5-2a_4\right)=\frac{1}{30}\left(\frac{5}{6}-\frac{1}{6}\right)=\frac{1}{45}$$

$$n=5\rightarrow\ a_7=\frac{1}{42}\left(30a_6-3a_5\right)=\frac{1}{30}\left(\frac{2}{3}-\frac{1}{8}\right)=\frac{13}{1008}$$

$$\vdots\quad\vdots\quad\vdots\quad\vdots$$

$$\boxed{y(x)=x+\frac{1}{6}x^3+\frac{1}{12}x^4+\frac{1}{24}x^5+\frac{1}{45}x^6+\frac{13}{1008}x^7+\cdots}$$

Exercise

Find the series solution to the initial value problem

$$\left(x^2+1\right)y''+2xy'=0; \quad y(0)=0, \quad y'(0)=1$$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$$\left(x^2+1\right)y'' + 2xy' = 0$$

$$\left(x^2+1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2} + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}2(n+1)a_{n+1}x^{n+1} = 0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}2na_n x^n = 0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n + 2a_2 + 6a_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n + 2a_1 x + \sum_{n=2}^{\infty}2na_n x^n = 0$$

$$2a_2 + \left(6a_3 + 2a_1\right)x + \sum_{n=2}^{\infty}\left[\left(n(n-1)+2n\right)a_n + (n+2)(n+1)a_{n+2}\right]x^n = 0$$

Given: $y(0) = 0 = a_0, \quad y'(0) = 1 = a_1$

$$2a_2 + \left(6a_3 + 2a_1\right)x = 0 \quad \rightarrow \quad \begin{cases} a_2 = 0 \\ a_3 = -\frac{1}{3}a_1 = -\frac{1}{3} \end{cases}$$

$$n(n+1)a_n + (n+2)(n+1)a_{n+2} = 0$$

$$\rightarrow \ a_{n+2} = -\frac{n}{n+2}a_n \qquad n = 2,3,\ldots$$

$$a_4 = -\frac{1}{2}a_2 = 0 \qquad\qquad a_5 = -\frac{3}{5}a_3 = \frac{1}{5}$$

$$a_6 = -\frac{2}{3}a_4 = 0 \qquad\qquad a_7 = -\frac{5}{7}a_5 = -\frac{1}{7}$$

$$\underline{y(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots}$$

Exercise

Find the series solution to the initial value problem

$$\left(x^2-1\right)y'' + 3xy' + xy = 0\ ; \qquad y(0) = 4, \qquad y'(0) = 6$$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2-1\right)y''+3xy'+xy=0$$

$$\left(x^2-1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+3x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+x\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}3(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}a_n x^{n+1}=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}\left[(3n+3)a_{n+1}+a_n\right]x^{n+1}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=1}^{\infty}\left(3na_n+a_{n-1}\right)x^n-\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\left(3a_1+a_0\right)x+\sum_{n=2}^{\infty}\left(3na_n+a_{n-1}\right)x^n-2a_2-6a_3x-\sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}x^n=0$$

$$-2a_2+\left(3a_1+a_0-6a_3\right)x+\sum_{n=2}^{\infty}\left[\left(n^2+2n\right)a_n+a_{n-1}-(n+2)(n+1)a_{n+2}\right]x^n=0$$

Given: $y(0)=4=a_0,\quad y'(0)=6=a_1$

$$\begin{cases}-2a_2=0 & \rightarrow \boxed{a_2=0}\\ 3a_1+a_0-6a_3=0 & \rightarrow \boxed{a_3=\frac{22}{6}=\frac{11}{3}}\end{cases}$$

$$\left(n^2+2n\right)a_n+a_{n-1}-(n+2)(n+1)a_{n+2}=0$$

$$\boxed{a_{n+2}=\frac{\left(n^2+2n\right)a_n+a_{n-1}}{(n+1)(n+2)}}$$

$$n=2\ \rightarrow\ a_4=\frac{8a_2+a_1}{12}=\frac{6}{12}=\frac{1}{2}$$

$$n=3 \to a_5 = \frac{1}{20}\left(15a_3 + a_2\right) = \frac{1}{20}(55) = \frac{11}{4}$$

$$n=4 \to a_6 = \frac{1}{30}\left(24a_4 + a_3\right) = \frac{1}{30}\left(12 + \frac{11}{3}\right) = \frac{47}{90}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = 4 + 6x + \frac{11}{3}x^3 + \frac{1}{2}x^4 + \frac{11}{4}x^5 + \frac{47}{90}x^6 + \cdots}$$

Exercise

Find the series solution to the initial value problem

$$\left(2+x^2\right)y'' - xy' + 4y = 0 \qquad y(0) = -1 \qquad y'(0) = 3$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\left(2+x^2\right)y'' - xy' + 4y = 0$$

$$\left(2+x^2\right)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} na_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} na_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$2(n+2)(n+1)a_{n+2} + n(n-1)a_n - na_n + 4a_n = 0$$

$$2(n+2)(n+1)a_{n+2} + \left(n^2 - 2n + 4\right)a_n = 0$$

$$\boxed{a_{n+2} = -\frac{n^2 - 2n + 4}{2(n+2)(n+1)}a_n}$$

$$a_0 = y(0) = -1 \qquad\qquad a_1 = y'(0) = 3$$

$n=0 \rightarrow \quad a_2 = -\frac{4}{4}a_0 = 1 \qquad n=1 \rightarrow \quad a_3 = -\frac{3}{12}a_1 = -\frac{1}{4}(3) = -\frac{3}{4}$

$n=2 \rightarrow \quad a_4 = -\frac{4}{24}a_2 = -\frac{1}{6} \qquad n=3 \rightarrow \quad a_5 = -\frac{7}{40}a_3 = -\frac{7}{40}\left(-\frac{3}{4}\right) = \frac{21}{160}$

$$y(x) = -1 + 3x + x^2 - \frac{3}{4}x^3 - \frac{1}{6}x^4 + \frac{21}{160}x^5 + \cdots$$

Exercise

Find the series solution to the initial value problem

$$\left(2 - x^2\right)y'' - xy' + 4y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(2 - x^2\right)y'' - xy' + 4y = 0$$

$$\left(2 - x^2\right)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+1)(n+2)a_{n+2} - \left(n^2 - n + n - 4\right)a_n\right] x^n = 0$$

$$2(n+1)(n+2)a_{n+2} - (n-2)(n+2)a_n = 0$$

$$a_{n+2} = \frac{n-2}{2(n+1)} a_n$$

$a_0 = y(0) = 1 \qquad\qquad a_1 = y'(0) = 0$

$n = 0 \to \quad a_2 = \frac{-2}{2} a_0 = -1 \qquad\qquad n = 1 \to \quad a_3 = -\frac{1}{4} a_1 = 0$

$n = 2 \to \quad a_4 = 0 \qquad\qquad n = 3 \to \quad a_5 = {*}a_3 = 0$

$\vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots$

$$\underline{y(x) = 1 - x^2}$$

Exercise

Find the series solution to the initial value problem

$$\left(4 - x^2\right) y'' + 2y = 0 \qquad y(0) = 0 \qquad y'(0) = 1$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(4 - x^2\right) y'' + 2y = 0$$

$$\left(4 - x^2\right) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 4(n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[4(n+1)(n+2) a_{n+2} - \left(n^2 - n - 2\right) a_n \right] x^n = 0$$

$$4(n+1)(n+2) a_{n+2} - (n+1)(n-2) a_n = 0$$

$$\underline{a_{n+2} = \frac{n-2}{4(n+2)} a_n}$$

$a_0 = y(0) = 0$ $\qquad a_1 = y'(0) = 1$

$n = 0 \rightarrow \quad a_2 = \frac{-2}{8} a_0 = 0$ $\qquad n = 1 \rightarrow \quad a_3 = -\frac{1}{12} a_1 = -\frac{1}{12}$

$n = 2 \rightarrow \quad a_4 = 0$ $\qquad n = 3 \rightarrow \quad a_5 = \frac{1}{20} a_3 = -\frac{1}{240}$

$\vdots \quad \vdots \quad \vdots$ $\qquad n = 5 \rightarrow \quad a_7 = \frac{3}{28} a_5 = -\frac{1}{2,240}$

$\vdots \quad \vdots \quad \vdots$

$$y(x) = x - \frac{1}{12}x^3 - \frac{1}{240}x^5 - \frac{1}{2240}x^7 - \frac{1}{16,128}x^9 - \cdots$$

Exercise

Find a power series solution. $\left(x^2 - 4\right)y'' + 3xy' + y = 0$; $\quad y(0) = 4, \quad y'(0) = 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2 - 4\right)y'' + 3xy' + y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 4\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 3x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} 4(n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[\left(n^2 - n + 3n + 1\right) a_n - 4(n+2)(n+1) a_{n+2}\right] x^n = 0$$

$$\left(n^2 + 2n + 1\right) a_n - 4(n+2)(n+1) a_{n+2} = 0$$

$$4(n+2)(n+1)a_{n+2} = (n+1)^2 a_n$$

$$\boxed{a_{n+2} = \frac{n+1}{4(n+2)} a_n}$$

$$a_0 = y(0) = 4 \qquad\qquad a_1 = y'(0) = 1$$

$$n = 0 \quad \to \quad a_2 = \frac{1}{8}a_0 = \frac{1}{2} \qquad\qquad n = 1 \quad \to \quad a_3 = \frac{2}{4\cdot 3}a_1 = \frac{1}{6}$$

$$n = 2 \quad \to \quad a_4 = \frac{3}{16}a_2 = \frac{3}{32} \qquad\qquad n = 3 \quad \to \quad a_5 = \frac{1}{5}a_3 = \frac{1}{30}$$

$$n = 4 \quad \to \quad a_6 = \frac{5}{24}a_4 = \frac{5}{256} \qquad\qquad n = 5 \quad \to \quad a_7 = \frac{6}{4\cdot 7}a_5 = \frac{1}{140}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = 4 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{3}{32}x^4 + \frac{1}{30}x^5 + \frac{5}{256}x^6 + \frac{1}{140}x^7 + \cdots}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y'' + 2xy' - 2y = 0$; $y(0) = 0,\quad y'(0) = 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y'' + 2xy' - 2y = 0$$

$$x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2x\sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+\left(n^2+n-2\right)a_n\right]x^n=0$$

$$(n+1)(n+2)a_{n+2}+(n-1)(n+2)a_n=0$$

$$\boxed{a_{n+2}=-\frac{n-1}{n+1}a_n}$$

$$a_0=y(0)=0 \qquad\qquad a_1=y'(0)=1$$

$$n=0 \;\rightarrow\; a_2=a_0 \qquad\qquad n=1 \;\rightarrow\; a_3=0$$

$$n=2 \;\rightarrow\; a_4=-\tfrac{1}{3}a_2=-\tfrac{1}{3}a_0 \qquad\qquad n=3 \;\rightarrow\; a_5=0$$

$$n=4 \;\rightarrow\; a_6=-\tfrac{3}{5}a_4=\tfrac{1}{5}a_0 \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n=6 \;\rightarrow\; a_8=-\tfrac{5}{7}a_6=-\tfrac{1}{7}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n}=\frac{(-1)^n}{2n-1}a_0$$

$$\boxed{y(x)=a_1x+a_0\left(1+x^2-\tfrac{1}{3}x^4+\tfrac{1}{5}x^6-\tfrac{1}{7}x^8+\cdots\right)}$$

$$y(x)=a_1x+a_0\left(1+x\left(x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7+\cdots\right)\right)$$

$$\boxed{=a_1x+a_0\left(1+x\tan^{-1}x\right)}$$

Exercise

Find a power series solution. $\left(2x-x^2\right)y''-6(x-1)y'-4y=0$; $y(1)=0,\quad y'(1)=1$

Solution

Let $z=x-1 \;\Rightarrow\; \begin{cases} x=z+1 \\ dz=dx \end{cases}$

$$\left(2x-x^2\right)y''-6(x-1)y'-4y=0$$

$$\left(2z+2-z^2-2z-1\right)y''-6zy'-4y=0$$

$$\left(1-z^2\right)y''-6zy'-4y=0$$

$$y(z)=\sum_{n=0}^{\infty}a_nz^n$$

$$y'(z)=\sum_{n=1}^{\infty} na_n z^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n$$

$$y''(z)=\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n$$

$$\left(1-z^2\right)y''-6zy'-4y=0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}-z^2\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}-6z\sum_{n=1}^{\infty} na_n z^{n-1}-4\sum_{n=0}^{\infty} a_n z^n=0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}-\sum_{n=2}^{\infty} n(n-1)a_n z^{n}-\sum_{n=1}^{\infty} 6na_n z^{n}-4\sum_{n=0}^{\infty} a_n z^n=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}z^n-\sum_{n=0}^{\infty} n(n-1)a_n z^{n}-\sum_{n=0}^{\infty} 6na_n z^{n}-4\sum_{n=0}^{\infty} a_n z^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}-\left(n^2+5n+4\right)a_n\right]z^n=0$$

$$(n+1)(n+2)a_{n+2}-(n+1)(n+4)a_n=0$$

$$\boxed{a_{n+2}=\frac{n+4}{n+2}a_n}$$

Given: $y(1)=0=a_0,\quad y'(1)=1=a_1$

$a_0=0$

$n=0 \quad \to \quad a_2=2a_0=0$

$n=2 \quad \to \quad a_4=\frac{3}{2}a_2=0$

$n=4 \quad \to \quad a_6=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$a_1=1$

$n=1 \quad \to \quad a_3=\frac{5}{3}a_1=\frac{5}{3}$

$n=3 \quad \to \quad a_5=\frac{7}{5}a_3=\frac{7}{3}$

$n=5 \quad \to \quad a_7=\frac{9}{7}a_5=\frac{9}{3}$

$n=7 \quad \to \quad a_9=\frac{11}{9}a_7=\frac{11}{3}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{2n+1}=\frac{2n+3}{3}$$

$$y(z)=z+\frac{5}{3}z^3+\frac{7}{3}z^5+3z^7+\frac{11}{3}z^9+\cdots+\frac{2n+3}{3}z^{2n+1}+\cdots$$

$$\boxed{y(x)=(x-1)+\frac{5}{3}(x-1)^3+\frac{7}{3}(x-1)^5+3(x-1)^7+\frac{11}{3}(x-1)^9+\cdots}$$

Exercise

Find a power series solution. $\left(x^2-6x+10\right)y''-4(x-3)y'+6y=0$; $y(3)=2,\quad y'(3)=0$

Solution

Let $z=x-3 \Rightarrow \begin{cases} x=z+3 \\ dz=dx \end{cases}$

$$\left(x^2-6x+10\right)y''-4(x-3)y'+6y=0$$

$$\left(z^2+6z+9-6z-18+10\right)y''-4zy'+6y=0$$

$$\left(z^2+1\right)y''-4zy'+6y=0$$

$$y(z)=\sum_{n=0}^{\infty}a_n z^n$$

$$y'(z)=\sum_{n=1}^{\infty}na_n z^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n$$

$$y''(z)=\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n$$

$$\left(z^2+1\right)y''-4zy'+6y=0$$

$$z^2\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}+\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}-4z\sum_{n=1}^{\infty}na_n z^{n-1}+6\sum_{n=0}^{\infty}a_n z^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n z^n+\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}-\sum_{n=1}^{\infty}4na_n z^n+\sum_{n=0}^{\infty}6a_n z^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n z^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n-\sum_{n=0}^{\infty}4na_n z^n+\sum_{n=0}^{\infty}6a_n z^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+\left(n^2-5n+6\right)a_n\right]z^n=0$$

$$(n+1)(n+2)a_{n+2}+(n-2)(n-3)a_n=0$$

$$a_{n+2}=-\frac{(n-2)(n-3)}{(n+1)(n+2)}a_n$$

Given: $y(3)=2=a_0,\quad y'(3)=0=a_1$

$$a_0 = 2 \qquad\qquad a_1 = 0$$

$$n=0 \;\to\; a_2 = -3a_0 = -6 \qquad n=1 \;\to\; a_3 = -\frac{2}{4}a_1 = 0$$

$$n=2 \;\to\; a_4 = 0 \qquad n=3 \;\to\; a_5 = 0$$

$$n=4 \;\to\; a_6 = 0 \qquad n=5 \;\to\; a_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(z) = 2 - 6z^2$$

$$\boxed{y(x) = 2 - 6(x-3)^2}$$

Exercise

Find a power series solution. $\left(4x^2 + 16x + 17\right)y'' - 8y = 0\,; \quad y(-2) = 1, \quad y'(-2) = 0$

Solution

Let $\quad z = x + 2 \quad \Rightarrow \quad \begin{cases} x = z - 2 \\ dz = dx \end{cases}$

$$\left(4z^2 - 16z + 16 + 16z - 32 + 17\right)y'' - 8y = 0$$

$$\boxed{\left(4z^2 + 1\right)y'' - 8y = 0}$$

$$y(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$y'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

$$y''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n$$

$$\left(4z^2 + 1\right)y'' - 8y = 0$$

$$4z^2 \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} - 8\sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=2}^{\infty} 4n(n-1) a_n z^n + \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} - 8\sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} 4n(n-1) a_n z^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n - 8\sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+\left(4n^2-4n-8\right)a_n\right]z^n=0$$

$$(n+1)(n+2)a_{n+2}+4(n+1)(n-2)a_n=0$$

$$\boxed{a_{n+2}=-\frac{4(n-2)}{n+2}a_n}$$

Given: $y(-2)=1=a_0,\quad y'(-2)=0=a_1$

$a_0=1$	$a_1=0$
$n=0 \ \rightarrow\ a_2=\frac{8}{2}a_0=4$	$n=1 \ \rightarrow\ a_3=\frac{4}{3}a_1=0$
$n=2 \ \rightarrow\ a_4=-0a_2=0$	$n=3 \ \rightarrow\ a_5=0$
$n=4 \ \rightarrow\ a_6=0$	$n=5 \ \rightarrow\ a_7=0$
$\vdots\ \vdots\ \vdots\ \vdots\ \vdots$	$\vdots\ \vdots\ \vdots\ \vdots\ \vdots$

$$y(z)=1+4z^2$$

$$\boxed{y(x)=1+4(x+2)^2}$$

Exercise

Find a power series solution. $\left(x^2+6x\right)y''+(3x+9)y'-3y=0\ ;\quad y(-3)=0,\quad y'(-3)=2$

Solution

Let $\quad z=x+3 \ \Rightarrow \begin{cases} x=z-3 \\ dz=dx \end{cases}$

$$\left(x^2+6x\right)y''+(3x+9)y'-3y=0$$

$$\left(z^2-6z+9+6z-18\right)y''+3zy'-3y=0$$

$$\boxed{\left(z^2-9\right)y''+3zy'-3y=0}$$

$$y(z)=\sum_{n=0}^{\infty}a_nz^n$$

$$y'(z)=\sum_{n=1}^{\infty}na_nz^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n$$

$$y''(z)=\sum_{n=2}^{\infty}n(n-1)a_nz^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n$$

$$\left(z^2-9\right)y''+3zy'-3y=0$$

$$z^2\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}-9\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}+3z\sum_{n=1}^{\infty}na_n z^{n-1}-3\sum_{n=0}^{\infty}a_n z^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n z^n-\sum_{n=2}^{\infty}9n(n-1)a_n z^{n-2}+\sum_{n=1}^{\infty}3na_n z^n-\sum_{n=0}^{\infty}3a_n z^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n z^n-\sum_{n=0}^{\infty}9(n+2)(n+1)a_{n+2}z^n+\sum_{n=0}^{\infty}3na_n z^n-\sum_{n=0}^{\infty}3a_n z^n=0$$

$$\sum_{n=0}^{\infty}\left[-9(n+2)(n+1)a_{n+2}+\left(n^2-n+3n-3\right)a_n\right]z^n=0$$

$$-9(n+2)(n+1)a_{n+2}+(n+3)(n-1)a_n=0$$

$$a_{n+2}=\frac{(n+3)(n-1)}{9(n+1)(n+2)}a_n$$

Given: $y(-3)=0=a_0, \quad y'(-3)=2=a_1$

$a_0=0$

$n=0 \quad \rightarrow \quad a_2=\frac{-3}{18}a_0=0$

$n=2 \quad \rightarrow \quad a_4=\frac{5}{108}a_2=0$

$n=4 \quad \rightarrow \quad a_6=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$a_1=2$

$n=1 \quad \rightarrow \quad a_3=0a_1=0$

$n=3 \quad \rightarrow \quad a_5=0$

$n=5 \quad \rightarrow \quad a_7=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$y(z)=2z$

$y(x)=2x+6$

Exercise

Find the series solution near the given value $\qquad y''-(x-2)y'+2y=0\,;\quad$ *near* $x=2$

Solution

$$y=\sum_{n=0}^{\infty}a_n(x-2)^n$$

$$y'=\sum_{n=1}^{\infty}na_n(x-2)^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-2)^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n (x-2)^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} (x-2)^n$$

$$y'' - (x-2)y' + 2y = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} (x-2)^n - (x-2)\sum_{n=0}^{\infty} (n+1)a_{n+1} (x-2)^n + 2\sum_{n=0}^{\infty} a_n (x-2)^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} (x-2)^n - \sum_{n=0}^{\infty} (n+1)a_{n+1} (x-2)^{n+1} + \sum_{n=0}^{\infty} 2a_n (x-2)^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} + 2a_n\right](x-2)^n - \sum_{n=0}^{\infty} (n+1)a_{n+1} (x-2)^{n+1} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} + 2a_n\right](x-2)^n - \sum_{n=1}^{\infty} na_n (x-2)^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left[(n+1)(n+2)a_{n+2} + 2a_n\right](x-2)^n - \sum_{n=1}^{\infty} na_n (x-2)^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left[(n+1)(n+2)a_{n+2} - (n-2)a_n\right](x-2)^n = 0$$

For $n = 0 \quad \to \quad 2a_2 + 2a_0 = 0 \quad \Rightarrow \quad \boxed{a_2 = -a_0}$

$$(n+1)(n+2)a_{n+2} - (n-2)a_n = 0$$

$$\boxed{a_{n+2} = \frac{n-2}{(n+1)(n+2)}a_n}$$

a_0	a_1
$n = 0 \to a_2 = -a_0$	$n = 1 \to a_3 = -\frac{1}{6}a_1$
$n = 2 \to a_4 = 0$	$n = 3 \to a_5 = \frac{1}{20}a_3 = -\frac{1}{120}a_1$
$n = 4 \to a_6 = 0$	$n = 5 \to a_7 = \frac{3}{42}a_5 = -\frac{1}{1680}a_1$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = a_0\left(1 - (x-2)^2\right) + a_1\left((x-2) - \tfrac{1}{6}(x-2)^3 - \tfrac{1}{120}(x-2)^5 - \tfrac{1}{1680}(x-2)^7 - \cdots\right)}$$

Exercise

Find the series solution near the given value $y''+(x-1)^2y'-4(x-1)y=0$; *near* $x=1$

Solution

$$y=\sum_{n=0}^{\infty}a_n(x-1)^n$$

$$y'=\sum_{n=1}^{\infty}na_n(x-1)^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n$$

$$y''=\sum_{n=2}^{\infty}n(n-1)a_n(x-1)^{n-2}=\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n$$

$$y''+(x-1)^2y'-4(x-1)y=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n+(x-1)^2\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n-4(x-1)\sum_{n=0}^{\infty}a_n(x-1)^n=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^{n+2}-\sum_{n=0}^{\infty}4a_n(x-1)^{n+1}=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n+\sum_{n=2}^{\infty}(n-1)a_{n-1}(x-1)^n-\sum_{n=1}^{\infty}4a_{n-1}(x-1)^n=0$$

$$2a_2+6a_3(x-1)+\sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n+\sum_{n=2}^{\infty}(n-1)a_{n-1}(x-1)^n$$

$$-4a_0(x-1)-\sum_{n=2}^{\infty}4a_{n-1}(x-1)^n=0$$

$$2a_2+\left(6a_3-4a_0\right)(x-1)+\sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2}+(n-5)a_{n-1}\right](x-1)^n=0$$

$$\begin{cases}2a_2=0 & \rightarrow \underline{a_2=0|}\\ 6a_3-4a_0=0 & \rightarrow \underline{a_3=\frac{2}{3}a_0|}\end{cases}$$

$$(n+1)(n+2)a_{n+2}+(n-5)a_{n-1}=0$$

$$\underline{a_{n+2}=-\frac{(n-5)}{(n+1)(n+2)}a_{n-1}|}$$

$$a_0 \qquad\qquad a_1 \qquad\qquad a_2 = 0$$

$$n=1 \rightarrow a_3 = \tfrac{2}{3}a_0 \qquad n=2 \rightarrow a_4 = \tfrac{1}{4}a_1 \qquad n=3 \rightarrow a_5 = \tfrac{2}{20}a_2 = 0$$

$$n=4 \rightarrow a_6 = \frac{1}{30}a_3 = \frac{1}{45}a_0 \qquad n=5 \rightarrow a_7 = 0 \qquad n=6 \rightarrow a_8 = 0$$

$$n=7 \rightarrow a_9 = -\frac{2}{8\cdot 9}a_6 = -\frac{1}{1,620}a_0 \qquad n=8 \rightarrow a_{10} = 0 \qquad n=9 \rightarrow a_{11} = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(x) = a_0\left(1 + \tfrac{2}{3}(x-1)^3 + \tfrac{1}{45}(x-1)^6 - \tfrac{1}{1,620}(x-1)^9 + \cdots\right) + a_1\left((x-1) + \tfrac{1}{4}(x-1)^4\right)$$

$$y(x) = a_0\sum_{n=0}^{\infty}\frac{(-1)^n 4(x-1)^{3n}}{3^n(3n-1)(3n-4)n!} + a_1\left((x-1) + \tfrac{1}{4}(x-1)^4\right)$$

Exercise

Find the series solution near the given value $y'' + (x-1)y = e^x$; *near* $x=1$

Solution

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y' = \sum_{n=1}^{\infty} na_n (x-1)^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n (x-1)^{n-2} = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n$$

$$y'' + (x-1)y = e^{x-1+1}$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n + (x-1)\sum_{n=0}^{\infty}a_n(x-1)^n = e\cdot e^{x-1} \qquad e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n + \sum_{n=0}^{\infty}a_n(x-1)^{n+1} = e\cdot\sum_{n=0}^{\infty}\frac{(x-1)^n}{n!}$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n + \sum_{n=1}^{\infty}a_{n-1}(x-1)^n = e\cdot\sum_{n=0}^{\infty}\frac{(x-1)^n}{n!}$$

$$2a_2+\sum_{n=1}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n+\sum_{n=1}^{\infty}a_{n-1}(x-1)^n=e+e\cdot\sum_{n=1}^{\infty}\frac{(x-1)^n}{n!}$$

$$2a_2+\sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2}+a_{n-1}\right](x-1)^n=e+e\cdot\sum_{n=1}^{\infty}\frac{(x-1)^n}{n!}$$

$$2a_2=e\quad\to\quad \underline{a_2=\frac{e}{2}}$$

$$(n+1)(n+2)a_{n+2}+a_{n-1}=\frac{e}{n!}$$

$$\boxed{a_{n+2}=\frac{e}{(n+1)(n+2)\,n!}-\frac{1}{(n+1)(n+2)}a_{n-1}}$$

a_0

$n=1\ \to\ a_3=\frac{e}{6}-\frac{1}{6}a_0$

$n=4\ \to\ a_6=\frac{e}{720}-\frac{1}{30}a_3=-\frac{11e}{720}+\frac{1}{180}a_0$

$\vdots\quad\vdots\quad\vdots\quad\vdots$

a_1

$n=2\to\ a_4=\frac{e}{24}-\frac{1}{12}a_1$

$n=5\to\ a_7=\frac{e}{5040}-\frac{1}{42}a_4=\frac{e}{1260}+\frac{1}{504}a_0$

$\vdots\quad\vdots\quad\vdots\quad\vdots$

$a_2=\frac{e}{2}$

$n=3\ \to\ a_5=\frac{e}{120}-\frac{1}{20}a_2=\frac{e}{120}-\frac{e}{40}=-\frac{e}{60}$

$n=6\ \to\ a_8=\frac{e}{40,320}-\frac{1}{56}a_5=\frac{e}{40,320}+\frac{e}{3360}=\frac{13e}{40,320}$

$\vdots\quad\vdots\quad\vdots\quad\vdots$

$$\begin{aligned}y(x)&=a_0+(x-1)a_1+\frac{e}{2}(x-1)^2+\left(\frac{e}{6}-\frac{1}{6}a_0\right)(x-1)^3+\left(\frac{e}{24}-\frac{1}{12}a_1\right)(x-1)^4-\frac{e}{60}(x-1)^5+\cdots\\&=a_0+(x-1)a_1+\frac{e}{2}(x-1)^2+\frac{e}{6}(x-1)^3+-\frac{1}{6}a_0(x-1)^3+\frac{e}{24}(x-1)^4\\&\qquad-\frac{1}{12}a_1(x-1)^4-\frac{e}{60}(x-1)^5+\cdots\end{aligned}$$

$$\boxed{\begin{aligned}y(x)=&\ e\left(\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3+\frac{1}{24}(x-1)^4-\frac{1}{60}(x-1)^5+\cdots\right)\\&+a_0\left(1-\frac{1}{6}(x-1)^3+\cdots\right)+a_1\left((x-1)-\frac{1}{12}(x-1)^4+\cdots\right)\end{aligned}}$$

Exercise

Find the series solution near the given value

$$y''+xy'+(2x-1)y=0\ ;\qquad near\ x=-1\qquad y(-1)=2,\qquad y'(-1)=-2$$

Solution

$t=x+1\quad\to\quad x=t-1$

$$y=\sum_{n=0}^{\infty}a_n t^n$$

$$y'=\sum_{n=1}^{\infty}na_n t^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n$$

$$y''=\sum_{n=2}^{\infty}n(n-1)a_n t^{n-2}=\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}t^n$$

$$y''+xy'+(2x-1)y=0$$

$$y''+(t-1)y'+(2t-3)y=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}t^n+(t-1)\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n+(2t-3)\sum_{n=0}^{\infty}a_n t^n=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}t^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}t^{n+1}-\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n+\sum_{n=0}^{\infty}2a_n t^{n+1}-\sum_{n=0}^{\infty}3a_n t^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}-(n+1)a_{n+1}-3a_n\right]t^n+\sum_{n=0}^{\infty}\left[(n+1)a_{n+1}+2a_n\right]t^{n+1}=0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}-(n+1)a_{n+1}-3a_n\right]t^n+\sum_{n=1}^{\infty}\left[na_n+2a_{n-1}\right]t^n=0$$

$$2a_2-a_1-3a_0+\sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2}-(n+1)a_{n+1}-3a_n\right]t^n+\sum_{n=1}^{\infty}\left[na_n+2a_{n-1}\right]t^n=0$$

$$2a_2-a_1-3a_0+\sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2}-(n+1)a_{n+1}+(n-3)a_n+2a_{n-1}\right]t^n=0$$

$$2a_2-a_1-3a_0=0\quad\rightarrow\quad \boxed{a_2=\tfrac{1}{2}\left(a_1+3a_0\right)}$$

$$(n+1)(n+2)a_{n+2}-(n+1)a_{n+1}+(n-3)a_n+2a_{n-1}=0$$

$$\boxed{a_{n+2}=\frac{1}{n+2}a_{n+1}-\frac{n-3}{(n+1)(n+2)}a_n-\frac{2}{(n+1)(n+2)}a_{n-1}}$$

Given: $t=x+1$

$y(x=-1)=y(t=0)=2=a_0,\qquad y'(x=-1)=y(t=0)=-2=a_1$

$$a_2=\tfrac{1}{2}\left(a_1+3a_0\right)=\tfrac{1}{2}(-2+6)\ \underline{=2}$$

$$n=1 \rightarrow a_3 = \frac{1}{3}a_2 + \frac{1}{3}a_1 - \frac{1}{3}a_0 = \frac{2}{3} - \frac{2}{3} - \frac{2}{3} = \underline{-\frac{2}{3}}$$

$$n=2 \rightarrow a_4 = \frac{1}{4}a_3 + \frac{1}{12}a_2 - \frac{1}{6}a_1 = -\frac{1}{6} + \frac{1}{6} + \frac{1}{3} = \underline{\frac{1}{3}}$$

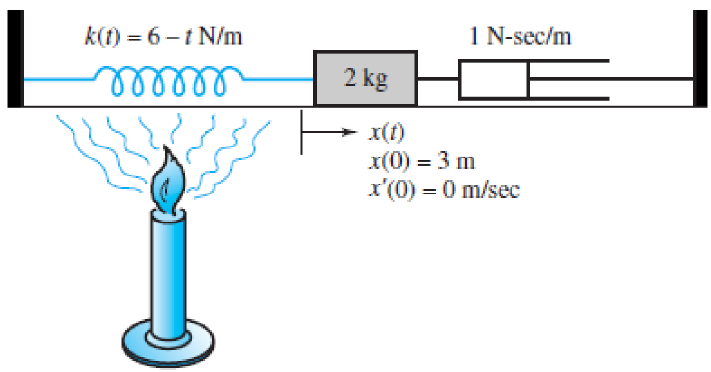
$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(t) = 2 - 2t + 3t^2 - \frac{1}{3}t^3 + \frac{1}{3}t^4 + \cdots$$

$$\underline{y(x) = 2 - 2(x+1) + 3(x+1)^2 - \frac{1}{3}(x+1)^3 + \frac{1}{3}(x+1)^4 + \cdots}$$

Exercise

As a spring is heated, its spring "constant" decreases. Suppose the spring is heated so that the spring "constant" at time t is $k(t) = 6 - t$ *N/m*.



If the unforced mass-spring system has mass $m = 2$ *kg* and a damping constant $b = 1$ *N-sec/m* with initial conditions $x(0) = 3$ *m* and $x'(0) = 0$ *m/sec*, then the displacement $x(t)$ is governed by the initial value problem

$$2x''(t) + x'(t) + (6-t)x(t) = 0 \; ; \quad x(0) = 3, \quad x'(0) = 0$$

Find at least the first four nonzero terms in a power series expansion about $t = 0$ for the displacement.

Solution

$$x(t) = \sum_{n=0}^{\infty} a_n t^n$$

$$x'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n$$

$$x''(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n$$

$$2x'' + x' + (6-t)x = 0$$

$$2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n+(6-t)\sum_{n=0}^{\infty}a_nt^n=0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}t^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n+\sum_{n=0}^{\infty}6a_nt^n-\sum_{n=0}^{\infty}a_nt^{n+1}=0$$

$$\sum_{n=0}^{\infty}\left[2(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}+6a_n\right]t^n-\sum_{n=1}^{\infty}a_{n-1}t^n=0$$

$$4a_2+a_1+6a_0+\sum_{n=1}^{\infty}\left[2(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}+6a_n\right]t^n-\sum_{n=1}^{\infty}a_{n-1}t^n=0$$

$$4a_2+a_1+6a_0+\sum_{n=1}^{\infty}\left[2(n+1)(n+2)a_{n+2}+(n+1)a_{n+1}+6a_n-a_{n-1}\right]t^n=0$$

Given: $x(0)=3=a_0,\quad x'(0)=0=a_1$

$$4a_2+a_1+6a_0=0\quad\rightarrow\quad \boxed{a_2=-\frac{9}{2}}$$

$$2(n+1)(n+2)a_{n+2}+(n+1)a_{n+1}+6a_n-a_{n-1}=0$$

$$\boxed{a_{n+2}=\frac{a_{n-1}-6a_n-(n+1)a_{n+1}}{2(n+1)(n+2)}}$$

$$n=1\ \rightarrow\ a_3=\frac{1}{12}\left(a_0-6a_1-2a_2\right)=\frac{1}{12}(3+9)=1$$

$$n=2\ \rightarrow\ a_4=\frac{1}{24}\left(a_1-6a_2-3a_3\right)=\frac{1}{24}(27-3)=1$$

$$\boxed{x(t)=3-\frac{9}{2}t^2+t^3+t^4+\cdots}$$

Solution Section 4.3 – Legendre's Equation

Exercise

Establish the recursion formula using the following two steps

a) Differentiate both sides of equation

$$g(t,\ x)=\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n \text{ with respect to } t \text{ to show that}$$

$$(x-t)\sum_{n=0}^{\infty}P_n(x)t^n=\left(1-2xt+t^2\right)\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$

b) Equate the coefficients of t^n in this equation to show that

$$P_1(x)=xP_0(x) \qquad \textbf{\textit{and}} \qquad (n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x) \quad \textit{for } n\ge 1$$

Solution

a) Let: $\left(1-2xt+t^2\right)^{-1/2}=\sum_{n=0}^{\infty}P_n(x)t^n$

Differentiate both sides with respect to *t*: $\left(\left(1-2xt+t^2\right)^{-1/2}\right)'=\left(\sum_{n=0}^{\infty}P_n(x)t^n\right)'$

$$-\frac{1}{2}(-2x+2t)\left(1-2xt+t^2\right)^{-3/2}=\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$

$$(x-t)\left(1-2xt+t^2\right)^{-1/2}=\left(1-2xt+t^2\right)\sum_{n=1}^{\infty}nP_n(x)t^{n-1} \qquad \textit{\textbf{Multiply both sides by: }} 1-2xt+t^2$$

$$(x-t)\sum_{n=0}^{\infty}P_n(x)t^n=\left(1-2xt+t^2\right)\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$

b) $$(x-t)\sum_{n=0}^{\infty}P_n(x)t^n=\sum_{n=0}^{\infty}xP_n(x)t^n-\underbrace{\sum_{n=0}^{\infty}P_n(x)t^{n+1}}_{n=n+1}$$

$$=\sum_{n=0}^{\infty}xP_n(x)t^n-\sum_{n=1}^{\infty}P_{n-1}(x)t^n$$

$$\left(1-2xt+t^2\right)\sum_{n=1}^{\infty} nP_n(x)t^{n-1} = \sum_{n=1}^{\infty} nP_n(x)t^{n-1} - \sum_{n=1}^{\infty} 2xnP_n(x)t^n + \sum_{n=1}^{\infty} nP_n(x)t^{n+1}$$

$$= \sum_{n=0}^{\infty}(n+1)P_{n+1}(x)t^n - \sum_{n=1}^{\infty} 2nxP_n(x)t^n + \sum_{n=2}^{\infty}(n-1)P_{n-1}(x)t^n$$

Thus,

$$\sum_{n=0}^{\infty} xP_n(x)t^n - \sum_{n=1}^{\infty} P_{n-1}(x)t^n = \sum_{n=0}^{\infty}(n+1)P_{n+1}(x)t^n - \sum_{n=1}^{\infty} 2nxP_n(x)t^n + \sum_{n=2}^{\infty}(n-1)P_{n-1}(x)t^n$$

Therefore;

$$0 = \left[xP_0(x) - P_1(x)\right]t^0 + \left[xP_1(x) - P_0(x) - 2P_2(x) + 2xP_1(x)\right]t^1$$

$$+\sum_{n=0}^{\infty}\left[xP_n(x) - P_{n-1}(x) - (n+1)P_{n+1}(x) + 2nxP_n(x) - (n-1)P_{n-1}(x)\right]t^n$$

$$0 = \left[xP_0(x) - P_1(x)\right]t^0 + \left[3xP_1(x) - P_0(x) - 2P_2(x)\right]t^1$$

$$+\sum_{n=0}^{\infty}\left[(2n+1)xP_n(x) - nP_{n-1}(x) - (n+1)P_{n+1}(x)\right]t^n$$

That implies:

$$xP_0(x) - P_1(x) = 0 \quad\Rightarrow\quad P_1(x) = xP_0(x)$$

$$3xP_1(x) - P_0(x) - 2P_2(x) = 0 \quad\Rightarrow\quad 2P_2(x) = P_0(x) - 3xP_1(x)$$

$$(2n+1)xP_n(x) - nP_{n-1}(x) - (n+1)P_{n+1}(x) = 0$$

$$\Rightarrow\quad (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

If n = 1 then: $2P_2(x) = 3xP_1(x) - P_0(x)$ √

Exercise

Show that $P_{2n+1}(0)=0$ *and* $P_{2n}(0)=(-1)^n \dfrac{2n!}{2^{2n}(n!)^2}$

Solution

Given the formula:

$$nP_n(x)=(2n-1)xP_{n-1}(x)-(n-1)P_{n-2}(x) \qquad \textit{for } n\ge 2$$

By letting $x=0$, then the formula can be:

$$nP_n(0)=-(n-1)P_{n-2}(0)$$

Replacing n with $2n$, then

$$2nP_{2n}(0)=-(2n-1)P_{2n-2}(0)$$

$$P_{2n}(0)=\frac{1-2n}{2n}P_{2n-2}(0)$$

$$P_2(0)=\frac{1-2}{2}P_0(0)=-\frac{1}{2}P_0(0)$$

$$P_4(0)=\frac{1-4}{4}P_2(0)=\frac{1-2}{2}\cdot\frac{1-4}{4}P_0(0)=\frac{1\cdot 3}{2^2\cdot 1\cdot 2}P_0(0)$$

$$P_6(0)=\frac{1-6}{6}P_4(0)=\frac{1-2}{2}\cdot\frac{1-4}{4}\cdot\frac{1-6}{6}P_0(0)=-\frac{1\cdot 3\cdot 5}{2^3\cdot 1\cdot 2\cdot 3}P_0(0)$$

$$\vdots \qquad \vdots \quad \vdots$$

$$P_{2n}(0)=\frac{1-2}{2}\cdot\frac{1-4}{4}\cdot\frac{1-6}{6}\cdots\frac{1-2n}{2n}P_0(0)$$

$$=(-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\cdot 1\cdot 2\cdot 3\cdots n}P_0(0)$$

$$1\cdot 3\cdot 5\cdots(2n-1)=\frac{1\cdot 2\cdot 3\cdot 4\cdots(2n-1)(2n)}{2\cdot 4\cdot 6\cdots(2n)}$$

$$=\frac{(2n)!}{2^n n!}$$

$$=(-1)^n\frac{(2n)!}{2^n\cdot(n!)^2}P_0(0)$$

With $P_0(0)=1$

$$\boxed{P_{2n}(0)=(-1)^n\frac{(2n)!}{2^n\cdot(n!)^2}}$$

Exercise

Show that $P_n'(0) = (-1)^{n+1} P_n'(-1) = \frac{n(n+1)}{2}$

Hint: Use Legendre's equation $\left(1-x^2\right)y'' - 2xy' + n(n+1)y = 0$

Solution

Because $P_n(x)$ is a solution of Legendre's equation, then

$$\left(1-x^2\right)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$$

Let $x = 1$, then

$$-2P_n'(1) + n(n+1)P_n(1) = 0$$

$$P_n'(1) = \frac{n(n+1)}{2}P_n(1)$$

Let $x = -1$, then

$$2P_n'(-1) + n(n+1)P_n(-1) = 0$$

$$P_n'(-1) = -\frac{n(n+1)}{2}P_n(-1)$$

However, $P_n(1) = P_n(-1) = 1$

$$\boxed{(-1)^{n+1} P_n'(-1) = \frac{n(n+1)}{2}}$$

Exercise

The differential equation $y'' + xy = 0$ is called ***Airy's equation***, and its solutions are called ***Airy functions***. Find the series for the solutions y_1 and y_2 where $y_1(0) = 1$ and $y_1'(0) = 0$, while $y_2(0) = 0$ and $y_2'(0) = 1$. What is the radius of convergence for these two series?

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + xy = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}a_{n-1} x^{n} = 0$$

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}a_{n-1} x^{n} = 0$$

$$2a_2 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_{n-1}\right]x^{n} = 0$$

$$2a_2 = 0 \quad \text{or} \quad (n+2)(n+1)a_{n+2} + a_{n-1} = 0$$

$$a_2 = 0 \quad \text{or} \quad a_{n+2} = \frac{-a_{n-1}}{(n+2)(n+1)} \qquad n \geq 1$$

$$a_3 = \frac{-a_0}{3\cdot 2} \qquad a_4 = -\frac{a_1}{4\cdot 3} \qquad a_5 = -\frac{a_2}{5\cdot 4} = 0$$

$$a_6 = -\frac{a_3}{6\cdot 5} = \frac{a_0}{6\cdot 5\cdot 3\cdot 2} \qquad a_7 = -\frac{a_4}{7\cdot 6} = \frac{a_1}{7\cdot 6\cdot 4\cdot 3} \qquad a_8 = -\frac{a_5}{8\cdot 7} = 0$$

$$a_9 = -\frac{a_6}{9\cdot 8} = -\frac{a_0}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2} \qquad a_{10} = -\frac{a_7}{10\cdot 9} = -\frac{a_1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3} \qquad a_{11} = 0$$

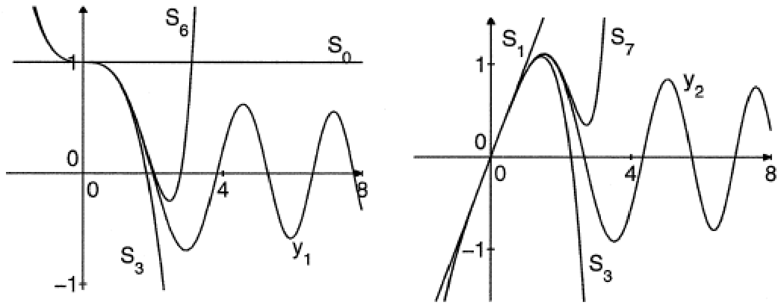
$$\vdots \quad \vdots \qquad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \qquad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots$$

$$a_{3n} = \frac{(-1)^n a_0}{(2\cdot 3)\cdot(5\cdot 6)\cdots(3n-1)(3n)} \qquad a_{3n+1} = \frac{(-1)^n a_1}{(3\cdot 4)\cdot(6\cdot 7)\cdots(3n)(3n+1)} \qquad a_{3n+2} = 0$$

$$y(x) = a_0\left[1 - \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 - \cdots\right] + a_1\left[x - \frac{1}{3\cdot 4}x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}x^7 - \cdots\right]$$

$$y_1(x) = 1 - \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 - \cdots = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^{3n}}{(2\cdot 3)\cdot(5\cdot 6)\cdots(3n-1)(3n)}$$

$$y_2(x) = x - \frac{1}{3\cdot 4}x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}x^7 - \cdots = x + \sum_{n=1}^{\infty}\frac{(-1)^n x^{3n+1}}{(3\cdot 4)\cdot(6\cdot 7)\cdots(3n)(3n+1)}$$



Exercise

The Hermite equation of order α is $\quad y'' - 2xy' + 2\alpha y = 0$

a) Find the general solution is $y(x) = a_0 y_1(x) + a_1 y_2(x)$

Show that $y_1(x)$ is a polynomial if α is an even integer, whereas $y_2(x)$ is a polynomial if α is an odd integer.

b) When $\alpha = n$, use $y_1(x)$ to find polynomial solutions for $n=0$, $n=2$, and $n=4$, then use $y_2(x)$ to find polynomial solutions for $n=1$, $n=3$, and $n=5$.

c) The Hermite polynomial of degree n is denoted by $H_n(x)$. It is the nth-degree polynomial solution of Hermite's equation, multiplied by a suitable constant so that the coefficient of x^n is 2^n. Use part (b) to show the first six Hermite polynomials are

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

d) A general formula for the Hermite polynomials is

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

Verify that this formula does in fact give an nth-degree polynomial.

Solution

a) $y(x) = \sum\limits_{n=0}^{\infty} a_n x^n$

$$y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''-2xy'+2\alpha y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+2\alpha\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}2(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}2\alpha a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n-\sum_{n=1}^{\infty}2na_n x^n+\sum_{n=0}^{\infty}2\alpha a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}-2(n-\alpha)a_n\right]x^n=0$$

$$(n+1)(n+2)a_{n+2}-2(n-\alpha)a_n=0$$

$$\boxed{a_{n+2}=\frac{2(n-\alpha)}{(n+1)(n+2)}a_n}$$

a_0

$$n=0\ \rightarrow\ a_2=-\frac{2\alpha}{2}a_0$$

$$n=2\ \rightarrow\ a_4=\frac{2(2-\alpha)}{3\cdot 4}a_2=-\frac{2^2\alpha(2-\alpha)}{4!}a_0$$

$$n=4\ \rightarrow\ a_6=\frac{2(4-\alpha)}{5\cdot 6}a_4=-\frac{2^3\alpha(2-\alpha)(4-\alpha)}{6!}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x)=1-\frac{2\alpha}{2!}x^2-\frac{2^2(2-\alpha)}{4!}x^4-\frac{2^3\alpha(2-\alpha)(4-\alpha)}{6!}x^6-\cdots$$

$$=1-\frac{2\alpha}{2!}x^2+\frac{2^2(\alpha-2)}{4!}x^4-\frac{2^3\alpha(\alpha-2)(\alpha-4)}{6!}x^6+\cdots$$

a_1

$$n=1\ \rightarrow\ a_3=\frac{2(1-\alpha)}{6}a_1=\frac{2(1-\alpha)}{3!}a_1$$

$$n=3 \;\rightarrow\; a_5 = \frac{2(3-\alpha)}{4\cdot 5}a_3 = \frac{2^2(1-\alpha)(3-\alpha)}{5!}a_1$$

$$n=5 \;\rightarrow\; a_7 = \frac{2(3-\alpha)}{6\cdot 7}a_5 = \frac{2^3(1-\alpha)(3-\alpha)(5-\alpha)}{7!}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = x + \frac{2(1-\alpha)}{3!}x^3 + \frac{2^2(1-\alpha)(3-\alpha)}{5!}x^5 + \frac{2^3(1-\alpha)(3-\alpha)(5-\alpha)}{7!}x^7 + \cdots$$

$$= x - \frac{2(\alpha-1)}{3!}x^3 + \frac{2^2(\alpha-1)(\alpha-3)}{5!}x^5 - \frac{2^3(\alpha-1)(\alpha-3)(\alpha-5)}{7!}x^7 + \cdots$$

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

$$= a_0\left(1 - \frac{2\alpha}{2!}x^2 + \frac{2^2\alpha(\alpha-2)}{4!}x^4 - \frac{2^3\alpha(\alpha-2)(\alpha-4)}{6!}x^6 + \cdots\right)$$

$$+ a_1\left(x - \frac{2(\alpha-1)}{3!}x^3 + \frac{2^2(\alpha-1)(\alpha-3)}{5!}x^5 - \frac{2^3(\alpha-1)(\alpha-3)(\alpha-5)}{7!}x^7 + \cdots\right)$$

$$= a_0 + a_0\sum_{m=1}^{\infty}(-1)^m \frac{2^m\alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!}x^{2m}$$

$$+ a_1 x + a_1\sum_{m=1}^{\infty}(-1)^m \frac{2^m(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!}x^{2m+1}$$

$$\boxed{y_1(x) = 1 + \sum_{m=1}^{\infty}(-1)^m \frac{2^m\alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!}x^{2m}}$$

$$\boxed{y_2(x) = x + \sum_{m=1}^{\infty}(-1)^m \frac{2^m(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!}x^{2m+1}}$$

b) $n=\alpha=0 \;\rightarrow\; y_1(x)\ \boxed{=1}$

$n=\alpha=2 \;\rightarrow\; y_1(x) = 1-\alpha x^2\ \boxed{=1-2x^2}$

$n=\alpha=4 \;\rightarrow\; y_1(x) = 1-\alpha x^2 + \dfrac{\alpha(\alpha-2)}{6}x^4\ \boxed{=1-4x^2+\frac{4}{3}x^4}$

$n=\alpha=1 \;\rightarrow\; y_2(x)\ \boxed{=x}$

$n=\alpha=3 \;\rightarrow\; y_2(x) = x - \dfrac{2(\alpha-1)}{3!}x^3\ \boxed{=x-\frac{2}{3}x^3}$

$$n=\alpha=5 \quad \rightarrow \quad y_2(x)=x-\frac{2(\alpha-1)}{3!}x^3+\frac{2^2(\alpha-1)(\alpha-3)}{5!}x^5 \;\; \underline{=x-\frac{4}{3}x^3+\frac{4}{15}x^5}$$

c) $H_0(x)=2^0\cdot 1=1$

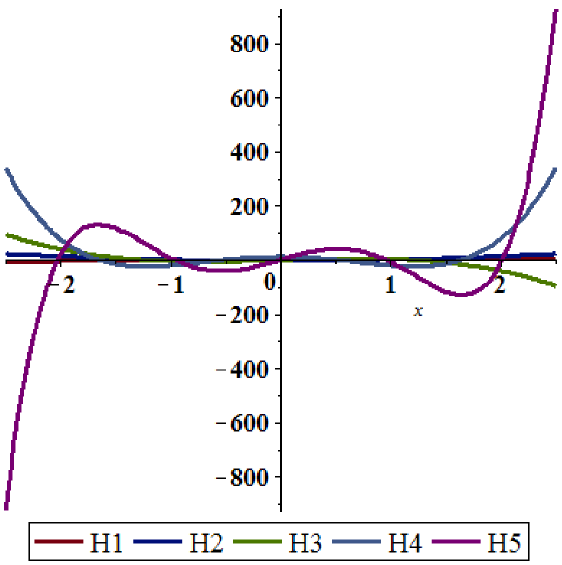
$H_1(x)=2^1\cdot x=2x$

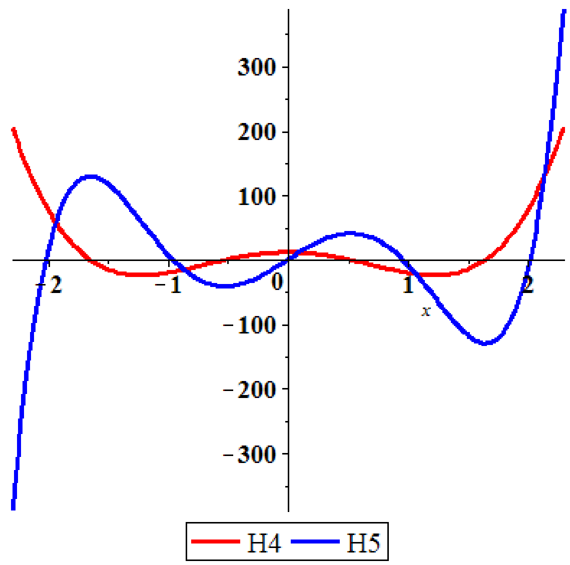
$H_2(x)=-2\left(1-2x^2\right)=4x^2-2$

$H_3(x)=-2^2\cdot 3\left(x-\frac{2}{3}x^3\right)=8x^3-12x$

$H_4(x)=2^2\cdot 3\left(1-4x^2+\frac{4}{3}x^4\right)=16x^4-48x^2+12$

$H_5(x)=2^3\cdot 3\cdot 5\left(\frac{4}{15}x^5-\frac{4}{3}x^3+x\right)=32x^5-160x^3+120x$





d) $\frac{d}{dx}\left(e^{-x^2}\right)=-2xe^{-x^2}$

$\frac{d^2}{dx^2}\left(e^{-x^2}\right)=-2\frac{d}{dx}\left(xe^{-x^2}\right)=-2\left(1-2x^2\right)e^{-x^2}$

$\frac{d^3}{dx^3}\left(e^{-x^2}\right)=2\frac{d}{dx}\left(\left(2x^2-1\right)e^{-x^2}\right)=2\left(4x-4x^3+2x\right)e^{-x^2}=\left(12x-8x^3\right)e^{-x^2}$

$\frac{d^4}{dx^4}\left(e^{-x^2}\right)=4\frac{d}{dx}\left(\left(3x-2x^3\right)e^{-x^2}\right)=4\left(3-6x^2-6x^2+4x^4\right)e^{-x^2}=\left(16x^4-48x^2+12\right)e^{-x^2}$

$H_1(x)=-e^{x^2}\frac{d}{dx}\left(e^{-x^2}\right)=2xe^{x^2}e^{-x^2}=2x$ √

$H_2(x)=e^{x^2}\frac{d^2}{dx^2}\left(e^{-x^2}\right)=-2e^{x^2}\left(1-2x^2\right)e^{-x^2}=4x^2-2$ √

$H_3(x)=e^{x^2}\frac{d^3}{dx^3}\left(e^{-x^2}\right)=e^{x^2}\left(12x-8x^3\right)e^{-x^2}=12x-8x^3$ √

$$H_4(x) = e^{x^2}\frac{d^4}{dx^4}\left(e^{-x^2}\right) = e^{x^2}\left(16x^4 - 48x^2 + 12x\right)e^{-x^2} = 16x^4 - 48x^2 + 12 \quad \checkmark$$

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

Exercise

Rodrigues's Formula is given by: $\quad P_n(x) = \dfrac{1}{n!\ 2^n}\dfrac{d^n}{dx^n}\left(x^2 - 1\right)^n$

For the *n*th-degree Legendre polynomial.

a) Show that $v = \left(x^2 - 1\right)^n$ satisfies the differential equation $\left(1 - x^2\right)v' + 2nxv = 0$

Differentiate each side of this equation to obtain

$$\left(1 - x^2\right)v'' + 2(n-1)xv' + 2nv = 0$$

b) Differentiate each side of the last equation *n* times in succession to obtain

$$\left(1 - x^2\right)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0$$

which satisfies Legendre's equation of order *n*.

c) Show that the coefficient of x^n in *u* is $\dfrac{(2n)!}{n!}$; then state why this proves Rodrigues. Formula.

Note: That the coefficient of x^n in $P_n(x)$ is $\dfrac{(2n)!}{2^n(n!)^2}$

$$u = v^{(n)} = D^n\left(x^2 - 1\right)^n$$

Solution

a) $v = \left(x^2 - 1\right)^n \quad \rightarrow \quad v' = 2nx\left(x^2 - 1\right)^{n-1}$

$$v' = 2nx\left(x^2 - 1\right)^{n-1}$$

$$\left(1 - x^2\right)v' + 2nxv = -2nx\left(x^2 - 1\right)\left(x^2 - 1\right)^{n-1} + 2nx\left(x^2 - 1\right)^n$$

$$= -2nx\left(x^2 - 1\right)^n + 2nx\left(x^2 - 1\right)^n$$

$$\underline{= 0}$$

$$\frac{d}{dx}\left(\left(1 - x^2\right)v' + 2nxv\right) = 0$$

$$\left(1 - x^2\right)v'' - 2xv' + 2nxv' + 2nv = 0$$

$$\boxed{\left(1-x^2\right)v''+2(n-1)xv'+2nv=0}$$

b) $\frac{d}{dx}\left(\left(1-x^2\right)v''-2xv'+2nxv'+2nv\right)=0$

$$\left(1-x^2\right)v^{(3)}-2xv''-2v'-2xv''+2nxv''+2nv'+2nv'$$

$$\boxed{\left(1-x^2\right)v^{(3)}+2(n-2)xv''+2(2n-1)v'=0}$$

$n=1 \quad \to \quad \left(1-x^2\right)v^{(3)}-2xv''+2v'=0$

$$\boxed{\left(1-x^2\right)v^{(n+2)}-2xv^{(n+1)}+n(n+1)v^{(n)}=0} \quad \checkmark$$

$$\frac{d}{dx}\left(\left(1-x^2\right)v^{(3)}+2x(n-2)v''+2(2n-1)v'\right)=0$$

$$\left(1-x^2\right)v^{(4)}-2xv^{(3)}+2x(n-2)v^{(3)}+2(n-2)v''+2(2n-1)v''=0$$

$$\left(1-x^2\right)v^{(4)}+2x(n-3)v^{(3)}+6(n-1)v''=0$$

$$\boxed{\left(1-x^2\right)v^{(4)}+2(n-3)xv^{(3)}+3(2n-2)v''=0}$$

$n=2 \quad \to \quad \left(1-x^2\right)v^{(4)}-2xv^{(3)}+3\cdot 2v''=0$

$$\boxed{\left(1-x^2\right)v^{(n+2)}-2xv^{(n+1)}+n(n+1)v^{(n)}=0} \quad \checkmark$$

$$\frac{d}{dx}\left(\left(1-x^2\right)v^{(4)}+2(n-3)xv^{(3)}+3(2n-2)v''\right)=0$$

$$\left(1-x^2\right)v^{(5)}-2xv^{(4)}+2(n-3)xv^{(4)}+2(n-3)v^{(3)}+3(2n-2)v^{(3)}=0$$

$$\left(1-x^2\right)v^{(5)}+(2n-6-2)xv^{(4)}+(2n-6+6n-6)v^{(3)}=0$$

$$\left(1-x^2\right)v^{(5)}+(2n-8)xv^{(4)}+(8n-12)v^{(3)}=0$$

$$\boxed{\left(1-x^2\right)v^{(5)}+2(n-4)xv^{(4)}+4(2n-3)v^{(3)}=0}$$

$n=3 \quad \to \quad \left(1-x^2\right)v^{(5)}-2xv^{(4)}+4\cdot 3v^{(3)}=0$

$$\boxed{\left(1-x^2\right)v^{(n+2)}-2xv^{(n+1)}+n(n+1)v^{(n)}=0} \quad \checkmark$$

After m differentiations:

$$\left(1-x^2\right)v^{(m+2)}+2\left(n-(m+1)\right)xv^{(m+1)}+(m+1)(2n-m)v^{(m)}=0$$

If we let $m=n$, then

$$\boxed{\left(1-x^2\right)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0}$$

Let assume that $\left(1-x^2\right)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0$ is true.

We need to prove that next derivative is also true.

$$\frac{d}{dx}\left(\left(1-x^2\right)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)}\right) = 0$$

$$\left(1-x^2\right)v^{(n+3)} - 2xv^{(n+2)} - 2v^{(n+1)} - 2xv^{(n+2)} + (2n-n)(n+1)v^{(n+1)} = 0$$

$$\left(1-x^2\right)v^{(n+3)} - 4xv^{(n+2)} + (2n-n-2)(n+1)v^{(n+1)} = 0$$

$$\left(1-x^2\right)v^{(n+3)} - 2(2)xv^{(n+2)} + (n-1)(n+2)v^{(n+1)} = 0$$

$$\left(1-x^2\right)v^{(n+3)} + 2(n-n-2)xv^{(n+2)} + (n-1)(n+2)v^{(n+1)} = 0$$

$$\left(1-x^2\right)v^{(n+3)} + 2(n-(n+2))xv^{(n+2)} + (2n-(n+1))((n+1)+1)v^{(n+1)} = 0$$

If we let $m = n+1$, then

$$\left(1-x^2\right)v^{(m+2)} + 2(n-(m+1))xv^{(m+1)} + (2n-m)(m+1)v^{(m)} = 0$$

If we let $m = n$, then

$$\left(1-x^2\right)v^{(n+2)} + 2(n-n-1)xv^{(m+1)} + (2n-n)(n+1)v^{(n)} = 0$$

$$\boxed{\left(1-x^2\right)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0} \quad \checkmark$$

c) $u = v^{(n)} = D^n\left(x^2-1\right)^n$

$$= \frac{d^n}{dx^n}\left(x^{2n} - nx^{2n-1} + \cdots - 1\right)$$

$$= 2n(2n-1)\cdots(2n-(n-1))x^n - \frac{d^n}{dx^n}\left(nx^{2n-1} + \cdots - 1\right)$$

$$= \frac{(2n)!}{n!}x^n - \frac{d^n}{dx^n}\left(nx^{2n-1} + \cdots - 1\right)$$

Since $u = v^{(n)}$ satisfies Legendre's equation of order n, $\frac{u}{2^n n!}$

From the notes:

The solution of Legendre polynomial of degree ***n*** is

$$P_n(x) = \sum_{k=0}^{K}(-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!}x^{n-2k}$$

For the highest order, the result when:

$$k = 0 \quad \rightarrow \quad P_n(x) = \frac{(2n)!}{2^n (n!)^2} x^n$$

$$\frac{u}{2^n n!} = \frac{(2n)!}{2^n (n!)^2} x^n + \cdots$$

$$P_n(x) = \frac{1}{n!\ 2^n} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n$$

Solution Section 4.4 – Solution about Singular Points

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^2y'' + 3y' - xy = 0$$

Solution

$$y'' + \frac{3}{x^2}y' - \frac{x}{x^2}y = 0$$

$$P(x) = \frac{3}{x^2} \qquad Q(x) = -\frac{x}{x^2}$$

For $P(x) = \frac{3}{x^2} \rightarrow \underline{x = 0}$

$\therefore\ p(x)$ is analytic except at $\underline{x = 0}$

For $Q(x) = -\frac{x}{x^2} = -\frac{1}{x}$

$\therefore\ q(x)$ is not analytic at $\underline{x = 0}$

The singular point is: $\underline{x = 0}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^2 + x\right)y'' + 3y' - 6xy = 0$$

Solution

$$y'' + \frac{3}{x(x+1)}y' - \frac{6}{x+1}y = 0$$

$$P(x) = \frac{3}{x(x+1)} \qquad Q(x) = -\frac{6x}{x(x+1)}$$

For $P(x) = \frac{3}{x(x+1)} \rightarrow \underline{x = 0, -1}$

$\therefore\ p(x)$ is analytic except at $\underline{x = 0, -1}$

For $q(x) = -\frac{6x}{x(x+1)} \rightarrow x = 0,\ -1$

$q(x) = -\frac{6x}{x(x+1)} = -\frac{6}{x+1}$; is actually analytic at $x = 0$

$\therefore\ q(x)$ is analytic except at $\underline{x = -1}$

The singular points are: $\underline{x = 0,\ -1}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^2-1\right)y''+(1-x)y'+\left(x^2-2x+1\right)y=0$$

Solution

$$(x-1)(x+1)y''+(1-x)y'+(x-1)^2 y=0$$

$$y''-\frac{1}{x+1}y'+\frac{x-1}{x+1}y=0$$

$$P(x)=\frac{1-x}{x^2-1} \qquad Q(x)=\frac{(x-1)^2}{x^2-1}$$

For $p(x)=\dfrac{1-x}{(x-1)(x+1)} \quad \rightarrow \quad \underline{x=-1,\ 1}$

$p(x)=\dfrac{1-x}{(x-1)(x+1)}=-\dfrac{1}{x+1}$; is actually analytic at $x=1$

$\therefore\ p(x)$ is analytic except at $\underline{x=-1}$

For $q(x)=\dfrac{(x-1)^2}{(x-1)(x+1)} \quad \rightarrow \quad \underline{x=-1,\ 1}$

$q(x)=\dfrac{(x-1)^2}{(x-1)(x+1)}=\dfrac{x-1}{x+1}$; is actually analytic at $x=1$

$\therefore\ q(x)$ is analytic except at $\underline{x=-1}$

The singular point is: $\underline{x=-1}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$e^x y''-\left(x^2-1\right)y'+2xy=0$$

Solution

$$y''-\frac{x^2-1}{e^x}y'+\frac{2x}{e^x}y=0$$

$$P(x)=-\frac{x^2-1}{e^x} \qquad Q(x)=\frac{2x}{e^x}$$

Since $e^x \neq 0$, there are ***no*** singular points.

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\ln(x-1)y'' + (\sin 2x)y' - e^x y = 0$$

Solution

$$y'' + \frac{\sin 2x}{\ln(x-1)}y' - \frac{e^x}{\ln(x-1)}y = 0$$

$$P(x) = \frac{\sin 2x}{\ln(x-1)} \qquad Q(x) = -\frac{e^x}{\ln(x-1)}$$

$$\ln(x-1) = 0 \quad \to \quad x-1=1 \;\Rightarrow\; \underline{x=2}$$

The singular point is: $\underline{x \le 1,\ x = 2}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$xy'' + x(1-x)^{-1}y' + (\sin x)y = 0$$

Solution

$$y'' + \frac{x}{x(1-x)}y' + \frac{\sin x}{x}y = 0$$

$$P(x) = \frac{x}{x(1-x)} \qquad Q(x) = \frac{\sin x}{x}$$

For $p(x) = \frac{x}{x(1-x)} \quad \to \quad \underline{x = 0,\ 1}$

$p(x) = \frac{x}{x(1-x)} = \frac{1}{1-x}$; is actually analytic at $x = 0$

$\therefore\ p(x)$ is analytic except at $\underline{x=1}$

For $q(x) = \frac{\sin x}{x} = \frac{x - \frac{1}{3!}x^3 + \cdots}{x} = 1 - \frac{1}{3!}x^2 + \cdots$ is analytic everywhere ($x = 0$ is removable).

The only singular point is $\underline{x=1}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x(x+3)^2 y'' - y = 0$$

Solution

$$y'' - \frac{1}{x(x+3)^2} y = 0$$

$$P(x) = 0 \qquad Q(x) = -\frac{1}{x(x+3)^2}$$

For $q(x) = -\frac{1}{x(x+3)^2} \quad \to \quad \underline{x = 0, \ -3}\rfloor$, is analytic elsewhere

The *Regular* singular points are $\underline{x = 0, \ -3}\rfloor$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^2-9\right)^2 y'' + (x+3)y' + 2y = 0$$

Solution

$$y'' + \frac{x+3}{\left(x^2-9\right)^2} y' + \frac{2}{\left(x^2-9\right)^2} y = 0$$

$$P(x) = \frac{x+3}{\left(x^2-9\right)^2} \qquad Q(x) = \frac{2}{\left(x^2-9\right)^2}$$

For $P(x) = \frac{x+3}{\left(x^2-9\right)^2} \quad \to \quad \underline{x = \pm 3}\rfloor$

$$p(x) = \frac{x+3}{\left((x+3)(x-3)\right)^2} = \frac{(x+3)(x-3)}{(x+3)(x-3)^2}; \text{ is analytic at } x = -3$$

For $Q(x) = \frac{2\left(x^2-9\right)^2}{\left(x^2-9\right)^2} \quad \to \quad \underline{x = \pm 3}\rfloor$

$\therefore \ q(x)$ is analytic at $\underline{x = \pm 3}\rfloor$

The *Regular* singular point: $\underline{x = -3}\rfloor$, and *Irregular* singular point: $\underline{x = 3}\rfloor$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$y'' - \frac{1}{x} y' + \frac{1}{(x-1)^3} y = 0$$

Solution

$$y'' - \frac{1}{x}y' + \frac{1}{(x-1)^3}y = 0$$

$$P(x) = -\frac{1}{x} \qquad Q(x) = \frac{1}{(x-1)^3}$$

For $P(x) = -\frac{1}{x} \quad \rightarrow \quad \underline{x = 0 \,|}$

$p(x) = \frac{x}{x} = 1$ is analytic at $\underline{x = 0|}$

For $Q(x) = \frac{1}{(x-1)^3} \quad \rightarrow \quad \underline{x = 1\,|}$

$q(x) = \frac{(x-1)^2}{(x-1)^3} = \frac{1}{x-1}$ is *not* an analytic at $x = 1$

The *Regular* singular point: $\underline{x = 0\,|}$, and *Irregular* singular point: $\underline{x = 1\,|}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^3 + 4x\right)y'' - 2xy' + 6y = 0$$

Solution

$$y'' - \frac{2x}{x\left(x^2+4\right)}y' + \frac{6}{x\left(x^2+4\right)}y = 0$$

$$P(x) = -\frac{2x}{x\left(x^2+4\right)} \qquad Q(x) = \frac{6}{x\left(x^2+4\right)}$$

For $P(x) = -\frac{2x}{x\left(x^2+4\right)} \quad \rightarrow \quad x = 0, \ \pm 2i$

$p(x) = -\frac{2}{x^2+4}$ is analytic at $x = \pm 2i$

For $Q(x) = \frac{6}{x\left(x^2+4\right)} \quad \rightarrow \quad x = 0, \ \pm 2i$ is analytic

The *Regular* singular points: $\underline{x = 0, \ \pm 2i\,|}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^2(x-5)^2 y'' + 4xy' + \left(x^2-25\right)y = 0$$

Solution

$$y'' + \frac{4x}{x^2(x-5)^2}y' + \frac{x^2-25}{x^2(x-5)^2}y = 0$$

$$P(x) = \frac{4x}{x^2(x-5)^2} \qquad Q(x) = \frac{x^2-25}{x^2(x-5)^2}$$

For $P(x) = \dfrac{4x}{x^2(x-5)^2} \quad \to \quad x = 0,\ 5$

$p(x) = \dfrac{4x}{x^2(x-5)^2} = x(x-5)\dfrac{4}{x(x-5)^2}$ is not an analytic at $x = 5$

For $Q(x) = \dfrac{(x-5)(x+5)}{x^2(x-5)^2} \quad \to \quad x = 0,\ 5$

$q(x) = x^2(x-5)^2\dfrac{x+5}{x^2(x-5)}$ is an analytic at $x = 0,\ 5$

The *Regular* singular point: $\boxed{x = 0}$, and *Irregular* singular point: $\boxed{x = 5}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^2+x-6\right)y'' + (x+3)y' + (x-2)y = 0$$

Solution

$$y'' + \frac{x+3}{x^2+x-6}y' + \frac{x-2}{x^2+x-6}y = 0$$

$$P(x) = \frac{x+3}{(x+3)(x-2)} \qquad Q(x) = \frac{x-2}{(x+3)(x-2)}$$

For $P(x) = \dfrac{x+3}{(x+3)(x-2)} \quad \to \quad x = -3,\ 2$

$p(x) = \dfrac{1}{x-2}$ is an analytic at $x = 2$

For $Q(x) = \dfrac{x-2}{(x+3)(x-2)} \quad \to \quad x = -3,\ 2$

$q(x) = \dfrac{1}{x+3}$ is an analytic at $x = -3$

The *Regular* singular points: $\boxed{x = -3,\ 2}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x\left(x^2+1\right)^2 y''+y=0$$

Solution

$$y''+\frac{1}{x\left(x^2+1\right)^2}y=0$$

$$P(x)=0 \qquad Q(x)=\frac{1}{x\left(x^2+1\right)^2}$$

For $Q(x)=\dfrac{1}{x\left(x^2+1\right)^2} \quad \to \quad x=0,\ \pm i$

$$q(x)=x^2\left(x^2+1\right)^2\frac{1}{x\left(x^2+1\right)^2} \text{ is an analytic at } x=0,\ \pm i$$

The *Regular* singular points: $\underline{x=0,\ \pm i}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^3\left(x^2-25\right)(x-2)^2 y''+3x(x-2)y'+7(x+5)y=0$$

Solution

$$y''+\frac{3x(x-2)}{x^3\left(x^2-25\right)(x-2)^2}y'+\frac{7(x+5)}{x^3\left(x^2-25\right)(x-2)^2}y=0$$

$$P(x)=\frac{3x(x-2)}{x^3(x-5)(x+5)(x-2)^2} \qquad Q(x)=\frac{7(x+5)}{x^3(x-5)(x+5)(x-2)^2}$$

For $P(x)=\dfrac{3x(x-2)}{x^3(x-5)(x+5)(x-2)^2} \quad \to \quad x=0,\ \pm 5,\ 2$

$$p(x)=\frac{3x(x-5)(x+5)(x-2)}{x^2(x-5)(x+5)(x-2)} \text{ is not an analytic at } x=0$$

For $Q(x)=\dfrac{7(x+5)}{x^3(x-5)(x+5)(x-2)^2} \quad \to \quad x=0,\ \pm 5,\ 2$

$$q(x)=\frac{3x^2(x-5)^2(x+5)^2(x-2)^2}{x^3(x-5)(x-2)^2} \text{ is not an analytic at } x=0$$

The *Regular* singular point: $\underline{x=2,\ \pm 5}$, and *Irregular* singular point: $\underline{x=0}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^3-2x^2-3x\right)^2 y''+x(x-3)^2 y'-(x+1)y=0$$

Solution

$$y''+\frac{x(x-3)^2}{\left(x^3-2x^2-3x\right)^2}y'-\frac{x+1}{\left(x^3-2x^2-3x\right)^2}y=0$$

$$P(x)=\frac{x(x-3)^2}{x^2(x-3)^2(x+1)^2} \qquad Q(x)=-\frac{x+1}{x^2(x-3)^2(x+1)^2}$$

For $P(x)=\dfrac{x(x-3)^2}{x^2(x-3)^2(x+1)^2} \quad \rightarrow \quad x=0,\ -1,\ 3$

$$p(x)=\frac{1}{x(x+1)^2} \text{ is not an analytic at } x=-1$$

For $Q(x)=\dfrac{x+1}{x^2(x-3)^2(x+1)^2} \quad \rightarrow \quad x=0,\ -1,\ 3$

$$q(x)=\frac{x^2(x-3)^2(x+1)^2}{x^2(x-3)^2(x+1)} \text{ is an analytic at } x=0,\ -1,\ 3$$

The *Regular* singular point: $\underline{x=0,\ 3}$, and *Irregular* singular point: $\underline{x=-1}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(1-x^2\right)y''+(\tan x)y'+x^{5/3}y=0$$

Solution

$$y''+\frac{\tan x}{1-x^2}y'+\frac{x^{5/3}}{1-x^2}y=0$$

$$P(x)=\frac{\tan x}{1-x^2} \qquad Q(x)=\frac{x^{5/3}}{1-x^2}$$

For $P(x)=\frac{\tan x}{1-x^2} \quad \to \quad x=\pm 1$

$\tan x=\pm\infty \quad \to \quad x=\pm\frac{\pi}{2}$ (Vertical Asymptotes).

For $Q(x)=\frac{x^{5/3}}{1-x^2} \quad \to \quad x=\pm 1$ is not analytic

The second derivatices doesn't exist at $x=0$

The *Regular* singular point: $\underline{x=0,\ \pm 1,\ \pm\frac{\pi}{2}}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x(x-1)^2(x+2)y''+x^2y'-\left(x^3+2x-1\right)y=0$$

Solution

$$y''+\frac{x^2}{x(x-1)^2(x+2)}y'-\frac{x^3+2x-1}{x(x-1)^2(x+2)}y=0$$

$$P(x)=\frac{x}{(x-1)^2(x+2)} \quad \& \quad Q(x)=-\frac{x^3+2x-1}{x(x-1)^2(x+2)}$$

For $P(x)=\frac{x}{(x-1)^2(x+2)} \quad \to \quad x=1,\ -2$

$$p_0=\lim_{x\to 1}(x-1)P(x)=\lim_{x\to 1}\frac{x}{(x-1)(x+2)}=\infty \quad \text{is not analytic}$$

$$p_0=\lim_{x\to -2}(x+2)P(x)=\lim_{x\to -2}\frac{x}{(x-1)^2}=-\frac{2}{9}$$

For $Q(x)=-\frac{x^3+2x-1}{x(x-1)^2(x+2)} \quad \to \quad x=0,\ 1,\ -2$

$$q_0=\lim_{x\to 0}x^2Q(x)=\lim_{x\to 0}\frac{x\left(x^3+2x-1\right)}{(x-1)^2(x+2)}=0$$

$$q_0=\lim_{x\to 1}(x-1)^2Q(x)=\lim_{x\to 1}\frac{x^3+2x-1}{x(x+2)}=\frac{2}{3}$$

$$q_0=\lim_{x\to -2}(x+2)^2Q(x)=-\lim_{x\to -2}\frac{\left(x^3+2x-1\right)(x+2)}{x(x-1)^2}=0$$

The *Regular* singular point: $\underline{x=0,\ -2}$, and *Irregular* singular point: $\underline{x=1}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^4\left(x^2+1\right)(x-1)^2 y''+4x^3(x-1)y'+(x+1)y=0$$

Solution

$$y''+\frac{4x^3(x-1)}{x^4\left(x^2+1\right)(x-1)^2}y'+\frac{x+1}{x^4\left(x^2+1\right)(x-1)^2}y=0$$

$$P(x)=\frac{4}{x\left(x^2+1\right)(x-1)} \quad \& \quad Q(x)=\frac{x+1}{x^4\left(x^2+1\right)(x-1)^2}$$

For $P(x)=\dfrac{4}{x\left(x^2+1\right)(x-1)} \quad \to \quad \underline{x=0,\ 1,\ \pm i}$

$$p_0=\lim_{x\to 0} xP(x)=\lim_{x\to 0}\frac{4}{\left(x^2+1\right)(x-1)}\underline{=-4}$$

$$p_0=\lim_{x\to 1}(x-1)P(x)=\lim_{x\to 1}\frac{4}{x\left(x^2+1\right)}\underline{=2}$$

$$p_0=\lim_{x\to i}(x-i)P(x)=\lim_{x\to i}\frac{4}{x(x-1)(x+i)}=-\frac{2}{i-1}=-\frac{2}{i-1}\frac{i+1}{i+1}\underline{=i+1}$$

$$p_0=\lim_{x\to -i}(x+i)P(x)=\lim_{x\to -i}\frac{4}{x(x-1)(x-i)}=\frac{2}{i-1}=\frac{2}{i-1}\frac{i+1}{i+1}\underline{=-i-1}$$

For $Q(x)=\dfrac{x+1}{x^4\left(x^2+1\right)(x-1)^2} \quad \to \quad x=0,\ 1,\ \pm i$

$$q_0=\lim_{x\to 0} x^2Q(x)=\lim_{x\to 0}\frac{x+1}{x^2\left(x^2+1\right)(x-1)^2}\underline{=\infty} \text{ is not analytic}$$

$$q_0=\lim_{x\to 1}(x-1)^2Q(x)=\lim_{x\to 0}\frac{x+1}{x^4\left(x^2+1\right)}\underline{=1}$$

$$q_0=\lim_{x\to \pm i}\left(x^2+1\right)^2Q(x)=\lim_{x\to \pm i}\frac{(x+1)\left(x^2+1\right)}{x^2(x-1)^2}\underline{=0}$$

The *Regular* singular point: $\underline{x=0,\ \pm i}$, and *Irregular* singular point: $\underline{x=0}$

Exercise

Determine whether $x=0$ is an ordinary point, singular point, or irregular singular point of the given differential equation $\quad xy''+(1-\cos x)y'+x^2y=0$

Solution

$$y'' + \frac{1-\cos x}{x} y' + xy = 0$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\frac{1-\cos x}{x} = \frac{1}{x}\left(1 - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right)$$

$$= \boxed{\frac{x}{2} - \frac{x^3}{24} + \frac{x^5}{720} - \cdots}, \quad \text{is analytic at } x = 0$$

$\therefore$ $x = 0$ is an ordinary point of the differential equation.

Exercise

Determine whether $x = 0$ is an ordinary point, singular point, or irregular singular point of the given differential equation $\left(e^x - 1 - x\right) y'' + xy = 0$

Solution

$$x^2 y'' + x^2 \frac{x}{e^x - 1 - x} y = 0$$

$$x^2 y'' + \frac{x^3}{e^x - 1 - x} y = 0$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^x - 1 - x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots - 1 - x$$

$$= \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$\frac{x^3}{e^x - 1 - x} = \frac{1}{\frac{1}{x^3}\left(\frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)}$$

$$= \frac{1}{\frac{1}{2x} + \frac{1}{6} + \frac{x}{24} + \cdots}$$

$\therefore$ $x = 0$ is a regular singular point of the differential equation

Exercise

Find the Frobenius series solutions of $2x^2 y'' + 3xy' - \left(1 + x^2\right) y = 0$

Solution

$$y'' + \frac{3}{2}\frac{1}{x}y' - \frac{1}{2}\frac{1+x^2}{x^2}y = 0$$ ***Divide each term by*** $2x^2$

Therefore, $x = 0$ is a regular singular point, and that $p_0 = \frac{3}{2}, \quad q_0 = -\frac{1}{2}$

$p(x) \equiv \frac{3}{2}, \quad q(x) = -\frac{1}{2} - \frac{1}{2}x^2$ are polynomials.

The Frobenius series will converge for all $x > 0$. The indicial equation is

$$r(r-1) + \frac{3}{2}r - \frac{1}{2} = r^2 + \frac{1}{2}r - \frac{1}{2} = (r+1)\left(r - \frac{1}{2}\right) = 0$$

So the roots are $r_1 = \frac{1}{2}$ *and* $r_2 = -1$.

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2}\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{-1}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2 y'' + 3xy' - \left(1 + x^2\right) y = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 3x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} - x^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(2n+2r-2) a_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-2) + 3(n+r) - 1\right] a_n x^{n+r} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r+1) - 1\right] a_n x^{n+r} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$\left(2r^2 + r - 1\right) a_0 + \left(2r^2 + 5r + 2\right) a_1 x +$$

$$\sum_{n=2}^{\infty} \left[(n+r)(2n+2r+1) - 1\right] a_n x^{n+r} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$\left(2r^2+r-1\right)a_0+\left(2r^2+5r+2\right)a_1x+\sum_{n=2}^{\infty}\left(\left[(n+r)(2n+2r+1)-1\right]a_n-a_{n-2}\right)x^{n+r}=0$$

For $n=0 \rightarrow \left(2r^2+r-1\right)a_0=0$

$$r=-1 \quad or \quad r=\frac{1}{2} \quad \checkmark$$

For $n=1 \rightarrow \left(2r^2+5r+2\right)a_1=0$

$$\cancel{r=-2 \quad or \quad r=-\frac{1}{2}} \quad \text{Therefore } a_1=0$$

$$\left[(n+r)(2n+2r+1)-1\right]a_n-a_{n-2}=0$$

$$a_n=\frac{1}{(n+r)\left(2(n+r)+1\right)-1}a_{n-2}$$

$$=\frac{1}{2(n+r)^2+(n+r)-1}a_{n-2} \qquad for \ \ n\geq 2$$

$r=\frac{1}{2}$

$$a_n=\frac{1}{2\left(n+\frac{1}{2}\right)^2+\left(n+\frac{1}{2}\right)-1}a_{n-2}=\frac{1}{2n^2+3n}a_{n-2}$$

$$a_2=\frac{1}{14}a_0$$

$$a_3=\frac{1}{24}a_1=0$$

$$a_4=\frac{1}{44}a_2=\frac{1}{616}a_0$$

$$a_5=0$$

$$a_6=\frac{1}{90}a_4=\frac{1}{55440}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x)=a_0x^{1/2}\left(1+\frac{x^2}{14}+\frac{x^4}{616}+\frac{x^6}{55,440}+\cdots\right)$$

$r=-1$

$$b_n=\frac{1}{2(n-1)^2+(n-1)-1}b_{n-2}$$

$$b_2=\frac{1}{2}b_0$$

$$b_3=\frac{1}{9}b_1=0$$

$$b_4=\frac{1}{20}b_2=\frac{1}{40}b_0$$

$$b_5=0$$

$$b_6=\frac{1}{54}b_4=\frac{1}{2160}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x)=b_0x^{-1}\left(1+\frac{x^2}{2}+\frac{x^4}{40}+\frac{x^6}{2160}+\cdots\right)$$

$$y(x)=C_1\left(1+\sum_{n=0}^{\infty}\frac{4}{n!\ 1\cdot 3\cdot 5\cdots(2n-1)}x^n\right)+C_2x^{1/2}\left(1+\sum_{n=0}^{\infty}\frac{4^n}{n!\ 3\cdot 5\cdots(2n+1)}x^n\right)$$

Exercise

Find the Frobenius series solutions of $2x^2y'' - xy' + \left(1+x^2\right)y = 0$

Solution

$$y'' - \frac{1}{2}\frac{1}{x}y' + \frac{1}{2}\frac{1+x^2}{x^2}y = 0 \qquad \textit{Divide each term by } 2x^2$$

Therefore, $x=0$ is a *regular singular* point, and that $p_0 = -\frac{1}{2}, \quad q_0 = \frac{1}{2}$

$p(x) = -\frac{1}{2}, \quad q(x) = \frac{1}{2} + \frac{1}{2}x^2$ are polynomials.

The Frobenius series will converge for all $x > 0$. The indicial equation is

$$r(r-1) - \frac{1}{2}r + \frac{1}{2} = r^2 - \frac{3}{2}r + \frac{1}{2} = \frac{1}{2}\left(2r^2 - 3r + 1\right) = 0$$

So the roots are $\underline{r_1 = \frac{1}{2} \quad \textit{and} \quad r_2 = 1}$.

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2}\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^1\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$2x^2y'' - xy' + \left(1+x^2\right)y = 0$$

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r} + x^2\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2) - (n+r) + 1\right]a_n x^{n+r} + \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-3) + 1\right]a_n x^{n+r} + \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0$$

$$\left(2r^2-3r+1\right)a_0+\left((1+r)(2r-1)+1\right)a_1x+$$

$$\sum_{n=2}^{\infty}\left[(n+r)(2n+2r-3)+1\right]a_nx^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}=0$$

$$\left(2r^2-3r+1\right)a_0+\left(2r^2+r\right)a_1x+\sum_{n=2}^{\infty}\left[\left((n+r)(2n+2r-3)+1\right)a_n+a_{n-2}\right]x^{n+r}=0$$

For $n=0 \quad\rightarrow\quad \left(2r^2-3r+1\right)a_0=0 \quad\Rightarrow\quad \boxed{r=1 \;\; or \;\; r=\frac{1}{2}}$ $\sqrt{}$

For $n=1 \quad\rightarrow\quad \left(2r^2+r\right)a_1=0 \quad\Rightarrow r \;\boxed{\cancel{=0,-\frac{1}{2}}}$ Therefore $\boxed{a_1=0}$

$$\left((n+r)(2n+2r-3)+1\right)a_n+a_{n-2}=0$$

$$\boxed{a_n=-\frac{1}{(n+r)(2n+2r-3)+1}a_{n-2}} \qquad for \;\; n\geq 2$$

$r=\frac{1}{2}$

$$a_n=-\frac{1}{\left(n+\frac{1}{2}\right)(2n-2)+1}a_{n-2}=\boxed{-\frac{1}{2n^2-n}a_{n-2}}$$

$n=2 \;\rightarrow\; a_2=-\frac{1}{6}a_0$ $\qquad n=3 \;\rightarrow\; a_3=-\frac{1}{15}a_1=0$

$n=4 \;\rightarrow\; a_4=-\frac{1}{28}a_2=\frac{1}{168}a_0$ $\qquad n=5 \;\rightarrow\; a_5=0$

$n=6 \;\rightarrow\; a_6=-\frac{1}{66}a_4=-\frac{1}{11,088}a_0$

$$y_1(x)=a_0x^{1/2}\left(1-\frac{x^2}{6}+\frac{x^4}{168}-\frac{x^6}{11,088}+\cdots\right)$$

$$\boxed{=a_0\left(x^{1/2}-\frac{1}{6}x^{5/2}+\frac{1}{168}x^{9/2}-\frac{1}{11,088}x^{13/2}+\cdots\right)}$$

$r=1$

$$b_n=-\frac{1}{(n+1)(2n-1)+1}b_{n-2}=\boxed{-\frac{1}{2n^2+n}b_{n-2}}$$

$n=2 \;\rightarrow\; b_2=-\frac{1}{10}b_0$ $\qquad n=3 \;\rightarrow\; b_3=-\frac{1}{21}b_1=0$

$n=4 \;\rightarrow\; b_4=-\frac{1}{36}b_2=\frac{1}{360}b_0$ $\qquad n=5 \;\rightarrow\; b_5=0$

$n=6 \;\rightarrow\; b_6=-\frac{1}{78}b_4=-\frac{1}{28,080}b_0$ $\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_2(x) = b_0 x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} - \frac{x^6}{28,080} + \cdots\right)$$

$$= b_0\left(x - \frac{1}{10}x^3 + \frac{1}{360}x^5 - \frac{1}{28,080}x^7 + \cdots\right)$$

$$y(x) = a_0\left(x^{1/2} - \frac{1}{6}x^{5/2} + \frac{1}{168}x^{9/2} - \frac{1}{11,088}x^{13/2} + \cdots\right) + b_0\left(x - \frac{1}{10}x^3 + \frac{1}{360}x^5 - \frac{1}{28,080}x^7 + \cdots\right)$$

Exercise

Find the general solution to the equation $2xy'' + (1+x)y' + y = 0$

Solution

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' + (1+x)y' + y = 0$$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + (1+x)\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty}c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)c_n x^{n-1}x^r + \sum_{n=0}^{\infty}(n+r)c_n x^{n-1}x^r + \sum_{n=0}^{\infty}(n+r)c_n x^n x^r + \sum_{n=0}^{\infty}c_n x^n x^r = 0$$

$$x^r\left(\sum_{n=0}^{\infty}(n+r)(2n+2r-2+1)c_n x^{n-1} + \sum_{n=0}^{\infty}(n+r+1)c_n x^n\right) = 0$$

$$x^r\left(\sum_{n=0}^{\infty}(n+r)(2n+2r-1)c_n x^{n-1} + \sum_{n=0}^{\infty}(n+r+1)c_n x^n\right) = 0$$

$$x^r\left(c_0 r(2r-1)x^{-1}+\underbrace{\sum_{n=1}^{\infty}c_n(n+r)(2n+2r-1)x^{n-1}}_{k=n-1}+\underbrace{\sum_{n=0}^{\infty}(n+r+1)c_n x^n}_{k=n}\right)=0$$

$$x^r\left(c_0 r(2r-1)x^{-1}+\sum_{k=0}^{\infty}c_{k+1}(r+k+1)(2k+2r+1)x^k+\sum_{k=0}^{\infty}c_k(r+k+1)x^k\right)=0$$

$$x^r\left(c_0 r(2r-1)x^{-1}+\sum_{k=0}^{\infty}\left[c_{k+1}(r+k+1)(2k+2r+1)+c_k(r+k+1)\right]x^k\right)=0$$

$$\begin{cases} c_0 r(2r-1)=0 & \Rightarrow \quad \boxed{r=0} \quad \boxed{r=\frac{1}{2}} \\ c_{k+1}(r+k+1)(2k+2r+1)+c_k(r+k)=0 & \Rightarrow \quad \boxed{c_{k+1}=-\frac{r+k+1}{(r+k+1)(2k+2r+1)}c_k} \end{cases}$$

$r=0$

$$c_{k+1}=-\frac{1}{2k+1}c_k$$

$$c_1=-\frac{1}{1}c_0$$

$$c_2=-\frac{1}{3}c_1=\frac{1}{3}c_0$$

$$c_3=-\frac{1}{5}c_2=-\frac{1}{1\cdot3\cdot5}c_0$$

$$c_4=-\frac{1}{7}c_3=\frac{1}{1\cdot3\cdot5\cdot7}c_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$c_n=\frac{(-1)^n}{1\cdot3\cdot5\cdots(2n-1)}c_0$$

$r=\frac{1}{2}$

$$c_{k+1}=-\frac{k+\frac{3}{2}}{\left(k+\frac{3}{2}\right)(2k+2)}c_k=-\frac{1}{2(k+1)}c_k$$

$$c_1=-\frac{1}{2}c_0$$

$$c_2=-\frac{1}{2\cdot2}c_1=\frac{1}{2\cdot2\cdot2}c_0$$

$$c_3=-\frac{1}{2\cdot3}c_2=-\frac{1}{2^3(2\cdot3)}c_0=-\frac{1}{2^3\cdot3!}c_0$$

$$c_4=-\frac{1}{2\cdot4}c_3=\frac{1}{2^4\cdot4!}c_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$c_n=\frac{(-1)^n}{2^n n!}c_0$$

$$y_1(x)=c_0\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{1\cdot3\cdot5\cdots(2n-1)}x^n\right) \qquad y_2(x)=c_0x^{1/2}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n!}x^n\right)$$

$$y(x)=C_1y_1(x)+C_2y_2(x)$$

$$=C_1\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{1\cdot3\cdot5\cdots(2n-1)}x^n\right)+C_2\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}x^{n+1/2}\right)$$

Exercise

Find the Frobenius series solutions of $xy'' + 2y' + xy = 0$

Solution

$$\frac{1}{x}\left(xy'' + 2y' + xy = 0\right)$$

$$y'' + 2\frac{1}{x}y' + \frac{x^2}{x^2}y = 0$$

$\therefore\ x = 0$ is a regular singular point with $p_0 = 2$ and $q_0 = 0$

The indicial equation is: $r(r-1) + 2r = r(r+1) = 0 \quad \rightarrow \quad \underline{r = 0,\ -1}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2}\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^1\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$xy'' + 2y' + xy = 0$$

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + x\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}2(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1) + 2(n+r)\right]a_n x^{n+r-1} + \sum_{n=2}^{\infty}a_{n-2}x^{n+r-1} = 0$$

$$r(r+1)a_0 x^{r-1} + (r+1)(r+2)a_1 x^r + \sum_{n=2}^{\infty}(n+r)(n+r+1)a_n x^{n+r-1} + \sum_{n=2}^{\infty}a_{n-2}x^{n+r-1} = 0$$

For $n = 0 \quad \rightarrow \quad r(r+1)a_0 = 0 \quad \Rightarrow \quad \underline{r = 0 \ \text{or}\ r = -1}$ $\surd$

For $n = 1 \quad \rightarrow \quad (r+1)(r+2)a_1 = 0 \quad \Rightarrow \quad \cancel{r = -1,\ -2} \qquad \therefore\ \underline{a_1 = 0}$

$$(n+r)(n+r+1)a_n + a_{n-2} = 0$$

$$\boxed{a_n = -\frac{1}{(n+r)(n+r+1)} a_{n-2}}$$

$$r = 0 \quad \rightarrow \quad \boxed{a_n = -\frac{1}{n(n+1)} a_{n-2}}$$

$$n = 2 \rightarrow a_2 = -\frac{1}{2\cdot 3} a_0 \qquad\qquad n = 3 \rightarrow a_3 = -\frac{1}{12} a_1 = 0$$

$$n = 4 \rightarrow a_4 = -\frac{1}{4\cdot 5} a_2 = \frac{1}{5!} a_0 \qquad\qquad n = 5 \rightarrow a_5 = 0$$

$$n = 6 \rightarrow a_6 = -\frac{1}{6\cdot 7} a_4 = -\frac{1}{7!} a_0 \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x) = a_0 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \right)$$

$$\boxed{= \frac{a_0}{x} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right)}$$

$$r = -1 \quad \rightarrow \quad \boxed{b_n = -\frac{1}{n(n-1)} b_{n-2}}$$

$$n = 2 \rightarrow b_2 = -\frac{1}{2\cdot 1} b_0 \qquad\qquad n = 3 \rightarrow b_3 = -\frac{1}{6} b_1 = 0$$

$$n = 4 \rightarrow b_4 = -\frac{1}{4\cdot 3} b_2 = \frac{1}{4!} b_0 \qquad\qquad n = 5 \rightarrow b_5 = 0$$

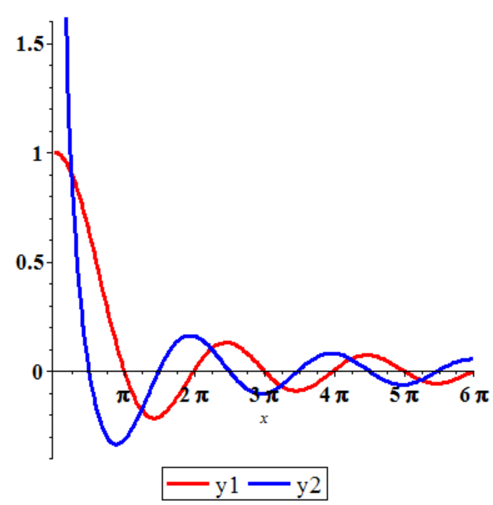
$$n = 6 \rightarrow b_6 = -\frac{1}{6\cdot 5} b_4 = -\frac{1}{6!} b_0 \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{-1} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right)}$$

$$y(x) = \frac{a_0}{x} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right) + \frac{b_0}{x} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right)$$

$$\boxed{= a_0 \frac{\sin x}{x} + b_0 \frac{\cos x}{x}}$$



Exercise

Find the Frobenius series solutions of $2xy'' - y' + 2y = 0$

Solution

$$\left(\frac{x}{2}\right)2xy'' - \left(\frac{x}{2}\right)y' + \left(\frac{x}{2}\right)2y = 0$$

$$x^2y'' - \frac{1}{2}xy' + xy = 0$$

That implies to $p(x) = -\frac{1}{2}$ and $q(x) = x$, both are analytic.

Hence, $x_0 = 0$ is a regular point

The indicial equation is: $r(r-1) - \frac{1}{2}r = r\left(r - \frac{3}{2}\right) = 0 \quad \rightarrow \quad \underline{r = 0, \ \frac{3}{2}}|$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{3/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$2xy'' - y' + 2y = 0$$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + 2\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[2(n+r)(n+r-1) - (n+r)\right]a_n x^{n+r-1} + \sum_{n=1}^{\infty}2a_{n-1}x^{n+r-1} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-3)a_n + 2a_{n-1}\right]x^{n+r-1} = 0$$

$$(n+r)(2n+2r-3)a_n + 2a_{n-1} = 0$$

$$\boxed{a_n = -\frac{2}{(n+r)(2n+2r-3)}a_{n-1}}$$

$r=0 \quad \rightarrow \quad \boxed{a_n = -\frac{2}{n(2n-3)}a_{n-1}}$

$n=1 \ \rightarrow \ a_1 = 2a_0$

$n=2 \ \rightarrow \ a_2 = -a_1 = -2a_0$

$n=3 \ \rightarrow \ a_3 = -\frac{2}{9}a_2 = \frac{4}{9}a_0$

$n=4 \ \rightarrow \ a_4 = -\frac{1}{10}a_3 = -\frac{2}{45}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1+2x-2x^2+\frac{4}{9}x^3-\frac{2}{45}x^4+\cdots\right)}$$

$r=\frac{3}{2} \quad \rightarrow \quad \boxed{b_n = -\frac{1}{n\left(n+\frac{3}{2}\right)}b_{n-1}}$

$n=1 \ \rightarrow \ b_1 = -\frac{2}{5}b_0$

$n=2 \ \rightarrow \ b_2 = -\frac{1}{7}b_1 = \frac{2}{35}b_0$

$n=3 \ \rightarrow \ b_3 = -\frac{2}{27}b_2 = -\frac{4}{945}b_0$

$n=4 \ \rightarrow \ b_4 = -\frac{1}{22}b_3 = \frac{2}{20,790}b_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_2(x) = b_0x^{3/2}\left(1-\frac{2}{5}x+\frac{2}{35}x^2-\frac{4}{945}x^3+\frac{2}{20,790}x^4-\cdots\right)$$

$$= b_0\sqrt{x}\left(x-\frac{2}{5}x^2+\frac{2}{35}x^3-\frac{4}{945}x^4+\frac{2}{20,790}x^5-\cdots\right)$$

$$\boxed{y(x) = a_0\left(1+2x-2x^2+\frac{4}{9}x^3-\frac{2}{45}x^4+\cdots\right)+b_0\sqrt{x}\left(x-\frac{2}{5}x^2+\frac{2}{35}x^3-\frac{4}{945}x^4+\frac{2}{20,790}x^5-\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2xy''+5y'+xy=0$

Solution

$\left(\frac{x}{2}\right)2xy''+5\left(\frac{x}{2}\right)y'+\left(\frac{x}{2}\right)xy=0$

$x^2y''+\frac{5}{2}xy'+\frac{1}{2}x^2y=0$

That implies to $p(x)=\frac{5}{2}$ and $q(x)=\frac{1}{2}x^2$, both are analytic.

Hence, $x_0 = 0$ is a regular point

The indicial equation is: $r(r-1)+\frac{5}{2}r = r^2+\frac{3}{2}r = 0 \quad \rightarrow \quad \underline{r=0,\ -\frac{3}{2}}|$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{-3/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$2xy''+5y'+xy=0$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty}5(n+r)a_n x^{n+r-1} + x\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}5(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty}\left[2(n+r)(n+r-1)+5(n+r)\right]a_n x^{n+r-1} + \sum_{n=2}^{\infty}a_{n-2}x^{n+r-1} = 0$$

$$r(2r+3)a_0 + (r+1)(2r+5)a_1 + \sum_{n=2}^{\infty}(n+r)(2n+2r+3)a_n x^{n+r-1} + \sum_{n=2}^{\infty}a_{n-2}x^{n+r-1} = 0$$

$$r(2r+3)a_0 + (r+1)(2r+5)a_1 + \sum_{n=2}^{\infty}\left[(n+r)(2n+2r+3)a_n + a_{n-2}\right]x^{n+r-1} = 0$$

For $n=0 \quad \rightarrow \quad r(2r+3)a_0 = 0 \quad \Rightarrow \quad \underline{r=0 \ \text{ or } \ r=-\frac{3}{2}}|$ √

For $n=1 \quad \rightarrow \quad (r+1)(2r+5)a_1 = 0 \quad \Rightarrow \quad \underline{\cancel{r=-1,\ -\frac{5}{2}}}| \quad \rightarrow \quad \underline{a_1 = 0}|$

$$(n+r)(2n+2r+3)a_n + a_{n-2} = 0$$

$$\underline{a_n = -\frac{1}{(n+r)(2n+2r+3)}a_{n-2}}|$$

$$r=0 \quad \rightarrow \quad \boxed{a_n = -\frac{1}{n(2n+3)}a_{n-2}}$$

$$n=2 \;\rightarrow\; a_2 = -\frac{1}{14}a_0 \qquad\qquad n=3 \;\rightarrow\; a_3 = -\frac{1}{27}a_1 = 0$$

$$n=4 \;\rightarrow\; a_4 = -\frac{1}{88}a_2 = \frac{1}{616}a_0 \qquad\qquad n=5 \;\rightarrow\; a_5 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0\left(1-\frac{1}{14}x^2+\frac{1}{616}x^4-\cdots\right)}$$

$$r=-\frac{3}{2} \quad \rightarrow \quad b_n = -\frac{1}{2n\left(n-\frac{3}{2}\right)}b_{n-2} \;\boxed{= -\frac{1}{n(2n-3)}b_{n-2}}$$

$$n=2 \;\rightarrow\; b_2 = -\frac{1}{2}b_0 \qquad\qquad n=3 \;\rightarrow\; b_3 = -\frac{1}{9}b_1 = 0$$

$$n=4 \;\rightarrow\; b_4 = -\frac{1}{20}b_2 = \frac{1}{40}b_0 \qquad\qquad n=5 \;\rightarrow\; b_5 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{-3/2}\left(1-\frac{1}{2}x^2+\frac{1}{40}x^3-\cdots\right)$$

$$\boxed{= b_0\left(x^{-3/2}-\frac{1}{2}x^{1/2}+\frac{1}{40}x^{3/2}-\cdots\right)}$$

$$\boxed{y(x) = a_0\left(1-\frac{1}{14}x^2+\frac{1}{616}x^4-\cdots\right)+b_0\left(x^{-3/2}-\frac{1}{2}x^{1/2}+\frac{1}{40}x^{3/2}-\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $4xy''+\frac{1}{2}y'+y=0$

Solution

$$\left(\frac{x}{4}\right)4xy''+\frac{1}{2}\left(\frac{x}{4}\right)y'+\left(\frac{x}{4}\right)y=0$$

$$x^2y''+\frac{1}{8}xy'+\frac{1}{4}x^2y=0$$

$$y''+\frac{1}{8x}y'+\frac{1}{4}y=0$$

That implies to $p(x)=\frac{1}{8x}$ and $q(x)=\frac{1}{4}$

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{1}{8x} \;\boxed{=\frac{1}{8}}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\frac{1}{8} \;\boxed{=0}$$

The indicial equation is: $r(r-1)+\frac{1}{8}r = r^2-\frac{7}{8}r=0 \quad \rightarrow \quad \boxed{r=0,\ \frac{7}{8}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{7/8} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$8xy'' + y' + 2y = 0$$

$$8x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + 2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 8(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[8(n+r)(n+r-1) + (n+r)\right] a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1} = 0$$

$$r(8r-7)a_0 + \sum_{n=0}^{\infty} (n+r)(8n+8r-7) a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1} = 0$$

$$r(8r-7)a_0 + \sum_{n=0}^{\infty} \left[(n+r)(8n+8r-7) a_n + 2a_{n-1}\right] x^{n+r-1} = 0$$

For $n = 0 \rightarrow r(8r-7)a_0 = 0 \Rightarrow \underline{r = 0, \ \tfrac{7}{8}}$ √

$$(n+r)(8n+8r-7)a_n + 2a_{n-1} = 0$$

$$\boxed{a_n = -\frac{2}{(n+r)(8n+8r-7)} a_{n-1}}$$

$$r = 0 \rightarrow \boxed{a_n = -\frac{2}{n(8n-7)} a_{n-1}}$$

$n = 1 \rightarrow a_1 = -2a_0$

$n = 2 \rightarrow a_2 = -\frac{1}{9} a_1 = \frac{2}{9} a_0$

$n = 3 \rightarrow a_3 = -\frac{2}{51} a_2 = -\frac{4}{459} a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \cdots\right)}$$

$$r = \frac{7}{8} \quad \rightarrow \quad b_n = -\frac{2}{\left(n + \frac{7}{8}\right)(8n)} b_{n-1} = \boxed{-\frac{2}{n(8n+7)} b_{n-1}}$$

$$n = 1 \ \rightarrow \ b_1 = -\frac{2}{15} b_0$$

$$n = 2 \ \rightarrow \ b_2 = -\frac{1}{23} b_1 = \frac{2}{345} b_0$$

$$n = 3 \ \rightarrow \ b_3 = -\frac{2}{93} b_2 = -\frac{4}{32,085} b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{7/8}\left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32,085}x^3 + \cdots\right)}$$

$$\boxed{y(x) = a_0\left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \cdots\right) + b_0 x^{7/8}\left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32,085}x^3 + \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2x^2y'' - xy' + \left(x^2 + 1\right)y = 0$

Solution

$$\frac{1}{2}2x^2y'' - \frac{1}{2}xy' + \frac{1}{2}\left(x^2 + 1\right)y = 0$$

$$x^2y'' - \frac{1}{2}xy' + \frac{1}{2}\left(x^2 + 1\right)y = 0$$

$$y'' - \frac{1}{2x}y' + \left(\frac{1}{2} + \frac{1}{2x^2}\right)y = 0$$

That implies to $p(x) = -\frac{1}{2x}$ and $q(x) = \frac{1}{2} + \frac{1}{2x^2}$.

$$p_0 = \lim_{x \to 0} xp(x) = -\lim_{x \to 0} x\frac{1}{2x} = \boxed{-\frac{1}{2}}$$

$$q_0 = \lim_{x \to 0} x^2q(x) = \lim_{x \to 0} x^2\left(\frac{1}{2} + \frac{1}{2x^2}\right) = \lim_{x \to 0}\left(\frac{1}{2}x^2 + \frac{1}{2}\right) = \boxed{\frac{1}{2}}$$

The indicial equation is: $r(r-1) - \frac{1}{2}r + \frac{1}{2} = r^2 - \frac{3}{2}r + \frac{1}{2} = 0 \ \rightarrow \ \boxed{r = 1, \ \frac{1}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^1\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$2x^2y''-xy'+\left(x^2+1\right)y=0$$

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}-x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\left(x^2+1\right)\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r}-\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}+\sum_{n=0}^{\infty}a_n x^{n+r+2}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left(2(n+r)(n+r-1)-(n+r)+1\right)a_n x^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}$$

$$\left(r(2r-3)+1\right)a_0+\left((r+1)(2r-1)+1\right)a_1+\sum_{n=2}^{\infty}\left((n+r)(2n+2r-3)+1\right)a_n x^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}=0$$

$$\left(2r^2-3r+1\right)a_0+\left(2r^2+r\right)a_1+\sum_{n=2}^{\infty}\left[\left((n+r)(2n+2r-3)+1\right)a_n+a_{n-2}\right]x^{n+r}=0$$

For $n=0 \quad\rightarrow\quad \left(2r^2-3r+1\right)a_0=0 \quad\Rightarrow\quad \boxed{r=1,\ \frac{1}{2}}$ √

For $n=1 \quad\rightarrow\quad \left(2r^2+r\right)a_1=0 \quad\Rightarrow\quad \boxed{\cancel{r=0,\ -\frac{1}{2}}} \quad\rightarrow\quad \boxed{a_1=0}$

$$\left((n+r)(2n+2r-3)+1\right)a_n+a_{n-2}=0$$

$$\boxed{a_n=-\frac{1}{(n+r)(2n+2r-3)+1}a_{n-2}}$$

$r=1 \quad\rightarrow\quad a_n=-\frac{1}{(n+1)(2n-1)+1}a_{n-2}=\boxed{-\frac{1}{2n^2+n}a_{n-2}}$

$n=2 \rightarrow a_2=-\frac{1}{10}a_0$ $\qquad$ $n=3 \rightarrow a_3=-\frac{1}{21}a_1=0$

$n=4 \rightarrow a_4=-\frac{1}{36}a_2=\frac{1}{360}a_0$ $\qquad$ $n=5 \rightarrow a_5=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$ $\qquad$ $\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_1(x) = a_0 x\left(1 - \frac{1}{10}x^2 + \frac{1}{360}x^4 - \cdots\right)$$

$$= a_0\left(x - \frac{1}{10}x^3 + \frac{1}{360}x^5 - \cdots\right)$$

$$r = \frac{1}{2} \quad \rightarrow \quad b_n = -\frac{1}{\left(n+\frac{1}{2}\right)(2n-2)+1}b_{n-2} = -\frac{1}{2n^2 - n}b_{n-2}$$

$n = 2 \ \rightarrow \ b_2 = -\frac{1}{6}b_0$ $\qquad n = 3 \ \rightarrow \ a_3 = -\frac{1}{15}a_1 = 0$

$n = 4 \ \rightarrow \ b_4 = -\frac{1}{28}b_2 = \frac{1}{168}b_0$ $\qquad n = 5 \ \rightarrow \ a_5 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_2(x) = b_0 x^{1/2}\left(1 - \frac{1}{6}x^2 + \frac{1}{168}x^4 - \cdots\right)$$

$$y_1(x) = a_0\left(x - \frac{1}{10}x^3 + \frac{1}{360}x^5 - \cdots\right) + b_0 x^{1/2}\left(1 - \frac{1}{6}x^2 + \frac{1}{168}x^4 - \cdots\right)$$

Exercise

Find the Frobenius series solutions of $3xy'' + (2-x)y' - y = 0$

Solution

$$\frac{x}{3}3xy'' + \frac{x}{3}(2-x)y' - \frac{x}{3}y = 0$$

$$x^2y'' + \left(\frac{2}{3}x - \frac{1}{3}x^2\right)y' - \frac{x}{3}y = 0$$

$$y'' + \left(\frac{2}{3x} - \frac{1}{3}\right)y' - \frac{1}{3x}y = 0$$

That implies to $p(x) = \frac{2}{3x} - \frac{1}{3}$ and $q(x) = -\frac{1}{3x}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(\frac{2}{3x} - \frac{1}{3}\right) = \lim_{x\to 0}\left(\frac{2}{3} - \frac{1}{3}x\right) = \frac{2}{3}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = -\lim_{x\to 0} x^2\frac{1}{3x} = \lim_{x\to 0}\frac{x}{3} = 0$$

The indicial equation is: $r(r-1) + \frac{2}{3}r = r^2 - \frac{1}{3}r = 0 \quad \rightarrow \quad r = 0, \ \frac{1}{3}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/3}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$3xy'' + (2-x)y' - y = 0$$

$$3x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + (2-x)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}3(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}2(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(3n+3r-3)+2(n+r)\right]a_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r+1)a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(3n+3r-1)a_n x^{n+r-1} - \sum_{n=1}^{\infty}(n+r)a_{n-1} x^{n+r-1} = 0$$

$$r(3r-1)a_0 + \sum_{n=1}^{\infty}(n+r)(3n+3r-1)a_n x^{n+r-1} - \sum_{n=1}^{\infty}(n+r)a_{n-1} x^{n+r-1} = 0$$

$$r(3r-1)a_0 + \sum_{n=1}^{\infty}\left[(n+r)(3n+3r-1)a_n - (n+r)a_{n-1}\right]x^{n+r-1} = 0$$

For $n=0 \quad \rightarrow \quad r(3r-1)a_0 = 0 \quad \Rightarrow \quad \boxed{r = 0, \ \frac{1}{3}}$ $\checkmark$

$$(n+r)(3n+3r-1)a_n - (n+r)a_{n-1} = 0$$

$$\boxed{a_n = \frac{1}{3n+3r-1}a_{n-1}}$$

$$r = 0 \quad \rightarrow \quad \boxed{a_n = \frac{1}{3n-1}a_{n-1}}$$

$$n=1 \ \rightarrow \ a_1 = \frac{1}{2}a_0$$

$$n=2 \ \rightarrow \ a_2 = \frac{1}{5}a_1 = \frac{1}{10}a_0$$

$$n=3 \ \rightarrow \ a_3 = \frac{1}{8}a_2 = \frac{1}{80}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0\left(1+\frac{1}{2}x+\frac{1}{10}x^2+\frac{1}{80}x^3+\cdots\right)}$$

$$r=\frac{1}{3} \quad \rightarrow \quad \boxed{b_n = \frac{1}{3n}b_{n-1}}$$

$$n=1 \ \rightarrow \ b_1 = \frac{1}{3}b_0$$

$$n=2 \ \rightarrow \ b_2 = \frac{1}{6}b_1 = \frac{1}{18}b_0$$

$$n=3 \ \rightarrow \ b_3 = \frac{1}{9}b_2 = \frac{1}{162}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/3}\left(1+\frac{1}{3}x+\frac{1}{18}x^2+\frac{1}{162}x^3+\cdots\right)}$$

$$\boxed{y(x) = a_0\left(1+\frac{1}{2}x+\frac{1}{10}x^2+\frac{1}{80}x^3+\cdots\right)+b_0 x^{1/3}\left(1+\frac{1}{3}x+\frac{1}{18}x^2+\frac{1}{162}x^3+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2xy'' - (3+2x)y' + y = 0$

Solution

$$\frac{x}{2}2xy'' - \frac{x}{2}(3+2x)y' + \frac{x}{2}y = 0$$

$$x^2y'' - \left(\frac{3}{2}x + x^2\right)y' + \frac{1}{2}xy = 0$$

$$y'' - \left(\frac{3}{2x}+1\right)y' + \frac{1}{2x}y = 0$$

That implies to $p(x) = -\frac{3}{2x} - 1$ and $q(x) = \frac{1}{2x}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(-\frac{3}{2x}-1\right) = \lim_{x\to 0}\left(-\frac{3}{2}-x\right) \boxed{= -\frac{3}{2}}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\frac{1}{2x} = \lim_{x\to 0}\frac{x}{2} \boxed{= 0}$$

The indicial equation is: $r(r-1) - \frac{3}{2}r = r^2 - \frac{5}{2}r = 0 \quad \rightarrow \quad \boxed{r = 0, \ \frac{5}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{5/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' - (3+2x)y' + y = 0$$

$$2x\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - (3+2x)\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} - \sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+r)(n+r-1) - 3(n+r)\right] a_n x^{n+r-1} - \sum_{n=0}^{\infty} (2n+2r-1) a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(2n+2r-5) a_n x^{n+r-1} - \sum_{n=1}^{\infty} (2n+2r-3) a_{n-1} x^{n+r-1} = 0$$

$$r(2r-5)a_0 + \sum_{n=1}^{\infty} (n+r)(2n+2r-5) a_n x^{n+r-1} - \sum_{n=1}^{\infty} (2n+2r-3) a_{n-1} x^{n+r-1} = 0$$

$$r(2r-5)a_0 + \sum_{n=1}^{\infty} \left[(n+r)(2n+2r-5) a_n - (2n+2r-3) a_{n-1}\right] x^{n+r-1} = 0$$

For $n = 0 \quad \rightarrow \quad r(2r-5)a_0 = 0 \quad \Rightarrow \quad \underline{r = 0, \ \frac{5}{2}}$ √

$$(n+r)(2n+2r-5)a_n - (2n+2r-3)a_{n-1} = 0$$

$$\boxed{a_n = \frac{2n+2r-3}{(n+r)(2n+2r-5)} a_{n-1}}$$

$$r = 0 \quad \rightarrow \quad \boxed{a_n = \frac{2n-3}{n(2n-5)} a_{n-1}}$$

$$n = 1 \ \rightarrow \ a_1 = \frac{1}{3} a_0$$

$$n = 2 \ \rightarrow \ a_2 = -\frac{1}{2} a_1 = -\frac{1}{6} a_0$$

$$n = 3 \ \rightarrow \ a_3 = -a_2 = -\frac{1}{6} a_0$$

$$n = 4 \ \rightarrow \ a_4 = \frac{5}{12} a_3 = -\frac{5}{72} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0\left(1+\frac{1}{3}x-\frac{1}{6}x^2-\frac{1}{6}x^3-\frac{5}{72}x^4-\cdots\right)}$$

$$r=\frac{5}{2} \quad \to \quad b_n = \frac{2n+2}{2n\left(n+\frac{5}{2}\right)}b_{n-1} \quad \boxed{=\frac{2n+2}{n(2n+5)}b_{n-1}}$$

$$n=1 \ \to \ b_1 = \frac{4}{7}b_0$$

$$n=2 \ \to \ b_2 = \frac{1}{3}b_1 = \frac{4}{21}b_0$$

$$n=3 \ \to \ b_3 = \frac{8}{33}b_2 = \frac{32}{693}b_0$$

$$n=4 \ \to \ b_4 = \frac{5}{26}b_3 = \frac{80}{9,009}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = b_0x^{5/2}\left(1+\frac{4}{7}x+\frac{4}{21}x^2+\frac{32}{693}x^3+\frac{80}{9,009}x^4+\cdots\right)}$$

$$\boxed{y(x) = a_0\left(1+\frac{1}{3}x-\frac{1}{6}x^2-\frac{1}{6}x^3-\frac{5}{72}x^4-\cdots\right)+b_0x^{5/2}\left(1+\frac{4}{7}x+\frac{4}{21}x^2+\frac{32}{693}x^3+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $xy''+(x-6)y'-3y=0$

Solution

$$xxy''+x(x-6)y'-3xy=0$$

$$x^2y''+\left(x^2-6x\right)y'-3xy=0$$

$$y''+\left(1-\frac{6}{x}\right)y'-\frac{3}{x}y=0$$

That implies to $p(x)=1-\frac{6}{x}$ and $q(x)=-\frac{3}{x}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(1-\frac{6}{x}\right) = \lim_{x\to 0}(x-6) \ \boxed{=-6}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = -\lim_{x\to 0} x^2\frac{3}{x} = -\lim_{x\to 0} 3x \ \boxed{=0}$$

The indicial equation is: $r(r-1)-6r=r^2-7r=0 \quad \to \quad \boxed{r=0,\ 7}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=x^0\sum_{n=0}^{\infty}a_nx^n \quad \textit{and} \quad y_2(x)=x^7\sum_{n=0}^{\infty}b_nx^n$$

$$y=\sum_{n=0}^{\infty}a_nx^{n+r}$$

$$y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$xy'' + (x-6)y' - 3y = 0$$

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + (x-6)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - 3\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}6(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty}3a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)-6(n+r)\right]a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r-3)a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-7)a_n x^{n+r-1} + \sum_{n=1}^{\infty}(n+r-4)a_{n-1} x^{n+r-1} = 0$$

$$r(r-7)a_0 + \sum_{n=1}^{\infty}(n+r)(n+r-7)a_n x^{n+r-1} + \sum_{n=1}^{\infty}(n+r-4)a_{n-1} x^{n+r-1} = 0$$

$$r(r-7)a_0 + \sum_{n=1}^{\infty}\left[(n+r)(n+r-7)a_n + (n+r-4)a_{n-1}\right]x^{n+r-1} = 0$$

For $n = 0 \quad \rightarrow \quad r(r-7)a_0 = 0 \quad \Rightarrow \quad \boxed{r = 0,\ 7}$ √

$$(n+r)(n+r-7)a_n + (n+r-4)a_{n-1} = 0$$

$$\boxed{a_n = -\frac{n+r-4}{(n+r)(n+r-7)}a_{n-1}}$$

$$\boxed{r = 0 \quad \rightarrow \quad a_n = -\frac{n-4}{n(n-7)}a_{n-1}}$$

$n = 1 \rightarrow a_1 = -\frac{1}{2}a_0$

$n = 2 \rightarrow a_2 = -\frac{1}{5}a_1 = \frac{1}{10}a_0$

$n = 3 \rightarrow a_3 = -\frac{1}{12}a_2 = -\frac{1}{120}a_0$

$n = 4 \rightarrow a_4 = 0a_3 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3\right)}$$

$$\boxed{r = 7 \quad \to \quad b_n = -\frac{n+3}{n(n+7)}b_{n-1}}$$

$$n = 1 \ \to \ b_1 = -\frac{1}{2}b_0$$

$$n = 2 \ \to \ b_2 = -\frac{5}{18}b_1 = \frac{5}{36}b_0$$

$$n = 3 \ \to \ b_3 = -\frac{1}{5}b_2 = -\frac{1}{36}b_0$$

$$n = 4 \ \to \ b_4 = -\frac{7}{44}b_3 = \frac{7}{1{,}584}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0x^7\left(1 - \frac{1}{2}x + \frac{5}{36}x^2 - \frac{1}{36}x^3 + \frac{7}{1{,}584}x^4 - \cdots\right)}$$

$$\boxed{y(x) = a_0\left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3\right) + b_0x^7\left(1 - \frac{1}{2}x + \frac{5}{36}x^2 - \frac{1}{36}x^3 + \frac{7}{1{,}584}x^4 - \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $x(x-1)y'' + 3y' - 2y = 0$

Solution

$$\frac{1}{x}x(x-1)y'' + 3\frac{1}{x}y' - 2\frac{1}{x}y = 0$$

$$(x-1)y'' + \frac{3}{x}y' - \frac{2}{x}y = 0$$

That implies to $p(x) = \frac{3}{x}$ and $q(x) = -\frac{2}{x}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{3}{x} \underline{= 3}|$$

$$q_0 = \lim_{x\to 0} x^2q(x) = -\lim_{x\to 0} x^2\frac{2}{x} = -\lim_{x\to 0} 2x \underline{= 0}|$$

The indicial equation is: $-r(r-1) + 3r = -r^2 + 4r = 0 \quad \to \quad \underline{r = 0,\ 4}|$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_nx^n \quad \text{and} \quad y_2(x) = x^4\sum_{n=0}^{\infty} b_nx^n$$

$$y = \sum_{n=0}^{\infty} a_nx^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x(x-1)y'' + 3y' - 2y = 0$$

$$\left(x^2 - x\right) \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 3\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1}$$

$$+ \sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) - 2\right] a_n x^{n+r} - \sum_{n=0}^{\infty} \left[(n+r)(n+r-1) - 3(n+r)\right] a_n x^{n+r-1} = 0$$

$$\sum_{n=1}^{\infty} \left[(n-1+r)(n+r-2) - 2\right] a_{n-1} x^{n+r-1} - \sum_{n=0}^{\infty} (n+r)(n+r-4) a_n x^{n+r-1} = 0$$

$$\sum_{n=1}^{\infty} \left[(n-1+r)(n+r-2) - 2\right] a_{n-1} x^{n+r-1} - r(r-4)a_0 - \sum_{n=1}^{\infty} (n+r)(n+r-4) a_n x^{n+r-1} = 0$$

$$-r(r-4)a_0 + \sum_{n=1}^{\infty} \left[\left((n+r-1)(n+r-2) - 2\right) a_{n-1} - (n+r)(n+r-4) a_n\right] x^{n+r-1} = 0$$

For $n = 0 \quad \to \quad -r(r-4)a_0 = 0 \quad \Rightarrow \quad \underline{r = 0, \ \ 4}$ √

$$\left((n+r-1)(n+r-2) - 2\right) a_{n-1} - (n+r)(n+r-4) a_n = 0$$

$$\boxed{a_n = \frac{(n+r-1)(n+r-2) - 2}{(n+r)(n+r-4)} a_{n-1}}$$

$$r = 0 \quad \to \quad \boxed{a_n = \frac{(n-1)(n-2) - 2}{n(n-4)} a_{n-1}}$$

$n = 1 \ \to \ a_1 = \frac{2}{3} a_0$

$n = 2 \ \to \ a_2 = \frac{1}{2} a_1 = \frac{1}{3} a_0$

$$n=3 \;\rightarrow\; a_3 = \frac{0}{3}a_2 = 0$$

$$n=4 \;\rightarrow\; a_4 = 0a_3 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0\left(1+\frac{2}{3}x+\frac{1}{3}x^2\right)}$$

$$\boxed{r=4 \quad \rightarrow \quad b_n = \frac{(n+3)(n+2)-2}{n(n+4)}b_{n-1}}$$

$$n=1 \;\rightarrow\; b_1 = 2b_0$$

$$n=2 \;\rightarrow\; b_2 = \frac{3}{2}b_1 = 3b_0$$

$$n=3 \;\rightarrow\; b_3 = \frac{28}{21}b_2 = 4b_0$$

$$n=4 \;\rightarrow\; b_4 = \frac{5}{4}b_3 = 5b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0x^4\left(1+2x+3x^2+4x^3+5x^4+\cdots\right)}$$

$$\boxed{y(x) = a_0\left(1+\frac{2}{3}x+\frac{1}{3}x^2\right) + b_0\left(x^4+2x^5+3x^6+4x^7+5x^8+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $x^2y'' - \left(x-\frac{2}{9}\right)y = 0$

Solution

$$x^2y'' - \left(x-\frac{2}{9}\right)y = 0$$

$$y'' - \left(\frac{1}{x} - \frac{2}{9x^2}\right)y = 0$$

That implies to $p(x)=0$ and $q(x) = \frac{2}{9x^2} - \frac{1}{x}$.

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\left(\frac{2}{9x^2}-\frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{2}{9}-x\right) = \boxed{\frac{2}{9}}$$

The indicial equation is: $r(r-1)+\frac{2}{9} = r^2 - r + \frac{2}{9} = 0$

$$9r^2 - 9r + 2 = 0 \quad \rightarrow \quad r = \frac{9\pm 3}{18} = \boxed{\frac{1}{3},\ \frac{2}{3}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/3}\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{2/3}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x^2 y'' - \left(x - \tfrac{2}{9}\right) y = 0$$

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - \left(x - \tfrac{2}{9}\right) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} + \sum_{n=0}^{\infty} \tfrac{2}{9} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + \tfrac{2}{9}\right] a_n x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0$$

$$\left(r^2 - r + \tfrac{2}{9}\right) a_0 + \sum_{n=1}^{\infty} \left[(n+r)(n+r-1) + \tfrac{2}{9}\right] a_n x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0$$

$$\left(r^2 - r + \tfrac{2}{9}\right) a_0 + \sum_{n=1}^{\infty} \left[\left((n+r)(n+r-1) + \tfrac{2}{9}\right) a_n - a_{n-1}\right] x^{n+r} = 0$$

For $n = 0 \quad \rightarrow \quad \left(r^2 - r + \tfrac{2}{9}\right) a_0 = 0 \quad \Rightarrow \quad \boxed{r = \tfrac{1}{3},\ \tfrac{2}{3}}$ √

$$\left((n+r)(n+r-1) + \tfrac{2}{9}\right) a_n - a_{n-1} = 0$$

$$\boxed{a_n = \frac{1}{(n+r)(n+r-1) + \frac{2}{9}} a_{n-1}}$$

$$r = \tfrac{1}{3} \quad \rightarrow \quad a_n = \frac{1}{\left(n + \frac{1}{3}\right)\left(n - \frac{2}{3}\right) + \frac{2}{9}} a_{n-1}$$

$$= \frac{1}{n^2 - \frac{1}{3} n} a_{n-1}$$

$$= \boxed{\frac{3}{3n^2 - n} a_{n-1}}$$

$$n = 1 \;\rightarrow\; a_1 = \frac{3}{2} a_0$$

$$n = 2 \;\rightarrow\; a_2 = \frac{3}{10} a_1 = \frac{9}{20} a_0$$

$$n = 3 \;\rightarrow\; a_3 = \frac{1}{8} a_2 = \frac{9}{160} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x^{1/3}\left(1 + \frac{3}{2}x + \frac{9}{20}x^2 + \frac{9}{160}x^3 + \cdots\right)}$$

$$r = \frac{2}{3} \quad \rightarrow \quad b_n = \frac{1}{\left(n + \frac{2}{3}\right)\left(n - \frac{1}{3}\right) + \frac{2}{9}} b_{n-1}$$

$$= \boxed{\frac{3}{3n^2 + n} b_{n-1}}$$

$$n = 1 \;\rightarrow\; b_1 = \frac{3}{4} b_0$$

$$n = 2 \;\rightarrow\; b_2 = \frac{3}{14} b_1 = \frac{9}{56} b_0$$

$$n = 3 \;\rightarrow\; b_3 = \frac{1}{10} b_2 = \frac{9}{560} b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{2/3}\left(1 + \frac{3}{4}x + \frac{9}{56}x^2 + \frac{9}{560}x^3 + \cdots\right)}$$

$$\boxed{y(x) = a_0 x^{1/3}\left(1 + \frac{3}{2}x + \frac{9}{20}x^2 + \frac{9}{160}x^3 + \cdots\right) + b_0 x^{2/3}\left(1 + \frac{3}{4}x + \frac{9}{56}x^2 + \frac{9}{560}x^3 + \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $x^2 y'' + x(3 + x)y' - 3y = 0$

Solution

$$\frac{1}{x^2} x^2 y'' + \frac{1}{x^2} x(3 + x)y' - 3\frac{1}{x^2} y = 0$$

$$y'' + \left(\frac{3}{x} + 1\right)y' - \frac{3}{x^2} y = 0$$

That implies to $p(x) = \frac{3}{x} + 1$ and $q(x) = -\frac{3}{x^2}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(\frac{3}{x} + 1\right) = \lim_{x\to 0} (3 + x) \; \underline{= 3}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = -\lim_{x\to 0} x^2 \frac{3}{x^2} \; \underline{= -3}$$

The indicial equation is: $r(r-1)+3r-3=r^2+2r-3=0 \quad \to \quad \underline{r=1,\ -3|}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=x\sum_{n=0}^{\infty}a_n x^n \quad \textit{and} \quad y_2(x)=x^{-3}\sum_{n=0}^{\infty}b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$x^2y''+x(3+x)y'-3y=0$$

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\left(3x+x^2\right)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-3\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}3(n+r)a_n x^{n+r}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1}-\sum_{n=0}^{\infty}3a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)+3(n+r)-3\right]a_n x^{n+r}+\sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r}=0$$

$$\left(r^2+2r-3\right)a_0+\sum_{n=1}^{\infty}\left[(n+r)(n+r+2)-3\right]a_n x^{n+r}+\sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r}=0$$

$$\left(r^2+2r-3\right)a_0+\sum_{n=1}^{\infty}\left[\left((n+r)(n+r+2)-3\right)a_n+(n+r-1)a_{n-1}\right]x^{n+r}=0$$

For $n=0 \quad \to \quad \left(r^2+2r-3\right)a_0=0 \quad \Rightarrow \quad \underline{r=1,\ -3|}$ √

$$\left((n+r)(n+r+2)-3\right)a_n+(n+r-1)a_{n-1}=0$$

$$\boxed{a_n=-\frac{n+r-1}{(n+r)(n+r+2)-3}a_{n-1}}$$

$$r=1 \quad \to \quad a_n=-\frac{n}{(n+1)(n+3)-3}a_{n-1}$$

$$\boxed{=-\frac{n}{n^2+4n}a_{n-1}}$$

$$n=1 \rightarrow a_1 = -\tfrac{1}{5}a_0$$

$$n=2 \rightarrow a_2 = -\frac{2}{12}a_1 = \frac{1}{30}a_0$$

$$n=3 \rightarrow a_3 = -\frac{3}{21}a_2 = -\frac{1}{210}a_0$$

$$n=4 \rightarrow a_4 = -\frac{4}{32}a_3 = \frac{1}{1,680}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x\left(1-\tfrac{1}{5}x+\frac{1}{30}x^2-\frac{1}{210}x^3+\frac{1}{1,680}x^4-\cdots\right)}$$

$$r=-3 \rightarrow b_n = -\frac{n-4}{(n-3)(n-1)-3}b_{n-1}$$

$$\boxed{= -\frac{n-4}{n^2-4n}b_{n-1}}$$

$$n=1 \rightarrow b_1 = -b_0$$

$$n=2 \rightarrow b_2 = -\tfrac{2}{4}b_1 = \tfrac{1}{2}b_0$$

$$n=3 \rightarrow b_3 = -\tfrac{1}{3}b_2 = -\tfrac{1}{6}b_0$$

$$n=4 \rightarrow b_4 = 0b_3 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{-3}\left(1-x+\tfrac{1}{2}x^2-\tfrac{1}{6}x^3\right)}$$

$$\boxed{y(x) = a_0 x\left(1-\tfrac{1}{5}x+\frac{1}{30}x^2-\frac{1}{210}x^3+\frac{1}{1,680}x^4-\cdots\right)+b_0 x^{-3}\left(1-x+\tfrac{1}{2}x^2-\tfrac{1}{6}x^3\right)}$$

Exercise

Find the Frobenius series solutions of $x^2y'' + \left(x^2-2x\right)y' + 2y = 0$

Solution

$$\frac{1}{x^2}x^2y'' + \frac{1}{x^2}\left(x^2-2x\right)y' + 2\frac{1}{x^2}y = 0$$

$$y'' + \left(1-\tfrac{2}{x}\right)y' + \frac{2}{x^2}y = 0$$

That implies to $p(x) = 1-\frac{2}{x}$ and $q(x) = \frac{2}{x^2}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(1-\tfrac{2}{x}\right) = \lim_{x\to 0}(x-2) \; \underline{=-2}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{2}{x^2} \underline{=2|}$$

The indicial equation is: $r(r-1)-2r+2 = r^2-3r+2=0 \quad \to \quad \underline{r=1,\ 2|}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^2\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$x^2y'' + \left(x^2-2x\right)y' + 2y = 0$$

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \left(x^2-2x\right)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1} - \sum_{n=0}^{\infty}2(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)-2(n+r)+2\right]a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-3)+2\right]a_n x^{n+r} + \sum_{n=1}^{\infty}(n+r-1)a_{n-1} x^{n+r} = 0$$

$$\left(r^2-3r+2\right)a_0 + \sum_{n=1}^{\infty}\left[(n+r)(n+r-3)+2\right]a_n x^{n+r} + \sum_{n=1}^{\infty}(n+r-1)a_{n-1} x^{n+r} = 0$$

$$\left(r^2-3r+2\right)a_0 + \sum_{n=1}^{\infty}\left[\left((n+r)(n+r-3)+2\right)a_n + (n+r-1)a_{n-1}\right]x^{n+r} = 0$$

For $n=0 \quad \to \quad \left(r^2-3r+2\right)a_0 = 0 \quad \Rightarrow \quad \underline{r=1,\ 2|} \quad \surd$

$$\left((n+r)(n+r-3)+2\right)a_n + (n+r-1)a_{n-1} = 0$$

$$a_n = -\frac{n+r-1}{(n+r-1)(n+r-2)}a_{n-1} \underline{= -\frac{1}{n+r-2}a_{n-1}}$$

$$r = 2 \quad \to \quad a_n = \underline{= -\frac{1}{n}a_{n-1}}$$

$$n = 1 \ \to \ a_1 = -a_0$$

$$n = 2 \ \to \ a_2 = -\frac{1}{2}a_1 = \frac{1}{2}a_0$$

$$n = 3 \ \to \ a_3 = -\frac{1}{3}a_2 = -\frac{1}{3!}a_0$$

$$n = 4 \ \to \ a_4 = -\frac{1}{4}a_3 = \frac{1}{4!}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 x\left(1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \cdots\right)}$$

$$r = 1 \quad \to \quad b_n \underline{= -\frac{1}{n-1}b_{n-1}}$$

Since $n \neq 1$

$$y(x) = a_0 x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots\right) + b_0 x \ln x\left(-x + x^2 - \frac{x^3}{2} + \frac{x^4}{6} - \cdots\right)$$
$$+ x \ln x\left(1 - x + \frac{x^3}{4} - \frac{5}{36}x^4 + \cdots\right)$$

Exercise

Find the Frobenius series solutions of $x^2y'' + \left(x^2 + 2x\right)y' - 2y = 0$

Solution

$$\frac{1}{x^2}x^2y'' + \frac{1}{x^2}\left(x^2 + 2x\right)y' - 2\frac{1}{x^2}y = 0$$

$$y'' + \left(1 + \frac{2}{x}\right)y' - \frac{2}{x^2}y = 0$$

That implies to $p(x) = 1 + \frac{2}{x}$ and $q(x) = -\frac{2}{x^2}$.

$$p_0 = \lim_{x \to 0} xp(x) = \lim_{x \to 0} x\left(1 + \frac{2}{x}\right) = \lim_{x \to 0} (x + 2) \underline{= 2}$$

$$q_0 = \lim_{x \to 0} x^2 q(x) = -\lim_{x \to 0} x^2 \frac{2}{x^2} \underline{= -2}$$

The indicial equation is: $r(r-1) + 2r - 2 = r^2 + r - 2 = 0 \quad \to \quad \underline{r = 1, \ -2}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x\sum_{n=0}^{\infty} a_n x^n \quad and \quad y_2(x) = x^{-2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x^2 y'' + \left(x^2 + 2x\right) y' - 2y = 0$$

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \left(x^2 + 2x\right)\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r+1} + \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r} - 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + 2(n+r) - 2\right] a_n x^{n+r} + \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^{n+r} = 0$$

$$\left(r^2 + r - 2\right) a_0 + \sum_{n=1}^{\infty} \left[(n+r)(n+r+1) - 2\right] a_n x^{n+r} + \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^{n+r} = 0$$

$$\left(r^2 + r - 2\right) a_0 + \sum_{n=0}^{\infty} \left[\left((n+r)(n+r+1) - 2\right) a_n + (n+r-1) a_{n-1}\right] x^{n+r} = 0$$

For $n = 0 \quad \to \quad \left(r^2 + r - 2\right) a_0 = 0 \quad \Rightarrow \quad \underline{r = 1,\ 2|}$ √

$$\left((n+r)^2 + (n+r) - 2\right) a_n + (n+r-1) a_{n-1} = 0$$

$$a_n = -\frac{n+r-1}{(n+r-1)(n+r+2)} a_{n-1} = -\frac{1}{n+r+2} a_{n-1}$$

$$r = 1 \quad \to \quad a_n = = -\frac{1}{n+3} a_{n-1}$$

$$n = 1 \ \to \ a_1 = -\frac{1}{4} a_0$$

$$n = 2 \ \to \ a_2 = -\frac{1}{5} a_1 = \frac{1}{20} a_0$$

$$n = 3 \ \to \ a_3 = -\frac{1}{6} a_2 = -\frac{1}{120} a_0$$

$n=4 \ \rightarrow \ a_4 = -\frac{1}{7}a_3 = \frac{1}{840}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0 x\left(1 - \frac{1}{4}x + \frac{1}{20}x^2 - \frac{1}{120}x^3 + \frac{1}{840}x^4 - \cdots\right)}$$

$r=2 \quad \rightarrow \quad b_n == \boxed{-\frac{1}{n+4}b_{n-1}}$

$n=1 \ \rightarrow \ b_1 = -\frac{1}{5}b_0$

$n=2 \ \rightarrow \ b_2 = -\frac{1}{6}b_1 = \frac{1}{30}b_0$

$n=3 \ \rightarrow \ b_3 = -\frac{1}{7}b_2 = -\frac{1}{210}b_0$

$n=4 \ \rightarrow \ b_4 = -\frac{1}{8}b_3 = \frac{1}{1,680}b_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = b_0 x^2\left(1 - \frac{x}{5} + \frac{x^2}{30} - \frac{x^3}{210} + \frac{x^4}{1,680} - \cdots\right)}$$

$$\boxed{y(x) = a_0 x\left(1 - \frac{x}{4} + \frac{x^2}{20} - \frac{x^3}{120} + \frac{x^4}{840} - \cdots\right) + b_0 x^2\left(1 - \frac{x}{5} + \frac{x^2}{30} - \frac{x^3}{210} + \frac{x^4}{1,680} - \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2xy'' + 3y' - y = 0$

Solution

$\frac{x}{2}2xy'' + 3\frac{x}{2}y' - \frac{x}{2}y = 0$

$x^2y'' + \frac{3}{2}xy' - \frac{1}{2}xy = 0$

$y'' + \frac{3}{2x}y' - \frac{1}{2x}y = 0$

That implies to $p(x) = \frac{3}{2x}$ and $q(x) = -\frac{1}{2x}$

$p_0 = \lim\limits_{x\to 0} xp(x) = \lim\limits_{x\to 0} x\frac{3}{2x} = \frac{3}{2}$

$q_0 = \lim\limits_{x\to 0} x^2 q(x) = \lim\limits_{x\to 0} x^2\frac{1}{2x} = 0$

The indicial equation is: $r(r-1) + \frac{3}{2}r = r^2 + \frac{1}{2}r = 0 \quad \rightarrow \quad \boxed{r = 0, \ -\frac{1}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{-1/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' + 3y' - y = 0$$

$$2x\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+r)(n+r-1) a_n + 3(n+r) \right] x^{n+r-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$r(2r+1)a_0 + \sum_{n=1}^{\infty} (n+r)(2n+2r+1) a_n x^{n+r-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$r(2r+1)a_0 + \sum_{n=1}^{\infty} \left[(n+r)(2n+2r+1) a_n - a_{n-1} \right] x^{n+r-1} = 0$$

For $n = 0 \quad \to \quad r(2r+1)a_0 = 0 \quad \Rightarrow \quad \boxed{r = 0, \quad -\frac{1}{2}}$ $\checkmark$

$$(n+r)(2n+2r+1)a_n - a_{n-1} = 0$$

$$\boxed{a_n = \frac{1}{(n+r)(2n+2r+1)} a_{n-1}}$$

$$\boxed{r = 0 \quad \to \quad a_n = \frac{1}{n(2n+1)} a_{n-1}}$$

$n = 1 \ \to \ a_1 = \frac{1}{3} a_0$

$n = 2 \ \to \ a_2 = \frac{1}{15} a_1 = \frac{1}{30} a_0$

$n = 3 \ \to \ a_3 = \frac{1}{21} a_2 = \frac{1}{630} a_0$

$n = 4 \ \to \ a_4 = \frac{1}{36} a_3 = \frac{1}{22,680} a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_1(x) = a_0\left(1+\frac{1}{3}x+\frac{1}{30}x^2+\frac{1}{630}x^3+\frac{1}{22,680}x^4+\cdots\right)$$

$$r=-\frac{1}{2} \quad \rightarrow \quad b_n = \frac{1}{n(2n-1)}b_{n-1}$$

$$n=1 \rightarrow b_1 = b_0$$

$$n=2 \rightarrow b_2 = \frac{1}{6}b_1 = \frac{1}{6}b_0$$

$$n=3 \rightarrow b_3 = \frac{1}{15}b_2 = \frac{1}{90}b_0$$

$$n=4 \rightarrow b_4 = \frac{1}{28}b_3 = \frac{1}{2,520}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0x^{-1/2}\left(1+x+\frac{1}{6}x^2+\frac{1}{90}x^3+\frac{1}{2,520}x^4+\cdots\right)$$

$$y(x) = a_0\left(1+\frac{1}{3}x+\frac{1}{30}x^2+\frac{1}{630}x^3+\frac{1}{22,680}x^4+\cdots\right)+b_0x^{-1/2}\left(1+x+\frac{1}{6}x^2+\frac{1}{90}x^3+\frac{1}{2,520}x^4+\cdots\right)$$

$$y(x) = a_0\sum_{n=0}^{\infty}\frac{x^n}{n!(2n+1)!!}+\frac{b_0}{\sqrt{x}}\left(1+\sum_{n=0}^{\infty}\frac{x^n}{n!(2n-1)!!}\right)$$

Exercise

Find the Frobenius series solutions of $2xy''-y'-y=0$

Solution

$$\frac{1}{2x}2xy''-\frac{1}{2x}y'-\frac{1}{2x}y=0$$

$$y''-\frac{1}{2x}y'-\frac{1}{2x}y=0$$

That implies to $p(x)=-\frac{1}{2x}$ and $q(x)=\frac{1}{2x}$

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{-1}{2x} = -\frac{1}{2}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\frac{-1}{2x} = -\lim_{x\to 0}\frac{x}{2} = 0$$

The indicial equation is: $\quad r(r-1)-\frac{1}{2}r = r^2-\frac{3}{2}r = 0 \quad \rightarrow \quad r = 0, \ \frac{3}{2}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty}a_nx^n \quad \textit{and} \quad y_2(x) = x^{3/2}\sum_{n=0}^{\infty}b_nx^n$$

$$y=\sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy''-y'-y=0$$

$$2x\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}-\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}-\sum_{n=0}^{\infty} a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1}-\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}-\sum_{n=0}^{\infty} a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty} \left[2(n+r)(n+r-1) a_n-(n+r)\right] x^{n+r-1}-\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}=0$$

$$r(2r-3)a_0+\sum_{n=1}^{\infty} (n+r)(2n+2r-3) a_n x^{n+r-1}-\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}=0$$

$$r(2r-3)a_0+\sum_{n=1}^{\infty} \left[(n+r)(2n+2r-3) a_n-a_{n-1}\right] x^{n+r-1}=0$$

For $n=0 \quad \to \quad r(2r-3)a_0=0 \quad \Rightarrow \quad \boxed{r=0,\ \frac{3}{2}}$ $\surd$

$$(n+r)(2n+2r-3)a_n-a_{n-1}=0$$

$$\boxed{a_n=\frac{1}{(n+r)(2n+2r-3)}a_{n-1}}$$

$$\boxed{r=0 \quad \to \quad a_n=\frac{1}{n(2n-3)}a_{n-1}}$$

$n=1 \ \to\ a_1=-a_0$

$n=2 \ \to\ a_2=\frac{1}{2}a_1=-\frac{1}{2}a_0$

$n=3 \ \to\ a_3=\frac{1}{9}a_2=-\frac{1}{18}a_0$

$n=4 \ \to\ a_4=\frac{1}{20}a_3=-\frac{1}{360}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_1(x) = a_0\left(1 - x - \frac{1}{2}x^2 - \frac{1}{18}x^3 - \frac{1}{360}x^4 - \cdots\right)$$

$$r = \frac{3}{2} \quad \rightarrow \quad b_n = \frac{1}{n(2n+3)}b_{n-1}$$

$$n = 1 \ \rightarrow \ b_1 = \frac{1}{5}b_0$$

$$n = 2 \ \rightarrow \ b_2 = \frac{1}{14}b_1 = \frac{1}{70}b_0$$

$$n = 3 \ \rightarrow \ b_3 = \frac{1}{27}b_2 = \frac{1}{1890}b_0$$

$$n = 4 \ \rightarrow \ b_4 = \frac{1}{44}b_3 = \frac{1}{83{,}160}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{3/2}\left(1 + \frac{1}{5}x + \frac{1}{70}x^2 + \frac{1}{1890}x^3 + \frac{1}{83{,}160}x^4 + \cdots\right)$$

$$y(x) = a_0\left(1 - x - \frac{1}{2}x^2 - \frac{1}{18}x^3 - \frac{1}{360}x^4 + \cdots\right) + b_0 x^{3/2}\left(1 + \frac{1}{5}x + \frac{1}{70}x^2 + \frac{1}{1{,}890}x^3 + \frac{1}{83{,}160}x^4 - \cdots\right)$$

Exercise

Find the Frobenius series solutions of $2xy'' + (1+x)y' + y = 0$

Solution

$$\frac{1}{2x}2xy'' + \frac{1}{2x}(1+x)y' + \frac{1}{2x}y = 0$$

$$y'' + \left(\frac{1}{2x} + \frac{1}{2}\right)y' + \frac{1}{2x}y = 0$$

That implies to $p(x) = \frac{1}{2x} + \frac{1}{2}$ and $q(x) = \frac{1}{2x}$

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(\frac{1}{2x} + \frac{1}{2}\right) = \lim_{x\to 0}\left(\frac{1}{2} + \frac{1}{2}x\right) = \frac{1}{2}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2\frac{1}{2x} = \lim_{x\to 0}\frac{x}{2} = 0$$

The indicial equation is: $r(r-1) + \frac{1}{2}r = r^2 - \frac{1}{2}r = 0 \quad \rightarrow \quad r = 0, \ \frac{1}{2}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' + (1+x) y' + y = 0$$

$$2x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + (1+x) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+r)(n+r-1) + (n+r) \right] a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r+1) a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-1) \right] a_n x^{n+r-1} + \sum_{n=1}^{\infty} (n+r) a_{n-1} x^{n+r-1} = 0$$

$$r(2r-1) a_0 + \sum_{n=1}^{\infty} (n+r)(2n+2r-1) a_n x^{n+r-1} + \sum_{n=1}^{\infty} (n+r) a_{n-1} x^{n+r-1} = 0$$

$$r(2r-1) a_0 + \sum_{n=1}^{\infty} \left[(n+r)(2n+2r-1) a_n + (n+r) a_{n-1} \right] x^{n+r-1} = 0$$

For $n = 0 \quad \rightarrow \quad r(2r-1) a_0 = 0 \quad \Rightarrow \quad \underline{r = 0, \ \frac{1}{2}} \quad \surd$

$$(n+r)(2n+2r-1) a_n + (n+r) a_{n-1} = 0$$

$$\underline{a_n = -\frac{1}{2n+2r-1} a_{n-1}}$$

$$\underline{r = 0 \quad \rightarrow \quad a_n = -\frac{1}{2n-1} a_{n-1}}$$

$n = 1 \ \rightarrow \ a_1 = -a_0$

$n = 2 \ \rightarrow \ a_2 = \frac{1}{3} a_1 = -\frac{1}{3} a_0$

$n = 3 \ \rightarrow \ a_3 = -\frac{1}{5} a_2 = \frac{1}{15} a_0$

$n = 4 \ \rightarrow \ a_4 = -\frac{1}{7} a_3 = -\frac{1}{105} a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1 - x + \frac{1}{3}x^2 - \frac{1}{15}x^3 + \frac{1}{105}x^4 - \cdots\right)}$$

$$\boxed{r = \frac{1}{2} \quad \rightarrow \quad b_n = -\frac{1}{2n}b_{n-1}}$$

$$n = 1 \ \rightarrow \ b_1 = -\frac{1}{2}b_0$$

$$n = 2 \ \rightarrow \ b_2 = -\frac{1}{4}b_1 = \frac{1}{8}b_0$$

$$n = 3 \ \rightarrow \ b_3 = -\frac{1}{6}b_2 = -\frac{1}{48}b_0$$

$$n = 4 \ \rightarrow \ b_4 = -\frac{1}{8}b_3 = -\frac{1}{384}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/2}\left(1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4 + \cdots\right)}$$

$$\boxed{y(x) = a_0\left(1 - x + \frac{1}{3}x^2 - \frac{1}{15}x^3 + \frac{1}{105}x^4 - \cdots\right) + b_0 x^{1/2}\left(1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4 + \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2xy'' + \left(1 - 2x^2\right)y' - 4xy = 0$

Solution

$$\frac{x}{2}2xy'' + \frac{x}{2}\left(1 - 2x^2\right)y' - \frac{x}{2}4xy = 0$$

$$x^2y'' + \left(\frac{1}{2}x - x^3\right)y' + 2x^2y = 0$$

$$y'' + \left(\frac{1}{2x} - x\right)y' + 2y = 0$$

That implies to $p(x) = \frac{1}{2x} - x$ and $q(x) = 2$

$$p_0 = \lim_{x \to 0} xp(x) = \lim_{x \to 0} x\left(\frac{1}{2x} - x\right) = \lim_{x \to 0}\left(\frac{1}{2} - x^2\right) = \frac{1}{2}$$

$$q_0 = \lim_{x \to 0} x^2q(x) = \lim_{x \to 0} 2x^2 = 0$$

The indicial equation is: $r(r-1) + \frac{1}{2}r = r^2 - \frac{1}{2}r = 0 \quad \rightarrow \quad \boxed{r = 0, \ \frac{1}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$2xy''+\left(1-2x^2\right)y'-4xy=0$$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\left(1-2x^2\right)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-4x\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-\sum_{n=0}^{\infty}2(n+r)a_n x^{n+r+1}-\sum_{n=0}^{\infty}4a_n x^{n+r+1}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2)+(n+r)\right]a_n x^{n+r-1}-\sum_{n=0}^{\infty}(2n+2r+4)a_n x^{n+r+1}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2)+(n+r)\right]a_n x^{n+r-1}-\sum_{n=2}^{\infty}(2n+2r)a_{n-2} x^{n+r-1}=0$$

$$r(2r-1)a_0+(r+2)(2r+1)a_1+\sum_{n=2}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1}-\sum_{n=2}^{\infty}(2n+2r)a_{n-2}x^{n+r-1}=0$$

$$r(2r-1)a_0+(r+2)(2r+1)a_1+\sum_{n=2}^{\infty}\left[(n+r)(2n+2r-1)a_n-2(n+r)a_{n-2}\right]x^{n+r-1}=0$$

For $n=0 \quad \rightarrow \quad r(2r+1)a_0=0 \quad \Rightarrow \quad \boxed{r=0,\ -\tfrac{1}{2}}$ √

For $n=1 \quad \rightarrow \quad (r+2)(2r+1)a_1=0 \quad \Rightarrow \quad \cancel{\boxed{r=-2,\ -\tfrac{1}{2}}} \qquad \boxed{a_1=0}$

$$(n+r)(2n+2r-1)a_n-2(n+r)a_{n-2}=0$$

$$\boxed{a_n=\frac{2}{2n+2r-1}a_{n-2}}$$

$$\boxed{r=0 \quad \rightarrow \quad a_n=\frac{2}{2n-1}a_{n-2}}$$

$n=2 \ \rightarrow \ a_2=\frac{2}{3}a_0 \qquad\qquad n=3 \ \rightarrow \ a_3=\frac{2}{5}a_1=0$

$n=4 \ \rightarrow \ a_4=\frac{2}{7}a_2=\frac{4}{21}a_0 \qquad\qquad n=5 \ \rightarrow \ a_5=\frac{2}{9}a_3=0$

$n=6 \rightarrow a_6 = \frac{2}{11}a_4 = \frac{8}{231}a_0$ $\qquad n=7 \rightarrow a_7 = 0$

$n=8 \rightarrow a_8 = \frac{2}{15}a_6 = \frac{16}{3,465}a_0$ $\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\underline{y_1(x) = a_0\left(1 + \frac{2}{3}x^2 + \frac{4}{21}x^4 + \frac{8}{231}x^6 + \frac{16}{3465}x^8 + \cdots\right)}$$

$$\underline{r = \frac{1}{2} \quad \rightarrow \quad b_n = \frac{1}{n}b_{n-2}}$$

$n=2 \rightarrow b_2 = \frac{1}{2}b_0$ $\qquad n=3 \rightarrow b_3 = \frac{1}{3}b_1 = 0$

$n=4 \rightarrow b_4 = \frac{1}{4}b_2 = \frac{1}{8}b_0$ $\qquad n=5 \rightarrow b_5 = \frac{1}{5}b_3 = 0$

$n=6 \rightarrow b_6 = \frac{1}{6}b_4 = \frac{1}{48}b_0$ $\qquad n=7 \rightarrow b_7 = 0$

$n=8 \rightarrow b_8 = \frac{1}{8}b_6 = \frac{1}{384}b_0$ $\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\underline{y_2(x) = b_0 x^{1/2}\left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \frac{1}{384}x^8 + \cdots\right)}$$

$$\underline{y(x) = a_0\left(1 + \frac{2}{3}x^2 + \frac{4}{21}x^4 + \frac{8}{231}x^6 + \frac{16}{3465}x^8 + \cdots\right) + b_0 x^{1/2}\left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \frac{1}{384}x^8 + \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2x^2y'' + xy' - \left(1 + 2x^2\right)y = 0$

Solution

$$\frac{1}{2}2x^2y'' + \frac{1}{2}xy' - \frac{1}{2}\left(1 + 2x^2\right)y = 0$$

$$x^2y'' + \frac{1}{2}xy' - \frac{1}{2}\left(1 + 2x^2\right)y = 0$$

$$y'' + \frac{1}{2x}y' - \left(\frac{1}{2x^2} + 1\right)y = 0$$

That implies to $p(x) = \frac{1}{2x}$ and $q(x) = -\frac{1}{2x^2} - 1$

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{1}{2x} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\left(-\frac{1}{2x^2} - 1\right) = \lim_{x\to 0} x^2\left(-\frac{1}{2} - x^2\right) = -\frac{1}{2}$$

The indicial equation is: $\quad r(r-1) + \frac{1}{2}r - \frac{1}{2} = r^2 - \frac{1}{2}r - \frac{1}{2} = 0 \quad \rightarrow \quad \underline{r = 1, \ -\frac{1}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^1 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2 y'' + xy' - \left(1+2x^2\right) y = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \left(1+2x^2\right) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} 2a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-2) + (n+r) - 1\right] a_n x^{n+r} - \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r} = 0$$

$$\left(2r^2 - r - 1\right) a_0 + \left((r+1)(2r+1) - 1\right) a_1 + \sum_{n=2}^{\infty} \left[(n+r)(2n+2r-1) - 1\right] a_n x^{n+r} - \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r} = 0$$

$$\left(2r^2 - r - 1\right) a_0 + \left(2r^2 + 3r\right) a_1 + \sum_{n=2}^{\infty} \left[\left((n+r)(2n+2r-1) - 1\right) a_n - 2a_{n-2}\right] x^{n+r} = 0$$

For $n = 0 \quad \rightarrow \quad \left(2r^2 - r - 1\right) a_0 = 0 \quad \Rightarrow \quad \boxed{r = 1, \ -\frac{1}{2}}$ $\checkmark$

For $n = 1 \quad \rightarrow \quad r(2r+3) a_1 = 0 \quad \Rightarrow \quad \cancel{r = 0, \ -\frac{3}{2}} \qquad \boxed{a_1 = 0}$

$$\left((n+r)(2n+2r-1) - 1\right) a_n - 2a_{n-2} = 0$$

$$\boxed{a_n = \frac{2}{(n+r)(2n+2r-1) - 1} a_{n-2}}$$

$$\boxed{r = 1 \quad \rightarrow \quad a_n = \frac{2}{(n+1)(2n+1) - 1} a_{n-2} = \frac{2}{2n^2 + 3n} a_{n-2}}$$

$$n=2 \rightarrow a_2 = \frac{2}{14}a_0 = \frac{1}{7}a_0 \qquad n=3 \rightarrow a_3 = \frac{2}{27}a_1 = 0$$

$$n=4 \rightarrow a_4 = \frac{2}{44}a_2 = \frac{1}{154}a_0 \qquad n=5 \rightarrow a_5 = \frac{2}{65}a_3 = 0$$

$$n=6 \rightarrow a_6 = \frac{2}{90}a_4 = \frac{1}{6,390}a_0 \qquad n=7 \rightarrow a_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x) = a_0 x\left(1+\frac{1}{7}x^2+\frac{1}{154}x^4+\frac{1}{6,930}x^6+\cdots\right)$$

$$r=-\frac{1}{2} \quad \rightarrow \quad b_n = \frac{2b_{n-2}}{\left(n-\frac{1}{2}\right)(2n-2)-1} = \frac{2}{2n^2-3n}b_{n-2}$$

$$n=2 \rightarrow b_2 = \frac{2}{2}b_0 = b_0 \qquad n=3 \rightarrow b_3 = \frac{2}{9}b_1 = 0$$

$$n=4 \rightarrow b_4 = \frac{2}{20}b_2 = \frac{1}{10}b_0 \qquad n=5 \rightarrow b_5 = \frac{2}{35}b_3 = 0$$

$$n=6 \rightarrow b_6 = \frac{2}{54}b_4 = \frac{1}{270}b_0 \qquad n=7 \rightarrow b_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{-1/2}\left(1+x^2+\frac{1}{10}x^4+\frac{1}{270}x^6+\cdots\right)$$

$$y(x) = a_0 x\left(1+\frac{1}{7}x^2+\frac{1}{154}x^4+\frac{1}{6,930}x^6+\cdots\right)+b_0 x^{-1/2}\left(1+x^2+\frac{1}{10}x^4+\frac{1}{270}x^6+\cdots\right)$$

Exercise

Find the Frobenius series solutions of $2x^2y''+xy'-\left(3-2x^2\right)y=0$

Solution

$$\frac{1}{2x^2}2x^2y''+\frac{1}{2x^2}xy'-\frac{1}{2x^2}\left(3-2x^2\right)y=0$$

$$y''+\frac{1}{2x}y'-\frac{1}{2x^2}\left(3-2x^2\right)y=0$$

That implies to $p(x)=\frac{1}{2x}$ and $q(x)=-\frac{1}{2x^2}\left(3-2x^2\right)$

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{1}{2x} = \frac{1}{2}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = -\lim_{x\to 0} x^2\left(\frac{1}{2x^2}\left(3-2x^2\right)\right) = -\lim_{x\to 0}\left(\frac{1}{2}\left(3-2x^2\right)\right) = \frac{3}{2}$$

The indicial equation is: $r(r-1)+\frac{1}{2}r-\frac{3}{2}=r^2-\frac{1}{2}r-\frac{3}{2}=0 \quad \rightarrow \quad r=-1,\ \frac{3}{2}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{-1}\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{3/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2 y'' + xy' - \left(3-2x^2\right) y = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \left(3-2x^2\right)\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} 3a_n x^{n+r} + \sum_{n=0}^{\infty} 2a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-2) + (n+r) - 3\right] a_n x^{n+r} + \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r} = 0$$

$$\left(2r^2 - r - 3\right) a_0 + \left((r+1)(2r+1) - 3\right) a_1 + \sum_{n=2}^{\infty} \left[(n+r)(2n+2r-1) - 3\right] a_n x^{n+r} + \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r} = 0$$

$$\left(2r^2 - r - 3\right) a_0 + \left(2r^2 + 3r - 2\right) a_1 + \sum_{n=2}^{\infty} \left[\left((n+r)(2n+2r-1) - 3\right) a_n + 2a_{n-2}\right] x^{n+r} = 0$$

For $n=0 \quad \rightarrow \quad \left(2r^2 - r - 3\right) a_0 = 0 \quad \Rightarrow \quad \boxed{r = -1,\ \frac{3}{2}}$ √

For $n=1 \quad \rightarrow \quad \left(2r^2 + 3r - 2\right) a_1 = 0 \quad \Rightarrow \quad \boxed{\cancel{r = -2,\ \frac{1}{2}}} \qquad \boxed{a_1 = 0}$

$$\left((n+r)(2n+2r-1) - 3\right) a_n + 2a_{n-2} = 0$$

$$\boxed{a_n = -\frac{2}{(n+r)(2n+2r-1) - 3} a_{n-2}}$$

$$\boxed{r = -1 \quad \rightarrow \quad a_n = -\frac{2}{(n-1)(2n-3) - 3} a_{n-2} = -\frac{2}{2n^2 - 5n} a_{n-2}}$$

$n = 2 \ \rightarrow \ a_2 = a_0 \qquad\qquad n = 3 \ \rightarrow \ a_3 = -\frac{2}{3} a_1 = 0$

$n=4 \;\rightarrow\; a_4 = -\frac{2}{12}a_2 = -\frac{1}{6}a_0$ $\qquad$ $n=5 \;\rightarrow\; a_5 = -\frac{2}{25}a_3 = 0$

$n=6 \;\rightarrow\; a_6 = -\frac{2}{252}a_4 = \frac{1}{126}a_0$ $\qquad$ $n=7 \;\rightarrow\; a_7 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\underline{y_1(x) = a_0 x^{-1}\left(1 + x^2 - \frac{1}{6}x^4 + \frac{1}{126}x^6 - \cdots\right)}$$

$$r = \frac{3}{2} \;\rightarrow\; b_n = -\frac{2b_{n-2}}{\left(n+\frac{3}{2}\right)(2n+2)-3} = -\frac{2}{2n^2+5n}b_{n-2}$$

$n=2 \;\rightarrow\; b_2 = -\frac{2}{18}b_0 = -\frac{1}{9}b_0$ $\qquad$ $n=3 \;\rightarrow\; b_3 = -\frac{2}{33}b_1 = 0$

$n=4 \;\rightarrow\; b_4 = -\frac{2}{568}b_2 = \frac{1}{234}b_0$ $\qquad$ $n=5 \;\rightarrow\; b_5 = 0$

$n=6 \;\rightarrow\; b_6 = \frac{2}{102}b_4 = \frac{1}{11{,}934}b_0$ $\qquad$ $n=7 \;\rightarrow\; b_7 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\underline{y_2(x) = b_0 x^{3/2}\left(1 - \frac{1}{9}x^2 + \frac{1}{234}x^4 - \frac{1}{11{,}934}x^6 + \cdots\right)}$$

$$\underline{y(x) = a_0 x^{-1}\left(1 + x^2 - \frac{1}{6}x^4 + \frac{1}{126}x^6 - \cdots\right) + b_0 x^{3/2}\left(1 - \frac{1}{9}x^2 + \frac{1}{234}x^4 - \frac{1}{11{,}934}x^6 + \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $3xy'' + 2y' + 2y = 0$

Solution

$$\frac{x}{3}3xy'' + 2\frac{x}{3}y' + 2\frac{x}{3}y = 0$$

$$x^2y'' + \frac{2}{3}xy' + \frac{2}{3}xy = 0$$

$$y'' + \frac{2}{3x}y' + \frac{2}{3x}y = 0$$

That implies to $p(x) = \frac{2}{3x}$ and $q(x) = \frac{2}{3x}$

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{2}{3x} = \underline{\frac{2}{3}}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\frac{2}{3x} = \lim_{x\to 0} \frac{2}{3}x = \underline{0}$$

The indicial equation is: $r(r-1) + \frac{2}{3}r = r^2 - \frac{1}{3}r = 0 \;\rightarrow\; \underline{r = 0,\ \frac{1}{3}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/3} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$3xy'' + 2y' + 2y = 0$$

$$3x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 2\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(3n+3r-3) + 2(n+r)\right] a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1} = 0$$

$$r(3r-1)a_n + \sum_{n=1}^{\infty} (n+r)(3n+3r-1) a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1} = 0$$

$$r(3r-1)a_n + \sum_{n=1}^{\infty} \left[(n+r)(3n+3r-1) a_n + 2a_{n-1}\right] x^{n+r-1} = 0$$

For $n = 0 \quad \rightarrow \quad r(3r-1)a_0 = 0 \quad \Rightarrow \quad \boxed{r = 0, \ \frac{1}{3}} \quad \surd$

$$(n+r)(3n+3r-1)a_n + 2a_{n-1} = 0$$

$$\boxed{a_n = -\frac{2}{(n+r)(3n+3r-1)} a_{n-1}}$$

$$\boxed{r = 0 \quad \rightarrow \quad a_n = -\frac{2}{3n^2 - n} a_{n-1}}$$

$n = 1 \ \rightarrow \ a_1 = -a_0$

$n = 2 \ \rightarrow \ a_2 = -\frac{1}{5} a_1 = \frac{1}{5} a_0$

$n = 3 \ \rightarrow \ a_3 = -\frac{2}{24} a_2 = -\frac{1}{60} a_0$

$$n=4 \;\to\; a_4 = -\frac{2}{44}a_3 = \frac{1}{1320}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x^0\left(1 - x + \frac{1}{5}x^2 - \frac{1}{60}x^3 + \frac{1}{1320}x^4 - \cdots\right)}$$

$$\boxed{r = \frac{1}{3} \quad \to \quad b_n = -\frac{2}{3n^2+n}b_{n-1}}$$

$$n=1 \;\to\; b_1 = -\frac{1}{2}b_0$$

$$n=2 \;\to\; b_2 = -\frac{2}{14}b_1 = \frac{1}{14}b_0$$

$$n=3 \;\to\; b_3 = -\frac{2}{30}b_2 = -\frac{1}{210}b_0$$

$$n=4 \;\to\; b_4 = -\frac{2}{52}b_3 = \frac{1}{5460}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/3}\left(1 - \frac{1}{2}x + \frac{1}{14}x^2 - \frac{1}{210}x^3 + \frac{1}{5460}x^4 - \cdots\right)}$$

$$\boxed{y(x) = a_0\left(1 - x + \frac{1}{5}x^2 - \frac{1}{60}x^3 + \frac{1}{1320}x^4 - \cdots\right) + b_0 x^{1/3}\left(1 - \frac{1}{2}x + \frac{1}{14}x^2 - \frac{1}{210}x^3 + \frac{1}{5460}x^4 - \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $3x^2y'' + 2xy' + x^2y = 0$

Solution

$$\frac{1}{3}3x^2y'' + 2\frac{1}{3}xy' + \frac{1}{3}x^2y = 0$$

$$x^2y'' + \frac{2}{3}xy' + \frac{1}{3}x^2y = 0$$

$$y'' + \frac{2}{3x}y' + \frac{1}{3}y = 0$$

That implies to $p(x) = \frac{2}{3x}$ and $q(x) = \frac{1}{3}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{2}{3x} \boxed{= \frac{2}{3}}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\frac{1}{3} \boxed{= 0}$$

The indicial equation is: $r(r-1) + \frac{2}{3}r = r^2 - \frac{1}{3}r = 0 \quad \to \quad \boxed{r = 0,\ \frac{1}{3}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/3}\sum_{n=0}^{\infty} b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$3x^2y''+2xy'+x^2y=0$$

$$3x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+2x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+x^2\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}3(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}2(n+r)a_n x^{n+r}+\sum_{n=0}^{\infty}a_n x^{n+r+2}=0$$

$$\sum_{n=0}^{\infty}\big[(n+r)(3n+3r-3)+2(n+r)\big]a_n x^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}=0$$

$$r(3r-1)a_0+(r+1)(3r+2)a_1+\sum_{n=2}^{\infty}(n+r)(3n+3r-1)a_n x^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}=0$$

$$r(3r-1)a_0+(r+1)(3r+2)a_1+\sum_{n=2}^{\infty}\big[(n+r)(3n+3r-1)a_n+a_{n-2}\big]x^{n+r}=0$$

For $n=0 \quad\to\quad r(3r-1)a_0=0 \quad\Rightarrow\quad \boxed{r=0,\ \frac{1}{3}}$ √

For $n=1 \quad\to\quad (r+1)(3r+2)a_1=0 \quad\Rightarrow\quad \boxed{\cancel{r=-1,\ -\frac{2}{3}}} \qquad \boxed{a_1=0}$

$$(n+r)(3n+3r-1)a_n+a_{n-2}=0$$

$$\boxed{a_n=-\frac{1}{(n+r)(3n+3r-1)}a_{n-2}}$$

$$\boxed{r=0 \quad\to\quad a_n=-\frac{1}{n(3n-1)}a_{n-2}}$$

$n=2 \ \to\ a_2=-\frac{1}{10}a_0$

$n=3 \ \to\ a_3=-\frac{1}{24}a_1=0$

$n=4 \ \to\ a_4=-\frac{1}{44}a_2=\frac{1}{440}a_0$

$n=5 \ \to\ a_5=-\frac{1}{70}a_3=0$

$n=6 \ \to\ a_6=-\frac{1}{102}a_4=-\frac{1}{44,880}a_0$

$n=7 \ \to\ a_7=0$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x) = a_0 x^0\left(1 - \frac{1}{10}x^2 + \frac{1}{440}x^4 - \frac{1}{44,880}x^6 + \cdots\right)$$

$$r = \frac{1}{3} \quad \rightarrow \quad b_n = -\frac{1}{\left(n+\frac{1}{3}\right)(3n)}b_{n-2} = -\frac{1}{n(3n+1)}b_{n-2}$$

$$n = 2 \ \rightarrow \ b_2 = -\frac{1}{14}b_0 \qquad\qquad n = 3 \ \rightarrow \ b_3 = -\frac{1}{30}b_1 = 0$$

$$n = 4 \ \rightarrow \ b_4 = -\frac{1}{72}b_2 = \frac{1}{728}b_0 \qquad\qquad n = 5 \ \rightarrow \ b_5 = 0$$

$$n = 6 \ \rightarrow \ b_6 = -\frac{1}{114}b_4 = -\frac{1}{82,992}b_0 \qquad\qquad n = 7 \ \rightarrow \ b_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{1/3}\left(1 - \frac{1}{14}x^2 + \frac{1}{728}x^4 - \frac{1}{82,992}x^6 + \cdots\right)$$

$$y(x) = a_0\left(1 - \frac{1}{10}x^2 + \frac{1}{440}x^4 - \frac{1}{44,880}x^6 + \cdots\right) + b_0 x^{1/3}\left(1 - \frac{1}{14}x^2 + \frac{1}{728}x^4 - \frac{1}{82,992}x^6 + \cdots\right)$$

Exercise

Find the Frobenius series solutions of $3x^2y'' - xy' + y = 0$

Solution

$$\frac{1}{3}3x^2y'' - \frac{1}{3}xy' + \frac{1}{3}y = 0$$

$$x^2y'' - \frac{1}{3}xy' + \frac{1}{3}y = 0$$

$$y'' - \frac{1}{3x}y' + \frac{1}{3x^2}y = 0$$

That implies to $p(x) = -\frac{1}{3x}$ and $q(x) = \frac{1}{3x^2}$.

$$p_0 = \lim_{x\to 0} xp(x) = -\lim_{x\to 0} x\frac{1}{3x} = -\frac{1}{3}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\frac{1}{3x^2} = \frac{1}{3}$$

The indicial equation is: $\quad r(r-1) - \frac{1}{3}r + \frac{1}{3} = r^2 - \frac{4}{3}r + \frac{1}{3} = 0 \quad \rightarrow \quad r = 1, \ \frac{1}{3}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/3}\sum_{n=0}^{\infty} b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$3x^2y''-xy'+y=0$$

$$3x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}-x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}3(n+r)(n+r-1)a_n x^{n+r}-\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(3n+3r-3)-(n+r)+1\right]a_n x^{n+r}=0$$

Since neither of λ, then let assume $a_n=0, \quad n\ge 1$

$$y_1(x)=x\sum_{n=0}^{\infty}a_n x^n \underline{=a_0x|}$$

$$y_2(x)=x^{1/3}\sum_{n=0}^{\infty}b_n x^n \underline{=b_0x^{1/3}|}$$

$$\underline{y(x)=a_0x+b_0x^{1/3}|}$$

Exercise

Find the Frobenius series solutions of $4xy''+2y'+y=0$

Solution

$$\frac{x}{4}4xy''+2\frac{x}{4}y'+\frac{x}{4}y=0$$

$$x^2y''+\frac{1}{2}xy'+\frac{x}{4}y=0$$

$$y''+\frac{1}{2x}y'+\frac{1}{4x}y=0$$

That implies to $p(x)=\frac{1}{2x}$ and $q(x)=\frac{1}{4x}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{1}{2x} = \boxed{\frac{1}{2}}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{1}{4x} = \lim_{x\to 0} \frac{x}{4} = \boxed{0}$$

The indicial equation is: $r(r-1)+\frac{1}{2}r = r^2 - \frac{1}{2}r = 0 \quad \to \quad \boxed{r = 0,\ \frac{1}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$4xy'' + 2y' + y = 0$$

$$4x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 2\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 4(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(4n+4r-4) + 2(n+r)\right] a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$r(4r-2)a_n + \sum_{n=1}^{\infty} (n+r)(4n+4r-2) a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$2r(2r-1)a_n + \sum_{n=1}^{\infty} \left[2(n+r)(2n+2r-1)a_n + a_{n-1}\right] x^{n+r-1} = 0$$

For $n = 0 \quad \to \quad 2r(2r-1)a_0 = 0 \quad \Rightarrow \quad \boxed{r = 0,\ \frac{1}{2}}$ ✓

$$2(n+r)(2n+2r-1)a_n + a_{n-1} = 0$$

$$\boxed{a_n = -\frac{1}{2(n+r)(2n+2r-1)} a_{n-1}}$$

$$\boxed{r=0 \quad \rightarrow \quad a_n = -\frac{1}{2n(2n-1)}a_{n-1}}$$

$$n=1 \;\rightarrow\; a_1 = -\frac{1}{2}a_0$$

$$n=2 \;\rightarrow\; a_2 = -\frac{1}{12}a_1 = \frac{1}{24}a_0$$

$$n=3 \;\rightarrow\; a_3 = -\frac{1}{30}a_2 = -\frac{1}{720}a_0$$

$$n=4 \;\rightarrow\; a_4 = -\frac{1}{42}a_3 = \frac{1}{30,240}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x^0\left(1-\frac{1}{2}x+\frac{1}{24}x^2-\frac{1}{720}x^3+\frac{1}{30,240}x^4-\cdots\right)}$$

$$\boxed{r=\frac{1}{2} \quad \rightarrow \quad b_n = -\frac{1}{2\left(n+\frac{1}{2}\right)(2n)}b_{n-1} = -\frac{1}{4n^2+2n}b_{n-1}}$$

$$n=1 \;\rightarrow\; b_1 = -\frac{1}{6}b_0$$

$$n=2 \;\rightarrow\; b_2 = -\frac{1}{20}b_1 = \frac{1}{120}b_0$$

$$n=3 \;\rightarrow\; b_3 = -\frac{1}{42}b_2 = -\frac{1}{5040}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/2}\left(1-\frac{1}{6}x+\frac{1}{120}x^2-\frac{1}{5,040}x^3+\cdots\right)}$$

$$\boxed{y(x) = a_0\left(1-\frac{1}{2}x+\frac{1}{24}x^2-\frac{1}{720}x^3+\frac{1}{30,240}x^4-\cdots\right)+b_0 x^{1/2}\left(1-\frac{1}{6}x+\frac{1}{120}x^2-\frac{1}{5,040}x^3+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $6x^2y''+7xy'-\left(x^2+2\right)y=0$

Solution

$$\frac{1}{6}6x^2y''+\frac{1}{6}7xy'-\frac{1}{6}\left(x^2+2\right)y=0$$

$$x^2y''+\frac{7}{6}xy'-\frac{1}{6}\left(x^2+2\right)y=0$$

$$y''+\frac{7}{6x}y'-\frac{1}{6x^2}\left(x^2+2\right)y=0$$

That implies to $p(x)=\frac{7}{6x}$ and $q(x)=-\frac{1}{6x^2}\left(x^2+2\right)$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{7}{6x} = \frac{7}{6}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = -\lim_{x\to 0} x^2 \frac{1}{6x^2}\left(x^2+2\right) = -\lim_{x\to 0}\left(\frac{1}{6}x^2+\frac{1}{3}\right) = -\frac{1}{3}$$

The indicial equation is: $r(r-1)+\frac{7}{6}r-\frac{1}{3} = r^2+\frac{1}{6}r-\frac{1}{3} = 0$

$6r^2+r-2=0 \quad \to \quad r=\frac{-1\pm 7}{12} \quad r=\frac{1}{2},\ -\frac{2}{3}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2}\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{-2/3}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$6x^2y''+7xy'-\left(x^2+2\right)y=0$$

$$6x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+7x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-x^2\sum_{n=0}^{\infty}a_n x^{n+r}-2\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}6(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}7(n+r)a_n x^{n+r}-\sum_{n=0}^{\infty}a_n x^{n+r+2}-\sum_{n=0}^{\infty}2a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[6(n+r)(n+r-1)+7(n+r)-2\right]a_n x^{n+r}-\sum_{n=2}^{\infty}a_{n-2}x^{n+r}=0$$

$$\left(6r^2+r-2\right)a_0+\left((r+1)(6r+7)-2\right)a_1+\sum_{n=2}^{\infty}\left[(n+r)(6n+6r+1)-2\right]a_n x^{n+r}-\sum_{n=2}^{\infty}a_{n-2}x^{n+r}=0$$

$$\left(6r^2+r-2\right)a_0+\left(6r^2+13r+5\right)a_1+\sum_{n=2}^{\infty}\left[\left((n+r)(6n+6r+1)-2\right)a_n-a_{n-2}\right]x^{n+r}=0$$

For $n=0 \quad \to \quad \left(6r^2+r-2\right)a_0=0 \quad \Rightarrow \quad r=\frac{1}{2},\ -\frac{2}{3}$ √

For $n=1 \rightarrow \left(6r^2+13r+5\right)a_1=0 \Rightarrow r=\frac{-13\pm 7}{12}=\cancel{-\frac{1}{2}, -\frac{5}{3}}$ $\quad \boxed{a_1=0}$

$$\left((n+r)(6n+6r+1)-2\right)a_n-a_{n-2}=0$$

$$\boxed{a_n=\frac{1}{(n+r)(6n+6r+1)-2}a_{n-2}}$$

$$\boxed{r=\frac{1}{2} \rightarrow a_n=\frac{1}{n(6n+7)}a_{n-2}}$$

$n=2 \rightarrow a_2=\frac{1}{38}a_0$ $\qquad$ $n=3 \rightarrow a_3=\frac{1}{75}a_1=0$

$n=4 \rightarrow a_4=\frac{1}{124}a_2=\frac{1}{4,712}a_0$ $\qquad$ $n=5 \rightarrow a_5=\frac{1}{185}a_3=0$

$n=6 \rightarrow a_6=\frac{1}{258}a_4=\frac{1}{1,215,696}a_0$ $\qquad$ $n=7 \rightarrow a_7=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x)=a_0x^{1/2}\left(1+\frac{1}{38}x^2+\frac{1}{4,712}x^4+\frac{1}{1,215,696}x^6+\cdots\right)}$$

$$\boxed{r=-\frac{2}{3} \rightarrow b_n=\frac{1}{n(6n-7)}b_{n-2}}$$

$n=2 \rightarrow b_2=\frac{1}{10}b_0$ $\qquad$ $n=3 \rightarrow b_3=\frac{1}{33}b_1=0$

$n=4 \rightarrow b_4=\frac{1}{68}b_2=\frac{1}{680}b_0$ $\qquad$ $n=5 \rightarrow b_5=0$

$n=6 \rightarrow b_6=\frac{1}{174}b_4=\frac{1}{118,320}b_0$ $\qquad$ $n=7 \rightarrow b_7=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x)=b_0x^{-2/3}\left(1+\frac{1}{10}x^2+\frac{1}{680}x^4+\frac{1}{118,320}x^6+\cdots\right)}$$

$$\boxed{y(x)=a_0x^{1/2}\left(1+\frac{x^2}{38}+\frac{x^4}{4,712}+\frac{x^6}{1,215,696}+\cdots\right)+b_0x^{-2/3}\left(1+\frac{x^2}{10}+\frac{x^4}{680}+\frac{x^6}{118,320}+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $xy''+y'+2y=0$

Solution

$x\times \quad xy''+y'+2y=0$

$$x^2y''+xy'+2xy=0$$

$$y''+\frac{1}{x}y'+\frac{2}{x}y=0$$

That implies to $p(x)=\frac{1}{x}$ and $q(x)=\frac{2}{x}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{1}{x} \underline{=1|}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{2}{x} = \lim_{x\to 0} 2x \underline{=0|}$$

The indicial equation is: $r^2 + (1-1)r = 0 \quad \to \quad \underline{r_{1,2} = 0|}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = y_1(x)\ln|x| + \sum_{n=0}^{\infty} c_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} a_n x^n \qquad\qquad (r = r_1 = 0)$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n+r-1} = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$xy'' + y' + 2y = 0$

$$x\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n^2 a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n^2 a_n x^{n-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n-1} = 0$$

$$\sum_{n=1}^{\infty} n^2 a_n x^{n-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n-1} = 0$$

$$\sum_{n=1}^{\infty} \left[n^2 a_n + 2a_{n-1}\right] x^{n-1} = 0$$

$$n^2 a_n + 2a_{n-1} = 0 \quad \Rightarrow \quad \underline{a_n = -\frac{2}{n^2} a_{n-1}|}$$

$$n = 1 \ \to\ a_1 = -2a_0$$

$$n=2 \;\rightarrow\; a_2 = -\frac{2}{2^2}a_1 = a_0 = \frac{2^2}{2^2}a_0$$

$$n=3 \;\rightarrow\; a_3 = -\frac{2}{9}a_2 = -\frac{2^3}{(2\cdot 3)^2}a_0 = -\frac{2}{9}a_0$$

$$n=4 \;\rightarrow\; a_4 = -\frac{2}{4^2}a_3 = \frac{2^4}{(4!)^2}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = \frac{(-1)^n 2^n}{(n!)^2}a_0}$$

$$\boxed{y_1(x) = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n 2^n}{(n!)^2}x^n}$$

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty}a_n x^n \quad (a_0 = 1)$$

$$y_2(x) = y_1(x)\ln x + \sum_{n=0}^{\infty}c_n x^n$$

$$y_2(x) = y_1(x)\ln x + x^{r_1}\sum_{n=0}^{\infty}c_n x^n$$

$$xy_2'' + y_2' + 2y_2 = 0$$

$$x\left(y_1(x)\ln x + \sum_{n=0}^{\infty}c_n x^n\right)'' + \left(y_1(x)\ln x + \sum_{n=0}^{\infty}c_n x^n\right)' + 2\left(y_1(x)\ln x + \sum_{n=0}^{\infty}c_n x^n\right) = 0$$

$$x\left(y_1'\ln x + \frac{1}{x}y_1\right)' + x\sum_{n=0}^{\infty}n(n-1)c_n x^{n-2} + y_1'\ln x + \frac{1}{x}y_1 + \sum_{n=0}^{\infty}nc_n x^{n-1} + 2y_1\ln x + \sum_{n=0}^{\infty}2c_n x^n = 0$$

$$x\left(y_1''\ln x + \frac{1}{x}y_1' - \frac{1}{x^2}y_1 + \frac{1}{x}y_1'\right) + y_1'\ln x + \frac{1}{x}y_1 + 2y_1\ln x$$

$$+\sum_{n=0}^{\infty}n(n-1)c_n x^{n-1} + \sum_{n=0}^{\infty}nc_n x^{n-1} + \sum_{n=0}^{\infty}2c_n x^n = 0$$

$$xy_1''\ln x + 2y_1' - \frac{1}{x}y_1 + \frac{1}{x}y_1 + \left(y_1' + 2y_1\right)\ln x + \sum_{n=0}^{\infty}n^2 c_n x^{n-1} + \sum_{n=0}^{\infty}2c_n x^n = 0$$

$$\left(xy_1'' + y_1' + 2y_1\right)\ln x + 2y_1' + \sum_{n=0}^{\infty}n^2 c_n x^{n-1} + \sum_{n=1}^{\infty}2c_{n-1}x^{n-1} = 0$$

Since: $xy_1'' + y_1' + 2y_1 = 0$

$$2y_1' + \sum_{n=1}^{\infty}n^2 c_n x^{n-1} + \sum_{n=1}^{\infty}2c_{n-1}x^{n-1} = 0$$

$$2y_1' + \sum_{n=1}^{\infty}\left(n^2 c_n + 2c_{n-1}\right)x^{n-1} = 0$$

$$2\sum_{n=1}^{\infty}(-1)^n \frac{2^n}{(n!)^2} n x^{n-1} + \sum_{n=1}^{\infty}\left(n^2 c_n + 2c_{n-1}\right)x^{n-1} = 0$$

$$\sum_{n=1}^{\infty}\left[(-1)^n \frac{2^{n+1}}{(n-1)!n!} + n^2 c_n + 2c_{n-1}\right]x^{n-1} = 0$$

$$n^2 c_n + 2c_{n-1} + (-1)^n \frac{2^{n+1}}{(n-1)!n!} = 0$$

$$\boxed{c_n = -\frac{2}{n^2}c_{n-1} + (-1)^{n+1}\frac{2^{n+1}}{(n-1)!n!\,n^2}}$$

$n = 1 \ \to \ c_1 = -2c_0 + 4$ $\qquad$ $c_0 = 0 \ \to \ c_1 = 4$

$n = 2 \ \to \ c_2 = -\frac{1}{2}c_1 - 1$ $\qquad$ $c_2 = -3$

$n = 3 \ \to \ c_3 = -\frac{2}{9}c_2 + \frac{16}{12(9)}$ $\qquad$ $c_3 = \frac{22}{27}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = y_1(x)\ln|x| + 4x - 3x^2 + \frac{22}{27}x^3 - \cdots}$$

$$y_2(x) = y_1(x)\ln x + x^{r_1}\sum_{n=0}^{\infty} c_n x^n$$

Exercise

Find the Frobenius series solutions of $xy'' - y = 0$

Solution

$x\times \quad xy'' - y = 0$

$x^2 y'' - xy = 0$

$y'' - \frac{1}{x}y = 0$

That implies to $p(x) = 0$ and $q(x) = -\frac{1}{x}$.

$p_0 = \lim_{x\to 0} xp(x) \ \underline{= 0}$

$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2\left(-\frac{1}{x}\right) = -\lim_{x\to 0} x \ \underline{= 0}$

The indicial equation is: $\quad r^2 - r = 0 \quad \to \quad \underline{r_1 = 1,\ r_2 = 0}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1} \quad \text{and} \quad y_2(x) = \alpha y_1(x)\ln|x| + \sum_{n=0}^{\infty} c_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} a_n x^{n+1} \qquad (r = r_1 = 1)$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n+r-1} = \sum_{n=0}^{\infty} n a_n x^n$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} = \sum_{n=0}^{\infty} n(n+1) a_n x^{n-1}$$

$$xy'' - y = 0$$

$$x\sum_{n=0}^{\infty} n(n+1) a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} n(n+1) a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} n(n+1) a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\sum_{n=1}^{\infty} n(n+1) a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\sum_{n=1}^{\infty} \left[n(n+1) a_n - a_{n-1} \right] x^n = 0$$

$$n(n+1) a_n - a_{n-1} = 0 \quad \Rightarrow \quad \boxed{a_n = \frac{1}{n(n+1)} a_{n-1}}$$

$$n = 1 \;\rightarrow\; a_1 = \tfrac{1}{2} a_0$$

$$n = 2 \;\rightarrow\; a_2 = \tfrac{1}{6} a_1 = a_0 = \frac{1}{(2)3!} a_0$$

$$n = 3 \;\rightarrow\; a_3 = \frac{1}{3\cdot 4} a_2 = \frac{1}{(2\cdot 3)4!} a_0$$

$$n = 4 \;\rightarrow\; a_4 = \frac{1}{4\cdot 5} a_3 = \frac{1}{4!5!} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = \frac{1}{n!(n+1)!} a_0}$$

$$y_1(x)=x\sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^n=\sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^{n+1}$$

$$y_1(x)=x^{r_1}\sum_{n=0}^{\infty}a_nx^n \quad (a_0=1)$$

$$y_2(x)=\alpha y_1\ln x+\sum_{n=0}^{\infty}d_nx^n$$

$$y_2(x)=\alpha y_1(x)\ln x+\sum_{n=0}^{\infty}d_nx^n \quad (d_0=1)$$

$$y_2'=\alpha y_1'\ln x+\frac{\alpha}{x}y_1+\sum_{n=0}^{\infty}nd_nx^{n-1}$$

$$xy_2''-y_2=0$$

$$x\left(\alpha y_1'\ln x+\frac{\alpha}{x}y_1+\sum_{n=0}^{\infty}nd_nx^{n-1}\right)'-\left(\alpha y_1\ln x+\sum_{n=0}^{\infty}d_nx^n\right)=0$$

$$x\left(\alpha y_1'\ln x+\frac{\alpha}{x}y_1\right)'+x\sum_{n=0}^{\infty}n(n-1)d_nx^{n-2}-\alpha y_1\ln x-\sum_{n=0}^{\infty}d_nx^n=0$$

$$x\left(\alpha y_1''\ln x+\frac{\alpha}{x}y_1'-\frac{\alpha}{x^2}y_1+\frac{\alpha}{x}y_1'\right)-\alpha y_1(x)\ln x+\sum_{n=0}^{\infty}n(n-1)d_nx^{n-1}-\sum_{n=0}^{\infty}d_nx^n=0$$

$$\alpha\left(2y_1'-\frac{1}{x}y_1\right)+\alpha\left(xy_1''-y_1\right)\ln x+\sum_{n=0}^{\infty}n(n+1)d_{n+1}x^n-\sum_{n=0}^{\infty}d_nx^n=0$$

Since: $xy_1''-y_1=0$

$$y_1(x)=\sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^{n+1} \quad\to\quad y_1'=\sum_{n=1}^{\infty}\frac{n+1}{n!(n+1)!}x^n$$

$$\alpha\left(2y_1'-\frac{1}{x}y_1\right)+\sum_{n=0}^{\infty}\left[n(n+1)d_{n+1}-d_n\right]x^n=0$$

$$\alpha\left(2\sum_{n=1}^{\infty}\frac{n+1}{n!(n+1)!}x^n-\frac{1}{x}\sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^{n+1}\right)+\sum_{n=0}^{\infty}\left[n(n+1)d_{n+1}-d_n\right]x^n=0$$

$$\alpha\left(\sum_{n=1}^{\infty}\frac{2n+2}{n!(n+1)!}x^n-\sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^n\right)+\sum_{n=0}^{\infty}\left[n(n+1)d_{n+1}-d_n\right]x^n=0$$

$$\alpha\sum_{n=1}^{\infty}\frac{2n+1}{n!(n+1)!}x^n+\sum_{n=0}^{\infty}\left[n(n+1)d_{n+1}-d_n\right]x^n=0$$

$$\sum_{n=0}^{\infty}\left[n(n+1)d_{n+1}-d_n\right]x^n=-\alpha\sum_{n=1}^{\infty}\frac{2n+1}{n!(n+1)!}x^n$$

$$\boxed{n(n+1)d_{n+1}-d_n=-\alpha\frac{2n+1}{n!(n+1)!}} \qquad (d_0=1)$$

$$n=0 \ \rightarrow \ -d_0=-\alpha \qquad \Rightarrow \ \boxed{\alpha=d_0=1}$$

$$d_{n+1}=\frac{1}{n(n+1)}\left(d_n-\frac{2n+1}{n!(n+1)!}\right)$$

$$n=1 \ \rightarrow \ d_2=\frac{1}{2}\left(d_1-\frac{3}{2}\right)=\boxed{\frac{1}{2}d_1-\frac{3}{4}}$$

$$n=2 \ \rightarrow \ d_3=\frac{1}{6}\left(d_2-\frac{5}{12}\right)=\frac{1}{6}\left(\frac{1}{2}d_1-\frac{3}{4}-\frac{5}{12}\right)=\boxed{\frac{1}{12}d_1-\frac{7}{36}}$$

$$n=3 \ \rightarrow \ d_4=\frac{1}{12}\left(d_3-\frac{7}{144}\right)=\frac{1}{12}\left(\frac{1}{12}d_1-\frac{7}{36}-\frac{7}{144}\right)=\boxed{\frac{1}{144}d_1-\frac{35}{1,728}}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

If we let $d_1=0$

$$\boxed{y_2(x)=y_1(x)\ln x+1-\frac{3}{4}x^2-\frac{7}{36}x^3-\cdots} \qquad y_2(x)=y_1(x)\ln x+\sum_{n=0}^{\infty}d_nx^n$$

Exercise

Find the Frobenius series solutions of $2x(1-x)y''+(1+x)y'-y=0$

Solution

$$xy''+\frac{x+1}{2(1-x)}y'-\frac{1}{2(1-x)}y=0$$

$$x^2y''+\frac{1}{2}\frac{x(x+1)}{1-x}y'-\frac{x}{2(1-x)}y=0$$

$$y''+\frac{1}{2}\frac{x+1}{x(1-x)}y'-\frac{1}{2x(1-x)}y=0$$

That implies to $p(x)=\frac{1}{2}\frac{x+1}{x(1-x)}$ and $q(x)=-\frac{1}{2x(1-x)}$.

$$p_0=\lim_{x\to 0}xp(x)=\frac{1}{2}\lim_{x\to 0}x\frac{x+1}{x(1-x)}=\frac{1}{2}\lim_{x\to 0}\frac{x+1}{1-x}\boxed{=\frac{1}{2}}$$

$$q_0=\lim_{x\to 0}x^2q(x)=-\frac{1}{2}\lim_{x\to 0}\frac{x}{1-x}\boxed{=0}$$

The indicial equation is: $r^2+\left(\frac{1}{2}-1\right)r=r^2-\frac{1}{2}r=0 \ \rightarrow \ \boxed{r=0,\ \frac{1}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x)=x^{1/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y=\sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$2x(1-x)y''+(1+x)y'-y=0$$

$$\left(2x-2x^2\right)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+(1+x)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r-1}-\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}-\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}\left[-2(n+r)(n+r-1)+n+r-1\right]a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}(n+r-1)(-2(n+r)+1)a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1}+\sum_{n=1}^{\infty}(n+r-2)(-2n-2r+3)a_{n-1} x^{n+r-1}=0$$

$$r(2r-1)a_0 x^{r-1}+\sum_{n=1}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1}-\sum_{n=1}^{\infty}(n+r-2)(2n+2r-3)a_{n-1} x^{n+r-1}=0$$

$$r(2r-1)a_0 x^{r-1}+\sum_{n=1}^{\infty}\left[(n+r)(2n+2r-1)a_n-(n+r-2)(2n+2r-3)a_{n-1}\right]x^{n+r-1}=0$$

For $n=0 \quad \rightarrow \quad r(2r-1)a_0=0 \quad \Rightarrow \underline{r=0,\ \tfrac{1}{2}}$ √

$$(n+r)(2n+2r-1)a_n-(n+r-2)(2n+2r-3)a_{n-1}=0$$

$$\boxed{a_n = \frac{(n+r-2)(2n+2r-3)}{(n+r)(2n+2r-1)} a_{n-1}}$$

$$\boxed{r=0 \quad \rightarrow \quad a_n = \frac{(n-2)(2n-3)}{n(2n-1)} a_{n-1}}$$

$n=1 \ \rightarrow \ a_1 = a_0$

$n=2 \ \rightarrow \ a_2 = 0a_1 = 0$

$n=3 \ \rightarrow \ a_3 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0(1+x)}$$

$$\boxed{r=\frac{1}{2} \quad \rightarrow \quad b_n = \frac{\left(n-\frac{3}{2}\right)(2n-2)}{2n\left(n+\frac{1}{2}\right)} b_{n-1} = \frac{(2n-3)(n-1)}{n(2n+1)} b_{n-1}}$$

$n=1 \ \rightarrow \ b_1 = 0b_0 = 0$

$n=2 \ \rightarrow \ b_2 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = b_0 x^{1/2}}$$

$$\boxed{y(x) = a_0(1+x) + b_0\sqrt{x}}$$

Exercise

Find the Frobenius series solutions of $x^2 y'' + \left(x^2 + \frac{1}{2}x\right)y' + xy = 0$

Solution

$$y'' + \left(1 + \frac{1}{2x}\right)y' + \frac{1}{x}y = 0$$

That implies to $p(x) = 1 + \frac{1}{2x}$ and $q(x) = \frac{1}{x}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(1+\frac{1}{2x}\right) = \lim_{x\to 0}\left(x+\frac{1}{2}\right) = \underline{\frac{1}{2}}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x = \underline{0}$$

The indicial equation is: $r^2 + \left(\frac{1}{2}-1\right)r = r^2 - \frac{1}{2}r = 0 \quad \rightarrow \quad \underline{r = 0, \ \frac{1}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=\sum_{n=0}^{\infty}a_n x^n \quad \text{and} \quad y_2(x)=x^{1/2}\sum_{n=0}^{\infty}b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$x^2y''+\left(x^2+\tfrac{1}{2}x\right)y'+xy=0$$

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\left(x^2+\tfrac{1}{2}x\right)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+x\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1}\sum_{n=0}^{\infty}\tfrac{1}{2}(n+r)a_n x^{n+r}+\sum_{n=0}^{\infty}a_n x^{n+r+1}=0$$

$$\sum_{n=0}^{\infty}(n+r+1)a_n x^{n+r+1}+\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)+\tfrac{1}{2}(n+r)\right]a_n x^{n+r}=0$$

$$\sum_{n=1}^{\infty}(n+r)a_{n-1}x^{n+r}+\sum_{n=0}^{\infty}(n+r)\left(n+r-\tfrac{1}{2}\right)a_n x^{n+r}=0$$

$$\sum_{n=1}^{\infty}(n+r)a_{n-1}x^{n+r}+r\left(r-\tfrac{1}{2}\right)a_0+\sum_{n=1}^{\infty}(n+r)\left(n+r-\tfrac{1}{2}\right)a_n x^{n+r}=0$$

$$r\left(r-\tfrac{1}{2}\right)a_0+\sum_{n=1}^{\infty}\left[(n+r)\left(n+r-\tfrac{1}{2}\right)a_n+(n+r)a_{n-1}\right]x^{n+r}=0$$

For $n=0 \quad \rightarrow \quad r(2r-1)a_0=0 \quad \Rightarrow \quad r=0,\ \tfrac{1}{2}$ √

$$(n+r)\left(n+r-\tfrac{1}{2}\right)a_n+(n+r)a_{n-1}=0$$

$$a_n=-2\frac{(n+r)}{(n+r)(2n+2r-1)}a_{n-1}$$

$$r=0 \quad \rightarrow \quad a_n=-\frac{2n}{n(2n-1)}a_{n-1}=-\frac{2}{2n-1}a_{n-1}$$

$$n=1 \rightarrow a_1=-2a_0$$

$$n=2 \ \to \ a_2 = -\frac{2}{3}a_1 = \frac{4}{3}a_0$$

$$n=3 \ \to \ a_3 = -\frac{2}{5}a_2 = -\frac{8}{15}a_0$$

$$n=4 \ \to \ a_4 = -\frac{2}{7}a_3 = \frac{16}{105}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0\left(1-2x+\frac{4}{3}x^2-\frac{8}{15}x^3+\frac{16}{105}x^4-\cdots\right)}$$

$$\boxed{r=\frac{1}{2} \ \to \ b_n = -\frac{2\left(n+\frac{1}{2}\right)}{\left(n+\frac{1}{2}\right)(2n)}b_{n-1} = -\frac{1}{n}b_{n-1}}$$

$$n=1 \ \to \ b_1 = -b_0$$

$$n=2 \ \to \ b_2 = -\frac{1}{2}b_1 = \frac{1}{2}b_0$$

$$n=3 \ \to \ b_3 = -\frac{1}{3}b_2 = -\frac{1}{6}b_0$$

$$n=4 \ \to \ b_4 = -\frac{1}{4}b_3 = \frac{1}{24}b_0$$

$$n=5 \ \to \ b_5 = -\frac{1}{5}b_4 = -\frac{1}{120}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/2}\left(1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{120}x^5+\cdots\right)}$$

$$\boxed{y(x) = a_0\left(1-2x+\frac{4}{3}x^2-\frac{8}{15}x^3+\frac{16}{105}x^4-\cdots\right)+b_0\sqrt{x}\left(1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{120}x^5+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $18x^2y''+3x(x+5)y'-(10x+1)y=0$

Solution

$$y''+\frac{x+5}{6x}y'-\frac{10x+1}{18x^2}y=0$$

That implies to $p(x)=\frac{x+5}{6x}$ and $q(x)=-\frac{10x+1}{18x^2}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(\frac{x+5}{6x}\right) = \lim_{x\to 0}\frac{x+5}{6} = \boxed{\frac{5}{6}}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = -\lim_{x\to 0}\frac{10x+1}{18} = \boxed{-\frac{1}{18}}$$

The indicial equation is: $r^2+\left(\frac{5}{6}-1\right)r-\frac{1}{18} = r^2-\frac{1}{6}r-\frac{1}{18}=0$

$$18r^2-3r-1=0 \quad \to \quad \underline{r=-\frac{1}{6},\ \frac{1}{3}}\Big|$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=x^{-1/6}\sum_{n=0}^{\infty}a_n x^n \quad \textit{and} \quad y_2(x)=x^{1/3}\sum_{n=0}^{\infty}b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$18x^2y''+3x(x+5)y'-(10x+1)y=0$$

$$18x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\left(3x^2+15x\right)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-(10x+1)\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}18(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}3(n+r)a_n x^{n+r+1}+\sum_{n=0}^{\infty}15(n+r)a_n x^{n+r}$$

$$-\sum_{n=0}^{\infty}10a_n x^{n+r+1}-\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[18(n+r)(n+r-1)+15(n+r)-1\right]a_n x^{n+r}+\sum_{n=0}^{\infty}(3n+3r-10)a_n x^{n+r+1}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(18n+18r-3)-1\right]a_n x^{n+r}+\sum_{n=1}^{\infty}(3n+3r-13)a_{n-1}x^{n+r}=0$$

$$\left(r(18r-3)-1\right)a_0x^r+\sum_{n=1}^{\infty}\left[(n+r)(18n+18r-3)-1\right]a_n x^{n+r}+\sum_{n=1}^{\infty}(3n+3r-13)a_{n-1}x^{n+r}=0$$

$$\left(r(18r-3)-1\right)a_0x^r+\sum_{n=1}^{\infty}\left[\left((n+r)(18n+18r-3)-1\right)a_n+(3n+3r-13)a_{n-1}\right]x^{n+r}=0$$

For $n=0 \quad \to \quad \left(18r^2-3r-1\right)a_0=0 \quad \Rightarrow \quad \underline{r=-\frac{1}{6},\ \frac{1}{3}}\Big| \quad \checkmark$

$$\left((n+r)(18n+18r-3)-1\right)a_n+(3n+3r-13)a_{n-1}=0$$

$$\boxed{a_n = -\frac{3n+3r-13}{(n+r)(18n+18r-3)-1}a_{n-1}}$$

$$\boxed{r=-\frac{1}{6} \quad\rightarrow\quad a_n = -\frac{3n-\frac{1}{2}-13}{\left(n-\frac{1}{6}\right)(18n-6)-1}a_{n-1} = -\frac{1}{2}\frac{6n-27}{(6n-1)(3n-1)-1}a_{n-1}}$$

$$n=1 \;\rightarrow\; a_1 = -\frac{1}{2}\frac{-21}{9}a_0 = \frac{7}{6}a_0$$

$$n=2 \;\rightarrow\; a_2 = -\frac{1}{2}\frac{-15}{54}a_1 = \frac{5}{36}\frac{7}{6}a_0 = \frac{35}{216}a_0$$

$$n=3 \;\rightarrow\; a_3 = -\frac{1}{2}\frac{-9}{135}a_2 = \frac{1}{30}\frac{35}{216}a_0 = \frac{7}{1,296}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x^{-1/6}\left(1+\frac{7}{6}x+\frac{35}{216}x^2+\frac{7}{1,296}x^3+\cdots\right)}$$

$$\boxed{r=\frac{1}{3} \quad\rightarrow\quad b_n = -\frac{3n-12}{\left(n+\frac{1}{3}\right)(18n+3)-1}b_{n-1} = -\frac{3(n-4)}{(3n+1)(6n+1)-1}b_{n-1}}$$

$$n=1 \;\rightarrow\; b_1 = -\frac{-9}{27}b_0 = \frac{1}{3}b_0$$

$$n=2 \;\rightarrow\; b_2 = \frac{6}{90}b_1 = \frac{1}{15}\frac{1}{3}b_0 = \frac{1}{45}b_0$$

$$n=3 \;\rightarrow\; b_3 = \frac{3}{189}b_2 = \frac{1}{63}\frac{1}{45}b_0 = \frac{1}{2,835}b_0$$

$$n=4 \;\rightarrow\; b_4 = 0b_3 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/3}\left(1+\frac{1}{3}x+\frac{1}{45}x^2+\frac{1}{2835}x^3\right)}$$

$$\boxed{y(x) = a_0\frac{1}{x^{1/6}}\left(1+\frac{7}{6}x+\frac{35}{216}x^2+\frac{7}{1296}x^3+\cdots\right)+b_0 x^{1/3}\left(1+\frac{1}{3}x+\frac{1}{45}x^2+\frac{1}{2835}x^3\right)}$$

Exercise

Find the Frobenius series solutions of $2x^2y'' + 7x(x+1)y' - 3y = 0$

Solution

$$y'' + \frac{7}{2}\frac{x+1}{x}y' - \frac{3}{2x^2}y = 0$$

That implies to $p(x) = \frac{7}{2}\frac{x+1}{x}$ and $q(x) = -\frac{3}{2x^2}$.

$$p_0 = \lim_{x\to 0} xp(x) = \frac{7}{2}\lim_{x\to 0} x\left(\frac{x+1}{x}\right) = \frac{7}{2}\lim_{x\to 0}(x+1) = \boxed{\frac{7}{2}}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = -\lim_{x\to 0} \frac{3}{2} = \boxed{-\frac{3}{2}}$$

The indicial equation is: $r^2 + \left(\frac{7}{2} - 1\right) r - \frac{3}{2} = r^2 + \frac{5}{2} r - \frac{3}{2} = 0$

$$2r^2 + 5r - 3 = 0 \quad \to \quad \boxed{r = -3,\ \frac{1}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{-3} \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2 y'' + 7x(x+1) y' - 3y = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \left(7x^2 + 7x\right) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - 3 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} 7(n+r) a_n x^{n+r+1} + \sum_{n=0}^{\infty} 7(n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} 3a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-2) + 7(n+r) - 3\right] a_n x^{n+r} + \sum_{n=0}^{\infty} 7(n+r) a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r+5) - 3\right] a_n x^{n+r} + \sum_{n=1}^{\infty} 7(n+r-1) a_{n-1} x^{n+r} = 0$$

$$\left(2r^2 + 5r - 3\right) a_0 x^r + \sum_{n=1}^{\infty} \left[(n+r)(2n+2r+5) - 3\right] a_n x^{n+r} + \sum_{n=1}^{\infty} 7(n+r-1) a_{n-1} x^{n+r} = 0$$

$$\left(2r^2 + 5r - 3\right) a_0 x^r + \sum_{n=1}^{\infty} \left[\left((n+r)(2n+2r+5) - 3\right) a_n + 7(n+r-1) a_{n-1}\right] x^{n+r} = 0$$

For $n = 0 \quad \to \quad \left(2r^2 + 5r - 3\right) a_0 = 0 \quad \Rightarrow \quad \boxed{r = -3,\ \frac{1}{2}}$ √

$$\left((n+r)(2n+2r+5) - 3\right) a_n + 7(n+r-1) a_{n-1} = 0$$

$$\boxed{a_n = -\frac{7(n+r-1)}{(n+r)(2n+2r+5)-3}a_{n-1}}$$

$$\boxed{r=-3 \quad \to \quad a_n = -\frac{7(n-4)}{(n-3)(2n-1)-3}a_{n-1}}$$

$$n=1 \ \to \ a_1 = -\frac{21}{5}a_0$$

$$n=2 \ \to \ a_2 = -\frac{14}{6}a_1 = -\frac{7}{3}\left(-\frac{21}{5}\right)a_0 = \frac{49}{5}a_0$$

$$n=3 \ \to \ a_3 = -\frac{-7}{-3}a_2 = -\frac{7}{3}\frac{49}{5}a_0 = -\frac{343}{15}a_0$$

$$n=4 \ \to \ a_4 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x^{-3}\left(1-\frac{21}{5}x+\frac{49}{5}x^2-\frac{343}{15}x^3\right)}$$

$$\boxed{r=\frac{1}{2} \quad \to \quad b_n = -\frac{7\left(n-\frac{1}{2}\right)}{\left(n+\frac{1}{2}\right)(2n+6)-3}b_{n-1} = -\frac{7}{2}\frac{2n-1}{(2n+1)(n+3)-3}b_{n-1}}$$

$$n=1 \ \to \ b_1 = -\frac{7}{2}\frac{1}{9}b_0 = -\frac{7}{18}b_0$$

$$n=2 \ \to \ b_2 = -\frac{7}{2}\frac{3}{22}b_1 = -\frac{21}{44}\frac{-7}{18}b_0 = \frac{49}{264}b_0$$

$$n=3 \ \to \ b_3 = -\frac{7}{2}\frac{5}{39}b_2 = -\frac{35}{78}\frac{49}{264}b_0 = -\frac{1,715}{20,592}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/2}\left(1-\frac{7}{18}x+\frac{49}{264}x^2-\frac{1,715}{20,592}x^3+\cdots\right)}$$

$$\boxed{y(x) = a_0\frac{1}{x^3}\left(1-\frac{21}{5}x+\frac{49}{5}x^2-\frac{343}{15}x^3\right)+b_0\sqrt{x}\left(1-\frac{7}{18}x+\frac{49}{264}x^2-\frac{1,715}{20,592}x^3+\cdots\right)}$$

Exercise

Find the Frobenius series solutions: $x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$ (*Gauss' Hypergeometric*)

Solution

$$y'' + \frac{c-(a+b+1)x}{x(1-x)}y' - \frac{ab}{x(1-x)}y = 0$$

That implies to $p(x) = \dfrac{c-(a+b+1)x}{x(1-x)}$ and $q(x) = -\dfrac{ab}{x(1-x)}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(\frac{c-(a+b+1)x}{x(1-x)}\right) = \lim_{x\to 0}\left(\frac{c-(a+b+1)x}{1-x}\right) \boxed{= c}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = -\lim_{x\to 0} x^2 \frac{ab}{x(1-x)} = -\lim_{x\to 0}\frac{abx}{1-x} \underline{=0|}$$

$$p_1 = \lim_{x\to 1}(x-1)\,p(x) = \lim_{x\to 1}(x-1)\left(\frac{c-(a+b+1)x}{x(1-x)}\right) = \lim_{x\to 1}\left(-\frac{c-(a+b+1)x}{x}\right) \underline{=a+b+1-c|}$$

$$q_1 = \lim_{x\to 1}(x-1)^2 q(x) = -\lim_{x\to 1}(x-1)^2\frac{ab}{x(1-x)} = \lim_{x\to 1}\frac{ab}{x}(x-1) \underline{=0|}$$

The *Regular* singular points: $\underline{x = 0,\ 1\,|}$

The indicial equation is: $r(r-1) - cr = r^2 + (c-1)r = 0 \quad \to \quad \underline{r = 0,\ 1-c\,|}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1-c}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$x(1-x)y'' + \left[c-(a+b+1)x\right]y' - aby = 0$$

$$\left(x-x^2\right)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \left[c-(a+b+1)x\right]\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - ab\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}c(n+r)a_n x^{n+r-1}$$

$$-\sum_{n=0}^{\infty}(a+b+1)(n+r)a_n x^{n+r} - ab\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1) + c(n+r)\right]a_n x^{n+r-1}$$

$$-\sum_{n=0}^{\infty}\left[(n+r)(n+r-1) + (a+b+1)(n+r) + ab\right]a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1+c)a_n x^{n+r-1}-\sum_{n=1}^{\infty}\left[(n+r-1)(n+r-2+a+b+1)+ab\right]a_{n-1}x^{n+r-1}=0$$

$$r(r+c-1)a_0 x^{r-1}+\sum_{n=1}^{\infty}(n+r)(n+r-1+c)a_n x^{n+r-1}$$

$$-\sum_{n=1}^{\infty}\left[(n+r-1)(n+r-1+a+b)+ab\right]a_{n-1}x^{n+r-1}=0$$

$$r(r+c-1)a_0 x^{r-1}+\sum_{n=1}^{\infty}\left[(n+r)(n+r-1+c)a_n-\left((n+r-1)(n+r-1+a+b)+ab\right)a_{n-1}\right]x^{n+r-1}=0$$

For $n=0 \quad \rightarrow \quad r(r+c-1)a_0=0 \quad \Rightarrow \quad \underline{r=0,\ 1-c}$ √

$$(n+r)(n+r-1+c)a_n-\left((n+r-1)(n+r-1+a+b)+ab\right)a_{n-1}=0$$

$$(n+r)(n+r-1+c)a_n=\left((n+r-1)(n+r-1+a+b)+ab\right)a_{n-1}$$

$$\underline{a_n=\frac{(n+r-1)(n+r-1+a+b)+ab}{(n+r)(n+r-1+c)}a_{n-1}}$$

$$r=0 \quad \rightarrow \quad \underline{a_n=\frac{(n-1)(n-1+a+b)+ab}{n(n-1+c)}a_{n-1}}$$

$$n=1 \rightarrow a_1=\frac{ab}{c}a_0$$

$$n=2 \rightarrow a_2=\frac{1+a+b+ab}{2\cdot(c+1)}a_1=\frac{(a+1)(b+1)}{2\cdot(c+1)}a_1=\frac{ab(a+1)(b+1)}{2\cdot c\cdot(c+1)}a_0$$

$$n=3 \rightarrow a_3=\frac{4+2a+2b+ab}{3\cdot(c+2)}a_2=\frac{(a+2)(b+2)}{3\cdot(c+2)}a_2=\frac{a(a+1)(a+2)\cdot b(b+1)(b+2)}{2\cdot3\cdot c(c+1)(c+2)}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\rightarrow a_n=\frac{a(a+1)(a+2)\cdots(a+n-1)\bullet b(b+1)(b+2)\cdots(b+n-1)}{n!\cdot c(c+1)(c+2)\cdots(c+n-1)}a_0$$

$$y_1(x)=a_0\left(1+\frac{ab}{c}x+\frac{ab(a+1)(b+1)}{2\cdot c\cdot(c+1)}x^2+\frac{a(a+1)(a+2)\cdot b(b+1)(b+2)}{2\cdot3\cdot c(c+1)(c+2)}x^3+\cdots\right)$$

$$\underline{=a_0\left(1+\sum_{n=0}^{\infty}\frac{a(a+1)\cdots(a+n-1)\bullet b(b+1)\cdots(b+n-1)}{n!\cdot c(c+1)\cdots(c+n-1)}x^n\right)}$$

$$r=1-c \quad \rightarrow \quad \underline{b_n=\frac{(n-c)(n-c+a+b)+ab}{n(n+1-c)}b_{n-1}}$$

$$n=1 \;\rightarrow\; b_1 = \frac{(1-c)(1-c+a+b)+ab}{2-c}b_0$$

$$n=2 \;\rightarrow\; b_2 = \frac{(2-c)(2-c+a+b)+ab}{2(3-c)}b_1 = \frac{(1-c)(1-c+a+b)+ab}{2-c}b_0$$

$$= \frac{\left((2-c)(2-c+a+b)+ab\right)\left((1-c)(1-c+a+b)+ab\right)}{2(2-c)(3-c)}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\rightarrow b_n = \frac{\left((n-c)(n-c+a+b)+ab\right)\cdots\left((1-c)(1-c+a+b)+ab\right)}{2(2-c)(3-c)\ \cdots(n+1-c)}b_0$$

$$y_2(x) = b_0 x^{1-c}\left(\begin{array}{l} 1+\dfrac{(1-c)(1-c+a+b)+ab}{2-c}x \\ +\dfrac{\left((2-c)(2-c+a+b)+ab\right)\left((1-c)(1-c+a+b)+ab\right)}{2(2-c)(3-c)}x^2+\cdots \end{array}\right)$$

$$y(x) = a_0\left(1+\sum_{n=0}^{\infty}\frac{a(a+1)\cdots(a+n-1)\cdot b(b+1)\cdots(b+n-1)}{n!\cdot c(c+1)\cdots(c+n-1)}x^n\right)$$
$$+b_0 x^{1-c}\left(1+\sum_{n=0}^{\infty}\frac{\left((n-c)(n-c+a+b)+ab\right)\cdots\left((1-c)(1-c+a+b)+ab\right)}{2(2-c)(3-c)\ \cdots(n+1-c)}x^n\right)$$

Solution

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + \left(x^2 - \upsilon^2\right)y = 0$: $\quad x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$

Solution

$\upsilon^2 = \frac{1}{9} \rightarrow \upsilon = \frac{1}{3}$

The general solution is: $\boxed{y(x) = c_1 J_{1/3}(x) + c_2 J_{-1/3}(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + \left(x^2 - \upsilon^2\right)y = 0$: $\quad x^2y'' + xy' + \left(x^2 - 1\right)y = 0$

Solution

$\upsilon^2 = 1 \rightarrow \upsilon = 1$

The general solution is: $\boxed{y(x) = c_1 J_1(x) + c_2 Y_1(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + \left(x^2 - \upsilon^2\right)y = 0$: $\quad 4x^2y'' + 4xy' + \left(4x^2 - 25\right)y = 0$

Solution

$\upsilon^2 = \frac{25}{4} \rightarrow \upsilon = \pm\frac{5}{2}$

The general solution is: $\boxed{y(x) = c_1 J_{5/2}(x) + c_2 J_{-5/2}(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + \left(x^2 - \upsilon^2\right)y = 0$: $\quad 16x^2y'' + 16xy' + \left(16x^2 - 1\right)y = 0$

Solution

$\upsilon^2 = \frac{1}{15} \rightarrow \upsilon = \pm\frac{1}{4}$

The general solution is: $\underline{y(x) = c_1 J_{1/4}(x) + c_2 J_{-1/4}(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2 y'' + xy' + \left(x^2 - \upsilon^2\right) y = 0$: $\quad xy'' + y' + xy = 0$

Solution

$\upsilon^2 = 0 \ \rightarrow \ \upsilon = 0$

The general solution is: $\underline{y(x) = c_1 J_0(x) + cY_0(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2 y'' + xy' + \left(x^2 - \upsilon^2\right) y = 0$: $\quad xy'' + y' + \left(x - \frac{4}{x}\right) y = 0$

Solution

$\upsilon^2 = 4 \ \rightarrow \ \upsilon = 2$

The general solution is: $\underline{y(x) = c_1 J_2(x) + c_2 Y_2(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2 y'' + xy' + \left(\alpha^2 x^2 - \upsilon^2\right) y = 0$: $\quad x^2 y'' + xy' + \left(9x^2 - 4\right) y = 0$

Solution

$$\begin{cases} \alpha^2 = 9 \ \rightarrow \ \alpha = 3 \\ \upsilon^2 = 4 \ \rightarrow \ \upsilon = 2 \end{cases}$$

The general solution is: $\underline{y(x) = c_1 J_2(3x) + c_2 Y_2(3x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2 y'' + xy' + \left(\alpha^2 x^2 - \upsilon^2\right) y = 0$: $\quad x^2 y'' + xy' + \left(36x^2 - \frac{1}{4}\right) y = 0$

Solution

$$\begin{cases} \alpha^2 = 36 \rightarrow \alpha = 6 \\ \upsilon^2 = \frac{1}{4} \rightarrow \upsilon = \frac{1}{2} \end{cases}$$

The general solution is: $\underline{y(x) = c_1 J_{1/2}(6x) + c_2 J_{-1/2}(6x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + (\alpha^2x^2 - \upsilon^2)y = 0$: $\qquad x^2y'' + xy' + \left(25x^2 - \frac{4}{9}\right)y = 0$

Solution

$$\begin{cases} \alpha^2 = 25 \rightarrow \alpha = 5 \\ \upsilon^2 = \frac{4}{9} \rightarrow \upsilon = \frac{2}{3} \end{cases}$$

The general solution is: $\underline{y(x) = c_1 J_{2/3}(5x) + c_2 J_{-2/3}(5x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + (\alpha^2x^2 - \upsilon^2)y = 0$: $\qquad x^2y'' + xy' + (2x^2 - 64)y = 0$

Solution

$$\begin{cases} \alpha^2 = 2 \rightarrow \alpha = \sqrt{2} \\ \upsilon^2 = 64 \rightarrow \upsilon = 8 \end{cases}$$

The general solution is: $\underline{y(x) = c_1 J_8(\sqrt{2}x) + c_2 Y_8(\sqrt{2}x)}$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$4x^2y'' + 8xy' + (x^4 - 3)y = 0$$

Solution

$$\frac{1}{4}\times \quad 4x^2y'' + 8xy' + (x^4 - 3)y = 0$$

$$x^2y'' + 2xy' + \left(-\frac{3}{4} + \frac{1}{4}x^4\right)y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + (B + Cx^p)y = 0$$

$A=2,\quad B=-\frac{3}{4},\quad C=\frac{1}{4},\quad p=4$

$\alpha=\frac{1-2}{2}=-\frac{1}{2},\quad \beta=\frac{4}{2}=2,\quad k=\frac{2\sqrt{\frac{1}{4}}}{4}=\frac{1}{4},\quad \upsilon=\frac{\sqrt{1+3}}{4}=\frac{1}{2}$

$y(x)=x^{-1/2}\left[c_1J_{1/2}\left(\frac{1}{4}x^2\right)+c_2J_{-1/2}\left(\frac{1}{4}x^2\right)\right]$ $\qquad y(x)=x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right)+c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$

$=x^{-1/2}\left(c_1\sqrt{\frac{2}{\pi z}}\sin z+c_2\sqrt{\frac{2}{\pi z}}\cos z\right)$ $\qquad =c_1\left(\frac{2}{\pi x}\right)^{1/2}\sin x+c_2\left(\frac{2}{\pi x}\right)^{1/2}\cos x$

$=x^{-1/2}\left(c_1\frac{2}{x}\sqrt{\frac{2}{\pi}}\sin\frac{x^2}{4}+c_2\frac{2}{x}\sqrt{\frac{2}{\pi}}\cos\frac{x^2}{4}\right)$ $\qquad z=kx^{\beta}=\frac{x^2}{4}$

$=x^{-3/2}\left(C_1\sqrt{\frac{2}{\pi}}\sin\frac{x^2}{4}+C_2\sqrt{\frac{2}{\pi}}\cos\frac{x^2}{4}\right)$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y''+9xy=0$$

Solution

$x^2\times\quad y''+9xy=0$

$x^2y''+9x^3y=0$ $\qquad x^2\frac{d^2y}{dx^2}+Ax\frac{dy}{dx}+\left(B+Cx^p\right)y=0$

$A=0,\quad B=0,\quad C=9,\quad p=3$

$\alpha=\frac{1}{2},\quad \beta=\frac{3}{2},\quad k=2,\quad \upsilon=\frac{1}{3}$ $\qquad \alpha=\frac{1-A}{2},\ \beta=\frac{p}{2},\ k=\frac{2\sqrt{C}}{p},\ \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}$

$y(x)=x^{1/2}\left[c_1J_{1/3}\left(2x^{3/2}\right)+c_2J_{-1/3}\left(2x^{3/2}\right)\right]$ $\qquad y(x)=x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right)+c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy''+(x-3)y=0$$

Solution

$x\times\quad xy''-3y+xy=0$

$x^2y''-3xy+x^2y=0$ $\qquad x^2\frac{d^2y}{dx^2}+Ax\frac{dy}{dx}+\left(B+Cx^p\right)y=0$

$A=-3,\quad B=0,\quad C=1,\quad p=2$

$\alpha = 2, \quad \beta = 1, \quad k = 1, \quad \upsilon = \frac{\sqrt{16}}{2} = 2$ $\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2-4B}}{p}$

$$\boxed{y(x) = x^2\left[c_1 Y_2(x) + c_2 J_2(x)\right]} \qquad y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' + \left(4x^3 - 1\right)y = 0$$

Solution

$x\times \quad xy'' - y + 4x^3 y = 0$

$x^2y'' - xy + 4x^4 y = 0 \qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$

$A = -1, \quad B = 0, \quad C = 4, \quad p = 4$

$\alpha = 1, \quad \beta = 2, \quad k = 1, \quad \upsilon = \frac{1}{2}$ $\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2-4B}}{p}$

$$y(x) = x\left[c_1 J_{1/2}\left(x^2\right) + c_2 J_{-1/2}\left(x^2\right)\right] \qquad y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

$$= x\left(c_1\frac{1}{x}\sqrt{\frac{2}{\pi}}\sin x^2 + c_2\frac{1}{x}\sqrt{\frac{2}{\pi}}\cos x^2\right) \qquad y(z) = x^{\alpha}\left(c_1\left(\frac{2}{\pi z}\right)^{1/2}\sin z + c_2\left(\frac{2}{\pi z}\right)^{1/2}\cos z\right)$$

$$= c_1\sqrt{\frac{2}{\pi}}\sin x^2 + c_2\sqrt{\frac{2}{\pi}}\cos x^2 \qquad z = kx^{\beta} = x^2$$

$$\boxed{= C_1\sin x^2 + C_2\cos x^2}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2y'' + xy' - \left(\frac{1}{4} + x^2\right)y = 0$$

Solution

$x^2y'' + xy' + \left(-\frac{1}{4} - x^2\right)y = 0 \qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$

$A = 1, \quad B = -\frac{1}{4}, \quad C = -1, \quad p = 2$

$\alpha = 0, \quad \beta = 1, \quad k = i, \quad \upsilon = \frac{1}{2}$ $\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2-4B}}{p}$

$y(x)=c_1 I_{1/2}(x)+c_2 I_{-1/2}(x)$

$y(x)=x^{\alpha}\left[c_1 I_{\upsilon}\left(kx^{\beta}\right)+c_2 I_{-\upsilon}\left(kx^{\beta}\right)\right]$

$$\boxed{y(x)=c_1\sqrt{\frac{2}{\pi x}}\sinh x+c_2\sqrt{\frac{2}{\pi x}}\cosh x}$$

$I_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sinh x$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy''+(2x+1)y'+(2x+1)y=0$$

Solution

$x\times \quad xy''+(2x+1)y'+(2x+1)y=0$

$x^2y''+x(2x+1)y'+\left(2x^2+x\right)y=0$

Let $Y=ye^{x} \quad \to \ y=Ye^{-x}$

$x^2\left(Y''-2Y'+Y\right)e^{-x}+x(2x+1)\left(Y'-Y\right)e^{-x}+\left(2x^2+x\right)Ye^{-x}=0$

$x^2Y''-2x^2Y'+x^2Y+\left(2x^2+x\right)Y'-\left(2x^2+x\right)Y+\left(2x^2+x\right)Y=0$

$x^2Y''+xY'+x^2Y=0$

$x^2\frac{d^2y}{dx^2}+Ax\frac{dy}{dx}+\left(B+Cx^p\right)y=0$

$A=1, \quad B=0, \quad C=1, \quad p=2$

$\alpha=0, \quad \beta=1, \quad k=1, \quad \upsilon=0$

$\alpha=\frac{1-A}{2}, \ \beta=\frac{p}{2}, \ k=\frac{2\sqrt{C}}{p}, \ \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}$

$Y(x)=c_1J_0(x)+c_2Y_0(x)$

$y(x)=x^{\alpha}\left[c_1J_0\left(kx^{\beta}\right)+c_2Y_0\left(kx^{\beta}\right)\right]$

$$\boxed{y(x)=\left(c_1J_0(x)+c_2Y_0(x)\right)e^{-x}}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy''-y'-xy=0$$

Solution

$x\times \quad xy''-y'-xy=0$

$x^2y''-xy'-x^2y=0$

$x^2\frac{d^2y}{dx^2}+Ax\frac{dy}{dx}+\left(B+Cx^p\right)y=0$

$A=-1, \quad B=0, \quad C=-1=i, \quad p=2$

Let $Y=\frac{y}{x}$ & $X=ix$

$$y = xY \quad \& \quad x = -iX$$

$$x^2\left(2Y' + xY''\right) - x\left(Y + xY'\right) - x^3 Y = 0$$

$$x^3 Y'' + x^2 Y' - x\left(x^2 + 1\right)Y = 0$$

$$x^2 Y'' + xY' - \left(x^2 + 1\right)Y = 0$$

$$-X^2 Y'' - iXY' - \left(-X^2 + 1\right)Y = 0$$

$$X^2 Y'' + XY' + \left(X^2 - 1\right)Y = 0$$

$$x^2 \frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$A = 1, \quad B = -1, \quad C = 1, \quad p = 2$

$\alpha = 0, \quad \beta = 1, \quad k = 1, \quad \upsilon = 1$

$$\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$Y = Z_1(X)$$

$$y(x) = xZ_1(ix)$$

$$= x\left(c_1 I_1(x) + c_2 K_1(x)\right)$$

$$y(x) = x^{\alpha}\left[c_1 J_0\left(kx^{\beta}\right) + c_2 Y_0\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^4 y'' + a^2 y = 0$$

Solution

$$\frac{1}{x^2}\times \quad x^4 y'' + a^2 y = 0$$

$$x^2 y'' + \frac{a^2}{x^2} y = 0$$

Let $Y = \dfrac{y}{\sqrt{x}} \quad \to y = \sqrt{x}\, Y$

$$X = \frac{a}{x} \quad \to \quad x = \frac{a}{X}$$

$$X^2 Y'' + XY' + \left(X^2 - K^2\right)Y = 0$$

$$Y = x^{-1/2} y$$

$$Y' = -\frac{1}{2}x^{-3/2} y + x^{-1/2} y'$$

$$Y'' = \frac{3}{4}x^{-5/2} y - x^{-3/2} y' + x^{-1/2} y''$$

$$x^2\left(x^{-1/2} y'' - x^{-3/2} y' + \frac{3}{4}x^{-5/2} y\right) + x\left(-\frac{1}{2}x^{-3/2} y + x^{-1/2} y'\right) + \left(x^2 - K^2\right)x^{-1/2} y = 0$$

$$x^{3/2}y'' - x^{1/2}y' + \frac{3}{4}x^{-1/2}y - \frac{1}{2}x^{-1/2}y + x^{1/2}y' + \left(x^2 - K^2\right)x^{-1/2}y = 0$$

$$x^{3/2}y'' + \left(x^2 - K^2 + \frac{1}{4}\right)x^{-1/2}y = 0 \quad \times x^{1/2}$$

$$x^2y'' + \left(x^2 - K^2 + \frac{1}{4}\right)y = 0$$

$$x^2 - K^2 + \frac{1}{4} = x^2$$

$$-K^2 + \frac{1}{4} = 0 \quad \to \quad K^2 = \frac{1}{4}$$

$$X^2Y'' + XY' + \left(X^2 - \frac{1}{4}\right)Y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = 1, \quad B = -\frac{1}{4}, \quad C = 1, \quad p = 2$$

$$\alpha = 0, \quad \beta = 1, \quad k = 1, \quad \upsilon = \frac{1}{2} \qquad\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$Y = Z_{1/2}(X)$$

$$y(x) = \sqrt{x}Z_{1/2}\left(\frac{a}{x}\right) \qquad\qquad y(x) = x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right) + c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

$$= \sqrt{x}\left(c_1J_{1/2}\left(\frac{a}{x}\right) + c_2J_{-1/2}\left(\frac{a}{x}\right)\right) \qquad\qquad y(z) = x^{\alpha}\left(c_1\left(\frac{2}{\pi z}\right)^{1/2}\sin z + c_2\left(\frac{2}{\pi z}\right)^{1/2}\cos z\right)$$

$$= \sqrt{x}\left(c_1\sqrt{\frac{2x}{\pi a}}\sin\frac{a}{x} + c_2\sqrt{\frac{2x}{\pi a}}\cos\frac{a}{x}\right) \qquad\qquad z = kX^{\beta} = \frac{a}{x}$$

$$= x\left(c_1\sqrt{\frac{2}{\pi a}}\sin\frac{a}{x} + c_2\sqrt{\frac{2}{\pi a}}\cos\frac{a}{x}\right)$$

$$\boxed{= x\left(C_1\sin\frac{a}{x} + C_2\cos\frac{a}{x}\right)}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y'' - x^2y = 0$$

Solution

$$x^2\times \quad y'' - x^2y = 0$$

$$x^2y'' - x^4y = 0$$

$$A = 0, \quad B = 0, \quad C = -1, \quad p = 4$$

$$\alpha = \frac{1}{2}, \quad \beta = 1, \quad k = \frac{i}{2}, \quad \upsilon = 0 \qquad\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

Let $Y = \frac{y}{\sqrt{x}} \quad \to y = \sqrt{x}\,Y$

$$X=\frac{1}{2}ix^2 \quad\to\quad x^2=-2iX$$

$$X^2Y''+XY'+\left(X^2-K^2\right)Y=0$$

$$Y=x^{-1/2}y$$

$$Y'=-\frac{1}{2}x^{-3/2}y+x^{-1/2}y'$$

$$Y''=\frac{3}{4}x^{-5/2}y-x^{-3/2}y'+x^{-1/2}y''$$

$$x^2\left(x^{-1/2}y''-x^{-3/2}y'+\frac{3}{4}x^{-5/2}y\right)+x\left(-\frac{1}{2}x^{-3/2}y+x^{-1/2}y'\right)+\left(x^2-K^2\right)x^{-1/2}y=0$$

$$x^{3/2}y''-x^{1/2}y'+\frac{3}{4}x^{-1/2}y-\frac{1}{2}x^{-1/2}y+x^{1/2}y'+\left(x^2-K^2\right)x^{-1/2}y=0$$

$$x^{3/2}y''+\left(x^2-K^2+\frac{1}{4}\right)x^{-1/2}y=0 \quad \times x^{1/2}$$

$$x^2y''+\left(x^2-K^2+\frac{1}{4}\right)y=0$$

$$x^2-K^2+\frac{1}{4}=-x^4$$

$$K=\frac{1}{4} \quad\to\quad K^2=\frac{1}{16}$$

$$X^2Y''+XY'+\left(X^2-\frac{1}{16}\right)Y=0$$

$$A=1, \quad B=-\frac{1}{16}, \quad C=1, \quad p=2$$

$$\alpha=0, \quad \beta=1, \quad k=1, \quad \upsilon=\frac{1}{4}$$

$$\alpha=\frac{1-A}{2}, \ \beta=\frac{p}{2}, \ k=\frac{2\sqrt{C}}{p}, \ \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}$$

$$Y=Z_{1/4}(X)$$

$$y(x)=\sqrt{x}Z_{\frac{1}{4}}\left(\frac{i}{2}x^2\right)$$

$$\boxed{=\sqrt{x}\left(c_1I_{\frac{1}{4}}\left(\frac{x^2}{2}\right)+c_2I_{-\frac{1}{4}}\left(\frac{x^2}{2}\right)\right)}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2y''-xy'+\left(1+x^2\right)y=0$$

Solution

$$x^2y''-xy'+\left(1+x^2\right)y=0$$

$$x^2\frac{d^2y}{dx^2}+Ax\frac{dy}{dx}+\left(B+Cx^p\right)y=0$$

$$A=-1, \quad B=1, \quad C=1, \quad p=2$$

$\alpha = 1, \quad \beta = 1, \quad k = 1, \quad \upsilon = 0$ $\qquad$ $\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$

$$\underline{y(x) = x\left[c_1 J_0(x) + c_2 Y_0(x)\right]}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' + 3y' + xy = 0$$

Solution

$x \times \quad xy'' + 3y' + xy = 0$

$x^2y'' + 3xy' + x^2y = 0$ $\qquad$ $x^2 \frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$

$A = 3, \quad B = 0, \quad C = 1, \quad p = 2$

$\alpha = -1, \quad \beta = 1, \quad k = 1, \quad \upsilon = 1$ $\qquad$ $\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$

$$\underline{y(x) = x^{-1}\left[c_1 J_1(x) + c_2 Y_1(x)\right]}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' - y' + 36x^3y = 0$$

Solution

$x \times \quad xy'' - y' + 36x^3y = 0$

$x^2y'' - xy' + 36x^4y = 0$ $\qquad$ $x^2 \frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$

$A = -1, \quad B = 0, \quad C = 36, \quad p = 4$

$\alpha = 1, \quad \beta = 2, \quad k = 3, \quad \upsilon = \frac{1}{2}$ $\qquad$ $\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$

$\underline{y(x) = x\left[c_1 J_{1/2}\left(3x^2\right) + c_2 J_{-1/2}\left(3x^2\right)\right]}$ $\qquad$ $y(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2y'' - 5xy' + (8+x)y = 0$$

Solution

$$x^2y'' - 5xy' + (8+x)y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = -5, \quad B = 8, \quad C = 1, \quad p = 1$$

$$\alpha = 3, \quad \beta = \frac{1}{2}, \quad k = 2, \quad \upsilon = 2 \qquad\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\boxed{y(x) = x^3\left[c_1J_2\left(2x^{1/2}\right) + c_2Y_2\left(2x^{1/2}\right)\right]} \qquad\qquad y(x) = x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right) + c_2Y_{\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$36x^2y'' + 60xy' + \left(9x^3 - 5\right)y = 0$$

Solution

$$x^2y'' + \frac{5}{3}xy' + \left(\frac{1}{4}x^3 - \frac{5}{36}\right)y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = \frac{5}{3}, \quad B = -\frac{5}{36}, \quad C = \frac{1}{4}, \quad p = 3$$

$$\alpha = -\frac{1}{3}, \quad \beta = \frac{3}{2}, \quad k = \frac{1}{3}, \quad \upsilon = \frac{\sqrt{\left(-\frac{2}{3}\right)^2 + \frac{5}{9}}}{3} = \frac{1}{3} \qquad\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\boxed{y(x) = x^{-1/3}\left[c_1J_{1/3}\left(\frac{1}{3}x^{3/2}\right) + c_2J_{-1/3}\left(\frac{1}{3}x^{3/2}\right)\right]} \qquad\qquad y(x) = x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right) + c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$16x^2y'' + 24xy' + \left(1 + 144x^3\right)y = 0$$

Solution

$$x^2y'' + \frac{3}{2}xy' + \left(\frac{1}{16} + 9x^3\right)y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = \frac{3}{2}, \quad B = \frac{1}{16}, \quad C = 9, \quad p = 3$$

$$\alpha = -\frac{1}{4}, \quad \beta = \frac{3}{2}, \quad k = 2, \quad \upsilon = \frac{\sqrt{\left(-\frac{1}{2}\right)^2 - \frac{1}{4}}}{3} = 0 \qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\underline{y(x) = x^{-1/4}\left[c_1 J_0\left(2x^{3/2}\right) + c_2 Y_0\left(2x^{3/2}\right)\right]} \qquad y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 Y_{\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2 y'' + 3xy' + \left(1 + x^2\right) y = 0$$

Solution

$$x^2 y'' + 3xy' + \left(1 + x^2\right) y = 0 \qquad x^2 \frac{d^2 y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right) y = 0$$

$$A = 3, \quad B = 1, \quad C = 1, \quad p = 2$$

$$\alpha = -1, \quad \beta = 1, \quad k = 1, \quad \upsilon = \frac{\sqrt{(-2)^2 - 4}}{3} = 0 \qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\underline{y(x) = x^{-1}\left[c_1 J_0(x) + c_2 Y_0(x)\right]} \qquad y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 Y_{\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$4x^2 y'' - 12xy' + (15 + 16x) y = 0$$

Solution

$$x^2 y'' - 3xy' + \left(\frac{15}{4} + 4x\right) y = 0 \qquad x^2 \frac{d^2 y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right) y = 0$$

$$A = -3, \quad B = \frac{15}{4}, \quad C = 4, \quad p = 1$$

$$\alpha = 2, \quad \beta = \frac{1}{2}, \quad k = 4, \quad \upsilon = \frac{\sqrt{(4)^2 - 15}}{1} = 1 \qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\underline{y(x) = x^{2}\left[c_1 J_1\left(4x^{1/2}\right) + c_2 Y_1\left(4x^{1/2}\right)\right]} \qquad y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 Y_{\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$16x^2y'' - \left(5 - 144x^3\right)y = 0$$

Solution

$$x^2y'' + \left(9x^3 - \frac{5}{16}\right)y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = 0, \quad B = -\frac{5}{16}, \quad C = 9, \quad p = 3$$

$$\alpha = \frac{1}{2}, \quad \beta = \frac{3}{2}, \quad k = 2, \quad \upsilon = \frac{\sqrt{1+\frac{5}{4}}}{3} = \frac{1}{2} \qquad\qquad \alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\boxed{y(x) = x^{1/2}\left[c_1J_{1/2}\left(2x^{3/2}\right) + c_2J_{-1/2}\left(2x^{3/2}\right)\right]} \qquad\qquad y(x) = x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right) + c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$2x^2y'' + 3xy' - \left(28 - 2x^5\right)y = 0$$

Solution

$$x^2y'' + \frac{3}{2}xy' + \left(x^5 - 14\right)y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = \frac{3}{2}, \quad B = -14, \quad C = 1, \quad p = 5$$

$$\alpha = -\frac{1}{4}, \quad \beta = \frac{5}{2}, \quad k = \frac{2}{5}, \quad \upsilon = \frac{\sqrt{\left(-\frac{1}{2}\right)^2 + 56}}{5} = \frac{\frac{15}{2}}{5} = \frac{3}{2} \qquad \alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\boxed{y(x) = x^{-1/4}\left[c_1J_{3/2}\left(\frac{2}{5}x^{5/2}\right) + c_2J_{-3/2}\left(\frac{2}{5}x^{5/2}\right)\right]} \qquad y(x) = x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right) + c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y'' + x^4y = 0$$

Solution

$$x^2 \times \quad y'' + x^4y = 0$$

$$x^2y'' + x^6y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$A=0, \quad B=0, \quad C=1, \quad p=6$

$\alpha=\frac{1}{2}, \quad \beta=3, \quad k=\frac{1}{3}, \quad \upsilon=\frac{1}{6}$ $\qquad \alpha=\frac{1-A}{2},\ \beta=\frac{p}{2},\ k=\frac{2\sqrt{C}}{p},\ \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}$

$\underline{y(x)=x^{1/2}\left[c_1 J_{1/6}\left(\frac{1}{3}x^3\right)+c_2 J_{-1/6}\left(\frac{1}{3}x^3\right)\right]}$ $\qquad y(x)=x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right)+c_2 J_{-\upsilon}\left(kx^{\beta}\right)\right]$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y''+4x^3y=0$$

Solution

$x^2\times \quad y''+4x^3y=0$

$x^2y''+4x^5y=0$ $\qquad x^2\frac{d^2y}{dx^2}+Ax\frac{dy}{dx}+\left(B+Cx^p\right)y=0$

$A=0, \quad B=0, \quad C=4, \quad p=5$

$\alpha=\frac{1}{2}, \quad \beta=\frac{5}{2}, \quad k=\frac{4}{5}, \quad \upsilon=\frac{1}{5}$ $\qquad \alpha=\frac{1-A}{2},\ \beta=\frac{p}{2},\ k=\frac{2\sqrt{C}}{p},\ \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}$

$\underline{y(x)=x^{1/2}\left[c_1 J_{1/5}\left(\frac{4}{5}x^{5/2}\right)+c_2 J_{-1/5}\left(\frac{4}{5}x^{5/2}\right)\right]}$ $\qquad y(x)=x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right)+c_2 J_{-\upsilon}\left(kx^{\beta}\right)\right]$

Exercise

Find a Frobenius solution of Bessel's equation of order zero $x^2y''+xy'+x^2y=0$

Solution

$$y''+\frac{1}{x}y'+y=0$$

Therefore, $x=0$ is a regular singular point, and that $p_0=1, \quad q_0=0$ and $p(x)\equiv 1, \quad q(x)=x^2$.

The indicial equation is: $r(r-1)+r=r^2=0 \quad \rightarrow \quad \boxed{r=0}$

There is only one Frobenius series solution: $y(x)=\sum_{n=0}^{\infty}a_nx^n$

$$y=\sum_{n=0}^{\infty}a_nx^n$$

$$y'=\sum_{n=0}^{\infty}na_nx^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$$

$$x^2 y'' + xy' + x^2 y = 0$$

$$x^2 \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=0}^{\infty} na_n x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} [n(n-1)+n]a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\sum_{n=0}^{\infty} n^2 a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0 \quad a_4 = -\frac{a_2}{4^2} = \frac{a_0}{2^2 \cdot 4^2}$$

$$0 + a_1 x + \sum_{n=2}^{\infty} n^2 a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$a_1 x + \sum_{n=2}^{\infty} \left(n^2 a_n + a_{n-2}\right) x^n = 0$$

$$a_1 = 0 \quad \to \quad a_{n(odd)} = 0$$

$$n^2 a_n + a_{n-2} = 0 \quad \Rightarrow \quad a_n = -\frac{a_{n-2}}{n^2} \quad (n \geq 2)$$

$$a_2 = -\frac{a_0}{4}$$

$$a_6 = -\frac{a_4}{6^2} = -\frac{a_0}{2^2 \cdot 4^2 \cdot 6^2}$$

$$a_{2n} = \frac{(-1)^n}{2^2 \cdot 4^2 \cdots (2n)^2} a_0 = \frac{(-1)^n}{2^{2n} \cdot (n!)^2} a_0$$

The choice $a_0 = 1$ gives us the Bessel function of order zero of the first kind.

$$J_0(x) = \frac{(-1)^n x^{2n}}{2^{2n} \cdot (n!)^2} \; \underline{= 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots}$$

Exercise

Derive the formula $\quad x J'_{\upsilon}(x) = \upsilon J_{\upsilon}(x) - x J_{\upsilon+1}(x)$

Solution

$$x J_{\upsilon}(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$x J'_{\upsilon}(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\upsilon)}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\upsilon)}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= 2\sum_{n=0}^{\infty} \frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon J_{v}(x)$$

$$= \upsilon J_{v}(x) + x \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!\,\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \upsilon J_{v}(x) + x \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n!\,\Gamma(2+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon+1}$$

$$\underline{= \upsilon J_{\upsilon}(x) - x J_{\upsilon+1}(x)} \quad \checkmark$$

Exercise

Derive the formula $\quad x J'_{\upsilon}(x) = -\upsilon J_{\upsilon}(x) + x J_{\upsilon-1}(x)$

Solution

$$x J'_{\upsilon}(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\upsilon)}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= 2\sum_{n=0}^{\infty} \frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$-\upsilon J_{\upsilon}(x)+xJ_{\upsilon-1}(x)=-\upsilon\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}+x\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$=-\sum_{n=0}^{\infty}\frac{(-1)^n\upsilon}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$=-\sum_{n=0}^{\infty}\frac{(-1)^n\upsilon}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}+\sum_{n=0}^{\infty}\frac{(-1)^n(\upsilon+n)}{n!\Gamma(1+\upsilon+n)}2\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(-\upsilon+2n+2\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$=x\sum_{n=0}^{\infty}\frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$\underline{=xJ'_{\upsilon}(x)\,\big|}\quad\checkmark$$

Exercise

Derive the formula $\quad 2\upsilon J'_{\upsilon}(x)=xJ_{\upsilon+1}(x)+xJ_{\upsilon-1}(x)$

Solution

From previous proofs:

$$\begin{array}{rl} & xJ'_{\upsilon}(x)=\upsilon J_{\upsilon}(x)-xJ_{\upsilon+1}(x) \\ - & xJ'_{\upsilon}(x)=-\upsilon J_{\upsilon}(x)+xJ_{\upsilon-1}(x) \\ \hline & 0=2\upsilon J_{\upsilon}(x)-xJ_{\upsilon+1}(x)-xJ_{\upsilon-1}(x) \end{array}$$

$$\underline{2\upsilon J'_{\upsilon}(x)=xJ_{\upsilon+1}(x)+xJ_{\upsilon-1}(x)\,\big|}\quad\checkmark$$

Exercise

Prove that $\quad \frac{d}{dx}\left[x^{\upsilon+1}J_{\upsilon+1}(x)\right]=x^{\upsilon+1}J_{\upsilon}(x)$

Solution

$$\frac{d}{dx}\left[x^{\upsilon+1}J_{\upsilon+1}(x)\right]=\frac{d}{dx}\left[x^{\upsilon+1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+1+n)}\left(\frac{x}{2}\right)^{2n+\upsilon+1}\right]$$

$$=\frac{d}{dx}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\upsilon+n+2)}\left(\frac{x}{2}\right)^{2n+2\upsilon+2}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(2n+2\upsilon+2)}{n!\Gamma(\upsilon+n+2)}\left(\frac{x}{2}\right)^{2n+2\upsilon+1}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n 2(n+\upsilon+1)}{n!2\Gamma(\upsilon+n+2)}\left(\frac{x}{2}\right)^{2n+2\upsilon+1} \qquad 2\Gamma(\upsilon+n+2)=2(\upsilon+n+1)\Gamma(\upsilon+n+1)$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n 2(n+\upsilon+1)}{n!\,2(\upsilon+n+1)\Gamma(\upsilon+n+1)}\left(\frac{x}{2}\right)^{2n+2\upsilon+1}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(\upsilon+n+1)}\left(\frac{x}{2}\right)^{2n+2\upsilon+1}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(\upsilon+n+1)}\left(\frac{x}{2}\right)^{2n+\upsilon}\left(\frac{x}{2}\right)^{\upsilon+1}$$

$$=x^{\upsilon+1}J_{\upsilon}(x) \quad \checkmark$$

Exercise

Show that $y=\sqrt{x}\,J_{3/2}(x)$ is a solution of $x^2y''+\left(x^2-2\right)y=0$

Solution

$$x^2y''+\left(x^2-2\right)y=0$$

$J_{3/2}(x)$ is the solution of Bessel's equation of order $\frac{3}{2}$:

$$x^2J''_{3/2}(x)+xJ'_{3/2}(x)+\left(x^2-\tfrac{9}{4}\right)J_{3/2}(x)=0$$

$$x^2\left(\sqrt{x}\,J_{3/2}(x)\right)''+\left(x^2-2\right)\sqrt{x}\,J_{3/2}(x)=$$

$$=x^2\left[-\tfrac{1}{4}x^{-3/2}J_{3/2}(x)+x^{-1/2}J'_{3/2}(x)+x^{1/2}J''_{3/2}(x)\right]+\left(x^2-2\right)\sqrt{x}\,J_{3/2}(x)$$

$$=-\tfrac{1}{4}x^{1/2}J_{3/2}(x)+x^{3/2}J'_{3/2}(x)+x^{5/2}J''_{3/2}(x)+x^{5/2}J_{3/2}(x)-2\sqrt{x}\,J_{3/2}(x)$$

$$= \sqrt{x}\left[x^2 J''_{3/2}(x) + x J'_{3/2}(x) + \left(x^2 - \frac{9}{4}\right) J_{3/2}(x)\right]$$

$$\underline{= 0}$$

Exercise

Show that $\quad 4J''_{\upsilon}(x) = J_{\upsilon-2}(x) - 2J_{\upsilon}(x) + J_{\upsilon+2}(x)$

Solution

$$J_{\upsilon}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$x J'_{\upsilon}(x) = x\sum_{n=0}^{\infty} \frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= 2\sum_{n=0}^{\infty} \frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$-\upsilon J_{\upsilon}(x) + x J_{\upsilon-1}(x) = -\upsilon\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + x\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n \upsilon}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n \upsilon}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \sum_{n=0}^{\infty} \frac{(-1)^n(\upsilon+n)}{n!\Gamma(1+\upsilon+n)}2\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(-\upsilon+2n+2\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= x\sum_{n=0}^{\infty} \frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$\underline{= x J'_{\upsilon}(x)}$$

$$x\,J'_{\upsilon}(x) = x\sum_{n=0}^{\infty}\frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= 2\sum_{n=0}^{\infty}\frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon\,J_v(x)$$

$$= \upsilon\,J_v(x) + x\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-1)!\,\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \upsilon\,J_v(x) + x\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n!\,\Gamma(2+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon+1}$$

$$\underline{= \upsilon\,J_{\upsilon}(x) - x\,J_{\upsilon+1}(x)\,\big|}$$

$$x\,J'_{\upsilon}(x) = -\upsilon\,J_{\upsilon}(x) + x\,J_{\upsilon-1}(x)$$

$$\begin{array}{rr} & x\,J'_{\upsilon}(x) = \upsilon\,J_{\upsilon}(x) - x\,J_{\upsilon+1}(x) \\ + & x\,J'_{\upsilon}(x) = -\upsilon\,J_{\upsilon}(x) + x\,J_{\upsilon-1}(x) \\ \hline & 2x\,J'_{\upsilon}(x) = -x\,J_{\upsilon+1}(x) + x\,J_{\upsilon-1}(x) \end{array}$$

$$J'_{\upsilon}(x) = \frac{1}{2}\left(J_{\upsilon-1}(x) - J_{\upsilon+1}(x)\right)$$

$$J''_{\upsilon}(x) = \frac{1}{2}\left(J'_{\upsilon-1}(x) - J'_{\upsilon+1}(x)\right)$$

$$J'_{\upsilon}(x) = \frac{1}{2}\left(J_{\upsilon-1}(x) - J_{\upsilon+1}(x)\right) \quad \to (\upsilon = \upsilon - 1) \quad J'_{\upsilon-1}(x) = \frac{1}{2}\left(J_{\upsilon-2}(x) - J_{\upsilon}(x)\right)$$

$$J'_{\upsilon}(x) = \frac{1}{2}\left(J_{\upsilon-1}(x) - J_{\upsilon+1}(x)\right) \quad \to (\upsilon = \upsilon + 1) \quad J'_{\upsilon+1}(x) = \frac{1}{2}\left(J_{\upsilon}(x) - J_{\upsilon+2}(x)\right)$$

$$J''_{\upsilon}(x) = \frac{1}{2}\left(J'_{\upsilon-1}(x) - J'_{\upsilon+1}(x)\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}J_{\upsilon-2}(x) - \frac{1}{2}J_{\upsilon}(x) - \frac{1}{2}J_{\upsilon}(x) + \frac{1}{2}J_{\upsilon+2}(x)\right)$$

$$= \frac{1}{4}\left(J_{\upsilon-2}(x) - 2J_{\upsilon}(x) + J_{\upsilon+2}(x)\right)$$

$$\underline{4J''_{\upsilon}(x) = J_{\upsilon-2}(x) - 2J_{\upsilon}(x) + J_{\upsilon+2}(x)\,\big|} \quad \checkmark$$

Exercise

Show that $y = x^{1/2} w\left(\frac{2}{3}\alpha x^{3/2}\right)$ is a solution of Airy's differential equation $y'' + \alpha^2 xy = 0$, $x > 0$, whenever w is a solution of Bessel's equation of order $\frac{2}{3}$, that is, $t^2 w'' + tw' + \left(t^2 - \frac{1}{9}\right)w = 0$, $t > 0$.

[*Hint*: After differentiating, substituting, and simplifying, then let $t = \frac{2}{3}\alpha x^{3/2}$].

Solution

$$y = x^{1/2} w\left(\frac{2}{3}\alpha x^{3/2}\right)$$

$$y' = \frac{1}{2}x^{-1/2} w\left(\frac{2}{3}\alpha x^{3/2}\right) + x^{1/2}\left(\alpha x^{1/2}\right) w'\left(\frac{2}{3}\alpha x^{3/2}\right)$$

$$= \alpha x w'\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{1}{2}x^{-1/2} w\left(\frac{2}{3}\alpha x^{3/2}\right)$$

$$y'' = \alpha x\left(\alpha x^{1/2}\right) w''\left(\frac{2}{3}\alpha x^{3/2}\right) + \alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{1}{2}x^{-1/2}\left(\alpha x^{1/2}\right) w'\left(\frac{2}{3}\alpha x^{3/2}\right) - \frac{1}{4}x^{-3/2} w\left(\frac{2}{3}\alpha x^{3/2}\right)$$

$$= \alpha^2 x^{3/2} w''\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{3}{2}\alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right) - \frac{1}{4}x^{-3/2} w\left(\frac{2}{3}\alpha x^{3/2}\right)$$

$$y'' + \alpha^2 xy = 0$$

$$\alpha^2 x^{3/2} w''\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{3}{2}\alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right) - \frac{1}{4}x^{-3/2} w\left(\frac{2}{3}\alpha x^{3/2}\right) + \alpha^2 x^{3/2} w\left(\frac{2}{3}\alpha x^{3/2}\right) = 0$$

$$\alpha^2 x^{3/2} w''\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{3}{2}\alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right) + \left(\alpha^2 x^{3/2} - \frac{1}{4x^{3/2}}\right) w\left(\frac{2}{3}\alpha x^{3/2}\right) = 0$$

$$t = \frac{2}{3}\alpha x^{3/2} \quad \rightarrow \quad \alpha x^{3/2} = \frac{3}{2}t$$

$$\frac{3}{2}\frac{\alpha}{t}\left[t^2 w''(t) + tw'(t) + \left(t^2 - \frac{1}{9}\right)w(t)\right] = 0$$

$$\boxed{t^2 w'' + tw' + \left(t^2 - \frac{1}{9}\right)w = 0} \quad \checkmark$$

Exercise

Use the relation $\Gamma(x+1) = x\Gamma(x)$ and if p is nonnegative integer, then show that

$$J_\upsilon(x) = \frac{1}{\Gamma(\upsilon+1)}\left(\frac{x}{2}\right)^\upsilon \left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)}\left(\frac{x}{2}\right)^{2n}\right]$$

Solution

$$J_\upsilon(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(\upsilon+n+1)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

Given: $\Gamma(x+1) = x\Gamma(x)$

$$\Gamma(\upsilon+n+1)=(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)\Gamma(\upsilon+n)$$

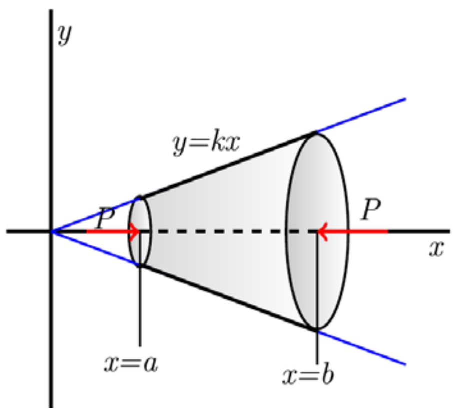
$$J_\upsilon(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)\Gamma(\upsilon+n)}\left(\frac{x}{2}\right)^{2n}\left(\frac{x}{2}\right)^{\upsilon}$$

$$=\frac{1}{\Gamma(\upsilon+1)}\left(\frac{x}{2}\right)^{\upsilon}\left[1+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)}\left(\frac{x}{2}\right)^{2n}\right] \quad \checkmark$$

Exercise

A linearly tapered rod with circular cross section, subject to an axial force *P* of compression. Its deflection curve $y=y(x)$ satisfies the endpoint value problem

$$EIy''+Py=0\ ; \quad y(a)=y(b)=0 \quad (1)$$



Here, however, the moment of inertia $I=I(x)$ of the cross section at *x* is given by

$$I(x)=\frac{1}{4}\pi(kx)^4=I_0\left(\frac{x}{b}\right)^4 \quad (2)$$

Where $I_0=I(b)$, the value of *I* at $x=b$. Substitution of $I(x)$ in the differential equation (1) yields to the eigenvalue problem

$$x^4y''+\lambda y=0\ ; \quad y(a)=y(b)=0 \quad (3)$$

Where $\lambda=\mu^2=\dfrac{Pb^4}{EI_0}$

a) Show that the general solution of $x^4y''+\mu^2y=0$ is $\quad y(x)=x\left(A\cos\dfrac{\mu}{x}+B\sin\dfrac{\mu}{x}\right)$

b) Conclude that the *n*th eigenvalue is given by $\mu_n=n\pi\dfrac{ab}{L}$, where $L=b-a$ is the length of the rod, and hence that the *n*th buckling force is

$$P_n=\frac{n^2\pi^2}{L^2}\left(\frac{a}{b}\right)^2EI_0$$

Solution

a) $x^{-2}\times \quad x^4y'' + \mu^2 y = 0$

$$x^2 y'' + \mu^2 x^{-2} y = 0$$

$$A = 0, \quad B = 0, \quad C = \mu^2, \quad p = -2$$

$$\alpha = \frac{1}{2}, \quad \beta = -1, \quad k = \mu, \quad \upsilon = \frac{1}{2} \qquad\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$y(x) = x^{1/2}\left[c_1 J_{1/2}\left(\mu x^{-1}\right) + c_2 J_{-1/2}\left(\mu x^{-1}\right)\right] \qquad y(x) = x^{\alpha}\left(c_1 J_{1/2}\left(kx^{\beta}\right) + c_2 J_{-1/2}\left(kx^{\beta}\right)\right)$$

$$= \sqrt{x}\left(c_1\sqrt{\frac{2x}{\pi\mu}}\cos\left(\frac{\mu}{x}\right) + c_2\sqrt{\frac{2x}{\pi\mu}}\sin\left(\frac{\mu}{x}\right)\right)$$

$$= x^{\alpha}\left(c_1\left(\frac{2}{\pi k x^{\beta}}\right)^{1/2}\sin\left(kx^{\beta}\right) + c_2\left(\frac{2}{\pi k x^{\beta}}\right)^{1/2}\cos\left(kx^{\beta}\right)\right)$$

$$= x\left(c_1\sqrt{\frac{2}{\pi\mu}}\cos\left(\frac{\mu}{x}\right) + c_2\sqrt{\frac{2}{\pi\mu}}\sin\left(\frac{\mu}{x}\right)\right) \qquad A = c_1\sqrt{\frac{2}{\pi\mu}}, \quad B = c_2\sqrt{\frac{2}{\pi\mu}}$$

$$\boxed{= x\left(A\cos\left(\frac{\mu}{x}\right) + B\sin\left(\frac{\mu}{x}\right)\right)}$$

b) ***Given***: $\mu_n = n\pi\frac{ab}{L}; \quad y(a) = y(b) = 0, \quad L = b - a$

$$\begin{cases} y(a) = a\left(A\cos\left(\frac{\mu}{a}\right) + B\sin\left(\frac{\mu}{a}\right)\right) = 0 \\ y(b) = b\left(A\cos\left(\frac{\mu}{b}\right) + B\sin\left(\frac{\mu}{b}\right)\right) = 0 \end{cases}$$

$$\begin{cases} A\cos\left(\frac{\mu}{a}\right) + B\sin\left(\frac{\mu}{a}\right) = 0 \\ A\cos\left(\frac{\mu}{b}\right) + B\sin\left(\frac{\mu}{b}\right) = 0 \end{cases} \qquad (a,\ b \neq 0)$$

$$\Delta = \begin{vmatrix} \cos\frac{\mu}{a} & \sin\frac{\mu}{a} \\ \cos\frac{\mu}{b} & \sin\frac{\mu}{b} \end{vmatrix}$$

$$= \cos\frac{\mu}{a}\sin\frac{\mu}{b} - \sin\frac{\mu}{a}\cos\frac{\mu}{b}$$

$$= \sin\left(\frac{\mu}{b} - \frac{\mu}{a}\right)$$

$$= \sin\left(\frac{b-a}{ab}\mu\right)$$

$$\boxed{= \sin\left(\frac{L}{ab}\mu\right)}$$

$$\lambda = \mu^2 = \frac{Pb^4}{EI_0}$$

$$P = \frac{EI_0}{b^4}\mu^2$$

$$= \frac{EI_0}{b^4}\left(n\pi\frac{ab}{L}\right)^2$$

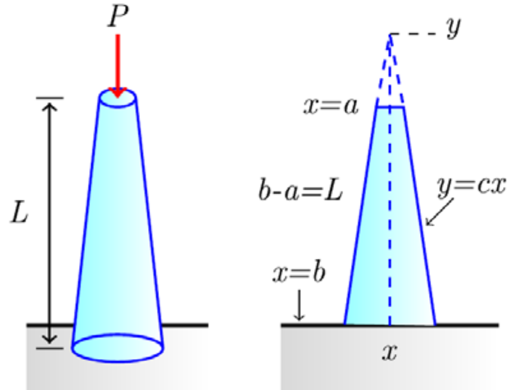
$$\boxed{= \frac{n^2\pi^2}{L^2}\left(EI_0\right)\left(\frac{a}{b}\right)^2}$$

Exercise

When a constant vertical compressive force or load *P* was applied to a thin column of uniform cross section, the deflection $y(x)$ was a solution of the boundary-value problem

$$EI\frac{d^2y}{dy^2} + Py = 0\ ; \quad y(0) = 0, \quad y(L) = 0$$

The assumption here is that the column is hinged at both ends. The column will buckle or deflect only when the compression force is a critical load P_n



a) Let assume that the column is of length *L*, is hinged at both ends, has circular cross sections, and is tapered. If the column, a truncated cone, has a linear taper $y = cx$ in cross section, the moment of inertia of a cross section with respect to an axis perpendicular to the $xy-plane$ is $I = \frac{1}{4}\pi r^4$, where $r = y$ and $y = cx$. Hence we can write $I(x) = I_0\,(x\ b)^4$, where $I_0 = I(b) = \frac{1}{4}\pi(cb)^4$. Substituting $I(x)$ into the differential equation, we see that the deflection in this case is determine from the *BVP*?

$$x^4\frac{d^2y}{dx^2} + \lambda y = 0\ ; \quad y(a) = 0, \quad y(b) = 0$$

Where $\lambda = Pb^4EI_0$

Find the critical loads P_n for the tapered column. Use an appropriate identity to express the buckling modes $y_n(x)$ as a single function.

b) Plot the graph of the first buckling mode $y_1(x)$ corresponding to the Euler load P_1 when $b=11$ and $a=1$

Solution

c) $x^{-2}\times \quad x^4y''+\lambda y=0$

$$x^2y''+\lambda x^{-2}y=0$$

$A=0,\quad B=0,\quad C=\lambda,\quad p=-2$

$$\alpha=\frac{1}{2},\quad \beta=-1,\quad k=\sqrt{\lambda},\quad \upsilon=\frac{1}{2} \qquad \alpha=\frac{1-A}{2},\ \beta=\frac{p}{2},\ k=\frac{2\sqrt{C}}{p},\ \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}$$

$$y(x)=x^{1/2}\left[c_1J_{1/2}\left(\sqrt{\lambda}x^{-1}\right)+c_2J_{-1/2}\left(\sqrt{\lambda}x^{-1}\right)\right] \qquad y(x)=x^{\alpha}\left(c_1J_{1/2}\left(kx^{\beta}\right)+c_2J_{-1/2}\left(kx^{\beta}\right)\right)$$

$$=\sqrt{x}\left(c_1\sqrt{\frac{2x}{\pi\sqrt{\lambda}}}\cos\left(\frac{\sqrt{\lambda}}{x}\right)+c_2\sqrt{\frac{2x}{\pi\sqrt{\lambda}}}\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right)$$

$$=x^{\alpha}\left(c_1\sqrt{\frac{2}{\pi kx^{\beta}}}\sin\left(kx^{\beta}\right)+c_2\sqrt{\frac{2}{\pi kx^{\beta}}}\cos\left(kx^{\beta}\right)\right)$$

$$=x\left(c_1\sqrt{\frac{2x}{\pi\sqrt{\lambda}}}\cos\left(\frac{\sqrt{\lambda}}{x}\right)+c_2\sqrt{\frac{2x}{\pi\sqrt{\lambda}}}\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right) \qquad A=c_1\sqrt{\frac{2}{\pi\sqrt{\lambda}}},\quad B=c_2\sqrt{\frac{2}{\pi\sqrt{\lambda}}}$$

$$=x\left(A\cos\left(\frac{\sqrt{\lambda}}{x}\right)+B\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right)$$

Given: $\lambda=Pb^4EI_0$; $y(a)=y(b)=0,\quad L=b-a$

$$\begin{cases} y(a)=a\left(A\cos\left(\frac{\sqrt{\lambda}}{a}\right)+B\sin\left(\frac{\sqrt{\lambda}}{a}\right)\right)=0\\ y(b)=b\left(A\cos\left(\frac{\sqrt{\lambda}}{b}\right)+B\sin\left(\frac{\sqrt{\lambda}}{b}\right)\right)=0\end{cases}$$

$$\begin{cases} A\cos\left(\frac{\sqrt{\lambda}}{a}\right)+B\sin\left(\frac{\sqrt{\lambda}}{a}\right)=0\\ A\cos\left(\frac{\sqrt{\lambda}}{b}\right)+B\sin\left(\frac{\sqrt{\lambda}}{b}\right)=0\end{cases} \qquad (a,\ b\neq 0)$$

$$\Delta=\begin{vmatrix}\cos\frac{\sqrt{\lambda}}{a} & \sin\frac{\sqrt{\lambda}}{a}\\ \cos\frac{\sqrt{\lambda}}{b} & \sin\frac{\sqrt{\lambda}}{b}\end{vmatrix}$$

$$=\cos\frac{\sqrt{\lambda}}{a}\sin\frac{\sqrt{\lambda}}{b}-\sin\frac{\sqrt{\lambda}}{a}\cos\frac{\sqrt{\lambda}}{b}$$

$$=\sin\left(\frac{\sqrt{\lambda}}{b}-\frac{\sqrt{\lambda}}{a}\right)$$

$$= \sin\left(\frac{b-a}{ab}\sqrt{\lambda}\right)$$

$$= \sin\left(\frac{L}{ab}\sqrt{\lambda}\right) = 0$$

$$\frac{L}{ab}\sqrt{\lambda} = n\pi \quad \to \quad \sqrt{\lambda} = \frac{n\pi ab}{L} \qquad (n \in \mathbb{N})$$

$$\lambda = \frac{n^2\pi^2a^2b^2}{L^2} = Pb^4EI_0$$

$$P_n = \frac{n^2\pi^2}{L^2}\left(EI_0\right)\left(\frac{a}{b}\right)^2$$

If we let $B = -A\dfrac{\sin\frac{\sqrt{\lambda}}{a}}{\cos\frac{\sqrt{\lambda}}{a}}$

$$y(x) = x\left(A\cos\left(\frac{\sqrt{\lambda}}{x}\right) + B\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right)$$

$$= x\left(A\cos\left(\frac{\sqrt{\lambda}}{x}\right) - A\frac{\sin\frac{\sqrt{\lambda}}{a}}{\cos\frac{\sqrt{\lambda}}{a}}\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right)$$

$$= \frac{A}{\cos\frac{\sqrt{\lambda}}{a}}x\left(\cos\frac{\sqrt{\lambda}}{a}\cos\left(\frac{\sqrt{\lambda}}{x}\right) - \sin\frac{\sqrt{\lambda}}{a}\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right) \qquad \frac{A}{\cos\frac{\sqrt{\lambda}}{a}} = C$$

$$= Cx\sin\left(\frac{\sqrt{\lambda}}{x} - \frac{\sqrt{\lambda}}{a}\right)$$

$$= Cx\sin\sqrt{\lambda}\left(\frac{1}{x} - \frac{1}{a}\right)$$

$$y_n(x) = Cx\sin\sqrt{\lambda}\left(\frac{1}{x} - \frac{1}{a}\right) \qquad \left(\sqrt{\lambda} = \frac{n\pi ab}{L}\right)$$

$$= Cx\sin\frac{n\pi ab}{L}\left(\frac{1}{x} - \frac{1}{a}\right)$$

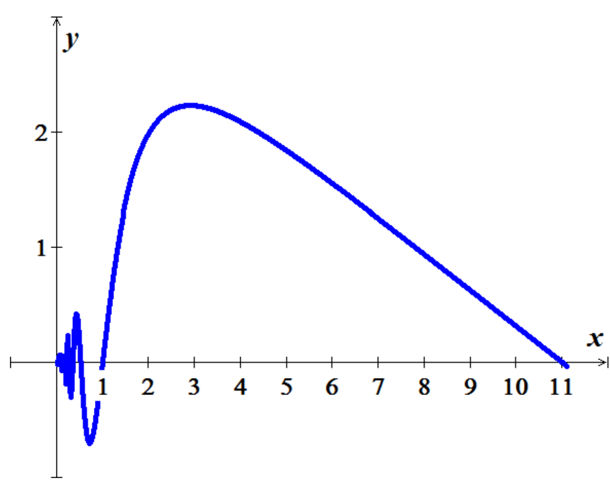
$$= Cx\sin\frac{n\pi b}{L}\left(\frac{a}{x} - 1\right)$$

$$= C_1x\sin\frac{n\pi b}{L}\left(1 - \frac{a}{x}\right)$$

d) ***Given***: $n = 1, \quad a = 1, \quad b = 11$

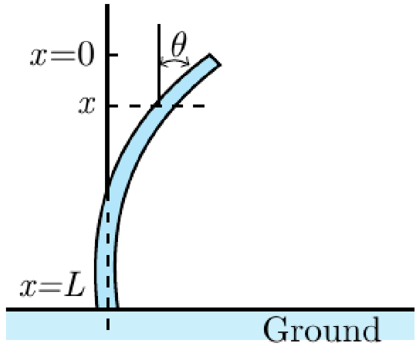
Let $C_1 = 1$

$$y_1(x) = x\sin\frac{11\pi}{10}\left(1 - \frac{1}{x}\right)$$



Exercise

For a practical application, we now consider the problem of determining when a uniform vertical column will buckle under its own weight (after, perhaps, being nudged laterally just a bit by a passing breeze). We take $x=0$ at the free top end of the column and $x=L>0$ at its bottom; we assume that the bottom is rigidly imbedded in the ground, perhaps in concrete.



Denote the angular deflection of the column at the point x by $\theta(x)$. From the theory of elasticity it follows that

$$EI\frac{d^2\theta}{dx^2}+g\rho x\theta=0$$

Where E is the Young's modulus of the material of the column,

I is its cross-sectional moment of inertia

ρ is the linear density of the column

g is gravitational acceleration.

For physical reasons – no bending at the free top of the column and no deflection at its imbedded bottom – the boundary conditions are $\theta'(0)=0, \quad \theta(L)=0$

Determine the general equation of the length L.

Solution

$$EI\theta''+g\rho x\theta=0$$

$$\theta''+\frac{g\rho}{EI}x\theta=0$$

Let $\lambda=\frac{g\rho}{EI}=\gamma^2$

$x^2\times \quad \theta''+\gamma^2 x\theta=0$

$x^2\theta''+\gamma^2x^3\theta=0$; $\theta'(0)=0, \quad \theta(L)=0$

$A=0, \quad B=0, \quad C=\gamma^2, \quad p=3$

$\alpha=\frac{1}{2}, \quad \beta=\frac{3}{2}, \quad k=\frac{2\gamma}{3}, \quad \upsilon=\frac{1}{3}$ $\qquad \alpha=\frac{1-A}{2},\ \beta=\frac{p}{2},\ k=\frac{2\sqrt{C}}{p},\ \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}$

$\theta(x)=x^{1/2}\left[c_1J_{1/3}\left(\frac{2}{3}\gamma x^{3/2}\right)+c_2J_{-1/3}\left(\frac{2}{3}\gamma x^{3/2}\right)\right]$ $\qquad y(x)=x^{\alpha}\left(c_1J_{\nu}\left(kx^{\beta}\right)+c_2J_{-\nu}\left(kx^{\beta}\right)\right)$

$$J_{1/3}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma\left(1+\frac{1}{3}+n\right)}\left(\frac{x}{2}\right)^{2n+\frac{1}{3}}$$

$$J_{\upsilon}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma\left(n+\frac{4}{3}\right)}\left(\frac{x}{2}\right)^{2n+\frac{1}{3}}$$

$$J_{\upsilon}(x)=\frac{x^{\upsilon}}{2^{\upsilon}\Gamma(\upsilon+1)}\left\{1-\frac{x^2}{2(2\upsilon+2)}+\frac{x^4}{2\cdot 4\cdot(2\upsilon+2)(2\upsilon+4)}-\cdots\right\}$$

$$=\frac{x^{1/3}}{2^{1/3}\Gamma\left(\frac{4}{3}\right)}\left\{1-\frac{x^2}{2\left(\frac{2}{3}+2\right)}+\frac{x^4}{2\cdot 4\cdot\left(\frac{2}{3}+2\right)\left(\frac{2}{3}+4\right)}-\cdots\right\}$$

$$=\frac{x^{1/3}}{2^{1/3}\Gamma\left(\frac{4}{3}\right)}\left\{1-\frac{3x^2}{2^3}+\frac{3^2x^4}{112\times 2^3}-\cdots\right\}$$

$$J_{1/3}\left(\frac{2}{3}\gamma x^{3/2}\right)=\frac{1}{2^{1/3}\Gamma\left(\frac{4}{3}\right)}\left(\frac{2}{3}\gamma x^{3/2}\right)^{1/3}\left\{1-\frac{3}{2^3}\left(\frac{2}{3}\gamma x^{3/2}\right)^2+\frac{3^2}{896}\left(\frac{2}{3}\gamma x^{3/2}\right)^4-\cdots\right\}$$

$$=\frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}x^{1/2}\left\{1-\frac{1}{12}\gamma^2x^3+\frac{1}{504}\gamma^4x^6-\cdots\right\}$$

$$J_{-1/3}\left(\frac{2}{3}\gamma x^{3/2}\right)=\frac{1}{2^{-1/3}\Gamma\left(\frac{4}{3}\right)}\left(\frac{2}{3}\gamma x^{3/2}\right)^{-1/3}\left\{1-\frac{1}{2\left(2-\frac{1}{3}\right)}\left(\frac{2}{3}\gamma x^{3/2}\right)^2+\frac{1}{8\left(2-\frac{1}{3}\right)\left(4-\frac{1}{3}\right)}\left(\frac{2}{3}\gamma x^{3/2}\right)^4-\cdots\right\}$$

$$=\frac{3^{1/3}}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)}x^{-1/2}\left\{1-\frac{1}{6}\gamma^2x^3+\frac{1}{180}\gamma^4x^6-\cdots\right\}$$

$$\theta(x)=x^{1/2}\left[c_1J_{1/3}\left(\frac{2}{3}\gamma x^{3/2}\right)+c_2J_{-1/3}\left(\frac{2}{3}\gamma x^{3/2}\right)\right]$$

$$=x^{1/2}\left[c_1\frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}x^{1/2}\left\{1-\frac{1}{12}\gamma^2x^3+\frac{1}{504}\gamma^4x^6-\cdots\right\}+c_2\frac{3^{1/3}}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)}x^{-1/2}\left\{1-\frac{1}{6}\gamma^2x^3+\frac{1}{180}\gamma^4x^6-\cdots\right\}\right]$$

$$=c_1\frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}\left\{x-\frac{1}{12}\gamma^2x^4+\frac{1}{504}\gamma^4x^7-\cdots\right\}+c_2\frac{3^{1/3}}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)}\left\{1-\frac{1}{6}\gamma^2x^3+\frac{1}{180}\gamma^4x^6-\cdots\right\}$$

Given: $\theta(L)=0,\quad \theta'(0)=0$

$$\theta'(x)=c_1\frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}\left\{1-\frac{1}{3}\gamma^2x^3+\frac{1}{72}\gamma^4x^6-\cdots\right\}+\frac{3^{1/3}}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)}\left\{\frac{1}{2}\gamma^2x^2+\frac{1}{30}\gamma^4x^5-\cdots\right\}$$

$$\theta'(0)=c_1\frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}=0\quad\to\quad c_1=0$$

$$\frac{3^{1/3}c_2}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)}\left\{1-\frac{1}{6}\gamma^2L^3+\frac{1}{180}\gamma^4L^6-\cdots\right\}=0$$

$$c_2J_{-1/3}\left(\frac{2}{3}\gamma L^{3/2}\right)=0 \quad\rightarrow\quad J_{-1/3}\left(\frac{2}{3}\gamma L^{3/2}\right)=0$$

$$J_{-1/3}\left(z=\frac{2}{3}\gamma L^{3/2}\right)=0$$

Using MatLab:

```
fzero('besselj(-1/3,z)',2)
syms z y
fplot(besselj(-1/3, z))
axis([0 10 -0.5 1.1])
grid on
ylabel('J_-1/3(z)')
legend('J_-1/3')
```

$z = 1.8664$

$$z=\frac{2}{3}\gamma L^{3/2} \quad\rightarrow\quad L=\left(\frac{3z}{2\gamma}\right)^{2/3}$$

$$L=\left(\frac{3(1.86635)}{2\sqrt{\frac{g\rho}{EI}}}\right)^{2/3}$$

$$\approx 1.986352\left(\frac{EI}{g\rho}\right)^{1/3}$$