Solution   *Section* **2.1 – Introducing the Derivative**

Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P; then graph the curve and the tangent line.
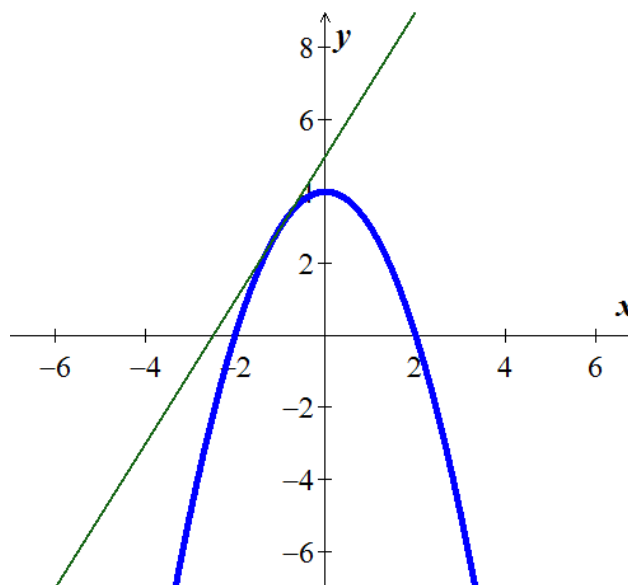
$$y = 4 - x^2; \quad P(-1, \ 3)$$

Solution

$$m = \lim_{h \to 0} \frac{4 - (x+h)^2 - \left(4 - x^2\right)}{h}$$

$$= \lim_{h \to 0} \frac{4 - (-1+h)^2 - \left(4 - (-1)^2\right)}{h}$$

$$= \lim_{h \to 0} \frac{4 - \left(1 - 2h + h^2\right) - (4 - 1)}{h}$$

$$= \lim_{h \to 0} \frac{4 - 1 + 2h - h^2 - 3}{h}$$

$$= \lim_{h \to 0} \frac{2h - h^2}{h}$$

$$= \lim_{h \to 0} (2 - h)$$

$$\underline{= 2}$$

$$y - y_1 = m\left(x - x_1\right)$$

At $(-1, \ 3) \Rightarrow y - 3 = 2\left(x - (-1)\right)$

$$y - 3 = 2x + 2$$

$$\underline{y = 2x + 5}$$

Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P; then graph the curve and the tangent line.
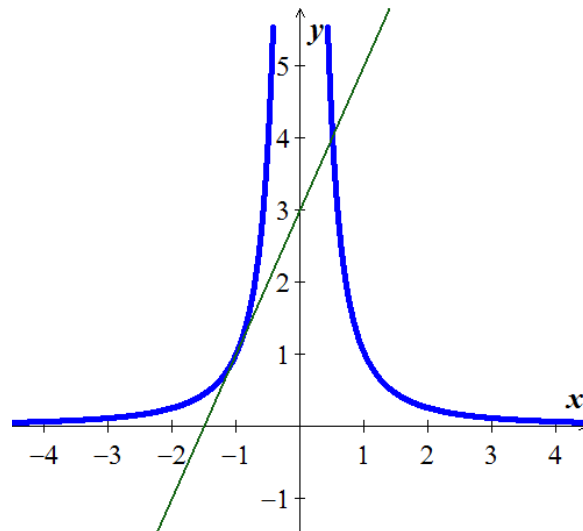
$$y = \frac{1}{x^2}; \quad P(-1,\, 1)$$

Solution

$$m = \lim_{h \to 0} \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{(-1+h)^2} - \frac{1}{1} \right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{1 - \left(1 - 2h + h^2\right)}{(-1+h)^2} \right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{1 - 1 + 2h - h^2}{(-1+h)^2} \right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{2h - h^2}{(-1+h)^2} \right)$$

$$= \lim_{h \to 0} \frac{h}{h}\left(\frac{2 - h}{(-1+h)^2} \right)$$

$$= \lim_{h \to 0} \left(\frac{2 - h}{(-1+h)^2} \right)$$

$$= \frac{2 + 0}{(-1+0)^2}$$

$$= 2$$

At $(-1,\, 3) \Rightarrow y - 1 = 2\left(x - (-1)\right)$

$$y - 1 = 2x + 2$$

$$y = 2x + 3$$

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Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P; then graph the curve and the tangent line.

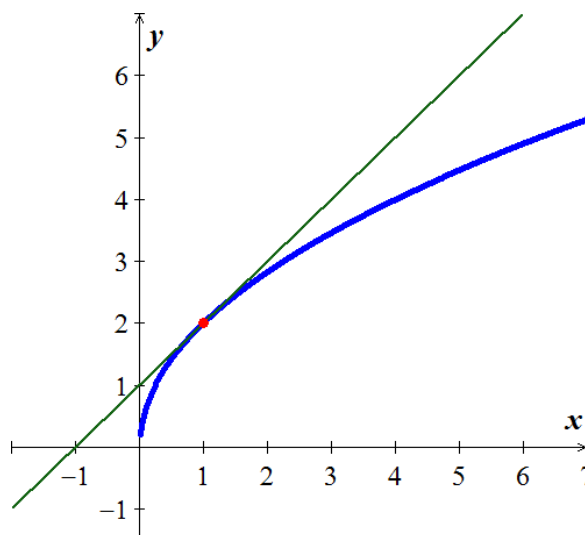$$f(x) = 2\sqrt{x}; \quad P(1,\ 2)$$

Solution

$$m = \lim_{h \to 0} \frac{2\sqrt{x+h} - 2\sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{2\sqrt{1+h} - 2\sqrt{x}}{h} \cdot \frac{2\sqrt{1+h} + 2}{2\sqrt{1+h} + 2}$$

$$= \lim_{h \to 0} \frac{4(1+h) - 4}{h\left(2\sqrt{1+h} + 2\right)}$$

$$= \lim_{h \to 0} \frac{4 + 4h - 4}{h\left(2\sqrt{1+h} + 2\right)}$$

$$= \lim_{h \to 0} \frac{4h}{h\left(2\sqrt{1+h} + 2\right)}$$

$$= \lim_{h \to 0} \frac{4}{2\sqrt{1+h} + 2}$$

$$= \frac{4}{2 + 2}$$

$$\underline{= 1}$$

At $(1,\ 2) \Rightarrow y - 2 = (x - 1)$ $\qquad y - y_1 = m\left(x - x_1\right)$

$$y - 2 = x - 1$$

$$\underline{y = x + 1}$$

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Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P; then graph the curve and the tangent line.

$$f(x) = x^3 + 3x; \quad P(1, 4)$$

Solution

$$m = \lim_{h \to 0} \frac{(x+h)^3 + 3(x+h) - \left(x^3 + 3x\right)}{h}$$

$$= \lim_{h \to 0} \frac{1 + 3h + 3h^2 + h^3 + 3 + 3h - (4)}{h}$$

$$= \lim_{h \to 0} \frac{3h + 3h^2 + h^3 + 3h}{h}$$

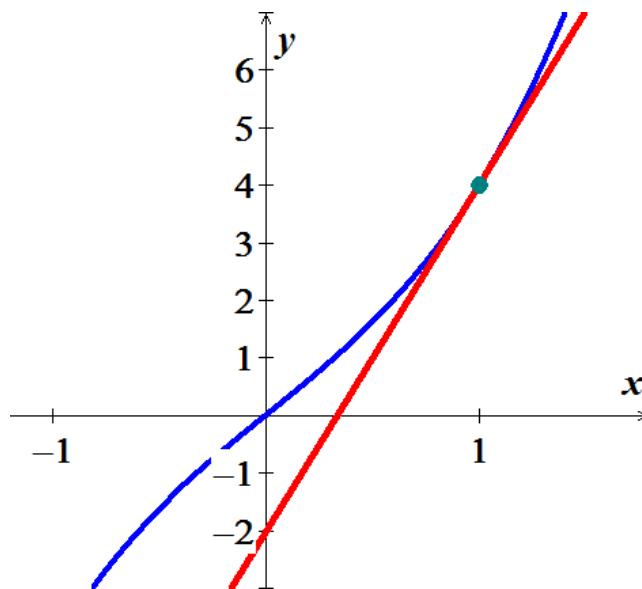$$= \lim_{h \to 0} \frac{3h^2 + h^3 + 6h}{h}$$

$$= 6$$

At $(1, 4)$

$$y - 4 = 6(x - 1) \qquad\qquad y - y_1 = m(x - x_1)$$

$$y - 4 = 6x - 6$$

$$y = 6x - 2$$

Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P; then graph the curve and the tangent line.
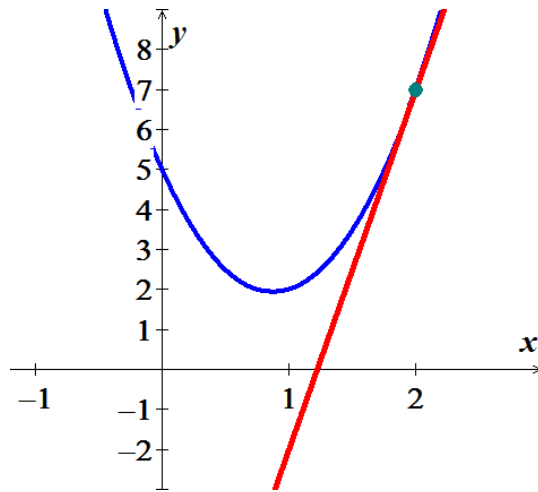
$$f(x) = 4x^2 - 7x + 5; \quad P(2, 7)$$

Solution

$$m = \lim_{h \to 0} \frac{4(x+h)^2 - 7(x+h) + 5 - 4x^2 + 7x - 5}{h}$$

$$= \lim_{h \to 0} \frac{4\left(x^2 + 2xh + h^2\right) - 7x - 7h - 4x^2 + 7x}{h}$$

$$= \lim_{h \to 0} \frac{4x^2 + 8xh + 4h^2 - 7h - 4x^2}{h}$$

$$= \lim_{h \to 0} \frac{8xh + 4h^2 - 7h}{h}$$

$$= \lim_{h \to 0} \left(8xh + 4h - 7\right)$$

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$$= 8x - 7|$$

At $(2, 7)$ → $m = 9|$

$$y = 9(x-2) + 7 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$= 9x - 11|$$

Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P; then graph the curve and the tangent line.

$$f(x) = 5x^3 + x; \quad P(1, 6)$$

Solution

$$m = \lim_{h \to 0} \frac{5(x+h)^3 + (x+h) - 5x^3 - x}{h}$$

$$= \lim_{h \to 0} \frac{5\left(x^3 + 3x^2 h + 3xh^2 + h^3\right) + h - 5x^3}{h}$$

$$= \lim_{h \to 0} \frac{15x^2 h + 15xh^2 + 5h^3 + h}{h}$$

$$= \lim_{h \to 0} \left(15x^2 + 15xh + 5h^2 + 1\right)$$

$$= 15x^2 + 1\Big|_{(1, 6)}$$

$$= 16|$$

$$y = 16(x - 1) + 6 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$= 16x - 10|$$

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Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P; then graph the curve and the tangent line.
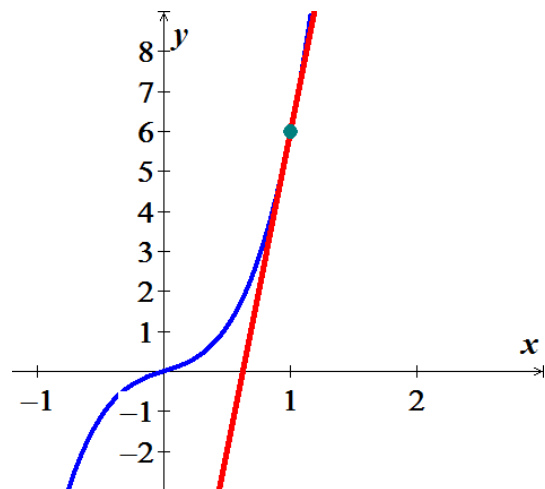
$$f(x) = \frac{x+3}{2x+1}; \quad P(0,\ 3)$$

Solution

$$m = \lim_{h \to 0} \frac{1}{h}\left[\frac{x+h+3}{2x+2h+1} - \frac{x+3}{2x+1}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{2x^2 + 2hx + 6x + x + h + 3 - 2x^2 - 2hx - x - 6x - 6h - 3}{(2x+2h+1)(2x+1)}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{-5h}{(2x+2h+1)(2x+1)}\right)$$

$$= \lim_{h \to 0} \left(\frac{-5}{(2x+2h+1)(2x+1)}\right)$$

$$= \frac{-5}{(2x+1)^2}\Bigg|_{(0,\ 3)}$$

$$= \underline{-5}$$

$$\underline{y = -5x + 3} \qquad\qquad y = m\left(x - x_1\right) + y_1$$

Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P; then graph the curve and the tangent line.
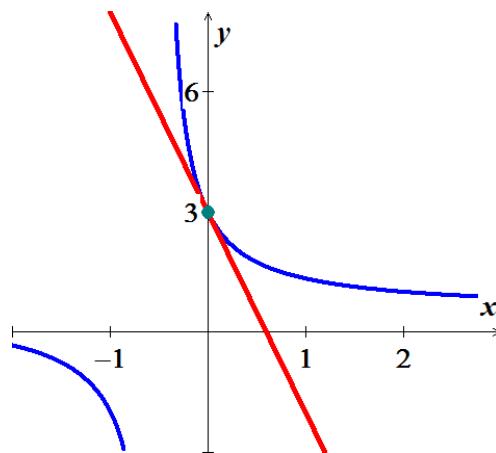
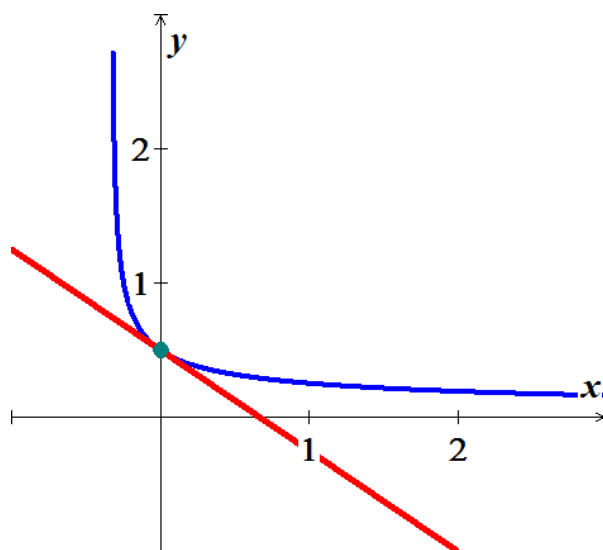$$f(x) = \frac{1}{2\sqrt{3x+1}}; \quad P\left(0,\ \tfrac{1}{2}\right)$$

Solution

$$m = \lim_{h \to 0} \frac{1}{h}\left[\frac{1}{2\sqrt{3x+3h+1}} - \frac{1}{2\sqrt{3x+1}}\right]$$

$$= \frac{1}{2}\lim_{h \to 0}\frac{1}{h}\left(\frac{\sqrt{3x+1} - \sqrt{3x+3h+1}}{\sqrt{3x+3h+1}\ \sqrt{3x+1}}\right) = \frac{0}{0}$$

$$= \frac{1}{2}\lim_{h \to 0}\frac{1}{h}\left(\frac{\sqrt{3x+1} - \sqrt{3x+3h+1}}{\sqrt{3x+3h+1}\ \sqrt{3x+1}}\right)\frac{\sqrt{3x+1} + \sqrt{3x+3h+1}}{\sqrt{3x+1} + \sqrt{3x+3h+1}}$$

$$= \frac{1}{2}\lim_{h \to 0}\frac{1}{h}\left(\frac{3x+1-3x-3h-1}{\sqrt{3x+3h+1}\ \sqrt{3x+1}\left(\sqrt{3x+1} + \sqrt{3x+3h+1}\right)}\right)$$

$$= \frac{1}{2}\lim_{h\to 0}\frac{1}{h}\left(\frac{-3h}{\sqrt{3x+3h+1}\ \sqrt{3x+1}\left(\sqrt{3x+1}+\sqrt{3x+3h+1}\right)}\right)$$

$$= -\frac{3}{2}\lim_{h\to 0}\left(\frac{1}{\sqrt{3x+3h+1}\ \sqrt{3x+1}\left(\sqrt{3x+1}+\sqrt{3x+3h+1}\right)}\right)$$

$$= -\frac{3}{2}\frac{1}{(3x+1)\left(2\sqrt{3x+1}\right)}$$

$$= -\frac{3}{4}\frac{1}{(3x+1)^{3/2}}\bigg|_{\left(0,\ \frac{1}{2}\right)}$$

$$= -\frac{3}{4}\bigg|$$

$$y = -\frac{3}{4}x+\frac{1}{2}\bigg| \qquad\qquad y = m\left(x-x_1\right)+y_1$$

Exercise

Find the slope of the curve $y = 1 - x^2$ at the point $x = 2$

Solution

$$m = \lim_{h\to 0}\frac{1-(x+h)^2-\left(1-x^2\right)}{h}$$

$$= \lim_{h\to 0}\frac{1-(2+h)^2-\left(1-2^2\right)}{h}$$

$$= \lim_{h\to 0}\frac{1-\left(4+4h+h^2\right)-(-3)}{h}$$

$$= \lim_{h \to 0} \frac{1 - 4 - 4h - h^2 + 3}{h}$$

$$= \lim_{h \to 0} \frac{-4h - h^2}{h}$$

$$= \lim_{h \to 0} \left(-4 - h \right)$$

$$\underline{= -4}$$

Exercise

Find the slope of the curve $y = \dfrac{1}{x-1}$ at the point $x = 3$

Solution

$$m = \lim_{h \to 0} \frac{\dfrac{1}{x+h-1} - \dfrac{1}{x-1}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{1}{3+h-1} - \dfrac{1}{3-1}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{1}{2+h} - \dfrac{1}{2}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\frac{2 - 2 - h}{2+h} \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\frac{-h}{2+h} \right)$$

$$= \lim_{h \to 0} \left(\frac{-1}{2+h} \right)$$

$$\underline{= -\frac{1}{2}}$$

Exercise

Find the slope of the curve $y = \dfrac{x-1}{x+1}$ at the point $x = 0$

Solution

$$m = \lim_{h \to 0} \frac{\dfrac{x+h-1}{x+h+1} - \dfrac{x-1}{x+1}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\frac{0+h-1}{0+h+1} - \frac{0-1}{0+1} \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\frac{h-1}{h+1} + 1 \right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{h-1+h+1}{h+1}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{2h}{h+1}\right)$$

$$= \lim_{h \to 0} \left(\frac{2}{h+1}\right)$$

$$= 2$$

Exercise

Find equations of all lines having slope -1 that are tangent to the curve $y = \dfrac{1}{x-1}$

Solution

$$m = \lim_{h \to 0} \frac{\dfrac{1}{x+h-1} - \dfrac{1}{x-1}}{h}$$

$$-1 = \lim_{h \to 0} \frac{\dfrac{1}{x+h-1} - \dfrac{1}{x-1}}{h}$$

$$-1 = \lim_{h \to 0} \frac{1}{h}\left(\frac{x-1-(x+h-1)}{x+h-1}\right) \qquad \textit{Cross multiplication}$$

$$-1 = \lim_{h \to 0} \frac{1}{h}\left(\frac{x-1-x-h+1}{x+h-1}\right)$$

$$-1 = \lim_{h \to 0} \frac{1}{h}\left(\frac{-h}{x+h-1}\right)$$

$$-1 = \lim_{h \to 0} \left(\frac{-1}{x+h-1}\right)$$

$$-1 = \frac{-1}{x-1}$$

$$-x+1 = -1$$

$$\boxed{x = 2}$$

$$\left\lfloor y = \frac{1}{x-1} = \frac{1}{2-1} = 1 \right\rfloor$$

At $(2,\ 1) \Rightarrow y-1 = -1(x-2)$

$$y-1 = -x+2$$

$$\underline{y = -x+3}$$

Exercise

What is the rate of change of the area of a circle $\left(A = \pi r^2 \right)$ with respect to the radius when the radius is $r = 3$?

Solution

$$m = \lim_{h \to 0} \frac{\pi(3+h)^2 - \pi(3)^2}{h}$$

$$= \lim_{h \to 0} \frac{\pi\left(9 + 6h + h^2\right) - 9\pi}{h}$$

$$= \lim_{h \to 0} \frac{9\pi + 6\pi h + \pi h^2 - 9\pi}{h}$$

$$= \lim_{h \to 0} \frac{6\pi h + \pi h^2}{h}$$

$$= \lim_{h \to 0} \frac{\pi h(6+h)}{h}$$

$$= \lim_{h \to 0} \pi(6+h)$$

$$\underline{= 6\pi}$$

Exercise

Find the slope of the tangent to the curve $y = \dfrac{1}{\sqrt{x}}$ at the point where $x = 4$

Solution

$$m = \lim_{h \to 0} \frac{\dfrac{1}{\sqrt{x+h}} - \dfrac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{\sqrt{4} - \sqrt{4+h}}{2\sqrt{4+h}} \right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{2 - \sqrt{4+h}}{2\sqrt{4+h}} \right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{2 - \sqrt{4+h}}{2\sqrt{4+h}} \cdot \frac{2 + \sqrt{4+h}}{2 + \sqrt{4+h}} \right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{4 - (4+h)}{2\sqrt{4+h}\left(2 + \sqrt{4+h}\right)} \right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{-h}{2\sqrt{4+h}\left(2 + \sqrt{4+h}\right)} \right)$$

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$$= \lim_{h \to 0} \left(\frac{-1}{2\sqrt{4+h}\left(2+\sqrt{4+h}\right)} \right)$$

$$= \frac{-1}{2\sqrt{4}\left(2+\sqrt{4}\right)}$$

$$= \frac{-1}{2(2)(2+2)}$$

$$= \frac{-1}{16}$$

Exercise

Fin the values of the derivatives of the function $f(x) = 4 - x^2$. Then find the values of $f'(-3), f'(0), f'(1)$

Solution

$$\frac{f(x+h)-f(x)}{h} = \frac{4-(x+h)^2 - \left(4 - x^2\right)}{h}$$

$$= \frac{4 - \left(x^2 + 2xh + h^2\right) - \left(4 - x^2\right)}{h}$$

$$= \frac{4 - x^2 - 2xh - h^2 - 4 + x^2}{h}$$

$$= \frac{-2xh - h^2}{h}$$

$$= -2x - h$$

$$f'(x) = \lim_{h \to 0} (-2x - h) = -2x$$

$$f'(-3) = 6 \qquad f'(0) = 0 \qquad f'(1) = -2$$

Exercise

Fin the values of the derivatives of the function $r(s) = \sqrt{2s+1}$. Then find the values of $r'(0), r'\left(\frac{1}{2}\right), r'(1)$

Solution

$$r'(s) = \lim_{h \to 0} \frac{r(s+h) - r(s)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2(s+h)+1} - \sqrt{2s+1}}{h}$$

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$$= \lim_{h \to 0} \frac{\sqrt{2s+2h+1} - \sqrt{2s+1}}{h} \cdot \frac{\sqrt{2s+2h+1} + \sqrt{2s+1}}{\sqrt{2s+2h+1} + \sqrt{2s+1}}$$

$$= \lim_{h \to 0} \frac{2s+2h+1-(2s+1)}{h\left(\sqrt{2s+2h+1} + \sqrt{2s+1}\right)}$$

$$= \lim_{h \to 0} \frac{2s+2h+1-2s-1}{h\left(\sqrt{2s+2h+1} + \sqrt{2s+1}\right)}$$

$$= \lim_{h \to 0} \frac{2h}{h\left(\sqrt{2s+2h+1} + \sqrt{2s+1}\right)}$$

$$= \lim_{h \to 0} \frac{2}{\sqrt{2s+2h+1} + \sqrt{2s+1}}$$

$$= \frac{2}{\sqrt{2s+1} + \sqrt{2s+1}}$$

$$= \frac{2}{2\sqrt{2s+1}}$$

$$= \frac{1}{\sqrt{2s+1}}$$

$$r'(0) = \frac{1}{\sqrt{2(0)+1}} = \underline{1 \rfloor}$$

$$r'\left(\tfrac{1}{2}\right) = \frac{1}{\sqrt{2\tfrac{1}{2}+1}} = \underline{\frac{1}{\sqrt{2}}} \rfloor$$

$$r'(1) = \frac{1}{\sqrt{2(1)+1}} = \underline{\frac{1}{\sqrt{3}}} \rfloor$$

Exercise

Find the derivative of $f(x) = 3x^2 - 2x$

<u>**Solution**</u>

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3(x+\Delta x)^2 - 2(x+\Delta x) - (3x^2 - 2x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3(x^2 + \Delta x^2 + 2x\Delta x) - 2x - 2\Delta x - 3x^2 + 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2 + 3\Delta x^2 + 6x\Delta x - 2x - 2\Delta x - 3x^2 + 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3\Delta x^2 + 6x\Delta x - 2\Delta x}{\Delta x}$$

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$$= \lim_{\Delta x \to 0} 3\Delta x + 6x - 2$$

$$\underline{= 6x - 2}$$

Exercise

Find the derivative of y with the respect to t for the function $y = \frac{4}{t}$

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta t \to 0} \frac{\frac{4}{t + \Delta t} - \frac{4}{t}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\frac{4t - 4(t + \Delta t)}{t(t + \Delta t)}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{4t - 4(t + \Delta t)}{t(t + \Delta t)}$$

$$= \lim_{\Delta t \to 0} \frac{-4\Delta t}{t(t + \Delta t)\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{-4}{t(t + \Delta t)}$$

$$\underline{= -\frac{4}{t^2}}$$

Exercise

Find the derivative of $\frac{dy}{dx}$ if $y = 2x^3$

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{2(x + \Delta x)^3 - 2x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2\left(x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3\right) - 2x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x^3 + 6x^2\Delta x + 6x(\Delta x)^2 + 3(\Delta x)^3 - 2x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x\left(6x^2 + 6x(\Delta x) + 3(\Delta x)^2\right)}{\Delta x}$$

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$$= \lim_{\Delta x \to 0} \left(6x^2 + 6x(\Delta x) + 3(\Delta x)^2 \right)$$

$$\underline{= 6x^2}$$

Exercise

Differentiate the function $y = \frac{x+3}{1-x}$ and find the slope of the tangent line at the given value of the independent variable.

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{\frac{x+\Delta x+3}{1-x-\Delta x} - \frac{x+3}{1-x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{1}{\Delta x} \right) \left(\frac{(x+\Delta x+3)(1-x)-(x+3)(1-x-\Delta x)}{(1-x-\Delta x)(1-x)} \right)$$

$$= \lim_{\Delta x \to 0} \left(\frac{1}{\Delta x} \right) \left(\frac{x+\Delta x+3-x^2-x\Delta x-3x-\left(x-x^2-x\Delta x+3-3x-3\Delta x\right)}{(1-x-\Delta x)(1-x)} \right)$$

$$= \lim_{\Delta x \to 0} \left(\frac{1}{\Delta x} \right) \left(\frac{x+\Delta x+3-x^2-x\Delta x-3x-x+x^2+x\Delta x-3+3x+3\Delta x}{(1-x-\Delta x)(1-x)} \right)$$

$$= \lim_{\Delta x \to 0} \left(\frac{1}{\Delta x} \right) \left(\frac{4\Delta x}{(1-x-\Delta x)(1-x)} \right)$$

$$= \lim_{\Delta x \to 0} \frac{4}{(1-x-\Delta x)(1-x)}$$

$$= \frac{4}{(1-x)(1-x)}$$

$$\underline{= \frac{4}{(1-x)^2}}$$

Exercise

Find the equation of the tangent line to $f(x) = x^2 + 1$ that is parallel to $2x + y = 0$

Solution

$$2x + y = 0 \Rightarrow y = -2x \Rightarrow \text{slope} = -2$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 + 1 - (x^2 + 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + \Delta x^2 + 2x\Delta x + 1 - x^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x^2 + 2x\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \Delta x + 2x = 2x$$

$$f' = 2x = -2 \implies x = -1 \implies f(-1) = (-1)^2 + 1 = 2 \quad \to (-1, 2)$$

The line equation is given by $\quad y = m\left(x - x_1\right) + y_1$

$$y = -2(x + 1) + 2$$

$$\underline{y = -2x} \ |$$

Exercise

Use the definition of limits to find the derivative: $f(x) = \dfrac{3}{\sqrt{x}}$ $\quad y - 2 = -2x - 2$

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(\dfrac{3}{\sqrt{x + \Delta x}}\right) - \left(\dfrac{3}{\sqrt{x}}\right)}{\Delta x} \cdot \frac{\sqrt{x} \cdot \sqrt{x + \Delta x}}{\sqrt{x} \cdot \sqrt{x + \Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{3\sqrt{x} - 3\sqrt{x + \Delta x}}{\Delta x \left(\sqrt{x} \cdot \sqrt{x + \Delta x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{3\left(\sqrt{x} - \sqrt{x + \Delta x}\right)}{\Delta x \left(\sqrt{x} \cdot \sqrt{x + \Delta x}\right)} \cdot \frac{\sqrt{x} + \sqrt{x + \Delta x}}{\sqrt{x} + \sqrt{x + \Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{3\left(x - (x + \Delta x)\right)}{\Delta x \left(\sqrt{x} \cdot \sqrt{x + \Delta x}\right)\left(\sqrt{x} + \sqrt{x + \Delta x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{-3\Delta x}{\Delta x \left(\sqrt{x} \cdot \sqrt{x + \Delta x}\right)\left(\sqrt{x} + \sqrt{x + \Delta x}\right)}$$

$$= \frac{-3}{x\left(2\sqrt{x}\right)}$$

$$\left.= \frac{-3}{2x^{3/2}}\right|$$

Exercise

Use the definition of limits to find the derivative: $f(x) = \sqrt{x+2}$

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+2} - \sqrt{x+2}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x+2} + \sqrt{x+2}}{\sqrt{x+\Delta x+2} + \sqrt{x+2}}$$

$$= \lim_{\Delta x \to 0} \frac{x+\Delta x+2-(x+2)}{\Delta x\left(\sqrt{x+\Delta x+2} + \sqrt{x+2}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{x+\Delta x+2} + \sqrt{x+2}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x+2} + \sqrt{x+2}}$$

$$\left.= \frac{1}{2\sqrt{x+2}}\right|$$

Exercise

Suppose the height s of an object (in m) above the ground after t seconds is approximated by the function

$$s = -4.9t^2 + 25t + 1$$

    a) Make a table showing the average velocities of the object from time $t=1$ to $t=1+h$, for
        $h = 0.01,\ 0.001,\ 0.0001,\ and\ 0.00001$.

    b) Use the table in part (*a*) to estimate the instantaneous velocity of the object at $t=1$.

    c) Use limits to verify your estimate in part (*b*).

Solution

    a) $\dfrac{f(1+h) - f(1)}{h} = \dfrac{1}{h}\left[-4.9(1+h)^2 + 25(1+h) + 1 + 4.9 - 25 - 1\right]$

$$= \frac{1}{h}\left[-4.9 - 9.8h - 4.9h^2 + 25h + 4.9\right]$$

$$= \frac{1}{h}\left(-4.9h^2 + 15.2h\right)$$

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$$= 15.2 - 4.9h \,|$$

h	$\dfrac{f(1+h)-f(1)}{h}$
0.01	15.151
0.001	15.1951
0.0001	15.1995
0.00001	15.2
0.000001	15.2

b) $f'(1) = \lim\limits_{h \to 0} \dfrac{f(1+h)-f(1)}{h}$

$\approx 15.2 \; m/sec \,|$

c) $f'(1) = \lim\limits_{h \to 0} \dfrac{f(1+h)-f(1)}{h}$

$= 15.2 \; m/sec \,|$

Exercise

Suppose the following graph represents the number of bacteria in a culture t hours after the start of an experiment.
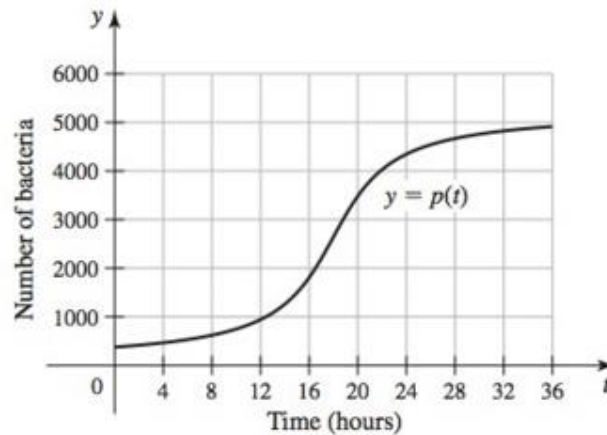
a) At approximately what time is the instantaneous growth rate the greatest, for $0 \le t \le 36$? Estimate the growth rate at this time.

b) At approximately what time is the instantaneous growth rate the least, for $0 \le t \le 36$? Estimate the growth rate at this time.

c) What is the average growth rate over the interval $0 \le t \le 36$?

Solution

a) $t = \dfrac{36}{2} = 18$

Point the rate $= \dfrac{N(20)-N(16)}{20-16}$

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$$= \frac{2500 - 1900}{4}$$

$$= 400 \ bacteria/hr$$

b) It is smallest at $t = 0 \quad or \quad t = 36$

$$\frac{N(36) - N(32)}{4} = \frac{4900 - 4800}{4}$$

$$= 25 \ bacteria/hr$$

c) Growth rate $= \dfrac{N(36) - N(0)}{36}$

$$\approx \frac{4900 - 400}{36}$$

$$= 125 \ bacteria/hr$$