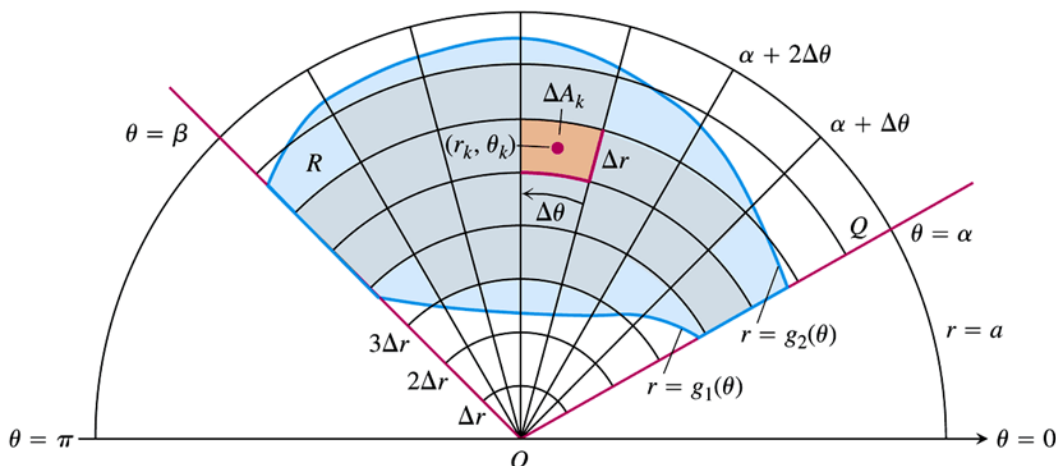


Section 3.3 – Double Integrals in Polar Coordinates

Integrals in Polar Coordinates



$$S_n = \sum_{k=1}^n f(r_k, \theta_k) \Delta A_k$$

If f is continuous throughout R , this sum will approach a limit as Δr and $\Delta \theta$ go to zero. The limit is called the double integral of f over R .

$$\lim_{n \rightarrow \infty} S_n = \iint_R f(r, \theta) dA$$

However, the area of a wedge-shaped sector of a circle having radius r and angle θ is

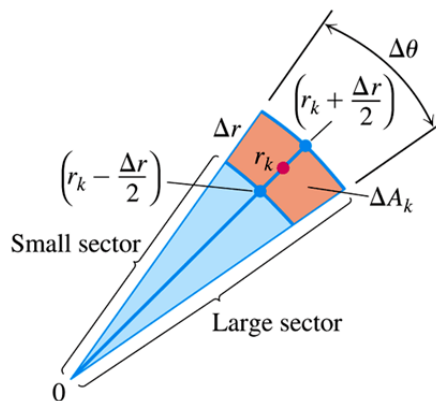
$$A = \frac{1}{2} \theta \cdot r^2$$

$$\text{Inner radius: } \frac{1}{2} \left(r_k - \frac{\Delta r}{2} \right)^2 \cdot \Delta \theta$$

$$\text{outer radius: } \frac{1}{2} \left(r_k + \frac{\Delta r}{2} \right)^2 \cdot \Delta \theta$$

$$\Delta A_k = \left(\text{area of large sector} \right) - \left(\text{area of small sector} \right)$$

$$\text{Leads to the formula: } \Delta A_k = r_k \Delta r \Delta \theta$$

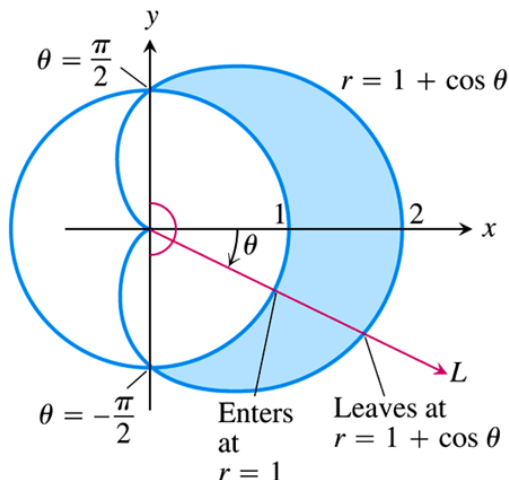


Example

Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$.

Solution

The sketch of the region:



From the graph, we can find the r - limits of integration. A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos \theta$

θ - limits of integration: The rays from the origin that intersect R run from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$. The integral is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^{1+\cos \theta} f(r, \theta) r \, dr \, d\theta$$

Area in Polar Coordinates

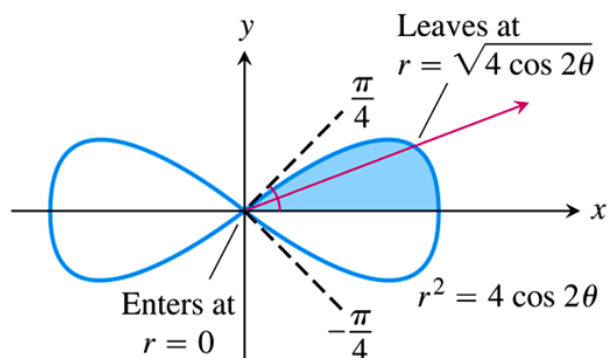
The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r \, dr \, d\theta$$

Example

Find the area enclosed by the lemniscate $r^2 = 4 \cos 2\theta$

Solution



From the graph, we can determine the lemniscate limits of integration, and the total area is 4 times the first-quadrant portion, since it has a form of symmetry.

$$\begin{aligned} A &= 4 \int_0^{\pi/4} \int_0^{\sqrt{4 \cos 2\theta}} r \, dr \, d\theta \\ &= 4 \int_0^{\pi/4} \left. \frac{1}{2} r^2 \right|_0^{\sqrt{4 \cos 2\theta}} d\theta \\ &= 4 \int_0^{\pi/4} (2 \cos 2\theta) d\theta \\ &= 4 \int_0^{\pi/4} \cos 2\theta \, d(2\theta) \\ &= 4 \sin 2\theta \Big|_0^{\pi/4} \\ &= 4 \sin \frac{\pi}{2} \\ &= \underline{4 \text{ unit}^2} \end{aligned}$$

Changing Cartesian Integrals into Polar Integrals

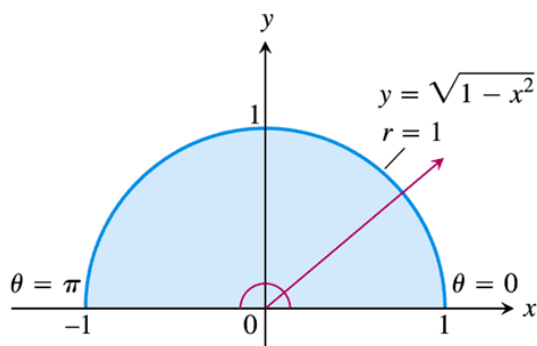
$$\iint_R f(x, y) dx dy = \iint_G f(r \cos \theta, r \sin \theta) \textcolor{red}{r} dr d\theta$$

Example

Evaluate $\iint_R e^{x^2+y^2} dy dx$

Where R is the semicircular region bounded by the x -axis and the curve $y = \sqrt{1-x^2}$

Solution

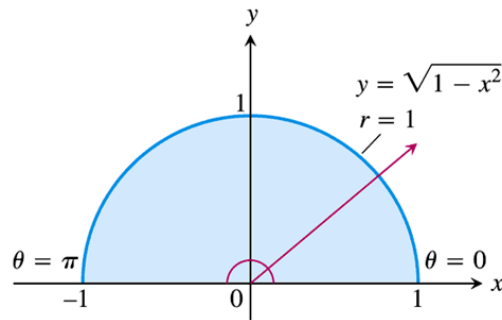


$$\begin{aligned} \iint_R e^{x^2+y^2} dy dx &= \int_0^\pi \int_0^1 e^{r^2} r dr d\theta & d(r^2) &= 2r dr \\ &= \frac{1}{2} \int_0^\pi d\theta \int_0^1 e^{r^2} d(r^2) \\ &= \frac{1}{2} \theta \bigg|_0^\pi e^{r^2} \bigg|_0^1 \\ &= \frac{1}{2} \int_0^\pi (e-1) d\theta \\ &= \frac{\pi}{2} (e-1) \end{aligned}$$

Example

Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx$

Solution



Since: $0 \leq x \leq 1 \rightarrow$ interior of $x^2 + y^2 = 1$ and in QI

Let: $r^2 = x^2 + y^2$ with $0 \leq r \leq 1$

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx &= \int_0^{\pi/2} \int_0^1 (r^2) r dr d\theta \\ &= \int_0^{\pi/2} d\theta \int_0^1 r^3 dr \\ &= \theta \left|_0^{\pi/2} \frac{1}{4} r^4 \right|_0^1 \\ &= \frac{\pi}{8} \end{aligned}$$

○ Or we can use the integral table to solve it

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx = \int_0^1 \left[x^2 \sqrt{1-x^2} + \frac{1}{3} (1-x^2)^{3/2} \right] dx$$

Example

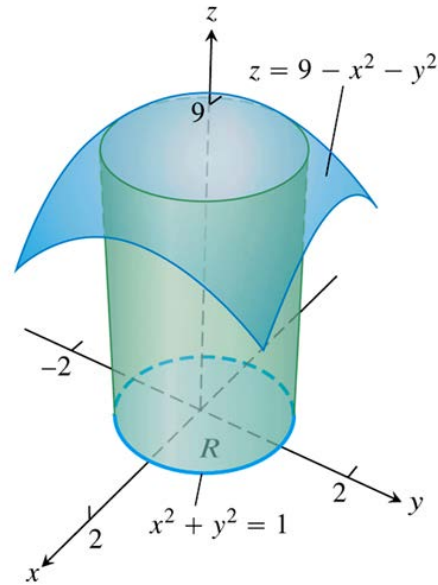
Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy -plane.

Solution

The region of integration R is the unit circle:

$$x^2 + y^2 = 1 \rightarrow r = 1, \quad 0 \leq \theta \leq 2\pi$$

$$\begin{aligned} \text{Volume} &= \int_0^{2\pi} \int_0^1 (9 - r^2) r \, dr \, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^1 (9r - r^3) \, dr \\ &= 2\pi \left(\frac{9}{2} r^2 - \frac{1}{4} r^4 \right) \bigg|_0^1 \\ &= 2\pi \left(\frac{9}{2} - \frac{1}{4} \right) \\ &= \underline{\underline{\frac{17\pi}{2} \text{ unit}^3}} \end{aligned}$$



Example

Using the polar integration, find the area of the region R in the xy -plane enclosed by the circle $x^2 + y^2 = 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.

Solution

The $y = \sqrt{3}x$ has a slope of $\sqrt{3} = \tan \theta \Rightarrow \underline{\underline{\theta = \frac{\pi}{3}}}$

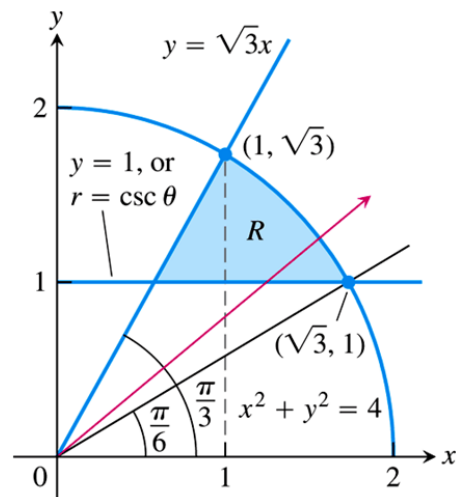
Line $y = 1$ intersects $x^2 + y^2 = 4$

when $x^2 + 1 = 4 \rightarrow x = \sqrt{3}$.

A line from origin to $(\sqrt{3}, 1)$ has a slope of

$$\frac{1}{\sqrt{3}} = \tan \theta \rightarrow \underline{\underline{\theta = \frac{\pi}{6}}}$$

$$\therefore \underline{\underline{\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}}}$$



The polar coordinate r varies from the horizontal line $y = 1$ to the circle $x^2 + y^2 = 4$.

Substituting $r \sin \theta$ for y :

$$y = 1 \rightarrow r \sin \theta = 1$$

$$r = \frac{1}{\sin \theta} = \csc \theta$$

The radius of the circle is 2.

$$\therefore \csc \theta \leq r \leq 2$$

$$\begin{aligned} \text{Area} &= \int_{\pi/6}^{\pi/3} \int_{\csc \theta}^2 r dr d\theta \\ &= \int_{\pi/6}^{\pi/3} \left. \frac{1}{2} r^2 \right|_{\csc \theta}^2 d\theta \\ &= \frac{1}{2} \int_{\pi/6}^{\pi/3} (4 - \csc^2 \theta) d\theta \\ &= \frac{1}{2} \left(4\theta + \cot \theta \right) \Big|_{\pi/6}^{\pi/3} \\ &= \frac{1}{2} \left[\frac{4\pi}{3} + \frac{1}{\sqrt{3}} - \left(\frac{4\pi}{6} + \sqrt{3} \right) \right] \\ &= \frac{1}{2} \left(\frac{2\pi}{3} + \frac{\sqrt{3}}{3} - \sqrt{3} \right) \\ &= \frac{1}{2} \left(\frac{2\pi - 2\sqrt{3}}{3} \right) \\ &= \frac{\pi - \sqrt{3}}{3} \text{ unit}^2 \end{aligned}$$

Exercises Section 3.3 – Double Integrals in Polar Coordinates

(1 – 16) Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

1. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} dy dx$

2. $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) dx dy$

3. $\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy dx$

4. $\int_0^6 \int_0^y x dx dy$

5. $\int_{-1}^0 \int_{-\sqrt{1-x^2}}^0 \frac{2}{1+\sqrt{x^2+y^2}} dy dx$

6. $\int_0^{\ln 2} \int_0^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2+y^2}} dx dy$

7. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1) dx dy$

8. $\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2 + y^2)^2} dy dx$

9. $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2 dy dx}{(1+x^2+y^2)^2}$

10. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1) dx dy$

11. $\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2} dx dy$

12. $\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2+y^2} dy dx$

13. $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2 + y^2)^{3/2} dy dx$

14. $\int_{-4}^4 \int_0^{\sqrt{16-y^2}} (16-x^2-y^2) dx dy$

15. $\int_0^{\pi/4} \int_0^{\sec \theta} r^3 dr d\theta$

16. $\int_0^{\pi/2} \int_1^\infty \frac{\cos \theta}{r^3} r dr d\theta$

17. Find the area of the region cut from the first quadrant by the curve $r = 2(2 - \sin 2\theta)^{1/2}$

18. Find the area of the region lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$

19. Find the area enclosed by one leaf of the rose $r = 12 \cos 3\theta$

20. Find the area of the region common to the interiors of the cardioids $r = 1 + \cos \theta$ and $r = 1 - \cos \theta$

21. Find the area of the region bounded by all leaves of the rose $r = 3 \cos 2\theta$

22. Find the area of the region inside both the circles $r = 2$ and $r = 4 \cos \theta$

23. Find the area of the region that lies inside both the cardioids $r = 2 - 2 \cos \theta$ and $r = 2 + 2 \cos \theta$
24. Find the area of the annular region $\{(r, \theta): 1 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$
25. Find the area of the region bounded by the cardioid $r = 2(1 - \sin \theta)$
26. Find the area of the region bounded by all leaves of the rose $r = 2 \cos 3\theta$
27. Find the area of the region inside both the cardioid $r = 1 - \cos \theta$ and the circle $r = 1$
28. Find the area of the region inside both the cardioid $r = 1 + \sin \theta$ and the cardioid $r = 1 + \cos \theta$
29. Find the area of the region bounded by the spiral $r = 2\theta$, for $0 \leq \theta \leq \pi$, and the x -axis.
30. Find the area of the region inside the limaçon $r = 1 + \frac{1}{2} \cos \theta$
31. Find the area of the region bounded by $r = 2 \sin 2\theta$
32. Find the area of the region bounded by $r^2 = 2 \sin 2\theta$
33. Find the area of the region outside the circle $r = 1$ and inside the rose $r = 2 \sin 3\theta$ in QI
34. Find the area of the region outside the circle $r = \frac{1}{2}$ and inside the circle $r = 1 + \cos \theta$
35. Integrate $f(x, y) = \frac{\ln(x^2 + y^2)}{\sqrt{x^2 + y^2}}$ over the region $1 \leq x^2 + y^2 \leq e$
36. The region enclosed by the lemniscates $r^2 = 2 \cos 2\theta$ is the base of a solid right cylinder whose top is bounded by the sphere $z = \sqrt{2 - r^2}$. Find the cylinder's volume.
37. Evaluate $\iint_R (x + y) dA$; R is the disk bounded by circle $r = 4 \sin \theta$
38. Find the volume of the solid bounded above by the paraboloid $z = 2 - x^2 - y^2$ and below by the plane $z = 1$
39. Find the volume of the solid bounded above by the paraboloid $z = 8 - x^2 - 3y^2$ and below by the hyperbolic paraboloid $z = x^2 - y^2$
- (40 – 51) Evaluate the integral over R using polar coordinates
40. $\iint_R (x^2 + y^2) dA$; $R = \{(r, \theta): 0 \leq r \leq 4, 0 \leq \theta \leq 2\pi\}$

41. $\iint_R 2xy dA; \quad R = \left\{ (r, \theta) : 1 \leq r \leq 3, \quad 0 \leq \theta \leq \frac{\pi}{2} \right\}$
42. $\iint_R 2xy \, dA; \quad R = \left\{ (x, y) : x^2 + y^2 \leq 9, \quad y \geq 0 \right\}$
43. $\iint_R \frac{dA}{1 + x^2 + y^2}; \quad R = \left\{ (r, \theta) : 1 \leq r \leq 2, \quad 0 \leq \theta \leq \pi \right\}$
44. $\iint_R \frac{dA}{\sqrt{16 - x^2 - y^2}}; \quad R = \left\{ (x, y) : x^2 + y^2 \leq 4, \quad y \geq 0 \right\}$
45. $\iint_R \frac{dA}{\sqrt{16 - x^2 - y^2}}; \quad R = \left\{ (x, y) : x^2 + y^2 \leq 4, \quad x, y \geq 0 \right\}$
46. $\iint_R e^{-x^2 - y^2} dA; \quad R = \left\{ (x, y) : x^2 + y^2 \leq 9 \right\}$
47. $\iint_R \sqrt{x^2 + y^2} \, dA; \quad R = \left\{ (x, y) : y \leq x \leq 1, \quad 0 \leq y \leq 1 \right\}$
48. $\iint_R \sqrt{x^2 + y^2} \, dA; \quad R = \left\{ (x, y) : 1 \leq x^2 + y^2 \leq 2 \right\}$
49. $\iint_R \frac{dA}{(x^2 + y^2)^{5/2}}; \quad R = \left\{ (r, \theta) : 1 \leq r \leq \infty, \quad 0 \leq \theta \leq 2\pi \right\}$
50. $\iint_R e^{-x^2 - y^2} dA; \quad R = \left\{ (r, \theta) : 0 \leq r \leq \infty, \quad 0 \leq \theta \leq \frac{\pi}{2} \right\}$
51. $\iint_R \frac{dA}{(1 + x^2 + y^2)^2}; \quad R \in QI$

52. Which bowl holds more water if it is filled to a depth of four units?

- a) The paraboloid $z = x^2 + y^2$, for $0 \leq z \leq 4$
- b) The cone $z = \sqrt{x^2 + y^2}$, for $0 \leq z \leq 4$
- c) The hyperboloid $z = \sqrt{1 + x^2 + y^2}$, for $1 \leq z \leq 5$

- d) To what weight (above the bottom of the bowl) must the cone and paraboloid bowls be filled to hold the same volume of water as the hyperboloid bowl filled to a depth of 4 units ($1 \leq z \leq 5$)

53. Consider the surface $z = x^2 - y^2$

a) Find the region in the xy -plane in polar coordinates for which $z \geq 0$.

b) Let $R = \left\{ (r, \theta) : 0 \leq r \leq a, -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \right\}$, which is a sector of a circle of radius a . Find the volume of the region below the hyperbolic paraboloid and above the region R .

54. A cake is shaped like a hemisphere of radius 4 with its base on the xy -plane. A wedge of the cake is removed by making two slices from the center of the cake outward, perpendicular to the xy -plane and separated by an angle of φ .

a) Use a double integral to find the volume of the slice for $\varphi = \frac{\pi}{4}$.

b) Suppose the cake is sliced by a plane perpendicular to the xy -plane at $x = a > 0$. Let D be the smaller of the two pieces produced. For what value of a is the volume of D equal to the volume in part (a)?

55. Suppose the density of a thin plate represented by the region R is $\rho(r, \theta)$ (in units of mass per

area). The mass of the plate is $\iint_R \rho(r, \theta) dA$. Find the mass of the thin half annulus

$R = \left\{ (r, \theta) : 1 \leq r \leq 4, 0 \leq \theta \leq \pi \right\}$ with a density $\rho(r, \theta) = 4 + r \sin \theta$

56. An important integral in statistics associated with the normal distribution is $I = \int_{-\infty}^{\infty} e^{-x^2} dx$. It is evaluated in the following steps.

a) Assume that
$$I = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2 - y^2} dx dy$$

Where we have chosen the variables of integration to be x and y and then written the product as an iterated integral. Evaluate this integral in polar coordinates and show that $I = \sqrt{\pi}$. Why is the solution $I = -\sqrt{\pi}$ rejected?

b) Evaluate $\int_0^{\infty} e^{-x^2} dx$, $\int_0^{\infty} x e^{-x^2} dx$, and $\int_0^{\infty} x^2 e^{-x^2} dx$.

57. For what values of p does the integral $\iint_R \frac{k}{(x^2 + y^2)^p} dA$ exist in the following cases?

a) $R = \{(r, \theta): 1 \leq r \leq \infty, 0 \leq \theta \leq 2\pi\}$

b) $R = \{(r, \theta): 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$

58. Consider the integral $\iint_R \frac{k}{(1 + x^2 + y^2)^2} dA$ where $R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq a\}$

a) Evaluate I for $a = 1$.

b) Evaluate I for arbitrary $a > 0$.

c) Let $a \rightarrow \infty$ in part (b) to find I over the infinite strip $R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq \infty\}$

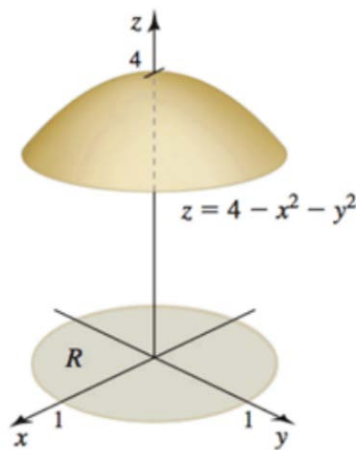
59. In polar coordinates an equation of an ellipse with eccentricity $0 < e < 1$ and semimajor axis a is

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta}$$

a) Write the integral that gives the area of the ellipse.

b) Show that the area of an ellipse is πab , where $b^2 = a^2(1 - e^2)$

(60 – 63) Find the volume of the solid below the paraboloid $z = 4 - x^2 - y^2$ and above



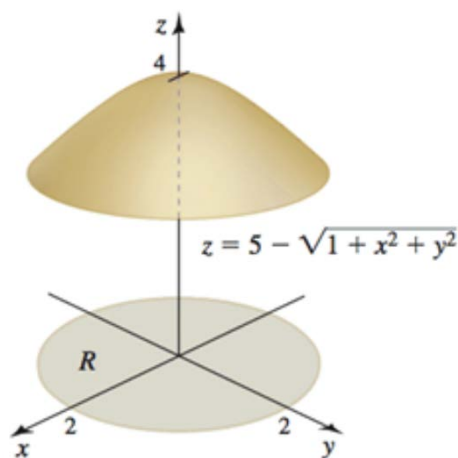
60. $R = \{(r, \theta): 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$

61. $R = \{(r, \theta): 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$

62. $R = \{(r, \theta): 1 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$

63. $R = \{(r, \theta): 1 \leq r \leq 2, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$

(64 – 67) Find the volume of the solid below the hyperboloid $z = 5 - \sqrt{1 + x^2 + y^2}$ and above



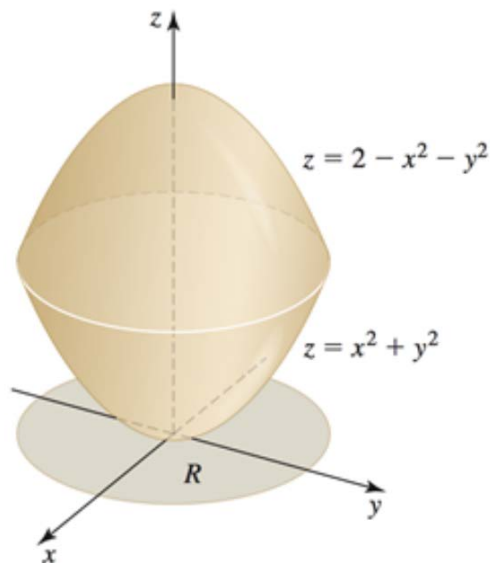
64. $R = \{(r, \theta) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$

65. $R = \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq \pi\}$

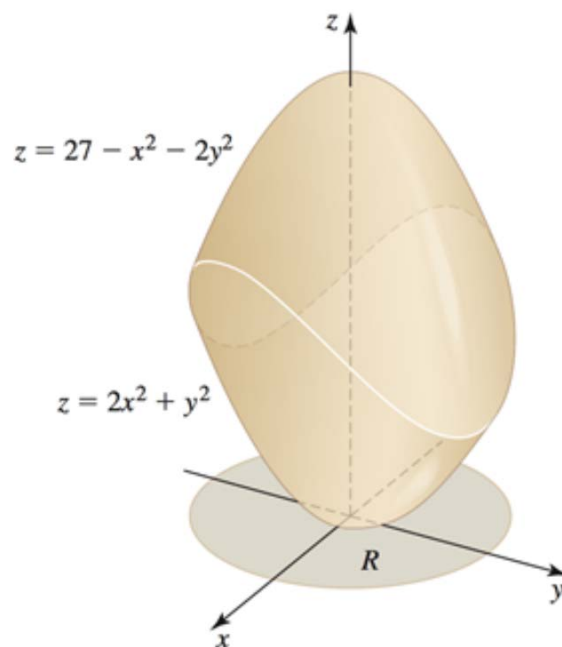
66. $R = \{(r, \theta) : \sqrt{3} \leq r \leq 2\sqrt{2}, 0 \leq \theta \leq 2\pi\}$

67. $R = \{(r, \theta) : \sqrt{3} \leq r \leq \sqrt{15}, -\frac{\pi}{2} \leq \theta \leq \pi\}$

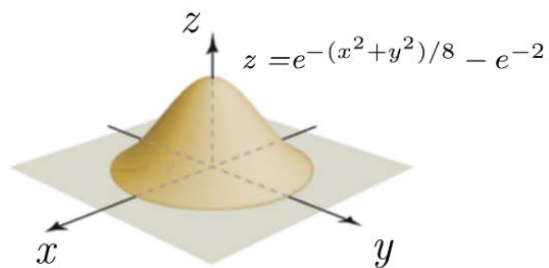
68. Find the volume of the solid between the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$



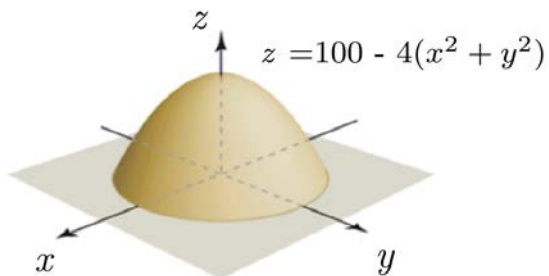
69. Find the volume of the solid between the paraboloids $z = 2x^2 + y^2$ and $z = 27 - x^2 - 2y^2$



70. Find the volume of island $z = e^{-(x^2+y^2)/8} - e^{-2}$



71. Find the volume of island $z = 100 - 4(x^2 + y^2)$



72. Find the volume of island $z = 25 - \sqrt{x^2 + y^2}$

