Solution Section 2.2 – Trigonometric Integrals

Exercise

Evaluate the integral $\int \sin^4 2x \cos 2x dx$

Solution

$$d(\sin 2x) = 2\cos 2x dx \quad \Rightarrow \quad \frac{1}{2} d(\sin 2x) = \cos 2x dx$$

$$\int \sin^4 2x \cos 2x dx = \frac{1}{2}\int \sin^4 2x \; d(\sin 2x)$$

$$= \frac{1}{10}\sin^5 2x + C$$

Exercise

Evaluate the integral $\int \sin^5 \frac{x}{2} dx$

Solution

$$\sin^5 \frac{x}{2} = \left(\sin^2 \frac{x}{2}\right)^2 \sin \frac{x}{2}$$

$$= \left(1 - \cos^2 \frac{x}{2}\right)^2 \sin \frac{x}{2}$$

$$= \left(1 - 2\cos^2 \frac{x}{2} + \cos^4 \frac{x}{2}\right)\sin \frac{x}{2}$$

$$d\left(\cos \frac{x}{2}\right) = -\frac{1}{2}\sin \frac{x}{2} dx \quad \rightarrow \quad -2d\left(\cos \frac{x}{2}\right) = \sin \frac{x}{2} dx$$

$$\int \sin^5 \frac{x}{2} dx = -2\int \left(1 - 2\cos^2 \frac{x}{2} + \cos^4 \frac{x}{2}\right) d\left(\cos \frac{x}{2}\right)$$

$$= -2\left(\cos \frac{x}{2} - \frac{2}{3}\cos^3 \frac{x}{2} + \frac{1}{5}\cos^5 \frac{x}{2}\right) + C$$

$$= -2\cos \frac{x}{2} + \frac{4}{3}\cos^3 \frac{x}{2} - \frac{2}{5}\cos^5 \frac{x}{2} + C$$

Exercise

Evaluate the integral $\int \cos^3 2x \sin^5 2x \; dx$

Solution

$$\int \cos^3 2x \sin^5 2x \; dx = \int \left(\cos^2 2x\right)\cos 2x \sin^5 2x \; dx \qquad d(\sin 2x) = 2\cos 2x \; dx$$

$$= \int \left(1 - \sin^2 2x\right) \sin^5 2x \ \left(\tfrac{1}{2} d \sin 2x\right)$$

$$= \frac{1}{2} \int \left(\sin^5 2x - \sin^7 2x\right) \ \left(d \sin 2x\right)$$

$$= \frac{1}{2}\left(\frac{1}{6}\sin^6 2x - \frac{1}{8}\sin^8 2x\right) + C$$

$$= \frac{1}{12}\sin^6 2x - \frac{1}{16}\sin^8 2x + C$$

Exercise

Evaluate the integral $\int 8\cos^4 2\pi x \, dx$

Solution

$$\int 8\cos^4 2\pi x \, dx = 8\int \left(\cos 2\pi x\right)^4 dx \qquad \cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$= 8\int \left(\frac{1+\cos 4\pi x}{2}\right)^2 dx$$

$$= 2\int \left(1+\cos 4\pi x\right)^2 dx$$

$$= 2\int \left(1 + 2\cos 4\pi x + \cos^2 4\pi x\right) dx$$

$$= 2\int dx + 4\int \cos 4\pi x dx + 2\int \cos^2 4\pi x dx$$

$$= 2x + 4\frac{1}{4\pi}\cos 4\pi x + 2\int \frac{1+\cos 8\pi x}{2} dx$$

$$= 2x + \frac{1}{\pi}\cos 4\pi x + \int \left(1+\cos 8\pi x\right) dx$$

$$= 2x + \frac{1}{\pi}\sin 4\pi x + x + \frac{1}{8\pi}\sin 8\pi x + C$$

$$= 3x + \frac{1}{\pi}\sin 4\pi x + \frac{1}{8\pi}\sin 8\pi x + C$$

Exercise

Evaluate the integral $\int 16\sin^2 x\cos^2 x dx$

Solution

$$\int 16\sin^2 x\cos^2 x dx = 16\int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx \qquad \cos^2\alpha = \frac{1+\cos 2\alpha}{2} \quad \sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$= 4\int \left(1-\cos^2 2x\right)dx$$

$$= 4\int \left(1-\frac{1+\cos 4x}{2}\right)dx$$

$$= 4\int \frac{1-\cos 4x}{2}dx$$

$$= 2\left(x-\frac{1}{4}\sin 4x\right)+C$$

$$= 2x-\frac{1}{2}(2\sin 2x\cos 2x)+C$$

$$= 2x-(2\sin x\cos x)\left(2\cos^2 x-1\right)+C$$

$$\boxed{= 2x-4\sin x\cos^3 x+2\sin x\cos x+C}$$

Exercise

Evaluate the integral $\displaystyle\int \sec x\tan^2 x\, dx$

Solution

$$\int \sec x\tan^2 x\, dx = \int \sec x\tan x\tan x\, dx \qquad u=\tan x \quad dv=\sec x\tan x dx$$

$$du=\sec^2 x dx \quad v=\sec x$$

$$\int \sec x\tan^2 x\, dx = \tan x\sec x - \int \sec x\sec^2 x\, dx$$

$$= \tan x\sec x - \int \sec x\left(1+\tan^2 x\right) dx$$

$$= \tan x\sec x - \left[\int \sec x\, dx + \int \sec x\tan^2 x\, dx\right]$$

$$= \tan x\sec x - \ln\left|\sec x+\tan x\right| - \int \sec x\tan^2 x\, dx$$

$$\int \sec x\tan^2 x\, dx + \int \sec x\tan^2 x\, dx = \tan x\sec x - \ln\left|\sec x+\tan x\right|$$

$$2\int \sec x\tan^2 x\, dx = \tan x\sec x - \ln\left|\sec x+\tan x\right|$$

$$\int \sec x\tan^2 x\, dx = \boxed{\frac{1}{2}\tan x\sec x - \frac{1}{2}\ln\left|\sec x+\tan x\right| + C}$$

Exercise

Evaluate the integral $\int \sec^2 x \tan^2 x\, dx$

Solution

$$\int \sec^2 x \tan^2 x dx = \int \tan^2 x\, d(\tan x) \qquad\qquad d(\tan x) = \sec^2 x dx$$

$$= \frac{1}{3}\tan^3 x + C$$

Exercise

Evaluate the integral $\int \sec^4 x \tan^2 x\, dx$

Solution

$$\int \sec^4 x \tan^2 x dx = \int \sec^2 x \sec^2 x \tan^2 x\, dx \qquad\qquad d(\tan x) = \sec^2 x dx \quad \sec^2 x = 1 + \tan^2 x$$

$$= \int \left(1 + \tan^2 x\right)\tan^2 x\, d(\tan x)$$

$$= \int \left(\tan^2 x + \tan^4 x\right)\, d(\tan x)$$

$$= \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$$

Exercise

Evaluate the integral $\int e^x \sec^3\left(e^x\right)\, dx$

Solution

$$u = \sec\left(e^x\right) \qquad\qquad dv = \sec\left(e^x\right)e^x dx$$

$$du = \sec\left(e^x\right)\tan\left(e^x\right)e^x dx \qquad v = \int \sec\left(e^x\right)d\left(e^x\right) = \tan\left(e^x\right)$$

$$\int e^x \sec^3\left(e^x\right)\, dx = \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec\left(e^x\right)\tan^2\left(e^x\right)e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec\left(e^x\right)\left(\sec^2\left(e^x\right) - 1\right)e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \left(\sec^3\left(e^x\right) - \sec\left(e^x\right)\right)e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec^3\left(e^x\right)e^x dx + \int \sec\left(e^x\right)e^x dx \qquad d\left(e^x\right) = e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec^3\left(e^x\right)e^x dx + \int \sec\left(e^x\right)d\left(e^x\right)$$

$$\int \sec^3\left(e^x\right)e^x dx = \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec^3\left(e^x\right)e^x dx + \ln\left|\sec\left(e^x\right) + \tan\left(e^x\right)\right|$$

$$2\int \sec^3\left(e^x\right)e^x dx = \sec\left(e^x\right)\tan\left(e^x\right) + \ln\left|\sec\left(e^x\right) + \tan\left(e^x\right)\right| + C$$

$$\int \sec^3\left(e^x\right)e^x dx = \underline{\tfrac{1}{2}\sec\left(e^x\right)\tan\left(e^x\right) + \tfrac{1}{2}\ln\left|\sec\left(e^x\right) + \tan\left(e^x\right)\right| + C}$$

Exercise

Evaluate $\int \sin^4 x \cos^2 x\, dx$

Solution

$$\int \sin^4 x \cos^2 x\, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{8}\int \left(1 - 2\cos 2x + \cos^2 2x\right)\left(1+\cos 2x\right) dx$$

$$= \frac{1}{8}\int \left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right) dx$$

$$= \frac{1}{8}\int \left(1 - \cos 2x - \frac{1}{2} - \frac{1}{2}\cos 4x\right) dx + \frac{1}{8}\int \cos^2 2x \cos 2x\, dx$$

$$= \frac{1}{8}\int \left(\frac{1}{2} - \cos 2x - \frac{1}{2}\cos 4x\right) dx + \frac{1}{16}\int \left(1 - \sin^2 2x\right)\, d\left(\sin 2x\right)$$

$$= \frac{1}{8}\left(\frac{1}{2}x - \frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x\right) + \frac{1}{16}\sin 2x - \frac{1}{48}\sin^3 2x + C$$

$$= \underline{\frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C}$$

Exercise

Evaluate $\int \tan^3 x \sec^4 x\, dx$

Solution

$$\int \tan^3 x \sec^4 x\, dx = \int \tan^3 x\left(1 + \tan^2 x\right)\sec^2 x\, dx \qquad \sec^2 x = 1 + \tan^2 x$$

$$= \int \left(\tan^3 x + \tan^5 x \right) d(\tan x) \qquad d(\tan x) = \sec^2 x dx$$

$$\boxed{= \frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + C}$$

Exercise

Evaluate $\displaystyle\int \sin 3x \cos 7x \, dx$

Solution

$$\int \sin 3x \cos 7x \, dx = \frac{1}{2}\int \left(\sin(-4x) + \sin 10x \right) dx \qquad \sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x \right]$$

$$= \frac{1}{2}\int \left(-\sin 4x + \sin 10x \right) dx$$

$$= \frac{1}{2}\left(\frac{1}{4}\cos 4x - \frac{1}{10}\cos 10x \right) + C$$

$$\boxed{= \frac{1}{8}\cos 4x - \frac{1}{20}\cos 10x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sin 2x \cos 3x \, dx$

Solution

$$\int \sin 2x \cos 3x \, dx = \frac{1}{2}\int \left(\sin 5x + \sin(-x) \right) dx \qquad \sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta) \right]$$

$$= \frac{1}{2}\int \left(\sin 5x - \sin x \right) dx$$

$$= \frac{1}{2}\left(-\frac{1}{5}\cos 5x + \cos x \right) + C$$

$$\boxed{= \frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C}$$

Exercise

Evaluate the integral $\int \sin^2\theta \cos 3\theta \, d\theta$

Solution

$$\int \sin^2\theta \cos 3\theta \, d\theta = \int \frac{1-\cos 2\theta}{2}\cos 3\theta \, d\theta \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \frac{1}{2}\int \left(\cos 3\theta - \cos 2\theta \cos 3\theta\right) d\theta$$

$$= \frac{1}{2}\int \cos 3\theta \, d\theta - \frac{1}{2}\int \cos 2\theta \cos 3\theta \, d\theta$$

$$= \frac{1}{6}\sin 3\theta - \frac{1}{2}\int \frac{1}{2}\left(\cos(5\theta) + \cos(-\theta)\right) d\theta \qquad \cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$= \frac{1}{6}\sin 3\theta - \frac{1}{4}\left(\frac{1}{5}\sin 5\theta + \sin\theta\right) + C$$

$$\boxed{= \frac{1}{6}\sin 3\theta - \frac{1}{20}\sin 5\theta - \frac{1}{4}\sin\theta + C}$$

Exercise

Evaluate the integral $\int \cos^3\theta \sin 2\theta \, d\theta$

Solution

$$\int \cos^3\theta \sin 2\theta \, d\theta = \int \cos^3\theta \left(2\sin\theta\cos\theta\right) d\theta \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= -2\int \cos^4\theta \, d(\cos\theta) \qquad d(\cos\theta) = -\sin\theta d\theta$$

$$\boxed{= -\frac{2}{5}\cos^5\theta + C}$$

Exercise

Evaluate the integral $\int \sin\theta \sin 2\theta \sin 3\theta \, d\theta$

Solution $\qquad \sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$

$$\int \sin\theta \sin 2\theta \sin 3\theta \, d\theta = \int \frac{1}{2}\left(\cos(1-2)\theta - \cos(1+2)\theta\right) \sin 3\theta \, d\theta$$

$$= \frac{1}{2}\int \left(\cos(-\theta) - \cos(3\theta)\right) \sin 3\theta \, d\theta$$

$$= \frac{1}{2}\int \cos\theta \ \sin 3\theta \ d\theta - \frac{1}{2}\int \cos 3\theta \ \sin 3\theta \ d\theta$$

$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$

$$= \frac{1}{4}\int (\sin 4\theta + \sin 2\theta) \ d\theta - \frac{1}{4}\int (\sin 6\theta + \sin(0)) \ d\theta$$

$$= \frac{1}{4}\left(-\frac{1}{4}\cos 4\theta - \frac{1}{2}\cos 2\theta\right) + \frac{1}{24}\cos 6\theta + C$$

$$\boxed{= -\frac{1}{16}\cos 4\theta - \frac{1}{8}\cos 2\theta + \frac{1}{24}\cos 6\theta + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sin^3 x}{\cos^4 x}\,dx$

Solution

$$\int \frac{\sin^3 x}{\cos^4 x}\,dx = \int \frac{\sin^2 x \sin x}{\cos^4 x}\,dx$$

$\cos^2\alpha + \sin^2\alpha = 1$

$$= \int \frac{\left(1-\cos^2 x\right)\sin x}{\cos^4 x}\,dx$$

$$= -\int \left(\frac{1}{\cos^4 x} - \frac{\cos^2 x}{\cos^4 x}\right) d(\cos x)$$

$$= -\int \left(\cos^{-4} x - \cos^{-2} x\right) d(\cos x)$$

$$= -\left(-\frac{1}{3}\cos^{-3} x + \cos^{-1} x\right) + C$$

$$= \frac{1}{3}\frac{1}{\cos^3 x} - \frac{1}{\cos x} + C$$

$$\boxed{= \frac{1}{3}\csc^3 x - \csc x + C}$$

Exercise

Evaluate the integral $\displaystyle\int x\cos^3 x\ dx$

Solution

$$\int x\cos^3 x dx = \int x\cos^2 x \cos x\ dx$$

$$= \int x\left(1-\sin^2 x\right)\cos x\ dx$$

$\cos^2\alpha + \sin^2\alpha = 1$

$$= \int x\cos x\,dx - \int x\sin^2 x\cos x\,dx$$

$u = x \qquad dv = \cos x dx \qquad\qquad u = x \qquad dv = \sin^2 x\cos x dx$

$du = dx \qquad v = \sin x \qquad\qquad du = dx \qquad v = \frac{1}{3}\sin^3 x$

$$= x\sin x - \int \sin x\,dx - \left(\frac{1}{3}x\sin^3 x - \frac{1}{3}\int \sin^3 x\,dx\right)$$

$$= x\sin x + \cos x - \frac{1}{3}x\sin^3 x + \frac{1}{3}\int \sin^2 x\sin x\,dx$$

$$= x\sin x + \cos x - \frac{1}{3}x\sin^3 x - \frac{1}{3}\int \left(1-\cos^2 x\right)d\left(\cos x\right)$$

$$= x\sin x + \cos x - \frac{1}{3}x\sin^3 x - \frac{1}{3}\left(\cos x - \frac{1}{3}\cos^3 x\right) + C$$

$$= x\sin x + \cos x - \frac{1}{3}x\sin^3 x - \frac{1}{3}\cos x + \frac{1}{9}\cos^3 x + C$$

$$= x\sin x + \frac{2}{3}\cos x - \frac{1}{3}x\sin^3 x + \frac{1}{9}\cos^3 x + C$$

Exercise

Evaluate the integral $\displaystyle\int \sin^3 x\cos^4 x\,dx$

Solution

$$\int \sin^3 x\cos^4 x\,dx = \int \sin^2 x\cos^4 x\ \sin x\,dx$$

$$= -\int \left(1-\cos^2 x\right)\cos^4 x\,d\left(\cos x\right)$$

$$= \int \left(\cos^6 x - \cos^4 x\right)\,d\left(\cos x\right)$$

$$= \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$$

Exercise

Evaluate the integral $\displaystyle\int \cos^4 x\,dx$

Solution

$$\int \cos^4 x\,dx = \frac{1}{4}\int \left(1+\cos 2x\right)^2 dx \qquad\qquad \cos^2 x = \frac{1}{2}\left(1+\cos 2x\right)$$

$$= \frac{1}{4}\int \left(1+2\cos 2x+\cos^2 2x\right)dx$$

$$= \frac{1}{4}\int\left(1+2\cos 2x+\frac{1}{2}+\frac{1}{2}\cos 4x\right)dx$$

$$= \frac{1}{4}\int\left(\frac{3}{2}+2\cos 2x+\frac{1}{2}\cos 4x\right)dx$$

$$\boxed{= \frac{1}{4}\left(\frac{3}{2}x+\sin 2x+\frac{1}{8}\sin 4x\right)+C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx$

Solution

$$\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx = \int \frac{\tan^2 x\tan x}{(\sec x)^{1/2}}\frac{\sec x}{\sec x}dx \qquad 1+\tan^2\alpha = \sec^2\alpha$$

$$= \int (\sec x)^{-3/2}\left(\sec^2 x-1\right)d(\sec x)$$

$$= \int \left((\sec x)^{1/2}-(\sec x)^{-3/2}\right)d(\sec x)$$

$$\boxed{= \frac{2}{3}(\sec x)^{3/2}+2(\sec x)^{-1/2}+C}$$

Exercise

Evaluate the integral $\displaystyle\int \sec^4 3x\tan^3 3x\,dx$

Solution

$$\int \sec^4 3x\tan^3 3x\,dx = \int \sec^2 3x\tan^3 3x\ \sec^2 3x\,dx$$

$$= \frac{1}{3}\int\left(1+\tan^2 3x\right)\tan^3 3x\,d(\tan 3x)$$

$$= \frac{1}{3}\int\left(\tan^3 3x+\tan^5 3x\right)\,d(\tan 3x)$$

$$= \frac{1}{3}\left(\frac{1}{4}\tan^4 3x+\frac{1}{6}\tan^6 3x\right)+C$$

$$\boxed{= \frac{1}{12}\tan^4 3x+\frac{1}{18}\tan^6 3x+C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sec x}{\tan^2 x}\,dx$

Solution

$$\int \frac{\sec x}{\tan^2 x}\,dx = \int \frac{1}{\cos x}\frac{\cos^2 x}{\sin^2 x}\,dx$$

$$= \int \frac{\cos x}{\sin^2 x}\,dx$$

$$= \int \frac{1}{\sin^2 x}\,d(\sin x)$$

$$= -\frac{1}{\sin x} + C$$

$$\underline{= -\csc x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sin 5x \cos 4x\; dx$

Solution

$$\int \sin 5x \cos 4x\; dx = \frac{1}{2}\int (\sin x + \sin 9x)\,dx \qquad \sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$= \frac{1}{2}\left(-\cos x - \frac{1}{9}\cos x9x\right) + C$$

$$\underline{= \frac{1}{2} - \cos x - \frac{1}{18}\cos x9x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sin x \cos^5 x\; dx$

Solution

$$\int \sin x \cos^5 x\; dx = -\int \cos^5 x\; d(\cos x)$$

$$\underline{= -\frac{1}{6}\cos^6 x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sin^4 x \cos^3 x\; dx$

Solution

$$\int \sin^4 x\cos^3 x\, dx = \int \sin^4 x\left(1-\sin^2 x\right)\, d(\sin x)$$

$$= \int \left(\sin^4 x - \sin^6 x\right)\, d(\sin x)$$

$$= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$$

Exercise

Evaluate the integral $\displaystyle\int \sin^7 2x\ \cos 2x\, dx$

Solution

$$\int \sin^7 2x\ \cos 2x\, dx = \frac{1}{2}\int \sin^7 2x\, d(\sin 2x)$$

$$= \frac{1}{16}\sin^8 2x + C$$

Exercise

Evaluate the integral $\displaystyle\int \sin^3 2x\sqrt{\cos 2x}\, dx$

Solution

$$\int \sin^3 2x\sqrt{\cos 2x}\, dx = -\frac{1}{2}\int \left(1-\cos^2 2x\right)(\cos 2x)^{1/2}\, d(\cos 2x)$$

$$= -\frac{1}{2}\int \left((\cos 2x)^{1/2} - (\cos 2x)^{5/2}\right)\, d(\cos 2x)$$

$$= -\frac{1}{2}\left(\frac{2}{3}(\cos 2x)^{3/2} - \frac{2}{7}(\cos 2x)^{7/2}\right) + C$$

$$= \frac{1}{7}(\cos 2x)^{7/2} - \frac{1}{3}(\cos 2x)^{3/2} + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\cos^5\theta}{\sqrt{\sin\theta}}\, d\theta$

Solution

$$\int \frac{\cos^5\theta}{\sqrt{\sin\theta}}\, d\theta = \int (\sin\theta)^{-1/2}\left(1-\sin^2\theta\right)^2\, d(\sin\theta)$$

$$= \int (\sin\theta)^{-1/2}\left(1-2\sin^2\theta+\sin^4\theta\right)\, d(\sin\theta)$$

$$=\int\left((\sin\theta)^{-1/2}-2(\sin\theta)^{3/2}+(\sin\theta)^{7/2}\right)\,d(\sin\theta)$$

$$\underline{=2(\sin\theta)^{1/2}-\tfrac{1}{5}(\sin\theta)^{5/2}+\tfrac{2}{9}(\sin\theta)^{9/2}+C}$$

Exercise

Evaluate $\quad \displaystyle\int \sin^4 6\theta\ d\theta$

Solution

$$\int \sin^4 6\theta\ d\theta=\int\left(\frac{1-\cos 12\theta}{2}\right)^2 d\theta \qquad \sin^2\alpha=\tfrac{1}{2}(1-\cos 2\alpha)$$

$$=\frac{1}{4}\int\left(1-2\cos 12\theta+\cos^2 12\theta\right)d\theta \qquad \cos^2\alpha=\tfrac{1}{2}(1+\cos 2\alpha)$$

$$=\frac{1}{4}\int\left(1-2\cos 12\theta+\frac{1}{2}+\frac{1}{2}\cos 24\theta\right)d\theta$$

$$\underline{=\frac{1}{4}\left(\frac{3}{2}\theta-\frac{1}{6}\sin 12\theta+\frac{1}{48}\sin 24\theta\right)+C}$$

Exercise

Evaluate $\quad \displaystyle\int \cos^2 3x\ dx$

Solution

$$\int \cos^2 3x\ dx=\frac{1}{2}\int(1+\cos 6x)\,dx \qquad \cos^2\alpha=\tfrac{1}{2}(1+\cos 2\alpha)$$

$$\underline{=\frac{1}{2}\left(x+\frac{1}{6}\sin 6x\right)+C}$$

Exercise

Evaluate $\quad \displaystyle\int x^2\sin^2 x\ dx$

Solution

$$\int x^2\sin^2 x\ dx=\frac{1}{2}\int x^2(1-\cos 2x)\,dx$$

$$=\frac{1}{2}\int\left(x^2-x^2\cos 2x\right)dx$$

$$\underline{=\frac{1}{2}\left(\frac{1}{3}x^3+\frac{1}{2}x^2\sin 2x+\frac{1}{2}x\cos 2x-\frac{1}{4}\sin 2x\right)+C}$$

		$\int \cos 2x$
+	x^2	$\frac{1}{2}\sin 2x$
−	$2x$	$-\frac{1}{4}\cos 2x$
+	2	$-\frac{1}{8}\sin 2x$

Exercise

Evaluate $\displaystyle\int \sin^3 3x\, dx$

Solution

$$\int \sin^3 3x\, dx = \int \sin^2 3x(\sin 3x)\, dx \qquad d(\cos 3x) = -3\sin 3x\, dx \qquad \cos^2\alpha + \sin^2\alpha = 1$$

$$= -\frac{1}{3}\int \left(1-\cos^2 3x\right)\, d(\cos 3x)$$

$$= -\frac{1}{3}\left(\cos 3x - \frac{1}{3}\cos^3 3x\right) + C$$

$$= \frac{1}{9}\cos^3 3x - \frac{1}{3}\cos 3x + C$$

Exercise

Evaluate $\displaystyle\int \sin^3 x\cos^2 x\, dx$

Solution

$$\int \sin^3 x\cos^2 x\, dx = \int \sin^2 x\cos^2 x\, \sin x dx \qquad d(\cos x) = -\sin x\, dx$$

$$= -\int \left(1-\cos^2 x\right)\cos^2 x\, d(\cos x)$$

$$= \int \left(\cos^4 x - \cos^2 x\right)\, d(\cos x)$$

$$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$$

Exercise

Evaluate $\displaystyle\int \cos^3 \frac{x}{3}\, dx$

Solution

$$\int \cos^3 \frac{x}{3}\, dx = \int \cos^2 \frac{x}{3}\cos\frac{x}{3}\, dx$$

$$= 3\int \left(1-\sin^2\frac{x}{3}\right)\, d\left(\sin\frac{x}{3}\right)$$

$$= 3\sin\frac{x}{3} - \sin^3\frac{x}{3} + C$$

Exercise

Evaluate $\displaystyle\int \sec^4 2x\, dx$

Solution

$$\int \sec^4 2x\, dx = \int \left(1+\tan^2 2x\right)\sec^2 2x\, dx$$

$$= \frac{1}{2}\int \left(1+\tan^2 2x\right)\, d\left(\tan 2x\right)$$

$$\boxed{= \frac{1}{2}\tan 2x + \frac{1}{6}\tan^3 2x + C}$$

Exercise

Evaluate $\displaystyle\int \sec^3 \pi x\, dx$

Solution

$$u = \sec \pi x \qquad dv = \sec^2 \pi x\, dx$$

$$du = \pi \sec \pi x \tan \pi x \qquad v = \frac{1}{\pi}\tan \pi x$$

$$\int \sec^3 \pi x\, dx = \frac{1}{\pi}\sec \pi x \tan \pi x - \int \sec \pi x \tan^2 \pi x\, dx$$

$$= \frac{1}{\pi}\sec \pi x \tan \pi x - \int \sec \pi x\left(\sec^2 \pi x - 1\right)\, dx$$

$$\int \sec^3 \pi x\, dx = \frac{1}{\pi}\sec \pi x \tan \pi x - \int \sec^3 \pi x\, dx + \int \sec \pi x\, dx$$

$$2\int \sec^3 \pi x\, dx = \frac{1}{\pi}\sec \pi x \tan \pi x + \ln\left|\sec \pi x + \tan \pi x\right| + C_1$$

$$\boxed{\int \sec^3 \pi x\, dx = \frac{1}{2\pi}\sec \pi x \tan \pi x + \frac{1}{2}\ln\left|\sec \pi x + \tan \pi x\right| + C}$$

Exercise

Evaluate $\displaystyle\int \tan^6 3x\, dx$

Solution

$$\int \tan^6 3x\, dx = \int \left(\sec^2 3x - 1\right)\tan^4 3x\, dx$$

$$= \frac{1}{3}\int \tan^4 3x\, d\left(\tan 3x\right) - \int \left(\sec^2 3x - 1\right)\tan^2 3x\, dx$$

$$= \frac{1}{15}\tan^5 3x - \int \sec^2 3x \tan^2 3x\ dx + \int \tan^2 3x\ dx$$

$$= \frac{1}{15}\tan^5 3x - \frac{1}{3}\int \tan^2 3x\ d(\tan 3x) + \int \left(\sec^2 3x - 1\right)\ dx$$

$$= \frac{1}{15}\tan^5 3x - \frac{1}{9}\tan^3 3x + \frac{1}{3}\int\ d(\tan 3x) - \int\ dx$$

$$\boxed{= \frac{1}{15}\tan^5 3x - \frac{1}{9}\tan^3 3x + \frac{1}{3}\tan 3x - x + C}$$

Exercise

Evaluate $\displaystyle\int \tan^3 \frac{\pi x}{2}\ \sec^2 \frac{\pi x}{2}\ dx$

Solution

$$\int \tan^3 \frac{\pi x}{2}\ \sec^2 \frac{\pi x}{2}\ dx = \frac{2}{\pi}\int \tan^3 \frac{\pi x}{2}\ \ d\left(\tan \frac{\pi x}{2}\right)$$

$$\boxed{= \frac{1}{2\pi}\tan^4 \frac{\pi x}{2} + C}$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\, dx$

Solution

$$\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\, dx = \int_{\pi/6}^{\pi/3} (\sin x)^{-1/2}\left(1 - \sin^2 x\right)\ d(\sin x)$$

$$= \int_{\pi/6}^{\pi/3} \left((\sin x)^{-1/2} - (\sin x)^{3/2}\right)\ d(\sin x)$$

$$= 2(\sin x)^{1/2} - \frac{2}{5}(\sin x)^{5/2}\Big|_{\pi/6}^{\pi/3}$$

$$= 2\left(\frac{\sqrt{3}}{2}\right)^{1/2} - \frac{2}{5}\left(\frac{\sqrt{3}}{2}\right)^{5/2} - 2\left(\frac{1}{2}\right)^{1/2} + \frac{2}{5}\left(\frac{1}{2}\right)^{5/2}$$

$$= \sqrt[4]{3}\sqrt{2} - \frac{3}{10}\frac{\sqrt[4]{3}}{\sqrt{2}} - \sqrt{2} + \frac{\sqrt{2}}{20}$$

$$\boxed{= \frac{\sqrt{2}}{20}\left(17\sqrt[4]{3} - 19\right)}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/4} \tan^4 x dx$

Solution

$$\int_0^{\pi/4} \tan^4 x dx = \int_0^{\pi/4} \tan^2 x\left(\sec^2 x - 1\right)dx$$

$$= \int_0^{\pi/4} \tan^2 x\left(\sec^2 x - 1\right)dx$$

$$= \int_0^{\pi/4} \tan^2 x \sec^2 x \, dx - \int_0^{\pi/4} \tan^2 x \, dx$$

$$= \int_0^{\pi/4} \tan^2 x \, d(\tan x) - \int_0^{\pi/4} \left(\sec^2 x - 1\right) \, dx$$

$$= \frac{1}{3}\tan^3 x - \tan x + x \Big|_0^{\pi/4}$$

$$= \frac{1}{3} - 1 + \frac{\pi}{4}$$

$$= \frac{\pi}{4} - \frac{2}{3}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/2} \cos^7 x \, dx$

Solution

$$\int_0^{\pi/2} \cos^7 x \, dx = \int_0^{\pi/2} \left(\cos^2 x\right)^3 \, d(\sin x)$$

$$= \int_0^{\pi/2} \left(1 - \sin^2 x\right)^3 \, d(\sin x)$$

$$= \int_0^{\pi/2} \left(1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x\right) \, d(\sin x)$$

$$= \left(\sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x\right)_0^{\pi/2}$$

$$= \frac{3}{5} - \frac{1}{7}$$

$$= \frac{16}{37}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/2} \cos^9 \theta \, d\theta$

Solution

$$\int_0^{\pi/2} \cos^9 \theta \, d\theta = \int_0^{\pi/2} \left(1-\sin^2 x\right)^4 \, d(\sin x)$$

$$= \int_0^{\pi/2} \left(1-4\sin^2 x+6\sin^4 x-4\sin^6 x+\sin^8 x\right) \, d(\sin x)$$

$$= \left(\sin x-\frac{4}{3}\sin^3 x+\frac{6}{5}\sin^5 x-\frac{4}{7}\sin^7 x+\frac{1}{9}\sin^9 x\right)_0^{\pi/2}$$

$$= 1-\frac{4}{3}+\frac{6}{5}-\frac{4}{7}+\frac{1}{9}$$

$$= \frac{128}{315}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/2} \sin^5 x \, dx$

Solution

$$\int_0^{\pi/2} \sin^5 x \, dx = \int_0^{\pi/2} \left(1-\cos^2 x\right)^2 \, d(\cos x)$$

$$= \int_0^{\pi/2} \left(1-2\cos^2 x+\cos^4 x\right) \, d(\cos x)$$

$$= \left(\cos x-\frac{2}{3}\cos^3 x+\frac{1}{5}\cos^5 x\right)_0^{\pi/2}$$

$$= -1+\frac{2}{3}-\frac{1}{5}$$

$$= -\frac{8}{15}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/6} 3\cos^5 3x \, dx$

Solution

$$\int_0^{\pi/6} 3\cos^5 3x \, dx = \int_0^{\pi/6} 3\left(\cos^2 3x\right)^2 \cos 3x \, dx$$

$$= \int_0^{\pi/6} \left(1-\sin^2 3x\right)^2 d\left(\sin 3x\right)$$

$$= \int_0^{\pi/6} \left(1-2\sin^2 3x+\sin^4 3x\right) d\left(\sin 3x\right)$$

$$= \left[\sin 3x - \frac{2}{3}\sin^2 3x + \frac{1}{5}\sin^4 3x\right]_0^{\pi/6}$$

$$= \sin\frac{\pi}{2} - \frac{2}{3}\sin^2\frac{\pi}{2} + \frac{1}{5}\sin^4\frac{\pi}{2} - 0$$

$$= 1 - \frac{2}{3} + \frac{1}{5}$$

$$\boxed{= \frac{8}{15}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta d\theta$

Solution

$$\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta d\theta = \int_0^{\pi/2} \sin^2 2\theta \left(\cos^2 2\theta\right) \cos 2\theta d\theta \qquad d\left(\sin 2\theta\right) = 2\cos 2\theta d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^2 2\theta \left(1-\sin^2 2\theta\right) d\left(\sin 2\theta\right)$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(\sin^2 2\theta - \sin^4 2\theta\right) d\left(\sin 2\theta\right)$$

$$= \frac{1}{2}\left[\frac{1}{3}\sin^3 2\theta - \frac{1}{5}\sin^5 2\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{1}{3}\sin^3 \pi - \frac{1}{5}\sin^5 \pi - 0\right)$$

$$\boxed{= 0}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}} dx$

Solution

$$\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}} dx = \int_0^{2\pi} \sin\frac{x}{2} dx \qquad \left|\sin\left(\frac{\alpha}{2}\right)\right| = \sqrt{\frac{1-\cos\alpha}{2}}$$

$$= \left[-2\cos\frac{x}{2}\right]_0^{2\pi}$$

$$= -2(\cos\pi - \cos 0)$$

$$\underline{= 2|}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi} \sqrt{1-\cos^2\theta}\, d\theta$

Solution

$$\int_0^{\pi} \sqrt{1-\cos^2\theta}\, d\theta = \int_0^{\pi} |\sin\theta|\, d\theta$$

$$= [-\cos\theta]_0^{\pi}$$

$$= -\cos\pi + \cos 0$$

$$\underline{= 2|}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/6} \sqrt{1+\sin x}\; dx$

Solution

$$\int_0^{\pi/6} \sqrt{1+\sin x}\; dx = \int_0^{\pi/6} \sqrt{1+\sin x}\,\frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}\; dx$$

$$= \int_0^{\pi/6} \frac{\sqrt{1-\sin^2 x}}{\sqrt{1-\sin x}}\; dx \qquad \cos x = \sqrt{1-\sin^2 x}$$

$$= \int_0^{\pi/6} \frac{\cos x}{\sqrt{1-\sin x}}\; dx \qquad d(1-\sin x) = -\cos x dx$$

$$= -\int_0^{\pi/6} (1-\sin x)^{-1/2}\; d(1-\sin x)$$

$$= -2\left[(1-\sin x)^{1/2}\right]_0^{\pi/6}$$

$$= -2\left(\sqrt{1-\sin\frac{\pi}{6}} - 1\right)$$

$$= -2\left(\sqrt{1-\frac{1}{2}} - 1\right)$$

$$= -2\left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= -2\left(\frac{\sqrt{2}}{2} - 1\right)$$

$$\underline{= 2 - \sqrt{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi}^{\pi} \left(1 - \cos^2 x\right)^{3/2} dx$

Solution

$$\int_{-\pi}^{\pi} \left(1 - \cos^2 x\right)^{3/2} dx = \int_{-\pi}^{\pi} \left(\sin^2 x\right)^{3/2} dx$$

$$= \int_{-\pi}^{\pi} \left|\sin^3 x\right| dx$$

$$= -\int_{-\pi}^{0} \sin^3 x \, dx + \int_{0}^{\pi} \sin^3 x \, dx \qquad \sin^2 x = 1 - \cos^2 x$$

$$= -\int_{-\pi}^{0} \left(1 - \cos^2 x\right) \sin x \, dx + \int_{0}^{\pi} \left(1 - \cos^2 x\right) \sin x \, dx \qquad d(\cos x) = -\sin x dx$$

$$= \int_{-\pi}^{0} \left(1 - \cos^2 x\right) d(\cos x) - \int_{0}^{\pi} \left(1 - \cos^2 x\right) d(\cos x)$$

$$= \left[\cos x - \tfrac{1}{3}\cos^3 x\right]_{-\pi}^{0} - \left[\cos x - \tfrac{1}{3}\cos^3 x\right]_{0}^{\pi}$$

$$= \left(1 - \tfrac{1}{3} - \left(-1 + \tfrac{1}{3}\right)\right) - \left(-1 + \tfrac{1}{3} - \left(1 - \tfrac{1}{3}\right)\right)$$

$$= 1 - \tfrac{1}{3} + 1 - \tfrac{1}{3} + 1 - \tfrac{1}{3} + 1 - \tfrac{1}{3}$$

$$= 4 - \tfrac{4}{3}$$

$$\underline{= \frac{8}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/4}^{\pi/2} \csc^4 \theta d\theta$

Solution

$$\int_{\pi/4}^{\pi/2} \csc^4 \theta d\theta = \int_{\pi/4}^{\pi/2} \left(1+\cot^2\theta\right)\csc^2\theta d\theta \qquad \csc^2\theta = 1+\cot^2\theta$$

$$= \int_{\pi/4}^{\pi/2} \csc^2\theta d\theta + \int_{\pi/4}^{\pi/2} \cot^2\theta \csc^2\theta d\theta \qquad d(\cot\theta) = -\csc^2\theta d\theta$$

$$= \int_{\pi/4}^{\pi/2} \csc^2\theta d\theta - \int_{\pi/4}^{\pi/2} \cot^2\theta \, d(\cot\theta)$$

$$= \left[-\cot\theta - \frac{1}{3}\cot^3\theta\right]_{\pi/4}^{\pi/2}$$

$$= -\left(\cot\frac{\pi}{2} + \frac{1}{3}\cot^3\frac{\pi}{2} - \cot\frac{\pi}{4} - \frac{1}{3}\cot^3\frac{\pi}{4}\right)$$

$$= -\left(0 + \frac{1}{3}(0) - 1 - \frac{1}{3}\right)$$

$$= \frac{4}{3}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx$

Solution

$$\int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos 0 - \cos 6x) \, dx \qquad \sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} (1 - \cos 6x) \, dx$$

$$= \frac{1}{2}\left[x - \frac{1}{6}\sin 6x\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left(\pi - \frac{1}{6}\sin 6\pi - \left(-\pi - \frac{1}{6}\sin(-6\pi)\right)\right)$$

$$= \frac{1}{2}(\pi + \pi)$$

$$= \pi$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x \cos 7x\, dx$

Solution

$$\int_{-\pi/2}^{\pi/2} \cos x \cos 7x dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(\cos 8x + \cos(-6x)\right) dx \qquad \cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(\cos 8x + \cos 6x\right) dx$$

$$= \frac{1}{2}\left[\frac{1}{6}\sin 6x + \frac{1}{8}\sin 8x\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{1}{6}\sin(3\pi) + \frac{1}{8}\sin(4\pi) - \frac{1}{6}\sin(-3\pi) - \frac{1}{8}\sin(-4\pi)\right)$$

$$= 0$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi} 8\sin^4 y \cos^2 y\, dy$

Solution

$$\int_0^{\pi} 8\sin^4 y \cos^2 y\, dy = 8\int_0^{\pi} \left(\frac{1-\cos 2y}{2}\right)^2 \left(\frac{1+\cos 2y}{2}\right) dy$$

$$= \int_0^{\pi} \left(1 - 2\cos 2y + \cos^2 2y\right)\left(1 + \cos 2y\right) dy$$

$$= \int_0^{\pi} \left(1 - 2\cos 2y + \cos^2 2y + \cos 2y - 2\cos^2 2y + \cos^3 2y\right) dy$$

$$= \int_0^{\pi} \left(1 - \cos 2y - \cos^2 2y + \cos^3 2y\right) dy$$

$$= \int_0^{\pi} \left(1 - \cos 2y - \frac{1}{2} - \frac{1}{2}\cos 4y\right) dy + \int_0^{\pi} \cos^2 2y \cos 2y\, dy$$

$$= \int_0^{\pi} \left(\frac{1}{2} - \cos 2y - \frac{1}{2}\cos 4y\right) dy + \frac{1}{2}\int_0^{\pi} \left(1 - \sin^2 2y\right) d\left(\sin 2y\right)$$

$$= \left[\frac{1}{2}y - \frac{1}{2}\sin 2y - \frac{1}{8}\sin 4y + \frac{1}{2}\left(\sin 2y - \frac{1}{3}\sin^3 2y\right)\right]_0^{\pi}$$

$$= \frac{\pi}{2}$$

Exercise

Evaluate $\displaystyle\int_0^{\pi/2} \cos^{10}\theta\, d\theta$

Solution

$$\int_0^{\pi/2} \cos^{10}\theta\, d\theta = \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^5 d\theta$$

$$= \frac{1}{32}\int_0^{\pi/2} \left(1+5\cos 2\theta+10\cos^2 2\theta+10\cos^3 2\theta+5\cos^4 2\theta+\cos^5 2\theta\right) d\theta$$

$$= \frac{1}{32}\int_0^{\pi/2} \left(\begin{array}{l} 1+5\cos 2\theta+5+5\cos 4\theta+\frac{5}{4}(1+\cos 4\theta)^2 \\ \quad +\left(10+\cos^2 2\theta\right)\cos^3 2\theta \end{array}\right) d\theta$$

$$= \frac{1}{32}\int_0^{\pi/2} \left(6+5\cos 2\theta+5\cos 4\theta+\frac{5}{4}\left(1+2\cos 4\theta+\cos^2 4\theta\right)\right) d\theta$$

$$+\frac{5}{16}\int_0^{\pi/2} \cos^3 2\theta\, d\theta+\frac{1}{32}\int_0^{\pi/2} \cos^5 2\theta\, d\theta$$

$$= \frac{1}{32}\int_0^{\pi/2} \left(\frac{63}{8}+5\cos 2\theta+\frac{11}{2}\cos 4\theta+\frac{5}{8}\cos 8\theta\right) d\theta$$

$$+\frac{5}{32}\int_0^{\pi/2} \left(1-\sin^2 2\theta\right) d(\sin 2\theta)+\frac{1}{64}\int_0^{\pi/2} \left(1-\sin^2 2\theta\right)^2 d(\sin 2\theta)$$

$$= \left[\frac{1}{32}\left(\frac{63}{8}\theta+\frac{5}{2}\sin 2\theta+\frac{11}{8}\sin 4\theta+\frac{5}{64}\sin 8\theta\right)+\frac{5}{32}\left(\sin 2\theta-\frac{1}{3}\sin^3 2\theta\right)\right]_0^{\pi/2}$$

$$+\frac{1}{64}\int_0^{\pi/2} \left(1-2\sin^2 2\theta+\sin^4 2\theta\right) d(\sin 2\theta)$$

$$= \frac{1}{32}\left(\frac{63\pi}{16}\right)+\frac{1}{64}\left(\sin 2\theta-\frac{2}{3}\sin^3 2\theta+\frac{1}{5}\sin^5 2\theta\right)\Big|_0^{\pi/2}$$

$$= \boxed{\frac{63\,\pi}{512}}$$

$$\int_0^{\pi/2} \cos^{10} x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\left(\frac{9}{10}\right)\left(\frac{\pi}{2}\right) = \boxed{\frac{63\,\pi}{512}}$$

$$\int_0^{\pi/2} \cos^n x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

Exercise

Evaluate $\displaystyle\int_0^{\pi/2} \cos^7 x\, dx$

Solution

$$\int_0^{\pi/2} \cos^7 x\, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right) = \boxed{\frac{16}{35}}$$

$$\int_0^{\pi/2} \cos^n \theta\, d\theta = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$$

Exercise

Evaluate $\displaystyle\int_0^{\pi/2} \cos^9 x\, dx$

Solution

$$\int_0^{\pi/2} \cos^9 x\, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\left(\frac{8}{9}\right) = \boxed{\frac{128}{315}}$$

$$\int_0^{\pi/2} \cos^n \theta\, d\theta = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$$

Exercise

Evaluate $\displaystyle\int_0^{\pi/2} \sin^5 x\, dx$

Solution

$$\int_0^{\pi/2} \sin^5 x\, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right) = \boxed{\frac{8}{15}}$$

$$\int_0^{\pi/2} \cos^n \theta\, d\theta = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$$

Exercise

Evaluate $\displaystyle\int_0^{\pi/2} \sin^6 x\, dx$

Solution

$$\int_0^{\pi/2} \sin^6 x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{\pi}{2}\right) = \boxed{\frac{5\pi}{32}}$$

$$\int_0^{\pi/2} \sin^n x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

Exercise

Evaluate $\displaystyle\int_0^{\pi/2} \sin^8 x\, dx$

Solution

$$\int_0^{\pi/2} \sin^8 x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\left(\frac{\pi}{2}\right) = \boxed{\frac{35\pi}{256}}$$

$$\int_0^{\pi/2} \sin^n x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

Exercise

Find the area of the region bounded by the graphs of $y = \tan x$ and $y = \sec x$ on the interval $\left[0, \ \frac{\pi}{4}\right]$

Solution

$$A = \int_0^{\pi/4} (\sec x - \tan x)\, dx$$

$$= \ln|\sec x + \tan x| + \ln|\cos x| \Big|_0^{\pi/4}$$

$$= \ln\left(\sqrt{2}+1\right) + \ln\frac{\sqrt{2}}{2} - 0$$

$$= \ln\left(\frac{\sqrt{2}}{2}\left(\sqrt{2}+1\right)\right)$$

$$= \ln\left(1 + \frac{\sqrt{2}}{2}\right)$$

Exercise

Find the area of the region bounded by the graphs of the equations

$$y = \sin x, \quad y = \sin^3 x, \quad x = 0, \quad x = \frac{\pi}{2}$$

Solution

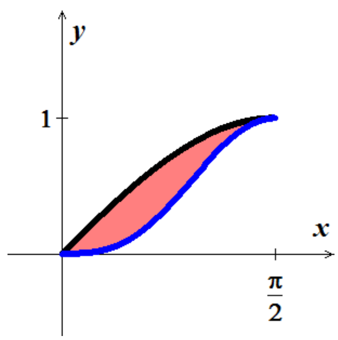
$$A = \int_0^{\pi/2} \left(\sin x - \sin^3 x\right) dx$$

$$= \int_0^{\pi/2} \sin x \, dx - \int_0^{\pi/2} \sin^3 x \, dx$$

$$= -\cos x \Big|_0^{\pi/2} - \frac{2}{3}$$

$$= 1 - \frac{2}{3}$$

$$= \frac{1}{3}$$



Exercise

Find the area of the region bounded by the graphs of the equations $y = \sin^2 \pi x, \quad y = 0, \quad x = 0, \quad x = 1$

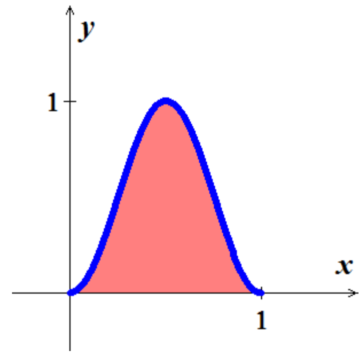
Solution

$$A = \int_0^1 \sin^2 \pi x \, dx$$

$$= \frac{1}{2}\int_0^1 (1+\cos 2\pi x)\ dx$$

$$= \frac{1}{2}\left(x + \frac{1}{2\pi}\sin 2\pi x\right)\Big|_0^1$$

$$\underline{= \frac{1}{2}}$$



Exercise

Find the area of the region bounded by the graphs of the equations

$$y = \cos^2 x, \quad y = \sin^2 x, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$$

Solution

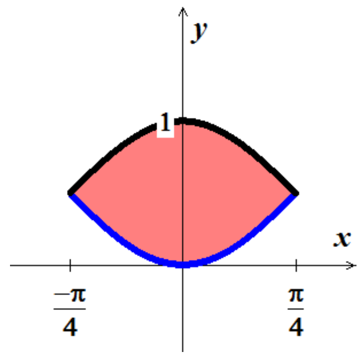
$$A = \int_{-\pi/4}^{\pi/4} \left(\cos^2 x - \sin^2 x\right) dx$$

$$= \int_{-\pi/4}^{\pi/4} \cos 2x\ dx$$

$$= \frac{1}{2}\sin 2x \Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{2}(1+1)$$

$$\underline{= 1}$$



Exercise

Find the area of the region bounded by the graphs of the equations

$$y = \cos^2 x, \quad y = \sin x \cos x, \quad x = -\frac{\pi}{2}, \quad x = \frac{\pi}{4}$$

Solution

$$A = \int_{-\pi/2}^{\pi/4} \left(\cos^2 x - \sin x \cos x\right) dx$$

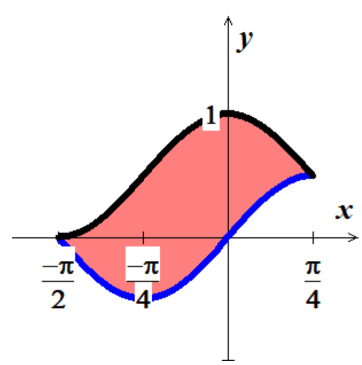
$$= \int_{-\pi/2}^{\pi/4} \left(\frac{1}{2} + \frac{1}{2}\cos 2x - \frac{1}{2}\sin 2x\right) dx$$

$$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x + \frac{1}{2}\cos 2x\right)\Big|_{-\pi/2}^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2} + \frac{\pi}{2} + \frac{1}{2}\right)$$

$$= \frac{1}{2}\left(\frac{3\pi}{4} + 1\right)$$

$$\underline{= \frac{3\pi}{8} + \frac{1}{2}}$$

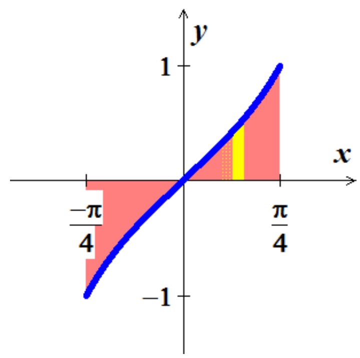


Exercise

Find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the *x-axis* $y = \tan x, \quad y = 0, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$

Solution

Disks Method:



$$V = 2\pi \int_0^{\pi/4} \tan^2 x \, dx$$

$$= 2\pi \int_0^{\pi/4} \left(\sec^2 x - 1\right) dx$$

$$= 2\pi \left(\tan x - x\right) \Big|_0^{\pi/4}$$

$$= 2\pi \left(1 - \frac{\pi}{4}\right)$$

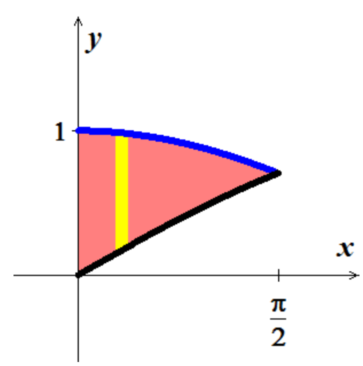
$$= 2\pi - \frac{1}{2}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the *x-axis* $\quad y = \cos\frac{x}{2}, \quad y = \sin\frac{x}{2}, \quad x = 0, \quad x = \frac{\pi}{2}$

Solution

$$V = \pi \int_0^{\pi/2} \left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) dx \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$



$$= \pi \int_0^{\pi/2} \cos x \, dx$$

$$= \pi \sin x \Big|_0^{\pi/2}$$

$$= \pi$$

Exercise

Find the ***volume*** of the solid generated by revolving the region bounded by the graphs of the equations about the *x-axis*, then find the ***centroid*** of the region

$$y = \sin x, \quad y = 0, \quad x = 0, \quad x = \pi$$

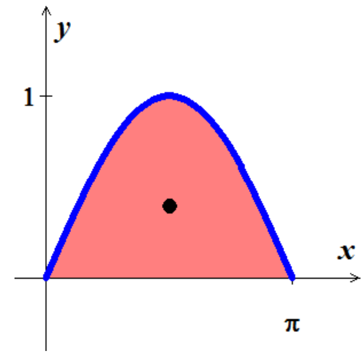
Solution

$$V = \pi \int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x) \, dx$$

$$= \frac{\pi}{2} \left(x - \frac{1}{2} \sin 2x \right) \Big|_0^{\pi}$$

$$= \frac{\pi^2}{2}$$



$$A = \int_0^{\pi} \sin x \, dx$$

$$= -\cos x \Big|_0^{\pi}$$

$$= -(-1 - 1)$$

$$= 2$$

		$\int \sin x$
+	x	$-\cos x$
−	1	$-\sin x$

$$\bar{x} = \frac{1}{A} \int_0^{\pi} x \sin x \, dx$$

$$= \frac{1}{2} (-x \cos x + \sin x) \Big|_0^{\pi}$$

$$= \frac{\pi}{2}$$

$$\bar{y} = \frac{1}{2A} \int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{1}{8} \int_0^{\pi} (1 - \cos 2x) \, dx$$

$$= \frac{1}{8} \left(x - \frac{1}{2} \sin 2x \right) \Big|_0^{\pi}$$

$$= \frac{\pi}{8}$$

$$(\bar{x}, \bar{y}) = \left(\frac{1}{2}, \ \frac{\pi}{8} \right)$$

Exercise

Find the ***volume*** of the solid generated by revolving the region bounded by the graphs of the equations about the *x-axis*, then find the ***centroid*** of the region

$$y = \cos x, \quad y = 0, \quad x = 0, \quad x = \frac{\pi}{2}$$

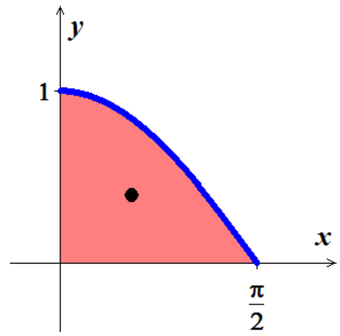
Solution

$$V = \pi \int_0^{\pi/2} \cos^2 x \, dx$$

$$= \frac{\pi}{2} \int_0^{\pi/2} (1 + \cos 2x) \, dx$$

$$= \frac{\pi}{2} \left(x + \frac{1}{2} \sin 2x \right) \Big|_0^{\pi/2}$$

$$= \frac{\pi^2}{4}$$



$$A = \int_0^{\pi/2} \cos x \, dx$$

$$= \sin x \Big|_0^{\pi/2}$$

$$= 1$$

		$\int \cos x$
+	x	$\sin x$
−	1	$-\cos x$

$$\bar{x} = \frac{1}{A} \int_0^{\pi/2} x \cos x \, dx$$

$$= (x \sin x + \cos x) \Big|_0^{\pi/2}$$

$$= \frac{\pi}{2} - 1$$

$$\bar{y} = \frac{1}{2A} \int_0^{\pi/2} \cos^2 x \, dx$$

$$= \frac{1}{4} \int_0^{\pi/2} (1 + \cos 2x) \, dx$$

$$= \frac{1}{4} \left(x + \frac{1}{2} \sin 2x \right) \Big|_0^{\pi/2}$$

$$= \frac{\pi}{8}$$

$$(\bar{x}, \bar{y}) = \left(\frac{\pi - 2}{2}, \frac{\pi}{8} \right)$$