Solution   Section 3.7 – Phase Plane Portraits & Applications

Exercise

Sketch a rough approximation of a solution in each region determined by the half-line solutions. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

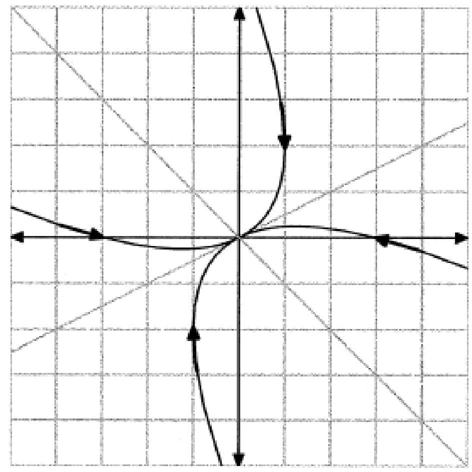
$$y(t) = C_1 e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Solution

Both eigenvalues are negative, so the equilibrium point at the origin is a sink.

Solutions dive toward the origin to the slow exponential solution, $e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Solutions dive toward the origin to the fast exponential solution, $e^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.



Exercise

Sketch a rough approximation of a solution in each region determined by the half-line solutions. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

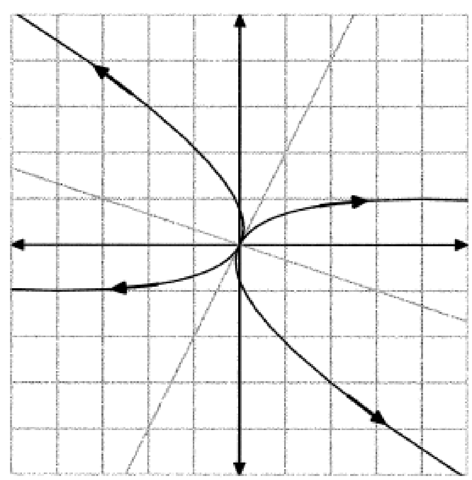
$$y(t) = C_1 e^{t}\begin{pmatrix} -1 \\ -2 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Solution

Both eigenvalues are positive, so the equilibrium point at the origin is a source.

Solutions emanate from the origin tangent to the slow exponential solution, $e^{t}(-1,\ -2)^T$.

Solutions emanate from the origin to the fast exponential solution, $e^{2t}(3,\ -1)^T$.

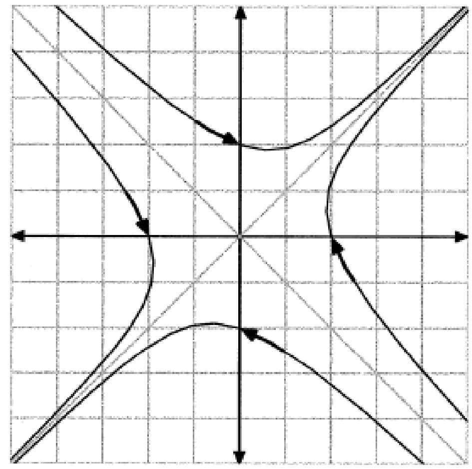


Exercise

Sketch a rough approximation of a solution in each region determined by the half-line solutions. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

$$y(t) = C_1 e^{t}\begin{pmatrix}1\\1\end{pmatrix} + C_2 e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix}$$

Solution



One eigenvalue is negative and the other positive. So the equilibrium point on the origin is a saddle.

As $t \to +\infty$, solutions parallel the exponential solution $e^{t}(1,\ 1)^T$

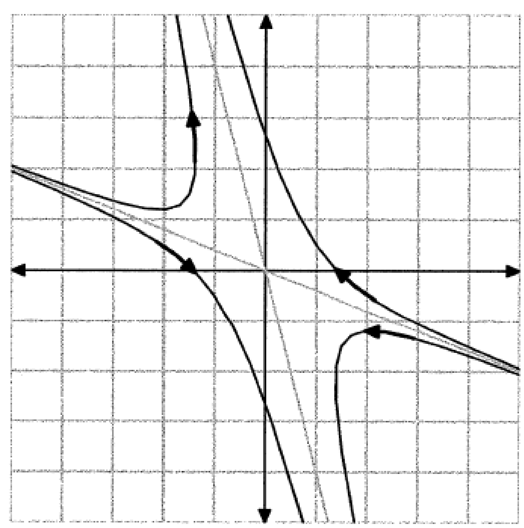
As $t \to -\infty$, solutions parallel the exponential solution $e^{-2t}(1,\ -1)^T$

Exercise

Sketch a rough approximation of a solution in each region determined by the half-line solutions. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

$$y(t) = C_1 e^{-t}\begin{pmatrix}-5\\2\end{pmatrix} + C_2 e^{2t}\begin{pmatrix}-1\\4\end{pmatrix}$$

Solution



One eigenvalue is negative and the other positive. So the equilibrium point on the origin is a saddle.

As $t \to +\infty$, solutions parallel the exponential solution $e^{2t}(-1,\ 4)^T$

As $t \to -\infty$, solutions parallel the exponential solution $e^{-t}(-5,\ 2)^T$

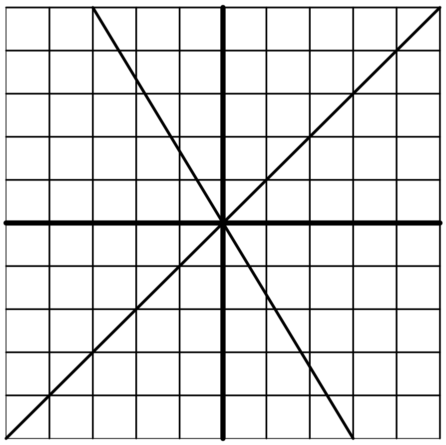
Exercise

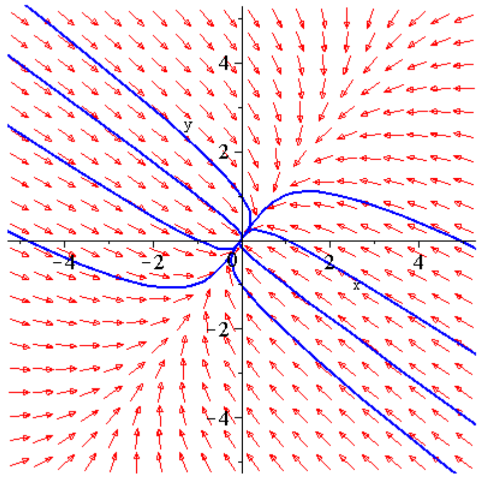
Sketch a rough approximation of the given system. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

$$y' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} y$$

Solution

Asymptotically stable sink at the center





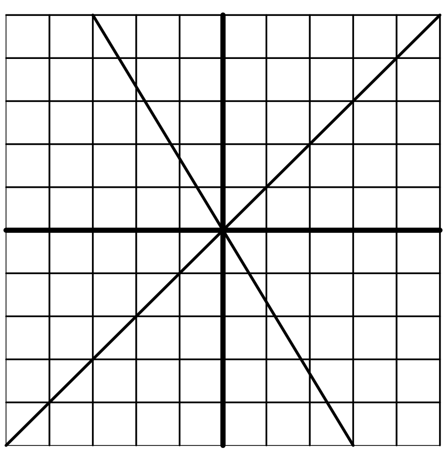
Exercise

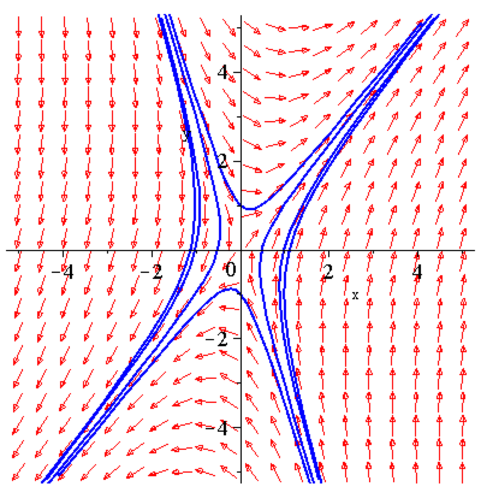
Sketch a rough approximation of the given system. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

$$y' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} y$$

Solution

Saddle point at (0, 0); semi-stable





Exercise

Calculate the eigenvalues to determine the behavior of the system whether the equilibrium point at the origin is the center, a spiral sink or a source. Calculate and sketch the vector generated by the right-hand side of the system at the point (1, 0). Use this to help sketch the solution trajectory for the system passing through the point (1, 0).

$$y' = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} y$$

Solution

Equilibrium point at the origin is the center

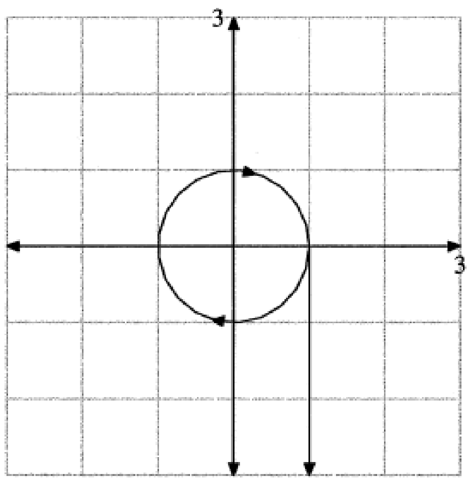
$A = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$ has a trace $T = 0$ and determinant $D = 9$.

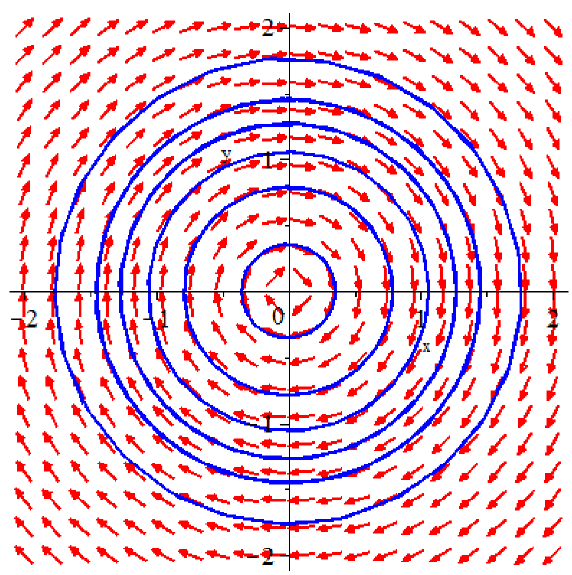
$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -\lambda & 3 \\ -3 & -\lambda \end{vmatrix} \\ &= \lambda^2 + 9 = 0 \end{aligned}$$

Therefore; the eigenvalues are: $\lambda_1 = 3i \quad and \quad \lambda_2 = -3i$

$$\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

$\therefore$ The motion is clockwise.





Exercise

Calculate the eigenvalues to determine the behavior of the system whether the equilibrium point at the origin is the center, a spiral sink or a source. Calculate and sketch the vector generated by the right-hand side of the system at the point (1, 0). Use this to help sketch the solution trajectory for the system passing through the point (1, 0).

$$y' = \begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix} y$$

Solution

$$A = \begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix}$$

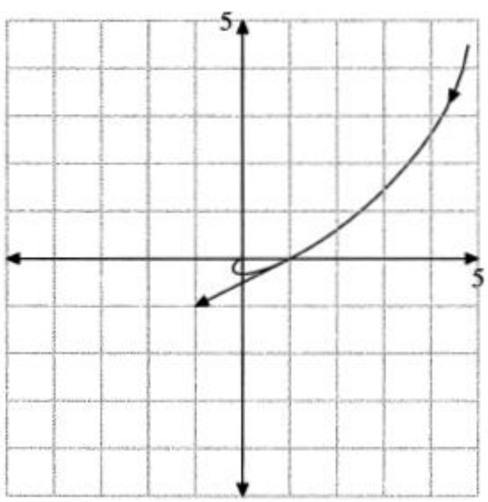
$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -2-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} \\ &= (-2-\lambda)(-\lambda) + 2 \\ &= \lambda^2 + 2\lambda + 2 = 0 \end{aligned}$$

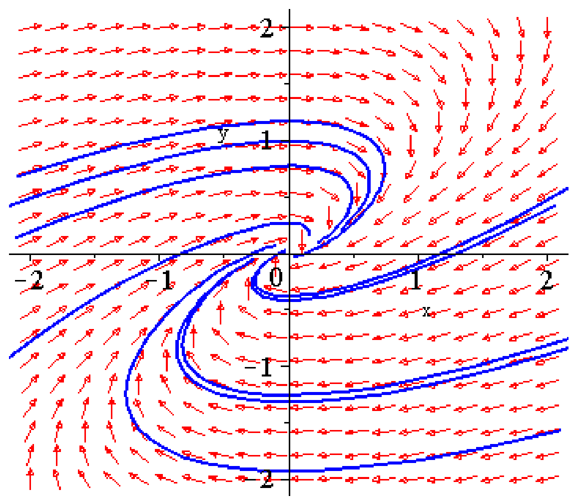
Therefore; the eigenvalues are: $\lambda_1 = -1 + i \quad and \quad \lambda_2 = -1 - i$

Because both the real part of the eigenvalues is negative, the equilibrium point at the origin is a spiral sink

$$\begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$\therefore$ The motion is clockwise.





Exercise

Calculate the eigenvalues to determine the behavior of the system whether the equilibrium point at the origin is the center, a spiral sink or a source. Calculate and sketch the vector generated by the right-hand side of the system at the point (1, 0). Use this to help sketch the solution trajectory for the system passing through the point (1, 0).

$$y' = \begin{pmatrix} 7 & -10 \\ 4 & -5 \end{pmatrix} y$$

Solution

$$A = \begin{pmatrix} 7 & -10 \\ 4 & -5 \end{pmatrix}$$

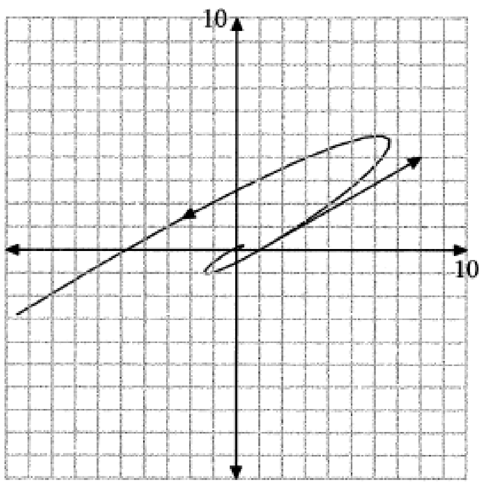
$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 7-\lambda & -10 \\ 4 & -5-\lambda \end{vmatrix} \\ &= (7-\lambda)(-5-\lambda) + 40 \\ &= \lambda^2 - 2\lambda + 5 = 0 \end{aligned}$$

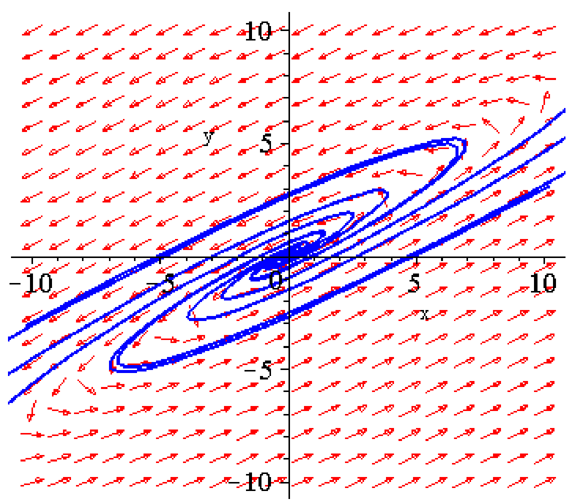
Therefore; the eigenvalues are: $\lambda_1 = 1 + 2i \quad and \quad \lambda_2 = 1 - 2i$

Because both the real part of the eigenvalues is positive, the equilibrium point at the origin is a spiral source.

$$\begin{pmatrix} 7 & -10 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

$\therefore$ The motion is counterclockwise.





Exercise

Calculate the eigenvalues to determine the behavior of the system whether the equilibrium point at the origin is the center, a spiral sink or a source. Calculate and sketch the vector generated by the right-hand side of the system at the point (1, 0). Use this to help sketch the solution trajectory for the system passing through the point (1, 0).

$$y' = \begin{pmatrix} -4 & 8 \\ -4 & 4 \end{pmatrix} y$$

Solution

$$A = \begin{pmatrix} -4 & 8 \\ -4 & 4 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -4-\lambda & 8 \\ -4 & 4-\lambda \end{vmatrix}$$

$$= (4-\lambda)(-4-\lambda) + 32$$

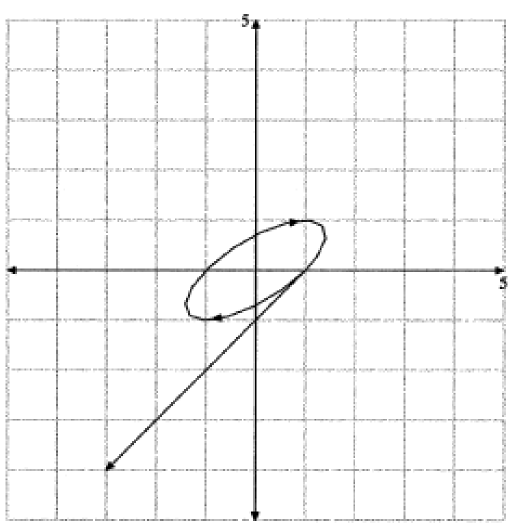
$$= \lambda^2 + 16 = 0$$

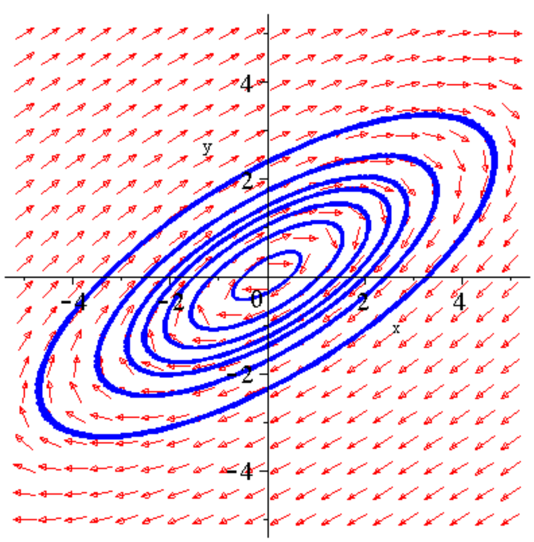
Therefore; the eigenvalues are: $\lambda_1 = -4i \quad and \quad \lambda_2 = 4i$

Because both the real part of the eigenvalues is zero, the equilibrium point at the origin is a center.

$$\begin{pmatrix} -4 & 8 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$$

$\therefore$ The motion is clockwise.





Exercise

Calculate the eigenvalues to determine the behavior of the system whether the equilibrium point at the origin is the center, a spiral sink or a source. Calculate and sketch the vector generated by the right-hand side of the system at the point (1, 0). Use this to help sketch the solution trajectory for the system passing through the point (1, 0).

$$y' = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} y$$

Solution

$$A = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix}$$

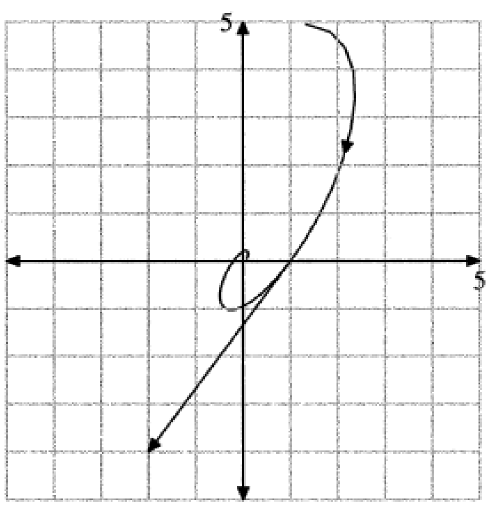
$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -3-\lambda & 2 \\ -4 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)(-3-\lambda) + 8 \\ &= \lambda^2 + 2\lambda + 5 = 0 \end{aligned}$$

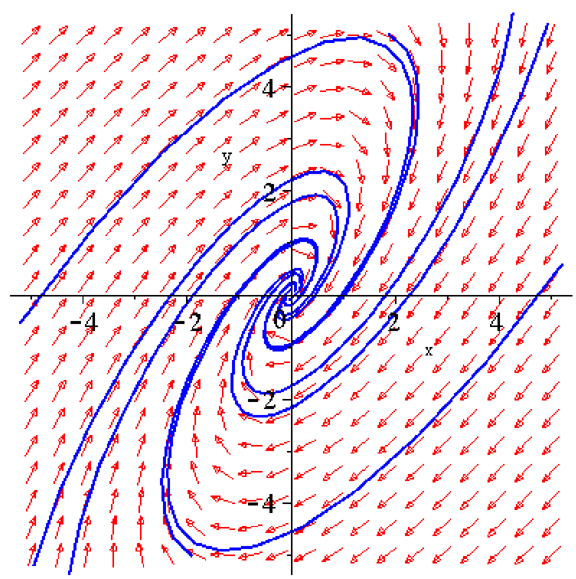
Therefore; the eigenvalues are: $\lambda_1 = -1 + 2i$ *and* $\lambda_2 = -1 - 2i$

Because both the real part of the eigenvalues is negative, the equilibrium point at the origin is a spiral sink.

$$\begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$$

$\therefore$ The motion is clockwise.





Exercise

For the given system $y' = \begin{pmatrix} -1 & 6 \\ -3 & 8 \end{pmatrix} y$

a) Sketch a rough approximation of the given system. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

b) Find the solution of the initial-value problem $y(0) = (0,\ 1)^T$

Solution

a) $|A - \lambda I| = \begin{vmatrix} -1-\lambda & 6 \\ -3 & 8-\lambda \end{vmatrix} = (-1-\lambda)(8-\lambda) + 18$

$$= -8 + \lambda - 8\lambda + \lambda^2 + 18$$

$$= \lambda^2 - 7\lambda + 10 \underline{\ = 0}$$

Thus, the eigenvalues are: $\lambda_1 = 2$ *and* $\lambda_2 = 5$

For $\lambda_1 = 2 \quad \Rightarrow (A - 2I)V_1 = 0$

$$\begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} -3x + 6y = 0 \\ -3x + 6y = 0 \end{cases} \Rightarrow -3x = -6y \rightarrow \boxed{x = 2y}$$

The eigenvector is: $V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \rightarrow y_1(t) = e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

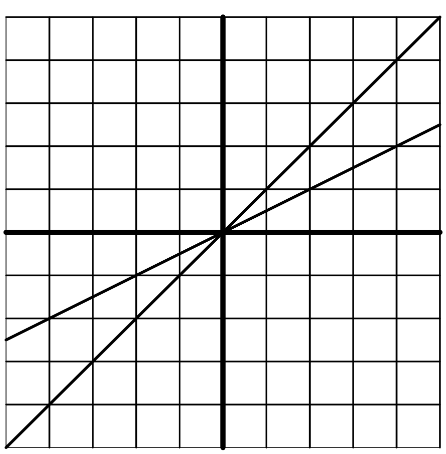
For $\lambda_2 = 5 \quad \Rightarrow (A - 5I)V_2 = 0$

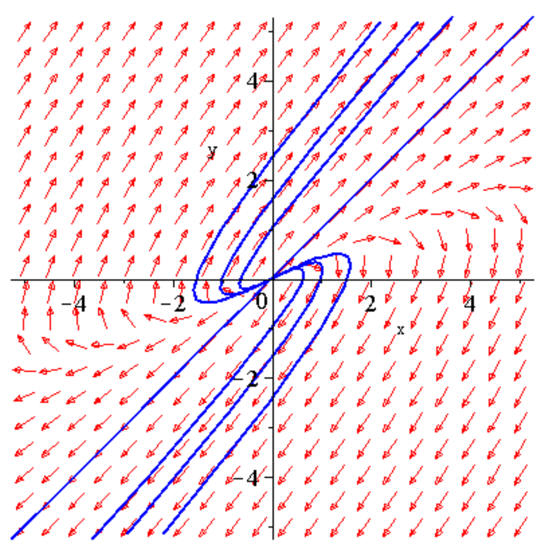
$$\begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} -6x + 6y = 0 \\ -3x + 3y = 0 \end{cases} \Rightarrow -6x = -6y \rightarrow \boxed{x = y}$$

The eigenvector is: $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \rightarrow y_2(t) = e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Therefore, the final solution can be written as: $\quad y(t) = C_1 e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Unstable at the center (source)





b) $y(0) = C_1 e^{2(0)}\begin{pmatrix}2\\1\end{pmatrix} + C_2 e^{5(0)}\begin{pmatrix}1\\1\end{pmatrix}$

$$\begin{pmatrix}1\\-2\end{pmatrix} = C_1\begin{pmatrix}2\\1\end{pmatrix} + C_2\begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}2C_1 + C_2\\C_1 + C_2\end{pmatrix}$$

$$\to \begin{cases}2C_1 + C_2 = 1\\C_1 + C_2 = -2\end{cases} \xrightarrow{rref} C_1 = 3 \quad C_2 = -5$$

$$\boxed{y(t) = 3e^{2t}\begin{pmatrix}2\\1\end{pmatrix} - 5e^{5t}\begin{pmatrix}1\\1\end{pmatrix}}$$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = x_1 + 2x_2,\quad x_2' = 2x_1 + x_2$

Solution

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}' = \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix}1-\lambda & 2\\2 & 1-\lambda\end{vmatrix} = (1-\lambda)^2 - 4$$

$$= \lambda^2 - 2\lambda - 3 = 0$$

The distinct real eigenvalues: $\boxed{\lambda_1 = -1, \quad \lambda_2 = 3}$

For $\lambda_1 = -1 \quad \Rightarrow \quad (A+I)V_1 = 0$

$$\begin{pmatrix}2 & 2\\2 & 2\end{pmatrix}\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad \Rightarrow \quad 2x_1 + 2y_1 = 0 \to y_1 = -x_1$$

$$\to V_1 = \begin{pmatrix}1\\-1\end{pmatrix} \quad \Rightarrow \quad x_1(t) = \begin{pmatrix}1\\-1\end{pmatrix}e^{-t}$$

For $\lambda_2 = 3 \quad \Rightarrow \quad (A-3I)V_2 = 0$

$$\begin{pmatrix}-2 & 2\\2 & -2\end{pmatrix}\begin{pmatrix}x_2\\y_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad \Rightarrow \quad -2x_2 + 2y_2 = 0 \to x_2 = y_2$$

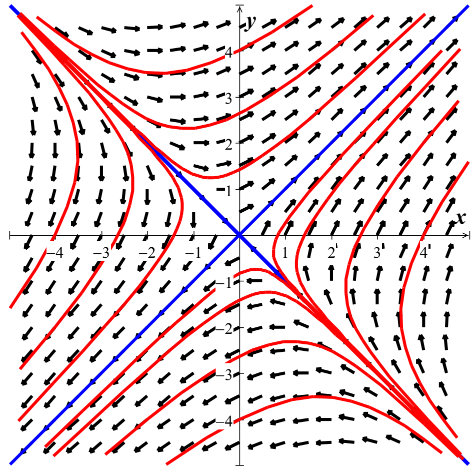
$$\to V_2 = \begin{pmatrix}1\\1\end{pmatrix} \quad \Rightarrow \quad x_2(t) = \begin{pmatrix}1\\1\end{pmatrix}e^{3t}$$

$$x_1(t)=\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} \quad x_2(t)=\begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$$

Using Wronskian: $\begin{vmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{vmatrix} = 2e^{2t} \neq 0$

The general solution: $\boxed{x(t)=C_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t}+C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t}}$

OR $\begin{cases} x_1(t)=C_1e^{-t}+C_2e^{3t} \\ x_2(t)=-C_1e^{-t}+C_2e^{3t} \end{cases}$



Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = 2x_1 + 3x_2, \quad x_2' = 2x_1 + x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)-6$$

$$= \lambda^2 - 3\lambda - 4 = 0$$

The distinct real eigenvalues: $\boxed{\lambda_1 = -1, \quad \lambda_2 = 4}$

For $\lambda_1 = -1 \;\Rightarrow\; (A+I)V_1 = 0$

$$\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\Rightarrow\; 3x_1 + 3y_1 = 0 \;\rightarrow\; y_1 = -x_1$$

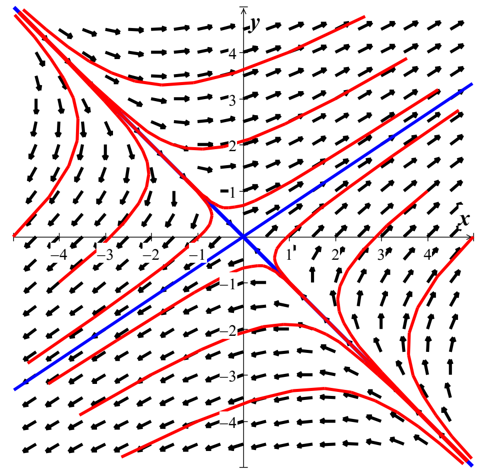
$$\rightarrow V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \;\Rightarrow\; x_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t}$$

For $\lambda_2 = 4 \;\Rightarrow\; (A-4I)V_2 = 0$

$$\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\Rightarrow\; 2x_2 = 3y_2$$

$$\rightarrow V_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \;\Rightarrow\; x_2(t) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{4t}$$

$$x_1(t) = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} \quad x_2(t) = \begin{pmatrix} 3e^{4t} \\ 2e^{4t} \end{pmatrix}$$



The general solution: $\boxed{x(t) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}}$

OR $\begin{cases} x_1(t) = C_1 e^{-t} + 3C_2 e^{4t} \\ x_2(t) = -C_1 e^{-t} + 2C_2 e^{4t} \end{cases}$

Exercise

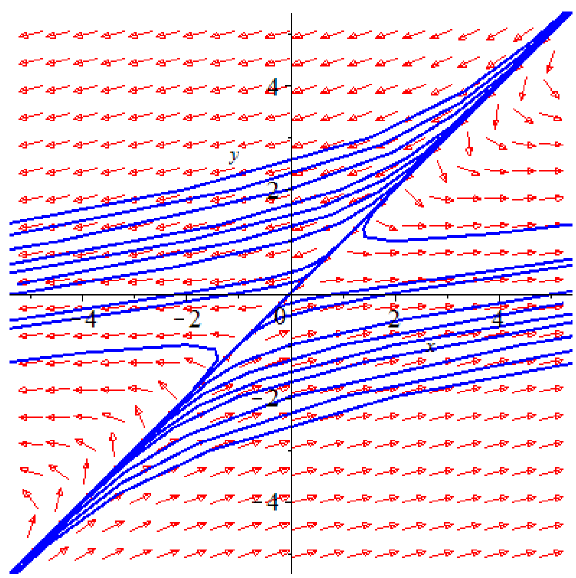
Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = 6x_1 - 7x_2, \quad x_2' = x_1 - 2x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 6-\lambda & -7 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 5 = 0$$



The distinct real eigenvalues: $\boxed{\lambda_1 = -1, \quad \lambda_2 = 5}$

For $\lambda_1 = -1 \quad \Rightarrow \quad (A + I)V_1 = 0$

$$\begin{pmatrix} 7 & -7 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad x_1 = y_1$$

$$\to V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad x_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

For $\lambda_2 = 5 \quad \Rightarrow \quad (A - 5I)V_2 = 0$

$$\begin{pmatrix} 1 & -7 \\ 1 & -7 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad x_2 = 7y_2$$

$$\to V_2 = \begin{pmatrix} 7 \\ 1 \end{pmatrix} \quad \Rightarrow \quad x_2(t) = \begin{pmatrix} 7 \\ 1 \end{pmatrix} e^{5t}$$

The general solution: $\boxed{x(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 7 \\ 1 \end{pmatrix} e^{5t}}$

OR $\begin{cases} x_1(t) = C_1 e^{-t} + 7C_2 e^{5t} \\ x_2(t) = C_1 e^{-t} + C_2 e^{5t} \end{cases}$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = -3x_1 + 4x_2, \quad x_2' = 6x_1 - 5x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -3 & 4 \\ 6 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} -3-\lambda & 4 \\ 6 & -5-\lambda \end{vmatrix} = \lambda^2 + 8\lambda - 9 = 0$$

The distinct real eigenvalues: $\lambda_1 = -9, \quad \lambda_2 = 1$

For $\lambda_1 = -9 \quad \Rightarrow \quad (A+9I)V_1 = 0$

$$\begin{pmatrix} 6 & 4 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 6x_1 = -4y_1$$

$$\to V_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad \Rightarrow \quad x_1(t) = \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{-9t}$$

For $\lambda_2 = 1 \quad \Rightarrow \quad (A-I)V_2 = 0$

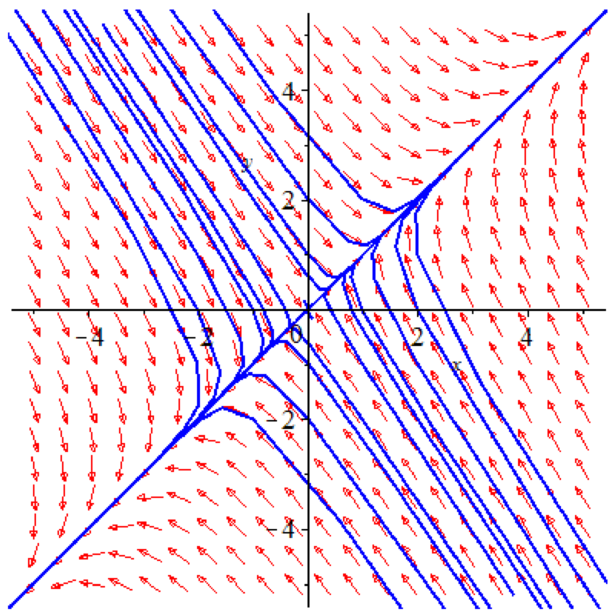
$$\begin{pmatrix} -4 & 4 \\ 6 & -6 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad x_2 = y_2$$

$$\to V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad x_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$$

The general solution:

$$x(t) = C_1\begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{-9t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$$

$$\begin{cases} x_1(t) = 2C_1 e^{-9t} + C_2 e^{t} \\ x_2(t) = -3C_1 e^{-9t} + C_2 e^{t} \end{cases}$$



Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = x_1 - 5x_2, \quad x_2' = x_1 - x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

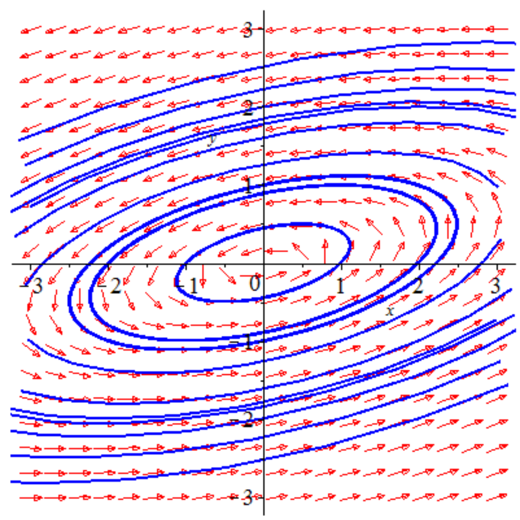
$$\begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + 4 = 0$$

The distinct real eigenvalues: $\lambda_{1,2} = \pm 2i$

For $\lambda = 2i \quad \Rightarrow \quad (A - \lambda I)V = 0$

$$\begin{pmatrix} 1-2i & -5 \\ 1 & -1-2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad (1-2i)x - 5y = 0 \ \rightarrow \ (1-2i)x = 5y$$

$$\rightarrow V = \begin{pmatrix} 5 \\ 1-2i \end{pmatrix}$$



$$x(t) = \begin{pmatrix} 5 \\ 1-2i \end{pmatrix} e^{2it} \qquad e^{ait} = \cos at + i \sin at$$

$$= \begin{pmatrix} 5 \\ 1-2i \end{pmatrix}(\cos 2t + i \sin 2t)$$

$$= \begin{pmatrix} 5\cos 2t + 5i \sin 2t \\ \cos 2t + 2\sin 2t + i(\sin 2t - 2\cos 2t) \end{pmatrix}$$

$$\begin{cases} x_1(t) = 5C_1 \cos 2t + 5C_2 \sin 2t \\ x_2(t) = C_1(\cos 2t + 2\sin 2t) + C_2(\sin 2t - 2\cos 2t) \\ \qquad\quad = (C_1 - 2C_2)\cos 2t + (2C_1 + C_2)\sin 2t \end{cases}$$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = -3x_1 - 2x_2, \quad x_2' = 9x_1 + 3x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -3 & -2 \\ 9 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

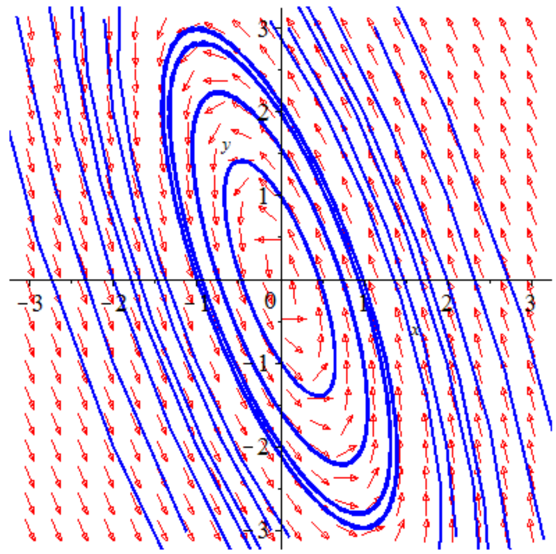
$$\begin{vmatrix} -3-\lambda & -2 \\ 9 & 3-\lambda \end{vmatrix} = \lambda^2 + 9 = 0$$

The distinct real eigenvalues: $\boxed{\lambda_{1,2} = \pm 3i}$

For $\lambda = 3i \quad \Rightarrow \quad (A - \lambda I)V = 0$

$$\begin{pmatrix} -3-3i & -2 \\ 9 & 3-3i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad (-3-3i)x - 2y = 0 \ \rightarrow \ (3+3i)x = -2y$$

$$\rightarrow V = \begin{pmatrix} -2 \\ 3+3i \end{pmatrix}$$



$$x(t) = \begin{pmatrix} -2 \\ 3+3i \end{pmatrix} e^{3it} \qquad e^{ait} = \cos at + i \sin at$$

$$= \begin{pmatrix} -2 \\ 3+3i \end{pmatrix}(\cos 3t + i \sin 3t)$$

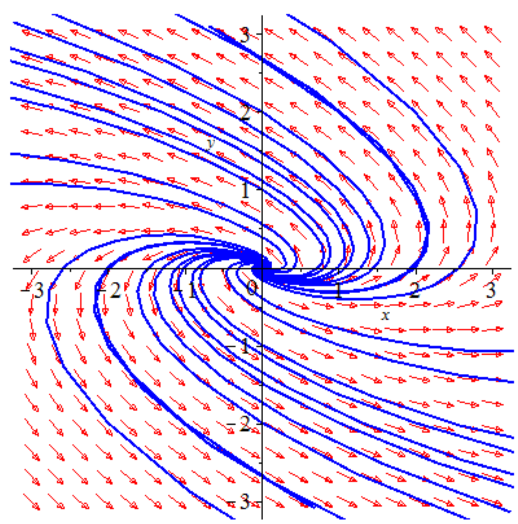
$$= \boxed{\begin{pmatrix} -2\cos 3t - 2i \sin 3t \\ 3\cos 3t - 3\sin 3t + i(3\sin 3t + 3\cos 3t) \end{pmatrix}}$$

$$\begin{cases} x_1(t) = -2C_1 \cos 3t - 2C_2 \sin 3t \\ x_2(t) = 3C_1(\cos 3t - \sin 3t) + 3C_2(\sin 3t + \cos 3t) \\ \qquad = 3(C_1 + C_2)\cos 3t + 3(C_2 - C_1)\sin 3t \end{cases}$$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = x_1 - 5x_2, \quad x_2' = x_1 + 3x_2$

Solution



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & -5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 1-\lambda & -5 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 8 = 0$$

The distinct real eigenvalues: $\boxed{\lambda_{1,2} = 2 \pm 2i}$

For $\lambda = 2 + 2i \quad \Rightarrow \quad (A - \lambda I)V = 0$

$$\begin{pmatrix} -1-2i & -5 \\ 1 & 1-2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad (1+2i)x = -5y$$

$$\rightarrow V = \begin{pmatrix} -5 \\ 1+2i \end{pmatrix}$$

$$\begin{aligned} x(t) &= \begin{pmatrix} -5 \\ 1+2i \end{pmatrix} e^{(2+2i)t} \\ &= \begin{pmatrix} -5 \\ 1+2i \end{pmatrix} e^{2t} e^{2it} \\ &= \begin{pmatrix} -5 \\ 1+2i \end{pmatrix} e^{2t} (\cos 2t + i \sin 2t) \\ &= \underline{\begin{pmatrix} -5\cos 2t - 5i \sin 2t \\ \cos 2t - 2\sin 2t + i(2\cos 2t + \sin 2t) \end{pmatrix} e^{2t} \Big|} \end{aligned}$$

$$\begin{cases} x_1(t) = \left(-5C_1 \cos 2t - 2C_2 \sin 2t\right) e^{2t} \\ x_2(t) = \left[C_1 (\cos 2t - 2\sin 2t) + C_2 (2\cos 2t + \sin 2t)\right] e^{2t} \\ \qquad\;\; = \left[\left(C_1 + 2C_2\right)\cos 2t + \left(C_2 - 2C_1\right)\sin 2t\right] e^{2t} \end{cases}$$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = 5x_1 - 9x_2$, $x_2' = 2x_1 - x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 5 & -9 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 5-\lambda & -9 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 13 = 0$$

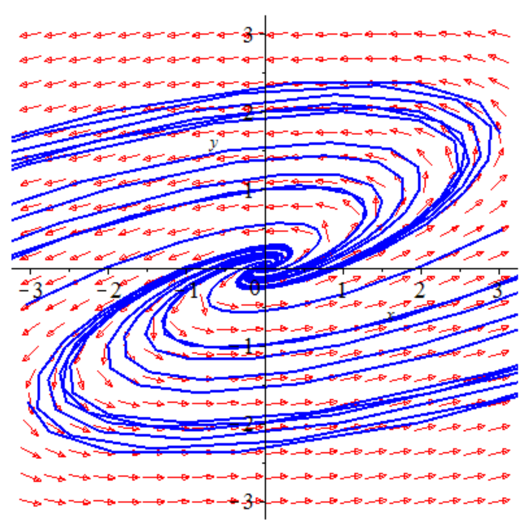
The distinct real eigenvalues: $\underline{\lambda_{1,2} = 2 \pm 3i \Big|}$

For $\lambda = 2 + 3i \;\Rightarrow\; (A - \lambda I)V = 0$

$$\begin{pmatrix} 3-3i & -9 \\ 2 & -3-3i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\Rightarrow\; 3(1-i)x = 9y$$

$$\rightarrow V = \begin{pmatrix} 3 \\ 1-i \end{pmatrix}$$

$$x(t) = \begin{pmatrix} 3 \\ 1-i \end{pmatrix} e^{(2+3i)t} = \begin{pmatrix} 3 \\ 1-i \end{pmatrix} e^{2t} e^{3it}$$



$$=\begin{pmatrix}3\\1-i\end{pmatrix}e^{2t}(\cos 3t+i\sin 3t)$$

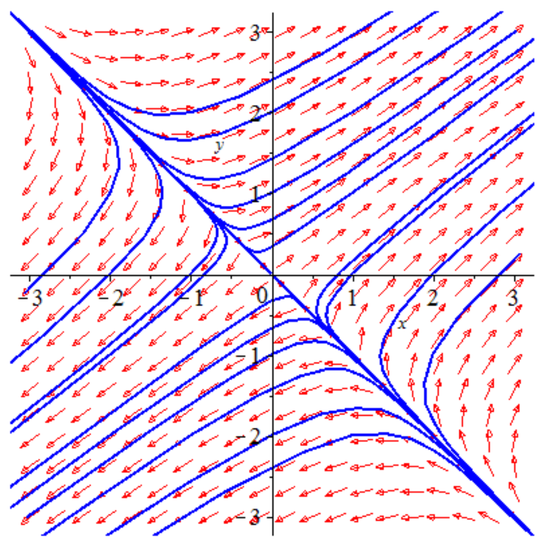
$$=\begin{pmatrix}3\cos 3t+3i\sin 3t\\ \cos 3t+\sin 3t+i(\sin 3t-\cos 3t)\end{pmatrix}e^{2t}$$

$$\begin{cases}x_1(t)=\left(3C_1\cos 3t+3C_2\sin 3t\right)e^{2t}\\ x_2(t)=\left[C_1(\cos 3t+\sin 3t)+C_2(\sin 3t-\cos 3t)\right]e^{2t}\\ \qquad\;\; =\left[\left(C_1-C_2\right)\cos 3t+\left(C_1+C_2\right)\sin 3t\right]e^{2t}\end{cases}$$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1'=3x_1+4x_2,\quad x_2'=3x_1+2x_2;\quad x_1(0)=x_2(0)=1$

Solution



$$\begin{pmatrix}x_1\\x_2\end{pmatrix}'=\begin{pmatrix}3&4\\3&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix}3-\lambda&4\\3&2-\lambda\end{vmatrix}=\lambda^2-5\lambda-6=0$$

The distinct real eigenvalues: $\lambda_1=-1,\quad \lambda_2=6$

For $\lambda_1=-1\ \Rightarrow\ (A+I)V_1=0$

$$\begin{pmatrix}4&4\\3&3\end{pmatrix}\begin{pmatrix}x_1\\y_1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\ \Rightarrow\ x_1=-y_1$$

$$\to V_1=\begin{pmatrix}1\\-1\end{pmatrix}\ \Rightarrow\ x_1(t)=\begin{pmatrix}1\\-1\end{pmatrix}e^{-t}$$

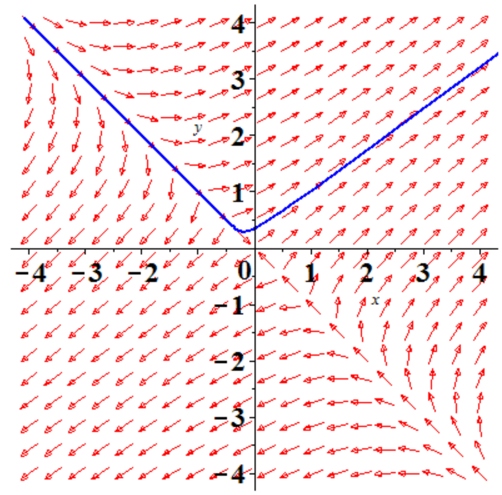
For $\lambda_2=6\ \Rightarrow\ (A-6I)V_2=0$

$$\begin{pmatrix}-3&4\\3&-4\end{pmatrix}\begin{pmatrix}x_2\\y_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\ \Rightarrow\ 3x_2=4y_2$$

$$\to V_2=\begin{pmatrix}4\\3\end{pmatrix}\ \Rightarrow\ x_2(t)=\begin{pmatrix}4\\3\end{pmatrix}e^{6t}$$

The general solution: $x(t)=C_1\begin{pmatrix}1\\-1\end{pmatrix}e^{-t}+C_2\begin{pmatrix}4\\3\end{pmatrix}e^{6t}$

$$\begin{cases} x_1(t) = C_1 e^{-t} + 4C_2 e^{6t} \\ x_2(t) = -C_1 e^{-t} + 3C_2 e^{6t} \end{cases}$$



Given: $\begin{cases} x_1(0) = C_1 + 4C_2 = 1 \\ x_2(0) = -C_1 + 3C_2 = 1 \end{cases}$

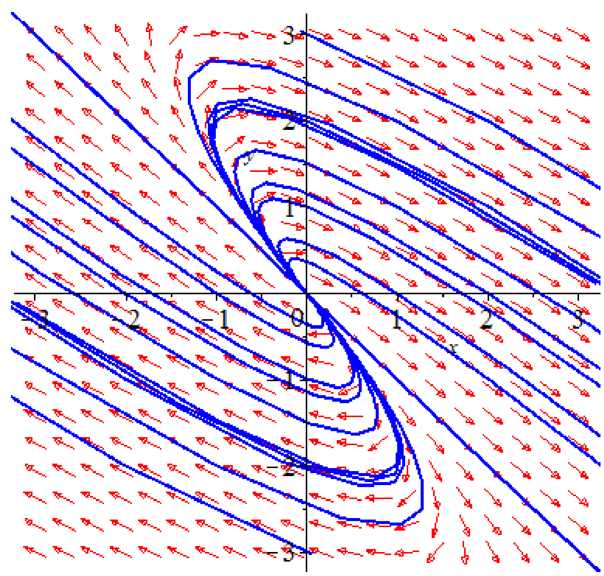
$$\rightarrow \boxed{C_2 = \frac{2}{7},\ C_1 = -\frac{1}{7}}$$

$$\begin{cases} x_1(t) = -\frac{1}{7}e^{-t} + \frac{8}{7}e^{6t} \\ x_2(t) = \frac{1}{7}e^{-t} + \frac{6}{7}e^{6t} \end{cases}$$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = 9x_1 + 5x_2, \quad x_2' = -6x_1 - 2x_2; \quad x_1(0) = 1,\ x_2(0) = 0$

Solution



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 9 & 5 \\ -6 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 12 = 0$$

The distinct real eigenvalues: $\boxed{\lambda_1 = 3, \quad \lambda_2 = 4}$

For $\lambda_1 = 3 \quad \Rightarrow \quad (A - 3I)V_1 = 0$

$$\begin{pmatrix} 6 & 5 \\ -6 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 6x_1 = -5y_1$$

$$\rightarrow V_1 = \begin{pmatrix} 5 \\ -6 \end{pmatrix} \quad \Rightarrow \quad x_1(t) = \begin{pmatrix} 5 \\ -6 \end{pmatrix} e^{3t}$$

For $\lambda_2 = 4 \quad \Rightarrow \quad (A - 4I)V_2 = 0$

$$\begin{pmatrix} 5 & 5 \\ -6 & -6 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad x_2 = -y_2$$

$$\rightarrow V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad x_2(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$$

The general solution:

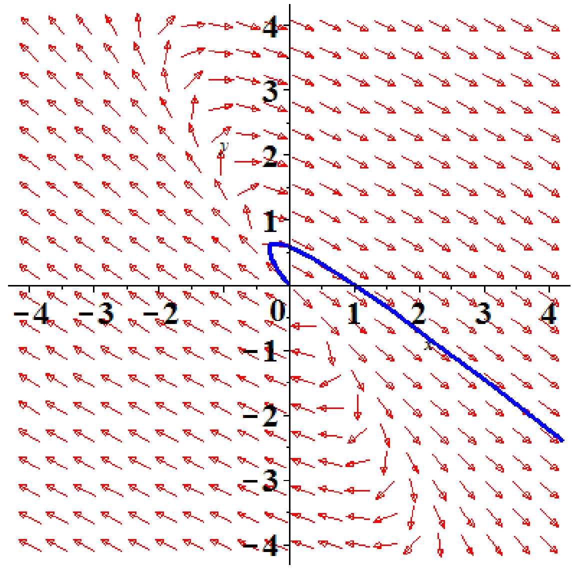
$$x(t)=C_1\begin{pmatrix}5\\-6\end{pmatrix}e^{3t}+C_2\begin{pmatrix}1\\-1\end{pmatrix}e^{4t}$$

$$\begin{cases}x_1(t)=5C_1e^{3t}+C_2e^{4t}\\ x_2(t)=-6C_1e^{3t}-C_2e^{4t}\end{cases}$$

Given: $\begin{cases}x_1(0)=5C_1+C_2=1\\ x_2(0)=-6C_1-C_2=0\end{cases}$

$$\rightarrow \underline{C_1=-1,\ \ C_2=6}|$$

$$\begin{cases}x_1(t)=-5e^{3t}+6e^{4t}\\ x_2(t)=6e^{3t}-6e^{4t}\end{cases}$$



Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1'=2x_1-5x_2,\quad x_2'=4x_1-2x_2;\quad x_1(0)=2,\ x_2(0)=3$

Solution

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}'=\begin{pmatrix}2&-5\\4&-2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

The characteristic equation:

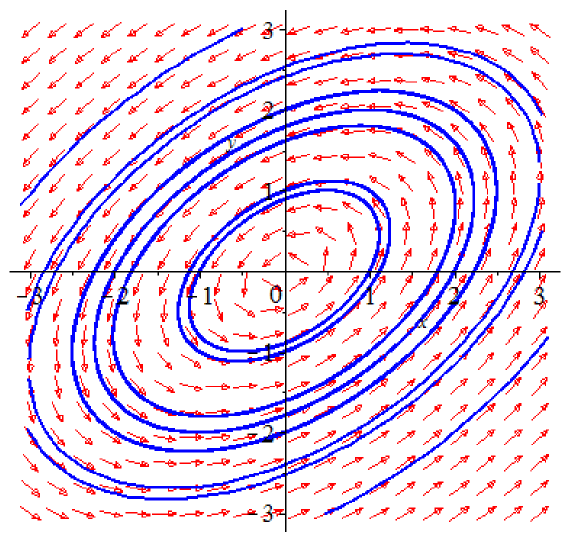
$$\begin{vmatrix}2-\lambda&-5\\4&-2-\lambda\end{vmatrix}=\lambda^2+16=0$$

The distinct real eigenvalues: $\underline{\lambda=\pm 4i}|$

For $\lambda=4i \quad\Rightarrow\quad (A-\lambda I)V=0$

$$\begin{pmatrix}2-4i&-5\\4&-2-4i\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad\Rightarrow\quad (2-4i)x=5y$$

$$\rightarrow V=\begin{pmatrix}5\\2-4i\end{pmatrix}$$



$$x(t)=\begin{pmatrix}5\\2-4i\end{pmatrix}e^{4it}\qquad\qquad e^{ait}=\cos at+i\sin at$$

$$=\begin{pmatrix}5\\2-4i\end{pmatrix}(\cos 4t+i\sin 4t)$$

$$=\underline{\begin{pmatrix}5\cos 4t+5i\sin 4t\\2\cos 4t+4\sin 4t+i(2\sin 4t-4\cos 4t)\end{pmatrix}}|$$

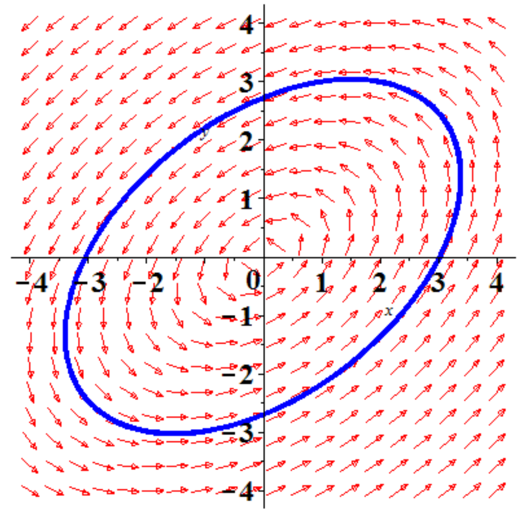
$$\begin{cases} x_1(t) = 5C_1\cos 4t + 5C_2\sin 4t \\ x_2(t) = C_1(2\cos 4t + 4\sin 4t) + C_2(2\sin 4t - 4\cos 4t) \end{cases}$$

Given: $x_1(0) = 2, \quad x_2(0) = 3$

$$\begin{cases} x_1(0) = 5C_1 = 2 \\ x_2(0) = 2C_1 - 4C_2 = 3 \end{cases}$$

$$\rightarrow \underline{C_1 = \frac{2}{5}, \quad C_2 = -\frac{11}{20}}$$

$$\begin{cases} x_1(t) = 2\cos 4t - \frac{11}{4}\sin 4t \\ x_2(t) = \frac{2}{5}(2\cos 4t + 4\sin 4t) - \frac{11}{20}(2\sin 4t - 4\cos 4t) \end{cases}$$

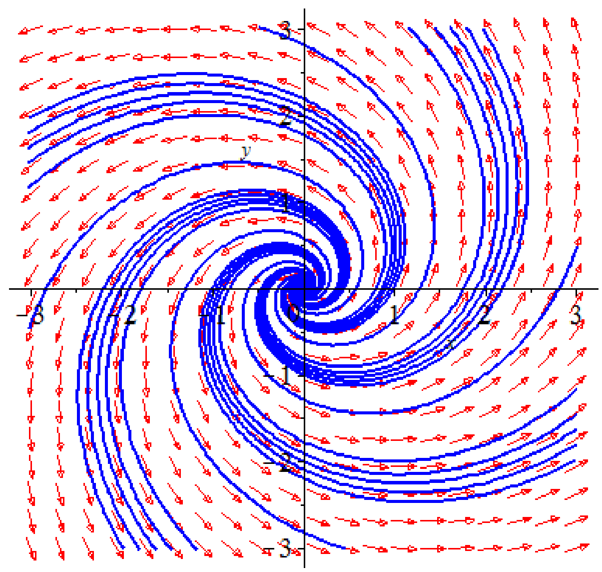


$$\begin{cases} x_1(t) = 2\cos 4t - \frac{11}{4}\sin 4t \\ x_2(t) = 3\cos 4t + \frac{1}{2}\sin 4t \end{cases}$$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = x_1 - 2x_2, \quad x_2' = 2x_1 + x_2; \quad x_1(0) = 0, \ x_2(0) = 4$

Solution



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5 = 0$$

The distinct real eigenvalues: $\underline{\lambda = 1 \pm 2i}$

For $\lambda = 1 - 2i \quad \Rightarrow \quad (A - \lambda I)V = 0$

$$\begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad (2i)x = 2y$$

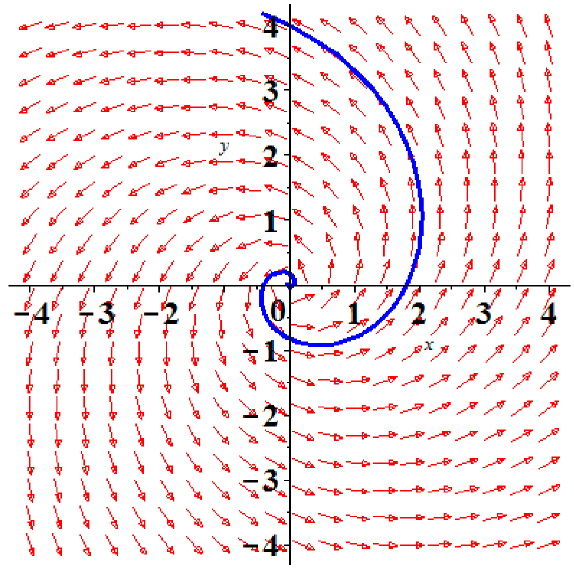
$$\rightarrow V = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$x(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(1-2i)t} \qquad\qquad e^{ait} = \cos at + i\sin at$$

$$= \begin{pmatrix} 1 \\ i \end{pmatrix}(\cos 2t - i\sin 2t)\ e^t$$

$$= \boxed{\begin{pmatrix} \cos 2t - i\sin 2t \\ \sin 2t + i\cos 2t \end{pmatrix} e^t}$$

$$\begin{cases} x_1(t) = \left(C_1 \cos 2t - C_2 \sin 2t\right) e^t \\ x_2(t) = \left(C_1 \sin 2t + C_2 \cos 2t\right) e^t \end{cases}$$



Given: $x_1(0) = 0, \quad x_2(0) = 4$

$$\begin{cases} x_1(0) = C_1 = 0 \\ x_2(0) = C_2 = 4 \end{cases}$$

$$\begin{cases} x_1(t) = -4e^t \sin 2t \\ x_2(t) = 4e^t \cos 2t \end{cases}$$

Exercise

Find the general solution $\quad x_1' = x_1 - 2x_2, \quad x_2' = 3x_1 - 4x_2; \quad x_1(0) = -1,\ x_2(0) = 2$

Solution

$$A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$$

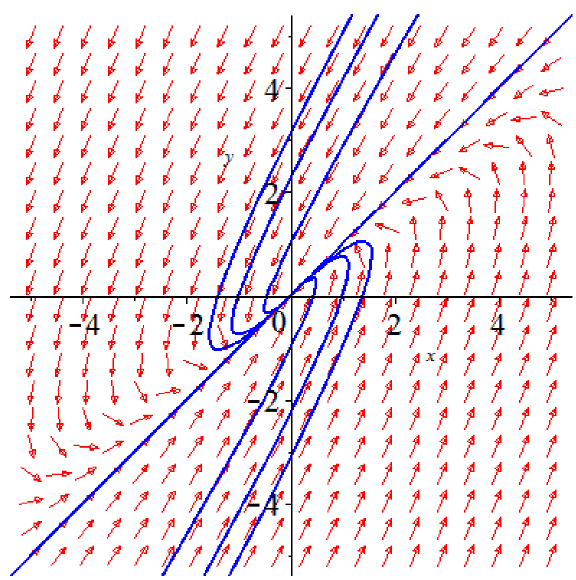
$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{vmatrix}$$

$$= \lambda^2 + 3\lambda + 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1 \quad and \quad \lambda_2 = -2$

For $\lambda_1 = -1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = y \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



For $\lambda_2 = -2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to 3x = 2y \quad V_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\boxed{y(t) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + C_2\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{-2t}}$$

$$y(0) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\begin{cases} C_1 + 2C_2 = -1 \\ C_1 + 3C_2 = 2 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1 \quad \Delta_1 = \begin{vmatrix} -1 & 2 \\ 2 & 3 \end{vmatrix} = -7 \quad \Delta_2 = \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 3$$

$$\boxed{C_1 = -7, \quad C_2 = 3}$$

$$y(t) = -7\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + 3\begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}$$

$$\begin{cases} y_1(t) = -7e^{-t} + 6e^{-2t} \\ y_2(t) = -7e^{-t} + 9e^{-2t} \end{cases}$$

Exercise

Find the general solution $\quad x_1' = -0.5x_1 + 2x_2, \quad x_2' = -2x_1 - 0.5x_2; \quad x_1(0) = -2, \; x_2(0) = 2$

Solution

$$A = \begin{pmatrix} -\frac{1}{2} & 2 \\ -2 & -\frac{1}{2} \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\frac{1}{2} - \lambda & 2 \\ -2 & -\frac{1}{2} - \lambda \end{vmatrix}$$

$$= \lambda^2 + \lambda + \frac{17}{4} = 0$$

$$= 4\lambda^2 + 4\lambda + 17 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = -\frac{1}{2} \pm 2i$

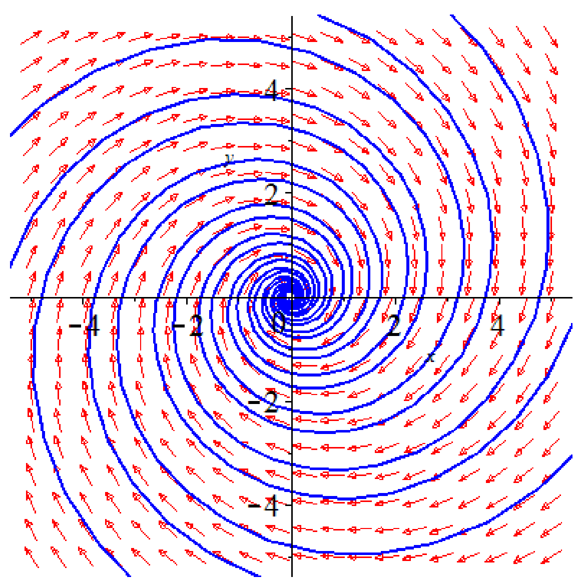
For $\lambda_1 = -\frac{1}{2} - 2i \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2i & 2 \\ -2 & 2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x = iy \qquad V_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$x(t) = \begin{pmatrix} i \\ 1 \end{pmatrix} e^{\left(-\frac{1}{2} - 2i\right)t} \qquad e^{ait} = \cos at + i\sin at$$

$$= \begin{pmatrix} i \\ 1 \end{pmatrix} (\cos 2t - i\sin 2t)\, e^{-t/2}$$

$$= \boxed{\begin{pmatrix} \sin 2t + i\cos 2t \\ \cos 2t - i\sin 2t \end{pmatrix} e^{-t/2}}$$



$$\boxed{y(t) = C_1\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^{-t/2} + C_2\begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^{-t/2}}$$

$$y(0) = C_1\begin{pmatrix}0\\1\end{pmatrix} + C_2\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-2\\2\end{pmatrix}$$

$$\underline{C_1 = 2, \quad C_2 = -2}$$

$$\begin{cases} y_1(t) = (2\sin 2t - 2\cos 2t)e^{-t/2} \\ y_2(t) = (2\cos 2t + 2\sin 2t)e^{-t/2} \end{cases}$$

Exercise

Find the general solution $\quad x_1' = 1.25x_1 + 0.75x_2, \quad x_2' = 0.75x_1 + 1.25x_2; \quad x_1(0) = -2, \ x_2(0) = 1$

Solution

$$A = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} \frac{5}{4} - \lambda & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} - \lambda \end{vmatrix}$$

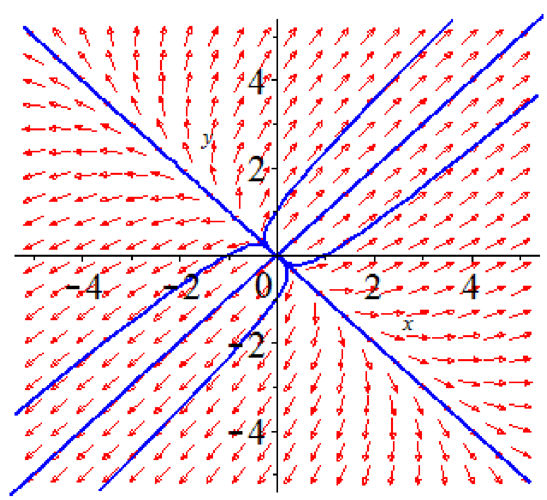
$$= \lambda^2 - \frac{5}{2}\lambda + 1$$

$$= 2\lambda^2 - 5\lambda + 2 = 0 \qquad \lambda_{1,2} = \frac{5 \pm 3}{4}$$

Thus, the eigenvalues are: $\lambda_1 = \frac{1}{2} \quad and \quad \lambda_2 = 2$

For $\lambda_1 = \frac{1}{2} \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} \end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad \to x = -y \qquad V_1 = \begin{pmatrix}-1\\1\end{pmatrix}$$



For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{3}{4} \end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad \to x = y \qquad V_2 = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\underline{y(t) = C_1\begin{pmatrix}-1\\1\end{pmatrix}e^{t/2} + C_2\begin{pmatrix}1\\1\end{pmatrix}e^{2t}}$$

$$y(0) = C_1\begin{pmatrix}-1\\1\end{pmatrix} + C_2\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-2\\1\end{pmatrix}$$

$$\begin{cases} -C_1 + C_2 = -2 \\ C_1 + C_2 = 1 \end{cases} \qquad \Delta = \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} = -4 \quad \Delta_1 = \begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix} = -5 \quad \Delta_2 = \begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix} = 1$$

$$\underline{C_1 = \frac{3}{2}, \quad C_2 = -\frac{1}{2}}$$

$$y(t) = \frac{3}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{t/2} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

$$\begin{cases} y_1(t) = -\frac{3}{2}e^{t/2} - \frac{1}{2}e^{2t} \\ y_2(t) = \frac{3}{2}e^{t/2} - \frac{1}{2}e^{2t} \end{cases}$$

Exercise

Find the general solution of the given system.

$$x_1' = 4x_1 + x_2 + 4x_3, \quad x_2' = x_1 + 7x_2 + x_3, \quad x_3' = 4x_1 + x_2 + 4x_3$$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 4-\lambda & 1 & 4 \\ 1 & 7-\lambda & 1 \\ 4 & 1 & 4-\lambda \end{vmatrix} = (4-\lambda)^2(7-\lambda) + 8 - 112 + 16\lambda - 8 + 2\lambda$$

$$= \left(16 - 8\lambda + \lambda^2\right)(7-\lambda) + 18\lambda - 112$$

$$= -\lambda^3 + 15\lambda^2 - 54\lambda = 0$$

The distinct real eigenvalues: $\underline{\lambda_1 = 0; \quad \lambda_2 = 6; \quad \lambda_3 = 9}$

For $\lambda_1 = 0 \quad \Rightarrow \quad (A - 0I)V_1 = 0$

$$\begin{pmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} 4a_1 + b_1 + 4c_1 = 0 \\ a_1 + 7b_1 + c_1 = 0 \end{cases}$$

$$\text{Let } b_1 = 0 \quad \Rightarrow \quad a_1 = -c_1 = 1 \quad \rightarrow V_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For $\lambda_2 = 6 \quad \Rightarrow \quad (A - 6I)V_2 = 0$

$$\begin{pmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \color{red}{rref} \quad \Rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_2 = c_2 \\ b_2 = -2c_2 \end{cases} \to V_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 9 \quad \Rightarrow \quad (A - 9I)V_3 = 0$

$$\begin{pmatrix} -5 & 1 & 4 \\ 1 & -2 & 1 \\ 4 & 1 & -5 \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_3 = c_3 \\ b_3 = c_3 \end{cases} \to V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x_1(t) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad x_2(t) = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} e^{6t} \quad x_3(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{9t}$$

$$\begin{cases} x_1(t) = \color{blue}{C_1 + C_2 e^{6t} + C_3 e^{9t}} \\ x_2(t) = \color{blue}{-2C_2 e^{6t} + C_3 e^{9t}} \\ x_3(t) = \color{blue}{-C_1 - C_2 e^{6t} + C_3 e^{9t}} \end{cases}$$

Exercise

Find the general solution of the given system.

$$x_1' = x_1 + 2x_2 + 2x_3, \quad x_2' = 2x_1 + 7x_2 + x_3, \quad x_3' = 2x_1 + x_2 + 7x_3$$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 7-\lambda & 1 \\ 2 & 1 & 7-\lambda \end{vmatrix} = (1-\lambda)(7-\lambda)^2 + 8 - 28 + 4\lambda - 1 + \lambda - 28 + 4\lambda$$

$$= (1-\lambda)\left(49 - 14\lambda + \lambda^2\right) + 9\lambda - 49$$

$$= -\lambda^3 + 15\lambda^2 - 54\lambda \color{red}{= 0}$$

The distinct real eigenvalues: $\lambda_1 = 0; \quad \lambda_2 = 6; \quad \lambda_3 = 9$

For $\lambda_1 = 0 \quad \Rightarrow \quad (A - 0I)V_1 = 0$

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \color{red}{rref} \Rightarrow \quad \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_1 = -4c_1 \\ b_1 = c_1 \end{cases} \quad \to V_1 = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 6 \quad \Rightarrow \quad (A - 6I)V_2 = 0$

$$\begin{pmatrix} -5 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \color{red}{rref} \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_2 = 0 \\ b_2 = -c_2 \end{cases} \quad \to V_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

For $\lambda_3 = 9 \quad \Rightarrow \quad (A - 9I)V_3 = 0$

$$\begin{pmatrix} -8 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_3 = \frac{1}{2}c_3 \\ b_3 = c_3 \end{cases} \to V_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$x_1(t) = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} \quad x_2(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{6t} \quad x_3(t) = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} e^{9t}$$

$$\begin{cases} x_1(t) = \color{blue}{-4C_1 + C_3 e^{9t}} \\ x_2(t) = \color{blue}{C_1 + C_2 e^{6t} + 2C_3 e^{9t}} \\ x_3(t) = \color{blue}{C_1 - C_2 e^{6t} + 2C_3 e^{9t}} \end{cases}$$

Exercise

Find the general solution of the given system.

$$x_1' = 4x_1 + x_2 + x_3, \quad x_2' = x_1 + 4x_2 + x_3, \quad x_3' = x_1 + x_2 + 4x_3$$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 4-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = (4-\lambda)^3 + 1 + 1 - 3(4-\lambda)$$

$$= 64 - 48\lambda + 12\lambda^2 - \lambda^3 - 10 + 3\lambda$$

$$= -\lambda^3 + 12\lambda^2 - 45\lambda + 54 \color{red}{= 0}$$

The distinct real eigenvalues: $\underline{\lambda_{1,2}=3;\quad \lambda_3=6}|$

For $\lambda_1=3 \quad\Rightarrow\quad (A-3I)V_1=0$

$$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad\Rightarrow\quad a_1+b_1+c_1=0 \qquad \to V_1=\begin{pmatrix}1\\-1\\0\end{pmatrix}\ \to V_2=\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

For $\lambda_3=6 \quad\Rightarrow\quad (A-6I)V_3=0$

$$\begin{pmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{pmatrix}\begin{pmatrix}a_3\\b_3\\c_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad rref\quad\Rightarrow\quad \begin{pmatrix}1&0&-1\\0&1&-1\\0&0&0\end{pmatrix}\to\begin{cases}a_3=c_3\\b_3=c_3\end{cases}\to V_3=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$x_1(t)=\begin{pmatrix}1\\-1\\0\end{pmatrix}e^{3t}\quad x_2(t)=\begin{pmatrix}1\\0\\-1\end{pmatrix}e^{3t}\quad x_3(t)=\begin{pmatrix}1\\1\\1\end{pmatrix}e^{6t}$$

$$\begin{cases}x_1(t)=C_1e^{3t}+C_2e^{3t}+C_3e^{6t}\\ x_2(t)=-C_1e^{3t}\qquad\quad +C_3e^{6t}\\ x_3(t)=\qquad\quad -C_2e^{3t}+C_3e^{6t}\end{cases}$$

Exercise

Find the general solution of the given system.

$$x_1'=5x_1+x_2+3x_3,\quad x_2'=x_1+7x_2+x_3,\quad x_3'=3x_1+x_2+5x_3$$

Solution

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}'=\begin{pmatrix}5&1&3\\1&7&1\\3&1&5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix}5-\lambda&1&3\\1&7-\lambda&1\\3&1&5-\lambda\end{vmatrix}=(7-\lambda)(5-\lambda)^2+6-9(7-\lambda)-5+\lambda-5+\lambda$$

$$=(7-\lambda)\left(25-10\lambda+\lambda^2\right)-67+11\lambda$$

$$=-\lambda^3+17\lambda^2-84\lambda+108=0$$

The distinct real eigenvalues: $\underline{\lambda_1=2;\quad \lambda_2=6;\quad \lambda_3=9}|$

For $\lambda_1=2 \quad\Rightarrow\quad (A-2I)V_1=0$

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 3 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \Rightarrow \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} a_1 = -c_1 \\ b_1 = 0 \end{cases} \quad \rightarrow V_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 6 \quad \Rightarrow \quad (A-6I)V_2 = 0$

$$\begin{pmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} a_2 = c_2 \\ b_2 = -2c_2 \end{cases} \quad \rightarrow V_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 9 \quad \Rightarrow \quad (A-9I)V_3 = 0$

$$\begin{pmatrix} -4 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} a_3 = c_3 \\ b_3 = c_3 \end{cases} \rightarrow V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x_1(t) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{2t} \quad x_2(t) = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{6t} \quad x_3(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{9t}$$

$$\begin{cases} x_1(t) = -C_1 e^{2t} + C_2 e^{6t} + C_3 e^{9t} \\ x_2(t) = \qquad -2C_2 e^{6t} + C_3 e^{9t} \\ x_3(t) = C_1 e^{2t} + C_2 e^{6t} + C_3 e^{9t} \end{cases}$$

Exercise

Find the general solution of the given system.

$$x_1' = 5x_1 - 6x_3, \quad x_2' = 2x_1 - x_2 - 2x_3, \quad x_3' = 4x_1 - 2x_2 - 4x_3$$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 5-\lambda & 0 & -6 \\ 2 & -1-\lambda & -2 \\ 4 & -2 & -4-\lambda \end{vmatrix} = (-1-\lambda)\left(-20-\lambda+\lambda^2\right) - 24\lambda - 20 + 4\lambda$$

$$= -\lambda^3 + \lambda = 0$$

The distinct real eigenvalues: $\lambda_1 = -1; \quad \lambda_2 = 0; \quad \lambda_3 = 1$

For $\lambda_1 = -1 \quad \Rightarrow \quad (A+I)V_1 = 0$

$$\begin{pmatrix} 6 & 0 & -6 \\ 2 & 0 & -2 \\ 4 & -2 & -3 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \Rightarrow \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_1 = c_1 \\ b_1 = \frac{1}{2}c_1 \end{cases} \quad \to V_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 0 \quad \Rightarrow \quad (A - 0I)V_2 = 0$

$$\begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & -\frac{6}{5} \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_2 = \frac{6}{5}c_2 \\ b_2 = \frac{2}{5}c_2 \end{cases} \quad \to V_2 = \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix}$$

For $\lambda_3 = 1 \quad \Rightarrow \quad (A - I)V_3 = 0$

$$\begin{pmatrix} 4 & 0 & -6 \\ 2 & -2 & -2 \\ 4 & -2 & -5 \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_3 = \frac{3}{2}c_3 \\ b_3 = \frac{1}{2}c_3 \end{cases} \to V_3 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$x_1(t) = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{-t} \quad x_2(t) = \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} \quad x_3(t) = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} e^{t}$$

$$\begin{cases} x_1(t) = 2C_1 e^{-t} + 6C_2 + 3C_3 e^{t} \\ x_2(t) = C_1 e^{-t} + 2C_2 + C_3 e^{t} \\ x_3(t) = 2C_1 e^{-t} + 5C_2 + 2C_3 e^{t} \end{cases}$$

Exercise

Find the general solution of the given system.

$$x_1' = 3x_1 + 2x_2 + 2x_3, \quad x_2' = -5x_1 - 4x_2 - 2x_3, \quad x_3' = 5x_1 + 5x_2 + 3x_3$$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 3 & 2 & 2 \\ -5 & -4 & -2 \\ 5 & 5 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 3-\lambda & 2 & 2 \\ -5 & -4-\lambda & -2 \\ 5 & 5 & 3-\lambda \end{vmatrix} = (3-\lambda)^2(-4-\lambda) - 20 - 50 - 10(-4-\lambda) + 20(3-\lambda)$$

$$= \left(9 - 6\lambda + \lambda^2\right)(-4 - \lambda) + 30 - 10\lambda$$

$$= -\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$$

The distinct real eigenvalues: $\boxed{\lambda_1 = -2; \quad \lambda_2 = 1; \quad \lambda_3 = 3}$

For $\lambda_1 = -2 \quad \Rightarrow \quad (A + 2I)V_1 = 0$

$$\begin{pmatrix} 5 & 2 & 2 \\ -5 & -2 & -2 \\ 5 & 5 & 5 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} a_1 = 0 \\ b_1 = -c_1 \end{cases} \quad \rightarrow V_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1 \quad \Rightarrow \quad (A - I)V_2 = 0$

$$\begin{pmatrix} 2 & 2 & 2 \\ -5 & -5 & -2 \\ 5 & 5 & 2 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} a_2 = -b_2 \\ c_2 = 0 \end{cases} \quad \rightarrow V_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

For $\lambda_3 = 3 \quad \Rightarrow \quad (A - 3I)V_3 = 0$

$$\begin{pmatrix} 0 & 2 & 2 \\ -5 & -7 & -2 \\ 5 & 5 & 0 \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} a_3 = c_3 \\ b_3 = -c_3 \end{cases} \rightarrow V_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

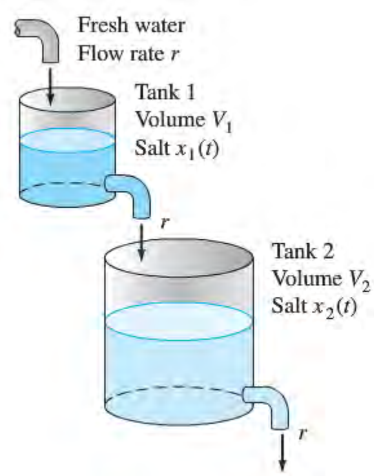
$$x_1(t) = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{-2t} \quad x_2(t) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{t} \quad x_3(t) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{3t}$$

$$\begin{cases} x_1(t) = \qquad\qquad C_2 e^{t} + C_3 e^{3t} \\ x_2(t) = -C_1 e^{-2t} - C_2 e^{t} - C_3 e^{3t} \\ x_3(t) = C_1 e^{-2t} \qquad\qquad + C_3 e^{3t} \end{cases}$$

Exercise

Find the amount $x_1(t),\ x_2(t)$ of salt in each tank at time $t \ge 0$, with $x_1(0) = 15\ lb \quad x_2(0) = 0$. If $V_1 = 50\ gal,\quad V_2 = 25\ gal,\quad r = 10\ gal/\min$

Solution



$$\begin{cases} x_1' = -k_1 x_1 \\ x_2' = k_1 x_1 - k_2 x_2 \end{cases} \qquad \text{where } k_i = \frac{r}{v_i} \quad i = 1,2$$

$$k_1 = \frac{10}{50} = .2 \quad k_2 = \frac{10}{25} = .4 \rightarrow \begin{cases} x_1' = -.2x_1 \\ x_2' = .2x_1 - .4x_2 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -.2 & 0 \\ .2 & -.4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{with} \quad x(0) = \begin{pmatrix} 15 \\ 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -.2 - \lambda & 0 \\ .2 & -.4 - \lambda \end{vmatrix} = (-.2 - \lambda)(-.4 - \lambda) = 0$$

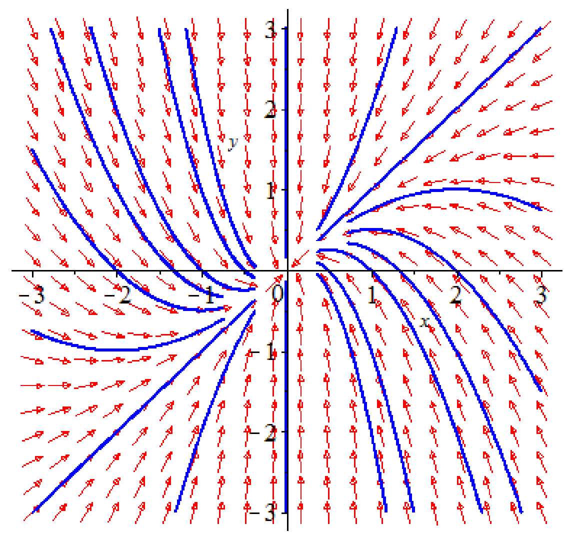
The eigenvalues are: $\underline{\lambda_1 = -.4 \quad \lambda_2 = -.2|}$

For $\lambda_1 = -.4 \quad \Rightarrow \quad (A + .4I)V_1 = 0$

$$\begin{pmatrix} .2 & 0 \\ .2 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a_1 = 0 \quad \rightarrow \quad V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -.2 \quad \Rightarrow \quad (A + .2I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 \\ .2 & -.2 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a_2 = b_2 \quad \rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



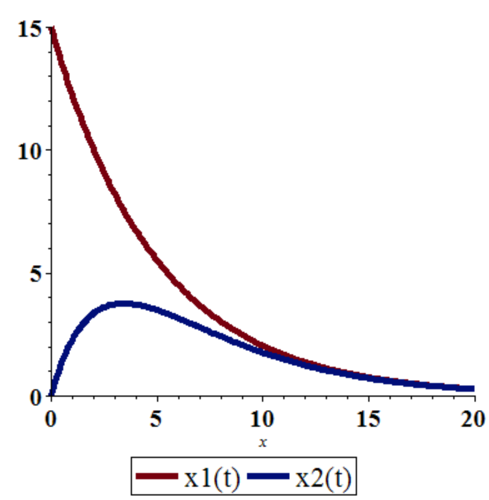
$$\Rightarrow x(t) = C_1\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-.4t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-.2t}$$

The general solution:

$$\begin{cases} x_1(t) = C_2 e^{-.2t} \\ x_2(t) = C_1 e^{-.4t} + C_2 e^{-.2t} \end{cases}$$

$$\begin{cases} x_1(0) = C_2 = 15 \\ x_2(0) = C_1 + C_2 = 0 \end{cases} \Rightarrow \quad \underline{C_2 = 15,\ C_1 = -15|}$$

$$\begin{cases} x_1(t) = 15e^{-.2t} \\ x_2(t) = 15e^{-.2t} - 15e^{-.4t} \end{cases}$$



***Tank* 2**: $x_2'(t) = -3e^{-.2t} + 6e^{-.4t} = 0$

$$e^{-.2t} = 2e^{-.4t}$$

$$\ln e^{-.2t} = \ln\left(2e^{-.4t}\right)$$

$$-.2t = \ln(2) - .4t$$

$$\lfloor t = \frac{1}{.2}\ln 2 = 5\ln 2 \rfloor$$

The maximum values of salt in tank 2 is: $x_2(t = 5\ln 2) = 15e^{-.2(5\ln 2)} - 15e^{-.4(5\ln 2)}$

$$= 15\left(2^{-1} - 2^{-2}\right)$$

$$= 3.75 \ \ lb.$$

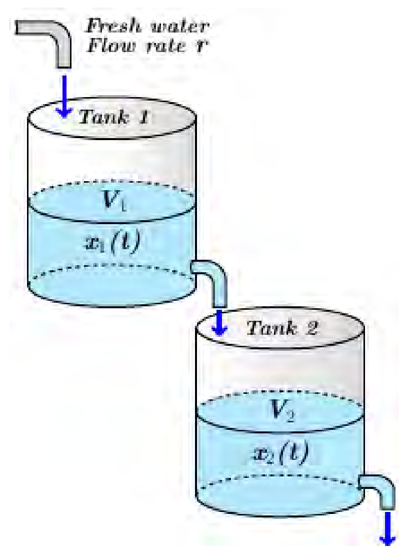
There is no maximum values of salt in tank 1.

$$x_1'(t) = -3e^{-.2t} \neq 0$$

Exercise

Find the amount $x_1(t)$, $x_2(t)$ of salt in each tank at time $t \geq 0$, with $x_1(0) = 15\ lb$ $\quad x_2(0) = 0$. If $V_1 = 25\ gal$, $\quad V_2 = 40\ gal$, $\quad r = 10\ gal/\min$

Solution



$$\begin{cases} x_1' = -k_1 x_1 \\ x_2' = k_1 x_1 - k_2 x_2 \end{cases} \qquad \text{where } k_i = \frac{r}{v_i} \quad i = 1,2$$

$$k_1 = \frac{10}{25} = .4 \quad k_2 = \frac{10}{40} = .25 \ \rightarrow \begin{cases} x_1' = -.4x_1 \\ x_2' = .4x_1 - .25x_2 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -.4 & 0 \\ .4 & -.25 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{with} \quad x(0) = \begin{pmatrix} 15 \\ 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -.4-\lambda & 0 \\ .4 & -.25-\lambda \end{vmatrix} = (-.25-\lambda)(-.4-\lambda) = 0$$

The eigenvalues are: $\lambda_1 = -.4 \quad \lambda_2 = -.25$

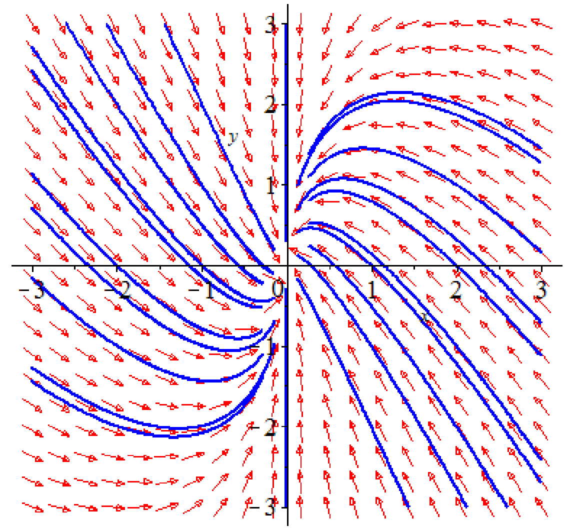
For $\lambda_1 = -.4 \quad \Rightarrow \quad (A + .4I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 \\ .4 & .15 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \Rightarrow \ .4a_1 = -.15b_1 \quad \rightarrow \quad V_1 = \begin{pmatrix} 3 \\ -8 \end{pmatrix}$$

For $\lambda_2 = -.25 \quad \Rightarrow \quad (A + .25I)V_2 = 0$

$$\begin{pmatrix} .15 & 0 \\ .4 & 0 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a_2 = 0 \quad \to \quad V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow x(t) = C_1 \begin{pmatrix} 3 \\ -8 \end{pmatrix} e^{-.4t} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-.25t}$$



The general solution:

$$\begin{cases} x_1(t) = 3C_1 e^{-.4t} \\ x_2(t) = -8C_1 e^{-.4t} + C_2 e^{-.25t} \end{cases}$$

$$\begin{cases} x_1(0) = 3C_1 = 15 \\ x_2(0) = -8C_1 + C_2 = 0 \end{cases} \Rightarrow \boxed{C_1 = 5,\ C_2 = 40}$$

$$\begin{cases} x_1(t) = 15e^{-.4t} \\ x_2(t) = -40e^{-.4t} + 40e^{-.25t} \end{cases}$$

There is no maximum values of salt in tank 1.

$$x_1'(t) = -6e^{-.4t} \neq 0$$

Tank 2: $x_2'(t) = 16e^{-.4t} - 10e^{-.25t} = 0$

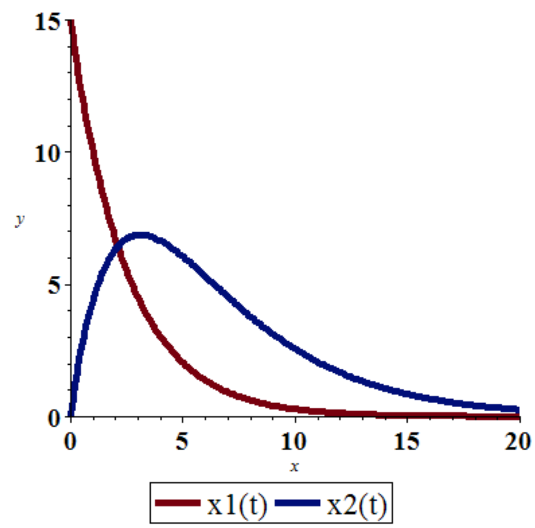
$$8e^{-.4t} = 5e^{-.25t}$$

$$\ln\left(e^{-.4t}\right) = \ln\left(\frac{5}{8}e^{-.25t}\right)$$

$$-.4t = \ln\left(\frac{5}{8}\right) - .25t$$

$$-.15t = \ln\left(\frac{5}{8}\right)$$

$$\boxed{t = \frac{1}{.15}\ln\frac{8}{5} = \frac{20}{3}\ln\frac{8}{5}}$$



The maximum values of salt in tank 2 is:

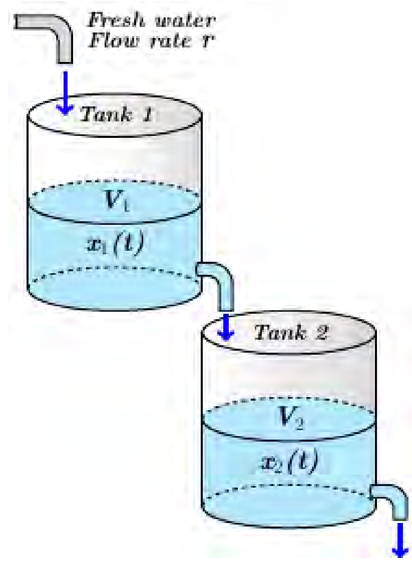
$$x_2\left(t = \frac{20}{3}\ln\frac{8}{5}\right) = -40e^{-.4\left(\frac{20}{3}\ln\frac{8}{5}\right)} + 40e^{-.25\left(\frac{20}{3}\ln\frac{8}{5}\right)}$$

$$\boxed{= 6.85 \ \ lb.}$$

Exercise

Find the amount $x_1(t)$, $x_2(t)$ of salt in each tank at time $t \ge 0$, with $x_1(0) = 15\ lb \quad x_2(0) = 0$. If $V_1 = 50\ gal,\quad V_2 = 25\ gal,\quad r = 5\ gal/\min$

Solution



$$\begin{cases} x_1' = -\frac{5}{50}x_1 \\ x_2' = \frac{5}{50}x_1 - \frac{5}{25}x_2 \end{cases} \to \begin{cases} x_1' = -\frac{1}{10}x_1 \\ x_2' = \frac{1}{10}x_1 - \frac{1}{5}x_2 \end{cases}$$

$$|A - \lambda I| = \begin{vmatrix} -\frac{1}{10} - \lambda & 0 \\ \frac{1}{10} & -\frac{1}{5} - \lambda \end{vmatrix} = \left(\frac{1}{10} + \lambda\right)\left(\frac{1}{5} + \lambda\right) = 0$$

The eigenvalues are: $\boxed{\lambda_1 = -\frac{1}{10} \quad \lambda_2 = -\frac{1}{5}}$

For $\lambda_1 = -\frac{1}{10} \quad \Rightarrow \quad (A + \lambda_1 I)V_1 = 0$

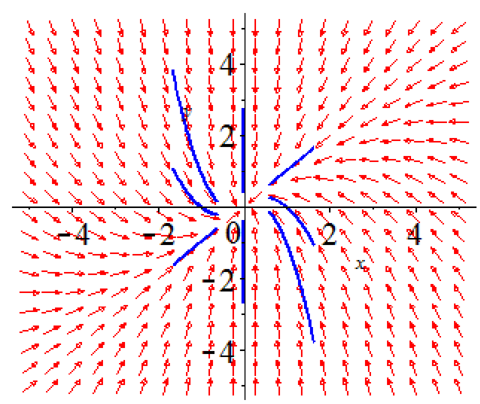
$$\begin{pmatrix} 0 & 0 \\ \frac{1}{10} & -\frac{1}{10} \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a_1 = b_1 \quad \to \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -\frac{1}{5} \quad \Rightarrow \quad (A + \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} \frac{1}{10} & 0 \\ \frac{1}{10} & 0 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a_2 = 0 \quad \to \quad V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow x(t) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t/10} + C_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-t/5}$$

The general solution: $\begin{cases} x_1(t) = C_1 e^{-t/10} \\ x_2(t) = C_1 e^{-t/10} + C_2 e^{-t/5} \end{cases}$



$$\begin{cases} x_1(0) = C_2 = 15 \\ x_2(0) = C_1 + C_2 = 0 \end{cases} \Rightarrow \boxed{C_2 = 15,\ C_1 = -15}$$

$$\begin{cases} x_1(t) = 15e^{-t/10} \\ x_2(t) = 15e^{-t/10} - 15e^{-t/5} \end{cases}$$

***Tank* 1**: $x_1'(t) = -\frac{3}{2}e^{-t/10} \neq 0$

There is ***no*** maximum values of salt in tank 1.

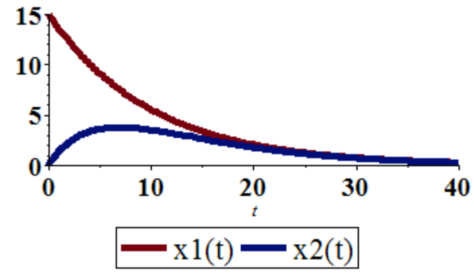
Tank 2: $x_2'(t) = -\frac{3}{2}e^{-t/10} + 3e^{-t/5} = 0$

$$e^{-t/10} = 2e^{-t/5}$$

$$\ln\left(e^{-t/10}\right) = \ln\left(2e^{-t/5}\right)$$

$$-\frac{1}{10}t = \ln(2) - \frac{1}{5}t$$

$$\underline{t = 10\ln 2}$$



The maximum values of salt in tank 2 is:

$$x_2\left(t = \ln 2^{10}\right) = 15e^{-\frac{1}{10}\ln 2^{10}} - 15e^{-\frac{1}{5}\ln 2^{10}}$$

$$= 15\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$\underline{= 3.75 \ \ lb.}$$

Exercise

Find the amount $x_1(t), x_2(t)$ of salt in each tank at time $t \geq 0$, with $x_1(0) = 15\ lb \quad x_2(0) = 0$. If $V_1 = 25\ gal, \quad V_2 = 40\ gal, \quad r = 5\ \ gal\ /\ min$

Solution

$$\begin{cases} x_1' = -\frac{5}{25}x_1 \\ x_2' = \frac{5}{25}x_1 - \frac{5}{40}x_2 \end{cases} \rightarrow \begin{cases} x_1' = -\frac{1}{5}x_1 \\ x_2' = \frac{1}{5}x_1 - \frac{1}{8}x_2 \end{cases}$$

$$|A - \lambda I| = \begin{vmatrix} -\frac{1}{5} - \lambda & 0 \\ \frac{1}{5} & -\frac{1}{8} - \lambda \end{vmatrix} = \left(\frac{1}{5} + \lambda\right)\left(\frac{1}{8} + \lambda\right) = 0$$

The eigenvalues are: $\underline{\lambda_1 = -\frac{1}{5} \quad \lambda_2 = -\frac{1}{8}}$

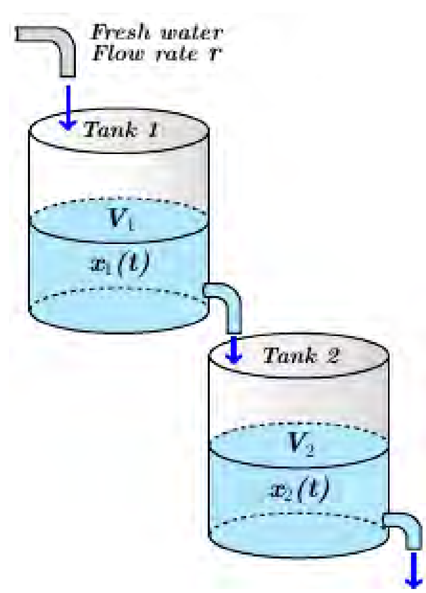
For $\lambda_1 = -\frac{1}{5} \quad \Rightarrow \quad \left(A + \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 0 & 0 \\ \frac{1}{5} & \frac{3}{40} \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 8a_1 = -3b_1 \quad \rightarrow \quad V_1 = \begin{pmatrix} 3 \\ -8 \end{pmatrix}$$

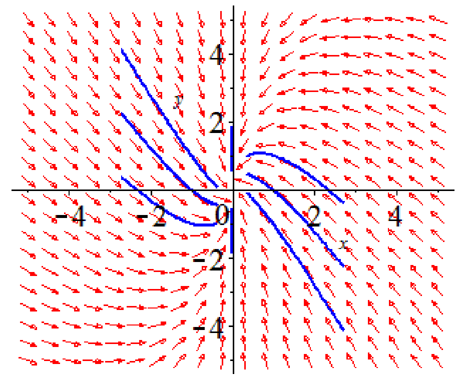
For $\lambda_2 = -\frac{1}{8} \quad \Rightarrow \quad \left(A + \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -\frac{3}{40} & 0 \\ \frac{1}{5} & 0 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a_2 = 0 \quad \rightarrow \quad V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow x(t) = C_1\begin{pmatrix} 3 \\ -8 \end{pmatrix}e^{-t/5} + C_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-t/8}$$



The general solution: $\begin{cases} x_1(t) = 3C_1 e^{-t/5} \\ x_2(t) = -8C_1 e^{-t/5} + C_2 e^{-t/8} \end{cases}$



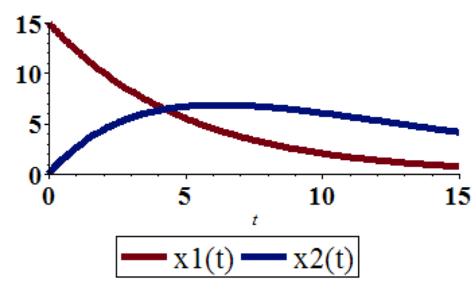
$$\begin{cases} x_1(0) = 3C_1 = 15 \\ x_2(0) = -8C_1 + C_2 = 0 \end{cases} \Rightarrow \boxed{C_1 = 5,\ C_2 = 40}$$

$$\begin{cases} x_1(t) = 15e^{-t/5} \\ x_2(t) = -40e^{-t/5} + 40e^{-t/8} \end{cases}$$

Tank **1**: $x_1'(t) = 15e^{-t/5} \neq 0$

There is ***no*** maximum values of salt in tank 1.

Tank **2**: $x_2'(t) = 8e^{-t/5} - 5e^{-t/8} = 0$



$$\ln\left(8e^{-t/5}\right) = \ln\left(5e^{-t/8}\right)$$

$$\ln 8 - \frac{1}{5}t = \ln 5 - \frac{1}{8}t$$

$$\frac{3}{40}t = \ln 8 - \ln 5$$

$$\boxed{t = \frac{40}{3}\ln\frac{8}{5}}$$

The maximum values of salt in tank 2 is:

$$x_2\left(t = \frac{40}{3}\ln\frac{8}{5}\right) = -40e^{-\frac{8}{3}\ln\frac{8}{5}} + 40e^{-\frac{5}{3}\ln\frac{8}{5}}$$

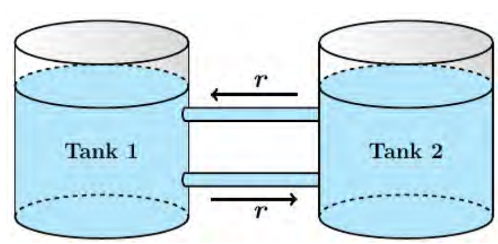
$$= 40\left(-\left(\frac{5}{8}\right)^{8/3} + \left(\frac{5}{8}\right)^{5/3}\right)$$

$$\boxed{= 6.85\ \ lb.}$$

Exercise

Find the amount $x_1(t),\ x_2(t)$ of salt in each tank at time $t \geq 0$, with $x_1(0) = 15\ lb \quad x_2(0) = 0$. If $V_1 = 50\ gal,\quad V_2 = 25\ gal,\quad r = 10\ gal/\min$

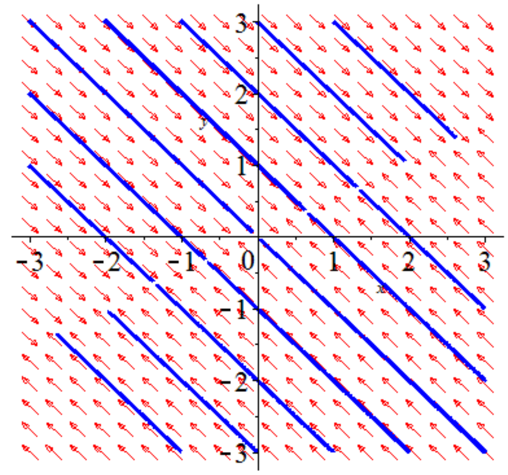
Solution



$$\begin{cases} x_1' = -k_1 x_1 + k_2 x_2 \\ x_2' = k_1 x_1 - k_2 x_2 \end{cases} \qquad \text{where } k_i = \frac{r}{V_i} \quad i = 1, 2$$

$$k_1 = \frac{10}{50} = .2 \quad k_2 = \frac{10}{25} = .4 \rightarrow \begin{cases} x_1' = -.2x_1 + .4x_2 \\ x_2' = .2x_1 - .4x_2 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -.2 & .4 \\ .2 & -.4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad with \quad x(0) = \begin{pmatrix} 15 \\ 0 \end{pmatrix}$$



$$|A - \lambda I| = \begin{vmatrix} -.2-\lambda & .4 \\ .2 & -.4-\lambda \end{vmatrix}$$

$$= (-.2-\lambda)(-.4-\lambda) - .08$$

$$= \lambda^2 + .6\lambda = 0$$

The eigenvalues are: $\lambda_1 = -.6 \quad \lambda_2 = 0$

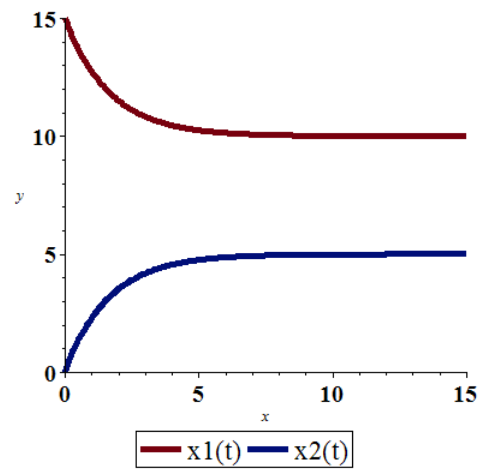
For $\lambda_1 = -.6 \quad \Rightarrow \quad (A + .6I)V_1 = 0$

$$\begin{pmatrix} .4 & .4 \\ .2 & .2 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow .4a_1 = -.4b_1 \quad \to \quad V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda_2 = 0 \quad \Rightarrow \quad (A - 0I)V_2 = 0$

$$\begin{pmatrix} -.2 & .4 \\ .2 & -.4 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow .2a_2 = .4b_2 \quad \to \quad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow x(t) = C_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-.6t} + C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$



The general solution:

$$\begin{cases} x_1(t) = C_1 e^{-0.6t} + 2C_2 \\ x_2(t) = -C_1 e^{-0.6t} + C_2 \end{cases}$$

$$\begin{cases} x_1(0) = C_1 + 2C_2 = 15 \\ x_2(0) = -C_1 + C_2 = 0 \end{cases} \quad \Rightarrow \quad C_1 = 5, \ C_2 = 5$$

$$\begin{cases} x_1(t) = 10 + 5e^{-0.6t} \\ x_2(t) = 5 - 5e^{-0.6t} \end{cases}$$

Exercise

Find the amount $x_1(t),\ x_2(t)$ of salt in each tank at time $t \ge 0$, with $x_1(0) = 15\ lb \quad x_2(0) = 0$. If

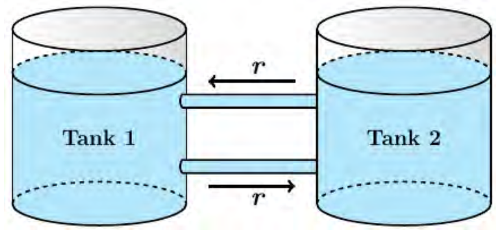
$V_1 = 25\ gal, \quad V_2 = 40\ gal, \quad r = 10\ gal/\min$

Solution

$$\begin{cases} x_1' = -k_1x_1 + k_2x_2 \\ x_2' = k_1x_1 - k_2x_2 \end{cases} \qquad \text{where } k_i = \frac{r}{v_i} \quad i = 1,2$$

$$k_1 = \frac{10}{25} = .4 \quad k_2 = \frac{10}{40} = .25 \ \rightarrow \begin{cases} x_1' = -.4x_1 + .25x_2 \\ x_2' = .4x_1 - .25x_2 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -.4 & .25 \\ .4 & -.25 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad with \quad x(0) = \begin{pmatrix} 15 \\ 0 \end{pmatrix}$$



$$|A - \lambda I| = \begin{vmatrix} -.4-\lambda & .25 \\ .4 & -.25-\lambda \end{vmatrix}$$

$$= (-.25-\lambda)(-.4-\lambda) - .1$$

$$= \lambda^2 + .65\lambda = 0$$

The eigenvalues are: $\underline{\lambda_1 = 0 \quad \lambda_2 = -.65}$

For $\lambda_1 = 0 \ \Rightarrow \ (A - 0I)V_1 = 0$

$$\begin{pmatrix} -.4 & .25 \\ .4 & -.25 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow .4a_1 = .25b_1 \ \rightarrow \ V_1 = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

For $\lambda_2 = -.65 \ \Rightarrow \ (A + .65I)V_2 = 0$

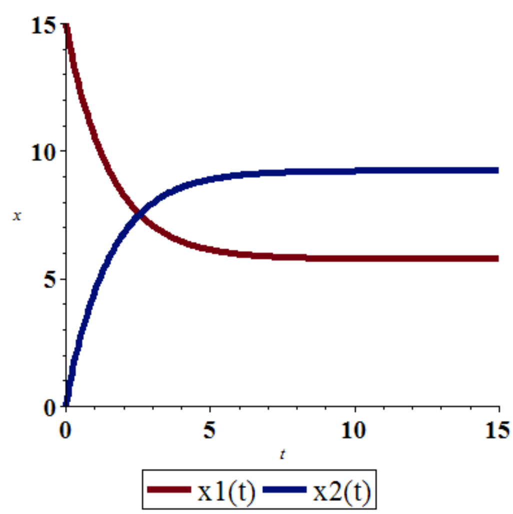
$$\begin{pmatrix} .25 & .25 \\ .4 & .4 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow .25a_2 = -.25b_2 \ \rightarrow \ V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow x(t) = C_1\begin{pmatrix} 5 \\ 8 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-.65t}$$

The general solution: $\begin{cases} x_1(t) = 5C_1 + C_2e^{-0.65t} \\ x_2(t) = 8C_1 - C_2e^{-0.65t} \end{cases}$

$$\begin{cases} x_1(0) = 5C_1 + C_2 = 15 \\ x_2(0) = 8C_1 - C_2 = 0 \end{cases} \Rightarrow \underline{C_1 = \frac{15}{13}, \ C_2 = \frac{120}{13}}$$

$$\begin{cases} x_1(t) = \frac{15}{13}\left(5 + 8e^{-0.6t}\right) \\ x_2(t) = \frac{120}{13}\left(1 - e^{-0.6t}\right) \end{cases}$$



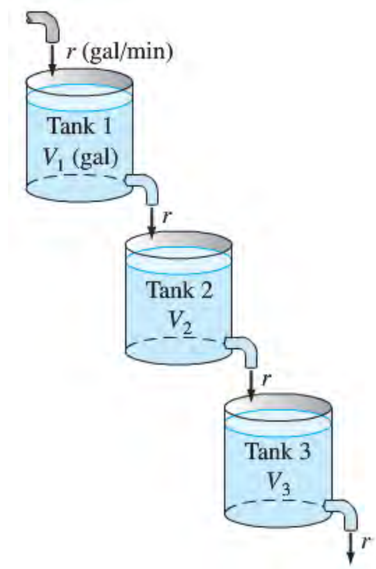
Exercise

Find the amount $x_1(t),\ x_2(t),\ x_3(t)$ of salt in each tank at time $t \ge 0$, if

$V_1 = 30\ gal,\quad V_2 = 15\ gal,\quad V_3 = 10\ gal,\quad r = 30\ gal/\min\quad x_1(0) = 27\ lb\quad x_2(0) = x_3(0) = 0$

Solution

$$\begin{cases} x_1' = -k_1 x_1 \\ x_2' = k_1 x_1 - k_2 x_2 \\ x_3' = \qquad k_2 x_2 - k_3 x_3 \end{cases} \qquad \text{where } k_i = \frac{r}{V_i} \quad i = 1, 2, 3$$



$$k_1 = \frac{30}{30} = 1 \quad k_2 = \frac{30}{15} = 2 \quad k_3 = \frac{30}{10} = 3$$

$$\rightarrow \begin{cases} x_1' = -x_1 \\ x_2' = x_1 - 2x_2 \\ x_3' = \qquad 2x_2 - 3x_3 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad with \quad x(0) = \begin{pmatrix} 27 \\ 0 \\ 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 0 & 0 \\ 1 & -2-\lambda & 0 \\ 0 & 2 & -3-\lambda \end{vmatrix} = (-1-\lambda)(-2-\lambda)(-3-\lambda) = 0$$

The eigenvalues are: $\underline{\lambda_1 = -3 \quad \lambda_2 = -2 \quad \lambda_3 = -1}$

For $\lambda_1 = -3 \;\Rightarrow\; (A + 3I)V_1 = 0$

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2a_1 = 0 \rightarrow a_1 = 0 \\ a_1 = -b_1 \rightarrow b_1 = 0 \end{cases} \rightarrow V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow x_1(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-3t}$$

For $\lambda_2 = -2 \;\Rightarrow\; (A + 2I)V_2 = 0$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a_2 = 0 \\ 2b_2 = c_2 \end{cases} \rightarrow V_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \Rightarrow x_2(t) = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} e^{-2t}$$

For $\lambda_3 = -1 \;\Rightarrow\; (A + I)V_3 = 0$

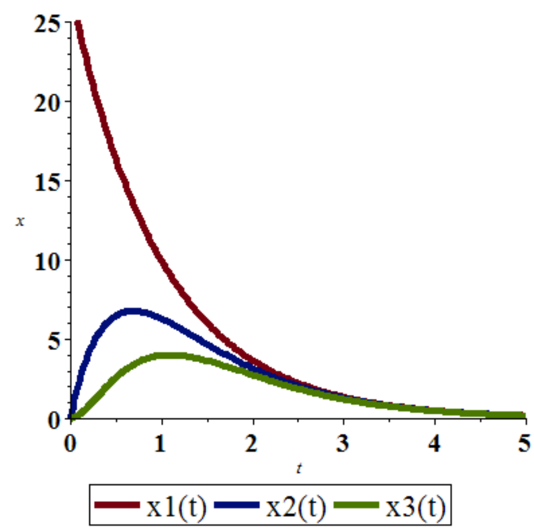
$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a_3 = b_3 \\ 2b_3 = 2c_3 \end{cases} \rightarrow V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow x_3(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-t}$$

$$\Rightarrow\ x(t) = C_1\begin{pmatrix}0\\0\\1\end{pmatrix}e^{-3t} + C_2\begin{pmatrix}0\\1\\2\end{pmatrix}e^{-2t} + C_3\begin{pmatrix}1\\1\\1\end{pmatrix}e^{-t}$$

$$\begin{cases} x_1(t) = & & & C_3e^{-t} \\ x_2(t) = & & C_2e^{-2t} & +C_3e^{-t} \\ x_3(t) = & C_1e^{-3t} & +2C_2e^{-2t} & +C_3e^{-t} \end{cases}$$

$$\begin{cases} 27 = C_3 \\ 0 = C_2 + C_3 \\ 0 = C_1 + 2C_2 + C_3 \end{cases} \rightarrow \begin{cases} \underline{C_3 = 27} \\ \underline{C_2 = -27} \\ C_1 = -27 - 2(-27) = \underline{27} \end{cases}$$

$$\begin{cases} x_1(t) = 27e^{-t} \\ x_2(t) = 27e^{-t} - 27e^{-2t} \\ x_3(t) = 27e^{-t} - 54e^{-2t} + 27e^{-3t} \end{cases}$$



***Tank* 2**: $x_2'(t) = -27e^{-t} + 54e^{-2t} = 0$

$e^{-t} = 2e^{-2t} \quad \Rightarrow \quad -t = \ln 2 - 2t$

$\underline{t = \ln 2}$

The maximum values of salt in tank 2 is:

$$x_2(\ln 2) = 27\left(e^{-\ln 2} - e^{-2\ln 2}\right) = 27\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{27}{4}\ lbs$$

***Tank* 3**: $x_3'(t) = 27\left(-e^{-t} + 4e^{-2t} - 3e^{-3t}\right) = 0$

$$e^{3t}\left(-e^{-t} + 4e^{-2t} - 3e^{-3t}\right) = 0$$

$$e^{2t} - 4e^t + 3 = 0 \quad \begin{cases} e^t = 1 \rightarrow t = 0 \\ e^t = 3 \rightarrow \underline{t = \ln 3} \end{cases}$$

The maximum values of salt in tank 3 is:

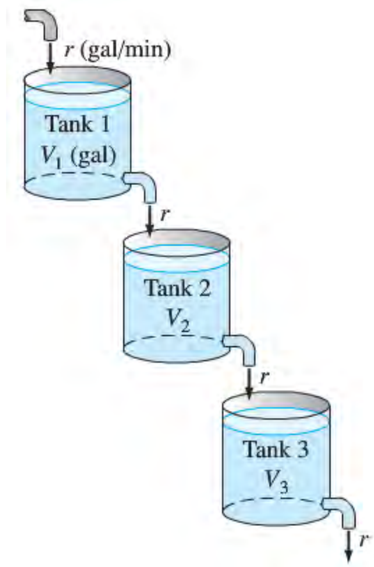
$$x_3(\ln 3) = 27\left(e^{-\ln 3} - 2e^{-2\ln 3} + e^{-3\ln 3}\right) = 27\left(\frac{1}{3} - \frac{2}{9} + \frac{1}{27}\right) \quad \underline{= 4\ lbs}$$

Exercise

Find the amount $x_1(t),\ x_2(t),\ x_3(t)$ of salt in each tank at time $t \ge 0$, if

$V_1 = 20\ gal,\quad V_2 = 30\ gal,\quad V_3 = 60\ gal,\quad r = 60\ gal/\min \quad x_1(0) = 45\ lb \quad x_2(0) = x_3(0) = 0$

Solution



$$\begin{cases} x_1' = -k_1x_1 \\ x_2' = k_1x_1 - k_2x_2 \\ x_3' = \qquad k_2x_2 - k_3x_3 \end{cases} \qquad \text{where } k_i = \frac{r}{V_i} \quad i = 1,2,3$$

$$k_1 = \frac{60}{20} = 3 \quad k_2 = \frac{60}{30} = 2 \quad k_3 = \frac{60}{60} = 1$$

$$\to \begin{cases} x_1' = -3x_1 \\ x_2' = 3x_1 - 2x_2 \\ x_3' = \qquad 2x_2 - x_3 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -3 & 0 & 0 \\ 3 & -2 & 0 \\ 0 & 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{with} \quad x(0) = \begin{pmatrix} 45 \\ 0 \\ 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -3-\lambda & 0 & 0 \\ 3 & -2-\lambda & 0 \\ 0 & 2 & -1-\lambda \end{vmatrix} = (-3-\lambda)(-2-\lambda)(-1-\lambda) = 0$$

The eigenvalues are: $\underline{\lambda_1 = -3 \quad \lambda_2 = -2 \quad \lambda_3 = -1}|$

For $\lambda_1 = -3 \ \Rightarrow\ (A + 3I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 3a_1 = -b_1 & \to a_1 = 1 \\ 2c_1 = -2b_1 & \to b_1 = -3 \\ & \quad c_1 = 3 \end{cases} \to V_1 = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} \Rightarrow x_1(t) = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} e^{-3t}$$

For $\lambda_2 = -2 \ \Rightarrow\ (A + 2I)V_2 = 0$

$$\begin{pmatrix} -1 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a_2 = 0 \\ 2b_2 = -c_2 \end{cases} \to V_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \Rightarrow x_2(t) = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} e^{-2t}$$

For $\lambda_3 = -1 \ \Rightarrow\ (A + I)V_3 = 0$

$$\begin{pmatrix} -2 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a_3 = b_3 = 0 \ \to V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow x_3(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t}$$

$$\Rightarrow\ x(t)=C_1\begin{pmatrix}1\\-3\\3\end{pmatrix}e^{-3t}+C_2\begin{pmatrix}0\\1\\-2\end{pmatrix}e^{-2t}+C_3\begin{pmatrix}0\\0\\1\end{pmatrix}e^{-t}$$

$$\begin{cases}x_1(t)=C_1e^{-3t}\\ x_2(t)=-3C_1e^{-3t}+C_2e^{-2t}\\ x_3(t)=3C_1e^{-3t}-2C_2e^{-2t}+C_3e^{-t}\end{cases}$$

With *initial* values

$$\begin{cases}45=C_1\\ 0=-3C_1+C_2\\ 0=3C_1-2C_2+C_3\end{cases}\ \rightarrow\ \begin{cases}\underline{C_1=45}\\ \underline{C_2=135}\\ \underline{C_3}=-3(45)+2(-135)\underline{=135}\end{cases}$$

$$\begin{cases}x_1(t)=45e^{-3t}\\ x_2(t)=-135e^{-3t}+135e^{-2t}\\ x_3(t)=135e^{-3t}-270e^{-2t}+135e^{-t}\end{cases}$$

Tank 2: $x_2'(t)=3e^{-3t}-2e^{-2t}=0$

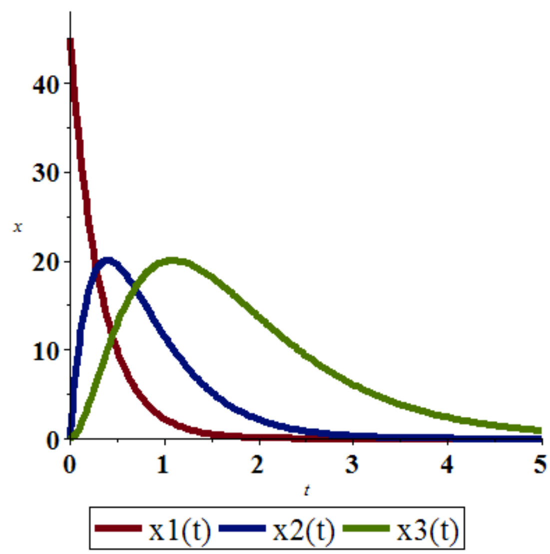
$1.5e^{-3t}=e^{-2t}\quad\Rightarrow\quad \ln 1.5-3t=-2t$

$\underline{t=\ln 1.5}$

The maximum values of salt in tank 2 is:

$x_2(\ln 1.5)=135\left(-e^{-3\ln 1.5}+e^{-2\ln 1.5}\right)=135\left(-\frac{8}{27}+\frac{4}{9}\right)$

$\underline{=20\ lbs}$



Tank 3: $x_3'(t)=135\left(-3e^{-3t}+4e^{-2t}-e^{-t}\right)=0$

$e^{3t}\left(-3e^{-3t}+4e^{-2t}-e^{-t}\right)=0$

$-3+4e^{t}-e^{2t}=0\quad \begin{cases}e^{t}=1\rightarrow t=0\\ e^{t}=3\rightarrow \underline{t=\ln 3}\end{cases}$

The maximum values of salt in tank 3 is:

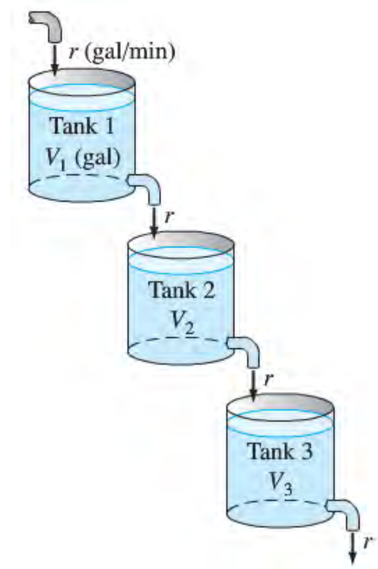
$x_2(\ln 3)=135\left(e^{-3\ln 3}-2e^{-2\ln 3}+e^{-\ln 3}\right)=135\left(\frac{1}{27}-\frac{2}{9}+\frac{1}{3}\right)$

$\underline{=20\ lbs}$

Exercise

Find the amount $x_1(t),\ x_2(t),\ x_3(t)$ of salt in each tank at time $t \ge 0$, if

$V_1 = 15\ gal,\quad V_2 = 10\ gal,\quad V_3 = 30\ gal,\quad r = 60\ gal/\min\quad x_1(0) = 45\ lb\quad x_2(0) = x_3(0) = 0$



Solution

$$\begin{cases} x_1' = -k_1x_1 \\ x_2' = k_1x_1 - k_2x_2 \\ x_3' = \qquad\quad k_2x_2 - k_3x_3 \end{cases} \qquad \text{where } k_i = \frac{r}{V_i} \quad i = 1,2,3$$

$$k_1 = \frac{60}{15} = 4 \quad k_2 = \frac{60}{10} = 6 \quad k_3 = \frac{60}{30} = 2$$

$$\to \begin{cases} x_1' = -4x_1 \\ x_2' = 4x_1 - 6x_2 \\ x_3' = \qquad\quad 6x_2 - 2x_3 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -4 & 0 & 0 \\ 4 & -6 & 0 \\ 0 & 6 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \textit{with} \quad x(0) = \begin{pmatrix} 45 \\ 0 \\ 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -4-\lambda & 0 & 0 \\ 4 & -6-\lambda & 0 \\ 0 & 6 & -2-\lambda \end{vmatrix} = (-4-\lambda)(-6-\lambda)(-2-\lambda) = 0$$

The eigenvalues are: $\underline{\lambda_1 = -4 \quad \lambda_2 = -6 \quad \lambda_3 = -2}$

For $\lambda_1 = -4 \quad \Rightarrow \quad (A + 4I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 4 & -2 & 0 \\ 0 & 6 & 2 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 4a_1 = 2b_1 \ \to a_1 = 1 \\ 2c_1 = -6b_1 \ \to b_1 = 2 \\ \qquad\qquad\qquad c_1 = -6 \end{cases} \to V_1 = \begin{pmatrix} 1 \\ 2 \\ -6 \end{pmatrix} \Rightarrow x_1(t) = \begin{pmatrix} 1 \\ 2 \\ -6 \end{pmatrix} e^{-4t}$$

For $\lambda_2 = -6 \quad \Rightarrow \quad (A + 6I)V_2 = 0$

$$\begin{pmatrix} 2 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 6 & 4 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a_2 = 0 \\ 6b_2 = -4c_2 \end{cases} \to V_2 = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} \Rightarrow x_2(t) = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} e^{-6t}$$

For $\lambda_3 = -2 \quad \Rightarrow \quad (A + 2I)V_3 = 0$

$$\begin{pmatrix} -2 & 0 & 0 \\ 4 & -4 & 0 \\ 0 & 6 & 0 \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a_3 = b_3 = 0 \ \to V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow x_3(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-2t}$$

$$\Rightarrow\ x(t)=C_1\begin{pmatrix}1\\2\\-6\end{pmatrix}e^{-4t}+C_2\begin{pmatrix}0\\2\\-3\end{pmatrix}e^{-6t}+C_3\begin{pmatrix}0\\0\\1\end{pmatrix}e^{-2t}$$

$$\begin{cases}x_1(t)=C_1e^{-4t}\\ x_2(t)=2C_1e^{-4t}+2C_2e^{-6t}\\ x_3(t)=-6C_1e^{-4t}-3C_2e^{-6t}+C_3e^{-2t}\end{cases}$$

With *initial* values

$$\begin{cases}45=C_1\\ 0=2C_1+2C_2\\ 0=-6C_1-3C_2+C_3\end{cases}\quad\rightarrow\begin{cases}\boxed{C_1=45}\\ \boxed{C_2=-45}\\ \boxed{C_3=6(45)+3(-45)=135}\end{cases}$$

$$\begin{cases}x_1(t)=45e^{-4t}\\ x_2(t)=90e^{-4t}-90e^{-6t}\\ x_3(t)=-270e^{-4t}+135e^{-6t}+135e^{-2t}\end{cases}$$

Tank 2: $x_2'(t)=-360e^{-4t}+540e^{-6t}=0$

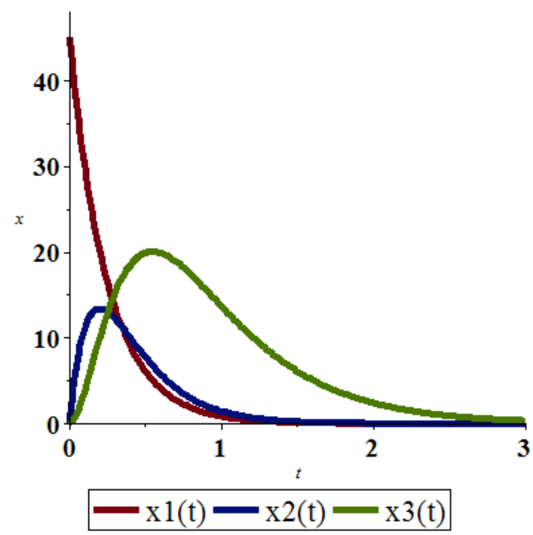
$$2e^{-4t}=3e^{-6t}\quad\Rightarrow\quad \ln(2)-4t=\ln(3)-6t$$

$$\boxed{t=\frac{1}{2}\ln 1.5}$$

The maximum values of salt in tank 2 is:

$$x_2\left(\frac{1}{2}\ln 1.5\right)=90\left(e^{-2\ln 1.5}-e^{-3\ln 1.5}\right)=90\left(\frac{4}{9}-\frac{8}{27}\right)$$

$$\boxed{=13.3\ lbs}$$



Tank 3: $x_3'(t)=135\left(8e^{-4t}-6e^{-6t}-2e^{-2t}\right)=0$

$$-2e^{-6t}\left(4e^{2t}-3-e^{4t}\right)=0$$

$$e^{4t}-4e^{2t}+3=0\qquad\begin{cases}e^{2t}=1\rightarrow t=0\\ e^{2t}=3\rightarrow \boxed{t=\frac{1}{2}\ln 3}\end{cases}$$

The maximum values of salt in tank 3 is:

$$x_2\left(\frac{1}{2}\ln 3\right)=135\left(-2e^{-2\ln 3}+e^{-3\ln 3}+e^{-\ln 3}\right)$$

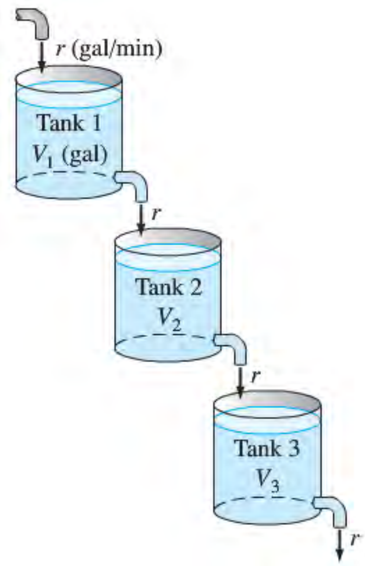
$$=135\left(-\frac{2}{9}+\frac{1}{27}+\frac{1}{3}\right)$$

$$\boxed{=20\ lbs}$$

Exercise

If $V_1 = 20\ gal$, $V_2 = 40\ gal$, $V_3 = 50\ gal$, $r = 10\ gal/\min$ and the initial amounts of salt in 3 brine tanks, in lbs, are $x_1(0) = 15$ $x_2(0) = x_3(0) = 0$. Find the amount of salt in each tank at time $t \ge 0$.

<u>***Solution***</u>



$$\begin{cases} x_1' = -k_1 x_1 \\ x_2' = k_1 x_1 - k_2 x_2 \\ x_3' = \quad k_2 x_2 - k_3 x_3 \end{cases} \qquad \text{where } k_i = \frac{r}{v_i} \quad i = 1,2,3$$

$$k_1 = \frac{10}{20} = .5 \quad k_2 = \frac{10}{40} = .25 \quad k_3 = \frac{10}{50} = .2$$

$$\begin{cases} x_1' = -.5x_1 \\ x_2' = .5x_1 - .25x_2 \\ x_3' = \quad .25x_2 - .2x_3 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -.5 & 0 & 0 \\ .5 & -.25 & 0 \\ 0 & .25 & -.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad with\ x(0) = \begin{pmatrix} 15 \\ 0 \\ 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -.5-\lambda & 0 & 0 \\ .5 & -.25-\lambda & 0 \\ 0 & .25 & -.2-\lambda \end{vmatrix}$$

$$= (-.5-\lambda)(-.25-\lambda)(-.2-\lambda) = 0$$

The eigenvalues are: $\lambda_1 = -.5 \quad \lambda_2 = -.25 \quad \lambda_3 = -.2$

For $\lambda_1 = -.5 \Rightarrow (A + .5I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ .5 & .25 & 0 \\ 0 & .25 & .3 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} .5a_1 + .25b_1 = 0 \rightarrow 2a_1 = -b_1 \\ .25b_1 + .3c_1 = 0 \rightarrow 6c_1 = -5b_1 \end{cases}$$

$$\rightarrow V_1 = \begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix} \Rightarrow x_1(t) = \begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix} e^{-.5t}$$

For $\lambda_2 = -.25 \Rightarrow (A + .25I)V_2 = 0$

$$\begin{pmatrix} -.25 & 0 & 0 \\ .5 & 0 & 0 \\ 0 & .25 & .05 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a_2 = 0 \\ .25b_2 + .05c_2 = 0 \rightarrow c_2 = -5b_2 \end{cases}$$

$$\rightarrow V_2 = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} \quad \Rightarrow \quad x_2(t) = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} e^{-.25t}$$

For $\lambda_3 = -.2 \quad \Rightarrow \quad (A + .2I)V_3 = 0$

$$\begin{pmatrix} -.3 & 0 & 0 \\ .5 & -.05 & 0 \\ 0 & .25 & 0 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} a_3 = 0 \\ b_3 = 0 \\ 0c_3 = 0 \rightarrow c_3 = 1 \end{cases}$$

$$\rightarrow V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad x_3(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-.2t}$$

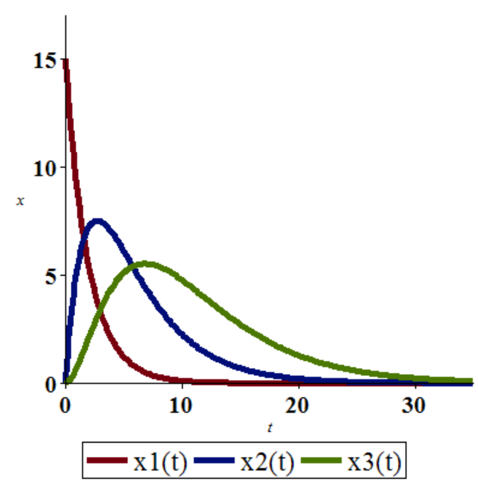
$$\Rightarrow x(t) = C_1 \begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix} e^{-.5t} + C_2 \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} e^{-.25t} + C_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-.2t}$$

$$\begin{cases} x_1(t) = 3C_1 e^{-.5t} \\ x_2(t) = -6C_1 e^{-.5t} + C_2 e^{-.25t} \\ x_3(t) = 5C_1 e^{-.5t} - 5C_2 e^{-.25t} + C_3 e^{-.2t} \end{cases}$$

With *initial* values

$$\begin{cases} 15 = 3C_1 \\ 0 = -6C_1 + C_2 \\ 0 = 5C_1 - 5C_2 + C_3 \end{cases} \rightarrow \begin{cases} \boxed{5 = C_1} \\ \boxed{C_2 = 30} \\ C_3 = -5(5) + 5(30) = \boxed{125} \end{cases}$$

$$\begin{cases} x_1(t) = 15e^{-.5t} \\ x_2(t) = -30e^{-.5t} + 30e^{-.25t} \\ x_3(t) = 25e^{-.5t} - 150e^{-.25t} + 125e^{-.2t} \end{cases}$$



Exercise

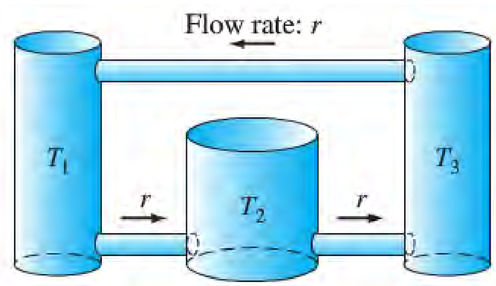
If $V_1 = 50\ gal$, $V_2 = 25\ gal$, $V_3 = 50\ gal$, $r = 10\ gal/\min$, find the amount $x_1(t), x_2(t), x_3(t)$ of salt in each tank at time $t \ge 0$

Solution

$$\begin{cases} x_1' = -k_1 x_1 \qquad + k_3 x_3 \\ x_2' = k_1 x_1 - k_2 x_2 \\ x_3' = \qquad k_2 x_2 - k_3 x_3 \end{cases} \qquad \text{where } k_i = \frac{r}{v_i} \quad i = 1,2,3$$

$$k_1 = \frac{10}{50} = .2 \quad k_1 = \frac{10}{25} = .4 \quad k_1 = \frac{10}{50} = .2$$

$$\begin{cases} x_1' = -.2x_1 \qquad + .2x_3 \\ x_2' = .2x_1 - .4x_2 \\ x_3' = \qquad .4x_2 - .2x_3 \end{cases}$$



$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -.2 & 0 & .2 \\ .2 & -.4 & 0 \\ 0 & .4 & -.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -.2-\lambda & 0 & .2 \\ .2 & -.4-\lambda & 0 \\ 0 & .4 & -.2-\lambda \end{vmatrix}$$

$$= (-.2-\lambda)(-.4-\lambda)(-.2-\lambda) + (.2)(.2)(.4)$$

$$= -\lambda^3 - .8\lambda^2 - .2\lambda$$

$$= -\lambda\left(\lambda^2 + .8\lambda + .2\right) = 0 \qquad \lambda^2 + .8\lambda + .2 = 0 \quad \lambda = \frac{-.8 \pm \sqrt{.64 - .8}}{2} = -.4 \pm .2i$$

The eigenvalues are: $\underline{\lambda_1 = 0 \quad \lambda_{2,3} = -.4 \pm .2i}|$

For $\lambda_1 = 0 \quad \Rightarrow \quad (A - 0I)V_1 = 0$

$$\begin{pmatrix} -.2 & 0 & .2 \\ .2 & -.4 & 0 \\ 0 & .4 & -.2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -.2a + .2c = 0 \ \rightarrow \ a = c \\ .2a - .4b = 0 \ \rightarrow \ a = 2b \end{cases}$$

$$\rightarrow V_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad \Rightarrow \quad x_1(t) = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

For $\lambda = -.4 - .2i \quad \Rightarrow \quad \left(A + (.4 + .2i)\right)V_2 = 0$

$$\begin{pmatrix} .2+.2i & 0 & .2 \\ .2 & .2i & 0 \\ 0 & .4 & .2+.2i \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} (.2+.2i)a = -.2c \\ .2a = -.2ib \end{cases}$$

$$\text{Let } b = i \quad \Rightarrow a = 1 \quad c = -1 - i$$

$$\to V_2 = \begin{pmatrix} 1 \\ i \\ -1-i \end{pmatrix} \quad \Rightarrow \quad x_{2,3}(t) = \begin{pmatrix} 1 \\ i \\ -1-i \end{pmatrix} e^{-.4t} e^{-.2ti}$$

$$x_{2,3}(t) = \begin{pmatrix} 1 \\ i \\ -1-i \end{pmatrix} e^{-.4t}\left(\cos(.2t) - i\sin(.2t)\right)$$

$$= \begin{pmatrix} \cos .2t - i\sin .2t \\ \sin .2t + i\cos .2t \\ -\cos .2t - \sin .2t - i(\cos .2t - \sin .2t) \end{pmatrix} e^{-.4t}$$

$$x_1(t) = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad x_2(t) = \begin{pmatrix} \cos .2t \\ \sin .2t \\ -\cos .2t - \sin .2t \end{pmatrix} e^{-.4t} \quad x_3(t) = \begin{pmatrix} -\sin .2t \\ \cos .2t \\ \sin .2t - \cos .2t \end{pmatrix} e^{-.4t}$$

$$\begin{cases} x_1(t) = 2C_1 + \left(C_2 \cos 0.2t - C_3 \sin 0.2t\right) e^{-.4t} \\ x_2(t) = C_1 + \left(C_2 \sin 0.2t + C_3 \cos 0.2t\right) e^{-.4t} \\ x_3(t) = 2C_1 + \left(\left(-C_2 - C_3\right)\cos 0.2t + \left(C_3 - C_2\right)\sin 0.2t\right) e^{-.4t} \end{cases}$$

Exercise

Two large tanks, each holding 100 *L* of liquid, are interconnected by pipes, with the liquid following from tank *A* into tank *B* at a rate of 3 *L/min* and from *B* to into *A* at a rate of 1 *L/min*.

The liquid inside each tank is kept well stirred. A brine solution with a concentration of 0.2 *kg/L* of salt flows into tank *A* at a rate of 6 *L/min*. The diluted solution flows out the system from tank *A* at 4 *L/min* and from tank *B* at 2 *L/min*. If, initially, tank *A* contains pure water and tank *B* contains 20 *kg* of salt, determine the mass of salt in each tank at time $t \geq 0$.

Solution

For Tank ***A***:

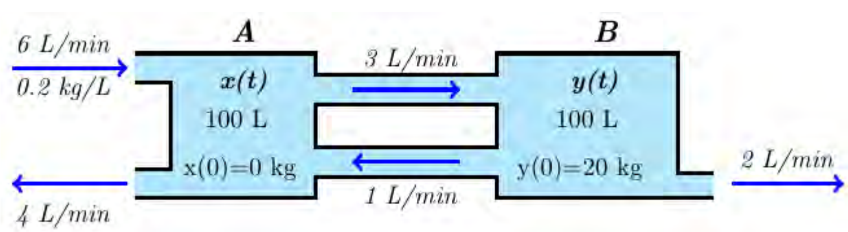
$$\frac{dx}{dt} = 0.2\frac{kg}{L}\left(6\frac{L}{min}\right) + \frac{1}{100}\frac{L/min}{L}\,y(kg) - \frac{3}{100}x - \frac{4}{100}x$$

$$= -\frac{7}{100}x + \frac{1}{100}y + \frac{6}{5}$$

For Tank ***B***:

$$\frac{dy}{dt} = \frac{3}{100}x - \frac{1}{100}y - \frac{2}{100}y$$

$$= \frac{3}{100}x - \frac{3}{100}y$$



$$\begin{cases} x' = -\frac{7}{100}x + \frac{1}{100}y + \frac{6}{5} \\ y' = \frac{3}{100}x - \frac{3}{100}y \end{cases}$$

$$\left|A - \lambda I\right| = \begin{vmatrix} -\frac{7}{100} - \lambda & \frac{1}{100} \\ \frac{3}{100} & -\frac{3}{100} - \lambda \end{vmatrix} \qquad A = \begin{pmatrix} -\frac{7}{100} & \frac{1}{100} \\ \frac{3}{100} & -\frac{3}{100} \end{pmatrix}$$

$$= \frac{21}{10^4} + \frac{1}{10}\lambda + \lambda^2 - \frac{3}{10^4}$$

$$= \lambda^2 + \frac{1}{10}\lambda + \frac{18}{10^4} = 0 \qquad 5\times 10^3\lambda^2 + 500\lambda + 9 = 0$$

The eigenvalues are: $\boxed{\lambda_{1,2} = \frac{-5 \pm \sqrt{7}}{100}}$

For $\lambda_1 = -\frac{5}{100} - \frac{\sqrt{7}}{100} \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

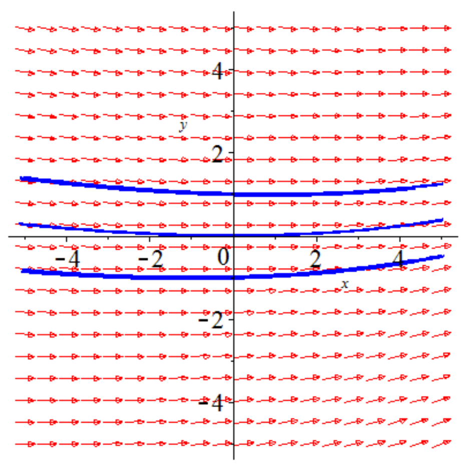
$$\begin{pmatrix} -\frac{2}{100} + \frac{\sqrt{7}}{100} & \frac{1}{100} \\ \frac{3}{100} & \frac{2}{100} + \frac{\sqrt{7}}{100} \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 3a_1 = -\left(2 + \sqrt{7}\right)b_1 \rightarrow V_1 = \begin{pmatrix} 2 + \sqrt{7} \\ -3 \end{pmatrix}$$

For $\lambda_2 = -\frac{5}{100} + \frac{\sqrt{7}}{100} \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -\frac{2}{100} - \frac{\sqrt{7}}{100} & \frac{1}{100} \\ \frac{3}{100} & \frac{2}{100} - \frac{\sqrt{7}}{100} \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 3a_2 = -\left(2 - \sqrt{7}\right)b_2 \rightarrow V_2 = \begin{pmatrix} 2 - \sqrt{7} \\ -3 \end{pmatrix}$$

The homogeneous solution: $= C_1\begin{pmatrix} 2 + \sqrt{7} \\ -3 \end{pmatrix}e^{\frac{-5-\sqrt{7}}{100}t} + C_2\begin{pmatrix} 2 - \sqrt{7} \\ -3 \end{pmatrix}e^{\frac{-5+\sqrt{7}}{100}t}$

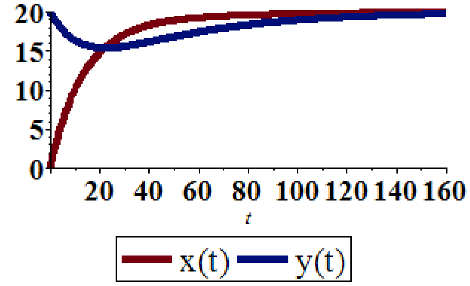
$$\begin{cases} x_h(t) = C_1\left(2 + \sqrt{7}\right)e^{\frac{-5-\sqrt{7}}{100}t} + C_2\left(2 - \sqrt{7}\right)e^{\frac{-5+\sqrt{7}}{100}t} \\ y_h(t) = -3C_1e^{\frac{-5-\sqrt{7}}{100}t} - 3C_2e^{\frac{-5+\sqrt{7}}{100}t} \end{cases}$$



$$\begin{cases} -\frac{7}{100}a_1 + \frac{1}{100}a_2 = -\frac{6}{5} \\ \frac{3}{100}a_1 - \frac{3}{100}a_2 = 0 \end{cases} \rightarrow \begin{cases} -7a_1 + a_2 = -120 \\ a_1 - a_2 = 0 \end{cases}$$

$$\boxed{a_1 = 20, \quad a_2 = 20} \rightarrow \begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} 20 \\ 20 \end{pmatrix}$$

$$\begin{cases} x(t)=C_1\left(2+\sqrt{7}\right)e^{\frac{-5-\sqrt{7}}{100}t}+C_2\left(2-\sqrt{7}\right)e^{\frac{-5+\sqrt{7}}{100}t}+20 \\ y(t)=-3C_1e^{\frac{-5-\sqrt{7}}{100}t}-3C_2e^{\frac{-5+\sqrt{7}}{100}t}+20 \end{cases}$$



$$\begin{cases} x(0)=C_1\left(2+\sqrt{7}\right)+C_2\left(2-\sqrt{7}\right)+20=0 \\ y(0)=-3C_1-3C_2+20=20 \end{cases}$$

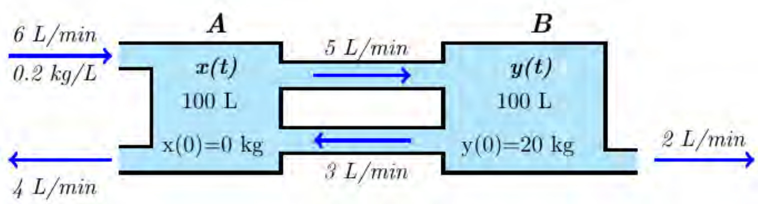
$$\begin{cases} \left(2+\sqrt{7}\right)C_1+\left(2-\sqrt{7}\right)C_2=-20 \\ C_1+C_2=0 \end{cases} \quad \rightarrow C_1=-C_2$$

$$\left(2+\sqrt{7}-2+\sqrt{7}\right)C_1=-20 \quad \rightarrow \quad \boxed{C_1=-\frac{10}{\sqrt{7}}} \quad \boxed{C_2=\frac{10}{\sqrt{7}}}$$

$$\begin{cases} x(t)=-\frac{10}{\sqrt{7}}\left(2+\sqrt{7}\right)e^{\frac{-5-\sqrt{7}}{100}t}+\frac{10}{\sqrt{7}}\left(2-\sqrt{7}\right)e^{\frac{-5+\sqrt{7}}{100}t}+20 \\ y(t)=\frac{30}{\sqrt{7}}e^{\frac{-5-\sqrt{7}}{100}t}-\frac{30}{\sqrt{7}}e^{\frac{-5+\sqrt{7}}{100}t}+20 \end{cases}$$

Exercise

Two large tanks, each holding 100 *L* of liquid, are interconnected by pipes, with the liquid following from tank *A* into tank *B* at a rate of 5 *L/min* and from *B* to into *A* at a rate of 3 *L/min*.



The liquid inside each tank is kept well stirred. A brine solution with a concentration of 0.2 *kg/L* of salt flows into tank *A* at a rate of 6 *L/min*. The diluted solution flows out the system from tank *A* at 4 *L/min* and from tank *B* at 2 *L/min*. If, initially, tank *A* contains pure water and tank *B* contains 20 *kg* of salt, determine the mass of salt in each tank at time $t \ge 0$.

Solution

Tank ***A***:

$$\frac{dx}{dt}=0.2\frac{kg}{L}\left(6\frac{L}{min}\right)+\frac{3\ L/min}{100\ L}y(kg)-\frac{5}{100}x-\frac{4}{100}x$$

$$=-\frac{9}{100}x+\frac{3}{100}y+\frac{6}{5}$$

Tank ***B***:

$$\frac{dy}{dt}=\frac{5}{100}x-\frac{3}{100}y-\frac{2}{100}y$$

$$= \frac{1}{20}x - \frac{1}{20}y$$

$$\begin{cases} x' = -\frac{9}{100}x + \frac{3}{100}y + \frac{6}{5} \\ y' = \frac{1}{20}x - \frac{1}{20}y \end{cases}$$

$$|A - \lambda I| = \begin{vmatrix} -\frac{9}{100} - \lambda & \frac{3}{100} \\ \frac{1}{20} & -\frac{1}{20} - \lambda \end{vmatrix} \qquad A = \begin{pmatrix} -\frac{9}{100} & \frac{3}{100} \\ \frac{1}{20} & -\frac{1}{20} \end{pmatrix}$$

$$= \frac{9}{2{,}000} + \frac{14}{100}\lambda + \lambda^2 - \frac{3}{2{,}000}$$

$$= \frac{14}{100}\lambda + \lambda^2 + \frac{3}{1{,}000} = 0 \qquad 10^3\lambda^2 + 140\lambda + 3 = 0$$

The eigenvalues are: $\lambda_{1,2} = \frac{-140 \pm 20\sqrt{19}}{2000} = \frac{-7 \pm \sqrt{19}}{100}$

For $\lambda_1 = \frac{-7}{100} - \frac{\sqrt{19}}{100} \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -\frac{2}{100} + \frac{\sqrt{19}}{100} & \frac{3}{100} \\ \frac{1}{20} & \frac{1}{50} + \frac{\sqrt{19}}{100} \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left(-2 + \sqrt{19}\right)a_1 = -3b_1 \quad \to \quad V_1 = \begin{pmatrix} 3 \\ 2 - \sqrt{19} \end{pmatrix}$$

For $\lambda_2 = -\frac{7}{100} + \frac{\sqrt{19}}{100} \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -\frac{2}{100} - \frac{\sqrt{19}}{100} & \frac{3}{100} \\ \frac{1}{20} & \frac{1}{50} - \frac{\sqrt{19}}{100} \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left(-2 - \sqrt{19}\right)a_2 = -3b_2 \quad \to \quad V_2 = \begin{pmatrix} 3 \\ 2 + \sqrt{19} \end{pmatrix}$$

The homogeneous solution: $X(t) = C_1\begin{pmatrix} 3 \\ 2 - \sqrt{19} \end{pmatrix}e^{\frac{-7-\sqrt{19}}{100}t} + C_2\begin{pmatrix} 3 \\ 2 + \sqrt{19} \end{pmatrix}e^{\frac{-7+\sqrt{19}}{100}t}$

$$\begin{cases} x_h(t) = 3C_1 e^{\frac{-7-\sqrt{19}}{100}t} + 3C_2 e^{\frac{-7+\sqrt{19}}{100}t} \\ y_h(t) = \left(2 - \sqrt{19}\right)C_1 e^{\frac{-7-\sqrt{19}}{100}t} + C_2\left(2 + \sqrt{19}\right)e^{\frac{-7+\sqrt{19}}{100}t} \end{cases}$$

$$\begin{cases} -\frac{9}{100}a_1 + \frac{3}{100}a_2 = -\frac{6}{5} \\ \frac{1}{20}a_1 - \frac{1}{20}a_2 = 0 \end{cases} \quad \to \quad \begin{cases} -3a_1 + a_2 = -40 \\ a_1 - a_2 = 0 \end{cases}$$

$$\underline{a_1 = \frac{40}{2} = 20, \quad a_2 = 20} \quad \to \quad \begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} 20 \\ 20 \end{pmatrix}$$

$$\begin{cases} x(t) = 3C_1 e^{\frac{-7-\sqrt{19}}{100}t} + 3C_2 e^{\frac{-7+\sqrt{19}}{100}t} + 20 \\ y(t) = \left(2-\sqrt{19}\right)C_1 e^{\frac{-7-\sqrt{19}}{100}t} + C_2\left(2+\sqrt{19}\right)e^{\frac{-7+\sqrt{19}}{100}t} + 20 \end{cases}$$

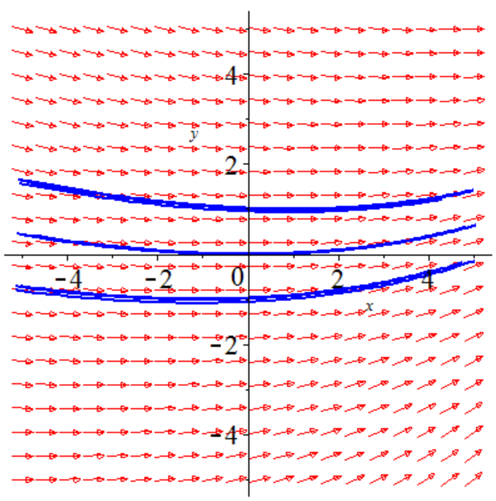
$$\begin{cases} x(0) = 3C_1 + 3C_2 + 20 = 0 \\ y(0) = \left(2-\sqrt{19}\right)C_1 + \left(2+\sqrt{19}\right)C_2 + 20 = 20 \end{cases}$$

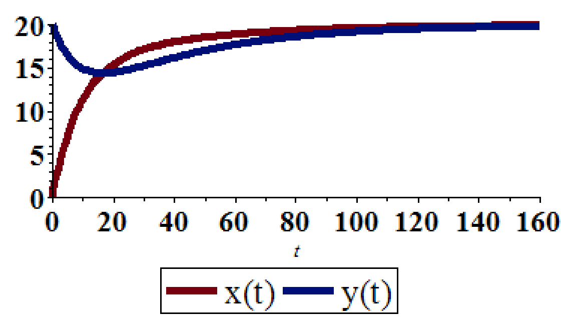
$$\begin{cases} 3C_1 + 3C_2 = -20 \\ \left(2-\sqrt{19}\right)C_1 + \left(2+\sqrt{19}\right)C_2 = 0 \end{cases} \qquad C_1 = -\frac{20\left(2+\sqrt{19}\right)}{6\sqrt{19}} = \boxed{-\frac{10\left(2+\sqrt{19}\right)}{3\sqrt{19}}}$$

$$C_2 = \frac{20\left(2-\sqrt{19}\right)}{6\sqrt{19}} = \boxed{\frac{20-10\sqrt{19}}{3\sqrt{19}}}$$

$$\begin{cases} x(t) = -\frac{20+10\sqrt{19}}{\sqrt{19}} e^{\frac{-7-\sqrt{19}}{100}t} + \frac{-20+10\sqrt{19}}{\sqrt{19}} e^{\frac{-7+\sqrt{19}}{100}t} + 20 \\ y(t) = -10\left(2-\sqrt{19}\right)\frac{2+\sqrt{19}}{3\sqrt{19}} e^{\frac{-7-\sqrt{19}}{100}t} + 10\left(2+\sqrt{19}\right)\frac{2-\sqrt{19}}{3\sqrt{19}} e^{\frac{-7+\sqrt{19}}{100}t} + 20 \end{cases}$$

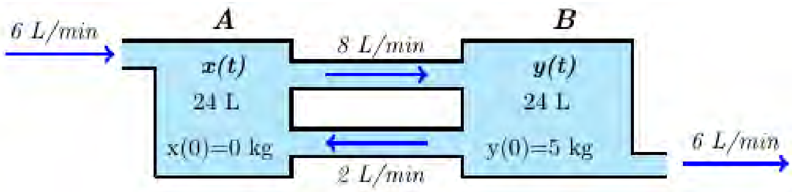
$$\begin{cases} x(t) = -\frac{20+10\sqrt{19}}{\sqrt{19}} e^{\frac{-7-\sqrt{19}}{100}t} + \frac{-20+10\sqrt{19}}{\sqrt{19}} e^{\frac{-7+\sqrt{19}}{100}t} + 20 \\ y(t) = \frac{50}{\sqrt{19}} e^{\frac{-7-\sqrt{19}}{100}t} - \frac{50}{\sqrt{19}} e^{\frac{-7+\sqrt{19}}{100}t} + 20 \end{cases}$$





Exercise

Two large tanks, each holding 24 *L* of liquid, are interconnected by pipes, with the liquid following from tank *A* into tank *B* at a rate of 8 *L/min* and from *B* to into *A* at a rate of 2 *L/min*.



The liquid inside each tank is kept well stirred. A brine solution flows into tank *A* at a rate of 6 *L/min*. The diluted solution flows out the system from tank *B* at 6 *L/min*. If, initially, tank *A* contains pure water and tank *B* contains 5 *kg* of salt, determine the mass of salt in each tank at time $t \ge 0$.

Solution

Tank ***A***: $\quad \dfrac{dx}{dt} = -\dfrac{8}{24}x + \dfrac{2}{24}y$

Tank ***B***: $\quad \dfrac{dy}{dt} = \dfrac{8}{24}x - \dfrac{2}{24}y - \dfrac{6}{24}y$

$$\begin{cases} x' = -\frac{1}{3}x + \frac{1}{12}y \\ y' = \frac{1}{3}x - \frac{1}{3}y \end{cases}$$

$$|A - \lambda I| = \begin{vmatrix} -\frac{1}{3} - \lambda & \frac{1}{12} \\ \frac{1}{3} & -\frac{1}{3} - \lambda \end{vmatrix} \qquad\qquad A = \begin{pmatrix} -\frac{1}{3} & \frac{1}{12} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

$$= \lambda^2 + \frac{2}{3}\lambda + \frac{1}{9} - \frac{1}{36}$$

$$= \lambda^2 + \frac{2}{3}\lambda + \frac{1}{12} = 0 \quad \to \quad 12\lambda^2 + 8\lambda + 1 = 0$$

The eigenvalues are: $\lambda_{1,2} = \dfrac{-8 \pm 4}{24} \quad \to \quad \underline{\lambda_{1,2} = -\frac{1}{2},\ -\frac{1}{6}}$

For $\lambda_1 = -\frac{1}{2} \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} \frac{1}{6} & \frac{1}{12} \\ \frac{1}{3} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a_1 = -\frac{1}{2}b_1 \quad \to \quad V_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = -\frac{1}{6} \quad \Rightarrow \quad (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -\frac{1}{6} & \frac{1}{12} \\ \frac{1}{3} & -\frac{1}{6} \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a_2 = \frac{1}{2}b_2 \quad \to \quad V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$X(t) = C_1\begin{pmatrix} -1 \\ 2 \end{pmatrix}e^{-t/2} + C_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-t/6}$$

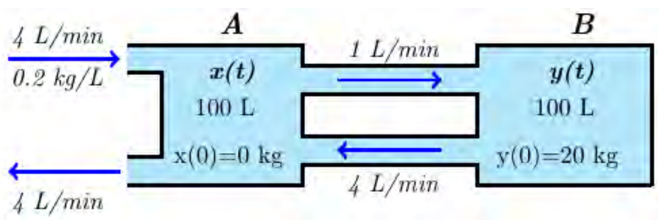
$$X(0) = C_1\begin{pmatrix}-1\\2\end{pmatrix} + C_2\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}0\\5\end{pmatrix}$$

$$\begin{cases}-C_1 + C_2 = 0\\ 2C_1 + 2C_2 = 5\end{cases} \qquad \underline{C_1 = \frac{5}{4}, \quad C_2 = \frac{5}{4}}$$

$$\begin{cases}x(t) = -\frac{5}{4}e^{-t/2} + \frac{5}{4}e^{-t/6}\\ y(t) = \frac{5}{2}e^{-t/2} + \frac{5}{2}e^{-t/6}\end{cases}$$

Exercise

Two large tanks, each holding 100 L of liquid, are interconnected by pipes, with the liquid following from tank A into tank B at a rate of 1 L/min and from B to into A at a rate of 4 L/min.



The liquid inside each tank is kept well stirred. A brine solution flows into tank A at a rate of 4 L/min. The diluted solution flows out the system from tank A at 4 L/min. If, initially, tank A contains pure water and tank B contains 20 kg of salt, determine the mass of salt in each tank at time $t \ge 0$.

Solution

Tank ***A***: $\quad \frac{dx}{dt} = 0.2\frac{kg}{L}\left(4\frac{L}{min}\right) + \frac{4\ L/min}{100\ L}y(kg) - \frac{1}{100}x - \frac{4}{100}x$

Tank ***B***: $\quad \frac{dy}{dt} = \frac{1}{100}x - \frac{4}{100}y$

$$\begin{cases}x' = -\frac{1}{20}x + \frac{1}{25}y + \frac{4}{5}\\ y' = \frac{1}{100}x - \frac{1}{25}y\end{cases}$$

$$|A - \lambda I| = \begin{vmatrix}-\frac{1}{20} - \lambda & \frac{1}{25}\\ \frac{1}{100} & -\frac{1}{25} - \lambda\end{vmatrix} \qquad A = \begin{pmatrix}-\frac{1}{20} & \frac{1}{25}\\ \frac{1}{100} & -\frac{1}{25}\end{pmatrix}$$

$$= \lambda^2 + \frac{9}{100}\lambda + \frac{1}{625} = 0$$

$$= \lambda^2 + \frac{9}{100}\lambda + \frac{1}{625} = 0 \quad \rightarrow \quad 2500\lambda^2 + 225\lambda + 4 = 0$$

The eigenvalues are: $\lambda_{1,2} = \frac{-225 \pm 25\sqrt{17}}{5,000} = \frac{-9 \pm \sqrt{17}}{200}$

For $\lambda_1 = -\frac{9}{200} - \frac{\sqrt{17}}{200} \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} \frac{-1+\sqrt{17}}{200} & \frac{1}{25} \\ \frac{1}{100} & \frac{1+\sqrt{17}}{200} \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2a_1 = -\left(1+\sqrt{17}\right)b_1 \quad \to \quad V_1 = \begin{pmatrix} 1+\sqrt{17} \\ -2 \end{pmatrix}$$

For $\lambda_2 = -\frac{9}{200} + \frac{\sqrt{17}}{200} \quad \Rightarrow \quad \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} \frac{-1-\sqrt{17}}{200} & \frac{1}{25} \\ \frac{1}{100} & \frac{1-\sqrt{17}}{200} \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2a_2 = \left(-1+\sqrt{17}\right)b_2 \quad \to \quad V_2 = \begin{pmatrix} -1+\sqrt{17} \\ 2 \end{pmatrix}$$

$$X_h(t) = C_1\begin{pmatrix} 1+\sqrt{17} \\ -2 \end{pmatrix} e^{\frac{-9-\sqrt{17}}{200}t} + C_2\begin{pmatrix} -1+\sqrt{17} \\ 2 \end{pmatrix} e^{\frac{-9+\sqrt{17}}{200}t}$$

$$\begin{cases} x_h(t) = \left(1+\sqrt{17}\right)C_1 e^{\frac{-9-\sqrt{17}}{200}t} + \left(-1+\sqrt{17}\right)C_2 e^{\frac{-9+\sqrt{17}}{200}t} \\ y_h(t) = -2C_1 e^{\frac{-9-\sqrt{17}}{200}t} + 2C_2 e^{\frac{-9+\sqrt{17}}{200}t} \end{cases}$$

$$\begin{cases} -\frac{1}{20}c_1 + \frac{1}{25}c_2 = -\frac{4}{5} \\ \frac{1}{100}c_1 - \frac{1}{25}c_2 = 0 \end{cases} \quad \to \quad \begin{cases} -5c_1 + 4c_2 = -80 \\ c_1 - 4c_2 = 0 \end{cases}$$

$$c_1 = 20, \quad c_2 = 5 \quad \to X_p = \begin{pmatrix} 20 \\ 5 \end{pmatrix}$$

$$\begin{cases} x(t) = \left(1+\sqrt{17}\right)C_1 e^{\frac{-9-\sqrt{17}}{200}t} + \left(-1+\sqrt{17}\right)C_2 e^{\frac{-9+\sqrt{17}}{200}t} + 20 \\ y(t) = -2C_1 e^{\frac{-9-\sqrt{17}}{200}t} + 2C_2 e^{\frac{-9+\sqrt{17}}{200}t} + 5 \end{cases}$$

$$\begin{cases} x(0) = \left(1+\sqrt{17}\right)C_1 + \left(-1+\sqrt{17}\right)C_2 + 20 = 0 \\ y(0) = -2C_1 + 2C_2 + 5 = 20 \end{cases}$$

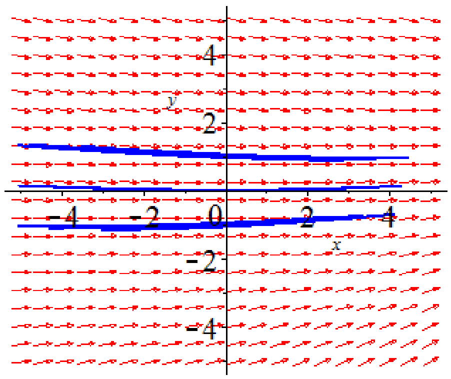
$$\begin{cases} \left(1+\sqrt{17}\right)C_1 + \left(-1+\sqrt{17}\right)C_2 = -20 \\ -2C_1 + 2C_2 = 15 \end{cases}$$

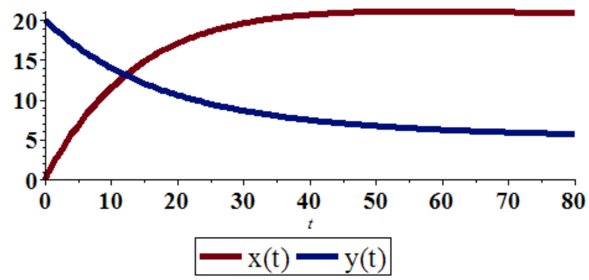
$$\Delta = \begin{vmatrix} 1+\sqrt{17} & -1+\sqrt{17} \\ -2 & 2 \end{vmatrix} = 4\sqrt{17} \qquad \Delta_1 = \begin{vmatrix} -20 & -1+\sqrt{17} \\ 15 & 2 \end{vmatrix} = -25 - 15\sqrt{17}$$

$$C_1 = -\frac{25+15\sqrt{17}}{4\sqrt{17}} \qquad C_2 = \frac{-25+15\sqrt{17}}{4\sqrt{17}}$$

$$\begin{cases} x(t) = -\dfrac{25+15\sqrt{17}}{4\sqrt{17}}\left(1+\sqrt{17}\right)e^{\frac{-9-\sqrt{17}}{200}t} + \dfrac{-25+15\sqrt{17}}{4\sqrt{17}}\left(-1+\sqrt{17}\right)C_2 e^{\frac{-9+\sqrt{17}}{200}t} + 20 \\ y(t) = \dfrac{25+15\sqrt{17}}{2\sqrt{17}}e^{\frac{-9-\sqrt{17}}{200}t} + \dfrac{-25+15\sqrt{17}}{2\sqrt{17}}e^{\frac{-9+\sqrt{17}}{200}t} + 5 \end{cases}$$

$$\begin{cases} x(t) = -\dfrac{70+10\sqrt{17}}{\sqrt{17}}e^{\frac{-9-\sqrt{17}}{200}t} + \dfrac{70-10\sqrt{17}}{\sqrt{17}}C_2 e^{\frac{-9+\sqrt{17}}{200}t} + 20 \\ y(t) = \dfrac{25+15\sqrt{17}}{2\sqrt{17}}e^{\frac{-9-\sqrt{17}}{200}t} + \dfrac{-25+15\sqrt{17}}{2\sqrt{17}}e^{\frac{-9+\sqrt{17}}{200}t} + 5 \end{cases}$$





Exercise

Two 1,000 *liter* tanks are with salt water. Tank ***A*** contains 800 *liters* of water initially containing 20 *grams* of salt dissolved in it and Tank ***B*** contains 1,000 *liters* of water initially containing 80 *grams* of salt dissolved in it. Salt water with a concentration of $\frac{1}{2}$ g/L of salt enters Tank ***A*** at a rate of 4 L/hr. Fresh water enters Tank ***B*** at a rate of 7 L/hr. Through a connecting pipe water flows from Tank ***B*** into Tank ***A*** at a rate of 10 L/hr. Through a different connecting pipe 14 L/hr flows out of Tank ***A*** and 11 L/hr are drained out of the pipe (and hence out of the system completely) and only 3 L/hr flows back into Tank ***B***. Find the amount of salt in each tank at any time.

Solution

Tank ***A***: $\dfrac{dx}{dt} = \dfrac{1}{2}\dfrac{g}{L}\left(4\dfrac{L}{hr}\right) + \dfrac{10\ L/hr}{1000\ L}y(g) - \dfrac{14}{800}x$

Tank ***B***: $\dfrac{dy}{dt} = 0\dfrac{g}{L}\left(7\dfrac{L}{hr}\right) + \dfrac{3}{800}x - \dfrac{10}{1000}y$

$$\begin{cases} x' = -\dfrac{7}{400}x + \dfrac{1}{100}y + 2 & x(0) = 20 \\ y' = \dfrac{3}{800}x - \dfrac{1}{100}y & y(0) = 80 \end{cases}$$

$$|A-\lambda I| = \begin{vmatrix} -\frac{7}{400}-\lambda & \frac{1}{100} \\ \frac{3}{800} & -\frac{1}{100}-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} -\frac{7}{400} & \frac{1}{100} \\ \frac{3}{800} & -\frac{1}{100} \end{pmatrix}$$

$$=\lambda^2+\frac{11}{400}\lambda+\frac{11}{8\times 10^4}=0 \qquad \rightarrow\ 8\times 10^4\lambda^2+2200\lambda+11=0$$

The eigenvalues are: $\lambda_{1,2}=\frac{-2200\pm 200\sqrt{33}}{16\times 10^4}=\frac{-11\pm\sqrt{33}}{800}$

For $\lambda_1=\frac{-11-\sqrt{33}}{800} \quad \Rightarrow \quad (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} -\frac{3}{800}+\frac{\sqrt{33}}{800} & \frac{1}{100} \\ \frac{3}{800} & \frac{3}{800}+\frac{\sqrt{33}}{800} \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 3a_1=-\left(3+\sqrt{33}\right)b_1$$

$$\Rightarrow \quad V_1=\begin{pmatrix} 3+\sqrt{33} \\ -3 \end{pmatrix}$$

For $\lambda_2=\frac{-11+\sqrt{33}}{800} \quad \Rightarrow \quad (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix} -\frac{3}{800}-\frac{\sqrt{33}}{800} & \frac{1}{100} \\ \frac{3}{800} & \frac{3}{800}-\frac{\sqrt{33}}{800} \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 3a_2=-\left(3-\sqrt{33}\right)b_2$$

$$\Rightarrow \quad V_2=\begin{pmatrix} -3+\sqrt{33} \\ 3 \end{pmatrix}$$

$$X_h(t)=C_1\begin{pmatrix} 3+\sqrt{33} \\ -3 \end{pmatrix}e^{\frac{-11-\sqrt{33}}{800}t}+C_2\begin{pmatrix} -3+\sqrt{33} \\ 3 \end{pmatrix}e^{\frac{-11+\sqrt{33}}{800}t}$$

$$\begin{cases} x_h(t)=\left(3+\sqrt{33}\right)C_1e^{\frac{-11-\sqrt{33}}{800}t}+\left(-3+\sqrt{33}\right)C_2e^{\frac{-11+\sqrt{33}}{800}t} \\ y_h(t)=-3C_1e^{\frac{-11-\sqrt{33}}{800}t}+3C_2e^{\frac{-11+\sqrt{33}}{800}t} \end{cases}$$

$$\begin{cases} -\frac{7}{400}c_1+\frac{1}{100}c_2=-2 \\ \frac{3}{800}c_1-\frac{1}{100}c_2=0 \end{cases} \quad \rightarrow \quad \begin{cases} -7c_1+4c_2=-800 \\ 3c_1-8c_2=0 \end{cases}$$

$$\Delta=\begin{vmatrix} -7 & 4 \\ 3 & -8 \end{vmatrix}=44 \quad \Delta_1=\begin{vmatrix} -800 & 4 \\ 0 & -8 \end{vmatrix}=6400 \quad \Delta_2=\begin{vmatrix} -7 & -800 \\ 3 & 0 \end{vmatrix}=2400$$

$$c_1=\frac{1600}{11},\quad c_2=\frac{600}{11} \quad \rightarrow X_p=\begin{pmatrix} \frac{1600}{11} \\ \frac{600}{11} \end{pmatrix}$$

$$\begin{cases} x(t)=\left(3+\sqrt{33}\right)C_1e^{\frac{-11-\sqrt{33}}{800}t}+\left(-3+\sqrt{33}\right)C_2e^{\frac{-11+\sqrt{33}}{800}t}+\frac{1600}{11} \\ y(t)=-3C_1e^{\frac{-11-\sqrt{33}}{800}t}+3C_2e^{\frac{-11+\sqrt{33}}{800}t}+\frac{600}{11} \end{cases}$$

Given: $x(0)=20 \quad y(0)=80$

$$\begin{cases} x(0)=\left(3+\sqrt{33}\right)C_1+\left(-3+\sqrt{33}\right)C_2+\dfrac{1600}{11}=20 \\ y(0)=-3C_1+3C_2+\dfrac{600}{11}=80 \end{cases}$$

$$\begin{cases} \left(3+\sqrt{33}\right)C_1+\left(-3+\sqrt{33}\right)C_2=-\dfrac{1380}{11} \\ -3C_1+3C_2=\dfrac{280}{11} \end{cases}$$

$$\Delta=\begin{vmatrix} 3+\sqrt{33} & -3+\sqrt{33} \\ -3 & 3 \end{vmatrix}=6\sqrt{33} \quad \Delta_1=\begin{vmatrix} -\dfrac{1380}{11} & -3+\sqrt{33} \\ \dfrac{280}{11} & 3 \end{vmatrix}=-\dfrac{3300}{11}-\dfrac{280\sqrt{33}}{11}=-300-\dfrac{280\sqrt{33}}{11}$$

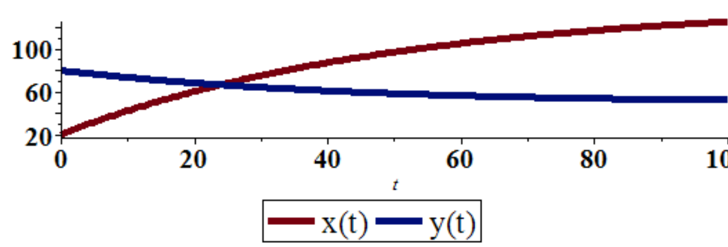
$$\Delta_2=\begin{vmatrix} 3+\sqrt{33} & -\dfrac{1380}{11} \\ -3 & \dfrac{280}{11} \end{vmatrix}=-300+\dfrac{280\sqrt{33}}{11}$$

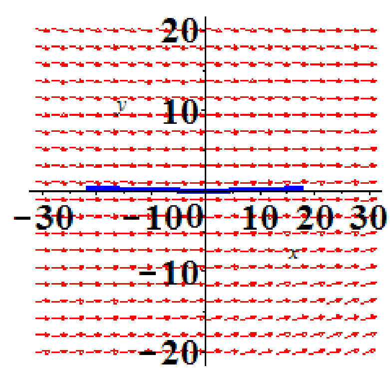
$$C_1=\left(-300-\frac{280\sqrt{33}}{11}\right)\frac{1}{6\sqrt{33}}=-\frac{50\sqrt{33}}{33}-\frac{140}{33}$$

$$C_2=\left(-300+\frac{280\sqrt{33}}{11}\right)\frac{1}{6\sqrt{33}}=-\frac{50\sqrt{33}}{33}+\frac{140}{33}$$

$$\begin{cases} x(t)=\dfrac{1}{33}\left(3+\sqrt{33}\right)\left(-50\sqrt{33}-140\right)e^{\frac{-11-\sqrt{33}}{800}t}+\dfrac{1}{33}\left(-3+\sqrt{33}\right)\left(-50\sqrt{33}+140\right)e^{\frac{-11+\sqrt{33}}{800}t}+\dfrac{1600}{11} \\ y(t)=-3\left(-\dfrac{50\sqrt{33}}{33}-\dfrac{140}{33}\right)e^{\frac{-11-\sqrt{33}}{800}t}+3\left(-\dfrac{50\sqrt{33}}{33}+\dfrac{140}{33}\right)e^{\frac{-11+\sqrt{33}}{800}t}+\dfrac{600}{11} \end{cases}$$

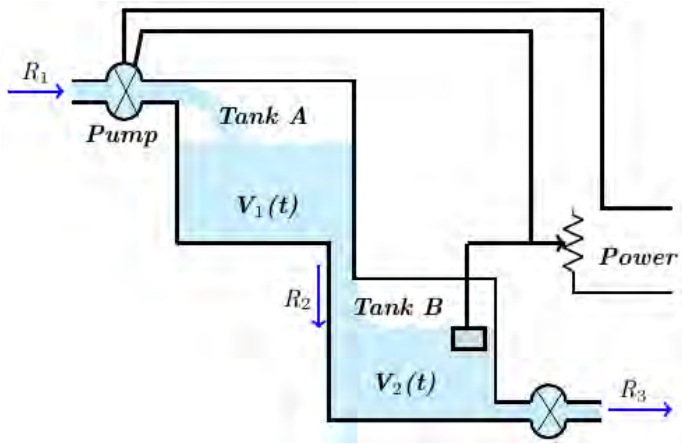
$$\begin{cases} x(t)=-\dfrac{10}{33}\left(207+29\sqrt{33}\right)e^{\frac{-11-\sqrt{33}}{800}t}+\dfrac{10}{33}\left(-207+29\sqrt{33}\right)e^{\frac{-11+\sqrt{33}}{800}t}+\dfrac{1600}{11} \\ y(t)=\dfrac{1}{11}\left(50\sqrt{33}+140\right)e^{\frac{-11-\sqrt{33}}{800}t}+\dfrac{1}{11}\left(-50\sqrt{33}+140\right)e^{\frac{-11+\sqrt{33}}{800}t}+\dfrac{600}{11} \end{cases}$$





Exercise

Many physical and biological systems involve time delays. A pure time delay has its output the same as its input but shifted in time. A more common type of delay is pooling delay. Here the level of fluid in tank *B* determines the rate at which fluid enters tank *A*. Suppose this rate is given by $R_1(t)=\alpha\left(V-V_2(t)\right)$, where α and *V* are positive constants and $V_2(t)$ is the volume of fluid in tank *B* at time *t*.



a) If the outflow rate R_3 from tank *B* is constant and the flow rate R_2 from tank *A* into tank *B* is $R_2(t)=KV_1(t)$ is the volume of fluid in tank *A* at time *t*, then show that this feedback system is governed by the system

$$\begin{cases} \dfrac{dV_1}{dt}=\alpha\left(V-V_2(t)\right)-KV_1(t) \\ \dfrac{dV_2}{dt}=KV_1(t)-R_3 \end{cases}$$

b) Find a general solution for the system in part (*a*) when $\alpha=5\ min^{-1}$, $V=20\ L$, $K=2\ min^{-1}$, and $R_3=10\ \ L/min$.

c) Using the general solution obtained in part (*b*), what can be said about the volume of fluid in each of the tanks as $t\to+\infty$?

Solution

a) *Tank **A***: $\dfrac{dV_1}{dt}=R_1(t)-R_2(t)$

$$=\alpha\left(V-V_2(t)\right)-KV_1(t)$$

*Tank **B***: $\dfrac{dV_2}{dt}=R_2(t)-R_3(t)$

$$=KV_1(t)-R_3$$

b) ***Given***: $\alpha=5\ min^{-1}$, $V=20\ L$, $K=2\ min^{-1}$, $R_3=10\ \ L/min$

$$\begin{cases} \dfrac{dV_1}{dt} = 5(20 - V_2) - 2V_1 \\ \dfrac{dV_2}{dt} = 2V_1 - 10 \end{cases}$$

$$\begin{cases} V_1' = -2V_1 - 5V_2 + 100 \\ V_2' = 2V_1 - 10 \end{cases}$$

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & -5 \\ 2 & -\lambda \end{vmatrix} \qquad A = \begin{pmatrix} -2 & -5 \\ 2 & 0 \end{pmatrix}$$

$$= \lambda^2 + 2\lambda + 10 = 0$$

The eigenvalues are: $\lambda_{1,2} = -1 \pm 3i$

For $\lambda_1 = -1 - 3i \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -1+3i & -5 \\ 2 & 1+3i \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow (-1+3i)x_1 = 5y_1 \quad \to \quad V_1 = \begin{pmatrix} 5 \\ -1+3i \end{pmatrix}$$

The solution is: $x_1(t) = \begin{pmatrix} 5 \\ -1+3i \end{pmatrix}$

$$z(t) = \begin{pmatrix} 5 \\ -1+3i \end{pmatrix} e^{-(1+3i)t}$$

$$= \left(\begin{pmatrix} 5 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 3 \end{pmatrix} \right)(\cos 3t + i \sin 3t)e^{-t}$$

$$= \left(\begin{pmatrix} 5 \\ -1 \end{pmatrix}\cos 3t - \begin{pmatrix} 0 \\ 3 \end{pmatrix}\sin 3t + i\left(\begin{pmatrix} 5 \\ -1 \end{pmatrix}\sin 3t + \begin{pmatrix} 0 \\ 3 \end{pmatrix}\cos 3t \right) \right)e^{-t}$$

$$= \left(\begin{pmatrix} 5\cos 3t \\ -\cos 3t - 3\sin 3t \end{pmatrix} + i\begin{pmatrix} 5\sin 3t \\ -\sin 3t + 3\cos 3t \end{pmatrix} \right)e^{-t}$$

$$V_h(t) = C_1\begin{pmatrix} 5\cos 3t \\ -\cos 3t - 3\sin 3t \end{pmatrix}e^{-t} + C_2\begin{pmatrix} 5\sin 3t \\ -\sin 3t + 3\cos 3t \end{pmatrix}e^{-t}$$

$$\begin{cases} 2a_1 + 5a_2 = 100 \\ 2a_1 = 10 \end{cases} \quad \to \quad a_1 = 5,\ a_2 = 18 \quad V_p = \begin{pmatrix} 5 \\ 18 \end{pmatrix}$$

$$\begin{cases} V_1(t) = (5C_1 \cos 3t + 5C_2 \sin 3t)e^{-t} + 5 \\ V_2(t) = ((3C_2 - C_1)\cos 3t - (3C_1 + C_2)\sin 3t)e^{-t} + 18 \end{cases}$$

c) $\lim\limits_{t\to\infty} V_1(t) = \lim\limits_{t\to\infty}\left((5C_1 \cos 3t + 5C_2 \sin 3t)e^{-t} + 5 \right) \underline{= 5\ L}$

$$\lim_{t\to\infty} V_2(t) = \lim_{t\to\infty}\left(\left(\left(3C_2 - C_1\right)\cos 3t - \left(3C_1 + C_2\right)\sin 3t\right)e^{-t} + 18\right)$$

$$\underline{= 18\ L|}$$

Exercise

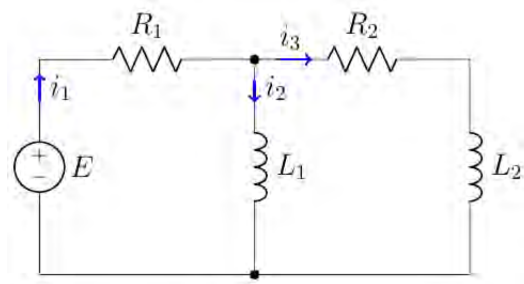
The electrical network shown below

a) Find the system equations for the currents $i_2(t)$ and $i_3(t)$

b) Solve the system for the given: $R_1 = 2\ \Omega,\quad R_2 = 3\ \Omega,\quad L_1 = 1\ h,\quad L_2 = 1\ h,\quad E = 60\ V$, with the initial values $i_2(0) = 0\quad \&\quad i_3(0) = 0$

c) Determine the current $i_1(t)$

Solution



a) $i_1 = i_2 + i_3$

$$\begin{cases} R_1 i_1 + L_1 i_2' = E(t) \\ R_1 i_1 + R_2 i_3 + L_2 i_3' = E(t) \end{cases}$$

$$\begin{cases} L_1 i_2' + R_1\left(i_2 + i_3\right) = E(t) \\ L_2 i_3' + R_1\left(i_2 + i_3\right) + R_2 i_3 = E(t) \end{cases}$$

$$\begin{cases} i_2' = -\frac{R_1}{L_1} i_2 - \frac{R_1}{L_1} i_3 + \frac{1}{L_1} E(t) \\ i_3' = -\frac{R_1}{L_2} i_2 - \frac{1}{L_2}\left(R_1 + R_2\right) i_3 + \frac{1}{L_2} E(t) \end{cases}$$

b) $\begin{cases} i_2' = -2i_2 - 2i_3 + 60 \\ i_3' = -2i_2 - 5i_3 + 60 \end{cases}$

$$A = \begin{pmatrix} -2 & -2 \\ -2 & -5 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & -2 \\ -2 & -5-\lambda \end{vmatrix}$$

$$= \lambda^2 + 7\lambda + 6 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1$ *and* $\lambda_2 = -6$

For $\lambda_1 = -1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = -2y|} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

For $\lambda_2 = -6 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{2x = y|} \quad \Rightarrow V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$i_h(t) = C_1\begin{pmatrix} 2 \\ -1 \end{pmatrix}e^{-t} + C_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-6t}$$

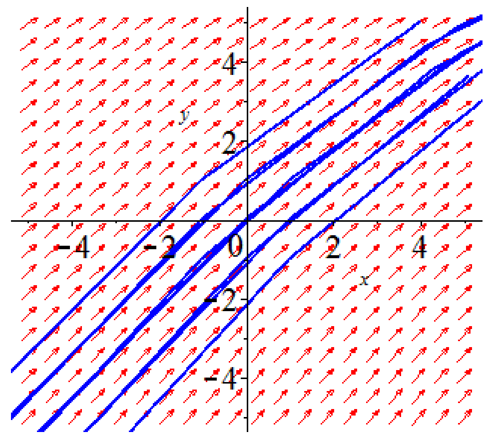
$$\begin{cases} -2a_1 - 2a_2 = -60 \\ 2a_1 + 5a_2 = 60 \end{cases} \quad a_1 = 30 \quad a_2 = 0 \quad \to \quad x_p = \begin{pmatrix} 30 \\ 0 \end{pmatrix}$$

$$i(t) = C_1\begin{pmatrix} 2 \\ -1 \end{pmatrix}e^{-t} + C_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-6t} + \begin{pmatrix} 30 \\ 0 \end{pmatrix}$$

$$i(0) = C_1\begin{pmatrix} 2 \\ -1 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 30 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

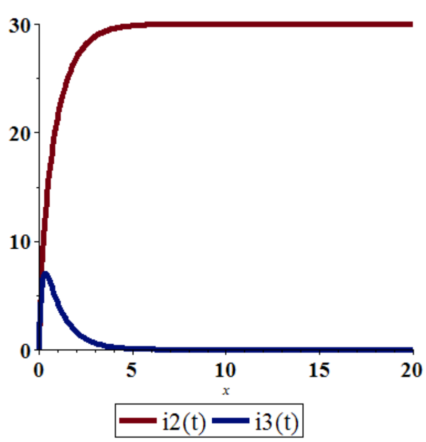
$$\begin{cases} 2C_1 + C_2 + 30 = 0 \\ -C_1 + 2C_2 = 0 \end{cases} \quad \underline{C_1 = -12 \quad C_2 = -6|}$$

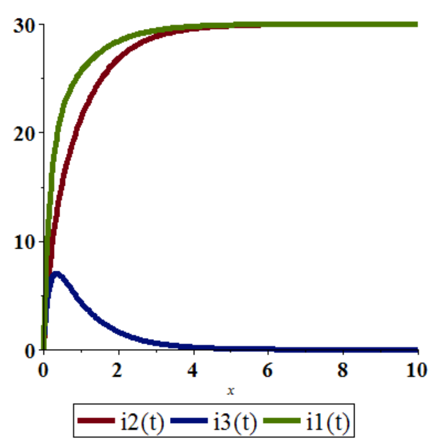
$$\begin{cases} i_2(t) = -24e^{-t} - 6e^{-6t} + 30 \\ i_3(t) = 12e^{-t} - 12e^{-6t} \end{cases}$$



c) $i_1(t) = i_2(t) + i_3(t)$

$$\underline{= -12e^{-t} - 18e^{-6t} + 30|}$$





Exercise

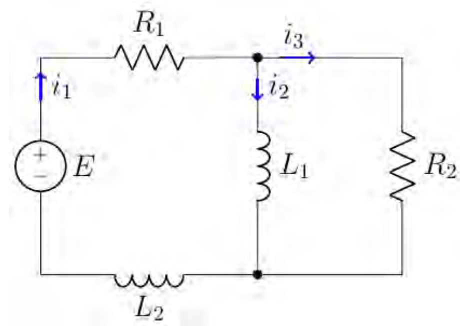
The electrical network shown below

a) Find the system equations for the currents $i_1(t)$ and $i_2(t)$

b) Solve the system for the given: $R_1 = 8\ \Omega,\quad R_2 = 3\ \Omega,\quad L_1 = 1\ h,\quad L_2 = 1\ h,\quad E = 100\sin t\ V$, with the initial values $i_1(0) = 0\quad \&\quad i_2(0) = 0$

c) Determine the current $i_3(t)$

Solution



a) $\begin{cases} R_1 i_1 + L_1 i_2' + L_2 i_1' = E(t) \\ R_1 i_1 + R_2 i_3 + L_2 i_1' = E(t) \end{cases}$

$$\begin{cases} L_1 i_2' + L_2 i_1' = -R_1 i_1 + E \\ L_2 i_1' = -R_1 i_1 - R_2 i_3 + E \end{cases}$$

$$\begin{cases} L_2 i_1' = -R_1 i_1 - R_2\left(i_1 - i_2\right) + E \\ L_1 i_2' = R_2\left(i_1 - i_2\right) \end{cases}$$

$$\begin{cases} i_1' = -\dfrac{1}{L_2}\left(R_1 + R_2\right) i_1 + \dfrac{R_2}{L_2} i_2 + \dfrac{E}{L_2} \\ i_2' = \dfrac{R_2}{L_1} i_1 - \dfrac{R_2}{L_1} i_2 \end{cases}$$

b) $\begin{cases} i_1' = -11 i_1 + 3 i_2 + 100\sin t \\ i_2' = 3 i_1 - 3 i_2 \end{cases}$

$$A = \begin{pmatrix} -11 & 3 \\ 3 & -3 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -11-\lambda & 3 \\ 3 & -3-\lambda \end{vmatrix}$$

$$= \lambda^2 - 14\lambda + 24 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -2$ *and* $\lambda_2 = -12$

For $\lambda_1 = -2 \quad \Rightarrow \left(A - \lambda_1 I\right) V_1 = 0$

$$\begin{pmatrix} -9 & 3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{3x = y|} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

For $\lambda_2 = -12 \quad \Rightarrow \left(A - \lambda_2 I\right) V_2 = 0$

$$\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x=-3y|} \quad \Rightarrow \quad V_2=\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$i_h(t)=C_1\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-2t}+C_2\begin{pmatrix} -3 \\ 1 \end{pmatrix}e^{-12t}$$

$$\varphi(t)=\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-2t}+\begin{pmatrix} -3 \\ 1 \end{pmatrix}e^{-12t}$$

$$=\begin{pmatrix} e^{-2t} & -3e^{-12t} \\ 3e^{-2t} & e^{-12t} \end{pmatrix}$$

$$\varphi^{-1}(t)=\frac{1}{e^{-14t}+9e^{-14t}}\begin{pmatrix} e^{-12t} & 3e^{-12t} \\ -3e^{-2t} & e^{-2t} \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{10}e^{2t} & \frac{3}{10}e^{2t} \\ -\frac{3}{10}e^{12t} & \frac{1}{10}e^{12t} \end{pmatrix}$$

$$\varphi^{-1}\begin{pmatrix} 100\sin t \\ 0 \end{pmatrix}=\begin{pmatrix} \frac{1}{10}e^{2t} & \frac{3}{10}e^{2t} \\ -\frac{3}{10}e^{12t} & \frac{1}{10}e^{12t} \end{pmatrix}\begin{pmatrix} 100\sin t \\ 0 \end{pmatrix}$$

$$=\begin{pmatrix} 10e^{2t}\sin t \\ -30e^{12t}\sin t \end{pmatrix}$$

		$\int \sin t$
+	e^{at}	$-\cos t$
−	ae^{at}	$-\sin t$
+	a^2e^{at}	

$$X=\int \varphi^{-1}\begin{pmatrix} 100\sin t \\ 0 \end{pmatrix}dt$$

$$=\int \begin{pmatrix} 10e^{2t}\sin t \\ -30e^{12t}\sin t \end{pmatrix}dt$$

$$\int e^{at}\sin t\ dt=e^{at}\left(-\cos t+a\sin t\right)-a^2\int e^{at}\sin t\ dt$$

$$\int e^{at}\sin t\ dt=\frac{1}{1+a^2}e^{at}\left(-\cos t+a\sin t\right)$$

$$=\begin{pmatrix} 2e^{2t}\left(2\sin t-\cos t\right) \\ -\frac{6}{29}e^{12t}\left(12\sin t-\cos t\right) \end{pmatrix}$$

$$i_p(t)=\varphi X=\begin{pmatrix} e^{-2t} & -3e^{-12t} \\ 3e^{-2t} & e^{-12t} \end{pmatrix}\begin{pmatrix} 2e^{2t}\left(2\sin t-\cos t\right) \\ -\frac{6}{29}e^{12t}\left(12\sin t-\cos t\right) \end{pmatrix}$$

$$= \begin{pmatrix} 4\sin t - 2\cos t + \frac{216}{29}\sin t - \frac{18}{29}\cos t \\ 12\sin t - 6\cos t - \frac{72}{29}\sin t + \frac{6}{29}\cos t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{332}{29}\sin t - \frac{76}{29}\cos t \\ \frac{276}{29}\sin t - \frac{168}{29}\cos t \end{pmatrix}$$

$$i(t) = C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-12t} + \begin{pmatrix} \frac{332}{29}\sin t - \frac{76}{29}\cos t \\ \frac{276}{29}\sin t - \frac{168}{29}\cos t \end{pmatrix}$$

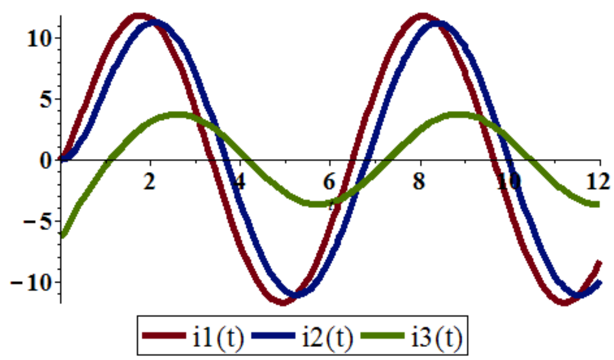
$$i(0) = C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{76}{29} \\ -\frac{168}{29} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} C_1 - 3C_2 = \frac{76}{29} \\ 3C_1 + C_2 = \frac{168}{29} \end{cases} \qquad \Delta = \begin{vmatrix} 1 & -3 \\ 3 & 1 \end{vmatrix} = 10 \quad \Delta_1 = \begin{vmatrix} \frac{76}{29} & -3 \\ \frac{168}{29} & 1 \end{vmatrix} = 20$$

$$\underline{C_1 = 2, \quad C_2 = -\frac{6}{29}}$$

$$i(t) = \begin{pmatrix} 2 \\ 6 \end{pmatrix} e^{-2t} + \begin{pmatrix} \frac{18}{29} \\ -\frac{6}{29} \end{pmatrix} e^{-12t} + \begin{pmatrix} \frac{332}{29}\sin t - \frac{76}{29}\cos t \\ \frac{276}{29}\sin t - \frac{168}{29}\cos t \end{pmatrix}$$

$$\begin{cases} i_1(t) = 2e^{-2t} + \frac{18}{29}e^{-12t} + \frac{332}{29}\sin t - \frac{76}{29}\cos t \\ i_2(t) = 6e^{-2t} - \frac{6}{29}e^{-12t} + \frac{276}{29}\sin t - \frac{168}{29}\cos t \end{cases}$$



c) $i_3(t) = i_1(t) - i_2(t)$

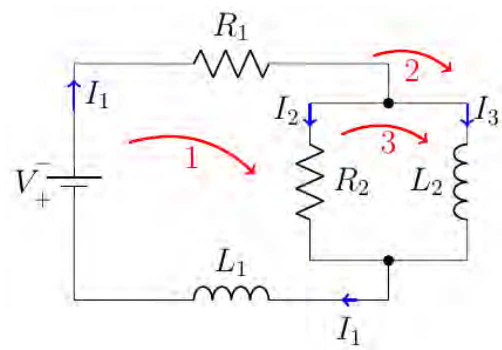
$$\underline{= -4e^{-2t} + \frac{24}{29}e^{-12t} + \frac{56}{29}\sin t - \frac{92}{29}\cos t}$$

Exercise

Find a system of differential equations and solve for the currents in the given network:

$R_1 = 2\ \Omega, \quad R_2 = 1\ \Omega, \quad L_1 = 0.2\ H, \quad L_2 = 0.1\ H, \quad V = 6\ V$

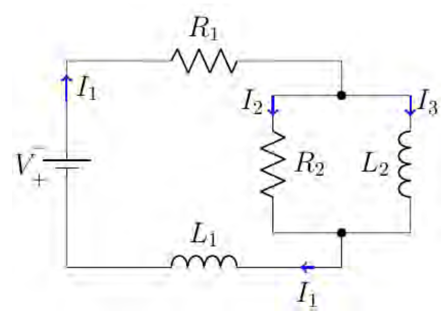
With initial values: $I_1(0) = I_2(0) = I_3(0) = 0$



Solution

$$\begin{cases} R_1 I_1 + R_2 I_2 + L_1 I_1' = V & (1) \\ R_1 I_1 + L_2 I_3' + L_1 I_1' = V & (2) \\ L_2 I_3' - R_2 I_2 = 0 & (3) \end{cases}$$

$$\begin{cases} 2I_1 + I_2 + 0.2I_1' = 6 \\ 2I_1 + 0.1I_3' + 0.2I_1' = 6 \\ 0.1I_3' - I_2 = 0 \end{cases}$$



$$\begin{cases} I_1' = -10I_1 - 5I_2 + 30 \\ I_3' + 2I_2' + 2I_3' = -20I_1 + 60 \\ I_3' = 10I_2 \end{cases} \qquad I_1 = I_2 + I_3$$

$$\begin{cases} I_1' = -10I_1 - 5I_2 + 30 \\ I_2' = -10I_1 - 15I_2 + 30 \\ I_3' = 10I_2 \end{cases}$$

$$|A - \lambda I| = \begin{vmatrix} -10-\lambda & -5 & 0 \\ -10 & -15-\lambda & 0 \\ 0 & 10 & -\lambda \end{vmatrix} \qquad A = \begin{pmatrix} -10 & -5 & 0 \\ -10 & -15 & 0 \\ 0 & 10 & 0 \end{pmatrix}$$

$$= -150\lambda - 25\lambda^2 - \lambda^3 + 50\lambda$$

$$= -\lambda\left(\lambda^2 + 25\lambda + 100\right) = 0$$

Thus, the eigenvalues are: $\lambda_1 = -20, \ \lambda_2 = -5 \ \ and \ \ \lambda_3 = 0$

For $\lambda_1 = -20 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 10 & -5 & 0 \\ -10 & 5 & 0 \\ 0 & 10 & 20 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{matrix} 2x = y \\ y = -2z \end{matrix} \quad \Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -5 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -5 & -5 & 0 \\ -10 & -10 & 0 \\ 0 & 10 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{matrix} x = -y \\ 2y = -z \end{matrix} \quad \Rightarrow \quad V_2 = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

For $\lambda_3 = 0 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -10 & -5 & 0 \\ -10 & -15 & 0 \\ 0 & 10 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{matrix} x = 0 \\ y = 0 \end{matrix} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$I_h = C_1\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}e^{-20t} + C_2\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}e^{-5t} + C_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

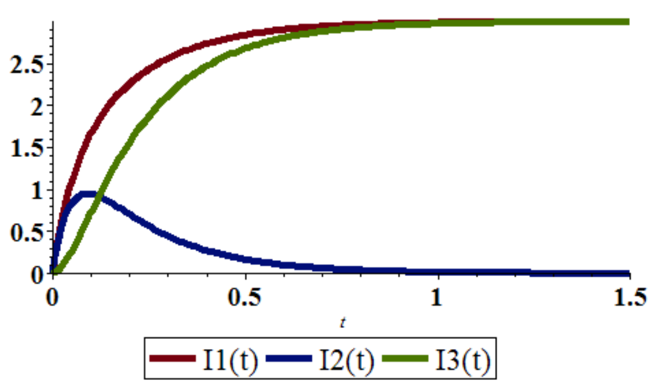
$$\begin{cases} 10a_1 + 5a_2 = 30 \\ 10a_1 + 15a_2 = 30 \\ 10a_2 = 0 \end{cases} \rightarrow \quad \underline{a_1 = 3,\ a_2 = 0,\ a_3 = 0|} \quad \rightarrow \quad I_p = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

$$I(t) = \begin{pmatrix} -C_1e^{-20t} - C_2e^{-5t} \\ -2C_1e^{-20t} + C_2e^{-5t} \\ C_1e^{-20t} - 2C_2e^{-5t} + +C_3 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

$$I(0) = \begin{pmatrix} -C_1 - C_2 + 3 \\ -2C_1 + C_2 \\ C_1 - 2C_2 + C_3 + a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} C_1 + C_2 = 3 \\ 2C_1 = C_2 \\ C_1 - 2C_2 + C_3 = 0 \end{cases} \rightarrow \quad C_1 = 1,\ C_2 = 2,\ C_3 = 3$$

$$\begin{cases} I_1(t) = -e^{-20t} - 2e^{-5t} + 3 \\ I_2(t) = -2e^{-20t} + 2e^{-5t} \\ I_3(t) = e^{-20t} - 4e^{-5t} + 3 \end{cases}$$



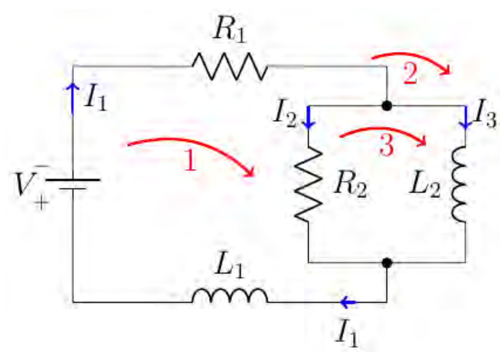
Exercise

Find a system of differential equations and solve for the currents in the given network:

$R_1 = 2\ \Omega, \quad R_2 = 1\ \Omega, \quad L_1 = 0.1\ H, \quad L_2 = 0.2\ H, \quad V = 6\ V$

With initial values: $I_1(0) = I_2(0) = I_3(0) = 0$

Solution



$$\begin{cases} R_1 I_1 + R_2 I_2 + L_1 I_1' = V & (1) \\ R_1 I_1 + L_2 I_3' + L_1 I_1' = V & (2) \\ L_2 I_3' - R_2 I_2 = 0 & (3) \end{cases}$$

$$\begin{cases} 2I_1 + I_2 + 0.1I_1' = 6 \\ 2I_1 + 0.2I_3' + 0.1I_1' = 6 \\ 0.2I_3' - I_2 = 0 \end{cases}$$

$$\begin{cases} I_1' = -20I_1 - 10I_2 + 60 \\ 2I_3' + I_2' + I_3' = -20I_1 + 60 \\ I_3' = 5I_2 \end{cases} \qquad I_1 = I_2 + I_3$$

$$\begin{cases} I_1' = -20I_1 - 10I_2 + 60 \\ I_2' = -20I_1 - 15I_2 + 60 \\ I_3' = 5I_2 \end{cases}$$

$$|A - \lambda I| = \begin{pmatrix} -20-\lambda & -10 & 0 \\ -20 & -15-\lambda & 0 \\ 0 & 5 & -\lambda \end{pmatrix} \qquad A = \begin{pmatrix} -20 & -10 & 0 \\ -20 & -15 & 0 \\ 0 & 5 & 0 \end{pmatrix}$$

$$= -\lambda^3 - 35\lambda^2 - 100\lambda = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = \dfrac{-35 \pm 5\sqrt{33}}{2}$ *and* $\lambda_3 = 0$

For $\lambda_1 = -\dfrac{35}{2} - \dfrac{5\sqrt{33}}{2} \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -\frac{5}{2} + \frac{5\sqrt{33}}{2} & -10 & 0 \\ -20 & \frac{5}{2} + \frac{5\sqrt{33}}{2} & 0 \\ 0 & 5 & \frac{35}{2} + \frac{5\sqrt{33}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{aligned} &\left(-1+\sqrt{33}\right)x = 4y \\ &2y = -\left(7+\sqrt{33}\right)z \end{aligned}$$

$$x = -\frac{4\left(7+\sqrt{33}\right)}{-1+\sqrt{33}} = \frac{1}{8}\left(-40 - 8\sqrt{33}\right) = -5 - \sqrt{33} \qquad \Rightarrow \quad V_1 = \begin{pmatrix} -5-\sqrt{33} \\ -7-\sqrt{33} \\ 2 \end{pmatrix}$$

For $\lambda_2 = -\frac{35}{2} + \frac{5\sqrt{33}}{2} \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -\frac{5}{2} - \frac{5\sqrt{33}}{2} & -10 & 0 \\ -20 & \frac{5}{2} - \frac{5\sqrt{33}}{2} & 0 \\ 0 & 5 & \frac{35}{2} - \frac{5\sqrt{33}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{matrix} -\left(1+\sqrt{33}\right)x = 4y \\ 2y = \left(-7+\sqrt{33}\right)z \end{matrix}$$

$$x = -\frac{4\left(-7+\sqrt{33}\right)}{1+\sqrt{33}} \frac{1-\sqrt{33}}{1-\sqrt{33}} = \frac{1}{8}\left(-40+8\sqrt{33}\right) = -5+\sqrt{33} \quad \Rightarrow \quad V_2 = \begin{pmatrix} -5+\sqrt{33} \\ -7+\sqrt{33} \\ 2 \end{pmatrix}$$

For $\lambda_3 = 0 \quad \Rightarrow (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -20 & -10 & 0 \\ -20 & -15 & 0 \\ 0 & 5 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{matrix} x = 0 \\ y = 0 \end{matrix} \quad \Rightarrow \quad V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$I_h(t) = C_1 \begin{pmatrix} -5-\sqrt{33} \\ -7-\sqrt{33} \\ 2 \end{pmatrix} e^{-\left(35+5\sqrt{33}\right)t/2} + C_2 \begin{pmatrix} -5+\sqrt{33} \\ -7+\sqrt{33} \\ 2 \end{pmatrix} e^{-\frac{5}{2}\left(7-\sqrt{33}\right)t} + C_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} 20a_1 10a_2 = 60 \\ 20a_1 + 15a_2 = 60 \\ 50a_2 = 0 \end{cases} \rightarrow \quad \underline{a_1 = 3,\ a_2 = 0,\ a_3 = 0} \quad \rightarrow \quad I_p = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

$$I(t) = \begin{pmatrix} -\left(5+\sqrt{33}\right)C_1 e^{-\frac{5}{2}\left(7+\sqrt{33}\right)t} + \left(-5+\sqrt{33}\right)C_2 e^{-\frac{5}{2}\left(7-\sqrt{33}\right)t} \\ -\left(7+\sqrt{33}\right)C_1 e^{-\frac{5}{2}\left(7+\sqrt{33}\right)t} + \left(-7+\sqrt{33}\right)C_2 e^{-\frac{5}{2}\left(7-\sqrt{33}\right)t} \\ 2C_1 e^{-\frac{5}{2}\left(7+\sqrt{33}\right)t} + 2C_2 e^{-\frac{5}{2}\left(7-\sqrt{33}\right)t} + C_3 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

$$I(0) = \begin{pmatrix} -\left(5+\sqrt{33}\right)C_1 + \left(-5+\sqrt{33}\right)C_2 + 3 \\ -\left(7+\sqrt{33}\right)C_1 + \left(-7+\sqrt{33}\right)C_2 \\ 2C_1 + 2C_2 + C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-\left(5+\sqrt{33}\right)C_1 + \left(-5+\sqrt{33}\right)C_2 = -3$$

$$-\left(7+\sqrt{33}\right)C_1 + \left(-7+\sqrt{33}\right)C_2 = 0$$

$$2C_1 + 2C_2 + C_3 = 0$$

$$\Delta = \begin{vmatrix} -5-\sqrt{33} & -5+\sqrt{33} \\ -7-\sqrt{33} & -7+\sqrt{33} \end{vmatrix} = 4\sqrt{33} \qquad \Delta_1 = \begin{vmatrix} -3 & -5+\sqrt{33} \\ 0 & -7+\sqrt{33} \end{vmatrix} = 21-3\sqrt{33}$$

$$\Delta_2 = \begin{vmatrix} -5-\sqrt{33} & -3 \\ -7-\sqrt{33} & 0 \end{vmatrix} = -3\left(7+\sqrt{33}\right)$$

$$\boxed{C_1 = \frac{-33+7\sqrt{33}}{44},\ \ C_2 = \frac{-33-7\sqrt{33}}{44},\ \ C_3 = 3}$$

$$I(t) = \begin{pmatrix} -\left(5+\sqrt{33}\right)\frac{-33+7\sqrt{33}}{44}e^{-\frac{5}{2}\left(7+\sqrt{33}\right)t} + \left(-5+\sqrt{33}\right)\frac{-33-7\sqrt{33}}{44}e^{-\frac{5}{2}\left(7-\sqrt{33}\right)t} + 3 \\ -\left(7+\sqrt{33}\right)\frac{-33+7\sqrt{33}}{44}e^{-\frac{5}{2}\left(7+\sqrt{33}\right)t} - \frac{4}{11}e^{-\frac{5}{2}\left(7-\sqrt{33}\right)t} \\ \frac{-33+7\sqrt{33}}{22}e^{-\frac{5}{2}\left(7+\sqrt{33}\right)t} + \frac{-33-7\sqrt{33}}{22}e^{-\frac{5}{2}\left(7-\sqrt{33}\right)t} + 3 \end{pmatrix}$$

$$\begin{cases} I_1(t) = -\frac{33+\sqrt{33}}{22}e^{-\frac{5}{2}\left(7+\sqrt{33}\right)t} + \frac{-33+\sqrt{33}}{22}e^{-\frac{5}{2}\left(7-\sqrt{33}\right)t} + 3 \\ I_2(t) = -\frac{4}{11}e^{-\frac{5}{2}\left(7+\sqrt{33}\right)t} + \frac{4}{11}e^{-\frac{5}{2}\left(7-\sqrt{33}\right)t} \\ I_3(t) = \frac{-33+7\sqrt{33}}{22}e^{-\frac{5}{2}\left(7+\sqrt{33}\right)t} + \frac{-33-7\sqrt{33}}{22}e^{-\frac{5}{2}\left(7-\sqrt{33}\right)t} + 3 \end{cases}$$

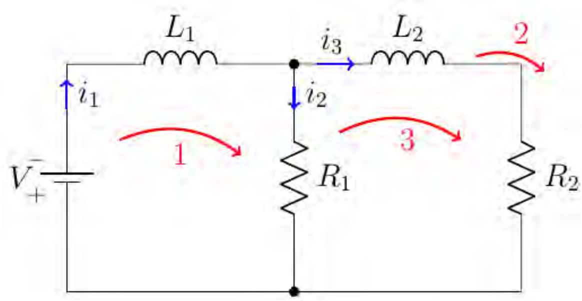
Exercise

Find a system of differential equations and solve for the currents in the given network:

$R_1 = 10\ \Omega,\quad R_2 = 20\ \Omega,\quad L_1 = 0.005\ H,\quad L_2 = 0.01\ H,\quad V = 50\ V$

With initial values: $\quad i_1(0) = i_2(0) = i_3(0) = 0$

Solution



$$\begin{cases} L_1 i_1' + R_1 i_2 = V & (1) \\ L_1 i_1' + L_2 i_3' + R_2 i_3 = V & (2) \\ L_2 i_3' + R_2 i_3 - R_1 i_2 = 0 & (3) \end{cases}$$

$$\begin{cases} 0.005 i_1' + 10 i_2 = 50 \\ 0.005 i_1' + 0.01 i_3' + 20 i_3 = 50 \\ 0.01 i_3' + 20 i_3 - 10 i_2 = 0 \end{cases}$$

$$\begin{cases} i_1' = -2{,}000 i_2 + 10^4 \\ i_1' + 2i_3' = -4{,}000 i_3 + 10^4 \\ i_3' = 10^3 i_2 - 2{,}000 i_3 \end{cases} \qquad i_1 = i_2 + i_3 \ \rightarrow\ i_1' = i_2' + i_3'$$

$$\begin{cases} i_1' = -2{,}000 i_2 + 10^4 \\ i_2' + 3i_3' = -4{,}000 i_3 + 10^4 \\ i_3' = 10^3 i_2 - 2{,}000 i_3 \end{cases}$$

$$\begin{cases} i_1' = -2{,}000 i_2 + 10^4 \\ i_2' = -3{,}000 i_2 + 2{,}000 i_3 + 10^4 \\ i_3' = 10^3 i_2 - 2{,}000 i_3 \end{cases}$$

$$\left|A-\lambda I\right| = \begin{vmatrix} -\lambda & -2000 & 0 \\ 0 & -3000-\lambda & 2000 \\ 0 & 1000 & -2000-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 0 & -2000 & 0 \\ 0 & -3000 & 2000 \\ 0 & 1000 & -2000 \end{pmatrix}$$

$$= -\lambda^3 - 6\times 10^6 \lambda - 5000\lambda^2 + 2\times 10^6 \lambda$$

$$= -\lambda\left(\lambda^2 + 5000\lambda + 4\times 10^6\right) = 0$$

Thus, the eigenvalues are: $\lambda_1 = -4000,\ \lambda_2 = -1000,\ and\ \lambda_3 = 0$

For $\lambda_1 = -4{,}000 \quad \Rightarrow \left(A-\lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 4000 & -2000 & 0 \\ 0 & 1000 & 2000 \\ 0 & 1000 & 2000 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{matrix} 2x = y \\ y = -2z \end{matrix} \quad \Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -1{,}000 \quad \Rightarrow \left(A-\lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 1000 & -2000 & 0 \\ 0 & -2000 & 2000 \\ 0 & 1000 & -1000 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{matrix} x = 2y \\ y = z \end{matrix} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 0 \quad \Rightarrow \left(A-\lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} 0 & -2000 & 0 \\ 0 & -3000 & 2000 \\ 0 & 1000 & -2000 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{matrix} y = 0 \\ z = 0 \end{matrix} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$i_h = C_1\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}e^{-4000t} + C_2\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}e^{-1000t} + C_3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

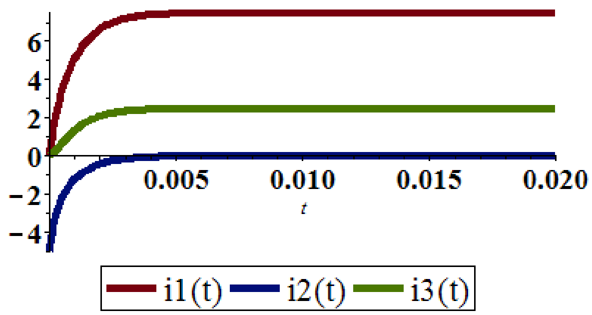
$$\begin{cases} 2{,}000 a_2 = 10^4 \\ 3{,}000 a_2 - 2{,}000 a_3 = 10^4 \\ 10^3 a_2 - 2{,}000 a_3 = 0 \end{cases} \quad \rightarrow \quad \underline{a_1 = 0,\ a_2 = 5,\ a_3 = \tfrac{5}{2}} \quad \rightarrow \quad i_p = \begin{pmatrix} 0 \\ 5 \\ \frac{5}{2} \end{pmatrix}$$

$$i(t)=C_1\begin{pmatrix}-1\\-2\\1\end{pmatrix}e^{-4000t}+C_2\begin{pmatrix}2\\1\\1\end{pmatrix}e^{-1000t}+C_3\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\5\\\frac{5}{2}\end{pmatrix}$$

$$i(0)=\begin{pmatrix}-C_1+2C_2+C_3\\-2C_1+C_2+5\\C_1+C_2+\frac{5}{2}\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{cases}-C_1+2C_2+C_3=0 & C_3=\frac{15}{2}\\ -2C_1+C_2=-5\\ -C_1-C_2=\frac{5}{2} & \rightarrow\ C_1=\frac{5}{6},\ C_2=-\frac{10}{3}\end{cases}$$

$$\begin{cases}i_1(t)=-\frac{5}{6}e^{-4000t}-\frac{20}{3}e^{-1000t}+\frac{15}{2}\\ i_2(t)=-\frac{5}{3}e^{-4000t}-\frac{10}{3}e^{-1000t}+5\\ i_3(t)=\frac{5}{6}e^{-4000t}-\frac{10}{3}e^{-1000t}+\frac{5}{2}\end{cases}$$



Exercise

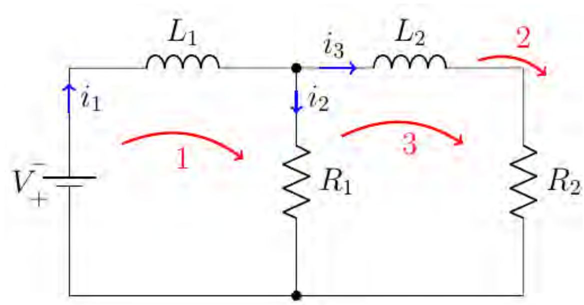
Find a system of differential equations and solve for the currents in the given network:

$R_1=10\ \Omega,\quad R_2=40\ \Omega,\quad L_1=10\ H,\quad L_2=30\ H,\quad V=20\ V$

With initial values: $i_1(0)=i_2(0)=i_3(0)=0$

Solution

$$\begin{cases}L_1i_1'+R_1i_2=V & (1)\\ L_1i_1'+L_2i_3'+R_2i_3=V & (2)\\ L_2i_3'+R_2i_3-R_1i_2=0 & (3)\end{cases}$$



$$\begin{cases} 10i_1' + 10i_2 = 20 \\ 10i_1' + 30i_3' + 40i_3 = 20 \\ 30i_3' + 40i_3 - 10i_2 = 0 \end{cases}$$

$$\begin{cases} i_1' + i_2 = 2 \\ i_2' = -4i_3' - 4i_3 + 2 \\ i_3' = \frac{1}{3}i_2 - \frac{4}{3}i_3 \end{cases} \qquad i_1 = i_2 + i_3 \quad \rightarrow \quad i_1' = i_2' + i_3'$$

$$\begin{cases} i_1' = -i_2 + 2 \\ i_2' = -\frac{4}{3}i_2 + \frac{4}{3}i_3 + 2 \\ i_3' = \frac{1}{3}i_2 - \frac{4}{3}i_3 \end{cases}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & -1 & 0 \\ 0 & -\frac{4}{3} - \lambda & \frac{4}{3} \\ 0 & \frac{1}{3} & -\frac{4}{3} - \lambda \end{vmatrix} \qquad A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & \frac{1}{3} & -\frac{4}{3} \end{pmatrix}$$

$$= -\lambda^3 - \frac{16}{9}\lambda - \frac{8}{3}\lambda^2 + \frac{4}{9}\lambda$$

$$= -\frac{1}{3}\lambda\left(3\lambda^2 + 8\lambda + 4\right) = 0$$

Thus, the eigenvalues are: $\lambda_1 = -2, \ \lambda_2 = -\frac{2}{3}, \ and \ \lambda_3 = 0$

For $\lambda_1 = -2 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{2}{3} & \frac{4}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{matrix} 2x = y \\ y = -2z \end{matrix} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

For $\lambda_2 = -\frac{2}{3} \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} \frac{2}{3} & -1 & 0 \\ 0 & -\frac{2}{3} & \frac{4}{3} \\ 0 & \frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{matrix} 2x = 3y \\ y = 2z \end{matrix} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 0 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & \frac{1}{3} & -\frac{4}{3} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{matrix} y = 0 \\ z = 0 \end{matrix} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$i_h = C_1\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}e^{-2t} + C_2\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}e^{-2t/3} + C_3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} a_2 = 2 \\ \frac{4}{3}a_2 - \frac{4}{3}a_3 = 2 \\ a_2 - 4a_3 = 0 \end{cases} \rightarrow \quad \underline{a_1 = 0,\ a_2 = 2,\ a_3 = \frac{1}{2}} \quad \rightarrow \quad i_p = \begin{pmatrix} 0 \\ 2 \\ \frac{1}{2} \end{pmatrix}$$

$$i(t) = \begin{pmatrix} C_1e^{-2t} + 3C_2e^{-2t/3} + C_3 \\ 2C_1e^{-2t} + 2C_2e^{-2t/3} \\ -C_1e^{-2t} + C_2e^{-2t/3} \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ \frac{1}{2} \end{pmatrix}$$

$$i(0) = \begin{pmatrix} C_1 + 3C_2 + C_3 \\ 2C_1 + 2C_2 + 2 \\ -C_1 + C_2 + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} C_1 + 3C_2 + C_3 = 0 & C_3 = \frac{5}{2} \\ 2C_1 + 2C_2 = -2 & \\ -C_1 + C_2 = -\frac{1}{2} & \rightarrow C_1 = -\frac{1}{4},\ C_2 = -\frac{3}{4} \end{cases}$$

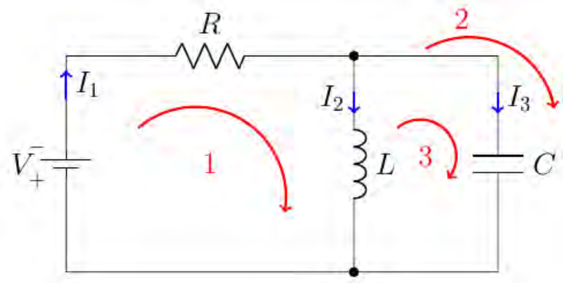
$$\begin{cases} i_1(t) = -\frac{1}{4}e^{-2t} - \frac{9}{4}e^{-2t/3} + \frac{5}{2} \\ i_2(t) = -\frac{1}{2}e^{-2t} - \frac{3}{2}e^{-2t/3} + 2 \\ i_3(t) = \frac{1}{4}e^{-2t} - \frac{3}{4}e^{-2t/3} + \frac{1}{2} \end{cases}$$

Exercise

Find a system of differential equations and determine the charge on the capacitor and the currents in the given network with initial values: $I_1(0)=I_2(0)=I_3(0)=0$

$R=20\ \Omega,\quad L=1\ H,\quad C=\frac{1}{160}\ F,\quad V=5\ V,\quad q(0)=2\ C$

Solution



$$\begin{cases} RI_1+LI_2'=V & (1) \\ RI_1+\frac{1}{C}q=V & (2) \\ -LI_2'+\frac{1}{C}q=0 & (3) \end{cases}$$

$$\begin{cases} 20I_1+I_2'=5 \\ 20I_1+160q=5 \\ -I_2'+160q=0 \end{cases}$$

$I_1=I_2+I_3 \quad (I_3=q')$

$I_1=I_2+q'$

$$\begin{cases} 20I_2+20q'+160q=5 \\ 160q-I_2'=0 \end{cases} \rightarrow \quad I_2=\frac{1}{4}-q'-8q$$

$160q-\left(\frac{1}{4}-q'-8q\right)'=0$

$160q+q''+8q'=0$

$q''+8q'+160q=0$

$\lambda^2+8\lambda+160=0 \quad \rightarrow \quad \lambda_{1,2}=-4\pm 12i$

$q(t)=e^{-4t}\left(C_1\cos 12t+C_2\sin 12t\right)$

$q(0)=2 \quad \rightarrow \quad \underline{C_1=2}$

$q'=e^{-4t}\left(-4C_1\cos 12t-4C_2\sin 12t-12C_1\sin 12t+12C_2\cos 12t\right)$

$q'(0)=I_3(0)=0 \quad \rightarrow \quad -4C_1+12C_2=0 \quad \Rightarrow \quad \underline{C_2=\frac{2}{3}}$

$\underline{q(t)=e^{-4t}\left(2\cos 12t+\frac{2}{3}\sin 12t\right)}$

$I_3(t)=e^{-4t}\left(-8\cos 12t-\frac{8}{3}\sin 12t-24\sin 12t+8\cos 12t\right) \qquad I_3=q'$

$\underline{=-\frac{80}{3}e^{-4t}\sin 12t}$

$I_2(t)=\frac{1}{4}-q'-8q \qquad (I_3=q')$

$$= \frac{1}{4} + \frac{80}{3}e^{-4t}\sin 12t - 8e^{-4t}\left(2\cos 12t + \frac{2}{3}\sin 12t\right)$$

$$\boxed{= \frac{1}{4} + \frac{64}{3}e^{-4t}\sin 12t - 16e^{-4t}\cos 12t}$$

$$I_1(t) = I_2(t) + I_3(t)$$

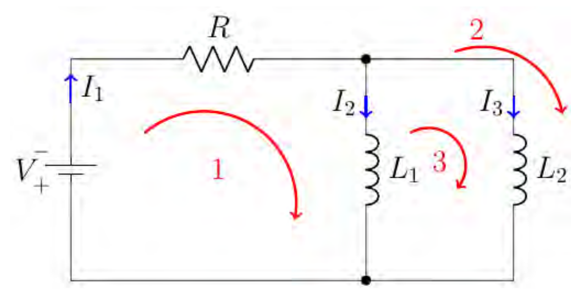
$$\boxed{= \frac{1}{4} - \frac{16}{3}e^{-4t}\sin 12t - 16e^{-4t}\cos 12t}$$

Exercise

Find a system of differential equations and solve for the currents in the given network with initial values:

$I_1(0) = I_2(0) = I_3(0) = 0 \qquad R = 10\ \Omega, \quad L_1 = 0.02\ H, \quad L_2 = 0.025\ H, \quad V = 10\ V$

Solution



$$\begin{cases} RI_1 + L_1I'_2 = V & (1) \\ RI_1 + L_2I'_3 = V & (2) \\ L_2I'_3 - L_1I'_2 = 0 & (3) \end{cases}$$

$$\begin{cases} 10I_1 + 0.02I'_2 = 10 \\ 10I_1 + 0.025I'_3 = 10 \\ 0.025I'_3 - 0.02I'_2 = 0 \end{cases}$$

$$\begin{cases} I'_2 = -500I_1 + 500 \\ I'_3 = -400I_1 + 400 \\ 0.025I'_1 - 0.025I'_2 - 0.02I'_2 = 0 \end{cases}$$

$$I_1 = I_2 + I_3 \quad \to \quad I'_1 = I'_2 + I'_3$$

$$\begin{cases} I'_2 = -500I_1 + 500 \\ I'_3 = -400I_1 + 400 \\ 0.025I'_1 = 0.045\left(-500I_1 + 500\right) \end{cases}$$

$$\begin{cases} I'_1 = -900I_1 + 900 \\ I'_2 = -500I_1 + 500 \\ I'_3 = -400I_1 + 400 \end{cases}$$

$$I'_1 + 900I_1 = 900$$

$$e^{\int 900dt} = e^{900t}$$

$$\int 900e^{900t}dt = e^{900t}$$

$$I_1(t) = e^{-900t}\left(e^{900t} + C_1\right)$$

$$=1+C_1e^{-900t}$$

$$I_1(0)=0 \quad \rightarrow \quad \boxed{C_1=-1}$$

$$I_1(t)=\boxed{1-e^{-900t}}$$

$$I_2'=-500I_1+500$$

$$=500e^{-900t}$$

$$I_2(t)=\int 500e^{-900t}dt$$

$$=-\frac{5}{9}e^{-900t}+C_2$$

$$I_2(0)=0 \quad \rightarrow \quad \boxed{C_2=\frac{5}{9}}$$

$$I_2(t)=\boxed{\frac{5}{9}-\frac{5}{9}e^{-900t}}$$

$$I_3(t)=1-e^{-900t}-\frac{5}{9}+\frac{5}{9}e^{-900t} \qquad I_1=I_2+I_3$$

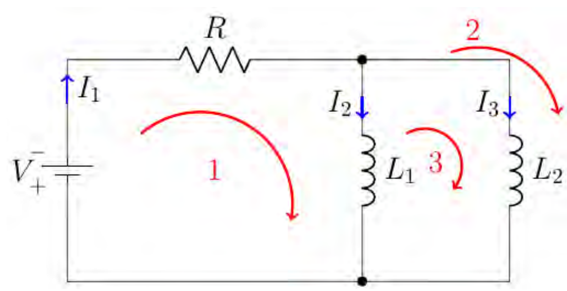
$$=\boxed{\frac{4}{9}-\frac{4}{9}e^{-900t}}$$

Exercise

Find a system of differential equations and solve for the currents in the given network with initial values:

$I_1(0)=I_2(0)=I_3(0)=0 \qquad R=10\ \Omega,\quad L_1=2\ H,\quad L_2=25\ H,\quad V=20\ V$

Solution



$$\begin{cases} RI_1+L_1I_2'=V & (1) \\ RI_1+L_2I_3'=V & (2) \\ L_2I_3'-L_1I_2'=0 & (3) \end{cases}$$

$$\begin{cases} 10I_1+2I_2'=20 \\ 10I_1+25I_3'=20 \\ 25I_3'-2I_2'=0 \end{cases}$$

$$\begin{cases} I_2'=-5I_1+10 \\ I_3'=-\frac{2}{5}I_1+\frac{4}{5} \\ 25I_1'-25I_2'-2I_2'=0 \end{cases} \qquad I_1=I_2+I_3 \quad \rightarrow \quad I_1'=I_2'+I_3'$$

$$\begin{cases} I_1' = -\frac{27}{5}I_1 + \frac{54}{5} \\ I_2' = -5I_1 + 10 \\ I_3' = -\frac{2}{5}I_1 + \frac{4}{5} \end{cases}$$

$$I_1' + \frac{27}{5}I_1 = \frac{54}{5}$$

$$e^{\int \frac{27}{5}dt} = e^{\frac{27}{5}t}$$

$$\int \frac{54}{5}e^{\frac{27}{5}t} = 2e^{\frac{27}{5}t}$$

$$I_1(t) = e^{-\frac{27}{5}t}\left(2e^{\frac{27}{5}t} + C_1\right)$$

$$= 2 + C_1 e^{-\frac{27}{5}t}$$

$$I_1(0) = 0 \quad \rightarrow \quad \boxed{C_1 = -2}$$

$$I_1(t) \boxed{= 2 - 2e^{-\frac{27}{5}t}}$$

$$I_2' = 10e^{-\frac{27}{5}t} \qquad\qquad I_2' = -5I_1 + 10$$

$$I_2(t) = 10\int e^{-\frac{27}{5}t}dt$$

$$= -\frac{50}{27}e^{-\frac{27}{5}t} + C_2$$

$$I_2(0) = 0 \quad \rightarrow \quad \boxed{C_2 = \frac{50}{27}}$$

$$I_2(t) \boxed{= \frac{50}{27} - \frac{50}{27}e^{-\frac{27}{5}t}}$$

$$I_3(t) = 2 - 2e^{-\frac{27}{5}t} - \frac{50}{27} + \frac{50}{27}e^{-\frac{27}{5}t} \qquad\qquad I_3 = I_1 - I_2$$

$$\boxed{= \frac{4}{27} - \frac{4}{27}e^{-\frac{27}{5}t}}$$

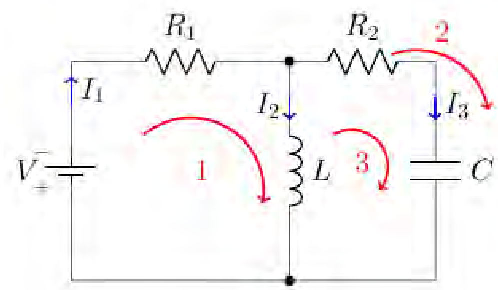
Exercise

Find a system of differential equations and solve for the currents in the given network with initial values:

$I_1(0) = I_2(0) = I_3(0) = 0 \qquad R_1 = 10\ \Omega, \quad R_2 = 5\ \Omega, \quad L = 20\ H, \quad C = \frac{1}{30}\ F, \quad V = 10\ V$

Solution

$$\begin{cases} R_1 I_1 + L I_2' = V & (1) \\ R_1 I_1 + R_2 I_3 + \frac{1}{C} q = V & (2) \\ R_2 I_3 + \frac{1}{C} q - L I_2' = 0 & (3) \end{cases}$$



$$\begin{cases} 10 I_1 + 20 I_2' = 10 \\ 10 I_1 + 5 I_3 + 30 q = 10 \\ 5 I_3 + 30 q - 20 I_2' = 0 \end{cases}$$

$$\begin{cases} I_1 + 2 I_2' = 1 \\ 2 I_1 + I_3 + 6q = 2 \\ I_3 + 6q - 4 I_2' = 0 \end{cases}$$

$$I_1 = I_2 + I_3 \quad (I_3 = q') \quad \rightarrow \quad I_1 = I_2 + q'$$

$$\begin{cases} 2 I_2 + 3q' + 6q = 2 & \rightarrow \ I_2 = \frac{2}{3} - \frac{3}{2} q' - 3q \\ q' + 6q - 4 I_2' = 0 & (4) \end{cases}$$

$$(4) \ \rightarrow \ q' + 6q - 4\left(\frac{2}{3} - \frac{3}{2} q' - 3q\right)' = 0$$

$$6q'' + 13q' + 6q = 0$$

$$6\lambda^2 + 13\lambda + 6 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-13 \pm 5}{12} \quad \underline{\lambda_{1,2} = -\frac{3}{2}, \ -\frac{2}{3}|}$$

$$q(t) = C_1 e^{-3t/2} + C_2 e^{-2t/3}$$

$$2 I_2(0) + 3q'(0) + 6q(0) = 2 \quad \rightarrow \quad q(0) = \frac{1}{3}$$

$$q(0) = \frac{1}{3} \quad \rightarrow \quad C_1 + C_2 = \frac{1}{3}$$

$$q'(t) = -\frac{3}{2} C_1 e^{-3t/2} - \frac{2}{3} C_2 e^{-2t/3}$$

$$q'(0) = I_3(0) = 0 \quad \rightarrow \quad -\frac{3}{2} C_1 - \frac{2}{3} C_2 = 0 \quad \Rightarrow \quad 9C_1 + 4C_2 = 0$$

$$\begin{cases} 3C_1 + 3C_2 = 1 \\ 9C_1 + 4C_2 = 0 \end{cases} \qquad \Delta = \begin{vmatrix} 3 & 3 \\ 9 & 4 \end{vmatrix} = -15 \quad \Delta_1 = \begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = 4 \quad \Delta_2 = \begin{vmatrix} 3 & 1 \\ 9 & 0 \end{vmatrix} = -9$$

$$\underline{C_1 = -\frac{4}{15}, \quad C_2 = \frac{3}{5}|}$$

$$\boxed{q(t) = -\frac{4}{15}e^{-3t/2} + \frac{3}{5}e^{-2t/3}}$$

$$\boxed{I_3(t) = \frac{2}{5}e^{-3t/2} - \frac{2}{5}e^{-2t/3}} \qquad I_3 = q'$$

$$2I_1 + I_3 + 6q = 2$$

$$I_1(t) = 1 - 3q - \frac{1}{2}I_3$$

$$= 1 + \frac{4}{5}e^{-3t/2} - \frac{9}{5}e^{-2t/3} - \frac{1}{5}e^{-3t/2} + \frac{1}{5}e^{-2t/3}$$

$$\boxed{= 1 + \frac{3}{5}e^{-3t/2} - \frac{8}{5}e^{-2t/3}}$$

$$I_2(t) = I_1 - I_3$$

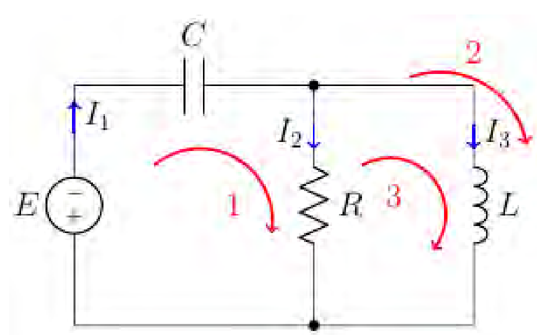
$$\boxed{= 1 + \frac{1}{5}e^{-3t/2} - \frac{6}{5}e^{-2t/3}}$$

Exercise

Find a system of differential equations and solve for the currents in the given network with initial values:

$I_1(0) = I_2(0) = I_3(0) = 0 \qquad R = 1\,\Omega, \quad L = 0.5\ H, \quad C = 0.5\ F, \quad E = \cos 3t\ V$

Solution



$$\begin{cases} \frac{1}{C}q + RI_2 = E & (1) \\ \frac{1}{C}q + LI_3' = E & (2) \\ LI_3' - RI_2 = 0 & (3) \end{cases}$$

$$\begin{cases} 2q + I_2 = \cos 3t \\ 2q + \frac{1}{2}I_3' = \cos 3t \\ \frac{1}{2}I_3' - I_2 = 0 \end{cases} \qquad I_1 = I_2 + I_3 = q'$$

$$\begin{cases} 2q + \frac{1}{2}I_3' = \cos 3t & \rightarrow \quad q = \frac{1}{2}\cos 3t - \frac{1}{4}I_3' \\ \frac{1}{2}I_3' - q' + I_3 = 0 \end{cases}$$

$$\frac{1}{2}I_3' - \left(\frac{1}{2}\cos 3t - \frac{1}{4}I_3'\right)' + I_3 = 0$$

$$\frac{1}{2}I_3' + \frac{3}{2}\sin 3t + \frac{1}{4}I_3'' + I_3 = 0$$

$$I_3'' + 2I_3' + 4I_3 = -6\sin 3t$$

$$\lambda^2 + 2\lambda + 4 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -1 \pm \sqrt{3}}$$

$$\boxed{I_h = \left(C_1 \cos\sqrt{3}t + C_2 \sin\sqrt{3}t\right)e^{-t}}$$

$$I_p = A\cos 3t + B\sin 3t$$

$$I'_p = -3A\sin 3t + 3B\cos 3t$$

$$I''_p = -9A\cos 3t - 9B\sin 3t$$

$$-9A\cos 3t - 9B\sin 3t - 6A\sin 3t + 6B\cos 3t + 4A\cos 3t + 4B\sin 3t = -6\sin 3t$$

$$\begin{cases} \cos 3t & -5A + 6B = 0 \\ \sin 3t & -6A - 5B = -6 \end{cases} \qquad \boxed{A = \frac{36}{61}, \quad B = \frac{30}{61}}$$

$$I_3(t) = \left(C_1 \cos\sqrt{3}t + C_2 \sin\sqrt{3}t\right)e^{-t} + \frac{36}{61}\cos 3t + \frac{30}{61}\sin 3t$$

$$I_3(0) = 0 \quad \rightarrow \quad \boxed{C_1 = -\frac{36}{61}}$$

$$I'_3(t) = \left(-C_1 \cos\sqrt{3}t - C_2 \sin\sqrt{3}t - C_1\sqrt{3}\sin\sqrt{3}t + C_2\sqrt{3}\cos\sqrt{3}t\right)e^{-t} - \frac{108}{61}\sin 3t + \frac{90}{61}\cos 3t$$

$$I'_3(0) = 0 \quad \rightarrow \quad -C_1 + \sqrt{3}C_2 + \frac{90}{61} = 0 \quad \Rightarrow \quad \boxed{C_2 = -\frac{126}{61\sqrt{3}}}$$

$$\boxed{I_3(t) = \left(-\frac{36}{61}\cos\sqrt{3}t - \frac{42\sqrt{3}}{61}\sin\sqrt{3}t\right)e^{-t} + \frac{36}{61}\cos 3t + \frac{30}{61}\sin 3t}$$

$$\frac{1}{2}I'_3 - I_2 = 0$$

$$I_2(t) = \frac{1}{2}I'_3$$

$$= \left(\frac{18}{61}\cos\sqrt{3}t + \frac{21\sqrt{3}}{61}\sin\sqrt{3}t + \frac{18\sqrt{3}}{61}\sin\sqrt{3}t - \frac{63}{61}\cos\sqrt{3}t\right)e^{-t} - \frac{54}{61}\sin 3t + \frac{45}{61}\cos 3t$$

$$\boxed{= \left(\frac{39\sqrt{3}}{61}\sin\sqrt{3}t - \frac{45}{61}\cos\sqrt{3}t\right)e^{-t} - \frac{54}{61}\sin 3t + \frac{45}{61}\cos 3t}$$

$$I_1(t) = I_2 + I_3$$

$$\boxed{= \left(-\frac{3\sqrt{3}}{61}\sin\sqrt{3}t - \frac{81}{61}\cos\sqrt{3}t\right)e^{-t} - \frac{14}{61}\sin 3t + \frac{81}{61}\cos 3t}$$

Exercise

Derive three equations for the unknown currents $I_1, I_2,$ *and* I_3 with thegiven values of the given electric circuit shown below, then find the general solution

$R_1 = R_2 = 1\,\Omega$, $C = 1\,F$, and $L = 1\,H$.

Solution

Applying Kirchhoff's voltage law to Loops 1 and 2.

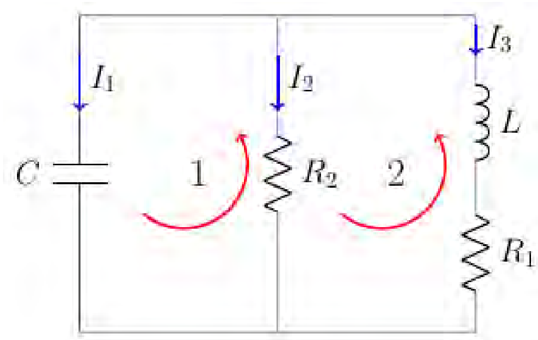
Loop 1: $\dfrac{q}{C} - R_2 I_2 = 0$

Loop 2: $R_2 I_2 - R_1 I_3 - L I_3' = 0$

$-I_1 - I_2 - I_3 = 0 \quad \Rightarrow \quad I_1 + I_2 + I_3 = 0$

$I_1 = \dfrac{dq}{dt}$

$R_1 = R_2 = 1\ \Omega$, $C = 1\ F$, and $L = 1\ H$



$$\begin{cases} q = I_2 & (1) \\ I_2 = I_3 + I_3' & (2) \\ q' + I_2 + I_3 = 0 & (3) \end{cases}$$

$(1) \ \rightarrow \ q' = I_2'$

$$\begin{cases} (3) & I_2' = -I_2 - I_3 \\ (2) & I_3' = I_2 - I_3 \end{cases}$$

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$= \lambda^2 + 2\lambda + 2 = 0$$

The eigenvalues are: $\lambda_{1,2} = -1 \pm i$

For $\lambda_1 = -1 + i \quad \Rightarrow \quad (A + \lambda_1 I) V_1 = 0$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a_1 = i b_1 \quad \rightarrow \quad V_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$z(t) = \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(-1+i)t}$$

$$= \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) (\cos t + i \sin t) e^{-t}$$

$$= \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t + i \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t \right) \right) e^{-t}$$

$$= \left(\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \right) e^{-t}$$

$$I_h = C_1 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^{-t}$$

$$\boxed{\begin{cases} I_2(t) = (-C_1 \sin t + C_2 \cos t) e^{-t} \\ I_3(t) = (C_1 \cos t + C_2 \sin t) e^{-t} \end{cases}}$$

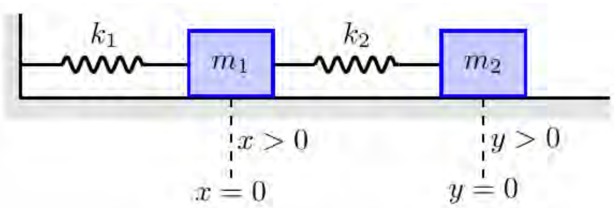
$$I_1(t) = -I_2(t) - I_3(t)$$

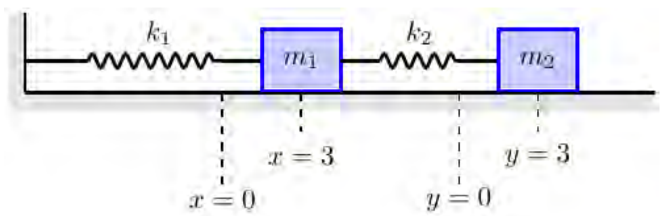
$$= \left(C_1 \sin t - C_2 \cos t - C_1 \cos t - C_2 \sin t\right)e^{-t}$$

$$\boxed{= \left(\left(C_1 - C_2\right)\sin t - \left(C_1 + C_2\right)\cos t\right)e^{-t}}$$

Exercise

On a smooth horizontal surface $m_1 = 2\ kg$ is attached to a fixed wall by a spring with spring constant $k_1 = 4\ N/m$. Another mass $m_2 = 1\ kg$ is attached to the first object by a spring with spring constant $k_2 = 2\ N/m$. The object are aligned horizontally so that the springs are their natural lengths. If both objects are displaced 3 m to the right of their equilibrium positions and then released, what are the equations of motion for the two objects?





Solution

Applying Hooke's law:

$$F_1 = -k_1 x$$

$$F_2 = k_2(y - x)$$

$$F_3 = -k_2(y - x)$$

Applying Newton's second law:

$$\begin{cases} m_1 x'' = -k_1 x + k_2(y - x) & (1) \\ m_2 y'' = -k_2(y - x) & (2) \end{cases}$$

$$\begin{cases} m_1 x'' = -\left(k_1 + k_2\right)x + k_2 y \\ m_2 y'' = k_2 x - k_2 y \end{cases}$$

Given: $m_1 = 2\ kg$, $m_2 = 1\ kg$, $k_1 = 4\ N/m$, and $k_2 = 2\ N/m$

$$\begin{cases} 2x'' = -6x + 2y \\ y'' = 2x - 2y \end{cases}$$

$$\begin{cases} x'' = -3x + y \\ y'' = 2x - 2y \end{cases}$$

$$\left|A - \lambda^2 I\right| = \begin{vmatrix} -3 - \lambda^2 & 1 \\ 2 & -2 - \lambda^2 \end{vmatrix} \qquad A = \begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix}$$

$$= \lambda^4 + 5\lambda^2 + 4 = 0 \quad \rightarrow \quad \lambda^2 = -1, \ -4$$

The eigenvalues are: $\lambda_{1,2} = \pm i \quad \lambda_{3,4} = \pm 2i$

Using Euler's formula

$z_1(t) = e^{it} = \cos t + i\sin t \quad \& \quad z_2(t) = e^{2it} = \cos 2t + i\sin 2t$

$x(t) = C_1\cos t + C_2\sin t + C_3\cos 2t + C_4\sin 2t$

Given: $x(0) = 3 \quad x'(0) = 0$

$$x(0) = C_1 + C_3 = 3 \quad (3)$$

$$x'(t) = -C_1\sin t + C_2\cos t - 2C_3\sin 2t + 2C_4\cos 2t$$

$$x'(0) = C_2 + 2C_4 = 0 \quad (4)$$

$$\begin{cases} x'' = -3x + y \\ y'' = 2x - 2y \end{cases} \rightarrow \quad y = x'' + 3x$$

$$y(t) = -C_1\cos t - C_2\sin t - 4C_3\cos 2t - 4C_4\sin 2t + 3C_1\cos t + 3C_2\sin t + 3C_3\cos 2t + 3C_4\sin 2t$$

$$= 2C_1\cos t + 2C_2\sin t - C_3\cos 2t - C_4\sin 2t$$

Given: $y(0) = 3 \quad y'(0) = 0$

$$y(0) = 2C_1 - C_3 = 3 \quad (5)$$

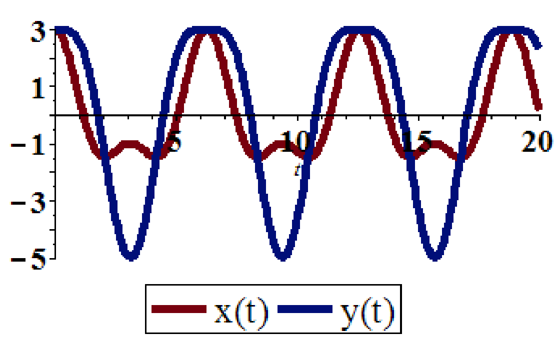
$$y'(t) = -2C_1\sin t + 2C_2\cos t + 2C_3\sin 2t - 2C_4\cos 2t$$

$$y'(0) = 2C_2 - 2C_4 = 0 \quad (6)$$

$$\begin{cases} (3) \quad C_1 + C_3 = 3 \\ (5) \quad 2C_1 - C_3 = 3 \end{cases} \rightarrow \quad \boxed{C_1 = 2, \ C_3 = 1}$$

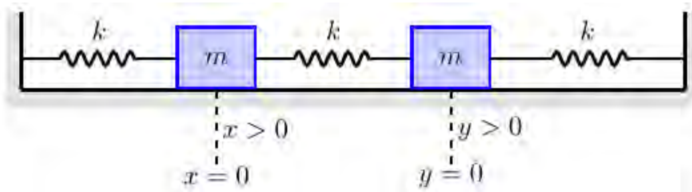
$$\begin{cases} (4) \quad C_2 + 2C_4 = 0 \\ (6) \quad 2C_2 - C_4 = 0 \end{cases} \rightarrow \quad \boxed{C_2 = C_4 = 0}$$

$$\boxed{\begin{cases} x(t) = 2\cos t + \cos 2t \\ y(t) = 4\cos t - \cos 2t \end{cases}}$$



Exercise

Three identical springs with spring constant k and two identical masses m are attached in a straight line with the ends of the outside springs fixed.



a) Determine and interpret the normal modes of the system.

b) Given the values $m = 2\ kg$, and $k = 2\ N/m$ with initial value $x(0)=1,\quad x'(0)=0,\quad y(0)=1,\quad y'(0)=0$. what are the equations of motion for the two objects?

c) Given the values $m = 2\ kg$, and $k = 2\ N/m$ with initial value $x(0)=1,\quad x'(0)=0,\quad y(0)=-1,\quad y'(0)=0$. what are the equations of motion for the two objects?

d) Given the values $m = 2\ kg$, and $k = 2\ N/m$ with initial value $x(0)=1,\quad x'(0)=0,\quad y(0)=2,\quad y'(0)=0$. what are the equations of motion for the two objects?

Solution

a) Applying Newton's second law:

$$\begin{cases} mx'' = -kx + k(y-x) & (1) \\ my'' = -k(y-x) - ky & (2) \end{cases}$$

$$\begin{cases} mx'' = -2kx + ky \\ my'' = kx - 2ky \end{cases}$$

$$\begin{cases} x'' = -2\frac{k}{m}x + \frac{k}{m}y \\ y'' = \frac{k}{m}x - 2\frac{k}{m}y \end{cases}$$

$$\left|A - \lambda^2 I\right| = \begin{vmatrix} -2\frac{k}{m} - \lambda^2 & \frac{k}{m} \\ \frac{k}{m} & -2\frac{k}{m} - \lambda^2 \end{vmatrix} \qquad A = \begin{pmatrix} -2\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -2\frac{k}{m} \end{pmatrix}$$

$$= \lambda^4 + 4\frac{k}{m}\lambda^2 + 3\frac{k^2}{m^2} = 0 \quad \to \quad \lambda^2 = -2\frac{k}{m} \pm \frac{k}{m} = -3\frac{k}{m},\ -\frac{k}{m}$$

The eigenvalues are: $\lambda_{1,2} = \pm\omega i\sqrt{3} \quad \lambda_{3,4} = \pm\omega i \qquad \omega = \sqrt{\frac{k}{m}}$

Using Euler's formula

$$z_1(t) = e^{\omega i t\sqrt{3}} = \cos\omega\sqrt{3}t + i\sin\omega\sqrt{3}t \quad \& \quad z_2(t) = e^{\omega i t} = \cos\omega t + i\sin\omega t$$

$$x(t) = C_1\cos\omega\sqrt{3}t + C_2\sin\omega\sqrt{3}t + C_3\cos\omega t + C_4\sin\omega t$$

$$x'' = -2\frac{k}{m}x + \frac{k}{m}y \quad \to \quad y = \frac{m}{k}x'' + 2x$$

$$y(t) = -3C_1 \cos \omega\sqrt{3}t - 3C_2 \sin \omega\sqrt{3}t - C_3 \cos \omega t - C_4 \sin \omega t$$
$$+ 2C_1 \cos \omega\sqrt{3}t + 2C_2 \sin \omega\sqrt{3}t + 2C_3 \cos \omega t + 2C_4 \sin \omega t$$

$$\frac{m}{k}\omega^2 = \frac{m}{k}\frac{k}{m} = 1$$

$$y(t) = -C_1 \cos \omega\sqrt{3}t - C_2 \sin \omega\sqrt{3}t + C_3 \cos \omega t + C_4 \sin \omega t$$

$$\begin{cases} x(t) = C_1 \cos\left(\omega\sqrt{3}\right)t + C_2 \sin\left(\omega\sqrt{3}\right)t + C_3 \cos \omega t + C_4 \sin \omega t \\ y(t) = -C_1 \cos\left(\omega\sqrt{3}\right)t - C_2 \sin\left(\omega\sqrt{3}\right)t + C_3 \cos \omega t + C_4 \sin \omega t \end{cases}$$

The normal angular frequencies are ω and $\sqrt{3}\,\omega$.

If we let $C_1 = C_2 = 0$, that implies $x(t) = y(t)$, where oscillating at the angular frequency $\omega = \sqrt{\frac{k}{m}}$. So, the two masses are moving as if they are a single block of mass $2m$, forced by a *double spring* with a spring constant given by $2k$.

If $C_3 = C_4 = 0$, that implies $x(t) = -y(t)$. Which are two mirror-image systems, each with a mass m and a *spring and a half* with spring constant $k + 2k = 3k$. (The half-spring will be twice as stiff.)

b) ***Given***: $m = 2\ kg$, and $k = 2\ N/m$ $\qquad \omega = \sqrt{\frac{k}{m}} = 1$

$$x(0) = 1, \quad x'(0) = 0, \quad y(0) = 1, \quad y'(0) = 0$$

$$x(0) = C_1 + C_3 = 1 \quad (3)$$

$$x'(t) = -\sqrt{3}C_1 \sin \sqrt{3}t + \sqrt{3}C_2 \cos \sqrt{3}t - C_3 \sin t + C_4 \cos t$$

$$x'(0) = \sqrt{3}C_2 + C_4 = 0 \quad (4)$$

$$y(t) = -C_1 \cos \sqrt{3}t - C_2 \sin \sqrt{3}t + C_3 \cos t + C_4 \sin t$$

$$y(0) = -C_1 + C_3 = 1 \quad (5)$$

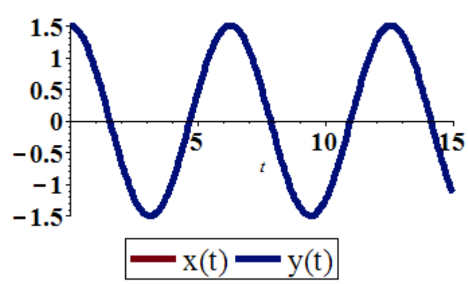
$$y'(t) = \sqrt{3}C_1 \sin \sqrt{3}t - \sqrt{3}C_2 \cos \sqrt{3}t - C_3 \sin t + C_4 \cos t$$

$$y'(0) = -\sqrt{3}C_2 + C_4 = 0 \quad (6)$$

$$\begin{cases} (3) & C_1 + C_3 = 1 \\ (5) & -C_1 + C_3 = 1 \end{cases} \rightarrow \underline{C_1 = 0,\ C_3 = 1|}$$

$$\begin{cases} (4) & \sqrt{3}C_2 + C_4 = 0 \\ (6) & -\sqrt{3}C_2 + C_4 = 0 \end{cases} \rightarrow \underline{C_2 = C_4 = 0|}$$

$$\begin{cases} x(t) = \frac{3}{2}\cos t \\ y(t) = \frac{3}{2}\cos t \end{cases}$$



c) ***Given***: $m = 2\ kg$, and $k = 2\ N/m$ $\qquad \omega = \sqrt{\frac{k}{m}} = 1$

$x(0) = 1, \quad x'(0) = 0, \quad y(0) = -1, \quad y'(0) = 0$

$x(0) = C_1 + C_3 = 1 \quad (3)$

$$x'(t) = -\sqrt{3}C_1 \sin\sqrt{3}t + \sqrt{3}C_2 \cos\sqrt{3}t - C_3 \sin t + C_4 \cos t$$

$x'(0) = \sqrt{3}C_2 + C_4 = 0 \quad (4)$

$y(t) = -C_1 \cos\sqrt{3}t - C_2 \sin\sqrt{3}t + C_3 \cos t + C_4 \sin t$

$y(0) = -C_1 + C_3 = -1 \quad (5)$

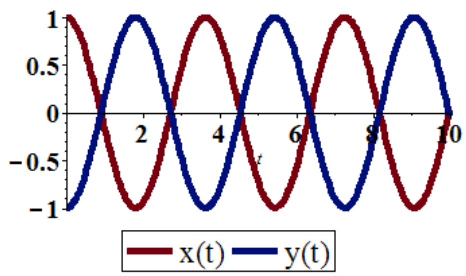
$$y'(t) = \sqrt{3}C_1 \sin\sqrt{3}t - \sqrt{3}C_2 \cos\sqrt{3}t - C_3 \sin t + C_4 \cos t$$

$y'(0) = -\sqrt{3}C_2 + C_4 = 0 \quad (6)$

$$\begin{cases} (3) & C_1 + C_3 = 1 \\ (5) & -C_1 + C_3 = -1 \end{cases} \rightarrow \underline{C_1 = 1, \ C_3 = 0|}$$

$$\begin{cases} (4) & \sqrt{3}C_2 + C_4 = 0 \\ (6) & -\sqrt{3}C_2 + C_4 = 0 \end{cases} \rightarrow \underline{C_2 = C_4 = 0|}$$

$$\begin{cases} x(t) = \cos\left(\sqrt{3}\right)t \\ y(t) = -\cos\left(\sqrt{3}\right)t \end{cases}$$



d) ***Given***: $m = 2\ kg$, and $k = 2\ N/m$ $\qquad \omega = \sqrt{\frac{k}{m}} = 1$

$x(0) = 1, \quad x'(0) = 0, \quad y(0) = 2, \quad y'(0) = 0$

$x(0) = C_1 + C_3 = 1 \quad (3)$

$$x'(t) = -\sqrt{3}C_1 \sin\sqrt{3}t + \sqrt{3}C_2 \cos\sqrt{3}t - C_3 \sin t + C_4 \cos t$$

$x'(0) = \sqrt{3}C_2 + C_4 = 0 \quad (4)$

$y(t) = -C_1 \cos\sqrt{3}t - C_2 \sin\sqrt{3}t + C_3 \cos t + C_4 \sin t$

$y(0) = -C_1 + C_3 = 2 \quad (5)$

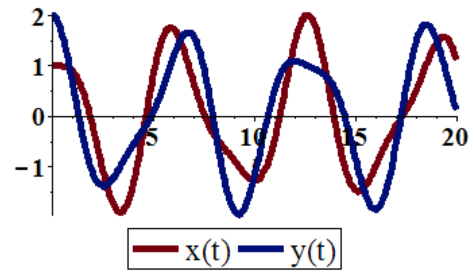
$$y'(t) = \sqrt{3}C_1 \sin\sqrt{3}t - \sqrt{3}C_2 \cos\sqrt{3}t - C_3 \sin t + C_4 \cos t$$

$y'(0) = -\sqrt{3}C_2 + C_4 = 0 \quad (6)$

$$\begin{cases} (3) & C_1 + C_3 = 1 \\ (5) & -C_1 + C_3 = 2 \end{cases} \rightarrow \underline{C_1 = -\frac{1}{2},\ C_3 = \frac{3}{2}}$$

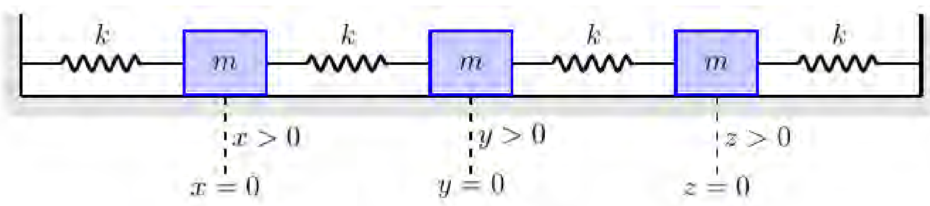
$$\begin{cases} (4) & \sqrt{3}C_2 + C_4 = 0 \\ (6) & -\sqrt{3}C_2 + C_4 = 0 \end{cases} \rightarrow \underline{C_2 = C_4 = 0}$$

$$\begin{cases} x(t) = -\frac{1}{2}\cos\left(\sqrt{3}\right)t + \frac{3}{2}\cos t \\ y(t) = \frac{1}{2}\cos\left(\sqrt{3}\right)t + \frac{3}{2}\cos t \end{cases}$$



Exercise

Four springs with the same spring constant and three equal masses are attached in a straight line on a horizontal frictionless surface.



a) What are the equations of motion for the three objects?

b) Determine the normal frequencies for the system, describe the three normal modes of vibration.

Solution

a)
$$\begin{cases} mx'' = -kx + k(y-x) \\ my'' = -k(y-x) + k(z-y) \\ mz'' = -k(z-y) - kz \end{cases}$$

$$\begin{cases} mx'' = -2kx + ky \\ my'' = kx - 2ky + kz \\ mz'' = ky - 2kz \end{cases}$$

$$\begin{cases} x'' = -2\frac{k}{m}x + \frac{k}{m}y & (1) \\ y'' = \frac{k}{m}x - 2\frac{k}{m}y + \frac{k}{m}z & (2) \\ z'' = \frac{k}{m}y - 2\frac{k}{m}z & (3) \end{cases}$$

$$\left|A - \lambda^2 I\right| = \begin{vmatrix} -2\frac{k}{m} - \lambda^2 & \frac{k}{m} & 0 \\ \frac{k}{m} & -2\frac{k}{m} - \lambda^2 & \frac{k}{m} \\ 0 & \frac{k}{m} & -2\frac{k}{m} - \lambda^2 \end{vmatrix} \qquad A = \begin{pmatrix} -2\frac{k}{m} & \frac{k}{m} & 0 \\ \frac{k}{m} & -2\frac{k}{m} & \frac{k}{m} \\ 0 & \frac{k}{m} & -2\frac{k}{m} \end{pmatrix}$$

$$= -\left(\frac{2k}{m} + \lambda^2\right)^3 + 2\left(\frac{k}{m}\right)^2\left(\frac{2k}{m} + \lambda^2\right)$$

$$= -\left(\frac{2k}{m} + \lambda^2\right)\left(\lambda^4 + 4\frac{k}{m}\lambda^2 + 4\left(\frac{k}{m}\right)^2 - 2\left(\frac{k}{m}\right)^2\right) \qquad \omega = \sqrt{\frac{k}{m}}$$

$$= \left(2\omega^2 + \lambda^2\right)\left(\lambda^4 + 4\omega^2\lambda^2 + 2\omega^4\right) = 0$$

$$\lambda^4 + 4\omega^2\lambda^2 + 2\omega^4 = 0 \rightarrow \lambda^2 = -2\omega^2 \pm 2\omega^2\sqrt{2}$$

The eigenvalues are: $\lambda_{1,2} = \pm\omega i\sqrt{2}$ $\quad \lambda_{3,4} = \pm i\omega\sqrt{2+\sqrt{2}}$ $\quad \lambda_{3,4} = \pm i\omega\sqrt{2-\sqrt{2}}$

Using Euler's formula

$$\begin{cases} Z_1(t) = e^{\omega it\sqrt{2}} = \cos\left(\omega\sqrt{2}\right)t + i\sin\left(\omega\sqrt{2}\right)t \\ Z_2(t) = e^{\omega i\sqrt{2+\sqrt{2}}t} = \cos\omega\sqrt{2+\sqrt{2}}t + i\sin\omega\sqrt{2+\sqrt{2}}t \\ Z_3(t) = e^{\omega i\sqrt{2-\sqrt{2}}t} = \cos\omega\sqrt{2-\sqrt{2}}t + i\sin\omega\sqrt{2-\sqrt{2}}t \end{cases}$$

$$x(t) = C_1\cos\omega\sqrt{2}t + C_2\sin\omega\sqrt{2}t + C_3\cos\omega\sqrt{2+\sqrt{2}}t + C_4\sin\omega\sqrt{2+\sqrt{2}}t$$
$$+ C_5\cos\omega\sqrt{2-\sqrt{2}}t + C_6\sin\omega\sqrt{2-\sqrt{2}}t$$

$$x'' = -2\omega^2C_1\cos\omega\sqrt{2}t - 2\omega^2C_2\sin\omega\sqrt{2}t$$
$$-\left(2+\sqrt{2}\right)\omega^2C_3\cos\omega\sqrt{2+\sqrt{2}}t - \left(2+\sqrt{2}\right)\omega^2C_4\sin\omega\sqrt{2+\sqrt{2}}t$$
$$-\left(2-\sqrt{2}\right)\omega^2C_5\cos\omega\sqrt{2-\sqrt{2}}t - \left(2-\sqrt{2}\right)\omega^2C_6\sin\omega\sqrt{2-\sqrt{2}}t$$

$$(1) \quad y = \frac{1}{\omega^2}x'' + 2x \qquad \omega^2 = \frac{k}{m}$$

$$y(t) = -\sqrt{2}C_3\cos\omega\sqrt{2+\sqrt{2}}t - \sqrt{2}C_4\sin\omega\sqrt{2+\sqrt{2}}t$$
$$+ \sqrt{2}C_5\cos\omega\sqrt{2-\sqrt{2}}t + \sqrt{2}C_6\sin\omega\sqrt{2-\sqrt{2}}t$$

$$y'' = 2\left(1+\sqrt{2}\right)\omega^2C_3\cos\omega\sqrt{2+\sqrt{2}}t + 2\left(1+\sqrt{2}\right)\omega^2C_4\sin\omega\sqrt{2+\sqrt{2}}t$$
$$-2\left(1+\sqrt{2}\right)\omega^2C_5\cos\omega\sqrt{2-\sqrt{2}}t - 2\left(1+\sqrt{2}\right)\omega^2C_6\sin\omega\sqrt{2-\sqrt{2}}t$$

$$(2) \quad z = \frac{1}{\omega^2}y'' - x + 2y$$

$$z(t) = -C_1\cos\omega\sqrt{2}t - C_2\sin\omega\sqrt{2}t + C_3\cos\omega\sqrt{2+\sqrt{2}}t + C_4\sin\omega\sqrt{2+\sqrt{2}}t$$
$$+ C_5\cos\omega\sqrt{2-\sqrt{2}}t + C_6\sin\omega\sqrt{2-\sqrt{2}}t$$

b) If we let $C_3 = C_4 = C_5 = C_6 = 0$, that has the mode $x(t) = -z(t)$ & $y(t) \equiv 0$

The normal frequency: $\dfrac{\omega\sqrt{2}}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{2k}{m}}$

If we let $C_1 = C_2 = C_5 = C_6 = 0$, that has the mode $x(t) = z(t) = -\frac{1}{\sqrt{2}} y(t)$

The normal frequency: $\frac{\omega\sqrt{2+\sqrt{2}}}{2\pi} = \frac{1}{2\pi}\sqrt{\left(2+\sqrt{2}\right)\frac{k}{m}}$

If we let $C_1 = C_2 = C_3 = C_4 = 0$, that has the mode $x(t) = z(t) = \frac{1}{\sqrt{2}} y(t)$

The normal frequency: $\frac{\omega\sqrt{2-\sqrt{2}}}{2\pi} = \frac{1}{2\pi}\sqrt{\left(2-\sqrt{2}\right)\frac{k}{m}}$

Exercise

Two springs and two masses are attached in a straight line on a horizontal frictionless surface. The system is set in motion by holding the mass m_2 at it equilibrium position and pulling the mass m_1 to the left of its equilibrium position a distance 1 *m* and them releasing both masses.

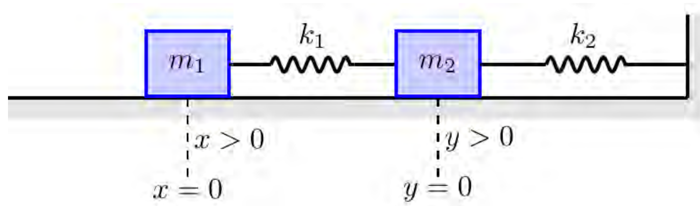
a) Express Newton's law for the system and determine the equations of motion for the two masses if $m_1 = 1\ kg$, $m_2 = 2\ kg$, $k_1 = 4\ N/m$, and $k_2 = \frac{10}{3}\ N/m$

b) Express Newton's law for the system and determine the equations of motion for the two masses if $m_1 = 1\ kg$, $m_2 = 1\ kg$, $k_1 = 3\ N/m$, and $k_2 = 2\ N/m$

Solution

Applying Newton's second law:

$$\begin{cases} m_1 x'' = k_1(y-x) \\ m_2 y'' = -k_1(y-x) - k_2 y \end{cases}$$



$$\begin{cases} m_1 x'' = -k_1 x + k_1 y \\ m_2 y'' = k_1 x - (k_1 + k_2) y \end{cases}$$

Given: $x(0) = -1, \quad x'(0) = 0, \quad y(0) = 0, \quad y'(0) = 0$

a) ***Given***: $m_1 = 1\ kg$, $m_2 = 2\ kg$, $k_1 = 4\ N/m$, and $k_2 = \frac{10}{3}\ N/m$

$$\begin{cases} x'' = -4x + 4y \\ 2y'' = 4x - \frac{22}{3} y \end{cases}$$

$$\begin{cases} x'' = -4x + 4y & (1) \\ y'' = 2x - \frac{11}{3} y & (2) \end{cases}$$

$$\left|A - \lambda^2 I\right| = \begin{vmatrix} -4-\lambda^2 & 4 \\ 2 & -\frac{11}{3} - \lambda^2 \end{vmatrix} \qquad A = \begin{pmatrix} -4 & 4 \\ 2 & -\frac{11}{3} \end{pmatrix}$$

$$= \lambda^4 + \frac{11}{3}\lambda^2 + 4\lambda^2 + \frac{44}{3} - 8$$

$$= \lambda^4 + \frac{23}{3}\lambda^2 + \frac{20}{3} = 0 \quad \rightarrow \quad 3\lambda^4 + 23\lambda^2 + 20 = 0$$

$$\boxed{\lambda^2 = \frac{-23 \pm 17}{6}}$$

The eigenvalues are: $\lambda_{1,2} = \pm i\sqrt{\frac{20}{3}} \quad \lambda_{3,4} = \pm i$

Using Euler's formula

$$z_1(t) = e^{it\sqrt{\frac{20}{3}}} = \cos\sqrt{\frac{20}{3}}t + i\sin\sqrt{\frac{20}{3}}t \quad \& \quad z_2(t) = e^{it} = \cos t + i\sin t$$

$$x(t) = C_1\cos\sqrt{\frac{20}{3}}t + C_2\sin\sqrt{\frac{20}{3}}t + C_3\cos t + C_4\sin t$$

Given: $x(0) = -1, \quad x'(0) = 0$

$$x(0) = C_1 + C_3 = -1 \quad (3)$$

$$x'(t) = -C_1\sqrt{\frac{20}{3}}\sin\sqrt{\frac{20}{3}}t + C_2\sqrt{\frac{20}{3}}\cos\sqrt{\frac{20}{3}}t - C_3\sin t + C_4\cos t$$

$$x'(0) = \sqrt{\frac{20}{3}}C_2 + C_4 = 0 \quad (4)$$

$$(1) \quad \rightarrow \quad y = \tfrac{1}{4}x'' + x$$

$$y(t) = -\tfrac{5}{3}C_1\cos\sqrt{\frac{20}{3}}t - \tfrac{5}{3}C_2\sin\sqrt{\frac{20}{3}}t - \tfrac{1}{4}C_3\cos t - \tfrac{1}{4}C_4\sin t$$
$$+ C_1\cos\sqrt{\frac{20}{3}}t + C_2\sin\sqrt{\frac{20}{3}}t + C_3\cos t + C_4\sin t$$

$$y(t) = -\tfrac{2}{3}C_1\cos\sqrt{\frac{20}{3}}t - \tfrac{2}{3}C_2\sin\sqrt{\frac{20}{3}}t + \tfrac{3}{4}C_3\cos t + \tfrac{3}{4}C_4\sin t$$

Given: $y(0) = 0, \quad y'(0) = 0$

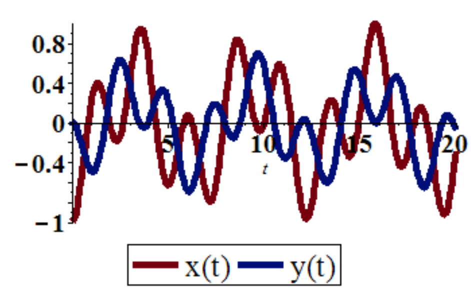
$$y(0) = -\tfrac{2}{3}C_1 + \tfrac{3}{4}C_3 = 0 \quad (5)$$

$$y' = \tfrac{2}{3}\sqrt{\frac{20}{3}}C_1\sin\left(\sqrt{\frac{20}{3}}t\right) - \tfrac{2}{3}\sqrt{\frac{20}{3}}C_2\cos\left(\sqrt{\frac{20}{3}}t\right) - \tfrac{3}{4}C_3\sin t + \tfrac{3}{4}C_4\cos t$$

$$y'(0) = -\tfrac{2}{3}\sqrt{\frac{20}{3}}C_2 + \tfrac{3}{4}C_4 = 0 \quad (6)$$

$$\begin{array}{ll} (3) & C_1 + C_3 = -1 \\ (5) & -8C_1 + 9C_3 = 0 \end{array} \quad \rightarrow \quad \boxed{C_1 = -\frac{9}{17}, \ C_3 = -\frac{8}{17}}$$

$$\begin{array}{ll} (4) & \sqrt{\frac{20}{3}}C_2 + C_4 = 0 \\ (6) & -\tfrac{2}{3}C_2 + \tfrac{3}{4}C_4 = 0 \end{array} \quad \rightarrow \quad \boxed{C_2 = C_4 = 0}$$



$$\boxed{\begin{cases} x(t) = -\frac{9}{17}\cos\sqrt{\frac{20}{3}}t - \frac{8}{17}\cos t \\ y(t) = \frac{6}{17}\cos\sqrt{\frac{20}{3}}t - \frac{6}{17}\cos t \end{cases}}$$

b) ***Given***: $m_1 = m_2 = 1, \quad k_1 = 3, \quad k_2 = 2$

$$\begin{cases} x'' = -3x + 3y & (7) \\ y'' = 3x - 5y & (8) \end{cases}$$

$$\left|A - \lambda^2 I\right| = \begin{vmatrix} -3-\lambda^2 & 3 \\ 3 & -5-\lambda^2 \end{vmatrix} \qquad A = \begin{pmatrix} -3 & 3 \\ 3 & -5 \end{pmatrix}$$

$$= \lambda^4 + 8\lambda^2 + 6 = 0 \quad \rightarrow \quad \underline{\lambda^2 = -4 \pm \sqrt{10}}$$

The eigenvalues are: $\lambda_{1,2} = \pm\left(4+\sqrt{10}\right)i \quad \lambda_{3,4} = \pm\left(4-\sqrt{10}\right)i$

Using Euler's formula

$$z_1(t) = \cos\left(4+\sqrt{10}\right)t + i\sin\left(4+\sqrt{10}\right)t \quad \& \quad z_2(t) = \cos\left(4-\sqrt{10}\right)t + i\left(4-\sqrt{10}\right)\sin t$$

$$x(t) = C_1\cos\left(4+\sqrt{10}\right)t + C_2\sin\left(4+\sqrt{10}\right)t + C_3\cos\left(4-\sqrt{10}\right)t + C_4\sin\left(4-\sqrt{10}\right)t$$

Given: $x(0) = -1, \quad x'(0) = 0$

$$x(0) = C_1 + C_3 = -1 \quad (9)$$

$$\begin{aligned} x(t) = &-\left(4+\sqrt{10}\right)C_1\sin\left(4+\sqrt{10}\right)t + \left(4+\sqrt{10}\right)C_2\cos\left(4+\sqrt{10}\right)t \\ &-\left(4-\sqrt{10}\right)C_3\sin\left(4-\sqrt{10}\right)t + \left(4-\sqrt{10}\right)C_4\cos\left(4-\sqrt{10}\right)t \end{aligned}$$

$$x'(0) = \left(4+\sqrt{10}\right)C_2 + \left(4-\sqrt{10}\right)C_4 = 0 \quad (10)$$

$$(7) \quad \rightarrow \quad y = \tfrac{1}{3}x'' + x$$

$$\begin{aligned} y(t) = &-\tfrac{1}{3}\left(4+\sqrt{10}\right)C_1\cos\left(4+\sqrt{10}\right)t - \tfrac{1}{3}\left(4+\sqrt{10}\right)C_2\sin\left(4+\sqrt{10}\right)t \\ &-\tfrac{1}{3}\left(4-\sqrt{10}\right)C_3\cos\left(4-\sqrt{10}\right)t - \tfrac{1}{3}\left(4-\sqrt{10}\right)C_4\sin\left(4-\sqrt{10}\right)t \\ &+C_1\cos\left(4+\sqrt{10}\right)t + C_2\sin\left(4+\sqrt{10}\right)t + C_3\cos\left(4-\sqrt{10}\right)t + C_4\sin\left(4-\sqrt{10}\right)t \end{aligned}$$

$$\begin{aligned} y(t) = &-\tfrac{1}{3}\left(1+\sqrt{10}\right)C_1\cos\left(4+\sqrt{10}\right)t - \tfrac{1}{3}\left(1+\sqrt{10}\right)C_2\sin\left(4+\sqrt{10}\right)t \\ &-\tfrac{1}{3}\left(1-\sqrt{10}\right)C_3\cos\left(4-\sqrt{10}\right)t - \tfrac{1}{3}\left(1-\sqrt{10}\right)C_4\sin\left(4-\sqrt{10}\right)t \end{aligned}$$

Given: $y(0) = 0, \quad y'(0) = 0$

$$y(0) = -\tfrac{1}{3}\left(1+\sqrt{10}\right)C_1 - \tfrac{1}{3}\left(1-\sqrt{10}\right)C_3 = 0 \quad (11)$$

$$\begin{aligned} y' = &\tfrac{1}{3}\left(1+\sqrt{10}\right)\left(4+\sqrt{10}\right)C_1\sin\left(4+\sqrt{10}\right)t - \tfrac{1}{3}\left(1+\sqrt{10}\right)\left(4+\sqrt{10}\right)C_2\cos\left(4+\sqrt{10}\right)t \\ &+\tfrac{1}{3}\left(1-\sqrt{10}\right)\left(4+\sqrt{10}\right)C_3\sin\left(4-\sqrt{10}\right)t - \tfrac{1}{3}\left(1-\sqrt{10}\right)\left(4+\sqrt{10}\right)C_4\cos\left(4-\sqrt{10}\right)t \end{aligned}$$

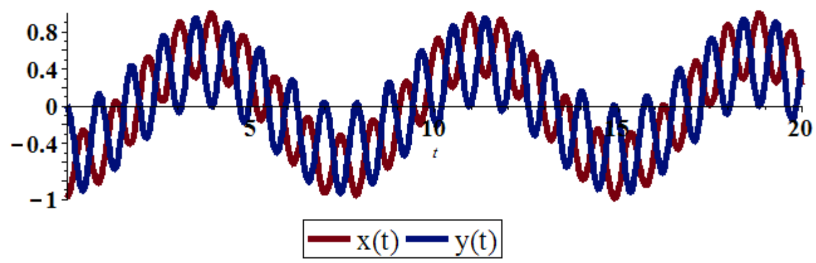
$$y'(0) = -\tfrac{1}{3}\left(14+5\sqrt{10}\right)C_2 + \tfrac{1}{3}\left(6+3\sqrt{10}\right)C_4 = 0 \quad (12)$$

$$\begin{array}{ll} (9) & C_1 + C_3 = -1 \\ (11) & \left(1+\sqrt{10}\right)C_1 + \left(1-\sqrt{10}\right)C_3 = 0 \end{array} \quad \rightarrow \quad \underline{C_1 = \frac{1-\sqrt{10}}{2\sqrt{10}}, \quad C_3 = -\frac{1+\sqrt{10}}{2\sqrt{10}}}$$

$$(10) \qquad \left(4+\sqrt{10}\right)C_2 + \left(4-\sqrt{10}\right)C_4 = 0$$

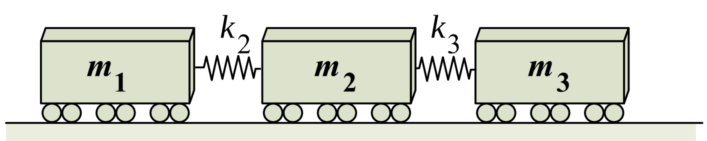
$$(12) \qquad -\tfrac{1}{3}\left(14+5\sqrt{10}\right)C_2 + \tfrac{1}{3}\left(6+3\sqrt{10}\right)C_4 = 0 \qquad \rightarrow \quad \underline{C_2 = C_4 = 0}$$

$$\begin{cases} x(t) = \dfrac{1-\sqrt{10}}{2\sqrt{10}}\cos\left(4+\sqrt{10}\right)t - \dfrac{1+\sqrt{10}}{2\sqrt{10}}\cos\left(4-\sqrt{10}\right)t \\ y(t) = \dfrac{3}{2\sqrt{10}}\cos\left(4+\sqrt{10}\right)t - \dfrac{3}{2\sqrt{10}}\cos\left(4-\sqrt{10}\right)t \end{cases}$$



Exercise

Three railway cars are connected by buffer springs that react when compressed, but disengage instead of stretching.



Given that $k_2 = k_3 = k = 3000\ lb/ft$ and $m_1 = m_3 = 750\ lbs$ *and* $m_2 = 500\ lbs$

Suppose that the leftmost car is moving to the right with velocity v_0 and at time $t = 0$ strikes the other 2 cars. The corresponding initial conditions are:

$$x_1(0) = x_2(0) = x_3(0) = 0$$

$$x_1'(0) = v_0 \quad x_2'(0) = x_3'(0) = 0$$

Solution

$$m_1 x_1'' = k_2\left(x_2 - x_1\right)$$

$$m_2 x_2'' = -k_2\left(x_2 - x_1\right) + k_3\left(x_3 - x_2\right)$$

$$m_3 x_3'' = -k_3\left(x_3 - x_2\right)$$

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}\vec{x}'' = \begin{pmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{pmatrix}\vec{x}$$

$$\begin{pmatrix} 750 & 0 & 0 \\ 0 & 500 & 0 \\ 0 & 0 & 750 \end{pmatrix}\vec{x}'' = \begin{pmatrix} -3000 & 3000 & 0 \\ 3000 & -6000 & 3000 \\ 0 & 3000 & -3000 \end{pmatrix}\vec{x} \qquad \begin{pmatrix} 750 & 0 & 0 \\ 0 & 500 & 0 \\ 0 & 0 & 750 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{750} & 0 & 0 \\ 0 & \frac{1}{500} & 0 \\ 0 & 0 & \frac{1}{750} \end{pmatrix}$$

$$\vec{x}'' = \begin{pmatrix} \frac{1}{750} & 0 & 0 \\ 0 & \frac{1}{500} & 0 \\ 0 & 0 & \frac{1}{750} \end{pmatrix}\begin{pmatrix} -3000 & 3000 & 0 \\ 3000 & -6000 & 3000 \\ 0 & 3000 & -3000 \end{pmatrix}\vec{x}$$

$$= \begin{pmatrix} -4 & 4 & 0 \\ 6 & -12 & 6 \\ 0 & 4 & -4 \end{pmatrix}\vec{x} \qquad A = \begin{pmatrix} -4 & 4 & 0 \\ 6 & -12 & 6 \\ 0 & 4 & -4 \end{pmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} -4-\lambda & 4 & 0 \\ 6 & -12-\lambda & 6 \\ 0 & 4 & -4-\lambda \end{vmatrix}$$

$$= (-4-\lambda)^2(-12-\lambda) - 24(-4-\lambda) - 24(-4-\lambda)$$

$$= (-4-\lambda)\left[48+16\lambda+\lambda^2-48\right]$$

$$= \lambda(-4-\lambda)(\lambda+16) = 0$$

The eigenvalues are: $\lambda_1 = 0 \rightarrow \omega_1 = 0, \quad \lambda_2 = -4 \rightarrow \omega_2 = 2, \quad \lambda_3 = -16 \rightarrow \omega_3 = 4$

For $\lambda_1 = 0 \ (\omega_1 = 0) \ \Rightarrow \ (A - 0I)V_1 = 0$

$$\begin{pmatrix} -4 & 4 & 0 \\ 6 & -12 & 6 \\ 0 & 4 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} a=b \\ \\ b=c \end{matrix} \rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1(t) = (a_1 + b_1 t)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -4 \ (\omega_2 = 2) \ \Rightarrow \ (A + 4I)V_2 = 0$

$$\begin{pmatrix} 0 & 4 & 0 \\ 6 & -8 & 6 \\ 0 & 4 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} b=0 \\ a=-c \end{matrix} \rightarrow V_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \vec{x}_2(t) = (a_2\cos 2t + b_2\sin 2t)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For $\lambda_3 = -16 \ (\omega_3 = 4) \ \Rightarrow \ (A + 16I)V_3 = 0$

$$\begin{pmatrix} 12 & 4 & 0 \\ 6 & 4 & 6 \\ 0 & 4 & 12 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 3a=-b \\ \\ b=-3c \end{matrix} \rightarrow V_3 = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_3(t) = (a_3\cos 4t + b_3\sin 4t)\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$$

$$\vec{x}(t) = a_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b_1 t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + a_2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\cos 2t + b_2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\sin 2t + a_3\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}\cos 4t + b_3\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}\sin 4t$$

$$\begin{cases} \vec{x}_1(t) = a_1 + b_1 t + a_2\cos 2t + b_2 \sin 2t + a_3\cos 4t + b_3 \sin 4t \\ \vec{x}_2(t) = a_1 + b_1 t - 3a_3\cos 4t - 3b_3\sin 4t \\ \vec{x}_3(t) = a_1 + b_1 t - a_2\cos 2t - b_2\sin 2t + a_3\cos 4t + b_3\sin 4t \end{cases}$$

Applying the initial values

$$\begin{aligned} &\vec{x}_1(0) = a_1 + a_2 + a_3 = 0 \\ &\vec{x}_2(0) = a_1 - 3a_3 = 0 \qquad\qquad a_1 = 3a_3 \qquad\qquad \Rightarrow \boxed{a_1 = a_2 = a_3 = 0} \\ &\vec{x}_3(0) = a_1 - a_2 + a_3 = 0 \quad (1)\,\&\,(3) \to 2a_1 + 2a_3 = 0 \end{aligned}$$

$$\begin{cases} \vec{x}_1(t) = b_1 t + b_2\sin 2t + b_3\sin 4t \\ \vec{x}_2(t) = b_1 t - 3b_3\sin 4t \\ \vec{x}_3(t) = b_1 t - b_2\sin 2t + b_3 \sin 4t \end{cases}$$

$$\begin{cases} \vec{x}_1'(t) = b_1 + 2b_2\cos 2t + 4b_3\cos 4t \\ \vec{x}_2'(t) = b_1 - 12b_3\cos 4t \\ \vec{x}_3'(t) = b_1 - 2b_2\cos 2t + 4b_3\cos 4t \end{cases}$$

$$\begin{cases} \vec{x}_1'(0) = b_1 + 2b_2 + 4b_3 = v_0 \\ \vec{x}_2'(0) = b_1 \qquad - 12b_3 = 0 \\ \vec{x}_3'(0) = b_1 - 2b_2 + 4b_3 = 0 \end{cases} \to \begin{aligned} &\\ &b_1 = 12b_3 \\ &2b_2 = 16b_3 \end{aligned} \to \quad 12b_3 + 16b_3 + 4b_3 = v_0 \quad \Rightarrow \begin{aligned} &b_3 = \tfrac{1}{32}v_0 \\ &b_1 = \tfrac{3}{8}v_0 \\ &b_2 = \tfrac{1}{4}v_0 \end{aligned}$$

$$\begin{cases} \vec{x}_1(t) = \frac{1}{32}v_0(12t + 8\sin 2t + \sin 4t) \\ \vec{x}_2(t) = \frac{1}{32}v_0(12t \qquad - 3\sin 4t) \\ \vec{x}_3(t) = \frac{1}{32}v_0(12t - 8\sin 2t + \sin 4t) \end{cases} \qquad \begin{cases} \vec{x}_1'(t) = \frac{1}{32}v_0(12 + 16\cos 2t + 4\cos 4t) \\ \vec{x}_2'(t) = \frac{1}{32}v_0(12 - 12\cos 4t) \\ \vec{x}_3'(t) = \frac{1}{32}v_0(12 - 16\cos 2t + 4\cos 4t) \end{cases}$$

For these equations to hold, only when the 2 buffer springs remain compressed; that is, while both

$$x_2 - x_1 < 0 \quad \textit{and} \quad x_3 - x_2 < 0$$

$$\begin{aligned} x_2(t) - x_1(t) &= \tfrac{1}{32}v_0(12t - 3\sin 4t) - \tfrac{1}{32}v_0(12t + 8\sin 2t + \sin 4t) \\ &= \tfrac{1}{32}v_0(-8\sin 2t - 4\sin 4t) \\ &= -\tfrac{1}{8}v_0(2\sin 2t + 2\sin 2t\cos 2t) \\ &= -\tfrac{1}{4}v_0\sin 2t(1 + \cos 2t) < 0 \end{aligned}$$

$$\sin 2t = 0 \Rightarrow (2t = 0, \pi) \to \boxed{t = 0, \ \tfrac{\pi}{2}} \qquad \cos 2t = -1 \to (2t = \pi) \to \boxed{t = \tfrac{\pi}{2}}$$

$$x_2 - x_1 < 0 \ \Rightarrow \ t \in \left(0, \ \tfrac{\pi}{2}\right)$$

$$x_3(t)-x_2(t)=\frac{1}{32}v_0(12t-8\sin 2t+\sin 4t)-\frac{1}{32}v_0(12t-3\sin 4t)$$
$$=\frac{1}{32}v_0(-8\sin 2t+4\sin 4t)$$
$$=-\frac{1}{8}v_0(2\sin 2t-2\sin 2t\cos 2t)$$
$$=-\frac{1}{4}v_0(\sin 2t)(1-\cos 2t)<0$$

$$\sin 2t=0\Rightarrow(2t=0,\pi)\rightarrow \underline{t=0,\ \frac{\pi}{2}}\quad \cos 2t=1\rightarrow(2t=0)\rightarrow \underline{t=0}$$

$$x_3-x_2<0\ \Rightarrow\ t\in\left(0,\ \frac{\pi}{2}\right)$$

$x_2-x_1<0$ *and* $x_3-x_2<0$ until $\underline{t=\frac{\pi}{2}\approx 1.57\ sec}$

$$x_1\left(\frac{\pi}{2}\right)=x_2\left(\frac{\pi}{2}\right)=x_3\left(\frac{\pi}{2}\right)=\frac{1}{32}v_0\left(12\frac{\pi}{2}\right)\ \underline{=\frac{3\pi}{16}v_0}$$

$$x_1'\left(\frac{\pi}{2}\right)=x_2'\left(\frac{\pi}{2}\right)=0\quad x_3'\left(\frac{\pi}{2}\right)=\frac{1}{32}v_0(32)\ \underline{=v_0}$$

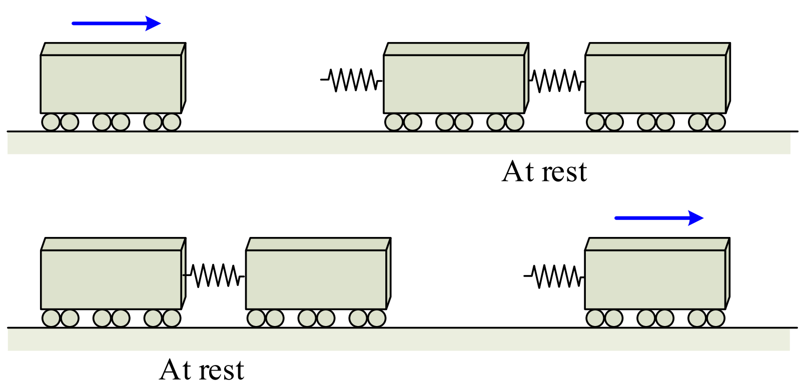
We conclude that the 3 railway cars remain engaged and moving to the right until disengagement occurs at time $t=\frac{\pi}{2}$.

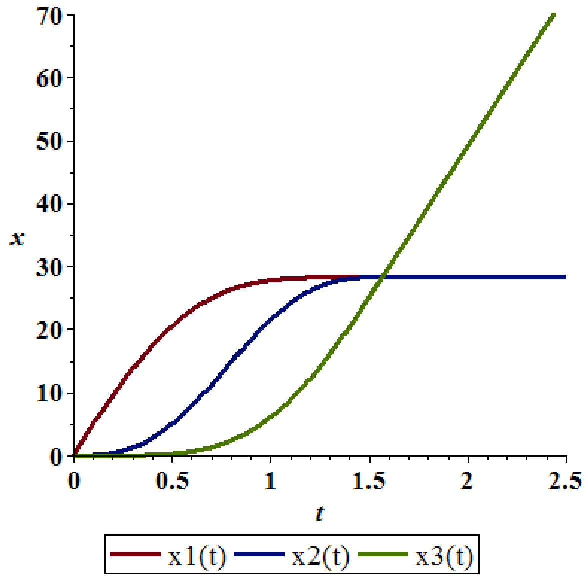
At $t>\frac{\pi}{2}$

$$x_1(t)=x_2(t)=\frac{3\pi}{16}v_0$$

$$\frac{3\pi}{16}v_0=v_0\left(\frac{\pi}{2}-\beta\right)\ \rightarrow\ \beta=\frac{\pi}{2}-\frac{3\pi}{16}=\frac{5\pi}{16}$$

$$x_3(t)=v_0\left(t-\frac{5\pi}{16}\right)\ \underline{=v_0t-\frac{5\pi}{16}v_0}$$





Exercise

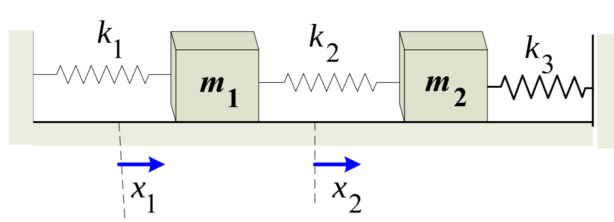
Consider the mass-and-spring system shown below and with the given masses and spring constants values. Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

$$m_1 = m_2 = 1; \quad k_1 = 0,\ k_2 = 2,\ k_3 = 0 \quad (no\ walls)$$

Solution

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 \end{cases} \Rightarrow \begin{cases} x_1'' = -2x_1 + 2x_2 \\ x_2'' = 2x_1 - 2x_2 \end{cases}$$

$$x'' = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\vec{x} \quad \to \quad A = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$



$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}$$
$$= (-2-\lambda)^2 - 4$$
$$= \lambda^2 + 4\lambda = 0$$

The eigenvalues are: $\lambda_1 = 0, \quad \lambda_2 = -4$

The natural frequencies: $\omega_1 = 0 \quad and \quad \omega_2 = \sqrt{-(-4)} = 2$

For $\lambda_1 = 0 \quad \Rightarrow \quad (A - 0I)V_1 = 0$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = b \ \to \ V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \vec{x}_1(t) = (a_1 + b_1 t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -4 \quad \Rightarrow \quad (A + 4I)V_2 = 0$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = -b \quad \rightarrow V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \Rightarrow \vec{x}_2(t) = \left(a_2 \cos 2t + b_2 \sin 2t\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1 + b_1 t + a_2 \cos 2t + b_2 \sin 2t \\ \vec{x}_2(t) = a_1 + b_1 t - a_2 \cos 2t - b_2 \sin 2t \end{cases}$$

In the degenerate natural mode with frequency $\omega_1 = 0$ the 2 masses move by translation without oscillating. At frequency $\omega_2 = 2$ they oscillate in opposite directions with equal amplitudes.

Exercise

Consider the mass-and-spring system shown below and with the given masses and spring constants values. Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

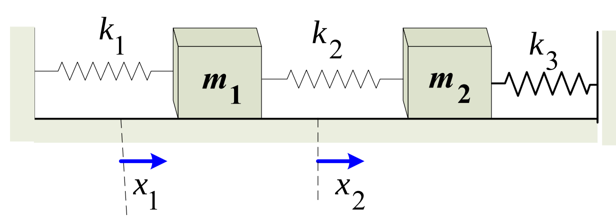
$$m_1 = m_2 = 1; \quad k_1 = 1,\ k_2 = 2,\ k_3 = 1$$

Solution

$$\begin{cases} m_1 x_1'' = -\left(k_1 + k_2\right) x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - \left(k_2 + k_3\right) x_2 \end{cases} \Rightarrow \begin{cases} x_1'' = -3x_1 + 2x_2 \\ x_2'' = 2x_1 - 3x_2 \end{cases}$$

$$x'' = \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}\vec{x} \quad \rightarrow \quad A = \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -3-\lambda & 2 \\ 2 & -3-\lambda \end{vmatrix} \\ &= (-3-\lambda)^2 - 4 \\ &= \lambda^2 + 4\lambda + 5 = 0 \end{aligned}$$



The eigenvalues are: $\lambda_1 = -1, \quad \lambda_2 = -5$

The natural frequencies: $\omega_1 = 1 \quad and \quad \omega_2 = \sqrt{5}$

For $\lambda_1 = -1 \quad \Rightarrow \quad (A + I)V_1 = 0$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = b \quad \rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \vec{x}_1(t) = \left(a_1 \cos t + b_1 \sin t\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -5 \quad \Rightarrow \quad (A + 5I)V_2 = 0$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = -b \quad \rightarrow V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \Rightarrow \vec{x}_2(t) = \left(a_2 \cos t\sqrt{5} + b_2 \sin t\sqrt{5}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1 \cos t + b_1 \sin t + a_2 \cos t\sqrt{5} + b_2 \sin t\sqrt{5} \\ \vec{x}_2(t) = a_1 \cos t + b_1 \sin t - a_2 \cos t\sqrt{5} - b_2 \sin t\sqrt{5} \end{cases}$$

In the degenerate natural mode with frequency $\omega_1 = 1$ the 2 masses move in the same direction with equal amplitudes of oscillation. At frequency $\omega_2 = \sqrt{5}$ they oscillate in opposite directions with equal amplitudes.

Exercise

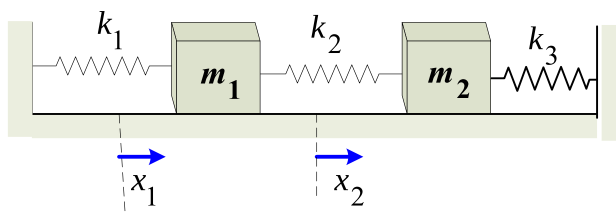
Consider the mass-and-spring system shown below and with the given masses and spring constants values. Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

$$m_1 = m_2 = 1; \quad k_1 = 2,\ k_2 = 1,\ k_3 = 2$$

Solution

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 \end{cases} \Rightarrow \begin{cases} x_1'' = -3x_1 + x_2 \\ x_2'' = x_1 - 3x_2 \end{cases}$$

$$x'' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\vec{x} \quad \rightarrow \quad A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$$



$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} \\ &= (-3-\lambda)^2 - 1 \\ &= \lambda^2 + 4\lambda + 8 = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = -2, \quad \lambda_2 = -4$

The natural frequencies: $\omega_1 = \sqrt{2}$ *and* $\omega_2 = 2$

For $\lambda_1 = -2 \Rightarrow (A + 2I)V_1 = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = b \quad \rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1(t) = \left(a_1 \cos t\sqrt{2} + b_1 \sin t\sqrt{2}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -4 \Rightarrow (A + 4I)V_2 = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = -b \quad \rightarrow V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \vec{x}_2(t) = \left(a_2 \cos 2t + b_2 \sin 2t\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1 \cos t\sqrt{2} + b_1 \sin t\sqrt{2} + a_2 \cos 2t + b_2 \sin 2t \\ \vec{x}_2(t) = a_1 \cos t\sqrt{2} + b_1 \sin t\sqrt{2} - a_2 \cos 2t - b_2 \sin 2t \end{cases}$$

In the degenerate natural mode with frequency $\omega_1 = \sqrt{2}$ the 2 masses move in the same direction with equal amplitudes of oscillation. At frequency $\omega_2 = 2$ they oscillate in opposite directions with equal amplitudes.

Exercise

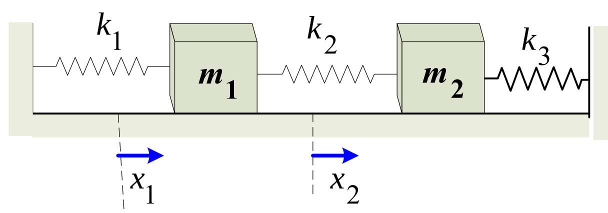
Consider the mass-and-spring system shown below and with the given masses and spring constants values. Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

$$m_1 = 1,\ m_2 = 2;\quad k_1 = 2,\ k_2 = k_3 = 4$$

Solution

$$\begin{cases} m_1 x_1'' = -\left(k_1 + k_2\right) x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - \left(k_2 + k_3\right) x_2 \end{cases} \Rightarrow \begin{cases} x_1'' = -6x_1 + 4x_2 \\ 2x_2'' = 4x_1 - 8x_2 \end{cases} \to \begin{cases} x_1'' = -6x_1 + 4x_2 \\ x_2'' = 2x_1 - 4x_2 \end{cases}$$

$$x'' = \begin{pmatrix} -6 & 4 \\ 2 & -4 \end{pmatrix} \vec{x} \quad \to \quad A = \begin{pmatrix} -6 & 4 \\ 2 & -4 \end{pmatrix}$$



$$\left|A - \lambda I\right| = \begin{vmatrix} -6-\lambda & 4 \\ 2 & -4-\lambda \end{vmatrix}$$

$$= \left(-6-\lambda\right)\left(-4-\lambda\right) - 8$$

$$= \lambda^2 + 10\lambda + 16 = 0$$

The eigenvalues are: $\lambda_1 = -2, \quad \lambda_2 = -8$

The natural frequencies: $\omega_1 = \sqrt{2} \quad and \quad \omega_2 = 2\sqrt{2}$

For $\lambda_1 = -2 \quad \Rightarrow \quad \left(A + 2I\right)V_1 = 0$

$$\begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = b$$

$$\to \ V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \vec{x}_1(t) = \left(a_1 \cos t\sqrt{2} + b_1 \sin t\sqrt{2}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -8 \quad \Rightarrow \quad \left(A + 8I\right)V_2 = 0$

$$\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = -2b$$

$$\to \ V_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \Rightarrow \vec{x}_2(t) = \left(a_2 \cos t\sqrt{8} + b_2 \sin t\sqrt{8}\right)\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1 \cos t\sqrt{2} + b_1 \sin t\sqrt{2} + 2a_2 \cos t\sqrt{8} + 2b_2 \sin t\sqrt{8} \\ \vec{x}_2(t) = a_1 \cos t\sqrt{2} + b_1 \sin t\sqrt{2} - a_2 \cos t\sqrt{8} - b_2 \sin t\sqrt{8} \end{cases}$$

In the degenerate natural mode with frequency $\omega_1 = \sqrt{2}$ the 2 masses move in the same direction with equal amplitudes of oscillation. At frequency $\omega_2 = \sqrt{8}$ they oscillate in opposite directions with amplitude of oscillation of m_1 twice that of m_2.

Exercise

Consider the mass-and-spring system shown below and with the given masses and spring constants values. The mass-and-spring system is set in motion from rest $x_1'(0) = x_2'(0) = 0$ in its equilibrium position $x_1(0) = x_2(0) = 0$.

Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

For the given external forces $F_1(t)$ and $F_2(t)$ acting on the masses m_1 and m_2, respectively.

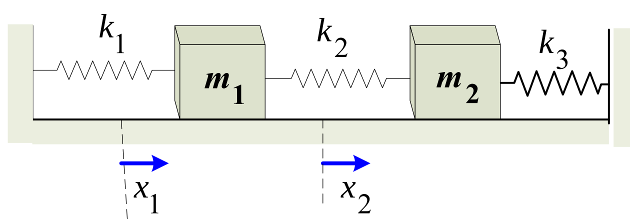
Find the resulting motion of the system and describe it as a superposition of oscillations at three different frequencies.

$$m_1 = m_2 = 1; \quad k_1 = 1,\ k_2 = 4,\ k_3 = 1 \quad F_1(t) = 96\cos 5t,\ F_2(t) = 0$$

Solution

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 + 96\cos 5t \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 \end{cases} \Rightarrow \begin{cases} x_1'' = -5x_1 + 4x_2 + 96\cos 5t \\ x_2'' = 4x_1 - 5x_2 \end{cases}$$

$$A = \begin{pmatrix} -5 & 4 \\ 4 & -5 \end{pmatrix}$$



$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -5-\lambda & 4 \\ 4 & -5-\lambda \end{vmatrix} \\ &= (-5-\lambda)^2 - 16 \\ &= \lambda^2 + 10\lambda + 9 = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = -1, \quad \lambda_2 = -9$

The natural frequencies: $\omega_1 = 1 \quad \omega_2 = 3 \quad \omega_3 = 5$

For $\lambda_1 = -1 \quad \Rightarrow \quad (A + I)V_1 = 0$

$$\begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = b \quad \to V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1(t) = \left(a_1 \cos t + b_1 \sin t\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -9 \quad \Rightarrow \quad (A + 9I)V_2 = 0$

$$\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = -b \quad \to V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \vec{x}_2(t) = \left(a_2 \cos 3t + b_2 \sin 3t\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1 \cos t + b_1 \sin t + a_2 \cos 3t + b_2 \sin 3t + c_1 \cos 5t \\ \vec{x}_2(t) = a_1 \cos t + b_1 \sin t - a_2 \cos 3t - b_2 \sin 3t + c_2 \cos 5t \end{cases}$$

$$\begin{cases} \vec{x}_1''(t) = -a_1 \cos t - b_1 \sin t - 9a_2 \cos 3t - 9b_2 \sin 3t - 25c_1 \cos 5t \\ \vec{x}_2''(t) = -a_1 \cos t - b_1 \sin t + 9a_2 \cos 3t + 9b_2 \sin 3t - 25c_2 \cos 5t \end{cases}$$

$$x_1'' = -5x_1 + 4x_2 + 96\cos 5t$$

$$-a_1 \cos t - b_1 \sin t - 9a_2 \cos 3t - 9b_2 \sin 3t - 25c_1 \cos 5t =$$

$$-5a_1\cos t - 5b_1\sin t - 5a_2\cos 3t - 5b_2\sin 3t - 5c_1\cos 5t$$
$$+4a_1\cos t + 4b_1\sin t - 4a_2\cos 3t - 4b_2\sin 3t + 4c_2\cos 5t + 96\cos 5t$$
$$-25c_1\cos 5t = -5c_1\cos 5t + 4c_2\cos 5t + 96\cos 5t$$
$$-20c_1 - 4c_2 = 96 \;\rightarrow\; \underline{5c_1 + c_2 = -24}$$

$$x_2'' = 4x_1 - 5x_2$$
$$-25c_2\cos 5t = 4c_1\cos 5t - 5c_2\cos 5t \qquad \rightarrow\; \underline{c_1 = -5c_2}$$
$$5(-5c_2) + c_2 = -24 \;\Rightarrow\; \underline{c_2 = 1,\; c_1 = -5}$$

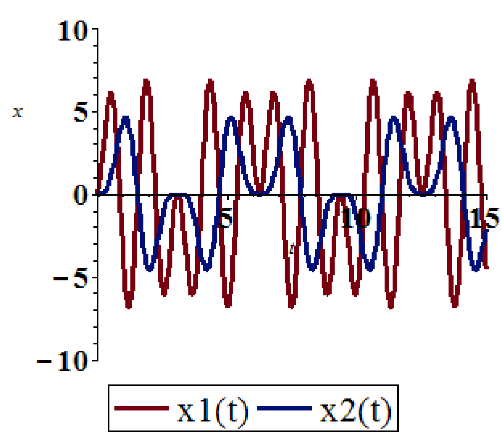
$$\begin{cases} \vec{x}_1(t) = a_1\cos t + b_1\sin t + a_2\cos 3t + b_2\sin 3t - 5\cos 5t \\ \vec{x}_2(t) = a_1\cos t + b_1\sin t - a_2\cos 3t - b_2\sin 3t + \cos 5t \end{cases}$$

Given initial values: $x_1'(0) = x_2'(0) = 0$ and $x_1(0) = x_2(0) = 0$.

$$\begin{cases} \vec{x}_1(0) = a_1 + a_2 - 5 = 0 \\ \vec{x}_2(0) = a_1 - a_2 + 1 = 0 \end{cases} \rightarrow\; \underline{a_1 = 2,\; a_2 = 3}$$

$$\begin{cases} \vec{x}_1'(0) = b_1 + 3b_2 = 0 \\ \vec{x}_2'(0) = b_1 - 3b_2 = 0 \end{cases} \rightarrow\; \underline{b_1 = b_2 = 0}$$

$$\boxed{\begin{cases} \vec{x}_1(t) = 2\cos t + 3\cos 3t - 5\cos 5t \\ \vec{x}_2(t) = 2\cos t - 3\cos 3t + \cos 5t \end{cases}}$$



At frequency $\omega_1 = 1$ the 2 masses move in the same direction with equal amplitudes of oscillation.

At frequency $\omega_2 = 3$ the 2 masses move in the opposite direction with equal amplitudes of oscillation.

At frequency $\omega_3 = 5$ they oscillate in opposite directions with amplitude of oscillation of m_1 **5** times that of m_2.

Exercise

Consider the mass-and-spring system shown below and with the given masses and spring constants values. The mass-and-spring system is set in motion from rest $x_1'(0) = x_2'(0) = 0$ in its equilibrium position $x_1(0) = x_2(0) = 0$.

Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

For the given external forces $F_1(t)$ and $F_2(t)$ acting on the masses m_1 and m_2, respectively.

Find the resulting motion of the system and describe it as a superposition of oscillations at three different frequencies.

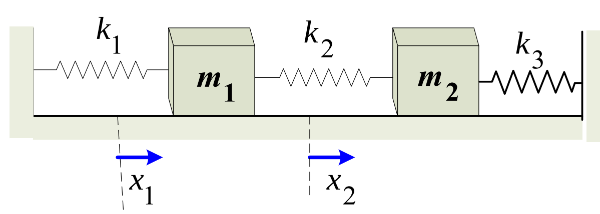
$$m_1 = 1,\ m_2 = 2;\quad k_1 = 1,\ k_2 = k_3 = 2;\quad F_1(t) = 0,\ F_2(t) = 120\cos 3t$$

Solution

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 + 120\cos 3t \end{cases}$$

$$\begin{cases} x_1'' = -3x_1 + 2x_2 \\ 2x_2'' = 2x_1 - 4x_2 + 120\cos 3t \end{cases}$$

$$\to \begin{cases} x_1'' = -3x_1 + 2x_2 \\ x_2'' = x_1 - 2x_2 + 60\cos 3t \end{cases} \qquad A = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}$$



$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} \\ &= (-3-\lambda)(-2-\lambda) - 2 \\ &= \lambda^2 + 5\lambda + 4 = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = -1,\quad \lambda_2 = -4$

The natural frequencies: $\omega_1 = 1\quad \omega_2 = 2\quad \omega_3 = 3$

For $\lambda_1 = -1 \quad \Rightarrow \quad (A + I)V_1 = 0$

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = b \quad \to V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \vec{x}_1(t) = (a_1 \cos t + b_1 \sin t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -4 \quad \Rightarrow \quad (A + 4I)V_2 = 0$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = -2b \quad \to V_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \Rightarrow \vec{x}_2(t) = (a_2 \cos 2t + b_2 \sin 2t)\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1 \cos t + b_1 \sin t + 2a_2 \cos 2t + 2b_2 \sin 2t + c_1 \cos 3t \\ \vec{x}_2(t) = a_1 \cos t + b_1 \sin t - a_2 \cos 2t - b_2 \sin 2t + c_2 \cos 3t \end{cases}$$

$$\begin{cases} \vec{x}''_{1p} = -9c_1 \cos 3t \\ \vec{x}''_{2p} = -9c_2 \cos 3t \end{cases}$$

$x''_1 = -3x_1 + 2x_2$

$$-9c_1 \cos 3t = -3c_1 \cos 3t + 2c_2 \cos 3t \quad \Rightarrow \quad -6c_1 = 2c_2 \quad \rightarrow \quad \underline{-3c_1 = c_2|}$$

$x''_2 = x_1 - 2x_2 + 60\cos 3t$

$$-9c_2 \cos 3t = c_1 \cos 3t - 2c_2 \cos 3t + 60\cos 3t \quad \Rightarrow \quad \underline{c_1 + 7c_2 = -60|}$$

$$c_1 + 7\left(-3c_1\right) = -60 \quad \Rightarrow \quad \underline{c_1 = 3, \ c_2 = -9|}$$

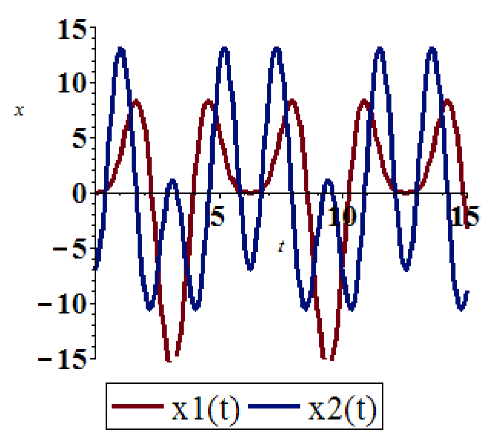
$$\begin{cases} \vec{x}_1(t) = a_1 \cos t + b_1 \sin t + 2a_2 \cos 2t + 2b_2 \sin 2t + 3\cos 3t \\ \vec{x}_2(t) = a_1 \cos t + b_1 \sin t - a_2 \cos 2t - b_2 \sin 2t - 9\cos 3t \end{cases}$$

Given initial values: $x'_1(0) = x'_2(0) = 0$ and $x_1(0) = x_2(0) = 0$.

$$\begin{cases} \vec{x}_1(0) = a_1 + 2a_2 + 3 = 0 \\ \vec{x}_2(0) = a_1 - a_2 - 9 = 0 \end{cases} \quad \rightarrow \quad \underline{a_1 = 5, \ a_2 = -4|}$$

$$\begin{cases} \vec{x}'_1(0) = b_1 + 4b_2 = 0 \\ \vec{x}'_2(0) = b_1 - 2b_2 = 0 \end{cases} \quad \rightarrow \quad \underline{b_1 = b_2 = 0|}$$

$$\begin{cases} \vec{x}_1(t) = 5\cos t - 8\cos 2t + 3\cos 3t \\ \vec{x}_2(t) = 5\cos t + 4\cos 2t - 9\cos 3t \end{cases}$$



At frequency $\omega_1 = 1$ the 2 masses oscillate in the same direction with equal amplitudes.

At frequency $\omega_2 = 2$ the 2 masses oscillate in opposite directions with equal amplitudes of m_1 ***twice*** that of m_2.

At frequency $\omega_3 = 3$ they oscillate in opposite directions with amplitude of oscillation of m_1 **3** times that of m_2.

Exercise

Consider the mass-and-spring system shown below and with the given masses and spring constants values. The mass-and-spring system is set in motion from rest $x_1'(0) = x_2'(0) = 0$ in its equilibrium position $x_1(0) = x_2(0) = 0$.

Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

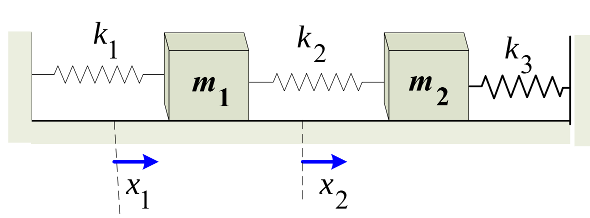
For the given external forces $F_1(t)$ and $F_2(t)$ acting on the masses m_1 and m_2, respectively.

Find the resulting motion of the system and describe it as a superposition of oscillations at three different frequencies.

$$m_1 = m_2 = 1; \quad k_1 = 4,\ k_2 = 6,\ k_3 = 4; \quad F_1(t) = 30\cos t,\ F_2(t) = 60\cos t$$

Solution

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 + 30\cos t \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 + 60\cos t \end{cases}$$



$$\begin{cases} x_1'' = -10x_1 + 6x_2 + 30\cos t \\ x_2'' = 6x_1 - 10x_2 + 60\cos t \end{cases}$$

$$A = \begin{pmatrix} -10 & 6 \\ 6 & -10 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -10-\lambda & 6 \\ 6 & -10-\lambda \end{vmatrix}$$

$$= (-10-\lambda)^2 - 36$$

$$= \lambda^2 + 20\lambda + 64 = 0$$

The eigenvalues are: $\lambda_1 = -4, \quad \lambda_2 = -16$

The natural frequencies: $\omega_1 = 2 \quad \omega_2 = 4 \quad \omega_3 = 1$

For $\lambda_1 = -4 \quad \Rightarrow \quad (A + 4I)V_1 = 0$

$$\begin{pmatrix} -6 & 6 \\ 6 & -6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = b \quad \to \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \vec{x}_1(t) = (a_1 \cos 2t + b_1 \sin 2t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -16 \quad \Rightarrow \quad (A + 16I)V_2 = 0$

$$\begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 2a = -3b$$

$$\to V_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad \Rightarrow \vec{x}_2(t) = (a_2 \cos 4t + b_2 \sin 4t)\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1 \cos 2t + b_1 \sin 2t + 3a_2 \cos 3t + 3b_2 \sin 3t + c_1 \cos t \\ \vec{x}_2(t) = a_1 \cos 2t + b_1 \sin 2t - 2a_2 \cos 3t - 2b_2 \sin 3t + c_2 \cos t \end{cases}$$

$$\begin{cases} \vec{x}''_{1p} = -c_1 \cos t \\ \vec{x}''_{2p} = -c_2 \cos t \end{cases}$$

$$x_1'' = -10x_1 + 6x_2 + 30\cos t$$

$$-c_1 \cos t = -10c_1 \cos t + 6c_2 \cos t + 30\cos t \quad \Rightarrow \ 9c_1 - 6c_2 = 30 \quad \Rightarrow \ \underline{3c_1 - 2c_2 = 10}$$

$$x_2'' = 6x_1 - 10x_2 + 60\cos t$$

$$-c_2 \cos t = 6c_1 \cos t - 10c_2 \cos t + 60\cos t \quad \Rightarrow \ -6c_1 + 9c_2 = 60 \quad \Rightarrow \ \underline{-2c_1 + 3c_2 = 20}$$

$$5c_1 = 70 \quad \Rightarrow \quad \underline{c_1 = 14, \ c_2 = 16}$$

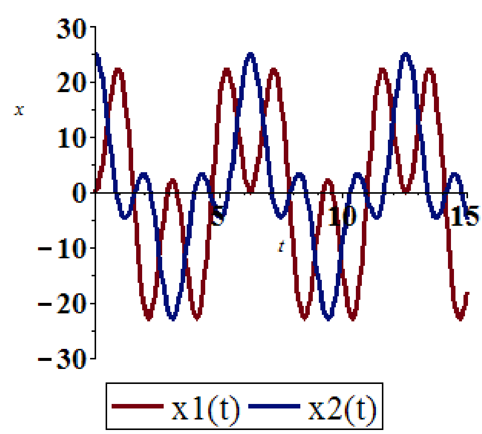
$$\begin{cases} \vec{x}_1(t) = a_1 \cos 2t + b_1 \sin 2t + 3a_2 \cos 3t + 3b_2 \sin 3t + 14\cos t \\ \vec{x}_2(t) = a_1 \cos 2t + b_1 \sin 2t - 2a_2 \cos 3t - 2b_2 \sin 3t + 16\cos t \end{cases}$$

Given initial values: $x_1'(0) = x_2'(0) = 0$ and $x_1(0) = x_2(0) = 0$.

$$\begin{cases} \vec{x}_1(0) = a_1 + 3a_2 + 14 = 0 \\ \vec{x}_2(0) = a_1 - 2a_2 + 16 = 0 \end{cases} \rightarrow \ \underline{a_1 = 1, \ a_2 = -5}$$

$$\begin{cases} \vec{x}_1'(0) = 2b_1 + 9b_2 = 0 \\ \vec{x}_2'(0) = 2b_1 - 6b_2 = 0 \end{cases} \rightarrow \ \underline{b_1 = b_2 = 0}$$

$$\begin{cases} \vec{x}_1(t) = \cos 2t - 15\cos 3t + 14\cos t \\ \vec{x}_2(t) = \cos 2t + 10\cos 3t + 16\cos t \end{cases}$$



At frequency $\omega_1 = 2$ the 2 masses oscillate in the same direction of m_1 ***twice*** that of m_2.

At frequency $\omega_2 = 3$ the 2 masses oscillate in opposite directions with equal amplitudes of m_1 **3** *times* that of m_2.

At frequency $\omega_3 = 1$ they oscillate in the same direction with equal amplitudes of oscillation.

Exercise

Consider a mass-and-spring system containing two masses $m_1 = m_2 = 1$ whose displacement functions $x(t)$ and $y(t)$ satisfy the differential equations

$$\begin{aligned} x'' &= -40x + 8y \\ y'' &= 12x - 60y \end{aligned}$$

a) Describe the two fundamental modes of free oscillation of the system.

b) Assume that the two masses start in motion with the initial conditions

$x(0)=19,\ x'(0)=12 \quad and \quad y(0)=3,\ y'(0)=6$

And are acted on by the same force, $F_1(t)=F_2(t)=-195\cos 7t$. Describe the resulting motion as a superposition of oscillations at three different frequencies.

Solution

a) $A=\begin{pmatrix} -40 & 8 \\ 12 & -60 \end{pmatrix}$

$$|A-\lambda I| = \begin{vmatrix} -40-\lambda & 8 \\ 12 & -60-\lambda \end{vmatrix}$$

$$=(-40-\lambda)(-60-\lambda)-96$$

$$=\lambda^2+100\lambda+144=0$$

The eigenvalues are: $\lambda_1=-36,\quad \lambda_2=-64$

The natural frequencies: $\omega_1=6 \quad \omega_2=8$

For $\lambda_1=-36 \quad \Rightarrow \quad (A+36I)V_1=0$

$$\begin{pmatrix} -4 & 8 \\ 12 & -24 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a=2b \to V_1=\begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1(t)=\left(a_1\cos 6t+b_1\sin 6t\right)\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2=-64 \quad \Rightarrow \quad (A+64I)V_2=0$

$$\begin{pmatrix} 24 & 8 \\ 12 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 3a=-b \to V_2=\begin{pmatrix} 1 \\ -3 \end{pmatrix} \Rightarrow \vec{x}_2(t)=\left(a_2\cos 8t+b_2\sin 8t\right)\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\begin{cases} \vec{x}(t)=2a_1\cos 6t+2b_1\sin 6t+a_2\cos 8t+b_2\sin 8t \\ \vec{y}(t)=a_1\cos 6t+b_1\sin 6t-3a_2\cos 8t-3b_2\sin 8t \end{cases}$$

In mode 1: At frequency $\omega_1=6,$ the 2 masses oscillate in the same direction of m_1 ***twice*** of m_2.

In mode 2: At frequency $\omega_2=8,$ the 2 masses oscillate in opposite directions of oscillation of m_1 **3** ***times*** that of m_2.

b) ***Given*** $x(0)=19,\ x'(0)=12 \quad y(0)=3,\ y'(0)=6$ and $F_1(t)=F_2(t)=-195\cos 7t$

$$\begin{aligned} x'' &= -40x+8y-195\cos 7t \\ y'' &= 12x-60y-195\cos 7t \end{aligned}$$

$$\begin{cases} \vec{x}(t) = 2a_1\cos 6t + 2b_1\sin 6t + a_2\cos 8t + b_2\sin 8t + c_1\cos 7t \\ \vec{y}(t) = a_1\cos 6t + b_1\sin 6t - 3a_2\cos 8t - 3b_2\sin 8t + c_2\cos 7t \end{cases}$$

$$\begin{cases} x''_p = -49c_1\cos 7t \\ y''_p = -49c_2\cos 7t \end{cases}$$

$x'' = -40x + 8y - 195\cos 7t$

$$-49c_1\cos 7t = -40c_1\cos 7t + 8c_2\cos 7t - 195\cos 7t \quad \Rightarrow \quad \boxed{9c_1 + 8c_2 = 195}$$

$y'' = 12x - 60y - 195\cos 7t$

$$-49c_2\cos 7t = 12c_1\cos 7t - 60c_2\cos 7t - 195\cos 7t \quad \Rightarrow \quad \boxed{12c_1 - 11c_2 = 195}$$

$$\Rightarrow \quad \boxed{c_1 = 19, \ c_2 = 3}$$

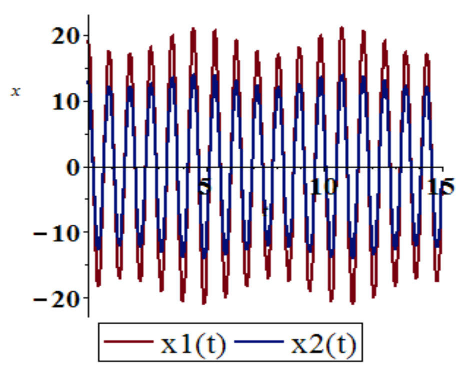
$$\begin{cases} \vec{x}(t) = 2a_1\cos 6t + 2b_1\sin 6t + a_2\cos 8t + b_2\sin 8t + 19\cos 7t \\ \vec{y}(t) = a_1\cos 6t + b_1\sin 6t - 3a_2\cos 8t - 3b_2\sin 8t + 3\cos 7t \end{cases}$$

$$\begin{cases} x(0) = 2a_1 + a_2 + 19 = 19 \\ y(0) = a_1 - 3a_2 + 3 = 3 \end{cases} \rightarrow \begin{cases} 2a_1 + a_2 = 0 \\ a_1 - 3a_2 = 0 \end{cases} \Rightarrow \quad \boxed{a_1 = 0, a_2 = 0}$$

$$\Rightarrow \begin{cases} x(t) = 2b_1\sin 6t + b_2\sin 8t + 19\cos 7t \\ y(t) = b_1\sin 6t - 3b_2\sin 8t + 3\cos 7t \end{cases}$$

$$\begin{cases} x'(0) = 12b_1 + 8b_2 = 12 \\ y'(0) = 6b_1 - 24b_2 = 6 \end{cases} \Rightarrow \quad \boxed{b_1 = 1, b_2 = 0}$$

$$\Rightarrow \boxed{\begin{cases} x(t) = 2\sin 6t + 19\cos 7t \\ y(t) = \sin 6t + 3\cos 7t \end{cases}}$$



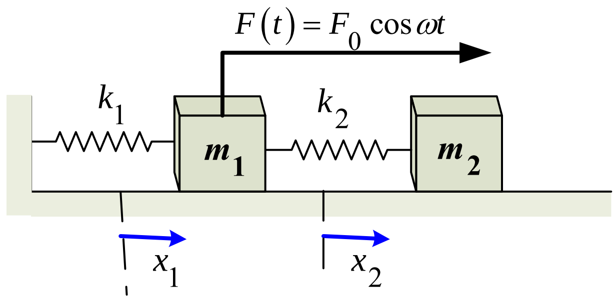
At frequency $\omega_1 = 6$, the 2 masses oscillate in the same direction with amplitude of motion of m_1 ***twice*** that of m_2.

At frequency $\omega_3 = 7$, the 2 masses oscillate in the same direction with amplitude of motion of m_1 being $\frac{19}{3}$ *times* that of m_2.

At frequency $\omega_2 = 8$, the expected oscillation is missing.

Exercise

Consider a mass-and-spring system shown below. Assume that $m_1 = 1;\quad k_1 = 50,\ k_2 = 10;\quad F_0 = 5$ in mks units, and that $\omega = 10$. Then find m_2 so that in the resulting steady periodic oscillations, the mass m_1 will remain at rest (!).



Thus the effect of the second mass-and-spring pair will be to neutralize the effect of the force on the first mass. This is an example of a dynamic damper. It has an electrical analogy that some cable companies use to prevent your reception of certain cable channels.

Solution

$$F(t) = F_0 \cos \omega t = 5\cos 10t$$

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 + 5\cos 10t \\ m_2 x_2'' = -k_2(x_2 - x_1) \end{cases}$$

$$\rightarrow \begin{cases} x_1'' = -60x_1 + 10x_2 + 5\cos 10t \\ m_2 x_2'' = 10x_1 - 10x_2 \end{cases}$$

$$\begin{cases} x_{1p} = c_1 \cos 10t \\ x_{2p} = c_2 \cos 10t \end{cases} \rightarrow \begin{cases} x_{1p}'' = -100c_1 \cos 10t \\ x_{2p}'' = -100c_2 \cos 10t \end{cases}$$

$$x_1'' = -60x_1 + 10x_2 + 5\cos 10t$$

$$-100c_1 \cos 10t = -60c_1 \cos 10t + 10c_2 \cos 10t + 5\cos 10t \quad \rightarrow \quad \underline{-40c_1 - 10c_2 = 5}$$

$$m_2 x_2'' = 10x_1 - 10x_2$$

$$-100m_2 c_2 \cos 10t = 10c_1 \cos 10t - 10c_2 \cos 10t \quad \rightarrow \quad \underline{c_1 - (1 - 10m_2)c_2 = 0}$$

$$-40(1 - 10m_2)c_2 - 10c_2 = 5 \qquad\qquad c_1 = (1 - 10m_2)c_2$$

$$390m_2 c_2 = 45 \quad \Rightarrow \quad \underline{c_2 = \frac{3}{26m_2}} \quad \rightarrow c_1 = (1 - 10m_2)\frac{3}{26m_2} = \underline{\frac{3}{26m_2} - \frac{15}{13}}$$

$$-40\left(\frac{3}{26m_2} - \frac{15}{13}\right) - 10\frac{3}{26m_2} = 5$$

$$-4.615 + 46.154m_2 - 1.154 = 5m_2$$

$41.154 m_2 = 5.769$

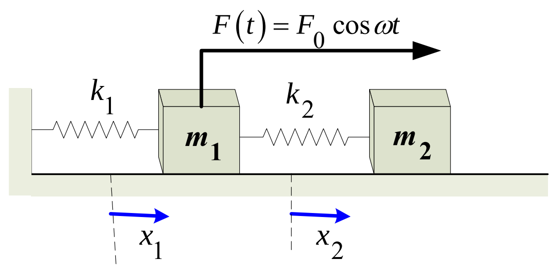
$\boxed{m_2 \approx 0.1 \ slug}$ $\quad \Rightarrow \quad c_1 = \frac{3}{26m_2} - \frac{15}{13} \approx 0 \quad c_2 = \frac{3}{26m_2} \approx 1.15$

Since $c_1 = 0$, so the mass m_1 remains at rest.

Exercise

Consider a mass-and-spring system shown below. Assume that

$m_1 = 2,\ m_2 = \frac{1}{2};\quad k_1 = 75,\ k_2 = 25;\quad F_0 = 100 \quad and \ \omega = 10$ (in *mks* units).



Find the solution of the system $M\vec{x}'' = K\vec{x} + F$ that satisfies the initial conditions $\vec{x}(0) = \vec{x}'(0) = \mathbf{0}$

Solution

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 + 100\cos 10t \\ m_2 x_2'' = -k_2(x_2 - x_1) \end{cases}$$

$$\begin{cases} 2x_1'' = -100x_1 + 25x_2 + 100\cos 10t \\ \frac{1}{2}x_2'' = 25x_1 - 25x_2 \end{cases}$$

$$\begin{cases} x_1'' = -50x_1 + \frac{25}{2}x_2 + 50\cos 10t \\ x_2'' = 50x_1 - 50x_2 \end{cases} \qquad \to A = \begin{bmatrix} -50 & \frac{25}{2} \\ 50 & -50 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -50-\lambda & \frac{25}{2} \\ 50 & -50-\lambda \end{vmatrix}$$

$$= (-50-\lambda)^2 - 625$$

$$= \lambda^2 + 100\lambda - 1875 = 0$$

The eigenvalues are: $\lambda_1 = -25, \quad \lambda_2 = -75$

The natural frequencies: $\omega_1 = 5 \quad \omega_2 = 5\sqrt{3}$

For $\lambda_1 = -25 \quad \Rightarrow \quad (A + 25I)V_1 = 0$

$$\begin{pmatrix} -25 & \frac{25}{2} \\ 50 & -25 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 2a = b \qquad \to \ V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\vec{x}_1(t) = \left(a_1 \cos 5t + b_1 \sin 5t\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = -75 \quad \Rightarrow \quad (A + 75I)V_2 = 0$

$$\begin{pmatrix} 25 & \frac{25}{2} \\ 50 & 25 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow 2a = -b \quad \to \quad V_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\Rightarrow \vec{x}_2(t) = \left(a_2 \cos 5t\sqrt{3} + b_2 \sin 5t\sqrt{3}\right)\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\begin{cases} x_1(t) = a_1 \cos 5t + b_1 \sin 5t + a_2 \cos 5t\sqrt{3} + b_2 \sin 5t\sqrt{3} \\ x_2(t) = 2a_1 \cos 5t + 2b_1 \sin 5t - 2a_2 \cos 5t\sqrt{3} - 2b_2 \sin 5t\sqrt{3} \end{cases}$$

$$\begin{cases} x_{1p} = c_1 \cos 10t \\ x_{2p} = c_2 \cos 10t \end{cases} \to \begin{cases} x''_{1p} = -100c_1 \cos 10t \\ x''_{2p} = -100c_2 \cos 10t \end{cases}$$

$$x''_1 = -50x_1 + \frac{25}{2}x_2 + 50\cos 10t$$

$$-100c_1 \cos 10t = -50c_1 \cos 10t + \frac{25}{2}c_2 \cos 10t + 50\cos 10t$$

$$\Rightarrow \; 50c_1 + \frac{25}{2}c_2 = -50 \quad \Rightarrow \; \underline{4c_1 + c_2 = -4|}$$

$$x''_2 = 50x_1 - 50x_2$$

$$-100c_2 = 50c_1 - 50c_2 \quad \Rightarrow \; \underline{c_1 + c_2 = 0|}$$

$$\underline{c_1 = -\frac{4}{3}, \; c_2 = \frac{4}{3}|}$$

$$\begin{cases} x_1(t) = a_1 \cos 5t + b_1 \sin 5t + a_2 \cos 5t\sqrt{3} + b_2 \sin 5t\sqrt{3} - \frac{4}{3}\cos 10t \\ x_2(t) = 2a_1 \cos 5t + 2b_1 \sin 5t - 2a_2 \cos 5t\sqrt{3} - 2b_2 \sin 5t\sqrt{3} + \frac{4}{3}\cos 10t \end{cases}$$

$$\begin{cases} x_1(0) = a_1 + a_2 - \frac{4}{3} = 0 \\ x_2(0) = 2a_1 - 2a_2 + \frac{4}{3} = 0 \end{cases} \quad \begin{cases} a_1 + a_2 = \frac{4}{3} \\ 2a_1 - 2a_2 = -\frac{4}{3} \end{cases} \quad \underline{a_1 = \frac{1}{3}, \; a_2 = 1|}$$

$$\begin{cases} x'_1(t) = -5a_1 \sin 5t + 5b_1 \cos 5t - 5a_2\sqrt{3}\sin 5t\sqrt{3} + 5b_2\sqrt{3}\cos 5t\sqrt{3} + \frac{40}{3}\sin 10t \\ x'_2(t) = -10a_1 \sin 5t + 10b_1 \cos 5t + 10a_2\sqrt{3}\sin 5t\sqrt{3} - 10b_2\sqrt{3}\cos 5t\sqrt{3} - \frac{40}{3}\sin 10t \end{cases}$$

$$\begin{cases} x'_1(0) = 5b_1 + 5\sqrt{3}b_2 = 0 \\ x'_2(0) = 10b_1 - 10\sqrt{3}b_2 = 0 \end{cases} \quad \Rightarrow \quad \underline{b_1 = b_2 = 0|}$$

$$\begin{cases} x_1(t) = \frac{1}{3}\cos 5t + \cos 5t\sqrt{3} - \frac{4}{3}\cos 10t \\ x_2(t) = \frac{2}{3}\cos 5t - 2\cos 5t\sqrt{3} + \frac{4}{3}\cos 10t \end{cases}$$

At frequency $\omega_1 = 5$, the 2 masses oscillate in the same direction with amplitude of motion of m_1 ***half*** that of m_2.

At frequency $\omega_2 = 5\sqrt{3}$, the 2 masses oscillate in opposite directions with amplitude of motion of m_1 being ***half*** that of m_2.

At frequency $\omega_3 = 10$ the 2 masses oscillate in opposite directions with equal amplitudes.

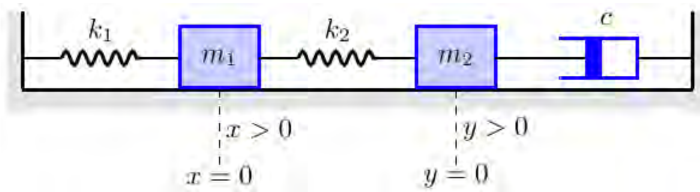
Exercise

Two springs, two masses, and a dashpot are attached in a straight line on a horizontal frictionless surface. The dashpot damping force on mass m_2, given by $F = -cy'$.

Determine the equations of motion for the two masses.

Solution

$$\begin{cases} m_1 x'' = -k_1 x + k_2(y - x) \\ m_2 y'' = -k_2(y - x) - cy' \end{cases}$$



$$\begin{cases} m_1 x'' = -(k_1 + k_2)x + k_2 y \\ m_2 y'' = k_2 x - k_2 y - cy' \end{cases}$$

Exercise

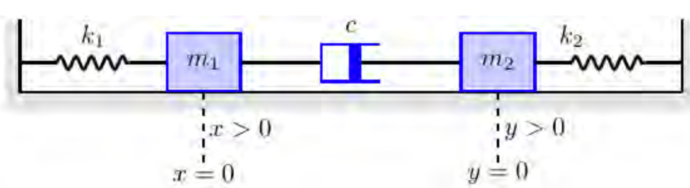
Two springs, two masses, and a dashpot are attached in a straight line on a horizontal frictionless surface. The system is set in motion by holding the mass m_2 at equilibrium position and pushing the mass m_1 to the left of its equilibrium position a distance 2 *m* and then releasing both masses. If $m_1 = m_2 = 1$ *kg* and $k_1 = k_2 = 1$ *N/m*, and $c = 1$ *N-sec*

Determine the equations of motion for the two masses

Solution

Given: $x(0) = -2, \quad x'(0) = 0, \quad y(0) = 0 \quad y'(0) = 0$

$$\begin{cases} m_1 x'' + k_1 x + c(x' - y') = 0 \\ m_2 y'' + k_2 y + c(y' - x') = 0 \end{cases}$$



$$\begin{cases} m_1 x'' = -k_1 x - c(x' - y') \\ m_2 y'' = -c(y' - x') - k_2 y \end{cases}$$

$$\begin{cases} x'' = -x - x' + y' \\ y'' = -y + x' - y' \end{cases}$$

Let $x_1 = x' \quad y_1 = y'$
$$\begin{cases} x' = x_1 \\ y' = y_1 \\ x_1' = -x - x_1 + y_1 \\ y_1' = -y + x_1 - y_1 \end{cases}$$

$$\begin{pmatrix} x \\ y \\ x_1 \\ y_1 \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ x_1 \\ y_1 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -1 & 0 & -1-\lambda & 1 \\ 0 & -1 & 1 & -1-\lambda \end{vmatrix}$$

$$= -\lambda\left[-\lambda\left(1 + 2\lambda + \lambda^2\right) - 1 - \lambda + \lambda\right] + 1 + \lambda + \lambda^2$$

$$= \lambda^4 + 2\lambda^3 + 2\lambda^2 + 2\lambda + 1 = 0 \quad \text{The eigenvalues: } \underline{\lambda = -1,\ -1,\ \pm i}$$

$$x(t) = \left(C_1 + C_2 t\right)e^{-t} + C_3 \cos t + C_4 \sin t$$

Given: $x(0) = -2, \quad x'(0) = 0$

$$x(0) = \underline{C_1 + C_3 = -2} \quad (1)$$

$$x'(t) = \left(C_2 - C_1 - C_2 t\right)e^{-t} - C_3 \sin t + C_4 \cos t$$

$$x'(0) = \underline{C_2 - C_1 + C_4 = 0} \quad (2)$$

$$x'' = \left(-2C_2 + C_1 + C_2 t\right)e^{-t} - C_3 \cos t - C_4 \sin t$$

$$x'' = -x - x' + y' \quad \to \quad y' = x'' + x' + x$$

$$y' = \left(C_1 - C_2\right)e^{-t} + C_2 t e^{-t} - C_3 \sin t + C_4 \cos t$$

$$y(t) = \left(C_2 - C_1\right)e^{-t} - C_2 (t+1)e^{-t} + C_3 \cos t + C_4 \sin t$$

Given: $y(0) = 0, \quad y'(0) = 0$

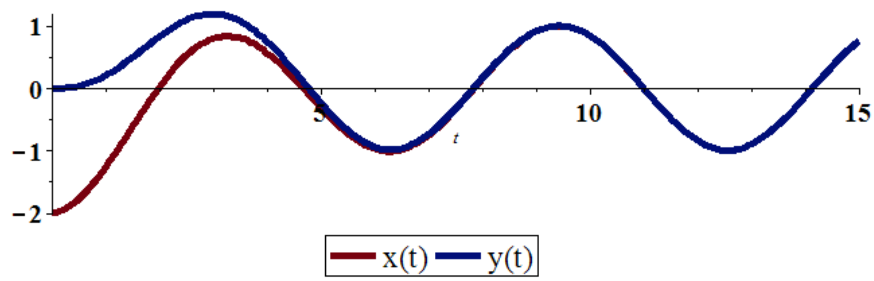
$$y(0) = \underline{-C_1 + C_3 = 0} \quad (3)$$

$$y'(0) = \underline{C_1 - C_2 + C_4 = 0} \quad (4)$$

$$\begin{cases} (1) & C_1 + C_3 = -2 \\ (3) & -C_1 + C_3 = 0 \end{cases} \quad \rightarrow \quad \underline{C_1 = -1, \ C_3 = -1|}$$

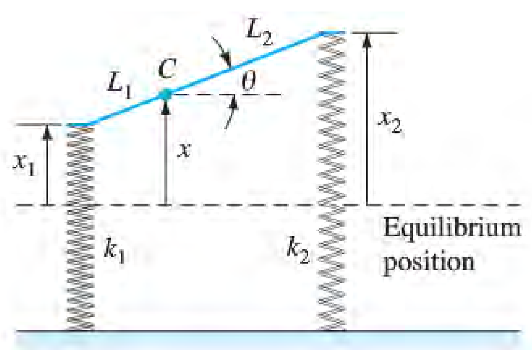
$$\begin{cases} (2) & C_2 + C_4 = -1 \\ (4) & -C_2 + C_4 = 1 \end{cases} \quad \rightarrow \quad \underline{C_2 = -1, \ C_4 = 0|}$$

$$\boxed{\begin{cases} x(t) = -(1+t)e^{-t} - \cos t \\ y(t) = (t+1)e^{-t} - \cos t \end{cases}}$$



Exercise

A car with two axles and with separate front and rear suspension systems.



We assume that the car body acts as would a solid bar of mass m and length $L = L_1 + L_2$. It has moment of inertia I about its center of mass C, which is at distance L_1 from the front of the car. The car has front and back suspension springs with Hooke's constants k_1 and k_2, respectively. When the car is in motion, let $x(t)$ denote the vertical displacement of the center of mass of the car from equilibrium; let $\theta(t)$ denote its angular displacement (in radians) from the horizontal. Then Newton's laws of motion for linear and angular acceleration can be used to derive the equations.

$$mx'' = -(k_1 + k_2)x + (k_1 L_1 - k_2 L_2)\theta$$

$$I\theta'' = (k_1 L_1 - k_2 L_2)x - \left(k_1 L_1^2 + k_2 L_2^2\right)\theta$$

Suppose that $m = 75\ slugs$ (the car weighs 2400 lb), $L_1 = 7\,ft$, $L_2 = 3\,ft$ (it's a rear engine car), $k_1 = k_2 = 2000\ \ lb/\ ft$, and $I = 1000\ ft.lb.s^2$.

a) Find the two natural frequencies ω_1 and ω_2 of the car.

b) Now suppose that the car is driven at a speed of $v\ \ ft/\sec$ along a washboard surface shaped like a sine curve with a wavelength of 40 ft. The result is a periodic force on the car with frequency $\omega = \frac{2\pi}{40}v = \frac{\pi}{20}v$. Resonance occurs when $\omega = \omega_1\ \ or\ \ \omega = \omega_2$. Find the corresponding two critical speeds of the car (in ft/sec)

Solution

a) $$\begin{cases} 75x'' = -4000x + 8000\theta \\ 1000\theta'' = 8000x - (98000 + 18000)\theta \end{cases}$$

$$\begin{cases} x'' = -\frac{160}{3}x + \frac{320}{3}\theta \\ \theta'' = 8x - 116\theta \end{cases} \quad \to A = \begin{bmatrix} -\frac{160}{3} & \frac{320}{3} \\ 8 & -116 \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -\frac{160}{3} - \lambda & \frac{320}{3} \\ 8 & -116 - \lambda \end{vmatrix} \\ &= \left(-\frac{160}{3} - \lambda\right)(-116 - \lambda) - \frac{2560}{3} \\ &= \lambda^2 + \frac{508}{3}\lambda - \frac{48640}{3} = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 \approx -41.8285, \quad \lambda_2 \approx -127.5049$

The natural frequencies: $\omega_1 \approx \underline{6.4675\ \ rad/\sec}$ $\quad \omega_2 \approx \underline{11.2918\ \ rad/\sec}$

$$\omega_1 = \frac{6.4675}{2\pi} \approx \underline{1.0293\ \ Hz} \quad \omega_2 = \frac{11.2918}{2\pi} \approx \underline{1.7971\ \ Hz}$$

b) $\omega = \frac{\pi}{20}v \quad \Rightarrow \quad v = \frac{20}{\pi}\omega$

$$v_1 = \frac{20}{\pi}\omega_1 = \frac{(20)(6.4675)}{\pi} \approx \underline{41\ \ ft/\sec} \qquad (41)(0.681818) \approx \underline{28\ \ mph}$$

$$v_2 = \frac{20}{\pi}\omega_2 = \frac{(20)(11.2918)}{\pi} \approx \underline{72\ \ ft/\sec} \qquad (72)(0.681818) \approx \underline{49\ \ mph}$$

Exercise

The system is taken as a model for an undamped car with the given parameters in *fps* units.

$m = 100; \quad I = 800; \quad L_1 = L_2 = 5; \quad k_1 = k_2 = 2000$

a) Find the two natural frequencies ω_1 and ω_2 of the car (in hertz).

b) Assume that his car is driven along a sinusoidal washboard surface with a wavelength of 40 *ft*. The result is a periodic force on the car with frequency $\omega = \frac{2\pi}{40}v = \frac{\pi}{20}v$. Resonance occurs when $\omega = \omega_1$ *or* $\omega = \omega_2$. Find the corresponding two critical speeds of the car (in *ft/sec*)

Solution

a) $$\begin{cases} mx'' = -\left(k_1 + k_2\right)x + \left(k_1 L_1 - k_2 L_2\right)\theta \\ I\theta'' = \left(k_1 L_1 - k_2 L_2\right)x - \left(k_1 L_1^2 + k_2 L_2^2\right)\theta \end{cases} \rightarrow \begin{cases} 100x'' = -4000x \\ 800\theta'' = -100{,}000\theta \end{cases}$$

$$\begin{cases} x'' = -40x \\ \theta'' = -125\theta \end{cases} \rightarrow A = \begin{bmatrix} -40 & 0 \\ 0 & -125 \end{bmatrix}$$

$$\left|A - \lambda I\right| = \begin{vmatrix} -40-\lambda & 0 \\ 0 & -125-\lambda \end{vmatrix} = (-40-\lambda)(-125-\lambda) = 0$$

The eigenvalues are: $\lambda_1 = -40, \quad \lambda_2 = -125$

The natural frequencies: $\omega_1 = \sqrt{40} \approx \underline{6.325 \ rad/\sec}$ $\quad \omega_2 = \sqrt{125} \ \underline{\approx 11.180 \ rad/\sec}$

$$\omega_1 = \frac{6.325}{2\pi} \underline{\approx 1.0067 \ Hz} \qquad \omega_2 = \frac{11.180}{2\pi} \underline{\approx 1.779 \ Hz}$$

b) $$v_1 = \frac{20}{\pi}\omega_1 = \frac{(20)(6.325)}{\pi} \ \underline{\approx 40.26 \ ft/\sec} \qquad (40.26)(0.681818) \ \underline{\approx 27 \ mph}$$

$$v_2 = \frac{20}{\pi}\omega_2 = \frac{(20)(11.180)}{\pi} \ \underline{\approx 71.18 \ ft/\sec} \qquad (71.18)(0.681818) \ \underline{\approx 49 \ mph}$$

Exercise

The system is taken as a model for an undamped car with the given parameters in *fps* units.

$m = 100; \quad I = 1000; \quad L_1 = 6, \ L_2 = 4; \quad k_1 = k_2 = 2000$

a) Find the two natural frequencies ω_1 and ω_2 of the car (in hertz).

b) Assume that his car is driven along a sinusoidal washboard surface with a wavelength of 40 *ft*. The result is a periodic force on the car with frequency $\omega = \frac{2\pi}{40}v = \frac{\pi}{20}v$. Resonance occurs when $\omega = \omega_1$ *or* $\omega = \omega_2$. Find the corresponding two critical speeds of the car (in *ft/sec*)

Solution

a) $\begin{cases} mx'' = -\left(k_1 + k_2\right)x + \left(k_1 L_1 - k_2 L_2\right)\theta \\ I\theta'' = \left(k_1 L_1 - k_2 L_2\right)x - \left(k_1 L_1^2 + k_2 L_2^2\right)\theta \end{cases} \quad \rightarrow \begin{cases} 100x'' = -4000x + 4000\theta \\ 1000\theta'' = 4000x - 104{,}000\theta \end{cases}$

$$\begin{cases} x'' = -40x + 40\theta \\ \theta'' = 4x - 104\theta \end{cases} \quad \rightarrow A = \begin{bmatrix} -40 & 40 \\ 4 & -104 \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -40-\lambda & 40 \\ 4 & -104-\lambda \end{vmatrix} \\ &= (-40-\lambda)(-104-\lambda) - 160 \\ &= \lambda^2 + 144\lambda + 4000 = 0 \end{aligned} \qquad \lambda_{1,2} = -72 \pm 4\sqrt{74}$$

The eigenvalues are: $\lambda_1 \approx -37.591, \quad \lambda_2 \approx -106.409$

The natural frequencies: $\omega_1 = \sqrt{37.591} \approx \underline{6.131 \ \ rad/\sec}$

$$= \frac{6.131}{2\pi} \approx \underline{.9758 \ \ Hz}$$

$$\omega_2 = \sqrt{106.409} \approx \underline{10.315 \ \ rad/\sec}$$

$$= \frac{10.315}{2\pi} \approx \underline{1.6417 \ \ Hz}$$

b) $v_1 = \frac{20}{\pi}\omega_1 = \frac{(20)(6.131)}{\pi} \approx \underline{39.03 \ \ ft/\sec}$

$$= (39.03)(0.681818) \approx \underline{27 \ \ mph}$$

$$v_2 = \frac{20}{\pi}\omega_2 = \frac{(20)(10.315)}{\pi} \approx \underline{65.67 \ \ ft/\sec}$$

$$= (65.67)(0.681818) \approx \underline{45 \ \ mph}$$

Exercise

The system is taken as a model for an undamped car with the given parameters in *fps* units.

$m = 100; \quad I = 800; \quad L_1 = L_2 = 5; \quad k_1 = 1000, \ \ k_2 = 2000$

a) Find the two natural frequencies ω_1 and ω_2 of the car (in hertz).

b) Assume that his car is driven along a sinusoidal washboard surface with a wavelength of 40 *ft*. The result is a periodic force on the car with frequency $\omega = \frac{2\pi}{40}v = \frac{\pi}{20}v$. Resonance occurs when $\omega = \omega_1$ *or* $\omega = \omega_2$. Find the corresponding two critical speeds of the car (in *ft/sec*)

Solution

a) $\begin{cases} mx'' = -\left(k_1 + k_2\right)x + \left(k_1L_1 - k_2L_2\right)\theta \\ I\theta'' = \left(k_1L_1 - k_2L_2\right)x - \left(k_1L_1^2 + k_2L_2^2\right)\theta \end{cases} \quad \to \begin{cases} 100x'' = -3000x - 5000\theta \\ 800\theta'' = -5000x - 75,000\theta \end{cases}$

$$\begin{cases} x'' = -30x - 50\theta \\ \theta'' = -\frac{25}{4}x - \frac{375}{4}\theta \end{cases} \to A = \begin{bmatrix} -30 & -50 \\ -\frac{25}{4} & -\frac{375}{4} \end{bmatrix}$$

$$\begin{aligned} \left|A - \lambda I\right| &= \begin{vmatrix} -30-\lambda & -50 \\ -\frac{25}{4} & -\frac{375}{4} - \lambda \end{vmatrix} \\ &= \left(-30-\lambda\right)\left(-\frac{375}{4} - \lambda\right) - \frac{625}{2} \\ &= \lambda^2 + \frac{495}{4}\lambda + 2500 = 0 \end{aligned}$$

$$\boxed{\lambda_{1,2} = \frac{-495 \pm 5\sqrt{3401}}{8}}$$

The eigenvalues are: $\boxed{\lambda_1 \approx -25.426, \quad \lambda_2 \approx -98.234}$

The natural frequencies: $\omega_1 = \sqrt{25.426} \approx \boxed{5.0424 \ \ rad/\sec}$

$$= \frac{5.0424}{2\pi} \boxed{\approx .8025 \ \ Hz}$$

$$\omega_2 = \sqrt{98.234} \ \boxed{\approx 9.9158 \ \ rad/\sec}$$

$$= \frac{9.9158}{2\pi} \boxed{\approx 1.5781 \ \ Hz}$$

b) $v_1 = \frac{20}{\pi}\omega_1 = \frac{(20)(5.0424)}{\pi} \ \boxed{\approx 32.10 \ \ ft/\sec}$

$$= (32.1)(0.681818) \ \boxed{\approx 22 \ \ mph}$$

$$v_2 = \frac{20}{\pi}\omega_2 = \frac{(20)(9.9158)}{\pi} \ \boxed{\approx 63.13 \ \ ft/\sec}$$

$$= (63.13)(0.681818) \ \boxed{\approx 43 \ \ mph}$$

Exercise

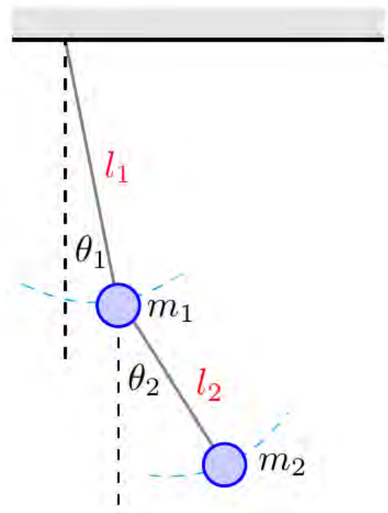
A double pendulum swinging in a vertical plane under the influence of gravity satisfies the system

$$\begin{cases}\left(m_1+m_2\right)\ell_1^2\theta_1''+m_2\ell_1\ell_2\theta_2''+\left(m_1+m_2\right)\ell_1 g\theta_1=0\\ m_2\ell_2^2\theta_2''+m_2\ell_1\ell_2\theta_1''+m_2\ell_2 g\theta_2=0\end{cases}$$

Where θ_1 and θ_2 are small angles.

Solve the system when $m_1=3\ kg,\quad m_2=2\ kg,\quad \ell_1=\ell_2=5\ m$

$$\theta_1(0)=\frac{\pi}{6},\quad \theta_2(0)=0,\quad \theta_1'(0)=\theta_2'(0)=0$$



Solution

$$\begin{cases}125\theta_1''+50\theta_2''=-25g\theta_1\\ 50\theta_2''+50\theta_1''=-10g\theta_2\end{cases}$$

$$\begin{cases}5\theta_1''+2\theta_2''=-9.8\theta_1\\ 5\theta_1''+5\theta_2''=-9.8\theta_2\end{cases}$$

$$\begin{cases}\theta_1''=\frac{9.8}{15}\left(-5\theta_1+2\theta_2\right)\\ \theta_2''=\frac{9.8}{3}\left(\theta_1-\theta_2\right)\end{cases}$$

$$\left|A-\lambda^2 I\right|=\frac{9.8}{3}\begin{vmatrix}-1-\lambda^2 & \frac{2}{5}\\ 1 & -1-\lambda^2\end{vmatrix}$$

$$=\lambda^4+2\lambda^2+\frac{3}{5}=0 \qquad\qquad 5\lambda^4+10\lambda^2+3=0$$

$$\lambda^2=\left(\frac{9.8}{3}\right)\left(-1\pm\frac{\sqrt{10}}{5}\right)$$

The eigenvalues are: $\lambda_{1,2}=\pm\sqrt{\frac{9.8\left(-5-\sqrt{10}\right)}{15}}=\pm i\sqrt{\frac{9.8\left(5+\sqrt{10}\right)}{15}},\quad \lambda_{3,4}=\pm i\sqrt{\frac{9.8\left(5-\sqrt{10}\right)}{15}}$

$$\theta_1(t)=C_1\cos\left(\sqrt{\frac{9.8}{15}\left(5+\sqrt{10}\right)}\,t\right)+C_2\sin\left(\sqrt{\frac{9.8}{15}\left(5+\sqrt{10}\right)}\,t\right)$$
$$+C_3\cos\left(\sqrt{\frac{9.8}{15}\left(5-\sqrt{10}\right)}\,t\right)+C_4\sin\left(\sqrt{\frac{9.8}{15}\left(5-\sqrt{10}\right)}\,t\right)$$

$$\theta_1'=-\sqrt{\frac{9.8}{15}(5+\sqrt{10})}C_1\sin\left(\sqrt{\frac{9.8}{15}(5+\sqrt{10})}\,t\right)+\sqrt{\frac{9.8}{15}(5+\sqrt{10})}C_2\cos\left(\sqrt{\frac{9.8}{15}(5+\sqrt{10})}\,t\right)$$
$$-\sqrt{\frac{9.8}{15}(5-\sqrt{10})}C_3\sin\left(\sqrt{\frac{9.8}{15}(5-\sqrt{10})}\,t\right)+\sqrt{\frac{9.8}{15}(5-\sqrt{10})}C_4\cos\left(\sqrt{\frac{9.8}{15}(5-\sqrt{10})}\,t\right)$$

$$\theta_1(0)=C_1+C_3=\frac{\pi}{6}$$

$$\theta_1'(0)=\sqrt{\frac{9.8}{15}(5+\sqrt{10})}C_2+\sqrt{\frac{9.8}{15}(5-\sqrt{10})}C_4=0$$

$$\frac{15}{9.8}\theta_1'' = -5\theta_1 + 2\theta_2$$

$$\theta_2 = \frac{15}{19.6}\theta_1'' + \frac{5}{2}\theta_1$$

$$\theta_1'' = -\frac{9.8}{15}\left(5+\sqrt{10}\right)C_1\cos\left(\sqrt{\frac{9.8}{15}\left(5+\sqrt{10}\right)}\,t\right) - \frac{9.8}{15}\left(5+\sqrt{10}\right)C_2\sin\left(\sqrt{\frac{9.8}{15}\left(5+\sqrt{10}\right)}\,t\right)$$
$$-\frac{9.8}{15}\left(5-\sqrt{10}\right)C_3\cos\left(\sqrt{\frac{9.8}{15}\left(5-\sqrt{10}\right)}\,t\right) - \frac{9.8}{15}\left(5-\sqrt{10}\right)C_4\sin\left(\sqrt{\frac{9.8}{15}\left(5-\sqrt{10}\right)}\,t\right)$$

$$\theta_2(t) = -\sqrt{10}C_1\cos\left(\sqrt{\frac{9.8}{15}\left(5+\sqrt{10}\right)}\,t\right) - \sqrt{10}C_2\sin\left(\sqrt{\frac{9.8}{15}\left(5+\sqrt{10}\right)}\,t\right)$$
$$+\sqrt{10}C_3\cos\left(\sqrt{\frac{9.8}{15}\left(5-\sqrt{10}\right)}\,t\right) + \sqrt{10}C_4\sin\left(\sqrt{\frac{9.8}{15}\left(5-\sqrt{10}\right)}\,t\right)$$

$$\theta_2' = \sqrt{10}\sqrt{\frac{9.8}{15}\left(5+\sqrt{10}\right)}C_1\sin\left(\sqrt{\frac{9.8}{15}\left(5+\sqrt{10}\right)}\,t\right) - \sqrt{10}\sqrt{\frac{9.8}{15}\left(5+\sqrt{10}\right)}C_2\cos\left(\sqrt{\frac{9.8}{15}\left(5+\sqrt{10}\right)}\,t\right)$$
$$-\sqrt{10}\sqrt{\frac{9.8}{15}\left(5-\sqrt{10}\right)}C_3\sin\left(\sqrt{\frac{9.8}{15}\left(5-\sqrt{10}\right)}\,t\right) + \sqrt{10}\sqrt{\frac{9.8}{15}\left(5-\sqrt{10}\right)}C_4\cos\left(\sqrt{\frac{9.8}{15}\left(5-\sqrt{10}\right)}\,t\right)$$

$$\theta_2(0) = \underline{-\sqrt{10}C_1 + \sqrt{10}C_3 = 0}$$

$$\theta_2'(0) = \underline{-\sqrt{10}\sqrt{\frac{9.8}{15}\left(5+\sqrt{10}\right)}C_2 + \sqrt{10}\sqrt{\frac{9.8}{15}\left(5-\sqrt{10}\right)}C_4 = 0}$$

$$\begin{cases} C_1 + C_3 = \frac{\pi}{6} \\ -\sqrt{10}C_1 + \sqrt{10}C_3 = 0 \end{cases} \rightarrow \underline{C_1 = C_3 = \frac{\pi}{12}}$$

$$\begin{cases} \sqrt{\frac{9.8}{15}\left(5+\sqrt{10}\right)}C_2 + \sqrt{\frac{9.8}{15}\left(5-\sqrt{10}\right)}C_4 = 0 \\ -\sqrt{10}\sqrt{\frac{9.8}{15}\left(5+\sqrt{10}\right)}C_2 + \sqrt{10}\sqrt{\frac{9.8}{15}\left(5-\sqrt{10}\right)}C_4 = 0 \end{cases} \rightarrow \underline{C_2 = C_4 = 0}$$

$$\begin{cases} \theta_1(t) = \frac{\pi}{12}\cos\left(\sqrt{\frac{9.8}{15}\left(5+\sqrt{10}\right)}\,t\right) + \frac{\pi}{12}\cos\left(\sqrt{\frac{9.8}{15}\left(5-\sqrt{10}\right)}\,t\right) \\ \theta_2(t) = -\frac{\pi\sqrt{10}}{12}\cos\left(\sqrt{\frac{9.8}{15}\left(5+\sqrt{10}\right)}\,t\right) + \frac{\pi\sqrt{10}}{12}\cos\left(\sqrt{\frac{9.8}{15}\left(5-\sqrt{10}\right)}\,t\right) \end{cases}$$

Exercise

The motion of a pair of identical pendulums coupled by a spring is modeled by the system

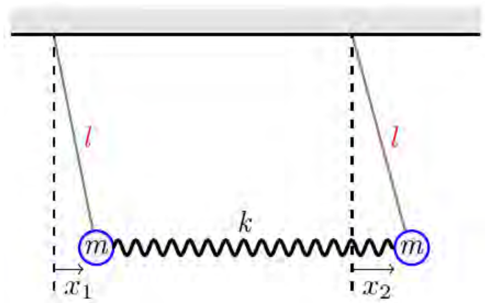
$$\begin{cases} mx_1'' = -\frac{mg}{\ell}x_1 - k\left(x_1 - x_2\right) \\ mx_2'' = -\frac{mg}{\ell}x_2 + k\left(x_1 - x_2\right) \end{cases}$$

For small displacements. Determine the two normal frequencies for the system.

Solution

$$\begin{cases} x_1'' = -\left(\frac{g}{\ell}+\frac{k}{m}\right)x_1 + \frac{k}{m}x_2 \\ x_2'' = \frac{k}{m}x_1 - \left(\frac{g}{\ell}+\frac{k}{m}\right)x_2 \end{cases}$$

$$A = \begin{pmatrix} -\frac{g}{\ell}-\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{g}{\ell}-\frac{k}{m} \end{pmatrix}$$



$$\begin{aligned} |A-\lambda I| &= \begin{vmatrix} -\left(\frac{g}{\ell}+\frac{k}{m}\right)-\lambda & \frac{k}{m} \\ \frac{k}{m} & -\left(\frac{g}{\ell}+\frac{k}{m}\right)-\lambda \end{vmatrix} \\ &= \lambda^2 + 2\left(\frac{g}{\ell}+\frac{k}{m}\right)\lambda + \left(\frac{g}{\ell}+\frac{k}{m}\right)^2 - \left(\frac{k}{m}\right)^2 \\ &= \lambda^2 + 2\left(\frac{g}{\ell}+\frac{k}{m}\right)\lambda + \left(\frac{g}{\ell}\right)^2 + 2\frac{kg}{m\ell} \end{aligned}$$

$$\begin{aligned} \lambda_{1,2} &= -\left(\frac{g}{\ell}+\frac{k}{m}\right) \pm \sqrt{4\left(\frac{g}{\ell}+\frac{k}{m}\right)^2 - 4\left(\frac{g}{\ell}\right)^2 - 8\frac{kg}{m\ell}} \\ &= -\left(\frac{g}{\ell}+\frac{k}{m}\right) \pm \frac{k}{m} \end{aligned}$$

The eigenvalues are: $\lambda_1 = -\frac{g}{\ell}, \quad \lambda_2 = -\frac{g}{\ell} - 2\frac{k}{m}$

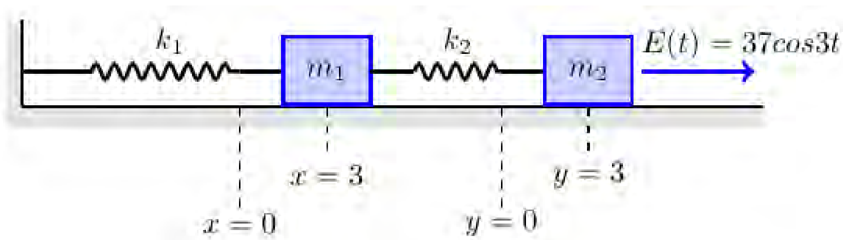
The natural frequencies:

$$\boxed{\omega_1 = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}}$$

$$\boxed{\omega_2 = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}+\frac{2k}{m}}}$$

Exercise

On a smooth horizontal surface $m_1 = 2\ kg$ is attached to a fixed wall by a spring with spring constant $k_1 = 4\ N/m$. Another mass $m_2 = 1\ kg$ is attached to the first object by a spring with spring constant $k_2 = 2\ N/m$. The object are aligned horizontally so that the springs are their natural lengths.



Suppose an external force $E(t) = 37\cos 3t$ is applied to the second object of mass 1 kg.

a) Find the general solution

b) Show that $x(t)$ satisfies the equation $x^{(4)}(t)+5x''(t)+4x(t)=37\cos 3t$

c) Find a general solution $x(t)$ to equation in part (*b*).

d) Substitute $x(t)$ to obtain a formula for $y(t)$

e) If both masses are displaced 2 *m* to the right of their equilibrium positions and then released, find the displacement functions $x(t)$ and $y(t)$

Solution

a) Applying Newton's second law:

$$\begin{cases} m_1 x''+k_1 x+k_2(x-y)=0 & (1) \\ m_2 y''+k_2(y-x)=E(t) & (2) \end{cases}$$

$$\begin{cases} m_1 x''=-(k_1+k_2)x+k_2 y \\ m_2 y''=k_2 x-k_2 y+37\cos 3t \end{cases}$$

Given: $m_1=2\ kg$, $m_2=1\ kg$, $k_1=4\ N/m$, and $k_2=2\ N/m$

$$\begin{cases} 2x''=-6x+2y \\ y''=2x-2y+37\cos 3t \end{cases}$$

$$\begin{cases} x''=-3x+y & (3) \\ y''=2x-2y+37\cos 3t & (4) \end{cases}$$

b) $\frac{d}{dx}\left(x''=-3x+y\right)$

$$x^{(3)}=-3x'+y'$$

$$\frac{d}{dx}\left(x^{(3)}=-3x'+y'\right)$$

$$x^{(4)}=-3x''+y'' \qquad (4)\to y''=2x-2y+37\cos 3t \qquad (3)\ \to\ y=x''+3x$$

$$x^{(4)}+3x''-2x+2(x''+3x)-37\cos 3t=0$$

$$x^{(4)}(t)+5x''(t)+4x(t)=37\cos 3t$$

c) $\lambda^4+5\lambda^2+4=0 \quad\to\quad \lambda^2=-1,\ -4$

The eigenvalues are: $\lambda_{1,2}=\pm i \quad \lambda_{3,4}=\pm 2i$

$$x_h=C_1\cos t+C_2\sin t+C_3\cos 2t+C_4\sin 2t$$

$$x_p=A\cos 3t$$

$$x'_p=-3A\sin 3t$$

$$x''_p=-9A\cos 3t$$

$$x'''_p=27A\sin 3t$$

$$x^{(4)}_p=81A\cos 3t$$

$$x^{(4)}(t)+5x''(t)+4x(t)=37\cos 3t$$

$$81A\cos 3t-45A\cos 3t+4A\cos 3t=37\cos 3t$$

$$40A=37 \quad \rightarrow \quad \boxed{A=\frac{37}{40}}$$

$$x_p=\frac{37}{40}\cos 3t$$

$$\boxed{x(t)=C_1\cos t+C_2\sin t+C_3\cos 2t+C_4\sin 2t+\frac{37}{40}\cos 3t}$$

d) $(3) \rightarrow y=x''+3x$

$$x'=-C_1\sin t+C_2\cos t-2C_3\sin 2t+2C_4\cos 2t-\frac{111}{40}\sin 3t$$

$$x''=-C_1\cos t-C_2\sin t-4C_3\cos 2t-4C_4\sin 2t-\frac{333}{40}\cos 3t$$

$$\boxed{y(t)=2C_1\cos t+2C_2\sin t-C_3\cos 2t-C_4\sin 2t-\frac{111}{20}\cos 3t}$$

e) ***Given***: $x(0)=2 \quad x'(0)=0$

$$x(0)=C_1+C_3+\frac{37}{40}=2$$

$$\boxed{C_1+C_3=\frac{43}{40} \quad (5)}$$

$$x'=-C_1\sin t+C_2\cos t-2C_3\sin 2t+2C_4\cos 2t-\frac{111}{40}\sin 3t$$

$$x'(0)=\boxed{C_2+2C_4=0 \quad (6)}$$

Given: $y(0)=2 \quad y'(0)=0$

$$y(0)=2C_1-C_3-\frac{111}{20}=2$$

$$\boxed{2C_1-C_3=\frac{151}{20} \quad (7)}$$

$$y'=-2C_1\sin t+2C_2\cos t-2C_3\sin 2t-2C_4\cos 2t+\frac{333}{20}\sin 3t$$

$$y'(0)=2C_2-2C_4=0 \quad (8)$$

$$\begin{cases} (5) & C_1+C_3=\frac{43}{40} \\ (7) & 2C_1-C_3=\frac{151}{20} \end{cases} \rightarrow \boxed{C_1=\frac{345}{120}=\frac{23}{8}, \ C_3=-\frac{216}{120}=-\frac{9}{5}}$$

$$\begin{cases} (6) & C_2+2C_4=0 \\ (8) & 2C_2-2C_4=0 \end{cases} \rightarrow \boxed{C_2=C_4=0}$$

$$\boxed{x(t)=\frac{23}{8}\cos t-\frac{9}{5}\cos 2t+\frac{37}{40}\cos 3t}$$

$$\boxed{y(t)=\frac{23}{4}\cos t+\frac{9}{5}\cos 2t-\frac{111}{20}\cos 3t}$$