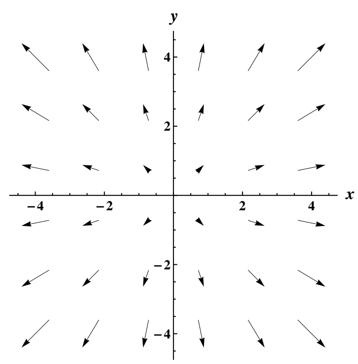
Chapter 17

Vector Calculus

17.1 Vector Fields

17.1.1 A vector field describes the motion of the air as a vector at each point in the room.

17.1.2



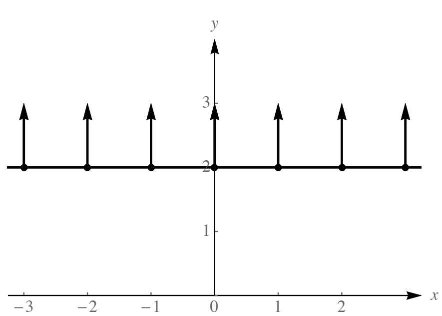
17.1.3 At selected points (a, b), plot the vector $\langle f(a, b), g(a, b)\rangle$.

17.1.4 The gradient of a function at a point is a vector describing the direction in which the value of the function is increasing most rapidly. The collection of these vectors over all points is a vector field.

17.1.5 The gradient field gives, at each point, the direction in which the temperature is increasing most rapidly and the amount of increase.

17.1.6 The length of $\mathbf{F}$ is $\left|\dfrac{\sqrt{2}\,\langle x, y\rangle}{\sqrt{x^2+y^2}}\right| = \dfrac{\sqrt{2}}{\sqrt{x^2+y^2}}\,|\langle x, y\rangle| = \dfrac{\sqrt{2}}{\sqrt{x^2+y^2}}\sqrt{x^2+y^2} = \sqrt{2}$.

17.1.7



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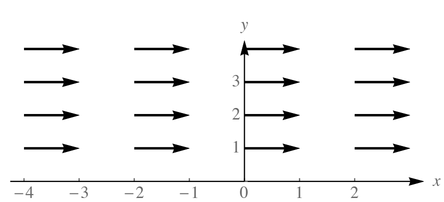

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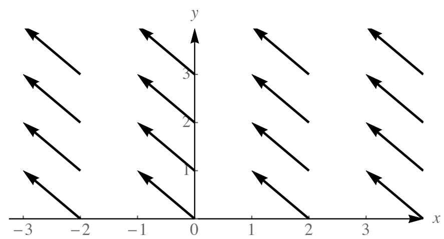

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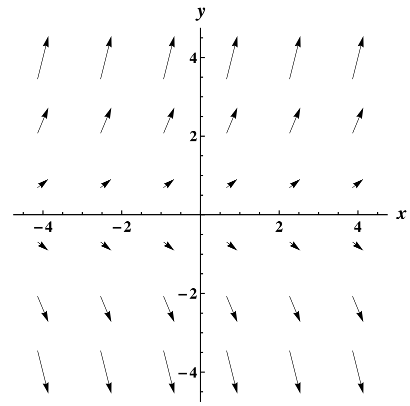

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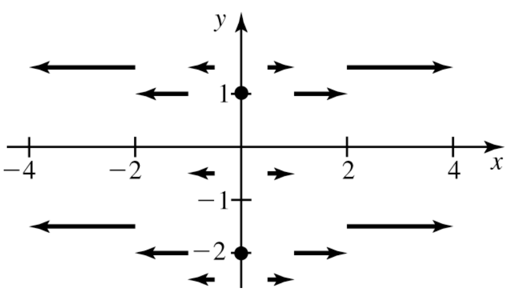

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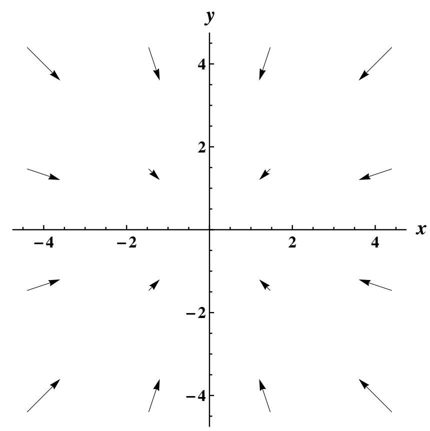

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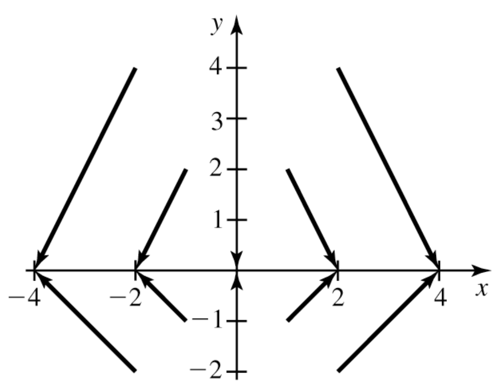

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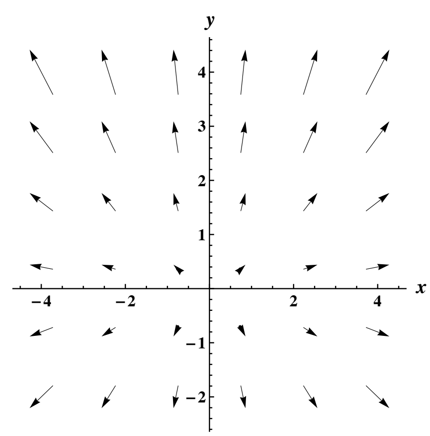

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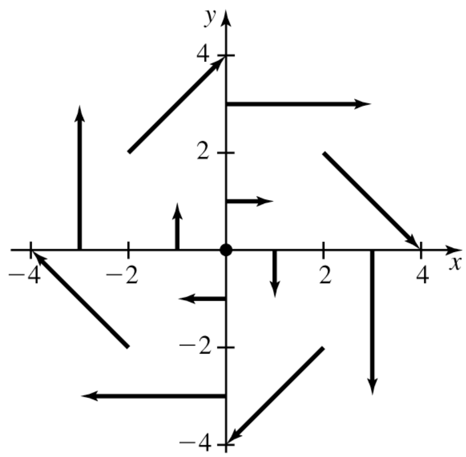

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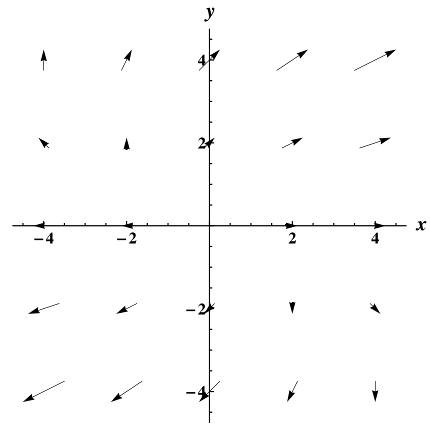
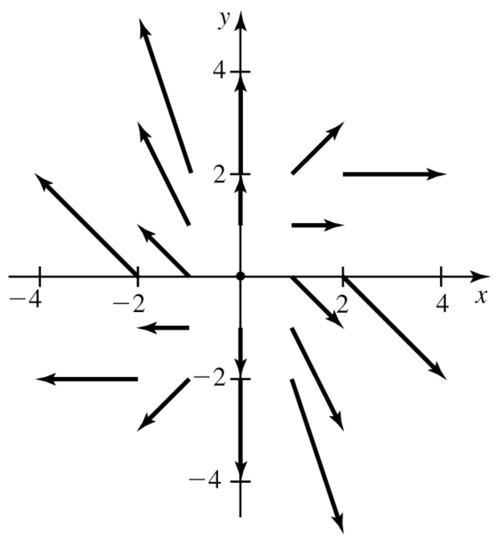


17.1.17



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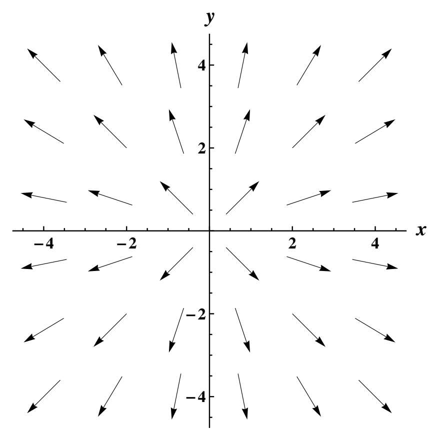

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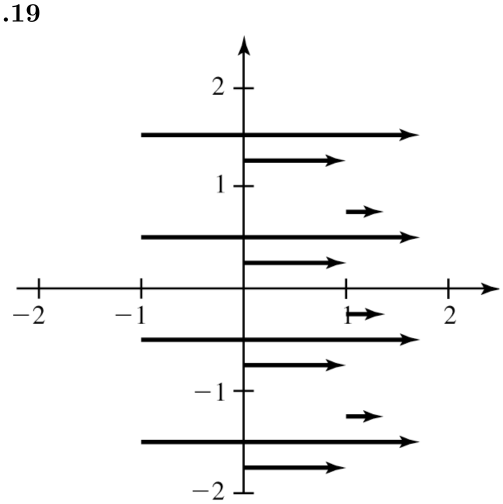

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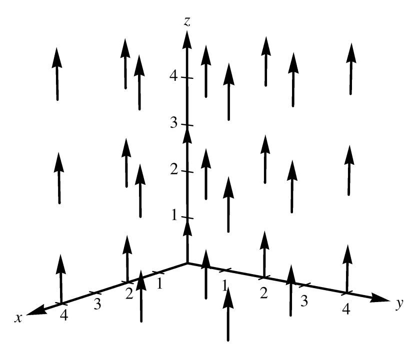

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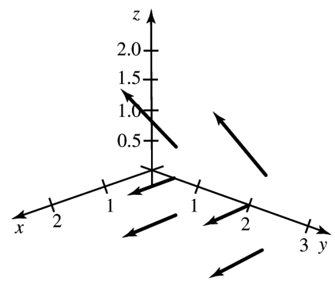

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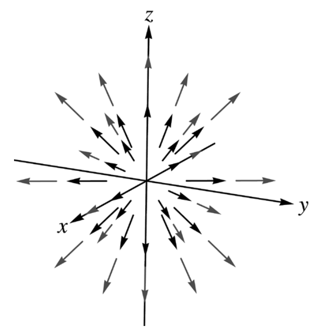

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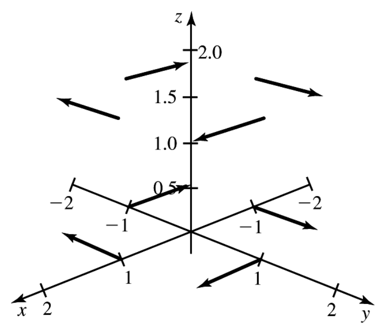


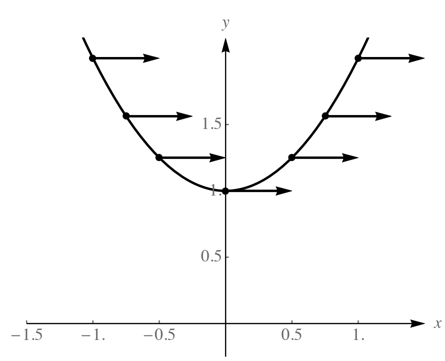
17.1.24 (a) corresponds to (D), since (D) has zero x component, and the y component increases as y does. (b) corresponds to (C), since the x component appears to be zero along the line $y = x$. (c) corresponds to (B) since the y component is zero on the x-axis and the x component is zero on the y axis. Finally, (d) corresponds to (A) since the x component is zero on the x-axis and the y component is zero on the y-axis.

17.1.25

a. Let $g(x, y) = y - x^2$. Then $\nabla g = \langle -2x, 1\rangle$. This is orthogonal to $\mathbf{F}$ when $\nabla g \cdot \mathbf{F} = 0$, or when $\frac{1}{2}(-2x) + 0 = 0$, or $x = 0$. So the point on C is $(0, 1)$

b. ∇g is never parallel to $\mathbf{F}$, because no nonzero multiple of $\left\langle \frac{1}{2}, 0\right\rangle$ can have a second component of 1.

c.

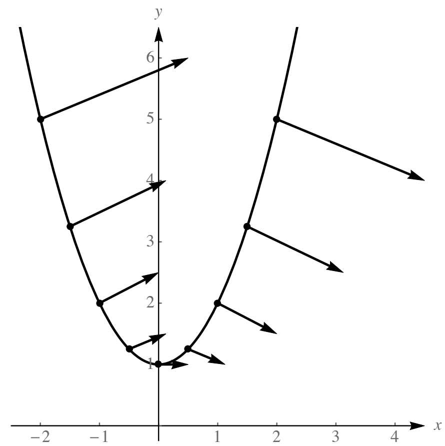


17.1.26

a. Let $g(x, y) = y - x^2$. Then $\nabla g = \langle -2x, 1\rangle$. This is perpendicular to $\mathbf{F}$ when $\nabla g \cdot \mathbf{F} = 0$, or $\frac{y}{2}(-2x) + \frac{-x}{2}(1) = 0$, or $-xy - \frac{x}{2} = 0$, so for $x = 0$ and $y = -\frac{1}{2}$. However, on our curve, $y \geq 1$, so we must have $x = 0$, in which case $y = 1$. So $\mathbf{F}$ is tangent to C at $(0, 1)$.

b. If $\mathbf{F} = k\nabla g$, then $y = -4kx$ and $x = -2k$. Substituting into the equation for C, we have $2x^2 - x^2 = 1$, so $x = \pm 1$ and $y = 2$. So $\mathbf{F}$ is normal to C at the points $(\pm 1, 2)$.

c.

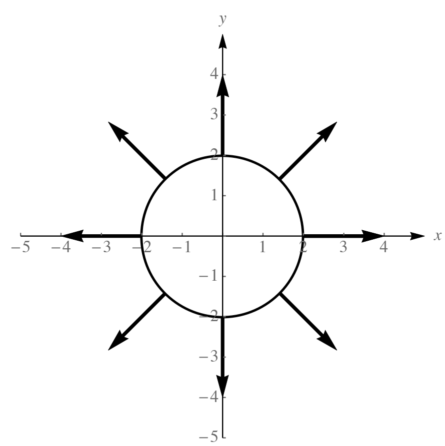


17.1.27

a. Let $g(x, y) = x^2+y^2$. Then $\nabla g = \langle 2x, 2y\rangle$. This is perpendicular to $\mathbf{F}$ when $\nabla g\cdot\mathbf{F} = 0$, or $2x^2+2y^2 = 0$, which doesn't occur on the circle C.

b. If $\mathbf{F} = k\nabla g$, then $\langle x, y\rangle = k\,\langle 2x, 2y\rangle$, which occurs for all points on the circle.

c.

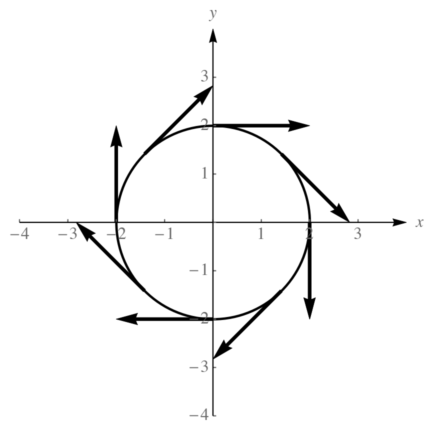


17.1.28

a. Let $g(x, y) = x^2+y^2$. Then $\nabla g = \langle 2x, 2y\rangle$. This is perpendicular to $\mathbf{F}$ when $\nabla g\cdot\mathbf{F} = 0$, or $2xy-2xy = 0$, which occurs for all points on the circle.

b. If $\mathbf{F} = k\nabla g$, then $\langle y, -x\rangle = k\,\langle 2x, 2y\rangle$, or $y = 2kx$ and $x = -2ky$. But then $y = 2kx = 2k(-2ky) = -4k^2y$, which doesn't occur. There are no points where $\mathbf{F}$ is normal to the curve.

c.

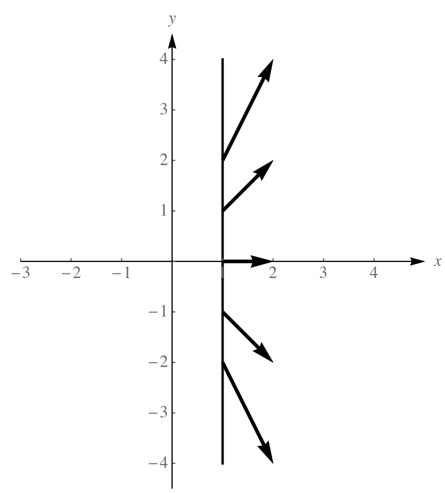


17.1.29

a. Let $g(x, y) = x$. Then $\nabla g = \langle 1, 0\rangle$. This is perpendicular to $\mathbf{F}$ for no points on C.

b. If $\mathbf{F} = k\nabla g$, then $y = 0$, so the point on C is $(1, 0)$.

c.

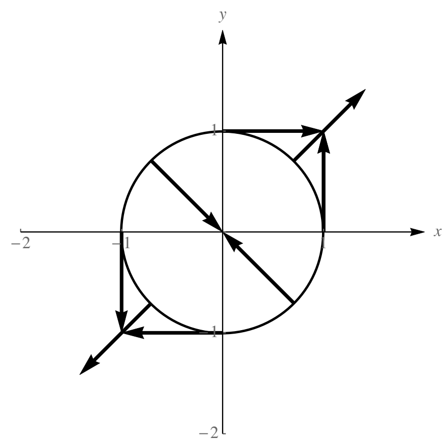


17.1.30

a. Let $g(x, y) = x^2 + y^2$. Then $\nabla g = \langle 2x, 2y\rangle$. This is perpendicular to $\mathbf{F}$ if $2xy + 2yx = 4xy = 0$, which occurs if $x = 0$ or $y = 0$. So the corresponding points on C are $(\pm 1, 0)$ and $(0, \pm 1)$.

b. If $\mathbf{F} = k\nabla g$, then $\langle y, x\rangle = k\,\langle 2x, 2y\rangle$, so $y = 2kx$ and $x = 2ky$. Then $y = 2kx = 2k(2ky)$, so $4k^2 = 1$ and $k = \pm\dfrac{1}{2}$. Then either $x = y$ or $x = -y$, so the points are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

c.



17.1.31 For example, $\mathbf{F} = \left\langle -y,\, x \right\rangle$ or $\mathbf{F} = \left\langle -1,\, 1 \right\rangle$.

17.1.32 For example, $\mathbf{F} = \left\langle y,\, 0 \right\rangle$.

17.1.33 For example, $\mathbf{F} = \frac{1}{\sqrt{x^2+y^2}}\left\langle x, y \right\rangle = \frac{\mathbf{r}}{|\mathbf{r}|}$, $\mathbf{F}\left(0,0\right) = \mathbf{0}$.

17.1.34 For example, $\mathbf{F} = \left\langle -y,\, x \right\rangle$. The magnitude is $\sqrt{x^2+y^2}$.

17.1.35 $\nabla\varphi = \left\langle \varphi_x, \varphi_y \right\rangle = \left\langle 2xy - y^2, x^2 - 2xy \right\rangle$.

17.1.36 $\nabla\varphi = \left\langle \varphi_x, \varphi_y \right\rangle = \left\langle \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}} \right\rangle = \frac{1}{2\sqrt{xy}}\left\langle y, x \right\rangle$.

17.1.37 $\nabla\varphi = \left\langle \varphi_x, \varphi_y \right\rangle = \left\langle 1/y, -x/y^2 \right\rangle$.

17.1.38 $\nabla\varphi = \left\langle \frac{1}{1+(x^2/y^2)} \cdot \frac{1}{y}, \frac{1}{1+(x^2/y^2)} \cdot \frac{-x}{y^2} \right\rangle = \left\langle \frac{y}{y^2+x^2}, \frac{-x}{y^2+x^2} \right\rangle = \frac{1}{x^2+y^2}\left\langle y, -x \right\rangle$.

17.1.39 $\nabla\varphi = \left\langle x, y, z \right\rangle = \mathbf{r}$.

17.1.40 $\nabla\varphi = \left\langle \frac{2x}{1+x^2+y^2+z^2}, \frac{2y}{1+x^2+y^2+z^2}, \frac{2z}{1+x^2+y^2+z^2} \right\rangle = \frac{2\,\mathbf{r}}{|\mathbf{r}|^2+1}$.

17.1.41 $\nabla\varphi = -\left(x^2+y^2+z^2\right)^{-3/2}\left\langle x,\, y,\, z \right\rangle = -\frac{\mathbf{r}}{|\mathbf{r}|^3}$.

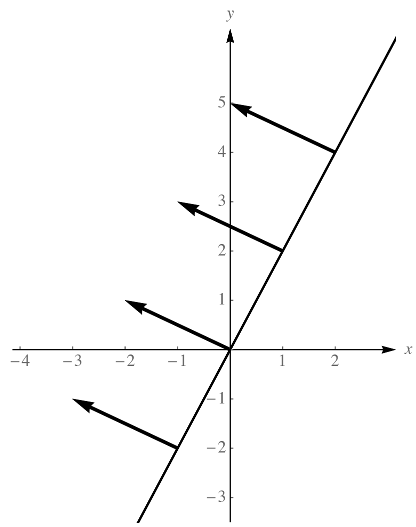
17.1.42 $\nabla\varphi = \left\langle e^{-z}\cos\left(x+y\right),\, e^{-z}\cos\left(x+y\right),\, -e^{-z}\sin\left(x+y\right) \right\rangle$.

17.1.43

a. $\mathbf{F} = \nabla\varphi = \langle -2, 1 \rangle$.

b. $\mathbf{F}(-1,-2) = \mathbf{F}(0,0) = \mathbf{F}(1,2) = \mathbf{F}(2,4) = \langle -2, 1 \rangle$.

c.

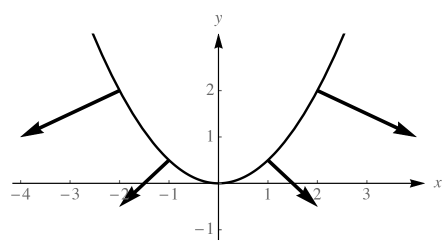


17.1.44

a. $\mathbf{F} = \nabla\varphi = \langle x, -1\rangle$.

b. $\mathbf{F}(-2, 2) = \langle -2, -1\rangle$. $\mathbf{F}(-1, 1/2) = \langle -1, -1\rangle$, $\mathbf{F}(1, 1/2) = \langle 1, -1\rangle$, $\mathbf{F}(2, 2) = \langle 2, -1\rangle$.

c.

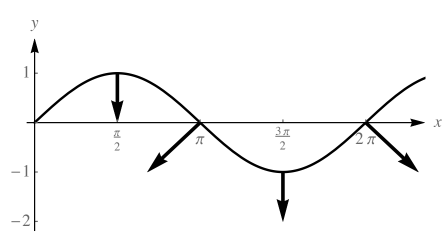


17.1.45

a. $\mathbf{F} = \nabla\varphi = \langle \cos x, -1\rangle$.

b. $\mathbf{F}(\pi/2, 1) = \langle 0, -1\rangle$. $\mathbf{F}(\pi, 0) = \langle -1, -1\rangle$, $\mathbf{F}(3\pi/2, -1) = \langle 0, -1\rangle$, $\mathbf{F}(2\pi, 0) = \langle 1, -1\rangle$.

c.

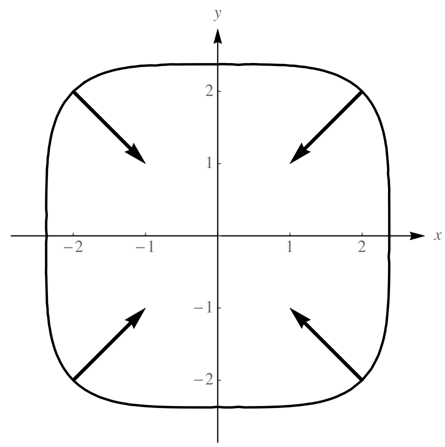


17.1.46

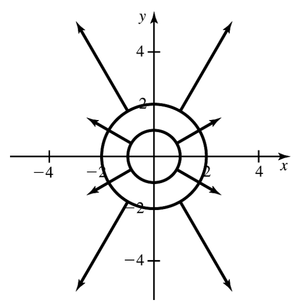
a. $\mathbf{F} = \nabla\varphi = \left\langle -x^3/8, -y^3/8\right\rangle$.

b. $\mathbf{F}(2, 2) = \langle -1, -1\rangle$. $\mathbf{F}(-2, 2) = \langle 1, -1\rangle$, $\mathbf{F}(-2, -2) = \langle 1, 1\rangle$, $\mathbf{F}(2, -2) = \langle -1, 1\rangle$.

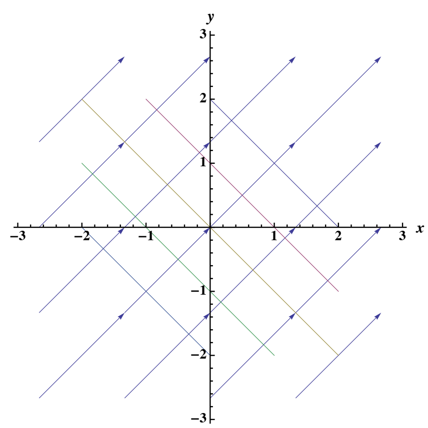
c.



17.1.47 The gradient field is $\langle \varphi_x, \varphi_y \rangle = \langle 2x,\, 2y \rangle$.



17.1.48 The gradient field is $\langle \varphi_x, \varphi_y \rangle = \langle 1, 1 \rangle$.



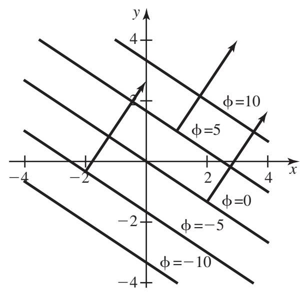
17.1.49

a. The gradient field is $\langle 2,\, 3 \rangle$.

b. The equipotential curve at $(1,\,1)$ is $2x + 3y = 5$, which is a line of slope $-\frac{2}{3}$ so has a tangent vector at $(x,\,y)$ parallel to $\langle 1,\, -\frac{2}{3} \rangle$. But $\langle 2,\, 3 \rangle \cdot \langle 1,\, -\frac{2}{3} \rangle = 0$, so the gradient field is normal to the equipotential line through $(1,\,1)$.

c. The equipotential curve at any point is a line of slope $-\frac{2}{3}$ and thus has a tangent vector at $(x,\,y)$ parallel to $\langle x,\, -\frac{2}{3}y \rangle$. The same argument as in part (b) shows that this is normal to the gradient field.

d.



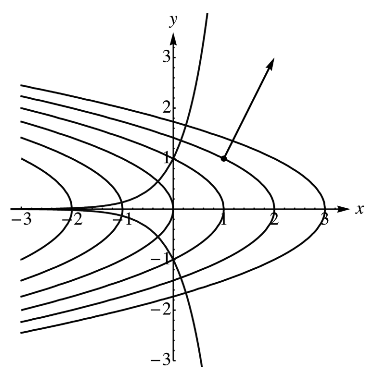
17.1.50

a. The gradient field is $\langle 1, 2y\rangle$.

b. At (1, 1), the tangent vector is parallel to $\langle -\varphi_y(1, 1), \varphi_x(1, 1)\rangle = \langle -2, 1\rangle$, which is normal to the gradient at (1, 1) (which is $\langle 1, 2\rangle$).

c. At (x, y), the tangent vector is parallel to $\langle -2y, 1\rangle$, and $\langle 1, 2y\rangle \cdot \langle -2y, 1\rangle = 0$, so the gradient is everywhere normal to the equipotential curves.

d.



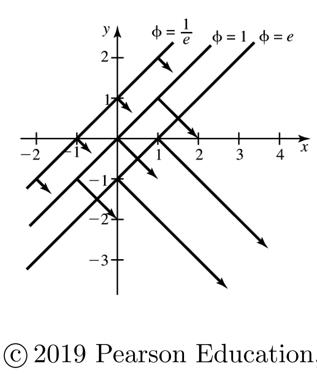
17.1.51

a. The gradient field is $\langle e^{x-y}, -e^{x-y}\rangle$.

b. At (1, 1), the tangent vector is parallel to $\langle -\varphi_y(1, 1), \varphi_x(1, 1)\rangle = \langle 1, 1\rangle$, which is normal to the gradient $\langle 1, -1\rangle$ at (1, 1).

c. At (x, y), the tangent vector is parallel to $\langle e^{x-y}, e^{x-y}\rangle$, and $\langle e^{x-y}, -e^{x-y}\rangle \cdot \langle e^{x-y}, e^{x-y}\rangle = 0$, so the gradient is everywhere normal to the equipotential curves.

d.



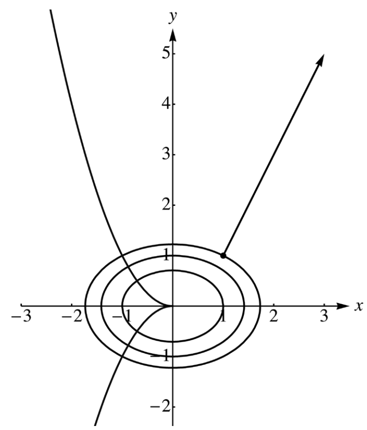
17.1.52

a. The gradient field is $\langle 2x,\, 4y\rangle$.

b. At $(1,\, 1)$, the tangent vector is parallel to $\langle -\varphi_y(1,\, 1),\, \varphi_x(1,\, 1)\rangle = \langle -4,\, 2\rangle$, which is normal to the gradient $\langle 2,\, 4\rangle$ at $(1,\, 1)$.

c. At $(x,\, y)$, the tangent vector is parallel to $\langle -4y,\, 2x\rangle$, which is normal to the gradient field.

d.



17.1.53

a. True. $(\varphi_1)_x = (\varphi_2)_x = 3x^2$, and $(\varphi_1)_y = (\varphi_2)_y = 1$.

b. False. It is constant in magnitude (magnitude 1) but not direction.

c. True. For example, it points outwards along the line $y = x$ but horizontally along the line $x = 0$.

17.1.54

a. $V(x,\, y) = k\left(x^2 + y^2\right)^{-1/2}$, so $\mathbf{E} = -\nabla V = -\langle V_x,\, V_y\rangle = k\left(x^2 + y^2\right)^{-3/2}\langle x,\, y\rangle$.

b. From the above formula, the field is a varying multiple of $\langle x,\, y\rangle$, which is a radial field pointing away from the origin. The radial component of $\mathbf{E}$ is thus $|\mathbf{E}| = k\left(x^2 + y^2\right)^{-3/2}\left|\langle x,\, y\rangle\right| = k\left(x^2 + y^2\right)^{-3/2}\left(x^2 + y^2\right)^{1/2} = \dfrac{k}{r^2}$.

c. The equipotential curves are curves of the form $\dfrac{k}{\sqrt{x^2 + y^2}} = C$ so that $\sqrt{x^2 + y^2} = \frac{k}{C}$ and the equipotential curves are circles. Thus the tangent vectors to the equipotential curves are proportional to $\left|\langle -y,\, x\rangle\right|$ and thus are normal to $\mathbf{E}$, which is proportional to $\left|\langle x,\, y\rangle\right|$.

17.1.55

a. $V_x = \dfrac{c\sqrt{x^2 + y^2}}{r_0} \cdot \left(-\dfrac{1}{2} r_0\left(x^2 + y^2\right)\right)$ and similarly, $V_y = -\frac{cy}{x^2+y^2}$, so that $\mathbf{E} = -\nabla V = \dfrac{c}{x^2 + y^2}\langle x,\, y\rangle = \dfrac{c}{|\mathbf{r}|^2}\mathbf{r} = \dfrac{c}{|\mathbf{r}|} \cdot \dfrac{\mathbf{r}}{|\mathbf{r}|}$.

b. From the above formula, the field is a varying multiple of $\langle x,\, y\rangle$, which is a radial field pointing away from the origin. The radial component of $\mathbf{E}$ is thus $|\mathbf{E}| = \dfrac{c}{x^2 + y^2}\sqrt{x^2 + y^2} = \dfrac{c}{\sqrt{x^2 + y^2}}$.

c. The equipotential curves are curves of the form $c\ln\left(\dfrac{r_0}{\sqrt{x^2+y^2}}\right) = K$, so are solutions to $\frac{r_0}{\sqrt{x^2+y^2}} = e^{cK} = C$ and thus of the form $\sqrt{x^2+y^2} = K_0$ for some constant K_0. Hence the equipotential curves are circles, so have tangent vectors proportional to $\langle -y,\, x\rangle$; these are clearly normal to $\mathbf{E}$, which is proportional to $\langle x,\, y\rangle$.

17.1.56

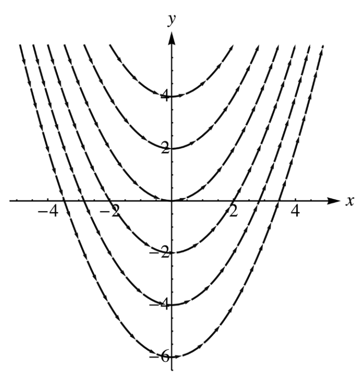
a. $U_x = \left(GMm\left(x^2+y^2+z^2\right)^{-1/2}\right)_x =$
$-\frac{1}{2}GMm\left(x^2+y^2+z^2\right)^{-3/2}(2x) = -GMmx\left(x^2+y^2+z^2\right)^{-3/2}$ and similarly for U_y and U_z. Thus $\mathbf{F} = -\nabla U = GMm\left(x^2+y^2+z^2\right)^{-3/2}\langle x, y, z\rangle$.

b. From the above formula the field is a varying multiple of $\langle x, y, z\rangle$, which is a radial field pointing away from the origin. The radial component of $\mathbf{F}$ is thus $|\mathbf{F}| = GMm\left(x^2+y^2+z^2\right)^{-3/2}\sqrt{x^2+y^2+z^2} = \dfrac{GMm}{r^2}$.

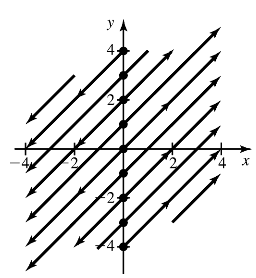
c. The equipotential surfaces are solutions to $GMm\sqrt{x^2+y^2+z^2} = K$ and so are spheres. The tangent plane at (x_0, y_0, z_0) is $U_x(x_0, y_0, z_0)(x-x_0) + U_y(x_0, y_0, z_0)(y-y_0) + U_z(x_0, y_0, z_0)(z-z_0)$ and so a normal to the plane is $\langle U_x, U_y, U_z\rangle$, which is proportional to $\mathbf{F}$.

17.1.57 The flow curve $y(x)$ of the vector field $\mathbf{F}$ at (x, y) is defined to be a continuous curve through (x, y) that is aligned with the vector field, i.e. whose tangent at (x, y) is given by $\mathbf{F}(x, y) = \langle f(x, y), g(x, y)\rangle$. The slope of the tangent line is then $\frac{g(x,y)}{f(x,y)}$, so this is $y'(x)$.

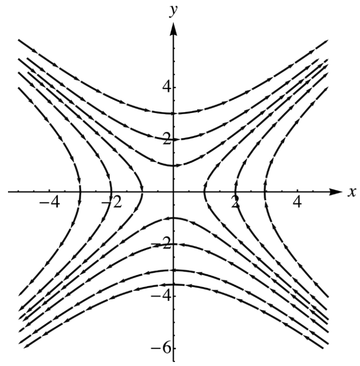
17.1.58 The streamlines satisfy $y'(x) = x$, so that $y(x) = \frac{1}{2}x^2 + C$.



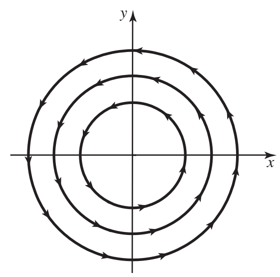
17.1.59 The streamlines satisfy $y'(x) = 1$, so that $y(x) = x + C$.



17.1.60 The streamlines satisfy $y'(x) = \dfrac{x}{y}$. But also $\dfrac{d}{dx}\left(y^2\right) = 2yy'(x)$, so that $y'(x) = \dfrac{\frac{d}{dx}\left(y^2\right)}{2y}$ and thus $\dfrac{d}{dx}\left(y^2\right) = 2x$. Thus $y^2 = x^2 + C$ and the streamlines are the hyperbolas $x^2 - y^2 = K$.



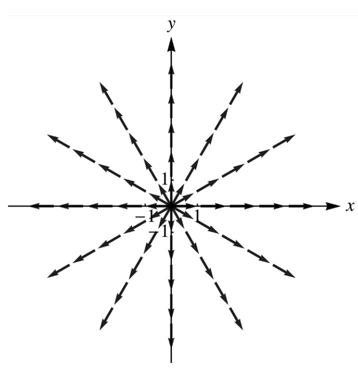
17.1.61 The streamlines satisfy $y'(x) = -\dfrac{x}{y}$. Because $\dfrac{d}{dx}\left(y^2\right) = 2yy'(x)$, we have $y'(x) = \dfrac{\frac{d}{dx}\left(y^2\right)}{2y}$ and thus $\dfrac{d}{dx}\left(y^2\right) = -2x$. Thus $y^2 = -x^2 + C$ and the streamlines are the circles $x^2 + y^2 = C$.



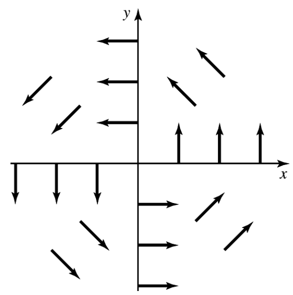
17.1.62 $\mathbf{u}_r$ is the unit vector forming an angle of θ with $\mathbf{i}$, so it is of unit length and proportional to $\left\langle \cos\theta, \sin\theta \right\rangle$. But $\cos^2(\theta) + \sin^2(\theta) = 1$, so this is in fact the unit vector. Thus $\mathbf{u}_r = \cos(\theta)\,\mathbf{i} + \sin(\theta)\,\mathbf{j}$. Similarly, $\mathbf{u}_\theta$ forms an angle of $\theta + \frac{\pi}{2}$ with $\mathbf{i}$, so it is the unit vector $\left\langle \cos\left(\theta + \frac{\pi}{2}\right), \sin\left(\theta + \frac{\pi}{2}\right) \right\rangle = \left\langle -\sin(\theta), \cos(\theta) \right\rangle$. The other two formulas can be found by solving these for $\mathbf{i}$, $\mathbf{j}$ as linear equations. For example, $\mathbf{u}_r = \cos(\theta)\,\mathbf{i} + \sin(\theta)\,\mathbf{j}$, $\mathbf{u}_\theta = -\sin(\theta)\,\mathbf{i} + \cos(\theta)\,\mathbf{j}$. Multiply the first equation by $\sin(\theta)$ and the second by $\cos(\theta)$ and add to obtain $\mathbf{u}_r \sin(\theta) + \mathbf{u}_\theta \cos(\theta) = \sin^2(\theta)\,\mathbf{j} + \cos^2(\theta)\,\mathbf{j} = \mathbf{j}$.

17.1.63 For $\theta = 0$, $\mathbf{u}_r$ is coincident with $\mathbf{i}$, and $\mathbf{u}_\theta$ with $\mathbf{j}$. From the formula $\mathbf{u}_r = \cos(0)\,\mathbf{i} + \sin(0)\,\mathbf{j} = \mathbf{i}$, $\mathbf{u}_\theta = -\sin(0)\,\mathbf{i} + \cos(0)\,\mathbf{j} = \mathbf{j}$. For $\theta = \frac{\pi}{2}$, the picture implies that we should have $\mathbf{u}_r = \mathbf{j}$, $\mathbf{u}_\theta = -\mathbf{i}$. From the formulas, $\mathbf{u}_r = \cos\left(\frac{\pi}{2}\right)\mathbf{i} + \sin\left(\frac{\pi}{2}\right)\mathbf{j} = \mathbf{j}$, $\mathbf{u}_\theta = -\sin\left(\frac{\pi}{2}\right)\mathbf{i} + \cos\left(\frac{\pi}{2}\right)\mathbf{j} = -\mathbf{i}$. For $\theta = \pi$, $\mathbf{u}_r$ is coincident with $-\mathbf{i}$, and $\mathbf{u}_\theta$ with $-\mathbf{j}$. From the formula $\mathbf{u}_r = \cos(\pi)\,\mathbf{i} + \sin(\pi)\,\mathbf{j} = -\mathbf{i}$, $\mathbf{u}_\theta = -\sin(\pi)\,\mathbf{i} + \cos(\pi)\,\mathbf{j} = -\mathbf{j}$. For $\theta = \frac{3\pi}{2}$, the picture implies that we should have $\mathbf{u}_r = -\mathbf{j}$, $\mathbf{u}_\theta = \mathbf{i}$. From the formulas, $\mathbf{u}_r = \cos\left(\frac{3\pi}{2}\right)\mathbf{i} + \sin\left(\frac{3\pi}{2}\right)\mathbf{j} = -\mathbf{j}$, $\mathbf{u}_\theta = -\sin\left(\frac{3\pi}{2}\right)\mathbf{i} + \cos\left(\frac{3\pi}{2}\right)\mathbf{j} = \mathbf{i}$.

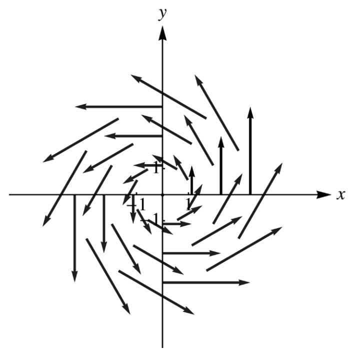
17.1.64 $\mathbf{F}(r, \theta) = \mathbf{u}_r = \cos(\theta)\,\mathbf{i} + \sin(\theta)\,\mathbf{j} = \dfrac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2}}\mathbf{j} = \dfrac{1}{\sqrt{x^2+y^2}}(x\mathbf{i} + y\mathbf{j})$.



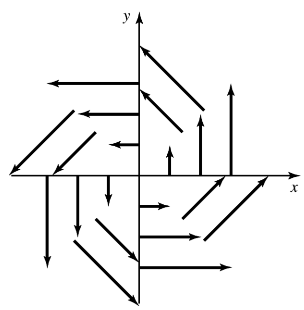
17.1.65 $\mathbf{F}(r,\,\theta) = \mathbf{u}_\theta = -\sin(\theta)\,\mathbf{i} + \cos(\theta)\,\mathbf{j} = -\dfrac{y}{\sqrt{x^2+y^2}}\mathbf{i} + \dfrac{x}{\sqrt{x^2+y^2}}\mathbf{j} = \dfrac{1}{\sqrt{x^2+y^2}}(-y\mathbf{i}+x\mathbf{j})$.



17.1.66 $\mathbf{F}(r,\,\theta) = r\,\mathbf{u}_\theta = \sqrt{x^2+y^2}\,(-\sin(\theta)\,\mathbf{i} + \cos(\theta)\,\mathbf{j}) = \sqrt{x^2+y^2}\dfrac{1}{\sqrt{x^2+y^2}}(-y\mathbf{i}+x\mathbf{j}) = \langle -y,\,x\rangle$.



17.1.67 $\mathbf{F}(x,y) = -y\,(\mathbf{u}_r\cos(\theta) - \mathbf{u}_\theta\sin(\theta)) + x\,(\mathbf{u}_r\sin(\theta) + \mathbf{u}_\theta\cos(\theta)) =$
$-r\sin(\theta)\,(\cos(\theta)\,\mathbf{u}_r - \sin(\theta)\,\mathbf{u}_\theta) + r\cos(\theta)\,(\sin(\theta)\cos(\theta)\,\mathbf{u}_r + \cos(\theta)\,\mathbf{u}_\theta) = r\left(\sin^2(\theta) + \cos^2(\theta)\right)\mathbf{u}_\theta = r\mathbf{u}_\theta.$



17.2 Line Integrals

17.2.1 A single-variable integral integrates along a segment while a line integral integrates along an arbitrary curve.

17.2.2 $|\mathbf{r}'(t)| = \sqrt{\left(\mathbf{r}'_x\right)^2 + \left(\mathbf{r}'_y\right)^2} = \sqrt{1+4t^2}$

17.2.3 $\mathbf{r}'(t) = \langle -\sin t, 1\rangle$, so $|\mathbf{r}'(t)| = \sqrt{1+\sin^2}$. The line integral is $\displaystyle\int_{\pi/2}^{\pi} \frac{\cos t\sqrt{1+\sin^2 t}}{t}\,dt$.

17.2.4 $\mathbf{r}(t) = \langle \sin \pi t, t\rangle$ for $0 \le t \le 3$.

17.2.5 A direction vector for the line is $\langle 5-1, 4-2, 0-3\rangle = \langle 4, 2, -3\rangle$. The line segment is given by $\mathbf{r}(t) = \langle 1+4t, 2+2t, 3-3t\rangle$ for $0 \le t \le 1$.

17.2.6 $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le \pi/2$.

17.2.7 $\mathbf{r}(t) = \left\langle t^2 + 1, t \right\rangle$ for $2 \le t \le 4$.

17.2.8 $\mathbf{F} = \langle \cos t - \sin t, \sin t - \cos t \rangle$.

17.2.9

a. $\mathbf{F} \cdot \mathbf{T} = \left\langle t, 2t^3 \right\rangle \cdot \left\langle 1, 3t^2 \right\rangle = t + 6t^5$. The integral is given by $\displaystyle\int_0^2 (t + 6t^5)\, dt$.

b. $\displaystyle\int_0^2 (t + 6t^5)\, dt = \left(\frac{t^2}{2} + t^6 \right) \Bigg|_0^2 = 2 + 64 = 66.$

17.2.10

a. Let $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ and $\mathbf{F} = \langle f, g \rangle = \langle 1, x \rangle = \langle 1, \cos t \rangle$. Then

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} (f(t)y'(t) - g(t)x'(t))\, dt = \int_0^{2\pi} (\cos t + \cos t \sin t)\, dt.$$

b. $\displaystyle\int_0^{2\pi} (\cos t + \cos t \sin t)\, dt = \left(\sin t + \frac{\sin^2 t}{2} \right) \Bigg|_0^{2\pi} = 0.$

17.2.11 $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ and $\displaystyle\int_C f\, dx + g\, dy + h\, dz$.

17.2.12 By reversing the orientation of C, we obtain the curve $-C$ from B to A. The value of the integral with the opposite orientation is the opposite of the value of the integral, so $\displaystyle\int_{-C} f\, ds = -10$.

17.2.13 $\displaystyle\int_{C_1} f\, ds + \int_{C_2} f\, ds = \int_C f\, ds = 10$, so $\displaystyle\int_{C_2} f\, ds = 7$.

17.2.14

a. Positive.

b. 0.

c. 0.

d. Negative.

e. 0.

f. Positive.

g. Negative.

h. 0.

17.2.15 Take the line integral of $\mathbf{F} \cdot \mathbf{T}$ along the curve using arc length parameterization.

17.2.16 The flux measures the degree to which the vector field is outwards normal to the curve C as C is traversed with a particular orientation.

17.2.17 $\mathbf{r}(s)$ is an arc length parameterization, so we have $\displaystyle\int_C xy\, ds = \int_0^{2\pi} \cos(s) \sin(s)\ ds = 0$.

17.2.18 With $\mathbf{r}(s) = \langle \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \rangle$, $|\mathbf{r}'(t)| = 1$, so that we have $\int_C \left(x^2 - 2y^2\right) ds = \int_0^4 \left(\frac{s^2}{2} - 2\frac{s^2}{2}\right) ds = -\int_0^4 \frac{s^2}{2}\, ds = -\frac{s^3}{6}\Big|_0^4 = -\frac{32}{3}$.

17.2.19 $|\mathbf{r}'(t)| = |\langle 3, 4\rangle| = \sqrt{9+16} = 5$, so $\int_C (2x+y)\, ds = \int_0^2 (6t+4t)5\, dt = 25t^2\Big|_0^2 = 100$.

17.2.20 $|\mathbf{r}'(t)| = \left|\langle 3t^2, 4\rangle\right| = \sqrt{9t^4+16}$, so $\int_C x\, ds = \int_0^1 t^3\sqrt{9t^4+16}\, dt = \frac{1}{36}\int_{16}^{25} u^{1/2}\, du = \frac{u^{3/2}}{54}\Big|_{16}^{25} = \frac{125-64}{54} = \frac{61}{54}$.

17.2.21 $|\mathbf{r}'(t)| = |\langle -2\sin t, 2\cos t\rangle| = 2$, so $\int_C xy^3\, ds = \int_0^{\pi/2} 32\cos t \sin^3 t\, dt = 32\int_0^1 u^3\, du = 8u^4\Big|_0^1 = 8$.

17.2.22 $|\mathbf{r}'(t)| = |\langle \cos t, 1\rangle| = \sqrt{\cos^2 t + 1}$, so

$$\int_C 3x\cos y\, ds = \int_0^{\pi/2} 3\sin t\cos t\sqrt{\cos^2 t+1}\, dt = -\frac{3}{2}\int_2^1 \sqrt{u}\, du = u^{3/2}\Big|_1^2 = 2^{3/2} - 1 = 2\sqrt{2} - 1.$$

17.2.23 $|\mathbf{r}'(t)| = |\langle -3\sin t, 3\cos t, 4\rangle| = \sqrt{25} = 5$, so

$$\int_C (y-z)\, ds = \int_0^{2\pi} 5(3\sin t - 4t)\, dt = 5(-3\cos t - 2t^2)\Big|_0^{2\pi} = 5(-3 - 8\pi^2 + 3) = -40\pi^2.$$

17.2.24 $|\mathbf{r}'(t)| = \sqrt{0 + 9\sin^2 t + 9\cos^2 t} = 3$, so $\int_C (x - y + 2z)\; ds = 3\int_0^{2\pi} (1 - 3\cos(t) + 6\sin t)\; dt = 6\pi$.

17.2.25 $\mathbf{r}(t) = \langle 4\cos t, 4\sin t\rangle$, $0 \le t \le 2\pi$. $|\mathbf{r}'(t)| = \sqrt{(-4\sin t)^2 + (4\cos t)^2} = 4$, so

$$\int_C \left(x^2+y^2\right) ds = \int_0^{2\pi} 4\left(16\cos^2 t + 16\sin^2 t\right)\; dt = \int_0^{2\pi} 64\, dt = 128\pi.$$

17.2.26 $\mathbf{r}(t) = \langle 5t, 5t\rangle$, $0 \le t \le 1$. $|\mathbf{r}'(t)| = \sqrt{(5)^2 + (5)^2} = 5\sqrt{2}$, so

$$\int_C \left(x^2+y^2\right)\, ds = \int_0^1 50t^2 \cdot 5\sqrt{2}\, dt = 250\sqrt{2}\int_0^1 t^2\, dt = \frac{250\sqrt{2}}{3}.$$

17.2.27 $\mathbf{r}(t) = \langle t, t\rangle$, $1 \le t \le 10$. $|\mathbf{r}'(t)| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$, so

$$\int_C \frac{x}{x^2+y^2}\, ds = \int_1^{10} \frac{t}{t^2+t^2}\cdot\sqrt{2}\, dt = \frac{\sqrt{2}}{2}\int_1^{10} \frac{1}{t}\, dt = \frac{\sqrt{2}}{2}\ln 10.$$

17.2.28] $\mathbf{r}(t) = \langle t, t^2\rangle$, $0 \le t \le 1$. $|\mathbf{r}'(t)| = \sqrt{(1)^2 + (2t)^2} = \sqrt{1+4t^2}$, so

$$\int_C (xy)^{1/3}\; ds = \int_0^1 \left(t^3\right)^{1/3}\; \sqrt{1+4t^2}\, dt = \int_0^1 t\, \sqrt{1+4t^2}\, dt = \frac{5\sqrt{5}-1}{12}.$$

17.2.29 $\mathbf{r}(t) = \langle 2\cos t, 4\sin t\rangle$, $0 \le t \le \pi/2$.
$|\mathbf{r}'(t)| = \sqrt{4\sin^2 t + 16\cos^2 t} = \sqrt{4 + 12\cos^2 t} = 2\sqrt{1+3\cos^2 t}$.

$$\int_C xy\, ds = \int_0^{\pi/2} 16\sin t\cos t\sqrt{1+3\cos^2 t}\, dt.$$

Let $u = (1 + 3\cos^2 t)\, dt$ so that $du = -6\cos t\sin t\, dt$. Substituting gives

$$\frac{8}{3}\int_1^4 u^{1/2}\, du = \frac{16}{9}\left(u^{3/2}\right)\Big|_1^4 = \frac{16}{9}(8-1) = \frac{112}{9}.$$

17.2.30 $\mathbf{r}_1(t) = \langle t-1, t\rangle$, $\mathbf{r}_2(t) = \langle t, 1-t\rangle$, $0 \le t \le 1$. $|\mathbf{r}_1'(t)| = \sqrt{2}$, $|\mathbf{r}_2'(t)| = \sqrt{2}$.

$$\int_C (2x-3y)\, ds = \int_0^1 (2(t-1)-3t)\, \sqrt{2}\, dt + \int_0^1 (2t - 3(1-t))\, \sqrt{2}\, dt = \sqrt{2}\int_0^1 (4t-5)\, dt = -3\sqrt{2}..$$

17.2.31 $|\mathbf{r}'(t)| = \sqrt{4\sin^2 t + 0 + 4\cos^2 t} = 2$, so

$$\int_C (x+y+z)\, ds = 2\int_0^\pi (2\cos t + 2\sin t)\, dt = 4(\sin t - \cos t)\Big|_0^\pi = 8.$$

17.2.32 Let $\mathbf{r}(t) = \langle 2t+1, 2t+4, 2t+1\rangle$, $0 \le t \le 1$. Then $|\mathbf{r}'(t)| = \sqrt{12} = 2\sqrt{3}$, so $\int_C \frac{xy}{z}\, ds = 2\sqrt{3}\int_0^1 \frac{(2t+1)(2t+4)}{(2t+1)}\, dt = 2\sqrt{3}\int_0^1 (2t+4)\, dt = 10\sqrt{3}$.

17.2.33 For the first segment, $\mathbf{r}(t) = \langle 3t, 2t, 6t\rangle$ for $0 \le t \le 1$, so $\mathbf{r}'(t) = |\langle 3,2,6\rangle| = 7$, so $\int_C dz\, ds = \int_0^1 7(18)t^2\, dt = 42t^3\Big|_0^1 = 42$.

For the second segment, $\mathbf{r}(t) = \langle 3+4t, 2+7t, 6+4t\rangle$ for $0 \le t \le 1$, so $\mathbf{r}'(t) = \langle 4,7,4\rangle$, and $|\mathbf{r}'(t)| = |\langle 4,7,4\rangle| = 9$, so $\int_C xz\, ds = \int_0^1 9(3+4t)(6+4t)\, dt = 18\int_0^1 (8t^2+18t+9)\, dt = 18\left(\frac{8}{3}t^3 + 9t^2 + 9t\right)\Big|_0^1 = 372$.

Therefore, for the union of the two segments, we have $\int_C xz\, ds = 42 + 372 = 414$.

17.2.34 $|\mathbf{r}'(t)| = \sqrt{9} = 3$, so $\int_C xe^{yz}\, ds = 3\int_0^2 t\, e^{-4t^2}\, dt = -\frac{3}{8}e^{-4t^2}\Big|_0^2 = \frac{3}{8}\left(1 - \frac{1}{e^{16}}\right)$.

17.2.35 Parameterize C by $\mathbf{r}(t) = \langle t, 2t^2\rangle$ for $0 \le t \le 3$. Then $|\mathbf{r}'(t)| = \sqrt{1+16t^2}$ and $\int_C \rho\, ds = \int_0^3 \left(1+2t^3\right)\sqrt{1+16t^2}\, dt \approx 409.5$.

17.2.36 $\mathbf{r}'(\theta) = \langle -\sin\theta, \cos\theta\rangle$, so that$|\mathbf{r}'(\theta)| = 1$ and $\int_C \rho\, ds = \int_0^\pi \left(\frac{2\theta}{\pi} + 1\right) d\theta = 2\pi$.

17.2.37 Let $\mathbf{r}(t) = \langle t+1, 4t+1\rangle$, $0 \le t \le 1$. Then $|\mathbf{r}'(t)| = \sqrt{17}$ and $\int_C (x+2y)\, ds = \int_0^1 ((t+1) + 2(4t+1))\cdot\sqrt{17}\, dt = \sqrt{17}\int_0^1 (9t+3)\, dt = \frac{15}{2}\sqrt{17}$. The length of the line segment is $\sqrt{17}$, so the average value is $\frac{15}{2}$.

17.2.38 Let $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$. Then $|\mathbf{r}'(t)| = 1$ and $\int_C (x\, e^y)\, ds = \int_0^{2\pi} \cos t\, e^{\sin t}\, dt = 0$. Thus, the average value is 0.

17.2.39 The length of the curve is the line integral of 1 along the curve. Then $\mathbf{r}'(t)$ simplifies to $\langle 5\cos(t/4), -5\sin(t/4), \frac{1}{2}\rangle$, and then

$$|\mathbf{r}'(t)| = \sqrt{25 + \frac{1}{4}} = \frac{1}{2}\sqrt{101}$$

so that the arc length is $\frac{1}{2}\int_0^2 \sqrt{101}\, dt = \sqrt{101}$.

17.2.40 The length of the curve is the line integral of 1 along the curve.
$|\mathbf{r}'(t)| = \sqrt{900\cos^2(t) + 1600\cos^2 t + 2500\sin^2 t} = 50$ so that $\int_C 1\,ds = 50\int_0^{2\pi} dt = 100\pi$.

17.2.41 $\mathbf{r}'(t) = \langle 4, 2t\rangle$, and $\mathbf{F}\cdot\mathbf{r}'(t) = \langle 4t, t^2\rangle \cdot \langle 4, 2t\rangle = 16t + 2t^3$, so that

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1 \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^1 \left(16t + 2t^3\right)\,dt = \frac{17}{2}.$$

17.2.42 $\mathbf{r}'(t) = \langle -4\sin t, 4\cos t\rangle$, so $\mathbf{F}\cdot\mathbf{r}'(t) = \langle -4\,\sin t, 4\cos t\rangle \cdot \langle -4\,\sin t, 4\cos t\rangle = 16$. Then $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^{\pi} \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{\pi} 16\,dt = 16\pi$.

17.2.43 Let $\mathbf{r}(t) = \langle 4t+1, 9t+1\rangle$, $0 \le t \le 1$; then $\mathbf{r}'(t) = \langle 4, 9\rangle$ and $\mathbf{F}\cdot\mathbf{r}'(t) = \langle 9t+1, 4t+1\rangle\cdot\langle 4, 9\rangle = 72t+13$. Then $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1 (72t+13)\,dt = 49$.

17.2.44 Let $\mathbf{r}(t) = \langle t, t^2\rangle$, $0 \le t \le 1$; then $\mathbf{r}'(t) = \langle 1, 2t\rangle$ and $\mathbf{F}\cdot\mathbf{r}'(t) = \langle -t^2, t\rangle \cdot \langle 1, 2t\rangle = t^2$. Then $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1 t^2\,dt = \frac{1}{3}$.

17.2.45 $\mathbf{r}'(t) = \langle 2t, 6t\rangle$; then $\mathbf{F}\cdot\mathbf{r}'(t) = \left(10t^4\right)^{-3/2}\langle t^2, 3t^2\rangle\cdot\langle 2t, 6t\rangle = \frac{20t^3}{10\sqrt{10}\,t^6} = \frac{2}{\sqrt{10}}t^{-3}$ and $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \frac{2}{\sqrt{10}}\int_1^2 t^{-3}\,dt = \frac{3}{40}\sqrt{10}$.

17.2.46 $\mathbf{r}'(t) = \langle 1, 4\rangle$, so that $\mathbf{F}\cdot\mathbf{r}'(t) = \frac{1}{17t^2}\langle t, 4t\rangle\cdot\langle 1, 4\rangle = \frac{1}{t}$. Then $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_1^{10}\frac{1}{t}\,dt = \ln(10)$.

17.2.47

a. For C_1, the vector field points "against" the curve for most of its length, and with larger magnitude, so the integral is negative.

b. For C_2, the vector field points with the curve for its entire length, so the integral is positive.

17.2.48

a. For C_1, the vector field points against the curve for its entire length, so the integral over C_1 is negative.

b. For C_2, the vector field points with the curve, so the integral over C_2 is positive.

17.2.49 $\mathbf{r}_1(t) = \langle 1-t, 2-2t\rangle$, $\mathbf{r}_2(t) = \langle 0, 4t\rangle$, $0 \le t \le 1$. Then $\mathbf{r}_1'(t) = \langle -1, -2\rangle$, $\mathbf{r}_2'(t) = \langle 0, 4\rangle$, so that $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^1 \langle 2-2t, 1-t\rangle\cdot\langle -1, -2\rangle\,dt + \int_0^1 \langle 4t, 0\rangle\cdot\langle 0, 4\rangle\,dt = \int_0^1 0\,dt = 0$.

17.2.50 $\mathbf{r}_1(t) = \langle t-1, 8t\rangle$, $\mathbf{r}_2(t) = \langle 2t, 8\rangle$, $0 \le t \le 1$. Then $\mathbf{r}_1'(t) = \langle 1, 8\rangle$, $\mathbf{r}_2'(t) = \langle 2, 0\rangle$, so that $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^1 \langle t-1, 8t\rangle\cdot\langle 1, 8\rangle\,dt + \int_0^1 \langle 2t, 8\rangle\cdot\langle 2, 0\rangle\,dt = \int_0^1 (69t-1)\,dt = \frac{67}{2}$.

17.2.51 $\mathbf{r}(t) = \langle 2t, 8t^2\rangle$, $0 \le t \le 1$. Then $\mathbf{r}'(t) = \langle 2, 16t\rangle$, so $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^1 \langle 8t^2, 2t\rangle\cdot\langle 2, 16t\rangle\,dt = \int_0^1 48t^2\,dt = 16$.

17.2.52 $\mathbf{r}(t) = \langle 2t+1, 8-4t\rangle$, $0 \le t \le 1$. Then $\mathbf{r}'(t) = \langle 2, -4\rangle$, so $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^1 \langle 8-4t, -1-2t\rangle \cdot \langle 2,-4\rangle\,dt = \int_0^1 20\,dt = 20$.

17.2.53 $\mathbf{r}'(t) = \langle -4\sin t, 4\cos t, -4\sin t\rangle$. Thus we have $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi} \langle 4\cos t, 4\sin t, 4\cos t\rangle \cdot \langle -4\sin t, 4\cos t, -4\sin t\rangle\,dt = \int_0^{2\pi} -16\sin t\cos(t)\,dt = 0$.

17.2.54 $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, \frac{1}{2\pi}\rangle$, so $\int_C \mathbf{F}\cdot\mathbf{r}'t\,dt = \int_0^{2\pi} \langle -2\sin t, 2\cos t, \dfrac{t}{2\pi}\rangle \cdot \langle -2\ \sin t, 2\cos t, \dfrac{1}{2\pi}\rangle\,dt = \int_0^{2\pi}\left(4+\dfrac{t}{4\pi^2}\right)dt = \dfrac{1}{2}+8\pi$.

17.2.55 Let $\mathbf{r}(t) = \langle t+1, t+1, t+1\rangle$, $0 \le t \le 9$, so that $\mathbf{r}'(t) = \langle 1,1,1\rangle$. Then $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^9 \dfrac{1}{\left(3(t+1)^2\right)^{3/2}}\langle t+1, t+1, t+1\rangle\cdot\langle 1,1,1\rangle\,dt = \dfrac{1}{\sqrt{3}}\int_0^9 \dfrac{1}{(t+1)^3}(t+1)\,dt = \dfrac{1}{\sqrt{3}}\int_0^9 \dfrac{1}{(t+1)^2}\,dt = \dfrac{3}{10}\sqrt{3}$.

17.2.56 Let $\mathbf{r}(t) = \langle 7t+1, 3t+1, t+1\rangle$, $0 \le t \le 1$, so that $\mathbf{r}'(t) = \langle 7,3,1\rangle$. Then $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^1 \dfrac{1}{(7t+1)^2+(3t+1)^2+(t+1)^2}\langle 7t+1, 3t+1, t+1\rangle\cdot\langle 7,3,1\rangle\,dt = \int_0^1 \dfrac{59t+11}{(7t+1)^2+(3t+1)^2+(t+1)^2}\,dt = \ln(2)+\dfrac{1}{2}\ln(7) = \ln\left(2\sqrt{7}\right)$.

17.2.57

a. Looking at the vector field, it appears that the vector field points counterclockwise just as much as it points clockwise at the boundary of the region, so we would expect the circulation to be zero.

b. $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t\rangle$, so $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi} \langle 2(\sin t-\cos t), 2\ \cos t\rangle\cdot\langle -2\sin(t), 2\cos t\rangle\,dt = 4\int_0^{2\pi}\left(\cos^2 t+\sin t\cos t-\sin^2 t\right)dt = 0$.

17.2.58

a. The circulation appears to be negative.

b. On C, $\mathbf{F} = \dfrac{\langle 4\sin t, -4\cos t\rangle}{\sqrt{16\cos^2 t+16\sin^2 t}} = \langle \sin t, -\cos t\rangle$, so $\mathbf{F}\cdot\mathbf{r}'(t) = \langle \sin t, -\cos t\rangle\cdot\langle -2\sin t, 4\cos t\rangle = -2\sin^2 t-4\cos^2 t = -2-2\cos^2 t$. So the circulation is $\int_0^{2\pi}(-2-2\cos^2 t)\,dt = \int_0^{2\pi}(-2-1-\cos 2t)\,dt = \int_0^{2\pi}(-3-\cos 2t)\,dt = -6\pi$.

17.2.59

a. Looking at the vector field, the inward-pointing vectors (in quadrants II and IV) appear larger than the outward-pointing vectors (in quadrants I and III). Thus we would expect the flux to be negative.

b. $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t\rangle$, so that $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi}(2(\sin t-\cos t)\cdot 2\cos t-2\cos t\cdot(-2\ \sin t))\,dt = -4\int_0^{2\pi}\left(\cos^2 t-2\sin t\cos t\right)dt = -4\pi$.

17.2.60

a. The flux appears to be 0.

b. We have $\mathbf{r}(t) = \langle 2\cos t, 4\sin t\rangle$, and on C, $\mathbf{F} = \dfrac{\langle 4\sin t, -4\cos t\rangle}{\sqrt{16\cos^2 t + 16\sin^2 t}} = \langle \sin t, -\cos t\rangle$, so the flux is $\displaystyle\int_0^{2\pi} (4\sin t\cos t - (-\cos t)(-2\sin t))\,dt = \int_0^{2\pi} 2\sin t\cos t\,dt = \sin^2 t\Big|_0^{2\pi} = 0.$

17.2.61

a. True. This is the definition of an arc length parameterization.

b. True. Let $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$. Then $\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$, and $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r}'(t)\;dt = \int_0^{2\pi} \langle \sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\,dt = \int_0^{2\pi} \left(\cos^2 t - \sin^2 t\right)\;dt = 0$. $\displaystyle\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} (\sin t\cdot\cos t - \cos t\,(-\sin t))\;dt = 2\int_0^{2\pi} \sin t\cos t\,dt = 0.$

c. True.

d. True. It is the line integral $\displaystyle\int_C \mathbf{F}\cdot\mathbf{n}\,ds$.

17.2.62 The work done on either path is simply $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r}'(t)\;dt$.

a. Here $\mathbf{r}(t) = \langle -t, 50\rangle$, for $-100 \le t \le 100$, so $\mathbf{r}'(t) = \langle -1, 0\rangle$ and $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r}'(t)\;dt = \int_{-100}^{100} -150\,dt = 30000.$

b. Here $\mathbf{r}(t) = \langle 100\cos t, 100\sin t\rangle$ and $\mathbf{r}'(t) = \langle -100\sin t, 100\cos t\rangle$, so we have $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r}'(t)\;dt = \int_0^{\pi} -15000\sin t\,dt = 30000$. The same amount of work is done along each path.

17.2.63

a. For the first path, the work done is $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r}'(t)\;dt = \int_{-100}^{100} -141\,dt = 28200$, and for the second path $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r}'(t)\;dt = \int_0^{\pi} (-14100\sin t - 5000\cos(t))\;dt = 28200,$
so again they are equal.

b. For the first path, the work done is $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r}'(t)\;dt = \int_{-100}^{100} -141\,dt = 28200$, while for the second path, $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r}'(t)\;dt = \int_0^{\pi} (-14100\sin t - 5000\cos t)\;dt = 28200$, so the amount of work is still equal along the two paths.

17.2.64

a. Let $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ for $0 \le t \le 2\pi$ so that $|\mathbf{r}'(t)| = 1$. Then $\displaystyle\int_C f\,ds = \int_0^{2\pi} (\cos t + 2\sin(t))\;dt = 0.$

b. Let $\mathbf{r}(t) = \langle \cos t, -\sin t\rangle$ for $0 \le t \le 2\pi$ so that $|\mathbf{r}'(t)| = 1$. Note that this parameterization traces out the unit circle, but clockwise. Then $\displaystyle\int_C f\,ds = \int_0^{2\pi} (\cos t - 2\sin(t))\;dt = 0.$

c. The two integrals are equal.

17.2.65

a. Let $\mathbf{r}(t) = \langle t, t^2 \rangle$ for $0 \le t \le 1$ so that $|\mathbf{r}'(t)| = \sqrt{4t^2+1}$, and $\displaystyle\int_C f\,ds = \int_0^1 t\sqrt{4t^2+1}\,dt = \frac{5\sqrt{5}-1}{12}$.

b. $\mathbf{r}(t) = \langle 1-t, (1-t)^2 \rangle$ for $0 \le t \le 1$. Then $|\mathbf{r}'(t)| = \sqrt{(-1)^2 + (-2(1-t))^2} = \sqrt{4t^2-8t+5}$ and $\displaystyle\int_C f\,ds = \int_0^1 (1-t)\sqrt{4t^2-8t+5}\,dt = \frac{5\sqrt{5}-1}{12}$.

c. The two integrals are equal.

17.2.66 Parameterize C_1 by $\mathbf{r}(t) = \langle 1-t, t \rangle$ for $0 \le t \le 1$. Then $\mathbf{r}'(t) = \langle -1, 1 \rangle$, and $\displaystyle\int_{C_1} \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1 ((-t)(-1) + (1-t)(1))\,dt = \int_0^1 1\,dt = 1$.

Parameterize C_2 by $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le \frac{\pi}{2}$. Then $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$, and $\displaystyle\int_{C_2} \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^{\pi/2} ((-\sin t)(-\sin t) + (\cos t)(\cos t))\,dt = \int_0^{\pi/2} 1\,dt = \frac{\pi}{2}$.

Finally, parameterize C_3 by the two paths $\mathrm{r}_1(t) = \langle 1-t, 0 \rangle$ and $\mathrm{r}_2(t) = \langle 0, t \rangle$ for $0 \le t \le 1$, so that $r_1'(t) = \langle -1, 0 \rangle$ and $r_2'(t) = \langle 0, 1 \rangle$. Then $\displaystyle\int_{C_3} \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1 ((0)(-1) + (1-t)(0))\,dt + \int_0^1 ((-t)(0) + (0)(1))\,dt = 0$. None of the three path integrals are equal.

17.2.67 Using the same parameterizations as for the previous problem, we have $\displaystyle\int_{C_1} \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1 ((t)(-1) + (1-t)(1))\,dt = \int_0^1 (1-2t)\,dt = 0$, $\displaystyle\int_{C_2} \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^{\pi/2} ((\sin t)(-\sin t) + (\cos t)(\cos t))\,dt = \int_0^{\pi/2} \left(\cos^2 t - \sin^2 t\right)\,dt = 0$, $\displaystyle\int_{C_3} \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1 ((0)(-1) + (1-t)(0))\,dt + \int_0^1 (t\cdot 0 + 0\cdot 1)\,dt = 0$. All three are equal to zero.

17.2.68 We have $dx = dt$, $dy = 2dt$, and $dz = 2t\,dt$. It follows that $\displaystyle\int_C x^2\,dx + dy + y\,dz = \int_0^3 (t^2+2+4t^2)\,dt = \int_0^3 (5t^2+2)\,dt = \left(\frac{5}{3}t^3 + 2t\right)\Big|_0^3 = 51$.

17.2.69 We have $dx = 2\,dt$, $dy = \cos t\,dt$, and $dz = -\sin t\,dt$. It follows that $\displaystyle\int_C x^3y\,dx + xz\,dy + (x+y)^2\,dz = \int_0^{4\pi} (16t^3\sin t + 2t\cos^2 t - (2t+\sin t)^2 \sin t)\,dt = 8\pi(48 + 7\pi - 128\pi^2)$, where the integral was calculated with a computer algebra system.

17.2.70 $\mathbf{r}(t) = \left\langle 3t^2, t \right\rangle$ for $1 \le t \le 3$. Therefore, $\displaystyle\int_C \frac{x^2}{y^4}\,ds = \int_1^3 \frac{9t^4}{t^4}\sqrt{36t^2+1}\,dt \approx 216.8$, where the integral was calculated with a computer algebra system.

17.2.71 $\mathbf{r}(t) = \langle 4\sin t, 4\cos t \rangle$, for $0 \le t \le \pi/2$. Then $\displaystyle\int_C \frac{y}{\sqrt{x^2+y^2}}\,dx - \frac{x}{\sqrt{x^2+y^2}}\,dy = \int_0^{\pi/2} (4\cos^2 t + 4\sin^2 t)\,dt = 4\cdot\frac{\pi}{2} = 2\pi$.

17.2.72 $\mathbf{r}(t) = \langle 1+2t, 2+6t, 4+9t\rangle$ for $0 \le t \le 1$. So $\int_C (x+y)\,dx + (x-y)\,dy + x\,dz = \int_0^1 2(3+8t) + 6(-1-4t) + 9(1+2t))\,dt = \int_0^1 (9+10t)\,dt = 14.$

17.2.73

a. $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t\rangle$, so the flux along the quarter circle is $\int_C \mathbf{F}\cdot\mathbf{n}\,ds =$
$\int_0^{\pi/2} \left((2\sin t)(2\cos t) - (2\cos t)(-2\sin t)\right)\,dt = \int_0^{\pi/2} 8\sin t\cos t\,dt = 4.$

b. With $\mathbf{r}'(t)$ as above, the flux is $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_{\pi/2}^{\pi} (8\sin t\,\cos t)\,dt = -4.$

c. Both the normal vectors and the vector field F in the third quadrant are the negatives of their values in the first quadrant, so their dot product is the same. Thus the flux is identical.

d. Both the normal vectors and the vector field F in the fourth quadrant are the negatives of their values in the second quadrant, so their dot product is the same. Thus the flux is identical.

e. The total flux is $4-4+4-4=0$.

17.2.74 Letting $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ and $\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$ as usual, we have $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt =$
$\int_0^{2\pi} \left(-b\sin^2 t + c\cos^2 t\right)\,dt = \pi\,(c-b)$, so that the circulation is zero only for $b=c$.

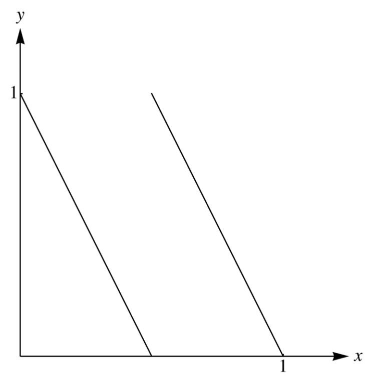
17.2.75 Let $\mathbf{r}(t) = \langle r\cos t, r\sin t\rangle$ for a circle of radius r, so that $\mathbf{r}'(t) = \langle -r\sin t, r\cos t\rangle$. Then $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi} \left((ar\cos t + br\sin t)(-r\sin t) + (cr\cos t + dr\sin t)(r\cos t)\right)\,dt =$
$r^2\int_0^{2\pi} \left(-a\sin t\cos t - b\sin^2 t + c\cos^2 t + d\sin t\cos t\right)\,dt = \pi\,(c-b)\,r^2$, so that the circulation is zero provided $b=c$.

17.2.76 Let $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$; then the flux is $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} (a\cos t\cos t - d\sin t\,(-\sin t))\,dt = (a+d)\,\pi$, so the flux is zero if $a=-d$.

17.2.77 Let $\mathbf{r}(t) = \langle r\cos t, r\sin t\rangle$ for a circle of radius r; then the flux is
$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} \left((ar\cos t + br\sin t)(r\cos t) - (cr\cos t + dr\sin t)(-r\sin t)\right)\,dt =$
$r^2\int_0^{2\pi} \left(a\cos^2 t + b\sin t\cos t + c\sin t\cos t + d\sin^2 t\right)\,dt = r^2\,(a+d)\,\pi$, so the flux is zero provided that $a=-d$.

17.2.78

a.



b. The gradient is $-50\mathbf{i} - 25\mathbf{j}$.

c. $\mathbf{F} = 50\mathbf{i} + 25\mathbf{j}$.

d. Parameterize the boundary C by $\mathbf{r}(t) = \langle 1, t\rangle$ for $0 \le t \le 1$. Then $\mathbf{r}'(t) = \langle 0, 1\rangle$ and $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1 (50\cdot 1 - 25\cdot 0)\,dt = 50$.

e. Parameterize the boundary C by $\mathbf{r}(t) = \langle 1-t, 1\rangle$ for $0 \le t \le 1$ (note: we do not use $\langle t, 1\rangle$ because we need counterclockwise orientation). Then $\mathbf{r}'(t) = \langle -1, 0\rangle$ and $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1 (50\cdot 0 - 25\cdot(-1))\,dt = 25$.

17.2.79 $\mathbf{r}'(t) = \langle 1, 1, 1\rangle$ and $|\mathbf{r}(t)| = t\sqrt{3}$, so the work is $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_1^a \frac{3t}{(t\sqrt{3})^p}\,dt = 3^{1-p/2}\int_1^a t^{1-p}\,dt$

a. For $p = 2$, we have $\int_1^a \frac{1}{t}dt = \ln a$.

b. The work is not finite.

c. For $p = 4$, we have $\frac{1}{3}\int_1^a t^{-3}\,dt = -\frac{1}{6t^2}\Big|_1^a = \frac{1}{6} - \frac{1}{6a^2}$.

d. As $a \to \infty$, the work approaches $\frac{1}{6}$.

e. For the general $p > 1$, the analysis above shows that the integral is (for $p \neq 2$) $3^{1-p/2}\frac{1}{2-p}t^{2-p}\Big|_1^a = \frac{3^{1-p/2}}{2-p}\left(-1 + a^{2-p}\right)$ while for $p = 2$ the integral is (from part (a)) $\ln a$.

e. This approaches a limit only for $p > 2$ (when $2 - p < 0$). This limit is $\frac{3^{1-p/2}}{p-2}$.

17.2.80

a. $\int_C f\,dx = \int_a^b (f(t)x'(t) + 0(y'(t)))\,dt = \int_a^b f(t)x'(t)\,dt$.

b. $\int_C g\,dy = \int_a^b (0(x'(t)) + g(t)(y'(t)))\,dt = \int_a^b g(t)y'(t)\,dt$.

c. $\displaystyle\int_C xy\,dx = \int_0^1 300t^2\,dt = 100.$

d. $\displaystyle\int_C xy\,dy = \int_{-1}^1 t^3\,dt = 0.$

17.2.81 We use four line segment parameterizations for the rectangle, all for $0 \le t \le 1$: $\mathbf{r}_1(t) = \langle at, 0\rangle$, so $\mathbf{r}_1'(t) = \langle a, 0\rangle$. $\mathbf{r}_2(t) = \langle a, bt\rangle$, so $\mathbf{r}_2'(t) = \langle 0, b\rangle$. $\mathbf{r}_3(t) = \langle a - at, b\rangle$, so $\mathbf{r}_3'(t) = \langle -a, 0\rangle$. $\mathbf{r}_4(t) = \langle 0, b - bt\rangle$, so $\mathbf{r}_4'(t) = \langle 0, -b\rangle$. Then

$$\int_C x\,dy = \int_0^1 (at\cdot 0 + a\cdot b + (a - at)\cdot 0 + 0\cdot(-b))\,dt = \int_0^1 ab\,dt = ab.$$

17.2.82 Parameterize C by $\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle$, so that $\mathbf{r}'(t) = \langle -a\sin t, a\cos t\rangle$. Then

$$-\int_C y\,dx = -\int_0^{2\pi} (a\sin t\cdot(-a\sin t))\,dt = a^2\int_0^{2\pi}\sin^2 t\,dt = \pi a^2.$$

17.3 Conservative Vector Fields

17.3.1 A simple curve has no self-intersections; the initial and terminal points of a closed curve are identical.

17.3.2 A region is connected, roughly speaking, if it consists of one piece. A simply connected region has the property that every closed loop can be contracted to a point.

17.3.3 If $\mathbf{F} = \langle f, g\rangle$ is a vector field in $\mathbb{R}^2$ and if $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$, then $\mathbf{F}$ is conservative.

17.3.4 If $\mathbf{F} = \langle f, g, h\rangle$ is a vector field in $\mathbb{R}^3$ and if $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$ and $\dfrac{\partial f}{\partial z} = \dfrac{\partial h}{\partial x}$ and $\dfrac{\partial g}{\partial z} = \dfrac{\partial h}{\partial y}$, then $\mathbf{F}$ is conservative.

17.3.5 Integrate f with respect to x to get an answer where the "constant" is actually a function of y. Take the partial with respect to y and equate with g to compute the constant.

17.3.6 The integral is $\varphi(B) - \varphi(A)$ where $\nabla\varphi =$F.

17.3.7 The integral is zero.

17.3.8

- There exists a potential function φ such that $\mathbf{F} = \nabla\varphi$.
- $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points A and B in R and all smooth oriented curves C from A to B (path independence).
- $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ on all simple smooth closed oriented curves C in R.

17.3.9 Conservative, because $1_y = 1_x = 0$.

17.3.10 Conservative, because $x_y = y_x = 0$.

17.3.11 Not Conservative, because $-y_y = -1$, but $x_x = 1$.

17.3.12 Not Conservative. $-y_y = -1$, but $(x + y)_x = 1$.

17.3.13 Conservative. $\dfrac{\partial}{\partial y}e^{-x}\cos(y) = -e^{-x}\sin(y) = \dfrac{\partial}{\partial x}e^{-x}\sin(y)$.

17.3.14 Not Conservative. $\frac{\partial}{\partial y}\left(2x^3 + xy^2\right) = 2xy$, and $\frac{\partial}{\partial x}\left(2y^3 - x^2y\right) = -2xy$.

17.3.15 Conservative. $f_y = z\cos xz = g_x$, $f_z = yz(-x\sin xz) + (\cos xz)y = -xyz\sin xz + y\cos xz = h_x$, and $g_z = x\cos xz = h_y$.

17.3.16 Not Conservative. Note that $f_z = -ye^{x-z}$ while $h_x = ye^{x-z}$.

17.3.17 Conservative. $\frac{\partial}{\partial y}(x) = \frac{\partial}{\partial x}(y) = 0$. A potential function is $\frac{x^2+y^2}{2}$.

17.3.18 Conservative. $\frac{\partial}{\partial y}(-y) = \frac{\partial}{\partial x}(-x) = -1$. A potential function is $-xy$.

17.3.19 Not Conservative. $\frac{\partial}{\partial y}\left(x^3 - xy\right) = -x$, and $\frac{\partial}{\partial x}\left(\frac{x^2}{2} + y\right) = x$.

17.3.20 Conserative. $\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{-2xy}{\left(x^2+y^2\right)^2} = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)$. Integrating $\frac{x}{x^2+y^2}$ with respect to x, we see that a potential function is $\frac{1}{2}\ln\left(x^2+y^2\right)$.

17.3.21 Conservative. $\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \frac{-xy}{\left(x^2+y^2\right)^{3/2}} = \frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}\right)$. Integrating $\frac{x}{\sqrt{x^2+y^2}}$ with respect to x, we obtain a potential function of $\sqrt{x^2+y^2}$.

17.3.22 Not Conservative. $f_z = 0$ while $h_x = 1$.

17.3.23 Conservative, because the mixed partials are pairwise equal. A potential function is $xz + y$.

17.3.24 Conservative, because the mixed partials are pairwise equal. A potential function is xyz.

17.3.25 Not Conservative. $g_z = e^z$ but $h_y = -e^z$.

17.3.26 Not Conservative. $g_z = -1$ and $h_y = 1$.

17.3.27 Conservative, because the mixed partials are pairwise equal. To find the potential function, integrate $y + z$ with respect to x to get $x(y+z) + f(y,z)$; differentiating with respect to y gives $x + z = x + f_y(y,z)$ so that $f_y(y,z) = z$ and$f(y,z) = yz$. Thus a potential function is $xy + yz + xz$.

17.3.28 Conservative, because the mixed partials are pairwise equal. A potential function is $\frac{1}{2}\ln\left(x^2+y^2+z^2\right)$.

17.3.29 Conservative, because the mixed partials are pairwise equal. A potential function is $\sqrt{x^2+y^2+z^2}$.

17.3.30 Conservative, because the mixed partials are pairwise equal (and zero). A potential function is $\frac{1}{4}x^4 + y^2 - \frac{1}{4}z^4$.

17.3.31

a. $\nabla\varphi = \langle y, x\rangle$, so $\int_C \nabla\varphi\cdot\mathbf{r}'(t)\,dt = \int_0^\pi \langle\sin t, \cos t\rangle\cdot\langle-\sin t, \cos t\rangle\,dt = \int_0^\pi \cos(2t)\,dt = 0$.

b. Because $\nabla\varphi$ is obviously conservative, the integral is simply $\varphi(\cos(\pi), \sin(\pi)) - \varphi(\cos(0), \sin(0)) = 0$.

17.3.32

a. $\nabla\varphi = \langle 1,3\rangle$, so $\displaystyle\int_C \nabla\varphi\cdot\mathbf{r}'(t)\,dt = \int_0^2 \langle 1,3\rangle\cdot\langle -1,1\rangle\,dt = \int_0^2 2\,dt = 4.$

b. The integral is $\varphi(0,2)-\varphi(2,0) = 6-2 = 4$. (Note that $\varphi(0,2)$ is φ evaluated at the point where $t=2$).

17.3.33

a. $\nabla\varphi = \langle x,y,z\rangle$, so $\displaystyle\int_C \nabla\varphi\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi} \langle \cos t,\sin t,\frac{t}{\pi}\rangle\cdot\langle -\sin t,\cos t,\frac{1}{\pi}\rangle\,dt = \int_0^{2\pi}\left(\frac{t}{\pi^2}\right)\,dt = 2.$

b. The integral is $\varphi(1,0,2)-\varphi(1,0,0) = \dfrac{5}{2}-\dfrac{1}{2} = 2.$

17.3.34

a. $\nabla\varphi = \langle y+z,x+z,x+y\rangle$, so $\displaystyle\int_C \nabla\varphi\cdot\mathbf{r}'(t)\,dt = \int_0^4 \langle 5t,4t,3t\rangle\cdot\langle 1,2,3\rangle\,dt = \int_0^4 22t\,dt = 176.$

b. The integral is $\varphi(4,8,12)-\varphi(0,0,0) = 176-0 = 176.$

17.3.35 $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(\mathbf{r}(2))-\varphi(\mathbf{r}(1)) = \varphi(3,6)-\varphi(1,2) = 10-7 = 3.$

17.3.36 $\displaystyle\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \varphi(6,4)-\varphi(1,2) = 20-7 = 13.$

17.3.37 $\displaystyle\int_C f\,dx+g\,dy = \varphi(1,2)-\varphi(6,4)+\varphi(3,6)-\varphi(1,2) = 7-20+10-7 = -10.$

17.3.38 $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = 0.$

17.3.39 Write $\varphi(x,y) = x^2+y^2$. Then using the Fundamental Theorem, this integral is equal to $\varphi(3,4)-\varphi(0,1) = 25-1 = 24$.

17.3.40 Write $\varphi(x,y,z) = x+y+z$. Then using the Fundamental Theorem, this integral is equal to $\varphi(3,0,7)-\varphi(1,-1,2) = 10-2 = 8$.

17.3.41 Write $\varphi(x,y) = e^{-x}\cos(y)$. Then using the Fundamental Theorem, this integral is equal to $\varphi(\ln 2,2\pi)-\varphi(0,0) = e^{-\ln 2}\cos(2\pi)-e^0\cos(0) = \dfrac{1}{2}-1 = -\dfrac{1}{2}$.

17.3.42 Write $\varphi(x,y,z) = 1+x^2yz$. Then using the Fundamental Theorem, this integral is equal to $\varphi(\cos(8\pi),\sin(8\pi),4\pi)-\varphi(\cos(0),\sin(0),0) = \varphi(1,0,4\pi)-\varphi(1,0,0) = 1-1 = 0$.

17.3.43 Let $\varphi(x,y,z) = x\cos(z-2y)$. The given integral is equal to $\varphi(\pi^2,\pi,\pi)-\varphi(0,0,0) = -\pi^2-0 = -\pi^2$.

17.3.44 Let $\varphi(x,y) = ye^x$. The given integral is equal to $\varphi(4,9)-\varphi(0,1) = 9e^4-1$.

17.3.45 Parameterize C by $\mathbf{r}(t) = \langle 4\cos t,4\sin t\rangle$, $0\le t\le 2\pi$. Then $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\langle 4\cos t,4\sin t\rangle\cdot\langle -4\sin t,4\cos t\rangle\,dt = \int_0^{2\pi} 0\,dt = 0.$

17.3.46 Parameterize C by $\mathbf{r}(t) = \langle 8\cos t,8\sin t\rangle$, $0\le t\le 2\pi$. Then $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\langle 8\sin t,8\cos t\rangle\cdot\langle -8\sin t,8\cos t\rangle\,dt = 8\int_0^{2\pi}\left(\cos^2 t-\sin^2(t)\right)\,dt = 0.$

17.3.47 Parameterize C by three paths, all for $0 \le t \le 1$: $\mathbf{r}_1(t) = \langle t, t-1\rangle$, so $\mathbf{r}_1'(t) = \langle 1, 1\rangle$. $\mathbf{r}_2(t) = \langle 1-t, t\rangle$, so $\mathbf{r}_2'(t) = \langle -1, 1\rangle$. $\mathbf{r}_3(t) = \langle 0, 1-2t\rangle$, so $\mathbf{r}_3'(t) = \langle 0, -2\rangle$. Then $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \langle t, t-1\rangle \cdot \langle 1, 1\rangle\, dt + \int_0^1 \langle 1-t, t\rangle \cdot \langle -1, 1\rangle\, dt + \int_0^1 \langle 0, 1-2t\rangle \cdot \langle 0, -2\rangle\, dt = \int_0^1 (t + (t-1) + (t-1) + t - 2(1-2t))\, dt = \int_0^1 (8t-4)\, dt = 0$.

17.3.48 Parameterize C by $\mathbf{r}(t) = \langle 3\cos t, 3\sin t\rangle$, $0 \le t \le 2\pi$. Then $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle 3\sin t, -3\cos t\rangle \cdot \langle -3\sin t, 3\cos t\rangle\, dt = -9\int_0^{2\pi} 1\, dt = -18\pi$. This integral is not zero because the vector field $\langle y, -x\rangle$ is not conservative: $\dfrac{\partial}{\partial y}(y) = 1$, while $\dfrac{\partial}{\partial x}(-x) = -1$.

17.3.49 Using the given parameterization,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle \cos t, \sin t, 2\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt = \int_0^{2\pi} 0\, dt = 0.$$

17.3.50 Using the given parameterization,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle \sin t - \cos t, 0, \cos t - \sin t\rangle \cdot \langle -\sin t, \cos t, -\sin t\rangle\, dt = \int_0^{2\pi} 0\, dt = 0.$$

17.3.51 $\oint_{C_1} \mathbf{F}\cdot d\mathbf{r} = 2 - 7 = -5$.

17.3.52 $\oint_{C_2} \mathbf{F}\cdot d\mathbf{r} = 9 - 0 = 9$.

17.3.53 $\mathbf{F}$ is a conservative vector field; a potential function can be found by integrating x^2 with respect to y to obtain $x^2y + f(x,z)$; differentiate with respect to z to get $f_z(x,z) = 2xz$, so that $f(x,z) = xz^2 + g(x)$. Thus the potential function is $x^2y + xz^2 + g(x)$; differentiating with respect to x gives $2xy + z^2 + g_x(x) = 2xy + z^2$, so that $g_x(x) = 0$ and we may take $g(x) = 0$. So if $\varphi = x^2y + xz^2$, then $\nabla\varphi = \mathbf{F}$ and thus the integral is zero because both sine and cosine, and thus φ, have the same values at the two endpoints of C.

17.3.54 If $x = x(t)$, $y = y(t)$, then $\oint_C e^{-x}(\cos y\, dx + \sin y\, dy) = \oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle e^{-x}\cos y, e^{-x}\sin y\rangle$. This is a conservative vector field, because $\dfrac{\partial}{\partial y}\left(e^{-x}\cos y\right) = -e^{-x}\sin y = \dfrac{\partial}{\partial x}\left(e^{-x}\sin y\right)$, so that the integral around the closed curve is zero.

17.3.55 The value of the integral is $\sin xy\Big|_{(0,0)}^{(2,\pi/4)} = \sin\pi/2 - \sin 0 = 1$.

17.3.56 Let $\varphi(x,y) = \dfrac{x^4}{4} + \dfrac{y^4}{4}$. The value of the integral is $\varphi(2,0) - \varphi(1,1) = 4 - \dfrac{1}{2} = \dfrac{7}{2}$.

17.3.57

a. False. Parametrize the curve by $x = 4\cos t + 1$, $y = 4\sin t$, for $0 \le t \le 2\pi$. Then $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle -4\sin t, 4\cos t + 1\rangle \cdot \langle -4\sin t, 4\cos t\rangle\, dt = \int_0^{2\pi} (16\sin^2 t + 16\cos^2 t + 4\cos t)\, dt = 32\pi \ne 0$.

b. True. This is because $\mathbf{F}$ is conservative.

c. True. If the vector field is $\langle a, b\rangle$, then a potential function is $ax + by$.

d. True. This is because $\frac{\partial}{\partial y} f(x) = \frac{\partial}{\partial x} g(y) = 0$.

e. True. This follows from the definitions.

17.3.58 $\oint_C ds$ is the length of the curve C, which is 2π. The other two integrals are zero, because they are the same as integrating the conservative vector fields $\langle 1, 0, 0\rangle$, and $\langle 0, 1, 0\rangle$ respectively around a closed curve.

17.3.59 This is a conservative vector field with potential function $\varphi(x, y) = \frac{1}{2}x^2 + 2y$, so the work is $\varphi(2, 4) - \varphi(0, 0) = 10$.

17.3.60 This is a conservative vector field with potential function $\varphi(x, y) = \frac{1}{2}\left(x^2 + y^2\right)$, so the work is $\varphi(3, -6) - \varphi(1, 1) = \frac{45}{2} - 1 = \frac{43}{2}$.

17.3.61 This is a conservative vector field with potential function $\varphi(x, y, z) = \frac{1}{2}\left(x^2 + y^2 + z^2\right)$, so the work is $\varphi(2, 4, 6) - \varphi(1, 2, 1) = 28 - 3 = 25$.

17.3.62 This is not a conservative vector field, because for example $\frac{\partial}{\partial z}\left(e^{x+y}\right) = 0$, while $\frac{\partial}{\partial x}\left(ze^{x+y}\right) = ze^{x+y}$. Parameterize C by $\langle -t, 2t, -4t\rangle$, $0 \le t \le 1$; then $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle e^t, e^t, -4te^t\rangle \cdot \langle -1, 2, -4\rangle\, dt = \int_0^1 \left(e^t + 16te^t\right)\, dt = e + 15$.

17.3.63

a. Negative.

b. Positive.

c. No. The value of the integral is not zero, so $\mathbf{F}$ is not conservative.

17.3.64 No.

17.3.65 $\mathbf{F} = \langle a, b, c\rangle$ is a conservative force field with potential function $\varphi(x, y, z) = ax + by + cz$, so the work done is $\varphi(B) - \varphi(A) = \mathbf{F} \cdot B - \mathbf{F} \cdot A = \mathbf{F} \cdot (B - A) = \mathbf{F} \cdot \overrightarrow{AB}$.

17.3.66

a. The acceleration is the time derivative of velocity, so Newton's second law says that $m\frac{d\mathbf{v}}{dt} = m\mathbf{a} = \mathbf{F} = -\nabla\varphi$

b. By the product rule, $\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} + \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = 2\frac{d\mathbf{v}}{dt} \cdot \mathbf{v}$, and the desired equation follows.

c. Multiplying part (a) by $\mathbf{v} = \mathbf{r}'$ and using (b), we have (letting C be a path from A to B) $m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = -\nabla\varphi \cdot \mathbf{r}' = \frac{1}{2}m\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v})$.

Thus, $\int_C -\nabla\varphi \cdot \mathbf{r}' = \frac{1}{2}m\int_A^B \frac{d}{dt}(\mathbf{v}(t) \cdot \mathbf{v}(t))\, dt$, so $\varphi(A) - \varphi(B) = \frac{1}{2}m(|\mathbf{v}|)^2 \Big|_A^B$. The last equality follows because the integrand is a conservative vector field. Thus $\frac{1}{2}m|\mathbf{v}(B)|^2 - \frac{1}{2}m|\mathbf{v}(A)|^2 = \varphi(A) - \varphi(B)$, so $\frac{1}{2}m|\mathbf{v}(B)|^2 + \varphi(B) = \frac{1}{2}m|\mathbf{v}(A)|^2 + \varphi(A)$.

17.3.67

a. Away from the origin (where the denominator of the force field equation is undefined), the force field is conservative because, for example,

$$\frac{\partial}{\partial y}GMm\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}=GMm\frac{2xy}{\left(x^2+y^2+z^2\right)^{5/2}}=\frac{\partial}{\partial x}GMm\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}.$$

b. A potential function for the force field is $\varphi\left(x,y,z\right)=GMm\left(x^2+y^2+z^2\right)^{-1/2}=GMm\frac{1}{|\mathbf{r}|}$.

c. The work done in moving the point from A to B, because the force field is conservative, is $\varphi\left(B\right)-\varphi\left(A\right)=GMm\left(\frac{1}{|B|}-\frac{1}{|A|}\right)=GMm\left(\frac{1}{r_2}-\frac{1}{r_1}\right)$.

d. Because the field is conservative, the work done does not depend on the path.

17.3.68 This vector field is $\mathbf{F}=\langle x,y,z\rangle\left(x^2+y^2+z^2\right)^{-p/2}$, so away from the origin is conservative with potential function $\varphi\left(x,y,z\right)=\frac{1}{2-p}\left(x^2+y^2+z^2\right)^{1-p/2}$ as long as $p\neq 2$. When $p=2$, the potential function is $\varphi=\frac{1}{2}\ln(x^2+y^2+z^2)$. The field is conservative at the origin if it is defined and if its potential function is defined, i.e. if both $-\frac{p}{2}$ and $1-\frac{p}{2}$ are nonnegative, which happens only if $p\leq 0$.

17.3.69

a. This field is

$$\mathbf{F}=\langle -y,x\rangle\left(x^2+y^2\right)^{-p/2},$$

and we have

$$\begin{aligned}\frac{\partial}{\partial y}\left(-y\left(x^2+y^2\right)^{-p/2}\right)&=-\left(x^2+y^2\right)^{-p/2}+py^2\frac{\left(x^2+y^2\right)^{-p/2}}{x^2+y^2}\\&=-\left(x^2+y^2\right)^{-p/2}+py^2\left(x^2+y^2\right)^{-1-p/2}=\frac{-x^2+(p-1)y^2}{(x^2+y^2)^{1+p/2}}.\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial x}\left(x\left(x^2+y^2\right)^{-p/2}\right)&=\left(x^2+y^2\right)^{-p/2}-px^2\frac{\left(x^2+y^2\right)^{-p/2}}{x^2+y^2}\\&=\left(x^2+y^2\right)^{-p/2}-px^2\left(x^2+y^2\right)^{-1-p/2}=\frac{-(p-1)x^2+y^2}{(x^2+y^2)^{1+p/2}}.\end{aligned}$$

For the force field to be conservative, these two would have to be equal. However, their difference is $\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=2\left(x^2+y^2\right)^{-p/2}-p\left(x^2+y^2\right)\left(x^2+y^2\right)^{-1-p/2}=2\left(x^2+y^2\right)^{-p/2}-p\left(x^2+y^2\right)^{-p/2}=(2-p)\left(x^2+y^2\right)^{-p/2}$ which is in general nonzero.

b. From the above formula, if $p=2$, then the mixed partials are equal, so that $\mathbf{F}$ is conservative.

c. For $p=2$, $\mathbf{F}=\frac{1}{x^2+y^2}\langle -y,x\rangle$. Integrating the x component of F with respect to x gives $\varphi=\tan^{-1}\left(\frac{y}{x}\right)$.

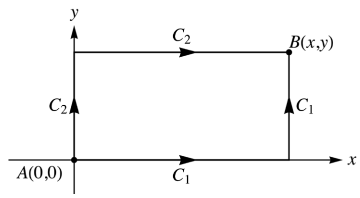
17.3.70

a. Because $\frac{\partial}{\partial y}\left(ax+by\right)=b$ and $\frac{\partial}{\partial x}\left(cx+dy\right)=c$, the field is conservative when $b=c$.

b. Because $\dfrac{\partial}{\partial y}\left(ax^2 - by^2\right) = -2by$ and $\dfrac{\partial}{\partial x}(cxy) = cy$, the field is conservative when $c = -2b$.

17.3.71

a.



b. Parameterize C_1 by two paths: $\mathrm{r}_1(t) = \langle t, 0\rangle,\ \ 0 \le t \le x$, and $\mathrm{r}_2(t) = \langle x, t\rangle,\ \ 0 \le t \le y$. Then $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^x \langle 2t, -t\rangle\cdot\langle 1, 0\rangle\, dt + \int_0^y \langle 2x - t, -x + 2t\rangle\cdot\langle 0, 1\rangle\, dt = \int_0^x 2t\, dt + \int_0^y (2t - x)\ dt = x^2 + y^2 - xy$.

c. Parameterize C_2 by the paths $\mathrm{r}_1(t) = \langle 0, t\rangle,\ \ 0 \le t \le y$, and $\mathrm{r}_2(t) = \langle t, y\rangle,\ \ 0 \le t \le x$. Then $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^y \langle -t, 2t\rangle\cdot\langle 0, 1\rangle\, dt + \int_0^x \langle 2t - y, -t + 2y\rangle\cdot\langle 1, 0\rangle\, dt = \int_0^y 2t\, dt + \int_0^x (2t - y)\ dt = x^2 + y^2 - xy$.

17.3.72 Using problem 71, we have the same paths $\mathbf{r}_1$, $\mathbf{r}_2$, and $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^x \langle 0, -t\rangle\cdot\langle 1, 0\rangle\, dt + \int_0^y \langle -t, -x\rangle\cdot \langle 0, 1\rangle\, dt = \int_0^x 0\, dt + \int_0^y (-x)\ dt = -xy$.

17.3.73 Using problem 71 and the same paths $\mathbf{r}_1$, $\mathbf{r}_2$, we have $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^x \langle t, 0\rangle\cdot\langle 1, 0\rangle\, dt + \int_0^y \langle x, t\rangle\cdot \langle 0, 1\rangle\, dt = \int_0^x t\, dt + \int_0^y t\, dt = \frac{1}{2}\left(x^2 + y^2\right)$.

17.3.74 Using problem 71 and the same paths $\mathbf{r}_1$, $\mathbf{r}_2$, note that $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|} = \langle x, y\rangle\left(x^2 + y^2\right)^{-1/2}$, and $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^x \langle 1, 0\rangle\cdot\langle 1, 0\rangle\, dt + \int_0^y \left\langle \frac{x}{\sqrt{x^2 + t^2}}, \frac{t}{\sqrt{x^2 + t^2}}\right\rangle\cdot\langle 0, 1\rangle\, dt = \int_0^x 1\, dt + \int_0^y \frac{t}{\sqrt{x^2 + t^2}}\, dt = x + \sqrt{x^2 + y^2} - x = \sqrt{x^2 + y^2}$.

17.3.75 Using problem 71 and the same paths $\mathbf{r}_1$, $\mathbf{r}_2$, we have $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^x \langle 2t^3, 0\rangle\cdot\langle 1, 0\rangle\, dt + \int_0^y \langle 2x^3 + xt^2, 2t^3 + x^2 t\rangle\cdot\langle 0, 1\rangle\, dt = \int_0^x 2t^3\, dt + \int_0^y \left(2t^3 + x^2 t\right)\ dt = \frac{1}{2}\left(x^4 + x^2y^2 + y^4\right)$.

17.4 Green's Theorem

17.4.1 As with the Fundamental Theorem of Calculus, it allows evaluation of the integral of a derivative by looking at the value of the underlying function on the boundary of a region (or, in the case of the Fundamental Theorem, an interval).

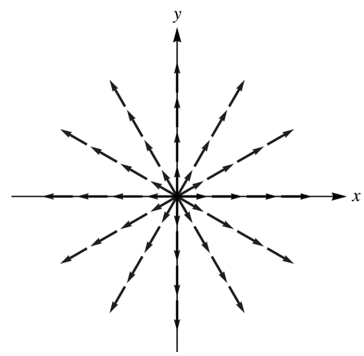
17.4.2 The line integral for flux corresponds to the double integral of the divergence; the line integral for circulation to the double integral of the curl.

17.4.3 The area is $\displaystyle\frac{1}{2}\oint_C (x\, dy - y\, dx)$ where C is the boundary of the region.

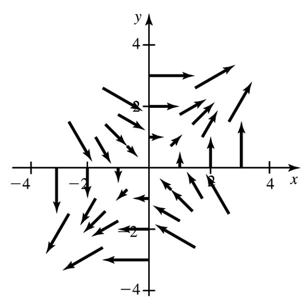
17.4.4 Because the curl being zero is an equivalent condition to the field being conservative.

17.4.5 Because the flux is the integrand in Green's theorem, so the integral vanishes.

17.4.6 A conservative vector field such as $\langle x, y\rangle$ will have zero curl:



17.4.7



17.4.8 A conservative and a source-free field each have functions (a potential function in the case of a conservative field; a stream function in the case of a source-free field) that closely reflect the vector field. The properties of the partials of these functions are such that the curl (or divergence, for a source-free field) vanish.

17.4.9

a. The two-dimensional curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 0 - 0 = 0$.

b. The two-dimensional divergence is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 1 + 1 = 2$.

c. The vector field is irrotational because its curl is zero.

d. The vector field is not source free because the divergence is not zero.

17.4.10

a. The two-dimensional curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = -1 - 1 = -2$.

b. The two-dimensional divergence is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 0 + 0 = 0$.

c. The vector field is not irrotational because its curl is not zero.

d. The vector field is source free because the divergence is zero.

17.4.11

a. The two-dimensional curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = -3 - 1 = -4$.

b. The two-dimensional divergence is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 0 + 0 = 0$.

c. The vector field is not irrotational because its curl is not zero.

d. The vector field is source free because the divergence is zero.

17.4.12

a. The two-dimensional curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = -2y - 2x$

b. The two-dimensional divergence is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 2x + 2y - 2x - 2y = 0$.

c. The vector field is not irrotational because its curl is not zero.

d. The vector field is source free because the divergence is zero.

17.4.13

a. The two-dimensional curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = y^2 + 4x^3 - 4x^3 = y^2$.

b. The two-dimensional divergence is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 12x^2y + 2xy$.

c. The vector field is not irrotational because its curl is not zero.

d. The vector field is not source free because the divergence is not zero.

17.4.14

a. The two-dimensional curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 12y - 1$.

b. The two-dimensional divergence is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 12x^2 + 12x$.

c. The vector field is not irrotational because its curl is not zero.

d. The vector field is not source free because the divergence is not zero.

17.4.15

a. The two-dimensional curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 1 - 0 = 1$, so $\mathbf{F}$ is not irrotational.

b. $\mathbf{r}_1(t) = \left\langle t, t^2 \right\rangle$ for $0 \le t \le 1$ and $\mathbf{r}_2(t) = \langle 1-t, 1-t \rangle$ for $0 \le t \le 1$.

c. $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2t^2 + 1)\, dt + \int_0^1 (t-2)\, dt = \left(\frac{2t^3}{3} + t \right) \Big|_0^1 + \left(\frac{t^2}{2} - 2t \right) \Big|_0^1 = \frac{5}{3} - \frac{3}{2} = \frac{1}{6}$.

$\displaystyle\int_0^1 \int_{x^2}^x 1\, dy\, dy = \int_0^1 (x - x^2)\, dx = \left(\frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_0^1 = \frac{1}{6}$.

d. The two-dimensional divergence is $0 + 0 = 0$, $\displaystyle\iint_R 0\, dA = 0$.

17.4.16

a. The two-dimensional curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 0 - 0 = 0$.

b. The two-dimensional divergence is $1 + 0 = 1$. The vector field is not source free.

c. $\mathbf{r}_1(t) = \left\langle t, 2t^2 - 2t \right\rangle$ for $0 \le t \le 1$ and $\mathbf{r}_2(t) = \langle 1 - t, 0 \rangle$ for $0 \le t \le 1$.

d. $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_{C_1} \mathbf{F}\cdot\mathbf{n}\,ds + \oint_{C_2} \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1 (4t^2-2t-1)\,dt + \int_0^1 dt = \left(\frac{4}{3}t^3 - t^2 - t\right)\Big|_0^1 + 1 = -\frac{2}{3} + 1 = \frac{1}{3}$.
$\displaystyle\int_0^1 \int_{2x^2-2x}^0 dy\,dx = 2\int_0^1 (x - x^2)\,dx = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{3}$.

17.4.17

a. The curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = -2 - 2 = -4$.

b. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi \langle 0, -2t\rangle \cdot \langle 1, 0\rangle\,dt + \int_\pi^0 \langle 2\sin t, -2t\rangle \cdot \langle 1, \cos t\rangle\,dt = \int_\pi^0 (2\sin t - 2t\cos t)\,dt = -8$.
$\displaystyle\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint_R (-4)\,dA = -4\int_0^\pi \sin x\,dx = -8$.

17.4.18

a. The curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 3 + 3 = 6$.

b. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \langle 0, 3t\rangle \cdot \langle 1, 0\rangle\,dt + \int_0^1 \langle -6t, 3 - 3t\rangle \cdot \langle -1, 2\rangle\,dt + \int_0^1 \langle -3\,(2 - 2t)\,, 0\rangle \cdot \langle 0, -2\rangle\,dt =$
$\displaystyle\int_0^1 (6t + 6 - 6t)\,dt = 6$.
$\displaystyle\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint_R 6\,dA = 6\int_0^1 (2 - 2x)\,dx = 6$.

17.4.19

a. The curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 2x + 2x = 4x$.

b. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^2 \langle 0, t^2\rangle \cdot \langle 1, 0\rangle\,dt + \int_2^0 \langle -2t^2(2-t), t^2\rangle \cdot \langle 1, 2-2t\rangle\,dt = \int_2^0 \left(-2t^2\,(2-t) + t^2\,(2-2t)\right)\,dt =$
$\displaystyle\int_2^0 \left(-4t^2 + 2t^3 + 2t^2 - 2t^3\right)\,dt = \int_2^0 -2t^2\,dt = -\frac{2t^3}{3}\Big|_2^0 = \frac{16}{3}$.
$\displaystyle\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint_R 4x\,dA = \int_0^2 \int_0^{2x-x^2} 4x\,dy\,dx = \int_0^2 (8x^2 - 4x^3)\,dx = \left(\frac{8x^3}{3} - x^4\right)\Big|_0^2 =$
$\displaystyle\frac{64}{3} - 16 = \frac{16}{3}$.

17.4.20

a. The curl is $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 2x$.

b. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle 0, \sin^2 t + \cos^2 t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt = \int_0^{2\pi} \cos t\, dt = 0.$

$$\iint\limits_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint\limits_R (2x)\, dA = \int_0^{2\pi}\int_0^1 2r\cos\theta\, r\, dr\, d\theta = 0.$$

17.4.21 Parameterize the boundary by $x = 5\cos t$, $y = 5\sin t$, $0 \le t \le 2\pi$. Then $dx = -5\sin t\, dt$, $dy = 5\cos t\, dt$, and the area is $\displaystyle\frac{1}{2}\oint_C x\, dy - y\, dx = \frac{1}{2}\int_0^{2\pi} ((5\cos t)\cdot(5\cos t) - (5\sin t)\cdot(-5\sin t))\, dt = \frac{25}{2}\int_0^{2\pi} 1\, dt = 25\pi.$

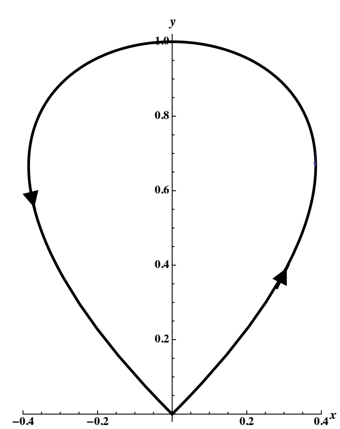
17.4.22 Parameterize the boundary by $x = 6\cos t$, $y = 4\sin t$, $0 \le t \le 2\pi$. Then $dx = -6\sin t\, dt$, $dy = 4\cos t\, dt$, and the area is $\displaystyle\frac{1}{2}\oint_C x\, dy - y\, dx = \frac{1}{2}\int_0^{2\pi} ((6\cos t)\cdot(4\cos t) - (4\sin t)\cdot(-6\sin t))\, dt = 12\int_0^{2\pi} 1\, dt = 24\pi.$

17.4.23 Parameterize the boundary by $x = 4\cos t$, $y = 4\sin t$, $0 \le t \le 2\pi$. Then $dx = -4\sin t\, dt$, $dy = 4\cos t\, dt$, and the area is $\displaystyle\frac{1}{2}\oint_C x\, dy - y\, dx = \frac{1}{2}\int_0^{2\pi} ((4\cos t)\cdot(4\cos t) - (4\sin t)\cdot(-4\sin t))\, dt = 8\int_0^{2\pi} 1\, dt = 16\pi.$

17.4.24 Note that C_1 can be parameterized by $x = -\frac{\sqrt{2}}{2}(1-t) + \frac{\sqrt{2}}{2}t$ and $y = \frac{\sqrt{2}}{2}$, $0 \le t \le 1$ while C_2 can be parametrized by $x = \cos t$ and $y = \sin t$ for $-\frac{\pi}{4} \le t \le \frac{\pi}{4}$. For C_1 we have $dy = 0$ and $dx = \sqrt{2}\, dt$. Thus $\displaystyle\frac{1}{2}\oint_{C_1} x\, dy - y\, dx = \frac{1}{2}\int_0^1 -\frac{\sqrt{2}}{2}\cdot\sqrt{2}\, dt = -\frac{1}{2}$. For C_2, we have $dx = -\sin t\, dt$ and $dy = \cos t\, dt$, and we have $\displaystyle\frac{1}{2}\oint_{C_2} x\, dy - y\, dx = \frac{1}{2}\int_{-\pi/4}^{\pi/4} dt = \frac{\pi}{4}$. Thus the area is $\frac{\pi}{4} - \frac{1}{2}$. As a quick check, note that the region could be thought of as a quarter circle of radius one minus a triangle with area $\frac{1}{2}$.

17.4.25 Traverse the first path from -2 to 2, then the second path back from 2 to -2. The area is then $\displaystyle\frac{1}{2}\oint_C x\, dy - y\, dx = \frac{1}{2}\int_{-2}^{2}\left(t\cdot(4t) - \left(2t^2\right)\cdot 1\right) dt + \frac{1}{2}\int_2^{-2}\left(t\cdot(-2t) - \left(12 - t^2\right)\cdot 1\right) dt = \frac{1}{2}\int_{-2}^{2} 2t^2\, dt + \frac{1}{2}\int_2^{-2}\left(-2t^2 - 12 + t^2\right) dt = 32.$

17.4.26



We have $dx = \left(1 - t^2 + t \cdot (-2t)\right)\, dt = \left(1 - 3t^2\right)\, dt$ and $dy = -2t\, dt$. We parameterize the curve from $t = 1$ to $t = -1$ so that we traverse the region counterclockwise; then the area is $\dfrac{1}{2}\displaystyle\oint_C x\, dy - y\, dx = \dfrac{1}{2}\int_1^{-1} \left(t\left(1-t^2\right)(-2t) - \left(1-t^2\right)\left(1-3t^2\right)\right)\, dt = \dfrac{8}{15}$.

17.4.27

a. The divergence is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 1 + 1 = 2$.

b. $\displaystyle\int_C \mathbf{F} \cdot \mathbf{n}\, ds = 4\int_0^{2\pi} (\cos t\,(\cos t) - \sin t\,(-\sin t))\, dt = 8\pi$.
$\displaystyle\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\, dA = \iint_R (2)\, dA = 2 \cdot 4\pi = 8\pi$.

17.4.28

a. The divergence is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 1 - 3 = -2$.

b. Parameterize the triangle by the three paths: $\mathbf{r}_1(t) = \langle t, 2t\rangle$, $\mathbf{r}_2(t) = \langle 1-t, 2\rangle$, and $\mathbf{r}_3(t) = \langle 0, 2-2t\rangle$, all for $0 \le t \le 1$. Then $\displaystyle\int_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_0^1 (2t + 3(2t))\, dt + \int_0^1 (0-6)\, dt + \int_0^1 (0+0)\, dt = \int_0^1 (8t-6)\, dt = \left(4t^2 - 6t\right)\Big|_0^1 = -2$.
$\displaystyle\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\, dA = \int_0^1 \int_{2x}^2 (-2)\, dy\, dx = \int_0^1 (-4+4x)\, dx = \left(-4x + 2x^2\right)\Big|_0^1 = -2$.

17.4.29

a. The divergence is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 2y + 0 = 2y$.

b. Parameterize the region by $\mathbf{r}_1(t) = \langle t, 0\rangle$ for $0 \le t \le 2$ and $\mathbf{r}_2(t) = \langle t, 2t - t^2\rangle$, for $0 \le t \le 2$ traversed backwards (from $t = 2$ to $t = 0$.) Then $\displaystyle\int_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_0^2 (0 - t^2)\, dt + \int_2^0 \left(2t(2t-t^2)(2-2t) - t^2\right)\, dt = \int_0^2 (-8t^2 + 12t^3 - 4t^4)\, dt = \left(\frac{-8t^3}{3} + 3t^4 - \frac{4t^5}{5}\right)\Big|_0^2 = \frac{16}{15}$.
$\displaystyle\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\, dA = \int_0^2 \int_0^{2x-x^2} 2y\, dy\, dx = \int_0^2 (2x - x^2)^2\, dx = \int_0^2 (4x^2 - 4x^3 + x^4)\, dx = \left(\frac{4x^3}{3} - x^4 + \frac{x^5}{5}\right)\Big|_0^2 = \frac{32}{3} - 16 + \frac{32}{5} = \frac{16}{15}$.

17.4.30

a. The divergence is $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 2x$.

b. $\displaystyle\int_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_0^{2\pi} \left((\sin^2 t + \cos^2 t)\cdot\cos t + 0 \cdot \sin t\right)\, dt = \int_0^{2\pi} \cos t\, dt = 0$.
$\displaystyle\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\, dA = \iint_R 2x\, dA = \int_0^{2\pi}\int_0^1 2\cos\theta \cdot r\, dr\, d\theta = 0$.

17.4.31 $\oint_C \langle 3y+1, 4x^2+3\rangle \cdot d\mathbf{r} = \int_0^2 \int_0^4 (8x-3)\,dx\,dy = \int_0^2 (4x^2-3x)\Big|_0^4\,dx = \int_0^2 52\,dx = 104.$

17.4.32 $\oint_C \langle \sin y, x\rangle \cdot d\mathbf{r} = \int_0^{\pi/2} \int_0^x (1-\cos y)\,dy\,dx = \int_0^{\pi/2} (y-\sin y)\Big|_0^x\,dx = \int_0^{\pi/2} (x-\sin x)\,dx = \left(\frac{x^2}{2}+\cos x\right)\Big|_0^{\pi/2} = \frac{\pi^2}{8}-1.$

17.4.33 $\oint_C xe^y\,dx + x\,dy = \int_0^2 \int_0^{x^2} (1-xe^y)\,dy\,dx = \int_0^2 (y-xe^y)\Big|_0^{x^2}\,dx = \int_0^2 \left(x^2+x-xe^{x^2}\right)\,dx = \left(\frac{x^3}{3}+\frac{x^2}{2}-\frac{1}{2}e^{x^2}\right)\Big|_0^2 = \frac{8}{3}+2-\frac{1}{2}e^4+\frac{1}{2} = \frac{31-3e^4}{6}.$

17.4.34 $\oint_C \frac{1}{1+y^2}\,dx + y\,dy = \int_0^1 \int_0^x \left(\frac{2y}{(1+y^2)^2}\right)\,dy\,dx = \int_0^1 \left(-\frac{1}{1+y^2}\right)\Big|_0^x\,dx = \int_0^1 \left(1-\frac{1}{1+x^2}\right)\,dx = \left(x-\tan^{-1}x\right)\Big|_0^1 = 1-\frac{\pi}{4} = \frac{4-\pi}{4}.$

17.4.35 The line integral, using the flux form of Green's theorem, is equal to

$$\iint_R \left(\frac{\partial}{\partial x}\left(2x+e^{y^2}\right)+\frac{\partial}{\partial y}\left(4y^2+e^{x^2}\right)\right)\,dA = \iint_R (2+8y)\,dA = \int_0^1 \int_0^1 (2+8y)\,dx\,dy$$
$$= \int_0^1 (2+8y)\,dy = 6.$$

17.4.36 Using the flux form of Green's theorem, the integral is equal to

$$\iint_R \left(\frac{\partial}{\partial x}(2x-3y)+\frac{\partial}{\partial y}(3x+4y)\right)\,dA = \iint_R 6\,dA = 6\times\text{area of } R = 6\pi.$$

17.4.37 Using the flux form of Green's theorem, the integral is equal to

$$\iint_R \left(\frac{\partial}{\partial x}(0)+\frac{\partial}{\partial y}(xy)\right)\,dA = \int_0^2 \int_0^{4-2x} x\,dy\,dx = \int_0^2 \left(4x-2x^2\right)\,dx = \frac{8}{3}.$$

17.4.38 Using the flux form of Green's theorem, the integral is equal to (note the leading minus sign to correct for the orientation)

$$-\iint_R \left(\frac{\partial}{\partial x}\left(x^2\right)+\frac{\partial}{\partial y}\left(2y^2\right)\right)\,dA = -\int_{-1}^1 \int_0^{\sqrt{1-x^2}} (2x+4y)\,dy\,dx$$
$$= -\int_{-1}^1 \left(2xy+2y^2\right)\Big|_{y=0}^{y=\sqrt{1-x^2}}\,dx = -\int_{-1}^1 \left(2x\sqrt{1-x^2}+2-2x^2\right)\,dx = -\frac{8}{3}.$$

17.4.39 Using the circulation form of Green's theorem, the integral is equal to

$$\iint_R \left(\frac{\partial}{\partial x}\left(4x+y^3\right)-\frac{\partial}{\partial y}\left(x^2+y^2\right)\right)\,dA = \iint_R (4-2y)\,dA = \int_0^\pi \int_0^{\sin(x)} (4-2y)\,dy\,dx = 8-\frac{\pi}{2}.$$

17.4.40 Using the flux form of Green's theorem, the integral is equal to

$$\iint_R \left(\frac{\partial}{\partial x}\left(e^{x-y}\right)+\frac{\partial}{\partial y}\left(e^{y-x}\right)\right)\,dA = \iint_R \left(e^{x-y}+e^{y-x}\right)\,dA = \int_0^1 \int_0^x \left(e^{x-y}+e^{y-x}\right)\,dy\,dx$$
$$= \int_0^1 \left(-e^{x-y}+e^{y-x}\right)\Big|_{y=0}^{y=x}\,dx = \int_0^1 \left(e^x-e^{-x}\right)\,dx = e+e^{-1}-2.$$

17.4.41

a. Using Green's theorem, the circulation is $\iint\limits_R \left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right)\, dA = \iint\limits_R 0\, dA = 0,$

b. Using Green's theorem, the flux is $\iint\limits_R \left(\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y)\right)\, dA = \iint\limits_R 2\, dA = 2 \cdot \text{area of } R = 2 \cdot \frac{1}{2}(4\pi - \pi) = 3\pi.$

17.4.42

a. Using Green's theorem, the circulation is $\iint\limits_R \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right)\, dA = \iint\limits_R 2\, dA = 2 \cdot \text{area of} R = 2 \cdot (9\pi - \pi) = 16\pi.$

b. Using Green's theorem, the flux is $\iint\limits_R \left(\frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x)\right)\, dA = 0.$

17.4.43

a. Using Green's theorem, the circulation is $\iint\limits_R \left(\frac{\partial}{\partial x}(x - 4y) - \frac{\partial}{\partial y}(2x + y)\right)\, dA = \iint\limits_R (1 - 1)\, dA = 0.$

b. Using Green's theorem, the flux is $\iint\limits_R \left(\frac{\partial}{\partial x}(2x + y) + \frac{\partial}{\partial y}(x - 4y)\right)\, dA = \iint\limits_R (-2)\, dA = -2 \cdot$ area of $R = -2 \cdot \frac{1}{4}(16\pi - \pi) = -\frac{15}{2}\pi.$

17.4.44

a. Using Green's theorem, the circulation is $\iint\limits_R \left(\frac{\partial}{\partial x}(-x + 2y) - \frac{\partial}{\partial y}(x - y)\right)\, dA = \iint\limits_R (0)\, dA = 0.$

b. Using Green's theorem, the flux is $\iint\limits_R \left(\frac{\partial}{\partial x}(x - y) + \frac{\partial}{\partial y}(-x + 2y)\right)\, dA = \iint\limits_R 3\, dA = 3 \cdot \text{area of } R = 6.$

17.4.45

a. Because $\mathbf{F}$ is conservative, the circulation on the boundary of R is zero.

b. $\mathbf{F} = \left(x^2 + y^2\right)^{-1/2} \langle x, y\rangle$, so the flux is $\iint\limits_R \left(\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right)\, dA =$
$\iint\limits_R \left(x^2 + y^2\right)^{-1/2}\, dA = \int_0^{\pi}\int_1^3 \frac{1}{r} r\, dr\, d\theta = \int_0^{\pi}\int_1^3 1\, dr\, d\theta = 2\pi.$

17.4.46

a. The circulation is $\iint\limits_R \left(\frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) - \frac{\partial}{\partial y}\left(\ln\left(x^2 + y^2\right)\right)\right)\, dA = \iint\limits_R \left(\frac{-y}{x^2+y^2} - \frac{2y}{x^2+y^2}\right)\, dA =$
$-3\iint\limits_R \frac{y}{x^2+y^2}\, dA = -3\int_0^{2\pi}\int_1^2 \frac{r\sin\theta}{r^2} r\, dr\, d\theta = -3\int_0^{2\pi}\int_1^2 2\sin\theta\, dr\, d\theta = -3\int_0^{2\pi} \sin\theta\, d\theta = 0.$

b. The flux is $\displaystyle\iint_R \left(\frac{\partial}{\partial x}\left(\ln\left(x^2+y^2\right)\right)+\frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{y}{x}\right)\right)\right)\,dA=\iint_R \frac{3x}{x^2+y^2}\,dA=$
$\displaystyle\int_0^{2\pi}\int_1^2 \frac{3r\cos\theta}{r^2}r\,dr\,d\theta=\int_0^{2\pi}\int_1^2 3\cos\theta\,dr\,d\theta=\int_0^{2\pi}3\cos(\theta)\,d\theta=0.$

17.4.47 Note that the region is the area between $x=y^2$ and$x=2-y^2$, which intersect at $y=\pm1$.

a. The circulation is

$$\begin{aligned}\iint_R \left(\frac{\partial}{\partial x}\left(x^2-y\right)-\frac{\partial}{\partial y}\left(x+y^2\right)\right)\,dA &= \iint_R (2x-2y)\,dA\\ &= \int_{-1}^{1}\int_{y^2}^{2-y^2}(2x-2y)\,dx\,dy\\ &= \int_{-1}^{1}\left(x^2-2xy\right)\Big|_{y^2}^{2-y^2}\,dy\\ &= \int_{-1}^{1}\left((2-y^2)^2-(4y-2y^3)-(y^4-2y^3)\right)\,dy\\ &= \int_{-1}^{1}(4-4y-4y^2+4y^3)\,dy\\ &= \left(4y-2y^2-\frac{4y^3}{3}+y^4\right)\Big|_{-1}^{1}=\frac{16}{3}.\end{aligned}$$

b. The flux is $\displaystyle\iint_R \left(\frac{\partial}{\partial x}\left(x+y^2\right)+\frac{\partial}{\partial y}\left(x^2-y\right)\right)\,dA=\iint_R 0\,dA=0.$

17.4.48

a. The circulation is

$$\begin{aligned}&\iint_R \left(\frac{\partial}{\partial x}(-\sin x)-\frac{\partial}{\partial y}(y\cos x)\right)\,dA=2\iint_R(-\cos x)\,dA\\ &=2\int_0^{\pi/2}\int_0^{\pi/2}(-\cos x)\,dy\,dx=\int_0^{\pi/2}(-\pi\cos x)\,dx=-\pi.\end{aligned}$$

b. The flux is

$$\begin{aligned}&\iint_R \left(\frac{\partial}{\partial x}(y\cos x)+\frac{\partial}{\partial y}(-\sin x)\right)\,dA=\iint_R(-y\sin x)\,dA\\ &=\int_0^{\pi/2}\int_0^{\pi/2}(-y\sin x)\,dy\,dx=-\frac{1}{8}\pi^2.\end{aligned}$$

17.4.49

a. True. This is the definition of work along a path.

b. False. Divergence corresponds to flux, so if the divergence is zero throughout a region, the flux is zero across the boundary.

c. True. This follows from Green's theorem.

17.4.50 Let $\mathbf{F} = \langle 1, 0\rangle$. By Green's theorem, $\oint_C 1\,dx = \oint_C 1\,dx + 0\,dy = \iint_R \left(\frac{\partial 0}{\partial x} - \frac{\partial 1}{\partial y}\right) dA = \iint_R 0\,dA =$ 0. Similarly, if we let $\mathbf{F} = \langle 0, 1\rangle$, we have $\oint_C 1\,dy = \oint_C 0\,dx + 1\,dy = \iint_R \left(\frac{\partial 1}{\partial x} - \frac{\partial 0}{\partial y}\right) dA = \iint_R 0\,dA = 0$.

17.4.51 Because $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = 0$, the integral is zero (because $\mathbf{F}$ is conservative.)

17.4.52 Let $f(x,y) = 0$, $g(x,y) = xy^2 + y^4$; then

$$\iint_R \left(2xy + 4y^3\right) dA = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA = \oint_C (0\,dy - g\,dx) = -\oint_C \left(xy^2 + y^4\right) dx$$

by Green's theorem, where C is the boundary of the triangle. To evaluate this line integral, parameterize C by three paths, all for $0 \le t \le 1$: $\mathbf{r}_1(t) = \langle t, 0\rangle$, so that $\mathbf{r}_1'(t) = \langle 1, 0\rangle$. $\mathbf{r}_2(t) = \langle 1-t, t\rangle$, so that $\mathbf{r}_2'(t) = \langle -1, 1\rangle$. $\mathbf{r}_3(t) = \langle 0, 1-t\rangle$, so that $\mathbf{r}_3'(t) = \langle 0, -1\rangle$. Then $-\oint_C \left(xy^2 + y^4\right) dx = -\int_0^1 \left(t\left(0^2\right) - 0^4\right) dt - \int_0^1 t^2(1-t)(-1)\,dt - \int_0^1 \left(0(1-t)^2 + t^4 \cdot 0\right) dt = \int_0^1 \left(t^4 - t^3 + t^2\right) dt = \frac{17}{60}$.

17.4.53 By Green's theorem,

$$\begin{aligned}\oint_C xy^2\,dx + \left(x^2y + 2x\right) dy &= \iint_R \left(\frac{\partial}{\partial x}\left(x^2y + 2x\right) - \frac{\partial}{\partial y}\left(xy^2\right)\right) dA \\ &= \iint_R (2xy + 2 - 2xy)\,dA = \iint_R 2\,dA = 2 \cdot \text{area of } A.\end{aligned}$$

17.4.54 Using the circulation form of Green's theorem, the integral is

$$\oint_C ay\,dx + bx\,dy = \iint_R (b-a)\,dA = (b-a) \cdot \text{area of } A.$$

17.4.55

a. The divergence is $\frac{\partial}{\partial x}(4) + \frac{\partial}{\partial y}(2) = 0$.

b. $\psi = 4y - 2x$.

17.4.56

a. The divergence is $\frac{\partial}{\partial x}\left(y^2\right) + \frac{\partial}{\partial y}\left(x^2\right) = 0 + 0 = 0$.

b. $\psi = \frac{1}{3}\left(y^3 - x^3\right)$.

17.4.57

a. The divergence is $\frac{\partial}{\partial x}\left(-e^{-x}\sin y\right) + \frac{\partial}{\partial y}\left(e^{-x}\cos y\right) = e^{-x}\sin y - e^{-x}\sin y = 0$.

b. $\psi = e^{-x}\cos y$.

17.4.58

a. The divergence is $\frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}(-2xy) = 2x - 2x = 0$.

b. $\psi = x^2 y$.

17.4.59

a. The curl and divergence are $\operatorname{curl}\mathbf{F} = \dfrac{\partial}{\partial x}\left(-e^x \sin y\right) - \dfrac{\partial}{\partial y}\left(e^x \cos y\right) = -e^x \sin y + e^x \sin y = 0$. $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}\left(e^x \cos y\right) + \frac{\partial}{\partial y}\left(-e^x \sin y\right) = e^x \cos y - e^x \cos y = 0$.

b. $\varphi(x,y) = e^x \cos(y)$. $\psi(x,y) = e^x \sin(y)$.

c. $\varphi_{xx} + \varphi_{yy} = \dfrac{\partial^2}{\partial x^2}\left(e^x \cos y\right) + \dfrac{\partial^2}{\partial y^2}\left(e^x \cos y\right) = e^x \cos y - e^x \cos y = 0$. $\psi_{xx} + \psi_{yy} = \frac{\partial^2}{\partial x^2}\left(e^x \sin y\right) + \frac{\partial^2}{\partial y^2}\left(e^x \sin y\right) = e^x \sin y - e^x \sin y = 0$.

17.4.60

a. The curl and divergence are $\operatorname{curl}\mathbf{F} = \dfrac{\partial}{\partial x}\left(y^3 - 3x^2 y\right) - \dfrac{\partial}{\partial y}\left(x^3 - 3xy^2\right) = -6xy + 6xy = 0$.
$\operatorname{div}\mathbf{F} = \dfrac{\partial}{\partial x}\left(x^3 - 3xy^2\right) + \dfrac{\partial}{\partial y}\left(y^3 - 3x^2 y\right) = 3x^2 - 3y^2 + 3y^2 - 3x^2 = 0$

b. $\varphi(x,y) = \dfrac{1}{4}\left(x^4 + y^4\right) - \dfrac{3}{2}x^2 y^2$. $\psi(x,y) = x^3 y - xy^3$.

c.

$$\begin{aligned}
\varphi_{xx} + \varphi_{yy} &= \frac{\partial^2}{\partial x^2}\left(\frac{1}{4}\left(x^4 + y^4\right) - \frac{3}{2}x^2 y^2\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{4}\left(x^4 + y^4\right) - \frac{3}{2}x^2 y^2\right) \\
&= \frac{\partial}{\partial x}\left(x^3 - 3xy^2\right) + \frac{\partial}{\partial y}\left(y^3 - 3x^2 y\right) = 3x^2 - 3y^2 + 3y^2 - 3x^2 = 0
\end{aligned}$$

$$\begin{aligned}
\psi_{xx} + \psi_{yy} &= \frac{\partial^2}{\partial x^2}\left(x^3 y - xy^3\right) + \frac{\partial^2}{\partial y^2}\left(x^3 y - xy^3\right) \\
&= \frac{\partial}{\partial x}\left(3x^2 y - y^3\right) + \frac{\partial}{\partial y}\left(x^3 - 3xy^2\right) = 6xy - 6xy = 0.
\end{aligned}$$

17.4.61

a. The curl and divergence are $\operatorname{curl}\mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{1}{2}\ln\left(x^2 + y^2\right)\right) - \dfrac{\partial}{\partial y}\left(\tan^{-1}\left(\dfrac{y}{x}\right)\right) = 0$.
$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) + \frac{\partial}{\partial y}\left(\frac{1}{2}\ln\left(x^2 + y^2\right)\right) = 0$.

b. $\varphi(x,y) = x\,\tan^{-1}\left(\dfrac{y}{x}\right) - y + \dfrac{1}{2}y\,\ln\left(x^2 + y^2\right)$. $\psi(x,y) = \int -\frac{1}{2}\ln\left(x^2 + y^2\right)\,dy =$
$y\tan^{-1}\left(\frac{y}{x}\right) - \frac{x}{2}\ln\left(x^2 + y^2\right) + x$.

c.

$$\begin{aligned}
\varphi_{xx} + \varphi_{yy} &= \frac{\partial^2}{\partial x^2}\left(x\tan^{-1}\left(\frac{y}{x}\right) - y + \frac{1}{2}y\,\ln\left(x^2 + y^2\right)\right) + \frac{\partial^2}{\partial y^2}\left(x\tan^{-1}\left(\frac{y}{x}\right) - y + \frac{1}{2}y\,\ln\left(x^2 + y^2\right)\right) \\
&= \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{y}{x}\right) - \frac{xy}{x^2 + y^2} + \frac{xy}{x^2 + y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x^2}{x^2 + y^2} - 1 + \frac{1}{2}\ln\left(x^2 + y^2\right) + \frac{y^2}{x^2 + y^2}\right) \\
&= \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) + \frac{\partial}{\partial y}\left(\frac{1}{2}\ln\left(x^2 + y^2\right)\right) = 0.
\end{aligned}$$

$$\begin{aligned}
\psi_{xx} + \psi_{yy} &= \frac{\partial^2}{\partial x^2}\left(y\tan^{-1}\left(\frac{y}{x}\right) - \frac{x}{2}\ln\left(x^2 + y^2\right) + x\right) + \frac{\partial^2}{\partial y^2}\left(y\tan^{-1}\left(\frac{y}{x}\right) - \frac{x}{2}\ln\left(x^2 + y^2\right) + x\right) \\
&= \frac{\partial}{\partial x}\left(-\frac{y^2}{x^2 + y^2} - \frac{x^2}{x^2 + y^2} - \frac{1}{2}\ln\left(x^2 + y^2\right) + 1\right) + \frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{y}{x}\right) + \frac{xy}{x^2 + y^2} - \frac{xy}{x^2 + y^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{1}{2}\ln\left(x^2 + y^2\right)\right) + \frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = 0.
\end{aligned}$$

17.4.62

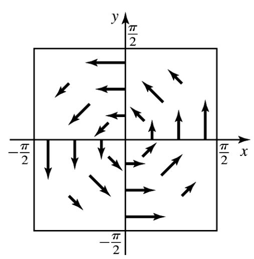
a. The curl and divergence are $\operatorname{curl} \mathbf{F} = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = 0$, and $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = 0$.

b. $\varphi(x,y) = \frac{1}{2}\ln\left(x^2+y^2\right)$. $\psi(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$.

c. $\varphi_{xx} + \varphi_{yy} = \frac{\partial^2}{\partial x^2}\left(\frac{1}{2}\ln\left(x^2+y^2\right)\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{2}\ln\left(x^2+y^2\right)\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{y^2-x^2}{\left(x^2+y^2\right)^2} + \frac{x^2-y^2}{\left(x^2+y^2\right)^2} = 0.$
$\psi_{xx} + \psi_{yy} = \frac{\partial^2}{\partial x^2}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) + \frac{\partial^2}{\partial y^2}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{2xy}{\left(x^2+y^2\right)^2} - \frac{2xy}{\left(x^2+y^2\right)^2} = 0.$

17.4.63

a. The velocity field is $\langle -4\cos x \sin y, 4\sin x \cos y\rangle$.



b. The field is source-free if its divergence is zero.

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(-4\cos x \sin y) + \frac{\partial}{\partial y}(4\sin x \cos y) = 4\sin x \sin y - 4\sin x \sin y = 0,$$

so the field is source-free.

c. The field is irrotational if its curl is zero.

$$\operatorname{curl} \mathbf{F} = \frac{\partial}{\partial x}(4\sin x \cos y) - \frac{\partial}{\partial y}(-4\cos x \sin y) = 4\cos x \cos y + 4\cos x \cos y = 8\cos x \cos y,$$

so the field is not irrotational.

d. Since the field is source-free, it has zero flux across the boundary.

e. The circulation around the boundary of the rectangle is (by Green's theorem) given by

$$\iint_R 8\cos x \cos y \, dA = \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} 8\cos x \cos y \, dy\, dx = \int_{-\pi/2}^{\pi/2} 16\cos x \, dx = 32.$$

17.4.64 If $f(x)$ is continuous, then the circulation form of Green's theorem says that

$$\oint_C \frac{f(x)}{c} dy = \frac{1}{c}\iint_R \frac{df}{dx}\, dA.$$

The right side of this equation evaluates to

$$\frac{1}{c}\iint_R \frac{df}{dx}\,dA = \frac{1}{c}\int_a^b \int_0^c \frac{df}{dx} dy\,dx = \int_a^b \frac{df}{dx}\,dx.$$

To evaluate the left side, parameterize the boundary of R with four paths, each for $0 \le t \le 1$: $\mathbf{r}_1(t) = \langle a + (b-a)\,t, 0\rangle$, so $\mathbf{r}_1'(t) = \langle b-a, 0\rangle$. $\mathbf{r}_2(t) = \langle b, c\,t\rangle$, so $\mathbf{r}_2'(t) = \langle 0, c\rangle$. $\mathbf{r}_3(t) = \langle b + (a-b)\,t, c\rangle$, so $\mathbf{r}_3'(t) = \langle a-b, 0\rangle$. $\mathbf{r}_4(t) = \langle a, c-ct\rangle$, so $\mathbf{r}_4'(t) = \langle 0, -c\rangle$. Then we evaluate $\mathbf{F}\cdot\mathbf{r}_i$ for each i and add:

$$\frac{1}{c}\oint_C f(x)\;dy = \frac{1}{c}\int_0^1 \left(f(a+(b-a)\,t)\cdot 0 + f(b)\cdot c + f(b+(a-b)\,t)\cdot 0 + f(a)\cdot(-c)\right)dt = f(b) - f(a).$$

17.4.65 If $f(x)$ is continuous, then the flux form of Green's theorem says that

$$\oint_C \frac{f(x)}{c} dx = \frac{1}{c}\iint_R \frac{df}{dx}\,dA.$$

The right side of this equation evaluates to

$$\frac{1}{c}\iint_R \frac{df}{dx}\,dA = \frac{1}{c}\int_a^b \int_0^c \frac{df}{dx} dy\,dx = \int_a^b \frac{df}{dx}\,dx.$$

To evaluate the left side, parameterize the boundary of R with four paths, each for $0 \le t \le 1$: $\mathbf{r}_1(t) = \langle a + (b-a)\,t, 0\rangle$, so $\mathbf{r}_1'(t) = \langle b-a, 0\rangle$. $\mathbf{r}_2(t) = \langle b, c\,t\rangle$, so $\mathbf{r}_2'(t) = \langle 0, c\rangle$. $\mathbf{r}_3(t) = \langle b + (a-b)\,t, c\rangle$, so $\mathbf{r}_3'(t) = \langle a-b, 0\rangle$. $\mathbf{r}_4(t) = \langle a, c-ct\rangle$, so $\mathbf{r}_4'(t) = \langle 0, -c\rangle$. Then we evaluate $\mathbf{F}\cdot\mathbf{r}_i$ for each i and add:

$$\oint_C \frac{f(x)}{c}\,dy = \frac{1}{c}\int_0^1 \left(0 + f(b)\cdot c + 0 + f(a)\cdot(-c)\right)dt = f(b) - f(a)$$

so that

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a).$$

17.4.66

a. The curl is $\dfrac{\partial}{\partial x}\left(\dfrac{x}{x^2+y^2}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{-y}{x^2+y^2}\right) = 0.$

b. Take a line integral around the unit circle, parameterized as $\langle \cos t, \sin t\rangle$. The circulation is then $\displaystyle\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \int_0^{2\pi} \left(-\sin t\,(-\sin t) + \cos t\cos t\right)\,dt = \int_0^{2\pi} 1\,dt = 2\pi.$

c. The vector field is not defined everywhere in R; specifically, it is undefined at the origin.

17.4.67

a. The divergence is $\dfrac{\partial}{\partial x}\left(\dfrac{x}{x^2+y^2}\right)+\dfrac{\partial}{\partial y}\left(\dfrac{y}{x^2+y^2}\right)=0.$

b. Take a line integral around the unit circle, parameterized as $\langle \cos t, \sin t\rangle$. The flux is then

$$\oint_C \frac{x}{x^2+y^2}dy+\frac{y}{x^2+y^2}dx=\int_0^{2\pi}(\cos t\cos t-\sin t\,(-\sin t))\,dt=\int_0^{2\pi}1\,dt=2\pi.$$

c. The vector field is not defined everywhere in R; specifically, it is undefined at the origin.

17.4.68

a. Green's theorem does not apply to a region including the origin because $\mathbf{F}$ is not defined at the origin.

b. $\displaystyle\iint_R \left(\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right)+\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right)dA=\iint_R \left(x^2+y^2\right)^{-1/2}\,dA=\int_0^{2\pi}\int_0^1 \frac{1}{r}r\,dr\,d\theta=2\pi.$

c. $\displaystyle\oint_C \frac{x}{\sqrt{x^2+y^2}}dy-\frac{y}{\sqrt{x^2+y^2}}dx=\int_0^{2\pi}(\cos t\,(\cos t)-\sin t\,(-\sin t))\,dt=\int_0^{2\pi}1\,dt=2\pi.$

d. They do agree. Because Green's theorem does not apply, there is no particular reason why they should.

17.4.69 Because ψ is a stream function, $d\psi=\psi_x\,dx+\psi_y\,dy$, so the flux integral is

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds=\int_C f\,dy-g\,dx=\int_C \psi_y\,dy-\psi_x\,dx=\int_C d\psi=\psi(B)-\psi(A),$$

so that the integral is independent of the path.

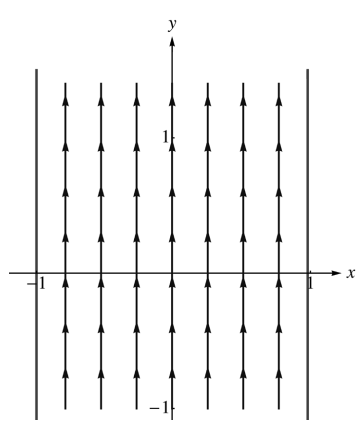
17.4.70 Showing that $\mathbf{F}$ is tangent to the level curves of the stream function is the same as showing that $\mathbf{F}$ is normal to the gradient of the stream function. But that gradient is $\langle\psi_x,\psi_y\rangle$, and

$$\mathbf{F}\cdot\langle\psi_x\,,\,\psi_y\rangle=\langle-\psi_y,\psi_x\rangle\cdot\langle\psi_x\,,\psi_y\rangle=0.$$

17.4.71 Showing that the level curves of φ and ψ are orthogonal is equivalent to showing that the gradients of φ and ψ are orthogonal. But $\nabla\varphi\cdot\nabla\psi=\langle f,g\rangle\cdot\langle -g,f\rangle=0.$

17.4.72

a. The stream function is found by taking $-\displaystyle\int\left(1-x^2\right)\,dx=\frac{1}{3}x^3-x$. A plot together with some streamlines is



b. The curl of F is $\frac{\partial}{\partial x}\left(1-x^2\right) = -2x$, so the curl on $x=0$ is 0; on $x=\frac{1}{4}$ it is $-\frac{1}{2}$; on $x=\frac{1}{2}$, it is -1, and on $x=1$, it is -2.

c. The circulation is (by Green's theorem) $\iint\limits_R (-2x)\ dA = \int_{-5}^{5}\int_{-1}^{1} (-2x)\ dx\, dy = \int_{-5}^{5} 0\, dy = 0$.

d. The curl is positive for negative x and negative for positive x. These cancel, giving a net circulation of zero. This can easily be seen from the picture - any circulation resulting from the top boundary ($y=1$) is cancelled by the circulation in the opposite direction resulting from the bottom boundary.

17.5 Divergence and Curl

17.5.1 The divergence is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$.

17.5.2 The divergence measures the expansion or contraction of the vector field at each point.

17.5.3 It means that the field has no sources or sinks.

17.5.4 The curl is $\nabla\times$F$=\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}$.

17.5.5 The curl indicates the axis and speed of rotation of a vector field at each point.

17.5.6 It means that the vector field is irrotational.

17.5.7 $\nabla\cdot(\nabla\times\mathbf{F}) = 0$; see Theorem 17.10.

17.5.8 Here $\mathbf{u}$ is a potential function, so $\nabla\times\nabla\mathbf{u}$ is the curl of a conservative vector field, which is 0.

17.5.9 $\frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(-3z) = 3$.

17.5.10 $\frac{\partial}{\partial x}(-2y) + \frac{\partial}{\partial y}(3x) + \frac{\partial}{\partial z}(z) = 1$.

17.5.11 $\frac{\partial}{\partial x}(12x) + \frac{\partial}{\partial y}(-6y) + \frac{\partial}{\partial z}(-6z) = 0$.

17.5.12 $\frac{\partial}{\partial x}\left(x^2yz\right) + \frac{\partial}{\partial y}\left(-xy^2z\right) + \frac{\partial}{\partial z}\left(-xyz^2\right) = 2xyz - 2xyz - 2xyz = -2xyz$.

17.5.13 $\frac{\partial}{\partial x}\left(x^2-y^2\right) + \frac{\partial}{\partial y}\left(y^2-z^2\right) + \frac{\partial}{\partial z}\left(z^2-x^2\right) = 2x+2y+2z$.

17.5.14 $\frac{\partial}{\partial x}\left(e^{y-x}\right) + \frac{\partial}{\partial y}\left(e^{z-y}\right) + \frac{\partial}{\partial z}\left(e^{x-z}\right) = -\left(e^{y-x}+e^{z-y}+e^{x-z}\right)$.

17.5.15 $\frac{\partial}{\partial x}\left(\frac{x}{1+x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{1+x^2+y^2}\right) + \frac{\partial}{\partial z}\left(\frac{z}{1+x^2+y^2}\right) = \frac{x^2+y^2+3}{\left(1+x^2+y^2\right)^2}$.

17.5.16 $\frac{\partial}{\partial x}(yz\sin(x)) + \frac{\partial}{\partial y}(xz\cos(y)) + \frac{\partial}{\partial z}(xy\cos(z)) = yz\cos(x) - xz\sin(y) - xy\sin(z)$.

17.5.17

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2+z^2}\right)+\frac{\partial}{\partial z}\left(\frac{z}{x^2+y^2+z^2}\right)\\
&=\frac{1}{\left(x^2+y^2+z^2\right)^2}\left(\left(z^2+y^2-x^2\right)+\left(x^2+z^2-y^2\right)+\left(x^2+y^2-z^2\right)\right)\\
&=\frac{1}{\left(x^2+y^2+z^2\right)^2}\left(x^2+y^2+z^2\right)=\frac{1}{|\mathbf{r}|^2}.
\end{aligned}$$

17.5.18

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}\right)+\frac{\partial}{\partial y}\left(\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}\right)+\frac{\partial}{\partial z}\left(\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}\right)\\
&=\frac{1}{\left(x^2+y^2+z^2\right)^{5/2}}\left(\left(z^2+y^2-2x^2\right)+\left(x^2+z^2-2y^2\right)+\left(x^2+y^2-2z^2\right)\right)=0.
\end{aligned}$$

17.5.19

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{x}{\left(x^2+y^2+z^2\right)^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{\left(x^2+y^2+z^2\right)^2}\right)+\frac{\partial}{\partial z}\left(\frac{z}{\left(x^2+y^2+z^2\right)^2}\right)\\
&=\frac{1}{\left(x^2+y^2+z^2\right)^3}\left(\left(z^2+y^2-3x^2\right)+\left(x^2+z^2-3y^2\right)+\left(x^2+y^2-3z^2\right)\right)\\
&=\frac{-1}{\left(x^2+y^2+z^2\right)^3}\left(x^2+y^2+z^2\right)=\frac{-1}{|\mathbf{r}|^4}.
\end{aligned}$$

17.5.20

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(x\left(x^2+y^2+z^2\right)\right)+\frac{\partial}{\partial y}\left(y\left(x^2+y^2+z^2\right)\right)+\frac{\partial}{\partial z}\left(z\left(x^2+y^2+z^2\right)\right)\\
&=\left(3x^2+y^2+z^2\right)+\left(x^2+3y^2+z^2\right)+\left(x^2+y^2+3z^2\right)=5\,|\mathbf{r}|^2.
\end{aligned}$$

17.5.21

a. At both P and Q, the arrows going away from the point are larger in both number and magnitude than those going in, so we would expect the divergence to be positive at both points.

b. The divergence is $\dfrac{\partial}{\partial \boldsymbol{x}}(x)+\dfrac{\partial}{\partial y}(x+y)=1+1=2$, so is positive everywhere.

c. The arrows all point roughly away from the origin, so we the flux is outward everywhere.

d. The net flux across C should be positive.

17.5.22

a. At P, the divergence should be positive, while at Q, the larger arrows point in towards Q, so the divergence should be negative.

b. The divergence is $\dfrac{\partial}{\partial \boldsymbol{x}}(x) + \dfrac{\partial}{\partial y}\left(y^2\right) = 1 + 2y$; at $P = (-1, 1)$, this is 3, while at $Q = (-1, -1)$, it is -1.

c. The flux is outward above the line $y = -1$ (approximately); below this line, the flux is inward across C.

d. The size of the arrows pointing outward at the top of the circle seems to roughly equal those pointing inward at the bottom, so the remaining outward-pointing arrows result in a net positive flux across C.

17.5.23

a. The axis of rotation is $\langle 1, 0, 0\rangle$, the x-axis. $\nabla \times \mathbf{F} = \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ x & y & z \end{vmatrix} = \nabla \times (-z\mathbf{j} + y\mathbf{k}) = (1+1)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = 2\mathbf{i}$. It is in the same direction as the axis of rotation.

b. The magnitude of the curl is $|2\mathbf{i}| = 2$.

17.5.24

a. The axis of rotation is $\langle 1, -1, 0\rangle$. $\nabla \times \mathbf{F} = \nabla \times \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ 1 & -1 & 0 \\ x & y & z \end{vmatrix} = \nabla \times (-z\mathrm{i} - z\mathrm{j} + (x+y)\,\mathrm{k}) = (1+1)\,\mathrm{i} + (-1-1)\,\mathrm{j} + (0-0)\,\mathrm{k} = 2\langle 1, -1, 0\rangle$

and the curl is in the same direction as the axis of rotation.

b. The magnitude of the curl is $2\,|\langle 1, -1, 0\rangle| = 2\sqrt{2}$

17.5.25

a. The axis of rotation is $\langle 1, -1, 1\rangle$.

$\nabla \times \mathbf{F} = \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ x & y & z \end{vmatrix} = \nabla \times (-(y+z)\,\mathbf{i} + (x-z)\,\mathbf{j} + (x+y)\,\mathbf{k}) = (1+1)\,\mathbf{i} + (-1-1)\,\mathbf{j} + (1+1)\,\mathbf{k} = 2\langle 1, -1, 1\rangle$, and the curl is in the same direction as the axis of rotation.

b. The magnitude of the curl is $2\,|\langle 1, -1, 1\rangle| = 2\sqrt{3}$.

17.5.26

a. The axis of rotation is $\langle 1, -2, -3\rangle$.

$\nabla \times \mathbf{F} = \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -3 \\ x & y & z \end{vmatrix} = \nabla \times ((3y-2z)\,\mathbf{i} + (-3x-z)\,\mathbf{j} + (2x+y)\,\mathbf{k}) = (1+1)\,\mathbf{i} + (-2-2)\,\mathbf{j} + (-3-3)\,\mathbf{k} = 2\langle 1, -2, -3\rangle$, and the curl is in the same direction as the axis of rotation.

b. The magnitude of the curl is $2\,|\langle 1, -2, -3\rangle| = 2\sqrt{14}$.

17.5.27 $\nabla \times \langle x^2 - y^2, xy, z\rangle = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (y+2y)\,\mathbf{k} = 3y\mathbf{k}$.

17.5.28 $\nabla \times \langle 0, z^2 - y^2, -yz \rangle = (-z - 2z)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = -3z\mathbf{i}$.

17.5.29 $\nabla \times \langle x^2 - z^2, 1, 2xz \rangle = (0-0)\,\mathbf{i} + (-2z - 2z)\,\mathbf{j} + (0-0)\,\mathbf{k} = -4z\mathbf{j}$.

17.5.30 $\nabla \times \langle x, y, z \rangle = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$.

17.5.31

$$\begin{aligned}
&\nabla \times \frac{1}{(x^2+y^2+z^2)^{3/2}}\langle x, y, z \rangle \\
&= \frac{1}{(x^2+y^2+z^2)^{5/2}}\left((-3yz + 3yz)\,\mathbf{i} + (-3xz + 3xz)\,\mathbf{j} + (-3xy + 3xy)\,\mathbf{k}\right) = \mathbf{0}.
\end{aligned}$$

17.5.32

$$\begin{aligned}
&\nabla \times \frac{1}{(x^2+y^2+z^2)^{1/2}}\langle x, y, z \rangle \\
&= \frac{1}{(x^2+y^2+z^2)^{3/2}}\left((-yz + yz)\,\mathbf{i} + (-xz + xz)\,\mathbf{j} + (-xy + xy)\,\mathbf{k}\right) = \mathbf{0}.
\end{aligned}$$

17.5.33

$$\begin{aligned}
&\nabla \times \langle z^2 \sin(y), xz^2 \cos y, 2xz \sin y \rangle \\
&= (2xz\cos y - 2xz\cos(y))\,\mathbf{i} + (2z\sin y - 2z \sin y)\,\mathbf{j} + \left(z^2\cos y - z^2 \cos y\right)\mathbf{k} = \mathbf{0}.
\end{aligned}$$

17.5.34

$$\begin{aligned}
\nabla \times \langle 3xz^3e^{y^2}, 2xz^3e^{y^2}, 3xz^2e^{y^2} \rangle &= \left(6xyz^2e^{y^2} - 6xz^2e^{y^2}\right)\mathbf{i} + \left(9xz^2e^{y^2} - 3z^2e^{y^2}\right)\mathbf{j} + \left(2z^3e^{y^2} - 6xyz^3e^{y^2}\right)\mathbf{k} \\
&= z^2e^{y^2}\left((6xy - 6x)\,\mathbf{i} + (9x - 3)\,\mathbf{j} + (2z - 6xyz)\,\mathbf{k}\right).
\end{aligned}$$

17.5.35 Simply compute it:

$$\begin{aligned}
&\left\langle \frac{\partial}{\partial x}\left(\frac{1}{(x^2+y^2+z^2)^{3/2}}\right), \frac{\partial}{\partial y}\left(\frac{1}{(x^2+y^2+z^2)^{3/2}}\right), \frac{\partial}{\partial z}\left(\frac{1}{(x^2+y^2+z^2)^{3/2}}\right)\right\rangle \\
&= \left\langle \frac{-3x}{(x^2+y^2+z^2)^{5/2}}, \frac{-3y}{(x^2+y^2+z^2)^{5/2}}, \frac{-3z}{(x^2+y^2+z^2)^{5/2}} \right\rangle = \frac{-3\mathbf{r}}{|\mathbf{r}|^5}.
\end{aligned}$$

17.5.36

$$\begin{aligned}
&\left\langle \frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2+z^2}\right), \frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2+z^2}\right), \frac{\partial}{\partial z}\left(\frac{1}{x^2+y^2+z^2}\right)\right\rangle \\
&= \left\langle \frac{-2x}{(x^2+y^2+z^2)^2}, \frac{-2y}{(x^2+y^2+z^2)^2}, \frac{-2z}{(x^2+y^2+z^2)^2} \right\rangle = \frac{-2\mathbf{r}}{|\mathbf{r}|^4}.
\end{aligned}$$

17.5.37 $\nabla\left(\frac{1}{|\mathbf{r}|^2}\right) = \frac{-2\mathbf{r}}{|\mathbf{r}|^4}$, from Problem 36; applying Theorem 17.8 we have

$$\nabla \cdot \nabla\left(\frac{1}{|\mathbf{r}|^2}\right) = -2\nabla \cdot \frac{\mathbf{r}}{|\mathbf{r}|^4} = -2\frac{3-4}{|\mathbf{r}|^4} = \frac{2}{|\mathbf{r}|^4}.$$

17.5.38

$$\begin{aligned}
\nabla(\ln|\mathbf{r}|) &= \nabla\left(\ln\left(\sqrt{x^2+y^2+z^2}\right)\right) = \frac{1}{2}\nabla\left(\ln\left(x^2+y^2+z^2\right)\right) \\
&= \frac{1}{2\left(x^2+y^2+z^2\right)}\langle 2x, 2y, 2z\rangle = \frac{\mathbf{r}}{|\mathbf{r}|^2}.
\end{aligned}$$

17.5.39

a. False. For example, $\mathbf{F} = \langle y, z, x \rangle$ has zero divergence yet is not constant.

b. False. For example, $\mathbf{F} = \langle x, y, z \rangle$ is a counterexample.

c. False. For example, consider the vector field $\langle 0, 1 - x^2 \rangle$.

d. False. For example, $\mathbf{F} = \langle x, 0, 0 \rangle$ has divergence 1.

e. False. For example, the curl of $\langle z, -z, y \rangle$ is $\langle 2, 1, 0 \rangle$.

17.5.40

a. $(\mathbf{F} \cdot \boldsymbol{\nabla})\, u = \left(\langle f, g, h \rangle \cdot \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \right) u = \left(f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} + h \frac{\partial}{\partial z} \right) u = f \frac{\partial u}{\partial x} + g \frac{\partial u}{\partial y} + h \frac{\partial u}{\partial z}.$

b. Because $\mathbf{F} = \langle 1, 1, 1 \rangle$, $\mathbf{F} \cdot \boldsymbol{\nabla} \left(xy^2 z^3 \right) = \frac{\partial}{\partial x} \left(xy^2 z^3 \right) + \frac{\partial}{\partial y} \left(xy^2 z^3 \right) + \frac{\partial}{\partial z} \left(xy^2 z^3 \right) = y^2 z^3 + 2xyz^3 + 3xy^2 z^2.$

17.5.41

a. No; divergence is a concept that applies to vector fields.

b. No; the gradient applies to functions.

c. Yes; this is the divergence of the gradient and is thus a scalar function.

d. No, because $\boldsymbol{\nabla} \cdot \varphi$ does not make sense (part (a)).

e. No; curl applies to vector fields, $\boldsymbol{\nabla} \times \varphi$ does not make sense.

f. No, because $\boldsymbol{\nabla} \cdot \mathrm{F}$ is a function, so that applying $\boldsymbol{\nabla} \cdot$ to it does not make sense.

g. Yes, this is the curl of a vector field and is thus a vector field.

h. No, because $\boldsymbol{\nabla} \cdot \mathrm{F}$ is a function, not a vector field.

i. Yes; this is the curl of the curl of a vector field and is thus a vector field.

17.5.42 Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$; then $\mathbf{F} = \mathbf{a} \times \mathbf{r} = (a_2 z - a_3 y)\, \mathbf{i} + (a_3 x - a_1 z)\, \mathbf{j} + (a_1 y - a_2 x)\, \mathbf{k}$, so that $\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{\partial}{\partial x} (a_2 z - a_3 y) + \frac{\partial}{\partial y} (a_3 x - a_1 z) + \frac{\partial}{\partial z} (a_1 y - a_2 x) = 0.$

17.5.43

a. $\langle 0, 1, 0 \rangle \times \langle 0, 1, 1 \rangle = \langle 1, 0, 0 \rangle$, so $\mathbf{F}$ points in the positive x-direction. $\langle 0, 1, 0 \rangle \times \langle 1, 1, 0 \rangle = \langle 0, 0, -1 \rangle$, so $\mathbf{F}$ points in the negative z-direction. $\langle 0, 1, 0 \rangle \times \langle 0, 1, -1 \rangle = \langle -1, 0, 0 \rangle$, so $\mathbf{F}$ points in the negative x-direction. $\langle 0, 1, 0 \rangle \times \langle -1, 1, 0 \rangle = \langle 0, 0, 1 \rangle$, so $\mathbf{F}$ points in the positive z-direction.

b. Note that these vectors circle the y-axis in the counterclockwise direction looking along $\mathbf{a}$ from head to tail.

17.5.44 Note that $\mathbf{a} \times \mathbf{r} = \langle 0, 1, 0 \rangle \times \langle x, y, z \rangle = \langle z, 0, -x \rangle$ is a rotational field whose vectors circle the y-axis in the counterclockwise direction looking along $\mathbf{a}$ from head to tail.

17.5.45 Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$; then $\mathbf{F} = \mathbf{a} \times \mathbf{r} = (a_2 z - a_3 y)\, \mathbf{i} + (a_3 x - a_1 z)\, \mathbf{j} + (a_1 y - a_2 x)\, \mathbf{k}$, so that $\boldsymbol{\nabla} \times \mathbf{F} = \frac{\partial}{\partial x} (a_1 + a_1)\, \mathbf{i} + \frac{\partial}{\partial y} (a_2 + a_2)\, \mathbf{j} + \frac{\partial}{\partial z} (a_3 + a_3)\, \mathbf{k} = 2\mathbf{a}.$

17.5.46 The field switches from inward-pointing to outward-pointing at points where it is tangent to the circle $x^2 + y^2 = 2$, i.e. where it is orthogonal to the normal to the circle. The normal to the circle at (x, y) is a multiple of (x, y), so we want to find x, y so that $\langle x, y \rangle \cdot \langle x^2, y \rangle = x^3 + y^2 = 0$ with $x^2 + y^2 = 2$. Thus $x^3 - x^2 + 2 = 0$. The solutions are $x = -1$ and $y = \pm 1$.

17.5.47 $\operatorname{div}\mathbf{F} = 2x + 2xyz + 2x = 2x\,(yz+2)$; this function clearly achieves its maximum magnitude at $(-1,1,1)$, $(-1,-1,-1)$, $(1,1,1)$, and $(1,-1,-1)$, where its magnitude is 6.

17.5.48 For $\mathbf{F} = \langle z, 0, -y\rangle$, $\operatorname{curl}\mathbf{F} = \langle -1, 1, 0\rangle$.

a. $\operatorname{scal}_{\langle 1,0,0\rangle}\langle -1,1,0\rangle = \langle -1,1,0\rangle \cdot \langle 1,0,0\rangle = -1$.

b. $\operatorname{scal}_{\langle 1/\sqrt{3},-1/\sqrt{3},1/\sqrt{3}\rangle}\langle -1,1,0\rangle = \langle -1,1,0\rangle \cdot \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle = -\frac{2}{\sqrt{3}}$.

c. We have $\operatorname{scal}_{\mathbf{n}}\langle -1,1,0\rangle = \langle -1,1,0\rangle \cdot \mathbf{n} = \sqrt{2}\cos\theta$, where θ is the angle between $\langle -1,1,0\rangle$ and $\mathbf{n}$. This is maximized when $\mathbf{n}$ points in the same direction as $\operatorname{curl}\mathbf{F}$. So $\mathbf{n} = \frac{\langle -1,1,0\rangle}{|\langle -1,1,0\rangle|} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\rangle$, in which case $\operatorname{scal}_{\mathbf{n}}\langle -1,1,0\rangle = \sqrt{2}$.

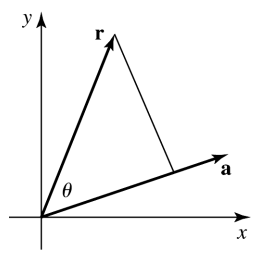
17.5.49 $\operatorname{curl}\mathbf{F} = \langle 0+2, 0+1, 0-1\rangle = \langle 2,1,-1\rangle$. If $\mathbf{n} = \langle a,b,c\rangle$, then $\operatorname{curl}\mathbf{F}\cdot\mathbf{n} = 0$ when $2a+b-c=0$ so that $c = 2a+b$; thus all such vectors are of the form $\langle a, b, 2a+b\rangle$, where a, b are real numbers.

17.5.50 $\mathbf{F} = \langle z, 0, 0\rangle$, or $\mathbf{F} = \langle 0, 0, -x\rangle$, so it is not unique.

17.5.51 $\mathbf{F} = \frac{1}{2}\langle y^2+z^2, 0, 0\rangle$ or $\mathbf{F} = \langle 0, -xy, -xz\rangle$, so it is not unique.

17.5.52

a. Looking at the picture, it is clear that the distance from P to $\mathbf{a}$ is $|\mathbf{r}|$.



b. The velocity field is $\mathbf{a}\times\mathbf{r}$, so the speed, which is the magnitude of velocity, is $|\mathbf{a}\times\mathbf{r}|$. Now $|\mathbf{a}\times\mathbf{r}| = |\mathbf{a}|\cdot|\mathbf{r}|\cdot\sin(\theta)\cdot\mathbf{n} = R\,|\mathbf{a}|\,\mathbf{n}$, where $\mathbf{n}$ is a vector normal to the plane determined by $\mathbf{a}$ and $\mathbf{r}$. Thus the motion of the particle is always perpendicular to this plane, so it rotates about the axis $\mathbf{a}$. It is moving at a speed $R\,|\mathbf{a}|$ around a circle of radius R, so its angular speed is $\frac{R\,|\mathbf{a}|}{R} = |\mathbf{a}|$.

c. Because $|\nabla\times\mathbf{v}| = 2\,|\mathbf{a}|$, it follows from part (b) that $\omega = |\mathbf{a}| = \frac{1}{2}\,|\nabla\times\mathbf{v}|$.

17.5.53 The curl of this vector field is $\langle 0,1,0\rangle$. The component of the curl along some unit vector $\mathbf{n}$ is $(\nabla\times\mathbf{F})\cdot\mathbf{n}$.

a. $\langle 0,1,0\rangle\cdot\langle 1,0,0\rangle = 0$, so the wheel does not spin.

b. $\langle 0,1,0\rangle\cdot\langle 0,1,0\rangle = 1$, so the wheel spins clockwise (looking towards positive y).

c. $\langle 0,1,0\rangle\cdot\langle 0,0,1\rangle = 0$, so the wheel does not spin.

17.5.54 The curl of the vector field is $\nabla\times\mathbf{v} = \langle -2,0,2\rangle$.

a. The wheel is placed with its axis in the direction $\langle 0,0,1\rangle$, so the component of velocity in that direction is $\langle -2,0,2\rangle\cdot\langle 0,0,1\rangle = 2$, and $\omega = \frac{1}{2}\cdot 2 = 1$.

b. The wheel is placed with its axis in the direction $\langle 0,1,0\rangle$, so the component of velocity in that direction is $\langle -2,0,2\rangle \cdot \langle 0,1,0\rangle = 0$, and the wheel does not turn.

c. The wheel is placed with its axis in the direction $\langle 1,0,0\rangle$, so the component of velocity in that direction is $\langle -2,0,2\rangle \cdot \langle 1,0,0\rangle = -2$, and $\omega = \frac{1}{2}\cdot|-2| = 1$.

17.5.55 The curl of the vector field is $\nabla\times\mathrm{v}= \langle -20,0,0\rangle$. Because the wheel is placed with its axis normal to the plane $x+y+z=1$, its axis must point in the direction $\langle 1,1,1\rangle$ (with unit vector $\frac{1}{\sqrt{3}}\langle 1,1,1\rangle$). Thus, the component of velocity along that direction is $\frac{1}{\sqrt{3}}\langle -20,0,0\rangle\cdot\langle 1,1,1\rangle = \frac{-20}{\sqrt{3}}$ and then ω is the absolute value of one half of that amount, or $\omega = \frac{10}{\sqrt{3}}$ or $\frac{5}{\pi\sqrt{3}} \approx 0.9189$ revolutions per time unit.

17.5.56

$$\mathbf{F} = -100k\nabla e^{-\sqrt{x^2+y^2+z^2}} = \frac{100ke^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\langle x,y,z\rangle.$$

Looking at the x component, its contribution to the divergence is

$$100k\frac{\partial}{\partial x}\left[\frac{xe^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right] = -100k\frac{\left(x^2\sqrt{x^2+y^2+z^2}-y^2-z^2\right)e^{-\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)^{3/2}}$$

and similarly for the y and z components. Thus the divergence is the sum of these three terms, which is

$$\begin{aligned}&-100k\frac{e^{-\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)^{3/2}}\left(\left(x^2+y^2+z^2\right)^{3/2}-2\left(x^2+y^2+z^2\right)\right)\\&=100k\frac{e^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\left(2-\sqrt{x^2+y^2+z^2}\right).\end{aligned}$$

17.5.57 $\mathbf{F} = -100k\nabla e^{-x^2+y^2+z^2} = -200ke^{-x^2+y^2+z^2}\langle -x,y,z\rangle$, so the divergence is

$$\begin{aligned}&-200k\left(\frac{\partial}{\partial x}\left(-xe^{-x^2+y^2+z^2}\right)+\frac{\partial}{\partial y}\left(ye^{-x^2+y^2+z^2}\right)+\frac{\partial}{\partial z}\left(ze^{-x^2+y^2+z^2}\right)\right)\\&=-200k\Big(-e^{-x^2+y^2+z^2}+2x^2e^{-x^2+y^2+z^2}+e^{-x^2+y^2+z^2}+2y^2e^{-x^2+y^2+z^2}+e^{-x^2+y^2+z^2}+2z^2e^{-x^2+y^2+z^2}\Big)\\&=-200k\left(e^{-x^2+y^2+z^2}+2\left(x^2+y^2+z^2\right)e^{-x^2+y^2+z^2}\right)\\&=-200k\left(2x^2+2y^2+2z^2+1\right)e^{-x^2+y^2+z^2}.\end{aligned}$$

17.5.58 $\mathbf{F} = -100k\nabla\left(1+\sqrt{1+x^2+y^2+z^2}\right) = -100k\left(x^2+y^2+z^2\right)^{-1/2}\langle x,y,z\rangle$, and thus the divergence is

$$\nabla\cdot\mathbf{F} = \frac{-200k}{\sqrt{x^2+y^2+z^2}}.$$

17.5.59

a. $\mathbf{F} = -\nabla\varphi = -GMm\Big\langle\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right],\frac{\partial}{\partial y}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right],\frac{\partial}{\partial z}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right]\Big\rangle =$
$-GMm\left(x^2+y^2+z^2\right)^{-3/2}\langle -x,-y,-z\rangle = GMm\left(x^2+y^2+z^2\right)^{-3/2}\langle x,y,z\rangle = GMm\frac{\mathbf{r}}{|\mathbf{r}|^3}.$

b. $\frac{\partial}{\partial y}x\left(x^2+y^2+z^2\right)^{-3/2} = -3xy\left(x^2+y^2+z^2\right)^{-5/2}$. Applying this pattern in computing the curl gives $\nabla \times \mathbf{F} = GMm\left(x^2+y^2+z^2\right)^{-5/2}\left((-3yz+3yz)\,\mathbf{i} + (-3xz+3xz)\,\mathbf{j} + (-3xy+3xy)\,\mathbf{k}\right) = \mathbf{0}$, so the field is irrotational.

17.5.60 Note: this is identical to the previous problem except for the constant.

a. $\mathbf{F} = -\nabla\varphi = -\frac{q}{4\pi\epsilon_0}\left\langle \frac{\partial}{\partial x}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right], \frac{\partial}{\partial y}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right], \frac{\partial}{\partial z}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right]\right\rangle =$ $-\frac{q}{4\pi\epsilon_0}\left(x^2+y^2+z^2\right)^{-3/2}\langle -x,-y,-z\rangle = \frac{q}{4\pi\epsilon_0}\left(x^2+y^2+z^2\right)^{-3/2}\langle x,y,z\rangle = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{|\mathbf{r}|^3}$.

b. $\frac{\partial}{\partial y}x\left(x^2+y^2+z^2\right)^{-3/2} = -3xy\left(x^2+y^2+z^2\right)^{-5/2}$. Applying this pattern in computing the curl gives $\nabla\times\mathbf{F} = \frac{q}{4\pi\epsilon_0}\left(x^2+y^2+z^2\right)^{-5/2}\left((-3yz+3yz)\,\mathbf{i} + (-3xz+3xz)\,\mathbf{j} + (-3xy+3xy)\,\mathbf{k}\right) = \mathbf{0}$, so the field is irrotational.

17.5.61 Using Exercise 40, we have

$$\rho\left(\langle\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t}\rangle + \left(u\frac{\partial}{\partial t} + v\frac{\partial}{\partial t} + w\frac{\partial}{\partial t}\right)\langle u,v,w\rangle\right) = -\langle\frac{\partial p}{\partial t}, \frac{\partial p}{\partial t}, \frac{\partial p}{\partial t}\rangle + \mu\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial t^2}\right)\langle u,v,w\rangle,$$

so that

$$\begin{aligned}
\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) &= -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\\
\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) &= -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)\\
\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) &= -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right).
\end{aligned}$$

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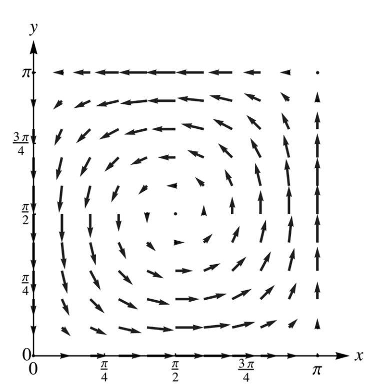
17.5.62

a. $\nabla\times\langle 2,-3y,5z\rangle = \mathbf{0}$ and $\nabla\times\langle y, x-z, -y\rangle = \mathbf{0}$ so they are both irrotational.

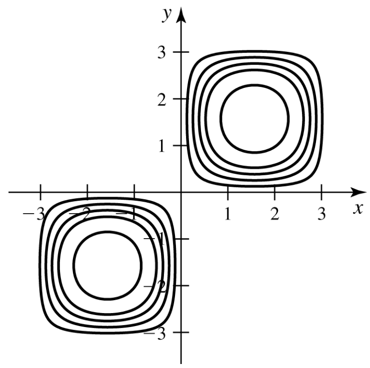
b. If ψ is defined as stated, then $\nabla^2\psi = \frac{\partial^2}{\partial x^2}\psi + \frac{\partial^2}{\partial y^2}\psi + \frac{\partial^2}{\partial z^2}\psi = -\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$ while $\zeta = \mathrm{k}\cdot\nabla\times\langle u,v,0\rangle = \mathrm{k}\cdot\left(-\frac{\partial v}{\partial z}\mathrm{i} + \frac{\partial u}{\partial z}\mathrm{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{k}\right) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ so that $\nabla^2\psi = -\zeta$ as desired.

c. $u = \frac{\partial\psi}{\partial y} = \sin(x)\cos(y)$ and $v = -\frac{\partial\psi}{\partial x} = -\cos(x)\sin(y)$. The velocity field looks like



d. The vorticity function is $\zeta = -\nabla^2\psi = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = \sin(x)\sin(y) + \sin(x)\sin(y) = 2\sin(x)\sin(y)$

The diagram shows level curves for ζ at $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, 1 and $\frac{3}{2}$ (from outer to inner).

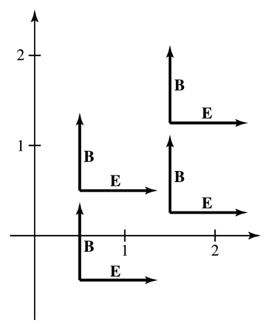


Using implicit differentiation (or from looking at the diagram), ζ achieves its maximum at $x = y = \dfrac{\pi}{2}$, where it has value 2, and its minimum on the boundary, where it is zero.

17.5.63

a. We have $\nabla \times \mathbf{B} = -\dfrac{\partial}{\partial z}(A\sin(kz - \omega t))\,\mathbf{i} + 0\mathbf{j} + \dfrac{\partial}{\partial x}(A\sin(kz-\omega t))\,\mathbf{k} = -Ak\cos(kz - \omega t)\,\mathbf{i}$.
$C\dfrac{\partial \mathbf{E}}{\partial t} = C\dfrac{\partial}{\partial t}(A\sin(kz-\omega t)\,\mathbf{i}) = -\omega\, CA\cos(kz-\omega t)\,\mathbf{i}$ so that the two are equal when $k = \omega C$, or $\omega = \dfrac{k}{C}$.

b.



17.5.64 Let $\mathbf{V} = \langle xy, -\frac{1}{2}y^2, 0\rangle$ and $\mathbf{W} = \langle 0, \frac{1}{2}y^2, 0\rangle$. Then $\nabla\cdot\mathbf{V} = 0$ and $\nabla\times\mathbf{W} = \mathbf{0}$.

17.5.65 Let $\mathbf{F} = \langle f, g, h\rangle$ and $\mathbf{G} = \langle k, m, n\rangle$. Then

a. $\nabla\cdot(\mathbf{F}+\mathbf{G}) = \nabla\cdot\langle f+k, g+m, h+n\rangle = \dfrac{\partial}{\partial x}(f+k) + \dfrac{\partial}{\partial y}(g+m) + \dfrac{\partial}{\partial z}(h+n) = \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} + \dfrac{\partial h}{\partial z} + \dfrac{\partial k}{\partial x} + \dfrac{\partial m}{\partial y} + \dfrac{\partial n}{\partial z} = \nabla\cdot\mathbf{F} + \nabla\cdot\mathbf{G}$.

b.

$$\begin{aligned}
&\nabla\times(\mathbf{F}+\mathbf{G})\\
&= \left(\frac{\partial}{\partial y}(h+n) - \frac{\partial}{\partial z}(g+m)\right)\mathbf{i} + \left(\frac{\partial}{\partial z}(f+k) - \frac{\partial}{\partial x}(h+n)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(g+m) - \frac{\partial}{\partial y}(f+k)\right)\mathbf{k}\\
&= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k} + \left(\frac{\partial n}{\partial y} - \frac{\partial m}{\partial z}\right)\mathbf{i} + \left(\frac{\partial k}{\partial z} - \frac{\partial n}{\partial x}\right)\mathbf{j} + \left(\frac{\partial m}{\partial x} - \frac{\partial k}{\partial y}\right)\mathbf{k}\\
&= \nabla\times\mathbf{F} + \nabla\times\mathbf{G}.
\end{aligned}$$

c. $\nabla \cdot (c\mathbf{F}) = \dfrac{\partial}{\partial x}(cf) + \dfrac{\partial}{\partial y}(cg) + \dfrac{\partial}{\partial z}(ch) = c\left(\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} + \dfrac{\partial h}{\partial z}\right) = c(\nabla \cdot \mathbf{F})$.

d. $\nabla \times (c\mathbf{F}) = \left(\dfrac{\partial}{\partial y}(ch) - \dfrac{\partial}{\partial z}(cg)\right)\mathbf{i} + \left(\dfrac{\partial}{\partial z}(cf) - \dfrac{\partial}{\partial x}(ch)\right)\mathbf{j} + \left(\dfrac{\partial}{\partial x}(cg) - \dfrac{\partial}{\partial y}(cf)\right)\mathbf{k} =$
$c\left[\left(\dfrac{\partial h}{\partial y} - \dfrac{\partial g}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial f}{\partial z} - \dfrac{\partial h}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}\right)\mathbf{k}\right] = c(\nabla \times \mathbf{F})$.

17.5.66 The statement is not true. The conditions imply that $\mathbf{F} - \mathbf{G}$ is irrotational and source-free, but this can happen with nonconstant vector fields. For example, if $\mathbf{F} = \langle x^2 - y^2, -2xy\rangle$ and $\mathbf{G} = \langle 2xy, x^2 - y^2\rangle$, then $\mathbf{F} - \mathbf{G}$ is irrotational and source-free (i.e. has zero curl and zero divergence). Clearly the two vector fields do not differ by a constant.

17.5.67 $\nabla \cdot (\varphi\mathbf{F}) = \nabla \cdot \langle \varphi f, \varphi g, \varphi h\rangle = \dfrac{\partial}{\partial x}(\varphi f) + \dfrac{\partial}{\partial y}(\varphi g) + \dfrac{\partial}{\partial z}(\varphi h) = \varphi\dfrac{\partial f}{\partial x} + f\dfrac{\partial \varphi}{\partial x} + \varphi\dfrac{\partial g}{\partial y} + g\dfrac{\partial \varphi}{\partial y} + \varphi\dfrac{\partial h}{\partial z} +$
$h\dfrac{\partial \varphi}{\partial z} = \varphi\left(\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} + \dfrac{\partial h}{\partial z}\right) + \langle f, g, h\rangle\langle \dfrac{\partial \varphi}{\partial x}, \dfrac{\partial \varphi}{\partial y}, \dfrac{\partial \varphi}{\partial z}\rangle = \varphi\nabla \cdot \mathbf{F} + \nabla\boldsymbol{\varphi} \cdot \mathbf{F}$

17.5.68 $\nabla \times (\varphi\mathbf{F}) = \nabla \times \langle \varphi f, \varphi g, \varphi h\rangle = \langle \dfrac{\partial}{\partial y}(\varphi h) - \dfrac{\partial}{\partial z}(\varphi g), \dfrac{\partial}{\partial z}(\varphi f) - \dfrac{\partial}{\partial x}(\varphi h), \dfrac{\partial}{\partial x}(\varphi g) - \dfrac{\partial}{\partial y}(\varphi f)\rangle =$
$\langle \varphi\dfrac{\partial h}{\partial y} + h\dfrac{\partial \varphi}{\partial y} - \varphi\dfrac{\partial g}{\partial z} - g\dfrac{\partial \varphi}{\partial z}, \varphi\dfrac{\partial f}{\partial z} + f\dfrac{\partial \varphi}{\partial z} - \varphi\dfrac{\partial h}{\partial x} - h\dfrac{\partial \varphi}{\partial x}, \varphi\dfrac{\partial g}{\partial x} + g\dfrac{\partial \varphi}{\partial x} - \varphi\dfrac{\partial f}{\partial y} - f\dfrac{\partial \varphi}{\partial y}\rangle = \langle \varphi\dfrac{\partial h}{\partial y} - \varphi\dfrac{\partial g}{\partial zy}, \varphi\dfrac{\partial f}{\partial z} -$
$\varphi\dfrac{\partial h}{\partial x}, \varphi\dfrac{\partial g}{\partial x} - \varphi\dfrac{\partial f}{\partial y}\rangle + \langle h\dfrac{\partial \varphi}{\partial y} - g\dfrac{\partial \varphi}{\partial z}, f\dfrac{\partial \varphi}{\partial z} - h\dfrac{\partial \varphi}{\partial x}, g\dfrac{\partial \varphi}{\partial x} - f\dfrac{\partial \varphi}{\partial y}\rangle = \varphi\nabla \times \mathbf{F} + \nabla\boldsymbol{\varphi} \times \mathbf{F}$

17.5.69 If $\mathbf{F} = \langle f, g, h\rangle$ and $\mathbf{G} = \langle k, m, n\rangle$, then

$$\begin{aligned}
\mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}) &= \langle k, m, n\rangle \cdot \langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\rangle \\
&\quad - \langle f, g, h\rangle \cdot \langle \frac{\partial n}{\partial y} - \frac{\partial m}{\partial z}, \frac{\partial k}{\partial z} - \frac{\partial n}{\partial x}, \frac{\partial m}{\partial x} - \frac{\partial k}{\partial y}\rangle \\
&= k\frac{\partial h}{\partial y} - k\frac{\partial g}{\partial z} + m\frac{\partial f}{\partial z} - m\frac{\partial h}{\partial x} + n\frac{\partial g}{\partial x} - n\frac{\partial f}{\partial y} \\
&\quad - f\frac{\partial n}{\partial y} + f\frac{\partial m}{\partial z} - g\frac{\partial k}{\partial z} + g\frac{\partial n}{\partial x} - h\frac{\partial m}{\partial x} + h\frac{\partial k}{\partial y} \\
&= n\frac{\partial g}{\partial x} + g\frac{\partial n}{\partial x} - h\frac{\partial m}{\partial x} - m\frac{\partial h}{\partial x} + h\frac{\partial k}{\partial y} + k\frac{\partial h}{\partial y} - f\frac{\partial n}{\partial y} - n\frac{\partial f}{\partial y} \\
&\quad + f\frac{\partial m}{\partial z} + m\frac{\partial f}{\partial z} - g\frac{\partial k}{\partial z} - k\frac{\partial g}{\partial z} \\
&= \frac{\partial}{\partial x}(gn - hm) + \frac{\partial}{\partial y}(hk - fn) + \frac{\partial}{\partial z}(fm - gk) = \nabla \cdot (\mathbf{F} \times \mathbf{G}).
\end{aligned}$$

17.5.70 First, $(\mathbf{G} \cdot \nabla)\mathbf{F} = \left(k\dfrac{\partial}{\partial x} + m\dfrac{\partial}{\partial y} + n\dfrac{\partial}{\partial z}\right)\langle f, g, h\rangle = \langle k\dfrac{\partial f}{\partial x} + m\dfrac{\partial f}{\partial y} + n\dfrac{\partial f}{\partial z}, k\dfrac{\partial g}{\partial x} + m\dfrac{\partial g}{\partial y} + n\dfrac{\partial g}{\partial z}, k\dfrac{\partial h}{\partial x} +$
$m\dfrac{\partial h}{\partial y} + n\dfrac{\partial h}{\partial z}\rangle$ and similarly for $(\mathbf{F} \cdot \nabla)\mathbf{G}$. Next, $\mathbf{G}(\nabla \cdot \mathbf{F}) = \langle k, m, n\rangle\left(\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} + \dfrac{\partial h}{\partial z}\right) = \langle k\dfrac{\partial f}{\partial x} + k\dfrac{\partial g}{\partial y} +$

$k\frac{\partial h}{\partial z}, m\frac{\partial f}{\partial x} + m\frac{\partial g}{\partial y} + m\frac{\partial h}{\partial z}, n\frac{\partial f}{\partial x} + n\frac{\partial g}{\partial y} + n\frac{\partial h}{\partial z}\rangle$ and similarly for $\mathbf{F}(\boldsymbol{\nabla} \cdot \mathbf{G})$. Thus

$$\begin{aligned}
&(\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} - \mathbf{G}(\boldsymbol{\nabla} \cdot \mathbf{F}) - (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G} + \mathbf{F}(\boldsymbol{\nabla} \cdot \mathbf{G}) \\
&= \langle k\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y} + n\frac{\partial f}{\partial z}, k\frac{\partial g}{\partial x} + m\frac{\partial g}{\partial y} + n\frac{\partial g}{\partial z}, k\frac{\partial h}{\partial x} + m\frac{\partial h}{\partial y} + n\frac{\partial h}{\partial z}\rangle \\
&\quad - \langle k\frac{\partial f}{\partial x} + k\frac{\partial g}{\partial y} + k\frac{\partial h}{\partial z}, m\frac{\partial f}{\partial x} + m\frac{\partial g}{\partial y} + m\frac{\partial h}{\partial z}, n\frac{\partial f}{\partial x} + n\frac{\partial g}{\partial y} + n\frac{\partial h}{\partial z}\rangle \\
&\quad - \langle f\frac{\partial k}{\partial x} + g\frac{\partial k}{\partial y} + h\frac{\partial k}{\partial z}, f\frac{\partial m}{\partial x} + g\frac{\partial m}{\partial y} + h\frac{\partial m}{\partial z}, f\frac{\partial n}{\partial x} + g\frac{\partial n}{\partial y} + h\frac{\partial n}{\partial z}\rangle \\
&\quad + \langle f\frac{\partial k}{\partial x} + f\frac{\partial m}{\partial y} + f\frac{\partial n}{\partial z}, g\frac{\partial k}{\partial x} + g\frac{\partial m}{\partial y} + g\frac{\partial n}{\partial z}, h\frac{\partial k}{\partial x} + h\frac{\partial m}{\partial y} + h\frac{\partial n}{\partial z}\rangle \\
&= \langle \frac{\partial}{\partial y}(fm - gk) - \frac{\partial}{\partial z}(hk - fn), \frac{\partial}{\partial z}(gn - hm) - \frac{\partial}{\partial x}(fm - gk), \frac{\partial}{\partial x}(hk - fn) - \frac{\partial}{\partial y}(gn - hm).
\end{aligned}$$

But $\mathbf{F} \times \mathbf{G} = \langle gn - hm, hk - fn, fm - gk\rangle$, so the above expression is indeed equal to $\boldsymbol{\nabla} \times (\mathbf{F} \times \mathbf{G})$.

17.5.71 Use the values of$(\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F}$ and $(\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G}$ from the previous problem. Then $\mathbf{G} \times (\boldsymbol{\nabla} \times \mathbf{F}) = \langle k, m, n\rangle \times \langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\rangle = \langle m\frac{\partial g}{\partial x} - m\frac{\partial f}{\partial y} - n\frac{\partial f}{\partial z} + n\frac{\partial h}{\partial x}, n\frac{\partial h}{\partial y} - n\frac{\partial g}{\partial z} - k\frac{\partial g}{\partial x} + k\frac{\partial f}{\partial y}, k\frac{\partial f}{\partial z} - k\frac{\partial h}{\partial x} - m\frac{\partial h}{\partial y} + m\frac{\partial g}{\partial z}\rangle$ and similarly for $\mathbf{F} \times (\boldsymbol{\nabla} \times \mathbf{G})$.

Thus

$$\begin{aligned}
&(\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} + (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G} + \mathbf{G} \times (\boldsymbol{\nabla} \times \mathbf{F}) + \mathbf{F} \times (\boldsymbol{\nabla} \times \mathbf{G}) \\
&= \langle k\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y} + n\frac{\partial f}{\partial z}, k\frac{\partial g}{\partial x} + m\frac{\partial g}{\partial y} + n\frac{\partial g}{\partial z}, k\frac{\partial h}{\partial x} + m\frac{\partial h}{\partial y} + n\frac{\partial h}{\partial z}\rangle \\
&\quad + \langle f\frac{\partial k}{\partial x} + g\frac{\partial k}{\partial y} + h\frac{\partial k}{\partial z}, f\frac{\partial m}{\partial x} + g\frac{\partial m}{\partial y} + h\frac{\partial m}{\partial z}, f\frac{\partial n}{\partial x} + g\frac{\partial n}{\partial y} + h\frac{\partial n}{\partial z}\rangle \\
&\quad + \langle m\frac{\partial g}{\partial x} - m\frac{\partial f}{\partial y} - n\frac{\partial f}{\partial z} + n\frac{\partial h}{\partial x}, n\frac{\partial h}{\partial y} - n\frac{\partial g}{\partial z} - k\frac{\partial g}{\partial x} + k\frac{\partial f}{\partial y}, k\frac{\partial f}{\partial z} - k\frac{\partial h}{\partial x} - m\frac{\partial h}{\partial y} + m\frac{\partial g}{\partial z}\rangle \\
&\quad + \langle g\frac{\partial m}{\partial x} - g\frac{\partial k}{\partial y} - h\frac{\partial k}{\partial z} + h\frac{\partial n}{\partial x}, h\frac{\partial n}{\partial y} - h\frac{\partial m}{\partial z} - f\frac{\partial m}{\partial x} + f\frac{\partial k}{\partial y}, f\frac{\partial k}{\partial z} - f\frac{\partial n}{\partial x} - g\frac{\partial n}{\partial y} + g\frac{\partial m}{\partial z}\rangle \\
&= \langle \frac{\partial}{\partial x}(fk + gm + hn), \frac{\partial}{\partial y}(fk + gm + hn), \frac{\partial}{\partial z}(fk + gm + hn)\rangle = \nabla(\mathbf{F} \cdot \mathbf{G}).
\end{aligned}$$

17.5.72

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{F}) &= \boldsymbol{\nabla} \times \langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\rangle \\
&= \langle \frac{\partial^2 g}{\partial y \partial x} - \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 h}{\partial z \partial x}, \frac{\partial^2 h}{\partial z \partial y} - \frac{\partial^2 g}{\partial z^2} - \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 h}{\partial x^2} - \frac{\partial^2 h}{\partial^2 y} - \frac{\partial^2 g}{\partial y \partial z}\rangle
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{F}) - (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{F} &= \boldsymbol{\nabla}\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\langle f, g, h\rangle \\
&= \langle \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial x \partial y} + \frac{\partial^2 h}{\partial x \partial z}, \frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 h}{\partial y \partial z}, \frac{\partial^2 f}{\partial z \partial x} + \frac{\partial^2 g}{\partial z \partial y} + \frac{\partial^2 h}{\partial z^2}\rangle \\
&\quad - \langle \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}, \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}, \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}\rangle.
\end{aligned}$$

The two expressions are equal after cancellations and noting that mixed partials are equal.

17.5.73

$$\begin{aligned}
&\nabla \cdot \frac{\langle x, y, z\rangle}{\left(x^2+y^2+z^2\right)^{p/2}} \\
&= \frac{(1-p)\,x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^{1+p/2}} + \frac{x^2+(1-p)\,y^2+z^2}{\left(x^2+y^2+z^2\right)^{1+p/2}} + \frac{x^2+y^2+(1-p)\,z^2}{\left(x^2+y^2+z^2\right)^{1+p/2}} \\
&= \frac{(3-p)\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{1+p/2}} = \frac{3-p}{\left(x^2+y^2+z^2\right)^{p/2}} = \frac{3-p}{|\mathbf{r}|^p}
\end{aligned}$$

17.5.74 $\boldsymbol{\nabla}\left(\dfrac{1}{|r|^p}\right) = \boldsymbol{\nabla}\left(\dfrac{1}{\left(x^2+y^2+z^2\right)^{p/2}}\right) = -\dfrac{p}{\left(x^2+y^2+z^2\right)^{1+p/2}}\langle x, y, z\rangle = -\dfrac{p\,\mathbf{r}}{|\mathbf{r}|^{p+2}}.$

17.5.75 $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\left(\dfrac{1}{|r|^p}\right) = \boldsymbol{\nabla}\cdot\left(-\dfrac{p\,\mathbf{r}}{|\mathbf{r}|^{p+2}}\right)$ by Exercise 72, and then by Exercise 71, $\boldsymbol{\nabla}\cdot\left(-\dfrac{p\,\mathbf{r}}{|\mathbf{r}|^{p+2}}\right) = -p\,\boldsymbol{\nabla}\cdot\dfrac{\mathbf{r}}{|\mathbf{r}|^{p+2}} = \dfrac{-p\,(3-(p+2))}{|\mathbf{r}|^{p+2}} = \dfrac{p\,(p-1)}{|\mathbf{r}|^{p+2}}.$

17.6 Surface Integrals

17.6.1 $\mathbf{r}\,(u, v) = \langle a\cos u, a\sin u, v\rangle$ where $0 \le u \le 2\pi$; $0 \le v \le h$.

17.6.2 $\mathbf{r}\,(u, v) = \left\langle \dfrac{av}{h}\cos u, \dfrac{av}{h}\sin u, v\right\rangle$ where $0 \le u \le 2\pi$; $0 \le v \le h$.

17.6.3 $\mathbf{r}\,(u, v) = \langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle$ where $0 \le u \le \pi$; $0 \le v \le 2\pi$.

17.6.4 A cone of height h and radius a has equation $a^2z^2 = h^2\left(x^2+y^2\right)$; thus $z_x = \dfrac{h^2x}{a^2z}$ and similarly for z_y, so compute

$$\begin{aligned}
\iint_S f\,(x, y, z)\;dS &= \iint_R f\left(x, y, \frac{h^2}{a^2}\sqrt{x^2+y^2}\right)\sqrt{\left(\frac{h^2x}{a^2z}\right)^2 + \left(\frac{h^2y}{a^2z}\right)^2 + 1}\,dA \\
&= \iint_R f\left(x, y, \frac{h^2}{a^2}\sqrt{x^2+y^2}\right)\sqrt{1+\frac{h^2}{a^2}}\,dA.
\end{aligned}$$

17.6.5 Use the parametric description from problem 3 and compute

$$\int_0^{\pi}\int_0^{2\pi} a^2 f\,(a\sin u\cos v, a\sin u\sin v, a\cos u)\sin u\,du\,dv.$$

17.6.6 It means that we can make a consistent choice of normal vectors such that when you walk along the surface, the direction of the normal vectors does not change discontinuously.

17.6.7 The usual orientation of a closed surface is that the normal vectors point outwards.

17.6.8 Because the vector field is vertical, the same amount of materials goes through the surface as through its projection on the xy-plane.

17.6.9 $\left\langle u, v, \dfrac{1}{3}\,(16-2u+4v)\right\rangle$, $|u| < \infty$, $|v| < \infty$.

17.6.10 $\langle 4\sin u\cos v, 4\sin u\sin v, 4\cos v\rangle$, $0 \le u \le \dfrac{\pi}{4}$, $0 \le v \le 2\pi$.

17.6.11 $\langle v\cos u, v\sin u, v\rangle$, $0 \le u \le 2\pi$, $2 \le v \le 8$.

17.6.12 $\left\langle \frac{v}{2}\cos u, \frac{v}{2}\sin u, v\right\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 4$.

17.6.13 $\langle 3\cos u, 3\sin u, v\rangle$, $0 \le u \le \dfrac{\pi}{2}$, $0 \le v \le 3$.

17.6.14 $\langle v, 6\cos u, 6\sin u\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 9$.

17.6.15 The segment of the plane $z = 2x + 3y - 1$ above $[1,3] \times [2,4]$.

17.6.16 The segment of the plane $z = 2 - y$ above $[0,2] \times [0,4]$.

17.6.17 The portion of the cone $z^2 = 16x^2 + 16y^2$ of height 12 and radius 3, where $y \ge 0$.

17.6.18 The cylinder $y^2 + z^2 = 36$ of radius 6 whose axis is the x-axis, for $0 \le x \le 2$.

17.6.19 Using the standard parametric description of the cylinder, we have $\mathbf{r}(u,v) = \langle 4\cos u, 4\sin u, v\rangle$ for $0 \le v \le 7$, $0 \le u \le \pi$. Then $|\mathbf{t}_u \times \mathbf{t}_v| = 4$ and the area is $\displaystyle\iint_S 1\,dS = \iint_R 4\,dA = \int_0^{\pi}\int_0^7 4\,dv\,du = 28\pi$.

17.6.20 The plane has the parametric description $\mathbf{r}(u,v) = \langle u, v, 3 - u - 3v\rangle$, $0 \le v \le 1$, $0 \le u \le 3 - 3v$. Then $\mathbf{t}_u \times \mathbf{t}_v = \langle 1, 0, -1\rangle \times \langle 0, 1, -3\rangle = \langle 1, 3, 1\rangle$, so that $|\mathbf{t}_u \times \mathbf{t}_v| = \sqrt{11}$. Then

$$\iint_S 1\,dS = \sqrt{11}\iint_R 1\,dA = \sqrt{11}\int_0^1\int_0^{3-3v} 1\,du\,dv = \sqrt{11}\int_0^1 (3 - 3v)\,dv = \frac{3\sqrt{11}}{2}.$$

17.6.21 The plane has parametric description $\mathbf{r}(u,v) = \langle u, v, 10 - u - v\rangle$, for $-2 \le u \le 2$, $-2 \le v \le 2$. Then $\mathbf{t}_u \times \mathbf{t}_v = \langle 1, 0, -1\rangle \times \langle 0, 1, -1\rangle = \langle 1, 1, 1\rangle$, so that $\displaystyle\iint_S 1\,dS = \sqrt{3}\iint_R 1\,dA = 16\sqrt{3}$.

17.6.22 Using the standard parametric description of the sphere with $0 \le u \le \dfrac{\pi}{2}$, $0 \le v \le 2\pi$, we have

$$|\mathrm{t}_u \times \mathrm{t}_v| = a^2 \sin u$$

so that

$$\iint_S 1\,dS = \iint_R 100\,\sin u\,dA = \int_0^{\pi/2}\int_0^{2\pi} 100\sin u\,dv\,du = 200\pi\int_0^{\pi/2}\sin u\,du = 200\pi.$$

17.6.23 Parameterize the cone by $\mathbf{r}(u,v) = \left\langle \dfrac{r}{h}v\cos u, \dfrac{r}{h}v\sin u, v\right\rangle$, for $0 \le v \le h$, $0 \le u \le 2\pi$; then $\mathbf{t}_u \times \mathbf{t}_v = \left\langle -\dfrac{r}{h}v\sin u, \dfrac{r}{h}v\cos u, 0\right\rangle \times \left\langle \dfrac{r}{h}\cos u, \dfrac{r}{h}\sin u, 1\right\rangle$ and $|\mathrm{t}_u \times \mathrm{t}_v| = \dfrac{r}{h^2}v\sqrt{h^2 + r^2}$. Then

$$\begin{aligned}\iint_S 1\,dS &= \frac{r\sqrt{h^2+r^2}}{h^2}\iint_R v\,dA = \frac{r\sqrt{h^2+r^2}}{h^2}\int_0^{2\pi}\int_0^h v\,dv\,du \\ &= \frac{r\sqrt{h^2+r^2}}{h^2}\int_0^{2\pi}\frac{1}{2}h^2\,du = \frac{2\pi r\sqrt{h^2+r^2}}{2} = \pi r\sqrt{h^2+r^2}.\end{aligned}$$

17.6.24 Using the standard parameterization of the sphere for $1 \le 2\cos u \le 2$, or $0 \le u \le \dfrac{\pi}{3}$, and $0 \le v \le 2\pi$, we obtain

$$\iint_S 1\,dS = \iint_R 4\,\sin u\,dA = 4\int_0^{2\pi}\int_0^{\pi/3}\sin u\,du\,dv = 4\int_0^{2\pi}(-\cos u)\Big|_{u=0}^{u=\pi/3}dv = 4\int_0^{2\pi}\frac{1}{2}\,dv = 4\pi.$$

17.6.25 Using the standard parameterization of the sphere for $0 \le u \le \dfrac{\pi}{2},\ 0 \le v \le 2\pi$, we obtain

$$\iint_S \left(x^2+y^2\right)\,dS = \int_0^{2\pi}\int_0^{\pi/2} 36\sin^2 u \cdot 36\sin u\,du\,dv = 1296\int_0^{2\pi}\int_0^{\pi/2}\sin^3 u\,du\,dv$$
$$= 1296\int_0^{2\pi}\frac{2}{3}\,dv = 1728\pi.$$

17.6.26 Use the standard parameterization of the cylinder, for $0 \le u \le 2\pi$, $0 \le v \le 3$; then

$$\iint_S y\,dS = \int_0^{2\pi}\int_0^3 3\sin u \cdot 3\,dv\,du = 27\int_0^{2\pi}\sin u\,du = 0.$$

17.6.27 Use the standard parameterization, for $0 \le u \le 2\pi$, $0 \le v \le 3$; then

$$\iint_S x\,dS = \int_0^{2\pi}\int_0^3 \cos u \cdot 1\,dv\,du = 3\int_0^{2\pi}\cos u\,du = 0.$$

17.6.28 Using the standard parameterization (with $u = \varphi$ and $v = \theta$) for $0 \le u \le \dfrac{\pi}{2}$ and $0 \le v \le \dfrac{\pi}{2}$, we have

$$\iint_R \cos u\,dA = \int_0^{\pi/2}\int_0^{\pi/2}\cos u \cdot \sin u\,du\,dv = \frac{\pi}{4}.$$

17.6.29 $z_x = 2$ and $z_y = 2$, so $\sqrt{z_x^2+z_y^2+1} = \sqrt{9} = 3$.

$$\int_0^2\int_0^{2x} 3\,dy\,dx = \int_0^2 6x\,dx = 3x^2\Big|_0^2 = 12.$$

17.6.30 $z_x = 1$ and $z_y = 3$, so $\sqrt{1+9+1} = \sqrt{11}$.

$$\int_0^{2\pi}\int_1^2 \sqrt{11}r\,dr\,d\theta = 2\pi\sqrt{11}\left(\frac{r^2}{2}\right)\Bigg|_1^2 = 3\pi\sqrt{11}.$$

17.6.31 $2z\,dz = 8x\,dx$, so $z_x = \dfrac{4x}{z}$; similarly, $z_y = \dfrac{4y}{z}$. Thus $\sqrt{z_x^2+z_y^2+1} = \sqrt{\dfrac{16x^2+16y^2+z^2}{z^2}} = \sqrt{\dfrac{20\left(x^2+y^2\right)}{4\left(x^2+y^2\right)}} = \sqrt{5}$. Further, this cone sits over $x^2+y^2=4$. Then $\displaystyle\iint_S 1\,dS = \iint_R \sqrt{5}\,dA = 4\pi\sqrt{5}$.

17.6.32 $z_x = x$ and $z_y = 0$, so that $\displaystyle\iint_S 1\,dS = \int_0^4\int_{-1}^1 \sqrt{x^2+1}\,dx\,dy = 4\sqrt{2}+4\ln(1+\sqrt{2})$.

17.6.33 $dz = 4x\,dx$ so that $z_x = 4x$ and similarly $z_y = 4y$. The paraboloid sits over $x^2+y^2=4$ Thus $\displaystyle\iint_S 1\,dS = \iint_R \sqrt{16x^2+16y^2+1}\,dA = \int_0^{2\pi}\int_0^2 r\sqrt{16r^2+1}\,dr\,d\theta = \frac{\left(65\sqrt{65}-1\right)\pi}{24}$.

17.6.34 We have $z_x = 2x$, $z_y = -2y$, so

$$\iint_S 1\,dS = \iint_R \sqrt{4x^2+4y^2+1}\,dA = \int_0^{\pi/2}\int_0^{\sqrt{2}} r\sqrt{4r^2+1}\,dr\,d\theta = \frac{13\pi}{12}.$$

17.6.35 $z_x = z_y = -1$, so $\displaystyle\iint\limits_S xy\,dS = \sqrt{3}\iint\limits_R xy\,dA = \sqrt{3}\int_0^2\int_0^{2-x} xy\,dy\,dx = \frac{2\sqrt{3}}{3}$.

17.6.36 $z_x = 2x$, $z_y = 2y$, and the paraboloid sits over $x^2+y^2=4$, so $\displaystyle\iint\limits_S \left(x^2+y^2\right)\,dS =$
$\displaystyle\iint\limits_R \left(x^2+y^2\right)\sqrt{4x^2+4y^2+1}\,dA = \int_0^{2\pi}\int_0^1 r^3\sqrt{4r^2+1}\,dr\,d\theta = 2\pi\int_0^1 r^3\sqrt{4r^2+1}\,dr$. Let $u = 4r^2+1$ so that $du = 8r\,dr$. Substituting gives

$$\begin{aligned} 2\pi\cdot\frac{1}{8}\int_1^5 \frac{u-1}{4}\sqrt{u}\,du &= \frac{\pi}{16}\int_1^5 (u^{3/2}-u^{1/2})\,du \\ &= \frac{\pi}{16}\left(\frac{2u^{5/2}}{5}-\frac{2u^{3/2}}{3}\right)\Bigg|_1^5 \\ &= \frac{\pi}{16}\left(10\sqrt{5}-\frac{10\sqrt{5}}{3}-\left(\frac{2}{5}-\frac{2}{3}\right)\right) \\ &= \frac{\pi}{16}\left(\frac{100\sqrt{5}}{15}+\frac{4}{15}\right) = \frac{\pi(1+25\sqrt{5})}{60}. \end{aligned}$$

17.6.37 $x^2+y^2+z^2=25$, so $z_x = --\dfrac{x}{z}$ and $z_y = -\dfrac{y}{z}$. Then $\displaystyle\iint\limits_S \left(25-x^2-y^2\right)\,dS =$
$\displaystyle\iint\limits_R \left(25-x^2-y^2\right)\sqrt{\frac{x^2+y^2+z^2}{z^2}}\,dA = 5\iint\limits_R \sqrt{25-x^2-y^2}\,dA = 5\int_0^{2\pi}\int_0^5 r\sqrt{25-r^2}\,dr\,d\theta = \frac{1250\pi}{3}$.

17.6.38 $z_x = -1$, $z_y = -2$, and the limits of integration are $0 \le y \le 4$, $0 \le x \le 8-2y$. Then $\displaystyle\iint\limits_S e^z\,dS =$
$\displaystyle\iint\limits_R e^{8-x-2y}\sqrt{6}\,dA = \sqrt{6}\int_0^4\int_0^{8-2y} e^{8-x-2y}\,dx\,dy = \frac{\sqrt{6}\left(e^8-9\right)}{2}$.

17.6.39 $\displaystyle\iint_S 1\,dS = \int_0^1\int_0^1 \sqrt{1+4+4}\,dx\,dy = 3$.
$\displaystyle\iint_S e^{2x+y+z-3}\,dS = 3\int_0^1\int_0^1 e^{2x+y+4-2x-2y-3}\,dx\,dy = 3\int_0^1\int_0^1 e^{1-y}\,dx\,dy = 3(e-1)$. So the average temperature is $\dfrac{3(e-1)}{3} = e-1$.

17.6.40 $z_x = -2x$, $z_y = -2y$. The paraboloid sits over $x^2+y^2=4$. Thus the area of the paraboloid is

$$\iint\limits_S 1\,dS = \iint\limits_R \sqrt{4x^2+4y^2+1}\,dA = \int_0^{2\pi}\int_0^2 r\sqrt{4r^2+1}\,dr\,d\theta \approx 36.1769,$$

and the integral of the square of the distance from the origin is

$$\begin{aligned} \iint\limits_S \left(x^2+y^2+z^2\right)\,dS &= \iint\limits_R \left(x^2+y^2+\left(4-x^2-y^2\right)^2\right)\sqrt{4x^2+4y^2+1}\,dA \\ &= \int_0^{2\pi}\int_0^2 r\left(r^2+\left(4-r^2\right)^2\right)\sqrt{4r^2+1}\,dr\,d\theta \approx 227.18. \end{aligned}$$

Thus the average squared distance is approximately 6.2797.

17.6.41 $z_x = -\dfrac{x}{z}$ and $z_y = -\dfrac{y}{z}$, so that $\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{\dfrac{x^2 + y^2 + z^2}{z^2}} = \left(1 - x^2 - y^2\right)^{-1/2}$. The area of the sphere is $\dfrac{1}{8} \cdot 4\pi = \dfrac{\pi}{2}$, and the integral of the function is

$$\iint_S xyz\, dS = \iint_R xy\left(1 - x^2 - y^2\right)^{1/2}\left(1 - x^2 - y^2\right)^{-1/2}\, dA = \int_0^{\pi/2}\int_0^1 r^3 \sin(\theta)\cos(\theta)\, dr\, d\theta = \frac{1}{8},$$

so that the average value is $\dfrac{1}{4\pi}$.

17.6.42 $z_x = \dfrac{x}{z}$ and $z_y = \dfrac{y}{z}$ so that $\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{2}$. The cone sits over $x^2 + y^2 = 4$. The area of the cone is $\displaystyle\iint_S 1\, dS = \sqrt{2}\iint_R dA = 4\pi\sqrt{2}$, and the integral of the temperature function is $\displaystyle\iint_S (100 - 25z)\, dS = \sqrt{2}\iint_R \left(100 - 25\sqrt{x^2 + y^2}\right) dA = \sqrt{2}\int_0^{2\pi}\int_0^2 r\,(100 - 25r)\, dr\, d\theta = \frac{800\pi\sqrt{2}}{3}$ so that the average temperature is $\dfrac{200}{3}$.

17.6.43 $z_x = z_y = -1$, so the normal vector is $\langle 1, 1, 1\rangle$, which points in the positive z-direction. Then $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R (0\cdot 1 + 0\cdot 1 - 1\cdot 1)\, dA = \int_0^4\int_0^{4-x}(-1)\, dy\, dx = -8.$

17.6.44 $z_x = -2$, $z_y = -5$, so the normal vector is $\langle 2, 5, 1\rangle$, which points in the positive z-direction.

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R (x\cdot 2 + y\cdot 5 + z\cdot 1)\, dA &= \int_0^2\int_0^{(10-5y)/2}(2x + 5y + (10 - 2x - 5y))\, dx\, dy \\ &= \int_0^2\int_0^{(10-5y)/2} 10\, dx\, dy = 50.\end{aligned}$$

17.6.45 We have $z = \sqrt{x^2 + y^2}$; then $\mathbf{n} = \left\langle -\dfrac{x}{z}, -\dfrac{y}{z}, 1\right\rangle$, which points upwards.
$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R \left(\left(-\frac{x}{z}\right)\cdot x + \left(-\frac{y}{z}\right)\cdot y + 1\cdot z\right) dA = \iint_R \left(z - \frac{x^2 + y^2}{z}\right) dA = \iint_R (z - z)\, dA = 0.$$

17.6.46 $z_x = 0$, $z_y = -\sin(y)$, so an upward-pointing normal is $\langle 0, \sin y, 1\rangle$. We have $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\, dS =$

$$\iint_R (2\sin y\cos y + xy)\, dA = \int_0^4\int_{-\pi}^{\pi}(2\sin y\,\cos y + xy)\, dy\, dx = 0.$$

17.6.47 An outward-pointing normal is $\dfrac{\mathbf{r}}{|\mathbf{r}|}$. The sphere has radius a, so the vector field is in fact $\dfrac{\mathbf{r}}{|\mathbf{r}|^3}$.
$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_S \frac{\mathbf{r}}{|\mathbf{r}|^3}\cdot\frac{\mathbf{r}}{|\mathbf{r}|}\, dS = \iint_S \frac{1}{|\mathbf{r}|^2} dS = \iint_S \frac{1}{a^2} dS = \frac{1}{a^2}\iint_S 1\, dS = \frac{1}{a^2}4\pi\, a^2 = 4\pi.$$

17.6.48 The parametric form is $\langle u, u^2, v\rangle$ for $0 \le u \le 1$, $0 \le v \le 4$. We have $\mathbf{t}_u = \langle 1, 2u, 0\rangle$ and $\mathbf{t}_v = \langle 0, 0, 1\rangle$, so that $\mathbf{t}_u \times \mathbf{t}_v = \langle 2u, -1, 0\rangle$; since we want normal vectors to point in the positive y direction, we choose$\langle -2u, 1, 0\rangle$ for the normal vector. Then $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R \left\langle -u^2, u, 1\right\rangle\cdot\langle -2u, 1, 0\rangle\, dA =$

$$\iint_R \left(u + 2u^3\right) dA = \int_0^4\int_0^1 \left(u + 2u^3\right) du\, dv = 4.$$

17.6.49

a. True. The formula in Theorem 17.12 gives $\displaystyle\iint_S f(x,y,z)\,dS = \iint_R f(x,y,10)\sqrt{0+0+1}\,dA$.

b. False. The formula in Theorem 17.12 gives

$$\iint_S f(x,y,z)\,dS = \iint_R f(x,y,x)\sqrt{1+0+1}\,dA = \sqrt{2}\iint_R f(x,y,x)\,dA.$$

c. True. Substituting $2u$ for u and $\sqrt{v}$ for v in the first parameterization gives $\langle\sqrt{v}\cos 2u, \sqrt{v}\sin 2u, v\rangle$, $0 \le 2u \le \pi$, $0 \le \sqrt{v} \le 2$. Simplifying the bounds conditions gives the second parameterization.

d. True. The standard parameterization is $\langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle$ for $0 \le u \le \pi$, $0 \le v \le 2\pi$. Then $\mathbf{t}_u \times \mathbf{t}_v = \left\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\cos u\sin u\right\rangle$, and it is easily seen that these are outward-pointing vectors, by considering various ranges for u and v.

17.6.50 $\nabla \ln|\mathbf{r}| = \nabla \ln\sqrt{x^2+y^2+z^2} = \dfrac{1}{|\mathbf{r}|^2}\langle x,y,z\rangle = \dfrac{1}{a^2}\langle x,y,z\rangle$ on the sphere of radius a; using the explicit description for the sphere, we have $\mathbf{n} = \left\langle \dfrac{x}{z}, \dfrac{y}{z}, 1\right\rangle$, so that

$$\iint_S \nabla\ln|\mathbf{r}|\cdot\mathbf{n}\,dS = \iint_R \frac{1}{a^2}\langle x,y,z\rangle\cdot\left\langle\frac{x}{z},\frac{y}{z},1\right\rangle dA = \frac{1}{a^2}\iint_R\left(\frac{x^2+y^2}{z}+z\right)dA =$$

$$\frac{1}{a^2}\iint_R\left(\frac{x^2+y^2+z^2}{z}\right)dA = \frac{1}{a^2}\iint_R \frac{a^2}{z}\,dA = \iint_R \frac{1}{z}\,dA = \int_0^{2\pi}\int_0^a \frac{1}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta = 2\pi a.$$

17.6.51 Parameterize the surface by $\langle 2\cos u, 2\sin u, v\rangle$ for $0 \le u \le 2\pi$, $0 \le v \le 8$. Then the normal vector has magnitude 2, and

$$\iint_S |\mathbf{r}|\,dS = \iint_R \sqrt{x^2+y^2+z^2}\,dA = 2\int_0^{2\pi}\int_0^8 \sqrt{4+v^2}\,dv\,du = 8\pi\left(4\sqrt{17}+\ln\left(4+\sqrt{17}\right)\right).$$

17.6.52 $z_x = 0, z_y = -1$, so

$$\iint_S xyz\,dS = \iint_R xyz\sqrt{2}\,dA = \sqrt{2}\int_0^{2\pi}\int_0^2 r\cdot r\cos\theta\cdot r\sin\theta\cdot(6-r\sin\theta)\,dr\,d\theta =$$

$$\sqrt{2}\int_0^{2\pi}\int_0^2 r^3\cos\theta\,\sin\theta\,(6-r\sin\theta)\,dr\,d\theta = 0.$$

17.6.53 The normal vector is $\langle x,0,z\rangle$, so

$$\iint_S \frac{1}{\sqrt{x^2+z^2}}\langle x,0,z\rangle\cdot\langle x,0,z\rangle\,dS = \iint_R \sqrt{x^2+z^2}\,dA = \int_{-2}^2\int_0^{2\pi} a\,dA = 8\pi a.$$

17.6.54 The two curves intersect where $x^2+y^2 = 16-x^2-y^2$, so on the plane $z = 2\sqrt{2}$. The projection on the xy-plane of the circle of intersection is $x^2+y^2=8$. The total surface area of the sphere, of radius 4, is 64π. Finally, the outward normals to the sphere are $\left\langle\dfrac{x}{z},\dfrac{y}{z},1\right\rangle$.

a. From part (b) below and the fact that the total surface area is 64π, we get an answer of $64\pi - 32\pi + 16\pi\sqrt{2} = 16\pi\left(2+\sqrt{2}\right)$.

b.

$$\iint_S 1\,dS = \iint_R \sqrt{\frac{x^2}{z^2}+\frac{y^2}{z^2}+1}\,dA = \iint_R \frac{4}{z}\,dA = \iint_R \frac{4}{\sqrt{16-x^2-y^2}}\,dA$$
$$= \int_0^{2\pi}\int_0^{\sqrt{8}} \frac{4r}{\sqrt{16-r^2}}\,dr\,d\theta = 16\pi\left(2-\sqrt{2}\right).$$

c. For the cone $z^2 = x^2+y^2$, we have $\sqrt{z_x^2+z_y^2+1} = \sqrt{2}$, so

$$\iint_S 1\,dS = \iint_R \sqrt{2}\,dA = \sqrt{2}\cdot\pi\cdot\left(\sqrt{8}\right)^2 = 8\pi\sqrt{2}.$$

17.6.55 The surface of the cylinder inside the sphere is defined parametrically by $\langle 1+\cos u, \sin u, v\rangle$ where $0 \le u \le 2\pi$ and also (because the z-coordinate must stay inside the sphere of radius 2),

$$0 \le v \le \sqrt{4-(1+\cos u)^2-\sin^2 u},$$

or

$$0 \le v \le \sqrt{2-2\cos u}.$$

The normal is $\langle \cos u, \sin u, 0\rangle$, which has magnitude 1, so we have

$$\iint_S 1\,dS = \iint_R 1\,dA = \int_0^{2\pi}\int_0^{\sqrt{2-2\cos u}} 1\,dv\,du = \int_0^{2\pi}\sqrt{2-2\cos u}\,du = 8.$$

17.6.56 We have $z = -\frac{c}{a}x + \left(-\frac{c}{b}\right)y + c$, so that $z_x = -\frac{c}{a}$, $z_y = -\frac{c}{b}$, and an upward-pointing normal is $\left\langle \frac{c}{a}, \frac{c}{b}, 1\right\rangle$. Then $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \left(\frac{c}{a}x+\frac{c}{b}y+z\right)dA = \iint_R \left(\frac{c}{a}x+\frac{c}{b}y+\left(-\frac{c}{a}x+\left(-\frac{c}{b}\right)y+c\right)\right)dA = \iint_R c\,dA$, which is c times the area of A. Recall that if the vector field is vertical, then the flux is equal to the area of the base. As c increases, the slope of the plane gets closer to vertical, so that the x and y components of the vector field $\langle x, y, z\rangle$ contribute more to the flux; also, the values of z get larger. Thus the flux increases as c does.

17.6.57

a. Using the standard parameterization, $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \left(\frac{-x^2}{z}+\frac{-y^2}{z}+z\right)dA = \iint_R 0\,dA = 0$, because $x^2+y^2 = z^2$, so that the flux is zero. This is due to the fact that the field $\mathbf{F}$ is aligned with the cone at all points on the cone.

b. $2z\,dz = \left(\frac{2x}{a^2}\right)dx$ so that $z_x = \frac{x}{a^2 z}$. Then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \left(\frac{-x^2}{a^2 z}+\frac{-y^2}{a^2 z}+z\right)dA = \iint_R \left(\frac{-a^2 z^2}{a^2 z}+z\right)dA = 0,$$

so the flux is again zero. This is because the flow is a radial flow, so is always tangent to this surface.

17.6.58 Parameterize the cone by $\mathbf{r}(u,v) = \left\langle \frac{a}{h} v\cos u, \frac{a}{h} v\sin u, v\right\rangle$for $0 \le v \le h$, $0 \le u \le 2\pi$; then $\mathbf{t}_u \times \mathbf{t}_v = \left\langle -\frac{a}{h} v\sin u, \frac{a}{h} v\cos u, 0\right\rangle \times \left\langle \frac{a}{h}\cos u, \frac{a}{h}\sin u, 1\right\rangle$ and $|\mathbf{t}_u \times \mathbf{t}_v| = \frac{a}{h^2} v\sqrt{h^2+a^2}$. Then

$$\iint_S 1\,dS = \frac{a\sqrt{h^2+a^2}}{h^2}\iint_R v\,dA = \frac{a\sqrt{h^2+a^2}}{h^2}\int_0^{2\pi}\int_0^h v\,dv\,du$$
$$= \frac{a\sqrt{h^2+a^2}}{h^2}\int_0^{2\pi}\frac{1}{2}h^2\,du = \frac{2\pi a\sqrt{h^2+a^2}}{2} = \pi a\sqrt{h^2+a^2}.$$

17.6.59 Because the cap has height h, the circle at the boundary of the cap has radius $\sqrt{a^2-(a-h)^2} = \sqrt{2ah-h^2}$, so that the equation of the base of the region is $x^2+y^2 = 2ah-h^2$. The outward normals to the sphere are $\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$. Thus $\iint_S 1\,dS = \iint_R \sqrt{\frac{x^2}{z^2}+\frac{y^2}{z^2}+1}\,dA = \iint_R \sqrt{\frac{a^2-z^2}{z^2}+1}\,dA = \iint_R \frac{a}{z}\,dA = \iint_R \frac{a}{\sqrt{a^2-x^2-y^2}}\,dA = \int_0^{2\pi}\int_0^{\sqrt{2ah-h^2}} \frac{ar}{\sqrt{a^2-r^2}}\,dr\,d\theta = 2\pi ah$. (See also problem 54(b)).

17.6.60 Using a parametric description, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \frac{1}{(x^2+y^2+z^2)^{p/2}}\langle x,y,z\rangle\cdot(\mathbf{t}_u\times\mathbf{t}_v)\,dA$$
$$= \frac{1}{a^p}\iint_R \langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle\cdot\left\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\cos u\sin u\right\rangle dA$$
$$= \frac{1}{a^p}\iint_R \left(a^3\sin^3 u\cos^2 v + a^3\sin^3 u\sin^2 v + a^3\cos^2 u\sin u\right) dA$$
$$= a^{3-p}\int_0^{\pi}\int_0^{2\pi}\left(\sin^3 u + \cos^2 u\sin u\right) dv\,du$$
$$= a^{3-p}\int_0^{\pi}\int_0^{2\pi}\sin u\,dv\,du = \frac{4\pi}{a^{p-3}}.$$

Using an explicit description, compute the flux on the upper half hemisphere and double it. There, for $z \ge 0$, we have $z_x = -\frac{x}{z}$, $z_y = -\frac{y}{z}$, so that

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \frac{1}{(x^2+y^2+z^2)^{p/2}}\left(\frac{x^2}{z}+\frac{y^2}{z}+z\right) dA$$
$$= \frac{1}{a^p}\iint_R \frac{a^2}{z}\,dA = a^{2-p}\int_0^{2\pi}\int_0^a \frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta = \frac{2\pi}{a^{p-3}}.$$

After doubling, we get the same answer.

17.6.61 $\mathbf{F} = -\nabla T = -\langle T_x, T_y, T_z\rangle = \langle 100e^{-x-y}, 100e^{-x-y}, 0\rangle$. Thus the flow is parallel to the two sides where $z = \pm 1$ so that the flux is zero there. We thus need only compute the flux on the remaining four sides. Parameterize the sides as

$S_1 : \langle -1, y, z\rangle$	$\mathbf{t}_y\times\mathbf{t}_z = \langle 1,0,0\rangle$
$S_2 : \langle 1, y, z\rangle$	$\mathbf{t}_y\times\mathbf{t}_z = \langle 1,0,0\rangle$
$S_3 : \langle x, -1, z\rangle$	$\mathbf{t}_x\times\mathbf{t}_z = \langle 0,-1,0\rangle$
$S_4 : \langle x, 1, z\rangle$	$\mathbf{t}_x\times\mathbf{t}_z = \langle 0,-1,0\rangle$

for $-1 \le x, y, z \le 1$.

We are looking for the outward flux, so we must choose outward normals, which are (respectively) $\langle -1,0,0\rangle$, $\langle 1,0,0\rangle$,$\langle 0,-1,0\rangle$, and $\langle 0,1,0\rangle$. Then

$$\iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS_1 = \iint_R -100e^{-x-y}\,dA = -100\int_{-1}^{1}\int_{-1}^{1} e^{1-y}\,dz\,dy = -200e^2+200$$
$$\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS_2 = \iint_R -100e^{-x-y}\,dA = 100\int_{-1}^{1}\int_{-1}^{1} e^{-1-y}\,dz\,dy = -200e^{-2}+200$$
$$\iint_{S_3} \mathbf{F}\cdot\mathbf{n}\,dS_3 = \iint_R -100e^{-x-y}\,dA = -100\int_{-1}^{1}\int_{-1}^{1} e^{-x+1}dz\,dx = -200e^2+200$$
$$\iint_{S_4} \mathbf{F}\cdot\mathbf{n}\,dS_4 = \iint_R -100e^{-x-y}\,dA = 100\int_{-1}^{1}\int_{-1}^{1} e^{-x-1}dz\,dx = -200e^{-2}+200$$

so that the total flux is $-400\left(e^2+e^{-2}-2\right) = -400\left(e-\dfrac{1}{e}\right)^2$.

17.6.62 $\mathbf{F} = -\nabla T = -\langle T_x, T_y, T_z\rangle = \left\langle 200xe^{-x^2-y^2-z^2}, 200ye^{-x^2-y^2-z^2}, 200ze^{-x^2-y^2-z^2}\right\rangle$. Thus integrating over the top half of the sphere gives

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_R \left\langle 200xe^{-x^2-y^2-z^2}, 200ye^{-x^2-y^2-z^2}, 200ze^{-x^2-y^2-z^2}\right\rangle\cdot\left\langle \frac{x}{z},\frac{y}{z},1\right\rangle dA\\ &= 200e^{-a^2}\iint_R\left(\frac{x^2}{z}+\frac{y^2}{z}+z\right)dA = 200a^2e^{-a^2}\iint_R \frac{1}{z}dA\\ &= 200a^2e^{-a^2}\int_0^{2\pi}\int_0^a \frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta = 400\pi a^3e^{-a^2},\end{aligned}$$

and because the vector field is symmetric, the answer is $800\pi a^3e^{-a^2}$.

17.6.63 $\mathbf{F} = -\nabla T = \dfrac{2}{x^2+y^2+z^2}\langle x,y,z\rangle$. Thus integrating on the top half of the sphere gives $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R\left(\dfrac{x^2}{z}+\dfrac{y^2}{z}+z\right)dA = 2\iint_R \dfrac{1}{z}dA = 2\int_0^{2\pi}\int_0^a \dfrac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta = 4\pi a$, and because the vector field is symmetric, the answer is $2\cdot 4\pi a = 8\pi a$.

17.6.64

a. $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \langle x,y,0\rangle\cdot\langle x,y,0\rangle\,dA = \iint_R \left(x^2+y^2\right)dA = a^2\int_0^{2\pi}\int_0^a r\,dr\,d\theta = \pi a^4$.

b. On the cylinder, this field is $\left(x^2+y^2\right)^{-p/2} = a^{-p}$ times as large as the field in part (a), so the flux is $4\pi La^{2-p}$.

c. As $a\to\infty$, this converges for $2-p\le 0$, or $p\ge 2$.

d. As $L\to\infty$, the flux never converges.

17.6.65

a. From problem 60, the outward flux across a sphere of radius b is $\dfrac{4\pi}{b^{p-3}}$, so the total flux across the concentric spheres when $p=0$ is $4\pi b^3 - 4\pi a^3 = 4\pi\left(b^3-a^3\right)$.

b. For $p = 3$, the flux across the sphere of radius b is 4π, so the net flux is zero across S.

17.6.66 By symmetry, $\overline{x} = \overline{y} = 0$. Since the shell has constant density, we assume the density is 1; then its mass is $2\pi a^2$, and $M_{xy} = \iint\limits_S z\,dS = \iint\limits_R z\sqrt{\left(\frac{-x}{z}\right)^2 + \left(\frac{-y}{z}\right)^2 + 1}\,dA = \iint\limits_R z\cdot\frac{a}{z}\,dA = a\int_0^{2\pi}\int_0^a r\,dr\,d\theta =$ $\pi\,a^3$ so that $\overline{z} = \frac{a}{2}$.

17.6.67 The cone is rotationally symmetric around the z axis, so $\overline{x} = \overline{y} = 0$. Parameterize the cone by $\left\langle \frac{r}{h}v\cos u, \frac{r}{h}v\sin u, v\right\rangle$. Then from problem 23, the surface area of the cone is $\pi r\sqrt{h^2+r^2}$, so its mass is $\rho\pi r\sqrt{h^2+r^2}$. Using the parameterization from that problem, $|\mathbf{t}_u \times \mathbf{t}_v| = \frac{r}{h^2}v\sqrt{h^2+r^2}$, so that $M_{xy} = \rho\iint\limits_S z\,dS = \rho\frac{r\sqrt{h^2+r^2}}{h^2}\iint\limits_R vz dA = \rho\frac{r\sqrt{h^2+r^2}}{h^2}\int_0^{2\pi}\int_0^h v^2\,dv\,du = \rho\frac{r\sqrt{h^2+r^2}}{h^2}\int_0^{2\pi}\frac{1}{3}h^3\,du =$ $\rho\frac{2\pi rh\sqrt{h^2+r^2}}{3}$ so that $\overline{z} = \frac{M_{xy}}{m} = \rho\frac{2\pi rh\sqrt{h^2+r^2}}{3}\cdot\frac{1}{\rho\pi r\sqrt{h^2+r^2}} = \frac{2h}{3}$.

17.6.68 Assume the shell has density 1. Then the mass of the shell, which is half the area of the entire cylinder, is πah. Further, by symmetry, $\overline{x} = \overline{y} = 0$. Use the parameterization $\langle a\cos u, v, a\sin u\rangle$; then $|\mathbf{t}_u \times \mathbf{t}_v| = a$ and $M_{xy} = \iint\limits_S z\,dS = a\iint\limits_R z\,dA = a\int_{-h/2}^{h/2}\int_0^{\pi} a\sin u\,du\,dv = 2a^2h$ and $\overline{z} = \frac{2}{\pi}a$.

17.6.69 Using the standard parameterization, the mass of the shell is

$$m = \iint\limits_S (1+z)\;dS = a\iint\limits_R (1+z)\;dA = a\int_0^{2\pi}\int_0^2 (1+v)\;dv\,du = 8\pi a.$$

The density does not depend on either x or y, and the cylinder is symmetric about the z axis, so $\overline{x} = \overline{y} = 0$. Then

$$M_{xy} = \iint\limits_S z\,(1+z)\;dS = a\iint\limits_R z\,(1+z)\;dA = a\int_0^{2\pi}\int_0^2 v\;(1+v)\;dv\,du = \frac{28\pi a}{3}.$$

Then $\overline{z} = \frac{7}{6}$.

17.6.70 $\mathbf{t}_u = \langle a\cos u\cos v, a\cos u\sin v, -a\sin u\rangle$ and $\mathbf{t}_v = \langle -a\sin u\sin v, a\sin u\cos v, 0\rangle$, and then $\mathbf{t}_u \times \mathbf{t}_v =$ $\left\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\cos u\sin u\right\rangle$ so that $|\mathbf{t}_u \times \mathbf{t}_v| = a^2\left|\left\langle \sin^2 u\cos v, \sin^2 u\sin v, \cos u\sin u\right\rangle\right| =$ $a^2\sqrt{\sin^4 u\cos^2 v + \sin^4 u\sin^2 v + \cos^2 u\sin^2 u} = a^2\sqrt{\sin^4 u + \sin^2 u\cos^2 u} = a^2\sin u$.

17.6.71 The explicit formula $z = g(x,y)$ becomes, on regarding x and y as parameters, the parametric form $\langle x, y, g(x,y)\rangle$, and now $\mathbf{t}_x = \langle 1, 0, z_x\rangle$ and $\mathbf{t}_y = \langle 0, 1, z_y\rangle$. Then $\mathbf{t}_x \times \mathbf{t}_y = \langle -z_x, -z_y, 1\rangle$, so that $|\mathbf{t}_x \times \mathbf{t}_y| = \sqrt{z_x^2 + z_y^2 + 1}$. Now the formula $\iint\limits_S f(x,y,z)\;dS = \iint\limits_R f(x,y,g(x,y))\sqrt{z_x^2+z_y^2+1}\,dA$ follows from the definition of the surface integral for parameterized surfaces.

17.6.72

a. Each point on the graph of f on $[a,b]$, say $f(x)$ becomes, in the surface of revolution, a circle of radius $f(x)$ with center on the x-axis. Letting $u = x$, that circle is then parameterized by $\langle f(u)\cos v, f(u)\sin v\rangle$ for $0 \le v \le 2\pi$, so the entire surface is parameterized by
$\langle u, f(u)\cos v, f(u)\sin v\rangle$, $a \le u \le b$, $0 \le v \le 2\pi$.

b. $\mathbf{t}_u = \langle 1, f'(u)\cos v, f'(u)\sin v\rangle$ and $\mathbf{t}_v = \langle 0, -f(u)\sin v, f(u)\cos v\rangle$, and then

$$\mathbf{t}_u \times \mathbf{t}_v = \langle f'(u)f(u), -f(u)\cos v, -f(u)\sin v\rangle$$

and

$$|\mathbf{t}_u \times \mathbf{t}_v| = \sqrt{f'(u)^2 f(u)^2 + f(u)^2\left(\sin^2 v + \cos^2 v\right)} = f(u)\sqrt{f'(u)^2 + 1}.$$

We have

$$\iint_S f(x,y,z)\,dS = \iint_R f(u)\sqrt{f'(u)^2+1}\,dA = \int_a^b \int_0^{2\pi} f(u)\sqrt{f'(u)^2+1}\,dv\,du$$
$$= 2\pi\int_a^b f(u)\sqrt{f'(u)^2+1}\,du.$$

c. The area of the surface is $2\pi\displaystyle\int_1^2 x^3\sqrt{9x^4+1}\,dx = \frac{\pi}{27}\left(145^{3/2} - 10^{3/2}\right)$.

d. The area of the surface is

$$2\pi\int_3^4 \left(25-x^2\right)^{1/2}\sqrt{\left(-x\left(25-x^2\right)^{-1/2}\right)^2+1}dx = 2\pi\int_3^4\left(25-x^2\right)^{1/2}\sqrt{\frac{x^2}{25-x^2}+1}\,dx$$
$$= 2\pi\int_3^4\left(25-x^2\right)^{1/2}\sqrt{\frac{25}{25-x^2}}\,dx = 2\pi\int_3^4 5\,dx = 10\pi.$$

17.6.73 We have $z = s(x,y)$, so a normal vector is $\langle -z_x, -z_y, 1\rangle$. Since we are interested in the downward flux, we choose a downward-pointing normal, which is $\langle s_x(x,y), s_y(x,y), -1\rangle$. Then $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,dS =$ $\displaystyle\iint_R \langle 0,0,-1\rangle\cdot\langle \mathrm{s}_x(x,y), s_y(x,y), -1\rangle\,dA = \iint_R 1\,dA$, which is the area of R. Since the vector field is constant and pointed downwards vertically, everything that goes through the surface is matched by something going through R.

17.6.74

a. Imagine the torus as built by starting with a circle of radius R in the xy plane, centered at the origin. From each point on this circle, parameterize as $\langle R\cos v, R\sin v, 0\rangle$, we can reach a circle of points on the surface by making it the center of a circle of radius r, parameterized by u. This second circle is drawn in a vertical plane that includes the z-axis. Each point of this second circle is thus in the plane determined by $\langle R\cos v, R\sin v, 0\rangle$ and the z-axis; its z-coordinate will be $r\sin u$ and its x and y-coordinates will then be (from its center) $r\cos v\cos u$ and $r\sin v\cos u$. Thus the set of points on the torus can be parameterized by the sum of these vectors, which is $\langle R\cos v, R\sin v, 0\rangle +$ $\langle r\cos v\cos u, r\sin v\cos u, r\sin u\rangle = \langle (R+r\cos u)\cos v, (R+r\cos u)\sin v, r\sin u\rangle$

b. From the parameterization above, we have

$$\mathbf{t}_u = \langle -r\sin u\cos v, -r\sin u\sin v, r\cos u\rangle$$
$$\mathbf{t}_v = \langle -(R+r\cos u)\sin v, (R+r\cos u)\cos v, 0\rangle$$

so that

$$\mathbf{t}_u\times\mathbf{t}_v = (R+r\cos u)\langle -r\cos u\cos v, -r\cos u\sin v, -r\sin u\rangle$$

and $|\mathbf{t}_u\times\mathbf{t}_v| = (R+r\cos u)\sqrt{r^2\cos^2 u\cos^2 v + r^2\cos^2 u\sin^2 v + r^2\sin^2 u} = r(R+r\cos u)$, so that the area of the torus is $\displaystyle\iint_S 1\,dS = \int_0^{2\pi}\int_0^{2\pi} r(R+r\cos u)\,du\,dv = 4\pi^2 Rr$.

17.6.75 The goal is to start with the surface area formula $A = \iint\limits_R \sqrt{z_x^2 + z_y^2 + 1}\, dA$ of Theorem 17.12 (where we have set $f(x, y, g(x, y)) = 1$) and derive the surface area formula $A = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx$ of section 6.6. Because S is generated by revolving the graph of f about the x-axis, we can use symmetry and take $R = \{(x, y) : a \le x \le b, 0 \le y \le f(x)\}$. The resulting surface area is then one-quarter of the desired surface area. The key observation is that the surface generated is given by $z^2 = f(x)^2 - y^2$ over the region R. It follows that $2zz_x = 2ff'$, or $z_x = \frac{ff'}{2}$ and $2zz_y = -2y$, or $z_y = \frac{-y}{z}$. Substituting, we have that the surface area is $A = \iint\limits_R \sqrt{z_x^2 + z_y^2 + 1}\, dA = 4\int_a^b \int_0^{f(x)} \sqrt{\left(\frac{ff'}{z}\right)^2 + \left(\frac{-y}{z}\right)^2 + 1}\, dy\, dx =$ $4\int_a^b \int_0^{f(x)} \sqrt{\frac{f^2 f'2 + y^2 + z^2}{z^2}}\, dy\, dx = 4\int_a^b \int_0^{f(x)} \sqrt{\frac{f^2 f'^2 + f^2}{f^2 - y^2}}\, dy\, dx$, because $z^2 = f^2 - y^2$.

Continuing, we have

$$\begin{aligned} A &= 4\int_a^b \int_0^{f(x)} \frac{f(x)\sqrt{1 + f'(x)^2}}{\sqrt{f(x)^2 - y}}\, dy\, dx = 4\int_a^b f(x)\sqrt{1 + f'(x)^2} \int_0^{f(x)} \frac{1}{\sqrt{f(x)^2 - y}}\, dy\, dx \\ &= 4\int_a^b f(x)\sqrt{1 + f'(x)^2} \left(\sin^{-1} \frac{y}{f(x)}\Big|_0^{f(x)}\right) dx = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx. \end{aligned}$$

17.7 Stokes' Theorem

17.7.1 It measures the circulation of the vector field $\mathbf{F}$ along the closed curve C.

17.7.2 It measures the accumulated rotation of the vector field $\mathbf{F}$ over the surface S.

17.7.3 It says that the circulation of a vector field along a closed curve is equal to the net circulation of the field over a surface whose boundary is that curve, so that either can be calculated from the other.

17.7.4 This is the fundamental theorem of line integrals - the integral of any conservative vector field around a closed curve is zero.

17.7.5 The line integral is $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_R \langle \sin t, -\cos t, 10\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dA = \int_0^{2\pi} (-1)\ dt = -2\pi$. For the surface integral, use the standard parameterization of the sphere; then

$$\mathbf{n} = \langle \sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u\rangle$$

and $\nabla \times \mathbf{F} = \langle 0, 0, -2\rangle$ so that $\iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \int_0^{2\pi} \int_0^{\pi/2} (-2\cos u\ \sin u)\ du\, dv = -2\pi$.

17.7.6 The line integral is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_R \langle 0, -2\cos t, 2\sin t\rangle \cdot \langle -2\ \sin t, 2\cos t, 0\rangle\, dA = \int_0^{2\pi} \left(-4\ \cos^2 t\right)\ dt = -4\pi.$$

$\nabla \times \mathbf{F} = \langle 1, 0, -1\rangle$. The outward normal to the sphere is $\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$, so the surface integral is

$$\iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint\limits_R \langle 1, 0, -1\rangle \cdot \left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle\ dA = \int_0^{2\pi} \int_0^2 \left(\frac{r^2 \cos\theta}{\sqrt{4 - r^2}} - r\right) dr\, d\theta = -4\pi.$$

17.7.7 The line integral is

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle 2\sqrt{2}\cos t, 2\sqrt{2}\sin t, 0\rangle \cdot \left\langle -2\sqrt{2}\sin t, 2\sqrt{2}\cos t, 0\right\rangle\, dt = \int_0^{2\pi} 0\, dt = 0.$$

For the surface integral, we have $\nabla\times\mathbf{F} = \mathbf{0}$ so that the surface integral is also zero.

17.7.8 The boundary of the region is the intersection of the sphere with the plane $z = 12$, which has the equation $x^2+y^2 = 25$ and $z = 12$. Thus the line integral is $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle 24, -4\cdot 5\cos t, 3\cdot 5\sin t\rangle \cdot \langle -5\sin t, 5\,\cos t, 0\rangle\, dt = \int_0^{2\pi}\left(-120\sin t - 100\cos^2 t\right)\, dt = -100\pi$.

The surface sits over $x^2+y^2 = 25$ and the normal to the sphere is $\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$. $\nabla\times\mathbf{F} = \langle 3, 2, -4\rangle$, so the surface integral is $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS = \int_0^{2\pi}\int_0^5 r\left(\frac{3r\cos\theta}{\sqrt{169-r^2}} + \frac{2r\sin\theta}{\sqrt{169-r^2}} - 4\right)\, dr\, d\theta = -100\pi$.

17.7.9 The boundary of the region is the intersection of the sphere with the plane $z = \sqrt{7}$, which has the equation $x^2+y^2 = 9$ and $z = \sqrt{7}$. Then the line integral is

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle 3\sin t - \sqrt{7}, \sqrt{7} - 3\cos t, 3\cos t - 3\,\sin t\rangle \cdot \langle -3\sin t, 3\cos t, 0\rangle\, dt$$
$$= \int_0^{2\pi}\left(-9\sin^2 t + 3\sqrt{7}\sin t + 3\sqrt{7}\cos t - 9\cos^2 t\right)\, dt = \int_0^{2\pi}\left(-9 + 3\sqrt{7}\sin t + 3\sqrt{7}\,\cos t\right)\, dt = -18\pi.$$

The surface sits over $x^2+y^2 = 9$ and the normal to the sphere is $\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$. $\nabla\times\mathbf{F} = \langle -2, -2, -2\rangle$, so the surface integral is

$$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS = -2\int_0^{2\pi}\int_0^3 r\left(\frac{r\cos\theta}{\sqrt{16-r^2}} + \frac{r\sin\theta}{\sqrt{16-r^2}} + 1\right)\, dr\, d\theta = -18\pi.$$

17.7.10 The boundary of the region can be parameterized by $\langle 4\cos t, 4\sin t, 6 - 4\sin t\rangle$, so the line integral is

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle -4\sin\ \mathrm{t}, 4\sin t - 4\cos t - 6, 4\sin t - 4\cos t\rangle \cdot \langle -4\sin t, 4\cos t, -4\,\cos t\rangle\, dt$$
$$= \int_0^{2\pi}\left(16\sin^2 t + 16\sin t\,\cos t - 16\cos^2 t - 24\cos t - 16\sin t\,\cos t + 16\cos^2 t\right)\, dt$$
$$= \int_0^{2\pi}\left(16\sin^2 t - 24\cos t\right)\, dt = 16\pi.$$

$\nabla\times\mathbf{F} = \langle 2, 1, 0\rangle$, and an outward pointing normal to the plane is $\langle 0, 1, 1\rangle$, so the surface integral is $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint_R 1\, dA = \int_0^{2\pi}\int_0^4 r\, dr\, d\theta = 16\pi$.

17.7.11 $\nabla\times\mathbf{F} = \langle 1, -1, -2\rangle$; for S take the disk $x^2+y^2 \le 12$ with upward-oriented normal vector $\langle 0, 0, 1\rangle$. Then $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint_R (-2)\, dA = -24\pi$.

17.7.12 $\nabla\times\mathbf{F} = \langle -1-x, 0, z-1\rangle$; for S take the region $x^2 + \frac{y^2}{4} \le 1$ in the plane $z = 1$, with normal vector $\langle 0, 0, 1\rangle$; then $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint_R (z-1)\, dA = 0$ because $z = 1$ on the region of integration.

17.7.13 $\nabla \times \mathbf{F} = \langle 0, -4z, 0\rangle$. For S take the plane in the first octant, which sits over $0 \le x \le 4$, $0 \le y \le 4 - x$. The upward-pointing normal to this plane is $\langle 1, 1, 1\rangle$. Then $\iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint\limits_R (-4z)\ dA = \int_0^4 \int_0^{4-x} (-4)\,(4 - x - y)\ dy\, dx = -\dfrac{128}{3}$.

17.7.14 $\nabla \times \mathbf{F} = \langle 2y - 2z, 0, 2y - 2x\rangle$. Take S to be the square bounded by C, with normal $\langle 0, 0, 1\rangle$. Then $\iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint\limits_R (2y - 2x)\ dA = \int_{-1}^1 \int_{-1}^1 (2y - 2x)\ dy\, dx = 0$.

17.7.15 $\nabla \times \mathbf{F} = \langle 2z, -1, -2y\rangle$. Take S to be the disk $\langle 3r\cos t, 4r\cos t, 5r\sin t\rangle$ for $0 \le r \le 1$, $0 \le t \le 2\pi$. $\mathbf{t}_r \times \mathbf{t}_t = \langle 20r, -15r, 0\rangle$ is a normal vector. Then $\iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint\limits_R \langle 2z, -1, -2y\rangle \cdot \langle 20r, -15r, 0\rangle\, dA = \int_0^{2\pi} \int_0^1 ((10r\ \sin t) \cdot 20r + 15r)\ dr\, dt = 15\pi$.

17.7.16 $\nabla \times \mathbf{F} = \mathbf{0}$, so the surface integral is zero.

17.7.17 The boundary of the surface is the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$, found by setting $z = 0$. Parameterize the path by $\mathbf{r}(t) = \langle 2\cos t, 3\sin t, 0\rangle$; then $\oint \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}'(t)\ dt = \int_0^{2\pi} (-4\cos t\sin t + 9\cos t\ \sin t)\ dt = \int_0^{2\pi} 5\cos t\sin t\, dt = 0$.

17.7.18 The boundary of the surface is on the plane $x = 0$, and it is the circle $y^2 + z^2 = 9$. Parameterize the circle by $\mathbf{r}(t) = \langle 0, 3\cos t, 3\sin t\rangle$; then $\mathbf{r}'(t) = \langle 0, -3\sin t, 3\cos t\rangle$. $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|} = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}\langle x, y, z\rangle$, and $\oint \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}'(t)\ dt = \dfrac{1}{3}\int_0^{2\pi} (-9\sin t\cos t + 9\ \sin t\cos t)\ dt = 0$.

17.7.19 The boundary of the surface is the intersection of the plane $x = 3$ with the sphere $x^2 + y^2 + z^2 = 25$, so is the circle $y^2 + z^2 = 16$ at $x = 3$. Parametrize the circle with $x = 3$, $y = 4\cos t$ and $z = 4\sin t$. We have $\mathbf{r}'(t) = \langle 0, -4\sin t, 4\cos t\rangle$, so $\oint \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle 8\cos t, -4\sin t, 3 - 4\cos t - 4\sin t\rangle \cdot \langle 0, -4\sin t, 4\cos t\rangle\, dt = \int_0^{2\pi} \left(16\sin^2 t + 12\cos t - 16\cos^2 t - 16\sin t\cos t\right)\ dt = 0$.

17.7.20 The boundary of the surface is given in the problem: $\mathbf{r}(t) = \langle \cos t, 2\sin t, \sqrt{3}\cos t.\rangle$; so $\mathbf{r}'(t) = \langle -\sin t, 2\cos\ t, -\sqrt{3}\sin t\rangle$ and the integral is

$$\begin{aligned}\oint \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle \cos t + 2\sin t, 2\ \sin t + \sqrt{3}\cos t, \left(1 + \sqrt{3}\right)\cos t\rangle \cdot \langle -\sin t, 2\cos t, -\sqrt{3}\ \sin t\rangle\, dt \\ &= \int_0^{2\pi} \left(-\sin t\cos t - 2\ \sin^2 t + 4\sin t\cos t + 2\sqrt{3}\cos^2 t - \left(3 + \sqrt{3}\right)\cos t\sin t\right)\ dt \\ &= \int_0^{2\pi} \left(-\sqrt{3}\sin t\cos t - 2\ \sin^2 t + 2\sqrt{3}\cos^2 t\right)\ dt = 2\pi\left(\sqrt{3} - 1\right).\end{aligned}$$

17.7.21 C is given by $\mathbf{r}(t) = \left\langle \cos t, \sin t, \cos^2 t\right\rangle$. Then

$$\begin{aligned}\mathbf{F} \cdot \mathbf{r}'(t) &= \left\langle \sin t, \cos^2 t - \cos t, -\sin t\right\rangle \cdot \langle -\sin t, \cos t, -2\sin t\cos t\rangle \\ &= -\sin^2 t + \cos^3 t - \cos^2 t + 2\sin^2 t\cos t = -1 + \cos^3 t + 2\sin^2 t\cos t.\end{aligned}$$

We have $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-1 + \cos^3 t + 2\sin^2 t \cos t)\, dt = -2\pi + \int_0^{2\pi} \cos t(2\sin^2 t + (1 - \sin^2 t))\, dt = -2\pi + \int_0^{2\pi} \cos t(\sin^2 t + 1)\, dt$. Let $u = \sin t$ so that $du = \cos t\, dt$. Then we have

$$-2\pi + \int_0^0 (u^2 + 1)\, du = -2\pi.$$

17.7.22 C is given by $\mathbf{r}(t) = \left\langle \frac{1}{2}\cos t, \frac{1}{2}\sin t, \frac{1}{2}\cos^2 t + \frac{1}{4}\sin^2 t \right\rangle = \left\langle \frac{1}{2}\cos t, \frac{1}{2}\sin t, \frac{1}{4}\cos^2 t + \frac{1}{4} \right\rangle$. Then $\mathbf{F} \cdot \mathbf{r}'(t) = \left\langle 2\cos t, -2\cos^2 t - 2, 2\sin t \right\rangle \cdot \left\langle -\frac{1}{2}\sin t, \frac{1}{2}\cos t, -\frac{1}{2}\cos t \sin t \right\rangle = -\sin t \cos t - 2\cos t$. We have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left(-\sin t \cos t - 2\cos t\right)\, dt$$
$$= -\int_0^{2\pi} \cos t\left(\sin t + 2\right)\, dt.$$

A substitution of $u = \sin t + 2$ reveals that the value of this integral is 0.

17.7.23 C is given by $\mathbf{r}(t) = \left\langle 2 + 2\cos t, 2\sin t, 2\sqrt{2 + 2\cos t} \right\rangle$. Then

$$\mathbf{F} \cdot \mathbf{r}'(t) = \left\langle 2\sin t, 1, 2\sqrt{2 + 2\cos t} \right\rangle \cdot . \left\langle -2\sin t, 2\cos t, \frac{-2\sin t}{\sqrt{2 + 2\cos t}} \right\rangle$$
$$= -4\sin^2 t + 2\cos t - 4\sin t.$$

We have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-4\sin^2 t + 2\cos t - 4\sin t)\, dt = \int_0^{2\pi} (-4\sin^2 t)\, dt = -4\pi.$$

17.7.24 C is given by $\mathbf{r}(t) = \left\langle 2\cos t, \sin t, 4 - 3\sin^2 t \right\rangle$. Then

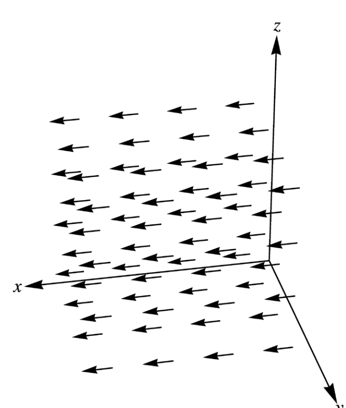
$$\mathbf{F} \cdot \mathbf{r}'(t) = \left\langle e^{2\cos t}, \frac{1}{4 - 3\sin^2 t}, \sin t \right\rangle \cdot \langle -2\sin t, \cos t, -6\sin t \cos t \rangle$$
$$= -2\sin t e^{2\cos t} + \frac{\cos t}{4 - 3\sin^2 t} - 6\sin^2 t \cos t.$$

We have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left(-2(\sin t)e^{2\cos t} + \frac{\cos t}{4 - 3\sin^2 t} - 6\sin^2 t \cos t\right)\, dt$$
$$= \int_0^{2\pi} -2\sin t e^{2\cos t}\, dt + \int_0^{2\pi} \frac{\cos t}{4 - 3\sin^2 t}\, dt - 6\int_0^{2\pi} \sin^2 t \cos t\, dt = 0 + 0 - 0 = 0,$$

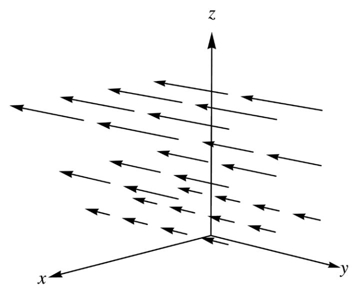
where the integrals are all evaluated either via the substitution $u = \sin t$ or $u = \cos t$.

17.7.25 $\boldsymbol{\nabla} \times \mathbf{v} = \langle 1, 0, 0 \rangle$. The curl looks like:



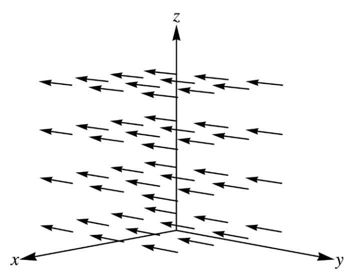
This means that the maximum rotation of the field is in the direction$\langle 1,0,0\rangle$. The rotation is counterclockwise looking in the negative x direction.

17.7.26 $\nabla \times \mathbf{v} = \langle 0, -2z, 0\rangle$. The curl looks like:



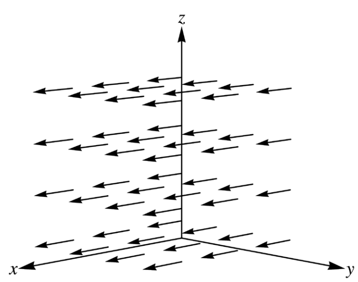
This means that the maximum rotation of the field is in the direction of the$y$-axis, and the amount of rotation, clockwise or counterclockwise, increases the further from the z-axis one gets. It rotates counterclockwise (viewed from the positive z-axis) for $z < 0$, and clockwise for $z > 0$.

17.7.27 $\nabla \times \mathbf{v} = \langle 0, -2, 0\rangle$. The curl looks like:



This means that the maximum rotation of the field is in the direction of the y-axis. It is constant at all points, and is clockwise viewed from the positive y-axis.

17.7.28 $\nabla \times \mathbf{v} = \langle 2, 0, 0\rangle$. The curl looks like:



The maximum rotation of the vector field is in the direction of the x-axis; it is constant at all points, and is counterclockwise viewed from the positive x-axis.

17.7.29

a. False. This is a rotation field with axis of rotation $\langle 1,1,2\rangle$, but $\langle 0,1,-1\rangle \cdot \langle 1,1,2\rangle \neq 0$, so the paddle wheel axis is not perpendicular to the axis of rotation.

b. False. It relates the curl of $\mathbf{F}$, not its flux.

c. True. This is because it is conservative: it is the gradient of $ax + F(x) + by + G(y) + cz + H(z)$, where F, G, H are the antiderivatives of f, g, h, respectively.

d. True. See Theorem 14.14.

17.7.30 This is a conservative vector field, with $\varphi = x^2 - y^2 + z^2$, so the integral $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C.

17.7.31 This is a conservative vector field, so the integral around any closed curve is zero.

17.7.32 This is a conservative vector field with $\varphi = x^3y + y^2z^2$, so the integral around any closed curve is zero.

17.7.33 This is a conservative vector field with $\varphi = xy^2z^3$, so the integral around any closed curve is zero.

17.7.34 The surface S is $\langle r\cos\varphi\cos t, r\sin t, r\sin\varphi\cos t\rangle$; computing the normal vector gives $\mathbf{t}_r \times \mathbf{t}_t = \langle -r\sin\varphi, 0, r\cos\varphi\rangle$. Thus the surface area is $\iint\limits_S 1\,dS = \int_0^1 \int_0^{2\pi} |\mathbf{t}_r \times \mathbf{t}_t|\,dt\,dr = \int_0^1 \int_0^{2\pi} r\,dt\,dr = \pi$. This makes sense because the surface is simply the unit circle inclined at the angle φ to the xy-plane.

17.7.35 $\mathbf{r}'(t) = \langle -\cos\varphi\sin t, \cos t, -\sin\varphi\sin t\rangle$, so that $|\mathbf{r}'(t)| = 1$. Then the length of C is $\int_C 1\,ds = \int_0^{2\pi} 1\,dt = 2\pi$, again as expected because C is just an inclined unit circle.

17.7.36 By Stokes' theorem, $\oint \mathbf{F} \cdot d\mathbf{r} = \iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS$, $\nabla \times \mathbf{F} = \langle 0, 0, 2\rangle$; using the normal $\mathbf{t}_r \times \mathbf{t}_t = \langle -r\sin\varphi, 0, r\cos\varphi\rangle$ we have $\iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS = \int_0^1 \int_0^{2\pi} 2r\cos\varphi\,dt\,dr = 2\pi\cos\varphi$. This is maximal for $\varphi = 0$, when it is 2π.

17.7.37 We have

$$\begin{aligned}\oint \mathbf{F} \cdot d\mathbf{r} &= \int_C \langle -y, -z, x\rangle \cdot \langle -\cos\varphi\sin t, \cos t, -\sin\varphi\ \sin t\rangle\,dt \\ &= \int_0^{2\pi} \left(\cos\varphi\sin^2 t - \sin\varphi\cos^2 t + \cos\varphi\sin\varphi\ \cos t\sin t\right)\,dt = \pi\left(\cos\varphi - \sin\varphi\right).\end{aligned}$$

This is maximal for $\varphi = 0$, when it is π.

17.7.38 $\nabla \times (\mathbf{a} \times \mathbf{r}) = \langle 2a_1, 2a_2, 2a_3\rangle$. Thus

$$\begin{aligned}\oint \mathbf{F} \cdot d\mathbf{r} &= \iint\limits_S \langle 2a_1, 2a_2, 2a_3\rangle \cdot \langle -r\sin\varphi, 0, r\cos\varphi\rangle\,dS \\ &= 2\int_0^{2\pi} \int_0^1 \left(-a_1 r\sin\varphi + a_3 r\cos\varphi\right)\,dr\,dt = 2\pi\left(a_3\cos\varphi - a_1\sin\varphi\right).\end{aligned}$$

This is a maximum when its derivative vanishes, i.e. when $a_3\cos\varphi - a_1\sin\varphi = 0$. Now, $\langle a_1, a_2, a_3\rangle$ points in the direction of the normal if their cross-product is zero, i.e. if $\langle a_1, a_2, a_3\rangle \times \langle -r\sin\varphi, 0, r\cos\varphi\rangle = \langle a_2\,r\cos\varphi, -a_3\,r\sin\varphi - a_1\,r\cos\varphi, a_2, r\sin\varphi\rangle = 0$. This happens when $a_2 = 0$ and $a_3\sin\varphi + a_1\cos\varphi = 0$.

17.7.39 $\nabla \times \mathbf{F} = \langle 3, 0, 0\rangle$. To evaluate the circulation around C, we instead (using Stokes' theorem) evaluate $\iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS$ for the surface of the disk of which C is a boundary. Note that $\mathbf{n} = \langle 1, 1, 1\rangle$, so that $\iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS = \iint\limits_R 3\,dA = 3 \cdot \text{area of } A = 48\pi$. From this calculation, it is clear that the result depended on the radius of the circle, because that affects the area of A, but not on the center of the circle.

17.7.40

a. $\nabla \times \mathbf{F} = \langle 1,1,0 \rangle$, so the integrand of the surface integral is just $\dfrac{x+y}{z}$; by symmetry, the integral vanishes on each level curve, so it vanishes altogether.

b. On the boundary of S, we have $z = 0$, so that $\mathbf{F} = \langle 0,0,2y+x \rangle$, and thus $\mathbf{r}'(t) = \langle -\sin t, \cos t, 0 \rangle$ so that the dot product and thus the line integral is zero.

17.7.41

a. The normal vectors point toward the z-axis on the curved surface of S and in the direction of $\langle 0,1,0 \rangle$ on the flat surface of S.

b. To evaluate the integral, we must add up the integrals on each of the surfaces. $\nabla \times \mathbf{F} = \langle 1,1,1 \rangle$. Let S_1 be the surface in the xz-plane, parameterized by $\langle x,0,z \rangle$ for $-2 \le x \le 2$,$x^2 \le z \le 4$; then the normal to S_1 is $\mathbf{t}_x \times \mathbf{t}_z = \langle 0,1,0 \rangle$, so that the integral over S_1 is

$$\iint_{S_1} \langle 1,1,1 \rangle \cdot \langle 0,-1,0 \rangle \, dS = \int_{-2}^{2} \int_{x^2}^{4} (-1) \; dy \, dx = -\int_{-2}^{2} \left(4 - x^2\right) \; dx = -\frac{32}{3}.$$

S_2 is the half of the paraboloid for $y \ge 0$, parameterized as $\langle r\cos u, r\sin u, r^2 \rangle$, $0 \le r \le 2, -\pi \le u \le 0$. The normal to S_2 is $\mathbf{t}_r \times \mathbf{t}_u = \langle -2r^2 \cos u, -2r^2 \, \sin u, r \rangle$. The integral over S_2 is

$$\iint_{S_1} \langle 1,1,1 \rangle \cdot \langle -2r^2 \cos u, -2r^2 \, \sin u, r \rangle \, dS = \int_0^2 \int_{-\pi}^{0} \left(-2r^2 \left(\cos u + \sin u\right) + r\right) \, du \, dr = \frac{32}{3} + 2\pi.$$

Thus the total is 2π.

c. The line integral is the sum of two line integrals:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}_2$$

where $C_1 = \langle t,0,4 \rangle$ for $-2 \le t \le 2$ and$C_2 = \langle 2\cos t, 2\sin t, 4 \rangle$ for $-\pi \le t \le 0$. Then

$$\begin{aligned}
\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 &= \int_{-2}^{2} \langle 2z+y, 2x+z, 2y+x \rangle \cdot \langle 1,0,0 \rangle \, dt = \int_{2}^{-2} 8 \, dt = -32 \\
\oint_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 &= \int_{-\pi}^{0} \langle 2z+y, 2x+z, 2y+x \rangle \cdot \langle -2\sin t, 2\,\cos t, 0 \rangle \, dt \\
&= \int_{-\pi}^{0} \left(-16\sin t - 4\sin^2 t + 8\,\cos t + 8\cos^2 t\right) \, dt = 32 + 2\pi,
\end{aligned}$$

so the total line integral is 2π.

17.7.42 We have, from Stokes' theorem and Ampere's Law, $\iint_S (\nabla \times \mathbf{B}) \cdot \mathbf{n} \, dS = \oint_C \mathbf{B} \cdot d\mathbf{r} = \mu I = \mu \iint_S \mathbf{J} \cdot \mathbf{n} \, dS$ Thus we have

$$\iint_S ((\nabla \times \mathbf{B}) - \mu \mathbf{J}) \cdot \mathbf{n} \, dS = 0$$

for all surfaces S bounded by any given closed curve C. It is clear that given the freedom to choose C and S, that it follows that the integrand is identically zero, i.e. that for any surface S,$((\nabla \times \mathbf{B}) - \mu \mathbf{J}) \cdot$n$= 0$. From this it is easy to see that we must have $\nabla \times \mathbf{B} = \mu \mathbf{J}$, since we are free to make the normal vector point in any direction at any given point by choosing S appropriately.

17.7.43 The boundary of the region is the circle C: $x^2+y^2=1$ for$z=0$. With the usual parameterization, we have

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle \cos t - \sin t, \sin t, -\cos t\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt$$
$$= \int_0^{2\pi} \left(-\sin t\cos t + \sin^2 t + \sin t\cos t\right)\, dt = \int_0^{2\pi} \sin^2 t\, dt = \pi.$$

So the integral is independent of a.

17.7.44

a. $\boldsymbol{\nabla}\times\mathbf{F} = \left\langle \frac{\partial}{\partial y}(ay) - \frac{\partial}{\partial z}(cx), \frac{\partial}{\partial z}(bz) - \frac{\partial}{\partial x}(ay), \frac{\partial}{\partial x}(cx) - \frac{\partial}{\partial y}(bz)\right\rangle = \langle a,b,c\rangle.$

b. The area of R is $\iint\limits_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint\limits_S \mathbf{n}\cdot\mathbf{n}\, dS = \iint\limits_R |\mathbf{n}|^2\, dA =$ area of R because $|\mathbf{n}|=1$, so that $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint\limits_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\, dS =$ area of R.

c. $\mathbf{r}'(t) = \langle 5\cos t, -13\sin t, 12\cos t\rangle$, so that $\mathbf{r}(t)\times\mathbf{r}'(t) = \langle 156, 0, -65\rangle$ and thus, because $\mathbf{r}(t)\times\mathbf{r}'(t)$ is constant, it points in a constant direction, so that $\mathbf{r}$ must lie in a plane.

d. By parts (b) and (c), we have a normal vector $\langle 156,0,-65\rangle$; its magnitude is 169, so we take $\mathbf{F} = \left\langle 0, -\frac{5x}{13}, \frac{12y}{13}\right\rangle$; then the area of R is

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \left\langle 0, -\frac{25}{13}\sin t, 12\cos t\right\rangle \cdot \langle 5\cos t, -13\sin t, 12\cos t\rangle\ dt$$
$$= \int_0^{2\pi} \left(25\sin^2 t + 144\cos^2 t\right)\, dt = 169\pi.$$

17.7.45

a. The boundary of this surface is the circle $x^2+y^2=1$ at $z=0$, so we choose instead the surface of the disk bounded by that circle. $\boldsymbol{\nabla}\times\mathbf{F} = \langle 2x, 0, -2z\rangle$, which is $\langle 2x,0,0\rangle$ at $z=0$, and the normal to the disk is $\langle 0,0,1\rangle$. Thus, the integral is equal to $\iint\limits_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint\limits_S 0\, dS = 0.$

b. With the usual parameterization of the boundary circle (and remembering that $z=0$), we have $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle 0,0,1\rangle\cdot\langle -\sin t, \cos t, 0\rangle\, dt = 0.$

17.7.46

a. Let C be the circle $x^2+y^2=a^2$. Parameterize the circle in the usual way; then $\mathbf{F} = \frac{1}{a^p}\langle x,y,0\rangle$ and $\mathbf{r}(t) = \langle a\cos t, a\sin t, 0\rangle$. Then $\oint_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{a^p}\int_0^{2\pi} \left(-a^2\sin t\cos t + a^2\sin t\cos t\right)\, dt = 0.$

b. Stokes' Theorem will apply when the vector field is defined throughout the disk of radius a, which happens only when $p\le 0$. In that case, $\boldsymbol{\nabla}\times\mathbf{F} = a^{-p}\langle 0,0,0\rangle$, so that the surface integral is zero.

17.7.47

a. $\boldsymbol{\nabla}\times\mathbf{F} = \left\langle 0-0, 0-0, \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2}\right\rangle = \mathbf{0}.$

b. Let C be the unit circle with the usual parameterization; then $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle -\sin t, \cos t, 0\rangle \cdot \langle -\sin t, \cos t, 0\rangle \, dt = 2\pi$.

c. The theorem does not apply because the vector field is not defined at the origin, which is inside the curve C. For example, the limit of the y-coordinate is different depending on the direction.

17.7.48

a. The circumference of the disk is $2\pi R$, so the average circulation is $\dfrac{1}{2\pi R} \iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS$

b. As R becomes small, because $\mathbf{F}$ and thus $\nabla \times \mathbf{F}$ are continuous, $\nabla \times \mathbf{F}$ can be made arbitrarily close to$(\nabla \times \mathbf{F})_P$ everywhere on S by taking R small enough. Approximately, then, $(\nabla \times \mathbf{F}) \cdot \mathbf{n} \approx (\nabla \times \mathbf{F})_P \cdot$n, so that

$$\frac{1}{2\pi R} \iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS \approx \frac{1}{2\pi R} \iint\limits_S (\nabla \times \mathbf{F})_P \cdot \mathbf{n} \, dS = (\nabla \times \mathbf{F})_P \cdot \mathbf{n} \frac{1}{2\pi R} \iint\limits_S 1 \, dS = (\nabla \times \mathbf{F})_P \cdot \mathbf{n}.$$

As R becomes smaller, the goodness of the approximation of $\nabla \times \mathbf{F}$ becomes better, so the value of the integral does as well.

17.7.49 By the chain rule, $\dfrac{df}{dy} = \dfrac{\partial f}{\partial y} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial y}$ and similarly for g, h, so

$$\begin{aligned} M_y &= \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + h z_{xy} + z_x \left(\frac{\partial h}{\partial y} + \frac{\partial h}{\partial z}\frac{\partial z}{\partial y}\right) = f_y + f_z z_y + h z_{xy} + z_x \left(h_y + h_z z_y\right) \\ N_x &= \frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} + h z_{yx} + z_y \left(\frac{\partial h}{\partial x} + \frac{\partial h}{\partial z}\frac{\partial z}{\partial x}\right) = g_x + g_z z_x + h z_{yx} + z_y \left(h_x + h_z z_x\right). \end{aligned}$$

17.7.50 One argument is quite simple: a closed surface has a closed (empty!) boundary, so the integral of $\mathbf{F}$ over that boundary is zero. Alternatively, choose any closed curve C dividing the surface into two pieces. On one half, the outward-pointing normals give a counterclockwise orientation to the boundary (viewed from above); on the other half, they give a clockwise orientation. Thus the integral $\iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} + \oint_{-C} \mathbf{F} \cdot d\mathbf{r}$, which is zero.

17.7.51 Let $\mathbf{F} = \langle 0, g(y,z), h(y,z)\rangle$ be a vector field in the yz-plane; for a region R in that plane, with boundary C, the normal is $\langle 1,0,0\rangle$. Now, $\nabla \times \mathbf{F} = \left\langle \dfrac{\partial h}{\partial y} - \dfrac{\partial g}{\partial z}, 0, 0 \right\rangle$, so by Stokes' theorem, $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \iint\limits_R \left(\dfrac{\partial h}{\partial y} - \dfrac{\partial g}{\partial z}\right) dA$.

17.8 Divergence Theorem

17.8.1 The surface integral measures the flow across the boundary.

17.8.2 The volume integral measures the net expansion or contraction of the vector field in the region.

17.8.3 The Divergence Theorem says that the flow across the boundary equals the net expansion or contraction of the field within the solid, so that either can be computed from the other.

17.8.4 Since $\nabla \cdot \langle 2z + y, -x, -2x\rangle = 0$, the net flux is zero.

17.8.5 Since $\nabla \cdot \langle x, y, z\rangle = 3$, the Divergence theorem says that the net outward flux is equal to

$$\iiint\limits_D 3\,dV = 3 \cdot \text{volume of } S = 32\pi.$$

17.8.6 From Example 4 (or Exercise 71 in section 14.5), the divergence is zero.

17.8.7 The outward fluxes must be equal, since by the Divergence theorem the net flux, which is the difference of the two, is zero.

17.8.8 Outward, since it is equal to the integral of $\operatorname{div}\mathbf{F}$ over the cube.

17.8.9 For the volume integral, $\boldsymbol{\nabla}\cdot\mathbf{F} = 9$, so that $\displaystyle\iiint\limits_D \boldsymbol{\nabla}\cdot\mathbf{F}\,dV = \iiint\limits_D 9\,dV = 9\cdot\frac{4}{3}2^3\pi = 96\pi.$

For the surface integral, with the usual parameterization of the sphere,

$$\begin{aligned}\mathbf{F}\cdot \mathrm{n} &= \langle 2a\sin u\cos v, 3a\sin u\sin v, 4a\,\cos u\rangle\cdot\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\cos u\sin u\rangle\\ &= 2a^3\sin^3 u\cos^2 v + 3a^3\sin^3 u\,\sin^2 v + 4a^3\cos^2 u\sin u\end{aligned}$$

and here $a = 2$, so that

$$\begin{aligned}\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS &= 8\iint\limits_R \left(2\sin^3 u\cos^2 v + 3\,\sin^3 u\sin^2 v + 4\cos^2 u\sin u\right)\,dA\\ &= 8\int_0^{2\pi}\int_0^{\pi}\left(2\sin^3 u\,\cos^2 v + 3\sin^3 u\sin^2 v + 4\cos^2 u\sin u\right)\,du\,dv = 96\pi.\end{aligned}$$

17.8.10 For the volume integral, $\boldsymbol{\nabla}\cdot\mathbf{F} = -3$, so that $\displaystyle\iiint\limits_D (-3)\;dV = -3\cdot\text{volume of} D = -24$

For the surface integral, we have six surfaces:

$$\begin{array}{ll} S_1 : x = -1 & \mathbf{n} = \langle -1, 0, 0\rangle\\ S_2 : x = 1 & \mathbf{n} = \langle 1, 0, 0\rangle\\ S_3 : y = -1 & \mathbf{n} = \langle 0, -1, 0\rangle\\ S_4 : y = 1 & \mathbf{n} = \langle 0, 1, 0\rangle\\ S_5 : z = -1 & \mathbf{n} = \langle 0, 0, -1\rangle\\ S_6 : z = 1 & \mathbf{n} = \langle 0, 0, 1\rangle \end{array}$$

and on each of those surfaces a simple computation shows that we have $\mathbf{F}\cdot\mathbf{n} = -1$. Thus

$$\sum_{i=1}^{6}\iiint\limits_{S_i}\mathbf{F}\cdot\mathbf{n}\,dS_i = \sum_{i=1}^{6}(-1\cdot\text{area of } S_i) = -24.$$

17.8.11 For the volume integral, $\boldsymbol{\nabla}\cdot\mathbf{F} = 0$, so the volume integral is zero. For the surface integral, the boundary ellipsoid can be parameterized by $\langle 2\sin u\cos v, 2\sqrt{2}\sin u\sin v, 2\sqrt{3}\cos u\rangle$, and $\mathbf{n} = \mathbf{t}_u\times\mathbf{t}_v = \langle 4\sqrt{6}\sin^2 u\cos v, 4\sqrt{3}\sin^2 u\sin v, 4\sqrt{2}\sin u\,\cos u\rangle$ so that

$$\begin{aligned}\mathbf{F}\cdot\mathbf{n} = \langle 2\sqrt{3}\cos u - 2\sqrt{2}\sin u\sin v, 2\sin u\cos v, -2\sin u\cos v\rangle\cdot&\\ \langle 4\sqrt{6}\sin^2 u\cos v, 4\sqrt{3}\sin^2 u\sin v, 4\sqrt{2}\sin u\cos u\rangle &= -8\sin^2 u\cos v\left(-2\sqrt{2}\cos u + \sqrt{3}\sin u\sin v\right)\end{aligned}$$

and then $\displaystyle\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^{\pi}\left(-8\sin^2 u\cos v\,\left(-2\sqrt{2}\,\cos u + \sqrt{3}\sin u\sin v\right)\right)\,du\,dv = 0.$

17.8.12 For the volume integral, $\nabla \cdot \mathbf{F} = 2(x+y+z)$, so that

$$\iiint\limits_D \nabla \cdot \mathbf{F}\, dV = 2\int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3} (x+y+z)\ dz\, dy\, dx = 0.$$

For the surface integral, we have six surfaces:

$$\begin{array}{ll} S_1 : x = -1 & \mathbf{n} = \langle -1, 0, 0\rangle \\ S_2 : x = 1 & \mathbf{n} = \langle 1, 0, 0\rangle \\ S_3 : y = -2 & \mathbf{n} = \langle 0, -1, 0\rangle \\ S_4 : y = 2 & \mathbf{n} = \langle 0, 1, 0\rangle \\ S_5 : z = -3 & \mathbf{n} = \langle 0, 0, -1\rangle \\ S_6 : z = 3 & \mathbf{n} = \langle 0, 0, 1\rangle \end{array}$$

A short computation shows that for S_1: $\mathbf{F}\cdot\mathbf{n} = -1$, for S_2: $\mathbf{F}\cdot\mathbf{n} = 1$, for S_3: $\mathbf{F}\cdot\mathbf{n} = -4$, for S_4: $\mathbf{F}\cdot\mathbf{n} = 4$, for S_5: $\mathbf{F}\cdot\mathbf{n} = -9$, and for S_6: $\mathbf{F}\cdot\mathbf{n} = 9$. Thus, the surface integral is zero.

17.8.13 $\nabla \cdot \mathbf{F} = 0$, so by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla\cdot$F is zero.

17.8.14 $\nabla \cdot \mathbf{F} = 0$, so by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla\cdot$F is zero.

17.8.15 $\nabla \cdot \mathbf{F} = 0$, so by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla\cdot$F is zero.

17.8.16 If $\mathbf{a} = \langle a, b, c\rangle$, then $\mathbf{a}\times\mathbf{r}$ is the field $\mathbf{F}$ in Exercise 15, so the net outward flux is zero.

17.8.17 By the Divergence theorem, we can compute the integral of $\nabla\cdot$F over the ball of radius $\sqrt{6}\cdot \nabla\cdot\mathbf{F} = 2$, so the volume integral is

$$\iiint\limits_D \nabla\cdot\mathbf{F}\, dV = 2\cdot \text{volume of sphere of radius } \sqrt{6} = 2\cdot\frac{4}{3}\pi\cdot 6\sqrt{6} = 16\pi\sqrt{6}.$$

17.8.18 $\nabla \cdot \mathbf{F} = 2x$, so by the Divergence theorem, the outward flux is

$$\iiint\limits_D 2x\, dV = \int_0^1\int_0^1\int_0^1 2x\, dx\, dy\, dz = 1.$$

17.8.19 $\nabla \cdot \mathbf{F} = 4$, so by the Divergence theorem, the outward flux is 4 times the volume of the tetrahedron, which is (by the formula for the volume of a pyramid), $\frac{1}{3}$ times the area of the base times the height, or $\frac{1}{6}$. So the outward flux is $\frac{2}{3}$.

17.8.20 $\nabla \cdot \mathbf{F} = 2(x+y+z)$, so by the Divergence theorem, the outward flux is

$$\iiint\limits_D 2(x+y+z)\ dV = 2\int_0^5\int_0^{2\pi}\int_0^{\pi} r\left(5\sin u\cos v + 5\sin u\ \sin v + 5\cos u\right)\ du\, dv\, dr = 0.$$

17.8.21 $\nabla \cdot \mathbf{F} = -4$, so by the Divergence theorem, the outward flux is -4 times the volume of the sphere, so is $-\dfrac{128}{3}\pi$.

17.8.22 $\nabla \cdot \mathbf{F} = 0$, so the outward flux is zero by the Divergence theorem.

17.8.23 $\nabla \cdot \mathbf{F} = 3$, so the outward flux is 3 times the volume of the paraboloid, which is

$$\int_0^2 \int_0^{2\pi} r\left(4 - r^2\right) \, d\theta \, dr = 8\pi,$$

so the outward flux is 24π.

17.8.24 $\nabla \cdot \mathbf{F} = 3$, so the outward flux is 3 times the volume of the cone. The area of the base of the cone is 16π, so the outward flux is $3 \cdot \frac{1}{3} \cdot 16\pi \cdot 4 = 64\pi$.

17.8.25 $\nabla \cdot \mathbf{F} = -3$, so the outward flux across the boundary of D is the outward flux across the sphere of radius 4 minus that across the sphere of radius 3, which is $-3 \cdot \frac{4}{3}\pi\left(4^3 - 2^3\right) = -224\pi$.

17.8.26 $\nabla \cdot \mathbf{F} = 4\,|\mathbf{r}|$, so the outward flux across a sphere of radius r is

$$\iiint\limits_D 4\sqrt{x^2 + y^2 + z^2} \, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{r} 4\rho\, \rho^2 \sin u \, d\rho \, du \, dv = 4\pi r^4.$$

Thus the net outward flux across the boundary of the given region is 60π.

17.8.27 $\nabla \cdot \mathbf{F} = \frac{2}{|\mathbf{r}|}$, so the outward flux across a sphere of radius r is

$$\iiint\limits_D \frac{2}{\sqrt{x^2 + y^2 + z^2}} \, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{r} \frac{2}{\rho} \rho^2 \sin u \, d\rho \, du \, dv = 4\pi r^2.$$

Thus the net outward flux across the boundary of the given region is 12π.

17.8.28 $\nabla \cdot \mathbf{F} = 0$, so the net outward flux is zero.

17.8.29 $\nabla \cdot \mathbf{F} = 2\,(x - y + z)$. The net outward flux is thus the difference in the outward flux across the two planes, so is

$$\begin{aligned}
&\iiint\limits_D 2\,(x - y + z) \; dV \\
&= 2\left(\int_0^4 \int_0^{4-x} \int_0^{4-x-y} (x - y + z) \; dz \, dy \, dx - \int_0^2 \int_0^{2-x} \int_0^{2-x-y} (x - y + z) \; dz \, dy \, dx\right) \\
&= 20.
\end{aligned}$$

17.8.30 $\nabla \cdot \mathbf{F} = 6$, so the net outward flux is 6 times the difference in the volumes of the cylinders, so is $6 \cdot (4\pi - \pi) \cdot 8 = 144\pi$.

17.8.31

a. False. For example, $\mathbf{F} = \langle y, 0, 0\rangle$ has $\nabla \cdot \mathbf{F} = 0$ at all points of the unit sphere, but the normal to the unit sphere, $\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$, is not perpendicular to $\mathbf{F}$ at all points.

b. False. For example, any rotation field has $\nabla \cdot \mathbf{F} = 0$, so that $\iint\limits_S \mathbf{F} \cdot \mathbf{n} \, dS = 0$ by the Divergence theorem, but $\mathbf{F}$ is not in general constant.

c. True. This is because it is bounded by $\iiint\limits_D 1 \, dV$.

17.8.32 $\nabla \cdot \mathbf{F} = 3$; we compute the outward flux from the Divergence theorem as (where S_1 is the upper hemisphere of radius a.)

$$\iiint\limits_{S_1} \nabla \cdot \mathbf{F}\, dV = 3 \iiint\limits_{S_1} 1\, dV = 3 \int_0^{2\pi} \int_0^{\pi/2} \int_0^a r^2 \sin u\, dr\, du\, dv = 2\pi a^3.$$

The the outward flux over the whole sphere is thus $4\pi a^3$.

17.8.33 Because $\nabla \cdot \mathbf{F} = 0$, the outward flux is zero from the Divergence theorem.

17.8.34 Because $\nabla \cdot \mathbf{F} = 0$, the outward flux is zero from the Divergence theorem.

17.8.35 $\nabla \cdot \mathbf{F} = 3 \sin y$, so the outward flux is

$$\iiint\limits_{S} 3 \sin y\, dV = \int_0^{\pi/2} \int_0^1 \int_0^x 3 \sin y\, dz\, dx\, dy = \frac{3}{2}.$$

17.8.36

a. We have $\mathbf{F} \cdot \mathbf{n} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p} \cdot \dfrac{\mathbf{r}}{|\mathbf{r}|} = \dfrac{|\mathbf{r}|^2}{|\mathbf{r}|^{p+1}} = |\mathbf{r}|^{1-p}$. Thus the surface integral is $\displaystyle\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = a^{1-p} \cdot$ area of sphere $= a^{1-p} \cdot 4\pi a^2 = 4\pi a^{3-p}$.

b. The conditions of the Divergence Theorem require that $\mathbf{F}$ be defined and have continuous partials everywhere inside the sphere; in particular, this must hold at the origin. Thus we must have $p \le -2$. Then the volume integral is

$$\iiint\limits_S \frac{3-p}{|\mathbf{r}|^p}\, dV = (3-p) \int_0^a \int_0^{2\pi} \int_0^{\pi} r^{-p} \cdot r^2 \sin u\, du\, dv\, dr = 4\pi a^{3-p}.$$

17.8.37

a. Either use Exercise 36(a), or compute $\mathbf{F} \cdot \mathbf{n} = \dfrac{\mathbf{r}}{|\mathbf{r}|} \cdot \dfrac{\mathbf{r}}{|\mathbf{r}|} = 1$, so the surface integral is

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \text{area of sphere} = 4\pi a^2.$$

b. $\nabla \cdot \mathbf{F} = 2\, |\mathbf{r}|^{-1}$, so if D is the shell between the spheres of radius ϵ and a, the volume integral is

$$\iiint\limits_S 2\, |\mathbf{r}|^{-1}\, dV = 2 \int_\epsilon^a \int_0^{2\pi} \int_0^{\pi} r \sin u\, du\, dv\, dr = 4\pi \left(a^2 - \epsilon^2\right)$$

and $\lim_{\epsilon \to 0} 4\pi \left(a^2 - \epsilon^2\right) = 4\pi a^2$.

17.8.38

a. $\dfrac{\partial}{\partial x} \varphi = \dfrac{x}{x^2 + y^2 + z^2}$, so that $\nabla \varphi = \dfrac{1}{|\mathbf{r}|^2} \langle x, y, z \rangle = \dfrac{\mathbf{r}}{|\mathbf{r}|^2}$.

b. $\mathbf{n} = \dfrac{\mathbf{r}}{|\mathbf{r}|^2}$, so that $\mathbf{F} \cdot \mathbf{n} = \dfrac{1}{|\mathbf{r}|^2}$. Then $\displaystyle\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \frac{1}{a^2} \int_0^{2\pi} \int_0^{\pi} a^3 \sin u\, du\, dv = 4\pi a$.

c. By Exercise 36, $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{1}{|\mathbf{r}|^2}$.

d. If D is the shell between the spheres of radius ϵ and$a$, the volume integral is

$$\iiint_D |\mathbf{r}|^{-2}\, dV = 2\int_\epsilon^a \int_0^{2\pi}\int_0^\pi \frac{1}{r^2}\, r\sin u\, du\, dv\, dr = 4\pi\,(a-\epsilon)$$

and $\lim\limits_{\epsilon\to 0} 4\pi\,(a-\epsilon) = 4\pi a$.

17.8.39

a. By Exercise 36, the flux of $\mathbf{E}$ across a sphere of radius a is $\dfrac{Q}{4\pi\epsilon_0}4\pi a^{3-3} = \dfrac{Q}{\epsilon_0}$.

b. The net outward flux across S is the difference of the fluxes across the inner and outer spheres; but by part (a), these are equal, so the net flux across S is zero.

c. The left-hand side is the flux across the boundary of D, while the right-hand side is the sum of the charge densities at each point of D. The statement says that the flux across the boundary, up to multiplication by a constant, is the sum of the charge densities in the region.

d. By the Divergence theorem, and using part (c),

$$\frac{1}{\epsilon_0}\iiint_D q\,(x,y,z)\; dV = \iint_S \mathrm{E}\cdot\mathbf{n}\, dS = \iiint_D \boldsymbol{\nabla}\cdot \mathrm{E}dV$$

and because this holds for all regions D, we conclude that $\boldsymbol{\nabla}\cdot\mathbf{E} = \dfrac{q\,(x,y,z)}{\epsilon_0}$.

e. $\nabla^2\varphi = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\varphi = \boldsymbol{\nabla}\cdot\mathbf{E} = \dfrac{q\,(x,y,z)}{\epsilon_0}$.

17.8.40

a. By Exercise 36, the flux of $\mathbf{F}$ across a sphere of radius a is $4\pi GMa^{3-p} = 4\pi GM$.

b. Since the outward flux across a sphere, from part (a), is independent of the radius of the sphere, the outward flux across the spheres of radii a and b are equal, so their difference, which is the net flux across the spherical shell bounded by them, is zero.

c. The left hand side is the flux across the boundary of D, while the right-hand side is the sum of the mass density inside D. The statement says that the flux across the boundary is determined by (is a constant multiple of) the sum of the mass density inside D.

d. By the Divergence theorem, and using part (c),

$$4\pi G\iiint_D \rho\,(x,y,z)\; dV = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_D \boldsymbol{\nabla}\cdot\mathbf{F}dV$$

and because this holds over all regions D, we have $\boldsymbol{\nabla}\cdot\mathbf{F} = 4\pi G\rho\,(x,y,z)$.

e. $\boldsymbol{\nabla}^2\varphi = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\varphi = \boldsymbol{\nabla}\cdot\mathbf{F} = 4\pi G\rho\,(x,y,z)$.

17.8.41 $\mathbf{F} = -\nabla\mathbf{T} = \langle -1,-2,-1\rangle$, so that $\boldsymbol{\nabla}\cdot\mathbf{F} = 0$ and the heat flux is zero.

17.8.42 $\mathbf{F} = -\nabla\mathbf{T} = \langle -2x,-2y,-2z\rangle$, so that $\boldsymbol{\nabla}\cdot\mathbf{F} = -6$, and the heat flux is -6 times the volume of the region, or -6.

17.8.43 $\mathbf{F} = -\nabla\mathbf{T} = \langle 0,0,e^{-z}\rangle$; then $\boldsymbol{\nabla}\cdot\mathbf{F} = -e^{-z}$. The heat flux is then

$$\int_0^1\int_0^1\int_0^1 -e^{-z}\, dx\, dy\, dz = e^{-1} - 1.$$

17.8.44 From Exercise 42, $\nabla \cdot \mathbf{F} = -6$, so the heat flux is -6 times the volume of the sphere, or -8π.

17.8.45 $\mathbf{F} = -\nabla \mathbf{T} = \langle 200xe^{-x^2-y^2-z^2}, 200ye^{-x^2-y^2-z^2}, 200ze^{-x^2-y^2-z^2}\rangle$. Then

$$\nabla \cdot \mathbf{F} = 200e^{-x^2-y^2-z^2}\left(3 - 2x^2 - 2y^2 - 2z^2\right)$$

so that

$$\iiint\limits_D \nabla \cdot \mathbf{F}\, dV = 200 \int_0^{2\pi} \int_0^{\pi} \int_0^a e^{-r^2}\left(3 - 2r^2\right) r^2 \sin u \, dr\, du\, dv = 800\pi a^3 e^{-a^2}.$$

17.8.46

a. By Exercise 36, the net flux across a sphere of radius a centered at the origin is $4\pi a^{3-p}$, which is independent of a only if $p = 3$.

b. In the general case, we have $\nabla \cdot \mathbf{F} = \dfrac{3-p}{|\mathbf{r}|^p}$, so

$$\begin{aligned}\iiint\limits_D \nabla \cdot \mathbf{F}\, dV &= \int_a^b \int_{\varphi_1}^{\varphi_2} \int_{\theta_1}^{\theta_2} \frac{3-p}{r^p} r^2 \sin u \, du\, dv\, dr \\ &= \left(a^{3-p} - b^{3-p}\right)\left(\varphi_1 \cos\theta_1 - \varphi_2 \cos\theta_1 - \varphi_1 \cos\theta_2 + \varphi_2 \cos\theta_2\right) \\ &= \left(a^{3-p} - b^{3-p}\right)\left(\varphi_1 - \varphi_2\right)\left(\cos\theta_1 - \cos\theta_2\right),\end{aligned}$$

and these are in general zero only if $3 - p = 0$.

17.8.47

a. $\varphi_x(x, y, z) = G'(\rho)\,\rho_x = G'(\rho) \cdot \dfrac{x}{\sqrt{x^2+y^2+z^2}} = G'(\rho)\,\dfrac{x}{\rho}$, so that $\nabla\varphi = \mathbf{F} = G'(\rho)\,\dfrac{\mathbf{r}}{\rho}$.

b. The normal to the sphere of radius a is $\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$, so on that sphere (where $\rho = a$)

$$\mathbf{F} \cdot \mathbf{n} = G'(a)\,\frac{\frac{x^2}{z} + \frac{y^2}{z} + z}{a} = G'(a)\,\frac{\frac{a^2-z^2}{z} + z}{a} = G'(a)\,\frac{a}{z},$$

and then the surface integral over the upper hemisphere is

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = aG'(a) \int_0^a \int_0^{2\pi} \frac{r}{\sqrt{a^2 - r^2}}\, d\theta\, dr = 2\pi a^2 G'(a),$$

so the total surface integral is twice that, or $4\pi a^2 G'(a)$.

c. By the Chain Rule,

$$\frac{\partial}{\partial x} G'(\rho)\,\frac{x}{\rho} = G''(\rho)\,\rho_x \frac{x}{\rho} + G'(\rho)\,\frac{y^2+z^2}{\rho^3}$$

so that (noting that $\rho_x = \frac{x}{\rho}$)

$$\nabla \cdot \mathbf{F} = G''(\rho)\left(\frac{x^2+y^2+z^2}{\rho^2}\right) + G'(\rho)\,\frac{2\left(x^2+y^2+z^2\right)}{\rho^3} = G''(\rho) + \frac{2\,G'(\rho)}{\rho}.$$

d. By the Divergence theorem, the flux is also given by

$$\begin{aligned}\iiint\limits_D \nabla \cdot \mathbf{F}\, dV &= \int_0^a \int_0^{\pi} \int_0^{2\pi} \rho^2 \sin u \left(G''(\rho) + \frac{2\,G'(\rho)}{\rho}\right) dv\, du\, d\rho \\ &= 2\pi \int_0^a \int_0^{\pi} \sin u \left(\rho^2\, G''(\rho) + 2\rho\, G'(\rho)\right) du\, d\rho \\ &= 2\pi \int_0^a (-\cos u)\left(\rho^2\, G''(\rho) + 2\rho\, G'(\rho)\right)\Big|_{u=0}^{u=\pi} d\rho \\ &= 4\pi \int_0^a \left(\rho^2\, G''(\rho) + 2\rho\, G'(\rho)\right) d\rho.\end{aligned}$$

It remains to evaluate this integral. Using integration by parts, we have

$$4\pi\left[\int_0^a\left(\rho^2 G''(\rho)+2\rho G'(\rho)\right)d\rho\right]=4\pi\left[\rho^2 G'(\rho)\Big|_{\rho=0}^{\rho=a}-\int_0^a 2\rho G'(\rho)\,d\rho\right]$$
$$=4\pi\left[a^2G'(a)-\int_0^a 2\rho G'(\rho)\,d\rho\right]$$

and the remaining integrals cancel, giving $4\pi a^2 G'(a)$ as the final result.

17.8.48

a. Rearrange the given equation and integrate over D to get

$$\iiint\limits_D u\,\nabla\cdot\mathbf{F}\,dS=\iiint\limits_D \nabla\cdot(u\,\mathbf{F})\,dS-\iiint\limits_D \mathbf{F}\cdot\nabla u\,dS.$$

By the Divergence theorem, the first term on the right side is equal to $\iint\limits_S u\,\mathbf{F}\cdot\mathbf{n}\,dS$ where S is the boundary of D. The result follows.

b. If you set $\mathbf{F}=\langle f(x),0,0\rangle$ and$u(x,y,z)=g(x)$, then

$$\begin{array}{ll}\nabla\cdot\mathbf{F}=f'(x) & u\nabla\cdot\mathbf{F}=f'(x)\,g(x)\\ u\mathbf{F}=\langle f(x)\,g(x),0,0\rangle & \nabla\cdot(u\mathbf{F})=(fg)'(x)\\ \nabla u=\langle g'(x),0,0\rangle & \mathbf{F}\cdot\nabla u=f(x)\,g'(x)\end{array}$$

so that

$$\iiint\limits_D f'(x)\,g(x)\,dV=\iiint\limits_D (fg)'(x)\,dV-\iiint\limits_D f(x)\,g'(x)\,dV=f(x)\,g(x)-\iiint\limits_D f(x)\,g'(x)\,dV,$$

which is the usual rule for integration by parts.

c. One approach is to set $u=1$ and $\mathbf{F}=\frac{1}{2}\langle x^2z^2,x^2y^2,y^2z^2\rangle$. Gauss' formula then gives

$$\iiint\limits_D\left(x^2y+y^2z+z^2x\right)=\frac{1}{2}\iint\limits_S\langle x^2z^2,x^2y^2,y^2z^2\rangle\,dS.$$

The integral on the right is computed by integrating over each face of the cube; on faces where x is constant, the normal is either $\langle 1,0,0\rangle$ or $\langle -1,0,0\rangle$ depending on whether x is 1 or 0; similarly for y and z. Considering x first, when $x=0$, this surface integral becomes 0 on that face (the integrand is x^2z^2 at $x=0$), while for $x=1$ it becomes z^2. In that case, the integral is $\frac{1}{3}$. This holds for each dimension, so the total integral on the right side is $3\cdot\frac{1}{3}=1$, and thus the integral on the left is $\frac{1}{2}$.

17.8.49 Suppose F$=\langle f,g\rangle$ where$f=f(x,y)$, $g=g(x,y)$, and suppose$u=u(x,y)$. Then $\nabla\cdot\mathbf{F}=f_x+g_y$, and $\nabla u=\langle u_x,u_y\rangle$. Then we have for this case

$$\iiint\limits_D u\,\nabla\cdot\mathbf{F}\,dS=\iint\limits_R u\,(f_x+g_y)\,dA$$
$$\iint\limits_S u\mathbf{F}\cdot\mathbf{n}\,dS=\oint_C u\mathbf{F}\cdot\mathbf{n}\,ds$$
$$\iiint\limits_D \mathbf{F}\cdot\nabla u\,dS=\iint\limits_R (f\,u_x+g\,u_y)\,dA,$$

and the result follows. Setting $u = 1$ then gives $\iint\limits_R (f_x + g_y)\, dA = \oint_C \mathbf{F}\cdot\mathbf{n}\, ds$, which is the flux form of Green's Theorem.

17.8.50 Apply the Divergence Theorem to the vector field $\mathbf{F} = u\nabla u$. By the product rule (Thm. 14.11), we have $\nabla\cdot(v\nabla v) = \nabla u\cdot\nabla v + u\nabla^2 v$, so the Divergence theorem says that

$$\iiint\limits_D \left(u\nabla^2 v + \nabla u\cdot\nabla v\right)\, dV = \iiint\limits_D \left(\nabla\cdot(v\nabla v)\right)\, dV = \iint\limits_S u\nabla v\cdot\mathbf{n}\, dS.$$

17.8.51 From Exercise 50, we have both

$$\iiint\limits_D \left(u\nabla^2 v + \nabla u\cdot\nabla v\right)\, dV = \iint\limits_S u\nabla v\cdot\mathbf{n}\, dS$$
$$\iiint\limits_D \left(v\nabla^2 u + \nabla v\cdot\nabla u\right)\, dV = \iint\limits_S v\nabla u\cdot\mathbf{n}\, dS,$$

where the second formula is obtained by switching u and v in the first formula. Subtracting the second from the first, and using the fact that the dot product is commutative and integrals are linear, we have the desired result:

$$\iiint\limits_D \left(u\nabla^2 v - v\nabla^2 u\right)\, dV = \iint\limits_S (u\nabla v - v\nabla u)\cdot\mathbf{n}\, dS.$$

17.8.52 A computation shows that $\nabla\varphi = \frac{-p\mathbf{r}}{|\mathbf{r}|^{p+2}}$. Thus the potential field $\nabla\varphi = 0$ for $p = 0$, so certainly $\nabla^2\varphi = 0$ as well. Otherwise, for $p = 1$, we have $\nabla\varphi = \frac{-\mathbf{r}}{|\mathbf{r}|^3}$; this is an inverse square field, and we have seen many times (e.g. Exercise 36(b)) that the divergence of such a field is zero. Thus $\nabla^2\varphi = 0$ for $p = 1$ as well.

17.8.53 The Divergence theorem applied to the field $\nabla\varphi$ says that

$$\iiint\limits_D \left(\nabla^2\varphi\right)\, dV = \iint\limits_S \nabla\varphi\cdot\mathbf{n}\, dS$$

and if φ is harmonic, the left side is zero.

17.8.54 Apply Green's First Identity (Exercise 49) to u and u to give

$$\iiint\limits_D \left(u\nabla^2 u + \nabla u\cdot\nabla u\right)\, dV = \iint\limits_S u\nabla u\cdot\mathbf{n}\, dS.$$

Because $\nabla^2 u = 0$ and $\nabla u\cdot\nabla u = |\nabla u|^2$, the result follows.

17.8.55 If $\mathbf{T}$ is a vector field $\langle t, u, v\rangle$, then by $\iint \mathbf{T}$, we mean $\left\langle \iint t, \iint u, \iint v\right\rangle$.

a. Let $\mathbf{F} = \langle f, g, h\rangle$ and suppose $\mathbf{n} = \langle n_1, n_2, n_3\rangle$. Then

$$\mathbf{n}\times\mathbf{F} = \langle n_2 h - n_3 g, n_3 f - n_1 h, n_1 g - n_2 f\rangle$$
$$\nabla\times\mathbf{F} = \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right\rangle$$

Considering first the $\mathbf{i}$ component of these vectors, note that for the vector field $\mathbf{F}_1 = \langle 0, h, -g\rangle$, the Divergence theorem says that

$$\iint\limits_S (n_2 h - n_3 g)\, dS = \iint\limits_S \langle 0, h, -g\rangle\cdot\langle n_1, n_2, n_3\rangle\, dS = \iint\limits_S \mathbf{F}_1\cdot\mathbf{n}\, dS = \iiint\limits_D (\nabla\cdot\mathbf{F}_1)\, dR$$
$$= \iint\limits_D \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\, dA$$

and similarly for the second and third components.

b. Similarly to part (a), note that

$$\mathbf{n} \times \boldsymbol{\nabla}\varphi = \mathbf{n} \times \langle \varphi_x, \varphi_y, \varphi_z \rangle = \langle n_2\varphi_z - n_3\varphi_y, n_3\varphi_x - n_1\varphi_z, n_1\varphi_y - n_2\varphi_x \rangle.$$

Looking first at the x component of this vector, we have

$$n_2\varphi_z - n_3\varphi_y = \langle 0, \varphi_z, -\varphi_y \rangle \cdot \langle n_1, n_2, n_3 \rangle = (\nabla \times \langle \varphi, 0, 0 \rangle) \cdot \langle n_1, n_2, n_3 \rangle$$

so that Stokes' theorem says that, writing $\mathbf{F} = \langle \varphi, 0, 0 \rangle$,

$$\iint_S (\nabla \times \langle \varphi, 0, 0 \rangle) \cdot \langle n_1, n_2, n_3 \rangle \, dS = \iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n} \, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \langle \varphi, 0, 0 \rangle \cdot d\mathbf{r} = \oint_C \varphi \, d\mathbf{r}.$$

and similarly for the second and third components.

Chapter Seventeen Review

1

a. False. The curl is $\dfrac{\partial}{\partial x}(x) - \dfrac{\partial}{\partial y}(-y) = 2$.

b. True. The curl of a conservative vector field is zero.

c. False. For example, $\langle -y, x \rangle$ and $\langle 0, 2x \rangle$ both have curl 2.

d. False. For example, $\langle x, 0, 0 \rangle$ and $\langle 0, y, 0 \rangle$ both have divergence 1.

e. True. By the Divergence theorem, the integral is equal to $\displaystyle\iiint_D \boldsymbol{\nabla} \cdot \mathbf{F} \, dV = \iiint_D 3 \, dV$.

2

a. Choice F, because this is a radial vector field whose magnitude increases with distance from the origin.

b. Choice E, because this is a rotational field.

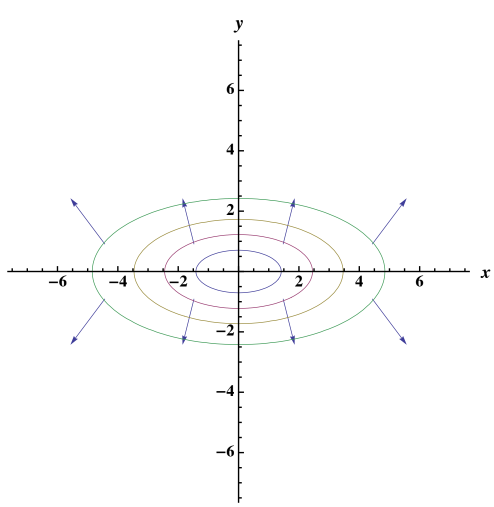
c. Choice D, because this is also a radial field, but the magnitude is constant.

d. Choice C, because this field is vertical along the line $y = x$.

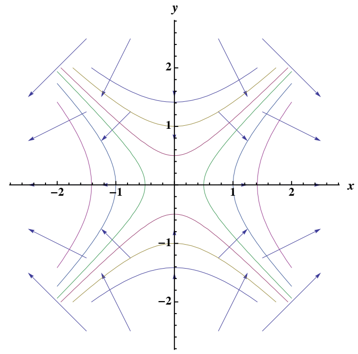
e. Choice B, because the magnitude decreases rapidly away from the origin.

f. Choice A, because this field is periodic.

3 $\nabla\varphi = \langle 2x, 8y \rangle$.



4 $\nabla\varphi = \langle x, -y\rangle$.



5 $\nabla\varphi = -\dfrac{\mathbf{r}}{|\mathbf{r}|^3}$

6 $\nabla\varphi = -e^{-x^2-y^2-z^2}\langle x, y, z\rangle$.

7 $\langle x, y\rangle$ is an outward normal; for (x, y) on the circle, $|\langle x, y\rangle| = \sqrt{x^2+y^2} = 2$, so the unit outward normal is $\dfrac{1}{2}\langle x, y\rangle$.

8 We have $|\mathbf{r}'(t)| = 5$, so the line integral is

$$\int_C \left(x^2 - 2xy + y^2\right)\, ds = 5\int_0^\pi \left(25\cos^2 t - 10\cos t\sin t + 25\sin^2 t\right)\, dt = 25\int_0^\pi (5 - 2\sin t\cos t)\, dt \,= 125\pi.$$

9 Here $|\mathbf{r}'(t)| = \sqrt{4+9+36} = \sqrt{49} = 7$, so

$$\int_C y\, e^{-xz}\, ds = 7\int_0^2 3te^{12t^2}\, dt = \frac{21}{24}e^{12t^2}\Big|_0^2 = \frac{7}{8}\left(e^{48}-1\right).$$

10 Parameterize the line by $\langle -3t, 1+6t, 2-3t\rangle$ for $0 \le t \le 1$. Then $|\mathbf{r}'(t)| = \sqrt{9+36+9} = \sqrt{54} = 3\sqrt{6}$, so

$$\int_C \left(xz - y^2\right)\, ds = 3\sqrt{6}\int_0^1 \left(3t\,(3t-2) - (6t+1)^2\right)\, dt = -57\sqrt{6}.$$

11 For the first parameterization we have $|\mathbf{r}'(t)| = 2$, so

$$\oint_C (x - 2y + 3z)\, ds = 2\int_0^{2\pi} (2\cos t - 4\sin t)\, dt = 0.$$

For the second parameterization we have $|\mathbf{r}'(t)| = \sqrt{16t^2\sin^2(t^2) + 16t^2\cos^2(t^2)} = 4t$, so

$$\oint_C (x - 2y + 3z)\, ds = \int_0^{\sqrt{2\pi}} \left(8t\cos t^2 - 16t\sin t^2\right)\, dt = 0.$$

12

a. Parameterize the path from P to Q by $\langle 1-t, t, 0\rangle$ for $0 \le t \le 1$; then the work done is $\int_C \langle 1, 2y, -4z\rangle \cdot d\mathbf{r} = \int_0^1 \langle 1, 2t, 0\rangle \cdot \langle -1, 1, 0\rangle\, dt = \int_0^1 (2t-1)\, dt = 0$.

b. Parameterize the path from P to O by $\langle 1-t, 0, 0\rangle$, and the path from O to Q by $\langle 0, t, 0\rangle$, both for $0 \le t \le 1$. Then the work done is $\int_C \langle 1, 2y, -4z\rangle\cdot d\mathbf{r} = \int_0^1 \left(\langle 1, 0, 0\rangle \cdot \langle -1, 0, 0\rangle + \langle 1, 2t, 0\rangle \cdot \langle 0, 1, 0\rangle\right) dt = \int_0^1 (2t-1)\, dt = 0$.

c. Parameterize the quarter circle by $\langle \cos t, \sin t, 0\rangle$ for $0 \le t \le \dfrac{\pi}{2}$; then the work done is $\displaystyle\int_C \langle 1, 2y, -4z\rangle \cdot d\mathbf{r} = \int_0^{\pi/2} \langle 1, 2\sin t, 0\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt = \int_0^{\pi/2} (2\cos t\,\sin t - \sin t)\, dt = 0.$

d. Yes, the work is independent of the path; this is a conservative vector field with potential function $x + y^2 - 2z^2$.

13 Parameterize the first path by $\mathbf{r}_1(t) = \langle 0, t, 0\rangle$, and the second by $\mathbf{r}_1(t) = \langle 0, 1, 4t\rangle$, both for $0 \le t \le 1$. Then $\mathbf{r}_1'(t) = \langle 0, 1, 0\rangle$ and $\mathbf{r}_2'(t) = \langle 0, 0, 4\rangle$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \big(\langle -t, 0, 0\rangle \cdot \langle 0, 1, 0\rangle + \langle -1, 4t, 0\rangle \cdot \langle 0, 0, 4\rangle\big)\, dt = 0.$$

14 $\mathbf{r}'(t) = \langle 2t, 6t, -2t\rangle$, so

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_1^2 \frac{1}{(11t^4)^{3/2}} \langle t^2, 3t^2, -t^2\rangle \cdot \langle 2t, 6t, -2t\rangle\, dt = 11^{-3/2}\int_1^2 t^{-6}\cdot 22t^3\, dt = 11^{-3/2}\int_1^2 22t^{-3}\, dt = \frac{3}{44}\sqrt{11}.$$

15 The circulation is

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \int_0^{2\pi} \langle 2\sin t - 2\cos t, 2\sin t\rangle \cdot \langle -2\sin t, 2\cos t\rangle\, dt = \int_0^{2\pi} \big(-4\sin^2 t + 8\sin t\cos t\big)\, dt = -4\pi.$$

The outward flux is

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{n}\, ds &= \int_0^{2\pi} ((2\sin t - 2\cos t)(2\cos t) - (2\sin t)(-2\sin t))\, dt \\ &= 4\int_0^{2\pi} \big(\sin t\,\cos t - \cos^2 t + \sin^2 t\big)\, dt = 0.\end{aligned}$$

16 The circulation is

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \int_0^{2\pi} \langle 2\cos t, 2\sin t\rangle \cdot \langle -2\sin t, 2\cos t\rangle\, dt = 0.$$

The outward flux is

$$\int_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_0^{2\pi} ((2\cos t)(2\cos t) - (2\sin t)(-2\sin t))\, dt = 8\pi.$$

17 The circulation is

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \frac{1}{4}\int_0^{2\pi} \langle 2\cos t, 2\sin t\rangle \cdot \langle -2\sin t, 2\cos t\rangle\, dt = 0.$$

The outward flux is

$$\int_C \mathbf{F}\cdot\mathbf{n}\, ds = \frac{1}{4}\int_0^{2\pi} ((2\cos t)(2\cos t) - (2\sin t)(-2\sin t))\, dt = 2\pi.$$

18 The circulation is

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \int_0^{2\pi} \langle 2\cos t - 2\sin t, 2\cos t\rangle \cdot \langle -2\sin t, 2\cos t\rangle\, dt = 4\int_0^{2\pi} (-\sin t\cos t + 1)\, dt = 4\pi.$$

The outward flux is

$$\int_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_0^{2\pi} ((2\cos t - 2\sin t)(2\cos t) - (2\cos t)(-2\sin t))\, dt = 4\int_0^{2\pi} \big(\cos^2 t\big)\, dt = 4\pi.$$

19 The normal to the plane $x = 0$ is $\langle 1, 0, 0\rangle$, so the flux is

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{-1/2}^{1/2} \int_{-L}^{L} v_0 \left(L^2 - y^2\right) dy\, dz = \frac{4}{3} v_0 L^3..$$

20 A potential function is xy^2.

21 A potential function is $xy + yz^2$.

22 A potential function is $e^x \cos y$.

23 A potential function is xye^z.

24

a. $\mathbf{F} = \langle 2xy, x^2\rangle$, so

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^3 \langle 2\left(9 - t^2\right) t, \left(9 - t^2\right)^2\rangle \cdot \langle -2t, 1\rangle\, dt \\ &= \int_0^3 \left(-4t^2 \left(9 - t^2\right) + \left(9 - t^2\right)^2\right) dt = \int_0^3 \left(9 - 5t^2\right)\left(9 - t^2\right) dt = 0\end{aligned}$$

b. Because $\mathbf{F} = \boldsymbol{\nabla}\varphi$, where $\varphi(x, y) = x^2 y$, we have $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(3) - \varphi(0) = 0 - 0 = 0.$

25

a. $\mathbf{F} = \langle yz, xz, xy\rangle$, so

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi \langle \frac{t}{\pi} \sin t, \frac{t}{\pi} \cos t, \sin t \cos t\rangle \cdot \langle -\sin t, \cos t, \frac{1}{\pi}\rangle\, dt \\ &= \int_0^\pi \left(\frac{t}{\pi}\left(\cos^2 t - \sin^2 t\right) + \frac{1}{\pi} \sin t \cos t\right) dt = 0.\end{aligned}$$

b. $\mathbf{F} = \boldsymbol{\nabla}(xyz) = \boldsymbol{\nabla}\varphi$, so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(\cos \pi \sin \pi) - \varphi\left(\cos 0 \sin 0 \cdot \frac{0}{\pi}\right) = 0 - 0 = 0.$$

26

a. Parameterize C by the four paths $\mathbf{r}_1(t) = \langle -1 + 2t, -1\rangle$, $\mathbf{r}_2(t) = \langle 1, -1 + 2t\rangle$, $\mathbf{r}_3(t) = \langle 1 - 2t, 1\rangle$, $\mathbf{r}_4(t) = \langle -1, 1 - 2t\rangle$, for $0 \le t \le 1$. Then $\mathbf{r}_1'(t) = \langle 2, 0\rangle$, $\mathbf{r}_2'(t) = \langle 0, 2\rangle$, $\mathbf{r}_3'(t) = \langle -2, 0\rangle$, $\mathbf{r}_4'(t) = \langle 0, -2\rangle$, and

$$\begin{aligned}&\int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^1 \left(\langle -1 + 2t, 1\rangle \cdot \langle 2, 0\rangle + \langle 1, 1 - 2t\rangle \cdot \langle 0, 2\rangle + \langle 1 - 2t, -1\rangle \cdot \langle -2, 0\rangle + \langle -1, 2t - 1\rangle \cdot \langle 0, -2\rangle\right) dt \\ &= \int_0^1 (4t - 2 + 2 - 4t + 4t - 2 + 2 - 4t)\, dt = \int_0^1 0\, dt = 0.\end{aligned}$$

b. $\mathbf{F} = \boldsymbol{\nabla}\varphi$, where $\varphi(x, y) = \frac{1}{2}\left(x^2 - y^2\right)$, so the integral around any closed curve is zero.

27

a. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle \sin t, 4, -\cos t\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt = \int_0^{2\pi} \left(-\sin^2 t + 4\cos t\right)\, dt = -\pi.$

b. The vector field is not conservative, since for example $\dfrac{\partial}{\partial y}(y) \neq \dfrac{\partial}{\partial z}(x)$.

28 For $p \neq 2$, $\mathbf{F} = \boldsymbol{\nabla}\varphi$ where $\varphi = \dfrac{-1}{(p-2)\,|\mathbf{r}|^{p-2}}$, while for $p = 2$, $\varphi = \dfrac{1}{2}\ln\left(|\mathbf{r}|^2\right)$, as can be seen by taking the gradient. Thus $\mathbf{F}$ is conservative on all of $\mathbb{R}^2$ for $p < 0$.

29 By the circulation form of Green's Theorem,

$$\oint_C xy^2\, dx + x^2 y\, dy = \iint_R \left(\frac{\partial}{\partial x}\left(x^2 y\right) - \frac{\partial}{\partial y}\left(xy^2\right)\right) dA = \iint_R (2xy - 2xy)\, dA = 0.$$

30 By the circulation form of Green's Theorem,

$$\oint_C \left(-3y + x^{3/2}\right) dx + \left(x - y^{2/3}\right) dy = \iint_R \left(\frac{\partial}{\partial x}\left(x - y^{2/3}\right) - \frac{\partial}{\partial y}\left(-3y + x^{3/2}\right)\right) dA = \iint_R 4\, dA = 4\pi.$$

31 By the circulation form of Green's Theorem,

$$\begin{aligned}\oint_C \left(x^3 + xy\right) dy + \left(2y^2 - 2x^2 y\right) dx &= \iint_R \left(\frac{\partial}{\partial x}\left(x^3 + xy\right) - \frac{\partial}{\partial y}\left(2y^2 - 2x^2 y\right)\right) dA \\ &= \iint_R \left(3x^2 + y - 4y + 2x^2\right) dA = \int_{-1}^{1}\int_{-1}^{1} \left(5x^2 - 3y\right) dy\, dx = \frac{20}{3}.\end{aligned}$$

32 By the flux form of Green's Theorem, $\oint_C 3x^3\, dy - 3y^3\, dx = \displaystyle\iint_R \left(9x^2 + 9y^2\right) dA = 9\int_0^4 \int_0^{2\pi} r^3\, d\theta\, dr =$ 1152π. Because the orientation is clockwise, the answer is -1152π.

33 The ellipse is $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$; parameterize it by $\mathbf{r}(t) = \langle x, y\rangle = \langle 4\cos t, 2\sin t\rangle$, $0 \le t \le 2\pi$. Then the area of the region is

$$\frac{1}{2}\oint_C \left((4\cos t)(2\cos t) - (2\sin t)(-4\sin t)\right) dt = \int_0^{2\pi} 4\left(\cos^2 t + \sin^2 t\right) dt = 8\pi.$$

34 $dx = -3\cos^2 t \sin t\, dt$, and$dy = 3\sin^2 t\cos t\, dt$, so the area of the hypocycloid is

$$\begin{aligned}\frac{1}{2}\oint_C x\, dy - y\, dx &= \frac{1}{2}\int_0^{2\pi} \left(\left(\cos^3 t\right)\left(3\sin^2 t\cos t\right) - \left(\sin^3 t\right)\left(-3\cos^2 t\sin t\right)\right) dt \\ &= \frac{3}{2}\int_0^{2\pi} \left(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t\right) dt = \frac{3\pi}{8}.\end{aligned}$$

35

a. $\mathbf{F} = \left(x^2 + y^2\right)^{-1/2}\langle x, y\rangle$, so the circulation is

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_R \left(\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right) dA \\ &= \iint_R \left(\frac{-xy}{\sqrt{x^2+y^2}} - \frac{-xy}{\sqrt{x^2+y^2}}\right) dA = 0.\end{aligned}$$

b. The flux is

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \left(\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right)+\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right) dA$$
$$= \iint_R \left(\frac{y^2}{(x^2+y^2)^{3/2}}+\frac{x^2}{(x^2+y^2)^{3/2}}\right) dA = \iint_R \left(\frac{1}{\sqrt{x^2+y^2}}\right) dA = \int_0^{\pi}\int_1^3 1\,dr\,d\theta = 2\pi.$$

36

a. The circulation is $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \left(\frac{\partial}{\partial x}(x\cos y) - \frac{\partial}{\partial y}(-\sin y)\right) dA = \int_0^{\pi/2}\int_0^{\pi/2} 2\cos y\,dy\,dx = \pi.$

b. The flux is $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \left(\frac{\partial}{\partial x}(-\sin y) + \frac{\partial}{\partial y}(x\cos y)\right) dA = \int_0^{\pi/2}\int_0^{\pi/2} (-x\sin y)\,dy\,dx = -\frac{1}{8}\pi^2.$

37

a. For $\mathbf{F}$ to be conservative, we must have $\frac{\partial}{\partial y}(ax+by) = \frac{\partial}{\partial x}(cx+dy)$, or $b=c$.

b. For $\mathbf{F}$ to be source-free, we must have $\frac{\partial}{\partial x}(ax+by) = -\frac{\partial}{\partial y}(cx+dy)$, or $a=-d$.

c. $\mathbf{F}$ is both conservative and source-free if $b=c$ and $a=-d$, i.e. if $\mathbf{F} = \langle ax+by, bx-ay\rangle$.

38 The divergence is $\frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0$. The curl is

$$\langle \frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz), \frac{\partial}{\partial z}(yz) - \frac{\partial}{\partial x}(xy), \frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(yz)\rangle = \mathbf{0}.$$

The field is both source-free and irrotational.

39 The divergence is $4\,|\mathbf{r}|$. The curl is

$$\langle \frac{\partial}{\partial y}(z\,|\mathbf{r}|) - \frac{\partial}{\partial z}(y\,|\mathbf{r}|), \frac{\partial}{\partial z}(x\,|\mathbf{r}|) - \frac{\partial}{\partial x}(z\,|\mathbf{r}|), \frac{\partial}{\partial x}(y\,|\mathbf{r}|) - \frac{\partial}{\partial y}(x\,|\mathbf{r}|)\rangle = \mathbf{0}.$$

The field is irrotational but not source-free.

40 The divergence is $\frac{\partial}{\partial x}(\sin xy) + \frac{\partial}{\partial y}(\cos yz) + \frac{\partial}{\partial z}(\sin xz) = y\cos xy - z\sin yz + x\cos xz$. The curl is $\langle y\sin yz, -z\cos xz, -x\cos xy\rangle$. The field is neither irrotational nor source-free.

41 The divergence is $\frac{\partial}{\partial x}\left(2xy+z^4\right) + \frac{\partial}{\partial y}\left(x^2\right) + \frac{\partial}{\partial z}\left(4xz^3\right) = 2y + 12xz^2$. The curl is

$$\langle \frac{\partial}{\partial y}\left(4xz^3\right) - \frac{\partial}{\partial z}\left(x^2\right), \frac{\partial}{\partial z}\left(2xy+z^4\right) - \frac{\partial}{\partial x}\left(4xz^3\right), \frac{\partial}{\partial x}\left(x^2\right) - \frac{\partial}{\partial y}\left(2xy+z^4\right)\rangle = \mathbf{0},$$

so the field is irrotational but not source-free.

42 $|\mathbf{r}|^4 = \left(x^2+y^2+z^2\right)^2$, so

$$\nabla\left(x^2+y^2+z^2\right)^{-2} = \left\langle -4x\left(x^2+y^2+z^2\right)^{-3}, -4y\left(x^2+y^2+z^2\right)^{-3}, -4z\left(x^2+y^2+z^2\right)^{-3}\right\rangle = -\frac{4\,\mathbf{r}}{|\mathbf{r}|^6}.$$

Now

$$\frac{\partial}{\partial x}\left(-4x\left(x^2+y^2+z^2\right)^{-3}\right) = -4\left(y^2+z^2-5x^2\right)\left(x^2+y^2+z^2\right)^{-4},$$

so that

$$\nabla\cdot\nabla\,|\mathbf{r}|^{-4} = -4\left(x^2+y^2+z^2\right)^{-4}\left(y^2+z^2-5x^2+x^2+z^2-5y^2+x^2+y^2-5z^2\right) = \frac{12}{|\mathbf{r}|^6}.$$

43

a. The curl is
$$\langle \frac{\partial}{\partial y}(-y) - \frac{\partial}{\partial z}(x), \frac{\partial}{\partial z}(z) - \frac{\partial}{\partial x}(-y), \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(z) \rangle = \langle -1, 1, 1 \rangle.$$
So the scalar component in the direction of $\langle 1,0,0\rangle$ is$\langle -1,1,1\rangle \cdot \langle 1,0,0\rangle = -1$,

b. The scalar component in the direction of $\left\langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ is$\left\langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \cdot \langle 1,0,0\rangle = 0$.

c. The scalar component of the curl is a maximum in the direction of the curl, i.e. in the direction $\langle -1,1,1\rangle$, whose unit vector is $\frac{1}{\sqrt{3}}\langle -1,1,1\rangle$.

44 The curl of the vector field is $\boldsymbol{\nabla} \times \mathbf{F} = \langle 0,0,2\rangle$, and the component of the curl along a unit vector $\mathbf{n}$ is thus $\langle 0,0,2\rangle \cdot \mathbf{n}$.

a. It does not spin, because $\langle 0,0,2\rangle \cdot \langle 1,0,0\rangle = 0$.

b. The scalar component of the curl in the direction $\langle 0,0,1\rangle$ is 2.

c. It spins the fastest when the paddle wheel is aligned with $\langle 0,0,1\rangle$.

45 Parameterize the sphere by $\langle 3\sin u\cos v, 3\sin u \sin v, 3\cos u\rangle$, $0 \le u \le \frac{\pi}{2}$; $0 \le v \le 2\pi$. Then $|\mathbf{n}| = 9\sin u$, so
$$\iint_S 1\,dS = \iint_R 9\sin u\,dA = \int_0^{2\pi}\int_0^{\pi/2} 9\sin u\,du\,dv = 18\pi.$$

46 Parameterize the surface by $\langle v\cos u, v\sin u, v\rangle$ for $2 \le v \le 4, 0 \le u \le 2\pi$. Then
$$\iint_S 1\,dS = \iint_R \sqrt{2}\,v\,dA = \sqrt{2}\int_2^4\int_0^{2\pi} v\,du\,dv = 12\pi\sqrt{2}.$$

47 The volume element is $\sqrt{1^2+1^2+1^2} = \sqrt{3}$, so the area is
$$\iint_S 1\,dS = \sqrt{3}\iint_R 1\,dA = \sqrt{3}\int_{-1}^1\int_{-1}^1 1\,dx\,dy = 4\sqrt{3}.$$

48 The volume element is $\sqrt{z_x^2+z_y^2+1} = \sqrt{2(x^2+y^2)+1}$, so the integral is
$$\iint_S 1\,dS = \iint_R \sqrt{2(x^2+y^2)+1}\,dA = \int_0^2\int_0^{2\pi} r\sqrt{2r^2+1}\,d\theta\,dr = \frac{26}{3}\pi.$$

49 The volume element for $z = 2-x-y$ is $\sqrt{3}$, so the integral is
$$\iint_S (1+yz)\,dS = \sqrt{3}\iint_R (1+yz)\,dA = \sqrt{3}\int_0^2\int_0^{2-x} (1+y(2-x-y))\,dy\,dx = \frac{8\sqrt{3}}{3}.$$

50 The normal to the curved surface of the cylinder at (x,y,z) is $\langle 0,y,z\rangle$, so
$$\iint_S \langle 0,y,z\rangle \cdot \mathbf{n}\,dS = a\iint_R \left(y^2+z^2\right)\,dA = a^2 \cdot \text{area of } R = 32\pi a^3.$$

51 Parameterize the curved surface using spherical coordinates, so that $|\mathbf{n}| = 4\sin u$; then for the curved surface we have

$$\iint_S (x - y + z)\, dS = 8\int_0^{2\pi}\int_0^{\pi/2} (\sin u\cos v - \sin u\sin v + \cos u)\sin u\, du\, dv = 8\pi.$$

For the planar surface, $\mathbf{n} = \langle 0, -1, 0\rangle$ so that $|\mathbf{n}| = 1$ and

$$\iint_S (x - y + z)\, dS = \iint_R (x - y + z)\, dA = \int_0^2\int_0^{2\pi} r(\cos\theta - \sin\theta)\, d\theta\, dr = 0$$

and the total integral is thus 8π.

52 The normal to the cylinder is $\langle x, y, 0\rangle$ with magnitude 1, so

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R \langle x, y, z\rangle\cdot\langle x, y, 0\rangle\, dA = \iint_R \left(x^2 + y^2\right)\, dA = \iint_R 1\, dA = 32\pi.$$

53 $\mathbf{F} = \left(x^2 + y^2 + z^2\right)^{-1/2}\langle x, y, z\rangle$; using spherical coordinates to parameterize the sphere gives

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\, dS &= \iint_R \frac{1}{a}\langle a\sin u\cos v, a\sin u\sin v, a\,\cos u\rangle\cdot\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\sin u\cos u\rangle\, dA\\ &= \iint_R a^2\sin u\, dA = \int_0^{2\pi}\int_0^{\pi} a^2\sin u\, du\, dv = 4\pi a^2.\end{aligned}$$

54

a. Using the explicit description, we have $\sqrt{z_z^2 + z_y^2 + 1} = \sqrt{4\left(x^2 + y^2\right) + 1}$, so

$$\iint_S 1\, dS = \iint_R \sqrt{4\left(x^2 + y^2\right) + 1}\, dA = \int_0^2\int_0^{2\pi} r\sqrt{4r^2 + 1}\, d\theta\, dr = \frac{\pi}{6}\left(17\sqrt{17} - 1\right).$$

b. Using the given parametric description, we have $|\mathbf{n}| = v\sqrt{1 + 4v^2}$, so

$$\iint_S 1\, dS = \iint_R v\sqrt{1 + 4v^2}\, dA = \int_0^{2\pi}\int_0^2 v\sqrt{1 + 4v^2}\, dv\, d\theta = \frac{\pi}{6}\left(17\sqrt{17} - 1\right).$$

c. Using the given parametric description, we have

$$|\mathbf{t}_u\times\mathbf{t}_v| = \left|\langle -\sqrt{v}\,\sin u, \sqrt{v}\cos u, 0\rangle\times\left\langle -\frac{1}{2}v^{-1/2}\cos u, \frac{1}{2}v^{-1/2}\sin u, 1\right\rangle\right| = \frac{1}{2}\sqrt{4v + 1}$$

so that

$$\iint_S 1\, dS = \frac{1}{2}\iint_R \sqrt{4v + 1}\, dA = \frac{1}{2}\int_0^{2\pi}\int_0^4 \sqrt{4v + 1}\, dv\, d\theta = \frac{\pi}{6}\left(17\sqrt{17} - 1\right).$$

55

a. The base of S is the surface where $z = 0$, or the circle $x^2 + y^2 = a^2$. Similarly, the base of the paraboloid is found by setting $z = 0$; simplifying gives again $x^2 + y^2 = a^2$. The high point of the hemisphere (maximum z-coordinate) occurs when $x = y = 0$; then $z = a$. Similarly, the high point on the paraboloid also occurs when $x = y = 0$ and again this gives $z = a$.

b. The graph of the paraboloid is inside that of the hemisphere everywhere, so we would expect it to have smaller surface area. We know that the surface area of the hemisphere is $4\pi a^2 \cdot \dfrac{1}{2} = 2\pi a^2$. For the paraboloid, we have $z_x = -\dfrac{2x}{a}$, $z_y = -\dfrac{2y}{a}$, so that $|\mathbf{n}| = \sqrt{\dfrac{4\left(x^2+y^2\right)}{a^2} + 1}$, so

$$\iint_S 1\,dS = \frac{1}{a}\iint_R \sqrt{4\left(x^2+y^2\right)+a^2}\,dA = \frac{1}{a}\int_0^a \int_0^{2\pi} r\sqrt{4r^2+a^2}\;d\theta\,dr = \frac{\left(5\sqrt{5}-1\right)}{6}\pi a^2,$$

which is in fact smaller than the area of the hemisphere.

c. $\mathbf{n} = \langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\cos u\sin u\rangle$, so

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_R \langle a\sin u\cos v, a\sin u\sin v, a\,\cos u\rangle \cdot \langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\cos u\sin u\rangle\,dA\\ &= a^3\int_0^{2\pi}\int_0^{\pi/2}\left(\sin^3 u\,\cos^2 v + \sin^3 u\sin^2 v + \cos^2 u\sin u\right)\,du\,dv\\ &= a^3\int_0^{2\pi}\int_0^{\pi/2}\left(\sin^3 u + \cos^2 u\sin u\right)\,du\,dv = a^3\int_0^{2\pi}\int_0^{\pi/2}\sin u\,du\,dv = 2\pi a^3.\end{aligned}$$

d. For the paraboloid, the parameterization is $\left\langle v\cos u, v\sin u, a - \dfrac{v^2}{a}\right\rangle$, $0 \le v \le a$, $0 \le$u$\le 2\pi$, and

$$\mathbf{n} = \langle -v\sin u, v\cos u, 0\rangle \times \left\langle \cos u, \sin u, -\frac{2v}{a}\right\rangle = \left\langle -\frac{1}{a}2v^2\cos u, -\frac{1}{a}2v^2\sin u, -v\right\rangle,$$

so that the outward-pointing normal is $\left\langle \dfrac{2v^2}{a}\cos u, \dfrac{2v^2}{a}\sin u, v\right\rangle$ and

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_R \left\langle v\cos u, v\sin u, a - \frac{v^2}{a}\right\rangle \cdot \left\langle \frac{2v^2}{a}\cos u, \frac{2v^2}{a}\sin u, v\right\rangle dA\\ &= \int_0^a\int_0^{2\pi}\left(\frac{2}{a}v^3 + av - \frac{v^3}{a}\right)\,du\,dv = \frac{3}{2}\pi a^3.\end{aligned}$$

56

a. For the given $\mathbf{r}$, we have $\dfrac{(a\cos u\sin v)^2}{a^2} + \dfrac{(b\,\sin u\sin v)^2}{b^2} + \dfrac{(c\cos v)^2}{c^2} = \cos^2 u\sin^2 v + \sin^2 u\sin^2 v + \cos^2 v = \sin^2 v + \cos^2 v = 1$

b. The normal vector is determined by $\mathbf{n} = \mathbf{t}_u \times\, \mathbf{t}_v = \langle -bc\cos u\sin^2 v, -ac\sin u\,\sin 2v, -ab\sin v\cos v\rangle$, so that the outward pointing normal is the negative of this vector. Then

$$\begin{aligned}\iint_S 1\,dS &= \int_0^{2\pi}\int_0^{\pi}\left|\langle bc\cos u\sin^2 v, ac\sin u\sin^2 v, ab\,\sin v\cos v\rangle\right|\;dv\,du\\ &= \int_0^{2\pi}\int_0^{\pi}\sqrt{b^2c^2\cos^2 u\,\sin^4 v + a^2c^2\sin^2 u\sin^4 v + a^2b^2\sin^2 v\,\cos^2 v}\,dv\,du.\end{aligned}$$

57 Let S be the disk in the xy-plane with radius 2 centered at the origin with upward-pointing normal vector $\mathbf{k}$. Then $\boldsymbol{\nabla}\times\mathbf{F} = \langle x-y, x-y, 0\rangle$, so

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\,dS = \iint_R \langle x-y, x-y, 0\rangle\cdot\langle 0,0,1\rangle\;dA = \iint_R 0\,dA = 0.$$

58 $\mathbf{F} = \langle u^2 - v^2, u, 2v\,(6 - 2u - v)\rangle$, $\mathbf{r}\,(u, v) = \langle u, v, 6 - 2u - v\rangle$, $\mathbf{t}_u \times \mathbf{t}_v = \langle 2, 1, 1\rangle$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \int_0^{6-2u} \left(2u^2 - 4v^2 + 12v - 4uv\right)\, dv\, du = \frac{39\sqrt{6}}{2}.$$

59 The boundary of this region is in the xy-plane, found by setting $z = 0$, so it is $x^2 + y^2 = 99$, the circle of radius $\sqrt{99}$ about the origin. Parameterize the circle in the usual way; then

$$\iint_S (\mathbf{\nabla} \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \langle 0, \sqrt{99}\cos t, \sqrt{99}\sin t\rangle \cdot \left\langle -\sqrt{99}\sin t, \sqrt{99}\cos t, 0\right\rangle dt$$
$$= \int_0^{2\pi} 99\cos^2 t\, dt = 99\pi.$$

60 The boundary of this region is the circle $x^2 + z^2 = 4$ for $y = 0$; parameterizing it in the usual way as $\langle 2\cos t, 0, 2\sin t\rangle$ gives

$$\iint_S (\mathbf{\nabla} \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left\langle 4\cos^2 t - 4\sin^2 t, 0, 4\sin t\cos t\right\rangle \cdot \langle -2\sin t, 0, 2\cos t\rangle\, dt$$
$$= 8\int_0^{2\pi} \left(\left(\sin^2 t - \cos^2 t\right)\, \sin t + \sin t\cos^2 t\right)\, dt = 8\int_0^{2\pi} \sin^3 t\, dt = 0.$$

61 By Stokes' theorem, the circulation around a closed curve C can be found by choosing a surface S of which C is the boundary; then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\mathbf{\nabla} \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

But for $\mathbf{F} = \mathbf{\nabla}\left(10 - x^2 + y^2 + z^2\right)$, $\mathbf{\nabla} \times \mathbf{F} = \mathbf{0}$, so the right-hand side is zero.

62 We have $\mathbf{\nabla} \cdot \mathbf{F} = -3$ so that

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \mathbf{\nabla} \cdot \mathbf{F}\, dV = -3 \cdot \text{volume of cube} = -3.$$

63 $\mathbf{\nabla} \cdot \mathbf{F} = x^2 + y^2 + z^2$, so

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \left(x^2 + y^2 + z^2\right)\, dV = \int_0^3 \int_0^{2\pi} \int_0^{\pi} r^2 \cdot r^2 \sin u\, du\, dv\, dr = \frac{972}{5}\pi.$$

64 $\mathbf{\nabla} \cdot \mathbf{F} = 2\,(x + y + z)$, so

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D 2\,(x + y + z)\, dV = \int_0^8 \int_0^2 \int_0^{2\pi} 2r\,(r\cos\theta + r\sin\theta + t)\, d\theta\, dr\, dt = 256\pi.$$

65 $\mathbf{\nabla} \cdot \mathbf{F} = 3\left(x^2 + y^2\right)$, so the outward flux across the boundary S of a hemisphere D of radius a is $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D 3\left(x^2 + y^2\right)\, dV = \int_0^a \int_0^{2\pi} \int_0^{\pi/2} 3r^2 \sin^2 u \cdot r^2 \sin u\, du\, dv\, dr = \frac{4}{5}\pi a^5$, so that the net flux across the region bounded by the hemispheres of radii 1 and 2 is $\frac{4}{5}\pi\,(32 - 1) = \frac{124}{5}\pi$.

66 $\mathbf{\nabla} \cdot \mathbf{F} = 0$, so the flux is zero across any surface that bounds a region where F is defined and differentiable; the given region does not include zero, so is one of these. Thus the net outward flux is zero.

67 Using the Divergence theorem, $\nabla \cdot \mathbf{F} = 2x + \sin y + 2y - 2\sin y + 2z + \sin y = 2(x+y+z)$, so that

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D 2(x+y+z)\, dV = \int_0^4 \int_0^1 \int_0^{1-x} 2(x+y+z)\, dy\, dx\, dz = \frac{32}{3}.$$

68

a. The normal vectors point outwards everywhere on S; that is, on the curved surface, they point upwards, and on the flat surface they point in the direction of negative x.

b. Parameterize C by two paths: $\mathbf{r}_1(t) = \langle a\cos t, a\sin t, 0\rangle$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$ and $\mathbf{r}_2(t) = \langle 0, a-2t, 0\rangle$ for $0 \le t \le a$. Then $\mathbf{r}_1'(t) = \langle -a\sin t, a\cos t, 0\rangle$ and $\mathbf{r}_2'(t) = \langle 0, -2, 0\rangle$. So

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{-\pi/2}^{\pi/2} \langle -a\sin t, a\cos t, a\sin t - 2a\cos t\rangle \cdot \langle -a\sin t, a\cos t, 0\rangle\, dt$$

$$+ \int_0^a \langle 2t-a, 0, a-2t\rangle \cdot \langle 0,-2,0\rangle dt = \int_{-\pi/2}^{\pi/2} a^2\, dt = \pi a^2.$$

c. $\nabla \times \mathbf{F} = \langle 2,4,2\rangle$. For the curved portion of S, using spherical coordinates, the normal vector is $\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\cos u\sin u\rangle$, and for the flat portion, the normal vector is$\langle -1,0,0\rangle$. Then $\iint_S (\nabla \times \mathbf{F})\cdot\mathbf{n}\, dS = \iint_{S_1} \left(2a^2\sin^2 u\cos v + 4a^2\sin^2 u\sin v + 2a^2\cos u\sin u\right) dS + \iint_{S_2} (-2)\, dS =$ $2a^2 \int_{-\pi/2}^{\pi/2}\int_0^{\pi/2} \left(\sin^2 u\cos v + 2\sin^2 u\sin v + \sin u\cos u\right) du\, dv - \pi a^2 = 2\pi a^2 - \pi a^2 = \pi a^2.$