Solution Section 2.1 – Definition of the Derivative

Exercise

Find the derivative of y with the respect to t for the function $y=\frac{4}{t}$

Solution

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{\Delta t\to 0}\frac{\frac{4}{t+\Delta t}-\frac{4}{t}}{\Delta t}$$

$$=\lim_{\Delta t\to 0}\frac{\frac{4t-4(t+\Delta t)}{t(t+\Delta t)}}{\Delta t}$$

$$=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\frac{4t-4(t+\Delta t)}{t(t+\Delta t)}$$

$$=\lim_{\Delta t\to 0}\frac{-4\Delta t}{t(t+\Delta t)\Delta t}$$

$$=\lim_{\Delta t\to 0}\frac{-4}{t(t+\Delta t)}$$

$$=-\frac{4}{t^2}$$

Exercise

Find the equation of the tangent line to $f(x)=x^2+1$ that is parallel to $2x+y=0$

Solution

$2x + y = 0 \Rightarrow y = -2x \Rightarrow$ slope $= -2$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2+1-(x^2+1)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{x^2+\Delta x^2+2x\Delta x+1-x^2-1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x^2 + 2x\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \Delta x + 2x = 2x$$

$$f' = 2x = -2 \quad \Rightarrow x = -1 \Rightarrow f(-1) = (-1)^2 + 1 = 2 \qquad \rightarrow \text{(-1, 2)}$$

The line equation is given by

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -2(x+1)$$

$$y - 2 = -2x - 2$$

$$y = -2x$$

Exercise

Use the definition of limits to find the derivative: $f(x) = \dfrac{3}{\sqrt{x}}$

<u>Solution</u>

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(\frac{3}{\sqrt{x+\Delta x}}\right) - \left(\frac{3}{\sqrt{x}}\right)}{\Delta x} \cdot \frac{\sqrt{x} \cdot \sqrt{x+\Delta x}}{\sqrt{x} \cdot \sqrt{x+\Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{3\sqrt{x} - 3\sqrt{x+\Delta x}}{\Delta x\left(\sqrt{x} \cdot \sqrt{x+\Delta x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{3\left(\sqrt{x} - \sqrt{x+\Delta x}\right)}{\Delta x\left(\sqrt{x} \cdot \sqrt{x+\Delta x}\right)} \cdot \frac{\sqrt{x} + \sqrt{x+\Delta x}}{\sqrt{x} + \sqrt{x+\Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{3\left(x - (x+\Delta x)\right)}{\Delta x\left(\sqrt{x} \cdot \sqrt{x+\Delta x}\right)\left(\sqrt{x} + \sqrt{x+\Delta x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{-3\Delta x}{\Delta x\left(\sqrt{x} \cdot \sqrt{x+\Delta x}\right)\left(\sqrt{x} + \sqrt{x+\Delta x}\right)}$$

$$= \frac{-3}{x\left(2\sqrt{x}\right)} = \frac{-3}{2x^{3/2}}$$

Exercise

Use the definition of limits to find the derivative: $f(x)=\sqrt{x+2}$

Solution

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x+2}-\sqrt{x+2}}{\Delta x}\cdot\frac{\sqrt{x+\Delta x+2}+\sqrt{x+2}}{\sqrt{x+\Delta x+2}+\sqrt{x+2}}$$

$$=\lim_{\Delta x\to 0}\frac{x+\Delta x+2-(x+2)}{\Delta x\left(\sqrt{x+\Delta x+2}+\sqrt{x+2}\right)}$$

$$=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x\left(\sqrt{x+\Delta x+2}+\sqrt{x+2}\right)}$$

$$=\lim_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x+2}+\sqrt{x+2}}$$

$$=\frac{1}{2\sqrt{x+2}}$$