Section 4.5 – Comparing Three or More Means

Analysis of Variance (**ANOVA**) is an inferential method used to test the equality of three or more population means.

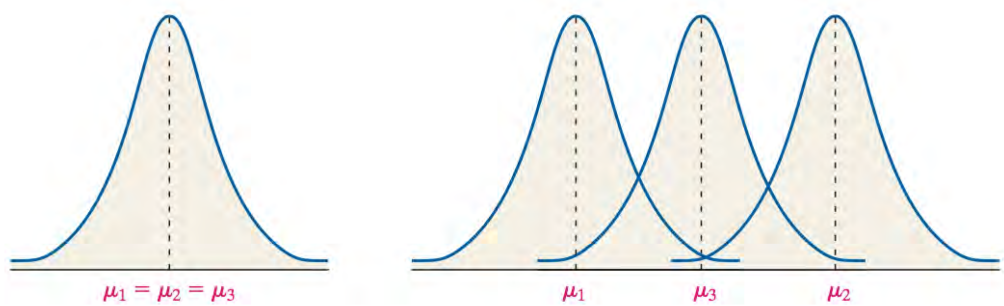
Requirements of a One-Way ANOVA Test

1. There must be *k* simple random samples; one from each of *k* populations or a randomized experiment with *k* treatments.
2. The *k* samples are independent of each other; that is, the subjects in one group cannot be related in any way to subjects in a second group.
3. The populations are normally distributed.
4. The populations must have the same variance; that is, each treatment group has the population variance σ^2

Testing a Hypothesis Regarding *k* = 3

$H_0 : \mu_1 = \mu_2 = \mu_3$

H_1 : At least one population mean is different from the others



Testing Using ANOVA

The methods of one-way ANOVA are ***robust***, so small departures from the normality requirement will not significantly affect the results of the procedure. In addition, the requirement of equal population variances does not need to be strictly adhered to, especially if the sample size for each treatment group is the same. Therefore, it is worthwhile to design an experiment in which the samples from the populations are roughly equal in size.

Verifying the Requirement of Equal Population Variance

The one-way ANOVA procedures may be used if the largest sample standard deviation is no more than twice the smallest sample standard deviation.

Example

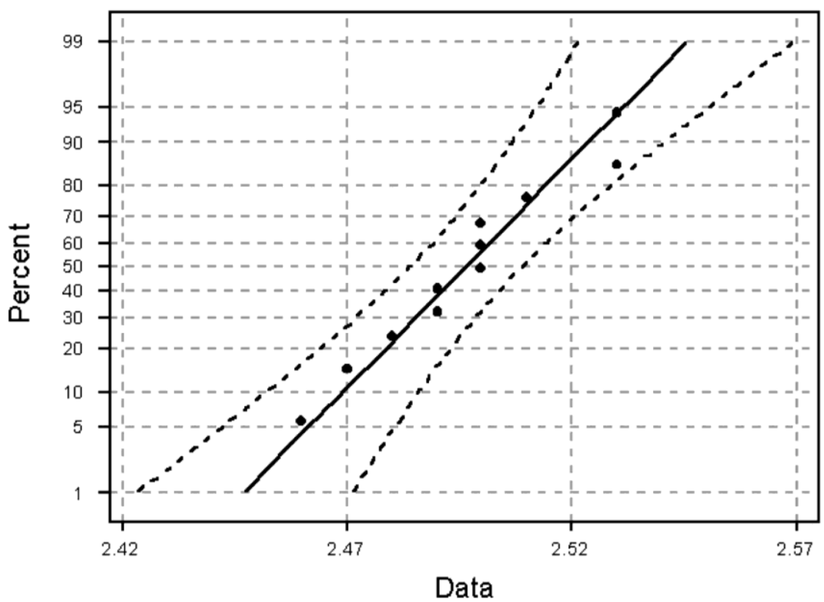
The following data represent the weight (in grams) of pennies minted at the Denver mint in 1990, 1995, and 2000. Verify that the requirements in order to perform a one-way ANOVA are satisfied.

1990	1995	2000
2.50	2.52	2.50
2.50	2.54	2.48
2.49	2.50	2.49
2.53	2.48	2.50
2.46	2.52	2.48
2.50	2.50	2.52
2.47	2.49	2.51
2.53	2.53	2.49
2.51	2.48	2.51
2.49	2.55	2.50
2.48	2.49	2.52

Solution

1. The 3 samples are simple random samples.
2. The samples were obtained independently.
3. Normal probability plots for the 3 years follow. All of the plots are roughly linear so the normality assumption is satisfied.

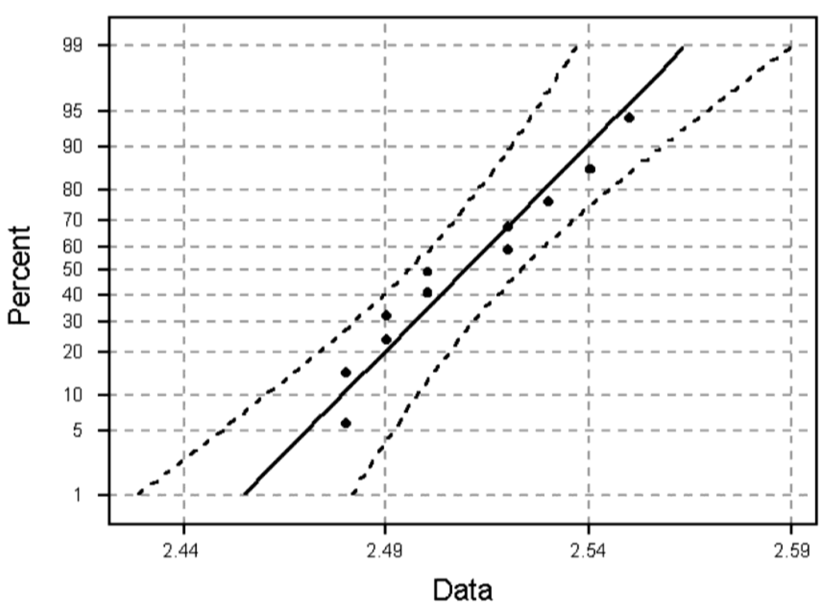
Normal Probability Plot for 1990



Mean: 2.49636

StDev: 0.0210077

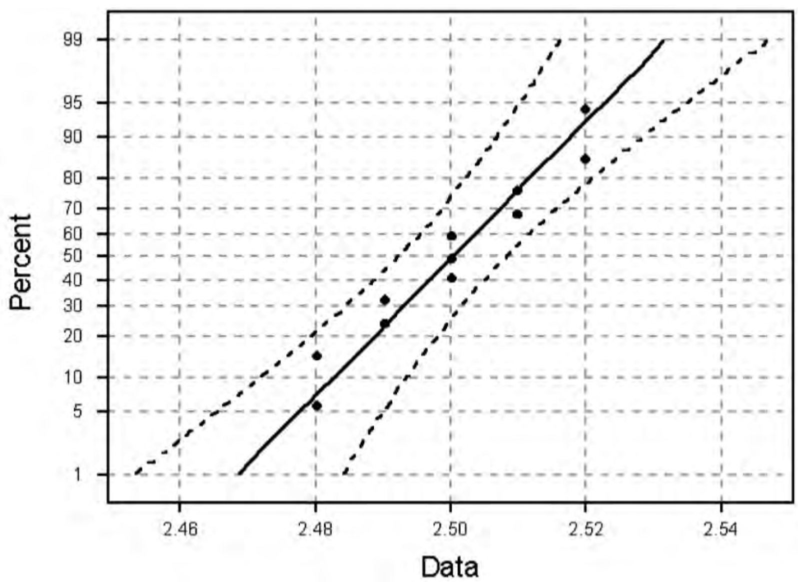
Normal Probability Plot for 1995



Mean: 2.50909

StDev: 0.0231417

Normal Probability Plot for 2000



Mean: 2.5

StDev: 0.0134840

The largest standard deviation is not more than twice the smallest standard deviation (2•0.0141 = 0.0282 > 0.02430) so the requirement of equal population variances is considered satisfied.

Variable	N	Mean	Median	TrMean	StDev	SE Mean
1990	11	2.4964	2.5000	2.4967	0.0220	0.0066
1995	11	2.5091	2.5000	2.5078	0.0243	0.0073
2000	11	2.5000	2.5000	2.5000	0.0141	0.0043

Variable	Minimum	Maximum	Q1	Q3
1990	2.4600	2.5300	2.4800	2.5130
1995	2.4800	2.5500	2.4900	2.5300
2000	2.4800	2.5200	2.4900	2.5100

The basic idea in one-way ANOVA is to determine if the sample data could come from populations with the same mean, μ, or suggests that at least one sample comes from a population whose mean is different from the others.

To make this decision, we compare the variability among the sample means to the variability within each sample.

We call the variability among the sample means the between-sample variability, and the variability of each sample the within-sample variability.

If the ***between-sample variability*** is large relative to the ***within-sample*** variability, we have evidence to suggest that the samples come from populations with different means.

ANOVA *F*-Test Statistic

The analysis of variance *F*-test statistic is given by

$$F_0 = \frac{\textit{between} - \textit{sample} \text{ variablity}}{\textit{within} - \textit{sample} \text{ var}\textit{iablity}}$$

$$= \frac{\textit{mean square due to treatments}}{\textit{mean square due to error}}$$

$$= \frac{MST}{MSE}$$

Computing the F-Test Statistic

***Step* 1**: Compute the sample mean of the combined data set by adding up all the observations and dividing by the number of observations. Call this value $\bar{x}$.

***Step* 2**: Find the sample mean for each sample (or treatment). Let $\bar{x}_1$ represent the sample mean of sample 1, $\bar{x}_2$ represent the sample mean of sample 2, and so on.

***Step* 3**: Find the sample variance for each sample (or treatment). Let s_1^2 represent the sample variance for sample 1, s_2^2 represent the sample variance for sample 2, and so on.

***Step* 4**: Compute the sum of squares due to treatments, SST, and the sum of squares due to error, SSE.

***Step* 5**: Divide each sum of squares by its corresponding degrees of freedom ($k - 1$ and $n - k$, respectively) to obtain the mean squares MST and MSE.

$$MST = \frac{\sum n_i \left(\bar{x}_i - \bar{x}\right)^2}{k-1} \qquad MSE = \frac{\sum \left(n_i - 1\right) s_i^2}{n-k}$$

***Step* 6**: Compute the *F*-test statistic: $F_0 = \dfrac{\textit{mean square due to treatments}}{\textit{mean square due to error}} = \dfrac{MST}{MSE}$

Example

Compute the *F*-test statistic for the penny data.

1990	**1995**	**2000**
2.50	2.52	2.50
2.50	2.54	2.48
2.49	2.50	2.49
2.53	2.48	2.50
2.46	2.52	2.48
2.50	2.50	2.52
2.47	2.49	2.51
2.53	2.53	2.49
2.51	2.48	2.51
2.49	2.55	2.50
2.48	2.49	2.52

Solution

1. $\bar{x} = \dfrac{2.50 + 2.50 + \cdots + 2.50 + 2.52}{33} = 2.5018$

2. $\bar{x}_{1990} = 2.4964 \quad \bar{x}_{1995} = 2.5091 \quad \bar{x}_{2000} = 2.5$

3. $s_{1990}^2 = \dfrac{(2.50 - 2.4964)^2 + \cdots + (2.48 - 2.4964)^2}{11-1} = 0.0005$

$s_{1995}^2 = \dfrac{(2.52 - 2.5091)^2 + \cdots + (2.49 - 2.5091)^2}{11-1} = 0.0006$

$$s^2_{200} = \frac{(2.50-2.5)^2 + \cdots + (2.52-2.5)^2}{11-1} \underline{=0.0002|}$$

4. $SST = 11(2.4964-2.5018)^2 + 11(2.5091-2.5018)^2 + 11(2.5-2.5018)^2 \underline{=0.0009|}$

$SSE = (11-1)(0.0005) + (11-1)(0.0006) + (11-1)(0.0002) \underline{=0.013|}$

5. $MST = \frac{SST}{k-1} = \frac{0.0009}{3-1} \underline{=0.0005|}$

$MSE = \frac{SSE}{n-k} = \frac{0.013}{33-3} \underline{=0.0004|}$

6. $F_0 = \frac{MST}{MSE} = \frac{0.0005}{0.0004} \underline{=1.25|}$

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Squares	*F*-Test Statistic
Treatment	0.0009	2	0.0005	1.25
Error	0.013	30	0.0004	
Total	0.0139	32		

Exercises Section 4.5 – Comparing Three or More Means

1. Fill in the ANOVA table

Source of Variation	***Sum of Squares***	***Degrees of Freedom***	***Mean Squares***	***F-Test Statistic***
Treatment	565	5		
Error	3560	32		
Total				

2. Fill in the ANOVA table

Source of Variation	***Sum of Squares***	***Degrees of Freedom***	***Mean Squares***	***F-Test Statistic***
Treatment	490	4		
Error	7267	21		
Total				

3. Determine the ***F***-test statistic based on the given summary statistics

Population	***Sample Size***	***Sample Mean***	***Sample Variance***
1	10	42	35
2	10	41	40
3	10	22	23

Compute $\overline{x}$, the sample mean of the combined data set, by adding up all the observations and dividing by the number pf the observations.

4. An engineer wants to know if the mean strengths of three concrete mix designs differ significantly. He randomly selects 9 cylinders that measure 6 inches in diameter and 12 inches in heights in which mixture *A* is poured, 9 cylinders of mixture *B*, and 9 cylinders of mixture *C*. After 28 days, he measures the strength (in pounds per square inch) of the cylinders. The results are presented in the table below.

Mixture A	***Mixture B***	***Mixture C***
3,980	4,070	4,130
4,040	4,340	3,820
3,760	4,620	4,020
3,870	3,730	4,150
3,990	4,870	4,190
4,090	4,120	3,840
3,820	4,640	3,750
3,940	4,180	3,990
4,080	3,850	4,320

a) Write the null and alternative hypotheses

b) Explain why the one-way ANOVA cannot be used to test these hypotheses

5. At a community college, the mathematics department has been experimenting with four different delivery mechanisms for content in their Elementary Statistics courses. One method is the traditional lecture (method I), the second is a hybrid format in which half the class time is online and the other half is face-to-face (method II), the third is online (method III), and the fourth is an emporium model from which students obtain their lectures and do their work in a lab with an instructor available for assistance (method IV). To assess the effectiveness of the four methods, students in each approach are given a final exam with the results shown in the accompanying table. Do the data suggest that any method has a different mean score from the others?

Method I	Method II	Method III	Method IV
76	88	78	89
81	52	60	90
85	77	73	79
68	73	70	62
88	64	62	83
73	38	82	75
80	57	74	54
65	63	80	70
60	83	53	80
92	65	46	94
83	78	84	76
51	64	80	78
71	87	78	81
63	92		
71			
65			

a) Write the null and alternative hypotheses

b) State the requirements that must be satisfied to use one-way ANOVA procedure

c) Assuming the requirements stated in part (b) are satisfied, use the following one-way ANOVA table to test the hypothesis of equal means at the $\alpha = 0.05$ level of significance.

d) Interpret the *P*-value.

e) Verify that the residuals are normally distributed