# Solution Section 1.5 – Continuity

### Exercise

Given the graphed function $f(x)$

*a)* Does $f(-1)$ exist?

*b)* Does $\lim\limits_{x \to -1^+} f(x)$ exist?

*c)* Does $\lim\limits_{x \to -1^+} f(x) = f(-1)$?

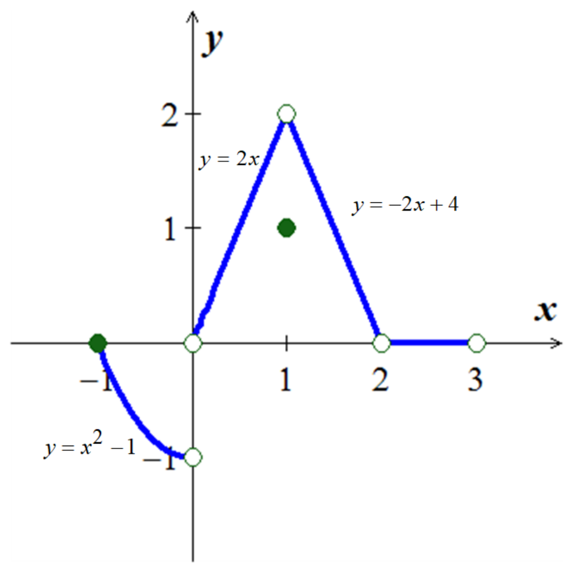
*d)* Is $f$ continuous at $x = -1$?

*e)* Does $f(1)$ exist?

*f)* Does $\lim\limits_{x \to 1} f(x)$ exist?

*g)* Does $\lim\limits_{x \to 1} f(x) = f(1)$?

*h)* Is $f$ continuous at $x = 1$?




**Solution**

***a)*** Yes $f(-1) = 0$

***b)*** Yes, $\lim\limits_{x \to -1^+} f(x) = 0$

***c)*** Yes

***d)*** Yes

***e)*** Yes, $f(1) = 1$

***f)*** Yes, $\lim\limits_{x \to 1} f(x) = 2$

***g)*** No

***h)*** No

### Exercise

At what points is the function $y = \dfrac{1}{x-2} - 3x$ continuous?

**Solution**

The function is continuous everywhere except when $x - 2 = 0 \Rightarrow x = 2$

### *Exercise*

At what points is the function $y=\dfrac{x+3}{x^2-3x-10}$ continuous?

**Solution**

The function is continuous everywhere except when $x^2-3x-10=0 \Rightarrow x=-2,\ 5$

### *Exercise*

At what points is the function $y=|x-1|+\sin x$ continuous?

**Solution**

The function is continuous everywhere

### *Exercise*

At what points is the function $y=\dfrac{x+2}{\cos x}$ continuous?

**Solution**

The function is continuous everywhere except when $\cos x=0 \Rightarrow x=\dfrac{\pi}{2}+n\pi, \quad n\in\mathbb{Z}$

### *Exercise*

At what points is the function $y=\tan\dfrac{\pi x}{2}$ continuous?

**Solution**

The function is continuous everywhere except when $x=2n-1, \quad n\in\mathbb{Z}$

### *Exercise*

At what points is the function $y=\dfrac{x\tan x}{x^2+1}$ continuous?

**Solution**

The function is continuous everywhere except when $x=(2n-1)\dfrac{\pi}{2}, \quad n\in\mathbb{Z}$

### *Exercise*

At what points is the function $y=\dfrac{\sqrt{x^4+1}}{1+\sin^2 x}$ continuous?

**Solution**

The function is continuous everywhere

### *Exercise*

At what points is the function $y = \sqrt{2x+3}$ continuous?

***Solution***

The function is continuous on the interval $2x+3 \ge 0 \quad \to \quad x \ge -\frac{3}{2} \Rightarrow \left[-\frac{3}{2},\ \infty\right)$, and discontinuous when $x < -\frac{3}{2}$

### *Exercise*

At what points is the function $y = \sqrt[4]{3x-1}$ continuous?

***Solution***

The function is continuous on the interval $3x-1 \ge 0 \to \left[\frac{1}{3},\ \infty\right)$, and discontinuous when $x < \frac{1}{3}$

### *Exercise*

At what points is the function $y = (2-x)^{1/5}$ continuous?

***Solution***

The function is continuous everywhere $\forall x$

### *Exercise*

Find $\lim\limits_{x\to\pi} \sin(x - \sin x)$, then is the function continuous at the point being approached?

***Solution***

$$\begin{aligned}\lim_{x\to\pi} \sin(x-\sin x) &= \sin(\pi - \sin\pi)\\ &= \sin(\pi - 0)\\ &= \sin(\pi)\\ &= 0\end{aligned}$$

The function is continuous at $x = \pi$

### *Exercise*

Find $\lim\limits_{x\to 0} \tan\left(\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right)$, then is the function continuous at the point being approached?

***Solution***

$$\begin{aligned}\lim_{x\to 0} \tan\left(\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right) &= \tan\left(\frac{\pi}{4}\cos\left(\sin(0)^{1/3}\right)\right)\\ &= \tan\left(\frac{\pi}{4}\cos(0)\right)\end{aligned}$$

$$= \tan\left(\frac{\pi}{4}\right)$$

$$\underline{= 1}$$ The function is continuous at $x = 0$

## *Exercise*

Find $\lim\limits_{t\to 0} \cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right)$, then is the function continuous at the point being approached?

***Solution***

$$\lim_{t\to 0} \cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right) = \cos\left(\frac{\pi}{\sqrt{19-3\sec 2(0)}}\right)$$

$$= \cos\left(\frac{\pi}{\sqrt{19-3}}\right)$$

$$= \cos\left(\frac{\pi}{\sqrt{16}}\right)$$

$$= \cos\left(\frac{\pi}{4}\right)$$

$$\underline{= \frac{\sqrt{2}}{2}}$$

$\therefore$ The function is continuous at $t = 0$

## *Exercise*

Explain why the equation $\cos x = x$ has at least one solution.

***Solution***

$$\cos x - x = 0$$

$$\begin{cases} if \quad x = -\frac{\pi}{2} & \to \cos\left(-\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) > 0 \\ if \quad x = \frac{\pi}{2} & \to \cos\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right) < 0 \end{cases}$$

$$\Rightarrow \cos x - x = 0$$

for some $x$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

According to the Intermediate Value Theorem, and the function $\cos x = x$ is continuous and has at least one solution.

### *Exercise*

Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval [–4, 4]

***Solution***

$$f(-4) = (-4)^3 - 15(-4) + 1 = -3$$

$$f(-2) = (-2)^3 - 15(-2) + 1 = 23$$

$$f(-1) = (-1)^3 - 15(-1) + 1 = 15$$

$$f(1) = (1)^3 - 15(1) + 1 = -13$$

$$f(4) = (4)^3 - 15(4) + 1 = 5$$

By the Intermediate Value Theorem, $f(x) = 0$ for some $x$ in each of the intervals $-4 < x < -1$, $-1 < x < 1$, and $1 < x < 4$. Thus, $x^3 - 15x + 1 = 0$ has three solutions in [–4, 4]. Since the polynomial of degree 3 can have at most 3 solutions, these are the solutions.

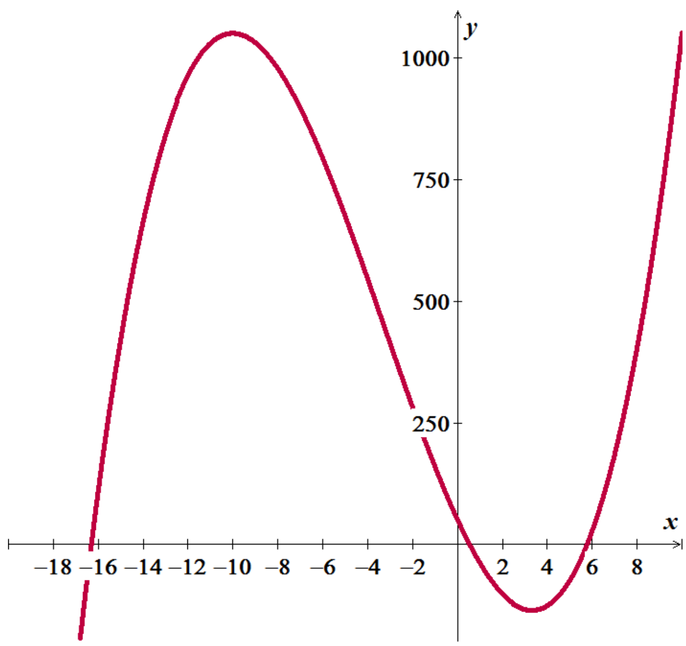
### *Exercise*

Show that the equation has three solutions in the given interval $x^3 + 10x^2 - 100x + 50 = 0;\quad (-20,\ 10)$

***Solution***

| $x$ | $y$ |
|---|---|
| –19 | –1299 |
| –18 | –742 |
| –17 | –273 |
| –16 | 114 |
| –15 | 425 |
| –14 | 666 |
| –13 | 962 |
| –12 | 1029 |
| –10 | 1050 |
| –9 | 1031 |
| –8 | 978 |
| –7 | 897 |
| –6 | 794 |
| –5 | 675 |
| –4 | 546 |

| | |
|---|---|
| $-3$ | 413 |
| $-2$ | 282 |
| $-1$ | 159 |
| 0 | 50 |
| 1 | $-39$ |
| 2 | $-102$ |
| 3 | $-133$ |
| 4 | $-126$ |
| 5 | $-75$ |
| 6 | 26 |



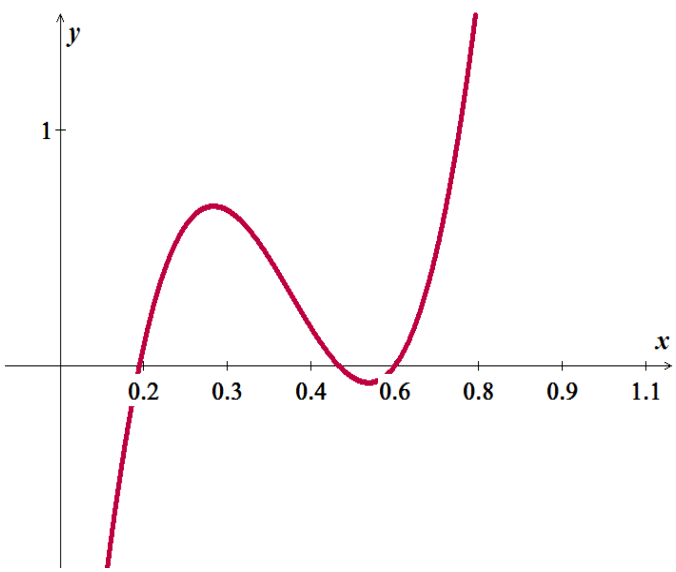

By the Intermediate Value Theorem, $f(x) = 0$ for some $x$ in each of the intervals $-17 < x < -16$, $0 < x < 1$, and $5 < x < 6$.

### *Exercise*

Show that the equation has three solutions in the given interval $70x^3 - 87x^2 + 32x - 3 = 0; \quad (0,\ 1)$

***Solution***

| $x$ | $y$ |
|---|---|
| .05 | –1.6 |
| .1 | –0.6 |
| .15 | 0.08 |
| .2 | .48 |
| .25 | .656 |
| .3 | .66 |
| .35 | .543 |
| .4 | .36 |
| .45 | .161 |
| .5 | 0 |
| .55 | –.07 |
| .6 | 0 |
| .65 | .266 |
| .7 | .78 |
| .75 | 1.6 |
| .8 | 2.76 |
| .85 | 4.33 |
| .9 | 6.36 |
| .95 | 8.9 |



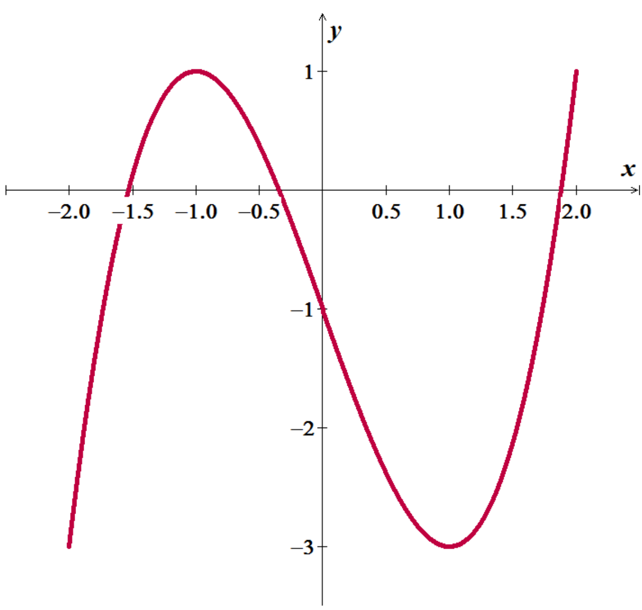

By the Intermediate Value Theorem, $f(x) = 0$ for some $x$ in each of the intervals $0.1 < x < 0.15$, $0.5 < x < 0.55$, and $0.55 < x < 0.6$.

## *Exercise*

Show that the equation has three solutions in the given interval $x^3 - 3x - 1 = 0; \quad [-2,\ 2]$

***Solution***

| $x$ | $y$ |
|---|---|
| –2 | –3.0 |
| –1.75 | –1.109 |
| –1.5 | 0.125 |
| –1.25 | 0.797 |
| –1.0 | 1 |
| –0.75 | 0.828 |
| –0.5 | 0.375 |
| –0.25 | –0.266 |
| 0 | –1.0 |
| 0.25 | –1.73 |
| 0.5 | –2.375 |
| 0.75 | –2.828 |
| 1.0 | –3.0 |
| 1.25 | –2.797 |
| 1.5 | –2.12 |
| 1.75 | –0.89 |
| 2. | 1.0 |



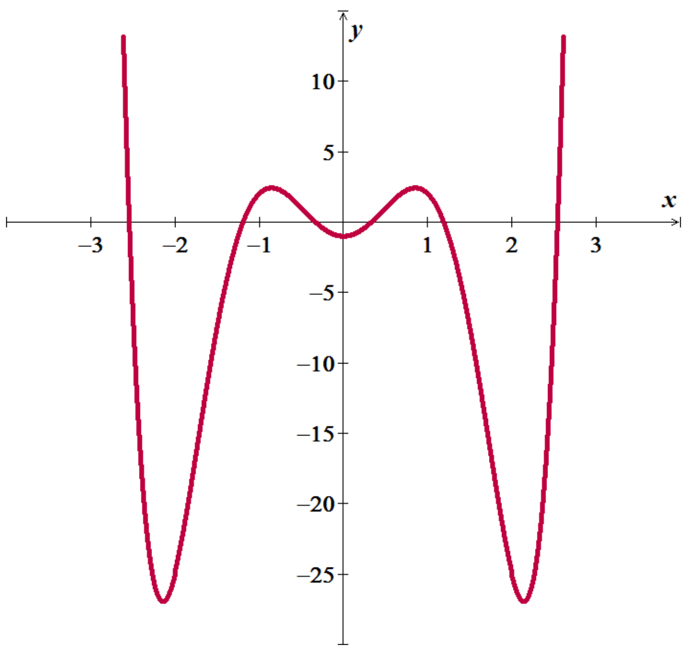

By the Intermediate Value Theorem, $f(x) = 0$ for some $x$ in each of the intervals $-1.75 < x < -1.5$, $-0.5 < x < -0.25$, and $1.75 < x < 2$.

## *Exercise*

Show that the equation has six solutions in the given interval $x^6 - 8x^4 + 10x^2 - 1 = 0;\quad [-3,\ 3]$

***Solution***

| $x$ | $y$ |
|---|---|
| –3.0 | 170.0 |
| –2.5 | –6.86 |
| –2.0 | –25.0 |
| –1.5 | –7.61 |
| –1.0 | 2.0 |
| –0.5 | 1.02 |
| 0.0 | –1.0 |
| 0.5 | 1.01 |
| 1.0 | 2.0 |
| 1.5 | –7.6 |
| 2.0 | –25.0 |
| 2.5 | –6.86 |
| 3.0 | 170.0 |




By the Intermediate Value Theorem, $f(x) = 0$ for some $x$ in each of the intervals $-3.0 < x < -2.5$, $-1.5 < x < -1.0$, $-0.5 \le x \le 0$, $-0.0 \le x \le 0.5$, $1.0 \le x \le 1.5$ and $2.5 < x < 3.0$.

## Exercise

If functions $f(x)$ and $g(x)$ are continuous for $0 \le x \le 1$, could $\dfrac{f(x)}{g(x)}$ possibly be discontinuous at a point of [0, 1]? Give reason for your answer.

***Solution***

Yes, if we can get a value of $g(x)$ is between [0, 1], $x = \frac{1}{2} \Rightarrow g(x) = 2x - 1$ and $f(x) = x$.

Then $\dfrac{f(x)}{g(x)} = \dfrac{x}{2x-1} \Rightarrow \dfrac{f(x)}{g(x)}$ is discontinuous at $x = \frac{1}{2}$

## Exercise

Suppose that a function $f$ is continuous on the closed interval [0, 1] and that $0 \le f(x) \le 1$ for every $x$ in [0, 1]. Show that there must exist a number $c$ in [0, 1] such that $f(c) = c$ (***c*** is called a ***fixed point*** of $f$).

***Solution***

Let $f(x) = x \Rightarrow f(0) = 0 \quad or \quad f(1) = 1$. In these cases, $c$ = 0 or $c$ =1.

Let $f(0) = a > 0 \quad and \quad f(1) = b < 1$ because $0 \le f(x) \le 1$.

Define $g(x) = f(x) - x \Rightarrow g$ is continuous on [0, 1].

$$\Rightarrow \begin{cases} g(0) = f(0) - 0 = a > 0 \\ g(1) = f(1) - 1 = b - 1 < 0 \end{cases}$$

By the Intermediate Value Theorem there is a number $c$ in [0, 1] such that

$$g(c) = 0 \Rightarrow f(c) - c = 0 \Rightarrow f(c) = c$$

## Exercise

Use the Intermediate Value Theorem to show that the equation $x^5 + 7x + 5 = 0$ has a solution in the interval $(-1,\ 0)$.

***Solution***

$$f(-1) = -1 - 7 + 5 = -3 < 0$$

$$f(0) = 5 > 0$$

By Intermediate value theorem, the function has a solution in $(-1,\ 0)$

## *Exercise*

The amount of an antibiotic (in *mg*) in the blood *t* hours after an intravenous line is opened is given by

$$m(t) = 100\left(e^{-0.1t} - e^{-0.3t}\right)$$

*a)* Use the Intermediate Value Theorem to show that the amount of drug is 30 *mg* at some time in the interval $[0,\ 5]$ and again at some time in the interval $[5,\ 15]$

*b)* Estimate the times at which $m = 30\ mg$

*c)* Is the amount of drug in the blood ever 50 *mg*?

**Solution**

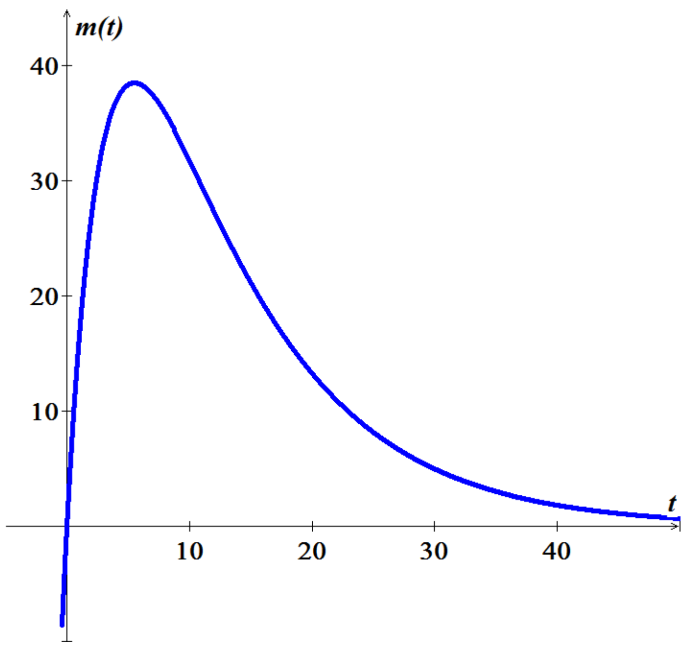
***a)*** $m(0) = 100(1-1) = 0$

$m(5) \approx 38.34 > 30$

$m(15) \approx 21.2 < 30$

30 is an intermediate value between for both $[0,\ 5]$ and $[5,\ 15]$.

***b)*** $m(t) = 100\left(e^{-0.1t} - e^{-0.3t}\right) = 30$




$$e^{-0.1t} - e^{-0.3t} = 0.3 \xrightarrow{\text{software}} \begin{cases} \underline{t_1 \approx 2.4} \\ \underline{t_2 \approx 10.8} \end{cases}$$

***c)*** No, peak is 38.5 (using the graph)

### Exercise

Determine whether the following functions are continuous at $a$. $f(x)=\dfrac{1}{x-5};\quad a=5$

***Solution***

$f(5)\ \nexists$

The function is continuous everywhere except @ $x=5$

### Exercise

Determine whether the following functions are continuous at $a$. $h(x)=\sqrt{x^2-9};\quad a=3$

***Solution***

$\lim\limits_{x\to 3^-} h(x)\ \nexists \quad \therefore\ h$ is discontinuous @ 3

### Exercise

Determine whether the following functions are continuous at $a$. $g(x)=\begin{cases} \dfrac{x^2-16}{x-4} & if \quad x\neq 4 \\ 9 & if \quad x=4 \end{cases};\quad a=4$

***Solution***

$$\lim_{x\to 4} g(x)=\lim_{x\to 4}\frac{(x-4)(x+4)}{x-4}=\lim_{x\to 4}(x+4)=8\neq 9=g(4)$$

$\therefore\ g$ is discontinuous @ 4

### Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $\quad f(x)=\sqrt{x^2-5}$

***Solution***

$\sqrt{x^2-5\geq 0} \quad\Rightarrow\quad x\leq -5\ \ \&\ \ x\geq 5$

The function is continuous at $-5$ to the left and right of $x=5$

### Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $\quad f(x)=e^{\sqrt{x-2}}$

***Solution***

The function is continuous at and to the right of $x=2$

### *Exercise*

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $f(x) = \dfrac{2x}{x^3 - 25x}$

***Solution***

The function is continuous everywhere except at $x = 0, \ \pm 5$

The function is continuous to the left of –5, then to the right of –5 to the left of 0, then to the right of 0 thru the left of 5 then to the tight of 5.

### *Exercise*

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $f(x) = \cos e^x$

***Solution***

The function is continuous everywhere.

### *Exercise*

Let $g(x) = \begin{cases} 5x - 2 & if \quad x < 1 \\ a & if \quad x = 1 \\ ax^2 + bx & if \quad x > 1 \end{cases}$

Determine values of the constants $a$ and $b$ for which $g(x)$ is continuous at $x = 1$

***Solution***

$$\lim_{x \to 1^-} g(x) = g(1)$$

$$= 5 - 2$$

$$\underline{= 3 = a}$$

$$\lim_{x \to 1^-} g(x) = g(1)$$

$$= a + b$$

$$= 3 + b = 3$$

$$\to \quad \underline{b = 0}$$