Exercise

Sketch the region of integration and evaluate the integral $\int_0^{\pi}\int_0^{x} x\sin y\ dydx$

Solution

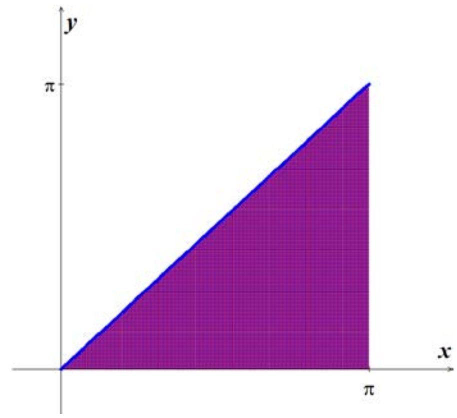
$$\int_0^{\pi}\int_0^{x} x\sin y\ dydx = \int_0^{\pi}\left[-x\cos y\right]_0^{x} dx$$

$$= \int_0^{\pi}\left[-x\cos x + x\right] dx$$

$$= \left[-(x\sin x + \cos x) + \frac{1}{2}x^2\right]_0^{\pi}$$

$$= -(-1) + \frac{1}{2}\pi^2 - (-1)$$

$$= \frac{\pi^2}{2} + 2$$



		$\int \cos x$
+	x	$\sin x$
–	1	$-\cos x$

Exercise

Sketch the region of integration and evaluate the integral $\int_0^{\pi}\int_0^{\sin x} ydydx$

Solution

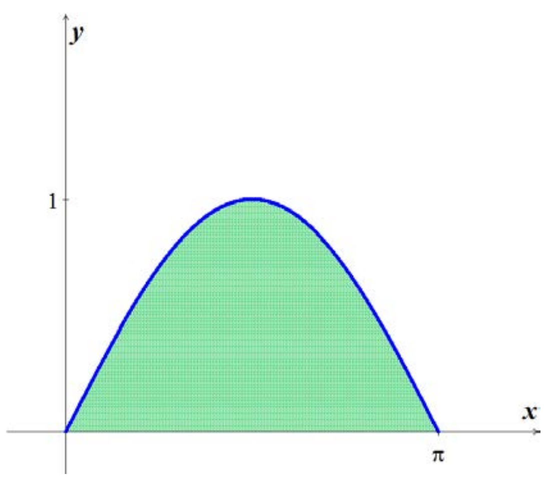
$$\int_0^{\pi}\int_0^{\sin x} ydydx = \int_0^{\pi}\left[\frac{1}{2}y^2\right]_0^{\sin x} dx$$

$$= \int_0^{\pi}\frac{1}{2}\sin^2 x\ dx$$

$$= \frac{1}{4}\int_0^{\pi}(1-\cos 2x)\ dx$$

$$= \frac{1}{4}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}$$

$$= \frac{\pi}{4}$$



Exercise

Sketch the region of integration and evaluate the integral $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dxdy$

Solution

$$\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dxdy = \int_1^{\ln 8}\left[e^{x+y}\right]_0^{\ln y} dy$$

$$= \int_1^{\ln 8}\left(e^{\ln y+y} - e^y\right)dy$$

$$= \int_1^{\ln 8}\left(e^{\ln y}e^y - e^y\right)dy$$

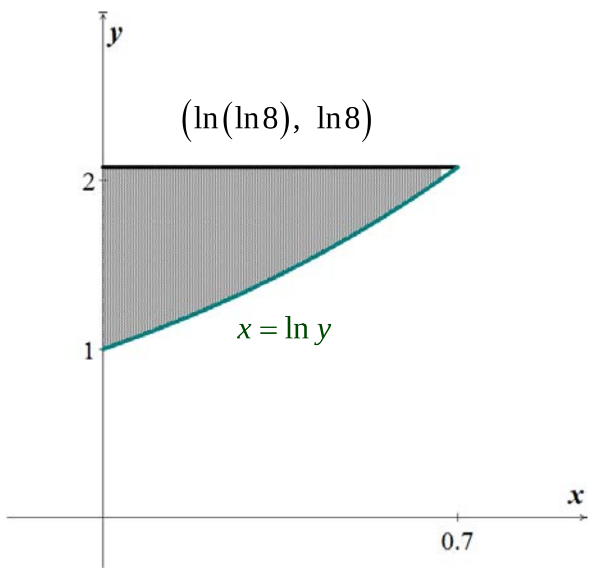
$$= \int_1^{\ln 8}\left(ye^y - e^y\right)dy$$

$$= \left[ye^y - e^y - e^y\right]_1^{\ln 8}$$

$$= (\ln 8)e^{\ln 8} - 2e^{\ln 8} - (e - 2e)$$

$$= 8\ln 8 - 16 - e$$

		$\int e^y$
+	y	e^y
−	1	e^y



Exercise

Sketch the region of integration and evaluate the integral $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}} dy dx$

Solution

$$\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}} dy dx = \frac{3}{2}\int_1^4 \left[\sqrt{x} e^{y/\sqrt{x}}\right]_0^{\sqrt{x}} dx$$

$$= \frac{3}{2}\int_1^4 \sqrt{x}(e-1) dx$$

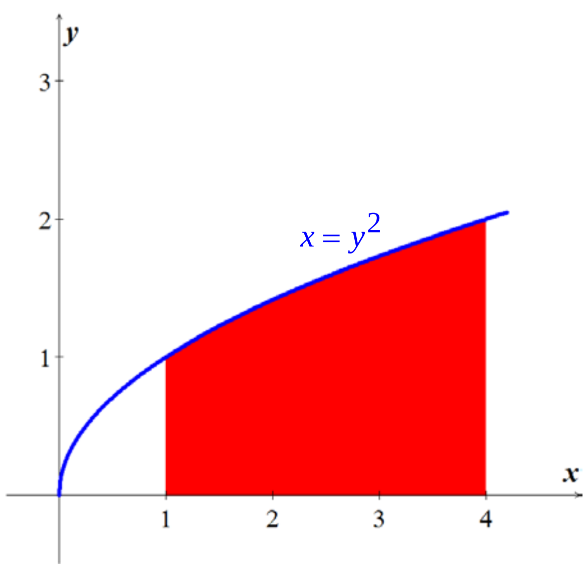
$$= \frac{3}{2}(e-1)\int_1^4 x^{1/2} dx$$

$$= \frac{3}{2}(e-1)\left[\frac{2}{3}x^{3/2}\right]_1^4$$

$$= (e-1)\left[x^{3/2}\right]_1^4$$

$$= (e-1)[8-1]$$

$$= 7(e-1)$$



Exercise

Integrate $f(x,y)=\frac{x}{y}$ over the region in the first quadrant bounded by the lines

$y=x, \quad y=2x, \quad x=1, \quad and \quad x=2$

Solution

$$
\begin{aligned}
\int_1^2\int_x^{2x}\frac{x}{y}\,dydx &= \int_1^2\left[x\ln y\right]_x^{2x}dx\\
&= \int_1^2 x\left(\ln 2x-\ln x\right)dx\\
&= \int_1^2 x\left(\ln\frac{2x}{x}\right)dx \qquad \textbf{Quotient Rule}: \ \ln M-\ln P=\ln\frac{M}{P}\\
&= \ln 2\int_1^2 x\,dx\\
&= \ln 2\left[\frac{1}{2}x^2\right]_1^2\\
&= \ln 2\left[\frac{1}{2}(4-1)\right]\\
&= \frac{3}{2}\ln 2
\end{aligned}
$$

Exercise

Integrate $f(x,y)=x^2+y^2$ over the triangular region with vertices $(0,0), \quad (1,0) \quad and \quad (0,1)$

Solution

$$
\begin{aligned}
\int_0^1\int_0^{1-x}\left(x^2+y^2\right)dydx &= \int_0^1\left[x^2y+\frac{1}{3}y^3\right]_0^{1-x}dx\\
&= \int_0^1\left[x^2(1-x)+\frac{1}{3}(1-x)^3\right]dx\\
&= \int_0^1\left[x^2-x^3+\frac{1}{3}(1-x)^3\right]dx\\
&= \left[\frac{1}{3}x^3-\frac{1}{4}x^4-\frac{1}{12}(1-x)^4\right]_0^1\\
&= \frac{1}{3}-\frac{1}{4}-0-\left(0-0-\frac{1}{12}\right)\\
&= \frac{1}{6}
\end{aligned}
$$

Exercise

Integrate $f(s,t) = e^s \ln t$ over the region in the first quadrant of the *st*-plane that lies above the curve $s = \ln t$ from $t = 1$ to $t = 2$.

Solution

$$\int_1^2 \int_0^{\ln t} e^s \ln t \, ds dt = \int_1^2 \left[e^s \ln t \right]_0^{\ln t} dt$$

$$= \int_1^2 (t \ln t - \ln t) dt$$

$$\begin{aligned} u &= \ln t & dv &= dt \\ du &= \frac{1}{t} dt & v &= t \end{aligned} \quad \to \quad \int \ln t = t \ln t - \int t \frac{1}{t} dt = t \ln t - t$$

$$\int t \ln t = \frac{1}{2} t^2 \ln t - \frac{1}{4} t^2$$

$$= \left[\frac{1}{2} t^2 \ln t - \frac{1}{4} t^2 - t \ln t + t \right]_1^2$$

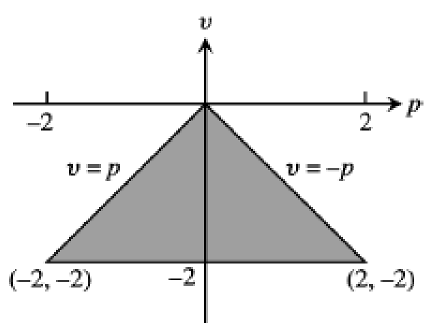
$$= 2 \ln 2 - 1 - 2 \ln 2 + 2 - \left(0 - \frac{1}{4} - 0 + 1 \right)$$

$$= \frac{1}{4}$$

Exercise

Evaluate $\displaystyle \int_{-2}^{0} \int_{v}^{-v} 2 dp dv$

Solution



$$\int_{-2}^{0} \int_{v}^{-v} 2 dp dv = 2 \int_{-2}^{0} [p]_v^{-v} dv$$

$$= -4 \int_{-2}^{0} v \, dv$$

$$= -2 \left[v^2 \right]_{-2}^{0}$$

$$= 8$$

Exercise

Evaluate $\displaystyle\int_{-\pi/3}^{\pi/3}\int_{0}^{\sec t} 3\cos t \; du dt$

Solution

$$\int_{-\pi/3}^{\pi/3}\int_{0}^{\sec t} 3\cos t \; du dt = \int_{-\pi/3}^{\pi/3} (3\cos t)\left[u\right]_0^{\sec t} dt$$

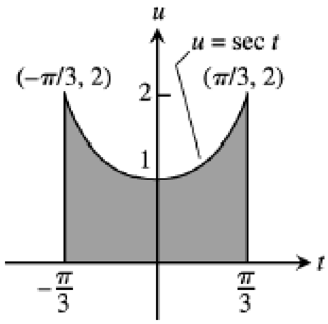
$$= \int_{-\pi/3}^{\pi/3} (3\cos t \sec t) dt \qquad \cos t \sec t = \cos t \frac{1}{\cos t} = 1$$

$$= \int_{-\pi/3}^{\pi/3} 3 dt$$

$$= 3t \Big|_{-\pi/3}^{\pi/3}$$

$$= 3\frac{2\pi}{3}$$

$$= 2\pi$$

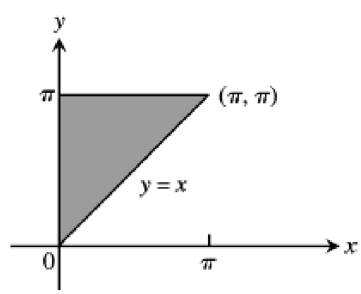


Exercise

Sketch the region of integration, reverse the order of integration, and evaluate the integral

$$\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y} dy dx$$

Solution



$$\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y} dy dx = \int_0^{\pi}\int_0^{y} \frac{\sin y}{y} dx dy$$

$$= \int_0^{\pi} \frac{\sin y}{y}\left[x\right]_0^{y} dy$$

$$= \int_0^{\pi} \frac{\sin y}{y}(y) dy$$

$$= \int_0^{\pi} \sin y \; dy$$

$$= -\cos y \Big|_0^{\pi}$$

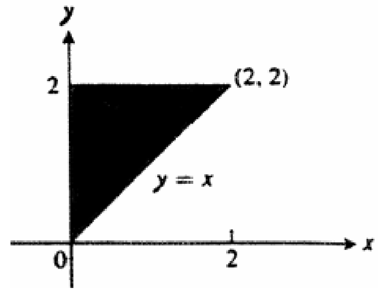
$$= -(-1-1)$$

$$= 2$$

Exercise

Sketch the region of integration, reverse the order of integration, and evaluate the integral

$$\int_0^2 \int_x^2 2y^2 \sin xy \; dydx$$



Solution

$$\begin{aligned}\int_0^2 \int_x^2 2y^2 \sin xy \; dydx &= \int_0^2 \int_0^y 2y^2 \sin xy \; dxdy \\ &= -2\int_0^2 \left[y\cos xy\right]_0^y \; dy \\ &= -2\int_0^2 \left(y\cos y^2 - y\right)dy \qquad u = y^2 \;\Rightarrow\; du = 2ydy \\ &= -\int_0^2 \cos u du + \int_0^2 2ydy \\ &= \left[-\sin y^2 + y^2\right]_0^2 \\ &= -\sin 4 + 4\end{aligned}$$

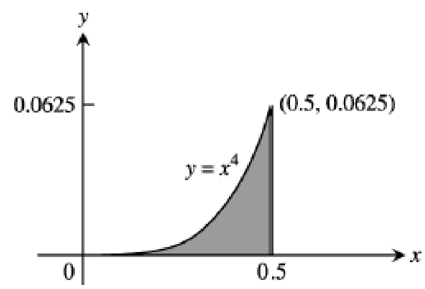
Exercise

Sketch the region of integration, reverse the order of integration, and evaluate the integral

$$\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos\left(16\pi x^5\right)dxdy$$

Solution

$$x = y^{1/4} \quad \Rightarrow \quad y = x^4$$



$$\begin{aligned}\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos\left(16\pi x^5\right)dxdy &= \int_0^{1/2} \int_0^{x^4} \cos\left(16\pi x^5\right)dydx \\ &= \int_0^{1/2} \cos\left(16\pi x^5\right)\left[y\right]_0^{x^4} \; dx \\ &= \int_0^{1/2} x^4\cos\left(16\pi x^5\right)dx \qquad u = 16\pi x^5 \;\rightarrow\; du = 80\pi x^4 dx\end{aligned}$$

$$=\frac{1}{80\pi}\int_0^{1/2}\cos u\ du$$

$$=\frac{1}{80\pi}\left[\sin 16\pi x^5\right]_0^{1/2}$$

$$=\frac{1}{80\pi}\left(\sin\frac{16\pi}{32}-0\right)$$

$$=\frac{1}{80\pi}$$

Exercise

Sketch the region of integration, reverse the order of integration, and evaluate the integral

$$\int_0^2\int_0^{4-x^2}\frac{xe^{2y}}{4-y}dydx$$

Solution

$$y=4-x^2 \quad\Rightarrow\quad x^2=4-y\to x=\sqrt{4-y}$$

$$\int_0^2\int_0^{4-x^2}\frac{xe^{2y}}{4-y}dydx=\int_0^4\int_0^{\sqrt{4-y}}\frac{xe^{2y}}{4-y}dxdy$$

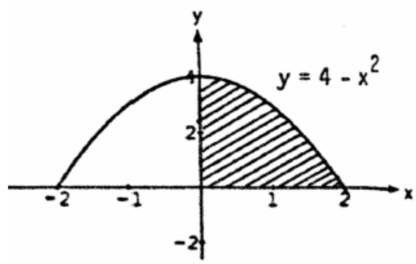
$$=\int_0^4\frac{e^{2y}}{4-y}\left[\frac{1}{2}x^2\right]_0^{\sqrt{4-y}}dy$$

$$=\frac{1}{2}\int_0^4\frac{e^{2y}}{4-y}(4-y)dy$$

$$=\frac{1}{2}\int_0^4 e^{2y}dy$$

$$=\frac{1}{4}\left[e^{2y}\right]_0^4$$

$$=\frac{1}{4}\left(e^8-1\right)$$



Exercise

Find the volume of the region bounded above the paraboloid $z=x^2+y^2$ and below by the triangle enclosed by the lines $y=x$, $x=0$, and $x+y=2$ in the *xy*-plane

Solution

$$V=\int_0^1\int_x^{2-x}\left(x^2+y^2\right)dydx$$

$y=x \qquad x+y=2\to y=2-x$

$x=0 \qquad y=x\to x+x=2\Rightarrow x=1$

$$=\int_0^1\left[x^2y+\frac{1}{3}y^3\right]_x^{2-x}dx$$

$$=\int_0^1\left(x^2(2-x)+\frac{1}{3}(2-x)^3-x^3-\frac{1}{3}x^3\right)dx$$

$$=\int_0^1\left(2x^2-x^3+\frac{1}{3}(2-x)^3-\frac{4}{3}x^3\right)dx$$

$$=\int_0^1\left(2x^2-\frac{7}{3}x^3\right)dx+\int_0^1\frac{1}{3}(2-x)^3\left(-d(2-x)\right)$$

$$=\left[\frac{2}{3}x^3-\frac{7}{12}x^4-\frac{1}{12}(2-x)^4\right]_0^1$$

$$=\left(\frac{2}{3}-\frac{7}{12}-\frac{1}{12}\right)-\left(-\frac{16}{12}\right)$$

$$=\frac{4}{3}$$

Exercise

Find the volume of the solid that is bounded above the cylinder $z=x^2$ and below by the region enclosed by the parabola $y=2-x^2$ and the line $y=x$ in the *xy*-plane

Solution

$$V=\int_{-2}^1\int_x^{2-x^2}x^2dydx$$

$$=\int_{-2}^1x^2[y]_x^{2-x^2}dx$$

$$=\int_{-2}^1x^2\left(2-x^2-x\right)dx$$

$$=\int_{-2}^1\left(2x^2-x^4-x^3\right)dx$$

$$=\left[\frac{2}{3}x^3-\frac{1}{5}x^5-\frac{1}{4}x^4\right]_{-2}^1 \qquad =\frac{2}{3}-\frac{1}{5}-\frac{1}{4}-\left(-\frac{15}{3}+\frac{32}{5}-\frac{16}{4}\right)$$

$$=\frac{63}{20}$$

Exercise

Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$ and the plane $z + y = 3$

Solution

$$V = \int_0^2 \int_0^{\sqrt{4-x^2}} (3-y)\,dy\,dx$$

$$= \int_0^2 \left[3y - \frac{1}{2}y^2\right]_0^{\sqrt{4-x^2}} dx$$

$$= \int_0^2 \left[3\sqrt{4-x^2} - \frac{1}{2}\left(4-x^2\right)\right] dx \qquad \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$= \left[\frac{3}{2}x\sqrt{4-x^2} + 6\sin^{-1}\left(\frac{x}{2}\right) - 2x + \frac{1}{6}x^3\right]_0^2$$

$$= 0 + 6\frac{\pi}{2} - 4 + \frac{8}{6} - (0)$$

$$= 3\pi - \frac{8}{3}$$

Exercise

Find the volume of the solid that is bounded on the front and back by the planes $x = 2$, and $x = 1$, on the sides by the cylinders $y = \pm\frac{1}{x}$ and above and below the planes $z = x + 1$ and $z = 0$.

Solution

$$V = \int_1^2 \int_{-1/x}^{1/x} (x+1)\,dy\,dx$$

$$= \int_1^2 (x+1)\left[y\right]_{-1/x}^{1/x} dx$$

$$= \int_1^2 (x+1)\left(\frac{2}{x}\right) dx$$

$$= 2\int_1^2 \left(1 + \frac{1}{x}\right) dx$$

$$= 2\left[x + \ln x\right]_1^2$$

$$= 2\left[2 + \ln 2 - 1\right]$$

$$= 2(1 + \ln 2)$$

Exercise

Find the area of the region enclosed by the coordinate axes and the line $x+y=2$.

Solution

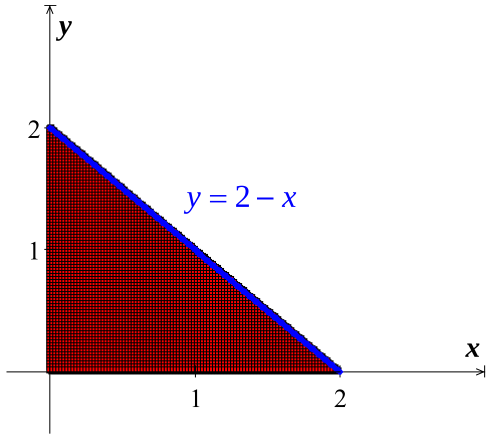
$$\int_0^2\int_0^{2-x} dydx = \int_0^2 [y]_0^{2-x}\,dx$$

$$= \int_0^2 (2-x)\,dx$$

$$= \left[2x-\frac{1}{2}x^2\right]_0^2$$

$$= 4-\frac{1}{2}(4)$$

$$\underline{=2}$$



Exercise

Find the area of the region enclosed by the lines $x=0$, $y=2x$, and $y=4$

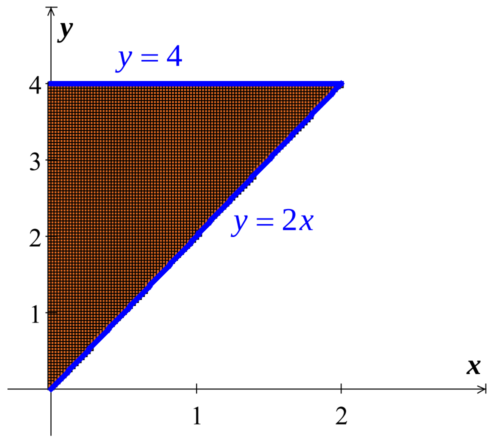
Solution

$$\int_0^2\int_{2x}^{4} dydx = \int_0^2 [y]_{2x}^{4}\,dx$$

$$= \int_0^2 (4-2x)\,dx$$

$$= \left[4x-x^2\right]_0^2$$

$$\underline{=4}$$



Exercise

Find the area of the region enclosed by the parabola $x=y-y^2$ and the line $y=-x$.

Solution

$$x=y-y^2=-y \quad \to \quad 2y-y^2=0 \Rightarrow \boxed{y=0,\ 2}$$

$$\int_0^2\int_{-y}^{y-y^2} dxdy = \int_0^2 [x]_{-y}^{y-y^2}\,dy$$

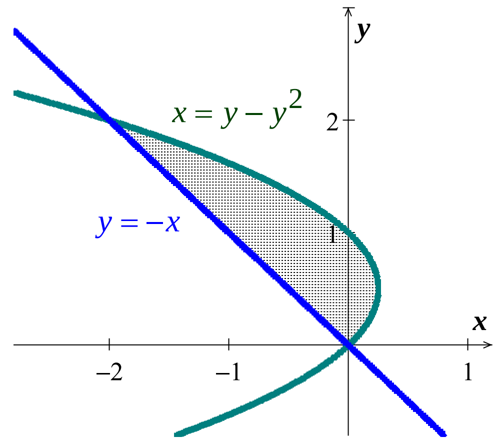
$$= \int_0^2 \left(y - y^2 + y \right) dy$$

$$= \int_0^2 \left(2y - y^2 \right) dy$$

$$= \left[y^2 - \frac{1}{3}y^3 \right]_0^2$$

$$= 4 - \frac{8}{3}$$

$$= \frac{4}{3}$$



Exercise

Find the area of the region enclosed by the curve $y = e^x$ and the lines $y = 0$, $x = 0$ and $x = \ln 2$.

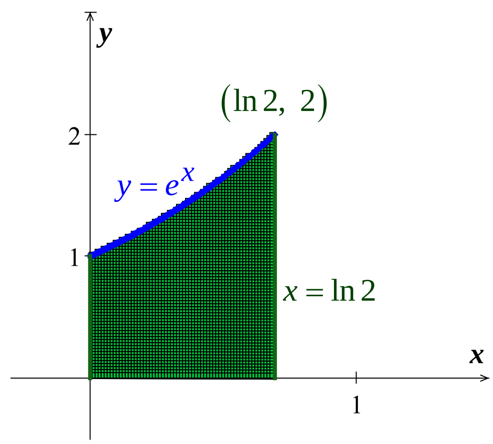
Solution

$$\int_0^{\ln 2} \int_0^{e^x} dydx = \int_0^{\ln 2} \left[y \right]_0^{e^x} dx$$

$$= \int_0^{\ln 2} e^x dx$$

$$= \left[e^x \right]_0^{\ln 2} \quad = 2 - 1$$

$$= 1$$



Exercise

Find the area of the region enclosed by the curve $y = \ln x$ and $y = 2\ln x$ and the lines $x = e$ in the first quadrant.

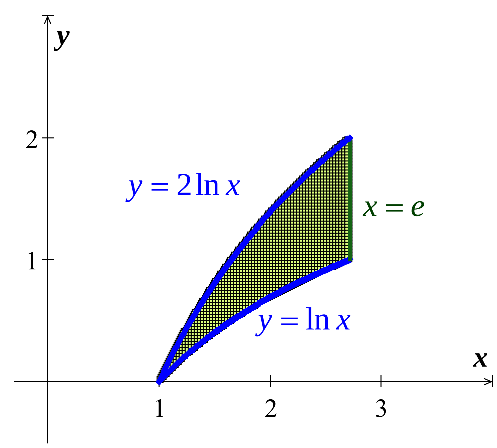
Solution

$$\int_1^e \int_{\ln x}^{2\ln x} dydx = \int_1^e \left[y \right]_{\ln x}^{2\ln x} dx$$

$$= \int_0^{\ln 2} \ln x \ dx$$

$$= \left[x\ln x - x \right]_1^e \quad = e - e - (0 - 1)$$

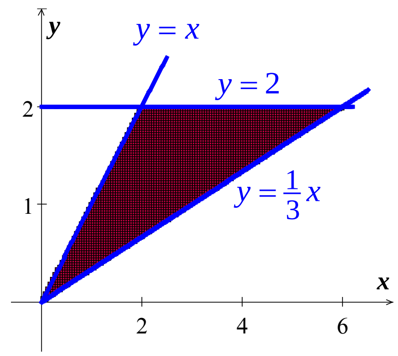
$$= 1$$



Exercise

Find the area of the region enclosed by the lines $y = x$, $y = \frac{x}{3}$, and $y = 2$

Solution



$$\int_0^2 \int_y^{3y} dxdy = \int_0^2 x \Big|_y^{3y} dy$$

$$= \int_0^2 (2y)\,dy$$

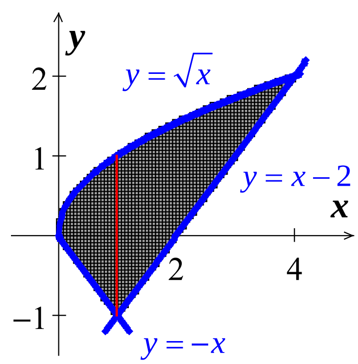
$$= y^2 \Big|_0^2$$

$$\underline{= 4}$$

Exercise

Find the area of the region enclosed by the lines $y = x - 2$ and $y = -x$ and the curve $y = \sqrt{x}$

Solution



$$\int_0^1 \int_{-x}^{\sqrt{x}} dydx + \int_1^4 \int_{x-2}^{\sqrt{x}} dydx = \int_0^1 y \Big|_{-x}^{\sqrt{x}} dx + \int_1^4 y \Big|_{x-2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left(\sqrt{x} - x\right) dx + \int_1^4 \left(\sqrt{x} - x + 2\right) dx$$

$$= \left[\frac{2}{3}x^{3/2} + \frac{1}{2}x^2\right]_0^1 + \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right]_1^4$$

$$= \frac{2}{3} + \frac{1}{2} + \frac{2}{3}4^{3/2} - 2 + 8 - \frac{2}{3} - \frac{1}{2} + 2$$

$$\underline{= \frac{13}{3}}$$

Exercise

Find the area of the region enclosed by the parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$

Solution

$$\int_{-1}^1 \int_{2y^2-2}^{y^2-1} dxdy = \int_{-1}^1 [y]_{2y^2-2}^{y^2-1} dy$$

$$= \int_{-1}^{1} \left(y^2 - 1 - 2y^2 + 2\right) dy$$

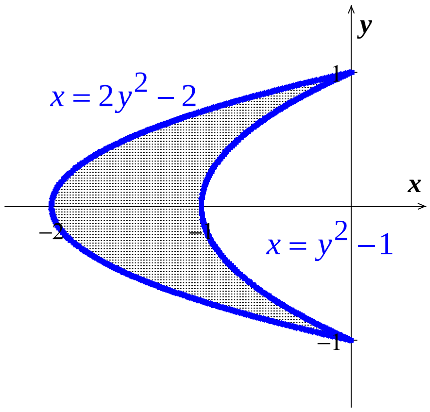
$$= \int_{-1}^{1} \left(1 - y^2\right) dy$$

$$= \left[y - \frac{1}{3} y^3 \right]_{-1}^{1}$$

$$= 1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)$$

$$= 2 - \frac{2}{3}$$

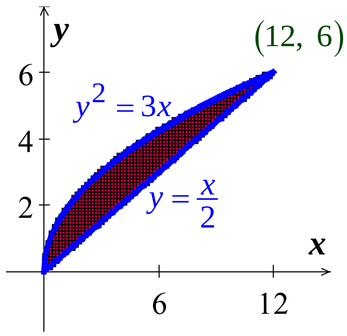
$$= \frac{4}{3}$$



Exercise

Find the area of the region $\int_0^6 \int_{y^2/3}^{2y} dxdy$

Solution



$$\int_0^6 \int_{y^2/3}^{2y} dxdy = \int_0^6 [x]_{y^2/3}^{2y} dy$$

$$= \int_0^6 \left(2y - \frac{1}{3}y^2\right) dy$$

$$= \left[y^2 - \frac{1}{9} y^3 \right]_0^6$$

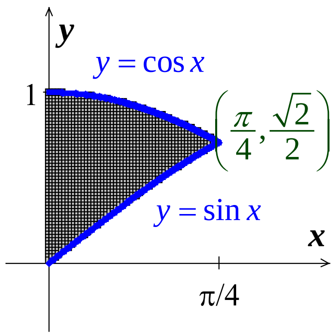
$$= 36 - \frac{1}{9}(216)$$

$$= 12$$

Exercise

Find the area of the region $\int_0^{\pi/4} \int_{\sin x}^{\cos x} dydx$

Solution



$$\int_0^{\pi/4} \int_{\sin x}^{\cos x} dydx = \int_0^{\pi/4} [y]_{\sin x}^{\cos x} dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x) dx$$

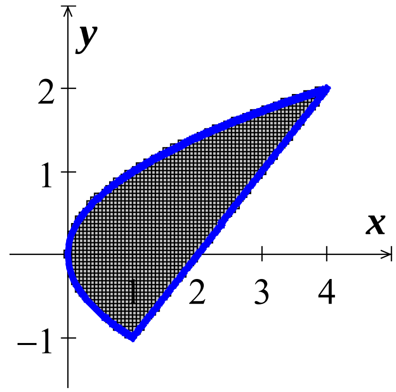
$$= \left[\sin x + \cos x\right]_0^{\pi/4}$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1)$$

$$= \sqrt{2} - 1$$

Exercise

Find the area of the region $\displaystyle\int_{-1}^{2}\int_{y^2}^{y+2} dxdy$



Solution

$$\int_{-1}^{2}\int_{y^2}^{y+2} dxdy = \int_{-1}^{2}\left(y+2-y^2\right)dy$$

$$= \left[\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3\right]_{-1}^{2}$$

$$= \left(2+4-\frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

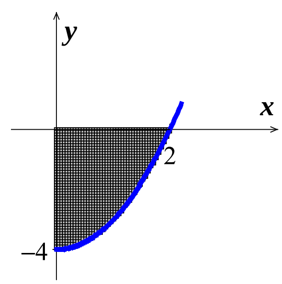
$$= \frac{9}{2}$$

Exercise

Find the area of the region $\displaystyle\int_{0}^{2}\int_{x^2-4}^{0} dydx + \int_{0}^{4}\int_{0}^{\sqrt{x}} dydx$

Solution

$$\int_{0}^{2}\int_{x^2-4}^{0} dydx + \int_{0}^{4}\int_{0}^{\sqrt{x}} dydx = \int_{0}^{2}\left(4-x^2\right)dx + \int_{0}^{4}\sqrt{x}\ dx$$

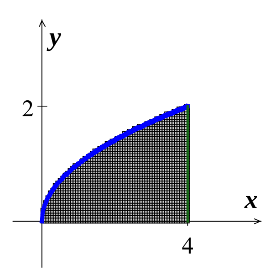


$$= \left[4x - \frac{1}{3}x^3\right]_0^2 + \frac{2}{3}\left[x^{3/2}\right]_0^4$$

$$= \left(8 - \frac{8}{3}\right) + \frac{2}{3}\left(4^{3/2}\right)$$

$$= \frac{16}{3} + \frac{16}{3}$$

$$= \frac{32}{3}$$



Exercise

Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2, \quad 0 \le y \le 2$

Solution

$$\text{Average height } = \frac{1}{4}\int_0^2 \int_0^2 \left(x^2 + y^2\right) dydx$$

$$= \frac{1}{4}\int_0^2 \left[x^2 y + \frac{1}{3}y^3\right]_0^2 dx$$

$$= \frac{1}{4}\int_0^2 \left(2x^2 + \frac{8}{3}\right) dx$$

$$= \frac{1}{4}\left[\frac{2}{3}x^3 + \frac{8}{3}x\right]_0^2$$

$$= \frac{1}{4}\left[\frac{2}{3}(8) + \frac{8}{3}(2)\right]$$

$$= \frac{1}{4}\left[\frac{16}{3} + \frac{16}{3}\right]$$

$$= \frac{8}{3}$$

Exercise

Find the average height of $f(x, y) = \frac{1}{xy}$ over the square $\ln 2 \le x \le 2\ln 2, \quad \ln 2 \le y \le 2\ln 2$

Solution

$$\text{Average height } = \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2} \int_{\ln 2}^{2\ln 2} \frac{1}{xy} dydx$$

$$= \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2} \frac{1}{x}\left[\ln y\right]_{\ln 2}^{2\ln 2} dx$$

$$= \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2} \frac{1}{x}\left(2\ln 2 - \ln 2\right) dx$$

$$= \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2} \frac{1}{x}\left(\ln 2\right) dx$$

$$= \frac{1}{\ln 2}\left[\ln x\right]_{\ln 2}^{2\ln 2}$$

$$= \frac{1}{\ln 2}\left(2\ln 2 - \ln 2\right)$$

$$= 1$$