Solution Section 1.6 – The Properties of Determinants

Exercise

Verify that $\det(AB) = \det(A)\det(B)$ when: $A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{pmatrix}$

Solution

$$AB = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{pmatrix}$$

$$\det(AB) = \begin{vmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{vmatrix}\begin{matrix} 9 & -1 \\ 31 & 1 \\ 10 & 0 \end{matrix}$$

$$= -170$$

$$\det(A) = \begin{vmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

$$= 10$$

$$\det(B) = \begin{vmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{vmatrix}$$

$$= -17$$

$$\det(AB) = \det(A)\det(B) \quad = -170 \quad \checkmark$$

Exercise

For which value(s) of $\boldsymbol{k}$ does A fail to be invertible? $A = \begin{bmatrix} k-3 & -2 \\ -2 & k-2 \end{bmatrix}$

Solution

For $\boldsymbol{A}$ to have an invertible the determinant cannot be equal to zero. To ***fail*** det(A) = 0.

$$|A| = \begin{vmatrix} k-3 & -2 \\ -2 & k-2 \end{vmatrix} = 0$$

$$(k-3)(k-2)-4=0$$

$$k^2-5k+6-4=0$$

$$k^2-5k+2=0$$

$$\boxed{k=\frac{5\pm\sqrt{17}}{2}}$$

Exercise

Without directly evaluating, show that $\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}=0$

Solution

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \quad R_2+R_1 \quad = \begin{vmatrix} a+b+c & b+c+a & c+b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \quad =0$$

It is equal to zero, since first row and third row are proportional.

$$\begin{vmatrix} a+b+c & b+c+a & c+b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \quad R_3-\frac{1}{a+b+c}R_1 \quad = \begin{vmatrix} a+b+c & b+c+a & c+b+a \\ a & b & c \\ 0 & 0 & 0 \end{vmatrix}$$

$$\boxed{=0}$$

Exercise

If the entries in every row of A add to zero, solve $\boldsymbol{Ax}=0$ to prove det $A=0$. If those entries add to one, show that det $(A-I)=0$. Does this mean det $A=I$?

Solution

If $\boldsymbol{x}=(1, 1, \dots, 1)$, then $\boldsymbol{Ax}$ = the sums of the rows of $\boldsymbol{A}$. Since every row of A add to zero, that implies $\boldsymbol{Ax}=0$. Since A has non-zero nullspace, it is not invertible and det $A=0$. If the entries in every row of A sum to one, then the entries in every row of $A-\mathrm{I}$ sum to zero. A – I has a non-zero nullspace and det $(A-\mathrm{I})=0$. This does not mean that det $A=\mathrm{I}$.

Example:

$A=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ every row of A adds up to zero

$$\Rightarrow \det A=-1\neq 1=\det I$$

Exercise

Does $\det(AB)=\det(BA)$ in general?

a) True or false if A and B are square n x n matrices?

b) True or false if A is m x n and B is n x m with $m \neq n$?

Solution

a) Matrices A and B are square matrices, then by the property:

$$\begin{aligned}\det(AB) &= \det(A)\det(B) \\ &= \det(B)\det(A) \\ &= \det(BA)\end{aligned}$$

Therefore; it is true for any A and B square matrices.

b) False, example if $A=\begin{pmatrix}1\\1\end{pmatrix} \quad B=\begin{pmatrix}1 & 1\end{pmatrix}$

$$\begin{aligned}AB &= \begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} \\ &= \begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\det(AB) &= \begin{vmatrix}1 & 1\\1 & 1\end{vmatrix} \\ &= 0\end{aligned}$$

$$BA=\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=(2)$$

$$\det(BA)=2$$

$$\det(AB) \neq \det(BA)$$

Exercise

True or false, with a reason if true or a counterexample if false:

a) The determinant of $I+A$ is 1 + det $\boldsymbol{A}$.

b) The determinant of ABC is $|A||B||C|$.

c) The determinant of 4A is $4|A|$

d) The determinant of $AB - BA$ is zero. (try an example)

e) If A is not invertible then AB is not invertible.

f) The determinant of $A - B$ equals to det A – det B.

Solution

a) ***False***, if $A=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$

$$\det(I+A)=\det\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$
$$=0$$

$$\det A=1$$

$$\begin{aligned}1+\det A&=1+1\\ &=2\\ &\neq \det(I+A)\end{aligned}$$

b) ***True***, $\det(ABC)=\det(A)\det(BC)=\det(A)\det(B)\det(C)$.

c) ***False***, in general $\det(4A)=4^n\det(A)$ if A is n x n.

d) ***False***, $A=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, $B=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

$$\begin{aligned}AB-BA&=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}-\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\\ &=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}-\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\\ &=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\det(AB-BA)&=\begin{vmatrix}0 & 1\\ -1 & 0\end{vmatrix}\\ &=1\neq 0\end{aligned}$$

e) False, any matrix is invertible, iff its determinant is nonzero. So det A = 0 which $\det(AB)=\det(A)\det(B)=0$. Therefore, AB can't be invertible.

f) $A=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \Rightarrow |A|=0$

$B=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \Rightarrow |B|=-1$

$\det(A)-\det(B)=0-(-1)\ =1$

$\det(A-B)=\det\begin{pmatrix}1 & -1\\ -1 & 0\end{pmatrix}\ =-1$

$\Rightarrow\ \det(A-B)\neq\det(A)-\det(B)$

Exercise

Use row operations to show the 3 by 3 "Vandermonde determinant" is

$$\det\begin{bmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{bmatrix}=(b-a)(c-a)(c-b)$$

Solution

$$\det\begin{bmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{bmatrix}=\det\begin{bmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{bmatrix}\begin{matrix}\\ R_2-R_1\\ R_3-R_1\end{matrix}$$

$$=\det\begin{bmatrix}1 & a & a^2\\ 0 & b-a & b^2-a^2\\ 0 & c-a & c^2-a^2\end{bmatrix}\qquad \textit{factor}\ \ (b-a)$$

$$=(b-a)\det\begin{bmatrix}1 & a & a^2\\ 0 & 1 & b+a\\ 0 & c-a & c^2-a^2\end{bmatrix}\ R_2-(c-a)R_2$$

$$(c-a)(c+a)-(b+a)(c-a)=(c-a)(c+a-b-a)$$

$$=(b-a)\det\begin{bmatrix}1 & a & a^2\\ 0 & 1 & b+a\\ 0 & 0 & (c-a)(c-b)\end{bmatrix}\quad \textbf{\textit{Multiply the main diagonal by (b - a)}}$$

$$\underline{=(b-a)(c-a)(c-b)}$$

Exercise

The inverse of a 2 by 2 matrix seems to have determinant = 1:

$$\det A^{-1}=\det\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}=\frac{ad-bc}{ad-bc}=1$$

What is wrong with this calculation? What is the correct $\det A^{-1}$

Solution

The $\det\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$ $(ad-bc)$ it is part of the determinant and it is not the solution.

$$\det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\det\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad-bc}\frac{1}{ad-bc}(ad-bc)$$

$$\boxed{= \frac{1}{ad-bc}}$$

Exercise

A *Hessenberg* matrix is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci's rule $|H_4| = |H_3| + |H_2|$. The same rule will continue for all sizes $|H_n| = |H_{n-1}| + |H_{n-2}|$. Which Fibonacci number is $|H_n|$?

$$H_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad H_3 = \begin{bmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad H_4 = \begin{bmatrix} 2 & 1 & & \\ \mathbf{1} & 2 & \mathbf{1} & \\ \mathbf{1} & 1 & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

Solution

$$|H_4| = 2C_{11} + 1C_{12}$$

The cofactor C_{11} for H_4 is the determinant $|H_3|$.

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

The cofactor $C_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$

$$= -\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= -\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$|H_2| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= -|H_3| + |H_2|$$

$$\begin{aligned}|H_4| &= 2C_{11} + 1C_{12} \\ &= 2|H_3| - |H_3| + |H_2| \\ &= |H_3| + |H_2|\end{aligned}$$

The actual number: $|H_2| = 3, \quad |H_3| = 5, \quad H_4 = 8$.

Since $|H_n|$ follows Fibonacci's rule $|H_{n-1}| + |H_{n-2}|$, it must be $|H_n| = F_{n+2}$.

Exercise

Evaluate $\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix}$

Solution

$$\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix} = -9 - (-6)$$

$$= -3$$

Exercise

Evaluate $\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix} = -6 - (0)$$

$$= -6$$

Exercise

Evaluate $\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix}$

Solution

$$\begin{aligned}\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix} &= x(8x) - 4x(2x) \\ &= 8x^2 - 8x^2 \\ &= 0\end{aligned}$$

Exercise

Evaluate $\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix} = 3x - 2x(4)$$

$$= 3x - 8x$$

$$\underline{= -5x}$$

Exercise

Evaluate $\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix}$

Solution

$$\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix} = \underline{-3x^4 - 2x}$$

Exercise

Evaluate $\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix}$

Solution

$$\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix} = \underline{-8a + 5b}$$

Exercise

Evaluate $\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}$

Solution

$$\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix} = 15 - 14$$

$$\underline{= 1}$$

Exercise

Evaluate $\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix} = 5 - 20$$

$$\underline{= -16}$$

Exercise

Evaluate $\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix}$

Solution

$$\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix} = 15 + 6$$

$$\underline{= 21}$$

Exercise

Evaluate $\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix}$

Solution

$$\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix} = -24 + 5$$

$$\underline{= -19}$$

Exercise

Evaluate $\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix}$

Solution

$$\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix} = 3 - 6$$

$$\underline{= -3}$$

Exercise

Evaluate $\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix}$

Solution

$$\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix} = 7 + 18$$

$$\underline{= 25}$$

Exercise

Evaluate $\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix} \underline{= 2\sqrt{5} + 6}$$

Exercise

Evaluate $\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix}$

Solution

$$\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix} = -\frac{3}{8} - \frac{1}{16}$$

$$\underline{= -\frac{7}{16}}$$

Exercise

Evaluate $\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix}$

Solution

$$\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix} = -1 + 1$$

$$\underline{= 0}$$

Exercise

Evaluate $\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix}$

Solution

$$\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix} = \frac{1}{2} + \frac{1}{6}$$

$$= \frac{2}{3}$$

Exercise

Evaluate $\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix}$

Solution

$$\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix} = x^2 - 4x^2$$

$$= -3x^2$$

Exercise

Evaluate $\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix} = 9x - x^3$$

Exercise

Evaluate $\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix} = 2x^2 + 3x$$

Exercise

Evaluate $\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix}$

Solution

$$\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix} = 4(x+2) - 6(x-2)$$
$$= 4x + 8 - 6x + 12$$
$$\underline{= -2x + 20}$$

Exercise

Evaluate $\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix}$

Solution

$$\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix} = -3x - 3 + 6x + 18$$
$$\underline{= -2x + 20}$$

Exercise

Evaluate $\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix} \begin{matrix} 3 & 0 \\ 2 & 1 \\ 2 & 5 \end{matrix}$$
$$= -3 + 0 + 0 - 0 + 75 - 0$$
$$\underline{= 72}$$

Exercise

Evaluate $\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix}$

Solution

$$\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix} \begin{matrix} 4 & 0 \\ 3 & -1 \\ 2 & -3 \end{matrix}$$

or $= 4\begin{vmatrix} -1 & 4 \\ -3 & 6 \end{vmatrix}$

$= -24 + 48$

$\underline{= 24}$

Exercise

Evaluate $\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix} \begin{matrix} 3 & 1 \\ -3 & -4 \\ -1 & 3 \end{matrix}$$

$= -60 + 15$

$\underline{= -45}$

Exercise

Evaluate $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix} \begin{matrix} 1 & 1 \\ 2 & 2 \\ 3 & -4 \end{matrix}$$

$= 10 + 6 - 8 - 6 + 8 - 10$

$\underline{= 0}$

Exercise

Evaluate $\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix} \begin{matrix} x & 0 \\ 2 & 1 \\ -3 & x \end{matrix}$$

$$= x - 2x - 3 - x^4$$

$$\underline{= -x^4 - x - 3}$$

Exercise

Evaluate $\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix} \begin{matrix} x & 1 \\ x^2 & x \\ 0 & x \end{matrix}$$

$$= x^2 - x^3 - x^3 - x^2$$

$$\underline{= -2x^3}$$

Exercise

Evaluate $\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix}$

Solution

$$\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix} = 0 + 126 + 48 - (-48 + 36 + 0)$$

$$\underline{= 90}$$

Exercise

Evaluate $\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix} = 0 - 4 - 16 - (-14 - 16 + 0)$$

$$\underline{= 10}$$

Exercise

Evaluate $\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix}$

Solution

$$\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix} \begin{matrix} 3 & 1 \\ -2 & 3 \\ 3 & 4 \end{matrix}$$

$$= -54 + 3 - 16 - 18 - 12 - 12$$

$$\underline{= -109}$$

Exercise

Evaluate $\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix} \begin{matrix} 2x & 1 \\ 0 & 4 \\ 3 & 0 \end{matrix}$$

$$= 16x + 3x + 12$$

$$\underline{= 19x + 12}$$

Exercise

Evaluate $\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix} \begin{matrix} 0 & x \\ x & x^2 \\ x & 7 \end{matrix}$$

$$= 5x^2 + 7x^2 - x^4 + 5x^2$$

$$= 17x^2 - x^4$$

Exercise

Evaluate $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix} \begin{matrix} 2 & x \\ -3 & 1 \\ 2 & 1 \end{matrix}$$

$$= 8 - 3 - 2 + 12x$$

$$= 12x + 3$$

Exercise

Evaluate $\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} \begin{matrix} 1 & x \\ 3 & 1 \\ 0 & -2 \end{matrix}$$

$$= 2 + 12 + 2 - 6x$$

$$= -6x + 16$$

Exercise

Evaluate $\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} - (-)\begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= 3 + 2 + 3 + 0$$

$$\underline{= 0}$$

Exercise

Find all the values of λ for which det(***A***) = 0: $A = \begin{bmatrix} \lambda - 1 & -2 \\ 1 & \lambda - 4 \end{bmatrix}$

Solution

$$\begin{vmatrix} \lambda - 1 & -2 \\ 1 & \lambda - 4 \end{vmatrix} = (\lambda - 1)(\lambda - 4) + 2$$

$$= \lambda^2 - 5\lambda + 4 + 2$$

$$= \lambda^2 - 5\lambda + 6 = 0$$ *Solve for λ.*

$$\underline{\lambda_{1,2} = -1,\ 6}$$

Exercise

Find all the values of λ for which det(***A***) = 0: $A = \begin{bmatrix} \lambda - 6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda - 4 \end{bmatrix}$

Solution

$$\begin{vmatrix} \lambda-6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda-4 \end{vmatrix} = \lambda(\lambda-6)(\lambda-4)+4(\lambda-6)$$

$$= \lambda\left(\lambda^2-10\lambda+24\right)+4\lambda-24$$

$$= \lambda^3-10\lambda^2+24\lambda+4\lambda-24$$

$$= \lambda^3-10\lambda^2+28\lambda-24$$

$$\lambda^3-10\lambda^2+28\lambda-24=0$$

$$\underline{\lambda_{1,2,3}=2,\ 2,\ 6}$$

Exercise

Prove that if a square matrix $\boldsymbol{A}$ has a column of zeros, then det($\boldsymbol{A}$) = 0

Solution

Consider a 3 by 3 matrix with a zero column, however to find the determinant we can interchange any column of that matrix; therefore:

$$\boldsymbol{A}=[a_{ij}]=\begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$$

By definition, the determinant of $\boldsymbol{A}$ using the cofactor:

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= 0\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} 0 & a_{23} \\ 0 & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} 0 & a_{22} \\ 0 & a_{32} \end{vmatrix}$$

$$\underline{=0}$$

Exercise

With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D| \quad \textit{but} \quad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D|-|C||B|$$

a) Why is the first statement true? Somehow B doesn't enter.

b) Show by example that equality fails (as shown) when C enters.

c) Show by example that the answer $\det(AD - CB)$ is also wrong.

Solution

a) If we don't pick any 0 entries, then the first two columns are picked from $\boldsymbol{A}$ and the last two rows are from *D*. We can't pick any columns or rows from *B*, because there aren't any left.

b) $\left|\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right| = -\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}$

$\underline{= -1}$

and $A = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0, \quad B = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0, \quad C = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0, \quad D = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$

c) Use the example from part (*b*): $1 \neq 0$

$\begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|$

Exercise

Show that the value of the following determinant is independent of θ.

$$\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{vmatrix}$$

Solution

$$\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{vmatrix} = \begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix}$$

$$= \sin^2\theta + \cos^2\theta$$

$$= \sin^2\theta - \left(-\cos^2\theta\right)$$

$$\underline{= 1}$$

Therefore, the determinant is independent of θ.

Exercise

Show that the matrices $A=\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ *and* $B=\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$ commute if and only if $\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix}=0$

Solution

$$AB=\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} d & e \\ 0 & f \end{pmatrix}=\begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix}$$

$$BA=\begin{pmatrix} d & e \\ 0 & f \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}=\begin{pmatrix} da & db+ec \\ 0 & fc \end{pmatrix}$$

$$AB=BA \quad \Rightarrow \quad \begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix}=\begin{pmatrix} da & db+ec \\ 0 & fc \end{pmatrix}$$

Iff $ae+bf=db+ec$

$$\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix}=b(d-f)-e(a-c)=bd-bf-ea+ec=0$$

$$\underline{bd+ec=bf+ae} \quad \checkmark$$

Exercise

Show that $\det(A)=\frac{1}{2}\begin{vmatrix} tr(A) & 1 \\ tr\left(A^2\right) & tr(A) \end{vmatrix}$ for every 2×2 matrix A.

Solution

Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \quad tr(A)=a+d$

$$A^2=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} \quad \Rightarrow \quad tr\left(A^2\right)=a^2+bc+bc+d^2$$

$$\det(A)=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc$$

$$\begin{aligned}\frac{1}{2}\begin{vmatrix} tr(A) & 1 \\ tr\left(A^2\right) & tr(A) \end{vmatrix}&=\frac{1}{2}\begin{vmatrix} a+d & 1 \\ a^2+bc+bc+d^2 & a+d \end{vmatrix}\\ &=\frac{1}{2}\left[(a+d)^2-\left(a^2+bc+bc+d^2\right)\right]\\ &=\frac{1}{2}\left(a^2+2ad+d^2-a^2-bc-bc-d^2\right)\end{aligned}$$

$$= \frac{1}{2}(2ad - 2bc)$$
$$= ad - bc$$
$$\boxed{= \det(A)}$$

Exercise

What is the maximum number of zeros that a 4×4 matrix can have without a zero determinant? Explain your reasoning.

Solution

The maximum number of zeros that a 4×4 matrix can have without a zero determinant is 12 zeros. If the main diagonal has nonzero entries and the rest are zero, then the determinant of the matrix is equal to the product of the main diagonal entries.

Exercise

Evaluate $\det(A)$, $\det(E)$, and $\det(AE)$. Then verify that $\det(A) \cdot \det(E) = \det(AE)$

$$A = \begin{bmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & & \\ & 3 & \\ & & 1 \end{bmatrix}$$

Solution

$$\det(A) = \begin{vmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{vmatrix}$$
$$= -40 + 18$$
$$\boxed{= -22}$$

$$\det(E) = \begin{vmatrix} 1 & & \\ & 3 & \\ & & 1 \end{vmatrix}$$
$$\boxed{= 3}$$

$$AE = \begin{bmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 3 & 3 \\ 0 & -6 & 0 \\ 3 & 3 & 5 \end{bmatrix}$$

$$\det(AE) = \begin{vmatrix} 4 & 3 & 3 \\ 0 & -6 & 0 \\ 3 & 3 & 5 \end{vmatrix}$$

$$= -120 + 54$$

$$\underline{= -66}$$

$$\det(A)\det(E) = (-22)(3)$$

$$\underline{= -66}$$

$$\det(A)\det(E) = \det(AE) \qquad \surd$$

Exercise

Show that $\begin{bmatrix} \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma \\ 1 & 1 & 1 \end{bmatrix}$ is not invertible for any values of α, β, γ

Solution

$$\begin{vmatrix} \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma \\ 1 & 1 & 1 \end{vmatrix} = \sin^2\alpha\cos^2\beta + \sin^2\beta\cos^2\gamma + \sin^2\gamma\cos^2\alpha - \sin^2\gamma\cos^2\beta - \sin^2\alpha\cos^2\gamma - \cos^2\alpha\sin^2\beta$$

$$= \sin^2\alpha\left(\cos^2\beta - \cos^2\gamma\right) + \cos^2\alpha\left(\sin^2\gamma - \sin^2\beta\right) + \sin^2\beta\cos^2\gamma - \sin^2\gamma\cos^2\beta$$

$$= \sin^2\alpha\left(\cos^2\beta - \cos^2\gamma\right) + \cos^2\alpha\left(1 - \cos^2\gamma - 1 + \cos^2\beta\right) + \left(1 - \cos^2\beta\right)\cos^2\gamma - \left(1 - \cos^2\gamma\right)\cos^2\beta$$

$$= \sin^2\alpha\left(\cos^2\beta - \cos^2\gamma\right) + \cos^2\alpha\left(\cos^2\beta - \cos^2\gamma\right) + \cos^2\gamma - \cos^2\gamma\cos^2\beta - \cos^2\beta + \cos^2\gamma\cos^2\beta$$

$$= \left(\sin^2\alpha + +\cos^2\alpha\right)\left(\cos^2\beta - \cos^2\gamma\right) + \cos^2\gamma - \cos^2\beta$$

$$= \cos^2\beta - \cos^2\gamma + \cos^2\gamma - \cos^2\beta$$

$$\underline{= 0}$$

Therefore, this matrix in not invertible.

Exercise

The determinant of a 2×2 matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det(A)=ad-bc$.

Assuming no rows swaps are required, perform elimination on A and show explicitly that $ad-bc$ is the product of the pivots.

Solution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \; aR_2-cR_1 \quad \rightarrow \quad \begin{pmatrix} a & b \\ 0 & ad-bc \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \; R_2-\frac{c}{a}R_1 \quad \rightarrow \quad \begin{pmatrix} a & b \\ 0 & d-\frac{bc}{a} \end{pmatrix}$$

$$\begin{vmatrix} a & b \\ 0 & d-\frac{bc}{a} \end{vmatrix} = a\left(d-\frac{bc}{a}\right)$$

$$=ad-bc$$

$$=\det(A)$$

Exercise

If A is a 7×7 matrix and let $\det(A)=17$. What is $\det\left(3A^2\right)$?

Solution

$$\det\left(A^2\right)=\det(A)\det(A)$$

$$=17^2$$

Multiplying a single row by 3 multiplies the determinant by 3.

Multiplying the whole 7×7 matrix by 3 multiplies all 7 rows by 3 $\Rightarrow$ 3^7.

$$\therefore \det\left(3A^2\right)=3^7\cdot 17^2$$

$$\underline{=632043}$$

Exercise

Explain without computations why the following determinant is equal to zero

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & 0 & 0 & 0 \\ d_1 & d_2 & 0 & 0 & 0 \\ e_1 & e_2 & 0 & 0 & 0 \end{vmatrix}$$

Solution

The determinant is equal to zero because there are too many zeros (as block 3×3).

Or

$d_1R_5 - e_1R_4 \quad \to R_5 \Rightarrow \quad 0 \quad d_1e_2 - e_1d_2 \quad 0 \quad 0 \quad 0$

Since row 5 is has zero entries, therefore the determinant is zero.

Exercise

Let A be an $n\times n$ real matrix.

a) Show that if $A^t = -A$ and n is odd, then $|A| = 0$.

b) Show that if $A^2 + I = 0$, then n must be even.

c) Does part (b) remain true for complex matrices?

Solution

a) Given: $A^t = -A$ and n is odd

$$\begin{aligned} |A| &= \left|A^t\right| \\ &= |-A| \\ &= (-1)^n |A| \qquad \text{Since } n \text{ is odd} \\ &= -|A| \end{aligned}$$

$|A| = -|A|$ only when $|A| = 0$

b) $A^2 + I = 0$

$A^2 = -I$

$$\begin{aligned} |A|^2 &= \left|A^2\right| \\ &= |-I| \\ &= (-1)^n \end{aligned}$$

If n is odd, then $\cancel{|A|^2 = -1}$ *impossible*

If n is even, then $|A|^2 = 1$

c) It can't be true because $|I| = -1 \in \mathbb{R}$

And A is real matrix, the determinant has to be a real number.

Exercise

Let A and C be $m \times m$ and $n \times n$ matrices, respectively.

a) Show that $\begin{vmatrix} A & B \\ 0 & C \end{vmatrix} = \begin{vmatrix} A & 0 \\ B & C \end{vmatrix} = |A||C|$

b) Evaluate

i. $\begin{vmatrix} I_m & 0 \\ 0 & I_n \end{vmatrix}$

ii. $\begin{vmatrix} 0 & I_m \\ I_n & 0 \end{vmatrix}$

iii. $\begin{vmatrix} I_m & B \\ 0 & I_n \end{vmatrix}$

iv. $\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix}$

c) Find a formula for $\begin{vmatrix} 0 & A \\ C & B \end{vmatrix}_{n\times n}$

Solution

a) If we let matrices B be $m \times n$ an **0** be $n \times m$, so the determinant of the matrix size will be $(m+n)\times(m+n)$, then

$$\begin{vmatrix} A & B \\ 0 & C \end{vmatrix} = \left|\begin{array}{c|c} A_{m\times m} & B_{m\times n} \\ \hline 0_{n\times m} & C_{n\times n} \end{array}\right|$$

$$= |A||C| - |B||0|$$

$$= |A||C| - 0$$

$$= |A||C|$$

If we let matrices B be $n\times m$ an $\mathbf{0}$ be $m\times n$, so the determinant of the matrix size will be $(m+n)\times(m+n)$, then

$$\begin{vmatrix} A & 0 \\ B & C \end{vmatrix} = \left|\begin{array}{c|c} A & 0 \\ \hline B & C \end{array}\right|$$

$$= |A||C| - |B||0|$$

$$= |A||C| - 0$$

$$= |A||C|$$

b) i - $$\begin{vmatrix} I_m & 0 \\ 0 & I_n \end{vmatrix} = \left|\begin{array}{c|c} I_m & 0_{m\times n} \\ \hline 0_{n\times m} & I_n \end{array}\right|$$

$$= |I_m||I_n| - 0$$

$$= 1 \bullet 1$$

$$= 1$$

ii - $$\begin{vmatrix} 0 & I_m \\ I_n & 0 \end{vmatrix} = \left|\begin{array}{c|c} 0_{m\times n} & I_m \\ \hline I_n & 0_{n\times m} \end{array}\right|$$

$$= -|I_m| \bullet (-1)|I_n|$$

$$= (-1)^{mn}$$

iii - $$\begin{vmatrix} I_m & B \\ 0 & I_n \end{vmatrix} = \left|\begin{array}{c|c} I_m & B_{m\times n} \\ \hline 0_{n\times m} & I_n \end{array}\right|$$

$$= |I_m||I_n| - 0$$

$$= 1 \bullet 1$$

$$= 1$$

iv - $$\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix}_{n\times n}$$

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \qquad \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1$$

$$\begin{array}{cccc} + & - & + & - \end{array}$$

$$\left|\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right|_{4\times 4} = -\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -(-1) = 1$$

$$\begin{array}{ccccc} + & - & + & - & + \end{array}$$

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix}_{5\times 5} = +\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}_{4\times 4} = 1$$

From that we can see that the signs are: $- \quad - \quad + \quad + \quad - \quad - \quad + \quad +$

$$\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix}_{n\times n} = (-1)^{\frac{n^2+3n}{2}}$$

c) $$\begin{vmatrix} 0 & A \\ C & B \end{vmatrix}_{n\times n} = \left|\begin{array}{c|c} 0_{m\times n} & A_{m\times m} \\ \hline C_{n\times n} & B_{n\times m} \end{array}\right|$$

$$= -\left|A_{m\times m}\right| \bullet (-1)\left|C_{n\times n}\right|$$

$$\underline{= (-1)^{mn}|A||C|}$$

Exercise

Let $f(x)=(p_1-x)(p_2-x)\dots(p_n-x)$ and let

$$\Delta_n=\begin{vmatrix} p_1 & a & a & \dots & a & a \\ b & p_2 & a & \dots & a & a \\ b & b & p_3 & \dots & a & a \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b & b & b & \cdots & p_{n-1} & a \\ b & b & b & \dots & b & p_n \end{vmatrix}$$

a) Show that, if $a\neq b$,

$$\Delta_n=\frac{bf(a)-af(b)}{b-a}$$

b) Show that, if $a=b$,

$$\Delta_n=a\sum_{i=1}^{n}f_i(a)+p_nf_n(a)$$

Where $f_i(a)$ means $f(a)$ with factor (p_i-a) missing.

c) Use part (*b*) to evaluate

$$\begin{vmatrix} a & b & b & \dots & b & b \\ b & a & b & \dots & b & b \\ b & b & a & \dots & b & b \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b & b & b & \cdots & a & b \\ b & b & b & \dots & b & a \end{vmatrix}_{n\times n}$$

Solution

a) $\Delta_n=\dfrac{bf(a)-af(b)}{b-a}$; with $a\neq b$

Using the mathematical Induction to prove the equality.

For $n=2$:

$$\Delta_2=\begin{vmatrix} p_1 & a \\ b & p_2 \end{vmatrix}$$

$$=p_1p_2-ab$$

$$\Delta_2=\frac{bf(a)-af(b)}{b-a}$$

$$=\frac{b\left(p_1-a\right)\left(p_2-a\right)-a\left(p_1-b\right)\left(p_2-b\right)}{b-a}$$

$$=\frac{bp_1p_2-abp_1-abp_2+a^2b-ap_1p_2+abp_1+abp_2-ab^2}{b-a}$$

$$=\frac{bp_1p_2+a^2b-ap_1p_2-ab^2}{b-a}$$

$$=\frac{(b-a)\,p_1p_2+ab(a-b)}{b-a}$$

$$=\frac{(b-a)\left(p_1p_2-ab\right)}{b-a}$$

$$\underline{=p_1p_2-ab\,\Big|}$$

For $n=2$, the proof is true.

Assume that is true for Δ_k

$$\Delta_k=\frac{bf(a)-af(b)}{b-a}$$

$$=\frac{b\left(\left(p_1-a\right)\ldots\left(p_{k-1}-a\right)\right)-a\left(\left(p_1-b\right)\ldots\left(p_{k-1}-b\right)\right)}{b-a}$$

$$f(x)=\left(p_1-x\right)\left(p_2-x\right)\ldots\left(p_k-x\right)$$

We need to prove it is also true for $\Delta_{k+1}\;\Rightarrow\;\Delta_{k+1}=\dfrac{bF(a)-aF(b)}{b-a}$

$$F(x)=\left(p_1-x\right)\left(p_2-x\right)\ldots\left(p_k-x\right)\left(p_{k+1}-x\right)$$

$$=f(x)\left(p_{k+1}-x\right)$$

$$\Delta_{k+1}=\frac{b\left(\left(p_1-a\right)\ldots\left(p_k-a\right)\left(p_{k+1}-a\right)\right)-a\left(\left(p_1-b\right)\ldots\left(p_k-b\right)\left(p_{k+1}-b\right)\right)}{b-a}$$

$$=\frac{bf(a)\left(p_{k+1}-a\right)-af(b)\left(p_{k+1}-b\right)}{b-a}$$

$$=\frac{bF(a)-aF(b)}{b-a}\qquad\checkmark$$

Δ_{k+1} is also true.

∴ by the mathematical induction, the proof is completed.

b) If $a=b\;\rightarrow\;\Delta_n=a\displaystyle\sum_{i=1}^{n}f_i(a)+p_nf_n(a)$

Where $f_i(a)$ means $f(a)$ with factor $(p_i - a)$ missing.

$$\Delta_n = (p_1 - a)\Delta_{n-1} + a(p_2 - b)\cdots(p_n - b) \qquad a = b$$

$$= (p_1 - a)\Delta_{n-1} + a(p_2 - a)\cdots(p_n - a) \qquad (p_1 - a) \text{ missing}$$

$$= (p_1 - a)\left[(p_2 - a)\Delta_{n-2} + aF_2(a)\right] + af_1(a)$$

$$= (p_1 - a)(p_2 - a)\Delta_{n-2} + a(p_1 - a)F_2(a) + af_1(a)$$

$$= (p_1 - a)(p_2 - a)\Delta_{n-2} + af_2(a) + af_1(a)$$

$$= (p_1 - a)(p_2 - a)(p_3 - a)\Delta_{n-3} + af_3(a) + af_2(a) + af_1(a)$$

$$= (p_1 - a)(p_2 - a)(p_3 - a)\Delta_{n-3} + a\left(f_3(a) + f_2(a) + f_1(a)\right)$$

$$\vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots$$

$$= (p_1 - a)\cdots(p_{n-2} - a)\Delta_2 + a\left(f_{n-2}(a) + \cdots + f_1(a)\right)$$

$$\Delta_2 = \begin{vmatrix} p_{n-1} & a \\ a & p_n \end{vmatrix}$$

$$= p_{n-1}p_n - a^2$$

$$= p_{n-1}p_n - ap_n + ap_n - a^2$$

$$= p_n(p_{n-1} - a) + a(p_n - a)$$

$$\Delta_n = \left[(p_1 - a)\cdots(p_{n-2} - a)\right]\left(p_n(p_{n-1} - a) + a(p_n - a)\right) + a\left(f_{n-2}(a) + \cdots + f_1(a)\right)$$

$$= p_n(p_1 - a)\cdots(p_{n-2} - a)(p_{n-1} - a) + a(p_1 - a)\cdots(p_{n-2} - a)(p_n - a) + a\sum_{i=1}^{n-1} f_i(a)$$

$$= p_n f_n(a) + af_{n-1}(a) + a\sum_{i=1}^{n-1} f_i(a)$$

$$= p_n f_n(a) + a\sum_{i=1}^{n} f_i(a)$$

c) $f_n(x) = (p_1 - x)(p_2 - x)\ldots(p_{n-1} - x)$

$f_n(b) = (a - b)(a - b)\ldots(a - b)$

$p_n = a$

$$\begin{vmatrix} a & b & b & \dots & b & b \\ b & a & b & \dots & b & b \\ b & b & a & \dots & b & b \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b & b & b & \cdots & a & b \\ b & b & b & \dots & b & a \end{vmatrix}_{n\times n} = p_n f_n(b) + b\sum_{i=1}^{n} f_i(b)$$

$$= a(a-b)^{n-1} + b\Big(\underbrace{(a-b)^{n-1} + \cdots + (a-b)^{n-1}}_{n-1}\Big)$$

$$= a(a-b)^{n-1} + b(n-1)(a-b)^{n-1}$$

$$\boxed{= \left[a+(n-1)b\right](a-b)^{n-1}}$$

Exercise

Let $A, B, C, D \in M_n(\mathbb{C})$

a) Show that when A is invertible: $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|\left|D - CA^{-1}B\right|$

b) Show that when $AC = CA$: $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|$

c) Can B and C on the right-hand side of the identity be switched?

d) Does part (b) remain true if the condition $AC = CA$ is dropped?

Solution

a) Since A in invertible, then A^{-1} exists and $AA^{-1} = A^{-1}A = I$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad A^{-1}R_1$$

$$\begin{pmatrix} I & A^{-1}B \\ C & D \end{pmatrix} \quad R_2 - CR_1$$

$$\begin{pmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix} \quad AR_1$$

$$\begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \left|\begin{array}{c|c} A & B \\ \hline 0 & D - CA^{-1}B \end{array}\right|$$

$$= |A|\left|D - CA^{-1}B\right|$$

b) When $AC = CA$

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|\left|D - CA^{-1}B\right|$$

$$= \left|AD - ACA^{-1}B\right| \qquad AC = CA$$

$$= \left|AD - C\left(AA^{-1}\right)B\right|$$

$$= |AD - CIB|$$

$$\underline{= |AD - CB|}$$

c) To switch B and C it is not necessary that $BC = CB$

Let $A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$AD = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$CB = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$BC = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$|AD - CB| = \left|\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right|$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

$$\underline{= 0}$$

$$|AD - BC| = \left|\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\right|$$

$$= \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}$$

$\underline{=1\,}|$

$$|AD - CB| \neq |AD - BC|$$

No, B and C on the right-hand side of the identity cannot be switched since $|AD - CB| \neq |AD - BC|$

d) No, since from previous part (c) D doesn't commute necessarily.