

Solution **Section 1.1 – The Binomial Theorem**

Exercise

Find the fifth term in the expansion $(x^3 + \sqrt{y})^{13}$

Solution

$$\begin{aligned}\binom{13}{4}(x^3)^9(\sqrt{y})^4 &= \frac{13!}{4!(13-4)!}x^{27}y^2 \\ &= \underline{715x^{27}y^2} \quad | \end{aligned}$$

Exercise

Find the term involving q^{10} in the binomial expansion $\left(\frac{1}{3}p + q^2\right)^{12}$

Solution

Given: $a = \frac{1}{3}p, \quad b = q^2, \quad n = 12$

$$q^{10} = (q^2)^5 = b^5$$

$$\begin{aligned}\binom{n}{k}a^{n-k}b^k &= \binom{12}{5}\left(\frac{1}{3}p\right)^{12-5}\left(q^2\right)^5 \\ &= \frac{12!}{5!(12-5)!}\left(\frac{1}{3}p\right)^7q^{10} \\ &= \underline{\frac{88}{243}p^7q^{10}} \quad | \end{aligned}$$

Exercise

Use the binomial theorem to expand and simplify: $(4x - y)^3$

Solution

$$\begin{aligned}(4x - y)^3 &= \binom{3}{0}(4x)^3(-y)^0 + \binom{3}{1}(4x)^2(-y)^1 + \binom{3}{2}(4x)^1(-y)^2 + \binom{3}{3}(4x)^0(-y)^3 \\ &= 64x^3 + 3(16x^2)(-y) + 3(4x)y^2 - y^3 \\ &= \underline{64x^3 - 48x^2y + 12xy^2 - y^3} \quad | \end{aligned}$$

Exercise

Use the binomial theorem to expand and simplify: $(x + y)^6$

Solution

$$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

Exercise

Use the binomial theorem to expand and simplify: $(a - b)^6$

Solution

$$(a - b)^6 = a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$$

Exercise

Use the binomial theorem to expand and simplify: $(x - y)^7$

Solution

$$(x - y)^7 = x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$$

Exercise

Use the binomial theorem to expand and simplify: $(a + b)^8$

Solution

$$(a + b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$$

Exercise

Use the binomial theorem to expand and simplify: $(3t - 5x)^4$

Solution

$$\begin{aligned}(3t - 5x)^4 &= (3t)^4 + 4(3t)^3(-5x)^1 + 6(3t)^2(-5x)^2 + 4(3t)^1(-5x)^3 + (-5x)^4 \\ &= \underline{81t^4 - 540t^3x + 1350t^2x^2 - 1500tx^3 + 625x^4}\end{aligned}$$

Exercise

Use the binomial theorem to expand and simplify: $\left(\frac{1}{3}x + y^2\right)^5$

Solution

$$\begin{aligned}\left(\frac{1}{3}x + y^2\right)^5 &= \left(\frac{1}{3}x\right)^5 + 5\left(\frac{1}{3}x\right)^4 y^2 + 10\left(\frac{1}{3}x\right)^3 (y^2)^2 + 10\left(\frac{1}{3}x\right)^2 (y^2)^3 + 5\frac{1}{3}x(y^2)^4 + (y^2)^5 \\ &= \frac{1}{243}x^5 + \frac{5}{81}x^4 y^2 + \frac{10}{27}x^3 y^4 + \frac{10}{9}x^2 y^6 + \frac{5}{3}xy^8 + y^{10} \quad | \end{aligned}$$

Exercise

Use the binomial theorem to expand and simplify: $\left(\frac{1}{x^2} + 3x\right)^6$

Solution

$$\begin{aligned}\left(\frac{1}{x^2} + 3x\right)^6 &= \left(x^{-2} + 3x\right)^6 \\ &= \left(x^{-2}\right)^6 + 6\left(x^{-2}\right)^5 (3x) + 15\left(x^{-2}\right)^4 (3x)^2 + 20\left(x^{-2}\right)^3 (3x)^3 \\ &\quad + 15\left(x^{-2}\right)^2 (3x)^4 + 15x^{-2}(3x)^5 + (3x)^6 \\ &= x^{-12} + 18x^{-9} + 135x^{-6} + 540x^{-3} + 1215 + 1458x^3 + 729x^6 \quad | \end{aligned}$$

Exercise

Use the binomial theorem to expand and simplify: $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^5$

Solution

$$\begin{aligned}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^5 &= \left(x^{1/2} + x^{-1/2}\right)^5 \\ &= \left(x^{1/2}\right)^5 + 5\left(x^{1/2}\right)^4 x^{-1/2} + 10\left(x^{1/2}\right)^3 \left(x^{-1/2}\right)^2 + 10\left(x^{1/2}\right)^2 \left(x^{-1/2}\right)^3 \\ &\quad + 5x^{1/2} \left(x^{-1/2}\right)^4 + \left(x^{-1/2}\right)^5 \\ &= x^{5/2} + 5x^2 x^{-1/2} + 10x^{3/2} x^{-1} + 10xx^{-3/2} + 5x^{1/2} x^{-2} + x^{-5/2} \\ &= x^{5/2} + 5x^{3/2} + 10x^{1/2} + 10x^{-1/2} + 5x^{-3/2} + x^{-5/2} \quad | \end{aligned}$$

ExerciseExpand and simplify: $(2y - 3)^4$ **Solution**

$$\begin{aligned}
 (2y - 3)^4 &= (2y)^4 + 4(2y)^3(-3) + 6(2y)^2(-3)^2 + 4(2y)(-3)^3 + (-3)^4 \\
 &= 16y^4 - 96y^3 + 216y^2 - 216y + 81
 \end{aligned}$$

ExerciseExpand and simplify: $(x + 2)^5$ **Solution**

$$\begin{aligned}
 (x + 2)^5 &= x^5 + 5x^4(2) + 10x^3(2)^2 + 10x^2(2)^3 + 5x(2)^4 + (2)^5 \\
 &= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32
 \end{aligned}$$

ExerciseExpand and simplify: $(x^2 - y^2)^6$ **Solution**

$$\begin{aligned}
 (x^2 - y^2)^6 &= (x^2)^6 + 6(x^2)^5(-y^2) + 15(x^2)^4(-y^2)^2 + 20(x^2)^3(-y^2)^3 \\
 &\quad + 15(x^2)^2(-y^2)^4 + 15(x^2)(-y^2)^5 + (-y^2)^6 \\
 &= x^{12} - 6x^{10}y^2 + 15x^8y^4 - 20x^6y^6 + 15x^4y^8 - 15x^2y^{10} + y^{12}
 \end{aligned}$$

ExerciseExpand and simplify: $(ax - by)^4$ **Solution**

$$\begin{aligned}
 (ax - by)^4 &= (ax)^4 + 4(ax)^3(-by) + 6(ax)^2(-by)^2 + 4(ax)(-by)^3 + (-by)^4 \\
 &= a^4x^4 - 4a^3x^3by + 6a^2x^2b^2y^2 - 4axb^3y^3 + b^4y^4
 \end{aligned}$$

Exercise

Expand and simplify: $(ax + by)^5$

Solution

$$\begin{aligned}(ax + by)^5 &= (ax)^5 + 5(ax)^4(by) + 10(ax)^3(by)^2 + 10(ax)^2(by)^3 + 5(ax)(by)^4 + (by)^5 \\ &= \underline{a^5x^5 + 5a^4x^4by + 10a^3x^3b^2y^2 + 10a^2x^2b^3y^3 + 5axb^4y^4 + b^5y^5}\end{aligned}$$

Exercise

Expand and simplify: $(\sqrt{x} - \sqrt{3})^4$

Solution

$$\begin{aligned}(\sqrt{x} - \sqrt{3})^4 &= (\sqrt{x})^4 + 4(\sqrt{x})^3(-\sqrt{3}) + 6(\sqrt{x})^2(-\sqrt{3})^2 + 4(\sqrt{x})(-\sqrt{3})^3 + (-\sqrt{3})^4 \\ &= \underline{x^2 - 4x\sqrt{3x} + 18x^2 - 13\sqrt{3x} + 9}\end{aligned}$$

Exercise

Expand and simplify: $(\sqrt{x} - \sqrt{2})^6$

Solution

$$\begin{aligned}(\sqrt{x} - \sqrt{2})^6 &= (\sqrt{x})^6 + 6(\sqrt{x})^5(-\sqrt{2}) + 15(\sqrt{x})^4(-\sqrt{2})^2 + 20(\sqrt{x})^3(-\sqrt{2})^3 \\ &\quad + 15(\sqrt{x})^2(-\sqrt{2})^4 + 15(\sqrt{x})(-\sqrt{2})^5 + (-\sqrt{2})^6 \\ &= \underline{x^3 - 6x^2\sqrt{2x} + 30x^2 - 40x\sqrt{2x} + 60x - 60\sqrt{2x} + 8}\end{aligned}$$

Exercise

Expand and simplify: $(2x - 1)^{12}$

Solution

$$\begin{aligned}(2x - 1)^{12} &= (2x)^{12} + 12(2x)^{11}(-1) + 66(2x)^{10}(-1)^2 + 240(2x)^9(-1)^3 + 535(2x)^8(-1)^4 \\ &\quad + 812(2x)^7(-1)^5 + 924(2x)^6(-1)^6 + 812(2x)^5(-1)^7 + 535(2x)^4(-1)^8 \\ &\quad + 240(2x)^3(-1)^9 + 66(2x)^2(-1)^{10} + 12(2x)(-1)^{11} + (-1)^{12} \\ &= \underline{4096x^{12} - 24576x^{11} + 67584x^{10} - 122880x^9 + 136960x^8 - 103936x^7} \\ &\quad + \underline{59136x^6 - 25984x^5 + 8560x^4 - 1920x^3 + 264x^2 - 24x + 1}\end{aligned}$$

Exercise

Expand and simplify: $\left(x - \frac{1}{x^2}\right)^9$

Solution

$$\begin{aligned}\left(x - \frac{1}{x^2}\right)^9 &= x^9 + 9x^8\left(-\frac{1}{x^2}\right) + 36x^7\left(-\frac{1}{x^2}\right)^2 + 84x^6\left(-\frac{1}{x^2}\right)^3 + 126x^5\left(-\frac{1}{x^2}\right)^4 + 126x^4\left(-\frac{1}{x^2}\right)^5 \\ &\quad + 84x^3\left(-\frac{1}{x^2}\right)^6 + 36x^2\left(-\frac{1}{x^2}\right)^7 + 9x\left(-\frac{1}{x^2}\right)^8 + \left(-\frac{1}{x^2}\right)^9 \\ &= \underline{x^9 - 9x^6 + 36x^3 - 84 + 126x^{-3} - 126x^{-6} + 84x^{-9} - 36x^{-12} + 9x^{-15} - x^{-18}}\end{aligned}$$

Exercise

Expand and simplify: $\left(\frac{2}{x} - 3y\right)^5$

Solution

$$\begin{aligned}\left(\frac{2}{x} - 3y\right)^5 &= \left(\frac{2}{x}\right)^5 + 5\left(\frac{2}{x}\right)^4(-3y) + 10\left(\frac{2}{x}\right)^3(-3y)^2 + 10\left(\frac{2}{x}\right)^2(-3y)^3 + 5\left(\frac{2}{x}\right)(-3y)^4 + (-3y)^5 \\ &= \underline{\frac{32}{x^5} - 240\frac{y}{x^4} + 720\frac{y^2}{x^3} - 1,080\frac{y^3}{x^2} + 810\frac{y^4}{x} - 243y^5}\end{aligned}$$

Exercise

Expand and simplify: $\left(3\sqrt{x} + \sqrt[4]{x}\right)^4$

Solution

$$\begin{aligned}\left(3\sqrt{x} + \sqrt[4]{x}\right)^4 &= \left(3\sqrt{x}\right)^4 + 4\left(3\sqrt{x}\right)^3\left(\sqrt[4]{x}\right) + 6\left(3\sqrt{x}\right)^2\left(\sqrt[4]{x}\right)^2 + 4\left(3\sqrt{x}\right)\left(\sqrt[4]{x}\right)^3 + \left(\sqrt[4]{x}\right)^4 \\ &= 81x^2 + 108x^{3/2}x^{1/4} + 54x\sqrt{x} + 12x^{1/2}x^{3/4} + x \\ &= 81x^2 + 108x^{7/4} + 54x\sqrt{x} + 12x^{5/4} + x \\ &= \underline{81x^2 + 108x\sqrt[4]{x^3} + 54x\sqrt{x} + 12x\sqrt[4]{x} + x}\end{aligned}$$

Exercise

Expand and simplify: $(x+1)^5$

Solution

$$(x+1)^5 = \underline{x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1}$$

Exercise

Expand and simplify: $(x-1)^5$

Solution

$$(x-1)^5 = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$$

Exercise

Expand and simplify: $(x-2)^6$

Solution

$$(x-2)^6 = x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64$$

Exercise

Expand and simplify: $\left(\frac{1}{x^3} - 2x\right)^5$

Solution

$$\begin{aligned} \left(\frac{1}{x^3} - 2x\right)^5 &= \frac{1}{x^{15}} - 10\frac{x}{x^{12}} + 10\frac{4x^2}{x^9} - 10\frac{8x^3}{x^6} + 5\frac{16x^4}{x^3} - 32x^5 \\ &= \frac{1}{x^{15}} - \frac{10}{x^{11}} + \frac{40}{x^7} - \frac{80}{x^3} + 80x - 32x^5 \end{aligned}$$

Exercise

Expand and simplify: $\left(\frac{1}{x} - 2x\right)^6$

Solution

$$\begin{aligned} \left(\frac{1}{x} - 2x\right)^6 &= \frac{1}{x^6} - 6\frac{1}{x^5}(2x) + 15\frac{1}{x^4}(2x)^2 - 20\frac{1}{x^3}(2x)^3 + 15\frac{1}{x^2}(2x)^4 - 6\frac{1}{x}(2x)^5 + (2x)^6 \\ &= \frac{1}{x^6} - \frac{12}{x^4} + \frac{60}{x^2} - 160 + 240x^2 - 192x^4 + 64x^6 \end{aligned}$$

Exercise

Expand and simplify: $(x^2 - 2y)^5$

Solution

$$(x^2 - 2y)^5 = x^{10} - 10x^8y + 40x^6y^2 - 80x^4y^3 + 80x^2y^4 - 32y^5$$

Exercise

Expand and simplify: $\left(\frac{2}{x} + 3\sqrt{x}\right)^4$

Solution

$$\begin{aligned}\left(\frac{2}{x} + 3\sqrt{x}\right)^4 &= \frac{16}{x^4} + \frac{32}{x^3}(3\sqrt{x}) + \frac{24}{x^2}(9x) + \frac{8}{x}(27x\sqrt{x}) + 81x^2 \\ &= \frac{16}{x^4} + \frac{96\sqrt{x}}{x^3} + \frac{216}{x} + 216\sqrt{x} + 81x^2\end{aligned}$$

Exercise

Expand and simplify: $(2x + 5y)^7$

Solution

$$\begin{aligned}(2x + 5y)^7 &= 128x^7 + 7(64x^6)(5y) + 21(32x^5)(25y^2) + 35(16x^4)(125y^3) \\ &\quad + 35(8x^3)(625y^4) + 21(4x^2)(3,125y^5) + 7(2x)(5^6y^6) + (5y)^7 \\ &= 128x^7 + 320x^6y + 16,800x^5y^2 + 70,000x^4y^3 + 175,000x^3y^4 + 262,500x^2y^5 \\ &\quad + 218,750xy^6 + 78,125y^7\end{aligned}$$

Exercise

Expand and simplify: $(2x - 3)^{11}$

Solution

$$\begin{aligned}(2x - 3)^{11} &= (2x)^{11} - 33(2x)^{10} + 495(2x)^9 - 4,995(2x)^8 + 350(3)^4(2x)^7 - 462(3)^5(2x)^6 \\ &\quad + 462(3)^6(2x)^5 - 350(3)^7(2x)^4 + 185(3)^8(2x)^3 - 55(3)^9(2x)^2 + 22(3)^{10}x - 3^{11} \\ &= 2,048x^{11} - 33,792x^{10} + 253,440x^9 - 1,278,720x^8 + 3,628,800x^7 - 7,185,024x^6 \\ &\quad + 462(3)^6 2^5x^5 - 5,600(3)^7x^4 + 1,480(3)^8x^3 - 220(3)^9x^2 + 22(3)^{10}x - 3^{11}\end{aligned}$$

Exercise

Expand and simplify: $(2x - 3y)^6$

Solution

$$(2x - 3y)^6 = 64x^6 - 576x^5y + 2,160x^4y^2 - 4,320x^3y^3 + 4,860x^2y^4 - 2,196xy^5 + 729y^6$$

Exercise

Expand and simplify: $(2x + 3y)^5$

Solution

$$(2x + 3y)^5 = 32x^5 + 240x^4y + 720x^3y^2 + 1,080x^2y^3 + 810xy^4 + 243y^5$$

Exercise

Expand and simplify: $(3x - 2y)^4$

Solution

$$(3x - 2y)^4 = 81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4$$

Exercise

Expand and simplify: $(x^2 + y^3)^3$

Solution

$$(x^2 + y^3)^3 = x^6 + 3x^4y^3 + 3x^2y^6 + y^9$$

Exercise

Expand and simplify: $(x^2 - y^2)^3$

Solution

$$(x^2 - y^2)^3 = x^6 - 3x^4y^2 + 3x^2y^4 - y^6$$

Exercise

Expand and simplify: $(2 + i)^6$

Solution

$$\begin{aligned}(2 + i)^6 &= 64 + 6(32)i + 15(16)i^2 + 20(8)i^3 + 15(4)i^4 + 12i^5 + i^6 \\ &= 64 + 192i - 240 - 160i + 60 + 12i - 1 \\ &= \underline{-117 + 44i} \quad | \end{aligned}$$

Exercise

Expand and simplify: $(2 - i)^6$

Solution

$$\begin{aligned}(2 - i)^6 &= 64 - 6(32)i + 15(16)i^2 - 20(8)i^3 + 15(4)i^4 - 12i^5 + i^6 \\ &= 64 - 192i - 240 + 160i + 60 - 12i - 1 \\ &= \underline{-117 - 44i} \quad | \end{aligned}$$

Exercise

Expand and simplify: $(\sqrt{2} + i)^5$

Solution

$$\begin{aligned}(\sqrt{2} + i)^5 &= 2\sqrt{2} + 20i + 20\sqrt{2}i^2 + 20i^3 + \sqrt{2}i^4 + i^5 \\ &= 2\sqrt{2} + 20i - 20\sqrt{2} - 20i + \sqrt{2} + i \\ &= \underline{-17\sqrt{2} + i} \quad | \end{aligned}$$

Exercise

Expand and simplify: $(3 - i)^4$

Solution

$$\begin{aligned}(3 - i)^4 &= 84 - 108i + 54i^2 - 12i^3 + i^4 \\ &= 84 - 108i - 54 + 12i + 1 \\ &= \underline{31 - 96i} \quad | \end{aligned}$$