Lecture One – Limits and Derivatives

Section 1.1 – Idea of Limits

Position Function

An object that is falling or vertically projected into the air has its height above the ground, s(t), in feet, given by

$$s(t) = -16t^2 + v_0 t + s_0$$

v_0 is the original velocity (initial velocity) of the object, in *feet* per *second*

t is the time that the object is in motion, in *second*

s_0 is the original height (initial height) of the object, in *feet*

The average rate is given by: $\frac{\Delta s}{\Delta t}$

Example

A rock breaks loose from the top of a tall cliff. What is its average speed

a) During the first 2 *sec* of fall?

b) During the 1-*sec* interval between second 1 and *second* 2?

Solution

Since the rock falls free (*down*) without any initial velocity or height. $\Rightarrow y(t) = 16t^2$

a) For the first 2 *sec*: *Average speed* $= \frac{\Delta y}{\Delta t}$

$$= \frac{y(2) - y(0)}{2 - 0}$$

$$= \frac{16(2)^2 - 16(0)^2}{2}$$

$$= \frac{64}{2}$$

$$= 32 \ ft/\sec$$

b) From 1 *sec* to 2 *sec*: *Average speed* $= \frac{y(2) - y(1)}{2 - 1}$

$$= \frac{16(2)^2 - 16(1)^2}{1}$$

$$= 48 \ ft/\sec$$

Example

Find the speed of a falling rock $\left(y(t)=16t^2\right)$ over a time interval $\left[t_0,\ t_0+h\right]$. Then find the average speed at 1 *sec* and 2 *sec*.

Solution

$$\frac{\Delta y}{\Delta t}=\frac{16\left(t_0+h\right)^2-16\left(t_0\right)^2}{\left(t_0+h\right)-t_0}$$

$$=\frac{16\left(t_0^2+2ht_0+h^2\right)-16t_0^2}{t_0+h-t_0}$$

$$=\frac{16t_0^2+32ht_0+16h^2-16t_0^2}{h}$$

$$=32\frac{ht_0}{h}+16\frac{h^2}{h}$$

$$\boxed{=32t_0+16h}$$

If $t_0=1\Rightarrow\frac{\Delta y}{\Delta t}=32(1)+16h=\boxed{32+16h}$

The average speed has the limiting value 32 *ft/sec* as h approaches 0.

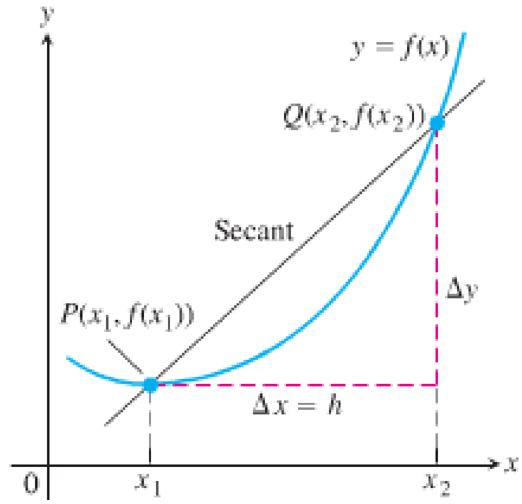
If $t_0=2\Rightarrow\frac{\Delta y}{\Delta t}=32(2)+16h=\boxed{64+16h}$

The average speed has the limiting value 64 *ft/sec* as h approaches 0.

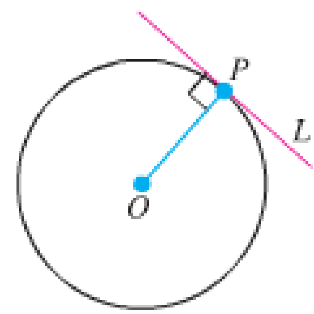
Average Rates of Changes and Secant Lines

The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0$$



Defining the Slope of a Curve

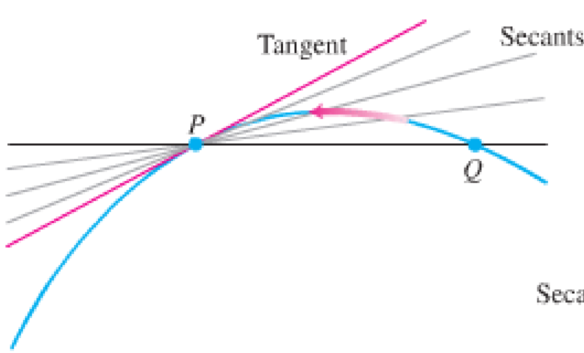


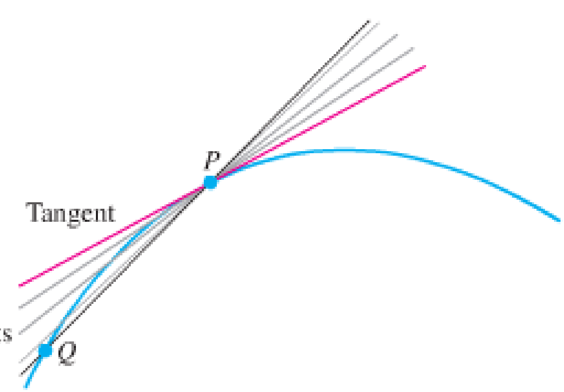
The slope of a line is the rate at which it rises or falls.

To define the tangency for general curves, we need an approach that makes the behavior of the secants through *P* and points *Q* as *Q* moves toward *P* along the curve:

1. Find the slope of the secant *PQ*.
2. Investigate the limiting value of the slope as *Q* approaches *P* along the curve.
3. If the limit exists, take it to be the slope of the curve at *P* and define the tangent to the curve at *P* to be the line through *P* with this slope.

$$m_{\tan} = \lim_{t \to a} = \frac{f(t) - f(a)}{t - a}$$





Example

Find the slope of the parabola $y = x^2$ at the point $P(2,\ 4)$. Write an equation for the tangent to the parabola at this point.

Solution

$$\textbf{\textit{Secant slope}} = \frac{\Delta y}{\Delta x} = \frac{f(x_1 + h) - f(x_1)}{h}$$

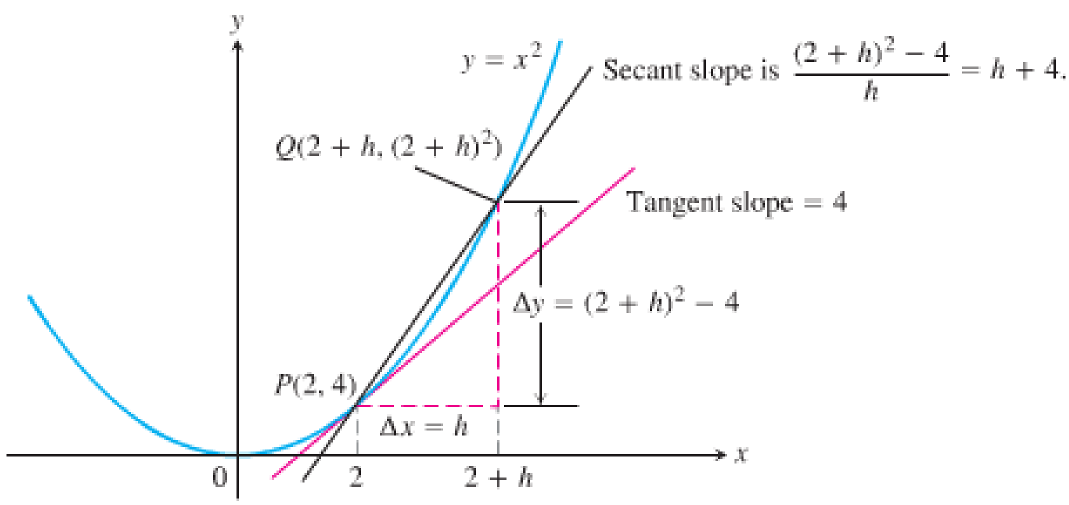
$$= \frac{f(2+h) - f(2)}{h}$$

$$= \frac{(2+h)^2 - 2^2}{h}$$

$$= \frac{4 + 4h + h^2 - 4}{h}$$

$$= \frac{4h}{h} + \frac{h^2}{h}$$

$$= 4 + h$$



As Q approaches P, h approaches 0. Then the secant slope $h + 4 \to 4 = \textit{slope}$

$$y = m(x - x_1) + y_1$$

$$y = 4(x - 2) + 4$$

$$y = 4x - 4$$

Exercises *Section* 1.1 – Idea of Limits

1. Find the average rate of change of the function $f(x) = x^3 + 1$ over the interval $[2,\ 3]$

2. Find the average rate of change of the function $f(x) = x^2$ over the interval $[-1,\ 1]$

3. Find the average rate of change of the function $f(t) = 2 + \cos t$ over the interval $[-\pi,\ \pi]$

4. Find the slope of $y = x^2 - 3$ at the point $P(2,\ 1)$ and an equation of the tangent line at this *P*.

5. Find the slope of $y = x^2 - 2x - 3$ at the point $P(2,\ -3)$ and an equation of the tangent line at this *P*.

6. Find the slope of $y = x^3$ at the point $P(2,\ 8)$ and an equation of the tangent line at this *P*.

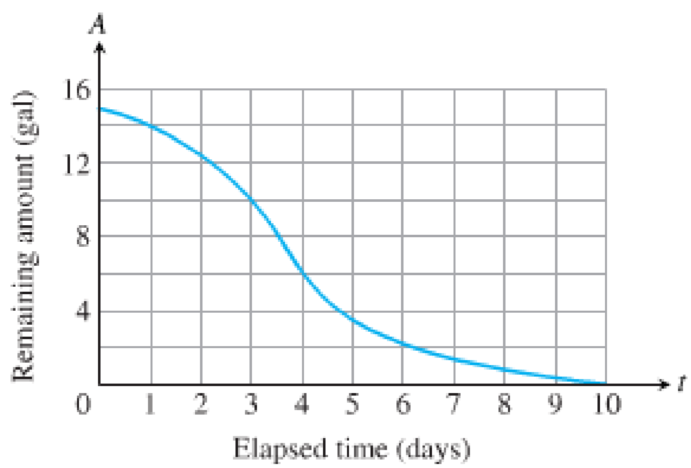
7. Make a table of values for the function $f(x) = \frac{x+2}{x-2}$ at the points

$$x = 1.2, \quad x = \frac{11}{10}, \quad x = \frac{101}{100}, \quad x = \frac{1001}{1000}, \quad x = \frac{10001}{10000}, \quad \text{and } x = 1$$

a) Find the average rate of change of $f(x)$ over the intervals $[1,\ x]$ for each $x \neq 1$ in the table

b) Extending the table if necessary, try to determine the rate of change of $f(x)$ at $x = 1$.

8. The accompanying graph shows the total amount of gasoline A in the gas tank of an automobile after being driven for *t* days.



a) Estimate the average rate of gasoline consumption over the time intervals

$[0,\ 3]$, $[0,\ 5]$, *and* $[7,\ 10]$

b) Estimate the instantaneous rate of gasoline consumption over the time $t = 1$, $t = 4$, *and* $t = 8$