Section 4.4 – Calculus in Polar Coordinates

Slope

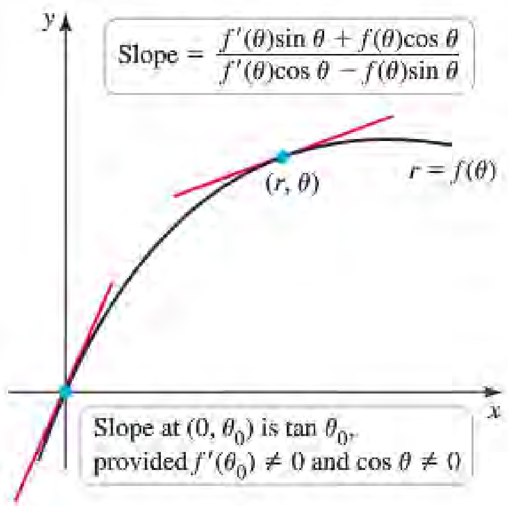
The slope of a polar curve $r = f(\theta)$ in the *xy*-plane is still given by $\frac{dy}{dx}$, which is not $r' = \frac{df}{d\theta}$

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\
&= \frac{\frac{d}{d\theta}\left(f(\theta)\cdot\sin\theta\right)}{\frac{d}{d\theta}\left(f(\theta)\cdot\cos\theta\right)} \\
&= \frac{\frac{df}{d\theta}\cdot\sin\theta + f(\theta)\cdot\cos\theta}{\frac{df}{d\theta}\cdot\cos\theta + f(\theta)\cdot\sin\theta}
\end{aligned}$$

$$\boxed{\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}}$$

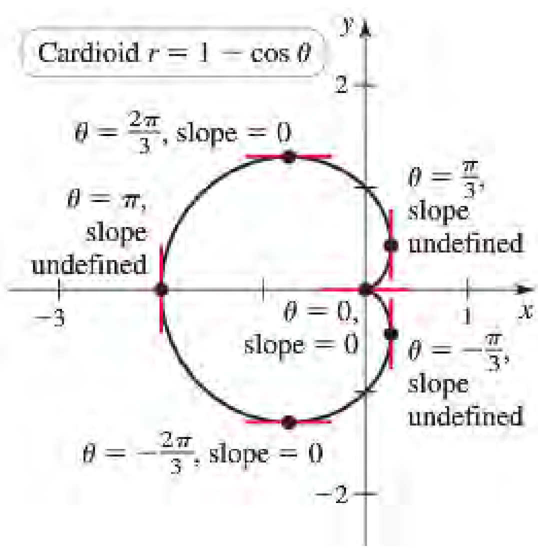


Example

Find the points on the interval $-\pi \le \theta \le \pi$ at which the cardioid $r = f(\theta) = 1 - \cos\theta$ has a vertical or horizontal tangent line.

Solution

$$\begin{aligned}
\frac{dy}{dx} &= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} \\
&= \frac{\sin\theta\sin\theta + (1-\cos\theta)\cos\theta}{\sin\theta\cos\theta - (1-\cos\theta)\sin\theta} \\
&= \frac{\sin^2\theta + \cos\theta - \cos^2\theta}{\sin\theta\cos\theta - \sin\theta + \cos\theta\sin\theta} \\
&= \frac{1-\cos^2\theta + \cos\theta - \cos^2\theta}{2\sin\theta\cos\theta - \sin\theta} \\
&= \frac{-2\cos^2\theta + \cos\theta + 1}{\sin\theta(2\cos\theta - 1)} \\
&= \frac{-(2\cos\theta + 1)(\cos\theta - 1)}{\sin\theta(2\cos\theta - 1)}
\end{aligned}$$



The points with a horizontal tangent line:

$$\frac{dy}{dx}=0$$

$$(2\cos\theta+1)(\cos\theta-1)=0$$

$$\rightarrow\begin{cases}\cos\theta=-\frac{1}{2} & \rightarrow \ \boxed{\theta=\pm\frac{2\pi}{3}} \\ \cos\theta=1 & \rightarrow \ \boxed{\cancel{\theta=0}}\end{cases} \quad \textit{numerator is } 0$$

The points with a Vertical tangent line:

$$\sin\theta(2\cos\theta-1)=0$$

$$\rightarrow\begin{cases}\sin\theta=0 & \rightarrow \ \boxed{\theta=\cancel{0},\pm\pi} \\ \cos\theta=\frac{1}{2} & \rightarrow \ \boxed{\theta=\pm\frac{\pi}{3}}\end{cases} \quad \textit{denominator is } 0$$

$$\frac{dy}{dx}=\lim_{\theta\to 0^+}\left(\frac{-2\cos^2\theta+\cos\theta+1}{2\cos\theta\sin\theta-\sin\theta}\right)=\frac{0}{0}$$

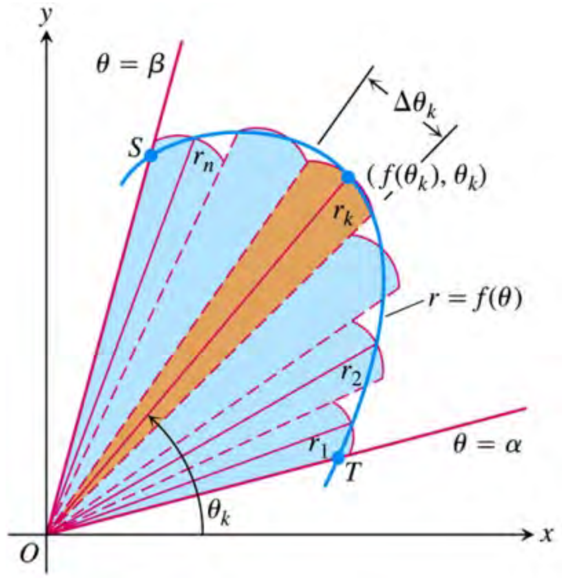
$$=\lim_{\theta\to 0^+}\left(\frac{4\cos\theta\sin\theta-\sin\theta}{-2\sin^2\theta+2\cos^2\theta-\cos\theta}\right)$$

$$=\frac{0}{1}$$

$\boxed{=0}$ Therefore, the curve has a slope of 0 at origin.

Area in the plane

The region OTS is bounded by the rays $\theta = \alpha$ *and* $\theta = \beta$ and the curve $r = f(\theta)$.



We approximate the region with *n* non-overlapping fan-shaped circular sectors based on a partition *P* of angle *TOS*. The typical sector has radius $r_k = f\left(\theta_k\right)$ and central angle of radian measure $\Delta\theta_k$. Its area is $\frac{\Delta\theta_k}{2\pi}$ times the area of a circle of radius r_k, or

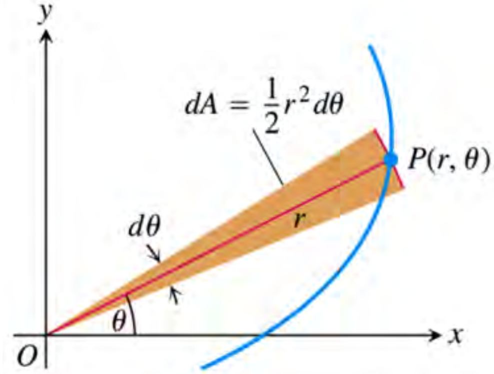
$$A_k = \tfrac{1}{2} r_k^2 \Delta\theta_k = \tfrac{1}{2}\left(f\left(\theta_k\right)\right)^2 \Delta\theta_k$$

Area of the Fan-Shaped Region between the Origin and the curve $r = f(\theta)$, $\alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \tfrac{1}{2}\left(f(\theta)\right)^2 d\theta$$

This is the integral of the ***area differential***

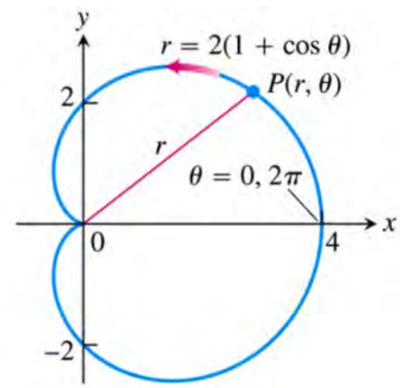
$$dA = \tfrac{1}{2} r^2 d\theta = \tfrac{1}{2}\left(f(\theta)\right)^2 d\theta$$



Example

Find the area of the region in the plane enclosed by the cardioid $r = 2(1+\cos\theta)$

Solution



$$A = \frac{1}{2}\int_0^{2\pi} r^2 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 4(1+\cos\theta)^2 d\theta$$

$$= \int_0^{2\pi} 2\left(1+2\cos\theta+\cos^2\theta\right) d\theta$$

$$= \int_0^{2\pi} \left(2+4\cos\theta+2\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \int_0^{2\pi} (3+4\cos\theta+\cos 2\theta)\, d\theta$$

$$= 3\theta + 4\sin\theta + \frac{1}{2}\sin 2\theta \Big|_0^{2\pi}$$

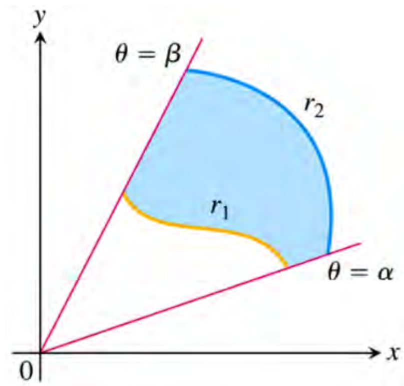
$$= 3(2\pi) + 4\sin(2\pi) + \frac{1}{2}\sin(2\pi) - \left(3(0) + 4\sin(0) + \frac{1}{2}\sin(0)\right)$$

$$= 6\pi \quad unit^2$$

Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta), \quad \alpha \le \theta \le \beta$

$$A = \frac{1}{2}\int_\alpha^\beta r_2^2\, d\theta \;-\; \frac{1}{2}\int_\alpha^\beta r_1^2\, d\theta$$

$$= \frac{1}{2}\int_\alpha^\beta \left(r_2^2 - r_1^2\right) d\theta$$



Example

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$

Solution

$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(r_2^2 - r_1^2\right) d\theta$$

$$= 2\frac{1}{2}\int_0^{\pi/2} \left(1^2 - (1 - \cos\theta)^2\right) d\theta$$

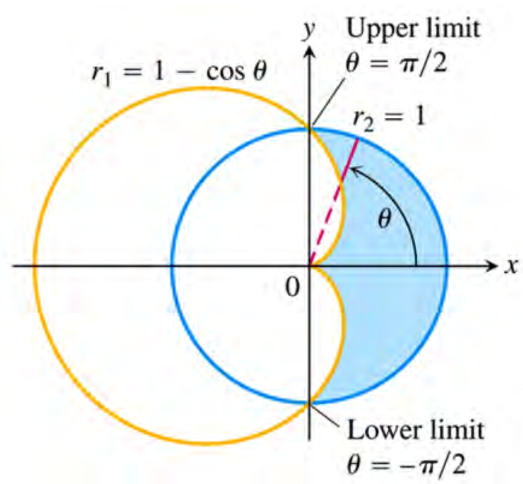
$$= \int_0^{\pi/2} \left(1 - \left(1 - 2\cos\theta + \cos^2\theta\right)\right) d\theta$$

$$= \int_0^{\pi/2} \left(2\cos\theta - \frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= 2\sin\theta - \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \Big|_0^{\pi/2}$$

$$= 2\sin\frac{\pi}{2} - \frac{1}{2}\frac{\pi}{2} - \frac{1}{4}\sin 2\frac{\pi}{2} - 0$$

$$= 2 - \frac{\pi}{4} \quad unit^2$$



Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example

Find the length of the cardioid $r = 1 - \cos\theta$

Solution

$$r = 1 - \cos\theta \quad \Rightarrow \quad \frac{dr}{d\theta} = \sin\theta$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1-\cos\theta)^2 + \sin^2\theta$$

$$= 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta \qquad \cos^2\theta + \sin^2\theta = 1$$

$$= 2 - 2\cos\theta$$

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta$$

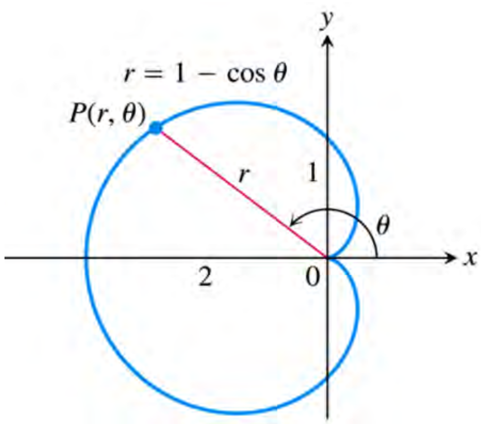
$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta$$

$$= -4\cos\frac{\theta}{2}\Big|_0^{2\pi}$$

$$= -4\left(\cos\frac{2\pi}{2} - \cos 0\right)$$

$$= -4(-1-1)$$

$$\underline{= 8 \quad unit}$$



Area of a surface of Revolution

Theorem

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$.

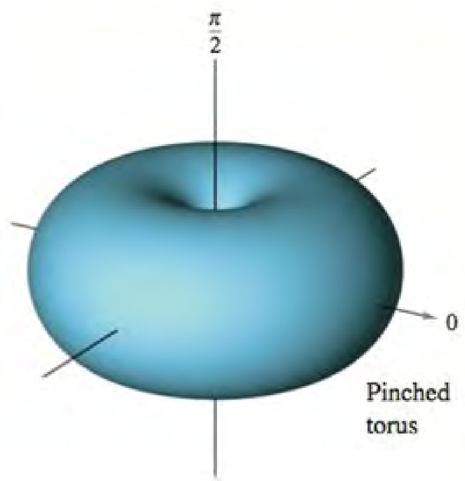
The area of the surface formed by revolving the graph of $r = f(\theta)$ about the indicated line

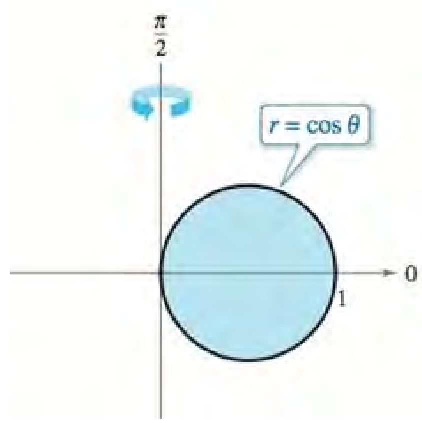
$$S = 2\pi \int_{\alpha}^{\beta} f(\theta)\sin\theta \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta \qquad \textit{About the polar axis}$$

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta)\cos\theta \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta \qquad \textit{About the line } \theta = \frac{\pi}{2}$$

Example

Find the area of the surface formed by revolving the circle $f(\theta) = \cos\theta$ about the line $\theta = \frac{\pi}{2}$





Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{\cos^2\theta + \sin^2\theta}$$

$$= 1$$

$$S = 2\pi \int_0^{\pi} \cos^2\theta\, d\theta$$

$$= \pi \int_0^{\pi} (1 + \cos 2\theta)\, d\theta$$

$$= \pi \left(\theta + \frac{1}{2}\sin 2\theta \right|_0^{\pi}$$

$$= \pi^2 \ \ unit^2$$

Exercises *Section* 4.4 – Calculus in Polar Coordinates

(1 – 4) Find the slopes of the curves at the given points. Sketch the curves along with their tangents at these points.

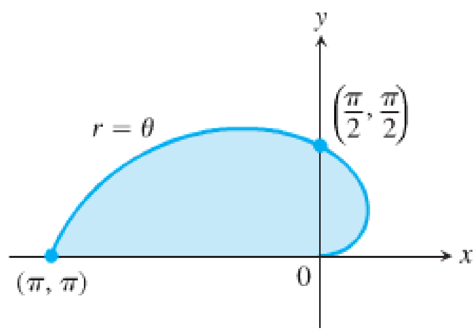
1. *Cardioid* $r=-1+\cos\theta;\quad \theta=\pm\frac{\pi}{2}$

2. *Cardioid* $r=-1+\sin\theta;\quad \theta=0,\ \pi$

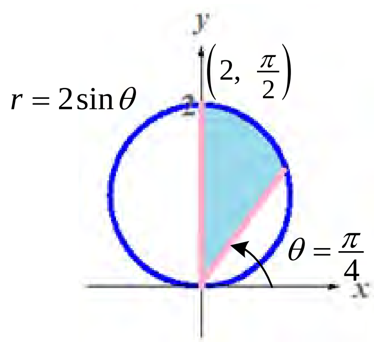
3. *Four – leaved rose* $r=\sin 2\theta;\quad \theta=\pm\frac{\pi}{4},\ \pm\frac{3\pi}{4}$

4. *Four – leaved rose* $r=\cos 2\theta;\quad \theta=0,\ \pm\frac{\pi}{2},\ \pi$

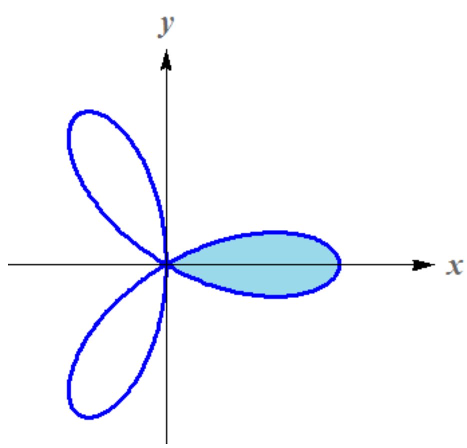
5. Find the area of the region bounded by the spiral $r=\theta$ *for* $0\le\theta\le\pi$



6. Find the area of the region bounded by the circle $r=2\sin\theta$ *for* $\frac{\pi}{4}\le\theta\le\frac{\pi}{2}$



7. Find the area of the region inside one leaf of the three-leaved rose $r=\cos 3\theta$



(**8 – 66**) Find the area of the region

8. Inside oval limaçon $r = 4 + 2\sin\theta$

9. Inside Cardioid $r = a(1 + \cos\theta), \quad a > 0$

10. Inside Six-leaved rose $r^2 = 2\sin 3\theta$

11. Inside curve $r = \sqrt{\cos\theta}$

12. Inside right lobe of $r = \sqrt{\cos 2\theta}$

13. Inside Cardioid $r = 4 + 4\sin\theta$

14. Inside Limaçon $r = 2 + \cos\theta$

15. Inside circle $r = 6$ above the line $r = 3\csc\theta$

16. Inside inner loop $r = \cos\theta - \frac{1}{2}$

17. Inside One leave of $r = \cos 3\theta$

18. Shared by the circles $r = 2\cos\theta \quad \textit{and} \quad r = 2\sin\theta$

19. Shared by the circle $r = 2$ and the cardioid $r = 2(1 - \cos\theta)$

20. Enclosed by the four-leaf rose $r = f(\theta) = 2\cos 2\theta$

21. Lies inside the circle $r = 1$ and outside the cardioid $r = 1 + \cos\theta$

22. Outside the circle $r = \frac{1}{2}$ and inside the circle $r = \cos\theta$

23. Outside the circle $r = \frac{1}{\sqrt{2}}$ and inside the curve $r = \sqrt{\cos\theta}$

24. Inside the circle $r = \frac{1}{\sqrt{2}}$ in QI and inside the right lobe of $r = \sqrt{\cos 2\theta}$

25. Inside the rose $r = 4\sin 2\theta$ and inside the circle $r = 2$

26. Inside the lemniscate $r^2 = 2\sin 2\theta$ and outside the circle $r = 1$

27. Inside all the leaves of the rose $r = 3\sin 2\theta$

28. Inside one leaf of the rose $r = \cos 5\theta$

29. A complete three-leaf rose $r = 2\cos 3\theta$

30. Inside the rose $r = 4\cos 2\theta$ and outside the circle $r = 2$

31. Bounded by the lemniscate $r^2 = 6\sin 2\theta$

32. Bounded by the limaçon $r = 2 - 4\sin\theta$

33. Bounded by the limaçon $r = 4 - 2\cos\theta$

34. Inside one leaf: $r = 2\sin 6\phi$

35. Between inner and outer: $r = 3 - 6\cos\phi$

36. Inner loop of $r = 1 + 2\cos\theta$

37. Inner loop of $r = 2 - 4\cos\theta$

38. Inner loop of $r = 1 + 2\sin\theta$

39. Inner loop of $r = 4 - 6\sin\theta$

40. Between the loops $r = 1 + 2\cos\theta$

41. Between the loops $r = 2(1 + 2\sin\theta)$

42. Between the loops $r = 3 - 6\sin\theta$

43. Between the loops $r = \frac{1}{2} + \cos\theta$

44. Inside $r = 2\cos\theta$ and outside $r = 1$

45. Inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$

46. Common interior of $r = 4\sin 2\theta$ and $r = 2$

47. Common interior of $r = 4\sin\theta$ and $r = 2$

48. Common interior of $r = 2\cos\theta$ and $r = 2\sin\theta$

49. Common interior of $r = 2(1 + \cos\theta)$ *and* $r = 2(1 - \cos\theta)$

50. Common interior of $r = 3 - 2\sin\theta$ *and* $r = -3 + 2\sin\theta$

51. Common interior of $r = 5 - 3\sin\theta$ *and* $r = 5 - 3\cos\theta$

52. Inside $r = a(1 + \cos\theta)$ and outside $r = a\cos\theta$

53. Inside $r = 2a\cos\theta$ and outside $r = a$

54. Common interior of $r = a(1 + \cos\theta)$ and $r = a\sin\theta$

55. Common interior of $r = a\cos\theta$ and $r = a\sin\theta$, where $a > 0$.

56. Enclosed by all the leaves of the rose $r = 3\sin 4\theta$

57. Enclosed by the limaçon $r = 2 - \cos\theta$

58. Inside limaçon $r = 2 + \cos\theta$ and outside the circle $r = 2$

59. Inside lemniscate $r^2 = 4\cos 2\theta$ and outside the circle $r = \frac{1}{2}$

60. Inside both cardioids $r = 1 - \cos\theta$ and $r = 1 + \cos\theta$

61. Inside the cardioid $r = 1 + \cos\theta$ and outside the cardioid $r = 1 - \cos\theta$

62. Inside both cardioids $r = 2 - 2\sin\theta$ and $r = 2 + 2\sin\theta$

63. common interior of $r = 2 - 2\sin\theta$ and $r = 2 + 2\sin\theta$

64. Inside both cardioids $r = 1 + \sin\theta$ and $r = 1 + \cos\theta$

65. Common interior $r=1$ and $r=\sqrt{2}\,\cos 2\theta$

66. Outside $r=1$ and inside $r=\sqrt{2}\,\cos 2\theta$

(**67 – 89**) Find the length of

67. $r=\theta^2, \quad 0\le\theta\le\sqrt{5}$

68. $r=\dfrac{e^\theta}{\sqrt{2}}, \quad 0\le\theta\le\pi$

69. $r=a\sin^2\left(\dfrac{\theta}{2}\right), \quad 0\le\theta\le\pi, \quad a>0$

70. $r=\dfrac{6}{1+\cos\theta}, \quad 0\le\theta\le\dfrac{\pi}{2}$

71. $r=\cos^3\left(\dfrac{\theta}{3}\right), \quad 0\le\theta\le\dfrac{\pi}{4}$

72. $r=\sqrt{1+\sin 2\theta}, \quad 0\le\theta\le\pi\sqrt{2}$

73. $r=8 \quad 0\le\theta\le 2\pi$

74. $r=a \quad 0\le\theta\le 2\pi$

75. $r=4\sin\theta \quad 0\le\theta\le\pi$

76. $r=2a\cos\theta \quad -\dfrac{\pi}{2}\le\theta\le\dfrac{\pi}{2}$

77. $r=1+\sin\theta \quad 0\le\theta\le 2\pi$

78. $r=8(1+\cos\theta) \quad 0\le\theta\le 2\pi$

79. $r=2\theta \quad 0\le\theta\le\dfrac{\pi}{2}$

80. $r=\sec\theta \quad 0\le\theta\le\dfrac{\pi}{3}$

81. $r=\dfrac{1}{\theta} \quad \pi\le\theta\le 2\pi$

82. $r=e^\theta \quad 0\le\theta\le\pi$

83. $r=5\cos\theta \quad \dfrac{\pi}{2}\le\theta\le\pi$

84. $r=3(1-\cos\theta) \quad 0\le\theta\le\pi$

85. $r=2\sin 6\phi$ one petal

86. Inner loop $r=3-6\cos\phi$

87. $r=e^{2\theta} \quad 0\le\theta\le 2$

88. $r=\cos\theta$

89. $r=a(1-\cos\theta)$

(**90 – 95**) Find the surface area bounded by

90. $r=6\cos\theta \quad 0\le\theta\le\dfrac{\pi}{2}$ *revolving about Polar axis*

91. $r=a\cos\theta \quad 0\le\theta\le\dfrac{\pi}{2}$ *revolving about* $\theta=\dfrac{\pi}{2}$

92. $r=e^{a\theta} \quad 0\le\theta\le\dfrac{\pi}{2}$ *revolving about* $\theta=\dfrac{\pi}{2}$

93. $r=a(1+\cos\theta) \quad 0\le\theta\le\pi$ *revolving about polar axis*

94. $r=1+4\cos\theta \quad 0\le\theta\le\dfrac{\pi}{2}$ *revolving about Polar axis*

95. $r=2\sin\theta \quad 0\le\theta\le\dfrac{\pi}{2}$ *revolving about* $\theta=\dfrac{\pi}{2}$

96. Find the surface area of the torus generated by revolving the circle given by $r=2$ about the line $r=5\sec\theta$

97. Find the surface area of the torus generated by revolving the circle given by $r = a$ about the line $r = b\sec\theta$, where $0 < a < b$

98. Let a and b be positive constants. Find the area of the region in the first quadrant bounded by the graph of the polar equation

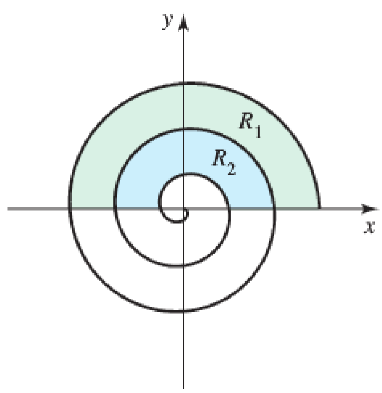
$$r = \frac{ab}{a\sin\theta + b\cos\theta}, \quad 0 \le \theta \le \frac{\pi}{2}$$

99. Assume m is a positive integer

a) Even number of leaves: what is the relationship between the total area enclosed by the $4m$-leaf rose $r = \cos(2m\theta)$ and m?

b) Odd number of leaves: what is the relationship between the total area enclosed by the $(2m+1)$-leaf rose $r = \cos((2m+1)\theta)$ and m?

100. Let R_n be the region bounded by the nth turn and the $(n+1)st$ turn of the spiral $r = e^{-\theta}$ in the first and second quadrants, for $\theta \ge 0$

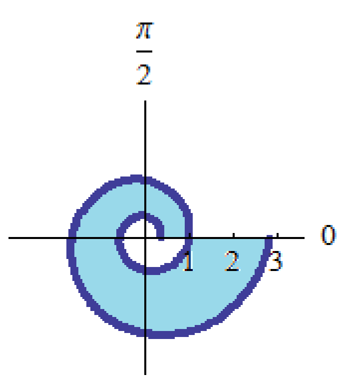


a) Find the area A_n of R_n.

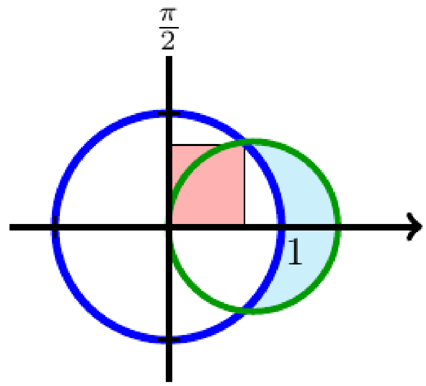
b) Evaluate $\lim\limits_{n\to\infty} A_n$

c) Evaluate $\lim\limits_{n\to\infty} \dfrac{A_{n+1}}{A_n}$

101. The curve represented by the equation $r = ae^{b\theta}$, where a and b are constants, is called a logarithmic spiral. The figure shows the graph of $r = e^{\theta/6}$. $-2\pi \le \theta \le 2\pi$. Find the area of the shaded region.

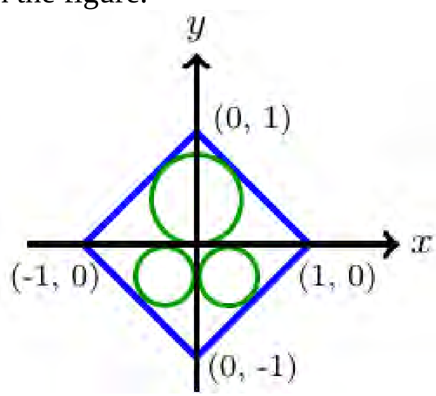


102. The larger circle in the figure is the graph of $r = 1$.



Find the polar equation of the smaller circle such that the shaded regrions are equal.

103. Find equations of the circles in the figure.

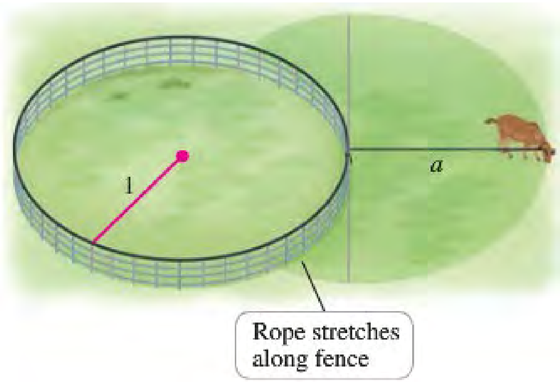


Determine whether the combined area of the circles is greater than or less than the area of the region inside the square but outside the circles.

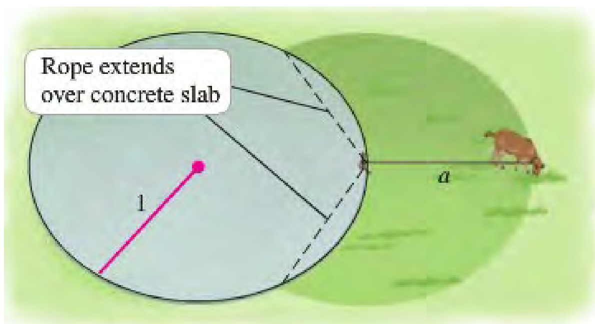
104. A circular corral of unit radius is enclosed by a fence. A goat inside the corral is tied to the fence with a rope of length $0 \le a \le 2$.

What is the area of the region (inside the corral) that the goat can graze? Check your answer with the special cases $a = 0$ and $a = 2$

105. A circular corral of unit radius is enclosed by a fence. A goat outside the corral is tied to the fence with a rope of length $0 \le a \le \pi$. What is the area of the grassy region (outside the corral) that the goat can reach?

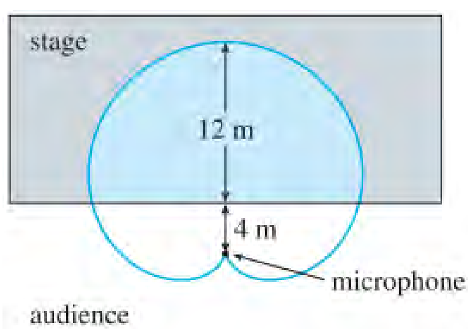


106. A circular concrete slab of unit radius is surrounded by grass. A goat is tied to the edge of the slab with a rope of length $0 \le a \le 2$.



What is the area of the grassy region that the goat can graze? Note that the rope can extend over the concrete slab. Check your answer with the special cases $a = 0$ and $a = 2$

107. When recording live performance, sound engineers often use a microphone with a cardioid pickup pattern because it suppresses noise from the audience. Suppose the microphone is placed 4 *m* from the front of the stage and the boundary of the optimal pickup region is given by the cardioid $r = 8 + 8\sin\theta$, where r if measured in meters and the microphone is at the pole.



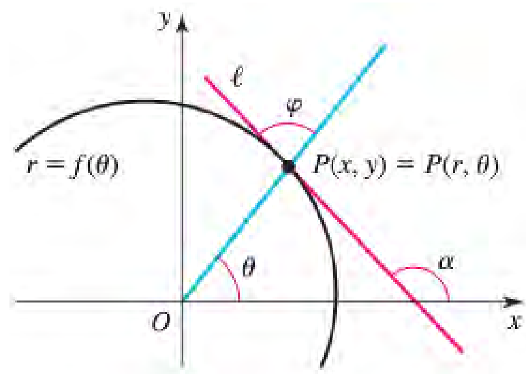
The musicians want to know the area they will have on stage within the optimal pickup range of the microphone, Answer their question.

108. The curve given by the parametric equations

$$x(t)=\frac{1-t^2}{1+t^2} \quad and \quad y(t)=\frac{t\left(1-t^2\right)}{1+t^2}$$

a) Find the rectangular equation of the strophoid.
b) Find a polar equation of the strophoid.
c) Sketch a graph of the strophoid.
d) Find the equations of the two tangent lines at the origin.
e) Find the points on the graph at which the tangent lines are horizontal.

109. Let a polar curve be described by $r=f(\theta)$ and let ℓ be the line tangent to the curve at the point $P(x,\ y)=P(r,\ \theta)$



a) Explain why $\tan\alpha=\frac{dy}{dx}$

b) Explain why $\tan\theta=\frac{y}{x}$

c) Let φ be the angle between ℓ and the line *O* and *P*. Prove that $\tan\varphi=\frac{f(\theta)}{f'(\theta)}$

d) Prove that the value of θ for which ℓ is parallel to the *x*-axis satisfy $\tan\theta=-\frac{f(\theta)}{f'(\theta)}$

e) Prove that the value of θ for which ℓ is parallel to the *y*-axis satisfy $\tan\theta=\frac{f'(\theta)}{f(\theta)}$