Section 2.5 – Derivatives as Rates of Change

Definition

The ***instantaneous rate of change*** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{x \to \infty} \frac{f(x_0 + h) - f(x)}{h}$$

Provided the limit exists.

Example

The area **A** of the circle is related to its diameter by the equation $A = \frac{\pi}{4} D^2$

How fast does the area change with respect to the diameter when the diameter is 10 *m*?

Solution

The rate of change of the area with respect to the diameter is

$$\frac{dA}{dD} = \frac{\pi}{4} \cdot 2D = \frac{\pi D}{2}$$

When $D = 10\ m$, the area is changing with respect to the diameter at the rate of

$$\frac{dA}{dD} = \frac{\pi(10)}{2} \approx 15.71\ m^2 / m$$

Motion along a Line: Displacement, Velocity, Speed, Acceleration, and *Jerk*

Suppose that an object is moving along a coordinate line (an *s*-axis), usually horizontal or vertical, so that we know its position *s* on that line as a function of time *t*:

$$s = f(t)$$

The ***displacement*** of the object over the time interval from t to $t + \Delta t$ is

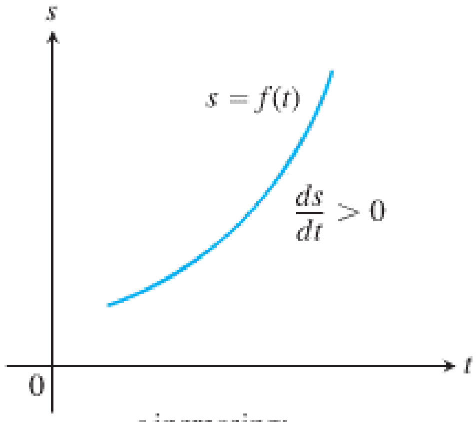
$$\Delta s = f(t + \Delta t) - f(t)$$

And the ***average velocity*** of the object over that time interval is

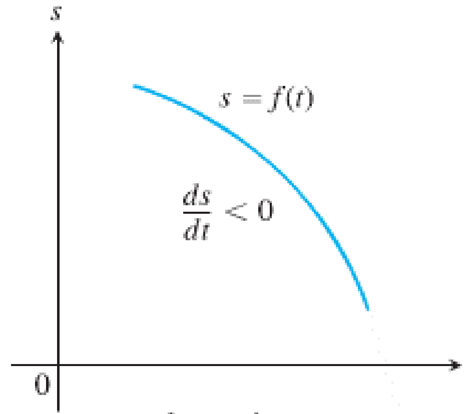
$$v_{avg} = \frac{displacement}{travel\ time} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

Definition

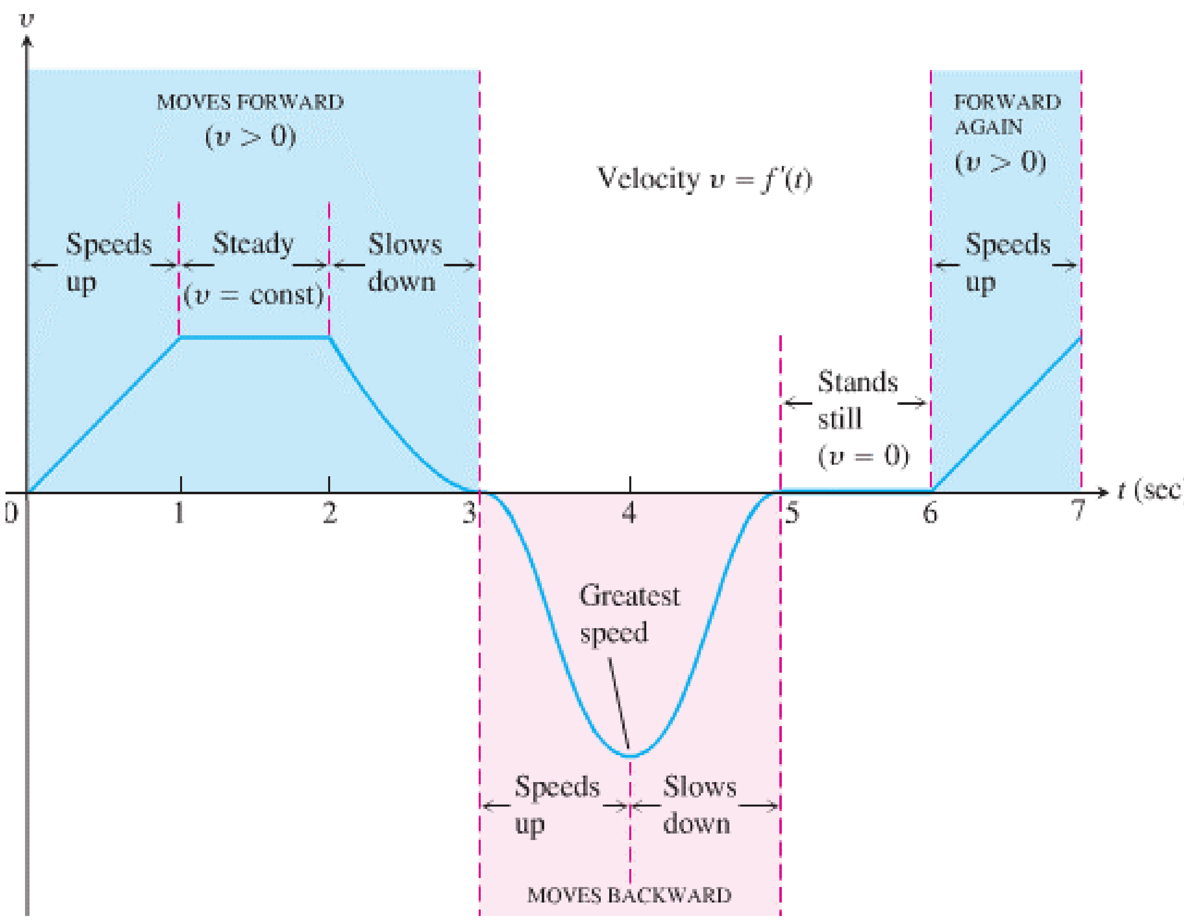
Speed is the absolute value of velocity $\qquad speed = |v(t)| = \left|\frac{ds}{dt}\right|$



s increasing:
positive slope so
moving upward



s decreasing:
negative slope so
moving downward



Definition

Acceleration is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Jerk is the derivative of acceleration with the respect to time

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

When a ride in a car is jerky, it is not that the accelerations involved are necessarily large but that the changes in acceleration are abrupt.

Example

The free fall of a heavy ball bearing released from rest at time $t = 0$ *sec*.

a) How many meters does the ball fall in the first 2 *sec*?

b) What is its velocity, speed, and acceleration when $t = 2$?

Solution

a) The metric free-fall equation is $s = 4.9t^2$.

During the first 2 *sec*: $s(2) = 4.9(2)^2 \ \underline{= 19.6\ m}$

b) At any time, the velocity is:

$$\begin{aligned} v &= \frac{ds}{dt} \\ &= \frac{d}{dt}\left(4.9t^2\right) \\ &\underline{= 9.8t} \end{aligned}$$

At $t = 2$, velocity: $v = 9.8(2) \ \underline{= 19.6\ m/\sec}$

$Speed = |v| \ \underline{= 19.6\ m/\sec}$

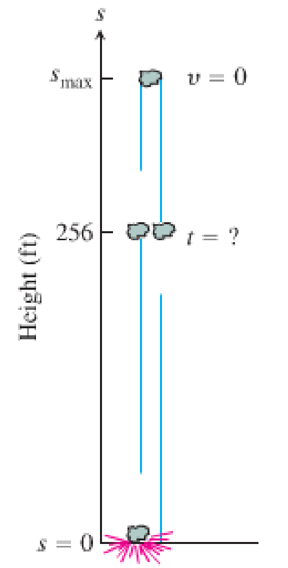
Acceleration:

$a(t) = v'(t) \ \underline{= 9.8\ m/\sec^2}$

Example

A dynamic blast blows a heavy rock straight up with a launch velocity of 160 *ft/sec* (about 109 *mph*).

It reaches a height of $s = 160t - 16t^2$ after *t sec*.



a) How high does the rock go?
b) What are the velocity and speed of the rock when it is 256 *ft* above the ground on the way up? On the way down?
c) What is the acceleration of the rock at any time *t* during its flight (after the blast)?
d) When does the rock hit the ground again?

Solution

a) At any time *t* during the rock's motion, its velocity is

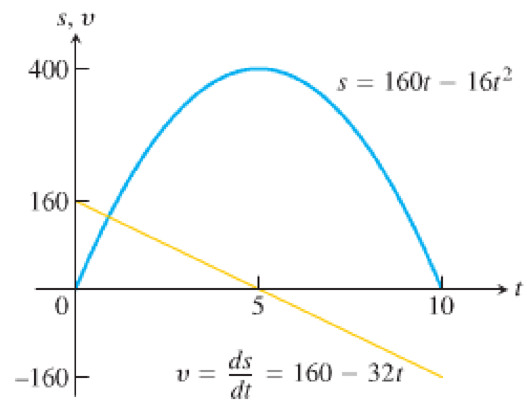
$$v = s' = 160 - 32t$$

The velocity is zero when it reaches maximum height:

$$v = 160 - 32t = 0$$

$$160 = 32t$$

$$t = \frac{160}{32} \boxed{= 5 \text{ sec}}$$



The rock's height at *t* = 5 sec is

$$s(t=5) = 160(5) - 16(5)^2 \boxed{= 400\ ft}$$

b) $s = 160t - 16t^2 = 256$

$$-16t^2 + 160t - 256 = 0 \quad \Rightarrow \quad t = 2\text{sec}, \quad t = 8\text{sec}$$

$$\begin{cases} t = 2\text{sec} & \rightarrow \quad v = 160 - 32(2) \boxed{= 96\ ft/\sec} \\ t = 8\text{sec} & \rightarrow \quad v = 160 - 32(8) \boxed{= -96\ ft/\sec} \end{cases}$$

The rock's speed is 96 *ft/sec*.

Since $v(t=2) > 0,$ the rock is moving upward and *s* is increasing.

$v(t=8) < 0,$ the rock is moving downward and *s* is decreasing.

c) Acceleration at any time is: $a = v' \boxed{= -32\ ft/\sec^2}$

d) $s = 160t - 16t^2 = 0$

$$t(160 - 16t) = 0 \quad \Rightarrow \quad t = 0,\ t = 10.$$

At *t* = 0, the blast occurred and the rock was thrown upward, it took 10 *sec* to return to ground.

Derivatives in Economics

Example

Suppose that it costs $C(x) = x^3 - 6x^2 + 15x$ dollars to produce x radiators when 8 to 30 radiators are produced and that $R(x) = x^3 - 3x^2 + 12x$ gives the dollar revenue from selling x radiators.
Your shop currently produces 10 radiators a day. About how much extra will it cost to produce one more radiator a day, and what is your estimated increase in revenue for selling 11 radiators a day?

Solution

The cost of producing one more radiator a day when 10 are produced is about $C'(10)$:

$$C'(x) = 3x^2 - 12x + 15$$

$$C'(x = 10) = 3(10)^2 - 12(10) + 15 = \underline{195}$$

The additional cost will be about \$195.00.

The marginal revenue is:

$$R'(x) = 3x^2 - 6x + 12$$

$$R'(x = 10) = 3(10)^2 - 6(10) + 12 = \underline{\$252.00}$$

If you increase sales to 11 radiators a day, the revenue is an additional of \$252.00.

Exercises Section 2.5 – Derivatives as Rates of Change

1. The position $s(t)=t^2-3t+2, \quad 0\le t\le 2$ of a body moving on a coordinate line, with s in meters and t in seconds.

a) Find the body's displacement and average velocity for the given time interval.
b) Find the body's speed and acceleration at the endpoints of the interval.
c) When, if ever, during the interval does the body change direction?

2. The position $s(t)=\frac{25}{t+5}, \quad -4\le t\le 0$ of a body moving on a coordinate line, with s in meters and t in seconds.

a) Find the body's displacement and average velocity for the given time interval.
b) Find the body's speed and acceleration at the endpoints of the interval.
c) When, if ever, during the interval does the body change direction?

3. At time t, the position of a body moving along the s-axis is $s=t^3-6t^2+9t$ ***m***.

a) Find the body's acceleration each time the velocity is zero.
b) Find the body's speed each time the acceleration is zero.
c) Find the total distance traveled by the body from $t = 0$ to $t = 2$.

4. A rock thrown vertically upward from the surface of the moon at a velocity of 24 *m/sec* (about 86 *km/h*) reaches a height of $s(t)=24t-0.8t^2$ *m* in *t sec*.

a) Find the rock's velocity and acceleration at time t. (The acceleration in this case is the acceleration of gravity on the moon.)
b) How long does it take the rock to reach its highest point?
c) How high does the rock go?
d) How long does it take the rock to reach half its maximum height?
e) How long is the rock aloft?

5. Had Galileo dropped a cannonball from the Tower of Pisa, 179 *feet* above the ground, the ball's height above the ground *t sec* into the fall would have been $s=179-16t^2$.

a) What would have been the ball's velocity, speed, and acceleration at time t?
b) About how long would it have taken the ball to hit the ground?
c) What would have been the ball's velocity at the moment of impact?

6. A toy rocket fired straight up into the air has height $s(t)=160t-16t^2$ *feet* after t seconds.

a) What is the rocket's initial velocity (when $t=0$)?
b) What is the acceleration when $t=3$?
c) At what time will the rocket hit the ground?
d) At what velocity will the rocket be traveling just as it smashes into the ground?

7. A helicopter is rising straight up in the air. Its distance from the ground t seconds after takeoff is $s(t) = t^2 + t$ *feet*

a) How long will it take for the helicopter to rise 20 *feet* ?

b) Find the velocity and the acceleration of the helicopter when it is 20 *feet* above the ground.

8. The position of a particle moving on a line is given by $s(t) = 2t^3 - 21t^2 + 60t, \quad t \geq 0$, where t is measured in *seconds* and s in *feet*.

a) What is the velocity after 3 *seconds* and after 6 *seconds*?

b) When the particle moving in the positive direction?

c) Find the total distance traveled by the particle during the first 7 *seconds*.

9. A small probe is launched vertically from the ground. After it reaches its high point, a parachute deploys and the probe descends to Earth. The height of the probe the ground is

$$s(t) = \frac{300t - 50t^2}{t^3 + 2} \quad \textit{for} \quad 0 \leq t \leq 6$$

a) Graph the height function and describe the motion of the probe.

b) Find the velocity of the probe.

c) Graph the velocity function and determine the approximate time at which the velocity is a maximum.

10. Suppose the cost of producing x lawn mowers is $C(x) = -0.02x^2 + 400x + 5000$

a) Determine the average and marginal costs for $x = 3000$ lawn mowers.

b) Interpret the meaning of your results in part (*a*)

11. Suppose a company produces fly rods. Assume $C(x) = -0.0001x^3 + 0.05x^2 + 60x + 800$ represents the cost of making x fly rods.

a) Determine the average and marginal costs for $x = 400$ fly rods.

b) Interpret the meaning of your results in part (*a*)

12. Suppose $p(t) = -1.7t^3 + 72t^2 + 7200t + 80,000$ is the population of a city t years after 1950.

a) Determine the average rate of growth of the city from 1950 to 2000.

b) What was the rate of growth of the city in 1990?