

Solution **Section 4.1 – System of linear Equations**

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$$

Solution

$$\begin{cases} 3x + 2y = -4 \\ 2 \times 2x - y = -5 \end{cases}$$

$$3x + 2y = -4$$

$$\frac{4x - 2y = -10}{7x = -14}$$

$$x = -2$$

$$y = 2x + 5$$

$$= -4 + 5$$

$$= 1$$

$$\text{Solution: } \underline{(-2, 1)}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$$

Solution

$$\begin{cases} -5 \times 2x + 5y = 7 \\ 2 \times 5x - 2y = -3 \end{cases}$$

$$-10x - 25y = -35$$

$$\frac{10x - 4y = -6}{-29y = -41}$$

$$y = \frac{41}{29}$$

$$x = \frac{1}{2} \left(7 - 5 \left(\frac{41}{29} \right) \right)$$

$$x = \frac{1}{2} \left(-\frac{2}{29} \right)$$

$$\underline{= -\frac{1}{29}}$$

$$\therefore \text{Solution: } \underline{\left(-\frac{1}{29}, \frac{41}{29}\right)}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$$

Solution

$$\begin{cases} 4x - 7y = -16 \\ -2 \times 2x + 5y = 9 \end{cases}$$

$$4x - 7y = -16$$

$$\begin{array}{r} -4x - 10y = -18 \\ \hline -17y = -34 \end{array}$$

$$\underline{y = 2}$$

$$x = \frac{9 - 5y}{2}$$

$$= \frac{9 - 10}{2}$$

$$\underline{= -\frac{1}{2}}$$

$$\therefore \text{Solution: } \underline{\left(-\frac{1}{2}, 2\right)}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$$

Solution

$$\begin{cases} 3x + 2y = 4 & (1) \\ 2x + y = 1 & (2) \end{cases}$$

$$(2) \rightarrow y = 1 - 2x \quad (3)$$

$$(1) \rightarrow 3x + 2 - 4x = 4$$

$$\underline{x = -2}$$

$$(3) \rightarrow y = 1 + 4$$

$$\underline{= 5}$$

$$\therefore \text{Solution: } \underline{(-2, 5)}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$$

Solution

$$\begin{cases} -2 \times & 3x + 4y = 2 \\ 3 \times & 2x + 5y = -1 \end{cases}$$

$$-6x - 8y = -4$$

$$\frac{6x + 15y = -3}{7y = -7}$$

$$\underline{y = -1}$$

$$2x = -1 + 5$$

$$x = \frac{4}{2}$$

$$\underline{= 2}$$

$$\therefore \text{Solution: } \underline{(2, -1)}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method) $\begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$

Solution

$$\begin{cases} 2 \times & 5x - 2y = 4 \\ & -10x + 4y = 7 \end{cases}$$

$$10x - 4y = 8$$

$$\frac{-10x + 4y = 7}{0 = 15} \quad (\text{impossible})$$

\therefore **No Solution**

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$$

Solution

$$\begin{cases} x - 4y = -8 & (1) \\ 5x - 20y = -40 & (2) \end{cases}$$

$$(1) \rightarrow x = 4y - 8$$

$$(2) \rightarrow 5(4y - 8) - 20y = -40$$

$$20y - 40 - 20y = -40$$

$$-40 = -40 \quad (\text{True})$$

$$\therefore \text{Solution: } \underline{x - 4y = -8}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$$

Solution

$$\begin{cases} 2x + y = 3 & (1) \\ x - y = 3 & (2) \end{cases}$$

$$(2) \rightarrow x = 3 + y \quad (3)$$

$$(1) \rightarrow 6 + 2y + y = 3$$

$$3y = -3$$

$$\underline{y = -1}$$

$$(3) \rightarrow \underline{x = 2}$$

$$\therefore \text{Solution: } \underline{(2, -1)}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$$

Solution

$$\begin{cases} 2x + 10y = -14 \\ 5 \times 7x - 2y = -16 \end{cases}$$

$$2x + 10y = -14$$

$$\begin{array}{r} 35x - 10y = -80 \\ \hline 37x = -94 \end{array}$$

$$x = -\frac{94}{37}$$

$$2y = 7\left(-\frac{94}{37}\right) + 16$$

$$y = -\frac{329}{37} + 8$$

$$= -\frac{33}{37}$$

$$\therefore \text{Solution: } \left(-\frac{94}{37}, -\frac{33}{37}\right)$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$$

Solution

$$\begin{cases} 3 \times 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$$

$$12x - 9y = 72$$

$$\begin{array}{r} -3x + 9y = -1 \\ \hline -9x = -71 \end{array}$$

$$x = \frac{71}{9}$$

$$3y = 4\left(\frac{71}{9}\right) - 24$$

$$y = \frac{284}{27} - 8$$

$$= \frac{68}{27} \Big|$$

$$\therefore \textbf{Solution:} \quad \underline{\left(\frac{71}{9}, \frac{68}{27} \right) \Big|}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$$

Solution

$$4x + 2y = 12$$

$$3x - 2y = 16$$

$$\underline{7x = 28}$$

$$\underline{x = 4 \Big|}$$

$$2y = 12 - 4(4)$$

$$y = -\frac{4}{2}$$

$$\underline{\underline{= -2 \Big|}}$$

$$\therefore \textbf{Solution:} \quad \underline{(4, -2) \Big|}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$$

Solution

$$x + 2y = -1$$

$$4x - 2y = 6$$

$$\underline{5x = 5}$$

$$\underline{x = 1 \Big|}$$

$$2y = -x - 1$$

$$y = -\frac{2}{2}$$

$$\underline{\underline{= -1 \Big|}}$$

$$\therefore \textbf{Solution:} \quad \underline{(1, -1) \Big|}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$$

Solution

$$x - 2y = 5$$

$$\begin{array}{r} -10x + 2y = 4 \\ \hline -9x = 9 \end{array}$$

$$\underline{x = -1}$$

$$2y = x - 5$$

$$y = -\frac{6}{2}$$

$$\underline{\underline{= -3}}$$

$$\therefore \text{Solution: } \underline{(-1, -3)}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases}$$

Solution

$$12x + 15y = -27$$

$$\begin{array}{r} 30x - 15y = -15 \\ \hline 42x = -42 \end{array}$$

$$\underline{x = -1}$$

$$15y = -27 - 12(-1)$$

$$y = -\frac{15}{15}$$

$$\underline{\underline{= -1}}$$

$$\therefore \text{Solution: } \underline{(-1, -1)}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$$

Solution

$$4x - 4y = -12$$

$$4x + 4y = -20$$

$$\hline 8x = -32$$

$$\underline{x = -4}$$

$$4y = 4(-4) + 12$$

$$y = -\frac{4}{4}$$

$$\underline{= -1}$$

$$\therefore \text{Solution: } \underline{(-4, -1)}$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 1 & 4 & 7 \\ 3 & 5 & 0 \end{array} \right] \quad R_2 - 3R_1$$

Solution

$$3 \quad 5 \quad 0$$

$$-3 \quad -12 \quad -21$$

$$\hline 0 \quad -7 \quad -21$$

$$\left[\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & -7 & -21 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array} \right] \quad R_2 - 2R_1$$

Solution

$$\begin{array}{rrr} 2 & 1 & -5 \\ -2 & 6 & -2 \\ \hline 0 & 7 & -7 \end{array}$$

$$\left[\begin{array}{rr|r} 1 & -3 & 1 \\ 0 & 7 & -7 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{rr|r} 1 & -3 & 3 \\ 5 & 2 & 19 \end{array} \right] \quad R_2 - 5R_1$$

Solution

$$\begin{array}{rrr} 5 & 2 & 19 \\ -5 & 15 & -15 \\ \hline 0 & 17 & -4 \end{array}$$

$$\left[\begin{array}{rr|r} 1 & -3 & 3 \\ 0 & 17 & -4 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{rr|r} 2 & -3 & 8 \\ -6 & 9 & 4 \end{array} \right] \quad R_2 + 3R_1$$

Solution

$$\begin{array}{rrr} -6 & 9 & 4 \\ 6 & -9 & 24 \\ \hline 0 & 0 & 28 \end{array}$$

$$\left[\begin{array}{rr|r} 2 & -3 & 8 \\ 0 & 0 & 28 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 2 & 3 & 11 \\ 1 & 2 & 8 \end{array} \right] \quad 2R_2 - R_1$$

Solution

$$\begin{array}{ccc} 2 & 4 & 16 \\ -2 & -3 & -11 \\ \hline 0 & 1 & 5 \end{array}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 11 \\ 0 & 1 & 5 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 3 & 5 & -13 \\ 2 & 3 & -9 \end{array} \right] \quad 3R_2 - 2R_1$$

Solution

$$\begin{array}{ccc} 6 & 9 & -27 \\ -6 & -10 & 26 \\ \hline 0 & -1 & -1 \end{array}$$

$$\left[\begin{array}{cc|c} 3 & 5 & -13 \\ 0 & -1 & -1 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 5 & 4 & 1 \end{array} \right] \quad R_3 - 5R_2$$

Solution

$$\begin{array}{cccc} 0 & 5 & 4 & 1 \\ 0 & -5 & 5 & -10 \\ \hline 0 & 0 & 9 & -9 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 9 & -9 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array} \right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

Solution

$$\begin{array}{cccc} 3 & 3 & -1 & 10 \\ -3 & 3 & -15 & 18 \\ \hline 0 & 6 & -16 & 28 \end{array} \quad \begin{array}{cccc} 1 & 3 & 2 & 5 \\ -1 & 1 & -5 & 6 \\ \hline 0 & 4 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 4 & -3 & 11 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 2 & 4 & 4 & 22 \\ -1 & -2 & 3 & 15 \end{array} \right] \quad \begin{array}{l} 3R_2 - 2R_1 \\ 3R_3 + R_1 \end{array}$$

Solution

$$\begin{array}{cccc} 6 & 12 & 12 & 66 \\ -6 & -4 & -2 & -2 \\ \hline 0 & 8 & 10 & 64 \end{array} \quad \begin{array}{cccc} -3 & -6 & 9 & 45 \\ 3 & 2 & 1 & 1 \\ \hline 0 & -4 & 10 & 46 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & 8 & 10 & 64 \\ 0 & -4 & 10 & 46 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & -4 & 2 & -7 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

Solution

$$\begin{array}{cccc} 2 & 1 & 1 & 3 \\ -2 & -2 & -2 & -4 \\ \hline 0 & -1 & -1 & -1 \end{array} \quad \begin{array}{cccc} 3 & -4 & 2 & -7 \\ -3 & -3 & -3 & -6 \\ \hline 0 & -7 & -1 & -13 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & -1 \\ 0 & -7 & -1 & -13 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ 2 & -3 & 5 & -1 & 0 \\ 1 & 0 & 3 & 1 & -4 \\ -4 & 3 & 2 & -1 & 3 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 + 4R_1 \end{array}$$

Solution

$$\begin{array}{ccccc} 2 & -3 & 5 & -1 & 0 \\ -2 & 4 & -2 & -6 & 4 \\ \hline 0 & 1 & 3 & -7 & 4 \end{array} \quad \begin{array}{ccccc} 1 & 0 & 3 & 1 & -4 \\ -1 & 2 & -1 & -3 & 2 \\ \hline 0 & 2 & 2 & -2 & -2 \end{array} \quad \begin{array}{ccccc} -4 & 3 & 2 & -1 & 3 \\ 4 & -8 & 4 & 12 & -8 \\ \hline 0 & -5 & 6 & 11 & -5 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ 0 & 1 & 3 & -7 & 4 \\ 0 & 2 & 2 & -2 & -2 \\ 0 & -5 & 6 & 11 & -5 \end{array} \right]$$

Exercise

Use the Gauss-Jordan method to solve the system

$$x - y + 5z = -6$$

$$3x + 3y - z = 10$$

$$x + 3y + 2z = 5$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array} \right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{ccc|c} 3 & 3 & -1 & 10 \\ -3 & 3 & -15 & 18 \\ \hline 0 & 6 & -16 & 28 \end{array} \quad \begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ -1 & 1 & -5 & 6 \\ \hline 0 & 4 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 4 & -3 & 11 \end{array} \right] \frac{1}{6}R_2$$

$$0 \quad 1 \quad -\frac{8}{3} \quad \frac{14}{3}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 4 & -3 & 11 \end{array} \right] \begin{array}{l} R_1 + R_2 \\ \\ R_3 - 4R_2 \end{array}$$

$$\begin{array}{ccc|c} 0 & 4 & -3 & 11 \\ 0 & -4 & \frac{32}{3} & -\frac{56}{3} \\ \hline 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array} \quad \begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ \hline 1 & 0 & \frac{7}{3} & -\frac{4}{3} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3} \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array} \right] \frac{3}{23}R_3$$

$$0 \quad 0 \quad 1 \quad -1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3} \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & 1 & -1 \end{array} \right] \begin{array}{l} R_1 - \frac{7}{3}R_3 \\ R_2 + \frac{8}{3}R_3 \\ \end{array}$$

$$\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3} \\ 0 & 0 & -\frac{7}{3} & \frac{7}{3} \\ \hline 1 & 0 & 0 & 1 \end{array} \quad \begin{array}{ccc|c} 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & \frac{8}{3} & -\frac{8}{3} \\ \hline 0 & 1 & 0 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Solution: $\underline{(1, 2, -1)}$

Exercise

Use the Gauss-Jordan method to solve the system

$$\begin{cases} 2x - y + 4z = -3 \\ x - 2y - 10z = -6 \\ 3x \quad \quad + 4z = 7 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & -3 \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array} \right] \frac{1}{2}R_1$$

$$\begin{array}{cccc} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array} \right] \begin{array}{l} R_2 - R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc|cccc} 1 & -2 & -10 & -6 & 3 & 0 & 4 & 7 \\ -1 & \frac{1}{2} & -2 & \frac{3}{2} & -3 & \frac{3}{2} & -6 & \frac{9}{2} \\ \hline 0 & -\frac{3}{2} & -12 & -\frac{9}{2} & 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array} \right] -\frac{2}{3}R_2$$

$$\begin{array}{cccc} 0 & 1 & 8 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & 1 & 8 & 3 \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array} \right] \begin{array}{l} R_1 + \frac{1}{2}R_2 \\ R_3 - \frac{3}{2}R_2 \end{array}$$

$$\begin{array}{cccc|cccc} 0 & \frac{3}{2} & -2 & \frac{23}{2} & 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} & 0 & \frac{1}{2} & 4 & \frac{3}{2} \\ \hline 0 & 0 & -14 & 7 & 1 & 0 & 6 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & -14 & 7 \end{array} \right] -\frac{1}{14}R_3$$

$$\begin{array}{cccc} 0 & 0 & 1 & -\frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array} \right] \begin{array}{l} R_1 - 6R_3 \\ R_2 - 8R_3 \end{array}$$

$$\begin{array}{cccc|cccc} 1 & 0 & 6 & 0 & 0 & 1 & 8 & 3 \\ 0 & 0 & -6 & 3 & 0 & 0 & -8 & 4 \\ \hline 1 & 0 & 0 & 3 & 0 & 1 & 0 & 7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array} \right]$$

Solution: $\left(3, 7, -\frac{1}{2} \right)$

Exercise

Use the Gauss-Jordan method to solve the system
$$\begin{cases} 4x + 3y - 5z = -29 \\ 3x - 7y - z = -19 \\ 2x + 5y + 2z = -10 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 4 & 3 & -5 & -29 \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array} \right] \frac{1}{4}R_1$$

$$1 \quad \frac{3}{4} \quad -\frac{5}{4} \quad -\frac{29}{4}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array} \right] \begin{array}{l} R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\begin{array}{ccc|c} 3 & -7 & -1 & -19 \\ -3 & -\frac{9}{4} & \frac{15}{4} & \frac{87}{4} \\ \hline 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \end{array}$$

$$\begin{array}{ccc|c} 2 & 5 & 2 & -10 \\ -2 & -\frac{3}{2} & \frac{5}{2} & \frac{29}{2} \\ \hline 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array} \right] -\frac{4}{37}R_2$$

$$0 \quad 1 \quad -\frac{11}{37} \quad -\frac{11}{37}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array} \right] \begin{array}{l} R_1 - \frac{3}{4}R_2 \\ R_3 - \frac{7}{2}R_2 \end{array}$$

$$\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{3}{4} & \frac{33}{148} & \frac{33}{148} \\ \hline 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \end{array}$$

$$\begin{array}{ccc|c} 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \\ 0 & -\frac{7}{2} & \frac{77}{72} & \frac{77}{72} \\ \hline 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array} \right] \frac{72}{401}R_3$$

$$0 \quad 0 \quad 1 \quad 1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & 1 & 1 \end{array} \right] \begin{array}{l} R_1 + \frac{38}{37}R_3 \\ R_2 + \frac{11}{37}R_3 \end{array}$$

$$\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 0 & \frac{38}{37} & \frac{38}{37} \\ \hline 1 & 0 & 0 & -6 \end{array}$$

$$\begin{array}{ccc|c} 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{11}{37} & \frac{11}{37} \\ \hline 0 & 1 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

Solution: $\underline{(-6, 0, 1)}$

Exercise

Use the Gauss-Jordan method to solve the system
$$\begin{cases} x + 2y - 3z = -15 \\ 2x - 3y + 4z = 18 \\ -3x + y + z = 1 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + 3R_1 \end{array}$$

$$\begin{array}{cccc} -2 & -4 & 6 & 30 \\ 2 & -3 & 4 & 18 \\ \hline 0 & -7 & 10 & 48 \end{array} \quad \begin{array}{cccc} 3 & 6 & -9 & -45 \\ -3 & 1 & 1 & 1 \\ \hline 0 & 7 & -8 & -44 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \end{array} \right] -\frac{1}{7}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 7 & -8 & -44 \end{array} \right] \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 - 7R_2 \end{array}$$

$$\begin{array}{cccc} 1 & 2 & -3 & -15 \\ 0 & -2 & \frac{20}{7} & \frac{96}{7} \\ \hline 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \end{array} \quad \begin{array}{cccc} 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \\ \hline 0 & 0 & 2 & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 2 & 4 \end{array} \right] \frac{1}{2}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 1 & 2 \end{array} \right] \begin{array}{l} R_1 + \frac{1}{7}R_3 \\ R_2 + \frac{10}{7}R_3 \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 0 & \frac{1}{7} & \frac{2}{7} \\ \hline 1 & 0 & 0 & -1 \end{array} \quad \begin{array}{cccc} 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & \frac{10}{7} & \frac{20}{7} \\ \hline 0 & 1 & 0 & -4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

Solution: $(-1, -4, 2)$

Exercise

Use the Gauss-Jordan method to solve the system
$$\begin{cases} x + 2y + 3z = 10 \\ 4x + 5y + 6z = 11 \\ 7x + 8y + 9z = 12 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 11 \\ 7 & 8 & 9 & 12 \end{array} \right] \begin{array}{l} \\ R_2 - 4R_1 \\ R_3 - 7R_1 \end{array} \quad \begin{array}{cccc} -4 & -8 & -12 & -40 \\ 4 & 5 & 6 & 11 \\ \hline 0 & -3 & -6 & -29 \end{array} \quad \begin{array}{cccc} -7 & -14 & -21 & -70 \\ 7 & 8 & 9 & 12 \\ \hline 0 & -6 & -12 & -58 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -3 & -6 & -29 \\ 0 & -6 & -12 & -58 \end{array} \right] \frac{1}{3}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & \frac{29}{3} \\ 0 & -6 & -12 & -58 \end{array} \right] R_3 + 6R_2 \quad \begin{array}{cccc} 0 & -6 & -12 & -58 \\ 0 & 6 & 12 & 58 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & \frac{29}{3} \\ 0 & 0 & 0 & 0 \end{array} \right]$$

let z be the variable

$$\text{From Row 1} \Rightarrow y + 2z = \frac{29}{3}$$

$$\underline{y = \frac{29}{3} - 2z}$$

$$\text{From Row 1} \Rightarrow x + 2y + 3z = 10$$

$$x = 10 - 2y - 3z$$

$$x = 10 - 2\left(\frac{29}{3} - 2z\right) - 3z$$

$$x = 10 - \frac{58}{3} + 4z - 3z$$

$$\underline{x = z - \frac{28}{3}}$$

$$\text{Solution: } \underline{\left(z - \frac{28}{3}, \frac{29}{3} - 2z, z \right)}$$

Exercise

Use the Gauss-Jordan method to solve the system
$$\begin{cases} 2x + y + 2z = 4 \\ 2x + 2y = 5 \\ 2x - y + 6z = 2 \end{cases}$$

Solution

$$\begin{aligned} & \left[\begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array} \right] \begin{array}{l} \frac{1}{2}R_1 \\ \\ \end{array} \quad \begin{array}{cccc} & & 1 & \frac{1}{2} \\ & & & 1 \\ & & & 2 \end{array} \\ & \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array} \quad \begin{array}{cccc} 2 & 2 & 0 & 5 \\ -2 & -1 & -2 & -4 \\ 0 & 1 & -2 & 1 \end{array} \quad \begin{array}{cccc} 2 & -1 & 6 & 2 \\ -2 & -1 & -2 & -4 \\ 0 & -2 & 4 & -2 \end{array} \\ & \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & -2 \end{array} \right] \begin{array}{l} \\ R_1 - \frac{1}{2}R_2 \\ R_3 + 2R_2 \end{array} \quad \begin{array}{cccc} 1 & \frac{1}{2} & 1 & 2 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 1 & 0 & 2 & \frac{3}{2} \end{array} \quad \begin{array}{cccc} 0 & -2 & 4 & -2 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array} \\ & \left[\begin{array}{ccc|c} 1 & 0 & 2 & \frac{3}{2} \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

From Row 3: $0 = 0$ is a true statement. Let z be the variable.

From Row 2: $y - 2z = 1$

$$\underline{y = 1 + 2z}$$

From Row 1: $x + 2z = \frac{3}{2}$

$$\underline{x = -2z + \frac{3}{2}}$$

\therefore **Solution:** $\underline{\left(-2z + \frac{3}{2}, 2z + 1, z \right)}$

Exercise

Use the Gauss-Jordan method to solve the system

$$\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array} \right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} -1 & -2 & 3 & 1 \\ 1 & 1 & 2 & 8 \\ \hline 0 & -1 & 5 & 9 \end{array}$$

$$\begin{array}{cccc} 3 & -7 & 4 & 10 \\ -3 & -3 & -6 & -24 \\ \hline 0 & -10 & -2 & -14 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array} \right] -R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array} \right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 10R_2 \end{array}$$

$$\begin{array}{cccc} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ \hline 1 & 0 & 7 & 17 \end{array}$$

$$\begin{array}{cccc} 0 & -10 & -2 & -14 \\ 0 & 10 & -50 & -90 \\ \hline 0 & 0 & -52 & -104 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array} \right] -\frac{1}{52}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array} \right] \begin{array}{l} R_1 - 7R_3 \\ R_2 + 5R_3 \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & 7 & 17 \\ 0 & 0 & -7 & -14 \\ \hline 1 & 0 & 0 & 3 \end{array}$$

$$\begin{array}{cccc} 0 & 1 & -5 & -9 \\ 0 & 0 & 5 & 10 \\ \hline 0 & 1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

\therefore **Solution:** (3, 1, 2)

Exercise

Use augmented elimination to solve linear system

$$2x - 5y + 3z = 1$$

$$x - 2y - 2z = 8$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 2 & -5 & 3 & 1 \end{array} \right] \quad R_2 - 2R_1$$

$$\begin{array}{cccc} 2 & -5 & 3 & 1 \\ -2 & 4 & 4 & -16 \\ \hline 0 & -1 & 7 & -15 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & -1 & 7 & -15 \end{array} \right] \quad -R_2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & 1 & -7 & 15 \end{array} \right] \quad R_1 + 2R_2$$

$$\begin{array}{cccc} 1 & -2 & -2 & 8 \\ 0 & 2 & -14 & 30 \\ \hline 1 & 0 & -16 & 38 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -16 & 38 \\ 0 & 1 & -7 & 15 \end{array} \right] \quad \begin{array}{l} \rightarrow x - 16z = 38 \\ \rightarrow y - 7z = 15 \end{array}$$

$$\begin{cases} x = 16z + 38 \\ y = 7z + 15 \end{cases}$$

$$\therefore \text{Solution: } \underline{(16z + 38, 7z + 15, z)}$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 5 \\ 1 & -1 & 1 & -2 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccc} 2 & 1 & -1 & 5 \\ -2 & -2 & -2 & -4 \\ \hline 0 & -1 & -3 & 1 \end{array}$$

$$\begin{array}{cccc} 1 & -1 & 1 & -2 \\ -1 & -1 & -1 & -2 \\ \hline 0 & -2 & 0 & -4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & 1 \\ 0 & -2 & 0 & -4 \end{array} \right] \quad \begin{array}{l} (2) \\ (1) \\ -2y = -4 \end{array}$$

$$\underline{y = 2}$$

$$(1) \rightarrow -y - 3z = 1$$

$$3z = -1 - 2$$

$$\underline{z = -1}$$

$$(2) \rightarrow x + y + z = 2$$

$$x = 2 - 2 + 1$$

$$\underline{\underline{=1}}$$

$$\therefore \text{Solution: } \underline{(1, 2, -1)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ -1 & -1 & 1 & 1 \\ 3 & -1 & 1 & 9 \end{array} \right] \begin{array}{l} \\ 2R_2 + R_1 \\ 2R_3 - 3R_1 \end{array} \quad \begin{array}{cccc} -2 & -2 & 2 & 2 \\ 2 & 1 & 1 & 9 \\ \hline 0 & -1 & 3 & 11 \end{array} \quad \begin{array}{cccc} 6 & -2 & 2 & 18 \\ -6 & -3 & -3 & -27 \\ \hline 0 & -5 & -1 & -9 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 0 & -1 & 3 & 11 \\ 0 & -5 & -1 & -9 \end{array} \right] R_3 - 5R_2 \quad \begin{array}{cccc} 0 & -5 & -1 & -9 \\ 0 & 5 & -15 & -55 \\ \hline 0 & 0 & -16 & -64 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 0 & -1 & 3 & 11 \\ 0 & 0 & -16 & -64 \end{array} \right] \begin{array}{l} (2) \\ (1) \\ -16z = -64 \end{array}$$

$$\underline{z = 4}$$

$$(1) \rightarrow -y + 3z = 11$$

$$y = 12 - 11$$

$$\underline{\underline{=1}}$$

$$(2) \rightarrow 2x + y + z = 9$$

$$2x = 9 - 1 - 4$$

$$\underline{\underline{x = 2}}$$

$$\therefore \text{Solution: } \underline{(2, 1, 4)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ -3 & 6 & 2 & 11 \end{array} \right] R_3 + 3R_1 \quad \begin{array}{cccc} -3 & 6 & 2 & 11 \\ 3 & 15 & -3 & -12 \\ \hline 0 & 21 & -1 & -1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ 0 & 21 & -1 & -1 \end{array} \right] R_3 - 7R_2 \quad \begin{array}{cccc} 0 & 21 & -1 & -1 \\ 0 & -21 & 7 & 7 \\ \hline 0 & 0 & 6 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ 0 & 0 & 6 & 6 \end{array} \right] \rightarrow \begin{array}{l} x + 5y - z = -4 \quad (2) \\ 3y - z = -1 \quad (1) \\ 6z = 6 \end{array}$$

$$\underline{z = 1}$$

$$(1) \rightarrow 3y = -1 + 1$$

$$\underline{y = 0}$$

$$(2) \rightarrow x = -4 + 1$$

$$\underline{x = -3}$$

$$\therefore \text{Solution: } \underline{(-3, 0, 1)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 2 & -3 & 2 & 10 \\ 3 & -1 & 1 & 9 \end{array} \right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \quad \begin{array}{cccc} 2 & -3 & 2 & 10 \\ -2 & -6 & -8 & -28 \\ \hline 0 & -9 & -6 & -18 \end{array} \quad \begin{array}{cccc} 3 & -1 & 1 & 9 \\ -3 & -9 & -12 & -42 \\ \hline 0 & -10 & -11 & -33 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -9 & -6 & -18 \\ 0 & -10 & -11 & -33 \end{array} \right] 9R_3 - 10R_2 \quad \begin{array}{cccc} 0 & -90 & -99 & -297 \\ 0 & 90 & 60 & 180 \\ \hline 0 & 0 & -39 & -117 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -9 & -6 & -18 \\ 0 & 0 & -39 & -117 \end{array} \right] \begin{array}{l} x + 3y + 4z = 14 \quad (3) \\ -9y - 6z = -18 \quad (2) \\ -39z = -117 \quad (1) \end{array}$$

$$(1) \rightarrow z = \frac{117}{39} \\ \underline{= 3}$$

$$(2) \rightarrow 9y = 18 - 6(3)$$

$$9y = 0$$

$$\underline{y = 0}$$

$$(3) \rightarrow x = 14 - 12$$

$$\underline{x = 2}$$

$$\therefore \text{Solution: } \underline{(2, 0, 3)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 3 & 2 & 1 & 8 \\ 2 & -3 & 2 & -16 \end{array} \right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \quad \begin{array}{cccc} 3 & 2 & 1 & 8 \\ -3 & -12 & 3 & -60 \\ 0 & -10 & 4 & -52 \end{array} \quad \begin{array}{cccc} 2 & -3 & 2 & -16 \\ -2 & -8 & 2 & -40 \\ 0 & -11 & 4 & -56 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 0 & -10 & 4 & -52 \\ 0 & -11 & 4 & -56 \end{array} \right] \begin{array}{l} \\ \\ 10R_3 - 11R_2 \end{array} \quad \begin{array}{cccc} 0 & -110 & 40 & -560 \\ 0 & 110 & -44 & 572 \\ 0 & 0 & -4 & 12 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 0 & -10 & 4 & -52 \\ 0 & 0 & -4 & 12 \end{array} \right] \begin{array}{l} x + 4y - z = 20 \quad (3) \\ -10y + 4z = -52 \quad (2) \\ -4z = 12 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -3}$$

$$(2) \rightarrow -10y = -52 + 12$$

$$-10y = -40$$

$$\underline{y = 4}$$

$$(3) \rightarrow x = 20 - 16 - 3$$

$$\underline{x = 1}$$

$$\therefore \text{Solution: } \underline{(1, 4, -3)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 2 & -1 & 7 \\ 2 & -3 & 2 & -1 \end{array} \right] R_3 - 2R_1 \quad \begin{array}{cccc} 2 & -3 & 2 & -1 \\ -2 & -4 & -2 & -34 \\ \hline 0 & -7 & 0 & -35 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 2 & -1 & 7 \\ 0 & -7 & 0 & -35 \end{array} \right] \begin{array}{l} x + 2y + z = 17 \quad (3) \\ 2y - z = 7 \quad (2) \\ -7y = -35 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{y = 5}$$

$$(2) \rightarrow \underline{z = 10 - 7} \\ \underline{= 3}$$

$$(3) \rightarrow \underline{x = 17 - 10 - 3} \\ \underline{= 4}$$

$$\therefore \text{Solution: } \underline{(4, 5, 3)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ -4 & 5 & 3 & 7 \\ -6 & 3 & 5 & -4 \end{array} \right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \quad \begin{array}{cccc} -4 & 5 & 3 & 7 \\ 4 & -12 & -14 & -6 \\ \hline 0 & -7 & -11 & 1 \end{array} \quad \begin{array}{cccc} -6 & 3 & 5 & -4 \\ 6 & -18 & -21 & -9 \\ \hline 0 & -15 & -16 & -13 \end{array}$$

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ 0 & -7 & -11 & 1 \\ 0 & -15 & -16 & -13 \end{array} \right] 7R_3 - 15R_1 \quad \begin{array}{cccc} 0 & -105 & -112 & -91 \\ 0 & 105 & 165 & -15 \\ \hline 0 & 0 & 53 & -106 \end{array}$$

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ 0 & -7 & -11 & 1 \\ 0 & 0 & 53 & -106 \end{array} \right] \begin{array}{l} -2x + 6y + 7z = 3 \quad (3) \\ -7y - 11z = 1 \quad (2) \\ 53z = -106 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -2}$$

$$(2) \rightarrow -7y = 1 - 22$$

$$-7y = -21$$

$$\underline{y = 3}$$

$$(3) \rightarrow -2x = 3 - 18 + 14$$

$$-2x = -1$$

$$\underline{x = \frac{1}{2}}$$

$$\therefore \text{Solution: } \underline{\left(\frac{1}{2}, 3, -2\right)}$$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 3 & -3 & 4 & 5 \\ 4 & -2 & 3 & 4 \end{array} \right] \quad \begin{array}{l} 2R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \quad \begin{array}{cccc} 6 & -6 & 8 & 10 \\ -6 & 3 & -3 & -3 \\ \hline 0 & -3 & 5 & 7 \end{array} \quad \begin{array}{cccc} 4 & -2 & 3 & 4 \\ -4 & 2 & -2 & -2 \\ \hline 0 & 0 & 1 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 0 & -3 & 5 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right] \quad \begin{array}{l} 2x - y + z = 1 \quad (2) \\ -3y + 5z = 7 \quad (1) \\ \underline{z = 2} \end{array}$$

$$(1) \rightarrow -3y = 7 - 10$$

$$-3y = -3$$

$$\underline{y = 1}$$

$$(3) \rightarrow 2x = 1 + 1 - 2$$

$$\underline{x = 0}$$

$$\therefore \text{Solution: } \underline{(0, 1, 2)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 3x - 4y + 4z = 7 \\ x - y - 2z = 2 \\ 2x - 3y + 6z = 5 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 3 & -4 & 4 & 7 \\ 2 & -3 & 6 & 5 \end{array} \right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \quad \begin{array}{cccc} 3 & -4 & 4 & 7 \\ -3 & 3 & 6 & -6 \\ \hline 0 & -1 & 10 & 1 \end{array} \quad \begin{array}{cccc} 2 & -3 & 6 & 5 \\ -2 & 2 & 4 & -4 \\ \hline 0 & -1 & 10 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & -1 & 10 & 1 \\ 0 & -1 & 10 & 1 \end{array} \right] \begin{array}{l} \\ \\ R_3 = R_2 \end{array} \quad \begin{array}{l} \rightarrow x - y - 2z = 2 \quad (2) \\ \rightarrow -y + 10z = 1 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{y = 10z - 1}$$

$$(2) \rightarrow \underline{x = 2 + 10z - 1 + 2z} \\ \underline{= 12z + 1}$$

$$\therefore \text{Solution: } \underline{(12z + 1, 10z - 1, z)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 2 & -1 & 1 & 4 \\ -1 & 1 & 1 & 4 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array} \quad \begin{array}{cccc} 2 & -1 & 1 & 4 \\ -2 & 4 & 2 & -4 \\ \hline 0 & 3 & 3 & 0 \end{array} \quad \begin{array}{cccc} -1 & 1 & 1 & 4 \\ 1 & -2 & -1 & 2 \\ \hline 0 & -1 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & 0 & 6 \end{array} \right] \begin{array}{l} x - 2y - z = 2 \quad (3) \\ 3y + 3z = 0 \quad (2) \\ -y = 6 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{y = -6}$$

$$(2) \rightarrow \underline{z = -y} \\ \underline{= 6}$$

$$(3) \rightarrow \underline{x = 2 - 12 + 6} \\ \underline{= -4}$$

$$\therefore \text{Solution: } \underline{(-4, -6, 6)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x + y + z = 3 \\ -y + 2z = 1 \\ -x + z = 0 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ -1 & 0 & 1 & 0 \end{array} \right] R_3 + R_1 \quad \begin{array}{cccc} -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 3 \\ \hline 0 & 1 & 2 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{array} \right] R_3 + R_2 \quad \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 0 & -1 & 2 & 1 \\ \hline 0 & 0 & 4 & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 4 & 4 \end{array} \right] \begin{array}{ll} x + y + z = 3 & (3) \\ -y + 2z = 1 & (2) \\ 4z = 4 & (1) \end{array}$$

$$(1) \rightarrow \underline{z=1}$$

$$(2) \rightarrow -y = 1 - 2$$

$$\underline{y=1}$$

$$(3) \rightarrow x = 3 - 1 - 1$$

$$\underline{\underline{=1}}$$

$$\therefore \text{Solution: } \underline{(1, 1, 1)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 3x + y + 3z = 14 \\ 7x + 5y + 8z = 37 \\ x + 3y + 2z = 9 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 3 & 1 & 3 & 14 \\ 7 & 5 & 8 & 37 \end{array} \right] \begin{array}{l} R_2 - 3R_1 \\ R_3 - 7R_1 \end{array} \quad \begin{array}{cccc} 3 & 1 & 3 & 14 \\ -3 & -9 & -6 & -27 \\ \hline 0 & -8 & -3 & -13 \end{array} \quad \begin{array}{cccc} 7 & 5 & 8 & 37 \\ -7 & -21 & -14 & -63 \\ \hline 0 & -16 & -6 & -26 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & -8 & -3 & -13 \\ 0 & -16 & -6 & -26 \end{array} \right] R_3 - 2R_2 \quad \begin{array}{cccc} 0 & -16 & -6 & -26 \\ 0 & 16 & 6 & 26 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & -8 & -3 & -13 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} x+3y+2z=9 \quad (2) \\ -8y-3z=-13 \quad (1) \end{array}$$

$$(1) \rightarrow -8y = 3z - 13$$

$$\underline{y = -\frac{3}{8}z + \frac{13}{8}}$$

$$(3) \rightarrow x = 9 - 3\left(\frac{13}{8} - \frac{3}{8}z\right) - 2z$$

$$= 9 - \frac{39}{8} + \frac{9}{8}z - 2z$$

$$\underline{= \frac{33}{8} - \frac{7}{8}z}$$

$$\therefore \text{Solution: } \underline{\left(\frac{33}{8} - \frac{7}{8}z, \frac{13}{8} - \frac{3}{8}z, z \right)}$$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 4x - 2y + z = 7 \\ x + y + z = -2 \\ 4x + 2y + z = 3 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 4 & -2 & 1 & 7 \\ 4 & 2 & 1 & 3 \end{array} \right] \quad \begin{array}{l} R_2 - 4R_1 \\ R_3 - 4R_1 \end{array} \quad \begin{array}{cccc} 4 & -2 & 1 & 7 \\ -4 & -4 & -4 & 8 \\ 0 & -6 & -3 & 15 \end{array} \quad \begin{array}{cccc} 4 & 2 & 1 & 3 \\ -4 & -4 & -4 & 8 \\ 0 & -2 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 15 \\ 0 & -2 & -3 & 11 \end{array} \right] \quad -3R_3 + R_2 \quad \begin{array}{cccc} 0 & 6 & 9 & -33 \\ 0 & -6 & -3 & 15 \\ 0 & 0 & 6 & -18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 15 \\ 0 & 0 & 6 & -18 \end{array} \right] \quad \begin{array}{l} x+y+z=-2 \quad (3) \\ -6y-3z=15 \quad (2) \\ 6z=-18 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -3}$$

$$(2) \rightarrow -6y = 15 - 9$$

$$\underline{y = -1}$$

$$(3) \rightarrow x = -2 + 1 + 3$$

$$\underline{= 2}$$

$$\therefore \text{Solution: } \underline{(2, -1, -3)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 2x - 2y + z = -4 \\ 6x + 4y - 3z = -24 \\ x - 2y + 2z = 1 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 2 & -2 & 1 & -4 \\ 6 & 4 & -3 & -24 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 6R_1 \end{array} \quad \begin{array}{ccc|c} 2 & -2 & 1 & -4 \\ -2 & 4 & -4 & -2 \\ \hline 0 & 2 & -3 & -6 \end{array} \quad \begin{array}{ccc|c} 6 & 4 & -3 & -24 \\ -6 & 12 & -12 & -6 \\ \hline 0 & 16 & -15 & -30 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 16 & -15 & -30 \end{array} \right] \begin{array}{l} \\ \\ R_3 - 8R_2 \end{array} \quad \begin{array}{ccc|c} 0 & 16 & -15 & -30 \\ 0 & -16 & 24 & 48 \\ \hline 0 & 0 & 9 & 18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 0 & 9 & 18 \end{array} \right] \begin{array}{l} x - 2y + 2z = 1 \quad (3) \\ 2y - 3z = -6 \quad (2) \\ 9z = 18 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = 2}$$

$$(2) \rightarrow 2y = -6 + 6$$

$$\underline{y = 0}$$

$$(3) \rightarrow x = 1 - 4$$

$$\underline{\underline{-3}}$$

$$\therefore \text{Solution: } \underline{(-3, 0, 2)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 9x + 3y + z = 4 \\ 16x + 4y + z = 2 \\ 25x + 5y + z = 2 \end{cases}$$

Solution

$$\begin{cases} z + 9x + 3y = 4 \\ z + 16x + 4y = 2 \\ z + 25x + 5y = 2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 1 & 16 & 4 & 2 \\ 1 & 25 & 5 & 2 \end{array} \right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \quad \begin{array}{ccc|c} 1 & 16 & 4 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 7 & 1 & -2 \end{array} \quad \begin{array}{ccc|c} 1 & 25 & 5 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 16 & 2 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 16 & 2 & -2 \end{array} \right] \quad 7R_3 - 16R_2$$

$$\begin{array}{cccc} 0 & 112 & 14 & -14 \\ 0 & -112 & -16 & 32 \\ \hline 0 & 0 & -2 & 18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 0 & -2 & 18 \end{array} \right] \quad \begin{array}{l} z + 9x + 3y = 4 \quad (3) \\ 7x + y = -2 \quad (2) \\ -2y = 18 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{y = -9}$$

$$(2) \rightarrow 7x = -2 + 9$$

$$\underline{= 1}$$

$$(3) \rightarrow z = 4 - 9 + 27$$

$$\underline{= 22}$$

$$\therefore \text{Solution: } \underline{(1, -9, 22)}$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 2 & -8 \\ 3 & -1 & -4 & 3 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 2 & -8 \\ -2 & -4 & 6 & -18 \\ \hline 0 & -5 & 8 & -26 \end{array} \quad \begin{array}{cccc} 3 & -1 & -4 & 3 \\ -3 & -6 & 9 & -27 \\ \hline 0 & -7 & 5 & -24 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & -7 & 5 & -24 \end{array} \right] \quad 5R_3 - 7R_2$$

$$\begin{array}{cccc} 0 & -35 & 25 & -120 \\ 0 & 35 & -56 & 182 \\ \hline 0 & 0 & -31 & 62 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & 0 & -31 & 62 \end{array} \right] \quad \begin{array}{l} x + 2y - 3z = 9 \quad (3) \\ -5y + 8z = -26 \quad (2) \\ -31z = 62 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -2}$$

$$(2) \rightarrow -5y = -26 + 16$$

$$-5y = 10$$

$$\underline{y = 2}$$

$$(3) \rightarrow x = 9 - 4 - 6$$

$$\underline{\underline{= -1}}$$

$$\therefore \text{Solution: } \underline{\underline{(-1, 2, -2)}}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x - 3z = -5 \\ 2x - y + 2z = 16 \\ 7x - 3y - 5z = 19 \end{cases}$$

Solution

$$\begin{array}{ccc|ccc} \begin{bmatrix} 1 & 0 & -3 & -5 \\ 2 & -1 & 2 & 16 \\ 7 & -3 & -5 & 19 \end{bmatrix} & \begin{matrix} R_2 - 2R_1 \\ R_3 - 7R_1 \end{matrix} & \begin{array}{ccc|ccc} 2 & -1 & 2 & 16 \\ -2 & 0 & 6 & 10 \\ \hline 0 & -1 & 8 & 26 \end{array} & \begin{array}{ccc|ccc} 7 & -3 & -5 & 19 \\ -7 & 0 & 21 & 35 \\ \hline 0 & -3 & 16 & 54 \end{array} \end{array}$$

$$\begin{array}{ccc|ccc} \begin{bmatrix} 1 & 0 & -3 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & -3 & 16 & 54 \end{bmatrix} & R_3 - 3R_2 & \begin{array}{ccc|ccc} 0 & -3 & 16 & 54 \\ 0 & 3 & -24 & -78 \\ \hline 0 & 0 & -8 & -24 \end{array} \end{array}$$

$$\begin{array}{ccc|ccc} \begin{bmatrix} 1 & 0 & -3 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & 0 & -8 & -24 \end{bmatrix} & \begin{matrix} x - 3z = -5 & (3) \\ -y + 8z = 26 & (2) \\ -8z = -24 & (1) \end{matrix} \end{array}$$

$$(1) \rightarrow \underline{\underline{z = 3}}$$

$$(2) \rightarrow -y = 26 - 24$$

$$\underline{\underline{y = -2}}$$

$$(3) \rightarrow x = -5 + 9$$

$$\underline{\underline{= 4}}$$

$$\therefore \text{Solution: } \underline{\underline{(4, -2, 3)}}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x + 2y - z = 5 \\ 2x - y + 3z = 0 \\ 2y + z = 1 \end{cases}$$

Solution

$$\begin{array}{ccc|ccc} \begin{bmatrix} 1 & 2 & -1 & 5 \\ 2 & -1 & 3 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix} & R_2 - 2R_1 & \begin{array}{ccc|ccc} 2 & -1 & 3 & 0 \\ -2 & -4 & 2 & -10 \\ \hline 0 & -5 & 5 & -10 \end{array} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 2 & 1 & 1 \end{array} \right] \quad 5R_3 + 2R_2 \quad \begin{array}{cccc} 0 & 10 & 5 & 5 \\ 0 & -10 & 10 & -20 \\ \hline 0 & 0 & 15 & -15 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 0 & 15 & -15 \end{array} \right] \quad \begin{array}{l} x + 2y - z = 5 \quad (3) \\ -5y + 5z = -10 \quad (2) \\ 15z = -15 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -1}$$

$$(2) \rightarrow -5y = -10 + 5 \\ \underline{y = 1}$$

$$(3) \rightarrow x = 5 - 2 - 1 \\ \underline{= 2}$$

$$\therefore \text{Solution: } \underline{(2, 1, -1)}$$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + y + z = 6 \\ 3x + 4y - 7z = 1 \\ 2x - y + 3z = 5 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 4 & -7 & 1 \\ 2 & -1 & 3 & 5 \end{array} \right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \quad \begin{array}{cccc} 3 & 4 & -7 & 1 \\ -3 & -3 & -3 & -18 \\ \hline 0 & 1 & -10 & -17 \end{array} \quad \begin{array}{cccc} 2 & -1 & 3 & 5 \\ -2 & -2 & -2 & -12 \\ \hline 0 & -3 & 1 & -7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -10 & -17 \\ 0 & -3 & 1 & -7 \end{array} \right] \quad R_3 + 3R_2 \quad \begin{array}{cccc} 0 & -3 & 1 & -7 \\ 0 & 3 & -30 & -51 \\ \hline 0 & 0 & -29 & -58 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -10 & -17 \\ 0 & 0 & -29 & -58 \end{array} \right] \quad \begin{array}{l} x + y + z = 6 \quad (3) \\ y - 10z = -17 \quad (2) \\ -29z = -58 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = 2}$$

$$(2) \rightarrow y = -17 + 20 \\ \underline{= 3}$$

$$(3) \rightarrow x = 6 - 3 - 2 \\ \underline{= 1}$$

$$\therefore \text{Solution: } \underline{(1, 3, 2)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 3x + 2y + 3z = 3 \\ 4x - 5y + 7z = 1 \\ 2x + 3y - 2z = 6 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 4 & -5 & 7 & 1 \\ 2 & 3 & -2 & 6 \end{array} \right] \begin{array}{l} \\ 3R_2 - 4R_1 \\ 3R_3 - 2R_1 \end{array} \quad \begin{array}{cccc} 12 & -15 & 21 & 3 \\ -12 & -8 & -12 & -12 \\ \hline 0 & -23 & 9 & -9 \end{array} \quad \begin{array}{cccc} 6 & 9 & -6 & 18 \\ -6 & -4 & -6 & -6 \\ \hline 0 & 5 & -12 & 12 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 0 & -23 & 9 & -9 \\ 0 & 5 & -12 & 12 \end{array} \right] 23R_3 + 5R_2 \quad \begin{array}{cccc} 0 & 115 & -276 & 276 \\ 0 & -115 & 45 & -45 \\ \hline 0 & 0 & -231 & 231 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 0 & -23 & 9 & -9 \\ 0 & 0 & -231 & 231 \end{array} \right] \begin{array}{l} 3x + 2y + 3z = 3 \quad (3) \\ -23y + 9z = -9 \quad (2) \\ -231z = 231 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -1}$$

$$(2) \rightarrow -23y = -9 + 9$$

$$\underline{y = 0}$$

$$(3) \rightarrow 3x = 3 + 3$$

$$\underline{x = 2}$$

$$\therefore \text{Solution: } \underline{(2, 0, -1)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x - 3y + z = 2 \\ 4x - 12y + 4z = 8 \\ -2x + 6y - 2z = -4 \end{cases}$$

Solution

$$\begin{cases} x - 3y + z = 2 \\ \frac{1}{4} \times 4x - 12y + 4z = 8 \\ -\frac{1}{2} \times -2x + 6y - 2z = -4 \end{cases}$$

$$\begin{cases} x - 3y + z = 2 \\ x - 3y + z = 2 \\ x - 3y + z = 2 \end{cases}$$

Since all three equations are the same.

∴ **Solution:** is the plane $\underline{x - 3y + z = 2}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x - 2y + z = -1 \\ x + 2y - z = 2 \\ 6x + 4y + 3z = 5 \end{cases}$

Solution

$$\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & -2 & 1 & -1 \\ 6 & 4 & 3 & 5 \end{array} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 6R_1 \end{array} \quad \begin{array}{ccc|c} 2 & -2 & 1 & -1 \\ -2 & -4 & 2 & -4 \\ \hline 0 & -6 & 3 & -5 \end{array} \quad \begin{array}{ccc|c} 6 & 4 & 3 & 5 \\ -6 & -12 & 6 & -12 \\ \hline 0 & -8 & 9 & -7 \end{array}$$

$$\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -6 & 3 & -5 \\ 0 & -8 & 9 & -7 \end{array} \begin{array}{l} \\ \\ 3R_3 - 4R_2 \end{array} \quad \begin{array}{ccc|c} 0 & -24 & 27 & -21 \\ 0 & 24 & -12 & 20 \\ \hline 0 & 0 & 15 & -1 \end{array}$$

$$\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -6 & 3 & -5 \\ 0 & 0 & 15 & -1 \end{array} \begin{array}{l} x + 2y - z = 2 \quad (3) \\ -6y + 3z = -5 \quad (2) \\ 15z = -1 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -\frac{1}{15}}$$

$$(2) \rightarrow -6y = -5 + \frac{1}{5}$$

$$-6y = -\frac{24}{5}$$

$$\underline{y = \frac{4}{5}}$$

$$(3) \rightarrow x = 2 - \frac{8}{5} - \frac{1}{15}$$

$$\underline{= \frac{1}{3}}$$

∴ **Solution:** $\underline{\left(\frac{1}{3}, \frac{4}{5}, -\frac{1}{15}\right)}$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 \\ x_2 - 3x_3 - x_4 = 0 \\ 3x_1 + 2x_3 - x_4 = 6 \\ -4x_1 + x_2 + 4x_3 + 2x_4 = -3 \end{cases}$$

Solution

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 3 & 0 & 2 & -1 & 6 \\ -4 & 1 & 4 & 2 & -3 \end{array} \right] \begin{array}{l} R_3 - 3R_1 \\ R_4 + 4R_1 \end{array}$$

$$\begin{array}{ccccc|ccccc} 3 & 0 & 2 & -1 & 6 & -4 & 1 & 4 & 2 & -3 \\ -3 & 15 & -6 & 6 & -12 & 4 & -20 & 8 & -8 & 16 \\ \hline 0 & 15 & -4 & 5 & -6 & 0 & -19 & 12 & -6 & 13 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 15 & -4 & 5 & -6 \\ 0 & -19 & 12 & -6 & 13 \end{array} \right] \begin{array}{l} R_3 - 15R_2 \\ R_4 + 19R_2 \end{array}$$

$$\begin{array}{ccccc|ccccc} 0 & 15 & -4 & 5 & -6 & 0 & -19 & 12 & -6 & 13 \\ 0 & -15 & 45 & 15 & 0 & 0 & 19 & -57 & -19 & 0 \\ \hline 0 & 0 & 41 & 20 & -6 & 0 & 0 & -45 & -25 & 13 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & -45 & -25 & 13 \end{array} \right] 41R_4 + 45R_2$$

$$\begin{array}{ccccc|ccccc} 0 & 0 & -1845 & -1025 & 533 & & & & & \\ 0 & 0 & 1845 & 900 & -270 & & & & & \\ \hline 0 & 0 & 0 & -125 & 263 & & & & & \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & 0 & -125 & 263 \end{array} \right] \begin{array}{l} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 \quad (4) \\ x_2 - 3x_3 - x_4 = 0 \quad (3) \\ 41x_3 + 20x_4 = -6 \quad (2) \\ -125x_4 = 263 \quad (1) \end{array}$$

$$(1) \rightarrow x_4 = -\frac{263}{125}$$

$$(2) \rightarrow 41x_3 = -6 + \frac{1,052}{25}$$

$$= \frac{902}{25}$$

$$x_3 = \frac{22}{25}$$

$$(3) \rightarrow x_2 = \frac{66}{25} - \frac{263}{125}$$

$$= \frac{67}{125}$$

$$(4) \rightarrow x_1 = 4 + \frac{67}{25} - \frac{44}{25} - \frac{526}{125}$$

$$\begin{aligned}
&= 4 + \frac{23}{25} - \frac{526}{125} \\
&= \frac{500 + 115 - 526}{125} \\
&= \frac{89}{125}
\end{aligned}$$

$$\therefore \text{Solution: } \left(\frac{89}{125}, \frac{67}{125}, \frac{22}{25}, -\frac{263}{125} \right)$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases}
x_1 + x_2 + x_3 + x_4 = 5 \\
x_1 + 2x_2 - x_3 - 2x_4 = -1 \\
x_1 - 3x_2 - 3x_3 - x_4 = -1 \\
2x_1 - x_2 + 2x_3 - x_4 = -2
\end{cases}$$

Solution

$$\left[\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 5 \\
1 & 2 & -1 & -2 & -1 \\
1 & -3 & -3 & -1 & -1 \\
2 & -1 & 2 & -1 & -2
\end{array} \right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - 2R_1 \end{array}$$

$$\begin{array}{ccccc}
1 & 2 & -1 & -2 & -1 \\
-1 & -1 & -1 & -1 & -5 \\
\hline
0 & 1 & -2 & -3 & -6
\end{array}$$

$$\begin{array}{ccccc}
1 & -3 & -3 & -1 & -1 \\
-1 & -1 & -1 & -1 & -5 \\
\hline
0 & -4 & -4 & -2 & -6
\end{array}$$

$$\begin{array}{ccccc}
2 & -1 & 2 & -1 & -2 \\
-2 & -2 & -2 & -2 & -10 \\
\hline
0 & -3 & 0 & -3 & -12
\end{array}$$

$$\left[\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 5 \\
0 & 1 & -2 & -3 & -6 \\
0 & -4 & -4 & -2 & -6 \\
0 & -3 & 0 & -3 & -12
\end{array} \right] \begin{array}{l} \\ \\ R_3 + 4R_2 \\ R_4 + 3R_2 \end{array}$$

$$\begin{array}{ccccc}
0 & -4 & -4 & -2 & -6 \\
0 & 4 & -8 & -12 & -24 \\
\hline
0 & 0 & -12 & -14 & -30
\end{array}$$

$$\begin{array}{ccccc}
0 & -3 & 0 & -3 & -12 \\
0 & 3 & -6 & -9 & -18 \\
\hline
0 & 0 & -6 & -12 & -30
\end{array}$$

$$\left[\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 5 \\
0 & 1 & -2 & -3 & -6 \\
0 & 0 & -12 & -14 & -30 \\
0 & 0 & -6 & -12 & -30
\end{array} \right] -2R_4 + R_3$$

$$\begin{array}{ccccc}
0 & 0 & 12 & 24 & 60 \\
0 & 0 & -12 & -14 & -30 \\
\hline
0 & 0 & 0 & 10 & 30
\end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & -12 & -14 & -30 \\ 0 & 0 & 0 & 10 & 30 \end{array} \right] \quad \begin{array}{l} x_1 + x_2 + x_3 + x_4 = 5 \quad (4) \\ x_2 - 2x_3 - 3x_4 = -6 \quad (3) \\ -12x_3 - 14x_4 = -30 \quad (2) \\ 10x_4 = 30 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{x_4 = 3}$$

$$(2) \rightarrow -12x_3 = -30 + 42 \\ = 12$$

$$\underline{x_3 = -1}$$

$$(3) \rightarrow x_2 = -6 - 2 + 9 \\ = 1$$

$$(4) \rightarrow x_1 = 5 - 1 + 1 - 3 \\ = 2$$

$$\therefore \text{Solution: } \underline{(2, 1, -1, 3)}$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 2x + 8y - z + w = 0 \\ 4x + 16y - 3z - w = -10 \\ -2x + 4y - z + 3w = -6 \\ -6x + 2y + 5z + w = 3 \end{cases}$$

Solution

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 4 & 16 & -3 & -1 & -10 \\ -2 & 4 & -1 & 3 & -6 \\ -6 & 2 & 5 & 1 & 3 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 + 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 26 & 2 & 4 & 3 \end{array} \right] \quad R_4 - \frac{13}{6}R_2$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 26 & 2 & 4 & 3 \end{array} \right] \quad \text{Interchange } R_2 \text{ and } R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & \frac{19}{3} & -\frac{14}{3} & 16 \end{array} \right] R_4 + \frac{19}{3}R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & 0 & -\frac{71}{3} & -\frac{142}{3} \end{array} \right] \begin{array}{l} 2x + 8y - z + w = 0 \quad (3) \\ 12y - 2z + 4w = -6 \quad (2) \\ -z - 3w = -10 \quad (1) \\ -\frac{71}{3}w = -\frac{142}{3} \rightarrow \underline{w = 2} \end{array}$$

$$(1) \rightarrow z = 10 - 3w = \underline{4}$$

$$(2) \rightarrow 12y = 2z - 4w - 6$$

$$\underline{y = -\frac{1}{2}}$$

$$(3) \rightarrow 2x = -8y + z - w$$

$$2x = 4 + 4 - 2$$

$$2x = 6$$

$$\underline{x = 3}$$

$$\therefore \text{Solution: } \underline{\left(3, -\frac{1}{2}, 4, 2 \right)}$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

Solution

$$\begin{cases} x_1 = -2x_2 \\ x_3 = -x_2 \end{cases}$$

$$2x_1 + x_2 + 3x_3 = 0$$

$$-4x_2 + x_2 - 3x_2 = 0 \rightarrow \underline{x_2 = 0}$$

$$\therefore \text{Solution: } \underline{(0, 0, 0)}$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 2x + 2y + 4z = 0 \\ -y - 3z + w = 0 \\ 3x + y + z + 2w = 0 \\ x + 3y - 2z - 2w = 0 \end{cases}$$

Solution

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & -1 & -3 & 1 & 0 \\ 3 & 1 & 1 & 2 & 0 \\ 1 & 3 & -2 & -2 & 0 \end{array} \right] \begin{array}{l} \\ -R_2 \\ 2R_3 - 3R_1 \\ 2R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & -4 & -10 & 4 & 0 \\ 0 & 4 & -8 & -4 & 0 \end{array} \right] \begin{array}{l} \\ R_3 + 4R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -20 & 0 & 0 \end{array} \right] \begin{array}{l} 2x + 2y - 4z = 0 \quad (1) \\ y + 3z - w = 0 \quad (2) \\ \rightarrow \underline{z = 0} \\ \end{array}$$

$$(2) \rightarrow \underline{y = w}$$

$$(1) \rightarrow 2x = -2y \quad \underline{x = -w}$$

$$\therefore \text{Solution: } \underline{(-w, w, 0, w)}$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 2x + z + w = 5 \\ y - w = -1 \\ 3x - z - w = 0 \\ 4x + y + 2z + w = 9 \end{cases}$$

Solution

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 3 & 0 & -1 & -1 & 0 \\ 4 & 1 & 2 & 1 & 9 \end{array} \right] \begin{array}{l} \\ \\ 2R_3 - 3R_1 \\ 2R_4 - 4R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -5 & -5 & -15 \\ 0 & 2 & 0 & -2 & -2 \end{array} \right] \quad R_4 - 2R_2$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -5 & -5 & -15 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} 2x + z + w = 5 \quad (1) \\ y - w = -1 \quad (2) \\ -5z - 5w = -15 \quad (3) \end{array}$$

$$(2) \rightarrow \underline{y = 1 + w}$$

$$(3) \rightarrow \underline{z = 3 - w}$$

$$(1) \rightarrow 2x = 5 - (3 - w) - w \Rightarrow \underline{x = 1}$$

$$\therefore \text{Solution: } \underline{(1, 1 + w, 3 - w, w)}$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 4y + z = 20 \\ 2x - 2y + z = 0 \\ x + z = 5 \\ x + y - z = 10 \end{cases}$$

Solution

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 10 & 10 \\ 2 & -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 5 & 5 \\ 0 & 4 & 1 & 20 & 20 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 10 & 10 \\ 0 & -4 & 3 & -20 & -20 \\ 0 & -1 & 2 & -5 & -5 \\ 0 & 4 & 1 & 20 & 20 \end{array} \right] \quad \begin{array}{l} 4R_3 - R_2 \\ R_4 + R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 10 & 10 \\ 0 & -4 & 3 & -20 & -20 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{array} \right] \quad \begin{array}{l} x + y = 10 \\ \rightarrow -4y = -20 \\ \rightarrow z = 0 \end{array}$$

$$\therefore \text{Solution: } \underline{(5, 5, 0)}$$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \quad \quad - 3w = -3 \end{cases}$$

Solution

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & -1 \\ -1 & 3 & -2 & -2 & -2 \\ 3 & 4 & -7 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array} \right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & -1 \\ 0 & 5 & -1 & -1 & -3 \\ 0 & -2 & -10 & 4 & 4 \\ 0 & 0 & 0 & -3 & -3 \end{array} \right] \begin{array}{l} \\ \\ 5R_3 + 2R_2 \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & -1 \\ 0 & 5 & -1 & -1 & -3 \\ 0 & 0 & -52 & -52 & -52 \end{array} \right] \begin{array}{l} x + 2y + z = -1 \quad (3) \\ 5y - z = -3 \quad (2) \\ -52z = -52 \quad (1) \end{array}$$

$$(1) \Rightarrow z = 1$$

$$(2) \Rightarrow 5y = -3 + 1 = -2 \rightarrow y = -\frac{2}{5}$$

$$(3) \Rightarrow x = -1 - 2(-\frac{2}{5}) - 1 = -\frac{3}{5}$$

$$\therefore \text{Solution: } \underline{\left(-\frac{3}{5}, -\frac{2}{5}, 1 \right)}$$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} 2u - 3v + w - x + y = 0 \\ 4u - 6v + 2w - 3x - y = -5 \\ -2u + 3v - 2w + 2x - y = 3 \end{cases}$$

Solution

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 4 & -6 & 2 & -3 & -1 & -5 \\ -2 & 3 & -2 & 2 & -1 & 3 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -3 & -5 \\ 0 & 0 & -1 & 1 & 0 & 3 \end{array} \right] \quad \begin{array}{l} 2u - 3v + w - x + y = 0 \quad (3) \\ -x - 3y = -5 \quad (2) \\ -w + x = 3 \quad (1) \end{array}$$

$$(2) \Rightarrow x = 5 - 3y$$

$$(1) \Rightarrow w = x - 3 = 2 - 3y$$

$$(3) \Rightarrow 2u = 3v - 2 + 3y + 5 - 3y - y = 3v - y + 3$$

$$u = \frac{3}{2}v - \frac{1}{2}y + \frac{3}{2}$$

$$\therefore \text{Solution: } \left(\frac{3}{2}v - \frac{1}{2}y + \frac{3}{2}, v, 2 - 3y, 5 - 3y, y \right)$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 6x_3 + 2x_4 - 4x_5 - 8x_6 = 8 \\ 3x_3 + x_4 - 2x_5 - 4x_6 = 4 \\ 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 + x_6 = 2 \\ 6x_1 - 9x_2 + 11x_4 - 19x_5 + 3x_6 = 1 \end{cases}$$

Solution

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 6 & -9 & 0 & 11 & -19 & 3 & 1 \end{array} \right] \quad R_4 - 3R_1$$

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & -3 & -1 & 2 & 0 & -5 \end{array} \right] \quad \begin{array}{l} R_3 - 2R_2 \\ R_4 + R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & -1 \end{array} \right] \quad \begin{array}{l} 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 = 2 - x_6 \\ 3x_3 + x_4 - 2x_5 = 4 + 4x_6 \end{array}$$

$$\rightarrow \underline{x_6 = \frac{1}{4}}$$

$$\begin{cases} 3x_3 = 5 - x_4 + 2x_5 \\ 2x_1 = \frac{7}{4} + 3x_2 - \frac{1}{3}(5 - x_4 + 2x_5) - 4x_4 + 7x_5 \end{cases}$$

$$\left\{ \begin{array}{l} x_3 = \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5 \\ 2x_1 = \frac{1}{12} + 3x_2 - \frac{11}{3}x_4 + \frac{19}{3}x_5 \end{array} \right|$$

$$\therefore \text{Solution: } \left(\frac{1}{24} + \frac{3}{2}x_2 - \frac{11}{6}x_4 + \frac{19}{6}x_5, x_2, \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5, x_4, x_5, \frac{1}{4} \right) |$$

Exercise

Use augmented elimination to solve linear system

$$\left\{ \begin{array}{l} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{array} \right.$$

Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{array} \right] \begin{array}{l} \\ 3R_2 - 5R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & -1 & 4 & 26 \\ 0 & 0 & 0 & 0 \end{array} \right] R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} 3x_1 + 2x_2 - x_3 = -15 \quad (3) \\ -x_2 + 11x_3 = 75 \quad (2) \\ -7x_3 = -49 \quad (1) \end{array}$$

$$(1) \rightarrow x_3 = 7$$

$$(2) \rightarrow x_2 = 77 - 75 = 2$$

$$(1) \rightarrow 3x_1 = -15 - 4 + 7 = -12$$

$$\underline{x_1 = -4}$$

$$\therefore \text{Solution: } \underline{(-4, 2, 7)} |$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ 5x_3 + 10x_4 + 15x_6 = 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$$

Solution

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array} \right] -R_2$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array} \right] \begin{array}{l} \\ R_3 - 5R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array} \right] \frac{1}{6}R_4 \text{ then interchanging row3 and row4}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] R_2 - 3R_3$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \begin{cases} x_1 + 3x_2 + 4x_4 + 2x_5 = 0 \\ x_3 + 2x_4 = 0 \\ + x_6 = \frac{1}{3} \end{cases}$$

The general solution of the system: $x_6 = \frac{1}{3}, \quad x_3 = -2x_4, \quad x_1 = -3x_2 - 4x_4 - 2x_5$

\therefore **Solution:** $\left(-3x_2 - 4x_4 - 2x_5, x_2, -2x_4, x_4, x_5, \frac{1}{3} \right)$

Exercise

At SnackMix, caramel corn worth \$2.50 per *pound* is mixed with honey roasted missed nuts worth \$7.50 per *pound* in order to get 20 *lbs.* of a mixture worth \$4.50 per *pound*. How much of each snack is used?

Solution

$$x + y = 20 \quad (1)$$

$$2.50x + 7.50y = 90 \quad (2)$$

$$(1) \quad y = 20 - x$$

$$(2) \quad 2.5x + 7.5(20 - x) = 90$$

$$2.5x + 150 - 7.5x = 90$$

$$-5x = 90 - 150$$

$$-5x = -60$$

$$x = \frac{-60}{-5} = 12$$

$$y = 20 - x$$

$$= 20 - 12$$

$$= 8$$

The mixture consists of 12 *lbs.* of caramel and 8 *lbs.* of nuts