Solution   *Section* **3.5 – Least Squares Analysis**

Exercise

Find the equation of the line that best fits the given points in the least-squares sense.

a) $\{(0, 2), (1, 2), (2, 0)\}$

b) $\{(1, 5), (2, 4), (3, 1), (4, 1), (5, -1)\}$

c) $\{(0, 1), (1, 3), (2, 4), (3, 4)\}$

d) $\{(-2, 0), (-1, 0), (0, 1), (1, 3), (2, 5)\}$

Solution

a) $\{(0, 2), (1, 2), (2, 0)\}$

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \quad \text{where } A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \quad x = \begin{bmatrix} m \\ b \end{bmatrix} \quad y = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$
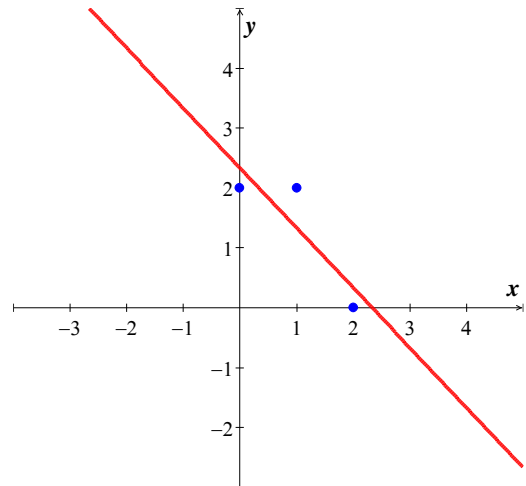
The normal equation formula: $A^T A x = A^T y$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

We have: $m = -1$ *and* $b = \dfrac{7}{3}$.

Thus, $y = -x + \dfrac{7}{3}$

b) $\{(1, 5), (2, 4), (3, 1), (4, 1), (5, -1)\}$

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{where } A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \quad x = \begin{bmatrix} m \\ b \end{bmatrix} \quad y = \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$
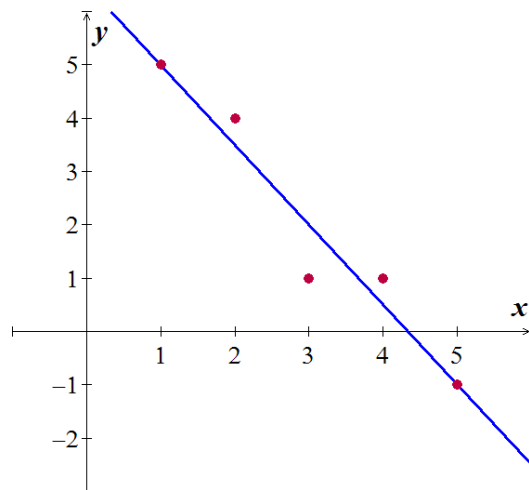
The normal equation: $A^T A x = A^T y$

34

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 55 & 15 \\ 15 & 5 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}$$

We have: $m = -\dfrac{3}{2}$ and $b = \dfrac{13}{2}$.

Thus, $\underline{y = -1.5x + 6.5}$

c) $\{(0,\ 1),\ (1,\ 3),\ (2,\ 4),\ (3,\ 4)\}$

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix} \qquad \text{where } A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \quad x = \begin{pmatrix} m \\ b \end{pmatrix} \quad y = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

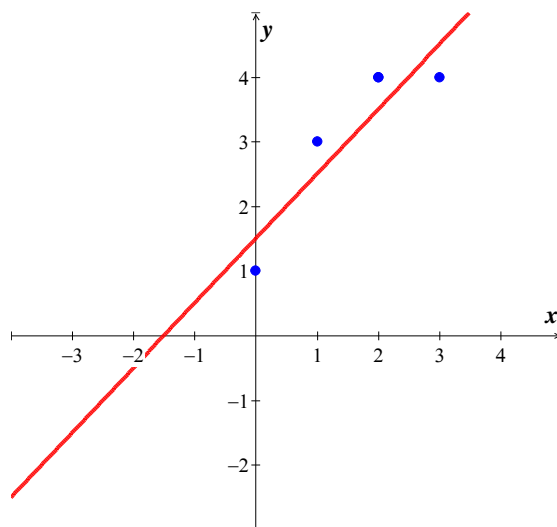The normal equation: $A^T A x = A^T y$

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 23 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 4 & -6 \\ -6 & 14 \end{pmatrix}\begin{pmatrix} 23 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$$

We have: $m = 1$ and $b = \dfrac{3}{2}$.

Thus, $\underline{y = x + 1.5}$

d) $\{(-2,\ 0),\ (-1,\ 0),\ (0,\ 1),\ (1,\ 3),\ (2,\ 5)\}$

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix} \qquad \text{where } A = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad x = \begin{pmatrix} m \\ b \end{pmatrix} \quad y = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

35

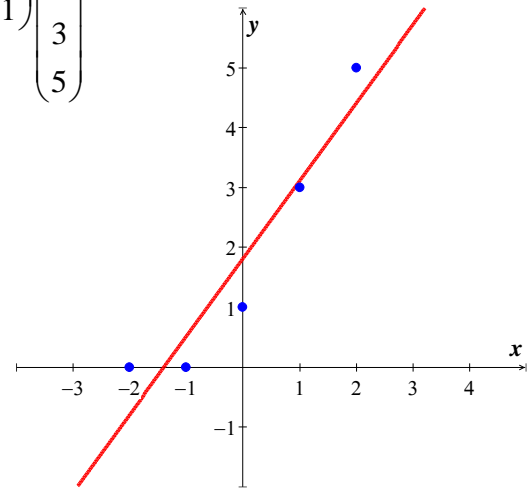The normal equation: $A^T A\boldsymbol{x} = A^T \boldsymbol{y}$

$$\begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 13 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{50} \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} 13 \\ 9 \end{pmatrix} = \begin{pmatrix} \frac{13}{10} \\ \frac{9}{5} \end{pmatrix}$$

We have: $m = 1.3 \quad and \quad b = 1.8$.

Thus, $y = 1.3x + 1.8$

Exercise

Find the orthogonal projection of the vector $\boldsymbol{u}$ on the subspace of $\boldsymbol{R}^4$ spanned by the vectors

a) $\boldsymbol{u} = (-3, -3, 8, 9); \quad \boldsymbol{v}_1 = (3, 1, 0, 1), \quad \boldsymbol{v}_2 = (1, 2, 1, 1), \quad \boldsymbol{v}_3 = (-1, 0, 2, -1)$

b) $\boldsymbol{u} = (6, 3, 9, 6); \quad \boldsymbol{v}_1 = (2, 1, 1, 1), \quad \boldsymbol{v}_2 = (1, 0, 1, 1), \quad \boldsymbol{v}_3 = (-2, -1, 0, -1)$

c) $\boldsymbol{u} = (-2, 0, 2, 4); \quad \boldsymbol{v}_1 = (1, 1, 3, 0), \quad \boldsymbol{v}_2 = (-2, -1, -2, 1), \quad \boldsymbol{v}_3 = (-3, -1, 1, 3)$

Solution

a) Let $A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix} \Rightarrow A^T A = \begin{pmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{pmatrix}$

$$A^T \boldsymbol{u} = \begin{pmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -3 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ 10 \end{pmatrix}$$

The normal solution is $A^T A\boldsymbol{x} = A^T \boldsymbol{u}$

$$\begin{pmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ 10 \end{pmatrix} \Rightarrow \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

So $proj_W \boldsymbol{u} = A\boldsymbol{x} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 4 \\ 0 \end{pmatrix}$

$\underline{proj_W \boldsymbol{u} = (-2,\ 3,\ 4,\ 0)}$

b) $\boldsymbol{u} = (6,\ 3,\ 9,\ 6);\quad \boldsymbol{v}_1 = (2,\ 1,\ 1,\ 1),\quad \boldsymbol{v}_2 = (1,\ 0,\ 1,\ 1),\quad \boldsymbol{v}_3 = (-2,\ -1,\ 0,\ -1)$

Let $A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \Rightarrow A^T A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix}$

$A^T \boldsymbol{u} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 9 \\ 6 \end{pmatrix} = \begin{pmatrix} 30 \\ 21 \\ -21 \end{pmatrix}$

The normal solution is $A^T A\boldsymbol{x} = A^T \boldsymbol{u}$

$\begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 30 \\ 21 \\ -21 \end{pmatrix} \Rightarrow \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$

So $proj_W \boldsymbol{u} = A\boldsymbol{x} = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 9 \\ 5 \end{pmatrix}$

$\underline{proj_W \boldsymbol{u} = (7,\ 2,\ 9,\ 5)}$

c) $\boldsymbol{u} = (-2,\ 0,\ 2,\ 4);\quad \boldsymbol{v}_1 = (1,\ 1,\ 3,\ 0),\quad \boldsymbol{v}_2 = (-2,\ -1,\ -2,\ 1),\quad \boldsymbol{v}_3 = (-3,\ -1,\ 1,\ 3)$

Let $A = \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix} \Rightarrow A^T A = \begin{pmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{pmatrix}$

$A^T \boldsymbol{u} = \begin{pmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix}$

The normal solution is $A^T A\boldsymbol{x} = A^T \boldsymbol{u}$

$$\begin{pmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix} \Rightarrow \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\dfrac{4}{5} \\ -\dfrac{8}{5} \\ \dfrac{8}{5} \end{pmatrix}$$

So $proj_W \boldsymbol{u} = A\boldsymbol{x} = \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} -\dfrac{4}{5} \\ -\dfrac{8}{5} \\ \dfrac{8}{5} \end{pmatrix} = \begin{pmatrix} -\dfrac{12}{5} \\ -\dfrac{4}{5} \\ \dfrac{12}{5} \\ \dfrac{16}{5} \end{pmatrix}$

$\underline{proj_W \boldsymbol{u} = \left(-\dfrac{12}{5}, \ -\dfrac{4}{5}, \ \dfrac{12}{5}, \ \dfrac{16}{5} \right)}$
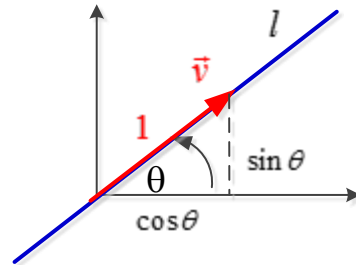
Exercise

Find the standard matrix for the orthogonal projection P of $\boldsymbol{R}^2$ on the line passes through the origin and makes an angle θ with the positive x-axis.

Solution

Since the line l in 2-dimensional, than we can take $\vec{v} = (\cos\theta, \ \sin\theta)$ as a basis for this subspace

$A = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

$[P] = A^T A = [\cos\theta \quad \sin\theta] \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$

Exercise

Hooke's law in physics states that the length x of a uniform spring is a linear function of the force y applied to it. If we express the relationship as $y = mx + b$, then the coefficient m is called the spring constant. Suppose a particular unstretched spring has a measured length of 6.1 *inches*.(i.e., $x = 6.1$ when $y = 0$). Forces of 2 pounds, 4 pounds, and 6 pounds are then applied to the spring, and the corresponding lengths are found to be 7.6 inches, 8.7 inches, and 10.4 inches. Find the spring constant.

Length
x

Force y
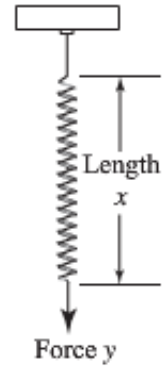
Solution

$$M = \begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$

The normal equation: $A^T A x = A^T y$

$$\begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 278.82 & 32.8 \\ 32.8 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 112.4 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{39.44} \begin{pmatrix} 4 & -32.8 \\ -32.8 & 278.2 \end{pmatrix} \begin{pmatrix} 112.4 \\ 12 \end{pmatrix} = \begin{pmatrix} 1.4 \\ -8.8 \end{pmatrix}$$

Thus, the estimated value of the spring constant is $\approx 1.4 \; pounds$.

Exercise

Prove: If A has a linearly independent column vectors, and if b is orthogonal to the column space of A, then the least squares solution of $Ax = b$ is $x = 0$.

Solution

If A has linearly independent column vectors, then $A^T A$ is invertible and the least squares solution of $Ax = b$ is the solution of $A^T Ax = A^T b$, but since b is orthogonal to the column space of A. $A^T b = 0$, so x is a solution of $A^T Ax = 0$. Thus $x = 0$ since $A^T A$ is invertible.

Exercise

Let A be an $m \times n$ matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of R^n onto the row space of A.

Solution

A^T will have linearly independent column vectors, and the column space A^T is the row space of A.

Thus, the standard matrix for the orthogonal projection of R^n onto the row space of A is

$$[P] = A^T \left[\left(A^T \right)^T A^T \right]^{-1} \left(A^T \right)^T = A^T \left(A A^T \right)^{-1} A$$

Exercise

Let W be the line with parametric equations $x = 2t, \quad t = -t, \quad z = 4t$

a) Find a basis for W.

b) Find the standard matrix for the orthogonal projection on W.

c) Use the matrix in part (b) to find the orthogonal projection of a point $P_0 \left(x_0, y_0, z_0 \right)$ on W.

d) Find the distance between the point $P_0 \left(2, 1, -3 \right)$ and the line W.

Solution

a) $W = span\{(2, -1, 4)\}$ so that the vector $(2, -1, 4)$ forms a basis for W (linear independence)

b) Let $A = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$

$$[P] = A \left(A^T A \right)^{-1} A^T$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \left(\begin{bmatrix} 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} [21]^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \frac{1}{21} \begin{bmatrix} 4 & -2 & 8 \\ -2 & 1 & -4 \\ 8 & -4 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{4}{21} & -\dfrac{2}{21} & \dfrac{8}{21} \\[2mm] -\dfrac{2}{21} & 1 & -\dfrac{4}{21} \\[2mm] \dfrac{8}{21} & -\dfrac{4}{21} & \dfrac{16}{21} \end{bmatrix}$$

c) $\quad \begin{bmatrix} \dfrac{4}{21} & -\dfrac{2}{21} & \dfrac{8}{21} \\[2mm] -\dfrac{2}{21} & 1 & -\dfrac{4}{21} \\[2mm] \dfrac{8}{21} & -\dfrac{4}{21} & \dfrac{16}{21} \end{bmatrix} \begin{bmatrix} x_0 \\[2mm] y_0 \\[2mm] z_0 \end{bmatrix} = \begin{bmatrix} \dfrac{4}{21}x_0 - \dfrac{2}{21}y_0 + \dfrac{8}{21}z_0 \\[2mm] -\dfrac{2}{21}x_0 + y_0 - \dfrac{4}{21}z_0 \\[2mm] \dfrac{8}{21}x_0 - \dfrac{4}{21}y_0 + \dfrac{16}{21}z_0 \end{bmatrix}$

d) $\quad \begin{bmatrix} \dfrac{4}{21} & -\dfrac{2}{21} & \dfrac{8}{21} \\[2mm] -\dfrac{2}{21} & 1 & -\dfrac{4}{21} \\[2mm] \dfrac{8}{21} & -\dfrac{4}{21} & \dfrac{16}{21} \end{bmatrix} \begin{bmatrix} 2 \\[2mm] 1 \\[2mm] -3 \end{bmatrix} = \begin{bmatrix} -\dfrac{6}{7} \\[2mm] \dfrac{3}{7} \\[2mm] -\dfrac{12}{7} \end{bmatrix}$

The distance between P_0 and W equals to the distance between P_0 and its projection on W.

The distance between $(2,\ 1,\ -3)$ and $\left(-\dfrac{6}{7},\ \dfrac{3}{7},\ -\dfrac{12}{7}\right)$ is

$$d = \sqrt{\left(2+\tfrac{6}{7}\right)^2 + \left(1-\tfrac{3}{7}\right)^2 + \left(-3+\tfrac{12}{7}\right)^2}$$

$$= \sqrt{\dfrac{400}{49} + \dfrac{16}{49} + \dfrac{81}{49}}$$

$$= \dfrac{\sqrt{497}}{7}$$

Exercise

In R^3, consider the line l given by the equations $\qquad x = t, \quad t = t, \quad z = t$

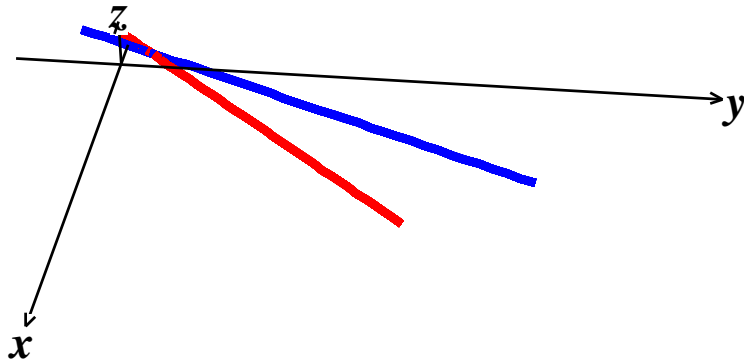And the line m given by the equations $\qquad x = s, \quad t = 2s - 1, \quad z = 1$

Let P be the point on l, and let Q be a point on m. Find the values of t and s that minimize the distance between the lines by minimizing the squared distance $\|P - Q\|^2$

Solution

When $t = 1 \implies$ *Let* $P = (1,\ 1,\ 1)$ is on line l

When $s = 1 \implies$ *Let* $Q = (1,\ 1,\ 1)$ is on line m

$\|P - Q\| = \sqrt{(1-1)^2 + (1-1)^2 + (1-1)^2} = 0 \ge 0$

Thus these are the values $P = (1, 1, 1)$ and $Q = (1, 1, 1)$ are the values for $s = t = 1$ that minimize the distance between the lines.

Exercise

Determine whether the statement is true or false,

a) If A is an $m \times n$ matrix, then $A^T A$ is a square matrix.

b) If $A^T A$ is invertible, then A is invertible.

c) If A is invertible, then $A^T A$ is invertible.

d) If $Ax = b$ is a consistent linear system, then $A^T Ax = A^T b$ is also consistent.

e) If $Ax = b$ is an inconsistent linear system, then $A^T Ax = A^T b$ is also inconsistent.

f) Every linear system has a least squares solution.

g) Every linear system has a unique least squares solution.

h) If A is an $m \times n$ matrix with linearly independent columns and b is in R^m, then $Ax = b$ has a unique least squares solution.

Solution

a) **True**; $A^T A$ is an $n \times n$ matrix

b) **False**; only square matrix has inverses, but $A^T A$ can be invertible when A is not square matrix.

c) **True**; if A is invertible, so is A^T , so the product $A^T A$ is also invertible

d) **True**

e) **False**; the system $A^T Ax = A^T b$ may be consistent

f) **True**

g) **False**; the least squares solution may involve a parameter

h) **True**; if A has linearly independent column vectors; then $A^T A$ is invertible, so $A^T Ax = A^T b$ has a unique solution