Solution Section 3.5 – Triple Integrals in Cylindrical and Spherical Coordinates

Exercise

Evaluate the cylindrical coordinate integral $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\ rdr\ d\theta$

Solution

$$\begin{aligned}
\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\ rdr\ d\theta &= \int_0^{2\pi}\int_0^1\left(\sqrt{2-r^2}-r\right)rdr\ d\theta \\
&= \int_0^{2\pi}\int_0^1\left(r\left(2-r^2\right)^{1/2}-r^2\right)drd\theta \qquad d\left(2-r^2\right)=-2rdr \\
&= \int_0^{2\pi}\left(\int_0^1\left(-\frac{1}{2}\left(2-r^2\right)^{1/2}d\left(2-r^2\right)-r^2dr\right)\right)d\theta \\
&= \int_0^{2\pi}\left[-\frac{1}{3}\left(2-r^2\right)^{3/2}-\frac{1}{3}r^3\right]_0^1 d\theta \\
&= \int_0^{2\pi}\left[\left(-\frac{1}{3}-\frac{1}{3}\right)-\left(-\frac{1}{3}2^{3/2}\right)\right]d\theta \\
&= \int_0^{2\pi}\left(-\frac{2}{3}+\frac{2^{3/2}}{3}\right)d\theta \\
&= \frac{2\sqrt{2}-2}{3}\left[\theta\right]_0^{2\pi} \\
&= 4\pi\frac{\sqrt{2}-1}{3}
\end{aligned}$$

Exercise

Evaluate the cylindrical coordinate integral $\displaystyle\int_0^{2\pi}\int_0^{\theta/(2\pi)}\int_0^{3+24r^2} dz\ rdr\ d\theta$

Solution

$$\int_0^{2\pi}\int_0^{\theta/(2\pi)}\int_0^{3+24r^2} dz\ rdr\ d\theta = \int_0^{2\pi}\int_0^{\theta/(2\pi)}\left(3+24r^2\right)rdr\ d\theta$$

$$=\int_0^{2\pi}\int_0^{\theta/(2\pi)}\left(3r+24r^3\right)dr\ d\theta$$

$$=\int_0^{2\pi}\left[\frac{3}{2}r^2+6r^4\right]_0^{\theta/(2\pi)} d\theta$$

$$=\int_0^{2\pi}\left[\frac{3}{8\pi^2}\theta^2+\frac{6}{16r^4}\theta^4\right]d\theta$$

$$=\left[\frac{1}{8\pi^2}\theta^3+\frac{3}{8r^4}\frac{1}{5}\theta^5\right]_0^{2\pi}$$

$$=\frac{1}{8\pi^2}8\pi^3+\frac{3}{8r^4}\frac{1}{5}32\pi^5$$

$$=\pi+\frac{12}{5}\pi$$

$$\underline{=\frac{17}{5}\pi}$$

Exercise

Evaluate the cylindrical coordinate integral $\displaystyle\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} zdz\ rdr\ d\theta$

Solution

$$\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} zdz\ rdr\ d\theta = \int_0^{\pi}\int_0^{\theta/\pi}\left[\frac{1}{2}z^2\right]_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} rdr\ d\theta$$

$$=\frac{1}{2}\int_0^{\pi}\int_0^{\theta/\pi}\left[9\left(4-r^2\right)-\left(4-r^2\right)\right]rdr\ d\theta$$

$$=\frac{1}{2}\int_0^{\pi}\int_0^{\theta/\pi}8\left(4-r^2\right)rdr\ d\theta$$

$$= 4\int_0^{\pi}\int_0^{\theta/\pi}\left(4r - r^3\right)dr\ d\theta$$

$$= 4\int_0^{\pi}\left[2r^2 - \frac{1}{4}r^4\right]_0^{\theta/\pi} d\theta$$

$$= 4\int_0^{\pi}\left(2\frac{\theta^2}{\pi^2} - \frac{1}{4}\frac{\theta^4}{\pi^4}\right)d\theta$$

$$= 4\left[\frac{2}{3}\frac{\theta^3}{\pi^2} - \frac{1}{20}\frac{\theta^5}{\pi^4}\right]_0^{\pi}$$

$$= 4\left[\frac{2}{3}\frac{\pi^3}{\pi^2} - \frac{1}{20}\frac{\pi^5}{\pi^4}\right]$$

$$= 4\left(\frac{2}{3}\pi - \frac{1}{20}\pi\right)$$

$$= 4\left(\frac{37}{60}\pi\right)$$

$$= \frac{37}{15}\pi$$

Exercise

Evaluate the cylindrical coordinate integral $\displaystyle\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2}\left(r^2\sin^2\theta + z^2\right)dz\ rdr\ d\theta$

Solution

$$\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2}\left(r^2\sin^2\theta + z^2\right)dz\ rdr\ d\theta = \int_0^{2\pi}\int_0^1\left[zr^2\sin^2\theta + \frac{1}{3}z^3\right]_{-1/2}^{1/2} rdr\ d\theta$$

$$= \int_0^{2\pi}\int_0^1\left[\frac{1}{2}r^2\sin^2\theta + \frac{1}{24} - \left(-\frac{1}{2}r^2\sin^2\theta - \frac{1}{24}\right)\right]rdrd\theta$$

$$= \int_0^{2\pi}\int_0^1\left(r^2\sin^2\theta + \frac{1}{12}\right)rdrd\theta$$

$$= \int_0^{2\pi}\int_0^1\left(r^3\sin^2\theta + \frac{1}{12}r\right)drd\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{4}r^4\sin^2\theta + \frac{1}{24}r^2\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{4}\sin^2\theta + \frac{1}{24}\right) d\theta \qquad \int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$= \left[\frac{1}{4}\left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right) + \frac{1}{24}\theta\right]_0^{2\pi}$$

$$= \left[\frac{\theta}{8} - \frac{1}{16}\sin 2\theta + \frac{1}{24}\theta\right]_0^{2\pi}$$

$$= \frac{2\pi}{8} + \frac{1}{24}2\pi$$

$$= \frac{\pi}{3}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3 dr\ dz\ d\theta$

Solution

$$\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3 dr\ dz\ d\theta = \int_0^{2\pi}\int_0^3 \left[\frac{1}{4}r^4\right]_0^{z/3} dz\ d\theta$$

$$= \frac{1}{324}\int_0^{2\pi}\int_0^3 z^4\ dz\ d\theta$$

$$= \frac{1}{324}\int_0^{2\pi}\left[\frac{1}{5}z^5\right]_0^3\ d\theta$$

$$= \frac{243}{1620}\int_0^{2\pi} d\theta$$

$$= \frac{3}{20}[\theta]_0^{2\pi}$$

$$= \frac{3\pi}{10}$$

Exercise

Evaluate the integral $\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} \left(r^2 \cos^2 \theta + z^2\right) r\ d\theta\ drdz$

Solution

$$\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} \left(r^2 \cos^2 \theta + z^2\right) r\ d\theta\ drdz = \int_0^1 \int_0^{\sqrt{z}} \left[r^2 \left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right) + z^2 \theta \right]_0^{2\pi} rdrdz$$

$$= \int_0^1 \int_0^{\sqrt{z}} \left(\pi r^2 + 2\pi z^2\right) rdrdz$$

$$= \int_0^1 \int_0^{\sqrt{z}} \left(\pi r^3 + 2\pi z^2 r\right) drdz$$

$$= \int_0^1 \left[\frac{1}{4}\pi r^4 + \pi z^2 r^2\right]_0^{\sqrt{z}} dz$$

$$= \int_0^1 \left(\frac{1}{4}\pi z^2 + \pi z^3\right) dz$$

$$= \left[\frac{1}{12}\pi z^3 + \frac{1}{4}\pi z^4\right]_0^1$$

$$= \frac{1}{12}\pi + \frac{1}{4}\pi$$

$$= \frac{\pi}{3}$$

Exercise

Evaluate the integral $\int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \int_0^{2\pi} (r\sin\theta + 1) r\ d\theta\ dz\ dr$

Solution

$$\int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \int_0^{2\pi} (r\sin\theta + 1) r\ d\theta dzdr = \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \left[-r\cos\theta + \theta\right]_0^{2\pi}\ r\ dzdr$$

$$= \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \left(-r + 2\pi - (-r)\right) r\ dzdr$$

$$= \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} 2\pi r\ dzdr$$

$$= 2\pi \int_0^2 r\left[z\right]_{r-2}^{\sqrt{4-r^2}}\ dr$$

$$= 2\pi \int_0^2 r\left[\left(4-r^2\right)^{1/2} - (r-2)\right] dr$$

$$= 2\pi \int_0^2 \left[r\left(4-r^2\right)^{1/2} - r^2 + 2r\right] dr \qquad d\left(4-r^2\right) = -2rdr$$

$$= 2\pi \left[-\frac{1}{3}\left(4-r^2\right)^{3/2} - \frac{1}{3}r^3 + r^2\right]_0^2$$

$$= 2\pi \left[-\frac{8}{3} + 4 - \left(-\frac{1}{3}(4)^{3/2}\right)\right]$$

$$= 2\pi\left(\frac{4}{3} + \frac{8}{3}\right)$$

$$\underline{= 8\pi}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{5}\int_0^{\frac{\pi}{2}}\int_0^3 r\cos\theta\ drd\theta dz$

Solution

$$\int_{-1}^{5}\int_0^{\frac{\pi}{2}}\int_0^3 r\cos\theta\ drd\theta dz = \int_{-1}^{5} dz \int_0^{\frac{\pi}{2}} \cos\theta d\theta \int_0^3 r\ dr$$

$$= z\Big|_{-1}^{5} \quad (\sin\theta)\Big|_0^{\frac{\pi}{2}} \quad \left(\frac{1}{2}r^2\right)\Big|_0^3$$

$$= \frac{1}{2}(5+1)(1)(9)$$

$$\underline{= 27}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\frac{\pi}{4}}\int_0^6\int_0^{6-r} rz\ dzdrd\theta$

Solution

$$\int_0^{\frac{\pi}{4}}\int_0^6\int_0^{6-r} rz\ dzdrd\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} d\theta \int_0^6 rz^2\Big|_0^{6-r}\ dr$$

$$= \frac{\pi}{8}\int_0^6 \left(36r - 12r^2 + r^3\right)dr$$

$$= \frac{\pi}{8}\left(18r^2 - 4r^3 + \frac{1}{4}r^4\right)\Big|_0^6$$

$$= \frac{\pi}{8}(648 - 864 + 324)$$

$$= \frac{27\pi}{2}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{2\cos^2\theta}\int_0^{4-r^2} r\sin\theta\ dzdrd\theta$

Solution

$$\int_0^{\frac{\pi}{2}}\int_0^{2\cos^2\theta}\int_0^{4-r^2} r\sin\theta\ dzdrd\theta = \int_0^{\frac{\pi}{2}}\int_0^{2\cos^2\theta} r\sin\theta\ z\Big|_0^{4-r^2}\ drd\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\cos^2\theta} \sin\theta\ \left(4r - r^3\right)drd\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta\ \left(2r^2 - \frac{1}{4}r^4\right)\Big|_0^{2\cos^2\theta}\ d\theta$$

$$= -\int_0^{\frac{\pi}{2}}\left(8\cos^4\theta - 4\cos^8\theta\right)d\left(\cos\theta\right)$$

$$= \left(\frac{4}{9}\cos^9\theta - \frac{8}{5}\cos^5\theta\right)\Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{4}{9} + \frac{8}{5}$$

$$= \frac{52}{45}$$

Exercise

Evaluate the integral $\int_0^4 \int_0^z \int_0^{\frac{\pi}{2}} re^r \ d\theta drdz$

Solution

$$\int_0^4 \int_0^z \int_0^{\frac{\pi}{2}} re^r \ d\theta drdz = \int_0^4 \int_0^z re^r \theta \Big|_0^{\frac{\pi}{2}} drdz$$

$$= \frac{\pi}{2} \int_0^4 \int_0^z re^r drdz$$

$$= \frac{\pi}{2} \int_0^4 \left(re^r - e^r \right) \Big|_0^z dz$$

$$= \frac{\pi}{2} \int_0^4 \left(ze^z - e^z + 1 \right) dz$$

$$= \frac{\pi}{2} \left(ze^z - e^z - e^z + z \right) \Big|_0^4$$

$$= \frac{\pi}{2} \left(4e^4 - 2e^4 + 4 + 2 \right)$$

$$= \frac{\pi}{2} \left(2e^4 + 6 \right)$$

$$= \pi \left(e^4 + 3 \right)$$

		$\int e^r$
+	r	e^r
−	1	e^r

Exercise

Evaluate the integral $\int_0^{\frac{\pi}{2}} \int_0^3 \int_0^{e^{-r^2}} r \ dzdrd\theta$

Solution

$$\int_0^{\frac{\pi}{2}} \int_0^3 \int_0^{e^{-r^2}} r \ dzdrd\theta = \int_0^{\frac{\pi}{2}} d\theta \int_0^3 rz \Big|_0^{e^{-r^2}} dr$$

$$= \frac{\pi}{2} \int_0^3 re^{-r^2} dr$$

$$= -\frac{\pi}{4} \int_0^3 e^{-r^2} d\left(-r^2\right)$$

$$= -\frac{\pi}{4} e^{-r^2} \Big|_0^3$$

$$\boxed{= \frac{\pi}{4}\left(1 - e^{-9}\right)}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{2\pi} \int_0^{\sqrt{5}} \int_0^{5-r^2} r\ dz\ dr\ d\theta$

Solution

$$\int_0^{2\pi} \int_0^{\sqrt{5}} \int_0^{5-r^2} r\ dz\ dr\ d\theta = \int_0^{2\pi} d\theta \int_0^{\sqrt{5}} r\ z \Big|_0^{5-r^2}\ dr$$

$$= 2\pi \int_0^{\sqrt{5}} \left(5r - r^3\right)\ dr$$

$$= 2\pi \left(\frac{5}{2} r^2 - \frac{1}{4} r^4\right) \Big|_0^{\sqrt{5}}$$

$$= 2\pi \left(\frac{25}{2} - \frac{25}{4}\right)$$

$$\boxed{= \frac{25\pi}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi} \int_0^{\cos\theta} \int_{2r^2}^{2r\cos\theta} r\ dz\ dr\ d\theta$

Solution

$$\int_0^{\pi} \int_0^{\cos\theta} \int_{2r^2}^{2r\cos\theta} r\ dz\ dr\ d\theta = \int_0^{\pi} \int_0^{\cos\theta} r\ z \Big|_{2r^2}^{2r\cos\theta}\ dr\ d\theta$$

$$= \int_0^{\pi} \int_0^{\cos\theta} \left(2r^2 \cos\theta - 2r^3\right) dr\ d\theta$$

$$= \int_0^{\pi} \left(\frac{2}{3} r^3 \cos\theta - \frac{1}{2} r^4\right) \Big|_0^{\cos\theta}\ d\theta$$

$$= \int_0^{\pi} \left(\frac{2}{3} \cos^4\theta - \frac{1}{2} \cos^4\theta\right) d\theta$$

$$=\frac{1}{6}\int_0^{\pi}\cos^4\theta\,d\theta$$

$$=\frac{1}{24}\int_0^{\pi}(1+\cos 2\theta)^2\,d\theta$$

$$=\frac{1}{24}\int_0^{\pi}\left(1+2\cos 2\theta+\cos^2 2\theta\right)d\theta$$

$$=\frac{1}{24}\int_0^{\pi}\left(\frac{3}{2}+2\cos 2\theta+\frac{1}{2}\cos 4\theta\right)d\theta$$

$$=\frac{1}{24}\left(\frac{3}{2}\theta+\sin 2\theta+\frac{1}{8}\sin 4\theta\right)\Big|_0^{\pi}$$

$$=\frac{\pi}{16}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$

Solution

$$\int_0^{\pi}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta=\int_0^{\pi}\int_0^{a\cos\theta} r\,z\Big|_0^{\sqrt{a^2-r^2}}\,dr\,d\theta$$

$$=\int_0^{\pi}\int_0^{a\cos\theta} r\left(a^2-r^2\right)^{1/2}dr\,d\theta$$

$$=-\frac{1}{2}\int_0^{\pi}\int_0^{a\cos\theta}\left(a^2-r^2\right)^{1/2}d\left(a^2-r^2\right)d\theta$$

$$=-\frac{1}{3}\int_0^{\pi}\left(a^2-r^2\right)^{3/2}\Big|_0^{a\cos\theta}\,d\theta$$

$$=-\frac{1}{3}\int_0^{\pi}\left[\left(a^2-a^2\cos^2\theta\right)^{3/2}-a^3\right]d\theta$$

$$=-\frac{1}{3}\int_0^{\pi}\left[a^3\left(1-\cos^2\theta\right)^{3/2}-a^3\right]d\theta$$

$$=-\frac{a^3}{3}\int_0^{\pi}\left[\left(\sin^2\theta\right)^{3/2}-1\right]d\theta$$

$$=\frac{a^3}{3}\int_0^{\pi}\left(1-\sin^3\theta\right)d\theta$$

$$=\frac{a^3}{3}\int_0^{\pi}d\theta-\frac{a^3}{3}\int_0^{\pi}\sin^2\theta\sin\theta\; d\theta$$

$$=\frac{a^3\pi}{3}+\frac{a^3}{3}\int_0^{\pi}\left(1-\cos^2\theta\right)\; d\left(\cos\theta\right)$$

$$=\frac{a^3\pi}{3}+\frac{a^3}{3}\left(\cos\theta-\frac{1}{3}\cos^3\theta\right)\Big|_0^{\pi}$$

$$=\frac{a^3\pi}{3}+\frac{a^3}{3}\left(-1+\frac{1}{3}-1+\frac{1}{3}\right)$$

$$=\frac{a^3\pi}{3}-\frac{4a^3}{9}$$

$$\boxed{=\frac{a^3}{9}\left(3\pi-4\right)}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a\cos\theta}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\; dz\; dr\; d\theta$

Solution

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a\cos\theta}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\; dz\; dr\; d\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a\cos\theta} rz\Big|_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}}\; dr\; d\theta$$

$$=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a\cos\theta} r\left(a^2-r^2\right)^{1/2}\; dr\; d\theta$$

$$=-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a\cos\theta}\left(a^2-r^2\right)^{1/2}\; d\left(a^2-r^2\right)d\theta$$

$$=-\frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(a^2-r^2\right)^{3/2}\Big|_0^{a\cos\theta}\; d\theta$$

$$=-\frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\left(a^2-a^2\cos^2\theta\right)^{3/2}-a^3\right)d\theta$$

$$= -\frac{2a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\left(\sin^2\theta\right)^{3/2}-1\right)d\theta$$

$$= -\frac{2a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\sin^3\theta-1\right)d\theta$$

$$= \frac{2a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta-\frac{2a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2\theta\sin\theta\;d\theta$$

$$= \frac{2a^3\pi}{3}+\frac{2a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1-\cos^2\theta\right)\;d\left(\cos\theta\right)$$

$$= \frac{2a^3\pi}{3}+\frac{2a^3}{3}\left(\cos\theta-\frac{1}{3}\cos^3\theta\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{2a^3\pi}{3}+\frac{2a^3}{3}\left(-1+\frac{1}{3}-1+\frac{1}{3}\right)$$

$$= \frac{2a^3\pi}{3}-\frac{8a^3}{9}$$

$$= \frac{2a^3}{9}(3\pi-4)$$

Exercise

Convert $\displaystyle\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} 3dz\; rdrd\theta, \qquad r\geq 0$

a) Rectangular coordinates with order of integration $dzdxdy$.

b) Spherical coordinates

c) Evaluate one of the integrals.

Solution

a) $z=r=\sqrt{x^2+y^2}$

$z=\sqrt{4-r^2}=\sqrt{4-x^2-y^2}$

$r\leq\sqrt{2} \quad\rightarrow\quad r^2\leq 2 \quad 0\leq\theta\leq 2\pi$

$x^2+y^2\leq 2 \quad\rightarrow\quad -\sqrt{2-y^2}\leq x\leq\sqrt{2-y^2}$

$x=0 \quad\rightarrow\quad -\sqrt{2}\leq y\leq\sqrt{2}$

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} 3dz\ rdrd\theta = \int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}}\int_{\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} 3\sqrt{x^2+y^2}\ dzdxdy$$

b) Spherical coordinates

$$\begin{cases} x=\rho\sin\varphi\cos\theta \\ y=\rho\sin\varphi\sin\theta \end{cases} \rightarrow \quad x^2+y^2=\rho^2\sin^2\varphi$$

$$0\le\theta\le 2\pi$$

$$z=\rho\cos\varphi=\sqrt{x^2+y^2}$$

$$\rho\cos\varphi=\rho\sin\varphi \quad\rightarrow\quad \boxed{\varphi=\frac{\pi}{4}}$$

$$\rho=\frac{r}{\sin\varphi}=\frac{r}{\sin\frac{\pi}{4}}=\frac{\sqrt{2}}{\frac{\sqrt{2}}{2}}=2$$

$$\boxed{0\le\rho\le 2}$$

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} 3dz\ rdrd\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{2} 3\rho^2\sin\varphi\ d\rho d\varphi d\theta$$

c) $$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} 3r\ dzdrd\theta = 3\int_0^{2\pi} d\theta\int_0^{\sqrt{2}} rz\Big|_r^{\sqrt{4-r^2}}\ dr$$

$$=3\theta\Big|_0^{2\pi}\int_0^{\sqrt{2}} r\left(\sqrt{4-r^2}-r\right)dr$$

$$=6\pi\int_0^{\sqrt{2}}\left(r\sqrt{4-r^2}-r^2\right)dr$$

$$=-3\pi\int_0^{\sqrt{2}}\left(4-r^2\right)^{1/2}d\left(4-r^2\right)-6\pi\int_0^{\sqrt{2}} r^2dr$$

$$=-2\pi\left(4-r^2\right)^{3/2}\Big|_0^{\sqrt{2}}-2\pi r^3\Big|_0^{\sqrt{2}}$$

$$=-2\pi\left(2\sqrt{2}-8\right)-4\pi\sqrt{2}$$

$$=-2\pi\left(2\sqrt{2}-8+2\sqrt{2}\right)$$

$$=-8\pi\left(\sqrt{2}-2\right)$$

$$\boxed{=8\pi\left(2-\sqrt{2}\right)}$$

Exercise

Convert the integral $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x}\left(x^2+y^2\right)dzdxdy$ to an equivalent integral in cylindrical coordinates and evaluate the result.

Solution

$$\int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{0}^{r\cos\theta} r^3 dzdrd\theta = \int_{-\pi/2}^{\pi/2}\int_{0}^{1} r^3\left[z\right]_0^{r\cos\theta} drd\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_{0}^{1} r^3 r\cos\theta drd\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{1}{5}r^5\cos\theta\right]_0^1 d\theta$$

$$= \frac{1}{5}\int_{-\pi/2}^{\pi/2}\cos\theta d\theta$$

$$= \frac{1}{5}\left[\sin\theta\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{5}(1+1)$$

$$= \frac{2}{5}$$

Exercise

Set up an integral in rectangular coordinates equivalent to the integral

$$\int_{0}^{\pi/2}\int_{1}^{\sqrt{3}}\int_{1}^{\sqrt{4-r^2}} r^3(\sin\theta\cos\theta)z^2\ dzdrd\theta$$

Arrange the order of integration to be z first, then y, then x.

Solution

$$\int_{0}^{\pi/2}\int_{1}^{\sqrt{3}}\int_{1}^{\sqrt{4-r^2}} r^2(\sin\theta\cos\theta)z^2\ dz\ rdr\ d\theta$$

$$r^2(\sin\theta\cos\theta)z^2 = (r\sin\theta)(r\cos\theta)z^2$$

$$= xyz^2$$

$$1 \le z \le \sqrt{4-r^2} \quad \rightarrow \quad 1 \le z \le \sqrt{4-x^2-y^2}$$

$1 \le r \le \sqrt{3}$

$$1 \le r^2 \le 3$$
$$1 \le x^2 + y^2 \le 3$$
$$1 - x^2 \le y^2 \le 3 - x^2$$
$$\sqrt{1-x^2} \le y \le \sqrt{3-x^2}$$

$0 \le \theta \le \frac{\pi}{2}$

$$\theta = 0 \to \begin{cases} r = 1 \;\Rightarrow\; x = r\cos\theta = 1 \\ r = \sqrt{3} \;\Rightarrow\; x = r\cos\theta = \sqrt{3} \end{cases}$$

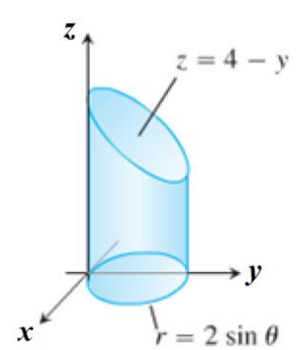
$$\theta = \frac{\pi}{2} \;\to\; x = r\cos\theta = 0$$

$$\int_0^{\pi/2}\int_1^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r^3(\sin\theta\cos\theta)z^2\, dzdrd\theta$$

$$= \int_0^1\int_{\sqrt{1-x^2}}^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} z^2yx\; dzdydx + \int_1^{\sqrt{3}}\int_0^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} z^2yx\; dzdydx$$

Exercise

Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)dzdrd\theta$ over the region D that is the right circular cylinder whose base is the circle $r = 2\sin\theta$ in the *xy*-plane and whose top lies in the plane $z = 4 - y$



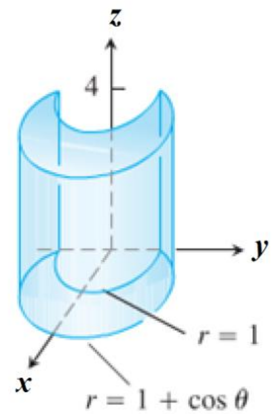
Solution

$$0 \le z \le 4 - y \Rightarrow 0 \le z \le 4 - r\sin\theta$$

$$\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{4-r\sin\theta} f(r,\theta,z)dz\; rdr\; d\theta$$

Exercise

Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dz\,dr\,d\theta$ over the region D which is the solid right cylinder whose base is the region in the xy-plane that lies inside the cardioid $r = 1+\cos\theta$ and outside the circle $r=1$ and whose top lies in the plane $z=4$



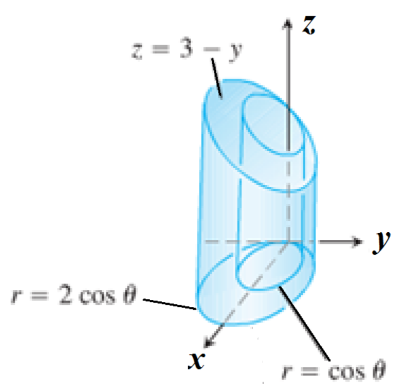
Solution

$$0 \le z \le 4 \quad 1 \le r \le 1+\cos\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta}\int_{0}^{4} f(r,\theta,z)\,dz\ r\,dr\ d\theta$$

Exercise

Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dz\,dr\,d\theta$ over the region D which is the solid right cylinder whose base is the region between the circles $r=\cos\theta$ and $r=2\cos\theta$ and whose top lies in the plane $z=3-y$

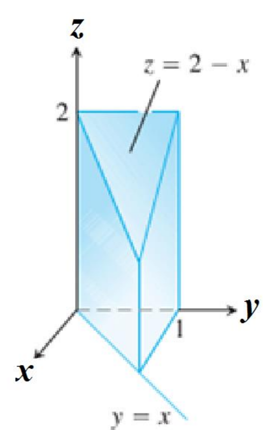


Solution

$$\int_{-\pi/2}^{\pi/2}\int_{\cos\theta}^{2\cos\theta}\int_{0}^{3-r\sin\theta} f(r,\theta,z)\,dz\ r\,dr\ d\theta$$

Exercise

Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dz\,dr\,d\theta$ over the region D which is the prism whose base is the triangle in the xy-plane bounded by the y-axis and the lines $y=x$ and $y=1$ and whose top lies in the plane $z=2-x$



Solution

$$0 \le z \le 2-x \quad \to 0 \le z \le 2 - r\cos\theta$$

$$y=1 \quad \to \quad r\sin\theta = 1 \to r = \frac{1}{\sin\theta} = \csc\theta$$

$$\int_{\pi/4}^{\pi/2}\int_{0}^{\csc\theta}\int_{0}^{2-r\sin\theta} f(r,\theta,z)\,dz\ r\,dr\ d\theta$$

Exercise

Evaluate the integrals in cylindrical coordinates. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^3\left(x^2+y^2\right)^{3/2}dzdydz$

Solution

$$\begin{cases} 0\le z\le 3 \\ 0\le y\le \sqrt{9-x^2} \end{cases} \rightarrow \quad 0\le r\le 3$$

$$0\le x\le 3 \quad (y\in QI) \quad \rightarrow 0\le\theta\le\frac{\pi}{2}$$

$$\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^3\left(x^2+y^2\right)^{3/2}dzdydz=\int_0^3\int_0^{\frac{\pi}{2}}\int_0^3\left(r^2\right)^{3/2}rdrd\theta dz$$

$$=\int_0^{\frac{\pi}{2}}d\theta\int_0^3dz\int_0^3r^4dr$$

$$=\theta\Big|_0^{\frac{\pi}{2}}\; z\Big|_0^3\;\frac{1}{5}r^5\Big|_0^3$$

$$=\theta\Big|_0^{\frac{\pi}{2}}\; z\Big|_0^3\;\frac{1}{5}r^5\Big|_0^3$$

$$=\frac{\pi}{2}(3)\frac{243}{5}$$

$$\boxed{=\frac{729\pi}{10}}$$

Exercise

Evaluate the integrals in cylindrical coordinates. $\displaystyle\int_{-2}^2\int_{-1}^1\int_0^{\sqrt{1-z^2}}\frac{1}{\left(1+x^2+z^2\right)^2}dxdydz$

Solution

$$0\le x\le\sqrt{1-z^2} \quad\rightarrow\quad 0\le r\le 1$$

$$-1\le y\le 1 \quad\rightarrow\quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$$

$$\int_{-2}^{2}\int_{-1}^{1}\int_{0}^{\sqrt{1-z^2}} \frac{1}{\left(1+x^2+z^2\right)^2}\,dxdydz = \int_{-2}^{2}dz\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_{0}^{1}\frac{1}{\left(1+r^2\right)^2}rdr$$

$$= z\Big|_{-2}^{2}\quad \theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\quad \frac{1}{2}\int_{0}^{1}\frac{1}{\left(1+r^2\right)^2}d\left(1+r^2\right)$$

$$= 4(\pi)\quad \left(-\frac{1}{2}\right)\frac{1}{1+r^2}\Big|_{0}^{1}$$

$$= -2\pi\left(\frac{1}{2}-1\right)$$

$$= \pi$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_{0}^{\pi}\int_{0}^{\pi}\int_{0}^{2\sin\phi}\rho^2\sin\phi\;d\rho\;d\phi\;d\theta$

Solution

$$\int_{0}^{\pi}\int_{0}^{\pi}\int_{0}^{2\sin\phi}\rho^2\sin\phi\;d\rho\;d\phi\;d\theta = \frac{1}{3}\int_{0}^{\pi}\int_{0}^{\pi}\sin\phi\left[\rho^3\right]_{0}^{2\sin\phi}\;d\phi\;d\theta$$

$$= \frac{8}{3}\int_{0}^{\pi}\int_{0}^{\pi}\sin^4\phi\;d\phi\;d\theta$$

$$\int \sin^4 x\;dx = \int\left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int\left(1-2\cos 2x+\cos^2 2x\right)dx$$

$$= \frac{1}{4}\int\left(1-2\cos 2x+\frac{1}{2}+\frac{1}{2}\cos 4x\right)dx$$

$$= \frac{1}{4}\int\left(\frac{3}{2}-2\cos 2x+\frac{1}{2}\cos 4x\right)dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x-\sin 2x+\frac{1}{8}\sin 4x\right)$$

$$= \frac{8}{3}\int_{0}^{\pi}\left[\frac{3}{8}\phi-\frac{1}{4}\sin 2\phi+\frac{1}{32}\sin 4\phi\right]_{0}^{\pi}d\theta$$

$$= \frac{8}{3}\int_{0}^{\pi}\left[\frac{3\pi}{8}\right]d\theta$$

$$= \pi[\theta]_0^{\pi}$$

$$\underline{= \pi^2}$$

Exercise

Evaluate the spherical coordinate integral $\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 (\rho\cos\phi)\rho^2\sin\phi\ d\rho\ d\phi\ d\theta$

Solution

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 (\rho\cos\phi)\rho^2\sin\phi\ d\rho d\phi d\theta = \int_0^{2\pi}\int_0^{\pi/4} (\cos\phi\sin\phi)\left[\frac{1}{4}\rho^4\right]_0^2\ d\phi d\theta$$

$$= 4\int_0^{2\pi}\int_0^{\pi/4} (\cos\phi\sin\phi)\ d\phi d\theta$$

$$= 4\int_0^{2\pi}\int_0^{\pi/4} \sin\phi\ d(\sin\phi) d\theta$$

$$= 2\int_0^{2\pi}\left[\sin^2\phi\right]_0^{\pi/4}\ d\theta$$

$$= \int_0^{2\pi} d\theta$$

$$\underline{= 2\pi}$$

Exercise

Evaluate the spherical coordinate integral $\int_0^{3\pi/2}\int_0^{\pi}\int_0^1 5\rho^3\sin^3\phi\ d\rho\ d\phi\ d\theta$

Solution

$$\int_0^{3\pi/2}\int_0^{\pi}\int_0^1 5\rho^3\sin^3\phi\ d\rho d\phi d\theta = \frac{5}{4}\int_0^{3\pi/2}\int_0^{\pi} \sin^3\phi\left[\rho^4\right]_0^1\ d\phi\ d\theta$$

$$= \frac{5}{4}\int_0^{3\pi/2}\int_0^{\pi} \sin^2\phi\sin\phi\ d\phi d\theta \qquad d(\cos\phi) = -\sin\phi$$

$$= -\frac{5}{4}\int_0^{3\pi/2}\int_0^{\pi} \left(1-\cos^2\phi\right)\ d(\cos\phi) d\theta$$

$$= -\frac{5}{4}\int_0^{3\pi/2}\left[\cos\phi - \frac{1}{3}\cos^3\phi\right]_0^{\pi} d\theta$$

$$= -\frac{5}{4}\int_0^{3\pi/2}\left(-1+\frac{1}{3}-\left(1-\frac{1}{3}\right)\right)d\theta$$

$$= \frac{5}{3}\int_0^{3\pi/2} d\theta$$

$$= \frac{5}{3}\left(\frac{3\pi}{2}\right)$$

$$= \frac{5\pi}{2}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\varphi} \rho^2 \sin\varphi \; d\rho d\varphi d\theta$

Solution

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\varphi} \rho^2 \sin\varphi \; d\rho d\varphi d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/2} \frac{1}{3}\sin\varphi \; \rho^3 \Big|_0^{2\cos\varphi} d\varphi$$

$$= \frac{8}{3}\theta \Big|_0^{2\pi} \int_0^{\pi/2} \sin\varphi \; \cos^3\varphi \; d\varphi$$

$$= -\frac{16\pi}{3}\int_0^{\pi/2} \cos^3\varphi \; d(\cos\varphi)$$

$$= -\frac{4\pi}{3}\cos^4\varphi \Big|_0^{\pi/2}$$

$$= \frac{4\pi}{3}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{\pi}\int_0^{\pi/4}\int_{2\sec\varphi}^{4\sec\varphi} \rho^2 \sin\varphi \; d\rho d\varphi d\theta$

Solution

$$\int_0^{\pi}\int_0^{\pi/4}\int_{2\sec\varphi}^{4\sec\varphi}\rho^2\sin\varphi\, d\rho d\varphi d\theta = \int_0^{\pi} d\theta \int_0^{\pi/4}\frac{1}{3}\sin\varphi\ \rho^3\Big|_{2\sec\varphi}^{4\sec\varphi}\ d\varphi$$

$$=\frac{1}{3}\theta\Big|_0^{\pi}\int_0^{\pi/4}\sin\varphi\left(64\sec^3\varphi-8\sec^3\varphi\right)d\varphi$$

$$=\frac{\pi}{3}\int_0^{\pi/4}\sin\varphi\left(56\sec^3\varphi\right)d\varphi$$

$$=-\frac{56\pi}{3}\int_0^{\pi/4}\cos^{-3}\varphi\ d\left(\cos\varphi\right)$$

$$=\frac{28\pi}{3}\frac{1}{\cos^2\varphi}\Big|_0^{\pi/4}$$

$$=\frac{28\pi}{3}(2-1)$$

$$=\frac{28\pi}{3}$$

Exercise

Evaluate the integral $\displaystyle\int_0^2\int_{-\pi}^0\int_{\pi/4}^{\pi/2}\rho^3\sin 2\phi\ d\phi\ d\theta\ d\rho$

Solution

$$\int_0^2\int_{-\pi}^0\int_{\pi/4}^{\pi/2}\rho^3\sin 2\phi\ d\phi d\theta d\rho = -\frac{1}{2}\int_0^2\int_{-\pi}^0\rho^3\left[\cos 2\phi\right]_{\pi/4}^{\pi/2}\,d\theta d\rho$$

$$=-\frac{1}{2}\int_0^2\int_{-\pi}^0\rho^3(-1-0)\,d\theta d\rho$$

$$=\frac{1}{2}\int_0^2\int_{-\pi}^0\rho^3 d\theta d\rho$$

$$=\frac{1}{2}\int_0^2\rho^3(\pi)\,d\rho$$

$$=\frac{\pi}{8}\left[\rho^4\right]_0^2$$

$$=2\pi$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2} 5\rho^4 \sin^3\phi \; d\rho \; d\theta \; d\phi$

Solution

$$\begin{aligned}
\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2} 5\rho^4 \sin^3\phi \; d\rho d\theta d\phi &= \int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2} \sin^3\phi \left[\rho^5\right]_{\csc\phi}^{2} \; d\theta d\phi \\
&= \int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2} \sin^3\phi \left(32 - \csc^5\phi\right) d\theta d\phi \\
&= \int_{\pi/6}^{\pi/2} \left(32\sin^3\phi - \csc^2\phi\right) [\theta]_{-\pi/2}^{\pi/2} \; d\phi \\
&= \pi \left(\int_{\pi/6}^{\pi/2} 32\sin^3\phi d\phi - \int_{\pi/6}^{\pi/2} \csc^2\phi d\phi \right) \\
&= 32\pi \int_{\pi/6}^{\pi/2} \sin^2\phi \sin\phi d\phi - \pi \int_{\pi/6}^{\pi/2} \csc^2\phi d\phi \\
&= 32\pi \int_{\pi/6}^{\pi/2} \left(1 - \cos^2\phi\right) \; d\left(\cos\phi\right) + \pi\left[\cot\phi\right]_{\pi/6}^{\pi/2} \\
&= 32\pi \left[\cos\phi - \tfrac{1}{3}\cos^3\phi\right]_{\pi/6}^{\pi/2} + \pi\left(-\sqrt{3}\right) \\
&= 32\pi \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}\right) - \pi\sqrt{3} \\
&= 12\pi\sqrt{3} - \pi\sqrt{3} \\
&= 11\pi\sqrt{3}
\end{aligned}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{3} \rho^2 \sin\phi \; d\rho d\phi d\theta$

Solution

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{3} \rho^2 \sin\phi \; d\rho d\phi d\theta = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\phi \; d\phi \int_0^{3} \rho^2 \; d\rho$$

$$= \theta \Big|_0^{2\pi} \quad \left(-\cos\phi\right)\Big|_0^{\frac{\pi}{4}} \quad \left(\frac{1}{3}\rho^3\right)\Big|_0^3$$

$$= (2\pi)\left(-\frac{1}{\sqrt{2}} + 1\right)(9)$$

$$\boxed{= 18\pi\left(1 - \frac{1}{\sqrt{2}}\right)}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^3 \rho^3 \cos\phi \sin\phi \; d\rho d\phi d\theta$

Solution

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^3 \rho^3 \cos\phi \sin\phi \; d\rho d\phi d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \; d\left(\sin\phi\right) \int_0^3 \rho^3 d\rho$$

$$= \theta \Big|_0^{2\pi} \quad \left(\frac{1}{2}\sin^2\phi\right)\Big|_0^{\pi/4} \quad \left(\frac{1}{4}\rho^4\right)\Big|_0^3$$

$$= (2\pi)\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{81}{4}\right)$$

$$\boxed{= \frac{81\pi}{8}}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{\sin\theta} 2\cos\phi \; \rho^2 \; d\rho d\theta d\phi$

Solution

$$\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{\sin\theta} 2\cos\phi \; \rho^2 \; d\rho d\theta d\phi = \frac{2}{3}\int_0^{\frac{\pi}{2}} \cos\phi d\phi \int_0^{\pi} \rho^3 \Big|_0^{\sin\theta} d\theta$$

$$= \frac{2}{3}\sin\phi \Big|_0^{\frac{\pi}{2}} \int_0^{\pi} \sin^3\theta \; d\theta$$

$$= \frac{2}{3}\int_0^{\pi} \sin^2\theta \sin\theta \; d\theta$$

$$= -\frac{2}{3}\int_0^{\pi} \left(1 - \cos^2\theta\right) \; d\left(\cos\theta\right)$$

$$= -\frac{2}{3}\left(\cos\theta - \frac{1}{3}\cos^3\theta\right)\Big|_0^{\pi}$$

$$= -\frac{2}{3}\left(-1 + \frac{1}{3} - 1 + \frac{1}{3}\right)$$

$$= -\frac{2}{3}\left(-\frac{4}{3}\right)$$

$$\boxed{= \frac{8}{9}}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{2} e^{-\rho^3}\rho^2\, d\rho d\theta d\varphi$

Solution

$$\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{2} e^{-\rho^3}\rho^2\, d\rho d\theta d\varphi = -\frac{1}{3}\int_0^{\frac{\pi}{2}} d\varphi \int_0^{\pi} d\theta \int_0^{2} e^{-\rho^3}\, d\left(-\rho^3\right)$$

$$= -\frac{1}{3}\left(\frac{\pi}{2}\right)(\pi)\, e^{-\rho^3}\Big|_0^{2}$$

$$= -\frac{\pi^2}{6}\left(e^{-8} - 1\right)$$

$$\boxed{= \frac{\pi^2}{6}\left(1 - e^{-8}\right)}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\varphi} \rho^2 \sin\phi\, d\rho d\phi d\theta$

Solution

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\varphi} \rho^2 \sin\phi\, d\rho d\phi d\theta = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\phi\left(\frac{1}{3}\rho^3\right)\Big|_0^{\cos\varphi} d\phi$$

$$= \frac{2\pi}{3}\int_0^{\frac{\pi}{4}} \sin\phi\left(\cos^3\varphi\right) d\phi$$

$$= -\frac{2\pi}{3}\int_0^{\frac{\pi}{4}} \left(\cos^3\varphi\right) d\left(\cos\phi\right)$$

$$= -\frac{\pi}{6}\left(\cos^4 \varphi\right)\Big|_0^{\frac{\pi}{4}}$$

$$= -\frac{\pi}{6}\left(\frac{1}{4} - 1\right)$$

$$= \frac{\pi}{8}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\int_0^{\cos\theta} \rho^2 \sin\varphi \cos\varphi \, d\rho d\theta d\varphi$

Solution

$$\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\int_0^{\cos\theta} \rho^2 \sin\varphi \cos\varphi \, d\rho d\theta d\varphi = \int_0^{\frac{\pi}{4}} \frac{1}{2}\sin 2\varphi \, d\varphi \int_0^{\frac{\pi}{4}} \frac{1}{3}\rho^3 \Big|_0^{\cos\theta} \, d\theta$$

$$= -\frac{1}{12}\cos 2\varphi \Big|_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \cos^3\theta \, d\theta$$

$$= -\frac{1}{12}(-1) \int_0^{\frac{\pi}{4}} \left(1 - \sin^2\theta\right) d(\sin\theta)$$

$$= \frac{1}{12} \left(\sin\theta - \frac{1}{3}\sin^3\theta\right)\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{12}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12}\right)$$

$$= \frac{5\sqrt{2}}{144}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{4} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$

Solution

$$\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{4} \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \int_0^{2\pi} d\theta \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin\varphi d\varphi \left(\frac{1}{3}\rho^3\right)\Big|_0^4$$

$$= \frac{64}{3}(2\pi)(-\cos\varphi)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$\boxed{= \frac{64\pi\sqrt{3}}{3}}$$

Exercise

Evaluate the spherical coordinate integral $\int_0^{2\pi}\int_0^{\pi}\int_0^{5} \rho^2 \sin\varphi \; d\rho d\varphi d\theta$

Solution

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{5} \rho^2 \sin\varphi \; d\rho d\varphi d\theta = \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \; \left(\frac{1}{3}\rho^3\right)\Big|_0^5$$

$$= \frac{125}{3}(2\pi)(-\cos\varphi)\Big|_0^{\pi}$$

$$\boxed{= \frac{500\pi}{3}}$$

Exercise

Evaluate the spherical coordinate integral $\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{\sin\theta} 2\cos\varphi \; \rho^2 \; d\rho d\theta d\varphi$

Solution

$$\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{\sin\theta} 2\cos\varphi \; \rho^2 \; d\rho d\theta d\varphi = \int_0^{\frac{\pi}{2}} \cos\varphi \; d\varphi \; \int_0^{\pi} \frac{2}{3}\rho^3 \Big|_0^{\sin\theta} \; d\theta$$

$$= \frac{2}{3}\sin\varphi\Big|_0^{\frac{\pi}{2}} \; \int_0^{\pi} \sin^3\theta \; d\theta$$

$$= \frac{2}{3} \int_0^{\pi} \sin^2\theta \sin\theta \; d\theta$$

$$= -\frac{2}{3} \int_0^{\pi} \left(1-\cos^2\theta\right) \; d(\cos\theta)$$

$$= -\frac{2}{3} \left(\cos\theta - \frac{1}{3}\cos^3\theta\right)\Big|_0^{\pi}$$

$$= -\frac{2}{3}\left(-1+\frac{1}{3}-1+\frac{1}{3}\right)$$

$$= \frac{8}{9}$$

Exercise

Evaluate the integral $\displaystyle\int_0^4 \int_0^{\frac{\sqrt{2}}{2}} \int_x^{\sqrt{1-x^2}} e^{-x^2-y^2}\, dydxdz$

Solution

$y = \sqrt{1-x^2} \quad \to \quad x^2 + y^2 = 1 = r^2$

$$\begin{cases} x = 0 = \cos\theta & \to \theta = \frac{\pi}{2} \\ x = \frac{\sqrt{2}}{2} = \cos\theta & \to \theta = \frac{\pi}{4} \end{cases} \quad \to \quad \frac{\pi}{4} \le \theta \le \frac{\pi}{2}$$

$0 \le z \le 4$

$$\int_0^4 \int_0^{\frac{\sqrt{2}}{2}} \int_x^{\sqrt{1-x^2}} e^{-x^2-y^2}\, dydxdz = \int_0^4 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 e^{-r^2}\, r\, drd\theta dz$$

$$= -\frac{1}{2} \int_0^4 dz \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_0^1 e^{-r^2}\, d\left(-r^2\right)$$

$$= -\frac{1}{2} z \Big|_0^4 \quad \theta \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \quad e^{-r^2} \Big|_0^1$$

$$= -\frac{1}{2}(4)\left(\frac{\pi}{4}\right)\left(e^{-1} - 1\right)$$

$$= \frac{\pi}{2}\left(1 - e^{-1}\right)$$

Exercise

Evaluate the integral $\displaystyle\int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{\sqrt{x^2+y^2}}^4 dzdydx$

Solution

$\sqrt{x^2 + y^2} \le z \le 4 \quad \to \quad r \le z \le 4$

$y = \sqrt{16 - x^2} \quad \to x^2 + y^2 = 16 = r^2$

$0 \le r \le 4$

$-4 \le x \le 4 \quad \to \quad 0 \le \theta \le 2\pi$

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{\sqrt{x^2+y^2}}^{4} dzdydx = \int_0^{2\pi}\int_0^4\int_r^4 dz\ rdrd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^4 z\Big|_r^4\ r\ dr$$

$$= 2\pi\int_0^4 \left(4r - r^2\right)\ dr$$

$$= 2\pi\left(2r^2 - \frac{1}{3}r^3\right)\Big|_0^4$$

$$= 2\pi\left(32 - \frac{64}{3}\right)$$

$$= \frac{64\pi}{3}$$

Exercise

Evaluate the integral $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{x^2+y^2}} \left(x^2+y^2\right)^{-1/2} dzdydx$

Solution

$$0 \le z \le \sqrt{x^2+y^2} \quad \to \quad 0 \le z \le r$$

$$y = \sqrt{9-x^2} \quad \to \quad x^2+y^2 = 9 = r^2 \qquad \underline{0 \le r \le 3}$$

$$\begin{cases} x = 0 = 3\cos\theta & \to \theta = \frac{\pi}{2} \\ x = 3 = 3\cos\theta & \to \theta = 0 \end{cases} \quad \to \quad 0 \le \theta \le \frac{\pi}{2}$$

$$\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{x^2+y^2}} \left(x^2+y^2\right)^{-1/2} dzdydx = \int_0^{\frac{\pi}{2}}\int_0^3\int_0^r \frac{1}{r} dz\ r\ drd\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^3 z\Big|_0^r\ dr$$

$$= \frac{\pi}{2}\int_0^3 r\ dr$$

$$= \frac{\pi}{4}r^2\Big|_0^3$$

$$= \frac{9\pi}{4}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{1}\int_{0}^{\frac{1}{2}}\int_{\sqrt{3}y}^{\sqrt{1-y^2}} \sqrt{x^2+y^2}\ dxdydz$

Solution

$$-1 \le z \le 1$$

$$y=\sqrt{1-x^2} \quad \to \ x^2+y^2=1=r^2 \qquad \underline{0\le r\le 1}$$

$$\begin{cases} y=0=\sin\theta & \to \theta=0 \\ y=\frac{1}{2}=\sin\theta & \to \theta=\frac{\pi}{6} \end{cases} \quad \to \quad 0\le\theta\le\frac{\pi}{6}$$

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{\frac{1}{2}}\int_{\sqrt{3}y}^{\sqrt{1-y^2}} \sqrt{x^2+y^2}\ dxdydz &= \int_{-1}^{1}\int_{0}^{\frac{\pi}{6}}\int_{0}^{1} r\ rdrd\theta dz \\ &= \int_{-1}^{1} dz \int_{0}^{\frac{\pi}{6}} d\theta \int_{0}^{1} r^2 dr \\ &= z\Big|_{-1}^{1}\ \ \theta\Big|_{0}^{\frac{\pi}{6}}\ \ \frac{1}{3}r^3\Big|_{0}^{1} \\ &= (2)\left(\frac{\pi}{6}\right)\left(\frac{1}{3}\right) \\ &= \frac{\pi}{9} \end{aligned}$$

Exercise

Evaluate $\displaystyle\iiint_D \left(x^2+y^2+z^2\right)^{5/2} dV$; D is the unit ball.

Solution

$$\begin{aligned}\iiint_D \left(x^2+y^2+z^2\right)^{5/2} dV &= \int_0^{2\pi}\int_0^{\pi}\int_0^{1}\left(\rho^2\right)^{5/2}\rho^2 \sin\varphi d\rho d\varphi d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \int_0^1 \rho^7\ d\rho \\ &= 2\pi\left(-\cos\varphi\right)\Big|_0^{\pi}\ \left(\frac{1}{8}\rho^8\right)\Big|_0^1 \\ &= 2\pi(2)\left(\frac{1}{8}\right)\end{aligned}$$

$$=\boxed{\frac{\pi}{2}}$$

Exercise

Evaluate $\iiint\limits_D e^{-\left(x^2+y^2+z^2\right)^{3/2}}\,dV$; D is the unit ball.

Solution

$$\iiint\limits_D e^{-\left(x^2+y^2+z^2\right)^{3/2}}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^1 e^{-\rho^3}\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi} d\theta \int_0^{\pi}\sin\varphi\, d\varphi \int_0^1 e^{-\rho^3}\, d\left(-\rho^3\right)$$

$$= -\frac{2\pi}{3}(-\cos\varphi)\Big|_0^{\pi}\left(e^{-\rho^3}\right)\Big|_0^1$$

$$= -\frac{2\pi}{3}(2)\left(e^{-1}-1\right)$$

$$=\boxed{\frac{4\pi}{3}\left(1-e^{-1}\right)}$$

Exercise

Evaluate $\iiint\limits_D \frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}\,dV$; D is the solid between the spheres of radius 1 and 2 centered at the origin.

Solution

$$\iiint\limits_D \left(x^2+y^2+z^2\right)^{-3/2}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_1^2 \left(\rho^{-3}\right)\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi}\sin\varphi\, d\varphi \int_1^2 \frac{1}{\rho}\, d\rho$$

$$= 2\pi(-\cos\varphi)\Big|_0^{\pi}(\ln\rho)\Big|_1^2$$

$$= 2\pi(2)(\ln 2)$$

$$=\boxed{4\pi\ln 2}$$

Exercise

Evaluate $\iiint\limits_D \left(x^2+y^2+z^2\right)^{-3/2} dV$, where D is the region in the first octant between two spheres of radius 1 and 2 centered at the origin.

Solution

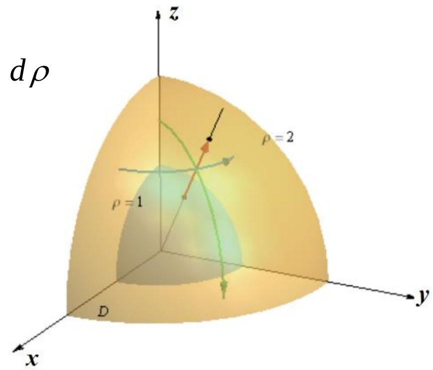
$$\iiint\limits_D \left(x^2+y^2+z^2\right)^{-3/2} dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \left(\rho^{-3}\right)\rho^2 \sin\varphi d\rho d\varphi d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin\varphi d\varphi \int_1^2 \frac{1}{\rho}\, d\rho$$

$$= \frac{\pi}{2}(-\cos\varphi)\Big|_0^{\pi/2} (\ln\rho)\Big|_1^2$$

$$= \frac{\pi}{2}(1)(\ln 2)$$

$$= \frac{\pi}{2}\ln 2$$



Exercise

Evaluate $\iiint\limits_D x^2 dV$; $D = \{(r,\ \theta,\ z):\ \ 0 \le r \le 1,\ \ 0 \le z \le 2r,\ \ 0 \le \theta \le 2\pi\}$

Solution

$$\iiint\limits_D x^2 dV = \int_0^{2\pi}\int_0^1\int_0^{2r} r^2\cos^2\theta\ r\ dz\ dr\ d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(1+\cos 2\theta)\, d\theta \int_0^1 r^3 z\Big|_0^{2r}\ dr$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} \int_0^1 2r^4\ dr$$

$$= (2\pi)\left(\frac{1}{5}r^5\right)\Big|_0^1$$

$$= \frac{2\pi}{5}$$

Exercise

Evaluate $\displaystyle\iiint_D dV$; $D = \left\{(r,\ \theta,\ z):\ \ 0 \le r \le 1,\ \ -\sqrt{4-r^2} \le z \le \sqrt{4-r^2},\ \ 0 \le \theta \le 2\pi\right\}$

Solution

$$\iiint_D dV = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\ dz\ dr\ d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 rz \Big|_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dr$$

$$= 2\pi \int_0^1 2r\left(4-r^2\right)^{1/2} dr$$

$$= -2\pi \int_0^1 \left(4-r^2\right)^{1/2} d\left(4-r^2\right)$$

$$= -\frac{4}{3}\pi\left(4-r^2\right)^{3/2} \Big|_0^1$$

$$= -\frac{4\pi}{3}\left(3^{3/2}-8\right)$$

$$\boxed{= \frac{4\pi}{3}\left(8-3\sqrt{3}\right)}$$

Exercise

Evaluate $\displaystyle\iiint_D dV$; $D = \left\{(r,\ \theta,\ z):\ \ 0 \le r \le 1,\ \ r \le z \le \sqrt{2-r^2},\ \ 0 \le \theta \le 2\pi\right\}$

Solution

$$\iiint_D dV = \int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} r\ dz\ dr\ d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 rz \Big|_r^{\sqrt{2-r^2}} dr$$

$$= 2\pi \int_0^1 \left(r\left(2-r^2\right)^{1/2} - r^2\right) dr$$

$$= -\pi \int_0^1 \left(2-r^2\right)^{1/2} d\left(2-r^2\right) - 2\pi \int_0^1 r^2 dr$$

$$= -\frac{2\pi}{3}\left(2-r^2\right)^{3/2} \Big|_0^1 - \frac{2\pi}{3} r^3 \Big|_0^1$$

$$= -\frac{2\pi}{3}\left(1-2\sqrt{2}\right) - \frac{2\pi}{3}$$

$$= -\frac{2\pi}{3} + \frac{4\pi\sqrt{2}}{3} - \frac{2\pi}{3}$$

$$= \frac{2\pi}{3}\left(\sqrt{2}-1\right)$$

Exercise

Evaluate $\iiint\limits_D dV$; $D = \left\{(r,\ \theta,\ z):\quad 0 \le r \le 1,\quad r^2 \le z \le \sqrt{2-r^2},\quad 0 \le \theta \le 2\pi\right\}$

Solution

$$\iiint\limits_D dV = \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 rz \Big|_{r^2}^{\sqrt{2-r^2}} dr$$

$$= 2\pi \int_0^1 \left(r\left(2-r^2\right)^{1/2} - r^3 \right) dr$$

$$= -\pi \int_0^1 \left(2-r^2\right)^{1/2} d\left(2-r^2\right) - 2\pi \int_0^1 r^3 dr$$

$$= -\frac{2\pi}{3}\left(2-r^2\right)^{3/2} \Big|_0^1 - \frac{\pi}{2} r^4 \Big|_0^1$$

$$= -\frac{2\pi}{3}\left(1-2\sqrt{2}\right) - \frac{\pi}{2}$$

$$= -\frac{2\pi}{3} + \frac{4\pi\sqrt{2}}{3} - \frac{\pi}{2}$$

$$= \left(\frac{4}{3}\sqrt{2} - \frac{7}{6}\right)\pi$$

Exercise

Evaluate $\iiint\limits_D dV$; $D=\left\{(r,\ \theta,\ z):\ \ 0\le r\le 4,\quad 2r\le z\le 24-r^2,\quad 0\le\theta\le 2\pi\right\}$

Solution

$$\iiint\limits_D dV=\int_0^{2\pi}\int_0^4\int_{2r}^{24-r^2} r\ dz\ dr\ d\theta$$

$$=\int_0^{2\pi}d\theta\ \int_0^4 rz\Big|_{2r}^{24-r^2}\ dr$$

$$=2\pi\int_0^4\left(24r-r^3-2r^2\right)dr$$

$$=2\pi\left(12r^2-\frac{1}{4}r^4-\frac{2}{3}r^3\right)\Big|_0^4$$

$$=2\pi\left(192-64-\frac{128}{3}\right)$$

$$=\frac{512\pi}{3}$$

Exercise

Evaluate $\iiint\limits_D y^2z^2dV$; $D=\left\{(\rho,\ \varphi,\ \theta):\ \ 0\le\rho\le 1,\quad 0\le\varphi\le\frac{\pi}{3},\quad 0\le\theta\le 2\pi\right\}$

Solution

$$x=\rho\sin\varphi\cos\theta\qquad y=\rho\sin\varphi\sin\theta\qquad z=\rho\cos\varphi$$

$$\iiint\limits_D y^2z^2dV=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^1\left(\rho^2\sin^2\varphi\sin^2\theta\right)\left(\rho^2\cos^2\varphi\right)\rho^2\sin\varphi\ d\rho d\varphi d\theta$$

$$=\int_0^{2\pi}\sin^2\theta\ d\theta\ \int_0^{\frac{\pi}{3}}\sin^3\varphi\cos^2\varphi\ d\varphi\ \int_0^1\rho^6\ d\rho$$

$$=\frac{1}{2}\int_0^{2\pi}(1-\cos 2\theta)\ d\theta\ \int_0^{\frac{\pi}{3}}\sin^2\varphi\cos^2\varphi\sin\varphi\ d\varphi\ \left(\frac{1}{7}\rho^7\right)\Big|_0^1$$

$$=\frac{1}{14}\left(\theta-\frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi}\ \int_0^{\frac{\pi}{3}}-\left(1-\cos^2\varphi\right)\cos^2\varphi\ d\left(\cos\varphi\right)$$

$$= \frac{\pi}{7} \int_0^{\frac{\pi}{3}} \left(\cos^4 \varphi - \cos^2 \varphi \right) d\left(\cos \varphi\right)$$

$$= \frac{\pi}{7} \left(\frac{1}{5} \cos^5 \varphi - \frac{1}{3} \cos^3 \varphi \right) \Big|_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{7} \left(\frac{1}{5} \left(\frac{1}{2} \right)^5 - \frac{1}{3}\frac{1}{8} - \frac{1}{5} + \frac{1}{3} \right)$$

$$= \frac{\pi}{7} \left(\frac{1}{160} - \frac{1}{24} + \frac{2}{15} \right)$$

$$= \frac{47\pi}{3360}$$

Exercise

Evaluate $\iiint\limits_D \left(x^2 + y^2\right) dV$; $D = \left\{ (\rho,\ \varphi,\ \theta):\ \ 2 \le \rho \le 3,\ \ \ 0 \le \varphi \le \pi,\ \ \ 0 \le \theta \le 2\pi \right\}$

Solution

$$x = \rho \sin \varphi \cos \theta \quad y = \rho \sin \varphi \sin \theta \quad z = \rho \cos \varphi$$

$$\iiint\limits_D \left(x^2 + y^2\right) dV = \int_0^{2\pi} \int_0^{\pi} \int_2^3 \left(\rho^2 \sin^2 \varphi \cos^2 \theta + \rho^2 \sin^2 \varphi \sin^2 \theta \right) \rho^2 \sin \varphi \ d\rho d\varphi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_2^3 \sin^2 \varphi \left(\cos^2 \theta + \sin^2 \theta \right) \rho^4 \sin \varphi \ d\rho d\varphi d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin^2 \varphi \sin \varphi \ d\varphi \int_2^3 \rho^4 \ d\rho$$

$$= 2\pi \int_0^{\pi} -\left(1 - \cos^2 \varphi\right) d\left(\cos \varphi\right) \ \left(\frac{1}{5} \rho^5 \right) \Big|_2^3$$

$$= \frac{2\pi}{5} \left(\frac{1}{3} \cos^3 \varphi - \cos \varphi \right) \Big|_0^{\pi} \ \ (243 - 32)$$

$$= \frac{422\pi}{5} \left(-\frac{1}{3} + 1 - \frac{1}{3} + 1 \right)$$

$$= \frac{1688\pi}{15}$$

Exercise

Evaluate $\iiint\limits_D y^2 dV$; $D = \left\{(\rho,\ \varphi,\ \theta):\ \ 0 \le \rho \le 3,\ \ \ 0 \le \varphi \le \pi,\ \ \ 0 \le \theta \le \pi\right\}$

Solution

$$x = \rho \sin\varphi \cos\theta \quad y = \rho \sin\varphi \sin\theta \quad z = \rho \cos\varphi$$

$$\iiint\limits_D y^2 dV = \int_0^{\pi}\int_0^{\pi}\int_0^{3}\left(\rho^2 \sin^2\varphi \sin^2\theta\right)\rho^2 \sin\varphi \ d\rho d\varphi d\theta$$

$$= \int_0^{\pi} \sin^2\theta \ d\theta \ \int_0^{\pi} \sin^2\varphi \sin\varphi \ d\varphi \ \int_0^{3} \rho^4 \ d\rho$$

$$= \frac{1}{2}\int_0^{\pi} (1-\cos 2\theta) \ d\theta \ \int_0^{\pi}\left(\cos^2\varphi - 1\right) \ d\left(\cos\varphi\right) \ \int_0^{3} \rho^4 \ d\rho$$

$$= \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi} \ \left(\frac{1}{3}\cos^3\varphi - \cos\varphi\right)\Big|_0^{\pi} \ \left(\frac{1}{5}\rho^5\right)\Big|_0^{3}$$

$$= \frac{1}{2}(\pi)\left(-\frac{1}{3}+1-\frac{1}{3}+1\right)\left(\frac{243}{5}\right)$$

$$= \frac{243\pi}{10}\left(\frac{4}{3}\right)$$

$$= \frac{162\pi}{5}$$

Exercise

Evaluate $\iiint\limits_D x e^{x^2+y^2+z^2} dV$; $D = \left\{(\rho,\ \varphi,\ \theta):\ \ 0 \le \rho \le 1,\ \ \ 0 \le \varphi \le \frac{\pi}{2},\ \ \ 0 \le \theta \le \frac{\pi}{2}\right\}$

Solution

$$x = \rho \sin\varphi \cos\theta \quad y = \rho \sin\varphi \sin\theta \quad z = \rho \cos\varphi$$

$$x^2 + y^2 + z^2 = \rho^2$$

$$\iiint\limits_D x e^{x^2+y^2+z^2} dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{1}\left(\rho \sin\varphi \cos\theta \ e^{\rho^2}\right)\rho^2 \sin\varphi \ d\rho d\varphi d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta d\theta \ \int_0^{\frac{\pi}{2}} \sin^2\varphi d\varphi \ \int_0^{1} \rho^3 e^{\rho^2} \ d\rho$$

$$u = \rho^2 \quad dv = \rho e^{\rho^2} d\rho = \frac{1}{2} e^{\rho^2} d\rho^2$$

$$du = 2\rho\, d\rho \qquad v = \frac{1}{2}e^{\rho^2}$$

$$\int \rho^3 e^{\rho^2}\, d\rho = \frac{1}{2}\rho^2 e^{\rho^2} - \int \rho e^{\rho^2}\, d\rho$$

$$= \frac{1}{2}\rho^2 e^{\rho^2} - \frac{1}{2}e^{\rho^2}$$

$$\iiint\limits_D x e^{x^2+y^2+z^2}\, dV = \sin\theta \Big|_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2\varphi)\, d\varphi \ \left(\frac{1}{2}\rho^2 e^{\rho^2} - \frac{1}{2}e^{\rho^2}\right)\Big|_0^1$$

$$= \frac{1}{2}(\varphi - \sin 2\varphi)\Big|_0^{\frac{\pi}{2}} \ \left(\frac{1}{2}e - \frac{1}{2}e + \frac{1}{2}\right)$$

$$= \frac{1}{4}\left(\frac{\pi}{2}\right)$$

$$\boxed{= \frac{\pi}{8}}$$

Exercise

Evaluate $\iiint\limits_D \sqrt{x^2+y^2+z^2}\, dV$; $D = \left\{(\rho,\ \varphi,\ \theta):\ 1 \le \rho \le 2,\quad 0 \le \varphi \le \frac{\pi}{4},\quad 0 \le \theta \le 2\pi\right\}$

Solution

$$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$$

$$x^2 + y^2 + z^2 = \rho^2$$

$$\iiint\limits_D \sqrt{x^2+y^2+z^2}\, dV = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^2 \rho^3 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\varphi\, d\varphi \int_1^2 \rho^3 d\rho$$

$$= 2\pi(-\cos\varphi)\Big|_0^{\frac{\pi}{4}} \ \left(\frac{1}{4}\rho^4\right)\Big|_1^2$$

$$= \frac{1}{2}\left(-\frac{\sqrt{2}}{2}+1\right)(16-1)$$

$$\boxed{= \frac{15\pi}{2}\left(1-\frac{\sqrt{2}}{2}\right)}$$

Exercise

Find the volume of the solid whose height is 4 and whose base is the disk $\{(r,\ \theta):\ \ 0 \le r \le 2\cos\theta\}$

Solution

Base is the disk $\Rightarrow\ 0 \le \theta \le \pi$

$0 \le z \le 4$

$$V = \int_0^4 \int_0^\pi \int_0^{2\cos\theta} r\ drd\theta dz$$

$$= \frac{1}{2}\int_0^4 dz \int_0^\pi r^2 \Big|_0^{2\cos\theta}\ d\theta$$

$$= 8\int_0^\pi \cos^2\theta\ d\theta$$

$$= 4\int_0^\pi (1+\cos 2\theta)\ d\theta$$

$$= 4\left(1+\frac{1}{2}\sin 2\theta\right)\Big|_0^\pi$$

$$\underline{= 4\pi\ \ unit^3}$$

Exercise

Find the volume of the solid in the first octant bounded by the cylinder $r = 1$ and the plane $z = x$

Solution

$0 \le z \le x = r\cos\theta \qquad\qquad 0 \le r \le 1$

first octant $0 \le \theta \le \frac{\pi}{2}$

$$V = \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{r\cos\theta} dz\ rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 z\Big|_0^{r\cos\theta} rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta d\theta \int_0^1 r^2 dr$$

$$= \sin\theta\Big|_0^{\frac{\pi}{2}}\ \left(\frac{1}{3}r^3\right)\Big|_0^1$$

$$= \boxed{\frac{1}{3}\ unit^3}$$

Exercise

Find the volume of the solid bounded by the cylinder $r = 1$ and $r = 2$ and the planes $z = 4 - x - y$ and $z = 0$

Solution

$r = 1$ and $r = 2 \quad \to \quad 1 \le r \le 2$

$z = 4 - x - y \to \quad 0 \le z \le 4 - r\cos\theta - r\sin\theta$

$0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi} \int_1^2 \int_0^{4 - r\cos\theta - \sin\theta} dz\ r dr d\theta$$

$$= \int_0^{2\pi} \int_1^2 z \Big|_0^{4 - r\cos\theta - r\sin\theta} r dr d\theta$$

$$= \int_0^{2\pi} \int_1^2 \left(4r - r^2 \cos\theta - r^2 \sin\theta\right) dr d\theta$$

$$= \int_0^{2\pi} \int_1^2 \left(4r - r^2 (\cos\theta + \sin\theta)\right) dr d\theta$$

$$= \int_0^{2\pi} \left(2r^2 - \frac{1}{3} r^3 (\cos\theta + \sin\theta)\right) \Big|_1^2 d\theta$$

$$= \int_0^{2\pi} \left(8 - \frac{8}{3}(\cos\theta + \sin\theta) - 2 + \frac{1}{3}(\cos\theta + \sin\theta)\right) d\theta$$

$$= \int_0^{2\pi} \left(6 - \frac{7}{3}(\cos\theta + \sin\theta)\right) d\theta$$

$$= \left(6\theta - \frac{7}{3}(\sin\theta - \cos\theta)\right) \Big|_0^{2\pi}$$

$$= 12\pi + \frac{7}{3} - \frac{7}{3}$$

$$= \boxed{12\pi\ unit^3}$$

Exercise

Find the volume of the solid D between the cone $z=\sqrt{x^2+y^2}$ and the inverted paraboloid $z=12-x^2-y^2$

Solution

$$\begin{cases} z=\sqrt{x^2+y^2}=r \\ z=12-x^2-y^2=12-r^2 \end{cases} \rightarrow \boxed{r\le z\le 12-r^2}$$

$$12-r^2=r \quad \rightarrow \quad r^2+r-12=0$$

$$\Rightarrow\ r=3,\ \cancel{-4} \quad \rightarrow \quad 0\le r\le 3$$

$$V=\int_0^{2\pi}\int_0^{3}\int_r^{12-r^2} dz\ rdrd\theta$$

$$=\int_0^{2\pi} d\theta\int_0^3 z\Big|_r^{12-r^2}\ rdr$$

$$=2\pi\int_0^3\left(12r-r^3-r^2\right)dr$$

$$=2\pi\left(6r^2-\frac{1}{4}r^4-\frac{1}{3}r^3\right)\Big|_0^3$$

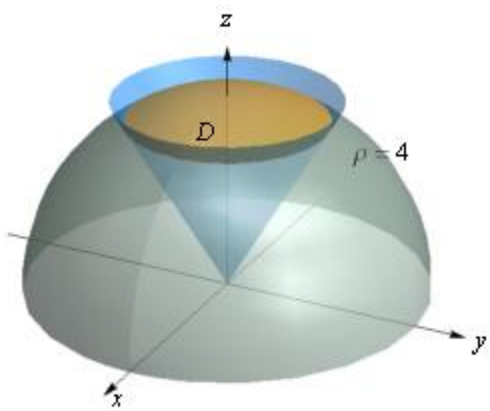
$$=2\pi\left(54-\frac{81}{4}-9\right)$$

$$\boxed{=\frac{99\pi}{2}\ unit^3}$$

Exercise

Find the volume of the solid region D that lies inside the cone $\phi=\frac{\pi}{6}$ and inside the sphere $\rho=4$

Solution



$$V=\int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{4}\rho^2\sin\varphi\ d\rho d\varphi d\theta$$

$$=\int_0^{2\pi} d\theta\int_0^{\frac{\pi}{6}}\sin\varphi d\varphi\int_0^4\rho^2\ d\rho$$

$$=2\pi\left(-\cos\varphi\right)\Big|_0^{\frac{\pi}{6}}\ \left(\frac{1}{3}\rho^3\right)\Big|_0^4$$

$$= \frac{2\pi}{3}\left(-\frac{\sqrt{3}}{2}+1\right)(64)$$

$$= \boxed{\frac{64\pi}{3}\left(2-\sqrt{3}\right)\ unit^3}$$

Exercise

Find the volume of the solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2, \ \ z \ge 0$

Solution

$$V = \int_0^{2\pi}\int_0^{\pi/2}\int_{\cos\phi}^{2} \rho^2 \sin\phi \ d\rho d\phi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \sin\phi\left[\rho^3\right]_{\cos\phi}^{2} \ d\phi d\theta \qquad d(\cos\phi) = -\sin\phi$$

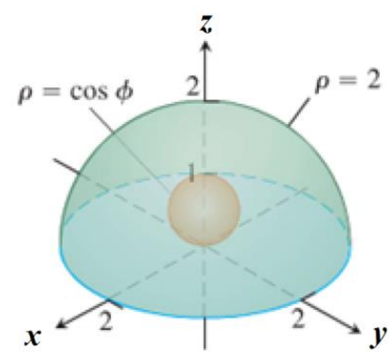
$$= -\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2}\left(8-\cos^3\phi\right) \ d(\cos\phi) d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi}\left[8\cos\phi - \frac{1}{4}\cos^4\phi\right]_0^{\pi/2} d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi}\left(8-\frac{1}{4}\right)d\theta$$

$$= \frac{31}{12}\left[\theta\right]_0^{2\pi}$$

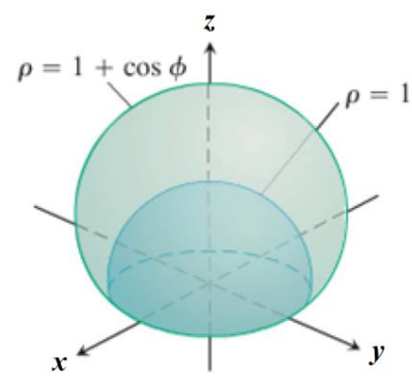
$$= \boxed{\frac{31\pi}{6}\ unit^3}$$



Exercise

Find the volume of the solid bounded below by the hemisphere $\rho = 1, \ \ z \ge 0$, and above the cardioid of revolution $\rho = 1+\cos\phi$

Solution



$$V = \int_0^{2\pi}\int_0^{\pi/2}\int_{1}^{1+\cos\phi} \rho^2 \sin\phi \ d\rho d\phi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \sin\phi\left[\rho^3\right]_{1}^{1+\cos\phi} \ d\phi d\theta$$

$$=\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2}\sin\phi\left[(1+\cos\phi)^3-1\right]d\phi d\theta$$

$$=-\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2}\left[(1+\cos\phi)^3-1\right]d(1+\cos\phi)d\theta$$

$$=-\frac{1}{3}\int_0^{2\pi}\left[\frac{1}{4}(1+\cos\phi)^4-(1+\cos\phi)\right]_0^{\pi/2}d\theta$$

$$=-\frac{1}{3}\int_0^{2\pi}\left[\frac{1}{4}-1-\left(\frac{1}{4}(2)^4-(1+1)\right)\right]d\theta$$

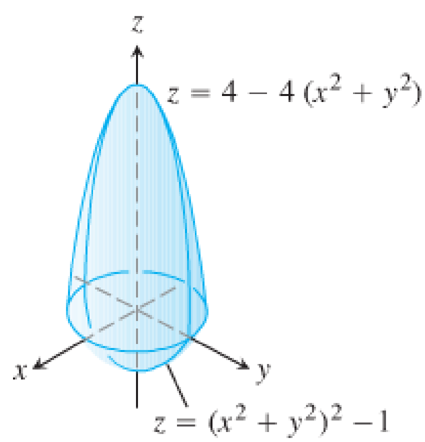
$$=\frac{11}{12}\int_0^{2\pi}d\theta$$

$$=\frac{11}{12}\left[\theta\right]_0^{2\pi}$$

$$=\frac{11\pi}{6}\ unit^3$$

Exercise

Find the volume of the solid



Solution

a) $\left(x^2+y^2\right)^2-1\le z\le 4-4\left(x^2+y^2\right)\quad ;\quad x^2+y^2=r^2$

$r^4-1\le z\le 4-4r$

$4-4r=0\to r=1 \qquad 0\le r\le 1$

$0\le\theta\le 2\pi \quad \to (4)\quad 0\le\theta\le\frac{\pi}{2}$

$$V=4\int_0^{\pi/2}\int_0^1\int_{r^4-1}^{4-4r^2}dz\ rdrd\theta$$

$$=4\int_0^{\pi/2}\int_0^1\left[\left(4-4r^2\right)-\left(r^4-1\right)\right]rdrd\theta$$

$$=4\int_0^{\pi/2}\int_0^1\left(5-4r^2-r^4\right)rdrd\theta$$

$$=4\int_0^{\pi/2}\int_0^1\left(5r-4r^3-r^5\right)drd\theta$$

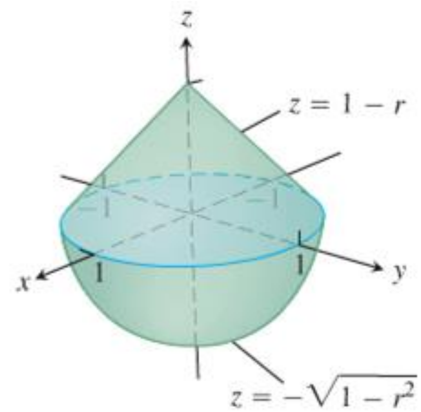
$$= 4\int_0^{\pi/2} \left[\frac{5}{2}r^2 - r^4 - \frac{1}{6}r^6\right]_0^1 d\theta$$

$$= 4\left(\frac{5}{2} - 1 - \frac{1}{6}\right)\int_0^{\pi/2} d\theta$$

$$= \frac{16}{3}\left[\theta\right]_0^{\pi/2}$$

$$\boxed{= \frac{8\pi}{3}\ unit^3}$$

b) $$V = 4\int_0^{\pi/2}\int_0^1\int_{-\sqrt{1-r^2}}^{1-r} dz\ rdrd\theta$$



$$= 4\int_0^{\pi/2}\int_0^1 \left[(1-r) + \sqrt{1-r^2}\right] rdrd\theta$$

$$= 4\int_0^{\pi/2}\int_0^1 \left[r - r^2 + r\left(1-r^2\right)^{1/2}\right] drd\theta$$

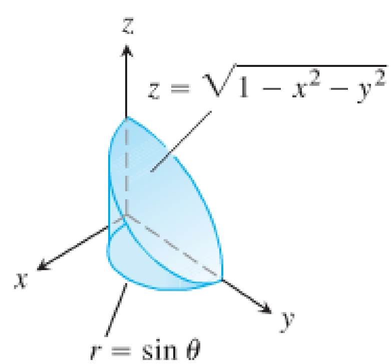
$$= 4\int_0^{\pi/2}\left(\left[\frac{1}{2}r^2 - \frac{1}{3}r^3\right]_0^1 - \frac{1}{2}\int_0^1 \left(1-r^2\right)^{1/2} d\left(1-r^2\right)\right) d\theta$$

$$= 4\int_0^{\pi/2}\left(\left(\frac{1}{2} - \frac{1}{3}\right) - \frac{1}{3}\left[\left(1-r^2\right)^{3/2}\right]_0^1\right) d\theta$$

$$= 4\left(\frac{1}{6} + \frac{1}{3}\right)\int_0^{\pi/2} d\theta$$

$$= 2\left[\theta\right]_0^{\pi/2}$$

$$\boxed{= \pi\ unit^3}$$



c) $0 \le z \le \sqrt{1-x^2-y^2} = \sqrt{1-r^2}$; $x^2 + y^2 = r^2$

$$V = \int_0^{\pi/2}\int_0^{\sin\theta}\int_0^{\sqrt{1-r^2}} dz\ rdrd\theta$$

$$= \int_0^{\pi/2}\int_0^{\sin\theta} \sqrt{1-r^2}\ rdrd\theta$$

$d\left(1-r^2\right) = -2rdr$

$$= -\frac{1}{2}\int_0^{\pi/2}\int_0^{\sin\theta} \left(1-r^2\right)^{1/2} d\left(1-r^2\right) d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left[\frac{2}{3}\left(1-r^2\right)^{3/2}\right]_0^{\sin\theta} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}\left[\left(1-\sin^2\theta\right)^{3/2}-1\right]d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}\left[\left(\cos^2\theta\right)^{3/2}-1\right]d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}\left(\cos^3\theta-1\right)d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}\cos^2\theta\cos\theta d\theta+\frac{1}{3}\int_0^{\pi/2}d\theta \qquad d\left(\sin\theta\right)=\cos\theta d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}\left(1-\sin^2\theta\right)d\left(\sin\theta\right)+\frac{\pi}{6}$$

$$= -\frac{1}{3}\left[\sin\theta-\frac{1}{3}\sin^3\theta\right]_0^{\pi/2}+\frac{\pi}{6}$$

$$= -\frac{1}{3}\left(1-\frac{1}{3}\right)+\frac{\pi}{6}$$

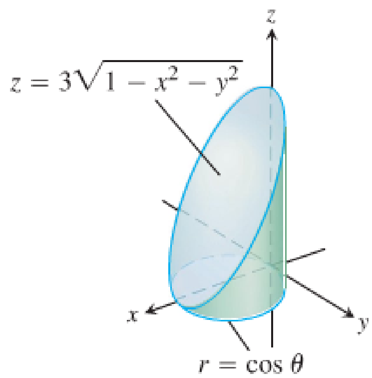
$$= -\frac{2}{9}+\frac{\pi}{6}$$

$$\boxed{= \frac{3\pi-4}{18}\ unit^3}$$

d) $$V = \int_0^{\pi/2}\int_0^{\cos\theta}\int_0^{3\sqrt{1-r^2}} dz\ rdrd\theta$$

$$= \int_0^{\pi/2}\int_0^{\cos\theta} 3r\sqrt{1-r^2}drd\theta \qquad d\left(1-r^2\right)=-2rdr$$

$$= -\frac{3}{2}\int_0^{\pi/2}\int_0^{\cos\theta}\left(1-r^2\right)^{1/2}drd\theta$$



$$= -\int_0^{\pi/2}\left[\left(1-r^2\right)^{3/2}\right]_0^{\cos\theta} d\theta$$

$$= -\int_0^{\pi/2}\left[\left(1-\cos^2\theta\right)^{3/2}-1\right]d\theta$$

$$= -\int_0^{\pi/2}\left(\sin^3\theta-1\right)d\theta$$

$$= -\int_0^{\pi/2} \sin^2\theta \sin\theta d\theta + \int_0^{\pi/2} d\theta \qquad\qquad d(\cos\theta) = -\sin\theta d\theta$$

$$= \int_0^{\pi/2} \left(1-\cos^2\theta\right) d(\cos\theta) + \left[\theta\right]_0^{\pi/2}$$

$$= \left[\cos\theta - \tfrac{1}{3}\cos^3\theta\right]_0^{\pi/2} + \frac{\pi}{2}$$

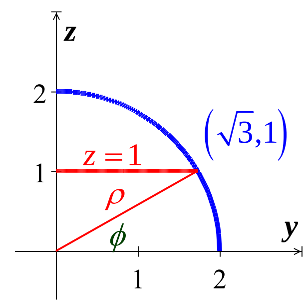
$$= -1 + \frac{1}{3} + \frac{\pi}{2}$$

$$= \frac{3\pi - 4}{6}\ unit^3$$

Exercise

Find the volume of the smaller region cut from the solid sphere $\rho \le 2$ by the plane $z = 1$

Solution



$$\cos\phi = \frac{z}{\rho} \Rightarrow \rho = \frac{1}{\cos\phi} = \sec\phi$$

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} \rho^2 \sin\phi \, d\rho d\phi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/3} \sin\phi \left[\rho^3\right]_{\sec\phi}^{2} \, d\phi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/3} \sin\phi \left[8 - \sec^3\phi\right] d\phi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/3} \left(8\sin\phi - \tan\phi\sec^2\phi\right) d\phi d\theta \qquad\qquad d(\tan\phi) = \sec^2\phi d\phi$$

$$= \frac{1}{3}\int_0^{2\pi}\left(\int_0^{\pi/3} 8\sin\phi \, d\phi - \int_0^{\pi/3} \tan\phi \, d(\tan\phi)\right) d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \left[-8\cos\phi - \tfrac{1}{2}\tan^2\phi\right]_0^{\pi/3} d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \left[-4 - \tfrac{1}{2}(3) - (-8-0)\right] d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \frac{5}{2} d\theta$$

$$= \frac{5}{6}[\theta]_0^{2\pi}$$

$$\boxed{= \frac{5\pi}{3}\ unit^3}$$

Exercise

Find the volume of the region bounded below by the paraboloid $z = x^2 + y^2$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2 + 1$

Solution

$$x^2 + y^2 \le z \le x^2 + y^2 + 1 \quad \rightarrow \quad r^2 \le z \le r^2 + 1$$

$$x^2 + y^2 = 1 = r^2 \quad \rightarrow \quad 0 \le r \le 1$$

$$0 \le \theta \le 2\pi$$

$$V = 4\int_0^{\pi/2} \int_0^1 \int_{r^2}^{r^2+1} dz\ rdrd\theta$$

$$= 4\int_0^{\pi/2} \int_0^1 \left[r^2 + 1 - r^2\right] rdrd\theta$$

$$= 4\int_0^{\pi/2} \int_0^1 rdrd\theta$$

$$= 2\int_0^{\pi/2} \left[r^2\right]_0^1 d\theta$$

$$= 2\int_0^{\pi/2} d\theta$$

$$= 2[\theta]_0^{\pi/2}$$

$$= 2\left(\frac{\pi}{2}\right)$$

$$\boxed{= \pi\ unit^3}$$

Exercise

Find the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$

Solution

$$V = 8\int_0^{\pi/2}\int_1^{\sqrt{2}}\int_0^{\sqrt{2-r^2}} dz\ r dr d\theta$$

$$= 8\int_0^{\pi/2}\int_1^{\sqrt{2}} r[z]_0^{\sqrt{2-r^2}}\ dr d\theta$$

$$= 8\int_0^{\pi/2}\int_1^{\sqrt{2}} r\sqrt{2-r^2}\ dr d\theta \qquad d\left(1-r^2\right) = -2rdr$$

$$= -4\int_0^{\pi/2}\int_1^{\sqrt{2}} \left(2-r^2\right)^{1/2}\ d\left(2-r^2\right) d\theta$$

$$= -\frac{8}{3}\int_0^{\pi/2}\left[\left(2-r^2\right)^{3/2}\right]_1^{\sqrt{2}}\ d\theta$$

$$= -\frac{8}{3}\int_0^{\pi/2}(-1)\,d\theta$$

$$= \frac{8}{3}[\theta]_0^{\pi/2}$$

$$= \frac{8}{3}\left(\frac{\pi}{2}\right)$$

$$\boxed{= \frac{4\pi}{3}\ \ unit^3}$$

Exercise

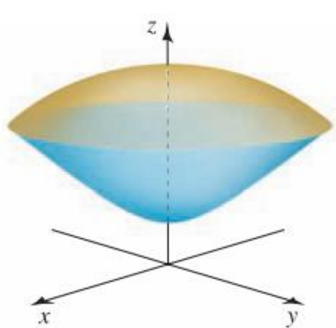
Find the volume of the solid between the sphere $x^2 + y^2 + z^2 = 19$ and the hyperboloid $z^2 - x^2 - y^2 = 1$ for $z > 0$

Solution

$$z = \sqrt{19 - x^2 - y^2} \qquad z = \sqrt{1 + x^2 + y^2}$$

$$19 - x^2 - y^2 = 1 + x^2 + y^2 \ \Rightarrow\ 2y^2 = 18 - 2x^2 \Rightarrow\ \ y = \pm\sqrt{9 - x^2}$$

$$9 - x^2 = 0 \quad \to \quad -3 \le x \le 3$$



$$V = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{\sqrt{1+x^2+y^2}}^{\sqrt{19-x^2-y^2}} 1\ dzdydx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left(\sqrt{19-x^2-y^2} - \sqrt{1+x^2+y^2}\right) dydx$$

Convert to ***Polar coordinates***

$$= \int_{0}^{2\pi}\int_{0}^{3} \left(\sqrt{19-r^2} - \sqrt{1+r^2}\right) r\ drd\theta$$

$$= \int_{0}^{2\pi} d\theta \left(-\frac{1}{2}\int_{0}^{3} \left(19-r^2\right)^{1/2} d\left(19-r^2\right) - \frac{1}{2}\int_{0}^{3}\left(1+r^2\right)^{1/2} d\left(1+r^2\right) \right)$$

$$= 2\pi \left(-\frac{1}{3}\left(19-r^2\right)^{3/2} - \frac{1}{3}\left(1+r^2\right)^{3/2} \right)_{0}^{3}$$

$$= -\frac{2}{3}\pi\left(10\sqrt{10} + 10\sqrt{10} - 19\sqrt{19} - 1\right)$$

$$= \frac{2\pi}{3}\left(1 + 19\sqrt{19} - 20\sqrt{10}\right)\ unit^3$$

Exercise

Evaluate the integral in cylindrical coordinates $\int_{0}^{2\pi}\int_{0}^{1}\int_{-1}^{1} r\ dzdrd\theta$

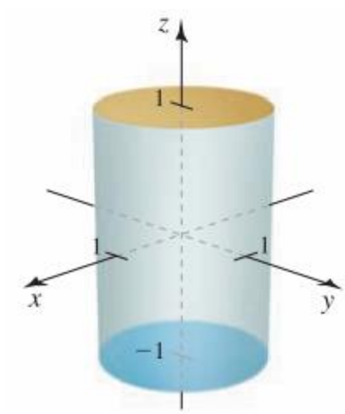
Solution

$$\int_{0}^{2\pi}\int_{0}^{1}\int_{-1}^{1} r\ dzdrd\theta = \int_{0}^{2\pi} d\theta \int_{0}^{1} r\ dr \int_{-1}^{1} dz$$

$$= (2\pi)\left(\frac{1}{2}r^2\right)\Big|_{0}^{1}\ z\Big|_{-1}^{1}$$

$$= (2\pi)\left(\frac{1}{2}\right)(2)$$

$$= 2\pi$$



Exercise

Evaluate the integral in cylindrical coordinates $\int_{0}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{0}^{9-3\sqrt{x^2+y^2}} dzdxdy$

Solution

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^{9-3\sqrt{x^2+y^2}} dzdxdy = \int_0^{\pi} \int_0^3 \int_0^{9-3r} r\ dzdrd\theta$$

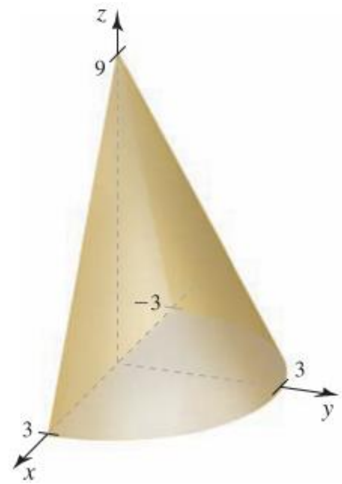
$$= \int_0^{\pi} d\theta \int_0^3 rz \Big|_0^{9-3r}\ dr$$

$$= \pi \int_0^3 \left(9r - 3r^2\right)\ dr$$

$$= \pi \left(\frac{9}{2}r^2 - r^3\right)\Big|_0^3$$

$$= \pi \left(\frac{81}{2} - 27\right)$$

$$= \frac{27}{2}\pi$$



Exercise

Evaluate the integral in cylindrical coordinates $\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{-1}^{1} \left(x^2 + y^2\right)^{3/2}\ dzdxdy$

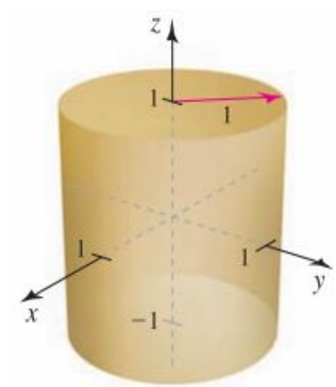
Solution

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{-1}^{1} \left(x^2 + y^2\right)^{3/2}\ dzdxdy = \int_0^{2\pi} \int_0^1 \int_{-1}^{1} r^3\ dz\ rdrd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 r^4 dr\ \left(z\right)\Big|_{-1}^{1}$$

$$= 2\pi \left(\frac{1}{5}r^5\right)\Big|_0^1 (2)$$

$$= \frac{4\pi}{5}$$

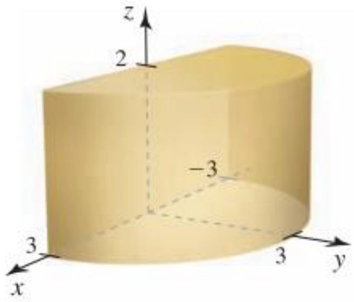


Exercise

Evaluate the integral in cylindrical coordinates $\displaystyle\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^2 \frac{1}{1+x^2+y^2}\ dzdydx$

Solution

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{2}\frac{1}{1+x^2+y^2}\,dzdydx=\int_{0}^{\pi}\int_{0}^{3}\int_{0}^{2}\frac{1}{1+r^2}\,dz\ rdrd\theta$$



$$=\frac{1}{2}\int_{0}^{\pi}d\theta\int_{0}^{3}\frac{1}{1+r^2}d\left(1+r^2\right)\ \left[z\right]\Big|_{0}^{2}$$

$$=\pi\ln\left(1+r^2\right)\Big|_{0}^{3}$$

$$=\pi\ln(10)$$

Exercise

Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the plane $z=0$ and the hyperboloid $z=\sqrt{17}-\sqrt{1+x^2+y^2}$

Solution

$$z=\sqrt{17}-\sqrt{1+x^2+y^2}=0\quad\rightarrow\quad 17=1+x^2+y^2$$

$$x^2+y^2=16=r^2$$

$$V=\int_{0}^{2\pi}\int_{0}^{4}\int_{0}^{\sqrt{17}-\sqrt{1+r^2}}1dz\ rdrd\theta$$

$$=\int_{0}^{2\pi}d\theta\int_{0}^{4}z\Big|_{0}^{\sqrt{17}-\sqrt{1+r^2}}rdr$$

$$=2\pi\int_{0}^{4}\left(\sqrt{17}-\sqrt{1+r^2}\right)rdr$$

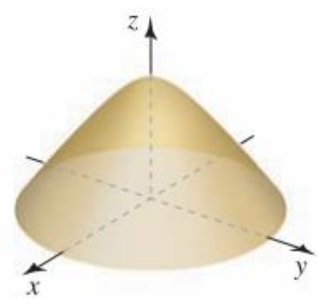
$$=2\pi\int_{0}^{4}\left(\sqrt{17}r-r\sqrt{1+r^2}\right)dr$$

$$=2\pi\left(\frac{1}{2}\sqrt{17}r^2\Big|_{0}^{4}-\frac{1}{2}\int_{0}^{4}\sqrt{1+r^2}\ d\left(1+r^2\right)\right)$$

$$=\pi\left(16\sqrt{17}-\frac{2}{3}\left(1+r^2\right)^{3/2}\Big|_{0}^{4}\right)$$

$$=\pi\left(16\sqrt{17}-\frac{2}{3}17\sqrt{17}+\frac{2}{3}\right)$$

$$=\pi\left(\frac{14\sqrt{17}+2}{3}\right)\ unit^3$$



$z=\sqrt{17}-\sqrt{1+x^2+y^2}$

Exercise

Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the plane $z = 25$ and the paraboloid $z = x^2 + y^2$

Solution

$$z = x^2 + y^2 = r^2 = 25 \ \rightarrow \ r = 5$$

$$V = \int_0^{2\pi} \int_0^5 \int_{r^2}^{25} 1dz \ rdrd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^5 z \Big|_{r^2}^{25} \ rdr$$

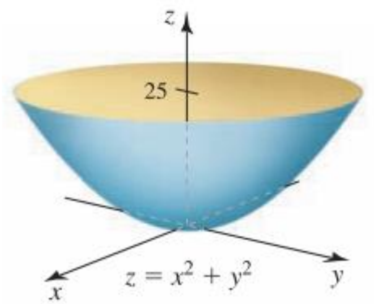
$$= 2\pi \int_0^5 \left(25 - r^2\right) r \ dr$$

$$= 2\pi \int_0^5 \left(25r - r^3\right) dr$$

$$= 2\pi \left(\frac{25}{2} r^2 - \frac{1}{4} r^4\right) \Big|_0^5$$

$$= 2\pi \left(\frac{1}{2} 5^4 - \frac{1}{4} 5^4\right)$$

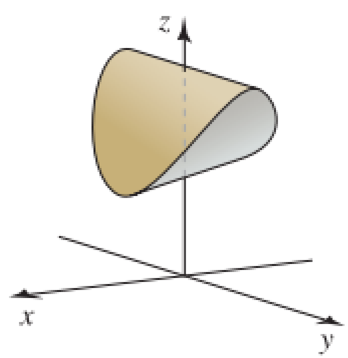
$$= \frac{625\pi}{2} \quad unit^3$$



Exercise

Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the parabolic cylinders $z = y^2 + 1$ and $z = 2 - x^2$

Solution



$$2 - x^2 - \left(y^2 + 1\right) = 1 - \left(x^2 + y^2\right)$$

$$z = y^2 + 1 = 2 - x^2 \ \rightarrow \ x^2 + y^2 = 1 = r^2$$

$$V = \int_0^{2\pi} \int_0^1 \left(1 - r^2\right) rdrd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 \left(r - r^3\right) dr$$

$$= 2\pi\left(\frac{1}{2}r^2 - \frac{1}{4}r^4\right)\Big|_0^1$$

$$= 2\pi\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{\pi}{2} \quad unit^3$$

Exercise

Evaluate the integral $\int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\sec\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$

Solution

$$\int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\sec\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/3} \frac{1}{3}\sin\varphi \ \rho^3 \Big|_0^{4\sec\varphi} \ d\varphi$$

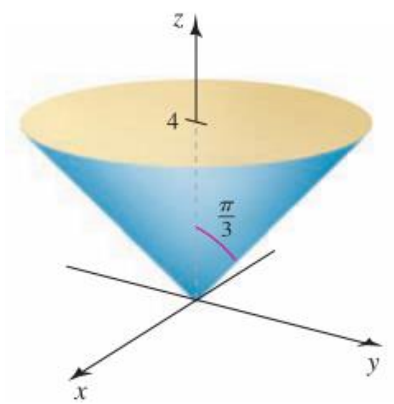
$$= \frac{128\pi}{3}\int_0^{\pi/3} \sin\varphi \ \sec^3\varphi \, d\varphi$$

$$= -\frac{128\pi}{3}\int_0^{\pi/3} \cos^{-3}\varphi \, d(\cos\varphi)$$

$$= \frac{64\pi}{3}\frac{1}{\cos^2\varphi}\Big|_0^{\pi/3}$$

$$= \frac{64\pi}{3}(4-1)$$

$$= 64\pi$$



Exercise

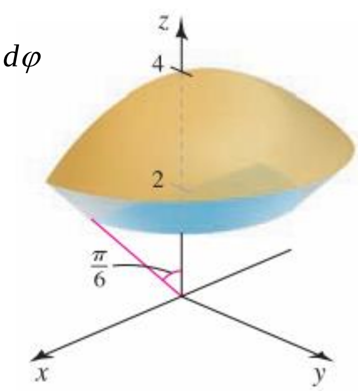
Evaluate the integral $\int_0^{\pi}\int_0^{\pi/6}\int_{2\sec\varphi}^{4} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$

Solution

$$\int_0^{\pi}\int_0^{\pi/6}\int_{2\sec\varphi}^{4} \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \frac{1}{3}\int_0^{\pi} d\theta \int_0^{\pi/6} \sin\varphi \ \rho^3 \Big|_{2\sec\varphi}^{4} \ d\varphi$$

$$= \frac{\pi}{3}\int_0^{\pi/6} \sin\varphi\left(64 - 8\sec^3\varphi\right) d\varphi$$

$$= \frac{8\pi}{3}\int_0^{\pi/6} \left(\cos^{-3}\varphi - 8\right) d(\cos\varphi)$$



$$= \frac{8\pi}{3}\left(\frac{-1}{2\cos^2\varphi} - 8\cos\varphi\right)\Big|_0^{\pi/6}$$

$$= \frac{8\pi}{3}\left(-\frac{2}{3} - 4\sqrt{3} + \frac{1}{2} + 8\right)$$

$$= \frac{8\pi}{3}\left(\frac{47}{3} - 4\sqrt{3}\right)$$

$$= \left(\frac{188}{9} - \frac{32}{3}\sqrt{3}\right)\pi$$

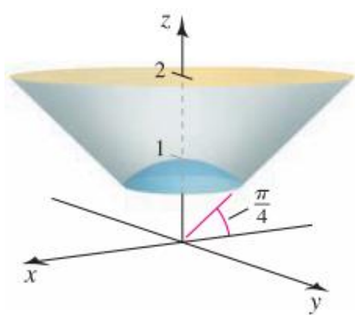
Exercise

Evaluate the integral $\int_0^{2\pi}\int_0^{\pi/4}\int_1^{2\sec\varphi}\left(\rho^{-3}\right)\rho^2\sin\varphi\; d\rho d\varphi d\theta$

Solution

$$\int_0^{2\pi}\int_0^{\pi/4}\int_1^{2\sec\varphi}\left(\rho^{-3}\right)\rho^2\sin\varphi\; d\rho d\varphi d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/4}\int_1^{2\sec\varphi} \sin\varphi\left(\frac{1}{\rho}\, d\rho\right)d\varphi$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\varphi \ln(\rho)\Big|_1^{2\sec\varphi} d\varphi$$



$$= 2\pi\int_0^{\pi/4} \sin\varphi\; \ln(2\sec\varphi)d\varphi$$

$$u = \ln(2\sec\varphi) \qquad dv = \sin\varphi d\varphi$$

$$du = \frac{2\sec\varphi\tan\varphi}{2\sec\varphi} = \tan\varphi \qquad v = -\cos\varphi$$

$$= 2\pi\left[-\cos\varphi\; \ln(2\sec\varphi)\Big|_0^{\pi/4} + \int_0^{\pi/4}\sin\varphi d\varphi\right]$$

$$= 2\pi\left(-\cos\varphi\; \ln(2\sec\varphi) - \cos\varphi\right)\Big|_0^{\pi/4}$$

$$= 2\pi\left(-\frac{\sqrt{2}}{2}\ln\left(2\sqrt{2}\right) - \frac{\sqrt{2}}{2} + \ln 2 + 1\right)$$

$$= 2\pi\left(\ln 2 - \frac{\sqrt{2}}{2}\ln\left(2\sqrt{2}\right) + 1 - \frac{\sqrt{2}}{2}\right)$$

Exercise

Evaluate the integral $\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_0^{2\csc\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$

Solution

$$\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_0^{2\csc\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \frac{1}{3}\int_0^{2\pi} d\theta \int_{\pi/6}^{\pi/3} \sin\varphi\left(\rho^3\right)\Big|_0^{2\csc\varphi} d\varphi$$

$$= \frac{16\pi}{3}\int_{\pi/6}^{\pi/3} \sin\varphi \csc^3\varphi \, d\varphi$$

$$= -\frac{16\pi}{3}\int_{\pi/6}^{\pi/3} \sin\varphi \csc\varphi \, d(\cot\varphi)$$

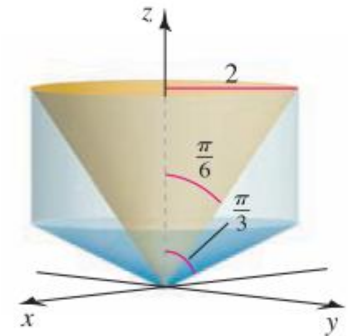
$$= -\frac{16\pi}{3}\int_{\pi/6}^{\pi/3} d(\cot\varphi)$$

$$= -\frac{16\pi}{3}\left(\cot\varphi\right)\Big|_{\pi/6}^{\pi/3}$$

$$= -\frac{16\pi}{3}\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right)$$

$$= \frac{32\pi}{3\sqrt{3}}$$

$$= \frac{32}{9}\pi\sqrt{3}$$



Exercise

Use the spherical coordinates to find the volume of a ball of radius $a > 0$

Solution

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \left(\rho^3\right)\Big|_0^a$$

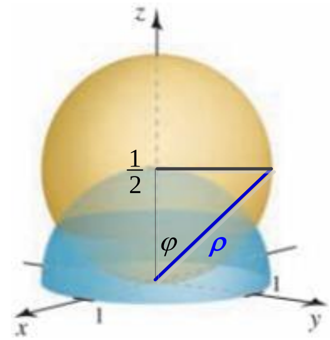
$$= \frac{2\pi}{3}a^3(-\cos\varphi)\Big|_0^{\pi}$$

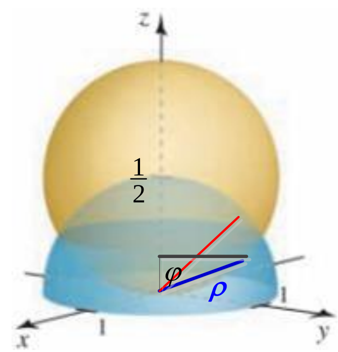
$$= \frac{4}{3}\pi a^3 \quad unit^3$$

Exercise

Use the spherical coordinates to find the volume of the solid bounded by the sphere $\rho = 2\cos\varphi$ and the hemisphere $\rho = 1,\ z \ge 0$

Solution





$$\rho = 2\cos\varphi = 1 \quad \to \quad \varphi = \frac{\pi}{3}$$

$$z = \frac{1}{2} \ \to \cos\varphi = \frac{1}{2}\frac{1}{\rho} \quad \to \quad \rho = \frac{1}{2}\sec\varphi$$

$$V = 2\int_0^{2\pi}\int_0^{\pi/3}\int_{\frac{1}{2}\sec\varphi}^{1} \rho^2 \sin\varphi \ d\rho d\varphi d\theta$$

$$= \frac{2}{3}\int_0^{2\pi} d\theta \int_0^{\pi/3} \sin\varphi\left(\rho^3\right)\Big|_{\frac{1}{2}\sec\varphi}^{1} \ d\varphi$$

$$= \frac{4\pi}{3}\int_0^{\pi/3} \sin\varphi\left(1 - \frac{1}{8}\sec^3\varphi\right) \ d\varphi$$

$$= \frac{4\pi}{3}\left(\int_0^{\pi/3} \sin\varphi \ d\varphi + \frac{1}{8}\int_0^{\pi/3} \cos^{-3}\varphi \ d(\cos\varphi)\right)$$

$$= \frac{4\pi}{3}\left(-\cos\varphi - \frac{1}{16}\frac{1}{\cos^2\varphi}\right)\Bigg|_0^{\pi/3}$$

$$= \frac{4\pi}{3}\left(-\frac{1}{2} - \frac{1}{4} + 1 + \frac{1}{16}\right)$$

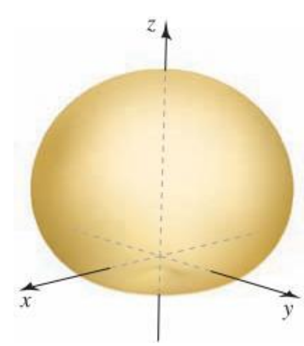
$$= \frac{5\pi}{12}$$

Exercise

Use the spherical coordinates to find the volume of the solid of revolution

$$D = \{(\rho, \varphi, \theta):\ 0 \le \rho \le 1 + \cos\varphi,\ 0 \le \varphi \le \pi,\ 0 \le \theta \le 2\pi\}$$

Solution



$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{1+\cos\varphi} \rho^2 \sin\varphi \ d\rho d\varphi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \rho^3 \Big|_0^{1+\cos\varphi} \ d\varphi$$

$$= \frac{2\pi}{3}\int_0^{\pi} \sin\varphi(1+\cos\varphi)^3 \ d\varphi$$

$$= -\frac{2\pi}{3}\int_0^{\pi} (1+\cos\varphi)^3 \ d(1+\cos\varphi)$$

$$= -\frac{\pi}{6}(1+\cos\varphi)^4 \Big|_0^{\pi}$$

$$= \frac{\pi}{6}2^4$$

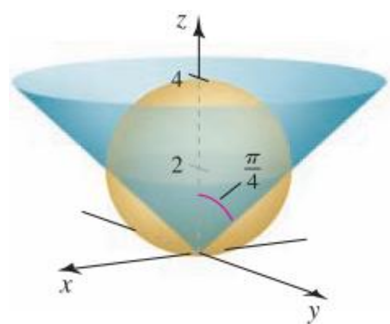
$$\boxed{= \frac{8}{3}\pi \quad unit^3}$$

Exercise

Use the spherical coordinates to find the volume of the solid outside the cone $\varphi = \frac{\pi}{4}$ and inside the sphere $\rho = 4\cos\varphi$

Solution

$$V = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{4\cos\varphi} \rho^2 \sin\varphi \ d\rho d\varphi d\theta$$



$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_{\pi/4}^{\pi/2} \sin\varphi\left(\rho^3\right)\Big|_0^{4\cos\varphi} \ d\varphi$$

$$= \frac{128}{3}\pi\int_{\pi/4}^{\pi/2} \sin\varphi\left(\cos^3\varphi\right) \ d\varphi$$

$$= \frac{128}{3}\pi\int_{\pi/4}^{\pi/2} \left(-\cos^3\varphi\right) \ d(\cos\varphi)$$

$$= \frac{32}{3}\pi\left(-\cos^4\varphi\right)\Big|_{\pi/4}^{\pi/2}$$

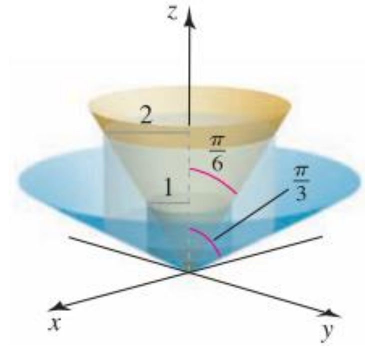
$$= \frac{32}{3}\pi\left(\frac{1}{4}\right)$$

$$\boxed{= \frac{8}{3}\pi \quad unit^3}$$

Exercise

Use the spherical coordinates to find the volume of the solid bounded by the cylinders $r=1$ and $r=2$, and the cone $\varphi=\frac{\pi}{6}$ and $\varphi=\frac{\pi}{3}$

Solution



$$V=\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_{\csc\varphi}^{2\csc\varphi}\rho^2\sin\varphi\,d\rho d\varphi d\theta$$

$$=\frac{1}{3}\int_0^{2\pi}d\theta\int_{\pi/6}^{\pi/3}\sin\varphi\left(\rho^3\right)\Big|_{\csc\varphi}^{2\csc\varphi}\,d\varphi$$

$$=\frac{14\pi}{3}\int_{\pi/6}^{\pi/3}\sin\varphi\left(\csc^3\varphi\right)\,d\varphi$$

$$=\frac{14\pi}{3}\int_{\pi/6}^{\pi/3}\csc^2\varphi\,d\varphi$$

$$=\frac{14\pi}{3}(-\cot\varphi)\Big|_{\pi/6}^{\pi/3}$$

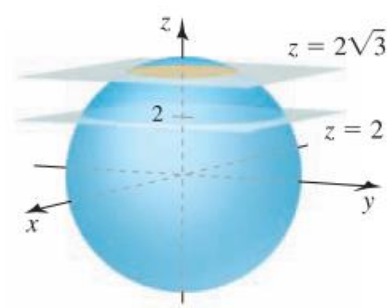
$$=\frac{14\pi}{3}\left(-\frac{1}{\sqrt{3}}+\sqrt{3}\right)$$

$$=\frac{14\pi}{3}\left(\frac{2}{\sqrt{3}}\right)$$

$$=\frac{28}{9}\pi\sqrt{3}\ \ unit^3$$

Exercise

Use the spherical coordinates to find the volume of the ball $\rho\le 4$ that lies between the planes $z=2$ and $z=2\sqrt{3}$



Solution

$$z=2\sqrt{3}\ \rightarrow\ \cos\varphi=\frac{2\sqrt{3}}{4}=\frac{\sqrt{3}}{2}\ \Rightarrow\ \varphi=\frac{\pi}{6}$$

$$z=2\ \rightarrow\ \cos\varphi=\frac{2}{4}=\frac{1}{2}\ \Rightarrow\ \varphi=\frac{\pi}{3}$$

$$V=\int_0^{2\pi}\int_0^{\pi/6}\int_{2\sqrt{3}\sec\varphi}^{4}\rho^2\sin\varphi\,d\rho d\varphi d\theta-\int_0^{2\pi}\int_0^{\pi/3}\int_{2\sec\varphi}^{4}\rho^2\sin\varphi\,d\rho d\varphi d\theta$$

$$=\frac{1}{3}\int_0^{2\pi}d\theta\int_0^{\pi/6}\sin\varphi\left(\rho^3\right)\Big|_{2\sqrt{3}\sec\varphi}^{4}\,d\varphi-\frac{1}{3}\int_0^{2\pi}d\theta\int_0^{\pi/3}\sin\varphi\left(\rho^3\right)\Big|_{2\sec\varphi}^{4}\,d\varphi$$

$$= \frac{2\pi}{3}\int_0^{\pi/6} \sin\varphi\left(64 - 24\sqrt{3}\sec^3\varphi\right)\, d\varphi - \frac{2\pi}{3}\int_0^{\pi/3} \sin\varphi\left(64 - 8\sec^3\varphi\right)\, d\varphi$$

$$= \frac{16\pi}{3}\int_0^{\pi/6} \left(3\sqrt{3}\cos^{-3}\varphi - 8\right)\, d\left(\cos\varphi\right) + \frac{16\pi}{3}\int_0^{\pi/3} \left(8 - \cos^{-3}\varphi\right)\, d\left(\cos\varphi\right)$$

$$= \frac{16\pi}{3}\left(-\frac{3\sqrt{3}}{2}\sec^2\varphi - 8\cos\varphi\right)\Big|_0^{\pi/6} + \frac{16\pi}{3}\left(8\cos\varphi + \frac{1}{2}\sec^2\varphi\right)\Big|_0^{\pi/3}$$

$$= \frac{16\pi}{3}\left(-2\sqrt{3} - 4\sqrt{3} + \frac{3\sqrt{3}}{2} + 8\right) + \frac{16\pi}{3}\left(4 + 2 - 8 - \frac{1}{2}\right)$$

$$= \frac{16\pi}{3}\left(-\frac{9\sqrt{3}}{2} + 8 - \frac{5}{2}\right)$$

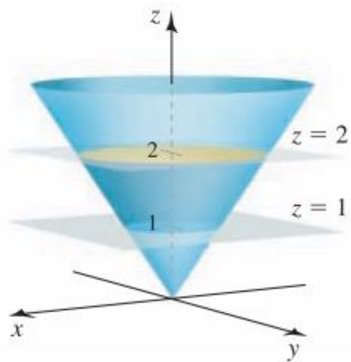
$$= \frac{8\pi}{3}\left(9\sqrt{3} - 11\right) \quad unit^3$$

Exercise

Use the spherical coordinates to find the volume of the solid inside the cone $z = \left(x^2 + y^2\right)^{1/2}$ that lies between the planes $z = 1$ and $z = 2$

Solution

$$z = 2 \quad \rightarrow \quad x^2 + y^2 = 4 = r^2 \quad \Rightarrow \quad \varphi = \tan^{-1}\frac{2}{2} = \frac{\pi}{4}$$



$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_{\sec\varphi}^{2\sec\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\varphi\left(\rho^3\right)\Big|_{2\sec\varphi}^{\sec\varphi} \, d\varphi$$

$$= \frac{2\pi}{3}\int_0^{\pi/4} \left(-7\sec^3\varphi\right)\, d\left(\cos\varphi\right)$$

$$= \frac{7\pi}{3}\left(\frac{1}{\cos^2\varphi}\right)\Big|_0^{\pi/4}$$

$$= \frac{7\pi}{3} \quad unit^3$$

Or: $Volume = \frac{1}{3}Ah = \frac{1}{3}\left(2^2\pi \times 2 - 1^2\pi \times 1\right) = \frac{7\pi}{3}$

Exercise

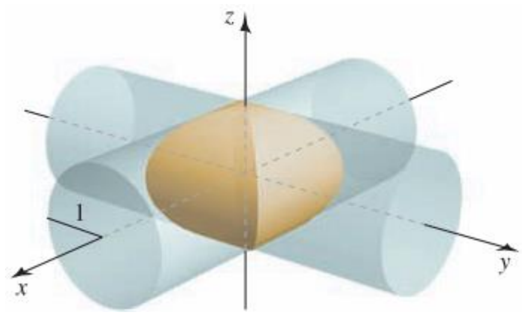
The *x*- and *y*-axes from the axes of two right circular cylinders with radius 1.
Find the volume of the solid that is common to the two cylinders.

Solution

Due to symmetry, this region is made up of *eight* identical pieces, one in each octant.

$$y=0 \quad \to \ x^2+z^2=1 \ \Rightarrow \ x=\sqrt{1-z^2}$$

$$//\ x\text{-}axis \quad \to \ y^2+z^2=1 \ \Rightarrow \ y=\sqrt{1-z^2}$$



$$V=8\int_0^1\int_0^{\sqrt{1-z^2}}\int_0^{\sqrt{1-z^2}} 1\ dydxdz$$

$$=8\int_0^1\int_0^{\sqrt{1-z^2}} \sqrt{1-z^2}\ dxdz$$

$$=8\int_0^1 \sqrt{1-z^2}\ x\Big|_0^{\sqrt{1-z^2}}\ dz$$

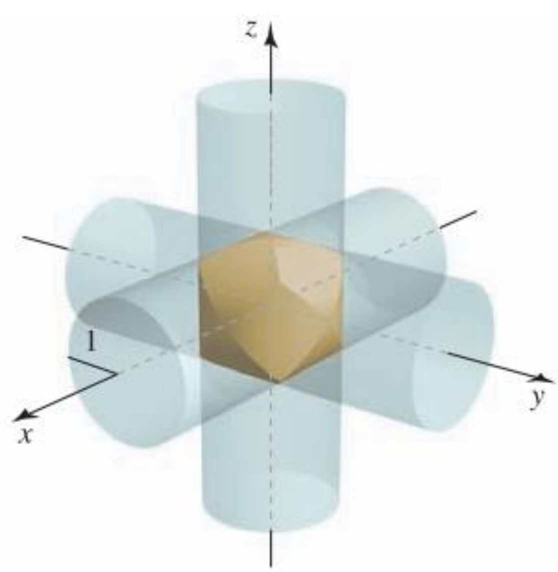
$$=8\int_0^1 \left(1-z^2\right)dz$$

$$=8\left(z-\frac{1}{3}z^3\right)\Big|_0^1$$

$$=\frac{16}{3}\quad unit^3$$

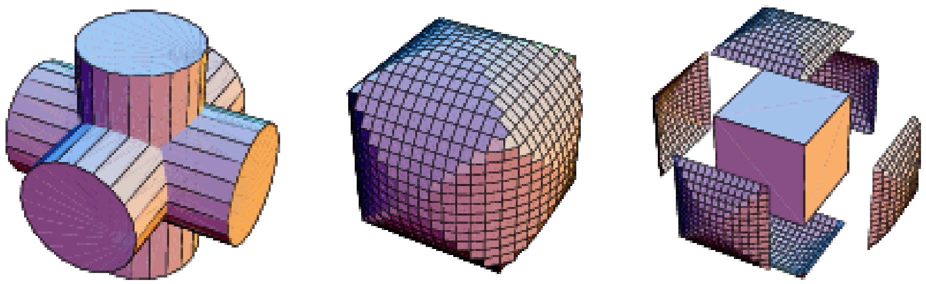
Exercise

The coordinate axes from the axes of three right circular cylinders with radius 1.



Find the volume of the solid that is common to the three cylinders.

Solution



Due to symmetry, this region is made up of *eight* identical pieces, one in each octant.

$y=0 \quad \to x^2+z^2=1 \Rightarrow x=\sqrt{1-z^2}$

$//\ z\text{-}axis \quad \to x^2+y^2=1 \Rightarrow y=\sqrt{1-x^2}$

If the particle starts at a point on the xz-plane for which $x<z$, then $\sqrt{1-z^2}<\sqrt{1-x^2}$

$$V=8\left(\int_0^{\frac{\sqrt{2}}{2}}\int_x^{\sqrt{1-x^2}}\int_0^{\sqrt{1-z^2}}1\ dydzdx+\int_0^{\frac{\sqrt{2}}{2}}\int_z^{\sqrt{1-z^2}}\int_0^{\sqrt{1-x^2}}1\ dydxdz\right)$$

$$=8\left(\int_0^{\frac{\sqrt{2}}{2}}\int_x^{\sqrt{1-x^2}}\sqrt{1-z^2}\ dzdx+\int_0^{\frac{\sqrt{2}}{2}}\int_z^{\sqrt{1-z^2}}\sqrt{1-x^2}\ dxdz\right)$$

$$=16\int_0^{\frac{\sqrt{2}}{2}}\int_x^{\sqrt{1-x^2}}\sqrt{1-z^2}\ dzdx$$

$$=16\int_0^{\frac{\pi}{4}}\int_0^1 r\sqrt{1-r^2\cos^2\theta}\ drd\theta \qquad\qquad w=r\cos\theta \Rightarrow dw=\cos\theta dr$$

$$=16\int_0^{\frac{\pi}{4}}\int_0^1 \frac{w}{\cos\theta}\sqrt{1-w^2}\ \frac{dw}{\cos\theta}d\theta$$

$$=-8\int_0^{\frac{\pi}{4}}\frac{1}{\cos^2\theta}\int_0^1\sqrt{1-w^2}\ d\left(1-w^2\right)d\theta$$

$$=-\frac{16}{3}\int_0^{\frac{\pi}{4}}\frac{1}{\cos^2\theta}\left(1-w^2\right)^{3/2}\Big|_0^1\ d\theta$$

$$=-\frac{16}{3}\int_0^{\frac{\pi}{4}}\frac{1}{\cos^2\theta}\left(1-r^2\cos^2\theta\right)^{3/2}\Big|_0^1\ d\theta$$

$$= -\frac{16}{3}\int_0^{\frac{\pi}{4}} \frac{1}{\cos^2\theta}\left(\left(1-\cos^2\theta\right)^{3/2} - 1\right)d\theta$$

$$= -\frac{16}{3}\int_0^{\frac{\pi}{4}} \frac{\sin^3\theta - 1}{\cos^2\theta}d\theta$$

$$= -\frac{16}{3}\int_0^{\frac{\pi}{4}} \left(\tan^2\theta\sin\theta - \sec^2\theta\right)d\theta$$

$$= -\frac{16}{3}\int_0^{\frac{\pi}{4}} \frac{\cos^2\theta - 1}{\cos^2\theta}d(\cos\theta) + \frac{16}{3}\tan\theta\Big|_0^{\frac{\pi}{4}}$$

$$= -\frac{16}{3}\int_0^{\frac{\pi}{4}} \left(1 - \frac{1}{\cos^2\theta}\right)d(\cos\theta) + \frac{16}{3}$$

$$= -\frac{16}{3}\left(\cos\theta + \frac{1}{\cos\theta}\right)\Big|_0^{\frac{\pi}{4}} + \frac{16}{3}$$

$$= -\frac{16}{3}\left(\frac{\sqrt{2}}{2} + \sqrt{2} - 2\right) + \frac{16}{3}$$

$$= -8\sqrt{2} + \frac{32}{3} + \frac{16}{3}$$

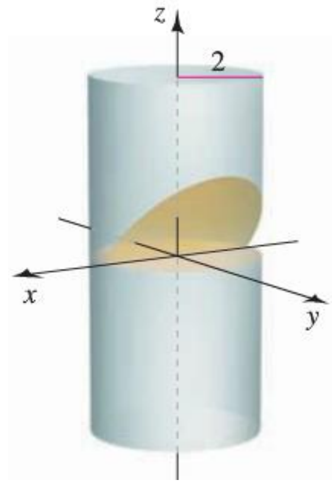
$$= 16 - 8\sqrt{2}$$

$$= 8\left(2 - \sqrt{2}\right) \quad unit^3$$

Exercise

Find the volume of one of the wedges formed when the cylinder $x^2 + y^2 = 4$ is cut by the planes $z = 0$ and $y = z$

Solution



$$x^2 + y^2 = 4 \quad \rightarrow \quad 0 \le r \le 2$$

$$z = 0 \quad y = z \quad \rightarrow 0 \le \theta \le \pi$$

$$z = y = r\sin\theta \quad \rightarrow \quad 0 \le z \le r\sin\theta$$

$$V = \int_0^2 \int_0^{\pi} \int_0^{r\sin\theta} r\ dz d\theta dr$$

$$= \int_0^2 \int_0^{\pi} rz \Big|_0^{r\sin\theta} \ d\theta dr$$

$$= \int_0^2 r^2 dr \int_0^\pi \sin\theta \, d\theta$$

$$= \frac{1}{3}r^3 \Big|_0^2 \; \left(-\cos\theta\right)\Big|_0^\pi$$

$$= \frac{8}{3}(1+1)$$

$$= \frac{16}{3} \; unit^3$$

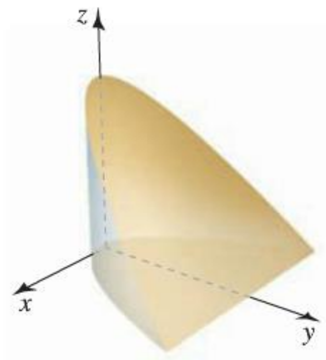
Exercise

Find the volume of the region inside the parabolic cylinder $y = x^2$ between the planes $z = 3 - y$ and $z = 0$

Solution

$z = 3 - y = 0 \;\rightarrow\; y = 3 \quad x^2 \le y \le 3$

$y = x^2 = 3 \;\rightarrow x = \pm\sqrt{3}$



$$V = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2}^{3} \int_0^{3-y} dzdydx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2}^{3} z \Big|_0^{3-y} \; dydx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2}^{3} (3-y)\, dydx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \left(3y - \frac{1}{2}y^2\right)\Big|_{x^2}^{3} \; dx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \left(9 - \frac{9}{2} - 3x^2 + \frac{1}{2}x^4\right) dx$$

$$= \left(\frac{9}{2}x - x^3 + \frac{1}{10}x^5\right)\Big|_{-\sqrt{3}}^{\sqrt{3}}$$

$$= 2\left(\frac{9}{2}\sqrt{3} - 3\sqrt{3} + \frac{3}{10}\sqrt{3}\right)$$

$$= 2\left(\frac{18}{10}\sqrt{3}\right)$$

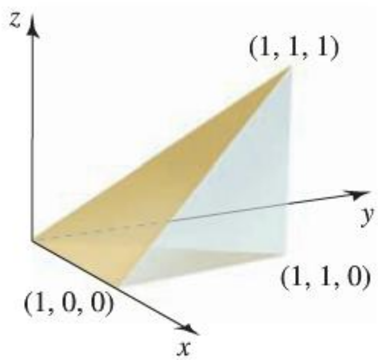
$$= \frac{18\sqrt{3}}{5} \; unit^3$$

Exercise

Find the volume of the tetrahedron with vertices $(0,\ 0,\ 0)$, $(1,\ 0,\ 0)$, $(1,\ 1,\ 0)$, and $(1,\ 1,\ 1)$

Solution

$$0 \le x \le 1$$
$$0 \le y \le x$$
$$0 \le z \le y$$

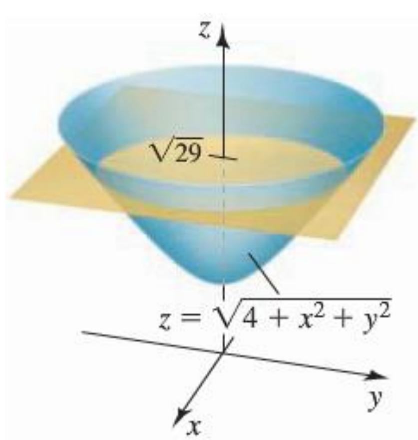


$$V = \int_0^1 \int_0^x \int_0^y dzdydx$$
$$= \int_0^1 \int_0^x z\Big|_0^y \ dydx$$
$$= \int_0^1 \int_0^x y\ dydx$$
$$= \int_0^1 \frac{1}{2}y^2\Big|_0^x \ dx$$
$$= \int_0^1 \frac{1}{2}x^2 dx$$
$$= \frac{1}{6}x^3\Big|_0^1$$
$$\boxed{= \frac{1}{6}\ unit^3}$$

Exercise

Find the volume of the region bounded by the plane $z = \sqrt{29}$ and the hyperboloid $z = \sqrt{4 + x^2 + y^2}$. Use integration in cylindrical coordinates.

Solution



$$z = \sqrt{4 + x^2 + y^2} = \sqrt{29}$$
$$4 + x^2 + y^2 = 29$$
$$x^2 + y^2 = 25 \quad \to \quad 0 \le r \le 5$$
$$0 \le \theta \le 2\pi$$

$$V = \int_0^{2\pi} \int_0^5 \int_{\sqrt{4+r^2}}^{\sqrt{29}} r\ dzdrd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^5 r\, z \Big|_{\sqrt{4+r^2}}^{\sqrt{29}} dr$$

$$= \theta \Big|_0^{2\pi} \int_0^5 r\left(\sqrt{29} - \sqrt{4+r^2}\right) dr$$

$$= 2\pi\sqrt{29}\int_0^5 r\, dr - 2\pi\int_0^5 r\left(4+r^2\right)^{1/2} dr$$

$$= \pi\sqrt{29}\, r^2 \Big|_0^5 - \pi\int_0^5 \left(4+r^2\right)^{1/2} d\left(4+r^2\right)$$

$$= 25\pi\sqrt{29} - \frac{2\pi}{3}\left(4+r^2\right)^{3/2}\Big|_0^5$$

$$= 25\pi\sqrt{29} - \frac{2\pi}{3}\left((29)^{3/2} - 8\right)$$

$$= 25\pi\sqrt{29} - \frac{58\pi}{3}\sqrt{29} + \frac{16\pi}{3}$$

$$\boxed{= \frac{17\pi}{3}\sqrt{29} + \frac{16\pi}{3}\ unit^3}$$

Exercise

Find the volume of the solid cylinder whose height is 4 and whose base is the disk $\{(r,\theta):\ 0 \le r \le 2\cos\theta\}$. Use integration in cylindrical coordinates

Solution

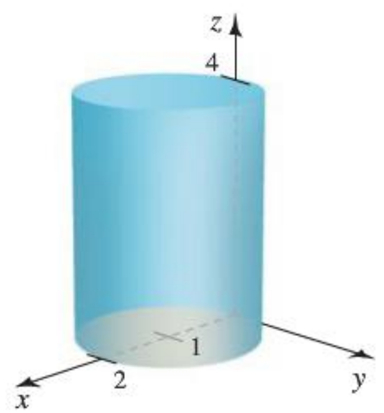
$$V = \int_0^4 \int_0^{\pi} \int_0^{2\cos\theta} r\, dr d\theta dz$$

$$= \int_0^4 dz \int_0^{\pi} \frac{1}{2} r^2 \Big|_0^{2\cos\theta} d\theta$$

$$= \frac{1}{2} z \Big|_0^4 \int_0^{\pi} 4\cos^2\theta\, d\theta$$

$$= 4\int_0^{\pi} (1+\cos 2\theta)\, d\theta$$

$$= 4\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi}$$



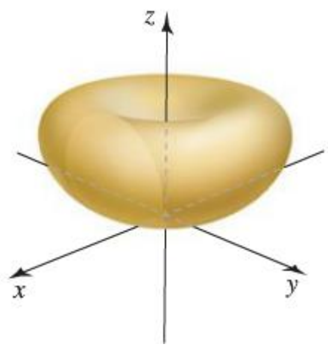
$$= 4\pi \ \ unit^3$$

Exercise

Use integration in spherical coordinates to find the volume of the rose petal of revolution

$$D = \left\{ (\rho, \varphi, \theta): \ 0 \le \rho \le 4\sin 2\varphi, \ 0 \le \varphi \le \frac{\pi}{2}, \ 0 \le \theta \le 2\pi \right\}$$

Solution



$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{4\sin 2\varphi} \rho^2 \sin\varphi \ d\rho d\varphi d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi \ \rho^3 \Big|_0^{4\sin 2\varphi} \ d\varphi$$

$$= \frac{1}{3} \theta \Big|_0^{2\pi} \int_0^{\frac{\pi}{2}} 64\sin\varphi \ \sin^3 2\varphi \ d\varphi$$

$$= \frac{128\pi}{3} \int_0^{\frac{\pi}{2}} 8\sin\varphi \ \sin^3\varphi \cos^3\varphi \ d\varphi$$

$$= \frac{1024\pi}{3} \int_0^{\frac{\pi}{2}} \sin^4\varphi \cos^2\varphi \cos\varphi \ d\varphi$$

$$= \frac{1024\pi}{3} \int_0^{\frac{\pi}{2}} \sin^4\varphi \left(1 - \sin^2\varphi\right) \ d(\sin\varphi)$$

$$= \frac{1024\pi}{3} \int_0^{\frac{\pi}{2}} \left(\sin^4\varphi - \sin^6\varphi\right) \ d(\sin\varphi)$$

$$= \frac{1024\pi}{3} \left(\frac{1}{5}\sin^5\varphi - \frac{1}{7}\sin^7\varphi \right) \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1024\pi}{3} \left(\frac{1}{5} - \frac{1}{7} \right)$$

$$= \frac{1024\pi}{3} \left(\frac{2}{35} \right)$$

$$= \frac{2048\pi}{105} \ \ unit^3$$

Exercise

Use integration in spherical coordinates to find the volume of the region above the cone $\varphi = \frac{\pi}{4}$ and inside the sphere $\rho = 4\cos\varphi$.

Solution

$$0 \le \varphi \le \frac{\pi}{4} \qquad 0 \le \rho \le 4\cos\varphi \qquad 0 \le \theta \le 2\pi$$

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{4\cos\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\varphi \rho^3 \Big|_0^{4\cos\varphi} \, d\varphi$$

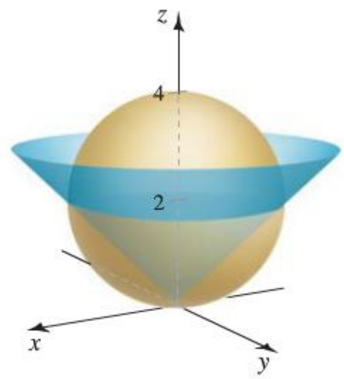
$$= \frac{128\pi}{3}\int_0^{\frac{\pi}{4}} \sin\varphi \cos^3\varphi \, d\varphi$$

$$= -\frac{128\pi}{3}\int_0^{\frac{\pi}{4}} \cos^3\varphi \, d(\cos\varphi)$$

$$= -\frac{32\pi}{3}\cos^4\varphi \Big|_0^{\frac{\pi}{4}}$$

$$= -\frac{32\pi}{3}\left(\frac{1}{4} - 1\right)$$

$$\underline{= 8\pi \ \ unit^3}$$



Exercise

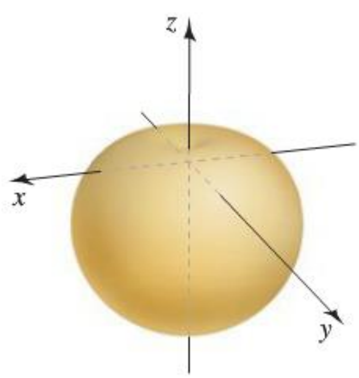
Find the volume of the cardioid of revolution $D = \left\{(\rho, \varphi, \theta): \ 0 \le \rho \le \frac{1-\cos\varphi}{2}, \ 0 \le \varphi \le \pi, \ 0 \le \theta \le 2\pi\right\}$

Solution

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{\frac{1-\cos\varphi}{2}} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \ \rho^3 \Big|_0^{\frac{1-\cos\varphi}{2}} \, d\varphi$$

$$= \frac{\pi}{12}\int_0^{\pi} \sin\varphi \ (1 - \cos\varphi)^3 \, d\varphi$$



$$= \frac{\pi}{12}\int_0^{\pi}(1-\cos\varphi)^3\, d(1-\cos\varphi)$$

$$= \frac{\pi}{48}(1-\cos\varphi)^4\Big|_0^{\pi}$$

$$= \frac{\pi}{48}(16)$$

$$\boxed{= \frac{\pi}{3}\ \ unit^3}$$

Exercise

A cake is shaped like a solid cone with radius 4 and height 2, with its base on the *xy*-plane. A wedge of the cake is removed by making two slices from the axis of the cone outward, perpendicular to the *xy*-plane separated by an angle of Q radians, where $0 < Q < 2\pi$

a) Find the volume of the slice for $Q = \frac{\pi}{4}$. Use geometry to check your answer.

b) Find the volume of the slice for $0 < Q < 2\pi$. Use geometry to check your answer.

Solution

Volume of a cone $= \frac{\pi}{3}(4)^2(2)$ $\quad V = \frac{\pi}{3}r^2h$

$$\boxed{= \frac{32\pi}{3}}$$

Equation of the cone in cylindrical coordinates is:

$$\begin{cases} r = 4 & \to z = 0 \\ r = 0 & \to z = 2 = h \end{cases}$$

$$m = \frac{2-0}{0-4} = -\frac{1}{2}$$

$$\boxed{z = -\frac{1}{2}r + 2}$$

a) $V = \int_0^{\frac{\pi}{4}}\int_0^4\int_0^{2-\frac{1}{2}r} r\ dzdrd\theta$

$$= \int_0^{\frac{\pi}{4}} d\theta \int_0^4 rz\Big|_0^{2-\frac{1}{2}r}\ dr$$

$$= \frac{\pi}{4}\int_0^4\left(2r - \frac{1}{2}r^2\right)dr$$

$$= \frac{\pi}{4}\left(r^2 - \frac{1}{6}r^3\right)\Big|_0^4$$

$$= \frac{\pi}{4}\left(16 - \frac{32}{3}\right)$$

$$\boxed{= \frac{4\pi}{3} \quad unit^3}$$

Since $Q = \frac{\pi}{4}$, then the volume of the slice is equal to $\frac{1}{8}$ of the cone volume

$$V = \frac{1}{8}\frac{32\pi}{3} \ \boxed{= \frac{4\pi}{3}}$$

b) $V = \int_0^Q \int_0^4 \int_0^{2-\frac{1}{2}r} r \, dz dr d\theta$

$$= \int_0^Q d\theta \int_0^4 rz \Big|_0^{2-\frac{1}{2}r} dr$$

$$= Q\int_0^4 \left(2r - \frac{1}{2}r^2\right) dr$$

$$= Q\left(r^2 - \frac{1}{6}r^3\right)\Big|_0^4$$

$$= Q\left(16 - \frac{32}{3}\right)$$

$$\boxed{= \frac{16}{3}Q}$$

Geometrically, since Q in radians, then $\frac{Q}{2\pi}$ of a circle.

$\therefore$ Volume of the slice is $\frac{Q}{2\pi}$ times of the curve.

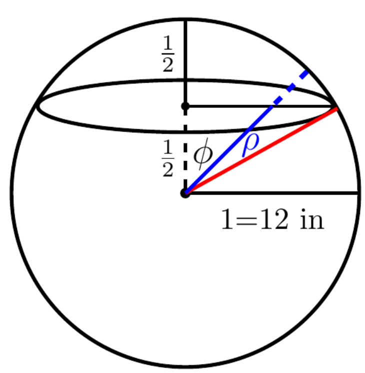
Exercise

A spherical fish tank with a radius of 1 *ft* is filled with water to a level 6 *in*. below the top of the tank.

a) Determine the volume and weight of the water in the fish tank. (The weight density of water is about 62.5 lb / ft^3.)

b) How much additional water must be added to completely fill the tank?

Solution



$$\varphi = \cos^{-1}\frac{6}{12} = \frac{\pi}{3}$$

$$0 \le \theta \le 2\pi$$

$$\cos\varphi = \frac{\frac{1}{2}}{\rho} \quad \to \quad \rho = \frac{1}{2}\sec\varphi$$

$$\frac{1}{2}\sec\varphi \le \rho \le 1$$

a) Volume of empty spherical cap:

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{\frac{1}{2}\sec\varphi}^{1} \rho^2 \sin\varphi \; d\rho d\varphi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{3}} \sin\varphi \; \rho^3 \Big|_{\frac{1}{2}\sec\varphi}^{1} \; d\varphi$$

$$= \frac{2\pi}{3}\int_0^{\frac{\pi}{3}} \sin\varphi \left(1 - \frac{1}{8}\sec^3\varphi\right) d\varphi$$

$$= \frac{2\pi}{3}\int_0^{\frac{\pi}{3}} \sin\varphi \; d\varphi - \frac{\pi}{12}\int_0^{\frac{\pi}{3}} \sin\varphi \cos^{-3}\varphi \; d\varphi$$

$$= -\frac{2\pi}{3}\cos\varphi \Big|_0^{\frac{\pi}{3}} + \frac{\pi}{12}\int_0^{\frac{\pi}{3}} \cos^{-3}\varphi \; d(\cos\varphi)$$

$$= -\frac{2\pi}{3}\left(\frac{1}{2} - 1\right) - \frac{\pi}{24}\cos^{-2}\varphi \Big|_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{3} - \frac{\pi}{24}(4-1)$$

$$= \frac{\pi}{3} - \frac{\pi}{8}$$

$$= \frac{5\pi}{24} \quad ft^3$$

Volume of a sphere is $\frac{4\pi}{3}$

$\therefore$ Volume of water $\frac{4\pi}{3} - \frac{5\pi}{24} \;\; = \frac{9\pi}{8} \quad ft^3$

Weights $= (6.25)\frac{9\pi}{8} \;\; \approx 220.893 \;\; lbs$

b) The addition water to fill the tank is $= \frac{5\pi}{24} \quad ft^3$

Exercise

A spherical cloud of electric charge has known charge density $Q(\rho)$, where ρ is the spherical coordinate. Find the total charge in the cloud in the following cases.

a) $Q(\rho)=\dfrac{2\times 10^{-4}}{\rho^4}, \quad 1\le \rho<\infty$

b) $Q(\rho)=\dfrac{2\times 10^{-4}}{1+\rho^3}, \quad 1\le \rho<\infty$

c) $Q(\rho)=2\times 10^{-4}e^{-0.01\rho^3}, \quad 0\le \rho<\infty$

Solution

a)
$$\int_0^{2\pi}\int_0^{\pi}\int_1^{\infty}\frac{2\times 10^{-4}}{\rho^4}\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta = 2\times 10^{-4}\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\, d\varphi\int_1^{\infty}\frac{1}{\rho^2}d\rho$$
$$= 2\times 10^{-4}(2\pi)(-\cos\varphi)\Big|_0^{\pi}\left(-\frac{1}{\rho}\right)\Big|_1^{\infty}$$
$$= 4\pi\times 10^{-4}(2)\ (1) \qquad \frac{1}{\rho}\xrightarrow[\rho\to\infty]{}0$$
$$\boxed{=8\pi\times 10^{-4}}$$

b)
$$\int_0^{2\pi}\int_0^{\pi}\int_1^{\infty}\frac{2\times 10^{-4}}{1+\rho^3}\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta = \frac{2}{3}\times 10^{-4}\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\, d\varphi\int_1^{\infty}\frac{1}{1+\rho^3}d\left(1+\rho^3\right)$$
$$= \frac{2}{3}\times 10^{-4}(2\pi)(-\cos\varphi)\Big|_0^{\pi}\left(\ln\left(1+\rho^3\right)\right)\Big|_1^{\infty}$$
$$\boxed{=\infty} \qquad \ln\left(1+\rho^3\right)\to\infty$$

c)
$$2\times 10^{-4}\int_0^{2\pi}\int_0^{\pi}\int_1^{\infty}e^{-0.01\rho^3}\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta =$$
$$= -\frac{2}{.003}\times 10^{-4}\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\, d\varphi\int_1^{\infty}e^{-0.01\rho^3}d\left(-0.01\rho^3\right)$$
$$= -\frac{2}{3}\times 10^{-2}(2\pi)(-\cos\varphi)\Big|_0^{\pi}\left(e^{-0.01\rho^3}\right)\Big|_1^{\infty}$$
$$= -\frac{4}{3}\pi\times 10^{-4}(2)\ (-1) \qquad \frac{1}{\rho}\xrightarrow[\rho\to\infty]{}0$$
$$\boxed{=\frac{8\pi}{3}\times 10^{-4}}$$

Exercise

A point mass *m* is a distance *d* from the center of a thin spherical shell of mass *M* and radius *R*. The magnitude of the gravitational force on the point mass is given by the integral

$$F(d) = \frac{GMm}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{(d - R\cos\phi)\sin\phi}{\left(R^2 + d^2 - 2Rd\cos\phi\right)^{3/2}}\,d\phi\, d\theta$$

Where *G* is the gravitational constant.

a) Use the change of variable $x = \cos\phi$ to evaluate the integral and show that if $d > R$, then $F(d) = \frac{GMm}{d^2}$, which means the force is the same as if the mass of the shell were concentrated at its center.

b) Show that is $d < r$ (the point mass is inside the shell), then $F = 0$.

Solution

a) $x = \cos\phi \quad \rightarrow \quad \sin\varphi = \sqrt{1 - x^2}$

$dx = -\sin\varphi d\varphi \quad \rightarrow \quad d\varphi = -\frac{dx}{\sqrt{1-x^2}}$

$$\begin{cases} \varphi = 0 & \rightarrow x = 1 \\ \varphi = \pi & \rightarrow x = -1 \end{cases}$$

$$F(d) = -\frac{GMm}{4\pi}\int_0^{2\pi} d\theta \int_{-1}^{1}\frac{(d - Rx)\sqrt{1-x^2}}{\left(R^2 + d^2 - 2xRd\right)^{3/2}}\frac{-dx}{\sqrt{1-x^2}}$$

$$= \frac{1}{2}GMm\int_{-1}^{1}\left(\frac{d}{\left(R^2 + d^2 - 2xRd\right)^{3/2}} - \frac{Rx}{\left(R^2 + d^2 - 2xRd\right)^{3/2}}\right)dx$$

$$= \frac{GMm}{2}\left(-\frac{1}{2R}\int_{-1}^{1}\frac{d\left(R^2 + d^2 - 2dRx\right)}{\left(R^2 + d^2 - 2dRx\right)^{3/2}} - \int_{-1}^{1}\frac{Rx}{\left(R^2 + d^2 - 2dRx\right)^{3/2}}dx\right)$$

$u = R^2 + d^2 - 2dRx \quad \rightarrow \quad du = -2dRdx$

$Rx = \frac{1}{2d}\left(R^2 + d^2 - u\right)$

$$\int\frac{Rx}{\left(R^2 + d^2 - 2dRx\right)^{3/2}}dx = \frac{1}{2d}\int\left(R^2 + d^2 - u\right)\left(u^{-3/2}\right)\left(-\frac{1}{2Rd}\right)du$$

$$= -\frac{1}{3Rd^2}\int\left(\left(R^2 + d^2\right)u^{-3/2} - u^{-1/2}\right)du$$

$$= -\frac{1}{4Rd^2}\left(-2\left(R^2+d^2\right)u^{-1/2} - 2u^{1/2}\right)$$

$$= \frac{1}{2Rd^2\sqrt{R^2+d^2-2dRx}}\left(R^2+d^2+R^2+d^2-2dRx\right)$$

$$= \frac{R^2+d^2-dRx}{Rd^2\sqrt{R^2+d^2-2dRx}}$$

$$F(d) = \frac{GMm}{2}\left(\frac{1}{R\sqrt{R^2+d^2-2dRx}} - \frac{R^2+d^2-dRx}{Rd^2\sqrt{R^2+d^2-2dRx}}\right)\Bigg|_{-1}^{1}$$

$$= \frac{GMm}{2}\left(\frac{dRx-R^2}{Rd^2\sqrt{R^2+d^2-2dRx}}\right)\Bigg|_{-1}^{1}$$

$$= \frac{GMm}{2}\left(\frac{Rd-R^2}{Rd^2\sqrt{R^2+d^2-2Rd}} - \frac{-Rd-R^2}{Rd^2\sqrt{R^2+d^2+2Rd}}\right)$$

$$= \frac{GMm}{2}\left(\frac{Rd-R^2}{Rd^2\sqrt{R^2+d^2-2Rd}} + \frac{Rd+R^2}{Rd^2\sqrt{R^2+d^2+2Rd}}\right)$$

$$= \frac{GMm}{2}\left(\frac{R(d-R)}{Rd^2\sqrt{(R-d)^2}} + \frac{R(d+R)}{Rd^2(R+d)}\right)$$

If $d > R$, then

$$F(d) = \frac{GMm}{2}\left(\frac{1}{d^2}+\frac{1}{d^2}\right)$$

$$\boxed{= \frac{GMm}{d^2}}$$

b) If $d < R$, then

$$F(d) = \frac{GMm}{2}\left(-\frac{1}{d^2}+\frac{1}{d^2}\right)$$

$$\boxed{= 0}$$

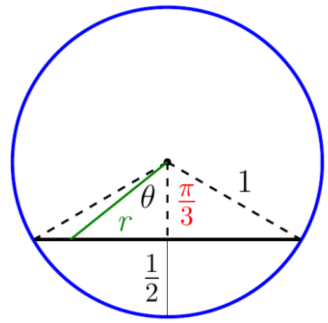
Exercise

Before a gasoline-powered engine is started, water must be drained from the bottom of the fuel tank. Suppose the tank is a right circular cylinder on its side with a length of 2 *ft* and a radius of 1 *ft*. If the water level is 6 *in*. above the lowest part of the tank, determine how much water must be drained from the tank.

Solution

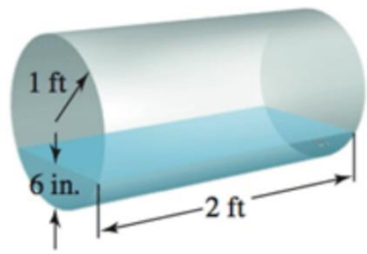
$$\cos\theta = \frac{\frac{1}{2}}{r} \quad \to \quad r = \frac{1}{2}\sec\theta$$

$$\theta = \cos^{-1}\frac{1}{2} = \pm\frac{\pi}{3}$$



$$V = \int_0^2 \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{\frac{1}{2}\sec\theta}^{1} r\ dr d\theta dz$$

$$= \frac{1}{2}\int_0^2 dz \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} r^2 \Big|_{\frac{1}{2}\sec\theta}^{1} \ d\theta$$



$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(1 - \frac{1}{4}\sec^2\theta\right) d\theta$$

$$= \left(\theta - \frac{1}{4}\tan\theta\right)\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{4} + \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

$$\boxed{= \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \quad ft^3}$$