Solution Section 2.5 – Directional Derivatives and the Gradient

Exercise

Find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point $f(x,y)=y-x, \quad (2,\ 1)$

Solution

$$\frac{\partial f}{\partial x}=-1, \quad \frac{\partial f}{\partial y}=1$$

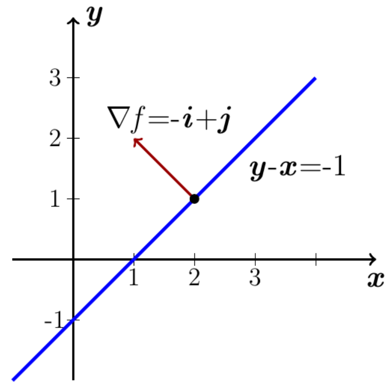
$$\nabla f = f_x\hat{i}+f_y\hat{j}$$

$$=-\hat{i}+\hat{j}$$

$$f(2,\ 1)=1-2$$

$$=-1$$

$-1=y-x$ is the level curve



Exercise

Find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point $f(x,y)=\ln\left(x^2+y^2\right), \quad (1,\ 1)$

Solution

$$\frac{\partial f}{\partial x}=\frac{2x}{x^2+y^2}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1)}=\frac{2}{1+1}=1$$

$$\frac{\partial f}{\partial y}=\frac{2y}{x^2+y^2}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,1)}=\frac{2}{1+1}=1$$

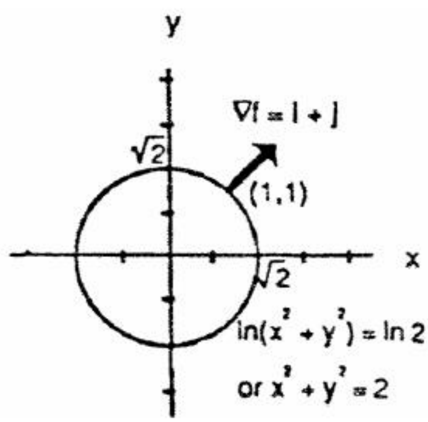
$$\nabla f = f_x\hat{i}+f_y\hat{j}$$

$$=\hat{i}+\hat{j}$$

$$f(1,\ 1)=\ln 2$$

$$\ln 2=\ln\left(x^2+y^2\right)$$

$\to \quad x^2+y^2=2$ is the level curve



Exercise

Find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point $f(x,y)=\sqrt{2x+3y}, \quad (-1,\ 2)$

Solution

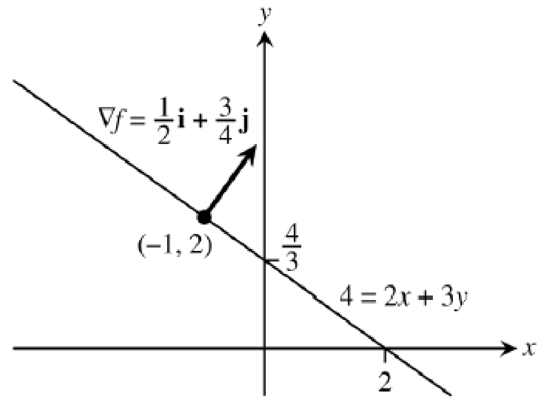
$$\frac{\partial f}{\partial x}=\frac{1}{\sqrt{2x+3y}}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(-1,2)}=\frac{1}{\sqrt{-2+6}}=\frac{1}{2}$$

$$\frac{\partial f}{\partial y}=\frac{3}{2\sqrt{2x+3y}}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(-1,2)}=\frac{3}{2\sqrt{-2+6}}=\frac{3}{4}$$

$$\nabla f=\frac{1}{2}\hat{i}+\frac{3}{4}\hat{j}$$



$$f(-1,\ 2)=\sqrt{2(-1)+3(2)}$$

$$\boxed{=2}$$

$2x+3y=4$ is the level curve

Exercise

Find ∇f at the given point $\quad f(x,y,z)=x^2+y^2-2z^2+z\ln x, \quad (1,\ 1,\ 1)$

Solution

$$\frac{\partial f}{\partial x}=2x+\frac{z}{x}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1,1)}=2+\frac{1}{1}=3$$

$$\frac{\partial f}{\partial y}=2y$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,1,1)}=2$$

$$\frac{\partial f}{\partial z}=-4z+\ln x$$

$$\left.\frac{\partial f}{\partial z}\right|_{(1,1,1)}=-4+\ln 1=-4$$

$$\boxed{\nabla f=3\hat{i}+2\hat{j}-4\hat{k}}$$

Exercise

Find ∇f at the given point $\quad f(x,y,z) = 2x^3 - 3\left(x^2 + y^2\right)z + \tan^{-1} xz, \quad (1,\ 1,\ 1)$

Solution

$$\frac{\partial f}{\partial x} = 6x^2 - 6xz + \frac{z}{1 + x^2z^2}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1,1)} = 6 - 6 + \frac{1}{1+1} = \frac{1}{2}$$

$$\frac{\partial f}{\partial y} = -6yz$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,1,1)} = -6$$

$$\frac{\partial f}{\partial z} = -3\left(x^2 + y^2\right) + \frac{z}{1 + x^2z^2}$$

$$\left.\frac{\partial f}{\partial z}\right|_{(1,1,1)} = -3(2) + \frac{1}{2} = -\frac{11}{2}$$

$$\nabla f = \frac{1}{2}\hat{i} - 6\hat{j} - \frac{11}{2}\hat{k}$$

Exercise

Find ∇f at the given point $\quad f(x,y,z) = e^{x+y}\cos z + (y+1)\sin^{-1} x, \quad \left(0,\ 0,\ \frac{\pi}{6}\right)$

Solution

$$\frac{\partial f}{\partial x} = e^{x+y}\cos z + \frac{y+1}{\sqrt{1 - x^2}}$$

$$\to \left.\frac{\partial f}{\partial x}\right|_{\left(0,0,\frac{\pi}{6}\right)} = e^0 \cos\frac{\pi}{6} + \frac{0+1}{\sqrt{1-0}}$$

$$= \frac{\sqrt{3}}{2} + 1$$

$$\frac{\partial f}{\partial y} = e^{x+y}\cos z + \sin^{-1} x$$

$$\to \left.\frac{\partial f}{\partial y}\right|_{\left(0,0,\frac{\pi}{6}\right)} = e^0 \cos\frac{\pi}{6} + 0$$

$$= \frac{\sqrt{3}}{2}$$

$$\frac{\partial f}{\partial z} = -e^{x+y}\sin z$$

$$\rightarrow \left.\frac{\partial f}{\partial z}\right|_{\left(0,0,\frac{\pi}{6}\right)} = -e^{0}\sin\frac{\pi}{6}$$

$$= -\frac{1}{2}$$

$$\nabla f = \left(\frac{\sqrt{3}}{2}+1\right)\hat{i} + \frac{\sqrt{3}}{2}\hat{j} - \frac{1}{2}\hat{k}$$

Exercise

Find the derivative of the function $f(x,y) = 2xy - 3y^2$ at $P_0(5,\ 5)$ in the direction of $\vec{v} = 4\hat{i} + 3\hat{j}$

Solution

$$\vec{u} = \frac{4\hat{i}+3\hat{j}}{\sqrt{16+9}} \qquad\qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}$$

$$f_x = 2y \quad \Rightarrow \quad f_x(5,5) = 10$$

$$f_y = 2x - 6y \quad \Rightarrow \quad f_y(5,5) = 10 - 30 = -20$$

$$\nabla f = 10\hat{i} - 20\hat{j}$$

$$\left(D_{\vec{u}}f\right)_{P_0} = \left(10\hat{i} - 20\hat{j}\right)\bullet\left(\frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}\right) \qquad\qquad \left(D_{\vec{u}}f\right)_{P_0} = \nabla f \bullet \vec{u}$$

$$= 10\left(\frac{4}{5}\right) - 20\left(\frac{3}{5}\right)$$

$$= 8 - 12$$

$$= -4$$

Exercise

Find the derivative of the function $f(x,y) = \dfrac{x-y}{xy+2}$ at $P_0(1,\ -1)$ in the direction of $\vec{v} = 12\hat{i} + 5\hat{j}$

Solution

$$\vec{u} = \frac{12\hat{i}+5\hat{j}}{\sqrt{144+25}} \qquad\qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{12}{13}\hat{i} + \frac{5}{13}\hat{j}$$

$$f_x = \frac{xy+2-y(x-y)}{(xy+2)^2}$$

$$= \frac{xy+2-xy+y^2}{(xy+2)^2}$$

$$= \frac{2+y^2}{(xy+2)^2}\Bigg|_{(1,-1)}$$

$$= \frac{2+1}{(-1+2)^2}$$

$$\underline{= 3}$$

$$f_y = \frac{-xy-2-x(x-y)}{(xy+2)^2}$$

$$= \frac{-2-x^2}{(xy+2)^2}\Bigg|_{(1,-1)}$$

$$= \frac{-2-1}{(-1+2)^2}$$

$$\underline{= -3}$$

$$\nabla f = 3\hat{i} - 3\hat{j}$$

$$\left(D_{\vec{u}} f\right)_{P_0} = \left(3\hat{i} - 3\hat{j}\right) \bullet \left(\frac{12}{13}\hat{i} + \frac{5}{13}\hat{j}\right) \qquad \left(D_{\vec{u}} f\right)_{P_0} = \nabla f \bullet \vec{u}$$

$$= \frac{36}{13} - \frac{15}{13}$$

$$\underline{= \frac{21}{13}}$$

Exercise

Find the derivative of the function $h(x,y) = \tan^{-1}\left(\frac{y}{x}\right) + \sqrt{3}\sin^{-1}\left(\frac{xy}{2}\right)$ at $P_0\ (1,\ 1)$ in the direction of $\vec{v} = 3\hat{i} - 2\hat{j}$

Solution

$$\vec{u} = \frac{3\hat{i} - 2\hat{j}}{\sqrt{9+4}} \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{3}{\sqrt{13}}\hat{i} - \frac{2}{\sqrt{13}}\hat{j}$$

$$h_x = \frac{-\frac{y}{x^2}}{\left(\frac{y}{x}\right)^2 + 1} + \sqrt{3}\frac{\frac{y}{2}}{\sqrt{1-\left(\frac{x^2y^2}{4}\right)}}$$

$$\to h_x\left(1,\ 1\right) = \frac{-1}{1+1} + \sqrt{3}\frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}}}$$

$$= -\frac{1}{2} + \sqrt{3}\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{2}$$

$$h_y = \frac{\frac{1}{x}}{\left(\frac{y}{x}\right)^2 + 1} + \sqrt{3}\frac{\frac{x}{2}}{\sqrt{1-\frac{x^2y^2}{4}}}$$

$$\to h_y\left(1,\ 1\right) = \frac{1}{2} + \sqrt{3}\frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}}}$$

$$= \frac{3}{2}$$

$$\nabla h = \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j}$$

$$\left(D_{\vec{u}}h\right)_{P_0} = \nabla h \bullet \vec{u}$$

$$= \left(\frac{1}{2}\hat{i} + \frac{3}{2}\hat{j}\right) \bullet \left(\frac{3}{\sqrt{13}}\hat{i} - \frac{2}{\sqrt{13}}\hat{j}\right)$$

$$= \frac{3}{2\sqrt{13}} - \frac{3}{\sqrt{13}}$$

$$= -\frac{3}{2\sqrt{13}}$$

Exercise

Find the derivative of the function $f\left(x, y, z\right) = xy + yz + zx$ at $P_0\left(1,\ -1,\ 2\right)$ in the direction of $\vec{v} = 3\hat{i} + 6\hat{j} - 2\hat{k}$

Solution

$$\vec{u} = \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{9+36+4}} \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}$$

$$f_x = y + z \quad \Rightarrow \quad f_x(1, -1, 2) = -1 + 2 = 1$$

$$f_y = x + z \quad \Rightarrow \quad f_y(1, -1, 2) = 1 + 2 = 3$$

$$f_z = y + x \quad \Rightarrow \quad f_z(1, -1, 2) = -1 + 1 = 0$$

$$\nabla f = \hat{i} + 3\hat{j}$$

$$\left(D_{\vec{u}} f\right)_{P_0} = \left(\hat{i} + 3\hat{j}\right) \bullet \left(\frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}\right) \qquad \left(D_{\vec{u}} f\right)_{P_0} = \nabla f \bullet \vec{u}$$

$$= \frac{3}{7} + \frac{18}{7}$$

$$= 3$$

Exercise

Find the derivative of the function $g(x, y, z) = 3e^x \cos yz$ at $P_0(0,\ 0,\ 0)$ in the direction of $\vec{v} = 2\hat{i} + \hat{j} - 2\hat{k}$

Solution

$$\vec{u} = \frac{2\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

$$g_y = -3ze^x \sin yz \Big|_{(0,0,0)}$$

$$= -3(0)e^0 \sin 0$$

$$= 0$$

$$g_x = 3e^x \cos yz \Big|_{(0,0,0)}$$

$$= 3e^0 \cos(0)$$

$$= 3$$

$$g_z = -3ye^x \sin yz \Big|_{(0,0,0)}$$

$$= -3(0)e^0 \sin 0$$

$$= 0$$

$$\nabla g = 3\hat{i}$$

$$\left(D_{\vec{u}}\, g\right)_{P_0} = \nabla g \bullet \vec{u}$$

$$= \left(3\hat{i}\right) \bullet \left(\tfrac{2}{3}\,\hat{i} + \tfrac{1}{3}\,\hat{j} - \tfrac{2}{3}\,\hat{k}\right)$$

$$\underline{= 2}$$

Exercise

Find the derivative of the function $h(x, y, z) = \cos xy + e^{yz} + \ln zx$ at $P_0\left(1,\ 0,\ \frac{1}{2}\right)$ in the direction of $\vec{v} = \hat{i} + 2\hat{j} + 2\hat{k}$

Solution

$$\vec{u} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1+4+4}} \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$\underline{= \tfrac{1}{3}\,\hat{i} + \tfrac{2}{3}\,\hat{j} + \tfrac{2}{3}\,\hat{k}}$$

$$h_x = -y \sin xy + \frac{1}{x}\Big|_{\left(1,0,\frac{1}{2}\right)}$$

$$= -(0)\sin(0) + \frac{1}{1}$$

$$\underline{= 1}$$

$$h_y = -x \sin xy + ze^{yz}\Big|_{\left(1,0,\frac{1}{2}\right)}$$

$$= -(1)\sin 0 + \frac{1}{2}e^0$$

$$\underline{= \frac{1}{2}}$$

$$h_z = ye^{yz} + \frac{1}{z}\Big|_{\left(1,0,\frac{1}{2}\right)}$$

$$= 0e^0 + \frac{1}{\frac{1}{2}}$$

$$\underline{= 2}$$

$$\nabla h = \hat{i} + \frac{1}{2}\,\hat{j} + 2\hat{k}$$

$$\left(D_{\vec{u}}\, h\right)_{P_0} = \nabla h \bullet \vec{u}$$

$$=\left(\hat{i}+\frac{1}{2}\hat{j}+2\hat{k}\right)\cdot\left(\frac{1}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}\right)$$

$$=\frac{1}{3}+\frac{1}{3}+\frac{4}{3}$$

$$\boxed{=2}$$

Exercise

Find the directions in which the function $f(x,y)=x^2+xy+y^2$ increase and decrease most rapidly at $P_0(-1,\ 1)$. Then find the derivatives of the function in these directions.

Solution

$$\left.\begin{aligned} f_x &= 2x+y \quad\Rightarrow\quad f_x(-1,1)=2(-1)+1=-1 \\ f_y &= x+2y \quad\Rightarrow\quad f_y(-1,1)=(-1)+2(1)=1 \end{aligned}\right\} \to\quad \nabla f=-\hat{i}+\hat{j}$$

$$\vec{u}=\frac{\nabla f}{|\nabla f|}=\frac{-\hat{i}+\hat{j}}{\sqrt{1+1}}$$

$$\boxed{=-\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}}$$

f *increases* most rapidly in the direction $\vec{u}=-\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}$

f *decreases* most rapidly in the direction $-\vec{u}=\frac{1}{\sqrt{2}}\hat{i}-\frac{1}{\sqrt{2}}\hat{j}$

$$\left(D_{\vec{u}}f\right)_{P_0}=\nabla f\cdot\vec{u}$$

$$=\left(-\hat{i}+\hat{j}\right)\cdot\left(-\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}\right)$$

$$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}$$

$$\boxed{=\sqrt{2}}$$

$$\boxed{\left(D_{-\vec{u}}f\right)_{P_0}=-\sqrt{2}}$$

Exercise

Find the directions in which the function $f(x,y)=x^2y+e^{xy}\sin y$ increase and decrease most rapidly at $P_0(1,\ 0)$. Then find the derivatives of the function in these directions.

Solution

$$f_x = 2xy + ye^{xy}\sin y\Big|_{(1,\ 0)}$$

$$= 2(1)(0) + 0e^0$$

$$\underline{= 0}$$

$$f_y = x^2 + xe^{xy}\sin y + e^{xy}\cos y\Big|_{(1,\ 0)}$$

$$= 1^2 + 0 + 1$$

$$\underline{= 2}$$

$$\nabla f = 2\hat{j}$$

$$\vec{u} = \frac{\nabla f}{|\nabla f|} = \hat{j}$$

f *increases* most rapidly in the direction $\vec{u} = \hat{j}$

f *decreases* most rapidly in the direction $-\vec{u} = -\hat{j}$

$$\left(D_{\vec{u}}\, f\right)_{P_0} = \nabla f \bullet \vec{u}$$

$$\underline{= 2}$$

$$\left(D_{-\vec{u}}\, f\right)_{P_0} = -2$$

Exercise

Find the directions in which the function $g(x, y, z) = xe^y + z^2$ increase and decrease most rapidly at $P_0\left(1,\ \ln 2,\ \frac{1}{2}\right)$. Then find the derivatives of the function in these directions.

Solution

$$g_x = e^y \quad \Rightarrow \quad g_x\left(1, \ln 2, \frac{1}{2}\right) = e^{\ln 2} = 2$$

$$g_y = xe^y \quad \Rightarrow \quad g_y\left(1, \ln 2, \frac{1}{2}\right) = e^{\ln 2} = 2 \quad \to \quad \nabla g = 2\hat{i} + 2\hat{j} + \hat{k}$$

$$g_z = 2z \quad \Rightarrow \quad g_z\left(1, \ln 2, \frac{1}{2}\right) = 2\left(\frac{1}{2}\right) = 1$$

$$g_x = e^y\Big|_{\left(1, \ln 2, \frac{1}{2}\right)}$$

$$= e^{\ln 2}$$

$$\underline{= 2}$$

$$g_y = xe^y\Big|_{\left(1, \ln 2, \frac{1}{2}\right)}$$

$$= e^{\ln 2}$$
$$\underline{= 2}$$

$$g_z = 2z \Big|_{\left(1, \ln 2, \frac{1}{2}\right)}$$
$$= 2\left(\frac{1}{2}\right)$$
$$\underline{= 1}$$

$$\nabla g = 2\hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{u} = \frac{\nabla g}{|\nabla g|} = \frac{2\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{4+4+1}}$$
$$\underline{= \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}}$$

g *increases* most rapidly in the direction $\vec{u} = \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$

g *decreases* most rapidly in the direction $-\vec{u} = -\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$

$$\left(D_{\vec{u}}\, g\right)_{P_0} = \nabla g \bullet \vec{u}$$
$$= \left(2\hat{i} + 2\hat{j} + \hat{k}\right) \bullet \left(\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}\right)$$
$$= \frac{4}{3} + \frac{4}{3} + \frac{1}{3}$$
$$\underline{= 3}$$

$$\underline{\left(D_{-\vec{u}}\, g\right)_{P_0} = -3}$$

Exercise

Find the directions in which the function $h(x, y, z) = \ln\left(x^2 + y^2 - 1\right) + y + 6z$ increase and decrease most rapidly at $P_0\,(1,\ 1,\ 0)$. Then find the derivatives of the function in these directions.

Solution

$$h_x = \frac{2x}{x^2 + y^2 - 1} \quad \Rightarrow \quad h_x(1,1,0) = \frac{2}{1+1-1} = 2$$

$$h_y = \frac{2y}{x^2 + y^2 - 1} + 1 \quad \Rightarrow \quad h_y(1,1,0) = \frac{2}{1+1-1} + 1 = 3$$

$$h_z = 6 \quad \Rightarrow \quad h_z(1,1,0) = 6$$

$$h_x = \left.\frac{2x}{x^2+y^2-1}\right|_{(1,1,0)}$$

$$= \frac{2}{1+1-1}$$

$$\boxed{=2}$$

$$h_y = \left.\frac{2y}{x^2+y^2-1}+1\right|_{(1,1,0)}$$

$$= \frac{2}{1+1-1}+1$$

$$\boxed{=3}$$

$$\boxed{h_z = 6}$$

$$\nabla h = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

$$\vec{u} = \frac{\nabla h}{|\nabla h|} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{4+9+36}}$$

$$\boxed{= \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}}$$

h *increases* most rapidly in the direction $\vec{u} = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$

h *decreases* most rapidly in the direction $-\vec{u} = -\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}$

$$\left(D_{\vec{u}}h\right)_{P_0} = \nabla h \bullet \vec{u}$$

$$= \left(2\hat{i}+3\hat{j}+6\hat{k}\right) \bullet \left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$$

$$= \frac{4}{7} + \frac{9}{7} + \frac{36}{7}$$

$$\boxed{=7}$$

$$\left(D_{-\boldsymbol{u}}h\right)_{P_0} \boxed{=-7}$$

Exercise

Sketch the curve $x^2 + y^2 = 4;\ \ \left(f(x,y)=c\right)$ together with ∇f and the tangent line at the point $\left(\sqrt{2},\ \sqrt{2}\right)$. Then write an equation for the tangent line.

Solution

$$f_x = 2x \quad \Rightarrow \quad f_x\left(\sqrt{2},\sqrt{2}\right) = 2\sqrt{2}$$

$$f_y = 2y \quad \Rightarrow \quad f_y\left(\sqrt{2},\sqrt{2}\right) = 2\sqrt{2}$$

$$\nabla f = 2\sqrt{2}\hat{i} + 2\sqrt{2}\hat{j}$$

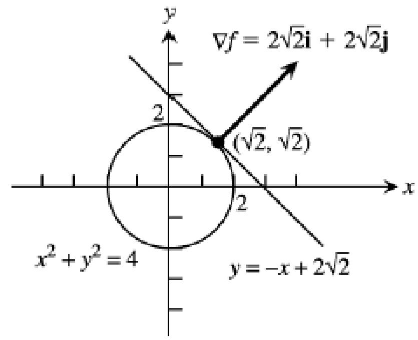
Tangent line:

$$2\sqrt{2}\left(x-\sqrt{2}\right)+2\sqrt{2}\left(y-\sqrt{2}\right)=0$$

$$2\sqrt{2}x-4+2\sqrt{2}y-4=0$$

$$2\sqrt{2}x+2\sqrt{2}y=8$$

$$\boxed{\sqrt{2}x+\sqrt{2}y=4}$$



Exercise

Sketch the curve $x^2 - y = 1$; $\left(f(x,y)=c\right)$ together with ∇f and the tangent line at the point $\left(\sqrt{2},\ 1\right)$. Then write an equation for the tangent line.

Solution

$$f_x = 2x \quad \Rightarrow \quad f_x\left(\sqrt{2},\ 1\right) = 2\sqrt{2}$$

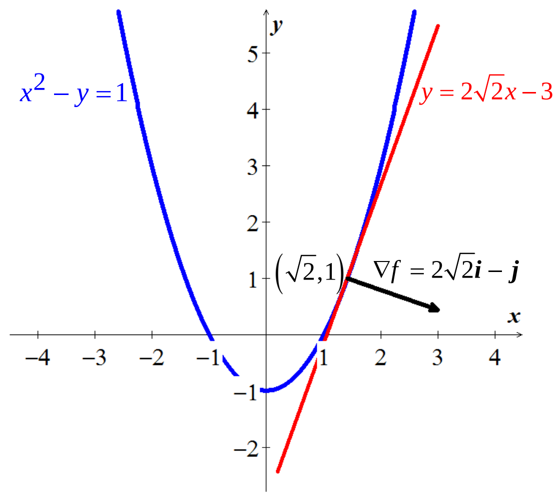
$$f_y = -1 \quad \Rightarrow \quad f_y\left(\sqrt{2},\ 1\right) = -1$$

$$\nabla f = 2\sqrt{2}\hat{i} - \hat{j}$$

Tangent line: $2\sqrt{2}\left(x-\sqrt{2}\right)-\left(y-1\right)=0$

$$2\sqrt{2}x-4-y+1=0$$

$$\boxed{y=2\sqrt{2}x-3}$$



Exercise

Sketch the curve $x^2 - xy + y^2 = 7$; $\left(f(x,y)=c\right)$ together with ∇f and the tangent line at the point $\left(-1,\ 2\right)$. Then write an equation for the tangent line.

Solution

$$f_x = 2x - y \quad \Rightarrow \quad f_x\left(-1,\ 2\right) = -4$$

$$f_y = -x + 2y \quad \Rightarrow \quad f_y\left(-1,\ 2\right) = 5$$

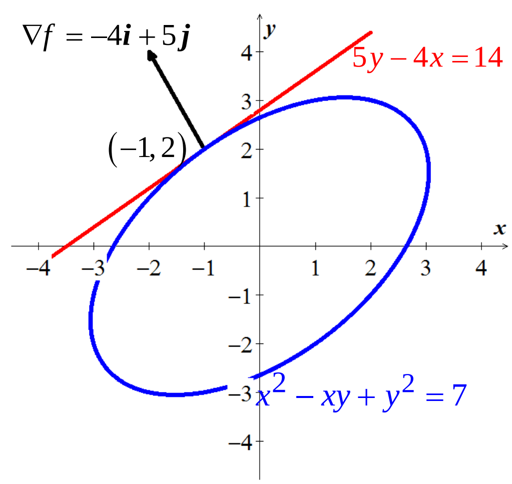
$$\rightarrow \nabla f = -4\hat{i} + 5\hat{j}$$

Tangent line:

$$-4(x+1)+5(y-2)=0$$

$$-4x+5y-14=0$$

$$\boxed{5y-4x=14}$$



Exercise

In what direction is the derivative of $f(x, y) = xy + y^2$ at $P(3,\ 2)$ equal to zero?

Solution

$$\begin{aligned} f_x &= y \\ f_y &= x + 2y \end{aligned} \quad \rightarrow \quad \nabla f = x\hat{i} + (x+2y)\hat{j}$$

$$\nabla f(3,\ 2) = 2\hat{i} + 7\hat{j}$$

A vector is orthogonal to ∇f is $\vec{v} = 7\hat{i} - 2\hat{j}$

$$\vec{u} = \frac{7\hat{i} - 2\hat{j}}{\sqrt{49+4}} \qquad\qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{7}{\sqrt{53}}\hat{i} - \frac{2}{\sqrt{53}}\hat{j}$$

$$-\vec{u} = -\frac{7}{\sqrt{53}}\hat{i} + \frac{2}{\sqrt{53}}\hat{j}$$

$\vec{u}$ and $-\vec{u}$ are the directions where the derivatives are zero.

Exercise

Compute the gradient of the function, evaluate it at the given point *P*, and evaluate the directional derivative at that point in the given direction

$$f(x,\ y) = x^2; \quad P = (1,\ 2); \quad \vec{u} = \left\langle \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}} \right\rangle$$

Solution

$$\nabla f = \langle 2x,\ 0 \rangle \qquad\qquad \nabla f = \left\langle f_x,\ f_y \right\rangle$$

$$\nabla f(1,\ 2) = \langle 2,\ 0 \rangle$$

$$\begin{aligned} \left(D_{\vec{u}} f\right)_{(1,\ 2)} &= \nabla f \Big|_{(1,\ 2)} \bullet \vec{u} \\ &= \langle 2,\ 0 \rangle \bullet \left\langle \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}} \right\rangle \\ &= \frac{2}{\sqrt{2}} \\ &= \sqrt{2} \end{aligned}$$

Exercise

Compute the gradient of the function, evaluate it at the given point P, and evaluate the directional derivative at that point in the given direction

$$f(x, y) = x^2 y^3; \quad P = (-1, 1); \quad \vec{u} = \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$$

Solution

$$\nabla f = \left\langle 2xy^3, \ 3x^2y^2 \right\rangle \qquad \nabla f = \left\langle f_x, f_y \right\rangle$$

$$\nabla f(-1, 1) = \langle -2, 3 \rangle$$

$$\begin{aligned}
\left(D_{\vec{u}} f\right)_{(-1, 1)} &= \nabla f \Big|_{(-1, 1)} \bullet \vec{u} \\
&= \langle -2, 3 \rangle \bullet \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle \\
&= -\frac{10}{13} + \frac{36}{13} \\
&= \frac{26}{3} \\
&= 2
\end{aligned}$$

Exercise

Compute the gradient of the function, evaluate it at the given point P, and evaluate the directional derivative at that point in the given direction

$$f(x, y) = \frac{x}{y^2}; \quad P = (0, 3); \quad \vec{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$

Solution

$$\nabla f = \left\langle \frac{1}{y^2}, \ -\frac{2x}{y^3} \right\rangle \qquad \nabla f = \left\langle f_x, f_y \right\rangle$$

$$\nabla f(0, 3) = \left\langle \frac{1}{9}, 0 \right\rangle$$

$$\begin{aligned}
\left(D_{\vec{u}} f\right)_{(0, 3)} &= \nabla f \Big|_{(0, 3)} \bullet \vec{u} \\
&= \left\langle \frac{1}{9}, 0 \right\rangle \bullet \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle \\
&= \frac{\sqrt{13}}{18}
\end{aligned}$$

Exercise

Compute the gradient of the function, evaluate it at the given point P, and evaluate the directional derivative at that point in the given direction

$$f(x,\ y)=\sqrt{2+x^2+2y^2};\quad P=(2,\ 1);\quad \vec{u}=\left\langle \frac{3}{5},\ \frac{4}{5}\right\rangle$$

Solution

$$\nabla f=\left\langle \frac{x}{\sqrt{2+x^2+2y^2}},\ \frac{2y}{\sqrt{2+x^2+2y^2}}\right\rangle \qquad \nabla f=\left\langle f_x,\ f_y\right\rangle$$

$$\begin{aligned}\nabla f(2,\ 1)&=\left\langle \frac{2}{\sqrt{8}},\ \frac{2}{\sqrt{8}}\right\rangle\\&=\left\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right\rangle\end{aligned}$$

$$\begin{aligned}\left(D_{\vec{u}}f\right)_{(2,\ 1)}&=\nabla f\Big|_{(2,\ 1)}\bullet\vec{u}\\&=\left\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right\rangle\bullet\left\langle \frac{3}{5},\ \frac{4}{5}\right\rangle\\&=\frac{3}{5\sqrt{2}}+\frac{4}{5\sqrt{2}}\\&=\frac{7}{5\sqrt{2}}\\&=\frac{7\sqrt{2}}{10}\end{aligned}$$

Exercise

Compute the gradient of the function, evaluate it at the given point P, and evaluate the directional derivative at that point in the given direction

$$f(x,\ y,\ z)=xy+yz+xz+4;\quad P=(2,\ -2,\ 1);\quad \vec{u}=\left\langle 0,\ -\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right\rangle$$

Solution

$$\nabla f=\left\langle y+z,\ x+z,\ y+x\right\rangle \qquad \nabla f=\left\langle f_x,\ f_y,\ f_z\right\rangle$$

$$\nabla f(2,\ -2,\ 1)=\langle -1,\ 3,\ 0\rangle$$

$$\begin{aligned}\left(D_{\vec{u}}f\right)_{(2,\ -2,\ 1)}&=\nabla f\Big|_{(2,\ -2,\ 1)}\bullet\vec{u}\\&=\langle -1,\ 3,\ 0\rangle\bullet\left\langle 0,\ -\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right\rangle\end{aligned}$$

$$= -\frac{3}{\sqrt{2}}$$

Exercise

Compute the gradient of the function, evaluate it at the given point P, and evaluate the directional derivative at that point in the given direction

$$f(x, y, z) = 1 + \sin(x + 2y - z); \quad P = \left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right); \quad \vec{u} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$$

Solution

$$\nabla f = \langle \cos(x + 2y - z), \ 2\cos(x + 2y - z), \ -\cos(x + 2y - z) \rangle \qquad \nabla f = \left\langle f_x, f_y, f_z \right\rangle$$

$$\nabla f\left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right) = \left\langle \cos\frac{2\pi}{3}, \ 2\cos\frac{2\pi}{3}, \ -\cos\frac{2\pi}{3} \right\rangle$$

$$= \left\langle -\frac{1}{2}, \ -1, \ \frac{1}{2} \right\rangle$$

$$\left(D_{\vec{u}} f\right)_{\left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right)} = \nabla f\Big|_{\left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right)} \bullet \vec{u}$$

$$= \left\langle -\frac{1}{2}, \ -1, \ \frac{1}{2} \right\rangle \bullet \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$$

$$= -\frac{1}{6} - \frac{2}{3} + \frac{1}{3}$$

$$= -\frac{1}{2}$$

Exercise

Find the direction in which f increases and decreases most rapidly at P_0 and find the derivative of f in each direction. Also, find the derivative of f at P_0 in the direction of the vector $\boldsymbol{v}$.

$$f(x, y) = \cos x \cos y, \quad P_0\left(\frac{\pi}{4}, \frac{\pi}{4}\right), \quad \vec{v} = 3\hat{i} + 4\hat{j}$$

Solution

$$\nabla f = -\sin x \cos y\,\hat{i} - \cos x \sin y\,\hat{j} \qquad \nabla f = f_x\,\hat{i} + f_y\,\hat{j}$$

$$\nabla f\Big|_{\left(\frac{\pi}{4}, \frac{\pi}{4}\right)} = -\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\hat{j}$$

$$= -\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j}$$

$$|\nabla f| = \sqrt{\frac{1}{4} + \frac{1}{4}}$$

$$= \boxed{\frac{\sqrt{2}}{2}}$$

$$\vec{u} = \frac{\nabla f}{|\nabla f|} = \sqrt{2}\left(-\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j}\right)$$

$$= -\frac{\sqrt{2}}{2}\hat{i} - \frac{\sqrt{2}}{2}\hat{j}$$

$\rightarrow f$ increases most rapidly in the direction of $\vec{u} = -\frac{\sqrt{2}}{2}\hat{i} - \frac{\sqrt{2}}{2}\hat{j}$

$\rightarrow f$ decreases most rapidly in the direction of $-\vec{u} = \frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j}$

$$\vec{u}_1 = \frac{3\hat{i} + 4\hat{j}}{\sqrt{9+16}} \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$$

$$\left(D_{\vec{u}} f\right)_{\left(\frac{\pi}{4},\ \frac{\pi}{4}\right)} = \nabla f\Big|_{\left(\frac{\pi}{4},\ \frac{\pi}{4}\right)} \bullet \vec{u}_1$$

$$= \left(-\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j}\right) \bullet \left(\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}\right)$$

$$= -\frac{3}{10} - \frac{4}{10}$$

$$= \boxed{-\frac{7}{10}}$$

Exercise

Find the direction in which f increases and decreases most rapidly at P_0 and find the derivative of f in each direction. Also, find the derivative of f at P_0 in the direction of the vector $\boldsymbol{v}$.

$$f(x, y) = x^2 e^{-2y}, \quad P_0\ (1,\ 0), \quad \vec{v} = \hat{i} + \hat{j}$$

Solution

$$\nabla f = 2xe^{-2y}\hat{i} - 2x^2 e^{-2y}\hat{j} \qquad \nabla f = f_x \hat{i} + f_y \hat{j}$$

$$\nabla f\Big|_{(1,\ 0)} = 2\hat{i} - 2\hat{j}$$

$$|\nabla f| = \sqrt{4+4}$$

$$= \boxed{2\sqrt{2}}$$

$$\vec{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{2\sqrt{2}}\left(2\hat{i} - 2\hat{j}\right)$$

$$= \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$$

$\to f$ increases most rapidly in the direction of $\vec{u} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

$\to f$ decreases most rapidly in the direction of $-\vec{u} = -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

$$\vec{u}_1 = \frac{\hat{i} + \hat{j}}{\sqrt{1+1}} \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

$$\left(D_{\vec{u}} f\right)_{(1,\ 0)} = \nabla f \Big|_{(1,\ 0)} \cdot \vec{u}_1$$

$$= \left(2\hat{i} - 2\hat{j}\right) \cdot \left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)$$

$$= \sqrt{2} - \sqrt{2}$$

$$\underline{= 0 \rfloor}$$

Exercise

Find the direction in which f increases and decreases most rapidly at P_0 and find the derivative of f in each direction. Also, find the derivative of f at P_0 in the direction of the vector ***v***.

$$f(x, y, z) = \ln(2x + 3y + 6z), \quad P_0(-1,\ -1,\ 1), \quad \vec{v} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

Solution

$$\nabla f = \frac{2}{2x+3y+6z}\hat{i} + \frac{3}{2x+3y+6z}\hat{j} + \frac{6}{2x+3y+6z}\hat{k} \qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$\nabla f \Big|_{(-1,\ -1,\ 1)} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

$$|\nabla f| = \sqrt{4 + 9 + 36}$$

$$\underline{= 7 \rfloor}$$

$$\vec{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)$$

$$= \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

$\to f$ increases most rapidly in the direction of $\vec{u} = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$

$\to f$ decreases most rapidly in the direction of $-\vec{u} = -\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}$

$$\vec{u}_1 = \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{4+9+36}} \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

$$\left(D_{\vec{u}} f\right)_{(-1,\ -1,\ 1)} = \nabla f \Big|_{(-1,\ -1,\ 1)} \bullet \vec{u}_1$$

$$= \left(2\hat{i} + 3\hat{j} + 6\hat{k}\right) \bullet \left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$$

$$= \frac{4}{7} + \frac{9}{7} + \frac{36}{7}$$

$$= 7$$

Exercise

Find the direction in which f increases and decreases most rapidly at P_0 and find the derivative of f in each direction. Also, find the derivative of f at P_0 in the direction of the vector $\boldsymbol{v}$.

$$f(x, y, z) = x^2 + 3xy - z^2 + 2y + z + 4, \quad P_0\ (0,\ 0,\ 0), \quad \vec{v} = \hat{i} + \hat{j} + \hat{k}$$

Solution

$$\nabla f = (2x + 3y)\hat{i} + (3x + 2)\hat{j} + (-2z + 1)\hat{k} \qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$\nabla f \Big|_{(0,\ 0,\ 0)} = 2\hat{j} + \hat{k}$$

$$|\nabla f| = \sqrt{4+1}$$

$$= \sqrt{5}$$

$$\vec{u} = \frac{\nabla f}{|\nabla f|} = \frac{2}{\sqrt{5}}\hat{j} + \frac{1}{\sqrt{5}}\hat{k}$$

$\rightarrow$ f increases most rapidly in the direction of $\vec{u} = \frac{2}{\sqrt{5}}\hat{j} + \frac{1}{\sqrt{5}}\hat{k}$

$\rightarrow$ f decreases most rapidly in the direction of $-\vec{u} = -\frac{2}{\sqrt{5}}\hat{j} - \frac{1}{\sqrt{5}}\hat{k}$

$$\vec{u}_1 = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{1+1+1}} \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

$$\left(D_{\vec{u}} f\right)_{(0,\ 0,\ 0)} = \nabla f \Big|_{(0,\ 0,\ 0)} \bullet \vec{u}_1$$

$$= \left(2\hat{j} + \hat{k}\right) \bullet \left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right)$$

$$= \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}$$

$$= \frac{3}{\sqrt{3}}$$

$$\boxed{= \sqrt{3}}$$

Exercise

Let $f(x, y) = \ln(1 + xy); \quad P = (2,\ 3)$

a) Find the unit vectors that give the direction of steepest ascent and steepest descent at P.
b) Find a unit vector that points in ta direction of no change.

Solution

a) $\nabla f = \frac{y}{1+xy}\hat{i} + \frac{x}{1+xy}\hat{j}$ $\qquad \nabla f = f_x \hat{i} + f_y \hat{j}$

$$\nabla f(2,\ 3) = \frac{3}{7}\hat{i} + \frac{2}{7}\hat{j}$$

$$= \frac{1}{7}\left(3\hat{i} + 2\hat{j}\right)$$

$$\vec{u} = \frac{3\hat{i} + 2\hat{j}}{\sqrt{9+4}}$$

$$\boxed{= \frac{3}{\sqrt{13}}\hat{i} + \frac{2}{\sqrt{13}}\hat{j}}$$

The direction of steepest ascent is the unit vector in the direction of $\vec{u} = \frac{3}{\sqrt{13}}\hat{i} + \frac{2}{\sqrt{13}}\hat{j}$

The direction of steepest descent is the unit vector in the direction of $-\vec{u} = -\frac{3}{\sqrt{13}}\hat{i} - \frac{2}{\sqrt{13}}\hat{j}$

b) The unit vectors that the point in the direction of no change are $\vec{v} = \pm\left(\frac{3}{\sqrt{13}}\hat{i} - \frac{2}{\sqrt{13}}\hat{j}\right)$

Since $\vec{u} \bullet \vec{v} = 0$

Exercise

Let $f(x, y) = \sqrt{4 - x^2 - y^2}; \quad P = (-1,\ 1)$

a) Find the unit vectors that give the direction of steepest ascent and steepest descent at P.
b) Find a unit vector that points in ta direction of no change.

Solution

a) $\nabla f = -\frac{x}{\sqrt{4-x^2-y^2}}\hat{i} - \frac{y}{\sqrt{4-x^2-y^2}}\hat{j}$ $\qquad \nabla f = f_x \hat{i} + f_y \hat{j}$

$$\nabla f(-1,\ 1) = \frac{1}{\sqrt{2}}\left(\hat{i} - \hat{j}\right)$$

$$\vec{u} = \frac{\hat{i} - \hat{j}}{\sqrt{1+1}}$$

$$= \frac{1}{\sqrt{2}}\left(\hat{i} - \hat{j}\right)$$

The direction of steepest ascent is the unit vector in the direction of $\vec{u} = \frac{\sqrt{2}}{2}\hat{i} - \frac{\sqrt{2}}{2}\hat{j}$

The direction of steepest descent is the unit vector in the direction of $-\vec{u} = -\frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j}$

b) The unit vectors that the point in the direction of no change are $\vec{v} = \pm\left(\frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j}\right)$

Since $\vec{u} \bullet \vec{v} = 0$

Exercise

Let $f(x,\ y) = 8 - 2x^2 - y^2$, for the level curves $f(x,\ y) = C$ and points $(a,\ b)$, compute the slope of the line tangent to the level curve at $(a,\ b)$ and verify that the tangent line is orthogonal to the gradient at that point.

$$f(x,\ y) = 5; \quad (a,\ b) = (1,\ 1)$$

Solution

$$\frac{dy}{dx} = -\frac{4x}{-2y} \qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= \frac{2x}{y}\Big|_{(1,\ 1)}$$

$$= -2$$

$m = -2 \quad f(x,\ y) = 5$

Tangent line has direction: $\quad \hat{i} - 2\hat{j}$

$$\nabla f = -4x\hat{i} - 2y\hat{j} \qquad \nabla f = f_x\hat{i} + f_y\hat{j}$$

$$\nabla f(1,\ 1) = -4\hat{i} - 2\hat{j}$$

$$\left(\hat{i} - 2\hat{j}\right) \bullet \left(-4\hat{i} - 2\hat{j}\right) = -4 + 4$$

$$= 0 \quad \checkmark$$

The gradient is orthogonal to the tangent direction.

Exercise

Let $f(x, y) = 8 - 2x^2 - y^2$, for the level curves $f(x, y) = C$ and points (a, b), compute the slope of the line tangent to the level curve at (a, b) and verify that the tangent line is orthogonal to the gradient at that point.

$$f(x, y) = 0; \quad (a, b) = (2, 0)$$

Solution

$$\frac{dy}{dx} = -\frac{4x}{-2y} \qquad\qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= \frac{2x}{y}\bigg|_{(2,\ 0)}$$

$$\boxed{= \infty}$$

Slope: $m = 0$

Tangent line has direction: $\hat{j}$

$$\nabla f = -4x\hat{i} - 2y\hat{j} \qquad\qquad \nabla f = f_x\hat{i} + f_y\hat{j}$$

$$\nabla f(2,\ 0) = -8\hat{i}$$

$$(\hat{j}) \cdot (-8\hat{i}) \ \boxed{= 0} \quad \checkmark$$

The gradient is orthogonal to the tangent direction.

Exercise

Find the direction in which the function $f(x, y) = 4x^2 - y^2$ has zero change at the point $(1,\ 1,\ 3)$. Express the directions in terms of unit vectors.

Solution

$$\nabla f = 8x\hat{i} - 2y\hat{j} \qquad\qquad \nabla f = f_x\hat{i} + f_y\hat{j}$$

$$\nabla f(1,\ 1,\ 3) = 8\hat{i} - 2\hat{j}$$

The unit vectors in the direction of no change are

$$\vec{u} = \pm\frac{8\hat{i} - 2\hat{j}}{\sqrt{64 + 4}}$$

$$= \pm\frac{2(4\hat{i} - \hat{j})}{2\sqrt{17}}$$

$$\boxed{= \pm\frac{1}{\sqrt{17}}(4\hat{i} - \hat{j})}$$

Exercise

An infinitely long charged cylinder of radius R with its axis along z-axis has an electric potential $V = k\ln\left(\frac{R}{r}\right)$, where r is the distance between a variable point $P(x,\ y)$ and the axis of the cylinder $\left(r^2 = x^2 + y^2\right)$ and k is a physical constant. The electric field at a point $(x,\ y)$ in the xy-plane is given by $\boldsymbol{E} = -\nabla V$, where ∇V is the two-dimensional gradient. Compute the electric field at a point $(x,\ y)$ with $r > R$.

Solution

$$V = k(\ln R - \ln r)$$

$$= k\left(\ln R - \ln\sqrt{x^2 + y^2}\right)$$

$$= k\left(\ln R - \frac{1}{2}\ln\left(x^2 + y^2\right)\right)$$

$$= \frac{1}{2}k\left(2\ln R - \ln\left(x^2 + y^2\right)\right)$$

$$= \frac{1}{2}k\left(\ln R^2 - \ln\left(x^2 + y^2\right)\right)$$

$$\vec{E} = -\nabla V$$

$$= -\frac{1}{2}k\left(-\frac{2x}{x^2 + y^2}\hat{i} - \frac{2y}{x^2 + y^2}\hat{j}\right)$$

$$= \frac{kx}{x^2 + y^2}\hat{i} + \frac{ky}{x^2 + y^2}\hat{j}$$