Section 2.4 – Cross Product

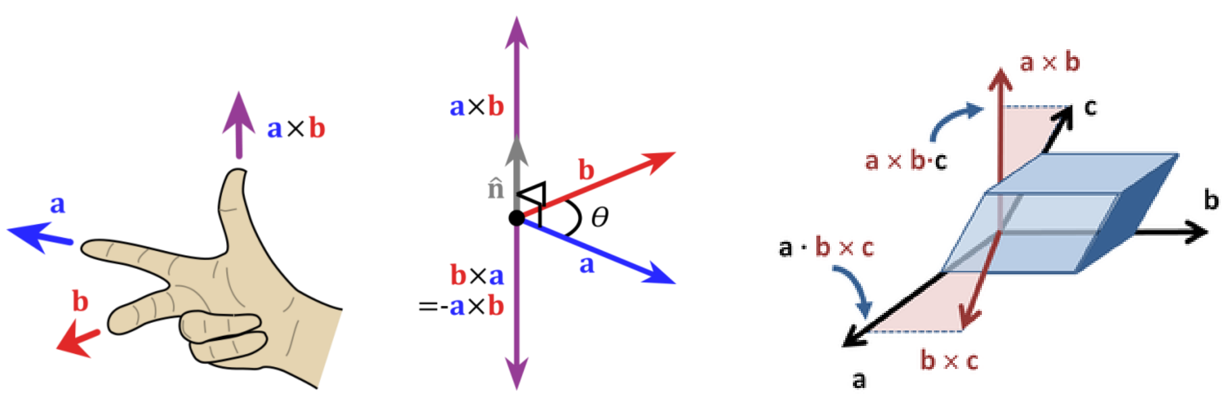
The *Cross* Product

To find a vector in 3-space that is perpendicular to two vectors; the type of vector multiplication that facilities this construction is the cross product.

Definition

The cross product of $\boldsymbol{u} = \left(u_1,\ u_2,\ u_3\right)$ and $\boldsymbol{v} = \left(v_1,\ v_2,\ v_3\right)$ is the vector

$$\begin{aligned}
\boldsymbol{u} \times \boldsymbol{v} &= \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\
&= \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} u_1 & v_2 \\ u_2 & v_1 \end{vmatrix} \boldsymbol{k} \\
&= \left(u_2 v_3 - u_3 v_2\right)\boldsymbol{i} - \left(u_1 v_3 - u_3 v_1\right)\boldsymbol{j} + \left(u_1 v_2 - u_2 v_1\right)\boldsymbol{k} \\
&\boxed{= \left(u_2 v_3 - u_3 v_2,\ \ u_3 v_1 - u_1 v_3,\ \ u_1 v_2 - u_2 v_1\right)}
\end{aligned}$$



- In 1773, ***Joseph Louis Lagrange*** introduced the component form of both the dot and cross products in order to study the tetrahedron in three dimensions. In 1843 the Irish mathematical physicist Sir ***William Rowan Hamilton*** introduced the quaternion product, and with it the terms "*vector*" and "*scalar*". Given two quaternions [0, $\boldsymbol{u}$] and [0, $\boldsymbol{v}$], where $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbf{R}^3$, their quaternion product can be summarized as $[-\boldsymbol{u}\cdot\boldsymbol{v},\ \ \boldsymbol{u}\times\boldsymbol{v}]$. ***James Clerk Maxwell*** used Hamilton's quaternion tools to develop his famous ***electromagnetism*** equations, and for this and other reasons quaternions for a time were an essential part of physics education.

Example

Find $\boldsymbol{u}\times\boldsymbol{v}$, where $\boldsymbol{u}=(1,\ 2,\ -2)$ and $\boldsymbol{v}=(3,\ 0,\ 1)$

Solution

$$\begin{bmatrix} 1 & 2 & -2 \\ 3 & 0 & 1 \end{bmatrix}$$

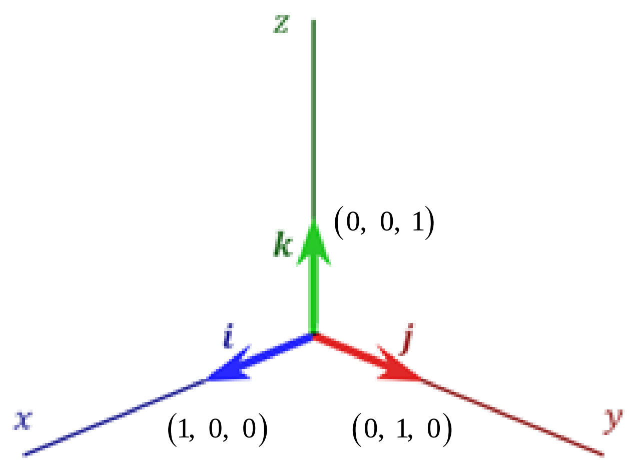
$$\boldsymbol{u}\times\boldsymbol{v}=\left(\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix},\ -\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix},\ \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}\right)$$

$$=(2,\ -7,\ -6)$$

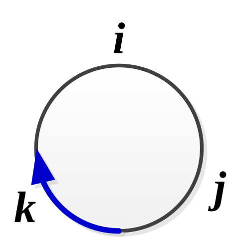
Example

Consider the vectors $\hat{\boldsymbol{i}}=(1,\ 0,\ 0)\quad \hat{\boldsymbol{j}}=(0,\ 1,\ 0)\quad \hat{\boldsymbol{k}}=(0,\ 0,\ 1)$

These vectors each have length of 1 and lie along the coordinate axes. They are called the ***standard unit vectors*** in 3-space.



For example: $(2,\ 3,\ -4)=2\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}-4\hat{\boldsymbol{k}}$



Note:

- ✓ $\hat{\boldsymbol{i}}\times\hat{\boldsymbol{i}}=\hat{\boldsymbol{j}}\times\hat{\boldsymbol{j}}=\hat{\boldsymbol{k}}\times\hat{\boldsymbol{k}}=\mathbf{0}$
- ✓ $\hat{\boldsymbol{j}}\times\hat{\boldsymbol{i}}=-\hat{\boldsymbol{k}},\quad \hat{\boldsymbol{k}}\times\hat{\boldsymbol{j}}=-\hat{\boldsymbol{i}},\quad \hat{\boldsymbol{i}}\times\hat{\boldsymbol{k}}=-\hat{\boldsymbol{j}}$
- ✓ $\hat{\boldsymbol{i}}\times\hat{\boldsymbol{j}}=\hat{\boldsymbol{k}},\quad \hat{\boldsymbol{j}}\times\hat{\boldsymbol{k}}=\hat{\boldsymbol{i}},\quad \hat{\boldsymbol{k}}\times\hat{\boldsymbol{i}}=\hat{\boldsymbol{j}}$

Properties

1. $\boldsymbol{u} \times \boldsymbol{v}$ reverses rows 2 and 3 in the determinant so it is equals $-(\boldsymbol{u} \times \boldsymbol{v})$

2. The cross product $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$, then $\boldsymbol{u}.(\boldsymbol{u} \times \boldsymbol{v}) = 0$

3. The cross product $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to $\boldsymbol{v}$, then $(\boldsymbol{u} \times \boldsymbol{v}).\boldsymbol{v} = 0$

4. The cross product of any vector with itself (two equal rows) is $\boldsymbol{u} \times \boldsymbol{u} = 0$.

5. Lagrange's identity: $\|\boldsymbol{u} \times \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 \ \|\boldsymbol{v}\|^2 - (\boldsymbol{u}.\boldsymbol{v})^2$

$$= \|\boldsymbol{u}\| \ \|\boldsymbol{v}\| \ |\sin\theta|$$

$$|\boldsymbol{u}.\boldsymbol{v}| = \|\boldsymbol{u}\| \ \|\boldsymbol{v}\| \ |\cos\theta|$$

Theorem

a) $\boldsymbol{u} \times \boldsymbol{v} = -(\boldsymbol{v} \times \boldsymbol{u})$

b) $\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} \times \boldsymbol{v}) + (\boldsymbol{u} \times \boldsymbol{w})$

c) $(\boldsymbol{u} + \boldsymbol{v}) \times \boldsymbol{w} = (\boldsymbol{u} \times \boldsymbol{w}) + (\boldsymbol{v} \times \boldsymbol{w})$

d) $k(\boldsymbol{u} \times \boldsymbol{v}) = (k\boldsymbol{u}) \times \boldsymbol{v} = \boldsymbol{u} \times (k\boldsymbol{v})$

e) $\boldsymbol{u} \times 0 = 0 \times \boldsymbol{u} = 0$

f) $\boldsymbol{u} \times \boldsymbol{u} = 0$

Definition

If $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are vectors in 3-space, then $\boxed{\boldsymbol{u} \bullet (\boldsymbol{v} \times \boldsymbol{w})}$ is called the ***scalar triple product*** of $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

Example

Calculate the scalar triple product $\vec{u} \bullet (\vec{u} \times \vec{v})$ of the vectors:

$$\boldsymbol{u} = -2\boldsymbol{i} + 6\boldsymbol{k} \quad \boldsymbol{v} = \boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k} \quad \boldsymbol{w} = -5\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$$

Solution

$$\vec{u} \bullet (\vec{u} \times \vec{v}) = \begin{vmatrix} -2 & 0 & 6 \\ 1 & -3 & 1 \\ -5 & -1 & 1 \end{vmatrix} \ \underline{= -92|}$$

Area of a Parallelogram

Theorem

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in 3-space, then $\|\boldsymbol{u}\times\boldsymbol{v}\|$ is equal to the area of the parallelogram determined by $\boldsymbol{u}$ and $\boldsymbol{v}$.

Example

Find the area of the triangle determined by the points $P_1(2,\ 2,\ 0),\ P_2(-1,\ 0,\ 2),\ and\ P_3(0,\ 4,\ 3)$.

Solution

The area of the triangle is $\frac{1}{2}$ the area of the parallelogram determined by the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$

$$\overrightarrow{P_1P_2}=(-1,\ 0,\ 2)-(2,\ 2,\ 0)=(-3,\ -2,\ 2)$$

$$\overrightarrow{P_1P_3}=(0,\ 4,\ 3)-(2,\ 2,\ 0)=(-2,\ 2,\ 3)$$

$$\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}=\left(\begin{vmatrix}-2 & 2\\ 2 & 3\end{vmatrix},\ -\begin{vmatrix}-3 & 2\\ -2 & 3\end{vmatrix},\ \begin{vmatrix}-3 & -2\\ -2 & 2\end{vmatrix}\right)$$

$$=(-10,\ 5,\ -10)$$

$$\text{Area }=\frac{1}{2}\left\|\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}\right\|$$

$$=\frac{1}{2}\sqrt{(-10)^2+5^2+(-10)^2}$$

$$=\frac{1}{2}(15)$$

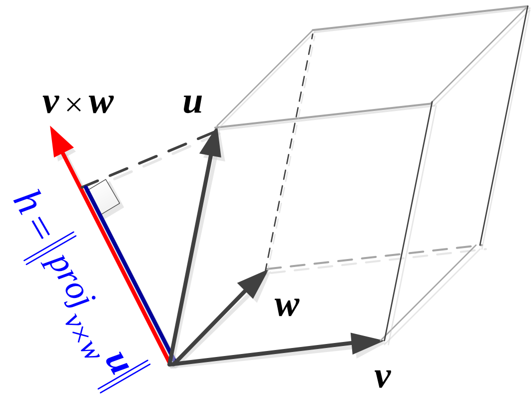
$$\boxed{=7.5}$$

Volume

The Volume of the Parallelepiped is

$$V = (area\ of\ \ base).(height) = \|\vec{v}\times\vec{w}\| \frac{|\vec{u}\bullet(\vec{v}\times\vec{w})|}{\|\vec{v}\times\vec{w}\|} \ = |\vec{u}\bullet(\vec{u}\times\vec{v})|$$

$$V = \left|\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}\right|$$



Theorem

If the vectors $\boldsymbol{u} = (u_1, u_2, u_3)$, $\boldsymbol{v} = (v_1, v_2, v_3)$, and $\boldsymbol{w} = (w_1, w_2, w_3)$ have the initial point, then they lie in the same plane if and only if

$$\boldsymbol{u}\bullet(\boldsymbol{v}\times\boldsymbol{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0$$

Example

Find the volume of the parallelepiped with sides $\boldsymbol{u} = (2,\ -6,\ 2)$, $\boldsymbol{v} = (0,\ 4,\ -2)$, and $\boldsymbol{w} = (2,\ 2,\ -4)$

Solution

$$V = \left|\det\begin{bmatrix} 2 & -6 & 2 \\ 0 & 4 & -2 \\ 2 & 2 & -4 \end{bmatrix}\right| \underline{=16|}$$

Exercises *Section* 2.4 – Cross Product

1. Prove when the cross product $\boldsymbol{u}\times\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$, then $\boldsymbol{u}.(\boldsymbol{u}\times\boldsymbol{v})=0$

2. Find $\boldsymbol{u}\times\boldsymbol{v}$, where $\boldsymbol{u}=(1,\ 2,\ -2)$ *and* $\boldsymbol{v}=(3,\ 0,\ 1)$ and show that $\boldsymbol{u}\times\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ and to $\boldsymbol{v}$.

3. Given $\boldsymbol{u}=(3,\ 2,\ -1)$, $\boldsymbol{v}=(0,\ 2,\ -3)$, *and* $\boldsymbol{w}=(2,\ 6,\ 7)$ Compute the vectors

a) $\boldsymbol{u}\times\boldsymbol{v}$
b) $\boldsymbol{v}\times\boldsymbol{w}$
c) $\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{w})$
d) $(\boldsymbol{u}\times\boldsymbol{v})\times\boldsymbol{w}$
e) $\boldsymbol{u}\times(\boldsymbol{v}-2\boldsymbol{w})$

4. Use the cross product to find a vector that is orthogonal to both

a) $\vec{u}=(-6,\ 4,\ 2)$, $\vec{v}=(3,\ 1,\ 5)$

b) $\vec{u}=(1,\ 1,\ -2)$, $\vec{v}=(2,\ -1,\ 2)$

c) $\vec{u}=(-2,\ 1,\ 5)$, $\vec{v}=(3,\ 0,\ -3)$

5. Find the area of the parallelogram determined by the given vectors

a) $\vec{u}=(1,\ -1,\ 2)$ *and* $\vec{v}=(0,\ 3,\ 1)$

b) $\vec{u}=(3,\ -1,\ 4)$ *and* $\vec{v}=(6,\ -2,\ 8)$

c) $\vec{u}=(2,\ 3,\ 0)$ *and* $\vec{v}=(-1,\ 2,\ -2)$

6. Find the area of the parallelogram with the given vertices

$P_1(3,\ 2)$, $P_2(5,\ 4)$, $P_3(9,\ 4)$, $P_4(7,\ 2)$

7. Find the area of the triangle with the given vertices:

a) $A(2,\ 0)$ $B(3,\ 4)$ $C(-1,\ 2)$

b) $A(1,\ 1)$ $B(2,\ 2)$ $C(3,\ -3)$

c) $P(2,\ 6,\ -1)$ $Q(1,\ 1,\ 1)$ $R=(4,\ 6,\ 2)$

8. *a*) Find the area of the parallelogram with edges $\boldsymbol{v}=(3,\ 2)$ and $\boldsymbol{w}=(1,\ 4)$

b) Find the area of the triangle with sides $\boldsymbol{v}$, $\boldsymbol{w}$, and $\boldsymbol{v}+\boldsymbol{w}$. Draw it.

c) Find the area of the triangle with sides $\boldsymbol{v}$, $\boldsymbol{w}$, and $\boldsymbol{v}-\boldsymbol{w}$. Draw it.

9. Find the volume of the parallelepiped with sides $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

a) $\boldsymbol{u}=(2,\ -6,\ 2)$, $\boldsymbol{v}=(0,\ 4,\ -2)$, $\boldsymbol{w}=(2,\ 2,\ -4)$

b) $\boldsymbol{u}=(3,\ 1,\ 2)$, $\boldsymbol{v}=(4,\ 5,\ 1)$, $\boldsymbol{w}=(1,\ 2,\ 4)$

10. Compute the scalar triple product $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w})$

a) $\boldsymbol{u}=(-2,\ 0,\ 6),\quad \boldsymbol{v}=(1,\ -3,\ 1),\quad \boldsymbol{w}=(-5,\ -1,\ 1)$

b) $\boldsymbol{u}=(-1,\ 2,\ 4),\quad \boldsymbol{v}=(3,\ 4,\ -2),\quad \boldsymbol{w}=(-1,\ 2,\ 5)$

c) $\boldsymbol{u}=(a,\ 0,\ 0),\quad \boldsymbol{v}=(0,\ b,\ 0),\quad \boldsymbol{w}=(0,\ 0,\ c)$

d) $\boldsymbol{u}=3\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}-5\hat{\boldsymbol{k}},\quad \boldsymbol{v}=\hat{\boldsymbol{i}}+4\hat{\boldsymbol{j}}-4\hat{\boldsymbol{k}},\quad \boldsymbol{w}=3\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$

e) $\boldsymbol{u}=(3,\ -1,\ 6)\quad \boldsymbol{v}=(2,\ 4,\ 3)\quad \boldsymbol{w}=(5,\ -1,\ 2)$

11. Use the cross product to find the sine of the angle between the vectors
$\boldsymbol{u}=(2,\ 3,\ -6),\quad \boldsymbol{v}=(2,\ 3,\ 6)$

12. Simplify $(\boldsymbol{u}+\boldsymbol{v})\times(\boldsymbol{u}-\boldsymbol{v})$

13. Prove Lagrange's identity: $\|\boldsymbol{u}\times\boldsymbol{v}\|^2=\|\boldsymbol{u}\|^2\ \|\boldsymbol{v}\|^2-(\boldsymbol{u}.\boldsymbol{v})^2$

14. Polar coordinates satisfy $x=r\cos\theta$ *and* $y=\sin\theta$. Polar area $J\ dr\ d\theta$ includes *J*:

$$J=\begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{bmatrix}=\begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix}$$

The two columns are orthogonal. Their lengths are ______. Thus *J* = ______.

15. Prove that $\|\vec{u}+\vec{v}\|=\|\vec{u}\|+\|\vec{v}\|$ if and only if $\vec{u}$ and $\vec{v}$ are parallel vectors.

16. State the following statements as True or False

a) The cross product of two nonzero vectors $\vec{u}$ and $\vec{v}$ is a nonzero vector if and only if $\vec{u}$ and $\vec{v}$ are not parallel.

b) A normal vector to a plane can be obtained by taking the cross product of two nonzero and noncollinear vectors lying in the plane.

c) The scalar triple product of $\vec{u}$, $\vec{v}$, and $\vec{w}$ determines a vector whose length is equal to the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$.

d) If $\vec{u}$ and $\vec{v}$ are vectors in 3-space, then $\|\vec{u}\times\vec{v}\|$ is equal to the area of the parallelogram determine by $\vec{u}$ and $\vec{v}$.

e) For all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in R^3, the vectors $(\vec{u}\times\vec{v})\times\vec{w}$ and $\vec{u}\times(\vec{v}\times\vec{w})$ are the same.

f) If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in R^3, where $\vec{u}$ is nonzero and $\vec{u}\times\vec{v}=\vec{u}\times\vec{w}$, then $\vec{v}=\vec{w}$