$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$$

$$n < n+1$$

$$\sqrt{n} < \sqrt{n+1}$$

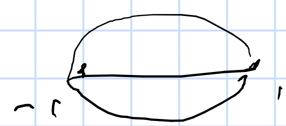
$$\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$$

$$u_n > u_{n+1} \checkmark$$

$$\frac{1}{\sqrt{n}} \longrightarrow 0 \checkmark$$

By the Alternating series, the given series converges

$$\sum_{n=1}^{\infty} \frac{1}{n} \cos n\pi = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$



$$n < n+1$$

$$\frac{1}{n} > \frac{1}{n+1}$$

$$u_n > u_{n+1} \checkmark$$

$$\frac{1}{n} \longrightarrow 0 \checkmark$$

By the alternating series, the given series converges

$$-1 \le \cos n\pi \le 1$$

$$-\frac{1}{n} \le \frac{\cos n\pi}{n} \le \frac{1}{n}$$

$$\frac{1}{n}$$

$$\frac{a_{n+1}}{a_n} = \frac{\cos(n+1)\pi}{n+1} \cdot \frac{n}{\cos n\pi} \to -1$$

$$\rho = |-1|$$

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

$$\int \frac{\ln x}{x^2}\,dx \qquad y = \ln x$$
$$x = e^y$$
$$dx = e^y\,dy$$

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \int_1^\infty y e^{-2y} e^y\,dy$$

$$= \int_1^\infty y e^{-y}\,dy \qquad \begin{matrix} y & -e^{-y} \\ 1 & e^{-y} \end{matrix}$$

$$= (-y-1)e^{-y}\Big|_1^\infty$$

$$= 0 - (-2)e^{-1}$$

$$= \frac{2}{e}$$

By the integral Test, the given series converges

$$\frac{\ln n}{n^2} \le \frac{1}{n^{3/2}} \qquad \ln n \ge 0$$

$$\frac{a_{n+1}}{a_n} = \frac{\ln(n+1)}{(n+1)^2} \cdot \frac{n^2}{\ln(n)}$$

$$= \frac{n^2}{(n+1)^2} \cdot \frac{\ln(n+1)}{\ln n}$$

$$\rho = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \qquad \frac{\infty}{\infty}$$

$$= 1 \cdot \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{n}{n+1}$$

$$= 1$$

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$$

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \int_2^{\infty} (\ln x)^{-2}\, d(\ln x)$$

$$= -\frac{1}{\ln x}\Big|_2^{\infty}$$

$$= -\left(0 - \frac{1}{\ln 2}\right)$$

$$= \boxed{\frac{1}{\ln 2}}$$

By the integral Test, the given series Converges

$$\sum_{k=1}^{\infty} \left(\frac{1}{\ln(k+1)}\right)^k$$

$$\sqrt[k]{\left(\frac{1}{\ln(k+1)}\right)^k} = \frac{1}{\ln(k+1)}$$

$$\rho = \lim_{k\to\infty} \frac{1}{\ln(k+1)}$$

$$= \frac{1}{\infty}$$

$$= 0$$

By the Root Test, the given series converges

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \frac{(n+1)}{(n+1)(n+1)^n} \quad n^n$$

$$= \left(\frac{n}{n+1}\right)^n$$

$$= \left(\frac{n+1}{n}\right)^{-n}$$

$$= \left(1 + \frac{1}{n}\right)^{-n}$$

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

$$\rho = \lim_{n\to\infty} \left(\left(1 + \frac{1}{n}\right)^n\right)^{-1}$$

$$= e^{-1}$$

$$= \frac{1}{e} < 1$$

By the Ratio Test, the given series converges

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = 1^\infty$$

$$\ln\left(\frac{n}{n+1}\right)^n = n \ln\left(\frac{n}{n+1}\right)$$

$$= \frac{\ln\left(\frac{n}{n+1}\right)}{1/n} \to \frac{0}{0}$$

$$\lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{1/n} = \frac{\frac{1}{(n+1)^2}}{\frac{n}{n+1}} \cdot \left(\frac{1}{-\frac{1}{n^2}}\right)$$

$$\sum_{k=1}^{\infty} \frac{2}{k^{3/2}}$$

$$p = \frac{3}{2} > 1$$

By p-series $(p = \frac{3}{2})$, the given series converges

$$\sum_{k=1}^{\infty} k^{-2/3} = \sum_{k=1}^{\infty} \frac{1}{k^{2/3}}$$

$$p = \frac{2}{3} < 1$$

By the p-series, the given series diverges

$$\sum_{n=1}^{\infty} 5\left(\frac{7}{8}\right)^n$$

$$|r| = \frac{7}{8} < 1$$

$$S = \frac{5}{1 - \frac{7}{8}}$$

$$= \underline{40}$$

By the Geometric series, the given series converges w/ sum $\underline{40}$

$$\frac{a_{n+1}}{a_n} = 5\left(\frac{7}{8}\right)^{n+1} \cdot \frac{1}{5\left(\frac{7}{8}\right)^n}$$

$$= \underline{\frac{7}{8} < 1}$$

$$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$$

$$\sqrt[n]{\frac{e^{2n}}{n^n}} = \frac{e^2}{n}$$

$$\rho = \lim_{n\to\infty} \frac{e^2}{n}$$

$$= 0 < 1$$

By the Root Test, the given series converges

$$\sum_{k=1}^{\infty} \frac{3}{2+e^k}$$

$$2+e^k > e^k$$

$$\frac{3}{2+e^k} < \frac{3}{e^k}$$

$$\sum 3 \left(\frac{1}{e}\right)^k \qquad |r| = \frac{1}{e} < 1$$

by Geometric series, it converges

By the Comparison Test, the given series converges

$$\int_1^{\infty} \frac{3}{2+e^x}\,dx \qquad u = 2+e^x$$

$$du = e^x\,dx$$

$$= (u-2)\,dx$$

$$= \int_1^{\infty} \frac{3}{u} \cdot \frac{du}{u-2}$$

$$= 3\int_1^{\infty} \frac{du}{u(u-2)} \qquad \frac{A}{u} + \frac{B}{u-2}$$

$$= \frac{-3}{2}\int_1^{\infty} \frac{du}{u} + \frac{3}{2}\int_1^{\infty} \frac{d(u-2)}{|u-2|} \qquad A+B=0$$

$$-2A = 1$$

$$= -\frac{3}{2}\ln|u| + \frac{3}{2}\ln|u-2| \Big|_1^{\infty}$$

$$= -\frac{3}{2}\ln(2+e^x) + \frac{3}{2}\ln e^x \Big|_1^{\infty}$$

$$= \frac{3}{2}\ln\frac{e^x}{2+e^x} \Big|_1^{\infty}$$

$$= \frac{3}{2}\left(\cancel{\ln 1} - \ln\frac{e}{2+e}\right)$$

3.7 Power Series

Defn A power series about $x=0$

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

A power series about $x=a$ (centre)

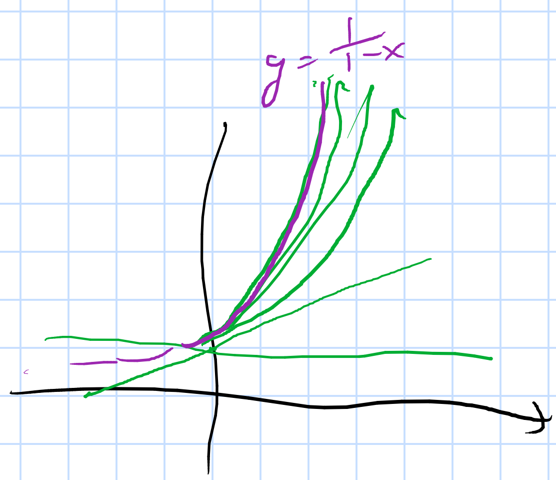
$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

$c_0, c_1, \ldots, c_n, \ldots$ coefficients

Ex $$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$

to converges $|x| < 1$



Ex

$$1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2+\dots+\left(-\frac{1}{2}\right)^n(x-2)^n+\dots$$

$$\sum\left(-\frac{1}{2}(x-2)\right)^n$$

$$r=-\frac{x-2}{2}$$

$$|r|=\left|\frac{x-2}{2}\right|<1$$

$$-1<\frac{x-2}{2}<1$$

$$-2<x-2<2$$

$$0<x<4$$

$$S=\frac{1}{1+\frac{x-2}{2}}=\frac{2}{x}$$

$$1-\frac{1}{2}(x-2)+\dots+\left(-\frac{1}{2}\right)^n(x-2)+\dots=\frac{2}{x}$$

$$P_0=1$$

$$P_1=1-\frac{1}{2}(x-2)=\frac{-1}{2}x+2$$

$$\vdots$$

Ex $\displaystyle \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

$x = 1$

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right|$$

$$= \frac{n}{n+1} |x| \longrightarrow |x| < 1$$

at $x = 1 \Rightarrow \displaystyle \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

$$n < n+1$$

$$\frac{1}{n} > \frac{1}{n+1}$$

$$u_n > u_{n+1} \checkmark$$

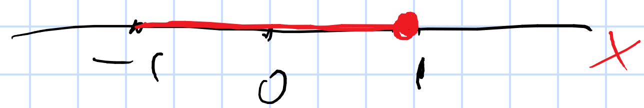
$$\frac{1}{n} \to 0$$

it converges by the alternating series

At : $x = -1 \Rightarrow \displaystyle \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$

p-series $p = 1$ diverges.

The series converges $-1 < x \leq 1$ + diverges elsewhere



Ex

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{2n-1}{2(n+1)-1} \cdot \left|\frac{x^{2n+1}}{x^{2n-1}}\right|$$

$$= \frac{2n-1}{2n+1} |x^2| \longrightarrow x^2$$

$$x^2 < 1 \Rightarrow -1 < x < 1$$

At $x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$

$$n < n+1$$
$$2n < 2n+2$$
$$2n-1 < 2n+1$$
$$\frac{1}{2n-1} > \frac{1}{2n+1}$$
$$u_n > u_{n+1} \checkmark$$

$$\frac{1}{2n-1} \; ? \; \frac{1}{2n+1}$$

$$\frac{1}{2n-1} \longrightarrow 0 \checkmark$$

It converges by Alternating series.

at $x = -1 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^{2n-1}}{2n-1} = \sum \frac{(-1)^{3n-2}}{2n-1}$

$$\frac{1}{2n-1} \longrightarrow 0$$

it converges by Alternating series

it converges $-1 \le x \le 1$ and diverges elsewhere

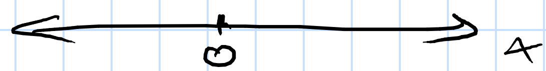
Ex $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1} + \frac{x^2}{2!}$ $\qquad 0! = 1$

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{n!}{(n+1)!} \cdot \left|\frac{x^{n+1}}{x^n}\right|$$

$$= \frac{1}{n+1} |x| \longrightarrow 0$$

$$\lim_{n \to \infty} \frac{1}{n+1} = 0$$

The given series converges for all x

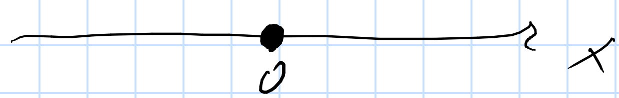


Ex $\sum n! \, x^n$

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{(n+1)!}{n!} \left|\frac{x^{n+1}}{x^n}\right|$$

$$= (n+1)|x| \longrightarrow \infty$$

The series diverges absolutely except $x=0$



$X - a \rightarrow$ centre : $x = a$

Radius of convergence ($\underline{R}$)

$$\sum c_n (x-a)^n$$

1- $|x-a| > R \rightarrow$ diverges

$|x-a| < R \rightarrow$ converges

2- series converges absolutely ($R = \infty$)

3- " diverges " ($R = 0$)

$-R < x - a < R$

 $\rightarrow$ interval of convergence

$a - R < x < a + R$

Defn If $L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$ where $R = \frac{1}{L}$

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Ex

Centre, Radius, & interval of converges

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n} = \sum \frac{2^n\left(x+\frac{5}{2}\right)^n}{(n^2+1)3^n}$$

$$2x+5=0 \Rightarrow x=\frac{-5}{2}$$

$$\text{centre}: \boxed{x=\frac{-5}{2}}$$

of convergence

$$\left|\frac{u_n}{u_{n+1}}\right| = \frac{2^n}{(n^2+1)3^n} \cdot \frac{((n+1)^2+1)\,3^{n+1}}{2^{n+1}}$$

$$= \frac{3}{2}\,\frac{(n+1)^2+1}{n^2+1} \longrightarrow \frac{3}{2}\,\frac{n^2}{n^2}$$

$$\boxed{R=\frac{3}{2}}$$ Radius of convergence.

$$-\frac{3}{2} < x+\frac{5}{2} < \frac{3}{2}$$

$$-\frac{5}{2}-\frac{3}{2} < x < -\frac{5}{2}+\frac{3}{2}$$

$$\boxed{-4 < x < -1}$$

At $x=-4 \Rightarrow$ $\cancel{\sum \left(\frac{2}{3}\right)^n \frac{\left(-4+\frac{5}{2}\right)^n}{n^2+1}}$

$$\sum \frac{(-8+5)^n}{(n^2+1)3^n} = \sum \frac{(-1)^n}{n^2+1} \qquad \frac{(-3)^n}{3^n}$$

$$n < n+1$$

$$n^2 < (n+1)^2$$

$$n^2+1 < (n+1)^2+1$$

$$\frac{1}{n^2+1} > \frac{1}{(n+1)^2+1}$$

$$u_n > u_{n+1} \quad \checkmark$$

$$\frac{1}{n^2+1} \longrightarrow 0 \quad \checkmark$$

it converges by Alternating series @ $x=-4$

At $x=-1$

$$\sum \frac{(-2+5)^n}{(n^2+1)3^n} = \sum \frac{1}{n^2+1}$$

$$\int_0^\infty \frac{dx}{x^2+1} = \arctan x \Big|_0^\infty$$

$$= \arctan(\infty) - \cancel{\arctan(0)}^{\,0}$$

$$= \frac{\pi}{2}$$

By the integral test, it converges @ $x=-1$

$\therefore$ The interval of converges $[-4, -1]$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Centre of convergence : $x=0$

$$R = \lim_{n\to\infty} \left|\frac{u_n}{u_{n+1}}\right|$$

$$= \lim_{n\to\infty} \frac{1}{n!} \cdot \frac{(n+1)!}{1}$$

$$= \lim_{n\to\infty} (n+1)$$

$$= \infty$$

Radius of convergence : ∞

interval of convergence : $(-\infty, \infty)$

$\forall x$

$$\sum_{n=0}^{\infty} n!\,x^n$$

centre of convergence, $x=0$

$$R = \lim_{n\to\infty} \left|\frac{u_n}{u_{n+1}}\right|$$

$$= \lim_{n\to\infty} \left(\frac{n!}{(n+1)!}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n+1}$$

$= 0$ radius of convergence

The series convergence only @ $x=0$ centre

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} = c_1 + 2 c_2 (x-a) + \cdots$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}$$

3.5