

SOLUTION**Section 3.1 – Sequences****Exercise**

Find the values of a_1 , a_2 , a_3 , and a_4 for $a_n = \frac{1-n}{n^2}$

Solution

$$a_1 = \frac{1-\textcolor{red}{1}}{\textcolor{red}{1}^2} = \underline{\textcolor{blue}{0}}$$

$$a_2 = \frac{1-\textcolor{red}{2}}{\textcolor{red}{2}^2} = \underline{-\frac{\textcolor{blue}{1}}{4}}$$

$$a_3 = \frac{1-\textcolor{red}{3}}{\textcolor{red}{3}^2} = \underline{-\frac{\textcolor{blue}{2}}{9}}$$

$$a_4 = \frac{1-\textcolor{red}{4}}{\textcolor{red}{4}^2} = \underline{-\frac{\textcolor{blue}{3}}{16}}$$

Exercise

Find the values of a_1 , a_2 , a_3 , and a_4 for $a_n = \frac{1}{n!}$

Solution

$$a_1 = \frac{1}{\textcolor{red}{1}!} = \underline{\textcolor{blue}{1}}$$

$$a_2 = \frac{1}{\textcolor{red}{2}!} = \underline{\frac{\textcolor{blue}{1}}{4}}$$

$$a_3 = \frac{1}{\textcolor{red}{3}!} = \underline{\frac{\textcolor{blue}{1}}{6}}$$

$$a_4 = \frac{1}{\textcolor{red}{4}!} = \underline{\frac{\textcolor{blue}{1}}{24}}$$

Exercise

Find the values of a_1 , a_2 , a_3 , and a_4 for $a_n = \frac{(-1)^{n+1}}{2n-1}$

Solution

$$a_1 = \frac{(-1)^{\textcolor{red}{1}+1}}{2(\textcolor{red}{1})-1} = \underline{\textcolor{blue}{1}}$$

$$a_2 = \frac{(-1)^{\textcolor{red}{2}+1}}{2(\textcolor{red}{2})-1} = \underline{-\frac{\textcolor{blue}{1}}{3}}$$

$$a_3 = \frac{(-1)^{\textcolor{red}{3}+1}}{2(\textcolor{red}{3})-1} = \underline{\frac{\textcolor{blue}{1}}{5}}$$

$$a_4 = \frac{(-1)^{4+1}}{2(\textcolor{red}{4})-1} = \underline{\underline{-\frac{1}{7}}}$$

Exercise

Find the values of a_1 , a_2 , a_3 , and a_4 for $a_n = 2 + (-1)^n$

Solution

$$a_1 = 2 + (-1)^{\textcolor{red}{1}} = \underline{\underline{1}}$$

$$a_2 = 2 + (-1)^{\textcolor{red}{2}} = \underline{\underline{3}}$$

$$a_3 = 2 + (-1)^{\textcolor{red}{3}} = \underline{\underline{1}}$$

$$a_4 = 2 + (-1)^{\textcolor{red}{4}} = \underline{\underline{3}}$$

Exercise

Find the values of a_1 , a_2 , a_3 , and a_4 for $a_n = \frac{2^n - 1}{2^n}$

Solution

$$a_1 = \frac{2^{\textcolor{red}{1}} - 1}{2^{\textcolor{red}{1}}} = \underline{\underline{\frac{1}{2}}}$$

$$a_2 = \frac{2^{\textcolor{red}{2}} - 1}{2^{\textcolor{red}{2}}} = \underline{\underline{\frac{3}{4}}}$$

$$a_3 = \frac{2^{\textcolor{red}{3}} - 1}{2^{\textcolor{red}{3}}} = \underline{\underline{\frac{7}{8}}}$$

$$a_4 = \frac{2^{\textcolor{red}{4}} - 1}{2^{\textcolor{red}{5}}} = \underline{\underline{\frac{15}{32}}}$$

Exercise

Write the first ten terms of the sequence $a_1 = 1$, $a_{n+1} = a_n + \frac{1}{2^n}$

Solution

$$\begin{aligned} a_2 &= a_1 + \frac{1}{2^{\textcolor{red}{1}}} \\ &= 1 + \frac{1}{2} \\ &= \underline{\underline{\frac{3}{2}}} \end{aligned}$$

$$\begin{aligned}
 a_3 &= a_2 + \frac{1}{2^{\color{red}2}} \\
 &= \frac{3}{2} + \frac{1}{4} \\
 &= \frac{7}{4} \quad \Big|
 \end{aligned}$$

$$\begin{aligned}
 a_4 &= a_3 + \frac{1}{2^{\color{red}3}} \\
 &= \frac{7}{4} + \frac{1}{8} \\
 &= \frac{15}{8} \quad \Big|
 \end{aligned}$$

$$\begin{aligned}
 a_5 &= a_4 + \frac{1}{2^{\color{red}4}} \\
 &= \frac{15}{8} + \frac{1}{16} \\
 &= \frac{31}{16} \quad \Big|
 \end{aligned}$$

$$\begin{aligned}
 a_6 &= a_5 + \frac{1}{2^{\color{red}5}} \\
 &= \frac{31}{16} + \frac{1}{32} \\
 &= \frac{63}{32} \quad \Big|
 \end{aligned}$$

$$\begin{aligned}
 a_7 &= a_6 + \frac{1}{2^{\color{red}6}} \\
 &= \frac{63}{32} + \frac{1}{64} \\
 &= \frac{127}{64} \quad \Big|
 \end{aligned}$$

$$\begin{aligned}
 a_8 &= a_7 + \frac{1}{2^{\color{red}7}} \\
 &= \frac{127}{64} + \frac{1}{128} \\
 &= \frac{255}{128} \quad \Big|
 \end{aligned}$$

$$\begin{aligned}
 a_9 &= a_8 + \frac{1}{2^{\color{red}8}} \\
 &= \frac{255}{128} + \frac{1}{256} \\
 &= \frac{511}{256} \quad \Big|
 \end{aligned}$$

$$a_{10} = a_9 + \frac{1}{2^{\color{red}9}}$$

$$= \frac{511}{256} + \frac{1}{512}$$

$$= \frac{1023}{512}$$

Exercise

Write the first ten terms of the sequence $a_1 = 1, \quad a_{n+1} = \frac{a_n}{n+1}$

Solution

$$a_1 = 1$$

$$a_2 = \frac{1}{1+1} = \frac{1}{2}$$

$$a_3 = \frac{\frac{1}{2}}{2+1} = \frac{1}{6}$$

$$a_4 = \frac{\frac{1}{6}}{3+1} = \frac{1}{24}$$

$$a_5 = \frac{\frac{1}{24}}{4+1} = \frac{1}{120}$$

$$a_6 = \frac{\frac{1}{120}}{5+1}$$

$$= \frac{1}{720}$$

$$a_7 = \frac{\frac{1}{720}}{6+1}$$

$$= \frac{1}{5040}$$

$$a_8 = \frac{\frac{1}{5040}}{7+1}$$

$$= \frac{1}{40,320}$$

$$a_9 = \frac{\frac{1}{40,320}}{8+1}$$

$$= \frac{1}{362,880}$$

$$a_{10} = \frac{\frac{1}{362,880}}{9+1}$$

$$= \frac{1}{3,628,800}$$

Exercise

Write the first ten terms of the sequence $a_1 = 2$, $a_2 = -1$, $a_{n+2} = \frac{a_{n+1}}{a_n}$

Solution

$$a_1 = 2, \quad a_2 = -1$$

$$a_3 = -\frac{1}{2}$$

$$a_4 = \frac{-\frac{1}{2}}{-1}$$

$$= \frac{1}{2}$$

$$a_5 = \frac{\frac{1}{2}}{-\frac{1}{2}}$$

$$= -1$$

$$a_6 = \frac{-1}{\frac{1}{2}}$$

$$= -2$$

$$a_7 = \frac{-2}{-1}$$

$$= 2$$

$$a_8 = \frac{2}{-2}$$

$$= -1$$

$$a_9 = \frac{-1}{2}$$

$$= -\frac{1}{2}$$

$$a_{10} = \frac{-\frac{1}{2}}{-1}$$

$$= \frac{1}{2}$$

Exercise

Find a formula for the n th term of the sequence $-1, 1, -1, 1, -1, \dots$

Solution

$$a_n = (-1)^n \quad n \in \mathbb{N}$$

Exercise

Find a formula for the n th term of the sequence $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \dots$

Solution

$$a_1 = 1 \quad r = -\frac{1}{4}$$

$$a_n = a_1 r$$

$$= -\frac{1}{4}$$

$$= \frac{(-1)^{n+1}}{n^2}$$

$$\underline{a_n = \frac{(-1)^{n+1}}{n^2}} \quad n \in \mathbb{N}$$

Exercise

Find a formula for the n th term of the sequence $\frac{1}{9}, \frac{2}{12}, \frac{2^2}{15}, \frac{2^3}{18}, \frac{2^4}{21}, \dots$

Solution

$$\underline{a_n = \frac{2^{n-1}}{3(n+2)}} \quad n \in \mathbb{N}$$

Exercise

Find a formula for the n th term of the sequence $-3, -2, -1, 0, 1, \dots$

Solution

$$d = -2 - (-3) = 1$$

$$a_n = a_1 + (n-1)d$$

$$= -3 + (n-1)(1)$$

$$= -3 + n - 1$$

$$\underline{= n - 4} \quad n \in \mathbb{N}$$

Exercise

Find a formula for the n th term of the sequence $\frac{1}{25}, \frac{8}{125}, \frac{27}{625}, \frac{64}{3125}, \frac{125}{15625}, \dots$

Solution

$$\frac{1}{5^2}, \frac{2^3}{5^3}, \frac{3^3}{5^4}, \frac{4^3}{5^5}, \frac{5^3}{5^6}, \dots$$

$$\underline{a_n = \frac{n^3}{5^{n+1}}} \quad n \in \mathbb{N}$$

Exercise

Find a formula for the n th term of the sequence 0, 1, 1, 2, 2, 3, 3, 4, ...

Solution

$$\underline{a_n = \frac{n - \frac{1}{2} + (-1)^n \left(\frac{1}{2}\right)}{2}} \quad n \in \mathbb{N}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{n + (-1)^n}{n}$$

Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n + (-1)^n}{n} &= \lim_{n \rightarrow \infty} \left(1 + \frac{(-1)^n}{n} \right) \\ &= \underline{1} \quad \Rightarrow \text{converges} \end{aligned}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{1 - 2n}{1 + 2n}$$

Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1 - 2n}{1 + 2n} &= \lim_{n \rightarrow \infty} \left(\frac{\frac{1}{n} - 2}{\frac{1}{n} + 2} \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{-2}{2} \right) \\ &= \underline{-1} \quad \text{The limit converges} \end{aligned}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{1-n^3}{70-4n^2}$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{1-n^3}{70-4n^2} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2} - n}{\frac{70}{n^2} - 4} \\&= \lim_{n \rightarrow \infty} \frac{0 - n}{0 - 4} \\&= \underline{\infty} \Rightarrow \text{diverges}\end{aligned}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \left(2 - \frac{1}{2^n}\right) \left(3 + \frac{1}{2^n}\right)$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left(2 - \frac{1}{2^n}\right) \left(3 + \frac{1}{2^n}\right) &= (2)(3) \\&= \underline{6} \Rightarrow \text{converges}\end{aligned}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = n\pi \cos(n\pi)$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} n\pi \cos(n\pi) &= \lim_{n \rightarrow \infty} n\pi (-1)^n \\&= \underline{\infty} \Rightarrow \text{diverges}\end{aligned}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = n - \sqrt{n^2 - n}$$

Solution

$$\begin{aligned}
\lim_{n \rightarrow \infty} n - \sqrt{n^2 - n} &= \lim_{n \rightarrow \infty} \left(n - \sqrt{n^2 - n} \right) \frac{n + \sqrt{n^2 - n}}{n + \sqrt{n^2 - n}} \\
&= \lim_{n \rightarrow \infty} \frac{n^2 - (n^2 - n)}{n + \sqrt{n^2 - n}} \\
&= \lim_{n \rightarrow \infty} \frac{n}{n + \sqrt{n^2 - n}} \\
&= \lim_{n \rightarrow \infty} \frac{1}{1 + \sqrt{1 - \frac{1}{n}}} \\
&= \frac{1}{2} \quad |
\end{aligned}$$

The given series *converges*.

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \sqrt{\frac{2n}{n+1}}$$

Solution

$$\begin{aligned}
\lim_{n \rightarrow \infty} \sqrt{\frac{2n}{n+1}} &= \sqrt{\lim_{n \rightarrow \infty} \frac{2}{1 + \frac{1}{n}}} \\
&= \sqrt{2} \quad |
\end{aligned}$$

The given series *converges*

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\sin^2 n}{2^n}$$

Solution

$$0 \leq \frac{\sin^2 n}{2^n} \leq \frac{1}{2^n} \quad \text{By the Sandwich Theorem for sequences}$$

$$\lim_{n \rightarrow \infty} \frac{\sin^2 n}{2^n} = 0 \quad |$$

The given series *converges*

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\ln n}{\ln 2n}$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{\ln n}{\ln 2n} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{2}{2n}} \\ &= 1\end{aligned}$$

The given series *converges*

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{3^n \cdot 6^n}{2^{-n} \cdot n!}$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{3^n \cdot 6^n}{2^{-n} \cdot n!} &= \lim_{n \rightarrow \infty} \frac{2^n \cdot 3^n \cdot 6^n}{n!} \\ &= \lim_{n \rightarrow \infty} \frac{36^n}{n!} \\ &= 0\end{aligned}$$

The given series *converges*

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \sqrt{n} \sin \frac{1}{\sqrt{n}}$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{\sin \frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} &= \lim_{n \rightarrow \infty} \frac{\left(-\frac{1}{2n^{3/2}}\right) \cos \frac{1}{\sqrt{n}}}{-\frac{1}{2n^{3/2}}} \\ &= \lim_{n \rightarrow \infty} \cos \frac{1}{\sqrt{n}} \\ &= \cos 0 \\ &= 1\end{aligned}$$

The given series *converges*

$$\text{or } \lim_{n \rightarrow \infty} \frac{\sin \frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{n^2}{2^n - 1}$$

Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \frac{n^2}{2^n - 1} && \text{L'Hôpital Rule} \\ &= \lim_{x \rightarrow \infty} \frac{2x}{(\ln 2) \cdot 2^x} \\ &= \lim_{x \rightarrow \infty} \frac{2}{(\ln 2)^2 \cdot 2^x} \\ &= 0 \end{aligned}$$

The sequence *converges*

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$\{c_n\} = \left\{ (-1)^n \frac{1}{n!} \right\}$$

Solution

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots n = 24 \cdot \underbrace{5 \cdot 6 \cdots n}_{n-4}$$

$$2^2 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdots 2 = 16 \cdot \underbrace{2 \cdot 2 \cdots 2}_{n-4}$$

$$\frac{-1}{2^n} \leq (-1)^n \frac{1}{n!} \leq \frac{1}{2^n} \quad n \geq 4$$

By the Squeeze Theorem

$$\lim_{n \rightarrow \infty} (-1)^n \frac{1}{n!} = 0$$

The sequence *converges*

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{5}{n+2}$$

Solution

$$\lim_{n \rightarrow \infty} \frac{5}{n+2} = 0$$

The sequence *converges*

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = 8 + \frac{5}{n}$$

Solution

$$\lim_{n \rightarrow \infty} \left(8 + \frac{5}{n}\right) = 8$$

The sequence *converges*

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = (-1)^n \left(\frac{n}{n+1}\right)$$

Solution

$$\lim_{n \rightarrow \infty} (-1)^n \left(\frac{n}{n+1}\right) \text{ does not exist (oscillates between } -1 \text{ and } 1)$$

The sequence *diverges*

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{1 + (-1)^n}{n^2}$$

Solution

$$\lim_{n \rightarrow \infty} \frac{1 + (-1)^n}{n^2} = 0$$

The sequence *converges*

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{10n^2 + 3n + 7}{2n^2 - 6}$$

Solution

$$\lim_{n \rightarrow \infty} \frac{10n^2 + 3n + 7}{2n^2 - 6} = \lim_{n \rightarrow \infty} \frac{10n^2}{2n^2} \\ = 5$$

The sequence *converges*

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\sqrt[3]{n}}{\sqrt[3]{n} + 1}$$

Solution

$$\lim_{n \rightarrow \infty} \frac{\sqrt[3]{n}}{\sqrt[3]{n} + 1} = 1$$

The sequence *converges*

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\ln(n^3)}{2n}$$

Solution

$$\lim_{n \rightarrow \infty} \frac{\ln(n^3)}{2n} = \lim_{n \rightarrow \infty} \frac{3 \ln(n)}{2n} \\ = \lim_{n \rightarrow \infty} \frac{3}{2} \frac{\frac{1}{n}}{1} \\ = 0$$

The sequence *converges*

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{5^n}{3^n}$$

Solution

$$\lim_{n \rightarrow \infty} \frac{5^n}{3^n} = \lim_{n \rightarrow \infty} \left(\frac{5}{3}\right)^n$$
$$\underline{= \infty}$$

The sequence *diverges*

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{(n+1)!}{n!}$$

Solution

$$\lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} = \lim_{n \rightarrow \infty} (n+1)$$
$$\underline{= \infty}$$

The sequence *diverges*

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{(n-2)!}{n!}$$

Solution

$$\lim_{n \rightarrow \infty} \frac{(n-2)!}{n!} = \lim_{n \rightarrow \infty} \frac{1}{n(n-1)}$$
$$\underline{= 0}$$

The sequence *converges*

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{n^p}{e^n}, \quad p > 0$$

Solution

$$\lim_{n \rightarrow \infty} \frac{n^p}{e^n} = 0$$

The sequence *converges* ($p > 0, n \geq 2$)

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = n \sin \frac{1}{n}$$

Solution

$$\lim_{n \rightarrow \infty} n \sin \frac{1}{n} = \lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}}$$

$$\text{Let } x = \frac{1}{n} \xrightarrow{n \rightarrow \infty} 0$$

$$= \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

$$\text{Since } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1$$

$$= 1$$

The sequence *converges*

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = 2^{1/n}$$

Solution

$$\lim_{n \rightarrow \infty} 2^{1/n} = 2^0$$

$$= 1$$

The sequence *converges*

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = -3^{-n}$$

Solution

$$\lim_{n \rightarrow \infty} -3^{-n} = \lim_{n \rightarrow \infty} \left(-\frac{1}{3^n} \right)$$

$$= 0$$

The sequence *converges*

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\sin n}{n}$$

Solution

$$\lim_{n \rightarrow \infty} \frac{\sin n}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} (\sin n)$$

$$= 0 \quad \text{since } \frac{1}{n} \rightarrow 0$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\cos \pi n}{n^2}$$

Solution

$$\lim_{n \rightarrow \infty} \frac{\cos \pi n}{n^2} = \lim_{n \rightarrow \infty} \frac{1}{n^2} (\cos \pi n)$$

$$= 0$$

The sequence ***converges***