Solution Section R.1 – Basic Algebra Review

Exercise

Expand and simplify: $(4x-y)^3$

Solution

$$(4x-y)^3=\binom{3}{0}(4x)^3(-y)^0+\binom{3}{1}(4x)^2(-y)^1+\binom{3}{2}(4x)^1(-y)^2+\binom{3}{3}(4x)^0(-y)^3$$
$$=64x^3+3\left(16x^2\right)(-y)+3(4x)y^2-y^3$$
$$=64x^3-48x^2y+12xy^2-y^3$$

Exercise

Expand and simplify: $\left(\sqrt{x}-\sqrt{3}\right)^4$

Solution

$$\left(\sqrt{x}-\sqrt{3}\right)^4=\left(\sqrt{x}\right)^4+4\left(\sqrt{x}\right)^3\left(-\sqrt{3}\right)+6\left(\sqrt{x}\right)^2\left(-\sqrt{3}\right)^2+4\left(\sqrt{x}\right)\left(-\sqrt{3}\right)^3+\left(-\sqrt{3}\right)^4$$
$$=x^2-4x\sqrt{3x}+18x^2-13\sqrt{3x}+9$$

Exercise

Expand and simplify: $(ax+by)^5$

Solution

$$(ax+by)^5=(ax)^5+5(ax)^4(by)+10(ax)^3(by)^2+10(ax)^2(by)^3+5(ax)(by)^4+(by)^5$$
$$=a^5x^5+5a^4x^4by+10a^3x^3b^2y^2+10a^2x^2b^3y^3+5axb^4y^4+b^5y^5$$

Exercise

Expand and simplify: $(x+y)^6$

Solution

$$(x+y)^6=x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$$

Exercise

Expand and simplify: $(2x+5y)^7$

Solution

$$(2x+5y)^7 = 128x^7 + 7\left(64x^6\right)(5y) + 21\left(32x^5\right)\left(25y^2\right) + 35\left(16x^4\right)\left(125y^3\right)$$
$$+35\left(8x^3\right)\left(625y^4\right) + 21\left(4x^2\right)\left(3{,}125y^5\right) + 7(2x)\left(5^6y^6\right) + (5y)^7$$
$$= 128x^7 + 320x^6y + 16{,}800x^5y^2 + 70{,}000x^4y^3 + 175{,}000x^3y^4 + 262{,}500x^2y^5$$
$$+ 218{,}750xy^6 + 78{,}125y^7$$

Exercise

Expand and simplify: $(a+b)^8$

Solution

$$(a+b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$$

Exercise

Expand and simplify: $\left(x-\frac{1}{x^2}\right)^9$

Solution

$$\left(x-\frac{1}{x^2}\right)^9 = x^9 + 9x^8\left(-\frac{1}{x^2}\right) + 36x^7\left(-\frac{1}{x^2}\right)^2 + 84x^6\left(-\frac{1}{x^2}\right)^3 + 126x^5\left(-\frac{1}{x^2}\right)^4 + 126x^4\left(-\frac{1}{x^2}\right)^5$$
$$+84x^3\left(-\frac{1}{x^2}\right)^6 + 36x^2\left(-\frac{1}{x^2}\right)^7 + 9x\left(-\frac{1}{x^2}\right)^8 + \left(-\frac{1}{x^2}\right)^9$$
$$= x^9 - 9x^6 + 36x^3 - 84 + 126x^{-3} - 126x^{-6} + 84x^{-9} - 36x^{-12} + 9x^{-15} - x^{-18}$$

Exercise

Find a line equation that passes through the given points $P(-4,\ 3)$ & $Q(2,\ -5)$

Solution

$$m = \frac{-5-3}{2+4} \qquad m = \frac{y_2 - y_1}{x_2 - x_1}$$

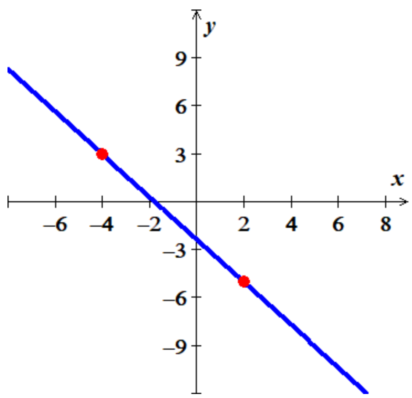
$$= \frac{-8}{6}$$

$$= -\frac{4}{3}$$

$$y = -\frac{4}{3}(x+4)+3 \qquad y = m\left(x - x_1\right) + y_1$$

$$= -\frac{4}{3}x - \frac{16}{3} + 3$$

$$= -\frac{4}{3}x - \frac{7}{3}$$



Exercise

Find a line equation that passes through the given points $P(8,\ 2)$ & $Q(3,\ 5)$

Solution

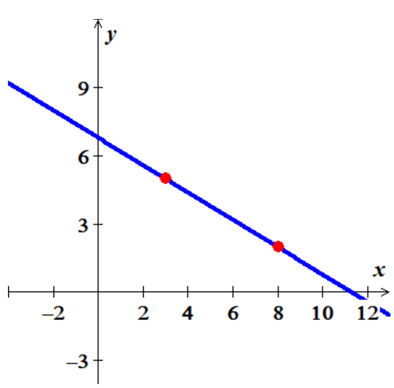
$$m = \frac{5-2}{3-8} \qquad m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= -\frac{3}{5}$$

$$y = -\frac{3}{5}(x-8)+2 \qquad y = m\left(x - x_1\right) + y_1$$

$$= -\frac{3}{5}x + \frac{24}{5} + 2$$

$$= -\frac{3}{5}x + \frac{34}{5}$$



Exercise

Find a line equation that passes through the given points $\left(2\sqrt{3},\ \sqrt{6}\right)$ & $\left(-\sqrt{3},\ 5\sqrt{6}\right)$

Solution

$$m = \frac{5\sqrt{6}-\sqrt{6}}{-\sqrt{3}-2\sqrt{3}} \qquad m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{4\sqrt{6}}{-3\sqrt{3}}$$

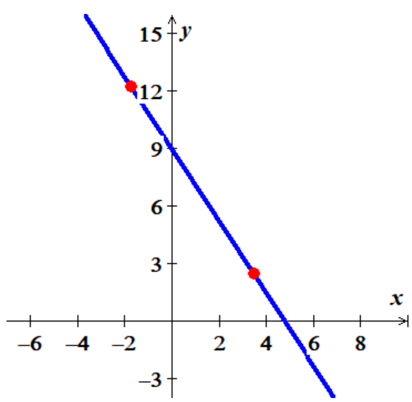
$$= -\frac{4}{3}\sqrt{\frac{6}{3}}$$

$$= -\frac{4\sqrt{2}}{3}$$

$$y = -\frac{4\sqrt{2}}{3}\left(x - 2\sqrt{3}\right) + \sqrt{6} \qquad y = m\left(x - x_1\right) + y_1$$

$$= -\frac{4\sqrt{2}}{3}x + \frac{8}{3}\sqrt{6} + \sqrt{6}$$

$$= -\frac{4\sqrt{2}}{3}x + \frac{11}{3}\sqrt{6}$$



Exercise

Find a line equation that passes through the given points $(-4,\ 9)$ & $(1,\ -3)$

Solution

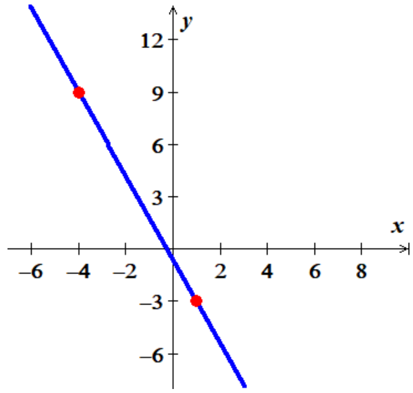
$$m = \frac{-3-9}{1+4} \qquad m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= -\frac{12}{5}$$

$$y = -\frac{12}{5}(x+4) + 9 \qquad y = m\left(x - x_1\right) + y_1$$

$$= -\frac{12}{5}x - \frac{48}{5} + 9$$

$$\underline{= -\frac{12}{5}x - \frac{3}{5}}$$



Exercise

Find the distance between the two given points $P(-4,\ 3)$ & $Q(2,\ -5)$

Solution

$$d = \sqrt{(-4-2)^2 + (3+5)^2}$$

$$= \sqrt{36+64}$$

$$= \sqrt{100}$$

$$\underline{= 10}$$

Exercise

Find the distance between the two given points $P(8,\ 2)$ & $Q(3,\ 5)$

Solution

$$d = \sqrt{(3-8)^2 + (5-2)^2}$$

$$= \sqrt{(-5)^2 + (3)^2}$$

$$= \sqrt{25+9}$$

$$\underline{= \sqrt{34}}$$

Exercise

Find the distance between the two given points $\left(2\sqrt{3},\ \sqrt{6}\right)$ & $\left(-\sqrt{3},\ 5\sqrt{6}\right)$

Solution

$$d=\sqrt{\left(2\sqrt{3}+\sqrt{3}\right)^2+\left(\sqrt{6}-5\sqrt{6}\right)^2}$$

$$=\sqrt{\left(3\sqrt{3}\right)^2+\left(-4\sqrt{6}\right)^2}$$

$$=\sqrt{9(3)+16(6)}$$

$$=\sqrt{123}$$

Exercise

Find the distance between the two given points $(-4,\ 9)$ & $(1,\ -3)$

Solution

$$d=\sqrt{(1+4)^2+(-3-9)^2}$$

$$=\sqrt{25+144}$$

$$=\sqrt{169}$$

$$=13$$

Exercise

Find the midpoint of the line segment with endpoints $P(-2,\ -1)$ & $Q(-8,\ 6)$

Solution

$$\left(\frac{-2-8}{2},\frac{-1+6}{2}\right)$$

$$\left(-5,\ \frac{5}{2}\right)$$

Exercise

Find the midpoint of the line segment with endpoints $P(8,\ 2)$ & $Q(3,\ 5)$

Solution

$$M\left(\frac{8+3}{2},\frac{2+5}{2}\right)$$

$$\underline{M\left(\frac{11}{2},\ \frac{7}{2}\right)}$$

Exercise

Find the midpoint of the line segment with endpoints $(1,\ 2)$ & $(7,\ -3)$

Solution

$$\left(\frac{1+7}{2},\frac{2-3}{2}\right)$$

$$\rightarrow \left(\frac{8}{2},\frac{-1}{2}\right)$$

$$\underline{\left(4,\ \frac{-1}{2}\right)}$$

Exercise

Find the midpoint of the line segment with endpoints $(7,\ -2)$ & $(9,\ 5)$

Solution

$$\left(\frac{7+9}{2},\frac{-2+5}{2}\right)$$

$$\underline{\left(8,\ \frac{3}{2}\right)}$$

Exercise

Write the standard form of the equation of the circle center $\left(-\sqrt{3},-\sqrt{3}\right)$, radius $\sqrt{3}$

Solution

$$\left(x+\sqrt{3}\right)^2+\left(y+\sqrt{3}\right)^2=\left(\sqrt{3}\right)^2$$

$$\underline{\left(x+\sqrt{3}\right)^2+\left(y+\sqrt{3}\right)^2=3}$$

Exercise

Write the standard form of the equation of the circle center $(-5,-3)$ and $r=\sqrt{5}$

Solution

$$\boxed{(x+5)^2+(y+3)^2=5}$$

Exercise

Write the standard form of the equation of the circle center $(6,\ -5)$ that passes through $(1,\ 7)$

Solution

$$\begin{aligned}\text{Radius} &= \sqrt{(1-6)^2+(7+5)^2}\\ &= \sqrt{5^2+12^2}\\ &= \sqrt{25+144}\\ &= \sqrt{139}\\ &= 13\end{aligned}$$

Equation of the circle: $\boxed{(x-6)^2+(y+5)^2=13}$ $\qquad (x-h)^2+(y-k)^2=r^2$

Exercise

Write the standard form of the equation of the circle:

Diameter whose endpoints are $(4,\ 4)$ and $(-2,\ 3)$

Solution

Center = midpoint of the endpoints

$$\begin{aligned}&= \left(\frac{4-2}{2},\frac{4+3}{2}\right)\\ &= \left(1,\ \frac{7}{2}\right)\end{aligned}$$

$$\begin{aligned}\text{Radius} &= \sqrt{(1-4)^2+\left(\frac{7}{2}-4\right)^2}\\ &= \sqrt{(-3)^2+\left(-\frac{1}{2}\right)^2}\\ &= \sqrt{9+\frac{1}{4}}\\ &= \sqrt{\frac{37}{4}}\end{aligned}$$

$9+\frac{1}{4}=\frac{4(9)+1}{4}=\frac{37}{4}$ $\qquad 9+\frac{1}{4}=9\frac{4}{4}+\frac{1}{4}=\frac{4(9)+1}{4}=\frac{37}{4}$

Equation of the circle: $\boxed{(x-1)^2 + \left(y-\frac{7}{2}\right)^2 = \frac{37}{4}}$

$(x-h)^2 + (y-k)^2 = r^2$

Solution — *Section R.2* – Solving Equations

Exercise

Solve: $3\left[2x-(4-x)+5\right]=7x-2$

Solution

$$6x-3(4-x)+15=7x-2$$
$$6x-12+3x+15=7x-2$$
$$9x+3=7x-2$$
$$2x=-5$$
$$\boxed{x=-\frac{5}{2}}$$

Exercise

Solve: $-4(2x-6)+8x=5x+24+x$

Solution

$$-8x+24+8x=6x+24$$
$$6x=0$$
$$\boxed{x=0}$$

Exercise

Solve: $-8(3x+4)+6x=4(x-8)+4x$

Solution

$$-24x-32+6x=4x-32+4x$$
$$-18x=8x$$
$$26x=0$$
$$\boxed{x=0}$$

Exercise

Solve: $\frac{1}{2}(4x+8)-16=-\frac{2}{3}(9x-12)$

Solution

$$6\times \quad \frac{1}{2}(4x+8)-16=-\frac{2}{3}(9x-12)$$

$3(4x+8)-96=-4(9x-12)$

$12x+24-96=-36x+48$

$12x+36x=48+72$

$48x=120$

$x=\dfrac{120}{48}$

$\quad \boxed{=\dfrac{5}{2}}$

Exercise

Solve: $\quad \frac{3}{4}(24-8x)-16=-\frac{2}{3}(6x-9)$

Solution

$12\times \quad \frac{3}{4}(24-8x)-16=-\frac{2}{3}(6x-9)$

$9(24-8x)-192=-8(6x-9)$

$216-72x-192=-48x+72$

$24x=24-72$

$24x=-48$

$\boxed{x=-2}$

Exercise

Solve: $\quad \dfrac{x-3}{4}=\dfrac{5}{14}-\dfrac{x+5}{7}$

Solution

$(28)\dfrac{x-3}{4}=(28)\dfrac{5}{14}-(28)\dfrac{x+5}{7}$ $\qquad LCD: 4 \quad 14 \quad 7 \to 28$

$7(x-3)=2(5)-4(x+5)$

$7x-21=10-4x-20$

$7x+4x=21-10$

$11x=11$

$\boxed{x=1}$

Exercise

Solve: $\dfrac{x+1}{4}=\dfrac{1}{6}+\dfrac{2-x}{3}$

Solution

$$12\frac{x+1}{4}=12\frac{1}{6}+12\frac{2-x}{3}$$

$$3(x+1)=2+4(2-x)$$

$$3x+3=2+8-4x$$

$$3x+4x=2+8-3$$

$$7x=7$$

$$\underline{x=1}$$

Exercise

Solve: $\dfrac{3x+2}{x-2}+\dfrac{1}{x}=\dfrac{-2}{x^2-2x}$

Solution

$$\text{Restriction: } \begin{cases} x-2\neq 0 \Rightarrow x\neq 2 \\ \qquad\qquad\quad x\neq 0 \end{cases}$$

$$x(x-2)\frac{3x+2}{x-2}+x(x-2)\frac{1}{x}=x(x-2)\frac{-2}{x^2-2x}$$

$$3x^2+2x+x-2=-2$$

$$3x^2+3x=0$$

$$3x(x+1)=0$$

$$3x=0 \qquad x+1=0$$

$$x=0 \qquad x=-1$$

$\boxed{x=-1}$ is the only solution

Exercise

Solve: $\dfrac{6}{x+1}-\dfrac{5}{x+2}=\dfrac{10}{x^2+3x+2}$

Solution

Restriction: $x\neq -1, -2$

$$6(x+2)-5(x+1)=10$$

$$6x+12-5x-5=10$$

$$\underline{x=3}$$

Exercise

Solve: $x^2=-25$

Solution

$$x=\pm\sqrt{-25}$$

$$\underline{=\pm 5i}$$

Exercise

Solve: $x^2=49$

Solution

$$\underline{x=\pm 7}$$

Exercise

Solve: $9x^2=100$

Solution

$$x^2=\frac{100}{9}$$

$$x=\pm\sqrt{\frac{100}{9}}$$

$$\underline{=\pm\frac{10}{3}}$$

Exercise

Solve: $4x^2+25=0$

Solution

$$4x^2=-25$$

$$x^2=-\frac{25}{4}$$

$$x=\pm\sqrt{-\frac{25}{4}}$$

$$= \pm\frac{5}{2}i$$

Exercise

Solve: $5x^2 - 45 = 0$

Solution

$$5x^2 = 45$$

$$x = \frac{45}{5}$$

$$x^2 = 9$$

$$x = \pm 3$$

Exercise

Solve: $(x-4)^2 = 12$

Solution

$$x - 4 = \pm\sqrt{12}$$

$$x = 4 \pm \sqrt{12} \qquad \sqrt{12} = \sqrt{4(3)} = 2\sqrt{3}$$

$$x = 4 \pm 2\sqrt{3}$$

Exercise

Solve: $(x+3)^2 = -16$

Solution

$$x + 3 = \pm\sqrt{-16}$$

$$x = -3 \pm 4i$$

Exercise

Solve: $x^2 + 8x + 15 = 0$

Solution

$$x = \frac{-8 \pm \sqrt{8^2 - 4(1)(15)}}{2(1)} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-8 \pm \sqrt{64-60}}{2}$$

$$= \frac{-8 \pm \sqrt{4}}{2}$$

$$= \frac{-8 \pm 2}{2}$$

$$= \begin{cases} \frac{-8+2}{2} = \frac{-6}{2} \underline{= -3|} \\ \frac{-8-2}{2} = \frac{-10}{2} \underline{= -5|} \end{cases}$$

Exercise

Solve: $x^2 + 5x + 2 = 0$

Solution

$$x = \frac{-5 \pm \sqrt{5^2 - 4(1)(2)}}{2(1)} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{25-8}}{2}$$

$$= \frac{-5 \pm \sqrt{17}}{2}$$

$$\underline{= \frac{-5}{2} \pm \frac{\sqrt{17}}{2}|}$$

Exercise

Solve: $x^2 + x - 12 = 0$

Solution

$$x = \frac{-1 \pm \sqrt{1+48}}{2} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm 7}{2}$$

$$= \begin{cases} \frac{-1-7}{2} \underline{= -4|} \\ \frac{-1+7}{2} \underline{= 3|} \end{cases}$$

$\therefore$ ***Solutions***: $\underline{x = -4,\ 3|}$

Exercise

Solve: $x^2 - 2x - 15 = 0$

Solution

$$x = \frac{2 \pm \sqrt{4+60}}{2} \qquad\qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm 8}{2}$$

$$= \begin{cases} \frac{2+8}{2} = 5 \\ \frac{2-8}{2} = -3 \end{cases}$$

$\therefore$ ***Solutions***: $x = -3,\ 5$

Exercise

Solve: $x(8x+1) = 3x^2 - 2x + 2$

Solution

$$8x^2 + x = 3x^2 - 2x + 2$$

$$5x^2 + 3x - 2 = 0$$

$$x = \frac{-3 \pm \sqrt{9+40}}{10} \qquad\qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-3 \pm 7}{2}$$

$$= \begin{cases} \frac{-3+7}{10} = \frac{2}{5} \\ \frac{-3-7}{10} = -1 \end{cases}$$

$\therefore$ ***Solutions***: $x = \frac{2}{5},\ -1$

Exercise

Solve: $3x^2 - x - 2 = 0$

Solution

$$3 - 1 - 2 = 0 \qquad a + b + c = 0$$

$$x = 1,\ -\frac{2}{3} \qquad\qquad x_1 = 1, \quad x_2 = \frac{c}{a}$$

Exercise

Solve: $3x^2 + x - 2 = 0$

Solution

$3 - 1 - 2 = 0 \qquad a - b + c = 0$

$x = -1,\ \frac{2}{3} \qquad\qquad x_1 = -1, \quad x_2 = -\frac{c}{a}$

Exercise

Solve: $2x^2 + 3x - 5 = 0$

Solution

$2 + 3 - 5 = 0 \qquad a + b + c = 0$

$x = 1,\ -\frac{5}{2} \qquad\qquad x_1 = 1, \quad x_2 = \frac{c}{a}$

Exercise

Solve: $2x^2 - 3x - 5 = 0$

Solution

$2 - (-3) - 5 = 0 \qquad a + b + c = 0$

$\therefore$ ***Solutions***: $x = -1,\ \frac{5}{2} \qquad x_1 = -1, \quad x_2 = -\frac{c}{a}$

Exercise

Solve $3x^3 + 2x^2 = 12x + 8$

Solution

$$3x^3 + 2x^2 - (12x + 8) = 0$$

$$x^2(3x + 2) - 4(3x + 2) = 0$$

$$(3x + 2)\left(x^2 - 4\right) = 0$$

$$3x + 2 = 0 \qquad\qquad x^2 - 4 = 0$$

$$3x = -2 \qquad\qquad x^2 = 4$$

$\underline{x = -\frac{2}{3}}$ $\qquad\qquad$ $\underline{x = \pm 2}$

$\therefore$ ***Solutions***: $\underline{x = -\frac{2}{3},\ \pm 2}$

Exercise

Solve: $\quad x^3 + x^2 - 4x - 4 = 0$

Solution

$$x^2(x+1) - 4(x+1) = 0$$

$$(x+1)\left(x^2 - 4\right) = 0$$

$x + 1 = 0$

$\underline{x = -1}$

$x^2 - 4 = 0$

$x^2 = 4$

$\underline{x = \pm 2}$

$\therefore$ ***Solutions***: $\underline{x = -1,\ \pm 2}$

Exercise

Solve: $x^3 - x^2 = 16x - 16$

Solution

$$x^3 - x^2 - 16x + 16 = 0$$

$$x^2(x-1) - 16(x-1) = 0$$

$$(x-1)\left(x^2 - 16\right) = 0$$

$x - 1 = 0$

$\underline{x = 1}$

$x^2 - 16 = 0$

$x^2 = 16$

$\underline{x = \pm 4}$

$\therefore$ ***Solutions***: $\underline{x = 1,\ \pm 4}$

Exercise

Solve $x^4+3x^2=10$

Solution

$$x^4+3x^2-10=0$$

$$\left(x^2+5\right)\left(x^2-2\right)=0$$

$x^2+5=0$	$x^2-2=0$
$x^2=-5$	$x^2=2$
$x=\pm i\sqrt{5}$	$x=\pm\sqrt{2}$

$\therefore$ ***Solutions***: $x=\pm i\sqrt{5},\ \pm\sqrt{2}$

Exercise

Solve: $x^4-4x^3+3x^2=0$

Solution

$$x^2\left(x^2-4x+3\right)=0$$

$x^2=0$	$x^2-4x+3=0$
$x=0,\ 0$	$x=1,\ 3$

$\therefore$ ***Solutions***: $x=0,\ 0,\ 1,\ 3$

Exercise

Solve: $x^4+6x^2-7=0$

Solution

$1+6-7=0$ $\qquad a+b+c=0$

$x^2=1,\ -7$ $\qquad x_1=1,\quad x_2=\frac{c}{a}$

$x^2=1$	$x^2=-7$
$x=\pm 1$	$x=\pm i\sqrt{7}$

$\therefore$ ***Solutions***: $x=\pm 1,\ \pm i\sqrt{7}$

Exercise

Solve: $3x^4 - x^2 - 2 = 0$

Solution

$3-1-2=0 \qquad a+b+c=0$

$x^2 = 1,\ -\frac{2}{3} \qquad\qquad x_1 = 1,\quad x_2 = \frac{c}{a}$

$x^2 = 1$

$x = \pm 1$

$x^2 = -\frac{2}{3}$

$x = \pm i\sqrt{\frac{2}{3}}\,\frac{\sqrt{3}}{\sqrt{3}}$

$= \pm i\frac{\sqrt{6}}{3}$

$\therefore$ ***Solutions***: $x = \pm 1,\ \pm i\frac{\sqrt{6}}{3}$

Exercise

Solve $x - 3\sqrt{x} - 4 = 0$

Solution

$\left(\sqrt{x}-4\right)\left(\sqrt{x}+1\right) = 0$

$\sqrt{x} - 4 = 0 \qquad \sqrt{x} + 1 = 0$

$\sqrt{x} = 4 \qquad \sqrt{x} = -1$ *Impossible*

$x = 16$

$\therefore$ ***Solutions***: $x = 16$

Exercise

Solve $\left(5x^2-6\right)^{1/4} = x$

Solution

$\left[\left(5x^2-6\right)^{1/4}\right]^4 = x^4$

$5x^2 - 6 = x^4$

$x^4 - 5x^2 + 6 = 0$

$$\left(x^2-3\right)\left(x^2-2\right)=0$$

$x^2=3 \qquad\qquad x^2=2$

$x=\pm\sqrt{3} \qquad\qquad x=\pm\sqrt{2}$

$\therefore$ ***Solutions***: $\boxed{x=\pm\sqrt{3},\ \pm\sqrt{2}}$

Exercise

Solve: $\sqrt[3]{6x-3}=3$

Solution

$6x-3=3^3$

$6x=27+3$

$x=\dfrac{30}{6}$

$\boxed{=5}$

Exercise

Solve: $\sqrt{2x+3}=5$

Solution

$2x+3=5^2 \qquad\qquad \sqrt[n]{u}=a \quad\rightarrow\quad u=a^n$

$2x=25-3$

$x=\dfrac{22}{2}$

$\boxed{=11}$

Check: $\sqrt{2(11)+3}\stackrel{?}{=}5$

$\sqrt{25}=5 \quad \checkmark$

$\therefore$ ***Solution*** set is: $\boxed{\{11\}}$

Exercise

Solve: $\sqrt{x-3}+6=5$

Solution

$\sqrt{x-3}=-1 \times$

∴ ***No*** solution.

Exercise

Solve: $\sqrt{3x-2}=4$

Solution

$3x-2=4^2$ $\qquad\qquad \sqrt[n]{u}=a \quad \rightarrow \quad u=a^n$

$3x=16+2$

$x=\dfrac{18}{3}$

$\underline{=6}$

Check: $\sqrt{3(6)-2}\overset{?}{=}4$

$\sqrt{16}=4$ ✓

∴ ***Solution*** set is: $\underline{\{6\}}$

Exercise

Solve: $\sqrt{2x+5}+11=6$

Solution

$\sqrt{2x+5}=-5$ ✗

∴ ***No*** solution.

Exercise

Solve: $\sqrt{x+2}+\sqrt{x-1}=3$

Solution

$\sqrt{x+2}=3-\sqrt{x-1}$

$x+2=\left(3-\sqrt{x-1}\right)^2$

$x+2=9-6\sqrt{x-1}+x-1$

$6\sqrt{x-1}=6$

$\sqrt{x-1}=1$

$x-1=1^2$

$\underline{x=2}$

Check:

$x = 2$

$\sqrt{4} + 1 \overset{?}{=} 3$

$2 + 1 = 3 \quad \checkmark$

$\therefore$ ***Solution***: $\underline{x = 2}$

Exercise

Solve: $\quad \sqrt{x+2} + \sqrt{3x+7} = 1$

Solution

$$\sqrt{x+2} = 1 - \sqrt{3x+7}$$

$$x + 2 = \left(1 - \sqrt{3x+7}\right)^2$$

$$x + 2 = 1 - 2\sqrt{3x+7} + 3x + 7$$

$$2\sqrt{3x+7} = 2x + 6$$

$$\sqrt{3x+7} = x + 3$$

$$3x + 7 = (x+3)^2$$

$$3x + 7 = x^2 + 6x + 9$$

$$x^2 + 3x + 2 = 0$$

$$\underline{x = -1, \ -2}$$

Check:

$x = -1$	$x = -2$
$\sqrt{-1+2} \overset{?}{=} 1 - \sqrt{-3+7}$	$\sqrt{-2+2} \overset{?}{=} 1 - \sqrt{-6+7}$
$1 \neq 1 - 2 \ \times$	$0 = 1 - 1 \quad \checkmark$

$\therefore$ ***Solution*** is: $\underline{x = -2}$

Exercise

Solve: $\quad |x| = -9$

Solution

$|x| = -9 \qquad$ Not True

$\therefore$ ***No*** Solution

Exercise

Solve: $|x| = 9$

Solution

$\therefore$ ***Solutions***: $\underline{x = \pm 9}$

Exercise

Solve: $|x-2| = 7$

Solution

$x - 2 = 7 \qquad x - 2 = -7$

$\underline{x = 9} \qquad \underline{x = -5}$

$\therefore$ ***Solutions***: $\underline{x = -5,\ 9}$

Exercise

Solve: $|x-2| = 0$

Solution

$x - 2 = 0$

$\therefore$ ***Solution***: $\underline{x = 2}$

Exercise

Solve: $2|x-6| = 8$

Solution

$x - 6 = 4$

$x - 6 = 4 \qquad x - 6 = -4$

$\underline{x = 10} \qquad \underline{x = 2}$

$\therefore$ ***Solutions***: $\underline{x = 2,\ 10}$

Exercise

Solve: $3|2x-1| = 21$

Solution

$|2x-1|=7$

$2x-1=7 \qquad 2x-1=-7$

$2x=8 \qquad 2x=-6$

$\underline{x=4} \qquad \underline{x=-3}$

$\therefore$ ***Solutions***: $\underline{x=-3,\ 4}$

Exercise

Solve: $2|3x-2|=14$

Solution

$|3x-2|=7$

$3x-2=7 \qquad 3x-2=-7$

$3x=9 \qquad 3x=-5$

$\underline{x=3} \qquad \underline{x=-\frac{5}{3}}$

$\therefore$ ***Solutions***: $\underline{x=-\frac{5}{3},\ 3}$

Exercise

Solve: $|3x-1|+2=16$

Solution

$|3x-1|=14$

$3x-1=14 \qquad 3x-1=-14$

$3x=15 \qquad 3x=-13$

$\underline{x=5} \qquad \underline{x=-\frac{13}{3}}$

$\therefore$ ***Solutions***: $\underline{x=-\frac{13}{3},\ 5}$

Exercise

Solve equation: $|x+1|=|1-3x|$

Solution

$x+1=-(1-3x) \qquad x+1=1-3x$

$$x+1=-1+3x \qquad x+3x=1-1$$
$$x-3x=-1-1 \qquad 4x=0$$
$$-2x=-2 \qquad x=0$$
$$x=1$$

∴ ***Solutions***: $\underline{x=0,\ 1}\,\big|$

Exercise

Solve: $|3x-1|=|x+5|$

Solution

$$3x-1=x+5 \qquad 3x-1=-(x+5)$$
$$2x=6 \qquad 3x-1=-x-5$$
$$\underline{x=3}\,\big| \qquad 4x=-4$$
$$\underline{x=-1}\,\big|$$

∴ ***Solutions***: $\underline{x=-1,\ 3}\,\big|$

Exercise

Solve: $|2x-4|=|x-1|$

Solution

$$2x-4=x-1 \qquad 2x-4=-x+1$$
$$\underline{x=3}\,\big| \qquad 3x=-5$$
$$\underline{x=-\frac{5}{3}}\,\Big|$$

∴ ***Solutions***: $\underline{x=-\frac{5}{3},\ 3}\,\Big|$

Exercise

Solve $-3x+5>-7$

Solution

$$-3x>-7-5$$
$$-3x>-12$$
$$\frac{-3}{-3}x<\frac{-12}{-3}$$

∴ ***Solutions***: $\underline{x<4}\,\big| \qquad (-\infty,\ 4)$

Exercise

Solve $4-3x\le 7+2x$

Solution

$$4-3x-4\le 7+2x-4$$

$$-3x\le 3+2x$$

$$-3x-2x\le 3+2x-2x$$

$$-5x\le 3$$

$\therefore$ ***Solutions***: $\underline{x\ge -\frac{3}{5}}$ *or* $\left[-\frac{3}{5},\ \infty\right)$

Exercise

Solve the inequality equation $4(x+1)+2\ge 3x+6$

Solution

$$4x+4+2\ge 3x+6$$

$\therefore$ ***Solutions***: $\underline{x\ge 0}$ *or* $[0,\ \infty)$

Exercise

Solve the inequality equation $8x+3>3(2x+1)+x+5$

Solution

$$8x+3>6x+3+x+5$$
$$8x+3>7x+8$$

$\therefore$ ***Solutions***: $\underline{x>5}$ *or* $(5,\ \infty)$

Exercise

Solve the inequality equation $5(3-x)\le 3x-1$

Solution

$$15-5x\le 3x-1$$
$$-8x\le -16$$
$$-x\le -2$$

$\therefore$ ***Solutions***: $\underline{x\ge 2}$ *or* $[2,\ \infty)$

Exercise

Solve $\dfrac{2x-5}{-8} \le 1-x$

Solution

$$(-8)\dfrac{2x-5}{-8} \ge (-8)(1-x)$$

$$2x-5 \ge -8+8x$$

$$2x-8x \ge -8+5$$

$$-6x \ge -3$$

$$\dfrac{-6}{-6}x \le \dfrac{-3}{-6}$$

∴ ***Solutions***: $\underline{x \le \frac{1}{2}}$ $\left(-\infty, \frac{1}{2}\right]$

Exercise

Solve the inequality equation $8(x+1) \le 7(x+5)+x$

Solution

$$8x+8 \le 7x+35+x$$

$$8x+8 \le 8x+35$$

$$8 \le 35$$

∴ ***Solutions***: $\underline{\mathbb{R}}$

Exercise

Solve the inequality equation $|x-2| < 1$

Solution

$$-1 < x-2 < 1$$

∴ ***Solutions***: $\underline{1 < x < 3}$

Exercise

Solve the inequality equation $|x+2| \ge 1$

Solution

$x+2 \le -1$ $\qquad$ $x+2 \ge 1$

$x \le -3$ $\qquad$ $x \ge -1$

$\therefore$ ***Solutions***: $\underline{x \le -3 \quad x \ge -1}$

Exercise

Solve the inequality equation $|2(x-1)+4| \le 8$

Solution

$$-8 \le 2x - 2 + 4 \le 8$$
$$-8 \le 2x + 2 \le 8$$
$$-10 \le 2x \le 6$$

$\therefore$ ***Solutions***: $\underline{-5 \le x \le 3}$

Exercise

Solve the inequality equation $\left|\dfrac{2x+6}{3}\right| > 2$

Solution

$$|2x+6| > 6$$

$$\begin{array}{ll} 2x+6 < -6 & \quad 2x+6 > 6 \\ 2x < -12 & \quad 2x > 0 \\ x < -6 & \quad x > 0 \end{array}$$

$\therefore$ ***Solutions***: $\underline{x < -6 \quad x > 0}$

Exercise

Solve $|12-9x| \ge -12$

Solution

$\therefore$ ***Solution*** set: $(-\infty, \infty)$ because the absolute value always greater than any negative number.

Exercise

Solve $|6-3x| < -11$

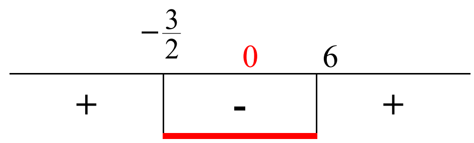
Solution

$\therefore$ ***No solution*** because the absolute value cannot be less than any negative number

Exercise

Solve: $2x^2 - 9x \le 18$

Solution



$$2x^2 - 9x - 18 \le 0$$
$$(2x+3)(x-6) \le 0$$

$\therefore$ ***Solutions***: $-\frac{3}{2} \le x \le 6$ $\quad \left[-\frac{3}{2},\ 6\right]$

Exercise

Solve the inequality: $x^2 - 5x + 4 > 0$

Solution

$$x^2 - 5x + 4 > 0$$
$$x = 1,\ 4$$

$\therefore$ ***Solutions***: $x < 1 \quad x > 4$ $\quad (-\infty, 1) \cup (4, \infty)$

Exercise

Solve the inequality equation $\quad x^2 + 7x + 10 < 0$

Solution

$$x^2 + 7x + 10 = 0$$

$$x = \frac{-7 \pm \sqrt{49 - 40}}{2}$$

$$= \frac{-7 \pm 3}{2}$$

$$= \begin{cases} \frac{-7-3}{2} = -5 \\ \frac{-7+3}{2} = -2 \end{cases}$$

$\therefore$ ***Solutions***: $-5 < x < 2$

Exercise

Solve the inequality equation $\quad x^3 - 3x^2 - 9x + 27 < 0$

Solution

$$x^3 - 3x^2 - 9x + 27 = 0$$

$$x^2(x-3) - 9(x-3) = 0$$

$$(x-3)(x^2-9) = 0$$

$$\rightarrow \begin{cases} x-3=0 \rightarrow \underline{x=3} \\ x^2-9=0 \rightarrow x^2=9 \rightarrow \underline{x=\pm 3} \end{cases}$$

$-\infty$	-3	3	∞
$-$	$+$	$+$	

$\therefore$ ***Solutions***: $\underline{x < -3}$ $\underline{(-\infty, -3)}$

Exercise

Solve the inequality equation $x^3 + 3x^2 \le x + 3$

Solution

$$x^3 + 3x^2 - x - 3 = 0$$

$$x^2(x+3) - (x+3) = 0$$

$$(x+3)(x^2-1) = 0$$

$$\begin{cases} x+3=0 \rightarrow x=-3 \\ x^2-1=0 \rightarrow x^2=1 \rightarrow x=\pm 1 \end{cases}$$

	-3		-1	0	1	
$-$		$+$		$-$		$+$

$\therefore$ ***Solutions***: $\underline{-1 < x < 0 \quad x > 1}$ $\underline{(-\infty, \ -3] \cup [-1, \ 1]}$

Exercise

Solve the inequality equation $x^4 - 20x^2 + 64 \le 0$

Solution

$$x^4 - 20x^2 + 64 = 0$$

$$x^2 = \frac{20 \pm \sqrt{400-256}}{2}$$

$$= \begin{cases} \frac{20-12}{2} = 4 \\ \frac{20+12}{2} = 16 \end{cases}$$

	-4	-2	0	2	4
$+$	$-$	$+$	$-$	$+$	

$$\begin{cases} x^2 = 4 \quad \rightarrow \quad \underline{x = \pm 2} \\ x^2 = 16 \quad \rightarrow \quad \underline{x = \pm 4} \end{cases}$$

∴ ***Solutions***: $-4 \le x \le -2 \quad 2 \le x \le 4$

Exercise

Solve the inequality equation $\quad x^4 - 10x^2 + 9 \ge 0$

Solution

$$x^4 - 10x^2 + 9 = 0$$

$$\begin{cases} x^2 = 1 & \rightarrow \quad x = \pm 1 \\ x^2 = 9 & \rightarrow \quad x = \pm 3 \end{cases}$$

	-3		-1	0	1		3	
$+$		$-$		$+$		$-$		$+$

∴ ***Solutions***: $x \le -3 \quad -1 \le x \le 1 \quad x \ge 3$

Exercise

Solve the inequality equation $\quad \dfrac{x+4}{x-1} < 0$

Solution

Restriction: $x \ne 1$

$$\frac{x+4}{x-1} = 0$$

$$x = -4$$

	-4		1	
$+$		$-$		$+$

∴ ***Solutions***: $-4 < x < 1$

Exercise

Solve the inequality equation $\quad \dfrac{x-2}{x+3} > 0$

Solution

Restriction: $x \ne -3$

$$\frac{x-2}{x+3} = 0$$

$$x = 2$$

	-3	0	2	
$+$		$-$		$+$

∴ ***Solutions***: $x < -3 \quad x > 2$

Exercise

Solve the inequality equation $\frac{x-4}{x+6} \le 1$

Solution

Restriction: $x \ne -6$

$\frac{x-4}{x+6} - 1 = 0$

$x - 4 - x - 6 = 0$

$-10 = 0$ ✗

	-6	**0**
$+$		$-$

$\therefore$ ***Solutions***: $x > -6$

Exercise

Solve the inequality equation $\frac{x}{2x+7} \ge 4$

Solution

Restriction: $x \ne -\frac{7}{2}$

$\frac{x}{2x+7} - 4 = 0$

$x - 8x - 28 = 0$

$7x = -28$

$x = -4$

	-4	**0**	$-\frac{7}{2}$	
$+$		$-$		$+$

$\therefore$ ***Solutions***: $x \le -4 \quad x > -\frac{7}{2}$

Exercise

Solve: $\frac{x-3}{x+4} \ge \frac{x+2}{x-5}$

Solution

Conditions: $x + 4 \ne 0 \to x \ne -4$ *and* $x - 5 \ne 0 \to x \ne 5$

$\frac{x-3}{x+4} - \frac{x+2}{x-5} = 0$

$\frac{0-3}{0+4} - \frac{0+2}{0-5} = \frac{-3}{4} - \frac{2}{-5} = \frac{-3}{4} + \frac{2}{5} = -$

$(x+4)(x-5)\left[\frac{x-3}{x+4} - \frac{x+2}{x-5}\right] = 0$

$(x-5)(x-3) - (x+4)(x+2) = 0$

$x^2 - 3x - 5x + 15 - (x^2 + 2x + 4x + 8) = 0$

$$x^2 - 3x - 5x + 15 - x^2 - 2x - 4x - 8 = 0$$
$$-14x + 7 = 0$$
$$-14x = -7$$
$$x = \frac{-7}{-14} = \frac{1}{2}$$

	-4	0	$1/2$	5	
$+$		$-$		$+$	$-$

∴ ***Solutions***: $\underline{x < -4 \quad \frac{1}{2} \le x < 5}\,\big|$ $\qquad$ $\underline{(-\infty, -4) \cup \left[\frac{1}{2}, 5\right)}\,\big|$

Solution Section R.3 – Functions

Exercise

$$f(x)=\begin{cases} 2+x & if \quad x<-4 \\ -x & if \quad -4\le x\le 2 \\ 3x & if \quad x>2 \end{cases}$$

Find: $f(-5), f(-1), f(0),$ *and* $f(3)$

Solution

a) $f(-5)=2-5=-3$

b) $f(-1)=-(-1)=1$

c) $f(0)=-0=0$

d) $f(3)=3(3)=9$

Exercise

$$f(x)=\begin{cases} -2x & if \quad x<-3 \\ 3x-1 & if \quad -3\le x\le 2 \\ -4x & if \quad x>2 \end{cases}$$

Find: $f(-5), f(-1), f(0),$ *and* $f(3)$

Solution

a) $f(-5)=-2(-5)=10$

b) $f(-1)=3(-1)-1=-4$

c) $f(0)=3(0)-1=-1$

d) $f(3)=-4(3)=-12$

Exercise

$$f(x)=\begin{cases} 3x+5 & if \quad x<0 \\ 4x+7 & if \quad x\ge 0 \end{cases}$$

Find

a) $f(0)$

b) $f(-2)$

c) $f(1)$

d) $f(3)+f(-3)$

e) Graph $f(x)$

Solution

a) $f(0)=4(0)+7$

$\qquad =7$

b) $f(-2) = 3(-2) + 5$

$\underline{= -1}$

c) $f(1) = 4(1) + 7$

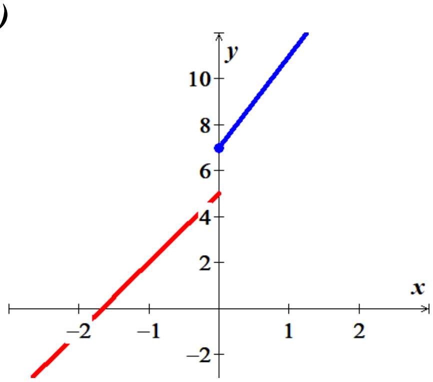
$\underline{= 11}$

d) $f(3) + f(-3) = 4(3) + 7 + 3(-3) + 5$

$= 12 + 7 - 9 + 5$

$\underline{= 15}$

e)



Exercise

Find the domain: $f(x) = 7x + 4$

Solution

Domain: $(-\infty, \infty)$

Exercise

Find the domain: $f(x) = |3x - 2|$

Solution

Domain: $\underline{\mathbb{R}}$

Exercise

Find the domain: $f(x) = 3x + \pi$

Solution

Domain: $\underline{\mathbb{R}}$

Exercise

Find the domain: $f(x)=-2x^2+3x-5$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x)=x^3-2x^2+x-3$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain $f(x)=4-\dfrac{2}{x}$

Solution

Domain: $x \neq 0$

Exercise

Find the domain $f(x)=\dfrac{1}{x^4}$

Solution

Domain: $x \neq 0$

Exercise

Find the domain $y=\dfrac{2}{x-3}$

Solution

Domain: $x \neq 3$

Exercise

Find the domain $f(x)=\dfrac{3x}{x+2}$

Solution

Domain: $x \neq -2$

Exercise

Find the domain $f(x)=x-\dfrac{2}{x-3}$

Solution

Domain: $\underline{x\neq 3}$

Exercise

Find the domain $f(x)=\dfrac{1}{2}x-\dfrac{8}{x+7}$

Solution

Domain: $\underline{x\neq -7}$

Exercise

Find the domain $f(x)=\dfrac{1}{x-3}-\dfrac{8}{x+7}$

Solution

Domain: $\underline{x\neq -7,\ 3}$

Exercise

Find the domain $f(x)=\dfrac{1}{x+4}-\dfrac{2x}{x-4}$

Solution

Domain: $\underline{x\neq \pm 4}$

Exercise

Find the domain $f(x)=\dfrac{x}{x^2+3x+2}$

Solution

$x^2+3x+2\neq 0 \qquad a-b+c=0 \ \rightarrow \ x=-1,\ -\dfrac{c}{a}$

Domain: $\underline{x\neq -1,\ -2}$

Exercise

Find the domain $f(x)=\dfrac{x^2}{x^2-5x+4}$

Solution

$$x^2 - 5x + 4 \neq 0 \qquad a+b+c=0 \ \rightarrow \ x = 1, \ \frac{c}{a}$$

Domain: $\underline{x \neq 1, \ 4}$

Exercise

Find the domain $\quad g(x) = \dfrac{2}{x^2 + x - 12}$

Solution

$$x^2 + x - 12 \neq 0$$

$$(x+4)(x-3) \neq 0$$

Domain: $\underline{x \neq -4, \ 3}$ $\quad \underline{(-\infty,-4)\cup(-4,3)\cup(3,\infty)}$

Exercise

Find the domain $\quad h(x) = \dfrac{5}{\frac{4}{x} - 1}$

Solution

$x \neq 0 \qquad \dfrac{4}{x} - 1 \neq 0$

$$\frac{4-x}{x} \neq 0$$

$$4 - x \neq 0$$

$$x \neq 4$$

Domain: $\underline{x \neq 0, \ 4}$ $\quad \underline{(-\infty,0)\cup(0,4)\cup(4,\infty)}$

Exercise

Find the domain $\quad y = \sqrt{x}$

Solution

$$x \geq 0$$

Domain: $\underline{x \geq 0}$ $\quad \underline{[0, \ \infty)}$

Exercise

Find the domain $\quad f(x) = \sqrt{3 - 2x}$

Solution

Domain: $\boxed{x \le \frac{3}{2}}$

Exercise

Find the domain $\quad f(x) = \sqrt{3+2x}$

Solution

Domain: $\boxed{x \ge -\frac{3}{2}}$

Exercise

Find the domain $\quad f(x) = \sqrt{6-3x}$

Solution

Domain: $\boxed{x \le 2}$

Exercise

Find the domain $\quad f(x) = \sqrt{2x+7}$

Solution

Domain: $\boxed{x \ge -\frac{7}{2}}$

Exercise

Find the domain $\quad f(x) = \sqrt{x^2 - 16}$

Solution

$$x^2 - 16 = 0$$
$$x^2 = 16$$
$$x = \pm 4$$

Domain: $\boxed{x \le -4 \quad x \ge 4}$

Exercise

Find the domain $f(x)=\sqrt{16-x^2}$

Solution

$x=\pm 4$

Domain: $\boxed{-4\le x\le 4}$

Exercise

Find the domain $f(x)=\sqrt{x^2-5x+4}$

Solution

$x^2-5x+4 \qquad a+b+c=0 \ \rightarrow \ x=1,\ \frac{c}{a}$

$x=1,\ 4$

Domain: $\boxed{x\le 1 \quad x\ge 4}$

Exercise

Find the domain $f(x)=\dfrac{\sqrt{x+1}}{x}$

Solution

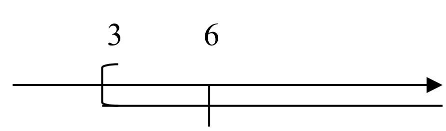
$x+1\ge 0 \qquad x\ne 0$

$x\ge -1$

Domain: $\boxed{x\ge -1 \quad x\ne 0}$ $\qquad \boxed{[-1,\ 0)\cup(0,\ \infty)}$

Exercise

Find the domain $g(x)=\dfrac{\sqrt{x-3}}{x-6}$



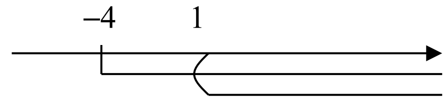
Solution

$\rightarrow \begin{cases} x\ge 3 \\ x\ne 6 \end{cases}$

Domain: $\boxed{x\ge 3 \quad x\ne 6}$ $\qquad \boxed{[3,\ 6)\cup(6,\ \infty)}$

Exercise

Find the domain $f(x)=\dfrac{\sqrt{x+4}}{\sqrt{x-1}}$



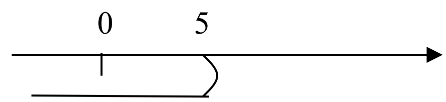
Solution

$$\to \begin{cases} x \ge -4 \\ x > 1 \end{cases}$$

Domain: $x > 1$ $(1,\ \infty)$

Exercise

Find the domain $f(x)=\dfrac{\sqrt{5-x}}{x}$



Solution

$x \le 5 \quad x \ne 0$

Domain: $x \le 5 \quad x \ne 0$ $(-\infty,\ 0)\cup(0,\ 5]$

Exercise

Find the domain $f(x)=\dfrac{x}{\sqrt{5-x}}$

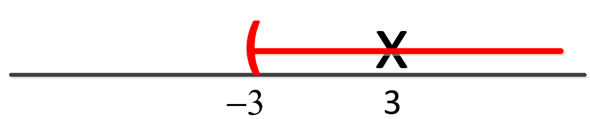
Solution

Domain: $x < 5$ $(-\infty,\ 5)$

Exercise

Find the domain of $f(x)=\dfrac{1}{(x-3)\sqrt{x+3}}$

Solution



$x-3 \ne 0 \qquad x+3>0$

$x \ne 3 \qquad x > -3$

Domain: $\{x \mid x > -3 \ \ and \ \ x \ne 3\}$

$(-3,\ 3)\cup(3,\ \infty)$

Exercise

Find the domain of $f(x)=\dfrac{\sqrt{x+2}}{\sqrt{x^2+3x+2}}$

Solution

$x^2+3x+2 \qquad a-b+c=0 \rightarrow x=-1,\ -\frac{c}{a}$

$x<-2 \quad x>-1$

$\sqrt{x+2} \rightarrow x\ge -2$

Domain: $\boxed{x>-1}$

Exercise

Find the domain of $f(x)=\dfrac{\sqrt{2x+3}}{x^2-6x+5}$

Solution

$x^2-6x+5 \qquad a+b+c=0 \rightarrow x=1,\ \frac{c}{a}$

$x\ne 1,\ 5$

$\sqrt{2x+3} \rightarrow x\ge -\frac{3}{2}$

Domain: $\boxed{x\ge -\frac{3}{2} \quad x\ne 1,\ 5}$

Exercise

Let $f(x)=2x^2+3$ and $g(x)=3x-4$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

Solution

a) $(f+g)(x)=2x^2+3+3x-4$

$=\boxed{2x^2+3x-1}$

Domain: $\boxed{\mathbb{R}}$

b) $(f-g)(x)=2x^2+3-(3x-4)$

$=2x^2+3-3x+4$

$=\boxed{2x^2-x+7}$

Domain: $\mathbb{R}$

c) $(fg)(x) = \left(2x^2+3\right)(3x-4)$

$$= 6x^2 + x - 12$$

Domain: $\mathbb{R}$

d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{2x^2+3}{3x-4}$

Domain: $x \neq -\frac{4}{3}$

Exercise

Let $f(x) = x^2 - 2x - 3$ and $g(x) = x^2 + 3x - 2$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\dfrac{f}{g}\right)(x)$

Solution

a) $(f+g)(x) = x^2 - 2x - 3 + x^2 + 3x - 2$

$$= 2x^2 + x - 5$$

Domain: $\mathbb{R}$

b) $(f-g)(x) = x^2 - 2x - 3 - x^2 - 3x + 2$

$$= -5x - 1$$

Domain: $\mathbb{R}$

c) $(fg)(x) = \left(x^2 - 2x - 3\right)\left(x^2 + 3x - 2\right)$

$$= x^4 + 3x^3 - 2x^2 - 2x^3 - 6x^2 + 4x - 3x^2 - 9x + 6$$

$$= x^4 + x^3 - 11x^2 - 5x + 6$$

Domain: $\mathbb{R}$

d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{x^2 - 2x - 3}{x^2 + 3x - 2}$

Domain: $x \neq \dfrac{-3 \pm \sqrt{17}}{2}$

Exercise

Given that $f(x)=x+1$ and $g(x)=\sqrt{x+3}$

a) Find $(f+g)(x)$

b) Find the domain of $(f+g)(x)$

c) Find: $(f+g)(6)$

Solution

a) $(f+g)(x)=f(x)+g(x)$

$$=x+1+\sqrt{x+3}$$

b) $x+3\ge 0 \to x\ge -3$

Domain $=[-3,\ \infty)$

c) $(f+g)(6)=6+1+\sqrt{6+3}$

$$=10$$

Exercise

Given $f(x)=x-3$ *and* $g(x)=x+3$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f\left(g(x)\right)=f(x+3)$ ***Domain***: $\mathbb{R}$

$$=(x-3)+3$$

$$=x$$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g\left(f(x)\right)=g(x-3)$ ***Domain***: $\mathbb{R}$

$$=(x+3)-3$$

$$=x$$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x)=\frac{2}{3}x$ *and* $g(x)=\frac{3}{2}x$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\frac{3}{2}x\right)$ ***Domain***: $\mathbb{R}$

$$=\frac{2}{3}\left(\frac{3}{2}x\right)$$

$$\underline{=x}$$ ***Domain***: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}$

b) $g(f(x))=g\left(\frac{2}{3}x\right)$ ***Domain***: $\mathbb{R}$

$$=\frac{3}{2}\left(\frac{2}{3}x\right)$$

$$\underline{=x}$$ ***Domain***: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}$

Exercise

Given $f(x)=x-1$ *and* $g(x)=3x^2-2x-1$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(3x^2-2x-1\right)$ ***Domain***: $\mathbb{R}$

$$=3(x-1)^2-2(x-1)-1$$

$$=3\left(x^2-2x+1\right)-2x+2-1$$

$$=3x^2-6x+3-2x+1$$

$$\underline{=3x^2-8x+4}$$ ***Domain***: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}$

b) $g(f(x))=g(x-1)$ ***Domain***: $\mathbb{R}$

$$=3x^2-2x-1-1$$

$$= 3x^2 - 2x - 2 \quad \text{Domain: } \mathbb{R}$$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = x^2 - 2$ *and* $g(x) = 4x - 3$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(4x - 3)$ ***Domain***: $\mathbb{R}$

$$= (4x-3)^2 - 2$$

$$= 16x^2 - 24x + 9 - 2$$

$$= 16x^2 - 24x + 7 \quad \text{Domain: } \mathbb{R}$$

Domain: $\mathbb{R}$

b) $g(f(x)) = g(x^2 - 2)$ ***Domain***: $\mathbb{R}$

$$= 4(x^2 - 2) - 3$$

$$= 4x^2 - 8 - 3$$

$$= 4x^2 - 11 \quad \text{Domain: } \mathbb{R}$$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = \sqrt{x}$ *and* $g(x) = x + 3$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(x + 3)$ ***Domain***: $\mathbb{R}$

$$= \sqrt{x+3} \quad \text{Domain: } x \ge -3$$

Domain: $x \ge -3$

b) $g(f(x)) = g(\sqrt{x})$ ***Domain***: $x \ge 0$

$\underline{= \sqrt{x} + 3}$ ***Domain***: $x \ge 0$

Domain: $\underline{x \ge 0}$

Exercise

Given $f(x) = \sqrt{x}$ *and* $g(x) = 2 - 3x$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(2-3x)$ ***Domain***: $\mathbb{R}$

$\underline{= \sqrt{2-3x}}$ ***Domain***: $x \le \frac{2}{3}$

Domain: $\underline{x \le \frac{2}{3}}$

b) $g(f(x)) = g(\sqrt{x})$ ***Domain***: $x \ge 0$

$\underline{= 2 - 3\sqrt{x}}$ ***Domain***: $x \ge 0$

Domain: $\underline{x \ge 0}$

Exercise

Given $f(x) = x^4$ *and* $g(x) = \sqrt[4]{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(\sqrt[4]{x})$ ***Domain***: $x \ge 0$

$= \left(\sqrt[4]{x}\right)^4$

$\underline{= x}$ ***Domain***: $\mathbb{R}$

Domain: $\underline{x \ge 0}$

b) $g(f(x)) = g(x^4)$ ***Domain***: $\mathbb{R}$

$= \sqrt[4]{x^4}$

$\underline{= x\,}|$ *Domain*: $\mathbb{R}$

Domain: $\underline{\mathbb{R}\,}|$

Exercise

Given $f(x) = x^n$ *and* $g(x) = \sqrt[n]{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\sqrt[n]{x}\right)$ *Domain*: $\begin{cases} \text{If } n \text{ is even} & \underline{x \geq 0}| \\ \text{If } n \text{ is odd} & \underline{\mathbb{R}}| \end{cases}$

$= \left(\sqrt[n]{x}\right)^n$

$\underline{= x\,}|$ *Domain*: $\mathbb{R}$

Domain: $\begin{cases} \text{If } n \text{ is even} & \underline{x \geq 0}| \\ \text{If } n \text{ is odd} & \underline{\mathbb{R}}| \end{cases}$

b) $g(f(x)) = g\left(x^n\right)$ *Domain*: $\mathbb{R}$

$= \sqrt[n]{x^n}$

$\underline{= x\,}|$ *Domain*: $\mathbb{R}$

Domain: $\underline{\mathbb{R}\,}|$

Exercise

Given $f(x) = x^2 - 3x$ *and* $g(x) = \sqrt{x+2}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\sqrt{x+2}\right)$ $x + 2 \geq 0 \Rightarrow x \geq -2$

$= \left(\sqrt{x+2}\right)^2 - 3\sqrt{x+2}$

$\underline{= x + 2 - 3\sqrt{x+2}\,}|$ $x + 2 \geq 0 \Rightarrow x \geq -2$

Domain: $\{x \mid x \geq -2\}$

b) $g(f(x)) = g\left(x^2 - 3x\right)$ $\qquad \mathbb{R}$

$= \sqrt{x^2 - 3x + 2}$ $\qquad x^2 - 3x + 2 \ge 0 \Rightarrow (x = 1,\ 2) \leftrightarrow x \le 1,\ x \ge 2$

Domain: $\{x \mid x \le 1,\ x \ge 2\}$

Exercise

Given $f(x) = \sqrt{x-2}$ *and* $g(x) = \sqrt{x+5}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\sqrt{x+5}\right)$ $\qquad x + 5 \ge 0 \Rightarrow x \ge -5$

$= \sqrt{\sqrt{x+5} - 2}$ $\qquad \sqrt{x+5} - 2 \ge 0 \Rightarrow \sqrt{x+5} \ge 2$

$$x + 5 \ge 4$$

$$x \ge -1$$

Domain: $\{x \mid x \ge -1\}$

b) $g(f(x)) = g\left(\sqrt{x-2}\right)$ $\qquad x - 2 \ge 0 \Rightarrow x \ge 2$

$= \sqrt{\sqrt{x-2} + 5}$

$\sqrt{x-2} + 5 \ge 0 \Rightarrow \sqrt{x-2} \ge -5$ *Always true when* $x \ge 2$

Domain: $\{x \mid x \ge 2\}$

Exercise

Given $f(x) = x^5 - 2$ *and* $g(x) = \sqrt[5]{x+2}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\sqrt[5]{x+2}\right)$ $\qquad$ ***Domain***: $\mathbb{R}$

$= \left(\sqrt[5]{x+2}\right)^5 - 2$

$= x + 2 - 2$

$= x$ | ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g\left(f(x)\right) = g\left(x^5 - 2\right)$ ***Domain***: $\mathbb{R}$

$= \sqrt[5]{x^5 - 2 + 2}$

$= \sqrt[5]{x^5}$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = 1 - x^2$ *and* $g(x) = \sqrt{x^2 - 25}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(\sqrt{x^2 - 25}\right)$ ***Domain***: $x \le -5 \quad x \ge 5$

$= 1 - \left(\sqrt{x^2 - 25}\right)^2$

$= 1 - \left(x^2 - 25\right)$

$= 1 - x^2 + 25$

$= 26 - x^2$ ***Domain***: $\mathbb{R}$

Domain: $x \le -5 \quad x \ge 5$

b) $g\left(f(x)\right) = g\left(1 - x^2\right)$ ***Domain***: $\mathbb{R}$

$= \sqrt{\left(1 - x^2\right)^2 - 25}$

$= \sqrt{1 - 2x^2 + x^4 - 25}$

$= \sqrt{x^4 - 2x^2 - 24}$

$$x^2 = \frac{2 \pm \sqrt{4 + 96}}{2}$$

$$=\begin{cases} \dfrac{2-10}{2}=-4 \times \\ \dfrac{2+10}{2}=6 \end{cases}$$

$x^2=6 \quad \rightarrow \quad \underline{x=\pm\sqrt{6}}$ **Domain**: $x \le -\sqrt{6} \quad x \ge \sqrt{6}$

Domain: $\underline{x \le -\sqrt{6} \quad x \ge \sqrt{6}}$

Exercise

Given $f(x)=\dfrac{1}{x-2}$ *and* $g(x)=\dfrac{x+2}{x}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\dfrac{x+2}{x}\right)$ **Domain**: $x \ne 0$

$$=\frac{1}{\frac{x+2}{x}-2}$$

$$=\frac{1}{\frac{x+2-2x}{x}}$$

$$=\frac{x}{2-x}$$

Domain: $x \ne 2$

Domain: $\underline{x \ne 0,\ 2}$ $(-\infty,\ 0)\cup(0,\ 2)\cup(2,\ \infty)$

b) $g(f(x))=g\left(\dfrac{1}{x-2}\right)$ **Domain**: $x \ne 2$

$$=\frac{\frac{1}{x-2}+2}{\frac{1}{x-2}}$$

$$=\frac{\frac{1+2x-4}{x-2}}{\frac{1}{x-2}}$$

$$\underline{=2x-3}$$

Domain: $\mathbb{R}$

Domain: $\underline{x \ne 2}$ $(-\infty,\ 2)\cup(2,\ \infty)$

Exercise

Given $f(x)=\dfrac{3x+5}{2}$ *and* $g(x)=\dfrac{2x-5}{3}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\frac{2x-5}{3}\right)$ **Domain**: $\mathbb{R}$

$$=\frac{3\frac{2x-5}{3}+5}{2}$$

$$=\frac{2x-5+5}{2}$$

$$=\frac{2x}{2}$$

$$\boxed{=x}$$

Domain: $\boxed{\mathbb{R}}$

b) $g(f(x))=g\left(\frac{3x+5}{2}\right)$ **Domain**: $\mathbb{R}$

$$=\frac{2\frac{3x+5}{2}-5}{3}$$

$$=\frac{3x+5-5}{3}$$

$$=\frac{3x}{3}$$

$$\boxed{=x}$$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x)=\dfrac{2x+3}{x+4}$ *and* $g(x)=\dfrac{-4x+3}{x-2}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\frac{-4x+3}{x-2}\right)$ **Domain**: $x\neq 2$

$$= \frac{2\frac{-4x+3}{x-2}+3}{\frac{4x+3}{x-2}+4}$$

$$= \frac{-8x+6+3x-6}{4x+3+4x-8}$$

$$= \frac{-5x}{-5}$$

$$= x$$ ***Domain***: $\mathbb{R}$

Domain: $x \neq 2$

b) $g\left(f(x)\right) = g\left(\frac{2x+3}{x+4}\right)$ ***Domain***: $x \neq -4$

$$= \frac{-4\frac{2x+3}{x+4}+3}{\frac{2x+3}{x+4}-2}$$

$$= \frac{-8x-12+3x+12}{2x+3-2x-8}$$

$$= \frac{-5x}{-5}$$

$$= x$$ ***Domain***: $\mathbb{R}$

Domain: $x \neq -4$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

$$f(x) = x^2 + 6x + 3$$

Solution

$$x = -\frac{6}{2(1)}$$

$$= -3$$

$$y = f(-3)$$

$$= (-3)^2 + 6(-3) + 3$$

$$= -6$$

Vertex point $(-3, -6)$

Line of *symmetry*: $x = -3$

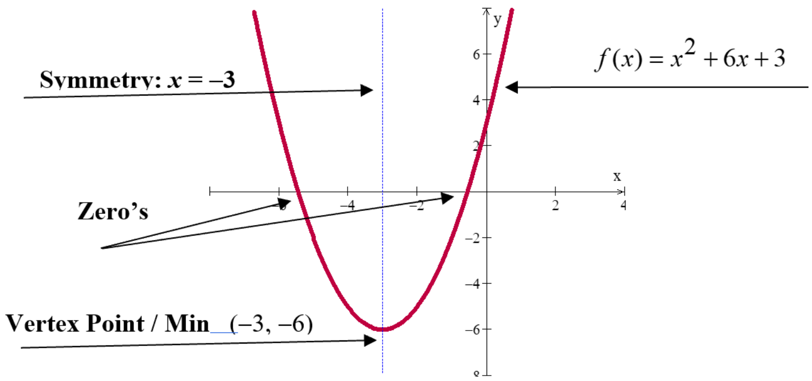
Minimum point, value $(-3, -6)$

$$x = \frac{-6 \pm \sqrt{36-12}}{2} = \frac{-6 \pm \sqrt{24}}{2} = \frac{-6 \pm 2\sqrt{6}}{2} = -3 \pm \sqrt{6}$$

$$x = \begin{cases} -3+\sqrt{6} \\ -3-\sqrt{6} \end{cases}$$

y-intercept $y = 3$

Range: $[-6, \infty)$ *Domain*: $(-\infty, \infty)$



Decreasing: $(-\infty, -3)$ *Increasing*: $(-3, \infty)$

$f(x) > 0 \quad \Rightarrow \quad x < -3-\sqrt{6} \quad \& \quad x > -3+\sqrt{6}$

$f(x) < 0 \quad \Rightarrow \quad -3-\sqrt{6} < x < -3+\sqrt{6}$

Exercise

For the function $f(x) = x^2 + 6x + 5$

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

Solution

$x = -\frac{6}{2}$ $\quad x = -\frac{b}{2a}$

$= -3$

$y = f(-3) = (-3)^2 + 6(-3) + 5$

$= -4$

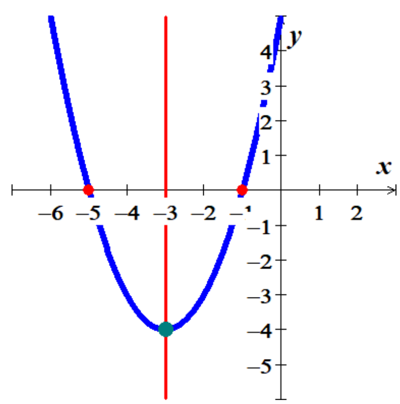
Vertex point: $(-3, -4)$

Axis of *symmetry*: $x = -3$

Minimum point @ $(-3, -4)$

$x^2 + 6x + 5 = 0$

$x = -5, \ -1$



$x = 0 \quad \to \quad \underline{y = 5|}$

Domain: $\mathbb{R}$ *Range*: $[-4, \infty)$

Increasing: $(-3, \infty)$ Decreasing: $(-\infty, -3)$

$f(x) > 0 \quad \Rightarrow \quad x < -5 \quad \& \quad x > -1$

$f(x) < 0 \quad \Rightarrow \quad -5 < x < -1$

Exercise

For the function $f(x) = -x^2 - 6x - 5$

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

Solution

$x = -\frac{-6}{-2}$ $x = -\frac{b}{2a}$

$= -3$

$y = f(-3)$

$= -9 + 18 - 5$

$= 4$

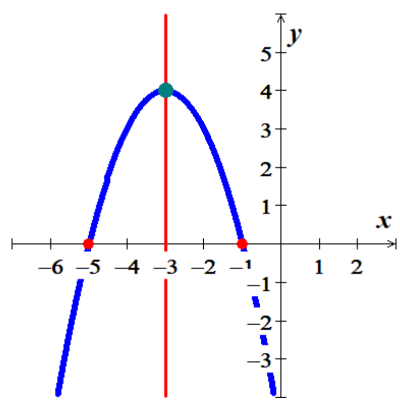
Vertex point: $(-3, 4)$

Axis of *symmetry*: $x = -3$

Maximum point @ $(-3, 4)$

$-\left(x^2 + 6x + 5\right) = 0 \quad \Rightarrow \quad x = -5, \ -1$

$x = 0 \quad \to \quad y = -5$



Domain: $\mathbb{R}$ *Range*: $(-\infty, 4]$

Increasing: $(-\infty, -3)$ *Decreasing*: $(-3, \infty)$

$f(x) > 0 \quad \Rightarrow \quad -5 < x < -1$

$f(x) < 0 \quad \Rightarrow \quad x < -5 \quad \& \quad x > -1$

Exercise

For the function $f(x) = x^2 - 4x + 2$

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

Solution

$x = -\frac{-4}{2}$ $\qquad\qquad\qquad x = -\frac{b}{2a}$

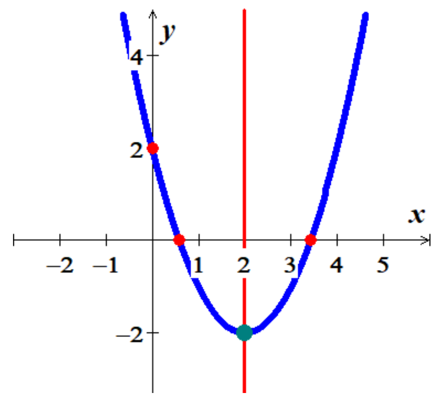
$= 2$

$f(2) = 4 - 8 + 2$

$= -2$

Vertex point: $(2,\ -2)$

Axis of *symmetry*: $x = 2$



Minimum point @ $(2,\ -2)$

$x^2 - 4x + 2 = 0$

$x = \frac{4 \pm \sqrt{8}}{2}$

$x = 2 \pm \sqrt{2}$

$x = 0 \quad \rightarrow \quad y = 2$

Domain: $\mathbb{R}$ $\qquad\qquad$ *Range*: $[-2,\ \infty)$

Increasing: $(2,\ \infty)$ $\qquad\qquad$ *Decreasing*: $(-\infty,\ 2)$

$f(x) > 0 \quad \Rightarrow \quad x < 2 - \sqrt{2} \quad \& \quad x > 2 + \sqrt{2}$

$f(x) < 0 \quad \Rightarrow \quad 2 - \sqrt{2} < x < 2 + \sqrt{2}$

Exercise

For the function $f(x) = -2x^2 + 16x - 26$

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

Solution

$x = -\frac{16}{-4}$ $\qquad\qquad\qquad x = -\frac{b}{2a}$

$= 4$

$f(4) = -32 + 64 - 26$

$= 6$

Vertex point: $\boxed{(4,\ 6)}$

Axis of *symmetry*: $\boxed{x = 4}$

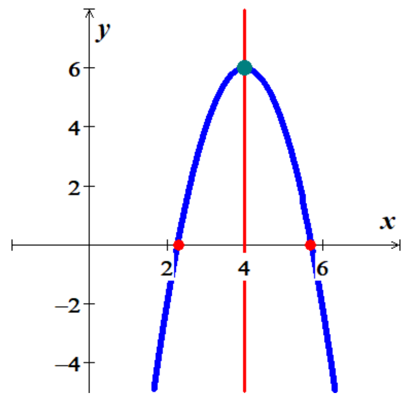
Maximum point @ $\boxed{(4,\ 6)}$

$-2x^2 + 16x - 26 = 0$

$x = \dfrac{-16 \pm \sqrt{128}}{-4}$

$\boxed{x = 4 \pm 2\sqrt{2}}$

$x = 0 \quad \to \quad \boxed{y = -26}$



Domain: $\boxed{\mathbb{R}}$ *Range*: $\boxed{(-\infty,\ 6]}$

Increasing: $\boxed{(-\infty,\ 4)}$ *Decreasing*: $\boxed{(4,\ \infty)}$

$f(x) > 0 \quad \Rightarrow \quad 4 - 2\sqrt{2} < x < 4 + 2\sqrt{2}$

$f(x) < 0 \quad \Rightarrow \quad x < 4 - 2\sqrt{2} \quad \& \quad x > 4 + 2\sqrt{2}$

Exercise

For the function $f(x) = x^2 + 4x + 1$

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

Solution

$x = -\dfrac{4}{2}$ $\qquad x = -\dfrac{b}{2a}$

$\boxed{= -2}$

$f(-2) = 4 - 8 + 1$

$\boxed{= -3}$

Vertex point: $\boxed{(-2,\ -3)}$

Axis of *symmetry*: $\boxed{x = -2}$

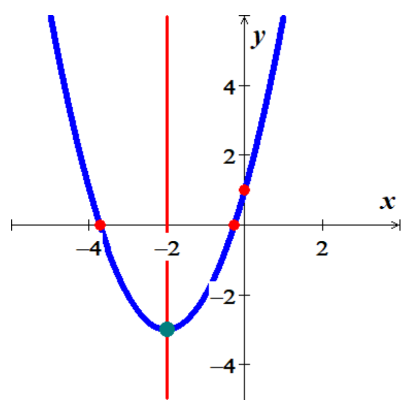
Minimum point @ $\boxed{(-2,\ -3)}$

$x^2 + 4x + 1 = 0$

$x = \dfrac{-4 \pm \sqrt{12}}{2}$

$\boxed{x = -2 \pm \sqrt{3}}$

$x = 0 \quad \to \quad \boxed{y = 1}$



Domain: $\mathbb{R}$ *Range*: $[-3, \infty)$

Increasing: $(-2, \infty)$ *Decreasing*: $(-\infty, -2)$

$f(x) > 0 \quad \Rightarrow \quad x < -2-\sqrt{2} \quad \& \quad x > -2+\sqrt{3}$

$f(x) < 0 \quad \Rightarrow \quad -2-\sqrt{3} < x < -2+\sqrt{3}$

Exercise

For the function $f(x) = x^2 + 6x - 1$

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

Solution

$$x = -\frac{6}{2} \qquad\qquad x = -\frac{b}{2a}$$

$$= -3$$

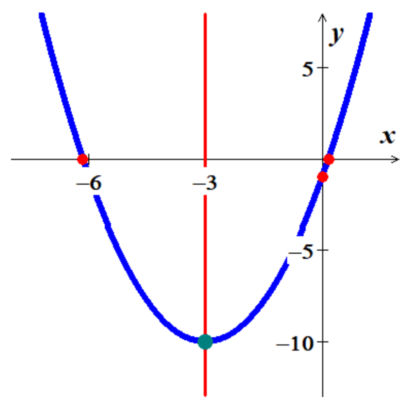
$$f(-3) = 9 - 18 - 1$$

$$= -10$$

Vertex point: $(-3, -10)$

Axis of *symmetry*: $x = -3$

Minimum point @ $(-3, -10)$



$x^2 + 6x - 1 = 0$

$$x = \frac{-6 \pm \sqrt{40}}{2}$$

$$x = -3 \pm \sqrt{10}$$

$x = 0 \quad \to \quad y = -1$

Domain: $\mathbb{R}$ *Range*: $[-10, \infty)$

Increasing: $(-3, \infty)$ *Decreasing*: $(-\infty, -3)$

$f(x) > 0 \quad \Rightarrow \quad x < -3-\sqrt{10} \quad \& \quad x > -3+\sqrt{10}$

$f(x) < 0 \quad \Rightarrow \quad -3-\sqrt{10} < x < -3+\sqrt{10}$

Exercise

For the function $f(x) = x^2 + 6x + 3$

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

Solution

$x = -\frac{6}{2}$ $\qquad x = -\frac{b}{2a}$

$\underline{= -3}$

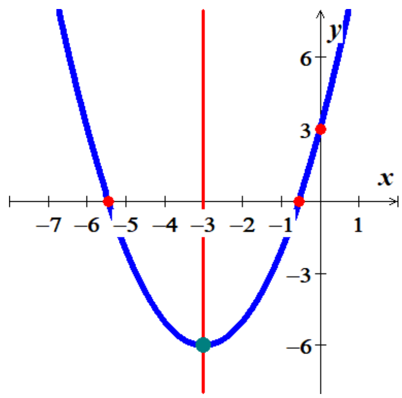
$f(-3) = 9 - 18 + 3$

$\underline{= -6}$

Vertex point: $\underline{(-3, -6)}$

Axis of *symmetry*: $\underline{x = -3}$

Minimum point @ $\underline{(-3, -6)}$



$x^2 + 6x + 3 = 0$

$x = \frac{-6 \pm \sqrt{24}}{2}$

$\underline{x = -3 \pm \sqrt{6}}$

$x = 0 \quad \to \quad \underline{y = 3}$

Domain: $\underline{\mathbb{R}}$ $\qquad$ *Range*: $\underline{[-6, \infty)}$

Increasing: $\underline{(-3, \infty)}$ $\qquad$ *Decreasing*: $\underline{(-\infty, -3)}$

$f(x) > 0 \quad \Rightarrow \quad x < -3 - \sqrt{6} \;\; \& \;\; x > -3 + \sqrt{6}$

$f(x) < 0 \quad \Rightarrow \quad -3 - \sqrt{6} < x < -3 + \sqrt{6}$

Exercise

For the function $f(x) = x^2 - 10x + 3$

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

Solution

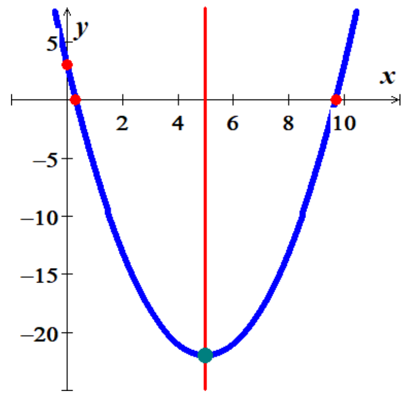
$x = -\frac{-10}{2}$ $\qquad x = -\frac{b}{2a}$

$\underline{= 5}$

$f(5) = 25 - 50 + 3$

$\underline{= -22}$

Vertex point: $\boxed{(5, \ -22)}$



Axis of *symmetry*: $\boxed{x = 5}$

Minimum point @ $\boxed{(5, \ -22)}$

$$x^2 - 10x + 3 = 0$$

$$x = \frac{10 \pm \sqrt{88}}{2}$$

$$\boxed{x = 5 \pm \sqrt{22}}$$

$$x = 0 \quad \to \quad \boxed{y = 3}$$

Domain: $\boxed{\mathbb{R}}$ *Range*: $\boxed{[-22, \ \infty)}$

Increasing: $\boxed{(5, \ \infty)}$ *Decreasing*: $\boxed{(-\infty, \ 5)}$

$$f(x) > 0 \quad \Rightarrow \quad x < 5 - \sqrt{22} \quad \& \quad x > 5 + \sqrt{22}$$

$$f(x) < 0 \quad \Rightarrow \quad 5 - \sqrt{22} \ < x < \ 5 + \sqrt{22}$$

Exercise

For the function $f(x) = x^2 - 3x + 4$

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

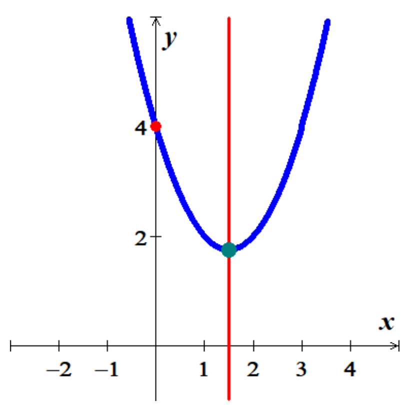
Solution

$$\boxed{x = \frac{3}{2}} \qquad x = -\frac{b}{2a}$$

$$f\left(\frac{3}{2}\right) = \frac{9}{4} - \frac{9}{2} + 4$$

$$\boxed{= \frac{7}{4}}$$

Vertex point: $\boxed{\left(\frac{3}{2}, \ \frac{7}{4}\right)}$



Axis of *symmetry*: $\boxed{x = \frac{3}{2}}$

Minimum point @ $\boxed{\left(\frac{3}{2}, \ \frac{7}{4}\right)}$

$$x^2 - 3x + 4 = 0$$

$$x = \frac{3 \pm \sqrt{-7}}{2} \qquad \mathbb{C}$$

$x = 0 \quad \to \quad \boxed{y = 4}$

Domain: $\boxed{\mathbb{R}}$ *Range*: $\boxed{\left[\frac{7}{4}, \infty\right)}$

Increasing: $\boxed{\left(\frac{3}{2}, \infty\right)}$ *Decreasing*: $\boxed{\left(-\infty, \frac{3}{2}\right)}$

$f(x) > 0 \quad \Rightarrow \quad \forall x$

$f(x) < 0 \quad \Rightarrow \quad$ *none*

Exercise

For the function $f(x) = x^2 - 4x - 5$

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

Solution

$\boxed{x = 2}$ $\qquad x = -\frac{b}{2a}$

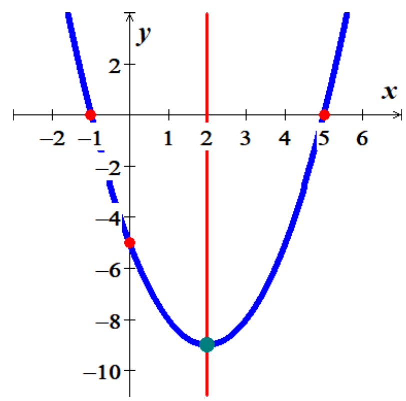
$f(2) = 4 - 8 - 5$

$\quad \boxed{= -9}$

Vertex point: $\boxed{(2, \ -9)}$

Axis of *symmetry*: $\boxed{x = 2}$

Minimum point @ $\boxed{(2, \ -9)}$



$x^2 - 4x - 5 = 0$

$\quad \boxed{x = -1, \ 5}$

$x = 0 \quad \to \quad \boxed{y = -5}$

Domain: $\boxed{\mathbb{R}}$ *Range*: $\boxed{[-9, \infty)}$

Increasing: $\boxed{(2, \infty)}$ *Decreasing*: $\boxed{(-\infty, 2)}$

$f(x) > 0 \quad \Rightarrow \quad x < -1 \ \ \& \ \ x > 5$

$f(x) < 0 \quad \Rightarrow \quad -1 < x < 5$

Exercise

For the function $f(x) = 2x^2 - 3x + 1$

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

Solution

$\boxed{x = \frac{3}{4}}$ $\qquad\qquad$ $x = -\frac{b}{2a}$

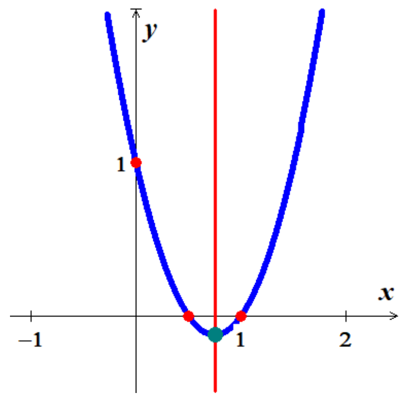
$f\left(\frac{3}{4}\right) = \frac{9}{8} - \frac{9}{4} + 1$

$= \boxed{-\frac{1}{8}}$

Vertex point: $\boxed{\left(\frac{3}{4},\ -\frac{1}{8}\right)}$

Axis of *symmetry*: $\boxed{x = \frac{3}{4}}$

Minimum point @ $\boxed{\left(\frac{3}{4},\ -\frac{1}{8}\right)}$



$2x^2 - 3x + 1 = 0$

$\boxed{x = 1,\ \frac{1}{2}}$

$x = 0 \quad \rightarrow \quad \boxed{y = 1}$

Domain: $\boxed{\mathbb{R}}$ $\qquad$ *Range*: $\boxed{\left[-\frac{1}{8},\ \infty\right)}$

Increasing: $\boxed{\left(\frac{3}{4},\ \infty\right)}$ $\qquad$ *Decreasing*: $\boxed{\left(-\infty,\ \frac{3}{4}\right)}$

$f(x) > 0 \quad \Rightarrow \quad x < -1 \quad \& \quad x > 5$

$f(x) < 0 \quad \Rightarrow \quad -1 < x < 5$

Exercise

For the function $f(x) = -x^2 - 4x + 5$

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

Solution

$\boxed{x = -2}$ $\qquad\qquad$ $x = -\frac{b}{2a}$

$f(-2) = -4 + 8 + 5$

$\boxed{= 9}$

Vertex point: $\boxed{(-2,\ 9)}$

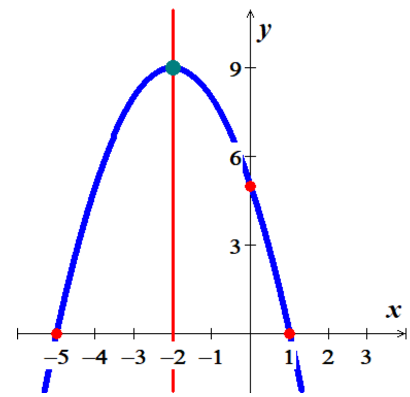
Axis of *symmetry*: $\boxed{x = -2}$

Maximum point @ $\boxed{(-2,\ 9)}$

$-x^2 - 4x + 5 = 0$

$\boxed{x = 1,\ -5}$

$x = 0 \ \rightarrow \ \boxed{y = 5}$



Domain: $\boxed{\mathbb{R}}$ *Range*: $\boxed{(-\infty,\ 9]}$

Increasing: $\boxed{(-\infty,\ -2)}$ *Decreasing*: $\boxed{(-2,\ \infty)}$

$f(x) > 0 \ \Rightarrow \ -5 < x < 1$

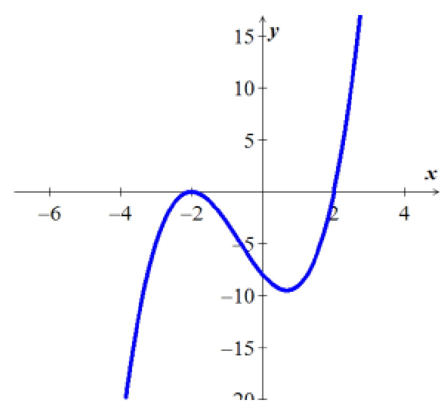
$f(x) < 0 \ \Rightarrow \ x < -5 \ \& \ x > 1$

Exercise

Let $f(x) = x^3 + 2x^2 - 4x - 8$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$f(x) = x^2(x+2) - 4(x+2)$$
$$= (x+2)\left(x^2 - 4\right)$$
$$= (x+2)(x+2)(x-2) = 0$$



The zeros are: 2, –2, –2

$-\infty$ -2	0 2	∞
–	–	+

$f(x) > 0 \quad \boxed{(2,\ \infty)}$

$f(x) < 0 \quad \boxed{(-\infty,\ -2) \cup (-2,\ 2)}$

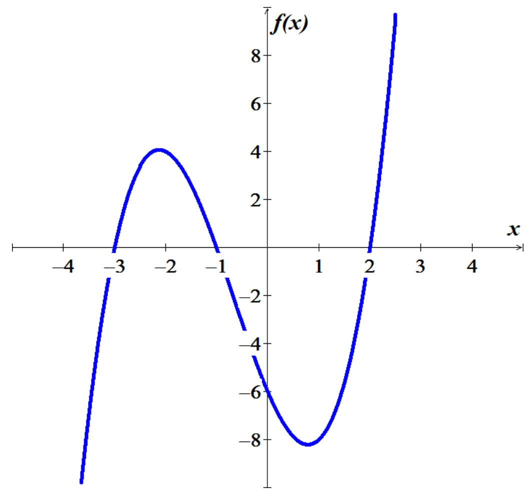
Exercise

Let $f(x) = x^3 + 2x^2 - 5x - 6$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$possibilities \ : \ \pm\left\{\frac{6}{1}\right\}$$
$$=\pm\{1,\ 2,\ 3,\ 6\}$$

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & & -1 & -1 & 6 \\ \hline & 1 & 1 & -6 & \boxed{0} \end{array} \rightarrow x^2+x-6=0$$



The zeros are: $x=-1,\ -3,\ 2$

	-3		-1		2	
$-$		$+$		$-$		$+$

$f(x)>0$ $(-3,\ -1)\cup(2,\ \infty)$

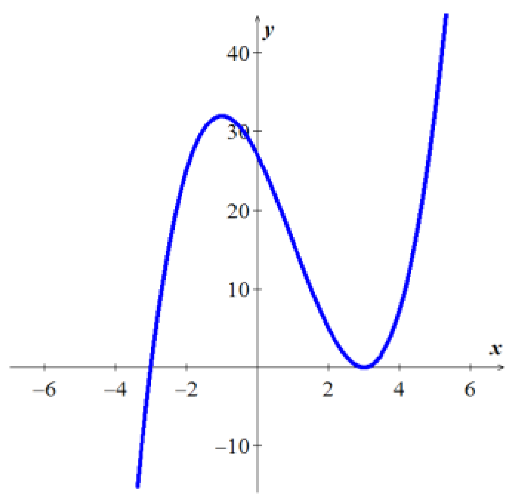
$f(x)<0$ $(-\infty,\ -3)\cup(-1,\ 2)$

Exercise

Let $f(x)=x^3-3x^2-9x+27$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

Solution

$$f(x)=x^2(x-3)-9(x-3)$$
$$=(x-3)\left(x^2-9\right)$$
$$=(x-3)(x-3)(x+3)\ =0$$



The zeros are: –3, 3 (multiplicity)

$-\infty$	-3	0	3	∞
$-$		$+$		$+$

$f(x)>0$ $(-3,\ 3)\cup(3,\ \infty)$

$f(x)<0$ $(-\infty,\ -3)$

Exercise

Let $f(x)=2x^3+11x^2-7x-6$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

Solution

$$possibilities:\pm\left\{\frac{6}{2}\right\}=\pm\left\{\frac{1,\ 2,\ 3,\ 6}{1,\ 2}\right\}$$
$$=\pm\left\{1,\ 2,\ 3,\ 5,\ \frac{1}{2},\ \frac{3}{2}\right\}$$

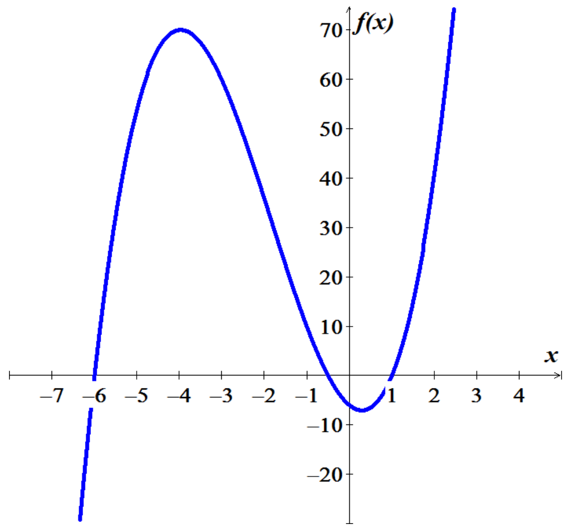
$$\begin{array}{r|rrrr} 1 & 2 & 11 & -7 & -6 \\ & & 2 & 13 & 6 \\ \hline & 2 & 13 & 6 & \boxed{0} \end{array} \quad \to\ 2x^2+13x+6=0$$

The zeros are: $x=1,\ -\frac{1}{2},\ -6$

	-6		$-\frac{1}{2}$		1	
$-$		$+$		$-$		$+$

$f(x)>0$ $\quad\left(-6,\ -\frac{1}{2}\right)\cup(1,\ \infty)$

$f(x)<0$ $\quad(-\infty,\ -6)\cup\left(-\frac{1}{2},\ 1\right)$



Exercise

Let $f(x)=x^3+8x^2+11x-20$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

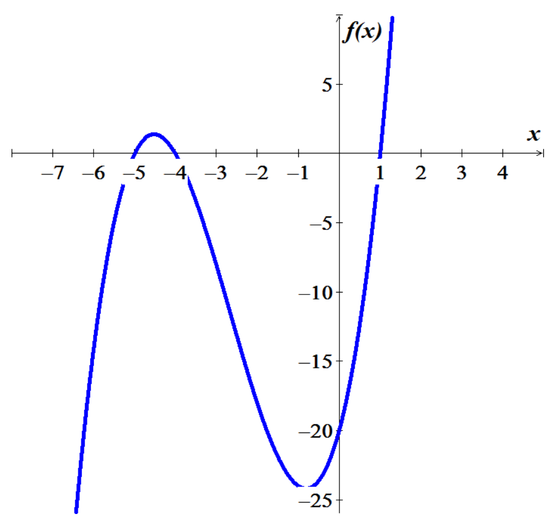
Solution

$$possibilities\ :\pm\left\{\frac{20}{1}\right\}=\pm\{1,\ 2,\ 4,\ 5,\ 20,\ 20\}$$

$$\begin{array}{r|rrrr} 1 & 1 & 8 & 11 & -20 \\ & & 1 & 9 & 20 \\ \hline & 1 & 9 & 20 & \boxed{0} \end{array} \quad \to\ x^2+9x+20=0$$

The zeros are: $x=-5,\ -4,\ 1$

	-5		-4		1	
$-$		$+$		$-$		$+$



$f(x) > 0 \quad \underline{(-5,\ -1) \cup (1,\ \infty)}$

$f(x) < 0 \quad \underline{(-\infty,\ -5) \cup (-4,\ 1)}$

Exercise

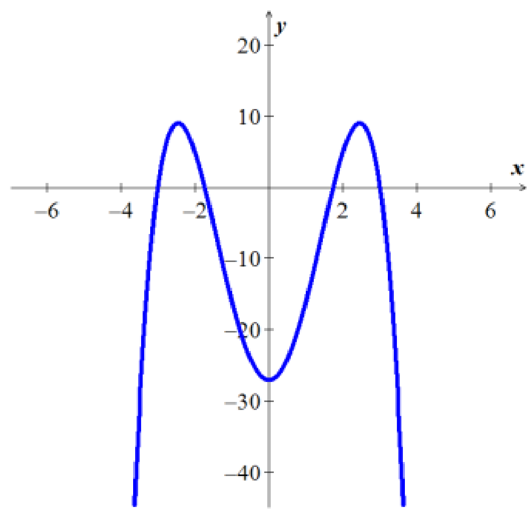
Let $f(x) = -x^4 + 12x^2 - 27$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$x^2 = \frac{-12 \pm \sqrt{36}}{-2}$$

$$= \begin{cases} \frac{-12-6}{-2} = 9 \\ \frac{-12+6}{-2} = 3 \end{cases}$$

$$\to \begin{cases} x^2 = 9 \\ x^2 = 3 \end{cases} \Rightarrow \begin{array}{l} x = \pm 3 \\ x = \pm\sqrt{3} \end{array}$$



-3		$-\sqrt{3}$		$\sqrt{3}$		3	
$-$	$+$	$-$	$+$	$-$			

$f(x) > 0 \quad \underline{\left(-3,\ -\sqrt{3}\right) \cup \left(\sqrt{3},\ 3\right)}$

$f(x) < 0 \quad \underline{(-\infty,\ -3) \cup \left(-\sqrt{3},\ \sqrt{3}\right) \cup (3,\ \infty)}$

Exercise

Let $f(x) = x^4 + x^2 - 2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

possibilities: $\pm\{1,\ 2\}$

$$\begin{array}{r|rrrrr} 1 & 1 & 0 & 1 & 0 & -2 \\ & & 1 & 1 & 2 & 1 \\ \hline -1 & 1 & 1 & 2 & 2 & 0 \\ & & -1 & 0 & -2 & \\ \hline & 1 & 0 & 2 & 0 & \end{array} \quad \to x^3 + x^2 + 2x + 1 = 0 \ \to \pm\{1,\ 2\}$$

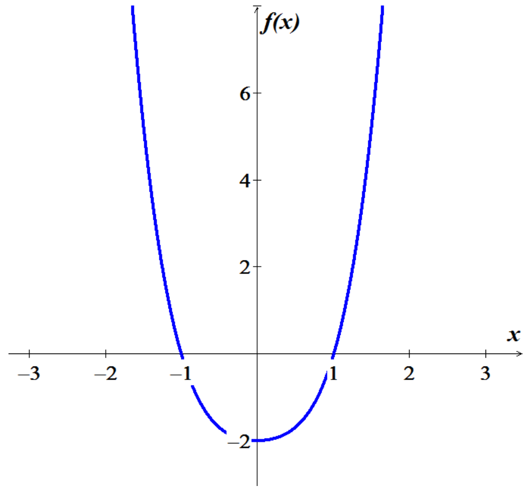
$\rightarrow x^2 + 2 = 0 \quad \Rightarrow \quad \underline{x = \pm i\sqrt{2}}$

The zeros are: $\underline{x = \pm 1}$

$-\infty$	-1		1		∞
	$+$		$-$		$+$

$f(x) > 0 \quad \underline{(-\infty,\ -1) \cup (1,\ \infty)}$

$f(x) < 0 \quad \underline{(-1,\ 1)}$



Exercise

Let $f(x) = x^4 - x^3 - 6x^2 + 4x + 8$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

possibilities : $\pm\{1,\ 2,\ 4,\ 8\}$

$$\begin{array}{r|rrrrr}
-1 & 1 & -1 & -6 & 4 & 8 \\
 & & -1 & 2 & 4 & -8 \\
\hline
-2 & 1 & -2 & -4 & 8 & 0 \\
 & & -2 & 8 & -8 & \\
\hline
 & 1 & -4 & 4 & 0 &
\end{array}$$

$\rightarrow x^3 - 2x^2 - 4x + 8 = 0$

$\rightarrow \pm\{1,\ 2,\ 4,\ 8\}$

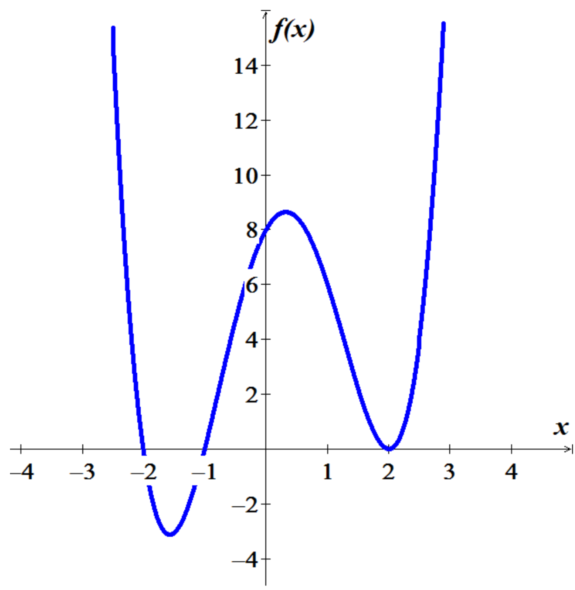
$\rightarrow x^2 - 4x + 4 = 0 \quad \Rightarrow \quad \underline{x = 2,\ 2}$

The zeros are: $\underline{x = -2,\ -1,\ 2,\ 2}$

	-2		-1		2	
$+$		$-$		$+$		$+$

$f(x) > 0 \quad \underline{(-\infty,\ -1) \cup (1,\ \infty)}$

$f(x) < 0 \quad \underline{(-1,\ 1)}$



Exercise

Let $f(x) = 2x^4 - x^3 - 5x^2 + 2x + 2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

possibilities: $\pm\left\{\frac{2}{2}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2}\right\}$

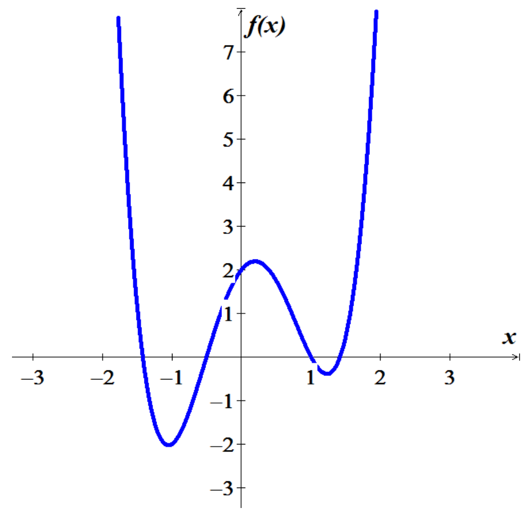
$$\begin{array}{r|rrrrr} 1 & 2 & -1 & -5 & 2 & 2 \\ & & 2 & 1 & -4 & -2 \\ \hline -\frac{1}{2} & 2 & 1 & -4 & -2 & 0 \\ & & -1 & 0 & 2 & \\ \hline & 2 & 0 & -4 & 0 & \end{array}$$

$\to 2x^3+x^2-4x-2=0$

$\to \pm\left\{1,\ 2,\ \frac{1}{2}\right\}$

$\to 2x^2-4=0 \Rightarrow \underline{x=\pm\sqrt{2}}$

The zeros are: $\underline{x=-\frac{1}{2},\ 1,\ -\sqrt{2},\ \sqrt{2}}$



	$-\sqrt{2}$	$-\frac{1}{2}$	1	$\sqrt{2}$	
+	–	+	–	+	

$f(x)>0$ $\underline{\left(-\infty,\ -\sqrt{2}\right)\cup\left(-\frac{1}{2},\ 1\right)\cup\left(\sqrt{2},\ \infty\right)}$

$f(x)<0$ $\underline{\left(-\sqrt{2},\ -\frac{1}{2}\right)\cup\left(1,\ \sqrt{2}\right)}$

Exercise

Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of $f(x)$ $\quad f(x)=6x^5+19x^4+x^3-6x^2$

Solution

$x^2\left(6x^3+19x^2+x-6\right)=0 \quad \to \quad \underline{x=0,\ 0}$

$6x^3+19x^2+x-6=0$

possibilities for $\frac{c}{d}$: $\pm\left\{\frac{6}{6}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{3}\right\}$

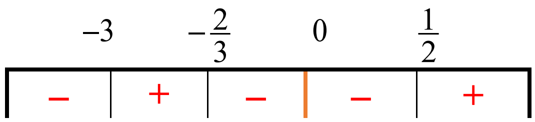
$$\begin{array}{r|rrrr} -3 & 6 & 19 & 1 & -6 \\ & & -18 & -3 & 6 \\ \hline & 6 & 1 & -2 & \boxed{0} \end{array}$$

$6x^2+x-2=0$

$$x=\frac{-1\pm\sqrt{1+48}}{12}$$

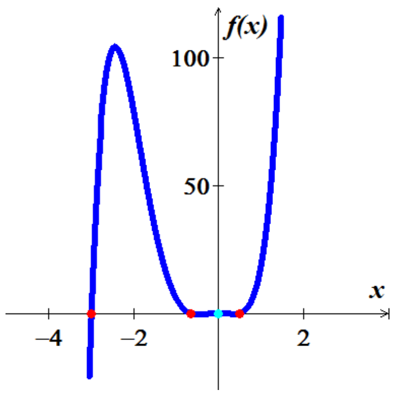
$$=\begin{cases} \frac{-1-7}{12}=-\frac{2}{3} \\ \frac{-1+7}{12}=\frac{1}{2} \end{cases}$$

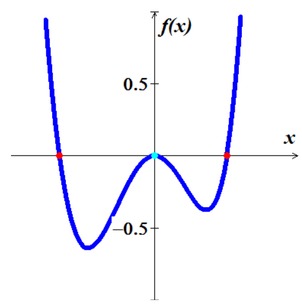
$\underline{x = 0,\ 0,\ -\frac{2}{3},\ -3,\ \frac{1}{2}}$



$f(x) > 0$ $\quad \left(-3,\ -\frac{2}{3}\right) \cup \left(\frac{1}{2},\ \infty\right)$

$f(x) < 0$ $\quad (-\infty,\ -3) \cup \left(-\frac{2}{3},\ 0\right) \cup \left(0,\ \frac{1}{2}\right)$





Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$ $\qquad f(x) = x^5 - x^4 - 7x^3 + 7x^2 + 12x - 12$

Solution

possibilities for $\frac{c}{d} : \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

1	1	−1	−7	7	12	−12	
		1	0	−7	0	12	
2	1	0	−7	0	12	0	$\rightarrow\ x^4 - 7x^2 - 12 = 0 \rightarrow \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$
		2	4	−6	−12		
−2	1	2	−3	−6	0		$\rightarrow\ x^3 + 2x^2 - 3x - 6 = 0 \rightarrow \pm\{1,\ 2,\ 3,\ 6\}$
		−2	0	6			
	1	0	−3	0			$\rightarrow\ x^2 - 3 = 0$

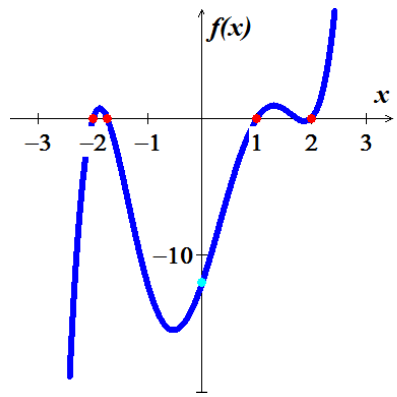
$x^2 = 3$

$\underline{x = -2,\ 1,\ 2,\ \pm\sqrt{3}}$

-2	$-\sqrt{3}$	1	$\sqrt{3}$	2	
$-$	$+$	$-$	$+$	$-$	$+$

$f(x) > 0 \quad \boxed{\left(-2,\ -\sqrt{3}\right) \cup \left(1,\ \sqrt{3}\right) \cup (2,\ \infty)}$

$f(x) < 0 \quad \boxed{(-\infty,\ -2) \cup \left(-\sqrt{3},\ 1\right) \cup \left(\sqrt{3},\ 2\right)}$



Exercise

Let $f(x) = x^5 - 7x^4 + 19x^3 - 37x^2 + 60x - 36$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$possibilities:\ \pm\left\{\frac{36}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12,\ 18,\ 36\}$

1	1	−7	19	−37	60	−36
		1	−6	13	−24	36
3	1	−6	13	−24	36	0
		3	−9	12	−36	
3	1	−3	4	−12	0	
		3	0	12		
	1	0	4	0		

$x^4 - 6x^3 + 13x^2 - 24x + 36 = 0 \ \rightarrow\ \pm\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

$x^3 - 3x^2 + 4x - 12 = 0 \ \rightarrow\ \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

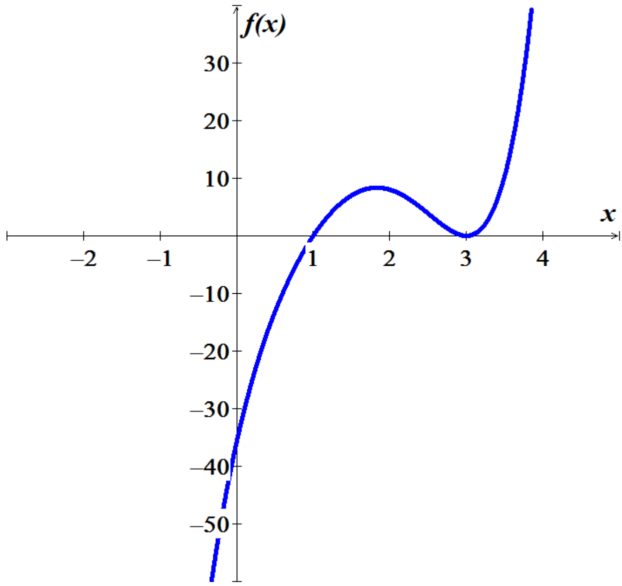
$x^2 + 4 = 0 \quad \Rightarrow \quad x = \pm 2i$

The zeros are: $\boxed{x = 1,\ 3,\ 3}$

1	3	
$-$	$+$	$+$

$f(x) > 0 \quad \boxed{(1,\ 3) \cup (3,\ \infty)}$

$f(x) < 0 \quad \boxed{(-\infty,\ 1)}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

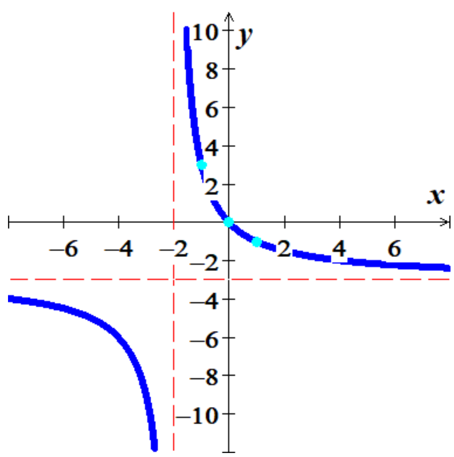
$$f(x) = \frac{-3x}{x+2}$$

Solution

VA: $x = -2$ ***HA***: $y = -3$

Hole: n/a ***OA***: n/a

x	y
0	0
1	–1
–1	3



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

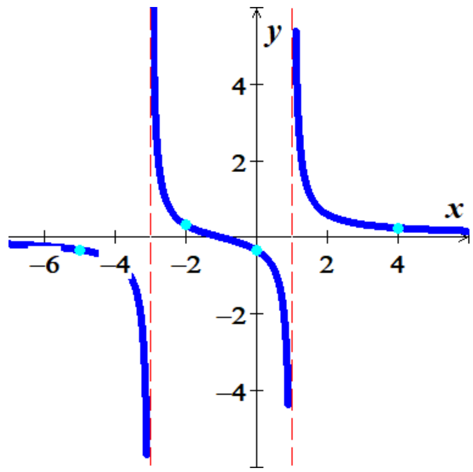
$$f(x) = \frac{x+1}{x^2+2x-3}$$

Solution

VA: $x = 1,\ x = -3$ ***HA***: $y = 0$

Hole: n/a ***Oblique asymptote***: n/a

x	y
–5	–0.33
–2	0.33
0	–1/3
4	0.24



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

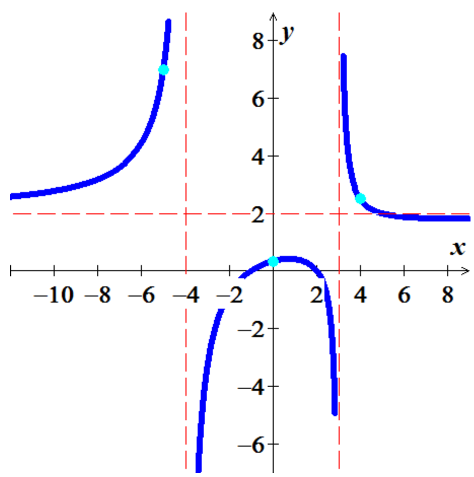
$$f(x) = \frac{2x^2 - 2x - 4}{x^2 + x - 12}$$

Solution

VA: $x = -4,\ 3$ ***HA***: $y = 2$

Hole: n/a ***OA***: n/a

x	y
–5	7
–2	–0.8
0	1/3
4	2.5
5	2



Exercise

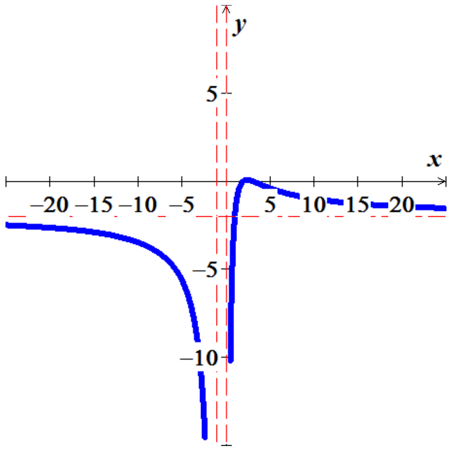
Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

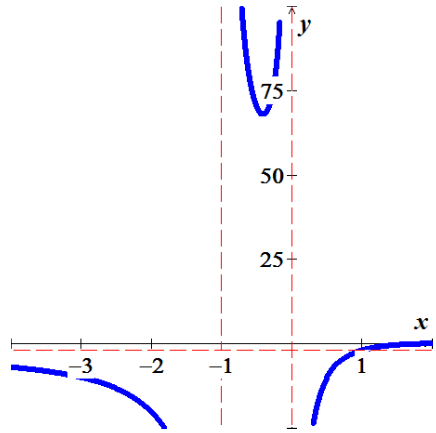
$$f(x) = \frac{-2x^2 + 10x - 12}{x^2 + x}$$

Solution

VA: $x=-1,\ 0$ ***HA***: $y=-2$

Hole: n/a ***OA***: n/a





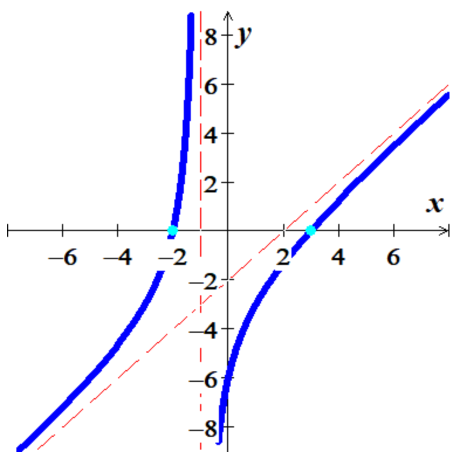
Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x^2-x-6}{x+1}$$

Solution

$$\begin{array}{r|l} & x-2 \\ \hline x+1 & x^2-x-6 \\ & \underline{x^2+x} \\ & \quad -2x-6 \\ & \quad \underline{-2x-2} \\ & \qquad \boxed{-4} \end{array}$$



VA: $x=-1$ ***HA***: n/a

Hole: n/a ***OA***: $y=x-2$

x	y
2	0
–2	0
0	–6

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{2x^2+x-6}{x^2+3x+2}$$

Solution

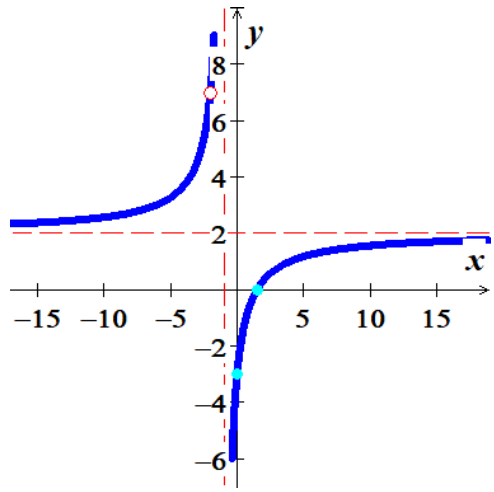
$$f(x)=\frac{(2x-3)(x+2)}{(x+1)(x+2)}$$

$$=\frac{2x-3}{x+1}$$

VA: $x=-1$ ***HA***: $y=2$

Hole: $(-2,\ 7)$ ***OA***: n/a

x	y
0	–3
$-\frac{3}{2}$	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x-1}{1-x^2}$$

Solution

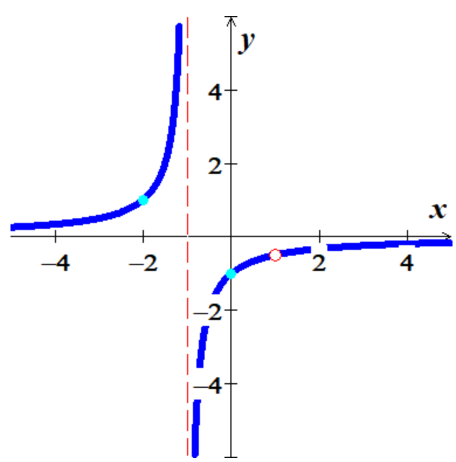
$$f(x)=\frac{x-1}{(x+1)(1-x)}$$

$$=-\frac{1}{x+1}$$

VA: $x=-1$ ***HA***: $y=0$

Hole: $\left(1,\ -\frac{1}{2}\right)$ ***OA***: n/a

x	y
0	–1
–2	1



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x^2+x-2}{x+2}$$

Solution

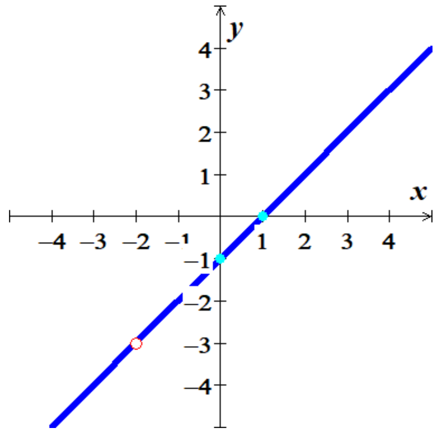
$$f(x)=\frac{(x+2)(x-1)}{x+2}$$

$$=x-1$$

VA: n/a ***HA***: n/a

Hole: $(-2,\ -3)$ ***OA***: n/a

x	y
0	–1
1	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x^3-2x^2-4x+8}{x-2}$$

Solution

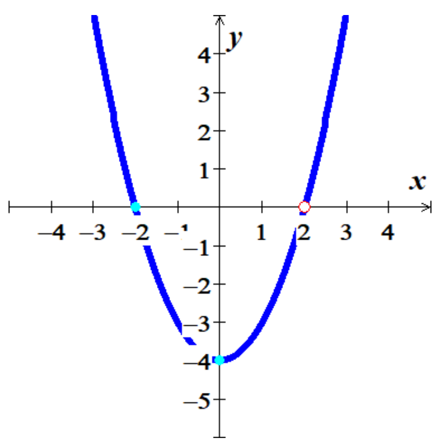
$$f(x)=\frac{\left(x^2-4\right)(x-2)}{x-2}$$

$$=x^2-4$$

VA: n/a ***HA***: n/a

Hole: $(2,\ 0)$ ***OA***: n/a

x	y
0	–4
–2	0



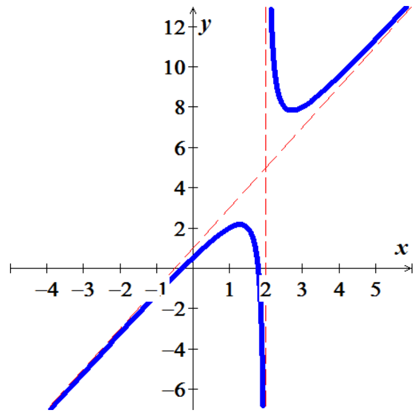
Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{2x^2-3x-1}{x-2}$$

Solution

$$\begin{array}{r|l} & 2x+1 \\ \hline x-2 & 2x^2-3x-1 \\ & \underline{-2x^2+4x} \\ & \qquad x-1 \\ & \qquad \underline{-x+2} \\ & \qquad\quad 1 \end{array}$$



$$\begin{aligned} f(x) &= \frac{2x^2-3x-1}{x-2} \\ &= (2x+1)+\frac{1}{x-2} \end{aligned}$$

VA: $x=2$ ***HA***: $y=1$

Hole: n/a ***OA***: $y=2x+1$

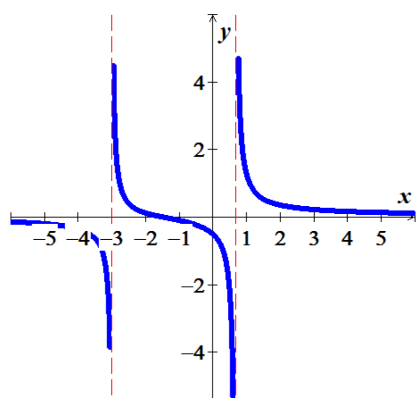
Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{2x+3}{3x^2+7x-6}$$

Solution

$$3x^2+7x-6=0 \quad\Rightarrow\quad x=-3,\ \frac{2}{3}$$



VA: $x=-3$ *and* $x=\frac{2}{3}$

HA: $y=0$

Hole: n/a

OA: n/a

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x^2-1}{x^2+x-6}$$

Solution

$$x^2+x-6=0 \quad\Rightarrow\quad x=-3,\ 2$$

VA: $x=-3$ *and* $x=2$ ***HA***: $y=1$

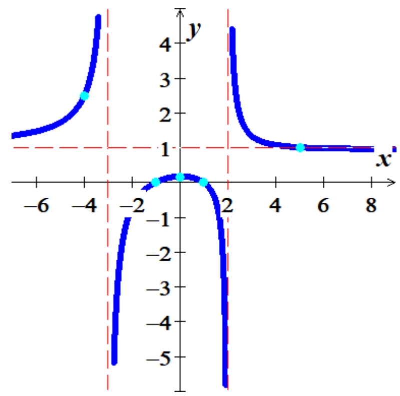
Hole: n/a ***OA***: n/a

$$1=\frac{x^2-1}{x^2+x-6}$$

$$x^2+x-6=x^2-1$$

$$x=5$$

x	y
0	$\frac{1}{6}$
5	1
±1	0
-4	$\frac{5}{2}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

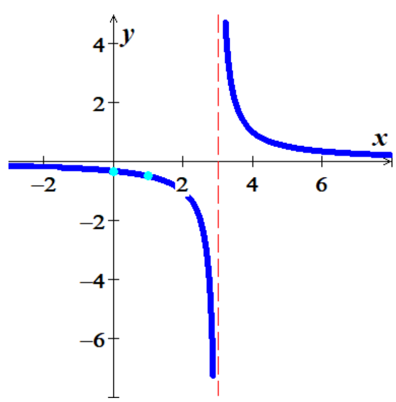
$$f(x)=\frac{1}{x-3}$$

Solution

VA: $x=3$ ***HA***: $y=0$

Hole: n/a ***OA***: n/a

x	y
0	$-\frac{1}{3}$
1	$-\frac{1}{2}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

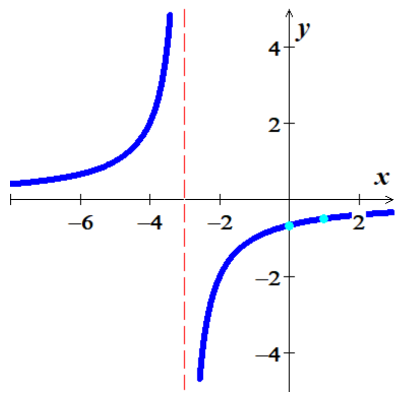
$$f(x)=\frac{-2}{x+3}$$

Solution

VA: $x=-3$ ***HA***: $y=0$

Hole: n/a ***OA:*** n/a

x	y
0	$-\frac{2}{3}$
1	$-\frac{1}{2}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

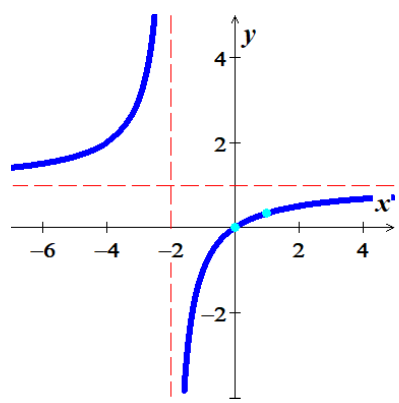
$$f(x)=\frac{x}{x+2}$$

Solution

VA: $x=-2$ ***HA***: $y=1$

Hole: n/a ***OA:*** n/a

x	y
0	0
1	$\frac{1}{3}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

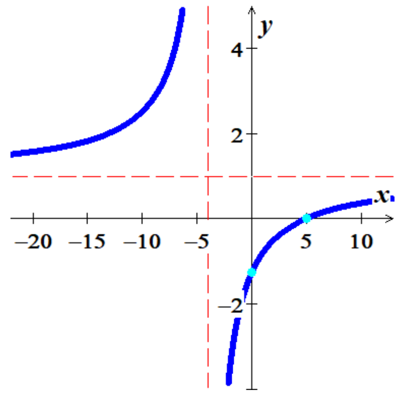
$$f(x)=\frac{x-5}{x+4}$$

Solution

VA: $x=-4$ ***HA***: $y=1$

Hole: n/a ***OA:*** n/a

x	y
0	$-\frac{5}{4}$
5	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{2x^2-2}{x^2-9}$$

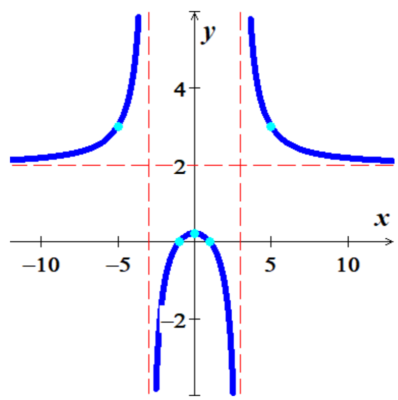
Solution

$x^2=9 \quad \rightarrow \quad \underline{x=\pm 3}|$

VA: $x=\pm 3$ ***HA***: $y=2$

Hole: n/a ***OA:*** n/a

x	y
0	$\frac{2}{9}$
± 1	0
± 5	3



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2-3}{x^2+4}$$

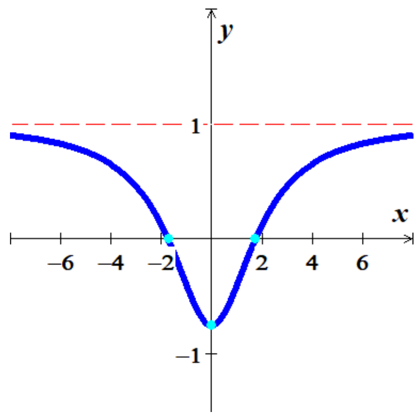
Solution

VA: n/a ***HA***: $y=1$

Hole: n/a ***OA:*** n/a

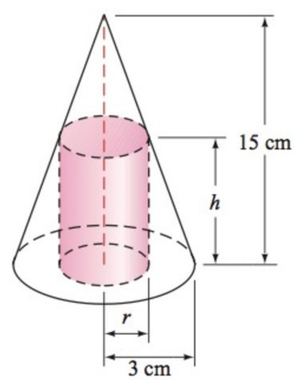
x	y

0	$-\frac{3}{4}$
$\pm\sqrt{3}$	0



Exercise

A cone has an altitude of 15 *cm* and a radius of 3 *cm*. A right circular cylinder of radius r and height h is inscribed in the cone. Use similar triangles to write h as a function of r.



Solution

$$\frac{15-h}{15}=\frac{r}{3}$$

$$15-h=5r$$

$$\boxed{h(r)=15-5r}$$

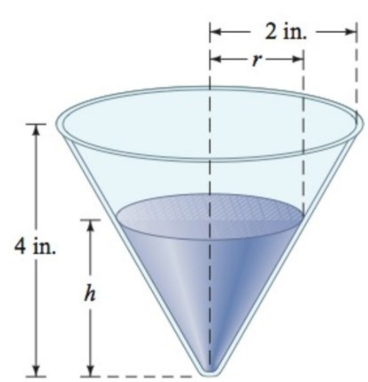
Exercise

Water is flowing into a conical drinking cup with an altitude of 4 *inches* am a radius of 2 *inches*.

a) Write the radius r of the surface of the water as a function of its depth h.

b) Write the volume V of the water as a function of its depth h.

Solution



a) $\frac{h}{4}=\frac{r}{2}$

$$\boxed{r(h)=\frac{1}{2}h}$$

b) $Area=\pi r^2$

$$V=\frac{1}{3}\pi r^2 h$$

$$=\frac{1}{3}\pi\left(\frac{h^2}{4}\right)h$$

$$=\frac{1}{12}\pi h^3$$

Exercise

A water tank has the shape of a right circular cone with height 16 *feet* and radius 8 *feet*. Water is running into the tank so that the radius r (in *feet*) of the surface of the water is given by $r = 1.5t$, where t is the time (in *minutes*) that the water has been running.

a) The area A of the surface of the water is $A = \pi r^2$. Find $A(t)$ and use it to determine the area of the surface of the water when $t = 2$ *minutes*.

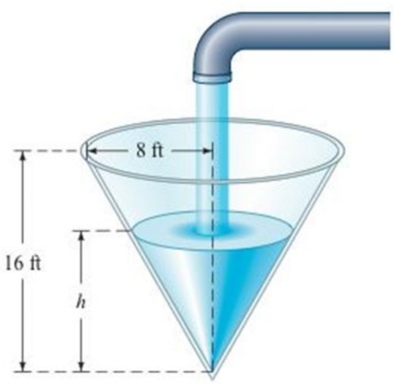
b) The volume V of the water is given by $V = \frac{1}{3}\pi r^2 h$. Find $V(t)$ and use it to determine the volume of the water when $t = 3$ *minutes*

Solution

c) $Area = \pi r^2$

$$A(t) = \pi\left(\frac{3}{2}t\right)^2$$

$$= \frac{9\pi}{4}t^2$$



d) $\frac{h}{16} = \frac{r}{8}$

$h = 2r$

$$V(t) = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi r^2(2r)$$

$$= \frac{2}{3}\pi r^3$$

$$= \frac{2}{3}\pi\left(\frac{3}{2}t\right)^3$$

$$= \frac{9}{4}\pi t^3$$

Exercise

The surface area S of a right circular cylinder is given by the formula $S = 2\pi rh + 2\pi r^2$. if the height is twice the radius, find each of the following.

a) A function $S(r)$ for the surface area as a function of r.

b) A function $S(h)$ for the surface area as a function of h.

Solution

Given: $h = 2r$

a) $S = 2\pi rh + 2\pi r^2$

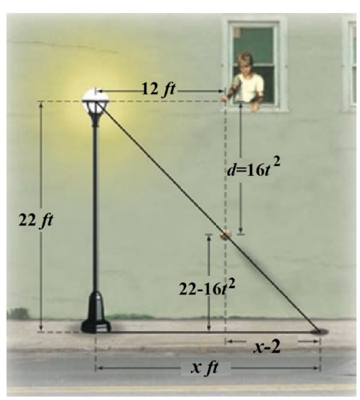
$$S(r) = 2\pi r(2r) + 2\pi r^2$$
$$= 4\pi r^2 + 2\pi r^2$$
$$= 6\pi r^2$$

b) $r = \frac{1}{2}h$

$$S(h) = 2\pi\left(\frac{1}{2}h\right)h + 2\pi\left(\frac{1}{2}h\right)^2$$
$$= \pi h^2 + \frac{1}{2}\pi h^2$$
$$= \frac{3}{2}\pi h^2$$

Exercise

The light from a lamppost casts a shadow from a ball that was dropped from a height of 22 *feet* above the ground. The distance d, in *feet*, the ball has dropped t *seconds* after it is released is given by $d(t) = 16t^2$. Find the distance x, in *feet*, of the shadow from the base of the lamppost as a function of time t.



Solution

$$\frac{22 - 16t^2}{22} = \frac{x - 12}{x}$$
$$\left(22 - 16t^2\right)x = 22(x - 12)$$
$$\left(22 - 16t^2\right)x = 22x - 264$$
$$\left(22 - 16t^2 - 22\right)x = -264$$
$$-16t^2 x = -264$$

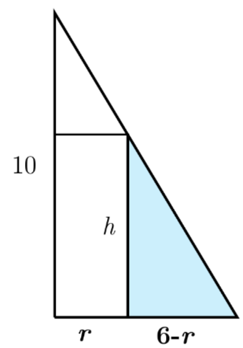
$$x(t) = \frac{33}{2t^2}$$

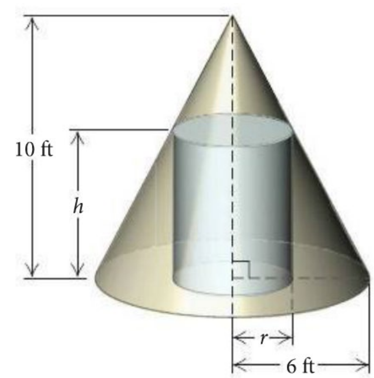
Exercise

A right circular cylinder of height h and a radius r is inscribed in a right circular cone with a height of 10 *feet* and a base with radius 6 *feet*.

a) Express the height h of the cylinder as a function of r.
b) Express the volume V of the cylinder as a function of r.
c) Express the volume V of the cylinder as a function of h.

Solution





a) $\dfrac{h}{10} = \dfrac{6-r}{6}$

$$h(r) = \frac{5}{3}(6-r)$$

b) $V = \pi r^2 h$

$$V(r) = \frac{5}{3}\pi r^2 (6-r)$$

$$= \frac{5}{3}\pi\left(6r^2 - r^3\right)$$

c) $\dfrac{3}{5}h = 6 - r$

$$r = 6 - \frac{3}{5}h$$

$$V = \pi r^2 h$$

$$V(h) = \pi\left(\frac{30-3h}{5}\right)^2 h$$

$$= \frac{1}{25}\pi h(30-3h)^2$$

Exercise

A child 4 *feet* tall is standing near a streetlamp that is 12 *feet* high. The light from the lamp casts a shadow.

a) Find the length l of the shadow as a function of the distance d of the child from the lamppost.
b) What is the domain of the function?
c) What is the length of the shadow when the child is 8 *feet* from the base of the lamppost?

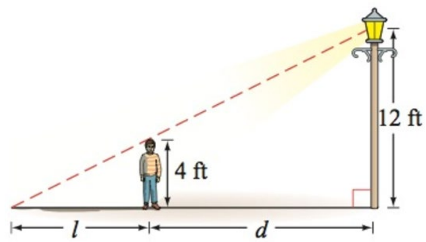
Solution

a) $\dfrac{l+d}{12}=\dfrac{l}{4}$

$l+d=3l$

$2l=d$

$\boxed{l(d)=\frac{1}{2}d}$



b) ***Domain***: $\{x \mid 0 \le d < \infty\}$

c) ***Given***: $d=8$

$\boxed{l=4 \text{ feet}}$

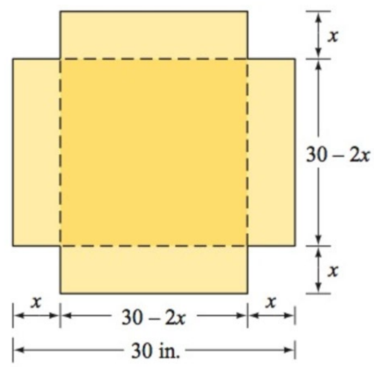
Exercise

An open box is to be made from a square piece of cardboard with the dimensions 30 *inches* by 30 *inches* by cutting out squares of area x^2 from each corner.

a) Express the volume V of the box as a function of x.

b) Determine the domain of V.

Solution



a) $V(x)=x(30-2x)^2$

$=x\left(900-120x+4x^2\right)$

$\boxed{=4x^3-120x^2+900x}$

b) $30-2x=0$

$\boxed{x=15}$

Domain: $\{x \mid 0 < x < 15\}$

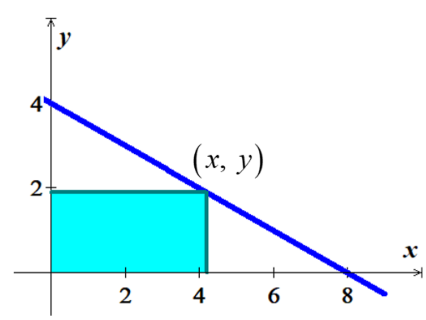
Exercise

A rectangle is bounded by the x- and y-*axis* of $y=-\frac{1}{2}x+4$

a) Find the area of the rectangle as a function of x.

b) What is the domain of this function?

Solution



a) $Area = xy$

$A(x)=x\left(-\frac{1}{2}x+4\right)$

$$= -\frac{1}{2}x^2 + 4x$$

b) $x\left(-\frac{1}{2}x + 4\right) = 0$

$x = 0 \quad x = 8$

Domain: $0 < x < 8$

Solution   Section R.4 – Exponential & Logarithm

Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given functions. Then, sketch the graph

$$f(x) = 2^x + 3$$

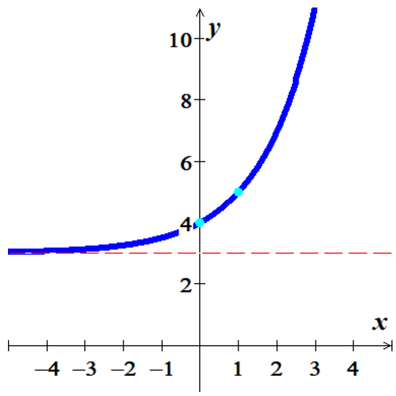
Solution

Asymptote: $y = 3$

Domain: $(-\infty, \infty)$

Range: $(3, \infty)$

x	$f(x)$
-1	3.5
0	4
1	5
2	7



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given functions. Then, sketch the graph

$$f(x) = 2^{3-x}$$

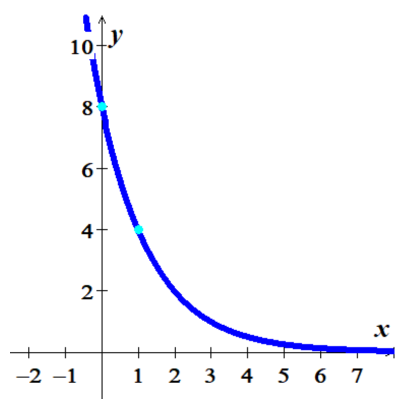
Solution

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

x	$f(x)$
1	4
2	2
0	8



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given functions. Then, sketch the graph

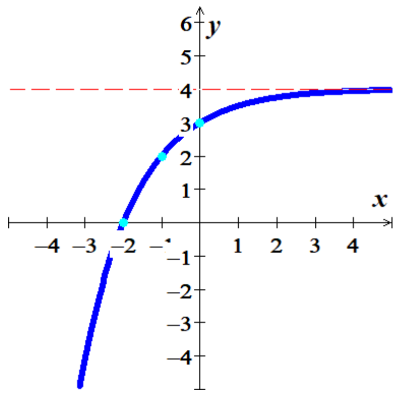
$$f(x) = -\left(\frac{1}{2}\right)^x + 4$$

Solution

Asymptote: $y = 4$

Domain: $(-\infty, \infty)$

Range: $(-\infty, 4)$



x	$f(x)$
–2	0
–1	2
0	3

Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given functions. Then, sketch the graph

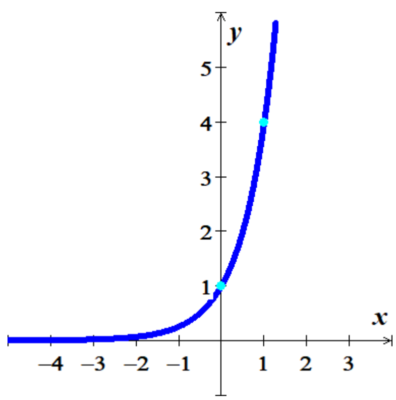
$$f(x) = 4^x$$

Solution

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$



x	$f(x)$
0	1
1	4

Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given functions. Then, sketch the graph

$$f(x) = 2 - 4^x$$

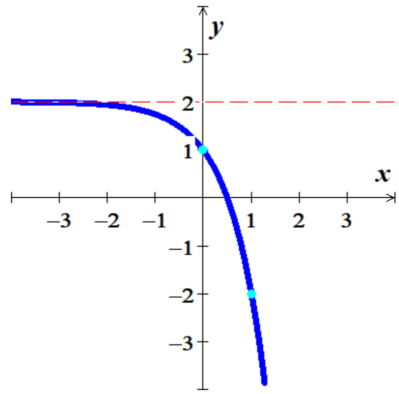
Solution

Asymptote: $y = 2$

Domain: $(-\infty,\ \infty)$

Range: $(-\infty,\ 2)$

x	$f(x)$
0	1
1	–2



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given functions. Then, sketch the graph

$$f(x) = -3 + 4^{x-1}$$

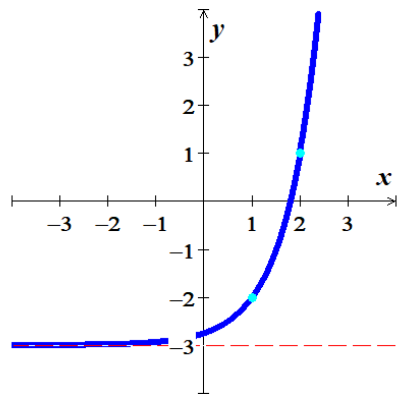
Solution

Asymptote: $y = -3$

Domain: $(-\infty,\ \infty)$

Range: $(-3,\ \infty)$

x	$f(x)$
1	–2
2	1



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given functions. Then, sketch the graph

$$f(x) = 1 + \left(\frac{1}{4}\right)^{x+1}$$

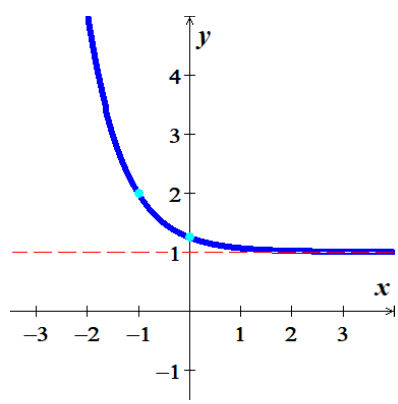
Solution

Asymptote: $y = 1$

Domain: $(-\infty,\ \infty)$

Range: $(1,\ \infty)$

x	$f(x)$
–1	2
0	$\frac{5}{4}$



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given functions. Then, sketch the graph

$$f(x) = e^{x-2}$$

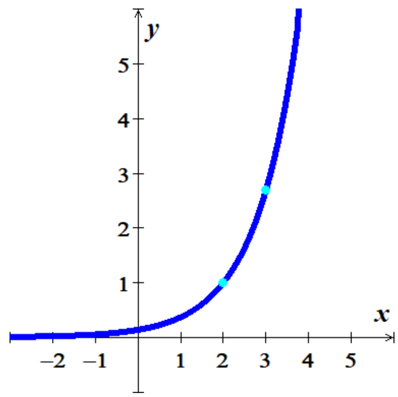
Solution

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

x	$f(x)$
2	1
3	2.7



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given functions. Then, sketch the graph

$$f(x) = 3 - e^{x-2}$$

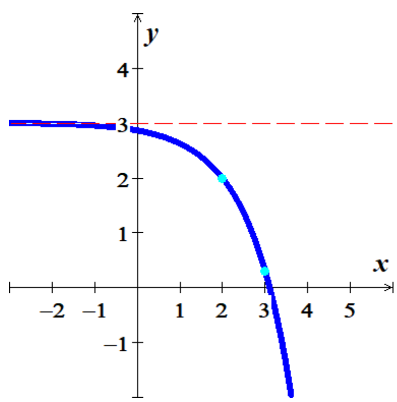
Solution

Asymptote: $y = 3$

Domain: $(-\infty, \infty)$

Range: $(-\infty, 3)$

x	$f(x)$
2	2
3	.3



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given functions. Then, sketch the graph

$$f(x) = e^{x+4}$$

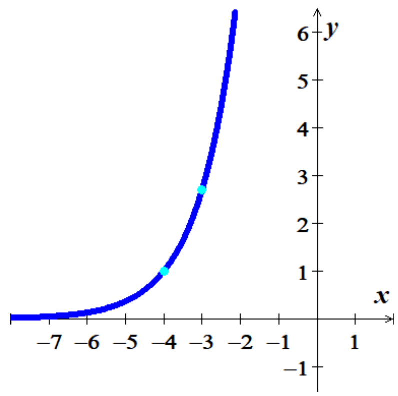
Solution

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

x	$f(x)$
–4	1
–3	2.7



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given functions. Then, sketch the graph

$$f(x) = 2 + e^{x-1}$$

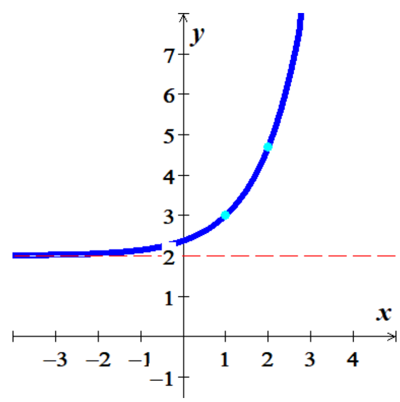
Solution

Asymptote: $y = 2$

Domain: $(-\infty, \infty)$

Range: $(2, \infty)$

x	$f(x)$
1	3
2	4.7



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given function. Then, sketch the graph

$f(x) = \log_4 (x-2)$

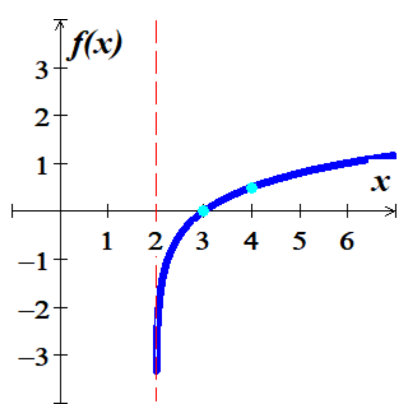
Solution

Asymptote: $x = 2$

Domain: $(2, \infty)$

Range: $(-\infty, \infty)$

x	$f(x)$
2	
3	0
4	.5



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given function. Then, sketch the graph

$f(x) = \log_4 |x|$

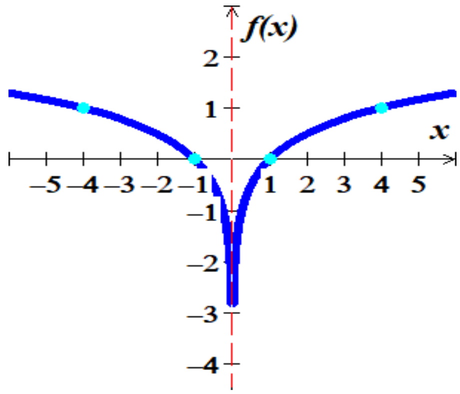
Solution

Asymptote: $x = 0$

Domain: $(-\infty,\ 0) \cup (0,\ \infty)$

Range: $(-\infty,\ \infty)$

x	*f*(*x*)
0	
±1	0
±4	1



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given function. Then, sketch the graph

$$f(x) = \left(\log_4 x\right) - 2$$

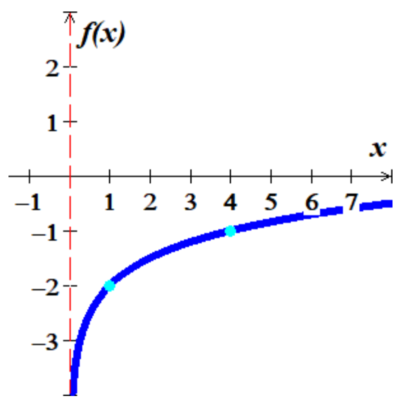
Solution

Asymptote: $x = 0$

Domain: $(0,\ \infty)$

Range: $(-\infty,\ \infty)$

x	*f*(*x*)
0	
1	0
4	–1



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given function. Then, sketch the graph

$$f(x) = \log(3 - x)$$

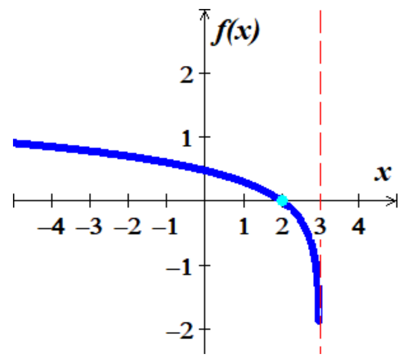
Solution

Asymptote: $x = 3$

Domain: $(-\infty,\ 3)$

Range: $(-\infty,\ \infty)$

x	$f(x)$
3	
2	0



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given function. Then, sketch the graph

$f(x) = 2 - \log(x+2)$

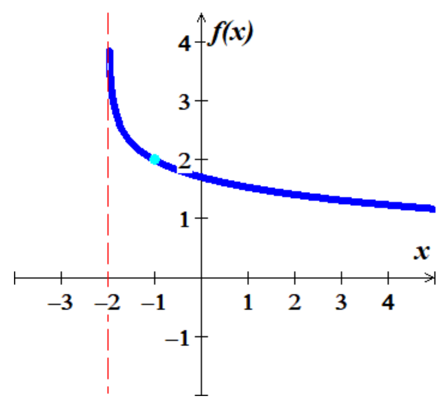
Solution

Asymptote: $x = -2$

Domain: $(-2,\ \infty)$

Range: $(-\infty,\ \infty)$

x	$f(x)$
–2	
–1	2



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given function. Then, sketch the graph

$f(x) = \ln(x-2)$

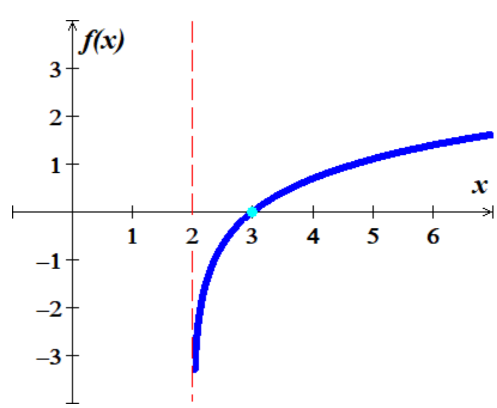
Solution

Asymptote: $x = 2$

Domain: $(2,\ \infty)$

Range: $(-\infty,\ \infty)$

x	$f(x)$
2	
3	0



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given function. Then, sketch the graph

$$f(x) = \ln(3-x)$$

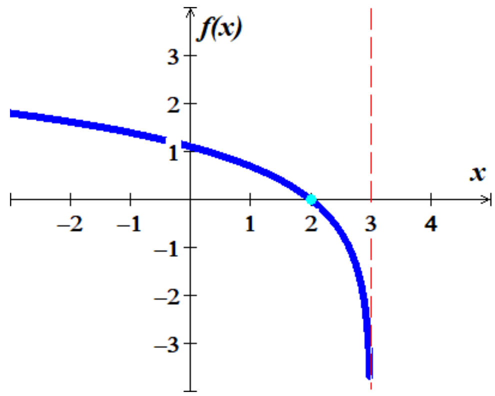
Solution

Asymptote: $x=3$

Domain: $(-\infty,\ 3)$

Range: $(-\infty,\ \infty)$

x	$f(x)$
3	
2	0



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given function. Then, sketch the graph

$$f(x) = 2+\ln(x+1)$$

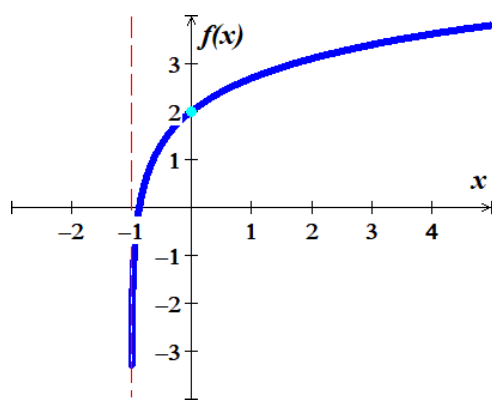
Solution

Asymptote: $x=-1$

Domain: $(-1,\ \infty)$

Range: $(-\infty,\ \infty)$

x	$f(x)$
–1	
0	2



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given function. Then, sketch the graph

$$f(x) = 1-\ln(x-2)$$

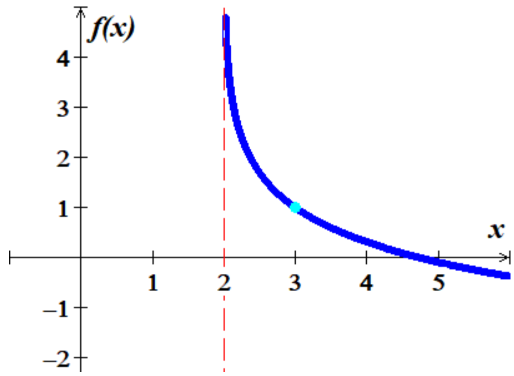
Solution

Asymptote: $x=2$

Domain: $(2,\ \infty)$

Range: $(-\infty,\ \infty)$

x	$f(x)$
2	
3	1



Exercise

Write the equation in its equivalent logarithmic form $2^6 = 64$

Solution

$$\boxed{6 = \log_2 64}$$

Exercise

Write the equation in its equivalent logarithmic form $5^4 = 625$

Solution

$$\boxed{4 = \log_5 625}$$

Exercise

Write the equation in its equivalent logarithmic form $5^{-3} = \frac{1}{125}$

Solution

$$\boxed{-3 = \log_5 \frac{1}{125}}$$

Exercise

Write the equation in its equivalent logarithmic form $\sqrt[3]{64} = 4$

Solution

$$64^{1/3} = 4$$

$$\boxed{\log_{64} = \frac{1}{3}}$$

Exercise

Write the equation in its equivalent logarithmic form $b^3 = 343$

Solution

$$\boxed{\log_b 343 = 3}$$

Exercise

Write the equation in its equivalent logarithmic form $8^y = 300$

Solution

$$\boxed{\log_8 300 = y}$$

Exercise

Write the equation in its equivalent logarithmic form: $\sqrt[n]{x} = y$

Solution

$$(x)^{1/n} = y$$

$$\boxed{\log_x (y) = \frac{1}{n}}$$

Exercise

Write the equation in its equivalent logarithmic form: $\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$

Solution

$$\boxed{\log_{\frac{2}{3}} \left(\frac{27}{8}\right) = -3}$$

Exercise

Write the equation in its equivalent logarithmic form: $\left(\frac{1}{2}\right)^{-5} = 32$

Solution

$$\boxed{\log_{\frac{1}{2}} (32) = -5}$$

Exercise

Write the equation in its equivalent logarithmic form: $e^{x-2} = 2y$

Solution

$$\underline{x-2 = \ln|2y|}$$

Exercise

Write the equation in its equivalent logarithmic form: $e = 3x$

Solution

$$\underline{1 = \ln|3x|}$$

Exercise

Write the equation in its equivalent logarithmic form: $\sqrt[3]{e^{2x}} = y$

Solution

$$e^{2x/3} = y$$

$$\underline{\frac{2x}{3} = \ln|y|}$$

Exercise

Write the equation in its equivalent exponential form $\log_5 125 = y$

Solution

$$\underline{5^y = 125}$$

Exercise

Write the equation in its equivalent exponential form $\log_4 16 = x$

Solution

$$\underline{16 = 4^x}$$

Exercise

Write the equation in its equivalent exponential form $\log_5 \frac{1}{5} = x$

Solution

$$\boxed{\frac{1}{5} = 5^x}$$

Exercise

Write the equation in its equivalent exponential form $\log_6 \sqrt{6} = x$

Solution

$$\boxed{\sqrt{6} = 6^x}$$

Exercise

Write the equation in its equivalent exponential form $\log_3 \frac{1}{\sqrt{3}} = x$

Solution

$$\boxed{3^{-1/2} = 3^x}$$

Exercise

Write the equation in its equivalent exponential form: $6 = \log_2 64$

Solution

$$6 = \log_2 64 \iff \boxed{2^6 = 64}$$

Exercise

Write the equation in its equivalent exponential form: $\log_{\sqrt{3}} 81 = 8$

Solution

$$\log_{\sqrt{3}} 81 = 8 \iff \boxed{81 = \left(\sqrt{3}\right)^8}$$

Exercise

Write the equation in its equivalent exponential form: $\log_4 \frac{1}{64} = -3$

Solution

$$\log_4 \frac{1}{64} = -3 \iff \boxed{\frac{1}{64} = x^{-3}}$$

Exercise

Write the equation in its equivalent exponential form: $\ln M = c$

Solution

$$\ln M = c \quad \Leftrightarrow \quad \underline{M = e^c}$$

Exercise

Simplify $\log_5 1$

Solution

$$\underline{\log_5 1 = 0}$$

Exercise

Simplify $\log_7 7^2$

Solution

$$\underline{\log_7 7^2 = 2}$$

Exercise

Simplify $3^{\log_3 8}$

Solution

$$\underline{3^{\log_3 8} = 8}$$

Exercise

Simplify $10^{\log 3}$

Solution

$$\underline{10^{\log 3} = 3}$$

Exercise

Simplify $e^{2+\ln 3}$

Solution

$$e^{2+\ln 3} = e^2 e^{\ln 3}$$
$$= 3e^2$$

Exercise

Simplify $\ln e^{-3}$

Solution

$$\ln e^{-3} = -3$$

Exercise

Simplify $\ln e^{x-5}$

Solution

$$\ln e^{x-5} = x-5$$

Exercise

Simplify $\log_b b^n$

Solution

$$\log_b b^n = n$$

Exercise

Simplify $\ln e^{x^2+3x}$

Solution

$$\ln e^{x^2+3x} = x^2+3x$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_5\left(\frac{125}{y}\right)$

Solution

$$\log_5\left(\frac{125}{y}\right)=\log_5 5^3-\log_5 y$$
$$\boxed{=3-\log_5 y}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b x^7$

Solution

$$\boxed{\log_b x^7 = 7\log_b x}$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln\sqrt[7]{x}$

Solution

$$\ln\sqrt[7]{x}=\ln x^{1/7}$$
$$\boxed{=\frac{1}{7}\ln x}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b\left(\frac{x^3y}{z^2}\right)$

Solution

$$\log_b\left(\frac{x^3y}{z^2}\right)=\log_b\left(x^3y\right)-\log_b z^2$$
$$=\log_b x^3+\log_b y-\log_b z^2$$
$$\boxed{=3\log_b x+\log_b y-2\log_b z}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b\left(\frac{\sqrt[3]{x}\,y^4}{z^5}\right)$

Solution

$$\log_b\left(\frac{\sqrt[3]{x}\,y^4}{z^5}\right)=\log_b\left(\sqrt[3]{x}\,y^4\right)-\log_b\left(z^5\right)$$

$$=\log_b\left(x^{1/3}\right)+\log_b\left(y^4\right)-\log_b\left(z^5\right)$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \sqrt[4]{\frac{m^8\,n^{12}}{a^3\,b^5}}$

Solution

$$\log_a \sqrt[4]{\frac{m^8\,n^{12}}{a^3\,b^5}}=\log_a\left(\frac{m^8n^{12}}{a^3b^5}\right)^{1/4} \qquad \textit{Power Rule}$$

$$=\frac{1}{4}\log_a\left(\frac{m^8n^{12}}{a^3b^5}\right) \qquad \textit{Quotient Rule}$$

$$=\frac{1}{4}\left[\log_a m^8n^{12}-\log_a a^3b^5\right] \qquad \textit{Product Rule}$$

$$=\frac{1}{4}\left[\log_a m^8+\log_a n^{12}-\left(\log_a a^3+\log_a b^5\right)\right] \qquad \textit{Power Rule}$$

$$=\frac{1}{4}\left[8\log_a m+12\log_a n-3-5\log_a b\right]$$

$$=2\log_a m+3\log_a n-\frac{3}{4}-\frac{5}{4}\log_a b$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \sqrt[3]{\frac{a^2\,b}{c^5}}$

Solution

$$\log_a \sqrt[3]{\frac{a^2\,b}{c^5}}=\log_a\left(\frac{a^2\,b}{c^5}\right)^{1/3} \qquad \textit{Convert the radical to power}$$

$$=\frac{1}{3}\log_a\left(\frac{a^2\,b}{c^5}\right) \qquad \textit{Power Rule}$$

$$=\frac{1}{3}\left[\log_a a^2\,b-\log_a c^5\right] \qquad \textit{Quotient Rule}$$

$$=\frac{1}{3}\left[\log_a a^2+\log_a b-\log_a c^5\right] \qquad \textit{Product Rule}$$

$$=\frac{1}{3}\left[2\log_a a+\log_a b-5\log_a c\right] \qquad \textit{Power Rule}$$

$$=\frac{2}{3}\log_a a + \frac{1}{3}\log_a b - \frac{5}{3}\log_a c$$

$$=\frac{2}{3} + \frac{1}{3}\log_a b - \frac{5}{3}\log_a c$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b\left(x^4\sqrt[3]{y}\right)$

Solution

$$\log_b\left(x^4\sqrt[3]{y}\right) = \log_b\left(x^4\right) + \log_b\left(\sqrt[3]{y}\right)$$

$$= \log_b\left(x^4\right) + \log_b\left(y^{1/3}\right)$$

$$= 4\log_b x + \frac{1}{3}\log_b y$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_5\left(\frac{\sqrt{x}}{25y^3}\right)$

Solution

$$\log_5\left(\frac{\sqrt{x}}{25y^3}\right) = \log_5\left(x^{1/2}\right) - \log_5\left(25y^3\right)$$

$$= \log_5\left(x^{1/2}\right) - \left[\log_5\left(5^2\right) + \log_5\left(y^3\right)\right]$$

$$= \log_5\left(x^{1/2}\right) - \log_5\left(5^2\right) - \log_5\left(y^3\right)$$

$$= \frac{1}{2}\log_5(x) - 2\log_5(5) - 3\log_5(y)$$

$$= \frac{1}{2}\log_5(x) - 2 - 3\log_5(y)$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln\left(x^2\sqrt{x^2+1}\right)$

Solution

$$\ln\left(x^2\sqrt{x^2+1}\right) = \ln x^2 + \ln\left(x^2+1\right)^{1/2}$$

$$= 2\ln x + \frac{1}{2}\ln\left(x^2+1\right)$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln\dfrac{x^2}{x^2+1}$

Solution

$$\ln\frac{x^2}{x^2+1} = \ln x^2 - \ln\left(x^2+1\right)$$

$$= 2\ln x - \ln\left(x^2+1\right)$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln\left(\dfrac{x^2(x+1)^3}{(x+3)^{1/2}}\right)$

Solution

$$\ln\left(\frac{x^2(x+1)^3}{(x+3)^{1/2}}\right) = \ln\left(x^2(x+1)^3\right) - \ln(x+3)^{1/2}$$

$$= \ln x^2 + \ln(x+1)^3 - \frac{1}{2}\ln(x+3)$$

$$= 2\ln x + 3\ln(x+1) - \frac{1}{2}\ln(x+3)$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln\sqrt{\dfrac{(x+1)^5}{(x+2)^{20}}}$

Solution

$$\ln\sqrt{\frac{(x+1)^5}{(x+2)^{20}}} = \ln\left(\frac{(x+1)^5}{(x+2)^{20}}\right)^{1/2}$$

$$= \frac{1}{2}\ln\left(\frac{(x+1)^5}{(x+2)^{20}}\right)$$

$$= \frac{1}{2}\left(\ln(x+1)^5 - \ln(x+2)^{20}\right)$$

$$= \frac{1}{2}\left(5\ln(x+1) - 20\ln(x+2)\right)$$

$$= \frac{5}{2}\ln(x+1) - 10\ln(x+2)$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln\frac{\left(x^2+1\right)^5}{\sqrt{1-x}}$

Solution

$$\ln\frac{\left(x^2+1\right)^5}{\sqrt{1-x}} = \ln\left(x^2+1\right)^5 - \ln(1-x)^{1/2}$$

$$= 5\ln\left(x^2+1\right) - \frac{1}{2}\ln(1-x)$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln\left(\sqrt[3]{\frac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}}\right)$

Solution

$$\ln\left(\sqrt[3]{\frac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}}\right) = \ln\left(\frac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}\right)^{1/3}$$

$$= \frac{1}{3}\ln\left(\frac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}\right)$$

$$= \frac{1}{3}\left(\ln\left(x(x+1)(x-2)\right) - \ln\left(\left(x^2+1\right)(2x+3)\right)\right)$$

$$= \frac{1}{3}\left(\ln x + \ln(x+1) + \ln(x-2) - \left(\ln\left(x^2+1\right) + \ln(2x+3)\right)\right)$$

$$= \frac{1}{3}\left(\ln x + \ln(x+1) + \ln(x-2) - \ln\left(x^2+1\right) - \ln(2x+3)\right)$$

$$= \frac{1}{3}\ln x + \frac{1}{3}\ln(x+1) + \frac{1}{3}\ln(x-2) - \frac{1}{3}\ln\left(x^2+1\right) - \frac{1}{3}\ln(2x+3)$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln\left(\sqrt{\dfrac{1}{x(x+1)}}\right)$

Solution

$$\begin{aligned}
\ln\left(\sqrt{\frac{1}{x(x+1)}}\right) &= \ln\left(\frac{1}{x(x+1)}\right)^{1/2}\\
&= \frac{1}{2}\left(\ln 1 - \ln\left(x(x+1)\right)\right)\\
&= -\frac{1}{2}\left(\ln x + \ln(x+1)\right)\\
&= -\frac{1}{2}\ln x - \frac{1}{2}\ln(x+1)
\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln\left(\sqrt{\left(x^2+1\right)(x-1)^2}\right)$

Solution

$$\begin{aligned}
\ln\left(\sqrt{\left(x^2+1\right)(x-1)^2}\right) &= \ln\left(\left(x^2+1\right)(x-1)^2\right)^{1/2}\\
&= \frac{1}{2}\ln\left(\left(x^2+1\right)(x-1)^2\right)\\
&= \frac{1}{2}\left(\ln\left(x^2+1\right) + \ln(x-1)^2\right)\\
&= \frac{1}{2}\left(\ln\left(x^2+1\right) + 2\ln(x-1)\right)\\
&= \frac{1}{2}\ln\left(x^2+1\right) + \ln(x-1)
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\log\left(x^3y^2\right) - 2\log\left(x\sqrt[3]{y}\right) - 3\log\left(\frac{x}{y}\right)$$

Solution

$$\begin{aligned}
\log\left(x^3y^2\right) - 2\log\left(x\sqrt[3]{y}\right) - 3\log\left(\frac{x}{y}\right) &= \log\left(x^3y^2\right) - \log\left(xy^{1/3}\right)^2 - \log\left(xy^{-1}\right)^3\\
&= \log\left(x^3y^2\right) - \left[\log\left(x^2y^{2/3}\right) + \log\left(x^3y^{-3}\right)\right]
\end{aligned}$$

$$= \log\left(x^3 y^2\right) - \log\left(x^2 y^{2/3} x^3 y^{-3}\right)$$

$$= \log\left(x^3 y^2\right) - \log\left(x^5 y^{-7/3}\right)$$

$$= \log\left(\frac{x^3 y^2}{x^5 y^{-7/3}}\right)$$

$$= \log\left(\frac{y^2 y^{7/3}}{x^2}\right)$$

$$= \log\left(\frac{y^{13/3}}{x^2}\right)$$

$$= \log\left(\frac{\sqrt[3]{y^{13}}}{x^2}\right)$$

$$\underline{= \log\left(\frac{y^4 \sqrt[3]{y}}{x^2}\right)}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$\ln y^3 + \frac{1}{3}\ln\left(x^3 y^6\right) - 5\ln y$

Solution

$$\ln y^3 + \frac{1}{3}\ln\left(x^3 y^6\right) - 5\ln y = \ln y^3 + \ln\left(x^3 y^6\right)^{1/3} - \ln y^5$$

$$= \ln y^3 + \ln\left(x^{3/3} y^{6/3}\right) - \ln y^5$$

$$= \ln y^3 + \ln\left(x y^2\right) - \ln y^5$$

$$= \ln\left(y^3 x y^2\right) - \ln y^5$$

$$= \ln\left(\frac{y^5 x}{y^5}\right)$$

$$\underline{= \ln x}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$4\ln x + 7\ln y - 3\ln z$

Solution

$$\begin{aligned} 4\ln x + 7\ln y - 3\ln z &= \ln x^4 + \ln y^7 - \ln z^3 \\ &= \ln\left(x^4 y^7\right) - \ln z^3 \\ &= \ln\left(\frac{x^4 y^7}{z^3}\right) \end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$\frac{1}{3}\left[5\ln(x+6) - \ln x - \ln(x^2-25)\right]$

Solution

$$\begin{aligned} \frac{1}{3}\left[5\ln(x+6) - \ln x - \ln(x^2-25)\right] &= \frac{1}{3}\left[5\ln(x+6) - \left(\ln x + \ln(x^2-25)\right)\right] \\ &= \frac{1}{3}\left[\ln(x+6)^5 - \ln x(x^2-25)\right] \\ &= \frac{1}{3}\left[\ln\frac{(x+6)^5}{x(x^2-25)}\right] \\ &= \ln\left(\frac{(x+6)^5}{x(x^2-25)}\right)^{1/3} \end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$\frac{2}{3}\left[\ln\left(x^2-4\right) - \ln\left(x+2\right)\right] + \ln(x+y)$

Solution

$$\begin{aligned} \frac{2}{3}\left[\ln\left(x^2-4\right) - \ln\left(x+2\right)\right] + \ln(x+y) &= \frac{2}{3}\left[\ln\frac{x^2-4}{x+2}\right] + \ln(x+y) \\ &= \frac{2}{3}\left[\ln\frac{(x+2)(x-2)}{x+2}\right] + \ln(x+y) \end{aligned}$$

$$= \frac{2}{3}\ln(x-2) + \ln(x+y)$$

$$= \ln(x-2)^{2/3} + \ln(x+y)$$

$$= \ln(x-2)^{2/3}(x+y)$$

$$= \ln(x+y)\sqrt[3]{(x-2)^2}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\frac{2}{3}\left[\ln\left(x^2-9\right) - \ln(x+3)\right] + \ln(x+y)$$

Solution

$$\frac{2}{3}\left[\ln\left(x^2-9\right) - \ln(x+3)\right] + \ln(x+y) = \frac{2}{3}\ln\frac{x^2-9}{x+3} + \ln(x+y)$$

$$= \frac{2}{3}\ln\frac{(x+3)(x-3)}{x+3} + \ln(x+y)$$

$$= \frac{2}{3}\ln(x-3) + \ln(x+y)$$

$$= \ln(x-3)^{2/3} + \ln(x+y)$$

$$= \ln\left((x-3)^{2/3}(x+y)\right)$$

$$= \ln\left((x+y)\sqrt[3]{(x-3)^2}\right)$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\frac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y$$

Solution

$$\frac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y = \log_b x^{1/4} - \log_b 5^2 - \log_b y^{10}$$

$$= \log_b x^{1/4} - \left[\log_b 5^2 + \log_b y^{10}\right]$$

$$= \log_b x^{1/4} - \log_b\left(5^2 y^{10}\right)$$

$$= \log_b \frac{\sqrt[4]{x}}{25y^{10}}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$2\ln(x+4) - \ln x - \ln\left(x^2-3\right)$

Solution

$$2\ln(x+4) - \ln x - \ln\left(x^2-3\right) = \ln(x+4)^2 - \left(\ln x + \ln\left(x^2-3\right)\right)$$

$$= \ln(x+4)^2 - \ln\left(x\left(x^2-3\right)\right)$$

$$= \ln \frac{(x+4)^2}{x\left(x^2-3\right)}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$\ln\left(x^2-25\right) - 2\ln(x+5) + \ln(x-5)$

Solution

$$\ln\left(x^2-25\right) - 2\ln(x+5) + \ln(x-5) = \ln\left(x^2-25\right) + \ln(x-5) - \ln(x+5)^2$$

$$= \ln \frac{(x-5)(x+5)(x-5)}{(x+5)^2}$$

$$= \ln\left(\frac{(x-5)^2}{x+5}\right)$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$5\log_a x - \frac{1}{2}\log_a(3x-4) - 3\log_a(5x+1)$

Solution

$$5\log_a x - \frac{1}{2}\log_a(3x-4) - 3\log_a(5x+1) = \log_a x^5 - \log_a(3x-4)^{1/2} - \log_a(5x+1)^3$$

$$= \log_a x^5 - \left[\log_a(3x-4)^{1/2} + \log_a(5x+1)^3\right]$$

$$= \log_a x^5 - \left[\log_a(3x-4)^{1/2}(5x+1)^3\right]$$

$$= \log_a \frac{x^5}{(3x-4)^{1/2}(5x+1)^3}$$

Exercise

Solve the equation: $2^x = 128$

Solution

$$2^x = 2^7$$

$$x = 7$$

Exercise

Solve the equation: $3^x = 243$

Solution

$$3^x = 3^5$$

$$x = 5$$

Exercise

Solve the equation: $5^x = 70$

Solution

$$x = \log_5 70$$

Exercise

Solve the equation: $2^{5x+3} = \frac{1}{16}$

Solution

$$2^{5x+3} = 2^{-4}$$

$5x+3=-4$

$5x=-7$

$\boxed{x=-\frac{7}{5}}$

Exercise

Solve the equation: $\left(\frac{2}{5}\right)^x=\frac{8}{125}$

Solution

$\left(\frac{2}{5}\right)^x=\left(\frac{2}{5}\right)^3$

$\boxed{x=3}$

Exercise

Solve the equation: $2^{3x-7}=32$

Solution

$$\begin{aligned}2^{3x-7}&=32\\&=2^5\end{aligned}$$

$3x-7=5$ *add 7 on both sides*

$3x=12$ *Divide by 3*

$\boxed{x=4}$

Exercise

Solve the equation: $4^{2x-1}=64$

Solution

$4^{2x-1}=4^3$

$2x-1=3$

$2x=4$

$\boxed{x=2}$

Exercise

Solve the equation: $2^{x+4}=8^{x-6}$

Solution

$$2^{x+4} = \left(2^3\right)^{x-6}$$

$$2^{x+4} = 2^{3x-18}$$

$$x + 4 = 3x - 18$$

$$2x = 22$$

$$\underline{x = 11}$$

Exercise

Solve the equation: $5^{x+4} = 4^{x+5}$

Solution

$$\ln 5^{x+4} = \ln 4^{x+5}$$

$$(x+4)\ln 5 = (x+5)\ln 4$$

$$x\ln 5 + 4\ln 5 = x\ln 4 + 5\ln 4$$

$$(\ln 5 - \ln 4)x = 5\ln 4 - 4\ln 5$$

$$\underline{x = \frac{5\ln 4 - 4\ln 5}{\ln 5 - \ln 4}}$$

Exercise

Solve the equation: $3^{x-1} = 7^{2x+5}$

Solution

$$\ln 3^{x-1} = \ln 7^{2x+5}$$

$$(x-1)\ln 3 = (2x+5)\ln 7$$

$$x\ln 3 - \ln 3 = 2x\ln 7 + 5\ln 7$$

$$x\ln 3 - 2x\ln 7 = \ln 3 + 5\ln 7$$

$$x(\ln 3 - 2\ln 7) = \ln 3 + 5\ln 7$$

$$\underline{x = \frac{\ln 3 + 5\ln 7}{\ln 3 - 2\ln 7}}$$

Exercise

Solve the equation: $3^{x+4} = 2^{1-3x}$

Solution

$\ln 3^{x+4} = \ln 2^{1-3x}$ ***'ln' both sides***

$(x+4)\ln 3 = (1-3x)\ln 2$ ***Power Rule***

$x\ln 3 + 4\ln 3 = \ln 2 - 3x\ln 2$ ***Distribute***

$x\ln 3 + 3x\ln 2 = \ln 2 - 4\ln 3$

$x(\ln 3 + 3\ln 2) = \ln 2 - 4\ln 3$

$x = \dfrac{\ln 2 - 4\ln 3}{\ln 3 + 3\ln 2}$

Exercise

Solve the equation: $e^x = 15$

Solution

$x = \ln 5$ ***Convert to Log***

Exercise

Solve the equation: $e^{x+1} = 20$

Solution

$x + 1 = \ln 20$ ***Convert to Log***

$x = -1 + \ln 20$

Exercise

Solve the equation: $e^{x\ln 3} = 27$

Solution

$x\ln 3 = \ln 27$ ***Convert to Log***

$x\ln 3 = \ln 3^3$

$x = \dfrac{3\ln 3}{\ln 3}$

$= 3$

Exercise

Solve the equation: $e^{x^2} = e^{7x-12}$

Solution

$$e^{x^2} = e^{7x-12}$$

$$x^2 = 7x - 12$$

$$x^2 - 7x + 12 = 0$$

$$\boxed{x = 3,\ 4}$$

Exercise

Solve the equation: $f(x) = xe^x + e^x$

Solution

$$xe^x + e^x = 0$$

$$e^x(x+1) = 0$$

$$e^x \neq 0 \qquad x + 1 = 0$$

$$\boxed{x = -1}\ (\textit{Only solution})$$

Exercise

Solve the equation $f(x) = x^3\left(4e^{4x}\right) + 3x^2e^{4x}$

Solution

$$x^3\left(4e^{4x}\right) + 3x^2e^{4x} = 0$$

$$x^2e^{4x}(4x+3) = 0$$

$$x^2 = 0 \qquad 4x + 3 = 0$$

$$x = 0,\ 0 \qquad x = -\frac{3}{4}$$

The solutions are: $\boxed{x = 0,\ 0,\ -\frac{3}{4}}$

Exercise

Solve the equation: $e^{2x} - 2e^x - 3 = 0$

Solution

$$\left(e^x\right)^2 - 2e^x - 3 = 0$$

$$\begin{cases} e^{x} = -1 \ \times \ \rightarrow \ \textit{Impossible} \\ e^{x} = 3 \ \rightarrow \ \boxed{x = \ln 3} \end{cases}$$

Exercise

Solve the equation: $e^{2x+1} \cdot e^{-4x} = 3e$

Solution

$$e^{2x+1-4x} = 3e$$

$$e^{-2x+1} = 3e$$

$$e^{-2x}e = 3e \qquad \textit{Divide by } e$$

$$e^{-2x} = 3$$

$$\ln e^{-2x} = \ln 3$$

$$-2x = \ln 3$$

$$\boxed{x = -\frac{1}{2}\ln 3}$$

Exercise

Solve the equation: $e^{2x} - 8e^{x} + 7 = 0$

Solution

$$\left(e^{x}\right)^2 - 8e^{x} + 7 = 0 \qquad a+b+c=0 \;\rightarrow\; x = 1,\ \frac{c}{a}$$

$$\begin{cases} e^{x} = 1 & \rightarrow \quad \boxed{x = 0} \\ e^{x} = 7 & \rightarrow \quad \boxed{x = \ln 7} \end{cases}$$

Exercise

Solve the equation without using the calculator: $e^{2x} + 2e^{x} - 15 = 0$

Solution

$$\left(e^{x}\right)^2 + 2e^{x} - 15 = 0 \qquad \textit{Solve for } e^{x}$$

$$e^{x} = 3 \qquad \cancel{e^{x} = -5} < 0$$

$$\boxed{x = \ln 3}$$

Exercise

Solve the equation: $e^{x} + e^{-x} - 6 = 0$

Solution

$$e^x e^x + e^x e^{-x} - e^x 6 = e^x 0$$

$$e^{2x} + 1 - 6e^x = 0$$

$$\left(e^x\right)^2 - 6e^x + 1 = 0$$

$$e^x = \frac{6 \pm \sqrt{36-4}}{2}$$

$$= \frac{6 \pm 4\sqrt{2}}{2}$$

$$e^x = 3 \pm 2\sqrt{2}$$

$$\boxed{x = \ln\left(3 \pm 2\sqrt{2}\right)}$$

Exercise

Solve the equation: $6\ln(2x) = 30$

Solution

$$\ln(2x) = \frac{30}{6}$$

$$\ln(2x) = 5$$

$$2x = e^5$$

$$\boxed{x = \frac{1}{2}e^5}$$

Exercise

Solve the equation: $\log_4(5+x) = 3$

Solution

$$5 + x = 4^3$$

$$x = 64 - 5$$

$$\boxed{= 59}$$

Check: $\log_4(5+59)$

Exercise

Solve the equation: $\log(4x - 18) = 1$

Solution

$$4x - 18 = 10$$

$4x = 28$

$\underline{x = 7}$

Exercise

Solve the equation: $\log_5 x + \log_5 (4x-1) = 1$

Solution

$\log_5 x(4x-1) = 1$

$4x^2 - x = 5$

$4x^2 - x - 5 = 0 \qquad a - b + c = 0 \quad \to \quad x = -1, \; -\frac{c}{a}$

$\underline{x = -\frac{5}{2}, \; 4}$

Check: $x = -\frac{5}{2} \quad \log_5\left(-\frac{5}{2}\right) + \log_5 (10-1)$ ✗

$x = 4 \quad \log_5 (4) + \log_5 (15)$

$\therefore$ ***Solution***: $\underline{x = 4}$

Exercise

Solve the equation: $\log x - \log(x+3) = 1$

Solution

$\log \frac{x}{x+3} = 1$

$\frac{x}{x+3} = 10$

$x = 10x + 30$

$9x = -30$

$\underline{x = -\frac{10}{3}}$

Check: $x = -\frac{10}{3} \quad \log\left(-\frac{10}{3}\right) - \log(x+3)$ ✗

$\therefore$ *No Solution*

Exercise

Solve the equation: $\log x + \log(x-9) = 1$

Solution

$\log x(x-9)=1$

$x^2-9x=10$

$x^2-9x-10=0 \qquad a-b+c=0 \;\rightarrow\; x=-1,\; -\frac{c}{a}$

$\underline{x=-1,\ 10}\rfloor$

Check: $x=-1 \quad \log(-1)+\log(x-9)$ ✗

$x=10 \quad \log(10)+\log(10-9)$

$\therefore$ ***Solution***: $\underline{x=10}\rfloor$

Exercise

Solve the equation: $\ln(4x+6)-\ln(x+5)=\ln x$

Solution

$\ln\frac{4x+6}{x+5}=\ln x$

$\frac{4x+6}{x+5}=x$

$4x+6=x^2+5x$

$x^2+x-6=0$

$\underline{x=-3,\ 2}\rfloor$

Check: $x=-3 \quad \ln(-6)-\ln(x+5)=\ln x$ ✗

$x=2 \quad \ln(14)-\ln(7)=\ln 2$

$\therefore$ ***Solution***: $\underline{x=2}\rfloor$

Exercise

Solve the equation: $\ln(5+4x)-\ln(x+3)=\ln 3$

Solution

$\ln\frac{5+4x}{x+3}=\ln 3$

$\frac{5+4x}{x+3}=3$

$5+4x=3x+9$

$\underline{x=4}\rfloor$

Check: $x=4 \quad \ln(21)-\ln(7)=\ln 3$

$\therefore$ ***Solution***: $\underline{x=4}\rfloor$

Exercise

Solve the equation: $\ln \sqrt[4]{x} = \sqrt{\ln x}$

Solution

Domain: $x \ge 1$

$$\ln x^{1/4} = \sqrt{\ln x}$$

$$\frac{1}{4}\ln x = \sqrt{\ln x}$$

$$\left(\frac{1}{4}\ln x\right)^2 = \left(\sqrt{\ln x}\right)^2$$

$$\frac{1}{6}\ln^2 x = \ln x$$

$$\ln^2 x = 6\ln x$$

$$\ln^2 x - 6\ln x = 0$$

$$(\ln x)(\ln x - 6) = 0$$

$$\begin{cases} \ln x = 0 & \rightarrow \quad x = 1 \\ \ln x = 6 & \rightarrow \quad x = e^6 \end{cases}$$

$\therefore$ ***Solution***: $x = 1,\ e^6$

Exercise

Solve the equation: $\sqrt{\ln x} = \ln \sqrt{x}$

Solution

Domain: $x \ge 1$

$$\sqrt{\ln x} = \ln x^{1/2}$$

$$\sqrt{\ln x} = \frac{1}{2}\ln x$$

$$\left(\sqrt{\ln x}\right)^2 = \left(\frac{1}{2}\ln x\right)^2$$

$$\ln x = \frac{1}{4}\ln^2 x$$

$$4\ln x = \ln^2 x$$

$$\ln^2 x - 4\ln x = 0$$

$$\ln x(\ln x - 4) = 0$$

$$\begin{cases} \ln x = 0 \ \rightarrow \ \underline{x = 1|} \\ \ln x = 4 \ \rightarrow \ \underline{x = e^4|} \end{cases}$$

$\therefore$ ***Solution***: $\underline{x = 1,\ e^4 \ |}$

Exercise

Solve the equation: $\log x^2 = (\log x)^2$

Solution

Domain: $\underline{x \ge 1\ |}$

$2\log x = (\log x)^2$

$(\log x)^2 - 2\log x = 0$

$\log x(\log x - 2) = 0$

$$\begin{cases} \log x = 0 \ \rightarrow \ \underline{x = 1|} \\ \log x = 2 \ \rightarrow \ \underline{x = 100|} \end{cases}$$

$\therefore$ ***Solution***: $\underline{x = 1,\ 100\ |}$

Exercise

Solve the equation: $\log x^3 = (\log x)^2$

Solution

Domain: $\underline{x \ge 1\ |}$

$3\log x = (\log x)^2$

$(\log x)^2 - 3\log x = 0$

$\log x(\log x - 3) = 0$

$$\begin{cases} \log x = 0 \ \rightarrow \ \underline{x = 1|} \\ \log x = 3 \ \rightarrow \ \underline{x = 10^3|} \end{cases}$$

Convert to exponential

$\therefore$ ***Solution***: $\underline{x = 1,\ 10^3\ |}$

Exercise

Solve the equation: $\ln(\ln x) = 2$

Solution

$\ln x = e^2$ *Convert to exponential*

$\therefore$ ***Solution***: $\underline{x = e^{e^2}}$

Exercise

Solve the equation: $\ln\left(e^{x^2}\right) = 64$

Solution

$e^{x^2} = e^{64}$ *Convert to exponential*

$x^2 = 64$

$\therefore$ ***Solution***: $\underline{x = \pm 8}$

Exercise

Solve the equation: $e^{\ln(x-1)} = 4$

Solution

$x - 1 = 4$

$\therefore$ ***Solution***: $\underline{x = 5}$

Exercise

Solve the equation: $\ln x^2 = \ln(12 - x)$

Solution

$\ln x^2 = \ln(12 - x)$

$x^2 = 12 - x$

$x^2 + x - 12 = 0$

$\underline{x = -4,\ 3}$

Check: $x = -4 \quad \ln(16) = \ln(16)$

$x = 3 \quad \ln(9) = \ln(12 - 3)$

$\therefore$ ***Solution***: $\underline{x = -4,\ 3}$

Exercise

Solve the equation $\ln x = 1 - \ln(x+2)$

Solution

$\ln x + \ln(x+2) = 1$

$\ln x(x+2) = 1$

$x^2 + 2x = e$

$x^2 + 2x - e = 0$

$$x = \frac{-2 \pm \sqrt{4+4e}}{2}$$

$$= \frac{-2 \pm 2\sqrt{1+e}}{2}$$

$$= \begin{cases} -1 - \sqrt{1+e} < 0 \\ -1 + \sqrt{1+e} > 0 \end{cases}$$

$\therefore$ ***Solution***: $\underline{x = -1 + \sqrt{1+e}}$

Exercise

Solve the equation $\ln x = 1 + \ln(x+1)$

Solution

$\ln x - \ln(x+1) = 1$

$\ln \frac{x}{x+1} = 1$

$\frac{x}{x+1} = e^1$

$x = (x+1)e$

$x = ex + e$

$x - ex = e$

$x(1-e) = e$

$x = \frac{e}{1-e} < 0$

$\therefore$ ***No solution***

Exercise

Solve the equation: $\log_3(x+3)+\log_3(x+5)=1$

Solution

Domain: $x>-3$

$\log_3(x+3)(x+5)=1$

$x^2+8x+15=3$

$x^2+8x+12=0$

$$x=\frac{-8\pm\sqrt{64-48}}{2}$$

$$=\begin{cases}\frac{-8-4}{2}=-6<-3 \times \\ \frac{-8+4}{2}=-2>-3\end{cases}$$

$\therefore$ ***Solution***: $\underline{x=-2}$

Exercise

Solve the equation: $\ln x=\frac{1}{2}\ln\left(2x+\frac{5}{2}\right)+\frac{1}{2}\ln 2$

Solution

Domain: $x>0$

$2\ln x=\ln\left(2x+\frac{5}{2}\right)+\ln 2$

$\ln x^2=\ln 2\left(2x+\frac{5}{2}\right)$

$x^2=4x+5$

$x^2-4x-5=0$ $\qquad a-b+c=0 \rightarrow x=-1,\ -\frac{c}{a}$

$\underline{x=-1,\ 5}$

$\therefore$ ***Solution***: $\underline{x=5}$

Exercise

Solve the equation $\ln(-4-x)+\ln 3=\ln(2-x)$

Solution

Domain: $x<5$

$\ln 3(-4-x)=\ln(2-x)$

$-12 - 3x = 2 - x$

$-12 - 2 = 3x - x$

$-14 = 2x$

$\therefore$ ***Solution***: $\underline{x = -7}$

Exercise

Lead shielding is used to contain radiation. The percentage of a certain radiation that can penetrate x *millimeters* of lead shielding is given by $I(x) = 100e^{-1.5x}$

a) What percentage of radiation will penetrate a lead shield that is 1 *millimeter* thick?

b) How many *millimeters* of lead shielding are required so that less than 0.02% of the radiations penetrates the shielding?

Solution

a) $I(1) = 100e^{-1.5}$

$\approx \underline{22.313}$

$\therefore$ The percentage of radiation will penetrate a lead shield is approximately 22.313%

b) $I(x) = 100e^{-1.5x} = .02$

$e^{-1.5x} = .02$

$-1.5x = \ln\left(2\times 10^{-4}\right)$

$x = -\frac{1}{1.5}\ln\left(2\times 10^{-4}\right)$

$\approx \underline{5.68 \ mm}$

Exercise

After a race, a runner's pulse rate R, in *beats* per *minute*, decreases according to the function

$$R(t) = 145e^{-0.092t}, \quad 0 \le t \le 15$$

Where t is measured in minutes.

a) Find the runner's pulse rate at the end of the race and 1 *minute* after the end of the race.

b) How long after the end of the race will the runner's pulse rate be 80 *beats* per *minute*?

Solution

a) $R(15) = 145e^{-0.092(15)}$

$\approx \underline{36.48}$

$$R(16) = 145e^{-0.092(16)}$$
$$\approx 33.27$$

b) $R(t) = 145e^{-0.092t} = 80$

$$e^{-0.092t} = \frac{80}{145}$$
$$-0.092t = \ln\frac{16}{29}$$
$$t = -\frac{1}{0.092}\ln\frac{16}{29}$$
$$\approx 6.46 \ \ min$$

Exercise

A can of soda at $79°F$ is placed in a refrigerator that maintains a constant temperature of $36°F$. The temperature T of the soda t *minutes* after it is placed in the refrigerator is given by

$$T(t) = 36 + 43e^{-0.058t}$$

a) Find the temperature of the soda 10 *minutes* after it is placed in the refrigerator.
b) When will the temperature of the soda be $45°F$

Solution

a) $T(10) = 36 + 43e^{-0.058(10)}$

$$\approx 60°F$$

b) $36 + 43e^{-0.058t} = 45$

$$43e^{-0.058t} = 9$$
$$e^{-0.058t} = \frac{9}{43}$$
$$-0.058t = \ln\frac{9}{43}$$
$$t = \frac{-1}{0.058}\ln\frac{9}{43}$$
$$\approx 27 \ \ min$$

Exercise

During surgery, a patient's circulatory system requires at least 50 *milligrams* of an anesthetic. The amount of anesthetic present *t hours* after 80 *milligrams* of anesthetic is administered is given by

$$T(t) = 80(0.727)^t$$

a) How much of the anesthetic is present in the patient's circulatory system 30 *minutes* after the anesthetic is administered?

b) How long can the operation last if the patient does not receive additional anesthetic?

Solution

a) $T\left(30 = \frac{1}{2}hr\right) = 80(0.727)^{1/2}$

$\approx 68 \;\; mg$

b) $T(t) = 80(0.727)^t = 50$

$(0.727)^t = \frac{5}{8}$

$t = \log_{.727}\left(\frac{5}{8}\right)$

$\approx 1.47 \;\; hrs$

$= 1\ hr \quad 28' \quad 12''$

Exercise

The following function models the average typing speed S, in *words* per *minute*, for a student who has been typing for *t months*.

$$S(t) = 5 + 29\ln(t+1), \quad 0 \le t \le 9$$

Use S to determine how long it takes the student to achieve an average speed of 65 *words* per *minute*.

Solution

$S(t) = 5 + 29\ln(t+1) = 65$

$29\ln(t+1) = 60$

$\ln(t+1) = \frac{60}{29}$

$t + 1 = e^{\frac{60}{29}}$

$t = e^{\frac{60}{29}} - 1$

$t \approx 7 \;\; months$

Exercise

A lawyer has determined that the number of people $P(t)$ in a city of 1.2 *million* people who have been exposed to a news item after *t days* is given by the function

$$P(t)=1{,}200{,}000\left(1-e^{-0.03t}\right)$$

a) How many days after a major crime has been reported has 40% of the population heard of the crime?

b) A defense lawyer knows it will be difficult to pick an unbiased jury after 80% of the population has heard of the crime. After how many days will 80% of the population have heard of the crime?

Solution

a) $P(t)=1{,}200{,}000\left(1-e^{-0.03t}\right)=(.4)(1{,}200{,}000)$

$$1-e^{-0.03t}=0.4$$

$$e^{-0.03t}=0.6$$

$$-0.03t=\ln(0.6)$$

$$t=-\frac{\ln(0.6)}{0.03}$$

$$\approx 17\ \ days$$

b) $P(t)=1{,}200{,}000\left(1-e^{-0.03t}\right)=\left(\frac{8}{100}\right)(1{,}200{,}000)$

$$1-e^{-0.03t}=\frac{2}{25}$$

$$e^{-0.03t}=1-\frac{2}{25}$$

$$-\frac{3}{100}t=\ln\left(\frac{23}{25}\right)$$

$$t=\frac{100}{3}\ln\left(\frac{25}{23}\right)$$

$$\approx 3\ \ days$$

Exercise

Newton's Law of Cooling states that is an object at temperature T_0 is placed into an environment at constant temperature *A*, then the temperature of the object, $T(t)$ (in degrees Fahrenheit), after *t minutes* is given by $T(t)=A+\left(T_0-A\right)e^{-kt}$, where *k* is a constant that depends on the object.

a) Determine the constant k for a canned soda drink that takes 5 *minutes* to cool from $75°F$ to $65°F$ after being placed in a refrigerator that maintains a constant temperature of $34°F$

b) What will be the temperature of the soda after 30 *minutes*?

c) When will the temperature of the soda drink be $36°F$?

Solution

a) $T(5) = 34 + (75-34)e^{-5k} = 65$

$41e^{-5k} = 31$

$e^{-5k} = \frac{31}{41}$

$-5k = \ln\left(\frac{31}{41}\right)$

$k = -\frac{1}{5}\ln\left(\frac{31}{41}\right)$

≈ 0.0559

b) $T(t) = 34 + 41e^{-0.0559t}$

$T(30) = 34 + 41e^{-0.0559(30)}$

$\approx 42°F$

c) $T(t) = 34 + 41e^{-0.0559t} = 36$

$41e^{-0.0559t} = 2$

$e^{-0.0559t} = \frac{2}{41}$

$-0.0559t = \ln\left(\frac{2}{41}\right)$

$t = -\frac{1}{0.0559}\ln\left(\frac{2}{41}\right)$

$\approx 54\ \text{min}$

Solution Section R.5 – Trigonometry

Exercise

Convert to radians

a) $256° \; 20'$ *b)* $-78.4°$ *c)* $330°$ *d)* $-60°$ *e)* $-225°$

Solution

a) $$256° \; 20' = 256° + \frac{20°}{60}$$
$$= 256° + \frac{2°}{6}$$
$$= \frac{1538°}{6} = \left(\frac{769}{3}\right)^{\circ}$$

$$\frac{769°}{3}\frac{\pi}{180°} = \frac{769\pi}{540}\ rad \approx 4.47\ rad$$

b) $$-78.4° = -78.4°\left(\frac{\pi}{180°}\right) rad$$
$$\approx -1.37\ rad$$

c) $$330° = 330°\left(\frac{\pi}{180°}\right) rad$$
$$= \frac{11\pi}{6}\ rad$$

d) $$-60° = -60°\left(\frac{\pi}{180°}\right) rad$$
$$= -\frac{\pi}{3}\ rad$$

e) $$-225° = -225°\left(\frac{\pi}{180°}\right) rad$$
$$= -\frac{5\pi}{4}\ rad$$

Exercise

Convert to degrees

a) $\frac{11\pi}{6}$ *b)* $-\frac{5\pi}{3}$ *c)* $\frac{\pi}{6}$ *d)* 2.4 *e)* $\frac{\pi}{3}$ *f)* $-\frac{5\pi}{12}$ *g)* -4π *h)* $\frac{7\pi}{13}$

Solution

a) $\frac{11\pi}{6}\ (rad) = \frac{11\pi}{6} \bullet \frac{180°}{\pi}$

$= 330°$

b) $-\frac{5\pi}{3}(rad) = -\frac{5\pi}{3} \bullet \frac{180°}{\pi}$

$= -300°$

c) $\frac{\pi}{6}(rad) = \frac{\pi}{6}\left(\frac{180}{\pi}\right)^{\circ}$

$= 30^{\circ}$

d) $2.4\ rad = 2.4 \bullet \frac{180°}{\pi}$

$= \frac{432°}{\pi}$

$\approx 137.5°$

e) $\frac{\pi}{3}(rad) = \frac{\pi}{3}\left(\frac{180}{\pi}\right)^{\circ}$

$= 60^{\circ}$

f) $-\frac{5\pi}{12}(rad) = -\frac{5\pi}{12}\left(\frac{180}{\pi}\right)^{\circ}$

$= -75^{\circ}$

g) $-4\pi(rad) = -4\pi\left(\frac{180}{\pi}\right)^{\circ}$

$= -720^{\circ}$

h) $\frac{7\pi}{13}(rad) = \frac{7\pi}{13}\left(\frac{180}{\pi}\right)^{\circ}$

$\approx 96.923^{\circ}$

Exercise

Prove the identity $\frac{\tan\theta\cot\theta}{\csc\theta} = \sin\theta$

Solution

$\frac{\tan\theta\cot\theta}{\csc\theta} = \frac{1}{\frac{1}{\sin\theta}}$ $\qquad$ $\tan\theta\cot\theta = 1$

$$\underline{= \sin\theta\,|}\quad \checkmark$$

Exercise

Prove the identity $\dfrac{\sec^2\theta}{\tan\theta} = \sec\theta\,\csc\theta$

Solution

$$\frac{\sec^2\theta}{\tan\theta} = \frac{\sec\theta\sec\theta}{\frac{\sin\theta}{\cos\theta}}$$

$$= \sec\theta\,\frac{1}{\cos\theta}\,\frac{\cos\theta}{\sin\theta}$$

$$= \sec\theta\,\frac{1}{\sin\theta}$$

$$\underline{= \sec\theta\,\csc\theta\,|}\quad \checkmark$$

Exercise

Prove the identity $\dfrac{\sec^2\theta}{\tan\theta} = \sec\theta\,\csc\theta$

Solution

$$\frac{\sec^2\theta}{\tan\theta} = \frac{\sec\theta\sec\theta}{\frac{\sin\theta}{\cos\theta}}$$

$$= \sec\theta\,\frac{1}{\cos\theta}\,\frac{\cos\theta}{\sin\theta}$$

$$= \sec\theta\,\frac{1}{\sin\theta}$$

$$\underline{= \sec\theta\,\csc\theta\,|}\quad \checkmark$$

Exercise

Prove the identity $\dfrac{\sin\theta}{\csc\theta} + \dfrac{\cos\theta}{\sec\theta} = 1$

Solution

$$\frac{\sin\theta}{\csc\theta} + \frac{\cos\theta}{\sec\theta} = \frac{\sin\theta}{\frac{1}{\sin\theta}} + \frac{\cos\theta}{\frac{1}{\cos\theta}}$$

$$= \sin^2\theta + \cos^2\theta$$

$$\underline{= 1\,|}\quad \checkmark$$

Exercise

Prove the identity $\cot\theta + \tan\theta = \csc\theta\sec\theta$

Solution

$$\cot\theta + \tan\theta = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta}\frac{1}{\cos\theta}$$

$$= \csc\theta\sec\theta \quad \checkmark$$

Exercise

Prove $\quad \tan x(\cos x + \cot x) = \sin x + 1$

Solution

$$\tan x(\cos x + \cot x) = \frac{\sin x}{\cos x}\left(\cos x + \frac{\cos x}{\sin x}\right)$$

$$= \cos x\frac{\sin x}{\cos x} + \frac{\sin x}{\cos x}\frac{\cos x}{\sin x}$$

$$= \sin x + 1 \quad \checkmark$$

Exercise

Prove $\quad \dfrac{\cos x}{1+\sin x} - \dfrac{1-\sin x}{\cos x} = 0$

Solution

$$\frac{\cos x}{1+\sin x} - \frac{1-\sin x}{\cos x} = \frac{\cos x}{\cos x}\frac{\cos x}{1-\sin x} - \frac{1+\sin x}{1+\sin x}\frac{1-\sin x}{\cos x}$$

$$= \frac{\cos^2 x - (1-\sin^2 x)}{\cos x(1+\sin x)}$$

$$= \frac{\cos^2 x - 1 + \sin^2 x)}{\cos x(1+\sin x)}$$

$$= \frac{1-1}{\cos x(1+\sin x)}$$

$$= \frac{0}{\cos x(1+\sin x)}$$

$$= 0 \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\tan^2 x = \sec^2 x - \sin^2 x - \cos^2 x$

Solution

$$\begin{aligned}
\sec^2 x - \sin^2 x - \cos^2 x &= \frac{1}{\cos^2 x} - \left(\sin^2 x + \cos^2 x\right) \\
&= \frac{1}{\cos^2 x} - 1 \\
&= \frac{1-\cos^2 x}{\cos^2 x} \\
&= \frac{\sin^2 x}{\cos^2 x} \\
&= \tan^2 x \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin\theta}{1+\sin\theta} - \dfrac{\sin\theta}{1-\sin\theta} = -2\tan^2\theta$

Solution

$$\begin{aligned}
\frac{\sin\theta}{1+\sin\theta} - \frac{\sin\theta}{1-\sin\theta} &= \sin\theta\left[\frac{1-\sin\theta-(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}\right] \\
&= \sin\theta\left[\frac{1-\sin\theta-1-\sin\theta}{1-\sin^2\theta}\right] \\
&= \sin\theta\left(\frac{-2\sin\theta}{\cos^2\theta}\right) \\
&= -2\frac{\sin^2\theta}{\cos^2\theta} \\
&= -2\tan^2\theta \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos x}{1+\sin x} + \dfrac{1+\sin x}{\cos x} = 2\sec x$

Solution

$$\begin{aligned}
\frac{\cos x}{1+\sin x} + \frac{1+\sin x}{\cos x} &= \frac{\cos^2 x + (1+\sin x)^2}{(1+\sin x)\cos x} \\
&= \frac{\cos^2 x + 1 + 2\sin x + \sin^2 x}{(1+\sin x)\cos x}
\end{aligned}$$

$$= \frac{2 + 2\sin x}{(1 + \sin x)\cos x}$$

$$= \frac{2(1 + \sin x)}{(1 + \sin x)\cos x}$$

$$= \frac{2}{\cos x}$$

$$= 2\sec x$$ ✓

Exercise

Prove the following equation is an identity: $\dfrac{\tan x + \cot x}{\tan x - \cot x} = \dfrac{1}{\sin^2 x - \cos^2 x}$

Solution

$$\frac{\tan x + \cot x}{\tan x - \cot x} = \frac{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}}{\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x}}$$

$$= \frac{\frac{\sin^2 x + \cos^2 x}{\cos x \sin x}}{\frac{\sin^2 x - \cos^2 x}{\cos x \sin x}}$$

$$= \frac{1}{\sin^2 x - \cos^2 x}$$ ✓

Exercise

Prove the following equation is an identity: $\sec x + \tan x = \dfrac{\cos x}{1 - \sin x}$

Solution

$$\sec x + \tan x = \frac{1}{\cos x} + \frac{\sin x}{\cos x}$$

$$= \frac{1 + \sin x}{\cos x}\frac{1 - \sin x}{1 - \sin x}$$

$$= \frac{1 - \sin^2 x}{\cos x(1 - \sin x)}$$

$$= \frac{\cos^2 x}{\cos x(1 - \sin x)}$$

$$= \frac{\cos x}{1 - \sin x}$$ ✓

Exercise

Prove the following equation is an identity: $\dfrac{\cos x}{\cos x - \sin x} = \dfrac{1}{1 - \tan x}$

Solution

$$\frac{\cos x}{\cos x - \sin x} = \frac{\frac{\cos x}{\cos x}}{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}}$$

$$= \boxed{\frac{1}{1 - \tan x}} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cot^2 x}{\csc x - 1} = \dfrac{1 + \sin x}{\sin x}$

Solution

$$\frac{\cot^2 x}{\csc x - 1} = \frac{\csc^2 x - 1}{\csc x - 1}$$

$$= \frac{(\csc x - 1)(\csc x + 1)}{\csc x - 1}$$

$$= \csc x + 1$$

$$= \frac{1}{\sin x} + 1$$

$$= \boxed{\frac{1 + \sin x}{\sin x}} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\sec^4 x - \tan^4 x = \sec^2 x + \tan^2 x$

Solution

$$\sec^4 x - \tan^4 x = \left(\sec^2 x + \tan^2 x\right)\left(\sec^2 x - \tan^2 x\right) \qquad a^2 - b^2 = (a - b)(a + b)$$

$$= \left(\sec^2 x + \tan^2 x\right)(1)$$

$$= \boxed{\sec^2 x + \tan^2 x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\left(1 + \tan^2 x\right)\left(1 - \sin^2 x\right) = 1$

Solution

$$\left(1+\tan^2 x\right)\left(1-\sin^2 x\right)=\sec^2 x\cos^2 x$$

$$=\frac{1}{\cos^2 x}\cos^2 x$$

$$\underline{=1}\ \checkmark$$

Exercise

Prove the following equation is an identity: $1-\dfrac{\cos^2 x}{1+\sin x}=\sin x$

Solution

$$1-\frac{\cos^2 x}{1+\sin x}=1-\frac{1-\sin^2 x}{1+\sin x}$$

$$=1-\frac{(1-\sin x)(1+\sin x)}{1+\sin x}$$

$$=1-(1-\sin x)$$

$$=1-1+\sin x$$

$$\underline{=\sin x}\ \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin(x-y)}{\sin x\cos y}=1-\cot x\tan y$

Solution

$$\frac{\sin(x-y)}{\sin x\cos y}=\frac{\sin x\cos y-\cos x\sin y}{\sin x\cos y}$$

$$=\frac{\sin x\cos y}{\sin x\cos y}-\frac{\cos x\sin y}{\sin x\cos y}$$

$$\underline{=1-\cot x\tan y}\ \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin(x-y)}{\sin x\sin y}=\cot y-\cot x$

Solution

$$\frac{\sin(x-y)}{\sin x\sin y}=\frac{\sin x\cos y-\cos x\sin y}{\sin x\sin y}$$

$$= \frac{\sin x \cos y}{\sin x \sin y} - \frac{\cos x \sin y}{\sin x \sin y}$$

$$= \frac{\cos y}{\sin y} - \frac{\cos x}{\sin x}$$

$$= \cot y - \cot x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\cos 3x = \cos^3 x - 3\cos x \sin^2 x$

Solution

$$\cos 3x = \cos(x + 2x)$$

$$= \cos x \cos 2x - \sin x \sin 2x$$

$$= \cos x\left(\cos^2 x - \sin^2 x\right) - \sin x(2\sin x \cos x)$$

$$= \cos^3 x - \sin^2 x \cos x - 2\sin^2 x \cos x$$

$$= \cos^3 x - 3\sin^2 x \cos x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\cos^4 x - \sin^4 x = \cos 2x$

Solution

$$\cos^4 x - \sin^4 x = \left(\cos^2 x - \sin^2 x\right)\left(\cos^2 x + \sin^2 x\right) \qquad (a-b)(a+b) = a^2 + b^2$$

$$= (\cos 2x)(1)$$

$$= \cos 2x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos 2x}{\cos^2 x} = \sec^2 x - 2\tan^2 x$

Solution

$$\frac{\cos 2x}{\cos^2 x} = \frac{1 - 2\sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} - \frac{2\sin^2 x}{\cos^2 x}$$

$$= \sec^2 x - 2 \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\tan^2 x(1+\cos 2x) = 1-\cos 2x$

Solution

$$\tan^2 x(1+\cos 2x) = \frac{\sin^2 x}{\cos^2 x}\left(1+2\cos^2 x - 1\right)$$

$$= \frac{\sin^2 x}{\cos^2 x}\left(2\cos^2 x\right)$$

$$= 2\sin^2 x$$

$$= 1-1+2\sin^2 x$$

$$= 1-\left(1-2\sin^2 x\right)$$

$$= 1-\cos 2x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos 2x}{\sin^2 x} = 2\cot^2 x - \csc^2 x$

Solution

$$\frac{\cos 2x}{\sin^2 x} = \frac{\cos^2 x - \sin^2 x}{\sin^2 x}$$

$$= \frac{\cos^2 x}{\sin^2 x} - \frac{\sin^2 x}{\sin^2 x}$$

$$= \cot^2 x - 1 \qquad \cot^2 x + 1 = \csc^2 x$$

$$= \cot^2 x + \cot^2 x - \csc^2 x$$

$$= 2\cot^2 x - \csc^2 x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $2\sin^2\left(\frac{x}{2}\right) = \dfrac{\sin^2 x}{1+\cos x}$

Solution

$$2\sin^2\left(\frac{x}{2}\right) = 2\frac{1-\cos x}{2}$$

$$= 1-\cos x \cdot \frac{1+\cos x}{1+\cos x}$$

$$= \frac{1-\cos^2 x}{1+\cos x}$$

$$= \boxed{\frac{\sin^2 x}{1+\cos x}} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\sec^2\left(\frac{x}{2}\right) = \frac{2\sec x + 2}{\sec x + 2 + \cos x}$

Solution

$$\sec^2\left(\frac{x}{2}\right) = \frac{1}{\cos^2\left(\frac{x}{2}\right)}$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}} \Rightarrow \cos^2\left(\frac{\alpha}{2}\right) = \frac{1+\cos\alpha}{2}$$

$$= \frac{1}{\frac{1+\cos x}{2}}$$

$$= \frac{2}{1+\cos x} \cdot \frac{1+\cos x}{1+\cos x}$$

$$= \frac{2+2\cos x}{1+2\cos x+\cos^2 x}$$

$$= \frac{2+2\cos x}{1+2\cos x+\cos^2 x} \cdot \frac{\frac{1}{\cos x}}{\frac{1}{\cos x}}$$

$$= \frac{\frac{2}{\cos x} + 2\frac{\cos x}{\cos x}}{\frac{1}{\cos x} + \frac{2\cos x}{\cos x} + \frac{\cos^2 x}{\cos x}}$$

$$= \boxed{\frac{2\sec x + 2}{\sec x + 2 + \cos x}} \quad \checkmark$$

Exercise

Find the solutions of the equation that are in the interval $[0,\ 2\pi)$: $2\sin^2 x = 1 - \sin x$

Solution

$$2\sin^2 x + \sin x - 1 = 0$$

$\sin x = -1$ $\boxed{x = \frac{3\pi}{2}}$	$\sin x = \frac{1}{2}$ $\boxed{x = \frac{\pi}{6};\quad x = \frac{5\pi}{6}}$

Exercise

Find the solutions of the equation that are in the interval $[0,\ 2\pi)$: $\tan^2 x \sin x = \sin x$

Solution

$$\tan^2 x \sin x - \sin x = 0$$

$$\sin x\left(\tan^2 x - 1\right) = 0$$

$\sin x = 0$ $x = 0;\quad x = \pi$	$\tan^2 x - 1 = 0 \Rightarrow \tan^2 x = 1$ $\tan x = \pm 1$ $x = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}$

Exercise

Find the solutions of the equation that are in the interval $[0,\ 2\pi)$: $1 - \sin x = \sqrt{3}\cos x$

Solution

$$(1-\sin x)^2 = \left(\sqrt{3}\cos x\right)^2$$

$$1 - 2\sin x + \sin^2 x = 3\cos^2 x$$

$$1 - 2\sin x + \sin^2 x = 3\left(1 - \sin^2 x\right)$$

$$1 - 2\sin x + \sin^2 x = 3 - 3\sin^2 x$$

$$1 - 2\sin x + \sin^2 x - 3 + 3\sin^2 x = 0$$

$$4\sin^2 x - 2\sin x - 2 = 0$$

$\sin x = 1$ $x = \frac{\pi}{2} \to (check)$ $1 - \sin\frac{\pi}{2} = \sqrt{3}\cos\frac{\pi}{2}$ $1 - (1) = \sqrt{3}(0)$ $0 = 0$ ✓	$\sin x = -\frac{1}{2}$ $x = \frac{7\pi}{6}$ $1 - \sin\frac{7\pi}{6} = \sqrt{3}\cos\frac{7\pi}{6}$ $1 - \left(-\frac{1}{2}\right) = \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right)$ $\frac{3}{2} = -\frac{3}{2}$	 $x = \frac{11\pi}{6}$ $1 - \sin\frac{11\pi}{6} = \sqrt{3}\cos\frac{11\pi}{6}$ $1 - \left(-\frac{1}{2}\right) = \sqrt{3}\left(\frac{\sqrt{3}}{2}\right)$ $\frac{3}{2} = \frac{3}{2}$ ✓

The solutions are: $x = \frac{\pi}{2},\ \frac{11\pi}{6}$

Exercise

Solve: $2\sin^2 x - \cos x - 1 = 0 \ \ if \ \ 0 \le x < 2\pi$

Solution

$$2\left(1-\cos^2 x\right) - \cos x - 1 = 0$$

$$2 - 2\cos^2 x - \cos x - 1 = 0$$

$$-2\cos^2 x - \cos x + 1 = 0$$

$\cos x = -1 \qquad\qquad \cos x = \frac{1}{2}$

$x = \pi \qquad\qquad x = \frac{\pi}{3},\ \frac{5\pi}{3}$

The solutions are: $x = \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}$

Exercise

Find the solutions of the equation that are in the interval $[0,\ 2\pi)$: $\sin x + \cos x \cot x = \csc x$

Solution

$\sin x + \cos x \frac{\cos x}{\sin x} = \frac{1}{\sin x}$ *Multiply by* **sinx** *both sides* $(\sin x \ne 0)$

$$\sin^2 x + \cos^2 x = 1$$

$1 = 1 \qquad$ ***(True)***

The solutions are: $x \in [0,\ 2\pi)$ except 0 and π.

Exercise

Find the solutions of the equation that are in the interval $[0,\ 2\pi)$: $2\sin^3 x + \sin^2 x - 2\sin x - 1 = 0$

Solution

$\sin^2 x(2\sin x + 1) - (2\sin x + 1) = 0$ *Factor by grouping*

$$(2\sin x + 1)\left(\sin^2 x - 1\right) = 0$$

$2\sin x + 1 = 0$ $\sin x = -\frac{1}{2}$	$\sin^2 x = 1$ $\sin x = \pm 1$

$\boxed{x = \frac{7\pi}{6}, \frac{11\pi}{6}}$	$\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2}}$

Exercise

Find the solutions of the equation that are in the interval $[0, 2\pi)$: $2\cos^2 t - 9\cos t = 5$

Solution

$$2\cos^2 t - 9\cos t - 5 = 0$$

$$(2\cos t + 1)(\cos t - 5) = 0$$

$$2\cos t + 1 = 0 \qquad \cos t - 5 = 0$$

$$\cos t = -\frac{1}{2} \qquad \cos t = 5$$

$$\cos t = -\frac{1}{2} \qquad \cos t = 5$$

$$\hat{t} = \cos^{-1}\left(-\frac{1}{2}\right) \qquad \textit{No solution}$$

$$\hat{t} = \frac{\pi}{3}$$

Negative sign $\rightarrow$ *cosine is in QII or QIII*

$$t = \pi - \frac{\pi}{3} \qquad t = \pi + \frac{\pi}{3}$$

$$t = \frac{2\pi}{3} \qquad t = \frac{4\pi}{3}$$

The solutions are: $\boxed{\frac{2\pi}{3}, \frac{4\pi}{3}}$

Exercise

Find the solutions of the equation that are in the interval $[0, 2\pi)$: $\tan^2 x + \tan x - 2 = 0$

Solution

$$\tan^2 x + \tan x - 2 = 0$$

$$\tan x = 1 \qquad \tan x = -2$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4} \qquad \hat{x} = \tan^{-1}(2) \approx 1.107 \qquad x \in QII,\ QIV$$

$$x = 2.034,\ 5.176$$

The solutions are: $\boxed{\frac{\pi}{4}, \frac{5\pi}{4}, 2.034, 5.176}$

Exercise

Find the solutions of the equation that are in the interval $[0,\ 2\pi)$: $\tan x+\sqrt{3}=\sec x$

Solution

$$\left(\tan x+\sqrt{3}\right)^2=(\sec x)^2$$

$$\tan^2 x+2\sqrt{3}\tan x+3=\sec^2 x$$

$$\tan^2 x+2\sqrt{3}\tan x+3=1+\tan^2 x$$

$$\tan^2 x+2\sqrt{3}\tan x+3-1-\tan^2 x=0$$

$$2\sqrt{3}\tan x+2=0$$

$$2\sqrt{3}\tan x=-2$$

$$\tan x=-\frac{2}{2\sqrt{3}}=-\frac{1}{\sqrt{3}}$$

$$x=\frac{5\pi}{6}\qquad or\qquad x=\frac{11\pi}{6}$$

$\tan\frac{5\pi}{6}+\sqrt{3}\overset{?}{=}\sec\frac{5\pi}{6}$ $-\frac{\sqrt{3}}{3}+\sqrt{3}\overset{?}{=}-\frac{2\sqrt{3}}{3}$ $\frac{2\sqrt{3}}{3}\neq-\frac{2\sqrt{3}}{3}$ ***False***	$\tan\frac{11\pi}{6}+\sqrt{3}\overset{?}{=}\sec\frac{11\pi}{6}$ $-\frac{\sqrt{3}}{3}+\sqrt{3}\overset{?}{=}\frac{2\sqrt{3}}{3}$ $\frac{2\sqrt{3}}{3}=\frac{2\sqrt{3}}{3}$

The solution is: $x=\frac{11\pi}{6}$

Exercise

Find the solutions of the equation that are in the interval $[0,\ 2\pi)$: $4\cos^2 x+4\sin x-5=0$

Solution

$$4\cos^2 x+4\sin x-5=0$$

$$4\left(1-\sin^2 x\right)+4\sin x-5=0$$

$$4-4\sin^2 x+4\sin x-5=0$$

$$-4\sin^2 x+4\sin x-1=0$$

$$4\sin^2 x-4\sin x+1=0$$

$$(2\sin x-1)^2=0$$

$\sin x = \frac{1}{2}$

The solutions are: $x = \frac{\pi}{6},\ \frac{5\pi}{6}$

Exercise

Find the amplitude, the period, the phase shift, and the vertical translation and sketch the graph of the equation $\quad y = 2\sin(x-\pi)$

Solution

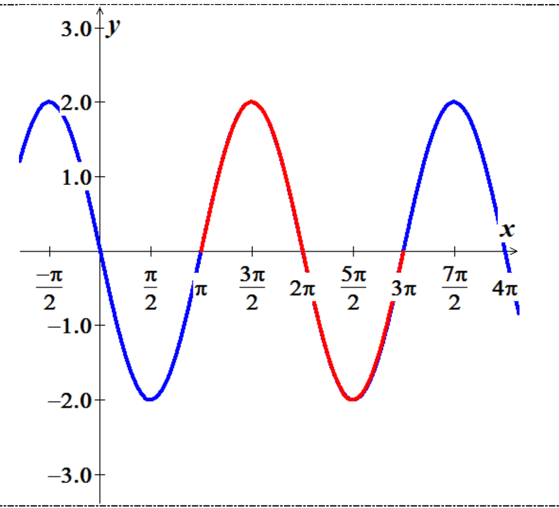
Amplitude: $|A| = 2$

Period: $P = 2\pi$

Phase Shift: $\phi = \pi$

VT: $y = 0$

	x	y
$0+\pi$	π	0
$\frac{1}{2}\pi+\pi$	$\frac{3\pi}{2}$	2
$\frac{3}{2}\pi+\pi$	2π	0
$\frac{3}{4}\pi+\pi$	$\frac{5\pi}{2}$	-2
$2\pi+\pi$	3π	0



Exercise

Find the amplitude, the period, the phase shift, and the vertical translation and sketch the graph of the equation $\quad y = \frac{2}{3}\sin\left(x+\frac{\pi}{2}\right)$

Solution

Amplitude: $|A| = \frac{2}{3}$

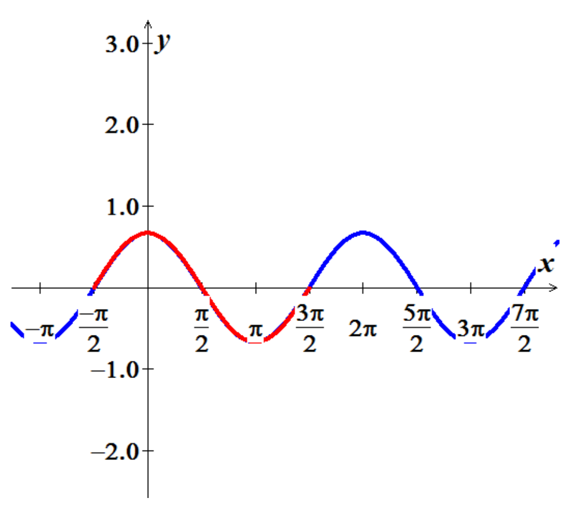
Period: $P = 2\pi$

Phase Shift:

$\phi = -\frac{\pi}{2}$

VT: $y = 0$

	x	y
$0-\frac{\pi}{2}$	$-\frac{\pi}{2}$	0
$\frac{1}{2}\pi-\frac{\pi}{2}$	0	$\frac{2}{3}$
$\frac{3}{2}\pi-\frac{\pi}{2}$	$\frac{\pi}{2}$	0
$\frac{3}{4}\pi-\frac{\pi}{2}$	π	$-\frac{2}{3}$
$2\pi-\frac{\pi}{2}$	$\frac{3\pi}{2}$	0



Exercise

Find the amplitude, the period, the phase shift, and the vertical translation and sketch the graph of the equation $y = 2 - \sin\left(3x - \frac{\pi}{5}\right)$

Solution

Amplitude: $|A| = 1$

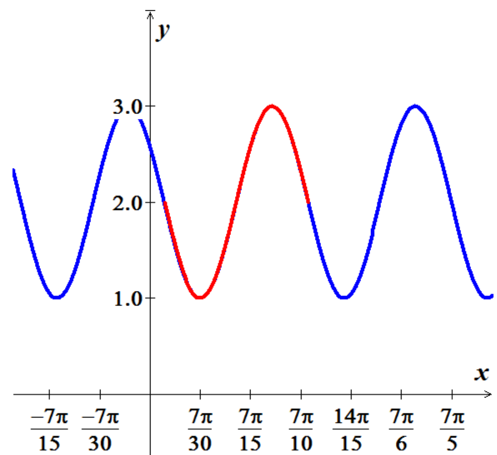
Period: $P = \frac{2\pi}{3}$

Phase Shift:

$\phi = \frac{\pi}{15}$

VT: $y = 2$

	x	y
$0 + \frac{\pi}{15}$	$\frac{\pi}{15}$	2
$\frac{\pi}{6} + \frac{\pi}{15}$	$\frac{7\pi}{30}$	1
$\frac{\pi}{3} + \frac{\pi}{15}$	$\frac{6\pi}{15}$	2
$\frac{\pi}{2} + \frac{\pi}{15}$	$\frac{17\pi}{30}$	3
$\frac{2\pi}{3} + \frac{\pi}{15}$	$\frac{11\pi}{15}$	2



Exercise

Find the amplitude, the period, and the phase shift and sketch the graph of the equation

$y = 2\sin\left(x - \frac{\pi}{3}\right)$

Solution

Amplitude: $|A| = 2$

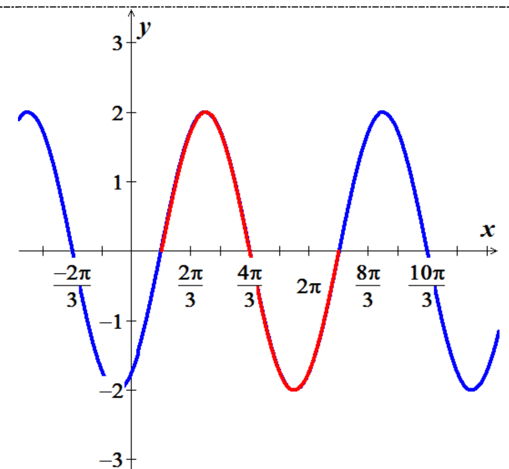
Period: $P = 2\pi$

Phase Shift:

$\phi = \frac{\pi}{3}$

VT: $y = 0$

	x	y
$0 + \frac{\pi}{3}$	$\frac{\pi}{3}$	0
$\frac{\pi}{2} + \frac{\pi}{3}$	$\frac{5\pi}{6}$	2
$\pi + \frac{\pi}{3}$	$\frac{4\pi}{3}$	0
$\frac{3\pi}{2} + \frac{\pi}{3}$	$\frac{11\pi}{6}$	−2
$2\pi + \frac{\pi}{3}$	$\frac{7\pi}{3}$	0



Exercise

Find the amplitude, the period, the phase shift, and the vertical translation and sketch the graph of the equation $y=-\frac{2}{3}\sin\left(3x-\frac{\pi}{2}\right)$

Solution

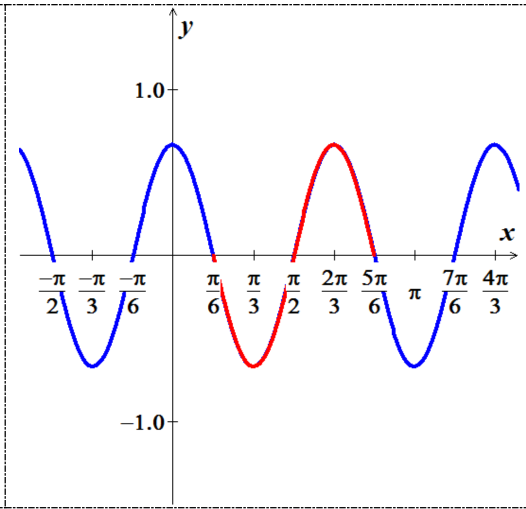
Amplitude: $|A|=\frac{2}{3}$

Period: $P=\frac{2\pi}{3}$

Phase Shift: $\phi=\frac{\pi}{6}$

VT: $y=0$

	x	y
$0+\frac{\pi}{6}$	$\frac{\pi}{6}$	0
$\frac{\pi}{6}+\frac{\pi}{6}$	$\frac{\pi}{3}$	$-\frac{2}{3}$
$\frac{\pi}{3}+\frac{\pi}{6}$	$\frac{\pi}{2}$	0
$\frac{\pi}{2}+\frac{\pi}{6}$	$\frac{2\pi}{3}$	$\frac{2}{3}$
$\frac{2\pi}{3}+\frac{\pi}{6}$	$\frac{5\pi}{6}$	0



Exercise

Find the amplitude, the period, and the phase shift and sketch the graph of the equation

$y=-2\sin(2\pi x+\pi)$

Solution

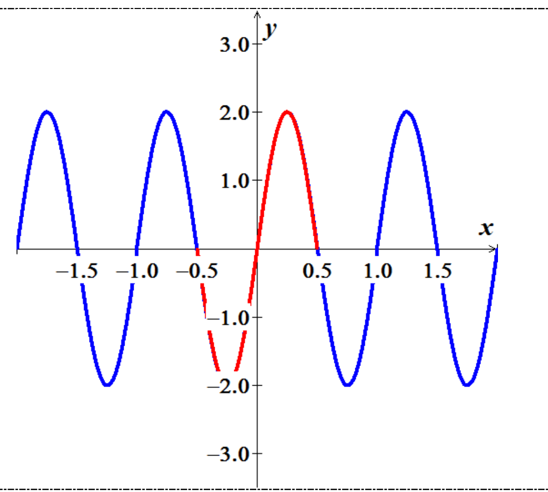
Amplitude: $|A|=2$

Period: $P=\frac{2\pi}{2\pi}=1$

Phase Shift: $\varphi=-\frac{1}{2}$

VT: $y=0$

	x	y
$0-\frac{1}{2}$	$-\frac{1}{2}$	0
$\frac{1}{4}-\frac{1}{2}$	$-\frac{1}{4}$	-2
$\frac{1}{2}-\frac{1}{2}$	0	0
$\frac{3}{4}-\frac{1}{2}$	$\frac{1}{4}$	2
$1-\frac{1}{2}$	$\frac{1}{2}$	0



Exercise

Find the amplitude, the period, the phase shift, and the vertical translation and sketch the graph of the equation $y = 3\cos\left[\frac{\pi}{2}\left(x - \frac{1}{2}\right)\right]$

Solution

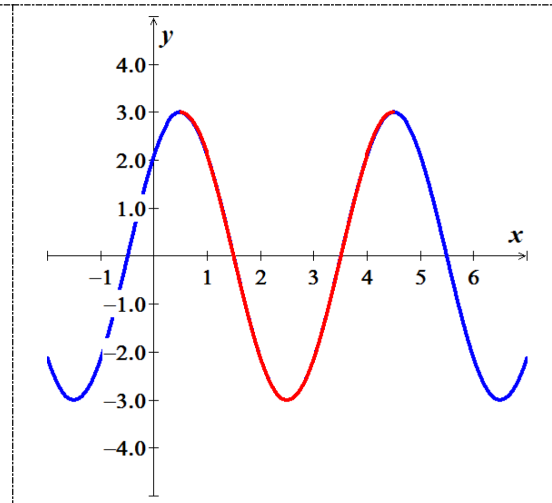
Amplitude: $|A| = 3$

Period: $P = \dfrac{2\pi}{\frac{\pi}{2}} = 4$

Phase Shift: $\varphi = \frac{1}{2}$

VT: $y = 0$

	x	y
$0+\frac{1}{2}$	$\frac{1}{2}$	3
$1+\frac{1}{2}$	$\frac{3}{2}$	0
$2+\frac{1}{2}$	$\frac{5}{2}$	-3
$3+\frac{1}{2}$	$\frac{7}{2}$	0
$4+\frac{1}{2}$	$\frac{9}{2}$	3



Exercise

Find the amplitude, the period, the phase shift, and the vertical translation and sketch the graph of the equation $y = -\cos\pi\left(x - \frac{1}{3}\right)$

Solution

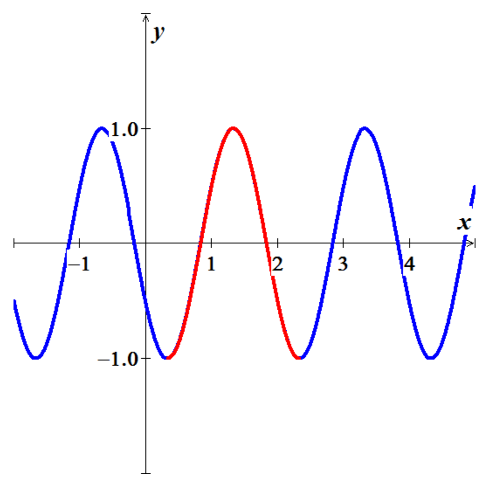
Amplitude: $|A| = 1$

Period: $P = 2$

Phase Shift: $\phi = \frac{1}{3}$

VT: $y = 0$

	x	y
$0+\frac{1}{3}$	$\frac{1}{3}$	-1
$\frac{1}{2}+\frac{1}{3}$	$\frac{5}{6}$	0
$1+\frac{1}{3}$	$\frac{4}{3}$	1
$\frac{3}{2}+\frac{1}{3}$	$\frac{11}{6}$	0
$2+\frac{1}{3}$	$\frac{7}{3}$	-1



Exercise

Find the amplitude, the period, the phase shift, and the vertical translation and sketch the graph of the equation $y = \frac{5}{2} - 3\cos\left(\pi x - \frac{\pi}{4}\right)$

Solution

Amplitude: $|A| = 3$

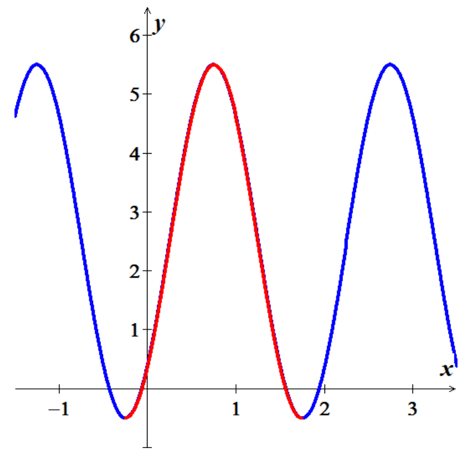
Period: $P = \frac{2\pi}{\pi} = 2$

Phase Shift:

$$\phi = -\frac{-\frac{\pi}{4}}{\pi} = \frac{1}{4}$$

VT: $y = \frac{5}{2}$

	x	y
$0 - \frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$
$\frac{1}{2} - \frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{2}$
$1 - \frac{1}{4}$	$\frac{3}{4}$	$\frac{11}{2}$
$\frac{3}{2} - \frac{1}{4}$	$\frac{5}{4}$	$\frac{5}{2}$
$2 - \frac{1}{4}$	$\frac{7}{4}$	$-\frac{1}{2}$



Exercise

Graph one complete cycle $y = -3 + \sin\left(\pi x + \frac{\pi}{2}\right)$

Solution

Amplitude: $|A| = 1$

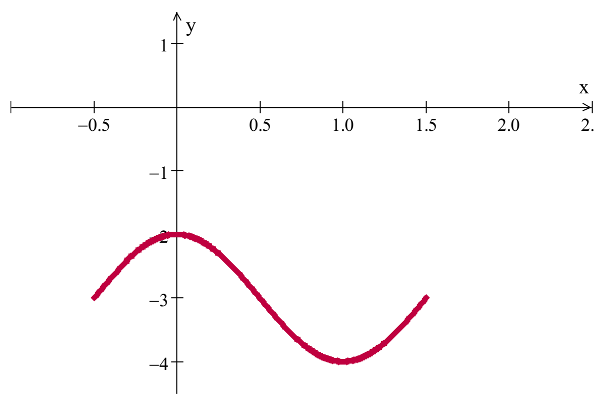
Period: $P = 2$

Phase Shift:

$$\varphi = -\frac{1}{2}$$

VT: $y = -3$

	x	y
$0 - \frac{1}{2}$	$-\frac{1}{2}$	-3
$\frac{1}{2} - \frac{1}{2}$	0	-2
$1 - \frac{1}{2}$	$\frac{1}{2}$	-3
$\frac{3}{2} - \frac{1}{2}$	1	-4
$2 - \frac{1}{2}$	$\frac{3}{2}$	-3



Exercise

Find the amplitude, the period, the phase shift, and the vertical translation and sketch the graph of the equation $y = 4\cos\left(\frac{1}{2}x + \frac{\pi}{2}\right)$

Solution

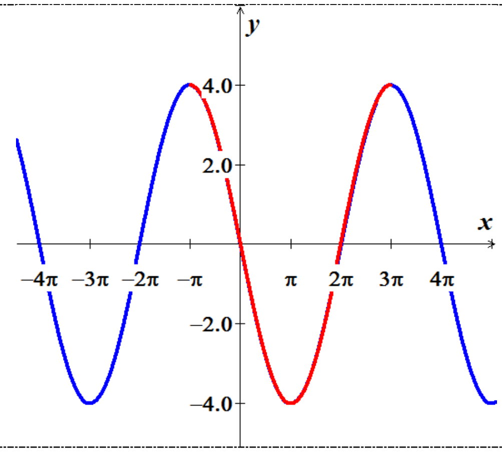
Amplitude: $|A| = 4$

Period: $P = 4\pi$

Phase Shift: $\phi = -\pi$

VT: $y = 0$

	x	y
$0-\pi$	$-\pi$	4
$\pi-\pi$	0	0
$2\pi-\pi$	π	-4
$3\pi-\pi$	2π	0
$4\pi-\pi$	3π	4



Exercise

Find the amplitude, the period, and the phase shift and sketch the graph of the equation
$y = -2\sin(2x - \pi) + 3$

Solution

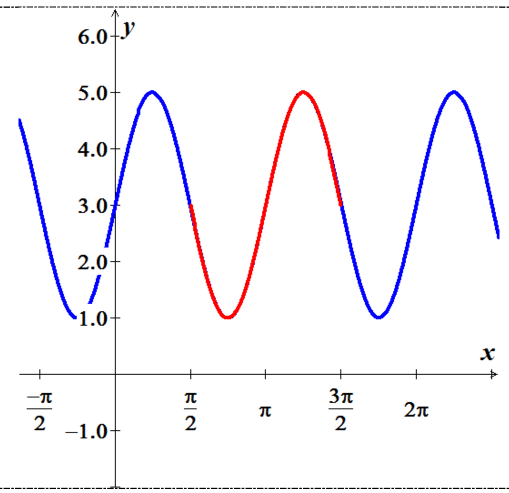
Amplitude: $|A| = 2$

Period: $P = \frac{2\pi}{2} = \pi$

Phase Shift: $\varphi = \frac{\pi}{2}$

VT: $y = 3$

	x	y
$0+\frac{\pi}{2}$	$\frac{\pi}{2}$	3
$\frac{\pi}{4}+\frac{\pi}{2}$	$\frac{3\pi}{4}$	1
$\frac{\pi}{2}+\frac{\pi}{2}$	π	3
$\frac{3\pi}{4}+\frac{\pi}{2}$	$\frac{5\pi}{4}$	5
$\pi+\frac{\pi}{2}$	$\frac{3\pi}{2}$	3



Exercise

Find the amplitude, the period, and the phase shift and sketch the graph of the equation
$y = 3\cos(x + 3\pi) - 2$

Solution

Amplitude: $|A| = 3$

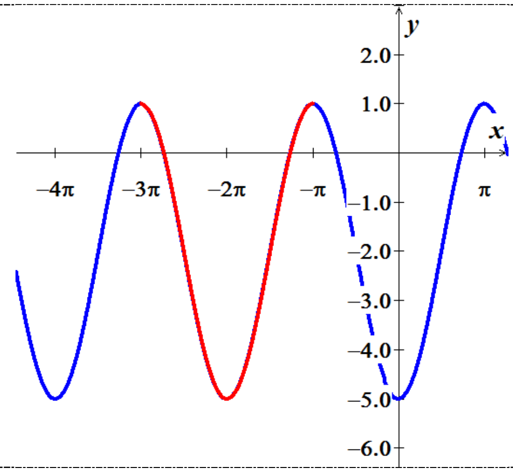
Period: $P = 2\pi$

Phase Shift:

$\varphi = -3\pi$

VT: $y = -2$

	x	y
$0 - 3\pi$	-3π	1
$\frac{\pi}{2} - 3\pi$	$-\frac{5\pi}{2}$	0
$\pi - 3\pi$	-2π	-5
$\frac{3\pi}{2} - 3\pi$	$-\frac{3\pi}{2}$	0
$2\pi - 3\pi$	$-\pi$	1



Exercise

Find the period, show the asymptotes, and sketch the graph of $y = 2\tan\left(2x + \frac{\pi}{2}\right)$

Solution

Amplitude: n/a

Period: $P = \frac{\pi}{2}$

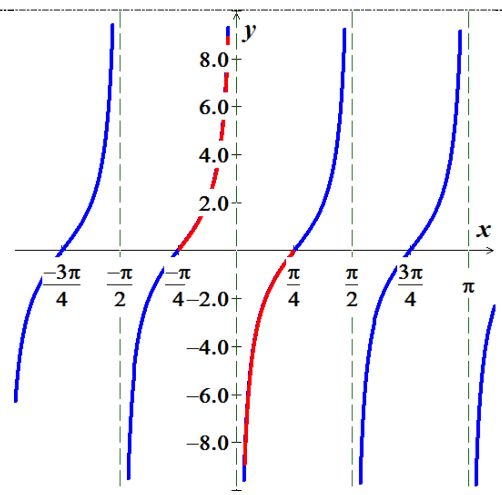
Phase Shift: $\varphi = -\frac{\pi}{4}$

VT: $y = 0$

Asymptotes: $x = n\frac{\pi}{2}$

$x = \ldots, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \ldots$

	x	y
$0 - \frac{\pi}{4}$	$-\frac{\pi}{4}$	0
$\frac{\pi}{8} - \frac{\pi}{4}$	$-\frac{\pi}{8}$	2
$\frac{\pi}{4} - \frac{\pi}{4}$	0	∞
$\frac{3\pi}{8} - \frac{\pi}{4}$	$\frac{\pi}{8}$	-2
$\frac{\pi}{2} - \frac{\pi}{4}$	$\frac{\pi}{4}$	0



Exercise

Find the period, show the asymptotes, and sketch the graph of $y=-\frac{1}{4}\tan\left(\frac{1}{2}x+\frac{\pi}{3}\right)$

Solution

Amplitude: n/a

Period: $P=2\pi$

Phase Shift:

$\varphi=-\frac{2\pi}{3}$

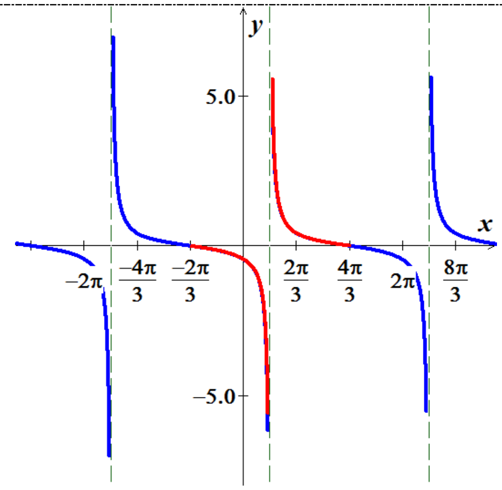
VT: $y=0$

Asymptotes:

$x=-\frac{5\pi}{3}+2n\pi$

$x=\ldots,-\frac{5\pi}{3},\ \frac{\pi}{3},\ \frac{7\pi}{3},\ldots$

	x	y
$0-\frac{2\pi}{3}$	$-\frac{2\pi}{3}$	0
$\frac{\pi}{2}-\frac{2\pi}{3}$	$-\frac{\pi}{6}$	$-\frac{1}{4}$
$\pi-\frac{2\pi}{3}$	$\frac{\pi}{3}$	∞
$\frac{3\pi}{2}-\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\frac{1}{4}$
$2\pi-\frac{2\pi}{3}$	$\frac{4\pi}{3}$	0



Exercise

Find the period, show the asymptotes, and sketch the graph of $y=2\cot\left(2x+\frac{\pi}{2}\right)$

Solution

Amplitude: n/a

Period: $P=\frac{\pi}{2}$

Phase Shift: $\varphi=-\frac{\pi}{4}$

VT: $y=0$

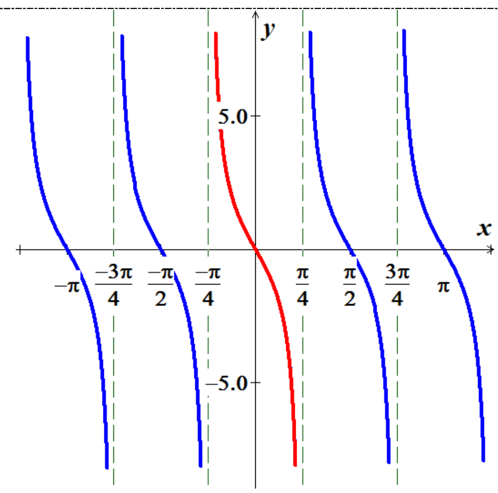
Asymptotes:

$2x+\frac{\pi}{2}=(2n+1)\pi$

$2x=2n\pi+\frac{\pi}{2}$

$x=\ldots,-\frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{7\pi}{4},\ldots$

	x	y
$0-\frac{\pi}{4}$	$-\frac{\pi}{4}$	∞
$\frac{\pi}{8}-\frac{\pi}{4}$	$-\frac{\pi}{8}$	2
$\frac{\pi}{4}-\frac{\pi}{4}$	0	0
$\frac{3\pi}{8}-\frac{\pi}{4}$	$\frac{\pi}{8}$	-2
$\frac{\pi}{2}-\frac{\pi}{4}$	$\frac{\pi}{4}$	∞



Exercise

Find the period, show the asymptotes, and sketch the graph of $y=-\frac{1}{2}\cot\left(\frac{1}{2}x+\frac{\pi}{4}\right)$

Solution

Amplitude: n/a

Period: $P=2\pi$

Phase Shift: $\varphi=-\frac{\pi}{2}$

VT: $y=0$

Asymptotes:

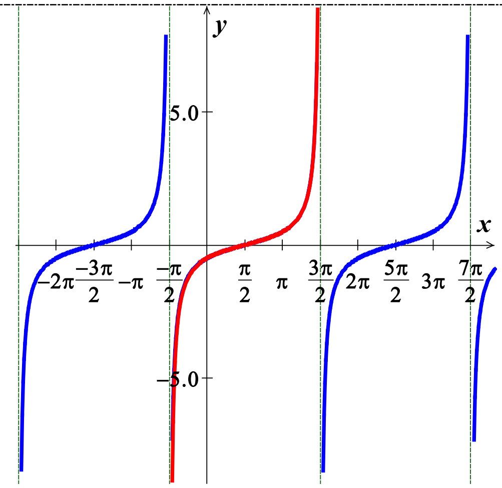
$\frac{1}{2}x+\frac{\pi}{4}=2n\pi+\pi$

$\frac{1}{2}x=2n\pi+\frac{3\pi}{4}$

$x=4n\pi+\frac{3\pi}{2}$

$x=\ldots,-\frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{11\pi}{2},$

	x	y
$0-\frac{\pi}{2}$	$-\frac{\pi}{2}$	∞
$\frac{\pi}{2}-\frac{\pi}{2}$	0	$-\frac{1}{2}$
$\pi-\frac{\pi}{2}$	$\frac{\pi}{2}$	0
$\frac{3\pi}{2}-\frac{\pi}{2}$	π	$\frac{1}{2}$
$2\pi-\frac{\pi}{2}$	$\frac{3\pi}{2}$	∞



Exercise

Graph over a 1-period interval $y=1-2\cot 2\left(x+\frac{\pi}{2}\right)$

Solution

Amplitude: n/a

Period: $P=\frac{\pi}{2}$

Phase Shift:

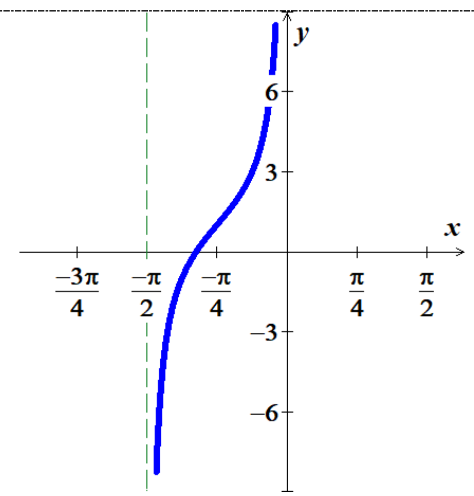
$\varphi=-\frac{\pi}{2}$

VT: $y=1$

Asymptotes:

$-\frac{\pi}{2}+n\pi$

	x	y
$0-\frac{\pi}{2}$	$-\frac{\pi}{2}$	∞
$\frac{\pi}{8}-\frac{\pi}{2}$	$-\frac{3\pi}{8}$	-1
$\frac{\pi}{4}-\frac{\pi}{2}$	$-\frac{\pi}{4}$	1
$\frac{3\pi}{8}-\frac{\pi}{2}$	$-\frac{\pi}{8}$	3
$\frac{\pi}{2}-\frac{\pi}{2}$	0	∞



Exercise

Graph one complete cycle $y=3+2\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)$

Solution

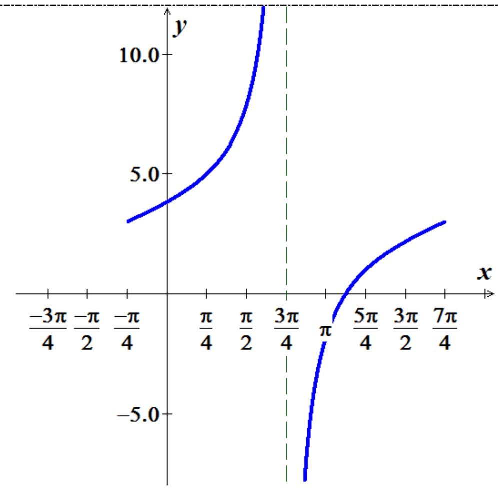
Amplitude: n/a

Period: $P=2\pi$

Phase Shift: $\varphi=-\frac{\pi}{4}$

VT: $y=3$

	x	y
$0-\frac{\pi}{4}$	$-\frac{\pi}{4}$	3
$\frac{\pi}{2}-\frac{\pi}{4}$	$\frac{\pi}{4}$	5
$\pi-\frac{\pi}{4}$	$\frac{3\pi}{4}$	∞
$\frac{3\pi}{2}-\frac{\pi}{4}$	$\frac{5\pi}{4}$	1
$2\pi-\frac{\pi}{4}$	$\frac{7\pi}{4}$	3



Exercise

Graph over a one-period interval $y=1-\frac{1}{2}\csc\left(x-\frac{3\pi}{4}\right)$

Solution

Amplitude: n/a

Period: $P=2\pi$

Phase Shift:

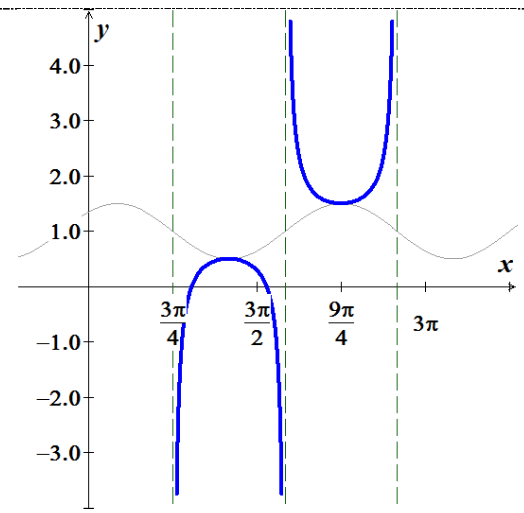
$\varphi=\frac{3\pi}{4}$

VT: $y=1$

Asymptotes:

$x=\frac{3\pi}{4}+2\pi n$

	x	$4\sin\left(\frac{1}{2}x-\frac{\pi}{4}\right)$
$0+\frac{3\pi}{4}$	$\frac{3\pi}{4}$	1
$\frac{\pi}{2}+\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{1}{2}$
$\pi+\frac{3\pi}{4}$	$\frac{7\pi}{4}$	1
$\frac{3\pi}{2}+\frac{3\pi}{4}$	$\frac{9\pi}{4}$	$-\frac{1}{2}$
$2\pi+\frac{3\pi}{4}$	$\frac{11\pi}{4}$	1



Exercise

Graph over a one-period interval $y = 2 + \frac{1}{4}\sec\left(\frac{1}{2}x - \pi\right)$

Solution

Amplitude: n/a

Period:

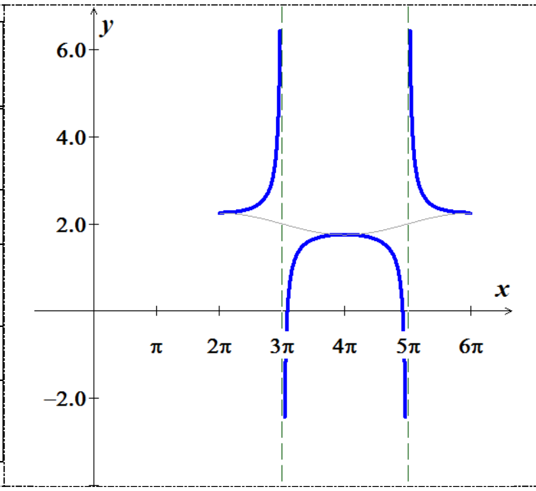
$P = 4\pi$

Phase Shift:

$\varphi = 2\pi$

VT: $y = 2$

	x	$2+\frac{1}{4}\cos\left(\frac{1}{2}x-\pi\right)$
$0+2\pi$	2π	$\frac{9}{4}$
$\pi+2\pi$	3π	2
$2\pi+2\pi$	4π	$\frac{7}{4}$
$3\pi+2\pi$	5π	2
$4\pi+2\pi$	6π	$\frac{9}{4}$



Exercise

Find the period, show the asymptotes, and sketch the graph of $y = -3\sec\left(\frac{1}{3}x + \frac{\pi}{3}\right)$

Solution

Amplitude: n/a

Period: $P = 6\pi$

Phase Shift: $\varphi = -\pi$

VT: $y = 0$

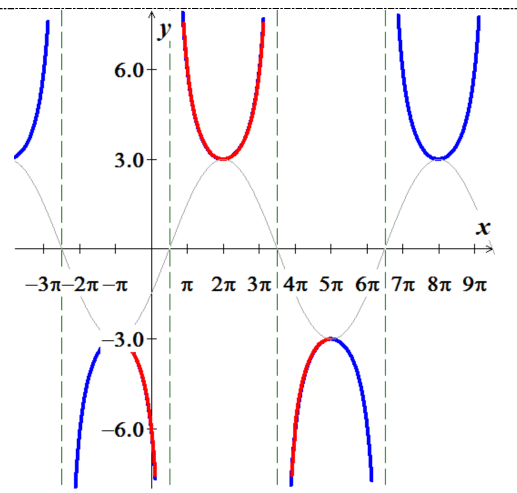
Asymptotes:

$$\frac{1}{3}x + \frac{\pi}{3} = n\pi + \frac{\pi}{2}$$

$$x = 3n\pi + \frac{\pi}{2}$$

$$x = \ldots,\ -\frac{5\pi}{2},\ \frac{\pi}{2},\ \frac{7\pi}{2},\ \ldots$$

	x	$-3\cos\left(\frac{1}{3}x+\frac{\pi}{3}\right)$
$0-\pi$	$-\pi$	-3
$\frac{3\pi}{2}-\pi$	$\frac{\pi}{2}$	0
$3\pi-\pi$	2π	3
$\frac{9\pi}{2}-\pi$	$\frac{7\pi}{2}$	0
$6\pi-\pi$	5π	-3



Exercise

Find the period, show the asymptotes, and sketch the graph of $y = 2\sec\left(2x - \frac{\pi}{2}\right)$

Solution

Amplitude: n/a

Period:

$P = \pi$

Phase Shift:

$\varphi = \frac{\pi}{4}$

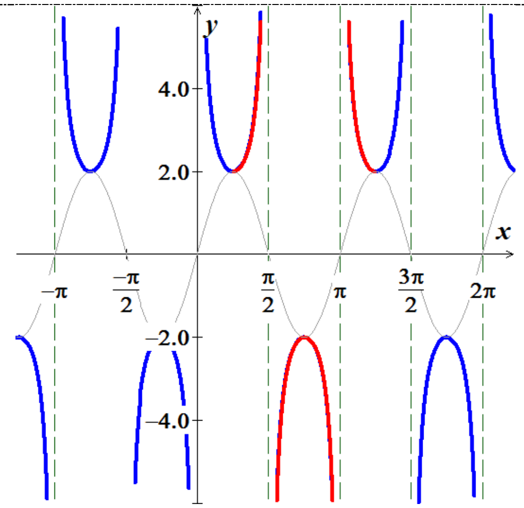
VT: $y = 0$

Asymptotes:

$2x - \frac{\pi}{2} = n\pi + \frac{\pi}{2}$

$x = \left(\frac{n+1}{2}\right)\pi$

	x	$2\cos\left(2x - \frac{\pi}{2}\right)$
$0 + \frac{\pi}{4}$	$\frac{\pi}{4}$	2
$\frac{\pi}{4} + \frac{\pi}{4}$	$\frac{\pi}{2}$	0
$\frac{\pi}{2} + \frac{\pi}{4}$	$\frac{3\pi}{4}$	-2
$\frac{3\pi}{4} + \frac{\pi}{4}$	2π	0
$\pi + \frac{\pi}{4}$	$\frac{5\pi}{4}$	2



Exercise

Convert to rectangular coordinates. $(4,\ 30°)$

Solution

$x = r\cos\theta$

$= 4\cos 30°$

$= 4\left(\frac{\sqrt{3}}{2}\right)$

$\underline{= 2\sqrt{3}}$

$y = r\sin\theta$

$= 4\sin 30°$

$= 4\left(\frac{1}{2}\right)$

$\underline{= 2}$

$\therefore$ The point $\left(2\sqrt{3},\ 2\right)$ in rectangular coordinates is equivalent to $\left(4,\ 30°\right)$ in polar coordinates.

Exercise

Convert to rectangular coordinates $\left(-\sqrt{2},\ \frac{3\pi}{4}\right)$.

Solution

$$x = -\sqrt{2}\cos\frac{3\pi}{4}$$
$$= -\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right)$$
$$= 1$$

$$y = -\sqrt{2}\sin\frac{3\pi}{4}$$
$$= -\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)$$
$$= -1$$

∴ The point $(1,\ -1)$ in rectangular coordinates is equivalent to $\left(-\sqrt{2},\ \frac{3\pi}{4}\right)$ in polar coordinates.

Exercise

Convert to rectangular coordinates $(3,\ 270°)$.

Solution

$$x = 3\cos 270°$$
$$= 3(0)$$
$$= 0$$

$$y = 3\sin 270°$$
$$= 3(-1)$$
$$= -3$$

∴ The point $(3,\ 270°)$ in polar coordinates is equivalent to $(0,\ -3)$ in rectangular coordinates.

Exercise

Convert to rectangular coordinates $(2,\ 60°)$

Solution

$$x = 2\cos 60°$$
$$= 2\left(\frac{1}{2}\right)$$
$$= 0$$

$$y = 2\sin 60^\circ$$
$$= 2\frac{\sqrt{3}}{2}$$
$$= \sqrt{3}$$

$\therefore$ The point $(2,\ 60^\circ)$ in polar coordinates is equivalent to $(1,\ \sqrt{3})$ in rectangular coordinates.

Exercise

Convert to rectangular coordinates $\left(\sqrt{2},\ -225^\circ\right)$

Solution

$$x = \sqrt{2}\cos(-225^\circ)$$
$$= \sqrt{2}\left(-\frac{1}{\sqrt{2}}\right)$$
$$= -1$$

$$y = \sqrt{2}\sin(-225^\circ)$$
$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\right)$$
$$= 1$$

$\therefore$ The point $\left(\sqrt{2},\ -225^\circ\right)$ in polar coordinates is equivalent to $(-1,\ 1)$ in rectangular coordinates.

Exercise

Convert to rectangular coordinates $\left(4\sqrt{3},\ -\frac{\pi}{6}\right)$

Solution

$$x = 4\sqrt{3}\cos\left(-\frac{\pi}{6}\right)$$
$$= 4\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)$$
$$= 6$$

$$y = 4\sqrt{3}\sin\left(-\frac{\pi}{6}\right)$$
$$= 4\sqrt{3}\left(-\frac{1}{2}\right)$$
$$= -2\sqrt{3}$$

$\therefore$ The point $\left(4\sqrt{3},\ -\frac{\pi}{6}\right)$ in polar coordinates is equivalent to $\left(6,\ -2\sqrt{3}\right)$ in rectangular coordinates.

Exercise

Convert to polar coordinates $(3,\ 3)$.

Solution

$$\begin{aligned} r &= \sqrt{3^2+3^2} \\ &= \sqrt{18} \\ &= 3\sqrt{2} \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{3}{3}\right) \\ &= \tan^{-1}(1) \\ &= 45^\circ \end{aligned}$$

$\therefore$ The point $(3,\ 3)$ in rectangular coordinates is equivalent to $\left(3\sqrt{2},\ 45^\circ\right)$ in polar coordinates.

Exercise

Convert to polar coordinates $(-2,\ 0)$.

Solution

$$\begin{aligned} r &= \pm\sqrt{4+0} \\ &= \pm\ 2 \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\frac{0}{-2} \\ &= 0^\circ \end{aligned}$$

$\therefore$ The point $(-2,\ 0)$ in rectangular coordinates is equivalent to $(-2,\ 0^\circ)$ $\ (2,\ 180^\circ)$ in polar coordinates.

Exercise

Convert to polar coordinates $(-1,\ \sqrt{3})$.

Solution

$$\begin{aligned} r &= \pm\sqrt{1+3} \\ &= \pm\ 2 \end{aligned}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$$
$$\underline{= 120^\circ}$$

$\therefore$ The point $\left(-1,\ \sqrt{3}\right)$ in rectangular coordinates is equivalent to $\left(2,\ 120^\circ\right)$ in polar coordinates.

Exercise

Convert to polar coordinates $\quad (-3,\ -3)\quad r \ge 0 \quad 0^\circ \le \theta < 360^\circ$

Solution

$$r = \sqrt{(-3)^2 + (-3)^2}$$
$$\underline{= 3\sqrt{2}}$$

$$\hat{\theta} = \tan^{-1}\left(\frac{3}{3}\right)$$
$$= \tan^{-1}(1)$$
$$\underline{= 45^\circ}$$

The angle is in quadrant III

Therefore, $\theta = 180^\circ + 45^\circ$

$$\underline{= 225^\circ}$$

$\therefore$ The point $(-3,\ 3)$ in rectangular coordinates is equivalent to $\left(3\sqrt{2},\ 225^\circ\right)$ in polar coordinates.

Exercise

Convert to polar coordinates $\quad \left(2,\ -2\sqrt{3}\right)\quad r \ge 0 \quad 0^\circ \le \theta < 360^\circ$

Solution

$$r = \sqrt{2^2 + \left(-2\sqrt{3}\right)^2}$$
$$\underline{= 4}$$

$$\hat{\theta} = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)$$
$$= \tan^{-1}\left(\sqrt{3}\right)$$
$$\underline{= 60^\circ}$$

The angle is in quadrant IV

Therefore, $\theta = 360° - 60°$

$= 300°$

$\therefore$ The point $\left(2,\ -2\sqrt{3}\right)$ in rectangular coordinates is equivalent to $(4,\ 300°)$ in polar coordinates.

Exercise

Convert to polar coordinates $\quad (-2,\ 0) \quad r \geq 0 \quad 0 \leq \theta < 2\pi$

Solution

$$r = \sqrt{(-2)^2 + 0^2}$$

$$= 2$$

$$\hat{\theta} = \tan^{-1}\left(\frac{0}{2}\right)$$

$$= 0$$

$$\theta = \pi$$

$\therefore$ The point $(-2,\ 0)$ in rectangular coordinates is equivalent to $(2,\ \pi)$ in polar coordinates.

Exercise

Write the equation in rectangular coordinates $r^2 = 4$

Solution

$$r^2 = 4$$

$$x^2 + y^2 = 4$$

Exercise

Write the equation in rectangular coordinates $r = 6\cos\theta$

Solution

$$r = 6\cos\theta$$

$$r = 6\frac{x}{r}$$

$$r^2 = 6x$$

$$x^2 + y^2 = 6x$$

Exercise

Write the equation in rectangular coordinates $r^2 = 4\cos 2\theta$

Solution

$$r^2 = 4\cos 2\theta$$

$$= 4\left(\cos^2\theta - \sin^2\theta\right) \qquad \cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r}$$

$$= 4\left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\right)$$

$$= 4\left(\frac{x^2 - y^2}{r^2}\right)$$

$$r^4 = 4\left(x^2 - y^2\right) \qquad r^2 = x^2 + y^2$$

$$\left(x^2 + y^2\right)^4 = 4x^2 - 4y^2$$

Exercise

Write the equation in rectangular coordinates $r(\cos\theta - \sin\theta) = 2$

Solution

$$r(\cos\theta - \sin\theta) = 2 \qquad \cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r}$$

$$r\left(\frac{x}{r} - \frac{y}{r}\right) = 2$$

$$r\left(\frac{x - y}{r}\right) = 2$$

$$x - y = 2$$

Exercise

Write the equation in rectangular coordinates $r^2 = 4\sin 2\theta$

Solution

$$r^2 = 4\sin 2\theta \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= 4(2\sin\theta\cos\theta) \qquad \cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r}$$

$$= 8\left(\frac{y}{r}\right)\left(\frac{x}{r}\right)$$

$$= 8\frac{xy}{r^2}$$

$$r^4 = 8xy \qquad\qquad r^2 = x^2 + y^2$$

$$\boxed{\left(x^2 + y^2\right)^2 = 8xy}$$

Exercise

Find an equation in x and y that has the same graph as polar equation. $r\sin\theta = -2$

Solution

$$r\sin\theta = -2 \qquad\qquad y = r\sin\theta$$

$$\boxed{y = -2}$$

Exercise

Find an equation in x and y that has the same graph as polar equation. $\theta = \frac{\pi}{4}$

Solution

$$\tan\theta = \tan\frac{\pi}{4}$$

$$\frac{y}{x} = 1$$

$$\boxed{y = x}$$

Exercise

Find an equation in x and y that has the same graph as polar $r^2\left(4\sin^2\theta - 9\cos^2\theta\right) = 36$

Solution

$$r^2\left(4\sin^2\theta - 9\cos^2\theta\right) = 36 \qquad\qquad \cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r}$$

$$r^2\left(4\frac{y^2}{r^2} - 9\frac{x^2}{r^2}\right) = 36$$

$$r^2\left(\frac{4y^2 - 9x^2}{r^2}\right) = 36$$

$$\boxed{4y^2 - 9x^2 = 36}$$

Exercise

Find an equation in x and y that has the same graph as polar $r^2\left(\cos^2\theta + 4\sin^2\theta\right) = 16$

Solution

$$r^2\left(\cos^2\theta + 4\sin^2\theta\right) = 16 \qquad \cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r}$$

$$r^2\left(\frac{x^2}{r^2} + 4\frac{y^2}{r^2}\right) = 16$$

$$r^2\left(\frac{x^2 + 4y^2}{r^2}\right) = 16$$

$$\underline{x^2 + 4y^2 = 16}$$

Exercise

Find an equation in x and y that has the same graph as polar $r(\sin\theta - 2\cos\theta) = 6$

Solution

$$r(\sin\theta - 2\cos\theta) = 6 \qquad \cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r}$$

$$r\left(\frac{y}{r} - 2\frac{x}{r}\right) = 6$$

$$r\left(\frac{y - 2x}{r}\right) = 6$$

$$\underline{y - 2x = 6}$$

Exercise

Find an equation in x and y that has the same graph as polar $r = 8\sin\theta - 2\cos\theta$

Solution

$$r = 8\sin\theta - 2\cos\theta \qquad \cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r}$$

$$r = 8\frac{y}{r} - 2\frac{x}{r}$$

$$r^2 = 8y - 2x \qquad r^2 = x^2 + y^2$$

$$\underline{x^2 + y^2 = 8y - 2x}$$

Exercise

Find an equation in x and y that has the same graph as polar $r = \tan\theta$

Solution

$$r = \tan\theta$$

$$x^2 + y^2 = \frac{y^2}{x^2}$$

$$x^4 + x^2y^2 = y^2$$

$$\boxed{\sqrt{x^2+y^2} = \frac{y}{x}}$$

Exercise

Find an equation in x and y that has the same graph as polar $r\left(\sin\theta + r\cos^2\theta\right) = 1$

Solution

$$r\left(\sin\theta + r\cos^2\theta\right) = 1 \qquad \cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r}$$

$$r\left(\frac{y}{r} + r\frac{x^2}{r^2}\right) = 1$$

$$r\left(\frac{y}{r} + \frac{x^2}{r}\right) = 1$$

$$r\left(\frac{y+x^2}{r}\right) = 1$$

$$\boxed{y + x^2 = 1}$$

Exercise

Find a polar equation that has the same graph as the equation in x and y. $y^2 = 6x$

Solution

$$y^2 = 6x \qquad x = r\cos\theta \quad y = r\sin\theta$$

$$(r\sin\theta)^2 = 6(r\cos\theta)$$

$$r^2\sin^2\theta = 6r\cos\theta$$

$$\boxed{r = 6\frac{\cos\theta}{\sin^2\theta}}$$

Exercise

Find a polar equation that has the same graph as the equation in x and y. $xy = 8$

Solution

$xy = 8$ $\qquad x = r\cos\theta \quad y = r\sin\theta$

$(r\cos\theta)(r\sin\theta) = 8$

$$r^2 = \frac{8}{\cos\theta\sin\theta}$$

Exercise

Write the equation in polar coordinates $x + y = 5$

Solution

$x + y = 5$

$r\cos\theta + r\sin\theta = 5$ $\qquad x = r\cos\theta \quad y = r\sin\theta$

$r(\cos\theta + \sin\theta) = 5$

$$r = \frac{5}{\cos\theta + \sin\theta}$$

Exercise

Write the equation in polar coordinates $x^2 + y^2 = 9$

Solution

$x^2 + y^2 = 9$ $\qquad r^2 = x^2 + y^2$

$r^2 = 9$

Exercise

Find a polar equation that has the same graph as the equation in x and y. $(x+2)^2 + (y-3)^2 = 13$

Solution

$x^2 + 4x + 4 + y^2 - 6y + 9 = 13$

$x^2 + 4x + y^2 - 6y = 13 - 9 - 4$

$x^2 + 4x + y^2 - 6y = 0$

$x^2 + y^2 = 6y - 4x$ $\qquad x = r\cos\theta \quad y = r\sin\theta$

$$r^2 = 6r\sin\theta - 4r\cos\theta$$

$$r^2 = r(6\sin\theta - 4\cos\theta) \qquad \textbf{\textit{Divide by r}}$$

$$\underline{r = 6\sin\theta - 4\cos\theta}$$

Exercise

Find a polar equation that has the same graph as the equation in x and y. $y^2 - x^2 = 4$

Solution

$$y^2 - x^2 = 4 \qquad x = r\cos\theta \quad y = r\sin\theta$$

$$r^2\sin^2\theta - r^2\cos^2\theta = 4$$

$$r^2\left(\sin^2\theta - \cos^2\theta\right) = 4 \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$r^2(-\cos 2\theta) = 4$$

$$\underline{r^2 = -\frac{4}{\cos 2\theta}}$$

Exercise

Write the equation in polar coordinates $x^2 + y^2 = 4x$

Solution

$$x^2 + y^2 = 4x \qquad r^2 = x^2 + y^2 \quad x = r\cos\theta$$

$$r^2 = 4r\cos\theta$$

$$\frac{r^2}{r} = \frac{4r\cos\theta}{r}$$

$$\underline{r = 4\cos\theta}$$

Exercise

Write the equation in polar coordinates $y = -x$

Solution

$$y = -x \qquad x = r\cos\theta \quad y = r\sin\theta$$

$$r\sin\theta = -r\cos\theta$$

$$\underline{\sin\theta = -\cos\theta}$$

Exercise

Write the equation in polar coordinates $x + y = 4$

Solution

$$x + y = 4$$

$$r\cos\theta + r\sin\theta = 4 \qquad x = r\cos\theta \quad y = r\sin\theta$$

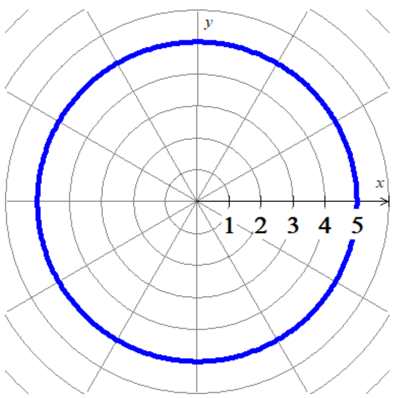
$$r(\cos\theta + \sin\theta) = 4$$

$$\boxed{r = \frac{4}{\cos\theta + \sin\theta}}$$

Exercise

Sketch the graph of the polar equation $r = 5$

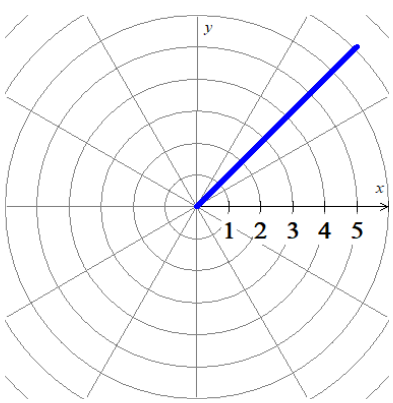
Solution



Exercise

Sketch the graph of the polar equation $\theta = \frac{\pi}{4}$

Solution

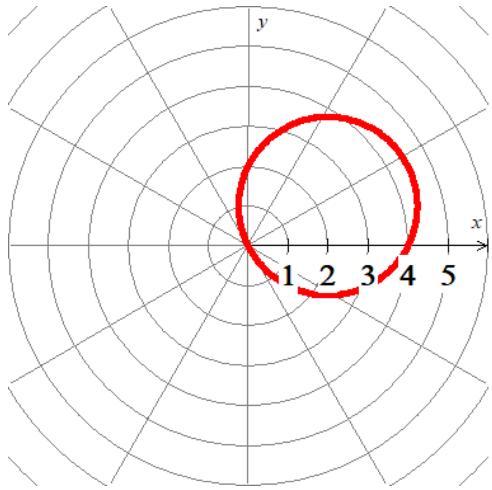


Exercise

Sketch graph $r = 4\cos\theta + 2\sin\theta$

Solution

θ	r
0	4
$\frac{\pi}{4}$	$3\sqrt{2}$
$\frac{\pi}{2}$	2
$\frac{3\pi}{4}$	$-\sqrt{2}$
π	-4
$\frac{3\pi}{2}$	-2

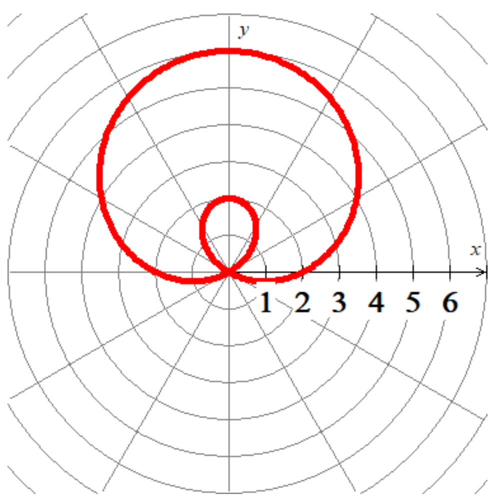


Exercise

Sketch the graph of the polar $r = 2 + 4\sin\theta$

Solution

θ	r
0	2
$\frac{\pi}{6}$	4
$\frac{\pi}{4}$	$2+2\sqrt{2}$
$\frac{\pi}{2}$	6
$\frac{5\pi}{6}$	4
π	2
$\frac{7\pi}{6}$	0
$\frac{3\pi}{2}$	-2
$\frac{11\pi}{6}$	0

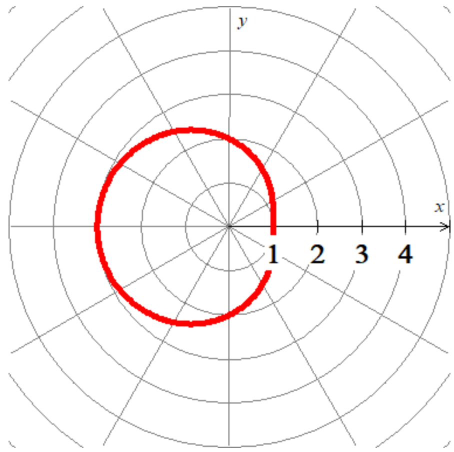


Exercise

Sketch the graph $r = 2 - \cos\theta$

Solution

θ	r
0	1
$\frac{\pi}{3}$	$\frac{3}{2}$
$\frac{\pi}{2}$	2
$\frac{2\pi}{3}$	$\frac{5}{2}$
π	3
$\frac{4\pi}{3}$	$\frac{5}{2}$
$\frac{3\pi}{2}$	2
$\frac{5\pi}{3}$	$\frac{3}{2}$



Exercise

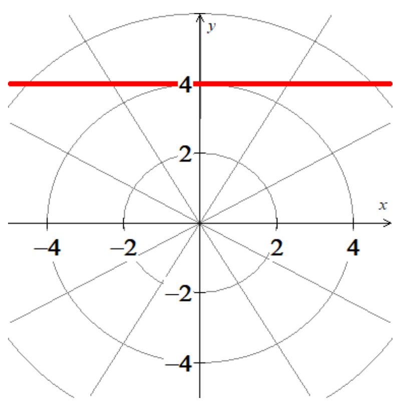
Sketch the graph $r = 4\csc\theta$

Solution

$$r = 4\csc\theta$$

$$= \frac{4}{\sin\theta}$$

$$r\sin\theta = 4 = y$$



Exercise

Sketch the graph $r^2 = 4\cos 2\theta$

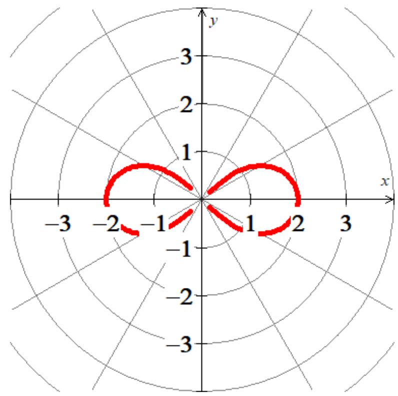
Solution

$$r^2 = 4\cos 2\theta \ \geq 0$$

$$-\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2}$$

$$-\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \quad \& \quad \frac{3\pi}{4} \le \theta \le \frac{5\pi}{4}$$

θ	r
0	2
$\frac{\pi}{6}$	$\sqrt{2}$
$\frac{\pi}{4}$	0
$\frac{3\pi}{4}$	0
π	2
$\frac{5\pi}{4}$	0
$\frac{7\pi}{4}$	0

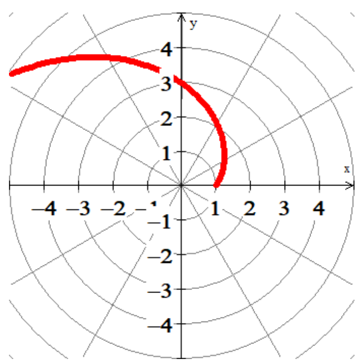


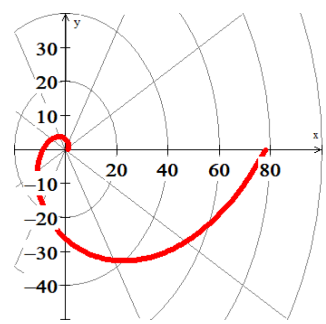
Exercise

Sketch the graph $r = 2^{\theta} \quad \theta \ge 0$

Solution

θ	r
0	1
$\frac{\pi}{2}$	$2^{\pi/2}$

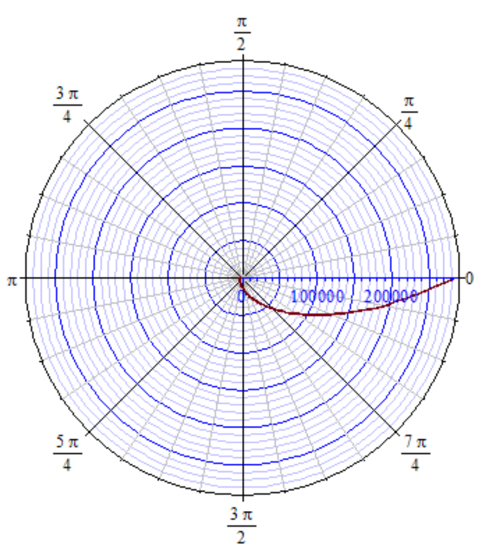


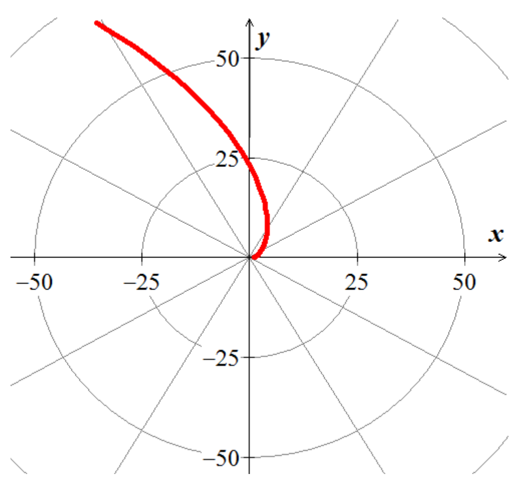


Exercise

Sketch the graph of the polar equation $r = e^{2\theta} \quad \theta \ge 0$

Solution

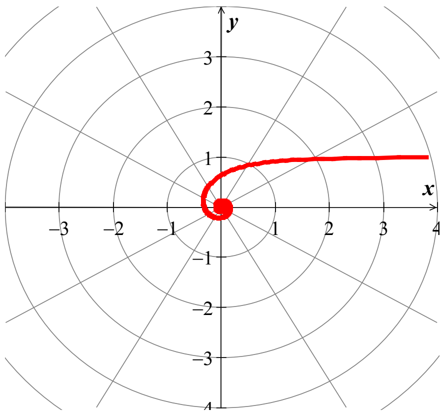


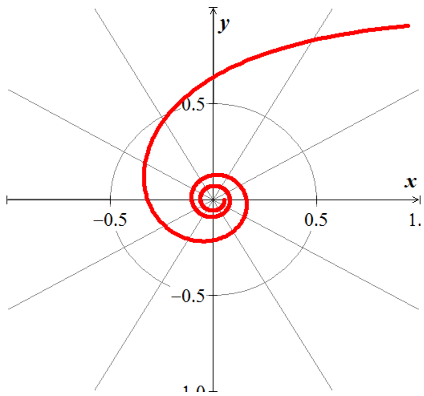


Exercise

Sketch the graph of the polar equation $r\theta = 1 \quad \theta > 0$

Solution

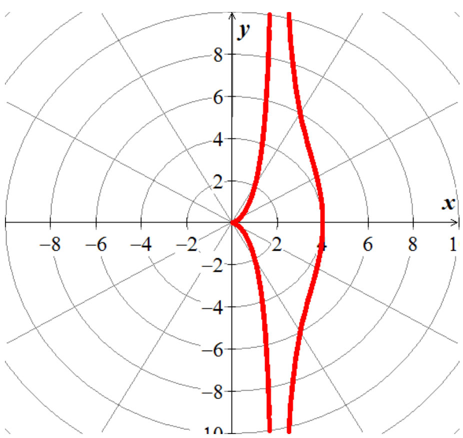




Exercise

Sketch the graph of the polar equation $r = 2 + 2\sec\theta$

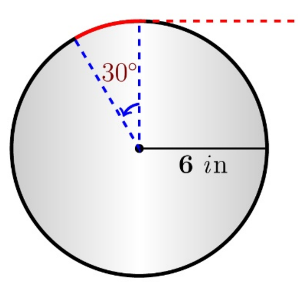
Solution



Exercise

A rope is being wound around a drum with radius 6 *inches*. How much rope will be wound around the drum if the drum is rotated through an angle of 30°?

Solution



$$s = 6\left(30^\circ \frac{\pi}{180^\circ}\right) \qquad s = r\theta$$

$$= \pi \ \ in$$

Exercise

The total arm and blade of a single windshield wiper was 10 *in*. long and rotated back and forth through an angle of 95°. The shaded region in the figure is the portion of the windshield cleaned by the 7-*in*. wiper blade. What is the area of the region cleaned?

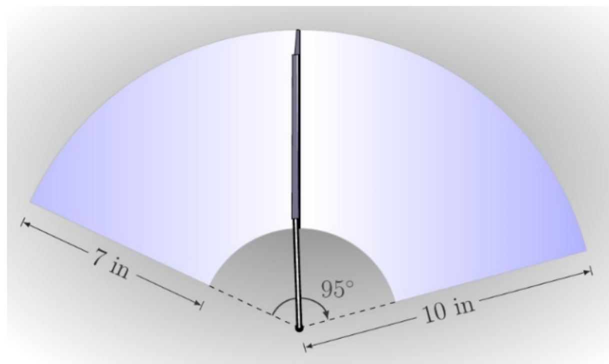
Solution

The total angle:

$$\theta = 95^\circ \frac{\pi}{180^\circ}$$

$$= \frac{19\pi}{36} \ rad$$

A_1 : The area of arm only (not cleaned by the blade).



$$A_1 = \frac{1}{2}(10-7)^2 \frac{19\pi}{36}$$

$$= \frac{19\pi}{8}$$

A_2 : The area of arm and the blade.

$$A_2 = \frac{1}{2}(10)^2 \frac{19\pi}{36}$$

$$= \frac{475\pi}{18}$$

The total cleaned area:

$$A = A_2 - A_1$$

$$= \frac{475\pi}{18} - \frac{19\pi}{8}$$

$$= \frac{1900-171}{72}\pi$$

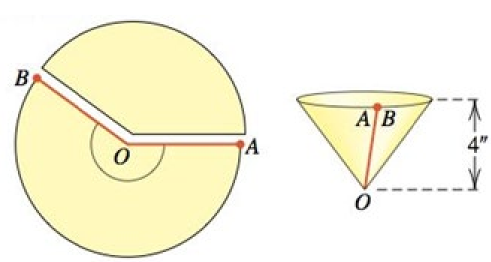
$$= \frac{1729\pi}{72} \ in^2 \qquad = 75.4 \ in^2$$

Exercise

A conical paper cup is constructed by removing a sector from a circle of radius 5 *inches* and attaching edge *OA* to *OB*. Find angle *AOB* so that the cap has a depth of 4 *inches*.

Solution

$$r^2 + 4^2 = 5^2 \ \rightarrow \ r = 3 \ in$$



The circumference of the rim of the cone is: $2\pi r = 6\pi$

$$\theta = \frac{s}{r} = \frac{6\pi}{5} \ rad$$

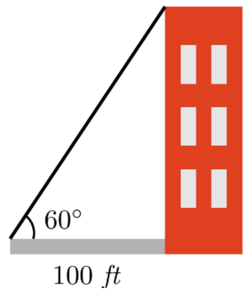
$$= \frac{6(180)}{5}$$

$$= 216° \,\big|$$

Exercise

The shadow of a vertical tower is 100 *feet* long when the angle of elevation of the sun is 60°. Find the height of the tower.

Solution



$$\tan 60° = \frac{h}{100}$$

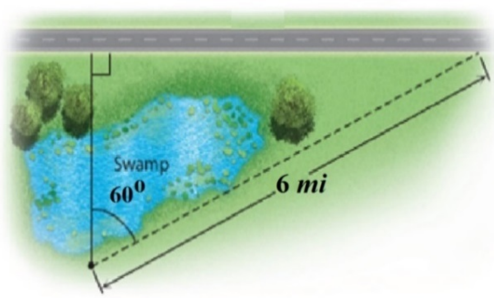
$$h = 100 \tan 60°$$

$$= 100\sqrt{3}\ ft \,\big|$$

Exercise

You were hiking directly toward a long straight road when you encountered a swamp. you turned 60° to the right and hiked 6 *mi* in that direction to reach the road. How far were you from the road when you encountered the swamp?

Solution



$$\cos 60° = \frac{d}{2}$$

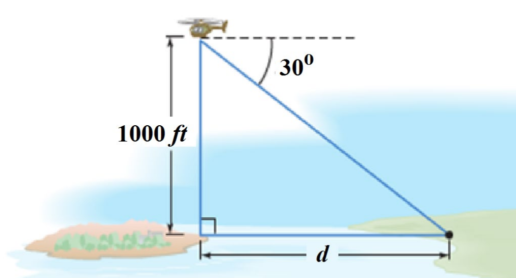
$$d = 6\left(\frac{1}{2}\right)$$

$$= 3\ miles \,\big|$$

Exercise

A helicopter hovers 1,000 *feet* above a small island. The angle of depression from the helicopter to point *P* on the coast is 30°. How far off the coast is the island?

Solution



$$\tan 30° = \frac{1,000}{d}$$

$$d = \frac{1,000}{\frac{1}{\sqrt{3}}}$$

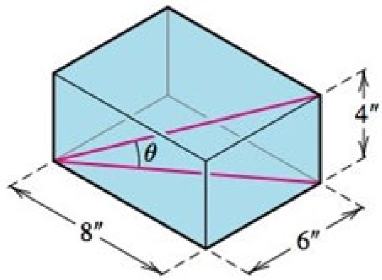
$$= 1,000\sqrt{3}\ feet \,\big|$$

∴The island is approximately 1,376 *feet* off the coast.

Exercise

A rectangular box has dimensions $8'' \times 6'' \times 4''$. Approximate, to the nearest tenth of a degree, the angle θ formed by a diagonal of the base and the diagonal of the box.

Solution



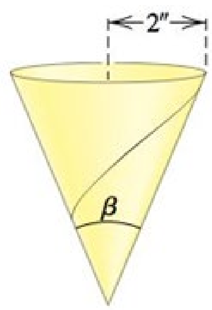
$$d = \sqrt{8^2 + 6^2}$$
$$= 10$$

$$\theta = \tan^{-1}\frac{4}{10}$$
$$\approx 21.8^\circ$$

Exercise

A conical paper cup has a radius of 2 *inches*, approximate, to the nearest degree, the angle β so that the cone will have a volume of $20\ in^3$.

Solution



$$V = \frac{1}{3}\pi r^2 h$$
$$= 20\ in^3$$

$$h = \frac{60}{\pi\left(2^2\right)}$$
$$= \frac{15}{\pi} \approx 4.77\ in$$

$$\tan\frac{\beta}{2} = \frac{2}{4.77}$$

$$\frac{\beta}{2} = \tan^{-1}\frac{2}{4.77}$$
$$\approx 22.75^\circ$$

$$\beta = 2\left(22.75^\circ\right)$$
$$\approx 45.5^\circ$$

Exercise

As a hot-air balloon rises vertically, its angle of elevation from a point P on level ground 100 *km* from the point Q directly underneath the balloon changes from $19^\circ\ 20'$ to $31^\circ\ 50'$. Approximately how far does the balloon rise during this period?

Solution

$$\tan(19° \ 20') = \frac{h_1}{100}$$

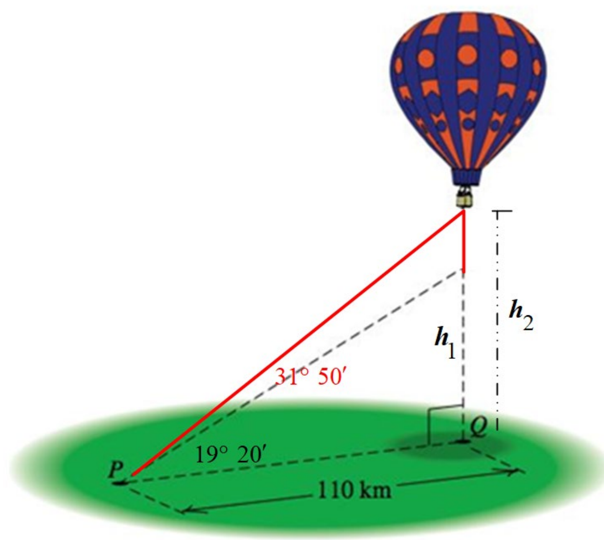
$$h_1 = 100\tan(19° \ 20')$$

$$\underline{\approx 38.59 \ km}$$

$$\tan(31° \ 50') = \frac{h_2}{100}$$

$$h_2 = 100\tan(31° \ 50')$$

$$\underline{\approx 68.29 \ km}$$



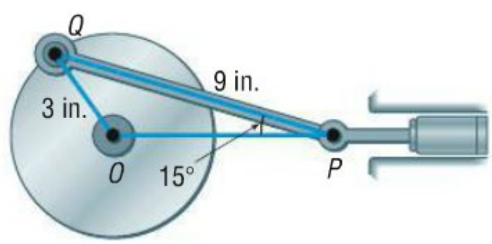
The change in elevation is:

$$h_2 - h_1 \approx 68.29 - 38.59$$

$$\underline{= 29.7 \ km}$$

Exercise

On a certain automobile, the crankshaft is 3 *inches* long and the connecting rod is 9 *inches* long. At the time when $\angle OPQ$ is 15°, how far is the piston P from the center O of the crankshaft?



Solution

$$\frac{\sin O}{9} = \frac{\sin 15°}{3}$$

$$\hat{O} = \sin^{-1}(3\sin 15°)$$

$$\underline{\approx 50.94°}$$

$$O = 50.94°$$
$$Q = 180° - 50.94° - 15°$$
$$= 114.06°$$

$$\frac{q}{\sin 114.06°} = \frac{3}{\sin 15°}$$

$$O = 180° - 50.94°$$
$$= 129.06°$$
$$Q = 180° - 129.06° - 15°$$
$$= 35.94°$$

$$\frac{q}{\sin 35.94°} = \frac{3}{\sin 15°}$$

$q = \dfrac{3\sin 114.06^\circ}{\sin 15^\circ}$

$\approx 10.58\ in$

$q = \dfrac{3\sin 35.94^\circ}{\sin 15^\circ}$

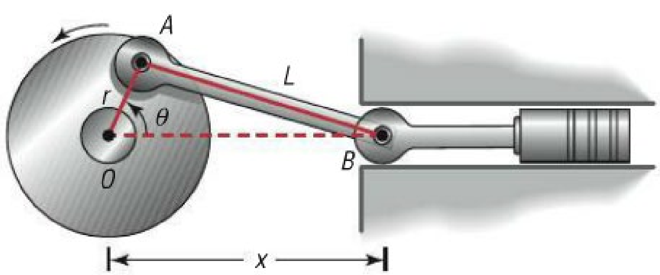
$\approx 6.80\ in$

The distance from the piston P to the center O of the crankshaft is approximately either 10.58 *inches* or 6.8 *inches*.

Exercise

Rod OA rotates about the fixed point O so that point A travels on a circle of radius r. Connected to point A is another rod AB of length $L > 2r$, and point B is connected to a piston. Show that the distance x between point O and point B is given by

$$x = r\cos\theta + \sqrt{r^2\cos^2\theta + L^2 - r^2}$$



Where θ is the angle of rotation of rod OA.

Solution

$L^2 = r^2 + x^2 - 2rx\cos\theta$ *Law of cosine*

$x^2 - 2rx\cos\theta + r^2 - L^2 = 0$

$x = \dfrac{2r\cos\theta \pm \sqrt{4r^2\cos^2\theta - 4\left(r^2 - L^2\right)}}{2}$ $\qquad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{2r\cos\theta \pm 2\sqrt{r^2\cos^2\theta - r^2 + L^2}}{2}$

$= r\cos\theta + \sqrt{r^2\cos^2\theta - r^2 + L^2}$ ✓

Exercise

Shown in the figure is a design for a rain gutter.

a) Express the volume V as a function of θ.

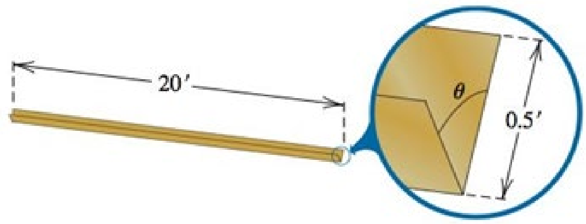
b) Approximate the acute angle θ that results in a volume of $2\ ft^3$

Solution

a) $Volume = 20 \times \left(Area\ of\ the\ sector - triangle \right)$

$$= 20 \times \left(\frac{1}{2}(0.5)^2 \sin\theta \right)$$

$$= 2.5\ \sin\theta$$



b) $\frac{5}{2}\sin\theta = 2$

$$\theta = \sin^{-1}\left(\frac{4}{5}\right)$$

$$\approx 53.13^\circ$$

Exercise

Shown in the figure is a plan for the top of a wing of a jet fighter.

a) Approximate angle ϕ.

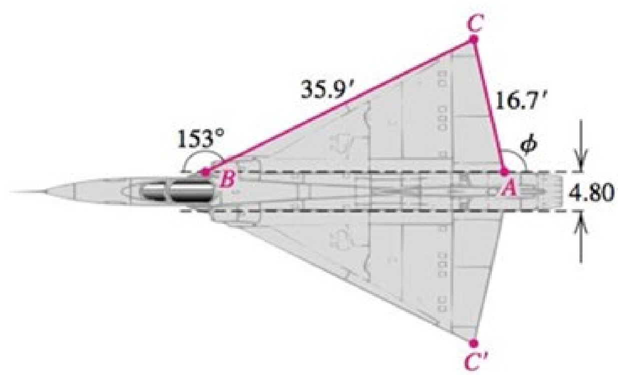
b) If the fuselage is 4.80 *feet* wide, approximate the wing span CC'.

c) Approximate the area of the triangle *ABC*.

Solution

a) $\angle ABC = 180^\circ - 153^\circ$

$$= 27^\circ$$



$$\frac{\sin A}{35.9} = \frac{\sin 27^\circ}{16.7}$$

$$A = \sin^{-1}\left(\frac{35.9}{16.7}\sin 27^\circ\right)$$

$$\approx 77.4^\circ$$

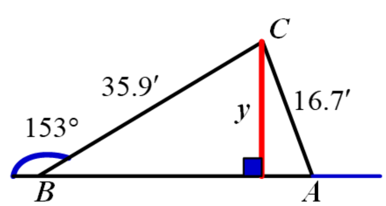
$$\phi = 180^\circ - 77.4^\circ$$

$$\approx 102.6^\circ$$

b) $\sin A = \frac{y}{16.7}$

$$y = 16.7 \sin 77.4^\circ$$

$$\approx 16.3$$



$$CC' = 2(16.3) + 4.8$$

$$\approx 37.4\ ft$$

c) $\angle ACB = 180^\circ - 27^\circ - 77.4^\circ$

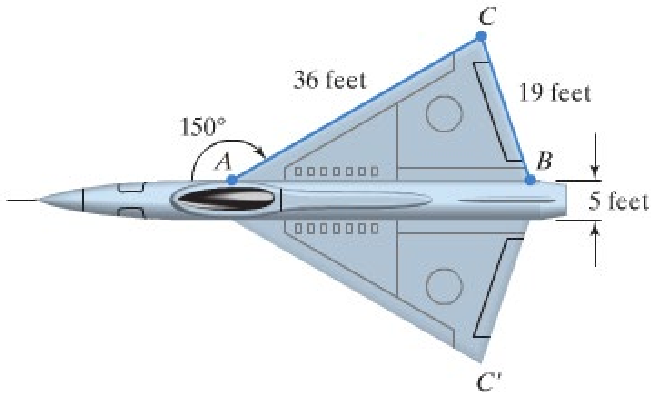
$$= 75.6^\circ$$

$$Area = \frac{1}{2}(35.9)(16.7)\sin 75.6^\circ$$

$$\approx 290.35\ ft^2$$

Exercise

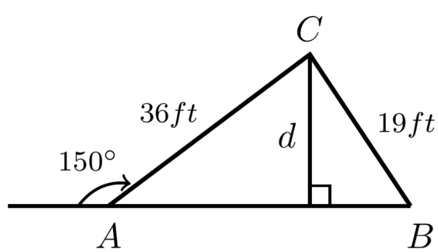
Shown in the figure is a plan for the top of a wing of a jet fighter. The fuselage is 5 *feet* wide. Find the wing span CC'



Solution

$$\angle BAC = 180° - 150°$$
$$= 30°$$

$$\sin A = \frac{d}{36}$$
$$d = 36 \sin 30°$$
$$= 36\left(\frac{1}{2}\right)$$
$$= 18\ ft$$



$$CC' = 2(18) + 5$$
$$= 41\ ft$$

Exercise

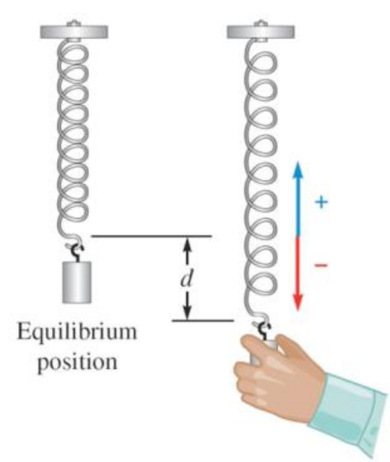
A mass attached to a spring oscillates upward and downward. The length L of the spring after t seconds is given by the function $L = 15 - 3.5\cos(2\pi t)$, where L is measured in *cm*.

a) Sketch the graph of this function for $0 \le t \le 5$

b) What is the length the spring when it is at equilibrium?

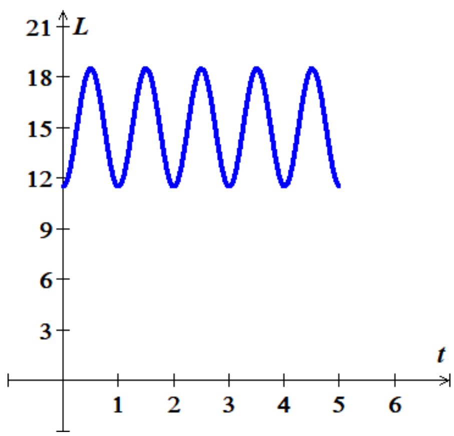
c) What is the length the spring when it is shortest?

d) What is the length the spring when it is longest?



Solution

a)



b) The length the spring when it is at equilibrium $L = 15\ cm$

c) $L = 15 - 3.5$

$\underline{= 11.5 \quad cm}$

d) $L = 15 + 3.5$

$\underline{= 18.5 \quad cm}$

Exercise

Based on years of weather data, the expected low temperature T (in °F) in Fairbanks, Alaska, can be approximated by

$$T = 36\sin\left(\frac{2\pi}{365}(t-101)\right) + 14$$

a) Sketch the graph T for $0 \le t \le 365$

b) Predict when the coldest day of the year will occur.

Solution

a)

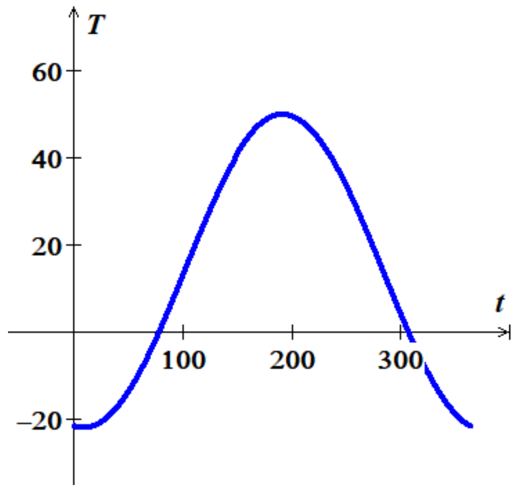
Amplitude: $|A| = 36$

Period: $P = 2\pi \frac{365}{2\pi} = 365$

Phase Shift: $\phi = 101$

VT: $y = 14$

x	y
101	14
$\frac{365}{4} + 101 = \frac{769}{4}$	50
$\frac{365}{2} + 101 = \frac{567}{2}$	14
$\frac{1095}{4} + 101 = \frac{1,499}{4}$	−22
$365 + 101 = 466$	14



b) From the table the coldest temperature is −22 °F at $t = \frac{1499}{4} = 374.75 > 365$

$t = 374.75 - 365$

$= 9.75 \ \ days$

Exercise

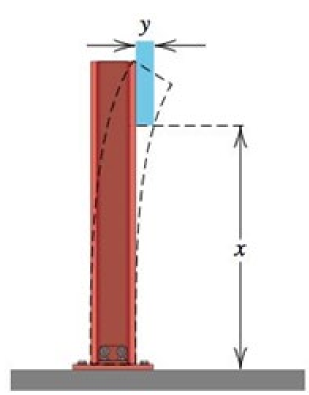
To simulate the response of a structure to an earthquake, an engineer must choose a shape for the initial displacement of the beams in the building. When the beam has length *L feet* and the maximum displacement is *a feet*, the equation

$$y = a - a\cos\frac{\pi}{2L}x$$

Has been used by engineers to estimate the displacement *y*. if $a = 1$ and $L = 10$, sketch the graph of the equation for $0 \le x \le 10$.

Solution

Given: $a = 1 \ \& \ L = 10$



$y = a - a\cos\frac{\pi}{2L}x$

$= 1 - \cos\left(\frac{\pi}{20}x\right)$

Amplitude: $|A| = 1$

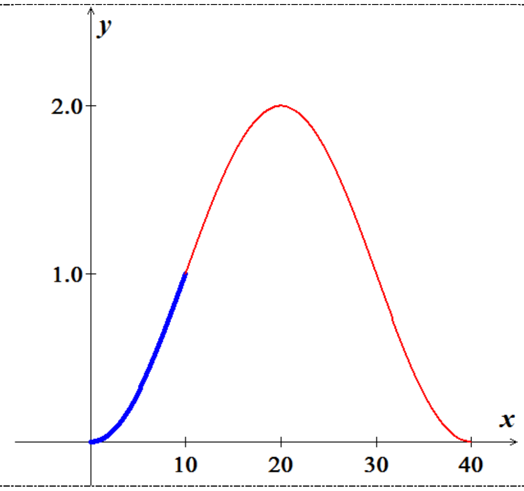
Period: $P = \dfrac{2\pi}{\frac{\pi}{20}} = 40$

Phase Shift: $\phi = 0$

VT: $y = 1$

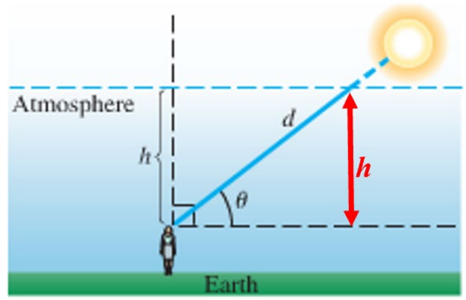
x	y
0	0
10	1
20	2
30	1
40	0

$0 \le x \le 10$



Exercise

The shortest path for the sun's rays through Earth's atmosphere occurs when the sun is directly overhead. Disregarding the curvature of Earth, as the sun moves lower on the horizon, the distance that sunlight passes through the atmosphere increases by a factor of $\csc\theta$, where θ is the angle of elevation of the sun. This increased distance reduces both the intensity of the sun and the amount of ultraviolet light that reached Earth's surface.



a) Verify that $d = h\csc\theta$

b) Determine θ when $d = 2h$

c) The atmosphere filters out the ultraviolet light that causes skin to burn, Compare the difference between sunbathing when $\theta = \frac{\pi}{2}$ and when $\theta = \frac{\pi}{3}$. Which measure gives less ultraviolet light?

Solution

a) $\sin\theta = \dfrac{h}{d}$

$= \dfrac{1}{\csc\theta}$

$\underline{d = h\csc\theta}$ (*cross-multiplication*)

b) $\sin\theta = \dfrac{h}{d}$

$$= \frac{h}{2h}$$

$$\boxed{= \frac{1}{2}}$$

$$\theta = \sin^{-1}\frac{1}{2}$$

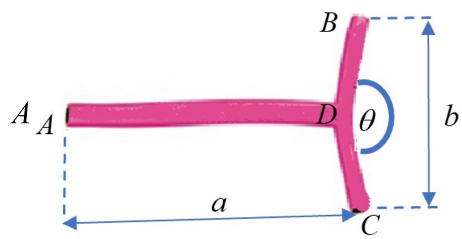
$$\boxed{= \frac{\pi}{6}}$$

c) $\begin{cases} \csc\frac{\pi}{2} = 1 \\ \csc\frac{\pi}{3} = \frac{2\sqrt{3}}{3} \approx 1.15 \end{cases}$

When the distance to the sun is lager $\left(\theta = \frac{\pi}{3}\right)$, there is less ultraviolet light reaching the earth's surface. In this case, sunlight passes through 15% more atmosphere.

Exercise

A common form of cardiovascular branching is bifurcation, in which an artery splits into two smaller blood vessels. The bifurcation angle θ is the angle formed by the two smaller arteries. The line through A and D bisects θ and is perpendicular to the line through B and C.



a) Show that the length l of the artery from A to B is given by $l = a + \frac{b}{2}\tan\frac{\theta}{4}$.

b) Estimate the length l from the three measurements $a = 10\ mm$, $b = 6\ mm$, and $\theta = 156°$.

Solution

a) $\tan\frac{\theta}{2} = \frac{\frac{b}{2}}{a - |AD|}$

$$|AD| = a - \frac{b}{2}\frac{1}{\tan\frac{\theta}{2}}$$

$$\sin\frac{\theta}{2} = \frac{b}{2}\frac{1}{|DB|}$$

$$|DB| = \frac{b}{2}\frac{1}{\sin\frac{\theta}{2}}$$

$$l = |AD| + |DB|$$

$$= a - \frac{b}{2}\frac{1}{\tan\frac{\theta}{2}} + \frac{b}{2}\frac{1}{\sin\frac{\theta}{2}}$$

$$= a + \frac{b}{2}\left(\frac{1}{\sin\frac{\theta}{2}} - \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right)$$

$$= a + \frac{b}{2}\left(\frac{1-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right) \qquad \tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$$

$$\boxed{= a + \frac{b}{2}\tan\frac{\theta}{4}}$$

b) ***Given***: $a = 10\ mm,\ b = 6\ mm,\ \theta = 156°$

$$l = 10 + \frac{6}{2}\tan\frac{156°}{4}$$

$$\boxed{= 10 + 3\tan 39°}$$

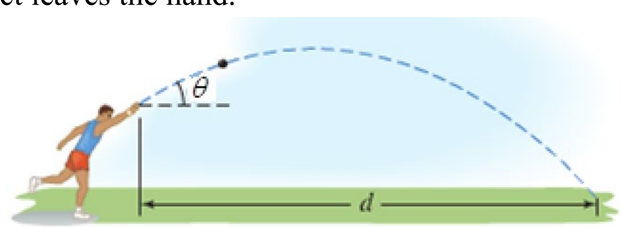
$$\boxed{\approx 12.43\ \ mm}$$

Exercise

Throwing events in track and field include the shot put, the discus throw, the hammer throw, and the javelin throw. The distance that the athlete can achieve depends on the initial speed of the object thrown and the angle above the horizontal at which the object leaves the hand. This angle is represented by θ. The distance, d, in *feet*, that the athlete throws is modeled by the formula

$$d = \frac{v_0^2}{16}\sin\theta\cos\theta$$

In which v_0 is the initial speed of the object thrown, in *feet* per *second*, and θ is the angle, in *degrees*, at which the object leaves the hand.



a) Use the identity to express the formula so that it contains the since function only.

b) Use the formula from part (*a*) to find the angle, θ, that produces the maximum distance, d, for a given initial speed, v_0.

Solution

a) $d = \dfrac{v_0^2}{16}\sin\theta\cos\theta \qquad\qquad \sin 2\theta = 2\sin\theta\cos\theta$

$= \dfrac{v_0^2}{16}\dfrac{1}{2}\sin 2\theta$

$\boxed{= \dfrac{v_0^2}{32}\sin 2\theta}$

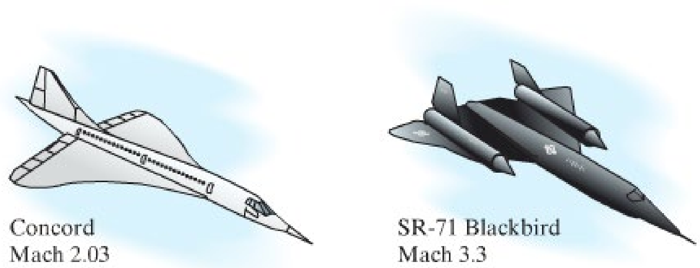
b) The maximum value of a sine function is 1 at $\frac{\pi}{2}$ on the interval $[0,\ 2\pi]$

$2\theta = \dfrac{\pi}{2}$

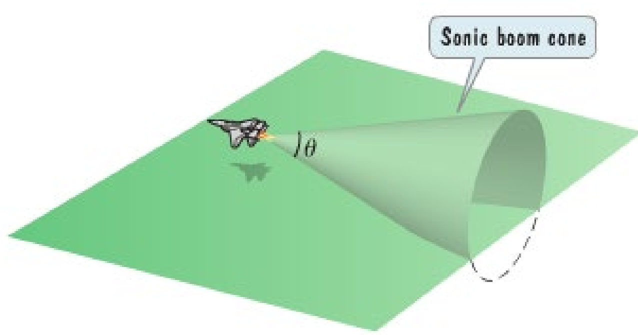
$\boxed{\theta = \dfrac{\pi}{4}}$

Exercise

The speed of a supersonic aircraft is usually represented by a Mach number. A Mach number is the speed of the aircraft, in *miles* per *hour*, divided by the speed of sound, approximately 740 *mph*. Thus, a plane flying at twice the speed of sound has a speed, *M*, of Mach 2.



If an aircraft has a speed greater than Mach 1, a sonic boom is heard, created by sound waves that form a cone with a vertex angle θ.



The relationship between the cone's vertex angle θ, and the Mach speed, *M*, of an aircraft that is flying faster than the speed of sound is given by

$$\sin\frac{\theta}{2} = \frac{1}{M}$$

a) If $\theta = \frac{\pi}{6}$, determine the Mach speed, *M*, of the aircraft. Express the speed as an exact value and as decimal to the nearest tenth.

b) If $\theta = \frac{\pi}{4}$, determine the Mach speed, *M*, of the aircraft. Express the speed as an exact value and as decimal to the nearest tenth.

Solution

a) At $\theta = \frac{\pi}{6}$

$$\begin{aligned}\sin\frac{\theta}{2} &= \sqrt{\frac{1}{2}(1-\cos\theta)}\\ &= \sqrt{\frac{1}{2}\left(1-\cos\frac{\pi}{6}\right)}\\ &= \sqrt{\frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right)}\\ &= \sqrt{\frac{2-\sqrt{3}}{4}}\\ &= \frac{1}{2}\sqrt{2-\sqrt{3}} = \frac{1}{M}\end{aligned}$$

$$\begin{aligned}M &= \frac{2}{\sqrt{2-\sqrt{3}}}\cdot\frac{\sqrt{2-\sqrt{3}}}{\sqrt{2-\sqrt{3}}}\\ &= \frac{2\sqrt{2-\sqrt{3}}}{2-\sqrt{3}}\cdot\frac{2+\sqrt{3}}{2+\sqrt{3}}\\ &= 2\left(2+\sqrt{3}\right)\sqrt{2-\sqrt{3}} \quad \approx 3.9\end{aligned}$$

b) At $\theta = \frac{\pi}{4}$

$$\begin{aligned}\sin\frac{\theta}{2} &= \sqrt{\frac{1}{2}(1-\cos\theta)}\\ &= \sqrt{\frac{1}{2}\left(1-\cos\frac{\pi}{4}\right)}\\ &= \sqrt{\frac{1}{2}\left(1-\frac{\sqrt{2}}{2}\right)}\\ &= \sqrt{\frac{2-\sqrt{2}}{4}}\\ &= \frac{1}{2}\sqrt{2-\sqrt{2}} = \frac{1}{M}\end{aligned}$$

$$M = \frac{2}{\sqrt{2-\sqrt{2}}}\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2-\sqrt{2}}}$$

$$= \frac{2\sqrt{2-\sqrt{2}}}{2-\sqrt{2}}\frac{2+\sqrt{2}}{2+\sqrt{2}}$$

$$= \frac{2\left(2+\sqrt{2}\right)\sqrt{2-\sqrt{2}}}{2}$$

$$= \left(2+\sqrt{2}\right)\sqrt{2-\sqrt{2}} \quad \approx 2.6$$

Solution Section R.6 – Inverse of a Function

Exercise

For the given function $f(x)=\dfrac{x}{x-2}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

d) Graph both functions (if $f^{-1}(x)$ exists)

Solution

a) $f(a)=f(b)$

$$\frac{a}{a-2}=\frac{b}{b-2}$$

$$ab-2a=ab-2b$$

$$-2a=-2b$$

$$\boxed{a=b} \quad \checkmark$$

$\therefore\ f(x)$ is one-to-one function.

b) $y=\dfrac{x}{x-2}$

$$x=\frac{y}{y-2}$$

$$xy-2x=y$$

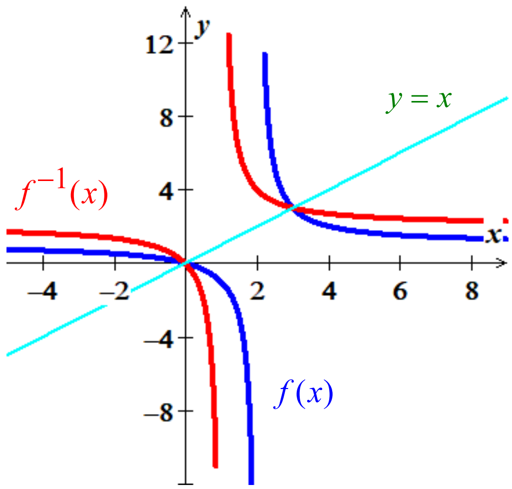
$$(x-1)y=2x$$

$$\boxed{f^{-1}(x)=\frac{2x}{x-1}}$$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\boxed{\mathbb{R}-\{2\}}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\boxed{\mathbb{R}-\{1\}}$

d)



Exercise

For the given function $f(x) = \dfrac{x+1}{x-1}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

d) Graph both functions (if $f^{-1}(x)$ exists)

Solution

a) $f(a) = f(b)$

$\dfrac{a+1}{a-1} = \dfrac{b+1}{b-1}$

$ab - a + b - 1 = ab - b + a - 1$

$-2a = -2b$

$\underline{a = b}$ ✓

$\therefore$ $f(x)$ is one-to-one function.

b) $y = \dfrac{x+1}{x-1}$

$x = \dfrac{y+1}{y-1}$

$xy - x = y + 1$

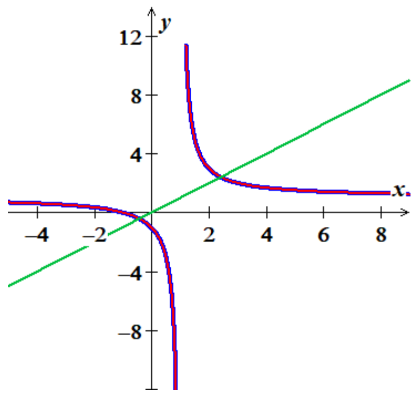
$(x-1)y = x + 1$

$\underline{f^{-1}(x) = \dfrac{x+1}{x-1}}$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\underline{\mathbb{R}-\{1\}}$ |

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\underline{\mathbb{R}-\{1\}}$ |

d)



$f(x)$

$f^{-1}(x)$

$y = x$

Exercise $f(x)=\dfrac{2x+1}{x+3}$

For the given function

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

d) Graph both functions (if $f^{-1}(x)$ exists)

Solution

a) $f(a)=f(b)$

$$\frac{2a+1}{a+3}=\frac{2b+1}{b+3}$$

$$2ab+6a+b+3=2ab+6b+a+3$$

$$5a=5b$$

$\underline{a=b}$ | ✓

$\therefore$ $f(x)$ is one-to-one function.

b) $y=\dfrac{2x+1}{x+3}$

$$x=\frac{2y+1}{y+3}$$

$$xy+3x=2y+1$$

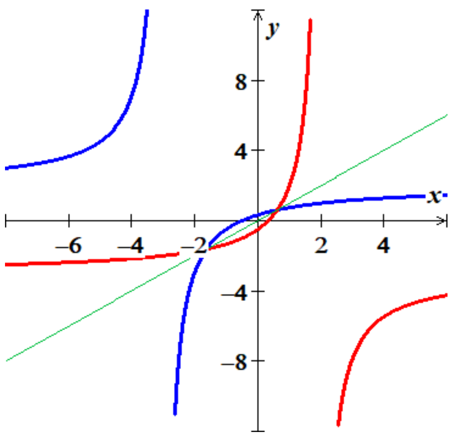
$$(x-2)y=-3x+1$$

$$\boxed{f^{-1}(x)=\frac{-3x+1}{x-2}}$$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\boxed{\mathbb{R}-\{-3\}}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\boxed{\mathbb{R}-\{2\}}$

d)



Exercise

For the given function $f(x)=\frac{3x-1}{x-2}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

d) Graph both functions (if $f^{-1}(x)$ exists)

Solution

a) $f(a)=f(b)$

$$\frac{3a-1}{a-2}=\frac{3b-1}{b-2}$$

$$3ab-6a-b+2=3ab-6b-a+2$$

$$-5a=-5b$$

$$\boxed{a=b}\quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

b) $y=\frac{3x-1}{x-2}$

$$x=\frac{3y-1}{y-2}$$

$$xy-2x=3y-1$$

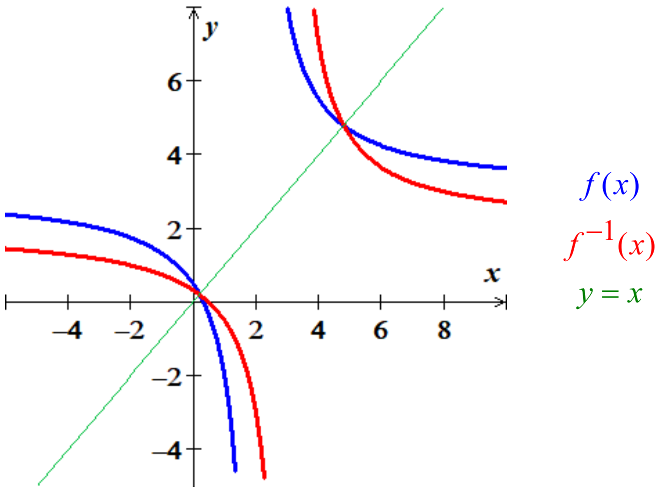
$(x-3)y = 2x-1$

$\boxed{f^{-1}(x) = \dfrac{2x-1}{x-3}}$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\boxed{\mathbb{R}-\{2\}}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\boxed{\mathbb{R}-\{3\}}$

d)



Exercise

For the given function $f(x) = \sqrt{x-1} \quad x \ge 1$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

d) Graph both functions (if $f^{-1}(x)$ exists)

Solution

a) $f(a) = f(b)$

$\sqrt{a-1} = \sqrt{b-1}$

$\left(\sqrt{a-1}\right)^2 = \left(\sqrt{b-1}\right)^2$

$a-1 = b-1$

$\boxed{a = b}$ ✓

$\therefore f(x)$ is one-to-one function.

b) $y = \sqrt{x-1}$

$$x = \sqrt{y-1}$$

$$x^2 = y - 1$$

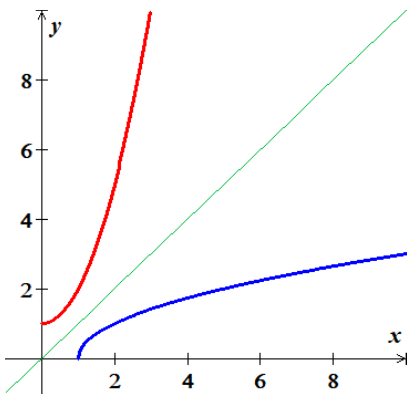
$$y = x^2 + 1$$

$$\underline{f^{-1}(x) = x^2 + 1 \quad x \geq 0}\,\big|$$

c) Domain of $f(x)$ = Range of $f^{-1}(x)$: $\underline{[1, \infty)}\,\big|$

Range of $f(x)$ = Domain of $f^{-1}(x)$: $\underline{[0, \infty)}\,\big|$

d)



$f(x)$

$f^{-1}(x)$

$y = x$

Exercise

For the given function $\quad f(x) = \sqrt{2-x} \quad x \leq 2$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

d) Graph both functions (if $f^{-1}(x)$ exists)

Solution

a) $f(a) = f(b)$

$$\sqrt{2-a} = \sqrt{2-b}$$

$$\left(\sqrt{2-a}\right)^2 = \left(\sqrt{2-b}\right)^2$$

$$2 - a = 2 - b$$

$$\underline{a = b}\,\big| \quad \checkmark$$

$\therefore$ $f(x)$ is one-to-one function.

b) $y=\sqrt{2-x}$

$x=\sqrt{2-y}$

$x^2=2-y$

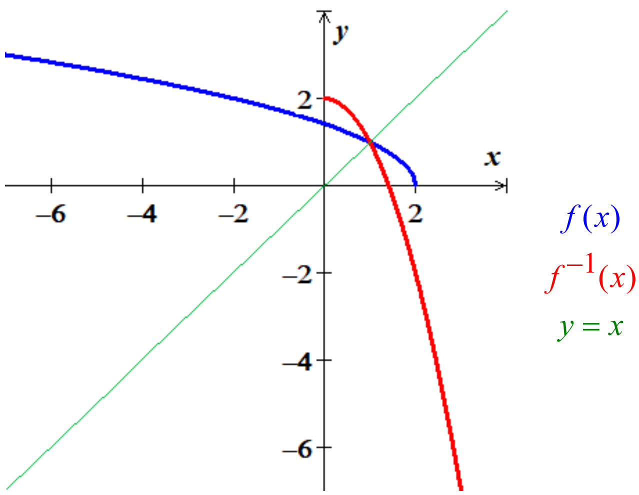
$y=2-x^2$

$\underline{f^{-1}(x)=2-x^2 \quad x\geq 0}$

c) Domain of $f(x)$ = Range of $f^{-1}(x)$: $\underline{(-\infty,\ 2]}$

Range of $f(x)$ = Domain of $f^{-1}(x)$: $\underline{[0,\ \infty)}$

d)



Exercise

For the given function $\quad f(x)=x^2+4x \quad x\geq -2$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

d) Graph both functions (if $f^{-1}(x)$ exists)

Solution

$x_{vertex}=-\frac{4}{2}$

$\underline{=-2}$

$f(-2)=4-8$

$\underline{=-4}$

$Vertex = (-2,\ -4)$

a) Since, $f(x)$ is a restricted function with $x \ge -2$.

$x = -2$ is the line symmetry, therefore; $f(x)$ is one-to-one function.

b) $y = x^2 + 4x$

$x = y^2 + 4y$

$y^2 + 4y - x = 0$

$$y = \frac{-4 \pm \sqrt{16 + 4x}}{2}$$

$$= \frac{-4 \pm 2\sqrt{4 + x}}{2}$$

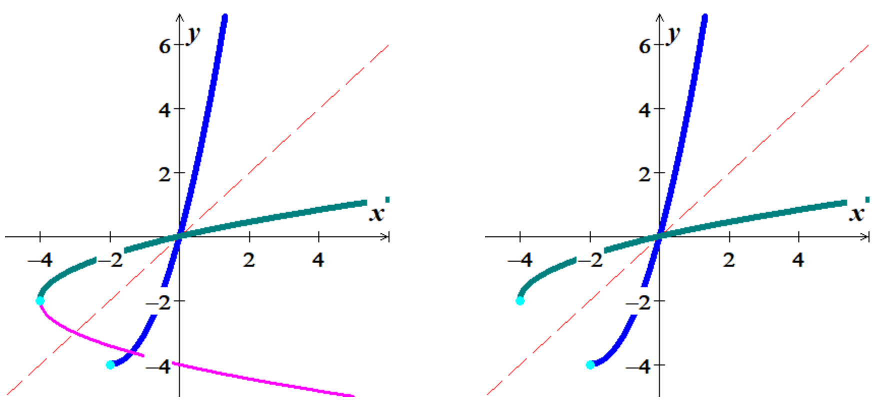
$$= -2 + \sqrt{x + 4}$$

$$\boxed{f^{-1}(x) = -2 + \sqrt{x + 4} \quad x \ge 0}$$

c) Domain of $f(x)$ = Range of $f^{-1}(x)$: $\boxed{[-2,\ \infty)}$

Range of $f(x)$ = Domain of $f^{-1}(x)$: $\boxed{[-4,\ \infty)}$

d)



Exercise

For the given function $f(x) = 3x + 5$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

d) Graph both functions (if $f^{-1}(x)$ exists)

Solution

a) $f(a) = f(b)$

$3a + 5 = 3b + 5$

$3a = 3b$

$a = b$

$\therefore\ f(x)$ is **1–1** **&** $f^{-1}(x)$ exists

b) $y = 3x + 5$

$x = 3y + 5$ *Interchange x and y*

$x - 5 = 3y$ *Solve for y*

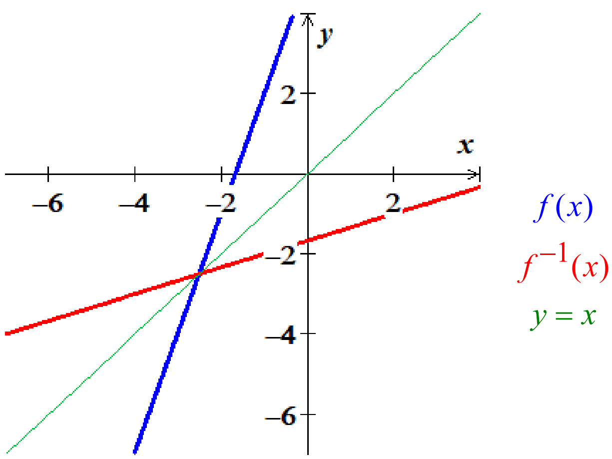
$\frac{x-5}{3} = y$

$f^{-1}(x) = \frac{x-5}{3}$

c) Domain of f^{-1} = Range of f: $\mathbb{R}$

Range of f^{-1} = Domain of f: $\mathbb{R}$

d)



Exercise

For the given function $f(x) = 2x^3 - 5$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

d) Graph both functions (if $f^{-1}(x)$ exists)

Solution

a) $f(a) = f(b)$

$2a^3 - 5 = 2b^3 - 5$

$a^3 = b^3$

$a = b$

$\therefore\ f(x)$ is **1–1** & $f^{-1}(x)$ exists

b) $y = 2x^3 - 5$

$y + 5 = 2x^3$

$\dfrac{y+5}{2} = x^3$

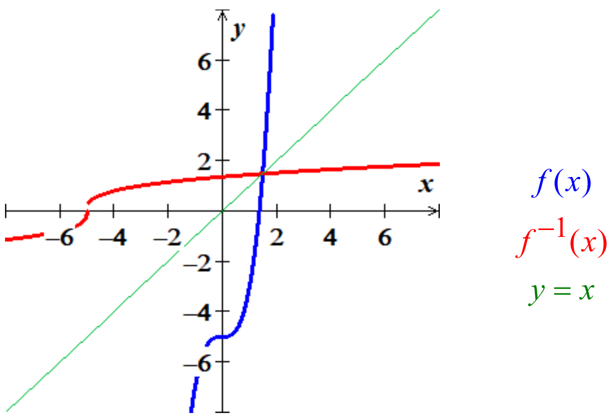
$x = \sqrt[3]{\dfrac{y+5}{2}}$

$\boxed{f^{-1}(x) = \sqrt[3]{\dfrac{x+5}{2}}}$

c) Domain of f^{-1} = Range of f: $\boxed{\mathbb{R}}$

Range of f^{-1} = Domain of f: $\boxed{\mathbb{R}}$

d)



Exercise

For the given function $f(x) = \sqrt{3 - x}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

d) Graph both functions (if $f^{-1}(x)$ exists)

Solution

a) $f(a) = f(b)$

$\left(\sqrt{3-a}\right)^2 = \left(\sqrt{3-b}\right)^2$

$3 - a = 3 - b$

$a = b$

$\therefore\ f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = \sqrt{3-x}$ $\qquad y \geq 0$

$y = \sqrt{3-x}$

$y^2 = 3 - x$

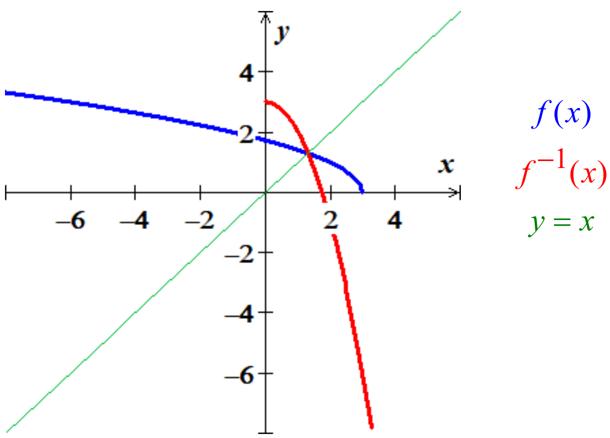
$x = 3 - y^2$ $\qquad x \geq 0$

$\boxed{f^{-1}(x) = 3 - x^2}$

c) Domain of f^{-1} = Range of f: $\boxed{(-\infty,\ 3]}$

Range of f^{-1} = Domain of f: $\boxed{[0,\ \infty)}$

d)



Exercise

For the given function $f(x) = \sqrt[3]{x} + 1$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

d) Graph both functions (if $f^{-1}(x)$ exists)

Solution

a) $f(a) = f(b)$

$\sqrt[3]{a} + 1 = \sqrt[3]{b} + 1$

$\left(\sqrt[3]{a}\right)^3 = \left(\sqrt[3]{b}\right)^3$

$a = b$

$\therefore\ f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = \sqrt[3]{x} + 1$

$y = \sqrt[3]{x} + 1$

$y - 1 = \sqrt[3]{x}$

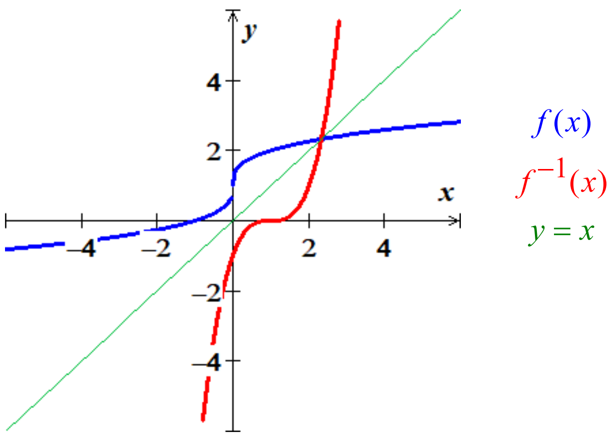
$(y-1)^3 = x$

$\boxed{f^{-1}(x) = (x-1)^3}$

c) Domain of f^{-1} = Range of f: $\boxed{\mathbb{R}}$

Range of f^{-1} = Domain of f: $\boxed{\mathbb{R}}$

d)



Exercise

For the given function $f(x) = \left(x^3 + 1\right)^5$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

d) Graph both functions (if $f^{-1}(x)$ exists)

Solution

a) $f(a) = f(b)$

$\left(a^3 + 1\right)^5 = \left(b^3 + 1\right)^5$

$a^3 + 1 = b^3 + 1$

$a^3 = b^3$

$a = b$

$\therefore\ f(x)$ is **1–1** **&** $f^{-1}(x)$ exists

b) $y = \left(x^3 + 1\right)^5$

$$y = \left(x^3 + 1\right)^5$$

$$\sqrt[5]{y} = x^3 + 1$$

$$\sqrt[5]{y} - 1 = x^3$$

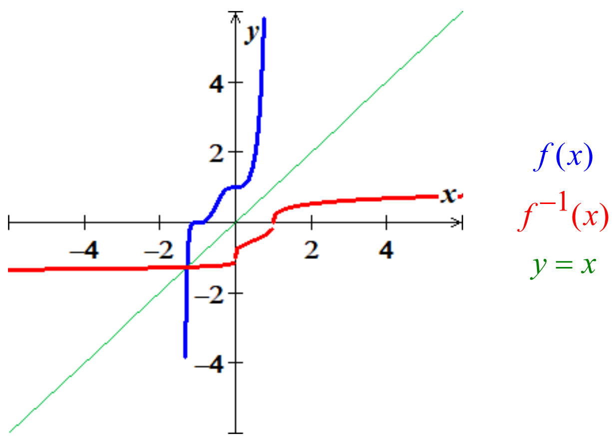
$$x = \sqrt[3]{\sqrt[5]{y} - 1}$$

$$f^{-1}(x) = \sqrt[3]{\sqrt[5]{x} - 1}$$

c) Domain of f^{-1} = Range of f: $\mathbb{R}$

Range of f^{-1} = Domain of f: $\mathbb{R}$

d)



Exercise

Find the exact value of the expression whenever it is defined: $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Solution

$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

Exercise

Find the exact value of the expression whenever it is defined: $\arccos\left(\frac{\sqrt{2}}{2}\right)$

Solution

$$\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

Exercise

Find the exact value of the expression whenever it is defined: $\arctan\left(-\frac{\sqrt{3}}{3}\right)$

Solution

$$\arctan\left(-\frac{\sqrt{3}}{3}\right) = -\arctan\left(\frac{\sqrt{3}}{3}\right)$$

$$= -\frac{\pi}{6}$$

Exercise

Evaluate without using a calculator: $\cos\left(\sin^{-1}\frac{1}{2}\right)$

Solution

$$\cos\left(\sin^{-1}\frac{1}{2}\right)$$

$$\sin\frac{\pi}{6} = \frac{1}{2}$$

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\cos\left(\sin^{-1}\frac{1}{2}\right) = \cos\frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2}$$

Exercise

Find the exact value of the expression whenever it is defined: $\arcsin\left(\sin\left(-\frac{\pi}{2}\right)\right)$

Solution

$$\arcsin\left(\sin\left(-\frac{\pi}{2}\right)\right) = -\frac{\pi}{2} \qquad -\frac{\pi}{2} \le -\frac{\pi}{2} \le \frac{\pi}{2}$$

Exercise

Find the exact value of the expression whenever it is defined: $\cos^{-1}\left(\cos\left(\frac{5\pi}{6}\right)\right)$

Solution

$$\cos^{-1}\left(\cos\left(\frac{5\pi}{6}\right)\right) = \frac{5\pi}{6} \qquad 0 \le \frac{5\pi}{6} \le \pi$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right)$

Solution

$$\tan^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right) = -\frac{\pi}{4} \qquad -\frac{\pi}{2} \le -\frac{\pi}{4} \le \frac{\pi}{2}$$

Exercise

Find the exact value of the expression whenever it is defined: $\sin\left(2\arccos\left(-\frac{3}{5}\right)\right)$

Solution

$$\sin\left(2\arccos\left(-\frac{3}{5}\right)\right) = \sin 2\alpha$$

$$= 2\sin\alpha\cos\alpha$$

$$\alpha = \arccos\left(-\frac{3}{5}\right) \quad \to \quad \cos\alpha = -\frac{3}{5}$$

$$\sin\alpha = \frac{3}{5}$$

$$\sin\left(2\arccos\left(-\frac{3}{5}\right)\right) = 2\frac{3}{5}\left(-\frac{3}{5}\right)$$

$$= -\frac{18}{25}$$

Exercise

Evaluate without using a calculator: $\cot\left(\tan^{-1}\frac{1}{2}\right)$

Solution

$$\alpha = \tan^{-1}\frac{1}{2}$$

$$\tan\alpha = \frac{1}{2}$$

$$\cot\alpha = \frac{1}{\tan\alpha}$$

$$= 2$$

Exercise

Evaluate without using a calculator: $\tan\left(\cos^{-1}\frac{3}{5}\right)$

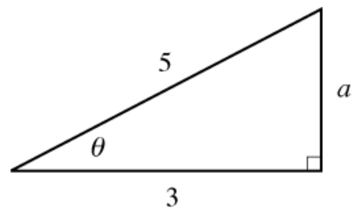
Solution

$$\tan\left(\cos^{-1}\frac{3}{5}\right)$$

$$5^2 = 3^2 + a^2 \quad \rightarrow \quad \underline{a = 4}$$

$$\tan\left(\cos^{-1}\frac{3}{5}\right) = \tan\theta$$

$$= \frac{4}{3}$$



Exercise

Find the exact value of the expression whenever it is defined: $\cos\left(\arctan\left(-\frac{3}{4}\right) - \arcsin\frac{4}{5}\right)$

Solution

$$\cos\left(\arctan\left(-\frac{3}{4}\right) - \arcsin\frac{4}{5}\right) = \cos(\alpha - \beta)$$

$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$\alpha = \arctan\left(-\frac{3}{4}\right)$ $\tan\alpha = -\frac{3}{4}$ $r = \sqrt{3^2 + 4^2} = 5$ $\sin\alpha = -\frac{3}{5} \qquad \cos\alpha = \frac{4}{5}$	$\beta = \arcsin\frac{4}{5}$ $\sin\beta = \frac{4}{5}$ $\Rightarrow \cos\beta = \frac{3}{5}$

$$\cos\left(\arctan\left(-\frac{3}{4}\right) - \arcsin\frac{4}{5}\right) = \frac{4}{5}\frac{3}{5} + \left(-\frac{3}{5}\right)\frac{4}{5}$$

$$= 0$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan\left(\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)\right)$

Solution

$$\tan\left(\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)\right) = \tan\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= \tan\left(\frac{\pi}{3}\right)$$

$$= \sqrt{3}$$

Exercise

Write an equivalent expression that involves x only for $\cos\left(\cos^{-1} x\right)$

Solution

$$\alpha = \cos^{-1} x$$

$$\cos\alpha = x$$

$$\cos\left(\cos^{-1} x\right) = \cos\alpha$$

$$= x$$

Exercise

Write an equivalent expression that involves x only for $\tan\left(\cos^{-1} x\right)$

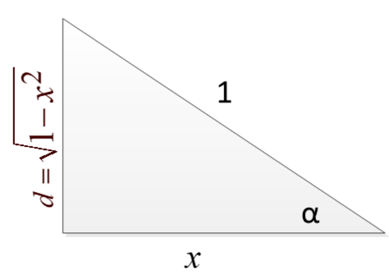
Solution

$$\alpha = \cos^{-1} x$$

$$\cos\alpha = x = \frac{x}{1}$$

$$x^2 + d^2 = 1 \Rightarrow d^2 = 1 - x^2$$

$$d = \sqrt{1-x^2}$$



$$\tan\left(\cos^{-1} x\right) = \tan\alpha$$

$$= \frac{\sqrt{1-x^2}}{x}$$

Exercise

Write an equivalent expression that involves x only for $\csc\left(\sin^{-1} \frac{1}{x}\right)$

Solution

$$\alpha = \sin^{-1} \frac{1}{x}$$

$\sin\alpha = \frac{1}{x}$

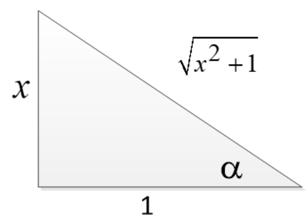
$\csc\left(\sin^{-1} x\right) = \csc\alpha = \frac{1}{\sin\alpha}$

$\qquad \boxed{= x}$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\sin\left(\tan^{-1} x\right)$

Solution



$\sin\left(\tan^{-1} x\right) = \sin\alpha$

$\alpha = \tan^{-1} x$

$\tan\alpha = x$

$\sin\left(\tan^{-1} x\right) = \sin\alpha$

$\qquad \boxed{= \frac{x}{\sqrt{x^2+1}}}$

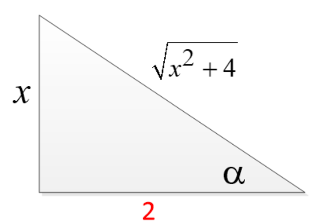
Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\sec\left(\sin^{-1}\frac{x}{\sqrt{x^2+4}}\right)$

Solution

$\alpha = \sin^{-1}\frac{x}{\sqrt{x^2+4}}$

$\sin\alpha = \frac{x}{\sqrt{x^2+4}}$



$\sqrt{\left(\sqrt{x^2+4}\right)^2 - x^2} = \sqrt{x^2+4-x^2} = \sqrt{4} = 2$

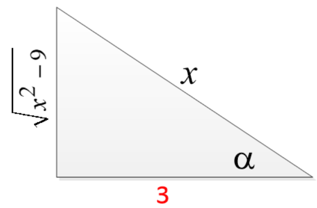
$\sec\left(\sin^{-1}\frac{x}{\sqrt{x^2+4}}\right) = \frac{1}{\cos\alpha}$

$\qquad \boxed{= \frac{2}{\sqrt{x^2+4}}}$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\cot\left(\sin^{-1}\frac{\sqrt{x^2-9}}{x}\right)$

Solution



$$\alpha = \sin^{-1}\frac{\sqrt{x^2-9}}{x}$$

$$\sin\alpha = \frac{\sqrt{x^2-9}}{x}$$

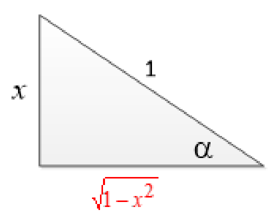
$$\cot\left(\sin^{-1}\frac{\sqrt{x^2-9}}{x}\right) = \cot\alpha$$

$$= \frac{3}{\sqrt{x^2-9}}$$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\sin\left(2\sin^{-1}x\right)$

Solution



$$\alpha = \sin^{-1}x$$

$$\sin\alpha = x$$

$$\sin\left(2\sin^{-1}x\right) = \sin 2\alpha$$

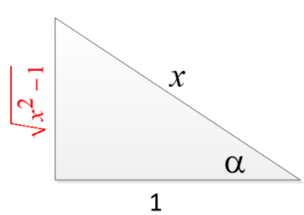
$$= 2\sin\alpha\cos\alpha$$

$$= 2x\sqrt{1-x^2}$$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\tan\left(\frac{1}{2}\cos^{-1}\frac{1}{x}\right)$

Solution



$$\alpha = \cos^{-1}\frac{1}{x}$$

$$\cos\alpha = \frac{1}{x}$$

$$\tan\left(\frac{1}{2}\cos^{-1}\frac{1}{x}\right) = \tan\left(\frac{\alpha}{2}\right)$$

$$=\frac{1-\cos\alpha}{\sin\alpha}$$

$$=\frac{1-\frac{1}{x}}{\frac{\sqrt{x^2-1}}{x}}$$

$$=\frac{\frac{x-1}{x}}{\frac{\sqrt{x^2-1}}{x}}$$

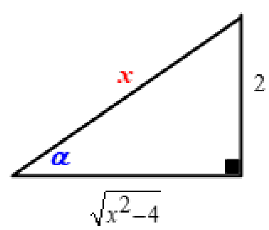
$$=\frac{x-1}{\sqrt{x^2-1}}$$

Exercise

Write the expression as an algebraic expression in x: $\sec\left(\tan^{-1}\frac{2}{\sqrt{x^2-4}}\right)$ $x>0$

Solution

$$\tan\alpha=\frac{2}{\sqrt{x^2-4}}$$



$$\sec\left(\tan^{-1}\frac{2}{\sqrt{x^2-4}}\right)=\sec\alpha$$

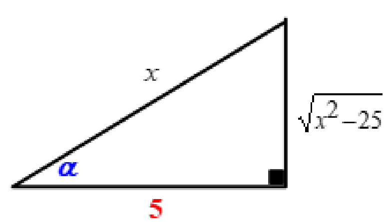
$$=\frac{x}{\sqrt{x^2-4}}$$

Exercise

Write the expression as an algebraic expression in x: $\sec\left(\sin^{-1}\frac{\sqrt{x^2-25}}{x}\right)$ $x>0$

Solution

$$\sin\alpha=\frac{\sqrt{x^2-25}}{x}$$



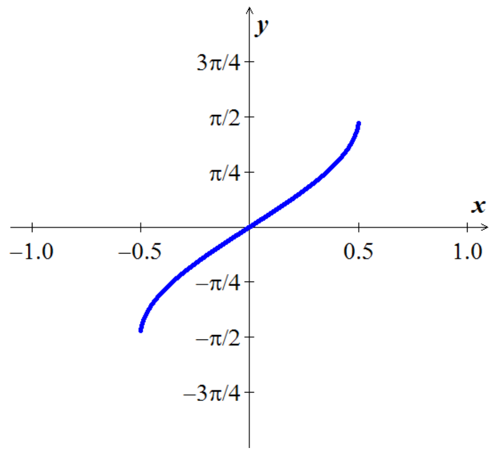
$$\sec\left(\sin^{-1}\frac{\sqrt{x^2-25}}{x}\right)=\sec\alpha$$

$$=\frac{x}{5}$$

Exercise

Sketch he graph of the equation: $y = \sin^{-1} 2x$

Solution



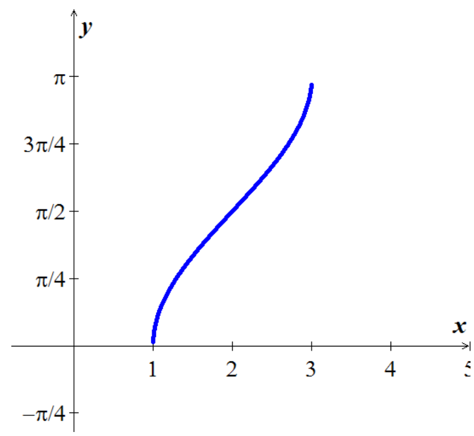
$$-\frac{\pi}{2} \le y \le \frac{\pi}{2} \quad \text{and} \quad -1 \le 2x \le 1$$

$$-\frac{1}{2} \le x \le \frac{1}{2}$$

Exercise

Sketch he graph of the equation: $y = \sin^{-1}(x-2) + \frac{\pi}{2}$

Solution



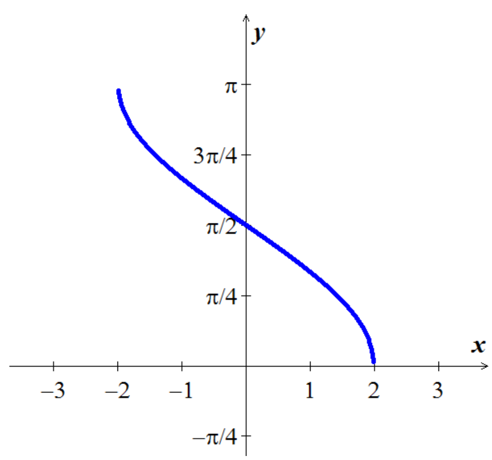
$$-\frac{\pi}{2} + \frac{\pi}{2} \le y \le \frac{\pi}{2} + \frac{\pi}{2} \quad \text{and} \quad -1 \le x - 2 \le 1$$

$$0 \le y \le \pi \quad \text{and} \quad 1 \le x \le 3$$

Exercise

Sketch he graph of the equation: $y = \cos^{-1} \frac{1}{2}x$

Solution



$$0 \le y \le \pi \quad \text{and} \quad -1 \le \frac{1}{2}x \le 1$$

$$-2 \le x \le 2$$