Solution Section 4.1 – Matrix Transformations from R^n to R^m

Exercise

Find the standard matrix for the transformation defined by the equations

a) $\begin{cases} w_1 = 2x_1 - 3x_2 + 4x_4 \\ w_2 = 3x_1 + 5x_2 - x_4 \end{cases}$

b) $\begin{cases} w_1 = 7x_1 + 2x_2 - 8x_3 \\ w_2 = \qquad\; - x_2 + 5x_3 \\ w_3 = 4x_1 + 7x_2 - x_3 \end{cases}$

c) $\begin{cases} w_1 = x_1 \\ w_2 = x_1 + x_2 \\ w_3 = x_1 + x_2 + x_3 \\ w_4 = x_1 + x_2 + x_3 + x_4 \end{cases}$

Solution

a) $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 & 4 \\ 3 & 5 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ The standard matrix is $\begin{bmatrix} 2 & -3 & 0 & 4 \\ 3 & 5 & 0 & -1 \end{bmatrix}$

b) $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ The standard matrix is $\begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix}$

c) $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ The standard matrix is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Exercise

Find the standard matrix for the operator T defined by the formula

a) $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$

b) $T(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3,\ x_1 + 5x_2,\ x_3)$

c) $T(x_1, x_2, x_3) = (4x_1,\ 7x_2,\ -8x_3)$

Solution

a) $T(x_1, x_2) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$

$$= \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The standard matrix is $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

b) $T(x_1, x_2, x_3) = \begin{bmatrix} x_1 + 2x_2 + x_3 \\ x_1 + 5x_2 \\ x_3 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The standard matrix is $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c) $T(x_1, x_2, x_3) = \begin{pmatrix} 4x_1 \\ 7x_2 \\ -8x_3 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

The standard matrix is $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -8 \end{pmatrix}$

Exercise

Find the standard matrix for the transformation T defined by the formula

a) $T\left(x_1, x_2\right)=\left(x_2,\ -x_1,\ x_1+3x_2,\ x_1-x_2\right)$

b) $T\left(x_1, x_2, x_3\right)=(0,\ 0,\ 0,\ 0,\ 0)$

c) $T\left(x_1, x_2, x_3, x_4\right)=\left(x_4,\ x_1,\ x_3,\ x_2,\ x_1-x_3\right)$

d) $T\left(x_1, x_2, x_3, x_4\right)=\left(7x_1+2x_2-x_3+x_4,\ x_2+x_3,\ -x_1\right)$

Solution

a) $T\left(x_1, x_2\right)=\begin{pmatrix} x_2 \\ -x_1 \\ x_1+3x_2 \\ x_1-x_2 \end{pmatrix}$ The matrix is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 3 \\ 1 & -1 \end{pmatrix}$

b) The matrix is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

c) The matrix is $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}$

d) $T\left(x_1,\ x_2,\ x_3,\ x_4\right)=\begin{bmatrix} 7x_1+2x_2-x_3+x_4 \\ x_2+x_3 \\ -x_1 \end{bmatrix}$

$$=\begin{bmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Solution   *Section* 4.2 – General Linear Transformations

Exercise

The matrix $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ gives a shearing transformation $T(x, y) = (x,\ 3x + y)$.

What happens to (1, 0) and (2, 0) on the *x*-axis.

What happens to the points on the vertical lines $x = 0$ and $x = a$?.

Solution

The points (1, 0) and (2, 0) on the *x*-axis transform by T to (1, 3) and (2, 6). The horizontal *x*-axis transforms to the straight line with slope 3 (going through (0, 0) of course). The points on the *y*-axis are not moved because $T(0, y) – (0, y)$. The *y* -axis is the line of eigenvectors of T with $\lambda = 1$.

The vertical line $x = a$ is moved up by $3a$, since $3a$ is added to the y component. This is ***shearing***. Vertical lines slide higher as you go from left to right.

Exercise

A nonlinear transformation T is invertible if every $\boldsymbol{b}$ in the output space comes from exactly one x in the input space. $T(\boldsymbol{x}) = \boldsymbol{b}$ always has exactly one solution. Which of these transformation (on real numbers $\boldsymbol{x}$ is invertible and what is T^{-1}? None are linear, not even T_3. When you solve $T(\boldsymbol{x}) = \boldsymbol{b}$, you are inverting

T: $T_1(x) = x^2 \quad T_2(x) = x^3 \quad T_3(x) = x + 9 \quad T_4(x) = e^x \quad T_5(x) = \frac{1}{x} \quad \textit{for nonzero } x\text{'s}$

Solution

T_1 is not invertible because $x^2 = 1 \to x = \pm 1$ and $x^2 = -1$ has no solution.

T_4 is not invertible because $e^x = -1$ has no solution.

T_2 is invertible. The solutions to $x^3 = b \to x = b^{1/3} = T_2^{-1}(b)$

T_3 is invertible. The solutions to $x + 9 = b \to x = b - 9 = T_3^{-1}(b)$

T_5 is invertible. The solutions to $\frac{1}{x} = b \to x = \frac{1}{b} = T_5^{-1}(b)$

Exercise

If S and T are linear transformations, is $S(T(\boldsymbol{v}))$ linear or quadratic?

a) If $S(\boldsymbol{v}) = \boldsymbol{v}$ and $T(\boldsymbol{v}) = \boldsymbol{v}$, then $S(T(\boldsymbol{v})) = \boldsymbol{v}$ or $\boldsymbol{v}^2$?

b) $S(\boldsymbol{w}_1 + \boldsymbol{w}_2) = S(\boldsymbol{w}_1) + S(\boldsymbol{w}_2)$ and $T(\boldsymbol{v}_1 + \boldsymbol{v}_2) = T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2)$ combine into

$$S(T(\boldsymbol{v}_1 + \boldsymbol{v}_2)) = S(\_\_\_\_\_) = \_\_\_\_ + \_\_\_\_\_\_$$

Solution

a) $S(T(v)) = S(v) = v$ **since $\boldsymbol{T}$ ($\boldsymbol{v}$) = $\boldsymbol{v}$**

b) $$S(T(\boldsymbol{v}_1 + \boldsymbol{v}_2)) = S(T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2))$$
$$= S(T(\boldsymbol{v}_1)) + S(T(\boldsymbol{v}_2))$$

It is quadratic.

Exercise

Find the range and kernel (like the column space and nullspace) of T:

a) $T(v_1, v_2) = (v_2, v_1)$

b) $T(v_1, v_2, v_3) = (v_1, v_2)$

c) $T(v_1, v_2) = (0,\ 0)$

d) $T(v_1, v_2) = (v_1, v_1)$

Solution

a) Range is the line $y = 0$, Kernel is the line $x = y$ in the xy plane.

b) Range is the xy plane, Kernel is the complementary line in $\mathbb{R}^3$.

c) Range is the point (0, 0), Kernel is plane

d) Range is the line $x = y$ in the xy plane, Kernel is the line $x = 0$.

Exercise

M is any 2 by 2 matrix and $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$. The transformation T is defined by $T(M)=AM$. What rules of matrix multiplication show that T is linear?

Solution

The distribution law and the association law for multiplication give the linearity

$$\begin{aligned} A(cM+dN) &= A(cM)+A(dN) \\ &= (Ac)M+(Ad)N \\ &= cA(M)+dA(N) \end{aligned}$$

Exercise

Which of these transformations satisfy $T(v+w)=T(v)+T(w)$ and which satisfy $T(cv)=cT(v)$?

a) $T(v)=\frac{v}{\|v\|}$

b) $T(v)=v_1+v_2+v_3$

c) $T(v)=(v_1, 2v_2, 3v_3)$

d) $T(v)=$ largest component of $\boldsymbol{v}$.

Solution

a) This is scaling the vector into a normal vector. This it is impossible that we get additivity, because the sums of normal vectors don't have to be normal. For example T(0, 1) and T(1, 0) for instance. However, true to its name this does have the scaling property. For $\boldsymbol{c}$ value, this value will be canceled from $\boldsymbol{v}$ and $\|\boldsymbol{v}\|$.

b) This satisfies both. One immediate way to see that it is matrix multiplication by [1, 1, 1], which is a linear operation and thus satisfies both properties.

c) This satisfies both. This a matrix multiplication by $\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix}$

d) Doesn't satisfy additivity [(0, 1) and (1, 0) still work]. Scaling doesn't work either, if we scale by -1 we now pick out the negative of the smallest component, which doesn't have to be related in any way to the largest component.

Exercise

Consider the basis $S=\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ for R^3, where $\boldsymbol{v}_1=(1,\ 1,\ 1)$ $\boldsymbol{v}_2=(1,\ 1,\ 0)$ $\boldsymbol{v}_3=(1,\ 0,\ 0)$ and let $T: R^3 \to R^3$ be the linear transformation for which

$$T(\boldsymbol{v}_1)=(2,\ -1,\ 4), \quad T(\boldsymbol{v}_2)=(3,\ 0,\ 1), \quad T(\boldsymbol{v}_3)=(-1,\ 5,\ 1)$$

Find a formula for $T(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$, and then use that formula to compute $T(2,\ 4,\ -1)$

Solution

$$\boldsymbol{u}=c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3$$

$$\begin{aligned}(x_1, x_2, x_3) &= c_1(1,\ 1,\ 1) + c_2(1,\ 1,\ 0) + c_3(1,\ 0,\ 0)\\ &= (c_1+c_2+c_3,\ c_1+c_2,\ c_1)\end{aligned}$$

$$\begin{cases} c_1+c_2+c_3=x_1\\ c_1+c_2 \qquad\ \ =x_2\\ c_1 \qquad\qquad\ \ =x_3\end{cases} \to \begin{cases} c_3=x_1-x_2\\ c_2=x_2-x_3\\ c_1=x_3\end{cases}$$

$$\begin{aligned}T(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) &= x_3T(\boldsymbol{v}_1)+(x_2-x_3)T(\boldsymbol{v}_2)+(x_1-x_2)T(\boldsymbol{v}_3)\\ &= x_3(2,-1,4)+(x_2-x_3)(3,0,1)+(x_1-x_2)(-1,5,1)\\ &= (2x_3+3x_2-3x_3-x_1+x_2,\ -x_3+5x_1-5x_2,\ 4x_3+x_2-x_3+x_1-x_2)\\ &= (-x_1+4x_2-x_3,\ 5x_1-5x_2-x_3,\ x_1+3x_3)\end{aligned}$$

$$\begin{aligned}T(2,\ 4,\ -1) &= (-2+16+1,\ 10-20+1,\ 2-3)\\ &= (15,\ -9,\ -1)\end{aligned}$$

Exercise

Consider the basis $S=\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ for R^3, where $\boldsymbol{v}_1=(1,\ 2,\ 1)$ $\boldsymbol{v}_2=(2,\ 9,\ 0)$ $\boldsymbol{v}_3=(3,\ 3,\ 4)$ and let $T: R^3 \to R^2$ be the linear transformation for which

$$T(\boldsymbol{v}_1)=(1,\ 0), \quad T(\boldsymbol{v}_2)=(-1,\ 1), \quad T(\boldsymbol{v}_3)=(0,\ 1)$$

Find a formula for $T(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$, and then use that formula to compute $T(7,13,7)$

Solution

$$\boldsymbol{u}=c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3$$

$$\begin{aligned}(x_1, x_2, x_3) &= c_1(1,\ 2,\ 1) + c_2(2,\ 9,\ 0) + c_3(3,\ 3,\ 4)\\ &= (c_1+2c_2+3c_3,\ 2c_1+9c_2+3c_3,\ c_1+4c_3)\end{aligned}$$

$$\begin{cases} c_1 + 2c_2 + 3c_3 = x_1 \\ 2c_1 + 9c_2 + 3c_3 = x_2 \\ c_1 \qquad\quad + 4c_3 = x_3 \end{cases} \rightarrow \begin{cases} c_1 + 7c_2 = x_2 - x_1 \\ \qquad\qquad 2c_1 + \frac{9}{7}x_2 - \frac{9}{7}x_1 - \frac{9}{7}c_1 + \frac{3}{4}x_3 - \frac{3}{4}c_1 = x_2 \\ c_1 + 4c_3 = x_3 \end{cases}$$

$$2c_1 - \frac{9}{7}c_1 - \frac{3}{4}c_1 = x_2 - \frac{9}{7}x_2 + \frac{9}{7}x_1 - \frac{3}{4}x_3$$

$$-\frac{1}{28}c_1 = \frac{9}{7}x_1 - \frac{2}{7}x_2 - \frac{3}{4}x_3$$

$$\underline{c_1 = -36x_1 + 8x_2 + 21x_3}$$

$$c_3 = \frac{1}{4}x_3 - \frac{1}{4}c_1 \quad \rightarrow \quad \underline{c_3 = 9x_1 - 2x_2 - 5x_3}$$

$$c_2 = \frac{1}{7}x_2 - \frac{1}{7}x_1 - \frac{1}{7}c_1 \quad \rightarrow \quad c_2 = \frac{1}{7}x_2 - \frac{1}{7}x_1 + \frac{36}{7}x_1 - \frac{8}{7}x_2 - 3x_3$$

$$\underline{c_2 = 5x_1 - x_2 - 3x_3}$$

$$\begin{aligned} T(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) &= (-36x_1 + 8x_2 + 21x_3)T(\boldsymbol{v}_1) + (5x_1 - x_2 - 3x_3)T(\boldsymbol{v}_2) \\ &\qquad + (9x_1 - 2x_2 - 5x_3)T(\boldsymbol{v}_3) \\ &= (-36x_1 + 8x_2 + 21x_3)(1,\ 0) + (5x_1 - x_2 - 3x_3)(-1,\ 1) + (9x_1 - 2x_2 - 5x_3)(0,\ 1) \\ &= (36x_1 - 8x_2 + 21x_3 - 5x_1 + x_2 + 3x_3,\ 5x_1 - x_2 - 3x_3 + 9x_1 - 2x_2 - 5x_3) \\ &= (41x_1 + 9x_2 + 24x_3,\ 14x_1 - 3x_2 - 8x_3) \end{aligned}$$

$$T(7,\ 13,\ 7) = (37(7) - 13(13) + 24(7),\ 8(7) + 3(13) - 8(7)) \ \underline{= (-2,\ 3)}$$

Exercise

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ be vectors in a vector space V, and let $T : V \rightarrow R^3$ be the linear transformation for which

$$T(\boldsymbol{v}_1) = (1,\ -1,\ 2), \quad T(\boldsymbol{v}_2) = (0,\ 3,\ 2), \quad T(\boldsymbol{v}_3) = (-3,\ 1,\ 2)$$

Find $T(2v_1 - 3v_2 + 4v_3)$

Solution

$$\begin{aligned} T(2v_1 - 3v_2 + 4v_3) &= 2T(v_1) - 3T(v_2) + 4T(v_3) \\ &= 2(1,\ -1,\ 2) - 3(0,\ 3,\ 2) + 4(-3,\ 1,\ 2) \\ &= (2,\ -2,\ 4) - (0,\ 9,\ 6) + (-12,\ 4,\ 8) \\ &\underline{= (-10,\ -7,\ 6)} \end{aligned}$$

Exercise

Let $T: R^2 \to R^2$ be the linear operation given by the formula $T(x, y) = (2x - y,\ -8x + 4y)$

Which of the following vectors are in $R(T)$

$\boldsymbol{a})\ (1,\ -4)\quad \boldsymbol{b})\ (5,\ 0)\quad \boldsymbol{c})\ (-3,\ 12)$

Solution

a) $T(x, y) = (2x - y,\ -8x + 4y) = (1,\ -4)$

$$\begin{cases} 2x - y = 1 \\ -8x + 4y = -4 \end{cases} \Leftrightarrow \left[\begin{array}{cc|c} 2 & -1 & 1 \\ -8 & 4 & -4 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right]$$

This is a consistent system, therefore (1, −4) is in $R(T)$

b) $T(x, y) = (2x - y,\ -8x + 4y) = (5,\ 0)$

$$\begin{cases} 2x - y = 5 \\ -8x + 4y = 0 \end{cases} \Leftrightarrow \left[\begin{array}{cc|c} 2 & -1 & 5 \\ -8 & 4 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{array}\right] \to 0 = 1$$

This is an inconsistent system, therefore (5, 0) is not in $R(T)$

c) $T(x, y) = (2x - y,\ -8x + 4y) = (-3,\ 12)$

$$\begin{cases} 2x - y = -3 \\ -8x + 4y = 12 \end{cases} \Leftrightarrow \left[\begin{array}{cc|c} 2 & -1 & -3 \\ -8 & 4 & 12 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \end{array}\right]$$

This is a consistent system, therefore (−3, 12) is in $R(T)$

Exercise

Let $T: R^2 \to R^2$ be the linear operation given by the formula $T(x,\ y) = (2x - y,\ -8x + 4y)$

Which of the following vectors are in $\ker(T)$

$\boldsymbol{a})\ (5,\ 10)\quad \boldsymbol{b})\ (3,\ 2)\quad \boldsymbol{c})\ (1,\ 1)$

Solution

a) $T(5,\ 10) = (10 - 10,\ -40 + 40) = (0,\ 0)$; therefore (5, 10) is in $\ker(T)$

b) $T(3,\ 2) = (6 - 2,\ -24 + 8) = (4,\ -16)$; therefore (3, 2) is not in $\ker(T)$

c) $T(1,\ 1) = (2 - 1,\ -8 + 4) = (1,\ -4)$; therefore (1, 1) is not in $\ker(T)$

Exercise

Let $T: R^4 \to R^3$ be the linear operation given by the formula

$T(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4) = (4x_1 + x_2 - 2x_3 - 3x_4,\ 2x_1 + x_2 + x_3 - 4x_4,\ 6x_1 - 9x_3 + 9x_4)$

Which of the following vectors are in $R(T)$ ***a***) $(0,\ 0,\ 6)$ ***b***) $(1,\ 3,\ 0)$ ***c***) $(2,\ 4,\ 1)$

Solution

a) $T(x_1, x_2, x_3, x_4) = (4x_1 + x_2 - 2x_3 - 3x_4,\ 2x_1 + x_2 + x_3 - 4x_4,\ 6x_1 - 9x_3 + 9x_4) = (0,\ 0,\ 6)$

$$\begin{cases} 4x_1 + x_2 - 2x_3 - 3x_4 = 0 \\ 2x_1 + x_2 + x_3 - 4x_4 = 0 \\ 6x_1 \qquad - 9x_3 + 9x_4 = 6 \end{cases}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 0 \\ 2 & 1 & 1 & -4 & 0 \\ 6 & 0 & -9 & 9 & 6 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & -\frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

This is a consistent system, therefore (0, 0, 6) is in $R(T)$

b) $T(x_1, x_2, x_3, x_4) = (4x_1 + x_2 - 2x_3 - 3x_4,\ 2x_1 + x_2 + x_3 - 4x_4,\ 6x_1 - 9x_3 + 9x_4) = (1,\ 3,\ 0)$

$$\begin{cases} 4x_1 + x_2 - 2x_3 - 3x_4 = 1 \\ 2x_1 + x_2 + x_3 - 4x_4 = 3 \\ 6x_1 \qquad - 9x_3 + 9x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 1 \\ 2 & 1 & 1 & -4 & 3 \\ 6 & 0 & -9 & 9 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & -\frac{3}{2} & 0 & -\frac{3}{2} \\ 0 & 1 & 4 & 0 & 10 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

This is a consistent system, therefore (1, 3, 0) is in $R(T)$

c) $T(x_1, x_2, x_3, x_4) = (4x_1 + x_2 - 2x_3 - 3x_4,\ 2x_1 + x_2 + x_3 - 4x_4,\ 6x_1 - 9x_3 + 9x_4) = (2,\ 4,\ 1)$

$$\begin{cases} 4x_1 + x_2 - 2x_3 - 3x_4 = 2 \\ 2x_1 + x_2 + x_3 - 4x_4 = 4 \\ 6x_1 \qquad - 9x_3 + 9x_4 = 1 \end{cases}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 2 \\ 2 & 1 & 1 & -4 & 4 \\ 6 & 0 & -9 & 9 & 1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & -\frac{3}{2} & 0 & -\frac{19}{12} \\ 0 & 1 & 4 & 0 & \frac{71}{6} \\ 0 & 0 & 0 & 1 & \frac{7}{6} \end{array}\right]$$

This is a consistent system, therefore (2, 4, 1) is in $R(T)$

Exercise

Let $T: R^4 \to R^3$ be the linear operation given by the formula

$$T(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4) = (4x_1 + x_2 - 2x_3 - 3x_4,\ 2x_1 + x_2 + x_3 - 4x_4,\ 6x_1 - 9x_3 + 9x_4)$$

Which of the following vectors are in $\ker(T)$ ***a***) $(3,\ -8,\ 2,\ 0)$ ***b***) $(0,\ 0,\ 0,\ 1)$ ***c***) $(0,\ -4,\ 1,\ 0)$

Solution

a) $T(3,\ -8,\ 2,\ 0) = (12-8-4,\ 6-8+2,\ 18-18) \underline{= (0,\ 0,\ 0)|}$

Therefore, (3, –8, 2, 0) is in $\ker(T)$

b) $T(0,\ 0,\ 0,\ 1) \underline{= (-3,\ -4,\ 9)|}$

Therefore, (0, 0, 0, 1) is ***not*** in $\ker(T)$

c) $T(0,\ -4,\ 1,\ 0) = (-4-2,\ -4+1,\ -9) \underline{= (-6,\ -3,\ -9)|}$

Therefore, (0, –4, 1, 0) is ***not*** in $\ker(T)$

Exercise

Determine if the given function T is a linear transformation

$$T: M_{22} \to M_{22} \quad by \quad T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2ab & 3cd \\ 0 & 0 \end{bmatrix}$$

Solution

Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$$\begin{aligned}
T(A+B) &= T\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \\
&= \begin{bmatrix} 2(a_1 + a_2)(b_1 + b_2) & 3(c_1 + c_2)(d_1 + d_2) \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2a_1b_1 + 2a_1b_2 + 2a_2b_1 + 2a_2b_2 & 3c_1d_1 + 3c_1d_2 + 3c_2d_1 + 3c_2d_2 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2a_1b_1 & 3c_1d_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2a_2b_2 & 3c_2d_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2a_1b_2 + 2a_2b_1 & 3c_1d_2 + 3c_2d_1 \\ 0 & 0 \end{bmatrix} \\
&= T(A) + T(B) + \begin{bmatrix} 2a_1b_2 + 2a_2b_1 & 3c_1d_2 + 3c_2d_1 \\ 0 & 0 \end{bmatrix} \\
&\underline{\neq T(A) + T(B)|}
\end{aligned}$$

Function T is NOT a linear transformation

Exercise

Determine if the given function T is a linear transformation

$$T: M_{22} \to M_{22} \quad by \quad T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+d & 0 \\ 0 & b+c \end{bmatrix}$$

Solution

Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$$\begin{aligned} T(A+B) &= T\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1+a_2+d_1+d_2 & 0 \\ 0 & b_1+b_2+c_1+c_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1+d_1 & 0 \\ 0 & b_1+c_1 \end{bmatrix} + \begin{bmatrix} a_2+d_2 & 0 \\ 0 & b_2+c_2 \end{bmatrix} \\ &= T(A)+T(B) \quad \checkmark \end{aligned}$$

$$\begin{aligned} T(kA) &= T\left(k\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{bmatrix}\right) \\ &= \begin{bmatrix} ka_1+kd_1 & 0 \\ 0 & kb_1+kc_1 \end{bmatrix} \\ &= \begin{bmatrix} k(a_1+d_1) & 0 \\ 0 & k(b_1+c_1) \end{bmatrix} \\ &= k\begin{bmatrix} a_1+d_1 & 0 \\ 0 & b_1+c_1 \end{bmatrix} \\ &= kT(A) \quad \checkmark \end{aligned}$$

Since $T(A+B) = T(A)+T(B)$ and $T(kA) = kT(A)$, then function T is a linear transformation.

Exercise

Determine if the given function T is a linear transformation where A is fixed 2×3 matrix

$$T : M_{22} \to M_{23} \quad by \quad T(B) = BA$$

Solution

$$\begin{aligned} T(B+C) &= (B+C)A \\ &= BA + CA \\ &= T(B) + T(C) \end{aligned}$$

$$\begin{aligned} T(rB) &= (rB)A \\ &= r(BA) \\ &= rT(B) \end{aligned}$$

Function T is a linear transformation

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T = f_A$. $\quad T(x, y, z) = (2x + y,\ x - y + z)$

Solution

Let $\boldsymbol{u} = (x_1, y_1, z_1)$ *and* $\boldsymbol{v} = (x_2, y_2, z_2)$

$$\begin{aligned} T(\boldsymbol{u}+\boldsymbol{v}) &= T(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2) \\ &= \left(2(x_1 + x_2) + y_1 + y_2,\ x_1 + x_2 - (y_1 + y_2) + z_1 + z_2\right) \\ &= (2x_1 + y_1 + 2x_2 + y_2,\ x_1 - y_1 + z_1 + x_2 - y_2 + z_2) \\ &= (2x_1 + y_1,\ x_1 - y_1 + z_1) + (2x_2 + y_2,\ x_2 - y_2 + z_2) \\ &= T(x_1, y_1, z_1) + T(x_2, y_2, z_2) \\ &= T(\boldsymbol{u}) + T(\boldsymbol{v}) \end{aligned}$$

$$\begin{aligned} T(r\boldsymbol{u}) &= T(rx_1,\ ry_1,\ rz_1) \\ &= (2rx_1 + ry_1,\ rx_1 - ry_1 + rz_1) \\ &= r(2x_1 + y_1,\ x_1 - y_1 + z_1) \\ &= rT(\boldsymbol{u}) \end{aligned}$$

Since $T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$ and $T(r\boldsymbol{u}) = rT(\boldsymbol{u})$, then function T is a linear transformation.

Domain: $T:\ \mathbb{R}^3 \to \mathbb{R}^2$

$$T(x, y, z) = (2x + y,\ x - y + z) \quad \Rightarrow \begin{pmatrix} 2x + y \\ x - y + z \end{pmatrix}$$

$$A = \begin{array}{c} \\ \end{array}\begin{array}{ccc} x & y & z \end{array}$$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

Exercise

Determine if the given function *T* is a linear transformation. Also give the domain and range of *T*.

$$T(x, y) = \left(x^2, y\right)$$

Solution

Let $\boldsymbol{u} = \left(x_1, y_1\right)$ *and* $\boldsymbol{v} = \left(x_2, y_2\right)$

$$\begin{aligned} T(\boldsymbol{u} + \boldsymbol{v}) &= T\left(x_1 + x_2, y_1 + y_2\right) \\ &= \left(\left(x_1 + x_2\right)^2,\ y_1 + y_2\right) \\ &= \left(x_1^2 + x_2^2 + 2x_1x_2,\ y_1 + y_2\right) \\ &= \left(x_1^2,\ y_1\right) + \left(x_2^2,\ y_2\right) + \left(2x_1x_2,\ 0\right) \\ &= T(\boldsymbol{u}) + T(\boldsymbol{v}) + \left(2x_1x_2,\ 0\right) \\ &\neq T(\boldsymbol{u}) + T(\boldsymbol{v}) \end{aligned}$$

The function *T* is ***not*** a linear transformation.

Domain: $T:\ \mathbb{R}^2 \to \mathbb{R}^2$

Exercise

Determine if the given function *T* is a linear transformation. Also give the domain and range of *T*; if *T* is linear, find the *A* such $T = f_A$.

$$T(x, y, z) = (z - x, z - y)$$

Solution

Let $\boldsymbol{u} = \left(x_1, y_1, z_1\right)$ *and* $\boldsymbol{v} = \left(x_2, y_2, z_2\right)$

$$T(\boldsymbol{u} + \boldsymbol{v}) = T\left(x_1 + x_2, y_1 + y_2, z_1 + z_2\right)$$

$$=\left(z_1+z_2-\left(x_1+x_2\right),\ z_1+z_2-\left(y_1+y_2\right)\right)$$

$$=\left(z_1+z_2-x_1-x_2,\ z_1+z_2-y_1-y_2\right)$$

$$=\left(z_1-x_1,\ z_1-y_1\right)+\left(z_2-x_2,\ z_2-y_2\right)$$

$$=T\left(x_1,y_1,z_1\right)+T\left(x_2,y_2,z_2\right)$$

$$=T(\boldsymbol{u})+T(\boldsymbol{v})$$

$$T(r\boldsymbol{u})=T\left(rx_1,\ ry_1,\ rz_1\right)$$

$$=\left(rz_1-rx_1,\ rz_1-ry_1\right)$$

$$=r\left(z_1-x_1,\ z_1-y_1\right)$$

$$=rT(\boldsymbol{u})$$

Since $T(\boldsymbol{u}+\boldsymbol{v})=T(\boldsymbol{u})+T(\boldsymbol{v})$ and $T(r\boldsymbol{u})=rT(\boldsymbol{u})$, then function T is a linear transformation.

Domain: $T:\ \mathbb{R}^3\to\mathbb{R}^2$

$$T(x,y,z)=(z-x,\ z-y)\Rightarrow\begin{pmatrix}-x+z\\-y+z\end{pmatrix}$$

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ A= & \begin{bmatrix}-1&0&1\\0&-1&1\end{bmatrix}\end{array}$$

Exercise

Show that the function $T:\ \mathrm{R}^3\to R^2$ given the formula $T\left(x_1,x_2,x_3\right)=\left(2x_1-x_2+x_3\ ,\ x_2-4x_3\right)$ is linear transformation

Solution

Let $\boldsymbol{u}=\left(x_1,x_2,x_3\right)$ *and* $\boldsymbol{v}=\left(y_1,y_2,y_3\right)$

$$T(\boldsymbol{u}+\boldsymbol{v})=T\left(x_1+y_1,\ x_2+y_2,\ x_3+y_3\right)$$

$$=\left(2\left(x_1+y_1\right)-x_2-y_2+x_3+y_3\ ,\ x_2+y_2-4\left(x_3+y_3\right)\right)$$

$$=\left(2x_1+2y_1-x_2-y_2+x_3+y_3\ ,\ x_2+y_2-4x_3-4y_3\right)$$

$$=\left(2x_1-x_2+x_3\ ,\ x_2-4x_3\right)+\left(2y_1-y_2+y_3\ ,\ y_2-4y_3\right)$$

$$=T(\boldsymbol{u})+T(\boldsymbol{v})$$

$$T(r\boldsymbol{u})=T\left(rx_1,\ rx_2,\ rx_3\right)$$

$$= \left(2rx_1 - rx_2 + rx_3 \ , \ rx_2 - 4rx_3\right)$$
$$= r\left(2x_1 - x_2 + x_3 \ , \ x_2 - 4x_3\right)$$
$$= rT(\boldsymbol{u})$$

Since $T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$ and $T(r\boldsymbol{u}) = rT(\boldsymbol{u})$, then function T is a linear transformation.

Solution Section 4.3 – LU−Decompositions

Exercise

What matrix E puts A into triangular form $EA=U$? Multiply by $E^{-1}=L$ to factor A into LU:

$$A=\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$$

Solution

$$A=\begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{pmatrix} R_3 - 3R_1 : \ell_{31}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{pmatrix} = U$$

$$E_{31}A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{pmatrix}$$

$$=\begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{pmatrix} = U$$

$$E=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

$$L=E^{-1}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$A=LU=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{pmatrix}$$

Exercise

Solve $Lc = b$ to find $\boldsymbol{c}$. Then solve $Ux = c$ to find $\boldsymbol{x}$. What was A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Solution

$$Lc = b$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

$$\to \begin{cases} \boxed{c_1 = 4} \\ c_1 + c_2 = 5 \Rightarrow \lfloor c_2 = 5 - 4 = 1 \rfloor \\ c_1 + c_2 + c_3 = 6 \Rightarrow \lfloor c_3 = 6 - 4 - 1 = 1 \rfloor \end{cases} \Rightarrow c = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

$$Ux = c$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x + y + z = 4 \\ y + z = 1 \\ z = 1 \end{cases} \to \begin{cases} x = 3 \\ y = 0 \end{cases} \Rightarrow x = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

$$Lc = b \Rightarrow LUx = b$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}}_{b}$$

Exercise

Find L and U for the symmetric matrix

$$A=\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get $A=LU$ with four pivots

Solution

$$A=\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

Exercise

For which c is $A=LU$ impossible – with three pivots?

$$A=\begin{pmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{pmatrix} R_2-3R_1$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & c-6 & 1 \\ 0 & 1 & 1 \end{pmatrix} R_3-R_1 \quad \to c-6\neq 0 \Rightarrow \boxed{c\neq 6}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & c-6 & 1 \\ 0 & 1 & 1 \end{pmatrix} \frac{1}{c-6}R_1$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{c-6} \\ 0 & 1 & 1 \end{pmatrix} R_3-R_2$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{c-6} \\ 0 & 0 & \frac{c-7}{c-6} \end{pmatrix} \to c-7 \neq 0 \Rightarrow \boxed{c \neq 7}$$

LU will be impossible for $c=6$ *and* $c=7$

Exercise

Find an LU-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix} \quad \frac{1}{2}R_1 \quad \boxed{\frac{1}{2} : \ell_1}$$

$$\to \begin{bmatrix} 1 & 4 \\ -1 & -1 \end{bmatrix} \quad R_2 + R_1 \quad \boxed{1 : \ell_{21}}$$

$$\begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \quad \frac{1}{3}R_2 \quad \boxed{\frac{1}{3} : \ell_2}$$

$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \quad \boldsymbol{U}$$

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & \frac{1}{3} \end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} \quad \boldsymbol{L}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \Rightarrow \begin{cases} 2y_1 = -2 & y_1 = -1 \\ -y_1 + 3y_2 = -2 & y_2 = -1 \end{cases}$$

$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} x_1 + 4x_2 = -1 \\ x_2 = -1 \end{cases} \to x_1 = 3$$

The solution: $x_1 = 3$ *and* $x_2 = -1$

Exercise

Find an *LU*-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} -5 & -10 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix}$$

Solution

$$\begin{bmatrix} -5 & -10 \\ 6 & 5 \end{bmatrix} \quad -\frac{1}{5}R_1 \quad \boxed{-\frac{1}{5}:\ell_1}$$

$$\to \begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix} \quad R_2 - 6R_1 \quad \boxed{-6:\ell_{21}}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & -7 \end{bmatrix} \quad -\frac{1}{7}R_2 \quad \boxed{-\frac{1}{7}:\ell_2}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \boldsymbol{U}$$

$$\begin{bmatrix} -\frac{1}{5} & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 0 \\ -6 & 1 \end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix} -5 & 0 \\ 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 0 \\ 6 & -\frac{1}{7} \end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}$$

$$\to \begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix} \quad \boldsymbol{L}$$

$$\begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix} \Rightarrow \begin{cases} -5y_1 = -10 & y_1 = 2 \\ 6y_1 - 7y_2 = 19 & y_2 = -1 \end{cases}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} x_1 + 2x_2 = 2 \\ x_2 = -1 \end{cases} \to x_1 = 4$$

The solution: $x_1 = 4$ *and* $x_2 = -1$

Exercise

Find an *LU*-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix} \quad \frac{1}{2}R_1 \quad \boxed{\frac{1}{2}:\ell_1}$$

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}$ $R_3 + R_1$ $\boxed{1:\ell_{31}}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ 0 & 4 & 1 \end{bmatrix}$ $-\frac{1}{2}R_2$ $\boxed{-\frac{1}{2}:\ell_{22}}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 4 & 1 \end{bmatrix}$ $R_3 - 4R_2$ $\boxed{-4:\ell_{32}}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & -4 & 1 \end{bmatrix} \xrightarrow{E_4^{-1}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix}$ $\frac{1}{5}R_3$ $\boxed{\frac{1}{5}:\ell_{32}}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & -4 & \frac{1}{5} \end{bmatrix} \xrightarrow{E_5^{-1}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix}$
$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ U	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix}$ L

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix} \rightarrow \begin{cases} 2y_1 = -4 \\ -2y_2 = -2 \\ y_1 + 4y_2 + 5y_3 = 6 \end{cases} \Rightarrow \begin{matrix} y_1 = -2 \\ y_2 = 1 \\ y_3 = 0 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{cases} x_1 - x_2 - x_3 = -2 \rightarrow x_1 = -1 \\ x_2 - x_3 = 1 \rightarrow x_2 = 1 \\ x_3 = 0 \end{cases}$$

Solution: $x_1 = -1, \quad x_2 = 1, \quad x_3 = 0$

Exercise

Find an LU-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$$

Solution

$\begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ $-\frac{1}{3}R_1$ $\boxed{-\frac{1}{3} : \ell_1}$	$\begin{bmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -4 & 2 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ $R_2 - R_1$ $\boxed{-1 : \ell_{21}}$	$\begin{bmatrix} -3 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix} -3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $\frac{1}{2}R_2$ $\boxed{\frac{1}{2} : \ell_{22}}$	$\begin{bmatrix} -3 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $R_3 - R_2$ $\boxed{-1 : \ell_{32}}$	$\begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{E_4^{-1}} \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ U	$\begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ L

$$\begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix} \rightarrow \begin{cases} -3y_1 = -33 \Rightarrow \underline{y_1 = 11} \\ y_1 + 2y_2 = 7 \Rightarrow \underline{y_2 = -2} \\ y_2 + y_3 = -1 \Rightarrow \underline{y_3 = 1} \end{cases}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \\ 1 \end{bmatrix} \rightarrow \begin{cases} x_1 - 4x_2 + 2x_3 = 11 \Rightarrow x_1 = 1 \\ x_2 = -2 \\ x_3 = 1 \end{cases}$$

Solution: $\underline{x_1 = 1, \quad x_2 = -2, \quad x_3 = 1}$

Exercise

Find an LU-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \\ 7 \end{bmatrix}$$

Solution

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad -R_1 \quad \boxed{-1 : \ell_{11}} \qquad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad R_2 - 2R_1 \quad \boxed{-2 : \ell_{21}} \qquad \begin{bmatrix} -1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 3 & 0 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad \tfrac{1}{3}R_2 \quad \boxed{\tfrac{1}{3} : \ell_{22}} \qquad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad R_3 + R_2 \quad \boxed{1 : \ell_{23}} \qquad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_4^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad \tfrac{1}{2}R_3 \quad \boxed{\tfrac{1}{2} : \ell_{23}} \qquad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_4^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad R_4 - R_3 \quad \boxed{-1 : \ell_{43}} \qquad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{E_5^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad \tfrac{1}{4}R_4 \quad \boxed{\tfrac{1}{4} : \ell_{43}} \qquad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & \frac{1}{4} \end{bmatrix} \xrightarrow{E_6^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{U}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix} \quad \boldsymbol{L}$$

For lower triangular: $\begin{bmatrix} -1 & 0 & 0 & 0 \\ -2 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & \frac{1}{4} \end{bmatrix} \xrightarrow{E^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \\ 7 \end{bmatrix} \rightarrow \begin{cases} -y_1 = 5 \rightarrow \underline{y_1 = -5} \\ 2y_1 + 3y_2 = -1 \rightarrow \underline{y_2 = 3} \\ -y_2 + 2y_3 = 3 \rightarrow \underline{y_3 = 3} \\ y_3 + 4y_4 = 7 \rightarrow \underline{y_4 = 1} \end{cases}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 3 \\ 1 \end{bmatrix} \rightarrow \begin{cases} x_1 - x_3 = -5 \rightarrow \underline{x_1 = -3} \\ x_2 + 2x_4 = 3 \rightarrow \underline{x_2 = 1} \\ x_3 + x_4 = 3 \rightarrow \underline{x_3 = 2} \\ \underline{x_4 = 1} \end{cases}$$

Solution: $\underline{x_1 = -3, \quad x_2 = 1, \quad x_3 = 2, \quad x_4 = 3}$

Solution Section 4.4 – Introduction to Eigenvalues

Exercise

Find the eigenvalues and eigenvectors of $A,\ A^2,\ A^{-1},\ and\ \ A+4I$:

$$A=\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad and \quad A^2=\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

Check the trace $\lambda_1+\lambda_2$ and the determinant $\lambda_1\lambda_2$ for A and also A^2.

Solution

For A:

$$\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix}$$

$$=(2-\lambda)^2-1$$

$$=\lambda^2-4\lambda+3=0$$

The eigenvalues of $\boldsymbol{A}$ are $\lambda_1=1 \quad and \quad \lambda_2=3$.

The trace of a square matrix A is the sum of the elements on the main diagonal: 2 + 2 agrees with 1+ 3. The det($\boldsymbol{A}$) = 3 agrees with the product $\lambda_1\lambda_2$.

The eigenvectors for $\boldsymbol{A}$ are:

$\lambda_1=1$: $\left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix} 2-1 & -1 \\ -1 & 2-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}\to\begin{cases} x-y=0 \\ -x+y=0 \end{cases}\Rightarrow x=y$$

Therefore the eigenvector $V_1=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2=3$: $\left(A-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}\to\begin{cases} -x-y=0 \\ -x-y=0 \end{cases}\Rightarrow x=-y$$

Therefore the eigenvector $V_2=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

For A^2:

$$\det\left(A^2-\lambda I\right)=\begin{vmatrix}5-\lambda & -4\\ -4 & 5-\lambda\end{vmatrix}=(5-\lambda)^2-16=\lambda^2-10\lambda+9=0$$

The eigenvalues of A^2 are $\lambda_1=1$ *and* $\lambda_2=9$. **Or** $\lambda_1=1^2=1$ *and* $\lambda_2=3^2=9$

$$\begin{cases} tr(A)=5+5=10\\ \lambda_1+\lambda_2=1+9=10\end{cases}\Rightarrow tr(A)=\lambda_1+\lambda_2$$

$$\begin{cases} \left|A^2\right|=\begin{vmatrix}5 & -4\\ -4 & 5\end{vmatrix}=9\\ \lambda_1\lambda_2=1(9)=9\end{cases}\Rightarrow \left|A^2\right|=\lambda_1\lambda_2$$

$\lambda_1=1:\left(A^2-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix}4 & -4\\ -4 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\to\begin{cases}4x-4y=0\\ -4x+4y=0\end{cases}\Rightarrow x=y$$

Therefore the eigenvector $V_1=\begin{pmatrix}1\\ 1\end{pmatrix}$

$\lambda_2=9:\left(A^2-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix}-4 & -4\\ -4 & -4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\to\begin{cases}-4x-4y=0\\ -4x-4y=0\end{cases}\Rightarrow x=y$$

Therefore the eigenvector $V_2=\begin{pmatrix}1\\ -1\end{pmatrix}$

For A^{-1}:

$$A^{-1}=\frac{1}{3}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}=\begin{pmatrix}\frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}$$

$$\det\left(A^{-1}-\lambda I\right)=\begin{vmatrix}\frac{2}{3}-\lambda & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}-\lambda\end{vmatrix}=\left(\frac{2}{3}-\lambda\right)^2-\frac{1}{9}=\lambda^2-\frac{4}{3}\lambda+\frac{1}{3}=0$$

The eigenvalues of A^{-1} are $\lambda_1=1$ *and* $\lambda_2=\frac{1}{3}$.

$\lambda_1=1:\left(A^{-1}-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix} \frac{2}{3}-1 & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3}-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -\frac{1}{3}x+\frac{1}{3}y=0 \\ \frac{1}{3}x-\frac{1}{3}y=0 \end{cases} \to \boxed{x=y}$$

Therefore the eigenvector $V_1=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2=\frac{1}{3}$: $\left(A^{-1}-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix} \frac{2}{3}-\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3}-\frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \frac{1}{3}x+\frac{1}{3}y=0 \\ \frac{1}{3}x+\frac{1}{3}y=0 \end{cases} \to \boxed{x=-y}$$

Therefore the eigenvector $V_2=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

For $A+4I$:

$$A+4I=\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}+\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}=\begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix}$$

$$\det\left(A^{-1}-\lambda I\right)=\begin{vmatrix} 6-\lambda & 1 \\ 1 & 6-\lambda \end{vmatrix}=(6-\lambda)^2-1=\lambda^2-12\lambda+35=0$$

The eigenvalues of A^{-1} are $\lambda_1=5$ *and* $\lambda_2=7$.

$\lambda_1=5$: $\left(A+4I-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix} 6-5 & 1 \\ 1 & 6-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x+y=0 \\ x+y=0 \end{cases} \to \boxed{x=-y}$$

Therefore the eigenvector $V_1=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2=7$: $\left(A+4I-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \begin{cases} -x + \frac{1}{3}y = 0 \\ x - y = 0 \end{cases} \to \boxed{x = y}$$

Therefore the eigenvector $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The eigenvalues $(A) = \lambda$

The eigenvalues $\left(A^2\right) = \lambda^2$

The eigenvalues $\left(A^{-1}\right) = \frac{1}{\lambda}$

Exercise

Show directly that the given vectors are eigenvectors of the given matrix. What are the corresponding eigenvalues

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$$

Solution

$$A\boldsymbol{v}_1 = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ -21 \end{bmatrix} = 7\begin{bmatrix} 1 \\ -3 \end{bmatrix} = 7\boldsymbol{v}_1$$

$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is an eigenvectors corresponding to the eigenvalue 7.

$$A\boldsymbol{v}_2 = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0\boldsymbol{v}_2$$

$\boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvectors corresponding to the eigenvalue 0.

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & -2 \\ -3 & 6-\lambda \end{vmatrix} \\ &= (1-\lambda)(6-\lambda) - 6 \\ &= 6 - 7\lambda + \lambda^2 - 6 \\ &= \lambda^2 - 7\lambda = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = 0$ *and* $\lambda_2 = 7$

Exercise

For which real numbers c does this matrix A have

$$A=\begin{pmatrix} 2 & -c \\ -1 & 2 \end{pmatrix}$$

a) Two real eigenvalues and eigenvectors.

b) A repeated eigenvalue with only one eigenvector

c) Two complex eigenvalues and eigenvectors.

Solution

$$\begin{aligned} \left|A-\lambda I\right| &= \begin{vmatrix} 2-\lambda & -c \\ -1 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)^2 - c \\ &= \lambda^2 - 4\lambda + 4 - c = 0 \end{aligned}$$

$$\lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-c)}}{2(1)}$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16+4c}}{2}$$

a) Two real eigenvalues and eigenvectors, when $16+4c>0 \ \rightarrow \ 4c>-16 \ \Rightarrow \ \boxed{c>-4}$

b) A repeated eigenvalue with only one eigenvector, when $16+4c=0 \ \Rightarrow \ \boxed{c=-4}$

c) Two complex eigenvalues and eigenvectors, when $16+4c<0 \ \Rightarrow \ \boxed{c<-4}$

Exercise

Find the eigenvalues of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{AB}$, and $\boldsymbol{BA}$:

$$A=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad AB=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad BA=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

a) The eigenvalues of $\boldsymbol{AB}$ (are equal to) (are not equal to) eigenvalues of $\boldsymbol{A}$ times eigenvalues of $\boldsymbol{B}$.

b) The eigenvalues of $\boldsymbol{AB}$ (are equal to) (are not equal to) eigenvalues of $\boldsymbol{BA}$.

Solution

Since $\boldsymbol{A}$ is a lower triangular, then $\lambda_1 = \lambda_2 = 1$

Since $\boldsymbol{B}$ is a upper triangular, then $\lambda_1 = \lambda_2 = 1$

$$\det(AB - I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 1 = \lambda^2 - 3\lambda + 1 = 0 \qquad \lambda_1 = \frac{3-\sqrt{5}}{2} \quad \lambda_2 = \frac{3+\sqrt{5}}{2}$$

$$\det(BA - I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1 = 0 \qquad \lambda_1 = \frac{3-\sqrt{5}}{2} \quad \lambda_2 = \frac{3+\sqrt{5}}{2}$$

a) The eigenvalues of $\boldsymbol{AB}$ are not equal to eigenvalues of $\boldsymbol{A}$ times eigenvalues of $\boldsymbol{B}$.

b) The eigenvalues of $\boldsymbol{AB}$ are equal to the eigenvalues of $\boldsymbol{BA}$.

Exercise

When $a+b=c+d$ show that (1, 1) is an eigenvector and find both eigenvalues of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Solution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix} = \begin{pmatrix} a+b \\ a+b \end{pmatrix} = (a+b)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

If $a+b=c+d=\lambda_1$

$tr(A) = a+d = \lambda_1 + \lambda_2$

$$\begin{aligned} \lambda_2 &= (a+d) - \lambda_1 \\ &= a+d-(a+b) \\ &= a+d-a-b \\ &= d-b \qquad \textit{or} \quad = a-c \end{aligned}$$

The eigenvalues for λ_2:

$$\begin{pmatrix} a-\lambda_2 & b \\ c & d-\lambda_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a-(a-c) & b \\ c & d-(d-b) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c & b \\ c & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \{cx + by = 0 \quad \Rightarrow \boxed{cx = -by}$$

The eigenvector: $V_2 = \begin{pmatrix} b \\ -c \end{pmatrix}$

Exercise

The eigenvalues of A equal to the eigenvalues of A^T. This is because $\det(A-\lambda I)$ equals $\det\left(A^T-\lambda I\right)$.

That is true because ______. Show by an example that the eigenvectors of A and A^T are not the same.

Solution

$$\det(A-\lambda I)=\det(A-\lambda I)^T=\det\left(A^T-(\lambda I)^T\right)=\det\left(A^T-\lambda I\right)$$

Therefore, A and A^T have the same eigenvalues.

Let consider the matrix: $A=\begin{pmatrix}0 & 1\\ 4 & 0\end{pmatrix} \quad\Rightarrow\quad A^T=\begin{pmatrix}0 & 4\\ 1 & 0\end{pmatrix}$

$$|A-\lambda I|=\begin{vmatrix}-\lambda & 1\\ 4 & -\lambda\end{vmatrix}=\lambda^2-4=0$$

The eigenvalues are: $\underline{\lambda=\pm 2|}$

For $\lambda=2$

$$(A-\lambda_1 I)V_1=\begin{pmatrix}-2 & 1\\ 4 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\rightarrow\begin{cases}-2x+y=0\\ 4x-2y=0\end{cases}\Rightarrow y=2x$$

$$V_1=\begin{pmatrix}1\\ 2\end{pmatrix}$$

$$\left(A^T-\lambda_1 I\right)V_1=\begin{pmatrix}-2 & 4\\ 1 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\rightarrow\begin{cases}-2x+4y=0\\ x-2y=0\end{cases}\Rightarrow x=2y$$

$$V_1=\begin{pmatrix}2\\ 1\end{pmatrix}$$

Exercise

Let $A=\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}$. Compute the eigenvalues and eigenvectors of A.

Solution

$$\det(A-\lambda I)=\begin{vmatrix}2-\lambda & -1\\ 1 & 2-\lambda\end{vmatrix}=(2-\lambda)^2+1=0$$

$$(2-\lambda)^2=-1$$

$$2-\lambda=\pm\sqrt{-1}=\pm i$$

The eigenvalues of $\boldsymbol{A}$ are: $\underline{\lambda=2\pm i|}$

For $\lambda_1=2-i\Rightarrow\left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix} 2-(2-i) & -1 \\ 1 & 2-(2-i) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} ix - y = 0 \\ x + iy = 0 \end{cases} \Rightarrow x = -iy$$

The eigenvector is: $V_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

For $\lambda_2 = 2 + i \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} -ix - y = 0 \\ x - iy = 0 \end{cases} \Rightarrow x = iy$$

The eigenvector is: $V_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

Exercise

Let $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

a) What is the characteristic polynomial for A (i.e. compute $\det\left(A - \lambda I\right)$?

b) Verify that 1 is an eigenvalue of A. What is a corresponding eigenvector?

c) What are the other eigenvalues of A?

Solution

a) $\det\left(A - \lambda I\right) = \begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix}$

$$= (2-\lambda)(1-\lambda)(-1-\lambda) - 2 + 9 - 3(1-\lambda) - 3(2-\lambda) + 2(-1-\lambda)$$

$$= \left(2 - 3\lambda + \lambda^2\right)(-1-\lambda) + 7 - 3 + 3\lambda - 6 + 3\lambda - 2 - 2\lambda$$

$$= -2 + 3\lambda - \lambda^2 - 2\lambda + 3\lambda^2 - \lambda^3 + 4\lambda - 4$$

$$\underline{= -\lambda^3 + 2\lambda^2 + 5\lambda - 6}$$

b) *If* $\lambda = 1 \quad \to -\lambda^3 + 2\lambda^2 + 5\lambda - 6 \overset{?}{=} 0$

$$-1^3 + 2(1)^2 + 5(1) - 6 \overset{?}{=} 0$$

$$-1 + 2 + 5 - 6 \overset{?}{=} 0$$

$$\boxed{0 = 0}$$

1 is an eigenvalue of A.

$$\begin{pmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0$$

$$\begin{pmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0 \rightarrow \begin{cases} x-2y+3z=0 \\ x \qquad +z=0 \\ x+3y-2z=0 \end{cases}$$

$$\Rightarrow \begin{cases} \boxed{x=-z} \\ 3y=2z-x=2z+z=3z \quad \Rightarrow \boxed{y=z} \end{cases}$$

The eigenvector for $\lambda=1$ is $V=\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

c) $-\lambda^3+2\lambda^2+5\lambda-6=0 \;\Rightarrow\; \lambda_1=1 \quad \lambda_2=-2 \quad \lambda_3=3$

Exercise

For the matrix:

$a)\ \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ $\quad b)\ \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$ $\quad c)\ \begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}$

$d)\ \begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$ $\quad e)\ \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$ $\quad f)\ \begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$

$g)\ \begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}$ $\quad h)\ \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\quad i)\ \begin{pmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

a)

i. $\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix}$

$= (3-\lambda)(-1-\lambda) - 0$

$= \lambda^2 - 2\lambda - 3$

The characteristic equation: $\lambda^2 - 2\lambda - 3$

ii. $\lambda^2 - 2\lambda - 3 = 0$

The eigenvalues are $\lambda_1 = -1$ *and* $\lambda_2 = 3$

iii. $\lambda_1 = -1 \rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & 0 \\ 8 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 4x = 0 \\ 8x = 0 \end{cases} \Rightarrow x = 0$$

Therefore the eigenvector $V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda_2 = 3 \rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 0 = 0 \\ 8x - 4y = 0 \end{cases} \Rightarrow 8x = 4y \rightarrow \boxed{2x = y}$$

Therefore the eigenvector $V_2 = \begin{pmatrix} x \\ 2x \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

b) For the matrix: $\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

i. $\det(A-\lambda I) = \begin{vmatrix} 10-\lambda & -9 \\ 4 & -2-\lambda \end{vmatrix}$

$= (10-\lambda)(-2-\lambda) + 36$

$= \lambda^2 - 8\lambda + 16$

$\Rightarrow$ The characteristic equation: $\lambda^2 - 8\lambda + 16$

ii. $\lambda^2 - 8\lambda + 16 = 0$

$\Rightarrow$ The eigenvalues are $\lambda_{1,2} = 4$

iii. $\lambda_1 = 4 \quad \rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 6x - 9y = 0 \\ 4x - 6y = 0 \end{cases} \Rightarrow \boxed{2x = 3y}$

Therefore the eigenvector $V_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$

c) For the matrix: $\begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}$

i. $\det(A-\lambda I) = \begin{vmatrix} -\lambda & 3 \\ 4 & -\lambda \end{vmatrix}$

$= \lambda^2 - 12$

$\Rightarrow$ The characteristic equation: $\lambda^2 - 12$

ii. $\lambda^2 - 12 = 0 \quad \Rightarrow \quad \lambda = \pm\sqrt{12}$

The eigenvalues are $\lambda_{1,2} = 4$

iii. For $\lambda_1 = \sqrt{12} \quad \rightarrow \quad \begin{pmatrix} -\sqrt{12} & 3 \\ 4 & -\sqrt{12} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -\sqrt{12} & 3 \\ 4 & -\sqrt{12} \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & -\frac{3}{\sqrt{12}} \\ 0 & 0 \end{pmatrix} \Rightarrow x - \frac{3}{\sqrt{12}} y = 0$

Therefore the eigenvector $V_1 = \begin{pmatrix} \frac{3}{\sqrt{12}} \\ 1 \end{pmatrix}$

For $\lambda_2 = -\sqrt{12} \quad \to \quad \begin{pmatrix} \sqrt{12} & 3 \\ 4 & \sqrt{12} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} \sqrt{12} & 3 \\ 4 & \sqrt{12} \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & \frac{3}{\sqrt{12}} \\ 0 & 0 \end{pmatrix} \Rightarrow x + \frac{3}{\sqrt{12}} y = 0$$

Therefore the eigenvector $V_2 = \begin{pmatrix} -\frac{3}{\sqrt{12}} \\ 1 \end{pmatrix}$

d) For the matrix $\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$

i. $\begin{vmatrix} -2-\lambda & -7 \\ 1 & 2-\lambda \end{vmatrix} = (-2-\lambda)(2-\lambda) + 7$

$$= -4 + \lambda^2 + 7$$

$$= \lambda^2 + 3$$

The characteristic equation: $\lambda^2 + 3 = 0$

ii. $\lambda^2 = -3 \quad \to \quad$ *The eigenvalues* $\lambda_{1,2} = \pm i\sqrt{3}$

iii. For $\lambda_1 = -i\sqrt{3}$, we have: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -2+i\sqrt{3} & -7 \\ 1 & 2+i\sqrt{3} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} \left(-2+i\sqrt{3}\right)x_1 - 7y_1 = 0 \\ x_1 + \left(2+i\sqrt{3}\right)y_1 = 0 \end{cases}$$

$$x_1 = -\left(2+i\sqrt{3}\right)y_1$$

Therefore the eigenvector $V_1 = \begin{pmatrix} 2+i\sqrt{3} \\ -1 \end{pmatrix}$

For $\lambda_2 = i\sqrt{3}$, we have: $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -2-i\sqrt{3} & -7 \\ 1 & 2-i\sqrt{3} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} \left(-2-i\sqrt{3}\right)x_2 - 7y_2 = 0 \\ x_2 + \left(2-i\sqrt{3}\right)y_2 = 0 \end{cases}$$

$$x_2 = -\left(2-i\sqrt{3}\right)y_2$$

Therefore the eigenvector $V_2 = \begin{pmatrix} 2-i\sqrt{3} \\ -1 \end{pmatrix}$

e) For the matrix: $\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

i. $\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda)(1-\lambda)+2(1-\lambda)$

$= (1-\lambda)\left[(4-\lambda)(1-\lambda)+2\right]$

$= (1-\lambda)\left(\lambda^2-5\lambda+6\right) \quad \Rightarrow$ The characteristic equation: $\underline{-\lambda^3+6\lambda^2-11\lambda+6}$

ii. $-\lambda^3+6\lambda^2-11\lambda+6=0 \quad \Rightarrow$ The eigenvalues are $\boxed{\lambda = 1,\ 2,\ 3}$

iii. $\lambda_1 = 1 \ \rightarrow \begin{pmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 3x_1 + x_3 = 0 \\ x_1 = 0 \end{cases} \Rightarrow x_1 = x_3 = 0$

Therefore the eigenvector $V_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 2 \ \rightarrow \begin{pmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x_1 + x_3 = 0 \\ -2x_1 - x_2 = 0 \\ -2x_1 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = -2x_1 \\ x_2 = -2x_1 \end{cases}$

Therefore the eigenvector $V_2 = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$

$\lambda_3 = 3 \ \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x_1 + x_3 = 0 \\ -2x_1 - 2x_2 = 0 \\ -2x_1 - 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = -x_1 \\ x_2 = -x_1 \end{cases}$

Therefore the eigenvector $V_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

f) For the matrix: $\begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$

i. $\det(A-\lambda I)=\begin{vmatrix} 3-\lambda & 0 & -5 \\ \frac{1}{5} & -1-\lambda & 0 \\ 1 & 1 & -2-\lambda \end{vmatrix}$

$$=(3-\lambda)(-1-\lambda)(-2-\lambda)-1+5(-1-\lambda)$$
$$=(3-\lambda)\left(\lambda^2+3\lambda+2\right)-1-5-5\lambda$$
$$=3\lambda^2+9\lambda+6-\lambda^3-3\lambda^2-2\lambda-6-5\lambda$$
$$=-\lambda^3+2\lambda$$

$\Rightarrow$ The characteristic equation: $-\lambda^3+2\lambda$

ii. $-\lambda^3+2\lambda=0 \quad\Rightarrow\quad$ The eigenvalues are $\boxed{\lambda=0,\ \pm\sqrt{2}}$

iii. $\lambda_1=-\sqrt{2} \quad\rightarrow\quad \begin{pmatrix} 3+\sqrt{2} & 0 & -5 \\ \frac{1}{5} & -1+\sqrt{2} & 0 \\ 1 & 1 & -2+\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\rightarrow\begin{cases} \left(3+\sqrt{2}\right)x_1-5x_3=0 \\ \frac{1}{5}x_1+\left(-1+\sqrt{2}\right)x_2=0 \\ x_1+x_2+\left(-2+\sqrt{2}\right)x_3=0 \end{cases}$

$$\rightarrow\begin{cases} x_3=\frac{3+\sqrt{2}}{5}x_1 \\ \left(-1+\sqrt{2}\right)x_2=-\frac{1}{5}x_1 \end{cases} \quad\Rightarrow x_2=-\frac{1}{5\left(-1+\sqrt{2}\right)}x_1$$

Therefore the eigenvector $V_1=\begin{pmatrix} 1 \\ \frac{1}{5\left(1-\sqrt{2}\right)} \\ \frac{3+\sqrt{2}}{5} \end{pmatrix}$

$$\lambda_2=0 \quad\rightarrow\quad \begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\rightarrow\begin{cases} 3x_1-5x_3=0 \\ \frac{1}{5}x_1-x_2=0 \\ x_1+x_2-2x_3=0 \end{cases} \Rightarrow\begin{cases} x_3=\frac{3}{5}x_1 \\ x_2=\frac{1}{5}x_1 \end{cases}$$

Therefore the eigenvector $V_2=\begin{pmatrix} 5 \\ \frac{1}{5} \\ \frac{3}{5} \end{pmatrix}$

$$\lambda_3=\sqrt{2} \;\rightarrow\; \begin{pmatrix}3-\sqrt{2} & 0 & -5\\ \frac{1}{5} & -1-\sqrt{2} & 0\\ 1 & 1 & -2-\sqrt{2}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\rightarrow\begin{cases}\left(3-\sqrt{2}\right)x_1-5x_3=0\\ \frac{1}{5}x_1+\left(-1-\sqrt{2}\right)x_2=0\\ x_1+x_2+\left(-2-\sqrt{2}\right)x_3=0\end{cases}$$

$$\rightarrow\begin{cases}x_3=\dfrac{3-\sqrt{2}}{5}x_1 0\\ \left(-1-\sqrt{2}\right)x_2=-\frac{1}{5}x_1\end{cases}\Rightarrow x_2=\frac{1}{5\left(1+\sqrt{2}\right)}x_1$$

Therefore the eigenvector $V_3=\begin{pmatrix}1\\ \\ \dfrac{1}{5\left(1+\sqrt{2}\right)}\\ \dfrac{3-\sqrt{2}}{5}\end{pmatrix}$

g) For the matrix: $\begin{pmatrix}-2 & 0 & 1\\ -6 & -2 & 0\\ 19 & 5 & -4\end{pmatrix}$

i. $\det\left(A-\lambda I\right)=\begin{vmatrix}-2-\lambda & 0 & 1\\ -6 & -2-\lambda & 0\\ 19 & 5 & -4-\lambda\end{vmatrix}$

$$=\left(-2-\lambda\right)^2\left(-4-\lambda\right)-30-19\left(-2-\lambda\right)$$

$$=\left(4+4\lambda+\lambda^2\right)\left(-4-\lambda\right)-30+38+19\lambda$$

$$=-16-16\lambda-4\lambda^2-4\lambda-4\lambda^2-\lambda^3+8+19\lambda$$

$$=-\lambda^3-8\lambda^2-\lambda-8$$

$\Rightarrow$ The characteristic equation: $\underline{-\lambda^3-8\lambda^2-\lambda-8}$

ii. $\lambda^3+8\lambda^2+\lambda+8=0 \quad\Rightarrow\quad \left(\lambda+8\right)\left(\lambda^2+1\right)=0$

$\lambda^3+8\lambda^2+\lambda+8=0 \quad\Rightarrow\quad$ The eigenvalues are $\boxed{\lambda_{1,2,3}=-8,\ \pm i}$

iii. $\lambda_1=-8 \;\rightarrow\; \begin{pmatrix}6 & 0 & 1\\ -6 & 6 & 0\\ 19 & 5 & 4\end{pmatrix}\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\begin{pmatrix} 6 & 0 & 1 \\ -6 & 6 & 0 \\ 19 & 5 & 4 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & \frac{1}{6} \\ 0 & 1 & \frac{1}{6} \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + \frac{1}{6} z_1 = 0 \\ y_1 + \frac{1}{6} z_1 = 0 \end{cases}$$

Therefore the eigenvector $V_1 = \begin{pmatrix} -\frac{1}{6} \\ -\frac{1}{6} \\ 1 \end{pmatrix}$

For $\lambda_2 = -i \quad \rightarrow \quad \begin{pmatrix} -2+i & 0 & 1 \\ -6 & -2+i & 0 \\ 19 & 5 & -4+i \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\rightarrow \begin{cases} (-2+i)x_2 + z_2 = 0 \\ -6x_2 + (-2+i)y_2 = 0 \\ 19x_2 + 5y_2 + (-4+i)z_2 = 0 \end{cases}$$

Therefore the eigenvector $V_2 = \begin{pmatrix} 1 \\ -\frac{12}{5} - i\frac{6}{5} \\ 2-i \end{pmatrix}$

For $\lambda_3 = i \quad \rightarrow \quad \begin{pmatrix} -2-i & 0 & 1 \\ -6 & -2-i & 0 \\ 19 & 5 & -4-i \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{cases} (-2-i)x_2 + z_2 = 0 \\ -6x_2 + (-2-i)y_2 = 0 \\ 19x_2 + 5y_2 + (-4-i)z_2 = 0 \end{cases}$$

Therefore the eigenvector $V_2 = \begin{pmatrix} 1 \\ -\frac{12}{5} + i\frac{6}{5} \\ 2+i \end{pmatrix}$

h) For the matrix: $\begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

i. $\det(A-\lambda I)=\begin{pmatrix} -\lambda & 0 & 2 & 0 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -2-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix}=(1-\lambda)\begin{vmatrix} -\lambda & 0 & 2 \\ 1 & -\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix}$

$$=(1-\lambda)\left(\lambda^2(-2-\lambda)+2+\lambda\right)$$

$$=(1-\lambda)\left(-\lambda^3-2\lambda^2+\lambda+2\right)$$

$$=\lambda^4+\lambda^3-3\lambda^2-\lambda+2$$

$\Rightarrow$ The characteristic equation: $\lambda^4+\lambda^3-3\lambda^2-\lambda+2$

ii. $\lambda^4+\lambda^3-3\lambda^2-\lambda+2=0 \quad \Rightarrow \quad$ The eigenvalues are $\boxed{\lambda=-2,-1,1,1}$

iii. $\lambda_1=-2 \quad \rightarrow \quad \begin{pmatrix} 2 & 0 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x_1+2x_3=0 \\ x_1+2x_2+x_3=0 \\ x_2=0 \\ x_4=0 \end{cases}$

$$\rightarrow \begin{cases} x_1=-x_3 \\ x_1=-x_3 \\ x_2=0 \\ x_4=0 \end{cases}$$

Therefore the eigenvector $V_1=\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_2=-1 \quad \rightarrow \quad \begin{pmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x_1+2x_3=0 \\ x_1+x_2+x_3=0 \\ x_2-x_3=0 \\ x_4=0 \end{cases}$

$$\rightarrow \begin{cases} x_1=-2x_3 \\ x_1=-x_2-x_3 \\ x_2=x_3 \end{cases}$$

Therefore the eigenvector $V_2=\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

$$\lambda_3 = 1 \quad \rightarrow \quad \begin{pmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -x_1 + 2x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_2 - 3x_3 = 0 \end{cases}$$

$$\rightarrow \begin{cases} x_1 = 2x_3 \\ x_1 = x_2 - x_3 \\ x_2 = 3x_3 \end{cases}$$

Therefore the eigenvector $V_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_4 = 1 \quad \rightarrow$ Therefore the eigenvector $V_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

i) For the matrix: $\begin{pmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

i. $\det(A - \lambda I) = \begin{pmatrix} 10-\lambda & -9 & 0 & 0 \\ 4 & -2-\lambda & 0 & 0 \\ 0 & 0 & -2-\lambda & -7 \\ 0 & 0 & 1 & 2-\lambda \end{pmatrix}$

$$= (10-\lambda)\begin{vmatrix} -2-\lambda & 0 & 0 \\ 0 & -2-\lambda & -7 \\ 0 & 1 & 2-\lambda \end{vmatrix} + 9\begin{vmatrix} 4 & 0 & 0 \\ 0 & -2-\lambda & -7 \\ 0 & 1 & 2-\lambda \end{vmatrix}$$

$$= (10-\lambda)\left[(-2-\lambda)^2(2-\lambda) + 7(-2-\lambda)\right] + 9\left[(4)(-2-\lambda)(2-\lambda) + 28\right]$$

$$= (10-\lambda)(-2-\lambda)\left(3+\lambda^2\right) + 9\left(4\lambda^2 + 12\right)$$

$$= \left(3+\lambda^2\right)\left(-8\lambda + \lambda^2 + 16\right)$$

$$= \left(3+\lambda^2\right)(\lambda - 4)^2$$

$\Rightarrow$ The characteristic equation: $\left(3+\lambda^2\right)(\lambda-4)^2$

ii. $\left(3+\lambda^2\right)(\lambda-4)^2 = 0 \quad \Rightarrow \quad$ The eigenvalues are $\boxed{\lambda = 4,\ 4,\ \pm i\sqrt{3}}$

iii. $\lambda_1 = 4 \rightarrow \begin{pmatrix} 6 & -9 & 0 & 0 \\ 4 & -6 & 0 & 0 \\ 0 & 0 & -6 & -7 \\ 0 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 6x_1 - 9x_2 = 0 \\ 4x_1 - 6x_2 = 0 \\ -6x_3 - 7x_4 = 0 \\ x_3 - 2x_4 = 0 \end{cases}$

$$\begin{cases} 6x_1 = 9x_2 \\ 6x_3 = -7x_4 \\ x_3 = 2x_4 \end{cases} \Rightarrow \begin{matrix} x_1 = \frac{3}{2}x_2 \\ x_3 = x_4 = 0 \end{matrix}$$

Therefore the eigenvector $V_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda_2 = 4 \rightarrow$ Therefore the eigenvector $V_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\lambda_3 = -i\sqrt{3} \rightarrow \begin{pmatrix} 10+i\sqrt{3} & -9 & 0 & 0 \\ 4 & -2+i\sqrt{3} & 0 & 0 \\ 0 & 0 & -2+i\sqrt{3} & -7 \\ 0 & 0 & 1 & 2+i\sqrt{3} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} \left(10+i\sqrt{3}\right)x_1 - 9x_2 = 0 \\ 4x_1 + \left(-2+i\sqrt{3}\right)x_2 = 0 \\ \left(-2+i\sqrt{3}\right)x_3 - 7x_4 = 0 \\ x_3 + \left(2+i\sqrt{3}\right)x_4 = 0 \end{cases}$$

$$\rightarrow \begin{cases} \left(10+i\sqrt{3}\right)x_1 = 9x_2 \\ 4x_1 = -\left(-2+i\sqrt{3}\right)x_2 \\ \left(-2+i\sqrt{3}\right)x_3 = 7x_4 \\ x_3 = -\left(2+i\sqrt{3}\right)x_4 \end{cases} \Rightarrow \begin{cases} x_1 = x_2 = 0 \\ x_3 = \dfrac{7}{-2+i\sqrt{3}}x_4\left(\dfrac{-2-i\sqrt{3}}{-2-i\sqrt{3}}\right) = -\left(2+i\sqrt{3}\right)x_4 \\ x_3 = -\left(2+i\sqrt{3}\right)x_4 \end{cases}$$

Therefore the eigenvector $V_3 = \begin{pmatrix} 0 \\ 0 \\ -\left(2+i\sqrt{3}\right) \\ 1 \end{pmatrix}$

$$\lambda_4 = i\sqrt{3} \quad \rightarrow \quad \begin{pmatrix} 10-i\sqrt{3} & -9 & 0 & 0 \\ 4 & -2-i\sqrt{3} & 0 & 0 \\ 0 & 0 & -2-i\sqrt{3} & -7 \\ 0 & 0 & 1 & 2-i\sqrt{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} \left(10-i\sqrt{3}\right)x_1 - 9x_2 = 0 \\ 4x_1 + \left(-2-i\sqrt{3}\right)x_2 = 0 \\ \left(-2-i\sqrt{3}\right)x_3 - 7x_4 = 0 \\ x_3 + \left(2-i\sqrt{3}\right)x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_2 = 0 \\ x_3 = \dfrac{7}{-2-i\sqrt{3}} x_4 \left(\dfrac{-2+i\sqrt{3}}{-2+i\sqrt{3}}\right) = \left(-2+i\sqrt{3}\right)x_4 \\ x_3 = -\left(2-i\sqrt{3}\right)x_4 \end{cases}$$

Therefore the eigenvector $V_4 = \begin{pmatrix} 0 \\ 0 \\ -2+i\sqrt{3} \\ 1 \end{pmatrix}$

j) For the matrix $\begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$

i. $\begin{vmatrix} -1-\lambda & 0 & 1 \\ -1 & 3-\lambda & 0 \\ -4 & 13 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2(3-\lambda) - 13 + 4(3-\lambda)$

$$= \left(\lambda^2 + 2\lambda + 1\right)(3-\lambda) - 13 + 12 - 4\lambda$$

$$= 3\lambda^2 + 6\lambda + 3 - \lambda^3 - 2\lambda^2 - \lambda - 1 - 4\lambda$$

$$= -\lambda^3 + \lambda^2 + \lambda + 2$$

The characteristic equation: $\underline{-\lambda^3 + \lambda^2 + \lambda + 2 = 0}|$

ii. $\rightarrow$ *The eigenvalues* $\underline{\lambda_{1,2,3} = 2, \ -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}}|$

iii. For $\lambda_1 = 2$, we have: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -3 & 0 & 1 \\ -1 & 1 & 0 \\ -4 & 13 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} -3x_1 + z_1 = 0 \\ -x_1 + y_1 = 0 \\ -4x_1 + 13y_1 - 3z_1 = 0 \end{cases} \quad \Rightarrow \begin{cases} z_1 = 3x_1 \\ y_1 = x_1 \end{cases}$$

If we let $x_1 = 1$; therefore the eigenvector $V_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

For $\lambda_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, we have: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -\frac{1}{2}+i\frac{\sqrt{3}}{2} & 0 & 1 \\ -1 & \frac{7}{2}+i\frac{\sqrt{3}}{2} & 0 \\ -4 & 13 & -\frac{1}{2}+i\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x_2 + z_2 = 0 \\ -x_2 + \left(\frac{7}{2}+i\frac{\sqrt{3}}{2}\right)y_2 = 0 \\ -4x_2 + 13y_2 + \left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)z_2 = 0 \end{cases}$$

$$\rightarrow \begin{cases} z_2 = -\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x_2 \\ y_2 = \left(\frac{2}{7+i\sqrt{3}}\right)x_2 \end{cases}$$

If we let $x_2 = 1$; therefore the eigenvector $V_2 = \begin{pmatrix} 1 \\ \frac{1-i\sqrt{3}}{2} \\ \frac{2}{7+i\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1-i\sqrt{3}}{2} \\ \frac{7-i\sqrt{3}}{26} \end{pmatrix}$

For $\lambda_3 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, we have: $\left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -\frac{1}{2}-i\frac{\sqrt{3}}{2} & 0 & 1 \\ -1 & \frac{7}{2}-i\frac{\sqrt{3}}{2} & 0 \\ -4 & 13 & -\frac{1}{2}-i\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)x_3 + z_3 = 0 \\ -x_3 + \left(\frac{7}{2}-i\frac{\sqrt{3}}{2}\right)y_3 = 0 \\ -4x_3 + 13y_3 + \left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)z_3 = 0 \end{cases}$$

$$\rightarrow \begin{cases} z_3 = -\left(\frac{1+i\sqrt{3}}{2}\right)x_3 \\ y_3 = \left(\frac{2}{7-i\sqrt{3}}\right)x_3 \end{cases}$$

If we let $x_3 = 1$; therefore the eigenvector $V_3 = \begin{pmatrix} 1 \\ \frac{1+i\sqrt{3}}{2} \\ \frac{2}{7-i\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1+i\sqrt{3}}{2} \\ \frac{7+i\sqrt{3}}{26} \end{pmatrix}$

Exercise

Find the eigenvalues of A^9 for $A=\begin{pmatrix} 1 & 3 & 7 & 11 \\ 0 & \frac{1}{2} & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

Solution

The eigenvalues are: $\lambda = 1, \ \frac{1}{2}, \ 0, \ 2$

The eigenvalues of A^9 are: $1^9 \ \underline{=1|}$ $\left(\frac{1}{2}\right)^9 \ \underline{=\frac{1}{512}|}$ $0^9 \ \underline{=0|}$ $2^9 \ \underline{=512|}$

Exercise

Find the eigenvalues of the matrices

$$A=\begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}, \quad A^2=\begin{pmatrix} 0.61 & 0.52 \\ 0.39 & 0.48 \end{pmatrix}, \quad A^\infty=\begin{pmatrix} 0.57143 & 0.57143 \\ 0.42857 & 0.42857 \end{pmatrix}, \quad \text{and} \ \ B=\begin{pmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{pmatrix}$$

Solution

The eigenvalues for A:

$$\begin{vmatrix} 0.7-\lambda & 0.4 \\ 0.3 & 0.6-\lambda \end{vmatrix} = (0.7-\lambda)(0.6-\lambda) - .12$$

$$= \lambda^2 - 1.3\lambda + .3 = 0 \qquad \lambda_{1,2} = 0.65 \pm 0.35$$

The eigenvalues are: $\lambda_1 = 1 \quad \lambda_2 = 0.3$

The eigenvalues for A^2: $\lambda_1 = 1^2 = 1 \quad \lambda_2 = 0.3^2 = 0.09$

The eigenvalues for A^∞: $\lambda^2 - \lambda = 0 \quad \lambda_1 = 1 \quad \lambda_2 = 0.3^\infty = 0$

The eigenvalues for B:

$$\begin{vmatrix} 0.3-\lambda & 0.6 \\ 0.7 & 0.4-\lambda \end{vmatrix} = \lambda^2 - .7\lambda - .3 = 0 \qquad \lambda_{1,2} = 0.35 \pm 0.65$$

The eigenvalues are: $\lambda_1 = 1 \quad \lambda_2 = -0.3$

Exercise

Given the matrix $\begin{bmatrix} -1 & -3 \\ -3 & 7 \end{bmatrix}$

a) Find the characteristic polynomial.
b) Find the eigenvalues
c) Find the bases for its eigenspaces
d) Graph the eigenspaces
e) Verify directly that $A\mathbf{v} = \lambda\mathbf{v}$, for all associated eigenvectors and eigenvalues.

Solution

a) $\begin{vmatrix} -1-\lambda & -3 \\ -3 & 7-\lambda \end{vmatrix} = (-1-\lambda)(7-\lambda) - 9$

$$= -7 - 6\lambda + \lambda^2 - 9$$
$$= \lambda^2 - 6\lambda - 16$$

The characteristic polynomial is $\underline{\lambda^2 - 6\lambda - 16 = 0}$

b) $\lambda^2 - 6\lambda - 16 = 0 \quad \Rightarrow \quad \lambda_1 = -2 \quad \text{and} \quad \lambda_2 = 8$

c) For $\lambda_1 = -2$, we have: $(A + 2I)V_1 = 0$

$$\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} x_1 - 3y_1 = 0 \\ -3x_1 + 9y_1 = 0 \end{cases} \quad \Rightarrow x_1 = 3y_1$$

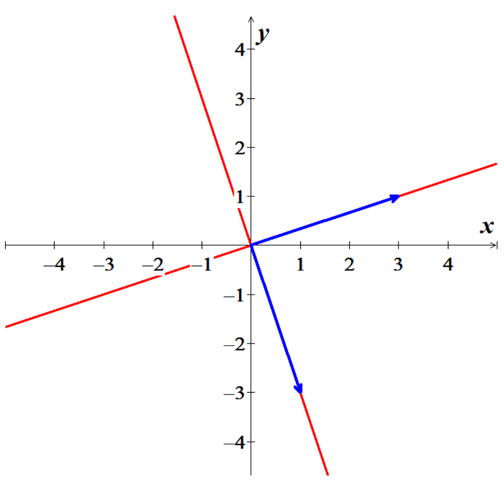
Therefore the eigenvector $V_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

For $\lambda_2 = 8$, we have: $(A + 8I)V_2 = 0$

$$\begin{pmatrix} -9 & -3 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} -9x_2 - 3y_2 = 0 \\ -3x_2 - y_2 = 0 \end{cases} \quad \Rightarrow y_2 = -3x_2$$

Therefore the eigenvector $V_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

d)



e) $AV_1 = \lambda V_1 \quad \to \quad \begin{pmatrix} -1 & -3 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = -2\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\begin{pmatrix} -6 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \end{pmatrix} \surd$

$AV_2 = \lambda V_2 \quad \to \quad \begin{pmatrix} -1 & -3 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = 8\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$\begin{pmatrix} 8 \\ -24 \end{pmatrix} = \begin{pmatrix} -6 \\ -24 \end{pmatrix} \surd$

Exercise

Given the matrix $\begin{bmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{bmatrix}$

a) Find the characteristic polynomial.
b) Find the eigenvalues
c) Find the bases for its eigenspaces
d) Graph the eigenspaces
e) Verify directly that $A\mathbf{v} = \lambda\mathbf{v}$, for all associated eigenvectors and eigenvalues.

Solution

a) $\begin{vmatrix} 5-\lambda & 0 & -4 \\ 0 & -3-\lambda & 0 \\ -4 & 0 & -1-\lambda \end{vmatrix} = (5-\lambda)(-3-\lambda)(-1-\lambda) - 16(-3-\lambda)$

$$= (5-\lambda)\left(3+4\lambda+\lambda^2\right) + 48 + 16\lambda$$
$$= 15 + 20\lambda + 5\lambda^2 - 3\lambda - 4\lambda^2 - \lambda^3 + 48 + 16\lambda$$
$$= -\lambda^3 + \lambda^2 + 33\lambda + 63$$

The characteristic polynomial is $\boxed{-\lambda^3 + \lambda^2 + 33\lambda + 63 = 0}$

b) $-\lambda^3 + \lambda^2 + 33\lambda + 63 = 0 \quad \Rightarrow \quad \lambda = -3,\ -3,\ 7$

c) For $\lambda_{1,2} = -3$, we have: $(A + 3I)V_1 = 0$

$$\begin{pmatrix} 8 & 0 & -4 \\ 0 & 0 & 0 \\ -4 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} 8x_1 - 4z_1 = 0 \\ \\ -4x_1 + 2z_1 = 0 \end{cases} \quad \Rightarrow z_1 = 2x_1$$

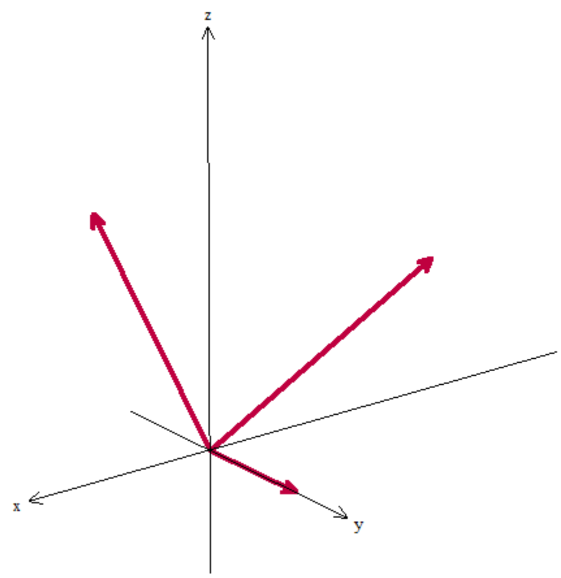
Therefore the eigenvector $V_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ *and* $V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_3 = 7$, we have: $(A-7I)V_3 = 0$

$$\begin{pmatrix} -2 & 0 & -4 \\ 0 & -10 & 0 \\ -4 & 0 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -2x_1 - 4z_1 = 0 \\ -10y_1 = 0 \\ -4x_1 - 8z_1 = 0 \end{cases} \Rightarrow x_1 = -2z_1 \text{ and } y_1 = 0$$

Therefore the eigenvector $V_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

d)



e) $AV_1 = \lambda V_1 \rightarrow \begin{pmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} -3 \\ 0 \\ -6 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ -6 \end{pmatrix} \checkmark$$

$AV_2 = \lambda V_2 \rightarrow \begin{pmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} \checkmark$$

$$AV_3 = \lambda V_3 \quad \rightarrow \quad \begin{pmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 7\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -14 \\ 0 \\ 7 \end{pmatrix} = \begin{pmatrix} -14 \\ 0 \\ 7 \end{pmatrix} \checkmark$$

Exercise

Given: $A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$. Compute A^{11}

Solution

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 7 & -1 \\ 0 & 1-\lambda & 0 \\ 0 & 15 & -2-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(1-\lambda)(-2-\lambda)$$

The eigenvalues are: $-1,\ 1,\ -2$

For $\lambda_1 = -1$, we have: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 7 & -1 \\ 0 & 2 & 0 \\ 0 & 15 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 7y_1 - z_1 = 0 \\ 2y_1 = 0 \\ 15y_1 - z_1 = 0 \end{cases} \quad \Rightarrow \begin{cases} z_1 = 7y_1 \\ y_1 = 0 \end{cases}$$

The eigenvector $V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

For $\lambda_2 = 1$, we have: $(A - I)V_2 = 0$

$$\begin{pmatrix} -2 & 7 & -1 \\ 0 & 0 & 0 \\ 0 & 15 & -3 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} -2x_2 + 7y_2 - z_2 = 0 \\ \\ 15y_2 - 3z_2 = 0 \end{cases} \quad \Rightarrow \begin{cases} 2x_2 = 7y_2 - z_2 \\ \\ 5y_2 = z_2 \end{cases}$$

If we let $y_2 = 1 \ \rightarrow z_2 = 5$ and $x_2 = \frac{7-5}{2} = 1$;

The eigenvector $V_2 = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$

For $\lambda_3 = -2$, we have: $(A+2I)V_3 = 0$

$$\begin{pmatrix} 1 & 7 & -1 \\ 0 & 3 & 0 \\ 0 & 15 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} x_3 + 7y_3 - z_3 = 0 \\ 3y_3 = 0 \\ 15y_3 = 0 \end{cases} \quad \Rightarrow \begin{cases} x_3 = -7y_3 + z_3 \\ y_3 = 0 \end{cases}$$

The eigenvector $V_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \Rightarrow D^{11} = \begin{pmatrix} (-1)^{11} & 0 & 0 \\ 0 & 1^{11} & 0 \\ 0 & 0 & (-2)^{11} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2048 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \Rightarrow P^{-1} = \begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

$$A^{11} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2048 \end{pmatrix}\begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & -2048 \\ 0 & 1 & 0 \\ 0 & 5 & -2048 \end{pmatrix}\begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 10237 & 2047 \\ 0 & 1 & 0 \\ 0 & 10245 & -2048 \end{pmatrix}$$

Exercise

The Lucas numbers are like Fibonacci numbers except they start with $L_1 = 1$ *and* $L_2 = 3$. Following the rule $L_{k+2} = L_{k+1} + L_k$. The next Lucas numbers are 4, 7, 11, 18. Show that the Lucas number $L_{100} = \lambda_1^{100} + \lambda_2^{100}$.

Solution

Let $u_k = \begin{pmatrix} L_{k+1} \\ L_k \end{pmatrix}$, the rule $\begin{matrix} L_{k+2} = L_{k+1} + L_k \\ L_{k+1} = L_{k+1} \end{matrix}$ becomes $u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k$. $\Rightarrow A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = -\lambda(1-\lambda) - 1 = \lambda^2 - \lambda - 1$$

The characteristic equation is $\lambda^2 - \lambda - 1 = 0$ and the solutions are

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618 \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -.618$$

$$(A - \lambda_1 I)v_1 = \begin{bmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to x_1 - \lambda_1 y_1 = 0 \Rightarrow v_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$$

$$(A - \lambda_2 I)v_2 = \begin{bmatrix} 1-\lambda_2 & 1 \\ 1 & -\lambda_2 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to x_2 - \lambda_2 y_2 = 0 \Rightarrow v_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$$

The linear combination: $c_1 v_1 + c_2 v_2 = u_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$\lambda_1 v_1 + \lambda_2 v_2 = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 + \lambda_2^2 \\ \lambda_1 + \lambda_2 \end{pmatrix} = \begin{bmatrix} \text{trace of } A^2 \\ \text{trace of } A \end{bmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

The solution $u_{100} = A^{99} u_1$

$$L_{100} = c_1 \lambda_1^{99} + c_2 \lambda_2^{99} = \lambda_1^{100} + \lambda_2^{100}$$

Exercise

Find all eigenvector matrices S that diagonalize A (rank 1) to give $S^{-1}AS = \Lambda$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

What is A^n? Which matrices B commute with A (so that $AB = BA$)

Solution

Since A has rank 1, its nullspace is a two-dimensional plane. Any vector with $x + y + z = 0$ solves $A\boldsymbol{x} = 0$. So $\lambda = 0$ is an eigenvalue with multiplicity 2. There are two independent eigenvectors. The other eigenvalues must be $\lambda = 3$ because the trace A is 1 + 1 + 1 = 3.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 + 2 - 3(1-\lambda) = -\lambda^3 + 3\lambda^2$$

The eigenvalues are $\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 3$.

The eigenvectors for $\lambda_3 = 3$ is:

$$(A - \lambda_3 I)v_3 = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \quad x_3 + y_3 - 2z_3$$

$$\text{if } z_3 = 1 \rightarrow x_3 + y_3 = 2 \rightarrow x_3 = y_3 = 1 \Rightarrow v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The eigenvectors for $\lambda_1 = \lambda_2 = 0$ are any two independent vectors in the plane of $x + y + z = 0$

The possible matrices S:

$$S = \begin{pmatrix} x & X & c \\ y & Y & c \\ -x-y & -X-Y & c \end{pmatrix} \text{ and } S^{-1}AS = \Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

where $c \neq 0$ *and* $xY \neq yX$.

The powers A^n come: $\quad A^2 = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = 3A$ and $A^n = 3^{n-1}A$

If $AB = BA$, all the column and row of $\boldsymbol{B}$ must be the same. One possible $\boldsymbol{B}$ is $\boldsymbol{A}$ itself, since $AA = AA$, $\boldsymbol{B}$ is any linear combination of permutation matrices.

Exercise

Determine whether the matrix is diagonalizable

$a)\ \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ $\qquad b)\ \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$

$c)\ \begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$ $\qquad d)\ \begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Solution

a) $\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = 0$

The only eigenvalue: $\lambda = 2$, the eigenvectors are:

$$(A-\lambda I)V_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \to \underset{x=0}{} \Rightarrow V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$S = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and the inverse doesn't exist. Therefore the matrix A is not diagonalizable.

b) $\det(A-\lambda I) = \begin{vmatrix} -3-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (-3-\lambda)(1-\lambda)+4$

$$= \lambda^2 + 2\lambda + 1 = 0$$

The only eigenvalue: $\lambda = -1$, the eigenvectors are:

$$(A-\lambda I)V_1 = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \to \begin{matrix} -2x+2y=0 \\ 2x-2y=0 \end{matrix} \to x = y \Rightarrow V_{1,2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$ (***linearly dependent***)

$S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \to$ the inverse doesn't exist. Therefore the matrix A is not diagonalizable.

This space is 1-dimensional, A does not have 2 linearly independent eigenvectors.

c) $\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 0 & 1 \\ -1 & 3-\lambda & 0 \\ -4 & 13 & -1-\lambda \end{vmatrix}$

$$\begin{aligned} &= (-1-\lambda)(3-\lambda)(-1-\lambda) - 13 + 4(3-\lambda) \\ &= \left(1+2\lambda+\lambda^2\right)(3-\lambda) - 13 + 12 - 4\lambda \\ &= 3 + 6\lambda + 3\lambda^2 - \lambda - 2\lambda^2 - \lambda^3 - 1 - 4\lambda \\ &= -\lambda^3 + \lambda^2 + \lambda + 2 \end{aligned}$$

The eigenvalues are: $\lambda_1 = 2, \quad \lambda_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \lambda_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

The eigenvector for $\lambda_1 = 2$ is:

$$\left(A-\lambda_1 I\right)V_1 = \begin{pmatrix} -3 & 0 & 1 \\ -1 & 1 & 0 \\ -4 & 13 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{matrix} -3x_1 + z_1 = 0 \\ -x_1 + y_1 = 0 \\ -4x_1 + 13y_1 - 3z_1 = 0 \end{matrix} \quad \Rightarrow \begin{matrix} z_1 = 3x_1 \\ y_1 = x_1 \end{matrix}$$

$$if \ \ x_1 = \tfrac{1}{3} \Rightarrow v_1 = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix}$$

d) Since the matrix is upper triangular with diagonal entries 2, 2, 3, and 3, the eigenvalues are 2 and 3 (each multiplicity of 2)

For $\lambda = 2 \quad \Rightarrow \quad \left(A-2I\right)V_1 = 0$

$$\begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{cases} -x_2 + x_4 = 0 \\ x_3 - x_4 = 0 \\ x_3 + 2x_4 = 0 \\ x_4 = 0 \end{cases} \quad \Rightarrow \quad x_2 = x_3 = x_4 = 0$$

$$V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ has dimension 1.}$$

For $\lambda = 3 \quad \Rightarrow \quad \left(A-3I\right)V_2 = 0$

$$\begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{cases} -x_1 - x_2 + x_4 = 0 \\ -x_2 + x_3 - x_4 = 0 \\ 2x_4 = 0 \end{cases} \to \begin{cases} x_1 = -x_2 \\ x_3 = x_2 \\ x_4 = 0 \end{cases}$$

$$V_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

The matrix is not diagonalizable since it has only 2 distinct eigenvectors. Note that showing that the geometric multiplicity of either eigenvalue is less than its algebraic multiplicity is sufficient to show that the matrix is not diagonalizable,

Exercise

Find a matrix P that diagonalizes A, and compute $P^{-1}AP$

a) $A=\begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}$ ***b***) $A=\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$ ***c***) $A=\begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Solution

a) $\det(A-\lambda I)=\begin{vmatrix} -14-\lambda & 12 \\ -20 & 17-\lambda \end{vmatrix}=\lambda^2-3\lambda+2$

The eigenvalues are: $\lambda_1=2 \quad \lambda_2=1$

For $\lambda_1=2 \quad \to \quad \begin{pmatrix} -16 & 12 \\ -20 & 15 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -16x_1+12y_1=0 \\ -20x_1+15y_1=0 \end{cases} \to 4x_1=3y_1$

Therefore the eigenvector: $V_1=\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

For $\lambda_2=1 \quad \to \quad \begin{pmatrix} -15 & 12 \\ -20 & 16 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -15x_2+12y_2=0 \\ -20x_2+16y_2=0 \end{cases} \to 5x_2=4y_2$

Therefore the eigenvector: $V_2=\begin{pmatrix} 4 \\ 5 \end{pmatrix}$

The eigenvectors: $P=\begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix} \quad \to \quad P^{-1}=\begin{pmatrix} -5 & 4 \\ 4 & -3 \end{pmatrix}$

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} -5 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} -14 & 12 \\ -20 & 17 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix} \\ &= \begin{pmatrix} -10 & 8 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

b) $\det(A-\lambda I)=\begin{vmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{vmatrix}=\lambda^2-1=0$

The eigenvalues are: $\lambda_1=-1, \quad \lambda_2=1$

For $\lambda_1=-1 \quad \to \quad \begin{pmatrix} 2 & 0 \\ 6 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2x_1=0 \\ 6x_1=0 \end{cases}$

Therefore the eigenvector: $V_1=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

For $\lambda_2 = 1 \rightarrow \begin{pmatrix} 0 & 0 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \\ 6x_2 - 2y_2 = 0 \end{cases} \rightarrow 3x_2 = y_2$

Therefore the eigenvector: $V_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

The eigenvectors: $P = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \rightarrow P^{-1} = \begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

c) $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (3-\lambda)^2(5-\lambda) - 4(5-\lambda)$

$$\begin{aligned} &= (5-\lambda)\left(\lambda^2 - 6\lambda + 9 - 4\right) \\ &= (5-\lambda)\left(\lambda^2 - 6\lambda + 5\right) \\ &= (5-\lambda)(\lambda-5)(\lambda-1) = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = 1, \quad \lambda_2 = 5, \quad \lambda_3 = 5$

The eigenvector for $\lambda_1 = 1$ is:

$$(A - \lambda_1 I)v_1 = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} 2x_1 - 2y_1 = 0 \\ \\ 4z_1 = 0 \end{matrix} \Rightarrow \begin{matrix} x_1 = y_1 \\ \\ z_1 = 0 \end{matrix}$$

$$\text{if } y_1 = 1 \rightarrow x_1 = 1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

The eigenvectors for $\lambda_{2,3} = 5$ is:

$$(A - \lambda I)v_2 = \begin{pmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} -2x_2 - 2y_2 = 0 \\ \\ \end{matrix} \Rightarrow \begin{matrix} x_2 = -y_2 \\ \\ 0z_2 = 0 \end{matrix}$$

$$\Rightarrow v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad and \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow P^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes *A* and determine $P^{-1}AP$

a) $A = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

b) $A = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix}$

c) $A = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

d) $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Solution

a) $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & -2 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^3$

The eigenvalues are: $\lambda_{1,2,3} = 3$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

b) $A = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix}$

The eigenvalues are: $\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 2$

$$P=\begin{pmatrix}1 & 0 & \frac{3}{4}\\ \frac{4}{3} & 0 & \frac{3}{4}\\ 1 & 0 & 1\end{pmatrix}$$

c) $A=\begin{pmatrix}-2 & 0 & 0 & 0\\ 0 & -2 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 1 & 3\end{pmatrix}$

The eigenvalues are: $\lambda_{1,2}=-2,\quad \lambda_{3,4}=3$

$$P=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

d) $\det(A-\lambda I)=\begin{vmatrix}-2-\lambda & 0 & 0 & 0\\ 0 & -2-\lambda & 5 & -5\\ 0 & 0 & 3-\lambda & 0\\ 0 & 0 & 0 & 3-\lambda\end{vmatrix}$

Since the matrix A is an upper triangular, then the eigenvalues are: $\lambda_{1,2}=-2\quad \lambda_{3,4}=3$

For $\lambda=-2 \quad\Rightarrow\quad (A+2I)V_1=0$

$$\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 5 & -5\\ 0 & 0 & 5 & 0\\ 0 & 0 & 0 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad\rightarrow\begin{cases}5x_3-5x_4=0\\ 5x_3=0\\ 5x_4=0\end{cases} \quad\Rightarrow\quad x_3=x_4=0$$

$$V_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \textit{and} \quad V_2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

For $\lambda=3 \quad\Rightarrow\quad (A-3I)V_2=0$

$$\begin{bmatrix}-5 & 0 & 0 & 0\\ 0 & -5 & 5 & -5\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad\rightarrow\quad \begin{cases}-5x_1=0\\ -5x_2+5x_3-5x_4=0\\ \end{cases} \quad\Rightarrow\begin{cases}x_1=0\\ x_2=x_3-x_4\end{cases}$$

$$V_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \textit{and} \quad V_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow P^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 2 & -2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \end{aligned}$$

Exercise

The 4 by 4 triangular Pascal matrix P_L and its inverse (alternating diagonals) are

$$P_L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \quad \textit{and} \quad P_L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Check that P_L and P_L^{-1} have the same eigenvalues. Find a diagonal matrix D with alternating signs that gives $P_L^{-1} = D^{-1}P_L D$, so P_L is similar to P_L^{-1}. Show that $P_L D$ with columns of alternating signs is its own inverse.

Since P_L and P_L^{-1} are similar they have the same Jordan form J. Find J by checking the number of independent eigenvectors of P_L with $\lambda = 1$.

Solution

The triangular matrices P_L and P_L^{-1} both have λ = 1, 1, 1, 1 on their main diagonals. Choose D with alternating 1 and −1 on its diagonal. D equals D^{-1}:

$$D^{-1}P_L D = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = P_L^{-1}$$

Check:

Changing signs in rows 1 and 3 of P_L, and columns 1 and 3, produces the four negative entries in P_L^{-1}. Multiply row i by $(-1)^i$ and column j by $(-1)^j$, which gives the alternating diagonals.

Then $P_L D = pascal(n,\ 1)$ has columns with alternating signs and equals its own inverse!

$$(P_L D)(P_L D) = P_L D^{-1} P_L D = P_L P_L^{-1} = I$$

P_L has only one line of eigenvectors $x = (0,\ 0,\ 0,\ x_4)$ with λ = 1. The rank of $P_L - I$ is certainly 3. So its Jordan form J has only one block (also with λ = 1):

P_L and P_L^{-1} are somehow similar to Jordan's $J = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Exercise

These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors (one from each block). But the block sizes don't match and they are not similar:

$$J = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \textit{and} \quad K = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

For any matrix M compare JM with MK. If they are equal show that M is not invertible. Then $M^{-1}JM = K$ is Impossible; J is not similar to K.

Solution

Let $M = (m_{ij})$, then

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix}$$

$$JM = \begin{pmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad and \quad MK = \begin{pmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{21} & m_{22} & 0 \\ 0 & m_{31} & m_{32} & 0 \\ 0 & m_{41} & m_{42} & 0 \end{pmatrix}$$

If $JM = MK$ then $m_{21} = m_{22} = m_{24} = m_{41} = m_{42} = m_{44} = 0$

Which in particular means that the second row is either a multiple of the fourth row, or the fourth row is all 0's. In either of these cases M is not invertible.

Suppose that J were similar to K. Then there would be some invertible matrix M such that $MK = JM$. But we just showed that in this case M is never invertible (contradiction). Thus J is not similar to K.

Exercise

If $\boldsymbol{x}$ is in the nullspace of A show that $M^{-1}\boldsymbol{x}$ is in the nullspace of $M^{-1}AM$.

The nullspaces of A and $M^{-1}AM$ have the same (vectors) (basis) (dimension)

Solution

$$x \in N(A) \Rightarrow Ax = 0 \Rightarrow M^{-1}AM\left(M^{-1}x\right) = 0 \Rightarrow M^{-1}x \in N\left(M^{-1}AM\right)$$

$$x \in N\left(M^{-1}AM\right) \Rightarrow M^{-1}AMx = 0 \Rightarrow AMx = 0 \Rightarrow Mx \in N(A)$$

So any vector in $N(A)$ *resp.* $N\left(M^{-1}AM\right)$ is a linear combination of those in $N\left(M^{-1}AM\right)$ *resp.* $N(A)$, hence is contained in it. That is, the two vector spaces consists of the same vectors.

Exercise

Prove that A^T is always similar to A (λ's are the same):

a) For one Jordan block J_i, find M_i so that $M_i^{-1}J_iM_i = J_i^T$.

b) For any J with blocks J_i, build M_0 from blocks so that $M_0^{-1}JM_0 = J^T$.

c) For any $A = MJM^{-1}$: Show that A^T is similar to J^T and so to J and so to A.

Solution

a) For one Jordan block J_i, then

$$\begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & \cdot^{\cdot^{\cdot}} & & & \\ 1 & & & & \end{pmatrix}\begin{pmatrix} \lambda & 1 & 0 & & 0 \\ & \lambda & 1 & 0 & \\ & & \lambda & & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}\begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & \cdot^{\cdot^{\cdot}} & & & \\ 1 & & & & \end{pmatrix} = \begin{pmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ 0 & 1 & \lambda & & \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix}$$

So J is similar to J^T

b) For any J with block J_i, that satisfies $J_i^T = M_i^{-1}J_iM_i$

Let M_0 be the block-diagonal matrix consisting of the $M_i's$ along the diagonal. Then

$$M_0^{-1}JM_0 = \begin{pmatrix} M_1^{-1} & & & \\ & M_2^{-1} & & \\ & & \ddots & \\ & & & M_n^{-1} \end{pmatrix}\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_n \end{pmatrix}\begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_n \end{pmatrix}$$

$$= \begin{pmatrix} M_1^{-1}J_1M_1 & & & \\ & M_2^{-1}J_2M_2 & & \\ & & \ddots & \\ & & & M_n^{-1}J_nM_n \end{pmatrix}$$

$$= \begin{pmatrix} J_1^T & & & \\ & J_2^T & & \\ & & \ddots & \\ & & & J_n^T \end{pmatrix}$$

$$= J^T$$

c) $A^T = \left(MJM^{-1}\right)^T = \left(M^{-1}\right)^T J^T M^T = \left(M^T\right)^{-1} J^T \left(M^T\right)$

So A^T is similar to J^T, which is similar to J, which is similar to A, Thus any matrix is similar to its transpose.

Exercise

Why are these statements all true?

a) If A is similar to B then A^2 is similar to B^2.

b) A^2 and B^2 can be similar when A and B are not similar.

c) $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ is similar to $\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ is not similar to $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

e) If we exchange rows 1 and 2 of A, and then exchange columns 1 and 2 the eigenvalues stay the same. In this case M =?

Solution

a) If A is similar to B then $A = M^{-1}BM$ for some M. Then $A^2 = M^{-1}B^2M$, so A^2 is similar to B^2.

b) Let $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $A^2 = B^2$ so they are similar but A is not similar to B because nothing but zero matrix.

c) $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

d) They are not similar because the first matrix has a plane of eigenvectors for the eigenvalues 3, while the second only has a line.

e) In order to exchange two rows of A we multiply on the left by

$$M = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

In order to exchange two columns we multiply on the right by the same M. As $M = M^{-1}$ the new matrix is similar to the old one, so the eigenvalues stay the same.

Exercise

If an n x n matrix A has all eigenvalues $\lambda = 0$ prove that A^n is the zero matrix.

Solution

Suppose that the Jordan Block has a size of $\boldsymbol{i}$ with eigenvalue 0. Then J^2 will have a diagonal of 1's two diagonals above the main diagonal and zeroes elsewhere. J^3 will have a diagonal of 1's three diagonals above the main diagonal and zeroes elsewhere. Therefore $J^i = 0$, since there is no diagonal $\boldsymbol{i}$ diagonals above the main diagonal. If A has all eigenvalues $\lambda = 0$ then A is similar to some matrix with Jordan block $J_1, \ldots, J_k$ with each J_i of size n_i and $\sum_{i=1}^{k} n_i = n$.

Each Jordan block will have eigenvalue of 0, so that $J_i^{n_i} = 0$, and thus $J_i^n = 0$

As A^n is similar to a block-diagonal matrix with blocks $J_1^n, J_2^n, \ldots, J_k^n$ and each of these is 0 we know that $A^n = 0$.

Another way, if A has all eigenvalues 0 this means that the characteristic polynomial of A must be x^n, as this is the only polynomial of degree n all of whose roots are 0. Thus $A^n = 0$ by the Cayley-Hamilton theorem.

Exercise

If A is similar to A^{-1}, must all the eigenvalues equal to 1 or −1?.

Solution

No

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Thus $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ is similar to $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^{-1}$

Exercise

Show that A and B are not similar matrices

a) $A=\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$ $\quad B=\begin{pmatrix}1 & 0\\ 3 & -2\end{pmatrix}$

b) $A=\begin{pmatrix}4 & -1\\ 2 & 4\end{pmatrix}$ $\quad B=\begin{pmatrix}4 & 1\\ 2 & 4\end{pmatrix}$

c) $A=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}$ $\quad B=\begin{pmatrix}1 & 2 & 0\\ \frac{1}{2} & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

Solution

a) $|A|=\begin{vmatrix}1 & 1\\ 1 & 2\end{vmatrix}=1 \quad |B|=\begin{vmatrix}1 & 0\\ 3 & -2\end{vmatrix}=-2$

$|A|\neq|B|$; therefore A and B are not similar

b) $|A|=\begin{vmatrix}4 & -1\\ 2 & 4\end{vmatrix}=18 \quad |B|=\begin{vmatrix}4 & 1\\ 2 & 4\end{vmatrix}=14$

$|A|\neq|B|$; therefore A and B are not similar

c) $|A|=\begin{vmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & 0 & 1\end{vmatrix}=1 \quad |B|=\begin{vmatrix}1 & 2 & 0\\ \frac{1}{2} & 1 & 0\\ 0 & 0 & 1\end{vmatrix}=0$

$|A|\neq|B|$; therefore A and B are not similar

Exercise

Prove that two similar matrices have the same determinant. Explain geometrically why this is reasonable.

Solution

Suppose that $A=PBP^{-1}$

Then $\det(A)=\det\left(PBP^{-1}\right)$ $\qquad |AB|=|A||B|$

$$=\det(P)\cdot\det(B)\cdot\det\left(P^{-1}\right)$$

$$=\det(B)\cdot\det(P)\cdot\det\left(P^{-1}\right)$$

$$=\det(B)\cdot\det\left(PP^{-1}\right)$$

$$=\det(B)\cdot\det(I)$$

$$= \det(B)$$

Geometric Explanation: The determinant tells us what Factor area changes when using a linear transformation. This "factor" doesn't care about the particular basis you use.

Exercise

Prove that two similar matrices have the same characteristic polynomial and thus the same eigenvalues. Explain geometrically why this is reasonable.

Solution

Suppose that $A = PBP^{-1}$

Then the characteristic polynomial is equal to $\det(A - \lambda I)$.

$$A - \lambda I = PBP^{-1} - \lambda\left(PIP^{-1}\right)$$

$$= P(B - \lambda I)P^{-1}$$

$$\det(A - \lambda I) = \det\left(P(B - \lambda I)P^{-1}\right)$$

$$= \det(P) \cdot \det(B - \lambda I) \cdot \det\left(P^{-1}\right)$$

$$= \det(B - \lambda I) \cdot \det(P) \cdot \det\left(P^{-1}\right)$$

$$= \det(B - \lambda I) \cdot \det\left(PP^{-1}\right) \qquad \det\left(PP^{-1}\right) = \det(I) = 1$$

$$= \det(B - \lambda I)$$

Geometric Explanation: At least in terms of the eigenvalues, these values are numbers λ such that there exists a vector $\boldsymbol{v} \neq 0$ such that the linear transformation T satisfies $T(\boldsymbol{v}) = \lambda \boldsymbol{v}$.

Exercise

Suppose that A is a matrix. Suppose that the linear transformation associated to A has two linearly independent eigenvectors. Prove that A is similar to a diagonal matrix.

Solution

Let T be the linear transformation associated with A. Consider the basis $\boldsymbol{v}_1, \boldsymbol{v}_2$ of the 2 linearly independent eigenvectors of A where λ_1, λ_2 the eigenvalues associated with. Then,

$$T\left(\boldsymbol{v}_1\right) = \lambda_1 \boldsymbol{v}_1 \quad \textit{and} \quad T\left(\boldsymbol{v}_2\right) = \lambda_2 \boldsymbol{v}_2$$

Let T be a matrix with respect to the basis $\boldsymbol{v}_1, \boldsymbol{v}_2$, then we obtain the matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

This completes the proof because A is similar to this diagonal matrix by definition.

Exercise

Prove that if A is a 2×2 matrix that has two distinct eigenvalues, then A is similar to a diagonal matrix.

Solution

Suppose A has 2 distinct eigenvalues λ_1, λ_2.

Let $\boldsymbol{v}_1 \neq 0$ be an eigenvector for λ_1.

Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2$ are not linearly independent, thus they are scalar multiples of each other.

So there exists $c \neq 0$ such that $c\boldsymbol{v}_1 = \boldsymbol{v}_2$. Then

$$\lambda_2\boldsymbol{v}_2 = A\boldsymbol{v}_2 = A\left(c\boldsymbol{v}_1\right) = c\left(A\boldsymbol{v}_1\right) = c\lambda_1\boldsymbol{v}_1 = \lambda_1 c\boldsymbol{v}_1 = \lambda_1\boldsymbol{v}_2$$

So that $\lambda_2\boldsymbol{v}_2 - \lambda_1\boldsymbol{v}_2 = 0 \quad \Rightarrow \quad \left(\lambda_2 - \lambda_1\right)\boldsymbol{v}_2 = 0$

But then $\lambda_2 = \lambda_1$ which contradicts the initial assumption.

Thus $\boldsymbol{v}_1, \boldsymbol{v}_2$ are linearly independent then $T\left(\boldsymbol{v}_1\right) = \lambda_1\boldsymbol{v}_1 \quad \text{and} \quad T\left(\boldsymbol{v}_2\right) = \lambda_2\boldsymbol{v}_2$

Let T be a matrix with respect to the basis $\boldsymbol{v}_1, \boldsymbol{v}_2$, then we obtain the matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

This completes the proof because A is similar to this diagonal matrix by definition.

Exercise

Suppose that the characteristic polynomial of a matrix has a double root. Is it true that the matrix has two linearly independent eigenvectors? Consider the example $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Is it true that matrices with equal characteristic polynomial are necessarily similar?

Solution

$$\left|A - \lambda I\right| = \begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix} = \lambda^2$$

The characteristic polynomial: $p(x) = x^2$ which has a double root $\left(\text{eigenvalue}: \ \lambda = 0\right)$.

$$\left(A - \lambda I\right)V = AV = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad x = 0$$

Therefore, the eigenvectors are vectors of the form $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ which can transform to $\begin{pmatrix} x \\ 0 \end{pmatrix}$

Thus matrices whose characteristic polynomials have a double root do not necessarily have 2 linear independent.

Let $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then the characteristic polynomial: $p(x) = x^2$ which has a double root $(eigenvalue: \ \lambda = 0)$. But they are not similar. The eigenvector is the $\mathbf{0}$ vector.

The linear transformation associated to the second matrix send every vector to $\mathbf{0}$. Thus the 2 matrices can't represent the same linear transformation.

Thus, matrices with equal characteristic polynomial are not necessarily similar.

Exercise

Show that the given matrix is not diagonalizable. $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

Solution

$$\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = 0$$

Since the determinant is 0, the inverse doesn't exist; therefore the matrix is not diagonalizable

Exercise

Determine if the given matrix is diagonalizable. If, so, find matrices S and $\Lambda(D)$ such that the given matrix equals $S\Lambda S^{-1}$

a) $\begin{bmatrix} 3 & 3 \\ 4 & 2 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{bmatrix}$

Solution

a) $\begin{vmatrix} 3-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 12 = \lambda^2 - 5\lambda - 6 = 0$

The eigenvalues $\lambda_1 = -1, \ \lambda_2 = 6$

For $\lambda = -1 \quad \Rightarrow \quad (A+I)V_1 = 0$

$$\begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad 4x_1 + 3y_1 = 0 \quad \Rightarrow x_1 = -\frac{3}{4}y_1$$

Therefore the eigenvector: $V_1 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

For $\lambda = 6 \quad \Rightarrow \quad (A - 6I)V_2 = 0$

$$\begin{pmatrix} -3 & 3 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad -3x_2 + 3y_2 = 0 \quad \Rightarrow x_2 = \frac{3}{4}y_2$$

Therefore the eigenvector: $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$S = \begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix} \quad \to \quad S^{-1} = \begin{bmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{3}{7} \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\begin{aligned} S\Lambda S^{-1} &= \begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{3}{7} \end{bmatrix} \\ &= \begin{bmatrix} 3 & 6 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{3}{7} \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3 \\ 4 & 2 \end{bmatrix} \\ &= A \end{aligned}$$

b) $$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & -1-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(-1-\lambda) + (2-\lambda)$$

$$\begin{aligned} &= (2-\lambda)\left((1-\lambda)(-1-\lambda)+1\right) \\ &= (2-\lambda)\left(-1+\lambda^2+1\right) \\ &= (2-\lambda)\lambda^2 = 0 \end{aligned}$$

The eigenvalues $\boxed{\lambda_1 = 2, \ \lambda_{2,3} = 0}$

The given matrix is not diagonalizable, since the eigenvalues are not distinct.

Solution Section 4.6 – Orthogonal Diagonalization

Exercise

Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$

a) $A=\begin{pmatrix}3&1\\1&3\end{pmatrix}$ *b)* $A=\begin{pmatrix}6&2\sqrt{3}\\2\sqrt{3}&7\end{pmatrix}$ *c)* $A=\begin{pmatrix}-2&0&-36\\0&-3&0\\-36&0&-23\end{pmatrix}$

d) $A=\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}$ *e)* $A=\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}$ *f)* $A=\begin{pmatrix}-7&24&0&0\\24&7&0&0\\0&0&-7&27\\0&0&24&7\end{pmatrix}$

Solution

a) $\det(A-\lambda I)=\begin{vmatrix}3-\lambda&1\\1&3-\lambda\end{vmatrix}$

$$=(3-\lambda)^2-1$$

$$=\lambda^2-6\lambda+8=0$$

The eigenvalues are: $\boxed{\lambda=2 \quad and \quad \lambda=4}$

For $\lambda_1=2$, we have: $(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\rightarrow\begin{cases}x+y=0\\x+y=0\end{cases}\Rightarrow x=-y$$

Therefore the eigenvector $V_1=\begin{pmatrix}-1\\1\end{pmatrix}$

For $\lambda_2=4$, we have: $(A-\lambda_2 I)V_2=0$

$$\begin{pmatrix}-1&1\\1&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\rightarrow\begin{cases}-x+y=0\\x-y=0\end{cases}\Rightarrow x=y$$

Therefore the eigenvector $V_2=\begin{pmatrix}1\\1\end{pmatrix}$

$$u_1=\frac{v_1}{\|v_1\|}=\frac{(-1,1)}{\sqrt{(-1)^2+1^2}}=\frac{(-1,1)}{\sqrt{2}}=\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$$

$$w_2=v_2-(v_2.u_1)u_1$$

$$=(1,1)-\left[(1,1)\cdot\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right]\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$$

$$= (1,1) - (0)\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$= (1,\ 1)$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{(1,\ 1)}{\sqrt{1^2+1^2}} = \boxed{\left(\frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{2}}\right)}$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$P^{-1}AP = P^T AP = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

b) $\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & 2\sqrt{3} \\ 2\sqrt{3} & 7-\lambda \end{vmatrix}$

$$= (6-\lambda)(7-\lambda) - 12$$

$$= \lambda^2 - 13\lambda + 30 = 0$$

The eigenvalues are: $\boxed{\lambda = 3 \quad and \quad \lambda = 10}$

For $\lambda_1 = 3$, we have: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3 & 2\sqrt{3} \\ 2\sqrt{3} & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & \frac{2}{\sqrt{3}} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = -\frac{2}{\sqrt{3}} y$$

Therefore the eigenvector $V_1 = \begin{pmatrix} -\frac{2}{\sqrt{3}} \\ 1 \end{pmatrix}$

For $\lambda_2 = 10$, we have: $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -4 & 2\sqrt{3} \\ 2\sqrt{3} & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & -\frac{\sqrt{3}}{2} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = \frac{\sqrt{3}}{2} y$$

Therefore the eigenvector $V_2 = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ 1 \end{pmatrix}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{\left(-\frac{2}{\sqrt{3}}, 1\right)}{\sqrt{\left(-\frac{2}{\sqrt{3}}\right)^2 + 1^2}} = \frac{\sqrt{3}}{\sqrt{7}}\left(-\frac{2}{\sqrt{3}}, 1\right) = \boxed{\left(-\frac{2}{\sqrt{7}}, \frac{\sqrt{3}}{\sqrt{7}}\right)}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + (1)^2}}\left(\frac{\sqrt{3}}{2},1\right) = \frac{2}{\sqrt{7}}\left(\frac{\sqrt{3}}{2},1\right) = \boxed{\left(\frac{\sqrt{3}}{\sqrt{7}}, \frac{2}{\sqrt{7}}\right)}$$

$$P = \begin{pmatrix} -\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{pmatrix}$$

$$P^{-1}AP = P^TAP = \begin{pmatrix} -\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{pmatrix}\begin{pmatrix} 6 & 2\sqrt{3} \\ 2\sqrt{3} & 7 \end{pmatrix}\begin{pmatrix} -\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 10 \end{pmatrix}$$

c) $$\det(A-\lambda I) = \begin{vmatrix} -2-\lambda & 0 & -36 \\ 0 & -3-\lambda & 0 \\ -36 & 0 & -23-\lambda \end{vmatrix}$$

$$\begin{aligned} &= (-2-\lambda)(-3-\lambda)(-23-\lambda) - (36)(-3-\lambda)(36) \\ &= -\left(6+5\lambda+\lambda^2\right)(23+\lambda) + 3888 + 1296\lambda \\ &= -138 - 115\lambda - 23\lambda - 6\lambda - 5\lambda^2 - \lambda^3 + 3888 + 1296\lambda \\ &= -\lambda^3 - 28\lambda^2 + 1175\lambda + 3750 = 0 \end{aligned}$$

The eigenvalues are: $\boxed{\lambda_1 = 25, \quad \lambda_2 = -3, \quad and \quad \lambda_3 = -50}$

For $\lambda_1 = 25$, we have: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -27 & 0 & -36 \\ 0 & -28 & 0 \\ -36 & 0 & -48 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -27x - 36z = 0 \\ \boxed{y=0} \\ -36x - 48z = 0 \end{cases} \Rightarrow 27x = 36z \Rightarrow \boxed{x = -\frac{4}{3}z}$$

Therefore the eigenvector $V_1 = \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}$

For $\lambda_2 = -3$, we have: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 1 & 0 & -36 \\ 0 & 0 & 0 \\ -36 & 0 & -20 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \boxed{x=0} \\ \boxed{z=0} \end{cases} \rightarrow \boxed{y=1}$$

Therefore the eigenvector $V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_3 = -25$, we have: $\left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} 48 & 0 & -36 \\ 0 & 47 & 0 \\ -36 & 0 & 27 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & -\frac{3}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} \boxed{x = \frac{3}{4}z} \\ \boxed{y = 0} \end{cases}$$

Therefore the eigenvector $V_3 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(-4,0,3)}{\sqrt{16+9}} = \frac{(-4,0,3)}{5} = \underline{\left(-\frac{4}{5},\ 0,\ \frac{3}{5}\right)|}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{(0,1,0)}{\sqrt{1^2}} = \underline{(0,\ 1,\ 0)|}$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{(3,0,4)}{\sqrt{3^2+4^2}} = \frac{(3,0,4)}{5} = \underline{\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)|}$$

$$P = \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

$$P^{-1}AP = P^T AP = \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}\begin{pmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{pmatrix}\begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 25 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -50 \end{pmatrix}$$

d) $$\det\left(A - \lambda I\right) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix}$$

$$= -\lambda\left[(1-\lambda)^2 - 1\right]$$

$$= -\lambda\left(\lambda^2 - 2\lambda\right) = 0$$

The eigenvalues are: $\boxed{\lambda_{1,2} = 0 \quad and \quad \lambda_3 = 2}$

For $\lambda_{1,2} = 0$, we have: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad x + y = 0 \Rightarrow \boxed{x = -y}$$

Therefore the eigenvector $V_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad V_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda_3 = 2$, we have: $\left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} -x + y = 0 \Rightarrow \boxed{x = y} \\ \boxed{z = 0} \end{cases}$$

Therefore the eigenvector $V_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(-1,1,0)}{\sqrt{1^2 + 1^2}} = \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{(0,0,1)}{\sqrt{1^2}} = (0,\ 0,\ 1)$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{(1,1,0)}{\sqrt{1^2 + 1^2}} = \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$P^{-1}AP = P^T AP = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$=\begin{pmatrix}0&0&0\\0&0&0\\0&0&2\end{pmatrix}$$

e) $\det(A-\lambda I)=\begin{vmatrix}2-\lambda&-1&-1\\-1&2-\lambda&-1\\-1&-1&2-\lambda\end{vmatrix}$

$$=(2-\lambda)^3-1-1-(2-\lambda)-(2-\lambda)-(2-\lambda)$$
$$=8-12\lambda+6\lambda^2-\lambda^3-2-6+3\lambda$$
$$=-\lambda^3+6\lambda^2-9\lambda=0$$

The eigenvalues are: $\lambda_1=0$ *and* $\lambda_{2,3}=3$

For $\lambda_1=0$, we have: $(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}\xrightarrow{rref}\begin{pmatrix}1&0&-1\\0&1&-1\\0&0&0\end{pmatrix}\Rightarrow\begin{cases}x_1-z_1=0\\y_1-z_1=0\end{cases}$$

Therefore the eigenvector $V_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$

For $\lambda_{2,3}=3$, we have: $(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}-1&-1&-1\\-1&-1&-1\\-1&-1&-1\end{pmatrix}\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\rightarrow -x_1-y_1-z_1=0\Rightarrow x_1=-y_1-z_1$$

Therefore the eigenvector $V_2=\begin{pmatrix}-1\\1\\0\end{pmatrix}$ $V_3=\begin{pmatrix}-1\\0\\1\end{pmatrix}$

$$u_1=\frac{V_1}{\|V_1\|}=\frac{(1,1,1)}{\sqrt{3}}=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

$$w_2=V_2-(V_2\bullet u_1)u_1$$

$$= (-1,\ 1,\ 0) - \left[(-1,\ 1,\ 0)\bullet\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (-1,\ 1,\ 0) - 0\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (-1,\ 1,\ 0)$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{(-1,\ 1,\ 0)}{\sqrt{2}} = \boxed{\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)}$$

$$w_3 = V_3 - \left(V_3\bullet u_2\right)u_2$$

$$= (-1,\ 0,\ 1) - \left[(-1,\ 0,\ 1)\bullet\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)\right]\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= (-1,\ 0,\ 1) - \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= (-1,\ 0,\ 1) - \left(-\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$= \left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)$$

$$u_3 = \frac{w_3}{\|w_3\|} = \frac{\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}} = \frac{1}{\frac{\sqrt{6}}{2}}\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right) = \boxed{\left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)}$$

$$P = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$$

$$P^T A P = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

f) $\det(A-\lambda I)=\begin{vmatrix} -7-\lambda & 24 & 0 & 0 \\ 24 & 7-\lambda & 0 & 0 \\ 0 & 0 & -7-\lambda & 24 \\ 0 & 0 & 24 & 7-\lambda \end{vmatrix}$

$$=(-7-\lambda)(7-\lambda)\left[(-7-\lambda)(7-\lambda)-24^2\right]-24^2\left[(-7-\lambda)(7-\lambda)-24^2\right]$$

$$=\left(\lambda^2-49\right)\left(\lambda^2-625\right)-576\left(\lambda^2-625\right)$$

$$=\left(\lambda^2-625\right)\left(\lambda^2-49-576\right)$$

$$=\left(\lambda^2-625\right)^2=0$$

The eigenvalues are: $\boxed{\lambda_{1,2}=25 \quad and \quad \lambda_{3,4}=-25}$

For $\lambda_{1,2}=25$, we have:

$$\begin{pmatrix} -32 & 24 & 0 & 0 \\ 24 & -18 & 0 & 0 \\ 0 & 0 & -32 & 24 \\ 0 & 0 & 24 & -18 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & -\frac{3}{4} & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1-\frac{3}{4}x_2=0 \\ x_3-\frac{3}{4}x_4=0 \end{cases}$$

Therefore the eigenvector $V_1=\begin{pmatrix} 0 \\ 0 \\ 3 \\ 4 \end{pmatrix} \quad V_2=\begin{pmatrix} 3 \\ 4 \\ 0 \\ 0 \end{pmatrix}$

For $\lambda_{3,4}=-25$, we have:

$$\begin{pmatrix} 18 & 24 & 0 & 0 \\ 24 & 32 & 0 & 0 \\ 0 & 0 & 18 & 24 \\ 0 & 0 & 24 & 32 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1+\frac{4}{3}x_2=0 \\ x_3+\frac{4}{3}x_4=0 \end{cases}$$

Therefore the eigenvector $V_3 = \begin{pmatrix} 0 \\ 0 \\ -4 \\ 3 \end{pmatrix}$ $V_4 = \begin{pmatrix} -4 \\ 3 \\ 0 \\ 0 \end{pmatrix}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(0,0,3,4)}{\sqrt{3^2+4^2}} = \underline{\left(0,\ 0,\ \frac{3}{5},\ \frac{4}{5}\right)}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{(3,4,0,0)}{\sqrt{3^2+4^2}} = \underline{\left(\frac{3}{5},\ \frac{4}{5},\ 0,\ 0\right)}$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{(0,0,-4,3)}{\sqrt{(-4)^2+3^2}} = \underline{\left(0,\ 0,\ -\frac{4}{5},\ \frac{3}{5}\right)}$$

$$u_4 = \frac{v_4}{\|v_4\|} = \frac{(-4,3,0,0)}{\sqrt{25}} = \underline{\left(-\frac{4}{5},\ \frac{3}{5},\ 0,\ 0\right)}$$

$$P = \begin{pmatrix} 0 & \frac{3}{5} & 0 & -\frac{4}{5} \\ 0 & \frac{4}{5} & 0 & \frac{3}{5} \\ \frac{3}{5} & 0 & -\frac{4}{5} & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} & 0 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} & 0 & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

$$P^T AP = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} & 0 & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} -7 & 24 & 0 & 0 \\ 24 & 7 & 0 & 0 \\ 0 & 0 & -7 & 24 \\ 0 & 0 & 24 & 7 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} & 0 & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -25 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & -25 & 0 \\ 0 & 0 & 0 & 25 \end{pmatrix}$$

g) For $A=\begin{bmatrix}3&1&0&0\\1&3&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$

$$\left|A-\lambda I\right|=\begin{vmatrix}3-\lambda&1&0&0\\1&3-\lambda&0&0\\0&0&-\lambda&0\\0&0&0&-\lambda\end{vmatrix}$$

$$=\lambda^2(3-\lambda)^2-\lambda^2$$

$$=\lambda^2\left(9-6\lambda+\lambda^2-1\right)$$

$$=\lambda^2\left(\lambda^2-6\lambda+8\right)$$

$$=\lambda^2(\lambda-2)(\lambda-4)$$

Therefore the matrix has eigenvalues $\underline{\lambda_{1,2,3,4}=0,\ 2,\ 4|}$

For $\lambda_{1,2}=0$, then $(A-0)V_1=0$

$$\begin{pmatrix}3&1&0&0\\1&3&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\y_1\\z_1\\w_1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\quad\to\quad\begin{cases}3x_1+y_1=0\\x_1+3y_1=0\end{cases}\quad\Rightarrow\ x_1=y_1=0$$

The eigenvectors are: $V_1=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad V_2=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

For $\lambda_3=2$, then $(A-2I)V_3=0$

$$\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&-2&0\\0&0&0&-2\end{pmatrix}\begin{pmatrix}x_3\\y_3\\z_3\\w_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\quad\to\quad\begin{cases}x_3+y_3=0\\x_3+y_3=0\\2z_3=0\\2w_3=0\end{cases}\quad\Rightarrow\ \begin{cases}x_3=-y_3\\z_3=w_3=0\end{cases}$$

The eigenvectors are: $V_3=\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$ or $V_3=\begin{pmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\\0\end{pmatrix}$

For $\lambda_4 = 4$, then $(A-4I)V_4 = 0$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x_4 \\ y_4 \\ z_4 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -x_4 + y_4 = 0 \\ x_4 - y_4 = 0 \\ -4z_4 = 0 \\ -4w_4 = 0 \end{cases} \Rightarrow \begin{cases} x_4 = y_4 \\ z_4 = w_4 = 0 \end{cases}$$

The eigenvectors are: $V_4 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}$

$$P = \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \rightarrow P^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Exercise

Find the eigenvalues of A and B and check the Orthogonality of their first two eigenvectors. Graph these eigenvectors to see the discrete sines and cosines:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The $-1, 2, -1$ pattern in both matrices is a "second derivative. Then $Ax = \lambda x$ and $Bx = \lambda x$ are like $\frac{d^2x}{dt^2} = \lambda x$. This has eigenvectors $x = \sin kt$ and $x = \cos kt$ that are the bases for Fourier series. The matrices lead to "discrete sines" and "discrete cosines" that are the bases for the discrete Fourier Transform. This DFT is absolutely central to all areas of digital signal processing.

Solution

The eigenvalues of A are $\lambda = 2 \pm \sqrt{2}$ *and* 2.

Their sum is 6 (the trace of A) and their product is 4 (the determinant).

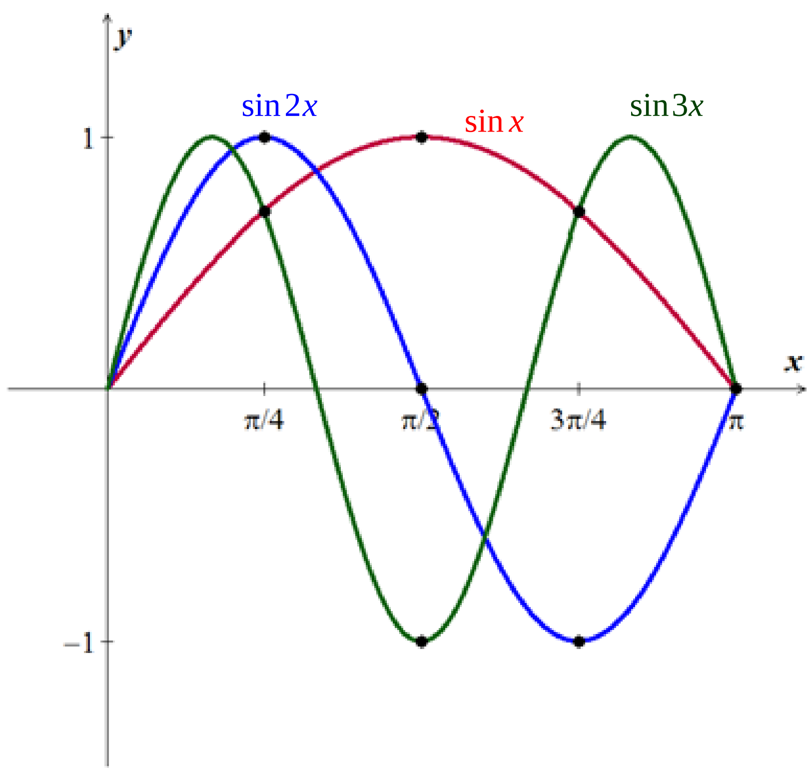
The eigenvector matrix S gives the "Discrete Sine Transform".

$$S = \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

$$V_1 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} \frac{\sqrt{2}}{2} \\ 1 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

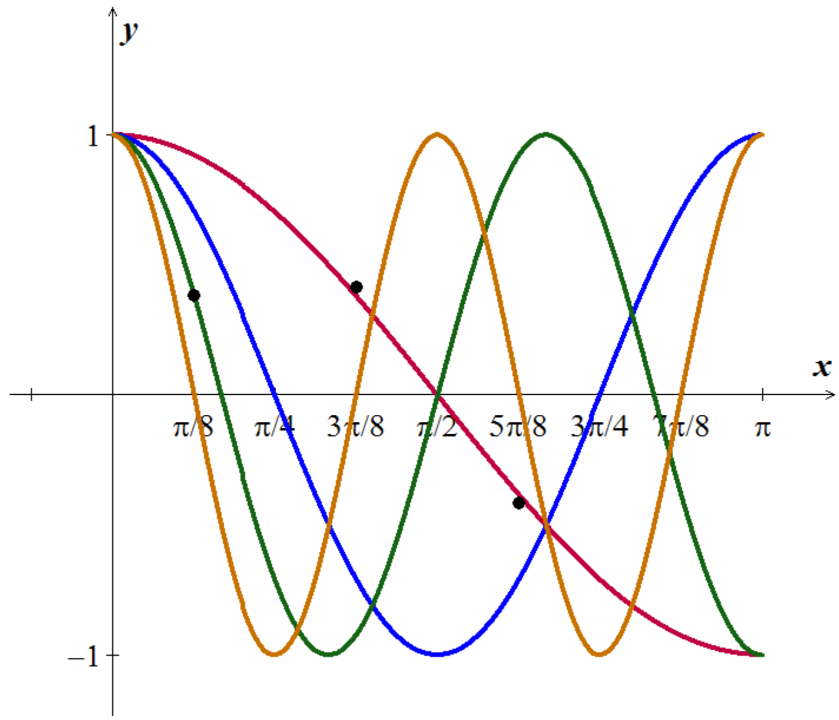
$$V_2 = \begin{pmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} \frac{\sqrt{2}}{2} \\ -1 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$



The eigenvalues of B are $\lambda = 2 \pm \sqrt{2},\ \ 2,\ \ 0$.

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \sqrt{2}-1 & -1 & 1-\sqrt{2} \\ 1 & 1-\sqrt{2} & -1 & \sqrt{2}-1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$



Exercise

Suppose $Ax = \lambda x$ and $Ay = 0y$ and $\lambda \neq 0$. Then y is in the nullspace and x is in the column space. They are perpendicular because _______. (why are these subspaces orthogonal?) If the second eigenvalue is a nonzero number β, apply this argument to $A - \beta I$. The eigenvalue moves to zero and the eigenvectors stay the same – so they are perpendicular.

Solution

Suppose that $A = A^T$ and $Ax = \lambda x$, $Ay = 0y$, and $\lambda \neq 0$. Then x is in the column space of A, and y is in the left nullspace of A since $N(A) = N\left(A^T\right)$. But $C(A)$ and $N\left(A^T\right)$ are orthogonal complements, so x and y are perpendicular.

If $Ay = \beta y$ with $\beta \neq \lambda$ then $(A - \beta I)x = (\lambda - \beta)x$ and $(A - \beta I)y = 0$. Since $\lambda - \beta \neq 0$ it follows that x is in the column space $A- \beta I$ and y is in the nullspace of $A- \beta I$, and $(A - \beta I)^T = A^T - \beta I^T = A - \beta I$, Therefore we can replace A with $A - \beta I$ in the argument of previous paragraph and it follows that x and y are perpendicular.

Exercise

True or false. Give a reason or a counterexample.

a) A matrix with real eigenvalues and eigenvectors is symmetric.

b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.

c) The inverse of a symmetric matrix is symmetric

d) The eigenvector matrix S of a symmetric matrix is symmetric.

e) A complex symmetric matrix has real eigenvalues.

f) If A is symmetric, then e^{iA} is symmetric.

g) If A is Hermitian, then e^{iA} is Hermitian.

Solution

a) ***False***. Let $A=\begin{pmatrix}1 & 2\\ 1 & 0\end{pmatrix}$

Then $\det(A-\lambda I)=\begin{vmatrix}1-\lambda & 2\\ 1 & -\lambda\end{vmatrix}=\lambda^2-\lambda-2=0$

So A has eigenvalues $\lambda_1=-1 \quad \lambda_2=2$

The eigenvectors are: $V_1=\begin{pmatrix}-1\\ 1\end{pmatrix} \quad V_2=\begin{pmatrix}2\\ 1\end{pmatrix}$ so both the eigenvalues and eigenvectors are real but A is not symmetric.

b) ***True***. If the matrix A has orthogonal eigenvectors $x_1, x_2, \cdots, x_n$ with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, we can define $s_i=\dfrac{x_i}{\|x_i\|}$ for all i; then $As_i=\lambda_i s_i$ for all i and the s_i are orthonormal. Then we can diagonalize A as: $A=S\Lambda S^{-1}$ where the i^{th} column of S is s_i, and Λ is the diagonal matrix, so $S^T=S^{-1}$ and $A=S\Lambda S^T$.

$$A^T=\left(S^T\right)^T\Lambda^T S^T=S\Lambda S^T=A$$

So A is symmetric.

c) ***True***. If A is symmetric then it can be diagonalized by an orthogonal matrix Q, $A=QDQ^{-1}$, and then $A^{-1}=QD^{-1}Q^{-1}=QD^{-1}Q^T$. Since D^{-1} is still a diagonal matrix, it follows:

$\left(A^{-1}\right)^T=QD^{-1}Q^T=A^{-1}$

d) ***False***. The eigenvalues of $A=\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}$ are: $\lambda_1=0 \quad \lambda_2=5$ and the eigenvectors are:

$V_1=\begin{pmatrix}2\\ -1\end{pmatrix} \quad V_2=\begin{pmatrix}1\\ 2\end{pmatrix}$. We can diagonalize A with eigenvector matrix $S=\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}$ which is not symmetric.

e) ***False***. For example $A = (i)$, the 1 by 1 matrix. The eigenvalue is $\boldsymbol{i}$, it is not a real number.

f) ***True***. $\left(e^{iA}\right)^T = e^{(iA)^T} = e^{iA}$

g) ***False***. $\left(e^{iA}\right)^H = e^{(iA)^H} = e^{-iA^H} = e^{-iA}$. It is typically not the same as e^{iA}. Taking $A = (1)$, the 1 by 1 matrix, would be a enough example because $e^{iA} = e^i$ which is not a real number.

Exercise

Find a symmetric matrix $\begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$ that has a negative eigenvalue.

a) How do you know it must have a negative pivot?

b) How do you know it can't have two negative eigenvalues?

Solution

a) The eigenvalues of that matrix are $1 \pm b\,/$ so take any $b < -1 \quad b > 1$. In this case, the determinant is $1 - b^2 < 0$.

b) The signs of the pivots coincide with the signs of the eigenvalues. Alternatively, the product of the pivots is the determinant, which is negative in this case. So, one of the two pivots must be negative.

c) The product of the eigenvalues equals the determinant, which is negative in this case. So, precisely one numbers cannot have a negative product.

Exercise

Which of these classes of matrices do A and B belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad B = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Which of these factorizations are possible for A and B: LU, QR, ADP^{-1}, QDQ^T?

Solution

Matrix A is invertible, orthogonal, a permutation matrix, diagonalizable, and Markov! (Everything but a projection).

Matrix A satisfies $A^2 = I$, $A = A^T$, and also $AA^T = I$, This means it is invertible, symmetric, and orthogonal. Since it is symmetric, it is diagonalizable (with real eigenvalues!). It is a permutation matrix by just looking at it. It is Markov since the columns add to 1. It is not a projection since $A^2 = I \neq A$.

All of the factorization are possible for it: LU and QR are always possible. PDP^{-1} is possible since it is diagonalizable, and QDQ^T is possible since it is symmetric.

Matrix B is a projection, diagonalizable, and Markov. It is not invertible, not orthogonal, and not a permutation.

B is a projection since $B^2 = B$, it is symmetric and thus diagonalizable, and it is Markov since the columns add to 1. It is not invertible since the columns are visibly linearly dependent, it is not orthogonal since the columns are far from orthonormal, and it's clearly not a permutation.

All the factorizations are possible for it: LU and QR are always possible. PDP^{-1} is possible since it is diagonalizable, and QDQ^T is possible since it is symmetric.

Exercise

Prove that A is any $m \times n$ matrix, then $A^T A$ has an orthonormal set of n eigenvectors

Solution

$\left(A^T A\right)^T = A^T \left(A^T\right)^T = A^T A$, then $A^T A$ is symmetric, therefore there is an eigenvector $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ for $A^T A$.

Let $A\vec{v}_1 = \lambda_1 \vec{v}_1 \quad$ *and* $\quad A\vec{v}_2 = \lambda_2 \vec{v}_2$

$A\vec{v}_1 \bullet \vec{v}_2 = \vec{v}_1 \bullet A^T \vec{v}_2 \qquad A\vec{x} \bullet \vec{y} = \vec{x} \bullet A^T \vec{y}$

$A\vec{v}_1 \bullet \vec{v}_2 = \vec{v}_1 \bullet A\vec{v}_2 \qquad A^T = A \ \ (\textit{Since } A \textit{ is a symmetric})$

$\lambda_1 \vec{v}_1 \bullet \vec{v}_2 = \vec{v}_1 \bullet \lambda_2 \vec{v}_2$

$\lambda_1 \left(\vec{v}_1 \bullet \vec{v}_2\right) = \lambda_2 \left(\vec{v}_1 \bullet \vec{v}_2\right)$

$\left(\lambda_1 - \lambda_2\right)\left(\vec{v}_1 \bullet \vec{v}_2\right) = 0 \qquad$ Since $\lambda_1 \neq \lambda_2$

Therefore; $\vec{v}_1 \bullet \vec{v}_2 = 0$

Then the vectors $A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_n$ are orthogonal

$$A\vec{v}_1 \bullet A\vec{v}_2 = \left(A\vec{v}_i\right)^T A\vec{v}_j = \vec{v}_i^{\,T} A^T A\vec{v}_j = \vec{v}_i \cdot \left(A^T A\vec{v}_j\right) = \vec{v}_i \cdot \left(\lambda_j \vec{v}_j\right) = \lambda_j \left(\vec{v}_i \cdot \vec{v}_j\right) = 0$$

Example

Construct a 3 by 3 matrix A with no zero entries whose columns are mutually perpendicular. Compute $A^T A$. Why is it a diagonal matrix?

Solution

Consider the matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ to be columns mutual perpendicular

Let assume $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{pmatrix}$ *then* $A^T = \begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \\ 2 & 1 & 2 \end{pmatrix}$ *or* $A = \begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$\left(A^T A\right)_{ij} = (\text{column } i \text{ of } A)(\text{column } j \text{ of } A)$$

Exercise

Assuming that $b \neq 0$, find a matrix that orthogonally diagonalizes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$

Solution

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} \\ &= (a-\lambda)^2 - b^2 \\ &= (a-\lambda-b)(a-\lambda+b) \\ &= (a-b-\lambda)(a+b-\lambda) = 0 \end{aligned}$$

Therefore the eigenvalues are: $\lambda_1 = a-b$ *and* $\lambda_2 = a+b$

Assume that $b \neq 0$.

For $\lambda_1 = a-b$, then $\left(A-(a-b)I\right)V_1 = 0$

$$\begin{pmatrix} b & b \\ b & b \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow bx_1 = -by_1$$

The eigenvectors are: $V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For $\lambda_2 = a+b$, then $\left(A-(a+b)I\right)V_2 = 0$

$$\begin{pmatrix} -b & b \\ b & -b \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad bx_2 = by_2$$

The eigenvectors are: $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Applying the Gram Schmidt process.

$$\boldsymbol{u}_1 = \frac{V_1}{\|V_1\|} = \frac{(-1,\ 1)}{\sqrt{2}} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\boldsymbol{w}_2 = V_2 - \frac{\langle \boldsymbol{u}_1, V_2 \rangle}{\|V_2\|^2} = (1,\ 1) - \frac{(-1,\ 1)\bullet(1,\ 1)}{2}(1,\ 1)$$

$$= (1,\ 1)$$

$$\boldsymbol{u}_2 = \frac{\boldsymbol{w}_2}{\|\boldsymbol{w}_2\|} = \frac{(1,\ 1)}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$