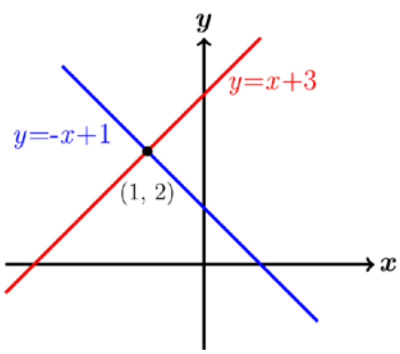
Lecture Four

Section 4.1 – System of linear Equations

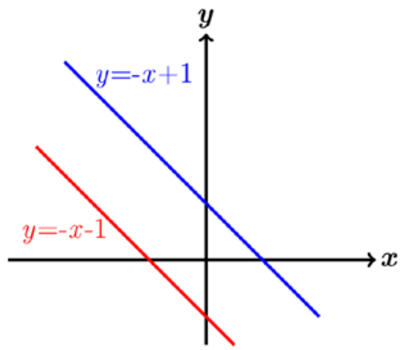
Solving Systems of Equations

1. Graphically
2. Substitution Method
3. Elimination Method

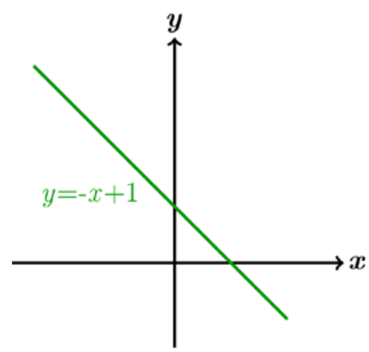
Graphically



One solution (lines intersect)
Consistent
Independent



No Solution (lines //)
Inconsistent
Independent



Infinite solution
Consistent
Dependent

Substitution Method

Solve: $\begin{cases} 3x+2y=11 & (1) \\ -x+y=3 & (2) \end{cases}$

Solution

$$\text{From } (2) \to y = x+3 \qquad (3)$$

$$(1) \Rightarrow 3x+2(x+3)=11$$

$$3x+2x+6=11$$

$$5x+6=11$$

$$5x+6-6=11-6$$

$$5x=5$$

$$x=1$$

$From\ (3) \to y = 1 + 3 = 4$

$Solution: \ \underline{(1,\ 4)}|$

Elimination Method

Solve: $\begin{cases} 3x - 4y = 1 & (1) \\ 2x + 3y = 12 & (2) \end{cases}$

Solution

$$\begin{array}{ll} -2\times) & 3x - 4y = 1 \\ 3\times) & 2x + 3y = 12 \end{array}$$

$$\begin{array}{r} -6x + 8y = -2 \\ \underline{6x + 9y = 36} \\ 17y = 34 \end{array}$$

$y = \frac{34}{17} = 2$

$$\begin{aligned} From\ \ (1) \Rightarrow 3x &= 1 + 4y \\ 3x &= 1 + 4(2) \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

$Solution: \ \underline{(3,\ 2)}|$

Matrices

$$\begin{array}{cc} & \textit{Column} \\ & \begin{array}{ccc} C_1 & C_2 & C_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} Row\ 1 \rightarrow R_1 \\ Row\ 2 \rightarrow R_2 \\ Row\ 3 \rightarrow R_3 \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

This is called Matrix (*Matrices*)

Each number in the array is an ***element*** or ***entry***

The matrix is said to be of order $m \times n$

m: numbers of rows,

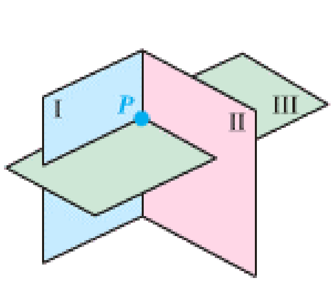
n: number of columns

When $m = n$, then matrix is said to be ***square***.

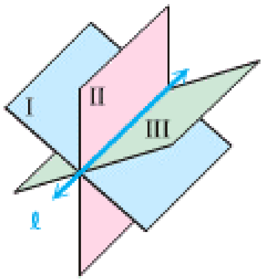
Given the system equations

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

The ***augmented matrix*** form is: $\left[\begin{array}{ccc|c} 3 & 1 & 2 & 31 \\ 1 & 1 & 2 & 19 \\ 1 & 3 & 2 & 25 \end{array}\right]$

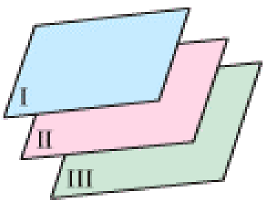


A single solution

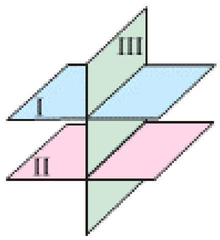


Points of a line in common

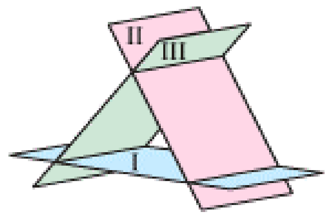
All points in common



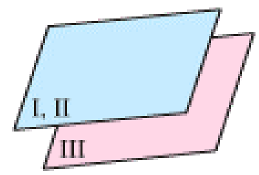
No points in common



No points in common



No points in common



No points in common

Gaussian Elimination

Example

Use the Gaussian elimination method to solve the system

$$\begin{aligned} 3x + y + 2z &= 31 \\ x + y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 3 & 1 & 2 & 31 \\ 1 & 3 & 2 & 25 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 1 & 2 & 31 \\ -3 & -3 & -6 & -57 \\ \hline 0 & -2 & -4 & -26 \end{array} \qquad \begin{array}{cccc} 1 & 3 & 2 & 25 \\ -1 & -1 & -2 & -19 \\ \hline 0 & 2 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & -2 & -4 & -26 \\ 0 & 2 & 0 & 6 \end{array}\right] \begin{array}{l} \\ -\frac{1}{2}R_2 \\ \\ \end{array} \qquad \begin{array}{cccc} 0 & 1 & 2 & 13 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 2 & 0 & 6 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_2 \end{array} \qquad \begin{array}{cccc} 0 & 2 & 0 & 6 \\ 0 & -2 & -4 & -26 \\ \hline 0 & 0 & -4 & -20 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -4 & -20 \end{array}\right] \begin{array}{l} \\ \\ -\frac{1}{4}R_3 \end{array} \qquad \begin{array}{cccc} 0 & 0 & 1 & 5 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array}\right] \Rightarrow \begin{array}{rl} x + y + 2z = 19 & (3) \\ y + 2z = 13 & (2) \\ z = 5 & (1) \end{array}$$

$(2) \Rightarrow y = 13 - 2z = 13 - 2(5) = 3$

$(3) \Rightarrow x = 19 - y - 2z = 19 - 3 - 10 = 6$

$\Rightarrow$ **(6, 3, 5)**

Gauss-Jordan Elimination

Example

Use the Gauss-Jordan method to solve the system

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 3 & 1 & 2 & 31 \\ 1 & 3 & 2 & 25 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 1 & 2 & 31 \\ -3 & -3 & -6 & -57 \\ \hline 0 & -2 & -4 & -26 \end{array} \qquad \begin{array}{cccc} 1 & 3 & 2 & 25 \\ -1 & -1 & -2 & -19 \\ \hline 0 & 2 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & -2 & -4 & -26 \\ 0 & 2 & 0 & 6 \end{array}\right] -\tfrac{1}{2}R_2 \qquad 0 \quad 1 \quad 2 \quad 13$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 2 & 0 & 6 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 2R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 2 & 0 & 6 \\ 0 & -2 & -4 & -26 \\ \hline 0 & 0 & -4 & -20 \end{array} \qquad \begin{array}{cccc} 1 & 1 & 2 & 19 \\ 0 & -1 & -2 & -13 \\ \hline 1 & 0 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -4 & -20 \end{array}\right] -\tfrac{1}{4}R_3 \qquad 0 \quad 0 \quad 1 \quad 5$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array}\right] R_2 - 2R_3 \qquad \begin{array}{cccc} 0 & 1 & 2 & 13 \\ 0 & 0 & -2 & -10 \\ \hline 0 & 1 & 0 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

***Solution*: (6, 3, 5)**

Example

Use the Gaussian elimination method to solve the system

$$\begin{aligned} 2x + \ \ y + 2z &= 4 \\ 2x + 2y \qquad &= 5 \\ 2x - \ \ y + 6z &= 2 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \ \tfrac{1}{2}R_1 \qquad\qquad 1 \quad \tfrac{1}{2} \quad 1 \quad 2$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array} \qquad \begin{array}{cccc} 2 & 2 & 0 & 5 \\ -2 & -1 & -2 & -4 \\ \hline 0 & 1 & -2 & 1 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 6 & 2 \\ -2 & -1 & -2 & -4 \\ \hline 0 & -2 & 4 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & -2 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 2R_2 \end{array} \qquad \begin{array}{cccc} 0 & -2 & 4 & -2 \\ 0 & 2 & -4 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{rl} x + \frac{1}{2}y + z = 2 & (3) \\ y - 2z = 1 & (2) \\ 0 = 0 & (1) \end{array}$$

From (1): $0 = 0$ is a true statement. Let ***z*** be the variable.

From (2): $y = 1 + 2z$

From (3): $x = -\frac{1}{2}y - z + 2$

$$x = -\frac{1}{2}(1 + 2z) - z + 2$$

$$x = -\frac{1}{2} - z - z + 2$$

$$x = -2z + \frac{3}{2}$$

Solution: $\left(-2z + \frac{3}{2},\ 2z + 1,\ z\right)$

Example

Use the Gaussian elimination method to solve the system

$$\begin{aligned} x + 2y - 5z &= -1 \\ 2x + 3y - 2z &= 2 \\ 3x + 5y - 7z &= 4 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & -1 \\ 2 & 3 & -2 & 2 \\ 3 & 5 & -7 & 4 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} 2 & 3 & -2 & 2 \\ -2 & -4 & 10 & 2 \\ \hline 0 & -1 & 8 & 4 \end{array} \qquad \begin{array}{cccc} 3 & 5 & -7 & 4 \\ -3 & -6 & 15 & 3 \\ \hline 0 & -1 & 8 & 7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & -1 \\ 0 & -1 & 8 & 4 \\ 0 & -1 & 8 & 7 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ \end{array}$$

$$\begin{array}{cccc} 0 & 1 & -8 & -4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & -1 \\ 0 & 1 & -8 & -4 \\ 0 & -1 & 8 & 7 \end{array}\right] \begin{array}{l} \\ \\ R_3 + R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -1 & 8 & 7 \\ 0 & 1 & -8 & -4 \\ \hline 0 & 0 & 0 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & -1 \\ 0 & 1 & -8 & -4 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

From Row 3: 0 = 3 is a False statement.

No Solution or ***Inconsistent***

Exercises *Section* 4.1 – System of linear Equations

(**1 – 15**) Use any method to solve the system equation (***elimination*** or ***substitution*** method)

1. $\begin{cases} 3x+2y=-4 \\ 2x-y=-5 \end{cases}$

2. $\begin{cases} 2x+5y=7 \\ 5x-2y=-3 \end{cases}$

3. $\begin{cases} 4x-7y=-16 \\ 2x+5y=9 \end{cases}$

4. $\begin{cases} 3x+2y=4 \\ 2x+y=1 \end{cases}$

5. $\begin{cases} 3x+4y=2 \\ 2x+5y=-1 \end{cases}$

6. $\begin{cases} 5x-2y=4 \\ -10x+4y=7 \end{cases}$

7. $\begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$

8. $\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

9. $\begin{cases} 2x+10y=-14 \\ 7x-2y=-16 \end{cases}$

10. $\begin{cases} 4x-3y=24 \\ -3x+9y=-1 \end{cases}$

11. $\begin{cases} 4x+2y=12 \\ 3x-2y=16 \end{cases}$

12. $\begin{cases} x+2y=-1 \\ 4x-2y=6 \end{cases}$

13. $\begin{cases} x-2y=5 \\ -10x+2y=4 \end{cases}$

14. $\begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$

15. $\begin{cases} 4x-4y=-12 \\ 4x+4y=-20 \end{cases}$

(**16 – 27**) Perform the matrix row operation (or operations) and write the new matrix.

16. $\left[\begin{array}{cc|c} 1 & 4 & 7 \\ 3 & 5 & 0 \end{array}\right] \quad R_2-3R_1$

17. $\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right] \quad R_2-2R_1$

18. $\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 5 & 2 & 19 \end{array}\right] \quad R_2-5R_1$

19. $\left[\begin{array}{cc|c} 2 & -3 & 8 \\ -6 & 9 & 4 \end{array}\right] \quad R_2+3R_1$

20. $\left[\begin{array}{cc|c} 2 & 3 & 11 \\ 1 & 2 & 8 \end{array}\right] \quad 2R_2-R_1$

21. $\left[\begin{array}{cc|c} 3 & 5 & -13 \\ 2 & 3 & -9 \end{array}\right] \quad 3R_2-2R_1$

22. $\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 5 & 4 & 1 \end{array}\right] \quad R_3-5R_2$

23. $\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right] \quad \begin{array}{l} R_2-3R_1 \\ R_3-R_1 \end{array}$

24. $\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 2 & 4 & 4 & 22 \\ -1 & -2 & 3 & 15 \end{array}\right] \quad \begin{array}{l} 3R_2-2R_1 \\ 3R_3+R_1 \end{array}$

25. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & -4 & 2 & -7 \end{array}\right] \quad \begin{array}{l} R_2-2R_1 \\ R_3-3R_1 \end{array}$

26. $\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ 2 & -3 & 5 & -1 & 0 \\ 1 & 0 & 3 & 1 & -4 \\ -4 & 3 & 2 & -1 & 3 \end{array}\right] \quad \begin{array}{l} R_2-2R_1 \\ R_3-R_1 \\ R_4+4R_1 \end{array}$

27. $\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ -3 & 6 & -3 & -9 & 6 \\ 2 & 1 & 2 & 3 & 4 \\ 5 & 3 & 2 & -1 & -7 \end{array}\right] \quad \begin{array}{l} R_2+3R_1 \\ R_3-2R_1 \\ R_4-5R_1 \end{array}$

(28 – 34) Use the Gauss-Jordan method to solve the system

28. $\begin{cases} x-y+5z=-6 \\ 3x+3y-z=10 \\ x+3y+2z=5 \end{cases}$

29. $\begin{cases} 2x-y+4z=-3 \\ x-2y-10z=-6 \\ 3x \quad +4z=7 \end{cases}$

30. $\begin{cases} 4x+3y-5z=-29 \\ 3x-7y-z=-19 \\ 2x+5y+2z=-10 \end{cases}$

31. $\begin{cases} x+2y-3z=-15 \\ 2x-3y+4z=18 \\ -3x+y+z=1 \end{cases}$

32. $\begin{cases} x+2y+3z=10 \\ 4x+5y+6z=11 \\ 7x+8y+9z=12 \end{cases}$

33. $\begin{cases} 2x+y+2z=4 \\ 2x+2y \quad =5 \\ 2x-y+6z=2 \end{cases}$

34. $\begin{cases} x_1+x_2+2x_3=8 \\ -x_1-2x_2+3x_3=1 \\ 3x_1-7x_2+4x_3=10 \end{cases}$

(35 – 69) Use augmented elimination to solve linear system

35. $\begin{cases} 2x-5y+3z=1 \\ x-2y-2z=8 \end{cases}$

36. $\begin{cases} x+y+z=2 \\ 2x+y-z=5 \\ x-y+z=-2 \end{cases}$

37. $\begin{cases} 2x+y+z=9 \\ -x-y+z=1 \\ 3x-y+z=9 \end{cases}$

38. $\begin{cases} 3y-z=-1 \\ x+5y-z=-4 \\ -3x+6y+2z=11 \end{cases}$

39. $\begin{cases} x+3y+4z=14 \\ 2x-3y+2z=10 \\ 3x-y+z=9 \end{cases}$

40. $\begin{cases} x+4y-z=20 \\ 3x+2y+z=8 \\ 2x-3y+2z=-16 \end{cases}$

41. $\begin{cases} 2y-z=7 \\ x+2y+z=17 \\ 2x-3y+2z=-1 \end{cases}$

42. $\begin{cases} -2x+6y+7z=3 \\ -4x+5y+3z=7 \\ -6x+3y+5z=-4 \end{cases}$

43. $\begin{cases} 2x-y+z=1 \\ 3x-3y+4z=5 \\ 4x-2y+3z=4 \end{cases}$

44. $\begin{cases} 3x-4y+4z=7 \\ x-y-2z=2 \\ 2x-3y+6z=5 \end{cases}$

45. $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

46. $\begin{cases} x+y+z=3 \\ -y+2z=1 \\ -x+z=0 \end{cases}$

47. $\begin{cases} 3x+y+3z=14 \\ 7x+5y+8z=37 \\ x+3y+2z=9 \end{cases}$

48. $\begin{cases} 4x-2y+z=7 \\ x+y+z=-2 \\ 4x+2y+z=3 \end{cases}$

49. $\begin{cases} 2x-2y+z=-4 \\ 6x+4y-3z=-24 \\ x-2y+2z=1 \end{cases}$

50. $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

51. $\begin{cases} 2x-y+2z=-8 \\ x+2y-3z=9 \\ 3x-y-4z=3 \end{cases}$

52. $\begin{cases} x \quad -3z=-5 \\ 2x-y+2z=16 \\ 7x-3y-5z=19 \end{cases}$

53. $\begin{cases} x+2y-z=5 \\ 2x-y+3z=0 \\ 2y+z=1 \end{cases}$

54. $\begin{cases} x+y+z=6 \\ 3x+4y-7z=1 \\ 2x-y+3z=5 \end{cases}$

55. $\begin{cases} 3x+2y+3z=3 \\ 4x-5y+7z=1 \\ 2x+3y-2z=6 \end{cases}$

56. $\begin{cases} x-3y+z=2 \\ 4x-12y+4z=8 \\ -2x+6y-2z=-4 \end{cases}$

57. $\begin{cases} 2x-2y+z=-1 \\ x+2y-z=2 \\ 6x+4y+3z=5 \end{cases}$

58. $\begin{cases} x_1-5x_2+2x_3-2x_4=4 \\ x_2-3x_3-x_4=0 \\ 3x_1+2x_3-x_4=6 \\ -4x_1+x_2+4x_3+2x_4=-3 \end{cases}$

59. $\begin{cases} x_1+x_2+x_3+x_4=5 \\ x_1+2x_2-x_3-2x_4=-1 \\ x_1-3x_2-3x_3-x_4=-1 \\ 2x_1-x_2+2x_3-x_4=-2 \end{cases}$

60. $\begin{cases} 2x+8y-z+w=0 \\ 4x+16y-3z-w=-10 \\ -2x+4y-z+3w=-6 \\ -6x+2y+5z+w=3 \end{cases}$

61. $\begin{cases} 2x_1+x_2+3x_3=0 \\ x_1+2x_2=0 \\ x_2+x_3=0 \end{cases}$

62. $\begin{cases} 2x+2y+4z=0 \\ -y-3z+w=0 \\ 3x+y+z+2w=0 \\ x+3y-2z-2w=0 \end{cases}$

63. $\begin{cases} 2x+z+w=5 \\ y-w=-1 \\ 3x-z-w=0 \\ 4x+y+2z+w=9 \end{cases}$

64. $\begin{cases} 4y+z=20 \\ 2x-2y+z=0 \\ x+z=5 \\ x+y-z=10 \end{cases}$

65. $\begin{cases} x-y+2z-w=-1 \\ 2x+y-2z-2w=-2 \\ -x+2y-4z+w=1 \\ 3x-3w=-3 \end{cases}$

66. $\begin{cases} 2u-3v+w-x+y=0 \\ 4u-6v+2w-3x-y=-5 \\ -2u+3v-2w+2x-y=3 \end{cases}$

67. $\begin{cases} 6x_3+2x_4-4x_5-8x_6=8 \\ 3x_3+x_4-2x_5-4x_6=4 \\ 2x_1-3x_2+x_3+4x_4-7x_5+x_6=2 \\ 6x_1-9x_2+11x_4-19x_5+3x_6=1 \end{cases}$

68. $\begin{cases} 3x_1+2x_2-x_3=-15 \\ 5x_1+3x_2+2x_3=0 \\ 3x_1+x_2+3x_3=11 \\ -6x_1-4x_2+2x_3=30 \end{cases}$

69. $\begin{cases} x_1+3x_2-2x_3+2x_5=0 \\ 2x_1+6x_2-5x_3-2x_4+4x_5-3x_6=-1 \\ 5x_3+10x_4+15x_6=5 \\ 2x_1+6x_2+8x_4+4x_5+18x_6=6 \end{cases}$

70. At Snack Mix, caramel corn worth \$2.50 per *pound* is mixed with honey roasted missed nuts worth \$7.50 per *pound* in order to get 20 *lbs*. of a mixture worth \$4.50 per *pound*. How much of each snack is used?

Section 4.2 – Matrix operations and Their Applications

Matrix Notation

The Matrix:

$$A=\begin{bmatrix}1 & 1 & 2\\ 3 & 1 & 2\\ 1 & 3 & 2\end{bmatrix} \text{ is called the coefficient matrix of the system}$$

The matrix is said to be of order *m x n*

m: numbers of rows,

n: number of columns

A matrix A with *m* rows and *n* columns can be written in a general form

$$A=\left[a_{ij}\right]=\begin{bmatrix}a_{11} & a_{12} & a_{13} & \cdots & a_{1n}\\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n}\\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n}\\ \vdots & \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}\end{bmatrix}$$

The matrix *A* above has 3 rows and 3 columns; therefore, the order of the matrix *A* is (3 *x* 3)

When $m = n$, then matrix is said to be ***square***.

The numbers in a matrix are called ***entries.***

Example

$$Let \quad A=\begin{bmatrix}5 & -2\\ -3 & \pi\\ 1 & 6\end{bmatrix}$$

a. What is the order of *A*?

3 rows and 2 columns $\Rightarrow$ *A* is 3 *x* 2

b. $a_{12}=-2$ $\qquad$ $a_{31}=1$

Equality of Matrices

Definition of Equality of Matrices

Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are equal if and only if they have the same order (size) $m \times n$ and if each pair corresponding elements is equal

$$a_{ij} = b_{ij} \text{ for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

Example

Find the values of the variables for which each statement is true, if possible.

a) $\begin{bmatrix} 2 & 1 \\ p & q \end{bmatrix} = \begin{bmatrix} x & y \\ -1 & 0 \end{bmatrix}$

$$x = 2, \; y = 1, \; p = -1, \; q = 0$$

b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$

can't be true

c) $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} w = 9 & x = 17 \\ 8 = y & -12 = z \end{bmatrix}$$

Matrix Addition and Subtraction

Given two m x n matrices $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ their sum is $A + B = \left[a_{ij} + b_{ij}\right]$

And their difference is $A - B = \left[a_{ij} - b_{ij}\right]$

The matrices have to be the ***same order***

Example

Find $\begin{bmatrix} -4 & 3 \\ 7 & -6 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & -4 \end{bmatrix}$

Solution

$$\begin{bmatrix} -4 & 3 \\ 7 & -6 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} -4+6 & 3+(-3) \\ 7+2 & -6+(-4) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 9 & -10 \end{bmatrix}$$

Example

Find $\begin{bmatrix} 5 & 4 \\ -3 & 7 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 8 \\ 6 & 0 \\ -5 & 3 \end{bmatrix}$

Solution

$$\begin{bmatrix} 5 & 4 \\ -3 & 7 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 8 \\ 6 & 0 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 5-(-4) & 4-8 \\ -3-6 & 7-0 \\ 0-(-5) & 1-3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -4 \\ -9 & 7 \\ 5 & -2 \end{bmatrix}$$

Example

Find $\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix}$

Solution

$$\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix} = \begin{bmatrix} 5-4 & -6+6 \\ 8+8 & 9-3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 16 & 6 \end{bmatrix}$$

Scalar Multiplication

The scalar product of a number k and a matrix A is denoted by $\boldsymbol{kA}$.

$$kA = k\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$$

Example

Find $5\begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix}$

Solution

$$5\begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2(5) & -3(5) \\ 0(5) & 4(5) \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -15 \\ 0 & 20 \end{bmatrix}$$

Example

Find $\frac{3}{4}\begin{bmatrix} 20 & 36 \\ 12 & -16 \end{bmatrix}$

Solution

$$\frac{3}{4}\begin{bmatrix} 20 & 36 \\ 12 & -16 \end{bmatrix} = \begin{bmatrix} 15 & 27 \\ 9 & -12 \end{bmatrix}$$

Example

Given: $A = \begin{bmatrix} -4 & 1 \\ 3 & 0 \end{bmatrix}$ $B = \begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix}$

Find:

a) $-6B$

b) $3A + 2B$

Solution

a) $-6B = -6\begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix}$

$$= \begin{bmatrix} -1(-6) & -2(-6) \\ 8(-6) & 5(-6) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 12 \\ -48 & -30 \end{bmatrix}$$

b) $3A + 2B = 3\begin{bmatrix} -4 & 1 \\ 3 & 0 \end{bmatrix} + 2\begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix}$

$$= \begin{bmatrix} -4(3) & 1(3) \\ 3(3) & 0(3) \end{bmatrix} + \begin{bmatrix} -1(2) & -2(2) \\ 8(2) & 5(2) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 3 \\ 9 & 0 \end{bmatrix} + \begin{bmatrix} -2 & -4 \\ 16 & 10 \end{bmatrix}$$

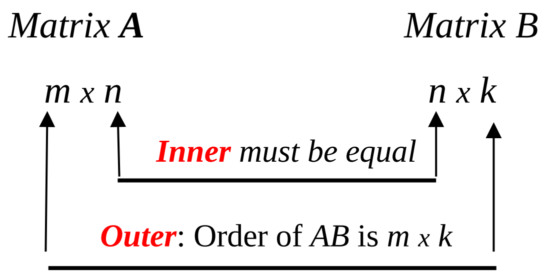
$$= \begin{bmatrix} -12-2 & 3-4 \\ 9+16 & 0+10 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & -1 \\ 25 & 10 \end{bmatrix}$$

Matrix Multiplication

Product of Two Matrices

Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix. To find the element in the i^{th} row and j^{th} column of the product matrix AB, multiply each element in the i^{th} row of A by the corresponding element in the j^{th} column of B, and then add these products. The product matrix AB is an $m \times k$ matrix.



$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$2x2 \qquad 2x2 \quad \rightarrow \quad 2x2$$

$$a_{11} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & \_ \\ \_ & \_ \end{bmatrix}$$

$$a_{12} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & af + bh \\ \_ & \_ \end{bmatrix}$$

$$a_{21} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ ce + dg & \_ \end{bmatrix}$$

$$a_{22} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ \_ & cf + dh \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Example

Given: $A=\begin{bmatrix} 1 & -3 \\ 7 & 2 \end{bmatrix} \quad B=\begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 4 & -1 \end{bmatrix}$

Find $\boldsymbol{AB}$ and $\boldsymbol{BA}$.

Solution

$$\boldsymbol{AB}=\begin{bmatrix} 1 & -3 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 4 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} 1(1)+(-3)3 & 1(0)+(-3)1 & 1(-1)+(-3)4 & 1(2)+(-3)(-1) \\ 7(1)+2(3) & 7(0)+2(1) & 7(-1)+2(4) & 7(2)+2(-1) \end{bmatrix}$$

$$=\begin{bmatrix} -8 & -3 & -13 & 5 \\ 13 & 2 & 1 & 12 \end{bmatrix}$$

$\boldsymbol{BA}$ can be found since: B: 2x4 and A: 2x2

Note: $AB \neq BA$

Example

Given: $A=\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \qquad B=\begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix}$

Find $\boldsymbol{AB}$.

Solution

$$AB=\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 1(4)+3(1) & 1(6)+3(0) \\ 2(4)+5(1) & 2(6)+5(0) \end{bmatrix}$$

$$=\begin{bmatrix} 7 & 6 \\ 13 & 12 \end{bmatrix}$$

Example

Given: $A=\begin{bmatrix}3 & 1 & -1\\ 2 & 0 & 3\end{bmatrix}$ $B=\begin{bmatrix}1 & 6\\ 3 & -5\\ -2 & 4\end{bmatrix}$ Find $\boldsymbol{AB}$.

Solution

$$AB=\begin{bmatrix}3 & 1 & -1\\ 2 & 0 & 3\end{bmatrix}\begin{bmatrix}1 & 6\\ 3 & -5\\ -2 & 4\end{bmatrix}$$

$$=\begin{bmatrix}3(1)+1(3)-1(-2) & 3(6)+1(-5)-1(4)\\ 2(1)+0(3)+3(-2) & 2(6)+0(-5)+3(4)\end{bmatrix}$$

$$=\begin{bmatrix}8 & 9\\ -4 & 24\end{bmatrix}$$

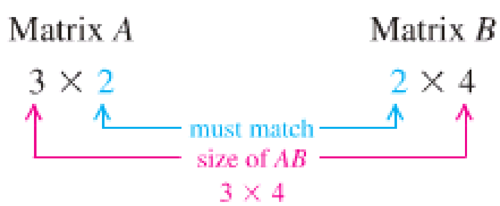
Example

Suppose A is a 3x2 matrix, while B is a 2x4 matrix.

a) Can the product AB be calculated?
b) If AB can be calculated, what size is it?
c) Can BA be calculated?
d) If BA can be calculated, what size is it?

Solution

a)



b) The product AB size is 3 x 4

c)

d) Can't be calculated

Example

Given: $A_{1x3} = \begin{bmatrix} 2 & 0 & 4 \end{bmatrix}$ $\quad B_{3x1} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$ $\quad$ Find ***AB*** and ***BA***.

Solution

A $\qquad\qquad$ ***B***

$1 \; x \; 3 \qquad\qquad 3 \; x \; 1$

Inner: Are equal

Outer: Order of *AB* is 1x1

AB = [2(1) + 0(3) + 4(7)]

= [30]

BA : 3*x*1 ---- 1*x*3

$$\boldsymbol{BA} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1(2) & 1(0) & 1(4) \\ 3(2) & 3(0) & 3(4) \\ 7(2) & 7(0) & 7(4) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 4 \\ 6 & 0 & 12 \\ 14 & 0 & 28 \end{bmatrix}$$

Example

Given: $\quad A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 & -1 & 6 \\ 0 & 5 & 4 & 1 \end{bmatrix}$ $\quad$ Find ***AB*** and ***BA***.

Solution

a) $\boldsymbol{AB} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \bullet \begin{bmatrix} 2 & 3 & -1 & 6 \\ 0 & 5 & 4 & 1 \end{bmatrix}$ $\qquad$ 2*x*2 --- 2*x*4

$$= \begin{bmatrix} 1(2)+3(0) & 1(3)+3(5) & 1(-1)+3(4) & 1(6)+3(1) \\ 0(2)+2(0) & 0(3)+2(5) & 0(-1)+2(4) & 0(6)+2(1) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 18 & 11 & 9 \\ 0 & 10 & 8 & 2 \end{bmatrix}$$

b) $\boldsymbol{BA} = \begin{bmatrix} 2 & 3 & -1 & 6 \\ 0 & 5 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ + Undefined $\quad$ 2*x*4 --- 2*x*2 (*Inner order are not equal* 2, 4)

Properties of Matrix

Addition and Scalar Multiplication

$A+B=B+A$	*Commutative Property of Addition*
$A+(B+C)=(A+B)+C$	*Associative Property of Addition*
$(kl)A=k(lA)$	*Associative Property of Scalar Multiplication*
$k(A+B)=kA+kB$	*Distributive Property*
$(k+l)A=kA+lA$	*Distributive Property*
$A+0=0+A=A$	*Additive Identity Property*
$A+(-A)=(-A)+A=0$	*Additive Inverse Property*

Multiplication

$A(BC)=(AB)C$	*Associative Property of Multiplication*
$A(B+C)=AB+AC$	*Distributive Property*
$(B+C)A=BA+CA$	*Distributive Property*

Exercises *Section* 4.2 – Matrix operations and Their Applications

(1 – 7) Find values for the variables so that the matrices are equal.

1. $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

2. $\begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$

3. $\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix} = \begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$

4. $\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$

5. $\begin{bmatrix} a+11 & 12z+1 & 5m \\ 11k & 3 & 1 \end{bmatrix} + \begin{bmatrix} 9a & 9z & 4m \\ 12k & 5 & 3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$

6. $\begin{bmatrix} x+2 & 3y+1 & 5z \\ 8w & 2 & 3 \end{bmatrix} + \begin{bmatrix} 3x & 2y & 5z \\ 2w & 5 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$

7. $\begin{bmatrix} 2x-3 & y-2 & 2z+1 \\ 5 & 2w & 7 \end{bmatrix} + \begin{bmatrix} 3x-3 & y+2 & z-1 \\ -5 & 5w+1 & 3 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$

8. Given $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix}$ Find: $A-B$, $3A+2B$

9. Given $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ $F = \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}$ Find: $3F+2A$

(10 – 22) Evaluate

10. $\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \\ 12 \end{bmatrix}$

11. $\begin{bmatrix} 5 & 8 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 9 & 1 \\ 4 & 2 & 5 \end{bmatrix}$

12. $\begin{bmatrix} -5 & 0 \\ 4 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix}$

13. $\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix}$

14. $\begin{bmatrix} -5 & 6 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ 5 & -8 \end{bmatrix}$

15. $\begin{bmatrix} 8 & 6 & -4 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -8 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix}$

17. $\begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix}\begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix}$

18. $\begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix} . \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix} . \begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}$

20. $\begin{bmatrix} -2 & -3 & -4 \\ 2 & -1 & 0 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$

21. $\begin{bmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{18} \\ \sqrt{3} & \sqrt{27} & 0 \end{bmatrix} \begin{bmatrix} 8 & -10 \\ 9 & 12 \\ 0 & 2 \end{bmatrix}$

22. $\begin{bmatrix} x & 2x+1 & 4 \\ 5 & x-1 & 8 \\ -2 & 3x & 2x+1 \end{bmatrix} + \begin{bmatrix} 2x-1 & -2x-1 & 4x \\ -5 & 6 & x+1 \\ -5 & 2 & -2x \end{bmatrix}$

(**23 – 33**) Find AB and BA, if possible

23. $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ $B = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}$

24. $A = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$ $B = \begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}$

25. $A = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$ $B = \begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}$

26. $A = \begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix}$

27. $A = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$

28. $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$

29. $A = \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$

30. $A = \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix}$

31. $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$

32. $A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix}$ $B = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

33. $A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}$ $B = \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}$

34. Given $A = \begin{bmatrix} -3 & 4 \\ 2 & -3 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 1 \\ 1 & -2 \\ 3 & -4 \end{bmatrix}$, Find

a) $A + B$
b) $A - B$
c) $3A$
d) $-2B$
e) $2A + 3B$
f) A^2
g) AB
h) BA

35. Given $A=\begin{bmatrix} 2 & -2 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}$ $B=\begin{bmatrix} -1 & 8 \\ 2 & -2 \\ -4 & 3 \end{bmatrix}$, Find

a) $A+B$
b) $A-B$
c) $3A$
d) $-2B$
e) $2A+3B$
f) A^2
g) AB
h) BA

36. Given $A=\begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}$ $B=\begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$, Find

a) $A+B$
b) $A-B$
c) $3A$
d) $-2B$
e) $2A+3B$
f) A^2
g) AB
h) BA

37. Given $A=\begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}$ $B=\begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$, Find

a) $A+B$
b) $A-B$
c) $3A$
d) $-2B$
e) $2A+3B$
f) A^2
g) AB
h) BA

38. Given $A=\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$ $B=\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$ $C=\begin{pmatrix} 4 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix}$ $D=\begin{pmatrix} -2 & 3 \\ 2 & -1 \\ 3 & 2 \end{pmatrix}$, Find

a) $4A-2B$
b) $3A+C$
c) $3A+B$
d) $2A-3B$
e) AB
f) BA
g) A^2
h) B^3
i) AC
j) CA
k) CD
l) DC

39. Given $A=\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$ $B=\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$ $C=\begin{pmatrix} 1 & 4 & 5 \\ -2 & 3 & 4 \\ -1 & 0 & -2 \end{pmatrix}$ $D=\begin{pmatrix} 2 & 4 & -2 \\ 0 & 3 & 5 \\ -3 & 1 & 1 \end{pmatrix}$, Find

a) $4A-2B$
b) $3A+C$
c) $3A+B$
d) $2A-3B$
e) AB
f) BA
g) A^2
h) B^3
i) AC
j) CB
k) CD
l) DC

40. A contractor builds three kinds of houses, models *A*, *B*, and *C*, with a choice of two styles, Spanish and contemporary. Matrix *P* shows the number of each kind of house planned for a new 100-home subdivision. The amounts for each of the exterior materials depend primarily on the style of the house. These amounts are shown in matrix *Q*. (concrete is in *cubic yards*, lumber in units of 1000 board *feet*, brick in 1000*s*, and shingles in units of 100 ft^2.) Matrix *R* gives the cost in dollars for each kind of material.

a) What is the total cost of these materials for each model?
b) How much of each of four kinds of material must be ordered
c) What is the total cost for exterior materials?

41. Mitchell Fabricators manufactures three styles of bicycle frames in its two plants. The following table shows the number of each style produced at each plant

	Mountain Bike	***Racing Bike***	***Touring Bike***
North Plant	150	120	100
South Plant	180	90	130

a) Write a 2 *x* 3 matrix ***A*** that represents the information in the table
b) The manufacturer increased production of each style by 20%. Find a Matrix ***M*** that represents the increased production figures.
c) Find the matrix ***A + M*** and tell what it represents

42. Sal's Shoes and Fred's Footwear both have outlets in California and Arizona. Sal's sells shoes for $80, sandals for $40, and boots for $120. Fred's prices are $60, $30, and $150 for shoes, sandals and boots, respectively. Half of all sales in California stores are shoes, 1/4 are *sandals*, and 1/4 are *boots*. In Arizona, the fractions are 1/5 *shoes*, 1/5 are *sandals*, and 3/5 are *boots*.
a) Write a 2 x 3 matrix called *P* representing prices for the two stores and three types of footwear.
b) Write a 2 x 3 matrix called *F* representing fraction of each type of footwear sold in each state.
c) Only one of the two products *PF* and *FP* is meaningful. Determine which one it is, calculate the product, and describe what the entries represent.

Section 4.3 – Multiplicative Inverses of Matrices

Identity Matrix

The $n \times n$ identity matrix with 1's on the main diagonal and 0's elsewhere and is denoted by

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2x2} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3x3} \qquad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{I} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

The Multiplicative Identity Matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then $AI = IA = A$

Example

$$A = \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution

$$AI = \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4(1)-7(0) & 4(0)-7(1) \\ -3(1)+2(0) & -3(0)+2(1) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix} = A$$

$$= A$$

Multiplicative inverse of a matrix

Multiplicative inverse of a matrix $A_{n\times n}$ and $A^{-1}_{n\times n}$ if exists, then:

$$A\cdot A^{-1} = A^{-1}\cdot A = I$$

Example

Show that B is Multiplicative inverse of A.

A = $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ B = $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

Solution

$$\boldsymbol{A.B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2(1)-1(1) & 2(-1)+1(2) \\ 1(1)+1(-1) & 1(-1)+1(2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{= I}$$

$$BA = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{= I}$$

$\therefore$ B is multiplicative inverse of a matrix A: $B = A^{-1}$

Finding Inverse matrix

To find inverse matrix using Gauss-Jordan method:

$$\left[A \middle| I\right] \rightarrow \left[I \middle| A^{-1}\right] \qquad \textit{where } A^{-1} \textit{ read as "A inverse"}$$

For 2 by 2 matrices (*only*)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

If $ad - bc = 0$, then A^{-1} doesn't exist

Example

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix} \Rightarrow A^{-1} = ?$$

Solution

$$A^{-1} = \frac{1}{(-1)(4) - (-2)(3)}\begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Example

$$A=\begin{bmatrix}3 & -2\\ -1 & 1\end{bmatrix} \Rightarrow A^{-1}=?$$

Solution

$$A^{-1}=\frac{1}{(3)(1)-(-2)(-1)}\begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix}$$

To find inverse matrix using Gauss-Jordan method:

$$\left[A \middle| I\right] \rightarrow \left[I \middle| A^{-1}\right] \qquad \textit{where } A^{-1} \textit{ read as "A inverse"}$$

Example

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \qquad \text{Find } A^{-1}$$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \end{array} \qquad \begin{array}{cccccc} -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 \\ \hline 0 & 2 & 5 & 1 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & -1 & 0 & 0 \\ \hline 0 & -1 & -2 & -1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ 2R_3 + R_2 \end{array} \qquad \begin{array}{cccccc} 0 & -2 & -4 & -2 & 0 & 2 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & -1 & 1 & 2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 - 2R_3 \\ R_2 - 5R_3 \\ \end{array} \qquad \begin{array}{cccccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 2 & -2 & -4 \\ \hline 0 & 0 & 0 & 3 & -2 & -4 \end{array} \qquad \begin{array}{cccccc} 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & 0 & -5 & 5 & -5 & -10 \\ \hline 0 & 2 & 0 & 6 & -4 & -10 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 2 & 0 & 6 & -4 & -10 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right] \quad \frac{1}{2} R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right]$$

$$\Rightarrow \qquad A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

Example

$$A=\begin{bmatrix}1&0&2\\-1&2&3\\1&-1&0\end{bmatrix} \qquad \text{Find } A^{-1}$$

Solution

$$\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\-1&2&3&0&1&0\\1&-1&0&0&0&1\end{array}\right] \begin{array}{l}\\R_2+R_1\\R_3-R_1\end{array}$$

$$\begin{array}{cccccc}-1&2&3&0&1&0\\1&0&2&1&0&0\\\hline 0&2&5&1&1&0\end{array} \qquad \begin{array}{cccccc}1&-1&0&0&0&1\\-1&0&-2&-1&0&0\\\hline 0&-1&-2&-1&0&1\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&2&5&1&1&0\\0&-1&-2&-1&0&1\end{array}\right] \ \tfrac{1}{2}R_2$$

$$0 \quad 1 \quad \tfrac{5}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad 0$$

$$\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&1&\frac{5}{2}&\frac{1}{2}&\frac{1}{2}&0\\0&-1&-2&-1&0&1\end{array}\right] \ R_3+R_2$$

$$\begin{array}{cccccc}0&-1&-2&-1&0&1\\0&1&\frac{5}{2}&\frac{1}{2}&\frac{1}{2}&0\\\hline 0&0&\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}&1\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&1&\frac{5}{2}&\frac{1}{2}&\frac{1}{2}&0\\0&0&\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}&1\end{array}\right] \ 2R_3$$

$$0 \quad 0 \quad 1 \quad -1 \quad 1 \quad 2$$

$$\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&1&\frac{5}{2}&\frac{1}{2}&\frac{1}{2}&0\\0&0&1&-1&1&2\end{array}\right] \begin{array}{l}R_1-2R_3\\R_2-\frac{5}{2}R_3\\ \end{array}$$

$$\begin{array}{cccccc}0&1&\frac{5}{2}&\frac{1}{2}&\frac{1}{2}&0\\0&0&-\frac{5}{2}&\frac{5}{2}&-\frac{5}{2}&-5\\\hline 0&1&0&3&-2&-5\end{array} \qquad \begin{array}{cccccc}1&0&2&1&0&0\\0&0&-2&2&-2&-4\\\hline 1&0&0&3&-2&-4\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&0&3&-2&-4\\0&1&0&3&-2&-5\\0&0&1&-1&1&2\end{array}\right]$$

$$\Rightarrow \qquad A^{-1}=\begin{bmatrix}3&-2&-4\\3&-2&-5\\-1&1&2\end{bmatrix}$$

Solving a System Using A^{-1}

To solve the matrix equation $AX = B$.

- $\boldsymbol{X}$: matrix of the variables
- $\boldsymbol{A}$: Coefficient matrix
- $\boldsymbol{B}$: Constant matrix

$$\begin{aligned}
AX &= B \\
A^{-1}(AX) &= A^{-1}B && \text{Multiply both side by } A^{-1} \\
\left(A^{-1}A\right)X &= A^{-1}B && \text{Associate property} \\
IX &= A^{-1}B && \text{Multiplicative inverse property} \\
X &= A^{-1}B && \text{Identity property}
\end{aligned}$$

Example

Solve the system using A^{-1}

$$\begin{cases} x + 2z = 6 \\ -x + 2y + 3z = -5 \\ x - y = 6 \end{cases} \qquad \text{Given } A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

Solution

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} \underset{=}{=} \underbrace{\begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix}}_{B}$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} 3(6)-2(-5)-4(6) \\ 3(6)-2(-5)-5(6) \\ -1(6)+1(-5)+2(6) \end{bmatrix} = \begin{bmatrix} 18+10-24 \\ 18+10-30 \\ -6-5+12 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

Solution: $\{(4, -2, 1)\}$

Example

Use the inverse of the coefficient matrix to solve the linear system

$$\begin{aligned} 2x - 3y &= 4 \\ x + 5y &= 2 \end{aligned}$$

Solution

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix} \qquad B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

The solution of the system is $(2, 0)$

Exercise Section 4.3 – Multiplicative Inverses of Matrices

Show that B is Multiplicative inverse of A

1. $A=\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$ $B=\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$

2. $A=\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ & $B=\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$

Find the inverse, if exists, of

3. $A=\begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

4. $A=\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$

5. $A=\begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$

6. $A=\begin{bmatrix} a & b \\ 3 & 3 \end{bmatrix}$

7. $A=\begin{bmatrix} -2 & a \\ 4 & a \end{bmatrix}$

8. $A=\begin{bmatrix} 4 & 4 \\ b & a \end{bmatrix}$

9. $A=\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$

10. $A=\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

11. $A=\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

12. $A=\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

13. $A=\begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$

14. $A=\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$

15. $A=\begin{pmatrix} -6 & 9 \\ 2 & -3 \end{pmatrix}$

16. $A=\begin{pmatrix} -2 & 7 \\ 0 & 2 \end{pmatrix}$

17. $A=\begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$

18. $A=\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

19. $A=\begin{pmatrix} 2 & 1 \\ a & a \end{pmatrix}$

20. $A=\begin{pmatrix} b & 3 \\ b & 2 \end{pmatrix}$

21. $A=\begin{pmatrix} 1 & a \\ 3 & a \end{pmatrix}$

22. $A=\begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}$

23. $A=\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$

24. $A=\begin{pmatrix} -3 & \frac{1}{2} \\ 6 & -1 \end{pmatrix}$

25. $A=\begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

26. $A=\begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{bmatrix}$

27. $A=\begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

28. $A=\begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$

29. $A=\begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{pmatrix}$

30. $A=\begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$

31. $A=\begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

32. $A=\begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 1 & 2 \end{pmatrix}$

33. $A=\begin{pmatrix}3 & 3 & 1\\ 1 & 2 & 1\\ 2 & -1 & 1\end{pmatrix}$

34. $A=\begin{pmatrix}-3 & 1 & -1\\ 1 & -4 & -7\\ 1 & 2 & 5\end{pmatrix}$

35. $A=\begin{pmatrix}1 & 1 & -3\\ 2 & -4 & 1\\ -5 & 7 & 1\end{pmatrix}$

36. $A=\begin{pmatrix}1 & 2 & -1\\ 3 & 5 & 3\\ 2 & 4 & 3\end{pmatrix}$

37. $A=\begin{bmatrix}-2 & -3 & 4 & 1\\ 0 & 1 & 1 & 0\\ 0 & 4 & -6 & 1\\ -2 & -2 & 5 & 1\end{bmatrix}$

38. $A=\begin{bmatrix}1 & -14 & 7 & 38\\ -1 & 2 & 1 & -2\\ 1 & 2 & -1 & -6\\ 1 & -2 & 3 & 6\end{bmatrix}$

39. $A=\begin{bmatrix}10 & 20 & -30 & 15\\ 3 & -7 & 14 & -8\\ -7 & -2 & -1 & 2\\ 4 & 4 & -3 & 1\end{bmatrix}$

State the conditions under which A^{-1} exists. Then find a formula for A^{-1}

40. $A=[x]$

41. $A=\begin{bmatrix}x & 0\\ 0 & y\end{bmatrix}$

42. $A=\begin{bmatrix}0 & 0 & x\\ 0 & y & 0\\ z & 0 & 0\end{bmatrix}$

43. $A=\begin{bmatrix}x & 1 & 1 & 1\\ 0 & y & 0 & 0\\ 0 & 0 & z & 0\\ 0 & 0 & 0 & w\end{bmatrix}$

44. Solve the system using A^{-1} $\begin{cases}x \qquad\quad + 2z = 6\\ -x+2y+3z=-5\\ x-y \qquad\quad = 6\end{cases}$ Given $A^{-1}=\begin{bmatrix}3 & -2 & -4\\ 3 & -2 & -5\\ -1 & 1 & 2\end{bmatrix}$

45. Solve the system using A^{-1}

$$\begin{cases}x+2y+5z=2\\ 2x+3y+8z=3\\ -x+\ y+2z=3\end{cases}$$

a) Write the linear system as a matrix equation in the form *AX*= *B*
b) Solve the system using the inverse that is given for the coefficient matrix

the inverse of $\begin{bmatrix}1 & 2 & 5\\ 2 & 3 & 8\\ -1 & 1 & 2\end{bmatrix}$ *is* $\begin{bmatrix}2 & -1 & -1\\ 12 & -7 & -2\\ -5 & 3 & 1\end{bmatrix}$

46. Solve the system using A^{-1}

$$\begin{cases} x - y + z = 8 \\ \quad 2y - z = -7 \\ 2x + 3y \quad\; = 1 \end{cases}$$

a) Write the linear system as a matrix equation in the form $AX = B$

b) Solve the system using the inverse that is given for the coefficient matrix

$$\textit{the inverse is } \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

(47– 75) Use the ***inverse*** of the coefficient matrix to solve the linear system

47. $\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$

48. $\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$

49. $\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$

50. $\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$

51. $\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$

52. $\begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$

53. $\begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$

54. $\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$

55. $\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$

56. $\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$

57. $\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$

58. $\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$

59. $\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$

60. $\begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases}$

61. $\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$

62. $\begin{cases} -2x + 3y = 4 \\ -3x + 4y = 5 \end{cases}$

63. $\begin{cases} x - 2y = 6 \\ 4x + 3y = 2 \end{cases}$

64. $\begin{cases} 2x - 3y = 7 \\ 4x + y = -7 \end{cases}$

65. $\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$

66. $\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$

67. $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

68. $\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$

69. $\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$

70. $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

71. $\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$

72. $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

73. $\begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$

Section 4.4 – Determinants and Cramer's Rule

Determinant of a 2 *x* 2 Matrix

Determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and is define as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example

Let $A = \begin{bmatrix} -3 & 4 \\ 6 & 8 \end{bmatrix}$. Find $|A|$

Solution

$$|A| = \begin{vmatrix} -3 & 4 \\ 6 & 8 \end{vmatrix}$$

$$= -3(8) - 4(6)$$

$$= -48$$

Example

Evaluate: $\begin{vmatrix} 2 & -3 \\ -4 & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & -3 \\ -4 & 1 \end{vmatrix} = 2(1) - (-3)(-4)$$

$$= 2 - 12$$

$$= -10$$

$$\boldsymbol{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor

For a square matrix $A = \left[a_{ij}\right]$, the minor M_{ij}. Of an element a_{ij} is the determinant of the matrix formed by deleting the i^{th} row and the j^{th} column of A.

Cofactor: $A_{ij} = (-1)^{i+j} M_{ij}$

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example

$A = \begin{pmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{pmatrix}$ Find the determinant of A.

Solution

$$|A| = \begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix}$$

$$= -8\begin{vmatrix} -6 & 7 \\ -3 & 5 \end{vmatrix} - 0\begin{vmatrix} 4 & 7 \\ -1 & 5 \end{vmatrix} + 6\begin{vmatrix} 4 & -6 \\ -1 & -3 \end{vmatrix}$$

$$= -8(-30 - (-21)) - 0 + 6(-12 - 6)$$

$$= -8(-9) + 6(-18)$$

$$= -36$$

Determinant Using Diagonal Method

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \qquad a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \quad (1)$$

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \qquad a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} \quad (2)$$

Determinant: $\boldsymbol{D = (1) - (2)}$

Example

Evaluate $\begin{vmatrix} 2 & -3 & -2 \\ -1 & -4 & -3 \\ -1 & 0 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & -3 & -2 \\ -1 & -4 & -3 \\ -1 & 0 & 2 \end{vmatrix}\begin{matrix} 2 & -3 \\ -1 & -4 \\ -1 & 0 \end{matrix} = 2(-4)(2) + (-3)(-3)(-1) + (-2)(-1)(0) - (-2)(-4)(-1) - (2)(-3)(0) - (-3)(-1)(2)$$

$$\underline{= -23}$$

Example

Evaluate $\begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix}\begin{matrix} -8 & 0 \\ 4 & -6 \\ -1 & -3 \end{matrix} = (-8)(-6)(-5) + 0(7)(-1) + 6(4)(-3) - 6(-6)(-1) - (-8)(7)(-3) - 0(4)(5)$$

$$\underline{= -36}$$

Example

Evaluate $\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix}\begin{matrix} x & 0 \\ 2 & x \\ -3 & x \end{matrix} = x^2 + 0 - 2x - (3x) - x^4 - 0$$

$$\underline{= -x^4 + x^2 - 5x}$$

Cramer's Rule

Given:

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

If $D \neq 0 \qquad x = \dfrac{D_x}{D} \qquad y = \dfrac{D_y}{D}$

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Example

Use Cramer's rule to solve the system

$$5x + 7y = -1$$
$$6x + 8y = 1$$

Solution

$$D = \begin{vmatrix} 5 & 7 \\ 6 & 8 \end{vmatrix} = -2$$

$$D_x = \begin{vmatrix} -1 & 7 \\ 1 & 8 \end{vmatrix} = -15$$

$$D_y = \begin{vmatrix} 5 & -1 \\ 6 & 1 \end{vmatrix} = 11$$

$$x = \frac{D_x}{D} = \frac{-15}{-2} = \frac{15}{2}$$

$$y = \frac{D_y}{D} = \frac{11}{-2} = -\frac{11}{2}$$

Solution: $\left(\frac{15}{2},\ -\frac{11}{2}\right)$

$$D_x = \begin{array}{ccccc} b_1 & a_{12} & a_{13} & b_1 & a_{12} \\ b_2 & a_{22} & a_{23} & b_2 & a_{22} \\ b_3 & a_{32} & a_{33} & b_3 & a_{32} \end{array}$$

$$D_x = b_1 a_{22} a_{33} + a_{12} a_{23} b_3 + a_{13} b_2 a_{32} - a_{13} a_{22} b_3 - b_1 a_{23} a_{32} - a_{12} b_2 a_{33}$$

$$D_y = \begin{array}{ccccc} a_{11} & b_1 & a_{13} & a_{11} & b_1 \\ a_{21} & b_2 & a_{23} & a_{21} & b_2 \\ a_{31} & b_3 & a_{33} & a_{31} & b_3 \end{array}$$

$$D_z = \begin{array}{ccccc} a_{11} & a_{12} & b_1 & a_{11} & a_{12} \\ a_{21} & a_{22} & b_2 & a_{21} & a_{22} \\ a_{31} & a_{32} & b_3 & a_{31} & a_{32} \end{array}$$

$$x = \frac{D_x}{D} \qquad y = \frac{D_y}{D} \qquad z = \frac{D_z}{D}$$

Example

Use Cramer's rule to solve the system

$x - 3y + 7z = 13$
$x + y + z = 1$
$x - 2y + 3z = 4$

Solution

$$D = \begin{vmatrix} 1 & -3 & 7 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = -10 \qquad D_x = \begin{vmatrix} 13 & -3 & 7 \\ 1 & 1 & 1 \\ 4 & -2 & 3 \end{vmatrix} = 20$$

$$D_y = \begin{vmatrix} 1 & 13 & 7 \\ 1 & 1 & 1 \\ 1 & 4 & 3 \end{vmatrix} = -6 \qquad D_z = \begin{vmatrix} 1 & -3 & 13 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{vmatrix} = -24$$

$$x = \frac{20}{-10} = -2 \qquad y = \frac{-6}{-10} = \frac{3}{5} \qquad z = \frac{-24}{-10} = \frac{12}{5}$$

Solution: $\left(-2, \ \frac{3}{5}, \ \frac{12}{5}\right)$

Exercises *Section* 4.4 – Determinants and Cramer's Rule

(1 – 34) Evaluate

1. $\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix}$

2. $\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix}$

3. $\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix}$

4. $\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix}$

5. $\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix}$

6. $\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix}$

7. $\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}$

8. $\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix}$

9. $\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix}$

10. $\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix}$

11. $\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix}$

12. $\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix}$

13. $\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix}$

14. $\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix}$

15. $\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix}$

16. $\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix}$

17. $\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix}$

18. $\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix}$

19. $\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix}$

20. $\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix}$

21. $\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix}$

22. $\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix}$

23. $\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix}$

24. $\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix}$

25. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix}$

26. $\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix}$

27. $\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix}$

28. $\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix}$

29. $\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix}$

30. $\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix}$

31. $\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix}$

32. $\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix}$

33. $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix}$

34. $\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix}$

(35 – 89) Use Cramer's rule to solve the system

35. $\begin{cases} 3x+2y=-4 \\ 2x-y=-5 \end{cases}$

36. $\begin{cases} 2x+5y=7 \\ 5x-2y=-3 \end{cases}$

37. $\begin{cases} 4x-7y=-16 \\ 2x+5y=9 \end{cases}$

38. $\begin{cases} 3x+2y=4 \\ 2x+y=1 \end{cases}$

39. $\begin{cases} 3x+4y=2 \\ 2x+5y=-1 \end{cases}$

40. $\begin{cases} 5x-2y=4 \\ -10x+4y=7 \end{cases}$

41. $\begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$

42. $\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

43. $\begin{cases} 2x+10y=-14 \\ 7x-2y=-16 \end{cases}$

44. $\begin{cases} 4x-3y=24 \\ -3x+9y=-1 \end{cases}$

45. $\begin{cases} 4x+2y=12 \\ 3x-2y=16 \end{cases}$

46. $\begin{cases} x+2y=-1 \\ 4x-2y=6 \end{cases}$

47. $\begin{cases} x-2y=5 \\ -10x+2y=4 \end{cases}$

48. $\begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$

49. $\begin{cases} 4x-4y=-12 \\ 4x+4y=-20 \end{cases}$

50. $\begin{cases} x+y=7 \\ x-y=3 \end{cases}$

51. $\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

52. $\begin{cases} 12x+3y=15 \\ 2x-3y=13 \end{cases}$

53. $\begin{cases} x-2y=5 \\ 5x-y=-2 \end{cases}$

54. $\begin{cases} 3x+2y=2 \\ 2x+2y=3 \end{cases}$

55. $\begin{cases} 4x-5y=17 \\ 2x+3y=3 \end{cases}$

56. $\begin{cases} x-3y=4 \\ 3x-4y=12 \end{cases}$

57. $\begin{cases} 2x-9y=5 \\ 3x-3y=11 \end{cases}$

58. $\begin{cases} 3x-4y=4 \\ x+y=6 \end{cases}$

59. $\begin{cases} 3x=7y+1 \\ 2x=3y-1 \end{cases}$

60. $\begin{cases} 2x=3y+2 \\ 5x=51-4y \end{cases}$

61. $\begin{cases} y=-4x+2 \\ 2x=3y-1 \end{cases}$

62. $\begin{cases} 3x=2-3y \\ 2y=3-2x \end{cases}$

63. $\begin{cases} x+2y-3=0 \\ 12=8y+4x \end{cases}$

64. $\begin{cases} 7x-2y=3 \\ 3x+y=5 \end{cases}$

65. $\begin{cases} 3x+2y-z=4 \\ 3x-2y+z=5 \\ 4x-5y-z=-1 \end{cases}$

66. $\begin{cases} x+y+z=2 \\ 2x+y-z=5 \\ x-y+z=-2 \end{cases}$

67. $\begin{cases} 2x+y+z=9 \\ -x-y+z=1 \\ 3x-y+z=9 \end{cases}$

68. $\begin{cases} 3y-z=-1 \\ x+5y-z=-4 \\ -3x+6y+2z=11 \end{cases}$

69. $\begin{cases} x+3y+4z=14 \\ 2x-3y+2z=10 \\ 3x-y+z=9 \end{cases}$

70. $\begin{cases} x+4y-z=20 \\ 3x+2y+z=8 \\ 2x-3y+2z=-16 \end{cases}$

71. $\begin{cases} -2x+6y+7z=3 \\ -4x+5y+3z=7 \\ -6x+3y+5z=-4 \end{cases}$

72. $\begin{cases} 2x-y+z=1 \\ 3x-3y+4z=5 \\ 4x-2y+3z=4 \end{cases}$

73. $\begin{cases} 3x-4y+4z=7 \\ x-y-2z=2 \\ 2x-3y+6z=5 \end{cases}$

74. $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

75. $\begin{cases} x+y+z=3 \\ -y+2z=1 \\ -x+z=0 \end{cases}$

76. $\begin{cases} 3x+y+3z=14 \\ 7x+5y+8z=37 \\ x+3y+2z=9 \end{cases}$

77. $\begin{cases} 4x-2y+z=7 \\ x+y+z=-2 \\ 4x+2y+z=3 \end{cases}$

78. $\begin{cases} 2y-z=7 \\ x+2y+z=17 \\ 2x-3y+2z=-1 \end{cases}$

79. $\begin{cases} 2x-2y+z=-4 \\ 6x+4y-3z=-24 \\ x-2y+2z=1 \end{cases}$

80. $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

81. $\begin{cases} 2x-y+2z=-8 \\ x+2y-3z=9 \\ 3x-y-4z=3 \end{cases}$

82. $\begin{cases} x \qquad -3z=-5 \\ 2x-y+2z=16 \\ 7x-3y-5z=19 \end{cases}$

83. $\begin{cases} x+2y-z=5 \\ 2x-y+3z=0 \\ 2y+z=1 \end{cases}$

84. $\begin{cases} x+y+z=6 \\ 3x+4y-7z=1 \\ 2x-y+3z=5 \end{cases}$

85. $\begin{cases} 3x+2y+3z=3 \\ 4x-5y+7z=1 \\ 2x+3y-2z=6 \end{cases}$

86. $\begin{cases} 4x+5y \qquad =2 \\ 11x+y+2z=3 \\ x+5y+2z=1 \end{cases}$

87. $\begin{cases} x-4y+z=6 \\ 4x-y+2z=-1 \\ 2x+2y-3z=-20 \end{cases}$

88. $\begin{cases} 2x-y+z=-1 \\ 3x+4y-z=-1 \\ 4x-y+2z=-1 \end{cases}$

89. $\begin{cases} -x_1-4x_2+2x_3+x_4=-32 \\ 2x_1-x_2+7x_3+9x_4=14 \\ -x_1+x_2+3x_3+x_4=11 \\ -x_1-2x_2+x_3-4x_4=-4 \end{cases}$

(**90 – 101**) Solve for x

90. $\begin{vmatrix} x & 3 \\ 2 & 1 \end{vmatrix}=12$

91. $\begin{vmatrix} x & 1 \\ 2 & x \end{vmatrix}=-1$

92. $\begin{vmatrix} 3 & x \\ x & 4 \end{vmatrix}=-13$

93. $\begin{vmatrix} x & 2 \\ 3 & x \end{vmatrix}=x$

94. $\begin{vmatrix} 4 & 6 \\ -2 & x \end{vmatrix}=32$

95. $\begin{vmatrix} x+2 & -3 \\ x+5 & -4 \end{vmatrix}=3x-5$

96. $\begin{vmatrix} x+3 & -6 \\ x-2 & -4 \end{vmatrix}=28$

97. $\begin{vmatrix} x & -3 \\ -1 & x \end{vmatrix}\ge 0$

98. $\begin{vmatrix} 2 & x & 1 \\ 1 & 2 & -1 \\ 3 & 4 & -2 \end{vmatrix}=-6$

99. $\begin{vmatrix} 1 & x & -3 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix}=8$

100. $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix}=39$

101. $\begin{vmatrix} x & 0 & 0 \\ 7 & x & 1 \\ 7 & 2 & 1 \end{vmatrix}=-1$

102. Find the quadratic function $f(x)=ax^2+bx+c$ for which $f(1)=-10, \quad f(-2)=-31, \quad f(2)=-19$. What is the function?

103. you wish to mix candy worth \$3.44 per pound with candy worth \$9.96 per pound to form 24 pounds of a mixture worth \$8.33 per pound.

a) Write the system equations?
b) How many pounds of each candy should you use?

104. Anne and Nancy use a metal alloy that is 17.76% copper to make jewelry. How many ounces of a 15% alloy must be mixed with a 19% alloy to form 100 ounces of the desired alloy?

105. A company makes 3 types of cable. Cable ***A*** requires 3 black, 3 white, and 2 red wires. ***B*** requires 1 black, 2 white, and 1 red. ***C*** requires 2 black, 1 white, and 2 red. They used 95 black, 100 white and 80 red wires.

a) Write the system equations?

b) How many of each cable were made?

106. A basketball fieldhouse seats 15,000. Courtside seats sell for \$8.00, end zone for \$6.00, and balcony for \$5.00. Total for a sell–out is \$86,000. If half the courtside and balcony and all end zone seats are sold, ticket sales total \$49,000.

a) Write the system equations?

b) How many of each type of seat are there?

107. A movie theater charges \$9.00 for adults and \$7.00 for senior citizens. On a day when 325 people paid admission, the total receipts were \$2,495.

a) Write the system equations?

b) How many who paid were adults? How many were seniors?

108. A Broadway theater has 500 seats, divided into orchestra, main, and balcony seating. Orchestra seats sell for \$150, main seats for \$135, and balcony seats for \$110. If all the seats are sold, the gross revenue to the theater is \$64,250. If all the main and balcony seats are sold, but only half the orchestra seats are sold, the gross revenue is \$56,750.

a) Write the system equations?

b) How many of each kind of seat are there?

109. A movie theater charges \$11 for adults, \$6.50 for children, and \$9 for senior citizens. One day the theater sold 405 tickets and collected \$3,315 in receipts. Twice as many children's tickets were sold as adult tickets.

a) Write the system equations?

b) How many adults, children, and senior citizens went to the theater that day?

110. Emma has \$20,000 to invest. As her financial planner, you recommend that she diversify into three investements: Treasure bills that yield 5% simple interest. Treasury bonds tht yield 7% simple interest, and corporate bonds that yield 10% simple interest. Emma wishes to earn \$1,390 per year in income. Also, Emma wants her investment in Treasury bills to be \$3,000 more than her investment in corporate bonds. How much money should Emma place in each investement?

111. A person invested \$17,000 for one year, part at 10%, part at 12%, and the remainder at 15%. The total annual income from these investements was \$2,110. The amount of money invested at 12% was \$1,000 less than the amounts invested at 10% and 15% combined. Find the amount invested at each rate.

112. At a production, 400 tickets were sold. The ticket prices were \$8, \$10, and \$12, and the total income from ticket sales was \$3,700. How many tickets of each type were sold if the combined number of \$8 and \$10 tickets sold was 7 times the number of \$12 tickets sold?

113. A certain brand of razor blades comes in packages if 6, 12, and 24 blades, costing \$2, \$3, and \$4 per package, respectively. A store sold 12 packages containing a total of 162 razor blades and took in \$35. How many packages of each type were sold?

114. A store sells cashews for \$5.00 per pound and peanuts for \$1.50 per pound. The manager decides to mix 30 pounds of peanuts with some cashews and sell the mixture for \$3.00 per pound.

a) Write the system equations?

b) How many pounds of cashews should be mixed with peanuts so that the mixture will produce the same revenue as selling the nuts separately?

115. A wireless store takes presale orders for a new smartphone and tablet. He gets 340 preorders for the smartphone and 250 preorders for the tablet. The combined value of the preorders is \$270,500.00. If the price of a smartphone and tablet together is \$965, how much does each device cost?

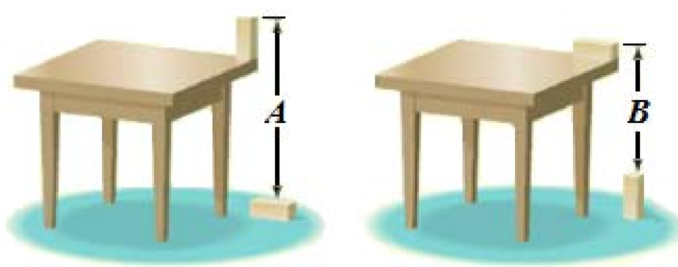
116. A restaurant manager wants to purchase 200 sets of dishes. One design costs \$25 per set, and another costs \$45 per set. If she has only \$7400 to spend, how many sets of each design should be order?

117. One group of people purchased 10 hot dogs and 5 soft drinks at a cost of \$35.00. A second bought 7 hot dogs and 4 soft drinks at a cost of \$25.25. What is the cost of a single hot dog and a single soft drink?

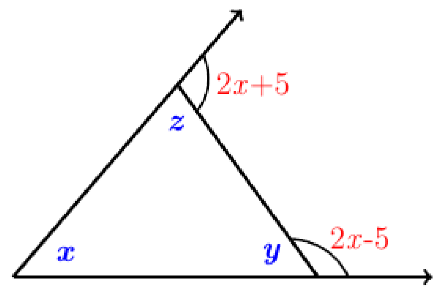
118. The sum of three times the first number, plus the second number, and twice the third number is 5. If 3 times the second number is subtracted from the sum of the first number and 3 times the third number, the result is 2. If the third number is subtracted from the sum of 2 times the first number and 3 times the second number, the result is 1. Find the three numbers.

119. The sum of three numbers is 16. The sum of twice the first number, 3 times the second number, and 4 times the third number is 46. The difference between 5 times the first number and the second number is 31. Find the three numbers.

120. Two blocks of wood having the same length and width are placed on the top and bottom of a table. Length *A* measure 32 *cm*. The blocks are rearranged. Length *B* measures 28 *cm*. Determine the height of the table.

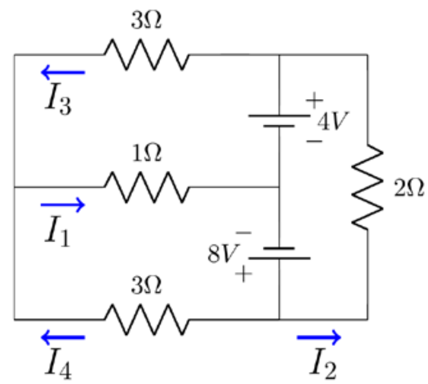


121. In the following triangle, the degree measures of the three interior angles and two of the exterior angles are represented with variables. Find the measure of each interior angle.



122. Three painters (Beth, Bill, and Edie), working together, can paint the exterior of a home in 10 *hours*. Bill and Edie together have painted similar house in 15 *hours*. One day, all three worked on this same kind of house for 4 *hours*, after which Edie left. Beth and Bill required 8 more *hours* to finish. Assuming no gain or loss in efficiency, how long should it take each person to complete such a job alone?

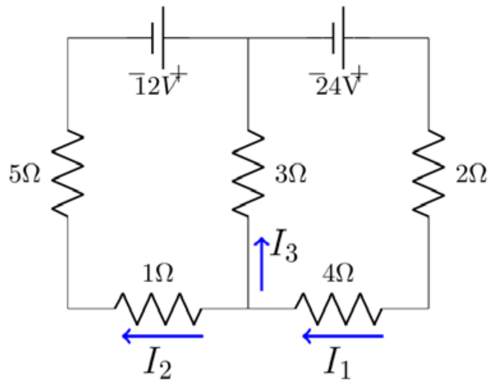
123. An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:



$$\begin{cases} I_1 = I_3 + I_4 \\ I_1 + 5I_4 = 8 \\ I_1 + 3I_3 = 4 \\ 8 - 4 - 2I_2 = 0 \end{cases}$$

Find the currents I_1, I_2, I_3, and I_4

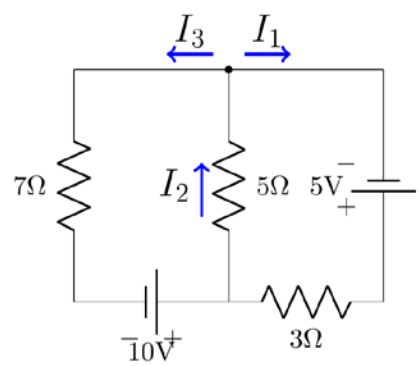
124. An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:



$$\begin{cases} I_1 = I_2 + I_3 \\ 24 - 6I_1 - 3I_3 = 0 \\ 12 + 24 - 6I_1 - 6I_2 = 0 \end{cases}$$

Find the currents I_1, I_2, and I_3

125. An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:



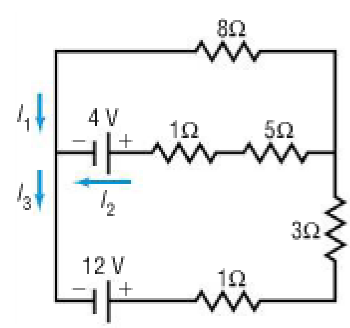
$$\begin{cases} I_2 = I_1 + I_3 \\ 5 - 3I_1 - 5I_2 = 0 \\ 10 - 5I_2 - 7I_3 = 0 \end{cases}$$

Find the currents I_1, I_2, and I_3

126. An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:

$$\begin{cases} I_3 = I_1 + I_2 \\ 6I_2 + 4I_3 = 8 \\ 8I_1 = 4 + 6I_2 \end{cases}$$

Find the currents I_1, I_2, and I_3



Section 4.5 – Partial Fraction Decomposition

1- Decompose $\frac{P}{Q}$, where Q has Only Non-repeated Linear Factor

Under the assumption that Q has only non-repeated linear factors, the polynomial Q has the form

$$Q(x)=\left(x-a_1\right)\left(x-a_2\right)\ \cdots\cdots\ \left(x-a_n\right)$$

Where no 2 of the number $a_1, a_2, \ldots, a_n$ are equal. In this case, the partial fraction decomposition of $\frac{P}{Q}$ is of the form

$$\frac{P}{Q}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}\ \cdots\cdots+\frac{A_n}{x-a_n}$$

Where the numbers $A_1, A_2, \ldots, A_n$ are to be determined.

Example

Write the partial fraction decomposition of $\dfrac{x}{x^2-5x+6}$

Solution

First factor the denominator, $x^2-5x+6=(x-2)(x-3)$

$$\frac{x}{x^2-5x+6}=\frac{A}{x-2}+\frac{B}{x-3}$$

$$\frac{x}{x^2-5x+6}=\frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

$$x=Ax-3A+Bx-2B$$

$$x=(A+B)x-3A-2B \qquad 1x+0=(A+B)x-3A-2B$$

$$x \qquad A+B=1$$

$$x^0 \qquad -3A-2B=0$$

$$A=\frac{\begin{vmatrix}1 & 1\\ 0 & -2\end{vmatrix}}{\begin{vmatrix}1 & 1\\ -3 & -2\end{vmatrix}}=\frac{-2}{1}\ \underline{=-2}\,|$$

$$B=1-(-2)\ \underline{=3}\,|$$

Therefore; $\underline{\dfrac{x}{x^2-5x+6}=\dfrac{-2}{x-2}+\dfrac{3}{x-3}}\,\Big|$

2- Decompose $\frac{P}{Q}$, where Q has Repeated Linear Factors

If a polynomial Q has a repeated linear factor, say $(x-a)^n$, $n \ge 2$ n is an integer, then in the partial fraction decomposition of $\frac{P}{Q}$, we allow for the terms

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} \cdots + \frac{A_n}{(x-a)^n}$$

Where the numbers $A_1, A_2, \ldots, A_n$ are to be determined.

Example

Write the partial fraction decomposition of $\frac{x+2}{x^3-2x^2+x}$

Solution

First factor the denominator, $x^3 - 2x^2 + x = x(x-1)^2$

$$\frac{x+2}{x^3-2x^2+x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$x+2 = A(x-1)^2 + Bx(x-1) + Cx$$

$$= Ax^2 - 2Ax + A + Bx^2 - Bx + Cx$$

$$\begin{array}{lll} x^2 & A+B=0 & \rightarrow B=-A \underline{=-2} \\ x & -2A-B+C=1 & \rightarrow \quad C=1+4-2 \underline{=3} \\ x^0 & \underline{A=2} & \end{array}$$

$$\frac{x+2}{x^3-2x^2+x} = \frac{2}{x} + \frac{-2}{x-1} + \frac{3}{(x-1)^2}$$

$$\frac{x+2}{x^3-2x^2+x} = \frac{2}{x} - \frac{2}{x-1} + \frac{3}{(x-1)^2}$$

Example

Write the partial fraction decomposition of $\dfrac{x^3-8}{x^2(x-1)^3}$

Solution

$$\frac{x^3-8}{x^2(x-1)^3}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{(x-1)^2}+\frac{E}{(x-1)^3}$$

$$x^3-8=Ax(x-1)^3+B(x-1)^3+Cx^2(x-1)^2+Dx^2(x-1)+Ex^2$$

Let $x=0 \ \rightarrow \ -8=B(-1)^3 \ \Rightarrow B=8$

$$x^3-8=Ax(x-1)^3+8(x-1)^3+Cx^2(x-1)^2+Dx^2(x-1)+Ex^2$$

Let $x=1 \ \rightarrow \ 1-8=E \ \Rightarrow E=-7$

$$x^3-8=Ax\left(x^3-3x^2+3x-1\right)+8\left(x^3-3x^2+3x-1\right)+Cx^2\left(x^2-2x+1\right)+Dx^2(x-1)-7x^2$$

$$x^3-8-8\left(x^3-3x^2+3x-1\right)+7x^2$$
$$=Ax^4-3Ax^3+3Ax^2-Ax+Cx^4-2Cx^3+Cx^2+Dx^3-Dx^2$$

$$x^3-8-8x^3+24x^2-24x+8+7x^2$$
$$=(A+C)x^4+(-3A-2C+D)x^3+(3A+C-D)x^2-Ax$$

$$-7x^3+31x^2-24x=(A+C)x^4+(-3A-2C+D)x^3+(3A+C-D)x^2-Ax$$

$$\rightarrow\begin{cases} A+C=0 & C=-A=-24 \\ -3A-2C+D=-7 & \\ 3A+C-D=31 & \\ -A=-24 & \rightarrow A=24 \end{cases} \qquad D=-7+3A+2C=-7+72-48=17$$

$$\frac{x^3-8}{x^2(x-1)^3}=\frac{24}{x}+\frac{8}{x^2}-\frac{24}{x-1}+\frac{17}{(x-1)^2}-\frac{7}{(x-1)^3}$$

3- Decompose $\frac{P}{Q}$, where Q has a Non-repeated Irreducible Quadratic Factor

If Q contains a no-repeated irreducible quadratic factor of the form ax^2+bx+c, then in the partial fraction decomposition of $\frac{P}{Q}$, we allow for the term

$$\frac{Ax+B}{ax^2+bx+c}$$

Where the numbers A and B are to be determined.

Example

Write the partial fraction decomposition of $\dfrac{3x-5}{x^3-1}$

Solution

$$\frac{3x-5}{x^3-1}=\frac{3x-5}{(x-1)\left(x^2+x+1\right)}$$

$$=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$$

$$3x-5=Ax^2+Ax+A+Bx^2+Cx-Bx-C$$

$$x^2 \qquad A+B=0 \qquad \to B=-A$$

$$x \qquad A-B+C=3 \qquad (1)$$

$$x^0 \qquad A-C=-5 \qquad \to C=A+5$$

$$(1)\to \quad A+A+A+5=3$$

$$3A=-2$$

$$A=-\frac{2}{3} \quad B=\frac{2}{3} \quad C=\frac{13}{3}$$

$$\frac{3x-5}{x^3-1}=\frac{-\frac{2}{3}}{x-1}+\frac{\frac{2}{3}x+\frac{13}{3}}{x^2+x+1}$$

$$=-\frac{2}{3}\frac{1}{x-1}+\frac{1}{3}\frac{2x+13}{x^2+x+1}$$

4- Decompose $\frac{P}{Q}$, where Q has a Repeated Irreducible Quadratic Factor

If Q contains a repeated irreducible quadratic factor of the form $\left(ax^2+bx+c\right)^n$, $n \geq 2$, n an integer,

then in the partial fraction decomposition of $\frac{P}{Q}$, we allow for the terms

$$\frac{A_1 x+B_1}{ax^2+bx+c}+\frac{A_2 x+B_2}{\left(ax^2+bx+c\right)^2}+\cdots+\frac{A_n x+B_n}{\left(ax^2+bx+c\right)^n}$$

Where the numbers $A_1, B_1, A_2, B_2, \ldots, A_n, B_n$ are to be determined.

Example

Write the partial fraction decomposition of $\dfrac{x^3+x^2}{\left(x^2+4\right)^2}$

Solution

$$\frac{x^3+x^2}{\left(x^2+4\right)^2}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{\left(x^2+4\right)^2}$$

$$x^3+x^2=(Ax+B)\left(x^2+4\right)+Cx+D$$

$$=Ax^3+4Ax+Bx^2+4B+Cx+D$$

x^3 $\quad \underline{A=1}$

x^2 $\quad \underline{B=1}$

x^1 $\quad 4A+C=0 \rightarrow C=-4A \underline{=-4}$

x^0 $\quad 4B+D=0 \rightarrow D=-4B \underline{=-4}$

$$\boxed{\frac{x^3+x^2}{\left(x^2+4\right)^2}=\frac{x+1}{x^2+4}+\frac{-4x-4}{\left(x^2+4\right)^2}}$$

Exercises *Section* 4.5 – Partial Fraction Decomposition

Write the partial fraction decomposition of each rational expression

1. $\dfrac{4}{x(x-1)}$

2. $\dfrac{3x}{(x+2)(x-1)}$

3. $\dfrac{1}{x\left(x^2+1\right)}$

4. $\dfrac{1}{(x+1)\left(x^2+4\right)}$

5. $\dfrac{x^2}{(x-1)^2(x+1)^2}$

6. $\dfrac{x+1}{x^2(x-2)^2}$

7. $\dfrac{x-3}{(x+2)(x+1)^2}$

8. $\dfrac{x^2+x}{(x+2)(x-1)^2}$

9. $\dfrac{10x^2+2x}{(x-1)^2\left(x^2+2\right)}$

10. $\dfrac{x^2+2x+3}{(x+1)\left(x^2+2x+4\right)}$

11. $\dfrac{x^2-11x-18}{x\left(x^2+3x+3\right)}$

12. $\dfrac{1}{(2x+3)(4x-1)}$

13. $\dfrac{x^2+2x+3}{\left(x^2+4\right)^2}$

14. $\dfrac{x^3+1}{\left(x^2+16\right)^2}$

15. $\dfrac{7x+3}{x^3-2x^2-3x}$

16. $\dfrac{x^2}{x^3-4x^2+5x-2}$

17. $\dfrac{x^3}{\left(x^2+16\right)^3}$

18. $\dfrac{4}{2x^2-5x-3}$

19. $\dfrac{2x+3}{x^4-9x^2}$

20. $\dfrac{x^2+9}{x^4-2x^2-8}$

21. $\dfrac{y}{y^2-2y-3}$

22. $\dfrac{x+3}{2x^3-8x}$

23. $\dfrac{x^2}{(x-1)\left(x^2+2x+1\right)}$

24. $\dfrac{3x^2+x+4}{x^3+x}$

25. $\dfrac{8x^2+8x+2}{\left(4x^2+1\right)^2}$

26. $\dfrac{1}{x^2+2x}$

27. $\dfrac{2x+1}{x^2-7x+12}$

28. $\dfrac{x^2+x}{x^4-3x^2-4}$

29. $\dfrac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{\left(\theta^2+1\right)^3}$

30. $\dfrac{3x^2+7x-2}{x^3-x^2-2x}$

31. $\dfrac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)}$

32. $\dfrac{5x^2-3x+2}{x^3-2x^2}$

33. $\dfrac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}$

34. $\dfrac{1}{x^2-5x+6}$

35. $\dfrac{1}{x^2-5x+5}$

36. $\dfrac{5x^2+20x+6}{x^3+2x^2+x}$

37. $\dfrac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}$

38. $\dfrac{8x^3+13x}{\left(x^2+2\right)^2}$

39. $\dfrac{1}{x^2-9}$

40. $\dfrac{2}{9x^2-1}$

41. $\dfrac{5}{x^2+3x-4}$

42. $\dfrac{3-x}{3x^2-2x-1}$

43. $\dfrac{x^2+12x+12}{x^3-4x}$

44. $\dfrac{5x-2}{(x-2)^2}$

Section 4.6 – Infinite Sequences and Summation Notation

An arbitrary ***infinite sequence*** may be denoted as follows:

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$$

An infinite sequence is a function whose domain is the set of positive integers.

Example

Find the first four terms and the tenth term of the sequence: $\left\{\dfrac{n}{n+1}\right\}$

Solution

$$n=1 \quad \to \quad \frac{1}{1+1}=\frac{1}{2}$$

$$n=2 \quad \to \quad \frac{2}{2+1}=\frac{2}{3}$$

$$n=3 \quad \to \quad \frac{3}{3+1}=\frac{3}{4}$$

$$n=4 \quad \to \quad \frac{4}{4+1}=\frac{4}{5}$$

$$n=10 \quad \Rightarrow \quad \frac{10}{11}$$

Example

Find the first four terms and the tenth term of the sequence: $\left\{2+(0.1)^n\right\}$

Solution

$$n=1 \quad \to \quad 2+0.1=2.1$$

$$n=2 \quad \to \quad 2+0.1^2=2.01$$

$$n=3 \quad \to \quad 2+0.1^3=2.001$$

$$n=4 \quad \to \quad 2+0.1^4=2.0001$$

$$n=10 \quad \Rightarrow \quad 2.0000000001$$

Example

Find the first four terms and the tenth term of the sequence: $\left\{(-1)^{n+1}\dfrac{n^2}{3n-1}\right\}$

Solution

$$n=1 \quad \to \quad (-1)^2\frac{1^2}{3(1)-1}=\frac{1}{2}$$

$$n=2 \quad \to \quad (-1)^3\frac{2^2}{3(2)-1}=-\frac{4}{5}$$

$$n=3 \quad \to \quad (-1)^4\frac{3^2}{3(3)-1}=\frac{9}{8}$$

$$n=4 \quad \to \quad (-1)^5\frac{4^2}{3(4)-1}=-\frac{16}{11}$$

$$n=10 \quad \Rightarrow \quad -\frac{100}{29}$$

Example

Find the first four terms and the tenth term of the sequence: $\{4\}$

Solution

$n=1 \quad \to \quad 4$

$n=2 \quad \to \quad 4$

$n=3 \quad \to \quad 4$

$n=4 \quad \to \quad 4$

$n=10 \quad \Rightarrow \quad 4$

Example

Find the first four terms of the recursively defined infinite sequence $a_1=3, \quad a_{n+1}=(n+1)a_n$

Solution

$$a_1=3$$

$$n=1 \quad \to \quad a_2=(1+1)a_1=2(3)=6$$

$$n=2 \quad \to \quad a_3=(2+1)a_2=3(6)=18$$

$$n=3 \quad \to \quad a_4=(3+1)a_3=4(18)=72$$

Summation Notation

To find the sum of many terms of an infinite sequence, it is easy to express using summation notation.

$$\overbrace{\sum_{\underbrace{n=1}_{\text{First value of } n}}^{5}}^{\text{Last value of } n} 2n+3 \leftarrow \text{Formula for each term}$$

Example

Find the sum: $\sum_{k=1}^{4} k^2(k-3)$

Solution

$$\sum_{k=1}^{4} k^2(k-3) = 1^2(1-3)+2^2(2-3)+3^2(3-3)+4^2(4-3)$$

$$= -2-4+0+16$$

$$= 10$$

Theorem on the Sum of a *Constant*

$$(1)\ \sum_{k=1}^{n} c = nc \qquad (2)\ \sum_{k=m}^{n} c = (n-m+1)c$$

Proof:

$$\sum_{k=1}^{n} c = \underbrace{c+c+\ldots+c}_{n} = nc$$

Example

Find the sum: $\sum_{k=10}^{20} 5$

Solution

$$\sum_{k=10}^{20} 5 = (20-10+1)5$$

$$= 55$$

Theorem on Sums

If $a_1, a_2, a_3, \ldots, a_n, \ldots$ and $b_1, b_2, b_3, \ldots, b_n, \ldots$ are infinite sequences, then for every positive integer ***n***,

(1) $\displaystyle\sum_{k=1}^{n}\left(a_k + b_k\right) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(2) $\displaystyle\sum_{k=1}^{n}\left(a_k - b_k\right) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

(3) $\displaystyle\sum_{k=1}^{n} ca_k = c\left(\sum_{k=1}^{n} a_k\right)$

Proof

$$\begin{aligned}\sum_{k=1}^{n}\left(a_k + b_k\right) &= \left(a_1 + b_1\right) + \left(a_2 + b_2\right) + \ldots + \left(a_n + b_n\right) \\ &= \left(a_1 + a_2 + \ldots + a_n\right) + \left(b_1 + b_2 + \ldots + b_n\right) \\ &= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k\end{aligned}$$

Example

Express the sum using summation notation $\quad 2^1 + 2^2 + 2^3 + \cdots + 2^{16}$

Solution

$$2^1 + 2^2 + 2^3 + \cdots + 2^{16} = \sum_{k=1}^{16} 2^k$$

Exercises *Section* 4.6 – Infinite Sequences and Summation Notation

(1 – 13) Find the first four terms and the eight term of the sequence:

1. $\{12-3n\}$

2. $\left\{\frac{3n-2}{n^2+1}\right\}$

3. $\{9\}$

4. $\left\{(-1)^{n-1}\frac{n+7}{2n}\right\}$

5. $\left\{\frac{2^n}{n^2+2}\right\}$

6. $\left\{(-1)^{n-1}\frac{n}{2n-1}\right\}$

7. $\left\{\frac{2^n}{3^n+1}\right\}$

8. $\left\{\frac{n^2}{2^n}\right\}$

9. $\left\{\frac{n}{e^n}\right\}$

10. $\{c_n\}=\left\{(-1)^{n+1}n^2\right\}$

11. $\{c_n\}=\left\{\frac{(-1)^n}{(n+1)(n+2)}\right\}$

12. $\{c_n\}=\left\{\left(\frac{4}{3}\right)^n\right\}$

13. $\{b_n\}=\left\{\frac{3^n}{n}\right\}$

14. Graph the sequence $\left\{\frac{1}{\sqrt{n}}\right\}$

15. Find the first four terms of the sequence of partial sums for the given sequence. $\left\{3+\frac{1}{2}n\right\}$

(16 – 27) Find the first five terms of the recursively defined infinite sequence

16. $a_1=2, \quad a_{k+1}=3a_k-5$

17. $a_1=-3, \quad a_{k+1}=a_k^2$

18. $a_1=5, \quad a_{k+1}=ka_k$

19. $a_1=2, \quad a_n=3+a_{n-1}$

20. $a_1=5, \quad a_n=2a_{n-1}$

21. $a_1=\sqrt{2}, \quad a_n=\sqrt{2+a_{n-1}}$

22. $a_1=2, \quad a_{n+1}=7-2a_n$

23. $a_1=128, \quad a_{n+1}=\frac{1}{4}a_n$

24. $a_1=2, \quad a_{n+1}=\left(a_n\right)^n$

25. $a_1=A, \quad a_n=a_{n-1}+d$

26. $a_1=A, \quad a_n=ra_{n-1}, \quad r\neq 0$

27. $a_1=2, \quad a_2=2; \quad a_n=a_{n-1}\bullet a_{n-2}$

(28 – 37) Express each sum using summation notation

28. $1+2+3+\ldots+20$

29. $1+2+3+\ldots+40$

30. $1^3+2^3+3^3+\ldots+8^3$

31. $1^2+2^2+3^2+\ldots+15^2$

32. $2^2+2^3+2^4+\ldots+2^{11}$

33. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{13}{14}$

34. $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\cdots+(-1)^6\frac{1}{3^6}$

35. $\frac{2}{3}-\frac{4}{9}+\frac{8}{27}-\cdots+(-1)^{12}\left(\frac{2}{3}\right)^{11}$

36. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{14}{14+1}$

37. $\frac{1}{e}+\frac{2}{e^2}+\frac{3}{e^3}+\cdots+\frac{n}{e^n}$

(**38 – 52**) Find the sum

38. $\sum_{k=1}^{5}(2k-7)$

39. $\sum_{k=0}^{5}k(k-2)$

40. $\sum_{k=1}^{5}(-3)^{k-1}$

41. $\sum_{k=253}^{571}\left(\frac{1}{3}\right)$

42. $\sum_{k=1}^{50}8$

43. $\sum_{k=1}^{40}k$

44. $\sum_{k=1}^{5}(3k)$

45. $\sum_{k=1}^{10}\left(k^3+1\right)$

46. $\sum_{k=1}^{24}\left(k^2-7k+2\right)$

47. $\sum_{k=6}^{20}\left(4k^2\right)$

48. $\sum_{k=1}^{16}\left(k^2-4\right)$

49. $\sum_{k=1}^{6}(10-3k)$

50. $\sum_{k=1}^{10}\left[1+(-1)^k\right]$

51. $\sum_{k=1}^{6}\frac{3}{k+1}$

52. $\sum_{k=137}^{428}2.1$

(**53 – 56**) Write out each sum

53. $\sum_{k=1}^{n}(k+2)$

54. $\sum_{k=1}^{n}k^2$

55. $\sum_{k=2}^{n}(-1)^k\ln k$

56. $\sum_{k=3}^{n}(-1)^{k+1}2^k$

57. $\sum_{k=0}^{n}\frac{1}{3^k}$

58. Fred has a balance of \$3,000 on his card which charges 1% interest per month on any unpaid balance. Fred can afford to pay \$100 toward the balance each month. His balance each month after making a \$100 payment is given by the recursively defined sequence

$$B_0=\$3{,}000 \quad B_n=1.01B_{n-1}-100$$

Determine Fred’s balance after making the first payment. That is, determine B_1

59. A pond currently has 2,000 trout in it. A fish hatchery decides to add an additional 20 trout each month. Is it also known that the trout population is grwoing at a rate of 3% per month. The size of the population after *n* months is given bu the recursively defined sequence

$$P_0=2{,}000 \quad P_n=1.03P_{n-1}+20$$

How many trout are in the pond after 2 months? That is, what is P_2?

60. Fred bought a car by taking out a loan for \$18,500 at 0.5% interest per month. Fred's normal monthly payment is \$434.47 per month, but he decides that he can afford to pay \$100 extra toward the balance each month. His balance each month is given by the recursively defined sequence

$$B_0 = \$18,500 \quad B_n = 1.005B_{n-1} - 534.47$$

Determine Fred's balance after making the first payment. That is, determine B_1

61. The Environmental Protection Agency (EPA) determines that Maple Lake has 250 *tons* of pollutant as a result of industrial waste and that 10% of the pollutant present is neuttralized by solar oxidation every year. The EPA imposes new pollution control laws that result in 15 *tons* of new pollutant entering the lake each year. The amount of pollutant in the lake after n years is given by the recursively defined sequence

$$P_0 = 250 \quad P_n = 0.9P_{n-1} + 15$$

Determine the amount of pollutant in the lake after 2 years? That is, what is P_2 ?

62. Let $$u_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n \sqrt{5}}$$

Define the nth term of a sequence

a) Show that $u_1 = 1$ and $u_2 = 1$

b) Show that $u_{n+2} = u_{n+1} + u_n$

c) Draw the conclusion that $\{u_n\}$ is a Fibonacci sequence

d) Find the first ten terms of the sequence from part (c)

Section 4.7 – Arithmetic and Geometric Sequences

Arithmetic Sequence

Definition of Arithmetic Sequence

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an arithmetic sequence if there is a real number $\boldsymbol{d}$ such that for every positive integer $\boldsymbol{k}$,

$$a_{k+1} = a_k + d$$

The number $d = a_{k+1} - a_k$ is called the ***common difference*** of the sequence.

Example

Show that the sequence: 1, 4, 7, 10, … , $3n - 2$, … is arithmetic, and find the common difference.

Solution

If $a_n = 3n - 2$, then for every positive integer k,

$$\begin{aligned} a_{k+1} - a_k &= [3(k+1) - 2] - (3k - 2) \\ &= 3k + 3 - 2 - 3k + 2 \\ &= 3 \end{aligned}$$

Hence, the given sequence is arithmetic with common difference 3.

The nth Term of an Arithmetic Sequence: $\boxed{a_n = a_1 + (n-1)d}$

Example

The first three terms of an arithmetic sequence are 20, 16.5, and 13. Find the fifteenth term.

Solution

The common difference is: $a_2 - a_1 = 16.5 - 20 = -3.5$

Substituting $a_1 = 20, \quad d = -3.5, \quad n = 15$ in the formula:

$$\begin{aligned} a_{15} &= a_1 + (15-1)d \\ &= 20 + (15-1)(-3.5) \\ &= -29 \end{aligned}$$

Example

The fourth term of an arithmetic sequence is 5, and the ninth term is 20. Find the sixth term.

Solution

Given: $a_4 = 5 \quad a_9 = 20$

$$\begin{cases} a_4 = a_1 + (4-1)d \\ a_9 = a_1 + (9-1)d \end{cases} \Rightarrow \begin{cases} 5 = a_1 + 3d \\ 20 = a_1 + 8d \end{cases}$$

$$\begin{matrix} a_{x1} = y1 \\ a_{x2} = y2 \end{matrix} \to d = \frac{y2 - y1}{x2 - x1}$$

$$\begin{array}{r} 20 = a_1 + 8d \\ -\ \ 5 = a_1 + 3d \\ \hline 15 = 5d \end{array}$$

$\underline{d = 3}$

$$\begin{aligned} a_1 &= 5 - 3d \\ &= 5 - 9 \\ &= -4 \end{aligned}$$

$$\begin{aligned} a_6 &= a_1 + (6-1)d \\ &= -4 + (5)3 \\ &= 11 \end{aligned}$$

Theorem

Formulas for S_n

If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an arithmetic sequence with common difference ***d***, then the *n*th partial sum S_n (that is, the sum of the first ***n*** terms) is given by either

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right] \quad or \quad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

Proof

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \ldots + a_n \\ &= a_1 + \left(a_1 + d\right) + \left(a_1 + 2d\right) + \ldots + \left(a_1 + (n-1)d\right) \\ &= a_1 + a_1 + \ldots + a_1 + \left[d + 2d + \ldots + (n-1)d\right] \\ &= na_1 + d\left[1 + 2 + \ldots + (n-1)\right] \\ &= \frac{2na_1 + (n-1)nd}{2} \\ &= \frac{n}{2}\left[2a_1 + (n-1)d\right] \end{aligned}$$

Using the formula of sum: $S_n = \frac{n(n+1)}{2}$

Example

Find the sum of all even integers from 2 through 100.

Solution

The arithmetic sequence: 2, 4, 6, …, 2n, …

Substituting $n=50, \quad a_1=2, \quad and \quad a_{50}=100$ in the formula:

$$S_n=\frac{50}{2}(2+100)$$

$$\underline{=2550}$$

Example

Express in terms of summation notation: $\frac{1}{4}+\frac{2}{9}+\frac{3}{14}+\frac{4}{19}+\frac{5}{24}+\frac{6}{29}$

Solution

$$\begin{array}{lll} \textit{Numerators}: & 1,2,3,4,5 & \textit{common difference } 1 \\ \textit{Denominators}: & 4,9,14,19,24,29 & \textit{common difference } 5 \end{array}$$

Using the formula for *n*th term:

$$a_n=a_1+(n-1)d=1+(n-1)1=n$$

$$a_n=a_1+(n-1)d=4+(n-1)5=4+5n-5=5n-1$$

Hence the *n*th term is:

$$\frac{1}{4}+\frac{2}{9}+\frac{3}{14}+\frac{4}{19}+\frac{5}{24}+\frac{6}{29}=\sum_{n=1}^{6}\frac{n}{5n-1}$$

Geometric Sequence

Definition of *Geometric* Sequence

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a geometric sequence if $a_1 \neq 0$ and if there is a real number $r \neq 0$ such that for every positive integer k.

$$a_{k+1} = a_k r$$

The number $r = \dfrac{a_{k+1}}{a_k}$ is called the ***common ratio*** of the sequence.

The formula for the n^th Term of a Geometric Sequence: $\boxed{a_n = a_1 r^{n-1}}$

The common ratio for: $6, \ -12, \ 24, \ -48, \ \ldots, \ (-2)^{n-1}(6), \ \ldots$ is $= \frac{-12}{6} = -2$

Example

A geometric sequence has first term 3 and common ratio $-\frac{1}{2}$. Find the first five terms and the tenth term.

Solution

$a_1 = 3$

$a_n = a_1 r^{n-1}$

$r = -\frac{1}{2}$ $\qquad a_2 = 3\left(-\frac{1}{2}\right) = -\frac{3}{2}$

$r^2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$ $\qquad a_3 = 3\left(\frac{1}{4}\right) = \frac{3}{4}$

$r^3 = \left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$ $\qquad a_4 = 3\left(-\frac{1}{8}\right) = -\frac{3}{8}$

$r^4 = \left(-\frac{1}{2}\right)^4 = \frac{1}{16}$ $\qquad a_5 = 3\left(\frac{1}{16}\right) = \frac{3}{16}$

$r^9 = \left(-\frac{1}{2}\right)^9 = -\frac{1}{512}$ $\qquad a_{10} = 3\left(-\frac{1}{512}\right) = -\frac{3}{512}$

Example

The third term of a geometric is 5, and the sixth term is –40. Find the eighth term.

Solution

Given: $a_3 = 5 \quad a_6 = -40$

$$a_n = a_1 r^{n-1}$$

$$\begin{cases} a_3 = a_1 r^{3-1} \\ a_6 = a_1 r^{6-1} \end{cases} \to \begin{cases} 5 = a_1 r^2 \\ -40 = a_1 r^5 \end{cases}$$

$$\frac{a_1 r^5}{a_1 r^2} = \frac{-40}{5}$$

$$\begin{aligned} a_{x1} &= y1 \\ a_{x2} &= y2 \end{aligned} \to \boldsymbol{r} = \left(\frac{y2}{y1}\right)^{\frac{1}{x2-x1}}$$

$$r^3 = -8$$

$$\underline{r = -2}$$

$$a_1 = \frac{5}{r^2}$$

$$= \frac{5}{(-2)^2}$$

$$= \frac{5}{4}$$

$$a_8 = \frac{5}{4}(-2)^7$$

$$\underline{= -160}$$

Theorem: Formula for S_n

The nth partial sum S_n of a geometric sequence with first term a_1 and common ratio $r \neq 1$ is

$$\boxed{S_n = a_1 \frac{1-r^n}{1-r}}$$

Proof

By definition, the nth partial sum S_n of a geometric sequence is:

$$\begin{aligned} S_n &= a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-2} + a_1 r^{n-1} \\ - \; rS_n &= a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-1} + a_1 r^n \\ \hline S_n - rS_n &= a_1 - a_1 r^n \end{aligned}$$

$$(1-r)S_n = a_1\left(1-r^n\right)$$

$$S_n = a_1 \frac{1-r^n}{1-r}$$

Example

If the sequence $1,\ 0.3,\ 0.09,\ .0027,\ \ldots$ is a geometric sequence, find the sum of the first five terms.

Solution

Given: $a_1 = 1$

$$r = \frac{0.3}{1} = 0.3, \quad n = 5$$

$$\begin{aligned} S_5 &= a_1 \frac{1-r^5}{1-r} \\ &= 1\frac{1-(0.3)^5}{1-0.3} \\ &= 1.4251 \end{aligned}$$

Theorem on the Sum of an Infinite Geometric Series

If $|r|<1$, then the infinite geometric series $a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1} + \ldots$ has the sum

$$S = \frac{a_1}{1-r}$$

Example

Find the sum S of the alternating infinite geometric series: to $\displaystyle\sum_{n=1}^{\infty} 3\left(-\frac{2}{3}\right)^{n-1}$

Solution

$$\sum_{n=1}^{\infty} 3\left(-\frac{2}{3}\right)^{n-1} = 3 - 2 + \frac{4}{3} - \frac{8}{9} + \ldots + 3\left(-\frac{2}{3}\right)^{n-1} + \ldots$$

$$\begin{aligned} S &= \frac{a_1}{1-r} \\ &= \frac{3}{1-\left(-\frac{2}{3}\right)} \\ &= \frac{3}{\frac{5}{3}} \\ &= \boxed{\frac{9}{5}} \end{aligned}$$

Example

Find a rational number that corresponds to $5.4\overline{27}$

Solution

$$\begin{aligned} 5.4\overline{27} &= 5.427272727\ldots \\ &= 5.4 + 0.027 + 0.00027 + .0000027 + \ldots \end{aligned}$$

$$a_1 = 0.027, \quad r = \frac{.00027}{.027} = 0.01$$

$$\begin{aligned} S &= 5.4 + \frac{a_1}{1-r} \\ &= \frac{54}{10} + \frac{.027}{1-.01} \\ &= \frac{54}{10} + \frac{.027}{.990} \\ &= \frac{54}{10} + \frac{27}{990} \end{aligned}$$

$$= \frac{54}{10} + \frac{3}{110}$$

$$= \boxed{\frac{597}{110}}$$

Example

Initially, a pendulum swings through an arc of 18 *inches*. On each successive swing, the length of the arc is 0.98 of the previous length.

a) What is the length of the arc of the 10th swing?

b) On which swing is the length of the arc first less than 12 inches?

c) After 15 swings, what total distance will the pendulum have swung?

d) When it stops, what total distance will the pendulum have swung?

Solution

a) The length of the first swing: $a_1 = 18$

The length of the second swing: $a_2 = 0.98a_1 = 0.98(18)$

$a_3 = 0.98a_2 = 0.98^2(18)$

The length of the arc of the 10th swing is:

$a_{10} = 0.98^9(18)$

$\approx 15.007 \ in$

b) $a_n = 18(0.98)^{n-1}$

$18(0.98)^{n-1} = 12 \quad \rightarrow \quad (0.98)^{n-1} = \frac{12}{18} = \frac{2}{3}$

$n - 1 = \log_{0.98}\left(\frac{2}{3}\right)$

$n = \log_{0.98}\left(\frac{2}{3}\right) + 1$

≈ 21.07

The length of the arc of the pendulum exceeds 12 inches on the 21st swing and the first less than 12 inches on the 22nd swing.

c) $L = 18 \cdot \frac{1 - 0.98^{15}}{1 - 0.98}$ $\qquad S_n = a_1 \frac{1 - r^n}{1 - r}$

$\approx 235.3 \ in$

d) $T = \frac{18}{1 - 0.98}$ $\qquad S_n = \frac{a_1}{1 - r}$

$= 900$

The pendulum will have swung a total of 900 *inches* when it finally stops.

Exercises *Section* 4.7 – Arithmetic and Geometric Sequences

1. Show that the sequence $-6,\ -2,\ 2,\ \ldots,\ 4n-10,\ \ldots$ is arithmetic, and find the common difference.

(**2 – 14**) Find the *n*th term, and the tenth term of the arithmetic sequence:

2. $2,\ 6,\ 10,\ 14,\ \ldots$

3. $3,\ 2.7,\ 2.4,\ 2.1,\ \ldots$

4. $-6,\ -4.5,\ -3,\ -1.5,\ \ldots$

5. $\ln 3,\ \ln 9,\ \ln 27,\ \ln 81,\ \ldots$

6. $a_1 = 2, \quad d = 3$

7. $a_1 = 5, \quad d = -3$

8. $a_1 = 1, \quad d = -\frac{1}{2}$

9. $a_1 = -2, \quad d = 4$

10. $a_1 = \sqrt{2}, \quad d = \sqrt{2}$

11. $a_1 = 0, \quad d = \pi$

12. $a_1 = 13, \quad d = 4$

13. $a_1 = -40, \quad d = 5$

14. $a_1 = -32, \quad d = 4$

(**15 – 26**) Find the common difference for the arithmetic sequence with the specified terms:

15. $a_4 = 14, \quad a_{11} = 35$

16. $a_{12};\ \ a_1 = 9.1, \quad a_2 = 7.5$

17. $a_1;\ \ a_8 = 47, \quad a_9 = 53$

18. $a_{10};\ \ a_2 = 1, \quad a_{18} = 49$

19. $a_{10};\ \ a_8 = 8, \quad a_{20} = 44$

20. $a_{12};\ \ a_8 = 4, \quad a_{18} = -96$

21. $a_8;\ \ a_{15} = 0, \quad a_{40} = -50$

22. $a_{20};\ \ a_9 = -5, \quad a_{15} = 31$

23. $a_n;\ \ a_8 = 8, \quad a_{20} = 44$

24. $a_n;\ \ a_8 = 4, \quad a_{18} = -96$

25. $a_n;\ \ a_{14} = -1, \quad a_{15} = 31$

26. $a_n;\ \ a_9 = -5, \quad a_{15} = 31$

Find the sum S_n of the arithmetic sequence that satisfies the conditions:

27. $a_1 = 40, \quad d = -3, \quad n = 30$

28. $a_7 = \frac{7}{3}, \quad d = -\frac{2}{3}, \quad n = 15$

29. Find the number of terms in the arithmetic sequence with the given conditions:

$a_1 = -2, \quad d = \frac{1}{4}, \quad S = 21$

30. Find the number of integers between 32 and 390 that are divisible by 6, find their sum.

(**31 – 44**) Find each arithmetic sum

31. $2 + 11 + 20 + \ldots + 16{,}058$

32. $60 + 64 + 68 + 72 + \cdots + 120$

33. $1 + 3 + 5 + \cdots + (2n-1)$

34. $2 + 4 + 6 + \cdots + 2n$

35. $2 + 5 + 8 + \cdots + 41$

36. $7 + 12 + 17 + \cdots + (2 + 5n)$

37. $73 + 78 + 83 + 88 + \cdots + 558$

38. $7 + 1 - 5 - 11 - \cdots - 299$

39. $-1 + 2 + 7 + \cdots + (4n-5)$

40. $5 + 9 + 13 + \cdots + 49$

41. $2 + 4 + 6 + \cdots + 70$

42. $1 + 3 + 5 + \cdots + 59$

43. $4 + 4.5 + 5 + 5.5 + \cdots + 100$

44. $8 + 8\frac{1}{4} + 8\frac{1}{2} + 8\frac{3}{4} + 9 + \cdots + 50$

45. Show that the given sequence is geometric, and find the common ratio

$5,\ -\frac{5}{4},\ \frac{5}{16},\ \dots,\ 5\left(-\frac{1}{4}\right)^{n-1},\ \dots$

(**46 – 61**) Find the ***n***th term, the fifth term, and the eighth term of the geometric sequence

46. $8,\ 4,\ 2,\ 1,\ \dots$

47. $300,\ -30,\ 3,\ -0.3,\ \dots$

48. $1,\ -\sqrt{3},\ 3,\ -3\sqrt{3},\ \dots$

49. $4,\ -6,\ 9,\ -13.5,\ \dots$

50. $1,\ -x^2,\ x^4,\ -x^6,\ \dots$

51. $10,\ 10^{2x-1},\ 10^{4x-3},\ 10^{6x-5},\ \dots$

52. $a_1 = 2, \quad r = 3$

53. $a_1 = 1, \quad r = -\frac{1}{2}$

54. $a_1 = -2, \quad r = 4$

55. $a_1 = \sqrt{2}, \quad r = \sqrt{2}$

56. $a_1 = 0, \quad r = \pi$

57. $\{s_n\} = \{3^n\}$

58. $\{s_n\} = \{(-5)^n\}$

59. $\{s_n\} = \left\{-3\left(\frac{1}{2}\right)^n\right\}$

60. $\{u_n\} = \left\{\frac{3^{n-1}}{2^n}\right\}$

61. $\{u_n\} = \left\{\frac{2^n}{3^{n-1}}\right\}$

62. Find all possible values of ***r*** for a geometric sequence with the two given terms $a_4 = 3, \quad a_6 = 9$

63. Find the sixth term of the geometric sequence whose first two terms are 4 and 6

(**64 – 71**) Find the specified term of the geometric sequence that that has 2 given terms

64. $a_{10}; \quad a_4 = 4, \quad a_7 = 12$

65. $a_6; \quad a_1 = 4, \quad a_2 = 6$

66. $a_7; \quad a_2 = 3, \quad a_3 = -\sqrt{3}$

67. $a_6; \quad a_2 = 3, \quad a_3 = -\sqrt{2}$

68. $a_5; \quad a_1 = 4, \quad a_2 = 7$

69. $a_9; \quad a_2 = 3, \quad a_5 = -81$

70. $a_7; \quad a_1 = -4, \quad a_3 = -1$

71. $a_8; \quad a_2 = 3, \quad a_4 = 6$

(**72 – 83**) Express the sum in terms of summation notation (Answers are not unique.)

72. $4 + 11 + 18 + 25 + 32$

73. $4 + 11 + 18 + \dots + 466$

74. $2 + 4 + 8 + 16 + 32 + 64 + 128$

75. $2 - 4 + 8 - 16 + 32 - 64$

76. $3 + 8 + 13 + 18 + 23$

77. $256 + 192 + 144 + 108 + \cdots$

78. $\frac{5}{13} + \frac{10}{11} + \frac{15}{9} + \frac{20}{7}$

79. $\frac{1}{4} - \frac{1}{12} + \frac{1}{36} - \frac{1}{108}$

80. $3 + \frac{3}{5} + \frac{3}{25} + \frac{3}{125} + \frac{3}{625}$

81. $\frac{3}{7} + \frac{6}{11} + \frac{9}{15} + \frac{12}{19} + \frac{15}{23} + \frac{18}{27}$

82. $\frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \cdots, \quad |x| < 3$

83. $2x + 4x^2 + 8x^3 + \cdots, \quad |x| < \frac{1}{2}$

(**84 – 97**) Find the sum of the infinite geometric series if it exists:

84. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$

85. $1.5+0.015+0.00015+\ldots$

86. $\sqrt{2}-2+\sqrt{8}-4+\ldots$

87. $256+192+144+108+\ldots$

88. $\frac{1}{4}+\frac{2}{4}+\frac{2^2}{4}+\frac{2^3}{4}+\ldots+\frac{2^{n-1}}{4}$

89. $\frac{3}{9}+\frac{3^2}{9}+\frac{3^3}{9}+\ldots+\frac{3^n}{9}$

90. $-1-2-4-8-\cdots-2^{n-1}$

91. $2+\frac{6}{5}+\frac{18}{25}+\ldots+2\left(\frac{3}{5}\right)^{n-1}$

92. $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$

93. $2+\frac{4}{3}+\frac{8}{9}+\frac{16}{27}+\ldots$

94. $2-\frac{1}{2}+\frac{1}{8}-\frac{1}{32}+\cdots$

95. $9+12+16+\frac{64}{3}+\cdots$

96. $8+12+18+27+\cdots$

97. $6+2+\frac{2}{3}+\frac{2}{9}+\cdots$

(**98 – 117**) Find the sum:

98. $\sum_{k=1}^{20}(3k-5)$

99. $\sum_{k=1}^{18}\left(\frac{1}{2}k+7\right)$

100. $\sum_{k=1}^{80}(2k-5)$

101. $\sum_{n=1}^{90}(3-2n)$

102. $\sum_{n=1}^{100}\left(6-\frac{1}{2}n\right)$

103. $\sum_{n=1}^{80}\left(\frac{1}{3}n+\frac{1}{2}\right)$

104. $\sum_{k=1}^{10}3^k$

105. $\sum_{k=1}^{9}\left(-\sqrt{5}\right)^k$

106. $\sum_{k=0}^{9}\left(-\frac{1}{2}\right)^{k+1}$

107. $\sum_{n=1}^{\infty}2\left(\frac{2}{3}\right)^{n-1}$

108. $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n}$

109. $\sum_{n=1}^{\infty}3\left(\frac{3}{2}\right)^{n}$

110. $\sum_{n=1}^{\infty}5\left(\frac{1}{4}\right)^{n-1}$

111. $\sum_{n=1}^{\infty}8\left(\frac{1}{3}\right)^{n-1}$

112. $\sum_{k=1}^{\infty}\frac{1}{2}\cdot 3^{k-1}$

113. $\sum_{k=1}^{\infty}6\left(-\frac{2}{3}\right)^{k-1}$

114. $\sum_{k=1}^{\infty}4\left(-\frac{1}{2}\right)^{k-1}$

115. $\sum_{k=8}^{14}\left(3^{k-7}+2j^2\right)$

116. The sum of the first 120 terms of $14,\ 16,\ 18,\ 20,\ \cdots$

117. The sum of the first 46 terms of $2,\ -1,\ -4,\ -7,\ \cdots$

(**118 – 124**) Find the rational number represented by the repeating decimal

118. $0.\overline{23}$ **120.** $2.4\overline{17}$ **122.** $5.\overline{146}$ **124.** $1.\overline{6124}$

119. $0.0\overline{71}$ **121.** $10.\overline{5}$ **123.** $3.2\overline{394}$

125. Find x so that $x+3$, $2x+1$, and $5x+2$ are consecutive terms of an arithmetic sequence.

126. Find x so that $2x$, $3x+2$, and $5x+3$ are consecutive terms of an arithmetic sequence.

127. Find x so that x, $x+2$, and $x+3$ are consecutive terms of a geometric sequence.

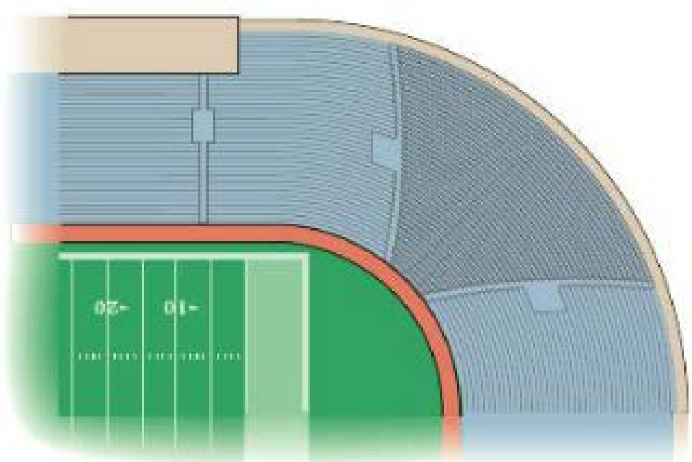
128. Find x so that $x-1$, x and $x+2$ are consecutive terms of a geometric sequence.

129. How many terms must be added in an arithmetic sequence whose first term is 11 and whose common difference is 3 to obtain a sum of 1092?

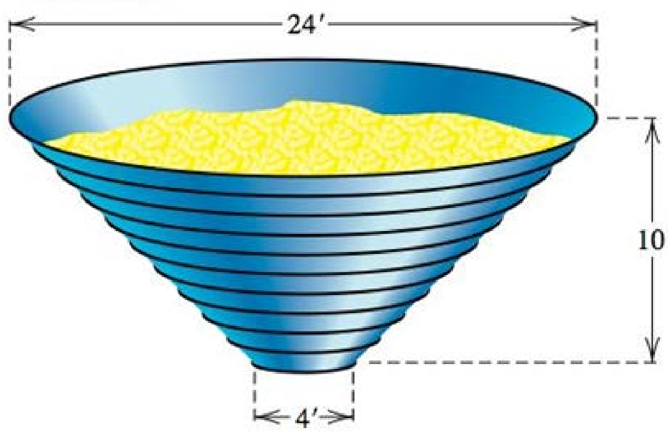
130. How many terms must be added in an arithmetic sequence whose first term is 78 and whose common difference is –4 to obtain a sum of 702?

131. The first ten rows of seating in a certain section of a stadium have 30 seats, 32 seats, 34 seats, and so on. The eleventh through the twentieth rows each contain 50 seats. Find the total number of seats in the section.

132. The corner section of a football stadium has 15 seats in the first row and 40 rows in all. Each successive row contains two additional seats. How many seats are in this section?



133. A gain bin is to be constructed in the shape of a frustum of a cone.



The bin is to be 10 *feet* tall with 11 metal rings positioned uniformly around it, from the 4-foot opening at the bottom to the 24-foot opening at the top. Find the total length of metal needed to make the rings.

134. A bicycle rider coasts downhill, traveling 4 *feet* the first second. In each succeeding second, the rider travels 5 *feet* farther than in the preceding second. If the rider reaches the bottom of the hill in 11 *seconds*, find the total distance traveled.

135. A contest will have five each prizes totaling \$5,000, and there will be a \$100 difference between successive prices. Find the first prize.

136. A Company is to distribute \$46,000 in bonuses to its top ten salespeople. The tenth salesperson on the list will receive \$1,000, and the difference in bonus money between successively ranked salesperson is to be constant. Find the bonus for each salesperson.

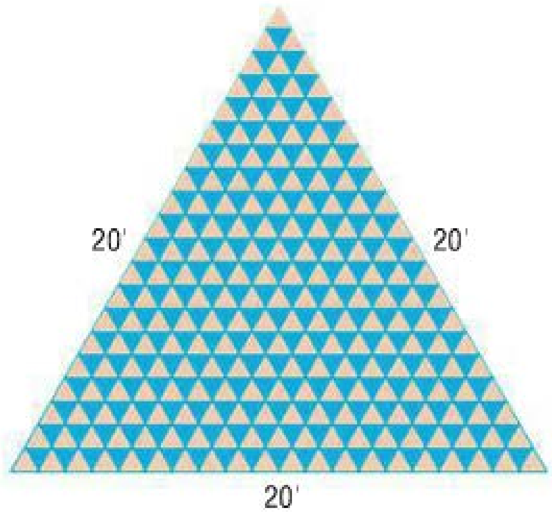
137. Assuming air resistance is negligible, a small object that is dropped from a hot air balloon falls 16 *feet* during the first second, 48 *feet* during the second second, 80 *feet* during the third second, 112 *feet* during the fourth second, and so on. Find an expression for the distance the object falls in n seconds.

138. A brick staircase has a total of 30 steps. The bottom step requires 100 bricks. Each successive step requires two fewer bricks than the prior step.

a) How many bricks are required for the top step?

b) How many bricks are required to build the staircase?

139. A mosaic is designed in the shape of an equilateral triangle, 20 feet on each side. Each tile in the mosaic is in the shape of an equilateral triangle, 12 inches to a side. The tiles are to alternate in color as shown below.



How many tiles of each color will be required?

Section 4.8 – Mathematical Induction

If n is a positive integer and we let P_n denote the mathematical statement $(xy)^n = x^n y^n$, we obtained the following ***infinite sequence*** of statements:

$$\begin{array}{ll} \textit{Statement } P_1: & (xy)^1 = x^1 y^1 \\ \textit{Statement } P_2: & (xy)^2 = x^2 y^2 \\ \textit{Statement } P_3: & (xy)^3 = x^3 y^3 \\ \quad\vdots & \quad\vdots \\ \textit{Statement } P_n: & (xy)^n = x^n y^n \\ \quad\vdots & \quad\vdots \end{array}$$

Principle of Mathematical Induction

If with each positive integer n there is associated a statement P_n then all the statements P_n are true, provided the following two conditions are satisfied.

1) P_1 is true.

2) Whenever k is a positive integer such that P_k is true, then P_{k+1} is also true.

Steps in Applying the Principle of Mathematical Induction

1) Show that P_1 is true.

2) Assume that P_k is true, and then prove that P_{k+1} is true.

Example

Use the mathematical induction to prove that for every positive integer n, the sum of the first n positive integers is:

$$\frac{n(n+1)}{2}$$

Solution

(1) For $n = 1 \Rightarrow \frac{1(1+1)}{2} = 1$; hence P_1 is true.

(2) Assume that P_k is true.

Thus the induction hypothesis is: $1+2+3+\ldots+k = \frac{k(k+1)}{2}$

For $k + 1$:

$$1+2+3+\ldots+k+(k+1) \stackrel{?}{=} \frac{(k+1)((k+1)+1)}{2}$$

$$1+2+3+\ldots+k+(k+1) = (1+2+3+\ldots+k)+(k+1)$$

$$= \frac{k(k+1)}{2}+(k+1) \qquad \textit{Induction hypothesis}$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2} \qquad \textit{Factor out } \mathbf{k+1}$$

$$= \frac{(k+1)((k+1)+1)}{2} \qquad \textit{Change form of } \mathbf{k+2}$$

This shows that P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed

Example

Prove that for every positive integer n,

$$1^2+3^2+\ldots+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow 1^2=\frac{1(2(1)-1)(2(1)+1)}{3}$

$$1=\frac{3}{3}$$

$1=1$ ✓ hence P_1 is true.

(2) $1^2+3^2+\ldots+(2k-1)^2=\frac{k(2k-1)(2k+1)}{3}$

For k + 1:

$$1^2+3^2+\ldots+(2k-1)^2+(2(k+1)-1)^2 \overset{?}{=} \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

$$1^2+3^2+\ldots+(2k-1)^2+(2(k+1)-1)^2 \overset{?}{=} \frac{(k+1)(2k+1)(2k+3)}{3}$$

$$1^2+3^2+\ldots+(2k-1)^2+[2k+2-1]^2=\frac{k(2k-1)(2k+1)}{3}+(2k+1)^2$$

$$=\frac{k(2k-1)(2k+1)+3(2k+1)^2}{3}$$

$$=\frac{(2k+1)[k(2k-1)+3(2k+1)]}{3}$$

$$=\frac{(2k+1)\left(2k^2-k+6k+3\right)}{3}$$

$$=\frac{(2k+1)\left(2k^2+5k+3\right)}{3}$$

$$=\frac{(2k+1)(k+1)(2k+3)}{3} \checkmark$$

This shows that P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed

Example

Prove that 2 is a factor of $n^2 + 5n$ for every positive integer n,

Solution

(1) For $\boldsymbol{n = 1} \Rightarrow n^2 + 5n = 1^2 + 5(1)$

$$= 6$$

$$= 2.3 \quad \checkmark$$

Thus, 2 is a factor of $n^2 + 5n$ for $n = 1$; hence P_1 is true.

(2) 2 is a factor of $k^2 + 5k \Leftrightarrow k^2 + 5k = 2p$

is 2 a factor of $(k+1)^2 + 5(k+1)$?

$$\begin{aligned}(k+1)^2 + 5(k+1) &= k^2 + 2k + 1 + 5k + 5 \\ &= k^2 + 5k + 2k + 6 \\ &= \left(k^2 + 5k\right) + 2(k+3) \\ &= 2p + 2(k+3) \\ &= 2.(p + k + 3) \quad \checkmark\end{aligned}$$

Thus, 2 is a factor of the last expression; hence P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed

Steps in Applying the Extended Principle of Mathematical Induction

1. Show that P_1 is true.
2. Assume that P_k is true with $k \geq j$, and then prove that P_{k+1} is true.

Example

Let a be a nonzero real number such that $a > -1$. Prove that $(1+a)^n > 1+na$ for every integer $n \geq 2$.

Solution

For $\boldsymbol{n = 1} \Rightarrow (1+a)^1 > 1+(1)a \Rightarrow P_1$ is false.

Step 1. For $\boldsymbol{n = 2} \Rightarrow (1+a)^2 \overset{?}{>} 1+(2)a$

$$1+2a+a^2 > 1+a \quad \checkmark$$

$\Rightarrow P_2$ is true.

Step 2. Assume that P_k is true $(1+a)^k > 1+ka$

We need to prove that P_{k+1} is true, that is $(1+a)^{k+1} > 1+(k+1)a$

$$(1+a)^{k+1} = (1+a)^k (1+a)^1$$
$$> (1+ka)(1+a)$$

$$(1+ka)(1+a) = 1+a+ka+ka^2$$
$$= 1+(a+ka)+ka^2$$
$$= 1+a(k+1)+ka^2$$
$$> 1+(k+1)a$$

$$(1+a)^{k+1} > (1+ka)(1+a)$$
$$> 1+(k+1)a \quad \checkmark$$

Thus, P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed

Exercises *Section* 4.8 – Mathematical Induction

1. Find all positive integers n for which the given statement is not true

a) $3^n > 6n$ *b)* $3^n > 2n+1$ *c)* $2^n > n^2$ *d)* $n! > 2n$

2. Prove that the statement is true for every positive integer n. $2+4+6+\ldots+2n = n(n+1)$

3. Prove that the statement is true for every positive integer n. $1+3+5+\ldots+(2n-1) = n^2$

4. Prove that the statement is true for every positive integer n. $2+7+12+\ldots+(5n-3) = \frac{1}{2}n(5n-1)$

5. Prove that the statement is true: $1+2\bullet 2+3\bullet 2^2+\ldots+n\bullet 2^{n-1} = 1+(n-1).2^n$

6. Prove that the statement is true: $1^2+2^2+3^2+\ldots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

7. Prove that the statement is true: $\dfrac{1}{1\bullet 2}+\dfrac{1}{2\bullet 3}+\dfrac{1}{3\bullet 4}+\ldots+\dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

8. Prove that the statement is true: $\dfrac{1}{2}+\dfrac{1}{2^2}+\dfrac{1}{2^3}+\ldots+\dfrac{1}{2^n} = 1-\dfrac{1}{2^n}$

9. Prove that the statement is true: $\dfrac{1}{1\cdot 4}+\dfrac{1}{4\cdot 7}+\dfrac{1}{7\cdot 10}+\cdots+\dfrac{1}{(3n-2)\cdot(3n+1)} = \dfrac{n}{3n+1}$

10. Prove that the statement is true: $\dfrac{4}{5}+\dfrac{4}{5^2}+\dfrac{4}{5^3}+\cdots+\dfrac{4}{5^n} = 1-\dfrac{1}{5^n}$

11. Prove that the statement is true: $1^3+2^3+3^3+\cdots+n^3 = \dfrac{n^2(n+1)^2}{4}$

12. Prove that the statement is true: $3+3^2+3^3+\ldots+3^n = \frac{3}{2}\left(3^n-1\right)$

13. Prove that the statement is true: $x^{2n}+x^{2n-1}y+\cdots+xy^{2n-1}+y^{2n} = \dfrac{x^{2n+1}-y^{2n+1}}{x-y}$

14. Prove that the statement is true: $5\bullet 6+5\bullet 6^2+5\bullet 6^3+\cdots+5\bullet 6^n = 6\left(6^n-1\right)$

15. Prove that the statement is true: $7\bullet 8+7\bullet 8^2+7\bullet 8^3+\cdots+7\bullet 8^n = 8\left(8^n-1\right)$

16. Prove that the statement is true: $3+6+9+\cdots+3n = \dfrac{3n(n+1)}{2}$

17. Prove that the statement is true: $5+10+15+\cdots+5n = \dfrac{5n(n+1)}{2}$

18. Prove that the statement is true: $1+3+5+\cdots+(2n-1) = n^2$

19. Prove that the statement is true: $4+7+10+\cdots+(3n+1) = \dfrac{n(3n+5)}{2}$

20. Prove that the statement is true for every positive integer n. $n < 2^n$

21. Prove that the statement is true for every positive integer n. 3 is a factor of $n^3 - n + 3$

22. Prove that the statement is true for every positive integer n. 4 is a factor of $5^n - 1$

23. Prove that the statement by mathematical induction: $\left(a^m\right)^n = a^{mn}$ (a and m are constant)

24. Prove that the statement by mathematical induction: $2^n > 2n \quad if \quad n \geq 3$

25. Prove that the statement by mathematical induction: If $0 < a < 1$, then $a^n < a^{n-1}$

26. Prove that the statement by mathematical induction: If $n \geq 4$, then $n! > 2^n$

27. Prove that the statement by mathematical induction: $3^n > 2n + 1 \quad if \quad n \geq 2$

28. Prove that the statement by mathematical induction: $2^n > n^2 \quad for \quad n > 4$

29. Prove that the statement by mathematical induction: $4^n > n^4 \quad for \quad n \geq 5$

30. A pile of n rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must moved one at a time, to a different peg with no ring ever placed on top of a smaller ring. Find the least number of moves that would be required. Prove your result by mathematical induction.