

Solution

Section 1.7 - Direction Fields; Existence and Uniqueness of Solutions

Exercise

Which of the initial value problems are guaranteed a unique solution. $y' = 4 + y^2$, $y(0) = 1$

Solution

$f(t, y) = 4 + y^2 \rightarrow f$ is continuous

$\frac{\partial f}{\partial y} = 2y$ is also continuous on the whole plane.

Hence the hypotheses are satisfied and guarantee a unique solution.

Exercise

Which of the initial value problems are guaranteed a unique solution? $y' = \sqrt{y}$, $y(4) = 0$

Solution

$f(t, y) = \sqrt{y} \Rightarrow y \geq 0$

$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}} \rightarrow y > 0$ (only)

Initial condition: $y(4) = 0 \Rightarrow y_0 = 0$ and $t_0 = 4$

Both f and $\frac{\partial f}{\partial y}$ are not continuous in the rectangle containing (t_0, y_0)

Hence the hypotheses are not satisfied.

Exercise

Which of the initial value problems are guaranteed a unique solution?

$y' = t \tan^{-1} y$, $y(0) = 2$

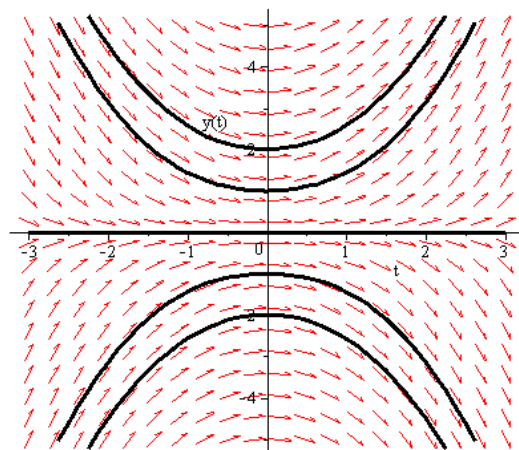
Solution

The right hand side of the equation is $f(t, y) = t \tan^{-1} y$,

which is continuous in the whole plane. $\frac{\partial f}{\partial y} = \frac{t}{t + y^2}$ is also

continuous in the whole plane.

Hence the hypotheses are satisfied and the theorem guarantees a unique solution.



Exercise

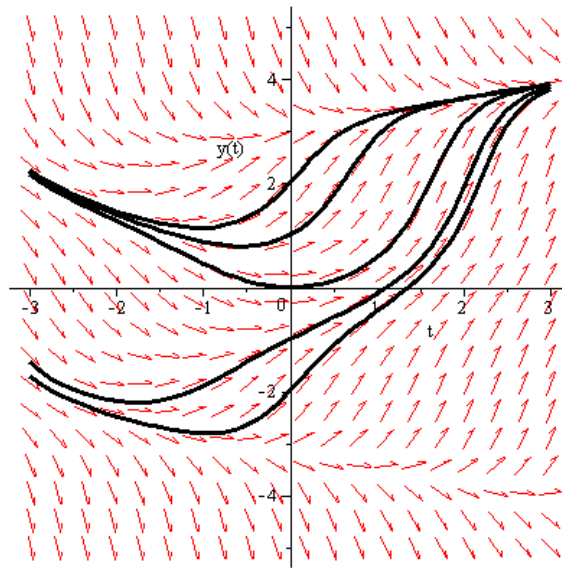
Which of the initial value problems are guaranteed a unique solution? $\omega' = \omega \sin \omega + s$, $\omega(0) = -1$

Solution

The right hand side of the equation is $f(s, \omega) = \omega \sin \omega + s$, which is continuous in the whole plane.

$\frac{\partial f}{\partial \omega} = \sin \omega + \omega \cos \omega$ is also continuous in the whole plane.

Hence the hypotheses are satisfied and the theorem guarantees a unique solution.



Exercise

Which of the initial value problems are guaranteed a unique solution? $x' = \frac{t}{x+1}$, $x(0) = 0$

Solution

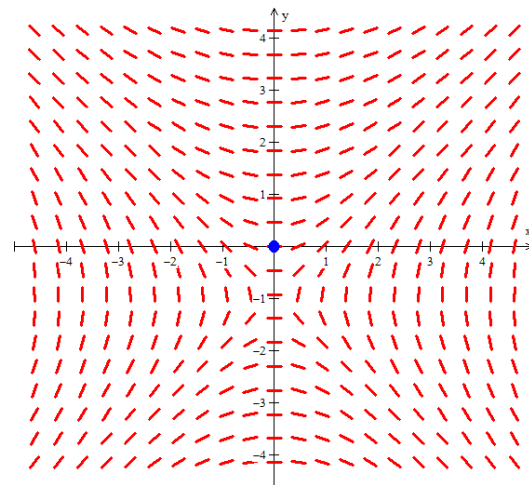
The right hand side of the equation is $f(t, x) = \frac{t}{x+1}$,

which is continuous in the whole plane, except where $x = -1$.

$\frac{\partial f}{\partial x} = -\frac{t}{(x+1)^2}$ is also continuous in the whole plane,

except where $x = -1$.

Hence the hypotheses are satisfied in a rectangle containing the initial point $(0, 0)$, so the theorem guarantees a unique solution.



Exercise

Which of the initial value problems are guaranteed a unique solution? $y' = \frac{1}{x}y + 2$, $y(0) = 1$

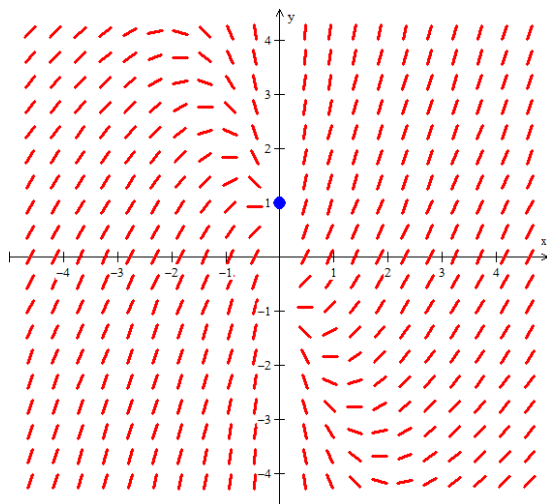
Solution

The right hand side of the equation is $f(x, y) = \frac{1}{x}y + 2$, which is continuous in the whole plane, except where $x = 0$.

Since the initial point is $(0, 1)$, f is discontinuous there.

Consequently, there is no rectangle containing this point in which f is continuous.

The hypotheses are not satisfied, so the theorem doesn't guarantee a unique solution.



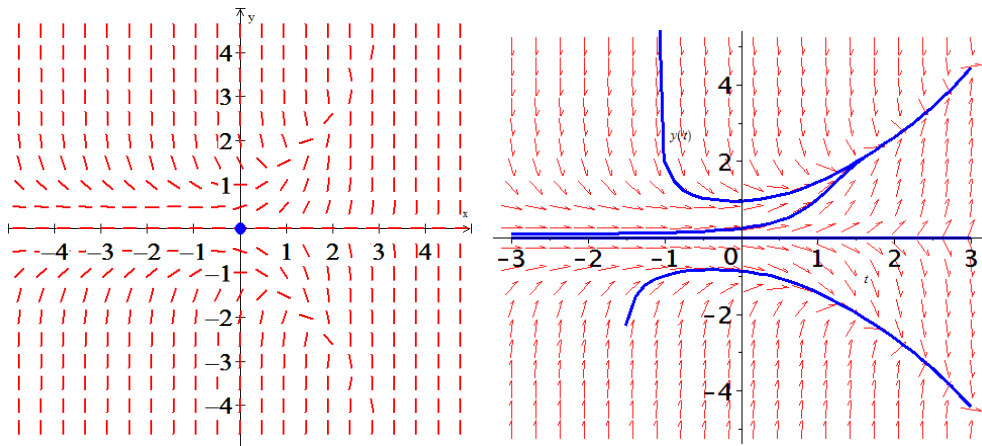
Exercise

Which of the initial value problems are guaranteed a unique solution? $y' = e^t y - y^3$, $y(0) = 0$

Solution

The right hand side of the equation is $f(t, y) = e^t y - y^3$, which is continuous in the whole plane.

$\frac{\partial f}{\partial y} = e^t - 3y^2$ is also continuous in the whole plane.



Hence the hypotheses are satisfied in a rectangle containing the initial point $(0, 0)$, so the theorem guarantees a unique solution.

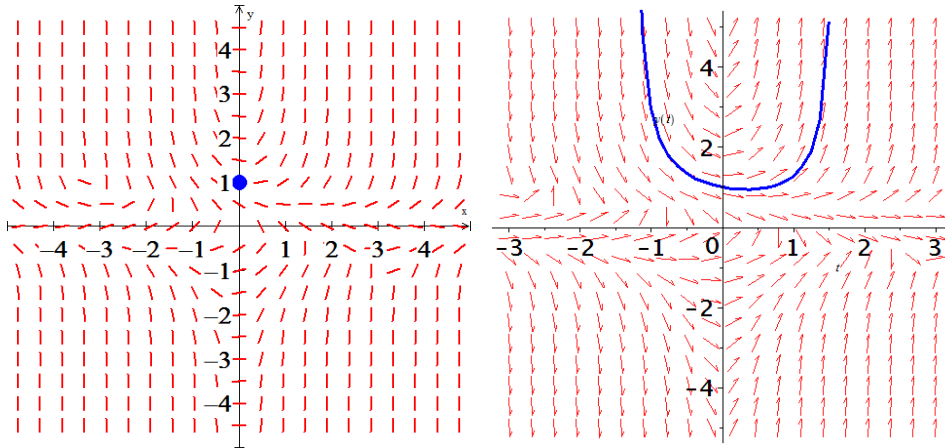
Exercise

Which of the initial value problems are guaranteed a unique solution? $y' = ty^2 - \frac{1}{3y+t}, \quad y(0)=1$

Solution

The right hand side of the equation is $f(t, y) = ty^2 - \frac{1}{3y+t}$, which is continuous in the whole plane.

$\frac{\partial f}{\partial y} = 2ty + \frac{3}{(3y+t)^2}$ is also continuous in the whole plane.



Hence the hypotheses are satisfied in a rectangle containing the initial point $(0, 1)$, so the theorem guarantees a unique solution.

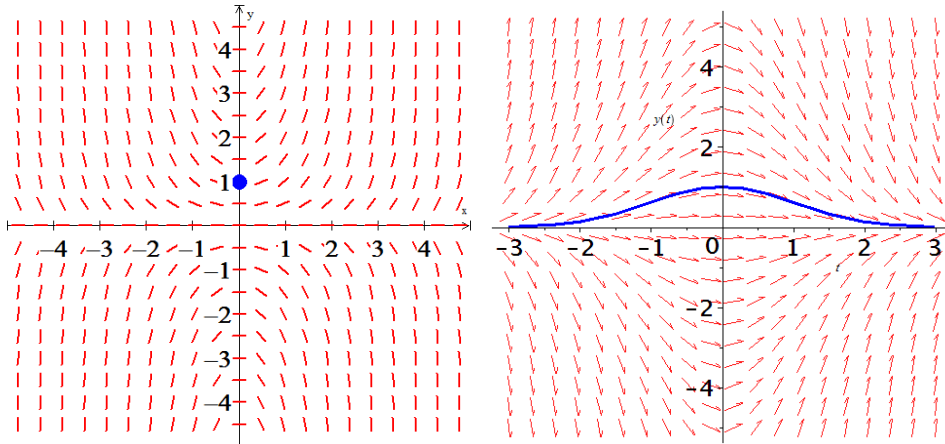
Exercise

Which of the initial value problems are guaranteed a unique solution? $y' = xy, \quad y(0)=1$

Solution

The right hand side of the equation is $f(x, y) = xy$, which is continuous in the whole plane.

$\frac{\partial f}{\partial y} = x$ is also continuous in the whole plane.



Hence the hypotheses are satisfied in a rectangle containing the initial point $(0, 1)$, so the theorem guarantees a unique solution.

Exercise

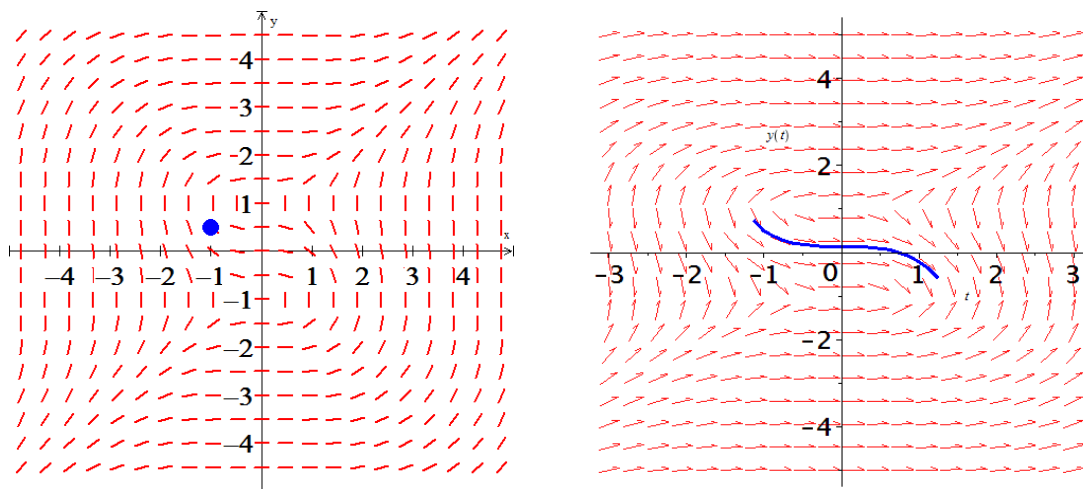
Which of the initial value problems are guaranteed a unique solution? $y' = -\frac{t^2}{1-y^2}, \quad y(-1) = \frac{1}{2}$

Solution

The right hand side of the equation is $f(t, y) = -\frac{t^2}{1-y^2}$, which is continuous in the whole plane, except where $y = \pm 1$.

$\frac{\partial f}{\partial y} = -\frac{2t^2 y}{(1-y^2)^2}$ is also continuous in the whole plane, except where $y = \pm 1$.

Since $t = -1 \rightarrow y_0 = \frac{1}{2} (\neq \pm 1)$



Hence the hypotheses are satisfied in a rectangle containing the initial point $(-1, \frac{1}{2})$, so the theorem guarantees a unique solution.

Exercise

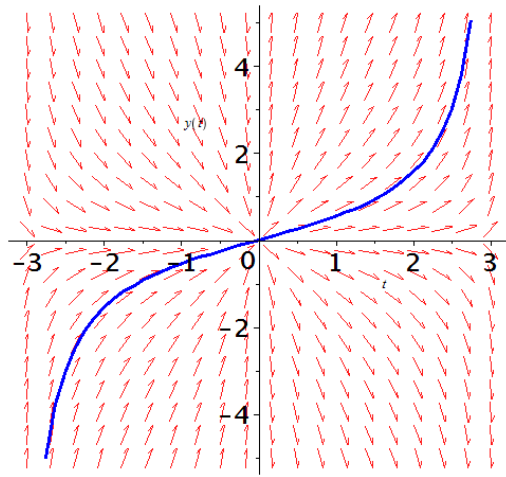
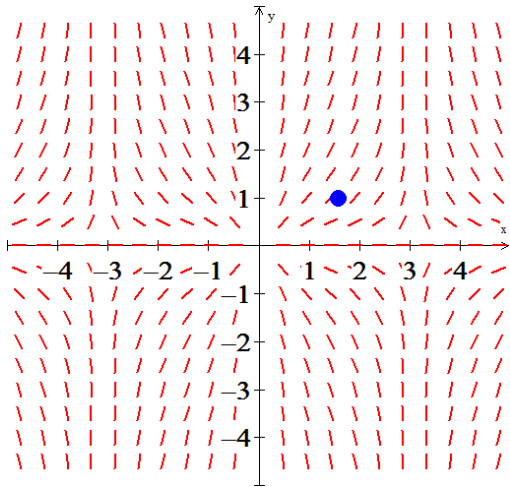
Which of the initial value problems are guaranteed a unique solution? $y' = \frac{y}{\sin t}, \quad y\left(\frac{\pi}{2}\right) = 1$

Solution

The right hand side of the equation is $f(t, y) = \frac{y}{\sin t}$, which is continuous in the whole plane, except where $t = n\pi$.

$\frac{\partial f}{\partial y} = \frac{1}{\sin t}$ is also continuous in the whole plane, except where $t = n\pi$

Since $t = \frac{\pi}{2} \rightarrow y_0 = 1 (\neq n\pi)$



Hence the hypotheses are satisfied in a rectangle containing the initial point $\left(\frac{\pi}{2}, 1\right)$, so the theorem guarantees a unique solution.

Exercise

Which of the initial value problems are guaranteed a unique solution? $y' = \sqrt{1 - y^2}$, $y(0) = 1$

Solution

The right hand side of the equation is $f(t, y) = \sqrt{1 - y^2}$, which is continuous in the whole plane except where $y < -1$ & $y > 1$.

$$\frac{\partial f}{\partial y} = -\frac{y}{\sqrt{1 - y^2}} \Big|_{y=1} = \infty \text{ undefined.}$$

So, the uniqueness theorem doesn't apply

$$\frac{dy}{dt} = \sqrt{1 - y^2}$$

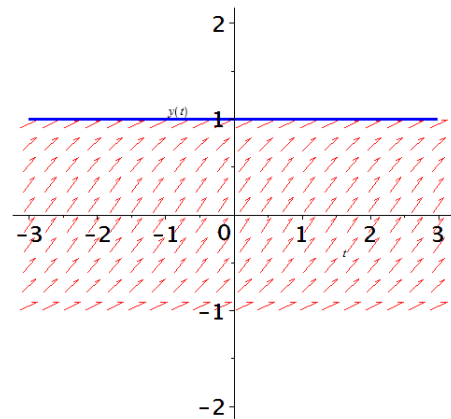
$$\int \frac{dy}{\sqrt{1 - y^2}} = \int dt$$

$$\arccos y = t + C$$

$$y(t) = \cos(t + C)$$

$$y(0) = 1 \rightarrow 1 = \cos C \Rightarrow C = 0 (= 2n\pi)$$

$$y(t) = \cos t \quad (2n - 1)\pi \leq t \leq 2\pi$$



Exercise

Show that $y(t) = 0$ and $y(t) = t^3$ are both solutions of the initial value problem $y' = 3y^{2/3}$, where $y(0) = 0$. Explain why this fact doesn't contradict Theorem

Solution

$$f(t, y) = 3y^{2/3}$$

$$f' = 2y^{-1/3} \text{ which is not continuous at } y = 0$$

Exercise

Use a numerical solver to sketch the solution of the given initial value problem

$$\frac{dy}{dt} = \frac{t}{y+1}, \quad y(2) = 0$$

- Where does your solver experience difficulty? why? Use the image of your solution to estimate the interval of existence.
- Find an explicit solution; then use your formula to determine the interval of existence. How does it compare with the approximation found in part (a).

Solution

$$a) \quad (y+1)dy = tdt$$

$$\int (y+1)dy = \int tdt$$

$$\frac{1}{2}y^2 + y = \frac{1}{2}t^2 + C$$

$$y^2 + 2y = t^2 + C$$

$$(0)^2 + 2(0) = 2^2 + C$$

$$0 = 4 + C$$

$$C = -4$$

$$y^2 + 2y = t^2 - 4$$

$$y^2 + 2y - t^2 + 4 = 0$$

Solve for y:

$$y = \frac{-2 \pm \sqrt{2^2 - 4(1)(-t^2 + 4)}}{2(1)} = \frac{-2 \pm \sqrt{4t^2 - 12}}{2} = \frac{-2 \pm 2\sqrt{t^2 - 3}}{2} \\ = -1 \pm \sqrt{t^2 - 3}$$

- b) The only solution is: $y = -1 + \sqrt{t^2 - 3}$ and $t^2 - 3 > 0 \Rightarrow t > \sqrt{3}$

The interval of the solution $(\sqrt{3}, \infty)$