

## ***Solution***      **Section 2.6 – Linear Independence**

### ***Exercise***

State the following statements as true or false

- a) If  $S$  is a linearly dependent set, then each vector in  $S$  is a linear combination of other vectors in  $S$ .
- b) Any set containing the zero vector is linearly dependent.
- c) The empty set is linearly dependent.
- d) Subsets of linearly dependent sets are linearly dependent.
- e) Subsets of linearly independent sets are linearly independent.
- f) If  $a_1 \vec{x}_1 + a_2 \vec{x}_2 + \dots + a_n \vec{x}_n = \vec{0}$  and  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$  are linearly independent, then the scalars  $a_i$  are zero

### **Solution**

- a) False
- b) True
- c) False
- d) False
- e) True
- f) True

### ***Exercise***

Given three independent vectors  $\vec{w}_1, \vec{w}_2, \vec{w}_3$ . Take combinations of those vectors to produce  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ . Write the combinations in a matrix form as  $V = WM$ .

$$\vec{v}_1 = \vec{w}_1 + \vec{w}_2$$

$$\vec{v}_2 = \vec{w}_1 + 2\vec{w}_2 + \vec{w}_3$$

$$\vec{v}_3 = \vec{w}_2 + c\vec{w}_3$$

$$\text{which is } \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix}$$

What is the test on a matrix  $V$  to see if its columns are linearly independent?

If  $c \neq 1$  show that  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are linearly independent.

If  $c = 1$  show that  $\vec{v}$ 's are linearly *dependent*.

### **Solution**

The nullspace of  $\mathbf{V}$  must contain only the *zero* vector. Then  $\vec{x} = (0, 0, 0)$  is the only combination of the columns that gives  $\mathbf{V}\vec{x} = \text{zero vector}$ .

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix} \quad \begin{array}{l} R_2 - R_1 \\ R_1 - R_2 \\ R_3 - R_2 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & c \end{bmatrix} \quad \begin{array}{l} R_1 - R_2 \\ R_3 - R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & c-1 \end{bmatrix}$$

If  $c \neq 1$ , then the matrix  $M$  is invertible. So if  $x$  is any nonzero vector we know that  $Mx$  is nonzero. Since  $\vec{w}$ 's are given as independent and  $WM\vec{x}$  is nonzero. Since  $V = WM$ , this says that  $x$  is not in the nullspace of  $\mathbf{V}$ . therefore;  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are independent.

If  $c = 1$ , that implies

$$\begin{cases} \vec{v}_1 = \vec{w}_1 + \vec{w}_2 \\ \vec{v}_2 = \vec{w}_1 + \vec{w}_2 + \vec{w}_2 + \vec{w}_3 \\ \vec{v}_3 = \vec{w}_2 + \vec{w}_3 \end{cases} \Rightarrow \begin{cases} \vec{v}_1 = \vec{w}_1 + \vec{w}_2 \\ \vec{v}_2 = \vec{v}_1 + \vec{v}_3 \\ \vec{v}_3 = \vec{w}_2 + \vec{w}_3 \end{cases}$$

$-\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = 0$ , which means that  $\vec{v}$ 's are linearly *dependent*.

The other way, the vector  $x = (1, -1, 1)$  is in that nullspace, and  $M\vec{x} = 0$ . Then certainly  $WM\vec{x} = 0$  which is the same as  $V\vec{x} = 0$ . So, the  $\vec{v}$ 's are dependent.

## Exercise

Find the largest possible number of independent vectors among

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \vec{v}_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad \vec{v}_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad \vec{v}_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

## Solution

Since  $\vec{v}_4 = \vec{v}_2 - \vec{v}_1$ ,  $\vec{v}_5 = \vec{v}_3 - \vec{v}_1$ , and  $\vec{v}_6 = \vec{v}_3 - \vec{v}_2$ , there are at most three

independent vectors among these: furthermore, applying row reduction to the matrix

$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$  gives three pivots, showing that  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are independent.

### Exercise

Show that  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are independent but  $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$  are dependent:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \vec{v}_4 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

Solve either  $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$  *or*  $A\vec{x} = \vec{0}$ . The  $v$ 's go in the columns of  $A$ .

### Solution

$$\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

This matrix has 3 pivots with rank of 3 equals to rows that implies the  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are independent.

$$\vec{v}_4 = \vec{v}_1 + \vec{v}_2 - 4\vec{v}_3 \quad \text{or} \quad \vec{v}_1 + \vec{v}_2 - 4\vec{v}_3 - \vec{v}_4 = \vec{0}$$

That shows that  $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$  are dependent.

### Exercise

Decide the dependence or independence of

- a) The vectors  $(1, 3, 2)$ ,  $(2, 1, 3)$ , and  $(3, 2, 1)$ .
- b) The vectors  $(1, -3, 2)$ ,  $(2, 1, -3)$ , and  $(-3, 2, 1)$ .

### Solution

- a) These are *linearly independent*.

$$x_1(1, 3, 2) + x_2(2, 1, 3) + x_3(3, 2, 1) = (0, 0, 0) \text{ only if } x_1 = x_2 = x_3 = 0$$

- b) These are *linearly dependent*:

$$1(1, -3, 2) + 1(2, 1, -3) + 1(-3, 2, 1) = (0, 0, 0)$$

### Exercise

Find two independent vectors on the plane  $x + 2y - 3z - t = 0$  in  $\mathbb{R}^4$ . Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

### Solution

This plane is the nullspace of the matrix  $A = \begin{pmatrix} 1 & 2 & -3 & -1 \end{pmatrix}$

$$x_1 + 2x_2 - 3x_3 - x_4 = 0$$

The pivot is 1<sup>st</sup> column, and the rest are 3 variables.

If  $x_2 = -1$   $x_3 = x_4 = 0 \Rightarrow x_1 = 2$ . The vector is  $(2, -1, 0, 0)$

If  $x_3 = 1$   $x_1 = x_4 = 0 \Rightarrow x_1 = 3$ . The vector is  $(3, 0, 1, 0)$

If  $x_4 = 1$   $x_1 = x_3 = 0 \Rightarrow x_1 = 1$ . The vector is  $(1, 0, 0, 1)$

The 3 vectors  $(2, -1, 0, 0)$ ,  $(3, 0, 1, 0)$ ,  $(1, 0, 0, 1)$  are linearly independent.

We can't find 4 independent vectors because the nullspace only has dimension 3 (have 3 variables).

### Exercise

Determine whether the vectors are linearly dependent or linearly independent in  $\mathbb{R}^3$

a)  $(4, -1, 2), (-4, 10, 2)$

c)  $(-3, 0, 4), (5, -1, 2), (1, 1, 3)$

b)  $(8, -1, 3), (4, 0, 1)$

d)  $(-2, 0, 1), (3, 2, 5), (6, -1, 1), (7, 0, -2)$

### Solution

a) The vector equation  $a(4, -1, 2) + b(-4, 10, 2) = (0, 0, 0)$

$$\left[ \begin{array}{cc|c} 4 & -4 & 0 \\ -1 & 10 & 0 \\ 2 & 2 & 0 \end{array} \right] \quad \frac{1}{4}R_1$$

$$\left[ \begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 10 & 0 \\ 2 & 2 & 0 \end{array} \right] \quad \begin{array}{l} R_2 + R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[ \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad 9R_1 + R_2$$

$$\left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad \frac{1}{9}R_2$$

$$\left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore, the system has only the trivial solution  $a = b = 0$ .

We conclude that the given set of vectors is linearly independent.

**b)**  $A(8, -1, 3) + b(4, 0, 1) = (0, 0, 0)$

$$\left[ \begin{array}{cc|c} 8 & 4 & 0 \\ -1 & 0 & 0 \\ 3 & 1 & 0 \end{array} \right] \quad \frac{1}{8}R_1$$

$$\left[ \begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ -1 & 0 & 0 \\ 3 & 1 & 0 \end{array} \right] \quad \begin{array}{l} R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[ \begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 \end{array} \right] \quad \begin{array}{l} R_1 - R_2 \\ R_3 + R_2 \end{array}$$

$$\left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array} \right] \quad 2R_2$$

$$\left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore, the system has only one trivial solution  $a = b = 0$ .

We conclude that the given set of vectors is linearly independent

**c)** The vector equation:

$$a(-3, 0, 4) + b(5, -1, 2) + c(1, 1, 3) = (0, 0, 0)$$

$$\left[ \begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{array} \right] \quad 3R_3 + 4R_1$$

$$\left[ \begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 26 & 13 & 0 \end{array} \right] \quad -R_2$$

$$\left[ \begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 26 & 13 & 0 \end{array} \right] \quad \begin{array}{l} R_1 - 5R_2 \\ R_3 - 26R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} -3 & 0 & 6 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 39 & 0 \end{array} \right] \quad \frac{1}{39}R_3$$

$$\left[ \begin{array}{ccc|c} -3 & 0 & 6 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \quad \begin{array}{l} R_1 - 6R_3 \\ R_2 + R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \quad -\frac{1}{3}R_1$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Therefore, the system has only the trivial solution  $a = b = c = 0$ .

We conclude that the given set of vectors is linearly independent.

**d)** The vector equation:

$$a(-2, 0, 1) + b(3, 2, 5) + c(6, -1, 1) + d(7, 0, -2) = (0, 0, 0)$$

$$\left[ \begin{array}{cccc|c} -2 & 3 & 6 & 7 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 1 & 5 & 1 & -2 & 0 \end{array} \right] \quad 2R_3 + R_1$$

$$\left[ \begin{array}{cccc|c} -2 & 3 & 6 & 7 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 13 & 8 & 3 & 0 \end{array} \right] \quad \begin{array}{l} 2R_1 - 3R_2 \\ 2R_3 - 13R_2 \end{array}$$

$$\left[ \begin{array}{cccc|c} -4 & 0 & 15 & 14 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 29 & 6 & 0 \end{array} \right] \quad \begin{array}{l} 29R_1 - 15R_3 \\ 29R_2 + R_3 \end{array}$$

$$\left[ \begin{array}{cccc|c} -116 & 0 & 0 & 316 & 0 \\ 0 & 58 & 0 & 6 & 0 \\ 0 & 0 & 29 & 6 & 0 \end{array} \right] \begin{array}{l} -\frac{1}{4}R_1 \\ \frac{1}{58}R_2 \\ \frac{1}{29}R_3 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & -\frac{79}{29} & 0 \\ 0 & 1 & 0 & \frac{3}{29} & 0 \\ 0 & 0 & 1 & \frac{6}{29} & 0 \end{array} \right]$$

Therefore, the system has nontrivial solutions  $a = \frac{79}{29}t$ ,  $b = -\frac{3}{29}t$ ,  $c = -\frac{6}{29}t$ ,  $d = t$

We conclude that the given set of vectors is linearly dependent.

### Exercise

Determine whether the vectors are linearly dependent or linearly independent in  $\mathbb{R}^4$

- a)  $\{(3, 8, 7, -3), (1, 5, 3, -1), (2, -1, 2, 6), (1, 4, 0, 3)\}$
- b)  $\{(0, 0, 2, 2), (3, 3, 0, 0), (1, 1, 0, -1)\}$
- c)  $\{(0, 3, -3, -6), (-2, 0, 0, -6), (0, -4, -2, -2), (0, -8, 4, -4)\}$
- d)  $\{(3, 0, -3, 6), (0, 2, 3, 1), (0, -2, -2, 0), (-2, 1, 2, 1)\}$
- e)  $\{(1, 3, -4, 2), (2, 2, -4, 0), (2, 3, 2, -4), (-1, 0, 1, 0)\}$
- f)  $\{(1, 3, -4, 2), (2, 2, -4, 0), (1, -3, 2, -4), (-1, 0, 1, 0)\}$
- g)  $\{(1, 0, 0, -1), (0, 1, 0, -1), (0, 1, 0, -1), (0, 0, 0, 1)\}$

### Solution

$$a) \det \begin{pmatrix} 3 & 1 & 2 & 1 \\ 8 & 5 & -1 & 4 \\ 7 & 3 & 2 & 0 \\ -3 & -1 & 6 & 3 \end{pmatrix} = 128 \neq 0$$

The system has only the trivial solution and the vectors are *linearly independent*.

$$b) k_1(0,0,2,2) + k_2(3,3,0,0) + k_3(1,1,0,-1) = (0,0,0,0)$$

$$\left[ \begin{array}{ccc|c} 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \end{array} \right]$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 - R_1 \\ R_4 - R_3 \end{array}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 + R_3$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 - R_2$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \frac{1}{2}R_3 \\ \frac{1}{3}R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$k_1 = k_2 = k_3 = 0$$

The system has only the trivial solution and the vectors are *linearly independent*.

$$c) \det \begin{pmatrix} 0 & -2 & 0 & 0 \\ 3 & 0 & -4 & -8 \\ -3 & 0 & -2 & 4 \\ -6 & -6 & -2 & -4 \end{pmatrix} = \underline{480 \neq 0}$$

The system has only the trivial solution and the vectors are *linearly independent*.

$$d) a(3, 0, -3, 6) + b(0, 2, 3, 1) + c(0, -2, -2, 0) + d(-2, 1, 2, 1) = (0, 0, 0, 0)$$

$$\left[ \begin{array}{cccc|c} 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & 1 & 0 \\ -3 & 3 & -2 & 2 & 0 \\ 6 & 1 & 0 & 1 & 0 \end{array} \right] \quad \begin{array}{l} R_3 + R_1 \\ R_4 - 2R_1 \end{array}$$



$$\left[ \begin{array}{cccc|c} 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & 1 & 0 \\ 0 & 3 & -2 & 0 & 0 \\ 0 & 1 & 0 & 5 & 0 \end{array} \right] \quad \begin{array}{l} 2R_3 - 3R_2 \\ 2R_4 - R_2 \end{array}$$

$$\left[ \begin{array}{cccc|c} 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & 1 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 2 & 12 & 0 \end{array} \right] \quad \begin{array}{l} R_2 + R_3 \\ R_4 - R_3 \end{array}$$

$$\left[ \begin{array}{cccc|c} 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 9 & 0 \end{array} \right] \quad \frac{1}{9}R_4$$

$$\left[ \begin{array}{cccc|c} 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right] \quad \begin{array}{l} R_1 + 2R_4 \\ R_2 + 2R_4 \\ R_3 + 3R_4 \end{array}$$

$$\left[ \begin{array}{cccc|c} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right] \quad \begin{array}{l} \frac{1}{3}R_1 \\ \frac{1}{2}R_2 \\ \frac{1}{2}R_3 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

Therefore, the system has only one trivial solution  $a = b = c = d = 0$ .

The given set of vectors is *linearly independent*

e)  $\{(1, 3, -4, 2), (2, 2, -4, 0), (2, 3, 2, -4), (-1, 0, 1, 0)\}$

$$\begin{vmatrix} 1 & 2 & 2 & -1 \\ 3 & 2 & 3 & 0 \\ -4 & -4 & 2 & 1 \\ 2 & 0 & -4 & 0 \end{vmatrix} = 28 \neq 0$$

The system has only the trivial solution and the vectors are *linearly independent*.

$$\mathcal{J} \quad \{(1, 3, -4, 2), (2, 2, -4, 0), (1, -3, 2, -4), (-1, 0, 1, 0)\}$$

$$\begin{vmatrix} 1 & 2 & 1 & -1 \\ 3 & 2 & -3 & 0 \\ -4 & -4 & 2 & 1 \\ 2 & 0 & -4 & 0 \end{vmatrix} \quad \underline{=0}$$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & 2 & -3 & 0 \\ -4 & -4 & 2 & 1 \\ 2 & 0 & -4 & 0 \end{pmatrix} \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 + 4R_1 \\ R_4 - 2R_1 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -4 & -6 & 3 \\ 0 & 4 & 6 & -3 \\ 0 & -4 & -6 & 2 \end{pmatrix} \quad \begin{array}{l} 2R_1 + R_2 \\ R_3 + R_2 \\ R_4 - R_2 \end{array}$$

$$\begin{pmatrix} 2 & 0 & -4 & 1 \\ 0 & -4 & -6 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & -4 & 1 \\ 0 & -4 & -6 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} R_1 - R_3 \\ R_2 - 3R_3 \end{array}$$

$$\begin{pmatrix} 2 & 0 & -4 & 0 \\ 0 & -4 & -6 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} \frac{1}{2}R_1 \\ -\frac{1}{4}R_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underline{x_4 = 0, \quad x_2 = -\frac{3}{2}x_3, \quad x_1 = 2x_3}$$

$\therefore$  The set is *linearly independent*.

$$g) \{(1, 0, 0, -1), (0, 1, 0, -1), (0, 1, 0, -1), (0, 0, 0, 1)\}$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 \end{vmatrix} \stackrel{=0}{=}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_3 + R_1$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_3 + R_2$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} x_1 = 0 \\ x_2 = -x_3 \\ x_4 = 0 \end{array}$$

$\therefore$  The set is *linearly independent*.

### ***Exercise***

a) Show that the three vectors  $\vec{v}_1 = (1, 2, 3, 4)$   $\vec{v}_2 = (0, 1, 0, -1)$   $\vec{v}_3 = (1, 3, 3, 3)$  form a linearly dependent set in  $\mathbb{R}^4$ .

b) Express each vector in part (a) as a linear combination of the other two.

### **Solution**

a) The vector equation:

$$k_1(1, 2, 3, 4) + k_2(0, 1, 0, -1) + k_3(1, 3, 3, 3) = (0, 0, 0, 0)$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ 3 & 0 & 3 & 0 \\ 4 & -1 & 3 & 0 \end{array} \right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 4R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{array} \right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 4R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{array} \right] \quad \begin{array}{l} \\ \\ \\ R_4 + R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The solution:  $k_1 = -t$ ,  $k_2 = -t$ ,  $k_3 = t$

Since the system has nontrivial solutions, the given set of vectors is *linearly dependent*.

**b)** Since  $k_1 = -t$ ,  $k_2 = -t$ ,  $k_3 = t$  and if we let  $t = 1$ , then  $-\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = 0$

$$\vec{v}_1 = -\vec{v}_2 + \vec{v}_3, \quad \vec{v}_2 = -\vec{v}_1 + \vec{v}_3, \quad \vec{v}_3 = \vec{v}_1 + \vec{v}_2$$

## Exercise

For which real values of  $\lambda$  do the following vectors form a linearly dependent set in  $\mathbf{R}^3$

$$\vec{v}_1 = \left( \lambda, -\frac{1}{2}, -\frac{1}{2} \right) \quad \vec{v}_2 = \left( -\frac{1}{2}, \lambda, -\frac{1}{2} \right) \quad \vec{v}_3 = \left( -\frac{1}{2}, -\frac{1}{2}, \lambda \right)$$

## Solution

$$k_1 \left( \lambda, -\frac{1}{2}, -\frac{1}{2} \right) + k_2 \left( -\frac{1}{2}, \lambda, -\frac{1}{2} \right) + k_3 \left( -\frac{1}{2}, -\frac{1}{2}, \lambda \right) = (0, 0, 0, 0)$$

$$\det \begin{pmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{pmatrix} = \frac{1}{4} (4\lambda^3 - 3\lambda - 1)$$

$$4\lambda^3 - 3\lambda - 1 = 0$$

For  $\lambda = 1$   $\lambda = -\frac{1}{2}$ , the determinant is zero and the vectors form a *linearly dependent* set.

### ***Exercise***

Show that if  $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$  is a linearly independent set of vectors, then so is every nonempty subset of  $S$ .

### **Solution**

Let  $\{\vec{v}_a, \vec{v}_b, \dots, \vec{v}_r\}$  be a nonempty subset of  $S$ .

If this set is linearly dependent, then there would be a nonzero solution  $(k_a, k_b, \dots, k_r)$  to

$$k_a \vec{v}_a + k_b \vec{v}_b + \dots + k_r \vec{v}_r = 0.$$

This can be expanded to a nonzero solution of

$k_1 \vec{v}_1 + k_2 \vec{v}_2 + \dots + k_n \vec{v}_n = 0$  by taking all other coefficients as 0. This contradicts the linear independence of  $S$ , so the subset must be *linearly independent*.

### ***Exercise***

Show that if  $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$  is a linearly independent set of vectors in a vector space  $V$ , and if  $\vec{v}_{r+1}, \dots, \vec{v}_n$  are vectors in  $V$  that are not in  $S$ , then  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r, \vec{v}_{r+1}, \dots, \vec{v}_n\}$  is also linearly dependent.

### **Solution**

If  $S$  is linearly dependent, then there is a nonzero solution  $(k_1, k_2, \dots, k_r)$  to

$$k_1 \vec{v}_1 + k_2 \vec{v}_2 + \dots + k_r \vec{v}_r = 0.$$

Thus  $(k_1, k_2, \dots, k_r, 0, 0, \dots, 0)$  is a nonzero solution to

$$k_1 \vec{v}_1 + k_2 \vec{v}_2 + \dots + k_r \vec{v}_r + k_{r+1} \vec{v}_{r+1} + \dots + k_n \vec{v}_n = 0$$

So, the set  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r, \vec{v}_{r+1}, \dots, \vec{v}_n\}$  is *linearly dependent*.

### Exercise

Show that  $\{\vec{v}_1, \vec{v}_2\}$  is linearly independent and  $\vec{v}_3$  does not lie in  $\text{span}\{\vec{v}_1, \vec{v}_2\}$ , then  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  is a linearly independent.

### Solution

If  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  are linearly dependent, there exist a nonzero solution to  $k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = 0$  with  $k_3 \neq 0$  (since  $\vec{v}_1$  and  $\vec{v}_2$  are linearly independent).

$$k_3\vec{v}_3 = -k_1\vec{v}_1 - k_2\vec{v}_2 \Rightarrow \vec{v}_3 = -\frac{k_1}{k_3}\vec{v}_1 - \frac{k_2}{k_3}\vec{v}_2 \text{ which contradicts that } \vec{v}_3 \text{ is not in span } \{\vec{v}_1, \vec{v}_2\}. \text{ Thus } \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \text{ is a linearly independent.}$$

### Exercise

By using the appropriate identities, where required, determine  $F(-\infty, \infty)$  are linearly dependent.

- a)  $6, 3\sin^2 x, 2\cos^2 x$       c)  $1, \sin x, \sin 2x$       e)  $\cos 2x, \sin^2 x, \cos^2 x$   
b)  $x, \cos x$       d)  $(3-x)^2, x^2 - 6x, 5$

### Solution

a) From the identity  $\sin^2 x + \cos^2 x = 1$

$$\begin{aligned} (-1)(6) + (2)(3\sin^2 x) + (3)(2\cos^2 x) &= -6 + 6(\sin^2 x + \cos^2 x) \\ &= 0 \end{aligned}$$

Therefore, the set is *linearly dependent*.

b)  $ax + b\cos x = 0$

$$\begin{aligned} x = 0 &\Rightarrow b = 0 \\ x = \frac{\pi}{2} &\Rightarrow a = 0 \end{aligned}$$

Therefore, the set is *linearly independent*.

c)  $a(1) + b\sin x + c\sin 2x = 0$

$$\begin{aligned} x = 0 &\Rightarrow a = 0 \\ x = \frac{\pi}{2} &\Rightarrow b = 0 \\ x = \frac{\pi}{4} &\Rightarrow c = 0 \end{aligned}$$

Therefore, the set is *linearly independent*.

$$d) (3-x)^2 = 9 - 6x + x^2$$

$$(3-x)^2 - (9 - 6x + x^2) = 0$$

$$(3-x)^2 - (x^2 - 6x) - 9 = 0$$

$$(1)(3-x)^2 + (-1)(x^2 - 6x) + \left(-\frac{9}{5}\right)5 = 0$$

Therefore, the set is linearly dependent.

e) By using the double angle:

$\cos 2x = \cos^2 x - \sin^2 x$  are linearly dependent.

### Exercise

$f_1(x) = \sin x$ ,  $f_2(x) = \cos x$  are linearly independent in  $F(-\infty, \infty)$  because neither function is a scalar multiple of the other. Confirm the linear independence using Wronski's test.

### Solution

$$\begin{aligned} \text{The Wronskian: } W(x) &= \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} \\ &= -\sin^2 x - \cos^2 x \\ &= -(\sin^2 x + \cos^2 x) \\ &= -1 \neq 0 \end{aligned}$$

$\sin x$  and  $\cos x$  are linearly independent

### Exercise

Show  $f_1(x) = e^x$ ,  $f_2(x) = xe^x$ ,  $f_3(x) = x^2e^x$  are linearly independent in  $F(-\infty, \infty)$

### Solution

$$\begin{aligned} W &= \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & e^x + xe^x & 2xe^x + x^2e^x \\ e^x & 2e^x + xe^x & 2e^x + 4xe^x + x^2e^x \end{vmatrix} & \begin{matrix} e^x \\ \text{factor } e^x \\ e^x \end{matrix} \\ &= e^{3x} \begin{vmatrix} 1 & x & x^2 \\ 1 & 1+x & 2x+x^2 \\ 1 & 2+x & 2+4x+x^2 \end{vmatrix} \end{aligned}$$

$$\begin{aligned}
&= e^{3x} \left[ (1+x)(2+4x+x^2) + 2x^2 + x^3 + 2x^2 + x^3 - x^2 - x^3 - (2x+x^2)(2+x) - 2x - 4x^2 - x^3 \right] \\
&= e^{3x} \left[ 2+4x+x^2+2x+4x^2+x^3-4x-2x^2-2x^2-x^3-2x-x^2 \right] \\
&= \underline{2e^{3x} \neq 0}
\end{aligned}$$

$\{e^x, xe^x, x^2e^x\}$  are linearly independent

### Exercise

Use the Wronskian to show that  $f_1(x) = \sin x$ ,  $f_2(x) = \cos x$ ,  $f_3(x) = x \cos x$  span a three-dimensional subspace of  $F(-\infty, \infty)$

### Solution

$$\begin{aligned}
\text{The Wronskian: } W(x) &= \begin{vmatrix} \sin x & \cos x & x \cos x \\ \cos x & -\sin x & \cos x - x \sin x \\ -\sin x & -\cos x & -2 \sin x - x \cos x \end{vmatrix} \\
&= 2 \sin^3 x + x \sin^2 x \cos x - \sin x \cos^2 x + x \sin^2 x \cos x - x \cos^3 x \\
&\quad - x \sin^2 x \cos x + \sin x \cos^2 x - x \sin^2 x \cos x + 2 \sin x \cos^2 x + x \cos^3 x \\
&= 2 \sin^3 x + 2 \sin x \cos^2 x \\
&= 2 \sin x (\sin^2 x + \cos^2 x) \\
&= \underline{2 \sin x}
\end{aligned}$$

Since  $\sin x \neq 0$  for all real  $x$  values, the vectors are *linearly independent*.

### Exercise

Show by inspection that the vectors are linearly dependent.

$$\vec{v}_1(4, -1, 3), \vec{v}_2(2, 3, -1), \vec{v}_3(-1, 2, -1), \vec{v}_4(5, 2, 3), \text{ in } \mathbb{R}^3$$

### Solution

$$\begin{bmatrix} 4 & 2 & -1 & 5 \\ -1 & 3 & 2 & 2 \\ 3 & -1 & -1 & 3 \end{bmatrix} \quad \begin{array}{l} \\ 4R_2 + R_1 \\ 4R_3 - 3R_1 \end{array}$$



$$\begin{bmatrix} 4 & 2 & -1 & 5 \\ 0 & 14 & 7 & 13 \\ 0 & -10 & -1 & -3 \end{bmatrix} \quad \begin{array}{l} 7R_1 - R_2 \\ 14R_3 + 10R_2 \end{array}$$

$$\begin{bmatrix} 28 & 0 & -14 & 22 \\ 0 & 14 & 7 & 13 \\ 0 & 0 & 56 & 88 \end{bmatrix} \quad \begin{array}{l} 4R_1 + R_3 \\ 8R_2 - R_3 \end{array}$$

$$\begin{bmatrix} 112 & 0 & 0 & 176 \\ 0 & 112 & 0 & 16 \\ 0 & 0 & 56 & 88 \end{bmatrix} \quad \begin{array}{l} \frac{1}{112}R_1 \\ \frac{1}{112}R_2 \\ \frac{1}{56}R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{11}{7} \\ 0 & 1 & 0 & \frac{1}{7} \\ 0 & 0 & 1 & \frac{11}{7} \end{bmatrix} \quad \begin{array}{l} \vec{v}_1 = -\frac{11}{7}\vec{v}_4 \\ \vec{v}_2 = -\frac{1}{7}\vec{v}_4 \\ \vec{v}_3 = -\frac{11}{7}\vec{v}_4 \end{array}$$

$$-\frac{11}{7}\vec{v}_1 - \frac{1}{7}\vec{v}_2 - \frac{11}{7}\vec{v}_3 + \vec{v}_4 = 0$$

$$\underline{7\vec{v}_4 = 11\vec{v}_1 + \vec{v}_2 + 11\vec{v}_3}$$

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$(2, -1, 3), (3, 4, 1), (2, -3, 4), \text{ in } \mathbb{R}^3$$

### Solution

$$a(2, -1, 3) + b(3, 4, 1) + c(2, -3, 4) = (0, 0, 0)$$

$$\begin{bmatrix} 2 & 3 & 2 & 0 \\ -1 & 4 & -3 & 0 \\ 3 & 1 & 4 & 0 \end{bmatrix} \quad \begin{array}{l} 2R_2 + R_1 \\ 2R_3 - 3R_1 \end{array}$$

$$\begin{bmatrix} 2 & 3 & 2 & 0 \\ 0 & 11 & -4 & 0 \\ 0 & -7 & 2 & 0 \end{bmatrix} \quad \begin{array}{l} 11R_1 - 3R_2 \\ 11R_3 + 7R_2 \end{array}$$

$$\begin{bmatrix} 22 & 0 & 34 & 0 \\ 0 & 11 & -4 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix} \quad -\frac{1}{6}R_3$$

$$\begin{bmatrix} 22 & 0 & 34 & 0 \\ 0 & 11 & -4 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} R_1 - 34R_3 \\ R_2 + 4R_3 \end{array}$$

$$\begin{bmatrix} 22 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} \frac{1}{22}R_1 \\ \frac{1}{11}R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The system has only the trivial solution  $a = b = c = 0$ .

$$\begin{vmatrix} 2 & 3 & 2 \\ -1 & 4 & -3 \\ 3 & 1 & 4 \end{vmatrix} = 32 - 27 - 2 - 24 + 6 + 12 \neq 0$$

The system has only the trivial solution and the vectors are *linearly independent*

### ***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)

$$(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1), \text{ in } \mathbb{R}^4$$

### **Solution**

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

The system has only the trivial solution and the vectors are *linearly independent*

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$A_1 \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \quad A_2 \begin{bmatrix} -2 & 4 \\ 1 & 0 \end{bmatrix}, \quad A_3 \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}, \quad \text{in } M_{22}$$

### Solution

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 2 & 4 & -1 & 0 \\ 3 & 1 & 2 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 8 & -7 & 0 \\ 0 & 7 & -7 & 0 \end{bmatrix} \quad \begin{array}{l} 4R_1 + R_2 \\ 8R_3 - 7R_2 \end{array}$$

$$\begin{bmatrix} 4 & 0 & 5 & 0 \\ 0 & 8 & -7 & 0 \\ 0 & 0 & -7 & 0 \end{bmatrix} \quad -\frac{1}{7}R_3$$

$$\begin{bmatrix} 4 & 0 & 5 & 0 \\ 0 & 8 & -7 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} R_1 - 5R_3 \\ R_2 + 7R_3 \end{array}$$

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} \frac{1}{4}R_1 \\ \frac{1}{8}R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The vectors are *linearly independent*

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix}, \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix}, \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} \right\} \text{ in } M_{2 \times 3}(\mathbb{R})$$

### Solution

$$\left[ \begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -3 & 7 & 3 & 0 \\ 2 & 4 & 11 & 0 \\ -4 & 6 & -1 & 0 \\ 0 & -2 & -3 & 0 \\ 5 & -7 & 2 & 0 \end{array} \right] \begin{array}{l} R_2 + 3R_1 \\ R_3 - 2R_1 \\ R_4 + 4R_1 \\ \\ R_6 - 5R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 10 & 15 & 0 \\ 0 & -6 & -9 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 8 & 12 & 0 \end{array} \right] \begin{array}{l} \\ R_3 + 5R_2 \\ R_4 - 3R_2 \\ R_5 - R_2 \\ R_6 + 4R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} \rightarrow a_1 = 3a_2 + 2a_3 \\ \rightarrow 2a_2 = -3a_3 \\ \\ \\ \\ \end{array}$$

$$\underline{a_2 = -\frac{3}{2}a_3}$$

$$\begin{aligned} a_1 &= -\frac{9}{2}a_3 + 2a_3 \\ &= -\frac{5}{2}a_3 \end{aligned} \underline{\hspace{1cm}}$$

It is *linearly dependent*.

$$\text{if } a_3 = -2 \quad a_2 = 3 \quad a_1 = 5$$

$$5 \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} + 3 \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix} - 2 \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

### ***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} \right\} \text{ in } M_{2 \times 2}(\mathbb{R})$$

### ***Solution***

$$\begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} = -2 \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}$$

$$2 \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\therefore$  Linearly dependent

### ***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} \right\} \text{ in } M_{2 \times 2}(\mathbb{R})$$

### ***Solution***

$$\begin{bmatrix} 1 & -1 \\ -2 & 1 \\ -1 & 2 \\ 4 & -4 \end{bmatrix} \quad \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 + R_1 \\ R_4 - 4R_1 \end{array}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$a_2 = 0 \rightarrow a_1 = a_2 = 0$$

$\therefore$  Linearly independent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix} \right\} \text{ in } M_{2 \times 2}(\mathbb{R})$$

### Solution

$$\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ -2 & 1 & 1 & -4 \\ 1 & 1 & 0 & 4 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 1 \\ 1 & 1 & -4 \\ 1 & 0 & 4 \end{vmatrix} - \begin{vmatrix} 0 & -1 & 1 \\ -2 & 1 & -4 \\ 1 & 1 & 4 \end{vmatrix} - 2 \begin{vmatrix} 0 & -1 & 2 \\ -2 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} \\ = -21 + 7 + 14 \\ = 0$$

$\therefore$  Linearly dependent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & -2 \end{pmatrix} \right\} \text{ in } M_{2 \times 2}(\mathbb{R})$$

### Solution

$$W = \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ -2 & 1 & 1 & 2 \\ 1 & 1 & 0 & -2 \end{vmatrix} = 24 \neq 0$$

$\therefore$  Linearly independent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\{e^x, \ln x\} \text{ in } \mathbb{R}$$

### Solution

$$W = \begin{vmatrix} e^x & \ln x \\ e^x & \frac{1}{x} \end{vmatrix}$$

$$\underline{= e^x \left( \frac{1}{x} - \ln x \right) \neq 0}$$

$\therefore$  Linearly independent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ x, \frac{1}{x} \right\} \text{ in } \mathbb{R}$$

### Solution

$$\begin{aligned} W &= \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix} \\ &= -\frac{1}{x} - \frac{1}{x} \\ &= \underline{-\frac{2}{x} \neq 0} \end{aligned}$$

$\therefore$  Linearly independent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\{1+x, 1-x\} \text{ in } P_2(\mathbb{R})$$

### Solution

$$\begin{aligned} W &= \begin{vmatrix} 1+x & 1-x \\ 1 & -1 \end{vmatrix} \\ &= -1 - x - 1 + x \\ &= \underline{-2 \neq 0} \end{aligned}$$

$\therefore$  Linearly independent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\{9x^2 - x + 3, 3x^2 - 6x + 5, -5x^2 + x + 1\} \text{ in } P_3(\mathbb{R})$$

### Solution

$$W = \begin{vmatrix} 9 & -1 & 3 \\ 3 & -6 & 5 \\ -5 & 1 & 1 \end{vmatrix} = \underline{-152 \neq 0}$$

$\therefore$  Linearly independent

$$\begin{aligned}
 W &= \begin{vmatrix} 9x^2 - x + 3 & 3x^2 - 6x + 5 & -5x^2 + x + 1 \\ 18x - 1 & 6x - 6 & -10x + 1 \\ 18 & 6 & -10 \end{vmatrix} \\
 &= (-60x - 60)(9x^2 - x + 3) + (-180x + 18)(3x^2 - 6x + 5) + (108x - 6)(-5x^2 + x + 1) \\
 &\quad - (108x - 108)(-5x^2 + x + 1) - (-60x + 6)(9x^2 - x + 3) - (-180x + 10)(3x^2 - 6x + 5) \\
 &\quad \begin{matrix} x^3 & -540 - 540 - 540 + 540 + 540 + 540 \\ x^2 & -540 + 60 + 54 + 1080 + 30 + 108 + 540 - 108 - 54 - 60 - 30 - 1080 \\ x^1 & 60 - 180 - 900 - 6 + 108 + 108 - 108 + 6 + 180 + 60 + 900 \\ x^0 & -180 + 90 - 6 + 108 - 18 - 50 \end{matrix} \\
 &= \underline{228x - 56 \neq 0}
 \end{aligned}$$

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\{-x^2, 1 + 4x^2\} \text{ in } P_3(\mathbb{R})$$

#### Solution

$$\begin{aligned}
 W &= \begin{vmatrix} -x^2 & 4x^2 + 1 \\ -2x & 8x \end{vmatrix} \\
 &= \underline{2x \neq 0}
 \end{aligned}$$

$\therefore$  Linearly independent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\{7x^2 + x + 2, 2x^2 - x + 3, -3x^2 + 4\} \text{ in } P_3(\mathbb{R})$$

#### Solution



$$W = \begin{vmatrix} 7 & 1 & 2 \\ 2 & -1 & 3 \\ -3 & 0 & 4 \end{vmatrix}$$

$$= -51 \neq 0$$

$\therefore$  Linearly independent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ 3x^2 + 3x + 8, 2x^2 + x, 2x^2 + 2x + 2, 5x^2 - 2x + 8 \right\} \text{ in } P_3(\mathbb{R})$$

### Solution

$$\begin{bmatrix} 3 & 2 & 2 & 5 \\ 3 & 1 & 2 & -2 \\ 8 & 0 & 2 & 8 \end{bmatrix} \quad \begin{array}{l} R_2 - R_1 \\ 3R_3 - 8R_1 \end{array}$$

$$\begin{bmatrix} 3 & 2 & 2 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & -16 & -10 & -16 \end{bmatrix} \quad R_3 + 16R_2$$

$$\begin{bmatrix} 3 & 2 & 2 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & -10 & 96 \end{bmatrix} \quad \begin{array}{l} 3a_1 = -2a_2 - 2a_3 - 5a_4 \\ a_2 = -7a_4 \\ a_3 = \frac{48}{5}a_4 \end{array}$$

$$3a_1 = 14a_4 - \frac{96}{5}a_4 - 5a_4$$

$$a_1 = \frac{3}{5}a_4$$

$\therefore$  Linearly dependent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ x^3 + 2x^2, -x^2 + 3x + 1, x^3 - x^2 + 2x - 1 \right\} \text{ in } P_3(\mathbb{R})$$

### Solution

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & -1 \\ 0 & 3 & 2 \\ 0 & 1 & -1 \end{bmatrix} \quad R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -3 \\ 0 & 3 & 2 \\ 0 & 1 & -1 \end{bmatrix} \quad \begin{array}{l} R_3 + 3R_2 \\ R_4 + R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & -7 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\begin{cases} a_3 = 0 \\ a_1 = -3a_3 = 0 \\ a_1 = -a_3 = 0 \end{cases}$$

$\therefore$  Linearly independent

### **Exercise**

Determine if the given vectors are linearly dependent or independent, (any method)

$$\{x^3 - x, 2x^2 - 4, -2x^3 + 3x^2 + 2x + 6\} \text{ in } P_3(\mathbb{R})$$

### **Solution**

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & 4 & 6 \end{bmatrix} \quad R_3 + R_1$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 4 & 6 \end{bmatrix} \quad R_4 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} 2a_2 = -3a_3 \\ a_1 = 2a_3 \end{cases}$$

$$\text{If } a_3 = 2 \rightarrow a_1 = 4 \quad a_2 = -3$$

$$\rightarrow 4(x^3 - x) - 3(2x^2 + 4) + 2(-2x^3 + 3x^2 + 2x + 6) = 0$$

$\therefore$  Linearly dependent

### Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ \begin{array}{l} x^4 - x^3 + 5x^2 - 8x + 6, \quad -x^4 + x^3 - 5x^2 + 5x - 3, \quad x^4 + 3x^2 - 3x + 5, \\ 2x^4 + 3x^3 + 4x^2 - x + 1, \quad x^3 - x + 2 \end{array} \right\} \text{ in } P_4(\mathbb{R})$$

### Solution

$$\begin{vmatrix} 1 & -1 & 1 & 2 & 0 \\ -1 & 1 & 0 & 3 & 1 \\ 5 & -5 & 3 & 4 & 0 \\ -8 & 5 & -3 & -1 & -1 \\ 6 & -3 & 5 & 1 & 2 \end{vmatrix} = -60 \neq 0$$

$\therefore$  Linearly dependent

### Exercise

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### Solution

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ -1 & 1 & 0 & 1 \\ 5 & -5 & 3 & 4 \\ -8 & 5 & -3 & 8 \\ 6 & -3 & 5 & 0 \end{bmatrix} \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 5R_1 \\ R_4 + 8R_1 \\ R_5 - 6R_1 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -6 \\ 0 & -3 & 5 & 24 \\ 0 & 3 & -1 & -12 \end{bmatrix} \quad R_5 + R_4$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -6 \\ 0 & -3 & 5 & 24 \\ 0 & 0 & 4 & 12 \end{bmatrix} \quad \begin{array}{l} R_3 + 2R_2 \\ R_5 - 4R_2 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 5 & 24 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \rightarrow a_1 = a_2 - a_3 - 2a_4 \\ \rightarrow a_3 = -3a_4 \\ \rightarrow 3a_2 = 5a_3 + 24a_4 \end{array}$$

$$\text{If } a_4 = -1 \rightarrow a_3 = 3 \quad a_2 = -3 \quad a_1 = -4$$

$$\begin{aligned} & -4(x^4 - x^3 + 5x^2 - 8x + 6) - 3(-x^4 + x^3 - 5x^2 + 5x - 3) \\ & + 3(x^4 + 3x^2 - 3x + 5) - (2x^4 + x^3 + 4x^2 + 8x) = 0 \end{aligned}$$

$\therefore$  Linearly dependent

### Exercise

Suppose that the vectors  $\vec{u}_1$ ,  $\vec{u}_2$ , and  $\vec{u}_3$  are linearly dependent. Are the vectors  $\vec{v}_1 = \vec{u}_1 + \vec{u}_2$ ,  $\vec{v}_2 = \vec{u}_1 + \vec{u}_3$ , and  $\vec{v}_3 = \vec{u}_2 + \vec{u}_3$  also linearly dependent?

(**Hint:** Assume that  $a_1 \vec{v}_1 + a_2 \vec{v}_2 + a_3 \vec{v}_3 = \vec{0}$ , and see what the  $a_i$ 's can be.)

### Solution

**Given:**  $\vec{u}_1$ ,  $\vec{u}_2$ , and  $\vec{u}_3$  are linearly dependent, then there are scalar  $b_1$ ,  $b_2$ , and  $b_3$  such that

$$b_1 \vec{u}_1 + b_2 \vec{u}_2 + b_3 \vec{u}_3 = \vec{0}.$$

Assume that  $a_1 \vec{v}_1 + a_2 \vec{v}_2 + a_3 \vec{v}_3 = \vec{0}$

$$a_1 (\vec{u}_1 + \vec{u}_2) + a_2 (\vec{u}_1 + \vec{u}_3) + a_3 (\vec{u}_2 + \vec{u}_3) = \vec{0}$$

$$a_1 \vec{u}_1 + a_1 \vec{u}_2 + a_2 \vec{u}_1 + a_2 \vec{u}_3 + a_3 \vec{u}_2 + a_3 \vec{u}_3 = \vec{0}$$

$$(a_1 + a_2)\vec{u}_1 + (a_1 + a_3)\vec{u}_2 + (a_2 + a_3)\vec{u}_3 = 0$$

If  $a_1 + a_2 = b_1$   $a_1 + a_3 = b_2$   $a_2 + a_3 = b_3$  and since  $\vec{u}_1$ ,  $\vec{u}_2$ , and  $\vec{u}_3$  are linearly dependent, therefore,  $\vec{v}_1$ ,  $\vec{v}_2$ , and  $\vec{v}_3$  are linearly dependent.

### Exercise

Show that the set  $F = \{1+t, t^2, t-2\}$  is a linearly independent subset of  $P_2$ .

### Solution

$$\begin{aligned} W &= \begin{vmatrix} 1+t & t^2 & t-2 \\ 1 & 2t & 1 \\ 0 & 2 & 0 \end{vmatrix} \\ &= 2t - 4 - 2 - 2t \\ &= \underline{-6 \neq 0} \quad \therefore \text{Linearly Independent.} \end{aligned}$$

$$\exists c_1, c_2, c_3 \text{ constants } \ni 0 = c_1(1+t) + c_2 t^2 + c_3(t-2)$$

$$\Rightarrow \begin{cases} t^0 & c_1 - 2c_3 = 0 \\ t & c_1 + c_3 = 0 \\ t^2 & c_2 = 0 \end{cases} \rightarrow \underline{c_1 = c_3 = 0}$$

Since the only solution to this system is the trivial one.  $F$  is Linearly Independent subset of  $P_2$

### Exercise

Suppose that  $A$  is linearly dependent set of vectors and  $B$  is any set containing  $A$ . Show that  $B$  must be linearly dependent.

### Solution

If  $A$  is linearly dependent, then there are vectors  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$  in  $A$  and  $\mathbb{R}, c_1, c_2, \dots, c_n$  with all not  $c_i \neq 0$  and  $c_1 \vec{x}_1 + c_2 \vec{x}_2 + \dots + c_n \vec{x}_n = \vec{0}$

If  $B$  any set that contains  $A$ , then this same relation holds in  $B$  set.  
 $B$  is also dependent.

### Exercise

Show that  $\{\sin t, \sin 2t, \cos t\}$  is a linearly independent, subset of  $C[0, 1]$ . Does it span  $C[0, 1]$

### Solution

$$\begin{aligned} W &= \begin{vmatrix} \sin t & \sin 2t & \cos t \\ \cos t & 2\cos 2t & -\sin t \\ -\sin t & -4\sin 2t & -\cos t \end{vmatrix} \\ &= -2\sin t \cos t \cos 2t + \sin^2 t \sin 2t - 4\cos^2 t \sin 2t + 2\sin t \cos t \cos 2t - 4\sin^2 t \sin 2t + \cos^2 t \sin 2t \\ &= \sin 2t - 4\sin 2t \\ &= -3\sin 2t \neq 0 \end{aligned}$$

$\therefore$  Linearly Independent.

$$a \sin t + b \sin 2t + d \cos t = 0$$

$$\text{If } \begin{cases} t = 0 & \rightarrow d = 0 \\ t = \frac{\pi}{2} & \rightarrow a = 0 \\ t = \frac{\pi}{4} & \rightarrow b = 0 \end{cases}$$

Since all the polynomials are in  $C[0, 1]$  and there is no other way that we can write them as linear combinations of  $\sin t$ ,  $\sin 2t$ , and  $\cos t$ .

The set can't possible span  $C[0, 1]$

### Exercise

Show that the set  $\{\sin(t+a), \sin(t+b), \sin(t+c)\}$  is linearly dependent on  $C[0, 1]$ .

### Solution

$$\begin{aligned} W &= \begin{vmatrix} \sin(t+a) & \sin(t+b) & \sin(t+c) \\ \cos(t+a) & \cos(t+b) & \cos(t+c) \\ -\sin(t+a) & -\sin(t+b) & -\sin(t+c) \end{vmatrix} \\ &= -\sin(t+a)\cos(t+b)\sin(t+c) - \sin(t+a)\cos(t+c)\sin(t+b) - \sin(t+b)\cos(t+a)\sin(t+c) \\ &\quad + \sin(t+a)\cos(t+b)\sin(t+c) + \sin(t+a)\cos(t+c)\sin(t+b) + \sin(t+b)\cos(t+a)\sin(t+c) \\ &= 0 \end{aligned}$$

$\therefore$  The set is linearly dependent on  $C[0, 1]$

$$k_1 \sin(t+a) + k_2 \sin(t+b) + k_3 \sin(t+c) = 0$$

$$\text{If } \begin{cases} t = -a & \rightarrow k_2 + k_3 = 0 \\ t = -b & \rightarrow k_1 + k_3 = 0 \\ t = -c & \rightarrow k_1 + k_2 = 0 \end{cases}$$

$$t = 0 \rightarrow k_1 \sin a + k_2 \sin b + k_3 \sin c = 0$$

$$t = \frac{\pi}{2} \rightarrow k_1 \cos a + k_2 \cos b + k_3 \cos c = 0$$

$$t = \pi \rightarrow -(k_1 \sin a + k_2 \sin b + k_3 \sin c) = 0$$

### ***Exercise***

Show that if  $\alpha_1, \alpha_2, \dots, \alpha_n$  are linearly independent and  $\alpha_1, \alpha_2, \dots, \alpha_n, \beta$  are linearly dependent, then  $\beta$  can be uniquely expressed as a linear combination of  $\alpha_1, \alpha_2, \dots, \alpha_n$

### **Solution**

Since,  $\alpha_1, \alpha_2, \dots, \alpha_n$  are linearly independent, then

$$a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_n \alpha_n = 0 \text{ when all } a_i = 0.$$

Let assume that:

$$a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_n \alpha_n = b\beta$$

$$a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_n \alpha_n - b\beta = 0$$

If  $b = 0$ , then  $a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_n \alpha_n = 0$  and  $\beta$  doesn't exist.

If  $b \neq 0$ , and  $\alpha_1, \alpha_2, \dots, \alpha_n, \beta$  are linearly dependent, then

$$\begin{aligned} \beta &= \frac{a_1}{b} \alpha_1 + \frac{a_2}{b} \alpha_2 + \dots + \frac{a_n}{b} \alpha_n \\ &= c_1 \alpha_1 + c_2 \alpha_2 + \dots + c_n \alpha_n \end{aligned}$$

$$\text{If } \frac{a_1}{b} = c_1 \Rightarrow \frac{a_1}{b} - c_1 = 0$$

$$\left( \frac{a_1}{b} - c_1 \right) \alpha_1 + \left( \frac{a_2}{b} - c_2 \right) \alpha_2 + \dots + \left( \frac{a_n}{b} - c_n \right) \alpha_n = 0$$

Then  $\frac{a_i}{b} - c_i = 0$  ( $1 \leq i \leq n$ ) since  $\alpha_1, \alpha_2, \dots, \alpha_n$  are linearly independent and contradict that  $\alpha_1, \alpha_2, \dots, \alpha_n, \beta$  are linearly dependent.

Therefore,  $\beta$  can be uniquely expressed as a linear combination of  $\alpha_1, \alpha_2, \dots, \alpha_n$  in the form

$$\beta = c_1 \alpha_1 + c_2 \alpha_2 + \dots + c_n \alpha_n$$

### ***Exercise***

Show that if  $\alpha_1, \alpha_2, \dots, \alpha_n$  are linearly dependent with  $(\alpha_1 \neq 0)$  if and only if there exists an integer  $k$  ( $1 < k \leq n$ ), such that  $\alpha_k$  is a linear combination of  $\alpha_1, \alpha_2, \dots, \alpha_{k-1}$

### **Solution**

Since,  $\alpha_1, \alpha_2, \dots, \alpha_n$  are linearly dependent, then if

$$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0$$

then there exists an  $\alpha_k \neq 0$  ( $1 < k \leq n$ )

If we let  $i > k$  where  $a_i = 0$ , then  $a_1\alpha_1 + a_2\alpha_2 + \dots + a_{k-1}\alpha_{k-1} - a_k\alpha_k = 0$

$$a_k\alpha_k = a_1\alpha_1 + a_2\alpha_2 + \dots + a_{k-1}\alpha_{k-1}$$