Solution   *Section* 3.1 – Definition of the Laplace Transform

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t)=3$

Solution

$$F(s)=\int_0^\infty 3e^{-st}dt$$

$$=\lim_{T\to\infty}\int_0^T 3e^{-st}dt$$

$$=\lim_{T\to\infty}\left(-\frac{3e^{-st}}{s}\right)_{t=0}^{T}$$

$$=\lim_{T\to\infty}\left(-\frac{3}{s}e^{-sT}+\frac{3}{s}\right) \qquad \lim_{T\to\infty}\left(e^{-sT}\right)=0$$

$$\boxed{=\frac{3}{s}}$$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t)=t$

Solution

$$F(s)=\int_0^\infty te^{-st}dt$$

		$\int e^{-st}dt$
+	t	$-\frac{1}{s}e^{-st}$
–	1	$\frac{1}{s^2}e^{-st}$

$$=\lim_{T\to\infty}\left(\left(-\frac{t}{s}-\frac{1}{s^2}\right)e^{-st}\right)_{t=0}^{T}$$

$$=\lim_{T\to\infty}\left(\left(-\frac{T}{s}-\frac{1}{s^2}\right)e^{-sT}+\frac{1}{s^2}\right) \qquad \lim_{T\to\infty}\left(e^{-sT}\right)=0$$

$$\boxed{=\frac{1}{s^2}}$$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t)=t^2$

Solution

		$\int e^{-st}dt$
+	t^2	$-\frac{1}{s}e^{-st}$
−	$2t$	$\frac{1}{s^2}e^{-st}$
+	2	$-\frac{1}{s^3}e^{-st}$

$$F(s)=\int_0^\infty t^2e^{-st}dt$$

$$=\left(-\frac{t^2}{s}-\frac{2t}{s^2}-\frac{2}{s^3}\right)e^{-st}\Big|_0^\infty$$

$$=\frac{2}{s^3}$$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t)=e^{6t}$

Solution

$$F(s)=\int_0^\infty e^{6t}e^{-st}dt$$

$$=\int_0^\infty e^{-(s-6)t}dt$$

$$=-\frac{e^{-(s-6)t}}{s-6}\Big|_0^\infty$$

$$=\frac{1}{s-6}\qquad with: s>6$$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t)=e^{-2t}$

Solution

$$F(s)=\int_0^\infty e^{-2t}e^{-st}dt$$

$$=\lim_{T\to\infty}\int_0^T e^{-(s+2)t}dt$$

$$=\lim_{T\to\infty}\left(\frac{-e^{-(s+2)t}}{s+2}\right)_{t=0}^T$$

$$= \lim_{T\to\infty}\left(-\frac{e^{-(s+2)T}}{s+2}+\frac{1}{s+2}\right) \qquad \lim_{T\to\infty}\left(e^{-(s+2)T}\right)=0$$

$$\boxed{=\frac{1}{s+2}} \qquad with: s>-2$$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t)=te^{-3t}$

Solution

$$F(s)=\int_0^\infty te^{-3t}e^{-st}dt$$

$$=\int_0^\infty te^{-(s+3)t}dt$$

		$\int e^{-(s+3)t}dt$
+	t	$-\frac{1}{s+3}e^{-(s+3)t}$
–	1	$\frac{1}{(s+3)^2}e^{-(s+3)t}$

$$F(s)=\left(-\frac{1}{s+3}te^{-(s+3)t}-\frac{1}{(s+3)^2}e^{-(s+3)t}\right)_0^\infty$$

$$\boxed{=\frac{1}{(s+3)^2}} \qquad with \;\; s>-3 \qquad e^{-\infty}=0 \quad e^0=1$$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t)=te^{3t}$

Solution

$$F(s)=\int_0^\infty te^{3t}e^{-st}dt$$

$$=\int_0^\infty te^{-(s-3)t}dt$$

		$\int e^{-(s-3)t}dt$
+	t	$-\frac{1}{s-3}e^{-(s-3)t}$
–	1	$\frac{1}{(s-3)^2}e^{-(s-3)t}$

$$F(s)=-\frac{1}{s-3}te^{-(s-3)t}-\frac{1}{(s-3)^2}e^{-(s-3)t}\Big|_0^\infty$$

$$\boxed{=\frac{1}{(s-3)^2}} \qquad with \;\; s>3 \qquad e^{-\infty}=0 \quad e^0=1$$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t)=e^{2t}\cos 3t$

Solution

$$F(s)=\int_0^{\infty}\left(e^{2t}\cos 3t\right)e^{-st}dt$$

$$=\int_0^{\infty}e^{-(s-2)t}\cos 3t\ dt$$

		$\int \cos 3t\ dt$
+	$e^{-(s-2)t}$	$\frac{1}{3}\sin 3t$
−	$-(s-2)e^{-(s-2)t}$	$-\frac{1}{9}\cos 3t$
+	$(s-2)^2e^{-(s-2)t}$	$-\frac{1}{9}\int \cos 3t$

$$\int e^{-(s-2)t}\cos 3t\ dt=\frac{1}{3}e^{-(s-2)t}\sin 3t-\frac{1}{9}(s-2)e^{-(s-2)t}\cos 3t-\frac{1}{9}(s-2)^2\int e^{-(s-2)t}\cos 3t\ dt$$

$$\left(1+\frac{1}{9}(s-2)^2\right)\int e^{-(s-2)t}\cos 3t\ dt=\frac{1}{3}e^{-(s-2)t}\sin 3t-\frac{1}{9}(s-2)e^{-(s-2)t}\cos 3t$$

$$\left(9+(s-2)^2\right)\int e^{-(s-2)t}\cos 3t\ dt=3e^{-(s-2)t}\sin 3t-(s-2)e^{-(s-2)t}\cos 3t$$

$$\int e^{-(s-2)t}\cos 3t\ dt=\frac{1}{9+(s-2)^2}\left[3e^{-(s-2)t}\sin 3t-(s-2)e^{-(s-2)t}\cos 3t\right]$$

$$F(s)=\left(\frac{3}{9+(s-2)^2}e^{-(s-2)t}\sin 3t-\frac{s-2}{9+(s-2)^2}e^{-(s-2)t}\cos 3t\right)_0^{\infty}$$

$$=\boxed{\frac{s-2}{9+(s-2)^2}}\qquad s>2$$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t)=\sin 3t$

Solution

$$F(s)=\int_0^{\infty}(\sin 3t)e^{-st}dt$$

		$\int \sin 3t\ dt$
+	e^{-st}	$-\frac{1}{3}\cos 3t$
−	$-se^{-st}$	$-\frac{1}{9}\sin 3t$
+	s^2e^{-st}	$-\frac{1}{9}\int \sin 3t$

$$\int \sin 3t\ e^{-st}dt=-\frac{1}{3}e^{-st}\cos 3t-\frac{s}{9}e^{-st}\sin 3t+\frac{s^2}{9}\int e^{-st}\sin 3t\ dt$$

$$\int \sin 3t\ e^{-st}dt+\frac{1}{9}s^2\int \sin 3t\ e^{-st}dt=-\frac{1}{3}e^{-st}\cos 3t-\frac{1}{9}se^{-st}\sin 3t$$

$$\left(9+s^2\right)\int \sin 3t\ e^{-st}dt=-(3\cos 3t-s\sin 3t)e^{-st}$$

$$\int \sin 3t\ e^{-st}dt=-\frac{3\cos 3t-s\sin 3t}{s^2+9}e^{-st}$$

$$F(s) = -\frac{3\cos 3t - s\sin 3t}{s^2+9}e^{-st}\Big|_0^\infty$$

$$= -0 + \frac{3\cos 3(0) - s\sin 3(0)}{s^2+9}e^{-s(0)}$$

$$\boxed{= \frac{3}{s^2+9}} \quad s > 0$$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t) = \sin 2t$

Solution

$$F(s) = \int_0^\infty (\sin 2t)e^{-st}dt$$

$$\int \sin 2t\ e^{-st}dt = -\frac{1}{2}e^{-st}\cos 2t - \frac{s}{4}e^{-st}\sin 2t + \frac{s^2}{4}\int e^{-st}\sin 2t\ dt$$

$$\left(4+s^2\right)\int \sin 2t\ e^{-st}dt = -(2\cos 2t - s\sin 2t)e^{-st}$$

$$\int \sin 2t\ e^{-st}dt = -\frac{2\cos 2t - s\sin 2t}{s^2+4}e^{-st}$$

		$\int \sin 2t\ dt$
+	e^{-st}	$-\frac{1}{2}\cos 2t$
−	$-se^{-st}$	$-\frac{1}{4}\sin 2t$
+	s^2e^{-st}	

$$F(s) = -\frac{2\cos 2t - s\sin 2t}{s^2+4}e^{-st}\Big|_0^\infty$$

$$= -0 + \frac{2\cos 2(0) - s\sin 2(0)}{s^2+4}e^{-s(0)}$$

$$\boxed{= \frac{2}{s^2+4}}$$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t) = \cos 2t$

Solution

$$F(s) = \int_0^\infty (\cos 2t)e^{-st}dt$$

		$\int \cos 2t\ dt$
+	e^{-st}	$\frac{1}{2}\sin 2t$
−	$-se^{-st}$	$-\frac{1}{4}\cos 2t$
+	s^2e^{-st}	

$$\int \cos 2t\ e^{-st}dt = \frac{1}{2}e^{-st}\sin 2t - \frac{s}{4}e^{-st}\cos 2t - \frac{s^2}{4}\int e^{-st}\cos 2t\ dt$$

$$\left(4+s^2\right)\int \cos 2t\ e^{-st}dt = (2\sin 2t - s\cos 2t)e^{-st}$$

$$\int \cos 2t\; e^{-st} dt = \frac{2\sin 2t - s\cos 2t}{s^2+4} e^{-st}$$

$$F(s) = \frac{2\sin 2t - s\cos 2t}{s^2+4} e^{-st} \Big|_0^\infty$$

$$= \boxed{\frac{s}{s^2+4}}$$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t) = \cos bt$

Solution

$$F(s) = \int_0^\infty (\cos bt) e^{-st} dt$$

$$\int \cos bt\; e^{-st} dt = \frac{1}{b} e^{-st} \sin bt - \frac{s}{b^2} e^{-st} \cos bt - \frac{s^2}{b^2} \int e^{-st} \cos bt\; dt$$

$$\left(b^2+s^2\right) \int \cos bt\; e^{-st} dt = (b\sin bt - s\cos bt) e^{-st}$$

		$\int \cos bt\; dt$
+	e^{-st}	$\frac{1}{b}\sin bt$
−	$-se^{-st}$	$-\frac{1}{b^2}\cos bt$
+	$s^2 e^{-st}$	

$$\int \cos bt\; e^{-st} dt = \frac{b\sin bt - s\cos bt}{s^2+b^2} e^{-st}$$

$$F(s) = \frac{b\sin bt - s\cos bt}{s^2+b^2} e^{-st} \Big|_0^\infty$$

$$= \boxed{\frac{s}{s^2+b^2}}$$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t) = e^{t+7}$

Solution

$$F(s) = \int_0^\infty e^{t+7} e^{-st} dt$$

$$= \int_0^\infty e^7 e^{-(s-1)t} dt$$

$$= -\frac{e^7}{s-1} e^{-(s-1)t} \Big|_0^\infty \qquad e^{-\infty} = 0 \quad e^0 = 1$$

$$=\boxed{\frac{e^{7}}{s-1}}$$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t)=e^{-2t-5}$

Solution

$$F(s)=\int_0^\infty e^{-2t-5}e^{-st}dt$$

$$=e^{-5}\int_0^\infty e^{-(s+2)t}dt$$

$$=-\frac{1}{e^5}\cdot\frac{1}{s+2}\left(e^{-(s+2)t}\right)_0^\infty \qquad e^{-\infty}=0 \quad e^{0}=1$$

$$=\boxed{\frac{1}{e^5}\cdot\frac{1}{s+2}}$$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t)=te^{4t}$

Solution

$$F(s)=\int_0^\infty te^{4t}e^{-st}dt$$

$$=\int_0^\infty te^{-(s-4)t}dt$$

$$=\left(-\frac{t}{s-4}-\frac{1}{(s-4)^2}\right)e^{-(s-4)t}\Big|_0^\infty$$

$$=\boxed{\frac{1}{(s-4)^2}}$$

	$\int e^{-(s-4)t}dt$
t	$-\frac{1}{s-4}e^{-(s-4)t}$
1	$\frac{1}{(s-4)^2}e^{-(s-4)t}$

Exercise

Use Definition of Laplace transform to find the Laplace transform of

$$f(t)=t^2e^{-2t}$$

Solution

$$F(s)=\int_0^\infty t^2e^{-2t}e^{-st}dt$$

$$= \int_0^\infty t^2 e^{-(s+2)t} dt$$

$$= \left(-\frac{t^2}{s+2} - \frac{2t}{(s+2)^2} - \frac{2}{(s+2)^3} \right) e^{-(s+2)t} \Big|_0^\infty$$

$$= \frac{2}{(s+2)^3}$$

	$\int e^{-(s+2)t} dt$
t^2	$-\frac{1}{s+2} e^{-(s+2)t}$
$2t$	$\frac{1}{(s+2)^2} e^{-(s+2)t}$
2	$-\frac{1}{(s+2)^3} e^{-(s+2)t}$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t) = e^{-t} \sin t$

Solution

$$F(s) = \int_0^\infty e^{-t} \sin t \ e^{-st} dt$$

$$= \int_0^\infty \sin t \ e^{-(s+1)t} dt$$

$$\int \sin t \ e^{-(s+1)t} dt = \left(-\cos t - (s+1) \sin t \right) e^{-(s+1)t} - (s+1)^2 \int \sin t \ e^{-(s+1)t} dt$$

$$\left((s+1)^2 + 1 \right) \int \sin t \ e^{-(s+1)t} dt = \left(-\cos t - (s+1) \sin t \right) e^{-(s+1)t}$$

$$\int_0^\infty \sin t \ e^{-(s+1)t} dt = -\frac{\cos t + (s+1)\sin t}{(s+1)^2 + 1} e^{-(s+1)t}$$

$$F(s) = -\frac{\cos t + (s+1)\sin t}{(s+1)^2 + 1} e^{-(s+1)t} \Big|_0^\infty$$

$$= \frac{1}{(s+1)^2 + 1}$$

	$\int \sin t \ dt$
$e^{-(s+1)t}$	$-\cos t$
$-(s+1)e^{-(s+1)t}$	$-\sin t$
$(s+1)^2 e^{-(s+1)t}$	$-\int \sin t \ dt$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t) = e^{2t} \cos 3t$

Solution

$$F(s) = \int_0^\infty e^{2t} \cos 3t \ e^{-st} dt$$

$$= \int_0^\infty \cos 3t \ e^{-(s-2)t} dt$$

$$\int \cos 3t\ e^{-(s-2)t}dt = \left(\tfrac{1}{3}\sin 3t + \tfrac{1}{9}(s-2)\cos 3t\right)e^{-(s-2)t} - \tfrac{1}{9}(s-2)^2 \int \cos 3t\ e^{-(s-2)t}dt$$

$$\left((s-2)^2+9\right)\int \sin t\ e^{-(s-2)t}dt = \left(3\sin 3t + (s-2)\cos 3t\right)e^{-(s-2)t}$$

$$\int_0^\infty \cos 3t\ e^{-(s-2)t}dt = \frac{3\sin 3t + (s-2)\cos 3t}{(s-2)^2+9}e^{-(s-2)t}$$

$$F(s) = \frac{3\sin 3t + (s-2)\cos 3t}{(s-2)^2+9}e^{-(s-2)t}\Big|_0^\infty$$

$$= \frac{s-2}{(s-2)^2+9}$$

		$\int \cos 3t\ dt$
+	$e^{-(s-2)t}$	$\frac{1}{3}\sin 3t$
−	$-(s-2)e^{-(s-2)t}$	$-\frac{1}{9}\cos 3t$
+	$(s-2)^2 e^{-(s-2)t}$	

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t) = e^{-t}\sin 2t$

Solution

$$F(s) = \int_0^\infty e^{-t}\sin 2t\ e^{-st}dt$$

$$= \int_0^\infty \sin 2t\ e^{-(s+1)t}dt$$

$$\int \sin 2t\ e^{-(s+1)t}dt = \left(-\tfrac{1}{2}\cos 2t - \tfrac{1}{4}(s+1)\sin 2t\right)e^{-(s+1)t} - \tfrac{1}{4}(s+1)^2\int \sin 2t\ e^{-(s+1)t}dt$$

$$\left((s+1)^2+4\right)\int \sin 2t\ e^{-(s+1)t}dt = -\left(2\cos 2t + (s+1)\sin 2t\right)e^{-(s+1)t}$$

$$\int_0^\infty \sin 2t\ e^{-(s+1)t}dt = -\frac{2\cos 2t + (s+1)\sin 2t}{(s+1)^2+4}e^{-(s+1)t}$$

$$F(s) = -\frac{2\cos 2t + (s+1)\sin 2t}{(s+1)^2+4}e^{-(s+1)t}\Big|_0^\infty$$

$$= \frac{2}{(s+1)^2+4}$$

		$\int \sin 2t\ dt$
+	$e^{-(s+1)t}$	$-\frac{1}{2}\cos t$
−	$-(s+1)e^{-(s+1)t}$	$-\frac{1}{4}\sin t$
+	$(s+1)^2 e^{-(s+1)t}$	

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t) = t\sin t$

Solution

$$F(s)=\int_0^\infty t\sin t\ e^{-st}dt$$

$$\int t\sin t\ e^{-st}dt=\left(-t\cos t+(1-st)\sin t\right)e^{-st}-s^2\int t\sin t\ e^{-st}dt+2s\int \sin t\ e^{-st}dt$$

$$\int \sin t\ e^{-st}dt=\left(-\cos t-s\sin t\right)e^{-st}-s^2\int \sin t\ e^{-st}dt$$

$$\left(s^2+1\right)\int \sin t\ e^{-st}dt=\left(-\cos t-s\sin t\right)e^{-st}$$

$$\int \sin t\ e^{-st}dt=-\frac{\cos t+s\sin t}{s^2+1}e^{-st}$$

	$\int \sin t\ dt$
te^{-st}	$-\cos t$
$(1-st)e^{-st}$	$-\sin t$
$\left(s^2t-2s\right)e^{-st}$	$-\int \sin t\ dt$

$$\left(s^2+1\right)\int t\sin t\ e^{-st}dt=\left(-t\cos t+(1-st)\sin t\right)e^{-st}-\frac{2s}{s^2+1}\left(\cos t+s\sin t\right)e^{-st}$$

$$\int t\sin t\ e^{-st}dt=\frac{1}{s^2+1}\left(-t\cos t+(1-st)\sin t\right)e^{-st}-\frac{2s}{\left(s^2+1\right)^2}\left(\cos t+s\sin t\right)e^{-st}$$

$$F(s)=\left(\frac{(1-st)\sin t-t\cos t}{s^2+1}-\frac{2s\left(\cos t+s\sin t\right)}{\left(s^2+1\right)^2}\right)e^{-st}\Big|_0^\infty$$

	$\int \sin t\ dt$
e^{-st}	$-\cos t$
$-se^{-st}$	$-\sin t$
s^2e^{-st}	$-\int \sin t\ dt$

$$=\frac{2s}{\left(s^2+1\right)^2}$$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t)=t\cos t$

Solution

$$F(s)=\int_0^\infty t\cos t\ e^{-st}dt$$

$$\int t\cos t\ e^{-st}dt=\left(t\sin t-(1-st)\cos t\right)e^{-st}-s^2\int t\cos t\ e^{-st}dt+2s\int \cos t\ e^{-st}dt$$

$$\int \cos t\ e^{-st}dt=\left(\sin t+s\cos t\right)e^{-st}-s^2\int \cos t\ e^{-st}dt$$

$$\left(s^2+1\right)\int \cos t\ e^{-st}dt=\left(\sin t+s\cos t\right)e^{-st}$$

$$\int \cos t\ e^{-st}dt=\frac{\sin t+s\cos t}{s^2+1}e^{-st}$$

	$\int \cos t\ dt$
te^{-st}	$\sin t$
$(1-st)e^{-st}$	$-\cos t$
$\left(s^2t-2s\right)e^{-st}$	

$$\left(s^2+1\right)\int t\cos t\, e^{-st}dt = \left(t\sin t-(1-st)\cos t\right)e^{-st} + \frac{2s(\sin t+s\cos t)}{s^2+1}e^{-st}$$

$$\int t\cos t\, e^{-st}dt = \frac{1}{s^2+1}\left(t\sin t-(1-st)\cos t\right)e^{-st} + \frac{2s(\sin t+s\cos t)}{\left(s^2+1\right)^2}e^{-st}$$

$$F(s)=\left(\frac{t\sin t-(1-st)\cos t}{s^2+1}+\frac{2s(\sin t+s\cos t)}{\left(s^2+1\right)^2}\right)e^{-st}\Bigg|_0^\infty$$

$$=\frac{1}{s^2+1}+\frac{2s^2}{\left(s^2+1\right)^2}$$

$$=\frac{-s^2-1+2s^2}{\left(s^2+1\right)^2}$$

$$=\frac{s^2-1}{\left(s^2+1\right)^2}$$

	$\int \cos t\; dt$
e^{-st}	$\sin t$
$-se^{-st}$	$-\cos t$
s^2e^{-st}	

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t)=2t^4$

Solution

$$F(s)=\int_0^\infty 2t^4\, e^{-st}dt$$

$$=2\left(-\frac{t^4}{s}-\frac{4t^3}{s^2}-\frac{12t^2}{s^3}-\frac{24t}{s^4}-\frac{24}{s^5}\right)e^{-st}\Bigg|_0^\infty$$

$$=2\left(0+\frac{24}{s^5}\right)$$

$$=\frac{48}{s^5}$$

		$\int e^{-st}dt$
+	t^4	$-\frac{1}{s}e^{-st}$
−	$4t^3$	$\frac{1}{s^2}e^{-st}$
+	$12t^2$	$-\frac{1}{s^3}e^{-st}$
−	$24t$	$\frac{1}{s^4}e^{-st}$
+	24	$-\frac{1}{s^5}e^{-st}$

Exercise

Use Definition of Laplace Transform to show the Laplace transform of $f(t)=\cos\omega t$ is $F(s)=\dfrac{s}{s^2+\omega^2}$

Solution

$$F(s)=\int_0^{\infty}(\cos\omega t)e^{-st}dt$$

$$\int \cos\omega t\ e^{-st}dt=\frac{1}{\omega}e^{-st}\sin\omega t-\frac{s}{\omega^2}e^{-st}\cos\omega t+\frac{s^2}{\omega^2}\int e^{-st}\cos\omega t\ dt$$

		$\int \cos\omega t\ dt$
+	e^{-st}	$\frac{1}{\omega}\sin\omega t$
−	$-se^{-st}$	$-\frac{1}{\omega^2}\cos\omega t$
+	s^2e^{-st}	$-\frac{1}{\omega^2}\int\cos\omega t$

$$\left(1-\frac{s^2}{\omega^2}\right)\int e^{-st}\cos\omega t\ dt=\frac{1}{\omega}e^{-st}\sin\omega t-\frac{s}{\omega^2}e^{-st}\cos\omega t$$

$$\left(\frac{\omega^2-s^2}{\omega^2}\right)\int e^{-st}\cos\omega t\ dt=\frac{1}{\omega}\left(\sin\omega t-\frac{s}{\omega}\cos\omega t\right)e^{-st}$$

$$\int e^{-st}\cos\omega t\ dt=\frac{\omega^2}{\omega^2-s^2}\frac{1}{\omega^2}(\omega\sin\omega t-s\cos\omega t)e^{-st}$$

$$=\frac{e^{-st}}{\omega^2-s^2}(\omega\sin\omega t-s\cos\omega t)$$

$$F(s)=\lim_{T\to\infty}\frac{e^{-st}}{\omega^2-s^2}(\omega\sin\omega t-s\cos\omega t)\Big|_0^T$$

$$=\lim_{T\to\infty}\left[\frac{e^{-sT}}{\omega^2-s^2}(\omega\sin\omega T-s\cos\omega T)-\frac{1}{\omega^2-s^2}(\omega\sin 0-s\cos 0)\right]$$

$$=0-\frac{1}{\omega^2-s^2}(-s)$$

$$\lim_{T\to\infty}e^{-sT}=\lim_{T\to\infty}\frac{1}{e^{-sT}}=0$$

$$=\frac{s}{s^2+\omega^2}\qquad s>0$$

Solution Section 3.2 – Basic Properties of the Laplace Transform

Exercise

Find the Laplace transform and defined the time domain of $y(t) = t^2 + 4t + 5$

Solution

$$\mathcal{L}\left(t^2 + 4t + 5\right)(s) = \mathcal{L}\left(t^2\right)(s) + 4\,\mathcal{L}(4t)(s) + 5\,\mathcal{L}(1)(s)$$

$$= \frac{2!}{s^3} + 4\frac{1}{s^2} + 5\frac{1}{s}$$

$$= \frac{2}{s^3} + \frac{4}{s^2} + \frac{5}{s}$$

$$= \frac{2 + 4s + 5s^2}{s^3} \qquad s > 0$$

Exercise

Find the Laplace transform and defined the time domain of $y(t) = -2\cos t + 4\sin 3t$

Solution

$$\mathcal{L}(-2\cos t + 4\sin 3t)(s) = -2\,\mathcal{L}(\cos t)(s) + 4\,\mathcal{L}(\sin 3t)(s)$$

$$= -2\frac{s}{s^2+1} + 4\frac{3}{s^2+9}$$

$$= \frac{-2s\left(s^2+9\right) + 12\left(s^2+1\right)}{\left(s^2+1\right)\left(s^2+9\right)}$$

$$= \frac{-2s^3 - 18s + 12s^2 + 12}{\left(s^2+1\right)\left(s^2+9\right)}$$

$$= \frac{-2s^3 + 12s^2 - 18s + 12}{\left(s^2+1\right)\left(s^2+9\right)} \qquad s > 0$$

Exercise

Find the Laplace transform and defined the time domain of $y(t) = 2\sin 3t + 3\cos 5t$

Solution

$$\mathcal{L}(2\sin 3t + 3\cos 5t)(s) = 2\,\mathcal{L}(\sin 3t)(s) + 3\,\mathcal{L}(\cos 5t)(s)$$

$$= 2\frac{3}{s^2+9} + 3\frac{s}{s^2+25}$$

$$= \frac{6s^2 + 150 + 3s^3 + 27s}{\left(s^2+9\right)\left(s^2+25\right)}$$

$$\boxed{= \frac{3s^3 + 6s^2 + 27s + 150}{\left(s^2+9\right)\left(s^2+25\right)}} \quad (s > 0)$$

Exercise

Find the Laplace transform and defined the time domain of $f(t) = 2t^4$

Solution

$$F(s) = \mathcal{L}\left(2t^4\right)(s)$$

$$= 2\mathcal{L}\left(t^4\right)(s)$$

$$= 2\frac{4!}{s^5}$$

$$\boxed{= \frac{48}{s^5}} \qquad s > 0$$

Exercise

Find the Laplace transform and defined the time domain of $f(t) = t^5$

Solution

$$\mathcal{L}\left(t^5\right)(s) = \frac{5!}{s^6}$$

$$\boxed{= \frac{120}{s^6}}$$

Exercise

Find the Laplace transform of $f(t) = 4t - 10$

Solution

$$F(s) = \mathcal{L}\{4t - 10\}$$

$$= \frac{4}{s^2} - \frac{10}{s}$$

$$\boxed{= \frac{4 - 10s}{s^2}}$$

Exercise

Find the Laplace transform of $f(t)=7t+3$

Solution

$$F(s)=\mathcal{L}\{7t+3\}$$

$$=\frac{7}{s^2}+\frac{3}{s}$$

$$\boxed{=\frac{7+3s}{s^2}}$$

Exercise

Find the Laplace transform of $f(t)=3t^4-2t^2+1$

Solution

$$F(s)=\mathcal{L}\left\{3t^4-2t^2+1\right\}$$

$$=\frac{72}{s^5}-\frac{4}{s^3}+\frac{1}{s}$$

$$\boxed{=\frac{s^4-4s^2+72}{s^5}}$$

Exercise

Find the Laplace transform of $f(t)=(t+1)^3$

Solution

$$\mathcal{L}\{f(t)\}=\mathcal{L}\left\{t^3+3t^2+3t+1\right\}$$

$$F(s)=\frac{6}{s^4}+\frac{6}{s^3}+\frac{3}{s^2}+\frac{1}{s}$$

$$\boxed{=\frac{s^3+3s^2+6s+6}{s^4}}$$

Exercise

Find the Laplace transform of $f(t)=(2t-1)^3$

Solution

$$F(s)=\mathcal{L}\left\{8t^3-12t^2+6t-1\right\}$$

$$=\frac{48}{s^4}-\frac{24}{s^3}+\frac{6}{s^2}-\frac{1}{s}$$

$$= \boxed{\frac{48-24s+6s^2-s^3}{s^4}}$$

Exercise

Find the Laplace transform of $f(t)=(t-1)^4$

Solution

$$F(s)=\mathcal{L}\left\{t^4-4t^3+6t^2-4t+1\right\}(s)$$

$$=\frac{4!}{s^5}-\frac{24}{s^4}+\frac{12}{s^3}-\frac{4}{s^2}+\frac{1}{s}$$

$$= \boxed{\frac{s^4-4s^3+12s^2-24s+24}{s^5}}$$

Exercise

Find the Laplace transform of $f(t)=t^2+6t-3$

Solution

$$\mathcal{L}\{f(t)\}=\mathcal{L}\left\{t^2+6t-3\right\}$$

$$F(s)=\frac{2}{s^3}+\frac{6}{s^2}-\frac{3}{s}$$

$$= \boxed{\frac{2s^2+6s-3}{s^3}}$$

Exercise

Find the Laplace transform of $f(t)=-4t^2+16t+9$

Solution

$$\mathcal{L}\{f(t)\}=\mathcal{L}\left\{-4t^2+16t+9\right\}$$

$$F(s)=-\frac{8}{s^3}+\frac{16}{s^2}+\frac{9}{s}$$

$$= \boxed{\frac{9s^2+16s-8}{s^3}}$$

Exercise

Find the Laplace transform of $f(t)=3t^2-e^{2t}$

Solution

$$F(s)=\mathcal{L}\left\{3t^2-e^{2t}\right\}(s)$$
$$=\frac{6}{s^3}-\frac{1}{s-2}$$
$$\boxed{=\frac{-s^3+6s-12}{s^3(s-2)}}$$

Exercise

Find the Laplace transform of $f(t)=t^2-e^{-9t}+9$

Solution

$$F(s)=\mathcal{L}\left\{t^2-e^{-9t}+9\right\}(s)$$
$$=\frac{2}{s^3}-\frac{1}{s+9}+\frac{9}{s}$$
$$\boxed{=\frac{-s^3+9s^2+2s+18}{s^3(s+9)}}$$

Exercise

Find the Laplace transform of $f(t)=6e^{-3t}-t^2+2t-8$

Solution

$$F(s)=\mathcal{L}\left\{6e^{-3t}-t^2+2t-8\right\}(s)$$
$$=\frac{6}{s+3}-\frac{1}{s^3}+\frac{2}{s^2}-\frac{8}{s}$$
$$=\frac{6s^3-s-3+2s^2+2s-8s^3-24s^2}{s^3(s+3)}$$
$$\boxed{=\frac{-2s^3-22s^2+s-3}{s^3(s+3)}}$$

Exercise

Find the Laplace transform of $f(t)=5-e^{2t}+6t^2$

Solution

$$F(s) = \mathcal{L}\left\{5 - e^{2t} + 6t^2\right\}(s)$$
$$= \frac{5}{s} - \frac{1}{s-2} + \frac{12}{s^3}$$
$$= \boxed{\frac{5s^2 - s^3 + 12s - 24}{s^3(s-2)}}$$

Exercise

Find the Laplace transform of $f(t) = t^2 e^{2t}$

Solution

$$f(t) = e^{2t} \xrightarrow{\mathcal{L}} F(s) = \frac{1}{s-2}$$

$$\mathcal{L}\left\{t^2 e^{2t}\right\}(s) = (-1)^2 Y''(s)$$ ***Using Derivative of a Laplace Transform Proposition***

$$= \frac{d}{ds}\left(\frac{-1}{(s-2)^2}\right)$$
$$= -\frac{(-1)2(s-2)}{(s-2)^4}$$
$$= \boxed{\frac{2}{(s-2)^3}}$$ ***OR Using Laplace Transform table***

Exercise

Find the Laplace transform of $f(t) = e^{-2t}(2t+3)$

Solution

$$f(t) = 2t + 3 \xrightarrow{\mathcal{L}} F(s) = 2\frac{1}{s^2} + 3\frac{1}{s}$$
$$= \frac{2+3s}{s^2}$$

$$\mathcal{L}\left\{e^{-2t}(2t+3)\right\} = Y(s+2)$$
$$= \frac{2+3(s+2)}{(s+2)^2}$$
$$= \boxed{\frac{3s+8}{(s+2)^2}}$$

Exercise

Find the Laplace transform of $f(t) = e^{-t}\left(t^2 + 3t + 4\right)$

Solution

$$y(t) = t^2 e^{-t} + 3te^{-t} + 4e^{-t}$$

$$Y(s) = \mathcal{L}\left(t^2 e^{-t}\right)(s) + 3\mathcal{L}\left(te^{-t}\right)(s) + 4\mathcal{L}\left(e^{-t}\right)(s) \qquad \mathcal{L}\left(t^n e^{-at}\right)(s) = \frac{n!}{(s+a))^{n+1}}$$

$$= \frac{2!}{(s+1)^3} + 3\frac{1}{(s+1)^2} + 4\frac{1}{s+1}$$

$$= \frac{2 + 3(s+1) + 4(s+1)^2}{(s+1)^3}$$

$$= \frac{2 + 3s + 3 + 4s^2 + 8s + 4}{(s+1)^3}$$

$$= \frac{4s^2 + 11s + 9}{(s+1)^3} \qquad (s > 0)$$

Exercise

Find the Laplace transform of $f(t) = 1 + e^{4t}$

Solution

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{1 + e^{4t}\right\}$$

$$F(s) = \frac{1}{s} + \frac{1}{s-4}$$

$$= \frac{2s-4}{s^2 - 4s}$$

Exercise

Find the Laplace transform of $y(t) = e^{2t}\cos 2t$

Solution

$$f(t) = \cos 2t \xrightarrow{\mathcal{L}} F(s) = \frac{s}{s^2 + 4}$$

$$y(t) = e^{2t}\cos 2t \xrightarrow{\mathcal{L}} Y(s) = F(s-2)$$

$$Y(s) = F(s-2)$$

$$= \frac{s-2}{(s-2)^2 + 4}$$

$$= \boxed{\frac{s-2}{s^2-4s+8}}$$

Exercise

Find the Laplace transform of $f(t) = t^3 - te^t + e^{4t}\cos t$

Solution

$$\mathcal{L}\left(e^{at}\cos \omega t\right) = \frac{s-a}{(s-a)^2+\omega^2} \qquad \mathcal{L}\left(t^n e^{-at}\right)(s) = \frac{n!}{(s+a)^{n+1}}$$

$$F(s) = \mathcal{L}\left\{t^3 - te^t + e^{4t}\cos t\right\}$$

$$= \boxed{\frac{6}{s^4} - \frac{1}{(s-1)^2} + \frac{s-4}{(s-4)^2+1}}$$

Exercise

Find the Laplace transform of $f(t) = t^2 - 3t - 2e^{-t}\sin 3t$

Solution

$$F(s) = \mathcal{L}\left\{t^2 - 3t - 2e^{-t}\sin 3t\right\} \qquad \mathcal{L}\left(e^{at}\sin \omega t\right) = \frac{\omega}{(s-a)^2+\omega^2}$$

$$= \boxed{\frac{6}{s^3} - \frac{3}{s^2} - \frac{6}{(s+1)^2+9}}$$

Exercise

Find the Laplace transform of $f(t) = \sin^2 t$

Solution

$$F(s) = \mathcal{L}\left\{\sin^2 t\right\}$$

$$= \frac{1}{2}\mathcal{L}\{1 - \cos 2t\}$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$= \boxed{\frac{4}{2s\left(s^2+4\right)}}$$

Exercise

Find the Laplace transform of $f(t)=e^{7t}\sin^2 t$

Solution

$$F(s)=\mathcal{L}\left\{e^{7t}\sin^2 t\right\}$$

$$=\frac{1}{2}\mathcal{L}\left\{e^{7t}-e^{7t}\cos 2t\right\} \qquad \mathcal{L}\left(e^{at}\cos\omega t\right)=\frac{s-a}{(s-a)^2+\omega^2}$$

$$=\frac{1}{2}\left(\frac{1}{s-7}-\frac{s-7}{(s-7)^2+4}\right)$$

$$\boxed{=\frac{2}{(s-7)\left((s-7)^2+4\right)}}$$

Exercise

Find the Laplace transform of $f(t)=t\sin^2 t$

Solution

$$F(s)=\mathcal{L}\left\{t\sin^2 t\right\}$$

$$=\frac{1}{2}\mathcal{L}\{t-t\cos 2t\} \qquad \mathcal{L}(t\cos\omega t)=\frac{s^2-\omega^2}{\left(s^2+\omega^2\right)^2}$$

$$\boxed{=\frac{1}{2}\frac{1}{s^2}-\frac{1}{2}\frac{s^2-4}{\left(s^2+4\right)^2}}$$

Exercise

Find the Laplace transform of $f(t)=\cos^3 t$

Solution

$$F(s)=\mathcal{L}\left\{\cos^3 t\right\}$$

$$=\frac{1}{2}\mathcal{L}\{\cos t(1+\cos 2t)\}$$

$$=\frac{1}{2}\mathcal{L}\{\cos t+\cos t\cos 2t\} \qquad \cos\alpha\cos\beta=\tfrac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$

$$=\frac{1}{2}\mathcal{L}\left\{\cos t+\tfrac{1}{2}\cos 3t+\tfrac{1}{2}\cos t\right\}$$

$$= \mathcal{L}\left\{\frac{3}{4}\cos t + \frac{1}{4}\cos 3t\right\}$$

$$\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2+\omega^2}$$

$$\boxed{= \frac{3s}{4\left(s^2+1\right)} + \frac{s}{4\left(s^2+9\right)}}$$

Exercise

Find the Laplace transform of $f(t) = te^{-t}\sin 2t$

Solution

$$F(s) = \mathcal{L}\left\{te^{-t}\sin 2t\right\}$$

$$\mathcal{L}\left\{te^{-at}\sin \omega t\right\} = \frac{2\omega(s+a)}{\left((s+a)^2+\omega^2\right)^2}$$

$$\boxed{= \frac{4(s+1)}{\left((s+1)^2+4\right)^2}}$$

Exercise

Find the Laplace transform of $f(t) = e^{2t}\cos 5t$

Solution

$$F(s) = \mathcal{L}\left\{e^{2t}\cos 5t\right\}$$

$$\mathcal{L}\left\{e^{-at}\cos \omega t\right\} = \frac{s+a}{(s+a)^2+\omega^2}$$

$$\boxed{= \frac{s-2}{(s-2)^2+25}}$$

Exercise

Find the Laplace transform of $f(t) = t^2 + e^t\sin 2t$

Solution

$$F(s) = \mathcal{L}\left\{t^2 + e^t\sin 2t\right\}$$

$$\mathcal{L}\left\{e^{at}\sin \omega t\right\} = \frac{\omega}{(s-a)^2+\omega^2}$$

$$\boxed{= \frac{2}{s^3} + \frac{2}{(s-1)^2+4}}$$

Exercise

Find the Laplace transform of $f(t) = e^{-t}\cos 3t + e^{6t} - 1$

Solution

$$F(s) = \mathcal{L}\left\{e^{-t}\cos 3t + e^{6t} - 1\right\}$$

$$\mathcal{L}\left\{e^{-at}\cos \omega t\right\} = \frac{s+a}{(s+a)^2 + \omega^2}$$

$$= \frac{s+1}{(s+1)^2 + 9} + \frac{1}{s-6} + \frac{1}{s}$$

Exercise

Find the Laplace transform of $f(t) = e^{-2t}\sin 2t + t^2 e^{3t}$

Solution

$$\mathcal{L}\left\{e^{at}\sin \omega t\right\} = \frac{\omega}{(s-a)^2 + \omega^2}$$

$$\mathcal{L}\left(t^n e^{-at}\right)(s) = \frac{n!}{(s+a)^{n+1}}$$

$$F(s) = \mathcal{L}\left\{e^{-2t}\sin 2t + t^2 e^{3t}\right\}$$

$$= \frac{2}{(s+2)^2 + 4} + \frac{2}{(s-3)^3}$$

Exercise

Find the Laplace transform of $f(t) = 2t^2 e^{-2t} - t + \cos 4t$

Solution

$$F(s) = \mathcal{L}\left\{2t^2 e^{-2t} - t + \cos 4t\right\}$$

$$\mathcal{L}\left(t^n e^{-at}\right)(s) = \frac{n!}{(s+a)^{n+1}}$$

$$= \frac{4}{(s+2)^3} - \frac{1}{s} - \frac{4}{s^2 + 4}$$

Exercise

Find the Laplace transform of $f(t) = t\sin 3t$

Solution

$$f(t) = \sin 3t \xrightarrow{\mathcal{L}} F(s) = \frac{3}{s^2 + 9}$$

$$\mathcal{L}\{t\sin 3t\}(s) = -Y'(s)$$

Using Derivative of a Laplace Transform Proposition

$$= -\frac{3(-2s)}{\left(s^2+9\right)^2}$$

$$\boxed{= \frac{6s}{\left(s^2+9\right)^2}}$$

Exercise

Find the Laplace transform of $\quad f(t) = t^2 \cos 2t$

Solution

$$f(t) = \cos 2t \xrightarrow{\mathcal{L}} F(s) = \frac{s}{s^2+4}$$

$$\mathcal{L}\left\{t^2 \cos 2t\right\}(s) = (-1)^2 Y''(s) \qquad \textit{Using Derivative of a Laplace Transform Proposition}$$

$$= \left(Y'(s)\right)'$$

$$= \frac{d}{ds}\left[\frac{(s^2+4)(1)-s(2s)}{\left(s^2+4\right)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{4-s^2}{\left(s^2+4\right)^2}\right]$$

$$= \frac{-2s\left(s^2+4\right)^2 - \left(4-s^2\right)(2)(2s)\left(s^2+4\right)}{\left(s^2+4\right)^4}$$

$$= \left(s^2+4\right)\frac{-2s\left(s^2+4\right) - 4s\left(4-s^2\right)}{\left(s^2+4\right)^4}$$

$$= \frac{-2s^3 - 8s - 16s + 4s^3}{\left(s^2+4\right)^3}$$

$$\boxed{= \frac{2s^3 - 24s}{\left(s^2+4\right)^3}}$$

Exercise

Find the Laplace transform of $f(t) = \left(1 + e^{-t}\right)^2$

Solution

$$\begin{aligned}
F(s) &= \mathcal{L}\left\{1 + 2e^{-t} + e^{-2t}\right\} \\
&= \frac{1}{s} + \frac{2}{s+1} + \frac{1}{s+2} \\
&= \frac{s^2 + 3s + 2 + 2s^2 + 4s + s^2 + s}{s(s+1)(s+2)} \\
&= \boxed{\frac{4s^2 + 8s + 2}{s(s+1)(s+2)}}
\end{aligned}$$

Exercise

Find the Laplace transform of $f(t) = \left(1 + e^{2t}\right)^2$

Solution

$$\begin{aligned}
F(s) &= \mathcal{L}\left\{1 + 2e^{2t} + e^{4t}\right\} \\
&= \frac{1}{s} + \frac{2}{s-2} + \frac{1}{s-4} \\
&= \boxed{\frac{4s^2 - 16s + 8}{s(s-2)(s-4)}}
\end{aligned}$$

Exercise

Find the Laplace transform of $f(t) = \left(e^{t} - e^{-t}\right)^2$

Solution

$$\begin{aligned}
F(s) &= \mathcal{L}\left\{e^{2t} - 2 + e^{-2t}\right\} \\
&= \frac{1}{s-2} - \frac{2}{s} + \frac{1}{s+2} \\
&= \frac{s^2 + 2s - s^2 + 8 + s^2 - 2s}{s\left(s^2 - 4\right)} \\
&= \boxed{\frac{s^2 + 8}{s\left(s^2 - 4\right)}}
\end{aligned}$$

Exercise

Find the Laplace transform of $f(t) = 4t^2 - 5\sin 3t$

Solution

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{4t^2 - 5\sin 3t\}$$

$$F(s) = \frac{2}{s^3} - \frac{15}{s^2+9}$$

$$\boxed{= \frac{-15s^3 + 2s^2 + 18}{s^5 + 9s^3}}$$

Exercise

Find the Laplace transform of $f(t) = \cos 5t + \sin 2t$

Solution

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos 5t + \sin 2t\}$$

$$F(s) = \frac{s}{s^2+25} + \frac{2}{s^2+4}$$

$$\boxed{= \frac{s^3 + 2s^2 + 4s + 50}{\left(s^2+4\right)\left(s^2+25\right)}}$$

Exercise

Find the Laplace transform of $f(t) = e^{3t}\sin 6t - t^3 + e^t$

Solution

$$F(s) = \mathcal{L}\left\{e^{3t}\sin 6t - t^3 + e^t\right\}$$

$$\mathcal{L}\left(e^{at}\sin\omega t\right) = \frac{\omega}{(s-a)^2 + \omega^2}$$

$$\boxed{= \frac{6}{(s-3)^2 + 36} - \frac{6}{s^4} + \frac{1}{s-1}}$$

Exercise

Find the Laplace transform of $f(t) = t^4 + t^2 - t + \sin\sqrt{2}t$

Solution

$$F(s) = \mathcal{L}\left\{t^4 + t^2 - t + \sin\sqrt{2}t\right\}$$

$$\mathcal{L}\left(e^{at}\sin\omega t\right) = \frac{\omega}{(s-a)^2 + \omega^2}$$

$$= \boxed{\frac{24}{s^5} + \frac{2}{s^3} - \frac{1}{s^2} + \frac{\sqrt{2}}{s^2+2}}$$

Exercise

Find the Laplace transform of $\quad f(t) = t^4 e^{5t} - e^t \cos\sqrt{7}t$

Solution

$$\mathcal{L}\left(t^n e^{-at}\right)(s) = \frac{n!}{(s+a)^{n+1}} \qquad\qquad \mathcal{L}\left(e^{at}\cos\omega t\right) = \frac{s-a}{(s-a)^2+\omega^2}$$

$$F(s) = \mathcal{L}\left\{t^4 e^{5t} - e^t \cos\sqrt{7}t\right\}$$

$$= \boxed{\frac{24}{(s-5)^5} - \frac{s-1}{(s-1)^2+7}}$$

Exercise

Find the Laplace transform of $\quad f(t) = e^{-2t}\cos\sqrt{3}t - t^2 e^{-2t}$

Solution

$$\mathcal{L}\left(t^n e^{-at}\right)(s) = \frac{n!}{(s+a)^{n+1}} \qquad\qquad \mathcal{L}\left(e^{at}\cos\omega t\right) = \frac{s-a}{(s-a)^2+\omega^2}$$

$$F(s) = \mathcal{L}\left\{e^{-2t}\cos\sqrt{3}t - t^2 e^{-2t}\right\}$$

$$= \boxed{\frac{s+2}{(s+2)^2+3} - \frac{2}{(s+2)^3}}$$

Exercise

Find the Laplace transform of $\quad f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$

Solution

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{6e^{-5t} + e^{3t} + 5t^3 - 9\right\}$$

$$\boxed{F(s) = \frac{6}{s+5} + \frac{1}{s-3} + \frac{5}{s^4} - \frac{9}{s}}$$

Exercise

Find the Laplace transform of $f(t) = 4\cos 4t - 9\sin 4t + 2\cos 10t$

Solution

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{4\cos 4t - 9\sin 4t + 2\cos 10t\}$$

$$F(s) = 4\frac{s}{s^2+4^2} - 9\frac{4}{s^2+4^2} + 2\frac{s}{s^2+10^2}$$

$$\boxed{= \frac{4s}{s^2+16} - \frac{36}{s^2+16} + \frac{2s}{s^2+100}}$$

Exercise

Find the Laplace transform of $f(t) = 3\sinh 2t + 3\sin 2t$

Solution

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{3\sinh 2t + 3\sin 2t\}$$

$$F(s) = 3\frac{2}{s^2-2^2} + 3\frac{2}{s^2+2^2}$$

$$\boxed{= \frac{6}{s^2-4} + \frac{6}{s^2+4}}$$

Exercise

Find the Laplace transform of $f(t) = e^{3t} + \cos 6t - e^{3t}\cos 6t$

Solution

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{e^{3t} + \cos 6t - e^{3t}\cos 6t\right\}$$

$$\boxed{F(s) = \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36}}$$

Exercise

Find the Laplace transform of $f(t) = t\cosh 3t$

Solution

$$f(t) = \cosh 3t \xrightarrow{\mathcal{L}} Y(s) = \frac{s}{s^2-9}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\cosh 3t\}$$

$$F(s) = -Y'(s)$$
$$= -\frac{s^2 - 9 - 2s^2}{\left(s^2 - 9\right)^2}$$
$$\boxed{= \frac{s^2 + 9}{\left(s^2 - 9\right)^2}}$$

Exercise

Find the Laplace transform of $f(t) = t^2 \sin 2t$

Solution

$$f(t) = \sin 2t \xrightarrow{\mathcal{L}} Y(s) = \frac{2}{s^2 + 4}$$
$$Y'(s) = -\frac{4s}{\left(s^2 + 4\right)^2}$$
$$Y''(s) = -4\frac{s^2 + 4 - 4s^2}{\left(s^2 + 4\right)^3} = \frac{12s^2 - 16}{\left(s^2 + 4\right)^3} \qquad \left(U^m V^n\right)' = U^{m-1} V^{n-1}\left(mU'V + nUV'\right)$$
$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{t^2 \sin 2t\right\}$$
$$F(s) = (-1)^2 Y''(s)$$
$$\boxed{= \frac{12s^2 - 16}{\left(s^2 + 4\right)^3}}$$

Exercise

Find the Laplace transform of $f(t) = \sinh kt$

Solution

$$F(s) = \mathcal{L}\{\sinh kt\}$$
$$= \frac{1}{2}\mathcal{L}\left\{e^{kt} - e^{-kt}\right\}$$
$$= \frac{1}{2}\left(\frac{1}{s - k} - \frac{1}{s + k}\right)$$
$$\boxed{= \frac{k}{s^2 - k^2}}$$

Exercise

Find the Laplace transform of $f(t) = \cosh kt$

Solution

$$F(s) = \mathcal{L}\{\cosh kt\}$$

$$= \frac{1}{2}\mathcal{L}\left\{e^{kt} + e^{-kt}\right\}$$

$$= \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right)$$

$$\boxed{= \frac{s}{s^2 - k^2}}$$

Exercise

Find the Laplace transform of $f(t) = e^t \sinh kt$

Solution

$$F(s) = \mathcal{L}\left\{e^t \sinh kt\right\}$$

$$= \frac{1}{2}\mathcal{L}\left\{e^t\left(e^{kt} - e^{-kt}\right)\right\}$$

$$= \frac{1}{2}\mathcal{L}\left\{e^{(k+1)t} - e^{-(k-1)t}\right\}$$

$$\boxed{= \frac{1}{2}\left(\frac{1}{s-(k+1)} - \frac{1}{s+(k-1)}\right)}$$

Exercise

Find the Laplace transform of $f(t) = e^{-t} \cosh kt$

Solution

$$F(s) = \mathcal{L}\left\{e^{-t} \cosh kt\right\}$$

$$= \frac{1}{2}\mathcal{L}\left\{e^{-t}\left(e^{kt} + e^{-kt}\right)\right\}$$

$$= \frac{1}{2}\mathcal{L}\left\{e^{(k-1)t} + e^{-(k-1)t}\right\}$$

$$\boxed{= \frac{1}{2}\left(\frac{1}{s-(k-1)} + \frac{1}{s+(k-1)}\right)}$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

$$y' + 2y = t\sin t, \quad with \quad y(0) = 1$$

Solution

Let $Y(s) = \mathcal{L}(y)(s)$, then

Left side;

$$\begin{aligned}\mathcal{L}(y' + 2y)(s) &= s\mathcal{L}(y)(s) - y(0) + 2\mathcal{L}(y)(s) \\ &= sY(s) - 1 + 2Y(s) \\ &= (s+2)Y(s) - 1\end{aligned}$$

Right side; $\quad f(t) = \sin t \xrightarrow{\mathcal{L}} F(s) = \dfrac{1}{s^2+1}$

$$\begin{aligned}\mathcal{L}\{t\sin t\}(s) &= -F'(s) \qquad \textit{Using Derivative of a Laplace Transform Proposition} \\ &= \frac{2s}{\left(s^2+1\right)^2}\end{aligned}$$

$$(s+2)Y(s) - 1 = \frac{2s}{\left(s^2+1\right)^2}$$

$$(s+2)Y(s) = \frac{2s}{\left(s^2+1\right)^2} + 1$$

$$Y(s) = \frac{2s}{(s+2)\left(s^2+1\right)^2} + \frac{1}{s+2}$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

$$y' - y = t^2 e^{-2t}, \quad with \quad y(0) = 0$$

Solution

Let $Y(s) = \mathcal{L}(y)(s)$, then

Left side;

$$\begin{aligned}\mathcal{L}(y' + 2y)(s) &= s\mathcal{L}(y)(s) - y(0) + 2\mathcal{L}(y)(s) \\ &= sY(s) - Y(s)\end{aligned}$$

$$=(s-1)Y(s)$$

Right side; $f(t)=e^{2t}\xrightarrow{\mathcal{L}}F(s)=\frac{1}{s-2}$

$\mathcal{L}\left\{t^2e^{2t}\right\}(s)=(-1)^2Y''(s)$ *Using Laplace Transform table*

$$=\frac{2}{(s-2)^3}$$

$$(s-1)Y(s)=\frac{2}{(s-2)^3}$$

$$\boxed{Y(s)=\frac{2}{(s-1)(s-2)^3}}$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}y)$. Solve the resulting equation for the Laplace transform of y.

$$y''+y'+2y=e^{-t}\cos 2t,\quad with\quad y(0)=1\ \ and\quad y'(0)=-1$$

Solution

$$\mathcal{L}(y''+y'+2y)(s)=\mathcal{L}\left(e^{-t}\cos 2t\right)$$

$$s^2\mathcal{L}(y)(s)-sy(0)-y'(0)+s\,\mathcal{L}(y)(s)-y(0)+2\,\mathcal{L}(y)(s)=\frac{s+1}{(s+1)^2+4}$$

$$s^2Y(s)-s+1+sY(s)-1+2Y(s)=\frac{s+1}{(s+1)^2+4}$$

$$\left(s^2+s+2\right)Y(s)-s=\frac{s+1}{s^2+2s+1+4}$$

$$\left(s^2+s+2\right)Y(s)=\frac{s+1}{s^2+2s+5}+s$$

$$Y(s)=\frac{s+1}{\left(s^2+2s+5\right)\left(s^2+s+2\right)}+\frac{s}{s^2+s+2}$$

$$=\frac{s+1+s\left(s^2+2s+5\right)}{\left(s^2+2s+5\right)\left(s^2+s+2\right)}$$

$$=\frac{s+1+s^3+2s^2+5s}{\left(s^2+2s+5\right)\left(s^2+s+2\right)}$$

$$\boxed{=\frac{s^3+2s^2+6s+1}{\left(s^2+2s+5\right)\left(s^2+s+2\right)}}$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of *y*.

$$y' - 5y = e^{-2t}, \quad with \quad y(0) = 1$$

Solution

$$\mathcal{L}(y' - 5y)(s) = \mathcal{L}\left(e^{-2t}\right)(s)$$

$$\mathcal{L}(y')(s) - 5\,\mathcal{L}(y)(s) = \frac{1}{s+2}$$

$$s\mathcal{L}(y)(s) - y(0) - 5\,\mathcal{L}(y)(s) = \frac{1}{s+2}$$

Let $Y(s) = \mathcal{L}(y)(s)$, then

$$sY(s) - 1 - 5Y(s) = \frac{1}{s+2}$$

$$(s-5)Y(s) = \frac{1}{s+2} + 1$$

$$Y(s) = \frac{1}{(s-5)(s+2)} + \frac{1}{(s-5)}$$

$$= \frac{1+s+2}{(s-5)(s+2)}$$

$$= \frac{s+3}{(s-5)(s+2)}$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of *y*.

$$y' - 4y = \cos 2t, \quad with \quad y(0) = -2$$

Solution

$$\mathcal{L}(y' - 4y)(s) = \mathcal{L}(\cos 2t)(s)$$

$$\mathcal{L}(y')(s) - 4\,\mathcal{L}(y)(s) = \frac{s}{s^2+4}$$

$$s\,\mathcal{L}(y)(s) - y(0) - 4\,\mathcal{L}(y)(s) = \frac{s}{s^2+4}$$

Let $Y(s) = \mathcal{L}(y)(s)$, then

$$sY(s) + 2 - 4Y(s) = \frac{s}{s^2+4}$$

$$(s-4)Y(s) = \frac{s}{s^2+4} - 2$$

$$Y(s)=\frac{s}{(s-4)\left(s^2+4\right)}-\frac{2}{s-4}$$

$$=\frac{s-2s^2-8}{(s-4)\left(s^2+4\right)}$$

$$\boxed{=\frac{-2s^2+s-8}{(s-4)\left(s^2+4\right)}}$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of *y*.

$$y''+2y'+2y=\cos 2t; \quad \text{with} \quad y(0)=1 \ \ \text{and} \quad y'(0)=0$$

Solution

$$\mathcal{L}\left(y''+2y'+2y\right)(s)=\mathcal{L}\left(\cos 2t\right)(s)$$

Let $Y(s)=\mathcal{L}(y)(s)$, then

$$s^2Y(s)-sy(0)-y'(0)+2\left(sY(s)-y(0)\right)+2Y(s)=\frac{s}{s^2+4}$$

$$s^2Y(s)-s+2sY(s)-2+2Y(s)=\frac{s}{s^2+4}$$

$$\left(s^2+2s+2\right)Y(s)=\frac{s}{s^2+4}+s+2$$

$$=\frac{s+s^3+2s^2+4s+8}{s^2+4}$$

$$=\frac{s^3+2s^2+5s+8}{s^2+4}$$

$$\boxed{Y(s)=\frac{s^3+2s^2+5s+8}{\left(s^2+4\right)\left(s^2+2s+2\right)}}$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of *y*.

$$y''+3y'+5y=t+e^{-t}; \quad \text{with} \quad y(0)=-1 \ \ \text{and} \quad y'(0)=0$$

Solution

$$\mathcal{L}\left(y''+3y'+5y\right)(s)=\mathcal{L}(t)(s)+\mathcal{L}\left(e^{-t}\right)(s)$$

$$s^2Y(s) - sy(0) - y'(0) + 3\left(sY(s) - y(0)\right) + 5Y(s) = \frac{1}{s^2} + \frac{1}{s+1}$$

$$s^2Y(s) + s + 3\left(sY(s) + 1\right) + 5Y(s) = \frac{s+1+s^2}{s^2(s+1)}$$

$$s^2Y(s) + s + 3sY(s) + 3 + 5Y(s) = \frac{s+1+s^2}{s^2(s+1)}$$

$$\begin{aligned}\left(s^2+3s+5\right)Y(s) &= \frac{s+1+s^2}{s^2(s+1)} - s - 3\\ &= \frac{s+1+s^2 - s^2(s+1)(s+3)}{s^2(s+1)}\\ &= \frac{s+1+s^2 - s^2\left(s^2+4s+3\right)}{s^2(s+1)}\\ &= \frac{s+1+s^2 - s^4 + 4s^3 + 3s^2}{s^2(s+1)}\\ &= \frac{-s^4 + 4s^3 + 4s^2 + s + 1}{s^2(s+1)}\end{aligned}$$

$$Y(s) = \frac{-s^4 + 4s^3 + 4s^2 + s + 1}{s^2(s+1)\left(s^2+3s+5\right)}$$

Solution *Section* 3.3 – Inverse Laplace Transform

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{1}{3s+2}$

Solution

$$Y(s)=\frac{1}{3}\frac{1}{s+2/3} \qquad \textit{Factor}$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{3}\frac{1}{s+2/3}\right\}$$

$$=\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2/3}\right\}$$

$$=\frac{1}{3}e^{-(2/3)t}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{2}{3-5s}$

Solution

$$Y(s)=-2\frac{1}{5s-3}$$

$$=-\frac{2}{5}\frac{1}{s-\frac{3}{5}}$$

Thus, by linearity;

$$y(t)=-\frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-\frac{3}{5}}\right\}$$

$$=-\frac{2}{5}e^{(3/5)t}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{1}{s^2+4}$

Solution

$$Y(s)=\frac{1}{2}\frac{2}{s^2+4}$$

$$y(t)=\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$=\frac{1}{2}\sin 2t$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\frac{3}{s^2}$

Solution

$$y(t)=3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$\boxed{=3t}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\frac{3s+2}{s^2+25}$

Solution

$$Y(s)=\frac{3s}{s^2+25}+\frac{2}{s^2+25}$$

$$=3\frac{s}{s^2+25}+\frac{2}{5}\frac{5}{s^2+25}$$

$$y(t)=3\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\}+\frac{2}{5}\,\mathcal{L}^{-1}\left\{\frac{5}{s^2+25}\right\}$$

$$\boxed{=3\cos 5t+\frac{2}{5}\sin 5t}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\frac{2-5s}{s^2+9}$

Solution

$$Y(s)=\frac{2}{s^2+9}-\frac{5s}{s^2+25}$$

$$=\frac{2}{3}\frac{3}{s^2+9}-5\frac{s}{s^2+9}$$

$$y(t)=\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}-5\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}$$

$$\boxed{=\frac{2}{3}\sin 3t-5\cos 3t}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\frac{5}{(s+2)^3}$

Solution

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s+a)^{n+1}}\right\}=t^n e^{-at} \qquad n=2 \quad a=2$$

$$Y(s)=\frac{5}{2!}\frac{2!}{(s+2)^3}$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{5}{(s+2)^3}\right\}$$

$$=\frac{5}{2}t^2e^{-2t}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{1}{(s-1)^6}$

Solution

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s+a)^{n+1}}\right\}=t^n e^{-at} \qquad n=5 \quad a=-1$$

$$Y(s)=\frac{1}{5!}\frac{5!}{(s-1)^6}$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{5!}\frac{5!}{(s-1)^6}\right\}$$

$$=\frac{1}{120}t^5e^{t}$$

Exercise

Find the inverse Laplace Transform of $\qquad Y(s)=\dfrac{4(s-1)}{(s-1)^2+4}$

Solution

$$\mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+\omega^2}\right\}=e^{-at}\cos\omega t \qquad a=-1 \quad \omega=2$$

$$Y(s)=4\frac{s-1}{(s-1)^2+4}$$

$$y(t)=\mathcal{L}^{-1}\left\{4\frac{s-1}{(s-1)^2+4}\right\}$$

$$=4e^{t}\cos 2t$$

Exercise

Find the inverse Laplace Transform of $Y(s) = \dfrac{2s-3}{(s-1)^2+5}$

Solution

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2s-3}{(s-1)^2+5}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2s-2-1}{(s-1)^2+5}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2(s-1)}{(s-1)^2+5} - \frac{1}{(s-1)^2+5}\right\}$$

$$= \mathcal{L}^{-1}\left\{2\frac{s-1}{(s-1)^2+5} - \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{(s-1)^2+5}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+\omega^2}\right\} = e^{-at}\cos \omega t$$

$$\mathcal{L}^{-1}\left\{\frac{\omega}{(s+a)^2+\omega^2}\right\} = e^{-at}\sin \omega t$$

$$= 2e^{t}\cos\sqrt{5}t - \frac{1}{\sqrt{5}}e^{t}\sin\sqrt{5}t$$

$$= e^{t}\left(2\cos\sqrt{5}t - \frac{\sqrt{5}}{5}\sin\sqrt{5}t\right)$$

Exercise

Find the inverse Laplace Transform of $Y(s) = \dfrac{2s-1}{(s+1)(s-2)}$

Solution

Use partial fraction $\dfrac{2s-1}{(s+1)(s-2)} = \dfrac{A}{s+1} + \dfrac{B}{s-2}$

$$= \frac{As-2A+Bs+B}{(s+1)(s-2)}$$

$$2s-1 = (A+B)s - 2A + B$$

$$\begin{cases} A+B=2 \\ -2A+B=-1 \end{cases} \Rightarrow A = B = 1$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2s-1}{(s+1)(s-2)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= e^{-t} + e^{2t}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{2s-2}{(s-4)(s+2)}$

Solution

$$\frac{2s-2}{(s-4)(s+2)}=\frac{A}{s-4}+\frac{B}{s+2}$$

$$=\frac{As+2A+Bs-4B}{(s-4)(s+2)}$$

$$2s-2=(A+B)s+2A-4B$$

$$\begin{cases} A+B=2 \\ 2A-4B=-2 \end{cases} \Rightarrow A=B=1$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{2s-2}{(s-4)(s+2)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s-4}+\frac{1}{s+2}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$=e^{4t}+e^{-2t}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{7s^2+3s+16}{(s+1)\left(s^2+4\right)}$

Solution

$$\frac{7s^2+3s+16}{(s+1)\left(s^2+4\right)}=\frac{A}{s+1}+\frac{Bs+C}{s^2+4}$$

$$=\frac{As^2+4A+Bs^2+Bs+Cs+C}{(s+1)\left(s^2+4\right)}$$

$$=\frac{(A+B)s^2+(B+C)s+4A+C}{(s+1)\left(s^2+4\right)}$$

$$7s^2+3s+16=(A+B)s^2+(B+C)s+4A+C$$

$$\begin{cases} A+B=7 \\ B+C=3 \\ 4A+C=16 \end{cases} \quad \begin{matrix} \to A-C=4 \\ 4A+C=16 \end{matrix} \quad \Rightarrow 5A=20 \to A=4 \quad B=3 \quad C=0$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{7s^2+3s+16}{(s+1)\left(s^2+4\right)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{4}{s+1} + \frac{3s}{s^2+4}\right\}$$

$$= 4\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$\underline{= 4e^{-t} + 3\cos 2t}$$

Exercise

Find the inverse Laplace Transform of $\quad Y(s) = \dfrac{1}{(s+2)^2\left(s^2+9\right)}$

Solution

$$\frac{1}{(s+2)^2\left(s^2+9\right)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{Cs+D}{s^2+9}$$

$$= \frac{A(s+2)\left(s^2+9\right) + Bs^2 + 9B + (Cs+D)(s^2+4s+4)}{(s+2)^2\left(s^2+9\right)}$$

$$= \frac{As^3 + 2As^2 + 9As + 18A + Bs^2 + 9B + Cs^3 + 4Cs^2 + 4Cs + Ds^2 + 4Ds + 4D}{(s+2)^2\left(s^2+9\right)}$$

$$1 = (A+C)s^3 + (2A+B+4C+D)s^2 + (9A+4C+4D)s + 18A + 9B + 4D$$

$$\begin{cases} A + C = 0 \\ 2A + B + 4C + D = 0 \\ 9A + 4C + 4D = 0 \\ 18A + 9B + 4D = 1 \end{cases} \Rightarrow A = \frac{4}{169} \quad B = \frac{1}{13} \quad C = -\frac{4}{169} \quad D = -\frac{5}{169}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2\left(s^2+9\right)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{4}{169}\frac{1}{s+2} + \frac{1}{13}\frac{1}{(s+2)^2} - \frac{1}{169}\frac{4s+5}{s^2+9}\right\}$$

$$= \frac{4}{169}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{1}{13}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} - \frac{4}{169}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - \frac{5}{169}\mathcal{L}^{-1}\left\{\frac{1}{3}\frac{3}{s^2+9}\right\}$$

$$\underline{= \frac{4}{169}e^{-2t} + \frac{1}{13}te^{-2t} - \frac{4}{169}\cos 3t - \frac{5}{507}\sin 3t}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{s}{(s+2)^2\left(s^2+9\right)}$

Solution

$$\frac{s}{(s+2)^2\left(s^2+9\right)}=\frac{A}{s+2}+\frac{B}{(s+2)^2}+\frac{Cs+D}{s^2+9}$$

$$=\frac{As^3+2As^2+9As+18A+Bs^2+9B+Cs^3+4Cs^2+4Cs+Ds^2+4Ds+4D}{(s+1)^2\left(s^2+9\right)}$$

$$s=(A+C)s^3+(2A+B+4C+D)s^2+(9A+4C+4D)s+18A+9B+4D$$

$$\begin{cases} A+C=0 \\ 2A+B+4C+D=0 \\ 9A+4C+4D=1 \\ 18A+9B+4D=0 \end{cases} \Rightarrow A=\frac{5}{169} \quad B=-\frac{2}{13} \quad C=-\frac{5}{169} \quad D=\frac{36}{169}$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2\left(s^2+9\right)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{5}{169}\frac{1}{s+2}-\frac{2}{13}\frac{1}{(s+2)^2}-\frac{1}{169}\frac{5s+36}{s^2+9}\right\}$$

$$=\frac{5}{169}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}-\frac{2}{13}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}-\frac{5}{169}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}-\frac{36}{169}\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$=\frac{5}{169}e^{-2t}-\frac{2}{13}te^{-2t}-\frac{5}{169}\cos 3t+\frac{12}{169}\sin 3t$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{1}{(s+1)^2\left(s^2-4\right)}$

Solution

$$\frac{1}{(s+1)^2\left(s^2-4\right)}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{Cs+D}{s^2-4}$$

$$=\frac{A(s+1)\left(s^2-4\right)+B\left(s^2-4\right)+(Cs+D)(s+1)^2}{(s+1)^2\left(s^2-4\right)}$$

$$1=As^3-4As+As^2-4A+Bs^2-4B+Cs^3+2Cs^2+Cs+Ds^2+2Ds+D$$

$$=(A+C)s^3+(A+B+2C+D)s^2+(-4A+C+2D)s-4A-4B+D$$

$$\begin{array}{l} s^3 \\ s^2 \\ s^1 \\ s^0 \end{array}\begin{cases} A+C=0 \\ A+B+2C+D=0 \\ -4A+C+2D=0 \\ -4A-4B+D=1 \end{cases} \qquad A=-\frac{2}{15} \quad B=\frac{1}{5} \qquad C=\frac{2}{15} \quad D=-\frac{1}{3}$$

$$Y(s) = -\frac{2}{15}\frac{1}{s+1} + \frac{1}{5}\frac{1}{(s+1)^2} + \frac{\frac{2}{15}s - \frac{1}{3}}{s^2-4}$$

$$= -\frac{2}{15}\frac{1}{s+1} + \frac{1}{5}\frac{1}{(s+1)^2} + \frac{2}{15}\frac{s}{s^2-4} - \frac{1}{3}\frac{1}{s^2-4}$$

$$y(t) = -\frac{2}{15}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{2}{15}\mathcal{L}^{-1}\left\{\frac{s}{s^2-4}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2-4}\right\}$$

$$= -\frac{2}{15}e^{-t} + \frac{1}{5}te^{-t} + \frac{2}{15}\cosh 2t - \frac{1}{6}\sinh 2t$$

$$= -\frac{2}{15}e^{-t} + \frac{1}{5}te^{-t} + \frac{2}{15}\frac{e^{2t}+e^{-2t}}{2} - \frac{1}{6}\frac{e^{2t}-e^{-2t}}{2}$$

$$= -\frac{2}{15}e^{-t} + \frac{1}{5}te^{-t} + \frac{1}{15}e^{2t} + \frac{1}{15}e^{-2t} - \frac{1}{12}e^{2t} + \frac{1}{12}e^{-2t}$$

$$= -\frac{2}{15}e^{-t} + \frac{1}{5}te^{-t} - \frac{1}{60}e^{2t} + \frac{3}{20}e^{-2t}$$

Exercise

Find the inverse Laplace Transform of $Y(s) = \dfrac{7s^2+20s+53}{(s-1)\left(s^2+2s+5\right)}$

Solution

$$\frac{7s^2+20s+53}{(s-1)\left(s^2+2s+5\right)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$$

$$7s^2+20s+53 = As^2+2As+5A+Bs^2-Bs+Cs-C$$

$$\begin{array}{l} s^2 \\ s^1 \\ s^0 \end{array}\begin{cases} A+B=7 \\ 2A-B+C=20 \\ 5A-C=53 \end{cases} \Rightarrow \begin{cases} A=10 \\ B=-3 \\ C=-3 \end{cases}$$

$$Y(x) = \frac{10}{s-1} + \frac{-3s-3}{s^2+2s+5}$$

$$= \frac{10}{s-1} - 3\frac{s+1}{s^2+2s+5}$$

$$y(t) = 10\mathcal{L}^{-1}\left\{\frac{10}{s-1}\right\} - 3\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\}$$

$$= 10e^{t} - 3e^{-t}\cos 2t$$

Exercise

Find the inverse Laplace Transform of $F(s)=\frac{1}{s^3}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$=\frac{1}{2}t^2$$

Exercise

Find the inverse Laplace Transform of $F(s)=\frac{1}{s^4}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$$

$$=\frac{1}{6}t^3$$

Exercise

Find the inverse Laplace Transform of $F(s)=\frac{1}{s^2}-\frac{48}{s^5}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^2}-\frac{48}{s^5}\right\}$$

$$=t-2t^4$$

Exercise

Find the inverse Laplace Transform of $F(s)=\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s-2}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s-2}\right\}$$

$$=t-1+e^{2t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\frac{4}{s}+\frac{4}{s^5}+\frac{1}{s-8}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{4}{s}+\frac{4}{s^5}+\frac{1}{s-8}\right\}$$

$$\boxed{=4+\frac{1}{6}t^4+e^{8t}}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\frac{1}{4s+1}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s+\frac{1}{4}}\right\}$$

$$\boxed{=\frac{1}{4}e^{-t/2}}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\frac{1}{5s-2}$

Solution

$$f(t)=\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-\frac{2}{5}}\right\}$$

$$\boxed{=\frac{1}{5}e^{-2t/5}}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\frac{s+1}{s^2+2}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}+\frac{1}{s^2+2}\right\}$$

$$\boxed{=\cos\sqrt{2}t+\frac{1}{2}\sin\sqrt{2}t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\frac{2s-6}{s^2+9}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{2s}{s^2+9} - \frac{6}{s^2+9}\right\}$$

$$\boxed{= 2\cos 3t - 2\sin 3t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{10s}{s^2+16}$

Solution

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{10s}{s^2+16}\right\}$$

$$\boxed{f(t) = 10\cos 4t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \left(\dfrac{2}{s} - \dfrac{1}{s^3}\right)^2$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{4}{s^2} - \frac{4}{s^4} + \frac{1}{s^6}\right\}$$

$$\boxed{= 4t - \frac{2}{3}t^3 + \frac{1}{5!}t^5}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{(s+1)^3}{s^4}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s^3+3s^2+3s+1}{s^4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{3}{s^2} + \frac{3}{s^3} + \frac{1}{s^4}\right\}$$

$$\boxed{= 1 + 3t + \frac{3}{2}t^2 + \frac{1}{6}t^3}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{(s+2)^2}{s^3}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{s^2+4s+4}{s^3}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{4}{s^2}+\frac{4}{s^3}\right\}$$

$$\underline{=1+4t+2t^2}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{1}{s^4-9}$

Solution

$$F(s)=\frac{1}{s^4-9}=\frac{A}{s-\sqrt{3}}+\frac{B}{s+\sqrt{3}}+\frac{Cs+D}{s^2+3} \qquad s^4-9=\left(s^2-3\right)\left(s^2+3\right)=\left(s-\sqrt{3}\right)\left(s+\sqrt{3}\right)\left(s^2+3\right)$$

$$A\left(s+\sqrt{3}\right)\left(s^2+3\right)+B\left(s-\sqrt{3}\right)\left(s^2+3\right)+Cs^3-3Cs+Ds^2-3D=1$$

$$As^3+3As+As^2\sqrt{3}+3A\sqrt{3}+Bs^3+3Bs-Bs^2\sqrt{3}-3B\sqrt{3}+Cs^3-3Cs+Ds^2-3D=1$$

$$\begin{cases} s^3 & A+B+C=0 \\ s^2 & \sqrt{3}A-\sqrt{3}B+D=0 \\ s^1 & 3A+3B-3C=0 \\ s^0 & 3\sqrt{3}A-3\sqrt{3}B-3D=1 \end{cases} \to \begin{matrix} \begin{cases} A+B+C=0 \\ A+B-C=0 \end{cases} \\ \begin{cases} \sqrt{3}A-\sqrt{3}B+D=0 \\ \sqrt{3}A-\sqrt{3}B-D=1 \end{cases} \end{matrix} \to \begin{cases} A+B=0 \\ 2\sqrt{3}A-2\sqrt{3}B=1 \end{cases}$$

$$\underline{A}=\frac{1}{4\sqrt{3}}=\underline{\frac{\sqrt{3}}{12}} \qquad \underline{B=-\frac{\sqrt{3}}{12}}$$

$$\underline{C}=A+B\underline{=0} \qquad \underline{D}=\sqrt{3}B-\sqrt{3}A\underline{=-\frac{1}{4}}$$

$$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{12}\frac{1}{s-\sqrt{3}}-\frac{\sqrt{3}}{12}\frac{1}{s+\sqrt{3}}-\frac{1}{4}\frac{\sqrt{3}}{\sqrt{3}}\frac{1}{s^2+3}\right\}$$

$$\underline{f(t)=\frac{\sqrt{3}}{12}\left(e^{\sqrt{3}t}-e^{-\sqrt{3}t}-\sin\sqrt{3}t\right)}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{1}{s^3+5s}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s\left(s^2+5\right)}\right\}$$

$$\frac{1}{s\left(s^2+5\right)}=\frac{A}{s}+\frac{Bs+C}{s^2+5}$$

$$1=As^2+5A+Bs^2+Cs$$

$$\begin{cases} s^2 & A+B=0 \quad \boxed{B=-\frac{1}{5}} \\ s & \boxed{C=0} \\ s^0 & 5A=1 \quad \boxed{A=\frac{1}{5}} \end{cases}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{5}\frac{1}{s}-\frac{1}{5}\frac{s}{s^2+5}\right\}$$

$$\boxed{=\frac{t}{5}+\frac{1}{5}\cos\sqrt{5}t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{5}{s^2+36}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{5}{s^2+36}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\}=\sin\omega t$$

$$\boxed{=\frac{5}{6}\sin 6t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{10s}{s^2+16}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{10s}{s^2+16}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\}=\cos\omega t$$

$$\boxed{=10\cos 4t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{4s}{4s^2+1}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{4s}{4s^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+\frac{1}{4}}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos\omega t$$

$$\boxed{= \cos\frac{1}{2}t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{1}{4s^2+1}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{4s^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{s}{s^2+\frac{1}{4}}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos\omega t$$

$$\boxed{= \frac{1}{4}\cos\frac{1}{2}t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{1}{s^2+3s}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s+3)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{3}\frac{1}{s} - \frac{1}{3}\frac{1}{s+3}\right\}$$

$$\boxed{= \frac{1}{3}t - \frac{1}{3}e^{-3t}}$$

$$\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$

$$1 = As + 3A + Bs$$

$s \quad A + B = 0 \quad \boxed{B = -\frac{1}{3}}$

$s^0 \quad 3A = 1 \quad \boxed{A = \frac{1}{3}}$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{s+1}{s^2-4s}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s+1}{s(s-4)}\right\}$$

$$= \mathcal{L}^{-1}\left\{-\frac{1}{4}\frac{1}{s} + \frac{5}{4}\frac{1}{s-4}\right\}$$

$$\boxed{= -\frac{1}{4}t + \frac{5}{4}e^{4t}}$$

$$\frac{s+1}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4}$$

$$s + 1 = As - 4A + Bs$$

$s \quad A + B = 1 \quad \boxed{B = \frac{5}{4}}$

$s^0 \quad -4A = 1 \quad \boxed{A = -\frac{1}{4}}$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{1}{s^3 + 5s}$

Solution

$$F(s) = \frac{1}{s^3+5s} = \frac{A}{s} + \frac{Bs+C}{s^2+5}$$

$$As^2 + 5A + Bs^2 + Cs = 1$$

$$\begin{cases} s^2 & A+B=0 \quad B=-\frac{1}{5} \\ s^1 & C=0 \\ s^0 & 5A=1 \quad A=\frac{1}{5} \end{cases}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{5}\frac{1}{s} - \frac{1}{5}\frac{s}{s^2+5}\right\}$$

$$= \frac{1}{5}\left(t - \cos\sqrt{5}t\right)$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{3}{s^2+9}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \sin\omega t$$

$$= \sin 3t$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{2}{s^2+4}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \sin\omega t$$

$$= \sin 2t$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{3}{(2s+5)^3}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{3}{(2s+5)^3}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3}{2^3\left(s+\frac{5}{8}\right)^3}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{n!}{(s+a)^{n+1}}\right\} = t^n e^{-at}$$

$$\boxed{= \frac{3}{16}t^2 e^{-5t/8}}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{6}{(s-1)^4}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{6}{(s-1)^4}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{n!}{(s+a)^{n+1}}\right\} = t^n e^{-at}$$

$$\boxed{= t^3 e^t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{5}{(s+2)^4}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{5}{(s+2)^4}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{n!}{(s+a)^{n+1}}\right\} = t^n e^{-at}$$

$$= \frac{5}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{(s+2)^4}\right\}$$

$$\boxed{= \frac{5}{6}t^3 e^{-2t}}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{s-1}{s^2-2s+5}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s-1}{s^2-2s+5}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+\omega^2}\right\} = e^{-at}\cos\omega t$$

$$\boxed{= e^{t}\cos 2t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{3s+2}{s^2+2s+10}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{3s+2}{s^2+2s+10}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3s+3-1}{(s+1)^2+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3(s+1)}{(s+1)^2+9} - \frac{1}{(s+1)^2+9}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+\omega^2}\right\} = e^{-at}\cos\omega t$$

$$\boxed{= 3e^{t}\cos 3t - \frac{1}{3}e^{-t}\sin 3t} \qquad \mathcal{L}^{-1}\left\{\frac{\omega}{(s+a)^2+\omega^2}\right\} = e^{-at}\sin\omega t$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{s}{s^2+2s-3}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s-1)(s+3)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s-1} + \frac{3}{4}\frac{1}{s+3}\right\}$$

$$\boxed{= -\frac{1}{4}e^{t} + \frac{3}{4}e^{-3t}}$$

$$\frac{s}{(s-1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+3}$$

$$s = As + 3A + Bs - B$$

$$\begin{cases} s & A+B=1 & \boxed{B=\frac{3}{4}} \\ s^0 & 3A-B=0 & \boxed{A=\frac{1}{4}} \end{cases}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{1}{s^2+2s-20}$

Solution

$$s^2+2s-20=0 \quad \to \quad \boxed{s_{1,2} = -1 \pm 2\sqrt{21}}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{\left(s+1+2\sqrt{21}\right)\left(s+1-2\sqrt{21}\right)}\right\}$$

$$\frac{1}{\left(s+1+2\sqrt{21}\right)\left(s+1-2\sqrt{21}\right)} = \frac{A}{s+1+2\sqrt{21}} + \frac{B}{s+1-2\sqrt{21}}$$

$$1 = sA + \left(1-2\sqrt{21}\right)A + sB + \left(1+2\sqrt{21}\right)B$$

$$\begin{cases} s & A+B=0 & A=-B \\ s^0 & \left(1-2\sqrt{21}\right)A+\left(1+2\sqrt{21}\right)B=1 & \rightarrow \left(1-2\sqrt{21}-1-2\sqrt{21}\right)A=1 \end{cases}$$

$$\boxed{A = -\frac{1}{4\sqrt{21}} \quad B = \frac{1}{4\sqrt{21}}}$$

$$= \mathcal{L}^{-1}\left\{-\frac{1}{4\sqrt{21}}\frac{1}{s+1+2\sqrt{21}} + \frac{1}{4\sqrt{21}}\frac{1}{s+1-2\sqrt{21}}\right\}$$

$$\boxed{= -\frac{1}{4\sqrt{21}}e^{\left(-1-2\sqrt{21}\right)t} + \frac{1}{4\sqrt{21}}e^{\left(-1+2\sqrt{21}\right)t}}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{s+1}{s^2+2s+10}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+10}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+\omega^2}\right\} = e^{-at}\cos\omega t$$

$$\boxed{= e^{-t}\cos 3t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{1}{s^2+4s+8}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+8}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+4}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{\omega}{(s+a)^2+\omega^2}\right\} = e^{-at}\sin\omega t$$

$$\boxed{= \frac{1}{2}e^{-2t}\sin 2t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{2s+16}{s^2+4s+13}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{2s+16}{s^2+4s+13}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2(s+2)+12}{(s+2)^2+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2(s+2)}{(s+2)^2+3^2} + \frac{4(3)}{(s+2)^2+3^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{\omega}{(s+a)^2+\omega^2}\right\} = e^{-at}\sin\omega t \qquad \mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+\omega^2}\right\} = e^{-at}\cos\omega t$$

$$\boxed{= 2e^{-2t}\cos 3t + 4e^{-2t}\sin 3t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{s-1}{2s^2+s+6}$

Solution

$$\frac{s-1}{2s^2+s+6} = \frac{s-1}{2\left(s^2+\frac{1}{2}s+3\right)}$$

$$= \frac{1}{2}\frac{s-1}{\left(s+\frac{1}{4}\right)^2+\frac{47}{16}}$$

$$= \frac{1}{2}\frac{s+\frac{1}{4}-\frac{5}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{47}{16}}$$

$$= \frac{1}{2}\left[\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{47}{16}} - \frac{5}{4}\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{47}{16}}\right]$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s-1}{2s^2+s+6}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{ \frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\left(\frac{\sqrt{47}}{4}\right)^2} - \frac{5}{4}\frac{4}{\sqrt{47}}\frac{\frac{\sqrt{47}}{4}}{\left(s+\frac{1}{4}\right)^2+\left(\frac{\sqrt{47}}{4}\right)^2} \right\}$$

$$\mathcal{L}^{-1}\left\{\frac{\omega}{(s+a)^2+\omega^2}\right\} = e^{-at}\sin\omega t \qquad \mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+\omega^2}\right\} = e^{-at}\cos\omega t$$

$$= \frac{1}{2}e^{-t/4}\cos\left(\frac{\sqrt{47}}{4}t\right) - \frac{5}{2\sqrt{47}}e^{-t/4}\sin\left(\frac{\sqrt{47}}{4}t\right)$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{s^2+1}{s^3-2s^2-8s}$

Solution

$$\frac{s^2+1}{s^3-2s^2-8s} = \frac{A}{s} + \frac{B}{s-4} + \frac{C}{s+2}$$

$$s^2+1 = As^2 - 2As - 8A + Bs^2 + 2Bs + Cs^2 - 4Cs$$

$$\begin{array}{l} s^2 \\ s^1 \\ s^0 \end{array} \begin{cases} A+B+C=1 \\ -2A+2B-4C=0 \\ -8A=1 \end{cases} \Rightarrow A = -\frac{1}{8} \quad B = \frac{17}{24} \quad C = \frac{5}{12}$$

$$F(s) = -\frac{1}{8}\frac{1}{s} + \frac{17}{24}\frac{1}{s-4} + \frac{5}{12}\frac{1}{s+2}$$

$$f(t) = \mathcal{L}^{-1}\left\{-\frac{1}{8}\frac{1}{s} + \frac{17}{24}\frac{1}{s-4} + \frac{5}{12}\frac{1}{s+2}\right\}$$

$$= -\frac{1}{8}t + \frac{17}{24}e^{4t} + \frac{5}{12}e^{-2t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{6s+3}{s^4+5s^2+4}$

Solution

$$F(s) = \frac{6s+3}{s^4+5s^2+4} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$6s+3 = As^3 + 4As + Bs^2 + 4B + Cs^3 + Cs + Ds^2 + D$$

$$\begin{cases} s^3 & A + C = 0 \\ s^2 & B + D = 0 \\ s^1 & 4A + C = 6 \\ s^0 & 4B + D = 3 \end{cases} \qquad \begin{array}{ll} A = 2 & B = 1 \\ C = -2 & D = -1 \end{array}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{2s}{s^2+1} + \frac{1}{s^2+1} - \frac{2s}{s^2+4} - \frac{1}{s^2+4}\right\}$$

$$\boxed{= 2\cos t + \sin t - 2\cos 2t - \frac{1}{2}\sin 2t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{s-3}{\left(s-\sqrt{3}\right)\left(s+\sqrt{3}\right)}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s-3}{\left(s-\sqrt{3}\right)\left(s+\sqrt{3}\right)}\right\}$$

$$\frac{s-3}{\left(s-\sqrt{3}\right)\left(s+\sqrt{3}\right)} = \frac{A}{s-\sqrt{3}} + \frac{B}{s+\sqrt{3}}$$

$$s - 3 = sA + \sqrt{3}A + sB - \sqrt{3}B$$

$$\begin{array}{ll} s & A + B = 1 \\ s^0 & \sqrt{3}A - \sqrt{3}B = -3 \end{array}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ \sqrt{3} & -\sqrt{3} \end{vmatrix} = -2\sqrt{3} \quad \Delta_A = \begin{vmatrix} 1 & 1 \\ -3 & -\sqrt{3} \end{vmatrix} = 3 - \sqrt{3} \quad \Delta_B = \begin{vmatrix} 1 & 1 \\ \sqrt{3} & -3 \end{vmatrix} = -3 - \sqrt{3}$$

$$\boxed{A = \frac{-3+\sqrt{3}}{2\sqrt{3}} = \frac{1-\sqrt{3}}{2} \quad B = \frac{3+\sqrt{3}}{2\sqrt{3}} = \frac{\sqrt{3}+1}{2}}$$

$$= \mathcal{L}^{-1}\left\{\frac{1-\sqrt{3}}{2}\frac{1}{s-\sqrt{3}} + \frac{1+\sqrt{3}}{2}\frac{1}{s+\sqrt{3}}\right\}$$

$$\boxed{= \frac{1-\sqrt{3}}{2}e^{\sqrt{3}t} + \frac{1+\sqrt{3}}{2}e^{\sqrt{3}t}}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{1}{\left(s^2+1\right)\left(s^2+4\right)}$

Solution

$$F(s)=\frac{1}{\left(s^2+1\right)\left(s^2+4\right)}=\frac{A}{s^2+1}+\frac{B}{s^2+4}$$

$$(A+B)s^2+4A+B=1$$

$$\begin{cases} A+B=0 \\ 4A+B=1 \end{cases} \rightarrow \quad \underline{A=\frac{1}{3};\ \ B=-\frac{1}{3}}$$

$$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{3}\frac{1}{s^2+1}-\frac{1}{3}\frac{2}{2}\frac{1}{s^2+4}\right\}$$

$$\underline{f(t)=\frac{1}{3}\sin t-\frac{1}{6}\sin 2t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{2s-4}{\left(s^2+s\right)\left(s^2+1\right)}$

Solution

$$F(s)=\frac{2s-4}{\left(s^2+s\right)\left(s^2+1\right)}=\frac{A}{s}+\frac{B}{s+1}+\frac{Cs+D}{s^2+1}$$

$$As^3+4As+As^2+4A+Bs^3+Bs+Cs^3+Cs^2+Ds^2+Ds=2s-4$$

$$\begin{cases} s^3 & A+B+C=0 \\ s^2 & A+C+D=0 \\ s^1 & 4A+B+D=2 \\ s^0 & 4A=-4 \rightarrow \underline{A=-1} \end{cases}$$

$$\rightarrow \begin{cases} B+C=1 & \rightarrow & & 6-D+1-D=1 \Rightarrow \underline{D=3} \\ C+D=1 & \rightarrow & C=1-D & \underline{\rightarrow C=-2} \\ B+D=6 & \rightarrow & B=6-D & \underline{\rightarrow B=3} \end{cases}$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{-1}{s}+\frac{3}{s+1}-\frac{2s}{s^2+1}+\frac{3}{s^2+1}\right\}$$

$$\underline{=-t+3e^{-t}-2\cos t+3\sin t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{s}{(s+2)\left(s^2+4\right)}$

Solution

$$F(s)=\frac{s}{(s+2)\left(s^2+4\right)}=\frac{A}{s+2}+\frac{Bs+C}{s^2+4}$$

$$As^2+4A+Bs^2+2Bs+Cs+2C=s$$

$$\begin{cases} s^2 & A+B=0 & \to A=-B & \boxed{A=-\frac{1}{4}} \\ s^1 & 2B+C=1 & \to C=1-2B & \boxed{C=\frac{1}{2}} \\ s^0 & 4A+2C=0 & \Rightarrow \quad -4B+2-4B=0 \Rightarrow & \boxed{B=\frac{1}{4}} \end{cases}$$

$$f(t)=\mathcal{L}^{-1}\left\{-\frac{1}{4}\frac{1}{s+2}+\frac{1}{4}\frac{s}{s^2+4}+\frac{1}{2}\frac{2}{2}\frac{1}{s^2+4}\right\}$$

$$\boxed{=-\frac{1}{4}e^{-2t}+\frac{1}{4}\cos 2t+\frac{1}{4}\sin 2t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{s^2+1}{s(s-1)(s+1)(s-2)}$

Solution

$$F(s)=\frac{s^2+1}{s(s-1)(s+1)(s-2)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+1}+\frac{D}{s-2}$$

$$A\left(s^2-1\right)(s-2)+Bs(s+1)(s-2)+Cs(s-1)(s-2)+Ds\left(s^2-1\right)=s^2+1$$

$$As^3-2As^2-As+2A+Bs^3-Bs^2-2Bs+Cs^3-3Cs^2+2Cs+Ds^3-Ds=s^2+1$$

$$\begin{array}{ll} s^3 & A+B+C+D=0 \\ s^2 & -2A-B-3C=1 \\ s^1 & -A-2B+2C-D=0 \\ s^0 & 2A=1 \to A=\frac{1}{2} \end{array} \quad \to \quad \begin{cases} B+C+D=-\frac{1}{2} & B=-1 \\ B+3C=-2 & \to C=-\frac{1}{3} \\ -2B+2C-D=\frac{1}{2} & D=\frac{5}{6} \end{cases}$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s}-\frac{1}{s-1}-\frac{1}{3}\frac{1}{s+1}+\frac{5}{6}\frac{1}{s-2}\right\}$$

$$\boxed{=\frac{1}{2}t-e^{t}-\frac{1}{3}e^{-t}+\frac{5}{6}e^{2t}}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{s}{(s-2)(s-3)(s-6)}$

Solution

$$F(s)=\frac{s}{(s-2)(s-3)(s-6)}=\frac{A}{s-2}+\frac{B}{s-3}+\frac{C}{s-6}$$

$$As^2-9As+18A+Bs^2-8Bs+12B+Cs^2-5Cs+6C=s$$

$$\begin{cases} s^2 & A+B+C=0 \\ s^1 & -9A-8B-5C=1 \\ s^0 & 18A+12B+6C=0 \end{cases}$$

$$\rightarrow \quad \boxed{A=\frac{1}{2} \quad B=-1 \quad C=\frac{1}{2}}$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s-2}-\frac{1}{s-3}+\frac{1}{2}\frac{1}{s-6}\right\}$$

$$\boxed{=\frac{1}{2}e^{2t}-e^{3t}+\frac{1}{2}e^{6t}}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{7s-1}{(s+1)(s+2)(s-3)}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{7s-1}{(s+1)(s+2)(s-3)}\right\}$$

$$\frac{7s-1}{(s+1)(s+2)(s-3)}=\frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{s-3}$$

$$7s-1=As^2-As-6A+Bs^2-2Bs-3B+Cs^2+3Cs+2C$$

$$\begin{cases} s^2 & A+B+C=0 \\ s & -A-2B+3C=7 \\ s^0 & -6A-3B+2C=-1 \end{cases}$$

$$\Delta=\begin{vmatrix}1 & 1 & 1\\ -1 & -2 & 3\\ -6 & -3 & 2\end{vmatrix}=-20 \quad \Delta_A=\begin{vmatrix}0 & 1 & 1\\ 7 & -2 & 3\\ -1 & -3 & 2\end{vmatrix}=-40 \quad \Delta_B=\begin{vmatrix}1 & 0 & 1\\ -1 & 7 & 3\\ -6 & -1 & 2\end{vmatrix}=60$$

$$\boxed{A=2, \quad B=-3, \quad C=1}$$

$$=\mathcal{L}^{-1}\left\{\frac{2}{s+1}-\frac{3}{s+2}+\frac{1}{s-3}\right\}$$

$$\boxed{=2e^{-t}-3e^{-2t}+e^{3t}}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{s^2+9s+2}{(s-1)^2(s+3)}$

Solution

$$\frac{s^2+9s+2}{(s-1)^2(s+3)}=\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{s+3}$$

$$s^2+9s+2=As^2+2As-3A+Bs+3B+Cs^2-2Cs+C$$

$$\begin{cases} s^2 & A+C=1 \\ s & 2A+B-2C=9 \\ s^0 & -3A+3B+C=2 \end{cases}$$

$$\Delta=\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & -2 \\ -3 & 3 & 1 \end{vmatrix}=16 \quad \Delta_A=\begin{vmatrix} 1 & 0 & 1 \\ 9 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}=32 \quad \Delta_B=\begin{vmatrix} 1 & 1 & 1 \\ 2 & 9 & -2 \\ -3 & 2 & 1 \end{vmatrix}=48$$

$$\underline{A=2, \quad B=3, \quad C=-1}$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{2}{s-1}+\frac{3}{(s-1)^2}-\frac{1}{s+3}\right\}$$

$$\underline{=2e^t+3te^t-e^{-3t}}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{2s^2+10s}{\left(s^2-2s+5\right)(s+1)}$

Solution

$$\frac{2s^2+10s}{\left(s^2-2s+5\right)(s+1)}=\frac{2s^2+10s}{\left((s-1)^2+4\right)(s+1)}=\frac{A(s-1)+B}{(s-1)^2+4}+\frac{C}{s+1}$$

$$2s^2+10s=As^2-A+Bs+B+Cs^2-2Cs+5C$$

$$\begin{cases} s^2 & A+C=2 \\ s & B-2C=10 \\ s^0 & -A+B+5C=0 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 5 \end{vmatrix} = 8 \quad \Delta_A = \begin{vmatrix} 2 & 0 & 1 \\ 10 & 1 & -2 \\ 0 & 1 & 5 \end{vmatrix} = 24 \quad \Delta_B = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 10 & -2 \\ -1 & 0 & 5 \end{vmatrix} = 64$$

$$\underline{A = 3, \quad B = 8, \quad C = -1}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{2s^2 + 10s}{\left(s^2 - 2s + 5\right)(s+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3(s-1)}{(s-1)^2 + 4} + \frac{4(2)}{(s-1)^2 + 4} - \frac{1}{s+1}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2 + \omega^2}\right\} = e^{-at}\cos\omega t \qquad \mathcal{L}^{-1}\left\{\frac{\omega}{(s+a)^2 + \omega^2}\right\} = e^{-at}\sin\omega t$$

$$\underline{= 3e^t \cos 2t + 4e^t \sin 2t - e^{-t}}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{s^2 - 26s - 47}{(s-1)(s+2)(s+5)}$

Solution

$$\frac{s^2 - 26s - 47}{(s-1)(s+2)(s+5)} = \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s+5}$$

$$s^2 - 26s - 47 = As^2 + 7As + 10A + Bs^2 + 4Bs - 5B + Cs^2 + Cs - 2C$$

$$\begin{cases} s^2 & A + B + C = 1 \\ s & 7A + 4B + C = -26 \\ s^0 & 10A - 5B - 2C = -47 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 7 & 4 & 1 \\ 10 & -5 & -2 \end{vmatrix} = -54 \quad \Delta_A = \begin{vmatrix} 1 & 1 & 1 \\ -26 & 4 & 1 \\ -47 & -5 & -2 \end{vmatrix} = 216 \quad \Delta_B = \begin{vmatrix} 1 & 1 & 1 \\ 7 & -26 & 1 \\ 10 & -47 & -2 \end{vmatrix} = 54$$

$$\underline{A = -4, \quad B = -1, \quad C = 6}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s^2 - 26s - 47}{(s-1)(s+2)(s+5)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{-4}{s-1} - \frac{1}{s+2} + \frac{6}{s+5}\right\}$$

$$\underline{= -4e^t - e^{-2t} + 6e^{-5t}}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{-s-7}{(s-1)(s+2)}$

Solution

$$\frac{-s-7}{(s-1)(s+2)}=\frac{A}{s-1}+\frac{B}{s+2}$$

$$-s-7=As+2A+Bs-B$$

$$\begin{cases} s & A+B=-1 \\ s^0 & 2A-B=-7 \end{cases}$$

$$\boxed{A=-\frac{8}{3},\quad B=\frac{5}{3}}$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{-s-7}{(s-1)(s+2)}\right\}$$

$$=\mathcal{L}^{-1}\left\{-\frac{8}{3}\frac{1}{s-1}+\frac{5}{3}\frac{1}{s+2}\right\}$$

$$\boxed{=-\frac{8}{3}e^{t}+\frac{5}{3}e^{-2t}}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{-8s^2-5s+9}{\left(s^2-3s+2\right)(s+1)}$

Solution

$$\frac{-8s^2-5s+9}{\left(s^2-3s+2\right)(s+1)}=\frac{-8s^2-5s+9}{(s-1)(s-2)(s+1)}$$

$$=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+1}$$

$$-8s^2-5s+9=As^2-As-2A+Bs^2-B+Cs^2-3Cs+2C$$

$$\begin{cases} s^2 & A+B+C=-8 \\ s & -A-3C=-5 \\ s^0 & -2A-B+2C=9 \end{cases}$$

$$\Delta=\begin{vmatrix}1&1&1\\-1&0&-3\\-2&-1&2\end{vmatrix}=6 \quad \Delta_A=\begin{vmatrix}-8&1&1\\-5&0&-3\\9&-1&2\end{vmatrix}=12 \quad \Delta_B=\begin{vmatrix}1&-8&1\\-1&-5&-3\\-2&9&2\end{vmatrix}=-66$$

$$\boxed{A=2,\quad B=-11,\quad C=1}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{-8s^2 - 5s + 9}{\left(s^2 - 3s + 2\right)(s+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s-1} - \frac{11}{s-2} + \frac{1}{s+1}\right\}$$

$$\underline{= 2e^t - 11e^{2t} + e^{-t}}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{-2s^2 + 8s - 14}{(s+1)\left(s^2 - 2s + 5\right)}$

Solution

$$\frac{-2s^2 + 8s - 14}{(s+1)\left(s^2 - 2s + 5\right)} = \frac{-2s^2 + 8s - 14}{(s-1)\left((s-1)^2 + 4\right)}$$

$$= \frac{A}{s-1} + \frac{B(s-1) + C}{(s-1)^2 + 4}$$

$$-2s^2 + 8s - 14 = As^2 - 2As + 5A + Bs^2 - 2Bs + B + Cs - C$$

$$\begin{cases} s^2 & A + B = -2 \\ s & -2A - 2B + C = 8 \\ s^0 & 5A + B - C = -14 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ -2 & -2 & 1 \\ 5 & 1 & -1 \end{vmatrix} = 4 \quad \Delta_A = \begin{vmatrix} -2 & 1 & 0 \\ 8 & -2 & 1 \\ -14 & 1 & -1 \end{vmatrix} = -8 \quad \Delta_B = \begin{vmatrix} 1 & -2 & 0 \\ -2 & 8 & 1 \\ 5 & -14 & -1 \end{vmatrix} = 0 \quad \Delta_C = \begin{vmatrix} 1 & 1 & -2 \\ -2 & -2 & 8 \\ 5 & 1 & -14 \end{vmatrix} = 16$$

$$\underline{A = -2, \quad B = 0, \quad C = 4}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{-2s^2 + 8s - 14}{(s+1)\left(s^2 - 2s + 5\right)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{-2}{s-1} + \frac{4}{(s-1)^2 + 2^2}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{\omega}{(s+a)^2 + \omega^2}\right\} = e^{-at}\sin\omega t$$

$$\underline{= -2e^t + 2e^t \sin 2t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{-5s-36}{(s+2)\left(s^2+9\right)}$

Solution

$$\frac{-5s-36}{(s+2)\left(s^2+9\right)}=\frac{A}{s+2}+\frac{Bs+C}{s^2+9}$$

$$-5s-36=As^2+9A+Bs^2+2Bs+Cs+2C$$

$$\begin{cases} s^2 & A+B=0 \\ s & 2B+C=-5 \\ s^0 & 9A+2C=-6 \end{cases}$$

$$\Delta=\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 9 & 0 & 2 \end{vmatrix}=13 \quad \Delta_A=\begin{vmatrix} 0 & 1 & 0 \\ -5 & 2 & 1 \\ -36 & 0 & 2 \end{vmatrix}=-26$$

$$\underline{A=-2, \quad B=2, \quad C=-9}$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{-2}{s+2}+\frac{2s}{s^2+3^2}-\frac{9}{s^2+3^2}\right\}$$

$$\underline{=-2e^{-2t}+2\cos 3t-3\sin 3t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{3s^2+5s+3}{s^4+s^3}$

Solution

$$\frac{3s^2+5s+3}{s^4+s^3}=\frac{3s^2+5s+3}{s^3(s+1)}$$

$$=\frac{A}{s^3}+\frac{B}{s^2}+\frac{C}{s}+\frac{D}{s+1}$$

$$3s^2+5s+3=As+A+Bs^2+Bs+Cs^3+Cs^2+Ds^3$$

$$\begin{cases} s^3 & C+D=0 & \underline{D=-1} \\ s^2 & B+C=3 & \underline{C=1} \\ s & A+B=5 & \underline{B=2} \\ s^0 & \underline{A=3} & \end{cases}$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{3s^2+5s+3}{s^4+s^3}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{3}{s^3}+\frac{2}{s^2}+\frac{1}{s}-\frac{1}{s+1}\right\}$$

$$=\frac{3}{2}t^2+2t+1-e^{-t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{7s^3-2s^2-3s+6}{s^3(s-2)}$

Solution

$$\frac{7s^3-2s^2-3s+6}{s^3(s-2)}=\frac{A}{s^3}+\frac{B}{s^2}+\frac{C}{s}+\frac{D}{s-2}$$

$$7s^3-2s^2-3s+6=As-2A+Bs^2-2Bs+Cs^3-2Cs^2+Ds^3$$

$$\begin{cases} s^3 & C+D=7 & \underline{D=6} \\ s^2 & B-2C=-2 & \underline{C=1} \\ s & A-2B=-3 & \underline{B=0} \\ s^0 & -2A=6 & \underline{A=-3} \end{cases}$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{7s^3-2s^2-3s+6}{s^3(s-2)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{-3}{s^3}+\frac{1}{s}+\frac{6}{s-2}\right\}$$

$$=-\frac{3}{2}t^2+1+6e^{2t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{7s^2-41s+84}{(s-1)\left(s^2-4s+13\right)}$

Solution

$$\frac{7s^2-41s+84}{(s-1)\left(s^2-4s+13\right)}=\frac{A}{s-1}+\frac{B(s-2)+C}{(s-2)^2+9}$$

$$7s^2-41s+84=As^2-4As+13A+Bs^2-3Bs+2B+Cs-C$$

$$\begin{cases} s^2 & A+B=7 \\ s & -4A-3B+C=-41 \\ s^0 & 13A+2B-C=84 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ -4 & -3 & 1 \\ 13 & 2 & -1 \end{vmatrix} = 10 \quad \Delta_A = \begin{vmatrix} 7 & 1 & 0 \\ -41 & -3 & 1 \\ 84 & 2 & -1 \end{vmatrix} = 50 \quad \Delta_B = \begin{vmatrix} 1 & 7 & 0 \\ -4 & -41 & 1 \\ 13 & 84 & -1 \end{vmatrix} = 20$$

$$\underline{A=5 \quad B=2 \quad C=-15}$$

$$f(t) = \mathcal{L}^{-1}\left\{ \frac{7s^2-41s+84}{(s-1)\left(s^2-4s+13\right)} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{5}{s-1} + \frac{2(s-2)}{(s-2)^2+3^2} - \frac{5(3)}{(s-2)^2+3^2} \right\}$$

$$\underline{= 5e^t + 2e^{2t}\cos 3t - 5e^{2t}\sin 3t}$$

Exercise

Find the inverse Laplace transform of $F(s) = \dfrac{6s-5}{s^2+7}$

Solution

$$F(s) = \frac{6s}{s^2+7} - \frac{5}{s^2+7}$$

$$= \frac{6s}{s^2+7} - \frac{\sqrt{7}}{\sqrt{7}}\frac{5}{s^2+7}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{ \frac{6s}{s^2+7} - \frac{5}{\sqrt{7}}\frac{\sqrt{7}}{s^2+7} \right\}$$

$$\underline{f(t) = 6\cos\sqrt{7}t - \frac{5}{\sqrt{7}}\sin\sqrt{7}t}$$

Exercise

Find the inverse Laplace transform of $F(s) = \dfrac{1-3s}{s^2+8s+21}$

Solution

$$s^2+8s+21 = s^2+8s+16-16+21$$

$$= (s+4)^2+5$$

$$F(s)=\frac{1-3s}{s^2+8s+21}$$

$$=\frac{1-3(s+4)+12}{(s+4)^2+5}$$

$$=\frac{13}{(s+4)^2+5}-3\frac{s+4}{(s+4)^2+5}$$

$$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{13}{(s+4)^2+5}-3\frac{s+4}{(s+4)^2+5}\right\}$$

$$\boxed{f(t)=\frac{13}{\sqrt{5}}e^{-4t}\sin\sqrt{5}t-3e^{-4t}\cos\sqrt{5}t}$$

Exercise

Find the inverse Laplace transform of $\quad F(s)=\dfrac{3s-2}{2s^2-6s-2}$

Solution

$$2s^2-6s-2=2\left(s^2-3s-1\right)$$

$$=2\left(s-\frac{3-\sqrt{13}}{2}\right)\left(s-\frac{3+\sqrt{13}}{2}\right)$$

$$=2\left[\left(s-\frac{3}{2}\right)^2-\frac{9}{4}-1\right]$$

$$=2\left(\left(s-\frac{3}{2}\right)^2-\frac{13}{4}\right)$$

$$F(s)=\frac{1}{2}\frac{3\left(s-\frac{3}{2}\right)+\frac{9}{2}-2}{\left(s-\frac{3}{2}\right)^2-\frac{13}{4}}$$

$$=\frac{3}{2}\frac{s-\frac{3}{2}}{\left(s-\frac{3}{2}\right)^2-\frac{13}{4}}+\frac{5}{4}\frac{1}{\left(s-\frac{3}{2}\right)^2-\frac{13}{4}}$$

$$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{3}{2}\frac{s-\frac{3}{2}}{\left(s-\frac{3}{2}\right)^2-\frac{13}{4}}+\frac{5}{2\sqrt{13}}\frac{\frac{\sqrt{13}}{2}}{\left(s-\frac{3}{2}\right)^2-\frac{13}{4}}\right\}$$

$$\boxed{f(t)=\frac{3}{2}e^{3t/2}\cosh\left(\frac{\sqrt{13}}{2}t\right)+\frac{5}{2\sqrt{13}}e^{3t/2}\sinh\left(\frac{\sqrt{13}}{2}t\right)}$$

Exercise

Find the inverse Laplace transform of $F(s)=\dfrac{s+7}{s^2-3s-10}$

Solution

$$F(s)=\frac{s+7}{(s+2)(s-5)}=\frac{A}{s+2}+\frac{B}{s-5}$$

$$s+7=As-5A+Bs+2B$$

$$\begin{cases} A+B=1 \\ -5A+2B=7 \end{cases} \rightarrow \boxed{A=-\frac{5}{7},\ \ B=\frac{12}{7}}$$

$$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{-\frac{5}{7}\frac{1}{s+2}+\frac{12}{7}\frac{1}{s-5}\right\}$$

$$\boxed{f(t)=-\frac{5}{7}e^{-2t}+\frac{12}{7}e^{5t}}$$

Exercise

Find the inverse Laplace transform of $F(s)=\dfrac{86s-78}{(s+3)(s-4)(5s-1)}$

Solution

$$F(s)=\frac{86s-78}{(s+3)(s-4)(5s-1)}=\frac{A}{s+3}+\frac{B}{s-4}+\frac{C}{5s-1}$$

$$86s-78=A(s-4)(5s-1)+B(s+3)(5s-1)+C(s+3)(s-4)$$

$$\begin{cases} s^2 & 5A+5B+C=0 \\ s & -21A+14B-C=86 \\ s^0 & 4A-3B-12C=-78 \end{cases}$$

$$\Delta=\begin{vmatrix} 5 & 5 & 1 \\ -21 & 14 & -1 \\ 4 & -3 & -12 \end{vmatrix}=-2128 \quad \Delta_A=\begin{vmatrix} 0 & 5 & 1 \\ 86 & 14 & -1 \\ -78 & -3 & -12 \end{vmatrix}=6384 \quad \Delta_B=\begin{vmatrix} 5 & 0 & 1 \\ -21 & 86 & -1 \\ 4 & -78 & -12 \end{vmatrix}=-4256$$

$$\boxed{A=-\frac{6384}{2128}=-3,\quad B=\frac{4256}{2128}=2,\quad C=5}$$

$$F(s)=-\frac{3}{s+3}+\frac{2}{s-4}+\frac{5}{5s-1}$$

$$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{-\frac{3}{s+3}+\frac{2}{s-4}+\frac{1}{s-\frac{1}{5}}\right\}$$

$$\boxed{f(t)=-3e^{-3t}+2e^{4t}+e^{t/5}}$$

Exercise

Find the inverse Laplace transform of $F(s)=\dfrac{2-5s}{(s-6)\left(s^2+11\right)}$

Solution

$$F(s)=\frac{2-5s}{(s-6)\left(s^2+11\right)}=\frac{A}{s-6}+\frac{Bs+C}{s^2+11}$$

$$2-5s=As^2+11A+Bs^2-6Bs+Cs-6C$$

$$\begin{cases} s^2 & A+B=0 \\ s & -6B+C=-5 \\ s^0 & 11A-6C=2 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -6 & 1 & -5 \\ 11 & 0 & -6 & 2 \end{array}\right)$$

$$\boxed{A=-\frac{28}{47}, \quad B=\frac{28}{47}, \quad C=-\frac{67}{47}}$$

$$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{-\frac{28}{47}\frac{1}{s-6}+\frac{28}{47}\frac{s}{s^2+11}-\frac{67}{47}\frac{1}{s^2+11}\frac{\sqrt{11}}{\sqrt{11}}\right\}$$

$$\boxed{f(t)=-\frac{28}{47}e^{6t}+\frac{28}{47}\cos\sqrt{11}t-\frac{67}{47\sqrt{11}}\sin\sqrt{11}t}$$

Exercise

Find the inverse Laplace transform of $F(s)=\dfrac{25}{s^3\left(s^2+4s+5\right)}$

Solution

$$F(s)=\frac{25}{s^3\left(s^2+4s+5\right)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{Ds+E}{s^2+4s+5}$$

$$25=As^4+4As^3+5As^2+Bs^3+4Bs^2+5Bs+Cs^2+4Cs+5C+Ds^4+Es^3$$

$$\begin{cases} s^4 & A+D=0 & \rightarrow \boxed{D=-\frac{11}{5}} \\ s^3 & 4A+B+E=0 & \rightarrow \boxed{E=-\frac{24}{5}} \\ s^2 & 5A+4B+C=0 & \rightarrow \boxed{A=\frac{11}{5}} \\ s & 5B+4C=0 & \rightarrow \boxed{B=-4} \\ s^0 & 5C=25 & \rightarrow \boxed{C=5} \end{cases}$$

$$F(s)=\frac{11}{5}\frac{1}{s}-\frac{4}{s^2}+\frac{5}{s^3}-\frac{1}{5}\frac{11s+24}{(s+2)^2-4+5}$$

$$=\frac{11}{5}\frac{1}{s}-\frac{4}{s^2}+\frac{5}{s^3}-\frac{1}{5}\frac{11(s+2)-22+24}{(s+2)^2+1}$$

$$=\frac{11}{5}\frac{1}{s}-\frac{4}{s^2}+\frac{5}{s^3}-\frac{1}{5}\frac{11(s+2)+2}{(s+2)^2+1}$$

$$=\frac{11}{5}\frac{1}{s}-\frac{4}{s^2}+\frac{5}{s^3}-\frac{11}{5}\frac{s+2}{(s+2)^2+1}-\frac{2}{5}\frac{1}{(s+2)^2+1}$$

$$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{11}{5}\frac{1}{s}-\frac{4}{s^2}+\frac{5}{s^3}-\frac{11}{5}\frac{s+2}{(s+2)^2+1}-\frac{2}{5}\frac{1}{(s+2)^2+1}\right\}$$

$$\boxed{f(t)=\frac{11}{5}-4t+\frac{5}{2}t^2-\frac{11}{5}e^{-2t}\cos t-\frac{2}{5}\sin t}$$

Exercise

Find the inverse Laplace transform of $F(s)=\dfrac{5e^{-6s}-3e^{-11s}}{(s+2)(s^2+9)}$

Solution

$$F(s)=\left(5e^{-6s}-3e^{-11s}\right)\frac{1}{(s+2)(s^2+9)}$$

$$=\left(5e^{-6s}-3e^{-11s}\right)G(s)$$

$$G(s)=\frac{1}{(s+2)(s^2+9)}=\frac{A}{s+2}+\frac{Bs+C}{s^2+9}$$

$$1=As^2+9A+Bs^2+2Bs+Cs+2C$$

$$\begin{cases} s^2 & A+B=0 \\ s & 2B+C=0 \\ s^0 & 9A+2C=1 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 9 & 0 & 2 & 1 \end{array}\right)$$

$$\boxed{A=\frac{1}{13},\quad B=-\frac{1}{13},\quad C=\frac{2}{13}}$$

$$\mathcal{L}^{-1}\{G(s)\}=\frac{1}{13}\mathcal{L}^{-1}\left\{\frac{1}{s+2}-\frac{s}{s^2+9}+\frac{2}{s^2+9}\right\}$$

$$g(t)=\frac{1}{13}\left(e^{-2t}-\cos 3t+\frac{2}{3}\sin 3t\right)$$

$$\boxed{f(t)=5u_6(t)g(t-6)-3u_{11}(t)g(t-11)}$$