Solution   *Section* 3.2 – Gaussian Elimination

Exercise

When elimination is applied to the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix}$

a) What are the first and second pivots?

b) What is the multiplier l_{21} in the first step (l_{21} times row 1 is subtracted from row 2)?

c) What entry in the 2, 2 position (instead of 9) would force an exchange of rows 2 and 3?

d) What is the multiplier $l_{31} = 0$, subtracting 0 times row 1 from row 3?

Solution

a) The first pivot is 3 and when 2 times row 1 is subtracted from row 2, the second pivot is revealed as 7.

$$\begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix} \begin{array}{c} \textit{subtract } 2 \textit{ times row.1} \\ \textit{from row.2} \end{array} \begin{bmatrix} 3 & 1 & 0 \\ 0 & \boxed{7} & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

b) The multiplier l_{21} in the first step is $\frac{6}{3} = 2$.

c) If we reduce the entry 9 to 2, that drop of 7 in the a_{22} position would force a row exchange.

$$\begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix} \begin{array}{c} \textit{subtract } 7 \textit{ times row.1} \\ \textit{from row.2} \end{array} \begin{bmatrix} 3 & 1 & 0 \\ -15 & \boxed{2} & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

d) The multiplier l_{31} is already zero because $a_{31} = 0$ and no needs row elimination.

Exercise

Use elimination to reach upper triangular matrices ***U***. Solve by back substitution or explain why this impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the $-x$ in equation (3).

$$\begin{cases} x+y+z=7 \\ x+y-z=5 \\ x-y+z=3 \end{cases} \qquad \begin{cases} x+y+z=7 \\ x+y-z=5 \\ -x-y+z=3 \end{cases}$$

Solution

For the *first* system:

$$\begin{matrix} x+y+z=7 & \textit{subtract eqn.1} & x+y+z=7 \\ x+y-z=5 & \textit{from eqn.2} & 0y-2z=-2 \\ x-y+z=3 & \textit{from eqn.3} & -2y-0z=-4 \end{matrix}$$

$$\begin{matrix} x+y+z=7 & & 1x+\ y+\ z=7 \\ x+y-z=5 & \textit{Exchange eqn.2} & -2y-0z=-4 \\ x-y+z=3 & \textit{and eqn.3} & -2z=-2 \end{matrix}$$

The solutions are: $z=1 \quad y=2 \quad x=4$ and the pivots are 1, -2, -2.

For the *second* system:

$$\begin{matrix} x+y+z=7 & \textit{Subtract eqn.1} & x+y+z=7 \\ x+y-z=5 & \textit{from eqn.2} & 0y-2z=-2 \\ -x-y+z=3 & \textit{Add eqn.1 to eqn.3} & 0y+2z=10 \end{matrix}$$

$$\begin{matrix} x+y+z=7 & & x+y+z=7 \\ 0y-2z=-2 & \textit{Add eqn.2} & 0y-2z=-2 \\ 0y+2z=10 & \textit{to eqn.3} & \boxed{0z=8} \end{matrix}$$

The three planes don’t meet. But if we change ‘3’ in the last equation to ‘-5’

$$\begin{matrix} x+y+z=7 & \textit{Subtract eqn.1} & x+y+z=7 \\ x+y-z=5 & \textit{from eqn.2} & 0y-2z=-2 \\ -x-y+z=-5 & \textit{Add eqn.1 to eqn.3} & 0y+2z=2 \end{matrix}$$

$$\begin{matrix} x+y+z=7 & x+y=6 \\ 0y-2z=-2 & \\ 0y+2z=10 & z=1 \end{matrix}$$

There are unique infinite many solutions!

The three planes now meet along a whole line.

Exercise

For which numbers a does the elimination break down (1) permanently (2) temporarily

$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= 6 \end{aligned}$$

Solve for x and y after fixing the second breakdown by a row change.

Solution

The matrix form is: $\begin{pmatrix} a & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$

If $a = 0$, the elimination brakes down temporarily.

$$\begin{pmatrix} 4 & 6 \\ 0 & \boxed{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$$

The system is in upper triangular form and entry row 2 column 2 is not equal to zero, therefore the system has a solution.

If $a \neq 0$,

$$\begin{pmatrix} a & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix} \qquad R_2 - \tfrac{4}{a} R_1$$

$$\begin{pmatrix} a & 3 \\ 0 & 6 - \frac{12}{a} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 + \frac{12}{a} \end{pmatrix}$$

$$6 - \frac{12}{a} = 0 \Rightarrow \frac{12}{a} = 6$$

$$\rightarrow \underline{\left| a = \tfrac{12}{6} = 2 \right|}$$

If $a = 2$,

$\begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \end{pmatrix}$, the system will fail and has no solution.

If $a \neq 2$;

$\begin{pmatrix} a & 3 \\ 0 & 6 - \frac{12}{a} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 + \frac{12}{a} \end{pmatrix}$, the system has a unique solution.

Exercise

Find the pivots and the solution for these four equations:

$$\begin{aligned} 2x + y \qquad &= 0 \\ x + 2y + z \quad &= 0 \\ y + 2z + t &= 0 \\ z + 2t &= 5 \end{aligned}$$

Solution

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} R_2 - \tfrac{1}{2}R_1$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 1.5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} R_3 - \tfrac{2}{3}R_2$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} R_4 - \tfrac{3}{4}R_3$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 5 \end{pmatrix} \begin{aligned} 2x = -y &\Rightarrow \lfloor x = -2\tfrac{1}{2} = -1 \rfloor \\ \tfrac{3}{2}y + z = 0 \Rightarrow y = -z\tfrac{2}{3} &= -(-3)\tfrac{2}{3} \to \boxed{y = 2} \\ \tfrac{4}{3}z + t = 0 \to \tfrac{4}{3}z = -t &\to \lfloor z = -4\tfrac{3}{4} = -3 \rfloor \\ \tfrac{5}{4}t = 5 &\to \boxed{t = 4} \end{aligned}$$

The pivots are diagonal entries and the solution is: $(-1,\ 2,\ -3,\ 4)$

Exercise

Look for a matrix that has row sums 4 and 8, and column sums 2 and s.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{array}{ll} a+b=4 & a+c=2 \\ c+d=8 & b+d=s \end{array}$$

The four equations are solvable only if s = ____. Then find two different matrices that have the correct row and column sums.

Solution

$$\begin{array}{rr} & a+b=4 \\ + & c+d=8 \\ \hline & a+c+b+d=12 \end{array}$$

$$2+s=12$$

$$\boxed{s=10}$$

Exercise

Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of A is a _______ of the first two rows. Find a third equation that can't be solved together with $x+y+z=0$ and $x-2y-z=1$

Solution

The system is singular if row 3 of A is a ***linear combination*** of the first two rows.

There are many possible of a third equation that can't be solved together with $x+y+z=0$ and $x-2y-z=1$.

$$\begin{array}{lr} 3 \text{ times } 1^{st} \text{ equation} & 3x+3y+3z \\ \quad \text{minus } 2nd & -x+2y+z \\ \hline & 2x+5y+4z=1 \end{array}$$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} -1 & -2 & 3 & 1 \\ 1 & 1 & 2 & 8 \\ \hline 0 & -1 & 5 & 9 \end{array} \qquad \begin{array}{cccc} 3 & -7 & 4 & 10 \\ -3 & -3 & -6 & -24 \\ \hline 0 & -10 & -2 & -14 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right] -R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 10R_2 \end{array}$$

$$\begin{array}{cccc} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ \hline 1 & 0 & 7 & 17 \end{array} \qquad \begin{array}{cccc} 0 & -10 & -2 & -14 \\ 0 & 10 & -50 & -90 \\ \hline 0 & 0 & -52 & -104 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right] -\frac{1}{52}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 - 7R_3 \\ R_2 + 5R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & 7 & 17 \\ 0 & 0 & -7 & -14 \\ \hline 1 & 0 & 0 & 3 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -5 & -9 \\ 0 & 0 & 5 & 10 \\ \hline 0 & 1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: $\boxed{(3,\ 1,\ 2)}$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \qquad\qquad - 3w = -3 \end{cases}$$

Solution

Solution: $\boxed{(w-1,\ 2z,\ z,\ w)}$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \qquad\qquad - 3w = -3 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ -1 & 3 & -2 & 1 \\ 3 & 4 & -7 & 10 \end{array}\right] \quad \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & -2 & -10 & -14 \end{array}\right] \quad \begin{array}{l} \\ \\ 5R_3 + 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & 0 & -52 & -52 \end{array}\right] \quad \begin{array}{rl} x + 2y + z = 8 & (3) \\ 5y - z = 9 & (2) \\ -52z = -52 & (1) \end{array}$$

$(1) \ \Rightarrow \ z = 1$

$(2) \ \Rightarrow \ 5y = 9 + 1 = 10 \rightarrow y = 2$

$(3) \ \Rightarrow \ x = 8 - 4 - 1 = 3$

$\therefore$ Solution: $(3,\ 2,\ 1)$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} 2u-3v+w-x+y=0 \\ 4u-6v+2w-3x-y=-5 \\ -2u+3v-2w+2x-y=3 \end{cases}$$

Solution

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 4 & -6 & 2 & -3 & -1 & -5 \\ -2 & 3 & -2 & 2 & -1 & 3 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3+R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -3 & -5 \\ 0 & 0 & -1 & 1 & 0 & 3 \end{array}\right] \qquad \begin{array}{rr} 2u-3v+w-x+y=0 & (3) \\ -x-3y=-5 & (2) \\ -w+x=3 & (1) \end{array}$$

$(2) \Rightarrow x=5-3y$

$(1) \Rightarrow w=x-3=2-3y$

$(3) \Rightarrow 2u=3v-2+3y+5-3y-y=3v-y+3$

$u=\frac{3}{2}v-\frac{1}{2}y+\frac{3}{2}$

$\therefore$ Solution: $\left(\frac{3}{2}v-\frac{1}{2}y+\frac{3}{2}, v, 2-3y, 5-3y, y\right)$

Exercise

Solve the given linear system by any method

$$\begin{cases} 2x_1+x_2+3x_3=0 \\ x_1+2x_2 \qquad =0 \\ \qquad x_2+x_3=0 \end{cases}$$

Solution

$$\begin{cases} x_1=-2x_2 \\ x_3=-x_2 \end{cases} \rightarrow -4x_2+x_2-3x_2=0 \Rightarrow x_2=0$$

Solution: $\boxed{(0,\ 0,\ 0)}$

Exercise

Solve the given linear system by any method $\begin{cases} 2x+2y+4z \quad\; =0 \\ \quad -y-3z+w=0 \\ 3x+y+z+2w=0 \\ x+3y-2z-2w=0 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & -2 & 0 \\ 0 & -1 & -3 & 1 & 0 \\ 3 & 1 & 1 & 2 & 0 \\ 2 & 2 & 4 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3-3R_1 \\ R_4-2R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & -2 & 0 \\ 0 & -1 & -3 & 1 & 0 \\ 0 & -8 & 7 & 8 & 0 \\ 0 & -4 & 8 & 4 & 0 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & -2 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & -8 & 7 & 8 & 0 \\ 0 & -4 & 8 & 4 & 0 \end{array}\right] \begin{array}{l} R_1-3R_2 \\ \\ R_3+8R_2 \\ R_4+4R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -11 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 31 & 0 & 0 \\ 0 & 0 & 20 & 0 & 0 \end{array}\right] \begin{array}{l} x+w=0 \\ y-w=0 \\ \to z=0 \\ \\ \end{array}$$

Solution: $\boxed{(-w,\ w,\ 0,\ w)}$

Exercise

Add 3 times the second row to the first of $\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$

Solution

$$E = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$EA=\begin{bmatrix}1&3&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}5&-1&5\\7&3&-2\\8&1&2\\6&0&-1\end{bmatrix}=\begin{bmatrix}27&8&-1\\7&3&-2\\8&1&2\\6&0&-1\end{bmatrix}$$

$$\begin{bmatrix}5&-1&5\\7&3&-2\\8&1&2\\6&0&-1\end{bmatrix}\ \ R_1+3R_2\ \ =\begin{bmatrix}27&8&-1\\7&3&-2\\8&1&2\\6&0&-1\end{bmatrix}$$

Exercise

Solve the system using Gaussian elimination $\begin{cases}3x_1+2x_2-x_3=-15\\5x_1+3x_2+2x_3=0\\3x_1+x_2+3x_3=11\\-6x_1-4x_2+2x_3=30\end{cases}$

Solution

$$\left[\begin{array}{ccc|c}3&2&-1&-15\\5&3&2&0\\3&1&3&11\\-6&-4&2&30\end{array}\right]\begin{array}{c}\\3R_2-5R_1\\R_3-R_1\\R_4+2R_1\end{array}$$

$$\left[\begin{array}{ccc|c}3&2&-1&-15\\0&-1&11&75\\0&-1&4&26\\0&0&0&0\end{array}\right]\begin{array}{c}\\\\R_3-R_2\\\\\end{array}$$

$$\left[\begin{array}{ccc|c}3&2&-1&-15\\0&-1&11&75\\0&0&-7&-49\\0&0&0&0\end{array}\right]\quad\begin{array}{ll}3x_1+2x_2-x_3=-15&(3)\\-x_2+11x_3=75&(2)\\-7x_3=-49&(1)\end{array}$$

$(1)\ \rightarrow\ x_3=7$

$(2)\ \rightarrow\ x_2=77-75=2$

$(1)\ \rightarrow\ 3x_1=-15-4+7=12\ \Rightarrow\ x_1=-4$

$\therefore$ Solution: $\boxed{(-4,2,7)}$

Exercise

For what value(s) of k, if any, does the system $\begin{cases} x+\ y-\ z=1 \\ 2x+3y+kz=3 \\ x+ky+3z=2 \end{cases}$ have

a) A unique solution?

b) Infinitely many solutions?

c) No solution?

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & k & 3 \\ 1 & k & 3 & 2 \end{array}\right] \begin{array}{l} R_2-2R_1 \\ R_3-R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & k-1 & 4 & 1 \end{array}\right] \ R_3-(k-1)R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & 0 & 4-(k-1)(k+2) & 2-k \end{array}\right] \quad \begin{array}{r} x=1-y+z \\ y=1-(k+2)z \\ \to\left(6-k^2-k\right)z=-(k-2) \end{array}$$

$$\begin{cases} z=-\dfrac{k-2}{-(k-2)(k+3)}=\dfrac{1}{k+3} \quad (k\neq 2,-3) \\ y=1-\dfrac{k+2}{k+3}=\dfrac{1}{k+3} \\ x=\dfrac{k+2}{k+3}+\dfrac{1}{k+3}=1 \end{cases}$$

a) Unique solution if $k\neq 2,-3$

b) Infinitely solution if $k=2$

c) No solution if $k=-3$