

Solution ***Section 2.3 – Orthogonality***

Exercise

Determine whether \vec{u} and \vec{v} are orthogonal

a) $\vec{u} = (-6, -2), \quad \vec{v} = (5, -7)$

b) $\vec{u} = (6, 1, 4), \quad \vec{v} = (2, 0, -3)$

c) $\vec{u} = (1, -5, 4), \quad \vec{v} = (3, 3, 3)$

d) $\vec{u} = (-2, 2, 3), \quad \vec{v} = (1, 7, -4)$

Solution

$$\begin{aligned} \text{a) } \vec{u} \cdot \vec{v} &= (-6)(5) + (-2)(-7) \\ &= -30 + 14 \\ &= \underline{-16 \neq 0} \end{aligned}$$

$\therefore \vec{u}$ and \vec{v} are not orthogonal

$$\begin{aligned} \text{b) } \vec{u} \cdot \vec{v} &= 6(2) + 1(0) + 4(-3) \\ &= \underline{0} \end{aligned}$$

$\therefore \vec{u}$ and \vec{v} are orthogonal

$$\begin{aligned} \text{c) } \vec{u} \cdot \vec{v} &= 1(3) - 5(3) + 4(3) \\ &= \underline{0} \end{aligned}$$

$\therefore \vec{u}$ and \vec{v} are orthogonal

$$\begin{aligned} \text{d) } \vec{u} \cdot \vec{v} &= -2(1) + 2(7) + 3(-4) \\ &= \underline{0} \end{aligned}$$

$\therefore \vec{u}$ and \vec{v} are orthogonal

Exercise

Determine whether the vectors form an orthogonal set

a) $\vec{v}_1 = (2, 3), \quad \vec{v}_2 = (3, 2)$

b) $\vec{v}_1 = (1, -2), \quad \vec{v}_2 = (-2, 1)$

c) $\vec{u} = (-4, 6, -10, 1) \quad \vec{v} = (2, 1, -2, 9)$

d) $\vec{u} = (a, b) \quad \vec{v} = (-b, a)$

e) $\vec{v}_1 = (-2, 1, 1), \quad \vec{v}_2 = (1, 0, 2), \quad \vec{v}_3 = (-2, -5, 1)$

$$f) \quad \vec{v}_1 = (1, 0, 1), \quad \vec{v}_2 = (1, 1, 1), \quad \vec{v}_3 = (-1, 0, 1)$$

$$g) \quad \vec{v}_1 = (2, -2, 1), \quad \vec{v}_2 = (2, 1, -2), \quad \vec{v}_3 = (1, 2, 2)$$

Solution

$$a) \quad \vec{v}_1 \cdot \vec{v}_2 = 2(3) + 3(2)$$

$$= 12 \neq 0$$

\therefore Vectors don't form an orthogonal set

$$b) \quad \vec{v}_1 \cdot \vec{v}_2 = 1(-2) - 2(1)$$

$$= -4 \neq 0$$

\therefore Vectors don't form an orthogonal set

$$c) \quad \vec{u} \cdot \vec{v} = -8 + 6 + 20 + 9$$

$$= 27 \neq 0$$

\therefore These vectors are not orthogonal

$$d) \quad \vec{u} \cdot \vec{v} = -ab + ab$$

$$= 0$$

\therefore These vectors are orthogonal

$$e) \quad \vec{v}_1 \cdot \vec{v}_2 = -2(1) + 1(0) + 1(2)$$

$$= 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = -2(-2) + 1(-5) + 1(1)$$

$$= 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = 1(-2) + 0(-5) + 2(1)$$

$$= 0$$

\therefore Vectors form an orthogonal set

$$f) \quad \vec{v}_1 \cdot \vec{v}_2 = 1(1) + 0(1) + 1(1)$$

$$= 2 \neq 0$$

\therefore Vectors don't form an orthogonal set

$$g) \quad \vec{v}_1 \cdot \vec{v}_2 = 2(2) - 2(1) + 1(-2)$$

$$= 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = 2(1) - 2(2) + 1(2)$$

$$= 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = 2(1) + 1(2) - 2(2)$$

$$\underline{= 0}$$

\therefore Vectors form an orthogonal set

Exercise

Find a unit vector that is orthogonal to both $\vec{u} = (1, 0, 1)$ and $\vec{v} = (0, 1, 1)$

Solution

Let $\vec{w} = (w_1, w_2, w_3)$ be the unit vector that is orthogonal to both \vec{u} and \vec{v} .

$$\vec{u} \cdot \vec{w} = 1(w_1) + 0(w_2) + 1(w_3)$$

$$= w_1 + w_3 = 0$$

$$\underline{w_3 = -w_1}$$

$$\vec{v} \cdot \vec{w} = 0(w_1) + 1(w_2) + 1(w_3)$$

$$= w_2 + w_3 = 0$$

$$\underline{w_3 = -w_2}$$

$$w_1 = w_2 = -w_3$$

The orthogonal vector to both \vec{u} and \vec{v} is $\vec{w} = (1, 1, -1)$, therefore the unit vector is

$$\frac{\vec{w}}{\|\vec{w}\|} = \frac{1}{\sqrt{1^2 + 1^2 + (-1)^2}}(1, 1, -1)$$

$$= \frac{1}{\sqrt{3}}(1, 1, -1)$$

$$= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right)$$

The possible vectors are: $\underline{\pm \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right)}$

Exercise

- Show that $\vec{v} = (a, b)$ and $\vec{w} = (-b, a)$ are orthogonal vectors.
- Use the result to find two vectors that are orthogonal to $\vec{v} = (2, -3)$.
- Find two unit vectors that are orthogonal to $(-3, 4)$

Solution

$$\begin{aligned}
 a) \quad \vec{v} \cdot \vec{w} &= a(-b) + b(a) \\
 &= -ab + ab \\
 &= \underline{0}
 \end{aligned}$$

\vec{v} and \vec{w} are orthogonal vectors.

$$b) \quad (2, 3) \text{ and } (-2, 3).$$

$$\begin{aligned}
 c) \quad \vec{u}_1 &= \frac{1}{\sqrt{4^2 + 3^2}}(4, 3) \\
 &= \underline{\left(\frac{4}{5}, \frac{3}{5}\right)}
 \end{aligned}$$

$$\begin{aligned}
 \vec{u}_2 &= -\frac{1}{\sqrt{4^2 + 3^2}}(4, 3) \\
 &= \underline{\left(-\frac{4}{5}, -\frac{3}{5}\right)}
 \end{aligned}$$

Exercise

Find the vector component of \vec{u} along \vec{a} and the vector component of \vec{u} orthogonal to

$$a) \quad \vec{u} = (6, 2), \quad \vec{a} = (3, -9)$$

$$d) \quad \vec{u} = (1, 1, 1), \quad \vec{a} = (0, 2, -1)$$

$$b) \quad \vec{u} = (3, 1, -7), \quad \vec{a} = (1, 0, 5)$$

$$e) \quad \vec{u} = (2, 1, 1, 2), \quad \vec{a} = (4, -4, 2, -2)$$

$$c) \quad \vec{u} = (1, 0, 0), \quad \vec{a} = (4, 3, 8)$$

$$f) \quad \vec{u} = (5, 0, -3, 7), \quad \vec{a} = (2, 1, -1, -1)$$

Solution

$$\begin{aligned}
 a) \quad \text{proj}_{\vec{a}} \vec{u} &= \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} \\
 &= \frac{6(3) + 2(-9)}{3^2 + (-9)^2} (3, -9) \\
 &= \frac{0}{90} (3, -9) \\
 &= \underline{(0, 0)}
 \end{aligned}$$

$$\begin{aligned}
 \vec{u} - \text{proj}_{\vec{a}} \vec{u} &= (6, 2) - (0, 0) \\
 &= \underline{(6, 2)}
 \end{aligned}$$

$$\begin{aligned}
 b) \quad \text{proj}_{\vec{a}} \vec{u} &= \frac{3(1) + 0 - 7(5)}{1^2 + 0 + 5^2} (1, 0, 5) \\
 &= \frac{-32}{26} (1, 0, 5)
 \end{aligned}$$

$$\text{proj}_{\vec{a}} \vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$$

$$= \left(-\frac{16}{13}, 0, -\frac{80}{13} \right) \Big|$$

$$\begin{aligned} \vec{u} - \text{proj}_{\vec{a}} \vec{u} &= (1, 0, 5) - \left(-\frac{16}{13}, 0, -\frac{80}{13} \right) \\ &= \left(\frac{55}{13}, 1, -\frac{11}{13} \right) \Big| \end{aligned}$$

$$\begin{aligned} c) \quad \text{proj}_{\vec{a}} \vec{u} &= \frac{1(4) + 0 + 0}{4^2 + 3^2 + 8^2} (4, 3, 8) \\ &= \frac{4}{89} (4, 3, 8) \\ &= \left(\frac{16}{89}, \frac{12}{89}, \frac{32}{89} \right) \Big| \end{aligned}$$

$$\text{proj}_{\vec{a}} \vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$$

$$\begin{aligned} \vec{u} - \text{proj}_{\vec{a}} \vec{u} &= (1, 0, 0) - \left(\frac{16}{89}, \frac{12}{89}, \frac{32}{89} \right) \\ &= \left(\frac{73}{89}, -\frac{12}{89}, -\frac{32}{89} \right) \Big| \end{aligned}$$

$$\begin{aligned} d) \quad \text{proj}_{\vec{a}} \vec{u} &= \frac{1(0) + 1(2) + 1(-1)}{0^2 + 2^2 + (-1)^2} (0, 2, -1) \\ &= \frac{1}{5} (0, 2, -1) \\ &= \left(0, \frac{2}{5}, -\frac{1}{5} \right) \Big| \end{aligned}$$

$$\text{proj}_{\vec{a}} \vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$$

$$\begin{aligned} \vec{u} - \text{proj}_{\vec{a}} \vec{u} &= (1, 1, 1) - \left(0, \frac{2}{5}, -\frac{1}{5} \right) \\ &= \left(1, \frac{3}{5}, \frac{6}{5} \right) \Big| \end{aligned}$$

$$\begin{aligned} e) \quad \text{proj}_{\vec{a}} \vec{u} &= \frac{2(4) + 1(-4) + 1(2) + 2(-2)}{4^2 + (-4)^2 + 2^2 + (-2)^2} (4, -4, 2, -2) \\ &= \frac{2}{40} (4, -4, 2, -2) \\ &= \left(\frac{1}{5}, -\frac{1}{5}, \frac{1}{10}, -\frac{1}{10} \right) \Big| \end{aligned}$$

$$\text{proj}_{\vec{a}} \vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$$

$$\begin{aligned} \vec{u} - \text{proj}_{\vec{a}} \vec{u} &= (2, 1, 1, 2) - \left(\frac{1}{5}, -\frac{1}{5}, \frac{1}{10}, -\frac{1}{10} \right) \\ &= \left(\frac{9}{5}, \frac{6}{5}, \frac{9}{10}, \frac{21}{10} \right) \Big| \end{aligned}$$

$$\begin{aligned}
 f) \quad \text{proj}_{\vec{a}} \vec{u} &= \frac{5(2)+0(1)-3(-1)+7(-1)}{2^2+1^2+(-1)^2+(-1)^2} (2, 1, -1, -1) & \text{proj}_{\vec{a}} \vec{u} &= \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} \\
 &= \frac{6}{7} (2, 1, -1, -1) \\
 &= \left(\frac{12}{7}, \frac{6}{7}, -\frac{6}{7}, -\frac{6}{7} \right) \\
 \vec{u} - \text{proj}_{\vec{a}} \vec{u} &= (5, 0, -3, 7) - \left(\frac{12}{7}, \frac{6}{7}, -\frac{6}{7}, -\frac{6}{7} \right) \\
 &= \left(\frac{23}{7}, -\frac{6}{7}, -\frac{15}{7}, \frac{55}{7} \right)
 \end{aligned}$$

Exercise

Project the vector \vec{v} onto the line through \vec{a} , check that $\vec{e} = \vec{u} - \text{proj}_{\vec{a}} \vec{u}$ is perpendicular to \vec{a} :

$$a) \quad \vec{v} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \text{ and } \vec{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$b) \quad \vec{v} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \text{ and } \vec{a} = \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}$$

$$c) \quad \vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } \vec{a} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Solution

$$\begin{aligned}
 a) \quad \text{proj}_{\vec{a}} \vec{v} &= \frac{1(1)+2(1)+2(1)}{1^2+1^2+1^2} (1, 1, 1) & \text{proj}_{\vec{a}} \vec{v} &= \frac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} \\
 &= \frac{5}{3} (1, 1, 1) \\
 &= \left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3} \right)
 \end{aligned}$$

$$\begin{aligned}
 \vec{e} &= \vec{v} - \text{proj}_{\vec{a}} \vec{v} \\
 &= (1, 2, 2) - \left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3} \right) \\
 &= \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right)
 \end{aligned}$$

$$\vec{e} \cdot \vec{a} = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right) \cdot (1, 1, 1)$$

$$= -\frac{2}{3} + \frac{1}{3} + \frac{1}{3}$$

$$\underline{=0}$$

\vec{e} is perpendicular to \vec{a}

$$b) \quad \text{proj}_{\vec{a}} \vec{v} = \frac{1(-1) + 3(-3) + 1(-1)}{(-1)^2 + (-3)^2 + (-1)^2} (-1, -3, -1)$$

$$\text{proj}_{\vec{a}} \vec{v} = \frac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$$

$$= \frac{-11}{11} (-1, -3, -1)$$

$$\underline{= (1, 3, 1)}$$

$$\vec{e} = \vec{v} - \text{proj}_{\vec{a}} \vec{v}$$

$$= (1, 3, 1) - (1, 3, 1)$$

$$\underline{= (0, 0, 0)}$$

$$\vec{e} \cdot \vec{a} = (0, 0, 0) \cdot (-1, -3, -1)$$

$$\underline{=0}$$

\vec{e} is perpendicular to \vec{a}

$$c) \quad \text{proj}_{\vec{a}} \vec{v} = \frac{1(1) + 1(2) + 1(2)}{(1)^2 + (2)^2 + (2)^2} (1, 2, 2)$$

$$\text{proj}_{\vec{a}} \vec{v} = \frac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$$

$$= \frac{5}{9} (1, 2, 2)$$

$$\underline{= \left(\frac{5}{9}, \frac{10}{9}, \frac{10}{9} \right)}$$

$$\vec{e} = \vec{v} - \text{proj}_{\vec{a}} \vec{v}$$

$$= (1, 1, 1) - \left(\frac{5}{9}, \frac{10}{9}, \frac{10}{9} \right)$$

$$\underline{= \left(\frac{4}{9}, -\frac{1}{9}, -\frac{1}{9} \right)}$$

$$\vec{e} \cdot \vec{a} = \left(\frac{4}{9}, -\frac{1}{9}, -\frac{1}{9} \right) \cdot (1, 2, 2)$$

$$= \frac{4}{9} - \frac{2}{9} - \frac{2}{9}$$

$$\underline{=0}$$

\vec{e} is perpendicular to \vec{a}

Exercise

Find the projection matrix $proj_{\vec{a}} \vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$ onto the line through $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Solution

$$\vec{a}^T \vec{a} = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 9$$

$$\begin{aligned} P &= \frac{1}{\vec{a}^T \vec{a}} \vec{a} \vec{a}^T \\ &= \frac{1}{9} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \end{pmatrix} \\ &= \frac{1}{9} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix} \end{aligned}$$

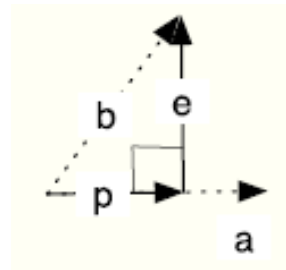
Exercise

Draw the projection of \vec{b} onto \vec{a} and also compute it from $proj_{\vec{a}} \vec{b} = \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$

$$\vec{b} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \quad \text{and} \quad \vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution

$$\begin{aligned} proj_{\vec{a}} \vec{b} &= \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} \\ &= \frac{\cos \theta(1) + \sin \theta(0)}{(1)^2 + 0} (1, 0) \\ &= \cos \theta (1, 0) \\ &= (\cos \theta, 0) \end{aligned}$$



$$\begin{aligned} \vec{e} &= \vec{b} - proj_{\vec{a}} \vec{b} \\ &= (\cos \theta, \sin \theta) - (\cos \theta, 0) \\ &= (0, \sin \theta) \end{aligned}$$

Exercise

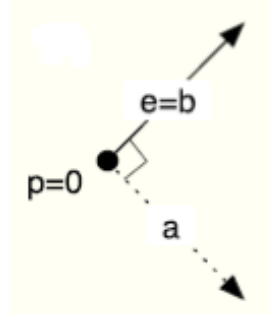
Draw the projection of \vec{b} onto \vec{a} and also compute it from $proj_{\vec{a}} \vec{b} = \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Solution

$$\begin{aligned} proj_{\vec{a}} \vec{b} &= \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} \\ &= \frac{1(1) + 1(-1)}{1^2 + (-1)^2} (1, -1) \\ &= \frac{0}{2} (1, -1) \\ &= \underline{(0, 0)} \end{aligned}$$

$$\begin{aligned} \vec{e} &= \vec{b} - proj_{\vec{a}} \vec{b} \\ &= (1, 1) - (0, 0) \\ &= \underline{(1, 1)} \end{aligned}$$



Exercise

Show that if \vec{v} is orthogonal to both \vec{w}_1 and \vec{w}_2 , then \vec{v} is orthogonal to $k_1 \vec{w}_1 + k_2 \vec{w}_2$ for all scalars k_1 and k_2 .

Solution

$$\begin{aligned} \vec{v} \cdot (k_1 \vec{w}_1 + k_2 \vec{w}_2) &= \vec{v} \cdot (k_1 \vec{w}_1) + \vec{v} \cdot (k_2 \vec{w}_2) \\ &= k_1 (\vec{v} \cdot \vec{w}_1) + k_2 (\vec{v} \cdot \vec{w}_2) \\ &= k_1 (0) + k_2 (0) \\ &= \underline{0} \end{aligned}$$

If \vec{v} is orthogonal to \vec{w}_1 & \vec{w}_2

$$\rightarrow \vec{v} \cdot \vec{w}_1 = \vec{v} \cdot \vec{w}_2 = 0$$

Exercise

- a) Project the vector $\vec{v} = (3, 4, 4)$ onto the line through $\vec{a} = (2, 2, 1)$ and then onto the plane that also contains $\vec{a}^* = (1, 0, 0)$.
- b) Check that the first error vector $\vec{v} - \vec{p}$ is perpendicular to \vec{a} , and the second error vector $\vec{v} - \vec{p}^*$ is also perpendicular to \vec{a}^* .

Solution

$$\begin{aligned} \text{a) } \text{proj}_{\vec{a}} \vec{v} &= \frac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} \\ &= \frac{3(2) + 4(2) + 4(1)}{(2)^2 + (2)^2 + (1)^2} (2, 2, 1) \\ &= \frac{18}{9} (2, 2, 1) \\ &= (4, 4, 2) \end{aligned}$$

The plane contains the vectors \vec{a} and \vec{a}^* is the column space of A .

$$\begin{aligned} A &= \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} \\ A^T A &= \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix} \\ (A^T A)^{-1} &= \begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix}^{-1} \\ &= \frac{1}{5} \begin{bmatrix} 1 & -2 \\ -2 & 9 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} P &= A (A^T A)^{-1} A^T \\ &= \frac{1}{5} \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 9 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & .8 & .4 \\ 0 & .4 & .2 \end{bmatrix} \end{aligned}$$

b) The error vector:

$$\vec{e} = \vec{v} - \vec{p}$$

$$= \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$$\begin{aligned} \vec{a} \cdot \vec{e} &= (2 \quad 2 \quad 1) \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \\ &= 2(-1) + 2(0) + 1(2) \\ &= 0 \end{aligned}$$

Therefore, \vec{e} is perpendicular to \vec{a}

$$p^* = P\vec{v}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & .8 & .4 \\ 0 & .4 & .2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 4.8 \\ 2.4 \end{pmatrix}$$

The error vector:

$$\vec{e}^* = \vec{v} - \vec{p}^*$$

$$= \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 4.8 \\ 2.4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -.8 \\ 1.6 \end{pmatrix}$$

$$\begin{aligned} \vec{a}^* \cdot \vec{e}^* &= (2 \quad 2 \quad 1) \begin{pmatrix} 0 \\ -.8 \\ 1.6 \end{pmatrix} \\ &= 2(0) + 2(-.8) + 1(1.6) \\ &= 0 \end{aligned}$$

Therefore, \vec{e}^* is perpendicular to \vec{a}^*

Exercise

Compute the projection matrices $\vec{a}\vec{a}^T / \vec{a}^T\vec{a}$ onto the lines through $\vec{a}_1 = (-1, 2, 2)$ and $\vec{a}_2 = (2, 2, -1)$. Multiply those projection matrices and explain why their product $P_1 P_2$ is what it is. Project $\vec{v} = (1, 0, 0)$ onto the lines \vec{a}_1 , \vec{a}_2 , and also onto $\vec{a}_3 = (2, -1, 2)$. Add up the three projections $P_1 + P_2 + P_3$.

Solution

For $\vec{a}_1 = (-1, 2, 2)$

$$\begin{aligned}\vec{a}_1 \vec{a}_1^T &= \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} -1 & 2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\vec{a}_1^T \vec{a}_1 &= \begin{pmatrix} -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \\ &= 9 \end{aligned}$$

$$\begin{aligned}P_1 &= \frac{\vec{a}_1 \vec{a}_1^T}{\vec{a}_1^T \vec{a}_1} \\ &= \frac{1}{9} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}\end{aligned}$$

For $\vec{a}_2 = (2, 2, -1)$

$$\begin{aligned}\vec{a}_2 \vec{a}_2^T &= \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 2 & 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\vec{a}_2^T \vec{a}_2 &= \begin{pmatrix} 2 & 2 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \\ &= 9 \end{aligned}$$

$$\begin{aligned}
P_2 &= \frac{\vec{a} \vec{a}^T}{\vec{a}^T \vec{a}} \\
&= \frac{1}{9} \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix} \\
P_1 P_2 &= \frac{1}{9} \left(\frac{1}{9} \right) \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix} \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix} \\
&= \frac{1}{81} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \underline{0}
\end{aligned}$$

This because \vec{a}_1 and \vec{a}_2 are perpendicular.

For $\vec{a}_3 = (2, -1, 2)$

$$\begin{aligned}
\vec{a}_3 \vec{a}_3^T &= \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
a_3^T a_3 &= \begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \\
&= \underline{9}
\end{aligned}$$

$$\begin{aligned}
P_3 &= \frac{\vec{a}_3 \vec{a}_3^T}{\vec{a}_3^T \vec{a}_3} \\
&= \frac{1}{9} \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
p_3 &= P_3 \vec{v} \\
&= \frac{1}{9} \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

$$= \frac{1}{9} \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ \frac{4}{9} \end{pmatrix}$$

$$p_1 = P_1 \vec{v}$$

$$= \frac{1}{9} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{9} \\ -\frac{2}{9} \\ -\frac{2}{9} \end{pmatrix}$$

$$p_2 = P_2 \vec{v}$$

$$= \frac{1}{9} \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{2}{9} \end{pmatrix}$$

$$p_1 + p_2 + p_3 = \begin{pmatrix} \frac{1}{9} \\ -\frac{2}{9} \\ -\frac{2}{9} \end{pmatrix} + \begin{pmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{2}{9} \end{pmatrix} + \begin{pmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ \frac{4}{9} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \underline{= \vec{v}} \quad |$$

The reason is that \vec{a}_3 is perpendicular to \vec{a}_1 and \vec{a}_2 .

Hence, when you compute the three projections of a vector and add them up you get back to the vector you start with.

Exercise

If $P^2 = P$ show that $(I - P)^2 = I - P$. When P projects onto the column space of A , $I - P$ projects onto the ____.

Solution

$$\begin{aligned} (I - P)^2 \vec{v} &= (I - P)(I - P)\vec{v} \\ &= (I - P)(I\vec{v} - P\vec{v}) \\ &= I^2\vec{v} - IP\vec{v} - PI\vec{v} + P^2\vec{v} \\ &= \vec{v} - P\vec{v} - P\vec{v} + P^2\vec{v} & P^2\vec{v} = P\vec{v} \quad \text{By given definition} \\ &= \vec{v} - P\vec{v} - P\vec{v} + P\vec{v} \\ &= \vec{v} - P\vec{v} \end{aligned}$$

$$(I - P)^2 \vec{v} = (I - P)\vec{v}$$

$$\underline{(I - P)^2 = (I - P)} \quad |$$

When P projects onto the column space of A , then $I - P$ projects onto the left nullspace.

Because $(I - P)^2 \vec{v} = (I - P)\vec{v}$; if $P\vec{v}$ is in the column space of A , then $\vec{v} - P\vec{v}$ is a vector perpendicular to $C(A)$.

Exercise

What linear combination of $(1, 2, -1)$ and $(1, 0, 1)$ is closest to $\vec{v} = (2, 1, 1)$?

Solution

$$\frac{1}{2}(1, 2, -1) + \frac{3}{2}(1, 0, 1) = (2, 1, 1)$$

So, this \vec{v} is actually in the span of the two given vectors.

Exercise

Show that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$ if and only if $\|\vec{u}\| = \|\vec{v}\|$

Solution

Suppose that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$. Then

$$\begin{aligned} 0 &= \langle \vec{u} - \vec{v}, \vec{u} + \vec{v} \rangle \\ &= (\vec{u} - \vec{v})^T (\vec{u} + \vec{v}) \\ &= (\vec{u}^T - \vec{v}^T)(\vec{u} + \vec{v}) \\ &= \vec{u}^T \vec{u} + \vec{u}^T \vec{v} - \vec{v}^T \vec{u} - \vec{v}^T \vec{v} \\ &= \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle - \langle \vec{v}, \vec{u} \rangle - \langle \vec{v}, \vec{v} \rangle & \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \\ &= \langle \vec{u}, \vec{u} \rangle - \langle \vec{v}, \vec{v} \rangle \end{aligned}$$

So $\langle \vec{u}, \vec{u} \rangle = \langle \vec{v}, \vec{v} \rangle$.

Therefore, $\|\vec{u}\|^2 = \|\vec{v}\|^2 \Rightarrow \|\vec{u}\| = \|\vec{v}\|$.

Suppose $\|\vec{u}\| = \|\vec{v}\|$. Then

$$\begin{aligned} \langle \vec{u} - \vec{v}, \vec{u} + \vec{v} \rangle &= (\vec{u} - \vec{v})^T (\vec{u} + \vec{v}) \\ &= (\vec{u}^T - \vec{v}^T)(\vec{u} + \vec{v}) \\ &= \vec{u}^T \vec{u} + \vec{u}^T \vec{v} - \vec{v}^T \vec{u} - \vec{v}^T \vec{v} \\ &= \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle - \langle \vec{v}, \vec{u} \rangle - \langle \vec{v}, \vec{v} \rangle & \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \\ &= \langle \vec{u}, \vec{u} \rangle - \langle \vec{v}, \vec{v} \rangle \\ &= \|\vec{u}\|^2 - \|\vec{v}\|^2 \\ &= 0 \end{aligned}$$

So, we can see that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$

We conclude that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$ if and only if $\|\vec{u}\| = \|\vec{v}\|$, as desired.

Exercise

Given $\vec{u} = (3, -1, 2)$ $\vec{v} = (4, -1, 5)$ and $\vec{w} = (8, -7, -6)$

- Find $3\vec{v} - 4(5\vec{u} - 6\vec{w})$
- Find $\vec{u} \cdot \vec{v}$ and then the angle θ between \vec{u} and \vec{v} .

Solution

$$a) \quad 3\vec{v} - 4(5\vec{u} - 6\vec{w}) = 3(4, -1, 5) - 4(5(3, -1, 2) - 6(8, -7, -6))$$

$$\begin{aligned}
&= (12, -3, 15) - 4((15, -5, 10) - (48, -42, -36)) \\
&= (12, -3, 15) - 4(-33, 37, 46) \\
&= (12, -3, 15) - (-132, 148, 184) \\
&= \underline{(144, -151, -169)}
\end{aligned}$$

$$\begin{aligned}
b) \quad \vec{u} \cdot \vec{v} &= (3, -1, 2) \cdot (1, 1, -1) \\
&= 3 - 1 - 2 \\
&= \underline{0} \\
\theta &= \underline{90^\circ}
\end{aligned}$$

Exercise

Given: $\vec{u} = (3, 1, 3)$ $\vec{v} = (4, 1, -2)$

- Compute the projection \vec{w} of \vec{u} on \vec{v}
- Find $\vec{p} = \vec{u} - \vec{v}$ and show that \vec{p} is perpendicular to \vec{v} .

Solution

$$\begin{aligned}
a) \quad \vec{w} &= \text{proj}_{\vec{v}} \vec{u} \\
&= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} \\
&= \frac{(3, 1, 3) \cdot (4, 1, -2)}{4^2 + 1^2 + (-2)^2} (4, 1, -2) \\
&= \frac{12 + 1 - 6}{21} (4, 1, -2) \\
&= \frac{7}{21} (4, 1, -2) \\
&= \frac{1}{3} (4, 1, -2) \\
&= \underline{\left(\frac{4}{3}, \frac{1}{3}, -\frac{2}{3} \right)}
\end{aligned}$$

$$\begin{aligned}
b) \quad \vec{p} &= (3, 1, 3) - \left(\frac{4}{3}, \frac{1}{3}, -\frac{2}{3} \right) \\
&= \left(\frac{5}{3}, \frac{2}{3}, \frac{11}{3} \right) \\
\vec{p} \cdot \vec{u} &= \left(\frac{5}{3}, \frac{2}{3}, \frac{11}{3} \right) \cdot (4, 1, -2) \\
&= \frac{20}{3} + \frac{2}{3} - \frac{22}{3} \\
&= \underline{0}
\end{aligned}$$

\vec{p} is perpendicular to \vec{v} .

Exercise

- a) Show that $\vec{v} = (a, b)$ and $\vec{w} = (-b, a)$ are orthogonal vectors
- b) Use the result in part (a) to find two vectors that are orthogonal to $\vec{v} = (2, -3)$
- c) Find two unit vectors that are orthogonal to $(-3, 4)$

Solution

$$\begin{aligned} \text{a) } \vec{u} \cdot \vec{v} &= -ab + ba \\ &= 0 \end{aligned}$$

The 2 vectors are orthogonal vectors.

$$\text{b) } \vec{v} = (2, -3)$$

$$\vec{w} = (-3, -2) \text{ and } \vec{w} = (3, 2)$$

$$\text{c) } (-3, 4)$$

$$\begin{aligned} \vec{u} &= \frac{(-3, 4)}{\sqrt{9+16}} \\ &= \left(-\frac{3}{5}, \frac{4}{5}\right) \end{aligned}$$

$$\vec{u}_1 = \left(\frac{4}{5}, \frac{3}{5}\right) \text{ and } \vec{u}_2 = \left(-\frac{4}{5}, -\frac{3}{5}\right)$$

Exercise

Show that $A(3, 0, 2)$, $B(4, 3, 0)$, and $C(8, 1, -1)$ are vertices of a right triangle. At which vertex is the right angle?

Solution

$$AB = (4-3, 3-0, 0-2) = (1, 3, -2)$$

$$AC = (5, 1, -3)$$

$$BC = (4, -2, -1)$$

$$AB \cdot AC = 5 + 3 + 6 = 14$$

$$AB \cdot BC = 4 - 6 + 2 = 0$$

$$AC \cdot BC = 20 - 2 + 3 = 21$$

The right triangle at point B

Exercise

Establish the identity: $\vec{u} \cdot \vec{v} = \frac{1}{4} \|\vec{u} + \vec{v}\|^2 - \frac{1}{4} \|\vec{u} - \vec{v}\|^2$

Solution

Let $\vec{u} = (u_1, u_2, \dots, u_n)$ and $\vec{v} = (v_1, v_2, \dots, v_n)$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

$$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= (u_1 + v_1)^2 + (u_2 + v_2)^2 + \dots + (u_n + v_n)^2 \\ &= u_1^2 + v_1^2 + 2u_1 v_1 + u_2^2 + v_2^2 + 2u_2 v_2 + \dots + u_n^2 + v_n^2 + 2u_n v_n \end{aligned}$$

$$\vec{u} - \vec{v} = (u_1 - v_1, u_2 - v_2, \dots, u_n - v_n)$$

$$\begin{aligned} \|\vec{u} - \vec{v}\|^2 &= (u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots + (u_n - v_n)^2 \\ &= u_1^2 + v_1^2 - 2u_1 v_1 + u_2^2 + v_2^2 - 2u_2 v_2 + \dots + u_n^2 + v_n^2 - 2u_n v_n \end{aligned}$$

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 &= u_1^2 + v_1^2 + 2u_1 v_1 + u_2^2 + v_2^2 + 2u_2 v_2 + \dots + u_n^2 + v_n^2 + 2u_n v_n \\ &\quad - (u_1^2 + v_1^2 - 2u_1 v_1 + u_2^2 + v_2^2 - 2u_2 v_2 + \dots + u_n^2 + v_n^2 - 2u_n v_n) \\ &= u_1^2 + v_1^2 + 2u_1 v_1 + u_2^2 + v_2^2 + 2u_2 v_2 + \dots + u_n^2 + v_n^2 + 2u_n v_n \\ &\quad - u_1^2 - v_1^2 + 2u_1 v_1 - u_2^2 - v_2^2 + 2u_2 v_2 - \dots - u_n^2 - v_n^2 + 2u_n v_n \\ &= 4u_1 v_1 + 4u_2 v_2 + \dots + 4u_n v_n \end{aligned}$$

$$\frac{1}{4} (\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2) = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

Therefore; $\vec{u} \cdot \vec{v} = \frac{1}{4} \|\vec{u} + \vec{v}\|^2 - \frac{1}{4} \|\vec{u} - \vec{v}\|^2$ is true.

2nd method:

$$\begin{aligned} \frac{1}{4} \|\vec{u} + \vec{v}\|^2 - \frac{1}{4} \|\vec{u} - \vec{v}\|^2 &= \frac{1}{4} [(\vec{u} + \vec{v})(\vec{u} + \vec{v}) - (\vec{u} - \vec{v})(\vec{u} - \vec{v})] \\ &= \frac{1}{4} [\vec{u}\vec{u} + 2\vec{u}\vec{v} + \vec{v}\vec{v} - (\vec{u}\vec{u} - 2\vec{u}\vec{v} + \vec{v}\vec{v})] \\ &= \frac{1}{4} [\vec{u}\vec{u} + 2\vec{u}\vec{v} + \vec{v}\vec{v} - \vec{u}\vec{u} + 2\vec{u}\vec{v} - \vec{v}\vec{v}] \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{4}(4\vec{u}\vec{v}) \\
 &= \vec{u} \cdot \vec{v}
 \end{aligned}$$

Exercise

Find the Euclidean inner product $\vec{u} \cdot \vec{v}$: $\vec{u} = (-1, 1, 0, 4, -3)$ $\vec{v} = (-2, -2, 0, 2, -1)$

Solution

$$\begin{aligned}
 \vec{u} \cdot \vec{v} &= 2 - 2 + 0 + 8 + 3 \\
 &= 11
 \end{aligned}$$

Exercise

Find the Euclidean distance between \vec{u} and \vec{v} : $\vec{u} = (3, -3, -2, 0, -3)$ $\vec{v} = (-4, 1, -1, 5, 0)$

Solution

$$\begin{aligned}
 d(\vec{u}, \vec{v}) &= \|\vec{u} - \vec{v}\| \\
 &= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots + (u_n - v_n)^2} \\
 &= \sqrt{(3 + 4)^2 + (-3 - 1)^2 + (-2 + 1)^2 + (0 - 5)^2 + (-3 - 0)^2} \\
 &= \sqrt{49 + 16 + 1 + 25 + 9} \\
 &= \sqrt{100} \\
 &= 10
 \end{aligned}$$

Exercise

Find for $\vec{v} = 2\hat{i} - 4\hat{j} + \sqrt{5}\hat{k}$, $\vec{u} = -2\hat{i} + 4\hat{j} - \sqrt{5}\hat{k}$

- $\vec{v} \cdot \vec{u}$, $|\vec{v}|$, $|\vec{u}|$
- The cosine of the angle between \vec{v} and \vec{u}
- The scalar component of \vec{u} in the direction of \vec{v}
- The vector $\text{proj}_{\vec{v}} \vec{u}$

Solution

$$\begin{aligned}
 a) \quad \vec{v} \cdot \vec{u} &= (2\hat{i} - 4\hat{j} + \sqrt{5}\hat{k}) \cdot (-2\hat{i} + 4\hat{j} - \sqrt{5}\hat{k}) \\
 &= -4 - 16 - 5 \\
 &= -25
 \end{aligned}$$

$$|\vec{v}| = \sqrt{2^2 + (-4)^2 + (\sqrt{5})^2}$$

$$\begin{aligned}
 &= \sqrt{4+16+5} \\
 &= \sqrt{25} \\
 &= 5
 \end{aligned}$$

$$\begin{aligned}
 |\vec{u}| &= \sqrt{(-2)^2 + 4^2 + (-\sqrt{5})^2} \\
 &= \sqrt{25} \\
 &= 5
 \end{aligned}$$

$$\begin{aligned}
 b) \quad \cos \theta &= \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} \\
 &= \frac{-25}{(5)(5)} \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 c) \quad |\vec{u}| \cos \theta &= (5)(-1) \\
 &= -5
 \end{aligned}$$

$$\begin{aligned}
 d) \quad \text{proj}_{\vec{v}} \vec{u} &= \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v} \\
 &= \left(\frac{-25}{5^2} \right) (2\hat{i} - 4\hat{j} + \sqrt{5}\hat{k}) \\
 &= -(2\hat{i} - 4\hat{j} + \sqrt{5}\hat{k}) \\
 &= -2\hat{i} + 4\hat{j} - \sqrt{5}\hat{k}
 \end{aligned}$$

Exercise

Find for $\vec{v} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{k}$, $\vec{u} = 5\hat{i} + 12\hat{j}$

- $\vec{v} \cdot \vec{u}$, $|\vec{v}|$, $|\vec{u}|$
- The cosine of the angle between \vec{v} and \vec{u}
- The scalar component of \vec{u} in the direction of \vec{v}
- The vector $\text{proj}_{\vec{v}} \vec{u}$

Solution

$$\begin{aligned}
 a) \quad \vec{v} \cdot \vec{u} &= \left(\frac{3}{5}\hat{i} + \frac{4}{5}\hat{k} \right) \cdot (5\hat{i} + 12\hat{j}) \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 |\vec{v}| &= \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} \\
 &= \sqrt{\frac{9}{25} + \frac{16}{25}} \\
 &= \sqrt{\frac{25}{25}} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 |\vec{u}| &= \sqrt{5^2 + 12^2} \\
 &= 13
 \end{aligned}$$

$$\begin{aligned}
 b) \quad \cos \theta &= \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} \\
 &= \frac{3}{(1)(13)} \\
 &= \frac{3}{13}
 \end{aligned}$$

$$\begin{aligned}
 c) \quad |\vec{u}| \cos \theta &= (13) \left(\frac{3}{13} \right) \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 d) \quad \text{proj}_{\vec{v}} \vec{u} &= \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v} \\
 &= \left(\frac{3}{1^2} \right) \left(\frac{3}{5} \hat{i} + \frac{4}{5} \hat{k} \right) \\
 &= \frac{9}{5} \hat{i} + \frac{12}{5} \hat{k}
 \end{aligned}$$

Exercise

Find for $\vec{v} = 2\hat{i} + 10\hat{j} - 11\hat{k}$, $\vec{u} = 2\hat{i} + 2\hat{j} + \hat{k}$

- $\vec{v} \cdot \vec{u}$, $|\vec{v}|$, $|\vec{u}|$
- The cosine of the angle between \vec{v} and \vec{u}
- The scalar component of \vec{u} in the direction of \vec{v}
- The vector $\text{proj}_{\vec{v}} \vec{u}$

Solution

$$\begin{aligned}
 a) \quad \vec{v} \cdot \vec{u} &= (2\hat{i} + 10\hat{j} - 11\hat{k}) \cdot (2\hat{i} + 2\hat{j} + \hat{k}) \\
 &= 4 + 20 - 11
 \end{aligned}$$

$$= 13 \mid$$

$$\begin{aligned} |\vec{v}| &= \sqrt{2^2 + 10^2 + (-11)^2} \\ &= \sqrt{4 + 100 + 121} \\ &= \sqrt{225} \\ &= 15 \mid \end{aligned}$$

$$\begin{aligned} |\vec{u}| &= \sqrt{2^2 + 2^2 + 1^2} \\ &= 3 \mid \end{aligned}$$

$$\begin{aligned} b) \quad \cos \theta &= \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} \\ &= \frac{13}{(3)(15)} \\ &= \frac{13}{45} \mid \end{aligned}$$

$$\begin{aligned} c) \quad |\vec{u}| \cos \theta &= (3) \left(\frac{13}{45} \right) \\ &= \frac{13}{15} \mid \end{aligned}$$

$$\begin{aligned} d) \quad \text{proj}_{\vec{v}} \vec{u} &= \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v} \\ &= \left(\frac{13}{15^2} \right) (2\hat{i} + 10\hat{j} - 11\hat{k}) \hat{j} \\ &= \frac{13}{225} (2\hat{i} + 10\hat{j} - 11\hat{k}) \mid \end{aligned}$$

Exercise

Find for $\vec{v} = 5\hat{i} + \hat{j}$, $\vec{u} = 2\hat{i} + \sqrt{17}\hat{j}$

$$a) \quad \vec{v} \cdot \vec{u}, \quad |\vec{v}|, \quad |\vec{u}|$$

b) The cosine of the angle between \vec{v} and \vec{u}

c) The scalar component of \vec{u} in the direction of \vec{v}

d) The vector $\text{proj}_{\vec{v}} \vec{u}$

Solution

$$\begin{aligned} a) \quad \vec{v} \cdot \vec{u} &= (5\hat{i} + \hat{j}) \cdot (2\hat{i} + \sqrt{17}\hat{j}) \\ &= 10 + \sqrt{17} \mid \end{aligned}$$

$$|\vec{v}| = \sqrt{25+1}$$

$$= \sqrt{26}$$

$$|\vec{u}| = \sqrt{4+17}$$

$$= \sqrt{21}$$

$$b) \quad \cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}$$

$$= \frac{10 + \sqrt{17}}{\sqrt{21}\sqrt{26}}$$

$$= \frac{10 + \sqrt{17}}{\sqrt{546}}$$

$$c) \quad |\vec{u}| \cos \theta = (\sqrt{21}) \left(\frac{10 + \sqrt{17}}{\sqrt{546}} \right)$$

$$= \frac{10 + \sqrt{17}}{\sqrt{26}}$$

$$d) \quad \text{proj}_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v}$$

$$= \left(\frac{10 + \sqrt{17}}{26} \right) (5\hat{i} + \hat{j})$$

Exercise

Find for $\vec{v} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right)$, $\vec{u} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}} \right)$

- $\vec{v} \cdot \vec{u}$, $|\vec{v}|$, $|\vec{u}|$
- The cosine of the angle between \vec{v} and \vec{u}
- The scalar component of \vec{u} in the direction of \vec{v}
- The vector $\text{proj}_{\vec{v}} \vec{u}$

Solution

$$a) \quad \vec{v} \cdot \vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}} \right)$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}$$

$$\begin{aligned}
 |\vec{v}| &= \sqrt{\frac{1}{2} + \frac{1}{3}} \\
 &= \frac{\sqrt{5}}{\sqrt{6}} \frac{\sqrt{6}}{\sqrt{6}} \\
 &= \frac{\sqrt{30}}{6}
 \end{aligned}$$

$$\begin{aligned}
 |\vec{u}| &= \sqrt{\frac{1}{2} + \frac{1}{3}} \\
 &= \frac{\sqrt{5}}{\sqrt{6}} \frac{\sqrt{6}}{\sqrt{6}} \\
 &= \frac{\sqrt{30}}{6}
 \end{aligned}$$

$$\begin{aligned}
 b) \quad \cos \theta &= \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} \\
 &= \frac{\frac{1}{6}}{\frac{\sqrt{30}}{6} \frac{\sqrt{30}}{6}} \\
 &= \frac{1}{6} \left(\frac{36}{30} \right) \\
 &= \frac{1}{5}
 \end{aligned}$$

$$\begin{aligned}
 c) \quad |\vec{u}| \cos \theta &= \left(\frac{\sqrt{30}}{6} \right) \left(\frac{1}{5} \right) \\
 &= \frac{\sqrt{30}}{30} \\
 &= \frac{1}{\sqrt{30}}
 \end{aligned}$$

$$\begin{aligned}
 d) \quad \text{proj}_{\vec{v}} \vec{u} &= \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v} \\
 &= \frac{1}{6} \left(\frac{36}{30} \right) \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right) \\
 &= \frac{1}{5} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right)
 \end{aligned}$$

Exercise

Suppose Ted weighs 180 *lb.* and he is sitting on an inclined plane that drops 3 *units* for every 4 horizontal units. The gravitational force vector is $\vec{F}_g = \begin{pmatrix} 0 \\ -180 \end{pmatrix}$.

- a) Find the force pushing Ted down the slope.
- b) Find the force acting to hold Ted against the slope

Solution

A vector parallel to the slope of the inclined plane is $\vec{v} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$.

- a) The vector of the force acting to push Ted down the slope is

$$\begin{aligned}\vec{F}_s &= \frac{\vec{v} \cdot \vec{F}_g}{|\vec{v}|^2} \vec{v} \\ &= \frac{(4, -3) \cdot (0, -180)}{16+9} (4, -3) \\ &= \frac{540}{25} (4, -3) \\ &= \left(\frac{432}{5}, -\frac{324}{5} \right)\end{aligned}$$

The magnitude of the force pushing Ted down the slope is

$$\begin{aligned}\|\vec{F}_s\| &= \sqrt{\left(\frac{432}{5}\right)^2 + \left(-\frac{324}{5}\right)^2} \\ &= \frac{540}{5} \\ &= 108 \text{ lb}\end{aligned}$$

- b) The vector of the force acting to hold Ted against the slope is

$$\begin{aligned}\vec{F}_p &= \vec{F}_g - \vec{F}_s \\ &= \begin{pmatrix} 0 \\ -180 \end{pmatrix} - \begin{pmatrix} \frac{432}{5} \\ -\frac{324}{5} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{432}{5} \\ -\frac{576}{5} \end{pmatrix} \\ \|\vec{F}_p\| &= \sqrt{\left(\frac{432}{5}\right)^2 + \left(\frac{576}{5}\right)^2} \\ &= \frac{720}{5} \\ &= 144 \text{ lb}\end{aligned}$$

Exercise

Prove that if two vectors \vec{u} and \vec{v} in \mathbb{R}^2 are orthogonal to nonzero vector \vec{w} in \mathbb{R}^2 , then \vec{u} and \vec{v} are scalar multiples of each other.

Solution

Since \vec{u} is orthogonal to $\vec{w} \rightarrow \vec{u} \cdot \vec{w} = 0$

\vec{v} is orthogonal to $\vec{w} \rightarrow \vec{v} \cdot \vec{w} = 0$

$$\Rightarrow \vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w} = 0$$

There exist $a \in \mathbb{R}$ such that $(a\vec{v}) \cdot \vec{w} = a(\vec{v} \cdot \vec{w}) = 0$

$$\vec{u} = a\vec{v}$$

$$\vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w} = 0 = (a\vec{v}) \cdot \vec{w}$$

Therefore, \vec{u} and \vec{v} are scalar multiples of each other