Solution   Section 5.7 – Mathematical Induction

Exercise

Find all positive integers n for which the given statement is not true

$a)\quad 3^n > 6n \qquad b)\quad 3^n > 2n+1 \qquad c)\quad 2^n > n^2 \qquad d)\quad n! > 2n$

Solution

a) $n=1 \quad 3<6$

$n=2 \quad 3^2 < 18$

$n=3, \quad 27 > 18$

The statement is true for all $n \ge 3 \quad 3^n > 6n$

The statement is not true for $\boxed{n=1,\ 2}$

b) $n=1; \quad 3=3$

$n=2; \quad 9>5$

The statement is true for all $n \ge 2 \quad 3^n > 2n+1$

The statement is not true for $\boxed{n=1}$

c) $n=1; \quad 2<4$

$n=2; \quad 4=4$

$n=3; \quad 8<9$

$n=4; \quad 16=16$

$n=5; \quad 32>25$

The statement is true for all $n \ge 5; \quad 2^n > n^2$

The statement is not true for $\boxed{n=1,\ 2,\ 3,\ 4}$

d) $n=1; \quad 1<2$

$n=2; \quad 2<4$

$n=3; \quad 6=6$

$n=4; \quad 12>8$

The statement is true for all $n \ge 4; \quad n! > 2n$

The statement is not true for $\boxed{n=1,\ 2,\ 3}$

Exercise

Prove that the statement is true for every positive integer n. $2+4+6+\ldots+2n=n(n+1)$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 2=1(1+1)=2$; hence P_1 is true.

(2) Assume $2+4+6+\ldots+2k=k(k+1)$ is true

$$\Rightarrow 2+4+6+\ldots+2k+2(k+1)=(k+1)(k+1+1)?$$

$$\begin{aligned} 2+4+6+\ldots+2k+2(k+1) &= 2+4+6+\ldots+2k+2(k+1) \\ &= k(k+1)+2(k+1) \\ &= (k+1)(k+2) \\ &= (k+1)(k+1+1) \ \checkmark \end{aligned}$$

***Factor* (k + 1)**

Hence P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $1+3+5+\ldots+(2n-1)=n^2$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 1 \overset{?}{=} 1^2=1$; hence P_1 is true.

(2) Assume $1+3+5+\ldots+(2k-1)=k^2$ is true

$$\Rightarrow 1+3+5+\ldots+(2(k+1)-1)=(k+1)^2?$$

$$\begin{aligned} 1+3+5+\ldots+(2k-1)+(2(k+1)-1) &= 1+3+5+\ldots+(2k-1)+(2k+2-1) \\ &= k^2+(2k+1) \\ &= k^2+2k+1 \\ &= (k+1)^2 \ \checkmark \end{aligned}$$

Hence P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $2+7+12+\ldots+(5n-3)=\frac{1}{2}n(5n-1)$

Solution

(1) For $\boldsymbol{n}=\mathbf{1} \Rightarrow 2\overset{?}{=}\frac{1}{2}(1)(5(1)-1)=\frac{1}{2}(4)=2$; hence P_1 is true.

(2) Assume $2+7+12+\ldots+(5k-3)=\frac{1}{2}k(5k-1)$ is true

$$2+7+12+\ldots+(5(k+1)-3)=\frac{1}{2}(k+1)(5(k+1)-1)?$$

$$\begin{aligned}2+7+12+\ldots+(5k-3)+(5(k+1)-3) &= 2+7+12+\ldots+(5k-3)+(5k+5-3)\\ &= \frac{1}{2}k(5k-1)+(5k+2)\frac{2}{2}\\ &= \frac{1}{2}\left[5k^2-k+10k+4\right]\\ &= \frac{1}{2}\left[5k^2-k+5k+5k+5-1\right]\\ &= \frac{1}{2}\left[k(5k-1+5)+5k+5-1\right]\\ &= \frac{1}{2}\left[(k+1)(5k+5-1)\right]\\ &= \frac{1}{2}\left[(k+1)(5(k+1)-1)\right] \quad \surd \quad P_{k+1} \text{ is also true.}\end{aligned}$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $1+2.2+3.2^2+\ldots+n.2^{n-1}=1+(n-1).2^n$

Solution

For $\boldsymbol{n}=\mathbf{1}$

$1\overset{?}{=}1+(1-1)2^1=1-0=1$

Hence P_1 is true.

$1+2.2+3.2^2+\ldots+k.2^{k-1}=1+(k-1).2^k$ is true

$$1+2.2+3.2^2+\ldots+k.2^{k-1}+(k+1).2^{(k+1)-1}=1+((k+1)-1).2^{k+1}?$$

$$\begin{aligned}1+2.2+3.2^2+\ldots+k.2^{k-1}+(k+1).2^{(k+1)-1} &= 1+(k-1).2^k+(k+1).2^{k+1-1}\\ &= 1+k.2^k-1.2^k+(k+1).2^k\\ &= 1+k.2^k-1.2^k+k.2^k+1.2^k\\ &= 1+2^1k.2^k\end{aligned}$$

$$=1+(k+0).2^{k+1}$$

$$=1+((k+1)-1).2^{k+1} \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. $1^2+2^2+3^2+\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6}$

Solution

For $\boldsymbol{n}$ **= 1**

$$1^2 \overset{?}{=} \frac{1(1+1)(2(1)+1)}{6}=\frac{1(2)(3)}{6}=\frac{6}{6}=1 \quad \surd$$

Hence P_1 is true.

$1^2+2^2+3^2+\ldots+k^2=\dfrac{k(k+1)(2k+1)}{6}$ is true

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}?$$

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$=\frac{(k+1)\left[k(2k+1)+6(k+1)\right]}{6}$$

$$=\frac{(k+1)\left[2k^2+k+6k+6\right]}{6}$$

$$=\frac{(k+1)\left[2k^2+7k+6\right]}{6}$$

$$=\frac{(k+1)((k+2)(2k+3))}{6}$$

$$=\frac{(k+1)((k+1+1)(2k+2+1))}{6}$$

$$=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. $\frac{1}{1\bullet 2}+\frac{1}{2\bullet 3}+\frac{1}{3\bullet 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

Solution

For $n = 1$

$$\frac{1}{1.2}\stackrel{?}{=}\frac{1}{1+1}=\frac{1}{2}=\frac{1}{1.2}\ \checkmark$$

Hence P_1 is true.

$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$ is true

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{(k+1)+1}?$$

$$\begin{aligned}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}&=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}\\&=\frac{k(k+2)+1}{(k+1)(k+2)}\\&=\frac{k^2+2k+1}{(k+1)(k+2)}\\&=\frac{(k+1)(k+1)}{(k+1)(k+2)}\\&=\frac{k+1}{(k+1+1)}\\&=\frac{k+1}{(k+1)+1}\ \checkmark\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$

Solution

For $n = 1$

$$\frac{1}{2}\stackrel{?}{=}1-\frac{1}{2}=\frac{1}{2}\ \checkmark$$

Hence, P_1 is true.

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}=1-\frac{1}{2^k} \text{ is true}$$

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^{k+1}}?$$

$$\begin{aligned}
\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}} &= 1-\frac{1}{2^k}+\frac{1}{2^{k+1}} \\
&= 1-\frac{1}{2^k}+\frac{1}{2^k\cdot 2} \\
&= \frac{2^{k+1}-2+1}{2^{k+1}} \\
&= \frac{2^{k+1}-1}{2^{k+1}} \\
&= \frac{2^{k+1}}{2^{k+1}}-\frac{1}{2^{k+1}} \\
&= 1-\frac{1}{2^{k+1}} \quad \checkmark
\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{(3n-2)\cdot(3n+1)}=\frac{n}{3n+1}$

Solution

For ***n* = 1**

$$\frac{1}{1\cdot 4}\overset{?}{=}\frac{1}{3(1)+1}=\frac{1}{4} \quad \checkmark$$

Hence, P_1 is true.

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1} \text{ is true}$$

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3(k+1)-2)(3(k+1)+1)}\overset{?}{=}\frac{k+1}{3(k+1)+1}$$

$$\begin{aligned}
\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3(k+1)-2)(3(k+1)+1)} &= \frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)} \\
&= \frac{3k^2+4k+1}{(3k+1)(3k+4)}
\end{aligned}$$

$$=\frac{(3k+1)(k+1)}{(3k+1)(3k+3+1)}$$

$$=\frac{k+1}{3(k+1)+1} \quad \checkmark$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\frac{4}{5}+\frac{4}{5^2}+\frac{4}{5^3}+\cdots+\frac{4}{5^n}=1-\frac{1}{5^n}$

Solution

For $\boldsymbol{n=1}$

$$\frac{4}{5}\overset{?}{=}1-\frac{1}{5}=\frac{4}{5} \quad \checkmark$$

Hence, P_1 is true.

$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}=1-\frac{1}{5^k}$ is true

$$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}+\frac{4}{5^{k+1}}\overset{?}{=}1-\frac{1}{5^{k+1}}$$

$$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}+\frac{4}{5^{k+1}}=1-\frac{1}{5^k}+\frac{4}{5^{k+1}}$$

$$=1-\left(\frac{1}{5^k}-\frac{4}{5^{k+1}}\right)$$

$$=1-\frac{5-4}{5^{k+1}}$$

$$=1-\frac{1}{5^{k+1}} \quad \checkmark$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$

Solution

For $\boldsymbol{n=1}$

$$1^3 \overset{?}{=} \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1 \quad \checkmark$$

Hence, P_1 is true.

$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}=1-\frac{1}{5^k}$ is true

$$1^3+2^3+\cdots+k^3+(k+1)^3 \overset{?}{=} \frac{(k+1)^2((k+1)+1)^2}{4}$$

$$\begin{aligned}
1^3+2^3+\cdots+k^3+(k+1)^3 &= \frac{k^2(k+1)^2}{4}+(k+1)^3 \\
&= \frac{k^2(k+1)^2+4(k+1)^3}{4} \\
&= \frac{(k+1)^2\left[k^2+4(k+1)\right]}{4} \\
&= \frac{(k+1)^2\left(k^2+4k+4\right)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k+1)^2((k+1)+1)^2}{4} \quad \checkmark
\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. $3+3^2+3^3+\ldots+3^n=\frac{3}{2}\left(3^n-1\right)$

Solution

For $\boldsymbol{n}$ **= 1**

$$3 \overset{?}{=} \frac{3}{2}\left(3^1-1\right)=\frac{3}{2}2=3 \quad \checkmark$$

Hence, P_1 is true.

$3+3^2+\cdots+3^k=\frac{3}{2}\left(3^k-1\right)$ is true

$$3+3^2+\cdots+3^k+3^{k+1} \overset{?}{=} \frac{3}{2}\left(3^{k+1}-1\right)$$

$$3+3^2+\cdots+3^k+3^{k+1} = \frac{3}{2}\left(3^k-1\right)+3^{k+1}$$
$$= \frac{1}{2}3^{k+1}-\frac{3}{2}+3^{k+1}$$
$$= \frac{3}{2}3^{k+1}-\frac{3}{2}$$
$$= \frac{3}{2}\left(3^{k+1}-1\right) \quad \checkmark$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $x^{2n}+x^{2n-1}y+\cdots+xy^{2n-1}+y^{2n} = \dfrac{x^{2n+1}-y^{2n+1}}{x-y}$

Solution

For $\boldsymbol{n} = \mathbf{1}$

$$x^2+xy+y^2 \overset{?}{=} \frac{x^3-y^3}{x-y}$$
$$= \frac{(x-y)\left(x^2+xy+y^2\right)}{x-y}$$
$$= x^2+xy+y^2 \quad \checkmark$$

Hence, P_1 is true.

$x^{2k}+x^{2k-1}y+\cdots+xy^{2k-1}+y^{2k} = \dfrac{x^{2k+1}-y^{2k+1}}{x-y}$ is true

$$x^{2(k+1)}+x^{2(k+1)-1}y+\cdots+xy^{2(k+1)-1}+y^{2(k+1)} \overset{?}{=} \frac{x^{2(k+1)+1}-y^{2(k+1)+1}}{x-y}$$

$$x^{2k+2}+x^{2k+1}y+\cdots+xy^{2k+1}+y^{2k+2} = x^2\left(x^{2k}+x^{2k-1}y+\cdots+y^{2k}\right)+xy^{2k+1}+y^{2k+2}$$
$$= x^2\left(\frac{x^{2k+1}-y^{2k+1}}{x-y}\right)+xy^{2k+1}+y^{2k+2}$$
$$= \frac{x^{2k+3}-x^2y^{2k+1}+x^2y^{2k+1}+xy^{2k+2}-xy^{2k+2}-y^{2(k+1)+1}}{x-y}$$
$$= \frac{x^{2(k+1)+1}-y^{2(k+1)+1}}{x-y} \quad \checkmark$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $5 \cdot 6 + 5 \cdot 6^2 + 5 \cdot 6^3 + \cdots + 5 \cdot 6^n = 6\left(6^n - 1\right)$

Solution

For $\boldsymbol{n} = \mathbf{1}$

$$5 \cdot 6 \stackrel{?}{=} 6\left(6^1 - 1\right) = 6(5) \quad \checkmark$$

Hence, P_1 is true.

$5 \cdot 6 + 5 \cdot 6^2 + \cdots + 5 \cdot 6^k = 6\left(6^k - 1\right)$ is true

$$5 \cdot 6 + 5 \cdot 6^2 + \cdots + 5 \cdot 6^k + 5 \cdot 6^{k+1} \stackrel{?}{=} 6\left(6^{k+1} - 1\right)$$

$$\begin{aligned}
5 \cdot 6 + 5 \cdot 6^2 + \cdots + 5 \cdot 6^k + 5 \cdot 6^{k+1} &= 6\left(6^k - 1\right) + 5 \cdot 6^{k+1} \\
&= 6^{k+1} - 6 + 5 \cdot 6^{k+1} \\
&= 6^{k+1}(1+5) - 6 \\
&= 6 \cdot 6^{k+1} - 6 \\
&= 6\left(6^{k+1} - 1\right) \quad \checkmark
\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $7 \cdot 8 + 7 \cdot 8^2 + 7 \cdot 8^3 + \cdots + 7 \cdot 8^n = 8\left(8^n - 1\right)$

Solution

For $\boldsymbol{n} = \mathbf{1}$

$$7 \cdot 8 \stackrel{?}{=} 8\left(8^1 - 1\right) = 8(7) \quad \checkmark$$

Hence, P_1 is true.

$7 \cdot 8 + 7 \cdot 8^2 + \cdots + 7 \cdot 8^k = 8\left(8^k - 1\right)$ is true

$$7 \cdot 8 + 7 \cdot 8^2 + \cdots + 7 \cdot 8^k + 7 \cdot 8^{k+1} \stackrel{?}{=} 8\left(8^{k+1} - 1\right)$$

$$\begin{aligned}
7 \cdot 8 + 7 \cdot 8^2 + \cdots + 7 \cdot 8^k + 7 \cdot 8^{k+1} &= 8\left(8^k - 1\right) + 7 \cdot 8^{k+1} \\
&= 8^{k+1} - 8 + 7 \cdot 8^{k+1}
\end{aligned}$$

$$= 8^{k+1}(1+7) - 8$$

$$= 8\left(8^{k+1} - 1\right) \quad \checkmark$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\quad 3 + 6 + 9 + \cdots + 3n = \dfrac{3n(n+1)}{2}$

Solution

For $\boldsymbol{n = 1}$

$$3 \overset{?}{=} \frac{3(1)(1+1)}{2} = 3 \quad \checkmark$$

Hence, P_1 is true.

$3 + 6 + 9 + \cdots + 3k = \dfrac{3k(k+1)}{2}$ is true

$$3 + 6 + 9 + \cdots + 3k + 3(k+1) \overset{?}{=} \frac{3(k+1)(k+2)}{2}$$

$$\begin{aligned} 3 + 6 + 9 + \cdots + 3k + 3(k+1) &= \frac{3k(k+1)}{2} + 3(k+1) \\ &= \frac{3k(k+1) + 6(k+1)}{2} \\ &= \frac{(k+1)(3k+6)}{2} \\ &= \frac{3(k+1)(k+2)}{2} \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\quad 5 + 10 + 15 + \cdots + 5n = \dfrac{5n(n+1)}{2}$

Solution

For $\boldsymbol{n = 1}$

$$5 \overset{?}{=} \frac{5(1)(1+1)}{2} = 5 \quad \checkmark$$

Hence, P_1 is true.

$$5+10+15+\cdots+5k = \frac{5k(k+1)}{2} \text{ is true}$$

$$5+10+15+\cdots+5k+5(k+1) \overset{?}{=} \frac{5(k+1)(k+2)}{2}$$

$$\begin{aligned} 5+10+15+\cdots+5k+5(k+1) &= \frac{5k(k+1)}{2} + 5(k+1) \\ &= \frac{5k(k+1)+10(k+1)}{2} \\ &= \frac{(k+1)(5k+10)}{2} \\ &= \frac{5(k+1)(k+2)}{2} \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $1+3+5+\cdots+(2n-1) = n^2$

Solution

For $\boldsymbol{n = 1}$

$$1 \overset{?}{=} 1^2 = 1 \quad \checkmark$$

Hence, P_1 is true.

$$1+3+5+\cdots+(2k-1) = k^2 \text{ is true}$$

$$1+3+5+\cdots+(2k-1)+(2(k+1)-1) \overset{?}{=} (k+1)^2$$

$$\begin{aligned} 1+3+5+\cdots+(2k-1)+(2(k+1)-1) &= k^2+2k+2-1 \\ &= k^2+2k+1 \\ &= (k+1)^2 \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $4+7+10+\cdots+(3n+1)=\frac{n(3n+5)}{2}$

Solution

For $\boldsymbol{n}$ **= 1**

$$4\overset{?}{=}\frac{1(3+5)}{2}=4 \quad \checkmark$$

Hence, P_1 is true.

$$4+7+10+\cdots+(3k+1)=\frac{k(3k+5)}{2} \text{ is true}$$

$$4+7+10+\cdots+(3k+1)+(3(k+1)+1)\overset{?}{=}\frac{(k+1)(3(k+1)+5)}{2}=\frac{(k+1)(3k+8)}{2}$$

$$\begin{aligned}4+7+10+\cdots+(3k+1)+(3k+4)&=\frac{k(3k+5)}{2}+3k+4\\&=\frac{3k^2+5k+6k+8}{2}\\&=\frac{3k^2+5k+3k+3k+8}{2}\\&=\frac{k(3k+8)+(3k+8)}{2}\\&=\frac{(3k+8)(k+1)}{2} \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction:

$$2+4+6+\cdots+2(n-1)+2n=n(n+1)$$

Solution

For $\boldsymbol{n}$ **= 1**

$$2\overset{?}{=}1(1+1)$$

$$2=2 \quad \checkmark$$

Hence, P_1 is true.

For $\boldsymbol{k}$: $2+4+6+\cdots+2(k-1)+2k=k(k+1)$

$$2+4+\cdots+2k+2(k+1)=(k+1)(k+2)$$

$$2+4+\cdots+2k+2(k+1)=k(k+1)+2(k+1)$$

$$=(k+1)(k+2) \quad \checkmark$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction:

$$1+(1+2)+(1+2+3)+\ \cdots\ +(1+2+\cdots+n)=\frac{n(n+1)(n+2)}{6}$$

Solution

For $\boldsymbol{n}$ **= 1**

$$1 \stackrel{?}{=} \frac{1(1+1)(1+2)}{6}$$

$$1 \stackrel{?}{=} \frac{(2)(3)}{6}$$

$$1=1 \quad \checkmark$$

Hence, P_1 is true.

For $\boldsymbol{k}$: $\quad 1+(1+2)+\ \cdots\ +(1+2+\cdots+k)=\dfrac{k(k+1)(k+2)}{6}$

Is P_{k+1}: $\quad 1+(1+2)+\ \cdots\ +(1+2+\cdots+k)+\big(1+2+\cdots+k+(k+1)\big) \stackrel{?}{=} \dfrac{(k+1)(k+2)(k+3)}{6}$

$$1+(1+2)+\ \cdots\ +(1+2+\cdots+k)+\big(1+2+\cdots+k+(k+1)\big)=\frac{k(k+1)(k+2)}{6}+\big(1+2+\cdots+k+(k+1)\big)$$

$$1+2+\cdots+n=\frac{1}{2}n(n+1)$$

$$1+2+\cdots+k+(k+1)=\frac{1}{2}k(k+1)+(k+1)$$

$$=(k+1)\left(\frac{1}{2}k+1\right)$$

$$=\frac{1}{2}(k+1)(k+2)$$

$$1+(1+2)+\ \cdots\ +(1+2+\cdots+k)+\big(1+2+\cdots+k+(k+1)\big)=\frac{k(k+1)(k+2)}{6}+\frac{1}{2}(k+1)(k+2)$$

$$=(k+1)(k+2)\left(\frac{k}{6}+\frac{1}{2}\right)$$

$$=\frac{(k+1)(k+2)(k+3)}{6} \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction: $1+2+3+\cdots+n<\frac{(2n+3)^2}{7}$

Solution

For $\boldsymbol{n}$ **= 1**

$$1 \overset{?}{<} \frac{(2+3)^2}{7}$$

$$1<\frac{25}{7} \quad >1 \quad \surd$$

Hence, P_1 is true.

For $\boldsymbol{k}$: $\quad 1+2+\cdots+k<\frac{(2k+3)^2}{7}$

Is P_{k+1}: $\quad 1+2+\cdots+k+(k+1)<\frac{(2(k+1)+3)^2}{7}$

$$<\frac{(2k+5)^2}{7} \ ? \qquad \frac{4k^2+20k+25}{7}$$

$$\begin{aligned}
1+2+\cdots+k+(k+1) &< \frac{(2k+3)^2}{7}+(k+1)\\
&=\frac{4k^2+12k+9+7k+7}{7}\\
&=\frac{1}{7}\left(4k^2+19k+16+k+9-k-9\right)\\
&=\frac{1}{7}\left(4k^2+20k+25-(k+9)\right)\\
&=\frac{(2k+5)^2}{7}-\frac{k+9}{7}\\
&<\frac{(2k+5)^2}{7} \quad \surd
\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction:

$$\frac{1}{2n} \le \frac{1\cdot 3\cdot 5\cdots(2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot(2n)}$$

Solution

For ***n* = 1**

$$\frac{1}{2} \le \frac{1}{2} \; \surd$$

Hence, P_1 is true.

For ***k***: $\quad \frac{1}{2k} \le \frac{1\cdots(2k-3)\cdot(2k-1)}{2\cdot\cdots(2k-2)\cdot(2k)}$

Is P_{k+1}: $\quad \frac{1}{2(k+1)} \le \frac{1\cdots(2k-1)\cdot(2k+1)}{2\cdot\cdots(2k)\cdot(2k+2)}$

$$\frac{1\cdots(2k-1)\cdot(2k+1)}{2\cdot\cdots(2k)\cdot(2k+2)} \ge \frac{1}{2(k+1)} \quad ?$$

$$\begin{aligned}
\frac{1\cdots(2k-1)\cdot(2k+1)}{2\cdot\cdots(2k)\cdot(2k+2)} &= \frac{1\cdots(2k-1)}{2\cdot\cdots(2k)}\frac{2k+1}{2k+2} \\
&\ge \frac{1}{2k}\cdot\frac{2k+1}{2k+2} \\
&= \frac{2k+1}{2k}\cdot\frac{1}{2(k+1)} \\
&= \left(1+\frac{1}{2k}\right)\cdot\frac{1}{2(k+1)} \\
&\ge \frac{1}{2(k+1)} \quad \surd
\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction: $\quad \frac{2n+1}{2n+2} \le \frac{\sqrt{n+1}}{\sqrt{n+2}}$

Solution

For ***n* = 1**

$$\frac{2+1}{2+2} \overset{?}{\le} \frac{\sqrt{1+1}}{\sqrt{1+2}}$$

$$\frac{3}{4} \overset{?}{\le} \frac{\sqrt{2}}{\sqrt{3}}$$

$$3\sqrt{3} \quad ? \quad 4\sqrt{2}$$ *Square both sides*

$$27 \quad \le \quad 32 \quad \surd$$

Hence, P_1 is true.

For ***k***: $\quad \frac{2k+1}{2k+2} \le \frac{\sqrt{k+1}}{\sqrt{k+2}}$

$$(2k+1)\sqrt{k+2} \le (2k+2)\sqrt{k+1}$$

Is P_{k+1}: $\quad \frac{2k+3}{2k+4} \overset{?}{\le} \frac{\sqrt{k+2}}{\sqrt{k+3}}$

$$(2k+3)\sqrt{k+3} \overset{?}{\le} (2k+4)\sqrt{k+2}$$

$$(2k+3) \le (2k+4)$$

$$\underline{\sqrt{k+3} \le \sqrt{k+2}}$$

$$(2k+3)\sqrt{k+3} \le (2k+4)\sqrt{k+2} \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction: $\quad n! < n^n \quad \textit{for} \quad n > 1$

Solution

For ***n* = 2**

$$2! \overset{?}{<} 2^2$$

$$2 < 4 \quad \surd$$

Hence, P_1 is true.

For ***k***: $\quad k! < k^k$

Is P_{k+1}: $\quad (k+1)! < (k+1)^{k+1}$

$$(k+1)! = k!\,(k+1)$$

$$< k^k\,(k+1)$$

$$k < k+1$$

$$k^k < (k+1)^k$$

$$< (k+1)^k \, (k+1)$$

$$= (k+1)^{k+1} \quad \checkmark$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction: $\left(a^m\right)^n = a^{mn}$ (a and m are constant)

Solution

For $\boldsymbol{n} = \mathbf{1}$

$$\left(a^m\right)^1 \stackrel{?}{=} a^{m(1)}$$

$$a^m = a^m \quad \checkmark$$

Hence, P_1 is true.

$\left(a^m\right)^k = a^{mk}$ is true

$$\left(a^m\right)^{(k+1)} \stackrel{?}{=} a^{m(k+1)}$$

$$\begin{aligned}\left(a^m\right)^{(k+1)} &= \left(a^m\right)^k a^m \\ &= a^{km} a^m \\ &= a^{km+m} \\ &= a^{m(k+1)} \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. $n < 2^n$

Solution

For $\boldsymbol{n} = \mathbf{1}$

$$1 \stackrel{?}{<} 2^1 \quad \checkmark$$

Hence, P_1 is true.

Assume that P_k is true $k < 2^k$

We need to prove that P_{k+1} is true, that is $k+1 < 2^{k+1}$

$$\begin{aligned} k+1 < k+k &= 2k \\ &< 2\cdot 2^k \\ &= 2^{k+1} \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. 3 is a factor of $n^3 - n + 3$

Solution

For $\boldsymbol{n} = \mathbf{1}$

$1^3 - 1 + 3 = 3 = 3(1) \quad \checkmark$

Hence, P_1 is true.

Assume that P_k is true 3 is a factor of $k^3 - k + 3$

We need to prove that P_{k+1} is true, that is $(k+1)^3 - (k+1) + 3$

$$\begin{aligned} (k+1)^3 - (k+1) + 3 &= k^3 + 3k^2 + 3k + 1 - k - 1 + 3 \\ &= \left(k^3 - k + 3\right) + 3k^2 + 3k \qquad\qquad k^3 - k + 3 = 3K \\ &= 3K + 3k^2 + 3k \\ &= 3\left(K + k^2 + k\right) \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. 4 is a factor of $5^n - 1$

Solution

For $\boldsymbol{n} = \mathbf{1}$

$5^{1} - 1 = 4 = 4(1)$ ✓

Hence, P_1 is true.

Assume that P_k is true 4 is a factor of $5^k - 1$

We need to prove that P_{k+1} is true, that is $5^{k+1} - 1$

$$\begin{aligned} 5^{k+1} - 1 &= 5^k 5^1 - 5 + 4 \\ &= 5\left(5^k - 1\right) + 4 \\ &= 5\left(5^k - 1\right) + 4 \end{aligned}$$

By the induction hypothesis, 4 is a factor of $5^k - 1$ and 4 is a factor of 4, so 4 is a factor of the $(k+1)$ term. ✓

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $2^n > 2n \quad if \quad n \geq 3$

Solution

For $\boldsymbol{n = 3}$

$2^3 \geq 2(3)$

$8 \geq 6$ ✓

Hence, P_3 is true.

Assume that P_k is true: $2^k > 2k$

We need to prove that $P_{k+1}: 2^{k+1} > 2(k+1)$ is true

$$\begin{aligned} 2^k &> 2k \\ 2^k \cdot 2 &> 2k \cdot 2 \\ 2^{k+1} &> 4k = 2k + 2k \qquad\qquad k \geq 3 \\ &> 2k + 2 \\ &= 2(k+1) \ ✓ \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: If $0 < a < 1$, then $a^n < a^{n-1}$

Solution

For $\boldsymbol{n} = \mathbf{1}$

$a^1 < a^{1-1}$

$a < 1$ √

since $0 < a < 1 \Rightarrow P_1$ is true.

Assume that P_k is true: $a^k < a^{k-1}$

We need to prove that P_{k+1} : $a^{k+1} < a^k$ is true

$$a^k < a^{k-1} \quad \to \quad a^k \cdot a < a^{k-1} \cdot a$$

$$a^{k+1} < a^k \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: If $n \geq 4$, then $n! > 2^n$

Solution

For $\boldsymbol{n} = \mathbf{4}$

$4! > 2^4$

$24 > 16$ √

Hence, P_4 is true.

Assume that P_k is true: $k! > 2^k$

We need to prove that P_{k+1} : $(k+1)! > 2^{k+1}$ is true

$$\begin{aligned}(k+1)! &= k! \cdot (k+1) \\ &> 2^k (k+1) && \textit{Since} \quad k \geq 4 \Rightarrow k+1 > 2 \\ &> 2^k \cdot 2 \\ &= 2^{k+1} \quad \surd\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $3^n > 2n+1 \quad if \quad n \ge 2$

Solution

For $\boldsymbol{n} = \mathbf{2}$

$$3^2 > 2(2)+1$$

$$9 > 5 \ \checkmark$$

Hence, P_2 is true.

Assume that P_k is true: $3^k > 2k+1$;

We need to prove that P_{k+1}: $3^{k+1} > 2(k+1)+1$ is true

$$\begin{aligned} 3^k > 2k+1 \quad \Rightarrow \quad 3^k \cdot 3 &> (2k+1)\cdot 3 \\ 3^{k+1} &> 6k+3 \\ &> 2k+2+1 \\ &= 2(k+1)+1 \ \checkmark \end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $2^n > n^2 \quad for \quad n > 4$

Solution

For $\boldsymbol{n} = \mathbf{5}$

$$2^5 > 5^2$$

$$32 > 25 \ \checkmark$$

Hence, P_5 is true.

Assume that P_k is true: $2^k > k^2$

We need to prove that P_{k+1}: $2^{k+1} > (k+1)^2$ is true

$$\begin{aligned} 2^k &> k^2 \\ 2^k \cdot 2 &> k^2 \cdot 2 \\ 2^{k+1} &> 2k^2 \\ &= k^2 + k^2 \qquad\qquad k < k+1 \ \Rightarrow \ k\cdot k > k+k+1 \Rightarrow \ k^2 > 2k+1 \\ &> k^2 + 2k+1 \end{aligned}$$

$= (k+1)^2$ ✓

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $4^n > n^4 \quad for \quad n \ge 5$

Solution

For ***n* = 5**

$4^5 > 5^4$

$1024 > 625$ ✓

Hence, P_5 is true.

Assume that P_k is true: $4^k > k^4$

We need to prove that P_{k+1} : $4^{k+1} > (k+1)^4$ is true

$4^k > k^4$

$4^k \cdot 4 > k^4 \cdot 4$

$4^{k+1} > 4k^4$

$k < k+1$

$4k > k+1$

$4k^4 > (k+1)^4$

$> (k+1)^4$ ✓

Hence P_{k+1} is true.

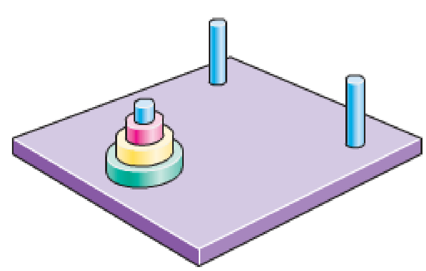
∴ By the mathematical induction, the given statement is true.

Exercise

A pile of *n* rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must be moved, one at a time, to a different peg with no ring ever placed on top of a smaller ring.

Find the least number of moves that would be required.

Prove your result by mathematical induction.



Solution

With 1 ring, 1 move is required.

With 2 rings, 3 moves are required $\Rightarrow\quad 3 = 2+1$

With 3 rings, 7 moves are required $\Rightarrow\quad 7 = 2^2+2+1$

With n rings, $2^{n-1}+\cdots+2^2+2^1+2^0 = 2^n - 1$ moves are required

For $\boldsymbol{n} = \mathbf{1}$

$$2^0 = 2^1 - 1 = 1 \;\checkmark$$

Hence, P_1 is true.

Assume that P_k is true: $2^{k-1}+\cdots+2^2+2^1+2^0 = 2^k - 1$

$$2^k + 2^{k-1}+\cdots+2^2+2^1+1 \stackrel{?}{=} 2^{k+1} - 1$$

$$\begin{aligned} 2^k + 2^{k-1}+\cdots+2^2+2^1+1 &= 2^k + 2^k - 1 \\ &= 2\cdot 2^k - 1 \\ &= 2^{k+1} - 1 \quad \checkmark \end{aligned}$$

$\therefore$ By the mathematical induction, the given statement is true.