Lecture One – Vectors and Vector-Values Functions

Solution   *Section* 1.1 – Vectors

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + z^2 = 4, \quad y = 0$

Solution

The circle $x^2 + z^2 = 4$ in the ***xz***-plane

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + y^2 = 4, \quad z = -2$

Solution

The circle $x^2 + y^2 = 4$ in the plane $z = -2$

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + y^2 + z^2 = 1, \quad x = 0$

Solution

The circle $y^2 + z^2 = 1$ in the ***yz***-plane

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + (y-1)^2 + z^2 = 4, \quad y = 0$

Solution

$$x^2 + (0-1)^2 + z^2 = 4 \quad \Rightarrow x^2 + z^2 = 3$$

The circle $x^2 + z^2 = 3$ in the ***xz***-plane

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2+y^2+z^2=4, \quad y=x$

Solution

The circle formed by the intersection of the sphere $x^2+y^2+z^2=4$ and the plane $y=x$

Exercise

Find the distance between points $P_1(1,\ 1,\ 1), \quad P_2(3,\ 3,\ 0)$

Solution

$$\begin{aligned}\left|P_1P_2\right| &= \sqrt{(3-1)^2+(3-1)^2+(0-1)^2} \\ &= \sqrt{4+4+1} \\ &= \sqrt{9} \\ &= 3\end{aligned}$$

Exercise

Find the distance between points $P_1(-1,\ 1,\ 5), \quad P_2(2,\ 5,\ 0)$

Solution

$$\begin{aligned}\left|P_1P_2\right| &= \sqrt{(2+1)^2+(5-1)^2+(0-5)^2} \\ &= \sqrt{9+16+25} \\ &= \sqrt{50} \\ &= 5\sqrt{2}\end{aligned}$$

Exercise

Find the distance between points $P_1(1,\ 4,\ 5), \quad P_2(4,\ -2,\ 7)$

Solution

$$\begin{aligned}\left|P_1P_2\right| &= \sqrt{(4-1)^2+(-2-4)^2+(7-5)^2} \\ &= \sqrt{9+36+4} \\ &= 7\end{aligned}$$

Exercise

Find the distance between points $P_1(3,\ 4,\ 5),\quad P_2(2,\ 3,\ 4)$

Solution

$$\begin{aligned}\left|P_1P_2\right| &= \sqrt{(2-3)^2+(3-4)^2+(4-5)^2}\\ &= \sqrt{1+1+1}\\ &= \sqrt{3}\end{aligned}$$

Exercise

Find the center and radii of the spheres $\quad x^2+y^2+z^2+4x-4z=0$

Solution

$$\left(x^2+4x\right)+y^2+\left(z^2-4z\right)=0$$

$$\left(x^2+4x+4\right)+y^2+\left(z^2-4z+4\right)=4+4$$

$$(x+2)^2+y^2+(z-2)^2=8$$

The center is at $(-2,\ 0,\ 2)$ and the radius is $\sqrt{8}=2\sqrt{2}$

Exercise

Find the center and radii of the spheres $\quad x^2+y^2+z^2-6y+8z=0$

Solution

$$x^2+\left(y^2-6y\right)+\left(z^2+8z\right)=0$$

$$x^2+\left(y^2-6y+\left(-\frac{6}{2}\right)^2\right)+\left(z^2+8z+\left(\frac{8}{2}\right)^2\right)=9+16$$

$$x^2+(y-3)^2+(z+4)^2=25$$

The center is at $(0,\ 3,\ -4)$ and the radius is 5

Exercise

Find the center and radii of the spheres $\quad 2x^2+2y^2+2z^2+x+y+z=9$

Solution

$$x^2+y^2+z^2+\frac{1}{2}x+\frac{1}{2}y+\frac{1}{2}z=\frac{9}{2}$$

$$\left(x^2+\frac{1}{2}x+\left(\frac{1}{2}\frac{1}{2}\right)^2\right)+\left(y^2+\frac{1}{2}y+\left(\frac{1}{4}\right)^2\right)+\left(z^2+\frac{1}{2}z+\left(\frac{1}{4}\right)^2\right)=\frac{9}{2}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}$$

$$\left(x+\frac{1}{4}\right)^2+\left(y+\frac{1}{4}\right)^2+\left(z+\frac{1}{4}\right)^2=\frac{9}{2}+\frac{3}{16}$$

$$\left(x+\frac{1}{4}\right)^2+\left(y+\frac{1}{4}\right)^2+\left(z+\frac{1}{4}\right)^2=\frac{75}{16}$$

The center is at $\left(-\frac{1}{4},\ -\frac{1}{4},\ -\frac{1}{4}\right)$ and the radius is $\frac{5\sqrt{3}}{4}$

Exercise

Find a formula for the distance from the point $P(x,\ y,\ z)$ to *x*-axis

Solution

The distance between $(x,\ y,\ z)$ and $(x,\ 0,\ 0)$ is:

$$d=\sqrt{(x-x)^2+(y-0)^2+(z-0)^2}$$
$$=\sqrt{y^2+z^2}$$

Exercise

Find a formula for the distance from the point $P(x,\ y,\ z)$ to *xy*-plane

Solution

The distance between $(x,\ y,\ z)$ and $(x,\ 0,\ z)$ is:

$$d=\sqrt{(x-x)^2+(y-0)^2+(z-z)^2}$$
$$=y$$

Exercise

Let $\boldsymbol{u}=\langle -3,\ 4\rangle$ *and* $\boldsymbol{v}=\langle 2,\ -5\rangle$. Find the component form and the magnitude if the vector

a) $3\boldsymbol{u}-4\boldsymbol{v}$ *b)* $-2\boldsymbol{u}$ *c)* $\boldsymbol{u}+\boldsymbol{v}$

Solution

a) $3\vec{u}-4\vec{v}=3\langle -3,\ 4\rangle-4\langle 2,\ -5\rangle$

$$= \boxed{\langle -17,\ 32\rangle}$$

b) $-2\vec{u} = -2\langle -3,\ 4\rangle$

$$= \boxed{\langle 6,\ -8\rangle}$$

c) $\vec{u} + \vec{v} = \langle -3,\ 4\rangle + \langle 2,\ -5\rangle$

$$= \boxed{\langle -1,\ -1\rangle}$$

Exercise

Let $\boldsymbol{u} = \langle 3,\ -2\rangle$ *and* $\boldsymbol{v} = \langle -2,\ 5\rangle$. Find the component form and the magnitude if the vector

a) $3\boldsymbol{u}$ *b)* $\boldsymbol{u} - \boldsymbol{v}$ *c)* $2\boldsymbol{u} - 3\boldsymbol{v}$ *d)* $-2\boldsymbol{u} + 5\boldsymbol{v}$ *e)* $-\frac{5}{13}\boldsymbol{u} + \frac{12}{13}\boldsymbol{v}$

Solution

a) $3\boldsymbol{u} = 3\langle 3,\ -2\rangle$

$$= \boxed{\langle 9,\ -6\rangle}$$

b) $\boldsymbol{u} - \boldsymbol{v} = \langle 3,\ -2\rangle - \langle -2,\ 5\rangle$

$$= \boxed{\langle 5,\ -7\rangle}$$

c) $2\boldsymbol{u} - 3\boldsymbol{v} = 2\langle 3,\ -2\rangle - 3\langle -2,\ 5\rangle$

$$= \langle 6,\ -4\rangle - \langle -6,\ 15\rangle$$

$$= \boxed{\langle 12,\ -19\rangle}$$

d) $-2\boldsymbol{u} + 5\boldsymbol{v} = -2\langle 3,\ -2\rangle + 5\langle -2,\ 5\rangle$

$$= \langle -6,\ 4\rangle + \langle -10,\ 25\rangle$$

$$= \boxed{\langle -14,\ 29\rangle}$$

e) $-\frac{5}{13}\boldsymbol{u} + \frac{12}{13}\boldsymbol{v} = -\frac{5}{13}\langle 3,\ -2\rangle + \frac{12}{13}\langle -2,\ 5\rangle$

$$= \langle -6,\ 4\rangle - \langle -10,\ 25\rangle$$

$$= \boxed{\langle 4,\ -21\rangle}$$

Exercise

Find scalars a, b, and c such that $\langle 2,\ 2,\ 2\rangle = a\langle 1,\ 1,\ 0\rangle + b\langle 0,\ 1,\ 1\rangle + c\langle 1,\ 0,\ 1\rangle$

Solution

$$\langle 2,\ 2,\ 2\rangle = a\langle 1,\ 1,\ 0\rangle + b\langle 0,\ 1,\ 1\rangle + c\langle 1,\ 0,\ 1\rangle$$

$$= \langle a+c,\ a+b,\ b+c \rangle$$

$$\begin{cases} a+c=2 \\ a+b=2 \\ b+c=2 \end{cases} \quad \begin{cases} c=2-a \\ b=2-a \\ \end{cases} \quad \begin{cases} \\ \\ 2a-4=2 \end{cases}$$

$$\boxed{a=b=c=1}$$

Exercise

Find the component form of the vector: The sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ where

$$A=(1,-1), \quad B=(2,0), \quad C=(-1,3), \quad and \quad D=(-2,2)$$

Solution

$$\overrightarrow{AB} = \langle 2-1,\ 0-(-1) \rangle = \langle 1,\ 1 \rangle$$

$$\overrightarrow{CD} = \langle -2-(-1),\ 2-3 \rangle = \langle -1,\ -1 \rangle$$

$$\overrightarrow{AB} + \overrightarrow{CD} = \langle 1,\ 1 \rangle + \langle -1,\ -1 \rangle$$

$$\boxed{= \langle 0,\ 0 \rangle}$$

Exercise

Find the component form of the vector: The unit vector that makes an angle $\theta = \frac{2\pi}{3}$ with the positive x-axis

Solution

$$\left\langle \cos\frac{2\pi}{3},\ \sin\frac{2\pi}{3} \right\rangle = \boxed{\left\langle -\frac{1}{2},\ \frac{\sqrt{3}}{2} \right\rangle}$$

Exercise

Find the component form of the vector: The unit vector obtained by rotating the vector $\langle 0,\ 1 \rangle$ $120°$ counterclockwise about the origin

Solution

The angle of unit vector $\langle 0,\ 1 \rangle$ is 90°, this unit vector rotates 120° which makes an angle of $90° + 120° = 210°$ with the positive x-axis

$$\langle \cos 210°,\ \sin 210° \rangle = \boxed{\left\langle -\frac{\sqrt{3}}{2},\ -\frac{1}{2} \right\rangle}$$

Exercise

Find the component form of the vector: The unit vector obtained by rotating the vector $\langle 1,\ 0\rangle$ $135°$ counterclockwise about the origin

Solution

The angle of unit vector $\langle 1,\ 0\rangle$ is $0°$, this unit vector rotates $135°$ which makes an angle of $0° + 135° = 135°$ with the positive x-axis

$$\langle \cos 135°,\ \sin 135°\rangle = \boxed{\left\langle -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right\rangle}$$

Exercise

Find the component form of the vector: The unit vector that makes an angle $\theta = \frac{\pi}{6}$ with the positive x-axis

Solution

$$\boxed{\left\langle \cos\frac{\pi}{6},\ \sin\frac{\pi}{6}\right\rangle = \left\langle \frac{\sqrt{3}}{2},\ \frac{1}{2}\right\rangle}$$

Exercise

Find the component form of the vector: The vector 5 units long in the direction opposite to the direction of $\frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{j}$

Solution

$$-5\left(\frac{1}{\sqrt{\frac{9}{25} + \frac{16}{25}}}\right)\left(\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}\right) = -5(1)\left(\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}\right)$$

$$\boxed{= -3\hat{i} - 4\hat{j}}$$

Exercise

Express the velocity vector $\boldsymbol{v} = \left(e^t \cos t - e^t \sin t\right)\boldsymbol{i} + \left(e^t \cos t + e^t \sin t\right)\boldsymbol{j}$ *when* $t = \ln 2$ in terms of its length and direction.

Solution

$$\vec{v}(t = \ln 2) = \left(e^{\ln 2}\cos(\ln 2) - e^{\ln 2}\sin(\ln 2)\right)\hat{i} + \left(e^{\ln 2}\cos(\ln 2) + e^{\ln 2}\sin(\ln 2)\right)\hat{j}$$

$$= \left(2\cos(\ln 2) - 2\sin(\ln 2)\right)\hat{i} + \left(2\cos(\ln 2) + 2\sin(\ln 2)\right)\hat{j}$$

$$Length = |v| = \sqrt{\left(2\cos(\ln 2) - 2\sin(\ln 2)\right)^2 + \left(2\cos(\ln 2) + 2\sin(\ln 2)\right)^2}$$

$$= 2\sqrt{\begin{array}{l}\cos^2(\ln 2) - 2\cos(\ln 2)\sin(\ln 2) + \sin^2(\ln 2) \\ \quad + \cos^2(\ln 2) + 2\cos(\ln 2)\sin(\ln 2) + \sin^2(\ln 2)\end{array}}$$

$$\boxed{= 2\sqrt{2}}$$

$$\text{Direction } = \frac{v}{|v|} = \frac{2\left(\left(\cos(\ln 2) - \sin(\ln 2)\right)\boldsymbol{i} + \left(\cos(\ln 2) + \sin(\ln 2)\right)\boldsymbol{j}\right)}{2\sqrt{2}}$$

$$\boxed{= \frac{\left(\cos(\ln 2) - \sin(\ln 2)\right)}{\sqrt{2}}\hat{i} + \frac{\left(\cos(\ln 2) + \sin(\ln 2)\right)}{\sqrt{2}}\hat{j}}$$

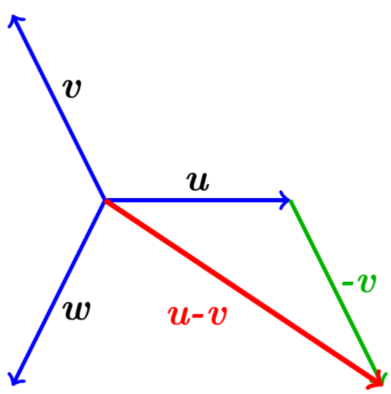
Exercise

Sketch the indicated vector

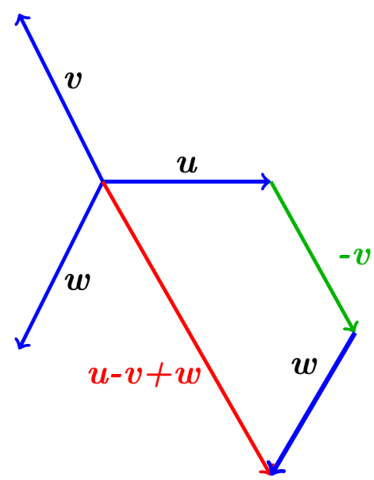
a) $\boldsymbol{u} - \boldsymbol{v}$ *b)* $2\boldsymbol{u} - \boldsymbol{v}$ *c)* $\boldsymbol{u} - \boldsymbol{v} + \boldsymbol{w}$

Solution

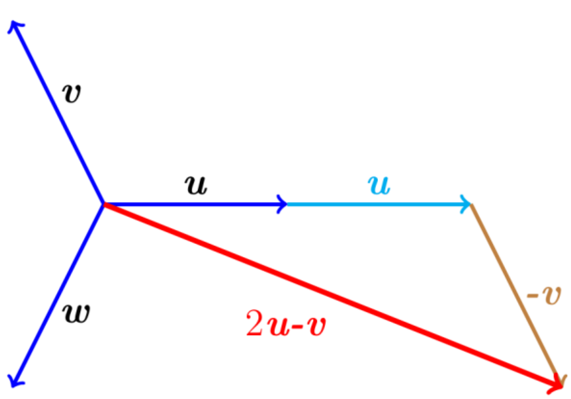
a)



b)



c)



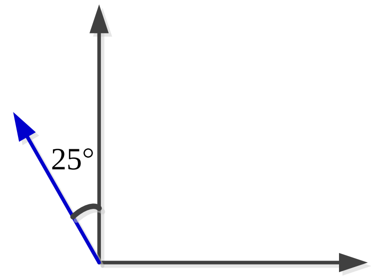
Exercise

An Airplane is flying in the direction 25° west of north at 800 *km/h*. Find the component form of the velocity of the airplane, assuming that the positive *x*-axis represents due east and the positive *y*-axis represents due north.

Solution

25° west of north is 25° + 90° = 115° north of east

$$800\langle \cos 115°,\ \sin 115°\rangle \approx \langle -338.095,\ 725.046\rangle$$



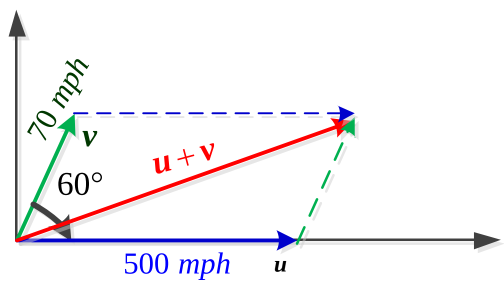
Exercise

A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What speed and direction should the jetliner have in order for the resultant vector to be 500 *mph* due east?

Solution

$\boldsymbol{u} = \langle x,\ y\rangle =$ the velocity of the airplane; $\boldsymbol{v}$ = the velocity of the tailwind

$$\boldsymbol{v} = \langle 70\cos 60°,\ 70\sin 60°\rangle$$
$$= \langle 35,\ 35\sqrt{3}\rangle$$



$$\boldsymbol{u} + \boldsymbol{v} = \langle 500,\ 0\rangle$$
$$\langle x,\ y\rangle + \langle 35,\ 35\sqrt{3}\rangle = \langle 500,\ 0\rangle$$
$$\langle x,\ y\rangle = \langle 500,\ 0\rangle - \langle 35,\ 35\sqrt{3}\rangle = \langle 765,\ -35\sqrt{3}\rangle$$
$$\boldsymbol{u} = \langle 765,\ -35\sqrt{3}\rangle$$

$$|\boldsymbol{u}| = \sqrt{465^2 + \left(-35\sqrt{3}\right)^2}$$
$$\approx 468.9\ mph$$

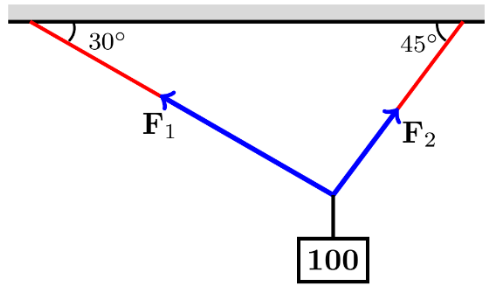
$$\theta = \tan^{-1}\frac{-35\sqrt{3}}{465}$$
$$\approx -7.4°$$

The direction is 7.4° south of east

Exercise

Consider a 100-N weight suspended by two wires. Find the magnitudes and components of the force vectors $\vec{F}_1$ *and* $\vec{F}_2$

Solution



$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos 30°,\ \left|\vec{F}_1\right|\sin 30° \right\rangle$$

$$= \left\langle -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right|,\ \frac{1}{2}\left|\vec{F}_1\right| \right\rangle$$

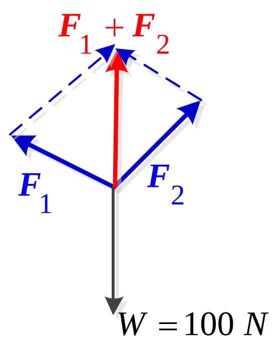
$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos 45°,\ \left|\vec{F}_2\right|\sin 45° \right\rangle$$

$$= \left\langle \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| \right\rangle$$

$$\vec{F}_1 + \vec{F}_2 = \langle 0,\ 100 \rangle$$

$$\left\langle -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right|,\ \frac{1}{2}\left|\vec{F}_1\right| \right\rangle + \left\langle \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| \right\rangle = \langle 0,\ 100 \rangle$$

$$\left\langle -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|,\ \frac{1}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| \right\rangle = \langle 0,\ 100 \rangle$$



$$\begin{cases} -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| = 0 \\ \frac{1}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| = 100 \end{cases}$$

$$\Rightarrow \quad \underline{\left|\vec{F}_1\right| \approx 73.205\ N} \quad \underline{\left|\vec{F}_2\right| \approx 89.658\ N}$$

$$\vec{F}_1 = \left\langle -\frac{\sqrt{3}}{2}(73.205),\ \frac{1}{2}(73.205) \right\rangle$$

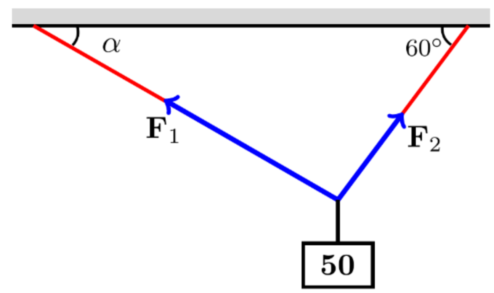
$$\underline{\approx \langle -63.397,\ 36.603 \rangle}$$

$$\vec{F}_2 = \left\langle \frac{\sqrt{2}}{2}(89.658),\ \frac{\sqrt{2}}{2}(89.658) \right\rangle$$

$$\underline{\approx \langle 63.397,\ 63.397 \rangle}$$

Exercise

Consider a 50-N weight suspended by two wires, If the magnitude of vector $\vec{F}_1 = 35\ N$, find the angle α and the magnitude of vector $\vec{F}_2$



Solution

$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos\alpha,\ \left|\vec{F}_1\right|\sin\alpha \right\rangle$$
$$= \left\langle -35\cos\alpha,\ 35\sin\alpha \right\rangle$$

$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos 60°,\ \left|\vec{F}_2\right|\sin 60° \right\rangle$$
$$= \left\langle \frac{1}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| \right\rangle$$

$$w = \langle 0,\ -50\rangle \quad \Rightarrow \quad \vec{F}_1 + \vec{F}_2 = \langle 0,\ 50\rangle$$

$$\left\langle -35\cos\alpha,\ 35\sin\alpha \right\rangle + \left\langle \frac{1}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| \right\rangle = \langle 0,\ 50\rangle$$

$$\left\langle -35\cos\alpha + \frac{1}{2}\left|\vec{F}_2\right|,\ 35\sin\alpha + \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| \right\rangle = \langle 0,\ 50\rangle$$

$$\rightarrow \begin{cases} -35\cos\alpha + \frac{1}{2}\left|\vec{F}_2\right| = 0 \\ 35\sin\alpha + \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| = 50 \end{cases} \qquad \rightarrow \begin{cases} \left|\vec{F}_2\right| = 70\cos\alpha \\ \end{cases}$$

$$35\sin\alpha + \frac{\sqrt{3}}{2}(70\cos\alpha) = 50$$
$$35\sqrt{3}\cos\alpha = 50 - 35\sin\alpha$$
$$\sqrt{3}\cos\alpha = \frac{10}{7} - \sin\alpha$$

$$\left(\sqrt{3}\cos\alpha\right)^2 = \left(\frac{10}{7} - \sin\alpha\right)^2$$
$$3\cos^2\alpha = \frac{100}{49} - \frac{20}{7}\sin\alpha + \sin^2\alpha$$
$$3\left(1 - \sin^2\alpha\right) = \frac{100}{49} - \frac{20}{7}\sin\alpha + \sin^2\alpha$$
$$3 - 3\sin^2\alpha - \frac{100}{49} + \frac{20}{7}\sin\alpha - \sin^2\alpha = 0$$
$$-4\sin^2\alpha + \frac{20}{7}\sin\alpha + \frac{47}{49} = 0$$

$$-196\sin^2\alpha + 140\sin\alpha + 47 = 0 \quad \Rightarrow \quad \sin\alpha = \frac{5 \pm 6\sqrt{2}}{14}$$

Since $\alpha > 0 \Rightarrow \sin\alpha > 0$

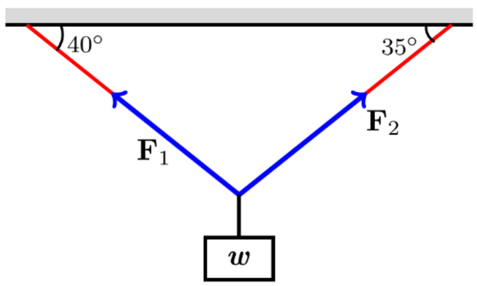
$\rightarrow \sin\alpha = \frac{5+6\sqrt{2}}{14} \approx 0.963$

$\boxed{\alpha \approx \sin^{-1}(0.963) = 74.42^\circ}$

$$\begin{aligned}\left|\vec{F}_2\right| &= 70\cos\alpha \\ &= 70\cos 74.42^\circ \\ &\approx \boxed{18.81\ N}\end{aligned}$$

Exercise

Consider a ***w***-N weight suspended by two wires, If the magnitude of vector $\vec{F}_2 = 100\ N$, find ***w*** and the magnitude of vector $\vec{F}_1$



Solution

$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos 40^\circ,\ \left|\vec{F}_1\right|\sin 40^\circ \right\rangle$

$$\begin{aligned}\vec{F}_2 &= \left\langle \left|\vec{F}_2\right|\cos 35^\circ,\ \left|\vec{F}_2\right|\sin 35^\circ \right\rangle \\ &= \langle 100(0.819),\ 100(0.5736)\rangle \\ &= \langle 81.915,\ 57.358\rangle\end{aligned}$$

$\vec{F}_1 + \vec{F}_2 = \langle 0,\ w\rangle$

$\left\langle -\left|\vec{F}_1\right|\cos 40^\circ,\ \left|\vec{F}_1\right|\sin 40^\circ \right\rangle + \langle 81.915,\ 57.358\rangle = \langle 0,\ w\rangle$

$\left\langle -\left|\vec{F}_1\right|\cos 40^\circ + 81.915,\ \left|\vec{F}_1\right|\sin 40^\circ + 57.358 \right\rangle = \langle 0,\ w\rangle$

$-\left|\vec{F}_1\right|\cos 40^\circ + 81.915 = 0$

$\left|\vec{F}_1\right|\cos 40^\circ = 81.915$

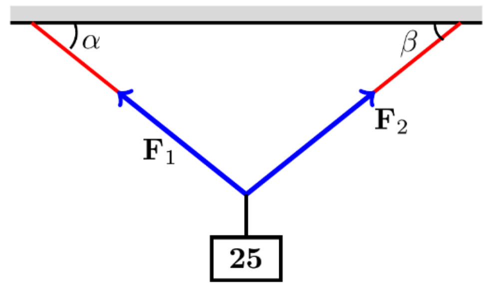
$$\begin{aligned}\left|\vec{F}_1\right| &= \frac{81.915}{\cos 40^\circ} \\ &\approx \boxed{106.933\ N}\end{aligned}$$

$$\begin{aligned}w &= \left|\vec{F}_1\right|\sin 40^\circ + 57.358 \\ &= 106.933\sin 40^\circ + 57.358 \\ &\approx \boxed{126.093\ \ N}\end{aligned}$$

Exercise

Consider a 25-N weight suspended by two wires, If the magnitude of vector $\vec{F}_1$ *and* $\vec{F}_2$ are both 75 N, then angles α and β are equal. Find α.

Solution



$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos\alpha,\ \left|\vec{F}_1\right|\sin\alpha \right\rangle$$
$$= \langle -75\cos\alpha,\ 75\sin\alpha \rangle$$

$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos\beta,\ \left|\vec{F}_2\right|\sin\beta \right\rangle$$
$$= \langle 75\cos\beta,\ 75\sin\beta \rangle$$

$$w = \langle 0,\ -25 \rangle \quad \Rightarrow \quad F_1 + F_2 = \langle 0,\ 25 \rangle$$

$$\langle -75\cos\alpha,\ 75\sin\alpha \rangle + \langle 75\cos\beta,\ 75\sin\beta \rangle = \langle 0,\ 25 \rangle \qquad \textit{since} \quad \alpha = \beta$$

$$\langle -75\cos\alpha + 75\cos\alpha,\ 75\sin\alpha + 75\sin\alpha \rangle = \langle 0,\ 25 \rangle$$

$$-75\cos\alpha + 75\cos\beta = 0 \quad \Rightarrow \quad \cos\alpha = \cos\beta$$

$$150\sin\alpha = 25$$

$$\sin\alpha = \frac{25}{150}$$

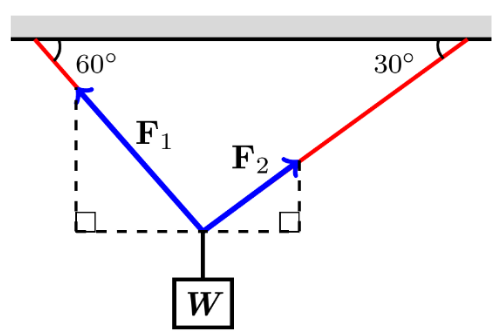
$$\alpha = \sin^{-1}\frac{25}{150}$$

$$\approx 9.59^\circ$$

Exercise

Consider a $W = 100\ N$ weight suspended by two wires. Find the magnitudes and components of the force vectors $\vec{F}_1$ *and* $\vec{F}_2$

Solution



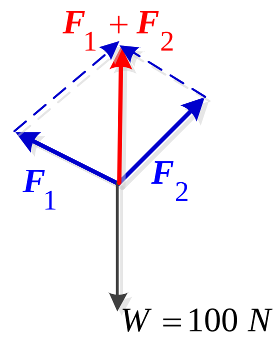
$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos 60^\circ,\ \left|\vec{F}_1\right|\sin 60^\circ \right\rangle$$
$$= \left\langle -\frac{1}{2}\left|\vec{F}_1\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_1\right| \right\rangle$$

$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos 30^\circ,\ \left|\vec{F}_2\right|\sin 30^\circ \right\rangle$$
$$= \left\langle \frac{\sqrt{3}}{2}\left|\vec{F}_2\right|,\ \frac{1}{2}\left|\vec{F}_2\right| \right\rangle$$

$$\vec{F}_1 + \vec{F}_2 = \langle 0,\ 100 \rangle$$

$$\left\langle -\frac{1}{2}\left|\vec{F}_1\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_1\right|\right\rangle + \left\langle \frac{\sqrt{3}}{2}\left|\vec{F}_2\right|,\ \frac{1}{2}\left|\vec{F}_2\right|\right\rangle = \langle 0,\ 100\rangle$$

$$\left\langle -\frac{1}{2}\left|\vec{F}_1\right| + \frac{\sqrt{3}}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_1\right| + \frac{1}{2}\left|\vec{F}_2\right|\right\rangle = \langle 0,\ 100\rangle$$



$$\begin{cases} -\frac{1}{2}\left|\vec{F}_1\right| + \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| = 0 \\ \frac{\sqrt{3}}{2}\left|\vec{F}_1\right| + \frac{1}{2}\left|\vec{F}_2\right| = 100 \end{cases}$$

$$\Delta = \begin{vmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix} = -1 \quad \Delta_1 = \begin{vmatrix} 0 & \frac{\sqrt{3}}{2} \\ 100 & \frac{1}{2} \end{vmatrix} = -50\sqrt{3} \quad \Delta = \begin{vmatrix} -\frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 100 \end{vmatrix} = -50$$

$$\Rightarrow \quad \boxed{\left|\vec{F}_1\right| = 50\sqrt{3}\ N} \quad \boxed{\left|\vec{F}_2\right| = 50\ N}$$

$$\vec{F}_1 = \left\langle -\frac{1}{2}\left(50\sqrt{3}\right),\ \frac{\sqrt{3}}{2}\left(50\sqrt{3}\right)\right\rangle$$

$$= \boxed{\left\langle -25\sqrt{3},\ 75\right\rangle}$$

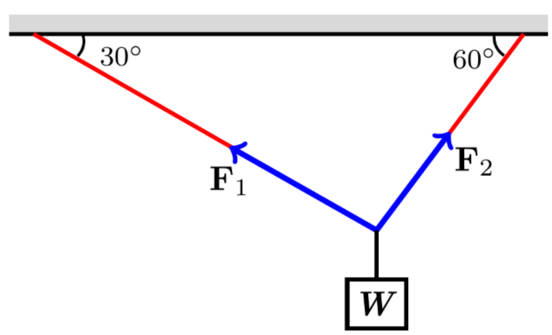
$$\vec{F}_2 = \left\langle \frac{\sqrt{3}}{2}(50),\ \frac{1}{2}(50)\right\rangle$$

$$= \boxed{\left\langle 25\sqrt{3},\ 25\right\rangle}$$

Exercise

Consider a $W = 50\ N$ weight suspended by two wires. Find the magnitudes and components of the force vectors $\vec{F}_1$ *and* $\vec{F}_2$

Solution



$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos 30°,\ \left|\vec{F}_1\right|\sin 30°\right\rangle$$

$$= \left\langle -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right|,\ \frac{1}{2}\left|\vec{F}_1\right|\right\rangle$$

$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos 60°,\ \left|\vec{F}_2\right|\sin 60°\right\rangle$$

$$= \left\langle \frac{1}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_2\right|\right\rangle$$

$$\vec{F}_1 + \vec{F}_2 = \langle 0,\ 50\rangle$$

$$\left\langle -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right|,\ \frac{1}{2}\left|\vec{F}_1\right|\right\rangle + \left\langle \frac{1}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_2\right|\right\rangle = \langle 0,\ 50\rangle$$

$$\left\langle -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right| + \frac{1}{2}\left|\vec{F}_2\right|,\ \frac{1}{2}\left|\vec{F}_1\right| + \frac{\sqrt{3}}{2}\left|\vec{F}_2\right|\right\rangle = \langle 0,\ 50\rangle$$

$$\begin{cases} -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right| + \frac{1}{2}\left|\vec{F}_2\right| = 0 \\ \frac{1}{2}\left|\vec{F}_1\right| + \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| = 50 \end{cases}$$

$$\Delta = \begin{vmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{vmatrix} = -1 \quad \Delta_1 = \begin{vmatrix} 0 & \frac{1}{2} \\ 50 & \frac{\sqrt{3}}{2} \end{vmatrix} = -25 \quad \Delta = \begin{vmatrix} -\frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 50 \end{vmatrix} = -25\sqrt{3}$$

$$\Rightarrow \boxed{\left|\vec{F}_1\right| = 25\ N} \quad \boxed{\left|\vec{F}_2\right| = 25\sqrt{3}\ N}$$

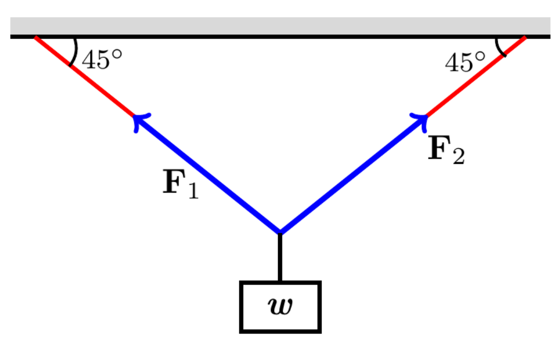
$$\boxed{\vec{F}_1 = \left\langle -\frac{25\sqrt{3}}{2},\ \frac{25}{2}\right\rangle}$$

$$\boxed{\vec{F}_2 = \left\langle \frac{25\sqrt{3}}{2},\ \frac{75}{2}\right\rangle}$$

Exercise

Consider a $W = 100\ N$ weight suspended by two wires. Find the magnitudes and components of the force vectors $\vec{F}_1$ *and* $\vec{F}_2$

Solution



$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos 45°,\ \left|\vec{F}_1\right|\sin 45°\right\rangle$$
$$= \left\langle -\frac{\sqrt{2}}{2}\left|\vec{F}_1\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_1\right|\right\rangle$$

$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos 45°,\ \left|\vec{F}_2\right|\sin 45°\right\rangle$$
$$= \left\langle \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|\right\rangle$$

$$\vec{F}_1 + \vec{F}_2 = \langle 0,\ 100\rangle$$

$$\left\langle -\frac{\sqrt{2}}{2}\left|\vec{F}_1\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_1\right|\right\rangle + \left\langle \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|\right\rangle = \langle 0,\ 100\rangle$$

$$\left\langle -\frac{\sqrt{2}}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|\right\rangle = \langle 0,\ 100\rangle$$

$$\begin{cases} -\frac{\sqrt{2}}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| = 0 \\ \frac{\sqrt{2}}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| = 100 \end{cases}$$

$$\Delta = \begin{vmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{vmatrix} = -1 \quad \Delta_1 = \begin{vmatrix} 0 & \frac{\sqrt{2}}{2} \\ 100 & \frac{\sqrt{2}}{2} \end{vmatrix} = -50\sqrt{2} \quad \Delta = \begin{vmatrix} -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 100 \end{vmatrix} = -50\sqrt{2}$$

$$\Rightarrow \quad \boxed{\left|\vec{F}_1\right| = 50\sqrt{2}\ \ N} \quad \boxed{\left|\vec{F}_2\right| = 50\sqrt{2}\ \ N}$$

$$\vec{F}_1 = \left\langle -\frac{\sqrt{2}}{2}\left(50\sqrt{2}\right),\ \frac{\sqrt{2}}{2}\left(50\sqrt{2}\right) \right\rangle$$

$$\boxed{= \langle -50,\ \ 50 \rangle}$$

$$\vec{F}_2 = \left\langle \frac{\sqrt{2}}{2}\left(50\sqrt{2}\right),\ \frac{\sqrt{2}}{2}\left(50\sqrt{2}\right) \right\rangle$$

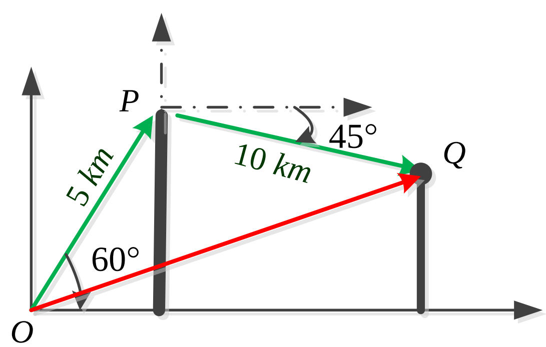
$$\boxed{= \langle 50,\ \ 50 \rangle}$$

Exercise

A bird flies from its nest 5 km in the direction 60° north east, where it stops to rest on a tree. It then flies 10 km in the direction due southeast and lands atop a telephone pole. Place an *xy*-coordinate system so that the origin is the bird's nest, the *x*-axis points east, and the *y*-axis points north.

a) At what point is the tree located?

b) At what point is the telephone pole?



Solution

a) $\overrightarrow{OP} = (5\cos 60°)\boldsymbol{i} + (5\sin 60°)\boldsymbol{j}$

$$= \frac{5}{2}\boldsymbol{i} + \frac{5\sqrt{3}}{2}\boldsymbol{j}$$

The tree is located at the point

$$\boxed{P = \left(\frac{5}{2},\ \frac{5\sqrt{3}}{2}\right)}$$

b) $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$

$$= \frac{5}{2}\hat{i} + \frac{5\sqrt{3}}{2}\hat{j} + (10\cos 315°)\hat{i} + (10\sin 315°)\hat{j}$$

$$= \frac{5}{2}\hat{i} + \frac{5\sqrt{3}}{2}\hat{j} + \left(10\frac{\sqrt{2}}{2}\right)\hat{i} + \left(10\left(-\frac{\sqrt{2}}{2}\right)\right)\hat{j}$$

$$=\left(\frac{5}{2}+5\sqrt{2}\right)\hat{i}+\left(\frac{5\sqrt{3}}{2}-\frac{10\sqrt{2}}{2}\right)\hat{j}$$

$$=\left(\frac{5+10\sqrt{2}}{2}\right)\hat{i}+\left(\frac{5\sqrt{3}-10\sqrt{2}}{2}\right)\hat{j}$$

The pole is located at the point $Q=\left(\frac{5+10\sqrt{2}}{2},\ \frac{5\sqrt{3}-10\sqrt{2}}{2}\right)$

Exercise

Suppose that *A*, *B*, and *C* are the corner points of the thin triangular plate of constant density.

a) Find the vector from *C* to the midpoint *M* of side *AB*.

b) Find the vector from *C* to the point that lies two-thirds of the way from *C* to *M* on the median *CM*.

c) Find the coordinates of the point in which the medians of $\triangle ABC$ intersect (this point is the plate's center of mass).

Solution

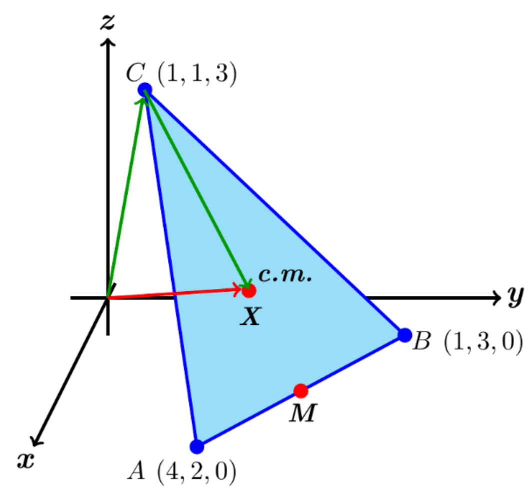
a) The midpoint of *AB* is:

$$M=\left(\frac{4+1}{2},\frac{2+3}{2},0\right)$$

$$=\left(\frac{5}{2},\ \frac{5}{2},\ 0\right)$$

$$\overrightarrow{CM}=\left(\frac{5}{2}-1\right)\hat{i}+\left(\frac{5}{2}-1\right)\hat{j}+(0-3)\hat{k}$$

$$=\frac{3}{2}\hat{i}+\frac{3}{2}\hat{j}-3\hat{k}$$



b) The desired vector is

$$\overrightarrow{CX}=\frac{2}{3}\overrightarrow{CM}$$

$$=\frac{2}{3}\left(\frac{3}{2}\hat{i}+\frac{3}{2}\hat{j}-3\hat{k}\right)$$

$$=\hat{i}+\hat{j}-2\hat{k}$$

c) The vector whose sum is the vector from the origin to *C* and the result of part (*b*) will terminate at the center of mass.

$$\overrightarrow{OX}=\overrightarrow{OC}+\overrightarrow{CX}$$

$$=\hat{i}+\hat{j}+3\hat{k}+\hat{i}+\hat{j}-2\hat{k}$$

$$=2\hat{i}+2\hat{j}+\hat{k}$$

Therefore; the center of mass point is $(2,\ 2,\ 1)$

Exercise

Show that a unit vector in the plane can be expressed as $\boldsymbol{u} = (\cos\theta)\boldsymbol{i} + (\sin\theta)\boldsymbol{j}$, obtained by rotating $\boldsymbol{i}$ through an angle θ in the counterclockwise direction. Explain why this form gives ***every*** *unit vector* in the plane.

Solution

Let $\boldsymbol{u}$ be any unit vector in the plane.

If $\boldsymbol{u}$ is positioned so that its initial point and terminal point is at (x, y), then $\boldsymbol{u}$ makes an angle θ with $\boldsymbol{i}$, measured in the *ccw* direction.

Since $|\boldsymbol{u}| = 1 \quad \Rightarrow \quad x = \cos\theta \quad and \quad y = \sin\theta$

That implies to: $\boldsymbol{u} = (\cos\theta)\hat{i} + (\sin\theta)\hat{j}$

Since $\boldsymbol{u}$ is any unit vector in the plane; this holds for every unit vector in the plane.

Exercise

Assume the positive x-*axis* points east and the positive y-*axis* points north.

a) An airliner flies northeast at a constant altitude at 550 *mi/hr* in calm air. Find *a* and *b* such that it velocity may be expressed in the form $\boldsymbol{v} = a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}}$

b) An airliner flies northeast at a constant altitude at 550 *mi/hr* relative to the air in a southerly crosswind $\boldsymbol{w} = \langle 0,\ 40\rangle$. Find the velocity of the airliner relative to the ground.

Solution

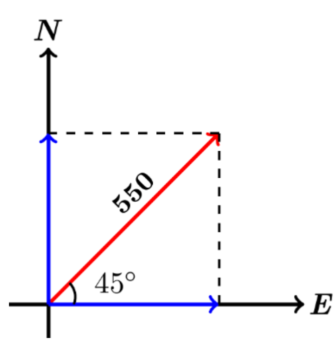
a) $\vec{v} = 550\langle -\cos 45°,\ \ \sin 45°\rangle$

$$= 550\left\langle -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right\rangle$$

$$\underline{= \langle -275\sqrt{2},\ \ 275\sqrt{2}\rangle}$$

b) $\vec{v} = 550\left\langle -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right\rangle + \langle 0,\ \ 40\rangle$

$$\underline{= \langle -275\sqrt{2},\ \ 275\sqrt{2}\rangle}$$



Exercise

Let $\overrightarrow{PQ}$ extended from $P(2,\ 0,\ 6)$ to $Q(2,\ -8,\ 5)$

a) Find the position vector equal to $\overrightarrow{PQ}$.

b) Find the midpoint *M* of the line segment PQ. Then find the magnitude of $\overrightarrow{PM}$.

c) Find a vector of length 8 with direction opposite that of $\overrightarrow{PQ}$.

Solution

a) $\overrightarrow{PQ} = \langle 2-2,\ -8-0,\ 5-6 \rangle$

$$\boxed{= \langle 0,\ -8,\ -1 \rangle}$$

b) $M = \left(\frac{2+2}{2},\ \frac{0-8}{2},\ \frac{6+5}{2} \right)$

$$\boxed{= \left(2,\ -4,\ \frac{11}{2} \right)}$$

$$\overrightarrow{PM} = \left\langle 0,\ -4,\ -\frac{1}{2} \right\rangle$$

$$\left| \overrightarrow{PM} \right| = \sqrt{16 + \frac{1}{4}}$$

$$\boxed{= \frac{1}{2}\sqrt{65}}$$

c) $\left| \overrightarrow{PQ} \right| = \sqrt{64+1}$

$$\boxed{= \sqrt{65}}$$

$$vector = \frac{-8}{\sqrt{65}} \langle 0,\ -8,\ -1 \rangle$$

$$\boxed{= \frac{8}{\sqrt{65}} \langle 0,\ 8,\ 1 \rangle}$$

Exercise

An object at the origin is acted on by the forces $\boldsymbol{F}_1 = -10\hat{\boldsymbol{i}} + 20\hat{\boldsymbol{k}}$, $\boldsymbol{F}_2 = 40\hat{\boldsymbol{j}} + 10\hat{\boldsymbol{k}}$, and $\boldsymbol{F}_3 = -50\hat{\boldsymbol{i}} + 20\hat{\boldsymbol{j}}$. Find the magnitude of the combined force and use a sketch to illustrate the direction of the combined force.

Solution

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$

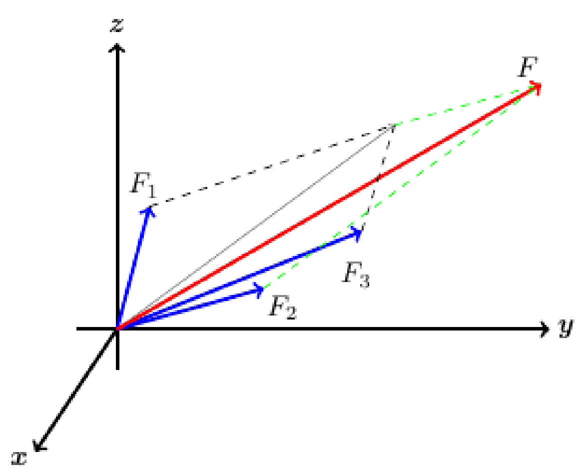
$$= -10\hat{i} + 20\hat{k} + 40\hat{j} + 10\hat{k} - 50\hat{i} + 20\hat{j}$$

$$= -60\hat{i} + 60\hat{j} + 30\hat{k}$$

$$\left| \vec{F} \right| = \sqrt{3600 + 3600 + 900}$$

$$= \sqrt{8100}$$

$$\boxed{= 90}$$



Exercise

A remote sensing probe falls vertically with a terminal of 60 *m/s* when it encounters a horizontal crosswind blowing north at 4 *m/s* and an updraft blowing vertically at 10 *m/s*. find the magnitude and direction of the resulting velocity relative to the ground.

Solution

The velocity relative to the ground is:

$$\langle 0,\ 4,\ 10-60\rangle = \langle 0,\ 4,\ -50\rangle$$

Magnitude: $\sqrt{16+2500} = \sqrt{2516}$

$$= 2\sqrt{629} \quad \approx 50.16\ \ m/s$$

$$\text{Direction } = \cos^{-1}\frac{4}{\sqrt{2516}}$$

$$\approx 85.4^\circ$$

Below the horizontal in the northerly horizontal direction.

Exercise

A small plane is flying north in calm air at 250 *mi/hr* when it is hit by a horizontal crosswind blowing northeast at 40 *mi/hr* and a 25 *mi/hr* downdraft. Find the resulting velocity and speed of the plane.

Solution

$$\text{Velocity vector } = \langle 250,\ 0,\ 0\rangle$$

$$\begin{aligned}\text{Crosswind } &= \langle 40\cos 45^\circ,\ 40\sin 45^\circ,\ 0\rangle \\ &= \langle 20\sqrt{2},\ 20\sqrt{2},\ 0\rangle\end{aligned}$$

$$\text{Downdraft } = \langle 0,\ 0,\ -25\rangle$$

$$\begin{aligned}\text{Resulting velocity } &= \langle 250,\ 0,\ 0\rangle + \langle 20\sqrt{2},\ 20\sqrt{2},\ 0\rangle + \langle 0,\ 0,\ -25\rangle \\ &= \langle 250+20\sqrt{2},\ 20\sqrt{2},\ -25\rangle\end{aligned}$$

$$\begin{aligned}\text{Speed } &= \sqrt{\left(250+20\sqrt{2}\right)^2 + 800 + 625} \\ &= \sqrt{62500 + 10^4\sqrt{2} + 800 + 1,425} \\ &= \sqrt{64,725 + 10^4\sqrt{2}} \\ &= 5\sqrt{2,589 + 400\sqrt{2}} \\ &= 280.83\ \ mph\end{aligned}$$