Solution Section 4.3 – Polar Coordinates

Exercise

Find the Cartesian coordinates of the following points (given in polar coordinates)

$$a)\ \left(\sqrt{2},\ \frac{\pi}{4}\right)\quad b)\ (1,\ 0)\quad c)\ \left(0,\ \frac{\pi}{2}\right)\quad d)\ \left(-\sqrt{2},\ \frac{\pi}{4}\right)$$

Solution

a) $\begin{cases} x = r\cos\theta = \sqrt{2}\cos\frac{\pi}{4} = 1 \\ x = r\sin\theta = \sqrt{2}\sin\frac{\pi}{4} = 1 \end{cases}$ ***Cartesian coordinates*** $(1,\ 1)$

b) $\begin{cases} x = r\cos\theta = 1\cos 0 = 1 \\ x = r\sin\theta = 1\sin 0 = 0 \end{cases}$ ***Cartesian coordinates*** $(1,\ 0)$

c) $\begin{cases} x = r\cos\theta = 0\cos\frac{\pi}{2} = 0 \\ x = r\sin\theta = 0\sin\frac{\pi}{2} = 0 \end{cases}$ ***Cartesian coordinates*** $(0,\ 0)$

d) $\begin{cases} x = r\cos\theta = -\sqrt{2}\cos\frac{\pi}{4} = -1 \\ x = r\sin\theta = -\sqrt{2}\sin\frac{\pi}{4} = -1 \end{cases}$ ***Cartesian coordinates*** $(-1,\ -1)$

Exercise

Find the polar coordinates, $0 \le \theta < 2\pi$ *and* $r \ge 0$, of the following points given in Cartesian coordinates

$$a)\ (1,\ 1)\quad b)\ (-3,\ 0)\quad c)\ \left(\sqrt{3},\ -1\right)\quad d)\ (-3,\ 4)$$

Solution

a) $\begin{cases} r = \sqrt{1^2 + 1^2} = \sqrt{2} \\ \theta = \tan^{-1}\frac{1}{1} = \frac{\pi}{4} \end{cases}$ ***Polar coordinates*** $\left(\sqrt{2},\ \frac{\pi}{4}\right)$

b) $\begin{cases} r = \sqrt{(-3)^2 + 0^2} = 3 \\ \theta = \tan^{-1}\frac{0}{-3} = \pi \end{cases}$ ***Polar coordinates*** $(3,\ \pi)$

c) $\begin{cases} r = \sqrt{\sqrt{3}^2 + (-1)^2} = 2 \\ \theta = \tan^{-1}\frac{-1}{\sqrt{3}} = \frac{11\pi}{6} \end{cases}$ ***Polar coordinates*** $\left(2,\ \frac{11\pi}{6}\right)$

d) $\begin{cases} r = \sqrt{(-3)^2 + 4^2} = 5 \\ \theta = \tan^{-1}\frac{4}{-3} = \pi - \arctan\left(\frac{4}{3}\right) \end{cases}$ ***Polar coordinates*** $\left(5,\ \pi - \arctan\left(\frac{4}{3}\right)\right)$

Exercise

Find the polar coordinates, $-\pi \le \theta < \pi$ *and* $r \ge 0$, of the following points given in Cartesian coordinates

$a)\ (-2,\ -2)\quad b)\ (0,\ 3)\quad c)\ \left(-\sqrt{3},\ 1\right)\quad d)\ (5,\ -12)$

Solution

a) $\begin{cases} r = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2} \\ \theta = \tan^{-1}\frac{-2}{-2} = -\frac{3\pi}{4} \end{cases}$ ***Polar coordinates*** $\left(2\sqrt{2}, -\frac{3\pi}{4}\right)$

b) $\begin{cases} r = \sqrt{0^2 + 3^2} = 3 \\ \theta = \tan^{-1}\frac{3}{0} = \frac{\pi}{2} \end{cases}$ ***Polar coordinates*** $\left(3,\ \frac{\pi}{2}\right)$

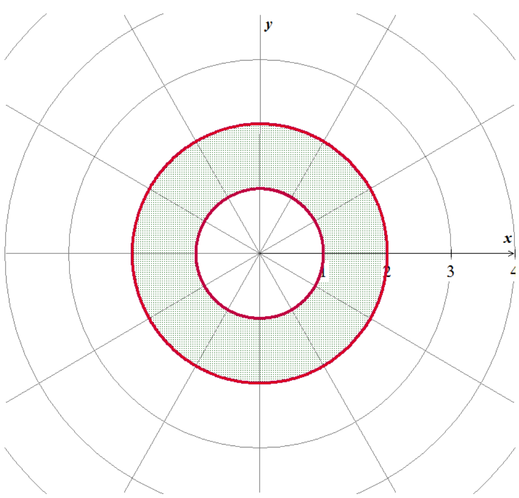
c) $\begin{cases} r = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = 2 \\ \theta = \tan^{-1}\frac{1}{-\sqrt{3}} = \frac{5\pi}{6} \end{cases}$ ***Polar coordinates*** $\left(2,\ \frac{5\pi}{6}\right)$

d) $\begin{cases} r = \sqrt{5^2 + (-12)^2} = 13 \\ \theta = \tan^{-1}\frac{-12}{5} = -\arctan\left(\frac{12}{5}\right) \end{cases}$ ***Polar coordinates*** $\left(13,\ -\arctan\left(\frac{12}{5}\right)\right)$

Exercise

Graph $1 \le r \le 2$

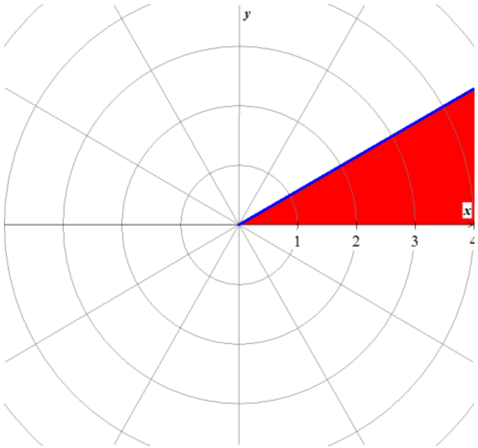
Solution



Exercise

Graph $\quad 0 \le \theta \le \frac{\pi}{6}, \quad r \ge 0$

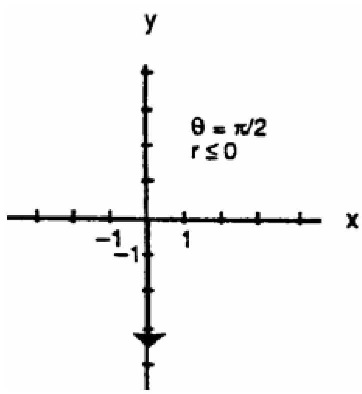
Solution



Exercise

Graph $\quad \theta = \frac{\pi}{2}, \quad r \le 0$

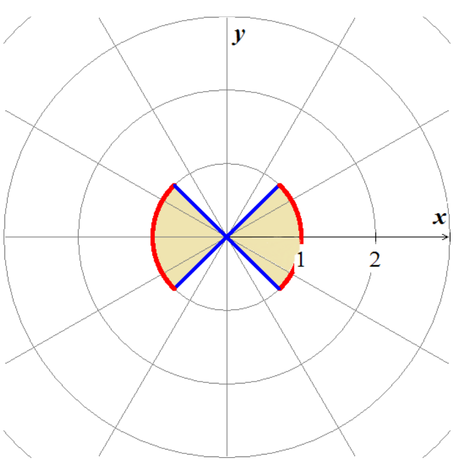
Solution



Exercise

Graph $\quad -\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}, \quad 0 \le r \le 1$

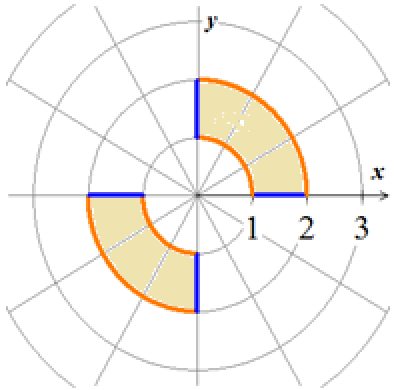
Solution



Exercise

Graph $0 \le \theta \le \frac{\pi}{2}, \quad 1 \le |r| \le 2$

Solution



Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r\cos\theta = 2$

Solution

$r\cos\theta = 2 \quad \Rightarrow \quad x = 2$, vertical line

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r\sin\theta = -1$

Solution

$r\sin\theta = -1 \quad \Rightarrow \quad y = -1$, horizontal line

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r = -3\sec\theta$

Solution

$$r = -3\sec\theta = -\frac{3}{\cos\theta} \quad \Rightarrow \quad r\cos\theta = -3$$

$x = -3$, vertical line through $(-3,\ 0)$

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r\cos\theta + r\sin\theta = 1$

Solution

$r\cos\theta + r\sin\theta = 1 \quad \Rightarrow \quad x + y = 1$, line with slope -1

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r^2 = 4r\sin\theta$

Solution

$$r^2 = 4r\sin\theta \quad \Rightarrow \quad x^2 + y^2 = 4y$$

$$x^2 + y^2 - 4y = 0$$

$$x^2 + y^2 - 4y + 4 = 4$$

$$x^2 + (y-2)^2 = 4$$

It is a circle with a center $C = (0,\ 2)$ and radius $r = 2$.

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r = \dfrac{5}{\sin\theta - 2\cos\theta}$

Solution

$$r = \frac{5}{\sin\theta - 2\cos\theta} \quad \Rightarrow \quad r\sin\theta - 2r\cos\theta = 5$$

$$y - 2x = 5 \quad \to y = 2x + 5$$

It is a line with slope $m = 2$ and intercept $b = 5$

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r = 4\tan\theta\sec\theta$

Solution

$$r = 4\tan\theta\sec\theta = 4\frac{\sin\theta}{\cos\theta}\frac{1}{\cos\theta} = 4\frac{\sin\theta}{\cos^2\theta} \quad \Rightarrow \quad r\cos^2\theta = 4\sin\theta$$

$$r^2\cos^2\theta = 4r\sin\theta$$

$x^2 = 4y \quad \to \quad \boxed{y = \frac{1}{4}x^2}$ It is a parabola with vertex (0, 0).

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r\sin\theta = \ln r + \ln\cos\theta$

Solution

$$r\sin\theta = \ln r + \ln\cos\theta$$

Power Rule

$$= \ln r\cos\theta$$

$\boxed{y = \ln x}$ Graph of the natural logarithm function

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $\cos^2\theta = \sin^2\theta$

Solution

$$\cos^2\theta = \sin^2\theta \quad \to \quad r^2\cos^2\theta = r^2\sin^2\theta$$

$$x^2 = y^2$$

$$\underline{y = \pm x}$$

The graph is 2 perpendicular lines through the origin with slopes –1 and 1,

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r = 2\cos\theta + 2\sin\theta$

Solution

$$r = 2\cos\theta + 2\sin\theta \quad \to \quad r^2 = 2r\cos\theta + 2r\sin\theta$$

$$x^2 + y^2 = 2x + 2y$$

$$x^2 - 2x + y^2 - 2y = 0$$

$$x^2 - 2x + 1 + y^2 - 2y + 1 = 1 + 1$$

$$(x-1)^2 + (y-1)^2 = 2$$

It is a circle with a center $C = (1,\ 1)$ and radius $r = \sqrt{2}$.

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r\sin\left(\frac{2\pi}{3} - \theta\right) = 5$

Solution

$$r\sin\left(\frac{2\pi}{3} - \theta\right) = 5 \quad \to \quad r\left(\sin\frac{2\pi}{3}\cos\theta - \cos\frac{2\pi}{3}\sin\theta\right) = 5$$

$$\frac{\sqrt{3}}{2}r\cos\theta + \frac{1}{2}r\sin\theta = 5$$

$$\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 5$$

$$\sqrt{3}x + y = 10$$

It is a line with slope $m = -\sqrt{3}$ and intercept $b = 10$

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r = \dfrac{4}{2\cos\theta - \sin\theta}$

Solution

$$2r\cos\theta - r\sin\theta = 4$$
$$2x - y = 4$$

The graph: Line $2x - y = 4$ with slope $m = 2$.

Exercise

Replace the Cartesian equation with equivalent polar equation $x = y$

Solution

$$x = y \quad \Rightarrow \quad r\cos\theta = r\sin\theta$$
$$\cos\theta = \sin\theta$$
$$\boxed{\theta = \frac{\pi}{4}}$$

Exercise

Replace the Cartesian equation with equivalent polar equation $x^2 - y^2 = 1$

Solution

$$x^2 - y^2 = 1 \quad \Rightarrow \quad r^2\cos^2\theta - r^2\sin^2\theta = 1$$
$$r^2\left(\cos^2\theta - \sin^2\theta\right) = 1$$
$$\boxed{r^2\cos 2\theta = 1}$$

Exercise

Replace the Cartesian equation with equivalent polar equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Solution

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \quad \Rightarrow \quad 4x^2 + 9y^2 = 36$$
$$\boxed{4r^2\cos^2\theta + 9r^2\sin^2\theta = 36}$$

Exercise

Replace the Cartesian equation with equivalent polar equation $xy = 1$

Solution

$$xy = 1 \quad \Rightarrow \quad r^2\cos\theta\sin\theta = 1 \qquad\qquad \sin 2\theta = 2\cos\theta\sin\theta$$
$$r^2\frac{1}{2}\sin 2\theta = 1$$
$$\boxed{r^2\sin 2\theta = 2}$$

Exercise

Replace the Cartesian equation with equivalent polar equation $x^2 + xy + y^2 = 1$

Solution

$$x^2 + xy + y^2 = 1 \quad \Rightarrow \quad r^2 + r^2 \cos\theta \sin\theta = 1$$

$$\boxed{r^2 (1 + \cos\theta \sin\theta) = 1}$$

Exercise

Replace the Cartesian equation with equivalent polar equation $x^2 + (y-2)^2 = 4$

Solution

$$x^2 + (y-2)^2 = 4 \quad \Rightarrow \quad x^2 + y^2 - 4y + 4 = 4$$

$$x^2 + y^2 - 4y = 0$$

$$r^2 - 4r\sin\theta = 0$$

$$r^2 = 4r\sin\theta$$

$$\boxed{r = 4\sin\theta}$$

Exercise

Replace the Cartesian equation with equivalent polar equation $(x+2)^2 + (y-5)^2 = 16$

Solution

$$(x+2)^2 + (y-5)^2 = 16 \quad \Rightarrow \quad x^2 + 4x + 4 + y^2 - 10y + 25 = 16$$

$$x^2 + 4x + y^2 - 10y = -13$$

$$r^2 + 4r\cos\theta - 10r\sin\theta = -13$$

$$\boxed{r^2 = -4r\cos\theta + 10r\sin\theta - 13}$$

Exercise

a) Show that every vertical line in the xy–plane has a polar equation of the form $r = a\sec\theta$

b) Find the analogous polar equation for horizontal lines in the xy–plane.

Solution

a) $x = a \quad \Rightarrow \quad r\cos\theta = a \qquad \to r = \dfrac{a}{\cos\theta} \boxed{= a\sec\theta}$

b) $y = b \quad \Rightarrow \quad r\sin\theta = b \qquad \to r = \dfrac{b}{\sin\theta} \boxed{= b\csc\theta}$

Exercise

Identify the symmetries of the curve. Then sketch the curve. $r = 2 - 2\cos\theta$

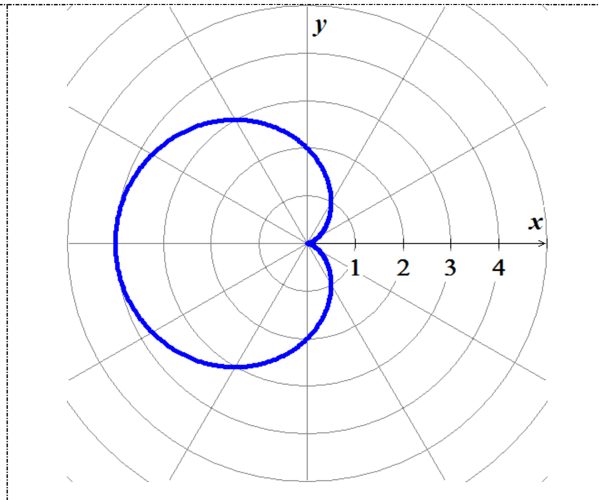
Solution

$2 - 2\cos(-\theta) = 2 - 2\cos\theta = r \quad \Rightarrow$ Symmetric about the *x*-axis

$\begin{cases} 2 - 2\cos(-\theta) \neq -r \\ 2 - 2\cos(\pi - \theta) = 2 + 2\cos\theta \neq r \end{cases} \quad \Rightarrow$ It is not symmetric about the *y*-axis

Therefore, it is *not* symmetric about the origin.

θ	$r = 2 - 2\cos\theta$
0	0
$\frac{\pi}{3}$	1
$\frac{\pi}{2}$	2
$\frac{2\pi}{3}$	1
π	4



Exercise

Identify the symmetries of the curve. Then sketch the curve. $r = 1 + \sin\theta$

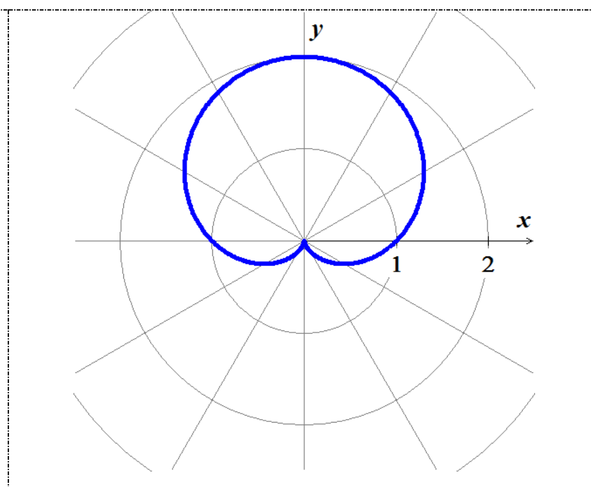
Solution

$\begin{cases} 1 + \sin(-\theta) = 1 - \sin\theta \neq r \\ 1 + \sin(\pi - \theta) = 1 + \sin\theta \neq -r \end{cases} \quad \Rightarrow$ It is not symmetric about the *x*-axis

$1 + \sin(\pi - \theta) = 1 + \sin\theta = r \quad \Rightarrow$ It is symmetric about the *y*-axis

Therefore, it is not symmetric about the origin.

θ	$r = 1 + \sin\theta$
$-\frac{\pi}{2}$	0
$-\frac{\pi}{4}$	.293
0	1
$\frac{\pi}{4}$	1.707
$\frac{\pi}{2}$	2



Exercise

Identify the symmetries of the curve. Then sketch the curve. $r = 2 + \sin\theta$

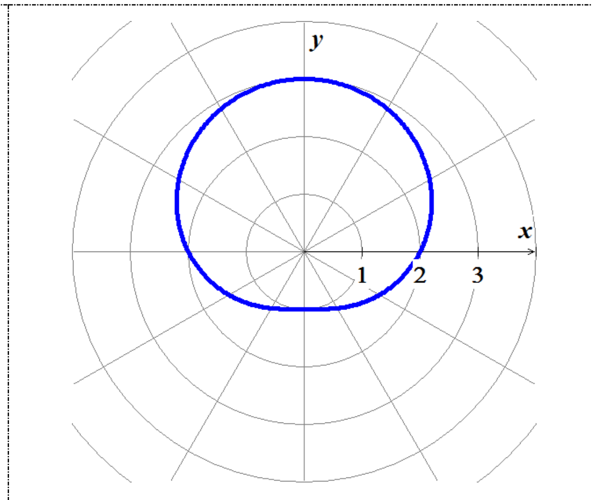
Solution

$$\begin{cases} 2+\sin(-\theta) = 2 - \sin\theta \neq r \\ 2+\sin(\pi-\theta) = 2+\sin\theta \neq -r \end{cases} \Rightarrow \text{ It is not symmetric about the } x\text{-axis}$$

$$2+\sin(\pi-\theta) = 2+\sin\theta = r \quad \Rightarrow \text{ It is symmetric about the } y\text{-axis}$$

Therefore, it is not symmetric about the origin.

θ	$r = 2+\sin\theta$
$-\frac{\pi}{2}$	1
$-\frac{\pi}{4}$	1.293
0	2
$\frac{\pi}{4}$	1.707
$\frac{\pi}{2}$	2.707



Exercise

Identify the symmetries of the curve. Then sketch the curve. $r^2 = \sin\theta$

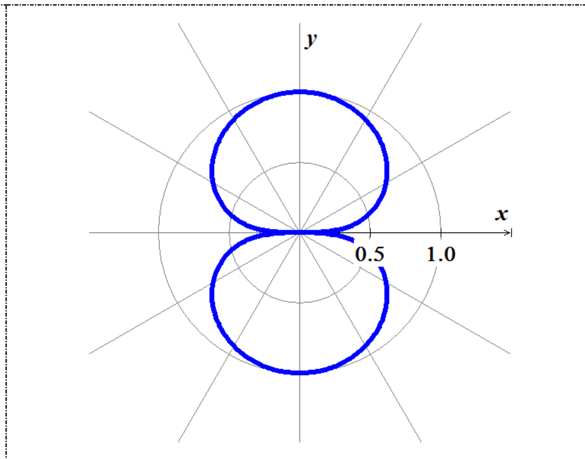
Solution

$$\sin(\pi-\theta) = \sin\theta = r^2 \quad \Rightarrow \text{ It is symmetric about the } x\text{-axis}$$

$$\sin(\pi-\theta) = \sin\theta = r^2 \quad \Rightarrow \text{ It is symmetric about the } y\text{-axis}$$

Therefore, it is symmetric about the origin.

θ	$r = \sqrt{\sin\theta}$
0	0
$\frac{\pi}{6}$	0.707
$\frac{\pi}{4}$	0.84
$\frac{\pi}{3}$	0.93
$\frac{\pi}{2}$	1



Exercise

Identify the symmetries of the curve. Then sketch the curve. $r^2 = -\sin\theta$

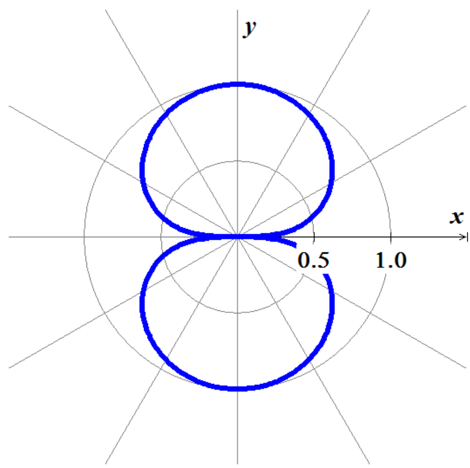
Solution

$-\sin(\pi - \theta) = -\sin\theta = r^2 \quad \Rightarrow$ It is symmetric about the *x*-axis

$-\sin(\pi - \theta) = -\sin\theta = r^2 \quad \Rightarrow$ It is symmetric about the *y*-axis

Therefore, it is symmetric about the origin

θ	$r^2 = -\sin\theta$
0	0
$\frac{\pi}{6}$	0.707
$\frac{\pi}{4}$	0.84
$\frac{\pi}{3}$	0.93
$\frac{\pi}{2}$	1



Exercise

Identify the symmetries of the curve. Then sketch the curve. $r^2 = -\cos\theta$

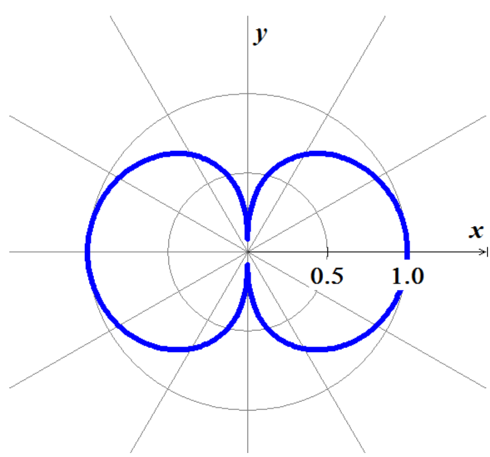
Solution

$-\cos(-\theta) = -\cos\theta = r^2 \quad \Rightarrow$ It is symmetric about the *x*-axis

$$\begin{cases} -\cos(-\theta) = -\cos\theta = r^2 \\ (-r)^2 = r^2 = -\cos\theta \end{cases} \quad \Rightarrow \text{ It is symmetric about the } y\text{-axis}$$

Therefore, it is symmetric about the origin

θ	$r = \sqrt{-\cos\theta}$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	0.7
$\frac{3\pi}{4}$	0.84
$\frac{5\pi}{6}$	0.93
	1

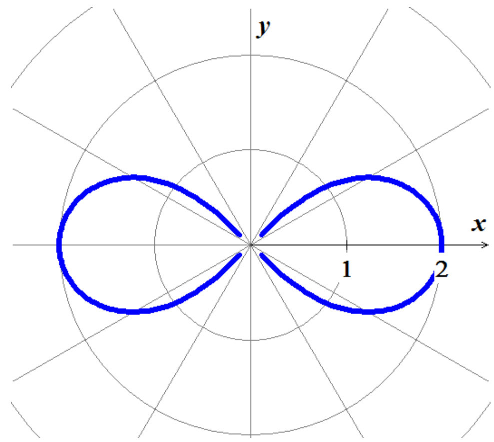


Exercise

Graph the lemniscate. What symmetries do these curves have? $r^2 = 4\cos 2\theta$

Solution

$\boldsymbol{\theta}$	$r^2 = 4\cos 2\theta$
0	2
$\frac{\pi}{12}$	1.8
$\frac{\pi}{6}$	1.4
$\frac{\pi}{4}$	0



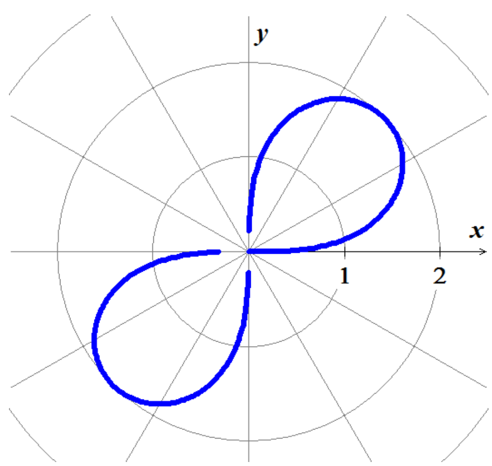
$(\pm r)^2 = 4\cos 2(-\theta) \Rightarrow r^2 = 4\cos 2\theta$ The graph is symmetric about the *x*-axis and the *y*-axis $\Rightarrow$ The graph is symmetric about the origin.

Exercise

Graph the lemniscate. What symmetries do these curves have? $r^2 = 4\sin 2\theta$

Solution

$\boldsymbol{\theta}$	$r^2 = 4\sin 2\theta$
0	0
$\frac{\pi}{12}$	1.4
$\frac{\pi}{6}$	1.8
$\frac{\pi}{4}$	2



$(\pm r)^2 = 4\sin 2\theta \Rightarrow r^2 = 4\sin 2\theta$ The graph is symmetric about the origin.

$4\sin 2(-\theta) = -4\sin 2\theta \neq r^2 \Rightarrow$ The graph is *not* symmetric about the *x*-axis

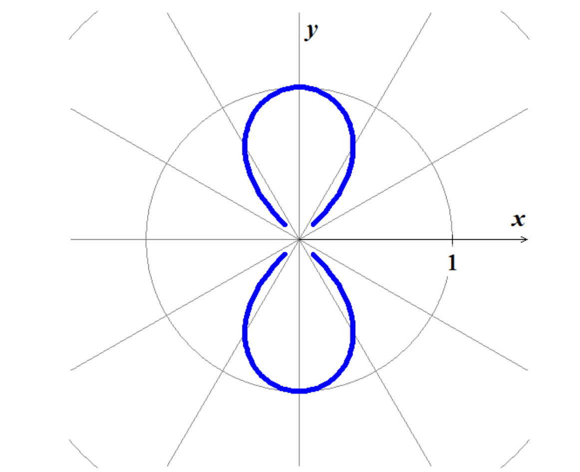
$4\sin 2(\pi - \theta) = 4\sin(2\pi - 2\theta) = 4\sin(-2\theta) = -4\sin 2\theta \neq r^2 \Rightarrow$ The graph is *not* symmetric about the *y*-axis.

Exercise

Graph the lemniscate. What symmetries do these curves have? $r^2 = -\cos 2\theta$

Solution

θ	$r^2 = -\cos 2\theta$
$\frac{\pi}{4}$	0
$\frac{\pi}{3}$	.7
$\frac{\pi}{2}$	1

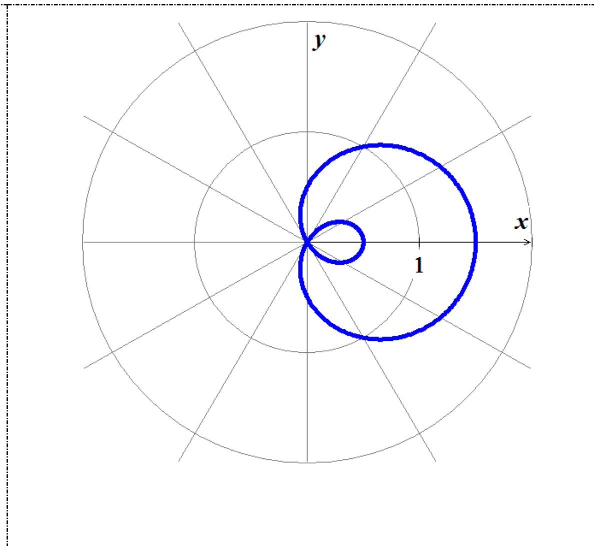


Exercise

Graph the limaçons is Old French for "snail". Equations for limaçons have the form $r = \frac{1}{2} + \cos\theta$

Solution

θ	$r = \frac{1}{2} + \cos\theta$
0	1.5
$\frac{\pi}{6}$	1.36
$\frac{\pi}{4}$	1.2
$\frac{\pi}{3}$	1
$\frac{\pi}{2}$	0.5
$\frac{3\pi}{4}$	–0.2
π	–0.5

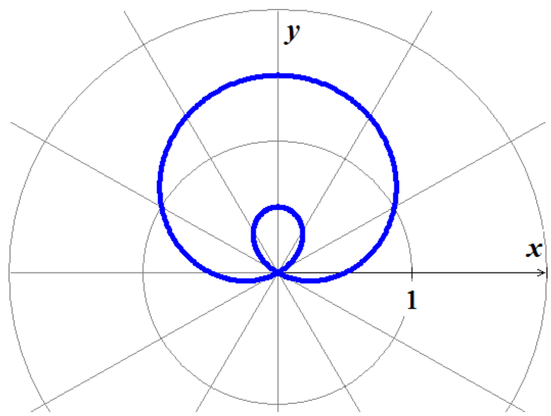


Exercise

Graph the limaçons is Old French for "snail". Equations for limaçons have the form $r = \frac{1}{2} + \sin\theta$

Solution

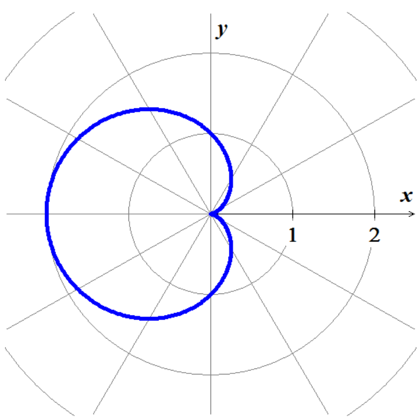
θ	$r = \frac{1}{2} + \sin\theta$
0	0.5
$\frac{\pi}{6}$	1
$\frac{\pi}{4}$	1.2
$\frac{\pi}{3}$	1.36
$\frac{\pi}{2}$	1.5
π	0.5
$\frac{5\pi}{4}$	−0.2
$\frac{3\pi}{2}$	−0.5



Exercise

Graph the limaçons is Old French for "snail". Equations for limaçons have the form $r = 1 - \cos\theta$

Solution

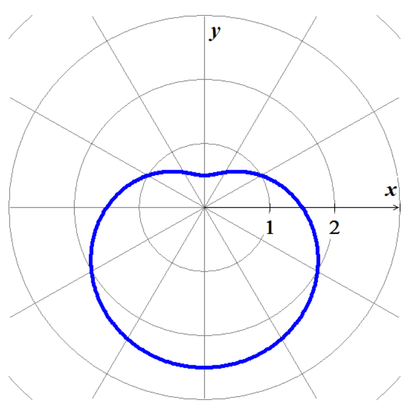


Exercise

Graph the limaçons is Old French for "snail".

Equations for limaçons have the form $r = \frac{3}{2} - \sin\theta$

Solution

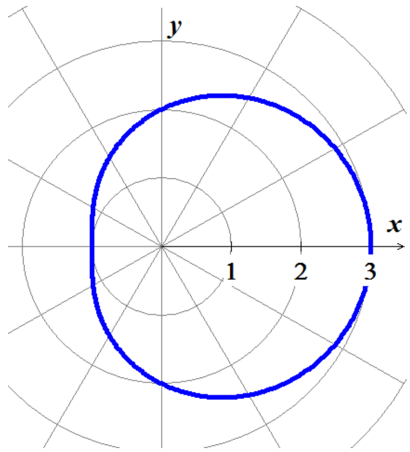


Exercise

Graph the limaçons is Old French for "snail". Equations for limaçons have the form $r = 2 + \cos\theta$

Solution

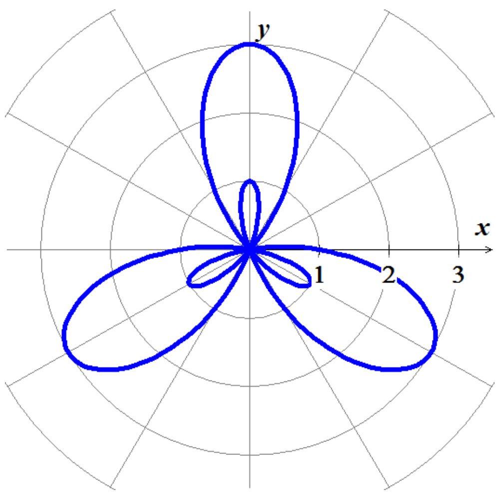
θ	$r = 2 + \cos\theta$
0	3
$\frac{\pi}{6}$	≈ 1.866
$\frac{\pi}{4}$	≈ 1.7
$\frac{\pi}{2}$	2
$\frac{3\pi}{4}$	≈ 1.29
π	1



Exercise

Graph the equation $r = 1 - 2\sin 3\theta$

Solution



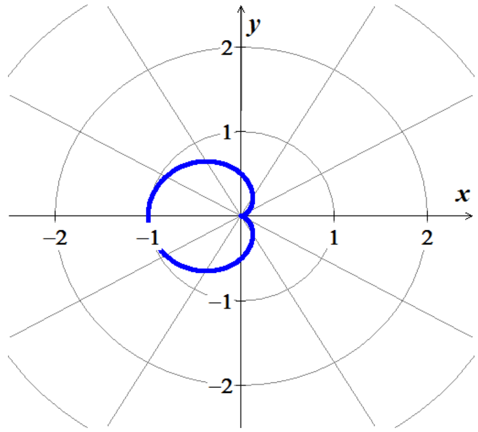
Exercise

Graph the equation $r = \sin^2 \frac{\theta}{2}$

Solution

$\sin^2\left(-\frac{\theta}{2}\right) = \sin^2\left(\frac{\theta}{2}\right) = r \quad \Rightarrow$ It is symmetric about the x-axis

θ	$r=\sin^2\dfrac{\theta}{2}$
0	0
$\frac{\pi}{3}$	0.25
$\frac{\pi}{2}$	0.5
$\frac{2\pi}{3}$	0.75
π	1

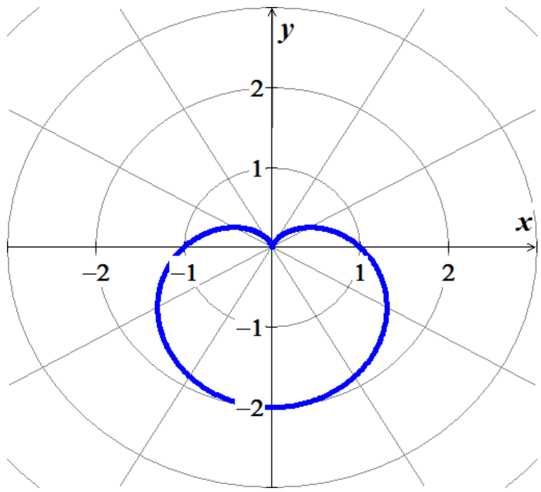


Exercise

Graph the equation $\quad r=1-\sin\theta$

Solution

θ	$r=1-\sin\theta$
0	1
$\frac{\pi}{6}$	0.5
$\frac{\pi}{4}$	≈ 0.3
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	≈ 0.3
π	1
$\frac{7\pi}{6}$	1.5
$\frac{3\pi}{2}$	2



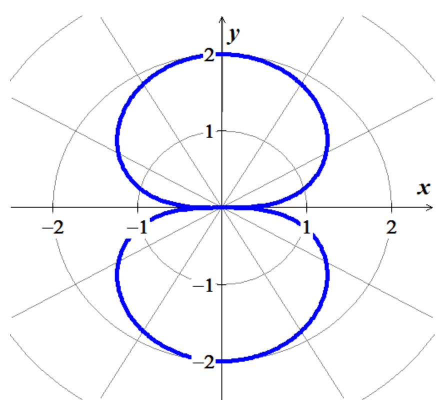
Exercise

Graph the equation $\quad r^2=4\sin\theta$

Solution

$4\sin(\pi-\theta)=4\sin\theta=r \quad \Rightarrow$ It is symmetric about the y-axis

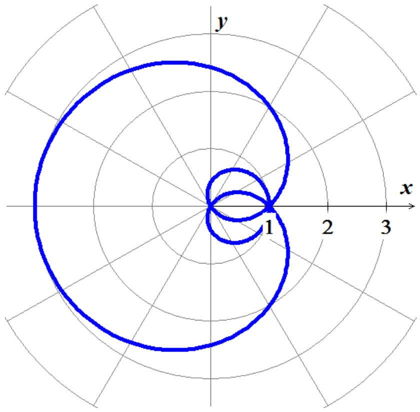
θ	$r=\pm 2\sqrt{\sin\theta}$
0	0
$\frac{\pi}{6}$	$\pm\sqrt{2}\approx\pm 1.4$
$\frac{\pi}{4}$	$\approx\pm 1.7$
$\frac{\pi}{3}$	$\approx\pm 1.9$
$\frac{\pi}{2}$	± 2



Exercise

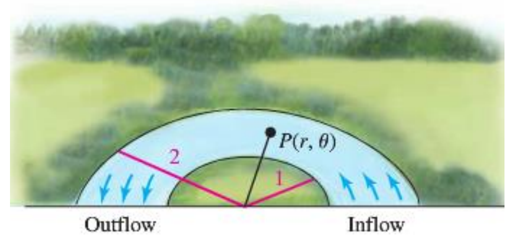
Graph the nephroid of Freeth equation $r = 1 + 2\sin\frac{\theta}{2}$

Solution



Exercise

Water flows in a shallow semicircular channel with inner and outer radii of 1 *m* and 2 *m*. At a point $P(r, \theta)$ in the channel, the flow is in the tangential direction (counterclockwise along circles), and it depends only on *r*, the distance from the center of the semicircles.



a) Express the region formed by the channel as a set in polar coordinates.

b) Express the inflow and outflow regions of the channel as sets in polar coordinates.

c) Suppose the tangential velocity of the water in *m/s* is given by $v(r) = 10r$, for $1 \le r \le 2$. Is the velocity greater at $\left(1.5,\ \frac{\pi}{4}\right)$ or $\left(1.2,\ \frac{3\pi}{4}\right)$? Explain.

d) Suppose the tangential velocity of the water is given by $v(r) = \frac{20}{r}$, for . Is the velocity greater $\left(1.8,\ \frac{\pi}{6}\right)$ or $\left(1.3,\ \frac{2\pi}{3}\right)$? Explain.

e) The total amount of water that flows through the channel (across a cross section of the channel $\theta = \theta_0$) is proportional to $\int_1^2 v(r)\,dr$. Is the total flow through the channel greater for the flow in part (*c*) or (*d*)?

Solution

a) The region is given by $\{(r,\ \theta):\ 1 \le r \le 2,\ \ 0 \le \theta \le \pi\}$

b) The inflow is given by $\{(r,\ \theta):\ 1 \le r \le 2,\ \ \theta = 0\}$

The outflow is given by $\{(r,\ \theta):\ 1 \le r \le 2,\ \ \theta = \pi\}$

c) The tangential velocity at $\left(1.5,\ \frac{\pi}{4}\right)$ is $v(1.5) = 10(1.5)\ \underline{= 15\ \ m/s}$

At $\left(1.2,\ \frac{3\pi}{4}\right)$ is $v(1.2) = 10(1.2)\ \underline{= 12\ \ m/s}$ So it is greater at 1.5.

d) The tangential velocity at $\left(1.8,\ \frac{\pi}{6}\right)$ is $v(1.8)=\frac{20}{1.8}\ \underline{\approx 11.11\ \ m/s}$

At $\left(1.3,\ \frac{2\pi}{3}\right)$ is $v(1.3)=\frac{20}{1.3}\ \underline{\approx 15.38\ \ m/s}$

So it is greater at 1.3.

e) $\displaystyle\int_1^2 v(r)\,dr=\int_1^2 10r\ dr$

$$=5r^2\Big|_1^2$$

$$\underline{=15}$$

$\displaystyle\int_1^2 v(r)\,dr=\int_1^2 \frac{20}{r}\ dr$

$$=20\ln r\Big|_1^2$$

$\underline{=20\ln 2}\quad \underline{\approx 13.86}$ So the flow in part (c) is greater.

Exercise

A simplified model assumes that the orbits of Earth and Mars are circular with radii of 2 and 3, respectively, and that Earth completes one orbit in one year while Mars takes two years. When $t=0$. Earth is at $(2,\ 0)$ and Mars is at $(3,\ 0)$; both orbit the Sum (at $(0,\ 0)$) in the counterclockwise direction. The position of Mars relative to Earth is given by the parametric equations

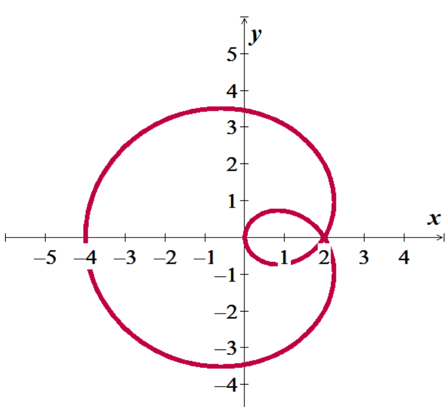
$$x=(3-4\cos\pi t)\cos\pi t+2,\quad y=(3-4\cos\pi t)\sin\pi t$$

a) Graph the parametric equations, for $0\le t\le 2$

b) Letting $r=3-4\cos\pi t$, explain why the path of Mars relative to Earth is a limaçon.

Solution

a)



b) $r=3-4\cos\pi t$ is a limaçon, and $x-2=r\cos\pi t$ and $y=r\,\text{sins}\,\pi t$ is a circle, and the composition of a limaçon and a circle is a limaçon.