#1 $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$

$$AB = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= I \checkmark$$
$$BA = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= I$$
$\therefore$ B is the inverse of A

#2/ $$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \frac{1}{6}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

#3/ $$\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}^{-1} = \frac{1}{7-6}\begin{pmatrix} 7 & -2 \\ -3 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 7 & -2 \\ -3 & -1 \end{pmatrix}$$

#4/ $$\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}^{-1} = \frac{1}{3-3}\,\cancel{A}\,\left(\frac{1}{0}\right).$$

57

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 4 & 0 & 1 & 0 \\ 3 & 6 & 5 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -3 & 1 & 0 \\ 0 & 3 & 2 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} 2R_1 - R_2 \\ \\ 2R_3 - 3R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 5 & -1 & 0 \\ 0 & 2 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & 3 & -3 & 2 \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 2 & 2 & -2 \\ 0 & 2 & 0 & -6 & 4 & -2 \\ 0 & 0 & 1 & 3 & -3 & 2 \end{array}\right] \begin{array}{l} \frac{1}{2} R_1 \\ \frac{1}{2} R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -3 & 2 & -1 \\ 0 & 0 & 1 & 3 & -3 & 2 \end{array}\right]$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{pmatrix}$$

6 $\begin{pmatrix} -4 & -6 \\ 2 & 3 \end{pmatrix} = \frac{1}{-12+12}\ \cancel{\gamma}\ \left(\frac{1}{0}\right).$

7 $A = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$ $\sin^2\theta + \cos^2\theta = 1.$

$\sin^2\theta - (-\cos^2\theta) = 1 \neq 0$

$A^{-1} = \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \quad R_2 - R_1$$

$$\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} \quad R_1 + R_2 \qquad E_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \quad -\tfrac{1}{2} R_2 \qquad E_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \quad R_1 \leftrightarrow R_2 \qquad E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$R_2 - R_1$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \tfrac{1}{2} R_2 \qquad E_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(solns are not unique)

8 A is invertible $\Rightarrow AA^{-1} = A^{-1}A = I$

$\Rightarrow (A^T)^{-1} \overset{?}{=} (A^{-1})^T$

$A^T(A^{-1})^T = (A^{-1}A)^T$

$= I^T$

$= I$

$(A^{-1})^T A^T = (AA^{-1})^T$

$= I^T$

$= I$ $\rightarrow$ A^T is the inverse of $(A^{-1})^T$

$\Rightarrow (A^T)^{-1} = (A^{-1})^T$

9/ C is invertible $\Rightarrow CC^{-1} = C^{-1}C = I$

if $CA = CB \Rightarrow A = B$

$CA = CB$

$C^{-1}(CA) = C^{-1}(CB)$

$(C^{-1}C)A = (C^{-1}C)B$

$IA = IB$

$A = B$ ✓

10/ if $A^2 = A \Rightarrow I - 2A = (I - 2A)^{-1}$

$(I-2A)(I-2A) = I - 2IA - 2IA + 4A^2$ $(A^2 = A)$

$= I - 4IA + 4A$ $(A = IA)$

$= I - 4IA + 4IA$

$= I$

$\rightarrow I - 2A = (I - 2A)^{-1}$

11. A is symmetric $\overset{?}{\Rightarrow}$ A^{-1} is symmetric

A is nonsingular $\Rightarrow$ A^{-1} is nonsingular too : $AA^{-1} = A^{-1}A = I$

A is symmetric $\Rightarrow$ $A^T = A$.

is $(A^{-1})^T = A^{-1}$?

$$A^{-1} = (A)^{-1}$$
$$= (A^T)^{-1} \qquad (A^T)^{-1} = (A^{-1})^T$$
$$= (A^{-1})^T \checkmark$$

$\Rightarrow$ A^{-1} is symmetric.

12. A, B, C square matrices

prove $ABC = I \Rightarrow B^{-1} = CA$?

$ABC = I \Rightarrow A$ is invertible $\Rightarrow AA^{-1} = I$

$I = AA^{-1} = ABC$

$\Rightarrow A^{-1} = BC$

$A^{-1}A = BCA$

$I = BCA \Rightarrow B$ is invertible $BB^{-1} = I$

$I = BB^{-1} = B(CA)$

$\Rightarrow B^{-1} = CA \checkmark$.

13/ $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

a) show $A^2 - 2A + 5I = 0$.

$A^2 = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

$= \begin{pmatrix} -3 & 4 \\ -4 & -3 \end{pmatrix}$

$A^2 - 2A + 5I = \begin{pmatrix} -3 & 4 \\ -4 & -3 \end{pmatrix} - 2\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$= 0$ ✓.

b) $A^{-1} = \frac{1}{5}(2I - A)$

$A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

$2I - A = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

$A^{-1} = \frac{1}{5}(2I - A)$ ✓

c) $A\left(\frac{1}{5}(2I - A)\right) = \frac{1}{5}(2IA - A^2)$

$A^2 - 2A + 5I = 0$

$5I = 2IA - A^2$

$A\left(\frac{1}{5}(2I - A)\right) = \frac{1}{5}(5I)$

$= I \Rightarrow A^{-1} = \frac{1}{5}(2I - A)$

14/ $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

15/

$$\begin{bmatrix} 3 & 0 & 1 \\ 6 & 1 & 1 \\ -3 & 1 & 0 \end{bmatrix} \begin{array}{l} R_2 - 2R_1 : l_{21} = 2 \\ R_3 + R_1 : l_{21} = -1 \end{array}$$

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad R_3 - R_2 \quad l_{32} = 1$$

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} = U \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 3 & 0 & 1 \\ 6 & 1 & 1 \\ -3 & 1 & 0 \end{pmatrix}} = \underset{L}{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}}$$