Lecture One

Section 1.1 – *Polynomials and Factoring*

Polynomials

Adding and Subtracting Polynomials

Properties of Real numbers

For all real numbers a, b, and c:

$a+b=b+a$ ***Commutative properties***
$ab=ba$

$(a+b)+c=a+(b+c)$ ***Associative properties***
$(ab)c=a(bc)$

$a(b+c)=ab+ac$ ***Distributive properties***

Add or subtract as indicated

a) $\left(8x^3-4x^2+6x\right)+\left(3x^3+5x^2-9x+8\right)$

$$\begin{aligned}\left(8x^3-4x^2+6x\right)+\left(3x^3+5x^2-9x+8\right)&=8x^3-4x^2+6x+3x^3+5x^2-9x+8\\&=\left(8x^3+3x^3\right)+\left(-4x^2+5x^2\right)+(6x-9x)+8\\&=11x^3+x^2-3x+8\end{aligned}$$

b) $\left(-4x^4+6x^3-9x^2-12\right)+\left(-3x^3+8x^2-11x+7\right)$

$$\begin{aligned}\left(-4x^4+6x^3-9x^2-12\right)+\left(-3x^3+8x^2-11x+7\right)&=-4x^4+6x^3-3x^3-9x^2+8x^2-11x-12+7\\&=-4x^4+3x^3-x^2-11x-5\end{aligned}$$

c) $\left(2x^2-11x+8\right)-\left(7x^2-6x+2\right)$

$$\begin{aligned}\left(2x^2-11x+8\right)-\left(7x^2-6x+2\right)&=2x^2-11x+8-7x^2+6x-2\\&=-5x^2-5x+6\end{aligned}$$

Multiply

a) $8x(6x-4)$

$$8x(6x-4) = 8x(6x) - 8x(4)$$
$$= 48x^2 - 32x$$

b) $(3p-2)\left(p^2+5p-1\right)$

$$(3p-2)\left(p^2+5p-1\right) = 3p^3+15p^2-3p-2p^2-10p+2$$
$$= 3p^3+13p^2-13p+2$$

c) $(x+2)(x+3)(x-4)$

$$(x+2)(x+3)(x-4) = \left(x^2+3x+2x+6\right)(x-4)$$
$$= \left(x^2+5x+6\right)(x-4)$$
$$= x^3+5x^2+6x-4x^2-20x-24$$
$$= x^3+x^2-14x-24$$

Find $(2m-5)(m+4)$

$$(2m-5)(m+4) = 2mm + 2m(4) - 5m - 5(4)$$
$$= 2m^2+8m-5m-20$$
$$= 2m^2+3m-20$$

Find $(2k-5)^2$

$$(2k-5)^2 = (2k-5)(2k-5)$$
$$= 4k^2-10k-10k+25$$
$$= 4k^2-20k+25$$

$$(a-b)^2 = a^2-2ab+b^2$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a-b)(a+b) = a^2-b^2$$

Perform the indicated operations: $2\left(3x^2+4x+2\right)-3\left(-x^2+4x-5\right)$

$$\begin{aligned}2\left(3x^2+4x+2\right)-3\left(-x^2+4x-5\right)&=6x^2+8x+4+3x^2-12x+15\\&=9x^2-4x+19\end{aligned}$$

Perform the indicated operations: $(3t-2y)(3t+5y)$

$$\begin{aligned}(3t-2y)(3t+5y)&=9t^2+15ty-6yt-10y^2\\&=9t^2+9yt-10y^2\end{aligned}$$

Perform the indicated operations: $(2a-4b)^2$

$$\begin{aligned}(2a-4b)^2&=(2a)^2-2(2a)(4b)+(4b)^2\\&=4a^2-16ab+16b^2\end{aligned}$$

$(a-b)^2=a^2-2ab+b^2$

Factoring

Prime Factorization

A process that allows us to write a composite number as a product of two or more prime numbers.

Tree

10

2 5

$10 = 2 \times 5$

$72 = 2 . 36$
$= 2 . 6 . 6$
$= 2 . 2 . 3 . 2 . 3$
$= 2^3 \, 3^2$

The Greatest Common Factor (GCF)

The largest factor that two or more numbers (or terms) have in common

Find GCF (18, 36)

18: 2. 9
2. 3. 3

36: 2. 18
2. 2. 3. 3

18: $2 \, 3^2 \rightarrow$ 1, 2, 3, 6, 9, **<u>18</u>**
36: $2^2 \, 3^2 \rightarrow$ 1, 2, 3, 4, 6, 9, 12, **<u>18</u>**, 36

GCF (18, 36) = 18 (is the greatest common factor)

Find GCF (27, 45)

$27 = 3^3$
$45 = \underline{3^2 \, 5}$
3^2

GCF (27, 45) = 9

Find GCF (40, 56)

$40 = 2^3 \, 5$
$56 = \underline{2^3 \, 7}$
2^3

GCF (40, 56) = 8

Find GCF (80, 60)

$80 = 2^4 \quad 5$
$60 = \underline{2^2 \, 3 \, 5}$
$2^2 \quad 5$

GCF (80, 60) = 20

Factor out the greatest common factor

12	2 . 2 . 3
18	2 . . 3 . 3
	2 . 3

a) $12p-18q$

$$12p-18q=6(2p-3q)$$

b) $8x^3-9x^2+15x$

$$8x^3-9x^2+15x=x\left(8x^2-9x+15\right)$$

Factoring Trinomial

Factor $y^2+8y+15$

Product 15	***Sum*** 8
15 *x* 1	15 + 1
3 *x* 5	3 + 5

$$y^2+8y+15=(y+3)(y+5)$$

Factor $4x^2+8xy-5y^2$

$$4x^2+8xy-5y^2=(2x-y)(2x+5y)$$

Special Factorization

$$a^2-b^2=(a-b)(a+b)$$
$$a^2+2ab+b^2=(a+b)^2$$
$$a^2-2ab+b^2=(a-b)^2$$
$$a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$$
$$a^3+b^3=(a+b)\left(a^2-ab+b^2\right)$$

Factor

a) $64p^2 - 49q^2$

$$\begin{aligned}64p^2 - 49q^2 &= (8p)^2 - (7q)^2 \\ &= (8p - 7q)(8p + 7q)\end{aligned}$$

b) $x^2 + 36$

$x^2 + 36$ *can't be factored (in real number) it is prime.*

c) $x^2 + 12x + 36$

$$x^2 + 12x + 36 = (x+6)^2$$

d) $9y^2 - 24yz + 16z^2$

$$\begin{aligned}9y^2 - 24yz + 16z^2 &= (3y)^2 - 2(3y)(4z) + (4z)^2 \\ &= (3y - 4z)^2\end{aligned}$$

e) $y^3 - 8$

$$\begin{aligned}y^3 - 8 &= y^3 - 2^3 \\ &= (y-2)\left(y^2 + 2y + 4\right)\end{aligned}$$

f) $m^3 + 125$

$$m^3 + 125 = (m+5)\left(m^2 - 5m + 25\right)$$

g) $8k^3 - 27z^3$

$$\begin{aligned}8k^3 - 27z^3 &= (2k)^3 - (3z)^3 \\ &= (2k - 3z)\left((2k)^2 + 6kz + (3z)^2\right) \\ &= (2k - 3z)\left(4k^2 + 6kz + 9z^2\right)\end{aligned}$$

h) $p^4 - 1$

$$\begin{aligned}p^4 - 1 &= \left(p^2\right)^2 - (1)^2 \\ &= \left(p^2 - 1\right)\left(p^2 + 1\right) \\ &= (p-1)(p+1)\left(p^2 + 1\right)\end{aligned}$$

Factor: $60m^4 - 120m^3n + 50m^2n^2$

$$60m^4 - 120m^3n + 50m^2n^2 = 10m^2\left(6m^2 - 12mn + 5n^2\right)$$

Factor: $y^2 - 4yz - 21z^2$

$$y^2 - 4yz - 21z^2 = (y + 3z)(y - 7z)$$

Factor: $4a^2 + 10a + 6$

$$\begin{aligned} 4a^2 + 10a + 6 &= 2\left(2a^2 + 5a + 3\right) \\ &= 2(2a + 3)(a + 1) \end{aligned}$$

Factor: $16a^4 - 81b^4$

$$\begin{aligned} 16a^4 - 81b^4 &= \left(4a^2\right)^2 - \left(9b^2\right)^2 \\ &= \left(4a^2 - 9b^2\right)\left(4a^2 + 9b^2\right) \\ &= \left((2a)^2 - (3b)^2\right)\left(4a^2 + 9b^2\right) \\ &= (2a - 3b)(2a + 3b)\left(4a^2 + 9b^2\right) \end{aligned}$$

Section 1.2 – *Exponents*

Integer Exponents

Definition of exponent

$$a^n = \underbrace{a \cdot a \cdot a \cdots \cdots a}_{n-times}$$

a appears as a factor n times

$$a^0 = 1 \qquad a^{-n} = \frac{1}{a^n}$$

$$a^m \cdot a^n = a^{m+n} \qquad \left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n}$$

$$\left(a^m\right)^n = a^{mn} \qquad \frac{a^m}{a^n} = a^{m-n}$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \qquad (ab)^m = a^m b^m$$

a) 6^0

$$6^0 = 1$$

b) $(-9)^0$

$$(-9)^0 = 1$$

c) 3^{-2}

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

d) $\left(\frac{3}{4}\right)^{-1}$

$$\left(\frac{3}{4}\right)^{-1} = \frac{4}{3}$$

a) $7^4.7^6$

$$7^4.7^6 = 7^{4+6} = 7^{10}$$

b) $\dfrac{9^{14}}{9^6}$

$$\frac{9^{14}}{9^6} = 9^{14-6} = 9^8$$

c) $\dfrac{r^9}{r^{17}}$

$$\frac{r^9}{r^{17}} = \frac{1}{r^{17-9}} = \frac{1}{r^8}$$

d) $\left(2m^3\right)^4$

$$\left(2m^3\right)^4 = (2)^4\left(m^3\right)^4$$
$$= 16m^{12}$$

e) $\left(\dfrac{x^2}{y^3}\right)^6$

$$\left(\frac{x^2}{y^3}\right)^6 = \frac{\left(x^2\right)^6}{\left(y^3\right)^6}$$
$$= \frac{x^{2.6}}{y^{3.6}}$$
$$= \frac{x^{12}}{y^{18}}$$

f) $\dfrac{a^{-3}b^5}{a^4b^{-7}}$

$$\frac{a^{-3}b^5}{a^4b^{-7}} = \frac{b^5b^7}{a^3a^4}$$
$$= \frac{b^{5+7}}{a^{4+3}}$$
$$= \frac{b^{12}}{a^7}$$

g) $p^{-1}+q^{-1}$

$$p^{-1}+q^{-1}=\frac{1}{p}+\frac{1}{q}$$
$$=\frac{1}{p}\frac{q}{q}+\frac{1}{q}\frac{p}{p}$$
$$=\frac{q+p}{pq}$$

h) $\dfrac{x^{-2}-y^{-2}}{x^{-1}-y^{-1}}$

$$\frac{x^{-2}-y^{-2}}{x^{-1}-y^{-1}}=\frac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}-\frac{1}{y}}$$
$$=\frac{\frac{y^2-x^2}{x^2y^2}}{\frac{y-x}{xy}}$$
$$=\frac{y^2-x^2}{x^2y^2}\cdot\frac{xy}{y-x}$$
$$=\frac{(y-x)(y+x)}{(xy)^2}\cdot\frac{xy}{y-x}$$
$$=\frac{y+x}{xy}$$

Calculations with exponents

a) $121^{1/2}=11$

b) $625^{1/4}=5$

c) $(-32)^{1/5}=-2$

d) $(-49)^{1/2}$ *is not a real number*

Rational Exponents

$$a^{m/n} = \left(a^{1/n}\right)^m$$

Calculations with Exponents

a) $27^{2/3}$ 27^(2/3)

$$\begin{aligned}27^{2/3} &= \left(27^{1/3}\right)^2 \\ &= \left(\left(3^3\right)^{1/3}\right)^2 \\ &= \left(3^{3 \cdot \frac{1}{3}}\right)^2 \\ &= (3)^2 \\ &= 9\end{aligned}$$

b) $32^{2/5}$ 32 ^ (2/5)

$$\begin{aligned}32^{2/5} &= \left(\left(2^5\right)^{1/5}\right)^2 \\ &= 2^2 \\ &= 4\end{aligned}$$

c) $64^{4/3}$ 64 ^ (4/3)

$$\begin{aligned}64^{4/3} &= \left(\left(4^3\right)^{1/3}\right)^4 \\ &= (4)^4 \\ &= 256\end{aligned}$$

Simplify

a) $\dfrac{y^{1/3}y^{5/3}}{y^3}$

$$\begin{aligned}\frac{y^{1/3}y^{5/3}}{y^3} &= \frac{y^{\frac{1}{3}+\frac{5}{3}}}{y^3}\\ &= \frac{y^{\frac{6}{3}}}{y^3}\\ &= \frac{y^2}{y^3}\\ &= \frac{1}{y^{3-2}}\\ &= \frac{1}{y}\end{aligned}$$

b) $m^{2/3}\left(m^{7/3}+7m^{1/3}\right)$

$$\begin{aligned}m^{2/3}\left(m^{7/3}+7m^{1/3}\right) &= m^{2/3}m^{7/3}+7m^{2/3}m^{1/3}\\ &= m^{\frac{2}{3}+\frac{7}{3}}+7m^{\frac{2}{3}+\frac{1}{3}}\\ &= m^{\frac{9}{3}}+7m^{\frac{3}{3}}\\ &= m^3+7m\end{aligned}$$

c) $\left(\dfrac{m^7n^{-2}}{m^{-5}n^2}\right)^{1/4}$

$$\begin{aligned}\left(\frac{m^7n^{-2}}{m^{-5}n^2}\right)^{1/4} &= \left(\frac{m^{7+5}}{n^{2+2}}\right)^{1/4}\\ &= \left(\frac{m^{12}}{n^4}\right)^{1/4}\\ &= \frac{\left(m^{12}\right)^{1/4}}{\left(n^4\right)^{1/4}}\\ &= \frac{m^{12/4}}{n^{4/4}}\\ &= \frac{m^3}{n}\end{aligned}$$

Simplify

a) $4m^{1/2}+3m^{3/2}$

$$4m^{1/2}+3m^{3/2}=m^{1/2}\left(4m^{1/2-1/2}+3m^{3/2-1/2}\right)$$
$$=m^{1/2}(4+3m)$$

b) $9x^{-2}-6x^{-3}$

$$9x^{-2}-6x^{-3}=3x^{-3}(3x-2)$$

c) $2\left(x^2+5\right)(3x-1)^{-1/2}+(3x-1)^{1/2}(2x)$

$$2\left(x^2+5\right)(3x-1)^{-1/2}+(3x-1)^{1/2}(2x)=2(3x-1)^{-1/2}\left[x^2+5+x(3x-1)\right]$$
$$=2(3x-1)^{-1/2}\left[x^2+5+3x^2-x\right]$$
$$=2(3x-1)^{-1/2}\left(4x^2-x+5\right)$$

Radicals

$$a^{1/n}=\sqrt[n]{a}$$

a) $\sqrt[4]{16}$

$$\sqrt[4]{16}=16^{1/4}=2$$

b) $\sqrt[5]{-32}=-2$

c) $\sqrt[3]{1000}$

$$\sqrt[3]{1000}=1000^{1/3}=10$$

d) $\sqrt[6]{\frac{64}{729}}$

$$\sqrt[6]{\frac{64}{729}}=\frac{\sqrt[6]{64}}{\sqrt[6]{729}}=\frac{2}{3}$$

Properties

$$\left(\sqrt[n]{a}\right)^n = a$$

$$\sqrt[n]{a^n} = \begin{cases} |a| & \text{if } n \text{ is even} \\ a & \text{if } n \text{ is odd} \end{cases}$$

$$\sqrt[n]{a}.\sqrt[n]{b} = \sqrt[n]{ab}$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

Simplify

a) $\sqrt{1000}$

$$\sqrt{1000} = \sqrt{100(10)}$$

$$= \sqrt{100}\sqrt{10}$$

$$= 10\sqrt{10}$$

b) $\sqrt{128}$

$$\sqrt{128} = \sqrt{64(2)}$$

$$= 8\sqrt{2}$$

c) $\sqrt{2}\sqrt{18}$

$$\sqrt{2}\sqrt{18} = \sqrt{2(18)}$$

$$= \sqrt{36}$$

$$= 6$$

d) $\sqrt[3]{54}$

$$\sqrt[3]{54} = \sqrt[3]{27(2)}$$

$$= 3\sqrt[3]{2}$$

e) $\sqrt{288m^5}$

$$\sqrt{288m^5} = \sqrt{144(2)m^4 m}$$

$$= 12m^2\sqrt{2m}$$

f) $2\sqrt{18}-5\sqrt{32}$

$$\begin{aligned}2\sqrt{18}-5\sqrt{32} &= 2\sqrt{9(2)}-5\sqrt{16(2)}\\ &= 6\sqrt{2}-20\sqrt{2}\\ &= -14\sqrt{2}\end{aligned}$$

Section 1.3 – Fractions and Rationalization

Fraction (*Basic*)

$$\frac{a}{b} = \frac{numerator}{denominator}$$

$\frac{a}{b} = \frac{c}{d} \iff ad = bc$ *Cross multiplication*

$$\frac{a}{b} = \frac{na}{nb} = \frac{an}{bn}$$

a) $\frac{5}{6} = \frac{25}{30}?$

$$\frac{5}{6} = \frac{5}{6}\frac{5}{5} = \frac{25}{30}$$

b) $\frac{16}{48} = \frac{1}{3}$

$$\frac{16}{48} = \frac{1}{3} \iff (16)(3) = (1)(48)$$

$$48 = 48$$

Simplify: $\frac{12}{18} = \frac{2.6}{2.9}$

$$= \frac{2.2.3}{2.3.3}$$

$$= \frac{2}{3}$$

Simplify: $\frac{36}{56} = \frac{2.18}{2.28}$

$$= \frac{18}{28}$$

$$= \frac{2.9}{2.14}$$

$$= \frac{9}{14}$$

If the denominators are the same $\Rightarrow$ add the numerators

$$\frac{3}{5}+\frac{4}{5}=\frac{3+4}{5}=\frac{7}{5}$$

If the denominators are the same $\Rightarrow$ subtract the numerators

$$\frac{4}{9}-\frac{2}{9}=\frac{4-2}{9}=\frac{2}{9}$$

If the denominators are not the same

$\Rightarrow$ Find Least Common Denominator (LCD) and convert so that the fractions have the same denominators

LCD: is the smallest whole number that is a multiple of each

$\frac{5}{8}+\frac{1}{12}$ LCD (8, 12)

$8 = 2^3$

$12 = \underline{2^2\ 3}$

$2^3\ 3 = 24$ LCD (8, 12) = 24

$$\frac{5}{8}+\frac{1}{12}=\frac{5}{8}\frac{3}{3}+\frac{1}{12}\frac{2}{2}$$
$$=\frac{15}{24}+\frac{2}{24}$$
$$=\frac{15+2}{24}$$
$$=\frac{17}{24}$$

$\frac{69}{75}-\frac{1}{50}$

LCD (75, 50)

$75 = 5^3$

$50 = \underline{2\ 5^2}$

$2\ 5^3 = 150$ LCD (75, 50) = 150

$$\frac{69}{75}-\frac{1}{50}=\frac{(69)(2)-(1)(3)}{150}$$
$$=\frac{138-3}{150}$$
$$=\frac{135}{150}$$
$$=\frac{9}{10}$$

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

$$\begin{aligned}\frac{2}{7}+\frac{3}{5}&=\frac{2(5)+3(7)}{7(5)}\\&=\frac{10+21}{35}\\&=\frac{31}{35}\end{aligned}$$

or $$\begin{aligned}\frac{2}{7}\frac{5}{5}+\frac{3}{5}\frac{7}{7}&=\frac{10}{35}+\frac{21}{35}\\&=\frac{10+21}{35}\\&=\frac{31}{35}\end{aligned}$$

$$\begin{aligned}\frac{5}{9}+\frac{3}{4}&=\frac{5(4)+3(9)}{9(4)}\\&=\frac{20+27}{36}\\&=\frac{47}{36}\end{aligned}$$

$$\begin{aligned}\frac{17}{15}+\frac{5}{12}&=\frac{17(12)+5(15)}{15(12)}\\&=\frac{204+75}{180}\\&=\frac{279}{180}\\&=\frac{31(9)}{20(9)}\\&=\frac{31}{20}\end{aligned}$$

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}=\frac{5(7)(9)+(3)(7)(9)+(3)(5)(9)+(3)(5)(7)}{(3)(5)(7)(9)}$$

$$=\frac{315+189+135+105}{945}$$

$$=\frac{744}{945}$$

$$=\frac{248}{315}\frac{3}{3}$$

$$=\frac{248}{315}$$

$$\frac{8}{9}+\frac{1}{12}+\frac{3}{16}$$

$$\frac{8}{9}+\frac{1}{12}+\frac{3}{16}=\frac{8(16)+1(12)+3(9)}{144}$$

$$\begin{cases} 9 = \quad\quad 3^2 \\ 12 = 2^2 \quad 3 \\ 16 = 2^4 \end{cases}$$

$$\text{LCD} \quad 2^4 \quad 3^2 = 144$$

$$=\frac{128+12+27}{144}$$

$$=\frac{167}{144}$$

$$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$$

$$\frac{2}{7}-\frac{3}{5}=\frac{2(5)-3(7)}{7(5)}=\frac{10-21}{35}=-\frac{11}{35}$$

$$\frac{a}{c}\frac{b}{d}=\frac{ab}{cd}$$

$$\frac{2}{7}\frac{3}{5}=\frac{6}{35}$$

$$\frac{a}{c}\div\frac{b}{d}=\frac{a}{c}\times\frac{d}{b}=\frac{ad}{cb}$$

$$\frac{2}{7}\div\frac{3}{5}=\frac{2}{7}\frac{5}{3}=\frac{10}{21}$$

Find:

1. $\frac{13}{21}+\frac{5}{21}=\frac{13+5}{21}=\frac{6}{7}$

2. $\frac{7}{12}-\frac{4}{15}=\frac{7(5)-4(4)}{60}=\frac{35-16}{60}=\frac{19}{60}$

3. $\frac{5}{8}+\frac{1}{2}=\frac{5+4}{8}=\frac{9}{8}$

4. $\frac{5}{8}+\frac{1}{2}+\frac{2}{3}=\frac{5(3)+1(12)+2(8)}{24}=\frac{43}{24}$

5. $\frac{7}{8}-\frac{1}{10}=\frac{7(5)-1(4)}{40}=\frac{31}{40}$

6. $\frac{11}{5}-\frac{31}{7}=-\frac{78}{35}$

7. $\frac{3}{4}\bullet\frac{3}{2}=\frac{9}{8}$

8. $\frac{3}{4}\bullet\frac{4}{3}\bullet\frac{2}{3}=\frac{2}{3}$

9. $\frac{3}{4}\div\frac{3}{2}=\frac{3}{4}\bullet\frac{2}{3}=\frac{2}{4}=\frac{1}{2}$

10. $\frac{14}{15}\div\frac{14}{3}=\frac{14}{15}\bullet\frac{3}{14}=\frac{1}{5}$

Operations with Fractions

A rational expression is proper if the degree of numerator is less than the degree of denominator
A rational expression is improper if the degrees of numerator is greater than or equal the degree of denominator

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

$$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$$

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right)=\frac{ac}{bd}$$

$$\frac{a/b}{c/d}=\frac{a}{b}\frac{d}{c}=\frac{ad}{bc}$$

$$\frac{a/b}{c}=\frac{a}{b}\frac{1}{c}=\frac{a}{bc}$$

$$\frac{ab}{ac}=\frac{b}{c}$$

$$\frac{ad+ac}{ad}=\frac{a(d+c)}{ad}=\frac{b+c}{d}$$

$$\frac{ab+cd}{ad} \quad stay$$

Example

Perform each indicated operation & simplify

a) $x+\frac{2}{x}=\frac{x^2+2}{x}$

b)
$$\frac{2}{x+1}-\frac{1}{2x+1}=\frac{2(2x+1)-1(x+1)}{(x+1)(2x+1)}$$
$$=\frac{4x+2-x-1}{(x+1)(2x+1)}$$
$$=\frac{3x+1}{(x+1)(2x+1)}$$

Example

Perform each indicated operation & simplify

a) $\dfrac{x}{x^2-4}-\dfrac{1}{x-2}=\dfrac{x-1(x+2)}{(x-2)(x+2)}$ $\qquad x^2-4=(x-2)(x+2)$

$$=\frac{x-x-2}{(x-2)(x+2)}$$

$$=\frac{-2}{(x-2)(x+2)}$$

b) $\dfrac{1}{3(x^2+2x)}-\dfrac{1}{3x}=\dfrac{1-1(x+2)}{3x(x+2)}$ $\qquad 3(x^2+2x)=3x(x+2)$

$$=\frac{1-x-2}{3x(x+2)}$$

$$=\frac{-x-1}{3x(x+2)}$$

Example

Perform each indicated operation & simplify

a) $\dfrac{\sqrt{x+2}-\frac{x}{4\sqrt{x+2}}}{x+2}=\left(\sqrt{x+2}-\dfrac{x}{4\sqrt{x+2}}\right)\div(x+2)$

$$=\left(\frac{4\sqrt{x+2}\sqrt{x+2}-x}{4\sqrt{x+2}}\right)\left(\frac{1}{x+2}\right)$$

$$=\frac{4(x+2)-x}{4(x+2)\sqrt{x+2}}$$

$$=\frac{4x+8-x}{4(x+2)\sqrt{x+2}}$$

$$=\frac{3x+8}{4(x+2)\sqrt{x+2}}$$

b) $\left(\dfrac{1}{x+\sqrt{x^2+4}}\right)\left(1+\dfrac{x}{\sqrt{x^2+4}}\right)=\dfrac{1}{x+\sqrt{x^2+4}}\dfrac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}}$

$$=\frac{1}{\sqrt{x^2+4}}$$

Example

Perform each indicated operation & simplify

$$\frac{-x\left(\frac{3x}{3\sqrt{x^2+4}}\right)+\sqrt{x^2+4}}{x^2}+\left(\frac{1}{x+\sqrt{x^2+4}}\right)\left(1+\frac{3x}{3\sqrt{x^2+4}}\right)$$

$$=\left(-\frac{3x^2}{3\sqrt{x^2+4}}+\sqrt{x^2+4}\right)\frac{1}{x^2}+\left(\frac{1}{x+\sqrt{x^2+4}}\right)\left(\frac{3\sqrt{x^2+4}+3x}{3\sqrt{x^2+4}}\right)$$

$$=\left(\frac{-3x^2+3\left(\sqrt{x^2+4}\right)^2}{3\sqrt{x^2+4}}\right)\frac{1}{x^2}+\left(\frac{1}{x+\sqrt{x^2+4}}\right)\left(\frac{3(\sqrt{x^2+4}+x)}{3\sqrt{x^2+4}}\right)$$

$$=\left(\frac{-3x^2+3(x^2+4)}{3\sqrt{x^2+4}}\right)\frac{1}{x^2}+\frac{3}{3\sqrt{x^2+4}}$$

$$=\frac{-3x^2+3x^2+12}{3\sqrt{x^2+4}}\frac{1}{x^2}+\frac{3}{3\sqrt{x^2+4}}$$

$$=\frac{12}{3\sqrt{x^2+4}}\frac{1}{x^2}+\frac{3}{3\sqrt{x^2+4}}$$

$$=\frac{12+3x^2}{3x^2\sqrt{x^2+4}}$$

$$=\frac{3(x^2+4)}{3x^2\left(x^2+4\right)^{1/2}}$$

$$=\frac{\sqrt{x^2+4}}{x^2}$$

Rationalization Techniques

1. If the denominator is $\sqrt{a}$, multiply by $\dfrac{\sqrt{a}}{\sqrt{a}}$

2. If the denominator is $\sqrt{a}-\sqrt{b}$, multiply by $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

3. If the denominator is $\sqrt{a}+\sqrt{b}$, multiply by $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}$

$$\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right)=a-b$$

Example

Simplify by rationalizing the denominator

a) $\dfrac{4}{\sqrt{3}}=\dfrac{4}{\sqrt{3}}\dfrac{\sqrt{3}}{\sqrt{3}}$

$=\dfrac{4\sqrt{3}}{3}$

b) $\dfrac{2}{\sqrt[3]{x}}=\dfrac{2}{\sqrt[3]{x}}\dfrac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}}$

$=\dfrac{2\sqrt[3]{x^2}}{x}$

c) $\dfrac{1}{1-\sqrt{2}}=\dfrac{1}{1-\sqrt{2}}\dfrac{1+\sqrt{2}}{1+\sqrt{2}}$

$=\dfrac{1+\sqrt{2}}{1-2}$

$=\dfrac{1+\sqrt{2}}{-1}$

$=-1-\sqrt{2}$

Example

Simplify $\sqrt{27}\sqrt{3}$

$$\begin{aligned}\sqrt{27}\sqrt{3} &= \sqrt{27(3)}\\ &= \sqrt{81}\\ &= 9\end{aligned}$$

Example

Simplify $\sqrt[4]{x^8 y^7 z^{11}}$

$$\sqrt[4]{x^8 y^7 z^{11}} = x^2 y z^2 \ \sqrt[4]{y^3 z^3}$$

Example

Simplify $\dfrac{5}{\sqrt{10}}$

$$\begin{aligned}\frac{5}{\sqrt{10}} &= \frac{5}{\sqrt{10}}\,\frac{\sqrt{10}}{\sqrt{10}}\\ &= \frac{5\sqrt{10}}{10}\\ &= \frac{\sqrt{10}}{2}\end{aligned}$$

Example

Simplify $\dfrac{5}{2-\sqrt{6}}$

$$\begin{aligned}\frac{5}{2-\sqrt{6}} &= \frac{5}{2-\sqrt{6}}\,\frac{2+\sqrt{6}}{2+\sqrt{6}}\\ &= \frac{5\left(2+\sqrt{6}\right)}{4-6}\\ &= -\frac{5}{2}\left(2+\sqrt{6}\right)\end{aligned}$$

Example

Simplify $\dfrac{1}{\sqrt{r}-\sqrt{3}}$

$$\begin{aligned}\frac{1}{\sqrt{r}-\sqrt{3}} &= \frac{1}{\sqrt{r}-\sqrt{3}}\,\frac{\sqrt{r}+\sqrt{3}}{\sqrt{r}+\sqrt{3}}\\ &= \frac{\sqrt{r}+\sqrt{3}}{r-3}\end{aligned}$$

Example

Rationalize the denominator or numerator

a) $\frac{5}{\sqrt{8}}$
$$= \frac{5}{\sqrt{8}}\frac{\sqrt{8}}{\sqrt{8}}$$
$$= \frac{5\sqrt{8}}{8}$$

b) $\frac{1}{\sqrt{6}-\sqrt{3}}$
$$= \frac{1}{\sqrt{6}-\sqrt{3}}\frac{\sqrt{6}+\sqrt{3}}{\sqrt{6}+\sqrt{3}}$$
$$= \frac{\sqrt{6}+\sqrt{3}}{\left(\sqrt{6}\right)^2-\left(\sqrt{3}\right)^2}$$
$$= \frac{\sqrt{6}+\sqrt{3}}{6-3} = \frac{\sqrt{6}+\sqrt{3}}{3}$$
$$= \frac{\sqrt{6}+\sqrt{3}}{3}$$

c) $\frac{1}{\sqrt{x}+\sqrt{x+2}}$
$$= \frac{1}{\sqrt{x}+\sqrt{x+2}}\frac{\sqrt{x}-\sqrt{x+2}}{\sqrt{x}-\sqrt{x+2}}$$
$$= \frac{\sqrt{x}-\sqrt{x+2}}{x-(x+2)}$$
$$= \frac{\sqrt{x}-\sqrt{x+2}}{x-x-2}$$
$$= \frac{\sqrt{x}-\sqrt{x+2}}{-2}$$
$$= \frac{\sqrt{x+2}-\sqrt{x}}{2}$$

Example

$$\frac{2}{x^2-4} - \frac{1}{x-2} = \frac{2-(x+2)}{(x-2)(x+2)}$$
$$= \frac{2-x-2}{(x-2)(x+2)}$$
$$= -\frac{x}{(x-2)(x+2)}$$

Example

$$-\frac{\sqrt{x^2+1}}{x^2}-\frac{1}{\sqrt{x^2+1}}=\frac{-\sqrt{x^2+1}\sqrt{x^2+1}-x^2}{x^2\sqrt{x^2+1}}$$

$$-\frac{\sqrt{x^2+1}}{x^2}\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}-\frac{1}{\sqrt{x^2+1}}\frac{x^2}{x^2}$$

$$=\frac{-\left(x^2+1\right)-x^2}{x^2\sqrt{x^2+1}}$$

$$=\frac{-x^2-1-x^2}{x^2\sqrt{x^2+1}}$$

$$=\frac{-2x^2-1}{x^2\sqrt{x^2+1}}$$

$$=-\frac{2x^2+1}{x^2\sqrt{x^2+1}}$$

Example

$$\left(\sqrt{x^2+1}-\frac{3x^3}{2\sqrt{x^2+1}}\right)\div\left(x^3+1\right)=\left(\frac{\sqrt{x^2+1}\left(2\sqrt{x^2+1}\right)-3x^3}{2\sqrt{x^2+1}}\right)\cdot\frac{1}{x^3+1}$$

$$=\frac{2\left(x^2+1\right)-3x^3}{2\left(x^3+1\right)\sqrt{x^2+1}}$$

$$=\frac{-3x^3+2x^2+2}{2\left(x^3+1\right)\sqrt{x^2+1}}$$

Exercises *Section* 1.3 – Fractions and Rationalization

1. Perform each indicated operation & simplify: $\frac{A}{x+1}-\frac{B}{x-1}+\frac{C}{x+2}$

2. Perform the operation and simplify: $-\frac{\sqrt{x^2+1}}{x^2}-\frac{1}{\sqrt{x^2+1}}$

3. Perform the operation and simplify: $\left(\sqrt{x^2+1}-\frac{3x^3}{2\sqrt{x^2+1}}\right)\div\left(x^3+1\right)$

4. Perform the operation and simplify: $\frac{6}{x(3x-2)}+\frac{5}{3x-2}-\frac{2}{x^2}$

5. Simplify the fraction: $\frac{\frac{2}{x+3}-\frac{2}{a+3}}{x-a}$

6. Simplify: $\frac{3x^2(2x+5)^{1/2}-x^3\left(\frac{1}{2}\right)(2x+5)^{-1/2}(2)}{\left[(2x+5)^{1/2}\right]^2}$

7. Simplify the expression: $\frac{\left(4x^2+9\right)^{1/2}(2)-(2x+3)\left(\frac{1}{2}\right)\left(4x^2+9\right)^{-1/2}(8x)}{\left[\left(4x^2+9\right)^{1/2}\right]^2}$

8. Simplify the expression: $\frac{\left(1-x^2\right)^{1/2}(2x)-x^2\left(\frac{1}{2}\right)\left(1-x^2\right)^{-1/2}(-2x)}{\left[\left(1-x^2\right)^{1/2}\right]^2}$

9. Simplify the expression: $\frac{\left(x^2+4\right)^{1/3}(3)-(3x)\left(\frac{1}{3}\right)\left(x^2+4\right)^{-2/3}(2x)}{\left[\left(x^2+4\right)^{1/3}\right]^2}$

10. Simplify the expression: $\frac{\left(x^2-5\right)^4(3x^2)-x^3(4)\left(x^2-5\right)^3(2x)}{\left[\left(x^2-5\right)^4\right]^2}$

11. Simplify the expression: $\dfrac{(3x+2)^{1/2}\left(\frac{1}{3}\right)(2x+3)^{-2/3}(2)-(2x+3)^{1/3}\left(\frac{1}{2}\right)(3x+2)^{-1/2}(3)}{\left[(3x+2)^{1/2}\right]^2}$

12. Simplify the expression: $\dfrac{\left(x^2+2\right)^3(2x)-x^2(3)\left(x^2+2\right)^2(2x)}{\left[\left(x^2+2\right)^3\right]^2}$

Section 1.4 – Equations and Application

Linear Equations

A ***linear equation*** in one variable is an equation that is equivalent to one of the form $mx+b=0$

Equation-Solving Principles

Addition Principle: If $a = b$ is true $\Rightarrow a+c=b+c$

Multiplication Principle: If $a = b$ is true $\Rightarrow ac=bc$

Solve the following equations

a) $x-2=3$

$$x-2+2=3+2$$
$$x=5$$

b) $\frac{x}{2}=3$

$$2\frac{x}{2}=(2)3$$
$$x=6$$

Solve: $2x-5+8=3x+2(2-3x)$

$$2x-5+8=3x+4-6x$$
$$2x+3=4-3x$$
$$2x+3-3+3x=4-3x-3+3x$$
$$5x=1$$

Divide both sides by 5

$$x=\frac{1}{5}$$

The Zero-Product Principle:

If $ab = 0$. then $a = 0$ or $b = 0$.

Solve $6x^2+7x=3$

$$6x^2+7x-3=0$$
$$(3x-1)(2x+3)=0$$

$$3x-1=0 \qquad 2x+3=0$$
$$3x=1 \qquad 2x=-3$$
$$x=\frac{1}{3} \qquad x=-\frac{3}{2}$$

Quadratic Formula

$$ax^2+bx+c=0 \quad \Rightarrow \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Solve $\quad x^2-4x-5=0 \qquad \Rightarrow \; a=1, \;\; b=-4, \;\; c=-5$

$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-5)}}{2(1)} \\ &= \frac{4\pm\sqrt{16+20}}{2} \\ &= \frac{4\pm\sqrt{36}}{2} \\ &= \frac{4\pm 6}{2} \end{aligned}$$

$$\begin{aligned} x &= \frac{4+6}{2} \\ &= \frac{10}{2} \\ &= 5 \end{aligned} \qquad\qquad \begin{aligned} x &= \frac{4-6}{2} \\ &= \frac{-2}{2} \\ &= -1 \end{aligned}$$

Solve $\quad x^2+1=4x$

$$x^2-4x+1=0$$

$$\begin{aligned} x &= \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)} \\ &= \frac{4\pm\sqrt{16-4}}{2} \\ &= \frac{4\pm\sqrt{12}}{2} \\ &= \frac{4\pm 2\sqrt{3}}{2} \\ &= \frac{2\left(2\pm\sqrt{3}\right)}{2} \\ &= 2\pm\sqrt{3} \end{aligned}$$

Equations with Fractions

Solve $\frac{r}{10}-\frac{2}{15}=\frac{3r}{20}-\frac{1}{5}$

10	2 5
15	3 5
20	2 2 5
5	5
	2 2 3 5 = 60

$$(60)\frac{r}{10}-(60)\frac{2}{15}=(60)\frac{3r}{20}-(60)\frac{1}{5}$$

$$6r-8=9r-12$$

$$6r-8+8-9r=9r-12+8-9r$$

$$-3r=-4$$

$$r=\frac{-4}{-3}=\frac{4}{3}$$

Solve $\frac{2}{x-3}+\frac{1}{x}=\frac{6}{x(x-3)}$ $\qquad x-3\neq 0$

Conditions: $x\neq 0,\ 3$

$$x(x-3)\frac{2}{x-3}+x(x-3)\frac{1}{x}=x(x-3)\frac{6}{x(x-3)}$$

$$2x+x-3=6$$

$$3x=9$$

$$x=3$$

Solve $\frac{1}{x-2}-\frac{3x}{x-1}=\frac{2x+1}{x^2-3x+2}$ $\qquad cond.\ x\neq 1,\ 2$

$$(x-2)(x-1)\frac{1}{x-2}-(x-2)(x-1)\frac{3x}{x-1}=(x-2)(x-1)\frac{2x+1}{x^2-3x+2}$$

$$x-1-3x(x-2)=2x+1$$

$$x-1-3x^2+6x-2x-1=0$$

$$-3x^2+5x-2=0$$

$$3x^2-5x+2=0$$

$$(x-1)(3x-2)=0$$

$x-1=0$ $\qquad$ $3x-2=0$

$x=1$ $\qquad$ $x=\frac{2}{3}$

Solution: $x=\frac{2}{3}$

Slopes and Equations of Lines

Slope of a line *(Definition)*

The slope of a line is defined as the vertical change (the *rise*) over the horizontal change (the *run*) as one travels along the line.

$$slope: \quad m = \frac{change\ in\ y}{change\ in\ x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Find the slope of the line through each pair point

a) $(7,6)$ *and* $(-4,5)$

$$m = \frac{5-6}{-4-7}$$

$$= \frac{-1}{-11}$$

$$= \frac{1}{11}$$

b) $(5,-3)$ *and* $(-2,-3)$

$$m = \frac{-3+3}{-2-5}$$

$$= \frac{0}{-7}$$

$$= 0$$

c) $(2,-4)$ *and* $(2,3)$

$$m = \frac{3+4}{2-2}$$

$= \frac{7}{0}$ Which is undefined $\rightarrow$ line is vertical.

Equations of a Line

$$y = mx + b$$

This *linear equation* is called the *slope-intercept form* of the equation of a line.

Point-Slope Form

$$y - y_1 = m(x - x_1)$$

Example

Find the equation of the line through $(0,\ -3)$ with slope $\frac{3}{4}$

Solution

$$y - y_1 = m(x - x_1)$$

$$y + 3 = \frac{3}{4}(x - 0)$$

$$y + 3 = \frac{3}{4}x$$

$$y = \frac{3}{4}x - 3$$

$$(4)y = (4)\frac{3}{4}x - (4)3$$

$$4y = 3x - 12$$

$$4y - 3x = -12$$

$$3x - 4y = 12$$

Example

Find the equation of the line that passes through the point $(3,\ -7)$ and has slope $\frac{5}{4}$

Solution

$$y - y_1 = m(x - x_1)$$

$$y + 7 = \frac{5}{4}(x - 3)$$

$$y + 7 = \frac{5}{4}x - \frac{15}{4}$$

$$y + 7 - 7 = \frac{5}{4}x - \frac{15}{4} - 7 \qquad -\frac{15}{4} - 7 = -\frac{15}{4} - 7\frac{4}{4} = -\frac{15}{4} - \frac{28}{4} = -\frac{43}{4}$$

$$y = \frac{5}{4}x - \frac{43}{4}$$

Parallel Lines (//)

Two lines are parallel if and only if they have the same slope, or they are both vertical. $m_1 = m_2$

Example

Find the equation of the line that passes through the point $(3,\ 5)$ and is parallel to the line $2x+5y=4$

<u>*Solution*</u>

$$2x+5y=4$$

$$5y=-2x+4$$

$$y=-\frac{2}{5}x+\frac{4}{5}$$

$$m_1 = m_2$$

$$Slope: m=-\frac{2}{5}$$

$$y-y_1 = m\left(x-x_1\right)$$

$$y-5=-\frac{2}{5}(x-3)$$

$$y-5=-\frac{2}{5}x+\frac{6}{5}$$

$$y-5+5=-\frac{2}{5}x+\frac{6}{5}+5$$

$$y=-\frac{2}{5}x+\frac{31}{5}$$

Perpendicular Lines (⊥)

Two lines are perpendicular if and only if the product of their slope is -1. $m_1 \cdot m_2 = -1$

Example

Find the slope of the line L perpendicular to the line having the equation $5x-y=4$

<u>*Solution*</u>

$$5x-y=4$$

$5x-4=y \rightarrow$ Slope = 5

Slope of the line L = $-\frac{1}{5}$

Linear Functions and Applications

Linear Function

A relationship f defined by

$$y = f(x) = mx + b$$

For real numbers m and b, is a ***linear function***

Example

Let $g(x) = -4x + 5$***. Find*** $g(3)$***,*** $g(0)$***,*** $g(-2)$***, and*** $g(b)$

<u>*Solution*</u>

$$g(x) = -4x + 5$$
$$g(\text{—}) = -4(\text{—}) + 5$$

$$\begin{aligned} g(3) &= -4(3) + 5 \\ &= -7 \end{aligned}$$

$$\begin{aligned} g(0) &= -4(0) + 5 \\ &= 5 \end{aligned}$$

$$\begin{aligned} g(-2) &= -4(-2) + 5 \\ &= 13 \end{aligned}$$

$$g(b) = -4b + 5$$

Cost Analysis

Definition

In a cost function of the form $C(x) = mx + b$ is the *linear cost function*

m: Represents the marginal cost per item

b: Fixed cost.

Example

The marginal cost to make x hatches of a prescription medication is \$10 per batch, while the cost to produce 100 batches is \$1500. Find the cost function $C(x)$, given that it is linear.

<u>*Solution*</u>

Since the cost function is linear $\Rightarrow C(x) = mx + b$

The marginal cost = slope $\Rightarrow m = 10$

$$C(x) = 10x + b$$
$$1500 = 10(100) + b$$
$$1500 = 1000 + b$$
$$b = 500$$

$$C(x) = 10x + 500$$

Break-Even Analysis

$R(x)$ *Revenue*

$P(x)$ *Profit*

$C(x)$ *Cost*

x *Units*

p *Price per unit*

$R(x) = p \cdot x$

$$\boxed{P(x) = R(x) - C(x)}$$

The number of units at which revenue just equals cost $\left(R(x) = C(x)\right)$ is the ***break-even quantity***.
The corresponding ordered pair gives the ***break-even point***.

Example

A firm producing poultry feed finds that the total cost $C(x)$ in dollars of producing and selling x units is given by

$$C(x) = 20x + 100$$

Management plans to charge \$24 per unit for the feed.

a) How many units must be sold for the firm to break even?

The revenue: $R(x) = px$
$\qquad = 24x$

Break-even: $R(x) = C(x)$

$$24x = 20x + 100$$
$$4x = 100$$
$$x = 25$$

b) What is the profit if 100 units of feed are sold?

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 24x - (20x + 100) \\ &= 24x - 20x - 100 \\ &= 4x - 100 \end{aligned}$$

$$\begin{aligned} P(100) &= 4(100) - 100 \\ &= 300 \end{aligned}$$

c) How many units must be sold to produce a profit of \$900?

$$900 = 4x - 100$$
$$1000 = 4x$$
$$x = 250$$

Exercises *Section* 1.4 – Equations and Application

1. Suppose that Greg, manager of a giant supermarket chain, has studied the supply and demand for watermelons. He has noticed that the demand increases as the price decreases. He has determined that the quantity (in thousands) demanded weekly q, and the price (in dollars) per watermelons, p, are related by the linear function.

$$p = D(q) = 9 - 0.75q \quad \text{Demand function}$$

a) Find the demand at a price of \$5.25 per watermelon and at a price of \$3.75 per watermelon.

b) Greg also noticed that the supply of watermelons decreased as the price decreased. Price p and supply q are related by the linear function

$$p = S(q) = 0.75q \qquad \text{Demand function}$$

Find the supply at a price of \$5.25 per watermelon and at a price of \$3.00 per watermelon.

c) Use the algebra to find the equilibrium quantity for the watermelon in example 2

2. In Recent years, the percentage of the U.S. population age 18 and older who smoke has decreased at a roughly constant rate, from 23.3% in 2000 to 20.9% in 2004.

a) Find the equation describing this linear relationship.

b) One of the objectives of Healthy People 2010 (a campaign of the U.S. Department of Health and Human Services) is to reduce the percentage of U.S. adults to smoke to 12% or less by the year 2010. If this decline in smoking continues at the same rate, will they meet this objective

3. The number of African Americans earning doctorate degrees has risen at an approximately constant rate from 1987 to 2005. The linear equation $y = 63.6x + 787$, where x represents the number of years since 1987, can be used to estimate the annual number of African Americans earning doctorate degrees. Determine this number in 2006.

Section 1.5 – Limits and Asymptotes

Definition of the Limit of a Function

If $f(x)$ becomes arbitrary close to a single number L as x approaches c from either side, then

$$\lim_{x \to c} f(x) = L$$

Which is read as "the limit of $f(x)$ as x approaches c is L."

Limit of a Polynomial Function

If p is a polynomial function and c is any real number, then

$$\lim_{x \to c} p(x) = p(c)$$

Example

Find the limit: $\lim_{x \to 1} (2x + 4)$

Solution

$$\lim_{x \to 1} (2x + 4) = 2*(1) + 4 = 6$$

Example

Find the limit: $\lim_{x \to 1} \frac{x^2 - 4}{x - 2}$

Solution

$$\lim_{x \to 1} \frac{x^2 - 4}{x - 2} = \frac{1^2 - 4}{1 - 2}$$

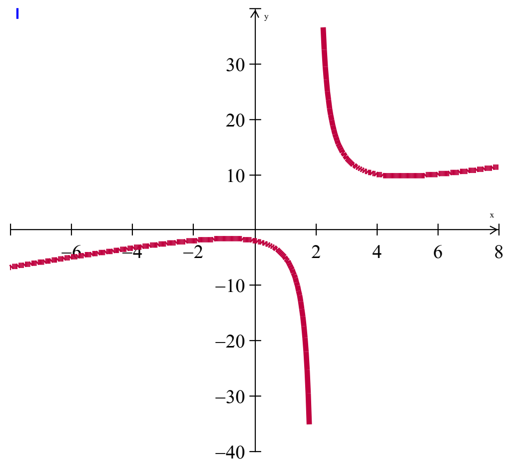
$$= \frac{-3}{-1}$$

$$= 3$$

Unbounded Behavior

Example

Find the limit: $\lim\limits_{x\to 2} \dfrac{x^2+4}{x-2}$



<u>*Solution*</u>

$$\lim_{x\to 2} \frac{x^2+4}{x-2} = \frac{2^2+4}{2-2}$$

$$= \frac{8}{0}$$

$$= \infty \quad (\textbf{\textit{Doesn't exist}})$$

$$\lim_{x\to 2^-} \frac{x^2+4}{x-2} = \frac{(-2)^2+4}{2^- -2}$$

$$= -\infty$$

$$\lim_{x\to 2^+} \frac{x^2+4}{x-2} = \frac{(-2)^2+4}{2^+ -2}$$

$$= +\infty$$

Example

Find the limit: $\lim\limits_{x\to 0} \frac{|x|}{x}$

<u>*Solution*</u>

$$\lim_{x\to 0} \frac{|x|}{x} = \frac{0}{0}$$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \frac{x}{-x} = -1$$

$$\lim_{x\to 0^+} \frac{|x|}{x} = \frac{x}{x} = 1$$

Doesn't exist

On-Sided limits

Example

Find the limit: $\lim\limits_{x\to 2^-} \frac{|x-2|}{x-2}$

<u>*Solution*</u>

$$\lim_{x\to 2^-} \frac{|x-2|}{x-2} = \frac{(x-2)}{-(x-2)} = -1$$

Find the limit: $\lim\limits_{x\to 2+} \frac{|x-2|}{x-2}$

$$\lim_{x\to 2^+} \frac{|x-2|}{x-2} = \frac{(x-2)}{(x-2)} = 1$$

Example

Find: $\lim\limits_{x\to 3} \frac{x^2-x-1}{\sqrt{x+1}}$

<u>*Solution*</u>

$$\lim_{x\to 3} \frac{x^2-x-1}{\sqrt{x+1}} = \frac{3^2-3-1}{\sqrt{3+1}}$$

$$= \frac{5}{2}$$

Example

Suppose $\lim\limits_{x \to 2} f(x) = 3$ and $\lim\limits_{x \to 2} g(x) = 4$

Find $\lim\limits_{x \to 2} \dfrac{[f(x)]^2}{\ln g(x)}$

<u>*Solution*</u>

$$\lim_{x \to 2} \frac{[f(x)]^2}{\ln g(x)} = \frac{\lim\limits_{x \to 2} [f(x)]^2}{\lim\limits_{x \to 2} \ln g(x)}$$

$$= \frac{\left[\lim\limits_{x \to 2} f(x)\right]^2}{\ln\left(\lim\limits_{x \to 2} g(x)\right)}$$

$$= \frac{[3]^2}{\ln(4)}$$

$$\approx \frac{9}{1.38629}$$

$$\approx 6.492$$

Example

Find: $\lim\limits_{x \to 2} \dfrac{x^2 + x - 6}{x - 2}$

<u>*Solution*</u>

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \frac{2^2 + 2 - 6}{2 - 2}$$

$$= \frac{0}{0}$$

$$\lim_{x \to 2} \frac{(x+3)(x-2)}{x - 2} = \lim_{x \to 2} (x+3)$$

$$= 5$$

Vertical Asymptotes and Infinite Limits

Definition

If $f(x)$ approaches infinity ($\pm\infty$) as x approaches c $(a \to c)$ from the right or from the left, then the line $x = c$ is a vertical asymptote of the graph *f*.

Example

Find each limit.

a. $$\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$$

$$\lim_{x\to 2^+} \frac{1}{x-2} = \infty$$

b. $$\lim_{x\to -3^-} \frac{-1}{x+3} = \infty$$

$$\lim_{x\to -3^+} \frac{-1}{x+3} = -\infty$$

Finding Vertical Asymptotes (***Think Domain***)

Example

$$f(x) = \frac{x+2}{x^2 - 2x}$$

Solution

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$\to x = 0,\ 2$$

Example

Find the vertical asymptote(s) of the graph of $f(x)=\dfrac{x+4}{x^2-4x}$

<u>*Solution*</u>

$$\begin{aligned}&x^2-4x=0\\&x(x-4)=0\\&\to x=0,\ 4\end{aligned}$$

Example

Find the vertical asymptote(s) of the graph of $f(x)=\dfrac{x^2+4x+3}{x^2-9}$

<u>*Solution*</u>

$$\begin{aligned}f(x)&=\frac{(x+3)(x-1)}{(x+3)(x-3)}\\&=\frac{(x-1)}{(x-3)}\end{aligned}$$

Vertical Asymptote (VA): $x = 3$

Hole: $x = -3$ (***undefined***)

Horizontal Asymptote

Definition

If *f* is a function and L_1 and L_2 are real numbers, the statements

$$\lim_{x\to\infty} f(x)=L_1 \qquad \textit{and} \qquad \lim_{x\to-\infty} f(x)=L_2$$

Denote limits at infinity. The lines $y=L_1$ and $y=L_2$ are ***horizontal asymptotes*** (***HA***) of the graph of *f*.

Example

Find the limit: $\displaystyle\lim_{x\to\infty}\left(2+\frac{5}{x^2}\right)$

<u>*Solution*</u>

$$\begin{aligned}\lim_{x\to\infty}\left(2+\frac{5}{x^2}\right)&=\lim_{x\to\infty}(2)+\lim_{x\to\infty}\left(\frac{5}{x^2}\right)\\&=2-5(0)\\&=2\end{aligned}$$

HA: $y=2$

Horizontal Asymptotes of Rational Functions

Let $f(x)=\frac{p(x)}{q(x)}=\frac{a_n x^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0}{b_m x^m+b_{m-1}x^{m-1}+\ldots+b_1x+b_0}=\frac{a_n x^n}{b_m x^m}$ be a rational function.

1. If the degree of numerator is less than of denominator $(n<m)\Rightarrow y=0$

   $$y=\frac{2x+1}{4x^2+5}\quad\Rightarrow \boxed{y=0}$$

2. If the degree of numerator is equal of denominator $(n=m)\Rightarrow y=\frac{a_n}{b_m}$

   $$y=\frac{2x^2+1}{4x^2+5}\quad\Rightarrow \boxed{y=\frac{2}{4}=\frac{1}{2}}$$

3. If the degree of numerator is greater than of denominator $(n>m)\Rightarrow$ No horizontal asymptote

   $$y=\frac{2x^3+1}{4x^2+5}\quad\Rightarrow No\ \ HA$$

Example

Find the vertical and horizontal asymptotes (if any) of

1. $f(x)=\frac{x^2+2x-15}{(x+3)(x-4)}$

   VA: $x=-3$ & $x=4$

   HA: $y=1$

2. $g(x)=\frac{3x^2-2x+7}{2x^2+5}$

   No **VA**

   HA: $y=\frac{3}{2}$

Slant or Oblique Asymptotes

When the degree of the numerator is one greater than the degree of the numerator, the graph has a slant or oblique asymptote. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

$$y=\frac{3x^2-1}{x+2}$$

$$\begin{array}{r}3x-6\\ x+2\overline{\big)\,3x^2+0x-1}\\ 3x^2+6x\quad\\ -6x-1\\ -6x-12\\ R=11\end{array}$$

$$y=\frac{3x^2-1}{x+2}=(3x-6)+\frac{11}{x+2}$$

The slant asymptote is the line ***y = 3x - 6***

Exercises *Section* 1.5 – Limits and Asymptotes

Find the limit:

1. $\lim\limits_{x\to 1}(2x^2 - x + 4)$

2. $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}$

3. $\lim\limits_{x\to 2}\dfrac{x^3-8}{x-2}$

4. $\lim\limits_{x\to 3}\dfrac{x^2+x-12}{x-3}$

5. $\lim\limits_{x\to 0}\dfrac{\sqrt{x+4}-2}{x}$

6. $\lim\limits_{x\to 0} f(x)$ $\qquad f(x) = \begin{cases} x^2+1 & x<0 \\ 2x+1 & x>0 \end{cases}$

7. $\lim\limits_{x\to -2}\dfrac{5}{x+2}$

8. $\lim\limits_{x\to 0}(3x-2)$

9. $\lim\limits_{x\to 3}\dfrac{\sqrt{x+1}-1}{x}$

10. $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}$

11. $\lim\limits_{x\to -2}\dfrac{|x+2|}{x+2}$

12. $\lim\limits_{x\to 2+}\dfrac{|x-2|}{x-2}$

Find the vertical and horizontal asymptotes (if any) of

13. $y=\dfrac{3x}{1-x}$

14. $y=\dfrac{x^2}{x^2+9}$

15. $y=\dfrac{x-2}{x^2-4x+3}$

16. $y=\dfrac{3}{x-5}$

17. $y = \dfrac{x^3 - 1}{x^2 + 1}$

18. $y = \dfrac{3x^2 - 27}{(x+3)(2x+1)}$

19. $y = \dfrac{x^3 + 3x^2 - 2}{x^2 - 4}$

20. $y = \dfrac{x-3}{x^2 - 9}$

21. $y = \dfrac{6}{\sqrt{x^2 - 4x}}$

22. $y = \dfrac{5x-1}{1-3x}$

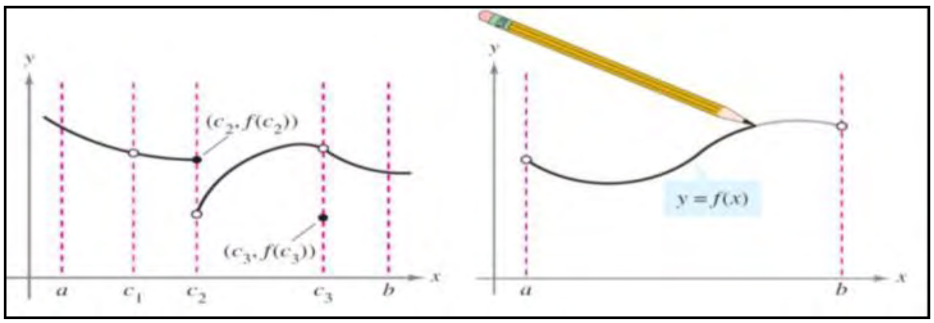
Section 1.6 – Continuity and Rates of Change

Definition of Continuity

Let c be a number in the interval (a, b), and let f be a function whose domain contains the interval (a, b). The function f is continuous at the point c if the following conditions are true.

1. $f(c)$ is defined
2. $\lim\limits_{x \to c} f(x)$ exists
3. $\lim\limits_{x \to c} f(x) = f(c)$

If f is continuous at every point in the interval (a, b), then it is continuous on an open interval (a, b)



The Continuity of Polynomial & Rational functions:

1- A Polynomial function is continuous @ every real number

2- A rational function is continuous @ every point in its domain

$x \neq c \Rightarrow$ Continuous $(-\infty, c)$ and (c, ∞)

Example

Find all values of x where the following function is discontinuous

$$f(x) = \begin{cases} x+1 & if \quad x < 1 \\ x^2 - 3x + 4 & if \quad 1 \le x \le 3 \\ 5 - x & if \quad x > 3 \end{cases}$$

<u>*Solution*</u>

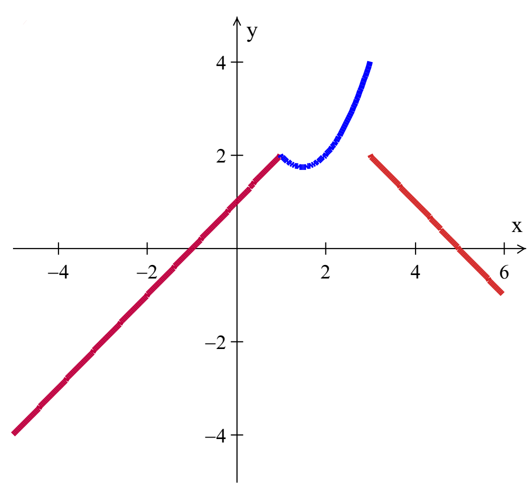
$\lim\limits_{x \to 1^-} (x+1) = 1 + 1 = 2$

$f(1) = (1)^2 - 3(1) + 4 = 2$

$f(3) = (3)^2 - 3(3) + 4 = 4$

$\lim\limits_{x \to 3^+} (5 - x) = 5 - 3 = 2$

So f is discontinue at $x = 3$



Example

a) $f(x)=\frac{1}{x-1}$

Consists of all real number except $x = 1$.

Or

Continuous on $(-\infty, 1)$ and $(1, \infty)$

b) $f(x)=\frac{x^2-4}{x-2}$

Continuous on $(-\infty, 2)$ and $(2, \infty)$

c) $f(x)=x^2-2x+3$

Continuous @ every real number

d) $f(x)=x^3+x$

Continuous @ every real number

If f is not continuous @ $x = c \Rightarrow$ function is said to have discontinuity @ c

$\Rightarrow$ This type of discontinuity falls into 2 categories:

1. Removable   *ex.* $\frac{x^2-4}{x-2}$ $\lim \frac{x^2-4}{x-2} = exists = \text{constant}$
2. Non-removable   *ex.* $\frac{1}{x-1}$ $\lim = doesn't\ exist = \pm\infty$

Definition: Continuity on Close Interval

Let f be defined on a closed interval $[a, b]$, if f is continuous on the open interval

$$\Rightarrow \lim_{x\to a^+} f(x) = f(a) \quad and \quad \lim_{x\to b^-} f(x) = f(b) \quad \Rightarrow f \text{ is continuous } [a,b]$$

Example

Discuss the continuity of $f(x) = \sqrt{x-2}$

Solution

Domain: $x - 2 \ge 0 \Rightarrow x \ge 2$

$$\Rightarrow \lim_{x\to 2^+} \sqrt{x-2} = 0 = f(2)$$

Example

Discuss the continuity of

$$f(x) = \begin{cases} x+2 & -1 \le x < 3 \\ 14 - x^2 & 3 \le x \le 5 \end{cases}$$

Solution

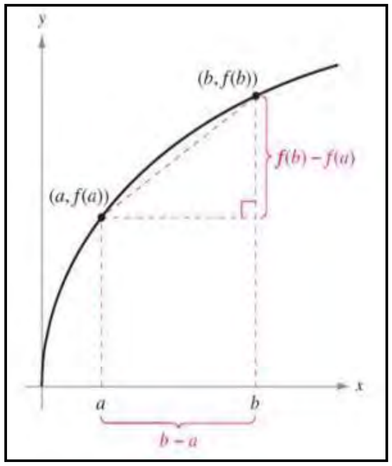
$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x+2) = 5$$

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} (14 - x^2) = 5$$

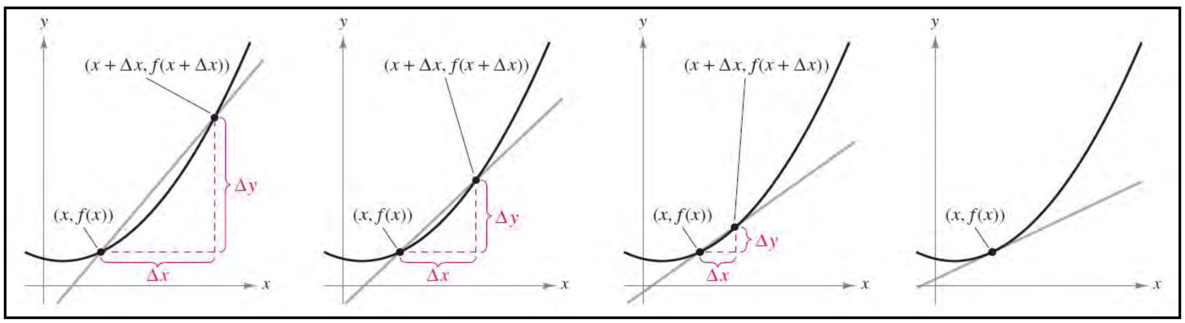
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (14 - x^2) = 5$$

Slope and Rate of change: Given $f(x)$ @ $(x, f(x))$

Definition of Average Rate of Change



$$\textit{Average Rate of Change} = \frac{f(b)-f(a)}{b-a} = \frac{\Delta y}{\Delta x}$$



Slope and limit process

The secant line contains the points $(c, f(c))$ and $(c+\Delta x, f(c+\Delta x))$. Using the slope formula, the slope of the secant line is

$$m_{\text{sec}} = \frac{f(x+\Delta x)-f(x)}{x+\Delta x-x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

If you allow Δx to become smaller, the secant line will change slope and become closer to the slope of the tangent line. If you were to allow Δx to equal 0, the secant line becomes the tangent line. Of course you cannot allow Δx to equal 0, because the slope of the secant line would then be undefined. But you could let Δx approach 0. As Δx approaches 0, the secant line becomes the tangent line to the graph at the point $(x, f(x))$. Thus the slope of the secant line becomes the slope of the tangent line at the point $(x, f(x))$ is given by:

$$m = \lim_{\Delta x \to 0} m_{\text{sec}} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad \textit{(Difference Quotient)}$$

As we know: $m = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Example

Cigarette consumption in the United Sates has been declining since reaching a peak around 1960. Per capita cigarette consumption since 1980 can be closely approximated by the function

$$f(t) = 3870(0.970)^t$$

Where t is the number of years since 1980. Find the average rate of change of per capita consumption from 1985 to 2005.

<u>*Solution*</u>

$$t = 1985 - 1980 = 5$$

$$f(t = 5) = 3870(0.970)^5 = 3323.3$$

$$f(t = 2005 - 1980 = 25) = 3870(0.970)^{25} = 1807.2$$

$$\frac{f(25) - f(5)}{25 - 5} = \frac{1807.2 - 3323.3}{20}$$
$$= -76$$

The average rate decreased at a rate of 76 cigarettes per year.

Definition of Instantaneous rate of Change

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

Example

The distance in feet of an object from a starting point is given by $s(t) = 2t^2 - 5t + 40$, where t is time in seconds.

a) Find the average velocity of the object from 2 sec to 4 sec.

The average velocity: $= \dfrac{s(4) - s(2)}{4 - 2}$

$$= \frac{\left(2(4)^2 - 5(4) + 40\right) - \left(2(2)^2 - 5(2) + 40\right)}{2}$$

$$= 7 \ \ ft / sec$$

c) Find the instantaneous velocity at 4 sec.

$$\lim_{b\to 4}\frac{s(b)-s(4)}{b-4}=\lim_{b\to 4}\frac{\left(2(b)^2-5(b)+40\right)-\left(2(4)^2-5(4)+40\right)}{b-4}$$

$$=\lim_{b\to 4}\frac{2b^2-5b+40-52}{b-4}$$

$$=\lim_{b\to 4}\frac{2b^2-5b-12}{b-4}$$

$$=\lim_{b\to 4}\frac{(2b+3)(b-4)}{b-4}$$

$$=\lim_{b\to 4}(2b+3)$$

$$=11\ \ ft/sec$$

Example

Find the slope of the graph of $f(x)=x^2$ at the point (2, 4)

<u>*Solution*</u>

$$m_{\sec}=\frac{f(2+\Delta x)-f(2)}{\Delta x}$$

$$=\frac{(2+\Delta x)^2-2^2}{\Delta x}$$

$$=\frac{4+\Delta x^2+4\Delta x-4}{\Delta x}$$

$$=\frac{\Delta x^2+4\Delta x}{\Delta x}$$

$$=\frac{\Delta x^2}{\Delta x}+\frac{4\Delta x}{\Delta x}$$

$$=\Delta x+4$$

$$m=\lim_{\Delta x\to 0} m_{\sec}$$

$$=\lim_{\Delta x\to 0}(\Delta x+4)$$

$$=4$$

Exercises *Section* 1.6 – Continuity and Rates of Change

Determine whether $f(x)$ is continuous on the entire number line. Explain your reasoning.

1. $f(x) = \dfrac{x}{x^2 - 1}$

2. $f(x) = \dfrac{x-5}{x^2 - 9x + 20}$

3. Find the slope of the graph of $f(x) = 2x + 5$

4. Find the slope of the graph of $f(x) = \sqrt{x}$