Lecture Three – Applications of Derivatives

Section 3.1 – Maxima and Minima

Definition

Let f be a function with Domain D. Then f has an ***absolute maximum*** value on D at a point c if

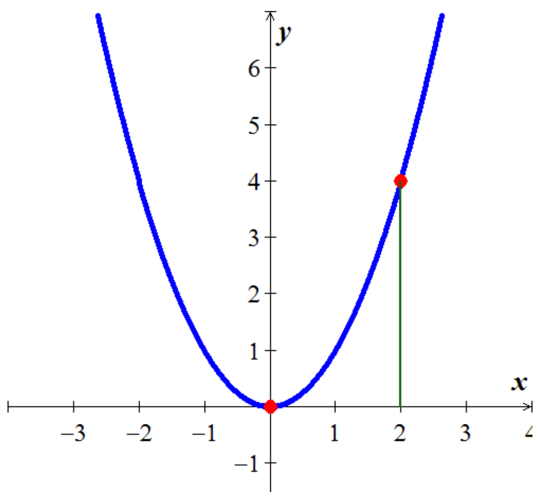
$$f(x) \le f(c) \quad \textit{for all } x \textit{ in } D$$

And an ***absolute maximum*** value on D at a point c if

$$f(x) \ge f(c) \quad \textit{for all } x \textit{ in } D$$

Maximum and minimum values are called ***extreme values*** of the function f.

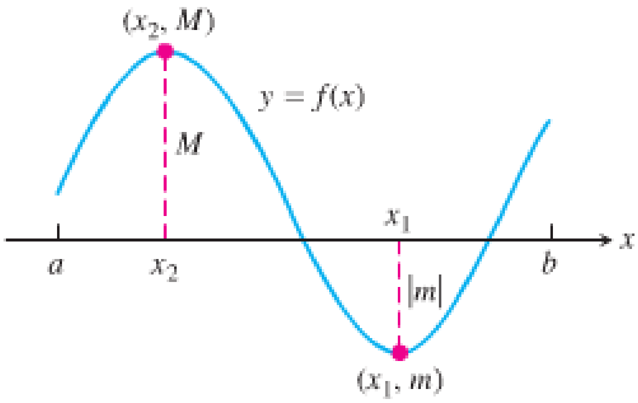
Absolute maxima or minima are also referred to as ***global*** maxima or minima.



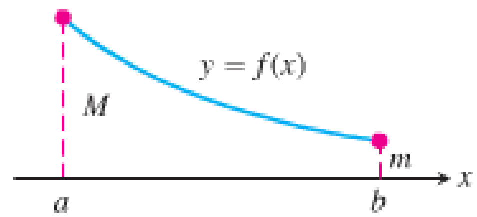
Function rule	***Domain D***	***Absolute extrema on D***
$y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
$y = x^2$	$[0, 2]$	Absolute minimum of 0 at $x = 0$. Absolute minimum of 4 at $x = 2$.
$y = x^2$	$(0, 2]$	No absolute minimum. Absolute minimum of 4 at $x = 2$.
$y = x^2$	$(0, 2)$	No absolute extrema.

Theorem **– The Extreme Value Theorem**

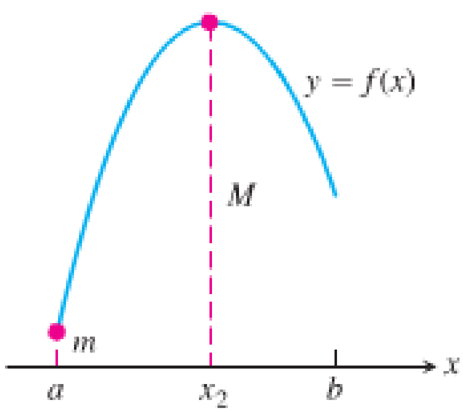
If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f\left(x_1\right) = m$, $f\left(x_2\right) = M$, and $m \le f\left(x\right) \le M$ for every other x in $[a, b]$.



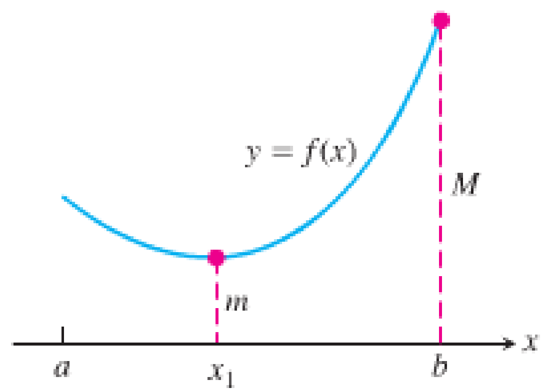
Maximum and minimum
at interior points



Maximum and minimum
at endpoints



Maximum at interior point,
minimum at endpoint

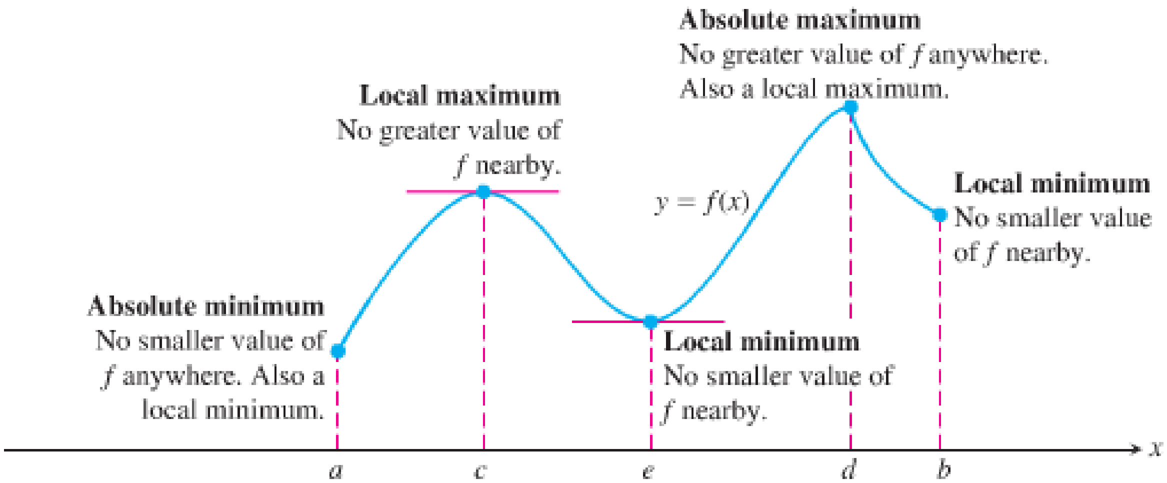


Minimum at interior point,
maximum at endpoint

Definitions

A function f has a ***local maximum*** (***LMAX***) value at a point c within its domain D if $f(x) \le f(c)$ for all $x \in D$ lying is some open interval containing c.

A function f has a ***local minimum*** (***LMIN***) value at a point c within its domain D if $f(x) \ge f(c)$ for all $x \in D$ lying is some open interval containing c.



An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood.

Finding Extrema

Theorem – The First Derivative Theorem for Local Extreme Values

If f has a ***local minimum*** or ***local maximum*** value at a point c of its domain D, and f' is defined at c, then

$$f'(c) = 0$$

Proof

For $f'(c) = 0$ at a local extremum, we need to show that $f'(c)$ can't be positive or negative at c.

$$\left.\begin{aligned} f'(c) &= \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \left\{ \frac{f(x) \le f(c)}{x - c > 0} \right\} \le 0 \\ f'(c) &= \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \left\{ \frac{f(x) \le f(c)}{x - c < 0} \right\} \ge 0 \end{aligned}\right. \Rightarrow f'(c) = 0$$

Definition

An interior point of the domain of a function f where f' is zero or undefined is a critical point of f.

How to find the Absolute Extrema of a continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

Example

Find the absolute maximum and minimum values of $f(x)=x^2$ on [–2, 1].

Solution

$$f'(x)=2x=0 \quad\Rightarrow\quad x=0$$

$$f(0)=0^2=0$$

Check: $f(-2)=(-2)^2=4$

$f(1)=(1)^2=1$

The function has an absolute maximum value of 4 at $x=-2$ and an absolute minimum value of 0 at $x=0$.

Example

Find the absolute maximum and minimum values of $g(t)=8t-t^4$ on [–2, 1].

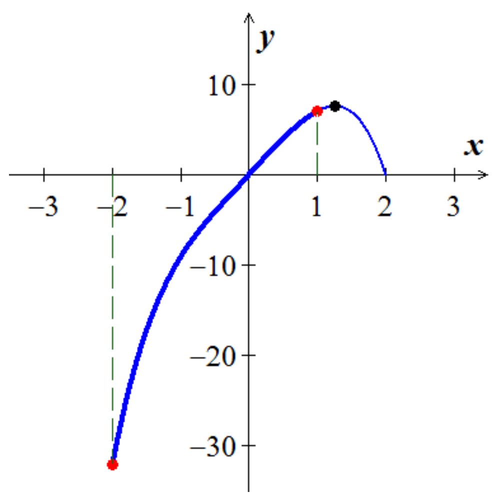
Solution

$$g'(t)=8-4t^3 \quad\Rightarrow\quad t^3=2\to \boxed{t=\sqrt[3]{2}} > 1$$

Check: $g(-2)=8(-2)-(-2)^4=-32$

$g(1)=8(1)-(1)^4=7$

The function has an absolute maximum value of 7 at $x=1$ and an absolute minimum value of –32 at $x=-2$.

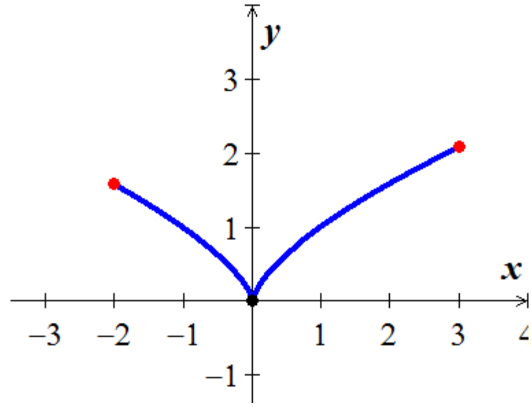


Example

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on [–2, 3].

Solution

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}} = 0 \quad \Rightarrow \quad \textit{Undefined}$$



Critical point: $f(0) = 0$

Endpoint values: $f(-2) = (-2)^{2/3} = \sqrt[3]{2^2} = \sqrt[3]{4}$

$f(3) = (3)^{2/3} = \sqrt[3]{3^2} = \sqrt[3]{9}$

Absolute ***MIN***: (0, 0)

Absolute ***MAX***: $\left(3,\ \sqrt[3]{9}\right)$

Critical Points ***(CP)*** or ***Critical Numbers***

The critical points for a function f are those numbers c in the domain of f for which $f'(c) = 0$ or $f'(c)$ doesn't exist. A critical point is a point whose *x*-coordinate is the critical point c, and whose *y*-coordinate is $f(c)$

$f(x) = x^2$

$\Rightarrow f'(x) = 2x = 0$

$\rightarrow \boxed{x = 0}$ is a Critical Number

$f(0) = 0$

Critical Point: $\boxed{(0,\ 0)}$

If $f'(x) = 0$ undefined

Exercises *Section* 3.1 – Maxima and Minima

Find the absolute maximum and minimum values of each function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

1. $f(x)=\frac{2}{3}x-5 \qquad -2\le x\le 3$

2. $f(x)=x^2-1 \qquad -1\le x\le 2$

3. $f(x)=-\dfrac{1}{x^2} \qquad 0.5\le x\le 2$

4. $f(x)=\sqrt{4-x^2} \qquad -2\le x\le 1$

5. $f(\theta)=\sin\theta \qquad -\frac{\pi}{2}\le\theta\le\frac{5\pi}{6}$

6. $g(x)=\sec x \qquad -\frac{\pi}{3}\le x\le\frac{\pi}{6}$

Find the absolute maximum and minimum values of each function (if they exist).

7. $f(x)=x^{4/3}, \quad -1\le x\le 8$

8. $f(\theta)=\theta^{3/5}, \quad -32\le\theta\le 1$

9. $f(x)=2^x\sin x \quad [-2,\ 6]$

10. $f(x)=\sec x \quad \left[-\frac{\pi}{4},\ \frac{\pi}{4}\right]$

11. $f(x)=x^3e^{-x} \quad [-1,\ 5]$

12. $f(x)=x\ln\left(\frac{x}{5}\right) \quad [0.1,\ 5]$

13. $f(x)=x^{8/3}-16x^{2/3} \quad [-1,\ 8]$

14. $f(x)=x^2-8x+10 \quad [0,\ 7]$

15. $f(x)=2(3-x), \quad [-1,\ 2]$

16. $f(x)=x^3-3x^2, \quad [0,\ 4]$

17. $f(x)=\frac{1}{3}x^3-2x^2+3x-4, \quad [-2,\ 5]$

18. $f(x)=\dfrac{1}{x+2}, \quad [-4,\ 1]$

19. $f(x)=\left(x^2+4\right)^{2/3}, \quad [-2,\ 2]$

20. $f(x)=\sin 2x+3 \quad \textit{on} \quad [-\pi,\ \pi]$

21. $f(x)=2x^3-3x^2-36x+12 \quad \textit{on} \quad (-\infty,\ \infty)$

22. $f(x)=4x^{1/2}-x^{5/2} \quad \textit{on} \quad [0,\ 4]$

23. $f(x)=2x\ln x+10 \quad \textit{on} \quad (0,\ 4)$

24. $f(x)=x\sin^{-1}x \quad \textit{on} \quad [-1,\ 1]$

Determine all critical points of each function

25. $y=x^2-6x+7$

26. $g(x)=(x-1)^2(x-3)^2$

27. $f(x)=\dfrac{x^2}{x-2}$

28. $g(x)=x^2-32\sqrt{x}$

Find the extreme values (absolute and local) of the function and where they occur

29. $f(x)=3x^2-4x+2$

30. $y=x^3-2x+4$

31. $y=\sqrt{x^2-1}$

32. $y=\dfrac{1}{\sqrt[3]{1-x^2}}$

33. $y=x^2\sqrt{3-x}$

34. $y=\dfrac{x+1}{x^2+2x+2}$

35. $y=x^{2/3}(x+2)$

36. $y=x\sqrt{4-x^2}$

37. $f(x)=\dfrac{e^x+e^{-x}}{2}$

38. $f(x)=\frac{1}{8}x^3-\frac{1}{2}x \quad [-1,\ 3]$

39. $f(x)=\dfrac{1}{x}-\ln x$

40. $f(x)=\sin x\cos x \quad [0,\ 2\pi]$

41. $f(x)=x-\tan^{-1}x$

42. Let $f(x) = (x-2)^{2/3}$

a) Does $f'(2)$ exist?

b) Show the only local extreme value of f occurs at $x = 2$.

c) Does the result in part (*b*) contradict the Extreme Value Theorem?

43. When a telephone wire is hung between two poles, the wire forms a U-shape curve called a Catenary. For instance, the function $y = 30\left(e^{x/60} + e^{-x/60}\right)$ $\quad -30 \le x \le 30$ models the shape of the telephone wire strung between two poles that are 60 *feet.* apart (x & y are measured in *feet*.). Show that the lowest point on the wire is midway between two poles. How much does the wire sag between the two poles?

44. You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 12 *feet* long and starts 3 *feet* from the wall you are sitting next to.

a) Show that your viewing angle is $\alpha = \cot^{-1}\frac{x}{15} - \cot^{-1}\frac{x}{3}$. If you are x *feet* from the front wall.

b) Find x so that α is as large as possible