

8.3 Double angle

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \quad (1) \\ &= 1 - 2\sin^2 A\end{aligned}$$

$$\sin 2A = 2\sin A \cos A$$

$$\begin{aligned}(1) \quad 2\cos^2 A &= 1 + \cos 2A \\ \cos^2 A &= \frac{1 + \cos 2A}{2} \quad \leftarrow\end{aligned}$$

$$\cos^2 \frac{A}{2} = \frac{1 + \cos A}{2}$$

$$\cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}} \quad Q?$$

$$\sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}}$$

Ex $\sin A = \frac{3}{5}$ AG QII $3, 4 \rightarrow 5$

$$\cos A = -\frac{4}{5} \quad \leftarrow$$

$$\begin{aligned}a) \quad \sin 2A &= 2\sin A \cos A \\ &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) \\ &= -\frac{24}{25}\end{aligned}$$

$$\begin{aligned}b) \quad \cos 2A &= \cos^2 A - \sin^2 A \\ &= \frac{16}{25} - \frac{9}{25} \\ &= \frac{7}{25}\end{aligned}$$

$$c) \tan 2A = -\frac{24}{7}$$

$$\frac{90^\circ}{2} < \frac{A}{2} < \frac{180^\circ}{2}$$

$$45^\circ < \frac{A}{2} < 90^\circ \Rightarrow \frac{A}{2} \in QI \leftarrow$$

$$d) \sin \frac{A}{2} = \sqrt{\frac{1}{2} \left(1 - \left(-\frac{4}{5} \right) \right)} \quad (\ominus)$$

$$\downarrow \quad \swarrow$$

$$\sqrt{\frac{1}{2} (1 - \cos A)} = \sqrt{\frac{1}{2} \left(1 + \frac{4}{5} \right)}$$

$$= \sqrt{\frac{1}{2} \cdot \frac{9}{5}}$$

$$= \frac{3}{\sqrt{10}}$$

$$\sqrt{\frac{1}{2} (1 + \cos A)}$$

$$e) \cos \frac{A}{2} = \sqrt{\frac{1}{2} \left(1 - \frac{4}{5} \right)}$$

$$= \sqrt{\frac{1}{2} \cdot \frac{1}{5}}$$

$$= \frac{1}{\sqrt{10}}$$

$$f) \tan \frac{A}{2} = 3 \quad \left(\frac{3}{4} \right)$$

Prove: $(\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta$

$$(\sin \theta + \cos \theta)^2 = \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta$$

$$= 1 + \sin 2\theta \quad \checkmark$$

Prove

prove

$$\sin 2x = \frac{2 \cot x}{1 + \cot^2 x} \quad \checkmark$$

$$\begin{aligned} \frac{2 \cot x}{1 + \cot^2 x} &= 2 \frac{\frac{\cos x}{\sin x}}{1 + \frac{\cos^2 x}{\sin^2 x}} \\ &= 2 \frac{\frac{\cos x}{\sin x}}{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} \quad \div \\ &= 2 \frac{\cos x}{\sin x} \cdot \frac{\sin^2 x}{1} \\ &= 2 \cos x \sin x \\ &= \sin 2x \quad \checkmark \end{aligned}$$

Handwritten notes: A red bracket under $\frac{2 \cot x}{1 + \cot^2 x}$ is labeled $\csc^2 x$. A red arrow points from this bracket to the $\sin^2 x$ term in the denominator of the third line.

Prove: $\cos 4x = \cos^4 x - 8 \cos^2 x + 1$

$$\begin{aligned} \cos 4x &= \cos(2(2x)) & \cos 2x &= 2 \cos^2 x - 1 \\ &= 2 \cos^2(2x) - 1 \\ &= 2 (\cos 2x)^2 - 1 \\ &= 2 (2 \cos^2 x - 1)^2 - 1 \\ &= 2 (4 \cos^4 x - 4 \cos^2 x + 1) - 1 \\ &= 8 \cos^4 x - 8 \cos^2 x + 2 - 1 \\ &= 8 \cos^4 x - 8 \cos^2 x + 1 \quad \checkmark \end{aligned}$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \quad A+B \quad \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan \frac{A}{2} = \frac{1 - \cos A}{\sin A} = \frac{\sin A}{1 + \cos A} \quad (\odot)$$

Prove: $\tan \theta = \frac{1 - \cos 2\theta}{\sin 2\theta}$

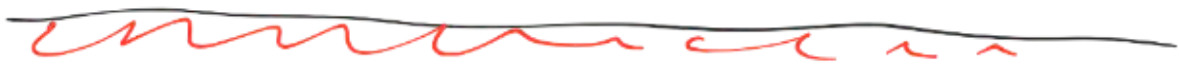
$$\begin{aligned} \frac{1 - \cos 2A}{\sin 2\theta} &= \frac{1 - (1 - 2\sin^2 \theta)}{2\sin \theta \cos \theta} \\ &= \frac{1 - 1 + 2\sin^2 \theta}{2\sin \theta \cos \theta} \\ &= \frac{2\sin^2 \theta}{2\sin \theta \cos \theta} \\ &= \frac{\sin \theta}{\cos \theta} \\ &= \tan \theta \quad \checkmark \end{aligned}$$

Prove: $\sin^2 \frac{x}{2} = \frac{\tan x - \sin x}{2 \tan x}$

$$\begin{aligned} \sin^2 \frac{x}{2} &= \frac{1}{2} (1 - \cos x) \cdot \frac{\tan x}{\tan x} \\ &= \frac{\tan x - \cos x \tan x}{2 \tan x} \\ &= \frac{\tan x - \cos x \frac{\sin x}{\cos x}}{2 \tan x} \\ &= \frac{\tan x - \sin x}{2 \tan x} \quad \checkmark \end{aligned}$$

Ex 20 Prove: $\sin 3x = \sin x (3\cos^2 x - \sin^2 x)$

$$\begin{aligned}
 \sin 3x &= \sin (x + 2x) \\
 &= \sin x \cos 2x + \cos x \sin 2x \\
 &= \sin x (\cos^2 x - \sin^2 x) + \cos x (2\sin x \cos x) \\
 &= \sin x \cos^2 x - \sin^3 x + 2\sin x \cos^2 x \\
 &= 3\sin x \cos^2 x - \sin^3 x \\
 &= \sin x (3\cos^2 x - \sin^2 x) \checkmark
 \end{aligned}$$



8.4 Solving Trig

[trig fcn] ~~of~~ a variable

Cosine & Sine \rightarrow 1 angle for ± 1

$$1 \rightarrow \cos 0, \sin \frac{\pi}{2}$$

$$-1 \rightarrow \cos \pi, \sin \frac{3\pi}{2}$$

otherwise each value \rightarrow 2 angles

$$\frac{\pi}{n} \quad \text{Q I}$$

$$\frac{(n-1)\pi}{n} \quad \text{Q II}$$

$$\frac{(n+1)\pi}{n} \quad \text{Q III}$$

$$\frac{(2n-1)\pi}{n} \quad \text{Q IV}$$



Ex $\sin \theta = \frac{1}{2}$ Q I, II

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6} \quad \underline{\underline{[0, 2\pi)}}$$

$$\theta = \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi$$

Solve: $\sin x \tan x = \sin x \quad [0, 2\pi)$

$$\sin x \tan x - \sin x = 0$$

$$\sin x (\tan x - 1) = 0$$

$$\sin x = 0$$

$$\tan x - 1 = 0 \quad \text{②}$$

$$\tan x = 1 \quad \left(\frac{\pi}{4}, \frac{5\pi}{4} \right)$$

$$x = 0, \pi, \frac{\pi}{4}, \frac{5\pi}{4}$$

$$x = n\pi, \frac{\pi}{4} + n\pi$$

Solve: $2\sin^2 t - \cos t - 1 = 0 \quad [0, 2\pi)$

$$2(1 - \cos^2 t) - \cos t - 1 = 0$$

$$2 - 2\cos^2 t - \cos t - 1 = 0$$

$$-2\cos^2 t - \cos t + 1 = 0$$

$$\cos t = \frac{1 \pm \sqrt{1+8}}{2(-2)} = \frac{1 \pm 3}{-4}$$

$$\cos t = -1$$

$$\cos t = \frac{1}{2}$$

$$t = \pi, \frac{\pi}{3}, \frac{5\pi}{3}$$

Solve $4\sin^2 x \tan x - \tan x = 0 \quad [0, 2\pi)$

$$\tan x (4 \sin^2 x - 1) = 0$$

$$\tan x = 0 \quad \frac{\sin x}{\cos x} \quad 4 \sin^2 x = 1$$

$$\downarrow$$

$$\sin^2 x = \frac{1}{4}$$

$$\sin x = \pm \frac{1}{2}$$

$$x = 0, \pi, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

Solve $\csc^4 2u - 4 = 0$

$$(\csc^2 2u - 2)(\csc^2 2u + 2) = 0$$

$$\csc^2 2u = 2 \quad \csc^2 2u = -2 \quad \#$$

$$\csc 2u = \pm \sqrt{2}$$

$$\frac{1}{\sin 2u} = \pm \sqrt{2}$$

$$\sin 2u = \pm \frac{1}{\sqrt{2}} \quad \frac{\pi}{4}$$

$$2u = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$u = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$$

$$\frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8} \quad \checkmark$$

Solve:

$$5 \sin \theta \tan \theta - 10 \tan \theta + 3 \sin \theta - 6 = 0$$

$$5 \tan \theta (\sin \theta - 2) + 3 (\sin \theta - 2) = 0$$

$$(\sin \theta - 2)(5 \tan \theta + 3) = 0$$

$$\sin \theta = 2 \quad | \quad \tan \theta = -\frac{3}{5}$$

#

$$\theta = \tan^{-1}\left(-\frac{3}{5}\right)$$

$$(-) \in \frac{\pi}{2} \text{ to } \frac{3\pi}{2}$$

$$\theta = \pi - \tan^{-1}\left(\frac{3}{5}\right)$$

$$\theta = 2\pi - \tan^{-1}\left(\frac{3}{5}\right)$$

#10 $2 \sin^2 x = 1 - \sin x$

$$a - b + c = 0 \\ x = -1, -\frac{c}{a}$$

$$2 \sin^2 x + \sin x - 1 = 0$$

$$\sin x = -1 \quad \sin x = \frac{1}{2}$$

$$x = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

$$ax^2 + bx + c = 0$$

$$a + b + c = 0 \rightarrow x = 1, \frac{c}{a}$$

$$a - b + c = 0 \quad x = -1, -\frac{c}{a}$$

#11 $2 \sin^3 x + \sin^2 x - 2 \sin x - 1 = 0$

$$\sin^2 x (2 \sin x + 1) - (2 \sin x + 1) = 0$$

$$(2 \sin x + 1)(\sin^2 x - 1) = 0$$

$$\sin x = -\frac{1}{2}$$

$$\sin^2 x = 1 \\ \sin x = \pm 1$$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}$$

$$24 / \sin \theta - \cos \theta = 1$$

$$(\sin \theta)^2 = (1 + \cos \theta)^2 \quad (\text{Method / Check!})$$

$$\frac{\sqrt{2}}{2} \sin \theta - \frac{\sqrt{2}}{2} \cos \theta = \frac{\sqrt{2}}{2}$$

$$\sin \frac{\pi}{4} \sin \theta - \cos \frac{\pi}{4} \cos \theta = \frac{\sqrt{2}}{2} \quad \left. \begin{array}{l} \cos \frac{\pi}{4} \sin \theta - \sin \frac{\pi}{4} \cos \theta \end{array} \right\}$$

$$-(\cos \frac{\pi}{4} \cos \theta - \sin \frac{\pi}{4} \sin \theta) = \frac{\sqrt{2}}{2}$$

$$\cos(\theta + \frac{\pi}{4}) = -\frac{\sqrt{2}}{2} \quad \left. \begin{array}{l} \cos(\frac{3\pi}{4}) \\ \cos(\frac{5\pi}{4}) \end{array} \right\}$$

$$\theta + \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\theta + \frac{\pi}{4} = \frac{5\pi}{4}$$

$$\theta = \frac{3\pi}{4} - \frac{\pi}{4}$$

$$\theta = \frac{5\pi}{4} - \frac{\pi}{4}$$

$$= \frac{\pi}{2}$$

$$= \pi$$

- By grouping

$$\begin{array}{c} \sin^2 x \\ \cos^2 x \end{array} \rightarrow \begin{array}{c} \sin^2 x + \cos^2 x \\ 1 - \cos^2 x \end{array}$$

$$a \sin^2 x + b \sin x + c = 0$$

$$(\quad)(\quad) = 0, \quad \sin x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$[0, 2\pi)$$

$$\#27 \quad 2 \sin^2 \theta - 2 \sin \theta - 1 = 0$$

$$\sin \theta = \frac{2 \pm \sqrt{4 + 8}}{4}$$

$$12 = 4 \cdot 3$$

$$= \frac{2 \pm 2\sqrt{3}}{4}$$

$$= \frac{1 \pm \sqrt{3}}{2}$$

$$\hat{\theta} = \sin^{-1} \frac{1 \pm \sqrt{3}}{2} \quad \underline{\text{4 Quad}}$$

$$\theta = \hat{\theta}$$

$$\theta = \pi - \hat{\theta}, \quad \bar{\theta} + \hat{\theta}, \quad 2\pi - \hat{\theta}$$
