Section 4.5 – Divergence and Curl

Green's Theorem out of the plane $\left(\mathbb{R}^2\right)$ and into space $\left(\mathbb{R}^3\right)$, it is done as follows:

- The circulation form of Green's Theorem relates a line integral over a simple closed oriented curve in the plane to a double integral over the enclosed region. Stoke's Theorem relates a line integral over a simple closed oriented curve in $\mathbb{R}^3$ to a double integral over a surface whose boundary is the same curve.

- The flux form of Green's Theorem relates a line integral over a simple closed oriented curve in the plane to a double integral over the enclosed region. Similarly, the Divergence Theorem relates an integral over a closed oriented surface in $\mathbb{R}^3$ to a triple integral over the region enclosed by the surface.

Definition

The divergence of a vector field $\vec{F} = \langle f,\ g,\ h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\begin{aligned} div\ \vec{F} &= \nabla \bullet \vec{F} \\ &= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \end{aligned}$$

If $\nabla \bullet \vec{F} = 0$, the vector field is ***source free***.

Example

Compute the divergence of the following vector fields

a) $\vec{F} = \langle x,\ y,\ z \rangle \quad (radial\ field)$

b) $\vec{F} = \langle -y,\ x - z,\ y \rangle \quad (rotation\ field)$

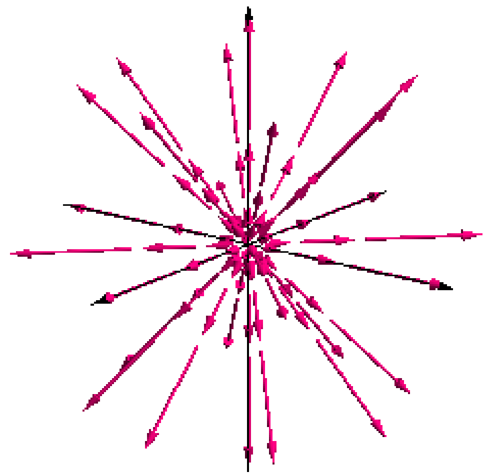
c) $\vec{F} = \langle -y,\ x,\ z \rangle \quad (spiral\ flow)$

Solution

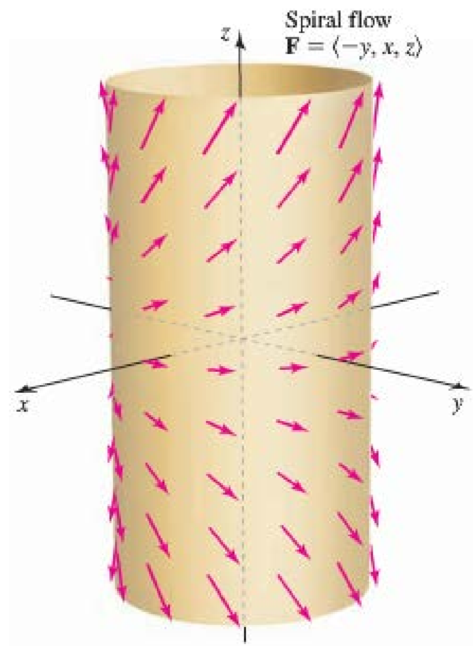
a) The divergence is

$$\begin{aligned} \nabla \bullet \vec{F} &= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \\ &= 1 + 1 + 1 \\ &= 3 \end{aligned}$$

Because the divergence is positive, the flow expands outward at all points



Radial field $\vec{F} = \langle x,\ y,\ z \rangle \quad (radial\ field)$



b) The divergence is

$$\nabla \bullet \vec{F} = \nabla \bullet \langle -y,\ x-z,\ y \rangle$$
$$= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$
$$\underline{= 0}$$

The field is source free.

c) The divergence is

$$\nabla \bullet \vec{F} = \nabla \bullet \langle -y,\ x,\ z \rangle$$
$$= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$
$$\underline{= 1}$$

The rotational part of the field in x and y does not contribute to the divergence. However, the z-component of the field produces a nonzero divergence.

Example

Compute the divergence of the radial vector field

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|} = \frac{(x,\ y,\ z)}{\sqrt{x^2 + y^2 + z^2}}$$

Solution

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}\right) = \frac{\left(x^2 + y^2 + z^2\right)^{1/2} - x^2\left(x^2 + y^2 + z^2\right)^{-1/2}}{x^2 + y^2 + z^2}$$

$$= \frac{y^2 + z^2}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2 + y^2 + z^2}}\right) = \frac{\left(x^2 + y^2 + z^2\right)^{1/2} - y^2\left(x^2 + y^2 + z^2\right)^{-1/2}}{x^2 + y^2 + z^2}$$

$$= \frac{x^2 + z^2}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) = \frac{\left(x^2 + y^2 + z^2\right)^{1/2} - z^2\left(x^2 + y^2 + z^2\right)^{-1/2}}{x^2 + y^2 + z^2}$$

$$= \frac{x^2 + y^2}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$\nabla \cdot \vec{F} = \frac{y^2 + z^2 + x^2 + z^2 + x^2 + y^2}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$= \frac{2\left(x^2 + y^2 + z^2\right)}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$= \frac{2}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{2}{|\vec{r}|}$$

Theorem

For a real number *p*, the divergence of the radial vector field

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|^p} = \frac{(x, y, z)}{\left(x^2+y^2+z^2\right)^{p/2}} \quad \to \quad \nabla \cdot \vec{F} = \frac{3-p}{|\vec{r}|^p}$$

Example

To gain some intuition about the divergence, consider the two-dimensional vector field $\vec{F} = \langle f, g \rangle = \langle x^2, y \rangle$ and a circle *C* of radius 2 centered at the origin.

a) Without computing it, determine whether the two-dimensional divergence is positive or negative at the point $Q(1, 1)$. Why?

b) Confirm tour conjecture in part (*a*) by computing the two-dimensional divergence at *Q*.

c) Based on part (*b*), over what regions within the circle is the divergence positive and over what regions within the circle is the divergence is negative?

d) By inspection of the figure, on what part of the circle is the flux across the boundary outward? Is the net flux out of the circle positive or negative?

Solution

a) At $Q(1, 1)$ the *x*-component and the *y*-component of the field are increasing $\left(f_x > 0 \ and \ g_y > 0\right)$, so the field is expanding at that point and the two-dimensional divergence is positive.

b)

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}(y)$$

$$= 2x + 1$$

$$\nabla \cdot \vec{F} \Big|_{Q(1,1)} = 3$$

∴ The divergence is 3.

c)

$$\nabla \cdot \vec{F} = 2x + 1 > 0 \quad \Rightarrow \quad x > -\frac{1}{2}$$

$$\nabla \cdot \vec{F} = 2x + 1 < 0 \quad \Rightarrow \quad x < -\frac{1}{2}$$

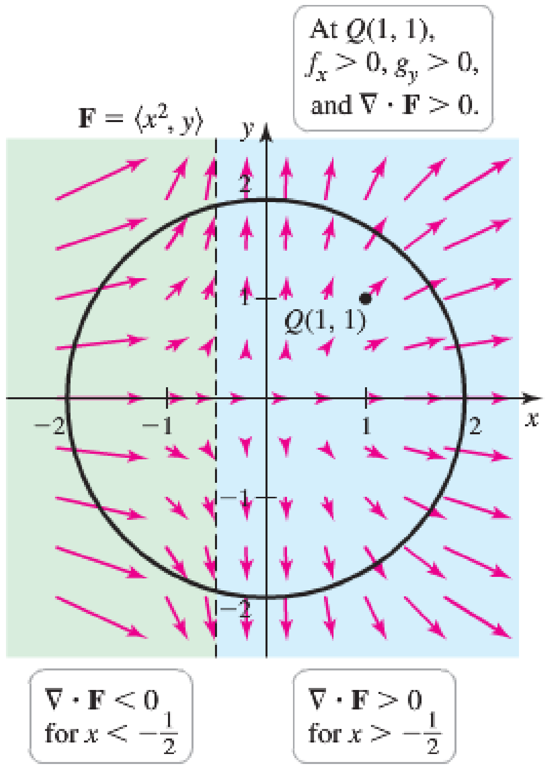
To the left of the line $x = -\frac{1}{2}$ the field is contracting and to the right of the line the field is expanding

d) It appears that the field is tangent to the circle at two points with $x \approx -\frac{1}{2}$.

For points on the circle with $x < -\frac{1}{2}$, the flow is into the circle.

For points on the circle with $x > -\frac{1}{2}$, the flow is out the circle.

It appears that the net outward flux across C is positive.
The points where the field changes from inward to outward may be determined exactly.



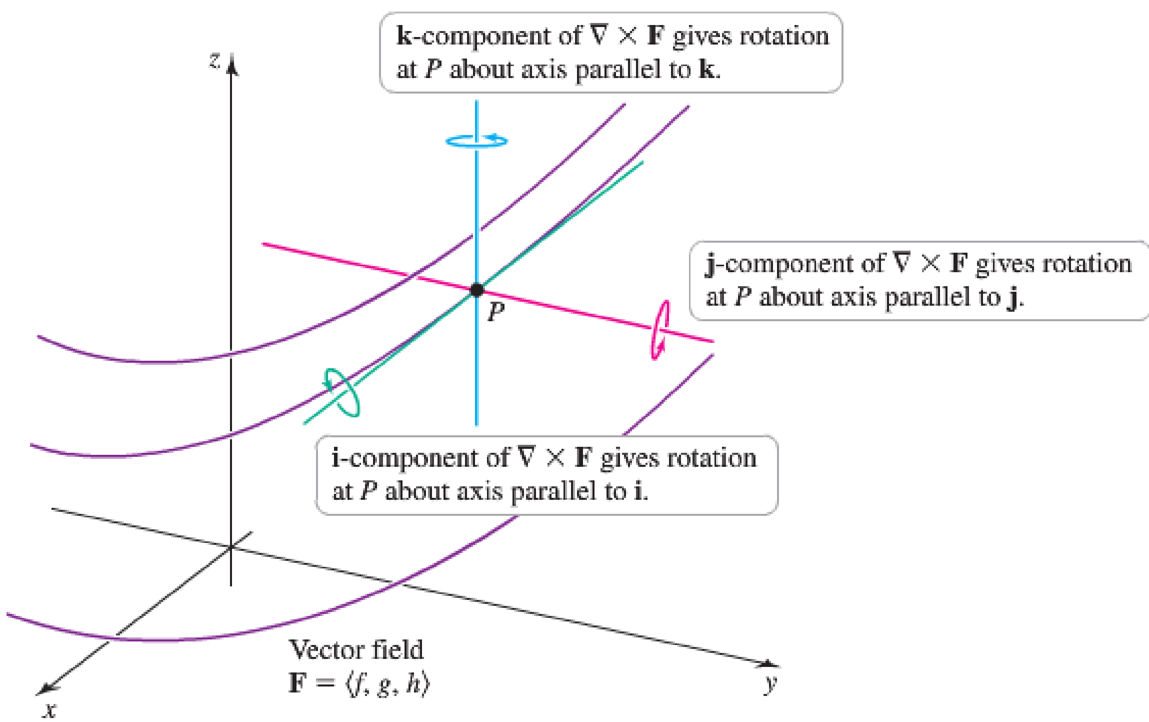
Curl

Definition

The curl of a vector field $\vec{F} = \langle f,\ g,\ h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$curl\ \vec{F} = \nabla \times \vec{F}$$
$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$
$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \hat{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \hat{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \hat{k}$$

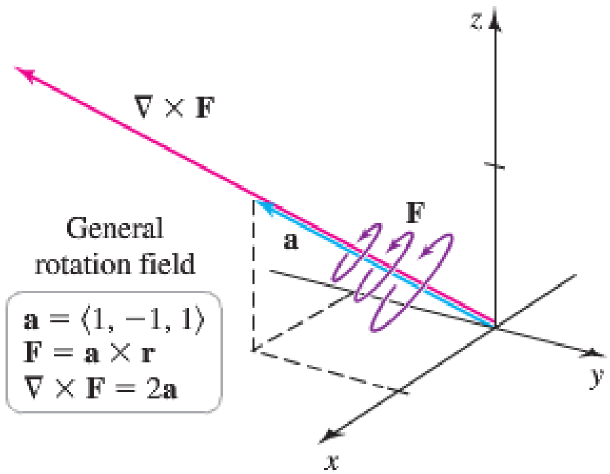
If $\nabla \times \vec{F} = 0$, the vector field is ***irrotational***.



Example

Consider the vector field $\vec{F} = \vec{a} \times \vec{r}$, where $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is a nonzero vector and $\vec{r} = \langle x, y, z \rangle$

$$\begin{aligned}\vec{F} &= \vec{a} \times \vec{r} \\ &= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} \\ &= \left(a_2 z - a_3 y\right)\hat{i} + \left(a_3 x - a_1 z\right)\hat{j} + \left(a_1 y - a_2 x\right)\hat{k}\end{aligned}$$



This vector field is a general rotation field in 3-dimensions.

Suppose a paddle wheel is placed in the vector field $\vec{F}$ at a point P with the axis of the wheel in the direction of a unit vector $\vec{n}$.

$$\left(\nabla \times \vec{F}\right) \bullet \vec{n} = \left(\nabla \times \vec{F}\right)\cos\theta \qquad (\boldsymbol{\vec{n}} = 1)$$

Where θ is the angle between $\nabla \times \vec{F}$ and $\vec{n}$.

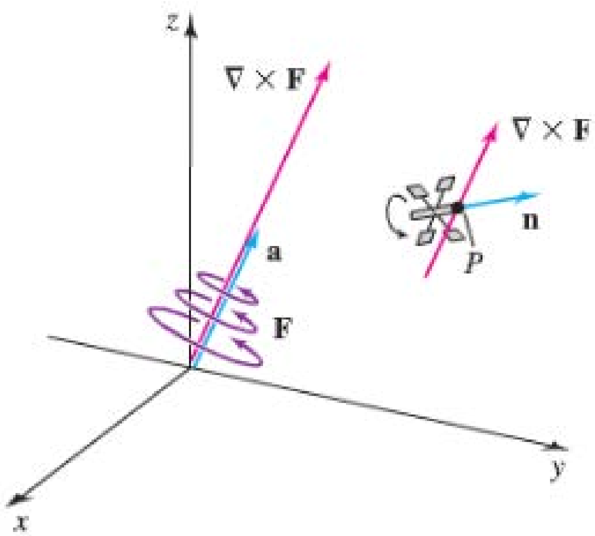
The scalar component is greatest in magnitude and the paddle wheel spins fastest when $\theta = 0$ *or* π; that is when $\nabla \times \vec{F}$ and $\vec{n}$ are parallel.

If the axis of the paddle wheel is orthogonal to $\nabla \times \vec{F}$ $\left(\theta = \pm\frac{\pi}{2}\right)$, the wheel doesn't spin.

General Rotation Vector Field

The general rotation vector field is $\vec{F} = \vec{a} \times \vec{r}$ where the nonzero constant vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is the axis of rotation and $\vec{r} = \langle x, y, z \rangle$. For all choices of $\vec{a}$, $\nabla \times \vec{F} = 2|\vec{a}|$ and $\nabla \cdot \vec{F} = 0$. The constant angular speed of the vector field is

$$\begin{aligned} \omega &= |\vec{a}| \\ &= \frac{1}{2}\left|\nabla \times \vec{F}\right| \end{aligned}$$



Paddle wheel at P with axis $\vec{n}$ measures rotation about $\vec{n}$.
Rotation is a maximum when $\nabla \times \vec{F}$ is parallel to $\vec{n}$.

Example

Compute the curl of the rotation field $\vec{F} = \vec{a} \times \vec{r}$ where $\vec{a} = \langle 1, -1, 1 \rangle$ is the axis of rotation and $\vec{r} = \langle x, y, z \rangle$. What is the direction and the magnitude of the curl?

Solution

$$\begin{aligned} \vec{F} &= \vec{a} \times \vec{r} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ x & y & z \end{vmatrix} \\ &= (-z - y)\hat{i} + (x - z)\hat{j} + (y + x)\hat{k} \end{aligned}$$

$$
\begin{aligned}
curl\ \vec{F} &= \nabla \times \vec{F} \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z-y & x-z & y+x \end{vmatrix} \\
&= 2\hat{i} - 2\hat{j} + 2\hat{k} \\
&= 2\vec{a}
\end{aligned}
$$

The direction of the curl is the direction of $\vec{a}$, which is the axis rotation.

The magnitude of $\nabla \times \vec{F} = |2\vec{a}| = 2\sqrt{3}$

Working with Divergence and Curl

Theorem

Suppose that $\vec{F}$ is a conservative vector field on an open region ***D*** of $\mathbb{R}^3$ Let $\vec{F} = \nabla\phi$, where ϕ is a potential function with continuous second partial derivatives on ***D***. Then $\nabla \times \vec{F} = \nabla \times \nabla\phi = 0$; that is, the curl of the gradient is the zero vector and $\vec{F}$ is irrotational.

$$
\begin{aligned}
\nabla \times \nabla\phi &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi_x & \phi_y & \phi_z \end{vmatrix} \\
&= \left(\phi_{zy} - \phi_{yz}\right)\hat{i} + \left(\phi_{xz} - \phi_{zx}\right)\hat{j} + \left(\phi_{yx} - \phi_{xy}\right)\hat{k} \\
&= 0
\end{aligned}
$$

Product Rule for the Divergence

Theorem

Let u be a scalar-valued function that is differentiable on a region D and let $\vec{F}$ be a vector field that us differentiable on D. Then

$$\nabla \bullet \left(u\vec{F}\right) = \nabla u \bullet \vec{F} + u\left(\nabla \bullet \vec{F}\right)$$

Example

Let $\vec{r} = \langle x,\ y,\ z \rangle$ and let $\phi = \dfrac{1}{|\vec{r}|} = \left(x^2 + y^2 + z^2\right)^{-1/2}$ be a potential function.

a) Find the associated gradient field $\vec{F} = \nabla\left(\dfrac{1}{|\vec{r}|}\right)$

b) Compute $\nabla \bullet \vec{F}$

Solution

a)

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{-1/2}$$

$$= -x\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$= -\frac{x}{|\vec{r}|^3}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + y^2 + z^2\right)^{-1/2}$$

$$= -y\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$= -\frac{y}{|\vec{r}|^3}$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial}{\partial z}\left(x^2 + y^2 + z^2\right)^{-1/2}$$

$$= -z\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$= -\frac{z}{|\vec{r}|^3}$$

$$\vec{F} = \nabla\left(\frac{1}{|\vec{r}|}\right)$$

$$= -\frac{\langle x,\ y,\ z \rangle}{|\vec{r}|^3}$$

$$= -\frac{\vec{r}}{|\vec{r}|^3}$$

This result reveals that $\vec{F}$ is an inverse square vector field and its potential function is $\phi = \dfrac{1}{|\vec{r}|}$

b) $\nabla \bullet \vec{F} = \nabla \bullet \left(-\dfrac{\vec{r}}{|\vec{r}|^3}\right)$

$$= -\nabla \bullet \vec{r}\,\frac{1}{|\vec{r}|^3} - \vec{r} \bullet \nabla \frac{1}{|\vec{r}|^3}$$

$$\nabla \bullet \vec{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \bullet \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1$$

$$\underline{= 3}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{|\vec{r}|^3}\right) = \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$= -3x\left(x^2 + y^2 + z^2\right)^{-5/2}$$

$$= -\frac{3x}{|\vec{r}|^5}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{|\vec{r}|^3}\right) = \frac{\partial}{\partial y}\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$= -3y\left(x^2 + y^2 + z^2\right)^{-5/2}$$

$$= -\frac{3y}{|\vec{r}|^5}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{|\vec{r}|^3}\right) = \frac{\partial}{\partial z}\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$= -3z\left(x^2 + y^2 + z^2\right)^{-5/2}$$

$$= -\frac{3z}{|\vec{r}|^5}$$

$$\nabla \frac{1}{|\vec{r}|^3} = -3\frac{x\hat{i} + y\hat{j} + z\hat{k}}{|\vec{r}|^5}$$

$$\underline{= -3\frac{\vec{r}}{|\vec{r}|^5}}$$

$$\nabla \bullet \vec{F} = \nabla \bullet \left(-\frac{\vec{r}}{|\vec{r}|^3}\right)$$

$$= -\nabla \bullet \vec{r}\,\frac{1}{|\vec{r}|^3} - \vec{r} \bullet \nabla \frac{1}{|\vec{r}|^3}$$

$$= -\nabla \bullet \vec{r}\,\frac{1}{|\vec{r}|^3} - \vec{r} \bullet \left(-3\frac{\vec{r}}{|\vec{r}|^5}\right)$$

$$= -\nabla \bullet \vec{r}\,\frac{1}{|\vec{r}|^3} + 3\frac{|\vec{r}|^2}{|\vec{r}|^5}$$

$$= -\frac{3}{|\vec{r}|^3} + \frac{3}{|\vec{r}|^3}$$

$$\underline{= 0}$$

Properties of a Conservative a Vector Field

Let $\vec{F}$ be a conservative vector field whose components have continuous second partial derivatives on an open connected region D in $\mathbb{R}^3$.

1. There exists a potential function ϕ such that $\vec{F} = \nabla\phi$
2. $\int_C \vec{F} \bullet d\vec{r} = \phi(B) - \phi(A)$ for all points A and B in ***D*** and all piecewise-smooth oriented curves C from A to B.
3. $\int_C \vec{F} \bullet d\vec{r} = 0$ on all simple piecewise-smooth closed oriented curves C in D.
4. $\nabla \times \vec{F} = 0$ at all points of D.

Exercises *Section* 4.5 – Divergence and Curl

(1 – 8) Find the divergence of the following vector fields

1. $\vec{F} = \langle 2x,\ 4y,\ -3z \rangle$

2. $\vec{F} = \langle -2y,\ 3x,\ z \rangle$

3. $\vec{F} = \left\langle x^2yz,\ -xy^2z,\ -xyz^2 \right\rangle$

4. $\vec{F} = \left\langle x^2 - y^2,\ y^2 - z^2,\ z^2 - x^2 \right\rangle$

5. $\vec{F} = \left\langle e^{-x+y},\ e^{-y+z},\ e^{-z+x} \right\rangle$

6. $\vec{F} = \langle yz\cos x,\ \ xz\cos y,\ \ xy\cos z \rangle$

7. $\vec{F} = \langle 12x,\ 4y,\ -3z \rangle$

8. $\vec{F} = \dfrac{\langle x,\ y,\ z \rangle}{1 + x^2 + y^2}$

(9 – 12) Calculate the divergence of the following radial fields. Express the result in terms of the position vector $\vec{r}$ and its length $|\vec{r}|$.

9. $\vec{F} = \dfrac{\langle x,\ y,\ z \rangle}{x^2 + y^2 + z^2} = \dfrac{\vec{r}}{|\vec{r}|^2}$

10. $\vec{F} = \langle x,\ y,\ z \rangle \left(x^2 + y^2 + z^2 \right) = \vec{r}\,|\vec{r}|^2$

11. $\vec{F} = \dfrac{\langle x,\ y,\ z \rangle}{\left(x^2 + y^2 + z^2 \right)^{3/2}} = \dfrac{\vec{r}}{|\vec{r}|^3}$

12. $\vec{F} = \dfrac{\langle x,\ y,\ z \rangle}{\left(x^2 + y^2 + z^2 \right)^2} = \dfrac{\vec{r}}{|\vec{r}|^4}$

(13 – 14) Consider the following vector fields, the circle C, and two points P and Q.

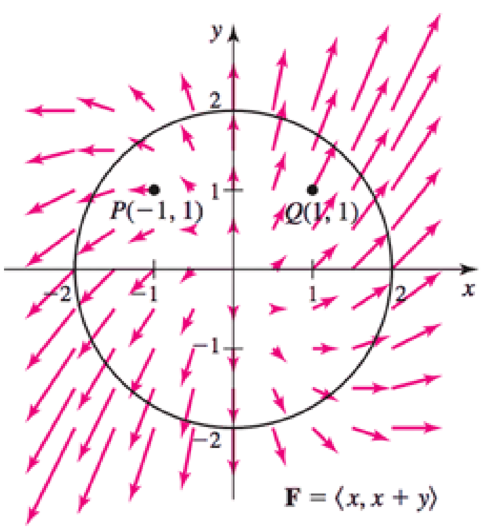
a) Without computing the divergence, does the graph suggest that the divergence is positive or negative at P and Q?

b) Compute the divergence and confirm your conjecture in part (*a*).

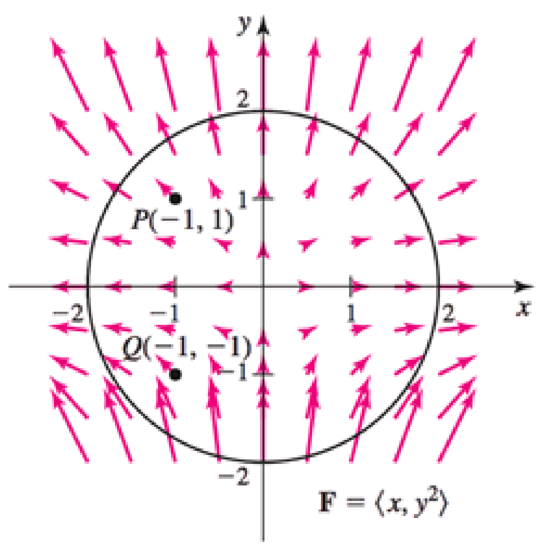
c) On what part of C is the flux outward? Inward?

d) Is the net outward flux across C positive or negative?

13. $\vec{F} = \langle x,\ \ x + y \rangle$



14. $\vec{F} = \left\langle x,\ \ y^2 \right\rangle$



(15–18) Consider the following vector fields, where $\vec{r} = \langle x,\ y,\ z\rangle$

a) Compute the curl field and verify that it has the same direction as the axis of rotation
b) Compute the magnitude of the curl of the field

15. $\vec{F} = \langle 1,\ 0,\ 0\rangle \times \vec{r}$

16. $\vec{F} = \langle 1,\ -1,\ 0\rangle \times \vec{r}$

17. $\vec{F} = \langle 1,\ -1,\ 1\rangle \times \vec{r}$

18. $\vec{F} = \langle 1,\ -2,\ -3\rangle \times \vec{r}$

(19–26) Compute the curl of the following vector fields

19. $\vec{F} = \left\langle x^2 - y^2,\ xy,\ z\right\rangle$

20. $\vec{F} = \left\langle 0,\ z^2 - y^2,\ -yz\right\rangle$

21. $\vec{F} = \left\langle z^2 \sin y,\ \ xz^2 \cos y,\ 2xz \sin y\right\rangle$

22. $\vec{F} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}} = \dfrac{\vec{r}}{|\vec{r}|^3}$

23. $\vec{F} = \vec{r} = \langle x,\ y,\ z\rangle$

24. $\vec{F} = \left\langle 3xz^3 e^{y^2},\ 2xz^3 e^{y^2},\ 3xz^2 e^{y^2}\right\rangle$

25. $\vec{F} = \left\langle x^2 - z^2,\ 1,\ 2xz\right\rangle$

26. $\vec{F} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2 + y^2 + z^2\right)^{1/2}} = \dfrac{\vec{r}}{|\vec{r}|}$

(27–30) Compute the divergence and curl of the following vector fields, state whether the field is *source-free* or *irrotational*.

27. $\vec{F} = \langle yz,\ xz,\ xy\rangle$

28. $\vec{F} = \vec{r}\,|\vec{r}| = \langle x,\ y,\ z\rangle \sqrt{x^2 + y^2 + z^2}$

29. $\vec{F} = \langle \sin xy,\ \ \cos yz,\ \ \sin xz\rangle$

30. $\vec{F} = \left\langle 2xy + z^4,\ x^2,\ 4xz^3\right\rangle$

31. Let $\vec{F} = \langle z,\ x,\ -y\rangle$

a) What are the components of curl $\vec{F}$ in the directions $\vec{n} = \langle 1,\ 0,\ 0\rangle$ and $\vec{n} = \left\langle 0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right\rangle$

b) In what direction is the scalar component of curl $\vec{F}$ a maximum?

32. Let $\vec{F} = \langle z,\ 0,\ -y\rangle$

c) What are the components of curl $\vec{F}$ in the directions $\vec{n} = \langle 1,\ 0,\ 0\rangle$ and $\vec{n} = \langle 1,\ -1,\ 1\rangle$

d) In what direction $\vec{n}$ is $\left(curl\ \vec{F}\right) \bullet \vec{n}$ a maximum?

33. Within the cube $\{(x,\ y,\ z):\ \ -1 \le x \le 1,\ \ -1 \le y \le 1,\ \ -1 \le z \le 1\}$, where does $div\vec{F}$ have the greatest magnitude when $\vec{F} = \left\langle x^2 - y^2,\ \ xy^2 z,\ \ 2xz\right\rangle$

34. Show that the general rotation field $\vec{F} = \vec{a} \times \vec{r}$, where $\vec{a}$ is a nonzero constant vector and $\vec{r} = \langle x,\ y,\ z \rangle$, has zero divergence.

35. Let $\vec{a} = \langle 0,\ 1,\ 0 \rangle$, $\vec{r} = \langle x,\ y,\ z \rangle$ and consider the rotation field $\vec{F} = \vec{a} \times \vec{r}$. Use the right-hand rule for cross product to find the direction of $\vec{F}$ at the points $(0,\ 1,\ 1)$, $(1,\ 1,\ 0)$, $(0,\ 1,\ -1)$, and $(-1,\ 1,\ 0)$

36. Find the exact points on the circle $x^2 + y^2 = 2$ at which the field $\vec{F} = \langle f,\ g \rangle = \langle x^2,\ y \rangle$ switches from pointing inward to outward on the circle, or vice versa.

37. Suppose a solid object in $\mathbb{R}^3$ has a temperature distribution given by $T(x,\ y,\ z)$. The heat flow vector field in the object is $\vec{F} = -k\nabla T$, where the conductivity $k > 0$ is a property of the material. Note that the heat flow vector points in the direction opposite to that of the gradient, which is the direction of greatest temperature decrease. The divergence of the heat flow vector is $\vec{F} = -k\nabla \bullet \nabla T = -k\nabla^2 T$ (the Laplacian of *T*). Compute the heat flow vector field and its divergence for the following temperature distribution.

a) $T(x,\ y,\ z) = 100\, e^{-\sqrt{x^2+y^2+z^2}}$

b) $T(x,\ y,\ z) = 100\, e^{-x^2+y^2+z^2}$

c) $T(x,\ y,\ z) = 100\left(1 + \sqrt{x^2 + y^2 + z^2}\right)$

38. Consider the rotational velocity field $\vec{v} = \langle -2y,\ 2z,\ 0 \rangle$

a) If a paddle is placed in the *xy*-plane with its axis normal to this plane, what is its angular speed?
b) If a paddle is placed in the *xz*-plane with its axis normal to this plane, what is its angular speed?
c) If a paddle is placed in the *yz*-plane with its axis normal to this plane, what is its angular speed?

39. Consider the rotational velocity field $\vec{v} = \langle 0,\ 10z,\ -10y \rangle$. If a paddle wheel is placed in the plane $x + y + z = 1$ with its axis normal to this plane, how fast does the paddle wheel spin (revolutions per unit time)?

40. The potential function for the gravitational force field due to a mass *M* at the origin acting on a mass *m* is $\phi = \dfrac{GMm}{|\vec{r}|}$, where $\vec{r} = \langle x,\ y,\ z \rangle$ is the position vector of the mass m and *G* is the gravitational constant.

a) Compute the gravitational force field $\vec{F} = -\nabla \phi$

b) Show that the field is irrotational; that is $\nabla \times \vec{F} = \vec{0}$

41. The potential function for the force field due to a charge q at the origin is $\phi = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{|\vec{r}|}$, where $\vec{r} = \langle x,\ y,\ z\rangle$ is the position vector of the mass m and G is the gravitational constant.

c) Compute the force field $\vec{F} = -\nabla\phi$

d) Show that the field is irrotational; that is $\nabla \times \vec{F} = \vec{0}$

42. The Navier-Stokes equation is the fundamental equation of fluid dynamics that models the motion of water in everything from bathtubs to oceans. In one of its many forms (incompressible, viscous flow), the equation is

$$\rho\left(\frac{\partial\vec{V}}{\partial t} + \left(\vec{V} \bullet \nabla\right)\vec{V}\right) = -\nabla p + \mu\left(\nabla \bullet \nabla\right)\vec{V}$$

In this notation $\vec{V} = \langle u,\ v,\ w\rangle$ is the three-dimensional velocity field, p is the (scalar) pressure, ρ is the constant density of the fluid, and μ is the constant viscosity. Write out the three component equations of this vector equation.

43. One of Maxwell's equations for electromagnetic waves is $\nabla \times \vec{B} = C\dfrac{\partial\vec{E}}{\partial t}$, where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, and C is a constant.

a) Show that the fields $\vec{E}(z,\ t) = A\sin(kz - \omega t)\,\hat{i}$ $\quad\quad$ $\vec{B}(z,\ t) = A\sin(kz - \omega t)\,\hat{j}$

Satisfy the equation for constants A, k, and ω, provided $\omega = \dfrac{k}{C}$

b) Make a rough sketch showing the directions of $\vec{E}$ and $\vec{B}$

44. Prove that for a real number p, with $\vec{r} = \langle x,\ y,\ z\rangle$, $\nabla \bullet \dfrac{\langle x,\ y,\ z\rangle}{|\vec{r}|^p} = \dfrac{3-p}{|\vec{r}|^p}$

45. Prove that for a real number p, with $\vec{r} = \langle x,\ y,\ z\rangle$, $\nabla\left(\dfrac{1}{|\vec{r}|^p}\right) = \dfrac{-p\vec{r}}{|\vec{r}|^{p+2}}$

46. Prove that for a real number p, with $\vec{r} = \langle x,\ y,\ z\rangle$, $\nabla \bullet \nabla\left(\dfrac{1}{|\vec{r}|^p}\right) = \dfrac{p(p-1)}{|\vec{r}|^{p+2}}$