Section 4.2 – Representing Relations

Representing Relations Using Matrices

A relation between finite sets can be represented using a zero-one matrix. Suppose that R is a relation from $A=\{a_1, a_2, a_3, \ldots, a_m\}$ to $B=\{b_1, b_2, b_3, \ldots, b_n\}$. The relation R can be represented by the matrix $M_a = \{m_{ij}\}$ where

$$m_{ij} = \begin{cases} 1 \ \ if \left(a_i, b_j\right) \in \boldsymbol{R} \\ 0 \ \ if \left(a_i, b_j\right) \notin \boldsymbol{R} \end{cases}$$

Example

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R the relation from A to B containing (a, b) if $a \in A,\ b \in B$, and $a > b$. What is the matrix representing R is $a_1 = 1,\ a_2 = 2,\ a_3 = 3,$ and $b_1 = 1,\ b_2 = 2$?

Solution

$R = \{(2, 1), (3, 1), (3, 2)\}$

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Example

Let $A=\{a_1, a_2, a_3\}$ and $B=\{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} ?$$

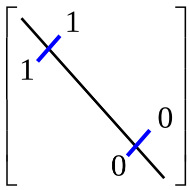
Solution

$$R = \left\{\left(a_1, b_2\right), \left(a_2, b_1\right), \left(a_2, b_3\right), \left(a_2, b_4\right), \left(a_3, b_1\right), \left(a_3, b_3\right), \left(a_3, b_5\right)\right\}$$

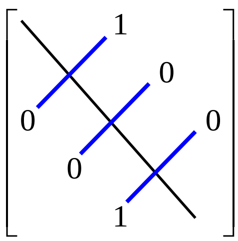
A relation R on A is ***reflexive*** if $(a, a) \in R$ whenever $a \in A$

$$M_R = \left(M_R\right)^t \qquad \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

A relation R on A is ***symmetric***



A relation R on A is ***antisymmetric*** iff $(a,\ b) \in R$ and $(b,\ a) \in R \ \Rightarrow\ a = b$



Example

Suppose that the relation R on the set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution

Because the diagonal elements are equal to 1, R is reflexive.

M_R is symmetric and it is not antisymmetric.

Relations Using Diagraphs

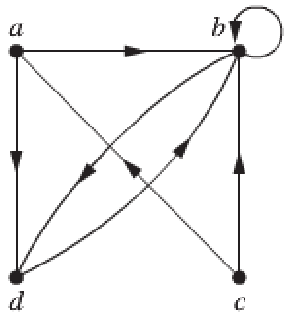
Definition

A directed ***graph***, or ***diagraph***, consists of a set V of ***vertices*** (or *nodes*) together with a set E ordered pairs of elements of V called ***edges*** (or *arcs*). The vertex a is called the ***initial*** vertex of the edge (a, b), and the vertex b is called the ***terminal*** vertex of this edge.

Example

Draw the directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), and (d, b)

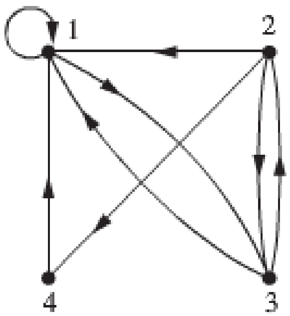
Solution



Example

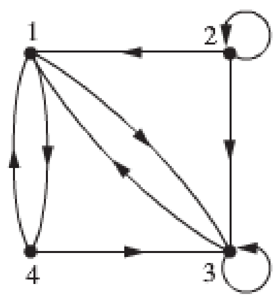
Draw the directed graph of the relation $R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$ on the set $\{1, 2, 3, 4\}$

Solution



Example

What are the ordered pairs in the relation R represented by the directed graph shown below



Solution

$R = \{(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)\}$

Exercises *Section* 4.2 – Representing Relations

1. Represent each of these relations on {1, 2, 3} with a matrix (with the elements of this set listed in increasing order). Then draw the directed graphs representing each relation

a) {(1, 1), (1, 2), (1, 3)}

b) {(1, 2), (2, 1), (2, 2), (3, 3)}

c) {(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}

d) {(1, 3), (3, 1)}

2. Represent each of these relations on {1, 2, 3, 4} with a matrix (with the elements of this set listed in increasing order). Then draw the directed graphs representing each relation

a) {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}

b) {(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)}

c) { (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)}

d) {(2, 4), (3, 1), (3, 2), (3, 4)}

3. List the ordered pairs in the relations on {1, 2, 3} corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order). Then draw the directed graphs representing each relation

$$a)\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad b)\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad c)\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

4. List the ordered pairs in the relations on {1, 2, 3, 4} corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order). Then draw the directed graphs representing each relation

$$a)\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad b)\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad c)\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

5. Let R be the relation represented by the matrix

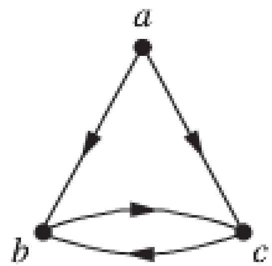
$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Find: $a)\ R^2 \quad b)\ R^3 \quad c)\ R^4$

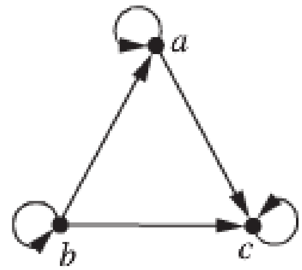
6. Draw the directed graph that represents the relation {(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)}

7. Determine whether the relations represented by the directed graphs are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive

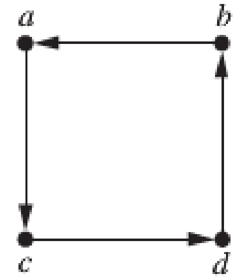
a)



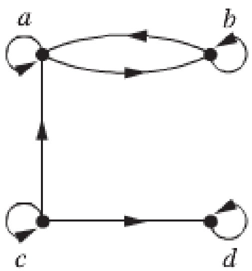
b)



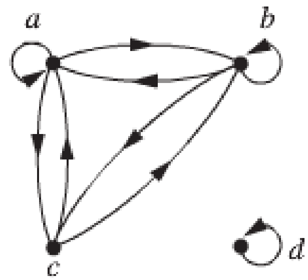
c)



d)



e)



f)