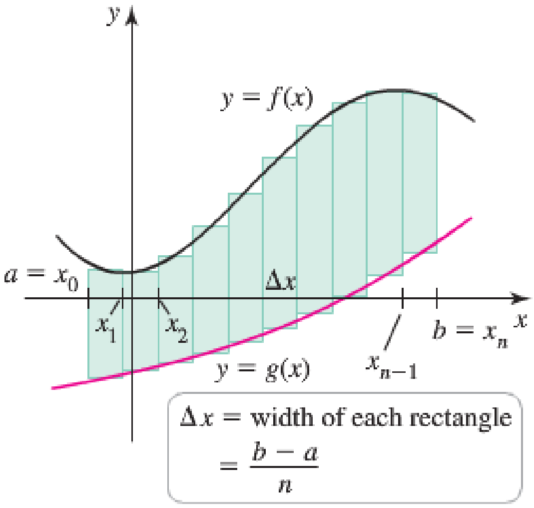
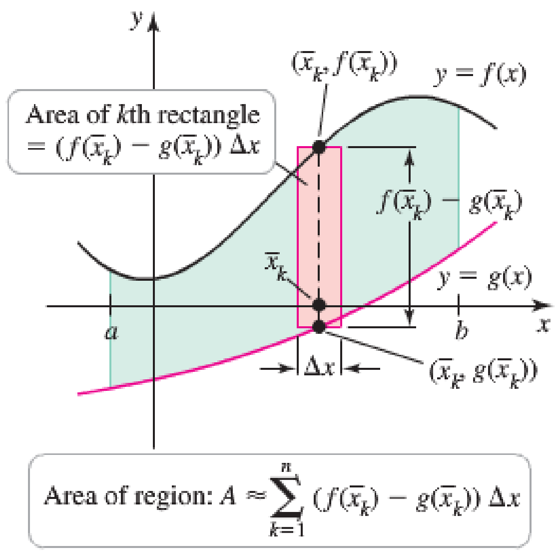
Section 1.2 – Region between Curves

Areas between Curves





Definition

If f and g are continuous with $f(x) \ge g(x)$ throughout $[a, b]$, then the ***area of the region between the curves*** $y = f(x)$ *and* $y = g(x)$ **from *a* to *b*** is:

$$A = \int_a^b \left[f(x) - g(x) \right] dx$$

Example

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution

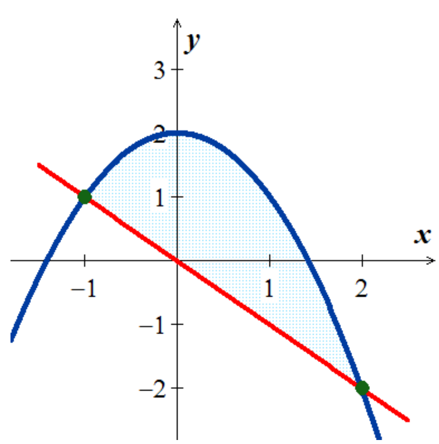
The limits of integrations are found by letting:

$$2 - x^2 = -x \qquad \Rightarrow x^2 - x - 2 = 0 \quad \rightarrow \quad \underline{x = -1,\ 2}$$

$$A = \int_{-1}^{2} \left[f(x) - g(x) \right] dx$$

$$= \int_{-1}^{2} \left[2 - x^2 - (-x) \right] dx$$

$$= \int_{-1}^{2} \left(2 - x^2 + x \right) dx$$



$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^{2}$$

$$= \left(4 - \frac{8}{3} + \frac{4}{2}\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right)$$

$$= \frac{9}{2} \ \ unit^2$$

Example

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below the x-axis and the line $y = x - 2$

Solution

$$\left(y = \sqrt{x}\right) \cap (y = 0) \ \ \rightarrow \ \ (0,\ 0)$$

$$\left(y = \sqrt{x}\right) \cap (y = x - 2) \ \ \rightarrow \ \ \sqrt{x} = x - 2$$

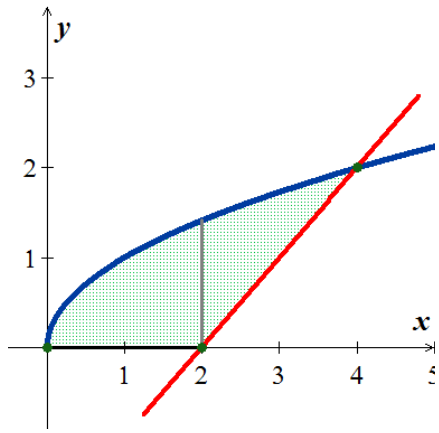
$$\left(\sqrt{x}\right)^2 = (x - 2)^2$$

$$x = x^2 - 4x + 4$$

$$x^2 - 5x + 4 = 0$$

$$\rightarrow \ \ x = \cancel{1},\ 4$$

$$(y = 0) \cap (y = x - 2) \ \ \rightarrow \ \ x = 2$$



$$\text{Total Area } = \int_0^2 \left[\sqrt{x} - 0\right] dx + \int_2^4 \left[\sqrt{x} - (-x + 2)\right] dx$$

$$= \left[\frac{2}{3} x^{3/2}\right]_0^2 + \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

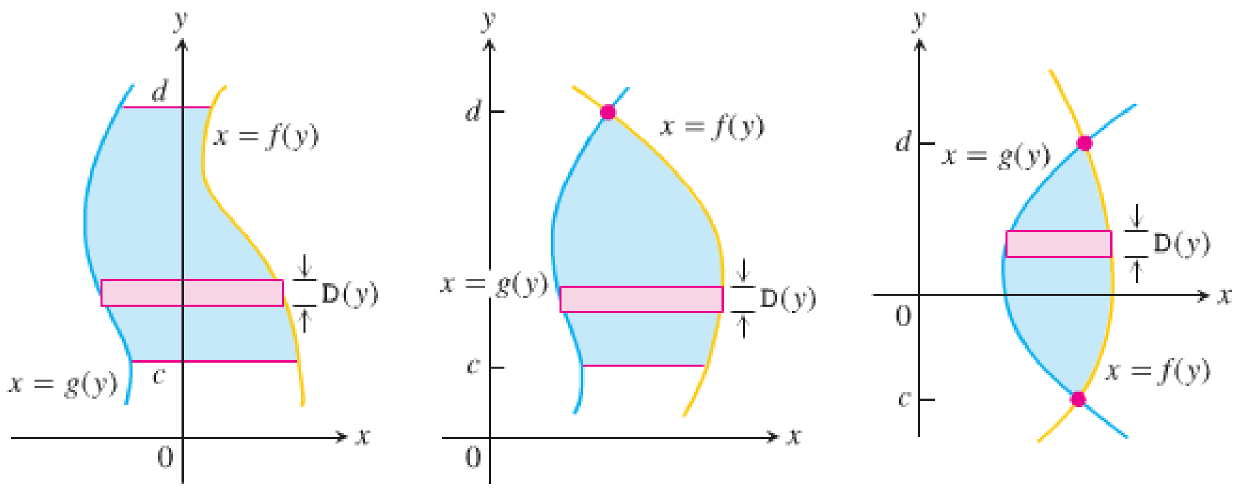
$$= \left[\frac{2}{3}\left(2^{3/2}\right) - 0\right] + \left(\frac{2}{3} 4^{3/2} - \frac{4^2}{2} + 2(4)\right) - \left(\frac{2}{3} 2^{3/2} - \frac{2^2}{2} + 2(2)\right)$$

$$= \frac{2}{3}\left(2^{3/2}\right) + \frac{2}{3} 4^{3/2} - \frac{16}{2} + 8 - \frac{2}{3} 2^{3/2} + \frac{4}{2} - 4$$

$$= \frac{2}{3}(8) - 2$$

$$= \frac{10}{3} \ \ unit^2$$

Integration with Respect to *y*



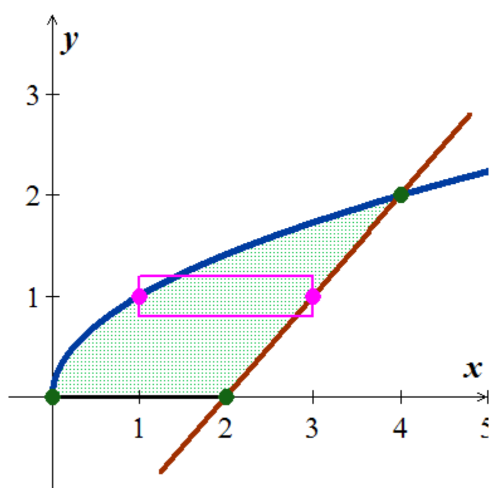
$$A=\int_c^d \left[f(y)-g(y)\right]dy$$

(*From right hand to left* hand)

Example

Find the area of the region by integrating with respect to y, in the first quadrant that is bounded above by $y=\sqrt{x}$ and below the *x*-axis and the line $y=x-2$.

Solution



$y=\sqrt{x} \quad \rightarrow \quad x=y^2$

$y=x-2 \quad \rightarrow \quad x=y+2$

$\left(x=y^2\right)\cap(y=0) \quad \rightarrow \quad (0,\ 0)$

$\left(x=y^2\right)\cap(x=y+2) \quad \rightarrow \quad y^2=y+2$

$y^2-y-2=0 \quad \rightarrow \quad y=-1,\ 2$

$(y=0)\cap(x=y+2) \quad \rightarrow \quad y=0$

$$A=\int_0^2\left[y+2-y^2\right]dy$$

$$=\left[\frac{y^2}{2}+2y-\frac{y^3}{3}\right]_0^2$$

$$=\frac{2^2}{2}+2(2)-\frac{2^3}{3}-0$$

$$\boxed{=\frac{10}{3}\ \ unit^2}$$

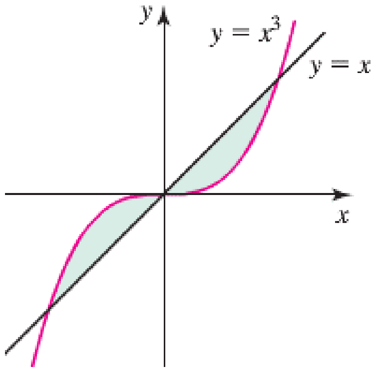
Exercises Section 1.2 – Region between Curves

Find the area of the region bounded by the graphs of

1. $y = 2x - x^2$ and $y = -3$
2. $y = 7 - 2x^2$ and $y = x^2 + 4$
3. $y = x^4 - 4x^2 + 4$ and $y = x^2$
4. $x = 2y^2, \quad x = 0, \quad and \quad y = 3$
5. $x = y^3 - y^2 \quad and \quad x = 2y$
6. $4x^2 + y = 4 \quad and \quad x^4 - y = 1$
7. $y = \sin\frac{\pi x}{2} \quad and \quad y = x$
8. $y = 3 - x^2$ *and* $y = 2x$
9. $y = x^2 - x - 2$ *and* x-axis
10. $y = \sqrt{x}, \quad y = x\sqrt{x}$
11. $y = x^{1/2} \quad and \quad y = x^3$
12. $x + 4y^2 = 4, \quad x + y^4 = 1, \quad x \ge 0$
13. $y = 2\sin x, \quad y = \sin 2x, \quad 0 \le x \le \pi$
14. $y = x^2 + 1$ *and* $y = x$ for $0 \le x \le 2$
15. $y = x^2 - 2x$ *and* $y = x$ on [0, 4]
16. $x = 1, \quad x = 2, \quad y = x^3 + 2, \quad y = 0$
17. $y = x^2 - 18, \quad y = x - 6$
18. $y = -x^2 + 3x + 1, \quad y = -x + 1$
19. $y = x, \quad y = 2 - x, \quad y = 0$
20. $y = \frac{4}{x^2}, \quad y = 0, \quad x = 1, \quad x = 4$
21. $f(x) = x^3 + 2x^2 - 3x, \quad g(x) = x^2 + 3x$
22. $y = \sec^2 x, \quad y = \tan^2 x, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$
23. $f(x) = -x^2 + 1, \quad g(x) = 2x + 4, \quad x = -1, x = 2$
24. $f(x) = \sqrt{x} + 3, \quad g(x) = \frac{1}{2}x + 3$
25. $f(x) = \sqrt[3]{x - 1}, \quad g(x) = x - 1$
26. $f(y) = y^2, \quad g(y) = y + 2$
27. $f(y) = y(2 - y), \quad g(y) = -y$
28. $f(y) = \frac{y}{\sqrt{16 - y^2}}, \quad g(y) = 0, \quad y = 3$
29. $f(y) = y^2 + 1, \quad g(y) = 0, \quad y = -1, \quad y = 2$
30. $f(x) = \frac{10}{x}, \quad x = 0, \quad y = 2, \quad y = 10$
31. $g(x) = \frac{4}{2 - x}, \quad y = 4, \quad x = 0$
32. $f(x) = \cos x, \quad g(x) = 2 - \cos x, \quad 0 \le x \le 2\pi$
33. $f(x) = \sin x, \quad g(x) = \cos 2x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{6}$
34. $f(x) = 2\sin x, \quad g(x) = \tan x, \quad -\frac{\pi}{3} \le x \le \frac{\pi}{3}$
35. $f(x) = \sec\frac{\pi x}{4}\tan\frac{\pi x}{4}, \quad g(x) = \left(\sqrt{2} - 4\right)x + 4, \quad x = 0$
36. $f(x) = xe^{-x^2}, \quad y = 0, \quad 0 \le x \le 1$
37. $f(x) = 2^x, \quad g(x) = \frac{3}{2}x + 1$

Determine the area of the shaded region in the following

38.



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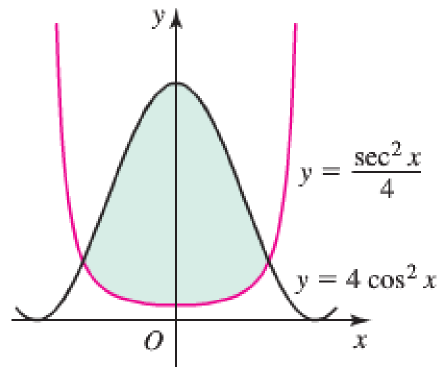

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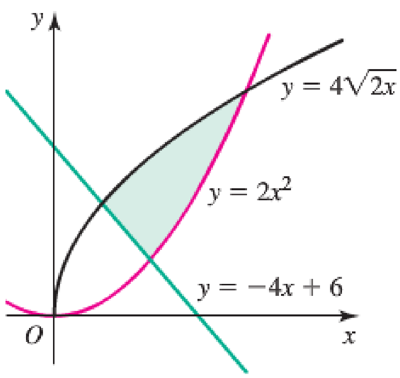
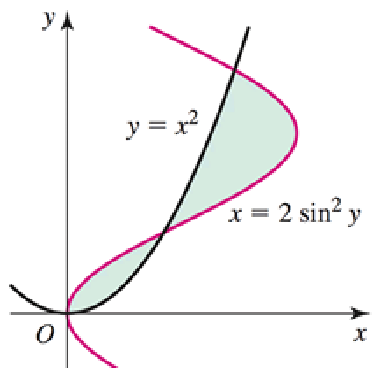
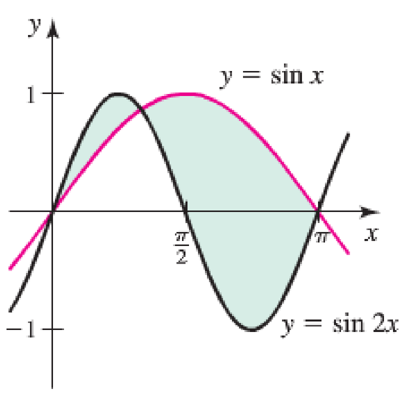


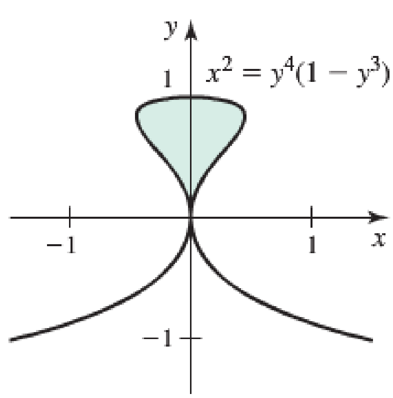
41.



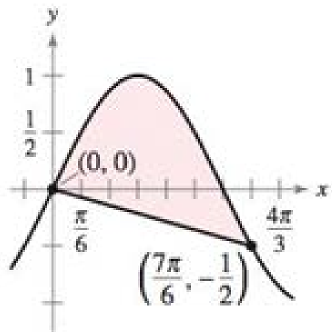
42. Determine the area of the shaded regions between $y = \sin x$ *and* $y = \sin 2x$, *for* $0 \le x \le \pi$



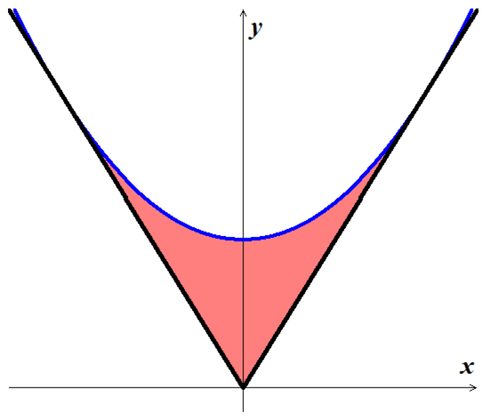
43. Determine the area of the shaded region bounded by the curve $x^2 = y^4\left(1 - y^3\right)$



44. Find the area between the graph of $y = \sin x$ and the line segment joining the points $(0,\ 0)$ and $\left(\frac{7\pi}{6},\ -\frac{1}{2}\right)$.



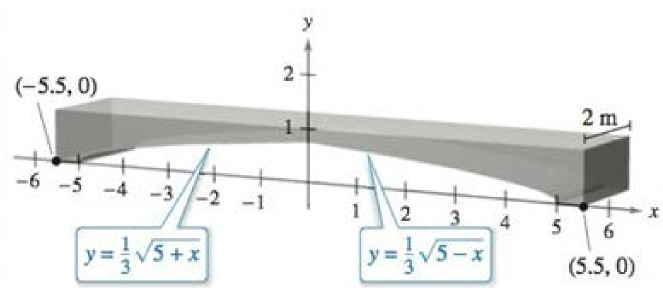
45. The surface of a machine part is the region between the graphs of $y_1 = |x|$ and $y_2 = 0.08x^2 + k$



a) Find k where the parabola is tangent to the graph of y_1

b) Find the area of the surface of the machine part.

46. Concrete sections for a new building have the dimensions (in meters) and shape shown in figure



a) Find the area of the face of the section superimposed on the rectangularcoordinate system.

b) Find the volume of concrete in one of the sections by multiplying the area in part (*a*) by 2 *meters*.

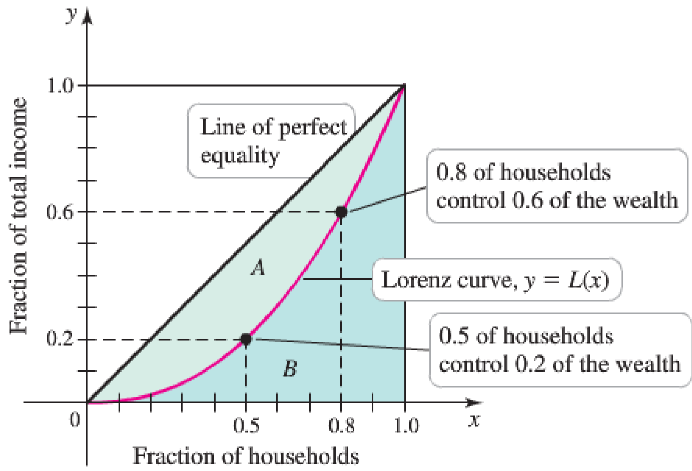
c) One cubic meter of concrete weighs 5,000 *pounds*. Find the weight of the section.

47. A Lorenz curve is given by $y = L(x)$, where $0 \le x \le 1$ represents the lowest fraction of the population of a society in terms of wealth and $0 \le y \le 1$ represents the fraction of the total wealth that is owned by that fraction of the society. For example, the Lorenz curve in the figure shows that $L(0.5) = 0.2$, which means that the lowest 0.5 (50%) of the society owns 0.2 (20%) of the wealth.

a) A Lorenz curve $y = L(x)$ is accompanied by the line $y = x$, called the ***line of perfect equality***. Explain why this line is given the name.

b) Explain why a Lorenz curve satisfies the conditions

$L(0)=0,\ L(1)=1,\ and\ L'(x)\geq 0\quad on\ [0,\ 1]$



c) Graph the Lorenz curves $L(x) = x^p$ corresponding to p = 1.1, 1.5, 2, 3, 4. Which value of p corresponds to the ***most*** equitable distribution of wealth (closest to the line of perfect equality)? Which value of p corresponds to the ***least*** equitable distribution of wealth? Explain.

d) The information in the Lorenz curve is often summarized in a single measure called the ***Gini index***, which is defined as follows. Let A be the area of the region between $y = x$ and $y = L(x)$ and Let B be the area of the region between $y = L(x)$ and the x-axis. Then the Gini index is

$G = \dfrac{A}{A+B}$. Show that $G = 2A = 1-2\int_0^1 L(x)\,dx$.

e) Compute the Gini index for the cases $L(x) = x^p$ and p = 1.1, 1.5, 2, 3, 4.

f) What is the smallest interval [a, b] on which values of the Gini index lie, for $L(x) = x^p$ with $p \geq 1$? Which endpoints of [a, b] correspond to the least and most equitable distribution of wealth?

g) Consider the Lorenz curve described by $L(x) = \dfrac{5x^2}{6} + \dfrac{x}{6}$. Show that it satisfies the conditions $L(0)=0,\ L(1)=1,\ and\ L'(x)\geq 0\quad on\ [0,\ 1]$. Find the Gini index for this function.