

Section 2.3 – Orthogonality

Definition

Two nonzero vectors \vec{u} and \vec{v} in \mathbb{R}^n are said to be **orthogonal** (or **perpendicular**) if their dot product is zero $\vec{u} \cdot \vec{v} = 0$.

We will also agree that the zero vector in \mathbb{R}^n is orthogonal to every vector in \mathbb{R}^n . A nonempty set of vectors in \mathbb{R}^n is called an **orthogonal set** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set of unit vectors is called an **orthonormal set**.

Example

The floor of your room (extended to infinity) is a subspace V . The line where two walls meet is a subspace W (one-dimensional). Those subspaces are orthogonal. Every vector up the meeting line is perpendicular to every vector on the floor. The origin $(0, 0, 0)$ is in the corner.

Example

Show that $\vec{u} = (-2, 3, 1, 4)$ and $\vec{v} = (1, 2, 0, -1)$ are orthogonal in \mathbb{R}^4

Solution

$$\begin{aligned}\vec{u} \cdot \vec{v} &= (-2)(1) + (3)(2) + (1)(0) + (4)(-1) \\ &= -2 + 6 + 0 - 4 \\ &= 0\end{aligned}$$

These vectors are orthogonal in \mathbb{R}^4

Standard Unit Vectors

$$\hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{k} = 0$$

Proof

$$\begin{aligned}\hat{i} \cdot \hat{j} &= (1, 0, 0) \cdot (0, 1, 0) \\ &= 0\end{aligned}$$

Normal

To specify slope and inclination is to use a nonzero vector \vec{n} , called a **normal**, that is orthogonal to the line or plane.

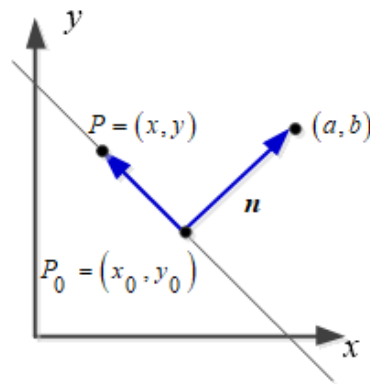
The line passes through a point $P_0(x_0, y_0)$ that has a normal $\vec{n} = (a, b)$

The plane through $P_0(x_0, y_0, z_0)$ that has a normal $\vec{n} = (a, b, c)$.

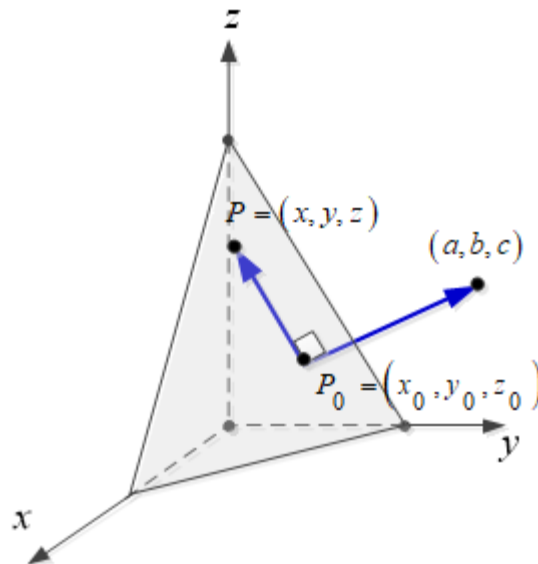
Both the line and the plane are represented by the vector equation

$$\vec{n} \cdot \overrightarrow{P_0P} = 0$$

The line equation: $a(x - x_0) + b(y - y_0) = 0$



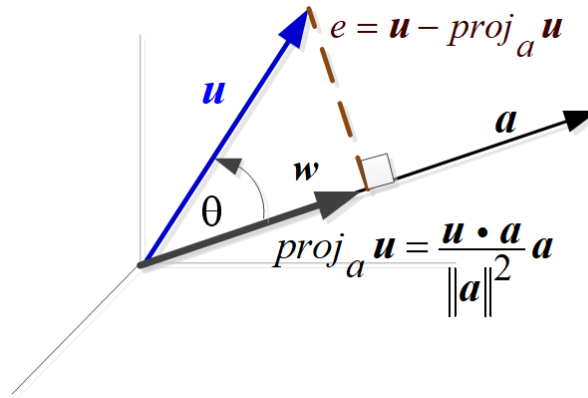
The plane equation: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$



Projections

Theorem Projection onto a line

If \vec{u} and \vec{a} are vectors in \mathbb{R}^n , and if $\vec{a} \neq 0$, then \vec{u} can be expressed in exactly one way in the form $\vec{u} = \vec{w} + \vec{e}$, where \vec{w} is a scalar multiple of \vec{a} and \vec{e} is orthogonal to \vec{a} .



The vector \vec{w} is called the **orthogonal projection** of \vec{u} on \vec{a} or sometimes **component** of \vec{u} along \vec{a} . The vector \vec{e} is called the vector **component** of \vec{u} **orthogonal** to \vec{a} (error vector and should be perpendicular to \vec{a})

$$proj_{\vec{a}} \vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} = \vec{p} \quad (\text{vector component of } \vec{u} \text{ along } \vec{a})$$

$$\vec{u} - proj_{\vec{a}} \vec{u} = \vec{u} - \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} \quad (\text{vector component of } \vec{u} \text{ orthogonal to } \vec{a})$$

The length is $\|proj_{\vec{a}} \vec{u}\| = \|\vec{u}\| \cos \theta$

$$\|proj_{\vec{a}} \vec{u}\| = \frac{|\vec{u} \cdot \vec{a}|}{\|\vec{a}\|}$$

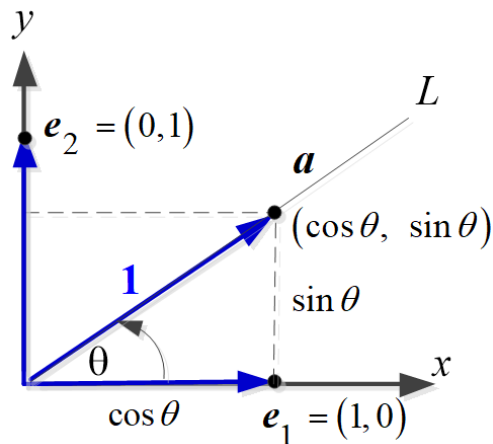
Special case: If $\vec{u} = \vec{a}$ then $\frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} = 1$. The projection of \vec{a} onto \vec{a} is itself.

Special case: If \vec{u} is perpendicular to \vec{a} then $\vec{u} \cdot \vec{a} = 0$. The projection is $\vec{p} = \vec{0}$.

Example

Find the orthogonal projections of the vectors $\hat{e}_1 = (1, 0)$ and $\hat{e}_2 = (0, 1)$ on the line L that makes an angle θ with the positive x -axis in \mathbb{R}^2

Solution



Let $\vec{a} = (\cos \theta, \sin \theta)$ be the unit vector along the line L .

$$\begin{aligned}\|\vec{a}\| &= \sqrt{\cos^2 \theta + \sin^2 \theta} \\ &= 1\end{aligned}$$

$$\begin{aligned}\hat{e}_1 \cdot \vec{a} &= (1, 0) \cdot (\cos \theta, \sin \theta) \\ &= (1)\cos \theta + (0)\sin \theta \\ &= \cos \theta\end{aligned}$$

$$\begin{aligned}proj_{\vec{a}} \hat{e}_1 &= \frac{\hat{e}_1 \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} \\ &= \frac{\cos \theta}{1} (\cos \theta, \sin \theta) \\ &= (\cos^2 \theta, \cos \theta \sin \theta)\end{aligned}$$

$$\begin{aligned}proj_{\vec{a}} \hat{e}_2 &= \frac{\hat{e}_2 \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} \\ &= \frac{(0, 1) \cdot (\cos \theta, \sin \theta)}{1} (\cos \theta, \sin \theta) \\ &= \sin \theta (\cos \theta, \sin \theta) \\ &= (\sin \theta \cos \theta, \sin^2 \theta)\end{aligned}$$

Example

Let $\vec{u} = (2, -1, 3)$ and $\vec{a} = (4, -1, 2)$. Find the vector component of \vec{u} along \vec{a} and the vector component of \vec{u} orthogonal to \vec{a} .

Solution

$$\begin{aligned} \text{proj}_{\vec{a}} \vec{u} &= \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} \\ &= \frac{(2, -1, 3) \cdot (4, -1, 2)}{\left(\sqrt{4^2 + (-1)^2 + 2^2}\right)^2} (4, -1, 2) \\ &= \frac{8+1+6}{21} (4, -1, 2) \\ &= \frac{15}{21} (4, -1, 2) \\ &= \frac{5}{7} (4, -1, 2) \\ &= \left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7} \right) \end{aligned}$$

The vector component of \vec{u} orthogonal to \vec{a} is

$$\begin{aligned} \vec{u} - \text{proj}_{\vec{a}} \vec{u} &= (2, -1, 3) - \left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7} \right) \\ &= \left(-\frac{6}{7}, -\frac{2}{7}, \frac{11}{7} \right) \end{aligned}$$

Theorem of Pythagoras in \mathbb{R}^n

If \vec{u} and \vec{v} are orthogonal vectors in \mathbb{R}^n with the Euclidean inner product, then

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

Proof

Since \vec{u} and \vec{v} are orthogonal, then $\vec{u} \cdot \vec{v} = 0$

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \\ &= \|\vec{u}\|^2 + 2(\vec{u} \cdot \vec{v}) + \|\vec{v}\|^2 \\ &= \|\vec{u}\|^2 + \|\vec{v}\|^2 \end{aligned}$$

Distance

Theorem

In \mathbb{R}^2 the distance D between the point $P_0 = (x_0, y_0)$ and the line $ax + by + c = 0$ is

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

In \mathbb{R}^3 the distance D between the point $P_0 = (x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Exercises Section 2.3 – Orthogonality

1. Determine whether \vec{u} and \vec{v} are orthogonal

a) $\vec{u} = (-6, -2), \vec{v} = (5, -7)$

c) $\vec{u} = (1, -5, 4), \vec{v} = (3, 3, 3)$

b) $\vec{u} = (6, 1, 4), \vec{v} = (2, 0, -3)$

d) $\vec{u} = (-2, 2, 3), \vec{v} = (1, 7, -4)$

2. Determine whether the vectors form an orthogonal set

a) $\vec{v}_1 = (2, 3), \vec{v}_2 = (3, 2)$

b) $\vec{v}_1 = (1, -2), \vec{v}_2 = (-2, 1)$

c) $\vec{u} = (-4, 6, -10, 1), \vec{v} = (2, 1, -2, 9)$

d) $\vec{u} = (a, b), \vec{v} = (-b, a)$

e) $\vec{v}_1 = (-2, 1, 1), \vec{v}_2 = (1, 0, 2), \vec{v}_3 = (-2, -5, 1)$

f) $\vec{v}_1 = (1, 0, 1), \vec{v}_2 = (1, 1, 1), \vec{v}_3 = (-1, 0, 1)$

g) $\vec{v}_1 = (2, -2, 1), \vec{v}_2 = (2, 1, -2), \vec{v}_3 = (1, 2, 2)$

3. Find a unit vector that is orthogonal to both $\vec{u} = (1, 0, 1)$ and $\vec{v} = (0, 1, 1)$

4. a) Show that $\vec{v} = (a, b)$ and $\vec{w} = (-b, a)$ are orthogonal vectors.

b) Use the result to find two vectors that are orthogonal to $\vec{v} = (2, -3)$.

c) Find two unit vectors that are orthogonal to $(-3, 4)$

5. Find the vector component of \vec{u} along \vec{a} and the vector component of \vec{u} orthogonal to \vec{a} .

a) $\vec{u} = (6, 2), \vec{a} = (3, -9)$

d) $\vec{u} = (1, 1, 1), \vec{a} = (0, 2, -1)$

b) $\vec{u} = (3, 1, -7), \vec{a} = (1, 0, 5)$

e) $\vec{u} = (2, 1, 1, 2), \vec{a} = (4, -4, 2, -2)$

c) $\vec{u} = (1, 0, 0), \vec{a} = (4, 3, 8)$

f) $\vec{u} = (5, 0, -3, 7), \vec{a} = (2, 1, -1, -1)$

6. Project the vector \vec{v} onto the line through \vec{a} , check that $\vec{e} = \vec{u} - \text{proj}_{\vec{a}} \vec{u}$ is perpendicular to \vec{a} :

a) $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\vec{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

b) $\vec{v} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ and $\vec{a} = \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}$

c) $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

7. Find the projection matrix $\text{proj}_{\vec{a}} \vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$ onto the line through $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

(8 – 9) Draw the projection of \vec{b} onto \vec{a} and also compute it from $proj_{\vec{a}} \vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$

8. $\vec{b} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$ and $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

9. $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

10. Show that if \vec{v} is orthogonal to both \vec{w}_1 and \vec{w}_2 , then \vec{v} is orthogonal to $k_1 \vec{w}_1 + k_2 \vec{w}_2$ for all scalars k_1 and k_2 .

11. a) Project the vector $\vec{v} = (3, 4, 4)$ onto the line through $\vec{a} = (2, 2, 1)$ and then onto the plane that also contains $\vec{a}^* = (1, 0, 0)$.

b) Check that the first error vector $\vec{v} - \vec{p}$ is perpendicular to \vec{a} , and the second error vector $\vec{v} - \vec{p}^*$ is also perpendicular to \vec{a}^* .

12. Compute the projection matrices $\vec{a}\vec{a}^T / \vec{a}^T \vec{a}$ onto the lines through $\vec{a}_1 = (-1, 2, 2)$ and $\vec{a}_2 = (2, 2, -1)$. Multiply those projection matrices and explain why their product $P_1 P_2$ is what it is. Project $\vec{v} = (1, 0, 0)$ onto the lines \vec{a}_1 , \vec{a}_2 , and also onto $\vec{a}_3 = (2, -1, 2)$. Add up the three projections $p_1 + p_2 + p_3$.

13. If $P^2 = P$ show that $(I - P)^2 = I - P$. When P projects onto the column space of A , $I - P$ projects onto the _____.

14. What linear combination of $(1, 2, -1)$ and $(1, 0, 1)$ is closest to $\vec{v} = (2, 1, 1)$?

15. Show that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$ if and only if $\|\vec{u}\| = \|\vec{v}\|$

16. Given $\vec{u} = (3, -1, 2)$ $\vec{v} = (4, -1, 5)$ and $\vec{w} = (8, -7, -6)$

a) Find $3\vec{v} - 4(5\vec{u} - 6\vec{w})$

b) Find $\vec{u} \cdot \vec{v}$ and then the angle θ between \vec{u} and \vec{v} .

17. Given: $\vec{u} = (3, 1, 3)$ $\vec{v} = (4, 1, -2)$

a) Compute the projection \vec{w} of \vec{u} on \vec{v}

b) Find $\vec{p} = \vec{u} - \vec{v}$ and show that \vec{p} is perpendicular to \vec{v} .

18. a) Show that $\vec{v} = (a, b)$ and $\vec{w} = (-b, a)$ are orthogonal vectors

b) Use the result in part (a) to find two vectors that are orthogonal to $\vec{v} = (2, -3)$

c) Find two unit vectors that are orthogonal to $(-3, 4)$

19. Show that $A(3, 0, 2)$, $B(4, 3, 0)$, and $C(8, 1, -1)$ are vertices of a right triangle. At which vertex is the right angle?
20. Establish the identity: $\vec{u} \cdot \vec{v} = \frac{1}{4}\|\vec{u} + \vec{v}\|^2 - \frac{1}{4}\|\vec{u} - \vec{v}\|^2$
21. Find the Euclidean inner product $\vec{u} \cdot \vec{v}$: $\vec{u} = (-1, 1, 0, 4, -3)$ $\vec{v} = (-2, -2, 0, 2, -1)$
22. Find the Euclidean distance between \vec{u} and \vec{v} : $\vec{u} = (3, -3, -2, 0, -3)$ $\vec{v} = (-4, 1, -1, 5, 0)$

(Exercises 22 – 26) Find

- $\vec{v} \cdot \vec{u}$, $|\vec{v}|$, $|\vec{u}|$
 - The cosine of the angle between \vec{v} and \vec{u}
 - The scalar component of \vec{u} in the direction of \vec{v}
 - The vector $\text{proj}_{\vec{v}} \vec{u}$
23. $\vec{v} = 2\hat{i} - 4\hat{j} + \sqrt{5}\hat{k}$, $\vec{u} = -2\hat{i} + 4\hat{j} - \sqrt{5}\hat{k}$
24. $\vec{v} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{k}$, $\vec{u} = 5\hat{i} + 12\hat{j}$
25. $\vec{v} = 2\hat{i} + 10\hat{j} - 11\hat{k}$, $\vec{u} = 2\hat{i} + 2\hat{j} + \hat{k}$
26. $\vec{v} = 5\hat{i} + \hat{j}$, $\vec{u} = 2\hat{i} + \sqrt{17}\hat{j}$
27. $\vec{v} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\right)$, $\vec{u} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}}\right)$
28. Suppose Ted weighs 180 *lb.* and he is sitting on an inclined plane that drops 3 *units* for every 4 horizontal units. The gravitational force vector is $\vec{F}_g = \begin{pmatrix} 0 \\ -180 \end{pmatrix}$.
- Find the force pushing Ted down the slope.
 - Find the force acting to hold Ted against the slope
29. Prove that if two vectors \vec{u} and \vec{v} in \mathbb{R}^2 are orthogonal to nonzero vector \vec{w} in \mathbb{R}^2 , then \vec{u} and \vec{v} are scalar multiples of each other.