Section 2.10 – Applications

Example

Use Laplace transform to find the solution to the initial value problem

$$y'' - y = e^{2t} \qquad y(0) = 0 \quad \textit{and} \quad y'(0) = 1$$

Solution

$$\mathcal{L}\{y'' - y\} = \mathcal{L}\left\{e^{2t}\right\}$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = \mathcal{L}\left\{e^{2t}\right\}$$

$$\left(s^2 Y(s) - sy(0) - y'(0)\right) - Y(s) = \frac{1}{s-2}$$

$$\left(s^2 - 1\right)Y(s) - 1 = \frac{1}{s-2}$$

$$\left(s^2 - 1\right)Y(s) = \frac{1}{s-2} + 1$$

$$(s-1)(s+1)Y(s) = \frac{s-1}{s-2}$$

$$\begin{aligned} Y(s) &= \frac{1}{(s+1)(s-2)} \\ &= \frac{A}{(s+1)} + \frac{B}{(s-2)} \\ &= \frac{(A+B)s + B - 2A}{(s+1)(s-2)} \end{aligned} \qquad \begin{cases} A + B = 0 \\ -2A + B = 1 \end{cases} \Rightarrow A = -\frac{1}{3};\ B = \frac{1}{3}$$

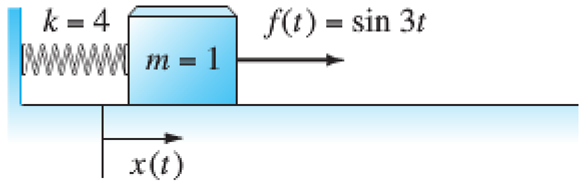
$$Y(s) = \frac{1}{3}\left[-\frac{1}{(s+1)} + \frac{1}{(s-2)}\right]$$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] \\ &= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s-2)} - \frac{1}{(s+1)}\right\} \\ &= \frac{1}{3}\left(e^{2t} - e^{-t}\right) \end{aligned}$$

Example

Solve the initial value problem $x'' + 4x = \sin 3t; \quad x(0) = x'(0) = 0.$

Such problem arises in the motion of a mass-and-spring system with external force as shown below.



Solution

$$\mathcal{L}\{x'' + 4x\} = \mathcal{L}\{\sin 3t\}$$

$$\mathcal{L}\{x''\} - \mathcal{L}\{4x\} = \mathcal{L}\{\sin 3t\}$$

$$\left(s^2 X(s) - sx(0) - x'(0)\right) + 4X(x) = \frac{3}{s^2 + 9}$$

$$\left(s^2 + 4\right) X(x) = \frac{3}{s^2 + 9}$$

$$X(x) = \frac{3}{\left(s^2 + 4\right)\left(s^2 + 9\right)} = \frac{As + B}{s^2 + 4} + \frac{Cs + D}{s^2 + 9}$$

$$3 = (As + B)\left(s^2 + 9\right) + (Cs + D)\left(s^2 + 4\right)$$

$$3 = (A + C)s^3 + (B + D)s^2 + (9A + 4C)s + 9B + 4D$$

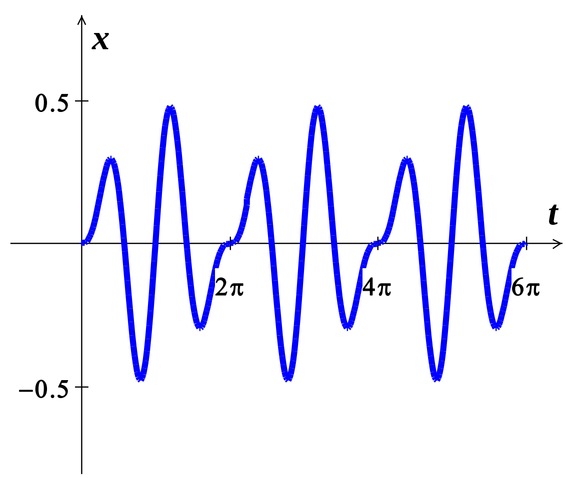
$$\begin{cases} A + C = 0 \\ B + D = 0 \\ 9A + 4C = 0 \\ 9B + 4D = 3 \end{cases} \Rightarrow \begin{cases} A = C = 0 \\ B = \frac{3}{5};\ D = -\frac{3}{5} \end{cases}$$

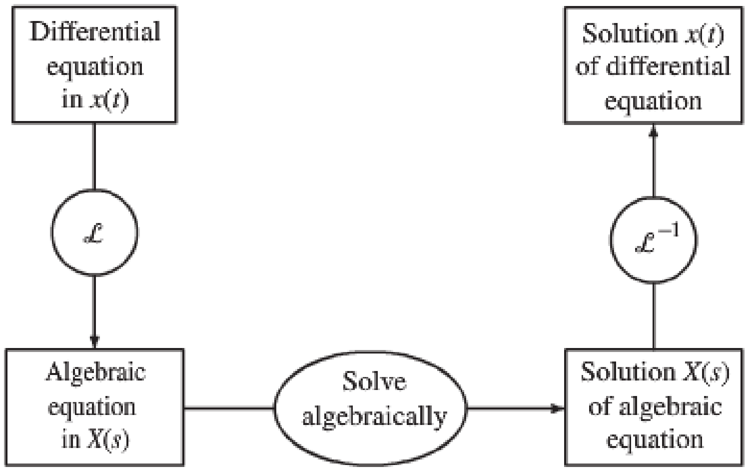
$$X(x) = \frac{3}{5}\frac{1}{s^2 + 4} - \frac{3}{5}\frac{1}{s^2 + 9} = \frac{3}{10}\frac{2}{s^2 + 4} - \frac{1}{5}\frac{3}{s^2 + 9}$$

$$x(t) = \mathcal{L}^{-1}[X(s)]$$

$$= \frac{3}{10}\mathcal{L}^{-1}\left\{\frac{2}{s^2 + 4}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{3}{s^2 + 9}\right\}$$

$$\boxed{= \frac{3}{10}\sin 2t - \frac{1}{5}\sin 3t}$$





Linear Systems

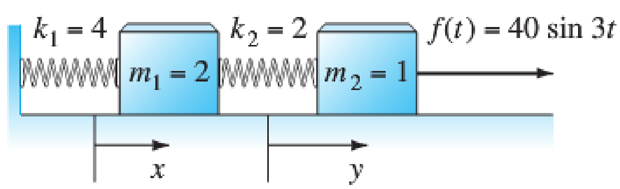
Laplace transforms are used frequently in engineering problems to solve linear system in which the coefficients are all constants. When initial conditions are specified, the Laplace transform reduces such a linear system of differential equations to a linear system of algebraic equations in which the unknowns are the transforms of the solution functions. The technique for a system is essentially the same as for a single linear differential equation with constant coefficients.

Example

Solve the system $\begin{cases} 2x'' = -6x + 2y \\ y'' = 2x - 2y + 40\sin 3t \end{cases}$

Subject to the initial conditions $x(0) = x'(0) = y(0) = y'(0) = 0$

Thus the force $f(t) = 40\sin 3t$ is applied to the second mass as shown below, beginning at time $t = 0$ when the system is at rest in its equilibrium position.



Solution

Let $X(s) = \mathcal{L}\{x(t)\}$ *and* $Y(s) = \mathcal{L}\{y(t)\}$

$$\begin{cases} \mathcal{L}\{2x''\} = \mathcal{L}\{-6x + 2y\} \\ \mathcal{L}\{y''\} = \mathcal{L}\{2x - 2y + 40\sin 3t\} \end{cases}$$

$$\begin{cases} 2\left(s^2 X(s) - sx(0) - x'(0)\right) = -6X(x) + 2Y(s) \\ s^2 Y(s) - sy(0) - y'(0) = 2X(s) - 2Y(s) + 40\dfrac{3}{s^2+9} \end{cases} \to \begin{cases} \left(s^2+3\right)X(s) - Y(s) = 0 \\ -2X(s) + \left(s^2+2\right)Y(s) = \dfrac{120}{s^2+9} \end{cases}$$

Solve the system using Cramer's rule

$$D=\begin{vmatrix} s^2+3 & -1 \\ -2 & s^2+2 \end{vmatrix}=s^4+5s^2+4=\left(s^2+1\right)\left(s^2+4\right)$$

$$D_X=\begin{vmatrix} 0 & -1 \\ \frac{120}{s^2+9} & s^2+2 \end{vmatrix}=\frac{120}{s^2+9} \quad\Rightarrow\quad X(s)=\frac{120}{\left(s^2+1\right)\left(s^2+4\right)\left(s^2+9\right)}$$

$$D_Y=\begin{vmatrix} s^2+3 & 0 \\ -2 & \frac{120}{s^2+9} \end{vmatrix}=\frac{120\left(s^2+3\right)}{s^2+9} \quad\Rightarrow\quad Y(s)=\frac{120\left(s^2+3\right)}{\left(s^2+1\right)\left(s^2+4\right)\left(s^2+9\right)}$$

$$X(s)=\frac{120}{\left(s^2+1\right)\left(s^2+4\right)\left(s^2+9\right)}=\frac{A}{s^2+1}+\frac{B}{s^2+4}+\frac{C}{s^2+9}$$

(*Nothing that the denominator factors are linear in* s^2)

$$120=A\left(s^2+4\right)\left(s^2+9\right)+B\left(s^2+1\right)\left(s^2+9\right)+C\left(s^2+1\right)\left(s^2+4\right)$$

$$=(A+B+C)s^4+(13A+10B+5C)s^2+36A+9B+4C$$

$$\begin{cases} A+B+C=0 \\ 13A+10B+5C=0 \\ 36A+9B+4C=120 \end{cases} \quad\Rightarrow\quad \begin{matrix} A=5 \\ B=-8 \\ C=3 \end{matrix}$$

$$X(s)=\frac{5}{s^2+1}-4\frac{2}{s^2+4}+\frac{3}{s^2+9} \quad\Rightarrow\quad \underline{x(t)=5\sin t-4\sin 2t+\sin 3t}$$

$$Y(s)=\frac{120s^2+360}{\left(s^2+9\right)\left(s^2+1\right)\left(s^2+4\right)}=\frac{A}{s^2+1}+\frac{B}{s^2+4}+\frac{C}{s^2+9}$$

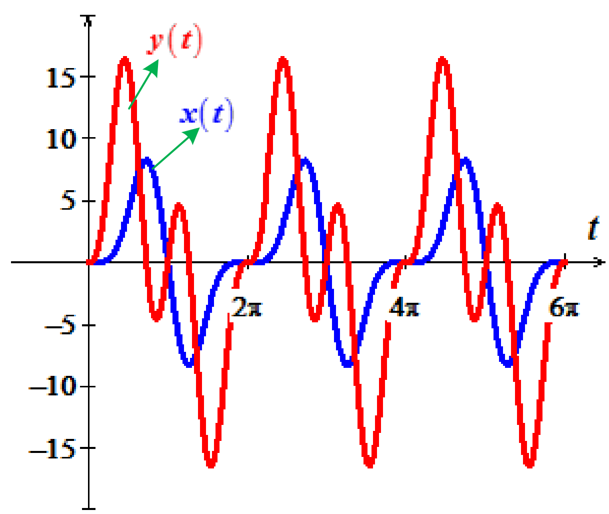
$$120s^2+360=A\left(s^2+4\right)\left(s^2+9\right)+B\left(s^2+1\right)\left(s^2+9\right)+C\left(s^2+1\right)\left(s^2+4\right)$$

$$=(A+B+C)s^4+(13A+10B+5C)s^2+36A+9B+4C$$

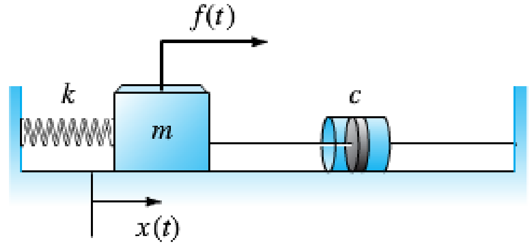
$$\begin{cases} A+B+C=0 \\ 13A+10B+5C=120 \\ 36A+9B+4C=360 \end{cases} \quad\Rightarrow\quad \begin{matrix} A=10 \\ B=8 \\ C=-18 \end{matrix}$$

$$Y(s)=10\frac{1}{s^2+1}+4\frac{1}{s^2+4}-6\frac{3}{s^2+9}$$

$$\Rightarrow\quad \underline{y(t)=10\sin t+4\sin 2t-6\sin 3t}$$



The transform Perspective



Consider the general constant-coefficient second-order equation as the equation of motion

$$mx'' + cx' + kx = f(t)$$

Of the familiar mass-spring-dashpot system

Then the transformed equation is

$$m\left[s^2 X(s) - sx(0) - x'(0)\right] + c\left[sX(s) - x(0)\right] + kX(s) = F(s)$$

This is an algebraic equation, in the unknown $X(s)$. This is the source of the power of the Laplace transform method:

- Linear ***differential*** equations are transformed into readily solved ***algebraic*** equations.

Example

Show that $\mathcal{L}\left\{te^{at}\right\} = \dfrac{1}{(s-a)^2}$

Solution

If $f(t) = te^{at}$, then $f(0) = 0$ and $f'(t) = e^{at} + te^{at}$

$$\mathcal{L}\left\{e^{at} + te^{at}\right\} = \mathcal{L}\left\{f'(t)\right\}$$

$$= s\mathcal{L}\left\{f(t)\right\}$$

$$= s\mathcal{L}\left\{te^{at}\right\}$$

It follows from the linearity of the transform that

$$\mathcal{L}\left\{e^{at}\right\} + a\mathcal{L}\left\{te^{at}\right\} = s\mathcal{L}\left\{te^{at}\right\}$$

$$\mathcal{L}\left\{e^{at}\right\} = (s-a)\mathcal{L}\left\{te^{at}\right\}$$

$$\mathcal{L}\left\{te^{at}\right\} = \frac{\mathcal{L}\left\{e^{at}\right\}}{s-a} = \frac{1}{(s-a)^2}$$

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}$$

Example

Find $\mathcal{L}\{t\sin at\}$

Solution

If $f(t)=t\sin at$, then $f(0)=0$

Then $f'(t)=\sin at+at\cos at \quad\Rightarrow\quad f'(0)=0$

$$f''(t)=a\cos at+a\cos at-a^2t\sin at=2a\cos at-a^2t\sin at$$

$$\mathcal{L}\{f''(t)\}=s^2\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f''(t)\}=2a\mathcal{L}\{\cos at\}-a^2\mathcal{L}\{t\sin at\}=s^2\mathcal{L}\{f(t)\}$$

$$2a\mathcal{L}\{\cos at\}=\left(s^2+a^2\right)\mathcal{L}\{t\sin at\}$$

$$\mathcal{L}\{t\sin at\}=\frac{2a}{s^2+a^2}\mathcal{L}\{\cos at\} \qquad \mathcal{L}\{\cos at\}=\frac{s}{s^2+a^2}$$

$$=\frac{2a}{s^2+a^2}\frac{s}{s^2+a^2}$$

$$=\frac{2as}{\left(s^2+a^2\right)^2}$$

Theorem

If $F(s)=\mathcal{L}\{f(t)\}$ exists for $s>c$, then $\mathcal{L}\left\{e^{at}f(t)\right\}$ exists for $s>a+c$, and

$$\mathcal{L}\left\{e^{at}f(t)\right\}=F(s-a) \quad\Leftrightarrow\quad \mathcal{L}^{-1}\{F(s-a)\}=e^{at}f(t)$$

Thus, the translation $s\to s-a$ in the transform corresponds to multiplication of the original function of t by e^{at}

$f(t)$	$F(s)$	
$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$	$(s>a)$
$e^{at}\cos\omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$	$(s>a)$
$e^{at}\sin\omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$	$(s>a)$

Resonance and Repeated Quadratic Factors

The following two inverse Laplace transforms are useful in inverting partial fractions that correspond to the case of repeated quadratic factors:

$$\mathcal{L}^{-1}\left\{\frac{s}{\left(s^2+\omega^2\right)^2}\right\}=\frac{1}{2\omega}t\sin\omega t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{\left(s^2+\omega^2\right)^2}\right\}=\frac{1}{2\omega^3}(\sin\omega t-\omega t\cos\omega t)$$

Example

Use the Laplace transforms to solve the initial value problem

$$x''+\omega_0^2x=F_0\sin\omega t;\quad x(0)=x'(0)=0$$

That determines the undamped forced oscillations of a mass on a spring

Solution

$$\mathcal{L}\left\{x''+\omega_0^2x\right\}=\left\{F_0\sin\omega t\right\}$$

$$s^2X(s)+\omega_0^2X(s)=\frac{F_0\omega}{s^2+\omega^2}$$

If $\omega\neq\omega_0$

$$X(s)=\frac{F_0\omega}{\left(s^2+\omega_0^2\right)\left(s^2+\omega^2\right)}=F_0\omega\left(\frac{As+B}{s^2+\omega_0^2}+\frac{Cs+D}{s^2+\omega^2}\right)$$

$$(As+B)\left(s^2+\omega^2\right)+(Cs+D)\left(s^2+\omega_0^2\right)=1$$

$$As^3+A\omega^2s+Bs^2+B\omega^2+Cs^3+C\omega_0^2s+Ds^2+D\omega_0^2=1$$

$$\begin{cases}A+C=0\\ B+D=0\\ \omega^2A+\omega_0^2C=0\\ \omega^2B+\omega_0^2D=1\end{cases}\rightarrow\begin{cases}A=C=0\\ B=\dfrac{1}{\omega^2-\omega_0^2}=-D\end{cases}$$

$$X(s)=\frac{F_0\omega}{\omega^2-\omega_0^2}\left(\frac{1}{s^2+\omega_0^2}-\frac{1}{s^2+\omega^2}\right)$$

$$\Rightarrow\ \boxed{x(t)=\frac{F_0\omega}{\omega^2-\omega_0^2}\left(\frac{1}{\omega_0}\sin\omega_0 t-\frac{1}{\omega}\sin\omega t\right)}$$

$$If\ \ \omega=\omega_0\ \ \Rightarrow X(s)=\frac{F_0\omega}{\left(s^2+\omega^2\right)^2}$$

$$\mathcal{L}^{-1}\frac{1}{\left(s^2+k^2\right)^2}=\frac{1}{2k^3}\left(\sin kt-kt\cos kt\right)$$

$$\Rightarrow\ \boxed{x(t)=\frac{F_0\omega}{2\omega_0^2}\left(\sin\omega_0 t-\omega_0 t\cos\omega_0 t\right)}$$

Exercises   *Section* 2.10 – Applications

Solve using the Laplace transform:

1. $y' + y = te^{t}, \quad y(0) = -2$

2. $y' - y = 2\cos 5t, \quad y(0) = 0$

3. $y' - y = 1 + te^{t}, \quad y(0) = 0$

4. $y' + 3y = e^{2t}, \quad y(0) = -1$

5. $y' + 4y = \cos t, \quad y(0) = 0$

6. $y' + 4y = e^{-4t}, \quad y(0) = 2$

7. $y' - 4y = t^2 e^{-2t}, \quad y(0) = 1$

8. $y' + 9y = e^{-t}, \quad y(0) = 0$

9. $y' + 16y = \sin 3t, \quad y(0) = 1$

10. $y'' - y = e^{2t}; \quad y(0) = 0, \quad y'(0) = 1$

11. $y'' - y = 2t; \quad y(0) = 0, \quad y'(0) = -1$

12. $y'' - y = t - 2\,; \quad y(2) = 3, \quad y'(2) = 0$

13. $y'' + y = t\,; \quad y(\pi) = y'(\pi) = 0$

14. $y'' - 2y' + 5y = -8e^{\pi - t}\,; \quad y(\pi) = 2, \ y'(\pi) = 12$

15. $y'' + y = t^2 + 2\,; \quad y(0) = 1, \quad y'(0) = -1$

16. $y'' + y = \sqrt{2}\sin\sqrt{2}t; \quad y(0) = 10, \quad y'(0) = 0$

17. $y'' + y = -2\cos 2t\,; \quad y(0) = 1, \quad y'(0) = -1$

18. $y'' - y' = e^{t}\cos t\,; \quad y(0) = 0, \quad y'(0) = 0$

19. $y'' + y' - y = t^3\,; \quad y(0) = 1, \quad y'(0) = 0$

20. $y'' - y' - 2y = 4t^2, \quad y(0) = 1, \quad y'(0) = 4$

21. $y'' - y' - 2y = e^{2t}\,; \quad y(0) = -1, \quad y'(0) = 0$

22. $y'' - y' - 2y = 0, \quad y(0) = -2, \quad y'(0) = 5$

23. $y'' - y' - 2y = -8\cos t - 2\sin t\,; \quad y\left(\frac{\pi}{2}\right) = 1, \quad y'\left(\frac{\pi}{2}\right) = 0$

24. $x'' - x' - 6x = 0\,; \quad x(0) = 2, \quad x'(0) = -1$

25. $y'' + 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 1$

26. $y'' + 2y' + y = t\,, \quad y(0) = -3, \quad y(1) = -1$

27. $y'' - 2y' - y = e^{2t} - e^{t}\,; \quad y(0) = 1, \quad y'(0) = 3$

28. $y'' - 2y' + y = 6t - 2\,; \quad y(-1) = 3, \quad y'(-1) = 7$

29. $y'' - 2y' + y = \cos t - \sin t\,; \quad y(0) = 1, \quad y'(0) = 3$

30. $y'' - 2y' + 5y = 0, \quad y(0) = 2, \quad y'(0) = 4$

31. $y'' - 2y' + 5y = 1 + t, \quad y(0) = 0, \quad y'(0) = 0$

32. $y'' + 3y' = -3t\,; \quad y(0) = -1, \quad y'(0) = 1$

33. $y'' + 3y' = t^3$; $y(0) = 0, \quad y'(0) = 0$

34. $y'' - 3y' + 2y = e^{-t}$, $y(1) = 0, \quad y'(1) = 0$

35. $y'' - 3y' + 2y = \cos t$; $y(0) = 0, \quad y'(0) = -1$

36. $y'' - 4y' + 4y = t^3 e^{2t}$, $y(0) = 0, \quad y'(0) = 0$

37. $y'' - 4y' + 4y = t^3$, $y(0) = 1, \quad y'(0) = 0$

38. $y'' - 4y = e^{-t}$; $y(0) = -1, \quad y'(0) = 0$

39. $y'' - 4y' = 6e^{3t} - 3e^{-t}$; $y(0) = 1, \quad y'(0) = -1$

40. $x'' + 4x' + 4x = t^2$; $x(0) = x'(0) = 0$

41. $y'' + 4y = 4t^2 - 4t + 10$; $y(0) = 0, \quad y'(0) = 3$

42. $y'' - 4y = 4t - 8e^{-2t}$; $y(0) = 0, \quad y'(0) = 5$

43. $y'' + 4y' = \cos(t-3) + 4t$, $y(3) = 0, \quad y'(3) = 7$

44. $y'' + 4y' + 8y = \sin t$, $y(0) = 1, \quad y'(0) = 0$

45. $y'' + 5y' - y = e^t - 1$; $y(0) = 1, \quad y'(0) = 1$

46. $y'' + 5y' - 6y = 21e^{t-1}$ $y(1) = -1, \quad y'(1) = 9$

47. $y'' + 5y' + 4y = 0$; $y(0) = 1, \quad y'(0) = 0$

48. $y'' + 6y = t^2 - 1$; $y(0) = 0, \quad y'(0) = -1$

49. $y'' - 6y' + 9y = t$, $y(0) = 0, \quad y'(0) = 1$

50. $y'' - 6y' + 13y = 0$, $y(0) = 0, \quad y'(0) = -3$

51. $y'' - 6y' + 15y = 2\sin 3t$, $y(0) = -1, \quad y'(0) = -4$

52. $y'' + 6y' + 9y = 0$, $y(0) = -1, \quad y'(0) = 6$

53. $y'' + 6y' + 5y = 12e^t$, $y(0) = -1, \quad y'(0) = 7$

54. $y'' - 7y' + 10y = 9\cos t + 7\sin t$; $y(0) = 5, \quad y'(0) = -4$

55. $y'' + 8y' + 25y = 0$, $y(\pi) = 0, \quad y'(\pi) = 6$

56. $y'' + 9y = 2\sin 2t$; $y(0) = 0, \quad y'(0) = -1$

57. $y'' + 9y = 3\sin 2t$; $y(0) = 0, \quad y'(0) = -1$

58. $y'' - 10y' + 9y = 5t$; $y(0) = -1, \quad y'(0) = 2$

59. $y'' + 16y = 2\sin 4t$; $y(0) = -\frac{1}{2}, \quad y'(0) = 0$

60. $2y'' + 3y' - 2y = te^{-2t}$, $y(0) = 0, \quad y'(0) = -2$

61. $2y'' + 20y' + 51y = 0$, $y(0) = 2, \quad y'(0) = 0$

62. $y^{(3)} + y' = e^t$, $y(0) = y'(0) = y''(0) = 0$

63. $2y^{(3)}+3y''-3y'-2y=e^{-t}$; $y(0)=0,\quad y'(0)=0,\quad y''(0)=1$

64. $y^{(3)}+2y''-y'-2y=\sin 3t$; $y(0)=0,\quad y'(0)=0,\quad y''(0)=1$

65. $y^{(3)}-y''+y'-y=0$; $y(0)=1,\quad y'(0)=1,\quad y''(0)=3$

66. $y^{(3)}+4y''+y'-6y=-12$; $y(0)=1,\quad y'(0)=4,\quad y''(0)=-2$

67. $y^{(3)}+3y''+3y'+y=0$; $y(0)=-4,\quad y'(0)=4,\quad y''(0)=-2$

68. $y^{(3)}-3y''+3y'-y=t^2e^t$, $y(0)=1,\quad y'(0)=2,\quad y''(0)=3$

69. $y^{(3)}+y''+3y'-5y=16e^{-t}$; $y(0)=0,\quad y'(0)=2,\quad y''(0)=-4$

70. $y'''+4y''+5y'+2y=10\cos t$, $y(0)=y'(0)=0,\quad y''(0)=3$

71. $y^{(4)}+2y''+y=4te^t$; $y(0)=y'(0)=y''(0)=y^{(3)}(0)=0$

72. $y^{(4)}-y=0$; $y(0)=1,\quad y'(0)=0,\quad y''(0)=0,\quad y^{(3)}(0)=0$

73. $y^{(4)}-4y=0$; $y(0)=1,\quad y'(0)=0,\quad y''(0)=-2,\quad y^{(3)}(0)=0$

74. $y^{(4)}-4y^{(3)}+6y''-4y'+y=0$; $y(0)=0,\quad y'(0)=1,\quad y''(0)=0,\quad y^{(3)}(0)=1$

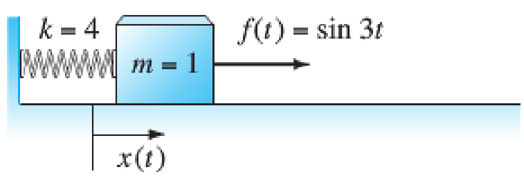
75. Given: $y''-4y'+3y=0,\quad y(0)=1\quad y'(0)=-1$

a) Show that the general solution is: $y(t)=C_1e^{3t}+C_2e^t$ and find C_1 *and* C_2

b) Use Laplace transform to solve the system

76. Solve the initial value problem $x''+4x=\sin 3t;\quad x(0)=x'(0)=0.$

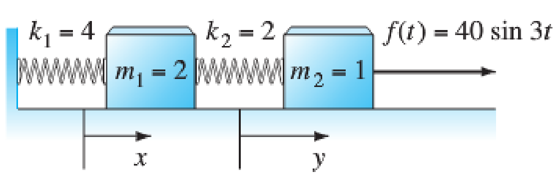
Such problem arises in the motion of a mass-and-spring system with external force as shown below.



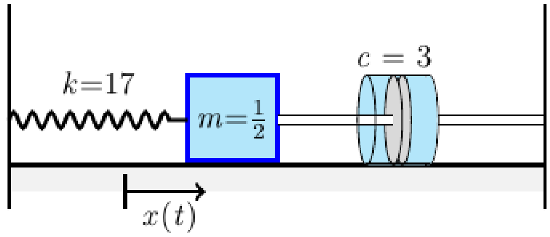
77. Solve the system
$$\begin{cases} 2x''=-6x+2y \\ y''=2x-2y+40\sin 3t \end{cases}$$

Subject to the initial conditions $x(0)=x'(0)=y(0)=y'(0)=0$

Thus the force $f(t)=40\sin 3t$ is applied to the second mass as shown below, beginning at time $t=0$ when the system is at rest in its equilibrium position.



78. Consider a mass-spring system with $m = \frac{1}{2}, \quad k = 17, \quad and\ c = 3$.



Let $x(t)$ be the displacement of the mass *m* from its equilibrium position. If the mass is set in motion with $x(0) = 3\ and\ x'(0) = 1$, find $x(t)$ for the resulting damped free oscillations.

79. A $4-lb$ weight stretches a spring 2 *feet*. The weight is released from rest 18 *inches* above the equilibrium position, and the resulting motion takes place in a medium offering a damping force numerically equal to $\frac{7}{8}$ times the instantaneous velocity. Use the Laplace transform to find the equation of motion $x(t)$.

80. Consider a mass-spring-dashpot system with $m = \frac{1}{2}, \quad k = 17, \quad c = 3, \quad and\ f(t) = 15\sin 2t$ with initial conditions $x(0) = x'(0) = 0$. Let $x(t)$ be the displacement of the mass *m* from its equilibrium position. Find the resulting transient motion and steady periodic motion of the mass.

81. A 8-*kg* mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by 3 *N-sec/m* and the spring constant is 40 *N/m*. If the mass is driven by an external force equal to $f(t) = 2\sin 2t \cos 2t\ N$. Find the solution.

82. A 2-*kg* mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $c = 8$ *kg/sec* and the spring constant is $k = 80$ *N/m*. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20e^{-t}\ N$. (t in *seconds*). Find the equation of motion.

83. A 10-*kg* mass is attached to a spring having a spring constant of 140 *N/m*. The mass is started in motion initially from the equilibrium position with an initial velocity 1 *m/sec* in the upward direction and with an applied external force $F(t) = 5\sin t$. If the force due to air resistance is $-90y'\ N$. Find the equation motion of the mass.

84. A 128-*lb* weight is attached to a spring having a spring constant of 64 *lb/ft*. The weight is started in motion initially by displacing it 6 *in* above the equilibrium position with no initial velocity and with an applied external force $F(t) = 8\sin 4t$. Assume no air resistance. Find the equation motion of the mass.

85. Find the motion of a damped mass-and-spring system with $m=1$, $c=2$, and $k=26$ under the influence of an external force $F(t)=82\cos 4t$ with $x(0)=6$ and $x'(0)=0$.

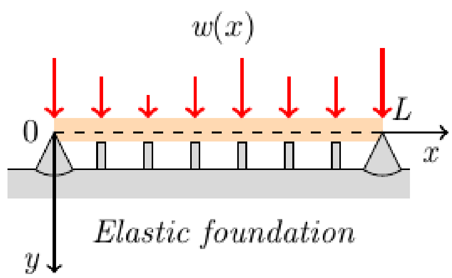
86. A spring with a mass of 2-*kg* has natural length $0.5\ m$. A force of $25.6\ N$ is required to maintain it stretched to a length of $0.7\ m$. The spring is immersed in a fluid with damping constant $c=40$. If the spring is started from the equilibrium position and is given a push to start it with initial velocity $0.6\ m/s$. Find the position of the mass at any time *t*.

87. A spring with a mass of 3-*kg* is held stretched $0.6\ m$ beyond its natural length by a force of $20\ N$. If the spring begins at its equilibrium and with initial velocity $1.2\ m/s$. Find the position of the mass.

88. A spring with a mass of 2-*kg* is held stretched $0.5\ m$, has damping constant 14, and a force of $6\ N$. If the spring is stretched 1 *m* beyond at its equilibrium and with no initial velocity. Find the position of the mass at any time *t*.

89. When a uniform beam is supported by an elastic foundation, the differential equation for its deflection $y(x)$ is

$$EI\frac{d^4y}{dx^4}+ky=w(x)$$



Where *k* is the modulus of the foundation and $-ky$ is the restoring force of the foundation that acts in the direction opposite to that of the load $w(x)$. For algebraic convenience suppose that the differential equation is written as

$$\frac{d^4y}{dx^4}+4a^4y=\frac{w(x)}{EI}$$

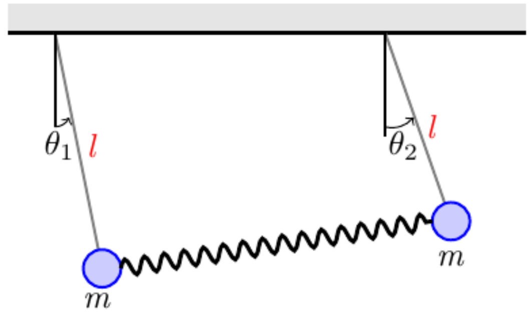
Where $a=\left(\frac{k}{4EI}\right)^{1/4}$. Assume $L=\pi$ and $a=1$. Find the deflection $y(x)$ of a beam that is supported on an elastic foundation when

a) The beam is simply supported at both ends and a constant load w_0 is uniformly distributed along its length,

b) The bean is embedded at both ends and $w(x)$ is concentrated load w_0 applied at $x=\frac{\pi}{2}$

90. Suppose two identical pendulums are coupled by means of a spring with constant k. when the displacement angles $\theta_1(t)$ and $\theta_2(t)$ are small, the system of linear differential equations describing the motion is

$$\begin{cases} \theta_1'' + \frac{g}{l}\theta_1 = -\frac{k}{m}(\theta_1 - \theta_2) \\ \theta_2'' + \frac{g}{l}\theta_2 = \frac{k}{m}(\theta_1 - \theta_2) \end{cases}$$



a) Use Laplace transform to solve the system when

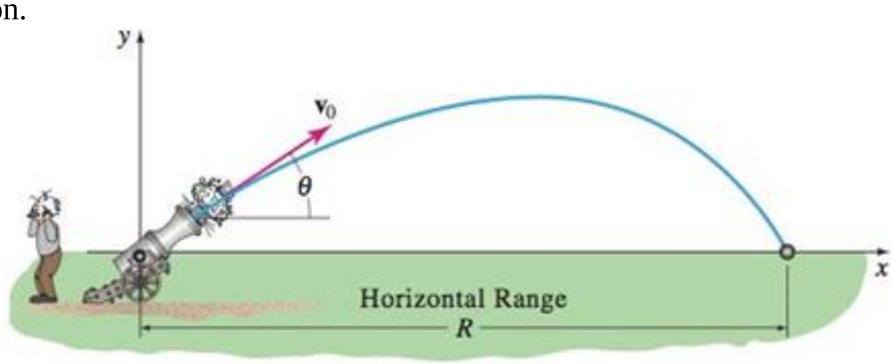
$$\theta_1'(0)=0 \quad \theta_1(0)=\theta_0 \quad \theta_2'(0)=0 \quad \theta_2(0)=\psi_0$$

Where θ_0 and ψ_0 constants. Let $\omega^2 = \frac{g}{l}, \ K = \frac{k}{m}$

b) Use the solution in part (*a*) to discuss the motion of the coupled pendulums in the special case when the initial conditions are $\theta_1'(0)=0, \ \ \theta_1(0)=\theta_0, \ \ \theta_2'(0)=\theta_0, \ \ \theta_2(0)=0$

c) Use the solution in part (*a*) to discuss the motion of the coupled pendulums in the special case when the initial conditions are $\theta_1'(0)=0, \ \ \theta_1(0)=\theta_0, \ \ \theta_2'(0)=-\theta_0, \ \ \theta_2(0)=0$

91. A projectile, such as the canon ball, has weight $w = mg$ and initial velocity $\boldsymbol{v}_0$ that is tangent to its path of motion.



If air resistance and all other forces except its weight are ignored, that motion of the projectile is describe by the system of linear differential equations:

$$\begin{cases} m\frac{d^2x}{dt^2} = 0 \\ m\frac{d^2y}{dt^2} = -mg \end{cases}$$

a) Use Laplace transform to solve the system when

$$x(0)=0 \quad x'(0)=v_0\cos\theta \quad y(0)=0 \quad y'(0)=v_0\sin\theta$$

Where $v_0 = |v|$ is constant and θ is the constant angle of elevation.

The solutions $x(t)$ and $y(t)$ are parametric equations of the trajectory of the projectile.

b) Use $x(t)$ in part (*a*) to eliminate the parameter *t* in $y(t)$. Use the resulting equation for *y* to show that the horizontal range *R* of the projectile is given by

$$R = \frac{v_0^2}{g}\sin 2\theta$$

c) From the formula in part (*b*), we see that *R* is a maximum when $\sin 2\theta = 1$ or when $\theta = \frac{\pi}{4}$. Show that the same range – less than the maximum– can be obtained by firing the gun at either of two complementary angles θ and $\frac{\pi}{2} - \theta$. The only difference is that the smaller angle results in a low trajectory whereas the larger angle fives a high trajectory.

d) Suppose $g = 32$ *ft/s*^{2}, $\theta = 30°$, and $v_0 = 300$ *ft/s*. Use part (*b*) to find the horizontal range of the projectile.

e) Find the time when the projectile hits the ground.

f) Use the parametric equations $x(t)$ and $y(t)$ in part (*a*) along with the numerical data in part (*d*) to plot the ballistic curve of the projectile.

g) Repeat with $\theta = 52°$ and $v_0 = 300$ *ft/s*.

h) Superimpose both curves part (*f*) & (*g*) on the same coordinate system.

92. Derive the system of differential equations describing the straight-line vertical motion of the coupled springs. Use the Laplace transform to solve the system when

$k_1 = 1,\ k_2 = 1,\ k_3 = 1,\ m_1 = 1,\ m_2 = 1$ and

$x_1(0) = 0,\ x_1'(0) = -1,\ x_2(0) = 0,\ x_2'(0) = 1$