Exercise 1

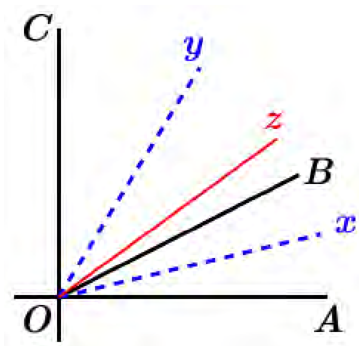
Ox and *Oy* are bisector to 2 adjacent acute angles, $\widehat{AOB}$ and $\widehat{BOC}$ where the difference is $36°$ and $\widehat{AOC}=90°$. *Oz* is the bisector of the angle $\widehat{xOy}$. Determine the angle $\widehat{BOz}$

Solution

$$\widehat{BOC}-\widehat{AOB}=36°$$
$$\underline{\widehat{BOC}+\widehat{AOB}=90°}$$
$$2\ \widehat{BOC}=126°$$

$$\widehat{BOC}=63°$$

$$\widehat{AOB}=27°$$



$$\widehat{xOB}=\frac{1}{2}\widehat{AOB}$$
$$=\frac{27}{2}^{\circ}$$

$$\widehat{BOy}=\frac{63}{2}^{\circ}$$

$$\widehat{xOy}=\widehat{xOB}+\widehat{BOy}$$
$$=\frac{1}{2}(63°+27°)$$
$$=45°$$

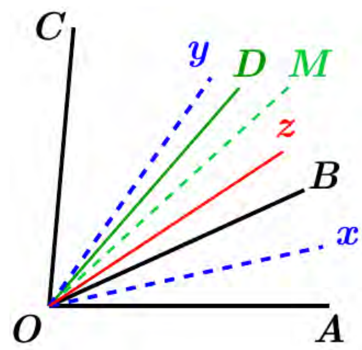
$$\widehat{xOz}=\frac{45}{2}^{\circ}$$

$$\widehat{BOZ}=\widehat{xOz}-\widehat{xOB}$$
$$=\frac{1}{2}(45°-27°)$$
$$=9°$$

Exercise 2

Ox and Oy are bisector to 2 adjacent acute angles, $\widehat{AOB}$ and $\widehat{BOC}$ where the difference is $36°$. Oz is the bisector of the angle $\widehat{xOy}$. Determine the angle $\widehat{BOz}$

Solution



Ox is the bisector $\widehat{AOB}$ $\quad (1)$

OB is the bisector $\widehat{AOD}$ $\quad (2)$

OM is the bisector $\widehat{AOC}$ $\quad (3)$

Oz is the bisector $\widehat{xOy}$ $\quad (4)$

Oy is the bisector $\widehat{BOC}$ $\quad (5)$

$$\widehat{BOC} - \widehat{AOB} = 36°$$
$$\widehat{BOC} - \widehat{BOD} = 36°$$
$$\widehat{DOC} = 36°$$

$$(3) \rightarrow \quad \widehat{AOM} = \frac{1}{2}\widehat{AOC}$$
$$= \frac{1}{2}\left(2\widehat{AOB} + \widehat{DOC}\right)$$
$$= \frac{1}{2}\left(2\widehat{AOB} + 36°\right)$$
$$= \widehat{AOB} + 18°$$

$$\widehat{BOM} = \widehat{AOM} - \widehat{AOB}$$
$$= \widehat{AOB} + 18° - \widehat{AOB}$$
$$= 18°$$

$$(1) \rightarrow \quad \widehat{BOx} = \frac{1}{2}\widehat{AOB}$$

$$(4) \rightarrow \quad \widehat{BOy} = \frac{1}{2}\widehat{BOC}$$

$$(1) + (4) \rightarrow \quad \widehat{xOy} = \frac{1}{2}\widehat{AOC}$$

$$(3) \rightarrow \quad \widehat{AOM} = \frac{1}{2}\widehat{AOC} = \widehat{xOy}$$

$$\widehat{BOz} = \widehat{xOz} - \widehat{xOB}$$
$$= \frac{1}{2}\left(\widehat{xOy} - \widehat{AOB}\right)$$
$$= \frac{1}{2}\left(\widehat{AOM} - \widehat{AOB}\right)$$
$$= \frac{1}{2}\widehat{BOM}$$
$$= 9°$$

Exercise 3

Four consecutive half-lines (segments): *OA*, *OB*, *OC*, and *OD* formed angles such as

$$\widehat{DOA} = \widehat{COB} = 2\widehat{AOB} \qquad \text{and} \qquad \widehat{COD} = 3\widehat{AOB}$$

Calculate the angles to demonstrate that the bisectors of $\widehat{AOB}$ and $\widehat{COD}$ are in a straight line.

Solution

$$\widehat{AOB} + \widehat{BOC} + \widehat{COD} + \widehat{DOA} = 360°$$

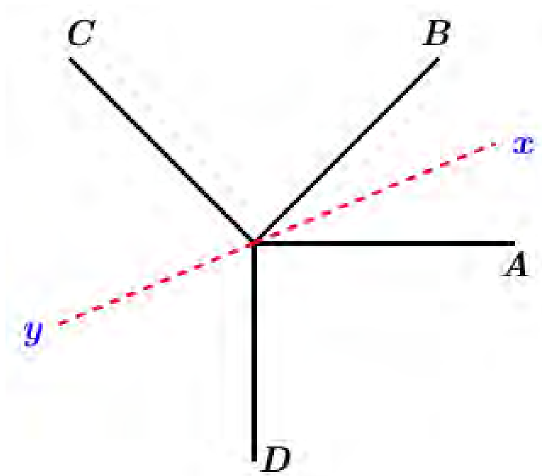
$$\widehat{AOB} + 2\widehat{AOB} + 3\widehat{AOB} + 2\widehat{AOB} = 360°$$

$$8\widehat{AOB} = 360°$$

$$\underline{\widehat{AOB} = 45°}\,\big|$$

$$\underline{\widehat{DOA} = \widehat{COB} = 90°}\,\big|$$

$$\underline{\widehat{COD} = 135°}\,\big|$$



Let:

Ox is the bisector $\widehat{AOB}$

Oy is the bisector $\widehat{COD}$

$$\widehat{xOy} = \widehat{xOB} + \widehat{BOC} + \widehat{COy}$$

$$= \frac{1}{2}\widehat{AOB} + 90° + \frac{1}{2}\widehat{COD}$$

$$= \frac{1}{2}(45° + 135°) + 90°$$

$$\underline{= 180°}\,\big|$$

Therefore; the bisectors of $\widehat{AOB}$ and $\widehat{COD}$ are in a straight line

Exercise 4

The segments *OA* and *OB* formed with *OX* the angles α and β.

a) Demonstrate that the bisector *OC* of the angle $\widehat{AOB}$ made with *OX* an angle $\frac{\alpha+\beta}{2}$.

b) Examine the cases where

i. $\alpha+\beta=90°$

ii. $\alpha+\beta=180°$

Solution

Given:

$\widehat{XOA}=\alpha$ & $\widehat{XOB}=\beta$

$$\widehat{AOC}=\frac{1}{2}\widehat{AOB}$$
$$=\frac{\beta-\alpha}{2}$$

a) $\widehat{XOC}=\widehat{XOA}+\widehat{AOC}$
$$=\alpha+\frac{\beta-\alpha}{2}$$
$$=\frac{\alpha+\beta}{2}$$

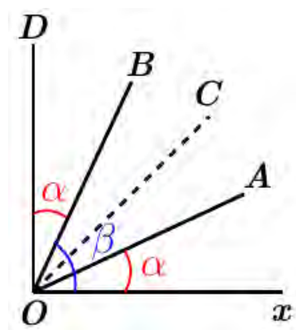
b) ***i.*** If $\alpha+\beta=90°$, then

$\widehat{XOC}=45°$

Let: $\widehat{XOD}=90°$ that implies *OC* is the bisector of $\widehat{XOD}$

Since *OC* is the bisector of $\widehat{AOB}$, then

$\widehat{BOD}=90°-\beta$ $\qquad\qquad$ $\beta=90°-\alpha$
$$=90°-90°+\alpha$$
$$=\alpha$$



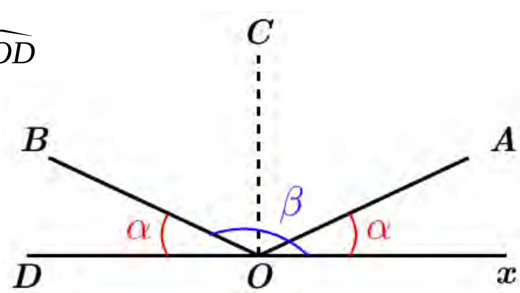
ii. If $\alpha+\beta=180°$, then

$\widehat{XOC}=90°$

Let: $\widehat{XOD}=180°$ that implies *OC* is the bisector of $\widehat{XOD}$

Since *OC* is the bisector of $\widehat{AOB}$, then

$\widehat{BOD}=180°-\beta$ $\qquad\qquad$ $\beta=180°-\alpha$
$$=180°-180°+\alpha$$
$$=\alpha$$



Exercise 5

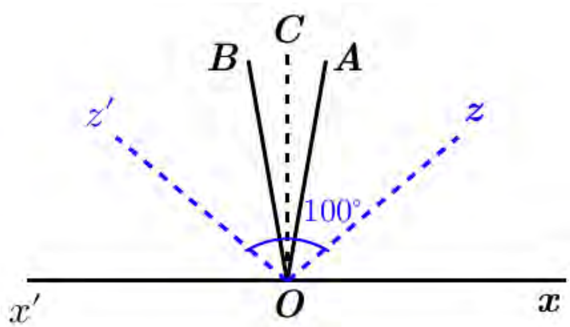
A point O takes on an infinite right $x'Ox$ be conducted the same side half-lines OA and OB, as well as the bisectors of angles $\widehat{xOA}$, $\widehat{AOB}$, and $\widehat{BOx'}$.

Calculate the angles of the figure such that the bisector of the angle $\widehat{AOB}$ is perpendicular to $x'Ox$ and the bisectors of the extreme angles formed an angle of $100°$.

Solution

Given: $\widehat{zOz'} = 100°$

$\widehat{xOC} = 90°$



OC is the bisector $\widehat{AOB}$

$\widehat{AOC} = \widehat{COB}$

Oz is the bisector $\widehat{xOA}$

$\widehat{xOz} = \widehat{zOA}$

Oz' is the bisector $\widehat{x'OB}$

$\widehat{x'Oz'} = \widehat{z'OB}$

$$\widehat{xOz} = \frac{180° - 100°}{2}$$

$$= 40°$$

$$\widehat{AOB} = 2\widehat{AOC}$$

$$= 2\left(90° - 2\widehat{xOz}\right)$$

$$= 2(90° - 80°)$$

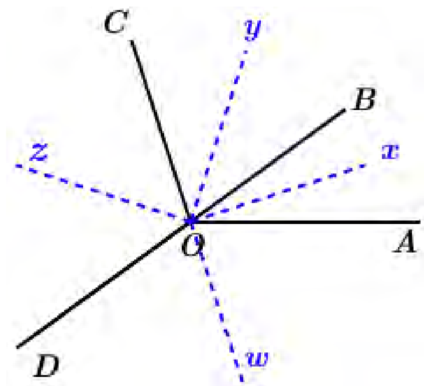
$$= 20°$$

Exercise 6

Four consecutive half-lines OA, OB, OC, and OD formed four adjacent consecutive angles which are between them like 1, 2, 3, 4.
Calculate the angles and the adjacent consecutive angles formed by their bisectors.

Solution

$$\widehat{AOB}+\widehat{BOC}+\widehat{COD}+\widehat{DOA}=360°$$
$$\widehat{AOB}+2\widehat{AOB}+3\widehat{AOB}+4\widehat{AOB}=360°$$
$$10\widehat{AOB}=360°$$



$$\underline{\widehat{AOB}=36°}\,\big|$$
$$\underline{\widehat{BOC}=72°}\,\big|$$
$$\underline{\widehat{COD}=108°}\,\big|$$
$$\underline{\widehat{DOA}=144°}\,\big|$$

$$\begin{aligned}\widehat{xOy}&=\frac{1}{2}\widehat{AOB}+\frac{1}{2}\widehat{BOC}\\&=\frac{1}{2}36°+\frac{1}{2}72°\\&=18°+36°\\&\underline{=54°}\,\big|\end{aligned}$$

$$\begin{aligned}\widehat{yOz}&=\frac{1}{2}\widehat{BOC}+\frac{1}{2}\widehat{COD}\\&=\frac{1}{2}72°+\frac{1}{2}108°\\&=36°+54°\\&\underline{=90°}\,\big|\end{aligned}$$

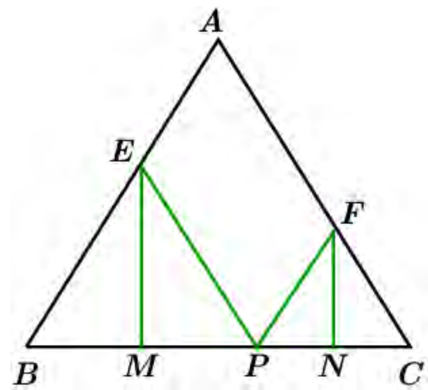
$$\begin{aligned}\widehat{zOw}&=\frac{1}{2}\widehat{COD}+\frac{1}{2}\widehat{DOA}\\&=\frac{1}{2}108°+\frac{1}{2}144°\\&=54°+72°\\&\underline{=126°}\,\big|\end{aligned}$$

$$\begin{aligned}\widehat{wOx}&=\frac{1}{2}\widehat{DOA}+\frac{1}{2}\widehat{AOB}\\&=\frac{1}{2}144°+\frac{1}{2}36°\\&=72°+18°\\&\underline{=90°}\,\big|\end{aligned}$$

Exercise 7

A point P is on the base BC of an isosceles triangle ABC. The two points M and N are the middle points of the segments PB and PC, respectively, which lead the perpendicular to the base BC; these perpendiculars meet AB in E, AC in F.
Demonstrate that the angle EPF is equal to A.

Solution



$$\widehat{BAC} = 180° - \widehat{ABC} - \widehat{ACB}$$

M is the middle of the segment BP and $EM \perp$ to BP, therefore

$$EB = EP \quad \& \quad \widehat{EBP} = \widehat{EPB}$$

N is the middle of the segment CP and $FN \perp$ to CP, therefore

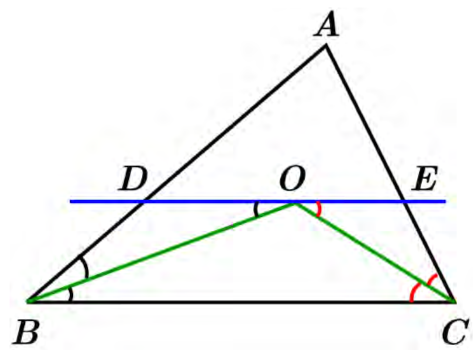
$$FP = FP \quad \& \quad \widehat{FPC} = \widehat{FCP}$$

$$\begin{aligned} \widehat{EPF} &= 180° - \widehat{CPF} - \widehat{BPE} \\ &= 180° - \widehat{PFC} - \widehat{PBE} \\ &= 180° - \widehat{ABC} - \widehat{ACB} \\ &= \widehat{A} \quad \surd \end{aligned}$$

Exercise 8

Given the triangle ABC and the bisectors BO and CO of the angles of the base, where the point O is the intersection of the 2 bisectors. A line DOE passes through the point O parallel to base BC.
Prove that $DE = DB + CE$

Solution



CO is the bisector of $\widehat{BCE} \Rightarrow \widehat{BCO} = \widehat{OCE}$

$OE \parallel BC \Rightarrow \widehat{COE} = \widehat{BOC}$

$\therefore \widehat{EOC} = \widehat{OCE} \rightarrow \underline{OE = EC}$

Similar; BO is the bisector of $\widehat{DBC} \Rightarrow \widehat{DBO} = \widehat{OBC}$

$DO \parallel BC \Rightarrow \widehat{DOB} = \widehat{OBC}$

$\therefore \widehat{DOB} = \widehat{OBC} \rightarrow \underline{DO = DB}$

$DE = DO + OE$

$\underline{= DB + CE}$

Exercise 9

A right triangle ABC at A with a height AH. We drop perpendiculars HE and HD from H to sides AB and AC respectively.

a) Prove that $DE = AH$

b) Prove that AM is perpendicular to DE, where M is the middle point of BC.

c) Prove that MN (N is the middle point of AB) and the segment Bx (parallel to DE) are intersect on AH.

d) Prove that AM and HD are intersect on Bx.

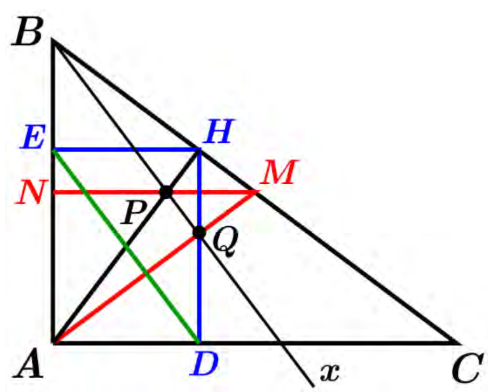
Solution

a) The triangles AEH and ADH are right triangles and angle A is right angle.
Then $AEHD$ is a rectangle.
Therefore, $DE = AH$

b) A middle point of a hypotenuse of a right triangle is the center of the circle of that triangle.
Therefore, $MC = MA = MB$

That implies to: $\widehat{MAC} = \widehat{MCA}$

From the rectangle $ADHE$: $\widehat{EAH} = \widehat{EDH}$



$$\widehat{EAH} + \widehat{HAM} + \widehat{MAC} = 90°$$

$$\widehat{HAM} + \widehat{MAC} = \widehat{HAC}$$

$$\widehat{EAH} + \widehat{HAC} = 90°$$

$$\widehat{EAH} + 90° - \widehat{MCA} = 90°$$

$$\widehat{EAH} = \widehat{MCA} = \widehat{EDH} = \widehat{MAC}$$

$$\widehat{ADE} + \widehat{EDH} = 90°$$

$$\widehat{ADE} + \widehat{MAD} = 90°$$

Therefore, AM is perpendicular to DE.

c) N is the middle point of $AB \Rightarrow NA = NB$

Bx parallel to $DE \Rightarrow \widehat{ABx} = \widehat{AED} = \widehat{EDH} = \widehat{EAH}$

Let point P the intersection of Bx and AH. Since $\widehat{ABP} = \widehat{BAP}$, then the triangle BPA is isosceles. PN is the perpendicular to AB as well MN. Which gives us that points M, P, N are on the same line.
Therefore, segment MN and AH intersect at point P.

d) Let Point Q be the intersection of AM and Bx.

$\widehat{ABQ} = \widehat{BAH}$ & $\widehat{BAQ} = \widehat{ABH}$

Then, the triangles BHA and BQA are equivalent, therefore $AQ \perp BQ$ with hypotenuse AB.

$HQ \parallel AB$, line HQ has to be perpendicular to AC.

AM and HD are intersecting on Bx at Q.

***Exercise* 10**

Given an isosceles triangle *ABC* with a peak at *A*. Extend base *BC* the length $CD = AB$, then extend *AB* of a length $BE = \frac{1}{2}BC$, at the end draw a line *EHF*, *H* is the middle point of *BC* and *F* is located on *AD*.

a) Prove that $\widehat{ADB} = \frac{1}{2}\widehat{ABC}$

b) Prove that $EA = HD$

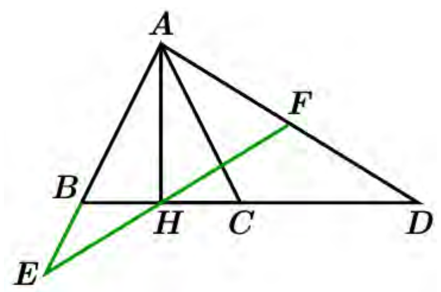
c) Prove that $FA = FD = FH$

d) Calculate the value of the angles $\widehat{AFH}$ and $\widehat{ADB}$ where $\widehat{BAC} = 58°$.

Solution

a) Triangle *ABC* is isosceles, then $\widehat{ABC} = \widehat{ACB}$

Since $CD = AB = AC$, then $\widehat{CAD} = \widehat{ADC}$



$$2\widehat{ADC} = 180° - \widehat{ACD}$$

$$2\widehat{ADC} = 180° - \left(180° - \widehat{ACB}\right)$$

$$2\widehat{ADC} = \widehat{ACB}$$

$$\widehat{ADB} = \frac{1}{2}\widehat{ACB}$$

$$= \frac{1}{2}\widehat{ABC}$$

b) $BE = \frac{1}{2}BC$ *H* the middle point of *BC*

$= HC$

$CD = AB$

$HC + CD = BE + AB$

$EA = HD$ √

c) $\widehat{ADH} = \frac{1}{2}\widehat{ABD}$

$$= \frac{1}{2}\left(180° - \widehat{HBE}\right)$$

$$= \frac{1}{2}\left(180° - 180° + 2\widehat{BHE}\right)$$

$$= \widehat{BHE}$$

$\Rightarrow$ $FD = FH$

$\widehat{AHF} = 90° - \widehat{FHD}$

$= 90° - \widehat{ADH}$ $(\Delta\ HDA)$

$= 90° - \left(90° - \widehat{HAF}\right)$

$= \widehat{HAF}$

$\Rightarrow \quad \underline{FA = FH}$

$FA = FD = FH$ √

d) $\widehat{BAC} = 58°$

$$\widehat{ADB} = \frac{1}{2}\widehat{ACB}$$
$$= \frac{1}{2}\left(\frac{1}{2}\left(180° - \widehat{BAC}\right)\right)$$
$$= \frac{1}{4}\left(180° - 58°\right)$$
$$= \frac{122}{4}^{\circ}$$
$$\underline{= \frac{61}{2}^{\circ}} \qquad \underline{= 30.5°}$$

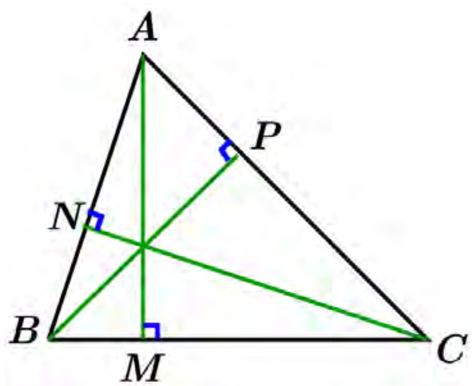
Triangle AFH is isosceles then,

$$\widehat{AFH} = 180° - \widehat{HFD}$$
$$= 180° - \left(180° - 2\widehat{FDH}\right)$$
$$= 2\widehat{FDH}$$
$$= 2\,\frac{61°}{2}$$
$$\underline{= 61°}$$

Exercise 11

Demonstrate that the heights of a triangle share the angles of triangle that equal to each other.

Solution



Consider the 2 right triangles *APB* and *ANC*, which they have the same angle *A*.

Therefore, $\widehat{ABP} = \widehat{ACN}$.

Similar, consider the 2 right triangles *BPC* and *AMC*, which they have the same angle *C*.

Therefore, $\widehat{MAC} = \widehat{CBP}$.

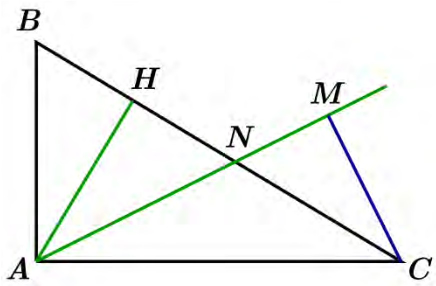
Similar, consider the 2 right triangles *BNC* and *AMB*, which they have the same angle *B*.

Therefore, $\widehat{BCN} = \widehat{BAM}$.

Exercise 12

A right triangle ABC at A where $AB < AC$, drop a perpendicular AH from A to the hypotenuse BC where $HN = HB$. From C drops a perpendicular CM at AN. Demonstrate that BC is the bisector of the angle $\widehat{ACM}$.

Solution



Consider the 2 right triangles ABC and ABH with a common angle B, then

$\widehat{BAH} = \widehat{ACB}$

Given: $HN = HB$, then $\widehat{HAN} = \widehat{BAH} = \widehat{ACB}$

$$\begin{aligned}\widehat{NAC} &= 90° - \widehat{HAB} - \widehat{HAN} \\ &= 90° - 2\widehat{HCA}\end{aligned}$$

Consider the 2 right triangles AHN and CMC, where $\widehat{HNA} = \widehat{MNC}$

Therefore, $\widehat{HAN} = \widehat{NCM}$

Since $\widehat{HAN} = \widehat{ACB}$

Then $\widehat{ACB} = \widehat{MCB}$

Therefore, BC is the bisector of the angle $\widehat{ACM}$

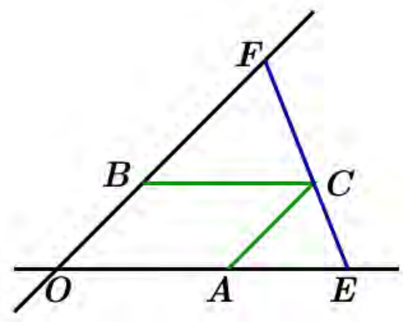
Exercise 13

On the sides of an angle that it takes the length *OA* and *OB*, so that $OA + OB = \ell$ (is given) and construct a parallelogram *OABC*. What is the place of the summit *C* of parallelogram?

Solution

Let segment *OE* extension of segment *OA* such that $OE = \ell$
Let segment *OF* extension of segment *OB* such that $OF = \ell$

Then, the triangle *OEF* is isosceles.



$$\widehat{OEF} = \widehat{OFE} = 90° - \frac{1}{2}\widehat{EOF}$$

$$OA + OB = \ell$$

$$\begin{cases} OA + AC = \ell \\ OA + AE = \ell \end{cases} \Rightarrow \quad AC = AE$$

$$\begin{cases} OB + BC = \ell \\ OB + BF = \ell \end{cases} \Rightarrow \quad BC = BF$$

$$\widehat{OEF} = \widehat{OFE} = \widehat{FCB} = \widehat{ACE}$$

Therefore, the point *C*, *E*, and *F* are aligned.

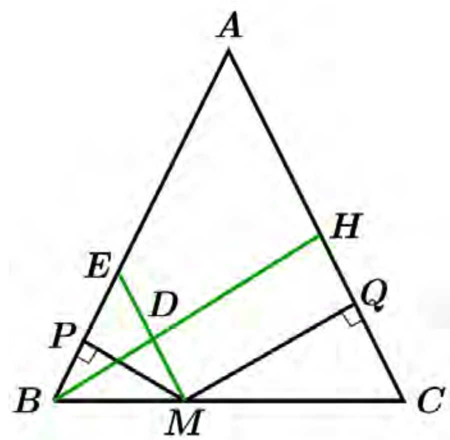
Exercise 14

Demonstrate that the sum of distances from a point *M* on the base *BC* of an isosceles triangle *ABC* to the sides equal a constant.

Solution

The shortest distance from a point to a line is the perpendicular from that point to the line.
Therefore, let:

$MP \perp AB$
$MQ \perp AC$



Let $BH \perp AC$ (Shortest distance from *B* to side *AC*.)

Let *D* be the point of intersection *ME* and *BH*.

Let $ME \,//\, AC$
Where the point *E* is the intersection of the lines *MD* and *AB*.

Since $MD \,//\, AC$ then $\widehat{DMB} = \widehat{ACB}$

Since triangle *ABC* is an isosceles triangle.
$\widehat{DMB} = \widehat{ACB} = \widehat{PBM}$

The right triangles *BPM* and *BDM* at *P* & *D* and have the same hypotenuse, then
$\Rightarrow \quad |MP| = |BD|$

$MD \,//\, HQ$ and $DH \,\&\, MQ \perp HQ$
$\Rightarrow \quad |MQ| = |DH|$

$$\begin{aligned} |MP| + |MQ| &= |BD| + |DH| \\ &= |BH| \\ &= \textbf{\textit{constant}} \end{aligned}$$

Therefore, the sum of distances from a point *M* on the base *BC* of an isosceles triangle *ABC* to the sides equal a constant.

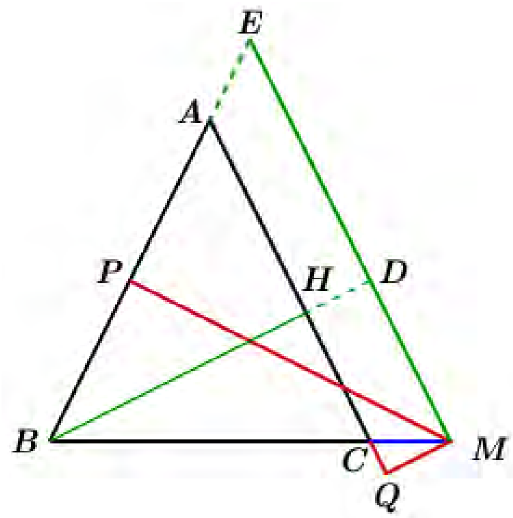
Exercise 15

Demonstrate that the difference of distances from a point M taken on the extension of the base BC of an isosceles triangle ABC to the sides equal a constant.

Solution

The shortest distance from a point to a line is the perpendicular from that point to the line. Therefore, let:

$MP \perp AB$
$MQ \perp AC$



Let $BH \perp AC$ (Shortest distance from B to side AC.)

Let D be the point of intersection ME and BH.

Let $ME \parallel AC$

Where the point E is the intersection of the extensions of the lines MD and AB.

Since $MD \parallel AC$ then $\widehat{DMB} = \widehat{ACB}$

Since triangle ABC is an isosceles triangle.

$\widehat{DMB} = \widehat{ACB} = \widehat{PBM}$

The right triangles BPM and BDM at P & D and have the same hypotenuse, then

$\Rightarrow \quad |MP| = |BD|$

$MD \parallel HQ$ and DH & $MQ \perp HQ$

$\Rightarrow \quad |MQ| = |DH|$

$$\begin{aligned} |MP| - |MQ| &= |BD| - |DH| \\ &= |BH| \\ &= \textbf{\textit{constant}} \end{aligned}$$

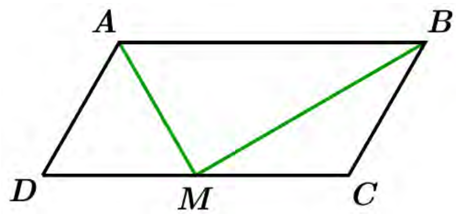
Therefore, the difference of distances from a point M taken on the extension of the base BC of an isosceles triangle ABC to the sides equal a constant.

Exercise 16

Consider a parallelogram *ABCD* in which $CD = 2AD$. In the joint *A* and *B* the middle *M* of *BC*.

Prove that the angle $\widehat{AMB}$ is a right angle.

Solution



Since the point *M* is the middle of side *BC*, then

$MD = MC = \frac{1}{2}CD$

$\Rightarrow \quad MD = AD = BC$

Therefore, the triangles *ADM* and *BCM* are isosceles at *D* and *C* respectively.
Which implies that $MA = MB$

Let *O* be the middle point of the side *AB*, and $OA = OB = AD$

O and *M* are middle of the parallelogram *ABCD*, that implies
$OM = BC = AD$

$\Rightarrow \quad OA = OB = OM$

The triangle *MAB* inscribed in a circle with center at *O* and diameter *AB*, that will imply that is a right triangle at the point *M*.

Or

$\widehat{AMD} = \frac{1}{2}\left(180° - \widehat{MDA}\right)$

$\widehat{BMC} = \frac{1}{2}\left(180° - \widehat{MCB}\right)$

$\widehat{ADM} + \widehat{MCB} = 180°$

$\widehat{DMA} + \widehat{AMB} + \widehat{BMC} = 180°$

$$\begin{aligned}
\widehat{AMB} &= 180° - \left(\widehat{BMC} + \widehat{DMA}\right)\\
&= 180° - \left(90° - \frac{1}{2}\widehat{MDA} + 90° - \frac{1}{2}\widehat{MCB}\right)\\
&= \frac{1}{2}\left(\widehat{MDA} + \widehat{MCB}\right)\\
&= \frac{1}{2}(180°)\\
&= 90°
\end{aligned}$$

Exercise 17

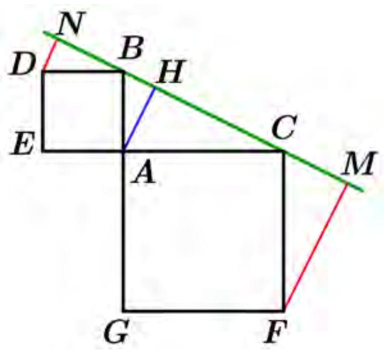
From the sides *AB* and *AC* of a right triangle *ABC* at *A*, draw two squares *ABDE* and *ACFG*. Then lead *DN* and *FM* perpendicular to the line *BC*.

a) Prove that $DN + FM = BC$

b) Prove that the points *D, A, F* on a straight line.

c) Prove that the lines *DE* and *FG* contribute on the extension of the height *AH*.

Solution



a) Let consider the 2 right triangles *DNB* & *BHA* at points *N* & *H* respectively, with $DB = AB$. Then

$$\widehat{HAB} = 90° - \widehat{ABH}$$
$$= 90° - \left(90° - \widehat{NBD}\right)$$
$$= \widehat{NBD}$$
$$\Rightarrow \quad \widehat{BDN} = \widehat{ABH}$$

$\therefore$ The 2 triangles are equals, which implies that $\underline{DN = BH}$

Similar, for the 2 right triangles *CMF* & *AHC* at points *M* & *H* respectively, with $AC = CF$. Then

$$\widehat{HAC} = 90° - \widehat{ACH}$$
$$= 90° - \left(90° - \widehat{MCF}\right)$$
$$= \widehat{MCF}$$
$$\Rightarrow \quad \widehat{ACH} = \widehat{CFM}$$

$\therefore$ The 2 triangles are equals, which implies that $\underline{FM = HC}$

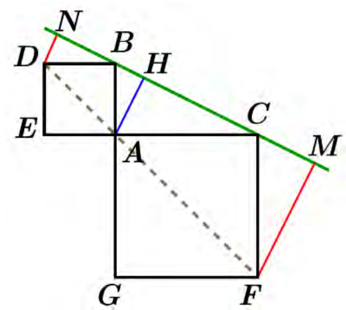
$$DN + FM = BH + HC$$
$$\underline{= BC} \quad \surd$$

b) Since *ABDE* is a square, then $\widehat{BAD} = 45°$

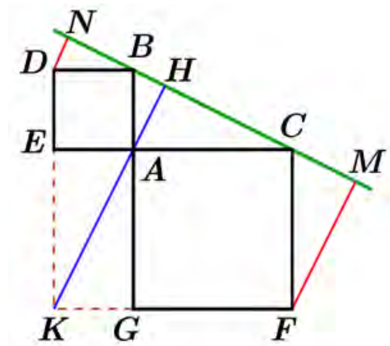
And *ACFG* is a square, then $\widehat{CAF} = 45°$

$$\widehat{DAF} = \widehat{DAB} + \widehat{BAC} + \widehat{CAF}$$
$$= 45° + 90° + 45°$$
$$\underline{= 180°}$$

$\therefore$ The points *D*, *A*, & *F* are on a straight line.



c) Let the point K be the intersection of the extension of the sides DE and FG.
Which will result of $GKEA$ is a rectangle with $AE = GK$ & $EK = AG$



Consider the 2 right triangles BAC & KGA at points A & G respectively with $AE = AB = GK$

$\widehat{ACB} = \widehat{GAK} = \widehat{ACH}$

From the right triangle AHC:

$\widehat{HAC} + \widehat{ACH} = 90°$

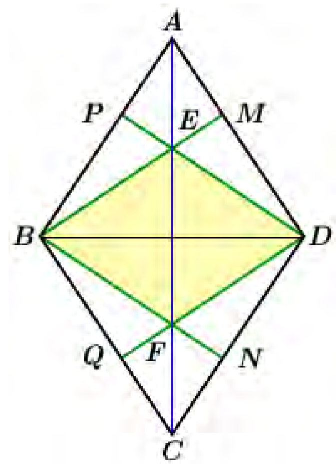
$\to \quad \widehat{HAC} + \widehat{KAG} = 90°$

$$\begin{aligned}\widehat{HAC} + \widehat{CAG} + \widehat{KAG} &= \left(\widehat{HAC} + \widehat{KAG}\right) + \widehat{CAG} \\ &= 90° + 90° \\ &= 180°\end{aligned}$$

∴ The points K, A, & H are on a straight line.

Exercise 18

Given a diamond *ABCD*; the peak *B* and *D*, the same the perpendiculars *BM*, *BN*, *DP*, *DQ* on opposite sides. These perpendiculars are intersected at *E* and *F*.
Demonstrate that the angles of the quadrilateral *BFDE* are equals to the diamond and which is a diamond itself.



Solution

From the right triangles *BPD* & *BMD*, that implies $\widehat{MBD} = \widehat{PDB}$

$\Rightarrow \quad \widehat{EBD} = \widehat{EDB}$

Similar, from the right triangles *BND* & *BQD*, that implies $\widehat{NBD} = \widehat{QDB}$

$\Rightarrow \quad \widehat{FBD} = \widehat{FDB}$

$\widehat{EBD} + \widehat{DBF} = \widehat{EDB} + \widehat{BDF}$

$\widehat{EBF} = \widehat{EDF}$

Since, $AC \perp BD$, then $EF \perp BD$

The 2 triangles *EBF* & *EDF* have *EF* as a common side and $\widehat{EBF} = \widehat{EDF}$, then

$\widehat{BEF} = \widehat{DEF} = \widehat{BFE} = \widehat{DFE}$

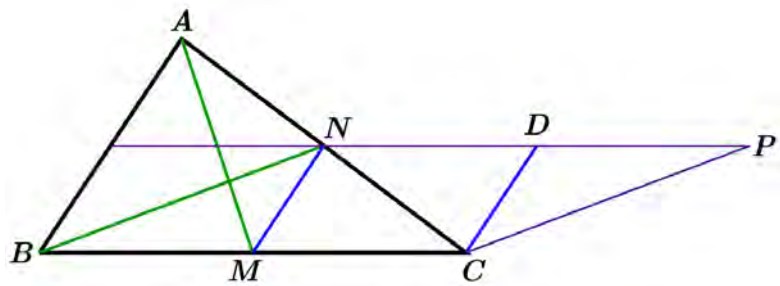
$\widehat{BED} = \widehat{BFD}$

Therefore, the angles of the quadrilateral *BFDE* are equals to the diamond and which is a diamond itself.

Exercise 19

In a triangle *ABC*, we trace the median *AM* and *BN* and from *N* a parallel to *BC*, from *C* a parallel to *BN*; that the two sides intersect at a point *P*. Let *D* be the middle point of the segment *PN*.
Prove that *CD* is parallel to *MN*.

Solution



Since the points *M* & *N* are middle of the sides *BC* & *AC* of the triangle *ABC*, then $MN \parallel AB$

Given: $NP \parallel MC$

$BN \parallel CP$

Since *M* & *D* are the middle points of the segments *BC* and *NP* respectively, then $BN \parallel CP \parallel MD$

Therefore, *BNPC* is a parallelogram, and $MC = ND$.

Since $MC = ND$ & $MN = CD$

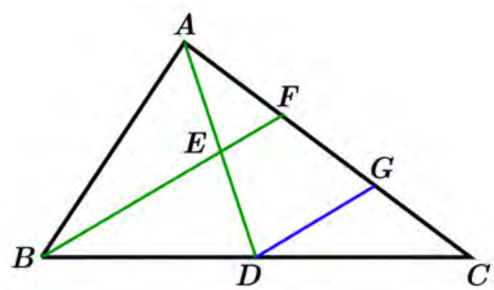
Therefore, *MCDC* is a parallelogram which implies to *CD* parallel to *MN*

Exercise 20

The median AD of a given triangle ABC to the side BC. The same the median BE to the side AD which intersect AC at a point F.

Prove that where $AF = \frac{1}{3}AC$

Solution



Let DG be parallel to segment BEF.

Given: E is the middle point of the segment $AD \Rightarrow AE = ED$

D is the middle point of the segment $BC \Rightarrow BD = DC$

Since $EF \,/\!/\, DG$, and $AE = ED$, that implies $AF = FG$

Consider the triangles CDG and CBF:

$EF \,/\!/\, DG$, and $CD = DB$, that implies $GC = FG$

That will imply to: $AF = FG = GC$

$$\begin{aligned} AC &= AF + FG + GC \\ &= 3AF \end{aligned}$$

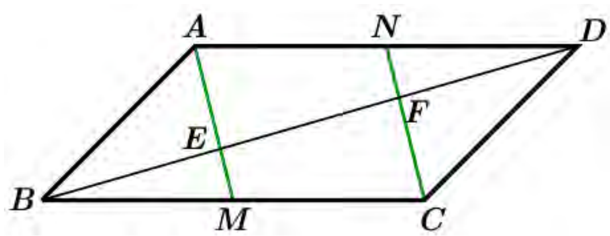
Therefore; $\boxed{AF = \frac{1}{3}AC}$

Exercise 21

In a parallelogram *ABCD*, from the points peak *A* and *C* joint the middle of opposite sides at *M* and *N* respectively.

Prove that the diagonal *BD* is divided in three equal parts.

Solution



M is the middle point of the segment *BC*, then $BM = CM$

N is the middle point of the segment *AD*, then $NA = ND$

From these, implies that $AM \parallel CN$.

From the triangles *BEM* & *BCF*, and since $ME \parallel CF$

It will give us that $BE = EF$

From the triangles *DFN* & *DEA*, and since $AE \parallel FN$

It will give us that $\Rightarrow \quad DF = EF$

Therefore, $BE = EF = DF$

$BD = BE + EF + FD$

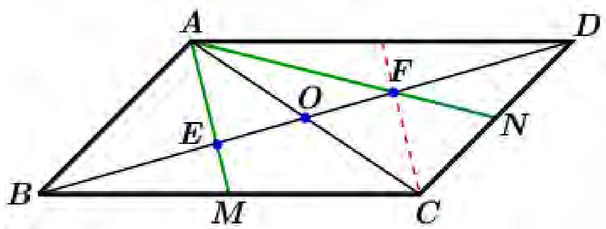
$\quad = 3BE$

Therefore, the diagonal *BD* is divided in three equal parts

Exercise 22

In a parallelogram *ABCD*, from the point peak *A*, extend to the middle of sides *BC* and *DC* at *M* and *N* respectively.
Prove that the diagonal *BD* is divided in three equal parts.

Solution



Let a point *E* be the intersection of the segments *AM* & *BD*.
Let a point *F* be the intersection of the segments *AN* & *BD*.

Le *O* be the intersection of both diagonal *AC* & *BD*.
From the triangles *BEM* & *BCF*, and since $ME \parallel CF$

$$\Rightarrow \quad BE = EF$$

Similar, $\Rightarrow \quad DF = EF$

$$BO = OF \quad \rightarrow \quad OE = OF$$

$$\begin{aligned} BO &= BE + EO \\ &= BE + \frac{1}{2}BE \\ &= \frac{3}{2}BE \end{aligned}$$

$$\begin{aligned} BE &= \frac{2}{3}BO \\ &= \frac{2}{3}\left(\frac{1}{2}BD\right) \\ &= \frac{1}{3}BD \end{aligned}$$

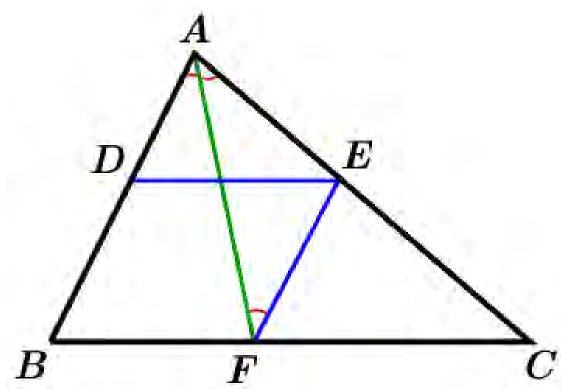
$$\begin{aligned} DF &= \frac{2}{3}DO \\ &= \frac{2}{3}\left(\frac{1}{2}BD\right) \\ &= \frac{1}{3}BD \end{aligned}$$

Therefore, the diagonal *BD* is divided in three equal parts

Exercise 23

Consider a triangle ABC with a bisector AF of the angle A. by F, we lead FE parallel to AB, and by E we lead ED parallel to BC.
Prove that $AE = BD$

Solution



Given: $\widehat{EAF} = \widehat{FAB}$

Since $FE \,//\, AB$, then

$$\widehat{FEC} = \widehat{BAE} = 2\widehat{EAF}$$

$$\widehat{AEF} = 180° - \widehat{FEC}$$
$$= 180° - 2\widehat{EAF}$$

Consider the triangle AEF:

$$\widehat{EAF} + \widehat{EFA} + \widehat{AEF} = 180°$$
$$\widehat{EAF} + \widehat{EFA} + 180° - 2\widehat{EAF} = 180°$$
$$\widehat{EFA} - \widehat{EAF} = 0°$$
$$\widehat{EFA} = \widehat{EAF}$$

∴ Triangle AEF is isosceles

$\Rightarrow$ $\underline{AE = EF}$

Given $DE \,//\, BF$ & $FE \,//\, DB$

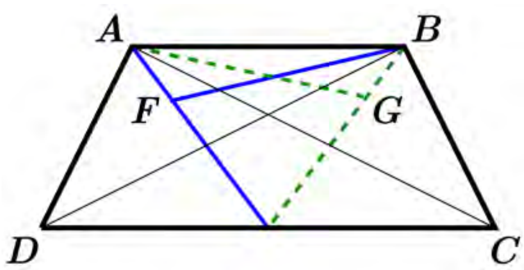
$FEDB$ is a parallelogram.

Then, $\underline{EF = DB = AE}$

Exercise 24

Given an isosceles trapezoid $ABCD$ $(AD = BC)$ with diagonals AC and BD. The bisector of angles $\widehat{DAB}$ and $\widehat{DBA}$ intersect in F, and the bisector of angles $\widehat{CBA}$ and $\widehat{CAB}$ intersect in G.
Demonstrate that FG is parallel to AB.

Solution



Consider the 2 triangles ABD & ABC:

- Both has the AB as common
- $AD = BC$

That implies to: $\widehat{ABD} = \widehat{CAB}$

Since BF is the bisector of the angle $\widehat{ABD}$

$\widehat{ABF} = \widehat{FBD}$

$$\Rightarrow \quad \widehat{ABF} = \frac{1}{2}\widehat{ABD}$$

$$= \frac{1}{2}\widehat{CAB}$$

$$= \frac{1}{2}\left(2\widehat{BAG}\right)$$

$$= \widehat{BAG}$$

$\widehat{ABF} = \widehat{BAG}$

From the 2 triangles AFB & AGB

- Both has the AB as common
- $\widehat{ABF} = \widehat{BAG}$

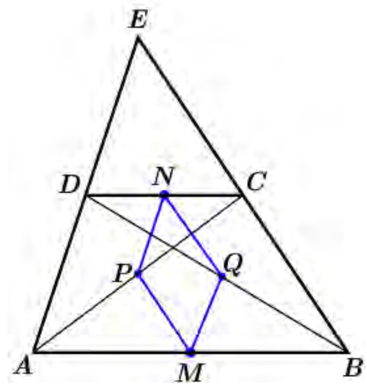
$FG \mathbin{/\!/} AB$

Exercise 25

Let M and N be the middle points of the bases AB and CD of a trapezoid $ABCD$.
Let P and Q be the middle points of the diagonals AC and BD respectively.

Demonstrate that the angles $\widehat{M}$ and $\widehat{N}$ of quadrilateral $MNPQ$ are equals to the angle formed by extending the sides not parallel to BC and AD, where intersect at point E.

Solution



Since N is the mid-point of the side DC, and P is the mid-point of the side AC, then

$$\Rightarrow \quad NP \parallel AD$$

Since M is the mid-point of the side AB, and Q is the mid-point of the side DB, then

$$\Rightarrow \quad QM \parallel AD$$

$$\therefore \quad NP \parallel QM \parallel AE$$

Since N is the mid-point of the side DC, and Q is the mid-point of the side DB, then

$$\Rightarrow \quad NQ \parallel CB$$

Since M is the mid-point of the side AB, and P is the mid-point of the side AC, then

$$\Rightarrow \quad MP \parallel CB$$

$$\therefore \quad NQ \parallel MP \parallel BE$$

$$\rightarrow \begin{cases} NP \parallel QM \\ NQ \parallel PM \end{cases}$$

$$\therefore \quad \widehat{PMQ} = \widehat{PNQ}$$

Exercise 26

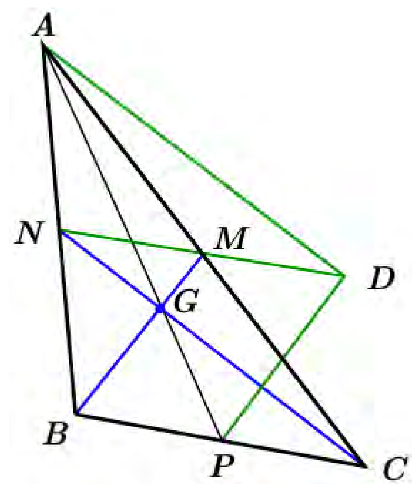
In a triangle *ABC*, the medians segment *BM* and *CN* intersect in right angles and the measurement are 3 and 6 units respectively.

1. Construct a geometrical to the triangle *ABC*.
2. In the trace of third median *AP* which leads *MN* extension such the distance $MD = MN$, which lead to the segments *AD* and *PD*. Calculate *AD* and *DP*.
3. What is the natural of the triangle *APD* ?

Solution

1. Since *M* and *N* are the middle point of the side's *AC* & *AB*, then

$$BG = \frac{2}{3}BM = \frac{2}{3}(3) = 2 \ \ units$$



Similar,

$$CG = \frac{2}{3}CN = \frac{2}{3}(6) = 4 \ \ units$$

Wish, we lead to: $GM = 1$ & $GN = 2$

We can construct 2 perpendicular lines intersect at a point *G*, then we use to measure the distance from the point *G* to get the points *B*, *C*, *M*, & *N*.
By extending the segment *BN* and *CM* with equal distance and which it will intersect at point *A*.

2. Since $ND \parallel BC$ & $MD = MN$

The parallelogram *BPDM*, $BP = MD = MN$

Then $PD = MB = 3 \ \ units$

$AD \parallel CN$ and *M* is the intersection of the diagonals of the parallelogram *ADCN*, then

$AD = CN = 6 \ \ units$

3. $PD \parallel BN$ & $MB \perp CN$, then $PD \perp CN$

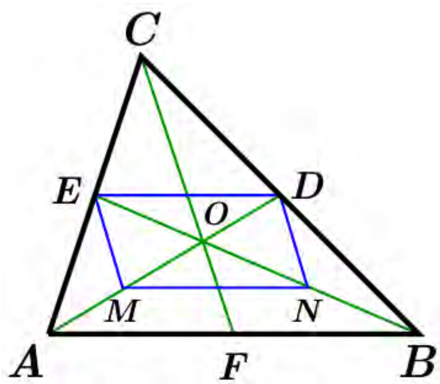
$AD \parallel CN$ & $PD \perp CN$, then $AD \perp PD$

Therefore, the triangle *ADP* is right triangle at point *D*.

Exercise 27

Inside the triangle *ABC*, the median *AD*, *BE*, and *CF* intersect at a point *O*. We take *M* the middle point of the segment *OA*, *N* the middle point of segment *OB*.
Show that *DEMN* is a parallelogram.

Solution



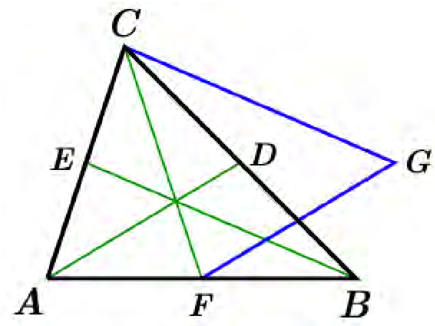
DE & *MN* are parallel to *AB* and equals to $\frac{1}{2}|AB|$

That implies to $ME \parallel DN$.

Therefore, *DEMN* is a parallelogram

Exercise 28

Inside the triangle *ABC*, the median *AD*, *BE*, and *CF* intersect at a point *O*. From the point *F*, draw *FG* parallel to *AD* and are equals, then joint *A* to *G*.



Show that $CG = BE$.

Solution

Given: $FG \parallel AD$ & $FG = AD$

Then, the quadrilateral *AFGD* is a parallelogram which it results to $DG \parallel AF$ & $DG = AF$.

$\rightarrow \quad DG \parallel BF$

Since *F* is the mid-point of the side *AB*, then $AF = DG = FB$.

Then, the quadrilateral *BFDG* is a parallelogram which it results to $FD \parallel BG$ & $DF = GB$.

So, $FD \parallel BG \parallel CE$

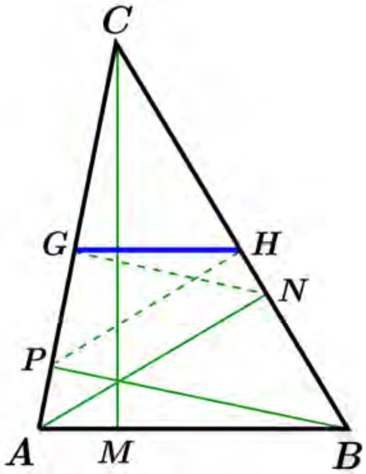
Given that D & F are midpoints, then $DF = \frac{1}{2}AC = CE$

And $CE \parallel BG$ & $DF = CE$, then *BGCE* is a parallelogram.

Therefore, $CG = BE$

Exercise 29

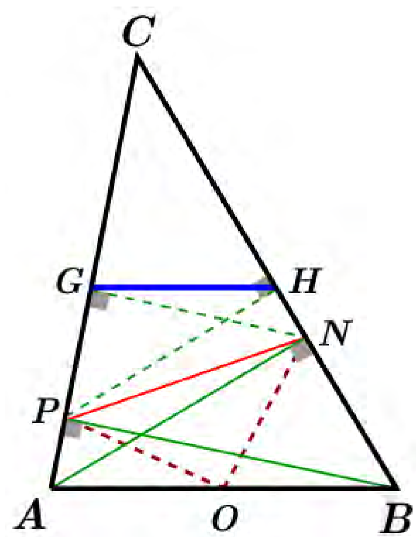
The height of a triangle *ABC* (each of the three lines passing through a vertex of the triangle and perpendicular to the opposite side to this vertex) are *AN*, *BP*, *CM*.



From *P*, let *PH* perpendicular to *BC*, same from *N*, let *NG* perpendicular to *AC*.
Show that *GH* is parallel to *AB*.

Solution

Let the point *O* be the middle of the segment *AB*.
Then *O* is the center of the 2 triangles *ANB* & *APB*.



The triangle *OBN* is isosceles, implies to $\widehat{ONB} = \widehat{OBN}$

The triangle *OPA* is isosceles, implies to $\widehat{OPA} = \widehat{OAP}$

$$\widehat{PON} = 180° - \left(\widehat{NOB} + \widehat{POA}\right)$$
$$= 180° - \left(180° - 2\widehat{NBO} + 180° - 2\widehat{OAP}\right)$$
$$= 2\widehat{B} + 2\widehat{A} - 180°$$

Consider the triangle *PON* with $OP = ON$, then

$$\widehat{OPN} = \widehat{ONP}$$
$$\widehat{OPN} = \frac{1}{2}\left(180° - \widehat{PON}\right)$$
$$= \frac{1}{2}\left(180° - 2\widehat{B} - 2\widehat{A} + 180°\right)$$
$$= 180° - \widehat{B} - \widehat{A}$$

$$\widehat{APN} = \widehat{APO} + \widehat{OPN}$$
$$= \widehat{A} + 180° - \widehat{B} - \widehat{A}$$
$$= 180° - \widehat{B}$$

$$\widehat{CPN} = 180° - \widehat{APN}$$
$$= 180° - 180° + \widehat{B}$$
$$\underline{= \widehat{B}\,|}$$

From the 2 right triangles CHP & CGN

$$\widehat{HPC} = \widehat{GNC}$$

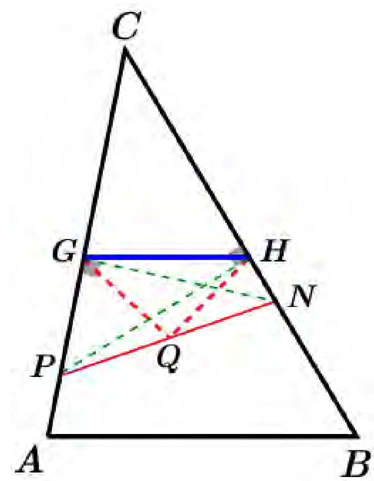
$$\begin{aligned}\widehat{GHN} &= 180° - \widehat{HGN} - \widehat{HNG} \\ &= 180° - \widehat{HGN} - \widehat{CPH}\end{aligned}$$

$$180° - \widehat{GHC} = 180° - \widehat{HGN} - \widehat{CPH}$$

$$\underline{\widehat{GHC} = \widehat{HGN} + \widehat{CPH}}\,\big|$$

Let Q be the middle point of the segment PN.
Since PGN & PHN are right triangle with the same hypothesis.
Then, the triangles HQN & GQN are isosceles.

$$\widehat{H} = \widehat{N} \quad \& \quad \widehat{G} = \widehat{P}$$



$$\begin{aligned}\widehat{GQH} &= 180° - \left(180° - 2\widehat{P} + 180° - 2\widehat{N}\right) \\ &= 2\widehat{P} + 2\widehat{N} - 180°\end{aligned}$$

Since $QG = QH \quad \Rightarrow \quad \widehat{QGH} = \widehat{QHG}$

$$\begin{aligned}\widehat{QGH} &= \frac{1}{2}\left(180° - \widehat{GQH}\right) \\ &= \frac{1}{2}\left(180° - 2\widehat{P} - 2\widehat{N} + 180°\right) \\ &= 180° - \widehat{P} - \widehat{N}\end{aligned}$$

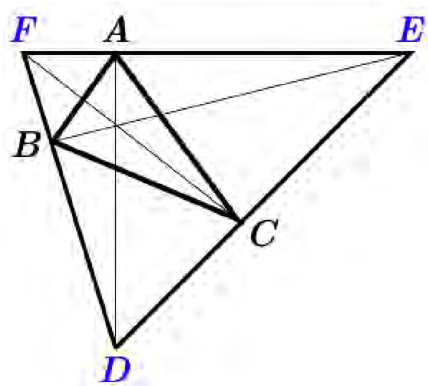
$$\begin{aligned}\widehat{HGN} &= \widehat{QGH} - \widehat{QGN} \\ &= 180° - \widehat{P} - \widehat{N} - 90° + \widehat{QGP} \\ &= 90° - \widehat{P} - \widehat{N} + \widehat{P} \\ &= 90° - \widehat{N} \\ &= \widehat{NPH}\end{aligned}$$

$$\begin{aligned}\widehat{CHG} &= \widehat{HGN} + \widehat{CPH} \\ &= \widehat{NPH} + \widehat{CPH} \\ &= \widehat{NPC} \\ &\underline{= \widehat{B}}\,\big|\end{aligned}$$

Therefore, $GH \parallel AB$

Exercise 30

From the top of a triangle, we lead the external bisectors of angles such that formed an outside triangle such that the top of the first are the feet of the second heights.



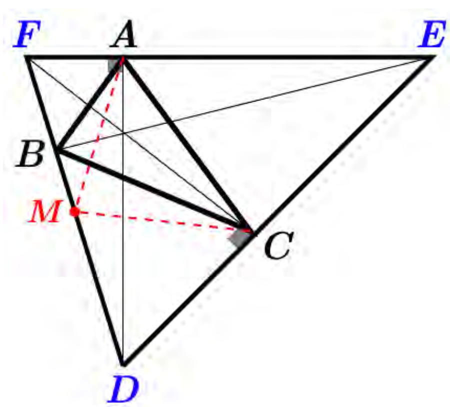
Solution

Let the triangle *DEF* where *DA*, *BE*, and *FC* are heights (perpendicular to sides).

Let the point *M* be the middle points of the same hypothenuse of the 2 right triangles *FAD* & *FCD*. Then, the 2 triangles inscribed the same circle with the center at point *M*.

$MF = MA = MC = MD$

$\widehat{MFA} = \widehat{MAF} \quad \& \quad \widehat{MCD} = \widehat{MDC}$



Therefore, the triangle *AMC* is isosceles.

$MA = MC \quad \& \quad \widehat{MAC} = \widehat{ACM}$

$$\begin{aligned}\widehat{AMC} &= 180° - \left(\widehat{FMA} + \widehat{CMD}\right)\\ &= 180° - \left(180° - 2\widehat{F} + 180° - 2\widehat{D}\right)\\ &= 2\widehat{F} + 2\widehat{D} - 180°\end{aligned}$$

$$\begin{aligned}\widehat{ACM} &= \frac{1}{2}\left(180° - \widehat{AMC}\right)\\ &= \frac{1}{2}\left(180° + 180° - 2\widehat{F} - 2\widehat{D}\right)\\ &= 180° - \widehat{F} - \widehat{D}\end{aligned}$$

$$\begin{aligned}\widehat{DCA} &= \widehat{DCM} + \widehat{MCA}\\ &= \widehat{D} + 180° - \widehat{F} - \widehat{D}\end{aligned}$$

$$180° - \widehat{ACE} = 180° - \widehat{F}$$

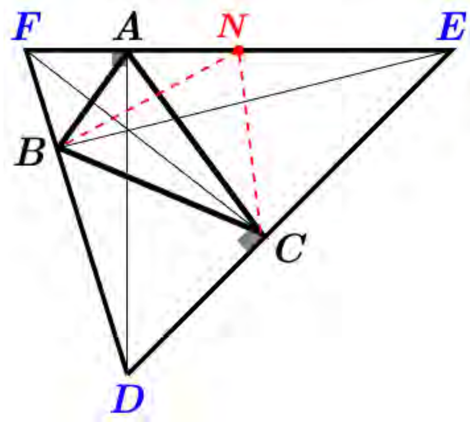
$$\widehat{ACE} = \widehat{F}$$

Similar,

Let the point *N* be the middle points of the same hypothenuse of the 2 right triangles *FCE* & *FBE*. Then, the 2 triangles inscribed the same circle with the center at point *N*.

$NF = NB = NC = NE$

$\widehat{NBF} = \widehat{BFN}$ & $\widehat{NEC} = \widehat{NCE}$



Therefore, the triangle NBC is isosceles.

$MA = MC$ & $\widehat{MAC} = \widehat{ACM}$

$$\begin{aligned}\widehat{BNC} &= 180° - \left(\widehat{FNB} + \widehat{CNE}\right)\\ &= 180° - \left(180° - 2\widehat{F} + 180° - 2\widehat{E}\right)\\ &= 2\widehat{F} + 2\widehat{E} - 180°\end{aligned}$$

$$\begin{aligned}\widehat{BCN} &= \frac{1}{2}\left(180° - \widehat{BNC}\right)\\ &= \frac{1}{2}\left(180° + 180° - 2\widehat{F} - 2\widehat{E}\right)\\ &= 180° - \widehat{F} - \widehat{E}\end{aligned}$$

$$\begin{aligned}\widehat{BCE} &= \widehat{BCN} + \widehat{NCE}\\ &= \widehat{E} + 180° - \widehat{F} - \widehat{E}\end{aligned}$$

$$180° - \widehat{BCD} = 180° - \widehat{F}$$

$$\widehat{BCD} = \widehat{F}$$

Then, $\widehat{ACE} = \widehat{F} = \widehat{BCD}$

Therefore, CF is the interior bisector of $\widehat{BCA}$ and DCE is the exterior bisector.

$$\begin{aligned}\widehat{MAC} &= \frac{1}{2}\left(180° - \widehat{AMC}\right)\\ &= \frac{1}{2}\left(180° + 180° - 2\widehat{F} - 2\widehat{D}\right)\\ &= 180° - \widehat{F} - \widehat{D}\end{aligned}$$

$$\begin{aligned}\widehat{FAC} &= \widehat{FAM} + \widehat{MAC}\\ &= \widehat{F} + 180 - \widehat{F} - \widehat{D}\end{aligned}$$

$$180° - \widehat{CAE} = 180° - \widehat{D}$$

$$\widehat{CAE} = \widehat{D}$$

Let the point P be the middle points of the same hypothenuse of the 2 right triangles DAE & BDE. Then, the 2 triangles inscribed the same circle with the center at point P.

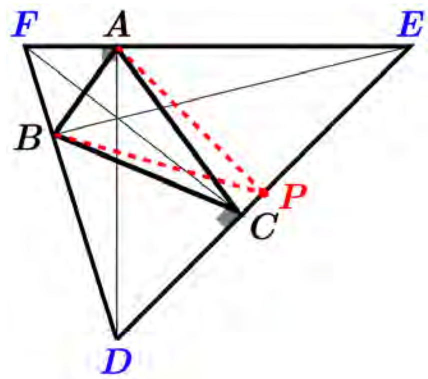
$PE = PA = PB = PD$

$\widehat{PAE} = \widehat{PAE}$ & $\widehat{PBD} = \widehat{PDB}$

Therefore, the triangle APB is isosceles.

$PA = PB$ & $\widehat{PAB} = \widehat{PBA}$

$$\widehat{APB} = 180° - \left(\widehat{DPB} + \widehat{APE}\right)$$
$$= 180° - \left(180° - 2\widehat{D} + 180° - 2\widehat{E}\right)$$
$$= 2\widehat{D} + 2\widehat{E} - 180°$$



$$\widehat{PAB} = \frac{1}{2}\left(180° - \widehat{APB}\right)$$
$$= \frac{1}{2}\left(180° + 180° - 2\widehat{D} - 2\widehat{E}\right)$$
$$= 180° - \widehat{D} - \widehat{E}$$

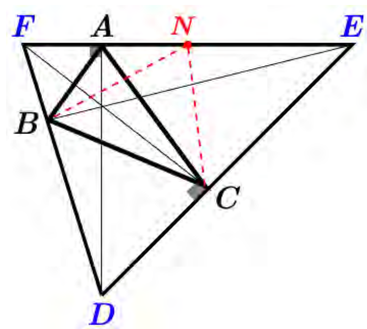
$$\widehat{BAE} = \widehat{PAE} + \widehat{PAB}$$
$$= \widehat{E} + 180° - \widehat{D} - \widehat{E}$$
$$180° - \widehat{FAB} = 180° - \widehat{D}$$

$$\widehat{FAB} = \widehat{D}$$

Then, $\widehat{FAB} = \widehat{D} = \widehat{CAE}$

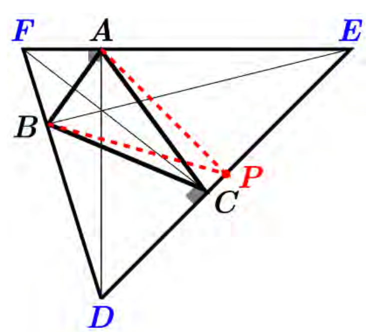
Therefore, AD is the interior bisector of $\widehat{BAC}$ and FAE is the exterior bisector.

$$\widehat{NBC} = \frac{1}{2}\left(180° - \widehat{BNC}\right)$$
$$= \frac{1}{2}\left(180° + 180° - 2\widehat{F} - 2\widehat{E}\right)$$
$$= 180° - \widehat{F} - \widehat{E}$$



$$\widehat{CBF} = \widehat{CBN} + \widehat{NBF}$$
$$= 180° - \widehat{F} - \widehat{E} + \widehat{F}$$
$$180° - \widehat{CBD} = 180° - \widehat{E}$$

$$\widehat{CBD} = \widehat{E}$$



$$\widehat{PBA} = \frac{1}{2}\left(180° - \widehat{APB}\right)$$
$$= \frac{1}{2}\left(180° + 180° - 2\widehat{D} - 2\widehat{E}\right)$$
$$= 180° - \widehat{D} - \widehat{E}$$

$$\widehat{ABD} = \widehat{DBP} + \widehat{PBA}$$
$$= \widehat{D} + 180° - \widehat{D} - \widehat{E}$$
$$180° - \widehat{ABF} = 180° - \widehat{E}$$

$$\widehat{ABF} = \widehat{E}$$

Then, $\widehat{CBD} = \widehat{E} = \widehat{ABF}$

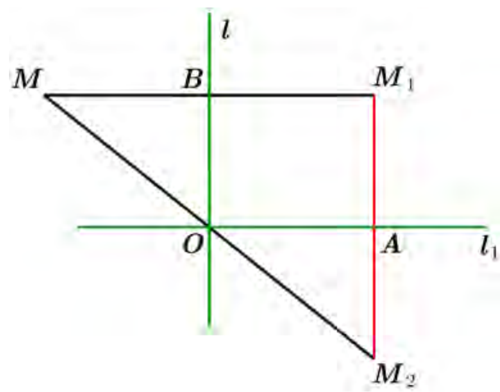
Therefore, BE is the interior bisector of $\widehat{ABC}$ and DBF is the exterior bisector.

Exercise 31

Consider a point O on a vertical line ℓ, a point M outside the line ℓ. We take the symmetries M_1 and M_2 from M across the line ℓ and the point O, respectively.

Demonstrate that the points M_1 and M_2 are symmetries with regard to a line perpendicular to the line ℓ_1 passing through the point O.

Solution



Since M_1 is the symmetry of M across the line ℓ, let B the middle point of the segment MM_1.

That implies to: $BM = BM_1$.

Similarly, the point O is the middle point of the segment MM_2.

That implies to: $OM = OM_2$.

Let A be the point intersection of the segment M_1M_2 and line ℓ_1.

Since $MM_1 \perp \ell$ and $\ell \perp \ell_1 \Rightarrow MM_1 // \ell_1$

From the right triangle MM_1M_2 Since O is the middle of MM_2 and $OA \perp MM_1$.

Therefore, the point A the middle point of the segment M_1M_2

Exercise 32

In a quadrilateral *ABCD* (*Kite*), the sides $AB = AD$, $\angle A = 135°$ and $\angle B = \angle D = 90°$.

1. Prove the symmetry in the figure.
2. Prove that the middles of the sides are the top of rectangle.
3. Prove there exists an interior of the given quadrilateral a point equidistant of 4 sides; determine these points.
4. On the same exterior bisector of angles *A, B, C, D*; they formed a quadrilateral

Solution

1. $C = 180° - 135°$
$\underline{= 45°}$

Since $AB = AD$, then the side *AC* is the bisector of $\widehat{BAC}$.
The 2 right triangles *ABC* & *ADC* are rights on *B* and *D* respectively, with same hypothenuse *AC*, therefore the figure is symmetric about the side *AC.*

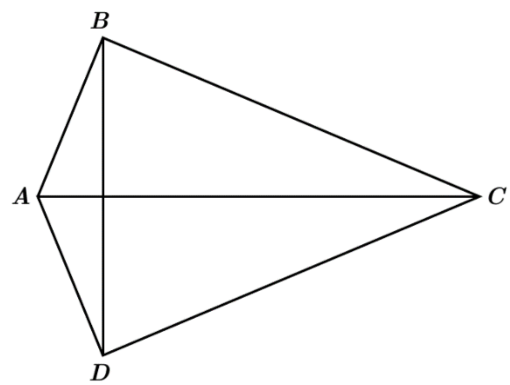
$\widehat{BAC} = \dfrac{135°}{2}$
$\underline{= 67.5°}$

$\widehat{BCA} = \dfrac{45°}{2}$
$\underline{= 22.5°}$

$\widehat{ABD} = 90° - 67.5°$
$\underline{= 22.5°}$

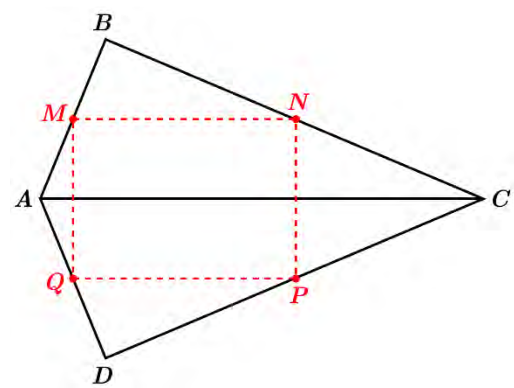
$\underline{\widehat{ADB} = 22.5°}$

$\widehat{DBC} = 90° - 22.5°$
$\underline{= 67.5°}$



2. Since $AB = AD$, then the triangle *BAD* is isosceles.

Let *M* and *Q* the middle points of the sides *AB* and *AD* respectively.
Therefore, the segment $MQ \parallel BD$.
Similar, the triangle BCD is isosceles, since $CB = CD$.
Let *N* and *P* the middle points of the sides *CA* and *CD* respectively.
Therefore, the segment $NP \parallel BD$.



The points *M* and *N* are the middle points of the sides *AB* and *CB* respectively in the right triangle *ABC.*
Therefore, the segment $MN \parallel AC$.

The points *P* and *Q* are the middle points of the sides *DC* and *DA* respectively in the right triangle *ADC.*
Therefore, the segment $PQ \parallel AC$.

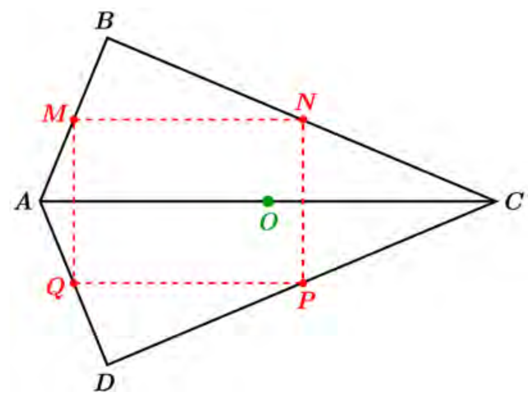
$PQ \parallel MN \parallel AC \quad \& \quad MQ \parallel NP \parallel BD$

That implies to $MQ \parallel NP$ and $MN \parallel PQ$, which implies that *MNPQ* is a parallelogram.

From the quadrilateral *ABCD*, $AC \perp BD$

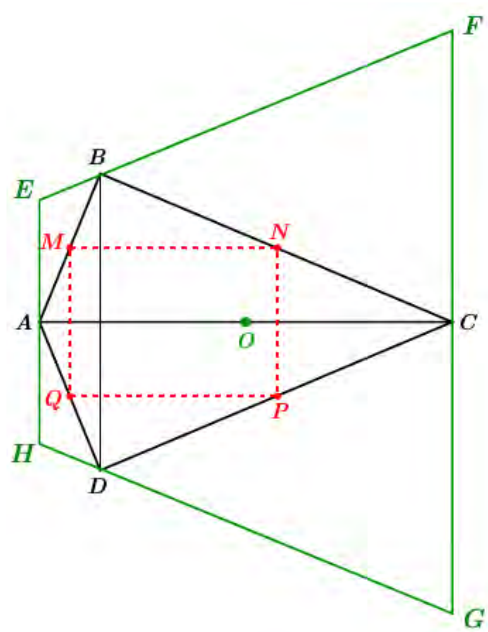
Therefore, *MNPQ* is rectangular.

3. Let the point *O* be the middle point of the segment *AC*.
Given that the 2 right triangles *ABC* & *ADC* have the same hypothesis segment *AC*.
The 4 point are inscribed in a circle with diameter *AC* where the radius is:
$OA = OB = OC = OD$



Therefore, the point *O* is an interior of the given quadrilateral which is a point equidistant of 4 sides.

4. Let two segments pass through the points *A* and *C* perpendicular to the segment *AC*, which are parallel to segment *BD*.



Let the segment *EF* passes through the point *B* such that:

$$\widehat{EBA} = \widehat{FBC} = \frac{90^\circ}{2} = 45^\circ$$

$$\widehat{EAB} = 90° - \widehat{BAC}$$
$$= 90° - 67.5°$$
$$\underline{= 22.5°}$$

$$\widehat{AEB} = 180° - \widehat{EBA} - \widehat{BAE}$$
$$= 180° - 45° - 22.5°$$
$$\underline{= 112.5°}$$

$$\widehat{BCF} = 90° - \widehat{ACB}$$
$$= 90° - 22.5°$$
$$\underline{= 67.5°}$$

$$\widehat{BFC} = 180° - \widehat{FBC} - \widehat{BCF}$$
$$= 180° - 45° - 67.5°$$
$$\underline{= 67.5°}$$

Exercise 33

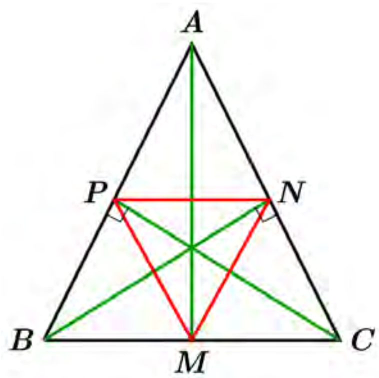
Given an isosceles triangle *ABC* with a peak at *A* with angle of $47°$.

The height of a triangle *ABC* of each of the three lines passing through a vertex of the triangle and perpendicular to the opposite side to this vertex, are *AM*, *BN*, *CP*.

What is the angle of the peak of the isosceles triangle *MNP* ?

Solution

The triangle *ABC* is isosceles triangle at *A* then the height *AM* is the bisector. So, the point *M* is the middle point of the side *BC*.



Since the 2 right triangles *BPC* & *BNC* have the same hypothesis segment *BC*.
There are inscribed in a circle with diameter *BC* with the center at point *M*.
Therefore, the triangle *MNP* is isosceles with a peak at the point *M*.

$$C = \widehat{MNC} = B = \widehat{BPM}$$

$$\begin{cases} \widehat{BMP} = 180° - 2B \\ \widehat{CMN} = 180° - 2C \end{cases}$$

$$B = \frac{180° - 47°}{2}$$
$$= \frac{133°}{2}$$

$$\begin{aligned} \widehat{PMN} &= 180° - \widehat{BMP} - \widehat{CMN} \\ &= 180° - 180° + 2B - 180° + 2B \\ &= 4B - 180° \\ &= 4\left(\frac{133°}{2}\right) - 180° \\ &= 266° - 180° \\ &= 86° \end{aligned}$$

Exercise 34

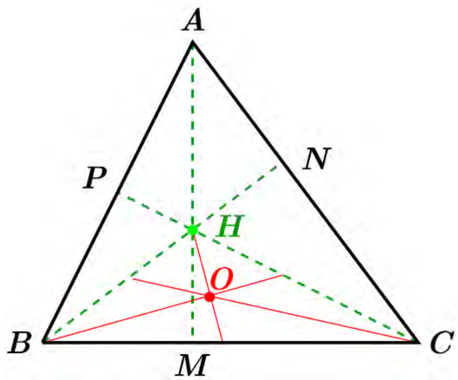
The angles A, B, and C of a triangle are:

$$A = 68^\circ \quad B = 62^\circ \quad C = 50^\circ$$

Let the point H be the intersection point of the heights of the triangle ABC.
The bisectors inside the triangle BHC intersect at a point O.

Find the angles: $\widehat{BOH}$, $\widehat{HOC}$, *and* $\widehat{COB}$

Solution



Triangle BNC:

$$\begin{aligned}\widehat{NBC} &= 90^\circ - C \\ &= 90^\circ - 50^\circ \\ &= 40^\circ\end{aligned}$$

Triangle BPC:

$$\begin{aligned}\widehat{PCB} &= 90^\circ - B \\ &= 90^\circ - 62^\circ \\ &= 28^\circ\end{aligned}$$

Triangle BHC:

$$\begin{aligned}\widehat{BHC} &= 180^\circ - C - \widehat{BPC} \\ &= 180^\circ - 40^\circ - 28^\circ \\ &= 112^\circ\end{aligned}$$

$$\begin{aligned}\widehat{OCB} &= \frac{1}{2}\widehat{BCP} \\ &= \frac{1}{2}(28^\circ) \\ &= 14^\circ\end{aligned}$$

$$\begin{aligned}\widehat{OBC} &= \frac{1}{2}\widehat{NBC} \\ &= \frac{1}{2}(40^\circ) \\ &= 20^\circ\end{aligned}$$

Triangle BOC:

$$\begin{aligned}\widehat{COB} &= 180^\circ - \widehat{OCB} - \widehat{OBC} \\ &= 180^\circ - 14^\circ - 20^\circ \\ &= 146^\circ\end{aligned}$$

$$\begin{aligned}\widehat{BHO} &= \frac{1}{2}\widehat{BHC} \\ &= \frac{1}{2}(112^\circ) \\ &= 56^\circ\end{aligned}$$

Triangle *BHO*:

$$\begin{aligned}\widehat{HOB} &= 180^\circ - \widehat{OBH} - \widehat{OHB} \\ &= 180^\circ - 20^\circ - 56^\circ \\ &= 104^\circ\end{aligned}$$

Exercise 35

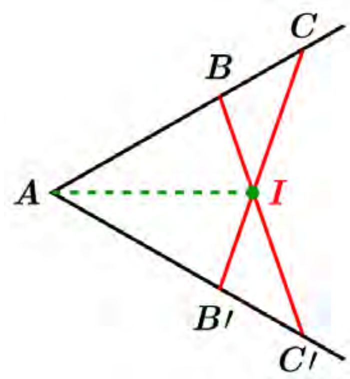
On the sides of an angle *A*, where the longest of $AB = AB'$ is $AC = AC'$

a) Prove that $BC' = B'C$

b) Let a point *I* be the intersection of the sides BC' and $B'C$. Prove that $IB = IB'$ and $IC = IC'$.

c) Prove that the point *I* is located on the bisector of angle *A*.

Solution



a) From the 2 triangles $AB'C$ and ABC', they have common:

- ✓ Common angle *A*.
- ✓ $AB = AB'$
- ✓ $AC = AC'$

The 2 triangles are equivalent.

Therefore, that implies to $BC' = B'C$.

b) From the 2 triangles IBC and $IB'C'$.

- ✓ $BC = B'C'$
- ✓ $\widehat{BIC} = \widehat{B'IC'}$
- ✓ $C = C'$

That implies to the 2 triangles IBC and $IB'C'$ are equivalent

Therefore, $IB = IB'$ and $IC = IC'$

c) Consider the 2 triangles ABI and $AB'I$

- ✓ $AB = AB'$
- ✓ $IB = IB'$
- ✓ Common side *AI*

That implies to

$$\widehat{IAB} = \widehat{IAB'}$$

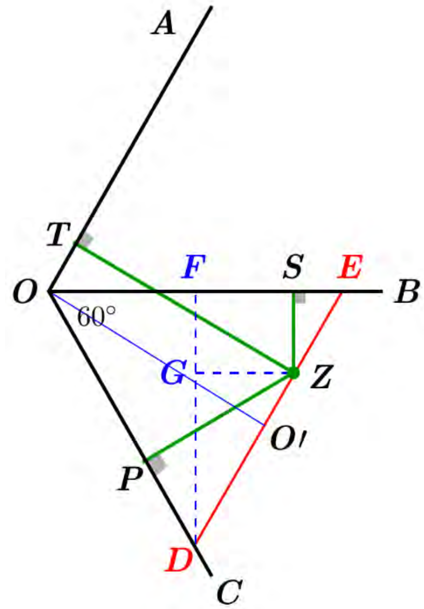
Therefore, *AI* is the ***bisector*** of angle *A*.

Exercise 36

Given two adjacent angles of $60°$, $\widehat{AOB}$ and $\widehat{BOC}$.

A point *Z* is located inside the angle $\widehat{BOC}$, where leads the perpendicular *ZP*, *ZT*, *ZS* on *OC*, *OA*, and *OB*.

Prove that $ZT = ZS + ZP$



Solution

Let the segment *DE* passing through the point *Z* and parallel to *OA*.

So that,

$\widehat{CDE} = 120°$

$\widehat{ODE} = 180° - 120°$

$= 60°$

And it is given that

$\widehat{DOE} = 60°$

Which implies that the triangle *ODE* is equilateral.

Let *OF* perpendicular to the side *OE*.

And since all the heights in the equilateral *ODE* triangle are equals $(OO' = DF)$.

Since side *ZT* is perpendicular to the line *OA*, then the points $OO'ZT$ is a rectangle with $OO' = ZT$

That implies to $DF = ZT$

Let the point *G* on the segment *DF* such that *ZG* perpendicular to *DF* and $ZS = GF$ from the rectangle *ZSFG*.

From the right triangle *DGZ* at *G*.

$\widehat{GDZ} = 30°$ (from the equilateral *ODE* triangle).

$\Rightarrow \widehat{GZD} = 60°$

Consider the 2 right triangles *DPZ* and *DGZ*, they have

- ✓ *ZD* common hypothenuse
- ✓ $\widehat{P} = \widehat{G} = 90°$
- ✓ $\widehat{PDZ} = \widehat{GZD} = 60°$

That imply to $GD = ZP$

$ZT = \overline{DF}$

$= \overline{DG} + \overline{GF}$

$= ZP + ZS$

Exercise 3

Solution