Solution   *Section* 1.4 – Lines and Curves in Space

Exercise

Find the parametric equation for the line through the point $P(3,\ -4,\ -1)$ parallel to the vector $\hat{i}+\hat{j}+\hat{k}$

Solution

$$\boxed{x=3+t,\quad y=-4+t,\quad z=-1+t}$$

Exercise

Find the parametric equation for the line through the points $P(1,\ 2,\ -1)$ and $Q(-1,0,1)$

Solution

The direction: $\overrightarrow{PQ}=-2\hat{i}-2\hat{j}+2\hat{k}$ and $P(1,\ 2,\ -1)$

$$\boxed{x=1-2t,\quad y=2-2t,\quad z=-1+2t}$$

Exercise

Find the parametric equation for the line through the points $P(-2,\ 0,\ 3)$ and $Q(3,\ 5,\ -2)$

Solution

The direction: $\overrightarrow{PQ}=5\hat{i}+5\hat{j}-5\hat{k}$ and $P(-2,\ 0,\ 3)$

$$\boxed{x=-2+5t,\quad y=5t,\quad z=3-5t}$$

Exercise

Find the parametric equation for the line through the origin parallel to the vector $2\hat{j}+\hat{k}$

Solution

The direction: $2\hat{i}+\hat{k}$ and $P(0,\ 0,\ 0)$

$$\boxed{x=0,\quad y=2t,\quad z=t}$$

Exercise

Find the parametric equation for the line through the point $P(3,\ -2,\ 1)$ parallel to the line

$x=1+2t,\quad y=2-t,\quad z=3t$

Solution

The direction: $2\hat{i} - \hat{j} + 3\hat{k}$ and $P(3, \ -2, \ 1)$

$\underline{x = 3 + 2t, \quad y = -2 - t, \quad z = 1 + 3t}|$

Exercise

Find the parametric equation for the line through $(2, \ 4, \ 5)$ perpendicular to the plane $3x + 7y - 5z = 21$

Solution

The direction: $3\hat{i} + 7\hat{j} - 5\hat{k}$ and $(2, 4, 5)$

$\underline{x = 2 + 3t, \quad y = 4 + 7t, \quad z = 5 - 5t}|$

Exercise

Find the parametric equation for the line through $(2, \ 3, \ 0)$ perpendicular to the vectors $\vec{u} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{v} = 3\hat{i} + 4\hat{j} + 5\hat{k}$

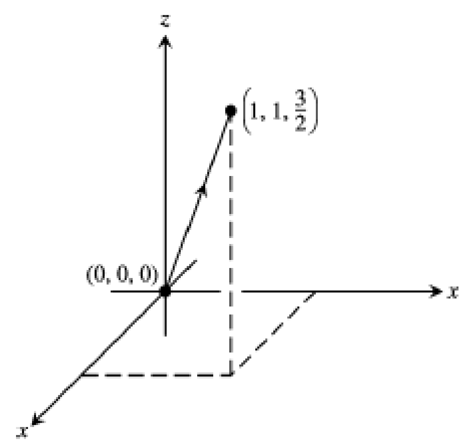
Solution

The direction: $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = -2\hat{i} + 4\hat{j} - 2\hat{k}$ and $(2, 3, 0)$

$\underline{x = 2 - 2t, \quad y = 3 + 4t, \quad z = -2t}|$

Exercise

Find the parameterization for the line segment joining the points. Draw coordinate axes and sketch the segment, indicate the direction on increasing t for the parametrization.



Solution

The direction: $\overrightarrow{PQ} = \hat{i} + \hat{j} + \frac{3}{2}\hat{k}$ and

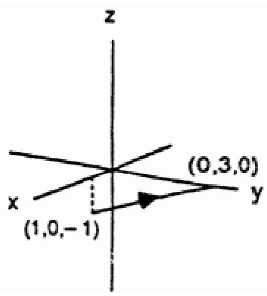
Exercise

Find the parameterization for the line segment joining the points $(1, \ 0, \ -1), \quad (0, \ 3, \ 0)$. Draw coordinate axes and sketch the segment, indicate the direction on increasing t for the parametrization.

Solution

The direction: $\overrightarrow{PQ} = -\hat{i} + 3\hat{j} + \hat{k}$ and $(1, \ 0, \ -1)$

$x = 1 - t, \quad y = 3t, \quad z = -1 + t, \quad 0 \le t \le 1$



Exercise

Find equation for the plane through normal to $\vec{n} = 3\hat{i} - 2\hat{j} - \hat{k}$

Solution

$$3(x-0) - 2(y-2) - (z+1) = 0$$

$$3x - 2y + 4 - z - 1 = 0$$

$$3x - 2y - z = -3$$

Exercise

Find equation for the plane through $(1, \ -1, \ 3)$ parallel to the plane $3x + y + z = 7$

Solution

$$3(x-1) + (y+1) + (z-3) = 0$$

$$3x - 3 + y + 1 + z - 3 = 0$$

$$3x + y + z = 5$$

Exercise

Find equation for the plane through $(1, \ 1, \ -1)$, $(2, \ 0, \ 2)$ and $(0, \ -2, \ 1)$

Solution

$$\overrightarrow{PQ} = \hat{i} - \hat{j} + 3\hat{k} \qquad \overrightarrow{PS} = -\hat{i} - 3\hat{j} + 2\hat{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PS} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = 7\hat{i} - 5\hat{j} - 4\hat{k} \ \text{ is normal to the plane.}$$

$$7(x-2) - 5(y+0) - 4(z-2) = 0$$

$$7x - 14 - 5y - 4z + 8 = 0$$

$$7x - 5y - 4z = 6$$

Exercise

Find equation for the plane through $P_0(2,\ 4,\ 5)$ perpendicular to the line $x = 5 + t, \quad y = 1 + 3t, \quad z = 4t$

Solution

$$x = 5 + t, \quad y = 1 + 3t, \quad z = 4t \quad \Rightarrow \quad \vec{n} = \hat{i} + 3\hat{j} + 4\hat{k}$$

$$1(x-2) + 3(y-4) + 4(z-5) = 0$$

$$x - 2 + 3y - 12 + 4z - 20 = 0$$

$$x + 3y + 4z = 34$$

Exercise

Find equation for the plane through $A(1,\ -2,\ 1)$ perpendicular to the vector from the origin to A.

Solution

$$\Rightarrow \quad \vec{n} = \hat{i} - 2\hat{j} + \hat{k}$$

$$1(x-1) - 2(y+2) + 1(z-1) = 0$$

$$x - 1 - 2y - 4 + z - 1 = 0$$

$$x - 2y + z = 6$$

Exercise

Find the point of intersection of the lines $x = 2t + 1, \quad y = 3t + 2, \quad z = 4t + 3$ and $x = s + 2, \quad y = 2s + 4, \quad z = -4s - 1$, and find the plane determined by these lines.

Solution

$$\begin{cases} x = 2t + 1 = s + 2 \\ y = 3t + 2 = 2s + 4 \\ z = 4t + 3 = -4s - 1 \end{cases} \Rightarrow \begin{cases} 2t - s = 1 \\ 3t - 2s = 2 \end{cases} \rightarrow \boxed{t = 0} \quad \boxed{s = -1}$$

$$z = 4t + 3 = -4s - 1 \quad \Rightarrow \quad 4(0) + 3 = -4(-1) - 1 \rightarrow 3 = 3 \ \surd \text{ (satisfied)}$$

The lines intersect when $t = 0$ and $s = -1 \Rightarrow$ The point of intersection $x = 1, \quad y = 2, \quad z = 3$

Therefore; the point is $P(1,\ 2,\ 3)$

The normal vectors: $\vec{n}_1 = 2\hat{i} + 3\hat{j} + 4\hat{k} \quad \vec{n}_2 = \hat{i} + 2\hat{j} - 4\hat{k}$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix} = -20\hat{i} + 12\hat{j} + \hat{k} \quad \boldsymbol{n}_1 \ \textit{and} \ \boldsymbol{n}_2 \text{ are directions of the lines.}$$

The plane containing the lines is represented by

$$-20(x-1)+12(y-2)+1(z-3)=0$$

$$\Rightarrow \quad \underline{-20x+12y+z=7}\,\rfloor$$

Exercise

Find the plane determined by the intersecting lines:

$$L_1 : x=-1+t, \quad y=2+t, \quad z=1-t; \quad -\infty < t < \infty$$

$$L_2 : x=1-4s, \quad y=1+2s, \quad z=2-2s; \quad -\infty < s < \infty$$

Solution

The normal vectors: $\vec{n}_1 = \hat{i} + \hat{j} - \hat{k} \quad \vec{n}_2 = -4\hat{i} + 2\hat{j} - 2\hat{k}$

$$\vec{n} = \vec{n}_1 \times \vec{n}_2$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{vmatrix}$$

$$= 6\hat{j} + 6\hat{k}$$

Let $t=0$ $L_1 : x=-1, \quad y=2, \quad z=1; \quad \Rightarrow \underline{P(-1,\ 2,\ 1)}\,\rfloor$

Therefore; the desired plane is:

$$0(x+1)+6(y-2)+6(z-1)=0$$

$$6y-12+6z-6=0$$

$$6y+6z=18 \quad \Rightarrow \quad \underline{y+z=3}\,\rfloor$$

Exercise

Find a plane through $P_0\ (2,\ 1,\ -1)$ and perpendicular to the line of intersection of the planes $2x+y-z=3, \quad x+2y+z=2$

Solution

The normal vectors: $\vec{n}_1 = 2\hat{i} + \hat{j} - \hat{k} \quad \vec{n}_2 = \hat{i} + 2\hat{j} + \hat{k}$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$= 3\hat{i} - 3\hat{j} + 3\hat{k}$ is the vector in the direction of the line of intersection of the planes.

$$\Rightarrow 3(x-2)-3(y-1)+3(z+1)=0$$

$$3x - 3y + 3z = 0$$

$$\boxed{x - y + z = 0} \quad \text{is the desired plane containing } P_0\left(2,\ 1,\ -1\right)$$

Exercise

Find the distance from the point to the plane $\left(0,\ 0,\ 12\right), \quad x = 4t, \quad y = -2t, \quad z = 2t$

Solution

At $t = 0 \Rightarrow P\left(0,\ 0,\ 0\right)$ and let $S\left(0,\ 0,\ 12\right)$

$\overrightarrow{PS} = 12\hat{k}$ and $\vec{v} = 4\hat{i} - 2\hat{j} + 2\hat{k}$

$$\overrightarrow{PS} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 12 \\ 4 & -2 & 2 \end{vmatrix}$$

$$= 24\hat{i} + 48\hat{j}$$

$$d = \frac{\left|\overrightarrow{PS} \times \vec{v}\right|}{\left|\vec{v}\right|}$$

$$= \frac{\sqrt{24^2 + 48^2}}{\sqrt{16 + 4 + 4}}$$

$$= \frac{24\sqrt{5}}{\sqrt{24}}$$

$$= \sqrt{5}\sqrt{24}$$

$$\boxed{= 2\sqrt{30}}$$

Exercise

Find the distance from the point to the plane $\left(2,\ 1,\ -1\right), \quad x = 2t, \quad y = 1 + 2t, \quad z = 2t$

Solution

At $t = 0 \Rightarrow P\left(0,\ 1,\ 0\right)$ and let $S\left(2,\ 1,\ -1\right)$

$\overrightarrow{PS} = 2\hat{i} - \hat{k}$ and $\vec{v} = 2\hat{i} + 2\hat{j} + 2\hat{k}$

$$\overrightarrow{PS} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -1 \\ 2 & 2 & 2 \end{vmatrix}$$

$$= 2\hat{i} - 6\hat{j} + 4\hat{k}$$

$$d = \frac{\sqrt{4+36+16}}{\sqrt{4+4+4}} \qquad\qquad d = \frac{\left|\overrightarrow{PS}\times\vec{v}\right|}{|\vec{v}|}$$

$$= \frac{\sqrt{56}}{\sqrt{12}}$$

$$= \frac{2\sqrt{14}}{2\sqrt{3}}$$

$$= \sqrt{\frac{14}{3}} \quad unit$$

Exercise

Find the distance from the point to the plane $(3, \ -1, \ 4), \quad x = 4 - t, \quad y = 3 + 2t, \quad z = -5 + 3t$

Solution

At $t = 0 \Rightarrow P(4, \ 3, \ -5)$ and let $S(3, \ -1, \ 4)$

$\overrightarrow{PS} = -\hat{i} - 4\hat{j} + 9\hat{k}$ and $\vec{v} = -\hat{i} + 2\hat{j} + 3\hat{k}$

$$\overrightarrow{PS}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 3 \\ -1 & 2 & 3 \end{vmatrix}$$

$$= -30\hat{i} - 6\hat{j} - 6\hat{k}$$

$$d = \frac{\sqrt{900+36+36}}{\sqrt{1+4+9}} \qquad\qquad d = \frac{\left|\overrightarrow{PS}\times\vec{v}\right|}{|\vec{v}|}$$

$$= \sqrt{\frac{972}{14}}$$

$$= \sqrt{\frac{486}{7}}$$

$$= \frac{9\sqrt{6}}{\sqrt{7}}\frac{\sqrt{7}}{\sqrt{7}}$$

$$= \frac{9\sqrt{42}}{7} \quad unit$$

Exercise

Find the distance from the point to the plane $(2, \ -3, \ 4), \quad x + 2y + 2z = 13$

Solution

$\Rightarrow P(13, \ 0, \ 0)$ and let $S(2, \ -3, \ 4)$

$\overrightarrow{PS} = -11\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{n} = \hat{i} + 2\hat{j} + 2\hat{k}$

$|\vec{n}| = \sqrt{1+4+4} \ \underline{= 3}|$

$d = \left|\left(-11\hat{i} - 3\hat{j} + 4\hat{k}\right) \bullet \left(\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right)\right| \qquad d = \left|\overrightarrow{PS} \bullet \frac{\vec{n}}{|\vec{n}|}\right|$

$= \left|-\frac{11}{3} - \frac{6}{3} + \frac{8}{3}\right|$

$\underline{= 3 \quad unit}|$

Exercise

Find the distance from the point to the plane $(0,\ 0,\ 0), \quad 3x + 2y + 6z = 6$

Solution

$3x + 2y + 6z = 6$

$3x + 2(0) + 6(0) = 6 \ \rightarrow \ \underline{x = 2}|$

$\Rightarrow \ P(2,\ 0,\ 0)$ and let $S(0,\ 0,\ 0)$

$\overrightarrow{PS} = -2\hat{i}$ and $\vec{n} = 3\hat{i} + 2\hat{j} + 6\hat{k}$

$\rightarrow |\boldsymbol{n}| = \sqrt{9+4+36} = \underline{7}|$

$d = \left|\left(-2\hat{i}\right) \bullet \left(\frac{3}{7}\hat{i} + \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}\right)\right| \qquad d = \frac{\left|\overrightarrow{PS} \bullet \vec{n}\right|}{|\vec{n}|}$

$\underline{= \frac{6}{7} \quad unit}|$

Exercise

Find the distance from the point to the plane $(0,\ 1,\ 1), \quad 4y + 3z = -12$

Solution

$\Rightarrow \ P(0,\ -3,\ 0)$ and let $S(0,\ 1,\ 1)$

$\overrightarrow{PS} = 4\hat{j} + \hat{k}$ and $\vec{n} = 4\hat{j} + 3\hat{k}$

$\rightarrow |\vec{n}| = \sqrt{16+9} \ \underline{= 5}|$

$d = \left|\left(4\hat{j} + \hat{k}\right) \bullet \left(\frac{4}{5}\hat{j} + \frac{3}{5}\hat{k}\right)\right| \qquad d = \frac{\left|\overrightarrow{PS} \bullet \vec{n}\right|}{|\vec{n}|}$

$= \left|\frac{16}{5} + \frac{3}{5}\right|$

$\underline{= \frac{19}{5} \quad unit}|$

Exercise

Find the distance from the point to the plane $(6,\ 0,\ -6),\quad x-y=4$

Solution

Let $y=0$, then the point P (4, 0 ,0) lies on the line $x-y=4$

$\overrightarrow{PS}=2\hat{i}-6\hat{k}$ and $\vec{n}=\hat{i}-\hat{j}$

$$d=\frac{|2+0+0|}{\sqrt{1+1}} \qquad\qquad d=\frac{\left|\overrightarrow{PS}\cdot\vec{n}\right|}{|\vec{n}|}$$

$$=\frac{2}{\sqrt{2}}$$

$$=\sqrt{2}$$

Exercise

Find the distance from the point to the plane $(3,\ 0,\ 10),\quad 2x+3y+z=2$

Solution

Let $y=z=0$, then the point P (1, 0 ,0) lies on the line $2x+3y+z=2$

$\overrightarrow{PS}=2\hat{i}+10\hat{k}$ and $\vec{n}=2\hat{i}+3\hat{j}+\hat{k}$

$$d=\frac{|4+0+10|}{\sqrt{4+9+1}} \qquad\qquad d=\frac{\left|\overrightarrow{PS}\cdot\vec{n}\right|}{|\vec{n}|}$$

$$=\frac{14}{\sqrt{14}}$$

$$=\sqrt{14}$$

Exercise

Find the distance from the point to the line $(2,\ 2,\ 0);\quad x=-t,\quad y=t,\quad z=-1+t$

Solution

The line passes through the point $P(0, 0, -1)$ parallel to $\vec{v}=-\hat{i}+\hat{j}+\hat{k}$

$\overrightarrow{PS}=2\hat{i}+2\hat{j}+\hat{k}$

$$\overrightarrow{PS}\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 2 & 1\\ -1 & 1 & 1\end{vmatrix}$$

$$=\hat{i}-3\hat{j}+4\hat{k}$$

$$d = \frac{\sqrt{1+9+16}}{\sqrt{1+1+1}} \qquad\qquad d = \frac{\left|\overrightarrow{PS} \times \vec{v}\right|}{|\vec{v}|}$$

$$= \frac{\sqrt{26}}{\sqrt{3}}$$

$$= \frac{\sqrt{78}}{3}$$

Exercise

Find the distance from the point to the line $(0,\ 4,\ 1);\quad x = 2+t,\quad y = 2+t,\quad z = t$

Solution

The line passes through the point $P(2, 2, 0)$ parallel to $\vec{v} = \hat{i} + \hat{j} + \hat{k}$

$$\overrightarrow{PS} = -2\hat{i} + 2\hat{j} + \hat{k}$$

$$\overrightarrow{PS} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \hat{i} + 3\hat{j} - 4\hat{k}$$

$$d = \frac{\sqrt{1+9+16}}{\sqrt{1+1+1}} \qquad\qquad d = \frac{\left|\overrightarrow{PS} \times \vec{v}\right|}{|\vec{v}|}$$

$$= \frac{\sqrt{26}}{\sqrt{3}}$$

$$= \frac{\sqrt{78}}{3}$$

Exercise

Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$

Solution

$$x + 2y + 6z = 1 \quad \Rightarrow \quad P(1,\ 0,\ 0)$$

$$x + 2y + 6z = 10 \quad \Rightarrow \quad S(10,\ 0,\ 0)$$

$$\overrightarrow{PS} = 9\hat{i} \ \text{ and } \ \vec{n} = \hat{i} + 2\hat{j} + 6\hat{k}$$

$$\rightarrow |\vec{n}| = \sqrt{1+4+36} \ = \sqrt{41}$$

$$d = \left|\left(9\hat{i}\right) \bullet \frac{1}{\sqrt{41}}\left(\hat{i} + 2\hat{j} + 6\hat{k}\right)\right| \qquad\qquad d = \left|\overrightarrow{PS} \bullet \frac{\vec{n}}{|\vec{n}|}\right|$$

$= \frac{1}{\sqrt{41}}|9|$

$\boxed{= \frac{9}{\sqrt{41}}}$

Exercise

Find the angle between the planes $x + y = 1, \quad 2x + y - 2z = 2$

Solution

The vectors: $\vec{n}_1 = \hat{i} + \hat{j}, \quad \vec{n}_2 = 2\hat{i} + \hat{j} - 2\hat{k}$ are normal to the planes.

The angle between them is:

$$\theta = \cos^{-1}\left(\frac{2+1}{\sqrt{1+1}\ \sqrt{4+1+4}}\right) \qquad \theta = \cos^{-1}\left(\frac{\vec{n}_1 \bullet \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right)$$

$$= \cos^{-1}\left(\frac{3}{3\sqrt{2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\boxed{= \frac{\pi}{4}}$$

Exercise

Find the angle between the planes $5x + y - z = 10, \quad x - 2y + 3z = -1$

Solution

The vectors: $\vec{n}_1 = 5\hat{i} + \hat{j} - \hat{k}, \quad \vec{n}_2 = \hat{i} - 2\hat{j} + 3\hat{k}$ are normal to the planes.

The angle between them is:

$$\theta = \cos^{-1}\left(\frac{5-2-3}{\sqrt{25+1+1}\sqrt{1+4+9}}\right) \qquad \theta = \cos^{-1}\left(\frac{\vec{n}_1 \bullet \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right)$$

$$= \cos^{-1}(0)$$

$$\boxed{= \frac{\pi}{2}}$$

Exercise

Find the angle between the planes $\quad x = 7, \quad x + y + \sqrt{2}z = -3$

Solution

The vectors: $\vec{n}_1 = \hat{i}, \quad \vec{n}_2 = \hat{i} + \hat{j} + \sqrt{2}\hat{k}$ are normal to the planes.

$$\theta = \cos^{-1}\left(\frac{1+0+0}{1\cdot\sqrt{1+1+2}}\right) \qquad \theta = \cos^{-1}\left(\frac{\vec{n}_1 \bullet \vec{n}_2}{\left|\vec{n}_1\right|\left|\vec{n}_2\right|}\right)$$

$$= \cos^{-1}\left(\frac{1}{2}\right)$$

$$\boxed{= \frac{\pi}{3}}$$

Exercise

Find the angle between the planes $x + y = 1, \quad y + z = 1$

Solution

The vectors: $\vec{n}_1 = \hat{i} + \hat{j}, \quad \vec{n}_2 = \hat{j} + \hat{k}$ are normal to the planes.

$$\theta = \cos^{-1}\left(\frac{0+1+1}{\sqrt{1+1}\cdot\sqrt{1+1}}\right) \qquad \theta = \cos^{-1}\left(\frac{\vec{n}_1 \bullet \vec{n}_2}{\left|\vec{n}_1\right|\left|\vec{n}_2\right|}\right)$$

$$= \cos^{-1}\left(\frac{1}{2}\right)$$

$$\boxed{= \frac{\pi}{3}}$$

Exercise

Find the point in which the line meets the plane $x = 1 - t, \quad y = 3t, \quad z = 1 + t; \quad 2x - y + 3z = 6$

Solution

$$2(1-t) - 3t + 3(1+t) = 6$$

$$2 - 2t - 3t + 3 + 3t = 6$$

$$-2t = 1$$

$$\boxed{t = -\frac{1}{2}}$$

$$x = 1 - \left(-\frac{1}{2}\right) = \frac{3}{2}, \quad y = 3\left(-\frac{1}{2}\right) = -\frac{3}{2}, \quad z = 1 + \left(-\frac{1}{2}\right) = \frac{1}{2}$$

$$\Rightarrow \boxed{P\left(\frac{3}{2},\ -\frac{3}{2},\ \frac{1}{2}\right)}$$

Exercise

Find the point in which the line meets the plane $x = 2, \quad y = 3 + 2t, \quad z = -2 - 2t; \quad 6x + 3y - 4z = -12$

Solution

$$12+3(3+2t)-4(-2-2t)=-12$$
$$12+9+6t+8+8t=-12$$
$$14t=-41$$
$$\boxed{t=-\frac{41}{14}}$$
$$x=2, \quad y=3+2\left(-\frac{41}{14}\right)=-\frac{20}{7}, \quad z=-2-2\left(-\frac{41}{14}\right)=\frac{27}{7}$$
$$\boxed{P\left(2,\ -\frac{20}{7},\ \frac{27}{7}\right)}$$

Exercise

Find an equation of the line through the point $(0,\ 1,\ 1)$ and parallel to the line $\boldsymbol{R}(t)=\langle 1+2t,\ 3-5t,\ 7+6t\rangle$

Solution

Direction: $\vec{v}=\langle 2,\ -5,\ 6\rangle$

Line: $\langle 0,\ 1,\ 1\rangle+t\langle 2,\ -5,\ 6\rangle$

$$\boxed{=\langle 2t,\ 1-5t,\ 1+6t\rangle}$$

Exercise

Find an equation of the line through the point $(0,\ 1,\ 1)$ that is orthogonal to both $\langle 0,\ -1,\ 3\rangle$ and $\langle 2,\ -1,\ 2\rangle$

Solution

Direction: $\langle 0,\ -1,\ 3\rangle\times\langle 2,\ -1,\ 2\rangle=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & -1 & 3\\ 2 & -1 & 2\end{vmatrix}$

$$=\langle 1,\ 6,\ 2\rangle$$

Line through $(0,\ 1,\ 1)$: $\begin{cases} x=t\\ y=1+6t\\ z=1+2t\end{cases}$

Exercise

Find an equation of the line through the point $(0,\ 1,\ 1)$ that is orthogonal to the vector $\langle -2,\ 1,\ 7\rangle$ and the *y-axis*

Solution

$(0,\ 1,\ 1)\ \perp\ \langle -2,\ 1,\ 7\rangle\ \ \&\ y-axis$

$$\text{Direction: } \langle -2,\ 1,\ 7\rangle \times \langle 0,\ 1,\ 0\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 7 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \langle -7,\ 0,\ -2\rangle$$

$$\text{Line through } (0,\ 1,\ 1): \begin{cases} x = -7t \\ y = 1 \\ z = 1 - 2t \end{cases}$$

Exercise

Suppose that $\boldsymbol{n}$ is normal to a plane and that $\boldsymbol{v}$ is parallel to the plane. Describe how you would find a vector $\boldsymbol{n}$ that is both perpendicular to $\boldsymbol{v}$ and parallel to the plane.

Solution

The desired vector is $\boldsymbol{n} \times \boldsymbol{v}$ or $\boldsymbol{v} \times \boldsymbol{n}$, since $\boldsymbol{n} \times \boldsymbol{v}$ is perpendicular to both $\boldsymbol{n}$ and $\boldsymbol{v}$, therefore, also parallel to the plane

Exercise

Given a point $(x_0,\ y_0,\ 0)$ and a vector $\boldsymbol{v} = \langle a,\ b,\ 0\rangle$ in $\mathbb{R}^3$, describe the set of points that satisfy the equation $\langle a,\ b,\ 0\rangle \times \langle x - x_0,\ y - y_0,\ 0\rangle = \mathbf{0}$. Use this result to determine an equation of a line in $\mathbb{R}^2$ passing through $(x_0,\ y_0)$ parallel to the vector $\langle a,\ b\rangle$

Solution

$$\langle a,\ b,\ 0\rangle \times \langle x - x_0,\ y - y_0,\ 0\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & 0 \\ x - x_0 & y - y_0 & 0 \end{vmatrix}$$

$$= \langle 0,\ 0,\ a(y - y_0) - b(x - x_0)\rangle$$

$$= \langle 0,\ 0,\ 0\rangle$$

$$a\left(y-y_0\right)-b\left(x-x_0\right)=0$$

$$a\left(y-y_0\right)=b\left(x-x_0\right)$$

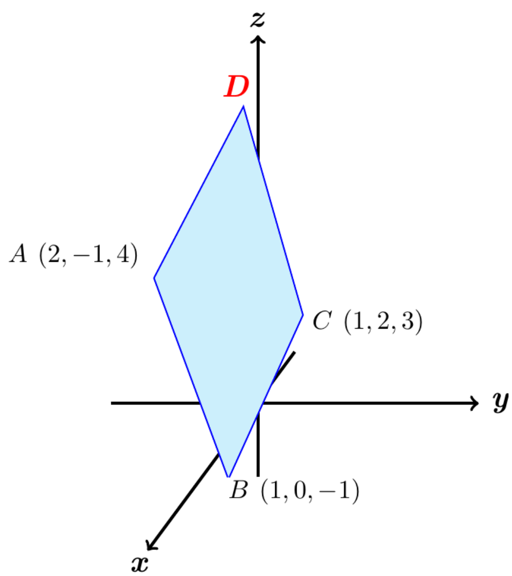
$$\frac{y-y_0}{x-x_0}=\frac{b}{a}=m \qquad (slope)$$

Equation of a line passing through $\left(x_0,\ y_0\right)$ parallel to the vector $\langle a,\ b\rangle$

$$\boxed{ay-bx=+ay_0-bx_0}$$

Exercise

The parallelogram has vertices at $A(2, -1, 4)$, $B(1, 0, -1)$, $C(1, 2, 3)$ and D. Find



a) The coordinates of D,

b) The cosine of the interior angle of B

c) The vector projection of $\overrightarrow{BA}$ onto $\overrightarrow{BC}$,

d) The area of the parallelogram,

e) An equation for the plane of the parallelogram,

f) The areas of the orthogonal projection of the parallelogram on the three coordinate planes.

Solution

a) $\overrightarrow{AB}=\langle 1-2,\ 0+1,\ -1-4\rangle$

$$=\langle -1,\ 1,\ -5\rangle$$

$$\overrightarrow{DC}=\langle 1-x,\ \ 2-y,\ \ 3-z\rangle$$

$$\overrightarrow{DC}=\overrightarrow{AB}$$

$$\langle 1-x,\ \ 2-y,\ \ 3-z\rangle = \langle -1,\ 1,\ \ -5\rangle$$

$$\begin{cases} 1-x=-1 & \rightarrow \quad x=2 \\ 2-y=1 & \rightarrow \quad y=1 \\ 3-z=-5 & \rightarrow \quad z=8 \end{cases}$$

$$\Rightarrow \quad \boxed{D = (2,\ 1,\ 8)}$$

b) $\overrightarrow{BA} = \langle 1,\ \ -1,\ 5\rangle$

$$\overrightarrow{BC} = \langle 1-1,\ 2-0,\ 3+1\rangle$$

$$= \langle 0,\ 2,\ 4\rangle$$

$$\cos B = \frac{\overrightarrow{BA} \bullet \overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\ \left|\overrightarrow{BC}\right|}$$

$$= \frac{\langle 1,\ \ -1,\ 5\rangle \bullet \langle 0,\ 2,\ 4\rangle}{\sqrt{1+1+25}\ \sqrt{4+16}}$$

$$= \frac{0-2+20}{\sqrt{27}\ \sqrt{20}}$$

$$= \frac{18}{3\sqrt{3}\ 2\sqrt{5}}$$

$$\boxed{= \frac{3}{\sqrt{15}}}$$

c) $proj_{\overrightarrow{BC}}\ \overrightarrow{BA} = \dfrac{\overrightarrow{BA} \bullet \overrightarrow{BC}}{\left|\overrightarrow{BC}\right|^2}\ \overrightarrow{BC}$

$$= \frac{\langle 1,\ \ -1,\ 5\rangle \bullet \langle 0,\ 2,\ 4\rangle}{4+16}\langle 0,\ 2,\ 4\rangle$$

$$= \frac{18}{20}\langle 0,\ 2,\ 4\rangle$$

$$= \frac{9}{10}\langle 0,\ 2,\ 4\rangle$$

$$\boxed{= \left\langle 0,\ \frac{9}{5},\ \frac{18}{5}\right\rangle}$$

d) $Area = \left|\overrightarrow{BA} \times \overrightarrow{BC}\right|$

$$= \left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 5 \\ 0 & 2 & 4 \end{matrix}\right\|$$

$$= \left|-14\hat{i} - 4\hat{j} + 2\hat{k}\right|$$

$$= \sqrt{196+16+4}$$

$$= \sqrt{216}$$

$$\boxed{= 6\sqrt{6}}$$

e) $\overrightarrow{BA} \times \overrightarrow{BC} = -14\hat{i} - 4\hat{j} + 2\hat{k} = \vec{n}$

$-14(x-1) - 4y + 2(z+1) = 0$

$-14x + 14 - 4y + 2z + 2 = 0$

$-14x - 4y + 2z = -16$

$$\boxed{7x + 2y - z = 8}$$

f) $\vec{n} = -14\hat{i} - 4\hat{j} + 2\hat{k}$

Area of the projection on $yz-plane$ $|\vec{n} \bullet \hat{i}|$ $\boxed{= 14}$

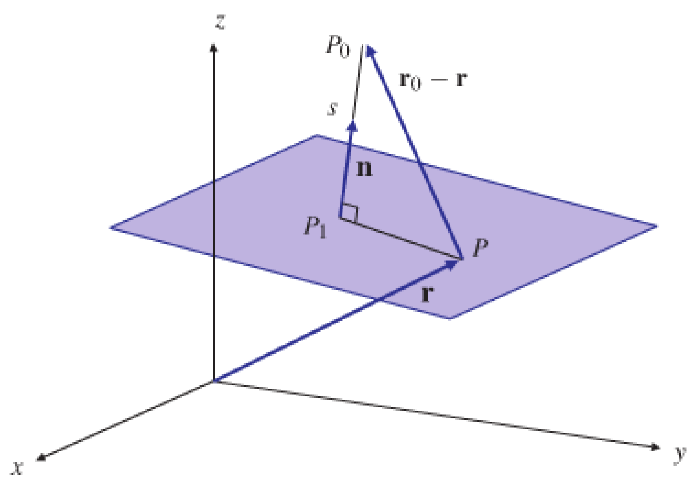
Area of the projection on $xz-plane$ $|\vec{n} \bullet \hat{j}|$ $\boxed{= 4}$

Area of the projection on $xy-plane$ $|\vec{n} \bullet \hat{k}|$ $\boxed{= 2}$

Exercise

a) Find the distance from the point $P_0\left(x_0, y_0, z_0\right)$ to the plane P having equation

$Ax + By + Cz = D$



b) What is the distance from $(2, \ -1, \ 3)$ to the plane $2x - 2y - z = 9$?

Solution

a) $\vec{n} = A\hat{i} + B\hat{j} + C\hat{k}$

$|\vec{n}| = \sqrt{A^2 + B^2 + C^2}$

Let $y = z = 0 \quad \to \quad P = \left(\frac{D}{A}, \ 0, \ 0\right)$

$\overrightarrow{PP_0} = \left\langle x_0 - \frac{D}{A}, \ y_0, \ z_0 \right\rangle$

$$d = \left(\left(x_0 - \frac{D}{A}\right)\hat{i} + y_0\hat{j} + z_0\hat{k}\right) \bullet \frac{A\hat{i} + B\hat{j} + C\hat{k}}{\sqrt{A^2 + B^2 + C^2}}$$

$$d = \overrightarrow{PP_0} \bullet \frac{\vec{n}}{|\vec{n}|}$$

$$= \frac{1}{\sqrt{A^2 + B^2 + C^2}}\left(\left(x_0 - \frac{D}{A}\right)A + y_0 B + z_0 C\right)$$

$$= \frac{Ax_0 + By_0 + Cz_0 - D}{\sqrt{A^2 + B^2 + C^2}}$$

b) Distance $(2,\ -1,\ 3)$ to $2x - 2y - z = 9$

$$d = \frac{|2(2) - 2(-1) + (-1)(3) - 9|}{\sqrt{4 + 4 + 1}}$$

$$= \frac{|-6|}{3}$$

$$= 2\ \ units$$