

## Section 5.5 – Infinite Sequences and Summation Notation

An arbitrary *infinite sequence* may be denoted as follows:

$$a_1, a_2, a_3, \dots, a_n, \dots$$

An infinite sequence is a function whose domain is the set of positive integers.

### ***Example***

Find the first four terms and the tenth term of the sequence:  $\left\{ \frac{n}{n+1} \right\}$

#### **Solution**

$$n = 1 \rightarrow \frac{1}{1+1} = \frac{1}{2}$$

$$n = 2 \rightarrow \frac{2}{2+1} = \frac{2}{3}$$

$$n = 3 \rightarrow \frac{3}{3+1} = \frac{3}{4}$$

$$n = 4 \rightarrow \frac{4}{4+1} = \frac{4}{5}$$

$$n = 10 \Rightarrow \frac{10}{11}$$

### ***Example***

Find the first four terms and the tenth term of the sequence:  $\left\{ 2 + (0.1)^n \right\}$

#### **Solution**

$$n = 1 \rightarrow 2 + 0.1 = 2.1$$

$$n = 2 \rightarrow 2 + 0.1^2 = 2.01$$

$$n = 3 \rightarrow 2 + 0.1^3 = 2.001$$

$$n = 4 \rightarrow 2 + 0.1^4 = 2.0001$$

$$n = 10 \Rightarrow 2.0000000001$$

**Example**

Find the first four terms and the tenth term of the sequence:  $\left\{(-1)^{n+1} \frac{n^2}{3n-1}\right\}$

**Solution**

$$n=1 \rightarrow (-1)^2 \frac{1^2}{3(1)-1} = \frac{1}{2}$$

$$n=2 \rightarrow (-1)^3 \frac{2^2}{3(2)-1} = -\frac{4}{5}$$

$$n=3 \rightarrow (-1)^4 \frac{3^2}{3(3)-1} = \frac{9}{8}$$

$$n=4 \rightarrow (-1)^5 \frac{4^2}{3(4)-1} = -\frac{16}{11}$$

$$n=10 \Rightarrow \underline{-\frac{100}{29}}$$

**Example**

Find the first four terms and the tenth term of the sequence:  $\{4\}$

**Solution**

$$n=1 \rightarrow 4$$

$$n=2 \rightarrow 4$$

$$n=3 \rightarrow 4$$

$$n=4 \rightarrow 4$$

$$n=10 \Rightarrow 4$$

**Example**

Find the first four terms of the recursively defined infinite sequence  $a_1 = 3, \quad a_{n+1} = (n+1)a_n$

**Solution**

$$\underline{a_1 = 3}$$

$$n=1 \rightarrow a_2 = (1+1)a_1 = 2(3) = \underline{6}$$

$$n=2 \rightarrow a_3 = (2+1)a_2 = 3(6) = \underline{18}$$

$$n=3 \rightarrow a_4 = (3+1)a_3 = 4(18) = \underline{72}$$

## Summation Notation

To find the sum of many terms of an infinite sequence, it is easy to express using summation notation.

$$\sum_{\substack{n=1 \\ \text{First value of } n}}^{\substack{5 \\ \text{Last value of } n}} 2n + 3 \quad \leftarrow \text{Formula for each term}$$

### Example

Find the sum:  $\sum_{k=1}^4 k^2(k-3)$

### Solution

$$\begin{aligned} \sum_{k=1}^4 k^2(k-3) &= 1^2(1-3) + 2^2(2-3) + 3^2(3-3) + 4^2(4-3) \\ &= -2 - 4 + 0 + 16 \\ &= \underline{10} \end{aligned}$$

### Theorem on the Sum of a *Constant*

$$(1) \sum_{k=1}^n c = nc \qquad (2) \sum_{k=m}^n c = (n - m + 1)c$$

**Proof:**

$$\sum_{k=1}^n c = \underbrace{c + c + \dots + c}_n = nc$$

### Example

Find the sum:  $\sum_{k=10}^{20} 5$

### Solution

$$\begin{aligned} \sum_{k=10}^{20} 5 &= (20 - 10 + 1)5 \\ &= \underline{55} \end{aligned}$$

### ***Theorem on Sums***

If  $a_1, a_2, a_3, \dots, a_n, \dots$  and  $b_1, b_2, b_3, \dots, b_n, \dots$  are infinite sequences, then for every positive integer  $n$ ,

$$(1) \quad \sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$$

$$(2) \quad \sum_{k=1}^n (a_k - b_k) = \sum_{k=1}^n a_k - \sum_{k=1}^n b_k$$

$$(3) \quad \sum_{k=1}^n c a_k = c \left( \sum_{k=1}^n a_k \right)$$

### **Proof**

$$\begin{aligned} \sum_{k=1}^n (a_k + b_k) &= (a_1 + b_1) + (a_2 + b_2) + \dots + (a_n + b_n) \\ &= (a_1 + a_2 + \dots + a_n) + (b_1 + b_2 + \dots + b_n) \\ &= \sum_{k=1}^n a_k + \sum_{k=1}^n b_k \end{aligned}$$

### ***Example***

Express the sum using summation notation  $2^1 + 2^2 + 2^3 + \dots + 2^{16}$

### **Solution**

$$2^1 + 2^2 + 2^3 + \dots + 2^{16} = \sum_{k=1}^{16} 2^k$$

## Exercises      Section 5.5 – Infinite Sequences and Summation Notation

(1 – 13) Find the first four terms and the eight term of the sequence:

1.  $\{12 - 3n\}$

6.  $\left\{(-1)^{n-1} \frac{n}{2n-1}\right\}$

10.  $\{c_n\} = \{(-1)^{n+1} n^2\}$

2.  $\left\{\frac{3n-2}{n^2+1}\right\}$

7.  $\left\{\frac{2^n}{3^n+1}\right\}$

11.  $\{c_n\} = \left\{\frac{(-1)^n}{(n+1)(n+2)}\right\}$

3.  $\{9\}$

8.  $\left\{\frac{n^2}{2^n}\right\}$

12.  $\{c_n\} = \left\{\left(\frac{4}{3}\right)^n\right\}$

4.  $\left\{(-1)^{n-1} \frac{n+7}{2n}\right\}$

9.  $\left\{\frac{n}{e^n}\right\}$

13.  $\{b_n\} = \left\{\frac{3^n}{n}\right\}$

14. Graph the sequence  $\left\{\frac{1}{\sqrt{n}}\right\}$

15. Find the first four terms of the sequence of partial sums for the given sequence.  $\left\{3 + \frac{1}{2}n\right\}$

(16 – 27) Find the first five terms of the recursively defined infinite sequence

16.  $a_1 = 2, \quad a_{k+1} = 3a_k - 5$

22.  $a_1 = 2, \quad a_{n+1} = 7 - 2a_n$

17.  $a_1 = -3, \quad a_{k+1} = a_k^2$

23.  $a_1 = 128, \quad a_{n+1} = \frac{1}{4}a_n$

18.  $a_1 = 5, \quad a_{k+1} = ka_k$

24.  $a_1 = 2, \quad a_{n+1} = (a_n)^n$

19.  $a_1 = 2, \quad a_n = 3 + a_{n-1}$

25.  $a_1 = A, \quad a_n = a_{n-1} + d$

20.  $a_1 = 5, \quad a_n = 2a_{n-1}$

26.  $a_1 = A, \quad a_n = ra_{n-1}, \quad r \neq 0$

21.  $a_1 = \sqrt{2}, \quad a_n = \sqrt{2 + a_{n-1}}$

27.  $a_1 = 2, \quad a_2 = 2; \quad a_n = a_{n-1} \cdot a_{n-2}$

(28 – 37) Express each sum using summation notation

28.  $1 + 2 + 3 + \dots + 20$

34.  $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \dots + (-1)^6 \frac{1}{3^6}$

29.  $1 + 2 + 3 + \dots + 40$

30.  $1^3 + 2^3 + 3^3 + \dots + 8^3$

35.  $\frac{2}{3} - \frac{4}{9} + \frac{8}{27} - \dots + (-1)^{12} \left(\frac{2}{3}\right)^{11}$

31.  $1^2 + 2^2 + 3^2 + \dots + 15^2$

36.  $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots + \frac{14}{14+1}$

32.  $2^2 + 2^3 + 2^4 + \dots + 2^{11}$

37.  $\frac{1}{e} + \frac{2}{e^2} + \frac{3}{e^3} + \dots + \frac{n}{e^n}$

33.  $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots + \frac{13}{14}$

(38 – 52) Find the sum

38.  $\sum_{k=1}^5 (2k - 7)$

43.  $\sum_{k=1}^{40} k$

48.  $\sum_{k=1}^{16} (k^2 - 4)$

39.  $\sum_{k=0}^5 k(k - 2)$

44.  $\sum_{k=1}^5 (3k)$

49.  $\sum_{k=1}^6 (10 - 3k)$

40.  $\sum_{k=1}^5 (-3)^{k-1}$

45.  $\sum_{k=1}^{10} (k^3 + 1)$

50.  $\sum_{k=1}^{10} [1 + (-1)^k]$

41.  $\sum_{k=253}^{571} \left(\frac{1}{3}\right)$

46.  $\sum_{k=1}^{24} (k^2 - 7k + 2)$

51.  $\sum_{k=1}^6 \frac{3}{k+1}$

42.  $\sum_{k=1}^{50} 8$

47.  $\sum_{k=6}^{20} (4k^2)$

52.  $\sum_{k=137}^{428} 2.1$

(53 – 56) Write out each sum

53.  $\sum_{k=1}^n (k + 2)$

55.  $\sum_{k=2}^n (-1)^k \ln k$

57.  $\sum_{k=0}^n \frac{1}{3^k}$

54.  $\sum_{k=1}^n k^2$

56.  $\sum_{k=3}^n (-1)^{k+1} 2^k$

58. Fred has a balance of \$3,000 on his card which charges 1% interest per month on any unpaid balance. Fred can afford to pay \$100 toward the balance each month. His balance each month after making a \$100 payment is given by the recursively defined sequence

$$B_0 = \$3,000 \quad B_n = 1.01B_{n-1} - 100$$

Determine Fred's balance after making the first payment. That is, determine  $B_1$

59. A pond currently has 2,000 trout in it. A fish hatchery decides to add an additional 20 trout each month. Is it also known that the trout population is growing at a rate of 3% per month. The size of the population after  $n$  months is given by the recursively defined sequence

$$P_0 = 2,000 \quad P_n = 1.03P_{n-1} + 20$$

How many trout are in the pond after 2 months? That is, what is  $P_2$ ?

60. Fred bought a car by taking out a loan for \$18,500 at 0.5% interest per month. Fred's normal monthly payment is \$434.47 per month, but he decides that he can afford to pay \$100 extra toward the balance each month. His balance each month is given by the recursively defined sequence

$$B_0 = \$18,500 \quad B_n = 1.005B_{n-1} - 534.47$$

Determine Fred's balance after making the first payment. That is, determine  $B_1$

61. The Environmental Protection Agency (EPA) determines that Maple Lake has 250 *tons* of pollutant as a result of industrial waste and that 10% of the pollutant present is neutralized by solar oxidation every year. The EPA imposes new pollution control laws that result in 15 *tons* of new pollutant entering the lake each year. The amount of pollutant in the lake after  $n$  years is given by the recursively defined sequence

$$P_0 = 250 \quad P_n = 0.9P_{n-1} + 15$$

Determine the amount of pollutant in the lake after 2 years? That is, what is  $P_2$ ?

62. Let 
$$u_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}$$

Define the  $n$ th term of a sequence

- Show that  $u_1 = 1$  and  $u_2 = 1$
- Show that  $u_{n+2} = u_{n+1} + u_n$
- Draw the conclusion that  $\{u_n\}$  is a Fibonacci sequence
- Find the first ten terms of the sequence from part (c)