Section 2.8 – Properties of the Normal Distribution

The *standard normal distribution* has three properties:

1. It is graph is bell-shaped.
2. It is mean is equal to 0 ($\mu = 0$).
3. It is standard deviation is equal to 1 ($\sigma = 1$).

Definitions

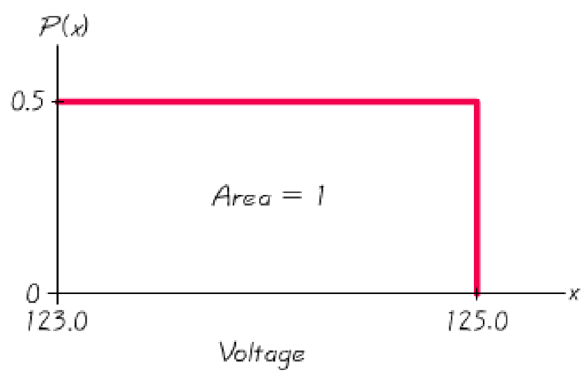
A continuous random variable has a ***uniform distribution*** if its values are spread evenly over the range of probabilities. The graph of a uniform distribution results in a rectangular shape.

A ***probability density function*** (***pdf***) is an equation used to compute probabilities of continuous random variables. It must satisfy the following two properties:

1. The total area under the graph of the equation over all possible values of the random variable must equal 1.
2. The height of the graph of the equation must be greater than or equal to 0 for all possible values of the random variable.

Example

The Power and Light Company provides electricity with voltage levels that are uniformly distributed between 123.0 volts and 125.0 volts. That is, any voltage amount between 123.0 volts and 125.0 volts is possible, and all of the possible values are equally likely. If we randomly select one of the voltage levels and represent its value by the random variable *x*, than *x* has a distribution that can me graphed as below



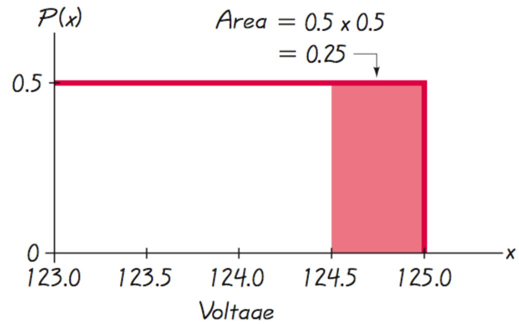
Uniform Distribution of Voltage Levels

Area and Probability

Because the total area under the density curve is equal to 1, there is a correspondence between ***area*** and ***probability***.

Example

Given the uniform distribution illustrated in the figure below, find the probability that a randomly voltage level is greater than 124.5 volts.



Shaded area represents voltage levels greater than 124.5 volts.

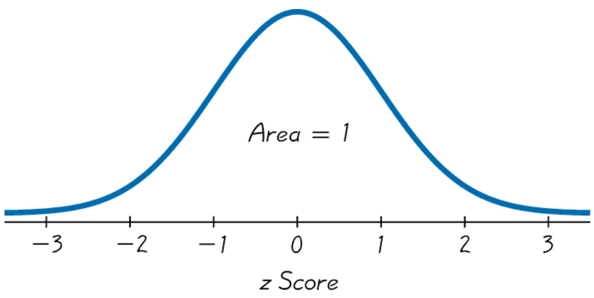
$$
\begin{aligned}
P(\text{voltage greater than } 124.5 \text{ volts}) &= \text{area of the shaded region} \\
&= 0.5 \times 0.5 \\
&= 0.25
\end{aligned}
$$

The probability of randomly selecting a voltage level greater than 124.5 volts is 0.25.

Standard Normal Distribution

Definition

The standard normal distribution is a normal probability distribution with $\mu = 0$ and $\sigma = 1$. The total area under its density curve is equal to 1.



Finding Probabilities: When Given z-scores

- Standard Normal distribution Table.

Using Table (Normal Distribution Table):

1. It is designed only for the *standard* normal distribution, which has a mean of 0 and a standard deviation of 1.
2. It is on two pages, with one page for *negative* z-scores and the other page for *positive* z-scores.
3. Each value in the body of the table is a *cumulative area from the left* up to a vertical boundary above a specific z-score.
4. When working with a graph, avoid confusion between z-scores and areas.

   z Score: Distance along horizontal scale of the standard normal distribution; refer to the leftmost column and top row of Normal Distribution Table.

   Area Region under the curve; refer to the values in the body of Normal Distribution Table.

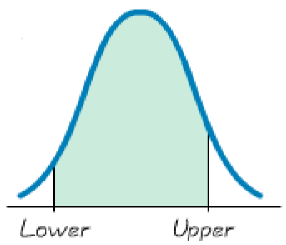
5. The part of the z-score denoting hundredths is found across the top.

- Formulas and Tables insert card
- Find areas for many different regions

Methods for Finding Normal Distribution Areas

TI-83/84 Plus Calculator

Gives area bounded on the left and bounded on the right by vertical lines above any specific values.



TI-83/84 Press 2ND VARS **[2: normal cdf ( ],** then enter the two z scores separated by a comma, as in (left z score, right z score).

Example

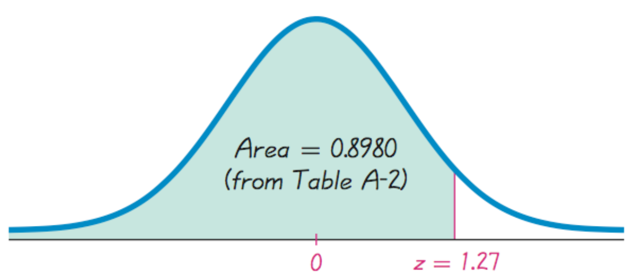
The Precision Scientific Instrument Company manufactures thermometers that are supposed to give readings of 0ºC at the freezing point of water. Tests on a large sample of these instruments reveal that at the freezing point of water, some thermometers give readings below 0º (denoted by negative numbers) and some give readings above 0º (denoted by positive numbers). Assume that the mean reading is 0ºC and the standard deviation of the readings is 1.00ºC. Also assume that the readings are normally distributed. If one thermometer is randomly selected, find the probability that, at the freezing point of water, the reading is less than 1.27º.

Solution

z	.00	.01	.02	.03	.04	.05	.06	.07
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292

$P(z < 1.27) = 0.8980$

The *probability* of randomly selecting a thermometer with a reading less than 1.27º is 0.8980. Or 89.80% will have readings below 1.27º.



Example

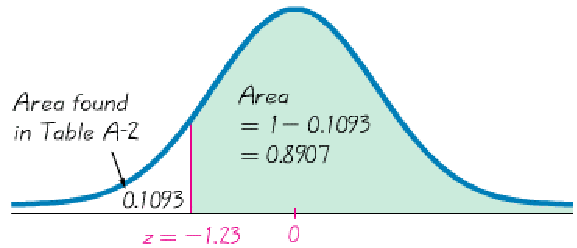
If thermometers have an average (mean) reading of 0 degrees and a standard deviation of 1 degree for freezing water, and if one thermometer is randomly selected, find the probability that it reads (at the freezing point of water) above –1.23 degrees.

Solution

$P(z > -1.23) = 1 - 0.1093 = \underline{= 0.8907}$

Probability of randomly selecting a thermometer with a reading above –1.23º is 0.8907.

89.07% of the thermometers have readings above –1.23 degrees.

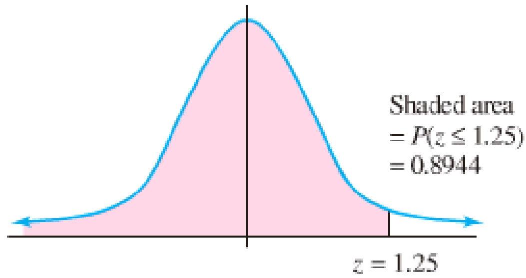


Example

Find the areas under the standard normal curve

a) The area to the ***left*** of $z = 1.25$

$A = P(z \le 1.25) \underline{= 0.8944}$ **(*Using left curve table*)**

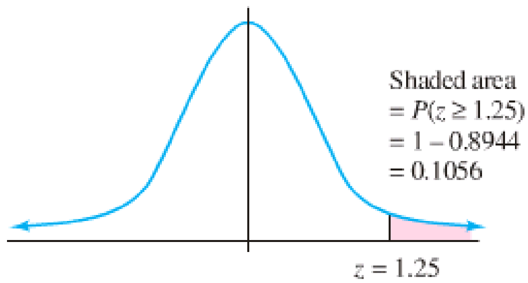


b) The area to the ***right*** of $z = 1.25$

$A = P(z \ge 1.25)$ **(*Using left curve table*)**

$= 1 - 0.8944$

$\underline{= 0.1056}$

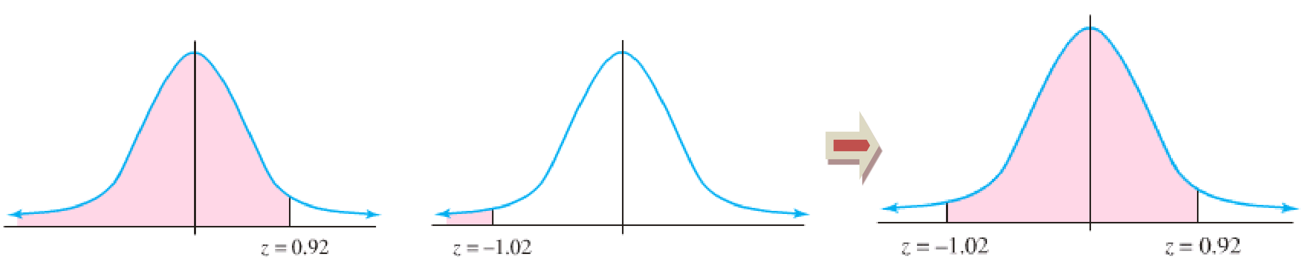


c) The area ***between*** $z = -1.02$ and

$A = P(-1.02 \le z \le 0.92)$ **(*Using left curve table*)**

$= 0.8212 - 0.1539$

$\underline{= 0.6673}$



Notation

$P(a < z < b)$ denotes the probability that the *z* score is between *a* and *b*.

$P(z > a)$ denotes the probability that the *z* score is greater than *a*.

$P(z < a)$ denotes the probability that the *z* score is less than *a*.

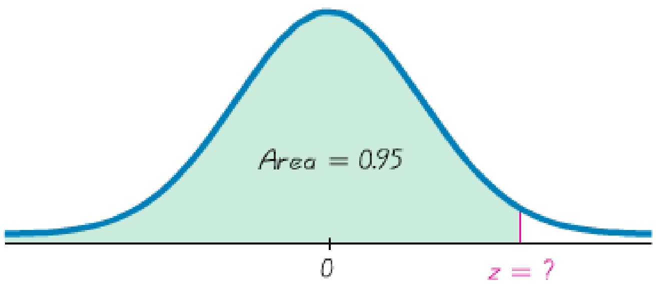
Finding a *z*–Score When Given a Probability Using Table

1. Draw a bell-shaped curve and identify the region under the curve that corresponds to the given probability. If that region is not a cumulative region from the left, work instead with a known region that is a cumulative region from the left.
2. Using the cumulative area from the left, locate the closest probability in the body of Table and identify the corresponding *z* score.

Example

With temperature readings at the freezing point of water that are normally distributed with a mean 0°C and a standard deviation of 1.00°C. Find the temperature corresponding to P_{95} , the 95th percentile. That is, find the temperature separating the bottom 95 from the top 5%.

Solution

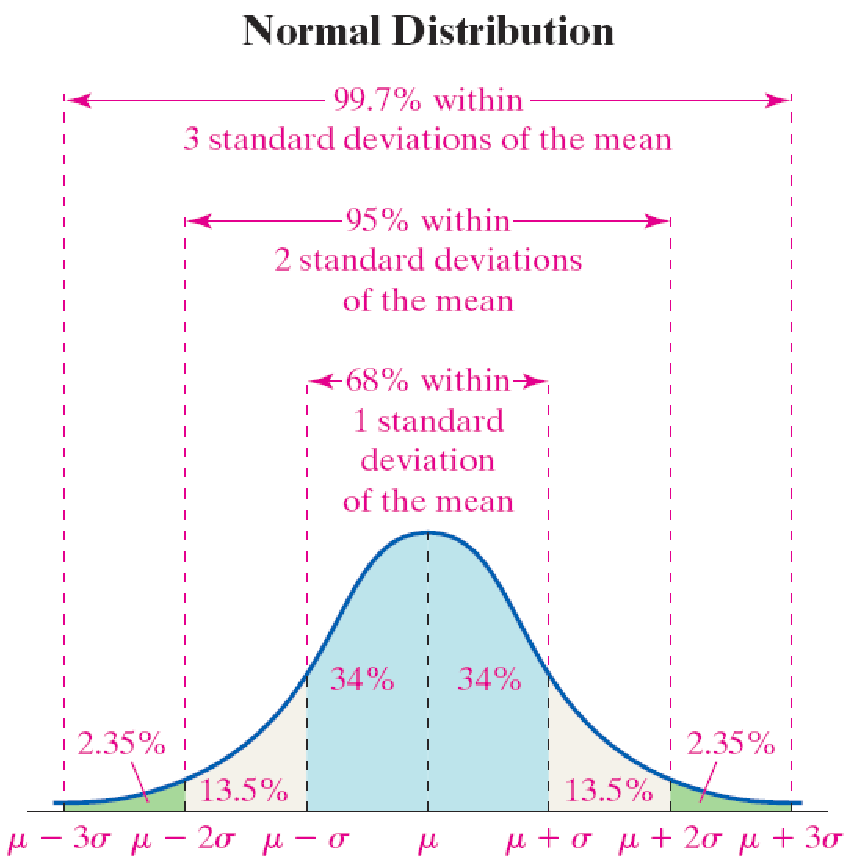


Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449

From the table, we find the areas of 0.9495 and 0.9505. The area 0.95 corresponds to a *z*-score of 1.645. When tested at freezing, 95% of the readings will be less than or equal to 1.645°C, and 5% of them will be greater than or equal to 1.645°C.

Properties of the Normal Density Curve

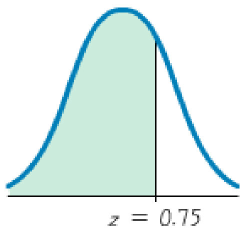
1. It is symmetric about its mean, μ.
2. Because ***mean*** = ***median*** = ***mode***, the curve has a single peak and the highest point occurs at $x = \mu$.
3. It has inflection points at $\mu - \sigma$ and $\mu - \sigma$
4. The area under the curve is 1.
5. The area under the curve to the right of μ equals the area under the curve to the left of μ, which equals 1/2.
6. As x increases without bound (gets larger and larger), the graph approaches, but never reaches, the horizontal axis. As x decreases without bound (gets more and more negative), the graph approaches, but never reaches, the horizontal axis.
7. The Empirical Rule: Approximately 68% of the area under the normal curve is between $x = \mu - \sigma$ and $x = \mu + \sigma$; approximately 95% of the area is between $x = \mu - 2\sigma$ and $x = \mu + 2\sigma$; approximately 99.7% of the area is between $x = \mu - 3\sigma$ and $x = \mu + 3\sigma$.



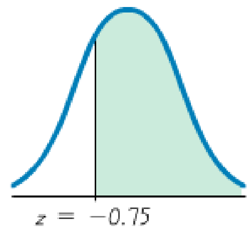
Exercises *Section* 2.8 – Properties of the Normal Distribution

1. Find the area shaded region. The graph depicts the standard distribution with mean 0 and standard deviation 1.

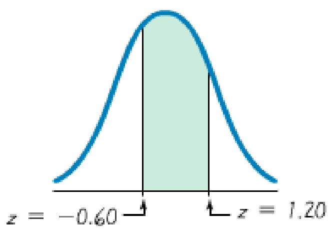
a)



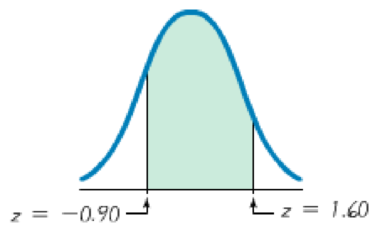
b)



c)

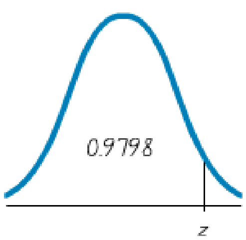


d)

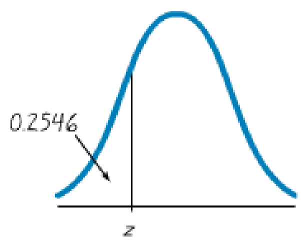


2. Find the indicated *z*-score. The graph depicts the standard distribution with mean 0 and standard deviation 1.

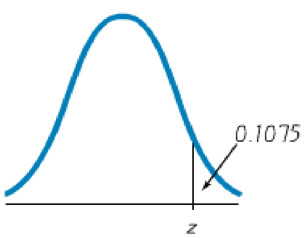
a)



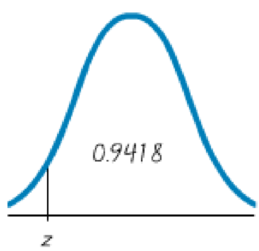
b)



c)



d)



3. Assume that thermometer readings are normally distributed with a mean of 0ºC and the standard deviation of the readings is 1.00ºC. A thermometer is randomly selected and tested. In each case, draw a sketch, and find the probability of each reading.

a) Less than –1.50
b) Less than –2.75
c) Less than 1.23
d) Greater than 2.22
e) Greater than 2.33
f) Greater than –1.75
g) Between 0.50 and 1.00
h) Between –3.00 and –1.00
i) Between –1.20 and 1.95
j) Between –2.50 and 5.00
k) Greater than 0
l) Less than 0

4. Assume that thermometer readings are normally distributed with a mean of 0ºC and the standard deviation of the readings is 1.00º*C*. A thermometer is randomly selected and tested. In each case, draw a sketch, and find the temperature reading corresponding to the given information.

a) Find P_{95}, the 95th percentile. This is the temperature separating the bottom 95% from the top 5%.

b) Find P_1, the 1st percentile. This is the temperature separating the bottom 1% from the top 99%.

c) If 2.5% of the thermometers are rejected because they have readings that are too high and another 2.5% are rejected because they have readings that are too low, find the 2 readings that are cutoff values separating the rejected thermometers from the others.

d) If 0.5% of the thermometers are rejected because they have readings that are too high and another 0.5% are rejected because they have readings that are too low, find the 2 readings that are cutoff values separating the rejected thermometers from the others.

5. For a standard normal distribution, find the percentage of data that are

a) Within 2 standard deviations of the mean.

b) More than 1 standard deviation away from the mean.

c) More than 1.96 standard deviations away from the mean.

d) Between $\mu - 3\sigma$ and $\mu + 3\sigma$.

e) More than 3 standard deviations away from the mean.