Section 1.8 – Applications

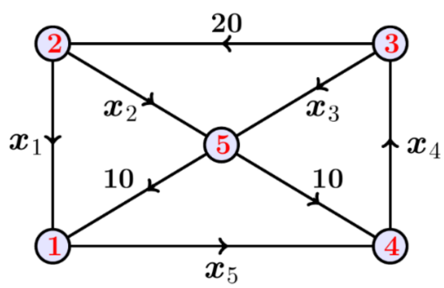
Network Analysis

Networks composed of branches and junctions are used as models in such fields as economics, traffic analysis, and electrical engineering.

In a network model, you assume that the total flow into a junction is equal to the total flow out of the junction

Example

Set up a system of linear equations to represent the network shown below. Then solve the system for $x_i, \quad i = 1,\ 2,\ 3,\ 4,\ 5.$



Solution

$1 \rightarrow \quad x_1 + 10 = x_5 \quad \Rightarrow \quad x_1 - x_5 = -10$

$2 \rightarrow \quad x_1 + x_2 = 20$

$3 \rightarrow \quad x_4 = x_3 + 20 \quad \Rightarrow \quad -x_3 + x_4 = 20$

$4 \rightarrow \quad x_4 = x_5 + 10 \quad \Rightarrow \quad x_4 - x_5 = 10$

$5 \rightarrow \quad x_2 + x_3 = 10 + 10 = 20$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 1 & 1 & 0 & 0 & 20 \end{array}\right) \quad R_2 - R_1$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 1 & 1 & 0 & 0 & 20 \end{array}\right) \quad R_5 - R_2$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 1 & 0 & -1 & -10 \end{array}\right) \quad R_5 + R_3$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 10 \end{array}\right) \quad R_5 - R_4$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{ll} \rightarrow x_1 - x_5 = -10 & \Rightarrow \boxed{x_1 = x_5 - 10} \\ \rightarrow x_2 + x_5 = 30 & \Rightarrow \boxed{x_2 = 30 - x_5} \\ \rightarrow -x_3 + x_4 = 20 & \Rightarrow \boxed{x_3 = x_5 - 10} \\ \rightarrow x_4 - x_5 = 10 & \Rightarrow \boxed{x_4 = x_5 + 10} \end{array}$$

Solution: $\boxed{\left(x_5 - 10,\ \ 30 - x_5,\ \ x_5 - 10,\ \ 10 + x_5,\ \ x_5\right)}$

2nd Method

$$\begin{vmatrix} 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= -1 + 1$$

$$\boxed{= 0}$$

Infinite solution:

$1 \rightarrow \ \boxed{x_1 = x_5 - 10}$

$2 \rightarrow \ x_2 = 20 - x_1 \ \boxed{= 30 - x_5}$

$4 \rightarrow \ \boxed{x_4 = x_5 + 10}$

$3 \rightarrow \ x_3 = x_4 - 20 \ \boxed{= x_5 - 10}$

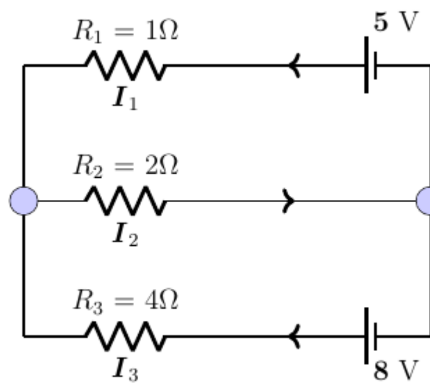
Electrical network

An electrical network is another type of network where analysis is commonly applied. An analysis of such a system uses two properties of electrical networks known as **Kirchhoff's Laws.**

- All the current flowing into a junction must flow out of it.
- The sum of the products *IR* (*I* is current and *R* is resistance) around a closed path is equal to the total voltage in the path.

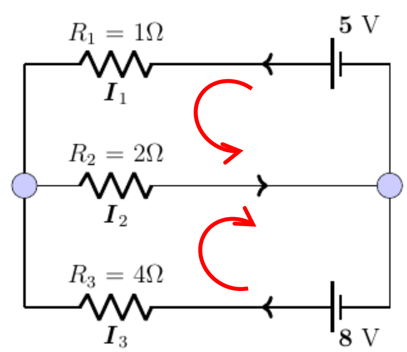
Example

Determine the currents I_1, I_2, and I_3 for the electrical network



Solution

$$I_2 = I_1 + I_3$$
$$I_1 + 2I_2 = 5$$
$$2I_2 + 4I_3 = 8$$



$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ I_1 + 2I_2 = 5 \\ I_2 + 2I_3 = 4 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 7 \qquad D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 5 & 2 & 0 \\ 4 & 1 & 2 \end{vmatrix} = 7 \qquad D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 5 & 0 \\ 0 & 4 & 2 \end{vmatrix} = 14 \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 5 \\ 0 & 1 & 4 \end{vmatrix} = 7$$

$\boxed{I_1 = 1 \ A}$ $\boxed{I_2 = 2 \ A}$ $\boxed{I_3 = 1 \ A}$

Cryptography

A ***cryptogram*** is a message written according to a secret code (the Greek word ***kryptos*** means "*hidden*"). One method of using matrix multiplication to ***encode*** and ***decode*** messages.

Let assign a number to each letter in the alphabet (with 0 assigned to a blank space), as shown

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

Example

Consider the invertible matrix: $A=\begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix}$

The message: ***MEET ME MONDAY***

a) Write the uncoded row matrices 1×3 for the message.

b) Use the matrix A to encode the message.

c) Decode a message from part *b*) given the matrix A.

Solution

a)

$$\begin{array}{ccccc} M\ E\ E & T\ \_\ M & E\ \_\ M & O\ N\ D & A\ Y\ \_ \\ \begin{bmatrix}13 & 5 & 5\end{bmatrix} & \begin{bmatrix}20 & 0 & 13\end{bmatrix} & \begin{bmatrix}5 & 0 & 13\end{bmatrix} & \begin{bmatrix}15 & 14 & 4\end{bmatrix} & \begin{bmatrix}1 & 25 & 0\end{bmatrix} \end{array}$$

b) Let encode the message ***MEET ME MONDAY***

$$\begin{bmatrix}13 & 5 & 5\end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}=\begin{bmatrix}13 & -26 & 21\end{bmatrix}$$

$$\begin{bmatrix}20 & 0 & 13\end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}=\begin{bmatrix}33 & -53 & -12\end{bmatrix}$$

$$\begin{bmatrix}5 & 0 & 13\end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}=\begin{bmatrix}18 & -23 & -42\end{bmatrix}$$

$$\begin{bmatrix}15 & 14 & 4\end{bmatrix}\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}=\begin{bmatrix}5 & -20 & 56\end{bmatrix}$$

$$\begin{bmatrix}1 & 25 & 0\end{bmatrix}\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}=\begin{bmatrix}-24 & 23 & 77\end{bmatrix}$$

The sequence of coded row matrices is

$$\begin{bmatrix}13 & -26 & -21\end{bmatrix}\ \begin{bmatrix}33 & -53 & -12\end{bmatrix}\ \begin{bmatrix}18 & -23 & -42\end{bmatrix}\ \begin{bmatrix}5 & -20 & 56\end{bmatrix}\ \begin{bmatrix}-24 & 23 & 77\end{bmatrix}$$

The cryptogram:

$$13\ \ -26\ \ -21\ \ 33\ \ -53\ \ -12\ \ 18\ \ -23\ \ -42\ \ 5\ \ -20\ \ 56\ \ -24\ \ 23\ \ 77$$

c) To decode a message given the matrix A.

$$|A|=\begin{vmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{vmatrix}=1$$

$$A^{-1}=\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}$$

With the cryptogram:

$$\begin{bmatrix}13 & -26 & -21\end{bmatrix}\ \begin{bmatrix}33 & -53 & -12\end{bmatrix}\ \begin{bmatrix}18 & -23 & -42\end{bmatrix}\ \begin{bmatrix}5 & -20 & 56\end{bmatrix}\ \begin{bmatrix}-24 & 23 & 77\end{bmatrix}$$

$$\begin{bmatrix}13 & -26 & 21\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}13 & 5 & 5\end{bmatrix}$$

$$\begin{bmatrix}33 & -53 & -1\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}20 & 0 & 13\end{bmatrix}$$

$$\begin{bmatrix}18 & -23 & -42\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}5 & 0 & 13\end{bmatrix}$$

$$\begin{bmatrix}5 & -20 & 56\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}15 & 14 & 4\end{bmatrix}$$

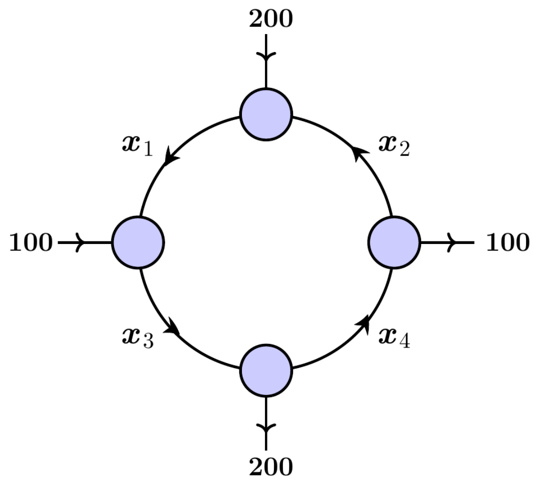
$$\begin{bmatrix} -24 & 23 & 77 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 25 & 0 \end{bmatrix}$$

The message is:

13	5	5	20	0	13	5	0	13	15	14	4	1	25	0
M	*E*	*E*	*T*	_	*M*	*E*	_	*M*	*O*	*N*	*D*	*A*	*Y*	_

Exercises *Section* 1.8 – Applications

1. The flow of traffic, in vehicles per hour, through a network of streets as is shown below



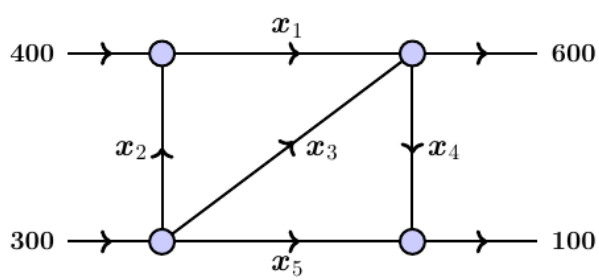
a) Solve this system for x_i, $i = 1, 2, 3, 4$.

b) Find the traffic flow when $x_4 = 0$.

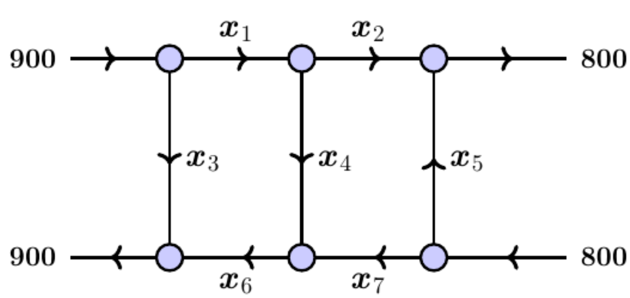
c) Find the traffic flow when $x_4 = 100$.

d) Find the traffic flow when $x_1 = 2x_2$.

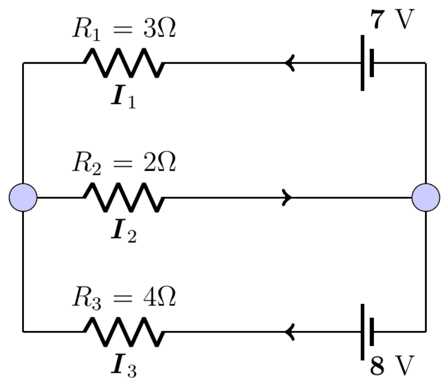
2. Through a network, Express x_n's in terms of the parameters ***s*** and ***t***.



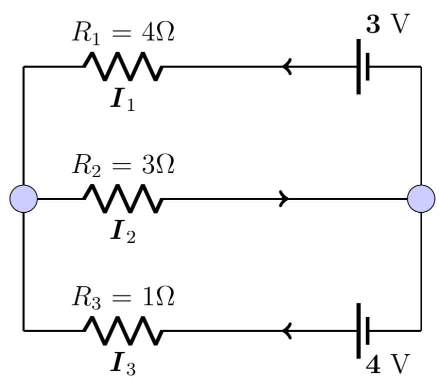
3. Water is flowing through a network of pipes. Express x_n's in terms of the parameters ***s*** and ***t***.



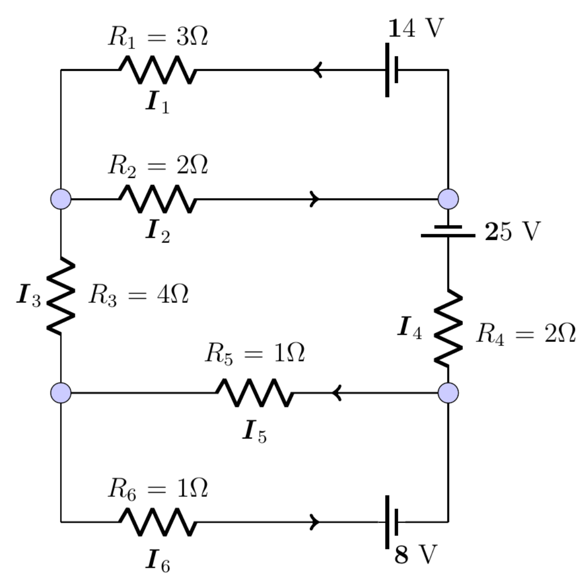
4. Determine the currents $I_1, I_2, and\ I_3$ for the electrical network shown below



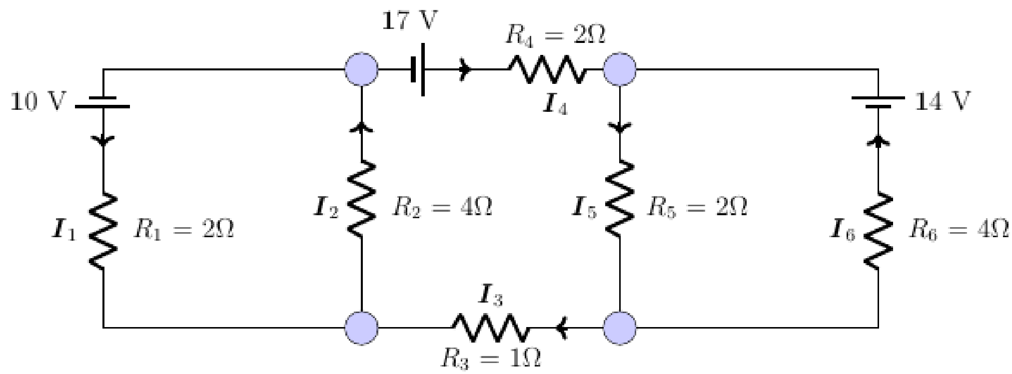
5. Determine the currents $I_1, I_2, and\ I_3$ for the electrical network shown below



6. Determine the currents $I_1, I_2, I_3, I_4, I_5, and\ I_6$ for the electrical network shown below



7. Determine the currents $I_1, I_2, I_3, I_4, I_5,$ *and* I_6 for the electrical network shown below



8. Consider the invertible matrix: $A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -2 & -2 & 7 \end{pmatrix}$

The message: ***ICEBERG DEAD AHEAD***

a) Write the uncoded for the message.
b) Use the matrix A to encode the message.
c) Decode a message from part *b*) given the matrix A.

9. You want to send the message: ***LINEAR ALGEBRA*** with a key word ***MATH***

a) Write the matrix A.
b) Write the uncoded for the message.
c) Use the matrix A to encode the message.
d) Decode a message from part *b*) given the matrix A.

10. Consider the invertible matrix: $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Decode the cryptogram 27 14 48 28 5 5 21 20 50 25

11. Consider the invertible matrix: $A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix}$

Decode the cryptogram

1 −5 11 19 −25 −45 11 −16 −28 20 −29 −27
12 −12 −53 40 −61 −35 8 −17 7