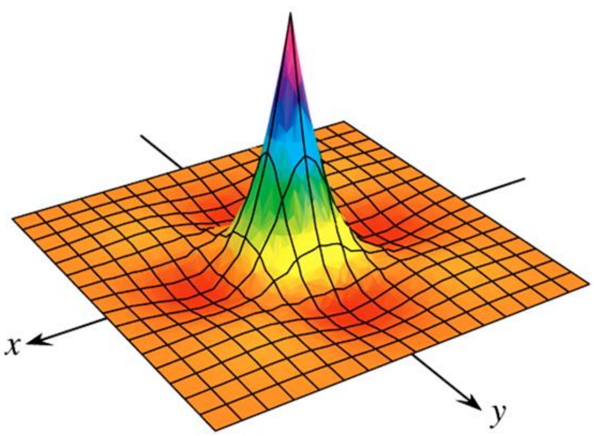
Section 2.7 – Maximum/Minimum Problems

Derivative Tests for Local Extreme Values

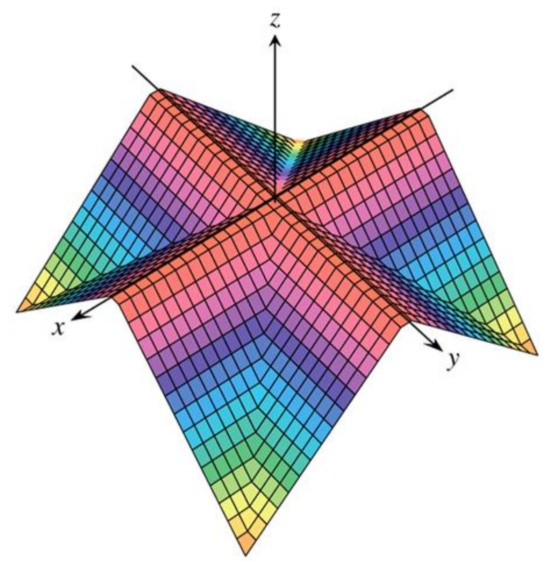
Definition

Let $f(x,y)$ be defined on a region R containing the point $(a,\ b)$. Then

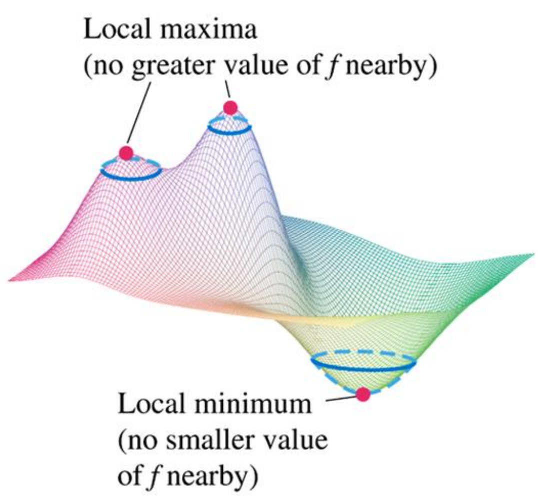
- $f(a,\ b)$ is a local maximum value of *f* if $f(a,\ b) \ge f(x,\ y)$ for all domain points $(x,\ y)$ in an open disk centered at $(a,\ b)$.
- $f(a,\ b)$ is a local minimum value of *f* if $f(a,\ b) \le f(x,\ y)$ for all domain points $(x,\ y)$ in an open disk centered at $(a,\ b)$.



$z = (\cos x)(\cos y)e^{-\sqrt{x^2+y^2}}$

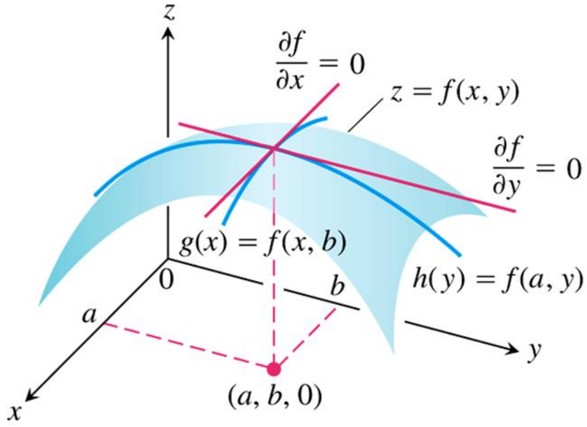


$z = \frac{1}{2}\left(\left||x|-|y|\right|-|x|-|y|\right)$



***Theorem* – First derivative Test for Local Extreme Values**

If $f(x,y)$ has a local maximum or minimum value at an interior point $(a,\ b)$ of its domain and if the first partial derivatives exist there, then $f_x(a,\ b)=0$ *and* $f_y(a,\ b)=0$

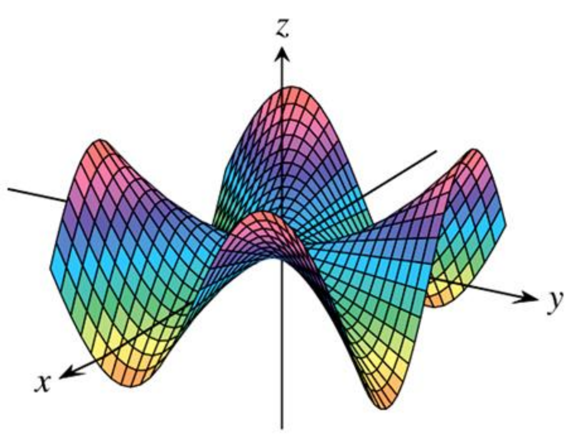


Definition

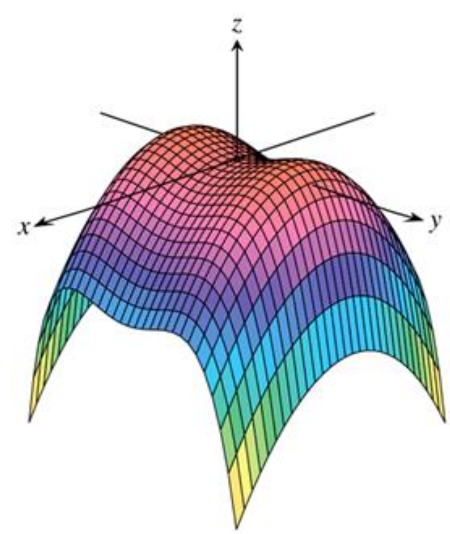
An interior point of the domain $f(x,y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a ***critical point*** of *f*.

Definition

A differentiable function $f(x,y)$ has a ***saddle point*** at a critical point $(a,\ b)$ if in every open disk centered at $(a,\ b)$ there are domain points (x,y) where $f(x,\ y)>f(a,\ b)$ and domain points (x,y) where $f(x,\ y)<f(a,\ b)$. The corresponding point $(a,\ b,\ f(a,\ b))$ on the surface $z=f(x,\ y)$ is called a saddle point of the surface.



$$z=\frac{xy\left(x^2-y^2\right)}{x^2+y^2}$$



$$z=y^2-y^4-x^2$$

Example

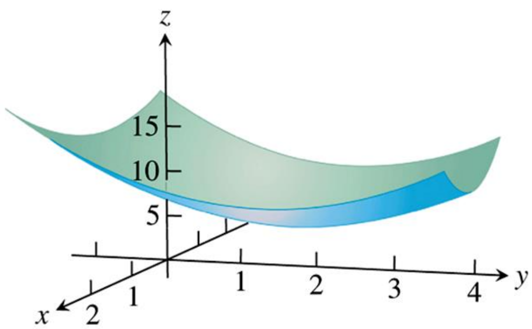
Find the local extreme values of $f(x,y)=x^2+y^2-4y+9$

Solution

The domain of *f* is the entire plane. The local extreme values occur:

$f_x = 2x = 0 \qquad f_y = 2y-4 = 0$

Therefore, the critical point is $(0,\ 2)$ and the value $f(0,2)=0+2^2-8+9=5$.



The critical point is a local minimum.

Example

Find the local extreme values of $f(x,y)=y^2-x^2$

Solution

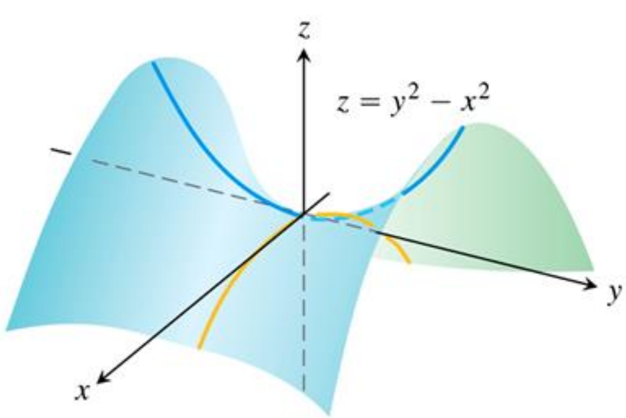
The domain of *f* is the entire plane.

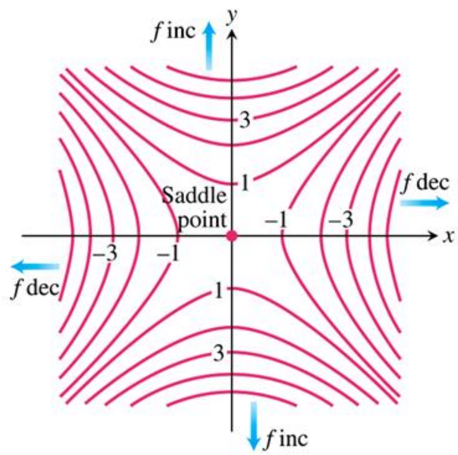
$f_x = -2x = 0 \qquad f_y = 2y = 0$

Therefore, the local extreme is the origin $(0,\ 0)$ and the value $f(0,\ 0)=0$.

$f(0,y)=y^2 \ge 0 \qquad f(x,0)=-x^2 \le 0$

The function has a saddle point at the origin and no local extreme values.





Theorem – Second Derivative Test for Local Extreme Values

Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

- f has a ***local maximum*** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
- f has a ***local minimum*** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
- f has a ***saddle point*** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).
- ***The test is inconclusive*** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of *f* at (a, b).

Example

Find the local extreme values of $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

Solution

$$f_x = y - 2x - 2 = 0 \qquad f_y = x - 2y - 2 = 0$$

$$\begin{cases} -2x + y = 2 \\ x - 2y = 2 \end{cases} \rightarrow \boxed{x = y = -2}$$

Therefore, the critical point is $(-2, \ -2)$

$$f_{xx} = -2 \quad f_{yy} = -2 \quad f_{xy} = 1$$

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - 1^2 = 3 > 0$$

$$f_{xx} = -2 < 0$$

The function *f* has a local maximum at $(-2, \ -2)$ and the value is

$$f(-2, \ -2) = (-2)(-2) - (-2)^2 - (-2)^2 - 2(-2) - 2(-2) + 4 \ \underline{= 8}$$

Example

Find the local extreme values of $f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$

Solution

$$f_x = -6x + 6y = 0 \quad \textit{and} \quad f_y = 6y - 6y^2 + 6x = 0$$

$$\begin{cases} -6x + 6y = 0 \\ 6y - 6y^2 + 6x = 0 \end{cases} \to \quad \begin{array}{l} x = y \\ \qquad\quad 6y - 6y^2 + 6y = -6y(y-2) = 0 \end{array}$$

$$\begin{cases} y = 0 = x & \boxed{(0,\ 0)} \\ y = 2 = x & \boxed{(2,\ 2)} \end{cases} \quad \text{are the critical points}$$

$$f_{xx} = -6 \quad f_y = 6 - 12y \quad f_{xy} = 6$$

$$\begin{aligned} f_{xx} f_{yy} - f_{xy}^2 &= (-6)(6-12y) - 6^2 \\ &= -36 + 72y - 36 \\ &= 72(y-1) \end{aligned}$$

At $(0,\ 0)$

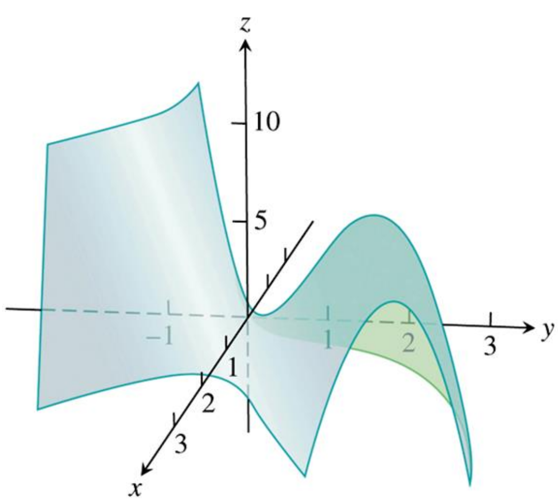
$$f_{xx} f_{yy} - f_{xy}^2 = -72 < 0$$

So, the function has a saddle point at the origin.

At $(2,\ 2)$

$$f_{xx} f_{yy} - f_{xy}^2 = 72 > 0 \ \text{ and } \ f_{xx} = -6 < 0$$

So, the function has a local maximum at $(2,\ 2)$ with a value of $f(2,\ 2) = 12 - 26 - 12 + 24 \underline{= 8}$



Absolute Maxima and Minima on Closed Bounded Regions

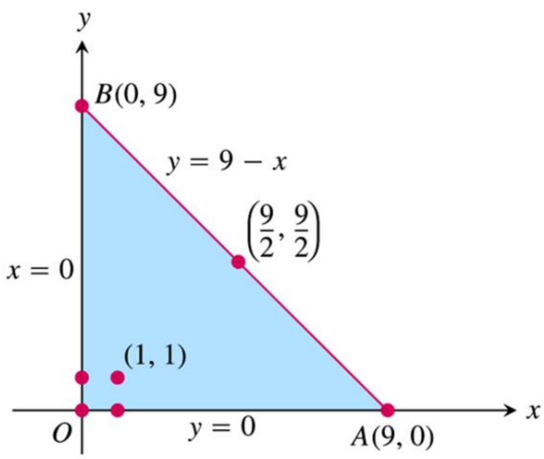
The absolute extrema of a continuous function $f(x,y)$ on a closed and bounded region *R* into three steps

1. *List the interior points of R* where *f* may have local maxima and minima and evaluate *f* at these points. These are the critical points of *f*.
2. *List the boundary points* of *R* where *f* may have local maxima and minima and evaluate *f* at these points.
3. *Look through the lists* for the maximum and minimum values of *f*. These will be the absolute maximum and minimum values of *f* on *R*. Since absolute maxima and minima are also local maxima and minima, the absolute maximum and minimum values of appear somewhere in the lists made in Steps 1 and 2

Example

Find the absolute maximum and minimum values of $f(x,\ y)=2+2x+2y-x^2-y^2$ on the triangular region in the first quadrant bounded by the lines $x=0, \quad y=0, \quad y=9-x$

Solution



$f_x=2-2x=0 \quad f_y=2-2y=0$

$x=1 \quad y=1$

The critical point is $(1,\ 1)$. The value of *f* is

$f(1,\ 1)=2+2+2-1-1=\underline{4}$

Boundary points:

i. On the segment *OA*, *y* = 0. The function

$f(x,\ 0)=2+2x-x^2$

This function is defined on the closed interval $0\le x\le 9$.

$$\begin{cases} x=0 & \rightarrow f(0,\ 0)=\underline{2} \\ x=9 & \rightarrow f(9,\ 0)=2+18-81=\underline{-61} \end{cases}$$

At the interior points where $f_x=0$. The only point is $x=1$ where $f(1,\ 0)=3$

ii. On the segment *OB*, *x* = 0. The function

$f(0,\ y)=2+2y-y^2$

$$f(0,\ 0)=2, \quad f(0,\ 9)=-61, \quad f(1,\ 0)=3$$

iii. Left the interior points of the segment *AB*. With $y=9-x$, then

$$\begin{aligned} f(x,\ y)&=2+2x+2(9-x)-x^2-(9-x)^2 \\ &=2+2x+18-2x-x^2-81+18x-x^2 \end{aligned}$$

$$= -2x^2 + 18x - 61$$

$$f'(x, 9-x) = -4x + 18 = 0 \quad \Rightarrow \quad \boxed{x = \frac{9}{2}}$$

At $x = \frac{9}{2} \quad \Rightarrow \quad y = 9 - x = \frac{9}{2}$

$$f\left(\frac{9}{2}, \frac{9}{2}\right) = 2 + 2\left(\frac{9}{2}\right) + 2\left(9 - \frac{9}{2}\right) - \left(\frac{9}{2}\right)^2 - \left(9 - \frac{9}{2}\right)^2$$

$$= -\frac{41}{2}$$

$\therefore\ 4,\ 2,\ -61,\ 3,\ -\frac{41}{2}$. The maximum is 4, which f assumes at (1, 1). The minimum is –61, which f assumes at (0, 9) and (9, 0).

Example

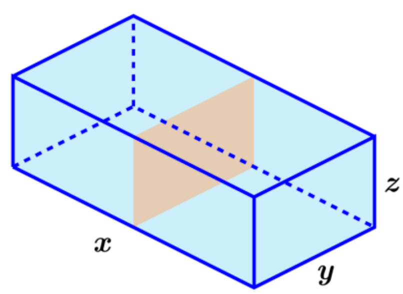
A delivery company accepts only rectangular boxes the sum of whose length and girth (perimeter of a cross-section) does not exceed 108 *in*. Find the dimensions of an acceptable box of largest volume.

Solution

Let x, y, and z represent the length, width, and height.

The girth is: $= 2y + 2z (= P)$

Volume: $V = xyz$



We want to maximize the volume of the box satisfying:

$$x + 2y + 2z = 108$$
$$x = 108 - 2y - 2z$$

$$V(y,z) = (108 - 2y - 2z)yz$$
$$= 108yz - 2y^2z - 2yz^2$$

$$V_y(y,z) = 108z - 4yz - 2z^2$$
$$= 2z(54 - 2y - z) = 0$$

$$V_z(y,z) = 108y - 2y^2 - 4yz$$
$$= 2y(54 - y - 2z) = 0$$

$$\begin{cases} 2z(54 - 2y - z) = 0 & \rightarrow \boxed{z = 0} \quad 54 - 2y - z = 0 \\ 2y(54 - y - 2z) = 0 & \rightarrow \boxed{y = 0} \quad 54 - y - 2z = 0 \end{cases}$$

$$\begin{cases} 2y + z = 54 \\ y + 2z = 54 \end{cases} \rightarrow \boxed{y = z = 18}$$

$$\begin{cases} if\ \ y=0 \quad 54-2y-z=0 \quad \Rightarrow z=54 \rightarrow \boxed{(0,\ 54)} \\ if\ \ z=0 \quad 54-y-2z=0 \quad \Rightarrow y=54 \rightarrow \boxed{(54,\ 0)} \end{cases}$$

$\therefore$ The critical points are: $(0,\ 0),\ (0,\ 54),\ (54,\ 0),\ (18,\ 18)$

At (0, 0): $V(0,0) = 108yz - 2y^2z - 2yz^2 \Big|_{(0,0)} = 0$

At (0, 54): $V(0,54) = 108yz - 2y^2z - 2yz^2 \Big|_{(0,54)} = 0$

At (54, 0): $V(54,0) = 108yz - 2y^2z - 2yz^2 \Big|_{(54,0)} = 0$

At (18, 18): $V(18,18) = 108yz - 2y^2z - 2yz^2 \Big|_{(18,18)} = 11664$

$V_{yy} = -4z, \quad V_{zz} = -4y, \quad V_{yz} = 108 - 4y - 4z$

$$\begin{aligned} V_{xx}V_{yy} - V_{xy}^2 &= (-4z)(-4y) - (108-4y-4z)^2 \\ &= \left[16yz - 16(27-y-z)^2\right]_{(18,18)} \\ &= 16(18)(18) - 16(27-18-18)^2 \\ &= 3888 > 0 \end{aligned}$$

$V_{yy}(18,18) = -4(18) < 0$

That implies (18, 18) give a maximum volume.

$x = 108 - 2(18) - 2(18) \ \underline{= 36}$

$V = xyz = 36(18)(18) \ \underline{= 11{,}664}$

The dimensions of the package are: $\boldsymbol{x}$ = 36 *in*., $\boldsymbol{y}$ = 18 *in*, $\boldsymbol{z}$ = 18 *in*.

The maximum volume is $11{,}664\ \ in^3$

Summary of Max–Min Tests

The extreme values of $f(x,y)$ can occur only at

i. ***Boundary points*** of the domain of *f*.

ii. ***Critical points*** (interior points where $f_x = f_y = 0$ or points where f_x *or* f_y fail to exist)

If the first- and second-order partial derivatives of *f* are continuous throughout a disk centered at a point $(a,\ b)$ and $f_x(a,\ b) = f_y(a,\ b) = 0$, the nature of $f(a,\ b)$ can be tested with the ***Second Derivative Test***:

i. $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at $(a,\ b)$ $\Rightarrow$ ***local maximum***

ii. $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at $(a,\ b)$ $\Rightarrow$ ***local minimum***

iii. $f_{xx} f_{yy} - f_{xy}^2 < 0$ at $(a,\ b)$ $\Rightarrow$ ***saddle point***

iv. $f_{xx} f_{yy} - f_{xy}^2 = 0$ at $(a,\ b)$ $\Rightarrow$ ***test is inconclusive.***

Exercises 2.7 – Maximum/Minimum Problems

Find all the local maxima, local minima, and saddle points of the function

1. $f(x,y)=x^2+xy+y^2+3x-3y+4$

2. $f(x,y)=x^2-xy+y^2+2x+2y-4$

3. $f(x,y)=x^3+y^3-3xy+15$

4. $f(x,y)=x^4-8x^2+3y^2-6y$

5. $f(x,y)=2xy-5x^2-2y^2+4x+4y-4$

6. $f(x,y)=x^2-4xy+y^2+6y+2$

7. $f(x,y)=2x^2+3xy+4y^2-5x+2y$

8. $f(x,y)=x^2-y^2-2x+4y+6$

9. $f(x,y)=\sqrt{56x^2-8y^2-16x-31}+1-8x$

10. $f(x,y)=1-\sqrt[3]{x^2+y^2}$

11. $f(x,y)=x^3+y^3+3x^2-3y^2-8$

12. $f(x,y)=4xy-x^4-y^4$

13. $f(x,y)=\dfrac{1}{x^2+y^2-1}$

14. $f(x,y)=\dfrac{1}{x}+xy+\dfrac{1}{y}$

15. $f(x,y)=y\sin x$

16. $f(x,y)=e^{2x}\cos y$

17. $f(x,y)=e^y-ye^x$

18. $f(x,y)=e^{-y}\left(x^2+y^2\right)$

19. $f(x,y)=2\ln x+\ln y-4x-y$

20. $f(x,y)=\ln(x+y)+x^2-y$

21. $f(x,\ y)=1+x^2+y^2$

22. $f(x,\ y)=x^2-6x+y^2+8y$

23. $f(x,\ y)=(3x-2)^2+(y-4)^2$

24. $f(x,\ y)=3x^2-4y^2$

25. $f(x,\ y)=x^4+y^4-16xy$

26. $f(x,\ y)=\frac{1}{3}x^3-\frac{1}{3}y^3+3xy$

27. $f(x,\ y)=x^4-2x^2+y^2-4y+5$

28. $f(x,\ y)=x^2+xy-2x-y+1$

29. $f(x,\ y)=x^2+6x+y^2+8$

30. $f(x,\ y)=e^{x^2y^2-2xy^2+y^2}$

Identify the critical points of the functions. Then determine whether each critical point corresponds to a local maximum, local minimum, or saddle point. State when your analysis is inconclusive.

31. $f(x,\ y)=x^4+y^4-16xy$

32. $f(x,\ y)=\frac{1}{3}x^3-\frac{1}{3}y^3+2xy$

33. $f(x,\ y)=xy(2+x)(y-3)$

34. $f(x,\ y)=10-x^3-y^3-3x^2+3y^2$

Find the absolute maximum and minimum values of the function on the specified region R.

35. $f(x,\ y)=\frac{1}{3}x^3-\frac{1}{3}y^3+2xy$ on the rectangle $R=\{(x,\ y):\ 0\le x\le 3,\ -1\le y\le 1\}$

36. $f(x,\ y)=x^4+y^4-4xy+1$ on the square $R=\{(x,\ y):\ -2\le x\le 2,\ -2\le y\le 2\}$

37. $f(x,\ y)=x^2y-y^3$ on the triangle $R=\{(x,\ y):\ 0\le x\le 2,\ 0\le y\le 2-x\}$

38. $f(x,\ y)=xy$ on the semicircular disk $R=\left\{(x,\ y):\ -1\le x\le 1,\ 0\le y\le \sqrt{1-x^2}\right\}$

39. $f(x, y) = x^2 + y^2 - 2y + 1; \quad R = \left\{(x, y):\ x^2 + y^2 \le 4\right\}$

40. $f(x, y) = 2x^2 + y^2; \quad R = \left\{(x, y):\ x^2 + y^2 \le 16\right\}$

41. $f(x, y) = 4 + 2x^2 + y^2; \quad R = \{(x, y):\ -1 \le x \le 1,\ -1 \le y \le 1\}$

42. $f(x, y) = 6 - x^2 - 4y^2; \quad R = \{(x, y):\ -2 \le x \le 2,\ -1 \le y \le 1\}$

43. $f(x, y) = 2x^2 - 4x + 3y^2 + 2; \quad R = \left\{(x, y):\ (x-1)^2 + y^2 \le 1\right\}$

44. $f(x, y) = -2x^2 + 4x - 3y^2 - 6y - 1; \quad R = \left\{(x, y):\ (x-1)^2 + (y+1)^2 \le 1\right\}$

45. $f(x, y) = \sqrt{x^2 + y^2 - 2x + 2}; \quad R = \left\{(x, y):\ x^2 + y^2 \le 4,\ y \ge 0\right\}$

46. $f(x, y) = \dfrac{-x^2 + 2y^2}{2 + 2x^2 y^2}$; R is the closed region bounded by the lines $y = x$, $y = 2x$, and $y = 2$

47. $f(x, y) = \sqrt{x^2 + y^2}$; R is the closed region bounded by the ellipse $\dfrac{x^2}{4} + y^2 = 1$

48. $f(x, y) = x^2 + y^2 - 4; \quad R = \left\{(x, y):\ x^2 + y^2 < 4\right\}$

49. $f(x, y) = x + 3y; \quad R = \{(x, y):\ |x| < 1,\ |y| < 2\}$

50. $f(x, y) = 2e^{-x-y}; \quad R = \{(x, y):\ x \ge 0,\ y \ge 0\}$

51. $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0, \quad y = 2, \quad y = 2x$ in the first quadrant.

52. $D(x,y) = x^2 - xy + y^2 + 1$ on the closed triangular plate bounded by the lines $x = 0, \quad y = 4, \quad y = x$ in the first quadrant.

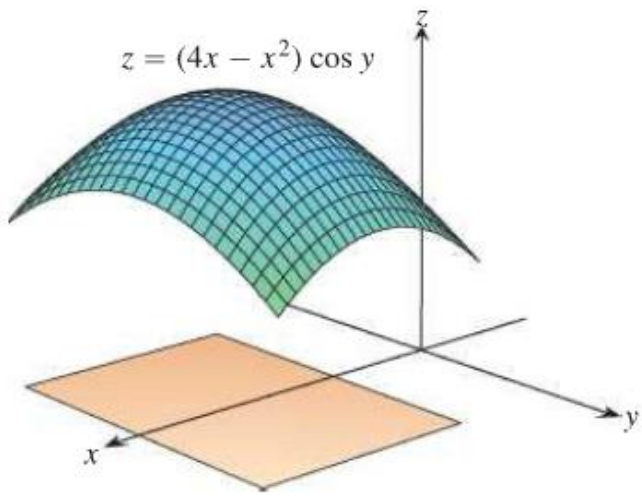
53. $T(x,y) = x^2 + xy + y^2 - 6x + 2$ on the triangular plate $0 \le x \le 5, \quad -3 \le y \le 0$.

54. Find the point on the graph of $z = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0$

55. Find the minimum distance from the point $(2,\ -1,\ 1)$ to the plane $x + y - z = 2$

56. Find the maximum value of $s = xy + yz + xz$ where $x + y + z = 6$

57. Find the absolute maxima and minima of the function $f(x,y) = \left(4x - x^2\right)\cos y$ on the triangular plate $1 \le x \le 3, \quad -\frac{\pi}{4} \le y \le \frac{\pi}{4}$.



58. Among all triangles with a perimeter of 9 *units*, find the dimensions of the triangle with the maximum area. It may be easiest to use Heron's formula, which states that the area of a triangle with side length a, b, and c is $A=\sqrt{s(s-a)(s-b)(s-c)}$, where $2s$ is the perimeter of the triangle.

59. Let P be a plane tangent to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at a point in the first octane. Let T be the tetrahedron in the first octant bounded by P and the coordinate planes $x=0$, $y=0$, and $z=0$. Find the minimum volume T. (the volume of a tetrahedron in one-third the area of the base times the height.)

60. Given three distinct noncollinear points A, B, and C in the plane, find the point P in the plane such the sum of the distances $|AP|+|BP|+|CP|$ is a minimum. Here is how to procees with three points, assuming that the triangle formed by the three points has no angle greater than $\left(120° = \frac{2\pi}{3}\right)$

a) Assume the coordinates of the threee given points are $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$. Let $d_1(x, y)$ be the distance between $A(x_1, y_1)$ and a variable point $P(x, y)$. Compute the gradient of d_1 and show that it is a unit vector pointing along the line between the two points.

b) Define d_2 and d_3 in a similar way and show that ∇d_2 and ∇d_3 are also unit vectors in the direction of line between the two points.

c) The goal is to minimize $f(x,y,z)=d_1+d_2+d_3$. Show that the condition $f_x = f_y = 0$ implies that $\nabla d_1 + \nabla d_2 + \nabla d_3 = 0$.

d) Explain why part (c) implies that the optimal point P has the property the the three line segments AP, BP, and CP all intersect symmetrically in angles of $\frac{2\pi}{3}$.

e) What is the optimal solution if one of the angles in the triangle is greater than $\frac{2\pi}{3}$ (draw a picture)?

f) Estimate the Steiner point for the three points $(0,\ 0)$, $(0,\ 1)$, $(2,\ 0)$

Show that the following two functions have two local maxima but no other extreme points (thus no saddle or basin between the mountains).

61. $f(x,\ y) = -\left(x^2 - 1\right)^2 - \left(x^2 - e^y\right)^2$

62. $f(x,\ y) = 4x^2 e^y - 2x^4 - e^{4y}$