

## ***Solution***      **Section 1.9 - Existence and Uniqueness of Solutions**

### ***Exercise***

Which of the initial value problems are guaranteed a unique solution.  $y' = 4 + y^2$ ,  $y(0) = 1$

### **Solution**

$f(t, y) = 4 + y^2 \rightarrow f$  is continuous

$\frac{\partial f}{\partial y} = 2y$  is also continuous on the whole plane.

Hence the hypotheses are satisfied and guarantee a unique solution.

### ***Exercise***

Which of the initial value problems are guaranteed a unique solution?  $y' = \sqrt{y}$ ,  $y(4) = 0$

### **Solution**

$f(t, y) = \sqrt{y} \Rightarrow y \geq 0$

$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}} \rightarrow y > 0$  (*only*)

Initial condition:  $y(4) = 0 \Rightarrow y_0 = 0$  and  $t_0 = 4$

Both  $f$  and  $\frac{\partial f}{\partial y}$  are not continuous in the rectangle containing  $(t_0, y_0)$

Hence the hypotheses are not satisfied.

### ***Exercise***

Which of the initial value problems are guaranteed a unique solution?  $y' = t \tan^{-1} y$ ,  $y(0) = 2$

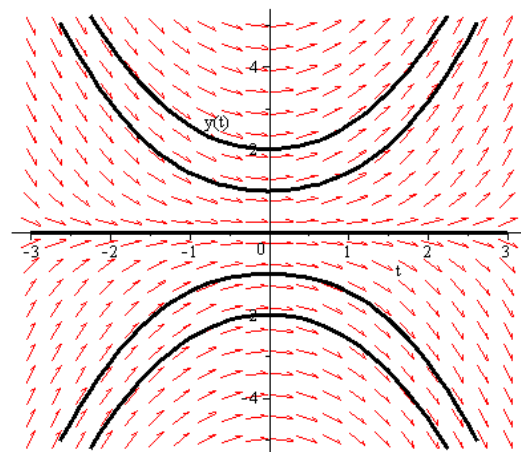
### **Solution**

The right hand side of the equation is  $f(t, y) = t \tan^{-1} y$ ,

which is continuous in the whole plane.  $\frac{\partial f}{\partial y} = \frac{t}{t + y^2}$  is also

continuous in the whole plane.

Hence the hypotheses are satisfied and the theorem guarantees a unique solution.



### Exercise

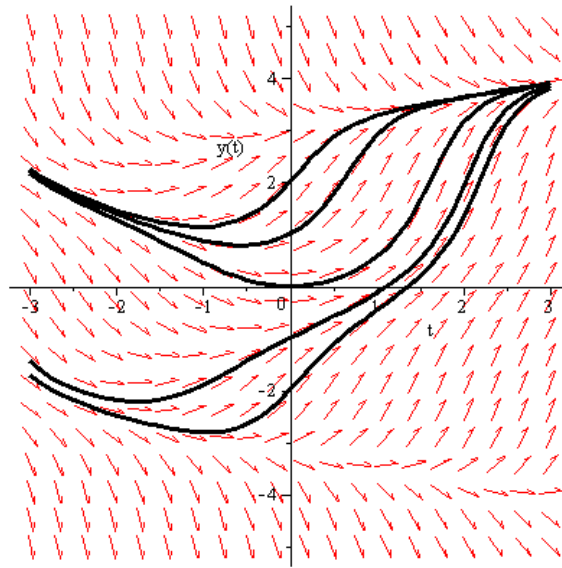
Which of the initial value problems are guaranteed a unique solution?  $\omega' = \omega \sin \omega + s$ ,  $\omega(0) = -1$

### Solution

The right hand side of the equation is  $f(s, \omega) = \omega \sin \omega + s$ , which is continuous in the whole plane.

$\frac{\partial f}{\partial \omega} = \sin \omega + \omega \cos \omega$  is also continuous in the whole plane.

Hence the hypotheses are satisfied and the theorem guarantees a unique solution.



### Exercise

Which of the initial value problems are guaranteed a unique solution?  $x' = \frac{t}{x+1}$ ,  $x(0) = 0$

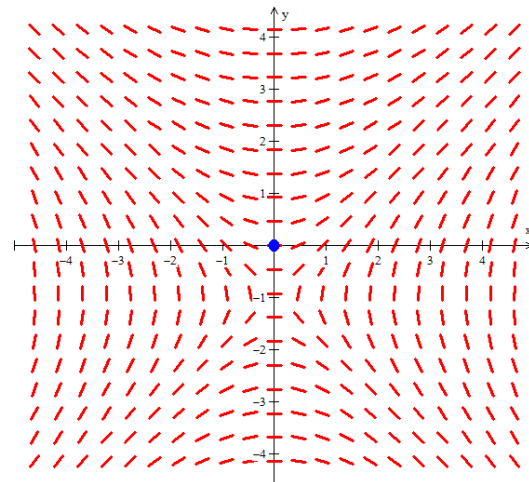
### Solution

The right hand side of the equation is  $f(t, x) = \frac{t}{x+1}$ , which is continuous in the whole plane, except where  $x = -1$ .

$\frac{\partial f}{\partial x} = -\frac{t}{(x+1)^2}$  is also continuous in the whole plane,

except where  $x = -1$ .

Hence the hypotheses are satisfied in a rectangle containing the initial point  $(0, 0)$ , so the theorem guarantees a unique solution.



### Exercise

Which of the initial value problems are guaranteed a unique solution?  $y' = \frac{1}{x}y + 2$ ,  $y(0) = 1$

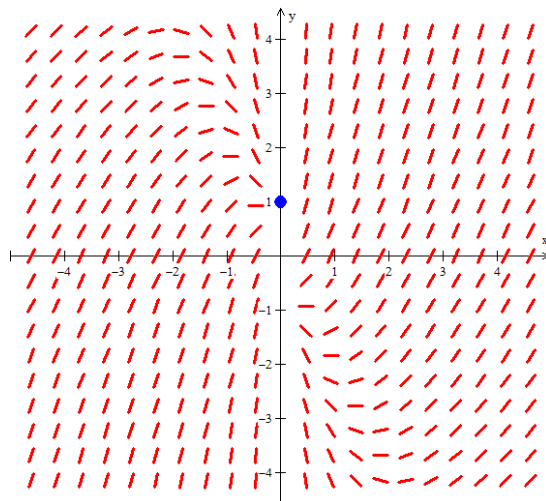
### Solution

The right hand side of the equation is  $f(x, y) = \frac{1}{x}y + 2$ , which is continuous in the whole plane, except where  $x = 0$ .

Since the initial point is  $(0, 1)$ ,  $f$  is discontinuous there.

Consequently there is no rectangle containing this point in which  $f$  is continuous.

The hypotheses are not satisfied, so the theorem doesn't guarantee a unique solution.



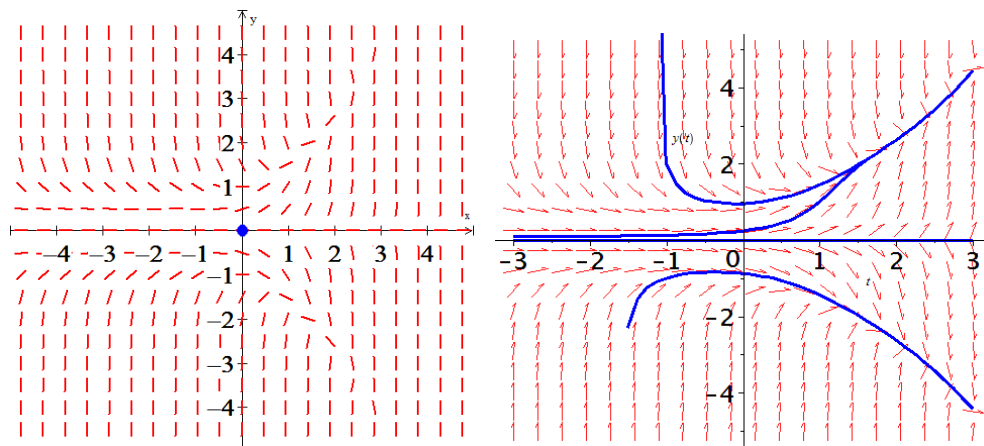
### Exercise

Which of the initial value problems are guaranteed a unique solution?  $y' = e^t y - y^3$ ,  $y(0) = 0$

### Solution

The right hand side of the equation is  $f(t, y) = e^t y - y^3$ , which is continuous in the whole plane.

$\frac{\partial f}{\partial y} = e^t - 3y^2$  is also continuous in the whole plane.



Hence the hypotheses are satisfied in a rectangle containing the initial point  $(0, 0)$ , so the theorem guarantees a unique solution.

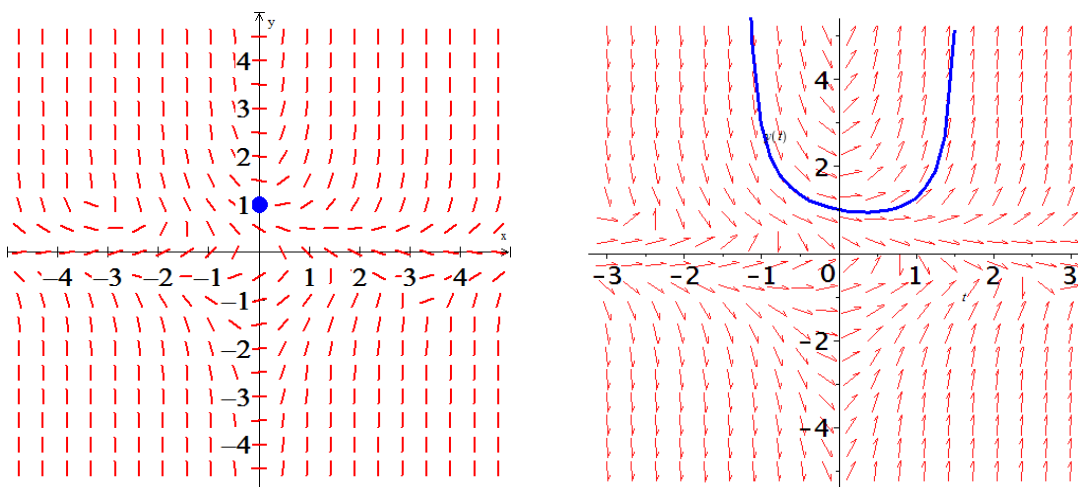
### Exercise

Which of the initial value problems are guaranteed a unique solution?  $y' = ty^2 - \frac{1}{3y+t}, \quad y(0)=1$

### Solution

The right hand side of the equation is  $f(t, y) = ty^2 - \frac{1}{3y+t}$ , which is continuous in the whole plane.

$\frac{\partial f}{\partial y} = 2ty + \frac{3}{(3y+t)^2}$  is also continuous in the whole plane.



Hence the hypotheses are satisfied in a rectangle containing the initial point  $(0, 1)$ , so the theorem guarantees a unique solution.

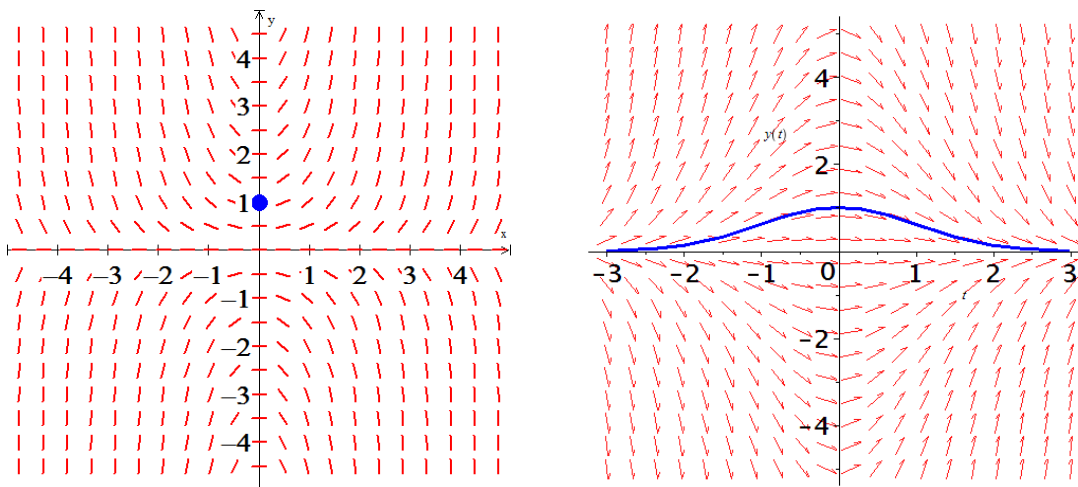
### Exercise

Which of the initial value problems are guaranteed a unique solution?  $y' = xy, \quad y(0)=1$

### Solution

The right hand side of the equation is  $f(x, y) = xy$ , which is continuous in the whole plane.

$\frac{\partial f}{\partial y} = x$  is also continuous in the whole plane.



Hence the hypotheses are satisfied in a rectangle containing the initial point  $(0, 1)$ , so the theorem guarantees a unique solution.

### Exercise

Which of the initial value problems are guaranteed a unique solution?  $y' = -\frac{t^2}{1-y^2}, \quad y(-1) = \frac{1}{2}$

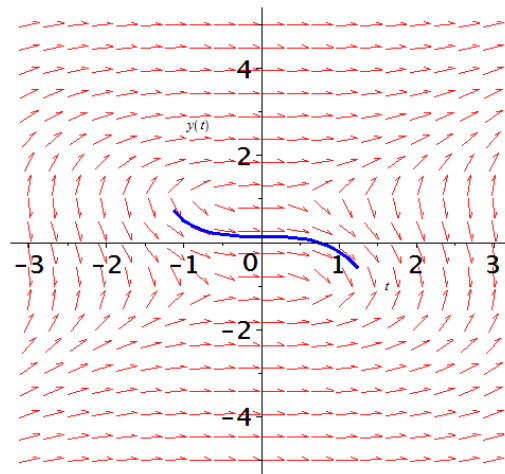
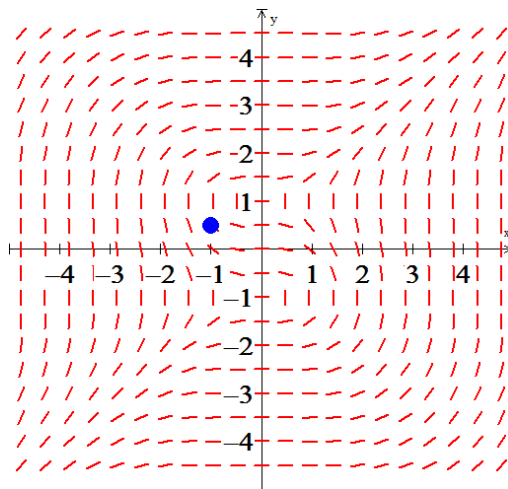
### Solution

The right hand side of the equation is  $f(t, y) = -\frac{t^2}{1-y^2}$ , which is continuous in the whole plane, except

where  $y = \pm 1$ .

$\frac{\partial f}{\partial y} = -\frac{2t^2 y}{(1-y^2)^2}$  is also continuous in the whole plane, except where  $y = \pm 1$ .

Since  $t = -1 \rightarrow y_0 = \frac{1}{2} (\neq \pm 1)$



Hence the hypotheses are satisfied in a rectangle containing the initial point  $(-1, \frac{1}{2})$ , so the theorem guarantees a unique solution.

### Exercise

Which of the initial value problems are guaranteed a unique solution?  $y' = \frac{y}{\sin t}, \quad y\left(\frac{\pi}{2}\right) = 1$

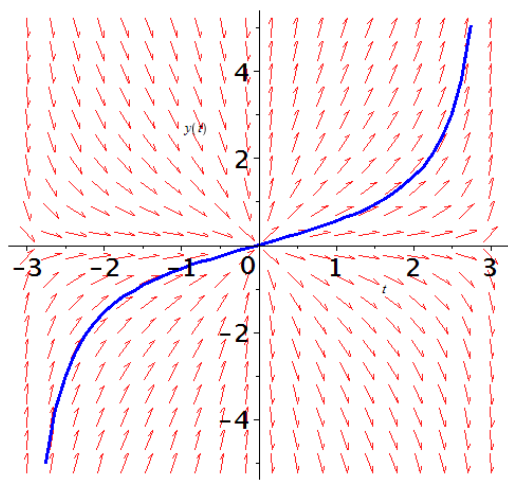
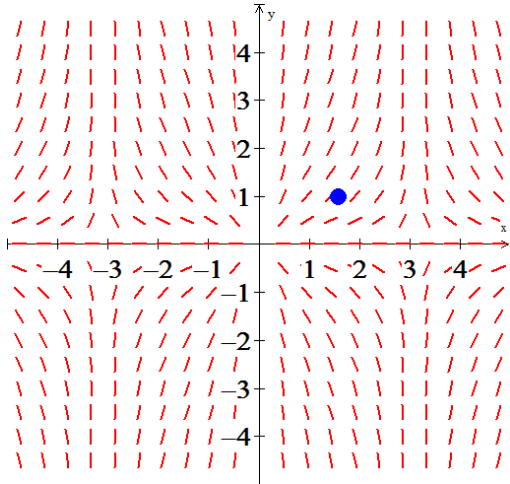
### Solution

The right hand side of the equation is  $f(t, y) = \frac{y}{\sin t}$ , which is continuous in the whole plane, except

where  $t = n\pi$ .

$\frac{\partial f}{\partial y} = \frac{1}{\sin t}$  is also continuous in the whole plane, except where  $t = n\pi$

Since  $t = \frac{\pi}{2} \rightarrow y_0 = 1$  ( $\neq n\pi$ )



Hence the hypotheses are satisfied in a rectangle containing the initial point  $\left(\frac{\pi}{2}, 1\right)$ , so the theorem guarantees a unique solution.

### Exercise

Which of the initial value problems are guaranteed a unique solution?  $y' = \sqrt{1-y^2}$ ,  $y(0) = 1$

### Solution

The right hand side of the equation is  $f(t, y) = \sqrt{1-y^2}$ , which is continuous in the whole plane except where  $y < -1$  &  $y > 1$ .

$$\frac{\partial f}{\partial y} = -\frac{y}{\sqrt{1-y^2}} \Big|_{y=1} = \infty \text{ undefined.}$$

So the uniqueness theorem doesn't apply

$$\frac{dy}{dt} = \sqrt{1-y^2}$$

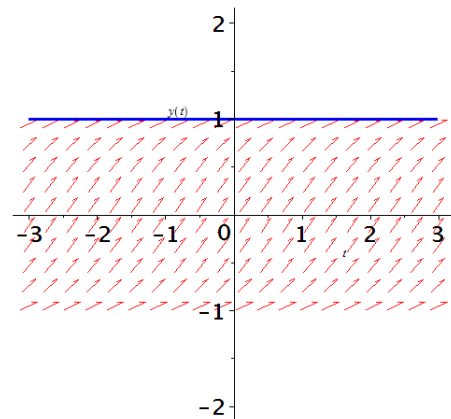
$$\int \frac{dy}{\sqrt{1-y^2}} = \int dt$$

$$\arccos y = t + C$$

$$y(t) = \cos(t + C)$$

$$y(0) = 1 \rightarrow 1 = \cos C \Rightarrow C = 0 (= 2n\pi)$$

$$y(t) = \cos t \quad (2n-1)\pi \leq t \leq 2\pi$$



### Exercise

Show that  $y(t) = 0$  and  $y(t) = t^3$  are both solutions of the initial value problem  $y' = 3y^{2/3}$ , where  $y(0) = 0$ . Explain why this fact doesn't contradict Theorem

### Solution

$$f(t, y) = 3y^{2/3}$$

$$f' = 2y^{-1/3} \text{ which is not continuous at } y = 0$$

### Exercise

Use a numerical solver to sketch the solution of the given initial value problem

$$\frac{dy}{dt} = \frac{t}{y+1}, \quad y(2) = 0$$

- Where does your solver experience difficulty? why? Use the image of your solution to estimate the interval of existence.
- Find an explicit solution; then use your formula to determine the interval of existence. How does it compare with the approximation found in part (a).

### Solution

$$a) \quad (y+1)dy = tdt$$

$$\int (y+1)dy = \int tdt$$

$$\frac{1}{2}y^2 + y = \frac{1}{2}t^2 + C$$

$$y^2 + 2y = t^2 + C$$

$$(0)^2 + 2(0) = 2^2 + C$$

$$0 = 4 + C$$

$$C = -4$$

$$y^2 + 2y = t^2 - 4$$

$$y^2 + 2y - t^2 + 4 = 0$$

Solve for y:

$$y = \frac{-2 \pm \sqrt{2^2 - 4(1)(-t^2 + 4)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4t^2 - 12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{t^2 - 3}}{2}$$

$$\underline{= -1 \pm \sqrt{t^2 - 3}}$$

**b)** The only solution is:  $y = -1 + \sqrt{t^2 - 3}$  and  $t^2 - 3 > 0 \Rightarrow t > \sqrt{3}$

The interval of the solution  $(\sqrt{3}, \infty)$