Solution Section 1.1 – Differential Equations & Solutions

Exercise

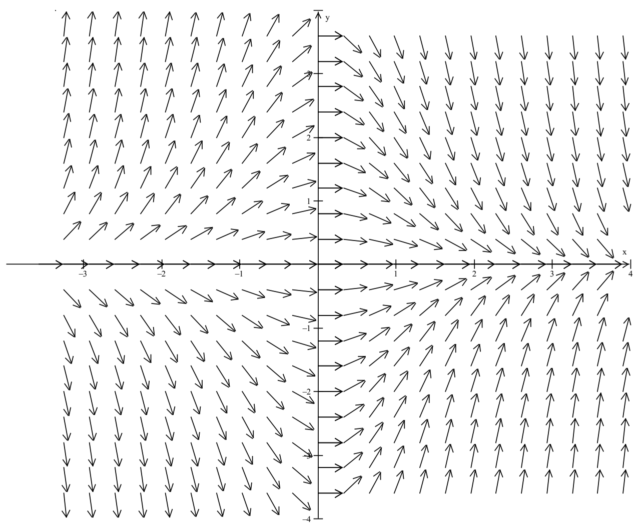
Show that $y(t) = Ce^{-(1/2)t^2}$ is a solution of the 1st order equation $y' = -ty$ *for* $-3 \le C \le 3$

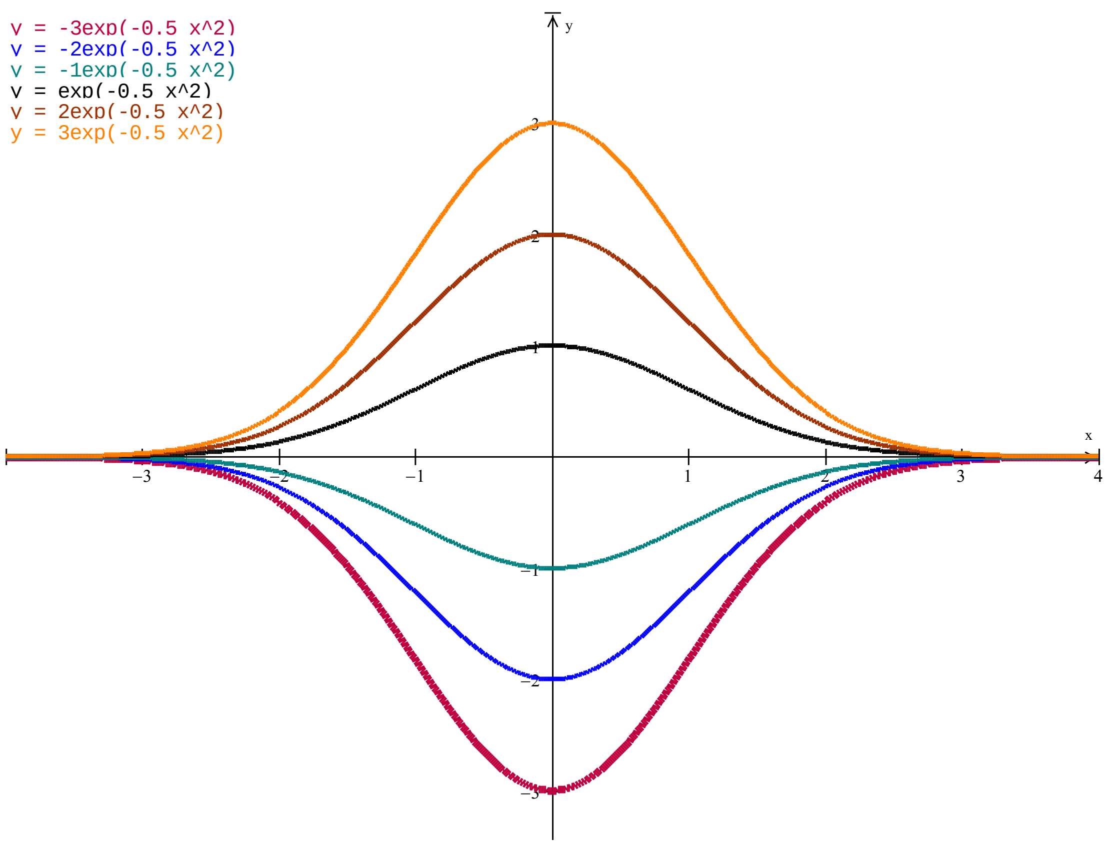
Solution

$$y' = -\frac{1}{2}2tCe^{-(1/2)t^2}$$

$$= -tCe^{-(1/2)t^2}$$

$$\boxed{= -ty}$$





Exercise

Show that $y(t)=\dfrac{4}{1+Ce^{-4t}}$ is a solution of the 1st order equation $y'=y(4-y)$

Solution

$$y'=\frac{d}{dt}\left(\frac{4}{1+Ce^{-4t}}\right)$$

$$=\frac{-4\left(Ce^{-4t}\right)'}{\left(1+Ce^{-4t}\right)^2}$$

$$=\frac{16Ce^{-4t}}{\left(1+Ce^{-4t}\right)^2}$$

$$=\frac{A}{1+Ce^{-4t}}+\frac{B}{\left(1+Ce^{-4t}\right)^2}$$

$$=\frac{A+ACe^{-4t}+B}{\left(1+Ce^{-4t}\right)^2}$$

$$\Rightarrow\begin{cases} A=16 \\ A+B=0\to B=-16\end{cases}$$

$$=\frac{16}{1+Ce^{-4t}}-\frac{16}{\left(1+Ce^{-4t}\right)^2}$$

$$=\frac{4}{1+Ce^{-4t}}\left(4-\frac{4}{1+Ce^{-4t}}\right)$$

$$\boxed{=y(4-y)}$$

$$y(4-y)=$$

$$=\frac{4}{1+Ce^{-4t}}\left[4-\frac{4}{1+Ce^{-4t}}\right]$$

$$=\frac{4}{1+Ce^{-4t}}\left[\frac{4+4Ce^{-4t}-4}{1+Ce^{-4t}}\right]$$

$$=\frac{4}{1+Ce^{-4t}}\left[\frac{4Ce^{-4t}}{1+Ce^{-4t}}\right]$$

$$=\frac{16Ce^{-4t}}{\left(1+Ce^{-4t}\right)^2}$$

Exercise

Show that $y(x)=x^{-3/2}$ is a solution of $4x^2y''+12xy'+3y=0$ for $x>0$

Solution

$$y(x)=x^{-3/2}$$

$$y'=-\frac{3}{2}x^{-5/2}$$

$$y''=\frac{15}{4}x^{-7/2}$$

$$4x^2y''+12xy'+3y\overset{?}{=}0$$

$$4x^2\left(\frac{15}{4}x^{-7/2}\right)+12x\left(-\frac{3}{2}x^{-5/2}\right)+3x^{-3/2}\overset{?}{=}0$$

$$15x^{-3/2}-18x^{-3/2}+3x^{-3/2}\overset{?}{=}0$$

$$0=0 \quad \checkmark$$

$\therefore \quad y(x)=x^{-3/2}$ is a solution of $4x^2y''+12xy'+3y=0$

Exercise

A general solution may fail to produce all solutions of a differential equation $y(t)=\dfrac{4}{1+Ce^{-4t}}$. Show that $y=0$ is a solution of the differential equation, but no value of C in the given general solution will produce this solution.

Solution

$$y(t)=0 \Rightarrow y'=0$$

$$y(4-y)=0(4-0)=0$$

Exercise

Use the given general solution to find a solution of the differential equation having the given initial condition. $\quad ty'+y=t^2, \quad y(t)=\frac{1}{3}t^2+\frac{C}{t}, \qquad y(1)=2$

Solution

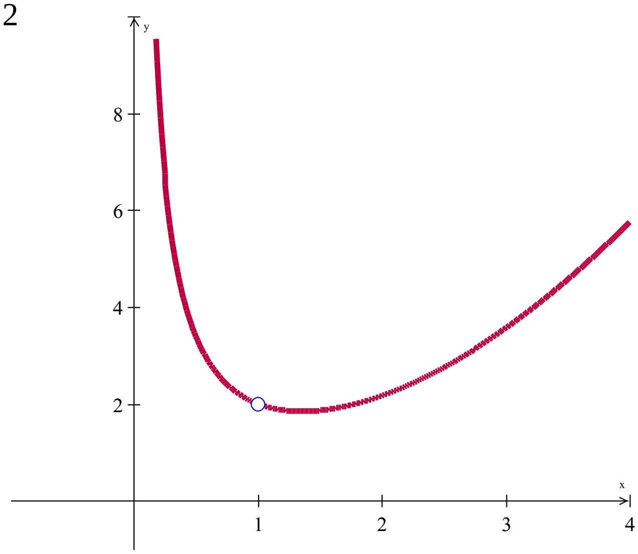
$$y(1)=2$$

$$y(1)=\frac{1}{3}(1)^2+\frac{C}{1}$$

$$2=\frac{1}{3}+C$$

$$C=2-\frac{1}{3}=\frac{5}{3}$$

$$\boxed{y(t)=\frac{1}{3}t^2+\frac{5}{3t}}$$



The interval of existence is $(0,\ \infty)$

Exercise

Show that $y(t)=2t-2+Ce^{-t}$ is a solution of the 1st order equation $y'+y=2t \ \ for \quad -3\le C\le 3$

Solution

$$y'+y=\left(2t-2+Ce^{-t}\right)'+2t-2+Ce^{-t}$$

$$= 2 - Ce^{-t} + 2t - 2 + Ce^{-t}$$
$$= 2t \quad \checkmark$$

Exercise

Use the given general solution to find a solution of the differential equation having the given initial condition. $y' + 4y = \cos t, \quad y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t + Ce^{-4t}, \qquad y(0) = -1$

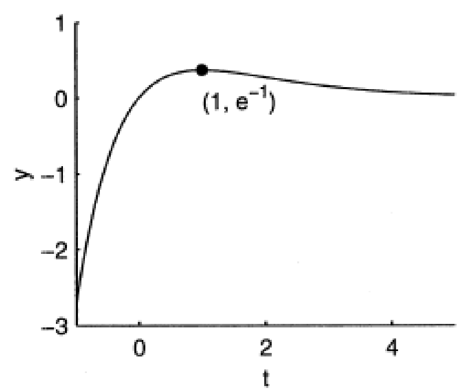
Solution

$$y(0) = \frac{4}{17}\cos(0) + \frac{1}{17}\sin(0) + Ce^{-4(0)}$$
$$-1 = \frac{4}{17} + C$$
$$C = -1 - \frac{4}{17} = -\frac{21}{17}$$
$$\boxed{y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t - \frac{21}{17}e^{-4t}}$$

Exercise

Use the given general solution to find a solution of the differential equation having the given initial condition. $ty' + (t+1)y = 2te^{-t}, \quad y(t) = e^{-t}\left(t + \frac{C}{t}\right), \quad y(1) = \frac{1}{e}$

Solution



$$y(1) = \frac{1}{e} = e^{-1}$$
$$y(1) = e^{-1}\left(1 + \frac{C}{1}\right)$$
$$e^{-1} = e^{-1}(1 + C)$$
$$1 = 1 + C$$

Hence, $C = 0$

The solution is: $y(t) = te^{-t}$

This function is defined and differentiable on the whole real line. Hence, the interval of existence is the whole real line.

Exercise

Use the given general solution to find a solution of the differential equation having the given initial condition. $y' = y(2+y), \quad y(t) = \frac{2}{-1 + Ce^{-2t}}, \quad y(0) = -3$

Solution

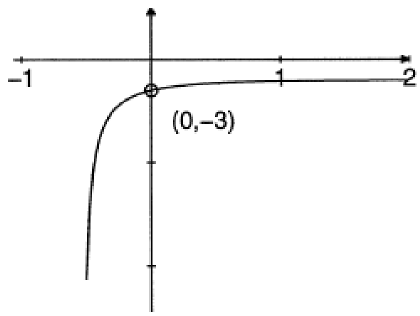
$$y(0)=\frac{2}{-1+Ce^{-2(0)}}$$

$$-3=\frac{2}{-1+C}$$

$$3-3C=2$$

$$-3C=-1$$

$$\boxed{C=\frac{1}{3}}$$



The solution is:

$$y(t)=\frac{2}{-1+\frac{1}{3}e^{-2t}}$$

$$\boxed{=\frac{6}{-3+e^{-2t}}}$$

Exercise

Find the values of $\boldsymbol{m}$ so that the function $y=e^{mx}$ is a solution of the given differential equation

a) $y'+2y=0$

b) $5y'-2y=0$

c) $y''-5y'+6y=0$

d) $2y''+7y'-4y=0$

Solution

$$y=e^{mx} \quad\Rightarrow\quad y'=me^{mx} \quad\Rightarrow\quad y''=m^2e^{mx}$$

a) $y'+2y=0$

$$me^{mx}+2e^{mx}=0 \quad\Rightarrow\quad (m+2)e^{mx}=0$$

$$\boxed{m=-2}$$

b) $5y'-2y=0$

$$5me^{mx}-2e^{mx}=0 \quad\Rightarrow\quad (5m-2)e^{mx}=0$$

$$\boxed{m=\frac{2}{5}}$$

c) $y''-5y'+6y=0$

$$m^2e^{mx}-5me^{mx}+6e^{mx}=0 \quad\Rightarrow\quad \left(m^2-5m+6\right)e^{mx}=0$$

$$\boxed{m=2,\ \ 3}$$

d) $2y''+7y'-4y=0$

$$2m^2e^{mx}+7me^{mx}-4e^{mx}=0 \quad\Rightarrow\quad \left(2m^2+7m-4\right)e^{mx}=0$$

$$\boxed{m=\frac{1}{2},\ \ -4}$$

Exercise

Let $x = c_1 \cos t + c_2 \sin t$ is 2-parameter family solutions of the second order differential equation of $x'' + x = 0$. Find a solution of the second-order consisting of this differential equation and the given initial conditions.

a) $x(0) = -1, \quad x'(0) = 8$

b) $x\left(\frac{\pi}{2}\right) = 0, \quad x'\left(\frac{\pi}{2}\right) = 1$

c) $x\left(\frac{\pi}{6}\right) = \frac{1}{2}, \quad x'\left(\frac{\pi}{6}\right) = 0$

d) $x\left(\frac{\pi}{4}\right) = \sqrt{2}, \quad x'\left(\frac{\pi}{4}\right) = 2\sqrt{2}$

Solution

$$x = c_1 \cos t + c_2 \sin t \quad \Rightarrow \quad x' = -c_1 \sin t + c_2 \cos t$$

a) $x(0) = -1 \quad \Rightarrow \quad \boxed{-1 = c_1}$

$x'(0) = 8 \quad \Rightarrow \quad \boxed{8 = c_2}$

b) $x\left(\frac{\pi}{2}\right) = 0 \quad \Rightarrow \quad \boxed{0 = c_2}$

$x'\left(\frac{\pi}{2}\right) = 1 \quad \Rightarrow \quad \boxed{-1 = c_1}$

c) $x\left(\frac{\pi}{6}\right) = \frac{1}{2} \quad \Rightarrow \quad \frac{\sqrt{3}}{2} c_1 + \frac{1}{2} c_2 = \frac{1}{2} \quad \rightarrow \quad \sqrt{3}\, c_1 + c_2 = 1$

$x'\left(\frac{\pi}{6}\right) = 0 \quad \Rightarrow \quad -\frac{1}{2} c_1 + \frac{\sqrt{3}}{2} c_2 = 0 \quad \rightarrow \quad -c_1 + \sqrt{3}\, c_2 = 0$

$\boxed{c_1 = \frac{\sqrt{3}}{4}}, \quad \boxed{c_2 = \frac{1}{4}}$

d) $x\left(\frac{\pi}{4}\right) = \sqrt{2} \quad \Rightarrow \quad \frac{\sqrt{2}}{2} c_1 + \frac{\sqrt{2}}{2} c_2 = \sqrt{2} \quad \rightarrow \quad c_1 + c_2 = 2$

$x'\left(\frac{\pi}{4}\right) = 2\sqrt{2} \quad \Rightarrow \quad \frac{\sqrt{2}}{2} c_1 + \frac{\sqrt{2}}{2} c_2 = 2\sqrt{2} \quad \rightarrow \quad -c_1 + c_2 = 4$

$\boxed{c_1 = -1} \quad \boxed{c_2 = 3}$

Exercise

Find values of r such that $y(x) = x^r$ is a solution of $x^2 y'' - 4xy' + 6y = 0$

Solution

$$y(x) = x^r \quad \Rightarrow \quad y' = rx^{r-1}$$

$$y'' = r(r-1)x^{r-2}$$

$$x^2 r(r-1)x^{r-2} - 4xrx^{r-1} + 6x^r = 0$$

$$r(r-1)x^r - 4rx^r + 6x^r = 0$$

$$r^2 - r - 4r + 6 = 0 \quad \text{since} \quad x^r \neq 0$$

$$r^2 - 5r + 6 = 0 \quad \rightarrow \quad \underline{r = 3,\ 2}$$

Exercise

Solve the differential equation $y' = 3x^2 - 2x + 4$

Solution

$$y(x) = \int \left(3x^2 - 2x + 4\right) dx$$

$$\underline{= x^3 - x^2 + 4x + C}$$

Exercise

Solve the differential equation $y'' = 2x + \sin 2x$

Solution

$$y' = \int (2x + \sin 2x) dx$$

$$= x^2 - \frac{1}{2}\cos 2x + C_1$$

$$y = \int \left(x^2 - \frac{1}{2}\cos 2x + C_1\right) dx$$

$$\underline{= \frac{1}{3}x^3 - \frac{1}{4}\sin 2x + C_1 x + C_2}$$

Exercise

Given the differential equation $x^2y'' - 2xy' + 2y = 4x^3$, is the given equation a solution to?

Solution

a) $y = 2x^3 + x^2$

$$y' = 6x^2 + 2x$$

$$y'' = 12x + 2$$

$$x^2y'' - 2xy' + 2y \stackrel{?}{=} 4x^3$$

$$x^2(12x + 2) - 2x\left(6x^2 + 2x\right) + 2\left(2x^3 + x^2\right) = 12x^3 + 2x^2 - 12x^3 - 4x^2 + 4x^3 + 2x^2$$

$$\underline{= 4x^3} \quad \checkmark$$

$y = 2x^3 + x^2$ is a solution.

b) $y = 2x + x^2$

$y' = 2 + 2x$

$y'' = 2$

$$x^2 y'' - 2xy' + 2y \stackrel{?}{=} 4x^3$$

$$x^2(2) - 2x(2+2x) + 2\left(2x + x^2\right) = 2x^2 - 4x - 4x^2 + 4x + 2x^2$$

$$= 0 \ \underline{\neq 4x^3}|$$

$y = 2x + x^2$ is ***not*** a solution.