

Solution

Section 4.8 – Divergence Theorem

Exercise

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement. $\vec{F} = \langle 2x, 3y, 4z \rangle \quad D = \{(x, y, z) : x^2 + y^2 + z^2 \leq 4\}$

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(4z) = 9$$

$$\begin{aligned} \iiint_D \nabla \cdot \vec{F} \, dV &= \iiint_D (9) \, dV \\ &= 9 \cdot \text{volume}(D) \\ &= 9 \cdot \frac{4\pi}{3} 2^3 \\ &= 96\pi \end{aligned}$$

$$x^2 + y^2 + z^2 = 4 = r^2$$

$$\text{volume}(D) = \frac{4\pi r^3}{3}$$

$$\vec{r} = \langle x, y, z \rangle$$

$$R = \{(\phi, \theta) : 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$$

$$= \langle 2 \sin \phi \cos \theta, 2 \sin \phi \sin \theta, 2 \cos \phi \rangle$$

$$\mathbf{t}_\phi = \langle 2 \cos \phi \cos \theta, 2 \cos \phi \sin \theta, -2 \sin \phi \rangle$$

$$\mathbf{t}_\theta = \langle -2 \sin \phi \sin \theta, 2 \sin \phi \cos \theta, 0 \rangle$$

$$\begin{aligned} \mathbf{t}_\phi \times \mathbf{t}_\theta &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 \cos \phi \cos \theta & 2 \cos \phi \sin \theta & -2 \sin \phi \\ -2 \sin \phi \sin \theta & 2 \sin \phi \cos \theta & 0 \end{vmatrix} \\ &= (4 \sin^2 \phi \cos \theta) \hat{i} + (4 \sin^2 \phi \sin \theta) \hat{j} + (4 \sin \phi \cos \phi \cos^2 \theta + 4 \sin \phi \cos \phi \sin^2 \theta) \hat{k} \\ &= \langle 4 \sin^2 \phi \cos \theta, 4 \sin^2 \phi \sin \theta, 4 \sin^2 \phi \cos \phi \rangle \end{aligned}$$

$$\vec{F} = \langle 2x, 3y, 4z \rangle$$

$$= \langle 2(2 \sin \phi \cos \theta), 3(2 \sin \phi \sin \theta), 4(2 \cos \phi) \rangle$$

$$\begin{aligned} \vec{F} \cdot (\mathbf{t}_\phi \times \mathbf{t}_\theta) &= \langle 4 \sin \phi \cos \theta, 6 \sin \phi \sin \theta, 8 \cos \phi \rangle \cdot \langle 4 \sin^2 \phi \cos \theta, 4 \sin^2 \phi \sin \theta, 4 \sin^2 \phi \cos \phi \rangle \\ &= 16 \sin^3 \phi \cos^2 \theta + 24 \sin^3 \phi \sin^2 \theta + 32 \sin \phi \cos^2 \phi \end{aligned}$$

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iint_R \nabla \vec{F} \cdot (\mathbf{t}_\phi \times \mathbf{t}_\theta) \, dA$$

$$\begin{aligned}
&= 8 \int_0^{2\pi} \int_0^\pi \left(2\cos^2 \theta \sin^3 \phi + 3\sin^2 \theta \sin^3 \phi + 4\sin \phi \cos^2 \phi \right) d\phi d\theta \\
&= 8 \int_0^{2\pi} \int_0^\pi \left((2\cos^2 \theta + 3\sin^2 \theta) \sin^2 \phi + 4\cos^2 \phi \right) \sin \phi d\phi d\theta \\
&= -8 \int_0^{2\pi} \int_0^\pi \left[(2\cos^2 \theta + 3\sin^2 \theta) (1 - \cos^2 \phi) + 4\cos^2 \phi \right] d(\cos \phi) d\theta \\
&= -8 \int_0^{2\pi} \left[(2\cos^2 \theta + 3(1 - \cos^2 \theta)) \left(\cos \phi - \frac{1}{3} \cos^3 \phi \right) + \frac{4}{3} \cos^3 \phi \right]_0^\pi d\theta \\
&= -8(2) \int_0^{2\pi} \left[\left(3 - \cos^2 \theta \right) \left(-\frac{2}{3} \right) - \frac{4}{3} \right] d\theta \\
&= -16 \int_0^{2\pi} \left[\frac{2}{3} \left(\frac{1 + \cos 2\theta}{2} \right) - \frac{10}{3} \right] d\theta \\
&= -\frac{16}{3} \int_0^{2\pi} [1 + \cos 2\theta - 10] d\theta \\
&= -\frac{16}{3} \left[\frac{1}{2} \sin 2\theta - 9\theta \right]_0^{2\pi} \\
&= -\frac{16}{3} (-18\pi) \\
&= \underline{96\pi}
\end{aligned}$$

Exercise

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement. $\vec{F} = \langle -x, -y, -z \rangle$ $D = \{(x, y, z) : |x| \leq 1, |y| \leq 1, |z| \leq 1\}$

Solution

$$\begin{aligned}
\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(-z) \\
&= -3
\end{aligned}$$

$$\begin{aligned}
\iiint_D \nabla \cdot \vec{F} dV &= \iiint_D (-3) dV \\
&= -3 \cdot \text{volume}(D) & \text{volume}(D) &= 2 \cdot 2 \cdot 2 \quad (\text{cube of distance between } -1 \text{ \& } 1 = 2) \\
&= -3 \cdot (2)^3 \\
&= \underline{-24}
\end{aligned}$$

Since the surface has a form of cube, therefore we have 6 surfaces

$$S_1 : x = -1 \rightarrow \vec{n}_1 = \langle -1, 0, 0 \rangle$$

$$\vec{F} \cdot \vec{n}_1 = \langle 1, -y, -z \rangle \cdot \langle -1, 0, 0 \rangle = -1$$

$$S_2 : x = 1 \rightarrow \vec{n}_2 = \langle 1, 0, 0 \rangle$$

$$\vec{F} \cdot \vec{n}_2 = \langle -1, -y, -z \rangle \cdot \langle 1, 0, 0 \rangle = -1$$

$$S_3 : y = -1 \rightarrow \vec{n}_3 = \langle 0, -1, 0 \rangle$$

$$\vec{F} \cdot \vec{n}_3 = \langle -x, 1, -z \rangle \cdot \langle 0, -1, 0 \rangle = -1$$

$$S_4 : y = 1 \rightarrow \vec{n}_4 = \langle 0, 1, 0 \rangle$$

$$\vec{F} \cdot \vec{n}_4 = \langle -x, -1, -z \rangle \cdot \langle 0, 1, 0 \rangle = -1$$

$$S_5 : z = -1 \rightarrow \vec{n}_5 = \langle 0, 0, -1 \rangle$$

$$\vec{F} \cdot \vec{n}_5 = \langle -x, -y, 1 \rangle \cdot \langle 0, 0, -1 \rangle = -1$$

$$S_6 : z = 1 \rightarrow \vec{n}_6 = \langle 0, 0, 1 \rangle$$

$$\vec{F} \cdot \vec{n}_6 = \langle -x, -y, -1 \rangle \cdot \langle 0, 0, 1 \rangle = -1$$

$$\begin{aligned} \iint_S \vec{F} \cdot \vec{n} \, dS &= \sum_{k=1}^6 \iint_{S_k} \vec{F} \cdot \vec{n}_k \, dS \\ &= 6 \int_{-1}^1 dz \int_{-1}^1 dy \int_{-1}^0 (-1) \, dx \\ &= 6 \left[z \right]_{-1}^1 \left[y \right]_{-1}^1 \left[-x \right]_{-1}^0 \\ &= 6(2)(2)(-1) \\ &= -24 \end{aligned}$$

Exercise

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement.

$$\vec{F} = \langle z - y, x, -x \rangle \quad D = \left\{ (x, y, z) : \frac{x^2}{4} + \frac{y^2}{8} + \frac{z^2}{12} \leq 1 \right\}$$

Solution

$$\begin{aligned} \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(z - y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(-x) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \iiint_D \nabla \cdot \vec{F} \, dV &= \iiint_D (0) \, dV \\ &= 0 \end{aligned}$$

$$\frac{x^2}{4} + \frac{y^2}{8} + \frac{z^2}{12} = 1 \rightarrow x^2 = 4, y^2 = 8, z^2 = 12$$

$$\mathbf{t} = \langle 2 \sin u \cos v, 2\sqrt{2} \sin u \sin v, 2\sqrt{3} \cos u \rangle$$

$$\mathbf{t}_u = \langle 2 \cos u \cos v, 2\sqrt{2} \cos u \sin v, -2\sqrt{3} \sin u \rangle \quad \mathbf{t}_v = \langle -2 \sin u \sin v, 2\sqrt{2} \sin u \cos v, 0 \rangle$$

$$\begin{aligned}
\vec{n} = \vec{t}_u \times \vec{t}_v &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\cos u \cos v & 2\sqrt{2} \cos u \sin v & -2\sqrt{3} \sin u \\ -2\sin u \sin v & 2\sqrt{2} \sin u \cos v & 0 \end{vmatrix} \\
&= \langle 4\sqrt{6} \sin^2 u \cos v, 4\sqrt{3} \sin^2 u \sin v, 4\sqrt{2} \sin u \cos u \cos^2 v + 4\sqrt{2} \sin u \cos u \sin^2 v \rangle \\
&= \langle 4\sqrt{6} \sin^2 u \cos v, 4\sqrt{3} \sin^2 u \sin v, 4\sqrt{2} \sin u \cos u \rangle
\end{aligned}$$

$$\begin{aligned}
\vec{F} &= \langle z - y, x, -x \rangle \\
&= \langle 2\sqrt{3} \cos u - 2\sqrt{2} \sin u \sin v, 2 \sin u \cos v, -2 \sin u \cos v \rangle
\end{aligned}$$

$$\begin{aligned}
\vec{F} \cdot \vec{n} &= \langle 2\sqrt{3} \cos u - 2\sqrt{2} \sin u \sin v, 2 \sin u \cos v, -2 \sin u \cos v \rangle \\
&\quad \cdot \langle 4\sqrt{6} \sin^2 u \cos v, 4\sqrt{3} \sin^2 u \sin v, 4\sqrt{2} \sin u \cos u \rangle \\
&= 24\sqrt{2} \cos u \sin^2 u \cos v - 16\sqrt{3} \sin^3 u \cos v \sin v + 8\sqrt{3} \sin^3 u \sin v \cos v - 8\sqrt{2} \sin^2 u \cos u \cos v \\
&= 16\sqrt{2} \cos u \sin^2 u \cos v - 8\sqrt{3} \sin^3 u \sin v \cos v
\end{aligned}$$

$$\begin{aligned}
\iint_S \vec{F} \cdot \vec{n} \, dS &= \int_0^{2\pi} \int_0^\pi (16\sqrt{2} \cos u \sin^2 u \cos v - 8\sqrt{3} \sin^3 u \sin v \cos v) \, du \, dv \\
&= 8 \left[\int_0^{2\pi} \int_0^\pi (2\sqrt{2} \sin^2 u \cos v) \, d(\sin u) + \int_0^\pi \sqrt{3} (1 - \cos^2 u) \sin v \cos v \, d(\cos u) \right] dv \\
&= 8 \left[\int_0^{2\pi} \frac{2\sqrt{2}}{3} \cos v \left[\sin^3 u \right]_0^\pi + \sqrt{3} \sin v \cos v \left(\cos u - \frac{1}{3} \cos^3 u \right) \Big|_0^\pi dv \right] \\
&= 8 \int_0^{2\pi} \left(\frac{2\sqrt{2}}{3} \cos v (0) + \sqrt{3} \sin v \cos v \left(-2 + \frac{2}{3} \right) \right) dv \\
&= -\frac{64\sqrt{3}}{3} \int_0^{2\pi} \sin v \, d(\sin v) \\
&= -\frac{64\sqrt{3}}{3} \left(\frac{1}{2} \sin v \right) \Big|_0^{2\pi} \\
&= 0
\end{aligned}$$

Exercise

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement. $\vec{F} = \langle x^2, y^2, z^2 \rangle$ $D = \{(x, y, z) : |x| \leq 1, |y| \leq 2, |z| \leq 3\}$

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) \\ &= 2x + 2y + 2z\end{aligned}$$

$$\begin{aligned}\iiint_D \nabla \cdot \vec{F} \, dV &= 2 \int_{-3}^3 \int_{-2}^2 \int_{-1}^1 (x + y + z) \, dx dy dz \\ &= 2 \int_{-3}^3 \int_{-2}^2 \int_{-1}^1 (x + y + z) \, dx dy dz \\ &= 2 \int_{-3}^3 \int_{-2}^2 \left(\frac{1}{2} x^2 + yx + zx \right)_{-1}^1 dy dz \\ &= 2 \int_{-3}^3 \int_{-2}^2 (2y + 2z) \, dy dz \\ &= 2 \int_{-3}^3 \left(y^2 + 2zy \right)_{-2}^2 dz \\ &= 2 \int_{-3}^3 (8z) \, dz \\ &= 2 \left[4z^2 \right]_{-3}^3 \\ &= 0\end{aligned}$$

$$S_1 : x = -1 \rightarrow \vec{n}_1 = \langle -1, 0, 0 \rangle$$

$$\begin{aligned}\vec{F} \cdot \vec{n}_1 &= \langle x^2, y^2, z^2 \rangle \cdot \langle -1, 0, 0 \rangle \\ &= -x^2 \Big|_{x=-1} \\ &= -1\end{aligned}$$

$$S_2 : x = 1 \rightarrow \vec{n}_2 = \langle 1, 0, 0 \rangle$$

$$\begin{aligned}\vec{F} \cdot \vec{n}_2 &= \langle x^2, y^2, z^2 \rangle \cdot \langle 1, 0, 0 \rangle \\ &= x^2 \Big|_{x=1} \\ &= 1\end{aligned}$$

$$\begin{aligned}
S_3 : y = -2 &\rightarrow \vec{n}_3 = \langle 0, -1, 0 \rangle \\
\vec{F} \cdot \vec{n}_3 &= \langle x^2, y^2, z^2 \rangle \cdot \langle 0, -1, 0 \rangle \\
&= -y^2 \Big|_{y=-2} \\
&= \underline{\underline{-4}}
\end{aligned}$$

$$\begin{aligned}
S_4 : y = 2 &\rightarrow \vec{n}_4 = \langle 0, 1, 0 \rangle \\
\vec{F} \cdot \vec{n}_4 &= \langle x^2, y^2, z^2 \rangle \cdot \langle 0, 1, 0 \rangle \\
&= y^2 \Big|_{y=2} \\
&= \underline{\underline{4}}
\end{aligned}$$

$$\begin{aligned}
S_5 : z = -3 &\rightarrow \vec{n}_5 = \langle 0, 0, -1 \rangle \\
\vec{F} \cdot \vec{n}_5 &= \langle x^2, y^2, z^2 \rangle \cdot \langle 0, 0, -1 \rangle \\
&= -z^2 \Big|_{z=-3} \\
&= \underline{\underline{-9}}
\end{aligned}$$

$$\begin{aligned}
S_6 : z = 3 &\rightarrow \vec{n}_6 = \langle 0, 0, 1 \rangle \\
\vec{F} \cdot \vec{n}_6 &= \langle x^2, y^2, z^2 \rangle \cdot \langle 0, 0, 1 \rangle \\
&= z^2 \Big|_{z=3} \\
&= \underline{\underline{9}}
\end{aligned}$$

$$\begin{aligned}
\iint_S \vec{F} \cdot \vec{n} \, dS &= \sum_{k=1}^6 \iint_{S_k} \vec{F} \cdot \vec{n}_k \, dS \\
&= \iint_S (-1 + 1 - 4 + 4 - 9 + 9) \, dS \\
&= \iint_S (0) \, dS \\
&= \underline{\underline{0}}
\end{aligned}$$

Exercise

Use the Divergence Theorem to compute the outward flux of the vector field $\vec{F} = \langle -x, x - y, x - z \rangle$ across S is the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \nabla \cdot \langle -x, x - y, x - z \rangle \\ &= \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(x - y) + \frac{\partial}{\partial z}(x - z) \\ &= -1 - 1 - 1 \\ &= \underline{-3}\end{aligned}$$

$$\begin{aligned}\iint_S \vec{F} \cdot \vec{n} \, dS &= \iiint_D \nabla \cdot \vec{F} \, dV \\ &= -3 \times (\text{Volume of the cube}) \\ &= -3 \times (1) \\ &= \underline{-3}\end{aligned}$$

Exercise

Use the Divergence Theorem to compute the outward flux of the vector field $\vec{F} = \frac{1}{3} \langle x^3, y^3, z^3 \rangle$ across S is the sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 9\}$

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{1}{3} \nabla \cdot \langle x^3, y^3, z^3 \rangle \\ &= \frac{1}{3} \left(\frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) \right) \\ &= \frac{1}{3} (3x^2 + 3y^2 + 3z^2) \\ &= \underline{x^2 + y^2 + z^2}\end{aligned}$$

$$\begin{aligned}\iint_S \vec{F} \cdot \vec{n} \, dS &= \iiint_D (x^2 + y^2 + z^2) \, dV \\ &= \int_0^{2\pi} \int_0^\pi \int_0^3 (r^2) r^2 \sin u \, dr \, du \, dv \\ &= \int_0^{2\pi} dv \int_0^\pi \sin u \, du \int_0^3 r^4 \, dr\end{aligned}$$

$$\begin{aligned}
&= (2\pi)(-\cos u) \Big|_0^\pi \left(\frac{1}{5} r^5 \right) \Big|_0^3 \\
&= (2\pi)(2) \left(\frac{243}{5} \right) \\
&= \frac{972\pi}{5}
\end{aligned}$$

Exercise

Use the Divergence Theorem to compute the outward flux of the vector field

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|^3} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}} \text{ across } D \text{ is the region between two spheres of radius 1 and 2 centered at } (5, 5, 5)$$

Solution

$$\vec{F} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\begin{aligned}
\frac{\partial}{\partial x} \vec{F} &= \frac{x^2 + y^2 + z^2 - 3x^2}{(x^2 + y^2 + z^2)^{5/2}} \\
&= \frac{-2x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial y} \vec{F} &= \frac{x^2 + y^2 + z^2 - 3y^2}{(x^2 + y^2 + z^2)^{5/2}} \\
&= \frac{x^2 - 2y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial z} \vec{F} &= \frac{x^2 + y^2 + z^2 - 3z^2}{(x^2 + y^2 + z^2)^{5/2}} \\
&= \frac{x^2 + y^2 - 2z^2}{(x^2 + y^2 + z^2)^{5/2}}
\end{aligned}$$

$$\nabla \cdot \vec{F} = \frac{1}{(x^2 + y^2 + z^2)^{5/2}} \left(-2x^2 + y^2 + z^2 + x^2 - 2y^2 + z^2 + x^2 + y^2 - 2z^2 \right)$$

$$= 0$$

So, the flux is zero across any surface that bounds a region where \vec{F} is defined and differentiable; the given region does not include zero.

Thus, the net outward flux is zero.

Exercise

Use the Divergence Theorem to compute the outward flux of the vector field $\vec{F} = \langle x^2, y^2, z^2 \rangle$; S is the cylinder $\{(x, y, z) : x^2 + y^2 = 4, 0 \leq z \leq 8\}$

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \nabla \cdot \langle x^2, y^2, z^2 \rangle \\ &= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) \\ &= 2x + 2y + 2z\end{aligned}$$

$$x^2 + y^2 = 4 \rightarrow 0 \leq r \leq 2$$

$$0 \leq z \leq 8, \quad 0 \leq \theta \leq 2\pi$$

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

$$\begin{aligned}\iint_S \vec{F} \cdot \vec{n} \, dS &= \iiint_D (2x + 2y + 2z) \, dV \\ &= 2 \int_0^8 \int_0^{2\pi} \int_0^2 (r \cos \theta + r \sin \theta + z) r \, dr d\theta dz \\ &= 2 \int_0^8 \int_0^{2\pi} \int_0^2 ((\cos \theta + \sin \theta) r^2 + zr) \, dr d\theta dz \\ &= 2 \int_0^8 \int_0^{2\pi} \left(\frac{1}{3} (\cos \theta + \sin \theta) r^3 + \frac{1}{2} z r^2 \right) \Big|_0^2 d\theta dz \\ &= 2 \int_0^8 \int_0^{2\pi} \left(\frac{8}{3} (\cos \theta + \sin \theta) + 2z \right) d\theta dz \\ &= 2 \int_0^8 \left(\frac{8}{3} (\sin \theta - \cos \theta) + 2z\theta \right) \Big|_0^{2\pi} dz \\ &= 2 \int_0^8 \left(-\frac{8}{3} + 4\pi z + \frac{8}{3} \right) dz\end{aligned}$$

$$\begin{aligned}
&= 4\pi \int_0^8 (2z) dz \\
&= 4\pi z^2 \Big|_0^8 \\
&= \underline{256\pi}
\end{aligned}$$

Exercise

Find the net outward flux of the field $\vec{F} = \langle 2z - y, x, -2x \rangle$ across the sphere of radius 1 centered at the origin.

Solution

$$\begin{aligned}
\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(2z - y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(-2x) \\
&= \underline{0}
\end{aligned}$$

So, by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla \cdot \vec{F}$ is zero.

Exercise

Find the net outward flux of the field $\vec{F} = \langle bz - cy, cx - az, ay - bx \rangle$ across any smooth closed surface \mathbb{R}^3 , where a , b , and c are constants.

Solution

$$\begin{aligned}
\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(bz - cy) + \frac{\partial}{\partial y}(cx - az) + \frac{\partial}{\partial z}(ay - bx) \\
&= \underline{0}
\end{aligned}$$

So, by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla \cdot \vec{F}$ is zero.

Exercise

Find the net outward flux of the field $\vec{F} = \langle z - y, x - z, y - x \rangle$ across the boundary of the cube $\{(x, y, z) : |x| \leq 1, |y| \leq 1, |z| \leq 1\}$

Solution

$$\begin{aligned}
\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(z - y) + \frac{\partial}{\partial y}(x - z) + \frac{\partial}{\partial z}(y - x) \\
&= \underline{0}
\end{aligned}$$

So, by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla \cdot \vec{F}$ is zero.

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface S . $\vec{F} = \langle x, -2y, 3z \rangle$; S is the sphere $\{(x, y, z): x^2 + y^2 + z^2 = 6\}$

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(3z) \\ &= 1 - 2 + 3 \\ &= 2\end{aligned}$$

The sphere has a radius $\sqrt{6}$, therefore the volume of the sphere is $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi 6\sqrt{6} = 8\pi\sqrt{6}$

$$\begin{aligned}\iiint_D \nabla \cdot \vec{F} \, dV &= 2 \cdot (\text{Volume of sphere}) \\ &= 2(8\pi\sqrt{6}) \\ &= 16\pi\sqrt{6}\end{aligned}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface S .

$\vec{F} = \langle x^2, 2xz, y^2 \rangle$; S is surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(2xz) + \frac{\partial}{\partial z}(y^2) \\ &= 2x\end{aligned}$$

$$\begin{aligned}\iiint_D \nabla \cdot \vec{F} \, dV &= \iiint_D 2x \, dV \\ &= \int_0^1 \int_0^1 \int_0^1 (2x) \, dx \, dy \, dz \\ &= \int_0^1 dz \int_0^1 dy \left[x^2 \right]_0^1\end{aligned}$$

$$= z \Big|_0^1 - y \Big|_0^1 \quad (1)$$

$$\underline{= 1}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface S .

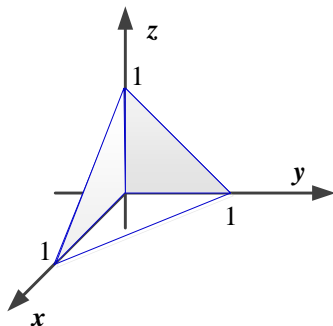
$\vec{F} = \langle x, 2y, z \rangle$; S is boundary of the tetrahedron in the first octant formed by the plane $x + y + z = 1$

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(z)$$

$$\underline{= 4}$$

So, by the Divergence Theorem, the net outward flux is 4 times the volume of the tetrahedron.



$$\text{Volume of the tetrahedron} = \frac{1}{3}(\text{area of the base})(\text{height})$$

$$\begin{aligned} \text{Area of the base} &= \frac{1}{2}(x)(y) \\ &= \frac{1}{2}(1)(1) \\ &\underline{= \frac{1}{2}} \end{aligned}$$

$$\begin{aligned} V &= \frac{1}{3}(\text{area of the base})(\text{height}) \\ &= \frac{1}{3}\left(\frac{1}{2}\right)(1) \\ &= \frac{1}{6} \end{aligned}$$

$$\iiint_D \nabla \cdot \vec{F} \, dV = 4\left(\frac{1}{6}\right)$$

$$\underline{= \frac{2}{3}}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface S .

$$\vec{F} = \langle x^2, y^2, z^2 \rangle; S \text{ is the sphere } \{(x, y, z): x^2 + y^2 + z^2 = 25\}$$

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) \\ &= 2(x + y + z)\end{aligned}$$

So, by the Divergence Theorem, the net outward flux is

$$\begin{aligned}\iiint_D \nabla \cdot \vec{F} \, dV &= 2 \iiint_D (x + y + z) \, dV \\ &= 2 \int_0^{2\pi} \int_0^\pi \int_0^5 5r (\sin \varphi \cos \theta + \sin \varphi \sin \theta + \cos \varphi) \, dr d\varphi d\theta \\ &= 5 \left[r^2 \right]_0^5 \int_0^{2\pi} \int_0^\pi (\sin \varphi \cos \theta + \sin \varphi \sin \theta + \cos \varphi) \, d\varphi d\theta \\ &= 125 \int_0^{2\pi} (-\cos \varphi \cos \theta - \cos \varphi \sin \theta + \sin \varphi) \Big|_0^\pi \, d\theta \\ &= 125 \int_0^{2\pi} 2(\cos \theta + \sin \theta) \, d\theta \\ &= 250 [\sin \theta - \cos \theta]_0^{2\pi} \\ &= 250 (0) \\ &= 0\end{aligned}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields across the given surface S or D .

$$\vec{F} = \langle y - 2x, x^3 - y, y^2 - z \rangle; S \text{ is the sphere } \{(x, y, z): x^2 + y^2 + z^2 = 4\}$$

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(y - 2x) + \frac{\partial}{\partial y}(x^3 - y) + \frac{\partial}{\partial z}(y^2 - z) \\ &= -2 - 1 - 1 \\ &= -4\end{aligned}$$

The net outward flux:

$$\begin{aligned}
 \iiint_D \nabla \cdot \vec{F} \, dV &= -4 \iiint_D dV \\
 &= -4 \times (\text{volume of the sphere}) \\
 &= -4 \times \left(\frac{4\pi}{3} 2^3 \right) \\
 &= \underline{-\frac{128}{3}\pi}
 \end{aligned}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface S .

$\vec{F} = \langle x, y, z \rangle$; S is the surface of the paraboloid $z = 4 - x^2 - y^2$, for $z \geq 0$, plus its base in the xy -plane

Solution

$$\begin{aligned}
 \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \\
 &= \underline{3}
 \end{aligned}$$

So, by the Divergence Theorem, the net outward flux is

$$\begin{aligned}
 \iiint_D \nabla \cdot \vec{F} \, dV &= \int_0^{2\pi} \int_0^2 (3) r z \, dr d\theta \\
 &= 3 \int_0^{2\pi} d\theta \int_0^2 r(4 - r^2) \, dr \\
 &= 3(2\pi) \left[2r^2 - \frac{1}{4}r^4 \right]_0^2 \\
 &= 6\pi(8 - 4) \\
 &= \underline{24\pi}
 \end{aligned}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface S .

$\vec{F} = \langle x, y, z \rangle$; S is the surface of the cone $z^2 = x^2 + y^2$, for $0 \leq z \leq 4$, plus its top surface in the plane $z = 4$

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \\ &= 3\end{aligned}$$

$$\begin{aligned}\text{Volume of a cone} &= \frac{1}{3}(\text{area of the base})(\text{height}) \\ &= \frac{1}{3}(\pi r^2)(4) \\ &= \frac{4\pi}{3}(16) \\ &= \frac{64\pi}{3}\end{aligned}$$

So, by the Divergence Theorem, the net outward flux is

$$\begin{aligned}\iiint_D \nabla \cdot \vec{F} \, dV &= (3)\left(\frac{64\pi}{3}\right) \\ &= 64\pi\end{aligned}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface D .

$\vec{F} = \langle z - x, x - y, 2y - z \rangle$; D is the region between the spheres of radius 2 and 4 centered at origin.

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(z - x) + \frac{\partial}{\partial y}(x - y) + \frac{\partial}{\partial z}(2y - z) \\ &= -3\end{aligned}$$

$$\begin{aligned}\text{Volume between 2 spheres} &= \frac{4}{3}\pi(4^3 - 2^3) \\ &= \frac{224}{3}\pi\end{aligned}$$

$$\begin{aligned}\iiint_D \nabla \cdot \vec{F} \, dV &= (-3)\left(\frac{224\pi}{3}\right) \\ &= -224\pi\end{aligned}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface D .

$\vec{F} = \vec{r} |\vec{r}| = \langle x, y, z \rangle \sqrt{x^2 + y^2 + z^2}$; D is the region between the spheres of radius 1 and 2 centered at origin.

Solution

$$\left(U^n V^m \right)' = U^{n-1} V^{m-1} (nU'V + mUV')$$

$$\begin{aligned} \frac{\partial}{\partial x} \left(x \left(x^2 + y^2 + z^2 \right)^{1/2} \right) &= \left(x^2 + y^2 + z^2 \right)^{-1/2} \left[x^2 + y^2 + z^2 + \frac{1}{2}(2x)(x) \right] \\ &= \left(2x^2 + y^2 + z^2 \right) \left(x^2 + y^2 + z^2 \right)^{-1/2} \end{aligned}$$

$$\begin{aligned} \nabla \cdot \vec{F} &= \frac{\partial}{\partial x} \left(x \left(x^2 + y^2 + z^2 \right)^{1/2} \right) + \frac{\partial}{\partial y} \left(y \left(x^2 + y^2 + z^2 \right)^{1/2} \right) + \frac{\partial}{\partial z} \left(z \left(x^2 + y^2 + z^2 \right)^{1/2} \right) \\ &= \left(x^2 + y^2 + z^2 \right)^{-1/2} \left(2x^2 + y^2 + z^2 + \color{red}{x^2} + \color{red}{2y^2} + \color{red}{z^2} + x^2 + y^2 + 2z^2 \right) \\ &= 4 \left(x^2 + y^2 + z^2 \right)^{-1/2} \left(x^2 + y^2 + z^2 \right) \\ &= 4 \sqrt{x^2 + y^2 + z^2} \\ &= \underline{4|\vec{r}|} \end{aligned}$$

$$\begin{aligned} \iiint_D \nabla \cdot \vec{F} \, dV &= \iiint_D 4 \sqrt{x^2 + y^2 + z^2} \, dV \\ &= \int_0^{2\pi} \int_0^\pi \int_0^r (4\rho) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \\ &= 4 \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi \, d\varphi \int_0^r \rho^3 \, d\rho \\ &= 4(2\pi) [-\cos \varphi]_0^\pi \left[\frac{1}{4} \rho^4 \right]_0^r \\ &= 4(2\pi)(2) \left(\frac{1}{4} r^4 \right) \\ &= 4\pi r^4 \\ &= 4\pi (2^4 - 1^4) \\ &= \underline{60\pi} \end{aligned}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface D .

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}; D \text{ is the region between the spheres of radius 1 and 2 centered at origin.}$$

Solution

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) &= \frac{x^2 + y^2 + z^2 - x^2}{(x^2 + y^2 + z^2)^{3/2}} \\ &= \frac{y^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{y}{\sqrt{x^2 + y^2 + z^2}} \right) &= \frac{x^2 + y^2 + z^2 - y^2}{x^2 + y^2 + z^2} \\ &= \frac{x^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) &= \frac{x^2 + y^2 + z^2 - z^2}{x^2 + y^2 + z^2} \\ &= \frac{x^2 + y^2}{(x^2 + y^2 + z^2)^{3/2}} \end{aligned}$$

$$\begin{aligned} \nabla \cdot \vec{F} &= \frac{y^2 + z^2 + x^2 + z^2 + x^2 + y^2}{(x^2 + y^2 + z^2)^{3/2}} \\ &= \frac{2(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{3/2}} \\ &= \frac{2}{\sqrt{x^2 + y^2 + z^2}} \\ &= \frac{2}{|\vec{r}|} \end{aligned}$$

$$\iiint_D \nabla \cdot \vec{F} \, dV = 2 \iiint_D \frac{1}{\sqrt{x^2 + y^2 + z^2}} \, dV$$

$$\begin{aligned}
&= 2 \int_0^{2\pi} \int_0^\pi \int_0^r \left(\frac{1}{\rho}\right) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \\
&= 2 \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi \, d\varphi \int_0^r \rho \, d\rho \\
&= 2(2\pi) [-\cos \varphi]_0^\pi \left[\frac{1}{2} \rho^2 \right]_0^r \\
&= 2(2\pi)(2) \left(\frac{1}{2} r^2 \right) \\
&= \underline{4\pi r^2}
\end{aligned}$$

D is the region between the spheres of radius 1 and 2 centered at origin

$$\begin{aligned}
&= 4\pi(2^2 - 1^2) \\
&= \underline{12\pi}
\end{aligned}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface D .

$\vec{F} = \langle z - y, x - z, 2y - x \rangle$; $D = \{(x, y, z) : 1 \leq |x| \leq 3, 1 \leq |y| \leq 3, 1 \leq |z| \leq 3\}$ is the region between two cubes

Solution

$$\begin{aligned}
\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(z - y) + \frac{\partial}{\partial y}(x - z) + \frac{\partial}{\partial z}(2y - x) \\
&= \underline{0}
\end{aligned}$$

Therefore, by the Divergence Theorem, the net outward flux is **zero**.

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields across the given surface S or D .

$\vec{F} = \langle y + z, x + z, x + y \rangle$; S consists of the faces of the cube $\{(x, y, z) : |x| \leq 1, |y| \leq 1, |z| \leq 1\}$

Solution

$$\begin{aligned}
\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(y + z) + \frac{\partial}{\partial y}(x + z) + \frac{\partial}{\partial z}(x + y) \\
&= \underline{0}
\end{aligned}$$

The net outward flux:

$$\iiint_D \nabla \cdot \vec{F} \, dV = \iiint_D (0) \, dV$$

$$\underline{= 0}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields

$\vec{F} = \langle x^2, -y^2, z^2 \rangle$; D is the region in the first octant between the planes $z = 4 - x - y$ and

$$z = 2 - x - y$$

Solution

$$\begin{aligned} \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(z^2) \\ &= 2x - 2y + 2z \\ &= 2(x - y + z) \end{aligned}$$

Top plane

$$0 \leq z \leq 4 - x - y$$

$$z = 4 - x - y = 0 \rightarrow 0 \leq y \leq 4 - x$$

$$y = 4 - x = 0 \rightarrow 0 \leq x \leq 4$$

Bottom plane

$$0 \leq z \leq 2 - x - y$$

$$z = 2 - x - y = 0 \rightarrow 0 \leq y \leq 2 - x$$

$$y = 2 - x = 0 \rightarrow 0 \leq x \leq 2$$

The net outward flux:

$$\begin{aligned} \iiint_D \nabla \cdot \vec{F} \, dV &= 2 \iiint_D (x - y + z) \, dV \\ &= 2 \int_0^4 \int_0^{4-x} \int_0^{4-x-y} (x - y + z) \, dz \, dy \, dx - 2 \int_0^2 \int_0^{2-x} \int_0^{2-x-y} (x - y + z) \, dz \, dy \, dx \\ &= 2 \int_0^4 \int_0^{4-x} \left((x - y)z + \frac{1}{2}z^2 \right) \Big|_0^{4-x-y} \, dy \, dx \\ &\quad - 2 \int_0^2 \int_0^{2-x} \left((x - y)z + \frac{1}{2}z^2 \right) \Big|_0^{2-x-y} \, dy \, dx \\ &= 2 \int_0^4 \int_0^{4-x} \left((x - y)(4 - x - y) + \frac{1}{2}(4 - x - y)^2 \right) \, dy \, dx \\ &\quad - 2 \int_0^2 \int_0^{2-x} \left((x - y)(2 - x - y) + \frac{1}{2}(2 - x - y)^2 \right) \, dy \, dx \end{aligned}$$

$$\begin{aligned}
&= 2 \int_0^4 \int_0^{4-x} \left(4x - x^2 - 4y + y^2 + 8 - 4x - 4y + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 \right) dy dx \\
&\quad - 2 \int_0^2 \int_0^{2-x} \left(2x - x^2 - 2y + y^2 + 2 - 2x + \frac{1}{2}x^2 + xy - 2y + \frac{1}{2}y^2 \right) dy dx \\
&= 2 \int_0^4 \int_0^{4-x} \left(-\frac{1}{2}x^2 + 8 - 8y + xy + \frac{3}{2}y^2 \right) dy dx \\
&\quad - 2 \int_0^2 \int_0^{2-x} \left(-\frac{1}{2}x^2 + 2 + xy - 4y + \frac{3}{2}y^2 \right) dy dx \\
&= 2 \int_0^4 \left(-\frac{1}{2}x^2 y + 8y - 4y^2 + \frac{1}{2}xy^2 + \frac{1}{2}y^3 \right) \Big|_0^{4-x} dx \\
&\quad - 2 \int_0^2 \left(-\frac{1}{2}x^2 y + 2y + \frac{1}{2}xy^2 - 2y^2 + \frac{1}{2}y^3 \right) \Big|_0^{2-x} dx \\
&= 2 \int_0^4 \left(-2x^2 + \frac{1}{2}x^3 + 32 - 8x + 32x - 4x^2 + 8x - 4x^2 + \frac{1}{2}x^3 + 32 - 24x + 6x^2 - \frac{1}{2}x^3 \right) dx \\
&\quad - 2 \int_0^2 \left(-x^2 + \frac{1}{2}x^3 + 4 - 2x + 2x - 2x^2 + \frac{1}{2}x^3 - 8 + 8x + 2x^2 + 4 - 6x + 3x^2 - \frac{1}{2}x^3 \right) dx \\
&= 2 \int_0^4 \left(8x - 4x^2 + \frac{1}{2}x^3 \right) dx - 2 \int_0^2 \left(\frac{1}{2}x^3 + 2x - 2x^2 \right) dx \\
&= 2 \left(4x^2 - \frac{4}{3}x^3 + \frac{1}{8}x^4 \right) \Big|_0^4 - 2 \left(\frac{1}{8}x^4 + x^2 - \frac{2}{3}x^3 \right) \Big|_0^2 \\
&= 2 \left(64 - \frac{256}{3} + 32 \right) - 2 \left(2 + 4 - \frac{16}{3} \right) \\
&= 2 \left(96 - \frac{256}{3} \right) - \frac{4}{3} \\
&= \frac{64}{3} - \frac{4}{3} \\
&= \underline{20}
\end{aligned}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface D .

$\vec{F} = \langle x, 2y, 3z \rangle$; D is the region between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ for $0 \leq z \leq 8$

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z) \\ &= 6\end{aligned}$$

Volume of the sphere $x^2 + y^2 = 4$

$$\begin{aligned}V &= \int_0^8 \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy \, dx \, dz \\ &= \int_0^8 dz \int_{-2}^2 y \Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \\ &= (8) \int_{-2}^2 2\sqrt{4-x^2} \, dx \\ &= 16 \left[\frac{x}{2} \sqrt{4-x^2} + 2 \sin^{-1} \frac{x}{2} \right]_{-2}^2 \\ &= 16 \left[2 \left(\frac{\pi}{2} \right) - 2 \left(-\frac{\pi}{2} \right) \right] \\ &= 32\pi\end{aligned}$$

Or $V_1 = z(\pi r^2) = 8(4\pi) = 32\pi$

Volume of the sphere $x^2 + y^2 = 1$

$$\begin{aligned}V &= \int_0^8 \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \, dx \, dz \\ &= \int_0^8 dz \int_{-1}^1 y \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \\ &= (8) \int_{-1}^1 2\sqrt{1-x^2} \, dx \\ &= 16 \left[\frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} \frac{x}{1} \right]_{-1}^1\end{aligned}$$

$$\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$$

$$\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$$

$$= 16 \left[\frac{1}{2} \left(\frac{\pi}{2} \right) - \frac{1}{2} \left(-\frac{\pi}{2} \right) \right]$$

$$= 8\pi$$

$$\text{Or } V_2 = z(\pi r^2) = 8(\pi) = 8\pi$$

Therefore, the net outward flux is $= 6(32\pi - 8\pi) = 144\pi$

Exercise

Compute the outward flux of the following vector field across the given surface

$\vec{F} = \langle x^2 e^y \cos z, -4xe^y \cos z, 2xe^y \sin z \rangle$; S is the boundary of the ellipsoid $\frac{x^2}{4} + y^2 + z^2 = 1$

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 e^y \cos z) + \frac{\partial}{\partial y}(-4xe^y \cos z) + \frac{\partial}{\partial z}(2xe^y \sin z)$$

$$= 2xe^y \cos z - 4xe^y \cos z + 2xe^y \cos z$$

$$= 0$$

Therefore, by the Divergence Theorem, the net outward flux is **zero**.

Exercise

Compute the outward flux of the following vector field across the given surface $\vec{F} = \langle -yz, xz, 1 \rangle$; S is the boundary of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{4} + z^2 = 1$

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(-yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(1)$$

$$= 0$$

Therefore, by the Divergence Theorem, the net outward flux is **zero**.

Exercise

Compute the outward flux of the following vector field across the given surface

$\vec{F} = \langle x \sin y, -\cos y, z \sin y \rangle$; S is the boundary of the region bounded by the planes $x = 1$, $y = 0$, $y = \frac{\pi}{2}$, $z = 0$, and $z = x$

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(x \sin y) + \frac{\partial}{\partial y}(-\cos y) + \frac{\partial}{\partial z}(z \sin y) \\ &= \sin y + \sin y + \sin y \\ &= \underline{3 \sin y}\end{aligned}$$

$$\begin{aligned}\iiint_D \nabla \cdot \vec{F} \, dV &= \int_0^{\pi/2} \int_0^1 \int_0^x 3 \sin y \, dz dx dy \\ &= \int_0^{\pi/2} \sin y \, dy \int_0^1 3z \Big|_0^x dx \\ &= -\cos y \Big|_0^{\pi/2} \int_0^1 3x \, dx \\ &= \frac{3}{2} x^2 \Big|_0^1 \\ &= \underline{\frac{3}{2}}\end{aligned}$$

Exercise

Compute the outward flux of the following vector field across the given surface

$\vec{F} = \langle x^2 + x \sin y, y^2 + 2 \cos y, z^2 + z \sin y \rangle$ across the surface S that is the boundary of the prism bounded by the planes $y = 1 - x$, $x = 0$, $y = 0$, $z = 0$, $z = 4$

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \nabla \cdot \langle x^2 + x \sin y, y^2 + 2 \cos y, z^2 + z \sin y \rangle \\ &= \frac{\partial}{\partial x}(x^2 + x \sin y) + \frac{\partial}{\partial y}(y^2 + 2 \cos y) + \frac{\partial}{\partial z}(z^2 + z \sin y) \\ &= 2x + \sin y + 2y - 2 \sin y + 2z + \sin y \\ &= \underline{2(x + y + z)}\end{aligned}$$

$$y = 1 - x, \quad x = 0, \quad y = 0, \quad z = 0, \quad z = 4$$

$$y = 1 - x = 0 \rightarrow x = 1$$

$$0 \leq y \leq 1 - x, \quad 0 \leq x \leq 1, \quad 0 \leq z \leq 4$$

$$\begin{aligned}\iint_S \vec{F} \cdot \vec{n} \, dS &= 2 \iiint_D (x + y + z) \, dV \\ &= 2 \int_0^4 \int_0^1 \int_0^{1-x} (x + y + z) \, dy dx dz\end{aligned}$$

$$\begin{aligned}
&= 2 \int_0^4 \int_0^1 \left(xy + \frac{1}{2} y^2 + zy \right) \Big|_0^{1-x} dx dz \\
&= 2 \int_0^4 \int_0^1 \left(x - x^2 + \frac{1}{2} - x + \frac{1}{2} x^2 + z - xz \right) dx dz \\
&= 2 \int_0^4 \int_0^1 \left(-\frac{1}{2} x^2 + \frac{1}{2} + z - xz \right) dx dz \\
&= 2 \int_0^4 \left(-\frac{1}{6} x^3 + \frac{1}{2} x + zx - \frac{1}{2} zx^2 \right) \Big|_0^1 dz \\
&= 2 \int_0^4 \left(-\frac{1}{6} + \frac{1}{2} + z - \frac{1}{2} z \right) dz \\
&= 2 \int_0^4 \left(\frac{1}{3} + \frac{1}{2} z \right) dz \\
&= 2 \left(\frac{1}{3} z + \frac{1}{4} z^2 \right) \Big|_0^4 \\
&= 2 \left(\frac{4}{3} + 4 \right) \\
&= \underline{\underline{\frac{32}{3}}}
\end{aligned}$$

Exercise

Compute the outward flux of the following vector field across the given surface $\vec{F} = \langle x, -2y, 4z \rangle$ out of the sphere S with $x^2 + y^2 + z^2 = a^2$, $a > 0$

Solution

$$\begin{aligned}
\nabla \cdot \vec{F} &= \nabla \cdot \langle x, -2y, 4z \rangle \\
&= 1 - 2 + 4 \\
&= \underline{\underline{3}}
\end{aligned}$$

$$\begin{aligned}
\iint_S \vec{F} \cdot \vec{n} \, dS &= 3 \iiint_D dV \\
&= 3 \times (\text{volume of a sphere}) \\
&= 3 \times \left(\frac{4\pi}{3} a^3 \right) \\
&= \underline{\underline{4\pi a^3}}
\end{aligned}$$

Exercise

Compute the outward flux of the following vector field across the given surface $\vec{F} = \langle ye^z, x^2e^z, xy \rangle$ out of the sphere S with $x^2 + y^2 + z^2 = a^2$, $a > 0$

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \nabla \cdot \langle ye^z, x^2e^z, xy \rangle \\ &= \frac{\partial}{\partial x}(ye^z) + \frac{\partial}{\partial y}(x^2e^z) + \frac{\partial}{\partial z}(xy) \\ &= 0\end{aligned}$$

$$\begin{aligned}\iint_S \vec{F} \cdot \vec{n} \, dS &= \iiint_D (0) \, dV \\ &= 0\end{aligned}$$

Exercise

Compute the outward flux of the following vector field across the given surface

$\vec{F} = \langle x^2 + y^2, y^2 - z^2, z \rangle$ of the sphere S with $x^2 + y^2 + z^2 = a^2$, $a > 0$

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \nabla \cdot \langle x^2 + y^2, y^2 - z^2, z \rangle \\ &= \frac{\partial}{\partial x}(x^2 + y^2) + \frac{\partial}{\partial y}(y^2 - z^2) + \frac{\partial}{\partial z}(z) \\ &= 2x + 2y + 1\end{aligned}$$

$$\begin{aligned}\iiint_D \nabla \cdot \vec{F} \, dV &= \iiint_D (2x + 2y + 1) \, dV \\ &= \int_0^{2\pi} \int_0^\pi \int_0^a (2a \sin \varphi \cos \theta + 2a \sin \varphi \sin \theta + 1) \rho^2 \sin \varphi \, d\rho d\varphi d\theta \\ &= \frac{1}{3} \int_0^{2\pi} \int_0^\pi \left(2a \sin^2 \varphi \cos \theta + 2a \sin^2 \varphi \sin \theta + \sin \varphi \right) \rho^3 \Big|_0^a \, d\varphi d\theta \\ &= \frac{a^3}{3} \int_0^{2\pi} \int_0^\pi \left(2a \sin^2 \varphi \cos \theta + 2a \sin^2 \varphi \sin \theta + \sin \varphi \right) d\varphi d\theta \\ &= \frac{a^3}{3} \int_0^{2\pi} \int_0^\pi \left(a(1 - \cos 2\varphi) \cos \theta + a(1 - \cos 2\varphi) \sin \theta + \sin \varphi \right) d\varphi d\theta\end{aligned}$$

$$\begin{aligned}
&= \frac{a^3}{3} \int_0^{2\pi} \int_0^\pi \left(a(\cos \theta + \sin \theta)(1 - \cos 2\varphi) + \sin \varphi \right) d\varphi d\theta \\
&= \frac{a^3}{3} \int_0^{2\pi} \left(a(\cos \theta + \sin \theta) \left(\varphi - \frac{1}{2} \sin 2\varphi \right) - \cos \varphi \right) \Big|_0^\pi d\theta \\
&= \frac{a^3}{3} \int_0^{2\pi} (a\pi(\cos \theta + \sin \theta) + 2) d\theta \\
&= \frac{a^3}{3} (a\pi(\sin \theta - \cos \theta) + 2\theta) \Big|_0^{2\pi} \\
&= \frac{a^3}{3} (-a\pi + 4\pi + a\pi) \\
&= \frac{4}{3} \pi a^3
\end{aligned}$$

Exercise

Compute the outward flux of the following vector field across the given surface

$\vec{F} = \langle x^3, 3yz^2, 3y^2z + x^2 \rangle$ out of the sphere S with $x^2 + y^2 + z^2 = a^2$, $a > 0$

Solution

$$\begin{aligned}
\nabla \cdot \vec{F} &= \nabla \cdot \langle x^3, 3yz^2, 3y^2z + x^2 \rangle \\
&= \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(3yz^2) + \frac{\partial}{\partial z}(3y^2z + x^2) \\
&= 3x^2 + 3z^2 + 3y^2 \\
\iiint_D \nabla \cdot \vec{F} \, dV &= 3 \iiint_D (x^2 + y^2 + z^2) \, dV \\
&= 3 \int_0^{2\pi} \int_0^\pi \int_0^a (\rho^2) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \\
&= 3 \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi \, d\varphi \int_0^a \rho^4 \, d\rho \\
&= 3(2\pi)(-\cos \varphi) \Big|_0^\pi \left(\frac{1}{5} \rho^5 \right) \Big|_0^a \\
&= 3(2\pi)(2) \left(\frac{1}{5} a^5 \right) \\
&= \frac{12}{5} \pi a^5
\end{aligned}$$

Exercise

$\vec{F} = \langle 2z, x, y^2 \rangle$; S is the surface of the paraboloid $z = 4 - x^2 - y^2$, for $z \geq 0$, and the xy -plane.

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \nabla \cdot \langle 2z, x, y^2 \rangle \\ &= \frac{\partial}{\partial x}(2z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y^2) \\ &= 0\end{aligned}$$

$$\begin{aligned}\iiint_D \nabla \cdot \vec{F} \, dV &= \iiint_D (0) \, dV \\ &= 0\end{aligned}$$

Exercise

$\vec{F} = \langle x, y^2, z \rangle$; S is the solid region bounded by the coordinate planes and the plane $2x + 2y + z = 6$.

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \nabla \cdot \langle x, y^2, z \rangle \\ &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z) \\ &= 1 + 2y + 1 \\ &= 2(1 + y)\end{aligned}$$

$$2x + 2y + z = 6 \rightarrow 0 \leq z \leq 6 - 2x - 2y$$

$$z = 6 - 2x - 2y = 0 \rightarrow 0 \leq x \leq 3 - y$$

$$x = 3 - y = 0 \rightarrow 0 \leq y \leq 3$$

$$\begin{aligned}\iiint_D \nabla \cdot \vec{F} \, dV &= 2 \iiint_D (1 + y) \, dV \\ &= 2 \int_0^3 \int_0^{3-y} \int_0^{6-2x-2y} (1 + y) \, dz \, dx \, dy \\ &= 2 \int_0^3 \int_0^{3-y} (1 + y) z \Big|_0^{6-2x-2y} \, dx \, dy \\ &= 2 \int_0^3 \int_0^{3-y} (1 + y)(6 - 2x - 2y) \, dx \, dy\end{aligned}$$

$$\begin{aligned}
&= 2 \int_0^3 \int_0^{3-y} (6 - 2x + 4y - 2xy - 2y^2) dx dy \\
&= 2 \int_0^3 \left(6x - x^2 + 4yx - yx^2 - 2y^2x \right) \Big|_0^{3-y} dy \\
&= 2 \int_0^3 \left(18 - 6y - 9 + 6y - y^2 + 12y - 4y^2 - 9y + 6y^2 - y^3 - 6y^2 + 2y^3 \right) dy \\
&= 2 \int_0^3 \left(9 + 3y - 5y^2 + y^3 \right) dy \\
&= 2 \left(9y + \frac{3}{2}y^2 - \frac{5}{3}y^3 + \frac{1}{4}y^4 \right) \Big|_0^3 \\
&= 2 \left(27 + \frac{27}{2} - 45 + \frac{81}{4} \right) \\
&= 2 \left(\frac{135}{4} - 18 \right) \\
&= \underline{\underline{\frac{63}{2}}}
\end{aligned}$$

Exercise

$\vec{F} = \langle x^2 + \sin z, xy + \cos z, e^y \rangle$; S is the solid region bounded by the cylinder $x^2 + y^2 = 4$, the plane $x + z = 6$, and the xy -plane.

Solution

$$\begin{aligned}
\nabla \cdot \vec{F} &= \nabla \cdot \langle x^2 + \sin z, xy + \cos z, e^y \rangle \\
&= \frac{\partial}{\partial x}(x^2 + \sin z) + \frac{\partial}{\partial y}(xy + \cos z) + \frac{\partial}{\partial z}(e^y) \\
&= 2x + x + 0 \\
&= \underline{\underline{3x}}
\end{aligned}$$

$$x^2 + y^2 = 4 \rightarrow 0 \leq r \leq 2$$

$$x^2 + y^2 = 4 \rightarrow 0 \leq \theta \leq 2\pi$$

$$x + z = 6 \rightarrow 0 \leq z \leq 6 - r \cos \theta$$

$$\begin{aligned}
\iiint_D \nabla \cdot \vec{F} dV &= 3 \iiint_D (x) dV \\
&= 3 \int_0^{2\pi} \int_0^2 \int_0^{6-r \cos \theta} (r \cos \theta) r dz dr d\theta
\end{aligned}$$

$$\begin{aligned}
&= 3 \int_0^{2\pi} \int_0^2 \left(r^2 \cos \theta \right) z \Big|_0^{6-r \cos \theta} dr d\theta \\
&= 3 \int_0^{2\pi} \int_0^2 \left(r^2 \cos \theta \right) (6 - r \cos \theta) dr d\theta \\
&= 3 \int_0^{2\pi} \int_0^2 \left(6r^2 \cos \theta - r^3 \cos^2 \theta \right) dr d\theta \\
&= 3 \int_0^{2\pi} \left(2r^3 \cos \theta - \frac{1}{4} r^4 \cos^2 \theta \right) \Big|_0^2 d\theta \\
&= 3 \int_0^{2\pi} \left(16 \cos \theta - 4 \cos^2 \theta \right) d\theta \\
&= 3 \int_0^{2\pi} (16 \cos \theta - 2 - 2 \cos 2\theta) d\theta \\
&= 3(16 \sin \theta - 2\theta - \sin 2\theta) \Big|_0^{2\pi} \\
&= 3(-4\pi) \\
&= \underline{12\pi}
\end{aligned}$$

Exercise

Compute the outward flux of the following vector field $\vec{F} = \langle 2x^3, 2y^3, 2z^3 \rangle$ out of the sphere S with $x^2 + y^2 + z^2 = 4$

Solution

$$\begin{aligned}
\nabla \cdot \vec{F} &= \nabla \cdot \langle 2x^3, 2y^3, 2z^3 \rangle \\
&= \frac{\partial}{\partial x}(2x^3) + \frac{\partial}{\partial y}(2y^3) + \frac{\partial}{\partial z}(2z^3) \\
&= \underline{6x^2 + 6z^2 + 6y^2} \\
\iiint_D \nabla \cdot \vec{F} dV &= 6 \iiint_D (x^2 + y^2 + z^2) dV \\
&= 6 \int_0^{2\pi} \int_0^\pi \int_0^2 (\rho^2) \rho^2 \sin \varphi d\rho d\varphi d\theta
\end{aligned}$$

$$\begin{aligned}
&= 6 \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_0^2 \rho^4 d\rho \\
&= 6(2\pi)(-\cos \varphi) \Big|_0^\pi \left(\frac{1}{5} \rho^5 \right) \Big|_0^2 \\
&= 6(2\pi)(2) \left(\frac{32}{5} \right) \\
&= \frac{768\pi}{5}
\end{aligned}$$

Exercise

Compute the outward flux of the following vector field $\vec{F} = \langle x, y, z \rangle$ out of the sphere S with $x^2 + y^2 + z^2 = 1$

Solution

$$\begin{aligned}
\nabla \cdot \vec{F} &= \nabla \cdot \langle x, y, z \rangle \\
&= 3
\end{aligned}$$

$$\begin{aligned}
\iiint_D \nabla \cdot \vec{F} dV &= 3 \iiint_D dV \\
&= 3 \times (\text{volume of a sphere}) \\
&= 3 \times \left(\frac{4\pi}{3} \right) \\
&= 4\pi
\end{aligned}$$

Exercise

Compute the outward flux of the following vector field $\vec{F} = \langle z, y, x \rangle$ out of the sphere S with $x^2 + y^2 + z^2 = 1$

Solution

$$\begin{aligned}
\nabla \cdot \vec{F} &= \nabla \cdot \langle z, y, x \rangle \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\iiint_D \nabla \cdot \vec{F} dV &= \iiint_D dV \\
&= \text{volume of a sphere } (r=1) \\
&= \frac{4\pi}{3}
\end{aligned}$$

Exercise

Compute the outward flux of the following vector field $\vec{F} = \langle xy, y^2 + e^{xz^2}, \sin(xy) \rangle$; S is the solid region bounded by the cylinder $z = 1 - x^2$, the planes $y + z = 2$, $z = 0$, and $y = 0$.

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \nabla \cdot \langle xy, y^2 + e^{xz^2}, \sin(xy) \rangle \\ &= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2 + e^{xz^2}) + \frac{\partial}{\partial z}(\sin(xy)) \\ &= y + 2y + 0 \\ &= 3y\end{aligned}$$

$$y + z = 2 \rightarrow 0 \leq y \leq 2 - z$$

$$z = 1 - x^2 \rightarrow 0 \leq z \leq 1 - x^2$$

$$z = 1 - x^2 = 0 \rightarrow -1 \leq x \leq 1$$

$$\begin{aligned}\iiint_D \nabla \cdot \vec{F} \, dV &= 3 \iiint_D (y) \, dV \\ &= 3 \int_{-1}^1 \int_0^{1-x^2} \int_0^{2-z} y \, dy \, dz \, dx \\ &= \frac{3}{2} \int_{-1}^1 \int_0^{1-x^2} y^2 \Big|_0^{2-z} \, dz \, dx \\ &= \frac{3}{2} \int_{-1}^1 \int_0^{1-x^2} (2-z)^2 \, dz \, dx \\ &= -\frac{3}{2} \int_{-1}^1 \int_0^{1-x^2} (2-z)^2 \, d(2-z) \, dx \\ &= -\frac{1}{2} \int_{-1}^1 (2-z)^3 \Big|_0^{1-x^2} \, dx \\ &= -\frac{1}{2} \int_{-1}^1 \left((1+x^2)^3 - 8 \right) \, dx \\ &= -\frac{1}{2} \int_{-1}^1 (-7 + 3x^2 + 3x^4 + x^6) \, dx\end{aligned}$$

$$\begin{aligned}
&= -\left(-7x + x^3 + \frac{3}{5}x^5 + \frac{1}{7}x^7\right)\bigg|_0^1 \\
&= 7 - 1 - \frac{3}{5} - \frac{1}{7} \\
&= 6 - \frac{26}{35} \\
&= \frac{184}{35}
\end{aligned}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields a

$\vec{F} = \langle x^2, y^2, z^2 \rangle$; across the boundary of an ellipsoid $x^2 + y^2 + 4(z-1)^2 \leq 4$

Solution

$$\begin{aligned}
\nabla \cdot \vec{F} &= \nabla \cdot \langle x^2, y^2, z^2 \rangle \\
&= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) \\
&= 2x + 2y + 2z
\end{aligned}$$

$$x^2 + y^2 + 4(z-1)^2 = 4 \rightarrow r^2 = 4(1 - (z-1)^2) \quad 0 \leq r \leq 2\sqrt{1 - (z-1)^2}$$

$$4(z-1)^2 = 4 \rightarrow (z-1)^2 = 1 \quad 0 \leq z \leq 2$$

$$\begin{aligned}
\iiint_D \nabla \cdot \vec{F} \, dV &= 2 \iiint_D (x + y + z) \, dV \\
&= 2 \int_0^{2\pi} \int_0^2 \int_0^{2\sqrt{1-(z-1)^2}} (2\cos\theta + 2\sin\theta + z) r \, dr \, dz \, d\theta \\
&= \int_0^{2\pi} \int_0^2 (2\cos\theta + 2\sin\theta + z) r^2 \bigg|_0^{2\sqrt{1-(z-1)^2}} dz \, d\theta \\
&= 4 \int_0^{2\pi} \int_0^2 (2\cos\theta + 2\sin\theta + z) (-z^2 + 2z) \, dz \, d\theta \\
&= 4 \int_0^{2\pi} \int_0^2 \left((2\cos\theta + 2\sin\theta) (-z^2 + 2z) - z^3 + 2z^2 \right) dz \, d\theta \\
&= 4 \int_0^{2\pi} \left((2\cos\theta + 2\sin\theta) \left(-\frac{1}{3}z^3 + z^2 \right) - \frac{1}{4}z^4 + \frac{2}{3}z^3 \right) \bigg|_0^2 d\theta
\end{aligned}$$

$$\begin{aligned}
&= 4 \int_0^{2\pi} \left((2 \cos \theta + 2 \sin \theta) \left(-\frac{8}{3} + 4 \right) - 4 + \frac{16}{3} \right) d\theta \\
&= 4 \int_0^{2\pi} \left(\frac{8}{6} (\cos \theta + \sin \theta) + \frac{4}{3} \right) d\theta \\
&= 4 \left(\frac{8}{6} (\sin \theta - \cos \theta) + \frac{4}{3} \theta \right) \Big|_0^{2\pi} \\
&= 4 \left(-\frac{8}{6} + \frac{8\pi}{3} + \frac{8}{6} \right) \\
&= \underline{\underline{\frac{32\pi}{3}}}
\end{aligned}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields a

$\vec{F} = \langle x^2, y^2, z^2 \rangle$; across the boundary of the tetrahedron $x + y + z \leq 3$ & $x, y, z \geq 0$

Solution

$$\begin{aligned}
\nabla \cdot \vec{F} &= \nabla \cdot \langle x^2, y^2, z^2 \rangle \\
&= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) \\
&= \underline{\underline{2x + 2y + 2z}}
\end{aligned}$$

$$x + y + z = 3 \rightarrow 0 \leq z \leq 3 - y - x$$

$$z = 3 - y - x = 0 \rightarrow 0 \leq y \leq 3 - x$$

$$y = 3 - x = 0 \rightarrow 0 \leq x \leq 3$$

$$\begin{aligned}
\iiint_D \nabla \cdot \vec{F} \, dV &= 2 \iiint_D (x + y + z) \, dV \\
&= 2 \int_0^3 \int_0^{3-x} \int_0^{3-y-x} (x + y + z) \, dz \, dy \, dx \\
&= 2 \int_0^3 \int_0^{3-x} \left(xz + yz + \frac{1}{2} z^2 \right) \Big|_0^{3-y-x} dy \, dx \\
&= 2 \int_0^3 \int_0^{3-x} \left(x(3-y-x) + y(3-y-x) + \frac{1}{2} (3-y-x)^2 \right) dy \, dx \\
&= 2 \int_0^3 \int_0^{3-x} \left(3x - xy - x^2 + 3y - y^2 - xy + \frac{1}{2} (9 - 6y - 6x + y^2 + 2xy + x^2) \right) dy \, dx
\end{aligned}$$

$$\begin{aligned}
&= 2 \int_0^3 \int_0^{3-x} \left(-xy - \frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{9}{2} \right) dy dx \\
&= 2 \int_0^3 \left(\frac{9}{2}y - \frac{1}{2}xy^2 - \frac{1}{2}x^2y - \frac{1}{6}y^3 \right) \Big|_0^{3-x} dx \\
&= \int_0^3 \left(9(3-x) - x(3-x)^2 - x^2(3-x) - \frac{1}{3}(3-x)^3 \right) dx \\
&= \int_0^3 \left(27 - 9x - 9x + 6x^2 - x^3 - 3x^2 + x^3 - 9 + 9x - 3x^2 + \frac{1}{3}x^3 \right) dx \\
&= \int_0^3 \left(18 - 9x + \frac{1}{3}x^3 \right) dx \\
&= \left(18x - \frac{9}{2}x^2 + \frac{1}{12}x^4 \right) \Big|_0^3 \\
&= 54 - \frac{81}{2} + \frac{27}{4} \\
&= \frac{81}{4}
\end{aligned}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields a

$\vec{F} = \langle x^2, y^2, z^2 \rangle$; across the boundary of the cylinder $x^2 + y^2 \leq 2y$ & $0 \leq z \leq 4$

Solution

$$\begin{aligned}
\nabla \cdot \vec{F} &= \nabla \cdot \langle x^2, y^2, z^2 \rangle \\
&= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) \\
&= 2x + 2y + 2z
\end{aligned}$$

$$x^2 + y^2 = 2y$$

$$x^2 = 2y - y^2 \rightarrow -\sqrt{2y - y^2} \leq x \leq \sqrt{2y - y^2}$$

$$(y-1)^2 = 1 \rightarrow 0 \leq y \leq 2$$

$$\iiint_D \nabla \cdot \vec{F} \, dV = 2 \iiint_D (x + y + z) \, dV$$

$$\begin{aligned}
&= 2 \int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int_0^4 (x+y+z) \, dz dx dy \\
&= 2 \int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \left(xz + yz + \frac{1}{2} z^2 \right) \Big|_0^4 \, dx dy \\
&= 2 \int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} (4x + 4y + 8) \, dx dy \\
&= 2 \int_0^2 \left(2x^2 + 4(y+2)x \right) \Big|_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} dy \\
&= 4 \int_0^2 \left(2y - y^2 + 2(y+2)\sqrt{2y-y^2} - 2y + y^2 + 2(y+2)\sqrt{2y-y^2} \right) dy \\
&= 16 \int_0^2 (y+2)\sqrt{2y-y^2} \, dy \\
&= 16 \int_0^2 (y+2)\sqrt{1-(y-1)^2} \, dy
\end{aligned}$$

$$\text{Let } u = y - 1 \rightarrow du = dy$$

$$y = u + 1 \rightarrow \begin{cases} y = 2 & \rightarrow u = 1 \\ y = 0 & \rightarrow u = -1 \end{cases}$$

$$= 16 \int_{-1}^1 (u+3)\sqrt{1-u^2} \, dy$$

$$\text{Let } u = \sin \theta \rightarrow du = \cos \theta d\theta$$

$$\rightarrow \begin{cases} u = 1 & \rightarrow \theta = \frac{\pi}{2} \\ u = -1 & \rightarrow \theta = -\frac{\pi}{2} \end{cases}$$

$$= 16 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3 + \sin \theta) \sqrt{1 - \sin^2 \theta} \cos \theta d\theta$$

$$= 16 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3 + \sin \theta) \cos \theta \cos \theta d\theta$$

$$= 16 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(3 \cos^2 \theta + \sin \theta \cos \theta \right) d\theta$$

$$\begin{aligned}
&= 16 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3}{2} + \frac{3}{2} \cos 2\theta + \frac{1}{2} \sin 2\theta \right) d\theta \\
&= 16 \left(\frac{3}{2} \theta + \frac{3}{4} \sin 2\theta - \frac{1}{4} \cos 2\theta \right) \Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= 16 \left(\frac{3\pi}{4} + \frac{3\pi}{4} \right) \\
&= \underline{24\pi}
\end{aligned}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields a

$\vec{F} = \langle x^2, y^2, z^2 \rangle$; across the boundary of a ball $(x-2)^2 + y^2 + (z-3)^2 \leq 9$

Solution

$$\begin{aligned}
\nabla \cdot \vec{F} &= \nabla \cdot \langle x^2, y^2, z^2 \rangle \\
&= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) \\
&= \underline{2x + 2y + 2z}
\end{aligned}$$

$$(x-2)^2 + y^2 + (z-3)^2 = 9$$

$$y^2 = 9 - (x-2)^2 - (z-3)^2$$

$$-\sqrt{9 - (x-2)^2 - (z-3)^2} \leq y \leq \sqrt{9 - (x-2)^2 - (z-3)^2}$$

$$y = 9 - (x-2)^2 - (z-3)^2 = 0 \rightarrow (x-2)^2 = 9 - (z-3)^2$$

$$2 - \sqrt{9 - (z-3)^2} \leq x \leq 2 + \sqrt{9 - (z-3)^2}$$

$$9 - (z-3)^2 = 0 \rightarrow z-3 = \pm 3 \quad 0 \leq z \leq 6$$

$$\begin{aligned}
\text{Flux} &= \iiint_D \nabla \cdot \vec{F} \, dV \\
&= 2 \iiint_D (x + y + z) \, dV \\
&= 2 \int_0^6 \int_{2-\sqrt{9-(z-3)^2}}^{2+\sqrt{9-(z-3)^2}} \int_{-\sqrt{9-(x-2)^2-(z-3)^2}}^{\sqrt{9-(x-2)^2-(z-3)^2}} (x + y + z) \, dy \, dx \, dz
\end{aligned}$$

$$\begin{aligned}
&= 2 \int_0^6 \int_{2-\sqrt{9-(z-3)^2}}^{2+\sqrt{9-(z-3)^2}} \left(xy + \frac{1}{2} y^2 + yz \right) \Bigg|_{\sqrt{9-(x-2)^2-(z-3)^2}}^{\sqrt{9-(x-2)^2-(z-3)^2}} dx dz \\
&= 4 \int_0^6 \int_{2-\sqrt{9-(z-3)^2}}^{2+\sqrt{9-(z-3)^2}} (x+z) \sqrt{9-(x-2)^2-(z-3)^2} dx dz
\end{aligned}$$

$$\text{Let } u = x - 2, \quad x = u + 2, \quad du = dx$$

$$v = z - 3, \quad z = v + 3, \quad dv = dz$$

$$\begin{cases} z = 6 \rightarrow v = 3 \\ z = 0 \rightarrow v = -3 \end{cases}$$

$$= 4 \int_{-3}^3 \int_{2-\sqrt{9-v^2}}^{2+\sqrt{9-v^2}} (u+v+5) \sqrt{9-(u^2+v^2)} dx dz$$

$$\text{Let } u = r \cos \theta = 3 \cos \theta$$

$$v = r \sin \theta = 3 \sin \theta$$

$$0 \leq r \leq 3, \quad 0 \leq \theta \leq 2\pi$$

$$\begin{aligned}
&= 4 \int_0^{2\pi} \int_0^3 (3 \cos \theta + 3 \sin \theta + 5) \sqrt{9-r^2} r dr d\theta \\
&= -2 \int_0^{2\pi} (3 \cos \theta + 3 \sin \theta + 5) d\theta \int_0^3 (9-r^2)^{1/2} d(9-r^2) \\
&= -\frac{4}{3} (3 \sin \theta - 3 \cos \theta + 5\theta) \Big|_0^{2\pi} (9-r^2)^{3/2} \Big|_0^3 \\
&= -\frac{4}{3} (10\pi) (-27) \\
&= \underline{360\pi}
\end{aligned}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields a

$\vec{F} = \langle x^4, -x^3 z^2, 4xy^2 z \rangle$; across the boundary of the cylinder $x^2 + y^2 = 1$ and the planes

$z = x + 2$ & $z = 0$

Solution

$$\begin{aligned}
\nabla \cdot \vec{F} &= \nabla \cdot \langle x^4, -x^3 z^2, 4xy^2 z \rangle \\
&= \frac{\partial}{\partial x} (x^4) + \frac{\partial}{\partial y} (-x^3 z^2) + \frac{\partial}{\partial z} (4xy^2 z)
\end{aligned}$$

$$\begin{aligned}
&= 4x^3 + 4xy^2 \\
&= 4x(x^2 + y^2) \Big|
\end{aligned}$$

$$x^2 + y^2 = 1 = r^2 \rightarrow 0 \leq r \leq 1$$

$$0 \leq \theta \leq 2\pi$$

$$z = x + 2 \rightarrow 0 \leq z \leq r \cos \theta + 2$$

$$\begin{aligned}
Flux &= \iiint_D \nabla \cdot \vec{F} \, dV \\
&= 4 \iiint_D x(x^2 + y^2) \, dV \\
&= 4 \int_0^{2\pi} \int_0^1 \int_0^{2+r \cos \theta} r \cos \theta (r^2) \, dz \, dr \, d\theta \\
&= 4 \int_0^{2\pi} \int_0^1 \int_0^{2+r \cos \theta} r^3 \cos \theta \, dz \, dr \, d\theta \\
&= 4 \int_0^{2\pi} \int_0^1 r^3 \cos \theta \, z \Big|_0^{2+r \cos \theta} \, dr \, d\theta \\
&= 4 \int_0^{2\pi} \int_0^1 r^3 \cos \theta (2 + r \cos \theta) \, dr \, d\theta \\
&= 4 \int_0^{2\pi} \int_0^1 (2r^3 \cos \theta + r^4 \cos^2 \theta) \, dr \, d\theta \\
&= 4 \int_0^{2\pi} \left(\frac{1}{2} r^4 \cos \theta + \frac{1}{5} r^5 \cos^2 \theta \right) \Big|_0^1 \, d\theta \\
&= 4 \int_0^{2\pi} \left(\frac{1}{2} \cos \theta + \frac{1}{5} \cos^2 \theta \right) \, d\theta \\
&= 4 \int_0^{2\pi} \left(\frac{1}{2} \cos \theta + \frac{1}{10} + \frac{1}{10} \cos 2\theta \right) \, d\theta \\
&= 4 \left(\frac{1}{2} \sin \theta + \frac{1}{10} \theta + \frac{1}{20} \sin 2\theta \right) \Big|_0^{2\pi} \\
&= \frac{4\pi}{5} \Big|
\end{aligned}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields a

$\vec{F} = \langle x^2 z^3, 2xyz^3, xz^4 \rangle$; S is the surface of the box with vertices $(\pm 1, \pm 2, \pm 3)$.

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \nabla \cdot \langle x^2 z^3, 2xyz^3, xz^4 \rangle \\ &= \frac{\partial}{\partial x}(x^2 z^3) + \frac{\partial}{\partial y}(2xyz^3) + \frac{\partial}{\partial z}(xz^4) \\ &= 2xz^3 + 2xz^3 + 4xz^3 \\ &= 8xz^3 \end{aligned}$$

$$\begin{aligned}\text{Flux} &= \iiint_D \nabla \cdot \vec{F} \, dV \\ &= \iiint_D (8xz^3) \, dV \\ &= 8 \int_{-1}^1 x \, dx \int_{-2}^2 y \, dy \int_{-3}^3 z^3 \, dz \\ &= 8 \left(\frac{1}{2} x^2 \right) \Big|_{-1}^1 (y) \Big|_{-2}^2 \left(\frac{1}{4} z^4 \right) \Big|_{-3}^3 \\ &= 0\end{aligned}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields a

$\vec{F} = \langle z \tan^{-1}(y^2), z^3 \ln(x^2 + 1), z \rangle$; across the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$ and is oriented upward.

Solution

$$\begin{aligned}\nabla \cdot \vec{F} &= \nabla \cdot \langle z \tan^{-1}(y^2), z^3 \ln(x^2 + 1), z \rangle \\ &= \frac{\partial}{\partial x}(z \tan^{-1}(y^2)) + \frac{\partial}{\partial y}(z^3 \ln(x^2 + 1)) + \frac{\partial}{\partial z}(z) \\ &= 1\end{aligned}$$

$$z = 2 - (x^2 + y^2) \rightarrow 1 \leq z \leq 2 - r^2$$

$$z = 2 - r^2 = 1 \rightarrow 0 \leq r \leq 1$$

$$0 \leq \theta \leq 2\pi$$

$$\begin{aligned}
Flux &= \iiint_D \nabla \cdot \vec{F} \, dV \\
&= \iiint_D (1) \, dV \\
&= \int_0^{2\pi} \int_0^1 \int_1^{2-r^2} r \, dz \, dr \, d\theta \\
&= \int_0^{2\pi} d\theta \int_0^1 r z \Big|_1^{2-r^2} dr \\
&= 2\pi \int_0^1 r(1-r^2) \, dr \\
&= 2\pi \int_0^1 (r-r^3) \, dr \\
&= 2\pi \left(\frac{1}{2}r^2 - \frac{1}{4}r^4 \right) \Big|_0^1 \\
&= 2\pi \left(\frac{1}{2} - \frac{1}{4} \right) \\
&= \frac{\pi}{2}
\end{aligned}$$

Exercise

Prove that $\nabla \left(\frac{1}{|\vec{r}|^4} \right) = -\frac{4\vec{r}}{|\vec{r}|^6}$ and use the result to prove that $\nabla \cdot \nabla \left(\frac{1}{|\vec{r}|^4} \right) = \frac{12}{|\vec{r}|^6}$

Solution

$$\begin{aligned}
|\vec{r}|^4 &= (x^2 + y^2 + z^2)^2 \\
\nabla \left(\frac{1}{|\vec{r}|^4} \right) &= \nabla \frac{1}{(x^2 + y^2 + z^2)^2} \\
&= -\frac{1}{(x^2 + y^2 + z^2)^3} \langle 4x, 4y, 4z \rangle \\
&= -\frac{4}{(|\vec{r}|^2)^3} \langle x, y, z \rangle
\end{aligned}$$

$$\boxed{= -\frac{4\vec{r}}{|\vec{r}|^6}} \quad \checkmark$$

$$\begin{aligned} \nabla \cdot \nabla \left(\frac{1}{|\vec{r}|^4} \right) &= \nabla \cdot \left(\frac{-4\vec{r}}{|\vec{r}|^4} \right) \\ &= -4 \nabla \cdot \left(\frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^3} \right) \\ &= -\frac{4}{(x^2 + y^2 + z^2)^4} (x^2 + y^2 + z^2 - 6x^2 + x^2 + y^2 + z^2 - 6y^2 + x^2 + y^2 + z^2 - 6z^2) \\ &= -\frac{4}{(x^2 + y^2 + z^2)^4} (-3x^2 - 3y^2 - 3z^2) \\ &= \frac{12}{(x^2 + y^2 + z^2)^4} (x^2 + y^2 + z^2) \\ &= \frac{12}{(x^2 + y^2 + z^2)^3} \\ &= \frac{12}{(\vec{r}^2)^3} \\ &= \frac{12}{|\vec{r}|^6} \end{aligned} \quad \checkmark$$

Exercise

Consider the radial vector field $\vec{F} = \frac{\vec{r}}{|\vec{r}|^p} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{p/2}}$. Let S be the sphere of radius a at the origin.

- Use the surface integral to show that the outward flux of \vec{F} across S is $4\pi a^{3-p}$. Recall that the unit normal to sphere is $\frac{\vec{r}}{|\vec{r}|}$.
- For what values of p does \vec{F} satisfy the conditions of the Divergence Theorem? For these values of p , use the fact the $\nabla \cdot \vec{F} = \frac{3-p}{|\vec{r}|^p}$ to compute the flux around S using the Divergence Theorem.

Solution

$$a) \quad \vec{F} = \frac{\vec{r}}{|\vec{r}|^p} \quad \& \quad \vec{n} = \frac{\vec{r}}{|\vec{r}|}$$

$$\begin{aligned} \vec{F} \cdot \vec{n} &= \frac{\vec{r}}{|\vec{r}|^p} \cdot \frac{\vec{r}}{|\vec{r}|} \\ &= \frac{|\vec{r}|^2}{|\vec{r}|^{p+1}} \\ &= |\vec{r}|^{1-p} \\ &= \left(\sqrt{x^2 + y^2 + z^2} \right)^{1-p} \\ &= \left(\sqrt{a^2} \right)^{1-p} \\ &= a^{1-p} \end{aligned}$$

$$\begin{aligned} \iint_S \vec{F} \cdot \vec{n} \, dS &= a^{1-p} \iint_S dS \\ &= a^{1-p} \times (\text{area of sphere}) \\ &= a^{1-p} \times 4\pi a^2 \\ &= 4\pi a^{3-p} \end{aligned} \quad \checkmark$$

b) The conditions of the Divergence Theorem require that \vec{F} be defined and have continuous partials everywhere inside the sphere; in particular, this must hold at the origin. Thus, we must have $p \leq -2$. Then the volume integral is:

$$\begin{aligned} \iiint_D \nabla \cdot \vec{F} \, dV &= \iiint_D \frac{3-p}{|\vec{r}|^p} \, dV \\ &= (3-p) \iiint_D r^{-p} \, dV \\ &= (3-p) \int_0^{2\pi} \int_0^\pi \int_0^a \rho^{-p} \rho^2 \sin \varphi \, d\rho d\varphi d\theta \\ &= (3-p) \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi \, d\varphi \int_0^a \rho^{2-p} \, d\rho \\ &= (3-p)(2\pi)(-\cos \varphi) \Big|_0^\pi \left(\frac{1}{3-p} \rho^{3-p} \right) \Big|_0^a \end{aligned}$$

$$= (3-p)(2\pi)(2) \left(\frac{1}{3-p} a^{3-p} \right)$$

$$= \underline{4\pi a^{3-p}}$$

Exercise

Consider the radial vector field $\vec{F} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}$.

- a) Evaluate a surface integral to show that $\iint_S \vec{F} \cdot \vec{n} \, dS = 4\pi a^2$, where S is the surface of a sphere of radius a centered at the origin.
- b) Note that the first partial derivatives of the components of \vec{F} are undefined at the origin, so the Divergence Theorem does not apply directly. Nevertheless, the flux across the sphere as computed in part (a) is finite. Evaluate the triple integral of the Divergence Theorem as an improper integral as follows. Integrate $\text{div} \vec{F}$ over the region between two spheres of radius a and $0 < \varepsilon < a$. Then let $\varepsilon \rightarrow 0^+$ to obtain the flux computed in part (a).

Solution

$$a) \quad \vec{F} = \frac{\vec{r}}{|\vec{r}|} \quad \& \quad \vec{n} = \frac{\vec{r}}{|\vec{r}|}$$

$$\vec{F} \cdot \vec{n} = \frac{\vec{r}}{|\vec{r}|} \cdot \frac{\vec{r}}{|\vec{r}|}$$

$$= \frac{|\vec{r}|^2}{|\vec{r}|^2}$$

$$= \underline{1}$$

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iint_S dS$$

$$= \text{area of sphere}$$

$$= \underline{4\pi a^2}$$

$$b) \quad \nabla \cdot \vec{F} = \nabla \cdot \left(\frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}} \right)$$

$$\frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) = \frac{x^2 + y^2 + z^2 - x^2}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\begin{aligned}
&= \frac{y^2 + z^2}{\left(x^2 + y^2 + z^2\right)^{3/2}} \\
\frac{\partial}{\partial x} \left(\frac{y}{\sqrt{x^2 + y^2 + z^2}} \right) &= \frac{x^2 + y^2 + z^2 - y^2}{x^2 + y^2 + z^2} \\
&= \frac{x^2 + z^2}{\left(x^2 + y^2 + z^2\right)^{3/2}} \\
\frac{\partial}{\partial x} \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) &= \frac{x^2 + y^2 + z^2 - z^2}{x^2 + y^2 + z^2} \\
&= \frac{x^2 + y^2}{\left(x^2 + y^2 + z^2\right)^{3/2}}
\end{aligned}$$

$$\begin{aligned}
\nabla \cdot \vec{F} &= \frac{y^2 + z^2 + x^2 + z^2 + x^2 + y^2}{\left(x^2 + y^2 + z^2\right)^{3/2}} \\
&= \frac{2\left(x^2 + y^2 + z^2\right)}{\left(x^2 + y^2 + z^2\right)^{3/2}} \\
&= \frac{2}{\sqrt{x^2 + y^2 + z^2}} \\
&= \frac{2}{|\vec{r}|}
\end{aligned}$$

$$\begin{aligned}
\iiint_D \nabla \cdot \vec{F} \, dV &= \iiint_D \frac{2}{|\vec{r}|} \, dV \\
&= 2 \int_0^{2\pi} \int_0^\pi \int_\varepsilon^a \frac{1}{\rho} \rho^2 \sin \varphi \, d\rho d\varphi d\theta \\
&= 2 \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi \, d\varphi \int_\varepsilon^a \rho \, d\rho \\
&= 4\pi \left(-\cos \varphi \right) \Big|_0^\pi \left(\frac{1}{2} \rho^2 \right) \Big|_\varepsilon^a \\
&= \underline{4\pi \left(a^2 - \varepsilon^2 \right)}
\end{aligned}$$

$$\lim_{\varepsilon \rightarrow 0} 4\pi \left(a^2 - \varepsilon^2 \right) = \underline{4\pi a^2}$$

Exercise

The electric field due to a point charge Q is $\mathbf{E} = \frac{Q}{4\pi\epsilon_0} \frac{\vec{r}}{|\vec{r}|^3}$, where $\vec{r} = \langle x, y, z \rangle$ and ϵ_0 is a constant

- a) Show that the flux of the field across a sphere of radius a centered at the origin is

$$\iint_S \mathbf{E} \cdot \vec{n} \, dS = \frac{Q}{\epsilon_0}$$

- b) Let S be the boundary of the region between two spheres centered at the origin with radii a and b with $a < b$. Use the Divergence Theorem to show that the net outward flux across S is zero.

- c) Suppose there is a distribution of charge within a region D . Let $q(x, y, z)$ be the charge density (charge per unit volume). Interpret the statement that

$$\iint_S \mathbf{E} \cdot \vec{n} \, dS = \frac{1}{\epsilon_0} \iiint_D q(x, y, z) \, dV$$

- d) Assuming \mathbf{E} satisfies the conditions of the Divergence Theorem, conclude from part (c) that

$$\nabla \cdot \mathbf{E} = \frac{q}{\epsilon_0}$$

- e) Because the electric force is conservative, it has a potential function ϕ . From part (d) conclude

$$\text{that } \nabla^2 \phi = \nabla \cdot \nabla \phi = \frac{q}{\epsilon_0}$$

Solution

$$\begin{aligned} \text{a) } \vec{F} \cdot \vec{n} &= \frac{\vec{r}}{|\vec{r}|^3} \cdot \frac{\vec{r}}{|\vec{r}|} \\ &= \frac{|\vec{r}|^2}{|\vec{r}|^4} \\ &= |\vec{r}|^{-2} \end{aligned}$$

$$\iint_S \vec{F} \cdot \vec{n} \, dS = |\vec{r}|^{-2}$$

$$\begin{aligned} \text{Area of sphere} &= |\vec{r}|^{-2} (4\pi r^2) \\ &= |a|^{-2} (4\pi a^2) \\ &= 4\pi \end{aligned}$$

$$\begin{aligned} \iint_S \mathbf{E} \cdot \vec{n} \, dS &= \frac{Q}{4\pi\epsilon_0} (4\pi) \\ &= \frac{Q}{\epsilon_0} \end{aligned}$$

- b) The net outward flux across S is the difference of the fluxes across the inner and outer sphere; by part (a), these are equal (independent of the radius), so the net flux across S is zero.
- c) The left-hand side is the flux across the boundary of D , while the right-hand side is the sum of the charge densities at each point of D .

$$\begin{aligned}\iint_S \mathbf{E} \cdot \vec{n} \, dS &= \frac{1}{\epsilon_0} \iiint_D q(x, y, z) \, dV \\ &= \frac{Q}{\epsilon_0} \\ \rightarrow \iiint_D q(x, y, z) \, dV &= Q\end{aligned}$$

The statement says that the flux across the boundary, up to multiplication by a constant, is the sum of the charge densities in the region.

$$\begin{aligned}d) \quad \frac{1}{\epsilon_0} \iiint_D q(x, y, z) \, dV &= \iint_S \mathbf{E} \cdot \vec{n} \, dS \\ &= \iiint_D \nabla \cdot \mathbf{E} \, dV\end{aligned}$$

This holds for all regions D .

Therefore; that implies that $\nabla \cdot \mathbf{E} = \frac{q(x, y, z)}{\epsilon_0}$

$$\begin{aligned}e) \quad \nabla^2 \phi &= \nabla \cdot \nabla \phi \\ &= \nabla \cdot \mathbf{E} \\ &= \frac{q}{\epsilon_0}\end{aligned}$$

Exercise

Fourier's Law of heat transfer (or heat conduction) states that the heat flow vector \vec{F} at a point is proportional to the negative gradient of the temperature that is, $\vec{F} = -k\nabla T$, which means that heat energy flows from hot regions to cold region. The constant $k > 0$ is called the *conductivity*, which has metric units of $J / m \cdot s \cdot K$. A temperature function for a region D is given. Find the net outward heat flux

$$\iint_S \vec{F} \cdot \vec{n} \, dS = -k \iint_S \nabla T \cdot \vec{n} \, dS \text{ across the boundary } S \text{ of } D. \text{ In some cases it may be easier to use the}$$

Divergence Theorem and evaluate a triple integral. Assume $k = 1$.

$$T(x, y, z) = 100 + x + 2y + z; \quad D = \{(x, y, z): 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$$

Solution

$$\begin{aligned}\nabla T &= T_x \hat{i} + T_y \hat{j} + T_z \hat{k} \\ &= \hat{i} + 2\hat{j} + \hat{k}\end{aligned}$$

$$\begin{aligned}\vec{F} &= -k\nabla T \\ &= k\langle -1, -2, -1 \rangle\end{aligned}$$

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(-k) + \frac{\partial}{\partial y}(-2k) + \frac{\partial}{\partial z}(-k) \\ &= 0\end{aligned}$$

Therefore, the heat flux is **zero**.

Exercise

Fourier's Law of heat transfer (or heat conduction) states that the heat flow vector \vec{F} at a point is proportional to the negative gradient of the temperature that is, $\vec{F} = -k\nabla T$, which means that heat energy flows from hot regions to cold region. The constant $k > 0$ is called the *conductivity*, which has metric units of $J / m \cdot s \cdot K$. A temperature function for a region D is given.

Find the net outward heat flux $\iint_S \vec{F} \cdot \vec{n} \, dS = -k \iint_S \nabla T \cdot \vec{n} \, dS$ across the boundary S of D . In some cases, it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume $k = 1$.

$$T(x, y, z) = 100 + x^2 + y^2 + z^2; \quad D = \{(x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$$

Solution

$$\begin{aligned}\nabla T &= T_x \hat{i} + T_y \hat{j} + T_z \hat{k} \\ &= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}\end{aligned}$$

$$\begin{aligned}\vec{F} &= -\nabla T \\ &= \langle -2x, -2y, -2z \rangle\end{aligned}$$

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(-2x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(-2z) \\ &= -6\end{aligned}$$

Therefore, the heat flux is **-6** times the volume of the region (or **-6**).

Exercise

Fourier's Law of heat transfer (or heat conduction) states that the heat flow vector \vec{F} at a point is proportional to the negative gradient of the temperature that is, $\vec{F} = -k\nabla T$, which means that heat energy flows from hot regions to cold region. The constant $k > 0$ is called the *conductivity*, which has metric units of $J / m \cdot s \cdot K$. A temperature function for a region D is given.

Find the net outward heat flux $\iint_S \vec{F} \cdot \vec{n} \, dS = -k \iint_S \nabla T \cdot \vec{n} \, dS$ across the boundary S of D . In some cases,

it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume $k = 1$.

$$T(x, y, z) = 100 + e^{-z}; \quad D = \{(x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$$

Solution

$$\nabla T = T_x \hat{i} + T_y \hat{j} + T_z \hat{k}$$

$$= -e^{-z} \hat{k}$$

$$\vec{F} = -\nabla T$$

$$= \langle 0, 0, e^{-z} \rangle$$

$$\begin{aligned} \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(e^{-z}) \\ &= -e^{-z} \end{aligned}$$

Therefore, the heat flux is

$$\begin{aligned} \int_0^1 \int_0^1 \int_0^1 (-e^{-z}) \, dx \, dy \, dz &= \int_0^1 (-e^{-z}) \, dz \int_0^1 dy \int_0^1 dx \\ &= e^{-z} \Big|_0^1 (1)(1) \\ &= e^{-1} - 1 \end{aligned}$$

Exercise

Fourier's Law of heat transfer (or heat conduction) states that the heat flow vector \vec{F} at a point is proportional to the negative gradient of the temperature that is, $\vec{F} = -k\nabla T$, which means that heat energy flows from hot regions to cold region. The constant $k > 0$ is called the *conductivity*, which has metric units of $J / m \cdot s \cdot K$. A temperature function for a region D is given.

Find the net outward heat flux $\iint_S \vec{F} \cdot \vec{n} \, dS = -k \iint_S \nabla T \cdot \vec{n} \, dS$ across the boundary S of D . In some cases,

it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume $k = 1$

$T(x, y, z) = 100e^{-x^2-y^2-z^2}$; D is the sphere of radius a centered at the origin.

Solution

$$\begin{aligned}\nabla T &= T_x \hat{i} + T_y \hat{j} + T_z \hat{k} \\ &= \left\langle -200xe^{-x^2-y^2-z^2}, -200ye^{-x^2-y^2-z^2}, -200ze^{-x^2-y^2-z^2} \right\rangle\end{aligned}$$

$$\begin{aligned}\vec{F} &= -\nabla T \\ &= 200 \left\langle xe^{-x^2-y^2-z^2}, ye^{-x^2-y^2-z^2}, ze^{-x^2-y^2-z^2} \right\rangle\end{aligned}$$

$$\begin{aligned}\nabla \cdot \vec{F} &= 200 \frac{\partial}{\partial x} \left(xe^{-x^2-y^2-z^2} \right) + 200 \frac{\partial}{\partial y} \left(ye^{-x^2-y^2-z^2} \right) + 200 \frac{\partial}{\partial z} \left(ze^{-x^2-y^2-z^2} \right) \\ &= 200e^{-x^2-y^2-z^2} (1-2x^2 + 1-2y^2 + 1-2z^2) \\ &= 200e^{-x^2-y^2-z^2} (3-2x^2-2y^2-2z^2)\end{aligned}$$

$$\begin{aligned}\iiint_D \nabla \cdot \vec{F} \, dV &= \iiint_D 200e^{-x^2-y^2-z^2} (3-2x^2-2y^2-2z^2) \, dV \\ &= 200 \int_0^{2\pi} \int_0^\pi \int_0^a e^{-\rho^2} (3-2\rho^2) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \\ &= 200 \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi \, d\varphi \int_0^a (3\rho^2 - 2\rho^4) e^{-\rho^2} \, d\rho \\ &= 200(2\pi) [-\cos \varphi]_0^\pi \left(\rho^3 e^{-\rho^2} \right)_0^a \quad \left((3\rho^2 - 2\rho^4) e^{-\rho^2} = \left(\rho^3 e^{-\rho^2} \right)' \right) \\ &= \underline{800\pi a^3 e^{-a^2}}\end{aligned}$$

Exercise

Consider the surface S consisting of the quarter-sphere $x^2 + y^2 + z^2 = a^2$, for $z \geq 0$ and $x \geq 0$, and the half disk in the yz -plane $y^2 + z^2 \leq a^2$, for $z \geq 0$. The boundary of S in the xy -plane is C , which consists of the semicircle $x^2 + y^2 = a^2$, for $x \geq 0$, and the line segment $[-a, a]$ on the y -axis, with a counterclockwise orientation. Let $\vec{F} = \langle 2z - y, x - z, y - 2x \rangle$

a) Describe the direction in which the normal vectors point on S .

b) Evaluate $\oint_C \vec{F} \cdot d\vec{r}$

c) Evaluate $\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$ and check for segment with part (b).

Solution

a) Normal vectors point outwards everywhere on S ; that is, on the curved surface. They point outwards and on the flat surface they point in the direction of negative x .

b) $\vec{F} = \langle 2z - y, x - z, y - 2x \rangle$ & $C: x^2 + y^2 = a^2 \quad [-a, a]$

Parameterized:

$$\vec{r}_1 = \langle a \cos t, a \sin t, 0 \rangle$$

$$\vec{r}_1' = \langle -a \sin t, a \cos t, 0 \rangle$$

$$\vec{F} = \langle 2z - y, x - z, y - 2x \rangle$$

$$= \langle -a \sin t, a \cos t, a \sin t - 2a \cos t \rangle$$

$$\vec{F} \cdot d\vec{r}_1 = \langle -a \sin t, a \cos t, a \sin t - 2a \cos t \rangle \cdot \langle -a \sin t, a \cos t, 0 \rangle$$

$$= a^2 \sin^2 t + a^2 \cos^2 t$$

$$= \underline{a^2}$$

For $0 \leq t \leq a$

$$\vec{r}_2 = \langle 0, a - 2t, 0 \rangle$$

$$\vec{r}_2' = \langle 0, -2, 0 \rangle$$

$$\vec{F} = \langle 2z - y, x - z, y - 2x \rangle$$

$$= \langle 2t - a, 0, a - 2t \rangle$$

$$\vec{F} \cdot d\vec{r}_2 = \langle 2t - a, 0, a - 2t \rangle \cdot \langle 0, -2, 0 \rangle$$

$$= \underline{0}$$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot d\vec{r}_1 + \oint_C \vec{F} \cdot d\vec{r}_2$$

$$\begin{aligned}
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2 dt + 0 \\
&= a^2 t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \pi a^2
\end{aligned}$$

$$\begin{aligned}
c) \quad \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z - y & x - z & y - 2x \end{vmatrix} \\
&= \langle 2, 4, 2 \rangle
\end{aligned}$$

Spherical: $\langle a \sin \varphi \cos \theta, a \sin \varphi \sin \theta, a \cos \varphi \rangle$

$$\mathbf{t}_\varphi = \langle a \cos \varphi \cos \theta, a \cos \varphi \sin \theta, -a \sin \varphi \rangle$$

$$\mathbf{t}_\theta = \langle -a \sin \varphi \sin \theta, a \sin \varphi \cos \theta, 0 \rangle$$

$$\vec{n}_1 = \mathbf{t}_\varphi \times \mathbf{t}_\theta$$

$$\begin{aligned}
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a \cos \varphi \cos \theta & a \cos \varphi \sin \theta & -a \sin \varphi \\ -a \sin \varphi \sin \theta & a \sin \varphi \cos \theta & 0 \end{vmatrix} \\
&= \langle a^2 \sin^2 \varphi \cos \theta, a^2 \sin^2 \varphi \sin \theta, a^2 \cos \varphi \sin \varphi \cos^2 \theta + a^2 \cos \varphi \sin \varphi \sin^2 \theta \rangle \\
&= \langle a^2 \sin^2 \varphi \cos \theta, a^2 \sin^2 \varphi \sin \theta, a^2 \cos \varphi \sin \varphi \rangle
\end{aligned}$$

$$\begin{aligned}
(\nabla \times \vec{F}) \cdot \vec{n}_1 &= \langle 2, 4, 2 \rangle \cdot \langle a^2 \sin^2 \varphi \cos \theta, a^2 \sin^2 \varphi \sin \theta, a^2 \cos \varphi \sin \varphi \rangle \\
&= \underline{2a^2 \sin^2 \varphi \cos \theta + 4a^2 \sin^2 \varphi \sin \theta + 2a^2 \cos \varphi \sin \varphi}
\end{aligned}$$

From part (a), the flat surface is pointing in x -negative direction

$$\vec{n}_2 = \langle -1, 0, 0 \rangle$$

$$\begin{aligned}
(\nabla \times \vec{F}) \cdot \vec{n}_2 &= \langle 2, 4, 2 \rangle \cdot \langle -1, 0, 0 \rangle \\
&= -2
\end{aligned}$$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \iint_{S_1} (2a^2 \sin^2 \varphi \cos \theta + 4a^2 \sin^2 \varphi \sin \theta + 2a^2 \cos \varphi \sin \varphi) \, dS - 2 \iint_{S_2} dS$$

$$\begin{aligned}
&= 2a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}(1 - \cos 2\varphi) \cos \theta + \frac{1}{2}(1 - \cos 2\varphi) \sin \theta + \cos \varphi \sin \varphi \right) d\varphi d\theta \\
&\quad - 2 \times (\text{semi-circle}) \\
&= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (\cos \theta - \cos 2\varphi \cos \theta + \sin \theta - \cos 2\varphi \sin \theta + \sin 2\varphi) d\varphi d\theta \\
&\quad - 2 \left(\frac{1}{2} \pi a^2 \right) \\
&= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left((\cos \theta + \sin \theta) \varphi - \frac{1}{2} (\cos \theta + \sin \theta) \sin 2\varphi - \frac{1}{2} \cos 2\varphi \right) \bigg|_0^{\frac{\pi}{2}} d\theta - \pi a^2 \\
&= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left((\cos \theta + \sin \theta) \frac{\pi}{2} + \frac{1}{2} + \frac{1}{2} \right) d\theta - \pi a^2 \\
&= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi}{2} (\cos \theta + \sin \theta) + 1 \right) d\theta - \pi a^2 \\
&= a^2 \left(\frac{\pi}{2} (\sin \theta - \cos \theta) + \theta \right) \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \pi a^2 \\
&= a^2 \left(\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} \right) - \pi a^2 \\
&= 2\pi a^2 - \pi a^2 \\
&= \pi a^2
\end{aligned}$$

Exercise

Let S be the hemisphere $x^2 + y^2 + z^2 = a^2$, for $z \geq 0$, and let T be the paraboloid $z = a - \frac{1}{a}(x^2 + y^2)$, for $z \geq 0$, where $a > 0$. Assume the surfaces have outward normal vectors.

- Verify that S and T have the same base $(x^2 + y^2 \leq a^2)$ and the same high point $(0, 0, a)$.
- Which surface has the greater area?
- Show that the flux of the radial field $\vec{F} = \langle x, y, z \rangle$ across S is $2\pi a^3$.
- Show that the flux of the radial field $\vec{F} = \langle x, y, z \rangle$ across T is $\frac{3\pi a^3}{2}$.

Solution

- S : base is the surface where $z = 0$

$$\Rightarrow x^2 + y^2 = a^2 \quad (\text{circle})$$

$$T: z = 0 = a - \frac{1}{a}(x^2 + y^2) \Rightarrow x^2 + y^2 = a^2$$

The high point of the hemisphere (maximum z -coordinate) occurs when

$$x = y = 0 \rightarrow z = a$$

$$\therefore x^2 + y^2 \leq a^2 \quad \checkmark$$

b) S : is the surface of hemisphere $\frac{1}{2}4\pi a^2 = 2\pi a^2$

For T : paraboloid

$$z_x = -\frac{2x}{a} \quad z_y = -\frac{2y}{a}$$

$$\vec{n} = \sqrt{z_x^2 + z_y^2 + 1}$$

$$= \sqrt{\frac{4x^2}{a^2} + \frac{4y^2}{a^2} + 1}$$

$$= \frac{1}{a} \sqrt{4(x^2 + y^2) + a^2}$$

$$\iint_S 1 \, dS = \frac{1}{a} \iint_S \sqrt{4(x^2 + y^2) + a^2} \, dS$$

$$= \frac{1}{a} \int_0^{2\pi} d\theta \int_0^a \sqrt{4r^2 + a^2} \, r \, dr$$

$$= \frac{\pi}{4a} \int_0^a (4r^2 + a^2)^{1/2} d(4r^2 + a^2)$$

$$= \frac{\pi}{6a} (4r^2 + a^2)^{3/2} \Big|_0^a$$

$$= \frac{\pi}{6a} \left((5a^2)^{3/2} - a^3 \right)$$

$$= \frac{\pi}{6a} (5^{3/2} a^3 - a^3)$$

$$= \frac{\pi}{6} (5\sqrt{5} - 1) a^2$$

$$= \frac{5\sqrt{5} - 1}{6} \pi a^2$$

$$< 2\pi a^2$$

\therefore Area of the paraboloid is smaller than the area of the hemisphere.

c) $\vec{F} = \langle x, y, z \rangle$

$$= \langle a \sin \varphi \cos \theta, a \sin \varphi \sin \theta, a \cos \varphi \rangle$$

$$\mathbf{t}_\varphi = \langle a \cos \varphi \cos \theta, a \cos \varphi \sin \theta, -a \sin \varphi \rangle$$

$$\mathbf{t}_\theta = \langle -a \sin \varphi \sin \theta, a \sin \varphi \cos \theta, 0 \rangle$$

$$\vec{n}_1 = \mathbf{t}_\varphi \times \mathbf{t}_\theta$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a \cos \varphi \cos \theta & a \cos \varphi \sin \theta & -a \sin \varphi \\ -a \sin \varphi \sin \theta & a \sin \varphi \cos \theta & 0 \end{vmatrix}$$

$$= \langle a^2 \sin^2 \varphi \cos \theta, a^2 \sin^2 \varphi \sin \theta, a^2 \cos \varphi \sin \varphi \cos^2 \theta + a^2 \cos \varphi \sin \varphi \sin^2 \theta \rangle$$

$$= \langle a^2 \sin^2 \varphi \cos \theta, a^2 \sin^2 \varphi \sin \theta, a^2 \cos \varphi \sin \varphi \rangle$$

$$\vec{F} \cdot \vec{n} = \langle a \sin \varphi \cos \theta, a \sin \varphi \sin \theta, a \cos \varphi \rangle \cdot \langle a^2 \sin^2 \varphi \cos \theta, a^2 \sin^2 \varphi \sin \theta, a^2 \cos \varphi \sin \varphi \rangle$$

$$= a^3 \sin^3 \varphi \cos^2 \theta + a^3 \sin^3 \varphi \sin^2 \theta + a^3 \cos^2 \varphi \sin \varphi$$

$$= a^3 \sin^3 \varphi (\cos^2 \theta + \sin^2 \theta) + a^3 \cos^2 \varphi \sin \varphi$$

$$= a^3 \sin \varphi (\sin^2 \varphi + \cos^2 \varphi)$$

$$= a^3 \sin \varphi$$

$$\iint_S \vec{F} \cdot \vec{n} \, dS = a^3 \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin \varphi \, d\varphi$$

$$= -2\pi a^3 (\cos \varphi) \Big|_0^{\frac{\pi}{2}}$$

$$= 2\pi a^3 \quad \checkmark$$

d) Across T : $z = a - \frac{1}{a}(x^2 + y^2)$

Parameterization:

$$\left\langle r \cos \theta, r \sin \theta, a - \frac{r^2}{a} \right\rangle$$

$$0 \leq r \leq a \quad 0 \leq \theta \leq 2\pi$$

$$\mathbf{t}_r = \left\langle \cos \theta, \sin \theta, -\frac{2}{a}r \right\rangle$$

$$\mathbf{t}_\theta = \langle -r \sin \theta, r \cos \theta, 0 \rangle$$

$$\vec{n} = \mathbf{t}_r \times \mathbf{t}_\theta$$

$$\begin{aligned}
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \theta & \sin \theta & -\frac{2}{a}r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} \\
&= \left\langle \frac{2r^2}{a} \cos \theta, \frac{2r^2}{a} \sin \theta, r \cos^2 \theta + r \sin^2 \theta \right\rangle \\
&= \left\langle \frac{2r^2}{a} \cos \theta, \frac{2r^2}{a} \sin \theta, r \right\rangle
\end{aligned}$$

$$\begin{aligned}
\vec{F} \cdot \vec{n} &= \left\langle r \cos \theta, r \sin \theta, a - \frac{r^2}{a} \right\rangle \cdot \left\langle \frac{2r^2}{a} \cos \theta, \frac{2r^2}{a} \sin \theta, r \right\rangle \\
&= \frac{2}{a} r^3 \cos^2 \theta + \frac{2r^2}{a} \sin^2 \theta + ar - \frac{1}{a} r^3 \\
&= \frac{2}{a} r^3 (\cos^2 \theta + \sin^2 \theta) + ar - \frac{1}{a} r^3 \\
&= \frac{2}{a} r^3 + ar - \frac{1}{a} r^3 \\
&= \frac{1}{a} (r^3 + a^2 r)
\end{aligned}$$

$$\begin{aligned}
\iint_S \vec{F} \cdot \vec{n} \, dS &= \frac{1}{a} \int_0^{2\pi} d\theta \int_0^a (r^3 + a^2 r) \, dr \\
&= \frac{2\pi}{a} \left(\frac{1}{4} r^4 + \frac{1}{2} a^2 r^2 \right) \Big|_0^a \\
&= \frac{2\pi}{a} \left(\frac{1}{4} a^4 + \frac{1}{2} a^4 \right) \\
&= \frac{2\pi}{a} \left(\frac{3}{4} a^4 \right) \\
&= \frac{3\pi a^3}{2} \quad \checkmark
\end{aligned}$$

Exercise

The gravitational force due to a point mass M is proportional to $\vec{F} = \frac{GM\vec{r}}{|\vec{r}|^3}$, where $\vec{r} = \langle x, y, z \rangle$ and G is the gravitational constant.

- a) Show that the flux force field across a sphere of radius a centered at the origin is

$$\iint_S \vec{F} \cdot \vec{n} \, dS = 4\pi GM$$

- b) Let S be the boundary of the region between two spheres centered at the origin of radius a and b with $a < b$. Use the Divergence Theorem to show that the net outward flux across S is zero.
- c) Suppose there is a distribution of mass within a region D containing the origin. Let $\rho = \rho(x, y, z)$ be the mass density (mass per unit volume). Interpret the statement that

$$\iint_S \vec{F} \cdot \vec{n} \, dS = 4\pi G \iiint_D \rho(x, y, z) \, dV$$

- d) Assuming \vec{F} satisfies the conditions of the Divergence Theorem, conclude from part (c) that $\nabla \cdot \vec{F} = 4\pi G\rho$
- e) Because the gravitational force is conservative, it has a potential function ϕ . From part (d) conclude that $\nabla^2 \phi = 4\pi G\rho$

Solution

- a) The unit normal to sphere is $\vec{n} = \frac{\vec{r}}{|\vec{r}|}$

$$\begin{aligned} \vec{F} \cdot \vec{n} &= GM \frac{\vec{r}}{|\vec{r}|^3} \cdot \frac{\vec{r}}{|\vec{r}|} \\ &= GM \frac{|\vec{r}|^2}{|\vec{r}|^4} \\ &= GM \frac{1}{|\vec{r}|^2} \\ &= GM \frac{1}{x^2 + y^2 + z^2} \\ &= \frac{GM}{a^2} \end{aligned}$$

$$\begin{aligned} \iint_S \vec{F} \cdot \vec{n} \, dS &= \frac{GM}{a^2} \iint_S dS \\ &= \frac{GM}{a^2} \times (\text{area of sphere}) \end{aligned}$$

$$= \frac{GM}{a^2} \times 4\pi a^2$$

$$= 4\pi GM \quad \checkmark$$

$$b) \quad \iint_S \vec{F} \cdot \vec{n} \, dS = 4\pi G \iiint_D \rho(x, y, z) \, dV$$

Since the outward flux across a sphere, from part (a), is independent of the radius of the sphere, the outward flux across the spheres of radii a and b are equal, so their difference, which is the net flux across the spherical shell bounded by them, is zero.

$$c) \quad \iint_S \vec{F} \cdot \vec{n} \, dS = 4\pi G \iiint_D \rho(x, y, z) \, dV \quad \rho = \rho(x, y, z)$$

The left-hand side is the flux across the boundary of D , while the right-hand side is the sum of the mass density inside D .

The statement says that the flux across the boundary is determined by (is a constant multiple of) the sum of the mass density inside D .

$$d) \quad 4\pi G \iiint_D \rho(x, y, z) \, dV = \iint_S \vec{F} \cdot \vec{n} \, dS$$

$$= \iiint_D \nabla \cdot \vec{F} \, dV$$

$$\iiint_D \nabla \cdot \vec{F} \, dV = \iiint_D 4\pi G \rho(x, y, z) \, dV$$

$$\nabla \cdot \vec{F} = 4\pi G \rho(x, y, z)$$

$$e) \quad \nabla^2 \phi = \nabla \cdot \nabla \phi$$

$$= \nabla \cdot \vec{F}$$

$$= 4\pi G \rho(x, y, z)$$

Exercise

Let \vec{F} be a radial field $\vec{F} = \frac{\vec{r}}{|\vec{r}|^p}$, where p is a real number and $\vec{r} = \langle x, y, z \rangle$. With $p = 3$, \vec{F} is an inverse square field.

- Show that the net flux across a sphere centered at the origin is independent of the radius of the sphere only for $p = 3$
- Explain the observation in part (a) by finding the flux of $\vec{F} = \frac{\vec{r}}{|\vec{r}|^p}$ across the boundaries of a spherical box $\{(\rho, \varphi, \theta) : a \leq \rho \leq b, \varphi_1 \leq \varphi \leq \varphi_2, \theta_1 \leq \theta \leq \theta_2\}$ for various values of p .

Solution

$$c) \quad \vec{F} = \frac{\vec{r}}{|\vec{r}|^p} \quad \& \quad \vec{n} = \frac{\vec{r}}{|\vec{r}|}$$

$$\begin{aligned} \vec{F} \cdot \vec{n} &= \frac{\vec{r}}{|\vec{r}|^p} \cdot \frac{\vec{r}}{|\vec{r}|} \\ &= \frac{|\vec{r}|^2}{|\vec{r}|^{p+1}} \\ &= |\vec{r}|^{1-p} \\ &= \left(\sqrt{x^2 + y^2 + z^2} \right)^{1-p} \\ &= \left(\sqrt{a^2} \right)^{1-p} \\ &= a^{1-p} \end{aligned}$$

$$\begin{aligned} \iint_S \vec{F} \cdot \vec{n} \, dS &= a^{1-p} \iint_S dS \\ &= a^{1-p} \times (\text{area of sphere}) \\ &= a^{1-p} \times 4\pi a^2 \\ &= 4\pi a^{3-p} \end{aligned}$$

If $p = 3$, then

$$\iint_S \vec{F} \cdot \vec{n} \, dS = 4\pi$$

Which is independent of the radius of the sphere.

$$\begin{aligned}
d) \quad \nabla \cdot \frac{\langle x, y, z \rangle}{|\vec{r}|^p} &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{p/2}} \\
&= \frac{\partial}{\partial x} \frac{x}{(x^2 + y^2 + z^2)^{p/2}} + \frac{\partial}{\partial y} \frac{y}{(x^2 + y^2 + z^2)^{p/2}} + \frac{\partial}{\partial z} \frac{z}{(x^2 + y^2 + z^2)^{p/2}} \\
&= \frac{x^2 + y^2 + z^2 - px^2}{(x^2 + y^2 + z^2)^{1+p/2}} + \frac{x^2 + y^2 + z^2 - py^2}{(x^2 + y^2 + z^2)^{1+p/2}} + \frac{x^2 + y^2 + z^2 - pz^2}{(x^2 + y^2 + z^2)^{1+p/2}} \\
&= \frac{3(x^2 + y^2 + z^2) - p(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{1+p/2}} \\
&= \frac{(3-p)(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{1+p/2}} \\
&= \frac{3-p}{(x^2 + y^2 + z^2)^{p/2}} \\
&= \frac{3-p}{|\vec{r}|^p}
\end{aligned}$$

$$\begin{aligned}
\iiint_D \nabla \cdot \vec{F} \, dV &= \iiint_D \frac{3-p}{|\vec{r}|^p} \, dV \\
&= (3-p) \iiint_D r^{-p} \, dV \\
&= (3-p) \int_{\theta_1}^{\theta_2} \int_{\varphi_1}^{\varphi_2} \int_a^b \rho^{-p} \rho^2 \sin \varphi \, d\rho d\varphi d\theta \\
&= (3-p) \int_{\theta_1}^{\theta_2} d\theta \int_{\varphi_1}^{\varphi_2} \sin \varphi \, d\varphi \int_a^b \rho^{2-p} \, d\rho \\
&= (3-p) \left(\theta \right) \Big|_{\theta_1}^{\theta_2} \left(-\cos \varphi \right) \Big|_{\varphi_1}^{\varphi_2} \left(\frac{1}{3-p} \rho^{3-p} \right) \Big|_a^b \\
&= (\theta_2 - \theta_1) (\cos \varphi_1 - \cos \varphi_2) (b^{3-p} - a^{3-p})
\end{aligned}$$

In general, for $\iiint_D \nabla \cdot \vec{F} \, dV = 0$, only if $3 - p = 0 \rightarrow \underline{p = 3}$

Exercise

Consider the potential function $\phi(x, y, z) = G(\rho)$, where G is any twice differentiable function and

$\rho = \sqrt{x^2 + y^2 + z^2}$; therefore, G depends only on the distance from the origin.

- Show that the gradient vector field associated with ϕ is $\vec{F} = \nabla \phi = G'(\rho) \frac{\vec{r}}{\rho}$, where $\vec{r} = \langle x, y, z \rangle$ and $\rho = |\vec{r}|$.
- Let S be the sphere of radius a centered at the origin and let D be the region enclosed by S . show that the flux of \vec{F} across S is $\iint_S \vec{F} \cdot \vec{n} \, dS = 4\pi a^2 G'(a)$.
- Show that $\nabla \cdot \vec{F} = \nabla \cdot \nabla \phi = \frac{2}{\rho} G'(\rho) + G''(\rho)$
- Use part (c) to show that the flux across S (as given in part (b)) is also obtained by the volume integral $\iiint_D \nabla \cdot \vec{F} \, dV$. (Hint: use spherical coordinates and integrate by parts.)

Solution

a) $\phi(x, y, z) = G(\rho)$

$$\begin{aligned} \nabla \phi(x, y, z) &= \langle \phi_x, \phi_y, \phi_z \rangle \\ &= \langle G'(\rho) \rho_x, G'(\rho) \rho_y, G'(\rho) \rho_z \rangle \\ &= G'(\rho) \langle \rho_x, \rho_y, \rho_z \rangle \\ &= G'(\rho) \left\langle \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right\rangle \\ &= \frac{G'(\rho)}{\sqrt{x^2 + y^2 + z^2}} \langle x, y, z \rangle \\ &= G'(\rho) \frac{\vec{r}}{\rho} \end{aligned}$$

$\therefore \vec{F} = \nabla \phi = G'(\rho) \frac{\vec{r}}{\rho}$

b) The sphere of radius a centered at the origin $\rightarrow x^2 + y^2 + z^2 = a^2$

$$2xdx + 2zdz = 0 \rightarrow z_x = -\frac{x}{z}$$

$$2ydy + 2zdz = 0 \rightarrow z_y = -\frac{y}{z}$$

$$\vec{n} = \left\langle \frac{x}{z}, \frac{y}{z}, 1 \right\rangle$$

$$\vec{F} \cdot \vec{n} = \frac{G'(\rho)}{\rho} \langle x, y, z \rangle \cdot \left\langle \frac{x}{z}, \frac{y}{z}, 1 \right\rangle$$

$$= \frac{G'(a)}{a} \left(\frac{x^2}{z} + \frac{y^2}{z} + z \right)$$

$$= \frac{G'(a)}{a} \left(\frac{x^2 + y^2 + z^2}{z} \right)$$

$$= \frac{G'(a)}{a} \left(\frac{a^2}{z} \right)$$

$$= G'(a) \frac{a}{z}$$

$$\iint_S \vec{F} \cdot \vec{n} \, dS = aG'(a) \iint_S \frac{1}{z} \, dS$$

$$= aG'(a) \int_0^{2\pi} \int_0^a \frac{1}{\sqrt{a^2 - r^2}} r \, dr d\theta$$

$$= -\frac{1}{2} aG'(a) \int_0^{2\pi} d\theta \int_0^a (a^2 - r^2)^{-1/2} d(a^2 - r^2)$$

$$= -2\pi aG'(a) (a^2 - r^2)^{1/2} \Big|_0^a$$

$$= 2\pi a^2 G'(a)$$

Since the surface area is *twice* the value

Total surface $= 4\pi a^2 G'(a)$

c) $\vec{F} = \nabla \phi = G'(\rho) \frac{\vec{r}}{\rho} \quad \rho = \sqrt{x^2 + y^2 + z^2}$

$$\nabla \cdot \vec{F} = \nabla \cdot \left(G'(\rho) \frac{\vec{r}}{\rho} \right)$$

$$= \nabla \cdot \frac{G'(\rho)}{\rho} \langle x, y, z \rangle$$

$$= \frac{\partial}{\partial x} \left(G'(\rho) \frac{x}{\rho} \right) + \frac{\partial}{\partial y} \left(G'(\rho) \frac{y}{\rho} \right) + \frac{\partial}{\partial z} \left(G'(\rho) \frac{z}{\rho} \right)$$

$$\begin{aligned}
\frac{\partial}{\partial x} \left(G'(\rho) \frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) &= G''(\rho) \rho_x \frac{x}{\rho} + G'(\rho) \frac{x^2 + y^2 + z^2 - x^2}{(x^2 + y^2 + z^2)^{3/2}} \\
&= G''(\rho) \frac{x}{\rho} \frac{x}{\rho} + G'(\rho) \frac{y^2 + z^2}{\rho^3} \\
&= G''(\rho) \frac{x^2}{\rho^2} + G'(\rho) \frac{y^2 + z^2}{\rho^3}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial y} \left(G'(\rho) \frac{y}{\sqrt{x^2 + y^2 + z^2}} \right) &= G''(\rho) \rho_y \frac{y}{\rho} + G'(\rho) \frac{x^2 + y^2 + z^2 - y^2}{(x^2 + y^2 + z^2)^{3/2}} \\
&= G''(\rho) \frac{y}{\rho} \frac{y}{\rho} + G'(\rho) \frac{y^2 + z^2}{\rho^3} \\
&= G''(\rho) \frac{y^2}{\rho^2} + G'(\rho) \frac{y^2 + z^2}{\rho^3}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial z} \left(G'(\rho) \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) &= G''(\rho) \rho_z \frac{z}{\rho} + G'(\rho) \frac{x^2 + y^2 + z^2 - z^2}{(x^2 + y^2 + z^2)^{3/2}} \\
&= G''(\rho) \frac{z}{\rho} \frac{z}{\rho} + G'(\rho) \frac{y^2 + x^2}{\rho^3} \\
&= G''(\rho) \frac{z^2}{\rho^2} + G'(\rho) \frac{y^2 + x^2}{\rho^3}
\end{aligned}$$

$$\begin{aligned}
\nabla \cdot \vec{F} &= G''(\rho) \frac{z^2}{\rho^2} + G'(\rho) \frac{y^2 + z^2}{\rho^3} + G''(\rho) \frac{y^2}{\rho^2} + G'(\rho) \frac{x^2 + z^2}{\rho^3} + G''(\rho) \frac{x^2}{\rho^2} + G'(\rho) \frac{y^2 + x^2}{\rho^3} \\
&= G''(\rho) \left(\frac{x^2 + y^2 + z^2}{\rho^2} \right) + G'(\rho) \frac{2(x^2 + y^2 + z^2)}{\rho^3} \\
&= \frac{\rho^2}{\rho^2} G''(\rho) + G'(\rho) \frac{2\rho^2}{\rho^3} \\
&= \underline{G''(\rho) + \frac{2}{\rho} G'(\rho)}
\end{aligned}$$

d) By the divergence theorem, the flux:

$$\iiint_D \nabla \cdot \vec{F} \, dV = \int_0^{2\pi} \int_0^\pi \int_0^a \rho^2 \sin \varphi \left(G''(\rho) + \frac{2}{\rho} G'(\rho) \right) d\rho d\varphi d\theta$$

$$\begin{aligned}
&= \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_0^a \rho^2 \left(G''(\rho) + \frac{2}{\rho} G'(\rho) \right) d\rho \\
&= 2\pi (-\cos \varphi) \Big|_0^\pi \int_0^a \rho^2 \left(G''(\rho) + \frac{2}{\rho} G'(\rho) \right) d\rho \\
&= 4\pi \int_0^a \left(\rho^2 G''(\rho) + 2\rho G'(\rho) \right) d\rho \quad \left(\rho^2 G'(\rho) \right)' = \rho^2 G''(\rho) + 2\rho G'(\rho) \\
&= 4\pi \int_0^a \left(\rho^2 G'(\rho) \right)' d\rho \\
&= 4\pi G'(\rho) \rho^2 \Big|_0^a \\
&= \underline{4\pi a^2 G'(a)}
\end{aligned}$$

Exercise

Prove Green's Identity for scalar-valued functions u and v defined on a region D :

$$\iiint_D (u \nabla^2 v - v \nabla^2 u) dV = \iint_S (u \nabla v - v \nabla u) \cdot \vec{n} dS$$

Solution

$$\nabla \cdot (u \nabla v) = \nabla u \cdot \nabla v + u \nabla^2 v$$

$$\begin{aligned}
\iiint_D (u \nabla^2 v + \nabla u \cdot \nabla v) dV &= \iiint_D (\nabla \cdot (u \nabla v)) dV \\
&= \iint_S u \nabla v \cdot \vec{n} dS
\end{aligned}$$

$$\begin{aligned}
\iiint_D (v \nabla^2 u + \nabla u \cdot \nabla v) dV &= \iiint_D (\nabla \cdot (v \nabla u)) dV \\
&= \iint_S v \nabla u \cdot \vec{n} dS
\end{aligned}$$

$$\iiint_D (u \nabla^2 v + \nabla u \cdot \nabla v) dV - \iiint_D (v \nabla^2 u + \nabla u \cdot \nabla v) dV = \iint_S u \nabla v \cdot \vec{n} dS - \iint_S v \nabla u \cdot \vec{n} dS$$

$$\begin{aligned} \iiint_D \left(u \nabla^2 v + \nabla u \cdot \nabla v - v \nabla^2 u + \nabla u \cdot \nabla v \right) dV &= \iint_S \left(u \nabla v \cdot \vec{n} - v \nabla u \cdot \vec{n} \right) dS \\ \iiint_D \left(u \nabla^2 v - v \nabla^2 u \right) dV &= \iint_S \left(u \nabla v - v \nabla u \right) \cdot \vec{n} dS \end{aligned}$$

Exercise

Prove the identity: $\iiint_D \nabla \times \vec{F} dV = \iint_S \left(\vec{n} \times \vec{F} \right) dS$

Solution

Let $\vec{F} = \langle f, g, h \rangle$ and $\vec{n} = \langle n_1, n_2, n_3 \rangle$, then:

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \\ &= \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle \\ \vec{F} \times \vec{n} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ f & g & h \\ n_1 & n_2 & n_3 \end{vmatrix} \\ &= \langle n_2 h - n_3 g, n_3 f - n_1 h, n_1 g - n_2 f \rangle \end{aligned}$$

$$\hat{i} \rightarrow n_2 h - n_3 g = \langle 0, h, -g \rangle \cdot \langle n_1, n_2, n_3 \rangle$$

$$\vec{F}_1 = \langle 0, h, -g \rangle$$

$$\begin{aligned} \iint_S (n_2 h - n_3 g) dS &= \iint_S \vec{F}_1 \cdot \vec{n} dS \\ &= \iiint_D (\nabla \cdot \vec{F}_1) dV \\ &= \iiint_D \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) dV \end{aligned}$$

$$\hat{j} \rightarrow n_3 f - n_1 h = \langle -h, 0, f \rangle \cdot \langle n_1, n_2, n_3 \rangle$$

$$\vec{F}_2 = \langle -h, 0, f \rangle$$

$$\begin{aligned} \iint_S (n_3 f - n_1 h) dS &= \iint_S \vec{F}_2 \cdot \vec{n} dS \\ &= \iiint_D (\nabla \cdot \vec{F}_2) dV \\ &= \iint_D \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) dA \end{aligned}$$

$$\hat{k} \rightarrow n_1 g - n_2 f = \langle g, -f, 0 \rangle \cdot \langle n_1, n_2, n_3 \rangle$$

$$\vec{F}_3 = \langle g, -f, 0 \rangle$$

$$\begin{aligned} \iint_S (n_1 g - n_2 f) dS &= \iint_S \vec{F}_3 \cdot \vec{n} dS \\ &= \iiint_D (\nabla \cdot \vec{F}_3) dV \\ &= \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA \end{aligned}$$

$$\iiint_D \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle dV = \iint_S \langle n_2 h - n_3 g, n_3 f - n_1 h, n_1 g - n_2 f \rangle dS$$

$$\iiint_D \nabla \times \vec{F} dV = \iint_S (\vec{n} \times \vec{F}) dS \quad \checkmark$$

Exercise

Prove the identity: $\iint_S (\vec{n} \times \nabla \varphi) dS = \oint_C \varphi d\vec{r}$

Solution

Let $\varphi = \langle \varphi_x, \varphi_y, \varphi_z \rangle$ and $\vec{n} = \langle n_1, n_2, n_3 \rangle$, then:

$$\vec{n} \times \nabla \varphi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ n_1 & n_2 & n_3 \\ \varphi_x & \varphi_y & \varphi_z \end{vmatrix}$$

$$= \langle n_2 \varphi_z - n_3 \varphi_y, n_3 \varphi_x - n_1 \varphi_z, n_1 \varphi_y - n_2 \varphi_x \rangle$$

$$\hat{i} \rightarrow n_2 \varphi_z - n_3 \varphi_y = \langle 0, \varphi_z, -\varphi_y \rangle \cdot \langle n_1, n_2, n_3 \rangle$$

$$= (\nabla \times \langle \varphi, 0, 0 \rangle) \cdot \langle n_1, n_2, n_3 \rangle$$

$$\vec{F}_1 = \langle \varphi, 0, 0 \rangle$$

$$\iint_S ((\nabla \times \langle \varphi, 0, 0 \rangle) \cdot \langle n_1, n_2, n_3 \rangle) dS = \iint_S (\nabla \times \vec{F}_1) \cdot \vec{n} dS$$

$$= \oint_C \vec{F}_1 \cdot d\vec{r}$$

$$= \oint_C \langle \varphi, 0, 0 \rangle \cdot d\vec{r}$$

$$= \oint_C \varphi \cdot d\vec{r}$$

$$\hat{j} \rightarrow n_3 \varphi_x - n_1 \varphi_z = \langle -\varphi_z, 0, \varphi_x \rangle \cdot \langle n_1, n_2, n_3 \rangle$$

$$= (\nabla \times \langle 0, \varphi, 0 \rangle) \cdot \langle n_1, n_2, n_3 \rangle$$

$$\vec{F}_2 = \langle 0, \varphi, 0 \rangle$$

$$\iint_S ((\nabla \times \langle 0, \varphi, 0 \rangle) \cdot \langle n_1, n_2, n_3 \rangle) dS = \iint_S (\nabla \times \vec{F}_2) \cdot \vec{n} dS$$

$$= \oint_C \vec{F}_2 \cdot d\vec{r}$$

$$= \oint_C \langle 0, \varphi, 0 \rangle \cdot d\vec{r}$$

$$= \oint_C \varphi \cdot d\vec{r}$$

$$\hat{k} \rightarrow n_1 \varphi_y - n_2 \varphi_x = \langle \varphi_y, -\varphi_x, 0 \rangle \cdot \langle n_1, n_2, n_3 \rangle$$

$$= (\nabla \times \langle 0, 0, \varphi \rangle) \cdot \langle n_1, n_2, n_3 \rangle$$

$$\vec{F}_3 = \langle 0, 0, \varphi \rangle$$

$$\iint_S \left((\nabla \times \langle 0, 0, \varphi \rangle) \cdot \langle n_1, n_2, n_3 \rangle \right) dS = \iint_S (\nabla \times \vec{F}_3) \cdot \vec{n} dS$$

$$= \oint_C \vec{F}_3 \cdot d\vec{r}$$

$$= \oint_C \langle 0, 0, \varphi \rangle \cdot d\vec{r}$$

$$= \oint_C \varphi \cdot d\vec{r}$$

$$\iint_S (\vec{n} \times \nabla \varphi) dS = \oint_C \varphi d\vec{r} \quad \checkmark$$