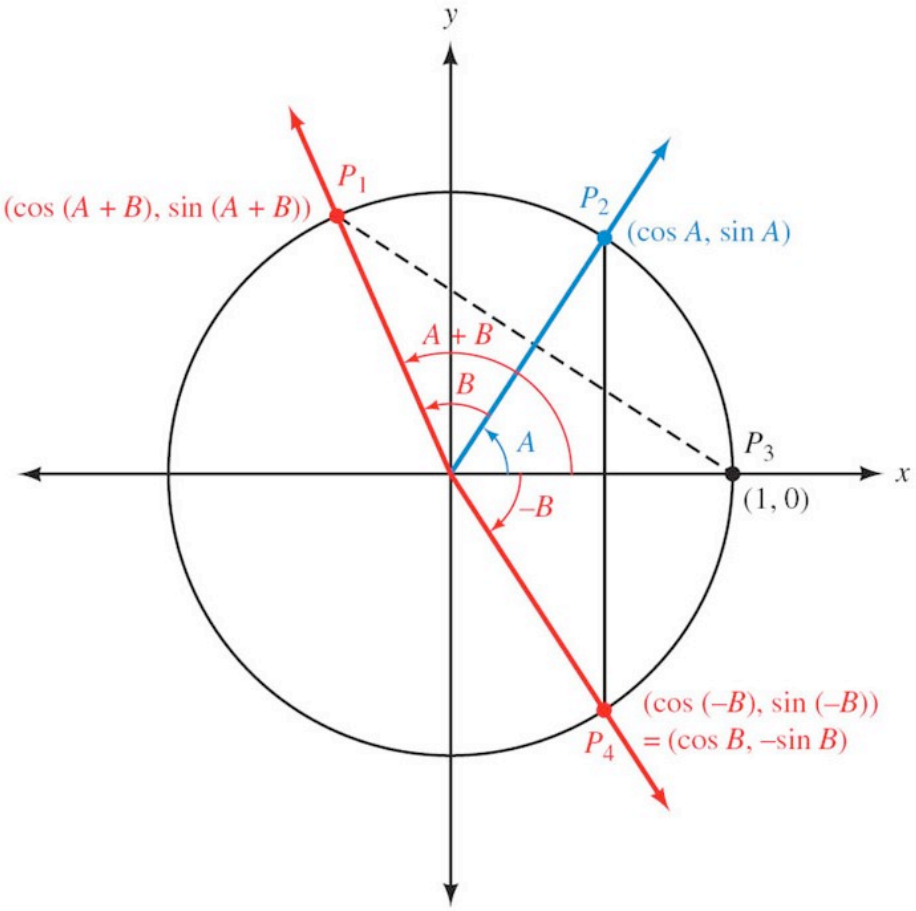
Section 3.2 – Sum and Difference Formulas



$P_1P_3 = P_2P_4$

$\left(P_1P_3\right)^2 = \left(P_2P_4\right)^2$ ***Distance between points***

$$\left[\cos(A+B)-1\right]^2 + \left[\sin(A+B)-0\right]^2 = \left(\cos A - \cos B\right)^2 + \left(\sin A + \sin B\right)^2$$

$$\cos^2(A+B) - 2\cos(A+B) + 1 + \sin^2(A+B) = \left(\cos A - \cos B\right)^2 + \left(\sin A + \sin B\right)^2$$

$$1 - 2\cos(A+B) + 1 = \cos^2 A - 2\cos B\cos A + \cos^2 B + \sin^2 A + 2\sin B\sin A + \sin^2 B$$

$$2 - 2\cos(A+B) = \cos^2 A + \sin^2 A + \cos^2 B + \sin^2 B - 2\cos B\cos A + 2\sin B\sin A$$

$$2 - 2\cos(A+B) = 1 + 1 - 2\cos B\cos A + 2\sin B\sin A$$

$$2 - 2\cos(A+B) = 2 - 2\cos B\cos A + 2\sin B\sin A$$

$$-2\cos(A+B) = -2\cos B\cos A + 2\sin B\sin A$$

$$\cos(A+B) = \cos B\cos A - \sin B\sin A$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

Example

Find the exact value for $\cos 75°$

Solution

$$\cos 75° = \cos(45° + 30°)$$

$$= \cos 45°\cos 30° - \sin 45°\sin 30°$$

$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

Example

Show that $\cos(x + 2\pi) = \cos x$

Solution

$$\cos(x + 2\pi) = \cos x\cos 2\pi - \sin x\sin 2\pi$$

$$= \cos x \cdot (1) - \sin x \cdot (0)$$

$$= \cos x$$

Example

Simplify: $\cos 3x\cos 2x - \sin 3x\sin 2x$

Solution

$$\cos 3x\cos 2x - \sin 3x\sin 2x = \cos(3x + 2x)$$

$$= \cos 5x$$

Example

Show that $\cos(90° - A) = \sin A$

Solution

$$\begin{aligned}\cos(90° - A) &= \cos 90° \cos A + \sin 90° \sin A \\ &= 0 \cdot \cos A + 1 \cdot \sin A \\ &= \sin A \end{aligned}$$

Example

Find the exact value of $\sin \frac{\pi}{12}$

Solution

$$\begin{aligned}\sin \frac{\pi}{12} &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \sin \frac{\pi}{4} \\ &= \frac{\sqrt{3}}{2} \frac{\sqrt{2}}{2} - \frac{1}{2} \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$$

Example

Find the exact value of $\cos 15°$

Solution

$$\begin{aligned}\cos 15° &= \cos\left(45° - 30°\right) \\ &= \cos\left(45°\right)\cos\left(30°\right) + \sin\left(45°\right)\sin\left(30°\right) \\ &= \frac{\sqrt{2}}{2} \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \frac{1}{2} \\ &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \\ &= \frac{\sqrt{6} + \sqrt{2}}{4} \end{aligned}$$

Example

If $\sin A = \frac{3}{5}$ with A in QI, and $\cos B = -\frac{5}{13}$ with B in QIII, find $\sin(A+B)$, $\cos(A+B)$, and $\tan(A+B)$

Solution

$\sin A = \frac{3}{5} \rightarrow A \in QI$ $\qquad$ $\cos B = -\frac{5}{13} \rightarrow B \in QIII$

$\cos A = \underline{\boxed{\frac{4}{5}}}$ $\qquad$ $\sin B = \underline{\boxed{-\frac{12}{13}}}$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$= \frac{3}{5}\left(-\frac{5}{13}\right) + \frac{4}{5}\left(-\frac{12}{13}\right)$$
$$= -\frac{15}{65} - \frac{48}{65}$$
$$= \underline{\boxed{-\frac{63}{65}}}$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$= \frac{4}{5}\left(-\frac{5}{13}\right) - \frac{3}{5}\left(-\frac{12}{13}\right)$$
$$= -\frac{20}{65} + \frac{36}{65}$$
$$= \underline{\boxed{\frac{16}{65}}}$$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$
$$= \underline{\boxed{-\frac{63}{16}}}$$

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}$$

$$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

$$=\frac{\dfrac{\sin A\cos B+\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B-\sin A\sin B}{\cos A\cos B}}$$

$$=\frac{\dfrac{\sin A\cos B}{\cos A\cos B}+\dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}-\dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$=\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{1-\dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}}$$

$$=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\boxed{\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}}$$

$$\boxed{\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}}$$

Example

If $\sin A=\frac{3}{5}$ with A in Q*I*, and $\cos B=-\frac{5}{13}$ with B in Q*III*, find $\tan(A+B)$

Solution

$$\tan A=\frac{3/5}{4/5} \qquad \tan A=\frac{\sin A}{\cos A}$$

$$=\boxed{\frac{3}{4}}$$

$$\tan B=\frac{-12/13}{-5/13} \qquad \tan B=\frac{\sin B}{\cos B}$$

$$=\boxed{\frac{12}{5}}$$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$=\frac{\frac{3}{4}+\frac{12}{5}}{1-\frac{3}{4}\frac{12}{5}}$$

$$= \frac{\frac{15}{20} + \frac{48}{20}}{1 - \frac{36}{20}}$$

$$= \frac{\frac{63}{20}}{-\frac{16}{20}}$$

$$= -\frac{63}{16}$$

Example

Establish the identity: $\dfrac{\cos(x-y)}{\sin x \sin y} = \cot x \cot y + 1$

Solution

$$\frac{\cos(x-y)}{\sin x \sin y} = \frac{\cos x \cos y + \sin x \sin y}{\sin x \sin y}$$

$$= \frac{\cos x \cos y}{\sin x \sin y} + \frac{\sin x \sin y}{\sin x \sin y}$$

$$= \cot x \cot y + 1 \quad \surd$$

Example

Establish the identity: $\cot(x+y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$

Solution

$$\cot(x+y) = \frac{\cos(x+y)}{\sin(x+y)}$$

$$= \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y}$$

$$= \frac{\frac{\cos x \cos y}{\sin x \sin y} - \frac{\sin x \sin y}{\sin x \sin y}}{\frac{\sin x \cos y}{\sin x \sin y} + \frac{\cos x \sin y}{\sin x \sin y}}$$

$$= \frac{\cot x \cot y - 1}{\cot x + \cot y} \quad \surd$$

Example

Establish the identity: $\sec(x-y)=\dfrac{\cos x\cos y-\sin x\sin y}{\cos^2 x-\sin^2 y}$

Solution

$$\sec(x-y)=\frac{1}{\cos(x-y)}\frac{\cos(x+y)}{\cos(x+y)}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{(\cos x\cos y+\sin x\sin y)(\cos x\cos y-\sin x\sin y)}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{\cos^2 x(1-\sin^2 y)-(1-\cos^2 x)\sin^2 y}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{\cos^2 x-\cos^2 x\sin^2 y-\sin^2 y+\cos^2 x\sin^2 y}$$

$$=\boxed{\frac{\cos x\cos y-\sin x\sin y}{\cos^2 x-\sin^2 y}}\quad\surd$$

Exercises *Section* 3.2 – Sum and Difference Formulas

1. Write the expression as a single trigonometric function $\sin 8x\cos x - \cos 8x\sin x$

2. Show that $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

3. If $\sin A = \frac{4}{5}\ (A \in QII)$, and $\cos B = -\frac{5}{13}\ (B \in QIII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

4. If $\sin A = \frac{3}{5}\ (A \in QII)$, and $\cos B = -\frac{12}{13}\ (B \in QIII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

5. If $\sin A = \frac{1}{\sqrt{5}}\ (A \in QI)$, and $\tan B = \frac{3}{4}\ (B \in QI)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

6. If $\sin A = \frac{3}{5}\ (A \in QII)$, and $\cos B = \frac{12}{13}\ (B \in QIV)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

7. If $\sin A = \frac{7}{25}\ (A \in QII)$, and $\cos B = -\frac{8}{17}\ (B \in QIII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

8. If $\cos A = -\frac{4}{5}\ (A \in QII)$, and $\sin B = \frac{24}{25}\ (B \in QII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

9. If $\cos A = \frac{15}{17}\ (A \in QI)$, and $\cos B = -\frac{12}{13}\ (B \in QII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

10. If $\sin A = -\frac{3}{5}\ (A \in QIV)$, and $\sin B = \frac{7}{25}\ (B \in QII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

11. If $\sec A = \sqrt{5}$ with A in QI, and $\sec B = \sqrt{10}$ with B in QI, find $\sec(A+B)$

(**12–30**) Prove the identity

12. $\dfrac{\sin(A-B)}{\cos A\cos B}=\tan A-\tan B$

13. $\sec(A+B)=\dfrac{\cos(A-B)}{\cos^2 A-\sin^2 B}$

14. $\dfrac{\cos 4\alpha}{\sin\alpha}-\dfrac{\sin 4\alpha}{\cos\alpha}=\dfrac{\cos 5\alpha}{\sin\alpha\cos\alpha}$

15. $\dfrac{\cos(x+y)}{\cos(x-y)}=\dfrac{\cot y-\tan x}{\cot y+\tan x}$

16. $\dfrac{\sin(x+y)}{\sin(x-y)}=\dfrac{\cot y+\cot x}{\cot y-\cot x}$

17. $\dfrac{\sin(x-y)}{\sin x\cos y}=1-\cot x\tan y$

18. $\dfrac{\sin(x-y)}{\sin x\sin y}=\cot y-\cot x$

19. $\dfrac{\cos(x+y)}{\cos x\sin y}=\cot y-\tan x$

20. $\dfrac{\sin(x+y)}{\cos(x-y)}=\dfrac{1+\cot x\tan y}{\cot x+\tan y}$

21. $\sin\left(\dfrac{\pi}{4}+x\right)+\sin\left(\dfrac{\pi}{4}-x\right)=\sqrt{2}\cos x$

22. $\cos(A+B)+\cos(A-B)=2\cos A\cos B$

23. $\sin(x-y)-\sin(y-x)=2\sin x\cos y-2\cos x\sin y$

24. $\cos(x-y)+\cos(y-x)=2\cos x\cos y+2\sin x\sin y$

25. $\tan(x+y)\tan(x-y)=\dfrac{\tan^2 x-\tan^2 y}{1-\tan^2 x\tan^2 y}$

26. $\dfrac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)}=\dfrac{1+\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$

27. $\sec(x+y)=\dfrac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\sin^2 y}$

28. $\csc(x-y)=\dfrac{\sin x\cos y+\cos x\sin y}{\sin^2 x-\sin^2 y}$

29. $\tan(x+y)+\tan(x-y)=\dfrac{2\tan x}{\cos^2 y\left(1-\tan^2 x\tan^2 y\right)}$

30. $\dfrac{\cos(x-y)}{\cos(x+y)}=\dfrac{1+\tan x\tan y}{1-\tan x\tan y}$

31. Common household current is called ***alternating current*** because the current alternates direction within the wires. The voltage V in a typical 115-*volt* outlet can be expressed by the function $V(t)=163\sin\omega t$ where ω is the angular speed (in *radians* per *second*) of the rotating generator at the electrical plant, and t is time measured in seconds.

a) It is essential for electric generators to rotate at precisely 60 cycles per second so household appliances and computers will function properly. Determine ω for these electric generators.

b) Determine a value of ϕ so that the graph of $V(t)=163\cos(\omega t-\phi)$ is the same as the graph of $V(t)=163\sin\omega t$