Exercise

Draw $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$, and $\vec{u}+2\vec{v}$ $\qquad \vec{u}=\begin{pmatrix}1\\0\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}1\\2\end{pmatrix}$

Solution

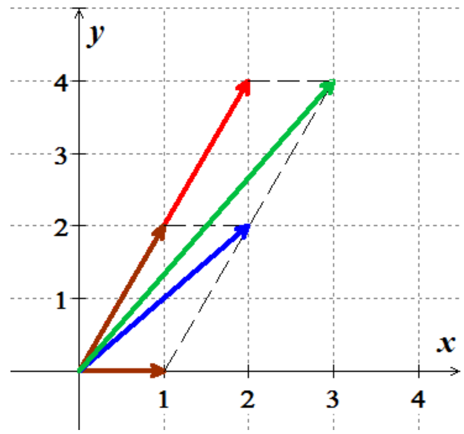
$$\vec{u}+\vec{v}=\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}1\\2\end{pmatrix}$$

$$=\begin{pmatrix}2\\2\end{pmatrix}$$

$$\vec{u}+2\vec{v}=\begin{pmatrix}1\\0\end{pmatrix}+2\begin{pmatrix}1\\2\end{pmatrix}$$

$$=\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}2\\4\end{pmatrix}$$

$$=\begin{pmatrix}3\\4\end{pmatrix}$$



Exercise

Draw $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$, and $\vec{u}+2\vec{v}$ $\qquad \vec{u}=\begin{pmatrix}-1\\0\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}1\\-1\end{pmatrix}$

Solution

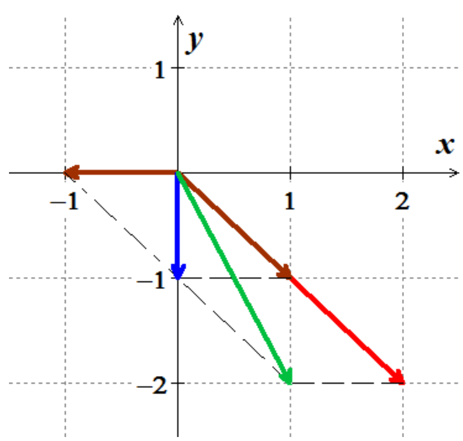
$$\vec{u}+\vec{v}=\begin{pmatrix}-1\\0\end{pmatrix}+\begin{pmatrix}1\\-1\end{pmatrix}$$

$$=\begin{pmatrix}0\\-1\end{pmatrix}$$

$$\vec{u}+2\vec{v}=\begin{pmatrix}-1\\0\end{pmatrix}+2\begin{pmatrix}1\\-1\end{pmatrix}$$

$$=\begin{pmatrix}-1\\0\end{pmatrix}+\begin{pmatrix}2\\-2\end{pmatrix}$$

$$=\begin{pmatrix}1\\-2\end{pmatrix}$$



Solution Section 2.2 – Norm, Dot product, and distance in R^n

Exercise

If $\|\vec{v}\| = 5$ and $\|\vec{w}\| = 3$, what are the smallest and largest possible values of $\|\vec{v} - \vec{w}\|$ and $\vec{v}.\vec{w}$?

Solution

$$\|\vec{v} - \vec{w}\| \le \|\vec{v}\| + \|\vec{w}\| = 5 + 3 = 8$$

$$\|\vec{v} - \vec{w}\| \ge \|\vec{v}\| - \|\vec{w}\| = 5 - 3 = 2$$

$$|\vec{v}.\vec{w}| = \|\vec{v}\|.\|\vec{w}\|.\cos\theta \le \|\vec{v}\|.\|\vec{w}\|$$

$$-\|\vec{v}\|.\|\vec{w}\| \le |\vec{v}.\vec{w}| \le \|\vec{v}\|.\|\vec{w}\|$$

$$-(3)(5) \le |\vec{v}.\vec{w}| \le (3)(5)$$

$$-15 \le |\vec{v}.\vec{w}| \le 15$$

The minimum value occurs when the dot product is a small as possible, $\vec{v}$ and $\vec{w}$ are parallel, but point in opposite directions. Thus, the smallest value is -15.

The maximum value occurs when the dot product is a large as possible, $\vec{v}$ and $\vec{w}$ are parallel and point in same direction. Thus, the largest value is 15.

Exercise

If $\|\vec{v}\| = 7$ and $\|\vec{w}\| = 3$, what are the smallest and largest possible values of $\|\vec{v} + \vec{w}\|$ and $\vec{v}.\vec{w}$?

Solution

$$\|\vec{v} + \vec{w}\| \le \|\vec{v}\| + \|\vec{w}\| = 7 + 3 = 10$$

$$\|\vec{v} + \vec{w}\| \ge \|\vec{v}\| - \|\vec{w}\| = 7 - 3 = 4$$

$$|\vec{v}.\vec{w}| \le \|\vec{v}\|.\|\vec{w}\|$$

$$-\|\vec{v}\|.\|\vec{w}\| \le |\vec{v}.\vec{w}| \le \|\vec{v}\|.\|\vec{w}\|$$

$$-(7)(3) \le |\vec{v}.\vec{w}| \le (7)(3)$$

$$-21 \le |\vec{v}.\vec{w}| \le 21$$

The minimum value occurs when the dot product is a small as possible, $\vec{v}$ and $\vec{w}$ are parallel, but point in opposite directions. Thus, the smallest value is -21. $\vec{v} = (7,\ 0,\ 0,\ \cdots)$ and

$\vec{w} = (-3,\ 0,\ 0,\ \cdots)$

The maximum value occurs when the dot product is a large as possible, $\vec{v}$ and $\vec{w}$ are parallel and point in same direction. Thus, the largest value is 21. $\vec{v} = (7,\ 0,\ 0,\ \cdots)$ and $\vec{w} = (3,\ 0,\ 0,\ \cdots)$

Exercise

Given that $cos(\alpha) = \frac{\vec{v}_1}{\|\vec{v}\|}$ and $sin(\alpha) = \frac{\vec{v}_2}{\|\vec{v}\|}$. Similarly, $cos(\beta) =$ ______ and $sin(\beta) =$ ______. The angle θ is $\beta - \alpha$. Substitute into the trigonometry formula $\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ for $cos(\beta - \alpha)$ to find $\cos\theta = \frac{\vec{v} \bullet \vec{w}}{\|\vec{v}\|.\|\vec{w}\|}$

Solution

$$\cos\beta = \frac{\vec{w}_1}{\|\vec{w}\|}$$

$$\sin\beta = \frac{\vec{w}_2}{\|\vec{w}\|}$$

$$\begin{aligned}
\cos(\beta - \alpha) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
&= \frac{\vec{v}_1}{\|\vec{v}\|}\frac{\vec{w}_1}{\|\vec{w}\|} + \frac{\vec{v}_2}{\|\vec{v}\|}\frac{\vec{w}_2}{\|\vec{w}\|} \\
&= \frac{\vec{v}_1\vec{w}_1 + \vec{v}_2\vec{w}_2}{\|\vec{v}\|.\|\vec{w}\|} \\
&= \frac{\vec{v} \bullet \vec{w}}{\|\vec{v}\|.\|\vec{w}\|}
\end{aligned}$$

Exercise

Can three vectors in the *xy* plane have $\vec{u} \bullet \vec{v} < 0$ and $\vec{v} \bullet \vec{w} < 0$ and $\vec{u} \bullet \vec{w} < 0$?

Solution

Let consider: $\vec{u} = (1,\ 0)$, $\vec{v} = \left(-\frac{1}{2},\ \frac{\sqrt{3}}{2}\right)$, $\vec{w} = \left(-\frac{1}{2},\ -\frac{\sqrt{3}}{2}\right)$

$$\vec{u} \bullet \vec{v} = (1)\left(-\frac{1}{2}\right) + 0 = -\frac{1}{2}$$

$$\begin{aligned}
\vec{v} \bullet \vec{w} &= \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) \\
&= \frac{1}{4} - \frac{3}{4} \\
&= -\frac{1}{2}
\end{aligned}$$

$$\vec{u} \bullet \vec{w} = (1)\left(-\frac{1}{2}\right) + (0)\left(-\frac{\sqrt{3}}{2}\right) = -\frac{1}{2} < 0$$

Yes, it is.

Exercise

Find the norm of $\vec{v}$, a unit vector that has the same direction as $\vec{v}$, and a unit vector that is oppositely directed.

a) $\vec{v} = (4, -3)$

b) $\vec{v} = (1, -1, 2)$

c) $\vec{v} = (-2, 3, 3, -1)$

Solution

a) $\|\vec{v}\| = \sqrt{4^2 + (-3)^2} = \boxed{5}$

Same direction unit vector: $\vec{u}_1 = \dfrac{\vec{v}}{\|\vec{v}\|} = \frac{1}{5}(4,-3) = \boxed{\left(\frac{4}{5}, -\frac{3}{5}\right)}$

Opposite direction unit vector: $\vec{u}_2 = -\dfrac{\vec{v}}{\|\vec{v}\|} = -\frac{1}{5}(4,-3) = \boxed{\left(-\frac{4}{5}, \frac{3}{5}\right)}$

b) $\|\vec{v}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \boxed{\sqrt{6}}$

Same direction unit vector:

$$\vec{u}_1 = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{6}}(1,-1,2) = \boxed{\left(\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)}$$

Opposite direction unit vector:

$$\vec{u}_2 = -\frac{\vec{v}}{\|\vec{v}\|} = -\frac{1}{\sqrt{6}}(1,-1,2) = \boxed{\left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right)}$$

c) $\|\vec{v}\| = \sqrt{(-2)^2 + (3)^2 + (3)^2 + (-1)^2} = \boxed{\sqrt{23}}$

Same direction unit vector:

$$\vec{u}_1 = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{23}}(-2,3,3,-1) = \boxed{\left(\frac{-2}{\sqrt{23}},\ \frac{3}{\sqrt{23}},\ \frac{3}{\sqrt{23}},\ -\frac{1}{\sqrt{23}}\right)}$$

Opposite direction unit vector:

$$\vec{u}_2 = -\frac{\vec{v}}{\|\vec{v}\|} = -\frac{1}{\sqrt{23}}(-2,3,3,-1) = \boxed{\left(\frac{2}{\sqrt{23}},\ -\frac{3}{\sqrt{23}},\ -\frac{3}{\sqrt{23}},\ \frac{1}{\sqrt{23}}\right)}$$

Exercise

Evaluate the given expression with $\boldsymbol{u} = (2, -2, 3)$, $\boldsymbol{v} = (1, -3, 4)$, and $\boldsymbol{w} = (3, 6, -4)$

a) $\|\vec{u}+\vec{v}\|$

b) $\|-2\vec{u}+2\vec{v}\|$

c) $\|3\vec{u}-5\vec{v}+\vec{w}\|$

d) $\|3\vec{v}\|-3\|\vec{v}\|$

e) $\|\vec{u}\|+\|-2\vec{v}\|+\|-3\vec{w}\|$

Solution

a)

$$\begin{aligned}\|\vec{u}+\vec{v}\| &= \|(2,-2,3)+(1,-3,4)\| \\ &= \|(3,\ -5,\ 7)\| \\ &= \sqrt{3^2+(-5)^2+7^2} \\ &= \boxed{\sqrt{83}}\end{aligned}$$

b)

$$\begin{aligned}\|-2\vec{u}+2\vec{v}\| &= \|(-4,\ 4,\ -6)+(2,\ -6,\ 8)\| \\ &= \|(-2,\ -2,\ 2)\| \\ &= \sqrt{(-2)^2+(-2)^2+2^2} \\ &= \sqrt{12} \\ &= \boxed{2\sqrt{3}}\end{aligned}$$

c)

$$\begin{aligned}\|3\vec{u}-5\vec{v}+\vec{w}\| &= \|(6,\ -6,\ 9)-(5,\ -15,\ 20)+(3,\ 6,\ -4)\| \\ &= \|(4,\ 15,\ -15)\| \\ &= \sqrt{(4)^2+(15)^2+(-15)^2} \\ &= \boxed{\sqrt{466}}\end{aligned}$$

d)

$$\begin{aligned}\|3\vec{v}\|-3\|\vec{v}\| &= \|(3,\ -9,\ 12)\|-3\|(1,\ -3,\ 4)\| \\ &= \sqrt{3^2+(-9)^2+12^2}-3\sqrt{1^2+(-3)^2+4^2} \\ &= \sqrt{234}-3\sqrt{26} \\ &= 3\sqrt{26}-3\sqrt{26} \\ &= \boxed{0}\end{aligned}$$

$$\|3v\|-3\|v\| = 3\|v\|-3\|v\| = \boxed{0}$$

e)

$$\begin{aligned}\|\vec{u}\|+\|-2\vec{v}\|+\|-3\vec{w}\| &= \|\vec{u}\|-2\|\vec{v}\|-3\|\vec{w}\| \\ &= \sqrt{2^2+(-2)^2+3^2}-2\sqrt{1^2+(-3)^2+4^2}-3\sqrt{3^2+6^2+(-4)^2} \\ &= \boxed{\sqrt{17}-2\sqrt{26}-3\sqrt{61}}\end{aligned}$$

Exercise

Let $\boldsymbol{v}$ = (1, 1, 2, –3, 1). Find all scalars k such that $\|k\vec{v}\| = 5$

Solution

$$\|k\vec{v}\| = |k|\|\vec{v}\|$$
$$= |k| \ \|(1,\ 1,\ 2,\ -3,\ 1)\|$$
$$= |k| \ \sqrt{1^2 + 1^2 + 2^2 + (-3)^2 + 1^2}$$
$$= |k|\sqrt{49}$$
$$= 7|k|$$

$$7|k| = 5 \quad \to \quad |k| = \tfrac{5}{7} \Rightarrow \boxed{k = \pm\tfrac{5}{7}}$$

Exercise

Find $\vec{u}\bullet\vec{v}$, $\vec{u}\bullet\vec{u}$, *and* $\vec{v}\bullet\vec{v}$

a) $\vec{u}$ = (3, 1, 4), $\vec{v}$ = (2, 2, –4)

b) $\vec{u}$ = (1, 1, 4, 6), $\vec{v}$ = (2, –2, 3, –2)

c) $\vec{u}$ = (2, –1, 1, 0, –2), $\vec{v}$ = (1, 2, 2, 2, 1)

Solution

a) $\vec{u}\bullet\vec{v} = (3,\ 1,\ 4)\bullet(2,\ 2,\ -4)$

$$= 3(2) + 1(2) + 4(-4)$$
$$= -8$$

$$\vec{u}\bullet\vec{u} = \|\vec{u}\|^2$$
$$= 3^2 + 1^2 + 4^2$$
$$= 26$$

$$\vec{v}\bullet\vec{v} = \|\vec{v}\|^2$$
$$= 2^2 + 2^2 + (-4)^2$$
$$= 24$$

b) $\vec{u}\bullet\vec{v} = (1,\ 1,\ 4,\ 6)\bullet(2,\ -2,\ 3,\ -2)$

$$= 1(2) + 1(-2) + 4(3) + 6(-2)$$
$$= 0$$

$$\vec{u}\bullet\vec{u} = \|\vec{u}\|^2$$

$$= 1^2 + 1^2 + 4^2 + 6^2$$

$$\underline{= 54|}$$

$$\vec{v}\bullet\vec{v} = \|\vec{v}\|^2$$

$$= 2^2 + (-2)^2 + 3^2 + (-2)^2$$

$$\underline{= 21|}$$

c) $\vec{u}\bullet\vec{v} = (2,\ -1,\ 1,\ 0,\ -2)\bullet(1,\ 2,\ 2,\ 2,\ 1)$

$$= 2(1) - 1(2) + 1(2) + 0(2) - 2(1)$$

$$\underline{= 0|}$$

$$\vec{u}\bullet\vec{u} = \|\vec{u}\|^2$$

$$= 2^2 + (-1)^2 + 1^2 + 0 + (-2)^2$$

$$\underline{= 10|}$$

$$\vec{v}\bullet\vec{v} = \|\vec{v}\|^2$$

$$= 1^2 + 2^2 + 2^2 + 2^2 + 1^2$$

$$\underline{= 14|}$$

Exercise

Find the Euclidean distance between ***u*** and ***v***, then find the angle between them

a) $\vec{u}$ = (3, 3, 3), $\vec{v}$ = (1, 0, 4)

b) $\vec{u}$ = (1, 2, –3, 0), $\vec{v}$ = (5, 1, 2, –2)

c) $\vec{u}$ = (0, 1, 1, 1, 2), $\vec{v}$ = (2, 1, 0, –1, 3)

Solution

a) $d = \|\vec{u} - \vec{v}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$$= \sqrt{(-2)^2 + (-3)^2 + (1)^2}$$

$$\underline{= \sqrt{14}|}$$

$$\cos\theta = \frac{\vec{u}\bullet\vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$= \frac{3(1) + 3(0) + 3(4)}{\sqrt{3^2 + 3^2 + 3^2}\sqrt{1^2 + 0^2 + 4^2}}$$

$$= \frac{15}{\sqrt{27}\sqrt{17}}$$

$$\theta = \cos^{-1}\left(\frac{15}{\sqrt{27}\sqrt{17}}\right) = \boxed{45.56°}$$

b) $d = \|\vec{u} - \vec{v}\| = \sqrt{(1-5)^2 + (-2-1)^2 + (-3-2)^2 + (-2-0)^2}$

$$= \sqrt{(-4)^2 + (-3)^2 + (-5)^2 + (-2)^2}$$

$$= \boxed{\sqrt{46}}$$

$$\cos\theta = \frac{\vec{u}\bullet\vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$= \frac{1(5) + 2(1) - 3(2) + 0(-2)}{\sqrt{1^2 + 2^2 + (-3)^2 + 0}\ \sqrt{5^2 + 1^2 + 2^2 + (-2)^2}}$$

$$= \frac{1}{\sqrt{14}\sqrt{34}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{14}\sqrt{34}}\right) = \boxed{87.37°}$$

c) $d = \|\vec{u} - \vec{v}\| = \sqrt{(0-2)^2 + (1-1)^2 + (1-0)^2 + (1-(-1))^2 + (2-3)^2}$

$$= \boxed{\sqrt{10}}$$

$$\cos\theta = \frac{\vec{u}\bullet\vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$= \frac{0(2) + 1(1) + 1(0) + 1(-1) + 2(3)}{\sqrt{0 + 1^2 + 1^2 + 1^2 + 2^2}\ \sqrt{2^2 + 1^2 + 0 + (-1)^2 + (3)^2}}$$

$$= \frac{6}{\sqrt{7}\sqrt{15}}$$

$$\theta = \cos^{-1}\left(\frac{6}{\sqrt{7}\sqrt{15}}\right) = \boxed{54.16°}$$

Exercise

Find a unit vector that has the same direction as the given vector

a) $(-4, -3)$ *b)* $\left(-3,\ 2,\ \sqrt{3}\right)$ *c)* $(1, 2, 3, 4, 5)$

Solution

a) $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{(-4,\ -3)}{\sqrt{(-4)^2 + (-3)^2}}$

$= \dfrac{(-4,\ -3)}{\sqrt{25}}$

$\underline{= \left(-\frac{4}{5},\ -\frac{3}{5}\right)}$

b) $\vec{u} = \dfrac{1}{\sqrt{(-3)^2 + (2)^2 + (\sqrt{3})^2}}\left(-3,\ 2,\ \sqrt{3}\right)$

$= \dfrac{1}{\sqrt{17}}\left(-3,\ 2,\ \sqrt{3}\right)$

$\underline{= \left(-\frac{3}{\sqrt{17}},\ \frac{2}{\sqrt{17}},\ \frac{\sqrt{3}}{\sqrt{17}}\right)}$

c) $\vec{u} = \dfrac{1}{\sqrt{1^2 + 2^2 + 3^2 + 4^2 + 5^2}}(1,\ 2,\ 3,\ 4,\ 5)$

$= \dfrac{1}{\sqrt{55}}(1,\ 2,\ 3,\ 4,\ 5)$

$\underline{= \left(\frac{1}{\sqrt{55}},\ \frac{2}{\sqrt{55}},\ \frac{3}{\sqrt{55}},\ \frac{4}{\sqrt{55}},\ \frac{5}{\sqrt{55}}\right)}$

Exercise

Find a unit vector that is oppositely to the given vector

a) $(-12, -5)$

b) $(3, -3, 3)$

c) $\left(-3, 1, \sqrt{6}, 3\right)$

Solution

a) $\vec{u} = -\dfrac{1}{\sqrt{(-12)^2 + (-5)^2}}(-12,\ -5)$

$= -\dfrac{1}{\sqrt{169}}(-12,\ -5)$

$\underline{= \left(\frac{12}{13},\ \frac{5}{13}\right)}$

b) $\vec{u} = -\dfrac{1}{\sqrt{(3)^2 + (-3)^2 + (3)^2}}(3,\ -3,\ 3)$

$$= -\frac{1}{\sqrt{27}}(3,\ -3,\ 3)$$

$$= -\frac{1}{3\sqrt{3}}(3,\ -3,\ 3)$$

$$\boxed{= \left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)}$$

c) $\vec{u} = -\frac{1}{\sqrt{(-3)^2 + 1^2 + \left(\sqrt{6}\right)^2 + 3^2}}\left(-3,\ 1,\ \sqrt{6},\ 3\right)$

$$= -\frac{1}{\sqrt{25}}\left(-3,\ 1,\ \sqrt{6},\ 3\right)$$

$$\boxed{= \left(\frac{3}{5},\ -\frac{1}{5},\ -\frac{\sqrt{6}}{5},\ -\frac{3}{5}\right)}$$

Exercise

Verify that the Cauchy-Schwarz inequality holds

a) $\vec{u} = (-3,\ 1,\ 0)$, $\vec{v} = (2, -1, 3)$

b) $\vec{u} = (0, 2, 2, 1)$, $\vec{v} = (1, 1, 1, 1)$

c) $\vec{u} = (1, 3, 5, 2, 0, 1)$, $\vec{v} = (0, 2, 4, 1, 3, 5)$

Solution

a) $\left|\vec{u} \bullet \vec{v}\right| = \left|(-3, 1, 0) \bullet (2, -1, 3)\right|$

$$= \left|-3(2) + 1(-1) + 0(3)\right|$$

$$= |-7|$$

$$\boxed{= 7}$$

$$\|\vec{u}\|\|\vec{v}\| = \sqrt{(-3)^2 + 1^2 + 0}\ \sqrt{(2)^2 + (-1)^2 + 3^2}$$

$$= \sqrt{10}\sqrt{14}$$

$$\boxed{\approx 11.83}$$

$\left|\vec{u} \bullet \vec{v}\right| \le \|\vec{u}\|\|\vec{v}\|$ Cauchy-Schwarz inequality holds

b) $\left|\vec{u} \bullet \vec{v}\right| = \left|(0,\ 2,\ 2,\ 1) \bullet (1,\ 1,\ 1,\ 1)\right|$

$$= |0 + 2 + 2 + 1|$$

$$\boxed{= 5}$$

$$\|\vec{u}\|\|\vec{v}\| = \sqrt{0+2^2+2^2+1^2}\ \sqrt{1^2+1^2+1^2+1^2}$$

$$= \sqrt{9}\sqrt{4}$$

$$= 6$$

$|\vec{u}\bullet\vec{v}| \le \|\vec{u}\|\|\vec{v}\|$ Cauchy-Schwarz inequality holds

c) $|\vec{u}\bullet\vec{v}| = |(1,\ 3,\ 5,\ 2,\ 0,\ 1)\bullet(0,\ 2,\ 4,\ 1,\ 3,\ 5)|$

$$= |0+6+20+2+0+5|$$

$$= 23$$

$$\|\vec{u}\|\|\vec{v}\| = \sqrt{1^2+3^2+5^2+2^2+0+1^2}\ \sqrt{0+2^2+4^2+1^2+3^2+5^2}$$

$$= \sqrt{40}\sqrt{55}$$

$$\approx 46$$

$|\vec{u}\bullet\vec{v}| \le \|\vec{u}\|\|\vec{v}\|$ Cauchy-Schwarz inequality holds

Exercise

Find $\boldsymbol{u}\bullet\boldsymbol{v}$ and then the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$ $\quad \boldsymbol{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$

Solution

$$\boldsymbol{u}\bullet\boldsymbol{v} = 3+0-2-1 = 0$$

$$\theta = \cos^{-1}\frac{0}{\sqrt{15}\sqrt{3}} = \cos^{-1}(0) = 90°$$

Exercise

Find the norm: $\|\boldsymbol{u}\|+\|\boldsymbol{v}\|$, $\|\boldsymbol{u}+\boldsymbol{v}\|$ for $\boldsymbol{u} = (3,\ -1,\ -2,\ 1,\ 4)$ $\quad \boldsymbol{v} = (1,\ 1,\ 1,\ 1,\ 1)$

Solution

$$\|\boldsymbol{u}\|+\|\boldsymbol{v}\| = \sqrt{3^2+(-1)^2+(-2)^2+1^2+4^2} + \sqrt{1+1+1+1+1}$$

$$= \sqrt{31}+\sqrt{5}$$

$$\|\boldsymbol{u}+\boldsymbol{v}\| = \|(4,\ 0,\ -1,\ 2,\ 5)\| = \sqrt{16+0+1+4+25}$$

$$= \sqrt{46}$$

Exercise

Find all numbers r such that: $\left\| r(1,\ 0,\ -3,\ -1,\ 4,\ 1) \right\| = 1$

Solution

$$r\sqrt{1+9+1+16+1} = \pm 1$$

$$r\sqrt{28} = \pm 1$$

$$r = \pm \frac{1}{2\sqrt{7}} = \pm \frac{\sqrt{7}}{14}$$

Exercise

Find the distance between $P_1(7,-5,1)$ and $P_2(-7,-2,-1)$

Solution

$$\begin{aligned} \left\| P_1P_2 \right\| &= \sqrt{(-7-7)^2 + (-2+5)^2 + (-1-1)^2} \\ &= \sqrt{14^2 + 3^2 + (-2)^2} \\ &= \sqrt{196+9+4} \\ &= \sqrt{209} \end{aligned}$$

Exercise

Given $\boldsymbol{u} = (1, -5, 4)$, $\boldsymbol{v} = (3, 3, 3)$

a) Find $\vec{u} \bullet \vec{v}$

b) Find the cosine of the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$.

Solution

a) $\vec{u} \bullet \vec{v} = 3 - 15 + 12 = 0$

b) $\cos\theta = \dfrac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\ \|\boldsymbol{v}\|} = 0$

Exercise

Let $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ *and* $\vec{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Find $\left\| \frac{1}{\|2\vec{u} + \vec{v}\|} (2\vec{u} + \vec{v}) \right\|$

Solution

Since, the unit vector equals to a vector $(2\vec{u} + \vec{v})$ divided by its magnitude.

Therefore, $\left\|\frac{1}{\|2\vec{u}+\vec{v}\|}\left(2\vec{u}+\vec{v}\right)\right\| = \frac{1}{\|2\vec{u}+\vec{v}\|}\|2\vec{u}+\vec{v}\| \underline{=1}$

Exercise

Let $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ *and* $\vec{v} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$. Find $\left\|\frac{1}{\|\vec{u}-\vec{v}\|}\left(\vec{u}-\vec{v}\right)\right\|$

Solution

$$\left\|\frac{1}{\|\vec{u}-\vec{v}\|}\left(\vec{u}-\vec{v}\right)\right\| = \frac{1}{\|\vec{u}-\vec{v}\|}\|\vec{u}-\vec{v}\| \underline{=1}$$

Exercise

Let $\vec{u} = \begin{pmatrix} 18 \\ 6 \end{pmatrix}$ *and* $\vec{v} = \begin{pmatrix} -11 \\ 12 \end{pmatrix}$. Find $\left\|\frac{1}{\|5\vec{u}+3\vec{v}\|}\left(5\vec{u}+3\vec{v}\right)\right\|$

Solution

$$\left\|\frac{1}{\|5\vec{u}+3\vec{v}\|}\left(5\vec{u}+3\vec{v}\right)\right\| = \frac{1}{\|5\vec{u}+3\vec{v}\|}\|5\vec{u}+3\vec{v}\| \underline{=1}$$

Exercise

Let $\vec{u} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$. Calculate the following:

a) $\vec{u}+\vec{v}$ *b)* $2\vec{u}+3\vec{v}$ *c)* $\vec{v}+\left(2\vec{u}-3\vec{v}\right)$ *d)* $\|\vec{u}\|$ *e)* $\|\vec{v}\|$ *f)* *unit vector of* $\vec{v}$

Solution

a) $\vec{u}+\vec{v} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

b) $2\vec{u}+3\vec{v} = 3\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + 3\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} 9 \\ 3 \\ -3 \end{pmatrix} + \begin{pmatrix} -6 \\ 3 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}$$

c) $\vec{v} + \left(2\vec{u} - 3\vec{v}\right) = \vec{v} + 2\vec{u} - 3\vec{v}$

$$= 2\vec{u} - 2\vec{v}$$

$$= 2\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} - 2\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 2 \\ -2 \end{pmatrix} - \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 10 \\ 0 \\ -6 \end{pmatrix}$$

d) $\|\vec{u}\| = \sqrt{3^2 + 1^2 + (-1)^2}$

$$= \sqrt{9+1+1}$$

$$\boxed{= \sqrt{11}}$$

e) $\|\vec{v}\| = \sqrt{(-2)^2 + 1^2 + 2^2}$

$$= \sqrt{4+1+4}$$

$$\boxed{= 3}$$

f) $\textit{unit vector of } \vec{v} = \dfrac{\vec{v}}{\|\vec{v}\|}$

$$= \frac{(-2,\ 1,\ 2)}{3}$$

$$\boxed{= \left(-\frac{2}{3},\ \frac{1}{3},\ \frac{2}{3}\right)}$$

Exercise

Let $\vec{u}=\begin{pmatrix}2\\-1\\0\\1\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}1\\4\\3\\1\end{pmatrix}$. Calculate the following:

a) $\vec{u}-\vec{v}$ *b)* $3\vec{u}-2\vec{v}$ *c)* $2(\vec{u}-\vec{v})+3\vec{u}$ *d)* $\|\vec{u}\|$ *e)* *unit vector of* $\vec{v}$

Solution

a) $\vec{u}-\vec{v}=\begin{pmatrix}2\\-1\\0\\1\end{pmatrix}-\begin{pmatrix}1\\4\\3\\1\end{pmatrix}$

$=\begin{pmatrix}1\\-5\\-3\\0\end{pmatrix}$

b) $3\vec{u}-2\vec{v}=3\begin{pmatrix}2\\-1\\0\\1\end{pmatrix}-2\begin{pmatrix}1\\4\\3\\1\end{pmatrix}$

$=\begin{pmatrix}6\\-3\\0\\3\end{pmatrix}-\begin{pmatrix}2\\8\\6\\2\end{pmatrix}$

$=\begin{pmatrix}4\\-11\\-6\\1\end{pmatrix}$

c) $2(\vec{u}-\vec{v})+3\vec{u}=2\vec{u}-2\vec{v}+3\vec{u}$

$=5\vec{u}-2\vec{v}$

$=5\begin{pmatrix}2\\-1\\0\\1\end{pmatrix}-2\begin{pmatrix}1\\4\\3\\1\end{pmatrix}$

$$=\begin{pmatrix}10\\-5\\0\\5\end{pmatrix}-\begin{pmatrix}2\\8\\6\\2\end{pmatrix}$$

$$=\begin{pmatrix}8\\-13\\-6\\3\end{pmatrix}$$

d) $\|\vec{u}\|=\sqrt{2^2+(-1)^2+0+1^2}$

$$=\sqrt{4+1+1}$$

$$=\sqrt{6}$$

e) $unit\ vector\ of\ \vec{v}=\dfrac{\vec{v}}{\|\vec{v}\|}$

$$=\frac{(1,\ 4,\ 3,\ 1)}{\sqrt{1+16+9+1}}$$

$$=\frac{(1,\ 4,\ 3,\ 1)}{\sqrt{27}}$$

$$=\left(\frac{1}{3\sqrt{3}},\ \frac{4}{3\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{3\sqrt{3}}\right)$$

Exercise

Let $\vec{u}=\begin{pmatrix}2\\1\\3\\0\\-1\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}1\\0\\2\\1\\-1\end{pmatrix}$. Calculate the following:

a) $\vec{v}-\vec{u}$ *b)* $\vec{u}+3\vec{v}$ *c)* $3(\vec{u}+\vec{v})-3\vec{u}$ *d)* $\|\vec{v}\|$ *e)* *unit vector of* $\vec{v}$

Solution

a) $\vec{v}-\vec{u}=\begin{pmatrix}1\\0\\2\\1\\-1\end{pmatrix}-\begin{pmatrix}2\\1\\3\\0\\-1\end{pmatrix}$

$$=\begin{pmatrix}-1\\-1\\-1\\1\\0\end{pmatrix}$$

b) $\vec{u}+3\vec{v}=\begin{pmatrix}2\\1\\3\\0\\-1\end{pmatrix}+3\begin{pmatrix}1\\0\\2\\1\\-1\end{pmatrix}$

$$=\begin{pmatrix}2\\1\\3\\0\\-1\end{pmatrix}+\begin{pmatrix}3\\0\\6\\3\\-3\end{pmatrix}$$

$$\boxed{=(5,\ 1,\ 9,\ 3,\ -4)}$$

c) $3(\vec{u}+\vec{v})-3\vec{u}=3\vec{u}+3\vec{v}-3\vec{u}$

$$=3\vec{v}$$

$$=3(1,\ 0,\ 2,\ 1,\ -1)$$

$$\boxed{=(3,\ 0,\ 6,\ 3,\ -3)}$$

d) $\|\vec{v}\|=\sqrt{1^2+0+2^2+1^2+(-1)^2}$

$$=\sqrt{1+4+1+1}$$

$$\boxed{=\sqrt{7}}$$

e) *unit vector of* $\vec{v}=\dfrac{\vec{v}}{\|\vec{v}\|}$

$$=\frac{(1,\ 0,\ 2,\ 1,\ -1)}{\sqrt{7}}$$

$$\boxed{=\left(\frac{1}{\sqrt{7}},\ 0,\ \frac{2}{\sqrt{7}},\ \frac{1}{\sqrt{7}},\ -\frac{1}{\sqrt{7}}\right)}$$

Exercise

Let $\vec{u}=\begin{pmatrix}2\\-1\\1\end{pmatrix}$, $\vec{v}=\begin{pmatrix}1\\2\\-2\end{pmatrix}$ and $\vec{w}=\begin{pmatrix}3\\2\\1\end{pmatrix}$. Calculate the following:

a) $\vec{u}\cdot\vec{v}$ *b)* $\vec{u}\cdot(\vec{v}+\vec{w})$ *c)* $(\vec{u}+2\vec{v})\cdot\vec{w}$ *d)* $\left\|(\vec{w}\cdot\vec{v})\vec{u}\right\|$

Solution

a) $\vec{u}\cdot\vec{v}=(2,\ -1,\ 1)\cdot(1,\ 2,\ -2)$

$=2-2-2$

$\underline{=-2}$

b) $\vec{u}\cdot(\vec{v}+\vec{w})=(2,\ -1,\ 1)\cdot\left[(1,\ 2,\ -2)+(3,\ 2,\ 1)\right]$

$=(2,\ -1,\ 1)\cdot(4,\ 4,\ -1)$

$=8-4-1$

$\underline{=3}$

c) $(\vec{u}+2\vec{v})\cdot\vec{w}=\left[(2,\ -1,\ 1)+2(1,\ 2,\ -2)\right]\cdot(3,\ 2,\ 1)$

$=(4,\ 3,\ -3)\cdot(3,\ 2,\ 1)$

$=12+6-3$

$\underline{=15}$

d) $\left\|(\vec{w}\cdot\vec{v})\vec{u}\right\|=\left|\vec{w}\cdot\vec{v}\right|\ \left\|\vec{u}\right\|$

$=\left|(3,\ 2,\ 1)\cdot(1,\ 2,\ -2)\right|\sqrt{2^2+(-1)^2+1^2}$

$=\left|3+4-2\right|\sqrt{4+1+1}$

$\underline{=5\sqrt{6}}$

Exercise

Let $\vec{u}=\begin{pmatrix}1\\3\\2\\1\end{pmatrix}$, $\vec{v}=\begin{pmatrix}-2\\5\\2\\-6\end{pmatrix}$ and $\vec{w}=\begin{pmatrix}4\\-1\\0\\-2\end{pmatrix}$. Calculate the following:

a) $\vec{u}\cdot\vec{v}$ *b)* $\vec{u}\cdot(\vec{v}+\vec{w})$ *c)* $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v})$ *d)* $\left\|(\vec{w}\cdot\vec{v})\vec{u}\right\|$

Solution

a) $\vec{u}\cdot\vec{v}=(1,\ 3,\ 2,\ 1)\cdot(-2,\ 5,\ 2,\ -6)$

$=-2+15+4-6$

$\underline{=11}$

b) $\vec{u}\cdot(\vec{v}+\vec{w}) = (1,\ 3,\ 2,\ 1)\cdot\left[(-2,\ 5,\ 2,\ -6)+(4,\ -1,\ 0,\ -2)\right]$

$$= (1,\ 3,\ 2,\ 1)\cdot(2,\ 4,\ 2,\ -8)$$

$$= 2+12+4-8$$

$$\underline{=10}$$

c) $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = \left[(1,\ 3,\ 2,\ 1)+(-2,\ 5,\ 2,\ -6)\right]\cdot\left[(1,\ 3,\ 2,\ 1)-(-2,\ 5,\ 2,\ -6)\right]$

$$= (-1,\ 8,\ 4,\ -5)\cdot(3,\ -2,\ 0,\ 7)$$

$$= -3-16+0-35$$

$$\underline{=-54}$$

Or $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u}-\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u}-\vec{v}\cdot\vec{v}$

$$= \vec{u}\cdot\vec{u}-\vec{v}\cdot\vec{v}$$

$$= (1+9+4+1)-(4+25+4+36)$$

$$= 15-69$$

$$\underline{=-54}$$

d) $\left\|(\vec{w}\cdot\vec{v})\vec{u}\right\| = |\vec{w}\cdot\vec{v}|\ \|\vec{u}\|$

$$= \left|(4,\ -1,\ 0,\ -2)\cdot(-2,\ 5,\ 2,\ -6)\right|\sqrt{1+9+4+1}$$

$$= |-8-5+12|\sqrt{15}$$

$$\underline{=\sqrt{15}}$$

Exercise

Let $\vec{u}=\begin{pmatrix}1\\0\\-2\\1\end{pmatrix}$, $\vec{v}=\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$ and $\vec{w}=\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$. Calculate the following:

a) $\vec{u}\cdot(\vec{v}+\vec{w})$ *b)* $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v})$ *c)* $(\vec{u}\cdot\vec{w})\vec{v}+(\vec{v}\cdot\vec{w})\vec{u}$ *d)* $(\vec{u}+2\vec{v})\cdot(\vec{u}-\vec{v})$

Solution

a) $\vec{u}\cdot(\vec{v}+\vec{w}) = (1,\ 0,\ -2,\ 1)\cdot\left[(0,\ 1,\ 1,\ 0)+(1,\ -1,\ -1,\ 1)\right]$

$$= (1,\ 0,\ -2,\ 1)\cdot(1,\ 0,\ 0,\ 1)$$

$$\underline{=2}$$

b) $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = \left[(1,\ 0,\ -2,\ 1)+(0,\ 1,\ 1,\ 0)\right]\cdot\left[(1,\ 0,\ -2,\ 1)-(0,\ 1,\ 1,\ 0)\right]$

$$= (1,\ 1,\ -1,\ 1)\cdot(1,\ -1,\ -3,\ 1)$$

$=1-1+3+1$

$\underline{= 4}\rfloor$

c) $(\vec{u}\cdot\vec{w})\vec{v}+(\vec{v}\cdot\vec{w})\vec{u}=\left[(1,\ 0,\ -2,\ 1)\cdot(1,\ -1,\ -1,\ 1)\right](0,\ 1,\ 1,\ 0)$

$\qquad +\left[(0,\ 1,\ 1,\ 0)\cdot(1,\ -1,\ -1,\ 1)\right](1,\ 0,\ -2,\ 1)$

$=4(0,\ 1,\ 1,\ 0)-2(1,\ 0,\ -2,\ 1)$

$\underline{=(-2,\ 4,\ 8,\ 2)}\rfloor$

d) $(\vec{u}+2\vec{v})\cdot(\vec{u}-\vec{v})=\left[(1,\ 0,\ -2,\ 1)+2(0,\ 1,\ 1,\ 0)\right]\cdot\left[(1,\ 0,\ -2,\ 1)-(0,\ 1,\ 1,\ 0)\right]$

$=(1,\ 2,\ 0,\ 1)\cdot(1,\ -1,\ -3,\ 1)$

$=1-2+1$

$\underline{=0}\rfloor$

Exercise

Suppose $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$ such that $\vec{u}\cdot\vec{v}=2$, $\vec{u}\cdot\vec{w}=-3$, and $\vec{v}\cdot\vec{w}=5$. If possible, calculate the following values:

a) $\vec{u}\cdot(\vec{v}+\vec{w})$	*d)* $\vec{w}\cdot(2\vec{v}-4\vec{u})$	*g)* $\vec{w}\cdot((\vec{u}\cdot\vec{w})\vec{u})$
b) $(\vec{u}+\vec{v})\cdot\vec{w}$	*e)* $(\vec{u}+\vec{v})\cdot(\vec{v}+\vec{w})$	*h)* $\vec{u}\cdot((\vec{u}\cdot\vec{v})\vec{v}+(\vec{u}\cdot\vec{w})\vec{w})$
c) $\vec{u}\cdot(2\vec{v}-\vec{w})$	*f)* $\vec{w}\cdot(5\vec{v}+\pi\vec{u})$	

Solution

a) $\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$

$=2+(-3)$

$\underline{=-1}\rfloor$

b) $(\vec{u}+\vec{v})\cdot\vec{w}=\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}$

$=-3+5$

$\underline{=2}\rfloor$

c) $\vec{u}\cdot(2\vec{v}-\vec{w})=\vec{u}\cdot(2\vec{v})-\vec{u}\cdot\vec{w}$

$=2\vec{u}\cdot\vec{v}-\vec{u}\cdot\vec{w}$

$=2(2)-(-3)$

$\underline{=7}\rfloor$

d) $\vec{w}\cdot(2\vec{v}-4\vec{u})=\vec{w}\cdot(2\vec{v})-\vec{w}\cdot(4\vec{u})$

$=2\vec{w}\cdot\vec{v}-4\vec{w}\cdot\vec{u}$

$=2(\vec{v}\cdot\vec{w})-4(\vec{u}\cdot\vec{w})$

$=2(5)-4(-3)$

$$= 22$$

e) $(\vec{u}+\vec{v})\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{v}+\vec{v}\cdot\vec{w}$

$$= 2+(-3)+\vec{v}\cdot\vec{v}+5$$

$$= 4+\vec{v}^2$$

f) $\vec{w}\cdot(5\vec{v}+\pi\vec{u}) = \vec{w}\cdot(5\vec{v})+\vec{w}\cdot(\pi\vec{u})$

$$= 5(\vec{w}\cdot\vec{v})+\pi(\vec{w}\cdot\vec{u})$$

$$= 5(\vec{v}\cdot\vec{w})+\pi(\vec{u}\cdot\vec{w})$$

$$= 5(5)+\pi(-3)$$

$$= 25-3\pi$$

g) $\vec{w}\cdot((\vec{u}\cdot\vec{w})\vec{u}) = \vec{w}\cdot((-3)\vec{u})$

$$= -3(\vec{w}\cdot\vec{u})$$

$$= -3(\vec{u}\cdot\vec{w})$$

$$= -3(-3)$$

$$= 9$$

h) $\vec{u}\cdot((\vec{u}\cdot\vec{v})\vec{v}+(\vec{u}\cdot\vec{w})\vec{w}) = \vec{u}\cdot(2\vec{v}+5\vec{w})$

$$= \vec{u}\cdot(2\vec{v})+\vec{u}\cdot(5\vec{w})$$

$$= 2\vec{u}\cdot\vec{v}+5\vec{u}\cdot\vec{w}$$

$$= 2(2)+5(-3)$$

$$= -11$$

Exercise

You are in an airplane flying from Chicago to Boston for a job interview. The compass in the cockpit of the plane shows that your plane is pointed due East, and the airspeed indicator on the plane shows that the plane is traveling through the air at 400 *mph*. there is a crosswind that affects your plane however, and the crosswind is blowing due South at 40 *mph*.

Given the crosswind you wonder; relative to the ground, in what direction are you really flying and how fast are you really traveling?

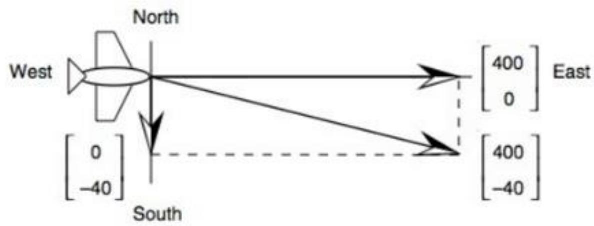
Solution

Let the air velocity of the plane be: $\vec{a} = \begin{pmatrix} 400 \\ 0 \end{pmatrix}$

The wind velocity be: $\vec{w} = \begin{pmatrix} 0 \\ -40 \end{pmatrix}$

The ground the velocity is:

$$\vec{g} = \vec{a} + \vec{w}$$
$$= \begin{pmatrix} 400 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -40 \end{pmatrix}$$
$$= \begin{pmatrix} 400 \\ -40 \end{pmatrix}$$



The magnitude: $\sqrt{400^2 + (-40)^2} \ \underline{= 402 \ \ mph}|$

The direction: $\theta = \tan^{-1}\frac{-40}{400} \ \underline{\approx 5.71^\circ}|$

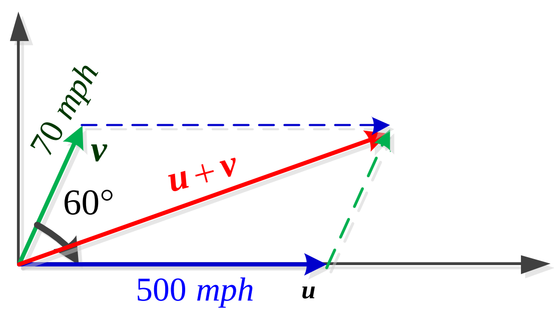
Exercise

A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What speed and direction should the jetliner have in order for the resultant vector to be 500 *mph* due east?

Solution

$\boldsymbol{u} = (x,\ y) =$ the velocity of the airplane

$\boldsymbol{v}$ = the velocity of the tailwind



$$\vec{v} = (70\cos 60^\circ,\ 70\sin 60^\circ)$$
$$= \left(35,\ 35\sqrt{3}\right)$$
$$\boldsymbol{u} + \boldsymbol{v} = (500,\ 0)$$
$$(x,\ y) + \left(35,\ 35\sqrt{3}\right) = (500,\ 0)$$
$$(x,\ y) = (500,\ 0) - \left(35,\ 35\sqrt{3}\right)$$
$$= \left(765,\ -35\sqrt{3}\right)$$
$$\underline{\boldsymbol{u} = \left(765,\ -35\sqrt{3}\right)}|$$
$$|\boldsymbol{u}| = \sqrt{465^2 + \left(-35\sqrt{3}\right)^2}$$
$$\underline{\approx 468.9 \ mph}|$$

$$\underline{|\theta} = \tan^{-1}\frac{-35\sqrt{3}}{465} \ \underline{\approx -7.4^\circ}|$$

∴ The direction is 7.4° south of east

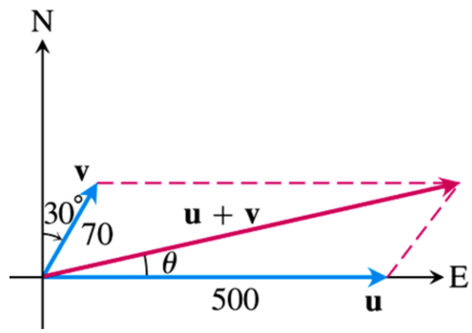
Example

A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?

Solution

$\boldsymbol{u}$ = the velocity of the airplane
$\boldsymbol{v}$ = the velocity of the tailwind

Given: $|u| = 500 \quad |v| = 70$



$$\boldsymbol{u} = \langle 500,\ 0 \rangle$$

$$\boldsymbol{v} = \langle 70\cos 60°,\ 70\sin 60° \rangle$$
$$= \langle 35,\ 35\sqrt{3} \rangle$$

$$\boldsymbol{u} + \boldsymbol{v} = \langle 535,\ 35\sqrt{3} \rangle = 535\boldsymbol{i} + 35\sqrt{3}\,\boldsymbol{j}$$

$$|\boldsymbol{u} + \boldsymbol{v}| = \sqrt{535^2 + \left(35\sqrt{3}\right)^2}$$
$$\approx 538.4 \text{ mph}$$

$$\theta = \tan^{-1}\frac{35\sqrt{3}}{535} \approx 6.5°$$

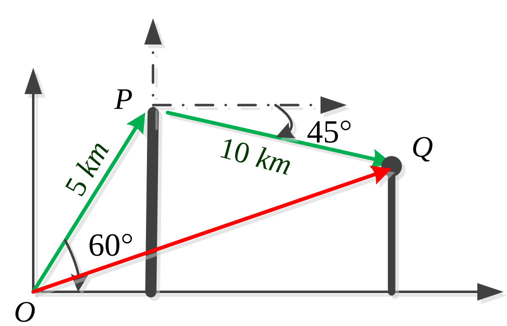
The ground speed of the airplane is about 538.4 *mph* 538.4 *mph*, and its direction is about 6.5° north of east.

Exercise

A bird flies from its nest 5 *km* in the direction 60° north east, where it stops to rest on a tree. It then flies 10 *km* in the direction due southeast and lands atop a telephone pole. Place an *xy*-coordinate system so that the origin is the bird's nest, the *x*-axis points east, and the *y*-axis points north.

a) At what point is the tree located?
b) At what point is the telephone pole?

Solution



a) $\overrightarrow{OP} = (5\cos 60°)\,\boldsymbol{i} + (5\sin 60°)\,\boldsymbol{j}$

$$= \frac{5}{2}\boldsymbol{i} + \frac{5\sqrt{3}}{2}\boldsymbol{j}$$

The tree is located at the point

$$P = \left(\frac{5}{2}, \ \frac{5\sqrt{3}}{2} \right)$$

b) $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$

$$= \frac{5}{2}\boldsymbol{i} + \frac{5\sqrt{3}}{2}\boldsymbol{j} + (10\cos 315°)\boldsymbol{i} + (10\sin 315°)\boldsymbol{j}$$

$$= \frac{5}{2}\boldsymbol{i} + \frac{5\sqrt{3}}{2}\boldsymbol{j} + \left(10\frac{\sqrt{2}}{2}\right)\boldsymbol{i} + \left(10\left(-\frac{\sqrt{2}}{2}\right)\right)\boldsymbol{j}$$

$$= \left(\frac{5}{2} + 5\sqrt{2}\right)\boldsymbol{i} + \left(\frac{5\sqrt{3}}{2} - \frac{10\sqrt{2}}{2}\right)\boldsymbol{j}$$

$$= \left(\frac{5+10\sqrt{2}}{2}\right)\boldsymbol{i} + \left(\frac{5\sqrt{3}-10\sqrt{2}}{2}\right)\boldsymbol{j}$$

The pole is located at the point $Q = \left(\frac{5+10\sqrt{2}}{2}, \ \frac{5\sqrt{3}-10\sqrt{2}}{2} \right)$

Exercise

Prove $\vec{u} \bullet \vec{u} = \|\vec{u}\|^2 \ \geq 0$

Solution

Let $\vec{u} = (u_1, \ u_2, \ \ldots, \ u_n)$

$$\vec{u} \bullet \vec{u} = (u_1, \ u_2, \ \ldots, \ u_n) \bullet (u_1, \ u_2, \ \ldots, \ u_n)$$

$$= u_1 u_1 + u_2 u_2 + \ldots + u_n u_n$$

$$= u_1^2 + u_2^2 + \ldots + u_n^2$$

$$\|\vec{u}\|^2 = \left(\sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}\right)^2$$

$$= u_1^2 + u_2^2 + \ldots + u_n^2$$

Thus, $\vec{u} \bullet \vec{u} = \|\vec{u}\|^2$

Each $u_i \in \mathbb{R}$, then $u_i^2 \geq 0$ for each $1 \leq i \leq n$, thus $u_1^2 + u_2^2 + \ldots + u_n^2 \geq 0$.

Hence, $\|\vec{u}\|^2 \geq 0$

Exercise

Prove, for any vectors and $\vec{v}$ in $\mathbb{R}^2$ and any scalars c and d,

$(c\vec{u}+d\vec{v})\bullet(c\vec{u}+d\vec{v}) = c^2\|\vec{u}\|^2 + 2cd\vec{u}\bullet\vec{v} + d^2\|\vec{v}\|^2$

Solution

$$\begin{aligned}
(c\vec{u}+d\vec{v})\bullet(c\vec{u}+d\vec{v}) &= (c\vec{u}+d\vec{v})\bullet c\vec{u} + (c\vec{u}+d\vec{v})\bullet d\vec{v} \\
&= c\vec{u}\bullet c\vec{u} + d\vec{v}\bullet c\vec{u} + c\vec{u}\bullet d\vec{v} + d\vec{v}\bullet d\vec{v} \\
&= c^2(\vec{u}\bullet\vec{u}) + cd(\vec{u}\bullet\vec{v}) + cd(\vec{u}\bullet\vec{v}) + d^2(\vec{v}\bullet\vec{v}) \\
&= c^2\|\vec{u}\|^2 + 2cd(\vec{u}\bullet\vec{v}) + d^2\|\vec{v}\|^2 \quad \checkmark
\end{aligned}$$

Exercise

Prove $\vec{u}\bullet(\vec{v}+\vec{w}) = \vec{u}\bullet\vec{v} + \vec{u}\bullet\vec{w}$

Solution

Let $\vec{u} = (u_1,\ u_2,\ \ldots,\ u_n)$, $\vec{v} = (v_1,\ v_2,\ \ldots,\ v_n)$, and $\vec{w} = (w_1,\ w_2,\ \ldots,\ w_n)$

$$\begin{aligned}
\vec{u}\bullet(\vec{v}+\vec{w}) &= (u_1,\ u_2,\ \ldots,\ u_n)\bullet\big((v_1,\ v_2,\ \ldots,\ v_n)+(w_1,\ w_2,\ \ldots,\ w_n)\big) \\
&= (u_1,\ u_2,\ \ldots,\ u_n)\bullet(v_1+w_1,\ v_2+w_2,\ \ldots,\ v_n+w_n) \\
&= u_1(v_1+w_1) + u_2(v_2+w_2) + \cdots + u_n(v_n+w_n) \\
&= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + \cdots + u_nv_n + u_nw_n \\
&= (u_1v_1 + u_2v_2 + \cdots + +u_nv_n) + (u_1w_1 + u_2w_2 + \cdots + u_nw_n) \\
&= (u_1,\ u_2,\ \ldots,\ u_n)\bullet(v_1,\ v_2,\ \ldots,\ v_n) + (u_1,\ u_2,\ \ldots,\ u_n)\bullet(w_1,\ w_2,\ \ldots,\ w_n) \\
&= \vec{u}\bullet\vec{v} + \vec{u}\bullet\vec{w} \quad \checkmark
\end{aligned}$$

Exercise

Prove Minkowski theorem: $\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$

Solution

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v})\bullet(\vec{u}+\vec{v}) \\
&= \vec{u}\bullet\vec{u} + 2\vec{u}\bullet\vec{v} + \vec{v}\bullet\vec{v} \\
&\le \|\vec{u}\|^2 + 2\|\vec{u}\|\ \|\vec{v}\| + \|\vec{v}\|^2
\end{aligned}$$

$$= \left(\|\vec{u}\| + \|\vec{v}\|\right)^2$$

$$\sqrt{\|\vec{u} + \vec{v}\|^2} \leq \sqrt{\left(\|\vec{u}\| + \|\vec{v}\|\right)^2}$$

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\| \qquad \checkmark$$