Solution Section 1.6 – Proof Methods and Strategy

Exercise

Prove that $n^2 + 1 \ge 2^n$ when n is a positive integer with $1 \le n \le 4$

Solution

$n = 1 \quad \to \quad 1^2 + 1 \ge 2^1 \Rightarrow 2 \ge 2 \quad \surd$

$n = 2 \quad \to \quad 2^2 + 1 \ge 2^2 \Rightarrow 5 \ge 4 \quad \surd$

$n = 3 \quad \to \quad 3^2 + 1 \ge 2^3 \Rightarrow 10 \ge 8 \quad \surd$

$n = 4 \quad \to \quad 4^2 + 1 \ge 2^4 \Rightarrow 17 \ge 16 \quad \surd$

Exercise

Prove that there are no positive perfect cubes less than 1000 that are the sum of the cubes of two positive integers.

Solution

The cubes are: 1, 8, 27, 64, 125, 216, 343, 512, and 729.

1 + 8 = 9, 1 + 27 = 28, 1 + 64, 1 + 125, …
8 + 8, 8 + 27, 8 + 64, 8 + 125, …
27 + 27, 27 + 64, 27 + 125, …
64 + 64, 64 + 125, 64 + 216, …
125 + 125, 125 + 216, …
216 + 216, 216 + 343, …
343 + 343, 343 + 512, 343 + 729
512 + 512, 512 + 729
729 + 729

None of them works.

We can conclude the no cube less than 1000 is the sum of two cubes.

Exercise

Prove that if x and y are real numbers, then $\max(x, y) + \min(x, y) = x + y$. (*Hint*: Use a proof by cases, with the two cases corresponding to $x \ge y$ and $x < y$, respectively.)

Solution

Suppose that $x \ge y$, then by definition $max(x, y) = x$ and $min(x, y) = y$. Therefore in this case $\max(x, y) + \min(x, y) = x + y$.

In the second case $x < y$, then by definition $max(x,\ y) = y$ and $min(x,\ y) = x$. Therefore in this case, $\max(x, y) + \min(x, y) = y + x = x + y$.

Hence in all cases, the equality holds.

Exercise

Prove the triangle inequality, which states that if *x* and *y* are real numbers, then $|x| + |y| \ge |x + y|$ (where $|x|$ represents the absolute value of *x*, which equals *x* if $x \ge 0$ and equals –*x* if $x < 0$

Solution

If *x* and *y* are both nonnegative, then $|x| + |y| = x + y = |x + y|$.

If *x* and *y* are both negative, then $|x| + |y| = (-x) + (-y) = -(x + y) = |x + y|$.

If $x \ge 0$ and $y < 0$, then there are two subcases to consider for *x* and –*y*:

Case 1: Suppose that $x \ge -y$, then $x + y \ge 0$. Therefore $x + y = |x + y|$, as desired.

$|x| + |y| = x + |y|$ is a positive number greater than *x*. Therefore $|x + y| < x < |x| + |y|$

Case 2: Suppose that $x < -y$, then $x + y < 0$. Therefore $|x + y| = -(x + y) = (-x) + (-y)$.

is a positive number less than or equal to –*y*. Therefore $|x + y| \le -y \le |x| + |y|$, as desired.

Exercise

Prove that either $2 \cdot 10^{500} + 15$ or $2 \cdot 10^{500} + 16$ is not a perfect square

Solution

A perfect square is a square of an integer

Rephrased: Show that a non-perfect square exists in the set $\left\{2 \cdot 10^{500} + 15,\ \ 2 \cdot 10^{500} + 16\right\}$

Proof: The only two perfect squares that differ by 1 are 0 and 1

Thus, any other numbers that differ by 1 cannot both be perfect squares

Thus, a non-perfect square must exist in any set that contains two numbers that differ by 1

Note that we didn't specify which one it was!

Exercise

Prove that there exists a pair of consecutive integers such that one of these integers is a perfect square and the other is a perfect cube.

Solution

$8 = 2^3 \qquad 9 = 3^2$

Exercise

Suppose that a and b are odd integers with $a \neq b$. Show there is a unique integer c such that $|a-c| = |b-c|$

Solution

The equation $|a-c| = |b-c|$ is equivalent to the disjunction of two equations:

$a-c = b-c$ *or* $a-c = -b+c$

Case: $a-c = b-c$ is equivalent to $a = b$, which contradicts the assumption $a \neq b$, so the original equation is equivalent to $a-c = -b+c$. By adding $b+c$ to both sides and dividing by 2, we see that this equation is equivalent to $c = \dfrac{a+b}{2}$. Thus there is a unique solution. Furthermore, this c is an integer, because the sum of the odd integers a and b is even.