Section 3.3 – Double-angle Formulas

$$\begin{aligned}\sin 2A &= \sin(A+A)\\ &= \sin A\cos A + \cos A\sin A\\ &= 2\sin A\cos A\end{aligned} \qquad\qquad \sin 2A \neq 2\sin A$$

Example

If $\sin A = \frac{3}{5}$ with A in QII, find $\sin 2A$

<u>Solution</u>

$$\begin{aligned}\cos A &= \pm\sqrt{1-\sin^2 A}\\ &= -\sqrt{1-\left(\frac{3}{5}\right)^2}\\ &= -\sqrt{1-\frac{9}{25}}\\ &= -\sqrt{\frac{25-9}{25}}\\ &= -\sqrt{\frac{16}{25}}\\ &= -\frac{4}{5}\end{aligned}$$

$$\begin{aligned}\sin 2A &= 2\sin A\cos A\\ &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)\\ &= -\frac{24}{5}\end{aligned}$$

Example

Prove $(\sin\theta+\cos\theta)^2 = 1+\sin 2\theta$

<u>Solution</u>

$$\begin{aligned}(\sin\theta+\cos\theta)^2 &= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta\\ &= \sin^2\theta+\cos^2\theta + 2\sin\theta\cos\theta\\ &= 1+2\sin\theta\cos\theta\\ &= 1+\sin 2\theta\end{aligned}$$

$$\begin{aligned}\cos 2A &= \cos(A+A)\\ &= \cos A\cos A - \sin A\sin A\\ &= \cos^2 A - \sin^2 A\end{aligned}$$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= \cos^2 A - \left(1-\cos^2 A\right)\\ &= \cos^2 A - 1 + \cos^2 A\\ &= 2\cos^2 A - 1\end{aligned}$$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= \left(1-\sin^2 A\right) - \sin^2 A\\ &= 1-\sin^2 A - \sin^2 A\\ &= 1-2\sin^2 A\end{aligned}$$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= 2\cos^2 A - 1\\ &= 1-2\sin^2 A\end{aligned}$$

Example

If $\sin A = \frac{1}{\sqrt{5}}$, find $\cos 2A$

<u>*Solution*</u>

$$\begin{aligned}\cos 2A &= 1-2\sin^2 A\\ &= 1-2\left(\frac{1}{\sqrt{5}}\right)^2\\ &= 1-2\cdot\frac{1}{5}\\ &= 1-\frac{2}{5}\\ &= \frac{3}{5}\end{aligned}$$

Example

Prove $\sin 2x = \dfrac{2\cot x}{1+\cot^2 x}$

<u>*Solution*</u>

$$\frac{2\cot x}{1+\cot^2 x} = \frac{2\frac{\cos x}{\sin x}}{1+\frac{\cos^2 x}{\sin^2 x}}$$

$$= \frac{2\frac{\cos x}{\sin x}}{\frac{\sin^2 x+\cos^2 x}{\sin^2 x}}$$

$$= 2\frac{\cos x}{\sin x}\frac{\sin^2 x}{\sin^2 x+\cos^2 x}$$

$$= 2\frac{\cos x}{1}\frac{\sin x}{1}$$

$$= 2\cos x\sin x$$

$$= \sin 2x$$

Example

Prove $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

<u>*Solution*</u>

$$\cos 4x = \cos(2.2x)$$

$$= 2\cos^2 2x - 1$$

$$= 2(\cos 2x)^2 - 1$$

$$= 2\left(2\cos^2 x - 1\right)^2 - 1$$

$$= 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1$$

$$= 8\cos^4 x - 8\cos^2 x + 2 - 1$$

$$= 8\cos^4 x - 8\cos^2 x + 1$$

$$\tan 2A = \tan(A + A)$$

$$= \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$\boxed{\tan 2A = \frac{2\tan A}{1 - \tan^2 A}}$$

Example

Simplify $\dfrac{2\tan 15^\circ}{1 - \tan^2 15^\circ}$

<u>Solution</u>

$$\frac{2\tan 15^\circ}{1 - \tan^2 15^\circ} = \tan(2 \cdot 15^\circ)$$

$$= \tan(30^\circ)$$

$$= \frac{1}{\sqrt{3}}$$

Example

Prove $\tan\theta = \dfrac{1 - \cos 2\theta}{\sin 2\theta}$

<u>Solution</u>

$$\frac{1 - \cos 2\theta}{\sin 2\theta} = \frac{1 - (1 - 2\sin^2\theta)}{2\sin\theta\cos\theta}$$

$$= \frac{1 - 1 + 2\sin^2\theta}{2\sin\theta\cos\theta}$$

$$= \frac{2\sin^2\theta}{2\sin\theta\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta}$$

Example

Given $\cos\theta = \frac{3}{5}$ and $\sin\theta < 0$, find $\sin 2\theta,\ \cos 2\theta,\ \textit{and}\ \tan 2\theta$

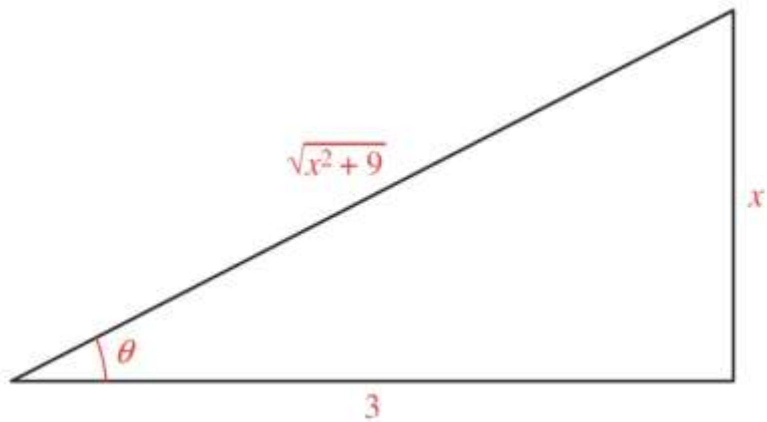
<u>*Solution*</u>

$\sin\theta = -\sqrt{1-\cos^2\theta}$ $= -\sqrt{1-\frac{9}{25}}$ $= -\sqrt{\frac{16}{25}}$ $= -\frac{4}{5}$	$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{3}$ $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ $= \frac{2\left(-\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2}$ $= \frac{-\frac{8}{3}}{1-\frac{16}{9}}$ $= \frac{-\frac{8}{3}}{-\frac{7}{9}}$ $= \left(-\frac{8}{3}\right)\left(-\frac{9}{7}\right)$ $= \frac{24}{7}$
$\sin 2\theta = 2\sin\theta\cos\theta$ $= 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right)$ $= -\frac{24}{25}$ $\cos 2\theta = \cos^2\theta - \sin^2\theta$ $= \left(\frac{3}{5}\right)^2 - \left(-\frac{4}{5}\right)^2$ $= \frac{9}{25} - \frac{16}{25}$ $= -\frac{7}{25}$	

Example

If $x = 3\tan\theta$, write the expression $\frac{\theta}{2} + \frac{\sin 2\theta}{4}$ in terms of just *x*.

<u>*Solution*</u>



$$x = 3\tan\theta \Rightarrow \tan\theta = \frac{x}{3}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{x}{3}\right)$$

$$\frac{\theta}{2} + \frac{\sin 2\theta}{4} = \frac{\theta}{2} + \frac{2\sin\theta\cos\theta}{4}$$

$$= \frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2}$$

$$= \frac{1}{2}(\theta + \sin\theta\cos\theta)$$

$$= \frac{1}{2}\left(\tan^{-1}x + \frac{x}{\sqrt{x^2+9}}\frac{3}{\sqrt{x^2+9}}\right)$$

$$= \frac{1}{2}\left(\tan^{-1}x + \frac{3x}{x^2+9}\right)$$

Exercises *Section* 3.3 – Double-angle Formulas

1. Let $\sin A = -\frac{3}{5}$ with A in QIII and find $\cos 2A$

2. Let $\cos x = \frac{1}{\sqrt{10}}$ with x in QIV and find $\cot 2x$

3. Verify: $(\cos x - \sin x)(\cos x + \sin x) = \cos 2x$

4. Verify: $\cot x \sin 2x = 1 + \cos 2x$

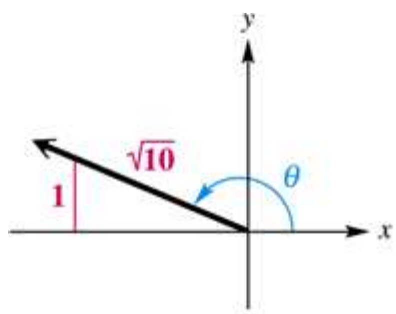
5. Prove: $\cot \theta = \dfrac{\sin 2\theta}{1 - \cos 2\theta}$

6. Simplify $\cos^2 7x - \sin^2 7x$

7. Write $\sin 3x$ in terms of $\sin x$

8. Find the values of the six trigonometric functions of θ if $\cos 2\theta = \frac{4}{5}$ *and* $90° < \theta < 180°$

9. Use a right triangle in QII to find the value of $\cos\theta$ *and* $\tan\theta$



10. Prove the following equation is an identity: $\sin 3x = \sin x\left(3\cos^2 x - \sin^2 x\right)$

11. Prove the following equation is an identity: $\cos 3x = \cos^3 x - 3\cos x \sin^2 x$

12. Prove the following equation is an identity: $\cos^4 x - \sin^4 x = \cos 2x$

13. Prove the following equation is an identity: $\sin 2x = -2\sin x \sin\left(x - \frac{\pi}{2}\right)$

14. Prove the following equation is an identity: $\dfrac{\sin 4t}{4} = \cos^3 t \sin t - \sin^3 t \cos t$

15. Prove the following equation is an identity: $\dfrac{\cos 2x}{\sin^2 x} = \csc^2 x - 2$

16. Prove the following equation is an identity: $\dfrac{\cos 2x + \cos 2y}{\sin x + \cos y} = 2\cos y - 2\sin x$

17. Prove the following equation is an identity: $\dfrac{\cos 2x}{\cos^2 x} = \sec^2 x - 2\tan^2 x$

18. Prove the following equation is an identity: $\sin 4x = (4\sin x\cos x)\left(2\cos^2 x - 1\right)$

19. Prove the following equation is an identity: $\cos 2y = \dfrac{1-\tan^2 y}{1+\tan^2 y}$

20. Prove the following equation is an identity: $\cos 4x = \cos^4 x - 6\sin^2 x\cos^2 x + \sin^4 x$

21. Prove the following equation is an identity: $\tan^2 x(1+\cos 2x) = 1-\cos 2x$

22. Prove the following equation is an identity: $\dfrac{\cos 2x}{\sin^2 x} = 2\cot^2 x - \csc^2 x$

23. Prove the following equation is an identity: $\tan x + \cot x = 2\csc 2x$

24. Prove the following equation is an identity: $\tan 2x = \dfrac{2}{\cot x - \tan x}$

25. Prove the following equation is an identity: $\dfrac{1-\tan x}{1+\tan x} = \dfrac{1-\sin 2x}{\cos 2x}$

26. Prove the following equation is an identity: $\sin 2\alpha \sin 2\beta = \sin^2(\alpha+\beta) - \sin^2(\alpha-\beta)$

27. Prove the following equation is an identity: $\cos^2(A-B) - \cos^2(A+B) = \sin 2A\ \sin 2B$