

Solution ***Section 2.4 – Properties of Division***

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x) = 2x^4 - x^3 + 7x - 12$; $p(x) = x^2 - 3$

Solution

$$\begin{array}{r} \overline{2x^4 - x^3 + 0x^2 + 7x - 12} \\ \underline{2x^4 - 6x^2} \\ -x^3 + 6x^2 + 7x \\ \underline{-x^3 + 3x} \\ 6x^2 + 4x - 12 \\ \underline{6x^2 - 18} \\ 4x + 6 \end{array}$$

$$\underline{Q(x) = 2x^2 - x + 6; \quad R(x) = 4x + 6}$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x) = 3x^3 + 2x - 4$; $p(x) = 2x^2 + 1$

Solution

$$\begin{array}{r} \overline{3x^3 + 0x^2 + 2x - 4} \\ \underline{3x^3 + \frac{3}{2}x} \\ \frac{1}{2}x - 4 \end{array}$$

$$\underline{Q(x) = \frac{3}{2}x; \quad R(x) = \frac{1}{2}x - 4}$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x) = 7x + 2$; $p(x) = 2x^2 - x - 4$

Solution

$$\begin{array}{r} \frac{2}{7}x - \frac{11}{49} \\ 7x + 2 \overline{) 2x^2 - x - 4} \\ \underline{2x^2 + \frac{4}{7}x} \\ -\frac{11}{7}x - 4 \\ \underline{-\frac{11}{7}x - \frac{22}{49}} \\ -\frac{174}{49} \end{array}$$

$$\underline{Q(x) = \frac{2}{7}x - \frac{11}{49}} \quad \left| \quad \underline{R(x) = -\frac{174}{49}} \right|$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x) = 9x + 4$; $p(x) = 2x - 5$

Solution

$$\begin{array}{r} \frac{9}{2} \\ 2x - 5 \overline{) 9x + 4} \\ \underline{9x - \frac{45}{2}} \\ -\frac{37}{2} \end{array}$$

$$\underline{Q(x) = \frac{9}{2}}; \quad \left| \quad \underline{R(x) = -\frac{37}{2}} \right|$$

Exercise

Use the remainder theorem to find $f(c)$: $f(x) = x^4 - 6x^2 + 4x - 8$; $c = -3$

Solution

$$\begin{aligned} f(-3) &= (-3)^4 - 6(-3)^2 + 4(-3) - 8 \\ &= 7 \end{aligned}$$

Exercise

Use the remainder theorem to find $f(c)$: $f(x) = x^4 + 3x^2 - 12$; $c = -2$

Solution

$$\begin{aligned} f(-2) &= (-2)^4 + 3(-2)^2 - 12 \\ &= 16 \end{aligned}$$

Exercise

Use the factor theorem to show that $x - c$ is a factor of $f(x)$: $f(x) = x^3 + x^2 - 2x + 12$; $c = -3$

Solution

$$\begin{aligned} f(-3) &= (-3)^3 + (-3)^2 - 2(-3) + 12 \\ &= 0 \end{aligned}$$

From the factor theorem; $x + 3$ is a factor of $f(x)$.

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $2x^3 - 3x^2 + 4x - 5$; $x - 2$

Solution

$$\begin{array}{r|rrrr} 2 & 2 & -3 & 4 & -5 \\ & & 4 & 2 & 12 \\ \hline & 2 & 1 & 6 & \boxed{7} \end{array}$$

$$\underline{Q(x) = 2x^2 + x + 6 \quad R(x) = 7}$$

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $5x^3 - 6x^2 + 15$; $x - 4$

Solution

$$\begin{array}{r|rrrr} 4 & 5 & -6 & 0 & 15 \\ & & 20 & 56 & 224 \\ \hline & 5 & 14 & 56 & \boxed{239} \end{array}$$

$$\underline{Q(x) = 5x^2 + 14x + 56 \quad R(x) = 239}$$

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $9x^3 - 6x^2 + 3x - 4$; $x - \frac{1}{3}$

Solution

$$\begin{array}{r|rrrr} \frac{1}{3} & 9 & -6 & 3 & -4 \\ & & 3 & -1 & \frac{2}{3} \\ \hline & 9 & -3 & 2 & \boxed{-\frac{10}{3}} \end{array}$$

$$\underline{Q(x) = 9x^2 - 3x + 2 \quad R(x) = -\frac{10}{3} \quad |}$$

Exercise

Use the synthetic division to find $f(c)$: $f(x) = 2x^3 + 3x^2 - 4x + 4$; $c = 3$

Solution

$$\begin{array}{r|rrrr} 3 & 2 & 3 & -4 & 4 \\ & & 9 & 36 & 93 \\ \hline & 3 & 12 & 32 & \boxed{97} \end{array}$$

$$\underline{f(3) = 97 \quad |}$$

Exercise

Use the synthetic division to find $f(c)$: $f(x) = 8x^5 - 3x^2 + 7$; $c = \frac{1}{2}$

Solution

$$\begin{array}{r|rrrrrr} \frac{1}{2} & 8 & 0 & 0 & -3 & 0 & 7 \\ & & 4 & 2 & 1 & -1 & -\frac{1}{2} \\ \hline & 8 & 4 & 2 & -2 & -1 & \boxed{\frac{13}{2}} \end{array}$$

$$\underline{f\left(\frac{1}{2}\right) = \frac{13}{2} \quad |}$$

Exercise

Use the synthetic division to find $f(c)$: $f(x) = x^3 - 3x^2 - 8$; $c = 1 + \sqrt{2}$

Solution

$$\begin{array}{r|rrrr}
 1 + \sqrt{2} & 3 & -3 & 0 & -8 \\
 & & 3 + 3\sqrt{2} & 6 + 3\sqrt{2} & 12 + 9\sqrt{2} \\
 \hline
 & 3 & 3\sqrt{2} & 6 + 3\sqrt{2} & \boxed{4 + 9\sqrt{2}}
 \end{array}$$

$$\underline{f(1 + \sqrt{2}) = 4 + 9\sqrt{2}}$$

Exercise

Use the synthetic division to show that c is a zero of $f(x)$: $f(x) = 3x^4 + 8x^3 - 2x^2 - 10x + 4$; $c = -2$

Solution

$$\begin{array}{r|rrrrr}
 -2 & 3 & 8 & -2 & -10 & 4 \\
 & & -6 & -4 & 12 & -4 \\
 \hline
 & 3 & 2 & -6 & 2 & \boxed{0}
 \end{array}$$

$$\underline{f(-2) = 0}$$

Exercise

Use the synthetic division to show that c is a zero of $f(x)$: $f(x) = 27x^4 - 9x^3 + 3x^2 + 6x + 1$; $c = -\frac{1}{3}$

Solution

$$\begin{array}{r|rrrrr}
 -\frac{1}{3} & 27 & -9 & 3 & 6 & 1 \\
 & & -9 & 6 & -3 & -1 \\
 \hline
 & 27 & -18 & 9 & 3 & \boxed{0}
 \end{array}$$

$$\underline{f\left(-\frac{1}{3}\right) = 0}$$

Exercise

Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

$$f(x) = kx^3 + x^2 + k^2x + 3k^2 + 11; \quad x + 2$$

Solution

$$\begin{array}{r|rrrr}
 -2 & k & 1 & k^2 & 3k^2 + 11 \\
 & & -2k & 4k - 2 & -2k^2 - 8k + 4 \\
 \hline
 & k & 1 - 2k & k^2 + 4k - 2 & k^2 - 8k + 15
 \end{array}$$

$$k^2 - 8k + 15 = 0 \Rightarrow \boxed{k = 3, 5}$$

Exercise

Find all solutions of the equation: $x^3 - x^2 - 10x - 8 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{8}{1} \right\} = \pm \{1, 2, 4, 8\}$

$$\begin{array}{r|rrrr}
 -1 & 1 & -1 & -10 & -8 \\
 & & -1 & 2 & 8 \\
 \hline
 & 1 & -2 & -8 & 0
 \end{array} \rightarrow x^2 - 2x - 8 = 0$$

The solutions are: $\boxed{x = -1, -2, 4}$

Exercise

Find all solutions of the equation: $x^3 + x^2 - 14x - 24 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{24}{1} \right\} = \pm \{1, 2, 3, 4, 6, 8, 12, 24\}$

$$\begin{array}{r|rrrr}
 -2 & 1 & 1 & -14 & -24 \\
 & & -2 & 2 & 24 \\
 \hline
 & 1 & -1 & -12 & 0
 \end{array} \rightarrow x^2 - x - 12 = 0$$

The solutions are: $\boxed{x = -2, -3, 4}$

Exercise

Find all solutions of the equation: $2x^3 - 3x^2 - 17x + 30 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{30}{2} \right\} = \pm \left\{ 1, 2, 3, 5, 6, 10, 15, 30, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{15}{2} \right\}$

$$\begin{array}{r|rrrr} 2 & 2 & -3 & -17 & 30 \\ & & 4 & 2 & -30 \\ \hline & 2 & 1 & -15 & \boxed{0} \end{array} \rightarrow 2x^2 + x - 15 = 0$$

The solutions are: $x = 2, -3, \frac{5}{2}$

Exercise

Find all solutions of the equation: $12x^3 + 8x^2 - 3x - 2 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{2}{12} \right\} = \pm \left\{ 1, 2, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{2}, \frac{1}{6}, \frac{1}{12} \right\}$

$$\begin{array}{r|rrrr} \frac{1}{2} & 12 & 8 & -3 & -2 \\ & & 6 & 7 & 2 \\ \hline & 12 & 14 & 4 & \boxed{0} \end{array} \rightarrow 12x^2 + 14x + 4 = 0$$

The solutions are: $x = \frac{1}{2}, -\frac{1}{2}, -\frac{2}{3}$

Exercise

Find all solutions of the equation: $x^3 + x^2 - 6x - 8 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{8}{1} \right\} = \pm \{1, 2, 4, 8\}$

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -6 & -8 \\ & & -2 & 2 & 8 \\ \hline & 1 & -1 & -4 & \boxed{0} \end{array} \rightarrow x^2 - x - 4 = 0$$

$$x = \frac{1 \pm \sqrt{1+16}}{2}$$

The solutions are: $x = -2, \frac{1 \pm \sqrt{17}}{2}$

Exercise

Find all solutions of the equation: $x^3 - 19x - 30 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{30}{1} \right\} = \pm \{1, 2, 3, 5, 6, 15, 30\}$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -19 & -30 \\ & & -2 & 4 & 30 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array} \rightarrow x^2 - 2x - 15$$

$$x = \frac{2 \pm \sqrt{4 + 60}}{2}$$

$$= \begin{cases} \frac{2-8}{2} = -3 \\ \frac{2+8}{2} = 5 \end{cases}$$

The solutions are: $x = -2, -3, 5$

Exercise

Find all solutions of the equation: $2x^3 + x^2 - 25x + 12 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{12}{2} \right\} = \pm \left\{ 1, 2, 3, 4, 6, 12, \frac{1}{2}, \frac{3}{2} \right\}$

$$\begin{array}{r|rrrr} 3 & 2 & 1 & -25 & 12 \\ & & 6 & 21 & -12 \\ \hline & 2 & 7 & -4 & \boxed{0} \end{array} \rightarrow 2x^2 + 7x - 4$$

$$x = \frac{-7 \pm \sqrt{49 + 32}}{4}$$

$$= \begin{cases} \frac{-7-9}{4} = -4 \\ \frac{-7+9}{4} = \frac{1}{2} \end{cases}$$

The solutions are: $x = -4, \frac{1}{2}, 3$

Exercise

Find all solutions of the equation: $3x^3 + 11x^2 - 6x - 8 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{8}{3} \right\} = \pm \left\{ 1, 2, 4, 8, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3} \right\}$

$$\begin{array}{r|rrrr} 1 & 3 & 11 & -6 & -8 \\ & & 3 & 14 & 8 \\ \hline & 3 & 14 & 8 & \boxed{0} \end{array} \rightarrow 3x^2 + 14x + 8$$

$$x = \frac{-14 \pm \sqrt{196 - 96}}{6}$$

$$= \begin{cases} \frac{-14-10}{6} = -4 \\ \frac{-14+10}{6} = -\frac{2}{3} \end{cases}$$

The solutions are: $x = -4, -\frac{2}{3}, 1$

Exercise

Find all solutions of the equation: $2x^3 + 9x^2 - 2x - 9 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{9}{2} \right\} = \pm \left\{ 1, 3, 9, \frac{1}{2}, \frac{3}{2}, \frac{9}{2} \right\}$

$$\begin{array}{r|rrrr} 1 & 2 & 9 & -2 & -9 \\ & & 2 & 11 & 9 \\ \hline & 2 & 11 & 9 & \boxed{0} \end{array} \rightarrow 2x^2 + 11x + 9$$

$$x = -1, -\frac{9}{2} \quad a - b + c = 0 \rightarrow x = -1, -\frac{c}{a}$$

The solutions are: $x = -\frac{9}{2}, -1, 1$

Exercise

Find all solutions of the equation: $x^3 + 3x^2 - 6x - 8 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{8}{1} \right\} = \pm \{1, 2, 4, 8\}$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & -6 & -8 \\ & & -1 & -2 & 8 \\ \hline & 1 & 2 & -8 & \boxed{0} \end{array} \rightarrow x^2 + 2x - 8 = 0$$

$$x = \frac{-2 \pm \sqrt{4 + 32}}{2}$$

$$= \begin{cases} \frac{-2-6}{2} = -4 \\ \frac{-2+6}{2} = 2 \end{cases}$$

The solutions are: $x = -4, -1, 2$

Exercise

Find all solutions of the equation: $3x^3 - x^2 - 6x + 2 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{2}{3} \right\} = \pm \left\{ 1, 2, \frac{1}{3}, \frac{2}{3} \right\}$

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -1 & -6 & 2 \\ & & 1 & 0 & -2 \\ \hline & 3 & 0 & -6 & \boxed{0} \end{array} \rightarrow 3x^2 - 6 = 0$$

$$x^2 = 2$$

The solutions are: $x = \frac{1}{3}, \pm\sqrt{2}$

Exercise

Find all solutions of the equation: $x^3 - 8x^2 + 8x + 24 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{24}{1} \right\} = \pm \{1, 2, 3, 4, 6, 8, 12, 24\}$

$$\begin{array}{r|rrrr} 6 & 1 & -8 & 8 & 24 \\ & & 6 & -12 & -24 \\ \hline & 1 & -2 & -4 & \boxed{0} \end{array} \rightarrow x^2 - 2x - 4 = 0$$

$$\begin{aligned} x &= \frac{2 \pm \sqrt{4 + 16}}{2} \\ &= \frac{2 \pm 2\sqrt{5}}{2} \end{aligned}$$

The solutions are: $x = 6, 1 \pm \sqrt{5}$

Exercise

Find all solutions of the equation: $x^3 - 7x^2 - 7x + 69 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{69}{1} \right\} = \pm \{1, 3, 23, 69\}$

$$\begin{array}{r|rrrr} -3 & 1 & -7 & -7 & 69 \\ & & -3 & 30 & -69 \\ \hline & 1 & -10 & 23 & \boxed{0} \end{array} \rightarrow x^2 - 10x + 23 = 0$$

$$x = \frac{10 \pm \sqrt{100 - 92}}{2}$$

$$= \frac{10 \pm 2\sqrt{2}}{2}$$

The solutions are: $x = -3, 5 \pm \sqrt{2}$

Exercise

Find all solutions of the equation: $x^3 - 3x - 2 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{2}{1} \right\} = \pm \{1, 2\}$

$$\begin{array}{c|cccc} -1 & 1 & 0 & -3 & -2 \\ & & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array} \rightarrow x^2 - x - 2 = 0$$

$$x = -1, 2 \quad a - b + c = 0 \rightarrow x = -1, -\frac{c}{a}$$

The solutions are: $x = -1, -1, 2$

Exercise

Find all solutions of the equation: $x^3 - 2x + 1 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \{1\}$

$$\begin{array}{c|cccc} 1 & 1 & 0 & -2 & 1 \\ & & 1 & 1 & 1 \\ \hline & 1 & 1 & -1 & \boxed{0} \end{array} \rightarrow x^2 + x - 1 = 0$$

$$x = \frac{-1 \pm \sqrt{5}}{2}$$

The solutions are: $x = 1, \frac{-1 \pm \sqrt{5}}{2}$

Exercise

Find all solutions of the equation: $x^3 - 2x^2 - 11x + 12 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{12}{1} \right\} = \pm \{1, 2, 3, 4, 6, 12\}$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -11 & 12 \\ & & 1 & -1 & 12 \\ \hline & 1 & -1 & -12 & \boxed{0} \end{array} \rightarrow x^2 - x - 12 = 0$$

$$x = \frac{1 \pm \sqrt{1+48}}{2}$$

$$= \begin{cases} \frac{1-7}{2} = -3 \\ \frac{1+7}{2} = 4 \end{cases}$$

The solutions are: $x = -3, 1, 4$

Exercise

Find all solutions of the equation: $x^3 - 2x^2 - 7x - 4 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{4}{1} \right\} = \pm \{1, 2, 4\}$

$$\begin{array}{r|rrrr} -1 & 1 & -2 & -7 & -4 \\ & & -1 & 3 & 4 \\ \hline & 1 & -3 & -4 & \boxed{0} \end{array} \rightarrow x^2 - 3x - 4 = 0$$

$$x = -1, 4 \quad a - b + c = 0 \rightarrow x = -1, -\frac{c}{a}$$

The solutions are: $x = -1, -1, 4$

Exercise

Find all solutions of the equation: $x^3 - 10x - 12 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{12}{1} \right\} = \pm \{1, 2, 3, 4, 6, 12\}$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -10 & -12 \\ & & -2 & 4 & 12 \\ \hline & 1 & -2 & -6 & \boxed{0} \end{array} \rightarrow x^2 - 2x - 6 = 0$$

$$\begin{aligned} x &= \frac{2 \pm \sqrt{4 + 24}}{2} \\ &= \frac{2 \pm 2\sqrt{7}}{2} \end{aligned}$$

The solutions are: $x = -2, 1 \pm \sqrt{7}$

Exercise

Find all solutions of the equation: $x^3 - 5x^2 + 17x - 13 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{13}{1} \right\} = \pm \{1, 13\}$

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 17 & -13 \\ & & 1 & -4 & 13 \\ \hline & 1 & -4 & 13 & \boxed{0} \end{array} \rightarrow x^2 - 4x + 13 = 0$$

$$\begin{aligned} x &= \frac{4 \pm \sqrt{16 - 52}}{2} \\ &= \frac{4 \pm 6i}{2} \end{aligned}$$

The solutions are: $x = 1, 2 \pm 3i$

Exercise

Find all solutions of the equation: $6x^3 + 25x^2 - 24x + 5 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{5}{6} \right\} = \pm \left\{ 1, 5, \frac{1}{2}, \frac{5}{2}, \frac{1}{3}, \frac{5}{3}, \frac{1}{6}, \frac{5}{6} \right\}$

$$\begin{array}{r|rrrr} -5 & 6 & 25 & -24 & 5 \\ & & -30 & 25 & -5 \\ \hline & 6 & -5 & 1 & \boxed{0} \end{array} \rightarrow 6x^2 - 5x + 1 = 0$$

$$x = \frac{5 \pm \sqrt{25 - 24}}{12}$$

$$= \begin{cases} \frac{5-1}{12} = \frac{1}{3} \\ \frac{5+1}{12} = \frac{1}{2} \end{cases}$$

The solutions are: $x = -5, \frac{1}{3}, \frac{1}{2}$

Exercise

Find all solutions of the equation: $8x^3 + 18x^2 + 45x + 27 = 0$

Solution

$$\begin{aligned} \text{possibilities: } \pm \left\{ \frac{27}{8} \right\} &= \pm \left\{ \frac{1, 3, 9, 27}{1, 2, 4, 8} \right\} \\ &= \pm \left\{ 1, 3, 9, 27, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{9}{2}, \frac{9}{4}, \frac{9}{8}, \frac{27}{2}, \frac{27}{4}, \frac{27}{8} \right\} \end{aligned}$$

$$\begin{array}{r|rrrr} -\frac{3}{4} & 8 & 18 & 45 & 27 \\ & & -6 & -9 & -27 \\ \hline & 8 & 12 & 36 & \boxed{0} \end{array} \rightarrow 8x^2 + 12x + 36 = 0$$

The solutions are: $x = -\frac{3}{4}, -\frac{3}{4} \pm i \frac{3\sqrt{7}}{4}$

Exercise

Find all solutions of the equation: $3x^3 - x^2 + 11x - 20 = 0$

Solution

$$\begin{aligned} \text{possibilities: } \pm \left\{ \frac{20}{3} \right\} &= \pm \left\{ \frac{1, 2, 4, 5, 10, 20}{1, 3} \right\} \\ &= \pm \left\{ 1, 2, 4, 5, 10, 20, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{10}{3}, \frac{20}{3} \right\} \end{aligned}$$

$$\begin{array}{r|rrrr} \frac{4}{3} & 3 & -1 & 11 & -20 \\ & & 4 & 4 & 20 \\ \hline & 3 & 3 & 15 & \boxed{0} \end{array} \rightarrow 3x^2 + 3x + 15 = 0$$

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{9 - 180}}{6} \\ &= \frac{-3 \pm 3\sqrt{19}}{6} \end{aligned}$$

The solutions are: $x = \frac{4}{3}, -\frac{1}{2} \pm i \frac{\sqrt{19}}{2}$

Exercise

Find all solutions of the equation: $x^4 - x^3 - 9x^2 + 3x + 18 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{18}{1} \right\} = \pm \{1, 2, 3, 6, 9, 18\}$

$$\begin{array}{r|rrrrr}
 -2 & 1 & -1 & -9 & 3 & 18 \\
 & & -2 & 6 & 6 & -18 \\
 \hline
 3 & 1 & -3 & -3 & 9 & 0 \\
 & & 3 & 0 & -9 & \\
 \hline
 & 1 & 0 & -3 & 0 &
 \end{array}
 \rightarrow x^3 - 3x^2 - 3x + 9 = 0 \rightarrow \pm \left\{ \frac{9}{1} \right\} = \pm \{1, 3, 9\}$$

$$\rightarrow x^2 - 3 = 0 \Rightarrow x = \pm \sqrt{3}$$

The solutions are: $x = -2, 3, \pm \sqrt{3}$

Exercise

Find all solutions of the equation: $2x^4 - 9x^3 + 9x^2 + x - 3 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{3}{2} \right\} = \pm \left\{ 1, 3, \frac{1}{2}, \frac{3}{2} \right\}$

$$\begin{array}{r|rrrrr}
 1 & 2 & -9 & 9 & 1 & -3 \\
 & & 2 & -7 & 2 & 3 \\
 \hline
 1 & 2 & -7 & 2 & 3 & 0 \\
 & & 2 & -5 & -3 & \\
 \hline
 & 2 & -5 & -3 & 0 &
 \end{array}
 \rightarrow 2x^3 - 7x^2 + 2x + 3 = 0 \rightarrow \pm \left\{ \frac{3}{2} \right\} = \pm \left\{ 1, 3, \frac{1}{2}, \frac{3}{2} \right\}$$

$$\rightarrow 2x^2 - 5x - 3 = 0$$

$$x = \frac{5 \pm \sqrt{25 + 24}}{4}$$

$$= \begin{cases} \frac{5-7}{4} = -\frac{1}{2} \\ \frac{5+7}{4} = 3 \end{cases}$$

The solutions are: $x = 1, 1, -\frac{1}{2}, 3$

Exercise

Find all solutions of the equation: $6x^4 + 5x^3 - 17x^2 - 6x = 0$

Solution

$$x(6x^3 + 5x^2 - 17x - 6) = 0 \rightarrow \underline{x=0}$$

$$\text{possibilities: } \pm \left\{ \frac{6}{6} \right\} = \pm \left\{ 1, 2, 3, 6, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{2}{3} \right\}$$

$$\begin{array}{r|rrrr} -2 & 6 & 5 & -17 & -6 \\ & & -12 & 14 & 6 \\ \hline & 6 & -7 & -3 & \boxed{0} \end{array} \rightarrow 6x^2 - 7x - 3 = 0$$

$$x = \frac{7 \pm \sqrt{49 + 72}}{12}$$

$$= \begin{cases} \frac{7-11}{12} = -\frac{1}{3} \\ \frac{7+11}{12} = \frac{3}{2} \end{cases}$$

$$\text{The solutions are: } \underline{x=0, -2, -\frac{1}{3}, \frac{3}{2}}$$

Exercise

Find all solutions of the equation: $x^4 - 2x^2 - 16x - 15 = 0$

Solution

$$\text{possibilities: } \pm \left\{ \frac{15}{1} \right\} = \pm \{1, 3, 5, 15\}$$

$$\begin{array}{r|rrrrr} -1 & 1 & 0 & -2 & -16 & -15 \\ & & -1 & 1 & 1 & 15 \\ \hline 3 & 1 & -1 & -1 & -15 & \boxed{0} \\ & & 3 & 6 & 15 & \\ \hline & 1 & 2 & 5 & \boxed{0} & \end{array} \rightarrow x^3 - x^2 - x - 15 = 0 \rightarrow \pm \left\{ \frac{15}{1} \right\} = \pm \{1, 3, 5, 15\}$$
$$\rightarrow x^2 + 2x + 5 = 0$$

$$x = \frac{-2 \pm \sqrt{-16}}{2}$$

$$= -1 \pm 2i$$

$$\text{The solutions are: } \underline{x=-1, 3, -1 \pm 2i}$$

Exercise

Find all solutions of the equation: $x^4 - 2x^3 - 5x^2 + 8x + 4 = 0$

Solution

possibilities: $\pm \left\{ \frac{4}{1} \right\} = \pm \{1, 2, 4\}$

$$\begin{array}{r|rrrrr} 2 & 1 & -2 & -5 & 8 & 4 \\ & & 2 & 0 & -10 & -4 \\ \hline -2 & 1 & 0 & -5 & -2 & 0 \\ & & -2 & 4 & 2 & \\ \hline & 1 & -2 & -1 & 0 & \end{array} \rightarrow x^3 - 5x - 2 = 0 \rightarrow \pm \left\{ \frac{2}{1} \right\} = \pm \{1, 2\}$$
$$\rightarrow x^2 - 2x - 1 = 0$$

$$x = \frac{2 \pm \sqrt{8}}{2}$$
$$= \frac{2 \pm 2\sqrt{2}}{2}$$

The solutions are: $x = -2, 2, 1 \pm \sqrt{2}$

Exercise

Find all solutions of the equation: $2x^4 - 17x^3 + 4x^2 + 35x - 24 = 0$

Solution

possibilities: $\pm \left\{ \frac{24}{2} \right\} = \pm \left\{ 1, 2, 3, 4, 6, 8, 12, 24, \frac{1}{2}, \frac{3}{2} \right\}$

$$\begin{array}{r|rrrrr} 1 & 2 & -17 & 4 & 35 & -24 \\ & & 2 & -15 & -11 & 24 \\ \hline 1 & 2 & -15 & -11 & 24 & 0 \\ & & 2 & -13 & 24 & \\ \hline & 2 & -13 & -24 & 0 & \end{array} \rightarrow 2x^3 - 15x^2 - 11x + 24 = 0 \rightarrow \pm \left\{ \frac{2}{1} \right\} = \pm \{1, 2\}$$
$$\rightarrow 2x^2 - 13x - 24 = 0$$

$$x = \frac{13 \pm \sqrt{169 + 192}}{4}$$

$$= \begin{cases} \frac{13-19}{4} = -\frac{3}{2} \\ \frac{13+19}{4} = 8 \end{cases}$$

The solutions are: $x = -\frac{3}{2}, 1, 1, 8$

Exercise

Find all solutions of the equation: $x^4 + x^3 - 3x^2 - 5x - 2 = 0$

Solution

possibilities: $\pm \left\{ \frac{2}{1} \right\} = \pm \{1, 2\}$

$$\begin{array}{r|rrrrr} -1 & 1 & 1 & -3 & -5 & -2 \\ & & -1 & 0 & 3 & 2 \\ \hline -1 & 1 & 0 & -3 & -2 & 0 \\ & & -1 & 1 & 2 & \\ \hline & 1 & -1 & -2 & 0 & \end{array} \rightarrow x^3 - 3x - 2 = 0 \rightarrow \pm \left\{ \frac{2}{1} \right\} = \pm \{1, 2\}$$
$$\rightarrow x^2 - x - 2 = 0$$

$$x = \frac{1 \pm \sqrt{9}}{2}$$
$$= \begin{cases} \frac{1-3}{2} = -1 \\ \frac{1+3}{2} = 2 \end{cases}$$

The solutions are: $x = -1, -1, -1, 2$

Exercise

Find all solutions of the equation: $6x^4 - 17x^3 - 11x^2 + 42x = 0$

Solution

$$x(6x^3 - 17x^2 - 11x + 42) = 0$$

$$x = 0 \quad 6x^3 - 17x^2 - 11x + 42 = 0$$

possibilities: $\pm \left\{ \frac{42}{6} \right\} = \pm \left\{ 1, 2, 3, 6, 7, 14, 21, 42, \frac{1}{2}, \frac{3}{2}, \frac{7}{2}, \frac{21}{2}, \frac{1}{3}, \frac{2}{3}, \frac{7}{3}, \frac{14}{3}, \frac{1}{6}, \frac{7}{6}, \frac{21}{6} \right\}$

$$\begin{array}{r|rrrr} 2 & 6 & -17 & -11 & 42 \\ & & 12 & -10 & -42 \\ \hline & 6 & -5 & -21 & 0 \end{array} \rightarrow 6x^2 - 5x - 21 = 0$$

$$x = \frac{5 \pm \sqrt{25 + 504}}{12}$$

$$= \begin{cases} \frac{5-23}{12} = -\frac{3}{2} \\ \frac{5+23}{12} = \frac{7}{3} \end{cases}$$

The solutions are: $x = -\frac{3}{2}, 0, 2, \frac{7}{3}$

Exercise

Find all solutions of the equation: $x^4 - 5x^2 - 2x = 0$

Solution

$$x(x^3 - 5x - 2) = 0$$

$$x = 0 \quad x^3 - 5x - 2 = 0$$

$$\text{possibilities: } \pm \left\{ \frac{2}{1} \right\} = \pm \{1, 2\}$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -5 & -2 \\ & & -2 & 4 & 2 \\ \hline & 1 & -2 & -1 & \boxed{0} \end{array} \rightarrow x^2 - 2x - 1 = 0$$

$$x = \frac{2 \pm \sqrt{4 + 4}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2}$$

The solutions are: $x = -2, 0, 1 \pm \sqrt{2}$

Exercise

Find all solutions of the equation: $3x^4 - 4x^3 - 11x^2 + 16x - 4 = 0$

Solution

$$\text{possibilities: } \pm \left\{ \frac{4}{3} \right\} = \pm \left\{ 1, 2, 4, \frac{1}{3}, \frac{2}{3}, \frac{4}{3} \right\}$$

$$\begin{array}{r|rrrrr} 1 & 3 & -4 & -11 & 16 & -4 \\ & & 3 & -1 & -12 & 4 \\ \hline 2 & 3 & -1 & -12 & 4 & \boxed{0} \\ & & 6 & 10 & -4 & \\ \hline & 3 & 5 & -2 & \boxed{0} & \end{array} \rightarrow 3x^3 - x^2 - 12x + 4 = 0 \rightarrow \pm \left\{ \frac{4}{3} \right\} = \pm \left\{ 1, 2, 4, \frac{1}{3}, \frac{2}{3}, \frac{4}{3} \right\}$$
$$\rightarrow 3x^2 + 5x - 2 = 0$$

$$x = \frac{-5 \pm \sqrt{25 + 24}}{6}$$

$$= \begin{cases} \frac{-5-7}{6} = -2 \\ \frac{-5+7}{6} = \frac{1}{3} \end{cases}$$

The solutions are: $x = -2, \frac{1}{3}, 1, 2$

Exercise

Find all solutions of the equation: $6x^4 + 23x^3 + 19x^2 - 8x - 4 = 0$

Solution

possibilities: $\pm \left\{ \frac{4}{6} \right\} = \pm \left\{ 1, 2, 4, \frac{1}{6}, \frac{2}{3}, \frac{2}{3} \right\}$

$$\begin{array}{r|rrrrr}
 -2 & 6 & 23 & 19 & -8 & -4 \\
 & & -12 & -22 & 6 & 4 \\
 \hline
 -2 & 6 & 11 & -3 & -2 & 0 \\
 & & -12 & 2 & 2 & \\
 \hline
 & 6 & -1 & -1 & 0 &
 \end{array} \rightarrow 6x^3 + 11x^2 - 3x - 2 = 0 \rightarrow \pm \left\{ \frac{2}{6} \right\} = \pm \left\{ 1, 2, \frac{1}{6}, \frac{1}{3} \right\}$$

$$\rightarrow 6x^2 - x - 1 = 0$$

$$\begin{aligned}
 x &= \frac{1 \pm \sqrt{25}}{12} \\
 &= \begin{cases} \frac{1-5}{12} = -\frac{1}{3} \\ \frac{1+5}{12} = \frac{1}{2} \end{cases}
 \end{aligned}$$

The solutions are: $x = -2, -2, -\frac{1}{3}, \frac{1}{2}$

Exercise

Find all solutions of the equation: $4x^4 - 12x^3 + 3x^2 + 12x - 7 = 0$

Solution

possibilities: $\pm \left\{ \frac{7}{4} \right\} = \pm \left\{ 1, 7, \frac{1}{2}, \frac{7}{2}, \frac{1}{4}, \frac{7}{4} \right\}$

$$\begin{array}{r|rrrrr}
 1 & 4 & -12 & 3 & 12 & -7 \\
 & & 4 & -8 & -5 & 7 \\
 \hline
 -1 & 4 & -8 & -5 & 7 & 0 \\
 & & -4 & 12 & -7 & \\
 \hline
 & 4 & -12 & 7 & 0 &
 \end{array} \rightarrow 4x^3 - 8x^2 - 5x + 7 = 0 \rightarrow \pm \left\{ \frac{7}{4} \right\} = \pm \left\{ 1, 7, \frac{1}{2}, \frac{7}{2}, \frac{1}{4}, \frac{7}{4} \right\}$$

$$\rightarrow 4x^2 - 12x + 7 = 0$$

$$\begin{aligned}
 x &= \frac{12 \pm \sqrt{144 - 112}}{8} \\
 &= \frac{12 \pm 4\sqrt{2}}{8}
 \end{aligned}$$

The solutions are: $x = -1, 1, \frac{3 \pm \sqrt{2}}{2}$

Exercise

Find all solutions of the equation: $2x^4 - 9x^3 - 2x^2 + 27x - 12 = 0$

Solution

$$\text{possibilities: } \pm \left\{ \frac{12}{2} \right\} = \pm \left\{ 1, 2, 3, 4, 6, 12, \frac{1}{2}, \frac{3}{2} \right\}$$

$$\begin{array}{r|rrrrr} 4 & 2 & -9 & -2 & 27 & -12 \\ & & 8 & -4 & -24 & 12 \\ \hline \frac{1}{2} & 2 & -1 & -6 & 3 & 0 \\ & & 1 & 0 & -3 & \\ \hline & 2 & 0 & -6 & 0 & \end{array} \rightarrow 2x^3 - x^2 - 6x + 3 = 0 \rightarrow \pm \left\{ \frac{3}{2} \right\} = \pm \left\{ 1, 3, \frac{1}{2}, \frac{3}{2} \right\}$$
$$\rightarrow 2x^2 - 6 = 0$$

$$x^2 = 3$$

$$\text{The solutions are: } x = \frac{1}{2}, 4, \pm \sqrt{3}$$

Exercise

Find all solutions of the equation: $2x^4 - 19x^3 + 51x^2 - 31x + 5 = 0$

Solution

$$\text{possibilities: } \pm \left\{ \frac{5}{2} \right\} = \pm \left\{ 1, 5, \frac{1}{2}, \frac{5}{2} \right\}$$

$$\begin{array}{r|rrrrr} 5 & 2 & -19 & 51 & -31 & 5 \\ & & 10 & -45 & 30 & -5 \\ \hline \frac{1}{2} & 2 & -9 & 6 & -1 & 0 \\ & & 1 & -4 & 1 & \\ \hline & 2 & -8 & 2 & 0 & \end{array} \rightarrow 2x^3 - 9x^2 + 6x - 1 = 0 \rightarrow \pm \left\{ \frac{1}{2} \right\}$$
$$\rightarrow 2x^2 - 8x + 2 = 0$$

$$x = \frac{8 \pm \sqrt{64 - 16}}{4}$$
$$= \frac{8 \pm 4\sqrt{3}}{4}$$

$$\text{The solutions are: } x = \frac{1}{2}, 5, 2 \pm \sqrt{3}$$

Exercise

Find all solutions of the equation: $4x^4 - 35x^3 + 71x^2 - 4x - 6 = 0$

Solution

$$\text{possibilities: } \pm \left\{ \frac{6}{4} \right\} = \pm \left\{ 1, 2, 3, 6, \frac{1}{2}, \frac{3}{2}, \frac{1}{4}, \frac{3}{4} \right\}$$

$$\begin{array}{r|rrrrr}
 3 & 4 & -35 & 71 & -4 & -6 \\
 & & 12 & -69 & 6 & 6 \\
 \hline
 -\frac{1}{4} & 4 & -23 & 2 & 2 & 0 \\
 & & -1 & 6 & -2 & \\
 \hline
 & 4 & -24 & 8 & 0 &
 \end{array} \rightarrow 4x^3 - 23x^2 + 2x + 2 = 0 \rightarrow \pm\left\{\frac{2}{4}\right\} = \pm\left\{1, 2, \frac{1}{2}, \frac{1}{4}\right\}$$

$$\rightarrow 4x^2 - 24x + 8 = 0$$

$$x^2 - 6x + 2 = 0$$

$$x = \frac{6 \pm \sqrt{36 - 8}}{2}$$

$$= \frac{6 \pm 2\sqrt{7}}{4}$$

The solutions are: $x = -\frac{1}{4}, 3, 3 \pm \sqrt{7}$

Exercise

Find all solutions of the equation: $2x^4 + 3x^3 - 4x^2 - 3x + 2 = 0$

Solution

$$\text{possibilities: } \pm\left\{\frac{2}{2}\right\} = \pm\left\{1, 2, \frac{1}{2}\right\}$$

$$\begin{array}{r|rrrrr}
 1 & 2 & 3 & -4 & -3 & 2 \\
 & & 2 & 5 & 1 & -2 \\
 \hline
 -1 & 2 & 5 & 1 & -2 & 0 \\
 & & -2 & -3 & 2 & \\
 \hline
 & 2 & 3 & -2 & 0 &
 \end{array} \rightarrow 2x^3 - 23x^2 + 2x - 2 = 0 \rightarrow \pm\left\{\frac{2}{2}\right\} = \pm\left\{1, 2, \frac{1}{2}\right\}$$

$$\rightarrow 2x^2 + 3x - 2 = 0$$

$$x = \frac{-3 \pm \sqrt{9 + 16}}{4}$$

$$= \begin{cases} \frac{-3 - 5}{4} = -2 \\ \frac{-3 + 5}{4} = \frac{1}{2} \end{cases}$$

The solutions are: $x = -2, -1, \frac{1}{2}, 1$

Exercise

Find all solutions of the equation: $x^4 + 3x^3 - 30x^2 - 6x + 56 = 0$

Solution

$$\text{possibilities for } \frac{c}{d} : \pm\left\{\frac{56}{1}\right\} = \pm\{1, 2, 4, 7, 8, 14, 28, 56\}$$

$$\begin{array}{r|rrrrr}
 4 & 1 & 3 & -3 & -6 & 56 \\
 & & 4 & 28 & -8 & -56 \\
 \hline
 -7 & 1 & 7 & -2 & -14 & 0 \\
 & & -7 & 0 & 14 & \\
 \hline
 & 1 & 0 & -2 & &
 \end{array} \rightarrow x^3 + 7x^2 - 2x - 14 = 0 \Rightarrow \frac{c}{d} = \pm \left\{ \frac{14}{1} \right\} = \pm \{1, 2, 7, 14\}$$

$$\rightarrow x^2 - 2 = 0$$

$$\Rightarrow x = \pm\sqrt{2}$$

The solutions are: $x = 4, -7, \pm\sqrt{2}$

Exercise

Find all solutions of the equation: $3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{6}{3} \right\} = \pm \left\{ 1, 2, 3, 6, \frac{1}{3}, \frac{2}{3} \right\}$

$$\begin{array}{r|rrrrrr}
 -1 & 3 & -10 & -6 & 24 & 11 & -6 \\
 & & -3 & 13 & -7 & -17 & 6 \\
 \hline
 -1 & 3 & -13 & 7 & 17 & -6 & 0 \\
 & & -3 & 16 & -23 & 6 & \\
 \hline
 2 & 3 & -16 & 23 & -6 & 0 & \\
 & & 6 & 20 & 6 & & \\
 \hline
 & 3 & -10 & 3 & 0 & &
 \end{array} \quad \begin{array}{l} x^4 - 13x^3 + 7x^2 + 17x - 6 = 0 \rightarrow \pm \left\{ \frac{6}{3} \right\} = \pm \left\{ 1, 2, 3, 6, \frac{1}{3}, \frac{2}{3} \right\} \\ \\ 3x^3 - 16x^2 + 26x - 6 = 0 \rightarrow \pm \left\{ 1, 2, 3, 6, \frac{1}{3}, \frac{2}{3} \right\} \\ \\ 3x^2 - 10x + 3 = 0 \end{array}$$

$$x = \frac{10 \pm \sqrt{100 - 36}}{6}$$

$$= \begin{cases} \frac{10-8}{6} = \frac{1}{3} \\ \frac{10+8}{6} = 3 \end{cases}$$

The solutions are: $x = -1, -1, \frac{1}{3}, 2, 3$

Exercise

Find all solutions of the equation: $6x^5 + 19x^4 + x^3 - 6x^2 = 0$

Solution

$$x^2 (6x^3 + 19x^2 + x - 6) = 0 \rightarrow x = 0, 0$$

$$6x^3 + 19x^2 + x - 6 = 0$$

possibilities for $\frac{c}{d} : \pm \left\{ \frac{6}{6} \right\} = \pm \left\{ 1, 2, 3, 6, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{2}{3} \right\}$

$$\begin{array}{r|rrrr} -3 & 6 & 19 & 1 & -6 \\ & & -18 & -3 & 6 \\ \hline & 6 & 1 & -2 & \boxed{0} \end{array} \quad 6x^2 + x - 2 = 0$$

$$x = \frac{-1 \pm \sqrt{1+48}}{12}$$

$$= \begin{cases} \frac{-1-7}{12} = -\frac{2}{3} \\ \frac{-1+7}{12} = \frac{1}{2} \end{cases}$$

The solutions are: $x = 0, 0, -\frac{2}{3}, -3, \frac{1}{2}$

Exercise

Find all solutions of the equation: $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = 0$

Solution

$$x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = (x+1)^5 = 0$$

possibilities for $\frac{c}{d} : \pm \{1\}$

$$\begin{array}{r|rrrrrr} -1 & 1 & 5 & 10 & 10 & 5 & 1 \\ & & -1 & -4 & -6 & -4 & -1 \\ \hline -1 & 1 & 4 & 6 & 4 & 1 & \boxed{0} \\ & & -1 & -3 & -3 & -1 \\ \hline -1 & 1 & 3 & 3 & 1 & \boxed{0} \\ & & -1 & -2 & -1 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array} \quad \begin{aligned} & \rightarrow x^4 + 4x^3 + 6x^2 + 4x + 1 = 0 \rightarrow \pm \{1\} \\ & \rightarrow x^3 + 3x^2 + 3x + 1 = 0 \rightarrow \pm \{1\} \\ & \rightarrow x^2 + 2x + 1 = 0 \end{aligned}$$

$$x^2 + 2x + 1 = (x+1)^2$$

The solutions are: $x = -1, -1, -1, -1, -1$

Exercise

Find all solutions of the equation: $x^5 - x^4 - 7x^3 + 7x^2 + 12x - 12 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\{1, 2, 3, 4, 6, 12\}$

1	1	-1	-7	7	12	-12	
		1	0	-7	0	12	
2	1	0	-7	0	12	0	$\rightarrow x^4 - 7x^2 - 12 = 0 \rightarrow \pm\{1, 2, 3, 4, 6, 12\}$
		2	4	-6	-12		
-2	1	2	-3	-6	0		$\rightarrow x^3 + 2x^2 - 3x - 6 = 0 \rightarrow \pm\{1, 2, 3, 6\}$
		-2	0	6			
	1	0	-3	0			$\rightarrow x^2 - 3 = 0$

$$x^2 = 3$$

The solutions are: $x = -2, 1, 2, \pm\sqrt{3}$

Exercise

Find all solutions of the equation: $x^5 - 2x^3 - 8x = 0$

Solution

$$x(x^4 - 2x^2 - 8) = 0$$

$$\underline{x = 0}$$

$$x^4 - 2x^2 - 8 = 0$$

$$x^2 = \frac{2 \pm \sqrt{4 + 32}}{2}$$

$$= \begin{cases} \frac{2-6}{2} = -2 \\ \frac{2+6}{2} = 4 \end{cases}$$

$$\begin{cases} x^2 = -2 \rightarrow x = \pm i\sqrt{2} \\ x^2 = 4 \rightarrow x = \pm 2 \end{cases}$$

The solutions are: $x = 0, \pm 2, \pm i\sqrt{2}$

Exercise

Find all solutions of the equation: $x^5 - 32 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \{1, 2, 4, 8, 16, 32\}$

$$\begin{array}{r|rrrrrr} 2 & 1 & 0 & 0 & 0 & 0 & -32 \\ & & 2 & 4 & 8 & 16 & 32 \\ \hline & 1 & 2 & 4 & 8 & 16 & 0 \end{array} \rightarrow x^4 + 2x^3 + 4x^2 + 8x + 16 = 0 \rightarrow \pm \{1, 2, 4, 8, 16\}$$

$$x^4 + 2x^3 + 4x^2 + 8x + 16 = 0$$

Cannot be solved using rational zero theorem.

Therefore; using program

$$\text{The solutions are: } x = 2, \frac{-1 - \sqrt{5} \pm i\sqrt{2}\sqrt{5 - \sqrt{5}}}{2}, \frac{-1 + \sqrt{5} \pm i\sqrt{2}\sqrt{5 - \sqrt{5}}}{2}$$

Exercise

Find all solutions of the equation: $3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{24}{3} \right\} = \pm \left\{ 1, 2, 3, 4, 6, 8, 12, 24, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3} \right\}$

$$\begin{array}{r|rrrrrrrr} 1 & 3 & -10 & -29 & 34 & 50 & -24 & -24 \\ & & 3 & -7 & -36 & -2 & 48 & 24 \\ \hline -1 & 3 & -7 & -36 & -2 & 48 & 24 & 0 \\ & & -3 & 10 & 26 & -24 & -24 & \\ \hline -2 & 3 & -10 & -26 & 24 & 24 & 0 & \\ & & -6 & 32 & -12 & -24 & & \\ \hline -\frac{2}{3} & 3 & -16 & 6 & 12 & 0 & & \\ & & -2 & 12 & -12 & & & \\ \hline & 3 & -18 & 18 & 0 & & & \end{array} \rightarrow \begin{array}{l} 3x^5 - 7x^4 - 36x^3 - 2x^2 + 48x + 24 = 0 \\ 3x^4 - 10x^3 - 26x^2 + 24x + 24 = 0 \\ 3x^3 - 16x^2 + 12x - 12 = 0 \\ 3x^2 - 18x + 18 = 0 \end{array}$$

$$x^2 - 6x + 6 = 0$$

$$\begin{aligned} x &= \frac{6 \pm \sqrt{36 - 24}}{2} \\ &= \frac{6 \pm 2\sqrt{3}}{2} \end{aligned}$$

$$\text{The solutions are: } x = -2, -1, 1, -\frac{2}{3}, 3 \pm \sqrt{3}$$

Exercise

Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k) = \frac{1}{6}(k^3 + 3k^2 + 2k)$$

Where k is the number of levels in the pyramid. If 220 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

Solution

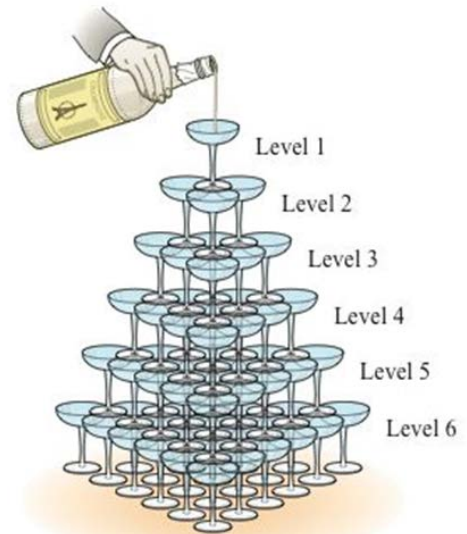
$$\frac{1}{6}(k^3 + 3k^2 + 2k) = 220$$

$$k^3 + 3k^2 + 2k - 1,320 = 0$$

$$\begin{array}{r|rrrr} 10 & 1 & 3 & 2 & -1320 \\ & & 10 & 130 & 1320 \\ \hline & 1 & 13 & 132 & 0 \end{array} \rightarrow k^2 + 13k + 132 = 0$$

$$k = \frac{-13 \pm \sqrt{-359}}{2} \quad \text{C}$$

The are **10** levels in the pyramid.



Exercise

Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k) = \frac{1}{6}(2k^3 + 3k^2 + k)$$

Where k is the number of levels in the pyramid. If 140 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

Solution

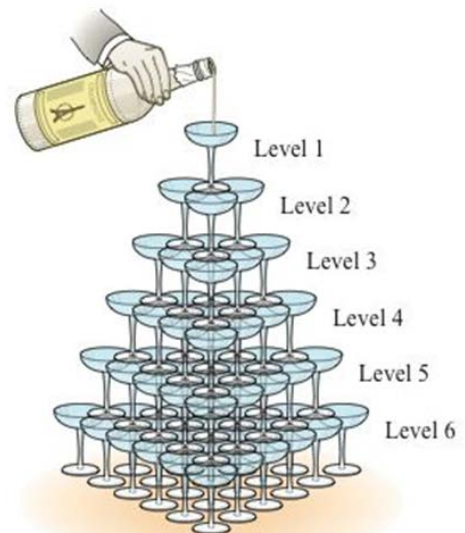
$$\frac{1}{6}(2k^3 + 3k^2 + k) = 150$$

$$2k^3 + 3k^2 + k - 840 = 0$$

$$\begin{array}{r|rrrr} 7 & 2 & 3 & 1 & -840 \\ & & 14 & 119 & 840 \\ \hline & 2 & 17 & 120 & 0 \end{array} \rightarrow 2k^2 + 17k + 120 = 0$$

$$k = \frac{-17 \pm \sqrt{-671}}{4} \quad \text{C}$$

The are **7** levels in the pyramid.



Exercise

A carbon dioxide cartridge for a paintball rifle has the shape of a right circular cylinder with a hemisphere at each end. The cylinder is 4 inches long, and the volume of the cartridge is $2\pi \text{ in}^3$.

The common interior radius of the cylinder and the hemispheres is denoted by x . Estimate the length of the radius x .

Solution

$$\text{Volume of the Cartridge} = 2 \times (\text{Volume of Hemisphere}) + \text{Volume of Cylinder}$$

$$\text{Volume of Sphere} = \frac{4}{3}\pi x^3$$

$$\text{Volume of Cylinder} = 4\pi x^2$$

$$\text{Volume of Cartridge} = \frac{4}{3}\pi x^3 + 4\pi x^2$$

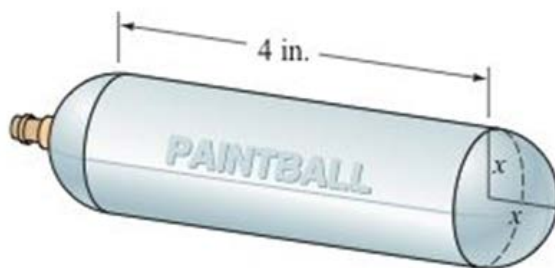
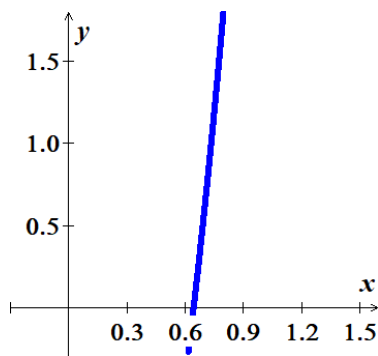
$$2\pi = \frac{4}{3}\pi x^3 + 4\pi x^2$$

$$2x^3 + 6x^2 = 3$$

$$2x^3 + 6x^2 - 3 = 0$$

Using Graph:

$$x \approx 0.64 \text{ in.}$$



Exercise

A propane tank has the shape of a circular cylinder with a hemisphere at each end. The cylinder is 6 feet long and the volume of the tank is $9\pi \text{ ft}^3$. Find the length of the radius x .

Solution

$$\text{Volume of the Cartridge} = 2 \times (\text{Volume of Hemisphere}) + \text{Volume of Cylinder}$$

$$\text{Volume of Sphere} = \frac{4}{3}\pi x^3$$

$$\text{Volume of Cylinder} = 6\pi x^2$$

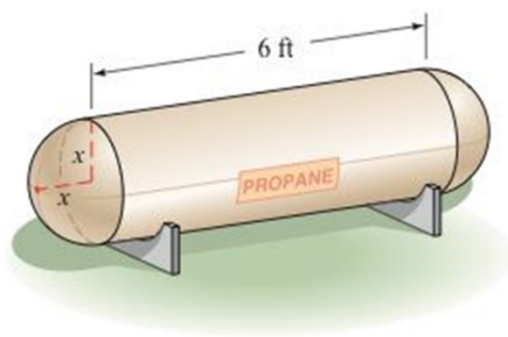
$$\text{Volume of Cartridge} = \frac{4}{3}\pi x^3 + 6\pi x^2$$

$$9\pi = \frac{4}{3}\pi x^3 + 6\pi x^2$$

$$27 = 4x^3 + 18x^2$$

$$4x^3 + 18x^2 - 27 = 0$$

$$\text{possibilities for } \frac{c}{d} : \pm \left\{ \frac{27}{4} \right\} = \pm \left\{ 1, 3, 9, 27, \frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{27}{2}, \frac{1}{4}, \frac{3}{4}, \frac{9}{4}, \frac{27}{4} \right\}$$



$$\begin{array}{c|cccc} -\frac{3}{2} & 4 & 18 & 0 & -27 \\ & & -6 & -18 & 27 \\ \hline & 4 & 12 & -18 & 0 \end{array} \rightarrow 4x^2 + 12x - 18 = 0$$

$$2x^2 + 6x - 9 = 0$$

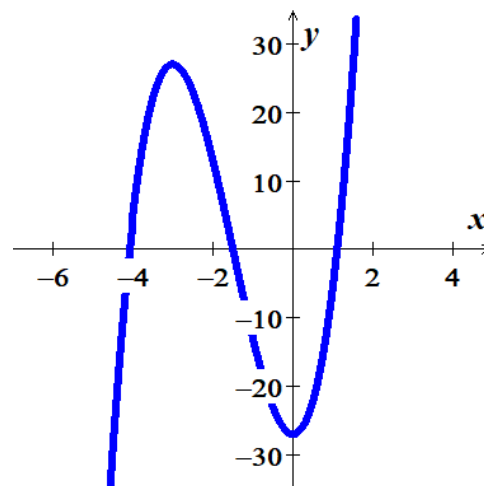
$$x = \frac{-6 \pm \sqrt{36 + 72}}{4}$$

$$= \frac{-6 \pm 6\sqrt{3}}{4}$$

$$= \frac{-3 \pm 3\sqrt{3}}{2}$$

$$x = \left[-\frac{3}{2}, \frac{-3-3\sqrt{3}}{2}, \frac{-3+3\sqrt{3}}{2} \right]$$

\therefore the length of the radius x is $\frac{-3+3\sqrt{3}}{2} \approx 1.1$ foot



Exercise

A cube measures n inches on each edge. If a slice 2 inches thick is cut from one face of the cube, the resulting solid has a volume of 567 in^3 . Find n .

Solution

$$\text{Volume} = n^2(n-2)$$

$$n^3 - 2n^2 = 567$$

$$n^3 - 2n^2 - 567 = 0$$

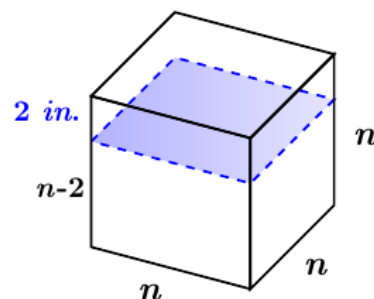
possibilities for $\frac{c}{d} := \pm\{1, 3, 7, 9, 21, 27, 63, 81, 189, 567\}$

$$\begin{array}{c|cccc} 9 & 1 & -2 & 0 & -567 \\ & & 9 & 63 & 567 \\ \hline & 1 & 7 & 63 & 0 \end{array} \rightarrow n^2 + 7n + 63 = 0$$

$$n = \frac{-7 \pm \sqrt{49 - 252}}{2}$$

$$= \frac{-7 \pm i\sqrt{203}}{2} \quad \times$$

$$\therefore n = 9$$



Exercise

A cube measures n inches on each edge. If a slice 1 inch thick is cut from one face of the cube and then a slice 3 inches thick is cut from another face of the cube, the resulting solid has a volume of 1560 in^3 . Find the dimensions of the original cube.

Solution

$$\text{Volume} = n(n-1)(n-3)$$

$$n^3 - 4n^2 + 3n = 1560$$

$$n^3 - 4n^2 + 3n - 1560 = 0$$

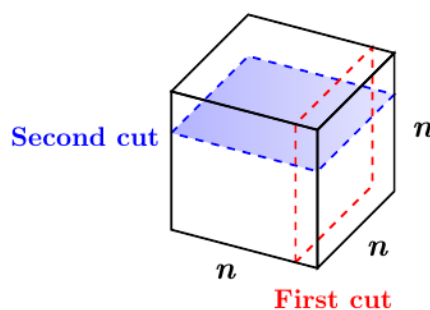
$$\text{possibilities for } \frac{c}{d} := \pm \left\{ \begin{array}{l} 1, 2, 4, 5, 6, 8, 10, 12, 13, 15, 20, 24, 26, 30, 39, \\ 40, 52, 60, 65, 78, 104, 120, 130, 156, 195, 260, 312, 390, 780, 1560 \end{array} \right\}$$

$$\begin{array}{r|rrrr} 13 & 1 & -4 & 3 & -1560 \\ & & 13 & 117 & 1560 \\ \hline & 1 & 9 & 120 & 0 \end{array} \rightarrow n^2 + 9n + 120 = 0$$

$$n = \frac{-9 \pm \sqrt{81 - 480}}{2}$$

$$= \frac{-9 \pm i\sqrt{399}}{2} \quad \times$$

$$\therefore n = 13$$



Exercise

For what value of x will the volume of the following solid be 112 in^3

Solution

$$\text{Volume of the bottom portion} = x^2(x+1)$$

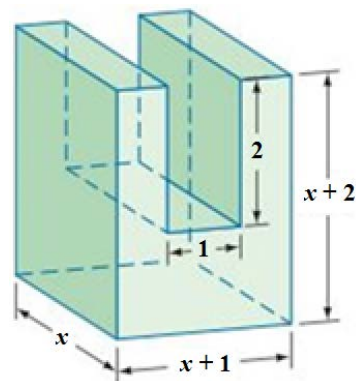
$$\begin{aligned} \text{Volume of one side portion} &= 2x\left(\frac{1}{2}x\right) \\ &= x^2 \end{aligned}$$

$$\text{Total Volume} = x^2(x+1) + 2x^2$$

$$112 = x^3 + 3x^2$$

$$x^3 + 3x^2 - 112 = 0$$

$$\text{possibilities for } \frac{c}{d} := \pm \{1, 2, 4, 8, 14, 28, 56, 112\}$$



$$\begin{array}{c|cccc} 4 & 1 & 3 & 0 & -112 \\ & & 4 & 28 & 112 \\ \hline & 1 & 7 & 28 & 0 \end{array} \rightarrow x^2 + 7x + 28 = 0$$

$$x = \frac{-7 \pm \sqrt{49 - 112}}{2}$$

$$= \frac{-7 \pm 3i\sqrt{7}}{2} \quad \times$$

$$\therefore \underline{x = 4}$$

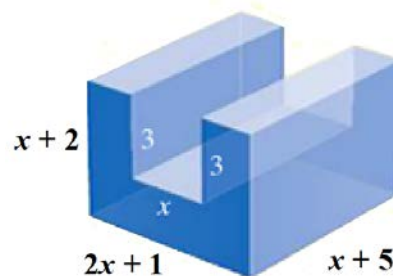
Exercise

For what value of x will the volume of the following solid be 208 in^3

Solution

$$\begin{aligned} \text{Volume of the bottom portion} &= (2x+1)(x+5)(x+2-3) \\ &= (2x^2 + 11x + 5)(x-1) \\ &= 2x^3 + 11x^2 + 5x - 2x^2 - 11x - 5 \\ &= 2x^3 + 9x^2 - 6x - 5 \end{aligned}$$

$$\begin{aligned} \text{Volume of one side portion} &= (3)\frac{1}{2}(2x+1-x)(x+5) \\ &= \frac{3}{2}(x+1)(x+5) \\ &= \frac{3}{2}(x^2 + 6x + 5) \end{aligned}$$



$$\text{Total Volume} = 2x^3 + 9x^2 - 6x - 5 + 2\left(\frac{3}{2}\right)(x^2 + 6x + 5)$$

$$208 = 2x^3 + 9x^2 - 6x - 5 + 3x^2 + 18x + 15$$

$$2x^3 + 12x^2 + 12x - 198 = 0$$

$$x^3 + 6x^2 + 6x - 99 = 0$$

possibilities for $\frac{c}{d} := \pm\{1, 3, 9, 11, 33, 99\}$

$$\begin{array}{c|cccc} 3 & 1 & 6 & 6 & -99 \\ & & 3 & 27 & 99 \\ \hline & 1 & 9 & 33 & 0 \end{array} \rightarrow x^2 + 9x + 33 = 0$$

$$x = \frac{-9 \pm \sqrt{81 - 132}}{2}$$

$$= \frac{-9 \pm i\sqrt{51}}{2} \quad \times$$

$$\therefore \underline{x = 3}$$

Exercise

The length of rectangular box is 1 *inch* more than twice the height of the box, and the width is 3 *inches* more than the height. If the volume of the box is 126 in^3 , find the dimensions of the box.

Solution

$$\text{Volume} = x(2x+1)(x+3)$$

$$2x^3 + 7x^2 + 3x = 126$$

$$2x^3 + 7x^2 + 3x - 126 = 0$$

$$\text{possibilities for } \frac{c}{d} := \pm \left\{ \frac{126}{2} \right\}$$

$$= \pm \left\{ 1, 2, 3, 6, 9, 14, 21, 42, 63, 126, \frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{21}{2}, \frac{63}{2} \right\}$$

$$\begin{array}{r|rrrr} 3 & 2 & 7 & 3 & -126 \\ & & 6 & 39 & 126 \\ \hline & 2 & 13 & 42 & 0 \end{array} \rightarrow 2x^2 + 13x + 42 = 0$$

$$x = \frac{-13 \pm \sqrt{169 - 336}}{4}$$

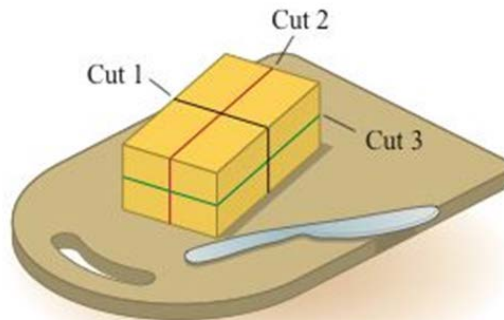
$$= \frac{-13 \pm i\sqrt{167}}{4} \quad \times$$

$$\therefore \underline{x=3}$$



Exercise

One straight cut through a thick piece of cheese produces two pieces. Two straight cuts can produce a maximum of four pieces. Two straight cuts can produce a maximum of four pieces. Three straight cuts can produce a maximum of eight pieces.



You might be inclined to think that every additional cut double number of pieces. However, for four straight cuts, you get a maximum of 15 pieces. The maximum number of pieces P that can be produced by n straight cuts is given by

$$P(n) = \frac{n^3 + 5n + 6}{6}$$

- a) Determine number of pieces that can be produced by five straight cuts.
 b) What is the fewest number of straight cuts that are needed to produce 64 pieces?

Solution

$$a) P(5) = \frac{5^3 + 25 + 6}{6} \\ = 26$$

$$b) \frac{n^3 + 5n + 6}{6} = 64$$

$$n^3 + 5n + 6 = 384$$

$$n^3 + 5n - 378 = 0$$

$$\text{possibilities for } \frac{c}{d} := \pm \{378\}$$

$$= \pm \{1, 2, 3, 6, 7, 9, 14, 18, 21, 27, 42, 54, 63, 126, 189, 378\}$$

$$\begin{array}{r|rrrr} 7 & 1 & 0 & 5 & -378 \\ & & 7 & 49 & 378 \\ \hline & 1 & 7 & 54 & 0 \end{array} \rightarrow n^2 + 7n + 54 = 0$$

$$n = \frac{-7 \pm \sqrt{49 - 216}}{2}$$

$$= \frac{-7 \pm i\sqrt{167}}{2} \quad \times$$

$$\therefore n = 7$$

Exercise

The number of ways one can select three cards from a group of n cards (the order of the selection matters), where $n \geq 3$, is given by $P(n) = n^3 - 3n^2 + 2n$. For a certain card trick, a magician has determined that there are exactly 504 ways to choose three cards from a given group. How many cards are in the group?

Solution

$$P(n) = n^3 - 3n^2 + 2n = 504$$

$$n^3 - 3n^2 + 2n - 504 = 0$$

$$\text{possibilities for } \frac{c}{d} := \pm \{504\}$$

$$= \pm \left\{ \begin{array}{l} 1, 2, 3, 4, 6, 7, 8, 9, 12, 14, 18, 21, \\ 24, 28, 36, 42, 56, 63, 72, 84, 126, 168, 252, 504 \end{array} \right\}$$

$$\begin{array}{r|rrrr} 9 & 1 & -3 & 2 & -504 \\ & & 9 & 54 & 504 \\ \hline & 1 & 6 & 56 & 0 \end{array} \rightarrow n^2 + 6n + 56 = 0$$

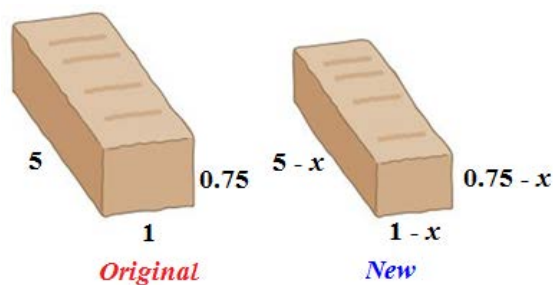
$$n = \frac{-6 \pm \sqrt{36 - 224}}{2}$$

$$= -3 \pm i\sqrt{47} \quad \times$$

$$\therefore \underline{n = 9}$$

Exercise

A nutrition bar in the shape of a rectangular solid measure 0.75 in. by 1 in. by 5 inches.



To reduce costs, the manufacturer has decided to decrease each of the dimensions of the nutrition bar by x inches, what value of x will produce a new bar with a volume that is 0.75 in^3 less than the present bar's volume.

Solution

$$\begin{aligned} V_{\text{original}} &= (5)(1)\left(\frac{3}{4}\right) \\ &= \frac{15}{4} \end{aligned}$$

$$V_{\text{new}} = (5-x)(1-x)\left(\frac{3}{4}-x\right) \quad \left(x < \frac{3}{4}\right)$$

$$\left(5-6x+x^2\right)\left(\frac{3-4x}{4}\right) = \frac{15}{4} - \frac{3}{4}$$

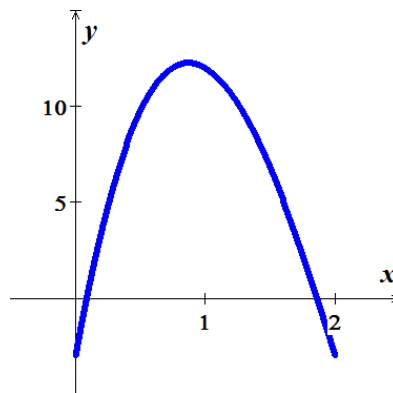
$$15 - 20x - 18x + 24x^2 + 3x^2 - 4x^3 = 4(3)$$

$$4x^3 - 27x^2 + 38x - 3 = 0$$

From graph table:

$$\begin{array}{rr} 0.08200 & -0.06334 \\ 0.08400 & 0.00386 \end{array}$$

$$\underline{x \approx 0.083 \text{ in.}}$$



Exercise

A rectangular box is square on two ends and has length plus girth of 81 *inches*. (Girth: distance *around* the box). Determine the possible lengths l ($l > w$) of the box if its volume is 4900 in^3 .

Solution

$$81 = l + 4w$$

$$l = 81 - 4w$$

$$V = lw^2$$

$$= (81 - 4w)w^2$$

$$-4w^3 + 81w^2 = 4900$$

$$4w^3 - 81w^2 + 4900 = 0$$

$$\text{possibilities for } \frac{c}{d} := \pm \left\{ \frac{4900}{4} \right\} = \pm \left\{ \begin{array}{l} 1, 2, 4, 7, 10, 14, 20, 28, 49, \\ 100, 175, 245, 350, 490, 700, 1225, 2450, 4900, \dots \end{array} \right\}$$

$$\begin{array}{r|rrrr} 14 & 4 & -81 & 0 & 4900 \\ & & 56 & -350 & -4900 \\ \hline & 4 & -25 & -350 & 0 \end{array} \rightarrow 4w^2 - 25w - 350 = 0$$

$$w = \frac{25 \pm \sqrt{625 + 5600}}{8}$$

$$= \frac{25 \pm 5\sqrt{249}}{8}$$

$$= \left\{ \begin{array}{l} \frac{25 - 5\sqrt{249}}{8} < 0 \\ \frac{25 + 5\sqrt{249}}{8} \approx 13 \end{array} \right.$$

$$l = 81 - 4(14) = 25$$

$$l = 81 - 4(13) = 29$$

\therefore the possible lengths l are around **25 in.** *or* **29 in.**

