Solution Section 3.1 – Integrals over Rectangular Regions

Exercise

Evaluate the iterated integral $\int_1^2 \int_0^4 2xy\ dydx$

Solution

$$\int_1^2 \int_0^4 2xy\ dydx = \int_1^2 x\left[y^2\right]_0^4 dx$$

$$= \int_1^2 16x\ dx$$

$$= 8\left(x^2\ \Big|_1^2\right.$$

$$= 8(4-1)$$

$$\underline{= 24}$$

Exercise

Evaluate the iterated integral $\int_0^2 \int_{-1}^1 (x-y)\ dydx$

Solution

$$\int_0^2 \int_{-1}^1 (x-y)\ dydx = \int_0^2 \left[xy - \frac{1}{2}y^2\right]_{-1}^1 dx$$

$$= \int_0^2 \left[x - \frac{1}{2} - \left(-x - \frac{1}{2}\right)\right] dx$$

$$= \int_0^2 2x\ dx$$

$$= x^2\ \Big|_0^2$$

$$\underline{= 4}$$

Exercise

Evaluate the iterated integral $\int_0^1 \int_0^1 \left(1 - \frac{x^2+y^2}{2}\right) dxdy$

Solution

$$\int_0^1 \int_0^1 \left(1 - \frac{x^2+y^2}{2}\right) dxdy = \int_0^1 \left[x - \frac{1}{6}x^3 - \frac{1}{2}y^2 x\right]_0^1 dy$$

$$= \int_0^1 \left(1 - \frac{1}{6} - \frac{1}{2}y^2\right) dy$$

$$= \int_0^1 \left(\frac{5}{6} - \frac{1}{2}y^2\right) dy$$

$$= \frac{5}{6}y - \frac{1}{6}y^3 \Big|_0^1$$

$$= \frac{5}{6} - \frac{1}{6}$$

$$= \frac{4}{6}$$

$$= \frac{2}{3}$$

Exercise

Evaluate the integral $\int_0^3 \int_{-2}^0 \left(x^2 y - 2xy\right) dydx$

Solution

$$\int_0^3 \int_{-2}^0 \left(x^2 y - 2xy\right) dydx = \int_0^3 \left(\frac{1}{2}x^2 y^2 - xy^2 \Big|_{-2}^0\right. dx$$

$$= \int_0^3 \left(-2x^2 + 4x\right) dx$$

$$= -\frac{2}{3}x^3 + 2x^2 \Big|_0^3$$

$$= -18 + 18$$

$$= 0$$

Exercise

Evaluate the integral $\int_0^1 \int_0^1 \frac{y}{1+xy}\, dxdy$

Solution

$$\int_0^1 \int_0^1 \frac{y}{1+xy} dxdy = \int_0^1 \int_0^1 \frac{d(1+xy)}{1+xy} dy \qquad d(1+xy) = ydx$$

$$= \int_0^1 \left(\ln|1+xy| \, \Big|_0^1 \right) dy$$

$$= \int_0^1 \ln|1+y| \; dy \qquad d(1+y) = dy$$

$$= (y+1)\ln|1+y| - (y+1) \, \Big|_0^1 \qquad \int \ln u \; du = u\ln u - u$$

$$= 2\ln 2 - 2 + 1$$

$$= 2\ln 2 - 1$$

Exercise

Evaluate the integral $\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y} dydx$

Solution

$$\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y} dydx = \int_0^{\ln 2} e^{2x} dx \int_1^{\ln 5} e^{y} dy$$

$$= \left(\frac{1}{2} e^{2x} \, \Big|_0^{\ln 2} \right) \left(e^{y} \, \Big|_1^{\ln 5} \right)$$

$$= \frac{1}{2}\left(e^{2\ln 2} - 1 \right)\left(e^{\ln 5} - e \right)$$

$$= \frac{1}{2}(4-1)(5-e)$$

$$= \frac{15}{2} - \frac{3}{2}e$$

Exercise

Evaluate the integral $\displaystyle\int_0^1\int_1^2 xye^x\ dydx$

Solution

$$\int_0^1\int_1^2 xye^x dydx = \int_0^1 xe^x\left(\frac{1}{2}y^2\ \Big|_1^5\ dx\right.$$

$$= \frac{3}{2}\int_0^1 xe^x dx$$

$$= \frac{3}{2}\left(xe^x - e^x\ \Big|_0^1\right.$$

$$= \frac{3}{2}(e - e + 1)$$

$$\boxed{= \frac{3}{2}}$$

Exercise

Evaluate the double integral $\displaystyle\int_\pi^{2\pi}\int_0^\pi (\sin x + \cos y)dxdy$

Solution

$$\int_\pi^{2\pi}\int_0^\pi (\sin x + \cos y)dxdy = \int_\pi^{2\pi}\left(-\cos x + x\cos y\ \Big|_0^\pi\ dy\right.$$

$$= \int_\pi^{2\pi}(1 + \pi\cos y + 1)dy$$

$$= \left(2y + \pi\sin y\ \Big|_\pi^{2\pi}\right.$$

$$= 4\pi - 2\pi$$

$$\boxed{= 2\pi}$$

Exercise

Evaluate the double integral $\displaystyle\int_1^2 \int_1^4 \frac{xy}{\left(x^2+y^2\right)^2}\, dxdy$

Solution

$$\int_1^2 \int_1^4 \frac{xy}{\left(x^2+y^2\right)^2} dxdy = \frac{1}{2}\int_1^2 \int_1^4 \frac{y}{\left(x^2+y^2\right)^2} d\left(x^2+y^2\right)dy$$

$$= -\frac{1}{2}\int_1^2 \frac{y}{x^2+y^2}\Big|_1^4 \ dy$$

$$= -\frac{1}{2}\int_1^2 \left(\frac{y}{16+y^2} - \frac{y}{1+y^2}\right) dy$$

$$= -\frac{1}{4}\int_1^2 \frac{d\left(16+y^2\right)}{16+y^2} + \frac{1}{4}\int_1^2 \frac{d\left(1+y^2\right)}{1+y^2}$$

$$= -\frac{1}{4}\ln\left(16+y^2\right) + \frac{1}{4}\ln\left(1+y^2\right)\Big|_1^2$$

$$= \frac{1}{4}\left(-\ln 20 + \ln 17 + \ln 5 - \ln 2\right)$$

$$= \frac{1}{4}\ln\left(\frac{17\times 5}{20\times 2}\right)$$

$$= \frac{1}{4}\ln\left(\frac{17}{8}\right)$$

Exercise

Evaluate the double integral $\displaystyle\int_1^3 \int_1^{e^x} \frac{x}{y}\, dydx$

Solution

$$\int_1^3 \int_1^{e^x} \frac{x}{y}\, dydx = \int_1^3 x\ln y\Big|_1^{e^x} \ dx$$

$$= \int_1^3 x(x)\,dx$$

$$= \frac{1}{3}x^3 \Big|_1^3$$

$$\boxed{= \frac{26}{3}}$$

Exercise

Evaluate the double integral $\displaystyle\int_1^2 \int_0^{\ln x} x^3 e^y \, dydx$

Solution

$$\int_1^2 \int_0^{\ln x} x^3 e^y \, dydx = \int_1^2 x^3 e^y \Big|_0^{\ln x} dx$$

$$= \int_1^2 x^3 (x-1)\, dx$$

$$= \int_1^2 \left(x^4 - x^3\right) dx$$

$$= \frac{1}{5}x^5 - \frac{1}{4}x^4 \Big|_1^2$$

$$= \frac{32}{5} - 4 - \frac{1}{5} + \frac{1}{4}$$

$$= \frac{31}{5} - \frac{15}{4}$$

$$\boxed{= \frac{49}{20}}$$

Exercise

Evaluate the double integral $\displaystyle\int_1^{10} \int_0^{1/y} ye^{xy} \, dxdy$

Solution

$$\int_1^{10} \int_0^{1/y} ye^{xy} \, dxdy = \int_1^{10} \int_0^{1/y} e^{xy} d\left(e^{xy}\right) dy \qquad d\left(e^{xy}\right) = ye^{xy} dx$$

$$= \int_1^{10} e^{xy} \Big|_0^{1/y} dy$$

$$= \int_1^{10} (e-1)\ dy$$

$$= (e-1)y \Big|_1^{10}$$

$$= (e-1)(10-1)$$

$$\underline{= 9(e-1)}$$

Exercise

Evaluate the double integral $\displaystyle\int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} xy\ dxdy$

Solution

$$\int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} xy\ dxdy = \frac{1}{2}\int_0^1 yx^2 \Big|_{\sqrt{y}}^{2-\sqrt{y}}\ dy$$

$$= \frac{1}{2}\int_0^1 y\left(\left(2-\sqrt{y}\right)^2 - y\right)\ dy$$

$$= \frac{1}{2}\int_0^1 y\left(4-4\sqrt{y}+y-y\right)dy$$

$$= 2\int_0^1 \left(y - y^{3/2}\right)dy$$

$$= 2\left(\frac{1}{2}y^2 - \frac{2}{5}y^{5/2}\right)\Big|_0^1$$

$$= 2\left(\frac{1}{2}-\frac{2}{5}\right)$$

$$\underline{= \frac{1}{5}}$$

Exercise

Evaluate the double integral $\displaystyle\int_0^1 \int_{x^2}^{x} \sqrt{x}\ dydx$

Solution

$$\int_0^1 \int_{x^2}^{x} \sqrt{x}\ dydx = \int_0^1 x^{1/2}\ y \Big|_{x^2}^{x}\ dx$$

$$= \int_0^1 x^{1/2}\left(x - x^2\right)\ dx$$

$$= \int_0^1 \left(x^{3/2} - x^{5/2}\right)\ dx$$

$$= \frac{2}{5}x^{5/2} - \frac{2}{7}x^{7/2}\ \Big|_0^1$$

$$= \frac{2}{5} - \frac{2}{7}$$

$$= \frac{4}{35}$$

Exercise

Evaluate the double integral $\displaystyle\int_0^{3/2} \int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y\ dxdy$

Solution

$$\int_0^{3/2} \int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y\ dxdy = \int_0^{3/2} yx\ \Big|_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}}\ dy$$

$$= 2\int_0^{3/2} y\ \sqrt{9-4y^2}\ dy$$

$$= -\frac{1}{4}\int_0^{3/2} \left(9-4y^2\right)^{1/2}\ d\left(9-4y^2\right)$$

$$= -\frac{1}{6}\left(9-4y^2\right)^{3/2}\ \Big|_0^{3/2}$$

$$= -\frac{1}{6}(-27)$$

$$= \frac{9}{2}$$

Exercise

Evaluate the double integral $\int_0^2 \int_0^{4-x^2} 2x\, dydx$

Solution

$$\int_0^2 \int_0^{4-x^2} 2x\, dydx = \int_0^2 2xy \Big|_0^{4-x^2} dx$$

$$= \int_0^2 \left(8x - 2x^3\right) dx$$

$$= 4x^2 - \frac{1}{2}x^4 \Big|_0^2$$

$$= 16 - 8$$

$$\underline{= 8}$$

Exercise

Evaluate the double integral $\int_0^1 \int_{2y}^{2} 4\cos\left(x^2\right) dxdy$

Solution

$$x = 2y \quad \to \quad y = \frac{x}{2}$$

$$\int_0^1 \int_{2y}^{2} 4\cos\left(x^2\right) dxdy = \int_0^2 \int_0^{x/2} 4\cos\left(x^2\right) dydx$$

$$= \int_0^2 4\cos\left(x^2\right) y \Big|_0^{x/2} dx$$

$$= \int_0^2 2x\cos\left(x^2\right) dx$$

$$= \int_0^2 \cos x^2 \; d\left(x^2\right)$$

$$= \sin x^2 \Big|_0^2$$

$$\underline{= \sin 4}$$

Exercise

Evaluate the double integral $\displaystyle\int_0^1\int_{\sqrt[3]{y}}^1 \frac{2\pi\sin\pi x^2}{x^2}\,dxdy$

Solution

$$x=\sqrt[3]{y}\ \rightarrow\ y=x^3$$

$$\int_0^1\int_{\sqrt[3]{y}}^1 \frac{2\pi\sin\pi x^2}{x^2}\,dxdy=\int_0^1\int_0^{x^3} \frac{2\pi\sin\pi x^2}{x^2}\,dydx$$

$$=\int_0^1 \frac{2\pi\sin\pi x^2}{x^2}\,y\,\Big|_0^{x^3}\,dx$$

$$=\int_0^1 2\pi x\sin\pi x^2\,dx$$

$$=\int_0^1 \sin\pi x^2\,d\left(\pi x^2\right)$$

$$=-\cos\pi x^2\,\Big|_0^1$$

$$=-(\cos\pi-\cos 0)$$

$$=2$$

Exercise

Evaluate the double integral over the given region R $\displaystyle\iint_R\left(6y^2-2x\right)dA\quad R:\ 0\le x\le 1,\quad 0\le y\le 2$

Solution

$$\iint_R\left(6y^2-2x\right)dA=\int_0^1\int_0^2\left(6y^2-2x\right)dydx$$

$$=\int_0^1\left(2y^3-2xy\,\Big|_0^2\right.dx$$

$$=\int_0^1(16-4x)dx$$

$$=16x-2x^2\,\Big|_0^1$$

$$=14$$

Exercise

Evaluate the double integral over the given region R $\displaystyle\iint_R \left(\frac{\sqrt{x}}{y^2}\right) dA \quad R: \ 0 \le x \le 4, \ \ 1 \le y \le 2$

Solution

$$\iint_R \left(\frac{\sqrt{x}}{y^2}\right) dA = \int_0^4 \int_1^2 \left(\frac{\sqrt{x}}{y^2}\right) dydx$$

$$= \int_0^4 \left(-\frac{\sqrt{x}}{y} \Big|_1^2\right) dx$$

$$= \int_0^4 -\sqrt{x}\left(\frac{1}{2}-1\right) dx$$

$$= \frac{1}{2}\int_0^4 x^{1/2}\, dx$$

$$= \frac{1}{3}x^{3/2} \Big|_0^4$$

$$= \frac{8}{3}$$

Exercise

Evaluate the double integral over the given region R $\displaystyle\iint_R y\sin(x+y)\, dA \quad R: \ -\pi \le x \le 0, \ \ 0 \le y \le \pi$

Solution

$$\iint_R y\sin(x+y)dA = \int_{-\pi}^0 \int_0^\pi y\sin(x+y)dxdy$$

		$\int \sin(x+y)$
+	y	$-\cos(x+y)$
−	1	$-\sin(x+y)$

$$= \int_{-\pi}^0 \left(-y\cos(x+y)+\sin(x+y) \Big|_0^\pi\right) dx$$

$$= \int_{-\pi}^0 \left[\sin(x+\pi)-\pi\cos(x+\pi)-\sin x\right] dx$$

$$= -\cos(x+\pi)-\pi\sin(x+\pi)+\cos x \Big|_{-\pi}^0$$

$$= -(-1) + 1 - (-1 - 1)$$

$$\underline{= 4}$$

Exercise

Evaluate the double integral over the given region R. $\iint_R e^{x-y}\,dA \quad R: \quad 0 \le x \le \ln 2, \quad 0 \le y \le \ln 2$

Solution

$$\iint_R e^{x-y}\,dA = \int_0^{\ln 2}\int_0^{\ln 2} e^{x-y}\,dydx$$

$$= \int_0^{\ln 2} \left(-e^{x-y} \Big|_0^{\ln 2}\right) dx$$

$$= \int_0^{\ln 2} \left(-e^{x-\ln 2} + e^x\right) dx$$

$$= -e^{x-\ln 2} + e^x \Big|_0^{\ln 2}$$

$$= -1 + e^{\ln 2} + e^{-\ln 2} - 1 \qquad e^{-\ln 2} = e^{\ln 2^{-1}} = 2^{-1} = \frac{1}{2}$$

$$= -2 + 2 + \frac{1}{2}$$

$$\underline{= \frac{1}{2}}$$

Exercise

Evaluate the double integral over the given region R. $\iint_R \frac{y}{x^2y^2+1}\,dA \quad R: \quad 0 \le x \le 1, \quad 0 \le y \le 1$

Solution

$$\iint_R \frac{y}{x^2y^2+1}\,dA = \int_0^1\int_0^1 \frac{y}{(xy)^2+1}dxdy \qquad \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} \qquad u = xy \to du = ydx$$

$$= \int_0^1 \left(\tan^{-1}(xy) \Big|_0^1 dy\right.$$

$$= \int_0^1 \tan^{-1} y\, dy \qquad\qquad \int \tan^{-1} ax dx = x \tan^{-1} ax - \frac{1}{2a} \ln\left(1 + a^2 x^2\right)$$

$$= y \tan^{-1} y - \frac{1}{2} \ln\left|1 + y^2\right| \Big|_0^1$$

$$= \tan^{-1} 1 - \frac{1}{2} \ln 2$$

$$= \frac{\pi}{4} - \frac{1}{2} \ln 2$$

Exercise

Evaluate $\iint_R x^{-1/2} e^y dA$; R is the region bounded by $x = 1$, $x = 4$, $y = \sqrt{x}$, and $y = 0$

Solution

$$\iint_R x^{-1/2} e^y dA = \int_1^4 \int_0^{\sqrt{x}} x^{-1/2} e^y \, dydx$$

$$= \int_1^4 x^{-1/2}\, e^y \Big|_0^{\sqrt{x}} dx$$

$$= \int_1^4 x^{-1/2} \left(e^{\sqrt{x}} - 1\right) dx$$

$$= \int_1^4 \frac{1}{\sqrt{x}} e^{\sqrt{x}}\, dx \; - \; \int_1^4 x^{-1/2}\, dx$$

$$= 2 \int_1^4 e^{\sqrt{x}} d\left(\sqrt{x}\right) \; - \; 2\sqrt{x} \Big|_1^4$$

$$= 2e^{\sqrt{x}} \Big|_1^4 - 2(2 - 1)$$

$$= 2\left(e^2 - e\right) - 2$$

$$= 2e^2 - 2e - 2$$

Exercise

Evaluate $\iint\limits_R \left(x^2+y^2\right)dA$; R is the region $\{(x,y):\ 0\le x\le 2,\ 0\le y\le x\}$

Solution

$$\iint\limits_R \left(x^2+y^2\right)dA=\int_0^2\int_0^x \left(x^2+y^2\right)dydx$$

$$=\int_0^2 \left(x^2y+\tfrac{1}{3}y^3\ \Big|_0^x\right)dx$$

$$=\int_0^2 \left(x^3+\tfrac{1}{3}x^3\right)dx$$

$$=\frac{4}{3}\int_0^2 x^3\,dx$$

$$=\frac{1}{3}x^4\ \Big|_0^2$$

$$=\frac{16}{3}$$

Exercise

Evaluate $\iint_R \frac{2y}{\sqrt{x^4+1}}dA$; *R is the region bounded by* $x=1,\quad x=2,\quad y=x^{3/2},\quad y=0$

Solution

$$\iint_R \frac{2y}{\sqrt{x^4+1}}dA=\int_1^2\int_0^{x^{3/2}} \frac{2y}{\sqrt{x^4+1}}dydx$$

$$=\int_1^2 \frac{1}{\sqrt{x^4+1}}\ y^2\ \Big|_0^{x^{3/2}}\ dx$$

$$=\int_1^2 \frac{x^3}{\sqrt{x^4+1}}dx$$

$$=\frac{1}{4}\int_1^2 \left(x^4+1\right)^{-1/2}d\left(x^4+1\right)$$

$$=\frac{1}{2}\sqrt{x^4+1}\,\Big|_1^2$$

$$=\boxed{\frac{1}{2}\left(\sqrt{17}-\sqrt{2}\right)}$$

Exercise

Integrate $f(x,y)=\frac{1}{xy}$ over the ***square*** $1\le x\le 2, \quad 1\le y\le 2$

Solution

$$\int_1^2\int_1^2 \frac{1}{xy}dydx=\int_1^2 \frac{1}{x}\left(\ln y\,\Big|_1^2\right.dx$$

$$=\int_1^2 \frac{1}{x}(\ln 2-\ln 1)\ dx$$

$$=\ln 2\int_1^2 \frac{1}{x}\ dx$$

$$=\ln 2\ \left(\ln x\,\Big|_1^2\right.$$

$$=\ln 2\cdot\ln 2$$

$$=\boxed{(\ln 2)^2}$$

Exercise

Integrate $f(x,y)=y\cos xy$ over the ***rectangle*** $0\le x\le \pi, \quad 0\le y\le 1$

Solution

$$\int_0^1\int_0^\pi y\cos(xy)\,dxdy=\int_0^1\left(\sin xy\,\Big|_0^\pi\right.dy$$

$$=\int_0^1 \sin(\pi y)\,dy$$

$$=-\frac{1}{\pi}\ \cos\pi y\,\Big|_0^1$$

$$=-\frac{1}{\pi}(-1-1)$$

$$=\boxed{\frac{2}{\pi}}$$

Exercise

Find the volume of the region bounded above the paraboloid $z = x^2 + y^2$ and below by the square $R: \quad -1 \le x \le 1, \quad -1 \le y \le 1$

Solution

$$V = \int_{-1}^{1} \int_{-1}^{1} \left(x^2 + y^2\right) dydx$$

$$= \int_{-1}^{1} \left(x^2 y + \frac{1}{3} y^3 \, \Big|_{-1}^{1} \, dx\right.$$

$$= \int_{-1}^{1} \left[x^2 + \frac{1}{3} - \left(-x^2 - \frac{1}{3}\right)\right] dx$$

$$= \int_{-1}^{1} \left(2x^2 + \frac{2}{3}\right) dx$$

$$= \frac{2}{3} x^3 + \frac{2}{3} x \, \Big|_{-1}^{1}$$

$$= \frac{2}{3} + \frac{2}{3} - \left(-\frac{2}{3} - \frac{2}{3}\right)$$

$$= \frac{8}{3} \; unit^3$$

Exercise

Find the volume of the region bounded above the plane $z = \frac{y}{2}$ and below by the rectangle

$R: \quad 0 \le x \le 4, \quad 0 \le y \le 2$

Solution

$$V = \int_{0}^{4} \int_{0}^{2} \frac{y}{2} \, dydx$$

$$= \int_{0}^{4} \left(\frac{1}{4} y^2 \, \Big|_{0}^{2} \, dx\right.$$

$$= \int_{0}^{4} (1) dx$$

$$= x \, \Big|_{0}^{4}$$

$$= 4 \; unit^3$$

Exercise

Find the volume of the region bounded above the surface $z = 4 - y^2$ and below by the rectangle $R:\ 0 \le x \le 1,\quad 0 \le y \le 2$

Solution

$$V = \int_0^1 \int_0^2 \left(4 - y^2\right) dydx$$

$$= \int_0^1 \left(4y - \frac{1}{3}y^3 \Big|_0^2\right. dx$$

$$= \int_0^1 \left(8 - \frac{8}{3}\right) dx$$

$$= \int_0^1 \frac{16}{3}\, dx$$

$$= \frac{16}{3}x \Big|_0^1$$

$$= \frac{16}{3}\quad unit^3$$

Exercise

Find the volume of the region bounded above the ellipitical paraboloid $z = 16 - x^2 - y^2$ and below by the square $R:\ 0 \le x \le 2,\quad 0 \le y \le 2$

Solution

$$V = \int_0^2 \int_0^2 \left(16 - x^2 - y^2\right) dydx$$

$$= \int_0^2 \left(16y - x^2 y - \frac{1}{3}y^3 \Big|_0^2\right. dx$$

$$= \int_0^2 \left(32 - 2x^2 - \frac{8}{3}\right) dx$$

$$= \int_0^2 \left(\frac{88}{3} - 2x^2\right) dx$$

$$= \frac{88}{3}x - \frac{2}{3}x^3 \Big|_0^2$$

$$= \frac{176}{3} - \frac{16}{3}$$

$$= \frac{160}{3} \ unit^3$$

Exercise

Evaluate $\int_0^{1/2} \left(\sin^{-1}[2x] - \sin^{-1} x\right) dx$ by converting it to a double integral.

Solution

$$0 \le x \le \frac{1}{2}$$

$$\begin{cases} \sin^{-1} 2x = y & \rightarrow \ 2x = \sin y \quad \Rightarrow x = \frac{1}{2}\sin y \\ \sin^{-1} x = y & \rightarrow \ x = \sin y \end{cases}$$

$$\begin{cases} x = 0 & \rightarrow y = 0 \\ x = \frac{1}{2} & \rightarrow \begin{cases} y = \sin^{-1} 1 = \frac{\pi}{2} \\ y = \sin^{-1} \frac{1}{2} = \frac{\pi}{6} \end{cases} \end{cases}$$

$$\int_0^{1/2} \left(\sin^{-1}(2x) - \sin^{-1} x\right) dx = \int_0^{\frac{\pi}{6}} \int_{\frac{1}{2}\sin y}^{\sin y} dxdy \ + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{\frac{1}{2}\sin y}^{\frac{1}{2}} dxdy$$

$$= \int_0^{\frac{\pi}{6}} x \Big|_{\frac{1}{2}\sin y}^{\sin y} dy \ + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \Big|_{\frac{1}{2}\sin y}^{\frac{1}{2}} dy$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} \sin y \ dy + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 - \sin y) dy$$

$$= -\frac{1}{2}\cos y \Big|_0^{\frac{\pi}{6}} \ + \frac{1}{2}(y + \cos y) \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= -\frac{1}{2}\left(\frac{\sqrt{3}}{2} - 1\right) + \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{6} - \frac{\sqrt{3}}{2}\right)$$

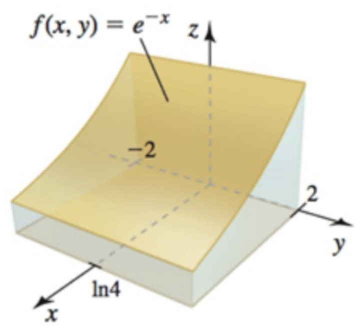
$$= \frac{1}{2} - \frac{\sqrt{3}}{4} + \frac{\pi}{6} - \frac{\sqrt{3}}{4}$$

$$\boxed{= \frac{\pi}{6} + \frac{1}{2} - \frac{\sqrt{3}}{2}}$$

Exercise

Find the volume of the solid beneath the cylinder $f(x, y) = e^{-x}$ and above the region $R = \{(x, y): \ 0 \le x \le \ln 4, \ \ -2 \le y \le 2\}$

Solution



$$V = \int_{-2}^{2} \int_{0}^{\ln 4} e^{-x}\, dxdy$$

$$= -\int_{-2}^{2} e^{-x} \Big|_{0}^{\ln 4} dy$$

$$= -\int_{-2}^{2} \left(\frac{1}{4} - 1\right) dy$$

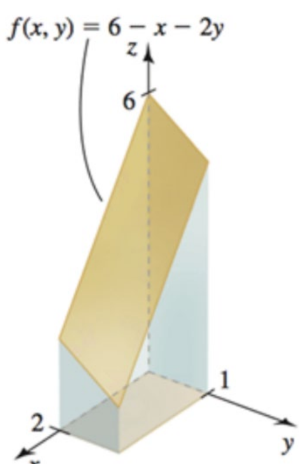
$$= \frac{3}{4} y \Big|_{-2}^{2}$$

$$\boxed{= 3 \ \ unit^3}$$

Exercise

Find the volume of the solid beneath the plane $f(x, y) = 6 - x - 2y$ and above the region $R = \{(x, y): \ 0 \le x \le 2, \ \ 0 \le y \le 1\}$

Solution



$$V = \int_{0}^{1} \int_{0}^{2} (6 - x - 2y)\, dxdy$$

$$= \int_{0}^{1} \left(6x - \frac{1}{2}x^2 - 2yx \Big|_{0}^{2} dy\right.$$

$$= \int_{0}^{1} (10 - 4y) dy$$

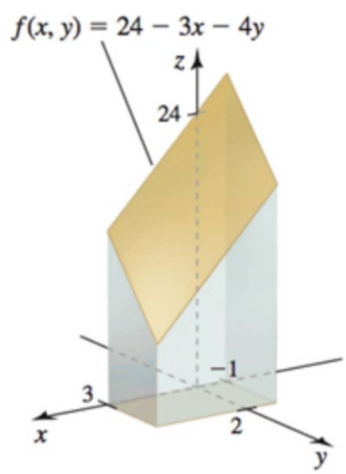
$$= 10y - 2y^2 \Big|_0^1$$

$$\underline{= 8 \ \ unit^3}$$

Exercise

Find the volume of the solid beneath the plane $f(x, y) = 24 - 3x - 4y$ and above the region $R = \{(x, y): \ -1 \le x \le 3, \ \ 0 \le y \le 2\}$

Solution



$$V = \int_{-1}^{3}\int_{0}^{2}(24 - 3x - 4y)\, dydx$$

$$= \int_{-1}^{3}\left(24y - 3xy - 2y^2 \Big|_0^2\right) dx$$

$$= \int_{-1}^{3}(48 - 6x - 8)dx$$

$$= \int_{-1}^{3}(40 - 6x)\, dx$$

$$= \left(40x - 3x^2\right)\Big|_{-1}^{3}$$

$$= 120 - 27 + 40 + 3$$

$$\underline{= 136 \ \ unit^3}$$

Exercise

Find the volume of the solid beneath the paraboloid $f(x, y) = 12 - x^2 - 2y^2$ and above the region $R = \{(x, y): \ 1 \le x \le 2, \ \ 0 \le y \le 1\}$

Solution

$$V = \int_{1}^{2}\int_{0}^{1}\left(12 - x^2 - 2y^2\right)dydx$$

$$= \int_{1}^{2}\left(12y - x^2y - \frac{2}{3}y^3 \Big|_0^1\right) dx$$

$$= \int_1^2 \left(12 - x^2 - \frac{2}{3}\right) dx$$

$$= \int_1^2 \left(\frac{34}{3} - x^2\right) dx$$

$$= \frac{34}{3}x - \frac{1}{3}x^3 \Big|_1^2$$

$$= \frac{68}{3} - \frac{8}{3} - \frac{34}{3} + \frac{1}{3}$$

$$= 9 \ \ unit^3$$