Lecture Three

Section 3.1 – Introduction to Linear Systems

Definition

A ***linear*** (algebraic) ***equation*** in n unknowns, $x_1, x_2, \ldots, x_n$, is an equation of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

Where $a_1, a_2, \ldots, a_n$ and b are real numbers.

Matrices

$$\begin{array}{c c} & \begin{array}{ccc} & \textit{Column} & \\ C_1 & C_2 & C_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} Row\ 1 \rightarrow R_1 \\ Row\ 2 \rightarrow R_2 \\ Row\ 3 \rightarrow R_3 \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

This is called Matrix (*Matrices*)

Each number in the array is an ***element*** or ***entry***

The matrix is said to be of order m x n

m: numbers of rows,

n: number of columns

When $m = n$, then matrix is said to be ***square***.

Given the system equations

$$\begin{aligned} 3x + y + 2z &= 31 \\ x + y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Write into an ***augmented matrix*** form

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 31 \\ 1 & 1 & 2 & 19 \\ 1 & 3 & 2 & 25 \end{array}\right]$$

Example

Given the linear equations

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases}$$

The solution to this system is $(3,\ 1)$, which means that 2 lines meeting at a single point.

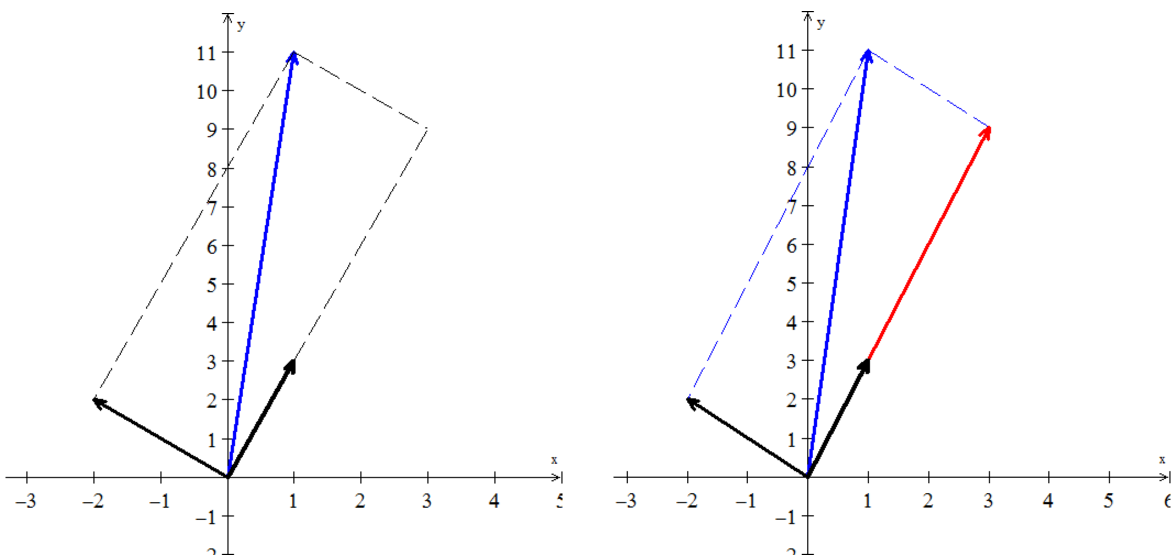
We can rewrite the system equation as linear combination:

$$x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

$$x.v_1 + y.v_2 = v$$

$$\begin{bmatrix} 1+x \\ 3+y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} \Rightarrow \begin{bmatrix} \color{red}{x=3} \\ \color{red}{y=9} \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 9 \end{bmatrix} = \color{red}{3}\begin{bmatrix} \color{blue}{1} \\ \color{blue}{3} \end{bmatrix}$$



Therefore, the side vectors are $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$ *and* $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$

The diagonal sum is $\begin{bmatrix} 3-2 \\ 9+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$

The linear combination is given by: $\color{red}{3}\begin{bmatrix} \color{blue}{1} \\ \color{blue}{3} \end{bmatrix} + \color{red}{1}\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$

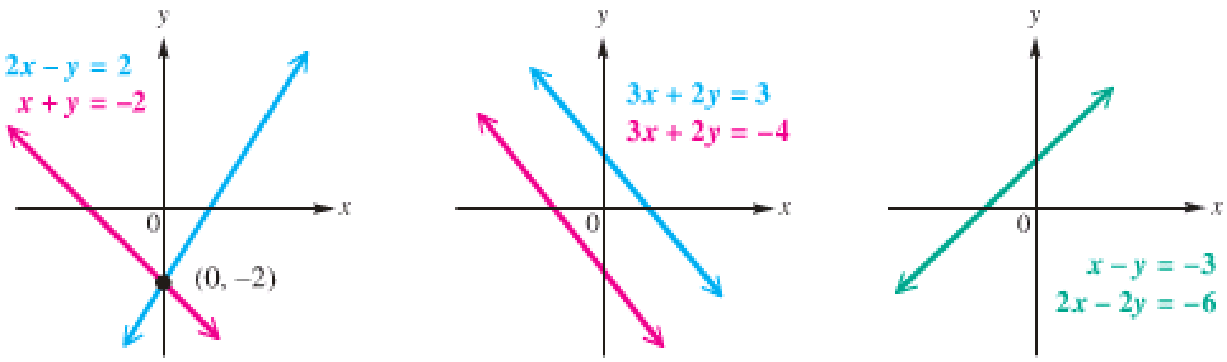
Thus, the solution is $x = 3 \quad y = 1$

Note

$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$ is called the coefficient matrix

The matrix form of the system is written as $Ax = b$

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$



One solution (lines intersect)	***No Solution (lines //)***	***Infinite solution***
Consistent	***Inconsistent***	***Consistent***
Independent	***Independent***	***Dependent***

Three Equations in 3 Unknowns

A linear equation in three unknowns *x, y, z*:

$$ax + by + cz = \alpha$$

A solution of the equation is an ordered triple of numbers $(x,\ y,\ z)$

If $a = b = c = 0,\ \text{and}\ \alpha = 0$, all ordered triples satisfy the equation $0x + 0y + 0z = 0$ (***infinitely many***)

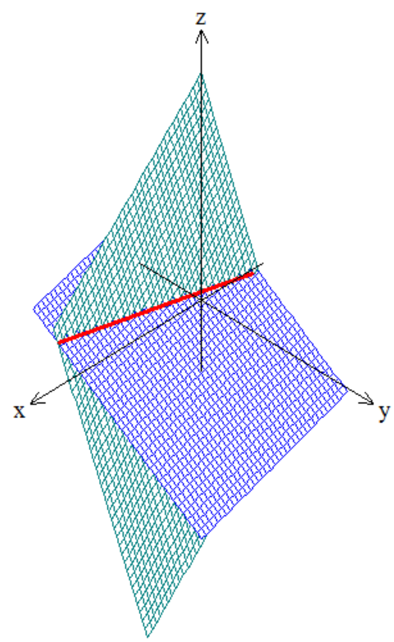
If $a = b = c = 0,\ \text{and}\ \alpha \neq 0$, no ordered triples satisfies the equation $0x + 0y + 0z \neq 0$ (***no solution***)

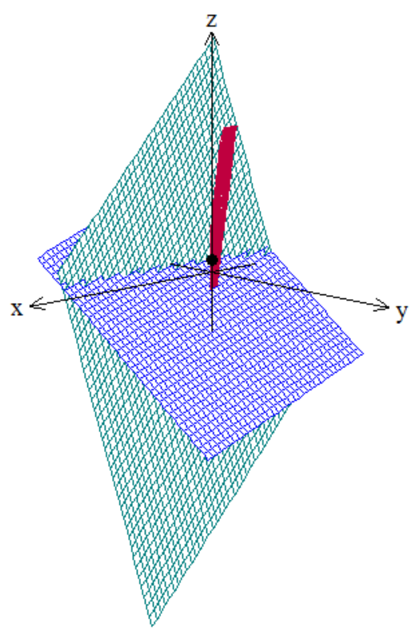
If *a, b, c*, not all 0, then the set of all ordered triples that satisfy the equation is a plane (in 3-space)

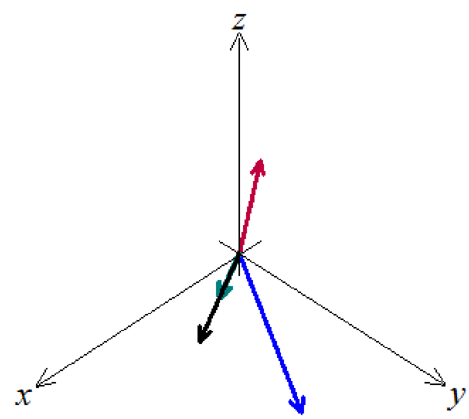
Example

Given the system equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 5y + 2z &= 4 \\ 6x - 3y + z &= 2 \end{aligned}$$







This system can be written as linear combination:

$$x\begin{bmatrix}1\\2\\6\end{bmatrix}+y\begin{bmatrix}2\\5\\-3\end{bmatrix}+z\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}6\\4\\2\end{bmatrix} \qquad \text{Let } b=\begin{bmatrix}6\\4\\2\end{bmatrix}$$

We want to multiply the three column vectors by x, y, z to produce $\boldsymbol{b}$.
The combination of the three vectors that produces
vector b is 2 times the third vector. $2(3,\ 2,\ 1)=(6,\ 4,\ 2)=b$
Therefore the coefficients that we need are $x=0, \quad y=0, \quad and\ z=2$.

$$0\begin{bmatrix}1\\2\\6\end{bmatrix}+0\begin{bmatrix}2\\5\\-3\end{bmatrix}+2\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}6\\4\\2\end{bmatrix}$$

Homogeneous Systems

The system of linear equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
\end{aligned}$$

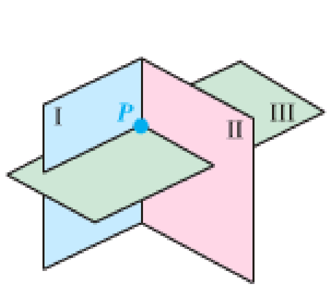
Is ***homogeneous*** if $b_1 = b_2 = \dots = b_m = 0$ otherwise, the system is ***nonhomogeous***.

A ***homogeneous*** system

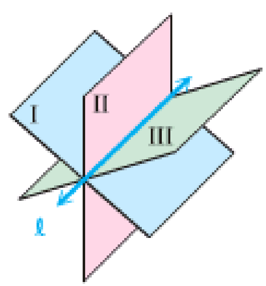
$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0
\end{aligned}$$

Always has at least one solution namely $x_1 = x_2 = \dots = x_n = 0$ called the ***trivial solution***

That is, homogeneous systems are always ***consistent***

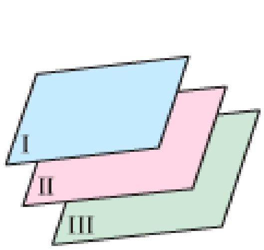


A single solution

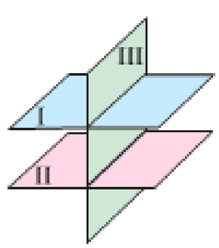


Points of a line in common

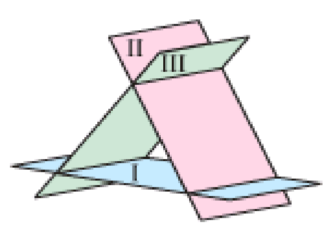
All points in common



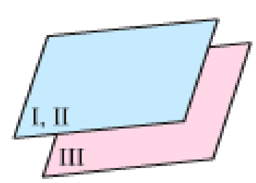
No points in common



No points in common



No points in common



No points in common

Exercises Section 3.1 – Introduction to Linear Systems

1. Find a solution for x, y, z to the system of equations

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3e+2\sqrt{2}+\pi \\ 6e+5\sqrt{2}+4\pi \\ 9e+8\sqrt{2}+7\pi \end{pmatrix}$$

2. Draw the two pictures in two planes for the equations: $x-2y=0, \quad x+y=6$

3. Normally 4 planes in 4-dimensional space meet at a ________. Normally 4 column vectors in 4-deimensional space can combine to produce b. what combinations of
$(1,\ 0,\ 0,\ 0), \quad (1,\ 1,\ 0,\ 0), \quad (1,\ 1,\ 1,\ 0), \quad (1,\ 1,\ 1,\ 1)$ produces $b=(3,\ 3,\ 3,\ 2)$?
What 4 equations for $x,\ y,\ z,\ w$ are you solving?

4. What 2 by 2 matrix A rotates every vector through 45° ?
The vector (1, 0) goes to $\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$. The vector (0, 1) goes to $\left(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$.
Those determine the matrix. Draw these particular vectors is the *xy*-plane and find A.

5. What two vectors are obtained by rotating the plane vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by 30° (*cw*) ?
Write a matrix A such that for every vector v in the plane, Av is the vector obtained by rotating v clockwise by 30°.
Find a matrix B such that for every 3-dimensional vector v, the vector Bv is the reflection of v through the plane $x+y+z=0$. *Hint* : $v=(1,\ 0,\ 0)$

6. In each part, find a system of linear equation corresponding to the given augmented matrix

a) $\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$ *b)* $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \\ 5 & -6 & 1 & 1 \\ -8 & 0 & 0 & 3 \end{bmatrix}$

7. Find the augmented matrix for the given system of linear equations.

a) $\begin{cases} -2x_1 = 6 \\ 3x_1 = 8 \\ 9x_1 = -3 \end{cases}$ *b)* $\begin{cases} 3x_1 - 2x_2 = -1 \\ 4x_1 + 5x_2 = 3 \\ 7x_1 + 3x_2 = 2 \end{cases}$ *c)* $\begin{cases} 2x_1 + 2x_3 = 1 \\ 3x_1 - x_2 + 4x_3 = 7 \\ 6x_1 + x_2 - x_3 = 0 \end{cases}$

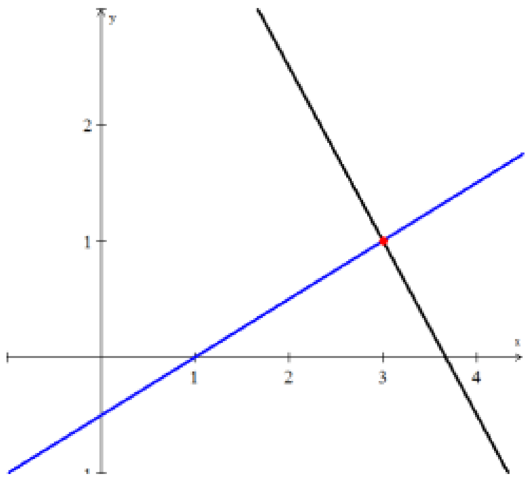
Section 3.2 – Gaussian Elimination

Elimination produces an ***upper triangular system***.

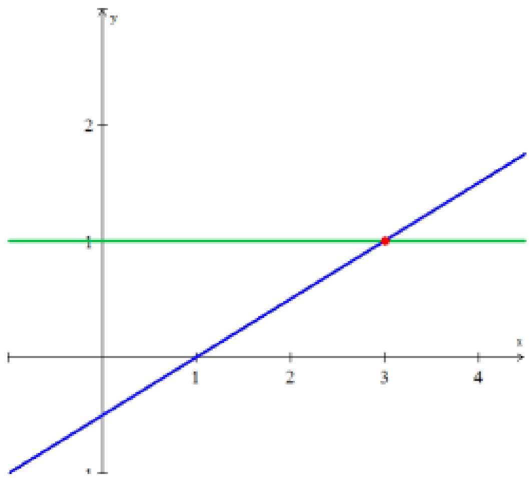
$$\begin{cases} x-2y=1 \\ 3x+2y=11 \end{cases} \Rightarrow \begin{cases} x-2y=1 & \textit{Mutliply by } 3 \\ \quad 8y=8 & \textit{and subtract} \end{cases}$$

The equation $8y=8$ *reveals* $y=1$

This process is called ***back substitution***.



Before elimination



After elimination

Definitions

Pivot: first nonzero in the row that does the elimination

Multiplier: (entry to eliminate) divide by pivot

$$\begin{array}{lll} 4x-8y=4 & \textit{Multiply equation } 1 \textit{ by } \frac{3}{4} & 4x-8y=4 \\ 3x+2y=11 & \textit{Subtract from equation } 2 & \quad 8y=8 \end{array}$$

The first pivot is 4 (the coefficient of x) and the multiplier is $l=\frac{3}{4}$

The pivots are on the diagonal of the triangle after elimination.

Reduced Row Echelon Form

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

Use the Gaussian elimination method to solve the system

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 3 & 1 & 2 & 31 \\ 1 & 3 & 2 & 25 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array} \qquad \begin{array}{cccc} 3 & 1 & 2 & 31 \\ -3 & -3 & -6 & -57 \\ \hline 0 & -2 & -4 & -26 \end{array} \qquad \begin{array}{cccc} 1 & 2 & 2 & 25 \\ -1 & -1 & -2 & -19 \\ \hline 0 & 2 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & -2 & -4 & -26 \\ 0 & 2 & 0 & 6 \end{array}\right] \ -\tfrac{1}{2}R_2 \qquad 0 \quad 1 \quad 2 \quad 13$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 2 & 0 & 6 \end{array}\right] \ R_3 - 2R_2 \qquad \begin{array}{cccc} 0 & 2 & 0 & 6 \\ 0 & -2 & -4 & -26 \\ \hline 0 & 0 & -4 & -20 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -4 & -20 \end{array}\right] \ -\tfrac{1}{4}R_3 \qquad 0 \quad 0 \quad 1 \quad 5$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array}\right] \Rightarrow \begin{array}{rl} x + y + 2z = 19 & (3) \\ y + 2z = 13 & (2) \\ z = 5 & (1) \end{array}$$

$(2) \Rightarrow y = 13 - 2z = 13 - 2(5) = 3$

$(3) \Rightarrow \ x = 19 - y - 2z = 19 - 3 - 10 = 6$

$\Rightarrow$ **(6, 3, 5)**

Example

Use Gauss-Jordan elimination to solve the linear system

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\quad &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\ 5x_3 + 10x_4 \qquad + 15x_6 &= 5 \\ 2x_1 + 6x_2 \qquad\quad + 8x_4 + 4x_5 + 18x_6 &= 6 \end{aligned}$$

Solution

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \quad \textit{Adding } (-2) \textit{ times the } 1st \textit{ row to the } 2nd \\ \\ R_4 - 2R_1 \quad \textit{Adding } (-2) \textit{ times the } 1st \textit{ row to the } 4th \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 5R_2 \\ R_3 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right] \begin{array}{l} \\ \\ \\ \frac{1}{6}R_4 \quad \textit{then interchanging row3 and row4} \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{cases} x_1 + 3x_2 \quad + 4x_4 + 2x_5 \quad = 0 \\ \qquad\qquad x_3 + 2x_4 \qquad\qquad = 0 \\ \qquad\qquad\qquad\qquad + x_6 = \frac{1}{3} \end{cases}$$

The general solution of the system: $x_6 = \frac{1}{3}, \quad x_3 = -2x_4, \quad x_1 = -3x_2 - 4x_4 - 2x_5$

Example

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$\begin{aligned} 2x+8y-z+w&=0\\ 4x+16y-3z-w&=-10\\ -2x+4y-z+3w&=-6\\ -6x+2y+5z+w&=3 \end{aligned}$$

Solution

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0\\ 4 & 16 & -3 & -1 & -10\\ -2 & 4 & -1 & 3 & -6\\ -6 & 2 & 5 & 1 & 3 \end{array}\right] \begin{array}{c} \\ R_2-2R_1\\ R_3+R_1\\ R_4+3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0\\ 0 & 12 & -2 & 4 & -6\\ 0 & 0 & -1 & -3 & -10\\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4-\frac{13}{6}R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0\\ 0 & 0 & -1 & -3 & -10\\ 0 & 12 & -2 & 4 & -6\\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \quad \textit{Interchange } R_2 \textit{ and } R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0\\ 0 & 12 & -2 & 4 & -6\\ 0 & 0 & -1 & -3 & -10\\ 0 & 0 & \frac{19}{3} & -\frac{14}{3} & 16 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4+\frac{19}{3}R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0\\ 0 & 12 & -2 & 4 & -6\\ 0 & 0 & -1 & -3 & -10\\ 0 & 0 & 0 & -\frac{71}{3} & -\frac{142}{3} \end{array}\right] \begin{array}{lll} 2x+8y-z+w=0 \rightarrow & 2x=-8y+z-w=6 & \Rightarrow \boxed{x=3}\\ 12y-2z+4w=-6 \rightarrow & 12y=2z-4w-6=-6 & \Rightarrow \boxed{y=-\frac{1}{2}}\\ -z-3w=-10 \rightarrow & z=10-3w=4 & \\ -\frac{71}{3}w=-\frac{142}{3} \rightarrow & \boxed{w=2} & \end{array}$$

Solution: $\left(3,\ -\frac{1}{2},\ 4,\ 2\right)$

Exercises *Section* 3.2 – Gaussian Elimination

1. When elimination is applied to the matrix $A=\begin{bmatrix}3 & 1 & 0\\ 6 & 9 & 2\\ 0 & 1 & 5\end{bmatrix}$

a) What are the first and second pivots?

b) What is the multiplier l_{21} in the first step (l_{21} times row 1 is subtracted from row 2)?

c) What entry in the 2, 2 position (instead of 9) would force an exchange of rows 2 and 3?

d) What is the multiplier $l_{31}=0$, subtracting 0 times row 1 from row 3?

2. Use elimination to reach upper triangular matrices U. Solve by back substitution or explain why this impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the $-x$ in equation (3).

$$\begin{cases}x+y+z=7\\ x+y-z=5\\ x-y+z=3\end{cases} \qquad \begin{cases}x+y+z=7\\ x+y-z=5\\ -x-y+z=3\end{cases}$$

3. For which numbers a does the elimination break down (1) permanently (2) temporarily

$$\begin{aligned}ax+3y&=-3\\ 4x+6y&=6\end{aligned}$$

Solve for x and y after fixing the second breakdown by a row change.

4. Find the pivots and the solution for these four equations:

$$\begin{aligned}2x+y\qquad\quad &=0\\ x+2y+z\qquad &=0\\ y+2z+t&=0\\ z+2t&=5\end{aligned}$$

5. Look for a matrix that has row sums 4 and 8, and column sums 2 and s.

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix} \qquad \begin{matrix}a+b=4 & a+c=2\\ c+d=8 & b+d=s\end{matrix}$$

The four equations are solvable only if s = ____. Then find two different matrices that have the correct row and column sums.

6. Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of A is a _______ of the first two rows. Find a third equation that can't be solved together with $x+y+z=0$ and $x-2y-z=1$

7. Solve the linear system by Gauss-Jordan elimination.

$a)\ \begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$

$b)\ \begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \qquad\qquad - 3w = -3 \end{cases}$

$c)\ \begin{cases} x + 2y + z = 8 \\ -x + 3y - 2z = 1 \\ 3x + 4y - 7z = 10 \end{cases}$

$d)\ \begin{cases} 2u - 3v + w - x + y = 0 \\ 4u - 6v + 2w - 3x - y = -5 \\ -2u + 3v - 2w + 2x - y = 3 \end{cases}$

8. Solve the given linear system by any method

$a)\ \begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 = 0 \\ x_2 + x_3 = 0 \end{cases}$

$b)\ \begin{cases} 2x + 2y + 4z = 0 \\ -y - 3z + w = 0 \\ 3x + y + z + 2w = 0 \\ x + 3y - 2z - 2w = 0 \end{cases}$

9. Add 3 times the second row to the first of

$$\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

10. Solve the system using Gaussian elimination $\begin{cases} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{cases}$

11. For what value(s) of *k*, if any, does the system $\begin{cases} x + y - z = 1 \\ 2x + 3y + kz = 3 \\ x + ky + 3z = 2 \end{cases}$ have

a) A unique solution?
b) Infinitely many solutions?
c) No solution?

Section 3.3 – Algebra of Matrices

$$A = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Equality of Matrices

Definition of Equality of Matrices

Two matrices ***A*** and ***B*** are equal if and only if they have the same order (size) $m \times n$ and if each pair corresponding elements is equal

$a_{ij} = b_{ij}$ for $i = 1, 2, \ldots, m$ ***and*** $j = 1, 2, \ldots, n$

Example

Find the values of the variables for which each statement is true, if possible.

a) $\begin{bmatrix} 2 & 1 \\ p & q \end{bmatrix} = \begin{bmatrix} x & y \\ -1 & 0 \end{bmatrix}$

$x = 2,\ y = 1,\ \ p = -1,\ \ q = 0$

b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$

can't be true

c) $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

$\Rightarrow \begin{bmatrix} w = 9 & x = 17 \\ 8 = y & -12 = z \end{bmatrix}$

Addition and *Subtraction* of Matrices

Definition

If $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ are $m\times n$ matrices, their sum $A+B$, is the $m\times n$ matrix obtained by adding the corresponding entries; that is

$$\left[a_{ij}\right]+\left[b_{ij}\right]=\left[a_{ij}+b_{ij}\right]$$

Matrices can be added if their shapes are the same, meaning have the same ***order***.

$$\begin{bmatrix}1 & 2\\ 3 & 4\\ 0 & 0\end{bmatrix}+\begin{bmatrix}2 & 2\\ 4 & 4\\ 9 & 9\end{bmatrix}=\begin{bmatrix}1+2 & 2+2\\ 3+4 & 4+4\\ 0+9 & 0+9\end{bmatrix}$$

$$=\begin{bmatrix}3 & 4\\ 7 & 8\\ 9 & 9\end{bmatrix}$$

Scalar Multiplication Matrices

Definition

If k is a scalar and $A=\left[a_{ij}\right]$ is an $m\times n$ matrices, then scalar product $\boldsymbol{kA}$ is the $m\times n$ matrix obtained by multiplying each entry of A by k; that is

$$k\left[a_{ij}\right]=\left[ka_{ij}\right]$$

$$kA=k\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$$

$$=\begin{bmatrix}ka_{11} & ka_{12}\\ ka_{21} & ka_{22}\end{bmatrix}$$

Example

$$2\begin{bmatrix}1 & 2\\ 3 & 4\\ 0 & 0\end{bmatrix}=\begin{bmatrix}(2)1 & (2)2\\ (2)3 & (2)4\\ (2)0 & (2)0\end{bmatrix}$$

$$=\begin{bmatrix}2 & 4\\ 6 & 8\\ 0 & 0\end{bmatrix}$$

Definition

If $A_1, A_2, \ldots, A_n$ are matrices of the same size, and if $c_1, c_2, \ldots, c_n$ are scalars, then expression of the form

$$c_1A_1 + c_2A_2 + \ldots + c_nA_n$$

Is called a ***linear combination*** of $A_1, A_2, \ldots, A_n$ with *coefficients* $c_1, c_2, \ldots, c_n$.

Matrix Multiplication

Product of Two Matrices

Let ***A*** be an $m \times n$ matrix and let ***B*** be an $n \times k$ matrix. To find the element in the i^{th} row and j^{th} column of the product matrix ***AB***, multiply each element in the i^{th} row of ***A*** by the corresponding element in the j^{th} column of ***B***, and then add these products. The product matrix ***AB*** is an $m \times k$ matrix.

*Matrix **A*** *Matrix B*

$m \times n$ $n \times k$

Inner *must be equal*

Outer: Order of AB is $m \times k$

- ✓ To multiply ***AB*** or dot product, if ***A*** has ***n*** columns, ***B*** must have ***n*** rows.
- ✓ Squares matrices can be multiplied if and only if (***iff***) they have the same size.
- ✓ The entry in row i and column j of AB is $\boxed{(row\ i\ of\ A).(col\ j\ of\ B)}$

The result: $\sum a_{ik}b_{kj}$

$$\underset{4\ by\ 5}{\begin{bmatrix} * & * & & & \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{i5} \\ * & & & & \\ * & & & & \end{bmatrix}} \underset{5\ by\ 6}{\begin{bmatrix} * & * & b_{1j} & * & * & * \\ & & b_{2j} & & & \\ & & \vdots & & & \\ & & \vdots & & & \\ & & b_{5j} & & & \end{bmatrix}} = \underset{4\ by\ 6}{\begin{bmatrix} & & * & & & \\ * & * & (AB)_{ij} & * & * & * \\ & & * & & & \\ & & * & & & \end{bmatrix}}$$

$$\boldsymbol{AB} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$2x2 \qquad 2x2 \quad \to \quad 2x2$$

$$a_{11} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & \_ \\ \_ & \_ \end{bmatrix}$$

$$a_{12} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & af + bh \\ \_ & \_ \end{bmatrix}$$

$$a_{21} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ ce + dg & \_ \end{bmatrix}$$

$$a_{22} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ \_ & cf + dh \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Example

Find: $\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}$

<u>Solution</u>

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1(5) + 1(1) & 1(6) + 1(0) \\ 2(5) - 1(1) & 2(6) - 1(0) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 \\ 9 & 12 \end{bmatrix}$$

Special Case

When A is a square matrix, then

$$A \quad times \quad A^2 = A^2 \quad times \quad A \quad \underline{= A^3}$$

$$A^p = AA \cdots A \qquad (p \ factors)$$

$$\left(A^p\right)\left(A^q\right) = A^{p+q}$$

$$\left(A^p\right)^q = A^{pq}$$

Block Multiplication

If the cuts between columns of ***A*** match the cuts between rows of ***B***, then the block multiplication of ***AB*** allowed.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \left[\begin{array}{c|c} a_{11}b_{11} + a_{12}b_{21} & a_{12}b_{21} + a_{12}b_{22} \\ \hline a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{21} + a_{22}b_{22} \end{array}\right]$$

Important special case

$$\begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix}$$

Matrix Form of the Equations

The coefficient matrix is $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}$

The equivalent matrix equation is in the form $AX = b$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Multiplication by ***rows*** $\quad AX = \begin{bmatrix} (row\ 1).X \\ (row\ 2).X \\ (row\ 3).X \end{bmatrix}$

Multiplication by ***columns*** $\quad AX = x\ (column\ 1) + y\ (column\ 2) + z\ (column\ 3)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Identity Matrix

The identity matrix is given by the form: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \boxed{Ix = x}$

Properties of Matrix

Addition and Scalar Multiplication

$A+B=B+A$	*Commutative Property of Addition*
$A+(B+C)=(A+B)+C$	*Associative Property of Addition*
$(kl)A=k(lA)$	*Associative Property of Scalar Multiplication*
$k(A+B)=kA+kB$	*Distributive Property*
$k(A-B)=kA-kB$	*Distributive Property*
$(k+l)A=kA+lA$	*Distributive Property*
$(k-l)A=kA-lA$	*Distributive Property*
$A+0=0+A=A$	*Additive Identity Property*
$A+(-A)=(-A)+A=0$	*Additive Inverse Property*
$k(AB)=kA(B)=A(kB)$	

Multiplication

$AB \neq BA$	*Commutative "**law**" is usually broken*
$A(BC)=(AB)C$	*Associative Property of Multiplication* (***Parentheses not needed***)
$A(B+C)=AB+AC$	*Distributive Property*
$(B+C)A=BA+CA$	*Distributive Property*
$A(B-C)=AB-AC$	*Distributive Property*
$(B-C)A=BA-CA$	*Distributive Property*

Exercises *Section* 3.3 – Algebra of Matrices

1. For the matrices: $A = \begin{bmatrix} p & 0 \\ q & r \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, when does $AB = BA$

2. A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

a) AB *b*) BA *c*) ABD *d*) DBA

e) ABC *f*) $ABCD$ *g*) $A(B + C)$

3. What rows or columns or matrices do you multiply to find.

a) The third column of AB?
b) The second column of AB?
c) The first row of AB?
d) The second row of AB?
e) The entry in row 3, column 4 of AB?
f) The entry in row 2, column 3 of AB?

4. Add AB to AC and compare with $A(B+C)$:

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad \textit{and} \quad C = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

5. True or False

a) If A^2 is defined then A is necessarily square.

b) If AB and BA are defined then A and B are square.

c) If AB and BA are defined then AB and BA are square.

d) If $AB = B$, then $A = I$

6. *a*) Find a nonzero matrix A such that $A^2 = 0$

b) Find a matrix that has $A^2 \neq 0$ but $A^3 = 0$

7. Suppose you solve $Ax = b$ for three special right sides b:

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If the three solutions x_1, x_2, x_3 are the columns of a matrix X, what is A times X?

8. Show that $(A+B)^2$ is different from $A^2 + 2AB + B^2$, when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

Write down the correct rule for $(A+B)(A+B) = A^2 + \underline{\hspace{3cm}} + B^2$

9. Find the product of the 2 matrices by rows or by columns:

a) $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

c) $\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

10. Given $A = \begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$ $B = \begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix}$ Find $A+B,\quad 2A,\quad \textit{and} \quad -B$

11. Given $A = \begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ Find $AB \quad \textit{and} \quad BA \quad \textit{if possible}$

12. Given $A = \begin{bmatrix} 5 & 3 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 4 & -2 \\ -2 & 0 \\ 9 & 1 \end{bmatrix}$ Find $AB \quad \textit{and} \quad BA \quad \textit{if possible}$

13. Given $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$ Find $AB \quad \textit{and} \quad BA \quad \textit{if possible}$

14. Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Compute the following (where possible):

a) $D+E$ ***b***) $D-E$ ***c***) $5A$ ***d***) $-7C$ ***e***) $2B-C$ ***g***) $-3(D+2E)$

Section 3.4 – Inverse Matrices

Definition

The matrix A is invertible if there exists a matrix A^{-1} such that:

$$A^{-1}A = AA^{-1} = I$$

where A^{-1} *read as* "A ***inverse***" and A has to be a ***square matrix***.

Not all matrices have inverses.

1. The inverse exists *iff* elimination produces n pivots (row exchanges allow).

2. The matrix A cannot have two different inverses.

3. If A is invertible, the one and only one solution to $Ax = B$ is $x = A^{-1}B$

$$AX = B$$

$$A^{-1}(AX) = A^{-1}B \qquad \textit{Multiply both side by } A^{-1}$$

$$\left(A^{-1}A\right)X = A^{-1}B \qquad \textit{Associate property}$$

$$IX = A^{-1}B \qquad \textit{Multiplicative inverse property}$$

$$X = A^{-1}B \qquad \textit{Identity property}$$

4. Suppose there is a ***nonzero*** vector x such that $Ax = 0$. Then A cannot have an inverse

5. A 2 by 2 matrix is invertible iff $ad - bc$ is not zero.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \textit{Only for 2 by 2 matrices}$$

If $ad - bc = 0$ is the determinant, then A^{-1} doesn't exist

The Inverse of a Product AB

Theorem

If an $n \times n$ matrix has an inverse, that inverse is unique.

Proof

Suppose that A has an inverse A^{-1} and B is a matrix such that $BA = I$

$$B = BI = B\left(AA^{-1}\right) = (BA)A^{-1} = IA^{-1} = A^{-1}$$

Theorem

If A and B are invertible then so is AB. The inverse of a product AB is $(AB)^{-1} = B^{-1}A^{-1}$

Proof

$$\begin{aligned}(AB)\left(B^{-1}A^{-1}\right) &= A\left(BB^{-1}\right)A^{-1} \\ &= (AI)A^{-1} \\ &= AA^{-1} \\ &= I\end{aligned}$$

Reverse Order

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

Theorem

If A is invertible and n is a nonnegative integer, then:

a) A^{-1} is *invertible* and $\left(A^{-1}\right)^{-1} = A$

b) A^{n} is *invertible* and $\left(A^{n}\right)^{-1} = A^{-n} = \left(A^{-1}\right)^{n}$

c) kA is *invertible* for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$

Proof

$$(kA)\left(k^{-1}A^{-1}\right) = k^{-1}(kA)A^{-1} = \left(k^{-1}k\right)AA^{-1} = (1)I = I$$

$$\left(k^{-1}A^{-1}\right)(kA) = k^{-1}\left(kA^{-1}\right)A = \left(k^{-1}k\right)A^{-1}A = (1)I = I$$

Finding A^{-1} using Gauss-Jordan Elimination

$$\left[A \middle| I\right] \rightarrow \left[I \middle| {\color{red}A^{-1}}\right]$$

Find A^{-1} *if* $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccccc} 2 & -2 & -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -2 & 0 & 0 \\ \hline 0 & -2 & -3 & -2 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 3 & 0 & 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & -3 & 0 & 0 \\ \hline 0 & 0 & -3 & -3 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \; -\tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \; -\tfrac{1}{3}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - \frac{3}{2}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & \frac{1}{3} \\ \hline 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{array} \qquad \begin{array}{cccccc} 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2} \\ \hline 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \qquad {\color{blue} A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}}$$

- ✓ Matrix ***A*** is ***symmetric*** across its main diagonal. So is A^{-1}
- ✓ Matrix ***A*** is ***tridiagonal*** (only three nonzero diagonals). But A^{-1} is a full matrix with no zeros. (another reason we don't compute A^{-1})

Singular *versus* Invertible

A^{-1} exists when A has a full set of n pivots. (Row exchanges allowed)

- With n pivots, elimination solves all the equations $Ax_i = b_i$. The columns x_i go into A^{-1}. Then $AA^{-1} = I$ is at least a ***right-inverse***.
- Elimination is really a sequence of multiplications.

Conclusion

- If A doesn't have n pivots, elimination will lead to a ***zero row***.
- Elimination steps are taken by an invertible M. So a row of MA is zero.
- If $AB = I$ then $MAB = M$. The zero row of MA, times B, gives a zero row of M.
- The invertible matrix M can't have a zero row! A must have n pivots if $AB = I$.

Elementary Matrices

Definition

Let ***e*** be an elementary row operation. Then the $n \times n$ ***elementary matrix*** E associated with ***e*** is the matrix obtained by applying ***e*** to the $n \times n$ identity matrix. Thus

$$E = eI$$

Example

a) $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \to$ *Multiply* R_2 *of* I *by* -3

b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to$ *Multiply the third row by* -5

c) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to$ *Interchange the first and second rows*

d) $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to$ *Add* -3 *times* R_1 *to* R_2

Theorem

Let e be an elementary operation and let E be the corresponding elementary matrix $E = e(I)$. Then for every $m \times n$ matrix A

$$e(A) = EA$$

That is, an elementary row operation can be performed on A by multiplying A on the left by the corresponding elementary matrix.

Example $m \times m$

Let $A = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$ $M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

$$MA = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 6 & 2 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$

This result can be obtained from A by multiplying the first row by 2.

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 & 1 \\ -6 & 2 & 1 & 5 \\ 1 & 0 & 2 & 4 \end{bmatrix}$$

This result can be obtained from A by interchanging rows 2 and 3.

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ 0 & -4 & 10 & 8 \end{bmatrix}$$

This result can be obtained from A by adding 3 times row 1 to row 3.

Uniqueness of Echelon Form

Two matrices A and B are row-equivalent if and only if they have the same reduced echelon form.

Proof

If A and B have the same reduced echelon form E, then A is row-equivalent to E and E is row-equivalent to B. It follows that A is row-equivalent to B.

Now Suppose A and B are row-equivalent. Let E_1 be a reduced echelon form of A and E_2 be a reduced echelon form of B. Then E_1 and E_2 are row equivalent.

Suppose $E_1 = IF_1 \quad and \quad E_2 = IF_2$. Since E_1 and E_2 are row equivalent, $E_2 = CE_1$ for some matrix C. This means $I = CI$ and $F_2 = CF_1$. But then $C = I \quad and \quad F_2 = F_1$.

Example

Show that the two matrices are row equivalent

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

Solution

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad \begin{matrix} R_1 + R_2 \\ R_2 - 2R_1 \end{matrix}$$

$$= \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

$$= B$$

Definition

A relationship ~ (equivalent) between elements of a set is called an equivalence relation if

- ✓ A ~ A is always true,
- ✓ A ~ B always implies B ~ A,
- ✓ A ~ B and B ~ C always implies A ~ C.

Exercises *Section* 3.4 – Inverse Matrices

1. Apply Gauss-Jordan method to find the inverse of this triangular "Pascal matrix"

Triangular Pascal matrix $A=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix}$

2. If A is invertible and $AB=AC$, prove that $B=C$

3. If $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$, find two matrices $B\neq C$ such that $AB=AC$

4. If A has ***row*** 1 + ***row*** 2 = ***row*** 3, show that A is not invertible

a) Explain why $Ax=(1,\ 0,\ 0)$ can't have a solution.

b) Which right sides $\left(b_1,\ b_2,\ b_3\right)$ might allow a solution to $Ax=b$

c) What happens to ***row*** 3 in elimination?

5. True or false (with a counterexample if false and a reason if true):

a) A 4 by 4 matrix with a row of zeros is not invertible.

b) A matrix with 1's down the main diagonal is invertible.

c) If A is invertible then A^{-1} is invertible.

d) If A is invertible then A^2 is invertible.

6. Do there exist 2 by 2 matrices A and B with real entries such that $AB-BA=I$, where I is the identity matrix?

7. If B is the inverse of A^2, show that AB is the inverse of A.

8. Find and check the inverses (assuming they exist) of these block matrices.

$\begin{bmatrix}I&0\\C&I\end{bmatrix}\quad\begin{bmatrix}A&0\\C&D\end{bmatrix}\quad\begin{bmatrix}0&I\\I&D\end{bmatrix}$

9. For which three numbers c is this matrix not invertible, and why not? $A=\begin{bmatrix}2&c&c\\c&c&c\\8&7&c\end{bmatrix}$

10. Find A^{-1} and B^{-1} (if they exist) by elimination. $A=\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}$ $\quad B=\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}$

11. Find A^{-1} using the Gauss-Jordan method, which has a remarkable inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

12. Find the inverse.

a) $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$

c) $\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

e) $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$

f) $\begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

g) $\begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}$

13. Show that A is not invertible for any values of the entries

$$A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

14. Prove that if A is an invertible matrix and B is row equivalent to A, then B is also invertible.

15. Determine if the given matrix has an inverse, and find the inverse if it exists. Check your answer by multiplying $A \cdot A^{-1} = I$

a) $\begin{bmatrix} 2 & 3 \\ -3 & -5 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}$

16. Show that the inverse of $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is $\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$

Section 3.5 – Determinants and Cramer's Rule

The determinant is a number that contains information about matrix. It is used to find formulas for inverse matrices, pivots, and solutions $A^{-1}b$.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ has inverse } A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Determinant of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is written $\det(A)$ *or* $|A|$ and is define as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinant is zero when the matrix has no inverse.

Properties of the Determinants

There are 3 basic properties (rules 1, 2, 3), by using those rules we can compute the determinant of any square matrix.

1. ***Determinant of the n by n identity matrix is* 1.**

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad \text{and} \quad \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} = 1$$

2. ***Determinant changes sign when 2 rows are exchanged.***

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -(ad - bc) \quad \Rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

3. ***Determinant is a linear function of each row separately***.

Multiply row 1 by any number t: $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Add row 1 of A to row 1 of A ′: $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

- ***For 2 by 2 determinants, if you expand to a rectangle, the determinants equal areas.***
- ***For n-dimensional, the determinants equal volumes.***

4. ***If 2 rows of A are equal, then*** $\det A = 0$.

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$$

5. ***Subtracting a multiple of one row from another row leaves*** $\det A$ ***unchanged.***

$$\begin{vmatrix} a & b \\ c-ta & d-ta \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

6. ***A matrix with a row of zeros has*** $\det A = 0$.

$$\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0 \quad and \quad \begin{vmatrix} 0 & 0 \\ b & c \end{vmatrix} = 0$$

7. ***If A is triangular then*** $\det A = a_{11}a_{22} \ \cdots \ a_{nn}$ ***= product of diagonal entries.***

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad \quad and \quad \begin{vmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{vmatrix} = a_{11}a_{22}\ldots a_{nn}$$

8. ***If A is singular then*** $\det A = 0$. ***If A is invertible then*** $\det A \neq 0$.

9. ***The determinant of AB*** $\det A$ ***is times*** $\det B$ ***:*** $|AB| = |A||B|$

10. ***The transpose*** A^T ***has the same determinant as A:*** $\det(A) = \det\left(A^T\right)$

- $\det(A+B) \neq \det(A) + \det(B)$

Big Formula for Determinants (Diagonal)

Determinant Using Diagonal Method

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix} \qquad a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \quad (1)$$

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix} \qquad -a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \quad (2)$$

Determinant: ***D*** **= (1) + (2)**

$$\textbf{det} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Example

Evaluate: $\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix} \begin{matrix} x & 0 \\ 2 & x \\ -3 & x \end{matrix} = x(x)(1) + 0\left(x^2\right)(2) + (-1)(2)(x) - (-1)(x)(-3) - x\left(x^2\right)(x) - 0(-3)(1)$$

$$= x^2 + 3x - 3x - x^4$$

$$= x^2 - x^4$$

Determinant by *Cofactors*

$$\boldsymbol{A}=[a_{ij}]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor

For a square matrix $A=\left[a_{ij}\right]$, the minor M_{ij}. Of an element a_{ij} is the ***determinant*** of the matrix formed by deleting the i^{th} row and the j^{th} column of A.

Example

Let $A=\begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$ Find M_{32}

Solution

$$M_{32}=\begin{vmatrix} 3 & \cancel{1} & -4 \\ 2 & \cancel{5} & 6 \\ \cancel{1} & \cancel{4} & \cancel{8} \end{vmatrix}$$

$$=\begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix}$$

$$=26$$

Theorem

The determinant is the dot product of any row $\boldsymbol{i}$ of $\boldsymbol{A}$ with its cofactors:

Cofactor Formula: $\boxed{C_{ij}=(-1)^{i+j}M_{ij}}$

$$|\boldsymbol{A}|=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}$$

$$=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example

Find the determinant of the matrix:

$$A = \begin{bmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{bmatrix}$$

Solution

$$|A| = \begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix}$$

$$= -8\begin{vmatrix} -6 & 7 \\ -3 & 5 \end{vmatrix} - 0\begin{vmatrix} 4 & 7 \\ -1 & 5 \end{vmatrix} + 6\begin{vmatrix} 4 & -6 \\ -1 & -3 \end{vmatrix}$$

$= -8(-30 - (-21)) - 0 + 6(-12 - 6)$

$= -8(-9) + 6(-18)$

$= -36$

✓ By the property of determinants, If A is triangular then $\det A = a_{11}a_{22} \ \cdots \ a_{nn}$ = product of diagonal entries.

Example

$$\begin{vmatrix} 2 & 7 & -3 & 8 & 3 \\ 0 & -3 & 7 & 5 & 1 \\ 0 & 0 & 6 & 7 & 6 \\ 0 & 0 & 0 & 9 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix} = (2)(-3)(6)(9)(4) \ \underline{= -1296}$$

Theorem

Let A be any n by n matrix.

a) If A' is the matrix that results when a single row of A is multiplied by a constant k, then $\det(A') = k\det(A)$.

b) If A' is the matrix that results when two rows of A are interchanged, then $\det(A') = -\det(A)$

c) If A' is the matrix that results when a multiple of one row of A is added to another row then $\det(A') = \det(A)$

Cramer's Rule

Theorem

If $AX = B$ is a system of a linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(B_1)}{\det(A)} \quad x_2 = \frac{\det(B_2)}{\det(A)}, \quad \cdots, \quad x_n = \frac{\det(B_n)}{\det(A)}$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & & a_{nn} \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

$$\det(B_1) = \begin{bmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & & & \\ \vdots & & & \\ b_n & a_{n2} & & a_{nn} \end{bmatrix}$$

Example

Use Cramer's rule to solve

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ -2x_1 + x_2 \qquad &= 0 \\ -4x_1 \qquad + x_3 &= 0 \end{aligned}$$

Solution

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 7$$

$$|B_1| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \qquad |B_2| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 2 \qquad |B_3| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 0 \end{vmatrix} = 4$$

$$x_1 = \frac{|B_1|}{|A|} = \frac{1}{7} \qquad x_2 = \frac{|B_2|}{|A|} = \frac{2}{7} \qquad x_3 = \frac{|B_3|}{|A|} = \frac{4}{7}$$

Solution: $\left(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right)$

Example

Use Cramer's Rule to solve.

$$\begin{aligned} x_1 + \quad\ \ + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

Solution

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} \quad \Rightarrow \quad \det(A) = 44$$

$$A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix} \quad \Rightarrow \quad \det(A_1) = -40$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix} \quad \Rightarrow \quad \det(A_2) = 72$$

$$A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix} \quad \Rightarrow \quad \det(A_3) = 152$$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = -\frac{10}{11}$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11}$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$

A Formula for A^{-1}

Theorem: ***Inverse of a matrix using its Adjoint***

The i, j entry of A^{-1} is the cofactor C_{ji} $\left(\textit{not } C_{ij}\right)$ divided by det(***A***):

$$\textbf{\textit{Formula for }} A^{-1}: \quad \left(A^{-1}\right)_{ij} = \frac{C_{ji}}{|A|} \quad \textit{and} \quad A^{-1} = \frac{C^T}{|A|}$$

$$\boxed{A^{-1} = \frac{1}{\det(A)} adj(A)}$$

Example

Find the inverse matrix of $A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ using its adjoint.

Solution

$$C_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1; \quad C_{12} = -\begin{vmatrix} -2 & 0 \\ -4 & 1 \end{vmatrix} = 2; \quad C_{13} = \begin{vmatrix} -2 & 1 \\ -4 & 0 \end{vmatrix} = 4$$

$$C_{21} = -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1; \quad C_{22} = \begin{vmatrix} 1 & 1 \\ -4 & 1 \end{vmatrix} = 5; \quad C_{23} = -\begin{vmatrix} 1 & 1 \\ -4 & 0 \end{vmatrix} = -4$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1; \quad C_{32} = -\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = -2; \quad C_{33} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3$$

$$C = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 5 & -2 \\ 4 & -4 & 3 \end{bmatrix} \text{ and } \det(A) = \tfrac{1}{7} \qquad \Rightarrow \quad A^{-1} = \tfrac{1}{7}\begin{bmatrix} 1 & -1 & -1 \\ 2 & 5 & -2 \\ 4 & -4 & 3 \end{bmatrix}$$

Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent

a) A is invertible

b) $A\boldsymbol{x} = 0$ has only the trivial solution

c) The reduced row echelon form of A is I_n

d) A can be expressed as a product of elementary matrices

e) $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $n \times 1$ matrix $\boldsymbol{b}$

f) $\det(A) \neq 0$

Exercises Section 3.5 – Determinants and Cramer's Rule

1. Verify that $\det(AB)=\det(A)\det(B)$ when: $A=\begin{pmatrix}2&1&0\\3&4&0\\0&0&2\end{pmatrix}$ *and* $B=\begin{pmatrix}1&-1&3\\7&1&2\\5&0&1\end{pmatrix}$

2. For which value(s) of $\boldsymbol{k}$ does A fail to be invertible? $A=\begin{bmatrix}k-3&-2\\-2&k-2\end{bmatrix}$

3. Without directly evaluating, show that $\begin{vmatrix}b+c&c+a&b+a\\a&b&c\\1&1&1\end{vmatrix}=0$

4. If the entries in every row of A add to zero, solve $\boldsymbol{Ax}=0$ to prove det $A = 0$. If those entries add to one, show that det $(A-I)=0$. Does this mean det $A = I$?

5. Does $\det(AB)=\det(BA)$ in general?

a) True or false if $\boldsymbol{A}$ and $\boldsymbol{B}$ are square n x n matrices?

b) True or false if $\boldsymbol{A}$ is m x n and B is n x m with $m\neq n$?

6. True or false, with a reason if true or a counterexample if false:

a) The determinant of $I+A$ is 1 + det $\boldsymbol{A}$.

b) The determinant of ABC is $|A||B||C|$.

c) The determinant of $4A$ is $4|A|$

d) The determinant of $AB-BA$ is zero. (try an example)

e) If A is not invertible then AB is not invertible.

f) The determinant of $A-B$ equals to det A – det B.

7. Use row operations to show the 3 by 3 "Vandermonde determinant" is

$$\det\begin{bmatrix}1&a&a^2\\1&b&b^2\\1&c&c^2\end{bmatrix}=(b-a)(c-a)(c-b)$$

8. The inverse of a 2 by 2 matrix seems to have determinant = 1:

$$\det A^{-1}=\det\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}=\frac{ad-bc}{ad-bc}=1$$

What is wrong with this calculation? What is the correct $\det A^{-1}$

9. A *Hessenberg* matrix is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci's rule $|H_4| = |H_3| + |H_2|$. The same rule will continue for all sizes $|H_n| = |H_{n-1}| + |H_{n-2}|$. Which Fibonacci number is $|H_n|$?

$$H_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad H_3 = \begin{bmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad H_4 = \begin{bmatrix} 2 & 1 & & \\ \mathbf{1} & 2 & \mathbf{1} & \\ \mathbf{1} & 1 & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

10. Evaluate the determinant:

a) $\begin{vmatrix} -1 & 7 \\ -8 & -3 \end{vmatrix}$

b) $\begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix}$

c) $\begin{vmatrix} k-1 & 2 \\ 4 & k-3 \end{vmatrix}$

d) $\begin{vmatrix} c & -4 & 3 \\ 2 & 1 & c^2 \\ 4 & c-1 & 2 \end{vmatrix}$

e) $\begin{vmatrix} 1 & 0 & 3 \\ 4 & 0 & -1 \\ 2 & 8 & 6 \end{vmatrix}$

f) $\begin{vmatrix} x & -3 & 9 \\ 2 & 4 & x+1 \\ 1 & x^2 & 3 \end{vmatrix}$

g) $\begin{vmatrix} 1 & 4 & -3 & 1 \\ 2 & 0 & 6 & 3 \\ 4 & -1 & 2 & 5 \\ 1 & 0 & -2 & 4 \end{vmatrix}$

h) $\begin{vmatrix} -3 & 1 & 2 \\ 6 & 2 & 1 \\ -9 & 1 & 2 \end{vmatrix}$

i) $\begin{vmatrix} 1 & 3 & 2 & -1 \\ 4 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 5 & 1 & 3 & -2 \end{vmatrix}$

j) $\begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & 1 & -2 & 0 \\ 1 & 0 & 3 & -3 \end{vmatrix}$

11. Find all the values of λ for which $\det(\boldsymbol{A}) = 0$

$$a)\ A = \begin{bmatrix} \lambda-1 & -2 \\ 1 & \lambda-4 \end{bmatrix} \qquad b)\ A = \begin{bmatrix} \lambda-6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda-4 \end{bmatrix}$$

12. Prove that if a square matrix $\boldsymbol{A}$ has a column of zeros, then $\det(\boldsymbol{A}) = 0$

13. With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D| \quad \textit{but} \quad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|$$

a) Why is the first statement true? Somehow B doesn't enter.

b) Show by example that equality fails (as shown) when C enters.

c) Show by example that the answer $\det(AD - CB)$ is also wrong.

14. Show that the value of the following determinant is independent of θ.

$$\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{vmatrix}$$

15. Show that the matrices $A=\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ *and* $B=\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$ commute if and only if $\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix}=0$

16. Show that $\det(A)=\frac{1}{2}\begin{vmatrix} tr(A) & 1 \\ tr(A^2) & tr(A) \end{vmatrix}$ for every 2×2 matrix A.

17. What is the maximum number of zeros that a 4×4 matrix can have without a zero determinant? Explain your reasoning.

18. Evaluate *det* ***A***, *det* ***E***, and *det* (***AE***). Then verify that (*det* ***A***)(*det* ***E***) = *det*(***AE***)

$$A=\begin{bmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix}, \qquad E=\begin{bmatrix} 1 & & \\ & 3 & \\ & & 1 \end{bmatrix}$$

19. Show that $\begin{bmatrix} \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma \\ 1 & 1 & 1 \end{bmatrix}$ is not invertible for any values of α, β, γ

20. Use Cramer's Rule with ratios $\frac{\det B_j}{\det A}$ to solve $A\boldsymbol{x} = b$. Also find the inverse matrix $A^{-1}=\frac{C^T}{\det A}$.

Why is the solution $\boldsymbol{x}$ is the first part the same as column 3 of A^{-1}? Which cofactors are involved in computing that column $\boldsymbol{x}$?

$$Ax=b \quad is \quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

21. Verify that $\det(AB)=\det(BA)$ and determine whether the equality $\det(A+B)=\det(A)+\det(B)$ holds

$$A=\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad and \quad B=\begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

22. Verify that $\det(kA)=k^n\det(A)$

$a)\quad A=\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}, \quad k=2$

$b)\quad A=\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}, \quad k=-2$

$c)\quad A=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}, \quad k=3$

23. Solve by using Cramer's rule

a) $\begin{cases} 7x-2y=3 \\ 3x+y=5 \end{cases}$

b) $\begin{cases} 4x+5y \qquad =2 \\ 11x+y+2z=3 \\ x+5y+2z=1 \end{cases}$

c) $\begin{cases} x-4y+z=6 \\ 4x-y+2z=-1 \\ 2x+2y-3z=-20 \end{cases}$

d) $\begin{cases} -x_1-4x_2+2x_3+x_4=-32 \\ 2x_1-x_2+7x_3+9x_4=14 \\ -x_1+x_2+3x_3+x_4=11 \\ -x_1-2x_2+x_3-4x_4=-4 \end{cases}$

e) $\begin{cases} 2x-y+z=-1 \\ 3x+4y-z=-1 \\ 4x-y+2z=-1 \end{cases}$

24. Show that the matrix A is invertible for all values of θ, then find A^{-1} using $A^{-1}=\frac{1}{\det(A)}adj(A)$

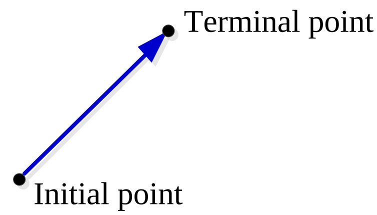
$$A=\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Section 3.6 – Vectors in 2-Space, 3-Space, and *n*-Space

Vectors in two dimensions are also called **2–*space***

Vectors in three dimensions are also called **3–*space*** by arrow

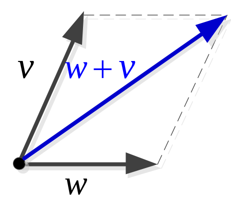
The direction of the arrowhead specifies the ***direction*** of the vector and the ***length*** of the arrow specifies the *magnitude*.



The tail of the arrow is called the ***initial point*** of the vector and the tip the ***terminal point***.

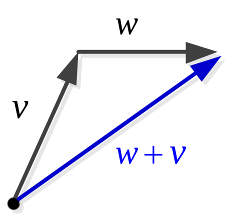
Parallelogram Rule for Vector Addition

If $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors in 2-space or 3-space that are positioned so their initial points coincides, then the vectors form adjacent sides of a parallelogram, and then the sum $\boldsymbol{v}+\boldsymbol{w}$ is the vector represented by the arrow from the common initial point of $\boldsymbol{v}$ and $\boldsymbol{w}$ to the opposite vertex of the parallelogram.

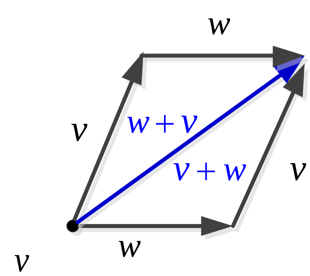


Triangle Rule for Vector Addition

If $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors in 2-space or 3-space that are positioned so the initial point of w is at the terminal point of $\boldsymbol{v}$, then the sum $\boldsymbol{v}+\boldsymbol{w}$ is represented by the arrow from the initial point of $\boldsymbol{v}$ to the terminal point of $\boldsymbol{w}$.



$$\boxed{\boldsymbol{v}+\boldsymbol{w}=\boldsymbol{w}+\boldsymbol{v}}$$



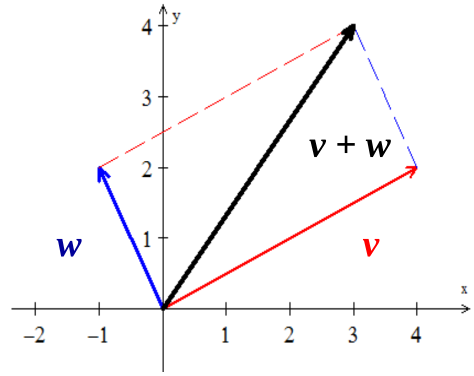
Special Cases:

- A $1\times n$ matrix $\boldsymbol{v}=\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ also written as $\boldsymbol{v}=\left(a_1, a_2, \ldots, a_n\right)$ is called a ***row vector***.
- A $m\times 1$ matrix $\boldsymbol{v}=\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ is called a ***column vector***.

The entries of a row or column vector are called the ***components*** of the vector.

Example of Sum and Difference of vectors

Consider the vector v is given by the component $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and represented by an arrow. The arrow goes from 4 units to the right and 2 units up.

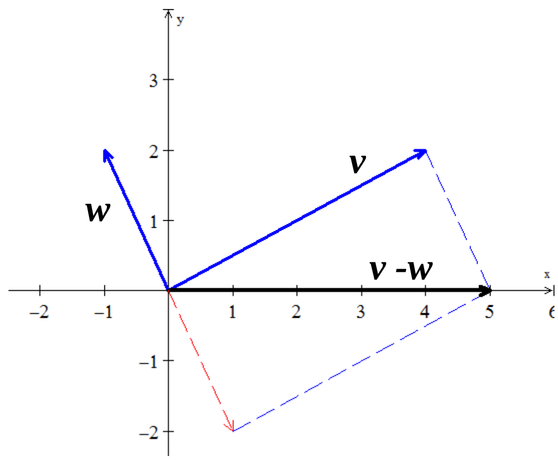


Consider anther vector $w = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Vector addition (head to tail) at the end of v, place the start of w.

The vector addition and w produces the diagonal of a parallelogram.

$$v + w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



$$v - w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

In 3-dimensional space, the arrow starts at the origin $(0,\ 0,\ 0)$, where the *xyz* axis meet.

$v = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is also written as $(1,\ 2,\ 2)$

<u>Notes</u>:

1. The picture of the combinations cu fills a line
2. The picture of the combinations $cu + dv$ fills a plane
3. The picture of the combinations $cu + dv + ew$ fills a 3-dimensional space.

Linear Combination

Definition

The sum of cv and dw is a linear combination of vectors v and w; c, d are constants.

4-Special linear Combinations:

$$\begin{aligned}
1v+1w &= \textit{sum of vectors}\\
1v-1w &= \textit{difference of vectors}\\
0v+0w &= \textit{zero vectors}\\
cv+0w &= \text{vector } cv \textit{ in the direction of } v
\end{aligned}$$

Vectors in Coordinate Systems

It is sometimes necessary to consider vectors whose initial are not at the origin. If $\overrightarrow{P_1P_2}$ denotes the vector with initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$, then the components of this vector are given by the formula

$$\boxed{\overrightarrow{P_1P_2} = (x_2 - x_1,\ y_2 - y_1)}$$

If $P_1(x_1, y_1, z_1)$ *and* $P_2(x_2, y_2, z_2)$

$$\boxed{\overrightarrow{P_1P_2} = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)}$$

Example

The components of the vector $v = \overrightarrow{P_1P_2}$ with initial point $P_1(2,\ -1,\ 4)$ and terminal point $P_2(7,\ 5,\ -8)$, find ***v***?

Solution

$$v = (7-2,\ 5-(-1),\ -8-4) = (5,\ 6,\ -12)$$

n – Space

The vector spaces are denoted by $\boldsymbol{R}^1, \boldsymbol{R}^2, \boldsymbol{R}^3, \boldsymbol{R}^4 \cdots$. Each space $\boldsymbol{R}^n$ consists of a whole collection of vectors.

Definition

The space $\boldsymbol{R}^n$ consists of all column vectors *v* with *n* components.

Example

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad (1,\ 2,\ 3,\ 0,\ 1) \qquad \begin{bmatrix} 1+i \\ 1-i \end{bmatrix}$$

$$\boldsymbol{R}^3 \qquad\qquad \boldsymbol{R}^5 \qquad\qquad \boldsymbol{C}^2$$

The one-dimensional space $\boldsymbol{R}^1$ is a line (like the *x*-axis)

The two essential vector operations go on inside the vector space that we can add any vectors in $\boldsymbol{R}^n$, and we can multiply any vector by any scalar. The ***result*** stays in the space.

A real vector space is a set of "***vectors***" together with rules for vector addition and for multiplication by real numbers. The addition and the multiplication must produce vectors that are in the space.

Here are three other spaces other than $\boldsymbol{R}^n$:

M The vector space of ***all real 2 by 2 matrices***.

F The vector space of ***all real functions*** $f(x)$.

Z The vector space that consists only of a ***zero vector***.

The zero vector in $\boldsymbol{R}^3$ is the vector (0, 0, 0).

Operation on Vectors in R^n

Definition

If n is a positive integer, then an ordered ***n-tuple*** is a sequence of real numbers $\left(v_1, v_2, \ldots, v_n\right)$. The set of all ordered n-tuples is called ***n-space*** and is denoted by $\boldsymbol{R}^n$

Definition

Vectors $\boldsymbol{v} = \left(v_1, v_2, \ldots, v_n\right)$ and $\boldsymbol{w} = \left(w_1, w_2, \ldots, w_n\right)$ in $\boldsymbol{R}^n$ are said to be ***equivalent*** (also called ***equal***) if

$$v_1 = w_1, \quad v_2 = w_2, \quad \ldots \quad v_n = w_n$$

We indicate this by $\boldsymbol{v} = \boldsymbol{w}$

Example

$$(a,b,c,d) = (1,-4,2,7)$$

Solution

$$Iff\ a = 1, \quad b = -4, \quad c = 2, \quad d = 7$$

Vector Space of Infinite Sequences of Real Numbers

If $\boldsymbol{v} = \left(v_1, v_2, \ldots, v_n\right)$ and $\boldsymbol{w} = \left(w_1, w_2, \ldots, w_n\right)$ are vectors in $\boldsymbol{R}^n$, and if $\boldsymbol{k}$ is any scalar, then we defined

$$\begin{aligned}
\boldsymbol{v} + \boldsymbol{w} &= \left(u_1, u_2, \ldots, u_n\right) + \left(w_1, w_2, \ldots, w_n\right) \\
&= \left(v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n\right)
\end{aligned}$$

$$\boldsymbol{kv} = \left(kv_1, kv_2, \ldots, kv_n\right)$$

$$-\boldsymbol{v} = \left(-v_1, -v_2, \ldots, -v_n\right)$$

$$\boldsymbol{w} - \boldsymbol{v} = \boldsymbol{w} + (-\boldsymbol{v}) = \left(w_1 - v_1, w_2 - v_2, \ldots, w_n - v_n\right)$$

The *Zero* Vector Space

Let V consist of a single object, which we denote by 0, and define

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \quad and \quad k\mathbf{0} = \mathbf{0}$$

Theorem

If $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are vectors in $\boldsymbol{R}^n$, and if $\boldsymbol{k}$ and $\boldsymbol{m}$ are scalars, then

a) $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$

b) $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$

c) $\boldsymbol{u}+0=0+\boldsymbol{u}=\boldsymbol{u}$

d) $\boldsymbol{u}+(-\boldsymbol{u})=0$

e) $k(\boldsymbol{u}+\boldsymbol{v})=k\boldsymbol{u}+k\boldsymbol{v}$

f) $(k+m)\boldsymbol{u}=k\boldsymbol{u}+m\boldsymbol{u}$

g) $k(m\boldsymbol{u})=(km)\boldsymbol{u}$

h) $1\boldsymbol{u}=\boldsymbol{u}$

i) $0\boldsymbol{v}=0$

j) $\boldsymbol{k}0=0$

k) $(-1)\boldsymbol{v}=-\boldsymbol{v}$

Proof: $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$

$$\begin{aligned}
(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w} &= \big((u_1,\ u_2,\ \ldots,\ u_n)+(v_1,\ v_2,\ \ldots,\ v_n)\big)+(w_1,\ w_2,\ \ldots,\ w_n)\\
&=(u_1+v_1,\ u_2+v_2,\ \ldots,\ u_n+v_n)+(w_1,\ w_2,\ \ldots,\ w_n)\\
&=\big((u_1+v_1)+w_1,\ (u_2+v_2)+w_2,\ \ldots,\ (u_n+v_n)+w_n\big)\\
&=\big(u_1+(v_1+w_1),\ u_2+(v_2+w_2),\ \ldots,\ u_n+(v_n+w_n)\big)\\
&=(u_1,\ u_2,\ \ldots,\ u_n)+\big((v_1,\ v_2,\ \ldots,\ v_n)+(w_1,\ w_2,\ \ldots,\ w_n)\big)\\
&=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})
\end{aligned}$$

Exercises *Section* 3.6 – Vectors in 2-Space, 3-Space, and *n*-Space

1. Sketch the following vectors with initial points located at the origin

a) $P_1(4,8)\quad P_2(3,7)$ *b)* $P_1(-1,0,2)\quad P_2(0,-1,0)$ *c)* $P_1(3,-7,2)\quad P_2(-2,5,-4)$

2. Find the components of the vector $\overrightarrow{P_1P_2}$

a) $P_1(3,5)\quad P_2(2,8)$ *b)* $P_1(5,-2,1)\quad P_2(2,4,2)$ *c)* $P_1(0,\ 0,\ 0)\quad P_2(-1,\ 6,\ 1)$

3. Find the terminal point of the vector that is equivalent to $\boldsymbol{u}$ = (1, 2) and whose initial point is $A(1,1)$

4. Find the initial point of the vector that is equivalent to $\boldsymbol{u}$ = (1, 1, 3) and whose terminal point is $B(-1,-1,2)$

5. Find a nonzero vector $\boldsymbol{u}$ with initial point $P(-1, 3, -5)$ such that

a) $\boldsymbol{u}$ has the same direction as $\boldsymbol{v}$ = (6, 7, –3)

b) $\boldsymbol{u}$ is oppositely directed as $\boldsymbol{v}$ = (6, 7, –3)

6. Let $\boldsymbol{u}$ = (–3, 1, 2), $\boldsymbol{v}$ = (4, 0, –8), and $\boldsymbol{w}$ = (6, –1, –4). Find the components

a) $v-w$ *c)* $5(v-4u)$ *e)* $(2u-7w)-(8v+u)$

b) $6u+2v$ *d)* $-3(v-8w)$ *f)* $-u+(v-4w)$

7. Let $\boldsymbol{u}$ = (2, 1, 0, 1, –1) and $\boldsymbol{v}$ = (–2, 3, 1, 0, 2). Find scalars *a* and *b* so that $a\boldsymbol{u}+b\boldsymbol{v}=(-8,\ 8,\ 3,\ -1,\ 7)$

8. Find all scalars $c_1,\ c_2,\ and\ c_3$ such that $c_1(1, 2, 0)+c_2(2, 1, 1)+c_3(0, 3, 1)=(0, 0, 0)$

9. Find the distance between the given points $\begin{bmatrix}5 & 1 & 8 & -1 & 2 & 9\end{bmatrix},\ \begin{bmatrix}4 & 1 & 4 & 3 & 2 & 8\end{bmatrix}$

10. Let *V* be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operation on $\boldsymbol{u}=(u_1,\ u_2)\quad \boldsymbol{v}=(v_1,\ v_2)$

$$\boldsymbol{u}+\boldsymbol{v}=(u_1+v_1+1,\ u_2+v_2+1)\qquad k\boldsymbol{u}=(ku_1,\ ku_2)$$

a) Compute $\boldsymbol{u}+\boldsymbol{v}$ and $k\boldsymbol{u}$ for $\boldsymbol{u}$ = (0, 4), $\boldsymbol{v}$ = (1, –3), and *k* = 2.

b) Show that (0, 0) ≠ **0**.

c) Show that (–1, –1) = 0.

d) Show that $\boldsymbol{u}+(-\boldsymbol{u})=0$ for $\boldsymbol{u}=(u_1,\ u_2)$

e) Find two vector space axioms that fail to hold.

Section 3.7 – Linear Dependence and Independence

There are n columns in an m by n matrix, and each column has m components. But the true ***dimension*** of the column space is not necessarily m or n. The dimension is measured by counting ***independent columns***.

- **Independent vectors** (*not too many*)
- **Spanning a space** (*not too few*)

Linear Independence (LI)

The column of A are ***linearly independent*** when the only solution to $Ax=0$ is $x=0$. ***No other combination*** Ax ***of the columns gives the zero vector***.

Definitions

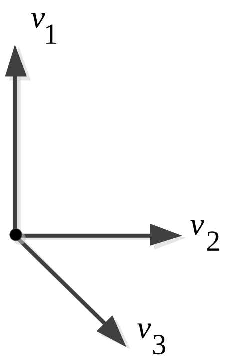
- A set of two or more vectors is ***linearly dependent*** if one vector in the set is a linear combination of the others. A set of one vector is ***linearly dependent*** if that one vector is the zero vector.
$$\vec{0}=0v_1+0v_2+\cdots+0v_k$$
- The sequence of vectors $v_1,\ v_2,\ v_3$ is ***linearly independent*** if the only combination that gives the zero vector is $0v_1+0v_2+\cdots+0v_k$. Thus linear independence means that:
$$x_1v_1+x_2v_2+\cdots+x_kv_k=0 \text{ only happens when all } x\text{'s are zero.}$$
- A (nonempty) set of vectors is ***linearly independent*** if it is not linearly dependent.
- If three vectors $w_1,\ w_2,\ w_3$ are in the same plane, they are dependent.

Theorem

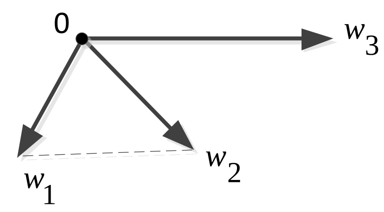
A set S with two or more vectors $S=\{v_1,\cdots,v_k\}$ is

a) Linearly dependent *iff* at least one of the vectors in S is expressible as a linear combination of the other vectors in S. There are numbers $c_1,\ \ldots,\ c_k$ at least one of which is nonzero, such that
$$c_1v_1+c_2v_2+\cdots+c_kv_k=0$$

b) Linearly independent *iff* no vector in S is expressible as a linear combination of the other vectors in S.



Independent vectors v_1, v_2, v_3



Dependent vectors w_1, w_2, w_3

The combination $w_1 - w_2 + w_3$ is (0, 0, 0)

Example

a) The vectors (1, 0) and (0, 1) are ***independent***.

b) The vectors (1, 1) and (1, 0.0001) are ***independent***.

c) The vectors (1, 1) and (2, 2) are ***dependent***.

d) The vectors (1, 1) and (0, 0) are ***dependent***.

Theorem

a) A finite set that contains $\mathbf{0}$ is linearly dependent.

b) A set with exactly one vector is linearly independent if and only if that vector is not $\mathbf{0}$.

c) A set with exactly two vectors is linearly independent *iff* neither vector is a scalar multiple of the other.

Theorem

Let S be a set $\boldsymbol{k}$ vectors in $\boldsymbol{R}^n$, then if $k > n$, S is ***linearly dependent***.

Example

v_1, v_2, v_3 are 3 vectors in $\boldsymbol{R}^2 \Rightarrow$ Linearly dependent.

Example

Determine whether the vectors $v_1 = (1,-2,3) \quad v_2 = (5,6,-1) \quad v_3 = (3,2,1)$ are linearly dependent or linearly independent in $\boldsymbol{R}^3$

Solution

$$k_1 v_1 + k_2 v_2 + k_3 v_3 = \mathbf{0}$$

$$k_1 (1,-2,3) + k_2 (5,6,-1) + k_3 (3,2,1) = (0,0,0)$$

$$\rightarrow \begin{cases} k_1 + 5k_2 + 3k_3 = 0 \\ -2k_1 + 6k_2 + 2k_3 = 0 \\ 3k_1 - k_2 + k_3 = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} k_1 + \frac{1}{2}k_3 = 0 \\ k_2 + \frac{1}{2}k_3 = 0 \end{matrix}$$

Solve the system equations: $k_1 = -\frac{1}{2}t, \quad k_2 = -\frac{1}{2}t, \quad k_3 = t$

This shows that the system has nontrivial solutions and hence that the vectors are linearly dependent.

2nd method to determine the linearly is to compute the determinant of the coefficient matrix

$$A = \begin{pmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{pmatrix}$$

$|A| = 0$ Which has nontrivial solutions and the vectors are linearly dependent.

Example

Determine whether the vectors are linearly dependent or linearly independent in $\boldsymbol{R}^4$

$$v_1 = (1,2,2,-1) \quad v_2 = (4,9,9,-4) \quad v_3 = (5,8,9,-5)$$

Solution

$$k_1 v_1 + k_2 v_2 + k_3 v_3 = \mathbf{0}$$

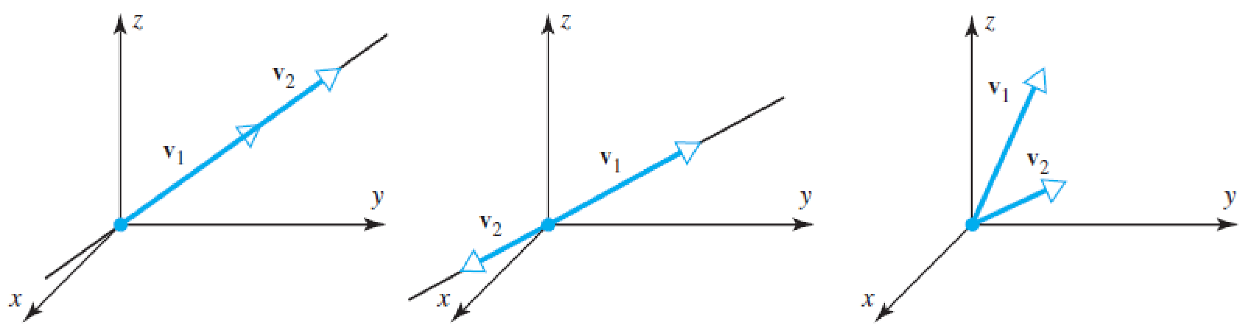
$$k_1 (1,2,2,-1) + k_2 (4,9,9,-4) + k_3 (5,8,9,-5) = (0,0,0)$$

Which yields the homogeneous linear system

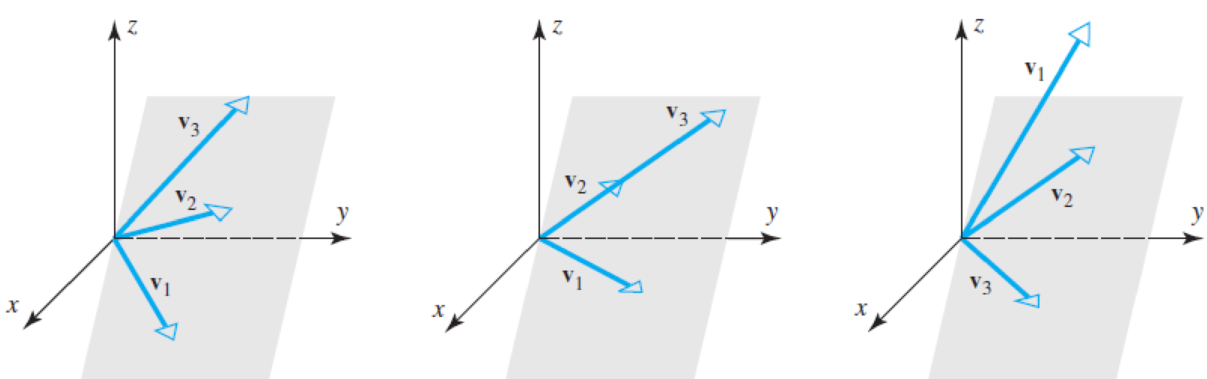
$$\rightarrow \begin{cases} k_1 + 4k_2 + 5k_3 = 0 \\ 2k_1 + 9k_2 + 8k_3 = 0 \\ 2k_1 + 9\,k_2 + 9k_3 = 0 \\ -k_1 - 4\,k_2 - 5k_3 = 0 \end{cases}$$

Solve the system equations: $k_1 = 0, \quad k_2 = 0, \quad k_3 = 0$ has a trivial solution.

The vectors $v_1,\ v_2,\ \textit{and}\ v_3$ are linearly independent.



(a) Linearly dependent $\quad$ (b) Linearly dependent $\quad$ (c) Linearly independent



(a) Linearly dependent $\quad$ (b) Linearly dependent $\quad$ (c) Linearly independent

Linear independence of Functions

Definition

If $\mathbf{f}_1 = f_1(x),\ \mathbf{f}_2 = f_2(x),\ \ldots,\ \mathbf{f}_n = f_n(x)$ are functions that are n - 1 times differentiable on the interval $(-\infty, \infty)$, the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & & f_n(x) \\ f_1'(x) & f_2'(x) & & f_n'(x) \\ & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the ***Wronskian*** of $f_1, f_2, \ldots, f_n$

Example

Use the Wronskian to show that $\mathbf{f}_1 = x,\quad \mathbf{f}_2 = \sin x$ are linearly independence

Solution

The Wronskian is

$$W(x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x \neq 0$$

This function is not identically zero. Thus the functions are linearly independent.

Example

Use the Wronskian to show that $\mathbf{f}_1 = 1,\quad \mathbf{f}_2 = e^x,\quad \mathbf{f}_3 = e^{2x}$ are linearly independence

Solution

The Wronskian is

$$W(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} = e^x 4e^{2x} - 2e^{2x}e^x = 2e^{3x} \neq 0$$

Thus the functions are linearly independent.

Exercises Section 3.7 – Linear Dependence and Independence

1. Given three independent vectors w_1, w_2, w_3. Take combinations of those vectors to produce v_1, v_2, v_3. Write the combinations in a matrix form as $V = WM$.

$$\begin{aligned} v_1 &= w_1 + w_2 \\ v_2 &= w_1 + 2w_2 + w_3 \\ v_1 &= \quad\;\; w_2 + cw_3 \end{aligned} \quad \text{which is } \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix}$$

What is the test on a matrix **V** to see if its columns are linearly independent?

If $c \ne 1$ show that v_1, v_2, v_3 are linearly independent.

If $c = 1$ show that $v's$ are linearly *dependent*.

2. Find the largest possible number of independent vectors among

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad v_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad v_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad v_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

3. Show that $v_1,\ v_2,\ v_3$ are independent but $v_1,\ v_2,\ v_3,\ v_4$ are dependent:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad v_4 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

Solve either $c_1v_1 + c_2v_2 + c_3v_3 = 0$ *or* $Ax = 0$. The v's go in the columns of ***A***.

4. Decide the dependence or independence of

a) The vectors (1, 3, 2) and (2, 1, 3) and (3, 2, 1).

b) The vectors $(1,\ -3,\ 2)$ and $(2,\ 1,\ -3)$ and $(-3,\ 2,\ 1)$.

5. Find two independent vectors on the plane $x + 2y - 3z - t = 0$ in $\boldsymbol{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

6. Determine whether the vectors are linearly dependent or linearly independent in $\boldsymbol{R}^3$

a) (4, –1, 2), (–4, 10, 2)

b) (8, –1, 3), (4, 0, 1)

c) (–3, 0, 4), (5, –1, 2), (1, 1, 3)

d) (–2, 0, 1), (3, 2, 5), (6, –1, 1), (7, 0, –2)

7. Determine whether the vectors are linearly dependent or linearly independent in $\boldsymbol{R}^4$

a) (3, 8, 7, –3), (1, 5, 3, –1), (2, –1, 2, 6), (1, 4, 0, 3)
b) (0, 0, 2, 2), (3, 3, 0, 0), (1, 1, 0, –1)
c) (0, 3, –3, –6), (–2, 0, 0, –6), (0, –4, –2, –2), (0, –8, 4, –4)
d) (3, 0, –3, 6), (0, 2, 3, 1), (0, –2, –2, 0), (–2, 1, 2, 1)

8. *a*) Show that the three vectors $v_1 = (1,2,3,4)$ $v_2 = (0,1,0,-1)$ $v_3 = (1,3,3,3)$ form a linearly dependent set in $\boldsymbol{R}^4$.

b) Express each vector in part (*a*) as a linear combination of the other two.

9. For which real values of λ do the following vectors form a linearly dependent set in $\boldsymbol{R}^3$

$$v_1 = \left(\lambda, -\tfrac{1}{2}, -\tfrac{1}{2}\right) \quad v_2 = \left(-\tfrac{1}{2}, \lambda, -\tfrac{1}{2}\right) \quad v_3 = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, \lambda\right)$$

10. Show that if $S = \{v_1,\ v_2,\ \ldots,\ v_n\}$ is a linearly independent set of vectors, then so is every nonempty subset of S.

11. Show that if $S = \{v_1,\ v_2,\ \ldots,\ v_r\}$ is a linearly dependent set of vectors in a vector space *V*, and if $v_{r+1},\ \ldots,\ v_n$ are vectors in *V* that are not in *S*, then $\{v_1,\ v_2,\ \ldots,\ v_r,\ v_{r+1},\ \ldots,\ v_n\}$ is also linearly dependent.

12. Show that $\{v_1,\ v_2\}$ is linearly independent and v_3 does not lie in span $\{v_1,\ v_2\}$, then $\{v_1,\ v_2,\ v_3\}$ is a linearly independent.

13. By using the appropriate identities, where required, determine $F(-\infty,\ \infty)$ are linearly dependent.

a) $6,\ 3\sin^2 x,\ 2\cos^2 x$
b) $x,\ \cos x$
c) $1,\ \sin x,\ \sin 2x$
d) $(3-x)^2,\ x^2-6x,\ 5$
e) $\cos 2x,\ \sin^2 x,\ \cos^2 x$

14. $f_1(x) = \sin x,\quad f_2(x) = \cos x$ are linearly independent in $F(-\infty,\ \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using Wrоński's test.

15. Use the Wronskian to show that $f_1(x) = \sin x,\quad f_2(x) = \cos x,\quad f_3(x) = x\cos x$ span a three-dimensional subspace of $F(-\infty,\ \infty)$

16. Show by inspection that the vectors are linearly dependent.

$\boldsymbol{v}_1(4,\ -1,\ 3),\ \ \boldsymbol{v}_2(2,\ 3,\ -1),\ \ \boldsymbol{v}_3(-1,\ 2,\ -1),\ \ \boldsymbol{v}_4(5,\ 2,\ 3),\ \ in\ \mathbb{R}^3$

17. Determine if the given vectors are linearly dependent or independent, (any method)

a) $(2,\ -1,\ 3),\ \ (3,\ 4,\ 1),\ \ (2,\ -3,\ 4),\ \ in\ \mathbb{R}^3$.

b) $(1,\ 0,\ 0,\ 0),\ \ (1,\ 1,\ 0,\ 0),\ \ (1,\ 1,\ 1,\ 0),\ \ (1,\ 1,\ 1,\ 1),\ \ in\ \mathbb{R}^4$.

c) $A_1\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix},\ \ A_2\begin{bmatrix} -2 & 4 \\ 1 & 0 \end{bmatrix},\ \ A_3\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix},\ \ in\ M_{22}$

18. Suppose that the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \textit{ and } \boldsymbol{u}_3$ are linearly dependent. Are the vectors $\boldsymbol{v}_1 = \boldsymbol{u}_1 + \boldsymbol{u}_2$, $\boldsymbol{v}_2 = \boldsymbol{u}_1 + \boldsymbol{u}_3$, and $\boldsymbol{v}_3 = \boldsymbol{u}_2 + \boldsymbol{u}_3$ also linearly dependent?

(***Hint***: Assume that $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 = 0$, and see what the $a_i\,'s$ can be.)

Section 3.8 – Dot Product and Orthogonality

Norm of a Vector

The ***length*** (or ***norm***) of a vector v is the square root of $v.v$

$$Length = \|v\| = \sqrt{v.v}$$

$$= \sqrt{x^2 + y^2} \qquad \textbf{2-\textit{dimension}}$$

$$= \sqrt{x^2 + y^2 + z^2} \qquad \textbf{3-\textit{dimension}}$$

Definition

If $\boldsymbol{v} = \left(v_1,\ v_2,\ \dots,\ v_n\right)$ is a vector in $\boldsymbol{R}^n$, then the norm of $\boldsymbol{v}$ (also called the length of $\boldsymbol{v}$ or the magnitude of $\boldsymbol{v}$) is denoted by $\|\boldsymbol{v}\|$, and is defined by the formula

$$\boxed{\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \dots + v_n^2}}$$

Example

Find the length of the vector $v = (1,\ 2,\ 3)$

Solution

$$\|v\| = \sqrt{1^2 + 2^2 + 3^2}$$

$$= \sqrt{14}$$

Theorem

If $\boldsymbol{v}$ is a vector in $\boldsymbol{R}^n$, and if $\boldsymbol{k}$ is any scalar, then:

a) $\|\boldsymbol{v}\| \ge 0$

b) $\|\boldsymbol{v}\| = 0 \quad iff \quad \boldsymbol{v} = 0$

c) $\|k\boldsymbol{v}\| = |k| \cdot \|\boldsymbol{v}\|$

Unit Vectors

Definition

A ***unit vector*** u is a vector whose length equals to one. Then $u.u = 1$

Divide any nonzero vector v by its length. Then $u = \frac{v}{\|v\|}$ is a unit vector in the same direction as v.

Example

Find the unit vector $\boldsymbol{u}$ that has the same direction as $\boldsymbol{v} = (2, 2, -1)$

Solution

$$\|\boldsymbol{v}\| = \sqrt{2^2 + 2^2 + (-1)^2} = \underline{3}$$

$$\begin{aligned} u &= \frac{v}{\|v\|} \\ &= \frac{1}{3}(2,\ 2,\ -1) \\ &= \underline{\left(\frac{2}{3},\ \frac{2}{3},\ -\frac{1}{3}\right)} \end{aligned}$$

$$\begin{aligned} \|u\| &= \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{-1}{3}\right)^2} \\ &= \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} \\ &= \sqrt{\frac{9}{9}} \\ &= \underline{1} \quad \checkmark \end{aligned}$$

Example of unit vectors

$$i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

In $\boldsymbol{R}^3$ $\quad \boldsymbol{i} = (1,\ 0,\ 0) \quad \boldsymbol{j} = (0,\ 1,\ 0)$ *and* $\boldsymbol{k} = (0,\ 0,\ 1)$

In general, these formulas can be defined as ***standard unit vector*** in $\boldsymbol{R}^n$

$$\boldsymbol{e}_1 = (1,\ 0,\ \cdots,\ 0), \quad \boldsymbol{e}_2 = (0,\ 1,\ \cdots,\ 0), \quad \ldots, \quad \boldsymbol{e}_, = (0,\ 0,\ \cdots,\ 1)$$

$$\boldsymbol{v} = (v_1,\ v_2,\ \ldots,\ v_n) = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + \ldots + v_n\boldsymbol{e}_n$$

Example $\quad (7,\ 3,\ -4,\ 5) = 7\boldsymbol{e}_1 + 3\boldsymbol{e}_2 - 4\boldsymbol{e}_3 + 5\boldsymbol{e}_4$

Distance in $\boldsymbol{R}^n$

Definition

If $\boldsymbol{u} = (u_1,\ u_2,\ \ldots,\ u_n)$ and $\boldsymbol{v} = (v_1,\ v_2,\ \ldots,\ v_n)$ are points in $\boldsymbol{R}^n$, then we denote the distance between u and v by $d(\boldsymbol{u},\ \boldsymbol{v})$ and define it to be

$$d(\boldsymbol{u},\ \boldsymbol{v}) = \|\boldsymbol{u}-\boldsymbol{v}\| = \sqrt{(u_1-v_1)^2+(u_2-v_2)^2+\ldots+(u_n-v_n)^2}$$

In $\boldsymbol{R}^2$ $\quad d = \left\|\overrightarrow{P_1P_2}\right\| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

In $\boldsymbol{R}^3$ $\quad d(\boldsymbol{u},\ \boldsymbol{v}) = \left\|\overrightarrow{P_1P_2}\right\| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

Dot Product

If ***u*** and ***v*** are nonzero vectors in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$, and if θ is he angle between ***u*** and ***v***, then the ***dot product*** (also called the ***Euclidean inner product***) of ***u*** and ***v*** is denoted by $\boldsymbol{u}\cdot\boldsymbol{v}$ and is defined as

$$\boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\ \|\boldsymbol{v}\|\ \cos\theta$$

Cosine Formula

If u and v are nonzero vectors that implies $\Rightarrow$ $\cos\theta = \dfrac{u.v}{\|u\|.\|v\|}$

Example

Find the dot product of the vectors ***u*** = (0, 0, 1) and ***v*** = (0, 2, 2) and have an angle of 45°.

Solution

$$\|u\| = 1 \quad and \quad \|v\| = \sqrt{0+2^2+2^2} = \sqrt{8} = 2\sqrt{2}$$

$$\begin{aligned} u\cdot v &= \|u\|\|v\|\cos\theta \\ &= (1)\left(2\sqrt{2}\right)\cos 45° \\ &= \left(2\sqrt{2}\right)\frac{1}{\sqrt{2}} \\ &= 2 \end{aligned}$$

Component Form of the Dot Product

The ***dot product*** or ***inner product*** of $v=\left(v_1,\ v_2\right)$ and $w=\left(w_1,\ w_2\right)$ is the number

$$vw=v_1w_1+v_2w_2$$

Example

Find the dot product of $v=(4,\ 2)$ and $w=(-1,\ 2)$

Solution

$\boxed{v.w=4.(-1)+2(2)=0}$

- ***For dot products, zero means that the 2 vectors are perpendicular*** (= 90°).

Example

Put a weight of 4 at the point $x=-1$ and weight of 2 at the point $x=2$. The *x*-axis will balance on the center point $x=0$.

Solution

The weight balance is $4(-1)+2(2)=0$ (*dot product*).

In 3-dimensionals the dot product:

$$\left(v_1,\ v_2,\ v_3\right).\left(w_1,\ w_2,\ w_3\right)=v_1w_1+v_2w_2+v_3w_3$$

Theorem

a) $\boldsymbol{u}\cdot\boldsymbol{v}=\boldsymbol{v}\cdot\boldsymbol{u}$

b) $\boldsymbol{u}\cdot\left(\boldsymbol{v}+\boldsymbol{w}\right)=\boldsymbol{u}\cdot\boldsymbol{v}+\boldsymbol{u}\cdot\boldsymbol{w}$

c) $\boldsymbol{u}\cdot\left(\boldsymbol{v}-\boldsymbol{w}\right)=\boldsymbol{u}\cdot\boldsymbol{v}-\boldsymbol{u}\cdot\boldsymbol{w}$

d) $\left(\boldsymbol{u}+\boldsymbol{v}\right)\cdot\boldsymbol{w}=\boldsymbol{u}\cdot\boldsymbol{w}+\boldsymbol{v}\cdot\boldsymbol{w}$

e) $\left(\boldsymbol{u}-\boldsymbol{v}\right)\cdot\boldsymbol{w}=\boldsymbol{u}\cdot\boldsymbol{w}-\boldsymbol{v}\cdot\boldsymbol{w}$

f) $k\left(\boldsymbol{u}\cdot\boldsymbol{v}\right)=\left(k\boldsymbol{u}\right)\cdot\boldsymbol{v}$

g) $k\left(\boldsymbol{u}\cdot\boldsymbol{v}\right)=\boldsymbol{u}\cdot\left(k\boldsymbol{v}\right)$

h) $\boldsymbol{v}\cdot\boldsymbol{v}\geq 0$ *and* $\boldsymbol{v}\cdot\boldsymbol{v}=0$ *iff* $\boldsymbol{v}=0$

i) $0\cdot\boldsymbol{v}=\boldsymbol{v}\cdot 0=0$

Right Angles

The dot product is $v.w = 0$ when v is ***perpendicular*** to w.

Proof

Perpendicular vectors: $\|v\|^2 + \|w\|^2 = \|v - w\|^2$

$$
\begin{aligned}
v_1^2 + v_2^2 + w_1^2 + w_2^2 &= \left(v_1 - w_1\right)^2 + \left(v_2 - w_2\right)^2 \\
&= v_1^2 - 2v_1w_1 + w_1^2 + v_2^2 - 2v_2w_2 + w_2^2 \\
&= v_1^2 + w_1^2 + v_2^2 + w_2^2 - 2\left(v_1w_1 + v_2w_2\right) \qquad v_1w_1 + v_2w_2 = 0 \quad \textit{dot product} \\
&= v_1^2 + w_1^2 + v_2^2 + w_2^2
\end{aligned}
$$

If u and U are unit vectors, then $u.U = \cos\theta$

Certainly,

$$|u.U| \le 1$$

$$-1 \le \cos\theta \le 1$$

$$-1 \le \textit{dot product} \le 1$$

Schwarz Inequality

If v and w are any vectors $\Rightarrow$ $\|v.w\| \le \|v\|.\|w\|$

Proof

The dot product of $v = (a,\ b)$ and $w = (b,\ a)$ is $2ab$ and both lengths are $\sqrt{a^2 + b^2}$.

Then, the Schwarz inequality says that: $2ab \le a^2 + b^2$

$$a^2 + b^2 - 2ab = \left(a - b\right)^2 \ge 0$$

$$a^2 + b^2 - 2ab \ge 0$$

$$a^2 + b^2 \ge 2ab$$

This proves the Schwarz inequality: $2ab \le a^2 + b^2 \Rightarrow \|v.w\| \le \|v\|.\|w\|$

Orthogonality

Definition

Two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\boldsymbol{R}^n$ are said to be ***orthogonal*** (or ***perpendicular***) if their dot product is zero $\boldsymbol{u.v} = 0$.

We will also agree that he zero vector in $\boldsymbol{R}^n$ is orthogonal to every vector in $\boldsymbol{R}^n$. A nonempty set of vectors $\boldsymbol{R}^n$ is called an ***orthogonal set*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set of unit vectors is called an ***orthonormal set***.

Example

The floor of your room (extended to infinity) is a subspace $\boldsymbol{V}$. The line where two walls meet is a subspace $\boldsymbol{W}$ (one-dimensional). Those subspaces are orthogonal. Every vector up the meeting line is perpendicular to every vector on the floor. The origin (0, 0, 0) is in the corner.

Example

Show that $\boldsymbol{u} = (-2, 3, 1, 4)$ and $\boldsymbol{v} = (1, 2, 0, -1)$ are orthogonal in $\boldsymbol{R}^4$

Solution

$$\begin{aligned} \boldsymbol{u.v} &= (-2)(1) + (3)(2) + (1)(0) + (4)(-1) \\ &= -2 + 6 + 0 - 4 \\ &= 0 \end{aligned}$$

These vectors are orthogonal in $\boldsymbol{R}^4$

Standard Unit Vectors

$$\boldsymbol{i} \bullet \boldsymbol{j} = \boldsymbol{i} \bullet \boldsymbol{k} = \boldsymbol{j} \bullet \boldsymbol{k} = \mathbf{0}$$

Proof

$$\boldsymbol{i} \bullet \boldsymbol{j} = (1, 0, 0) \bullet (0, 1, 0) = \mathbf{0}$$

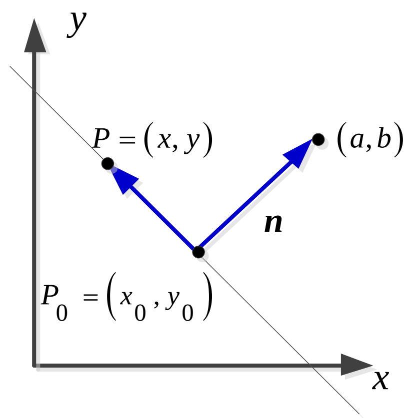
Normal

To specify slope and inclination is to use a nonzero vector ***n***, called a ***normal***, which is orthogonal to the line or plane.

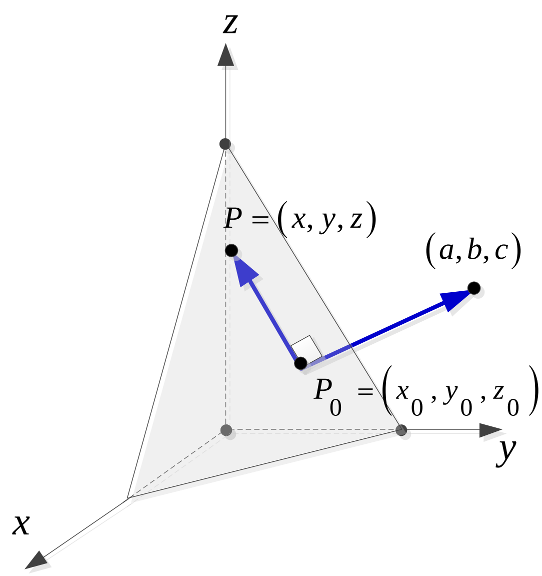
The line passes through a point $P_0\left(x_0, y_0\right)$ that has a normal ***n*** = (*a*, *b*) and the plane through $P_0\left(x_0, y_0, z_0\right)$ that has a normal ***n*** = (*a*, *b*, *c*). Both the line and the plane are represented by the vector equation

$$\boldsymbol{n} \bullet \overrightarrow{P_0 P} = 0$$

The line equation: $a\left(x - x_0\right) + b\left(y - y_0\right) = 0$



The plane equation: $a\left(x - x_0\right) + b\left(y - y_0\right) + c\left(z - z_0\right) = 0$



Exercises *Section* 3.8 – Dot Product and Orthogonality

1. If $\|\vec{v}\| = 5$ and $\|\vec{w}\| = 3$, what are the smallest and largest possible values of $\|\vec{v} - \vec{w}\|$ and $\vec{v}.\vec{w}$?

2. If $\|\vec{v}\| = 7$ and $\|\vec{w}\| = 3$, what are the smallest and largest possible values of $\|\vec{v} + \vec{w}\|$ and $\vec{v}.\vec{w}$?

3. Given that $\cos(\alpha) = \frac{v_1}{\|v\|}$ and $\sin(\alpha) = \frac{v_2}{\|v\|}$. Similarly, $\cos(\beta) =$ ______ and $\sin(\beta) =$ ______. The angle θ is $\beta - \alpha$. Substitute into the trigonometry formula $\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ for $\cos(\beta - \alpha)$ to find $\cos(\theta) = \frac{v.w}{\|v\|.\|w\|}$

4. Can three vectors in the *xy* plane have $u.v < 0$, $v.w < 0$ and $u.w < 0$?

5. Find the norm of *v*, a unit vector that has the same direction as *v*, and a unit vector that is oppositely directed.

a) $v = (4, -3)$

b) $v = (1, -1, 2)$

c) $v = (-2, 3, 3, -1)$

6. Evaluate the given expression with $\boldsymbol{u} = (2, -2, 3)$, $\boldsymbol{v} = (1, -3, 4)$, and $\boldsymbol{w} = (3, 6, -4)$

a) $\|u + v\|$

b) $\|-2u + 2v\|$

c) $\|3u - 5v + w\|$

d) $\|3v\| - 3\|v\|$

e) $\|u\| + \|-2v\| + \|-3w\|$

7. Let $\boldsymbol{v} = (1, 1, 2, -3, 1)$. Find all scalars *k* such that $\|kv\| = 5$

8. Find $\boldsymbol{u} \cdot \boldsymbol{v}$, $\boldsymbol{u} \cdot \boldsymbol{u}$, *and* $\boldsymbol{v} \cdot \boldsymbol{v}$

a) $u = (3, 1, 4)$, $v = (2, 2, -4)$

b) $u = (1, 1, 4, 6)$, $v = (2, -2, 3, -2)$

c) $u = (2, -1, 1, 0, -2)$, $v = (1, 2, 2, 2, 1)$

9. Find the Euclidean distance between $\boldsymbol{u}$ and $\boldsymbol{v}$, then find the angle between them

a) $u = (3, 3, 3)$, $v = (1, 0, 4)$

b) $u = (1, 2, -3, 0)$, $v = (5, 1, 2, -2)$

c) $u = (0, 1, 1, 1, 2)$, $v = (2, 1, 0, -1, 3)$

10. Find a unit vector that has the same direction as the given vector

a) $(-4, -3)$

b) $\left(-3,\ 2,\ \sqrt{3}\right)$

c) $(1, 2, 3, 4, 5)$

11. Find a unit vector that is oppositely to the given vector

a) $(-12, -5)$

b) $(3, -3, 3)$

c) $\left(-3,\ 1,\ \sqrt{6},\ 3\right)$

12. Verify that the Cauchy-Schwarz inequality holds

a) $u = (-3, 1, 0)$, $v = (2, -1, 3)$

b) $u = (0, 2, 2, 1)$, $v = (1, 1, 1, 1)$

c) $u = (1, 3, 5, 2, 0, 1)$, $v = (0, 2, 4, 1, 3, 5)$

13. Find $\boldsymbol{u} \bullet \boldsymbol{v}$ and then the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$ $\quad \boldsymbol{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$

14. Find the norm: $\|\boldsymbol{u}\| + \|\boldsymbol{v}\|$, $\|\boldsymbol{u} + \boldsymbol{v}\|$ for $\boldsymbol{u} = (3,\ -1,\ -2,\ 1,\ 4)$ $\quad \boldsymbol{v} = (1,\ 1,\ 1,\ 1,\ 1)$

15. Find all numbers r such that: $\|r(1,\ 0,\ -3,\ -1,\ 4,\ 1)\| = 1$

16. Find the distance between $P_1(7, -5, 1)$ and $P_2(-7, -2, -1)$

17. Given $\boldsymbol{u} = (1, -5, 4)$, $\boldsymbol{v} = (3, 3, 3)$

a) Find $\boldsymbol{u} \cdot \boldsymbol{v}$

b) Find the cosine of the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$.

18. Determine whether $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

a) $\boldsymbol{u} = (-6,\ -2)$, $\quad \boldsymbol{v} = (5,\ -7)$

b) $\boldsymbol{u} = (6,\ 1,\ 4)$, $\quad \boldsymbol{v} = (2,\ 0,\ -3)$

c) $\boldsymbol{u} = (1,\ -5,\ 4)$, $\quad \boldsymbol{v} = (3,\ 3,\ 3)$

d) $\boldsymbol{u} = (-2,\ 2,\ 3)$, $\quad \boldsymbol{v} = (1,\ 7,\ -4)$

19. Determine whether the vectors form an orthogonal set

a) $\boldsymbol{v}_1 = (2,\ 3)$, $\quad \boldsymbol{v}_2 = (3,\ 2)$

b) $\boldsymbol{v}_1 = (1,\ -2)$, $\quad \boldsymbol{v}_2 = (-2,\ 1)$

c) $\boldsymbol{u} = (-4,\ 6,\ -10,\ 1)$ $\quad \boldsymbol{v} = (2,\ 1,\ -2,\ 9)$

d) $\boldsymbol{u} = (a,\ b)$ $\quad \boldsymbol{v} = (-b,\ a)$

e) $\boldsymbol{v}_1 = (-2,\ 1,\ 1)$, $\quad \boldsymbol{v}_2 = (1,\ 0,\ 2)$, $\quad \boldsymbol{v}_3 = (-2,\ -5,\ 1)$

f) $\boldsymbol{v}_1 = (1,\ 0,\ 1)$, $\quad \boldsymbol{v}_2 = (1,\ 1,\ 1)$, $\quad \boldsymbol{v}_3 = (-1,\ 0,\ 1)$

g) $\boldsymbol{v}_1 = (2, -2, 1)$, $\quad \boldsymbol{v}_2 = (2, 1, -2)$, $\quad \boldsymbol{v}_3 = (1, 2, 2)$

20. Find a unit vector that is orthogonal to both $\boldsymbol{u} = (1, 0, 1)$ and $\boldsymbol{v} = (0, 1, 1)$

21. *a*) Show that $\boldsymbol{v} = (a, b)$ and $\boldsymbol{w} = (-b, a)$ are orthogonal vectors.

b) Use the result to find two vectors that are orthogonal to $\boldsymbol{v} = (2, -3)$.

c) Find two unit vectors that are orthogonal to $(-3, 4)$

22. Show that if $\boldsymbol{v}$ is orthogonal to both $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, then $\boldsymbol{v}$ is orthogonal to $k_1\boldsymbol{w}_1 + k_2\boldsymbol{w}_2$ for all scalars k_1 and k_2.

23. Show that $\vec{u}-\vec{v}$ is orthogonal to $\vec{u}+\vec{v}$ if and only if $\|\vec{u}\|=\|\vec{v}\|$

24. Given $\boldsymbol{u}=(3,\ -1,\ 2)$ $\boldsymbol{v}=(4,\ -1,\ 5)$ *and* $\boldsymbol{w}=(8,\ -7,\ -6)$

a) Find $3\boldsymbol{v}-4(5\boldsymbol{u}-6\boldsymbol{w})$

b) Find $\boldsymbol{u}\bullet\boldsymbol{v}$ and then the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$.

25. *a*) Show that $\boldsymbol{v}=(a, b)$ and $\boldsymbol{w}=(-b, a)$ are orthogonal vectors

b) Use the result in part (*a*) to find two vectors that are orthogonal to $\boldsymbol{v}=(2, -3)$

c) Find two unit vectors that are orthogonal to $(-3, 4)$

26. Show that A (3, 0, 2), .B (4, 3, 0), and C (8, 1, –1) are vertices of a right triangle. At which vertex is the right angle?

27. Establish the identity: $\boldsymbol{u}\bullet\boldsymbol{v}=\frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2-\frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2$

Section 3.9 – Eigenvalues and Eigenvectors

In many problems in science and mathematics, linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ come from steady state problems. Eigenvalues have their greatest importance in dynamic problems. The solution of $Ax = \lambda x$ *or* $\frac{dx}{dt} = Ax$ (is changing with time) has nonzero solutions. (***All matrices are square***)

Definition

Suppose A is an n x n matrix and

$$\boldsymbol{\lambda x} = \boldsymbol{Ax}$$

The values of λ are called eigenvalues of the matrix $\boldsymbol{A}$ and the nonzero vectors $\boldsymbol{x}$ in $\boldsymbol{R}^n$ are called the eigenvectors corresponding to that eigenvalue (λ).

- One of the meanings of the word "***eigen***" in German is "***proper***"; eigenvalues are also called ***proper values, characteristic values,*** or ***latent roots***.

Example

The vector $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 3$ since

$$Ax = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3x$$

Eigenvalues and eigenvectors have a useful geometric interpretation in $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$.

The equation for the *eigenvalues*

Let's rewrite the equation $Ax = \lambda x$.

$$Ax - \lambda x = 0 \qquad \lambda: \text{ are the eigenvalues and not a vector}$$

$$Ax - \lambda I x = 0$$

$$(A - \lambda I)x = 0$$

The matrix $A - \lambda I$ times the eigenvectors $\boldsymbol{x}$ is the zero vector. The eigenvectors makes up the nullspace of $A - \lambda I$.

Definition

The number λ is an eigenvalue of $\boldsymbol{A}$ if and only if $A - \lambda I$ is singular:

$$\boxed{\det(A - \lambda I) = 0}$$

This is called ***characteristic equation*** of $\boldsymbol{A}$; the scalars satisfying this equation are the eigenvalues of $\boldsymbol{A}$. when expanding the determinant $\det(A - \lambda I)$ is a polynomial in λ called the ***characteristic polynomial*** of $\boldsymbol{A}$.

Example

Find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

Solution

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{bmatrix}\right)$$

$$= (3-\lambda)(-\lambda) + 2$$

$$= \lambda^2 - 3\lambda + 2$$

The characteristic equation of $\boldsymbol{A}$ is:

$\lambda^2 - 3\lambda + 2 = 0 \ \Rightarrow\ \boxed{\lambda_1 = 1} \quad \boxed{\lambda_2 = 2}$; these are the eigenvalues of $\boldsymbol{A}$.

Theorem

If $\boldsymbol{A}$ is an n x n triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of $\boldsymbol{A}$ are the entries on the main diagonal of $\boldsymbol{A}$.

Example

Find the eigenvalues of the lower triangular matrix

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{3}{2} & 0 \\ 5 & -8 & -\frac{1}{4} \end{pmatrix}$$

Solution

The eigenvalues are: $\lambda = \frac{1}{2}, \quad \lambda = \frac{3}{2}, \quad \textit{and} \quad \lambda = -\frac{1}{4}$

Theorem

If $\boldsymbol{A}$ is an n x n matrix, the following are equivalent.

a) λ is an eigenvalue of $\boldsymbol{A}$.

b) The system of equations $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ has nontrivial solutions.

c) There is a nonzero vector $\boldsymbol{x}$ in $\boldsymbol{R}^n$ such that $Ax = \lambda x$.

d) λ is a real solution of the characteristic equation $\det(A - \lambda I) = 0$

Eigenvectors

To find the eigenvector $\boldsymbol{x}$, for each eigenvalue λ solve $(A-\lambda I)x=0$ *or* $Ax=\lambda x$

From the eigenvalues, the eigenvectors, in the form $\boldsymbol{V}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$, of the system can be determined by letting:

$$(A-\lambda_1 I)V_1 = 0 \qquad \text{and} \ (A-\lambda_2 I)V_2 = 0$$

Example

Find the eigenvalues and the eigenvectors of the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

Solution

$$\begin{aligned} \det(A-\lambda I) &= \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} \\ &= (1-\lambda)(4-\lambda)-4 \\ &= \lambda^2 - 5\lambda + 4 - 4 \\ &= \lambda^2 - 5\lambda \\ &= \lambda(\lambda-5) = \mathbf{0} \end{aligned}$$

The eigenvalues of $\boldsymbol{A}$ are: $\lambda_1 = 0 \quad \lambda_2 = 5$

For $\lambda_1 = 0$, we have:

$$(A-\lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 1-0 & 2 \\ 2 & 4-0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} x+2y=0 \\ 2x+4y=0 \end{cases} \Rightarrow x=-2y$$

If $y=-1 \Rightarrow x=2$, therefore the eigenvector $V_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Or $\begin{pmatrix} -2y \\ y \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \end{pmatrix} \Rightarrow V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ *or* $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

For $\lambda_2 = 5$, we have:

$$\left(A-\lambda_2 I\right)V_2 = 0$$

$$\begin{pmatrix} 1-5 & 2 \\ 2 & 4-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} -4x+2y=0 \\ 2x-y=0 \end{cases} \Rightarrow 2x = y$$

If $x = 1 \Rightarrow y = 2$, therefore the eigenvector $V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Power of a Matrix

Theorem

If k is a positive integer, λ is an eigenvalue of a matrix A, and $\boldsymbol{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\boldsymbol{x}$ is a corresponding eigenvector.

Example

Find the eigenvalues of A^7 for $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

Solution

$$\det\left(A-\lambda I\right) = \begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix} = \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

The eigenvalues of A: $\lambda = 1$ *and* $\lambda = 2$

The eigenvalues of A^7 are: $\boxed{\lambda = 1^7 = 1}$ *and* $\boxed{\lambda = 2^7 = 128}$

Theorem

A square matrix A is invertible *iff* $\lambda = 0$ is not an eigenvalue of A.

Summary

To solve the eigenvalue problem for an n by n matrix:

1. **1.** Compute the determinant of $A-\lambda I$. With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It is a polynomial in λ of degree n.
2. **2.** Find the roots of this polynomial, by solving $\det(A-\lambda I)=0$. The n roots are the n eigenvalues of A. They make $A-\lambda I$ singular.
3. **3.** For each eigenvalue λ, solve $(A-\lambda I)\mathbf{x}=\mathbf{0}$ ***to find an eigenvector x***.

Imaginary Eigenvalues

Example

Find the eigenvalues and the eigenvectors of the matrix $A=\begin{pmatrix}-2 & -1\\ 5 & 2\end{pmatrix}$

Solution

$$\begin{aligned}\det(A-\lambda I)&=\begin{vmatrix}-2-\lambda & -1\\ 5 & 2-\lambda\end{vmatrix}\\ &=(-2-\lambda)(2-\lambda)+5\\ &=\lambda^2-4+5\\ &=\lambda^2+1=0\end{aligned}$$

$$\Rightarrow \lambda^2=-1$$

The solutions are: $\lambda=\pm i$.

$\lambda_1=i:\left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix}-2-i & -1\\ 5 & 2-i\end{pmatrix}\begin{pmatrix}x_1\\ y_1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\to\begin{cases}(-2-i)x_1-y_1=0\\ 5x_1+(2-i)y_1=0\end{cases}\Rightarrow\begin{cases}(-2-i)x_1=y_1\\ 5x_1=-(2-i)y_1\end{cases}$$

Therefore the eigenvector $V_1=\begin{pmatrix}-1\\ 2+i\end{pmatrix}$

$\lambda_1=-i:\left(A-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix}-2+i & -1\\ 5 & 2+i\end{pmatrix}\begin{pmatrix}x_2\\ y_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\to\begin{cases}(-2+i)x_2-y_2=0\\ 5x_2+(2+i)y_2=0\end{cases}\Rightarrow\begin{cases}(-2+i)x_2=y_2\\ 5x_2=-(2+i)y_2\end{cases}$$

Therefore the eigenvector $V_2=\begin{pmatrix}1\\ -2+i\end{pmatrix}$

Example

Find the eigenvalues and the eigenvectors of the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 = -1 \quad \rightarrow \quad \lambda_1 = i \quad \lambda_2 = -i$$

The matrix $\boldsymbol{A}$ is a 90° rotation which has no real eigenvalues or eigenvectors.

No vector $\boldsymbol{Ax}$ stays in the same direction as $\boldsymbol{x}$ (except the zero vector which is useless).

If we add the eigenvalues together the result is zero which is the trace of $\boldsymbol{A}$.

$$\lambda_1 = i : \left(A - \lambda_1 I\right) V_1 = 0$$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -ix + y = 0 \\ -x - iy = 0 \end{cases} \Rightarrow x = -iy$$

Therefore the eigenvector $V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$$\lambda_2 = -i : \left(A - \lambda_2 I\right) V_2 = 0$$

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} ix + y = 0 \\ -x + iy = 0 \end{cases} \Rightarrow x = iy$$

Therefore the eigenvector $V_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

Exercises Section 3.9 – Eigenvalues and Eigenvectors

1. Find the eigenvalues and eigenvectors of $A,\ A^2,\ A^{-1},\ \ and\ \ A+4I$:

$$A=\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad and \quad A^2=\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

Check the trace $\lambda_1+\lambda_2$ and the determinant $\lambda_1\lambda_2$ for A and also A^2.

2. Show directly that the given vectors are eigenvectors of the given matrix. What are the corresponding eigenvalues

$$\boldsymbol{v}_1=\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \boldsymbol{v}_2=\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad A=\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$$

3. For which real numbers c does this matrix A have

$$A=\begin{pmatrix} 2 & -c \\ -1 & 2 \end{pmatrix}$$

a) Two real eigenvalues and eigenvectors.

b) A repeated eigenvalue with only one eigenvector

c) Two complex eigenvalues and eigenvectors.

4. Find the eigenvalues of ***A***, ***B***, ***AB***, and ***BA***:

$$A=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad AB=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad BA=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

a) The eigenvalues of ***AB*** (are equal to) (are not equal to) eigenvalues of ***A*** times eigenvalues of ***B***.

b) The eigenvalues of ***AB*** (are equal to) (are not equal to) eigenvalues of ***BA***.

5. When $a+b=c+d$ show that (1, 1) is an eigenvector and find both eigenvalues of

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

6. The eigenvalues of A equal to the eigenvalues of A^T. This is because $\det(A-\lambda I)$ equals $\det\left(A^T-\lambda I\right)$. That is true because _____. Show by an example that the eigenvectors of A and A^T are not the same.

7. Let $A=\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$. Compute the eigenvalues and eigenvectors of A.

8. Let $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

a) What is the characteristic polynomial for A (i.e. compute $\det(A - \lambda I)$?

b) Verify that 1 is an eigenvalue of A. What is a corresponding eigenvector?

c) What are the other eigenvalues of A?

9. For the following matrices:

$a)\ \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ $\quad b)\ \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$ $\quad c)\ \begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}$

$d)\ \begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$ $\quad e)\ \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$ $\quad f)\ \begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$

$g)\ \begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}$ $\quad h)\ \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\quad i)\ \begin{pmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

$j)\ \begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

10. Find the eigenvalues of A^9 for $A = \begin{pmatrix} 1 & 3 & 7 & 11 \\ 0 & \frac{1}{2} & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

11. Find the eigenvalues of the matrices

$$A = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0.61 & 0.52 \\ 0.39 & 0.48 \end{pmatrix}, \quad A^\infty = \begin{pmatrix} 0.57143 & 0.57143 \\ 0.42857 & 0.42857 \end{pmatrix}, \quad \textit{and} \quad B = \begin{pmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{pmatrix}$$

12. Given the matrix $\begin{bmatrix} -1 & -3 \\ -3 & 7 \end{bmatrix}$

a) Find the characteristic polynomial.

b) Find the eigenvalues

c) Find the bases for its eigenspaces

d) Graph the eigenspaces

e) Verify directly that $A\mathbf{v} = \lambda\mathbf{v}$, for all associated eigenvectors and eigenvalues.

13. Given the matrix $\begin{bmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{bmatrix}$

a) Find the characteristic polynomial.

b) Find the eigenvalues

c) Find the bases for its eigenspaces

d) Graph the eigenspaces

e) Verify directly that $A\mathbf{v} = \lambda\mathbf{v}$, for all associated eigenvectors and eigenvalues.

14. Given: $A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$. Compute A^{11}