Solution Section 3.2 – Extrema and the First-Derivative Test

Exercise

Find all relative extrema of the function $f(x) = 6x^3 - 15x^2 + 12x$

Solution

$$f' = 18x^2 - 30x + 12$$
$$= 6\left(3x^2 - 5x + 2\right)$$
$$= 0$$
$$x = 1, \ \frac{2}{3}$$

$$\begin{cases} x = 1 \to y = f(1) = 3 \\ x = \frac{2}{3} \to y = f\left(\frac{2}{3}\right) = \frac{28}{9} \end{cases} \qquad \left(\frac{2}{3}, \frac{28}{9}\right), \quad (1,3)$$

-∞	$\frac{2}{3}$	1	∞
$f'(0) > 0$ Increasing	$f'(.75) < 0$ Decreasing	$f'(2) > 0$ Increasing	

RMAX: $\left(\frac{2}{3}, \frac{28}{9}\right)$;

RMIN: $(1,3)$

Exercise

Find all relative Extrema of $f(x) = x^4 - 4x^3$ and Find the open intervals on which is increasing or decreasing

Solution

$$f'(x) = 4x^3 - 12x^2$$
$$= 4x^2(x-3) = 0$$
$$\Rightarrow x = 0, \ 3 \quad (CN)$$
$$x = 3 \to y = f(3) = -27$$

-∞	0	3	∞
$f'(-1) < 0$ Decreasing	$f'(1) < 0$ Decreasing	$f'(4) > 0$ Increasing	

RMIN: (3, -27);

Decreasing: (-∞, 3); Increasing: (3, ∞)

Exercise

Find all relative Extrema of $f(x) = 3x^{2/3} - 2x$ and Find the open intervals on which is increasing or decreasing

Solution

$$f'(x) = 2x^{-1/3} - 2$$

$$= 2\left(\frac{1}{x^{1/3}} - 1\right)$$

$$f'(x) = 2\left(\frac{1-x^{1/3}}{x^{1/3}}\right) = 0$$

$$\Rightarrow \begin{cases} x^{1/3} = 0 \to x = 0 \\ 1 - x^{1/3} = 0 \to x^{1/3} = 1 \Rightarrow x = 1 \end{cases}$$

$$\begin{cases} x = 0 \to y = 0 \\ x = 1 \to y = 1 \end{cases}$$

(0, 0) and (1, 1)

-∞ 0	0 1	1 ∞
$f'(-1) > 0$	$f'(\frac{1}{2}) < 0$	$f'(2) > 0$
Increasing	*Decreasing*	*Increasing*

RMAX: (0, 0); **RMIN**: (1, 1);

Increasing:(-∞, 0) and (1, ∞);

Decreasing: (0, 1)

Exercise

Find all relative Extrema as well as where the function is increasing and decreasing $y = \sqrt{4 - x^2}$

Solution

$$f'(x) = \frac{-x}{\sqrt{4-x^2}}$$

The critical values are $x = 0, \ \pm 2$, but the domain of the function is [-2,2]. We can't go outside of that interval to test.

Interval(s)	$(-2,0)$	$(0,2)$
Sign of f'	$f'(-1) > 0$	$f'(1) < 0$
Conclusion for f	increasing	decreasing

The function has a RMAX of $f(0) = 2$ @ $x = 0$. Some texts also consider $f(-2) = 0$ and $f(2) = 0$ as RMIN

Exercise

Find all relative Extrema as well as where the function is increasing and decreasing $f(x) = x\sqrt{x+1}$

Solution

$$f'(x) = \frac{3x+2}{2\sqrt{x+1}}$$

Critical numbers are $x = -\frac{2}{3}$ *and* $x = -1$, but the domain is $[-1, \infty)$.

Interval(s)	$(-1, -2/3)$	$(-2/3, \infty)$
Sign of f'	$f'(-0.9) < 0$	$f'(0) > 0$
Conclusion for f	*decreasing*	*increasing*

The function has a RMIN of $f\left(-\frac{2}{3}\right) = -\frac{2\sqrt{3}}{9}$ @ $x = -\frac{2}{3}$.

Some texts may also consider $f(-1) = 0$ as a RMAX.

Exercise

Find all relative Extrema as well as where the function is increasing and decreasing $f(x) = \dfrac{x}{x^2+1}$

Solution

$$f'(x) = -\frac{(x-1)(x+1)}{(x^2+1)^2}$$

Critical numbers are $x = -1$ & $x = 1$.

Interval(s)	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Sign of f'	$f'(-2) < 0$	$f'(0) > 0$	$f'(0) < 0$
Conclusion for f	*decreasing*	*increasing*	*decreasing*

The function has a RMIN of $f(-1) = -\frac{1}{2}$ @ $x = -1$.

The function has a RMAX of $f(1) = \frac{1}{2}$ @ $x = 1$.

Exercise

Find all relative Extrema as well as where the function is increasing and decreasing $f(x)=x^4-8x^2+9$

Solution

$$f'(x)=4x^3-16x$$

$$=4x\left(x^2-4\right)=0$$

$$\boxed{x=0} \qquad x^2-4=0$$

$$x^2=4 \Rightarrow \boxed{x=\pm 2}$$

$CN: \quad x=-2,\ 0,\ 2$

-∞ -2	0	2	∞
$f'(-3)<0$	$f'(-1)>0$	$f'(1)<0$	$f'(3)>0$
decreasing	*increasing*	*decreasing*	*increasing*

$x=-2 \quad \rightarrow f(-2)=-7$

$x=0 \quad \rightarrow f(0)=9$

$x=2 \quad \rightarrow f(2)=-7$

Increasing: $(-2,\ 0)\cup(2,\ \infty)$

Decreasing: $(-\infty,\ -2)\cup(0,\ 2)$

RMIN: $(-2,\ -7)\, and\, (2,\ -7)$

RMAX: $(0,\ 9)$

Exercise

Find all relative Extrema as well as where the function is increasing and decreasing $f(x) = 3xe^x + 2$

Solution

$$f'(x) = 3e^x + 3xe^x$$
$$= 3e^x(1+3x) = 0$$

$$1 + 3x = 0 \;\Rightarrow \boxed{x = -\frac{1}{3}} \quad (CN)$$

$$f\left(-\tfrac{1}{3}\right) = 3\left(-\tfrac{1}{3}\right)e^{-1/3} + 2 = 1.28$$

$-\infty$	$-\frac{1}{3}$	∞
$f'(-1) < 0$		$f'(0) > 0$
decreasing		*increasing*

Increasing: $\left(-\frac{1}{3},\ \infty\right)$

Decreasing: $\left(-\infty,\ -\frac{1}{3}\right)$

RMIN: $\left(-\frac{1}{3},\ 1.28\right)$

RMAX: *NA*

Exercise

Coughing forces the trachea to contract, which in turn affects the velocity of the air through the trachea. The velocity of the air during coughing can be modeled by: $v = k(R-r)r^2, \quad 0 \le r < R$ where k is a constant, R is the normal radius of the trachea (also a constant) and r is the radius of the trachea during coughing. What radius r will produce the maximum air velocity?

Solution

$$v = k\left(Rr^2 - r^3\right)$$

$$v' = k\left(2Rr - 3r^2\right)$$
$$= kr(2R - 3r) = 0$$

$$r = 0 \quad or \quad 2R - 3r = 0$$
$$r = 0 \quad or \quad r = (2/3)R$$

A trachea radius of zero minimizes air velocity (duh!). And a radius of 2/3 its normal size maximizes air flow.

Exercise

When a telephone wire is hung between two poles, the wire forms a U-shape curve called a Catenary. For instance, the function $y = 30\left(e^{x/60} + e^{-x/60}\right) \quad -30 \le x \le 30$ models the shape of the telephone wire strung between two poles that are 60 *ft* apart (x & y are measured in *ft*). Show that the lowest point on the wire is midway between two poles. How much does the wire sag between the two poles?

Solution

$$y' = 30\left(\frac{1}{60}e^{x/60} - \frac{1}{60}e^{-x/60}\right)$$

$$= \frac{1}{2}\left(e^{x/60} - e^{-x/60}\right)$$

Find the critical number(s)

$$y' = 0$$

$$\frac{1}{2}\left(e^{x/60} - e^{-x/60}\right) = 0$$

$$e^{x/60} - e^{-x/60} = 0$$

$$e^{x/60} = e^{-x/60}$$

$$\frac{x}{60} = -\frac{x}{60}$$

$$\Rightarrow x = 0$$

$$y(x = -30) = 30\left(e^{-30/60} + e^{-(-30)/60}\right) \approx 67.7\ ft$$

$$y(x = 0) = 30\left(e^{0} + e^{0}\right) = 30(2) = 60\ ft$$

$$y(x = 30) = 30\left(e^{30/60} + e^{-(30)/60}\right) \approx 67.7\ ft$$

Sag 7.7 *ft*

Exercise

The demand function for the product is modeled by $p = 50e^{-0.0000125x}$ where p is the price per unit in dollars and x is the number of units. What price will yield maximum revenue?

Solution

$$R = xp = 50xe^{-0.0000125x}$$

$$R' = 50e^{-0.0000125x} + (-0.0000125)50xe^{-0.0000125x}$$

$$R' = 50e^{-0.0000125x} - 0.000625xe^{-0.0000125x}$$

$$R' = e^{-0.0000125x}(50 - 0.000625x) = 0$$

$$50 - 0.000625x = 0$$

$$-0.000625x = -50$$

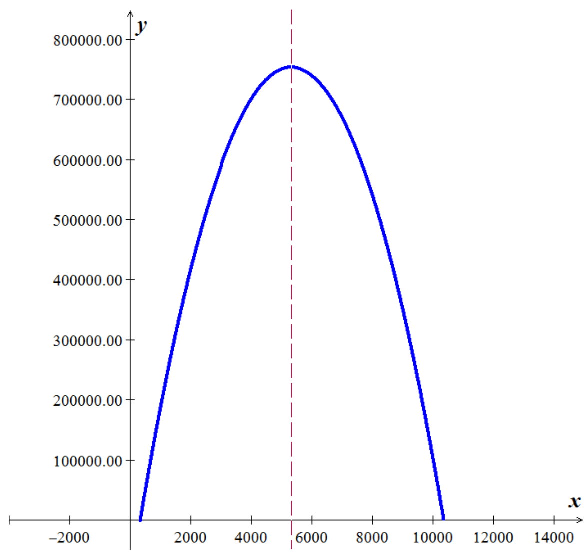
$$x = \frac{-50}{-0.000625} = 80000$$

$$p(x = 80000) = 50e^{-0.0000125(80000)}$$

$$\approx \$18.39 \ / \, unit$$

Exercise

The annual revenue and cost functions for a manufacturer of grandfather clocks are approximately $R(x) = 520x - 0.03x^2$ and $C(x) = 200x + 100{,}000$, where x denotes the number of clocks made. What is the maximum annual profit?



Solution

$$P(x) = R(x) - C(x)$$

$$= 520x - 0.03x^2 - (200x + 100{,}000)$$

$$= 520x - 0.03x^2 - 200x - 100{,}000$$

$$= -0.03x^2 + 320x - 100{,}000$$

$$P'(x) = -0.06x + 320 = 0$$

$$\Rightarrow -0.06x = -320$$

$$x = \frac{-320}{-0.06} = \underline{5333}|$$

$$P(x = 5333) = -0.03(5333)^2 + 320(5333) - 100{,}000$$

$$= \underline{\$753{,}333.33}|$$

Exercise

Find the number of units, x, that produces the maximum profit P, if $C(x) = 30 + 20x$ and $p = 32 - 2x$

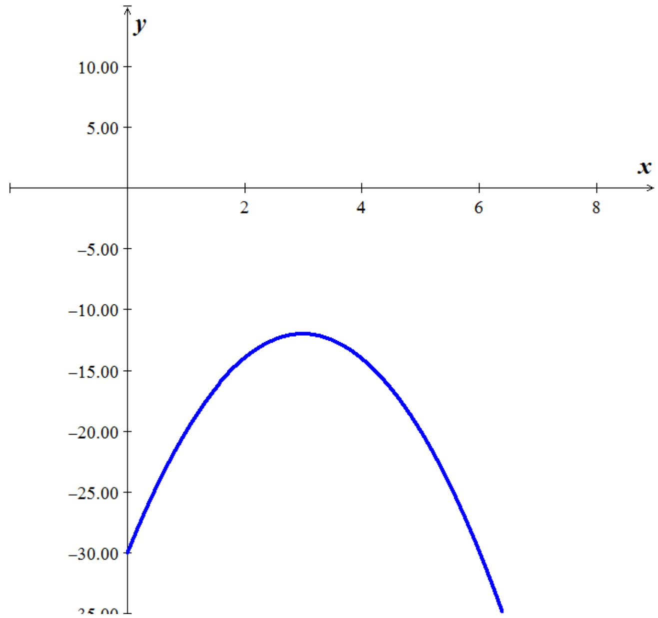
Solution

$$P(x) = R(x) - C(x)$$
$$= x \cdot p - (30 + 20x)$$
$$= x(32 - 2x) - 30 - 20x$$
$$= 32x - 2x^2 - 30 - 20x$$
$$= -2x^2 + 12x - 30$$

$$P'(x) = -4x + 12 = 0$$
$$\Rightarrow \boxed{x = 3}$$

$$P(x = 3) = -2(3)^2 + 12(3) - 30 = -12$$
$$\underline{= -12|} < 0$$

There is no profit.



Exercise

$P(x) = -x^3 + 15x^2 - 48x + 450, \quad x \ge 3$ is an approximation to the total profit (in thousands of dollars) from the sale of x hundred thousand tires. Find the number of hundred thousands of tires that must be sold to maximize profit.

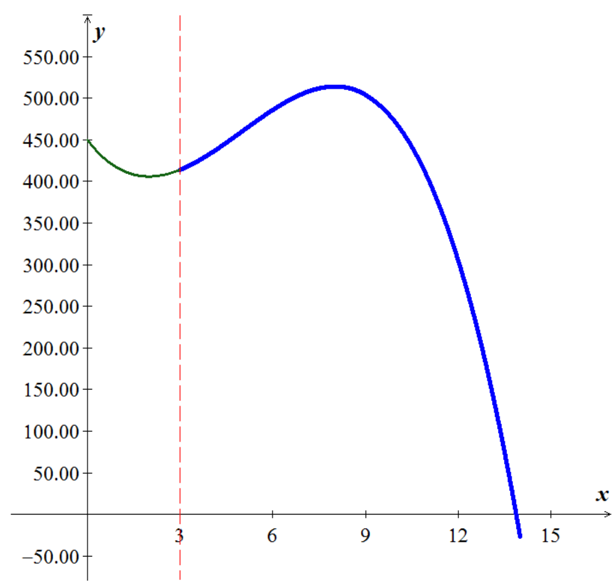
Solution

$$P'(x) = -3x^2 + 30x - 48 = 0$$
$$\Rightarrow x = 2,\ 8$$

$$Since\ x \ge 3 \quad \Rightarrow \boxed{x = 8}$$

$$P(x = 8) = -(8)^3 + 15(8)^2 - 48(8) + 450$$
$$\underline{= 541|}$$

The number of tires that must be sold to maximize profit is 800,000 tires



Exercise

$P(x) = -x^3 + 3x^2 + 360x + 5000; \quad 6 \le x \le 20$ is an approximation to the number of salmon swimming upstream to spawn, where x represents the water temperature in degrees Celsius. Find the temperature that produces the maximum number of salmon.

Solution

$$P'(x) = -3x^2 + 6x + 360 = 0$$

$$\Rightarrow x = 12, \quad -10\left(\textit{not in the interval}\right)$$

$$P(x=6) = -(6)^3 + 3(6)^2 + 360(6) + 5000 \underline{= 7052|}$$

$$P(x=20) = -(20)^3 + 3(20)^2 + 360(20) + 5000 \underline{= 5400|}$$

$$P(x=12) = -(12)^3 + 3(12)^2 + 360(12) + 5000 \underline{= 8024|}$$

12° is the temperature that produces the maximum number of salmon