Section 3.8 – Dot Product and Orthogonality

Norm of a Vector

The ***length*** (or ***norm***) of a vector v is the square root of $v.v$

$$Length = \|v\| = \sqrt{v.v}$$

$$= \sqrt{x^2 + y^2} \qquad \textbf{2-}\textbf{\textit{dimension}}$$

$$= \sqrt{x^2 + y^2 + z^2} \qquad \textbf{3-}\textbf{\textit{dimension}}$$

Definition

If $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ is a vector in $\boldsymbol{R}^n$, then the norm of $\boldsymbol{v}$ (also called the length of $\boldsymbol{v}$ or the magnitude of $\boldsymbol{v}$) is denoted by $\|\boldsymbol{v}\|$, and is defined by the formula

$$\boxed{\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \ldots + v_n^2}}$$

Example

Find the length of the vector $v = (1, 2, 3)$

Solution

$$\|v\| = \sqrt{1^2 + 2^2 + 3^2}$$

$$= \sqrt{14}$$

Theorem

If $\boldsymbol{v}$ is a vector in $\boldsymbol{R}^n$, and if $\boldsymbol{k}$ is any scalar, then:

a) $\|\boldsymbol{v}\| \ge 0$

b) $\|\boldsymbol{v}\| = 0 \quad iff \quad \boldsymbol{v} = 0$

c) $\|k\boldsymbol{v}\| = |k| \cdot \|\boldsymbol{v}\|$

Unit Vectors

Definition

A ***unit vector*** u is a vector whose length equals to one. Then $u.u = 1$

Divide any nonzero vector v by its length. Then $u = \frac{v}{\|v\|}$ is a unit vector in the same direction as v.

Example

Find the unit vector $\boldsymbol{u}$ that has the same direction as $\boldsymbol{v} = (2, 2, -1)$

Solution

$$\|\boldsymbol{v}\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

$$\begin{aligned} u &= \frac{v}{\|v\|} \\ &= \frac{1}{3}(2,\ 2,\ -1) \\ &= \left(\frac{2}{3},\ \frac{2}{3},\ -\frac{1}{3}\right) \end{aligned}$$

$$\begin{aligned} \|u\| &= \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{-1}{3}\right)^2} \\ &= \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} \\ &= \sqrt{\frac{9}{9}} \\ &= 1 \end{aligned}$$

Example of unit vectors

$$i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

In $\boldsymbol{R}^3$ $\quad \boldsymbol{i} = (1,\ 0,\ 0) \quad \boldsymbol{j} = (0,\ 1,\ 0) \quad$ *and* $\quad \boldsymbol{k} = (0,\ 0,\ 1)$

In general, these formulas can be defined as ***standard unit vector*** in $\boldsymbol{R}^n$

$$\boldsymbol{e}_1 = (1,\ 0,\ \cdots,\ 0), \quad \boldsymbol{e}_2 = (0,\ 1,\ \cdots,\ 0), \quad \ldots, \quad \boldsymbol{e}_{,} = (0,\ 0,\ \cdots,\ 1)$$

$$\boldsymbol{v} = \left(v_1,\ v_2,\ \ldots,\ v_n\right) = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + \ldots + v_n\boldsymbol{e}_n$$

Example $\quad (7,\ 3,\ -4,\ 5) = 7\boldsymbol{e}_1 + 3\boldsymbol{e}_2 - 4\boldsymbol{e}_3 + 5\boldsymbol{e}_4$

***Distance* in $\boldsymbol{R}^n$**

Definition

If $\boldsymbol{u}=\left(u_1,\, u_2,\, \ldots,\, u_n\right)$ and $\boldsymbol{v}=\left(v_1,\, v_2,\, \ldots,\, v_n\right)$ are points in $\boldsymbol{R}^n$, then we denote the distance between u and v by $d\left(\boldsymbol{u},\, \boldsymbol{v}\right)$ and define it to be

$$d\left(\boldsymbol{u},\, \boldsymbol{v}\right)=\left\|\boldsymbol{u}-\boldsymbol{v}\right\|=\sqrt{\left(u_1-v_1\right)^2+\left(u_2-v_2\right)^2+\ldots+\left(u_n-v_n\right)^2}$$

In $\boldsymbol{R}^2$ $\qquad d=\left\|\overrightarrow{P_1P_2}\right\|=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}$

In $\boldsymbol{R}^3$ $\qquad d\left(\boldsymbol{u},\, \boldsymbol{v}\right)=\left\|\overrightarrow{P_1P_2}\right\|=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2}$

Dot Product

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonzero vectors in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$, and if θ is he angle between $\boldsymbol{u}$ and $\boldsymbol{v}$, then the ***dot product*** (also called the ***Euclidean inner product***) of $\boldsymbol{u}$ and $\boldsymbol{v}$ is denoted by $\boldsymbol{u}\cdot\boldsymbol{v}$ and is defined as

$$\boldsymbol{u}\cdot\boldsymbol{v}=\left\|\boldsymbol{u}\right\|\ \left\|\boldsymbol{v}\right\|\ \cos\theta$$

Cosine Formula

If u and v are nonzero vectors that implies $\Rightarrow\ \cos\theta=\dfrac{u.v}{\|u\|.\|v\|}$

Example

Find the dot product of the vectors $\boldsymbol{u}$ = (0, 0, 1) and $\boldsymbol{v}$ = (0, 2, 2) and have an angle of 45°.

Solution

$$\|u\|=1 \quad \textit{and} \quad \|v\|=\sqrt{0+2^2+2^2}=\sqrt{8}=2\sqrt{2}$$

$$\begin{aligned} u\cdot v &= \|u\|\|v\|\cos\theta \\ &=(1)\left(2\sqrt{2}\right)\cos 45° \\ &=\left(2\sqrt{2}\right)\frac{1}{\sqrt{2}} \\ &=2 \end{aligned}$$

Component Form of the Dot Product

The ***dot product*** or ***inner product*** of $v = \left(v_1,\ v_2\right)$ and $w = \left(w_1,\ w_2\right)$ is the number

$$vw = v_1 w_1 + v_2 w_2$$

Example

Find the dot product of $v = (4,\ 2)$ and $w = (-1,\ 2)$

Solution

$\underline{v.w =} 4.(-1) + 2(2) = \underline{0}$

- ***For dot products, zero means that the 2 vectors are perpendicular*** (= 90°).

Example

Put a weight of 4 at the point $x = -1$ and weight of 2 at the point $x = 2$. The *x*-axis will balance on the center point $x = 0$.

Solution

The weight balance is $4(-1) + 2(2) = 0$ (*dot product*).

In 3-dimensionals the dot product:

$$\left(v_1,\ v_2,\ v_3\right).\left(w_1,\ w_2,\ w_3\right) = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Theorem

a) $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$

b) $\boldsymbol{u} \cdot \left(\boldsymbol{v} + \boldsymbol{w}\right) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$

c) $\boldsymbol{u} \cdot \left(\boldsymbol{v} - \boldsymbol{w}\right) = \boldsymbol{u} \cdot \boldsymbol{v} - \boldsymbol{u} \cdot \boldsymbol{w}$

d) $\left(\boldsymbol{u} + \boldsymbol{v}\right) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}$

e) $\left(\boldsymbol{u} - \boldsymbol{v}\right) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} - \boldsymbol{v} \cdot \boldsymbol{w}$

f) $k\left(\boldsymbol{u} \cdot \boldsymbol{v}\right) = \left(k\boldsymbol{u}\right) \cdot \boldsymbol{v}$

g) $k\left(\boldsymbol{u} \cdot \boldsymbol{v}\right) = \boldsymbol{u} \cdot \left(k\boldsymbol{v}\right)$

h) $\boldsymbol{v} \cdot \boldsymbol{v} \ge 0 \quad \textit{and} \quad \boldsymbol{v} \cdot \boldsymbol{v} = 0 \quad \textit{iff} \quad \boldsymbol{v} = 0$

i) $0 \cdot \boldsymbol{v} = \boldsymbol{v} \cdot 0 = 0$

Right Angles

The dot product is $v.w = 0$ when v is ***perpendicular*** to w.

Proof

Perpendicular vectors: $\|v\|^2 + \|w\|^2 = \|v - w\|^2$

$$
\begin{aligned}
v_1^2 + v_2^2 + w_1^2 + w_2^2 &= \left(v_1 - w_1\right)^2 + \left(v_2 - w_2\right)^2 \\
&= v_1^2 - 2v_1 w_1 + w_1^2 + v_2^2 - 2v_2 w_2 + w_2^2 \\
&= v_1^2 + w_1^2 + v_2^2 + w_2^2 - 2\left(v_1 w_1 + v_2 w_2\right) \qquad v_1 w_1 + v_2 w_2 = 0 \quad \textit{dot product} \\
&= v_1^2 + w_1^2 + v_2^2 + w_2^2
\end{aligned}
$$

If u and U are unit vectors, then $u.U = \cos\theta$

Certainly,

$$|u.U| \le 1$$

$$-1 \le \cos\theta \le 1$$

$$-1 \le \textit{dot product} \le 1$$

Schwarz Inequality

If v and w are any vectors $\Rightarrow$ $\|v.w\| \le \|v\|.\|w\|$

Proof

The dot product of $v = (a,\ b)$ and $w = (b,\ a)$ is $2ab$ and both lengths are $\sqrt{a^2 + b^2}$.

Then, the Schwarz inequality says that: $2ab \le a^2 + b^2$

$$a^2 + b^2 - 2ab = \left(a - b\right)^2 \ge 0$$

$$a^2 + b^2 - 2ab \ge 0$$

$$a^2 + b^2 \ge 2ab$$

This proves the Schwarz inequality: $2ab \le a^2 + b^2 \Rightarrow \|v.w\| \le \|v\|.\|w\|$

Orthogonality

Definition

Two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\boldsymbol{R}^n$ are said to be ***orthogonal*** (or ***perpendicular***) if their dot product is zero $\boldsymbol{u.v} = 0$.

We will also agree that he zero vector in $\boldsymbol{R}^n$ is orthogonal to every vector in $\boldsymbol{R}^n$. A nonempty set of vectors $\boldsymbol{R}^n$ is called an ***orthogonal set*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set of unit vectors is called an ***orthonormal set***.

Example

The floor of your room (extended to infinity) is a subspace $\boldsymbol{V}$. The line where two walls meet is a subspace $\boldsymbol{W}$ (one-dimensional). Those subspaces are orthogonal. Every vector up the meeting line is perpendicular to every vector on the floor. The origin (0, 0, 0) is in the corner.

Example

Show that $\boldsymbol{u} = (-2, 3, 1, 4)$ and $\boldsymbol{v} = (1, 2, 0, -1)$ are orthogonal in $\boldsymbol{R}^4$

Solution

$$\boldsymbol{u.v} = (-2)(1) + (3)(2) + (1)(0) + (4)(-1)$$
$$= -2 + 6 + 0 - 4$$
$$= 0$$

These vectors are orthogonal in $\boldsymbol{R}^4$

Standard Unit Vectors

$$\boldsymbol{i} \bullet \boldsymbol{j} = \boldsymbol{i} \bullet \boldsymbol{k} = \boldsymbol{j} \bullet \boldsymbol{k} = \mathbf{0}$$

Proof

$$\boldsymbol{i} \bullet \boldsymbol{j} = (1,0,0) \bullet (0,1,0) = \mathbf{0}$$

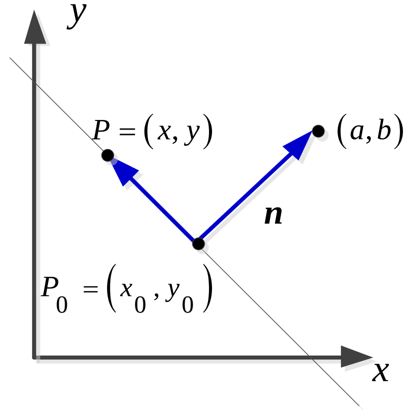
Normal

To specify slope and inclination is to use a nonzero vector ***n***, called a ***normal***, which is orthogonal to the line or plane.

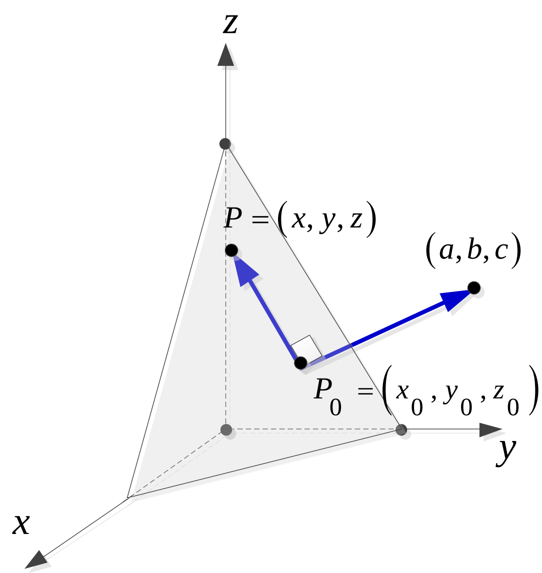
The line passes through a point $P_0\left(x_0, y_0\right)$ that has a normal ***n*** = (a, b) and the plane through $P_0\left(x_0, y_0, z_0\right)$ that has a normal ***n*** = (a, b, c). Both the line and the plane are represented by the vector equation

$$\boldsymbol{n} \bullet \overrightarrow{P_0 P} = 0$$

The line equation: $\quad a\left(x - x_0\right) + b\left(y - y_0\right) = 0$



The plane equation: $\quad a\left(x - x_0\right) + b\left(y - y_0\right) + c\left(z - z_0\right) = 0$



Exercises *Section* 3.8 – Dot Product and Orthogonality

1. If $\|\vec{v}\| = 5$ and $\|\vec{w}\| = 3$, what are the smallest and largest possible values of $\|\vec{v} - \vec{w}\|$ and $\vec{v}.\vec{w}$?

2. If $\|\vec{v}\| = 7$ and $\|\vec{w}\| = 3$, what are the smallest and largest possible values of $\|\vec{v} + \vec{w}\|$ and $\vec{v}.\vec{w}$?

3. Given that $\cos(\alpha) = \frac{v_1}{\|v\|}$ and $\sin(\alpha) = \frac{v_2}{\|v\|}$. Similarly, $\cos(\beta) =$ ______ and $\sin(\beta) =$ ______. The angle θ is $\beta - \alpha$. Substitute into the trigonometry formula $\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ for $\cos(\beta - \alpha)$ to find $\cos(\theta) = \frac{v.w}{\|v\|.\|w\|}$

4. Can three vectors in the *xy* plane have $u.v < 0$, $v.w < 0$ and $u.w < 0$?

5. Find the norm of *v*, a unit vector that has the same direction as *v*, and a unit vector that is oppositely directed.

a) $v = (4, -3)$

b) $v = (1, -1, 2)$

c) $v = (-2, 3, 3, -1)$

6. Evaluate the given expression with $\boldsymbol{u} = (2, -2, 3)$, $\boldsymbol{v} = (1, -3, 4)$, and $\boldsymbol{w} = (3, 6, -4)$

a) $\|u + v\|$ *c)* $\|3u - 5v + w\|$ *e)* $\|u\| + \|-2v\| + \|-3w\|$

b) $\|-2u + 2v\|$ *d)* $\|3v\| - 3\|v\|$

7. Let $\boldsymbol{v} = (1, 1, 2, -3, 1)$. Find all scalars *k* such that $\|kv\| = 5$

8. Find $\boldsymbol{u} \cdot \boldsymbol{v}$, $\boldsymbol{u} \cdot \boldsymbol{u}$, *and* $\boldsymbol{v} \cdot \boldsymbol{v}$

a) $u = (3, 1, 4)$, $v = (2, 2, -4)$

b) $u = (1, 1, 4, 6)$, $v = (2, -2, 3, -2)$

c) $u = (2, -1, 1, 0, -2)$, $v = (1, 2, 2, 2, 1)$

9. Find the Euclidean distance between $\boldsymbol{u}$ and $\boldsymbol{v}$, then find the angle between them

a) $u = (3, 3, 3)$, $v = (1, 0, 4)$

b) $u = (1, 2, -3, 0)$, $v = (5, 1, 2, -2)$

c) $u = (0, 1, 1, 1, 2)$, $v = (2, 1, 0, -1, 3)$

10. Find a unit vector that has the same direction as the given vector

a) $(-4, -3)$ *b)* $\left(-3,\ 2,\ \sqrt{3}\right)$ *c)* $(1, 2, 3, 4, 5)$

11. Find a unit vector that is oppositely to the given vector

a) $(-12, -5)$ *b)* $(3, -3, 3)$ *c)* $\left(-3,\ 1,\ \sqrt{6},\ 3\right)$

12. Verify that the Cauchy-Schwarz inequality holds

a) $u = (-3, 1, 0)$, $v = (2, -1, 3)$

b) $u = (0, 2, 2, 1)$, $v = (1, 1, 1, 1)$

c) $u = (1, 3, 5, 2, 0, 1)$, $v = (0, 2, 4, 1, 3, 5)$

13. Find $\boldsymbol{u}\bullet\boldsymbol{v}$ and then the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$ $\quad \boldsymbol{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$

14. Find the norm: $\|\boldsymbol{u}\| + \|\boldsymbol{v}\|$, $\|\boldsymbol{u}+\boldsymbol{v}\|$ for $\boldsymbol{u} = (3,\ -1,\ -2,\ 1,\ 4)$ $\quad \boldsymbol{v} = (1,\ 1,\ 1,\ 1,\ 1)$

15. Find all numbers r such that: $\|r(1,\ 0,\ -3,\ -1,\ 4,\ 1)\| = 1$

16. Find the distance between $P_1(7, -5, 1)$ and $P_2(-7, -2, -1)$

17. Given $\boldsymbol{u} = (1, -5, 4)$, $\boldsymbol{v} = (3, 3, 3)$

a) Find $\boldsymbol{u}\cdot\boldsymbol{v}$

b) Find the cosine of the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$.

18. Determine whether $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

a) $\boldsymbol{u} = (-6,\ -2)$, $\quad \boldsymbol{v} = (5,\ -7)$

b) $\boldsymbol{u} = (6,\ 1,\ 4)$, $\quad \boldsymbol{v} = (2,\ 0,\ -3)$

c) $\boldsymbol{u} = (1,\ -5,\ 4)$, $\quad \boldsymbol{v} = (3,\ 3,\ 3)$

d) $\boldsymbol{u} = (-2,\ 2,\ 3)$, $\quad \boldsymbol{v} = (1,\ 7,\ -4)$

19. Determine whether the vectors form an orthogonal set

a) $\boldsymbol{v}_1 = (2,\ 3)$, $\quad \boldsymbol{v}_2 = (3,\ 2)$

b) $\boldsymbol{v}_1 = (1,\ -2)$, $\quad \boldsymbol{v}_2 = (-2,\ 1)$

c) $\boldsymbol{u} = (-4,\ 6,\ -10,\ 1)$ $\quad \boldsymbol{v} = (2,\ 1,\ -2,\ 9)$

d) $\boldsymbol{u} = (a,\ b)$ $\quad \boldsymbol{v} = (-b,\ a)$

e) $\boldsymbol{v}_1 = (-2,\ 1,\ 1)$, $\quad \boldsymbol{v}_2 = (1,\ 0,\ 2)$, $\quad \boldsymbol{v}_3 = (-2,\ -5,\ 1)$

f) $\boldsymbol{v}_1 = (1,\ 0,\ 1)$, $\quad \boldsymbol{v}_2 = (1,\ 1,\ 1)$, $\quad \boldsymbol{v}_3 = (-1,\ 0,\ 1)$

g) $\boldsymbol{v}_1 = (2, -2, 1)$, $\quad \boldsymbol{v}_2 = (2, 1, -2)$, $\quad \boldsymbol{v}_3 = (1, 2, 2)$

20. Find a unit vector that is orthogonal to both $\boldsymbol{u} = (1, 0, 1)$ and $\boldsymbol{v} = (0, 1, 1)$

21. *a*) Show that $\boldsymbol{v} = (a, b)$ and $\boldsymbol{w} = (-b, a)$ are orthogonal vectors.

b) Use the result to find two vectors that are orthogonal to $\boldsymbol{v} = (2, -3)$.

c) Find two unit vectors that are orthogonal to $(-3, 4)$

22. Show that if $\boldsymbol{v}$ is orthogonal to both $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, then $\boldsymbol{v}$ is orthogonal to $k_1\boldsymbol{w}_1 + k_2\boldsymbol{w}_2$ for all scalars k_1 and k_2.

23. Show that $\vec{u}-\vec{v}$ is orthogonal to $\vec{u}+\vec{v}$ if and only if $\|\vec{u}\|=\|\vec{v}\|$

24. Given $\boldsymbol{u}=(3,\ -1,\ 2)$ $\boldsymbol{v}=(4,\ -1,\ 5)$ *and* $\boldsymbol{w}=(8,\ -7,\ -6)$

a) Find $3\boldsymbol{v}-4(5\boldsymbol{u}-6\boldsymbol{w})$

b) Find $\boldsymbol{u}\bullet\boldsymbol{v}$ and then the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$.

25. *a*) Show that $\boldsymbol{v} = (a, b)$ and $\boldsymbol{w} = (-b, a)$ are orthogonal vectors

b) Use the result in part (*a*) to find two vectors that are orthogonal to $\boldsymbol{v} = (2, -3)$

c) Find two unit vectors that are orthogonal to $(-3, 4)$

26. Show that A (3, 0, 2), .B (4, 3, 0), and C (8, 1, −1) are vertices of a right triangle. At which vertex is the right angle?

27. Establish the identity: $\boldsymbol{u}\bullet\boldsymbol{v}=\frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2-\frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2$