Section 2.3 – Derivatives of Products and Quotients

Product Rule

The derivative of the product of two differentiable functions is equal to the first function times the derivative of the second plus the second function times the derivative of the first,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$(f.g)' = f'.g + g'.f \qquad \boxed{(u.v)' = u'.v + v'.u}$$

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'gh + fg'h + fgh'$$

Example

Find the derivative of $f(x) = (2x+3)\left(3x^2\right)$

<u>*Solution*</u>

$$u = 2x+3 \quad v = 3x^2$$
$$u' = 2 \quad v' = 6x$$

$$\begin{aligned} f' &= u'v + v'u \\ &= (2)\left(3x^2\right) + (2x+3)(6x) \\ &= 12x^2 + 18x + 6x^2 \\ &= 18x^2 + 18x \\ &= 18x(x+1) \end{aligned}$$

$$f(x) = 6x^3 + 9x^2$$

Example

Find the derivative of $f(x) = \left(\sqrt{x}+3\right)\left(x^2-5x\right)$

<u>*Solution*</u>

$$u = x^{1/2} + 3 \quad v = x^2 - 5x$$
$$u' = \frac{1}{2}x^{-1/2} \quad v' = 2x - 5$$

$$\begin{aligned} f' &= \left(\frac{1}{2}x^{-1/2}\right)\left(x^2 - 5x\right) + (2x-5)\left(\sqrt{x}+3\right) \\ &= \frac{1}{2}x^{3/2} - \frac{5}{2}x^{1/2} + 2x^{3/2} - 5x^{1/2} + 6x - 15 \\ &= \frac{5}{2}x^{3/2} - \frac{15}{2}x^{1/2} + 6x - 15 \\ &= \frac{5}{2}x^{3/2} + 6x - \frac{15}{2}x^{1/2} - 15 \end{aligned}$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)g(x)-g'(x)f(x)}{[g(x)]^2}$$

$$=\frac{f'g-g'f}{g^2}$$

$$\boxed{\left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}}$$

Example

Find $f'(x)$ if $f(x)=\dfrac{2x-1}{4x+3}$

Solution

$$u=2x-1 \quad v=4x+3$$
$$u'=2 \qquad v'=4$$

$$f'(x)=\frac{(2)(4x+3)-(2x-1)(4)}{(4x+3)^2} \qquad \left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{8x+6-8x+4}{(4x+3)^2}$$

$$=\frac{10}{(4x+3)^2}$$

Example

Find the derivative of $y=\dfrac{x+4}{5x-2}$

Solution

$$u=x+4 \quad v=5x-2$$
$$u'=1 \qquad v'=5$$

$$y'=\frac{(1)(5x-2)-(5)(x+4)}{(5x-2)^2} \qquad \left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{5x-2-5x-20}{(5x-2)^2}$$

$$=-\frac{22}{(5x-2)^2}$$

Example

Find an equation of the tangent line to the graph of $y=\dfrac{x^2-4}{2x+5}$ when $x = 0$

<u>*Solution*</u>

$$u=x^2-4 \quad v=2x+5$$
$$u'=2x \qquad v'=2$$

$$y'=\frac{(2x)(2x+5)-(2)\left(x^2-4\right)}{(2x+5)^2} \qquad \left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{4x^2+10x-2x^2+8}{(2x+5)^2}$$

$$=\frac{2x^2+10x+8}{(2x+5)^2}$$

$$\Rightarrow x=0 \to y'=\frac{8}{25}=m$$

$$x=0 \to y=\frac{x^2-4}{2x+5}=-\frac{4}{5}$$

$$y-y_1=m\left(x-x_1\right)$$

$$\to y+\frac{4}{5}=\frac{8}{25}(x-0) \qquad \to \boxed{y=\frac{8}{25}x-\frac{4}{5}}$$

Combining the product and Quotient Rules

Example

Find the derivative of $y = \dfrac{(1+x)(2x-1)}{x-1}$

<u>*Solution*</u>

$$y' = \frac{(x-1)\frac{d}{dx}\left[(1+x)(2x-1)\right] - (1+x)(2x-1)\frac{d}{dx}[x-1]}{(x-1)^2}$$

$$= \frac{(x-1)\left[(1)(2x-1)+2(1+x)\right]-(1+x)(2x-1)(1)}{(x-1)^2}$$

$$= \frac{(x-1)(2x-1+2+2x)-(2x-1+2x^2-x)}{(x-1)^2}$$

$$= \frac{(x-1)(4x+1)-2x+1-2x^2+x}{(x-1)^2}$$

$$= \frac{4x^2+x-4x-1-2x+1-2x^2+x}{(x-1)^2}$$

$$= \frac{2x^2-4x}{(x-1)^2}$$

Or

$$y = \frac{(1+x)(2x-1)}{x-1}$$

$$= \frac{2x-1+2x^2-x}{x-1}$$

$$= \frac{2x^2+x-1}{x-1}$$

$$y' = \frac{(x-1)(4x+1)-(2x^2+x-1)(1)}{(x-1)^2}$$

$$= \frac{4x^2+x-4x-1-2x^2-x+1}{(x-1)^2}$$

$$= \frac{2x^2-4x}{(x-1)^2}$$

$$= \frac{2x(x-2)}{(x-1)^2}$$

Average Cost Function

To Study the effects of production levels on cost, economist use the average cost function $\overline{C}$, which is defined as

$$\overline{C} = \frac{C}{x}$$

Where $C = f(x)$ is the total cost function and x the number of units produced.

Marginal Average Cost Function $= \overline{C}\,'$

Example

Suppose the cost in dollars of manufacturing x hundred small motors is given by

$$C(x) = \frac{3x^2+120}{2x+1} \qquad 10 \le x \le 200$$

a) Find the average cost per hundred motors

$$\overline{C} = \frac{C}{x}$$

$$= \frac{3x^2+120}{2x+1}\cdot\frac{1}{x}$$

$$= \frac{3x^2+120}{2x^2+x}$$

b) Find the marginal average cost

$$\overline{C}\,' = \frac{(6x)\left(2x^2+x\right)-\left(3x^2+120\right)(4x+1)}{\left(2x^2+x\right)^2}$$

$$\overline{C}\,' = \frac{\left(3x^2+120\right)'\left(2x^2+x\right)-\left(3x^2+120\right)\left(2x^2+x\right)'}{\left(2x^2+x\right)^2}$$

$$= \frac{12x^3+6x^2-12x^3-3x^2-480x-120}{\left(2x^2+x\right)^2}$$

$$= \frac{3x^2-480x-120}{\left(2x^2+x\right)^2}$$

c) Average cost is generally minimized when the marginal average cost is zero. Find the level of production that minimizes average cost

$$\frac{3x^2-480x-120}{\left(2x^2+x\right)^2} = 0$$

$$3x^2-480x-120=0 \quad \Rightarrow x = -0.25,\ 160$$

16,000 *motors* will minimize average cost.

Exercises *Section* 2.3 – Derivatives of Products and Quotients

Find the derivative

1. $y=(x+1)\left(\sqrt{x}+2\right)$

2. $y=\left(4x+3x^2\right)(6-3x)$

3. $y=\left(\frac{1}{x}+1\right)(2x+1)$

4. $y=3x\left(2x^2+5x\right)$

5. $y=3\left(2x^2+5x\right)$

6. $y=\frac{x^2+4x}{5}$

7. $y=\frac{3x^4}{5}$

8. $y=\frac{3-\frac{2}{x}}{x+4}$

9. $g(x)=\frac{x^2-4x+2}{x^2+3}$

10. $f(x)=\frac{(3-4x)(5x+1)}{7x-9}$

11. $f(x)=x\left(1-\frac{2}{x+1}\right)$

12. $g(s)=\frac{s^2-2s+5}{\sqrt{s}}$

13. $f(x)=\frac{x+1}{\sqrt{x}}$

14. $f(x)=\frac{x^2}{2x+1}$

15. $f(x)=\frac{x^2-x}{x^3+1}$

16. $f(x)=\frac{2x}{x^2+3}$

17. $y=\frac{t^3-3t}{t^2-4}$

18. $f(x) = 5x^2(x^3 + 2)$

19. $f(x) = \frac{3x-4}{2x+3}$

20. $f(x) = \frac{3x+5}{x^2-3}$

21. $f(x) = (x^2-4)(x^2+5)$

22. Find an equation of the tangent line to the graph of $y = \frac{x^2-4}{2x+5}$ when $x = 0$

23. A company that manufactures bicycles has determined that a new employee can assemble $M(d)$ bicycles per day after *d* days of on-the-job training, where

$$M(d) = \frac{100d^2}{3d^2+10}$$

a) Find the rate of change function for the number of bicycles assembled with respect to time.

b) Find and interpret $M'(2)$ and $M'(5)$

24. A small business invests $25,000.00 in a new product. In addition, the product will cost $0.75 per unit to produce. Find the cost function and the average cost function. What is the limit of the average cost function as production increase?

25. A communications company has installed a new cable TV system in a city. The total number *N* (in thousands) of subscribers *t* months after the installation of the system is given by

$$N(t) = \frac{180t}{t+4}$$

a) Find $N'(t)$

b) Find $N(16)$ and $N'(16)$. Write a brief interpretation of these results.

c) Use the results from part (***b***) to estimate the total number of subscribers after 17 months.

26. One hour after a dose of x milligrams of a particular drug is administered to a person, the change in body temperature $T(x)$, in degrees Fahrenheit, is given approximately by

$$T(x) = x^2\left(1 - \frac{x}{9}\right) \quad 0 \le x \le 7$$

The rate $T'(x)$ at which T changes with respect to the size of the dosage *x* is called the sensitivity of the body to the dosage.

a) Find $T'(x)$

b) Find $T'(1)$, $T'(3)$, and $T'(6)$

27. According to economic theory, the supply x of a quantity in a free market increases as the price p increases. Suppose that the number x of DVD players a retail chain is willing to sell per week at a price of \$$p$ is given by

$$x = \frac{100p}{0.1p+1} \quad 10 \le p \le 70$$

a) Find $\frac{dx}{dp}$

b) Find the supply and the instantaneous rate of change of supply with respect to price is \$40. Write a brief interpretation of these results.

c) Use the results from part (***b***) to estimate the supply if the price is increased to \$41.