Section 1.4 – Quadratic Graphics

Quadratic Function

A function f is a ***quadratic function*** if $f(x) = ax^2 + bx + c$

Formula | **Example**

Vertex of a Parabola

The ***vertex*** of the graph of $f(x)$ is

$$V_x \text{ or } x_v = -\frac{b}{2a}$$

$$V_y \text{ or } y_v = f\left(-\frac{b}{2a}\right)$$

$$\text{Vertex Point } \left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right)$$

Example: $f(x) = x^2 - 4x - 2$

$$x = -\frac{b}{2a} = -\frac{-4}{2(1)} = 2$$

$$y = f\left(-\frac{b}{2a}\right) = f(2)$$

$$= (2)^2 - 4(2) - 2$$

$$= -6$$

Vertex point: $(2,\ -6)$

Axis of Symmetry:

$$x = V_x = -\frac{b}{2a}$$

Axis of Symmetry: $x = 2$

Minimum or Maximum Point

If $a > 0 \Rightarrow$ *f(x)* has a ***minimum*** point

If $a < 0 \Rightarrow$ *f(x)* has a ***maximum*** point

@ vertex point $(V_x,\ V_y)$

Minimum point @ $(2,\ -6)$

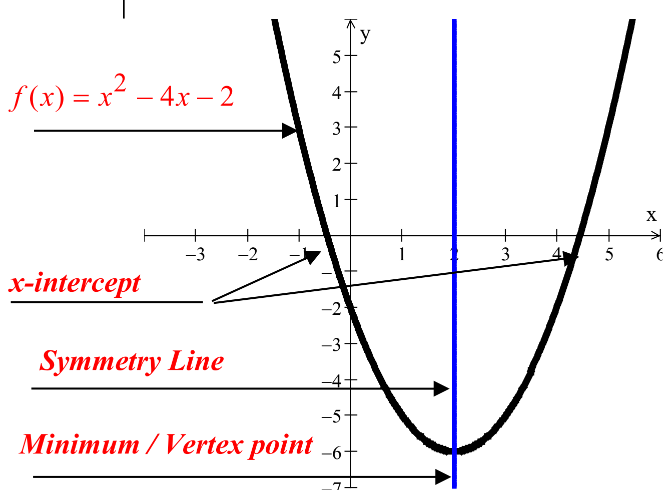
Range

If $a > 0 \Rightarrow [V_y,\ \infty)$

If $a < 0 \Rightarrow (-\infty,\ V_y]$

$[-6,\ \infty)$

Domain: $(-\infty,\ \infty)$



Example

For the graph of the function $f(x) = -x^2 - 2x + 8$

a. **Find the vertex point**

$$x = -\frac{-2}{2(-1)} = -1$$

$$y = f(-1) = -(-1)^2 - 2(-1) + 8 = 9$$

Vertex point $(-1, 9)$

b. **Find the line of symmetry:** $x = -1$

c. **State whether there is a maximum or minimum value *and* find that value**

Minimum point, value $(-1, 9)$

d. **Find the *x*-intercept**

$x = -4,\ 2$

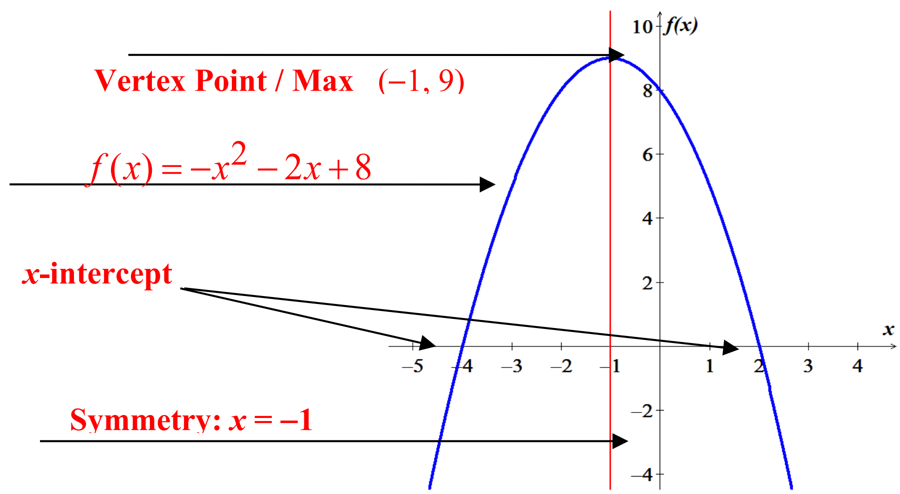
e. **Find the *y*-intercept**

$y = 8$

f. **Find the range and the domain of the function.**

Range: $(-\infty, 9]$ *Domain*: $(-\infty, \infty)$

g. **Graph the function and label, show part *a thru d* on the plot below**



h. **On what intervals is the function increasing? Decreasing?**

Increasing: $(-\infty, -1)$ *Decreasing*: $(-1, \infty)$

Example

Find the axis and vertex of the parabola having equation $f(x) = 2x^2 + 4x + 5$

Solution

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{4}{2(2)} \\ &= -1 \end{aligned}$$

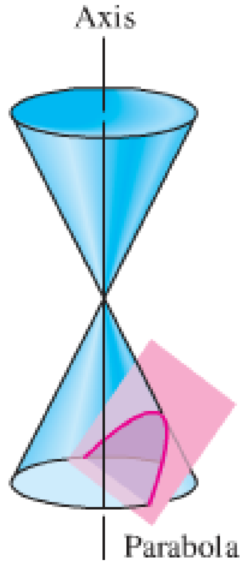
Axis of the parabola: $x = -1$

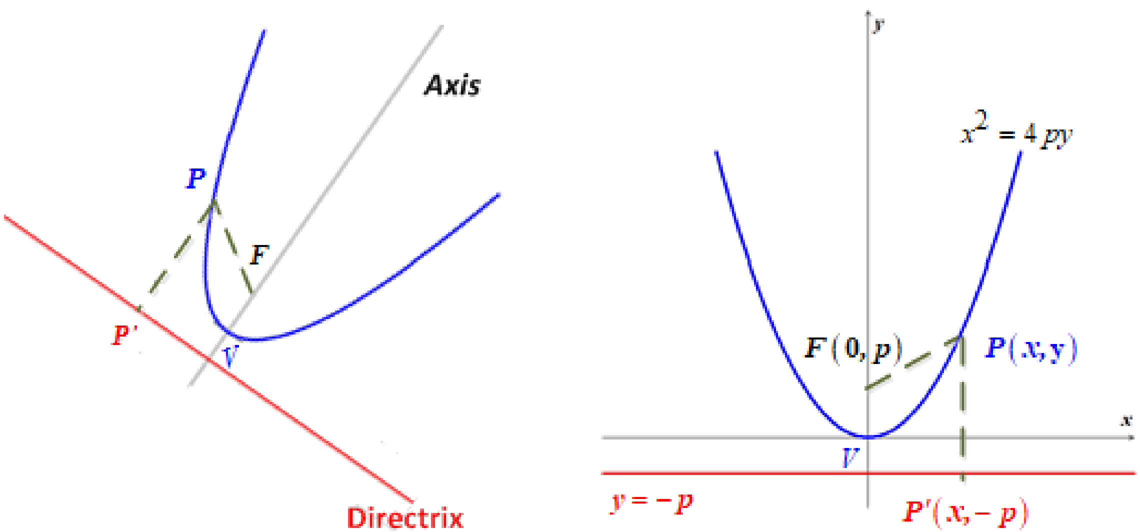
$$\begin{aligned} y &= f(-1) \\ &= 2(-1)^2 + 4(-1) + 5 \\ &= 3 \end{aligned}$$

Vertex point: $\boxed{(-1,\ 3)}$

Definition of a Parabola

A ***parabola*** is the set of all points in a plane equidistant from a fixed–point F (the ***focus***) and a fixed line l (the ***directrix***) that lie in the plane



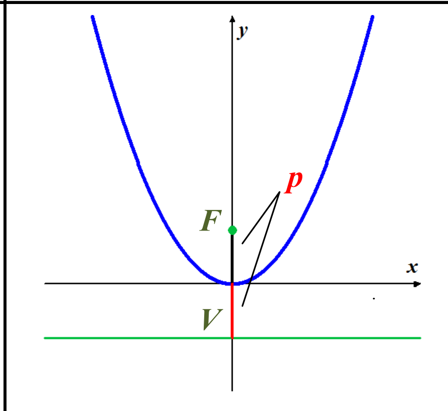
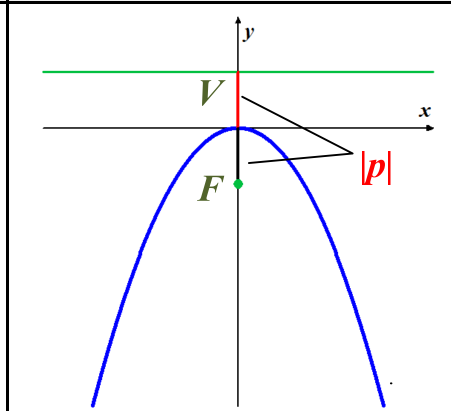
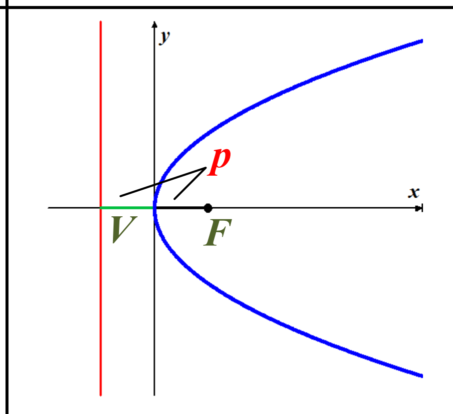
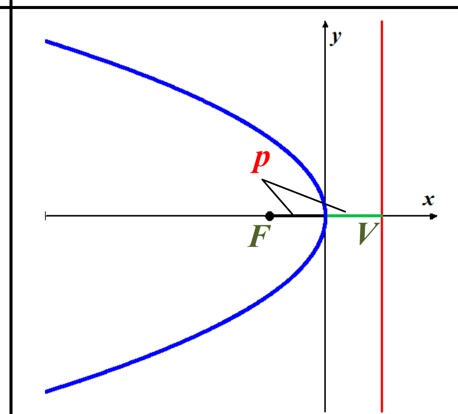
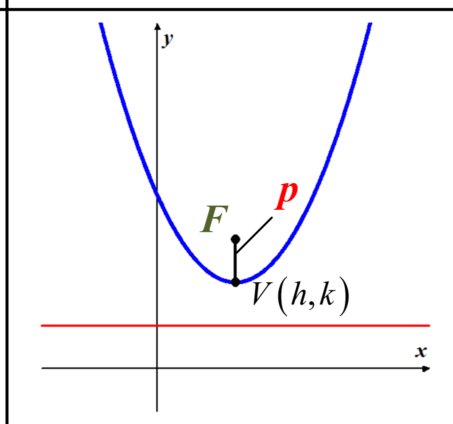
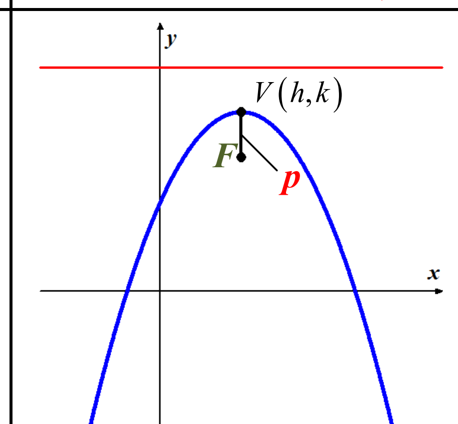
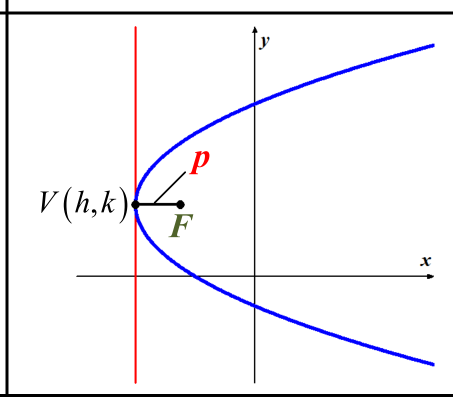
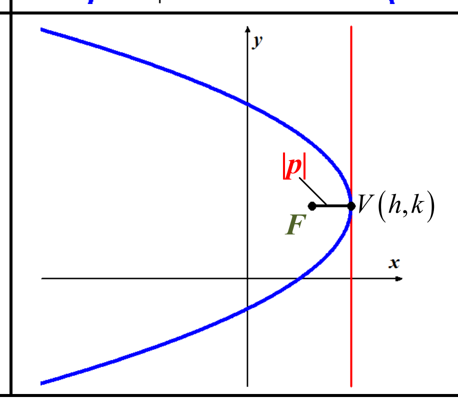


$$y = \frac{1}{4p}x^2 \quad or \quad x^2 = 4py \quad \rightarrow \begin{cases} \textbf{\textit{Focus}}: & F(0,\ p) \\ \textbf{\textit{Directrix}}: & y = -p \end{cases}$$

$$x = \frac{1}{4p}y^2 \quad or \quad y^2 = 4px \quad \rightarrow \begin{cases} \textbf{\textit{Focus}}: & F(p,\ 0) \\ \textbf{\textit{Directrix}}: & x = -p \end{cases}$$

The standard equation $y = ax^2 \quad or \quad x = ay^2$ is a parabola with vertex $V = (0, 0)$.

Moreover, $a = \dfrac{1}{4p} \quad or \quad p = \dfrac{1}{4a}$

Equation, focus, Directrix	Graph for $p>0$	Graph for $p<0$
$x^2 = 4py$ *or* $y = \frac{1}{4p}x^2$ Focus: $F(0,\ p)$ Directrix: $y = -p$		
$y^2 = 4px$ *or* $x = \frac{1}{4p}y^2$ Focus: $F(p,\ 0)$ Directrix: $x = -p$		
$(y-k)^2 = 4p(x-h)$ $x = ay^2 + by + c$ Focus: $F(h,\ k+p)$ Directrix: $x = h - p$		
$(x-h)^2 = 4p(y-k)$ $y = ax^2 + bx + c$ Focus: $F(h+p,\ k)$ Directrix: $y = k - p$		

Example

Find the focus and directrix of the parabola $y = -\frac{1}{6}x^2$.

Solution

Given: $a = -\frac{1}{6}$

$$p = \frac{1}{4a}$$

$$= \frac{1}{4\left(-\frac{1}{6}\right)}$$

$$= -\frac{6}{4}$$

$$\boxed{= -\frac{3}{2}}$$

The parabola opens downward and has focus $F\left(0,\ -\frac{3}{2}\right)$.

The directrix is the horizontal line $y = \frac{3}{2}$ which is a distance $\frac{3}{2}$ above V.

Example

a) Find an equation of a parabola that has vertex at the origin, open right, and passes through the point $P(7, -3)$.

b) Find the focus of the parabola.

Solution

a) An equation of a parabola with vertex at the origin that opens right has the form $x = ay^2$

$$7 = a(-3)^2$$

$$a = \frac{7}{9}$$

The equation is: $x = \frac{7}{9}y^2$

b) $p = \frac{1}{4a}$

$$= \frac{1}{4\left(\frac{7}{9}\right)}$$

$$\boxed{= \frac{9}{28}}$$

Thus, the focus has coordinate $\left(\frac{9}{28},\ 0\right)$

Example

Sketch the graph of $2x = y^2 + 8y + 22$

Solution

$$2x - 22 = y^2 + 8y$$

$$y^2 + 8y + \left(\tfrac{1}{2}8\right)^2 = 2x - 22 + \left(\tfrac{1}{2}8\right)^2$$

$$(y+4)^2 = 2x - 6$$

$$(y+4)^2 = 2(x-3)$$

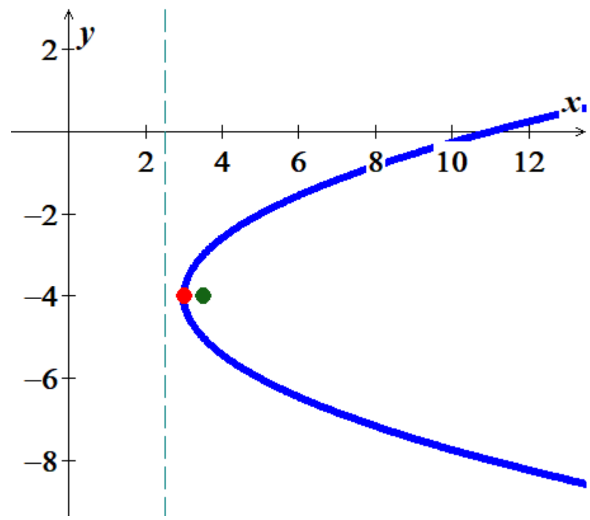
The vertex is $V(h,k) = V(3,\ -4)$

The focus is $F(h+p,k) = F\left(3+\frac{1}{2}, -4\right)$

$$\boxed{= F\left(\tfrac{7}{2},\ -4\right)}$$

The directrix is $x = h - p = 3 - \frac{1}{2}$

$$\boxed{= \tfrac{5}{2}}$$



Example

A parabola has vertex $V(-4, 2)$ and directrix $y = 5$. Express the equation of the parabola in the form $y = ax^2 + bx + c$

Solution

Directrix: $y = k - p \Rightarrow \quad p = k - y$

$$p = 2 - 5$$

$$\boxed{= -3}$$

$$(x-h)^2 = 4p(y-k)$$

$$(x+4)^2 = 4(-3)(y-2)$$

$$(x+4)^2 = -12(y-2)$$

$$x^2 + 8x + 16 = -12y + 24$$

$$x^2 + 8x + 16 - 24 = -12y + 24 - 24$$

$$-12y = x^2 + 8x - 8$$

$$\boxed{y = -\frac{1}{12}x^2 - \frac{2}{3}x + \frac{2}{3}}$$

Exercises *Section* 1.4 – Quadratic Graphics

(1 – 21) For the Given functions

a) Find the vertex point
b) Find the line of symmetry
c) State whether there is a *maximum* or *minimum* value *and* find that value
d) Find the zeros of $f(x)$
e) Find the y-intercept
f) Find the *range* and the *domain* of the function.
g) Graph the function and label, show part *a thru d*
h) On what intervals is the function *increasing*? *decreasing*?

1. $f(x)=x^2+6x+3$
2. $f(x)=x^2+6x+5$
3. $f(x)=-x^2-6x-5$
4. $f(x)=x^2-4x+2$
5. $f(x)=-2x^2+16x-26$
6. $f(x)=x^2+4x+1$
7. $f(x)=x^2-8x+5$
8. $f(x)=x^2+6x-1$
9. $f(x)=x^2+6x+3$
10. $f(x)=x^2-10x+3$
11. $f(x)=x^2-3x+4$
12. $f(x)=x^2-3x-4$
13. $f(x)=x^2-4x-5$
14. $f(x)=2x^2-3x+1$
15. $f(x)=-x^2-3x+4$
16. $f(x)=-2x^2+3x-1$
17. $f(x)=-2x^2-3x-1$
18. $f(x)=-x^2-4x+5$
19. $f(x)=-x^2+4x+2$
20. $f(x)=-3x^2+3x+7$
21. $f(x)=-x^2+2x-2$

(22 – 36) Find the vertex, focus, and directrix of the parabola. Sketch its graph.

22. $20x=y^2$
23. $2y^2=-3x$
24. $(x+2)^2=-8(y-1)$
25. $(x-3)^2=\frac{1}{2}(y+1)$
26. $(y+1)^2=-12(x+2)$
27. $y=x^2-4x+2$
28. $y^2+14y+4x+45=0$
29. $x^2+20y=10$
30. $x^2=16y$
31. $x^2=-\frac{1}{2}y$
32. $(y+1)^2=-4(x-2)$
33. $x^2+6x-4y+1=0$
34. $y^2+2y-x=0$
35. $y^2-4y+4x+4=0$
36. $x^2-4x-4y=4$

(37 – 45) Find an equation of the parabola that satisfies the given conditions

37. $Focus: F(2,0)$ $directrix: x=-2$
38. $Focus: F(0,-40)$ $directrix: y=4$
39. $Focus: F(-3,-2)$ $directrix: y=1$
40. $Vertex: V(3,-5)$ $directrix: x=2$
41. $Vertex: V(-2,3)$ $directrix: y=5$
42. $Vertex: V(-1,0)$ $focus: F(-4,0)$
43. $Vertex: V(1,-2)$ $focus: F(1,0)$
44. $Vertex: V(0,\ 1)$ $focus: F(0,\ 2)$
45. $Vertex: V(3,\ 2)$ $focus: F(-1,\ 2)$