Lecture Three

Section 3.1 – Inner Products

Definition

An ***inner product*** on a real vector space V is a function that associates a real number $\langle \vec{u}, \vec{v} \rangle$ with each pair of vectors in V in such a way that the following axioms are satisfies for all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in V and all scalars k.

1. $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ ***Symmetry axiom***
2. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ ***Additivity axiom***
3. $\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle$ ***Homogeneity axiom***
4. $\langle \vec{v}, \vec{v} \rangle \ge 0$ and $\langle \vec{v}, \vec{v} \rangle = 0$ iff $\vec{v} = 0$ ***Positivity axiom***

A real vector space with an inner product is called a ***real inner product space***.

$$\boxed{\langle \vec{u}, \vec{u} \rangle = \vec{u} \bullet \vec{v} = u_1 v_1 + u_2 v_2 + \cdots u_n v_n}$$

This is called the ***Euclidean inner product*** (or the ***standard inner product***)

Definition

If V is a real inner product space, then the norm (or length) of a vector $\vec{v}$ in V is denoted by $\|\vec{v}\|$ and is defined by

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

And the ***distance*** between two vectors is denoted by $d(\vec{u}, \vec{v})$ and is defined by

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{\langle \vec{u} - \vec{v}, \ \vec{u} - \vec{v} \rangle}$$

A vector of norm 1 is called a ***unit vector***.

Theorem

If ***u*** and ***v*** are vectors in a real inner product space V, and if k is a scalar, then:

a) $\|\vec{v}\| \ge 0$ with equality *iff* $\vec{v} = 0$

b) $\|k\vec{v}\| = |k| \|\vec{v}\|$

c) $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$

d) $d(\vec{u}, \vec{v}) \ge 0$ with equality *iff* $\vec{u} = \vec{v}$

Although the Euclidean inner product is the most important inner product on $\mathbb{R}^n$, there are various applications in which is desirable to modify it by weighing each term differently. More precisely, if $w_1, w_2, \ldots, w_n$ are positive real numbers, which we will call weighs, and if $\vec{u} = (u_1, u_2, \ldots, u_n)$ and are vectors in $\mathbb{R}^n$, then it can be shown that the formula

$$\langle \vec{u}, \vec{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \ldots + w_n u_n v_n$$

Defines an inner product on $\mathbb{R}^n$ that we call the ***weighted Euclidean inner product*** with weights $\boldsymbol{w_1, w_2, \ldots, w_n}$

Example

Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ be vectors in $\mathbb{R}^2$, verify that the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 3u_1 v_1 + 2u_2 v_2$ satisfies the four inner product axioms.

Solution

Axiom 1: $\langle \vec{u}, \vec{v} \rangle = 3u_1 v_1 + 2u_2 v_2$

$$= 3v_1 u_1 + 2v_2 u_2$$

$$= \langle \vec{v}, \vec{u} \rangle$$

Axiom 2: $\langle \vec{u} + \vec{v}, \vec{w} \rangle = 3(u_1 + v_1) w_1 + 2(u_2 + v_2) w_2$

$$= 3(u_1 w_1 + v_1 w_1) + 2(u_2 w_2 + v_2 w_2)$$

$$= 3u_1 w_1 + 3v_1 w_1 + 2u_2 w_2 + 2v_2 w_2$$

$$= (3u_1 w_1 + 2u_2 w_2) + (3v_1 w_1 + 2v_2 w_2)$$

$$= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$$

Axiom 3: $\langle k\vec{u}, \vec{v} \rangle = 3(ku_1) v_1 + 2(ku_2) v_2$

$$= k(3u_1 v_1 + 2u_2 v_2)$$

$$= k\langle \vec{u}, \vec{v} \rangle$$

Axiom 4: $\langle \vec{v}, \vec{v} \rangle = 3v_1 v_1 + 2v_2 v_2$

$$= 3v_1^2 + 2v_2^2 \ge 0$$

$$v_1 = v_2 = 0 \quad \textit{iff} \quad \vec{v} = \vec{0}$$

Exercises *Section* 3.1 – Inner Products

1. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (1,\ 1)$, $\vec{v} = (3,\ 2)$, $\vec{w} = (0,\ -1)$, and $k = 3$. Compute the following.

a) $\langle \vec{u}, \vec{v} \rangle$ *c)* $\langle \vec{u} + \vec{v}, \vec{w} \rangle$ *e)* $d(\vec{u}, \vec{v})$

b) $\langle k\vec{v}, \vec{w} \rangle$ *d)* $\|\vec{v}\|$ *f)* $\|\vec{u} - k\vec{v}\|$

2. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (1,\ 1)$, $\vec{v} = (3,\ 2)$, $\vec{w} = (0,\ -1)$ and $k = 3$. Compute the following for the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 2u_1v_1 + 3u_2v_2$.

a) $\langle \vec{u}, \vec{v} \rangle$ *c)* $\langle \vec{u} + \vec{v}, \vec{w} \rangle$ *e)* $d(\vec{u}, \vec{v})$

b) $\langle k\vec{v}, \vec{w} \rangle$ *d)* $\|\vec{v}\|$ *f)* $\|\vec{u} - k\vec{v}\|$

3. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (3,\ -2)$, $\vec{v} = (4,\ 5)$, $\vec{w} = (-1,\ 6)$, and $k = -4$. Verify the following.

a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

d) $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$

e) $\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$

4. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (3,\ -2)$, $\vec{v} = (4,\ 5)$, $\vec{w} = (-1,\ 6)$, and $k = -4$. Verify the following for the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 4u_1v_1 + 5u_2v_2$

a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

d) $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$

e) $\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$

5. Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$. Show that the following are inner product on $\mathbb{R}^2$ by verifying that the inner product axioms hold. $\langle \vec{u}, \vec{v} \rangle = 3u_1v_1 + 5u_2v_2$

6. Show that the following identity holds for the vectors in any inner product space

$$\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

7. Show that the following identity holds for the vectors in any inner product space

$$\langle \vec{u}, \vec{v} \rangle = \frac{1}{4}\left(\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2 \right)$$

8. Prove that $\|k\vec{v}\| = |k| \, \|\vec{v}\|$