Solution *Section* 2.7 – Implicit Differentiation

Exercise

Find $\frac{dy}{dx}$: $y^2 + x^2 - 2y - 4x = 4$

Solution

$$\frac{d}{dx}[y^2 + x^2 - 2y - 4x] = \frac{d}{dx}[4]$$

$$\frac{d}{dx}\left[y^2\right] + \frac{d}{dx}\left[x^2\right] - \frac{d}{dx}[2y] - \frac{d}{dx}[4x] = \frac{d}{dx}[4]$$

$$2y\frac{dy}{dx} + 2x - 2\frac{dy}{dx} - 4 = 0$$

$$2(y-1)\frac{dy}{dx} = 4 - 2x$$

$$(y-1)\frac{dy}{dx} = 2 - x$$

$$\boxed{\frac{dy}{dx} = \frac{2-x}{y-1}}$$

Exercise

Find $\frac{dy}{dx}$: $x^2y^2 - 2x = 3$

Solution

$$2xy^2 + 2x^2yy' - 2 = 0$$

$$2x^2yy' = 2 - 2xy^2$$

$$y' = \frac{2\left(1 - xy^2\right)}{2x^2y}$$

$$\boxed{\frac{dy}{dx} = \frac{1-xy^2}{x^2y}}$$

Exercise

Find $\frac{dy}{dx}$: $\quad x+\sqrt{x}\sqrt{y}=y^2$

Solution

$$\frac{d}{dx}\left(x+x^{1/2}y^{1/2}\right)=\frac{d}{dx}y^2$$

$$1+\frac{d}{dx}\left(x^{1/2}\right)y^{1/2}+x^{1/2}\frac{d}{dx}\left(y^{1/2}\right)=2y\frac{dy}{dx}$$

$$1+\frac{1}{2}x^{-1/2}y^{1/2}+x^{1/2}\frac{1}{2}y^{-1/2}\frac{dy}{dx}=2y\frac{dy}{dx}$$

$$1+\frac{y^{1/2}}{2x^{1/2}}+\frac{x^{1/2}}{2y^{1/2}}\frac{dy}{dx}=2y\frac{dy}{dx}$$

$$1+\frac{y^{1/2}}{2x^{1/2}}=2y\frac{dy}{dx}-\frac{x^{1/2}}{2y^{1/2}}\frac{dy}{dx}$$

$$\left(\frac{4y^{3/2}-x^{1/2}}{2y^{1/2}}\right)\frac{dy}{dx}=\frac{2x^{1/2}+y^{1/2}}{2x^{1/2}}$$

$$\frac{dy}{dx}=\frac{2x^{1/2}+y^{1/2}}{2x^{1/2}}\cdot\frac{2y^{1/2}}{4y^{3/2}-x^{1/2}}$$

$$=\frac{4x^{1/2}y^{1/2}+2y}{8x^{1/2}y^{3/2}-2x}$$ ***Divide every term by 2***

$$=\frac{2x^{1/2}y^{1/2}+y}{4x^{1/2}y^{3/2}-x}$$

Exercise

Find $\frac{dy}{dx}$: $\quad x^2y+xy^2=6$

Solution

$$\left(2xy+x^2\frac{dy}{dx}\right)+\left(y^2+2xy\frac{dy}{dx}\right)=0$$

$$x^2\frac{dy}{dx}+2xy\frac{dy}{dx}=-2xy-y^2$$

$$\left(x^2+2xy\right)\frac{dy}{dx}=-2xy-y^2$$

$$\frac{dy}{dx}=\frac{-2xy-y^2}{x^2+2xy}$$

Exercise

Find $\frac{dy}{dx}$: $\quad x^3 - xy + y^3 = 1$

Solution

$$3x^2 - \left(y + x\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$$

$$3x^2 - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$\left(3y^2 - x\right)\frac{dy}{dx} = y - 3x^2$$

$$\frac{dy}{dx} = \frac{y - 3x^2}{3y^2 - x}$$

Exercise

Find $\frac{dy}{dx}$: $\quad y^2 = \frac{x-1}{x+1}$

Solution

$$2yy' = \frac{1(x+1) - (1)(x-1)}{(x+1)^2}$$

$$2yy' = \frac{x+1-x+1}{(x+1)^2}$$

$$y' = \frac{2}{2y(x+1)^2}$$

$$\frac{dy}{dx} = \frac{1}{y(x+1)^2}$$

Exercise

Find $\frac{dy}{dx}$: $\quad (3xy+7)^2 = 6y$

Solution

$$2(3xy+7)(3y+3xy') = 6y'$$

$$6(3xy+7)(y+xy') = 6y'$$

Divide by 6 both sides

$$(3xy+7)(y+xy') = y'$$

$$3xy^2 + 3x^2yy' + 7y + 7xy' = y'$$

$$3x^2yy' + 7xy' - y' = -3xy^2 - 7y$$

$$\left(3x^2y+7x-1\right)y' = -\left(3xy^2+7y\right)$$

$$\boxed{\frac{dy}{dx} = -\frac{3xy^2+7y}{3x^2y+7x-1}}$$

Exercise

Find $\frac{dy}{dx}$: $\quad xy = \cot(xy)$

Solution

$$y + xy' = -\csc^2(xy)\ \left(y + xy'\right)$$

$$y + xy' = -y\csc^2(xy) - x\csc^2(xy)y'$$

$$x\csc^2(xy)y' + xy' = -y\csc^2(xy) - y$$

$$x\left(\csc^2(xy)+1\right)y' = -y\left(\csc^2(xy)+1\right)$$

$$y' = -\frac{y\left(\csc^2(xy)+1\right)}{x\left(\csc^2(xy)+1\right)}$$

$$\boxed{\frac{dy}{dx} = -\frac{y}{x}}$$

Exercise

Find $\frac{dy}{dx}$: $\quad x + \tan(xy) = 0$

Solution

$$1 + \sec^2(xy)\left(y + xy'\right) = 0$$

$$1 + y\sec^2(xy) + x\sec^2(xy)y' = 0$$

$$x\sec^2(xy)y' = -y\sec^2(xy) - 1$$

$$y' = -\frac{y\sec^2(xy)}{x\sec^2(xy)} - \frac{1}{x\sec^2(xy)}$$

$$\frac{dy}{dx} = -\frac{y}{x} - \frac{\cos^2 x}{x}$$

$$\boxed{= \frac{-y-\cos^2 x}{x}}$$

Exercise

Find $\frac{dy}{dx}$: $\quad x\cos(2x+3y) = y\sin x$

Solution

$$\cos(2x+3y) - \sin(2x+3y)(2x+3y') = y'\sin x + y\cos x$$

$$\cos(2x+3y) - 2x\sin(2x+3y) - 3\sin(2x+3y)y' = y'\sin x + y\cos x$$

$$\cos(2x+3y) - 2x\sin(2x+3y) - y\cos x = y'\sin x + 3\sin(2x+3y)y'$$

$$\cos(2x+3y) - 2x\sin(2x+3y) - y\cos x = y'\left(\sin x + 3\sin(2x+3y)\right)$$

$$y' = \frac{\cos(2x+3y) - 2x\sin(2x+3y) - y\cos x}{\sin x + 3\sin(2x+3y)}$$

Exercise

Find $\frac{dy}{dx}$: $\quad y = \frac{e^y}{1+\sin x}$

Solution

$$y(1+\sin x) = e^y$$

$$y'(1+\sin x) + y\cos x = y'e^y$$

$$y'\left(e^y - 1 - \sin x\right) = y\cos x$$

$$\frac{dy}{dx} = \frac{y\cos x}{e^y - 1 - \sin x}$$

Exercise

Find $\frac{dy}{dx}$: $\quad \sin x\cos(y-1) = \frac{1}{2}$

Solution

$$\cos x\cos(y-1) - y'\sin x\sin(y-1) = 0$$

$$y'\sin x\sin(y-1) = \cos x\cos(y-1)$$

$$y' = \frac{\cos x\cos(y-1)}{\sin x\sin(y-1)}$$

$$\frac{dy}{dx} = \cot x\cot(y-1)$$

Exercise

Find $\frac{dy}{dx}$: $\quad y\sqrt{x^2+y^2}=15$

Solution

$$y'\sqrt{x^2+y^2}+\frac{1}{2}y(2x+2yy')\left(x^2+y^2\right)^{-1/2}=0 \qquad \times\sqrt{x^2+y^2}$$

$$y'\left(x^2+y^2\right)+y(x+yy')=0$$

$$y'\left(x^2+y^2\right)+y^2y'=-xy$$

$$y'\left(x^2+2y^2\right)=-xy$$

$$\boxed{\frac{dy}{dx}=-\frac{xy}{x^2+2y^2}}$$

Exercise

Find $\frac{dr}{d\theta}$ $\quad r-2\sqrt{\theta}=\frac{3}{2}\theta^{2/3}+\frac{4}{3}\theta^{3/4}$

Solution

$$r-2\theta^{1/2}=\frac{3}{2}\theta^{2/3}+\frac{4}{3}\theta^{3/4}$$

$$\frac{dr}{d\theta}-2\frac{1}{2}\theta^{-1/2}=\frac{3}{2}\frac{2}{3}\theta^{-1/3}+\frac{4}{3}\frac{3}{4}\theta^{-1/4}$$

$$\boxed{\frac{dr}{d\theta}=\theta^{-1/3}+\theta^{-1/4}+\theta^{-1/2}}$$

Exercise

Find $\frac{dr}{d\theta}$ $\quad \sin(r\theta)=\frac{1}{2}$

Solution

$$\cos(r\theta)\left(\theta\frac{dr}{d\theta}+r\right)=0$$

$$\theta\frac{dr}{d\theta}+r=0 \qquad \cos(r\theta)\neq 0$$

$$\boxed{\frac{dr}{d\theta}=-\frac{r}{\theta}} \qquad \cos(r\theta)\neq 0$$

Exercise

Find $\dfrac{d^2y}{dx^2}$ $\qquad x^{2/3}+y^{2/3}=1$

Solution

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0 \qquad \textit{Multiply all terms by } \frac{3}{2}$$

$$x^{-1/3}+y^{-1/3}y'=0$$

$$y^{-1/3}y'=-x^{-1/3}$$

$$y'=-\frac{x^{-1/3}}{y^{-1/3}}$$

$$=-\frac{y^{1/3}}{x^{1/3}}=-\left(\frac{y}{x}\right)^{1/3}$$

$$y''=-\frac{1}{3}\left(\frac{y}{x}\right)^{-2/3}\left(\frac{xy'-y}{x^2}\right)$$

$$=-\frac{1}{3}\left(\frac{x}{y}\right)^{2/3}\left(\frac{-x\left(\frac{y}{x}\right)^{1/3}-y}{x^2}\right)=\frac{1}{3}\left(\frac{x^{4/3}y^{1/3}}{y^{2/3}x^2}+\frac{x^{2/3}y}{y^{2/3}x^2}\right)$$

$$=\frac{1}{3}\left(\frac{x}{y}\right)^{2/3}\left(\frac{x\frac{y^{1/3}}{x^{1/3}}+y}{x^2}\right)$$

$$=\frac{1}{3}\frac{x^{2/3}}{y^{2/3}}\frac{x^{2/3}y^{1/3}+y}{x^2}$$

$$=\frac{1}{3}\left(\frac{1}{y^{1/3}x^{2/3}}+\frac{y^{1/3}}{x^{4/3}}\right)$$

Exercise

Find $\dfrac{d^2y}{dx^2}$ $\qquad 2\sqrt{y}=x-y$

Solution

$$2\frac{1}{2}y^{-1/2}y'=1-y'$$

$$2\frac{1}{2}y^{-1/2}y' + y' = 1$$

$$\left(y^{-1/2}+1\right)y' = 1 \Rightarrow \boxed{y' = \frac{1}{y^{-1/2}+1}}$$

$$y' = \frac{1}{y^{-1/2}+1} = \frac{1}{\frac{1}{\sqrt{y}}+1} = \frac{\sqrt{y}}{1+\sqrt{y}}$$

$$\left(y^{-1/2}+1\right)y'' + \left(-\frac{1}{2}y^{-3/2}y'\right)y' = 0$$

$$\left(y^{-1/2}+1\right)y'' - \frac{1}{2}y^{-3/2}\left(y'\right)^2 = 0$$

$$\left(y^{-1/2}+1\right)y'' = \frac{1}{2}y^{-3/2}\left(\frac{1}{y^{-1/2}+1}\right)^2$$

$$y'' = \frac{1}{2}y^{-3/2}\frac{1}{\left(y^{-1/2}+1\right)^2}\frac{1}{y^{-1/2}+1}$$

$$= \frac{1}{2}y^{-3/2}\frac{1}{\left(\frac{1}{\sqrt{y}}+1\right)^3}$$

$$= \frac{1}{2}y^{-3/2}\frac{1}{\left(\frac{1+\sqrt{y}}{\sqrt{y}}\right)^3}$$

$$= \frac{1}{2}y^{-3/2}\frac{1}{\frac{\left(1+\sqrt{y}\right)^3}{\left(y^{1/2}\right)^3}}$$

$$= \frac{1}{2}y^{-3/2}\frac{y^{3/2}}{\left(1+\sqrt{y}\right)^3}$$

$$= \frac{1}{2\left(1+\sqrt{y}\right)^3}$$

Exercise

If $x^3 + y^3 = 16$, find the value of $\frac{d^2y}{dx^2}$ at the point (2, 2).

Solution

$$3x^2 + 3y^2y' = 0$$

$$3y^2y' = -3x^2$$

$$y^2y' = -x^2$$

$$2yy'y' + y^2y'' = -2x$$

$$y^2 y'' = -2x - 2y\left(y'\right)^2$$

$$y^2 y'' = -2x - 2y\left(\frac{-x^2}{y^2}\right)^2$$

$$y^2 y'' = -2x - 2\frac{x^4}{y^3}$$

$$y'' = -2\frac{x}{y^2} - 2\frac{x^4}{y^5}$$

$$= \frac{-2xy^3 - 2x^4}{y^5}$$

$$y''\Big|_{(2,2)} = \frac{-2(2)2^3 - 2(2)^4}{2^5}$$

$$= \frac{-2^5 - 2^5}{2^5}$$

$$= -2$$

Exercise

Find dy/dx: $x^2 - xy + y^2 = 4$ and evaluate the derivative at the given point $(0,-2)$

Solution

$$2x - \left(y + xy'\right) + 2yy' = 0$$

$$-y - xy' + 2yy' = -2x$$

$$\left(2y - x\right)y' = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

$$@(0,-2) \rightarrow \frac{dy}{dx} = \frac{-2 - 2(0)}{2(-2) - (0)}$$

$$= \frac{-2}{-4}$$

$$= \frac{1}{2}$$

Exercise

Find the slope of the curve $\left(x^2+y^2\right)^2=(x-y)^2$ at the point (–2, 1) and (–2, –1)

Solution

1 and -1

Exercise

Find the slope of the tangent line to the circle $x^2-9y^2=16$ at the point (5, 1)

Solution

$$2x-18y\frac{dy}{dx}=0$$

$$-18y\frac{dy}{dx}=-2x$$

$$\frac{dy}{dx}=\frac{-2x}{-18y}=\frac{x}{9y}$$

$$@\ (5,\ 1)\ \rightarrow\ \frac{dy}{dx}=\frac{5}{9(1)}\ =\frac{5}{9}$$

Exercise

Find the slope of the tangent line to the circle $x^2+y^2=25$ at the point (3, –4)

Solution

$$\frac{d}{dx}\left[x^2+y^2\right]=\frac{d}{dx}[25]$$

$$2x+2y\frac{dy}{dx}=0$$

$$2y\frac{dy}{dx}=-2x\quad\rightarrow\quad\frac{dy}{dx}=-\frac{x}{y}$$

Slope: $\frac{dy}{dx}=-\frac{3}{-4}\ =\frac{3}{4}$

Exercise

Find an equation of the line tangent to the following curves at the given point

$$y=3x^3+\sin x;\quad (0,\ 0)$$

Solution

$$m=y'=9x^2+\cos x\Big|_{(0,\ 0)}$$

$$=1$$

$$y = x$$ $\qquad\qquad y = m\left(x - x_1\right) + y_1$

Exercise

Find an equation of the line tangent to the following curves at the given point

$$y = \frac{4x}{x^2+3}; \quad (3,\ 1)$$

Solution

$$m = y' = \frac{4x^2 + 12 - 8x^2}{\left(x^2+3\right)^2}$$

$$= \frac{12 - 4x^2}{\left(x^2+3\right)^2}\Bigg|_{(3,\ 1)}$$

$$= \frac{-24}{144}$$

$$= -\frac{1}{6}$$

$$y = -\frac{1}{6}(x-3) + 1 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$= -\frac{1}{6}x + \frac{3}{2}$$

Exercise

Find an equation of the line tangent to the following curves at the given point

$$y + \sqrt{xy} = 6; \quad (1,\ 4)$$

Solution

$$y' + \frac{1}{2}\left(y + xy'\right)\frac{1}{\sqrt{xy}} = 0\Bigg|_{(1,\ 4)}$$

$$y' + \frac{1}{2}\left(4 + y'\right)\frac{1}{2} = 0$$

$$y' + \frac{1}{4}y' = -1$$

$$\frac{5}{4}y' = -1$$

$$m = y' = -\frac{4}{5}$$

$$y = -\frac{4}{5}(x-1) + 4 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$= -\frac{4}{5}x + \frac{24}{5}$$

Exercise

Find an equation of the line tangent to the following curves at the given point

$$x^2y + y^3 = 75; \quad (4,\ 3)$$

Solution

$$2xy + x^2y' + 3y^2y' = 0\Big|_{(4,\ 3)}$$

$$(16 + 27)y' = -24$$

$$\boxed{y' = -\frac{24}{43} = m}$$

$$y = -\frac{24}{43}(x-4) + 3 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$\boxed{= -\frac{24}{43}x + \frac{225}{43}}$$

Exercise

Find the equation of the tangent line to the circle $x^3 + y^3 = 9xy$ at the point (2, 4)

Solution

$$3x^2 + 3y^2y' = 9y + 9xy'$$

$$3y^2y' - 9xy' = 9y - 3x^2$$

$$\left(3y^2 - 9x\right)y' = 9y - 3x^2$$

$$y' = \frac{3\left(3y - x^2\right)}{3\left(y^2 - 3x\right)}$$

$$= \frac{3y - x^2}{y^2 - 3x}$$

$$\boxed{m\Big|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}}$$

$$y = \frac{4}{5}(x-2) + 4 \quad \Rightarrow \ y = \frac{4}{5}x - \frac{8}{5} + 4 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$\boxed{y = \frac{4}{5}x + \frac{12}{5}}$$

Exercise

Find the lines that are (***a***) tangent and (***b***) normal to the curve $x^2 + xy - y^2 = 1$ at the point (2, 3).

Solution

$$2x + y + xy' - 2yy' = 0$$

$$(x - 2y)y' = -2x - y$$

$$y' = \frac{-2x-y}{x-2y} = \frac{2x + y}{2y - x}$$

a) $tangent\ slope = y'\Big|_{(2,3)} = \frac{2(2)+3}{2(3)-2} = \frac{7}{4}$

$$y = \frac{7}{4}(x - 2) + 3 \qquad\qquad y = m(x - x_1) + y_1$$

$$y = \frac{7}{4}x - \frac{7}{2} + 3$$

$$\boxed{y = \frac{7}{4}x - \frac{1}{2}}$$

b) $normal\ slope = -\frac{4}{7}$

$$y = -\frac{4}{7}(x - 2) + 3 \qquad\qquad y = m(x - x_1) + y_1$$

$$y = \frac{4}{7}x - \frac{8}{7} + 3$$

$$\boxed{y = -\frac{4}{7}x + \frac{29}{7}}$$

Exercise

Find the lines that are (***a***) tangent and (***b***) normal to the curve $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$ at the point (–1, 0).

Solution

$$12x + 3y + 3xy' + 4yy' + 17y' = 0$$

$$(3x + 4y + 17)y' = -12x - 3y$$

$$y' = \frac{-12x - 3y}{3x + 4y + 17}$$

a) $tangent\ slope = y'\Big|_{(-1,0)} = \frac{-12(-1)-3(0)}{3(-1)+4(0)+17} = \boxed{\frac{6}{7}}$

$$y = \frac{6}{7}(x + 1) \Rightarrow \boxed{y = \frac{6}{7}x + \frac{6}{7}} \qquad\qquad y = m(x - x_1) + y_1$$

b) $normal\ slope = -\frac{7}{6}$

$$y = -\frac{7}{6}(x+1) \quad \Rightarrow \quad \boxed{y = -\frac{7}{6}x - \frac{7}{6}}$$

$$y = m\left(x - x_1\right) + y_1$$

Exercise

Find the lines that are (***a***) tangent and (***b***) normal to the curve $x^2\cos^2 y - \sin y = 0$ at the point (0, π).

Solution

$$2x\cos^2 y + x^2\left(2\cos y(-\sin y)y'\right) - (\cos y)y' = 0$$

$$\left(-2x^2\cos y\sin y - \cos y\right)y' = -2x\cos^2 y$$

$$y' = \frac{-2x\cos^2 y}{-\left(2x^2\sin y+1\right)\cos y} = \frac{2x\cos y}{2x^2\sin y+1}$$

***a)** tangent slope* $= y'\Big|_{(0,\pi)} = \dfrac{2(0)\cos(\pi)}{2(0)^2\sin(\pi)+1} = \boxed{0}$

$$y - \pi = 0(x-0) \quad \Rightarrow \quad \boxed{y = \pi}$$

***b)** normal slope* $= 0$

$$\Rightarrow \quad \boxed{x = 0}$$

Exercise

Suppose that *x* and *y* are both functions of *t*, which can be considered to represent time, and that *x* and *y* are related by the equation $xy^2 + y = x^2 + 17$

Suppose further that when *x* = 2 and *y* = 3, then $\frac{dx}{dt} = 13$. Find the value of the $\frac{dy}{dt}$ at that moment.

Solution

$$y^2\frac{dx}{dt} + 2xy\frac{dy}{dt} + \frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$3^2(13) + 2(2)(3)\frac{dy}{dt} + \frac{dy}{dt} = 2(2)(13)$$

$$117 + 12\frac{dy}{dt} + \frac{dy}{dt} = 52$$

$$13\frac{dy}{dt} = -65$$

$$\frac{dy}{dt} = \frac{-65}{13}$$

$$\boxed{= -5}$$

Exercise

A cone-shaped icicle is dripping from the roof. The radius of the icicle is decreasing at a rate of 0.2 *cm* per hour, while the length is increasing at a rate of 0.8 *cm* per *hour*. If the icicle is currently 4 *cm* in radius and 20 *cm* long, is the volume of the icicle increasing or decreasing and at what rate?

Solution

The volume of the cone is given by the formula: $V = \frac{1}{3}\pi r^2 h$.

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right]$$

Given the values:

$$\frac{dr}{dt} = -0.2 \qquad \frac{dh}{dt} = 0.8 \qquad r = 4 \qquad h = 20$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[2(4)(20)(-0.2) + 4^2(0.8)\right]$$

$$= -20$$

The volume is decreasing at a rate of 20 cm^3 per *hour*.

Solution Section 2.8 – Derivatives of Logarithmic & Exponential Functions

Exercise

Find the derivative of $y = \ln\sqrt{x+5}$

Solution

$$y = \ln(x+5)^{1/2}$$

$$= \frac{1}{2}\ln(x+5)$$

$$\boxed{y' = \frac{1}{2(x+5)}}$$

Exercise

Find the Derivatives of $y = (3x+7)\ln(2x-1)$

Solution

$$f = 3x+7 \qquad f' = 3$$

$$g = \ln(2x-1) \qquad g' = \frac{2}{2x-1}$$

$$\boxed{y' = 3\ln(2x-1) + \frac{2(3x+7)}{2x-1}}$$

Exercise

Find the Derivatives of $f(x) = \ln\sqrt[3]{x+1}$

Solution

$$f(x) = \ln(x+1)^{1/3}$$

$$= \frac{1}{3}\ln(x+1)$$

$$u = x+1 \Rightarrow \frac{du}{dx} = 1$$

$$f'(x) = \frac{1}{3}\frac{1}{x+1}$$

$$\boxed{= \frac{1}{3(x+1)}}$$

Exercise

Find the Derivatives of $f(x) = \ln\left[x^2\sqrt{x^2+1}\right]$

Solution

$$f(x) = \ln(x^2) + \ln\sqrt{x^2+1} \qquad \textit{Product Property}$$

$$f(x) = \ln(x^2) + \ln(x^2+1)^{1/2}$$

$$f(x) = 2\ln x + \tfrac{1}{2}\ln\left(x^2+1\right) \qquad \textit{Power Property}$$

$$f'(x) = 2\frac{1}{x} + \frac{1}{2}\frac{2x}{x^2+1} \qquad \textit{Differentiate}$$

$$= \frac{2}{x} + \frac{x}{x^2+1}$$

Exercise

Find the Derivatives of $y = \ln\dfrac{x^2}{x^2+1}$

Solution

$$y = \ln x^2 - \ln x^2 + 1$$

$$y' = \frac{2x}{x^2} - \frac{2x}{x^2+1}$$

$$= \frac{2}{x} - \frac{2x}{x^2+1}$$

Exercise

Find the Derivatives of $y = \ln\left[\dfrac{x^2(x+1)^3}{(x+3)^{1/2}}\right]$

Solution

$$y = \ln\left[x^2(x+1)^3\right] - \ln(x+3)^{1/2} \qquad \textit{Quotient Rule}$$

$$= \ln x^2 + \ln(x+1)^3 - \ln(x+3)^{1/2} \qquad \textit{Product Rule}$$

$$= 2\ln x + 3\ln(x+1) - \tfrac{1}{2}\ln(x+3) \qquad \textit{Power Rule}$$

$$y' = \frac{2}{x} + \frac{3}{x+1} - \frac{1}{2(x+3)}$$

Exercise

Find the Derivatives of $y = \ln(x^2 + 1)$

Solution

$$y' = \frac{2x}{x^2+1} \qquad\qquad (\ln U)' = \frac{U'}{U}$$

Exercise

Find the Derivatives of $f(x) = \ln(x^2 - 4)$

Solution

Let u = $x^2 - 4 \Rightarrow \frac{du}{dx} = 2x$

$$f'(x) = \frac{1}{x^2-4}(2x)$$

$$= \frac{2x}{x^2-4}$$

Exercise

Find the derivative $\quad f(x) = 2\ln\left(x^2 - 3x + 4\right)$

Solution

$$f'(x) = 2\frac{2x-3}{x^2-3x+4}$$

$$= \frac{4x-6}{x^2-3x+4}$$

Exercise

Find the derivative $\quad f(x) = 3\ln\left(1 + x^2\right)$

Solution

$$f'(x) = 3\frac{2x}{1+x^2}$$

$$\boxed{= \frac{6x}{1+x^2}}$$

Exercise

Find the derivative $\quad f(x) = (1+\ln x)^3$

Solution

$$f'(x) = 3(1+\ln x)^2 (1+\ln x)'$$

$$= 3(1+\ln x)^2 \left(\frac{1}{x}\right)$$

$$\boxed{= \frac{3}{x}(1+\ln x)^2}$$

Exercise

Find the derivative $\quad f(x) = (x - 2\ln x)^4$

Solution

$$f'(x) = 4(x-2\ln x)^3 (x-2\ln x)'$$

$$= 4(x-2\ln x)^3 \left(1 - \frac{2}{x}\right)$$

$$= 4(x-2\ln x)^3 \left(\frac{x-2}{x}\right)$$

$$\boxed{= \frac{4x-8}{x}(x-2\ln x)^3}$$

Exercise

Find the Derivatives of $f(x) = x^2 \ln x$

Solution

$$f' = x^2\left(\frac{1}{x}\right) + 2x\ln x \qquad\qquad (fg)' = f'g + fg'$$

$$= x + 2x\ln x$$

$$\boxed{= x(1+2\ln x)}$$

Exercise

Find the Derivatives of $f(x) = -\dfrac{\ln x}{x^2}$

Solution

$$f' = -\frac{x^2 \frac{d}{dx}[\ln x] - \ln x \frac{d}{dx}\left[x^2\right]}{\left(x^2\right)^2}$$

$$= -\frac{x^2 \frac{1}{x} - 2x \ln x}{x^4}$$

$$= -\frac{x - 2x \ln x}{x^4}$$

$$= -\frac{x(1 - 2\ln x)}{x^4}$$

$$= -\frac{1 - 2\ln x}{x^3}$$

Exercise

Find the derivative of $y = \ln\left(t^2\right)$

Solution

$$y' = \frac{\left(t^2\right)'}{t^2}$$

$$= \frac{2t}{t^2}$$

$$= \frac{2}{t}$$

Exercise

Find the derivative of $y = \ln(2\theta + 2)$

Solution

$$y' = \frac{2}{2\theta + 2}$$

$$= \frac{1}{\theta + 1}$$

Exercise

Find the derivative of $y=(\ln x)^3$

Solution

$$y'=3(\ln x)^2(\ln x)'=3(\ln x)^2\frac{1}{x}$$

$$\boxed{=\frac{3(\ln x)^2}{x}}$$

Exercise

Find the derivative of $y=x(\ln x)^2$

Solution

$$y'=(\ln x)^2+x\left(2(\ln x)\frac{1}{x}\right)$$

$$\boxed{=(\ln x)^2+2\ln x}$$

Exercise

Find the derivative of $y=\frac{x^4}{4}\ln x-\frac{x^4}{16}$

Solution

$$y'=\frac{4x^3}{4}\ln x+\frac{x^4}{4}\frac{1}{x}-\frac{4x^3}{16}$$

$$=x^3\ln x+\frac{1}{4}x^3-\frac{1}{4}x^3$$

$$\boxed{=x^3\ln x}$$

Exercise

Find the derivative of $y=\frac{1+\ln t}{t}$

Solution

$$y'=\frac{\frac{1}{t}t-(1+\ln t)}{t^2}$$

$$=\frac{1-1-\ln t}{t^2}$$

$$\boxed{=-\frac{\ln t}{t^2}}$$

Exercise

Find the derivative $f(x)=\dfrac{\ln x}{1+x}$

Solution

$$f'(x)=\frac{\left(\frac{1}{x}\right)(1+x)-\ln x}{(1+x)^2}$$

$$u=\ln x \quad v=1+x$$

$$u'=\frac{1}{x} \qquad v'=1$$

$$=\frac{1}{x}\frac{1+x-x\ln x}{(1+x)^2}$$

$$=\boxed{\frac{1+x-x\ln x}{x(1+x)^2}}$$

Exercise

Find the derivative $f(x)=\dfrac{2x}{1+\ln x}$

Solution

$$f'(x)=\frac{2(1+\ln x)-(2x)\frac{1}{x}}{(1+\ln x)^2}$$

$$u=2x \quad v=1+\ln x$$

$$u'=2 \qquad v'=\frac{1}{x}$$

$$=\frac{2+2\ln x-2}{(1+\ln x)^2}$$

$$=\boxed{\frac{2\ln x}{(1+\ln x)^2}}$$

Exercise

Find the derivative $f(x)=x^3\ln x$

Solution

$$u=x^3 \qquad v=\ln x$$

$$u'=3x^2 \qquad v'=\frac{1}{x}$$

$$f'(x)=3x^2\ln x+x^3\frac{1}{x}$$

$$=3x^2\ln x+x^2$$

$$=\boxed{(3\ln x+1)x^2}$$

Exercise

Find the derivative $f(x) = 6x^4 \ln x$

Solution

$u = 6x^4 \quad v = \ln x$

$u' = 24x^3 \quad v' = \frac{1}{x}$

$$f'(x) = 24x^3 \ln x + 6x^4 \frac{1}{x}$$

$$= 24x^3 \ln x + 6x^3$$

$$\boxed{= 6x^3(4\ln x + 1)}$$

Exercise

Find the derivative $f(x) = \ln x^8$

Solution

$$f(x) = \ln x^8 = 8\ln x$$

Power Rule

$$\boxed{f'(x) = \frac{8}{x}}$$

$(\ln x)' = \frac{1}{x}$

Exercise

Find the derivative $f(x) = 5x - \ln x^5$

Solution

$$f(x) = 5x - \ln x^5$$

$$= 5x - 5\ln x$$

Power Rule

$$\boxed{f'(x) = 5 - \frac{5}{x}}$$

$(\ln x)' = \frac{1}{x}$

Exercise

Find the derivative $f(x) = \ln x^{10} + 2\ln x$

Solution

$$f(x) = 10\ln x + 2\ln x$$

Power Rule

$$= 12\ln x$$

$$\boxed{f'(x) = \frac{12}{x}}$$

$(\ln x)' = \frac{1}{x}$

Exercise

Find the derivative $f(x) = \dfrac{\ln x}{2x+5}$

Solution

$$u = \ln x \quad v = 2x+5$$

$$u' = \frac{1}{x} \qquad v' = 2$$

$$f'(x) = \frac{\frac{1}{x}(2x+5) - (2)\ln x}{(2x+5)^2} \cdot \frac{x}{x}$$

$$= \frac{2x+5-2x\ln x}{x(2x+5)^2}$$

Exercise

Find the derivative $f(x) = -2\ln x + x^2 - 4$

Solution

$$f'(x) = -\frac{2}{x} + 2x$$

Exercise

Find the derivative of $y = \ln\left(\dfrac{1}{x\sqrt{x+1}}\right)$

Solution

$$y = \ln(1) - \ln\left(x\sqrt{x+1}\right)$$

$$= -\ln x - \ln(x+1)^{1/2}$$

$$= -\ln x - \frac{1}{2}\ln(x+1)$$

$$y' = -\frac{1}{x} - \frac{1}{2}\frac{1}{x+1}$$

$$= -\frac{2(x+1)+x}{2x(x+1)}$$

$$= -\frac{3x+2}{2x(x+1)}$$

Exercise

Find the derivative of $y = \ln\left(\ln\left(\ln x\right)\right)$

Solution

$$y' = \frac{1}{\ln(\ln x)} \cdot \left(\ln(\ln x)\right)'$$

$$= \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot (\ln x)'$$

$$= \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$$

$$= \frac{1}{x(\ln x)(\ln(\ln x))}$$

Exercise

Find the derivative of $y = \ln\left(\sec\left(\ln x\right)\right)$

Solution

$$y' = \frac{1}{\sec(\ln x)} \cdot \left(\sec(\ln x)\right)'$$

$$= \frac{1}{\sec(\ln x)} \cdot \left(\sec(\ln x)\tan(\ln x)\right) \cdot (\ln x)'$$

$$= \frac{\sec(\ln x)}{\sec(\ln x)}\tan(\ln x) \cdot \frac{1}{x}$$

$$= \frac{\tan(\ln x)}{x}$$

Exercise

Find the derivative of $y = \ln\left(\frac{\left(x^2+1\right)^5}{\sqrt{1-x}}\right)$

Solution

$$y = \ln\left(x^2+1\right)^5 - \ln(1-x)^{1/2}$$

$$= 5\ln\left(x^2+1\right) - \frac{1}{2}\ln(1-x)$$

$$y' = 5\frac{2x}{x^2+1} - \frac{1}{2}\frac{-1}{1-x}$$

$$= \frac{10x}{x^2+1} + \frac{1}{2(1-x)}$$

Exercise

Find the derivative of $y=\ln\sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$

Solution

$$\begin{aligned} y &= \frac{1}{2}\ln\left(\frac{(x+1)^5}{(x+2)^{20}}\right) \\ &= \frac{1}{2}\left[\ln(x+1)^5-\ln(x+2)^{20}\right] \\ &= \frac{1}{2}\left[5\ln(x+1)-20\ln(x+2)\right] \end{aligned}$$

$$\begin{aligned} y' &= \frac{1}{2}\left[5\frac{1}{x+1}-20\frac{1}{x+2}\right] \\ &= \frac{5}{2}\left[\frac{1}{x+1}-\frac{4}{x+2}\right] \\ &= \frac{5}{2}\left(\frac{x+2-4x-4}{(x+1)(x+2)}\right) \\ &= \frac{5}{2}\left[\frac{-3x-2}{(x+1)(x+2)}\right] \\ &= -\frac{5}{2}\frac{3x+2}{(x+1)(x+2)} \end{aligned}$$

Exercise

Find the derivative of $f(x)=e^{3x}$

Solution

$$f'(x)=3e^{3x}$$

Exercise

Find the derivative of $f(x)=e^{-2x^3}$

Solution

$$\begin{aligned} f'(x) &= e^{-2x^3}\left(-6x^2\right) \\ &= -\frac{6x^2}{e^{2x^3}} \end{aligned}$$

Exercise

Find the derivative of $f(x) = 4e^{x^2}$

Solution

$$f'(x) = 4e^{x^2}(2x)$$

$$\boxed{= 8xe^{x^2}}$$

Exercise

Find the derivative of $f(x) = x^2e^x$

Solution

$$f'(x) = e^x \frac{d}{dx}[x^2] + x^2 \frac{d}{dx}[e^x]$$

$$= e^x(2x) + x^2e^x$$

$$\boxed{= xe^x(2+x)}$$

Exercise

Find the derivative $f(x) = 2x^3e^x$

Solution

$$f'(x) = 6x^2e^x + 2x^3e^x$$

$$\boxed{= 2x^2e^x(3+x)}$$

$u = 2x^3 \quad v = e^x$

$u' = 6x^2 \quad v' = e^x$

Exercise

Find the derivative $f(x) = \dfrac{3e^x}{1+e^x}$

Solution

$$f'(x) = \frac{3e^x(1+e^x) - 3e^xe^x}{(1+e^x)^2}$$

$$= \frac{3e^x + 3e^{2x} - 3e^{2x}}{(1+e^x)^2}$$

$$\boxed{= \frac{3e^x}{(1+e^x)^2}}$$

$u = 3e^x \quad v = 1+e^x$

$u' = 3e^x \quad v' = e^x$

Exercise

Find the derivative $f(x) = 5e^{x} + 3x + 1$

Solution

$$\boxed{f'(x) = 5e^{x} + 3}$$

Exercise

Find the derivative of $f(x) = \frac{e^{x}+e^{-x}}{2}$

Solution

$$f(x) = \frac{1}{2}(e^{x} + e^{-x})$$

$$f'(x) = \frac{1}{2}\left(\frac{d}{dx}\left[e^{x}\right] + \frac{d}{dx}\left[e^{-x}\right]\right)$$

$$\boxed{= \frac{1}{2}\left(e^{x} - e^{-x}\right)}$$

Exercise

Find the derivative of $f(x) = \frac{e^{x}}{x^{2}}$

Solution

$$f'(x) = \frac{x^{2}e^{x} - e^{x}(2x)}{x^{4}}$$

$$= \frac{x^{2}e^{x} - 2xe^{x}}{x^{4}}$$

$$= \frac{xe^{x}(x-2)}{x^{4}}$$

$$\boxed{= \frac{e^{x}(x-2)}{x^{3}}}$$

Exercise

Find the derivative of $f(x) = x^{2}e^{x} - e^{x}$

Solution

$$f'(x) = e^{x}\frac{d}{dx}[x^{2}] + x^{2}\frac{d}{dx}[e^{x}] - \frac{d}{dx}[e^{x}]$$

$= e^x(2x) + x^2e^x - e^x$

$= e^x\left(x^2 + 2x - 1\right)$

Exercise

Find the derivative of $f(x) = (1 + 2x)e^{4x}$

Solution

$$f'(x) = (2)e^{4x} + (1 + 2x)(4e^{4x})$$

$$= 2e^{4x} + (1 + 2x)(4e^{4x})$$

$$= 2e^{4x}(1 + 2(1 + 2x))$$

$$= 2e^{4x}(1 + 2 + 4x)$$

$$= 2e^{4x}(3 + 4x)$$

Exercise

Find the derivative of $y = x^2e^{5x}$

Solution

$$y' = x^2\left(5e^{5x}\right) + 2x\left(e^{5x}\right)$$

$$= xe^{5x}(5x + 2)$$

Exercise

Find the derivative of $y = x^2e^{-2x}$

Solution

$$y' = 2xe^{-2x} - 2x^2e^{-2x}$$

$$= 2xe^{-2x}(1 - x)$$

Exercise

Find the derivative $f(x)=\dfrac{e^x}{x^2+1}$

Solution

$$f'(x)=\frac{e^x\left(x^2+1\right)-2xe^x}{\left(x^2+1\right)^2}$$

$u=e^x \quad v=x^2+1$

$u'=e^x \quad v'=2x$

$$=\boxed{\frac{\left(x^2+1-2x\right)e^x}{\left(x^2+1\right)^2}}$$

Exercise

Find the derivative $f(x)=\dfrac{1-e^x}{1+e^x}$

Solution

$$f'(x)=\frac{-e^x\left(1+e^x\right)-e^x\left(1-e^x\right)}{\left(1+e^x\right)^2}$$

$u=1-e^x \quad v=1+e^x$

$u'=-e^x \quad v'=e^x$

$$=\frac{-e^x-e^{2x}-e^x+e^{2x}}{\left(1+e^x\right)^2}$$

$$=\boxed{-\frac{2e^x}{\left(1+e^x\right)^2}}$$

Exercise

Find the derivative of $y=\dfrac{e^x+e^{-x}}{x}$

Solution

$$y=\frac{\left(e^x-e^{-x}\right)x-\left(e^x+e^{-x}\right)}{x^2}$$

$f=e^x+e^{-x} \quad g=x$

$f'=e^x-e^{-x} \quad g'=1$

$$=\frac{xe^x-xe^{-x}-e^x-e^{-x}}{x^2}$$

$$=\boxed{\frac{(x-1)e^x-(x+1)e^{-x}}{x^2}}$$

Exercise

Find the derivative of $y=\sqrt{e^{2x^2}+e^{-2x^2}}$

Solution

$$y=\sqrt{e^{2x^2}+e^{-2x^2}}=\left(e^{2x^2}+e^{-2x^2}\right)^{1/2}=U^{1/2}$$

$$U=e^{2x^2}+e^{-2x^2} \qquad \left(e^{2x^2}\right)'=\left(2x^2\right)'e^{2x^2}=4xe^{2x^2}$$

$$U'=4xe^{2x^2}-4xe^{-2x^2}$$

$$y'=\frac{1}{2}\left(4xe^{2x^2}-4xe^{-2x^2}\right)\left(e^{2x^2}+e^{-2x^2}\right)^{-1/2}$$

$$=\frac{1}{2}\frac{4x\left(e^{2x^2}-e^{-2x^2}\right)}{\left(e^{2x^2}+e^{-2x^2}\right)^{1/2}}$$

$$=\frac{2x\left(e^{2x^2}-e^{-2x^2}\right)}{\sqrt{e^{2x^2}+e^{-2x^2}}}$$

Exercise

Find the derivative of $y=\frac{x}{e^{2x}}$

Solution

$$y'=\frac{1\left(e^{2x}\right)-x\left(2e^{2x}\right)}{\left(e^{2x}\right)^2} \qquad f=x \quad g=e^{2x}$$

$$f'=1 \quad g'=2e^{2x}$$

$$=\frac{e^{2x}(1-2x)}{\left(e^{2x}\right)^2}$$

$$=\frac{1-2x}{e^{2x}}$$

Exercise

Find the derivative of $y = 3e^{5x^3+1}$

Solution

$$y' = 3(15x^2)e^{5x^3+1}$$

$$y' = 45x^2 e^{5x^3+1}$$

$$f = x^2 \qquad g = e^{5x^3+1}$$

$$f' = 2x \quad g' = 15x^2 e^{5x^3+1}$$

$$y'' = 45\left(2xe^{5x^3+1} + \left(x^2\right)15x^2 e^{5x^3+1}\right)$$

$$= 45e^{5x^3+1}\left(2x + 15x^4\right)$$

$$\underline{= 45xe^{5x^3+1}\left(2 + 15x^3\right)}$$

Exercise

Find the derivative of $\left(x^2 - 2x + 2\right)e^x$

Solution

$$y = \left(x^2 - 2x + 2\right)e^x$$

$$y' = (2x - 2)e^x + \left(x^2 - 2x + 2\right)e^x$$

$$= \left(2x - 2 + x^2 - 2x + 2\right)e^x$$

$$\underline{= x^2 e^x}$$

Exercise

Find the derivative of $e^\theta(\sin\theta + \cos\theta)$

Solution

$$\frac{d}{d\theta}e^\theta(\sin\theta + \cos\theta) = e^\theta(\sin\theta + \cos\theta) + e^\theta(\cos\theta - \sin\theta)$$

$$= e^\theta(\sin\theta + \cos\theta + \cos\theta - \sin\theta)$$

$$\underline{= 2e^\theta \cos\theta}$$

Exercise

Find the derivative of $\ln\left(3\theta e^{-\theta}\right)$

Solution

$\frac{d}{d\theta}\ln\left(3\theta e^{-\theta}\right)=\frac{\left(3\theta e^{-\theta}\right)'}{3\theta e^{-\theta}}$ $=3\frac{e^{-\theta}-\theta e^{-\theta}}{3\theta e^{-\theta}}$ $=\frac{e^{-\theta}(1-\theta)}{\theta e^{-\theta}}$ $=\frac{1-\theta}{\theta}$	$\ln\left(3\theta e^{-\theta}\right)=\ln(3)+\ln(\theta)+\ln\left(e^{-\theta}\right)$ $=\ln 3+\ln\theta-\theta$ $\frac{d}{d\theta}\ln\left(3\theta e^{-\theta}\right)=\frac{1}{\theta}-1$

Exercise

Find the derivative of $\theta^3 e^{-2\theta}\cos 5\theta$

Solution

$$\frac{dy}{d\theta}=\left(\theta^3\right)' e^{-2\theta}\cos 5\theta+\theta^3\left(e^{-2\theta}\right)'\cos 5\theta+\theta^3 e^{-2\theta}(\cos 5\theta)'$$

$$=3\theta^2 e^{-2\theta}\cos 5\theta-2\theta^3 e^{-2\theta}\cos 5\theta-5\theta^3 e^{-2\theta}\sin 5\theta$$

$$=\theta^3 e^{-2\theta}\left(3\cos 5\theta-2\theta\cos 5\theta-5\theta\sin\theta\right)$$

Exercise

Find the derivative of $\ln\left(\frac{\sqrt{\theta}}{1+\sqrt{\theta}}\right)$

Solution

$$\frac{d}{d\theta}\ln\left(\frac{\sqrt{\theta}}{1+\sqrt{\theta}}\right)=\frac{d}{d\theta}\left[\ln\theta^{1/2}-\ln\left(1+\sqrt{\theta}\right)\right]$$

$$=\frac{d}{d\theta}\left[\frac{1}{2}\ln\theta-\ln\left(1+\sqrt{\theta}\right)\right]$$

$$=\frac{1}{2}\frac{1}{\theta}-\frac{\frac{1}{2}\theta^{-1/2}}{1+\sqrt{\theta}}$$

$$=\frac{1}{2\theta}-\frac{1}{2}\frac{1}{\sqrt{\theta}\left(1+\sqrt{\theta}\right)}$$

$$= \frac{1}{2}\left(\frac{1}{\theta} - \frac{1}{\sqrt{\theta}\left(1+\sqrt{\theta}\right)}\right)$$

$$= \frac{1}{2}\frac{\sqrt{\theta}\left(1+\sqrt{\theta}\right) - \theta}{\theta\sqrt{\theta}\left(1+\sqrt{\theta}\right)}$$

$$= \frac{1}{2}\frac{\sqrt{\theta} + \theta - \theta}{\theta\sqrt{\theta}\left(1+\sqrt{\theta}\right)}$$

$$= \frac{1}{2}\frac{\sqrt{\theta}}{\theta\sqrt{\theta}\left(1+\sqrt{\theta}\right)}$$

$$= \frac{1}{2\theta\left(1+\sqrt{\theta}\right)}$$

Exercise

Find the derivative of $e^{(\cos t + \ln t)}$

Solution

$$e^{(\cos t+\ln t)} = e^{\cos t}e^{\ln t}$$
$$= t\, e^{\cos t}$$

$$\frac{d}{dt}e^{(\cos t+\ln t)} = \frac{d}{dt}\left(t\, e^{\cos t}\right)$$
$$= e^{\cos t} + te^{\cos t}(-\sin t)$$
$$= (1 - t\sin t)e^{\cos t}$$

Exercise

Find the derivative of $e^{\sin t}\left(\ln t^2 + 1\right)$

Solution

$$\frac{d}{dt}\overset{u}{e^{\sin t}}\overset{v}{\left(\ln t^2+1\right)} = \overset{u'}{e^{\sin t}\cos t}\overset{v}{\left(\ln t^2+1\right)} + \overset{v'}{\frac{2}{t}}\overset{u}{e^{\sin t}}$$
$$= e^{\sin t}\left[\left(\ln t^2+1\right)\cos t + \frac{2}{t}\right]$$

Exercise

Find the Derivatives of $y = e^{x^2} \ln x$

Solution

$$\boxed{y' = 2xe^{x^2} \ln x + \frac{e^{x^2}}{x}}$$

$f = e^{x^2} \qquad g = \ln x$

$f' = 2xe^{x^2} \qquad g' = \frac{1}{x}$

Exercise

Find the derivative $\quad f(x) = e^x + x - \ln x$

Solution

$$\boxed{f'(x) = e^x + 1 - \frac{1}{x}}$$

Exercise

Find the derivative $\quad f(x) = \ln x + 2e^x - 3x^2$

Solution

$$\boxed{f'(x) = \frac{1}{x} + 2e^x - 6x}$$

Exercise

Find the derivative $\quad f(x) = \ln x^2 + 4e^x$

Solution

$$f(x) = 2\ln x + 4e^x$$

$$\boxed{f'(x) = \frac{2}{x} + 4e^x}$$

Power Rule

$(\ln x)' = \frac{1}{x}$

Exercise

Find the Derivatives of $y = \ln \frac{1+e^x}{1-e^x}$

Solution

$$y = \ln\left(1+e^x\right) - \ln\left(1-e^x\right)$$

$$y' = \frac{e^x}{1+e^x} - \frac{-e^x}{1-e^x}$$

$$= \frac{e^x}{1+e^x} + \frac{e^x}{1-e^x}$$

$$= \frac{e^x - e^{2x} + e^x + e^{2x}}{\left(1+e^x\right)\left(1-e^x\right)}$$

$$= \frac{2e^x}{\left(1+e^x\right)\left(1-e^x\right)}$$

Exercise

Find the Derivatives of $y = \dfrac{\ln x}{e^{2x}}$

Solution

$$y' = \frac{e^{2x}(1/x) - \ln x(2e^{2x})}{e^{4x}}$$

$$= \frac{e^{2x} - 2x\ln x(e^{2x})}{e^{4x}}$$

$$= \frac{e^{2x}(1-2x\ln x)}{e^{4x}}$$

Exercise

Find the Derivatives of $f(x) = e^{2x}\ln\left(xe^x + 1\right)$

Solution

$$f = e^{2x} \qquad U = 2x \to U' = 2 \qquad f' = 2e^{2x}$$

$$g = \ln\left(xe^x + 1\right) \quad U = xe^x + 1 \to U' = e^x + xe^x \quad g' = \frac{e^x + xe^x}{xe^x + 1}$$

$$f'(x) = 2e^{2x}\ln\left(xe^x + 1\right) + e^{2x}\frac{e^x + xe^x}{xe^x + 1}$$

$$= e^{2x}\left[2\ln\left(xe^x + 1\right) + \frac{e^x(1+x)}{xe^x + 1}\right]$$

Exercise

Find the Derivatives of $f(x)=\dfrac{xe^x}{\ln\left(x^2+1\right)}$

Solution

$$f'(x)=\frac{e^x(1+x)\ln\left(x^2+1\right)-\frac{2x}{x^2+1}xe^x}{\left[\ln\left(x^2+1\right)\right]^2}$$

$u=xe^x \qquad u'=e^x+xe^x$

$v=\ln\left(x^2+1\right) \qquad v'=\frac{2x}{x^2+1}$

$$=\frac{e^x\left[(1+x)\ln\left(x^2+1\right)-\frac{2x^2}{x^2+1}\right]}{\left[\ln\left(x^2+1\right)\right]^2}$$

$$=\frac{e^x\left[\frac{\left(x^2+1\right)(1+x)\ln\left(x^2+1\right)-2x^2}{x^2+1}\right]}{\left[\ln\left(x^2+1\right)\right]^2}$$

$$=\frac{e^x\left[\left(x^2+1\right)(1+x)\ln\left(x^2+1\right)-2x^2\right]}{\left(x^2+1\right)\left[\ln\left(x^2+1\right)\right]^2}$$

Exercise

Find the Derivatives of $f(x)=xe^{-10x}$

Solution

$$f'(x)=e^{-10x}-10xe^{-10x}$$

Exercise

Find the Derivatives of $f(x)=x\ln^2 x$

Solution

$$f'(x)=\ln^2 x+x\left(2\frac{1}{x}\ln x\right)$$

$$=\ln^2 x+2\ln x$$

Exercise

Find the Derivatives of $f(x) = e^{-x} \ln x$

Solution

$$\boxed{f'(x) = -e^{-x} \ln x + \frac{e^{-x}}{x}}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt{x(x+1)}$

Solution

$$\ln y = \ln\left(x(x+1)\right)^{1/2} \quad = -\ln x - \tfrac{1}{2}\ln(x+1)$$

$$\frac{y'}{y} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x+1}\right)$$

$$\frac{y'}{y} = \frac{1}{2}\left(\frac{2x+1}{x(x+1)}\right)$$

$$y' = \frac{1}{2}\left(\frac{2x+1}{x(x+1)}\right) \bullet y$$

$$\boxed{= \frac{1}{2}\left(\frac{2x+1}{x(x+1)}\right)\sqrt{x(x+1)}}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt{\left(x^2+1\right)(x-1)^2}$

Solution

$$\ln y = \ln\left(\left(x^2+1\right)(x-1)^2\right)^{1/2}$$

$$= \frac{1}{2}\ln\left(\left(x^2+1\right)(x-1)^2\right)$$

$$= \frac{1}{2}\left[\ln\left(x^2+1\right) + \ln(x-1)^2\right]$$

$$= \frac{1}{2}\left[\ln\left(x^2+1\right) + 2\ln(x-1)\right]$$

$$= \frac{1}{2}\ln\left(x^2+1\right) + \ln(x-1)$$

$$\frac{y'}{y} = \frac{1}{2}\frac{2x}{x^2+1} + \frac{1}{x-1}$$

$$= \frac{x}{x^2+1} + \frac{1}{x-1}$$

$$= \frac{x(x-1) + \left(x^2+1\right)}{\left(x^2+1\right)(x-1)}$$

$$= \frac{x^2 - x + x^2 + 1}{\left(x^2+1\right)(x-1)}$$

$$= \frac{2x^2 - x + 1}{\left(x^2+1\right)(x-1)}$$

$$y' = \frac{2x^2 - x + 1}{\left(x^2+1\right)(x-1)} \cdot y$$

$$= \frac{2x^2 - x + 1}{\left(x^2+1\right)(x-1)} \sqrt{\left(x^2+1\right)(x-1)^2}$$

$$= \frac{2x^2 - x + 1}{\left(x^2+1\right)(x-1)} |x-1| \sqrt{\left(x^2+1\right)}$$

$$= \frac{\left(2x^2 - x + 1\right)|x-1|}{\left(x^2+1\right)(x-1)} \left(x^2+1\right)^{1/2}$$

$$\boxed{= \frac{\left(2x^2 - x + 1\right)|x-1|}{\sqrt{x^2+1}\,(x-1)}}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt{\frac{1}{t(t+1)}}$

Solution

$$y = \left(\frac{1}{t(t+1)}\right)^{1/2}$$

$$\ln y = \ln\left(\frac{1}{t(t+1)}\right)^{1/2}$$

$$\ln y = \frac{1}{2}\ln\left(\frac{1}{t(t+1)}\right)$$

$$= -\frac{1}{2}\ln\left(t(t+1)\right)$$

$$= -\frac{1}{2}\left[\ln t + \ln(t+1)\right]$$

$$\frac{y'}{y} = -\frac{1}{2}\left(\frac{1}{t} + \frac{1}{t+1}\right)$$

$$y' = -\frac{1}{2}\left(\frac{1}{t} + \frac{1}{t+1}\right)y$$

$$= -\frac{1}{2}\left(\frac{t+1+t}{t(t+1)}\right)\frac{1}{\left(t(t+1)\right)^{1/2}}$$

$$= -\frac{1}{2}\frac{2t+1}{\left(t(t+1)\right)^{3/2}}$$

$$\boxed{= -\frac{2t+1}{2\left(t^2+t\right)^{3/2}}}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \dfrac{\theta+5}{\theta\cos\theta}$

Solution

$$\ln y = \ln\left(\frac{\theta+5}{\theta\cos\theta}\right)$$

$$\ln y = \ln(\theta+5) - \ln(\theta\cos\theta)$$

$$\ln y = \ln(\theta+5) - \ln\theta - \ln(\cos\theta)$$

$$\frac{y'}{y} = \frac{1}{\theta+5} - \frac{1}{\theta} + \frac{\sin\theta}{\cos\theta}$$

$$y' = \left(\frac{1}{\theta+5} - \frac{1}{\theta} + \frac{\sin\theta}{\cos\theta}\right)y$$

$$\boxed{y' = \left(\frac{\theta+5}{\theta\cos\theta}\right)\left(\frac{1}{\theta+5} - \frac{1}{\theta} + \tan\theta\right)}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt[3]{\dfrac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}}$

Solution

$$\ln y = \ln\left(\frac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}\right)^{1/3}$$

$$= \frac{1}{3}\left[\ln x + \ln(x+1) + \ln(x-2) - \ln\left(x^2+1\right) - \ln(2x+3)\right]$$

$$\frac{y'}{y} = \frac{1}{3}\left[\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3}\right]$$

$$y' = \frac{1}{3} \cdot \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}} \cdot \left(\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3}\right)$$

Exercise

Find the derivative of $y = t^{1-e}$

Solution

$$y' = (1-e)t^{1-e-1}$$

$$= (1-e)t^{-e}$$

Exercise

Find the derivative of $y = 2^{\sin 3t}$

Solution

$$y = a^u \quad \Rightarrow y' = a^u \ln a \cdot (u')$$

$$y' = \left(2^{\sin 3t} \ln 2\right)(\cos 3t)(3)$$

$$= 3(\ln 2)\cos 3t\left(2^{\sin 3t}\right)$$

Exercise

Find the derivative of $y = \log_3 (1 + \theta \ln 3)$

Solution

$$y = \frac{\ln(1+\theta \ln 3)}{\ln 3}$$

$$y' = \frac{1}{\ln 3} \cdot \frac{\ln 3}{1+\theta \ln 3} \qquad\qquad y = \ln u \quad \Rightarrow y' = \frac{u'}{u}$$

$$= \frac{1}{1+\theta \ln 3}$$

Exercise

Find the derivative of $y = \log_{25} e^x - \log_5 \sqrt{x}$

Solution

$$y=\frac{\ln e^{x}}{\ln 25}-\frac{\ln x^{1/2}}{\ln 5}$$

$$=\frac{x}{2\ln 5}-\frac{1}{2}\frac{\ln x}{\ln 5}$$

$$=\frac{1}{2\ln 5}(x-\ln x)$$

$$y'=\frac{1}{2\ln 5}\left(1-\frac{1}{x}\right)$$

$$\boxed{=\frac{x-1}{2x\ln 5}}$$

Exercise

Find the derivative of $y=\log_3 r\cdot\log_9 r$

Solution

$$y=\frac{\ln r}{\ln 3}\cdot\frac{\ln r}{\ln 9}$$

$$=\frac{1}{\ln 3\cdot\ln 9}\cdot\ln^2 r$$

$$y'=\frac{1}{\ln 3\cdot\ln 9}\cdot(2\ln r)\left(\frac{1}{r}\right)$$

$$\boxed{=\frac{2\ln r}{r\cdot\ln 3\cdot\ln 9}}$$

Exercise

Find the derivative of $y=\log_7\left(\frac{\sin\theta\cos\theta}{e^{\theta}2^{\theta}}\right)$

Solution

$$y=\frac{\ln(\sin\theta)+\ln(\cos\theta)-\ln\left(e^{\theta}\right)-\ln\left(2^{\theta}\right)}{\ln 7}$$

$$=\frac{1}{\ln 7}\left[\ln(\sin\theta)+\ln(\cos\theta)-\theta-\theta\ln(2)\right]$$

$$y'=\frac{1}{\ln 7}\left[\frac{\cos\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta}-1-\ln(2)\right]$$

$$\boxed{=\frac{1}{\ln 7}(\cot\theta-\tan\theta-1-\ln 2)}$$

Exercise

Find the derivative of $y = 3\log_8\left(\log_2 t\right)$

Solution

$$y = 3\frac{\ln\left(\log_2 t\right)}{\ln 8} = \frac{3}{\ln 8}\ln\left(\frac{\ln t}{\ln 2}\right)$$

$$y' = \frac{3}{\ln 2^3}\left(\frac{1}{\frac{\ln t}{\ln 2}}\right)\left(\frac{1}{\ln 2}\cdot\frac{1}{t}\right)$$

$$= \frac{3}{2\ln 2}\left(\frac{\ln 2}{\ln t}\right)\left(\frac{1}{t\ln 2}\right)$$

$$= \frac{1}{t(\ln t)(\ln 2)}$$

Exercise

Find the derivative of $y = t\ \log_3\left(e^{(\sin t)(\ln 3)}\right)$

Solution

$$y = t\frac{\ln e^{(\sin t)(\ln 3)}}{\ln 3}$$

$$= \frac{1}{\ln 3}t(\sin t)(\ln 3)$$

$$= t\sin t$$

$$y' = \sin t + t\cos t$$

Exercise

Find the derivative of $f(x) = \log_3(x+8)$

Solution

$$f'(x) = \frac{1}{\ln 3}\left(\frac{1}{x+8}\right)$$

$$\frac{d}{dx}\left[\log_a u\right] = \left(\frac{1}{\ln a}\right)\left(\frac{1}{u}\right)\frac{du}{dx}$$

Exercise

Find the derivative of $f(x) = 2^{x^2-x}$

Solution

$$f'(x) = (2x-1)(\ln 2)2^{x^2-x}$$

$$\frac{d}{dx}[a^u] = a^u\ln(a)\frac{du}{dx}$$

Exercise

Use logarithmic differentiation to find the derivative of $y=(x+1)^x$

Solution

$$\ln y=\ln(x+1)^x=x\cdot\ln(x+1)$$

$$\frac{y'}{y}=\ln(x+1)+x\cdot\frac{1}{x+1}$$

$$\boxed{y'=(x+1)^x\left(\ln(x+1)+\frac{x}{x+1}\right)}$$

Exercise

Use logarithmic differentiation to find the derivative of $y=x^2+x^{2x}$

Solution

$$y-x^2=x^{2x}$$

$$\ln\left(y-x^2\right)=\ln x^{2x}=\underline{2x\ln x}$$

$$\frac{1}{y-x^2}(y'-2x)=2\ln x+2x\frac{1}{x}$$

$$y'-2x=\left(y-x^2\right)(2\ln x+2)$$

$$y'-2x=\left(x^2+x^{2x}-x^2\right)(2\ln x+2)$$

$$y'=2x^{2x}(\ln x+1)+2x$$

$$\boxed{=2\left(x^{2x}\ln x+x^{2x}+x\right)}$$

Exercise

Use logarithmic differentiation to find the derivative of $y=(\sin x)^x$

Solution

$$\ln y=\ln(\sin x)^x$$

$$u=x \quad v=\ln(\sin x)$$

$$\ln y=x\ln(\sin x)$$

$$u'=1 \qquad v'=\frac{\cos x}{\sin x}$$

$$\frac{y'}{y}=\ln(\sin x)+x\frac{\cos x}{\sin x}$$

$$y'=y(\ln(\sin x)+x\cot x)$$

$$\boxed{=(\sin x)^x\left[\ln(\sin x)+x\cot x\right]}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = x^{\sin x}$

Solution

$$\ln y = \ln x^{\sin x}$$

$$\ln y = \sin x \ln x$$

$$\frac{y'}{y} = \cos x \ln x + \frac{\sin x}{x}$$

$$\frac{y'}{y} = \frac{x\cos x \ln x + \sin x}{x}$$

$$y' = y\frac{x\cos x \ln x + \sin x}{x}$$

$$= x^{\sin x}\left[\frac{\sin x + x(\ln x)(\cos x)}{x}\right]$$

Exercise

Use logarithmic differentiation to find the derivative of $y = (\ln x)^{\ln x}$

Solution

$$\ln y = \ln(\ln x)^{\ln x}$$

$$\ln y = (\ln x)\ln(\ln x)$$

$$\frac{y'}{y} = \frac{1}{x}\ln(\ln x) + \ln x\frac{\frac{1}{x}}{\ln x}$$

$$y' = \left(\frac{1}{x}\ln(\ln x) + \frac{1}{x}\right)y$$

$$= \left(\frac{\ln(\ln x) + 1}{x}\right)(\ln x)^{\ln x}$$

Exercise

Find the second derivative of $y = 3e^{5x^3+1}$

Solution

$$y' = 45x^2 e^{5x^3+1}$$

$$y'' = \left(90x + 675x^5\right)e^{5x^3+1}$$

Exercise

Find the equations of the tangent lines to $f(x) = e^x$ at the points (0, 1)

Solution

$$f'(x) = e^x$$

$$(0,\ 1) \Rightarrow m = f'(x = 0)$$

$$= e^0$$

$$= 1$$

$$y - 1 = 1(x - 0) + 1 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$\underline{y = x + 1}$$

Exercise

Find the equations of the tangent lines to $f(x) = e^x$ at the points $(1,\ e)$

Solution

$$f'(x) = e^x$$

$$(1,\ e) \Rightarrow m = f'(x = 1) = e^1 = e$$

$$y = e(x - 1) + e \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$\underline{y = ex}$$

Exercise

Find the equations of the tangent lines to $y = 4xe^{-x} + 5$ at $x = 1$

Solution

$$y' = 4e^{-x} - 4xe^{-x} = 4e^{-x}(1 - x)$$

$$= 4e^{-x}(1 - x)$$

$$m = y'(x = 1)$$

$$= 4e^{-1}(1 - 1) = 0$$

$$\Rightarrow x = 1 \rightarrow y = 4e^{-1} + 5 \qquad \left(1,\ 4e^{-1} + 5\right)$$

$$y = 0\left(x - 1\right) + 4e^{-1} + 5 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$\underline{y = 4e^{-1} + 5}$$

Exercise

Find the equation of the tangent lines to $f(x) = 4e^{-8x}$ at the points (0, 4)

Solution

$$f'(x) = -32e^{-8x}$$

$$m = f'(0) = -32e^{-8(0)} = -32$$

$$y = -32(x - 0) + 4 \qquad\qquad y = m(x - x_1) + y_1$$

$$\boxed{y = -32x + 4}$$

Exercise

The following formula accurately models the relationship between the size of a certain type of tumor and the amount of time that it has been growing:

$$V(t) = 450(1 - e - 0.0022t)^3$$

where *t* is in months and $V(t)$ is measured in cubic centimeters. Calculate the rate of change of tumor volume at 80 months.

Solution

$$U = 1 - e - 0.0022t \qquad V = 450U^3$$

$$U' = -.0022 \qquad V' = 450(3)U^2U'$$

$$V'(t) = 450(3)(1 - e - 0.0022t)^2(-.0022)$$

$$\boxed{= 2.97(1 - e - 0.0022t)^2}$$

$$V'(t = 80) = 2.97(1 - e - 0.0022(80))^2$$

$$\boxed{\approx 10.66}$$

Exercise

A yeast culture at room temperature (68° *F*) is placed in a refrigerator set at a constant temperature of 38° *F*. After *t* hours, the temperature *T* of the culture is given approximately by

$$T = 30e^{-0.58t} + 38 \quad t \ge 0$$

What is the rate of change of temperature of the culture at the end of 1 *hour*? At the end of 4 *hours*?

Solution

$$T' = 30(-0.58)e^{-0.58t} \boxed{= -17.4e^{-0.58t}}$$

$$T'(1) = -17.4e^{-0.58(1)} \underline{\approx -9.74^\circ\ F/hr}$$

$$T'(4) = -17.4e^{-0.58(4)} \underline{\approx -1.71^\circ\ F/hr}$$

Exercise

A mathematical model for the average age of a group of people learning to type is given by

$$N(t) = 10 + 6\ln t \quad t \geq 1$$

Where $N(t)$ is the number of words per minute typed after *t hours* of instruction and practice (2 hours per day, 5 days per week). What is the rate of learning after 10 *hours* of instruction and practice? After 100 *hours*?

Solution

$$N'(t) = \frac{6}{t}$$

$$N'(10) = \frac{6}{10} \underline{= 0.6}$$

After 10 *hours* of instruction and practice, the rate of learning is 0.6 words/minute per hour of instruction and practice.

$$N'(100) = \frac{6}{100} \underline{= 0.06}$$

After 100 *hours* of instruction and practice, the rate of learning is 0.06 words/minute per hour of instruction and practice.

Exercise

The population of coyotes in the northwestern portion of Alabama is given by the formula $P(t) = \left(t^2 + 100\right)\ln(t+2)$, where *t* represents the time in years since 2000 (the year 2000 corresponds to $(t = 0)$ Find the rate of change of the coyote population in $2013\ (t = 13)$.

Solution

$$P'(t) = 2t\ln(t+2) + \frac{1}{t+2}\left(t^2 + 100\right) \qquad P' = f'g + g'f \qquad \begin{matrix} f = t^2 + 100 & g = \ln(t+2) \\ f' = 2t & g' = \dfrac{1}{t+2} \end{matrix}$$

$$= 2t\ln(t+2) + \frac{t^2 + 100}{t+2}$$

$$P'(t = 13) = 2(13)\ln(13+2) + \frac{13^2 + 100}{13+2} \qquad 2*13\ln(13+2) + (13\wedge 2 + 100)/(13+2)$$

$$\underline{\approx 88.34}$$

Solution — Section 2.9 – Derivatives of Inverse Trigonometric Functions

Exercise

Find the value of $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

Solution

$$\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) = \sin\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}$$

Exercise

Find the value of $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

Solution

$$\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \cot\left(-\frac{\pi}{3}\right)$$

$$= -\frac{1}{\sqrt{3}}$$

Exercise

Find the limit: $\lim\limits_{x\to -1^+} \cos^{-1} x$

Solution

$$\lim_{x\to -1^+} \cos^{-1} x = \cos^{-1}(-1)$$

$$= \pi$$

Exercise

Find the limit: $\lim\limits_{x\to -\infty} \tan^{-1} x$

Solution

$$\lim_{x\to -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

Exercise

Find the limit: $\lim\limits_{x\to\infty} \csc^{-1} x$

Solution

$$\lim_{x\to\infty} \csc^{-1} x = \lim_{x\to\infty} \sin^{-1}\left(\frac{1}{x}\right)$$

$$= \sin^{-1}\left(\frac{1}{\infty}\right)$$

$$\underline{= 0}$$

Exercise

Find the derivative $y = \cos^{-1}\left(\frac{1}{x}\right)$

Solution

$$y = \cos^{-1}\left(\frac{1}{x}\right)$$

$$= \sec^{-1}(x)$$

$$y' = \frac{1}{|x| \cdot \sqrt{x^2 - 1}}$$

Exercise

Find the derivative $y = \sin^{-1}\sqrt{2}t$

Solution

$$y' = \frac{\sqrt{2}}{\sqrt{1 - \left(\sqrt{2}t\right)^2}}$$

$$= \frac{\sqrt{2}}{\sqrt{1 - 2t^2}}$$

Exercise

Find the derivative $y = \sec^{-1}(5s)$

Solution

$$y' = \frac{5s}{|5s|\sqrt{(5s)^2 - 1}}$$

$$= \boxed{\frac{s}{|s|\sqrt{25s^2 - 1}}}$$

Exercise

Find the derivative $\quad y = \cot^{-1}\sqrt{t-1}$

Solution

$$y' = -\frac{\frac{1}{2}(t-1)^{-1/2}}{1 + \left[(t-1)^{1/2}\right]^2}$$

$$= -\frac{1}{2(t-1)^{1/2}(1+t-1)}$$

$$= \boxed{-\frac{1}{2t\sqrt{t-1}}}$$

Exercise

Find the derivative $\quad y = \ln\left(\tan^{-1} x\right)$

Solution

$$y' = \frac{\frac{1}{1+x^2}}{\tan^{-1} x}$$

$$= \boxed{\frac{1}{\left(1+x^2\right)\tan^{-1} x}}$$

Exercise

Find the derivative $y = \tan^{-1}(\ln x)$

Solution

$$y' = \frac{\frac{1}{x}}{1 + (\ln x)^2} \qquad \left(\tan^{-1} u\right)' = \frac{u'}{1+u^2}$$

$$= \boxed{\frac{1}{x\left[1 + (\ln x)^2\right]}}$$

Exercise

Find the derivative $y = \csc^{-1}\left(e^t\right)$

Solution

$$y' = -\frac{e^t}{\left|e^t\right|\sqrt{\left(e^t\right)^2 - 1}}$$

$$\boxed{= -\frac{1}{\sqrt{e^{2t} - 1}}}$$

Exercise

Find the derivative $y = x\sqrt{1-x^2} + \cos^{-1} x$

Solution

$$y' = \sqrt{1-x^2} + x\left(\frac{1}{2}\right)\left(1-x^2\right)^{-1/2}(-2x) - \frac{1}{\sqrt{1-x^2}}$$

$$= \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{1-x^2-x^2-1}{\sqrt{1-x^2}}$$

$$\boxed{= \frac{-2x^2}{\sqrt{1-x^2}}}$$

Exercise

Find the derivative $y = \ln\left(x^2+4\right) - x\tan^{-1}\left(\frac{x}{2}\right)$

Solution

$$y' = \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - x\frac{\frac{1}{2}}{1+\left(\frac{x}{2}\right)^2}$$

$$= \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{x}{2}\cdot\frac{1}{1+\frac{x^2}{4}}$$

$$= \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{x}{2}\cdot\frac{4}{4+x^2}$$

$$= \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{2x}{4+x^2}$$

$$= -\tan^{-1}\left(\frac{x}{2}\right)$$

Exercise

Find the derivative $\quad f(x) = \sin^{-1}\frac{1}{x}$

Solution

$$f'(x) = -\frac{1}{x^2}\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}$$

$$= \frac{-1}{|x|\sqrt{x^2-1}}$$

Exercise

Find the derivative $\quad \left.\frac{d}{dx}\left(x\sec^{-1}x\right)\right|_{x=\frac{2}{\sqrt{3}}}$

Solution

$$\frac{d}{dx}\left(x\sec^{-1}x\right) = \sec^{-1}x + \left.\frac{x}{x\sqrt{x^2-1}}\right|_{x=\frac{2}{\sqrt{3}}}$$

$$= \sec^{-1}\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{\frac{4}{3}-1}}$$

$$= \frac{\pi}{6} + \sqrt{3}$$

Exercise

Find the derivative $\quad \left.\frac{d}{dx}\left(\tan^{-1}e^{-x}\right)\right|_{x=0}$

Solution

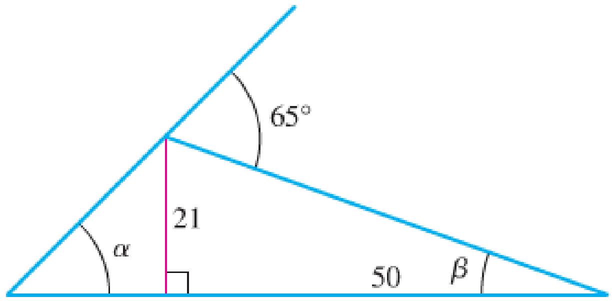
$$\frac{d}{dx}\left(\tan^{-1}e^{-x}\right) = \left.\frac{-e^{-x}}{1+e^{-2x}}\right|_{x=0}$$

$$= -\frac{1}{2}$$

Exercise

Find the angle α

Solution



$65° + (90° - \beta) + (90° - \alpha) = 180°$

$65° + 180° - \beta - \alpha = 180°$

$\beta + \alpha = 65° \quad \Rightarrow \quad \underline{\alpha = 65° - \beta}$

$\tan\beta = \frac{21}{50} \quad \Rightarrow \beta = \tan^{-1}\left(\frac{21}{50}\right) \approx 22.78°$

$\underline{\alpha} \approx 65° - 22.78°$

$\quad \underline{\approx 42.22°}$