Solution   *Section* 4.1 – Antiderivatives

Exercise

Find indefinite integral $\int v^2 dv$

Solution

$$\int v^2 dv = \boxed{\frac{v^3}{3} + C}$$

Exercise

Find indefinite integral $\int x^{1/2} dx$

Solution

$$\int x^{1/2} dx = \boxed{\frac{2}{3}x^{3/2} + C}$$

Exercise

Find indefinite integral $\int 4y^{-3} dy$

Solution

$$\int 4y^{-3} dy = 4\frac{y^{-2}}{-2} + C$$

$$\boxed{= -\frac{2}{y^2} + C}$$

Exercise

Find indefinite integral $\int (x^3 - 4x + 2) dx$

Solution

$$\int (x^3 - 4x + 2) dx = \frac{x^4}{4} - 4\frac{x^2}{2} + 2x + C$$

$$\boxed{= \frac{1}{4}x^4 - 2x^2 + 2x + C}$$

Exercise

Find indefinite integral $\displaystyle\int \left(3z^2 - 4z + 5\right) dz$

Solution

$$\int \left(3z^2 - 4z + 5\right) dz = 3\frac{z^3}{3} - 4\frac{z^2}{2} + 5z + C$$

$$= z^3 - 2z^2 + 5z + C$$

Exercise

Find indefinite integral $\displaystyle\int \left(x^2 - 1\right)^2 dx$

Solution

$$\int \left(x^2 - 1\right)^2 dx = \int \left(x^4 - 2x^2 + 1\right) dx \qquad (a-b)^2 = a^2 - 2ab + b^2$$

$$= \frac{1}{5}x^5 - \frac{2}{3}x^3 + x + C$$

Exercise

Find indefinite integral $\displaystyle\int \frac{x^2 + 1}{\sqrt{x}} dx$

Solution

$$\int \frac{x^2 + 1}{\sqrt{x}} dx = \int \left(\frac{x^2}{x^{1/2}} + \frac{1}{x^{1/2}}\right) dx$$

$$= \int \left(x^{3/2} + x^{-1/2}\right) dx$$

$$= \frac{x^{5/2}}{5/2} - \frac{x^{1/2}}{1/2} + C$$

$$= \frac{2}{5}x^{5/2} - 2x^{1/2} + C$$

Exercise

Find indefinite integral $\displaystyle\int \left(\sqrt[4]{x^3} + 1\right) dx$

Solution

$$\int\left(\sqrt[4]{x^3}+1\right)dx=\int\left(x^{3/4}+1\right)dx$$

$$\boxed{=\frac{4}{7}x^{7/4}+x+C}$$

Exercise

Find indefinite integral $\int\sqrt{x}\,(x+1)\,dx$

Solution

$$\int\sqrt{x}\,(x+1)\,dx=\int x^{1/2}(x+1)\,dx$$

$$=\int\left(x^{3/2}+x^{1/2}\right)dx$$

$$\boxed{=\frac{2}{5}x^{5/2}+\frac{2}{3}x^{3/2}+C}$$

Exercise

Find indefinite integral $\int(1+3t)\,t^2\,dt$

Solution

$$\int(1+3t)\,t^2\,dt=\int\left(t^2+3t^3\right)dt$$

$$\boxed{=\frac{1}{3}t^3+\frac{3}{4}t^4+C}$$

Exercise

Find indefinite integral $\int\frac{x^2-5}{x^2}dx$

Solution

$$\int\frac{x^2-5}{x^2}\,dx=\int\left(1-\frac{5}{x^2}\right)dx$$

$$=\int\left(1-5x^{-2}\right)dx$$

$$\boxed{=x+\frac{5}{x}+C}$$

Exercise

Find indefinite integral $\int(-40x+250)dx$

Solution

$$\int(-40x+250)dx \underline{=-20x^2+250x+C}$$

Exercise

Find indefinite integral $\int \frac{x+2}{\sqrt{x}}\,dx$

Solution

$$\begin{aligned}\int \frac{x+2}{\sqrt{x}}dx &= \int\left[\frac{x}{x^{1/2}}+\frac{2}{x^{1/2}}\right]dx\\ &= \int \frac{x}{x^{1/2}}dx+\int \frac{2}{x^{1/2}}dx\\ &= \int x^{1/2}dx+2\int x^{-1/2}dx\\ &= \frac{x^{3/2}}{3/2}+2\frac{x^{1/2}}{1/2}+C\\ &= \frac{2}{3}x^{3/2}+4x^{1/2}+C\end{aligned}$$

Exercise

Find indefinite integral $\int\left(\frac{1}{5}-\frac{2}{x^3}+2x\right)dx$

Solution

$$\begin{aligned}\int\left(\frac{1}{5}-\frac{2}{x^3}+2x\right)dx &= \int \frac{1}{5}dx-\int 2x^{-3}dx+\int 2xdx\\ &= \frac{x}{5}-2\frac{x^{-2}}{-2}+x^2+C\\ &= \frac{x}{5}+\frac{1}{x^2}+x^2+C\end{aligned}$$

Exercise

Find indefinite integral $\int\left(\sqrt{x}+\sqrt[3]{x}\right)dx$

Solution

$$\int\left(\sqrt{x}+\sqrt[3]{x}\right)dx=\int\left(x^{1/2}+x^{1/3}\right)dx$$

$$=\frac{x^{3/2}}{3/2}+\frac{x^{4/3}}{4/3}+C$$

$$=\frac{2}{3}x^{3/2}+\frac{3}{4}x^{4/3}+C$$

Exercise

Find indefinite integral $\int 2x\left(1-x^{-3}\right)dx$

Solution

$$\int 2x\left(1-x^{-3}\right)dx=\int\left(2x-2x^{-2}\right)dx$$

$$=x^2-2\frac{x^{-1}}{-1}+C$$

$$=x^2+\frac{2}{x}+C$$

Exercise

Find indefinite integral $\int\left(\frac{4+\sqrt{t}}{t^3}\right)dt$

Solution

$$\int\left(\frac{4+\sqrt{t}}{t^3}\right)dt=\int\left(\frac{4}{t^3}+\frac{t^{1/2}}{t^3}\right)dt$$

$$=\int\left(4t^{-3}+t^{-5/2}\right)dt$$

$$=4\frac{t^{-2}}{-2}+\frac{t^{-3/2}}{-3/2}+C$$

$$=-\frac{2}{t^2}-\frac{2}{3t^{3/2}}+C$$

Exercise

Find ech indefinite integral $\quad \int (-2\cos t)\, dt$

Solution

$$\int (-2\cos t)\, dt \underline{= -2\sin t + C}$$

Exercise

Find indefinite integral $\quad \int 7\sin\frac{\theta}{3} d\theta$

Solution

$$\int 7\sin\frac{\theta}{3}\, d\theta = 7\frac{-\cos\left(\frac{\theta}{3}\right)}{\frac{1}{3}} + C$$

$$\underline{= -21\cos\left(\frac{\theta}{3}\right) + C}$$

Exercise

Find indefinite integral $\quad \int \frac{2}{5}\sec\theta\tan\theta\, d\theta$

Solution

$$\int \frac{2}{5}\sec\theta\tan\theta\, d\theta \underline{= \frac{2}{5}\sec\theta + C}$$

Exercise

Find indefinite integral $\quad \int \left(4\sec x\tan x - 2\sec^2 x\right)dx$

Solution

$$\int \left(4\sec x\tan x - 2\sec^2 x\right)dx = 4\int (\sec x\tan x)dx - 2\int \left(\sec^2 x\right)dx$$

$$\underline{= 4\sec x - 2\tan x + C}$$

Exercise

Find indefinite integral $\int (2\cos 2x - 3\sin 3x)dx$

Solution

$$\int (2\cos 2x - 3\sin 3x)dx \underline{= \sin 2x + \cos 3x + C}$$

Exercise

Find indefinite integral $\int \left(1+\tan^2\theta\right)d\theta$

Solution

$$\int \left(1+\tan^2\theta\right)d\theta = \int \left(\sec^2\theta\right)d\theta$$

$$\underline{= \tan\theta + C}$$

Exercise

Find indefinite integral $\int \frac{\csc\theta}{\csc\theta - \sin\theta}d\theta$

Solution

$$\int \frac{\csc\theta}{\csc\theta - \sin\theta}d\theta = \int \frac{1}{1-\frac{\sin\theta}{\csc\theta}}d\theta \qquad \text{divide by } \csc\theta \ \& \ \csc\theta = \frac{1}{\sin\theta}$$

$$= \int \frac{1}{1-\sin^2\theta}d\theta \qquad \sin^2\theta + \cos^2\theta = 1 \Rightarrow 1-\sin^2\theta = \cos^2\theta$$

$$= \int \frac{1}{\cos^2\theta}d\theta$$

$$= \int \sec^2\theta d\theta$$

$$\underline{= \tan\theta + C}$$

Exercise

Evaluate the integral $\int \left(2e^x - 3e^{-2x}\right)dx$

Solution

$$\int\left(2e^{x}-3e^{-2x}\right)dx = \boxed{2e^{x}+\frac{3}{2}e^{-2x}+C}$$

Exercise

Evaluate $\int\frac{dx}{\sqrt{9-x^{2}}}$

Solution

$$\int\frac{dx}{\sqrt{9-x^{2}}} = \boxed{\sin^{-1}\left(\frac{x}{3}\right)+C}$$

Exercise

Evaluate $\int\frac{dx}{9+3x^{2}}$

Solution

$$\int\frac{dx}{9+3x^{2}} = \frac{1}{3}\int\frac{dx}{3+x^{2}} \qquad a^{2}=3 \ \rightarrow a=\sqrt{3}$$

$$= \frac{1}{3}\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+C$$

$$= \boxed{\frac{\sqrt{3}}{9}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+C}$$

Exercise

Find the integral $\int\frac{4x^{2}-3x+2}{x^{2}}dx$

Solution

$$\int\frac{4x^{2}-3x+2}{x^{2}}dx = \int\left(\frac{4x^{2}}{x^{2}}-\frac{3x}{x^{2}}+\frac{2}{x^{2}}\right)dx$$

$$= \int\left(4-\frac{3}{x}+2x^{-2}\right)dx$$

$$= 4x-3\ln|x|-2x^{-1}+C$$

$$= \boxed{4x-3\ln|x|-\frac{2}{x}+C}$$

Exercise

Find the integral $\int \left(x^8 - 3x^3 + 1\right) dx$

Solution

$$\int \left(x^8 - 3x^3 + 1\right) dx = \boxed{\frac{1}{9}x^9 - \frac{3}{4}x^4 + x + C}$$

Exercise

Find the integral $\int (2x+1)^2\, dx$

Solution

$$\int (2x+1)^2\, dx = \int \left(4x^2 + 4x + 1\right) dx$$

$$= \boxed{\frac{4}{3}x^3 + 2x^2 + x + C}$$

Exercise

Find the integral $\int \frac{x+1}{x} dx$

Solution

$$\int \frac{x+1}{x} dx = \int \left(1 + \frac{1}{x}\right) dx$$

$$= \boxed{x + \ln|x| + C}$$

Exercise

Find the integral $\int \left(\frac{1}{x^2} - \frac{2}{x^{5/2}}\right) dx$

Solution

$$\int \left(\frac{1}{x^2} - \frac{2}{x^{5/2}}\right) dx = \int \left(\frac{1}{x^2} - 2x^{-5/2}\right) dx$$

$$= -\frac{1}{x} + \frac{4}{3}x^{-3/2} + C$$

$$= \boxed{-\frac{1}{x} + \frac{4}{3x^{3/2}} + C}$$

Exercise

Find the integral $\displaystyle\int \frac{x^4 - 2\sqrt{x} + 2}{x^2}\,dx$

Solution

$$\int \frac{x^4 - 2\sqrt{x} + 2}{x^2}\,dx = \int \left(x^2 - 2x^{-3/2} + 2x^{-2}\right)dx$$

$$= \frac{1}{3}x^3 + 4x^{-1/2} - \frac{2}{x} + C$$

Exercise

Find the integral $\displaystyle\int (1 + \cos 3\theta)\,d\theta$

Solution

$$\int (1 + \cos 3\theta)\,d\theta = \theta + \frac{1}{3}\sin 3\theta + C$$

Exercise

Find the integral $\displaystyle\int 2\sec^2\theta\,d\theta$

Solution

$$\int 2\sec^2\theta\,d\theta = 2\tan\theta + C$$

Exercise

Find the integral $\displaystyle\int \sec 2x \tan 2x\,dx$

Solution

$$\int \sec 2x \tan 2x\,dx = \frac{1}{2}\sec 2x + C$$

Exercise

Find the integral $\int 2e^{2x}dx$

Solution

$$\int 2e^{2x}dx = \underline{e^{2x} + C}$$

Exercise

Find the integral $\int \frac{12}{x}dx$

Solution

$$\int \frac{12}{x}dx = \underline{12\ln|x| + C}$$

Exercise

Find the integral $\int \frac{dx}{\sqrt{1-x^2}}$

Solution

$$\int \frac{dx}{\sqrt{1-x^2}} = \underline{\sin^{-1}x + C}$$

Exercise

Find the integral $\int \frac{dx}{x^2+1}$

Solution

$$\int \frac{dx}{x^2+1} = \underline{\tan^{-1}x + C}$$

Exercise

Find the integral $\int \frac{1+\tan\theta}{\sec\theta}d\theta$

Solution

$$\int \frac{1+\tan\theta}{\sec\theta}d\theta = \int\left(\frac{1}{\sec\theta}+\frac{\tan\theta}{\sec\theta}\right)d\theta$$

$$= \int(\cos\theta+\sin\theta)d\theta$$

$$= \sin\theta - \cos\theta + C$$

Exercise

Find the integral $\int\left(\sqrt[4]{x^3}+\sqrt{x^5}\right)dx$

Solution

$$\int\left(\sqrt[4]{x^3}+\sqrt{x^5}\right)dx = \int\left(x^{3/4}+x^{5/2}\right)dx$$

$$= \frac{4}{7}x^{7/4}+\frac{2}{7}x^{7/2}+C$$

$$= \frac{4}{7}x\sqrt[4]{x^3}+\frac{2}{7}x^3\sqrt{x}+C$$

Exercise

Find the integral $\int\left(x^{-3}+7e^{5x}+\frac{4}{x}\right)dx$

Solution

$$\int\left(x^{-3}+7e^{5x}+\frac{4}{x}\right)dx = \frac{x^{-2}}{-2}+\frac{7}{5}e^{5x}+4\ln|x|+C$$

$$= -\frac{1}{2x^2}+\frac{7}{5}e^{5x}+4\ln|x|+C$$

Exercise

Find the integral $\int\left(\frac{2}{x}+\frac{x}{2}\right)dx$

Solution

$$\int\left(\frac{2}{x}+\frac{x}{2}\right)dx = 2\ln|x|+\frac{1}{4}x^2+C$$

Exercise

Find the integral $\int \frac{1}{ax} dx$

Solution

$$\int \frac{1}{ax} dx = \boxed{\frac{1}{a}\ln|x| + C}$$

Exercise

Find the integral $\int x\sqrt{x}\ dx$

Solution

$$\int x\sqrt{x}\ dx = \int x^{3/2}\ dx$$

$$= \boxed{\frac{2}{5}x^{5/2} + C}$$

Exercise

Find the integral $\int \left(\frac{2}{\sqrt{x}} + 2\sqrt{x}\right)dx$

Solution

$$\int \left(\frac{2}{\sqrt{x}} + 2\sqrt{x}\right)dx = \int \left(2x^{-1/2} + 2x^{1/2}\right)dx$$

$$= 4x^{1/2} + \frac{4}{3}x^{3/2} + C$$

$$= \boxed{4\sqrt{x} + \frac{4}{3}x\sqrt{x} + C}$$

Exercise

Find the integral $\int \left(x - 2x^2 + \frac{1}{2x}\right)dx$

Solution

$$\int \left(x - 2x^2 + \frac{1}{2x}\right)dx = \boxed{\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{2}\ln|x| + C}$$

Exercise

Find the integral $\int\left(\frac{7}{2x^3}-\sqrt[3]{x}\right)dx$

Solution

$$\int\left(\frac{7}{2x^3}-\sqrt[3]{x}\right)dx=\int\left(\frac{7}{2}x^{-3}-x^{1/3}\right)dx$$

$$=-\frac{7}{4}x^{-2}-\frac{3}{4}x^{4/3}+C$$

$$\boxed{=-\frac{7}{4x^2}-\frac{3}{4}x\sqrt[3]{x}+C}$$

Exercise

Find the integral $\int 3e^{-2x}dx$

Solution

$$\int 3e^{-2x}dx\ \boxed{=-\frac{3}{2}e^{-2x}+C}$$

Exercise

Find the integral $\int e^{-x}dx$

Solution

$$\boxed{\int e^{-x}dx=-e^{-x}+C}$$

Exercise

Find the integral $\int e\ dx$

Solution

$$\boxed{\int e\ dx=ex+C}$$

Exercise

Find the integral $\displaystyle\int \frac{7}{2e^{2x}}\,dx$

Solution

$$\int \frac{7}{2e^{2x}}\,dx = \frac{7}{2}\int e^{-2x}\,dx$$

$$= -\frac{7}{4}e^{-2x} + C$$

$$\boxed{= -\frac{7}{4e^{2x}} + C}$$

Exercise

Find the integral $\displaystyle\int -3\left(e^{2x}+1\right)dx$

Solution

$$\int -3\left(e^{2x}+1\right)dx \;\boxed{= -3\left(\frac{1}{2}e^{2x}+x\right)+C}$$

Exercise

Find the integral $\displaystyle\int \left(-3e^{-x}+2x-\frac{1}{2}e^{5x}\right)dx$

Solution

$$\int \left(-3e^{-x}+2x-\frac{1}{2}e^{5x}\right)dx \;\boxed{= 3e^{-x}+x^2-\frac{1}{10}e^{5x}+C}$$

Exercise

Find the integral $\displaystyle\int \left(\sqrt[4]{x^3}+1\right)\,dx$

Solution

$$\int \left(\sqrt[4]{x^3}+1\right)\,dx = \int \left(x^{3/4}+1\right)\,dx$$

$$\boxed{= \frac{4}{7}x^{7/4}+x+C}$$

Exercise

Find the integral $\int \left(5x^4 + 3x^2 + 2x + 5\right) dx$

Solution

$$\int \left(5x^4 + 3x^2 + 2x + 5\right) dx = x^5 + x^3 + x^2 + 5x + C$$

Exercise

Find the integral $\int \left(5x^{4/3} + 3x^{2/3} + 2x^{1/3}\right) dx$

Solution

$$\int \left(5x^{4/3} + 3x^{2/3} + 2x^{1/3}\right) dx = \frac{15}{7}x^{7/3} + \frac{9}{5}x^{5/3} + \frac{3}{2}x^{4/3} + C$$

Exercise

Find the integral $\int \left(5x^{-4/3} + 3x^{-2/3} + 2x^{-1/3}\right) dx$

Solution

$$\int \left(5x^{-4/3} + 3x^{-2/3} + 2x^{-1/3}\right) dx = -15x^{-1/3} + 9x^{1/3} + 3x^{2/3} + C$$

Exercise

Find the integral $\int \frac{x^4 - 3x^2 + 5}{x^4} dx$

Solution

$$\int \frac{x^4 - 3x^2 + 5}{x^4} dx = \int \left(1 - \frac{3}{x^2} + 5x^{-4}\right) dx$$

$$= x + \frac{3}{x} - \frac{5}{3x^3} + C$$

Exercise

Find the integral $\displaystyle\int \left(\frac{3}{x^7}-\frac{5}{x^6}\right)dx$

Solution

$$\int \left(\frac{3}{x^7}-\frac{5}{x^6}\right)dx = \int \left(3x^{-7}-5x^{-6}\right)dx$$

$$= -\frac{1}{2}x^{-6}+x^{-5}+C$$

$$= -\frac{1}{2x^6}+\frac{1}{x^5}+C$$

Exercise

Find the integral $\displaystyle\int \frac{x+8}{\sqrt{x}}\,dx$

Solution

$$\int \frac{x+8}{\sqrt{x}}\,dx = \int \left(x^{1/2}+8x^{-1/2}\right)dx$$

$$= \frac{2}{3}x^{3/2}+16x^{1/2}+C$$

Exercise

Find the integral $\displaystyle\int \frac{x^2+8}{\sqrt[3]{x}}\,dx$

Solution

$$\int \frac{x^2+8}{\sqrt[3]{x}}\,dx = \int \left(x^{5/3}+8x^{-1/3}\right)dx$$

$$= \frac{3}{8}x^{8/3}+12x^{2/3}+C$$

Exercise

Find the integral $\displaystyle\int \cos\left(\frac{5\pi}{3}x\right)dx$

Solution

$$\int \cos\left(\frac{5\pi}{3}x\right)dx = \frac{3}{5\pi}\sin\frac{5\pi}{3}x+C$$

Exercise

Find the integral $\int \sin\left(\frac{2x}{3}\right) dx$

Solution

$$\int \sin\left(\frac{2x}{3}\right) dx = -\frac{3}{2}\cos\frac{2x}{3} + C$$

Exercise

Find the integral $\int \left(5\cos x + 4\sin x + 3\sec^2 x\right) dx$

Solution

$$\int \left(5\cos x + 4\sin x + 3\sec^2 x\right) dx = 5\sin x - 4\cos x + 3\tan x + C$$

Exercise

Find the integral $\int \sec\theta\left(\sec\theta + \tan\theta\right) d\theta$

Solution

$$\int \sec\theta\left(\sec\theta + \tan\theta\right) d\theta = \int \left(\sec^2\theta + \sec\theta\tan\theta\right) d\theta$$

$$= \tan\theta + \sec\theta + C$$

Exercise

Find the integral $\int \left(\tan^2\theta + 1\right) d\theta$

Solution

$$\int \left(\tan^2\theta + 1\right) d\theta = \int \sec^2\theta \, d\theta$$

$$= \tan\theta + C$$

Exercise

Find the integral $\displaystyle\int \left(\cos^4\theta - \sin^4\theta\right)\, d\theta$

Solution

$$\int \left(\cos^4\theta - \sin^4\theta\right)\, d\theta = \int \left(\cos^2\theta - \sin^2\theta\right)\left(\cos^2\theta + \sin^2\theta\right)\, d\theta$$

$$= \int \cos 2\theta\, d\theta \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta \quad \cos^2\theta + \sin^2\theta = 1$$

$$= \frac{1}{2}\sin 2\theta + C$$

Exercise

Find the integral $\displaystyle\int \left(\cos^2\theta - \sin^2\theta\right)\, d\theta$

Solution

$$\int \left(\cos^2\theta - \sin^2\theta\right)\, d\theta = \int \cos 2\theta\, d\theta \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$= \frac{1}{2}\sin 2\theta + C$$

Exercise

Find the integral $\displaystyle\int \left(\cos^2\theta + \sin^2\theta\right)\, d\theta$

Solution

$$\int \left(\cos^2\theta + \sin^2\theta\right)\, d\theta = \int (1)\, d\theta$$

$$= \theta + C$$

Exercise

Find the integral $\displaystyle\int (\cos 2x \cos 4x - \sin 2x \sin 4x)\, dx$

Solution

$$\int (\cos 2x \cos 4x - \sin 2x \sin 4x)\, dx = \int \cos 6x\, dx$$

$$= \boxed{\frac{1}{6}\sin 6x + C}$$

Exercise

Find the integral $\displaystyle\int (\sin 2x \cos 4x - \cos 2x \sin 4x)\, dx$

Solution

$$\int (\sin 2x \cos 4x - \cos 2x \sin 4x)\, dx = \int \sin(-2x)\, dx$$

$$= -\int \sin 2x\, dx$$

$$= \boxed{\frac{1}{2}\cos 2x + C}$$

Exercise

Find the integral $\displaystyle\int (\sin 3x \cos 2x + \cos 3x \sin 2x)\, dx$

Solution

$$\int (\sin 3x \cos 2x + \cos 3x \sin 2x)\, dx = \int \sin 5x\, dx$$

$$= \boxed{-\frac{1}{5}\cos 5x + C}$$

Exercise

Find the integral $\displaystyle\int \cos 2x \sin 2x\, dx$

Solution

$$\int \cos 2x \sin 2x\, dx = \frac{1}{2}\int \sin 4x\, dx \qquad \sin\alpha = 2\sin\alpha\cos\alpha$$

$$= \boxed{-\frac{1}{8}\cos 4x + C}$$

Exercise

Find the integral $\displaystyle\int \left(2\cos^2 x - 1\right) dx$

Solution

$$\int \left(2\cos^2 x - 1\right)\, dx = \int \cos 2x \, dx \qquad \cos 2x = 2\cos^2 x - 1$$

$$= \boxed{\frac{1}{2}\sin 2x + C}$$

Exercise

Find the integral $\int \left(1 - 2\sin^2 x\right)\, dx$

Solution

$$\int \left(1 - 2\sin^2 x\right)\, dx = \int \cos 2x \, dx \qquad \cos 2x = 1 - 2\sin^2 x$$

$$= \boxed{\frac{1}{2}\sin 2x + C}$$

Exercise

Find the integral $\int e^{-5x}\, dx$

Solution

$$\int e^{-5x}\, dx = \boxed{-\frac{1}{5}e^{-5x} + C}$$

Exercise

Find the integral $\int 4e^{4x}\, dx$

Solution

$$\int 4e^{4x}\, dx = \boxed{e^{4x} + C}$$

Exercise

Find the integral $\int \left(2\sin\theta - 5e^{\theta}\right)\, d\theta$

Solution

$$\int \left(2\sin\theta - 5e^{\theta}\right)\, d\theta = \boxed{-2\cos\theta - 5e^{\theta} + C}$$

Exercise

Find the integral $\int \left(\frac{3}{x}+\sec^2 x\right)\, dx$

Solution

$$\int \left(\frac{3}{x}+\sec^2 x\right)\, dx \underline{= 3\ln|x| + \tan x + C}$$

Exercise

Find the integral $\int \left(\sin x + 2^x\right)\, dx$

Solution

$$\int \left(\sin x + 2^x\right)\, dx \underline{= -\cos x + \frac{2^x}{\ln 2} + C} \qquad \left(a^x\right)' = a^x \ln a$$

Exercise

Find the integral $\int \left(2x - 3^x\right)\, dx$

Solution

$$\int \left(2x - 3^x\right)\, dx \underline{= x^2 - \frac{3^x}{\ln 3} + C} \qquad \left(a^x\right)' = a^x \ln a$$

Exercise

Find the integral $\int \left(4x - \frac{3}{x} - \csc^2 x\right)\, dx$

Solution

$$\int \left(4x - \frac{3}{x} - \csc^2 x\right)\, dx \underline{= 2x^2 - 3\ln|x| + \cot x + C}$$

Exercise

Find the integral $\int \left(e^{4x} - \frac{3}{x} + 2\csc x \cot x\right)\, dx$

Solution

$$\int \left(e^{4x} - \frac{3}{x} + 2\csc x \cot x\right)\, dx \underline{= \frac{1}{4}e^{4x} - 3\ln|x| - 2\csc x + C}$$

Exercise

Find the integral $\int (a+b)e^{(a+b)x}\,dx$

Solution

$$\int (a+b)e^{(a+b)x}\,dx = \boxed{e^{(a+b)x} + C}$$

Exercise

Find the integral $\int \left(a^2 - b^2\right)e^{(a-b)x}\,dx$

Solution

$$\begin{aligned}\int \left(a^2 - b^2\right)e^{(a-b)x}\,dx &= \frac{a^2 - b^2}{a-b}e^{(a-b)x} + C \\ &= \boxed{(a+b)e^{(a-b)x} + C}\end{aligned}$$

Exercise

Find the function with the following property: $\frac{dy}{dx} = 2x - 7, \quad y(2) = 0$

Solution

$$\frac{dy}{dx} = 2x - 7$$

$$dy = (2x-7)\,dx$$

$$\int dy = \int (2x-7)\,dx$$

$$y = x^2 - 7x + C$$

At point (2, 0):

$$0 = 2^2 - 7(2) + C$$

$$0 = 4 - 14 + C$$

$$\to 0 = -10 + C$$

$$\boxed{C = 10}$$

$$\boxed{y(x) = x^2 - 7x + 10}$$

Exercise

Find the function with the following property: $\frac{dy}{dx}=\frac{1}{x^2}+x, \quad y(2)=1; \ \ x>0$

Solution

$$\frac{dy}{dx}=\frac{1}{x^2}+x$$

$$dy=\left(x^{-2}+x\right)dx$$

$$\int dy=\int\left(x^{-2}+x\right)dx$$

$$y=-x^{-1}+\frac{1}{2}x^2+C$$

$$1=-(2)^{-1}+\frac{1}{2}(2)^2+C$$

$$1+\frac{1}{2}-2=C$$

$$C=-\frac{1}{2}$$

$$y(x)=-\frac{1}{x}+\frac{1}{2}x^2-\frac{1}{2}$$

Exercise

Find the function with the following property: $\frac{ds}{dt}=1+\cos t, \quad s(0)=4$

Solution

$$\frac{ds}{dt}=1+\cos t$$

$$ds=(1+\cos t)dt$$

$$\int ds=\int(1+\cos t)dt$$

$$s=t+\sin t+C$$

$$4=0+\sin(0)+C$$

$$C=4$$

$$s(t)=t+\sin t+4$$

Exercise

Find the function with the following property: $\frac{ds}{dt} = \cos t + \sin t, \quad s(\pi) = 1$

Solution

$$\frac{ds}{dt} = \cos t + \sin t$$

$$ds = (\cos t + \sin t)\,dt$$

$$\int ds = \int (\cos t + \sin t)\,dt$$

$$s = \sin t - \cos t + C$$

$$1 = \sin \pi - \cos \pi + C$$

$$1 = 0 - (-1) + C$$

$$1 = 1 + C$$

$$\underline{C = 0}$$

$$\underline{s(t) = \sin t - \cos t}$$

Exercise

Find the function with the following property: $f'(x) = 3x^2 - 1 \quad \& \quad f(0) = 10$

Solution

$$f(x) = \int \left(3x^2 - 1\right) dx$$

$$= x^3 - x + C$$

$$f(0) = \underline{C = 10}$$

$$\underline{f(x) = x^3 - x + 10}$$

Exercise

Find the function with the following property: $f'(t) = \sin t + 2t \quad \& \quad f(0) = 5$

Solution

$$f(t) = \int (\sin t + 2t)\ dt$$

$$= -\cos t + t^2 + C$$

$$f(0) = -1 + C = 5$$

$$\boxed{C = 6}$$

$$\boxed{f(t) = -\cos t + t^2 + 6}$$

Exercise

Find the function with the following property: $f'(x) = x^2 + x^{-2}$ & $f(1) = 1$

Solution

$$f(x) = \int \left(x^2 + x^{-2}\right) dx$$

$$= \frac{1}{3}x^3 - \frac{1}{x} + C$$

$$f(1) = \frac{1}{3} - 1 + C = 1$$

$$\boxed{C = \frac{5}{3}}$$

$$\boxed{f(x) = \frac{1}{3}x^3 - \frac{1}{x} + \frac{5}{3}}$$

Exercise

Find the function with the following property: $f'(x) = \sin^2 x$ & $f(1) = 1$

Solution

$$f(x) = \int \sin^2 x\, dx \qquad \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$= \frac{1}{2}\int (1 - \cos 2x)\, dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$$

$$f(1) = \frac{1}{2} - \frac{1}{4}\sin 2 + C = 1$$

$$\boxed{C = \frac{1}{2} + \frac{1}{4}\sin 2}$$

$$\boxed{f(x) = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + \frac{1}{2} + \frac{1}{4}\sin 2}$$

Exercise

Derive the position function if a ball is thrown upward with initial velocity of 32 *feet* per second from an initial height of 48 *feet*. When does the ball hit the ground? With what velocity does the ball hit the ground?

Solution

$$s(t) = -16t^2 + 32t + 48$$

$$s(0) = 48$$

$$s'(0) = 32$$

$$s''(t) = -32$$

$$s'(t) = \int -32dt$$

$$= -32t + C_1$$

$$s'(0) = -32(0) + C_1 = 32$$

$$\Rightarrow \boxed{C_1 = 32}$$

$$s'(t) = -32t + 32$$

$$s(t) = \int (-32t + 32)\ dt$$

$$= -32\frac{t^2}{2} + 32t + C_2$$

$$s(0) = -32\frac{0^2}{2} + 32(0) + C_2 = 48$$

$$\Rightarrow \boxed{C_2 = 48}$$

$$s(t) = -16t^2 + 32t + 48 = 0$$

$$-t^2 + 2t + 3 = 0$$

$$\boxed{t_{1,2} = -1,\ 3}$$

The ball hits the ground in 3 *seconds*

The velocity: $v(t) = s'(t) = -32t + 32$

$$v(t = 3) = -32(3) + 32$$

$$\boxed{= -64\ \ ft/\sec^2}$$

Exercise

Suppose a publishing company has found that the marginal cost at a level of production of x thousand books is given by

$$\frac{dC}{dx}=\frac{50}{\sqrt{x}}$$

And that the fixed cost (the cost before the first book can be produced) is a \$25,000. Find the cost function $C(x)$.

Solution

$$\frac{dC}{dx}=\frac{50}{\sqrt{x}}=50x^{-1/2}$$

$$dC=50x^{-1/2}dx$$

$$\int dC=\int 50x^{-1/2}dx$$

$$\begin{aligned}C(x)&=50\frac{x^{1/2}}{1/2}+C\\&=50(2)x^{1/2}+C\\&=100\sqrt{x}+C\end{aligned}$$

$$25000=100\sqrt{0}+C$$ Before the first (x = 0) costs 25,000

$$25,000=C$$

$$C(x)=100\sqrt{x}+25,000$$

Exercise

Find the general solution of $F'(x)=4x+2$, and find the particular solution that satisfies the initial condition $F(1)=8$.

Solution

$$\begin{aligned}F(x)&=\int(4x+2)\,dx\\&=4\frac{x^2}{2}+2x+C\\&=2x^2+2x+C\end{aligned}$$

$$F(x)=2(1)^2+2(1)+C=8$$

$$2+2+C=8$$

$$4+C=8$$

$$C=4$$

$$F(x)=2x^2+2x+4$$

Exercise

The marginal cost function for producing x units of a product is modeled by

$$\frac{dC}{dx} = 28 - 0.02x$$

It costs \$40 to produce one unit. Find the cost of producing 200 units.

Solution

$$C = \int (28 - 0.02x)\, dx$$

$$= 28x - 0.02\frac{x^2}{2} + K$$

Cost \$40 for one unit

$$C(x = 1) = 40$$

$$C(x = 1) = 28(1) - 0.01(1)^2 + K = 40$$

$$\text{K} = 12.01$$

$$C(x) = -0.01x^2 + 28x + 12.01$$

$$C(200) = -0.01(200)^2 + 28(200) + 12.01$$

$$= \$5212.01$$

Solution Section 4.2 – Area under Curves

Exercise

Use finite approximations to estimate the area under the graph of the function using

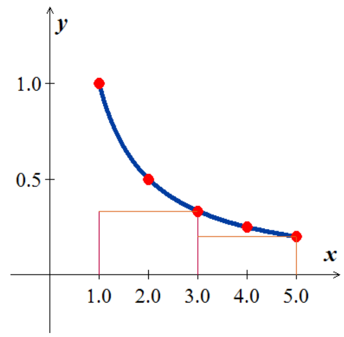
$$f(x)=\frac{1}{x} \quad between \quad x=1 \quad and \quad x=5$$

a) A lower sum with two rectangles of equal width
b) A lower sum with four rectangles of equal width
c) A upper sum with two rectangles of equal width
d) A upper sum with four rectangles of equal width

Solution

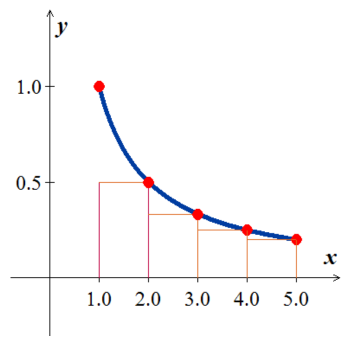
a) Using 2 lower rectangles: $\Delta x=\frac{5-1}{2}=2$

$$\begin{aligned}A &\approx \Delta x\left(f\left(x_1\right)+f\left(x_2\right)\right)\\ &\approx 2\cdot\left(f(3)+f(5)\right)\\ &\approx 2\cdot\left(\frac{1}{3}+\frac{1}{5}\right)\\ &\approx \boxed{\frac{16}{15}}\end{aligned}$$



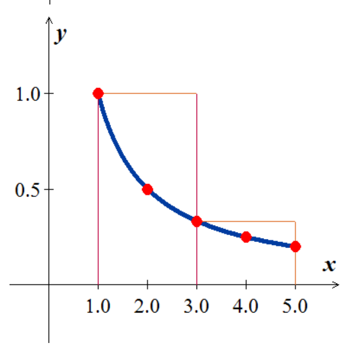
b) Using 4 lower rectangles: $\Delta x=\frac{5-1}{4}=1$

$$\begin{aligned}A &\approx 1\cdot\left(f(2)+f(3)+f(4)+f(5)\right)\\ &\approx 1\cdot\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)\\ &\approx \boxed{\frac{77}{60}}\end{aligned}$$



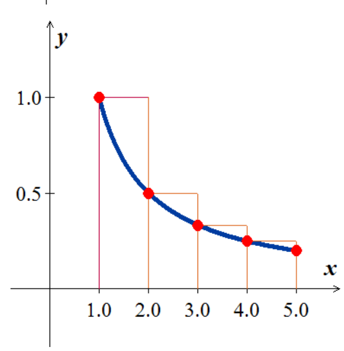
c) Using 2 upper rectangles: $\Delta x=\frac{5-1}{2}=2$

$$\begin{aligned}A &\approx 2\cdot\left(f(1)+f(3)\right)\\ &\approx 2\cdot\left(1+\frac{1}{3}\right)\\ &\approx \boxed{\frac{8}{3}}\end{aligned}$$



d) Using 4 lower rectangles: $\Delta x=\frac{5-1}{4}=1$

$$\begin{aligned}A &\approx 1\cdot\left(f(1)+f(2)+f(3)+f(4)\right)\\ &\approx 1\cdot\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\\ &\approx \boxed{\frac{25}{12}}\end{aligned}$$



Exercise

Use finite approximations to estimate the area under the graph of the function using

$$f(x) = 4 - x^2 \quad \textit{between} \quad x = -2 \quad \textit{and} \quad x = 2$$

a) A lower sum with two rectangles of equal width
b) A lower sum with four rectangles of equal width
c) A upper sum with two rectangles of equal width
d) A upper sum with four rectangles of equal width

Solution

a) Using 2 lower rectangles:

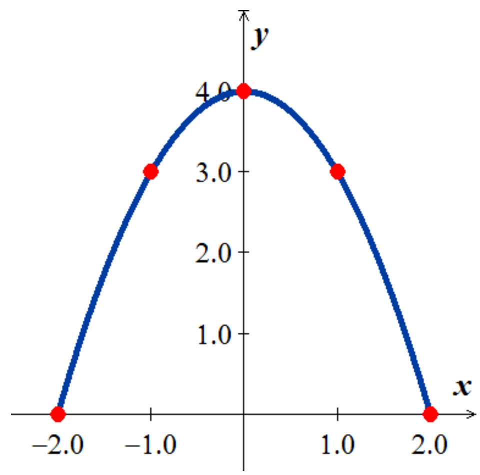
$$\Delta x = \frac{2-(-2)}{2} = 2$$

$$A \approx \Delta x\left(f\left(x_1\right) + f\left(x_2\right)\right)$$

$$\approx 2 \cdot \left(f(-2) + f(2)\right)$$

$$\approx 2 \cdot \left[\left(4-(-2)^2\right) + \left(4 - 2^2\right)\right]$$

$$= 0$$



b) Using 4 lower rectangles:

$$\Delta x = \frac{2-(-2)}{4} = 1$$

$$A \approx 1 \cdot \left(f(-2) + f(-1) + f(1) + f(2)\right)$$

$$\approx 1 \cdot (0 + 3 + 3 + 0)$$

$$= 6$$

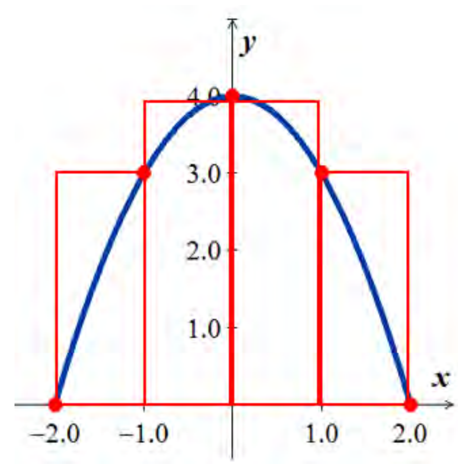
c) Using 2 upper rectangles:

$$\Delta x = \frac{2-(-2)}{2} = 2$$

$$A \approx 2 \cdot \left(f(0) + f(0)\right)$$

$$\approx 2 \cdot (4 + 4)$$

$$= 16$$



d) Using 4 lower rectangles:

$$\Delta x = \frac{2-(-2)}{4} = 1$$

$$A \approx 1 \cdot \left(f(-1) + f(0) + f(1) + f(2)\right)$$

$$\approx 1 \cdot (3 + 4 + 4 + 3)$$

$$= 14$$

Exercise

Use finite approximations to estimate the average value of *f* on the given interval by partitioning the interval into four subintervals of equal length and evaluating *f* at the subinterval midpoints.

$$f(t)=\frac{1}{2}+\sin^2 \pi t \quad on \quad [0,\ 2]$$

Solution

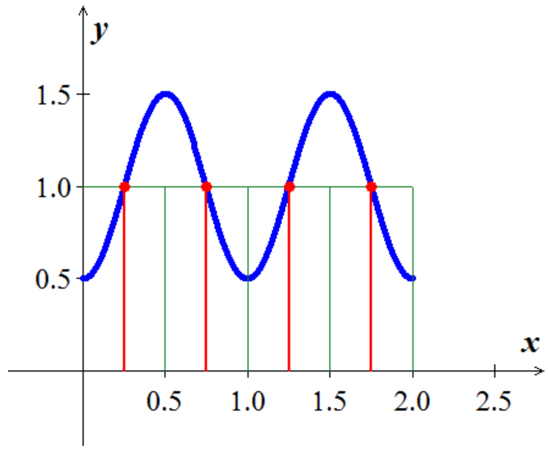
$$\Delta x=\frac{2-0}{4}=0.5$$

$$f(t=.25)=\frac{1}{2}+\sin^2(.25\pi)\ =1$$

$$f(\ \ =.75)=\frac{1}{2}+\sin^2(.75\pi)\ =1$$

$$f(t=1.25)=\frac{1}{2}+\sin^2(1.25\pi)\ =1$$

$$f(t=1.75)=\frac{1}{2}+\sin^2(1.75\pi)\ =1$$



$$A\approx .5\cdot\left(f(.25)+f(.75)+f(1.25)+f(1.75)\right)$$

$$=.5(1+1+1+1)$$

$$\underline{=2}$$

$$\text{Average value }\approx\frac{Area}{Length\ [0,\ 2]}$$

$$=\frac{2}{2}$$

$$\underline{=1}$$

Exercise

Use finite approximations to estimate the average value of *f* on the given interval by partitioning the interval into four subintervals of equal length and evaluating *f* at the subinterval midpoints.

$$f(t)=1-\left(\cos\frac{\pi t}{4}\right)^4 \quad on \quad [0,\ 4]$$

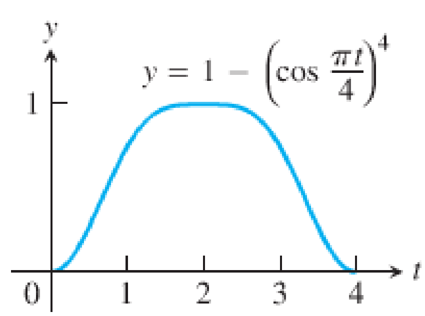
Solution

$$\Delta x=\frac{4-0}{4}=1$$

$$f(t=0.5)=1-\left(\cos\frac{0.5\pi}{4}\right)^4=0.27145$$

$$f(t=1.5)=1-\left(\cos\frac{1.5\pi}{4}\right)^4=0.97855$$

$$f(t=2.5)=1-\left(\cos\frac{2.5\pi}{4}\right)^4=0.97855$$



$$f(t=3.5)=1-\left(\cos\frac{3.5\pi}{4}\right)^4=0.27145$$

$$A\approx 1\cdot\left(f(.5)+f(1.5)+f(2.5)+f(3.5)\right)$$
$$=1(0.27145+0.97855+0.97855+0.27145)$$
$$\underline{=2.5}$$

$$\text{Average value }\approx\frac{Area}{Length\ [0,\ 2]}$$
$$=\frac{2.5}{4}$$
$$\underline{=0.625}$$

Exercise

Write the sums without sigma notation. Then evaluate: $\displaystyle\sum_{k=1}^{2}\frac{6k}{k+1}$

Solution

$$\sum_{k=1}^{2}\frac{6k}{k+1}=\frac{6}{2}+\frac{12}{3}$$
$$\underline{=7}$$

Exercise

Write the sums without sigma notation. Then evaluate: $\displaystyle\sum_{k=1}^{3}\frac{k-1}{k}$

Solution

$$\sum_{k=1}^{3}\frac{k-1}{k}=\frac{0}{1}+\frac{1}{2}+\frac{2}{3}$$
$$\underline{=\frac{7}{6}}$$

Exercise

Write the sums without sigma notation. Then evaluate: $\displaystyle\sum_{k=1}^{5}\sin k\pi$

Solution

$$\sum_{k=1}^{5} \sin k\pi = \sin \pi + \sin 2\pi + \sin 3\pi + \sin 4\pi + \sin 5\pi$$

$$= 0 + 0 + 0 + 0$$

$$\underline{= 0}$$

Exercise

Write the sums without sigma notation. Then evaluate: $\displaystyle\sum_{k=1}^{4} (-1)^k \cos k\pi$

Solution

$$\sum_{k=1}^{4} (-1)^k \cos k\pi = -\cos \pi + \cos 2\pi - \cos 3\pi + \cos 4\pi$$

$$= -(-1) + 1 - (-1) + 1$$

$$\underline{= 4}$$

Exercise

Write the following expression 1 + 2 + 4 + 8 + 16 + 32 in sigma notation

Solution

$$1 + 2 + 4 + 8 + 16 + 32 = \sum_{k=1}^{6} 2^{k-1}$$

$$1 + 2 + 4 + 8 + 16 + 32 = \sum_{k=0}^{5} 2^{k}$$

Exercise

Write the following expression 1 – 2 + 4 – 8 + 16 – 32 in sigma notation

Solution

$$1 - 2 + 4 - 8 + 16 - 32 = \sum_{k=1}^{6} (-2)^{k-1}$$

$$1-2+4-8+16-32=\sum_{k=0}^{5}(-1)^k\, 2^k$$

Exercise

Write the following expression $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ in sigma notation

Solution

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}=\sum_{k=1}^{4}\frac{1}{2^k}$$

Exercise

Write the following expression $-\frac{1}{5}+\frac{2}{5}-\frac{3}{5}+\frac{4}{5}-\frac{5}{5}$ in sigma notation

Solution

$$-\frac{1}{5}+\frac{2}{5}-\frac{3}{5}+\frac{4}{5}-\frac{5}{5}=\sum_{k=1}^{5}(-1)^k\frac{k}{5}$$

Exercise

Suppose that $\sum_{k=1}^{n} a_k=-5$ *and* $\sum_{k=1}^{n} b_k=6$. Find the value of $\sum_{k=1}^{n}\left(b_k-2a_k\right)$

Solution

$$\sum_{k=1}^{n}\left(b_k-2a_k\right)=\sum_{k=1}^{n} b_k-2\sum_{k=1}^{n} a_k$$

$$=6-2(-5)$$

$$\underline{=16}$$

Exercise

Evaluate the sums $\displaystyle\sum_{k=1}^{10} k^3$

Solution

$$\sum_{k=1}^{10} k^3 = \left(\frac{10(10+1)}{2}\right)^2$$

$$= 55^2$$

$$\underline{= 3,025}$$

Exercise

Evaluate the sums $\displaystyle\sum_{k=1}^{7} (-2k)$

Solution

$$\sum_{k=1}^{7} (-2k) = -2\sum_{k=1}^{7} k$$

$$= -2\left(\frac{7(7+1)}{2}\right)$$

$$\underline{= -56}$$

Exercise

Evaluate the sums $\displaystyle\sum_{k=1}^{5} \frac{\pi k}{15}$

Solution

$$\sum_{k=1}^{5} \frac{\pi k}{15} = \frac{\pi}{15}\sum_{k=1}^{5} k$$

$$= \frac{\pi}{15}\left(\frac{5(5+1)}{2}\right)$$

$$\underline{= \pi}$$

Exercise

Evaluate the sums $\displaystyle\sum_{k=1}^{5} k(3k+5)$

Solution

$$\sum_{k=1}^{5} k(3k+5) = \sum_{k=1}^{5} \left(3k^2+5k\right)$$

$$= 3\sum_{k=1}^{5} k^2 + 5\sum_{k=1}^{5} k$$

$$= 3\left(\frac{5(5+1)(2(5)+1)}{6}\right) + 5\frac{5(5+1)}{2}$$

$$= 240$$

Exercise

Evaluate the sums $\displaystyle\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$

Solution

$$\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3 = \frac{1}{225}\sum_{k=1}^{5} k^3 + \left(\sum_{k=1}^{5} k\right)^3$$

$$= \frac{1}{225}\left(\frac{5(5+1)}{2}\right)^2 + \left(\frac{5(5+1)}{2}\right)^3$$

$$= \frac{1}{225}\left(\frac{900}{4}\right) + \frac{27,000}{8}$$

$$= 1 + 3,375$$

$$= 3,376$$

Exercise

Evaluate the sums $\displaystyle\sum_{k=1}^{500} 7$

Solution

$$\sum_{k=1}^{500} 7 = 7(500)$$

$$= 3,500$$

Exercise

Evaluate the sums $\displaystyle\sum_{k=18}^{71} k(k-1)$

Solution

Let $n = (k-18)+1$

$= k-17$

$$\begin{cases} k=18 & \rightarrow n=1 \\ k=71 & \rightarrow n=54 \end{cases}$$

$\Rightarrow \underline{k=n+17}$

$$\sum_{k=18}^{71} k(k-1) = \sum_{n=1}^{54} (n+17)(n+17-1)$$

$$= \sum_{n=1}^{54} (n+17)(n+16)$$

$$= \sum_{n=1}^{54} \left(n^2+33n+272\right)$$

$$= \sum_{n=1}^{54} n^2 + 33\sum_{n=1}^{54} n + \sum_{n=1}^{54} 272$$

$$= \frac{54(54+1)(54(2)+1)}{6} + 33\cdot\frac{54(54+1)}{6} + 272(54)$$

$$= 117,648$$

Exercise

Evaluate the sums $\sum_{k=1}^{n}\left(\frac{1}{n}+2n\right)$

Solution

$$\sum_{k=1}^{n}\left(\frac{1}{n}+2n\right)=n\cdot\left(\frac{1}{n}+2n\right)$$

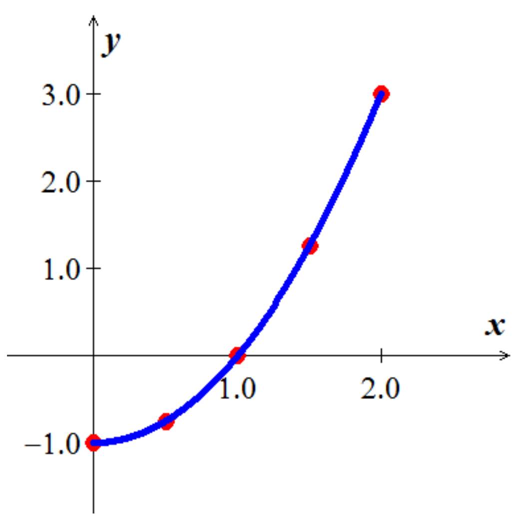
$$=1+2n^2$$

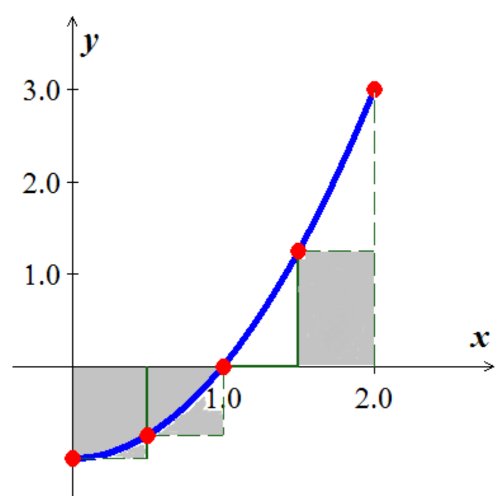
Exercise

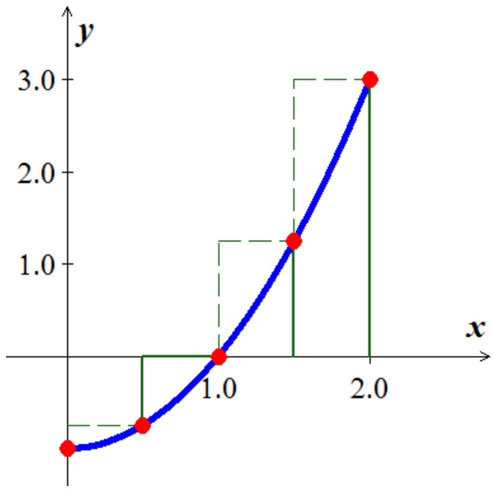
Graph the function $f(x)=x^2-1$ over the given interval [0, 2]. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\sum_{k=1}^{4} f\left(c_k\right)\Delta x_k$, given c_k is the

a) Left-hand endpoint
b) Right-hand endpoint
c) Midpoint of kth subinterval.

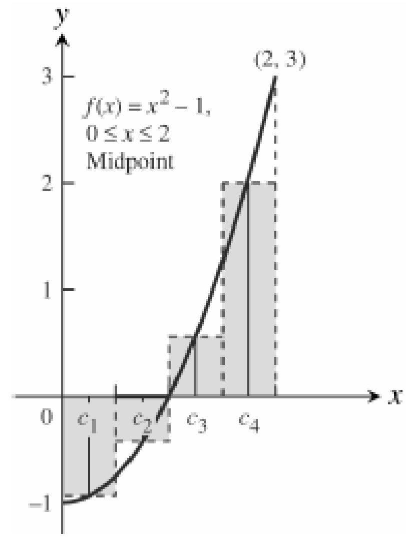
Solution







y
3.0
2.0
1.0
x
1.0
2.0



y
(2, 3)
3
$f(x) = x^2 - 1,$
$0 \leq x \leq 2$
Midpoint
2
1
0
c_1
c_2
c_3
c_4
x
−1

Solution — *Section* 4.4 – Fundamental Theorem of Calculus

Exercise

Evaluate the integral $\displaystyle\int_0^3 (2x+1)\,dx$

Solution

$$\int_0^3 (2x+1)\,dx = x^2 + x \Big|_0^3$$

$$= 3^2 + 3 - (0+0)$$

$$\underline{= 12}$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 x(x-3)\,dx$

Solution

$$\int_0^2 x(x-3)\,dx = \int_0^2 \left(x^2 - 3x\right)dx$$

$$= \frac{x^3}{3} - \frac{3x^2}{2} \Big|_0^2$$

$$= \left(\frac{2^3}{3} - \frac{3(2)^2}{2}\right) - \left(\frac{0^3}{3} - \frac{3(2)^2}{2}\right)$$

$$\underline{= -\frac{10}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^4 \left(3x - \frac{x^3}{4}\right)dx$

Solution

$$\int_0^4 \left(3x - \frac{x^3}{4}\right)dx = 3\frac{x^2}{2} - \frac{x^4}{16} \Big|_0^4$$

$$=\left(3\frac{(4)^2}{2}-\frac{(4)^4}{16}\right)-0$$

$$\underline{=8}$$

Exercise

Evaluate the integral $\displaystyle\int_{-2}^{2}\left(x^3-2x+3\right)dx$

Solution

$$\int_{-2}^{2}\left(x^3-2x+3\right)dx=\frac{x^4}{4}-x^2+3x\ \Big|_{-2}^{2}$$

$$=\left(\frac{(2)^4}{4}-(2)^2+3(2)\right)-\left(\frac{(-2)^4}{4}-(-2)^2+3(-2)\right)$$

$$\underline{=12}$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{1}\left(x^2+\sqrt{x}\right)dx$

Solution

$$\int_{0}^{1}\left(x^2+\sqrt{x}\right)dx=\frac{x^3}{3}+\frac{2}{3}x^{3/2}\ \Big|_{0}^{1}$$

$$=\left(\frac{(1)^3}{3}+\frac{2}{3}(1)^{3/2}\right)-0$$

$$\underline{=1}$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{\pi/3}4\sec u\tan u\ du$

Solution

$$\int_{0}^{\pi/3}4\sec u\tan u\ du=4\sec u\ \Big|_{0}^{\pi/3}$$

$$= 4\left(\sec\frac{\pi}{3} - \sec 0\right)$$

$$= 4(2-1)$$

$$\underline{= 4}$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/4}^{3\pi/4} \csc\theta \cot\theta \, d\theta$

Solution

$$\int_{\pi/4}^{3\pi/4} \csc\theta \cot\theta \, d\theta = -\csc\theta \Big|_{\pi/4}^{3\pi/4}$$

$$= -\left(\csc\frac{3\pi}{4} - \csc\frac{\pi}{4}\right)$$

$$= -\left(\sqrt{2} - \sqrt{2}\right)$$

$$\underline{= 0}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt$

Solution

$$\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt = \int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \pi t^{-2}\right) dt$$

$$= 4\tan t - \pi t^{-1} \Big|_{-\pi/3}^{-\pi/4}$$

$$= \left(4\tan\left(-\frac{\pi}{4}\right) - \pi\left(-\frac{4}{\pi}\right)\right) - \left(4\tan\left(-\frac{\pi}{3}\right) - \pi\left(-\frac{3}{\pi}\right)\right)$$

$$= (4(-1) + 4) - \left(4\left(-\sqrt{3}\right) + 3\right)$$

$$= -\left(-4\sqrt{3} + 3\right)$$

$$\underline{= 4\sqrt{3} - 3}$$

Exercise

Evaluate the integral $\displaystyle\int_{-3}^{-1} \frac{y^5-2y}{y^3}\,dy$

Solution

$$\int_{-3}^{-1} \frac{y^5-2y}{y^3}\,dy = \int_{-3}^{-1} \left(\frac{y^5}{y^3}-\frac{2y}{y^3}\right)dy$$

$$= \int_{-3}^{-1} \left(y^2-2y^{-2}\right)dy$$

$$= \frac{1}{3}y^3+2y^{-1}\Big|_{-3}^{-1}$$

$$= \left(\frac{1}{3}(-1)^3+\frac{2}{-1}\right)-\left(\frac{1}{3}(-3)^3+\frac{2}{-3}\right)$$

$$\boxed{= \frac{22}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_{1}^{8} \frac{\left(x^{1/3}+1\right)\left(2-x^{2/3}\right)}{x^{1/3}}\,dx$

Solution

$$\int_{1}^{8} \frac{\left(x^{1/3}+1\right)\left(2-x^{2/3}\right)}{x^{1/3}}\,dx = \int_{1}^{8} \frac{2x^{1/3}-x+2-x^{2/3}}{x^{1/3}}\,dx$$

$$= \int_{1}^{8} \left(2-x^{2/3}+2x^{-1/3}-x^{1/3}\right)dx$$

$$= 2x-\frac{3}{5}x^{5/3}+3x^{2/3}-\frac{3}{4}x^{4/3}\Big|_{1}^{8}$$

$$= \left(2(8)-\frac{3}{5}(8)^{5/3}+3(8)^{2/3}-\frac{3}{4}(8)^{4/3}\right)-\left(2(1)-\frac{3}{5}(1)^{5/3}+3(1)^{2/3}-\frac{3}{4}(1)^{4/3}\right)$$

$$= \left(-\frac{16}{5}\right)-\left(\frac{73}{20}\right)$$

$$\boxed{= -\frac{137}{20}}$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x}\,dx$

Solution

$$\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x}\,dx = \int_{\pi/2}^{\pi} \frac{2\sin x\cos x}{2\sin x}\,dx$$

$$= \int_{\pi/2}^{\pi} \cos x\,dx$$

$$= \sin x \Big|_{\pi/2}^{\pi}$$

$$= \sin\pi - \sin\frac{\pi}{2}$$

$$= -1$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{\pi/3} (\cos x + \sec x)^2\,dx$

Solution

$$\int_{0}^{\pi/3} (\cos x + \sec x)^2\,dx = \int_{0}^{\pi/3} \left(\cos^2 x + 2 + \sec^2 x\right)dx$$

$$= \int_{0}^{\pi/3} \left(\frac{1}{2} + \frac{1}{2}\cos 2x + 2 + \sec^2 x\right)dx$$

$$= \int_{0}^{\pi/3} \left(\frac{5}{2} + \frac{1}{2}\cos 2x + \sec^2 x\right)dx$$

$$= \frac{5}{2}x + \frac{1}{4}\sin 2x + \tan x \Big|_{0}^{\pi/3}$$

$$= \left(\frac{5}{2}\frac{\pi}{3} + \frac{1}{4}\sin\frac{2\pi}{3} + \tan\frac{\pi}{3}\right) - \left(\frac{5}{2}(0) + \frac{1}{4}\sin(2\cdot 0) + \tan(0)\right)$$

$$= \frac{5\pi}{6} + \frac{1}{4}\frac{\sqrt{3}}{2} + \sqrt{3}$$

$$= \frac{5\pi}{6} + \frac{9\sqrt{3}}{8}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi} \frac{1}{2}\left(\cos x + |\cos x|\right) dx$

Solution

$$\int_0^{\pi} \frac{1}{2}\left(\cos x + |\cos x|\right) dx = \int_0^{\pi/2} \frac{1}{2}\left(\cos x + \cos x\right) dx \ + \int_{\pi/2}^{\pi} \frac{1}{2}\left(\cos x - \cos x\right) dx$$

$$= \int_0^{\pi/2} \cos x \, dx$$

$$= \sin x \Big|_0^{\pi/2}$$

$$\underline{= 1}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{1} 2x\left(4 - x^2\right) dx$

Solution

$$\int_0^{1} 2x\left(4 - x^2\right) dx = \int_0^{1} \left(8x - 2x^3\right) dx$$

$$= 4x^2 - \frac{1}{2}x^4 \Big|_0^1$$

$$= 4 - \frac{1}{2}$$

$$\underline{= \frac{7}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{4} (8 - 2x)\, dx$

Solution

$$\int_0^{4} (8 - 2x)\, dx = 8x - x^2 \Big|_0^4$$

$$= 8(4) - (4)^2 - 0$$

$$\underline{= 16}$$

Exercise

Evaluate the integral $\displaystyle\int_0^4 \frac{1}{\sqrt{16-x^2}}\,dx$

Solution

$$\int_0^4 \frac{1}{\sqrt{16-x^2}}\,dx = \sin^{-1}\frac{x}{4}\Big|_0^4$$

$$= \sin^{-1}\frac{4}{4} - \sin^{-1}0 \qquad\qquad \sin^{-1}1 = \frac{\pi}{2}$$

$$\underline{= \frac{\pi}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-4}^2 (2x+4)\,dx$

Solution

$$\int_{-4}^2 (2x+4)\,dx = x^2 + 4x\Big|_{-4}^2$$

$$= 2^2 + 4(2) - \left(\left(-4^2\right) + 4(-4)\right)$$

$$= 4 + 8 - (16 - 16)$$

$$\underline{= 12}$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 (1-x)\,dx$

Solution

$$\int_0^2 (1-x)\,dx = x - \frac{1}{2}x^2\Big|_0^2$$

$$= 2 - \frac{1}{2}(2)^2 - 0$$

$$\underline{= 0}$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 \left(x^2-2\right)dx$

Solution

$$\int_0^2 \left(x^2-2\right)dx = \frac{1}{3}x^3 - 2x \Big|_0^2$$

$$= \frac{1}{3}(2)^3 - 2(2) - 0$$

$$= \frac{8}{3} - 4$$

$$\boxed{= -\frac{4}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/2} \cos x\ dx$

Solution

$$\int_0^{\pi/2} \cos x\ dx = \sin x \Big|_0^{\pi/2}$$

$$= \sin\frac{\pi}{2} - \sin 0$$

$$\boxed{=1}$$

Exercise

Evaluate the integral $\displaystyle\int_1^7 \frac{dx}{x} =$

Solution

$$\int_1^7 \frac{dx}{x} = \ln|x| \Big|_1^7$$

$$= \ln 7 - \ln 1$$

$$\boxed{= \ln 7}$$

Exercise

Evaluate the integral $\displaystyle\int_4^9 3\sqrt{x}\,dx$

Solution

$$\int_4^9 3\sqrt{x}\,dx = 2x^{3/2}\Big|_4^9$$

$$= 2\left((9)^{3/2} - (4)^{3/2}\right)$$

$$= 2(27-8)$$

$$\underline{= 38}$$

Exercise

Evaluate the integral $\displaystyle\int_{-2}^3 \left(x^2 - x - 6\right)dx$

Solution

$$\int_{-2}^3 \left(x^2 - x - 6\right)dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x\Big|_{-2}^3$$

$$= 9 - \frac{9}{2} - 18 - \left(\frac{8}{3} - 2 - 12\right)$$

$$= -\frac{27}{2} + \frac{46}{3}$$

$$\underline{= \frac{11}{6}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 \left(1 - \sqrt{x}\right)dx$

Solution

$$\int_0^1 \left(1 - \sqrt{x}\right)dx = \int_0^1 \left(1 - x^{1/2}\right)dx$$

$$= x - \frac{2}{3}x^{3/2}\Big|_0^1$$

$$= 1 - \frac{2}{3}$$

$$\underline{= \frac{1}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/4} 2\cos x\, dx$

Solution

$$\int_0^{\pi/4} 2\cos x\, dx = 2\sin x \Big|_0^{\pi/4}$$

$$= 2\left(\sin\frac{\pi}{4} - \sin 0\right)$$

$$= \sqrt{2}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi/4}^{7\pi/4} (\sin x + \cos x)\, dx$

Solution

$$\int_{-\pi/4}^{7\pi/4} (\sin x + \cos x)\, dx = -\cos x + \sin x \Big|_{-\pi/4}^{7\pi/4}$$

$$= -\cos\left(-\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right) - \left(-\cos\left(\frac{7\pi}{4}\right) + \sin\left(\frac{7\pi}{4}\right)\right)$$

$$= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$$

$$= -\sqrt{2} + \sqrt{2}$$

$$= 0$$

or since $-\frac{\pi}{4} = \frac{7\pi}{4}$ $\displaystyle\int_a^a f(x)\,dx = 0$

Exercise

Evaluate the integral $\displaystyle\int_0^{\ln 8} e^x\, dx$

Solution

$$\int_0^{\ln 8} e^x\, dx = e^x \Big|_0^{\ln 8}$$

$$= e^{\ln 8} - e^0$$

$$= 8 - 1$$

$$= 7$$

Exercise

Evaluate the integral $\displaystyle\int_1^4 \left(\frac{x-1}{x}\right)dx$

Solution

$$\int_1^4 \left(\frac{x-1}{x}\right)dx = \int_1^4 \left(1-\frac{1}{x}\right)dx$$

$$= x - \ln|x| \Big|_1^4$$

$$= 4 - \ln 4 - (1 - \ln 1)$$

$$= 4 - \ln 2^2 - 1$$

$$\boxed{= 3 - 2\ln 2}$$

Exercise

Evaluate the integral $\displaystyle\int_{-2}^{-1} \left(3e^{3x} + \frac{2}{x}\right)dx$

Solution

$$\int_{-2}^{-1} \left(3e^{3x} + \frac{2}{x}\right)dx = e^{3x} + 2\ln|x| \Big|_{-2}^{-1}$$

$$= e^{-3} + 2\ln|-1| - \left(e^{-6} - 2\ln|-2|\right)$$

$$= e^{-3} + 2\ln 1 - e^{-6} + 2\ln 2$$

$$\boxed{= e^{-3} - e^{-6} + 2\ln 2}$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 \frac{dx}{x^2+4}$

Solution

$$\int_0^2 \frac{dx}{x^2+4} = \frac{1}{2}\tan^{-1}\frac{x}{2} \Big|_0^2$$

$$= \frac{1}{2}\left(\tan^{-1}1 - \tan^{-1}0\right)$$

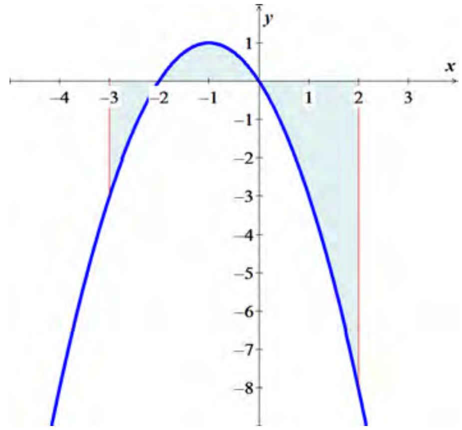
$$= \frac{1}{2}\left(\frac{\pi}{4} - 0\right)$$

$$\boxed{= \frac{\pi}{8}}$$

Exercise

Find the total area between the region and the x-axis $\quad y=-x^2-2x, \quad -3\le x\le 2$

Solution



$$-x^2-2x=0$$

$$-x(x+2)=0$$

$$\boxed{x=-2,\ 0}$$

$$A=-\int_{-3}^{-2}\left(-x^2-2x\right)dx+\int_{-2}^{0}\left(-x^2-2x\right)dx-\int_{0}^{2}\left(-x^2-2x\right)dx$$

$$=-\left(-\frac{1}{3}x^3-x^2\right)\Big|_{-3}^{-2}+\left(-\frac{1}{3}x^3-x^2\right)\Big|_{-2}^{0}-\left(-\frac{1}{3}x^3-x^2\right)\Big|_{0}^{2}$$

$$=-\left[\left(-\tfrac{1}{3}(-2)^3-(-2)^2\right)-\left(-\tfrac{1}{3}(-3)^3-(-3)^2\right)\right]+\left[-\left(-\tfrac{1}{3}(-2)^3-(-2)^2\right)\right]-\left[\left(-\tfrac{1}{3}(2)^3-(2)^2\right)\right]$$

$$=\frac{4}{3}+\frac{4}{3}+\frac{20}{3}$$

$$\boxed{=\frac{28}{3}\ \ unit^2}$$

Exercise

Find the total area between the region and the x-axis $y=x^3-3x^2+2x, \quad 0\le x\le 2$

Solution

$$x^3-3x^2+2x=0$$

$$\boxed{x=-2,\ 0}$$

$$A=\int_{0}^{1}\left(x^3-3x^2+2x\right)dx-\int_{1}^{2}\left(x^3-3x^2+2x\right)dx$$

$$=\left(\frac{1}{4}x^4-x^3+x^2\right|_{0}^{1}-\left(\frac{1}{4}x^4-x^3+x^2\right|_{1}^{2}$$

$$=\frac{1}{4}+\frac{1}{4}$$

$$\boxed{=\frac{1}{2}\ \ unit^2}$$

Exercise

Find the total area between the region and the x-axis $y = x^{1/3} - x, \quad -1 \le x \le 8$

Solution

$$x^{1/3} - x = 0$$
$$x^{1/3}\left(1 - x^{2/3}\right) = 0$$
$$\boxed{x = 0, \ \pm 1}$$

$$A = -\int_{-1}^{0}\left(x^{1/3} - x\right)dx + \int_{0}^{1}\left(x^{1/3} - x\right)dx - \int_{1}^{8}\left(x^{1/3} - x\right)dx$$
$$= -\left(\frac{3}{4}x^{4/3} - \frac{1}{2}x^2\right.\Big|_{-1}^{0} + \left(\frac{3}{4}x^{4/3} - \frac{1}{2}x^2\right.\Big|_{0}^{1} - \left(\frac{3}{4}x^{4/3} - \frac{1}{2}x^2\right.\Big|_{1}^{8}$$
$$= \left(\frac{3}{4} - \frac{1}{2}\right) + \left(\frac{3}{4} - \frac{1}{2}\right) - \left[(12 - 32) - \left(\frac{3}{4} - \frac{1}{2}\right)\right]$$
$$= \frac{1}{4} + \frac{1}{4} + \frac{81}{4}$$
$$\boxed{= \frac{83}{4} \ unit^2}$$

Exercise

Find the total area between the region and the x-axis $f(x) = x^2 + 1, \qquad 2 \le x \le 3$

Solution

$$Area = \int_{2}^{3}\left(x^2 + 1\right)dx$$
$$= \frac{1}{3}x^3 + x \ \Big|_{2}^{3}$$
$$= \left(\frac{1}{3}3^3 + 3\right) - \left(\frac{1}{3}2^3 + 2\right)$$
$$= (9 + 3) - \left(\frac{8}{3} + 2\right)$$
$$= 12 - \left(\frac{14}{3}\right)$$
$$\boxed{= \frac{22}{3} \ unit^2}$$

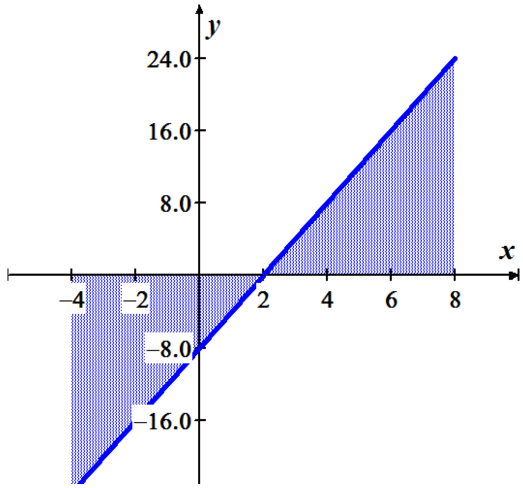
Exercise

Find the area of the region between the graph of $y = 4x - 8$ and the x-*axis*, for $-4 \le x \le 8$

Solution

$$y = 4x - 8 = 0$$

$$\underline{x = 2}$$



$$Area = -\int_{-4}^{2}(4x-8)\,dx + \int_{2}^{8}(4x-8)\,dx$$

$$= -\left(2x^2 - 8x \right|_{-4}^{2} + \left(2x^2 - 8x \right|_{2}^{8}$$

$$= -\big(8 - 16 - (32 + 32)\big) + \big(128 - 64 - (8 - 16)\big)$$

$$= 72 + 72$$

$$\underline{= 144 \ \ unit^2}$$

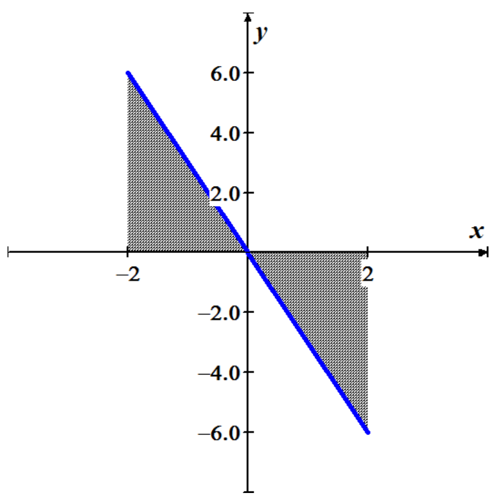
Exercise

Find the area of the region between the graph of $y = -3x$ and the x-*axis*, for $-2 \le x \le 2$

Solution

$$y = -3x = 0$$

$$\underline{x = 0}$$



$$Area = \int_{-2}^{0}(-3x)\,dx + \int_{0}^{2}(-3x)\,dx$$

$$= 2\int_{-2}^{0}(-3x)\,dx$$

$$= 3\left(-x^2 \right|_{-2}^{0}$$

$$= 3(0 + 4)$$

$$\underline{= 12 \ \ unit^2}$$

Exercise

Find the area of the region between the graph of $y = 3x + 6$ and the x-*axis*, for $0 \le x \le 6$

Solution

$y = 3x + 6 = 0$

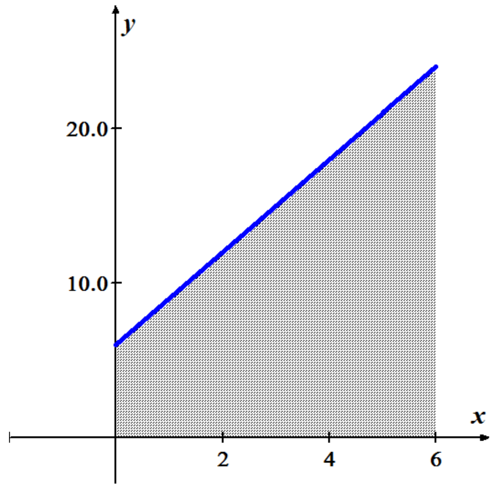
$\underline{x = -2}$

$$Area = \int_0^6 (3x+6)\,dx$$

$$= \frac{3}{2}x^2 + 6x \Big|_0^6$$

$$= \frac{3}{2}(36) + 36 - 0$$

$$\underline{= 90 \ \ unit^2}$$



Exercise

Find the area of the region between the graph of $y = 1 - |x|$ and the x-*axis*, for $-2 \le x \le 2$

Solution

$y = 1 - x = 0$

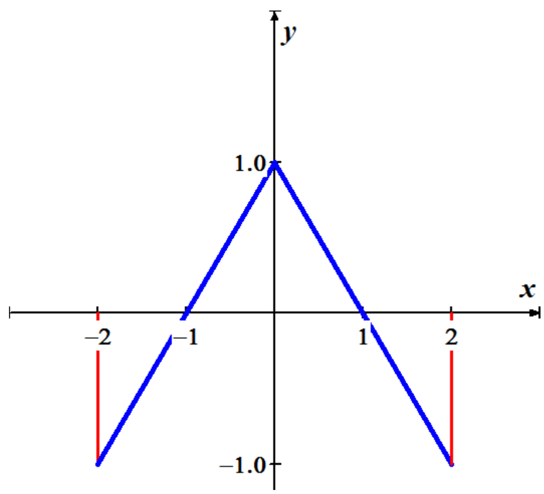
$\underline{x = 1}$

$$Area = 2\int_0^1 (1-x)\,dx - 2\int_1^2 (1-x)\,dx$$

$$= 2\left(x - \frac{1}{2}x^2\right)\Big|_0^1 - 2\left(x - \frac{1}{2}x^2\right)\Big|_1^2$$

$$= 4\left(1 - \frac{1}{2}\right)$$

$$\underline{= 2 \ \ unit^2}$$



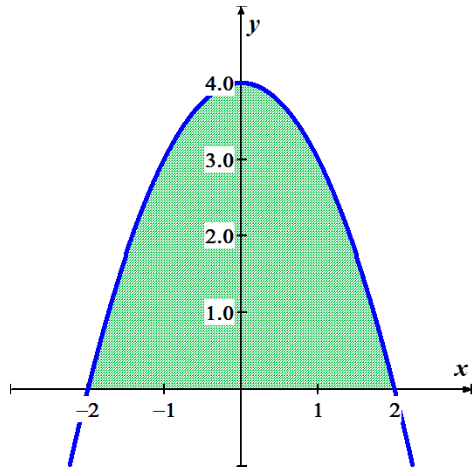
Exercise

Find the area of the region above the *x-axis* bounded by $y = 4 - x^2$

Solution

$$y = 4 - x^2 = 0$$

$$\underline{x = \pm 2}$$



$$\begin{aligned} Area &= \int_{-2}^{2} \left(4 - x^2\right) dx \\ &= 4x - \frac{1}{3}x^3 \Big|_{-2}^{2} \\ &= 8 - \frac{8}{3} - \left(-8 + \frac{8}{3}\right) \\ &= 2\left(\frac{16}{3}\right) \\ &= \frac{32}{3} \ unit^2 \end{aligned}$$

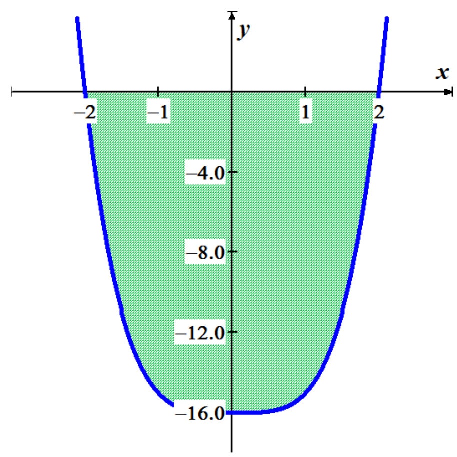
Exercise

Find the area of the region above the *x-axis* bounded by $y = x^4 - 16$

Solution

$$y = x^4 - 16 = 0$$

$$\underline{x = \pm 2}$$



$$\begin{aligned} Area &= -\int_{2}^{2} \left(x^4 - 16\right) dx \\ &= -\frac{1}{5}x^5 + 16x \Big|_{-2}^{2} \\ &= -\frac{32}{5} + 32 - \left(\frac{32}{5} - 32\right) \\ &= \frac{256}{5} \ unit^2 \end{aligned}$$

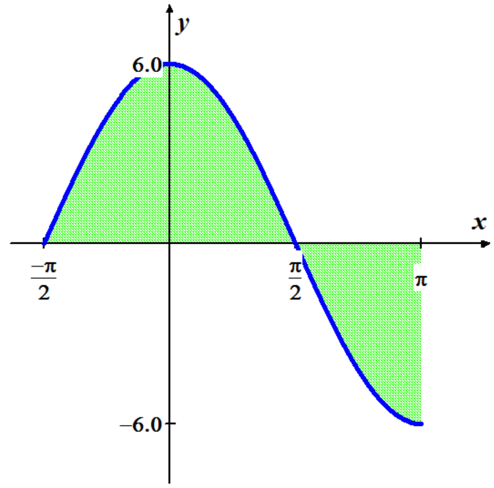
Exercise

Find the area of the region between the graph of $y = 6\cos x$ and the *x-axis*, for $-\frac{\pi}{2} \le x \le \pi$

Solution

$y = 6\cos x = 0$

$x = \pm\frac{\pi}{2}$



$$Area = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (6\cos x)\,dx + \int_{\frac{\pi}{2}}^{\pi} (-6\cos x)\,dx$$

$$= 6\sin x \Big|_{-\pi/2}^{\pi/2} - 6\sin x \Big|_{\pi/2}^{\pi}$$

$$= 6\left(\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right) - 6\left(\sin\pi - \sin\left(\frac{\pi}{2}\right)\right)$$

$$= 6(1+1) - 6(0-1)$$

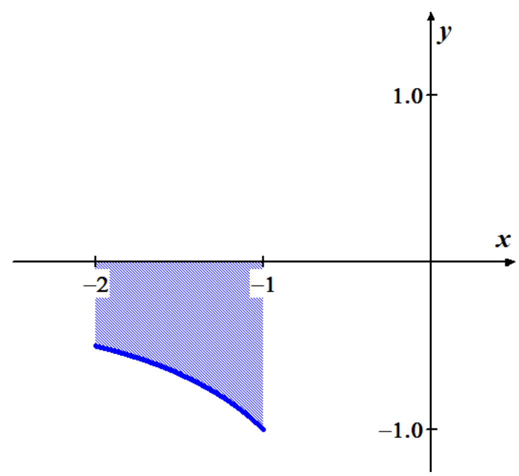
$$= 18 \ \ unit^2$$

Exercise

Find the area of the region between the graph of $f(x) = \frac{1}{x}$ and the *x-axis*, for $-2 \le x \le -1$

Solution

$$Area = -\int_{-2}^{-1} \frac{1}{x}\,dx$$



$$= -\ln|x| \Big|_{-2}^{-1}$$

$$= -\ln|-1| + \ln|-2|$$

$$= -\ln 1 + \ln 2$$

$$= \ln 2 \ \ unit^2$$

Exercise

Find the area of the region bounded by the graph of

$f(x) = x^2 - 4x + 3 \quad x\text{-}axis \ \ on \ 0 \le x \le 3$

Solution

$f(x) = x^2 - 4x + 3 = 0$

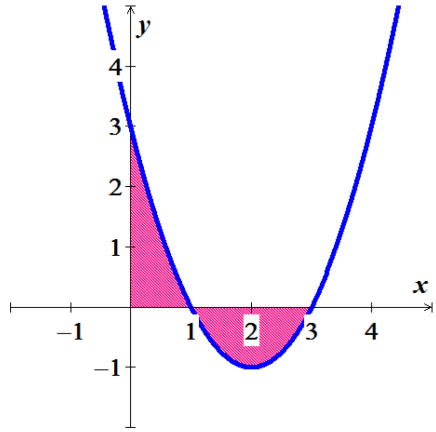
$x = 1, \ 3$

$$A=\int_0^1\left(x^2-4x+3\right)dx-\int_1^3\left(x^2-4x+3\right)dx$$

$$=\left(\frac{1}{3}x^3-2x^2+3x\,\right|_0^1 - \left(\frac{1}{3}x^3-2x^2+3x\,\right|_1^3$$

$$=\frac{1}{3}-2+3-\left(9-18+9-\frac{1}{3}+2-3\right)$$

$$=\frac{8}{3}\ unit^2$$



Exercise

Find the area of the region bounded by the graph of $f(x)=x^2+4x+3$ *x-axis on* $-3\le x\le 0$

Solution

$$f(x)=x^2+4x+3=0$$

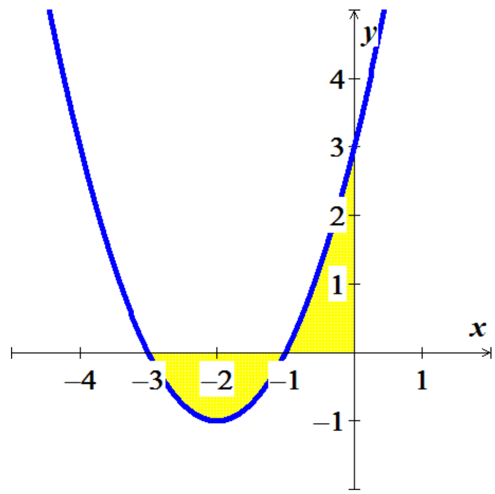
$$x=-1,\ -3$$

$$A=-\int_{-3}^{-1}\left(x^2+4x+3\right)dx+\int_{-1}^{0}\left(x^2+4x+3\right)dx$$

$$=-\left(\frac{1}{3}x^3+2x^2+3x\,\right|_{-3}^{-1} + \left(\frac{1}{3}x^3+2x^2+3x\,\right|_{-1}^{0}$$

$$=-\left(-\frac{1}{3}+2-3+9-18+9\right)+\frac{1}{3}-2+3$$

$$=\frac{8}{3}\ unit^2$$



Exercise

Find the area of the region bounded by the graph of $f(x)=x^2-3x+2$ *x-axis on* $0\le x\le 2$

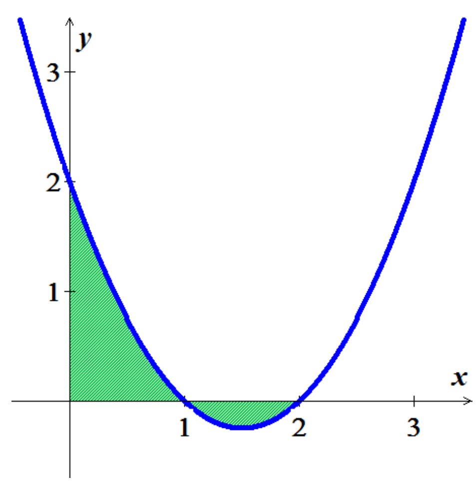
Solution

$$f(x)=x^2-3x+2=0$$

$$x=1,\ 2$$

$$A=\int_0^1\left(x^2-3x+2\right)dx-\int_1^2\left(x^2-3x+2\right)dx$$

$$=\frac{1}{3}x^3-\frac{3}{2}x^2+2x\,\Big|_0^1 - \left(\frac{1}{3}x^3-\frac{3}{2}x^2+2x\,\right|_1^2$$



$$=\frac{1}{3}-\frac{3}{2}+2-\left(\frac{8}{3}-6+4-\frac{1}{3}+\frac{3}{2}-2\right)$$

$$=-\frac{7}{6}+2-\left(\frac{8}{3}-2+\frac{7}{6}-2\right)$$

$$=1\ unit^2$$

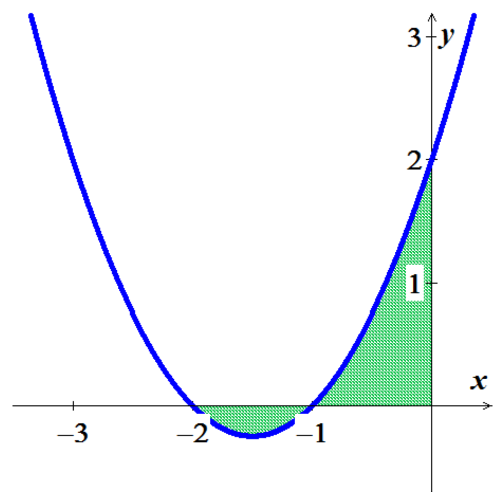
Exercise

Find the area of the region bounded by the graph of $f(x)=x^2+3x+2$ *x-axis on* $-2\le x\le 0$

Solution

$$f(x)=x^2+3x+2=0$$

$$x=-1,\ -2$$



$$A=-\int_{-2}^{-1}\left(x^2+3x+2\right)dx+\int_{-1}^{0}\left(x^2+3x+2\right)dx$$

$$=-\left(\frac{1}{3}x^3+\frac{3}{2}x^2+2x\ \right|_{-2}^{-1}+\left(\frac{1}{3}x^3+\frac{3}{2}x^2+2x\ \right|_{-1}^{0}$$

$$=-\left(\frac{8}{3}-6+4-\frac{1}{3}+\frac{3}{2}-2\right)+\frac{1}{3}-\frac{3}{2}+2$$

$$=1\ unit^2$$

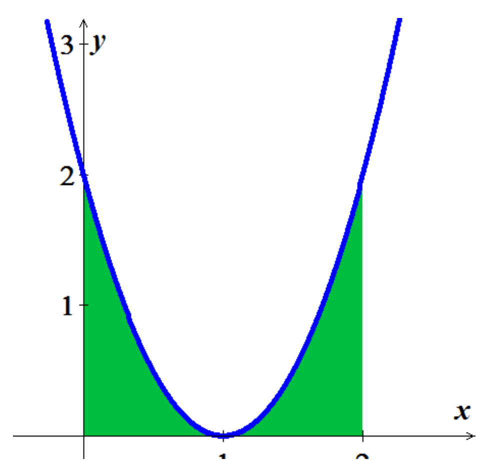
Exercise

Find the area of the region bounded by the graph of $f(x)=2x^2-4x+2$ *x-axis on* $0\le x\le 2$

Solution

$$f(x)=2x^2-4x+2=0$$

$$x=1$$



$$A=\int_{0}^{1}\left(2x^2-4x+2\right)dx+\int_{1}^{2}\left(2x^2-4x+2\right)dx$$

$$=\left(\frac{2}{3}x^3-2x^2+2x\ \right|_{0}^{1}+\left(\frac{2}{3}x^3-2x^2+2x\ \right|_{1}^{2}$$

$$=\frac{2}{3}-2+2+\frac{16}{3}-8+4-\frac{2}{3}+2-2$$

$$=\frac{4}{3}\ unit^2$$

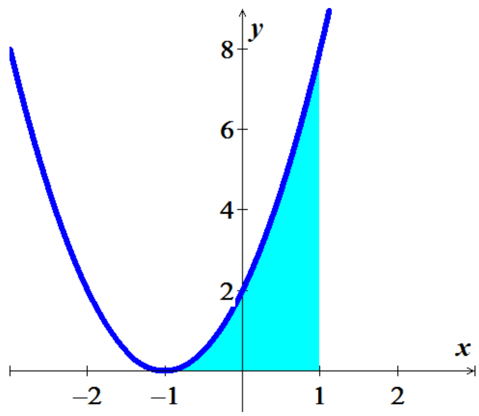
Exercise

Find the area of the region bounded by the graph of $f(x)=2x^2+4x+2$ *x-axis on* $-1\le x\le 1$

Solution

$$f(x)=2x^2+4x+2=0$$

$$\boxed{x=-1}$$



$$A=\int_{-1}^{1}\left(2x^2+4x+2\right)dx$$

$$=\frac{2}{3}x^3+2x^2+2x\Big|_{-1}^{1}$$

$$=\frac{2}{3}+2+2+\frac{2}{3}-2+2$$

$$\boxed{=\frac{16}{3}\ unit^2}$$

Exercise

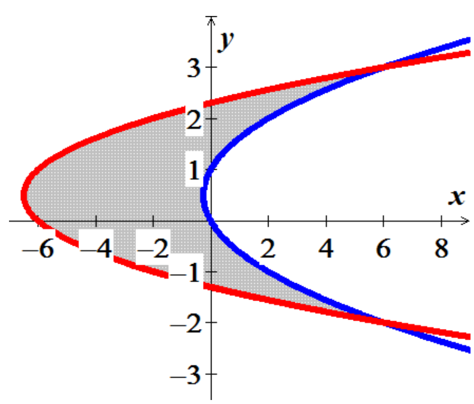
Find the area of the region bounded by the graphs of $x=y^2-y$ and $x=2y^2-2y-6$

Solution

$$x=2y^2-2y-6=y^2-y$$

$$y^2-y-6=0$$

$$\boxed{y=-2,\ 3}$$



$$A=\int_{-2}^{3}\left(y^2-y-2y^2+2y+6\right)dy$$

$$=\int_{-2}^{3}\left(-y^2+y+6\right)dy$$

$$=-\frac{1}{3}y^3+\frac{1}{2}y^2+6y\Big|_{-2}^{3}$$

$$=-9+\frac{9}{2}+18-\frac{8}{3}-2+12$$

$$=19+\frac{11}{6}$$

$$\boxed{=\frac{125}{6}\ unit^2}$$

Exercise

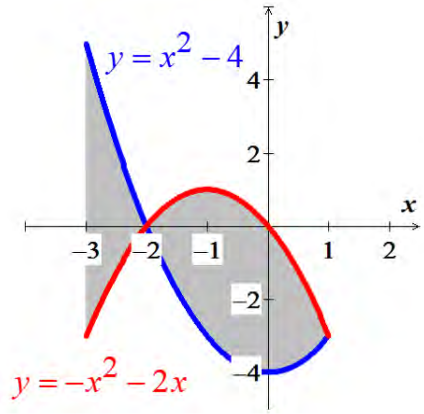
Find the area of the region bounded by the graphs of $y = x^2 - 4$ & $y = -x^2 - 2x$ on $-3 \le x \le 1$

Solution

$$y = x^2 - 4 = -x^2 - 2x$$

$$2x^2 + 2x - 4 = 0$$

$$\boxed{x = 1, \ -2}$$



$$A = \int_{-3}^{-2} \left(x^2 - 4 + x^2 + 2x\right) dx + \int_{-2}^{1} \left(-x^2 - 2x - x^2 + 4\right) dx$$

$$= \int_{-3}^{-2} \left(2x^2 + 2x - 4\right) dx + \int_{-2}^{1} \left(-2x^2 - 2x + 4\right) dx$$

$$= \left(\frac{2}{3}x^3 + x^2 - 4x \right|_{-3}^{-2} + \left(-\frac{2}{3}x^3 - x^2 + 4x \right|_{-2}^{1}$$

$$= -\frac{16}{3} + 12 + 18 - 21 + \left(-\frac{2}{3} + 3\right) - \frac{16}{3} + 12$$

$$= -\frac{34}{3} + 24$$

$$\boxed{= \frac{38}{3} \ \ unit^2}$$

Exercise

Compute the area of the region bounded by the graph of f and the x-*axis* on the given interval.

$$f(x) = \frac{1}{x^2 + 1} \quad on \quad \left[-1, \sqrt{3}\right]$$

Solution

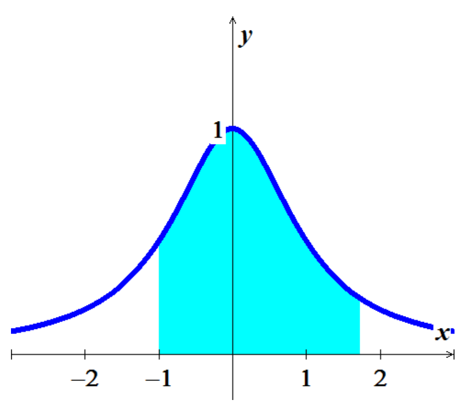
$$A = \int_{-1}^{\sqrt{3}} \frac{1}{x^2 + 1} dx \qquad\qquad \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$= \tan^{-1} x \Big|_{-1}^{\sqrt{3}}$$

$$= \tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(1)$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$\boxed{= \frac{\pi}{12} \ \ unit^2}$$

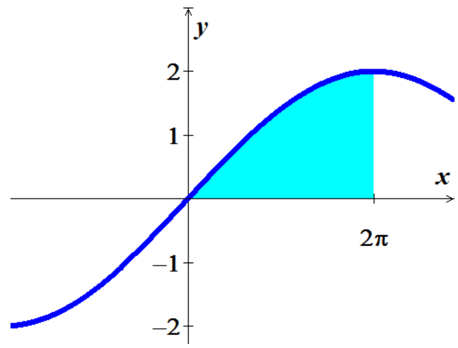


Exercise

Compute the area of the region bounded by the graph of f and the x-*axis* on the given interval.

$$f(x) = 2\sin\frac{x}{4} \quad on \quad [0,\ 2\pi]$$

Solution



$$A = \int_0^{2\pi} 2\sin\frac{x}{4}\, dx$$

$$= -8\cos\frac{x}{4}\Big|_0^{2\pi}$$

$$= -8(0-1)$$

$$\boxed{= 8 \ \ unit^2}$$

Exercise

Archimedes, inventor, military engineer, physicist, and the greatest mathematician of classical times in the Western world, discovered that the area under a parabolic arch is two-thirds the base times the height. Sketch the parabolic arch $y = h - \left(\frac{4h}{b^2}\right)x^2 \quad -\frac{b}{2} \le x \le \frac{b}{2}$, assuming that h and b are positive. Then use calculus to find the area of the region enclosed between the arch and the x-axis

Solution

$$A = \int_{-b/2}^{b/2} \left(h - \left(\frac{4h}{b^2}\right)x^2\right) dx$$

$$= hx - \frac{4h}{b^2}\frac{x^3}{3}\Big|_{-b/2}^{b/2}$$

$$= \left(\frac{hb}{2} - \frac{4h}{3b^2}\frac{b^3}{8}\right) - \left(-\frac{hb}{2} + \frac{4h}{3b^2}\frac{b^3}{8}\right)$$

$$= \left(\frac{hb}{2} - \frac{hb}{6}\right) - \left(-\frac{hb}{2} + \frac{hb}{6}\right)$$

$$= \frac{hb}{3} + \frac{hb}{3}$$

$$\boxed{= \frac{2}{3}bh \quad unit^2}$$

Exercise

Suppose that a company's marginal revenue from the manufacture and sale of eggbeaters is

$$\frac{dr}{dx} = 2 - \frac{2}{(x+1)^2}$$

Where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand eggbeaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

Solution

$$r = \int_0^3 \left(2 - 2(x+1)^{-2}\right) dx \qquad d(x+1) = dx$$

$$= \int_0^3 2dx - \int_0^3 2(x+1)^{-2}\, d(x+1)$$

$$= 2x + 2(x+1)^{-1} \Big|_0^3$$

$$= 6 + 2(4)^{-1} - 2$$

$$= 4.5$$

$\$4500.00$

Exercise

The height H (*feet*) of a palm tree after growing for t years is given by

$$H = \sqrt{t+1} + 5t^{1/3} \quad \text{for} \quad 0 \le t \le 8$$

a) Find the tree's height when $t = 0$, $t = 4$, and $t = 8$.

b) Find the tree's average height for $0 \le t \le 8$

Solution

a) $t = 0 \;\Rightarrow\; H = 1\ ft$

$t = 4 \;\Rightarrow\; H = 10.17\ ft$

$t = 8 \;\Rightarrow\; H = 13\ ft$

b) Average height $= \frac{1}{8-0} \int_0^8 \left(\sqrt{t+1} + 5t^{1/3}\right) dt \qquad d(t+1) = dt$

$$= \frac{1}{8} \int_0^8 (t+1)^{1/2}\, d(t+1) + \frac{5}{8} \int_0^8 t^{1/3} dt$$

$$= \frac{1}{12}(t+1)^{3/2} + \frac{15}{32}t^{4/3} \Big|_0^8$$

$$= \frac{1}{12}(9)^{3/2} + \frac{15}{32}(8)^{4/3} - \frac{1}{12}$$

$$= \frac{27}{12} + \frac{15}{2} - \frac{1}{12}$$

$$\boxed{= \frac{29}{3} \quad ft}$$

$$\boxed{\approx 9.67 \quad ft}$$

Solution Section 4.5 – Working with Integrals

Exercise

If f is an odd function, why is $\int_{-a}^{a} f(x)\,dx = 0$?

Solution

If $f(x)$ is an odd function then it is symmetric about the origin, which the region between $-a$ and a, there is as much area above the axis and under f as there is below the axis and above f.
Therefore, the net area must be 0.

Exercise

If f is an even function, why is $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

Solution

If f is an even function then it is symmetric about the y-axis, which the region that between $-a$ and 0 has the same net area as the region between 0 and a.

So $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

$$= 2\int_{0}^{a} f(x)\,dx$$

Exercise

Is x^{12} an even or odd function? Is $\sin\left(x^2\right)$ an even or odd function?

Solution

$f(x) = x^{12}$

$$f(-x) = (-x)^{12}$$
$$= x^{12}$$
$$= x^{12} = f(x)$$

Therefore; $f(x)$ is an *even* function.

$g(x) = \sin\left(x^2\right)$

$$g(-x) = \sin\left((-x)^2\right)$$
$$= \sin\left(x^2\right)$$
$$= g(x)$$

Therefore; $g(x)$ is also an even function.

Exercise

Use symmetry to evaluate the following integrals $\int_{-2}^{2} x^9\, dx$

Solution

Because x^9 is an *odd* function, then

$$\boxed{\int_{-2}^{2} x^9 dx = 0}$$

Exercise

Use symmetry to evaluate the following integrals $\int_{-200}^{200} 2x^5\, dx$

Solution

Because $2x^5$ is an *odd* function, then

$$\boxed{\int_{-200}^{200} 2x^5\, dx = 0}$$

Exercise

Use symmetry to evaluate the following integrals $\int_{-\pi/4}^{\pi/4} \cos x\, dx$

Solution

Because $\cos x$ is an even function, then

$$\int_{-\pi/4}^{\pi/4} \cos x\, dx = 2\int_{0}^{\pi/4} \cos x\, dx$$
$$= 2\sin x \Big|_{0}^{\pi/4}$$

$$= 2\left(\frac{\sqrt{2}}{2}\right)$$

$$\underline{= \sqrt{2}}$$

Exercise

Use symmetry to evaluate the following integrals $\displaystyle\int_{-2}^{2}\left(x^9 - 3x^5 + 2x^2 - 10\right)dx$

Solution

$$\int_{-2}^{2}\left(x^9 - 3x^5 + 2x^2 - 10\right)dx = \underbrace{\int_{-2}^{2}\left(x^9 - 3x^5\right)dx}_{\textbf{Odd}} + \underbrace{\int_{-2}^{2}\left(2x^2 - 10\right)dx}_{\textbf{Even}}$$

$$= 0 + 2\int_{0}^{2}\left(2x^2 - 10\right)dx$$

$$= 2\left(\frac{2}{3}x^3 - 10x\right)\Big|_0^2$$

$$= 2\left(\frac{16}{3} - 20\right)$$

$$\underline{= -\frac{88}{3}}$$

Exercise

Use symmetry to evaluate the following integrals $\displaystyle\int_{-\pi/2}^{\pi/2}\left(\cos 2x + \cos x \sin x - 3\sin x^5\right)\, dx$

Solution

$$\int_{-\pi/2}^{\pi/2}\left(\cos 2x + \cos x \sin x - 3\sin x^5\right)\, dx = \underbrace{\int_{-\pi/2}^{\pi/2}\cos 2x\, dx}_{\textbf{Even}} + \underbrace{\int_{-\pi/2}^{\pi/2}\left(\cos x \sin x - 3\sin x^5\right)dx}_{\textbf{Odd}}$$

$$= 2\int_{0}^{\pi/2}\cos 2x\, dx$$

$$= \sin 2x\Big|_0^{\pi/2}$$

$$\underline{= 0}$$

Exercise

Find the average value of the following functions on the given interval. $f(x) = x^3 \quad on \ [-1, 1]$

Solution

$$\text{Average value } = \frac{1}{1-(-1)} \int_{-1}^{1} x^3 \, dx$$

$$= \frac{1}{2}\frac{1}{4}x^4 \Big|_{-1}^{1}$$

$$\boxed{= 0}$$

Exercise

Find the average value of the following functions on the given interval. $f(x) = \frac{1}{x^2+1} \quad on \ [-1, 1]$

Solution

$$\text{Average value } = \frac{1}{1-(-1)} \int_{-1}^{1} \frac{1}{x^2+1} \, dx$$

$$= \frac{1}{2}\tan^{-1} x \Big|_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$\boxed{= \frac{\pi}{4}}$$

Exercise

Find the average value of the following functions on the given interval. $f(x) = \frac{1}{x} \quad on \ [1, e]$

Solution

$$\text{Average value } = \frac{1}{e-1} \int_{1}^{2} \frac{1}{x} \, dx$$

$$= \frac{1}{e-1} \ \left(\ln|x| \Big|_{1}^{e}\right.$$

$$= \frac{1}{e-1}(\ln e - 0)$$

$$\boxed{= \frac{1}{e-1}}$$

Exercise

Find the average value of the following functions on the given interval. $f(x)=e^{2x}$ *on* $[0,\ \ln 2]$

Solution

$$\text{Average value } = \frac{1}{\ln 2}\int_0^{\ln 2} e^{2x}\,dx$$

$$= \frac{1}{2}\ \frac{1}{\ln 2}e^{2x}\ \Big|_0^{\ln 2}$$

$$= \frac{1}{2\ln 2}\left(e^{2\ln 2}-1\right)$$

$$= \frac{1}{2\ln 2}(4-1)$$

$$= \frac{3}{2\ln 2}$$

Exercise

Suppose that $\int_0^4 f(x)\,dx = 10$ and $\int_0^4 g(x)\,dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_{-4}^{4} f(x)\,dx$$

Solution

f is an even function.

$$\int_{-4}^{4} f(x)\,dx = 2\int_0^4 f(x)\,dx$$

$$= 2(10)$$

$$= 20$$

Exercise

Suppose that $\int_0^4 f(x)dx = 10$ and $\int_0^4 g(x)dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_{-4}^{4} 3g(x)dx$$

Solution

g is an odd function

$$\int_0^4 g(x)dx = -\int_{-4}^{0} g(x)dx$$

$$\begin{aligned}\int_{-4}^{4} 3g(x)dx &= 3\int_{-4}^{0} g(x)dx + 3\int_0^4 g(x)dx \\ &= 3\int_{-4}^{0} g(x)dx - 3\int_{-4}^{0} g(x)dx \\ &= 0\end{aligned}$$

Exercise

Suppose that $\int_0^4 f(x)dx = 10$ and $\int_0^4 g(x)dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_0^1 8xf\left(4x^2\right)dx$$

Solution

$$\int_0^1 8xf\left(4x^2\right)dx = \int_0^1 f\left(4x^2\right)d\left(4x^2\right) \qquad d\left(4x^2\right) = 8xdx$$

$$\begin{cases} x = 1 & \rightarrow \ 4x^2 = 4 \\ x = 0 & \rightarrow \ 4x^2 = 0 \end{cases}$$

$$\begin{aligned}\int_0^1 8xf\left(4x^2\right)dx &= \int_0^4 f\left(4x^2\right)d\left(4x^2\right) \\ &= 10\end{aligned}$$

Exercise

Suppose that $\int_0^4 f(x)\,dx = 10$ and $\int_0^4 g(x)\,dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_{-2}^{2} 3x f(x)\,dx$$

Solution

f is an even function, that implies $xf(x)$ is an odd function.

$$\int_{-2}^{2} 3x f(x)\,dx = 0$$

Exercise

Suppose that $\int_0^4 f(x)\,dx = 10$ and $\int_0^4 g(x)\,dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_{-4}^{4} \left(4f(x) - 3g(x)\right)dx$$

Solution

f is an even function and g is an odd function.

$$\begin{aligned}\int_{-4}^{4} \left(4f(x) - 3g(x)\right)dx &= 4\int_{-4}^{4} f(x)\,dx - 3\int_{-4}^{4} g(x)\,dx \\ &= 8\int_{0}^{4} f(x)\,dx - 3(0) \\ &= 80\end{aligned}$$

Exercise

Suppose that f is an even function with $\int_0^8 f(x)\,dx = 9$. Evaluate the integral $\int_{-1}^{1} x f\left(x^2\right)dx$

Solution

f is an even function, that implies $xf(x)$ is an odd function.

$$\boxed{\int_{-1}^{1} x\, f\left(x^2\right)dx = 0}$$

Exercise

Suppose that f is an even function with $\int_0^8 f(x)\,dx = 9$. Evaluate the integral $\int_{-2}^{2} x^2 f\left(x^3\right)dx$

Solution

$$d\left(x^3\right) = 3x^2 dx$$

$$\begin{cases} x = 2 & \rightarrow\ x^3 = 8 \\ x = -2 & \rightarrow\ x^3 = -8 \end{cases}$$

$$\begin{aligned}
\int_{-2}^{2} x^2 f\left(x^3\right)dx &= \frac{1}{3}\int_{-8}^{8} f\left(x^3\right)d\left(x^3\right) \\
&= \frac{1}{3}\int_{-8}^{0} f\left(x^3\right)d\left(x^3\right) + \frac{1}{3}\int_{0}^{8} f\left(x^3\right)d\left(x^3\right) \qquad f \text{ is an } \textit{even} \text{ function} \\
&= \frac{2}{3}\int_{0}^{8} f\left(x^3\right)d\left(x^3\right) \\
&= \frac{2}{3}\cdot 9 \\
&\boxed{= 6}
\end{aligned}$$

Exercise

Suppose that p is a nonzero real number and f is an odd integrable function with $\int_0^1 f(x)\,dx = \pi$.

Evaluate the integral $\int_0^{\frac{\pi}{2p}} (\cos px)\, f(\sin px)\,dx$

Solution

$$\begin{cases} x = \frac{\pi}{2p} & \rightarrow\ \sin px = 1 \\ x = 0 & \rightarrow\ \sin px = 0 \end{cases}$$

$$\int_0^{\frac{\pi}{2p}} (\cos px)\, f(\sin px)\,dx = \frac{1}{p}\int_0^1 f(\sin px)\ d(\sin px) \qquad d(\sin px) = p\cos px\ dx$$

$$= \frac{\pi}{p}$$

Exercise

Suppose that p is a nonzero real number and f is an odd integrable function with $\int_0^1 f(x)\,dx = \pi$.

Evaluate the integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)\, f(\sin x)\,dx$

Solution

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos px)\, f(\sin px)\,dx = \frac{1}{p}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\sin px)\; d(\sin px) \qquad f \text{ is an odd function}$$

$$= 0$$

Exercise

The Gateway Arch in St. Louis is 630 *feet* high and has a 630-*ft* base. Its shape can be modeled by the parabola

$$y = 630\left(1-\left(\frac{x}{315}\right)^2\right)$$

Find the average height of the arch above the ground.

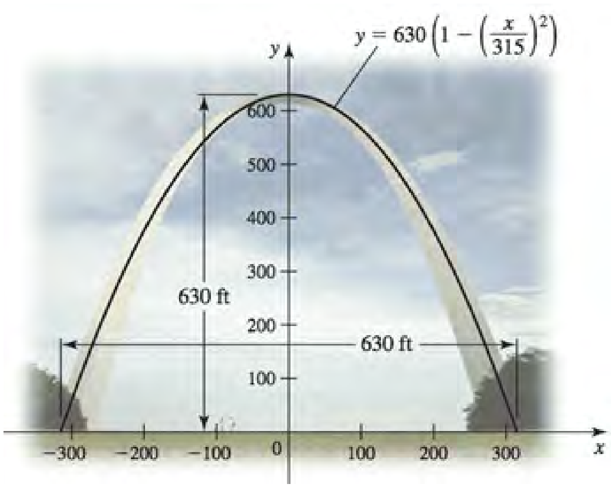
Solution

$$\text{Average height } = \frac{1}{630}\int_{-315}^{315} 630\left(1-\frac{1}{315^2}x^2\right)dx$$

$$= x - \frac{1}{315^2}\frac{x^3}{3}\Big|_{-315}^{315}$$

$$= 315 - 105 + 315 - 105$$

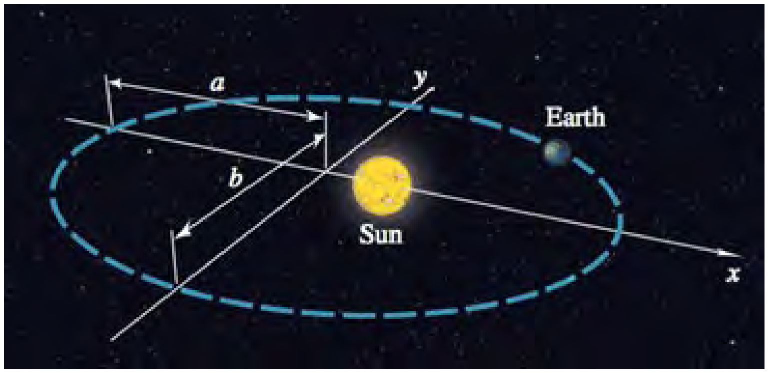
$$= 420\ \ ft$$



Exercise

The planets orbit the Sun in elliptical orbits with the Sun at one focus. The equation of an ellipse whose dimensions are 2 a in the x-direction and 2 b in the y-direction is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



a) Let d^2 denote the square of the distance from a planet to the center of the ellipse at (0, 0). Integrate over the interval $[-a,\ a]$ to show that the average value of d^2 is $\frac{a^2+2b^2}{3}$

b) Show that in the case of a circle ($a = b = R$), the average value in part (a) is R^2.

c) Assuming $0 < b <$ a, the coordinates of the Sun are $\left(\sqrt{a^2-b^2},\ 0\right)$. Let D^2 denote the square of the distance from the planet to the Sun. Integrate over the interval $[-a,\ a]$ to show that the average value of D^2 is $\frac{4a^2-b^2}{3}$.

Solution

a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=b^2-\frac{b^2}{a^2}x^2$$

$$d^2=x^2+y^2=x^2+b^2-\frac{b^2}{a^2}x^2$$

$$=b^2+\left(1-\frac{b^2}{a^2}\right)x^2$$

The average value of $d^2=\frac{1}{2a}\int_{-a}^{a}\left(b^2+\left(1-\frac{b^2}{a^2}\right)x^2\right)dx$

$$= \frac{1}{2a}\left(b^2 x + \frac{1}{3}\left(1 - \frac{b^2}{a^2}\right)x^3 \right|_{-a}^{a}$$

$$= \frac{1}{2a}\left(ab^2 + \frac{1}{3}\left(1 - \frac{b^2}{a^2}\right)a^3 + ab^2 + \frac{1}{3}\left(1 - \frac{b^2}{a^2}\right)a^3\right)$$

$$= \frac{1}{2a}\left(2ab^2 + \frac{2}{3}a^3 - \frac{2}{3}b^2 a\right)$$

$$\boxed{= \frac{2}{3}b^2 + \frac{a^2}{3}}$$

b) If $a = b = R$

The average value of $d^2 = \frac{2R^2}{3} + \frac{R^2}{3}$

$$\boxed{= R^2}$$

c) $D^2 = \left(x - \sqrt{a^2 - b^2}\right)^2 + y^2$

$$= x^2 - 2x\sqrt{a^2 - b^2} + a^2 - b^2 + b^2 - \frac{b^2}{a^2}x^2$$

$$= \left(1 - \frac{b^2}{a^2}\right)x^2 - 2x\sqrt{a^2 - b^2} + a^2$$

The average value of $D^2 = \frac{1}{2a}\int_{-a}^{a}\left(\left(1 - \frac{b^2}{a^2}\right)x^2 - 2x\sqrt{a^2 - b^2} + a^2\right)dx$

$$= \frac{1}{2a}\left(\frac{1}{3}\left(1 - \frac{b^2}{a^2}\right)x^3 - x^2\sqrt{a^2 - b^2} + a^2 x \right|_{-a}^{a}$$

$$= \frac{1}{2a}\left(\frac{1}{3}\left(1 - \frac{b^2}{a^2}\right)a^3 - a^2\sqrt{a^2 - b^2} + a^3 + \frac{1}{3}\left(1 - \frac{b^2}{a^2}\right)a^3 + a^2\sqrt{a^2 - b^2} + a^3\right)$$

$$= \frac{1}{3}\left(1 - \frac{b^2}{a^2}\right)a^2 + a^2$$

$$= \frac{1}{3}a^2 - \frac{1}{3}b^2 + a^2$$

$$\boxed{= \frac{4}{3}a^2 - \frac{1}{3}b^2}$$

Exercise

A particle moves along a line with a velocity given by $v(t)=5\sin \pi t$ starting with an initial position $s(0)=0$. Find the displacement of the particle between $t=0$ and $t=2$, which is given by $s(t)=\int_0^2 v(t)\,dt$. Find the distance traveled by the particle during this interval, which is $\int_0^2 |v(t)|\,dt$.

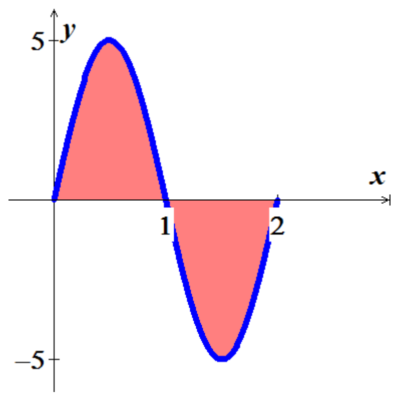
Solution

$$s(t)=\int_0^2 5\sin \pi t\; dt$$

$$=-\frac{5}{\pi}\cos \pi t\,\Big|_0^2$$

$$=-\frac{5}{\pi}(1-1)$$

$$\boxed{=0}$$



$$s(t)=\int_0^2 |5\sin \pi t|\; dt$$

$$=5\int_0^1 \sin \pi t\; dt\; + \;5\int_1^2 (-\sin \pi t)\; dt$$

$$=-\frac{5}{\pi}\cos \pi t\,\Big|_0^1 \;+\; \frac{5}{\pi}\cos \pi t\,\Big|_1^2$$

$$=-\frac{5}{\pi}(-1-1)+\frac{5}{\pi}(1+1)$$

$$=\frac{10}{\pi}+\frac{10}{\pi}$$

$$\boxed{=\frac{20}{\pi}}$$

Exercise

A baseball is launched into the outfield on a parabolic trajectory given by $y=0.01x(200-x)$. Find the average height of the baseball over the horizontal extent of its flight.

Solution

$$y=0.01x(200-x)=0 \quad \rightarrow \quad x=0,\; 200$$

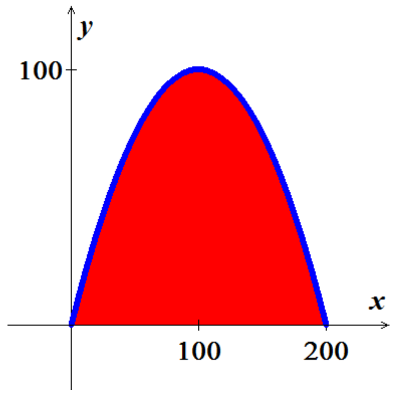
$$Avg=\frac{1}{200}\int_0^{200}\left(2x-0.01x^2\right)\,dx$$

$$= \frac{1}{200}\left(x^2 - \frac{1}{300}x^3\right)\Big|_0^{200}$$

$$= \frac{1}{200}\left(4\times10^4 - \frac{1}{300}8\times10^6\right)$$

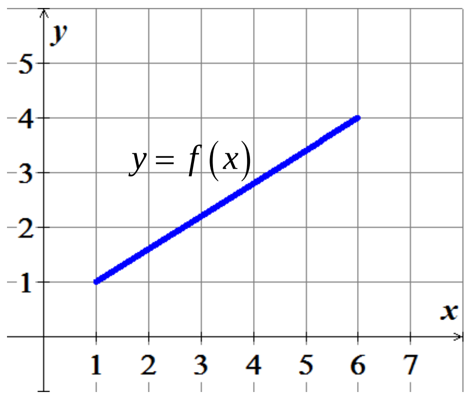
$$= \frac{4\times10^4}{200}\left(1-\frac{2}{3}\right)$$

$$= \boxed{\frac{200}{3}}$$



Exercise

Find the average value of f shown in the figure on the interval $[1, 6]$ and then find the point(s) c in $(1, 6)$ guaranteed to exist by the Mean Value Theorem for Integrals



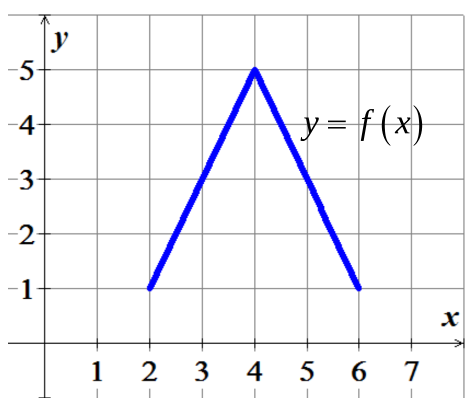
Solution

Since it is a straight line, then the average value is 2.5

The average value occurs at the midpoint of the interval which is $(3.5, 2.5)$

Exercise

Find the average value of f shown in the figure on the interval $[2, 6]$ and then find the point(s) c in $(2, 6)$ guaranteed to exist by the Mean Value Theorem for Integrals



Solution

Over interval $[2,\ 4]$; it is a straight line, then the average value is 3

Over interval $[4,\ 6]$; it is a straight line, then the average value is 3 .

Therefore; the overage is 3 over $[2,\ 6]$

Solution   *Section* 4.6 – Substitution Rule

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int 2(2x+4)^5\,dx, \quad u = 2x+4$$

Solution

Let $u = 2x+4 \quad \Rightarrow \quad du = 2xdx$

$$\int 2(2x+4)^5\,dx = \int u^5 du$$

$$= \frac{1}{6}u^6 + C$$

$$\boxed{= \frac{1}{6}(2x+4)^6 + C}$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int \frac{4x^3}{(x4+1)^2}\,dx, \quad u = x^4+1$$

Solution

Let $u = x^4 + 1$

$du = 4x^3 dx$

$$\int \frac{4x^3}{(x4+1)^2}\,dx = \int \frac{du}{u^2}$$

$$= -\frac{1}{u} + C$$

$$\boxed{= -\frac{1}{x^2+1} + C}$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int x\sin\left(2x^2\right)dx, \quad u = 2x^2$$

Solution

Let $u = 2x^2$

$$du = 4xdx \rightarrow \ \tfrac{1}{4}du = xdx$$

$$\int x\sin\left(2x^2\right)dx = \int \tfrac{1}{4}\sin u\, du$$

$$= -\frac{1}{4}\cos u + C$$

$$= -\frac{1}{4}\cos\left(2x^2\right) + C$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int 12\left(y^4 + 4y^2 + 1\right)^2\left(y^3 + 2y\right)dx, \quad u = y^4 + 4y^2 + 1$$

Solution

Let $u = y^4 + 4y^2 + 1$

$$du = \left(4y^3 + 8y\right)dx \rightarrow \ du = 4\left(y^3 + 2y\right)dx$$

$$\int 12\left(y^4 + 4y^2 + 1\right)^2\left(y^3 + 2y\right)dx = \int 12u^2\left(\tfrac{1}{4}du\right)$$

$$= 3\int u^2\ du$$

$$= 3\frac{u^3}{3} + C$$

$$= \left(y^4 + 4y^2 + 1\right)^3 + C$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int \csc^2 2\theta \cot 2\theta \, d\theta \quad \to \begin{cases} a) \text{ Using } u = \cot 2\theta \\ b) \text{ Using } u = \csc 2\theta \end{cases}$$

Solution

Let $u = \cot 2\theta \quad \Rightarrow \quad du = -2\csc^2 2\theta d\theta \to \quad -\frac{1}{2}du = \csc^2 2\theta dx$

$$\int \csc^2 2\theta \cot 2\theta \, d\theta = -\int \frac{1}{2}udu$$

$$= -\frac{1}{2}\frac{u^2}{2} + C$$

$$\boxed{= -\frac{1}{4}\cot^2 2\theta + C}$$

Let $u = \csc 2\theta$

$du = -2\csc 2\theta \cot 2\theta d\theta$

$-\frac{1}{2}du = \csc 2\theta \cot 2\theta dx$

$$\int \csc^2 2\theta \cot 2\theta \, d\theta = \int \csc 2\theta \left(\csc 2\theta \cot 2\theta \, d\theta\right)$$

$$= -\int \frac{1}{2}u \, du$$

$$= -\frac{1}{2}\frac{u^2}{2} + C$$

$$\boxed{= -\frac{1}{4}\csc^2 2\theta + C}$$

Exercise

Evaluate the integrals $\quad \displaystyle\int \frac{1}{\sqrt{5s+4}} ds$

Solution

Let $u = 5s + 4$

$du = 5ds$

$\frac{1}{5}du = ds$

$$\int \frac{1}{\sqrt{5s+4}} ds = \frac{1}{5}\int u^{-1/2} du$$

$$= \frac{1}{5}\frac{u^{1/2}}{1/2} + C$$

$$= \frac{2}{5}\sqrt{5s+4} + C$$

Exercise

Evaluate the integrals $\displaystyle\int \theta\,\sqrt[4]{1-\theta^2}\,d\theta$

Solution

Let $d\left(1-\theta^2\right) = -2\theta d\theta$

$$\int \theta\,\sqrt[4]{1-\theta^2}\,d\theta = -\frac{1}{2}\int \left(1-\theta^2\right)^{1/4} d\left(1-\theta^2\right)$$

$$= -\frac{2}{5}\left(1-\theta^2\right)^{5/4} + C$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}dx$

Solution

$$d\left(1+\sqrt{x}\right) = \frac{1}{2\sqrt{x}}dx$$

$$\int \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}\,dx = \int \frac{2}{\left(1+\sqrt{x}\right)^2}\,d\left(1+\sqrt{x}\right)$$

$$= -\frac{2}{1+\sqrt{x}} + C$$

Exercise

Evaluate the integrals $\displaystyle\int \tan^2 x \sec^2 x\,dx$

Solution

$$d\left(\tan x\right) = \sec^2 x\,dx$$

$$\int \tan^2 x \sec^2 x\, dx = \int \tan^2 x\, d(\tan x)$$

$$= \frac{1}{3}\tan^3 x + C$$

Exercise

Evaluate the integrals $\int \sin^5 \frac{x}{3} \cos \frac{x}{3}\, dx$

Solution

Let $d\left(\sin\left(\frac{x}{3}\right)\right) = \frac{1}{3}\cos\left(\frac{x}{3}\right)dx$

$$\int \sin^5 \frac{x}{3} \cos \frac{x}{3}\, dx = 3\int \sin^5 \frac{x}{3}\, d\left(\sin \frac{x}{3}\right)$$

$$= \frac{1}{2}\sin^6\left(\frac{x}{3}\right) + C$$

Exercise

Evaluate the integrals $\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2}\, dx$

Solution

Let $u = \tan\left(\frac{x}{2}\right)$

$$du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx \rightarrow \ 2du = \sec^2\left(\frac{x}{2}\right)dx$$

$$\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2}\, dx = 2\int u^7 du$$

$$= 2\frac{1}{8}u^8 + C$$

$$= \frac{1}{4}\tan^8 \frac{x}{2} + C$$

$$\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2}\, dx = 2\int \tan^7 \frac{x}{2}\, d\left(\tan \frac{x}{2}\right)$$

$$= \frac{1}{4}\tan^8 \frac{x}{2} + C$$

Exercise

Evaluate the integrals $\int r^4 \left(7 - \frac{r^5}{10}\right)^3 dr$

Solution

Let $d\left(7 - \frac{r^5}{10}\right) = -\frac{1}{2} r^4\, dr$

$$\int r^4\left(7-\frac{r^5}{10}\right)^3 dr = -2\int\left(7-\frac{r^5}{10}\right)^3 d\left(7-\frac{r^5}{10}\right)$$

$$= -\frac{1}{2}\left(7-\frac{r^5}{10}\right)^4 + C$$

Exercise

Evaluate the integrals $\int x^{1/2}\sin\left(x^{3/2}+1\right)dx$

Solution

$$d\left(x^{3/2}+1\right)=\frac{3}{2}x^{1/2}dx$$

$$\int x^{1/2}\sin\left(x^{3/2}+1\right)dx = \frac{2}{3}\int \sin\left(x^{3/2}+1\right)\, d\left(x^{3/2}+1\right)$$

$$= -\frac{2}{3}\cos\left(x^{3/2}+1\right)+C$$

Let $u = x^{3/2}+1$

$$du = \frac{3}{2}x^{1/2}dx$$

$$\frac{2}{3}du = x^{1/2}dx$$

$$\int x^{1/2}\sin\left(x^{3/2}+1\right)dx = \int \sin u\,\left(\frac{2}{3}du\right)$$

$$= \frac{2}{3}\int \sin u\;\; du$$

$$= \frac{2}{3}\left(-\cos u\right)+C$$

$$= -\frac{2}{3}\cos\left(x^{3/2}+1\right)+C$$

Exercise

Evaluate the integrals $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$

Solution

$$d\left(\csc\left(\frac{v-\pi}{2}\right)\right) = -\frac{1}{2}\csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)\,dv \qquad \frac{d}{dv}\left(\frac{v-\pi}{2}\right)=\frac{1}{2}$$

$$\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv = -\frac{1}{2}\int d\left(\csc\left(\frac{v-\pi}{2}\right)\right)$$

$$\boxed{= -2\csc\left(\frac{v-\pi}{2}\right) + C}$$

Let $u = \csc\left(\frac{v-\pi}{2}\right)$

$$du = -\frac{1}{2}\csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$$

$$-2du = \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$$

$$\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv = \int -2du$$

$$= -2u + C$$

$$\boxed{= -2\csc\left(\frac{v-\pi}{2}\right) + C}$$

Exercise

Evaluate the integrals $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}dt$

Solution

$$\int \frac{\sin(2t+1)}{\cos^2(2t+1)}dt = -\frac{1}{2}\int \frac{d(\cos(2t+1))}{\cos^2(2t+1)}$$

$$\boxed{= \frac{1}{2\cos(2t+1)} + C}$$

Exercise

Evaluate the integrals $\int \frac{\sec z\ \tan z}{\sqrt{\sec z}}dz$

Solution

$$d(\sec z) = \sec z \tan z\, dz$$

$$\int \frac{\sec z\ \tan z}{\sqrt{\sec z}}dz = \int (\sec z)^{-1/2}d(\sec z)$$

$$\boxed{= 2\sqrt{\sec z} + C}$$

Let $u = \sec z \quad \Rightarrow \quad du = \sec z \tan z dz$

$$\int \frac{\sec z \ \tan z}{\sqrt{\sec z}} dz = \int \frac{du}{u^{1/2}}$$

$$= \int u^{-1/2} du$$

$$= 2u^{1/2} + C$$

$$\boxed{= 2\sqrt{\sec z} + C}$$

Exercise

Evaluate the integrals $\int \frac{1}{\sqrt{t}} \cos\left(\sqrt{t} + 3\right) dt$

Solution

$$d\left(\sqrt{t} + 3\right) = \frac{1}{2\sqrt{t}} dt$$

$$\int \frac{1}{\sqrt{t}} \cos\left(\sqrt{t} + 3\right) dt = 2 \int \cos\left(\sqrt{t} + 3\right) d\left(\sqrt{t} + 3\right)$$

$$\boxed{= 2\sin\left(\sqrt{t} + 3\right) + C}$$

$$u = \sqrt{t} + 3$$

$$du = \frac{1}{2\sqrt{t}} dt$$

$$2du = \frac{1}{\sqrt{t}} dt$$

$$\int \frac{1}{\sqrt{t}} \cos\left(\sqrt{t} + 3\right) dt = \int (\cos u)(2du)$$

$$= 2 \int \cos u \ du$$

$$= 2\sin u + C$$

$$\boxed{= 2\sin\left(\sqrt{t} + 3\right) + C}$$

Exercise

Evaluate the integrals $\int \frac{1}{\theta^2} \sin\frac{1}{\theta} \cos\frac{1}{\theta} \ d\theta$

Solution

$$\frac{d}{d\theta}\left(\frac{1}{\theta}\right)=-\frac{1}{\theta^2}$$

$$d\left(\cos\frac{1}{\theta}\right)=\frac{1}{\theta^2}\sin\frac{1}{\theta}\ d\theta$$

$$\int \frac{1}{\theta^2}\ \sin\frac{1}{\theta}\cos\frac{1}{\theta}\ d\theta=\int \cos\frac{1}{\theta}\ d\left(\cos\frac{1}{\theta}\right)$$

$$\boxed{=\frac{1}{2}\cos^2\frac{1}{\theta}+C}$$

Let $u=\sin\frac{1}{\theta}$

$$du=\left(\cos\frac{1}{\theta}\right)\left(\frac{1}{\theta}\right)'$$

$$=\left(\cos\frac{1}{\theta}\right)\left(-\frac{1}{\theta^2}\right)d\theta$$

$$-du=\frac{1}{\theta^2}\cos\frac{1}{\theta}d\theta$$

$$\int \frac{1}{\theta^2}\ \sin\frac{1}{\theta}\cos\frac{1}{\theta}\ d\theta=-\int udu$$

$$=-\frac{1}{2}u^2+C$$

$$\boxed{=-\frac{1}{2}\sin^2\frac{1}{\theta}+C}$$

Exercise

Evaluate the integrals $\int t^3\left(1+t^4\right)^3\ dt$

Solution

$$d\left(1+t^4\right)=4t^3dt$$

$$\int t^3\left(1+t^4\right)^3\ dt=\frac{1}{4}\int\left(1+t^4\right)^3\ d\left(1+t^4\right)$$

$$\boxed{=\frac{1}{16}\left(1+t^4\right)^4+C}$$

$u=1+t^4$

$du=4t^3dt$

$$\frac{1}{4}du = t^3 dt$$

$$\int t^3\left(1+t^4\right)^3 dt = \frac{1}{4}\int u^3 du$$

$$= \frac{1}{4}\left(\frac{u^4}{4}\right) + C$$

$$\boxed{= \frac{1}{16}\left(1+t^4\right)^4 + C}$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}\,dx$

Solution

$$d\left(\frac{x^2-1}{x^2}\right) = d\left(1-x^{-2}\right)$$

$$= \frac{1}{x^3}dx$$

$$\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}\,dx = \int \left(1-\frac{1}{x^2}\right)^{1/2} d\left(1-\frac{1}{x^2}\right)$$

$$\boxed{= \frac{1}{3}\left(1-\frac{1}{x^2}\right)^{3/2} + C}$$

Let $u = \dfrac{x^2-1}{x^2}$

$$= 1-\frac{1}{x^2}$$

$$= 1-x^{-2}$$

$$du = 2x^{-3}dx$$

$$\frac{1}{2}du = \frac{1}{x^3}dx$$

$$\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}\,dx = \int u^{1/2}\left(\frac{1}{2}du\right)$$

$$= \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) + C$$

$$=\frac{1}{3}\left(1-\frac{1}{x^2}\right)^{3/2}+C$$

Exercise

Evaluate the integrals $\int x^3\sqrt{x^2+1}\,dx$

Solution

Let $u=x^2+1 \quad \Rightarrow \quad x^2=u-1$

$du=2xdx$

$\frac{1}{2}du=xdx$

$$\int x^3\sqrt{x^2+1}\,dx=\int x^2\sqrt{x^2+1}\,x\,dx$$

$$=\int (u-1)u^{1/2}\left(\frac{1}{2}du\right)$$

$$=\frac{1}{2}\int\left(u^{3/2}-u^{1/2}\right)du$$

$$=\frac{1}{2}\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)+C$$

$$=\frac{1}{5}\left(x^2+1\right)^{5/2}-\frac{1}{3}\left(x^2+1\right)^{3/2}+C$$

Exercise

Evaluate the integrals $\int \frac{x}{\left(x^2-4\right)^3}dx$

Solution

$d\left(x^2-4\right)=2xdx$

$$\int \frac{x}{\left(x^2-4\right)^3}\,dx=\frac{1}{2}\int\left(x^2-4\right)^{-3}d\left(x^2-4\right)$$

$$=-\frac{1}{4\left(x^2-4\right)^2}+C$$

$u = x^2 - 4$

$du = 2x dx$

$\frac{1}{2} du = x dx$

$$\int \frac{x}{\left(x^2-4\right)^3} dx = \frac{1}{2}\int u^{-3} du$$

$$= -\frac{1}{4}u^{-2} + C$$

$$= -\frac{1}{4\left(x^2-4\right)^2} + C$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}} dr$

Solution

$$d\left(\sqrt{3(2r-1)^2+6}\right) = \frac{1}{2}\frac{6(2)(2r-1)}{\sqrt{3(2r-1)^2+6}} dr$$

$$\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}} dr = \frac{1}{6}\int \cos\sqrt{3(2r-1)^2+6}\; d\left(\sqrt{3(2r-1)^2+6}\right)$$

$$= \frac{1}{6}\sin\sqrt{3(2r-1)^2+6} + C$$

Let $u = \sqrt{3(2r-1)^2+6}$

$$du = \frac{1}{2}\left(3(2r-1)^2+6\right)^{-1/2}\left(6(2r-1)(2)\right)dr$$

$$= \frac{6(2r-1)}{\left(3(2r-1)^2+6\right)^{1/2}} dr$$

$$\rightarrow \quad \frac{1}{6} du = \frac{2r-1}{\sqrt{3(2r-1)^2+6}} dr$$

$$\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}} dr = \int \cos u \left(\frac{1}{6} du\right)$$

$$=\frac{1}{6}\sin u+C$$

$$\boxed{=\frac{1}{6}\sin\sqrt{3(2r-1)^2+6}+C}$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\,d\theta$

Solution

$$d\left(\cos\sqrt{\theta}\right)=-\frac{1}{2\sqrt{\theta}}\sin\sqrt{\theta}\;d\theta \qquad\qquad \frac{d}{du}-u'\sin u$$

$$\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\;\sqrt{\cos^3\sqrt{\theta}}}\,d\theta=-2\int \cos^{3/2}\sqrt{\theta}\;\;d\left(\cos\sqrt{\theta}\right)$$

$$\boxed{=\frac{4}{\sqrt{\cos\sqrt{\theta}}}+C}$$

Let $u=\cos\sqrt{\theta}$

$$du=\left(-\sin\sqrt{\theta}\right)\left(\frac{1}{2\sqrt{\theta}}\right)d\theta$$

$$-2du=\frac{1}{\sqrt{\theta}}\sin\sqrt{\theta}d\theta$$

$$\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\,d\theta=\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\sqrt{\cos^3\sqrt{\theta}}}\,d\theta$$

$$=\int \frac{1}{u^{3/2}}(-2du)$$

$$=-2\int u^{-3/2}du$$

$$=-2\frac{u^{-1/2}}{-1/2}+C$$

$$\boxed{=\frac{4}{\sqrt{\cos\sqrt{\theta}}}+C}$$

Exercise

Evaluate the integrals. $\int 2x\sqrt{x^2-2}\,dx$

Solution

$$d\left(x^2-2\right)=2x\,dx$$

$$\int 2x\sqrt{x^2-2}\,dx=\int\left(x^2-2\right)^{1/2}\,d\left(x^2-2\right)$$

$$=\frac{2}{3}\left(x^2-2\right)^{3/2}+C$$

Exercise

Evaluate the integrals $\int x^3\left(3x^4+1\right)^2\,dx$

Solution

$$d\left(3x^4+1\right)=12x^3dx$$

$$\int x^3\left(3x^4+1\right)^2\,dx=\int\left(3x^4+1\right)^2\,d\left(3x^4+1\right)$$

$$=\frac{1}{36}\left(3x^4+1\right)^3+C$$

Exercise

Evaluate the integrals $\int 2\left(3x^4+1\right)^2\,dx$

Solution

$$\int 2\left(3x^4+1\right)^2\,dx=\int 2\left(9x^8+6x^4+1\right)dx \qquad (a+b)^2=a^2+2ab+b^2$$

$$=\int\left(18x^8+12x^4+2\right)\,dx$$

$$=2x^9+\frac{12}{5}x^5+2x+C$$

Exercise

Evaluate the integrals $\int 5x\sqrt{x^2-1}\,dx$

Solution

$$d\left(x^2-1\right)=2x\,dx$$

$$\int 5x\sqrt{x^2-1}\,dx=\frac{5}{2}\int\left(x^2-1\right)^{1/2}d\left(x^2-1\right)$$

$$=\frac{5}{3}\left(x^2-1\right)^{3/2}+C$$

$$u=x^2-1$$

$$du=2xdx$$

$$\Rightarrow \frac{1}{2}du=xdx$$

$$\int 5x\left(x^2-1\right)^{1/2}dx=5\int u^{1/2}\ \frac{1}{2}du \qquad \textit{Substitute for x and dx}$$

$$=5\int u^{1/2}\ \frac{1}{2}du$$

$$=\frac{5}{2}\int u^{1/2}du$$

$$=\frac{5}{2}\frac{u^{3/2}}{3/2}+C$$

$$=\frac{5}{3}u^{3/2}+C$$

$$=\frac{5}{3}\left(x^2-1\right)^{3/2}+C$$

Exercise

Find the integral $\int\left(x^2-1\right)^3(2x)dx$

Solution

$$\int\left(x^2-1\right)^3(2x)\,dx=\int\left(x^2-1\right)^3d\left(x^2-1\right)$$

$$=\frac{1}{4}\left(x^2-1\right)^4+C$$

Exercise

Find the integral $\displaystyle\int \frac{6x}{\left(1+x^2\right)^3}dx$

Solution

$$d\left(1+x^2\right)=2x\,dx$$

$$\int \frac{6x}{\left(1+x^2\right)^3}dx=3\int\left(1+x^2\right)^3 d\left(1+x^2\right)$$

$$=-\frac{3}{2}\left(1+x^2\right)^{-2}+C$$

$$\boxed{=-\frac{3}{2}\frac{1}{\left(1+x^2\right)^2}+C}$$

Exercise

Find the integral $\displaystyle\int u^3\sqrt{u^4+2}\;du$

Solution

$$d\left(u^4+2\right)=4u^3\,du$$

$$\int u^3\sqrt{u^4+2}\;du=\frac{1}{4}\int\left(u^4+2\right)^{1/2}d\left(u^4+2\right)$$

$$\boxed{=\frac{1}{6}\left(u^4+2\right)^{3/2}+C}$$

Exercise

Find the integral $\displaystyle\int \frac{t+2t^2}{\sqrt{t}}\;dt$

Solution

$$\int \frac{t+2t^2}{\sqrt{t}}\;dt=\int\left(\frac{t}{t^{1/2}}+2\frac{t^2}{t^{1/2}}\right)dt$$

$$=\int\left(t^{1/2}+2t^{3/2}\right)dt$$

$$\boxed{=\frac{2}{3}t^{3/2}+\frac{4}{5}t^{5/2}+C}$$

Exercise

Find the integral $\int \left(1+\frac{1}{t}\right)^3 \frac{1}{t^2}\, dt$

Solution

$$\int \left(1+\frac{1}{t}\right)^3 \frac{1}{t^2}\, dt = -\int \left(1+\frac{1}{t}\right)^3 d\left(1+\frac{1}{t}\right)$$

$$= -\frac{1}{4}\left(1+\frac{1}{t}\right)^4 + C$$

Exercise

Find the integral $\int \left(7-3x-3x^2\right)(2x+1)\, dx$

Solution

$$d\left(7-3x-3x^2\right) = \left(-3-6x^2\right)dx$$

$$= -3\left(1+2x^2\right)dx$$

$$\int \left(7-3x-3x^2\right)(2x+1)\, dx = -\frac{1}{3}\int \left(7-3x-3x^2\right)\left(7-3x-3x^2\right)\, dx$$

$$= -\frac{1}{6}\left(7-3x-3x^2\right)^2 + C$$

Exercise

Find the integral $\int \sqrt{x}\left(4-x^{3/2}\right)^2\, dx$

Solution

$$d\left(4-x^{3/2}\right) = -\frac{3}{2}x^{1/2}dx$$

$$\int \sqrt{x}\left(4-x^{3/2}\right)^2 dx = -\frac{2}{3}\int \left(4-x^{3/2}\right)^2 d\left(4-x^{3/2}\right)$$

$$= -\frac{2}{9}\left(4-x^{3/2}\right)^3 + C$$

$$u = 4-x^{3/2}$$

$$du = -\frac{3}{2}x^{1/2}dx$$

$$\rightarrow \quad -\frac{2}{3}du = \sqrt{x}dx$$

$$\int \sqrt{x}\left(4 - x^{3/2}\right)^2 \, dx = \int u^2\left(-\frac{2}{3}\right)du$$

$$= -\frac{2}{3}\int u^2 du$$

$$= -\frac{2}{9}u^3 + C$$

$$\boxed{= -\frac{2}{9}\left(4 - x^{3/2}\right)^3 + C}$$

Exercise

Find the integral $\displaystyle\int \frac{1}{\sqrt{x}+\sqrt{x+1}}\,dx$

Solution

$$\int \frac{1}{\sqrt{x}+\sqrt{x+1}}\,dx = \int \frac{1}{\sqrt{x}+\sqrt{x+1}}\,\frac{\sqrt{x}-\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}\,dx$$

$$= \int \frac{\sqrt{x}-\sqrt{x+1}}{x-(x+1)}\,dx$$

$$= -\int \left(x^{1/2} - (x+1)^{1/2}\right)dx$$

$$= -\left(\frac{2}{3}x^{3/2} - \frac{2}{3}(x+1)^{3/2}\right) + C$$

$$\boxed{= \frac{2}{3}(x+1)^{3/2} - \frac{2}{3}x^{3/2} + C}$$

Exercise

Find the integral $\displaystyle\int \sqrt{1-x}\,dx$

Solution

$$\int \sqrt{1-x}\,dx = -\int (1-x)^{1/2}\,d(1-x) \qquad d(1-x) = -dx$$

$$\boxed{= -\frac{2}{3}(1-x)^{3/2} + C}$$

Exercise

Find the integral $\displaystyle\int x\sqrt{x^2+4}\,dx$

Solution

$$\int \sqrt{x^2+4}\;x\,dx = \frac{1}{2}\int \left(x^2+4\right)^{1/2} d\left(x^2+4\right) \qquad d\left(x^2+4\right) = 2xdx$$

$$= \frac{1}{3}\left(x^2+4\right)^{3/2} + C$$

Exercise

Find the integral $\displaystyle\int \sin^2\left(\theta+\frac{\pi}{6}\right)d\theta$

Solution

$$\int \sin^2\left(\theta+\frac{\pi}{6}\right)d\theta = \frac{1}{2}\int\left(1-\cos\left(2\theta+\frac{\pi}{3}\right)\right)d\theta$$

$$= \frac{1}{2}\left(\theta-\frac{1}{2}\sin\left(2\theta+\frac{\pi}{3}\right)\right)+C$$

$$= \frac{\theta}{2}-\frac{1}{4}\sin\left(2\theta+\frac{\pi}{3}\right)+C$$

Exercise

Find the integral $\displaystyle\int \cos^2(8\theta)\,d\theta$

Solution

$$\int \cos^2(8\theta)\,d\theta = \frac{1}{2}\int\left(1+\cos(16\theta)\right)d\theta$$

$$= \frac{1}{2}\left(1+\frac{1}{16}\sin(16\theta)\right)+C$$

$$= \frac{1}{2}+\frac{1}{32}\sin(16\theta)+C$$

Exercise

Find the integral $\displaystyle\int \sin^2(2\theta)\,d\theta$

Solution

$$\int \sin^2(2\theta)\,d\theta = \frac{1}{2}\int \left(1-\cos(4\theta)\right)d\theta$$

$$= \frac{1}{2}\left(1-\frac{1}{4}\sin(4\theta)\right)+C$$

$$\boxed{= \frac{1}{2}-\frac{1}{8}\sin(4\theta)+C}$$

Exercise

Evaluate the integral $\int 8\cos^4 2\pi x\ dx$

Solution

$$\int 8\cos^4 2\pi x\ dx = 8\int (\cos 2\pi x)^4\,dx \qquad\qquad \cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$= 8\int \left(\frac{1+\cos 4\pi x}{2}\right)^2 dx$$

$$= 2\int (1+\cos 4\pi x)^2\,dx$$

$$= 2\int \left(1+2\cos 4\pi x+\cos^2 4\pi x\right)dx$$

$$= 2\int dx + 4\int \cos 4\pi x dx + 2\int \cos^2 4\pi x dx$$

$$= 2x + 4\frac{1}{4\pi}\cos 4\pi x + 2\int \frac{1+\cos 8\pi x}{2}dx$$

$$= 2x + \frac{1}{\pi}\cos 4\pi x + \int (1+\cos 8\pi x)\,dx$$

$$= 2x + \frac{1}{\pi}\sin 4\pi x + x + \frac{1}{8\pi}\sin 8\pi x + C$$

$$\boxed{= 3x + \frac{1}{\pi}\sin 4\pi x + \frac{1}{8\pi}\sin 8\pi x + C}$$

Exercise

Evaluate the integral $\int \sec x dx$

Solution

$$\int \sec x dx = \int \sec x\frac{\sec x+\tan x}{\sec x+\tan x}dx$$

$$= \int \frac{\sec^2 x + \sec\tan x}{\sec x + \tan x} dx \qquad d(\sec x + \tan x) = \left(\sec x \tan x + \sec^2 x\right) dx$$

$$= \int \frac{d(\sec x + \tan x)}{\sec x + \tan x}$$

$$\boxed{= \ln\left|\sec x + \tan x\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{1-4x^2}}$

Solution

Let $u = 2x$

$du = 2dx$

$\frac{1}{2} du = dx$

$$\int \frac{dx}{\sqrt{1-4x^2}} = \int \frac{dx}{\sqrt{1-(2x)^2}}$$

$$= \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{1}{2}\sin^{-1} u + C$$

$$\boxed{= \frac{1}{2}\sin^{-1}(2x) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{3-4x^2}}$

Solution

$a^2 = 3 \to a = \sqrt{3} \qquad u^2 = 4x^2 = (2x)^2 \to u = 2x \quad du = 2dx$

$$\int \frac{dx}{\sqrt{3-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{a^2-u^2}}$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\boxed{= \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-6}}$

Solution

$$a^2 = 6 \to a = \sqrt{6}$$

$$u^2 = e^{2x} \quad \to \; u = e^x$$

$$du = e^x dx$$

$$dx = \frac{du}{e^x} \; = \frac{du}{u}$$

$$\int \frac{dx}{\sqrt{e^{2x}-6}} = \int \frac{du}{u\sqrt{u^2-a^2}}$$

$$= \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

$$= \frac{1}{\sqrt{6}}\sec^{-1}\left|\frac{e^x}{\sqrt{6}}\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{4x-x^2}}$

Solution

$$4x - x^2 = -\left(x^2-4x\right) - 4 + 4 \qquad \textit{Using Completing the Square}$$

$$= -\left(x^2-4x+4\right) + 4$$

$$= 4 - (x-2)^2$$

$$a = 2$$

$$u = x-2 \quad \to \; du = dx$$

$$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}}$$

$$= \sin^{-1}\left(\frac{x-2}{2}\right) + C$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{4x^2+4x+2}$

Solution

$$\begin{aligned}4x^2+4x+2 &= 4\left(x^2+x\right)+2\\ &= 4\left(x^2+x+\frac{1}{4}\right)+2-4\left(\frac{1}{4}\right)\\ &= 4\left(x+\frac{1}{2}\right)^2+1\\ &= (2x+1)^2+1\end{aligned}$$

$a=1 \qquad u=2x+1 \quad \to du=2dx$

$$\begin{aligned}\int \frac{dx}{4x^2+4x+2} &= \int \frac{dx}{(2x+1)^2+1}\\ &= \frac{1}{2}\int \frac{du}{u^2+1}\\ &= \frac{1}{2}\cdot\frac{1}{1}\tan^{-1}\left(\frac{2x+1}{1}\right)+C\\ &= \frac{1}{2}\tan^{-1}(2x+1)+C\end{aligned}$$

$$\int \frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}$$

Exercise

Find the integral $\displaystyle\int \frac{1}{6x-5}dx$

Solution

$$\begin{aligned}\int \frac{1}{6x-5}dx &= \frac{1}{6}\int \frac{d(6x-5)}{6x-5}\\ &= \frac{1}{6}\ln|6x-5|+C\end{aligned}$$

Exercise

Find the integral $\displaystyle\int \frac{x^2+2x+3}{x^3+3x^2+9x+1}dx$

Solution

$$d\left(x^3+3x^2+9x+1\right)=\left(3x^2+6x+9\right)dx$$

$$\int \frac{x^2+2x+3}{x^3+3x^2+9x+1}\,dx=\frac{1}{3}\int \frac{d\left(x^3+3x^2+9x+1\right)}{x^3+3x^2+9x+1}$$

$$\boxed{=\frac{1}{3}\ln\left|x^3+3x^2+9x+1\right|+C}$$

Exercise

Find the integral $\displaystyle\int \frac{1}{x(\ln x)^2}dx$

Solution

$$d(\ln x)=\frac{1}{x}dx$$

$$\int \frac{1}{x(\ln x)^2}dx=\int \frac{1}{(\ln x)^2}d(\ln x)$$

$$\boxed{=-\frac{1}{\ln x}+C}$$

Exercise

Find the integral $\displaystyle\int \frac{x-3}{x+3}dx$

Solution

$$\int \frac{x-3}{x+3}dx=\int \left(1-\frac{6}{x+3}\right)dx$$

$$\boxed{=x-6\ln|x+3|+C}$$

Exercise

Find the indefinite integral. $\displaystyle\int \frac{3x}{x^2+4}\,dx$

Solution

$$d\left(x^2+4\right)=2x\,dx$$

$$\int \frac{3x}{x^2+4}\,dx=\frac{3}{2}\int \frac{1}{x^2+4}d\left(x^2+4\right)$$

$$=\frac{3}{2}\ln\left(x^2+4\right)+C$$

$u=x^2+4$

$du=2xdx$

$\frac{1}{2}du=xdx$

$$\int\frac{3x}{x^2+4}dx=\frac{1}{2}\int\frac{3}{u}du$$

$$=\frac{3}{2}\ln|u|+C$$

$$=\frac{3}{2}\ln\left(x^2+4\right)+C$$

Exercise

Evaluate the integral $\int\frac{dx}{2\sqrt{x}+2x}$

Solution

$$\int\frac{dx}{2\sqrt{x}+2x}=\int\frac{dx}{2\sqrt{x}\left(1+\sqrt{x}\right)}$$

$$u=1+\sqrt{x} \quad\Rightarrow\quad du=\frac{1}{2\sqrt{x}}dx$$

$$=\int\frac{du}{u}$$

$$=\ln u+C$$

$$=\ln\left(1+\sqrt{x}\right)+C$$

Exercise

Evaluate the integral $\int\frac{\sec x\,dx}{\sqrt{\ln\left(\sec x+\tan x\right)}}$

Solution

Let $u=\sec x+\tan x$

$$du=\left(\sec x\tan x+\sec^2 x\right)dx$$

$$=\sec x\left(\tan x+\sec x\right)dx$$

$$\sec x\,dx=\frac{du}{\tan x+\sec x}=\frac{du}{u}$$

$$\int \frac{\sec x dx}{\sqrt{\ln(\sec x+\tan x)}} = \int \frac{du}{u\sqrt{\ln u}}$$

$$= \int (\ln u)^{-1/2}\, d(\ln u) \qquad d(\ln u) = \frac{1}{u}du$$

$$= 2(\ln u)^{1/2} + C$$

$$= 2\sqrt{\ln(\sec x+\tan x)} + C$$

Exercise

Evaluate the integral $\int 8e^{(x+1)}dx$

Solution

$$d(x+1) = dx$$

$$\int 8e^{(x+1)}\, dx = 8\int e^{(x+1)}\, d(x+1)$$

$$= 8e^{(x+1)} + C$$

Exercise

Find the indefinite integral. $\int 4x\, e^{x^2}\, dx$

Solution

$$d\left(x^2\right) = 2xdx$$

$$\int 4x\, e^{x^2}\, dx = 2\int e^{x^2}\, d\left(x^2\right)$$

$$= 2e^{x^2} + C$$

Exercise

Evaluate the integral $\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}dr$

Solution

$$d\left(-\sqrt{r}\right) = -\frac{1}{2\sqrt{r}}dr$$

$$\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}dr = -2\int e^{-\sqrt{r}}\ d\left(-\sqrt{r}\right)$$

$$\underline{= -2e^{-\sqrt{r}} + C}$$

$$u = -r^{1/2}$$

$$du = -\frac{1}{2}r^{-1/2}dr$$

$$-2du = \frac{1}{r^{1/2}}dr$$

$$\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}dr = \int e^{u}\left(-2du\right)$$

$$= -2e^{u} + C$$

$$\underline{= -2e^{-\sqrt{r}} + C}$$

Exercise

Evaluate the integral $\int t^{3}e^{t^{4}}dt$

Solution

$$d\left(t^{4}\right) = 4t^{3}dt$$

$$\int t^{3}e^{t^{4}}dt = \frac{1}{4}\int e^{t^{4}}d\left(t^{4}\right)$$

$$= \underline{\frac{1}{4}e^{t^{4}} + C}$$

Exercise

Evaluate the integral $\int e^{\sec \pi t}\sec \pi\ \tan \pi t\ dt$

Solution

$$d\left(\sec \pi t\right) = \pi \sec \pi t \tan \pi t\ dt$$

$$\int e^{\sec \pi t}\sec \pi\ \tan \pi t\ dt = \frac{1}{\pi}\int e^{\sec \pi t}d\left(\sec \pi\right)$$

$$=\boxed{\frac{1}{\pi}e^{\sec \pi t}+C}$$

$u=\sec \pi t$

$du=\pi \sec \pi t \tan \pi t\ dt$

$\frac{1}{\pi}du=\sec \pi t \tan \pi t\ dt$

$$\int e^{\sec \pi t}\sec \pi\ \tan \pi t\ dt=\frac{1}{\pi}\int e^{u}du$$

$$=\frac{1}{\pi}e^{u}+C$$

$$=\boxed{\frac{1}{\pi}e^{\sec \pi t}+C}$$

Exercise

Find the integral $\int (2x+1)\, e^{x^2+x}\ dx$

Solution

$d\left(x^2+x\right)=(2x+1)dx$

$$\int (2x+1)\, e^{x^2+x}\ dx=\int e^{x^2+x}\ d\left(x^2+x\right)$$

$$=\boxed{e^{x^2+x}+C}$$

Exercise

Evaluate the integral $\int \frac{dx}{1+e^x}$

Solution

$$\int \frac{dx}{1+e^x}=\int \frac{e^{-x}}{e^{-x}}\frac{dx}{1+e^x}$$

$$=\int \frac{e^{-x}dx}{e^{-x}+1} \qquad d\left(e^{-x}+1\right)=-e^{-x}dx$$

$$=-\int \frac{1}{e^{-x}+1}\,d\left(e^{-x}+1\right)$$

$$=\boxed{-\ln\left(e^{-x}+1\right)+C}$$

Exercise

Find the integral $\displaystyle\int \frac{e^x}{1+e^x}\,dx$

Solution

$$d\left(e^x+1\right)=e^x dx$$

$$\int \frac{e^x}{1+e^x}\,dx=\int \frac{1}{1+e^x}\,d\left(1+e^x\right)$$

$$\boxed{=\ln(1+e^x)+C}$$

$$u=1+e^x$$

$$du=e^x dx$$

$$\int \frac{1}{u}du=\ln|u|+C$$

$$\boxed{=\ln(1+e^x)+C}$$

Exercise

Find the integral $\displaystyle\int \frac{2}{e^{-x}+1}dx$

Solution

$$\int \frac{2}{e^{-x}+1}dx=\int \frac{2}{e^{-x}+1}\frac{e^x}{e^x}dx$$

$$=2\int \frac{e^x}{1+e^x}dx$$

$$=2\int \frac{d(e^x+1)}{1+e^x}$$

$$\boxed{=2\ln(e^x+1)+C}$$

Exercise

Find the integral $\int \frac{1}{x^3} e^{1/4x^2}\, dx$

Solution

$$d\left(\frac{1}{4}x^{-2}\right) = -\frac{1}{2}x^{-3}dx$$

$$\int \frac{1}{x^3} e^{1/4x^2}\, dx = -2\int e^{1/4x^2}\, d\left(\frac{1}{4x^2}\right)$$

$$= -2e^{1/4x^2} + C$$

$$u = \frac{1}{4x^2} \quad = \frac{1}{4}x^{-2}$$

$$du = -\frac{1}{2}x^{-3}dx$$

$$-2du = \frac{1}{x^3}dx$$

$$\int e^u(-2)du = -2\int e^u du$$

$$= -2e^u + C$$

$$= -2e^{1/4x^2} + C$$

Exercise

Find the integral $\int \frac{e^{1/\sqrt{x}}}{x^{3/2}}\, dx$

Solution

$$d\left(\frac{1}{\sqrt{x}}\right) = -\frac{1}{2\, x^{3/2}}$$

$$\int \frac{e^{1/\sqrt{x}}}{x^{3/2}}\, dx = -2\int e^{1/\sqrt{x}}\, d\left(\frac{1}{\sqrt{x}}\right)$$

$$= -2e^{1/\sqrt{x}} + C$$

$$u = \frac{1}{\sqrt{x}} \quad = x^{-1/2}$$

$$du = -\frac{1}{2}x^{-3/2}dx$$

$$-2du = \frac{1}{x^{3/2}}dx$$

$$\int \frac{e^{1/\sqrt{x}}}{x^{3/2}}dx = \int e^{u}(-2du)$$

$$= -2\int e^{u}du$$

$$= -2e^{u} + C$$

$$\boxed{= -2e^{1/\sqrt{x}} + C}$$

Exercise

Find the integral $\int \frac{-e^{3x}}{2-e^{3x}}\,dx$

Solution

$$d\left(2-e^{3x}\right) = -3e^{3x}dx$$

$$\int \frac{-e^{3x}}{2-e^{3x}}\,dx = \frac{1}{3}\int \frac{1}{2-e^{3x}}\,d\left(2-e^{3x}\right)$$

$$\boxed{= \frac{1}{3}\ln\left|2-e^{3x}\right| + C}$$

Exercise

Evaluate the integral $\int \frac{7e^{7x}}{3+e^{7x}}dx$

Solution

$$d\left(3+e^{7x}\right) = 7e^{7x}dx$$

$$\int \frac{7e^{7x}}{3+e^{7x}}\,dx = \int \frac{1}{3+e^{7x}}d\left(3+e^{7x}\right)$$

$$\boxed{= \ln\left(3+e^{7x}\right) + C}$$

$$u = 3+e^{7x}$$

$$du = 7e^{7x}dx$$

$$\int \frac{7e^{7x}}{3+e^{7x}}\,dx = \int \frac{du}{u}$$

$$= \ln|u|$$

$$\boxed{= \ln\left(3+e^{7x}\right)+C}$$

Exercise

Find the integral $\displaystyle\int \frac{2\left(e^{x}-e^{-x}\right)}{\left(e^{x}+e^{-x}\right)^{2}}dx$

Solution

$$d\left(e^{x}+e^{-x}\right)=\left(e^{x}-e^{-x}\right)dx$$

$$\int \frac{2\left(e^{x}-e^{-x}\right)}{\left(e^{x}+e^{-x}\right)^{2}}dx = \int \frac{2}{\left(e^{x}+e^{-x}\right)^{2}}d\left(e^{x}-e^{-x}\right)$$

$$\boxed{=-\frac{2}{e^{x}+e^{-x}}+C}$$

$$u=e^{x}+e^{-x}$$

$$du=\left(e^{x}-e^{-x}\right)dx$$

$$\int \frac{2\left(e^{x}-e^{-x}\right)}{\left(e^{x}+e^{-x}\right)^{2}}dx = 2\int \frac{1}{u^{2}}du$$

$$=2\int u^{-2}du$$

$$=2\frac{u^{-1}}{-1}+C$$

$$=-2\frac{1}{u}+C$$

$$\boxed{=-\frac{2}{e^{x}+e^{-x}}+C}$$

Exercise

Evaluate the integral $\int \frac{3^x}{3-3^x}\,dx$

Solution

$$d\left(3-3^x\right)=\left(-3^x \ln 3\right)dx$$

$$\int \frac{3^x}{3-3^x}\,dx=-\frac{1}{\ln 3}\int \frac{1}{3-3^x}\,d\left(3-3^x\right)$$

$$\boxed{=-\frac{1}{\ln 3}\ln\left|3-3^x\right|+C}$$

Let $u=3-3^x$

$$du=\left(-3^x \ln 3\right)dx$$

$$-\frac{1}{\ln 3}du=3^x\,dx$$

$$\int \frac{3^x}{3-3^x}\,dx=-\frac{1}{\ln 3}\int \frac{du}{u}$$

$$=-\frac{1}{\ln 3}\ln|u|+C$$

$$\boxed{=-\frac{1}{\ln 3}\ln\left|3-3^x\right|+C}$$

Exercise

Find the integral $\int \left(6x+e^x\right)\sqrt{3x^2+e^x}\,dx$

Solution

$$d\left(3x^2+e^x\right)=\left(6x+e^x\right)dx$$

$$\int \left(6x+e^x\right)\sqrt{3x^2+e^x}\,dx=\int \left(3x^2+e^x\right)^{1/2}d\left(3x^2+e^x\right)$$

$$\boxed{=\frac{2}{3}\left(3x^2+e^x\right)^{3/2}+C}$$

$$u=3x^2+e^x$$

$$du=(6x+e^x)dx$$

$$\frac{du}{6x+e^x}=dx$$

$$\int \left(6x+e^x\right)\sqrt{3x^2+e^x}\,dx = \int \left(6x+e^x\right)\sqrt{u}\,\frac{du}{6x+e^x}$$

$$= \int u^{1/2}du$$

$$= \frac{2}{3}u^{3/2}+C$$

$$= \frac{2}{3}\left(3x^2+e^x\right)^{3/2}+C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x\,2^{x^2}}{1+2^{x^2}}\,dx$

Solution

$$d\left(1+2^{x^2}\right) = 2x(\ln 2)2^{x^2}\,dx$$

$$\int \frac{x\,2^{x^2}}{1+2^{x^2}}\,dx = \frac{1}{2\ln 2}\int \frac{1}{1+2^{x^2}}\,d\left(1+2^{x^2}\right)$$

$$= \frac{1}{2\ln 2}\ln\left(1+2^{x^2}\right)+C$$

Let $u = 1+2^{x^2}$

$$du = 2x2^{x^2}\ln(2)\,dx$$

$$\frac{du}{2\ln 2} = x2^{x^2}\,dx$$

$$\int \frac{x2^{x^2}}{1+2^{x^2}}dx = \frac{1}{2\ln 2}\int \frac{du}{u}$$

$$= \frac{1}{2\ln 2}\ln u + C$$

$$= \frac{1}{2\ln 2}\ln\left(1+2^{x^2}\right)+C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{x\left(\log_8 x\right)^2}$

Solution

$$\int \frac{dx}{x\left(\log_8 x\right)^2} = \int \frac{dx}{x\left(\frac{\ln x}{\ln 8}\right)^2}$$

$$= (\ln 8)^2 \int \frac{dx}{x(\ln x)^2} \qquad\qquad d(\ln x) = \frac{1}{x}dx$$

$$= (\ln 8)^2 \int \frac{d(\ln x)}{(\ln x)^2}$$

$$= -(\ln 8)^2 \frac{1}{\ln x} + C$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{x\sqrt{25x^2 - 2}}$

Solution

Let $u = 5x$

$du = 5dx$

$\frac{1}{5}du = dx$

$$\int \frac{dx}{x\sqrt{25x^2 - 2}} = \int \frac{du/5}{\frac{u}{5}\sqrt{u^2 - 2}}$$

$$= \int \frac{du}{u\sqrt{u^2 - \left(\sqrt{2}\right)^2}} \qquad\qquad \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}$$

$$= \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{u}{\sqrt{2}}\right| + C$$

$$= \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{5x}{\sqrt{2}}\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{6dr}{\sqrt{4-(r+1)^2}}$

Solution

$u = r+1 \quad \Rightarrow du = dr$

$a^2 = 4 \quad \to a = 2$

$$\int \frac{6dr}{\sqrt{4-(r+1)^2}} = 6\int \frac{du}{\sqrt{4-u^2}}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$= 6\sin^{-1}\frac{u}{2} + C$$

$$\boxed{= 6\sin^{-1}\left(\frac{r+1}{2}\right) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{2+(x-1)^2}$

Solution

$u = x-1 \quad \Rightarrow du = dx$

$a^2 = 2 \quad \to a = \sqrt{2}$

$$\int \frac{dx}{2+(x-1)^2} = \int \frac{du}{2+u^2}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$\boxed{= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{\sec^2 y\, dy}{\sqrt{1-\tan^2 y}}$

Solution

$u = \tan y \quad \Rightarrow du = \sec^2 y dy$

$a^2 = 1 \quad \to a = 1$

$$\int \frac{\sec^2 y\, dy}{\sqrt{1-\tan^2 y}} = \int \frac{du}{\sqrt{1-u^2}} \qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$= \sin^{-1}(u) + C$$

$$= \sin^{-1}(\tan y) + C$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{-x^2+4x-3}}$

Solution

$$\int \frac{dx}{\sqrt{-x^2+4x-3}} = \int \frac{dx}{\sqrt{1-x^2+4x-3-1}}$$

$$= \int \frac{dx}{\sqrt{1-\left(x^2-4x+4\right)}}$$

$$= \int \frac{dx}{\sqrt{1-(x+2)^2}} \qquad u = x+2 \quad \Rightarrow du = dx$$

$$= \int \frac{du}{\sqrt{1-u^2}} \int \frac{dx}{\sqrt{2x-x^2}} \qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$= \sin^{-1} u + C$$

$$= \sin^{-1}(x-2) + C$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{2x-x^2}}$

Solution

$$\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1+2x-x^2-1}}$$

$$= \int \frac{dx}{\sqrt{1-\left(x^2-2x+1\right)}}$$

$$= \int \frac{dx}{\sqrt{1-(x-1)^2}} \qquad u = x-1 \quad \Rightarrow du = dx$$

$$= \int \frac{du}{\sqrt{1-u^2}} \qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$\boxed{= \sin^{-1}(x-1) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x-2}{x^2-6x+10}dx$

Solution

$$\int \frac{x-2}{x^2-6x+10}dx = \int \frac{x-2}{x^2-6x+9+1}dx$$

$$= \int \frac{x-2-1+1}{(x-3)^2+1}dx$$

$$= \int \frac{x-3+1}{(x-3)^2+1}dx \qquad u = x-3 \quad \Rightarrow \quad du = dx$$

$$= \int \frac{u+1}{u^2+1}du$$

$$= \int \frac{u}{u^2+1}du + \int \frac{1}{u^2+1}du \qquad w = u^2+1 \quad \Rightarrow dw = 2udu \quad \to \frac{1}{2}dw = udu$$

$$= \frac{1}{2}\int \frac{dw}{w} + \int \frac{1}{u^2+1}du$$

$$= \frac{1}{2}\ln w + \tan^{-1}u + C$$

$$= \frac{1}{2}\ln\left((x-3)^2+1\right) + \tan^{-1}(x-3) + C$$

$$\boxed{= \frac{1}{2}\ln\left(x^2-6x+10\right) + \tan^{-1}(x-3) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

Solution

$$\int \frac{dx}{(x+1)\sqrt{x^2+2x}} = \int \frac{dx}{(x+1)\sqrt{x^2+2x+1-1}}$$

$$= \int \frac{dx}{(x+1)\sqrt{(x+1)^2-1}} \qquad \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u|$$

$$\boxed{= \sec^{-1}|x+1| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

Solution

$$\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}} = \int \frac{dx}{(x-2)\sqrt{x^2-4x+4-1}}$$

$$= \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}}$$

$$= \int \frac{du}{u\sqrt{u^2-1}} \qquad \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u|$$

$$= \sec^{-1} u + C$$

$$\boxed{= \sec^{-1}|x-2| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{e^{\cos^{-1}x}dx}{\sqrt{1-x^2}}$

Solution

$$\int \frac{e^{\cos^{-1}x}dx}{\sqrt{1-x^2}} = -\int e^{\cos^{-1}x}\, d\left(\cos^{-1}x\right) \qquad d\left(\cos^{-1}x\right) = \frac{-1}{\sqrt{1-x^2}}dx$$

$$\boxed{= -e^{\cos^{-1}x} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{\left(\sin^{-1} x\right)^2 dx}{\sqrt{1-x^2}}$

Solution

$$d\left(\sin^{-1} x\right) = \frac{dx}{\sqrt{1-x^2}}$$

$$\int \frac{\left(\sin^{-1} x\right)^2 dx}{\sqrt{1-x^2}} = \int \left(\sin^{-1} x\right)^2 d\left(\sin^{-1} x\right)$$

$$= \frac{1}{3}\left(\sin^{-1} x\right)^3 + C$$

Exercise

Evaluate $\displaystyle\int \frac{dy}{\left(\sin^{-1} y\right)\sqrt{1+y^2}}$

Solution

$$d\left(\sin^{-1} y\right) = \frac{dy}{\sqrt{1-y^2}}$$

$$\int \frac{dy}{\left(\sin^{-1} y\right)\sqrt{1+y^2}} = \int \frac{1}{\sin^{-1} y}\, d\left(\sin^{-1} y\right)$$

$$= \ln\left|\sin^{-1} y\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{1}{\sqrt{x}\,(x+1)\left(\left(\tan^{-1}\sqrt{x}\right)^2 + 9\right)} dx$

Solution

$$d\left(\tan^{-1}\sqrt{x}\right) = \frac{1}{2\sqrt{x}} \frac{1}{1+\left(\sqrt{x}\right)^2} dx$$

$$= \frac{1}{2\sqrt{x}\,(1+x)} dx$$

$$\int \frac{1}{\sqrt{x}\,(x+1)\left(\left(\tan^{-1}\sqrt{x}\right)^2+9\right)}dx = 2\int \frac{1}{\left(\tan^{-1}\sqrt{x}\right)^2+9}\, d\left(\tan^{-1}\sqrt{x}\right)$$

$$\boxed{= \frac{2}{3}\tan^{-1}\left(\frac{\tan^{-1}\sqrt{x}}{3}\right)+C}$$

Exercise

Evaluate the integral $\quad \int 2x\left(x^2+1\right)^4 dx$

Solution

$$d\left(x^2+1\right) = 2x\,dx$$

$$\int 2x\left(x^2+1\right)^4 dx = \int \left(x^2+1\right)^4 d\left(x^2+1\right)$$

$$\boxed{= \frac{1}{5}\left(x^2+1\right)^5 + C}$$

Exercise

Evaluate the integral $\quad \int 8x\cos\left(4x^2+3\right)dx$

Solution

$$d\left(4x^2+3\right) = 8x\,dx$$

$$\int 8x\cos\left(4x^2+3\right)dx = \int \cos\left(4x^2+3\right)d\left(4x^2+3\right)$$

$$\boxed{= \sin\left(4x^2+3\right) + C}$$

Exercise

Evaluate the integral $\quad \int \sin^3 x\cos x\; dx$

Solution

$$d(\sin x) = \cos x\; dx$$

$$\int \sin^3 x \cos x \, dx = \int \sin^3 x \, d(\sin x)$$

$$= \frac{1}{4}\sin^4 x + C$$

Exercise

Evaluate the integral $\int (6x+1)\sqrt{3x^2+x}\, dx$

Solution

$$d\left(3x^2+x\right) = (6x+1)\,dx$$

$$\int (6x+1)\sqrt{3x^2+x}\, dx = \int \left(3x^2+x\right)^{1/2} d\left(3x^2+x\right)$$

$$= \frac{2}{3}\left(3x^2+x\right)^{3/2} + C$$

Exercise

Evaluate the integral $\int 2x\left(x^2-1\right)^{99} dx$

Solution

$$d\left(x^2-1\right) = 2x\,dx$$

$$\int 2x\left(x^2-1\right)^{99} dx = \int \left(x^2-1\right)^{99} d\left(x^2-1\right)$$

$$= \frac{1}{100}\left(x^2-1\right)^{100} + C$$

Exercise

Evaluate the integral $\int xe^{x^2} dx$

Solution

$$\int xe^{x^2} dx = \frac{1}{2}\int e^{x^2} d\left(x^2\right) \qquad d\left(x^2\right) = 2xdx$$

$$= \frac{1}{2}e^{x^2} + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{2x^2}{\sqrt{1-4x^3}}dx$

Solution

$$d\left(1-4x^3\right) = -12x^2 dx$$

$$\int \frac{2x^2}{\sqrt{1-4x^3}}dx = -\frac{1}{6}\int \left(1-4x^3\right)^{-1/2} d\left(1-4x^3\right)$$

$$= -\frac{1}{3}\sqrt{1-4x^3} + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\left(\sqrt{x}+1\right)^4}{2\sqrt{x}}dx$

Solution

$$d\left(\sqrt{x}+1\right) = \frac{1}{2\sqrt{x}}dx$$

$$\int \frac{\left(\sqrt{x}+1\right)^4}{2\sqrt{x}}dx = \int \left(\sqrt{x}+1\right)^4 d\left(\sqrt{x}+1\right)$$

$$= \frac{1}{5}\left(\sqrt{x}+1\right)^5 + C$$

Exercise

Evaluate the integral $\displaystyle\int \left(x^2+x\right)^{10}(2x+1)dx$

Solution

$$d\left(x^2+x\right) = (2x+1)dx$$

$$\int \left(x^2+x\right)^{10}(2x+1)dx = \int \left(x^2+x\right)^{10} d\left(x^2+x\right)$$

$$= \frac{1}{11}\left(x^2+x\right)^{11} + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{10x-3}$

Solution

$$d(10x-3)=10dx$$

$$\int \frac{dx}{10x-3}=\frac{1}{10}\int \frac{d(10x-3)}{10x-3}$$

$$\boxed{=\ln|10x-3|+C}$$

Exercise

Evaluate the integral $\displaystyle\int x^3\left(x^4+16\right)^6 dx$

Solution

$$d\left(x^4+16\right)=4x^3dx$$

$$\int x^3\left(x^4+16\right)^6 dx=\frac{1}{4}\int \left(x^4+16\right)^6 d\left(x^4+16\right)$$

$$\boxed{=\frac{1}{28}\left(x^4+16\right)^7+C}$$

Exercise

Evaluate the integral $\displaystyle\int \sin^{10}\theta\ \cos\theta\ d\theta$

Solution

$$d(\sin\theta)=\cos\theta\, d\theta$$

$$\int \sin^{10}\theta\ \cos\theta\ d\theta=\int \sin^{10}\theta\ \ d(\sin\theta)$$

$$\boxed{=\frac{1}{11}\sin^{11}\theta+C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{1-9x^2}}$

Solution

$$d(3x) = 3dx$$

$$\int \frac{dx}{\sqrt{1-9x^2}} = \frac{1}{3}\int \frac{d(3x)}{\sqrt{1-(3x)^2}} \qquad\qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$= \frac{1}{3}\arcsin 3x + C$$

Exercise

Evaluate the integral $\displaystyle\int x^9 \sin x^{10}\, dx$

Solution

$$d\left(x^{10}\right) = 10x^9 dx$$

$$\int x^9 \sin x^{10}\, dx = \frac{1}{10}\int \sin x^{10}\, d\left(x^{10}\right)$$

$$= -\frac{1}{10}\cos x^{10} + C$$

Exercise

Evaluate the integral $\displaystyle\int \left(x^6-3x^2\right)^4 \left(x^5-x\right)dx$

Solution

$$d\left(x^6-3x^2\right) = 6\left(x^5-x\right)dx$$

$$\int \left(x^6-3x^2\right)^4 \left(x^5-x\right)dx = \frac{1}{6}\int \left(x^6-3x^2\right)^4\, d\left(x^6-3x^2\right)$$

$$= \frac{1}{30}\left(x^6-3x^2\right)^5 + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x}{x-2}\,dx$

Solution

$$\begin{array}{r} 1 \\ x-2\ \overline{)\,x\qquad} \\ \underline{-x+2} \\ 2 \end{array}$$

$$\int \frac{x}{x-2}\,dx = \int \left(1+\frac{2}{x-2}\right)dx$$

$$= x + 2\ln|x-2| + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{1+4x^2}$

Solution

$d(2x) = 2dx$

$$\int \frac{dx}{1+4x^2} = \frac{1}{2}\int \frac{d(2x)}{1+(2x)^2} \qquad\qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{1}{2}\arctan 2x + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{3}{1+25y^2}\,dy$

Solution

$d(5y) = 5dy$

$$\int \frac{3}{1+25y^2}\,dy = \frac{3}{5}\int \frac{d(5y)}{1+(5y)^2} \qquad\qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{3}{5}\arctan 5y + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{2}{x\sqrt{4x^2-1}}\,dx \quad \left(x > \frac{1}{2}\right)$

Solution

$$\int \frac{2}{x\sqrt{4x^2-1}}dx = \int \frac{d(2x)}{x\sqrt{(2x)^2-1}}$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|$$

$$\boxed{= \operatorname{arcsec}(2x)+C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{8x+6}{2x^2+3x}dx$

Solution

$$2d\left(2x^2+3x\right) = 2(4x+3)dx$$

$$\int \frac{8x+6}{2x^2+3x}dx = 2\int \frac{1}{2x^2+3x}d\left(2x^2+3x\right)$$

$$\boxed{= 2\ln\left|2x^2+3x\right|+C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x}{\sqrt{x-4}}dx$

Solution

$u = x-4 \quad \to \quad x = u+4$

$dx = du$

$$\int \frac{x}{\sqrt{x-4}}dx = \int \frac{u+4}{u^{1/2}}du$$

$$= \int \left(u^{1/2}+4u^{-1/2}\right)du$$

$$= \frac{2}{3}u^{3/2}+8u^{1/2}+C$$

$$\boxed{= \frac{2}{3}(x-4)^{3/2}+8(x-4)^{1/2}+C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^2}{(x+1)^4}dx$

Solution

$u = x+1 \quad \rightarrow \quad x = u-1$

$dx = du$

$$\int \frac{x^2}{(x+1)^4}\,dx = \int \frac{(u-1)^2}{u^4}\,du$$

$$= \int \frac{u^2 - 2u + 1}{u^4}\,du$$

$$= \int \left(\frac{1}{u^2} - 2u^{-3} + u^{-4}\right) du$$

$$= -\frac{1}{u} + u^{-2} - \frac{1}{3}u^{-3} + C$$

$$= -\frac{1}{x+1} + \frac{1}{(x+1)^2} - \frac{1}{3(x+1)^3} + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x}{\sqrt[3]{x+4}}\,dx$

Solution

$u = x+4 \quad \rightarrow \quad x = u-4$

$dx = du$

$$\int \frac{x}{\sqrt[3]{x+4}}\,dx = \int \frac{u-4}{u^{1/3}}\,du$$

$$= \int \left(u^{2/3} - 4u^{-1/3}\right) du$$

$$= \frac{3}{5}u^{5/3} - 6u^{2/3} + C$$

$$= \frac{3}{5}(x+4)^{5/3} - 6(x+4)^{2/3} + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

Solution

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{1}{e^x + e^{-x}}\,d\left(e^x + e^{-x}\right) \qquad d\left(e^x + e^{-x}\right) = \left(e^x - e^{-x}\right)dx$$

$$= \ln\left(e^x + e^{-x}\right) + C$$

Exercise

Evaluate the integral $\int x\sqrt[3]{2x+1}\,dx$

Solution

$$u = 2x+1 \quad \to \quad x = \frac{1}{2}(u-1)$$

$$dx = \frac{1}{2}du$$

$$\begin{aligned}\int x\sqrt[3]{2x+1}\,dx &= \int \frac{1}{2}(u-1)u^{1/3}\left(\frac{1}{2}du\right)\\ &= \frac{1}{4}\int\left(u^{4/3}-u^{1/3}\right)du\\ &= \frac{1}{4}\left(\frac{3}{7}u^{7/3}-\frac{3}{4}u^{4/3}\right)+C\\ &= \frac{3}{28}(2x+1)^{7/3}-\frac{3}{16}(2x+1)^{4/3}+C\end{aligned}$$

Exercise

Evaluate the integral $\int (x+1)\sqrt{3x+2}\,dx$

Solution

$$u = 3x+2 \quad \to \quad x = \frac{1}{3}(u-2)$$

$$dx = \frac{1}{3}du$$

$$\begin{aligned}\int (x+1)\sqrt{3x+2}\,dx &= \int\left(\frac{1}{3}u-2+1\right)u^{1/2}\ \frac{1}{3}du\\ &= \frac{1}{3}\int\left(\frac{1}{3}u-1\right)u^{1/2}\ du\\ &= \frac{1}{3}\int\left(\frac{1}{3}u^{3/2}-u^{1/2}\right)\ du\\ &= \frac{1}{3}\left(\frac{2}{15}u^{5/2}-\frac{2}{3}u^{3/2}\right)+C\\ &= \frac{2}{45}(3x+2)^{5/2}-\frac{2}{9}(3x+2)^{3/2}+C\end{aligned}$$

Exercise

Evaluate the integral $\int \sin^2 x\, dx$

Solution

$$\int \sin^2 x\, dx = \frac{1}{2}\int (1-\cos 2x)\,dx \qquad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$\boxed{= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C}$$

Exercise

Evaluate the integral $\int \sin^2\left(\theta + \frac{\pi}{6}\right) d\theta$

Solution

$$\int \sin^2\left(\theta+\frac{\pi}{6}\right)d\theta = \int \frac{1}{2}\left(1-\cos 2\left(\theta+\frac{\pi}{6}\right)\right)d\theta \qquad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$= \frac{1}{2}\int d\theta \;-\; \frac{1}{4}\int \cos\left(2\theta+\frac{\pi}{3}\right)\, d\left(2\theta+\frac{\pi}{3}\right) \qquad d\left(2\theta+\frac{\pi}{3}\right) = 2d\theta$$

$$\boxed{= \frac{1}{2}\theta - \frac{1}{4}\sin\left(2\theta+\frac{\pi}{3}\right) + C}$$

Exercise

Evaluate the integral $\int x\cos^2\left(x^2\right)dx$

Solution

$$d\left(x^2\right) = 2xdx$$

$$\int x\cos^2\left(x^2\right)dx = \frac{1}{2}\int \cos^2\left(x^2\right)\, d\left(x^2\right)$$

$$= \frac{1}{4}\int\left(1+\cos\left(2x^2\right)\right)\, d\left(x^2\right)$$

$$= \frac{1}{4}\int d\left(x^2\right) \;+\; \frac{1}{8}\int \cos\left(2x^2\right)\, d\left(2x^2\right) \qquad d\left(x^2\right) = 2d\left(x^2\right)$$

$$\boxed{= \frac{1}{4}x^2 + \frac{1}{8}\sin\left(2x^2\right) + C}$$

$$\int x\cos^2\left(x^2\right)dx = \frac{1}{2}\int x\left(1+\cos\left(2x^2\right)\right)\, dx \qquad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$= \frac{1}{2}\int x\, dx + \frac{1}{2}\int x\cos\left(2x^2\right)\, dx$$

$$= \frac{1}{4}x^2 + \frac{1}{8}\int \cos\left(2x^2\right)\, d\left(2x^2\right)$$

$$\boxed{= \frac{1}{4}x^2 + \frac{1}{8}\sin\left(2x^2\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sec 4x\ \tan 4x\ dx$

Solution

$$d\left(\sec 4x\right) = 4\sec 4x\ \tan 4x$$

$$\int \sec 4x\ \tan 4x\ dx = \frac{1}{4}\int d\left(\sec 4x\right)$$

$$\boxed{= \frac{1}{4}\sec 4x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sec^2 10x\ dx$

Solution

$$\int \sec^2 10x\ dx = \frac{1}{10}\int \sec^2 10x\ d\left(10x\right)$$

$$\boxed{= \frac{1}{10}\tan 10x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \left(\sin^5 x + 3\sin^3 x - \sin x\right)\cos x\ dx$

Solution

$$d\left(\sin x\right) = \cos x\, dx$$

$$\int \left(\sin^5 x + 3\sin^3 x - \sin x\right)\cos x \, dx = \int \left(\sin^5 x + 3\sin^3 x - \sin x\right) d(\sin x)$$

$$= \frac{1}{6}\sin^6 x + \frac{3}{4}\sin^4 x - \frac{1}{2}\sin^2 x + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\csc^2 x}{\cot^3 x} dx$

Solution

$d(\cot x) = -\csc^2 x \, dx$

$$\int \frac{\csc^2 x}{\cot^3 x} dx = -\int \cot^{-3} x \, d(\cot x)$$

$$= \frac{1}{2}\cot^{-2} x + C$$

$$= \frac{1}{2\cot^2 x} + C$$

$$= \frac{1}{2}\tan^2 x + C$$

Exercise

Evaluate the integral $\displaystyle\int \left(x^{3/2} + 8\right)^5 \sqrt{x} \, dx$

Solution

$d\left(x^{3/2} + 8\right) = \frac{3}{2} x^{1/2} dx$

$$\int \left(x^{3/2} + 8\right)^5 \sqrt{x} \, dx = \frac{2}{3}\int \left(x^{3/2} + 8\right)^5 \, d\left(x^{3/2} + 8\right)$$

$$= \frac{1}{9}\left(x^{3/2} + 8\right)^6 + C$$

Exercise

Evaluate the integral $\displaystyle\int \sin x \, \sec^8 x \, dx$

Solution

$d(\cos x) = -\sin x \, dx; \quad \sec x = \frac{1}{\cos x}$

$$\int \sin x \ \sec^8 x \, dx = -\int \cos^{-8} x \, d(\cos x)$$

$$= \frac{1}{7}\cos^{-7} x + C$$

$$= \frac{1}{7}\sec^7 x + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{e^{2x}}{e^{2x}+1} dx$

Solution

$$d\left(e^{2x}+1\right) = 2e^{2x} dx$$

$$\int \frac{e^{2x}}{e^{2x}+1} dx = \frac{1}{2}\int \frac{1}{e^{2x}+1} \, d\left(e^{2x}+1\right)$$

$$= \frac{1}{2}\ln\left(e^{2x}+1\right) + C$$

Exercise

Evaluate the integral $\displaystyle\int \sec^3\theta \ \tan\theta \ d\theta$

Solution

$$d(\sec\theta) = \tan\theta \sec\theta \, d\theta$$

$$\int \sec^3\theta \ \tan\theta \ d\theta = \int \sec^2\theta \ \sec\theta \tan\theta \ d\theta$$

$$= \int \sec^2\theta \ d(\sec\theta)$$

$$= \frac{1}{3}\sec^3\theta + C$$

Exercise

Evaluate the integral $\displaystyle\int x \ \sin^4 x^2 \ \cos x^2 \ dx$

Solution

$$d\left(\sin x^2\right) = 2x\cos x^2 \ dx$$

$$\int x\ \sin^4 x^2\ \cos x^2\ dx = \frac{1}{2}\int \sin^4 x^2\ \ d\left(\sin x^2\right)$$

$$= \frac{1}{10}\sin^5\left(x^2\right) + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{1+\sqrt{1+x}}}$

Solution

$$u = 1+\sqrt{1+x} \quad \rightarrow \quad \sqrt{1+x} = u-1$$

$$du = \frac{1}{2\sqrt{1+x}}dx$$

$$dx = 2(u-1)\,du$$

$$\int \frac{dx}{\sqrt{1+\sqrt{1+x}}} = 2\int \frac{(u-1)}{u^{1/2}}du$$

$$= 2\int \left(u^{1/2} - u^{-1/2}\right)du$$

$$= 2\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right) + C$$

$$= \frac{4}{3}\left(1+\sqrt{1+x}\right)^{3/2} - 4\left(1+\sqrt{1+x}\right)^{1/2} + C$$

Exercise

Evaluate the integral $\displaystyle\int \tan^{10} 4x\ \sec^2 4x\ dx$

Solution

$$d\left(\tan 4x\right) = 4\sec^2\left(4x\right)dx$$

$$\int \tan^{10} 4x\ \sec^2 4x\ dx = \frac{1}{4}\int \tan^{10} 4x\ \ d\left(\tan 4x\right)$$

$$= \frac{1}{44}\tan^{11} 4x + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^2}{x^3+27}dx$

Solution

$$d\left(x^3+27\right)=3x^2\ dx$$

$$\int \frac{x^2}{x^3+27}dx=\frac{1}{3}\int \frac{1}{x^3+27}d\left(x^3+27\right)$$

$$\boxed{=\frac{1}{3}\ln\left|x^3+27\right|+C}$$

Exercise

Evaluate the integral $\displaystyle\int y^2\left(3y^3+1\right)^4 dy$

Solution

$$d\left(3y^3+1\right)=9y^2dy$$

$$\int y^2\left(3y^3+1\right)^4 dy=\frac{1}{9}\int \left(3y^3+1\right)^4 d\left(3y^3+1\right)$$

$$\boxed{=\frac{1}{45}\left(3y^3+1\right)^5+C}$$

Exercise

Evaluate the integral $\displaystyle\int x\sin x^2\cos^8 x^2\ dx$

Solution

$$d\left(\cos x^2\right)=-2x\sin x^2\ dx$$

$$\int x\sin x^2\cos^8 x^2\ dx=-\frac{1}{2}\int \cos^8 x^2\ d\left(\cos x^2\right)$$

$$\boxed{=-\frac{1}{18}\cos^9\left(x^2\right)+C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

Solution

$$d\left(1+\cos^2 x\right) = -2\cos x \sin x\ dx$$

$$= -\sin 2x\ dx$$

$$\int \frac{\sin 2x}{1+\cos^2 x}\,dx = -\int \frac{1}{1+\cos^2 x}\,d\left(1+\cos^2 x\right)$$

$$= -\ln\left|1+\cos^2 x\right| + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$

Solution

$$d\left(\sin^{-1} x\right) = \frac{dx}{\sqrt{1-x^2}}$$

$$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx = \int \sin^{-1} x\ d\left(\sin^{-1} x\right)$$

$$= \frac{1}{2}\left(\sin^{-1} x\right)^2 + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\left(\tan^{-1} x\right)\left(1+x^2\right)}$

Solution

$$d\left(\tan^{-1} x\right) = \frac{dx}{1+x^2}$$

$$\int \frac{dx}{\left(\tan^{-1} x\right)\left(1+x^2\right)} = \int \frac{1}{\tan^{-1} x}\ d\left(\tan^{-1} x\right)$$

$$= \ln\left|\tan^{-1} x\right| + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\left(\tan^{-1} x\right)^5}{1+x^2}dx$

Solution

$$d\left(\tan^{-1} x\right) = \frac{dx}{1+x^2}$$

$$\int \frac{\left(\tan^{-1} x\right)^5}{1+x^2}dx = \int \left(\tan^{-1} x\right)^5 \, d\left(\tan^{-1} x\right)$$

$$= \frac{1}{6}\left(\tan^{-1} x\right)^6 + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{1}{x^2}\sin\frac{1}{x}\, dx$

Solution

$$d\left(\frac{1}{x}\right) = -\frac{1}{x^2}dx$$

$$\int \frac{1}{x^2}\sin\frac{1}{x}\, dx = -\int \sin\frac{1}{x}\, d\left(\frac{1}{x}\right)$$

$$= \cos\frac{1}{x} + C$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{2} x^2 e^{x^3+1} dx$

Solution

$$d\left(x^3+1\right) = 3x^2 dx$$

$$\int_{-1}^{2} x^2 e^{x^3+1}\, dx = \frac{1}{3}\int_{-1}^{2} e^{x^3+1}\, d\left(x^3+1\right)$$

$$= \frac{1}{3} e^{x^3+1} \Big|_{-1}^{2}$$

$$= \frac{1}{3}\left(e^9 - 1\right)$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 x^2 e^{x^3} dx$

Solution

$d\left(x^3\right) = 3x^2 dx$

$$\int_0^2 x^2 e^{x^3}\, dx = \frac{1}{3}\int_0^2 e^{x^3}\, d\left(x^3\right)$$

$$= \frac{1}{3}e^{x^3}\Big|_0^2$$

$$= \frac{1}{3}\left(e^8 - e\right)$$

Exercise

Evaluate the integral $\displaystyle\int_0^4 \frac{x}{x^2+1} dx$

Solution

$$\int_0^4 \frac{x}{x^2+1}\, dx = \frac{1}{2}\int_0^4 \frac{1}{x^2+1}\, d\left(x^2+1\right)$$

$$= \frac{1}{2}\ln\left(x^2+1\right)\Big|_0^4$$

$$= \frac{1}{2}(\ln 17 - \ln 1)$$

$$= \frac{1}{2}\ln 17$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{18\tan^2 x\sec^2 x}{\left(2+\tan^3 x\right)^2} dx$

a) $u = \tan x$, followed by $v = u^3$ *then by* $w = 2 + v$

b) $u = \tan^3 x$, followed by $v = 2 + u$

c) $u = 2 + \tan^3 x$

Solution

a) Let $u = \tan x \quad \Rightarrow \quad du = \sec^2 x\, dx$

$v = u^3 \quad \Rightarrow \quad dv = 3u^2 du$

$w = 2 + v \quad \Rightarrow \quad dw = dv$

$$\int \frac{18\tan^2 x \sec^2 x}{\left(2+\tan^3 x\right)^2} dx = \int \frac{18u^2 du}{\left(2+u^3\right)^2}$$

$$= \int \frac{6\, dv}{(2+v)^2}$$

$$= \int \frac{6dw}{w^2}$$

$$= 6\int w^{-2} dw$$

$$= 6\frac{w^{-1}}{-1} + C$$

$$= -\frac{6}{w} + C$$

$$= -\frac{6}{2+v} + C$$

$$= -\frac{6}{2+u^3} + C$$

$$\boxed{= -\frac{6}{2+\tan^3 x} + C}$$

b) $d\left(2+\tan^3 x\right) = 3\tan^2 x \sec^2 x\, dx$

$$\int \frac{18\tan^2 x \sec^2 x}{\left(2+\tan^3 x\right)^2} dx = \int \frac{6}{\left(2+\tan^3 x\right)^2} d\left(2+\tan^3 x\right)$$

$$\boxed{= -\frac{6}{2+\tan^3 x} + C}$$

Let $u = \tan^3 x \quad \Rightarrow \quad du = 3\tan^2 x \sec^2 x dx$

$v = 2 + u \quad \Rightarrow \quad dv = du$

$$\int \frac{18\tan^2 x \sec^2 x}{\left(2+\tan^3 x\right)^2} dx = \int \frac{6\, du}{(2+u)^2}$$

$$= \int \frac{6\, dv}{v^2}$$

$$= \int 6v^{-2}dv$$

$$= -6v^{-1} + C$$

$$= -\frac{6}{v} + C$$

$$= -\frac{6}{2+u} + C$$

$$\boxed{= -\frac{6}{2+\tan^3 x} + C}$$

c) Let $u = 2 + \tan^3 x$

$$du = 3\tan^2 x \sec^2 x\, dx$$

$$\frac{1}{3}du = \tan^2 x \sec^2 x\ dx$$

$$\int \frac{18\tan^2 x \sec^2 x}{\left(2+\tan^3 x\right)^2}dx = \int \frac{18}{u^2}\left(\frac{1}{3}du\right)$$

$$= 6\int u^{-2}du$$

$$= -6u^{-1} + C$$

$$= -\frac{6}{u} + C$$

$$\boxed{= -\frac{6}{2+\tan^3 x} + C}$$

Exercise

Evaluate: $\displaystyle\int_0^1 (2t+3)^3\, dt$

Solution

$$d(2t+3) = 2dt \ \rightarrow\ \frac{1}{2}d(2t+3) = dt$$

$$\int_0^1 (2t+3)^3\, dt = \frac{1}{2}\int_0^1 (2t+3)^3\, d(2t+3)$$

$$= \frac{1}{8}\ (2t+3)^4\ \Big|_0^1$$

$$= \frac{1}{8}\left[(2(1)+3)^4 - (2(0)+3)^4\right]$$

$$= \frac{1}{8}\left(5^4 - 3^4\right)$$

$$\underline{= 68}$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$

Solution

$$\int_0^2 \sqrt{4-x^2}\,dx = \left(\frac{1}{2}x\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right|_0^2$$

$\sqrt{4-x^2}$ is a semi-circle with center (0, 0) and radius = 2.

Since x from 0 to 2

Area = $\frac{1}{4}$(Area of this circle)

$$= \frac{1}{4}2\pi 2^2$$

$$\underline{= 2\pi \quad unit^2}$$

Exercise

Evaluate the integral $\displaystyle\int_0^3 \sqrt{y+1}\,dy$

Solution

$d(y+1) = dy$

$$\int_0^3 \sqrt{y+1}\,dy = \int_0^3 (y+1)^{1/2}\,d(y+1)$$

$$= \frac{2}{3}(y+1)^{3/2}\Big|_0^3$$

$$= \frac{2}{3}\left[(3+1)^{3/2} - (0+1)^{3/2}\right]$$

$$= \frac{2}{3}(8-1)$$

$$\underline{= \frac{14}{3}}$$

Exercise

Evaluate the integral $\int_{-1}^{1} r\sqrt{1-r^2}\ dr$

Solution

$$d\left(1-r^2\right) = -2rdr$$

$$\begin{aligned}\int_{-1}^{1} r\sqrt{1-r^2}\ dr &= -\frac{1}{2}\int_{-1}^{1}\left(1-r^2\right)^{1/2}\ d\left(1-r^2\right)\\ &= -\frac{1}{3}\left(1-r^2\right)^{3/2}\Big|_{-1}^{1}\\ &= -\frac{1}{3}\left[\left(1-(1)^2\right)^{3/2}-\left(1-(-1)^2\right)^{3/2}\right]\\ &= -\frac{1}{3}(0-0)\\ &= 0\end{aligned}$$

Exercise

Evaluate the integral $\int_{0}^{\pi/4} \tan x \sec^2 x\ dx$

Solution

$$d(\tan x) = \sec^2 x\ dx$$

$$\begin{aligned}\int_{0}^{\pi/4} \tan x \sec^2 x\ dx &= \int_{0}^{\pi/4} \tan x\ d(\tan x)\\ &= \frac{1}{2}\tan^2 x\ \Big|_{0}^{\pi/4}\\ &= \frac{1}{2}\left(1^2-0^2\right)\\ &= \frac{1}{2}\end{aligned}$$

Exercise

Evaluate the integral $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\ dx$

Solution

$$d(\cos x) = -\sin x\, dx$$

$$\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\, dx = -\int_{2\pi}^{3\pi} 3\cos^2 x\, d(\cos x)$$

$$= -\cos^3 x \Big|_{2\pi}^{3\pi}$$

$$= -\left((-1)^3 - 1^3\right)$$

$$\underline{= 2}$$

Exercise

Evaluate the integral $\int_0^1 t^3\left(1+t^4\right)^3\, dt$

Solution

$$d\left(1+t^4\right) = 4t^3\, dt$$

$$\int_0^1 t^3\left(1+t^4\right)^3\, dt = \frac{1}{4}\int_0^1 \left(1+t^4\right)^3 d\left(1+t^4\right)$$

$$= \frac{1}{16}\left(1+t^4\right)^4 \Big|_0^1$$

$$= \frac{1}{16}\left(2^4 - 1^4\right)$$

$$\underline{= \frac{15}{16}}$$

Exercise

Evaluate the integral $\int_0^1 \frac{r}{\left(4+r^2\right)^2}\, dr$

Solution

$$d\left(4+r^2\right) = 2rdr$$

$$\int_0^1 \frac{r}{\left(4+r^2\right)^2}\, dr = \frac{1}{2}\int_0^1 \frac{d\left(4+r^2\right)}{\left(4+r^2\right)^2}$$

$$= -\frac{1}{2}\left(\frac{1}{4+r^2}\right|_0^1$$

$$= -\frac{1}{2}\left(\frac{1}{5}-\frac{1}{4}\right)$$

$$\underline{= -\frac{1}{40}}\,\Big|$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 \frac{10\sqrt{v}}{\left(1+v^{3/2}\right)^2}\,dv$

Solution

$$d\left(1+v^{3/2}\right) = \frac{3}{2}\sqrt{v}\,dv$$

$$\int_0^1 \frac{10\sqrt{v}}{\left(1+v^{3/2}\right)^2}\,dv = \frac{20}{3}\int_0^1 \frac{1}{\left(1+v^{3/2}\right)^2}\,d\left(1+v^{3/2}\right)$$

$$= -\frac{20}{3}\left(\frac{1}{1+v^{3/2}}\right|_0^1$$

$$= -\frac{20}{3}\left(\frac{1}{2}-1\right)$$

$$\underline{= \frac{10}{3}}\,\Big|$$

Exercise

Evaluate the integral $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$

Solution

$$d\left(x^2+1\right) = 2x\,dx$$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx = 2\int_{-\sqrt{3}}^{\sqrt{3}} \left(x^2+1\right)^{-1/2}\,d\left(x^2+1\right)$$

$$= 4\sqrt{x^2+1}\,\Big|_{-\sqrt{3}}^{\sqrt{3}}$$

$$= 4(2-2)$$

$$\underline{=0}$$

Let $u = x^2 + 1$

$du = 2xdx \quad \rightarrow \quad \frac{1}{2}du = xdx$

$$\rightarrow \begin{cases} x = \sqrt{3} & \rightarrow u = 4 \\ x = -\sqrt{3} & \rightarrow u = 4 \end{cases}$$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}} dx = 4\int_4^4 \frac{1}{u}\left(\frac{1}{2}du\right)$$

$$\underline{=0}$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}} dx$

Solution

$$d\left(x^4+9\right) = 4x^3 dx$$

$$\int_0^1 \frac{x^3}{\sqrt{x^4+9}} dx = \frac{1}{4}\int_0^1 \left(x^4+9\right)^{-1/2} d\left(x^4+9\right)$$

$$= \frac{1}{2}\left(x^4+9\right)^{1/2} \Big|_0^1$$

$$= \frac{1}{2}\left(10^{1/2} - 9^{1/2}\right)$$

$$\underline{= \frac{\sqrt{10}-3}{2}}$$

$u = x^4 + 9$

$du = 4x^3 dx$

$\frac{1}{4}du = x^3 dx$

$$\begin{cases} x = 1 \rightarrow u = 10 \\ x = 0 \rightarrow u = 9 \end{cases}$$

$$\int_0^1 \frac{x^3}{\sqrt{x^4+9}} dx = \frac{1}{4}\int_3^{10} u^{-1/2} du$$

$$= \frac{1}{4}\left(2u^{1/2}\right|_{9}^{10}$$

$$= \frac{1}{2}\left(10^{1/2} - 9^{1/2}\right)$$

$$\underline{= \frac{\sqrt{10}-3}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/6} (1-\cos 3t)\sin 3t \; dt$

Solution

$$d(1-\cos 3t) = 3\sin 3t\, dt$$

$$\int_0^{\pi/6} (1-\cos 3t)\sin 3t \; dt = \frac{1}{3}\int_0^{\pi/6} (1-\cos 3t)\; d(1-\cos 3t)$$

$$= \frac{1}{6}\left(1-\cos 3t\right)^2 \Big|_0^{\pi/6}$$

$$= \frac{1}{6}\left(1^2 - 0^2\right)$$

$$\underline{= \frac{1}{6}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\; dt$

Solution

$$d\left(2+\tan\frac{t}{2}\right) = \frac{1}{2}\sec^2\frac{t}{2}\; dt$$

$$\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\; dt = 2\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right) d\left(2+\tan\frac{t}{2}\right)$$

$$= \left(2+\tan\frac{t}{2}\right)^2 \Big|_{-\pi/2}^{\pi/2}$$

$$= 3^2 - 1$$

$$\underline{= 8}$$

$$u = 2 + \tan\frac{t}{2}$$

$$du = \frac{1}{2}\sec^2\frac{t}{2}\ dt$$

$$2du = \sec^2\frac{t}{2}\ dt$$

$$\begin{cases} t = \frac{\pi}{2} & \to u = 3 \\ t = -\frac{\pi}{2} & \to u = 1 \end{cases}$$

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} \left(2 + \tan\frac{t}{2}\right)\sec^2\frac{t}{2}\ dt &= \int_1^3 u(2du) \\ &= 2\left(\frac{u^2}{2}\right|_1^3 \\ &= 3^2 - 1^2 \\ &= 8 \end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}dz$

Solution

$$d(4+3\sin z) = 3\cos z\, dz$$

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\ dz &= \frac{1}{3}\int_{-\pi}^{\pi} (4+3\sin z)^{-1/2}\ d(4+3\sin z) \\ &= \frac{2}{3}\sqrt{4+3\sin z}\ \Big|_{-\pi}^{\pi} \\ &= \frac{2}{3}(2-2) \\ &= 0 \end{aligned}$$

Let $u = 4+3\sin z$

$du = 3\cos z\ dz$

$\frac{1}{3}du = \cos z\ dz$

$$\begin{cases} z = \pi & \to u = 4 \\ z = -\pi & \to u = 4 \end{cases}$$

$$\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}} dz = \frac{1}{3}\int_{4}^{4} \frac{1}{\sqrt{u}} du$$

$$\underline{= 0}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi/2}^{0} \frac{\sin w}{(3+2\cos w)^2} dw$

Solution

$d(3+2\cos w) = -2\sin w\, dw$

$$\int_{-\pi/2}^{0} \frac{\sin w}{(3+2\cos w)^2} dw = -\frac{1}{2}\int_{-\pi/2}^{0} \frac{d(3+2\cos w)}{(3+2\cos w)^2} \qquad d\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$= \frac{1}{2}\left(\frac{1}{3+2\cos w}\right|_{-\pi/2}^{-}$$

$$= \frac{1}{2}\left(\frac{1}{5} - \frac{1}{3}\right)$$

$$\underline{= -\frac{1}{15}}$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{1} \sqrt{t^5+2t}\,\left(5t^4+2\right)dt$

Solution

$d\left(t^5+2t\right) = \left(5t^4+2\right)dt$

$$\int_{0}^{1} \sqrt{t^5+2t}\,\left(5t^4+2\right)dt = \int_{0}^{1} \left(t^5+2t\right)^{1/2}\, d\left(t^5+2t\right)$$

$$= \frac{2}{3}\left(t^5+2t\right)^{3/2}\Big|_{0}^{1}$$

$$= \frac{2}{3}\left(3^{3/2}\right)$$

$$\underline{= 2\sqrt{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_1^4 \frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2}$

Solution

$$d\left(1+\sqrt{y}\right)=\frac{1}{2\sqrt{y}}dy$$

$$\int_1^4 \frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2}=\int_1^4 \frac{1}{\left(1+\sqrt{y}\right)^2}\,d\left(1+\sqrt{y}\right)$$

$$=-\frac{1}{1+\sqrt{y}}\Bigg|_1^4$$

$$=-\left(\frac{1}{3}-\frac{1}{2}\right)$$

$$=\frac{1}{6}$$

Exercise

Evaluate the integral $\displaystyle\int_0^1\left(4y-y^2+4y^3+1\right)^{-2/3}\left(12y^2-2y+4\right)dy$

Solution

$$d\left(4y-y^2+4y^3+1\right)=\left(4-2y+12y^2\right)dy$$

$$\int_0^1\left(4y-y^2+4y^3+1\right)^{-2/3}\left(12y^2-2y+4\right)dy=\int_0^1\left(4y-y^2+4y^3+1\right)^{-2/3}d\left(4y-y^2+4y^3+1\right)$$

$$=3\left(4y-y^2+4y^3+1\right)^{1/3}\Bigg|_0^1$$

$$=3\ (2-1)$$

$$=3$$

Let $\quad u=4y-y^2+4y^3+1$

$$du=\left(4-2y+12y^2\right)dy$$

$$\rightarrow\begin{cases} y=1 & \rightarrow u=8 \\ y=0 & \rightarrow u=1 \end{cases}$$

$$\int_0^1 \left(4y - y^2 + 4y^3 + 1\right)^{-2/3}\left(12y^2 - 2y + 4\right)dy = \int_1^8 u^{-2/3}\,du$$

$$= 3u^{1/3}\Big|_1^8$$

$$= 3\left(8^{1/3} - 1^{1/3}\right)$$

$$\underline{= 3}$$

Exercise

Evaluate the integral $\int_0^5 |x-2|\,dx$

Solution

$$|x-2| = \begin{cases} x-2 & x > 2 \\ -(x-2) & x < 2 \end{cases}$$

$$\int_0^5 |x-2|\,dx = \int_0^2 -(x-2)\,dx + \int_2^5 (x-2)\,dx$$

$$= \left(-\frac{x^2}{2} + 2x\right)\Big|_0^2 + \left(\frac{x^2}{2} - 2x\right)\Big|_2^5$$

$$= -\frac{4}{2} + 4 - 0 + \left(\frac{25}{2} - 10 - \left(\frac{4}{2} - 4\right)\right)$$

$$= -2 + 4 + \frac{25}{2} - 10 - 2 + 4$$

$$= \frac{25}{2} - 6$$

$$\underline{= \frac{13}{2}}$$

Exercise

Evaluate the integral $\int_0^{\pi/2} e^{\sin x}\cos x\,dx$

Solution

$$d(\sin x) = \cos x\,dx$$

$$\int_0^{\pi/2} e^{\sin x}\cos x\,dx = \int_0^{\pi/2} e^{\sin x}\,d(\sin x)$$

$$= e^{\sin x} \Big|_0^{\pi/2}$$

$$= e^{\sin\frac{\pi}{2}} - e^{\sin 0}$$

$$= e^1 - e^0$$

$$\boxed{= e - 1}$$

Exercise

Evaluate $\displaystyle\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$

Solution

$$\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big|_{\sqrt{2}/2}^{\sqrt{3}/2}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$\boxed{= \frac{\pi}{12}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\, d\theta$

Solution

$$\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\, d\theta = \int_0^{\pi/3} \frac{d(1-4\cos\theta)}{1-4\cos\theta}$$

$$= \ln|1-4\cos\theta| \Big|_0^{\pi/3}$$

$$= \ln\left|1-4\cos\frac{\pi}{3}\right| - \ln|1-4\cos 0|$$

$$= \ln|-1| - \ln|-3|$$

$$= \ln 1 - \ln 3$$

$$= -\ln 3$$

$$\boxed{= \frac{1}{\ln 3}}$$

Exercise

Evaluate the integral $\displaystyle\int_1^2 \frac{2\ln x}{x}\,dx$

Solution

$$d(\ln x) = \frac{1}{x}dx$$

$$\int_1^2 \frac{2\ln x}{x}\,dx = 2\int_1^2 \ln x \; d(\ln x)$$

$$= (\ln x)^2 \Big|_1^2$$

$$= (\ln 2)^2$$

$$u = \ln x$$

$$du = \frac{1}{x}dx$$

$$\begin{cases} x = 1 & u = \ln 1 = 0 \\ x = 2 & u = \ln 2 \end{cases}$$

$$\int_1^2 \frac{2\ln x}{x}\,dx = \int_0^{\ln 2} 2u\,du$$

$$= u^2 \Big|_0^{\ln 2}$$

$$= (\ln 2)^2$$

Exercise

Evaluate the integral $\displaystyle\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$

Solution

$$d(\ln x) = \frac{1}{x}dx$$

$$\int_2^{16} \frac{dx}{2x\sqrt{\ln x}} = \frac{1}{2}\int_2^{16} (\ln x)^{-1/2} \; d(\ln x)$$

$$= \sqrt{\ln x}\,\Big|_2^{16}$$

$$= \sqrt{\ln 2^4} - \sqrt{\ln 2}$$

$$= 2\sqrt{\ln 2} - \sqrt{\ln 2}$$

$$= \sqrt{\ln 2}$$

$u = \ln x$

$du = \frac{1}{x}dx$

$$\begin{cases} x = 2 & u = \ln 2 \\ x = 16 & u = \ln 16 = \ln 2^4 \end{cases}$$

$$\int_2^{16} \frac{dx}{2x\sqrt{\ln x}} = \int_{\ln 2}^{4\ln 2} \frac{1}{2}u^{-1/2}\, du$$

$$= u^{1/2} \Big|_{\ln 2}^{4\ln 2}$$

$$= (4\ln 2)^{1/2} - (\ln 2)^{1/2}$$

$$= 2\sqrt{\ln 2} - \sqrt{\ln 2}$$

$$= \sqrt{\ln 2}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/2} \tan\frac{x}{2}\, dx$

Solution

$$d\cos\frac{x}{2} = -\frac{1}{2}\sin\frac{x}{2}dx$$

$$\int_0^{\pi/2} \tan\frac{x}{2}\, dx = \int_0^{\pi/2} \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\, dx$$

$$= \int_0^{\pi/2} \frac{-2}{\cos\frac{x}{2}}\, d\cos\frac{x}{2}$$

$$= -2\ln\left|\cos\frac{x}{2}\right| \Big|_0^{\pi/2}$$

$$= -2\left(\ln\left|\cos\frac{\pi}{4}\right| - \ln\left|\cos 0\right|\right)$$

$$= -2\left(\ln\left|\frac{1}{\sqrt{2}}\right| - \ln|1|\right)$$

$$= -2\ln\left(2^{-1/2}\right)$$

$$\underline{= \ln 2}$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/4}^{\pi/2} \cot x\, dx$

Solution

$$\int_{\pi/4}^{\pi/2} \cot x\, dx = \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x}\, dx$$

$$= \int_{\pi/4}^{\pi/2} \frac{d(\sin x)}{\sin x}$$

$$= \ln(\sin x)\Big|_{\pi/4}^{\pi/2}$$

$$= \ln 1 - \ln\frac{1}{\sqrt{2}}$$

$$= -\ln\frac{1}{\sqrt{2}}$$

$$\underline{= \ln\sqrt{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\ln 2}^{0} e^{-x}\, dx$

Solution

$$\int_{-\ln 2}^{0} e^{-x} dx = -e^{-x}\Big|_{-\ln 2}^{0}$$

$$= -\left(e^{0} - e^{\ln 2}\right)$$

$$= -(1-2)$$

$$\underline{= 1}$$

Exercise

Evaluate the integral $\int_{\pi/4}^{\pi/2}\left(1+e^{\cot\theta}\right)\csc^2\theta\, d\theta$

Solution

$d(\cot\theta) = -\csc\theta\, d\theta$

$$\begin{aligned}
\int_{\pi/4}^{\pi/2}\left(1+e^{\cot\theta}\right)\csc^2\theta\, d\theta &= -\int_{\pi/4}^{\pi/2}\left(1+e^{\cot\theta}\right) d(\cot\theta) \\
&= -\left(\cot\theta + e^{\cot\theta}\ \right|_{\pi/4}^{\pi/2} \\
&= -\left(e^0 - 1 - e\right) \\
&= e
\end{aligned}$$

Let $u = \cot\theta$

$du = -\csc^2\theta\, d\theta$

$$\begin{cases} \theta = \frac{\pi}{2} & \Rightarrow u = 0 \\ \theta = \frac{\pi}{4} & \Rightarrow u = 1 \end{cases}$$

$$\begin{aligned}
\int_{\pi/4}^{\pi/2}\left(1+e^{\cot\theta}\right)\csc^2\theta\, d\theta &= \int_{\pi/4}^{\pi/2}\csc^2\theta\, d\theta + \int_{\pi/4}^{\pi/2} e^{\cot\theta}\csc^2\theta\, d\theta \\
&= -\cot\theta\ \Big|_{\pi/4}^{\pi/2} + \int_0^1 e^u du \\
&= -\left(\cot\frac{\pi}{2} - \cot\frac{\pi}{4}\right) + e^u\ \Big|_0^1 \\
&= -(0-1) + e^1 - 1 \\
&= e
\end{aligned}$$

Exercise

Evaluate the integral $\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\, dx$

Solution

$d\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}dx$

$$\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int_1^4 2^{\sqrt{x}}\,d\left(\sqrt{x}\right)$$

$$= \frac{2}{\ln 2}\left(2^{\sqrt{x}}\right|_1^4$$

$$= \frac{2}{\ln 2}\,(4-2)$$

$$= \boxed{\frac{4}{\ln 2}}$$

Let $u = \sqrt{x}$

$$du = \frac{1}{2\sqrt{x}}dx$$

$$2du = \frac{1}{\sqrt{x}}dx$$

$$\begin{cases} x=1 & u=1 \\ x=4 & u=\sqrt{4}=2 \end{cases}$$

$$\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}dx = \int_1^2 2^u\,(2du)$$

$$= 2\int_1^2 2^u\,du$$

$$= 2\left(\frac{2^u}{\ln 2}\right|_1^2$$

$$= \frac{2}{\ln 2}\left(2^2 - 2^1\right)$$

$$= \frac{2}{\ln 2}(2)$$

$$= \boxed{\frac{4}{\ln 2}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t} \sec^2 t\ dt$

Solution

$$d(\tan t) = \sec^2 t\ dt$$

$$\int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t} \sec^2 t\ dt = \int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t}\ d(\tan t)$$

$$= \frac{1}{\ln\frac{1}{3}} \left(\frac{1}{3}\right)^{\tan t} \Big|_0^{\pi/4}$$

$$= -\ln 3\left(\frac{1}{3} - 1\right)$$

$$= \boxed{\frac{2}{3\ln 3}}$$

$u = \tan t$

$du = \sec^2 t\ dt$

$$\begin{cases} t = \frac{\pi}{4} & \rightarrow u = 1 \\ t = 0 & \rightarrow u = 0 \end{cases}$$

$$\int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t} \sec^2 t\ dt = \int_0^{1} \left(\frac{1}{3}\right)^{u} du$$

$$= \frac{1}{\ln\frac{1}{3}} \left(\frac{1}{3}\right)^{u} \Big|_0^{1}$$

$$= \frac{1}{-\ln 3}\left(\frac{1}{3} - 1\right)$$

$$= \frac{1}{-\ln 3}\left(\frac{-2}{3}\right)$$

$$= \boxed{\frac{2}{3\ln 3}}$$

Exercise

Evaluate the integral $\int_1^e x^{(\ln 2)-1}\ dx$

Solution

$$\int_1^e x^{(\ln 2)-1}\ dx = \frac{1}{\ln 2} x^{\ln 2} \Big|_1^e$$

$$= \frac{1}{\ln 2}\left(e^{\ln 2} - 1\right)$$

$$= \frac{1}{\ln 2}(2-1)$$

$$= \boxed{\frac{1}{\ln 2}}$$

Exercise

Evaluate the integral $\displaystyle\int_1^e \frac{2\ln 10 \log_{10} x}{x}\,dx$

Solution

$$\int_1^e \frac{2\ln 10 \log_{10} x}{x}\,dx = 2\ln 10\int_1^e \frac{1}{x}\frac{\ln x}{\ln 10}\,dx \qquad = 2\int_1^e \frac{\ln x}{x}dx \quad d(\ln x) = \frac{1}{x}dx$$

$$= 2\int_1^e \ln x \;\; d(\ln x)$$

$$= 2\left(\tfrac{1}{2}(\ln x)^2\right)\Big|_1^e$$

$$= (\ln e)^2 - (\ln 1)^2$$

$$= 1$$

Exercise

Evaluate the integral $\displaystyle\int_0^9 \frac{2\log_{10}(x+1)}{x+1}dx$

Solution

$$\int_0^9 \frac{2\log_{10}(x+1)}{x+1}\,dx = 2\int_0^9 \frac{1}{x+1}\frac{\ln(x+1)}{\ln 10}\,dx$$

$$= \frac{2}{\ln 10}\int_0^9 \frac{\ln(x+1)}{x+1}dx \qquad d(\ln(x+1)) = \frac{1}{x+1}dx$$

$$= \frac{2}{\ln 10}\int_0^9 \ln(x+1)\;\; d(x+1)$$

$$= \frac{2}{\ln 10}\left(\tfrac{1}{2}(\ln(x+1))^2\right)\Big|_0^9$$

$$= \frac{1}{\ln 10}\left[(\ln 10)^2 - (\ln 1)^2\right]$$

$$= \frac{1}{\ln 10}\left[(\ln 10)^2\right]$$

$$= \ln 10$$

Exercise

Evaluate the integral $\int_1^{e^x} \frac{1}{t}\, dt$

Solution

$$\int_1^{e^x} \frac{1}{t}\, dt = \ln|t| \Big|_1^{e^x}$$

$$= \ln\left|e^x\right| - \ln 1$$

$$\underline{= x}$$

Exercise

Evaluate the integral $\frac{1}{\ln a}\int_1^{x} \frac{1}{t}\, dt \quad x > 0$

Solution

$$\frac{1}{\ln a}\int_1^{x} \frac{1}{t}\, dt = \frac{1}{\ln a}\left(\ln|t| \Big|_1^{x}\right.$$

$$= \frac{1}{\ln a}(\ln x - \ln 1)$$

$$= \frac{\ln x}{\ln a}$$

$$\underline{= \log_a x}$$

Exercise

Evaluate the integral $\int_0^{\sqrt{\ln \pi}} 2x\, e^{x^2} \cos\left(e^{x^2}\right) dx$

Solution

$$\int_0^{\sqrt{\ln \pi}} 2x\, e^{x^2} \cos\left(e^{x^2}\right) dx = \int_0^{\sqrt{\ln \pi}} \cos\left(e^{x^2}\right) d\left(e^{x^2}\right)$$

$$= \sin\left(e^{x^2}\right) \Big|_0^{\sqrt{\ln \pi}}$$

$$= \sin \pi - \sin 1$$

$$\underline{= -\sin 1} \qquad \approx \underline{-0.84147}$$

Exercise

Evaluate $\displaystyle\int_0^{3\sqrt{2}/4} \frac{dx}{\sqrt{9-4x^2}}$

Solution

Let: $u = 2x$

$du = 2dx \quad \to \quad \frac{du}{2} = dx$

$$\begin{cases} x = \frac{3\sqrt{2}}{4} & \to u = \frac{3\sqrt{2}}{2} \\ x = 0 & \to u = 0 \end{cases}$$

$$\int_0^{3\sqrt{2}/4} \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int_0^{3\sqrt{2}/2} \frac{du}{\sqrt{9-u^2}}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$= \frac{1}{2}\sin^{-1}\frac{u}{3}\Big|_0^{3\sqrt{2}/2}$$

$$= \frac{1}{2}\left(\sin^{-1}\frac{\sqrt{2}}{2} - \sin^{-1}0\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi}{8}$$

Exercise

Evaluate $\displaystyle\int_{\pi/6}^{\pi 4} \frac{\csc^2 x}{1+(\cot x)^2}\,dx$

Solution

$d(\cot x) = -\csc^2 x\,dx$

$$\int_{\pi/6}^{\pi 4} \frac{\csc^2 x}{1+(\cot x)^2}\,dx = -\int_{\pi/6}^{\pi 4} \frac{1}{1+(\cot x)^2}\,d(\cot x)$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= -\arctan(\cot x)\Big|_{\pi/6}^{\pi/4}$$

$$= -\left(\arctan(1) - \arctan\left(\sqrt{3}\right)\right)$$

$$= -\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$$

$$\boxed{= \frac{\pi}{12}}$$

$u = \cot x \quad du = -\csc^2 x\, dx$

$a^2 = 1 \qquad \to a = 1$

$$\begin{cases} x = \frac{\pi}{4} & \to u = \cot\frac{\pi}{4} = 1 \\ x = \frac{\pi}{6} & \to u = \cot\frac{\pi}{6} = \sqrt{3} \end{cases}$$

$$\int_{\pi/6}^{\pi 4} \frac{\csc^2 x dx}{1 + (\cot x)^2} = -\int_{\sqrt{3}}^{1} \frac{du}{1+u^2}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= -\tan^{-1} u \Big|_{\sqrt{3}}^{1}$$

$$= -\left(\tan^{-1} 1 - \tan^{-1}\sqrt{3}\right)$$

$$= -\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$$

$$\boxed{= \frac{\pi}{12}}$$

Exercise

Evaluate $\displaystyle\int_1^{e^{\pi/4}} \frac{4dt}{t\left(1+\ln^2 t\right)}$

Solution

$d(\ln t) = \frac{1}{t}\, dt$

$$\int_1^{e^{\pi/4}} \frac{4dt}{t\left(1+\ln^2 t\right)} = 4\int_1^{e^{\pi/4}} \frac{1}{1+\ln^2 t}\, d(\ln t)$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= -\arctan(\ln t) \Big|_1^{e^{\pi/4}}$$

$\ln e^u = u$

$$= -\left(\arctan(0) - \arctan\left(\frac{\pi}{4}\right)\right)$$

$$\boxed{= 4\arctan\left(\frac{\pi}{4}\right)}$$

$u = \ln t \quad \Rightarrow \; du = \frac{dt}{t}$

$a^2 = 1 \quad \rightarrow \; a = 1$

$$\begin{cases} u = e^{\pi/4} & \rightarrow u = \ln e^{\pi/4} = \frac{\pi}{4} \\ \quad u = 1 & \quad \rightarrow u = \ln 1 = 0 \end{cases}$$

$$\int_1^{e^{\pi/4}} \frac{4dt}{t\left(1+\ln^2 t\right)} = 4\int_0^{\pi/4} \frac{du}{1+u^2}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \tan^{-1} u \Big|_0^{\pi/4}$$

$$= 4\left(\tan^{-1}\frac{\pi}{4} - \tan^{-1} 0\right)$$

$$\boxed{= 4\tan^{-1}\frac{\pi}{4}}$$

Exercise

Evaluate $\displaystyle\int_{1/2}^{1} \frac{6}{\sqrt{-4x^2+4x+3}}\, dx$

Solution

$$-4x^2+4x+3 = -4x^2+4x+3+1-1$$

$$= 4-4x^2+4x-1$$

$$= 4-\left(4x^2-4x+1\right)$$

$$= 2^2-(2x-1)^2$$

$$\int_{1/2}^{1} \frac{6}{\sqrt{-4x^2+4x+3}}\, dx = \int_{1/2}^{1} \frac{6}{\sqrt{2^2-(2x-1)^2}}\, dx$$

$u = 2x-1 \quad \Rightarrow du = 2dx \rightarrow \frac{du}{2} = dx$

$$= \int_{1/2}^{1} \frac{3}{\sqrt{2^2-u^2}}\, du$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$= 3\sin^{-1}\left(\frac{2x-1}{2}\right) \Big|_{1/2}^{1}$$

$$= 3\left(\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right)$$

$$= 3\left(\frac{\pi}{6} - 0\right)$$

$$\boxed{= \frac{\pi}{2}}$$

Exercise

Evaluate $\displaystyle\int_{2/\sqrt{3}}^{2} \frac{\cos\left(\sec^{-1} x\right)}{x\sqrt{x^2 - 1}}\,dx$

Solution

$$d\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2 - 1}}\,dx$$

$$\int_{2/\sqrt{3}}^{2} \frac{\cos\left(\sec^{-1} x\right)}{x\sqrt{x^2 - 1}}\,dx = \int_{2/\sqrt{3}}^{2} \cos\left(\sec^{-1} x\right) d\left(\sec^{-1} x\right)$$

$$= \sin\left(\sec^{-1} x\right)\Big|_{2/\sqrt{3}}^{2}$$

$$= \sin\left(\sec^{-1} 2\right) - \sin\left(\sec^{-1} \frac{2}{\sqrt{3}}\right)$$

$$= \sin\frac{\pi}{3} - \sin\frac{\pi}{6}$$

$$\boxed{= \frac{\sqrt{3} - 1}{2}}$$

$u = \sec^{-1} x \qquad du = \dfrac{dx}{x\sqrt{x^2 - 1}}$

$$\begin{cases} x = 2 & \rightarrow u = \sec^{-1} 2 = \frac{\pi}{3} \\ x = \frac{2}{\sqrt{3}} & \rightarrow u = \sec^{-1} \frac{2}{\sqrt{3}} = \frac{\pi}{6} \end{cases}$$

$$\int_{2/\sqrt{3}}^{2} \frac{\cos\left(\sec^{-1} x\right)}{x\sqrt{x^2 - 1}}\,dx = \int_{\pi/6}^{\pi/3} \cos u\, du$$

$$= \sin u\Big|_{\pi/6}^{\pi/3}$$

$$= \sin\frac{\pi}{3} - \sin\frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2}$$

$$\boxed{= \frac{\sqrt{3} - 1}{2}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^3 \frac{x}{\sqrt{25-x^2}}dx$

Solution

$d\left(25-x^2\right)=-2xdx$

$$\int_0^3 \frac{x}{\sqrt{25-x^2}}dx=-\frac{1}{2}\int_0^3\left(25-x^2\right)^{-1/2}d\left(25-x^2\right)$$

$$=-\sqrt{25-x^2}\,\Big|_0^3$$

$$=-(4-5)$$

$$\underline{=1}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^\pi \sin^2 5\theta\, d\theta$

Solution

$$\int_0^\pi \sin^2 5\theta\, d\theta=\frac{1}{2}\int_0^\pi (1-\cos 10\theta)\, d\theta \qquad \sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$$

$$=\frac{1}{2}\left(\theta-\frac{1}{10}\sin 10\theta\,\right|_0^\pi$$

$$\underline{=\frac{\pi}{2}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^\pi \left(1-\cos^2 3\theta\right) d\theta$

Solution

$$\int_0^\pi \left(1-\cos^2 3\theta\right) d\theta=\int_0^\pi\left(1-\frac{1}{2}-\cos 6\theta\right) d\theta \qquad \cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$$

$$=\int_0^\pi\left(\frac{1}{2}-\cos 6\theta\right) d\theta$$

$$= \frac{1}{2}\theta - \frac{1}{6}\sin 6\theta \Big|_0^{\pi}$$

$$= \frac{\pi}{2}$$

Exercise

Evaluate the definite integral $\displaystyle\int_2^3 \frac{x^2+2x-2}{x^3+3x^2-6x}dx$

Solution

$$d\left(x^3+3x^2-6x\right) = \left(3x^2+6x-6\right)dx$$

$$= 3\left(x^2+2x-2\right)dx$$

$$\int_2^3 \frac{x^2+2x-2}{x^3+3x^2-6x}dx = \frac{1}{3}\int_2^3 \frac{1}{x^3+3x^2-6x}\,d\left(x^3+3x^2-6x\right)$$

$$= \frac{1}{3}\ln\left|x^3+3x^2-6x\right| \Big|_2^3$$

$$= \frac{1}{3}(\ln 36 - \ln 8)$$

$$= \frac{1}{3}\left(\ln 6^2 - \ln 2^3\right)$$

$$= \frac{1}{3}(2\ln 6 - 3\ln 2)$$

$$= \frac{2}{3}\ln 6 - \ln 2$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\ln 2} \frac{e^x}{1+e^{2x}}dx$

Solution

$$d\left(e^x\right) = e^x dx$$

$$\int_0^{\ln 2} \frac{e^x}{1+e^{2x}}dx = \int_0^{\ln 2} \frac{1}{1+\left(e^x\right)^2}d\left(e^x\right)$$

$$= \arctan e^{x} \Big|_{0}^{\ln 2} \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \arctan e^{\ln 2} - \arctan 1$$

$$\underline{= \arctan 2 - \frac{\pi}{4}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_1^3 x\sqrt[3]{x^2-1}\,dx$

Solution

$$d\left(x^2-1\right) = 2x\,dx$$

$$\int_1^3 x\sqrt[3]{x^2-1}\,dx = \frac{1}{2}\int_1^3 \left(x^2-1\right)^{1/3} d\left(x^2-1\right)$$

$$= \frac{3}{8}\left(x^2-1\right)^{4/3}\Big|_1^3$$

$$= \frac{3}{8}\left(8^{4/3}-0\right)$$

$$= \frac{3}{8}\left(2^4\right)$$

$$\underline{= 6}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^2 (x+3)^3\,dx$

Solution

$$d(x+3) = dx$$

$$\int_0^2 (x+3)^3\,dx = \int_0^2 (x+3)^3\,d(x+3)$$

$$= \frac{1}{4}(x+3)^4\Big|_0^2$$

$$= \frac{1}{4}\left(5^4-3^4\right)$$

$$= \frac{1}{4}(625 - 81)$$

$$= \frac{544}{4}$$

$$\underline{= 136}$$

Exercise

Evaluate the definite integral $\displaystyle\int_{-2}^{2} e^{4x+8}dx$

Solution

$$d(4x+8) = 4dx$$

$$\int_{-2}^{2} e^{4x+8}\,dx = \frac{1}{4}\int_{-2}^{2} e^{4x+8}\,d(4x+8)$$

$$= \frac{1}{4}e^{4x+8}\Big|_{-2}^{2}$$

$$\underline{= \frac{1}{4}\left(e^{16} - 1\right)}$$

Exercise

Evaluate the definite integral $\displaystyle\int_{0}^{1} \sqrt{x}\left(\sqrt{x} + 1\right)dx$

Solution

$$\int_{0}^{1} \sqrt{x}\,\left(\sqrt{x} + 1\right)dx = \int_{0}^{1}\left(x + x^{1/2}\right)\,dx$$

$$= \frac{1}{2}x + \frac{2}{3}x^{3/2}\Big|_{0}^{1}$$

$$= \frac{1}{2} + \frac{2}{3}$$

$$\underline{= \frac{7}{6}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

Solution

$$\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\frac{x}{2}\Big|_0^1 \qquad\qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$= \sin^{-1}\frac{1}{2} - \sin^{-1}0$$

$$= \frac{\pi}{3}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^2 \frac{2x}{\left(x^2+1\right)^2}dx$

Solution

$$\int_0^2 \frac{2x}{\left(x^2+1\right)^2}dx = \int_0^2 \frac{1}{\left(x^2+1\right)^2}d\left(x^2+1\right)$$

$$= -\frac{1}{x^2+1}\Big|_0^2$$

$$= -\left(\frac{1}{5}-1\right)$$

$$= \frac{4}{5}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\pi/2} \sin^2\theta\ \cos\theta\ d\theta$

Solution

$$d\left(\sin\theta\right) = \cos\theta\ d\theta$$

$$\int_0^{\pi/2} \sin^2\theta\ \cos\theta\ d\theta = \int_0^{\pi/2} \sin^2\theta\ d\left(\sin\theta\right)$$

$$= \frac{1}{3}\sin^3\theta \Big|_0^{\pi/2}$$

$$\underline{= \frac{1}{3}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\pi/4} \frac{\sin x}{\cos^2 x}dx$

Solution

$$\int_0^{\pi/4} \frac{\sin x}{\cos^2 x}dx = -\int_0^{\pi/4} \frac{1}{\cos^2 x}d(\cos x)$$

$$= \frac{1}{\cos x}\Big|_0^{\pi/4}$$

$$\underline{= \sqrt{2} - 1}$$

Exercise

Evaluate the definite integral $\displaystyle\int_{1/3}^{1/\sqrt{3}} \frac{4}{9x^2+1}dx$

Solution

$$d(3x) = 3dx$$

$$\int_{1/3}^{1/\sqrt{3}} \frac{4}{9x^2+1}dx = \frac{4}{3}\int_{1/3}^{1/\sqrt{3}} \frac{1}{(3x)^2+1}d(3x)$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{4}{3}\arctan(3x)\Big|_{1/3}^{1/\sqrt{3}}$$

$$= \frac{4}{3}\left(\arctan\left(\sqrt{3}\right) - \arctan 1\right)$$

$$= \frac{4}{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$\underline{= \frac{\pi}{9}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\ln 4} \frac{e^x}{3+2e^x}\,dx$

Solution

$$\int_0^{\ln 4} \frac{e^x}{3+2e^x}\,dx = \frac{1}{2}\int_0^{\ln 4} \frac{1}{3+2e^x}\,d\left(3+2e^x\right)$$

$$= \frac{1}{2}\ln\left(3+2e^x\right)\Big|_0^{\ln 4}$$

$$= \frac{1}{2}\left(\ln\left(3+2e^{\ln 4}\right)-\ln 5\right)$$

$$= \frac{1}{2}(\ln 11-\ln 5)$$

$$= \frac{1}{2}\ln\frac{11}{5}$$

Exercise

Evaluate the definite integral $\displaystyle\int_{-\pi}^{\pi} \cos^2 x\,dx$

Solution

$$\int_{-\pi}^{\pi} \cos^2 x\,dx = \frac{1}{2}\int_{-\pi}^{\pi} (1+\cos 2x)\,dx$$

$$= \frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right)\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2}(\pi+\pi)$$

$$= \pi$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\pi/4} \cos^2 8\theta\,d\theta$

Solution

$$\int_0^{\pi/4} \cos^2 8\theta\,d\theta = \frac{1}{2}\int_0^{\pi/4} (1+\cos 16\theta)\,d\theta$$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$= \frac{1}{2}\left(x + \frac{1}{16}\sin 16\theta\right)\Big|_{0}^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right)$$

$$\boxed{= \frac{\pi}{8}}$$

Exercise

Evaluate the definite integral $\int_{-\pi/4}^{\pi/4} \sin^2 2\theta \, d\theta$

Solution

$$\int_{-\pi/4}^{\pi/4} \sin^2 2\theta \, d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} (1 - \cos 4\theta)\, d\theta \qquad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$= \frac{1}{2}\left(\theta - \frac{1}{4\theta}\sin 4\theta\right)\Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$\boxed{= \frac{\pi}{4}}$$

Exercise

Evaluate the definite integral $\int_{0}^{\pi/6} \frac{\sin 2x}{\sin^2 x + 2} dx$

Solution

$$d\left(\sin^2 x + 2\right) = 2\sin x \cos x \, dx$$

$$= \sin 2x \, dx$$

$$\int_{0}^{\pi/6} \frac{\sin 2x}{\sin^2 x + 2} dx = \int_{0}^{\pi/6} \frac{1}{\sin^2 x + 2} d\left(\sin^2 x + 2\right)$$

$$= \ln\left|\sin^2 x + 2\right| \Big|_{0}^{\pi/6}$$

$$= \ln\frac{9}{4} - \ln 2$$

$$\boxed{= \ln\frac{9}{8}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\pi/2} \sin^4\theta\, d\theta$

Solution

$$\int_0^{\pi/2} \sin^4\theta\, d\theta = \int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta \qquad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(1 - 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(1 - 2\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(\frac{3}{2} - 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{1}{4}\left(\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta\right. \Bigg|_0^{\pi/2}$$

$$= \frac{1}{4}\left(\frac{3}{2}\frac{\pi}{2}\right)$$

$$= \frac{3\pi}{16}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^1 x\sqrt{1-x^2}\, dx$

Solution

$$d\left(1-x^2\right) = -2x dx$$

$$\int_0^1 x\sqrt{1-x^2}\, dx = -\frac{1}{2}\int_0^1 \left(1-x^2\right)^{1/2} d\left(1-x^2\right)$$

$$= -\frac{1}{3}\left(1-x^2\right)^{3/2} \Bigg|_0^1$$

$$= -\frac{1}{3}(0-1)$$

$$= \frac{1}{3}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{1/4} \frac{x}{\sqrt{1-16x^2}}\,dx$

Solution

$d\left(1-16x^2\right) = -32x\,dx$

$$\int_0^{1/4} \frac{x}{\sqrt{1-16x^2}}\,dx = -\frac{1}{32}\int_0^{1/4} \left(1-16x^2\right)^{-1/2} d\left(1-16x^2\right)$$

$$= -\frac{1}{16}\left(1-16x^2\right)^{1/2} \Big|_0^{1/4}$$

$$= -\frac{1}{16}(0-1)$$

$$= \frac{1}{16}$$

Exercise

Evaluate the definite integral $\displaystyle\int_2^{3} \frac{x}{\sqrt[3]{x^2-1}}\,dx$

Solution

$d\left(x^2-1\right) = 2x\,dx$

$$\int_2^{3} \frac{x}{\sqrt[3]{x^2-1}}\,dx = \frac{1}{2}\int_2^{3} \left(x^2-1\right)^{-1/3} d\left(x^2-1\right)$$

$$= \frac{3}{4}\left(x^2-1\right)^{2/3} \Big|_2^{3}$$

$$= \frac{3}{4}\left(8^{2/3}-1\right)$$

$$= \frac{3}{4}(4-1)$$

$$= \frac{9}{4}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{6/5} \frac{dx}{25x^2+36}$

Solution

$$\int_0^{6/5} \frac{dx}{25x^2+36} = \int_0^{6/5} \frac{dx}{25\left(x^2+\frac{36}{25}\right)}$$

$$= \int_0^{6/5} \frac{dx}{25\left(x^2+\left(\frac{6}{5}\right)^2\right)}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{1}{25}\left(\frac{5}{6}\right)\tan^{-1}\frac{5x}{6}\Bigg|_0^{6/5}$$

$$= \frac{1}{30}\left(\tan^{-1}1 - \tan^{-1}0\right)$$

$$= \frac{1}{30}\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{120}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{2} x^3\sqrt{16-x^4}\, dx$

Solution

$$d\left(16-x^4\right) = -4x^3 dx$$

$$\int_0^{2} x^3\sqrt{16-x^4}\, dx = -\frac{1}{4}\int_0^{2}\left(16-x^4\right)^{1/2} d\left(16-x^4\right)$$

$$= -\frac{1}{6}\left(16-x^4\right)^{3/2}\Bigg|_0^{2}$$

$$= -\frac{1}{6}\left(0-4^3\right)$$

$$= \frac{32}{3}$$

Exercise

Evaluate the definite integral $\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^2 x} dx$

Solution

$d(\sin x) = \cos x\, dx$

$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^2 x}\, dx = \int_{\pi/4}^{\pi/2} \frac{1}{\sin^2 x}\, d(\sin x)$$

$$= -\frac{1}{\sin x}\Big|_{\pi/4}^{\pi/2}$$

$$= -\left(1-\sqrt{2}\right)$$

$$= \sqrt{2}-1$$

Exercise

Evaluate the definite integral $\int_{-1}^{1} (x-1)\left(x^2-2x\right)^7 dx$

Solution

$d\left(x^2-2x\right) = (2x-2)\,dx$

$= 2(x-1)\,dx$

$$\int_{-1}^{1} (x-1)\left(x^2-2x\right)^7 dx = \frac{1}{2}\int_{-1}^{1} \left(x^2-2x\right)^7 d\left(x^2-2x\right)$$

$$= \frac{1}{16}\left(x^2-2x\right)^8 \Big|_{-1}^{1}$$

$$= \frac{1}{16}\left(1-3^8\right)$$

$$= \frac{6560}{16}$$

$$= 410$$

Exercise

Evaluate the definite integral $\displaystyle\int_{-\pi}^{0} \frac{\sin x}{2+\cos x}dx$

Solution

$d(2+\cos x) = -\sin x \; dx$

$$\int_{-\pi}^{0} \frac{\sin x}{2+\cos x}dx = -\int_{-\pi}^{0} \frac{1}{2+\cos x}\; d(2+\cos x)$$

$$= -\ln|2+\cos x| \; \Big|_{-\pi}^{0}$$

$$= -(\ln 3 - \ln 1)$$

$$\underline{= -\ln 3}$$

Exercise

Evaluate the definite integral $\displaystyle\int_{0}^{1} \frac{(x+1)(x+2)}{2x^3+9x^2+12x+36}\; dx$

Solution

$$d\left(2x^3+9x^2+12x+36\right) = \left(6x^2+18x+12\right)dx$$

$$= 6\left(x^2+3x+2\right)dx$$

$$\int_{0}^{1} \frac{(x+1)(x+2)}{2x^3+9x^2+12x+36}\; dx = \int_{0}^{1} \frac{x^2+3x+2}{2x^3+9x^2+12x+36}\; dx$$

$$= \frac{1}{6}\int_{0}^{1} \frac{1}{2x^3+9x^2+12x+36}\; d\left(2x^3+9x^2+12x+36\right)$$

$$= \frac{1}{6}\ln\left|2x^3+9x^2+12x+36\right| \; \Big|_{0}^{1}$$

$$= \frac{1}{6}(\ln 59 - \ln 36)$$

$$\underline{= \frac{1}{6}\ln\frac{59}{36}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_1^2 \frac{4}{9x^2+6x+1}\,dx$

Solution

$$\int_1^2 \frac{4}{9x^2+6x+1}\,dx = \int_1^2 \frac{4}{(3x+1)^2}\,dx$$

$$= \frac{4}{3}\int_1^2 \frac{1}{(3x+1)^2}\,d(3x+1) \qquad d(3x+1) = 3xdx$$

$$= -\frac{4}{3}\left.\frac{1}{3x+1}\right|_1^2$$

$$= -\frac{4}{3}\left(\frac{1}{7}-\frac{1}{4}\right)$$

$$= -\frac{4}{3}\left(-\frac{3}{28}\right)$$

$$= \frac{1}{7}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\pi/4} e^{\sin^2 x}\sin 2x\,dx$

Solution

$$d\left(\sin^2 x\right) = 2\sin x\cos x\,dx$$

$$= \sin 2x\,dx$$

$$\int_0^{\pi/4} e^{\sin^2 x}\sin 2x\,dx = \int_0^{\pi/4} e^{\sin^2 x}\,d\left(\sin^2 x\right)$$

$$= \left. e^{\sin^2 x}\right|_0^{\pi/4}$$

$$= e^{\sin^2\frac{\pi}{4}} - e^{\sin^2 0}$$

$$= e^{\frac{1}{2}} - 1$$

$$= \sqrt{e} - 1$$

Exercise

Evaluate the definite integral $\int_0^1 x\sqrt{x+a}\,dx \quad (a>0)$

Solution

Let $u = x+a \quad \to \quad x = u-a$

$\Rightarrow \; du = dx$

$$\int_0^1 x\sqrt{x+a}\,dx = \int_0^1 (u-a)u^{1/2}\,du$$

$$= \int_0^1 \left(u^{3/2} - au^{1/2}\right)du$$

$$= \frac{2}{5}u^{5/2} - \frac{2}{3}au^{3/2}\Big|_0^1$$

$$= \frac{2}{5}(x+a)^{5/2} - \frac{2}{3}a(x+a)^{3/2}\Big|_0^1$$

$$= \frac{2}{5}(1+a)^{5/2} - \frac{2}{3}a(1+a)^{3/2} - \frac{2}{5}a^{5/2} + \frac{2}{3}a\left(a^{3/2}\right)$$

$$= \frac{2}{5}(1+a)^{5/2} - \frac{2}{3}a(1+a)^{3/2} - \frac{2}{5}a^{5/2} + \frac{2}{3}a^{5/2}$$

$$= \frac{2}{5}(1+a)^{5/2} - \frac{2}{3}a(1+a)^{3/2} + \frac{4}{15}a^{5/2}$$

$$= \frac{2}{5}(1+a)^2\sqrt{1+a} - \frac{2}{3}a(1+a)\sqrt{1+a} + \frac{4}{15}a^2\sqrt{a}$$

$$= \left(\frac{2}{5}(1+a)^2 - \frac{2}{3}\left(a+a^2\right)\right)\sqrt{1+a} + \frac{4}{15}a^2\sqrt{a}$$

Exercise

Evaluate the definite integral $\int_0^1 x\sqrt[p]{x+a}\,dx \quad (a>0)$

Solution

Let $u = x+a \quad \to \quad x = u-a$

$du = dx$

$$\int_0^1 x\sqrt[p]{x+a}\,dx = \int_0^1 (u-a)u^{1/p}\,du$$

$$= \int_0^1 \left(u^{1+1/p} - au^{1/p} \right) du$$

$$= \frac{p}{2p+1} u^{2+1/p} - \frac{p}{p+1} au^{1+1/p} \Big|_0^1$$

$$= \frac{p}{2p+1} (x+a)^{2+1/p} - \frac{p}{p+1} a(x+a)^{1+1/p} \Big|_0^1$$

$$= \frac{p}{2p+1} (1+a)^{2+1/p} - \frac{p}{p+1} a(1+a)^{1+1/p} - \frac{p}{2p+1} a^{2+1/p} + \frac{p}{p+1} a(a)^{1+1/p}$$

$$= \frac{p}{2p+1} (1+a)^2 \sqrt[p]{1+a} - \frac{p}{p+1} a(1+a) \sqrt[p]{1+a} - \frac{p}{2p+1} a^{2+1/p} + \frac{p}{p+1} a^{2+1/p}$$

$$= \left(\frac{p}{2p+1} (1+a)^2 - \frac{p}{p+1} a(1+a) \right) \sqrt[p]{1+a} + \left(\frac{p}{p+1} - \frac{p}{2p+1} \right) a^{2+1/p}$$

$$= \left(\frac{p}{2p+1} (1+a)^2 - \frac{p}{p+1} \left(a+a^2 \right) \right) \sqrt[p]{1+a} + \left(\frac{2p^2+p-p^2-p}{(p+1)(2p+1)} \right) a^{2+1/p}$$

$$\boxed{= \left(\frac{p}{2p+1} (1+a)^2 - \frac{p}{p+1} \left(a+a^2 \right) \right) \sqrt[p]{1+a} + \frac{p^2}{(p+1)(2p+1)} a^{2+1/p}}$$

Or

Let $u = \sqrt[p]{x+a} \quad \rightarrow \quad u^p = x+a$

$x = u^p - a \quad \rightarrow \quad dx = pu^{p-1} du$

$$\int_0^1 x \sqrt[p]{x+a}\, dx = \int_0^1 \left(u^p - a \right) \cdot u \cdot \left(pu^{p-1} \right) du$$

$$= p \int_0^1 \left(u^p - a \right) \cdot u^p\, du$$

$$= p \int_0^1 \left(u^{2p} - au^p \right) du$$

$$= p \left(\frac{1}{2p+1} \left(\sqrt[p]{x+a} \right)^{2p+1} - \frac{1}{p+1} a \left(\sqrt[p]{x+a} \right)^{p+1} \right) \Big|_0^1$$

$$= p \left(\frac{1}{2p+1} \left(\sqrt[p]{1+a} \right)^{2p+1} - \frac{1}{p+1} a \left(\sqrt[p]{1+a} \right)^{\frac{p+1}{p}} - \frac{1}{2p+1} \left(\sqrt[p]{a} \right)^{2p+1} + \frac{1}{p+1} a \left(\sqrt[p]{a} \right)^{p+1} \right)$$

$$= p\left(\begin{array}{l}\frac{1}{2p+1}(1+a)^{(2p+1)/p} - \frac{1}{p+1}a(1+a)^{(p+1)/p} \\ -\frac{1}{2p+1}(a)^{(2p+1)/p} + \frac{1}{p+1}a(a)^{(p+1)/p}\end{array}\right)$$

$$= p\left(\begin{array}{l}\frac{1}{2p+1}(1+a)^{(2p+1)/p} - \frac{1}{p+1}a(1+a)^{(p+1)/p} \\ -\frac{1}{2p+1}(a)^{(2p+1)/p} + \frac{1}{p+1}(a)^{(2p+1)/p}\end{array}\right)$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^1 x\sqrt{1-\sqrt{x}}\, dx$

Solution

$u = 1-\sqrt{x} \quad \rightarrow \quad x = (1-u)^2$

$dx = -2(1-u)du$

$$\int_0^1 x\sqrt{1-\sqrt{x}}\, dx = -2\int_0^1 (1-u)^2\; u^{1/2}\; (1-u)du$$

$$= -2\int_0^1 (1-u)^3\; u^{1/2}\; du$$

$$= -2\int_0^1 \left(1-3u+3u^2-u^3\right) u^{1/2}\; du$$

$$= -2\int_0^1 \left(u^{1/2}-3u^{3/2}+3u^{5/2}-u^{7/2}\right)\; du$$

$$= -2\left(\tfrac{2}{3}\left(1-\sqrt{x}\right)^{3/2} - \tfrac{6}{5}\left(1-\sqrt{x}\right)^{5/2} + \tfrac{6}{7}\left(1-\sqrt{x}\right)^{7/2} - \tfrac{2}{9}\left(1-\sqrt{x}\right)^{9/2}\right)\Bigg|_0^1$$

$$= -2\left(0-\tfrac{2}{3}+\tfrac{6}{5}-\tfrac{6}{7}+\tfrac{2}{9}\right)$$

$$= -2\left(-\tfrac{32}{315}\right)$$

$$= \boxed{\tfrac{34}{315}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^1 \sqrt{x - x\sqrt{x}}\ dx$

Solution

$u = 1 - \sqrt{x} \quad \to \quad x = (1-u)^2$

$\Rightarrow \ dx = -2(1-u)\,du$

$$\int_0^1 \sqrt{x - x\sqrt{x}}\ dx = \int_0^1 \sqrt{x\left(1-\sqrt{x}\right)}\ dx$$

$$= -2\int_0^1 \sqrt{(1-u)^2 u}\ (1-u)\,du$$

$$= -2\int_0^1 \sqrt{(1-u)^2 u}\ (1-u)\,du$$

$$= -2\int_0^1 (1-u)^2\sqrt{u}\ du$$

$$= -2\int_0^1 \left(1-2u+u^2\right)u^{1/2}\ du$$

$$= -2\int_0^1 \left(u^{1/2} - 2u^{3/2} + u^{5/2}\right)\ du$$

$$= -2\ \left(\frac{2}{3}\left(1-\sqrt{x}\right)^{3/2} - \frac{4}{5}\left(1-\sqrt{x}\right)^{5/2} + \frac{2}{7}\left(1-\sqrt{x}\right)^{7/2}\right)\Bigg|_0^1$$

$$= -2\left(0 - \frac{2}{3} + \frac{4}{5} - \frac{2}{7}\right)$$

$$= -2\left(\frac{-16}{105}\right)$$

$$= \frac{32}{105}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\pi/2} \frac{\cos\theta\sin\theta}{\sqrt{\cos^2\theta + 16}}\,d\theta$

Solution

$d\left(\cos^2\theta + 16\right) = -2\cos\theta\sin\theta\ d\theta$

$$\int_0^{\pi/2} \frac{\cos\theta \sin\theta}{\sqrt{\cos^2\theta+16}} d\theta = -\frac{1}{2}\int_0^{\pi/2} \left(\cos^2\theta+16\right)^{-1/2} d\left(\cos^2\theta+16\right)$$

$$= -\sqrt{\cos^2\theta+16}\ \Big|_0^{\pi/2}$$

$$= -\left(4-\sqrt{17}\right)$$

$$= \sqrt{17}-4$$

Exercise

Evaluate the definite integral $\displaystyle\int_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \frac{dx}{x\sqrt{25x^2-1}}$

Solution

$$\int_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \frac{dx}{x\sqrt{25x^2-1}} = \int_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \frac{d(5x)}{(5x)\sqrt{(5x)^2-1}}$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|$$

$$= \sec^{-1}(5x)\ \Big|_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}}$$

$$= \sec^{-1}(2) - \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$$

$$= \frac{\pi}{3} - \frac{\pi}{6}$$

$$= \frac{\pi}{6}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^4 \frac{x}{\sqrt{9+x^2}} dx$

Solution

$$d\left(9+x^2\right) = 2x\,dx$$

$$\int_0^4 \frac{x}{\sqrt{9+x^2}} dx = \frac{1}{2}\int_0^4 \left(9+x^2\right)^{-1/2} d\left(9+x^2\right)$$

$$= \sqrt{9+x^2}\ \Big|_0^4$$

$$= 5-3$$

$$\boxed{= 2}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\pi/4} \frac{\sin\theta}{\cos^3\theta}\, d\theta$

Solution

$d(\cos\theta) = -\sin\theta$

$$\int_0^{\pi/4} \frac{\sin\theta}{\cos^3\theta}\, d\theta = -\int_0^{\pi/4} \cos^{-3}\theta\, d(\cos\theta)$$

$$= \frac{1}{2}\frac{1}{\cos^2\theta}\ \Big|_0^{\pi/4}$$

$$= \frac{1}{2}(2-1)$$

$$\boxed{= \frac{1}{2}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^1 2x\left(4-x^2\right) dx$

Solution

$d\left(4-x^2\right) = -2xdx$

$$\int_0^1 2x\left(4-x^2\right) dx = -\int_0^1 \left(4-x^2\right) d\left(4-x^2\right)$$

$$= -\frac{1}{2}\left(4-x^2\right)^2\ \Big|_0^1$$

$$= -\frac{1}{2}(9-16)$$

$$\boxed{= \frac{7}{2}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^3 \frac{x^2+1}{\sqrt{x^3+3x+4}}dx$

Solution

$$d\left(x^3+3x+4\right)=\left(3x^2+3\right)dx$$

$$\begin{aligned}\int_0^3 \frac{x^2+1}{\sqrt{x^3+3x+4}}dx &= \frac{1}{3}\int_0^3 \left(x^3+3x+4\right)^{-1/2} d\left(x^3+3x+4\right)\\ &= \frac{2}{3}\sqrt{x^3+3x+4}\ \Big|_0^3\\ &= \frac{2}{3}\left(\sqrt{40}-2\right)\\ &= \frac{4}{3}\left(\sqrt{10}-1\right)\end{aligned}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^4 \frac{x}{x^2+1}dx$

Solution

$$d\left(x^2+1\right)=2xdx$$

$$\begin{aligned}\int_0^4 \frac{x}{x^2+1}dx &= \frac{1}{2}\int_0^4 \frac{1}{x^2+1}d\left(x^2+1\right)\\ &= \frac{1}{2}\ln\left(x^2+1\right)\Big|_0^4\\ &= \frac{1}{2}(\ln 17-\ln 1)\\ &= \frac{1}{2}\ln 17\end{aligned}$$

Exercise

Evaluate the definite integral $\displaystyle\int_1^{e^2} \frac{\ln x}{x}dx$

Solution

$$d(\ln x) = \frac{1}{x}dx$$

$$\int_1^{e^2} \frac{\ln x}{x} dx = \int_1^{e^2} \ln x \; d(\ln x)$$

$$= \frac{1}{2}(\ln x)^2 \Big|_1^{e^2}$$

$$= \frac{1}{2}\left(\left(\ln e^2\right)^2 - (\ln 1)^2\right)$$

$$= \frac{1}{2}(2)^2$$

$$= 2$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^3 \frac{x^2+1}{\sqrt{x^3+3x+4}} dx$

Solution

$$d\left(x^3+3x+4\right) = \left(3x^2+3\right)dx$$

$$= 3\left(x^2+1\right)dx$$

$$\int_0^3 \frac{x^2+1}{\sqrt{x^3+3x+4}} dx = \frac{1}{3}\int_0^3 \left(x^3+3x+4\right)^{-1/2} d\left(x^3+3x+4\right)$$

$$= \frac{2}{3}\sqrt{x^3+3x+4} \Big|_0^3$$

$$= \frac{2}{3}\left(\sqrt{40}-2\right)$$

$$= \frac{2}{3}\left(\sqrt{10}-1\right)$$

Exercise

Evaluate the definite integral $\displaystyle\int_{-\pi/4}^{\pi/4} \sin^2 2\theta \; d\theta$

Solution

$$\int_{-\pi/4}^{\pi/4} \sin^2 2\theta \, d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} (1-\cos 4\theta)\, d\theta \qquad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$= \frac{1}{2}\left(\theta - \frac{1}{4}\sin 4\theta\right) \Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$= \frac{\pi}{4}$$

Exercise

Evaluate the definite integral $\int_0^1 \left(y^3 + 6y^2 - 12y + 9\right)^{-1/2}\left(y^2 + 4y - 4\right)dy$

Solution

$$d\left(y^3 + 6y^2 - 12y + 9\right) = \left(3y^2 + 12y - 12\right)dy$$

$$= 3\left(y^2 + 4y - 4\right)dy$$

$$\int_0^1 \left(y^3 + 6y^2 - 12y + 9\right)^{-1/2}\left(y^2 + 4y - 4\right)dy$$

$$= \frac{1}{3}\int_0^1 \left(y^3 + 6y^2 - 12y + 9\right)^{-1/2}\left(y^3 + 6y^2 - 12y + 9\right)dy$$

$$= \frac{2}{3}\sqrt{y^3 + 6y^2 - 12y + 9}\Big|_0^1$$

$$= \frac{2}{3}(2-3)$$

$$= -\frac{2}{3}$$

Exercise

Solve the initial value problem $\frac{dy}{dt} = e^t \sin\left(e^t - 2\right), \quad y(\ln 2) = 0$

Solution

$$\frac{dy}{dt} = e^t \sin\left(e^t - 2\right)$$

$$y = \int e^t \sin\left(e^t - 2\right) dt$$

Let $u = e^t - 2 \quad \rightarrow du = e^t dt$

$$y = \int \sin u \ du$$

$$= -\cos u + C$$

$$= -\cos\left(e^t - 2\right) + C$$

$$y(\ln 2) = -\cos\left(e^{\ln 2} - 2\right) + C = 0$$

$$C = \cos(2-2)$$

$$= \cos(0)$$

$$\underline{= 1}$$

$$\underline{y(t) = -\cos\left(e^t - 2\right) + 1}$$

Exercise

Solve the initial value problem $\dfrac{dy}{dt} = e^{-t} \sec^2\left(\pi e^{-t}\right), \quad y(\ln 4) = \dfrac{2}{\pi}$

Solution

$$\frac{dy}{dt} = e^{-t} \sec^2\left(\pi e^{-t}\right)$$

$$y = \int e^{-t} \sec^2\left(\pi e^{-t}\right) dt$$

$$d\left(\pi e^{-t}\right) = -\pi e^{-t} dt$$

$$y = \int e^{-t} \sec^2\left(\pi e^{-t}\right) dt$$

$$= -\frac{1}{\pi} \int \sec^2\left(\pi e^{-t}\right) d\left(\pi e^{-t}\right)$$

$$= -\frac{1}{\pi} \tan\left(\pi e^{-t}\right) + C$$

$$y(\ln 4) = -\frac{1}{\pi} \tan\left(\pi e^{-\ln 4}\right) + C$$

$$\underline{= \frac{2}{\pi}}$$

$$\begin{aligned} C &= \frac{2}{\pi} + \frac{1}{\pi}\tan\left(\pi \cdot \frac{1}{4}\right) \\ &= \frac{2}{\pi} + \frac{1}{\pi} \\ &= \frac{3}{\pi} \end{aligned}$$

$$y(t) = -\frac{1}{\pi}\tan\left(\pi e^{-t}\right) + \frac{3}{\pi}$$

Exercise

Verify the integration formula: $\displaystyle\int \frac{\tan^{-1} x}{x^2}\,dx = \ln x - \frac{1}{2}\ln\left(1 + x^2\right) - \frac{\tan^{-1} x}{x} + C$

Solution

If $y = \ln x - \frac{1}{2}\ln\left(1 + x^2\right) - \frac{\tan^{-1} x}{x} + C$

$$dy = \left(\frac{1}{x} - \frac{1}{2}\frac{2x}{1+x^2} - \frac{\frac{x}{1+x^2} - \tan^{-1} x}{x^2} \right) dx$$

$$dy = \left(\frac{1}{x} - \frac{x}{1+x^2} - \frac{x - \left(1+x^2\right)\tan^{-1} x}{x^2\left(1+x^2\right)} \right) dx$$

$$dy = \left(\frac{x\left(1+x^2\right) - x^3 - x + \left(1+x^2\right)\tan^{-1} x}{x^2\left(1+x^2\right)} \right) dx$$

$$dy = \left(\frac{x + x^3 - x^3 - x + \left(1+x^2\right)\tan^{-1} x}{x^2\left(1+x^2\right)} \right) dx$$

$$dy = \frac{\left(1+x^2\right)\tan^{-1} x}{x^2\left(1+x^2\right)} dx$$

$$dy = \frac{\tan^{-1} x}{x^2} dx \qquad \checkmark$$

Which verifies the formula

Exercise

Verify the integration formula: $\displaystyle\int \ln\left(a^2+x^2\right)dx = x\ln\left(a^2+x^2\right) - 2x + 2a\tan^{-1}\frac{x}{a} + C$

Solution

If $y = x\ln\left(a^2+x^2\right) - 2x + 2a\tan^{-1}\frac{x}{a} + C$

$$dy = \left(\ln\left(a^2+x^2\right) + x\frac{2x}{a^2+x^2} - 2 + 2a\frac{\frac{1}{a}}{1+\frac{x^2}{a^2}}\right)dx$$

$$dy = \left(\ln\left(a^2+x^2\right) + \frac{2x^2}{a^2+x^2} - 2 + \frac{2}{\frac{a^2+x^2}{a^2}}\right)dx$$

$$dy = \left(\ln\left(a^2+x^2\right) + \frac{2x^2}{a^2+x^2} - 2 + \frac{2a^2}{a^2+x^2}\right)dx$$

$$dy = \left(\ln\left(a^2+x^2\right) + \frac{2x^2+2a^2}{a^2+x^2} - 2\right)dx$$

$$dy = \left(\ln\left(a^2+x^2\right) + \frac{2\left(x^2+a^2\right)}{a^2+x^2} - 2\right)dx$$

$$dy = \left(\ln\left(a^2+x^2\right) + 2 - 2\right)dx$$

$$dy = \ln\left(a^2+x^2\right)dx \qquad \checkmark$$

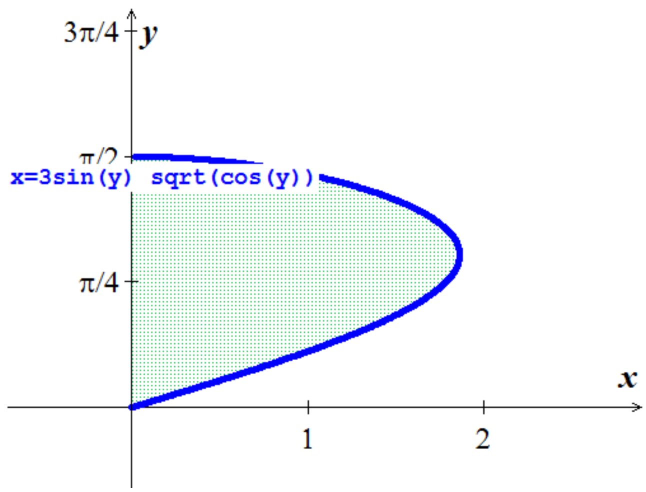
Which verifies the formula

Exercise

Find the area of the region bounded by the graphs of $x = 3\sin y\sqrt{\cos y}$, *and* $x = 0$, $0 \le y \le \frac{\pi}{2}$

Solution

$$d(\cos y) = -\sin y\, dy$$



$$A = \int_0^{\pi/2} \left(3\sin y\sqrt{\cos y} - 0\right)dx$$

$$= -3\int_0^{\pi/2} \cos^{1/2} y \ \ d(\cos y)$$

$$= -3\left(\frac{2}{3}\cos^{3/2} y \ \Big|_0^{\pi/2}\right.$$

$$= -2(0-1)$$

$$\underline{= 2 \ \ unit^2}$$

Exercise

Find the area of the region bounded by the graph of $f(x) = \dfrac{x}{\sqrt{x^2-9}}$ *on* $3 \le x \le 4$

Solution

$$A = \int_3^4 \frac{x}{\sqrt{x^2-9}}dx$$

$$= \frac{1}{2}\int_3^4 \left(x^2-9\right)^{-1/2} d\left(x^2-9\right) \qquad d\left(x^2-9\right) = 2xdx$$

$$= \sqrt{x^2-9}\ \Big|_3^4$$

$$= \sqrt{7} - 0$$

$$\underline{= \sqrt{7} \ \ unit^2}$$

Exercise

Find the area of the region bounded by the graph of $f(x) = \dfrac{x}{\sqrt{x^2-9}}$ and the *x-axis* between $x = 4$ and $x = 5$.

Solution

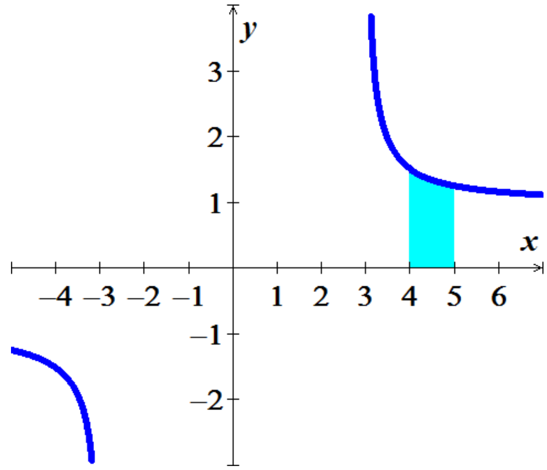
$d\left(x^2-9\right)=2x\,dx$

$$A=\int_4^5 \frac{x}{\sqrt{x^2-9}}\,dx$$

$$=\frac{1}{2}\int_4^5 \left(x^2-9\right)^{-1/2} d\left(x^2-9\right)$$

$$=\sqrt{x^2-9}\,\Big|_4^5$$

$$=4-\sqrt{7}\ \ unit^2$$



Exercise

Find the area of the region bounded by the graph of $f(x)=x\ \sin x^2$ and the *x-axis* between $x=0$ and $x=\sqrt{\pi}$.

Solution

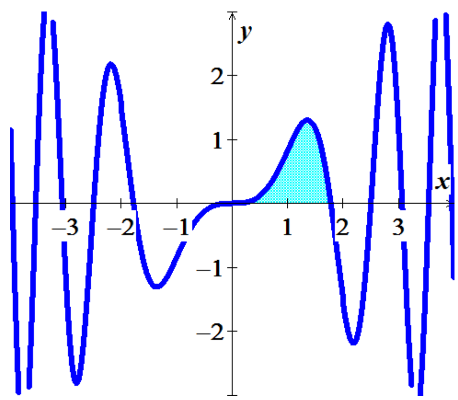
$$A=\int_0^{\sqrt{\pi}} x\ \sin x^2\ dx$$

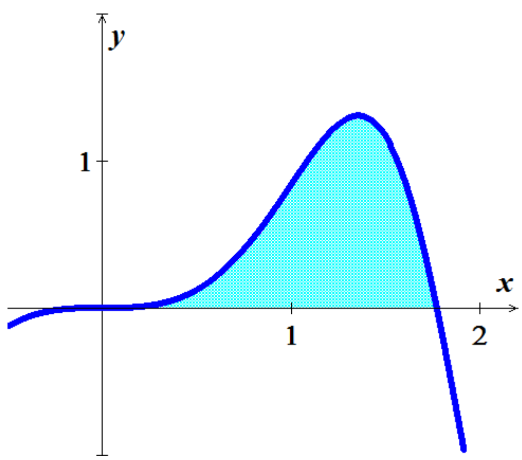
$$=\frac{1}{2}\int_0^{\sqrt{\pi}} \sin x^2\ d\left(x^2\right)$$

$$=-\frac{1}{2}\cos\left(x^2\right)\Big|_0^{\sqrt{\pi}}$$

$$=-\frac{1}{2}(-1-1)$$

$$=1\ \ unit^2$$





Exercise

Find the area of the region bounded by the graph of $f(\theta) = \cos\theta \ \sin\theta$ and the θ-*axis* between $\theta = 0$ and $\theta = \frac{\pi}{2}$.

Solution

$$A = \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta \, d(\sin\theta)$$

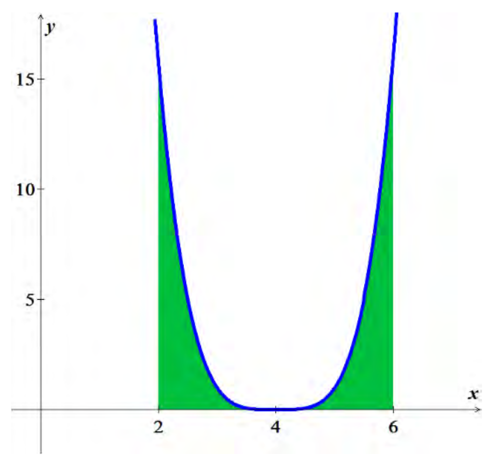
$$= \frac{1}{2}\sin^2\theta \Big|_0^{\pi/2}$$

$$= 1 \ \ unit^2$$

Exercise

Find the area of the region bounded by the graph of $f(x) = (x-4)^4$ and the x-*axis* between $x = 2$ and $x = 6$.

Solution



$$A = \int_2^6 (x-4)^4 \, dx$$

$$= \int_2^6 (x-4)^4 \, d(x-4)$$

$$= 2\left(\frac{1}{5}\right)(x-4)^5 \Big|_2^4$$

$$= \frac{64}{5} \ \ unit^2$$

Exercise

Perhaps the simplest change of variables is the shift or translation given by $u = x + c$, where c is a real number.

a) Prove that shifting a function does not change the net area under the curve, in the sense that

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(u)\,du$$

b) Draw a picture to illustrate this change of variables in the case that $f(x) = \sin x$, $a = 0$, $b = \pi$, and $c = \frac{\pi}{2}$

Solution

a) Let $u = x + c \quad \rightarrow \quad du = dx$

$$\begin{cases} x = b & \rightarrow \ u = b + c \\ x = a & \rightarrow \ u = a + c \end{cases}$$

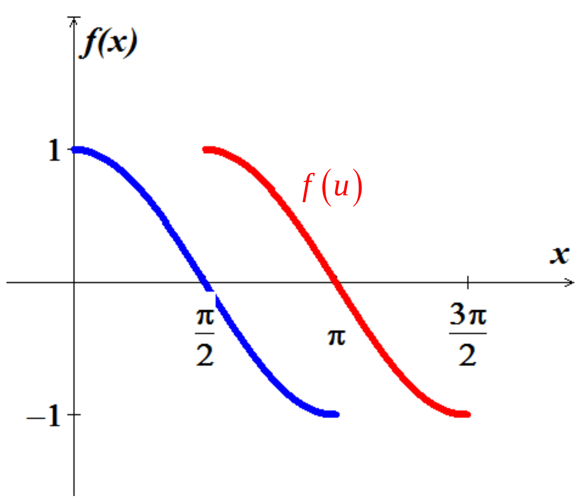
$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(u)\,du$$

b) Given: $\quad f(x) = \sin x, \quad a = 0, \quad b = \pi, \quad \& \quad c = \frac{\pi}{2}$

$$f(x+c) = \sin\left(x + \frac{\pi}{2}\right)$$

$$\begin{cases} b = \pi & \rightarrow f\left(\pi + \frac{\pi}{2}\right) = \sin\frac{3\pi}{2} = -1 \\ a = 0 & \rightarrow f\left(0 + \frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1 \end{cases}$$

$$f(u) \rightarrow \begin{cases} b + c = \frac{3\pi}{2} \\ a + c = \frac{\pi}{2} \end{cases}$$



Exercise

Another change of variables that can be interpreted geometrically is the scaling $u = cx$, where c is a real number. Prove and interpret the fact that

$$\int_a^b f(cx)\,dx = \frac{1}{c}\int_{ac}^{bc} f(u)\,du$$

Draw a picture to illustrate this change of variables in the case that $f(x) = \sin x$, $a = 0$, $b = \pi$, and $c = \frac{1}{2}$

Solution

Let $u = cx \quad \rightarrow \quad du = cdx$

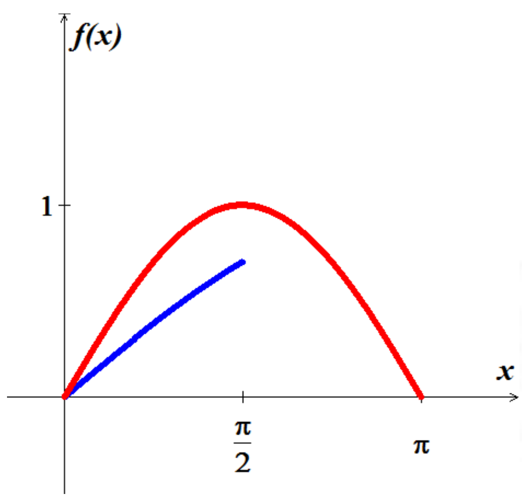
$$\begin{cases} x = b & \rightarrow \ u = bc \\ x = a & \rightarrow \ u = ac \end{cases}$$

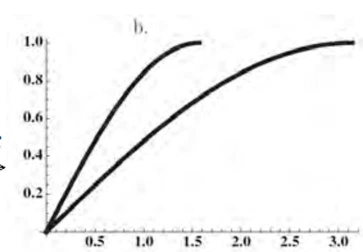
$$\int_a^b f(cx)\,dx = \frac{1}{c}\int_{ac}^{bc} f(u)\,du$$

Given: $f(x) = \sin x, \quad a = 0, \quad b = \pi, \quad \& \quad c = \frac{1}{2}$

$$f(cx) = f\left(\frac{x}{2}\right) = \sin\frac{x}{2}$$

$$\begin{cases} a = 0 & \rightarrow \ ac = 0 \\ b = \pi & \rightarrow \ bc = \frac{\pi}{2} \end{cases}$$





Exercise

The function f satisfies the equation $3x^4 - 48 = \int_2^x f(t)\,dt$. Find f and check your answer by substitution.

Solution

$$\frac{d}{dx}\left(3x^4-48\right)=\frac{d}{dx}\int_2^x f(t)\,dt$$

$$12x^3=f(x)$$

$$\int_2^x 12t^3\,dt=3t^4\Big|_2^x$$

$$=3x^4-3(2)^4$$

$$\boxed{=3x^4-48}\quad\checkmark$$

Exercise

Assume f' is continuous on $[2,\ 4]$, $\int_1^2 f'(2x)\,dx=10$, and $f(2)=4$. Evaluate $f(4)$.

Solution

$$\int_1^2 f'(2x)\,dx=f(2x)\Big|_1^2$$

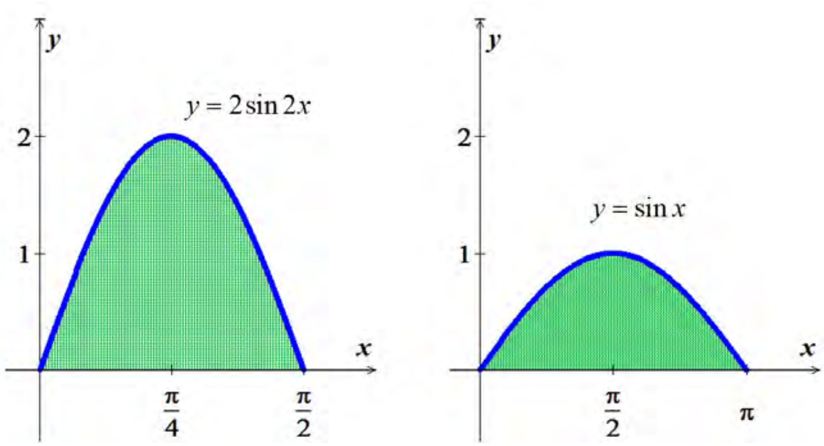
$$=f(4)-f(2)=10$$

$$f(4)-4=10$$

$$\boxed{f(4)=14}$$

Exercise

The area of the shaded region under the curve $y=2\sin 2x$ in



a) Equals the area on the shaded region under the curve $y=\sin x$

b) Explain why this is true without computation areas.

Solution

a) $A = \int_0^{\pi/2} 2\sin 2x\, dx$

$u = 2x \quad \to \quad du = 2dx$

$\begin{cases} x = \frac{\pi}{2} & \to \ u = \pi \\ x = 0 & \to \ u = 0 \end{cases}$

$= \int_0^{\pi} \sin u\, du$

$= \int_0^{\pi} \sin x\, dx$

b) Let A_1 = area of $\sin x \quad 0 \le x \le \pi$

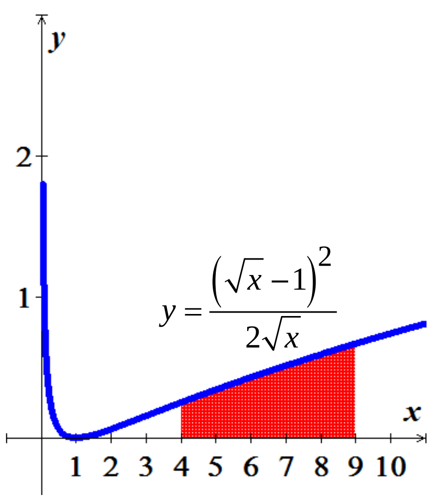
A_2 = area of $\sin 2x \quad 0 \le 2x \le \pi \quad \to \quad 0 \le x \le \frac{\pi}{2}$

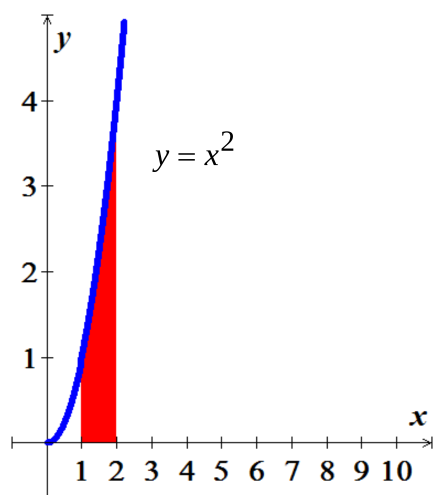
Area of $\ 0 \le x \le \frac{\pi}{2}$ is $\frac{1}{2}A_1$

$A_2 = 2\frac{1}{2}A_1 = A_1$ ✓

Exercise

The area of the shaded region under the curve $y = \frac{\left(\sqrt{x}-1\right)^2}{2\sqrt{x}}$ on the interval $[4,\ 9]$





a) Equals the area on the shaded region under the curve $y = x^2$ on the interval $[1,\ 2]$

b) Explain why this is true without computation areas.

Solution

a) Let $u=\sqrt{x}-1 \quad \rightarrow \quad x=(u+1)^2$

$dx = 2(u+1)du$

$$\begin{cases} x=9 & \rightarrow \ u=2 \\ x=4 & \rightarrow \ u=1 \end{cases}$$

$$A_1 = \int_4^9 \frac{\left(\sqrt{x}-1\right)^2}{2\sqrt{x}}\, dx$$

$$= \int_1^2 \frac{u^2}{2(u+1)}\, 2(u+1)du$$

$$= \int_1^2 u^2\, du \qquad \checkmark$$

$$= \frac{1}{3}\left(\sqrt{x}-1\right)^3 \Big|_4^9$$

$$= \frac{1}{3}\left(2^3-1\right)$$

$$= \frac{7}{3}$$

$$A_2 = \int_1^2 x^2\, dx$$

$$= \frac{1}{3}x^3 \Big|_1^2$$

$$= \frac{1}{3}\left(2^3-1\right)$$

$$= \frac{7}{3}$$

b) $\displaystyle\int_4^9 \frac{\left(\sqrt{x}-1\right)^2}{2\sqrt{x}}\, dx = \int_1^2 u^2\, du = \int_1^2 x^2\, dx \qquad \checkmark$

Exercise

The family of parabolas $y=\frac{1}{a}-\frac{x^2}{a^3}$, where $a>0$, has the property that for $x\ge 0$, the *x*-intercept is $(a,\ 0)$ and the *y*-intercept is $\left(0,\ \frac{1}{a}\right)$. Let $A(a)$ be the area of the region in the first quadrant bounded by the parabola and the $x\text{-}axis$. Find $A(a)$ and determine whether it is increasing, decreasing, or constant function of a.

Solution

Given: $y=\frac{1}{a}-\frac{x^2}{a^3}$ $\quad (a,\ 0)\ \&\ \left(0,\ \frac{1}{a}\right)$

$$A=\int_0^a\left(\frac{1}{a}-\frac{x^2}{a^3}\right)dx$$

$$=\frac{1}{a}x-\frac{1}{3}\frac{x^3}{a^3}\Big|_0^a$$

$$=1-\frac{1}{3}$$

$$\boxed{=\frac{2}{3}}$$

$A(a)=\frac{2}{3}$ is a constant function.

Exercise

Consider the right triangle with vertices $(0,\ 0)$, $(0,\ b)$, and $(a,\ 0)$, where $a>0$ and $b>0$. Show that the average vertical distance from points on the $x\text{-}axis$ to the hypotenuse is $\frac{b}{2}$, for all $a>0$.

Solution

$$y=\frac{b-0}{0-a}(x-0)+b \qquad\qquad y=m\left(x-x_0\right)+y_0$$

$$\boxed{=-\frac{b}{a}x+b}$$

Average vertical distance is:

$$\frac{1}{a-0}\int_0^a\left(-\frac{b}{a}x+b\right)dx=\frac{1}{a}\int_0^a\left(b-\frac{b}{a}x\right)dx$$

$$=\frac{1}{a}\left(bx-\frac{b}{2a}x^2\right)\Big|_0^a$$

$$=\frac{1}{a}\left(ba-\frac{b}{2a}a^2\right)$$

$$=b-\frac{b}{2}$$

$$\boxed{=\frac{b}{2}}$$

Exercise

Consider the integral $I=\int \sin^2 x\ \cos^2 x\ dx$

a) Find I using the identity $\sin 2x=2\sin x\cos x$

b) Find I using the identity $\cos^2 x=1-\sin^2 x$

c) Confirm that the results in part (a) and (b) are consistent and compare the work involved in each method.

Solution

a) $\sin 2x=2\sin x\cos x$

$$\sin^2 2x=4\sin^2 x\cos^2 x$$

$$\sin^2 x\cos^2 x=\frac{1}{4}\sin^2 2x$$

$$I=\int \sin^2 x\ \cos^2 x\ dx$$

$$=\frac{1}{4}\int \sin^2 2x\ dx \qquad \sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$$

$$=\frac{1}{8}\int (1-\cos 4x)\ dx$$

$$=\frac{1}{8}\left(x-\frac{1}{4}\sin 4x\right)+C$$

$$\boxed{=\frac{1}{8}x-\frac{1}{32}\sin 4x+C}$$

b) $\cos^2 x=1-\sin^2 x \qquad \sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$

$$=1-\frac{1}{2}+\frac{1}{2}\cos 2x$$

$$=\frac{1}{2}+\frac{1}{2}\cos 2x$$

$$I=\int \frac{1}{2}(1-\cos 2x)\ \frac{1}{2}(1+\cos 2x)\ dx \qquad \sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$$

$$=\frac{1}{4}\int \left(1-\cos^2 2x\right)\ dx \qquad \cos^2 x=1-\sin^2 x$$

$$=\frac{1}{4}\int \sin^2 2x\,dx \qquad \text{From part } (\boldsymbol{a})$$

$$\boxed{=\frac{1}{8}x-\frac{1}{32}\sin 4x+C}$$

c) The results from part *a* & *b* are consistent.

Exercise

Let $H(x)=\int_0^x \sqrt{4-t^2}\,dt$, for $-2\le x\le 2$.

a) Evaluate $H(0)$

b) Evaluate $H'(1)$

c) Evaluate $H'(2)$

d) Use geometry to evaluate $H(2)$

e) Find the value of *s* such that $H(x)=sH(-x)$

Solution

a) $H(0)=\int_0^0 \sqrt{4-t^2}\,dt$

$$\boxed{=0}$$

b) $H'(x)=\sqrt{4-x^2}\ \frac{d}{dx}(x)$

$$=\sqrt{4-x^2}$$

$$H'(1)\ \boxed{=\sqrt{3}}$$

c) $H'(2)=\sqrt{4-4}$

$$\boxed{=0}$$

d) $H(2)=\int_0^2 \sqrt{4-t^2}\,dt$ is the area inside a circle in the first quadrant of radius 2

$$=\frac{1}{4}\pi(2)^2$$

$$\boxed{=\pi}$$

e) $H(x)=\int_0^{-x} \sqrt{4-t^2}\,dt$ $\qquad$ $\sqrt{4-t^2}$ is an *even* function

$$= -\int_{-x}^{0} \sqrt{4-t^2}\ dt$$

$$= -H(x)$$

$\therefore\quad \boxed{s = -1}$

$t = 2\sin u$

$dt = 2\cos u\ du$

$\sqrt{4-t^2} = 2\cos u$

$$H(x) = \int_0^x \sqrt{4-t^2}\ dt$$

$$= \int_0^x 2\cos u\ 2\cos u\ du$$

$$= \int_0^x 4\cos^2 u\ du$$

$$= \int_0^x 2(1+\cos 2u)\ du$$

$$= 2\left(u + \frac{1}{2}\sin 2u\right.\Bigg|_0^x \qquad t = 2\sin u \ \rightarrow\ u = \sin^{-1}\frac{t}{2}$$

$$= 2\left(\sin^{-1}\frac{t}{2} + \sin u\cos u\right.\Bigg|_0^x \qquad \sqrt{4-t^2} = 2\cos u \ \rightarrow\ \cos u = \frac{1}{2}\sqrt{4-t^2}$$

$$= 2\left(\sin^{-1}\frac{t}{2} + \frac{t}{4}\sqrt{4-t^2}\right.\Bigg|_0^x$$

$$= 2\left(\sin^{-1}\frac{x}{2} + \frac{x}{4}\sqrt{4-x^2}\right)$$

$$\boxed{= 2\sin^{-1}\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2}}$$

Exercise

Evaluate the limits $\displaystyle\lim_{x\to 2}\frac{\int_2^x e^{t^2}\,dt}{x-2}$

Solution

$$\lim_{x\to 2}\frac{\int_2^x e^{t^2}\,dt}{x-2} = \frac{\int_2^2 e^{t^2}\,dt}{2-2} = \frac{0}{0}$$

$$= \lim_{x\to 2}\frac{e^{x^2}\frac{d}{dx}(x)}{1}$$

$$= \lim_{x\to 2} e^{x^2}$$

$$\underline{= e^4}$$

Exercise

Evaluate the limits $\displaystyle\lim_{x\to 1}\frac{\int_1^{x^2} e^{t^3}\,dt}{x-1}$

Solution

$$\lim_{x\to 1}\frac{\int_1^{x^2} e^{t^3}\,dt}{x-1} = \lim_{x\to 1}\frac{\int_1^{1} e^{t^3}\,dt}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to 1}\frac{2xe^{x^3}}{1}$$

$$\underline{= 2e}$$

Exercise

Prove that for nonzero constants a and b, $\displaystyle\int\frac{dx}{a^2x^2+b^2} = \frac{1}{ab}\tan^{-1}\left(\frac{ax}{b}\right)+C$

Solution

$$\int\frac{dx}{a^2x^2+b^2} = \int\frac{dx}{a^2\left(x^2+\left(\frac{b}{a}\right)^2\right)}$$

$$\int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{1}{a^2}\frac{a}{b}\tan^{-1}\frac{x}{\frac{b}{a}} + C$$

$$\boxed{= \frac{1}{ab}\tan^{-1}\left(\frac{ax}{b}\right) + C} \quad \checkmark$$

Exercise

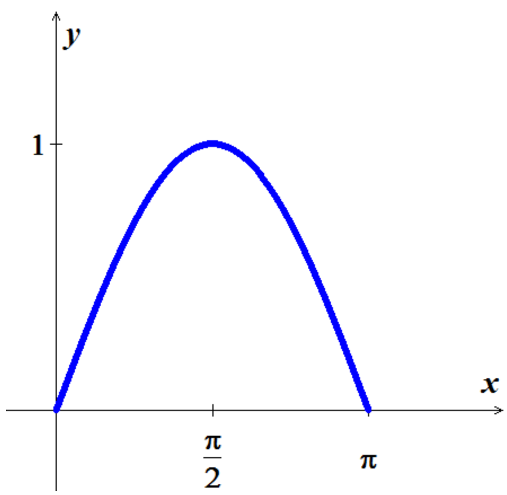
Let $a > 0$ be a real number and consider the family of functions $f(x) = \sin ax$ on the interval $\left[0, \frac{\pi}{a}\right]$.

a) Graph f, for $a = 1, 2, 3$.

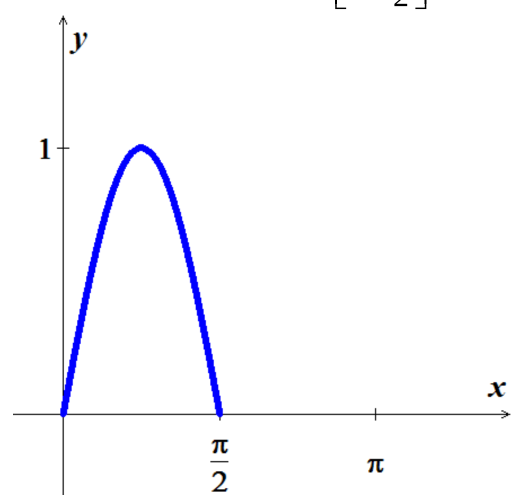
b) Let $g(a)$ be the area of the region bounded by the graph of f and the *x-axis* on the interval $\left[0, \frac{\pi}{a}\right]$. Graph g for $0 < a < \infty$. Is g an increasing function, a decreasing function, or neither?

Solution

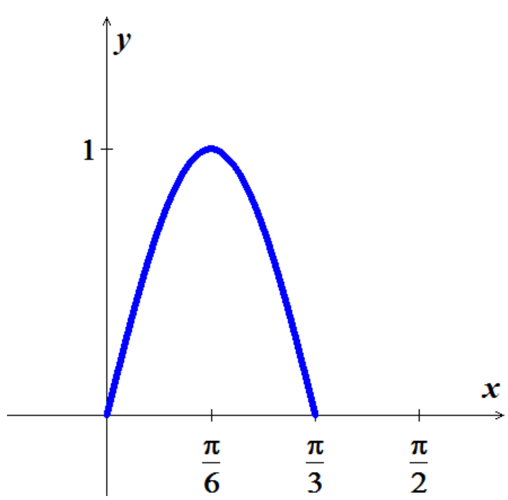
a) $a = 1 \to \quad f(x) = \sin x \quad x \in [0, \pi]$

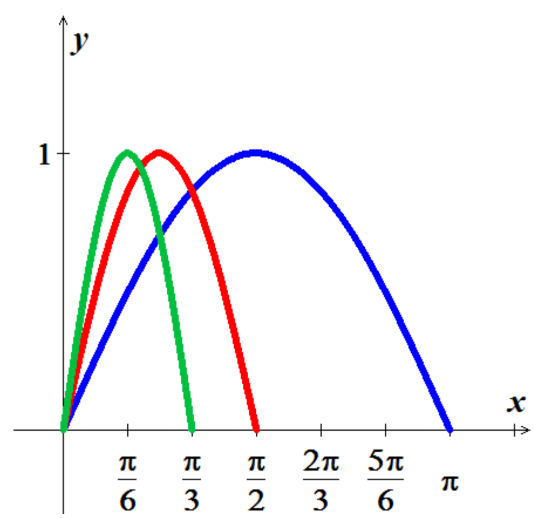


$a = 2 \to \quad f(x) = \sin 2x \quad x \in \left[0, \frac{\pi}{2}\right]$



$a = 3 \to \quad f(x) = \sin 3x \quad x \in \left[0, \frac{\pi}{3}\right]$





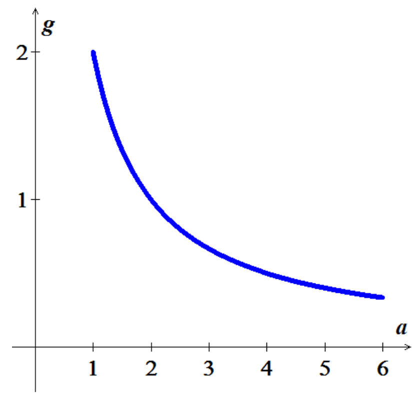
b) $g(x)=\int_0^{\pi/a}\sin ax\,dx$

$$=-\frac{1}{a}\cos ax\Big|_0^{\pi/a}$$

$$=-\frac{1}{a}(\cos\pi-\cos 0)$$

$$=-\frac{1}{a}(-1-1)$$

$$\boxed{=\frac{2}{a}}$$



The function is decreasing as $a\geq 1$ is increasing.

Exercise

Explain why if a function *u* satisfies the equation $u(x)+2\int_0^x u(t)\,dt=10$, then it also satisfies the equation $u'(x)+2u(x)=0$. Is it true that is *u* satisfies the second equation, then it satisfies the first equation?

Solution

$$\frac{d}{dx}u(x)+2\frac{d}{dx}\int_0^x u(t)\,dt=\frac{d}{dx}(10)$$

$$u'(x)+2\frac{d}{dx}u(x)\frac{d}{dx}x=0$$

$$u'(x)+2u(x)=0 \qquad \checkmark$$

Exercise

Let $f(x)=\int_0^x (t-1)^{15}(t-2)^9\,dt$

a) Find the interval on which f is increasing and the intervals on which f is decreasing.

b) Find the intervals on which f is concave up and the intervals on which f is concave down.

c) For what values of *x* does f have local minima? Local maxima?

d) Where are the inflection points of f?

Solution

a) $f'(x)=(x-1)^{15}(x-2)^9=0$

$CN: \ \boxed{x=1,\ 2}$

0	1		2	
	+	–	+	

Where $x=1$ is multiplicity of 15

$x=2$ is multiplicity of 9

Therefore, the sign will change.

f is increasing on $(-\infty,\ 1)\cup(2,\ \infty)$

f is decreasing on $(1,\ 2)$

b) $f''(x)=(x-1)^{14}(x-2)^{8}\left(15(x-2)+9(x-1)\right)$

$=(x-1)^{14}(x-2)^{8}(24x-39)=0$

$\boxed{x=1,\ 2,\ \frac{13}{8}}$

0	1	$\frac{13}{8}$	2
–	–	+	+

$(x-1)^{14}(x-2)^{8}\geq 0$ $\quad (always)$

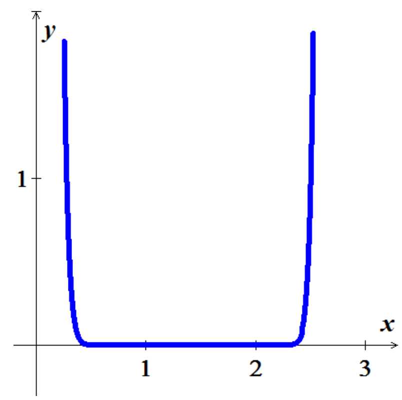
Concave up: $\left(\frac{13}{8},\ \infty\right)$

Concave down: $\left(-\infty,\ \frac{13}{8}\right)$

c) ***LMIN***: $(1,\ 0)$

LMAX: $(2,\ 0)$

d) *point of inflection*: $\boxed{x=\frac{13}{8}}$



Exercise

A company is considering a new manufacturing process in one of its plants. The new process provides substantial initial savings, with the savings declining with time *t* (in years) according to the rate-of-savings function

$$S'(t) = 100 - t^2$$

where $S'(t)$ is in thousands of dollars per year. At the same time, the cost of operating the new process increases with time *t* (in years), according to the rate-of-cost function (in thousands of dollars per year)

$$C'(t) = t^2 + \frac{14}{3}t$$

a) For how many years will the company realize savings?

b) What will be the net total savings during this period?

Solution

a) For how many years will the company realize savings?

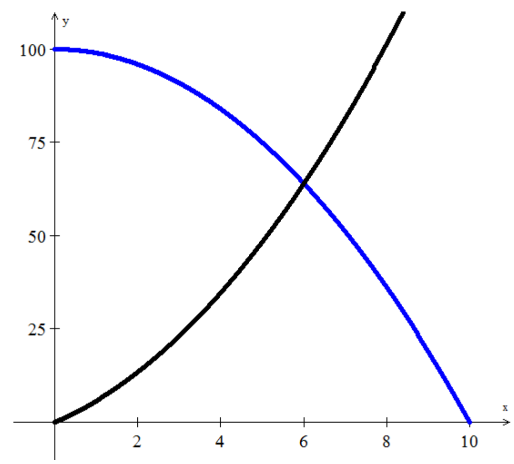
$$C'(t) = S'(t)$$

$$t^2 + \frac{14}{3}t = 100 - t^2$$

$$2t^2 + \frac{14}{3}t - 100 = 0$$

$$\rightarrow t = -\frac{25}{3} \text{ or } 6$$

$$\underline{t = 6}$$



The company should use this type for 6 years.

b) What will be the net total savings during this period?

$$\text{Total savings } = \int_0^6 \left(\left(100 - t^2\right) - \left(t^2 + \frac{14}{3}t\right)\right) dt$$

$$= \int_0^6 \left(100 - 2t^2 - \frac{14}{3}t\right) dt$$

$$= 100t - \frac{2}{3}t^3 - \frac{7}{3}t^2 \Big|_0^6$$

$$= 100(6) - \frac{2}{3}(6)^3 - \frac{7}{3}(6)^2 - \left(100(0) - \frac{2}{3}(0)^3 - \frac{7}{3}(0)^2\right)$$

$$\underline{= 372}$$

The company will save a total of \$372,000. Over the 6-*year* period

Exercise

Find the producers' surplus if the supply function for pork bellies is given by

$$S(x) = x^{5/2} + 2x^{3/2} + 50$$

Assume supply and demand are in equilibrium at $x = 16$.

Solution

The equilibrium price:

$$p_0 = S(x = 16) = 16^{5/2} + 2(16)^{3/2} + 50$$

$$\underline{= 1202}$$

$$\text{Producer's surplus} = \int_0^{x_0} \left(p_0 - S(x)\right) dx$$

$$= \int_0^{16} \left(1202 - \left(x^{5/2} + 2x^{3/2} + 50\right)\right) dx$$

$$= \int_0^{16} \left(1152 - x^{5/2} - 2x^{3/2}\right) dx$$

$$= 1152x - \tfrac{2}{7}x^{7/2} - \tfrac{4}{5}x^{5/2} \Big|_0^{16}$$

$$= \left(1152(16) - \tfrac{2}{7}(16)^{7/2} - \tfrac{4}{5}(16)^{5/2}\right) - \left(1152(0) - \tfrac{2}{7}(0)^{7/2} - \tfrac{4}{5}(0)^{5/2}\right)$$

$$\underline{= \$12{,}931.66}$$

The producers' surplus is $12,931.66

Exercise

An object moves along a line with a velocity in m/s given by $v(t) = 8\cos\left(\frac{\pi t}{6}\right)$. Its initial position is $s(0) = 0$.

a) Graph the velocity function.

b) The position of the object is given by $s(t) = \int_0^t v(y)\,dy$, for $t \geq 0$. Find the position function, for $t \geq 0$.

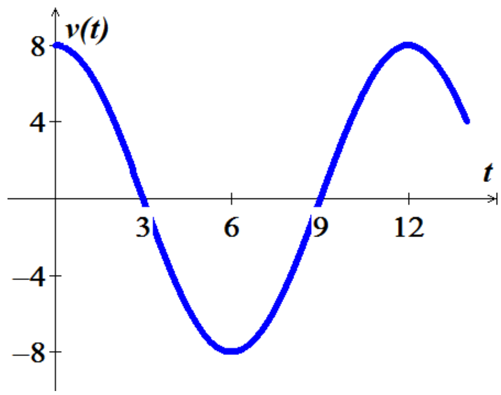
c) What is the period of the motion – that is, starting at any point, how long does it take the object to return to that position?

Solution

a) $v(t) = 8\cos\left(\frac{\pi t}{6}\right)$

$|A| = 8 \quad P = 12$

t	$v(t)$
0	8
3	0
6	−8
9	0
12	8



b) $s(t) = \int_0^t v(y)\,dy$

$$= \int_0^t 8\cos\left(\frac{\pi}{6}y\right)\,dy$$

$$= \frac{48}{\pi}\sin\left(\frac{\pi}{6}y\right)\Big|_0^t$$

$$= \frac{48}{\pi}\sin\frac{\pi}{6}t$$

c) Period: $P = \dfrac{2\pi}{\frac{\pi}{6}}$

$$= 12$$

Exercise

The population of a culture of bacteria has a growth rate given by $p'(t) = \dfrac{200}{(t+1)^r}$ bacteria per hour, for $t \geq 0$, where $r > 1$ is a real number. It is shown that the increase in the population over time interval $[0,\ t]$ is given by $\int_0^t p'(s)\,ds$. (note that the growth rate decreases in time, reflecting competition for space and food.)

a) Using the population model with $r = 2$, what is the increase in the population over the time interval $0 \leq t \leq 4$?

b) Using the population model with $r = 3$, what is the increase in the population over the time interval $0 \leq t \leq 6$?

c) Let ΔP be the increase in the population over a fixed time interval $[0,\ T]$. For fixed *T*, does ΔP increase or decrease with the parameter *r*? Explain.

d) A lab technician measures an increase in the population of 350 bacteria over the 10-*hr* period $[0,\ 10]$. Estimate the value of r that best fits this data point.

e) Use the population model in part (*b*) to find the increase in population over time interval $[0,\ T]$, for any $T>0$. If the culture is allowed to grow indefinitely $(T\to\infty)$, does the bacteria population increase without bound? Or does it approach a finite limit?

Solution

a) $r=2\ \&\ 0\le t\le 4$

$$\int_0^4 \frac{200}{(t+1)^2}\,dt=\int_0^4 \frac{200}{(t+1)^2}\,d(t+1)$$

$$=-\frac{200}{t+1}\Big|_0^4$$

$$=-(40-200)$$

$$\underline{=160}$$

b) $r=3\ \&\ 0\le t\le 6$

$$\int_0^6 \frac{200}{(t+1)^3}\,dt=200\int_0^6 (t+1)^{-3}\,d(t+1)$$

$$=-100\frac{1}{(t+1)^2}\Big|_0^6$$

$$=-100\left(\frac{1}{49}-1\right)$$

$$\underline{=\frac{4800}{49}}$$

c) $\Delta P=\int_0^T \frac{200}{(t+1)^r}\,dt$ decreases as r increases.

Because $\frac{200}{(t+1)^r}>\frac{200}{(t+1)^{r+1}}$

d) $\int_0^{10} \frac{200}{(t+1)^r}\,dt=350$

$$200\int_0^{10} (t+1)^{-r}\,d(t+1)=350$$

$$\frac{1}{1-r}(t+1)^{1-r}\Big|_0^{10} = \frac{7}{4}$$

$$\frac{1}{1-r}\left(11^{1-r}-1\right) = \frac{7}{4}$$

$$4(11)^{1-r} - 4 = 7 - r$$

$$4(11)^{1-r} + r - 44 = 0 \xrightarrow{\text{using software}} \underline{r \approx 1.278}$$

e) $\displaystyle\int_0^T \frac{200}{(t+1)^3}\,dt = -100\frac{1}{(t+1)^2}\Big|_0^T$

$$= -100\left(\frac{1}{(T+1)^2} - 1\right)$$

$$= 100 - \frac{100}{(T+1)^2}$$

$$\lim_{T\to\infty}\left(100 - \frac{100}{(T+1)^2}\right) \underline{= 100}$$

∴ The bacteria approach a finite limit of 100.

Exercise

Consider the function $f(x) = x^2 - 5x + 4$ and the area function $A(x) = \int_0^x f(t)\,dt$.

a) Graph f on the interval [0, 6].

b) Compute and graph *A* on the interval [0, 6].

c) Show that the local extrema of *A* occur at the zeros of f.

d) Give a geometric and analytical explanation for the observation in part (c).

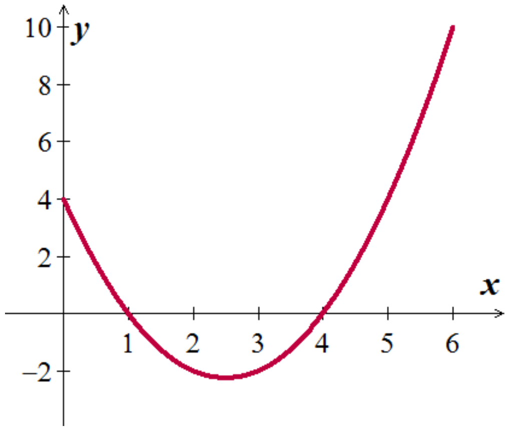
e) Find the approximate zeros of *A*, other than 0, and call them x_1 and x_2.

f) Find *b* such that the area bounded by the graph of f and the *x*-axis on the interval $\left[0,\ x_1\right]$ equals the area bounded by the graph of f and the *x*-axis on the interval $\left[x_1,\ b\right]$.

g) If f is an integrable function and $A(x) = \int_0^x f(t)\,dt$, is it always true that the local extrema of *A* occur at the zeros of f? Explain

Solution

a)

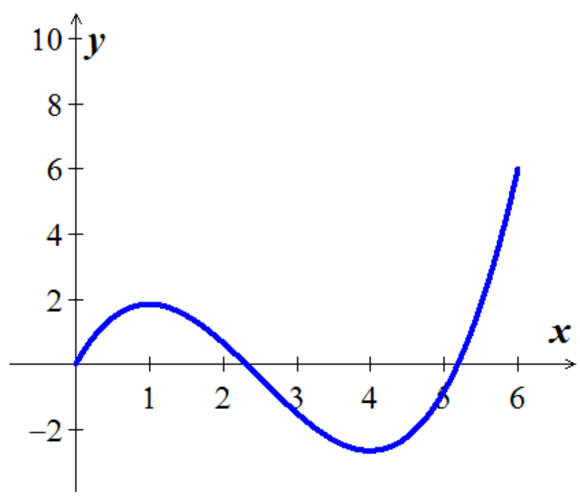


b) $A(x) = \int_0^x f(t)\,dt$

$= \int_0^x \left(t^2 - 5t + 4\right)dt$

$= \frac{1}{3}t^3 - \frac{5}{2}t^2 + 4t \Big|_0^x$

$= \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$



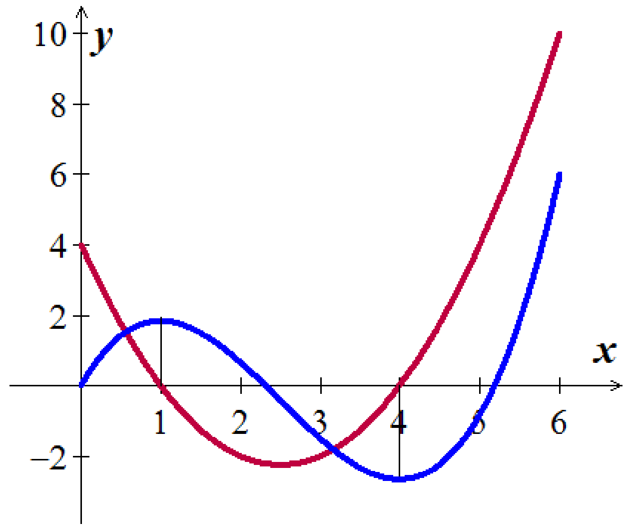
c) $f(x) = x^2 - 5x + 4 = 0$

$\rightarrow \quad \underline{x = 0,\ 4}\big|$

$A(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$

$A'(x) = f(x)$

The zeros of *f* are at 1 and 4, and *A* has a local maximum at *x* = 1 and local minimum at *x* = 4.



d) Since f is above the axis from 0 to 1, the net area A is increasing and switches to decreasing to the right of 1. When x is between 1 and 4, the function f is below x-axis (negative sign), the Area A is decreasing.

By the fundamental Theorem: $A'(x) = f(x)$, the zeros of f are critical points of A.

e) $A(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$

$$= \frac{1}{6}x\left(2x^2 - 15x + 24\right)$$

$$x = \frac{15 \pm \sqrt{33}}{4}$$

$$\rightarrow \begin{cases} x_1 = \frac{15-\sqrt{33}}{4} \approx 2.31386 \\ x_2 = \frac{15+\sqrt{33}}{4} \approx 5.18614 \end{cases}$$

f) $A_1 = \int_0^{x_1} f(x)\,dx$

$$= \int_0^1 f(x)\,dx + \left|\int_1^{x_1} f(x)\,dx\right|$$

$$= \left(\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x\right|_0^1 + \left|\left(\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x\right|_1^{x_1}\right|$$

$$= \left(\frac{1}{3} - \frac{5}{2} + 4\right) - 0 + \left|0 - \left(\frac{1}{3} - \frac{5}{2} + 4\right)\right|$$

$$= 2\left(\frac{1}{3} - \frac{5}{2} + 4\right)$$

$$= 2\left(\frac{11}{6}\right)$$

$$\boxed{= \frac{11}{3}}$$

$$A_2 = \left| \int_{x_1}^{x_2} f(x)\,dx \right| + \int_{x_2}^{b} f(x)\,dx$$

$$= \left| \left(\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x \right|_{x_1}^{x_2} \right| + \left(\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x \right|_{x_2}^{b}$$

$$= 0 + \left[\left(\frac{1}{3}b^3 - \frac{5}{2}b^2 + 4b \right) - 0 \right]$$

$$= \frac{1}{3}b^3 - \frac{5}{2}b^2 + 4b$$

Since $A_1 = A_2$

$$\frac{1}{3}b^3 - \frac{5}{2}b^2 + 4b = \frac{11}{3}$$

$$\frac{1}{3}b^3 - \frac{5}{2}b^2 + 4b - \frac{11}{3} = 0$$

$\rightarrow$ $\boxed{b = 5.744348}$ (and 2 complex numbers)

g) No, if the function is a piecewise function.

$$f(x) = \begin{cases} 1 & if \ \ 0 \le x \le 1 \\ -1 & if \ \ 1 \le x \le 2 \end{cases}$$

Then $A(x)$ has a maximum at $x = 1$ even though f is never zero.

This is a case where an extreme point occurs at a singular point rather than a stationary point.