Symmetry

Symmetries are used to classify geometrical objects; and each symmetry has a unique inverse, which is it itself a symmetry.

When studying symmetries of ordinary differential equations (ODE), Lie invented a method or a theory of Lie groups. Sophus Lie, Norwegian mathematician, produced a work on symmetries which is applied not only to solve differential equation, but also to fields like quantum mechanics, function theory, etc…

Let consider first order ODE of the form:

$$\frac{dy}{dx} = f(x, y) \qquad \text{(s.1)}$$

Few ODE have many symmetries, which they are discrete symmetries 'Lie Symmetries', are defined by changing their variables by a transformation of independent variable x and depend variable y, and they are written of the form:

$$\left(\tilde{x}, \tilde{y}\right) = (x+\varepsilon, y+\varepsilon) \qquad \text{(s.2)}$$

or

$$\tilde{x} = \tilde{x}\,(x, y; \varepsilon), \;\; \tilde{y} = \tilde{y}\,(x, y; \varepsilon) \qquad \text{(s.3)}$$

The equations s.2 and s.3 are considered point transformations that depend on arbitrary parameter ε; and the most obvious symmetries are rotations, reflections and scaling.

Example 1:

Consider the differential equation given by:

$$\frac{dy}{dx} = y \;\Rightarrow y = c_1\, e^x$$

Let use Lie symmetries: $\left(\tilde{x}, \tilde{y}\right) = (x+\varepsilon, y)$

Then: $\tilde{y} = y = c_1\, e^x = c_1\, e^{\tilde{x}-\varepsilon} = c_1\, e^{\tilde{x}}\, e^{-\varepsilon} = c_2\, e^{\tilde{x}}$ (where $c_2 = c_1\, e^{-\varepsilon}$

If ε is very small then we can assume that ε = 0, that implies to $c_2 = c_1$; the symmetry is trivial $\left(\tilde{x}, \tilde{y}\right) = (x, y)$

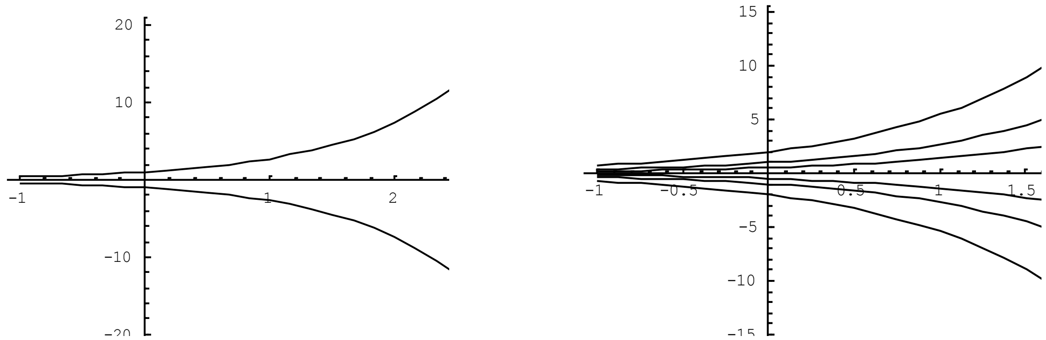

Fig.1: Discrete symmetry of the $y = c_1\, e^x$.

When $(\tilde{x}, \tilde{y})$ = (x, -y), the symmetry exists about x-axis and this symmetry is called a *discrete symmetry*.

Example 2:

Let consider the differential equation:

$$\frac{dy}{dx} = \frac{y}{x}$$

$\Rightarrow$ y = cx $\Rightarrow$ the solutions are straight lines passing thru the origin. And the most symmetry is rotations, reflections, and scaling.

$(\tilde{x}, \tilde{y})$ = (nx, ny), where $n \in \mathbb{N}$

Tangent Vector

The action of Lie group maps every point on an orbit to a point on the same orbit; under this action every orbit is invariant.

Consider an orbit through non-invariant point (x, y), the tangent vector to the orbit at the point $(\tilde{x}, \tilde{y})$ is given by:

$$\left(\xi(\tilde{x}, \tilde{y}),\, \eta(\tilde{x}, \tilde{y})\right) \qquad \text{(s.4)}$$

where

$$\frac{d\tilde{x}}{d\varepsilon} = \xi(\tilde{x}, \tilde{y}), \text{ and } \frac{d\tilde{y}}{d\varepsilon} = \eta(\tilde{x}, \tilde{y}) \qquad \text{(s.5)}$$

with η and ξ are independent of y'

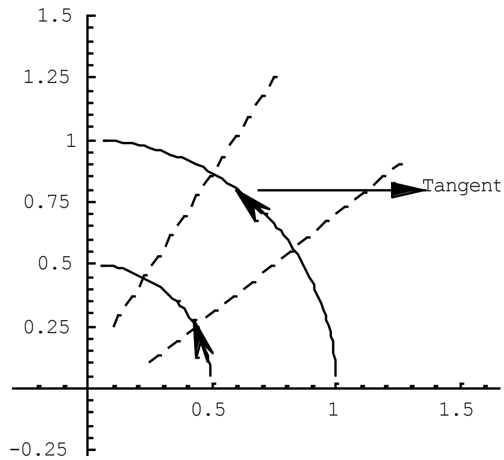

Fig.2: Tangent vector.

Therefore, the first order in ε, the Taylor series for the transformation (Lie group) action can be written as:

$$\begin{cases} \tilde{x} = x + \varepsilon\xi(x,y) + \cdots = x + \varepsilon\xi(x,y) + O(\varepsilon^2) = x + \varepsilon Xx + O(\varepsilon^2) \\ \tilde{y} = y + \varepsilon\eta(x,y) + \cdots = y + \varepsilon\eta(x,y) + O(\varepsilon^2) = x + \varepsilon Xy + O(\varepsilon^2) \end{cases} \quad \text{(s.6-a)}$$

$$\begin{cases} \tilde{x} = x + \varepsilon\xi(x,y) + \cdots = x + \varepsilon\xi(x,y) + O(\varepsilon^2) \\ \tilde{y} = y + \varepsilon\eta(x,y) + \cdots = y + \varepsilon\eta(x,y) + O(\varepsilon^2) \\ \tilde{y}' = y' + \varepsilon\eta'(x,y,y') + \cdots = y' + \varepsilon\eta'(x,y,y') + O(\varepsilon^2) \\ \vdots \qquad \vdots \qquad \vdots \\ \tilde{y}^{(k)} = y^{(k)} + \varepsilon\eta^{(k)}(x,y,y',\ldots,y^{(k)}) + O(\varepsilon^2) \end{cases} \quad \text{(s.6-b)}$$

Note: The superscript $\eta^{(k)}$ is not a derivative.

As an invariant point is mapped to itself by 'every Lie symmetry', therefore from eq.s.6, the point (x, y) in invariant only if the tangent vector is zero that is:

$$\xi(\tilde{x},\tilde{y}) = \eta(\tilde{x},\tilde{y}) = 0 \quad \text{(s.7)}$$

The set of a smooth vector field is the tangent vectors vary smoothly with (x, y). In particular, the tangent vector at (x, y) is:

$$(\xi(\tilde{x},\tilde{y}),\ \eta(\tilde{x},\tilde{y})) = \left(\left.\frac{d\tilde{x}}{d\varepsilon}\right|_{\varepsilon=0},\ \left.\frac{d\tilde{y}}{d\varepsilon}\right|_{\varepsilon=0} \right) \quad \text{(s.8)}$$

Therefore,

$$\eta' = \left.\frac{d\tilde{y}'}{d\varepsilon}\right|_{\varepsilon=0},\ldots,\ \eta^{(n)} = \left.\frac{d\tilde{y}^{(n)}}{d\varepsilon}\right|_{\varepsilon=0} \quad \text{(s.9)}$$

Infinitesimal Generator/Transformation:

Suppose the equation eq.s.1 has a 1-parameter Lie group of symmetries, and has a tangent vector of eq.s.4. Then the partial differential operator X is given by:

$$X = \xi(x,y)\partial_x + \eta(x,y)\partial_y \qquad (s.10)$$

Therefore, the operator X is called the *infinitesimal generator* of the Lie group (or transformation).
Where X in eq.s.10 can be rewritten as:

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta'\frac{\partial}{\partial y'} + \ \cdots + \eta^{(n)}\frac{\partial}{\partial y^{(n)}} \qquad (s.11)$$

The eq.s.11 is called *extension* (prolongation) up to the n^{th} derivative where:

$$\eta^{(n)} = (x, y, y', \ldots, y^{(n)}) = \frac{d^n}{dx}(\eta - y'\xi) + y^{(n+1)}\xi \qquad (s.12)$$

From the eq.s.12, we can derive the different approach to η:

$$\eta' = \frac{d\eta}{dx} - y'\frac{d\xi}{dx} = \frac{\partial\eta}{\partial x} - y'\left(\frac{\partial\eta}{\partial y} - \frac{d\xi}{dx}\right) - y'^2\frac{d\xi}{dy} \qquad (s.13)$$

or $\quad \eta^{(1)} = \eta' = \eta_x + (\eta_y - \xi_x)\, y' - \xi_y\, y'^2 \qquad (s.14)$

$$\begin{aligned}\eta^{(2)} &= \eta'' \\ &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})\, y' + (\eta_{yy} - 2\xi_{xy})\, y'^2 - \xi_{yy}\, y'^3 \\ &\quad + (\eta_y - 2\xi_x - 3\xi_y\, y')y''\end{aligned} \qquad (s.15)$$

$$\begin{aligned}\eta^{(3)} &= \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})\, y' + 3(\eta_{xyy} - \xi_{xxy})\, y'^2 + (\eta_{yyy} - 3\,\xi_{xyy})\, y'^3 \\ &\quad - \xi_{yyy}\, y'^4 + 3[\eta_{xy} - \xi_{xx} + (\eta_{yy} - 2\xi_{xy})\, y' - 2\xi_{yy}\, y'^2\,]\, y'' \\ &\quad - 3\,\xi_y\, y''^2 + [\eta_y - 3\,\xi_x - 4\,\xi_y\, y'\,]\, y'''\end{aligned} \qquad (s.16)$$

The infinitesimal generator affected by a change of coordinates. Suppose that (u, v) are the new coordinates, and let F(u, v) be an arbitrary function. Then by the chain rule:

$$\begin{aligned}X\,F(u,v) &= X\,F(u(x,y), v(x,y)) = \xi(u_x\,F_u + v_x\,F_v) + \eta(u_y\,F_u + v_yF_v) \\ &= (Xu)\,F_u + (X_v)\,F_v\end{aligned}$$

The infinitesimal generator in terms of the new coordinates is:

$$X = (Xu)\,\partial_u + (Xv)\,\partial_v \qquad (s.17)$$

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \qquad (s.18)$$

From These definitions and methods of the infinitesimal generator, we can apply them to find the Lie point symmetries.
In the next example, we will determine how to find these Lie point symmetries.

Example 3:

Let consider a second order ODE equation: $y'' = 0$
The linearized symmetry condition for this ODE is:

$\eta^{(2)} = 0$ from eq.s.14, when $y'' = 0$

Therefore:

1) $\eta_{xx} = 0$
2) $\xi_{yy} = 0 \quad \Rightarrow \xi_y = A(x) \quad \Rightarrow$ the general solution: $\xi(x, y) = A(x)\, y + B(x)$
3) $\eta_{yy} - 2\xi_{xy} = 0 \;\Rightarrow \eta_{yy} = 2\,\xi_{xy} = \xi_x\,[A(x)] = 2\,A'(x) \;\Rightarrow \eta_{yy} = 2\,A'(x)$
   $\Rightarrow \eta_y = 2\,A'(x)\, y + C(x)$
   $\Rightarrow \eta = 2\,A'(x)\, y^2 + C(x)\, y + D(x)$ (a)
4) $2\eta_{xy} - \xi_{xx} = 0$

From part-1, the solution: $\eta_x\,[A''(x)\, y^2 + C'(x)\, y + D'(x)] = 0$
$\Rightarrow A'''(x)\, y^2 + C''(x)\, y + D''(x) = 0$ (b)

From (4) $\Rightarrow 2\eta_{xy} = \xi_{xx} \Rightarrow 2\,[2A'(x)\, y + C(x)] = A'(x)\, y + B'(x)$

$\Rightarrow 2\,[2A''(x)\, y + C'(x)] = A''(x)\, y + B''(x)$

$\Rightarrow 3A''(x)\, y + 2C'(x) - B''(x) = 0$ (c)

When y = 0, then from (b): $D''(x) = 0 \;\Rightarrow D'(x) = d_1 \;\Rightarrow D(x) = d_1\, x + d_2$

And from (c): $2C'(x) - B''(x) = 0 \;\Rightarrow 2\,C'(x) = B''(x)$,

and $A''(x) = C''(x) = 0 \;\Rightarrow C'(x) = c_1 \;\Rightarrow C(x) = c_1\, x + c_2$

$\Rightarrow A'(x) = a_1 \;\Rightarrow A(x) = a_1\, x + a_2$

$\Rightarrow B''(x) = 2\,C'(x) = 2(c_1\, x + c_2) = 2c_1\, x + b_2 \;\Rightarrow B(x) = c_1\, x^2 + b_2\, x + b_3$

$\xi(x, y) = (a_1\, x + a_2)\, y + c_1\, x^2 + b_2\, x + b_3 = \alpha_1 + \alpha_3\, x + \alpha_5\, y + \alpha_7\, x^2 + \alpha_8\, xy$

From (a):

$\eta(x, y) = (a_1\, y^3 + (c_1\, x + c_2)y + d_1\, x + d_2 = \alpha_2 + \alpha_4\, x + \alpha_6\, y + \alpha_7\, xy + \alpha_8\, y^2$

Where α_i (i = 1, 2, …,8) are constants, and the general infinitesimal generator is:

$$X = \sum_{n=1}^{8} \alpha_n X_n$$

Where $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x\,\partial_x$, $X_4 = y\,\partial_y$, $X_5 = y\,\partial_x$,
$X_6 = x\,\partial_y$, $X_7 = x^2\,\partial_x + xy\,\partial_y$, $X_8 = xy\,\partial_x + y^2\,\partial_y$

Canonical Coordinates:

Given any 1-parameter transformation of symmetry:

$(u, v) = (u(x, y), v(x, y))$ such that: $(\tilde{u}, \tilde{v}) \equiv (u(\tilde{x}, \tilde{y}), v(\tilde{x}, \tilde{y})) = (r, v + \varepsilon)$

Where the tangent vector at the point (u, v) is $(0, 1)$ that is:

$$\left.\frac{d\tilde{u}}{d\varepsilon}\right|_{\varepsilon=0} = 0, \quad \left.\frac{d\tilde{v}}{d\varepsilon}\right|_{\varepsilon=0} = 1 \qquad \text{(s.19)}$$

By using the chain rule and the eq.(s.5), we can determine:

$$\begin{cases} \xi(x, y)u_X + \eta(x, y)u_y = 0 \\ \xi(x, y)v_X + \eta(x, y)v_y = 1 \end{cases} \qquad \text{(s.20)}$$

The change of coordinates should be invertible in some neighborhood of (x, y), with a condition is given by:

$$u_x\, v_y - u_y\, v_x \neq 0 \qquad \text{(s.21)}$$

Any functions of u(x, y) and v(x, y) satisfy the equations s.20 and s.21 are called a pair of *canonical coordinates*.

Let $X: V \to V$ be smooth Γ- equivalent vector field on V;

If X is equivalent then: $X(0) = X(\gamma(0)) = \gamma\, X(0)$, where $\gamma \in \Gamma$.

However, the zero is a subspace invariant since V contains no trivial factors.

Symmetry breaking bifurcations include branches with sub-maximal isotropy groups. Knowledge of sub-maximal branches of solution is extremely important in understanding the dynamics spawned in equivariant bifurcations. And we can define the isotropy subgroup of Γ at a point x by:

$$\Gamma_x = \{\, \gamma \in \Gamma \mid \gamma x = x \,\}$$

The isotropy subgroup of Γ at x is measure of the symmetry of the point x.

Geometry of representations:

Let consider that (V, Γ) is a finite dimension representation, and let assume that Γ has a subgroup H ($H \subset \Gamma$).

Then the fixed-point space:

$$V^H = \{ v \in V \mid \gamma v = v, \forall \gamma \in H\}; \text{ is a linear subspace of V.}$$

$$V_\tau = \{ v \in V \mid L(v) = \tau\}; V_\tau \text{ is the orbit stratum of the type } \tau.$$

Let assume $\Gamma = D_4$ acting on $\mathbb{R}^2$, let I group generated by reflection on x-axis and J group generated by reflection in the line x = y.

Both I & J are isomorphic to $\mathbb{Z}_2$ through they are not conjugate subgroups of D_4.
a = (I), b = (J) and (e) trivial isotropy type the lines $x = \pm y$ less than the origin and the open dense subset of $\mathbb{R}^2$ complementing the lines $xy(x^2 - y^2) = 0$

Let consider the $\mathbb{Z}_2$ equivariantly saddle stable system:

Degenerate intersections forced by symmetry:

$$V_a \cap V_b = (a, b)$$

And this intersection is not transversal.
Let X be a $\mathbb{Z}_2$ equivariant vector field on $\mathbb{R}^2$, which has two adjacent saddle points a < b on x-axis.

The mid-point of (a, b) plays a role, by dividing the unstable point ‘a’ and stable point ‘b’, which no longer intersect. However, since V_a and V_b lie on x-axis inside a symmetry subspace (1-dimension); therefore they are forced to have intersection $\mathbb{Z}_2$ by symmetry. This type of stable intersection a $\mathbb{Z}_2$ in a *transversal* $\cap$ (intersection), but not transversal.

It is possible to get a ‘cycle’ of saddle links between equilibria. This cycle is typically associated to a cycle of symmetry subspaces and persists under equivariant perturbations of x. This connection is called a *heteroclinic* cycle.

A cell as specified by a differential equation:

$$\dot{x} = f(x)$$

is defined on a finite dimensional linear phase space E. We always assume that **f** has an equilibrium at zero $\Rightarrow f(0) = 0$

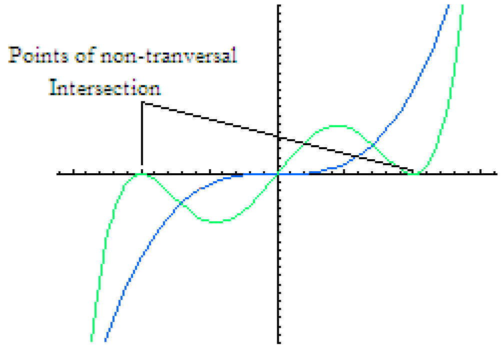

Fig.: Stratum-wise transversality.