SOLUTION *Section* 2.1 – Second-Order Linear Differential Equations

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation. $y'' + 2y' - 3y = 0$

Solution

Let $v = y' \quad \Rightarrow \quad v' = y''$

$y'' = -2y' + 3y$

$v' = -2v + 3y$

The following system of the first-order equations: $\begin{cases} y' = v \\ v' = -2v + 3y \end{cases}$

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation. $y'' + 3y' + 4y = 2\cos 2t$

Solution

Let $v = y' \quad \Rightarrow \quad v' = y''$

$y'' = -3y' - 4y + 2\cos 2t$

$v' = -3v - 4y + 2\cos 2t$

The following system of the first-order equations: $\begin{cases} y' = v \\ v' = -3v - 4y + 2\cos 2t \end{cases}$

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation. $y'' + 2y' + 2y = 2\sin 2\pi t$

Solution

Let $v = y' \quad \Rightarrow \quad v' = y''$

$y'' = -2y' - 2y + 2\sin 2\pi t$

$v' = -2v - 2y + 2\sin 2\pi t$

The following system of the first-order equations: $\begin{cases} y' = v \\ v' = -2v - 2y + 2\sin 2\pi t \end{cases}$

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation. $\quad y'' + \mu\left(t^2 - 1\right)y' + y = 0$

Solution

Let $v = y' \quad \Rightarrow \quad v' = y''$

$$y'' = -\mu\left(t^2 - 1\right)y' - y$$

$$v' = -\mu\left(t^2 - 1\right)v - y$$

The following system of the first-order equations:

$$\begin{cases} y' = v \\ v' = -\mu\left(t^2 - 1\right)v - y \end{cases}$$

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation. $\quad 4y'' + 4y' + y = 0$

Solution

Let $v = y' \quad \Rightarrow \quad v' = y''$

$$4y'' = -4y' - y$$

$$y'' = -y' - \frac{1}{4}y$$

$$v' = -v - \frac{1}{4}y$$

The following system of the first-order equations: $\begin{cases} y' = v \\ v' = -v - \frac{1}{4}y \end{cases}$

Exercise

Show that the functions $y_1(x) = e^{-3x}, \quad y_2(x) = \cos 2x, \quad y_3(x) = \sin 2x$ are linearly independent.

Solution

$$W = \begin{vmatrix} e^{-3x} & \cos 2x & \sin 2x \\ -3e^{-3x} & -2\sin 2x & 2\cos 2x \\ 9e^{-3x} & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$= 8e^{-3x}\sin^2 2x + 18e^{-3x}\cos^2 2x + 12e^{-3x}\sin 2x\cos 2x$$
$$+18e^{-3x}\sin^2 2x + 8e^{-3x}\cos^2 2x - 12e^{-3x}\sin 2x\cos 2x$$
$$\underline{= 26e^{-3x} \neq 0}|$$

$\therefore\ y_1, y_2, \text{ and } y_3$ are linearly independent.

Example

Determine whether $\left\{e^x,\ xe^x,\ (x+1)e^x\right\}$ is a set of linearly independent.

Solution

$$W = \begin{vmatrix} e^x & xe^x & (x+1)e^x \\ e^x & (x+1)e^x & (x+2)e^x \\ e^x & (x+2)e^x & (x+3)e^x \end{vmatrix}$$

$$= (x+1)(x+3)e^{3x} + x(x+2)e^{3x} + (x+1)(x+2)e^{3x} - (x+1)^2 e^{3x} - (x+2)^2 e^{3x} - x(x+3)e^{3x}$$
$$= \left(x^2 + 4x + 3 + x^2 + 2x + x^2 + 3x + 2 - x^2 - 2x - 1 - x^2 - 4x - 4 - x^2 - 3x\right)e^{3x}$$
$$\underline{= 0}|$$

Thus the set $\left\{e^x,\ xe^x,\ (x+1)e^x\right\}$ is linearly dependent.

Exercise

Use the Wronskian to show that are linearly independence $\quad y_1(x) = e^{-3x}, \quad y_2(x) = e^{3x}$

Solution

$$W(x) = \begin{vmatrix} e^{-3x} & e^{3x} \\ -3e^{-3x} & 3e^{3x} \end{vmatrix}$$
$$= 3 + 3$$
$$\underline{= 6 \ \neq 0}|$$

Thus the functions are linearly independent.

Exercise

Use the Wronskian to show that are linearly independence $\quad \mathbf{f}_1 = 1, \quad \mathbf{f}_2 = e^x, \quad \mathbf{f}_3 = e^{2x}$

Solution

The Wronskian is

$$W(x)=\begin{vmatrix} 1 & e^{x} & e^{2x} \\ 0 & e^{x} & 2e^{2x} \\ 0 & e^{x} & 4e^{2x} \end{vmatrix}=e^{x}4e^{2x}-2e^{2x}e^{x}=2e^{3x}\neq 0$$

Thus the functions are linearly independent.

Exercise

Use the Wronskian to show that are linearly independence $\left\{e^{x},\ xe^{x},\ (x+1)e^{x}\right\}$

Solution

$$W=\begin{vmatrix} e^{x} & xe^{x} & (x+1)e^{x} \\ e^{x} & (x+1)e^{x} & (x+2)e^{x} \\ e^{x} & (x+2)e^{x} & (x+3)e^{x} \end{vmatrix}$$

$$=(x+1)(x+3)e^{3x}+x(x+2)e^{3x}+(x+1)(x+2)e^{3x}-(x+1)^{2}e^{3x}-(x+2)^{2}e^{3x}-x(x+3)e^{3x}$$

$$=\left(x^{2}+4x+3+x^{2}+2x+x^{2}+3x+2-x^{2}-2x-1-x^{2}-4x-4-x^{2}-3x\right)e^{3x}$$

$$\underline{=0}$$

Thus the set $\left\{e^{x},\ xe^{x},\ (x+1)e^{x}\right\}$ is linearly dependent.

Exercise

Use the Wronskian to show that are linearly independence

$$y_1(x)=e^{-3x},\quad y_2(x)=\cos 2x,\quad y_3(x)=\sin 2x$$

Solution

$$W=\begin{vmatrix} e^{-3x} & \cos 2x & \sin 2x \\ -3e^{-3x} & -2\sin 2x & 2\cos 2x \\ 9e^{-3x} & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$=8e^{-3x}\sin^{2}2x+18e^{-3x}\cos^{2}2x+12e^{-3x}\sin 2x\cos 2x$$
$$+18e^{-3x}\sin^{2}2x+8e^{-3x}\cos^{2}2x-12e^{-3x}\sin 2x\cos 2x$$

$$\underline{=26e^{-3x}\neq 0}$$

$\therefore\ y_1,\ y_2,\ and\ y_3$ are linearly independent.

Exercise

Use the Wronskian to show that are linearly independence $y_1(x)=e^x, \quad y_2(x)=e^{2x}, \quad y_3(x)=e^{3x}$

Solution

$$W=\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$=18e^{6x}+3e^{6x}+4e^{6x}-2e^{6x}-12e^{6x}-9e^{6x}$$

$$\boxed{=2e^{6x} \quad \neq 0}$$

$\therefore\ y_1,\ y_2,\ \textit{and}\ y_3$ are linearly independent.

Exercise

Use the Wronskian to show that are linearly independence

$$y_1(x)=\cos^2 x, \quad y_2(x)=\sin^2 x, \quad y_3(x)=\sec^2 x, \quad y_4(x)=\tan^2 x$$

Solution

Since $\cos^2 x+\sin^2 x=1\ \&\ \sec^2 x=1+\tan^2 x$

$c_1\cos^2 x+c_2\sin^2 x+c_3\sec^2 x+c_4\tan^2 x=0$

Let: $c_1=c_2=0 \quad c_3=-1 \quad c_4=1$

$\cos^2 x+\sin^2 x-\sec^2 x+\tan^2 x=0$

The set of functions are linearly dependent.

Exercise

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0,\ 1)$

$$y_1(t)=\cos t\sin t, \quad y_2(t)=\sin 2t$$

Solution

$y_1(t)=cy_2(t)$

$\cos t\sin t=c\sin 2t \quad \rightarrow \quad \boxed{c=\frac{1}{2}}$

The given functions are linearly dependent.

Exercise

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0,\ 1)$

$$y_1(t) = e^{3t}, \quad y_2(t) = e^{-4t}$$

Solution

$$y_1(t) = cy_2(t)$$

$$e^{3t} = ce^{-4t} \quad \rightarrow \quad e^{7t} = c$$

Since an exponential function is strictly monotone, this is a contradiction.

Hence, given functions are linearly independent on $(0,\ 1)$

Exercise

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0,\ 1)$

$$y_1(t) = te^{2t}, \quad y_2(t) = e^{2t}$$

Solution

$$y_1(t) = cy_2(t)$$

$$te^{2t} = ce^{2t} \quad \rightarrow \quad \underline{c = t|}$$

Hence, given functions are linearly independent on $(0,\ 1)$

Exercise

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0,\ 1)$

$$y_1(t) = t^2\cos(\ln t), \quad y_2(t) = t^2\sin(\ln t)$$

Solution

$$y_1(t) = cy_2(t)$$

$$t^2\cos(\ln t) = ct^2\sin(\ln t) \quad \rightarrow \quad \cos(\ln t) = c\sin(\ln t) \quad \Rightarrow \quad \underline{c = \cot(\ln t)|}$$

Hence, given functions are linearly independent on $(0,\ 1)$

Exercise

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0,\ 1)$

$$y_1(t) = \tan^2 t - \sec^2 t, \quad y_2(t) = 3$$

Solution

$$y_1(t) = cy_2(t)$$

$$\tan^2 t - \sec^2 t = 3c \quad \rightarrow \quad -1 = 3c \Rightarrow \boxed{c = -\frac{1}{3}}$$

The given functions are linearly dependent.

Exercise

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0,\ 1)$

$$y_1(t) \equiv 0, \quad y_2(t) = e^t$$

Solution

$$y_1(t) = cy_2(t)$$

$$0 \equiv ce^t \quad \rightarrow \quad \boxed{c \equiv 0}$$

The given functions are linearly dependent.

Exercise

Find a particular solution satisfying the given initial conditions

$$y'' - 4y = 0\ ; \quad y_1(t) = e^{2t},\ y_2(t) = 2e^{-2t}; \quad y(0) = 1,\ y'(0) = -2$$

Solution

$$W = \begin{vmatrix} e^{2t} & 2e^{-2t} \\ 2e^{2t} & -4e^{-2t} \end{vmatrix}$$

$$\boxed{= -8 \neq 0}$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$y(t) = C_1 e^{2t} + 2C_2 e^{-2t} \qquad y(0) = 1 \quad \rightarrow \quad C_1 + 2C_2 = 1$$

$$y'(t) = 2C_1 e^{2t} - 4C_2 e^{-2t} \qquad y'(0) = -2 \quad \rightarrow \quad 2C_1 - 4C_2 = -2$$

$$\begin{cases} C_1 + 2C_2 = 1 \\ C_1 - 2C_2 = -1 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = -4 \quad \Delta_1 = \begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} = 0 \quad \Delta_2 = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

$$\boxed{C_1 = 0, \quad C_2 = \frac{1}{2}}$$

$$\boxed{y(t) = e^{-2t}}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$y'' - y = 0\ ; \quad y_1(t) = 2e^t,\ y_2(t) = e^{-t+3}; \quad y(-1) = 1,\ y'(-1) = 0$$

Solution

$$W = \begin{vmatrix} 2e^t & e^{-t+3} \\ 2e^t & -e^{-t+3} \end{vmatrix}$$

$$= -4e^3 \neq 0$$

$\therefore\ y_1\ \text{and}\ y_2$ are linearly independent.

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$y(t) = 2C_1 e^t + C_2 e^{-t+3} \qquad y(-1) = 1 \ \rightarrow\ 2C_1 e^{-1} + C_2 e^4 = 1$$

$$y'(t) = 2C_1 e^t - C_2 e^{-t+3} \qquad y'(-1) = 0 \ \rightarrow\ 2C_1 e^{-1} - C_2 e^4 = 0$$

$$\begin{cases} 2C_1 + e^5 C_2 = e \\ 2C_1 - e^5 C_2 = 0 \end{cases} \quad \rightarrow\ 4C_1 = e$$

$$C_1 = \frac{e}{4}, \quad C_2 = \frac{1}{2e^4}$$

$$y(t) = \frac{e}{4}e^t + \frac{1}{2e^4}e^{-t+3}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$y'' + y = 0\ ; \quad y_1(t) = 0,\ y_2(t) = \sin t; \quad y\left(\frac{\pi}{2}\right) = 1,\ y'\left(\frac{\pi}{2}\right) = 1$$

Solution

$$W = \begin{vmatrix} 0 & \sin t \\ 0 & \cos t \end{vmatrix}$$

$$= 0$$

$\therefore\ y_1\ \text{and}\ y_2$ are linearly dependent.

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$y(t) = C_2 \sin t \qquad y\left(\frac{\pi}{2}\right) = 1 \ \rightarrow\ C_2 = 1$$

$$y'(t) = C_2 \cos t \qquad\qquad y'\left(\frac{\pi}{2}\right) = 1 \;\rightarrow\; \cancel{0 = 1}$$

$$\underline{y(t) = C_2 \sin t}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$y'' + y = 0\;; \quad y_1(t) = \cos t,\; y_2(t) = \sin t; \quad y\left(\frac{\pi}{2}\right) = 1,\; y'\left(\frac{\pi}{2}\right) = 1$$

Solution

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$$

$$= \cos^2 t + \sin^2 t$$

$$\underline{= 1 \neq 0}$$

$\therefore\; y_1$ *and* y_2 are linearly independent.

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$y(t) = C_1 \cos t + C_2 \sin t \qquad y\left(\frac{\pi}{2}\right) = 1 \;\rightarrow\; \underline{C_2 = 1}$$

$$y'(t) = -C_1 \sin t + C_2 \cos t \qquad y'\left(\frac{\pi}{2}\right) = 1 \;\rightarrow\; \underline{C_1 = -1}$$

$$\underline{y(t) = -\cos t + \sin t}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$y'' - 4y' + 4y = 0\;; \quad y_1(t) = e^{2t},\; y_2(t) = te^{2t}; \quad y(0) = 2,\; y'(0) = 0$$

Solution

$$W = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & (1+2t)e^{2t} \end{vmatrix}$$

$$= (1 + 2t - 2t)e^{4t}$$

$$\underline{= e^{4t} \neq 0}$$

$\therefore\; y_1$ *and* y_2 are linearly independent.

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$y(t)=C_1e^{2t}+C_2te^{2t} \qquad y(0)=2 \;\rightarrow\; \boxed{C_1=2}$$

$$y'(t)=2C_1e^{2t}+C_2(1+2t)e^{2t} \qquad y'(0)=0 \;\rightarrow\; 2C_1+C_2=0 \;\rightarrow\; \boxed{C_2=-4}$$

$$\boxed{y(t)=2e^{2t}-4te^{2t}}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$2y''-y'=0\;; \quad y_1(t)=1,\;\; y_2(t)=e^{t/2}; \quad y(2)=0,\;\; y'(2)=2$$

Solution

$$W=\begin{vmatrix}1 & e^{t/2}\\ 0 & \frac{1}{2}e^{t/2}\end{vmatrix}$$

$$\boxed{=\frac{1}{2}e^{t/2}\;\neq 0}$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$$y(t)=C_1y_1(t)+C_2y_2(t)$$

$$y(t)=C_1+C_2e^{t/2} \qquad y(2)=0 \;\rightarrow\; C_1+eC_2=0$$

$$y'(t)=\frac{1}{2}C_2e^{t/2} \qquad y'(2)=2 \;\rightarrow\; \frac{1}{2}eC_2=2$$

$$\boxed{C_1=-4, \quad C_2=\frac{4}{e}}$$

$$\boxed{y(t)=-4+\frac{4}{e}e^{t/2}}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$y''-3y'+2y=0\;; \quad y_1(t)=2e^{t},\;\; y_2(t)=e^{2t}; \quad y(-1)=1,\;\; y'(-1)=0$$

Solution

$$W=\begin{vmatrix}2e^{t} & e^{2t}\\ 2e^{t} & 2e^{2t}\end{vmatrix}$$

$$\boxed{=2e^{3t}\;\neq 0}$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$y(t) = C_1 y_1(t) + C_2 y_2(t)$

$y(t) = 2C_1 e^t + C_2 e^{2t}$ $\qquad$ $y(-1) = 1 \quad \rightarrow \quad 2e^{-1}C_1 + e^{-2}C_2 = 1$

$y'(t) = 2C_1 e^t + 2C_2 e^{2t}$ $\qquad$ $y'(-1) = 0 \quad \rightarrow \quad 2e^{-1}C_1 + 2e^{-2}C_2 = 2$

$$\begin{cases} 2eC_1 + C_2 = e^2 \\ eC_1 + C_2 = e^2 \end{cases} \qquad \Delta = \begin{vmatrix} 2e & 1 \\ e & 1 \end{vmatrix} = e \quad \Delta_1 = \begin{vmatrix} e^2 & 1 \\ e^2 & 1 \end{vmatrix} = 0 \quad \Delta_2 = \begin{vmatrix} 2e & e^2 \\ e & e^2 \end{vmatrix} = e^3$$

$\boxed{C_1 = 0, \quad C_2 = e^2}$

$\boxed{y(t) = e^{2t+2}}$

Exercise

Find a particular solution satisfying the given initial conditions

$$ty'' + y' = 0\ ; \quad y_1(t) = \ln t, \ y_2(t) = \ln 3t; \quad y(3) = 0, \ y'(3) = 3$$

Solution

$$W = \begin{vmatrix} \ln t & \ln 3t \\ \frac{1}{t} & \frac{1}{t} \end{vmatrix}$$

$$\boxed{= \frac{1}{t}(\ln t - \ln 3t) \ \neq 0}$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$y(t) = C_1 y_1(t) + C_2 y_2(t)$

$y(t) = C_1 \ln t + C_2 \ln 3t$ $\qquad$ $y(3) = 0 \quad \rightarrow \quad (\ln 3)C_1 + (\ln 9)C_2 = 0$

$y'(t) = \dfrac{C_1}{t} + \dfrac{C_2}{t}$ $\qquad$ $y'(3) = 3 \quad \rightarrow \quad \frac{1}{3}(C_1 + C_2) = 3$

$$\begin{cases} (\ln 3)C_1 + (2\ln 3)C_2 = 0 \\ C_1 + C_2 = 9 \end{cases}$$

$$\Delta = \begin{vmatrix} \ln 3 & 2\ln 3 \\ 1 & 1 \end{vmatrix} = -\ln 3 \quad \Delta_1 = \begin{vmatrix} 0 & 2\ln 3 \\ 9 & 1 \end{vmatrix} = -18\ln 3 \quad \Delta_2 = \begin{vmatrix} \ln 3 & 0 \\ 1 & 9 \end{vmatrix} = 9\ln 3$$

$\boxed{C_1 = 18, \quad C_2 = -9}$

$\boxed{y(t) = 18\ln t - 9\ln 3t}$

Exercise

Find a particular solution satisfying the given initial conditions

$$t^2y'' - ty' - 3y = 0\ ; \quad y_1(t) = t^3,\ y_2(t) = -\frac{1}{t};\quad y(-1) = 0,\ y'(-1) = -2 \quad (t < 0)$$

Solution

$$W = \begin{vmatrix} t^3 & -\frac{1}{t} \\ 3t^2 & \frac{1}{t^2} \end{vmatrix}$$

$$\underline{= 4t \neq 0|}$$

$\therefore\ y_1\ and\ y_2$ are linearly independent.

$$y(t) = C_1y_1(t) + C_2y_2(t)$$

$$y(t) = C_1t^3 - \frac{C_2}{t} \qquad y(-1) = 0 \ \rightarrow\ -C_1 + C_2 = 0$$

$$y'(t) = 3C_1t^2 + \frac{C_2}{t^2} \qquad y'(-1) = -2 \ \rightarrow\ 3C_1 + C_2 = -2$$

$$\begin{cases} -C_1 + C_2 = 0 \\ 3C_1 + C_2 = -2 \end{cases} \qquad \Delta = \begin{vmatrix} -1 & 1 \\ 3 & 1 \end{vmatrix} = -4 \quad \Delta_1 = \begin{vmatrix} 0 & 1 \\ -2 & 1 \end{vmatrix} = 2 \quad \Delta_2 = \begin{vmatrix} -1 & 0 \\ 3 & -2 \end{vmatrix} = 2$$

$$\underline{C_1 = -\frac{1}{2},\quad C_2 = -\frac{1}{2}|}$$

$$\underline{y(t) = -\frac{1}{2}t^3 + \frac{1}{2t}\ |}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$y'' + \pi^2 y = 0\ ; \quad y_1(t) = \sin \pi t + \cos \pi t,\ y_2(t) = \sin \pi t - \cos \pi t;\quad y\left(\frac{1}{2}\right) = 1,\ y'\left(\frac{1}{2}\right) = 0$$

Solution

$$W = \begin{vmatrix} \sin \pi t + \cos \pi t & \sin \pi t - \cos \pi t \\ \pi \cos \pi t - \pi \sin \pi t & \pi \cos \pi t + \pi \sin \pi t \end{vmatrix}$$

$$= \pi \sin^2 \pi t + \pi \cos^2 \pi t + 2\pi \sin \pi t \cos \pi t - 2\pi \sin \pi t \cos \pi t + \pi \sin^2 \pi t + \pi \cos^2 \pi t$$

$$= 2\pi\left(\sin^2 \pi t + \cos^2 \pi t\right)$$

$$\underline{= 2\pi \neq 0|}$$

$\therefore\ y_1\ and\ y_2$ are linearly independent.

$y(t)=C_1y_1(t)+C_2y_2(t)$

$y(t)=C_1(\sin\pi t+\cos\pi t)+C_2(\sin\pi t-\cos\pi t)$ $\qquad y\left(\frac{1}{2}\right)=1 \quad\rightarrow\quad C_1+C_2=1$

$y'(t)=C_1(\pi\cos\pi t-\pi\sin\pi t)+C_2(\pi\cos\pi t+\pi\sin\pi t)$ $\qquad y'\left(\frac{1}{2}\right)=0 \quad\rightarrow\quad -\pi C_1+\pi C_2=0$

$$\begin{cases} C_1+C_2=1 \\ -C_1+C_2=0 \end{cases} \quad\rightarrow\quad \underline{C_1=\frac{1}{2},\quad C_2=\frac{1}{2}}$$

$$y(t)=\frac{1}{2}(\sin\pi t+\cos\pi t)+\frac{1}{2}(\sin\pi t-\cos\pi t)$$

$$\underline{=\sin\pi t}$$

Exercise

Find a particular solution satisfying the given initial conditions $\quad x^3y^{(3)}-x^2y''+2xy'-2y=0$

$y(1)=3,\quad y'(1)=2,\quad y''(1)=1 \qquad y_1(x)=x,\quad y_2(x)=x\ln x,\quad y_3(x)=x^2$

Solution

$$W=\begin{vmatrix} x & x\ln x & x^2 \\ 1 & \ln x+1 & 2x \\ 0 & \frac{1}{x} & 2 \end{vmatrix}$$

$$=2x\ln x+2x+x-2x-2x\ln x$$

$$\underline{=x\neq 0}$$

$\therefore\ y_1,\ y_2,\ and\ y_3$ are linearly independent.

$y(x)=C_1y_1(x)+C_2y_2(x)+C_3y_3(x)$

$y(x)=C_1x+C_2x\ln x+C_3x^2$ $\qquad y(1)=3 \quad\rightarrow\quad C_1+C_3=3$

$y'(x)=C_1+C_2(1+\ln x)+2C_3x$ $\qquad y'(1)=2 \quad\rightarrow\quad C_1+C_2+2C_3=2$

$y''(x)=\frac{C_2}{x}+2C_3$ $\qquad y''(1)=1 \quad\rightarrow\quad C_2+2C_3=1$

$$\begin{cases} C_1+C_3=3 \\ C_1+C_2+2C_3=2 \\ C_2+2C_3=1 \end{cases} \qquad \Delta=\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{vmatrix}=1 \quad \Delta_1=\begin{vmatrix} 3 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{vmatrix}=1$$

$$\underline{C_1=1,\quad C_2=-3,\quad C_3=2}$$

$$\underline{y(x)=x-3x\ln x+2x^2}$$

Exercise

Find a particular solution satisfying the given initial conditions $y^{(3)} + 2y'' - y' - 2y = 0$

$y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 0 \qquad y_1(x) = e^{x}, \quad y_2(x) = e^{-x}, \quad y_3(x) = e^{-2x}$

Solution

$$W = \begin{vmatrix} e^{x} & e^{-x} & e^{-2x} \\ e^{x} & -e^{-x} & -2e^{-2x} \\ e^{x} & e^{-x} & 4e^{-2x} \end{vmatrix}$$

$$= -4e^{-2x} - 2e^{-2x} + e^{-2x} + e^{-2x} + 2e^{-2x} - 4e^{-2x}$$

$$= -6e^{-2x} \neq 0$$

$\therefore\ y_1,\ y_2,\ and\ y_3$ are linearly independent. $\qquad y(x) = C_1y_1(x) + C_2y_2(x) + C_3y_3(x)$

$$y(x) = C_1e^{x} + C_2e^{-x} + C_3e^{-2x} \qquad y(0) = 1 \ \rightarrow\ C_1 + C_2 + C_3 = 1$$

$$y'(x) = C_1e^{x} - C_2e^{-x} - 2C_3e^{-2x} \qquad y'(0) = 2 \ \rightarrow\ C_1 - C_2 - 2C_3 = 2$$

$$y''(x) = C_1e^{x} + C_2e^{-x} + 4C_3e^{-2x} \qquad y''(0) = 0 \ \rightarrow\ C_1 + C_2 + 4C_3 = 0$$

$$\begin{cases} C_1 + C_2 + C_3 = 1 \\ C_1 - C_2 - 2C_3 = 2 \\ C_1 + C_2 + 4C_3 = 0 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix} = -9 \quad \Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 0 & 1 & 4 \end{vmatrix} = -12 \quad \Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -2 \\ 1 & 0 & 4 \end{vmatrix} = 0$$

$$C_1 = \frac{4}{3}, \quad C_2 = 0, \quad C_3 = -\frac{1}{3}$$

$$y(x) = \frac{4}{3}e^{x} - \frac{1}{3}e^{-2x}$$

Exercise

Find a particular solution satisfying the given initial conditions $y^{(3)} - 6y'' + 11y' - 6y = 0$

$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 3 \qquad y_1(x) = e^{x}, \quad y_2(x) = e^{2x}, \quad y_3(x) = e^{3x}$

Solution

$$W = \begin{vmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= 18e^{6x} + 3e^{6x} + 4e^{6x} - 2e^{6x} - 12e^{6x} - 9e^{6x}$$

$$= 2e^{6x} \quad \neq 0$$

$\therefore\ y_1, y_2, \text{ and } y_3$ are linearly independent.

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x)$$

$$y(x) = C_1 e^{x} + C_2 e^{2x} + C_3 e^{3x} \qquad y(0) = 0 \ \rightarrow \ C_1 + C_2 + C_3 = 0$$

$$y'(x) = C_1 e^{x} + 2C_2 e^{2x} + 3C_3 e^{3x} \qquad y'(0) = 0 \ \rightarrow \ C_1 + 2C_2 + 3C_3 = 0$$

$$y''(x) = C_1 e^{x} + 4C_2 e^{2x} + 9C_3 e^{3x} \qquad y''(0) = 3 \ \rightarrow \ C_1 + 4C_2 + 9C_3 = 3$$

$$\begin{cases} C_1 + C_2 + C_3 = 0 \\ C_1 + 2C_2 + 3C_3 = 0 \\ C_1 + 4C_2 + 9C_3 = 3 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2 \quad \Delta_1 = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 2 & 3 \\ 3 & 4 & 9 \end{vmatrix} = 3 \quad \Delta_2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 9 \end{vmatrix} = -6$$

$$C_1 = \frac{3}{2}, \quad C_2 = -3, \quad C_3 = \frac{3}{2}$$

$$y(x) = \frac{3}{2} e^{x} - 3e^{2x} + \frac{3}{2} e^{3x}$$

Exercise

Find a particular solution satisfying the given initial conditions $\quad y^{(3)} - 3y'' + 3y' - y = 0$

$y(0) = 2, \quad y'(0) = 0, \quad y''(0) = 0 \qquad y_1(x) = e^{x}, \quad y_2(x) = xe^{x}, \quad y_3(x) = x^2 e^{x}$

Solution

$$W = \begin{vmatrix} e^{x} & xe^{x} & x^2 e^{x} \\ e^{x} & (1+x)e^{x} & \left(2x + x^2\right)e^{x} \\ e^{x} & (2+x)e^{x} & \left(2 + 4x + x^2\right)e^{x} \end{vmatrix}$$

$$= \left(2 + 6x + 5x^2 + x^3 + 2x^2 + x^3 + 2x^2 + x^3 - x^2 - x^3 - 2x - 4x^2 - x^3 - 2x - 4x^2 - x^3\right)e^{3x}$$

$$= (2 + 2x)e^{3x} \quad \neq 0$$

$\therefore\ y_1, y_2, \text{ and } y_3$ are linearly independent.

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x)$$

$$y(x) = C_1 e^{x} + C_2 xe^{x} + C_3 x^2 e^{x} \qquad y(0) = 2 \ \rightarrow \ C_1 = 2$$

$$y'(x) = C_1 e^{x} + C_2 (1+x)e^{x} + C_3 \left(2x + x^2\right)e^{x} \qquad y'(0) = 0 \ \rightarrow \ C_1 + C_2 = 0$$

$$y''(x) = C_1 e^x + C_2(2+x)e^x + C_3\left(2+4x+x^2\right)e^x \qquad y''(0)=0 \quad \rightarrow \quad C_1 + 2C_2 + 2C_3 = 0$$

$$\boxed{C_1 = 2, \quad C_2 = -2, \quad C_3 = 1}$$

$$\boxed{y(x) = 2e^x - 2xe^x + x^2 e^x}$$

Exercise

Find a particular solution satisfying the given initial conditions $y^{(3)} - 5y'' + 8y' - 4y = 0$

$y(0) = 1, \quad y'(0) = 4, \quad y''(0) = 0 \qquad y_1(x) = e^x, \quad y_2(x) = e^{2x}, \quad y_3(x) = xe^{2x}$

Solution

$$W = \begin{vmatrix} e^x & e^{2x} & xe^{2x} \\ e^x & 2e^{2x} & (1+2x)e^{2x} \\ e^x & 4e^{2x} & (4+4x)e^{2x} \end{vmatrix}$$

$$= (8 + 8x + 4 + 8x + 4x - 2x - 4 - 8x - 4 - 4x)e^{5x}$$

$$\boxed{= (6x+4)e^{5x} \neq 0}$$

$\therefore\ y_1, y_2,$ *and* y_3 are linearly independent.

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x)$$

$$y(x) = C_1 e^x + C_2 e^{2x} + C_3 xe^{2x} \qquad y(0)=1 \quad \rightarrow \quad C_1 + C_2 = 1$$

$$y'(x) = C_1 e^x + 2C_2 e^{2x} + C_3(1+2x)e^{2x} \qquad y'(0)=4 \quad \rightarrow \quad C_1 + 2C_2 + C_3 = 4$$

$$y''(x) = C_1 e^x + 4C_2 e^{2x} + (4+4x)C_3 e^{2x} \qquad y''(0)=0 \quad \rightarrow \quad C_1 + 4C_2 + 4C_3 = 0$$

$$\begin{cases} C_1 + C_2 = 1 \\ C_1 + 2C_2 + C_3 = 4 \\ C_1 + 4C_2 + 4C_3 = 0 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 4 & 4 \end{vmatrix} = 1 \quad \Delta_1 = \begin{vmatrix} 1 & 1 & 0 \\ 4 & 2 & 1 \\ 0 & 4 & 4 \end{vmatrix} = -12 \quad \Delta_2 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 4 & 1 \\ 1 & 0 & 4 \end{vmatrix} = 13$$

$$\boxed{C_1 = -12, \quad C_2 = 13, \quad C_3 = -10}$$

$$\boxed{y(x) = -12e^x + 13e^{2x} - 10xe^{2x}}$$

Exercise

Find a particular solution satisfying the given initial conditions $y^{(3)} + 9y'' = 0$

$y(0)=3, \quad y'(0)=-1, \quad y''(0)=2 \qquad y_1(x)=1, \quad y_2(x)=\cos 3x, \quad y_3(x)=\sin 3x$

Solution

$$W = \begin{vmatrix} 1 & \cos 3x & \sin 3x \\ 0 & -3\sin 3x & 3\cos 3x \\ 0 & -9\cos 3x & -9\sin 3x \end{vmatrix}$$

$$= 27\sin^2 3x + 27\cos^2 3x$$

$$\boxed{= 27 \ \neq 0}$$

$\therefore\ y_1,\ y_2,\ \textit{and}\ y_3$ are linearly independent.

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x)$$

$y(x) = C_1 + C_2\cos 3x + C_3\sin 3x \qquad y(0)=3 \ \rightarrow\ C_1 + C_2 = 3$

$y'(x) = -3C_2\sin 3x + 3C_3\cos 3x \qquad y'(0)=-1 \ \rightarrow\ 3C_3 = -1$

$y''(x) = -9C_2\cos 3x - 9C_3\sin 3x \qquad y''(0)=0 \ \rightarrow\ -9C_2 = 0$

$$\boxed{C_1 = 3, \quad C_2 = 0, \quad C_3 = -\tfrac{1}{3}}$$

$$\boxed{y(x) = 3 - \tfrac{1}{3}\sin 3x}$$

Exercise

Find a particular solution satisfying the given initial conditions $y^{(3)} - 3y'' + 4y' - 2y = 0$

$y(0)=1, \quad y'(0)=0, \quad y''(0)=0 \qquad y_1(x)=e^x, \quad y_2(x)=e^x\cos x, \quad y_3(x)=e^x\sin x$

Solution

$$W = \begin{vmatrix} e^x & e^x\cos x & e^x\sin x \\ e^x & (\cos x - \sin x)e^x & (\sin x + \cos x)e^x \\ e^x & -2e^x\sin x & 2e^x\cos x \end{vmatrix}$$

$$= \left(2\cos^2 x - \sin x\cos x + \cos^2 x - 2\sin^2 x - \sin x\cos x + \sin^2 x + 2\sin^2 x + 2\sin x\cos x - 2\cos^2 x\right)e^{3x}$$

$$\boxed{= e^{3x} \ \neq 0}$$

$\therefore\ y_1,\ y_2,\ \textit{and}\ y_3$ are linearly independent.

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x)$$

$$y(x) = C_1 e^x + C_2 e^x \cos x + C_3 e^x \sin x \qquad y(0) = 1 \rightarrow C_1 + C_2 = 1$$

$$y'(x) = C_1 e^x + C_2 (\cos x - \sin x) e^x + C_3 (\sin x + \cos x) e^x \qquad y'(0) = 0 \rightarrow C_1 + C_2 + C_3 = 0$$

$$y''(x) = C_1 e^x - 2C_2 e^x \sin x + 2C_3 e^x \cos x \qquad y''(0) = 0 \rightarrow C_1 + 2C_3 = 0$$

$$\begin{cases} C_1 + C_2 = 1 \\ C_1 + C_2 + C_3 = 0 \\ C_1 + 2C_3 = 0 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 1 \quad \Delta_1 = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2$$

$$C_1 = 2, \quad C_2 = -1, \quad C_3 = -1$$

$$y(x) = 2e^x - e^x \cos x - e^x \sin x$$

Exercise

Find a particular solution satisfying the given initial conditions $x^3 y^{(3)} - 3x^2 y'' + 6xy' - 6y = 0$

$y(1) = 6, \quad y'(1) = 14, \quad y''(1) = 1 \qquad y_1(x) = x, \quad y_2(x) = x^2, \quad y_3(x) = x^3$

Solution

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$= 12x^3 + 2x^3 - 6x^3 - 6x^3$$

$$= 2x^3 \neq 0$$

$\therefore\ y_1, y_2, \text{ and } y_3$ are linearly independent.

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x)$$

$$y(x) = C_1 x + C_2 x^2 + C_3 x^3 \qquad y(1) = 1 \rightarrow C_1 + C_2 + C_3 = 6$$

$$y'(x) = C_1 + 2C_2 x + 3C_3 x^2 \qquad y'(1) = 14 \rightarrow C_1 + 2C_2 + 3C_3 = 14$$

$$y''(x) = 2C_2 + 6C_3 x \qquad y''(1) = 1 \rightarrow 2C_2 + 6C_3 = 1$$

$$\begin{cases} C_1 + C_2 + C_3 = 6 \\ C_1 + 2C_2 + 3C_3 = 14 \\ 2C_2 + 6C_3 = 1 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} = 2 \quad \Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 14 & 2 & 3 \\ 1 & 2 & 6 \end{vmatrix} = -19 \quad \Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 14 & 3 \\ 0 & 1 & 6 \end{vmatrix} = 46$$

$$\boxed{C_1 = -\frac{19}{2}, \quad C_2 = 23, \quad C_3 = -\frac{15}{2}}$$

$$\boxed{y(x) = -\frac{19}{2}x + 23x^2 - \frac{15}{2}x^3}$$

Exercise

Find a particular solution satisfying the given initial conditions $x^3y^{(3)} + 6x^2y'' + 4xy' - 4y = 0$

$y(1) = 1, \quad y'(1) = 5, \quad y''(1) = -11 \qquad y_1(x) = x, \quad y_2(x) = x^{-2}, \quad y_3(x) = x^{-2}\ln x$

Solution

$$W = \begin{vmatrix} x & x^{-2} & x^{-2}\ln x \\ 1 & -2x^{-3} & (1-2\ln x)x^{-3} \\ 0 & 6x^{-4} & (-5+6\ln x)x^{-4} \end{vmatrix}$$

$$= (10 - 12\ln x + 6 - 6 + 12\ln x + 5 - 6\ln x)x^{-6}$$

$$\boxed{= (15 - 6\ln x)x^{-6} \ \neq 0}$$

$\therefore\ y_1, y_2, \textit{ and } y_3$ are linearly independent.

$y(x) = C_1y_1(x) + C_2y_2(x) + C_3y_3(x)$

$y(x) = C_1x + C_2x^{-2} + C_3x^{-2}\ln x \qquad y(1) = 1 \ \rightarrow \ C_1 + C_2 = 1$

$y'(x) = C_1 - 2C_2x^{-3} + C_3x^{-3}(1 - 2\ln x) \qquad y'(1) = 5 \ \rightarrow \ C_1 - 2C_2 + C_3 = 5$

$y''(x) = 6C_2x^{-4} + C_3x^{-4}(-5 + 6\ln x) \qquad y''(1) = -11 \ \rightarrow \ 6C_2 - 5C_3 = -11$

$$\begin{cases} C_1 + C_2 = 1 \\ C_1 - 2C_2 + C_3 = 5 \\ 6C_2 - 5C_3 = -11 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 6 & -5 \end{vmatrix} = 9 \quad \Delta_1 = \begin{vmatrix} 1 & 1 & 0 \\ 5 & -2 & 1 \\ -11 & 6 & -5 \end{vmatrix} = 18 \quad \Delta_2 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 5 & 1 \\ 0 & -11 & -5 \end{vmatrix} = -9$$

$$\boxed{C_1 = 2, \quad C_2 = -1, \quad C_3 = 1}$$

$$\boxed{y(x) = 2x - x^{-2} + x^{-2}\ln x}$$

Exercise

When the values of a solution to a differential equation are specified at two different points, these conditions. (In contrast, initial conditions specify the values of a function and its derivative at the same point). The purpose of this is to show that for boundary value problems there is no existence-uniqueness theorem. Given that every solution to

$$y'' + y = 0 \qquad \text{is of the form} \qquad y(t) = c_1 \cos t + c_2 \sin t$$

Where c_1 and c_2 are arbitrary constants, show that

a) There is a unique solution to the given differential equation that satisfies the boundary conditions $y(0) = 2$ and $y\left(\frac{\pi}{2}\right) = 0$

b) There is no solution to given equation that satisfies $y(2) = 0$ and $y(\pi) = 0$

c) There are infinitely many solution to the given DE equation that satisfy $y(0) = 2$ and $y(\pi) = -2$

Solution

a) $\lambda^2 + 1 = 0 \quad \rightarrow \quad \lambda = \pm i$

$y(t) = c_1 \cos t + c_2 \sin t$

$y(0) = 2 \quad \rightarrow \quad 2 = c_1$

$y\left(\frac{\pi}{2}\right) = 0 \quad \rightarrow \quad 0 = c_2$

$y(t) = 2\cos t$

b) $y(0) = 2 \quad \rightarrow \quad 2 = c_1$

$y(\pi) = 0 \quad \rightarrow \quad 0 = -c_1$

This system is inconsistent, so there is no solution satisfying the given boundary.

c) $y(0) = 2 \quad \rightarrow \quad 2 = c_1$

$y(\pi) = -2 \quad \rightarrow \quad -2 = -c_1$

$y(t) = 2\cos t + c_2 \sin t$

Which has infinitely many solutions given c_2 is an arbitrary constant.