Solution Section 2.1 –Tangents and the Derivative at a point

Exercise

Find an equation for the tangent to the curve $y = 4 - x^2$ at the point $(-1,\ 3)$. Then sketch the curve and tangent together.

Solution

$$m = \lim_{h\to 0} \frac{4-(x+h)^2-\left(4-x^2\right)}{h}$$

$$= \lim_{h\to 0} \frac{4-(-1+h)^2-\left(4-(-1)^2\right)}{h}$$

$$= \lim_{h\to 0} \frac{4-\left(1-2h+h^2\right)-(4-1)}{h}$$

$$= \lim_{h\to 0} \frac{4-1+2h-h^2-3}{h}$$

$$= \lim_{h\to 0} \frac{2h-h^2}{h}$$

$$= \lim_{h\to 0} (2-h)$$

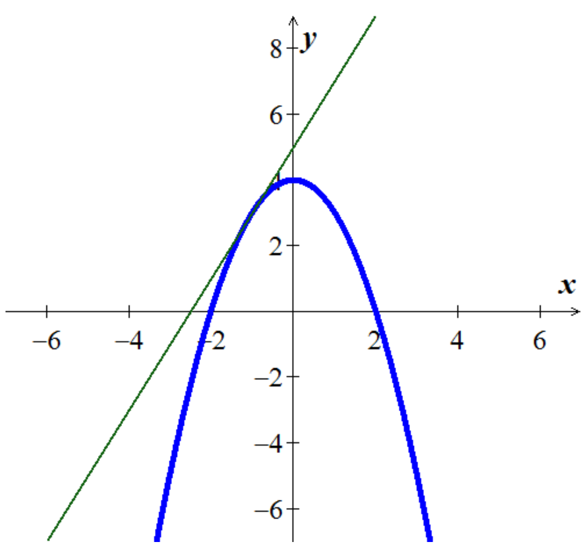
$$= 2$$

$$y - y_1 = m\left(x - x_1\right)$$

At $(-1,\ 3) \Rightarrow y-3 = 2\left(x-(-1)\right)$

$$y - 3 = 2x + 2$$

$$\boxed{y = 2x + 5}$$



Exercise

Find an equation for the tangent to the curve $y=\frac{1}{x^2}$ at the point $(-1,\ 1)$. Then sketch the curve and tangent together.

Solution

$$m=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{(-1+h)^2}-\frac{1}{1}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{1-\left(1-2h+h^2\right)}{(-1+h)^2}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{1-1+2h-h^2}{(-1+h)^2}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{2h-h^2}{(-1+h)^2}\right)$$

$$=\lim_{h\to 0}\frac{h}{h}\left(\frac{2-h}{(-1+h)^2}\right)$$

$$=\lim_{h\to 0}\left(\frac{2-h}{(-1+h)^2}\right)$$

$$=\frac{2+0}{(-1+0)^2}$$

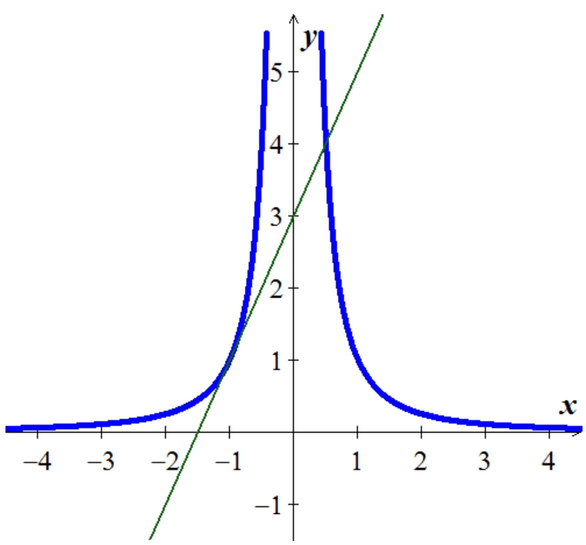
$$=2$$

$$y-y_1=m\left(x-x_1\right)$$

At $(-1,3)\Rightarrow y-1=2\left(x-(-1)\right)$

$$y-1=2x+2$$

$$\boxed{y=2x+3}$$



Exercise

Find the slope of the function $f(x)=2\sqrt{x}$ at the point $(1,\ 2)$. Then find an equation for the line tangent to the graph there.

Solution

$$m=\lim_{h\to 0}\frac{2\sqrt{x+h}-2\sqrt{x}}{h}$$

$$=\lim_{h\to 0}\frac{2\sqrt{1+h}-2\sqrt{x}}{h}\cdot\frac{2\sqrt{1+h}+2}{2\sqrt{1+h}+2}$$

$$=\lim_{h\to 0}\frac{4(1+h)-4}{h\left(2\sqrt{1+h}+2\right)}$$

$$=\lim_{h\to 0}\frac{4+4h-4}{h\left(2\sqrt{1+h}+2\right)}$$

$$=\lim_{h\to 0}\frac{4h}{h\left(2\sqrt{1+h}+2\right)}$$

$$=\lim_{h\to 0}\frac{4}{2\sqrt{1+h}+2}$$

$$=\frac{4}{2+2}$$

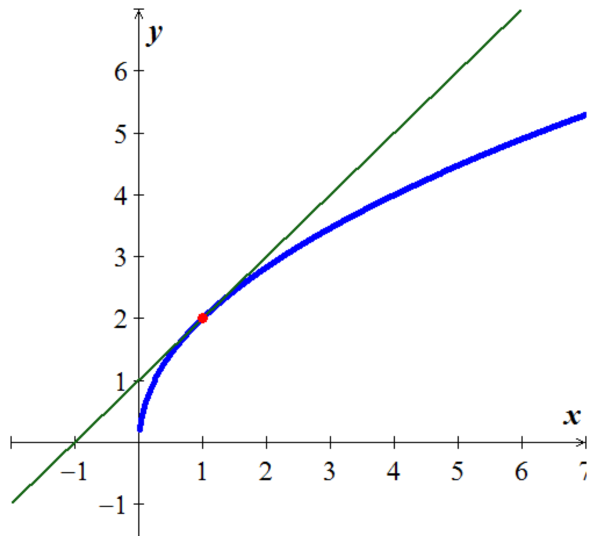
$$\underline{=1}$$

$$y-y_1=m\left(x-x_1\right)$$

At (1, 2) $\Rightarrow y-2=(x-1)$

$$y-2=x-1$$

$$\boxed{y=x+1}$$



Exercise

Find the slope of the function $f(x) = x^3 + 3x$ at the point $(1,\ 4)$. Then find an equation for the line tangent to the graph there.

Solution

$$m = \lim_{h \to 0} \frac{(x+h)^3 + 3(x+h) - \left(x^3 + 3x\right)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)^3 + 3(1+h) - \left(1^3 + 3(1)\right)}{h}$$

$$= \lim_{h \to 0} \frac{1 + 3h + 3h^2 + h^3 + 3 + 3h - (4)}{h}$$

$$= \lim_{h \to 0} \frac{3h + 3h^2 + h^3 + 3h}{h}$$

$$= \lim_{h \to 0} \frac{3h^2 + h^3 + 6h}{h}$$

$$= \lim_{h \to 0} \left(3h + h^2 + 6\right)$$

$$= 6$$

At $(1,\ 4) \Rightarrow y - 4 = 6(x - 1)$

$$y - 4 = 6x - 6$$

$$\boxed{y = 6x - 2}$$

Exercise

Find the slope of the curve $y = 1 - x^2$ at the point $x = 2$

Solution

$$m = \lim_{h \to 0} \frac{1 - (x+h)^2 - \left(1 - x^2\right)}{h}$$

$$= \lim_{h \to 0} \frac{1 - (2+h)^2 - \left(1 - 2^2\right)}{h}$$

$$= \lim_{h \to 0} \frac{1 - \left(4 + 4h + h^2\right) - (-3)}{h}$$

$$= \lim_{h \to 0} \frac{1 - 4 - 4h - h^2 + 3}{h}$$

$$= \lim_{h \to 0} \frac{-4h - h^2}{h}$$

$$= \lim_{h \to 0} (-4 - h)$$

$$\boxed{= -4}$$

Exercise

Find the slope of the curve $y = \frac{1}{x-1}$ at the point $x = 3$

Solution

$$m = \lim_{h \to 0} \frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{3+h-1} - \frac{1}{3-1}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{2-2-h}{2+h}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{-h}{2+h}\right)$$

$$= \lim_{h \to 0} \left(\frac{-1}{2+h}\right)$$

$$\boxed{= -\frac{1}{2}}$$

Exercise

Find the slope of the curve $y = \frac{x-1}{x+1}$ at the point $x = 0$

Solution

$$m = \lim_{h \to 0} \frac{\frac{x+h-1}{x+h+1} - \frac{x-1}{x+1}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{0+h-1}{0+h+1} - \frac{0-1}{0+1}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{h-1}{h+1} + 1\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{h-1+h+1}{h+1}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{2h}{h+1}\right)$$

$$= \lim_{h\to 0} \left(\frac{2}{h+1}\right)$$

$$\underline{=2}$$

Exercise

Find equations of all lines having slope -1 that are tangent to the curve $y=\dfrac{1}{x-1}$

Solution

$$m = \lim_{h\to 0} \frac{\frac{1}{x+h-1}-\frac{1}{x-1}}{h}$$

$$-1 = \lim_{h\to 0} \frac{\frac{1}{x+h-1}-\frac{1}{x-1}}{h}$$

$$-1 = \lim_{h\to 0} \frac{1}{h}\left(\frac{x-1-(x+h-1)}{x+h-1}\right)$$

Cross multiplication

$$-1 = \lim_{h\to 0} \frac{1}{h}\left(\frac{x-1-x-h+1}{x+h-1}\right)$$

$$-1 = \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{x+h-1}\right)$$

$$-1 = \lim_{h\to 0} \left(\frac{-1}{x+h-1}\right)$$

$$-1 = \frac{-1}{x-1}$$

$$-x+1=-1$$

$$\boxed{x=2}$$

$$y=\frac{1}{x-1}=\frac{1}{2-1}=1$$

$$\text{At } (2,\ 1) \Rightarrow y-1=-1(x-2)$$

$$y-1=-x+2$$

$$\boxed{y=-x+3}$$

Exercise

What is the rate of change of the area of a circle $\left(A=\pi r^2\right)$ with respect to the radius when the radius is $r=3$?

Solution

$$m=\lim_{h\to 0}\frac{\pi(3+h)^2-\pi(3)^2}{h}$$

$$=\lim_{h\to 0}\frac{\pi\left(9+6h+h^2\right)-9\pi}{h}$$

$$=\lim_{h\to 0}\frac{9\pi+6\pi h+\pi h^2-9\pi}{h}$$

$$=\lim_{h\to 0}\frac{6\pi h+\pi h^2}{h}$$

$$=\lim_{h\to 0}\frac{\pi h(6+h)}{h}$$

$$=\lim_{h\to 0}\ \pi(6+h)$$

$$=6\pi$$

Exercise

Find the slope of the tangent to the curve $y=\frac{1}{\sqrt{x}}$ at the point where $x=4$

Solution

$$m=\lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}$$

$$=\lim_{h\to 0}\ \frac{1}{h}\left(\frac{\sqrt{4}-\sqrt{4+h}}{2\sqrt{4+h}}\right)$$

$$=\lim_{h\to 0}\ \frac{1}{h}\left(\frac{2-\sqrt{4+h}}{2\sqrt{4+h}}\right)$$

$$=\lim_{h\to 0}\ \frac{1}{h}\left(\frac{2-\sqrt{4+h}}{2\sqrt{4+h}}\cdot\frac{2+\sqrt{4+h}}{2+\sqrt{4+h}}\right)$$

$$=\lim_{h\to 0}\ \frac{1}{h}\left(\frac{4-(4+h)}{2\sqrt{4+h}\left(2+\sqrt{4+h}\right)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{2\sqrt{4+h}\left(2+\sqrt{4+h}\right)}\right)$$

$$= \lim_{h\to 0} \left(\frac{-1}{2\sqrt{4+h}\left(2+\sqrt{4+h}\right)}\right)$$

$$= \frac{-1}{2\sqrt{4}\left(2+\sqrt{4}\right)}$$

$$= \frac{-1}{2(2)(2+2)}$$

$$= \frac{-1}{16}$$

Solution Section 2.2 –The Derivative as a Function

Exercise

Fin the values of the derivatives of the function $f(x)=4-x^2$. Then find the values of $f'(-3),\ f'(0),\ f'(1)$

Solution

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{4-(x+h)^2-\left(4-x^2\right)}{h} \\
&= \frac{4-\left(x^2+2xh+h^2\right)-\left(4-x^2\right)}{h} \\
&= \frac{4-x^2-2xh-h^2-4+x^2}{h} \\
&= \frac{-2xh-h^2}{h} \\
&= -2x-h
\end{aligned}$$

$$f'(x)=\lim_{h\to 0}(-2x-h)=-2x$$

$$f'(-3)=\underline{6} \qquad f'(0)=\underline{0} \qquad f'(1)=\underline{-2}$$

Exercise

Fin the values of the derivatives of the function $r(s)=\sqrt{2s+1}$. Then find the values of $r'(0),\ r'\left(\frac{1}{2}\right),\ r'(1)$

Solution

$$\begin{aligned}
r'(s) &= \lim_{h\to 0}\frac{r(s+h)-r(s)}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{2(s+h)+1}-\sqrt{2s+1}}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{2s+2h+1}-\sqrt{2s+1}}{h}\cdot\frac{\sqrt{2s+2h+1}+\sqrt{2s+1}}{\sqrt{2s+2h+1}+\sqrt{2s+1}} \\
&= \lim_{h\to 0}\frac{2s+2h+1-(2s+1)}{h\left(\sqrt{2s+2h+1}+\sqrt{2s+1}\right)} \\
&= \lim_{h\to 0}\frac{2s+2h+1-2s-1}{h\left(\sqrt{2s+2h+1}+\sqrt{2s+1}\right)}
\end{aligned}$$

$$= \lim_{h \to 0} \frac{2h}{h\left(\sqrt{2s+2h+1}+\sqrt{2s+1}\right)}$$

$$= \lim_{h \to 0} \frac{2}{\sqrt{2s+2h+1}+\sqrt{2s+1}}$$

$$= \frac{2}{\sqrt{2s+1}+\sqrt{2s+1}}$$

$$= \frac{2}{2\sqrt{2s+1}}$$

$$= \frac{1}{\sqrt{2s+1}}$$

$$r'(0) = \frac{1}{\sqrt{2(0)+1}} = \underline{1|}$$

$$r'\left(\tfrac{1}{2}\right) = \frac{1}{\sqrt{2\frac{1}{2}+1}} = \underline{\frac{1}{\sqrt{2}}|}$$

$$r'(1) = \frac{1}{\sqrt{2(1)+1}} = \underline{\frac{1}{\sqrt{3}}|}$$

Exercise

Find the derivative of $f(x) = 3x^2 - 2x$

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3(x+\Delta x)^2 - 2(x+\Delta x) - (3x^2 - 2x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3(x^2 + \Delta x^2 + 2x\Delta x) - 2x - 2\Delta x - 3x^2 + 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2 + 3\Delta x^2 + 6x\Delta x - 2x - 2\Delta x - 3x^2 + 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3\Delta x^2 + 6x\Delta x - 2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 3\Delta x + 6x - 2$$

$$\underline{= 6x - 2|}$$

Exercise

Find the derivative of y with the respect to t for the function $y = \frac{4}{t}$

Solution

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta t \to 0} \frac{\frac{4}{t+\Delta t} - \frac{4}{t}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\frac{4t-4(t+\Delta t)}{t(t+\Delta t)}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{4t-4(t+\Delta t)}{t(t+\Delta t)} \\
&= \lim_{\Delta t \to 0} \frac{-4\Delta t}{t(t+\Delta t)\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{-4}{t(t+\Delta t)} \\
&= -\frac{4}{t^2}
\end{aligned}
$$

Exercise

Find the derivative of $\frac{dy}{dx}$ *if* $y = 2x^3$

Solution

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{2(x+\Delta x)^3 - 2x^3}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2\left(x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3\right) - 2x^3}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x^3 + 6x^2\Delta x + 6x(\Delta x)^2 + 3(\Delta x)^3 - 2x^3}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x\left(6x^2 + 6x(\Delta x) + 3(\Delta x)^2\right)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(6x^2 + 6x(\Delta x) + 3(\Delta x)^2\right) \\
&= 6x^2
\end{aligned}
$$

Exercise

Differentiate the function $y = \dfrac{x+3}{1-x}$ and find the slope of the tangent line at the given value of the independent variable.

Solution

$$f'(x) = \lim_{\Delta x \to 0} \dfrac{\frac{x+\Delta x+3}{1-x-\Delta x} - \frac{x+3}{1-x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{1}{\Delta x}\right)\left(\frac{(x+\Delta x+3)(1-x)-(x+3)(1-x-\Delta x)}{(1-x-\Delta x)(1-x)}\right)$$

$$= \lim_{\Delta x \to 0} \left(\frac{1}{\Delta x}\right)\left(\frac{x+\Delta x+3-x^2-x\Delta x-3x-\left(x-x^2-x\Delta x+3-3x-3\Delta x\right)}{(1-x-\Delta x)(1-x)}\right)$$

$$= \lim_{\Delta x \to 0} \left(\frac{1}{\Delta x}\right)\left(\frac{x+\Delta x+3-x^2-x\Delta x-3x-x+x^2+x\Delta x-3+3x+3\Delta x}{(1-x-\Delta x)(1-x)}\right)$$

$$= \lim_{\Delta x \to 0} \left(\frac{1}{\Delta x}\right)\left(\frac{4\Delta x}{(1-x-\Delta x)(1-x)}\right)$$

$$= \lim_{\Delta x \to 0} \frac{4}{(1-x-\Delta x)(1-x)}$$

$$= \frac{4}{(1-x)(1-x)}$$

$$\boxed{= \frac{4}{(1-x)^2}}$$

Exercise

Find the equation of the tangent line to $f(x) = x^2 + 1$ that is parallel to $2x + y = 0$

Solution

$2x + y = 0 \Rightarrow y = -2x \Rightarrow \text{slope} = -2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + 1 - (x^2+1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + \Delta x^2 + 2x\Delta x + 1 - x^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x^2 + 2x\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \Delta x + 2x = 2x$$

$$f' = 2x = -2 \ \Rightarrow x = -1 \Rightarrow f(-1) = (-1)^2 + 1 = 2 \quad \to \text{(-1, 2)}$$

The line equation is given by

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -2(x + 1)$$

$$y - 2 = -2x - 2$$

$$\boxed{y = -2x}$$

Exercise

Use the definition of limits to find the derivative: $f(x) = \dfrac{3}{\sqrt{x}}$

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(\frac{3}{\sqrt{x + \Delta x}}\right) - \left(\frac{3}{\sqrt{x}}\right)}{\Delta x} \cdot \frac{\sqrt{x} \cdot \sqrt{x + \Delta x}}{\sqrt{x} \cdot \sqrt{x + \Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{3\sqrt{x} - 3\sqrt{x + \Delta x}}{\Delta x\left(\sqrt{x} \cdot \sqrt{x + \Delta x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{3\left(\sqrt{x} - \sqrt{x + \Delta x}\right)}{\Delta x\left(\sqrt{x} \cdot \sqrt{x + \Delta x}\right)} \cdot \frac{\sqrt{x} + \sqrt{x + \Delta x}}{\sqrt{x} + \sqrt{x + \Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{3\left(x - (x + \Delta x)\right)}{\Delta x\left(\sqrt{x} \cdot \sqrt{x + \Delta x}\right)\left(\sqrt{x} + \sqrt{x + \Delta x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{-3\Delta x}{\Delta x\left(\sqrt{x} \cdot \sqrt{x + \Delta x}\right)\left(\sqrt{x} + \sqrt{x + \Delta x}\right)}$$

$$= \frac{-3}{x\left(2\sqrt{x}\right)}$$

$$\boxed{= \frac{-3}{2x^{3/2}}}$$

Exercise

Use the definition of limits to find the derivative: $f(x) = \sqrt{x+2}$

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+2} - \sqrt{x+2}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x+2} + \sqrt{x+2}}{\sqrt{x+\Delta x+2} + \sqrt{x+2}}$$

$$= \lim_{\Delta x \to 0} \frac{x+\Delta x+2-(x+2)}{\Delta x\left(\sqrt{x+\Delta x+2} + \sqrt{x+2}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{x+\Delta x+2} + \sqrt{x+2}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x+2} + \sqrt{x+2}}$$

$$\boxed{= \frac{1}{2\sqrt{x+2}}}$$

Solution Section 2.3 – Differentiation Rules

Exercise

Find the derivative of $y = \frac{1}{x^3}$

Solution

$$y = x^{-3}$$

$$y' = -3x^{-3-1}$$

$$\underline{= -3x^{-4}} \qquad \text{or} \quad -\frac{3}{x^4}$$

Exercise

Find the derivative of $D_x\left(x^{4/3}\right)$

Solution

$$D_x\left(x^{4/3}\right) = \frac{4}{3}x^{1/3}$$

Exercise

Find the derivative of $y = \sqrt{z}$

Solution

$$\frac{dy}{dz} = \frac{d}{dz}\left[z^{1/2}\right]$$

$$= \frac{1}{2}z^{1/2-1}$$

$$= \frac{1}{2}z^{-1/2} \qquad \frac{1}{2z^{1/2}} \qquad \frac{1}{2\sqrt{z}}$$

Exercise

Find the derivative of $D_t(-8t)$

Solution

$$D_t(-8t)\ \underline{= -8}$$

Exercise

Find the derivative of $y = \frac{9}{4x^2}$

Solution

$$y = \frac{9}{4}x^{-2}$$

$$y' = \frac{9}{4}(-2)x^{-3}$$

$$= -\frac{9}{2x^3}$$

Exercise

Find the derivative of $y = 6x^3 + 15x^2$

Solution

$$y' = 18x^2 + 30x$$

Exercise

Find the first derivative of $y = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$

Solution

$$y' = 3(4)x^3 - 6(3)x^2 + \frac{2}{8}x + 0$$

$$= 12x^3 - 18x^2 + \frac{1}{4}x$$

Exercise

Find the derivative of $p(t) = 12t^4 - 6\sqrt{t} + \frac{5}{t}$

Solution

$$p(t) = 12t^4 - 6t^{1/2} + 5t^{-1}$$

$$p' = 48t^3 - 3t^{-1/2} - 5t^{-2}$$

$$= 48t^3 - \frac{3}{t^{1/2}} - \frac{5}{t^2}$$

Exercise

Find the derivative of $f(x)=\dfrac{x^3+3\sqrt{x}}{x}$

Solution

$$f(x)=\frac{x^3}{x}+3\frac{x^{1/2}}{x}=x^2+3x^{-1/2}$$

$$f'(x)=2x-\tfrac{3}{2}x^{-3/2}$$

$$=2x-\frac{3}{2x^{3/2}}$$

$$\boxed{=2x-\frac{3}{2\sqrt{x^3}}}$$

Exercise

Find the derivative of $y=\dfrac{x^3-4x}{\sqrt{x}}$

Solution

$$y=\frac{x^3}{x^{1/2}}-4\frac{x}{x^{1/2}}$$

$$=x^{5/2}-4x^{1/2}$$

$$y'=\tfrac{5}{2}x^{3/2}-4\tfrac{1}{2}x^{-1/2}$$

$$\boxed{=\tfrac{5}{2}x\sqrt{x}-2\frac{2}{\sqrt{x}}}$$

Exercise

Find the derivative of $f(x)=\left(4x^2-3x\right)^2$

Solution

$$f(x)=\left(4x^2-3x\right)^2 \qquad (a+b)^2=a^2+2ab+b^2$$

$$=16x^4-24x^3+9x^2$$

$$\boxed{f'=64x^3-72x^2+18x}$$

Exercise

Find the derivative of $y=(x+1)\left(\sqrt{x}+2\right)$

Solution

$$y'=(1)\left(x^{1/2}+2\right)+(x+1)\left(\frac{1}{2}x^{-1/2}\right)$$

$$=x^{1/2}+2+\frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2}$$

$$\boxed{=\frac{3}{2}x^{1/2}+\frac{1}{2}x^{-1/2}+2}$$

Exercise

Find the derivative of $y=\left(4x+3x^2\right)(6-3x)$

Solution

$$y'=\left(4x+3x^2\right)\frac{d}{dx}(6-3x)+(6-3x)\frac{d}{dx}\left(4x+3x^2\right)$$

$y=24x+6x^2-9x^3$

$$=\left(4x+3x^2\right)(-3)+(6-3x)(4+6x)$$

$$=-12x-9x^2+24+36x-12x-18x^2$$

$$\boxed{=-27x^2+12x+24}$$

Exercise

Find the derivative of $y=\left(\frac{1}{x}+1\right)(2x+1)$

Solution

$$y'=\left(x^{-1}+1\right)\frac{d}{dx}(2x+1)+(2x+1)\frac{d}{dx}\left(x^{-1}+1\right)$$

$$=\left(x^{-1}+1\right)(2)+(2x+1)\left(-x^{-2}\right)$$

$$=\frac{2}{x}+2+(2x+1)\left(-\frac{1}{x^2}\right)$$

$$=\frac{2}{x}+2-\frac{2x}{x^2}-\frac{1}{x^2}$$

$$=\frac{2}{x}+2-\frac{2}{x}-\frac{1}{x^2}$$

$$=2-\frac{1}{x^2}$$

$$\boxed{=\frac{2x^2-1}{x^2}}$$

Exercise

Find the derivative of $y = 3x\left(2x^2 + 5x\right)$

Solution

$$y = 6x^3 + 15x^2$$

$$\Rightarrow \underline{y' = 18x^2 + 30x}$$

Exercise

Find the derivative of $y = 3\left(2x^2 + 5x\right)$

Solution

$$y = 6x^2 + 15x$$

$$\Rightarrow \underline{y' = 12x + 15}$$

Exercise

Find the derivative of $y = \dfrac{x^2 + 4x}{5}$

Solution

$$y = \tfrac{1}{5}\left[x^2 + 4x\right]$$

$$\underline{y' = \tfrac{1}{5}(2x + 4)}$$

Exercise

Find the derivative of $y = \dfrac{3x^4}{5}$

Solution

$$y = \frac{3}{5}x^4$$

$$\underline{y' = \frac{12}{5}x^3}$$

Exercise

Find the derivative of $y=\dfrac{3-\frac{2}{x}}{x+4}$

Solution

$$y=\frac{\frac{3x-2}{x}}{x+4}$$

$$=\frac{3x-2}{x}\cdot\frac{1}{x+4}$$

$$=\frac{3x-2}{x^2+4x}$$

$$y'=\frac{\left(x^2+4x\right)(3)-(3x-2)(2x+4)}{\left[x(x+4)\right]^2}$$

$$=\frac{3x^2+12x-6x^2-12x+4x+8}{x^2(x+4)^2}$$

$$=\frac{-3x^2+4x+8}{x^2(x+4)^2}$$

Exercise

Find the derivative of $g(x)=\dfrac{x^2-4x+2}{x^2+3}$

Solution

$$g'=\frac{(2x-4)\left(x^2+3\right)-\left(x^2-4x+2\right)(2x)}{\left(x^2+3\right)^2}$$

$$=\frac{2x^3+6x-4x^2-12-2x^3+8x^2-4x}{\left(x^2+3\right)^2}$$

$$=\frac{4x^2+2x-12}{\left(x^2+3\right)^2}$$

Exercise

Find the derivative of $f(x)=\dfrac{(3-4x)(5x+1)}{7x-9}$

Solution

$$D_x\left[\frac{(3-4x)(5x+1)}{7x-9}\right]=\frac{\left[(-4)(5x+1)+(3-4x)(5)\right](7x-9)-(3-4x)(5x+1)(7)}{(7x-9)^2}$$

$$=\frac{\left[-20x-4+15-20x\right](7x-9)-\left(15x+3-20x^2-4x\right)(7)}{(7x-9)^2}$$

$$=\frac{(-40x+11)(7x-9)-7\left(-20x^2+11x+3\right)}{(7x-9)^2}$$

$$=\frac{-280x^2+360x+77x-99-140x^2-77x-21}{(7x-9)^2}$$

$$=\boxed{\frac{-420x^2+360x-120}{(7x-9)^2}}$$

Exercise

Find the derivative of $f(x)=x\left(1-\dfrac{2}{x+1}\right)$

Solution

$$f(x)=x-\frac{2x}{x+1}$$

$$\left(\frac{2x}{x+1}\right)' \Rightarrow \qquad \begin{array}{ll} f=2x & f'=2 \\ g=x+1 & g'=1 \end{array}$$

$$f'(x)=1-\frac{2(x+1)-2x}{(x+1)^2}$$

$$=1-\frac{2x+2-2x}{(x+1)^2}$$

$$=\boxed{1-\frac{2}{(x+1)^2}}$$

Exercise

Find the derivative of $g(s)=\dfrac{s^2-2s+5}{\sqrt{s}}$

Solution

$$g(s)=\frac{s^2}{s^{1/2}}-2\frac{s}{s^{1/2}}+\frac{5}{s^{1/2}}$$

$$=s^{3/2}-2s^{1/2}+5s^{-1/2}$$

$$g'(s)=\tfrac{3}{2}s^{1/2}-2\tfrac{1}{2}s^{-1/2}+5\left(-\tfrac{1}{2}\right)s^{-3/2}$$

$$=\tfrac{3}{2}s^{1/2}-s^{-1/2}-\tfrac{5}{2}s^{-3/2}$$

$$=\frac{3}{2}\sqrt{s}-\frac{1}{\sqrt{s}}-\frac{5}{2s^{3/2}}$$

$$=\frac{3}{2}\sqrt{s}-\frac{1}{\sqrt{s}}-\frac{5}{2s\sqrt{s}}$$

Exercise

Find the derivative of $f(x)=\dfrac{x+1}{\sqrt{x}}$

Solution

$$f(x)=\frac{x}{x^{1/2}}+\frac{1}{x^{1/2}}$$

$$=x^{1/2}+x^{-1/2}$$

$$f'(x)=\tfrac{1}{2}x^{-1/2}-\tfrac{1}{2}x^{-3/2}$$

$$=\frac{1}{2x^{1/2}}-\frac{1}{2x^{3/2}}$$

Exercise

Find the derivative of $f(x)=\left(\sqrt{x}+3\right)\left(x^2-5x\right)$

Solution

$$f'=\left(\frac{1}{2}x^{-1/2}\right)\left(x^2-5x\right)+\left(\sqrt{x}+3\right)(2x-5)$$

$$=\frac{1}{2}x^{3/2}-\frac{5}{2}x^{1/2}+2x^{3/2}-5x^{1/2}+6x-15$$

$$=\frac{5}{2}x^{3/2}-\frac{15}{2}x^{1/2}+6x-15$$

$$\boxed{=\frac{5}{2}x^{3/2}+6x-\frac{15}{2}x^{1/2}-15}$$

Exercise

Find the derivative of $y=(2x+3)\left(5x^2-4x\right)$

Solution

$$y=(2x+3)\left(5x^2-4x\right)$$

$$=10x^3-8x^2+15x^2-12x$$

$$=10x^3+7x^2-12x$$

$$\boxed{y'=30x^2+14x-12}$$

Exercise

Find the derivative of $y=\left(x^2+1\right)\left(x+5+\frac{1}{x}\right)$

Solution

$$y=x^3+5x^2+x+x+5+\frac{1}{x}$$

$$=x^3+5x^2+2x+5+x^{-1}$$

$$y'=3x^2+10x+2-x^{-2}$$

$$\boxed{=3x^2+10x+2-\frac{1}{x^2}}$$

Exercise

Find the derivative of $y=\dfrac{x+4}{5x-2}$

Solution

$$y'=\frac{(5x-2)\frac{d}{dx}[(x+4)]-(x+4)\frac{d}{dx}[(5x-2)]}{(5x-2)^2}$$

$$=\frac{(5x-2)(1)-(x+4)(5)}{(5x-2)^2}$$

$$=\frac{5x-2-5x-20}{(5x-2)^2}$$

$$\boxed{=-\frac{22}{(5x-2)^2}}$$

Exercise

Find the derivative of $z=\dfrac{4-3x}{3x^2+x}$

Solution

$u=4-3x \qquad v=3x^2+x$

$u'=-3 \qquad v'=6x+1$

$$z'=\frac{-3\left(3x^2+x\right)-(6x+1)(4-3x)}{\left(3x^2+x\right)^2} \qquad\qquad z'=\frac{u'v-v'u}{u^2}$$

$$=\frac{-9x^2-3x-\left(24x-18x^2+4-3x\right)}{\left(3x^2+x\right)^2}$$

$$=\frac{-9x^2-3x-21x+18x^2-4}{\left(3x^2+x\right)^2}$$

$$\boxed{=\frac{9x^2-24x-4}{\left(3x^2+x\right)^2}}$$

Exercise

Find the derivative of $y=(2x-7)^{-1}(x+5)$

Solution

$$\begin{aligned}
y' &= -(2x-7)^{-2}(2)(x+5)+(2x-7)^{-1} \\
&= -(2x-7)^{-2}(2x+10)+(2x-7)^{-1} \\
&= \left[-(2x-7)^{-2}(2x+10)+(2x-7)^{-1}\right]\frac{(2x-7)^2}{(2x-7)^2} \\
&= \frac{-2x-10+2x-7}{(2x-7)^2} \\
&= \boxed{\frac{-17}{(2x-7)^2}}
\end{aligned}$$

Exercise

Find the derivative of $f(x)=\dfrac{\sqrt{x}-1}{\sqrt{x}+1}$

Solution

$$u=x^{1/2}-1 \qquad v=x^{1/2}+1$$

$$u'=\tfrac{1}{2}x^{-1/2} \qquad v'=\tfrac{1}{2}x^{-1/2}$$

$$\begin{aligned}
f'(x) &= \frac{\frac{1}{2}x^{1/2}\left(x^{1/2}+1\right)-\frac{1}{2}x^{1/2}\left(x^{1/2}-1\right)}{\left(\sqrt{x}+1\right)^2} \\
&= \frac{1}{2}\frac{1+x^{-1/2}-1+x^{-1/2}}{\left(\sqrt{x}+1\right)^2} \\
&= \frac{1}{2}\frac{2x^{-1/2}}{\left(\sqrt{x}+1\right)^2} \\
&= \frac{1}{x^{1/2}\left(\sqrt{x}+1\right)^2} \\
&= \boxed{\frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)^2}}
\end{aligned}$$

Exercise

Find the derivative of $y = \dfrac{1}{\left(x^2-1\right)\left(x^2+x+1\right)}$

Solution

$$y = \frac{1}{x^4+x^3+x^2-x^2-x-1}$$

$$= \frac{1}{x^4+x^3-x-1}$$

$$y' = \frac{-\left(4x^3+3x^2-1\right)}{x^4+x^3-x-1} \qquad \left(\frac{1}{v}\right)' = -\frac{v'}{v^2}$$

$$= \frac{-4x^3-3x^2+1}{x^4+x^3-x-1}$$

Exercise

Find the first and second derivatives $y = -x^3+3$

Solution

$$y' = -3x^2 \qquad y'' = -6x$$

Exercise

Find the first and second derivatives $y = 3x^7-7x^3+21x^2$

Solution

$$y' = 21x^6-21x^2+42x \qquad y'' = 126x^5-42x+42$$

Exercise

Find the first and second derivatives $y = 6x^2-10x-\frac{1}{x}$

Solution

$$y' = 12x-10+\frac{1}{x^2} \qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$y'' = 12+\frac{-2x}{x^4}$$

$$= 12-\frac{2}{x^3}$$

Exercise

Find the first and second derivatives $y = \dfrac{x^2 + 5x - 1}{x^2}$

Solution

$$u = x^2 + 5x - 1 \quad v = x^2$$
$$u' = 2x + 5 \qquad v' = 2x$$

$$y' = \frac{(2x+5)x^2 - 2x\left(x^2 + 5x - 1\right)}{x^4}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$= \frac{(2x+5)x^2 - 2x\left(x^2 + 5x - 1\right)}{x^4}$$

$$= x\frac{(2x+5)x - 2\left(x^2 + 5x - 1\right)}{x^4}$$

$$= \frac{2x^2 + 5x - 2x^2 - 10x + 2}{x^3}$$

$$\boxed{= \frac{-5x+2}{x^3}}$$

$$u = -5x + 2 \qquad v = x^3$$
$$u' = -5 \qquad v' = 3x^2$$

$$y'' = \frac{(-5)x^3 - 3x^2(-5x+2)}{x^6}$$

$$= x^2\,\frac{-5x^3 + 15x - 6}{x^6}$$

$$\boxed{= \frac{-5x^3 + 15x - 6}{x^4}}$$

Exercise

Find the first and second derivatives $y = \dfrac{x^2+3}{(x-1)^3+(x+1)^3}$

Solution

$$(x-1)^3+(x+1)^3 = x^3-3x^2+3x-1+x^3+3x^2+3x+1$$

$$= 2x^3+6x$$

$$y = \frac{x^2+3}{2x^3+6x}$$

$$u = x^2+3 \qquad v = 2x^3+6x$$

$$u' = 2x \qquad v' = 6x^2+6$$

$$y' = \frac{2x\left(2x^3+6x\right)-\left(6x^2+6\right)\left(x^2+3\right)}{\left(2x^3+6x\right)^2} \qquad \left(\frac{u}{v}\right)' = \frac{u'v-v'u}{v^2}$$

$$= \frac{4x^4+12x^2-6x^4-18x^2-6x^2-18}{\left(2x^3+6x\right)^2}$$

$$= \frac{-2x^4-12x^2-18}{\left(2x^3+6x\right)^2}$$

$$= -2\frac{x^4+6x^2+9}{\left(2x^3+6x\right)^2}$$

$$u = x^4++6x^2+9 \qquad v = \left(2x^3+6x\right)^2$$

$$u' = 4x^3+12x \qquad v' = 2\left(2x^3+6x\right)\left(6x^2+6\right)$$

$$= 4x\left(x^2+3\right)$$

$$y'' = -2\frac{4x\left(x^2+3\right)\left(2x^3+6x\right)^2-2\left(2x^3+6x\right)\left(6x^2+6\right)\left(x^4+6x^2+9\right)}{\left(2x^3+6x\right)^4}$$

$$= -4\left(2x^3+6x\right)\frac{2x\left(2x^5+6x^3+6x^3+18x\right)-\left(6x^6+36x^4+54x^2+x^4+36x^2+54\right)}{\left(2x^3+6x\right)^4}$$

$$= -4\frac{4x^5 + 24x^3 + 36x^2 - 6x^6 - 37x^4 - 90x^2 - 54}{\left(2x^3 + 6x\right)^3}$$

$$= -4\frac{-6x^6 + 4x^5 - 37x^4 + 24x^3 - 54x^2 - 54}{\left(2x^3 + 6x\right)^3}$$

Exercise

Find an equation of the tangent line to the graph of $y = \frac{x^2 - 4}{2x + 5}$ when $x = 0$

Solution

$$y' = \frac{(2x+5)(2x) - (x^2 - 4)(2)}{(2x+5)^2} \qquad \left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$= \frac{4x^2 + 10x - 2x^2 + 8}{(2x+5)^2}$$

$$= \frac{2x^2 + 10x + 8}{(2x+5)^2}$$

$$\Rightarrow x = 0 \to y' = \frac{8}{25} = m$$

$$x = 0 \to y = \frac{x^2 - 4}{2x + 5} = -\frac{4}{5}$$

$$y - y_1 = m\left(x - x_1\right)$$

$$\to y + \frac{4}{5} = \frac{8}{25}(x - 0) \qquad \to y = \frac{8}{25}x - \frac{4}{5}$$

Exercise

Find an equation for the line perpendicular to the tangent to the curve $y = x^3 - 4x + 1$ at the point (2, 1).

Solution

$$y' = 3x^2 - 4$$

$$m = y'\big|_{x=2} = 3(2)^2 - 4 = 8$$

$$m_1 = -\frac{1}{8}$$

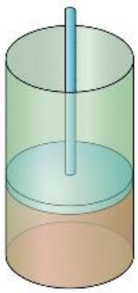
$$y - y_1 = m_1\left(x - x_1\right)$$

$$y - 1 = -\frac{1}{8}(x - 2)$$

$$y - 1 = -\frac{1}{8}x - \frac{1}{4}$$

$$\boxed{y = -\frac{1}{8}x - \frac{3}{4}}$$

Exercise



If gas in a cylinder is maintained at a constant temperature T, the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2}$$

In which a, b, n, and R are constants. Find $\frac{dP}{dV}$

Solution

$$\frac{dP}{dV} = \frac{d}{dV}\left(\frac{nRT}{V - nb}\right) - \frac{d}{dV}\left(\frac{an^2}{V^2}\right)$$

$$= -nRT\frac{(V - nb)'}{(V - nb)^2} - an^2\left(-\frac{2V}{V^4}\right)$$

$$= -nRT\frac{1}{(V - nb)^2} + an^2\left(\frac{2}{V^3}\right)$$

$$= -\frac{nRT}{(V - nb)^2} + \frac{2an^2}{V^3}$$

Solution   *Section* 2.4 – The Derivative as a Rate of Change

Exercise

The position $s(t)=t^2-3t+2, \quad 0\le t\le 2$ of a body moving on a coordinate line, with *s* in meters and *t* in seconds.

a) Find the body's displacement and average velocity for the given time interval.
b) Find the body's speed and acceleration at the endpoints of the interval.
c) When, if ever, during the interval does the body change direction?

Solution

a) Displacement: $\Delta s = s(2)-s(0)$

$$= 2^2-3(2)+2-\left(0^2-3(0)+2\right)$$

$$\underline{=-2\ m}$$

Average velocity $= \dfrac{\Delta s}{\Delta t}=\dfrac{-2}{2-0}=\underline{-1\ m/\sec}$

b) $v=\dfrac{ds}{dt}=2t-3$

$$\Rightarrow \begin{cases} |v(0)| = |-3| = \underline{3\ m/\sec} \\ |v(2)| = \underline{1\ m/\sec} \end{cases}$$

$$a=\dfrac{dv}{dt}=2 \quad \Rightarrow \quad a(0)=a(2)=\underline{2\ m/\sec^2}$$

c) $v=0 \quad \Rightarrow \quad 2t-3=0 \rightarrow \boxed{t=\dfrac{3}{2}}$

v is negative in the interval $0<t<\dfrac{3}{2}$

v is positive in the interval $\dfrac{3}{2}<t<2$

The body changes direction at $t=\dfrac{3}{2}$

Exercise

The position $s(t)=\frac{25}{t+5}, \quad -4\le t\le 0$ of a body moving on a coordinate line, with s in meters and t in seconds.

a) Find the body's displacement and average velocity for the given time interval.
b) Find the body's speed and acceleration at the endpoints of the interval.
c) When, if ever, during the interval does the body change direction?

Solution

a) Displacement: $\Delta s = s(0)-s(-4)$

$$\begin{aligned} &= \frac{25}{0+5}-\frac{25}{-4+5} \\ &= 5-25 \\ &= -20\ m \end{aligned}$$

Average velocity = $\frac{\Delta s}{\Delta t}=\frac{-20}{0-(-4)}=-5\ m/\sec$

b) $v=\frac{ds}{dt}=\frac{25(-1)}{(t+5)^2}=-\frac{25}{(t+5)^2}$

$$\Rightarrow \begin{cases} |v(-4)| = \left|-\frac{25}{(-4+5)^2}\right| = 25\ m/\sec \\ |v(0)| = \left|-\frac{25}{(0+5)^2}\right| = 1\ m/\sec \end{cases}$$

$$a=\frac{dv}{dt}=-\frac{-25\left[2(t+5)(1)\right]}{(t+5)^4}$$

$$=\frac{50}{(t+5)^3}$$

$$a(-4)=\frac{50}{(-4+5)^3}=50\ m/\sec^2$$

$$a(0)=\frac{50}{(0+5)^3}=\frac{2}{5}\ m/\sec^2$$

c) $v=0 \quad \Rightarrow \quad -\frac{25}{(t+5)^2}=0 \to \boxed{v<0}$

v is never equal to zero $\Rightarrow$ The body never changes direction.

Exercise

At time t, the position of a body moving along the s-axis is $s = t^3 - 6t^2 + 9t$ ***m***.

a) Find the body's acceleration each time the velocity is zero.
b) Find the body's speed each time the acceleration is zero.
c) Find the total distance traveled by the body from t = 0 to t = 2.

Solution

a) $v = s' = 3t^2 - 12t + 9 = 0 \quad \Rightarrow \quad \boxed{t=1} \quad \boxed{t=3}$

$$a = v' = 6t - 12 \quad \Rightarrow \quad \begin{cases} a(1) = 6 - 12 = \underline{-6\ m/\sec^2} \\ a(3) = 6(3) - 12 = \underline{6\ m/\sec^2} \end{cases}$$

The body is motionless but being accelerated left when t = 1, and motionless but being accelerated right when t = 3.

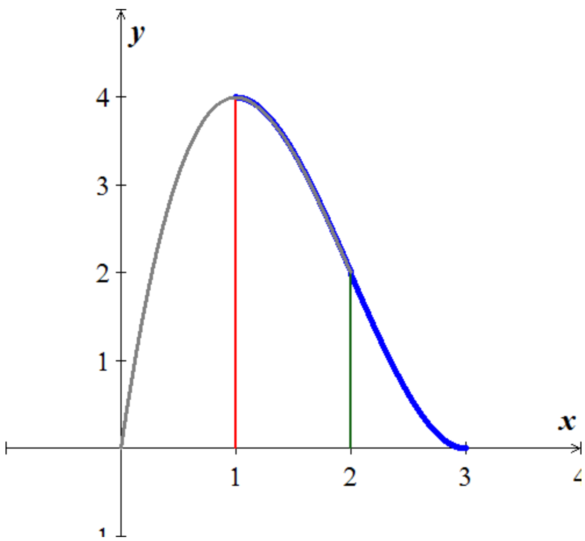
b) $a = 0 = 6t - 12 \quad \Rightarrow \quad \boxed{t=2}$

$$|v(2)| = \left|3(2)^2 - 12(2) + 9\right| = \underline{3\ m/\sec}$$

c) The body moves forward on $0 \le t < 1 \quad \rightarrow \quad d_1 = s(1) - s(0) = 1 - 6 + 9 = 4$

The body moves backward on $1 \le t < 2 \quad \rightarrow \quad d_2 = |s(2) - s(1)| = |2 - 4| = 2$

Total distance $= d_1 + d_2 = 4 + 2 = \underline{6\ m}$



Exercise

A rock thrown vertically upward from the surface of the moon at a velocity of 24 *m/sec* (about 86 *km/h*) reaches a height of $s = 24t - 0.8t^2$ *m* in *t sec*.

a) Find the rock's velocity and acceleration at time *t*. (The acceleration in this case is the acceleration of gravity on the moon.)

b) How long does it take the rock to reach its highest point?

c) How high does the rock go?

d) How long does it take the rock to reach half its maximum height?

e) How long is the rock aloft?

Solution

a) $v(t) = s' = 24 - 1.6t \ m/\sec$

$a(t) = v' = s'' = -1.6 \ m/\sec^2$

b) $v(t) = 0 = 24 - 1.6t \quad \Rightarrow \quad \underline{t = \frac{24}{1.6} = 15 \ \sec}$

c) $s(15) = 24(15) - 0.8(15)^2 = \underline{180 \ m}$

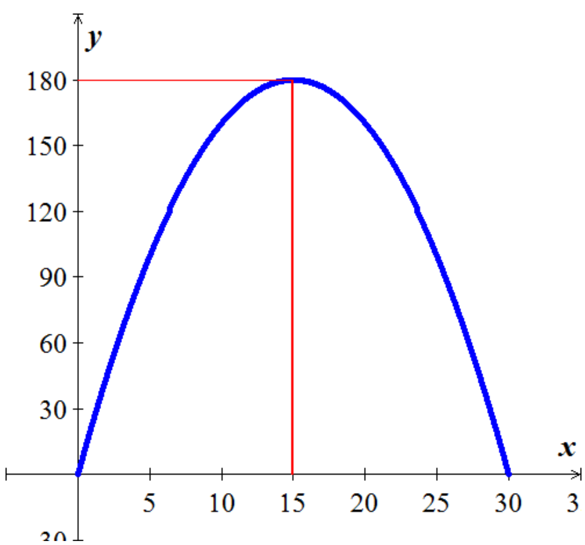
d) Since the maximum high is 180 *m*, then half is 90 *m*:

$s(t) = 24t - 0.8t^2 = 90$

$-0.8t^2 + 24t - 90 = 0 \quad \Rightarrow \quad t = 4.39 \quad t = 25.61$

It took 4.39 *sec* going up and 25.6 *sec* going down.

e) The rock took 30 sec to reach its highest point.



Exercise

Had Galileo dropped a cannonball from the Tower of Pisa, 179 *ft* above the ground, the ball's height above the ground *t sec* into the fall would have been $s = 179 - 16t^2$.

a) What would have been the ball's velocity, speed, and acceleration at time *t*?

b) About how long would it have taken the ball to hit the ground?

c) What would have been the ball's velocity at the moment of impact?

Solution

a) $v = s' = -32t$

$speed = |v| = 32t \ ft/\sec$

$a = -32 \ ft/\sec^2$

b) $s = 0 = 179 - 16t^2 \quad \Rightarrow \quad 16t^2 = 179$

$$t = \sqrt{\frac{179}{16}} \approx \underline{3.3 \ \sec}$$

When t = 3.3 sec $\Rightarrow v = -32t = -32(3.3) = \underline{-107 \ ft/\sec}$

Exercises Section 2.5 –Derivatives of Trigonometric Functions

Exercise

Find the derivative of $y = -10x + 3\cos x$

Solution

$$\boxed{y' = -10 - 3\sin x}$$

Exercise

Find the derivative of $y = \csc x - 4\sqrt{x} + 7$

Solution

$$y' = -\csc x \cot x - 4\left(\frac{1}{2}x^{-1/2}\right)$$

$$\boxed{= -\csc x \cot x - \frac{2}{\sqrt{x}}}$$

Exercise

Find the derivative of $y = x^2 \cos x$

Solution

$$y = 2x\cos x + x^2(-\sin x) \qquad (uv)' = u'v + v'u$$

$$\boxed{= 2x\cos x - x^2 \sin x}$$

Exercise

Find the derivative of $y = \csc x \cot x$

Solution

$$y' = (-\csc x \cot x)\cot x + \csc x\left(-\csc^2 x\right) \qquad (uv)' = u'v + uv'$$

$$= -\csc x \cot^2 x - \csc^3 x$$

$$\boxed{= -\csc x\left(\cot^2 x + \csc^2 x\right)}$$

Exercise

Find the derivative of $y=(\sin x+\cos x)\sec x$

Solution

$$u=\sin x+\cos x \qquad v=\sec x$$
$$u'=\cos x-\sin x \quad v'=\sec x\tan x$$

$$y'=(\cos x-\sin x)\sec x+(\sin x+\cos x)(\sec x\tan x)$$
$$=\sec x\left[\cos x-\sin x+(\sin x+\cos x)\frac{\sin x}{\cos x}\right]$$
$$=\sec x\left[\cos x-\sin x+\frac{\sin^2 x}{\cos x}+\sin x\right]$$
$$=\sec x\left[\cos x+\frac{\sin^2 x}{\cos x}\right]$$
$$=\sec x\left[\frac{\cos^2 x+\sin^2 x}{\cos x}\right]$$
$$=\sec x\left(\frac{1}{\cos x}\right)$$
$$=\sec x\sec x$$
$$=\sec^2 x$$

$$y=(\sin x+\cos x)\frac{1}{\cos x}$$
$$=\tan x+1$$
$$y'=\sec^2 x$$

Exercise

Find the derivative of $y=(\sec x+\tan x)(\sec x-\tan x)$

Solution

$$y=(\sec x+\tan x)(\sec x-\tan x)$$
$$=\sec^2 x-\tan^2 x$$
$$=1+\tan^2 x-\tan^2 x$$
$$=1$$
$$\boxed{y'=0}$$

$$y'=(\sec x+\tan x)'(\sec x-\tan x)+(\sec x+\tan x)(\sec x-\tan x)'$$
$$=\left(\sec x\tan x+\sec^2 x\right)(\sec x-\tan x)$$
$$+(\sec x+\tan x)\left(\sec x\tan x-\sec^2 x\right)$$
$$=\sec^2 x\tan x-\sec x\tan^2 x+\sec^3 x-\sec^2 x\tan x$$
$$+\sec^2 x\tan x-\sec^3 x+\sec x\tan^2 x-\sec^2 x\tan x$$
$$=0$$

Exercise

Find the derivative of $y = \dfrac{\cos x}{x} + \dfrac{x}{\cos x}$

Solution

$$y = \frac{\cos^2 x + x^2}{x\cos x}$$

$$u = \cos^2 x + x^2 \qquad\qquad v = x\cos x$$

$$u' = 2\cos x(-\sin x) + 2x \quad v' = \cos x - x\sin x$$

$$y' = \frac{(-2\cos x\sin x + 2x)x\cos x - (\cos x - x\sin x)\left(\cos^2 x + x^2\right)}{(x\cos x)^2} \qquad \left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$= \frac{-2x\sin x\cos^2 x + 2x^2\cos x - \cos^3 x - x^2\cos x + x\sin x\cos^2 x + x^3\sin x}{(x\cos x)^2}$$

$$= \frac{-x\sin x\cos^2 x - x^2\cos x - \cos^3 x + x^3\sin x}{(x\cos x)^2}$$

Exercise

Find the derivative of $y = x^2\cos x - 2x\sin x - 2\cos x$

Solution

$$y' = 2x\cos x - x^2\sin x - 2(\sin x + x\cos x) - 2(-\sin x)$$

$$= 2x\cos x - x^2\sin x - 2\sin x - 2x\cos x + 2\sin x$$

$$= -x^2\sin x$$

Exercise

Find the derivative of $y = (2 - x)\tan^2 x$

Solution

$$y' = -\tan^2 x + (2 - x)\left(2\tan x\sec^2 x\right)$$

$$= \tan x\left(-\tan x + 2(2 - x)\sec^2 x\right)$$

$$= \tan x\left(2(2 - x)\sec^2 x - \tan x\right)$$

Exercise

Find the derivative of $y = t^2 - \sec t + 1$

Solution

$$\boxed{y' = 2t - \sec t \tan t}$$

Exercise

Find the derivative of $y = \dfrac{1+\csc t}{1-\csc t}$

Solution

$$u = 1+\csc t \qquad v = 1-\csc t$$
$$u' = -\csc x \cot x \quad v' = \csc x \cot x$$

$$y' = \frac{(-\csc x \cot x)(1-\csc t) - (1+\csc t)(\csc x \cot x)}{(1-\csc t)^2}$$

$$= \frac{-\csc x \cot x + \csc^2 x \cot x - \csc x \cot x - \csc^2 x \cot x}{(1-\csc t)^2}$$

$$\boxed{= -\frac{2\csc x \cot x}{(1-\csc t)^2}}$$

Exercise

Find the derivative of $r = \theta \sin\theta + \cos\theta$

Solution

$$r' = \sin\theta + \theta\cos\theta - \sin\theta$$
$$\boxed{= \theta\cos\theta}$$

Exercise

Find the derivative of $p = \dfrac{\sin q + \cos q}{\cos q}$

Solution

$$u = \sin q + \cos q \qquad v = \cos q$$
$$u' = \cos q - \sin q \quad v' = -\sin q$$

$$p' = \frac{(\cos q - \sin q)\cos q - (-\sin q)(\sin q + \cos q)}{\cos^2 q}$$

$$= \frac{\cos^2 q - \sin q \cos q + \sin^2 q + \sin q \cos q}{\cos^2 q}$$

$$= \frac{\cos^2 q + \sin^2 q}{\cos^2 q}$$

$$= \frac{1}{\cos^2 q}$$

$$\boxed{= \sec^2 q}$$

Exercise

Find the derivative of $p = \dfrac{3q + \tan q}{q \sec q}$

Solution

$$u = 3q + \tan q \qquad v = q \sec q$$

$$u' = 3 + \sec^2 q \quad v' = \sec q + q \sec q \tan q$$

$$p' = \frac{\left(3 + \sec^2 q\right)(q \sec q) - (3q + \tan q)(\sec q + q \sec q \tan q)}{(q \sec q)^2} \qquad \left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$= \frac{3q \sec q + q \sec^3 q - 3q \sec q - 3q^2 \sec q \tan q - \tan q \sec q - q \sec q \tan^2 q}{q^2 \sec^2 q}$$

$$\boxed{= \frac{q \sec^3 q - 3q^2 \sec q \tan q - \tan q \sec q - q \sec q \tan^2 q}{q^2 \sec^2 q}}$$

Exercise

Find $y^{(4)}$ *if* $y = 9 \cos x$

Solution

$$y' = -9 \sin x$$

$$y'' = -9 \cos x$$

$$y''' = 9 \sin x$$

$$y^{(4)} = 9 \cos x$$

Exercise

Find $\frac{d^{999}}{dx^{999}}(\cos x)$

Solution

$$y' = -\sin x$$

$$y'' = -\cos x$$

$$y''' = \sin x$$

$$y^{(4)} = \cos x$$

$$999 = 249 \times 4 + 3 \quad \Rightarrow \quad \frac{d^{999}}{dx^{999}}(\cos x) = \frac{d^3}{dx^3}(\cos x) = \underline{\sin x}$$

Exercise

Find $\lim\limits_{x \to -\frac{\pi}{6}} \sqrt{1 + \cos(\pi \csc x)}$

Solution

$$\lim_{x \to -\frac{\pi}{6}} \sqrt{1 + \cos(\pi \csc x)} = \sqrt{1 + \cos\left(\pi \csc\left(-\frac{\pi}{6}\right)\right)}$$

$$= \sqrt{1 + \cos(\pi(-2))}$$

$$= \sqrt{1 + \cos(-2\pi)}$$

$$= \sqrt{1+1}$$

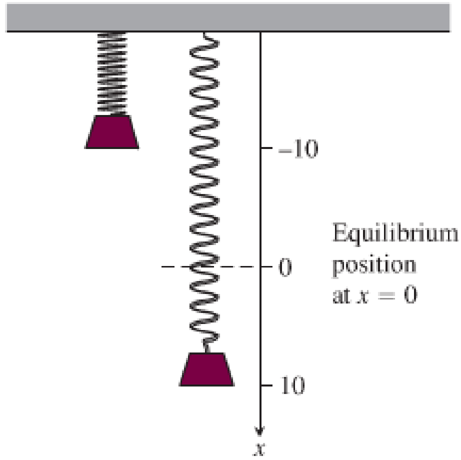
$$\underline{= \sqrt{2}}$$

Exercise

A weight is attached to a spring and reaches its equilibrium position $(x=0)$. It is then set in motion resulting in a displacement of

$$x = 10\cos t$$

Where x is measured in centimeters and t is measured in seconds.



a) Find the spring's displacement when

$$t=0, \quad t=\frac{\pi}{3}, \quad and \quad t=\frac{3\pi}{4}$$

b) Find the spring's velocity when

$$t=0, \quad t=\frac{\pi}{3}, \quad and \quad t=\frac{3\pi}{4}$$

Solution

a) $t=0 \quad \Rightarrow \quad x=10\cos 0=\underline{10\ cm}$

$t=\frac{\pi}{3} \quad \Rightarrow \quad x=10\cos\frac{\pi}{3}=10\left(\frac{1}{2}\right)=\underline{5\ cm}$

$t=\frac{3\pi}{4} \quad \Rightarrow \quad x=10\cos\frac{3\pi}{4}=10\frac{\sqrt{2}}{2}=\underline{5\sqrt{2}\ cm}$

b) $v=x'=-10\sin t$

$t=0 \quad \Rightarrow \quad x=-10\sin 0=\underline{0\ cm/\sec}$

$t=\frac{\pi}{3} \quad \Rightarrow \quad x=-10\sin\frac{\pi}{3}=10\left(\frac{\sqrt{3}}{2}\right)=\underline{5\sqrt{3}\ cm/\sec}$

$t=\frac{3\pi}{4} \quad \Rightarrow \quad x=-10\sin\frac{3\pi}{4}=-10\frac{\sqrt{2}}{2}=\underline{-5\sqrt{2}\ cm/\sec}$

Exercise

Assume that a particle's position on the x-axis is given by

$$x=3\cos t+4\sin t; \quad ft$$

a) Find the particle's position when $t=0, \quad t=\frac{\pi}{2}, \quad and \quad t=\pi$

b) Find the particle's velocity when $t=0, \quad t=\frac{\pi}{2}, \quad and \quad t=\pi$

Solution

a) $t=0 \quad \Rightarrow \quad x=3\cos 0+4\sin 0=\underline{3\ ft}$

$t=\frac{\pi}{2} \quad \Rightarrow \quad x=3\cos\frac{\pi}{2}+4\sin\frac{\pi}{2}=0+4=\underline{4\ ft}|$

$t=\pi \quad \Rightarrow \quad x=3\cos\pi+4\sin\pi=3(-1)+0=\underline{-3\ ft}|$

b) $v=x'=-3\sin t+4\cos t$

$t=0 \quad \Rightarrow \quad x=-3\sin 0+4\cos 0=\underline{4\ ft/\sec}|$

$t=\frac{\pi}{2} \quad \Rightarrow \quad x=-3\sin\frac{\pi}{2}+4\cos\frac{\pi}{2}=-3+0=\underline{-3\ ft/\sec}|$

$t=\pi \quad \Rightarrow \quad x=-3\sin\pi+4\cos\pi=0-4=\underline{-4\ ft/\sec}|$

Solution Section 2.6 – The Chain Rule

Exercise

Find the derivative of $y=\left(3x^4+1\right)^4\left(x^3+4\right)$

Solution

$$\begin{aligned}
y' &= 4\left(12x^3\right)\left(3x^4+1\right)^3\left(x^3+4\right)+3x^2\left(3x^4+1\right)^4 \\
&= 48x^3\left(3x^4+1\right)^3\left(x^3+4\right)+3x^2\left(3x^4+1\right)^4 \\
&= 3x^2\left(3x^4+1\right)^3\left[16x\left(x^3+4\right)+3x^4+1\right] \\
&= 3x^2\left(3x^4+1\right)^3\left[16x^4+64x+3x^4+1\right] \\
&= 3x^2\left(3x^4+1\right)^3\left[19x^4+64x+1\right]
\end{aligned}$$

Exercise

Find the derivative of $p(t)=\dfrac{(2t+3)^3}{4t^2-1}$

Solution

$$\left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$\begin{aligned}
P'(x) &= \frac{2(3)(2t+3)^2\left(4t^2-1\right)-8t(2t+3)^3}{\left(4t^2-1\right)^2} \\
&= \frac{(2t+3)^2\left[6\left(4t^2-1\right)-8t(2t+3)\right]}{\left(4t^2-1\right)^2} \\
&= \frac{(2t+3)^2\left[24t^2-6-16t^2-24t\right]}{\left(4t^2-1\right)^2} \\
&= \frac{(2t+3)^2\left(8t^2-24t-6\right)}{\left(4t^2-1\right)^2} \\
&= \frac{2(2t+3)^2\left(4t^2-12t-3\right)}{\left(4t^2-1\right)^2}
\end{aligned}$$

Exercise

Find the derivative of $y = \left(x^3 + 1\right)^2$

Solution

$$u = x^3 + 1 \quad \to y = u^2$$

$$\frac{d}{dx} y = \frac{dy}{du} \frac{du}{dx}$$

$$= 2u\left(3x^2\right)$$

$$y' = 2\left(x^3 + 1\right)\left(3x^2\right)$$

$$= 6x^2\left(x^3 + 1\right)$$

Exercise

Find the derivative of $y = \left(x^2 + 3x\right)^4$

Solution

$$u = x^2 + 3x$$

$$y' = n \quad (u)^{n-1} \quad \frac{d}{dx}[u]$$

$$= 4\left(x^2 + 3x\right)^3 \frac{d}{dx}\left[x^2 + 3x\right]$$

$$= 4\left(x^2 + 3x\right)^3 (2x + 3)$$

Exercise

Find the derivative of $y = \dfrac{4}{2x + 1}$

Solution

$$y = 4(2x + 1)^{-1}$$

$$y' = -4(2x + 1)^{-2}(2)$$

$$= -8(2x + 1)^{-2}$$

$$= -\frac{8}{(2x+1)^2}$$

Exercise

Find the derivative of $y = \dfrac{2}{(x-1)^3}$

Solution

$$y = 2(x-1)^{-3}$$

$$y' = 2(-3)(x-1)^{-4}(1)$$

$$\boxed{= -\frac{6}{(x-1)^4}}$$

Exercise

Find the derivative of $y = x^2\sqrt{x^2+1}$

Solution

$$y = x^2\left(x^2+1\right)^{1/2}$$

$$y' = x^2 \frac{d}{dx}\left[(x^2+1)^{1/2}\right] + (x^2+1)^{1/2}\frac{d}{dx}\left[x^2\right]$$

$$= x^2\left[\frac{1}{2}(x^2+1)^{-1/2}(2x)\right] + (x^2+1)^{1/2}[2x]$$

$$= x^3(x^2+1)^{-1/2} + 2x(x^2+1)^{1/2}$$

$$= \frac{(x^2+1)^{1/2}}{(x^2+1)^{1/2}}\left[x^3(x^2+1)^{-1/2} + 2x(x^2+1)^{1/2}\right]$$

$$= \frac{x^3(x^2+1)^{-1/2}(x^2+1)^{1/2} + 2x(x^2+1)^{1/2}(x^2+1)^{1/2}}{(x^2+1)^{1/2}}$$

$$= \frac{x^3+2x(x^2+1)}{(x^2+1)^{1/2}}$$

$$= \frac{x^3+2x^3+2x}{\sqrt{x^2+1}}$$

$$= \frac{3x^3+2x}{\sqrt{x^2+1}}$$

$$\boxed{= \frac{x(3x^2+2)}{\sqrt{x^2+1}}}$$

Exercise

Find the derivative of $y=\left(\frac{x+1}{x-5}\right)^2$

Solution

$$y'=2\left(\frac{x+1}{x-5}\right)\frac{d}{dx}\left[\frac{x+1}{x-5}\right]$$

$$=2\left(\frac{x+1}{x-5}\right)\left[\frac{(1)(x-5)-(1)(x+1)}{(x-5)^2}\right]$$

$$=2\left(\frac{x+1}{x-5}\right)\left(\frac{x-5-x-1}{(x-5)^2}\right)$$

$$=2\left(\frac{x+1}{x-5}\right)\left(\frac{-6}{(x-5)^2}\right)$$

$$=-\frac{12(x+1)}{(x-5)^3}$$

Exercise

Find the derivative of $s(t)=\sqrt{2t^2+5t+2}$

Solution

$$s(t)=\left(2t^2+5t+2\right)^{1/2}$$

$$U=2t^2+5t+2 \quad \rightarrow \quad U'=4t+5$$

$$s'(t)=\frac{1}{2}(4t+5)\left(2t^2+5t+2\right)^{-1/2}$$

$$\left(U^n\right)'=nU'U^{n-1}$$

$$=\frac{1}{2}\frac{4t+5}{\sqrt{2t^2+5t+2}}$$

Exercise

Find the derivative of $f(x)=\frac{1}{\left(x^2-3x\right)^2}$

Solution

$$f(x)=\left(x^2-3x\right)^{-2}$$

$$f'(x)=-2(2x-3)\left(x^2-3x\right)^{-3}$$

$$=-\frac{2(2x-3)}{\left(x^2-3x\right)^3}$$

Exercise

Find the derivative of $y = t^2\sqrt{t-2}$

Solution

$$f = t^2 \qquad f' = 2t$$
$$g = (t-2)^{1/2} \quad g' = \frac{1}{2}(t-2)^{-1/2}$$

$$y' = 2t\sqrt{t-2} + t^2\frac{1}{2}(t-2)^{-1/2}$$

$$= \left[2t(t-2)^{1/2} + t^2\frac{1}{2}(t-2)^{-1/2}\right]\frac{2(t-2)^{1/2}}{2(t-2)^{1/2}}$$

$$= \frac{4t(t-2)+t^2}{2(t-2)^{1/2}}$$

$$= \frac{4t^2-8t+t^2}{2\sqrt{t-2}}$$

$$\boxed{= \frac{5t^2-4t}{2\sqrt{t-2}}}$$

Exercise

Find the derivative of $y = \left(\frac{6-5x}{x^2-1}\right)^2$

Solution

$$f = 6-5x \quad f' = -5$$
$$g = x^2-1 \quad g' = 2x$$

$$y = 2\frac{-5\left(x^2-1\right)-2x(6-5x)}{\left(x^2-1\right)^2}\left(\frac{6-5x}{x^2-1}\right) \qquad \left(U^n\right)' = nU'U^{n-1}$$

$$= 2\frac{-5x^2+5-12x+10x^2}{\left(x^2-1\right)^3}(6-5x)$$

$$\boxed{= \frac{2\left(5x^2-12x+5\right)(6-5x)}{\left(x^2-1\right)^3}}$$

Exercise

Find the derivative of $y = 4x(3x+5)^5$

Solution

$$\begin{aligned}
y' &= 4(3x+5)^5 + 5(3)(3x+5)^4(4x) \\
&= 4(3x+5)^5 + 60x(3x+5)^4 \\
&= 4(3x+5)^4(3x+5+15x) \\
&= 4(3x+5)^4(18x+5)
\end{aligned}$$

Exercise

Find the derivative of $y = \left(3x^2 - 5x\right)^{1/2}$

Solution

$$u = 3x^2 - 5x \quad \& \quad y = u^{1/2}$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= \frac{1}{2}u^{-1/2}(6x-5) \\
&= \frac{1}{2}(6x-5)\left(3x^2-5x\right)^{-1/2} \\
&= \frac{6x-5}{2\left(3x^2-5x\right)^{1/2}}
\end{aligned}$$

Exercise

Find the derivative of $D_x\left(x^2+5x\right)^8$

Solution

$$\begin{aligned}
D_x\left(x^2+5x\right)^8 &= 8\left(x^2+5x\right)^7\left(x^2+5x\right)' \\
&= 8\left(x^2+5x\right)^7(2x+5) \\
&= 8(2x+5)\left(x^2+5x\right)^7
\end{aligned}$$

Exercise

Find the derivative of $y = \dfrac{(3x+2)^7}{x-1}$

Solution

$$y' = \frac{7(3)(3x+2)^6(x-1)-(1)(3x+2)^7}{(x-1)^2}$$

$$= \frac{(3x+2)^6(21x-21-3x-2)}{(x-1)^2}$$

$$= \frac{(3x+2)^6(18x-23)}{(x-1)^2}$$

Exercise

Find the derivative of $y = \left(\dfrac{x^2}{8} + x - \dfrac{1}{x}\right)^4$

Solution

$$y' = 4\left(\frac{x^2}{8} + x - \frac{1}{x}\right)^3\left(\frac{2x}{8} + 1 - \frac{-1}{x^2}\right)$$

$$= 4\left(\frac{x^2}{8} + x - \frac{1}{x}\right)^3\left(\frac{x}{4} + 1 + \frac{1}{x^2}\right)$$

$$= \left(x + 4 + \frac{4}{x^2}\right)\left(\frac{x^2}{8} + x - \frac{1}{x}\right)^3$$

Exercise

Find the derivative of $y = \sqrt{3x^2 - 4x + 6}$

Solution

$$y = \left(3x^2 - 4x + 6\right)^{1/2} = u^{1/2} \qquad u = 3x^2 - 4x + 6 \quad \Rightarrow \quad u' = 6x - 4$$

$$y' = \frac{1}{2}u^{1/2}u'$$

$$= \frac{1}{2}\left(3x^2 - 4x + 6\right)^{-1/2} 2(3x - 42)$$

$$= \frac{3x-2}{\sqrt{3x^2 - 4x + 6}}$$

Exercise

Find the derivative of $y = \cot\left(\pi - \frac{1}{x}\right)$

Solution

$$u = \pi - \frac{1}{x} \quad \rightarrow \quad u' = \frac{1}{x^2}$$

$$y' = -\csc^2\left(\pi - \frac{1}{x}\right)\left(\frac{1}{x^2}\right)$$

$$\boxed{= -\frac{1}{x^2}\csc^2\left(\pi - \frac{1}{x}\right)}$$

Exercise

Find the derivative of $y = 5\cos^{-4} x$

Solution

$$y = 5\cos^{-4} x \qquad\qquad u = \cos x \quad \rightarrow \quad u' = -\sin x$$

$$y' = 5u^{-5}u'$$

$$= 5(-4)\cos^{-5} x(-\sin x)$$

$$\boxed{= 20\sin x\cos^{-5} x}$$

Exercise

Find the derivative of $y = \sin\left(\frac{3\pi t}{2}\right) + \cos\left(\frac{3\pi t}{2}\right)$

Solution

$$y' = \frac{3\pi}{2}\cos\left(\frac{3\pi t}{2}\right) + \frac{3\pi}{2}\left(-\cos\left(\frac{3\pi t}{2}\right)\right)$$

$$= \frac{3\pi}{2}\cos\left(\frac{3\pi t}{2}\right) - \frac{3\pi}{2}\cos\left(\frac{3\pi t}{2}\right)$$

$$\boxed{= \frac{3\pi}{2}\left(\cos\left(\frac{3\pi t}{2}\right) - \cos\left(\frac{3\pi t}{2}\right)\right)}$$

Exercise

Find the derivative of $r = 6(\sec\theta - \tan\theta)^{3/2}$

Solution

$$r = 6(\sec\theta - \tan\theta)^{3/2} = 6u^{3/2} \quad \Rightarrow \quad u = \sec\theta - \tan\theta \ \rightarrow\ u' = \sec\theta\tan\theta - \sec^2\theta$$

$$\Rightarrow\ u = \sec\theta - \tan\theta \ \rightarrow\ u' = \sec\theta\tan\theta - \sec^2\theta$$

$$r' = 6\left(\frac{3}{2}\right)(\sec\theta - \tan\theta)^{3/2-1}\left(\sec\theta\tan\theta - \sec^2\theta\right)$$

$$= 9(\sec\theta - \tan\theta)^{1/2}\left(\sec\theta\tan\theta - \sec^2\theta\right)$$

$$\boxed{= 9\left(\sec\theta\tan\theta - \sec^2\theta\right)\sqrt{\sec\theta - \tan\theta}}$$

Exercise

Find the derivative of $g(x) = \dfrac{\tan 3x}{(x+7)^4}$

Solution

$$u = \tan 3x \qquad v = (x+7)^4$$

$$u' = 3\sec^2 3x \quad v' = 4(x+7)^3$$

$$g'(x) = \frac{\left(3\sec^2 3x\right)(x+7)^4 - 4(x+7)^3\tan 3x}{(x+7)^8} \qquad \left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$= \frac{(x+7)^3\left[3(x+7)\sec^2 3x - 4\tan 3x\right]}{(x+7)^8}$$

$$\boxed{= \frac{3(x+7)\sec^2 3x - 4\tan 3x}{(x+7)^8}}$$

Exercise

Find the derivative of $f(\theta) = \left(\dfrac{\sin\theta}{1+\cos\theta}\right)^2$

Solution

$$f'(\theta) = 2\left(\frac{\sin\theta}{1+\cos\theta}\right)\left(\frac{\sin\theta}{1+\cos\theta}\right)'$$

$$= \frac{2\sin\theta}{1+\cos\theta}\left(\frac{\cos\theta(1+\cos\theta)-(-\sin\theta)\sin\theta}{(1+\cos\theta)^2}\right)$$

$$= \frac{2\sin\theta}{1+\cos\theta}\left(\frac{\cos\theta+\cos^2\theta+\sin^2\theta}{(1+\cos\theta)^2}\right)$$

$$= \frac{2\sin\theta}{1+\cos\theta}\left(\frac{\cos\theta+1}{(1+\cos\theta)^2}\right)$$

$$= \frac{2\sin\theta}{(1+\cos\theta)^2}$$

Exercise

Find the derivative of $y = \sin^2(\pi t - 2)$

Solution

$$y' = 2\sin(\pi t - 2)(\sin(\pi t - 2))'$$

$$= 2\sin(\pi t - 2)(\pi\cos(\pi t - 2))$$

$$= 2\pi\sin(\pi t - 2)\cos(\pi t - 2)$$

Exercise

Find the derivative of $y = (t\tan t)^{10}$

Solution

$$y' = 10(t\tan t)^9(t\tan t)'$$

$$= 10(t\tan t)^9\left(\tan t + t\sec^2 t\right)$$

$$= 10(t\tan t)^9\tan t + 10t(t\tan t)^9\sec^2 t$$

$$= 10t^9\tan^{10} t + 10t^{10}\tan^9 t\sec^2 t$$

Exercise

Find the derivative of $y = \cos\left(5\sin\left(\frac{t}{3}\right)\right)$

Solution

$$y' = -\sin\left(5\sin\left(\frac{t}{3}\right)\right)\left(5\sin\left(\frac{t}{3}\right)\right)'$$

$$= -\sin\left(5\sin\left(\frac{t}{3}\right)\right)\left(5\frac{1}{3}\cos\left(\frac{t}{3}\right)\right)$$

$$\boxed{= -\frac{5}{3}\sin\left(5\sin\left(\frac{t}{3}\right)\right)\cos\left(\frac{t}{3}\right)}$$

Exercise

Find the derivative of $y = 4\sin\left(\sqrt{1+\sqrt{t}}\right)$

Solution

$$y' = 4\cos\left(\sqrt{1+\sqrt{t}}\right)\left(\sqrt{1+\sqrt{t}}\right)'$$

$$\left(\left(1+\sqrt{t}\right)^{1/2}\right)' = \frac{1}{2}\left(1+\sqrt{t}\right)^{-1/2}\left(t^{1/2}\right)'$$

$$= \frac{1}{2}\left(1+\sqrt{t}\right)^{-1/2}\left(\frac{1}{2}t^{-1/2}\right)$$

$$= \frac{1}{4}\frac{1}{\sqrt{t}\sqrt{1+\sqrt{t}}}$$

$$= \frac{1}{4}\frac{1}{\sqrt{t\left(1+\sqrt{t}\right)}}$$

$$y' = 4\cos\left(\sqrt{1+\sqrt{t}}\right)\left(\frac{1}{4}\frac{1}{\sqrt{t+t\sqrt{t}}}\right)$$

$$\boxed{= \frac{\cos\left(\sqrt{1+\sqrt{t}}\right)}{\sqrt{t+t\sqrt{t}}}}$$

Exercise

Find the derivative of $y = \tan^2\left(\sin^3 x\right)$

Solution

$$u = \sin^3 x \quad \Rightarrow \quad u' = 3\sin^2 x(\sin x) = 3\sin^2 x(\cos x)$$

$$y' = 2\tan\left(\sin^3 x\right).\left(\tan\left(\sin^3 x\right)\right)'$$

$$= 2\tan\left(\sin^3 x\right)\cdot\sec^2\left(\sin^3 x\right)\cdot\left(\sin^3 x\right)'$$

$$= 2\tan\left(\sin^3 x\right)\cdot\sec^2\left(\sin^3 x\right)\cdot\left(3\sin^2 x\cos x\right)$$

$$= 6\cos x\sin^2 x\cdot\tan\left(\sin^3 x\right)\cdot\sec^2\left(\sin^3 x\right)$$

Exercise

Find the second derivatives of $y = \left(1+\frac{1}{x}\right)^3$

Solution

$$y' = 3\left(1+\frac{1}{x}\right)^2\left(1+\frac{1}{x}\right)' \qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$= 3\left(1+\frac{1}{x}\right)^2\left(-\frac{1}{x^2}\right)$$

$$= -\frac{3}{x^2}\left(1+\frac{1}{x}\right)^2$$

$$y'' = \left(-\frac{3}{x^2}\right)'\left(1+\frac{1}{x}\right)^2 + \left(-\frac{3}{x^2}\right)\left(\left(1+\frac{1}{x}\right)^2\right)'$$

$$= \left(-\frac{-3(2x)}{x^4}\right)\left(1+\frac{1}{x}\right)^2 + \left(-\frac{3}{x^2}\right)\left(2\left(1+\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)\right)$$

$$= \frac{6}{x^3}\left(1+\frac{1}{x}\right)^2 + \frac{6}{x^4}\left(1+\frac{1}{x}\right)$$

$$= \frac{6}{x^3}\left(1+\frac{1}{x}\right)\left(1+\frac{1}{x}+\frac{1}{x}\right)$$

$$= \frac{6}{x^3}\left(1+\frac{1}{x}\right)\left(1+\frac{2}{x}\right)$$

Exercise

Find the second derivatives of $y = 9\tan\left(\frac{x}{3}\right)$

Solution

$$y' = 9\sec^2\left(\frac{x}{3}\right)\cdot\left(\frac{x}{3}\right)'$$

$$= 9\sec^2\left(\frac{x}{3}\right)\cdot\left(\frac{1}{3}\right)$$

$$= 3\sec^2\left(\frac{x}{3}\right)$$

$$y'' = 6\sec\left(\frac{x}{3}\right)\cdot\left(\sec\left(\frac{x}{3}\right)\right)'$$

$$= 6\sec\left(\frac{x}{3}\right)\cdot\frac{1}{3}\sec\left(\frac{x}{3}\right)\cdot\tan\left(\frac{x}{3}\right)$$

$$= 2\sec^2\left(\frac{x}{3}\right)\cdot\tan\left(\frac{x}{3}\right)$$

Exercise

Find the tangent line to the graph of $y = \sqrt[3]{(x+4)^2}$ when $x = 4$.

Solution

$$y = (x+4)^{2/3}$$

$$y' = \frac{2}{3}(x+4)^{-1/3}$$

$$= \frac{2}{3}\frac{1}{(x+4)^{1/3}}$$

$$= \frac{2}{3\sqrt[3]{x+4}}$$

$$x = 4 \to m = y' = \frac{2}{3\sqrt[3]{4+4}} = \frac{2}{3\sqrt[3]{2^3}} = \frac{2}{3(2)} = \frac{1}{3}$$

$$x = 4 \to y = \sqrt[3]{(4+4)^2} = 4$$

$$y - 4 = \frac{1}{3}(x-4)$$

$$y = \frac{1}{3}x - \frac{4}{3} + 4$$

$$y = \frac{1}{3}x + \frac{8}{3}$$

Solution Section 2.7 – Implicit Differentiation

Exercise

Find $\frac{dy}{dx}$ $\quad y^2+x^2-2y-4x=4$

Solution

$$\frac{d}{dx}[y^2+x^2-2y-4x]=\frac{d}{dx}[4]$$

$$\frac{d}{dx}\left[y^2\right]+\frac{d}{dx}\left[x^2\right]-\frac{d}{dx}[2y]-\frac{d}{dx}[4x]=\frac{d}{dx}[4]$$

$$2y\frac{dy}{dx}+2x-2\frac{dy}{dx}-4=0$$

$$2(y-1)\frac{dy}{dx}=4-2x$$

$$(y-1)\frac{dy}{dx}=2-x$$

$$\boxed{\frac{dy}{dx}=\frac{2-x}{y-1}}$$

Exercise

Find $\frac{dy}{dx}$ $\quad x^2y^2-2x=3$

Solution

$$2xy^2+2x^2yy'-2=0$$

$$2x^2yy'=2-2xy^2$$

$$y'=\frac{2\left(1-xy^2\right)}{2x^2y}$$

$$\boxed{\frac{dy}{dx}=\frac{1-xy^2}{x^2y}}$$

Exercise

Find $\frac{dy}{dx}$ $\quad x+\sqrt{x}\sqrt{y}=y^2$

Solution

$$\frac{d}{dx}\left(x+x^{1/2}y^{1/2}\right)=\frac{d}{dx}y^2$$

$$1+\frac{d}{dx}\left(x^{1/2}\right)y^{1/2}+x^{1/2}\frac{d}{dx}\left(y^{1/2}\right)=2y\frac{dy}{dx}$$

$$1+\frac{1}{2}x^{-1/2}y^{1/2}+x^{1/2}\frac{1}{2}y^{-1/2}\frac{dy}{dx}=2y\frac{dy}{dx}$$

$$1+\frac{y^{1/2}}{2x^{1/2}}+\frac{x^{1/2}}{2y^{1/2}}\frac{dy}{dx}=2y\frac{dy}{dx}$$

$$1+\frac{y^{1/2}}{2x^{1/2}}=2y\frac{dy}{dx}-\frac{x^{1/2}}{2y^{1/2}}\frac{dy}{dx}$$

$$\left(\frac{4y^{3/2}-x^{1/2}}{2y^{1/2}}\right)\frac{dy}{dx}=\frac{2x^{1/2}+y^{1/2}}{2x^{1/2}}$$

$$\frac{dy}{dx}=\frac{2x^{1/2}+y^{1/2}}{2x^{1/2}}\cdot\frac{2y^{1/2}}{4y^{3/2}-x^{1/2}}$$

$$=\frac{4x^{1/2}y^{1/2}+2y}{8x^{1/2}y^{3/2}-2x}$$ *Divide every term by 2*

$$\boxed{=\frac{2x^{1/2}y^{1/2}+y}{4x^{1/2}y^{3/2}-x}}$$

Exercise

Find $\frac{dy}{dx}$ $\quad x^2y+xy^2=6$

Solution

$$\left(2xy+x^2\frac{dy}{dx}\right)+\left(y^2+2xy\frac{dy}{dx}\right)=0$$

$$x^2\frac{dy}{dx}+2xy\frac{dy}{dx}=-2xy-y^2$$

$$\left(x^2+2xy\right)\frac{dy}{dx}=-2xy-y^2$$

$$\boxed{\frac{dy}{dx}=\frac{-2xy-y^2}{x^2+2xy}}$$

Exercise

Find $\frac{dy}{dx}$ $\quad x^3 - xy + y^3 = 1$

Solution

$$3x^2 - \left(y + x\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$$

$$3x^2 - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$\left(3y^2 - x\right)\frac{dy}{dx} = y - 3x^2$$

$$\boxed{\frac{dy}{dx} = \frac{y - 3x^2}{3y^2 - x}}$$

Exercise

Find $\frac{dy}{dx}$ $\quad y^2 = \frac{x-1}{x+1}$

Solution

$$2yy' = \frac{1(x+1) - (1)(x-1)}{(x+1)^2}$$

$$2yy' = \frac{x+1-x+1}{(x+1)^2}$$

$$y' = \frac{2}{2y(x+1)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{y(x+1)^2}}$$

Exercise

Find $\frac{dy}{dx}$ $\quad (3xy+7)^2 = 6y$

Solution

$$2(3xy+7)(3y+3xy') = 6y'$$

$$6(3xy+7)(y+xy') = 6y' \qquad \textit{Divide by 6 both sides}$$

$$(3xy+7)(y+xy') = y'$$

$$3xy^2 + 3x^2yy' + 7y + 7xy' = y'$$

$$3x^2yy' + 7xy' - y' = -3xy^2 - 7y$$

$$\left(3x^2y + 7x - 1\right)y' = -\left(3xy^2 + 7y\right)$$

$$\boxed{\frac{dy}{dx} = -\frac{3xy^2 + 7y}{3x^2y + 7x - 1}}$$

Exercise

Find $\frac{dy}{dx}$ $\quad xy = \cot(xy)$

Solution

$$y + xy' = -\csc^2(xy)\ \left(y + xy'\right)$$

$$y + xy' = -y\csc^2(xy) - x\csc^2(xy)y'$$

$$x\csc^2(xy)y' + xy' = -y\csc^2(xy) - y$$

$$x\left(\csc^2(xy) + 1\right)y' = -y\left(\csc^2(xy) + 1\right)$$

$$y' = -\frac{y\left(\csc^2(xy) + 1\right)}{x\left(\csc^2(xy) + 1\right)}$$

$$\boxed{\frac{dy}{dx} = -\frac{y}{x}}$$

Exercise

Find $\frac{dy}{dx}$ $\quad x + \tan(xy) = 0$

Solution

$$1 + \sec^2(xy)\left(y + xy'\right) = 0$$

$$1 + y\sec^2(xy) + x\sec^2(xy)y' = 0$$

$$x\sec^2(xy)y' = -y\sec^2(xy) - 1$$

$$y' = -\frac{y\sec^2(xy)}{x\sec^2(xy)} - \frac{1}{x\sec^2(xy)}$$

$$\frac{dy}{dx} = -\frac{y}{x} - \frac{\cos^2 x}{x} = \boxed{\frac{-y - \cos^2 x}{x}}$$

Exercise

Find $\frac{dy}{dx}$ $\quad x\cos(2x+3y) = y\sin x$

Solution

$$\cos(2x+3y) - \sin(2x+3y)(2x+3y') = y'\sin x + y\cos x$$

$$\cos(2x+3y) - 2x\sin(2x+3y) - 3\sin(2x+3y)y' = y'\sin x + y\cos x$$

$$\cos(2x+3y) - 2x\sin(2x+3y) - y\cos x = y'\sin x + 3\sin(2x+3y)y'$$

$$\cos(2x+3y) - 2x\sin(2x+3y) - y\cos x = y'\left(\sin x + 3\sin(2x+3y)\right)$$

$$\boxed{y' = \frac{\cos(2x+3y) - 2x\sin(2x+3y) - y\cos x}{\sin x + 3\sin(2x+3y)}}$$

Exercise

Find $\frac{dr}{d\theta}$ $\quad r - 2\sqrt{\theta} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4}$

Solution

$$r - 2\theta^{1/2} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4}$$

$$\frac{dr}{d\theta} - 2\frac{1}{2}\theta^{-1/2} = \frac{3}{2}\frac{2}{3}\theta^{-1/3} + \frac{4}{3}\frac{3}{4}\theta^{-1/4}$$

$$\boxed{\frac{dr}{d\theta} = \theta^{-1/3} + \theta^{-1/4} + \theta^{-1/2}}$$

Exercise

Find $\frac{dr}{d\theta}$ $\quad \sin(r\theta) = \frac{1}{2}$

Solution

$$\cos(r\theta)\left(\theta\frac{dr}{d\theta} + r\right) = 0$$

$$\theta\frac{dr}{d\theta} + r = 0 \qquad \cos(r\theta) \neq 0$$

$$\boxed{\frac{dr}{d\theta} = -\frac{r}{\theta}} \qquad \cos(r\theta) \neq 0$$

Exercise

Find $\dfrac{d^2y}{dx^2}$ $\qquad x^{2/3}+y^{2/3}=1$

Solution

$$\tfrac{2}{3}x^{-1/3}+\tfrac{2}{3}y^{-1/3}y'=0 \qquad \textit{Multiply all terms by } \tfrac{3}{2}$$

$$x^{-1/3}+y^{-1/3}y'=0$$

$$y^{-1/3}y'=-x^{-1/3}$$

$$y'=-\frac{x^{-1/3}}{y^{-1/3}}$$

$$y'=-\frac{y^{1/3}}{x^{1/3}}=-\left(\frac{y}{x}\right)^{1/3}$$

$$y''=-\tfrac{1}{3}\left(\frac{y}{x}\right)^{-2/3}\left(\frac{xy'-y}{x^2}\right)$$

$$=-\tfrac{1}{3}\left(\frac{x}{y}\right)^{2/3}\left(\frac{-x\left(\frac{y}{x}\right)^{1/3}-y}{x^2}\right)=\tfrac{1}{3}\left(\frac{x^{4/3}y^{1/3}}{y^{2/3}x^2}+\frac{x^{2/3}y}{y^{2/3}x^2}\right)$$

$$=\tfrac{1}{3}\left(\frac{x}{y}\right)^{2/3}\left(\frac{x\frac{y^{1/3}}{x^{1/3}}+y}{x^2}\right)$$

$$=\frac{1}{3}\frac{x^{2/3}}{y^{2/3}}\frac{x^{2/3}y^{1/3}+y}{x^2}$$

$$\boxed{=\frac{1}{3}\left(\frac{1}{y^{1/3}x^{2/3}}+\frac{y^{1/3}}{x^{4/3}}\right)}$$

Exercise

Find $\dfrac{d^2y}{dx^2}$ $\quad 2\sqrt{y} = x - y$

Solution

$$2\frac{1}{2}y^{-1/2}y' = 1 - y'$$

$$2\frac{1}{2}y^{-1/2}y' + y' = 1$$

$$\left(y^{-1/2} + 1\right)y' = 1 \Rightarrow \boxed{y' = \frac{1}{y^{-1/2}+1}}$$

$$y' = \frac{1}{y^{-1/2}+1} = \frac{1}{\frac{1}{\sqrt{y}}+1} = \frac{\sqrt{y}}{1+\sqrt{y}}$$

$$\left(y^{-1/2} + 1\right)y'' + \left(-\frac{1}{2}y^{-3/2}y'\right)y' = 0$$

$$\left(y^{-1/2} + 1\right)y'' - \frac{1}{2}y^{-3/2}\left(y'\right)^2 = 0$$

$$\left(y^{-1/2} + 1\right)y'' = \frac{1}{2}y^{-3/2}\left(\frac{1}{y^{-1/2}+1}\right)^2$$

$$y'' = \frac{1}{2}y^{-3/2}\frac{1}{\left(y^{-1/2}+1\right)^2}\frac{1}{y^{-1/2}+1}$$

$$= \frac{1}{2}y^{-3/2}\frac{1}{\left(\frac{1}{\sqrt{y}}+1\right)^3}$$

$$= \frac{1}{2}y^{-3/2}\frac{1}{\left(\frac{1+\sqrt{y}}{\sqrt{y}}\right)^3}$$

$$= \frac{1}{2}y^{-3/2}\frac{1}{\frac{\left(1+\sqrt{y}\right)^3}{\left(y^{1/2}\right)^3}}$$

$$= \frac{1}{2}y^{-3/2}\frac{y^{3/2}}{\left(1+\sqrt{y}\right)^3}$$

$$= \frac{1}{2\left(1+\sqrt{y}\right)^3}$$

Exercise

If $x^3 + y^3 = 16$, find the value of $\frac{d^2y}{dx^2}$ at the point (2, 2).

Solution

$$3x^2 + 3y^2 y' = 0$$

$$3y^2 y' = -3x^2$$

$$y^2 y' = -x^2$$

$$2yy'y' + y^2 y'' = -2x$$

$$y^2 y'' = -2x - 2y(y')^2$$

$$y^2 y'' = -2x - 2y\left(\frac{-x^2}{y^2}\right)^2$$

$$y^2 y'' = -2x - 2\frac{x^4}{y^3}$$

$$y'' = -2\frac{x}{y^2} - 2\frac{x^4}{y^5}$$

$$= \frac{-2xy^3 - 2x^4}{y^5}$$

$$y''\Big|_{(2,2)} = \frac{-2(2)2^3 - 2(2)^4}{2^5}$$

$$= \frac{-2^5 - 2^5}{2^5}$$

$$\underline{= -2}$$

Exercise

Find dy/dx: $x^2 - xy + y^2 = 4$ and evaluate the derivative at the given point $(0, -2)$

Solution

$$2x - (y + xy') + 2yy' = 0$$

$$-y - xy' + 2yy' = -2x$$

$$(2y - x)y' = y - 2x$$

$$\boxed{\frac{dy}{dx} = \frac{y - 2x}{2y - x}}$$

$$@(0,-2) \to \frac{dy}{dx} = \frac{-2-2(0)}{2(-2)-(0)}$$

$$= \frac{-2}{-4}$$

$$= \frac{1}{2}$$

Exercise

Find the slope of the curve $\left(x^2+y^2\right)^2 = (x-y)^2$ at the point $(-2, 1)$ and $(-2, -1)$

Solution

1 and -1

Exercise

Find the slope of the tangent line to the circle $x^2 - 9y^2 = 16$ at the point $(5, 1)$

Solution

$$2x - 18y\frac{dy}{dx} = 0$$

$$-18y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{-18y} = \frac{x}{9y}$$

$$@(5,1) \to \frac{dy}{dx} = \frac{5}{9(1)} = \frac{5}{9}$$

Exercise

Find the slope of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$

Solution

$$\frac{d}{dx}\left[x^2+y^2\right] = \frac{d}{dx}[25]$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Slope: $\frac{dy}{dx} = -\frac{3}{-4} = \frac{3}{4}$

Exercise

Find the equation of the tangent line to the circle $x^3 + y^3 = 9xy$ at the point (2, 4)

Solution

$$3x^2 + 3y^2 y' = 9y + 9xy'$$

$$3y^2 y' - 9xy' = 9y - 3x^2$$

$$\left(3y^2 - 9x\right)y' = 9y - 3x^2$$

$$y' = \frac{3\left(3y - x^2\right)}{3\left(y^2 - 3x\right)}$$

$$= \frac{3y - x^2}{y^2 - 3x}$$

$$m\Big|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}$$

$$y - 4 = \frac{4}{5}(x - 2) \quad \Rightarrow \quad y - 4 = \frac{4}{5}x - \frac{8}{5}$$

$$y = \frac{4}{5}x - \frac{8}{5} + 4$$

$$\boxed{y = \frac{4}{5}x + \frac{12}{5}}$$

Exercise

Find the lines that are (***a***) tangent and (***b***) normal to the curve $x^2 + xy - y^2 = 1$ at the point (2, 3).

Solution

$$2x + y + xy' - 2yy' = 0$$

$$(x - 2y)y' = -2x - y$$

$$y' = \frac{-2x - y}{x - 2y} = \frac{2x + y}{2y - x}$$

a) *tangent slope* $= y'\Big|_{(2,3)} = \frac{2(2) + 3}{2(3) - 2} = \frac{7}{4}$

$$y-3=\frac{7}{4}(x-2)\Rightarrow y-3=\frac{7}{4}x-\frac{7}{2}\Rightarrow \boxed{y=\frac{7}{4}x-\frac{1}{2}}$$

b) $normal\ slope=-\frac{4}{7}$

$$y-3=-\frac{4}{7}(x-2)\Rightarrow y-3=\frac{4}{7}x-\frac{8}{7}\Rightarrow \boxed{y=-\frac{4}{7}x+\frac{29}{7}}$$

Exercise

Find the lines that are (***a***) tangent and (***b***) normal to the curve $6x^2+3xy+2y^2+17y-6=0$ at the point (–1, 0).

Solution

$$12x+3y+3xy'+4yy'+17y'=0$$

$$(3x+4y+17)y'=-12x-3y$$

$$y'=\frac{-12x-3y}{3x+4y+17}$$

a) $tangent\ slope=y'\Big|_{(-1,0)}=\frac{-12(-1)-3(0)}{3(-1)+4(0)+17}=\frac{6}{7}\Big|$

$$y=\frac{6}{7}(x+1)\quad\Rightarrow\quad \boxed{y=\frac{6}{7}x+\frac{6}{7}}$$

b) $normal\ slope=-\frac{7}{6}$

$$y=-\frac{7}{6}(x+1)\quad\Rightarrow\quad \boxed{y=-\frac{7}{6}x-\frac{7}{6}}$$

Exercise

Find the lines that are (***a***) tangent and (***b***) normal to the curve $x^2\cos^2 y-\sin y=0$ at the point (0, π).

Solution

$$2x\cos^2 y+x^2\left(2\cos y(-\sin y)y'\right)-(\cos y)y'=0$$

$$\left(-2x^2\cos y\sin y-\cos y\right)y'=-2x\cos^2 y$$

$$y'=\frac{-2x\cos^2 y}{-\left(2x^2\sin y+1\right)\cos y}=\frac{2x\cos y}{2x^2\sin y+1}$$

a) $tangent\ slope=y'\Big|_{(0,\pi)}=\frac{2(0)\cos(\pi)}{2(0)^2\sin(\pi)+1}=0\Big|$

$$y-\pi=0(x-0) \quad \Rightarrow \quad \boxed{y=\pi}$$

b) $\quad normal\ slope = 0$

$$\Rightarrow \quad \boxed{x=0}$$

Exercise

Suppose that *x* and *y* are both functions of *t*, which can be considered to represent time, and that *x* and *y* are related by the equation

$$xy^2+y=x^2+17$$

Suppose further that when *x* = 2 and *y* = 3, then $\frac{dx}{dt}=13$. Find the value of the $\frac{dy}{dt}$ at that moment.

Solution

$$y^2\frac{dx}{dt}+2xy\frac{dy}{dt}+\frac{dy}{dt}=2x\frac{dx}{dt}$$

$$3^2(13)+2(2)(3)\frac{dy}{dt}+\frac{dy}{dt}=2(2)(13)$$

$$117+12\frac{dy}{dt}+\frac{dy}{dt}=52$$

$$13\frac{dy}{dt}=-65$$

$$\boxed{\frac{dy}{dt}=\frac{-65}{13}=-5}$$

Exercise

A cone-shaped icicle is dripping from the roof. The radius of the icicle is decreasing at a rate of 0.2 cm per hour, while the length is increasing at a rate of 0.8 cm per hour. If the icicle is currently 4 cm in radius and 20 cm long, is the volume of the icicle increasing or decreasing and at what rate?

Solution

The volume of the cone is given by the formula: $V=\frac{1}{3}\pi r^2 h$.

$$\frac{dV}{dt}=\frac{1}{3}\pi\left[2rh\frac{dr}{dt}+r^2\frac{dh}{dt}\right]$$

Given the values:

$$\frac{dr}{dt}=-0.2 \qquad \frac{dh}{dt}=0.8 \qquad r=4 \qquad h=20$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[2(4)(20)(-0.2) + 4^2(0.8)\right]$$
$$= -20$$

The volume is decreasing at a rate of 20 cm^3 per hour.

Solution   Section 2.8 – Related Rates

Exercise

If $y = x^2$ and $\frac{dx}{dt} = 3$, then what is $\frac{dy}{dt}$ *when* $x = -1$

Solution

..

$$= 2x(3)$$
$$= 6x$$

$$\left.\frac{dy}{dt}\right|_{x=-1} = 6(-1) \underline{= -6}$$

Exercise

If $x = y^3 - y$ and $\frac{dy}{dt} = 5$, then what is $\frac{dx}{dt}$ *when* $y = 2$

Solution

$$\frac{dx}{dt} = \frac{dx}{dy}\frac{dy}{dt}$$
$$= \left(3y^2 - 1\right)(5)$$
$$= 5\left(3y^2 - 1\right)$$

$$\left.\frac{dx}{dt}\right|_{y=2} = 5\left(3(2)^2 - 1\right) = \underline{55}$$

Exercise

A cube's surface area increases at the rate of 72 $in^2 / \sec$. At what rate is the cube's volume changing when the edge length is $x = 3$ *in*?

Solution

Cube's surface: $S = 6x^2$

$$\frac{dS}{dt} = 12x\frac{dx}{dt}$$

$$72 = 12x(3) \quad \Rightarrow \quad \underline{x = \frac{72}{26} = 2}$$

$$V = x^3 \quad \Rightarrow \quad \frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

$$\left.\frac{dV}{dt}\right|_{x=3} = 3(3)^2(2) = \underline{54\ in^2 / \sec}$$

Exercise

The radius r and height h of a right circular cone are related to the cone's volume V by the equation $V = \frac{1}{3}\pi r^2 h$.

a) How is $\frac{dV}{dt}$ related to $\frac{dh}{dt}$ if $\boldsymbol{r}$ is constant?

b) How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ if $\boldsymbol{h}$ is constant?

c) How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ and $\frac{dh}{dt}$ if neither $\boldsymbol{r}$ nor $\boldsymbol{h}$ is constant?

Solution

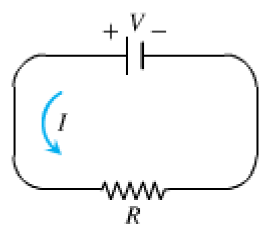
a) $\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt}$

b) $\frac{dV}{dt} = \frac{2}{3}\pi rh \frac{dr}{dt}$

c) $\frac{dV}{dt} = \frac{2}{3}\pi rh \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$

Exercise

The voltage $\boldsymbol{V}$ (volts), current $\boldsymbol{I}$ (amperes), and resistance $\boldsymbol{R}$ (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of $\frac{1}{3}\, amp/\sec$. Let t denote time in seconds.



a) What is the value of $\frac{dV}{dt}$?

b) What is the value of $\frac{dI}{dt}$?

c) What equation relates $\frac{dR}{dt}$ to $\frac{dV}{dt}$ and $\frac{dI}{dt}$?

d) Find the rate at which R is changing when V = 12 volts and I = 2 amp. Is R increasing or decreasing?

Solution

a) $\frac{dV}{dt} = \boxed{1\ volt/\sec}$

b) $\frac{dI}{dt} = \boxed{\frac{1}{3}\ amp/\sec}$

c) $\frac{dV}{dt} = R\frac{dI}{dt} + I\frac{dR}{dt}$

$I\frac{dR}{dt} = \frac{dV}{dt} - R\frac{dI}{dt}$ $\qquad V = IR \ \Rightarrow\ R = \frac{V}{I}$

$\frac{dR}{dt} = \frac{1}{I}\left(\frac{dV}{dt} - \frac{V}{I}\frac{dI}{dt}\right)$

d) $\frac{dR}{dt} = \frac{1}{2}\left((1) - \frac{12}{2}\left(-\frac{1}{3}\right)\right) = \frac{1}{2}(3) = \boxed{\frac{3}{2}\ ohms/\sec}$ $\quad$ $\boldsymbol{R}$ is increasing

Exercise

Let x and y be differentiable functions of t and let $s = \sqrt{x^2 + y^2}$ be the distance between the points $(x, 0)$ and $(0, y)$ in the xy–plane.

a) How is $\frac{ds}{dt}$ related to $\frac{dx}{dt}$ if y is constant?

b) How is $\frac{ds}{dt}$ related to $\frac{dx}{dt}$ and $\frac{dy}{dt}$ if neither x nor y is constant?

c) How is $\frac{dx}{dt}$ related to $\frac{dy}{dt}$ if s is constant?

Solution

$$s = \sqrt{x^2 + y^2} = \left(x^2 + y^2\right)^{1/2}$$

a) $$\frac{ds}{dt} = \frac{1}{2}\left(x^2 + y^2\right)^{-1/2}\left(2x\frac{dx}{dt}\right)$$

$$= \frac{x}{\sqrt{x^2+y^2}}\frac{dx}{dt}$$

b) $$\frac{ds}{dt} = \frac{1}{2}\left(x^2 + y^2\right)^{-1/2}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$$

$$= \frac{1}{\sqrt{x^2+y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$= \frac{x}{\sqrt{x^2+y^2}}\frac{dx}{dt} + \frac{y}{\sqrt{x^2+y^2}}\frac{dy}{dt}$$

c) $$s = \sqrt{x^2 + y^2} \quad \Rightarrow \quad s^2 = x^2 + y^2$$

$$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$2x\frac{dx}{dt} = -2y\frac{dy}{dt}$$

$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$$

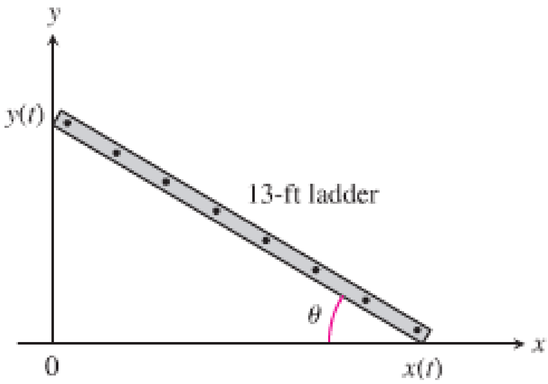
Exercise

A 13-ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 *ft* from the house, the base is moving at the rate of 5 *ft/sec*.

a) How fast is the top of the ladder sliding down the wall then?

b) At what rate is the area of the triangle formed by the ladder, wall, and the ground changing then?

c) At what rate is the angle θ between the ladder and the ground changing then?



Solution

Given: $L = 13\ ft \quad x = 12 \quad \frac{dx}{dt} = 5\ ft/\sec$

$$y = \sqrt{13^2 - 12^2} = 5$$

a) $x^2 + y^2 = 13^2$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$y\frac{dy}{dt} = -x\frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

$$= -\frac{12}{5}(5)$$

$$\underline{= -12\ ft/\sec|}$$ The ladder is sliding down the wall

b) Area of the triangle formed by the ladder and the walls is: $A = \frac{1}{2}xy$

$$\frac{dA}{dt} = \frac{1}{2}\left(y\frac{dx}{dt} + x\frac{dy}{dt}\right)$$

$$= \frac{1}{2}\left((5)(5) + (12)(-12)\right)$$

$$\underline{= -19.5\ ft^2/\sec|}$$

c) $\cos\theta = \frac{x}{13} \quad \Rightarrow \quad -\sin\theta\frac{d\theta}{dt} = \frac{1}{13}\frac{dx}{dt}$

$$\frac{d\theta}{dt} = -\frac{1}{13\sin\theta}\frac{dx}{dt}$$

$$= -\frac{1}{13\sin\theta}(5) \qquad \sin\theta = \frac{5}{13}$$

$$= -\frac{1}{13\left(\frac{5}{13}\right)}(5)$$

$$\underline{= -1\ rad/\sec|}$$

Exercise

A 13-*ft* ladder is leaning against a vertical wall when he begins pulling the foot of the ladder away from the wall at a rate of 0.5 *ft/s*. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 5 *ft* from the wall?

Solution

$$x^2 + h^2 = 13^2$$

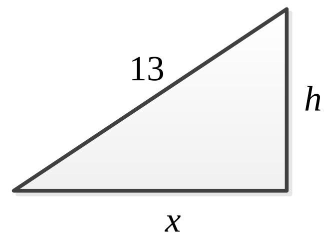
$$x^2 + h^2 = 169 \qquad \rightarrow \quad h = \sqrt{169 - 25} = 12$$

$$2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0$$

$$\frac{dh}{dt} = -\frac{x}{h}\frac{dx}{dt}$$

$$= -\frac{5}{12}(0.5)$$

$$= -\frac{5}{24}\ ft/\sec$$



So, the top of the ladder slides down the wall at $\frac{5}{24}\ ft/\sec$

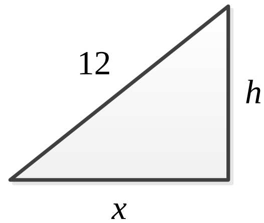
Exercise

A 12-*ft* ladder is leaning against a vertical wall when he begins pulling the foot of the ladder away from the wall at a rate of 0.2 *ft/s*. What is the configuration of the ladder at the instant that the vertical speed of the top of the ladder equals the horizontal speed of the foot of the ladder?

Solution

$$x^2 + h^2 = 144$$

$$2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0 \quad \Rightarrow \quad x\frac{dx}{dt} + h\frac{dh}{dt} = 0$$



The vertical speed of the top of the ladder equals the horizontal speed of the foot of the ladder.

$$\frac{dx}{dt} = 0.2 \quad \Rightarrow \quad \frac{dh}{dt} = -0.2$$

$$0.2x - 0.2h = 0 \quad \Rightarrow \quad x = h$$

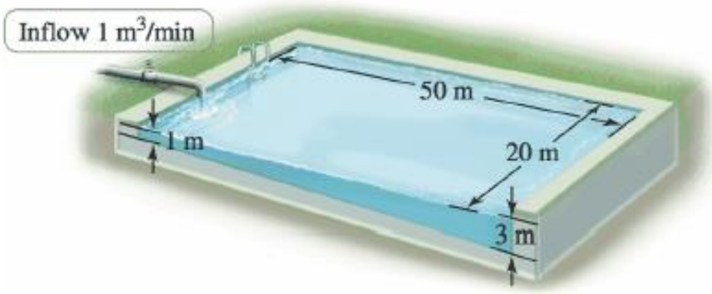
Since $x = h$, the triangle is forming a $(45° - 45° - 90°)$ with $x = h = 12\cos 45° = 6\sqrt{2}$

Exercise

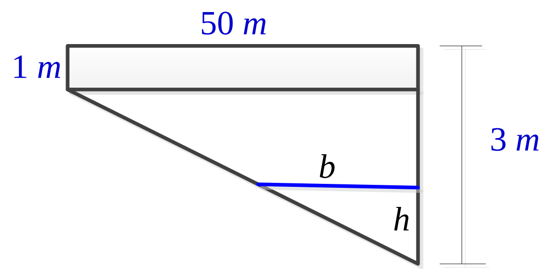
A swimming pool is 50 *m* long and 20 *m* wide. Its length decreases linearly along the length from 3 *m* to 1 *m*. It is initially empty and is filed at a rate of 1 $m^3 / \min$.

a) How fast is the water level rising 250 min after the filling begins?

b) How long will it take to fill the pool?



Solution



$$\frac{h}{2}=\frac{b}{50} \quad \Rightarrow \quad b=25h$$

The area of the side:

$$A=\frac{1}{2}bh=\frac{25}{2}h^2$$

$$For \quad 0\le h\le 2 \quad \Rightarrow \quad V(h)=12.5h^2(20)=250h^2$$

$$For \quad 2< h\le 3 \quad \Rightarrow \quad V(h)=250\times 2^2+50\times 20\times (h-2)=1000h-1000$$

a) $When \quad t=250 \ min \quad \Rightarrow \quad V=250 \ min\times 1\frac{m^3}{min}=250 \ m^3$

So $V(h)=250h^2=250 \quad \Rightarrow \quad h=1$

$$\frac{dV}{dt}=500h\frac{dh}{dt}=1\frac{m^3}{min}$$

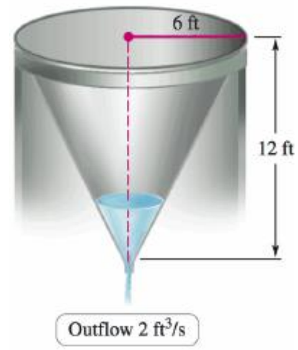
$$\frac{dh}{dt}=\frac{1}{500} \ \frac{m}{\text{min}} \ =.002 \ \frac{m}{\text{min}}$$

b) $V(h)=1000(3)-1000=2000 \ m^3$

Since $\frac{dV}{dt}=1\frac{m^3}{min}$, Then it will take 2,000 minutes.

Exercise

An inverted conical water tank with a height of 12 *ft* and a radius of 6 *ft* is drained through a hole in the vertex at a rate of 2 $ft^3/\sec$. What is the rate of change of the water depth when the water depth is 3 *ft*?

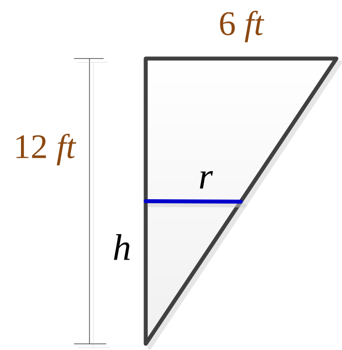


Solution

Given: $\frac{dV}{dt} = -2\frac{ft^3}{min}$

The water forms a cone with volume: $V = \frac{1}{3}\pi r^2 h$

From the triangles: $\frac{r}{6} = \frac{h}{12} \quad \Rightarrow \quad r = \frac{1}{2}h$



$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$= \frac{1}{12}\pi h^3$$

$$\frac{dV}{dt} = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$$

$$-2 = \frac{\pi 3^2}{4} \cdot \frac{dh}{dt}$$

$$\boxed{\frac{dh}{dt} = -\frac{8}{9\pi} \quad ft/s}$$

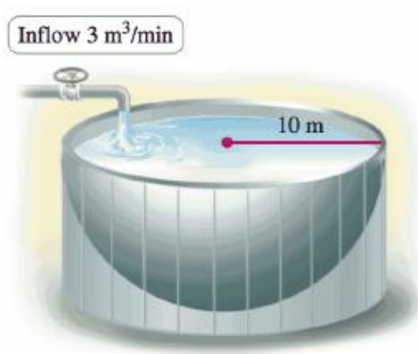
So, the depth of the water is decreasing at a rate of $\frac{8}{9\pi} \; ft/s$

Exercise

A hemispherical tank with a radius of 10 *m* is filled from an inflow pipe at a rate of $3\ m^3/\min$. (Hint: The volume of a cap of thickness *h* sliced from a sphere of radius *r* is $\frac{\pi h^2(3r-h)}{3}$).

a) How fast is the water level rising when the water level is 5 *m* from the bottom of the tank?

b) What is the rate of change of the surface area of the water when the water is 5 *m* deep?



Solution

Given: $\frac{dV}{dt} = 3\frac{m^3}{min}, \quad r = 10\ m$

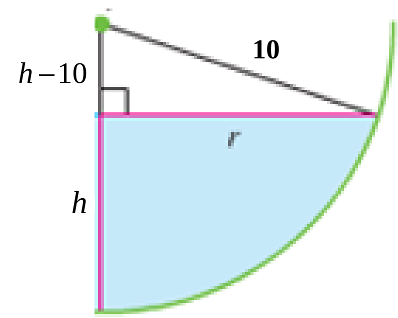
a) $V(h) = \frac{1}{3}\pi h^2(3r - h) = 10\pi h^2 - \frac{1}{3}\pi h^3$

$$\frac{dV}{dt} = \left(20\pi h - \pi h^2\right)\frac{dh}{dt}$$

When $h = 5\ m$

$$3 = (100 - 25)\pi \frac{dh}{dt}$$

$$\boxed{\frac{dh}{dt} = \frac{3}{75\pi}\ \ m/\min}$$



b) $S = \pi r^2 \quad \Rightarrow \quad \frac{dS}{dt} = 2\pi r \frac{dr}{dt}$

From the right triangle:

$$10^2 = r^2 + (10 - h)^2$$

$$100 = r^2 + 100 - 20h + h^2$$

$$20h = h^2 + r^2 \qquad\qquad h = 5, \quad r = \sqrt{100 - 25} = 5\sqrt{3}$$

$$20\frac{dh}{dt} = 2h\frac{dh}{dt} + 2r\frac{dr}{dt}$$

$$10\frac{dh}{dt} = h\frac{dh}{dt} + r\frac{dr}{dt}$$

$$5\sqrt{3}\frac{dr}{dt} = 10\frac{3}{75\pi} - 5\frac{3}{75\pi}$$

$$\frac{dr}{dt} = \frac{1}{5\sqrt{3}}\frac{15}{75\pi} = \frac{\sqrt{3}}{75\pi}$$

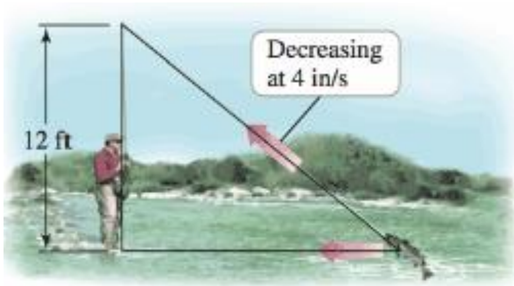
$$\frac{dS}{dt} = 2\pi r\frac{dr}{dt}$$

$$= 2\pi 5\sqrt{3}\frac{\sqrt{3}}{75\pi}$$

$$\boxed{= \frac{2}{5}\ \frac{m^2}{min}}$$

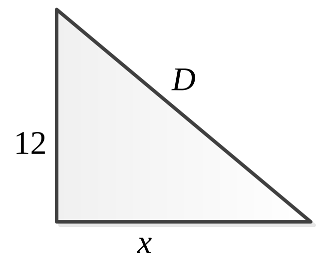
Exercise

A fisherman hooks a trout and reels in his line at 4 *in/sec*. Assume the trip of the fishing rod is 12 *ft* above the water directly above the fisherman and the fish is pulled horizontally directly towards the fisherman. Find the horizontal peed of the fish when it is 20 *ft* from the fisherman.



Solution

Let x be the distance between the fisherman's feet & th $m^3/\min$ e fish.



Let D be the distance between the fisherman's head & the fish.

Given: $\frac{dD}{dt} = -4\ in/s, \quad x = 20\ ft$

$$D^2 = x^2 + 144$$

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt}$$

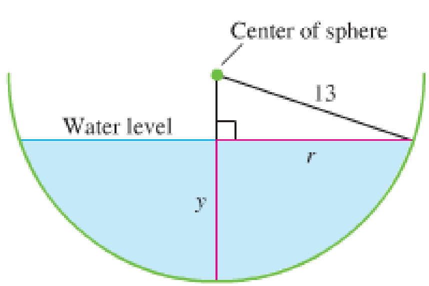
$$\frac{dx}{dt} = \frac{D}{x}\frac{dD}{dt} = \frac{\sqrt{20^2+144}}{20}(-4)$$

$$\boxed{\approx -4.66\ \frac{in}{s}}$$

The fish is moving toward the fisherman at about 4.66 inches per second.

Exercise

Water is flowing at the rate of 6 from a reservoir shaped like a hemispherical bowl of radius 13 ***m***. Answer the following questions, given that the volume of water in a hemispherical bowl of radius R is $V = \frac{\pi}{3}y^2(3R-y)$ when the water is y meters deep.



a) At what rate the water level changing when the water is 8 ***m*** deep?

b) What is the radius r of the water's surface when the water is y ***m*** deep?

c) At what rate is the radius r changing when the water is 8 ***m*** deep?

Solution

Given: $\frac{dV}{dt} = 6\ m^3 / \min \quad R = 13\ m$

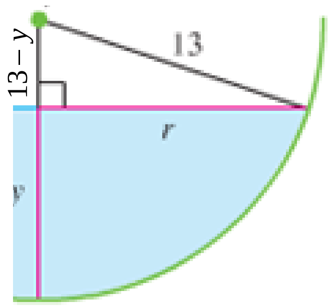
a) $V = \frac{\pi}{3} y^2 (3R - y) = \pi R y^2 - \frac{\pi}{3} y^3$

$$\frac{dV}{dt} = \left(2\pi R y - \pi y^2\right)\frac{dy}{dt} \qquad\qquad \textit{Factor } \pi y$$

$$\frac{dV}{dt} = \pi y(2R - y)\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{\pi(8)(2(13)-(8))}(-6) = \underline{-\frac{1}{24\pi}\ m / \min}$$

b) The hemispherical is on the circle: $r^2 + (13 - y)^2 = 13^2$



$$r^2 = 169 - \left(169 - 26y + y^2\right)$$

$$= 169 - 169 + 26y - y^2$$

$$= 26y - y^2$$

$$\boxed{r = \sqrt{26y - y^2}}$$

c) $r = \left(26y - y^2\right)^{1/2} \quad \Rightarrow \quad \frac{dr}{dt} = \frac{1}{2}\left(26y - y^2\right)^{-1/2}(26 - 2y)\frac{dy}{dt}$

$$= \frac{1}{2}\frac{26 - 2y}{\sqrt{26y - y^2}}\frac{dy}{dt}$$

$$\left.\frac{dr}{dt}\right|_{y=8} = \frac{1}{2}\frac{26 - 2(8)}{\sqrt{26(8) - (8)^2}}\left(-\frac{1}{24\pi}\right)$$

$$= -\frac{5}{288\pi}$$

$$= \underline{0.005526} \quad \textit{or} \quad \boxed{5.526 \times 10^{-3}}$$

Exercise

A spherical balloon is inflated with helium at the rate of $100\pi \ ft^3/\min$. How fast is the balloon's radius increasing at the instant the radius is 5 ft? How fast the surface area increasing?

Solution

Given: $\frac{dV}{dt} = 100\pi \ ft^3/\min \quad r = 5 \ ft$

If $V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

$$= \frac{1}{4\pi(5)^2}(100\pi)$$

$$\boxed{= 1 \ ft/\min}$$

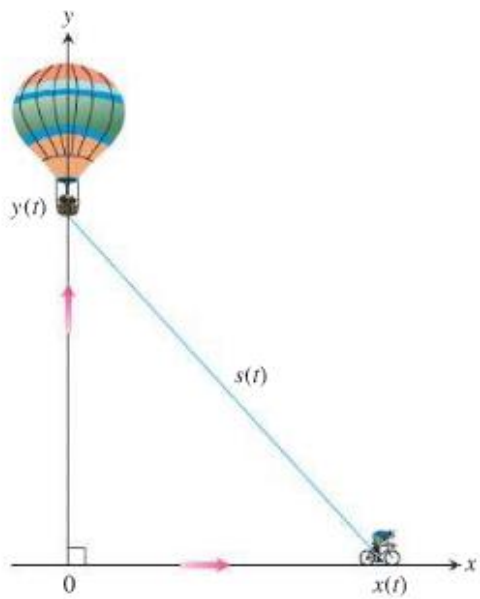
$$S = 4\pi r^2 \quad \Rightarrow \quad \frac{dS}{dt} = 8\pi r\frac{dr}{dt} = 8\pi(5)(1) = \boxed{40\pi \ ft^2/\min}$$

The rate of the surface area is increasing.

Exercise

A balloon rising vertically above a level, straight road at a constant rate of 1 *ft/sec*. Just when the balloon is 65 *ft* above the ground, a bicycle moving at a constant rate of 17 *ft/sec* passes under it. How fast is the distance $s(t)$ between the bicycle and the balloon increasing 3 sec later?

Solution



Given: $\frac{dy}{dt} = 1 \ ft/\sec \quad y = 65 \ ft \quad \frac{dx}{dt} = 17 \ ft/\sec$

Bicycle increasing 3 sec: $x = vt = 17(3) = \boxed{51 \ ft}$

$$s^2 = x^2 + y^2 \quad \Rightarrow \quad \boxed{s = \sqrt{51^2 + 65^2} \approx 83 \ ft}$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$= \frac{1}{83}\left(51(17) + 65(1)\right)$$

$$\boxed{\approx 11 \ ft/\sec}$$

Exercise

An observer stands 300 *ft* from the launch site of a hot-air balloon. The balloon is launched vertically and maintains a constant upward velocity of 20 *ft/sec*. what is the rate of change of the angle of elevation of the balloon when it is 400 *ft* from the ground? The angle of elevation is the angle θ between the observer's line of sight to the balloon and the ground.

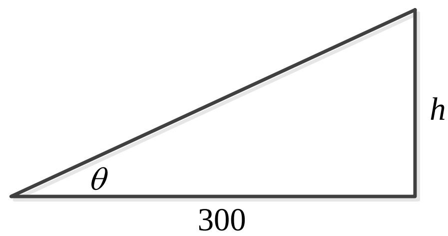
Solution

Given: $\frac{dh}{dt} = 20\ ft/s, \quad h = 400\ ft$

$$\tan\theta = \frac{h}{300} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{h}{300}\right)$$

$$\frac{d\theta}{dt} = \frac{1}{300\left(1+\left(\frac{h}{300}\right)^2\right)}\frac{dh}{dt} = \frac{1}{300\left(1+\left(\frac{400}{300}\right)^2\right)}(20)$$

$$\approx .024\ \ rad/sec$$



Exercise

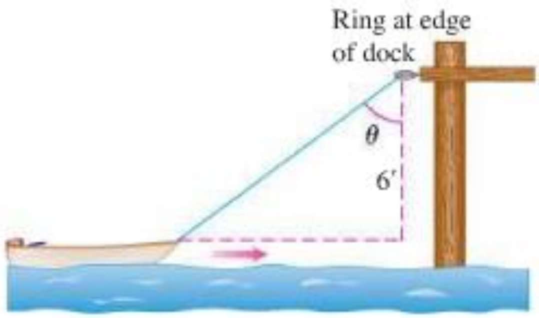
A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 *ft* above the bow. The rope is hauled in at rate of 2 *ft/sec*.

a) How fast is the boat approaching the dock when 10 *ft* of rope are out?

b) At what rate is the angle θ changing at this instant?

Solution

Given: $h = 6\ \ ft \quad \frac{ds}{dt} = -2\ \ ft/\sec$



a) $s = 10\,ft$

$$s^2 = x^2 + 6^2 \quad \Rightarrow \quad x = \sqrt{s^2 - 36}$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{s}{\sqrt{s^2-36}}\frac{ds}{dt}$$

$$\left.\frac{dx}{dt}\right|_{s=10} = \frac{10}{\sqrt{10^2-36}}(-2) = -2.5\ \ ft/\sec$$

b) $\cos\theta = \frac{6}{s} \quad \Rightarrow \quad -\sin\theta\frac{d\theta}{dt} = -\frac{6}{s^2}\frac{ds}{dt}$

$$\frac{d\theta}{dt} = \frac{6}{\sin\theta s^2}\frac{ds}{dt}$$

$$\sin\theta = \frac{x}{s} = \frac{\sqrt{10^2-36}}{10} = \frac{8}{10}$$

$$\frac{d\theta}{dt} = \frac{6}{(.8)10^2}(-2) = -0.15\ \ rad/\sec$$

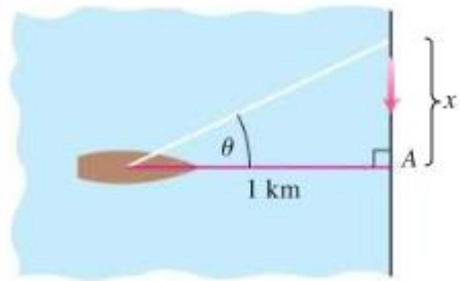
Exercise

The figure shows a boat 1 *km* offshore, sweeping the shore with a searchlight. The light turns at a constant rate $\frac{d\theta}{dt}=-0.6\ rad/\sec$.

a) How fast is the light moving along the shore when it reaches point *A*?

b) How many revolutions per minute is 0.6 *rad/sec*?

Solution



Given: $\frac{d\theta}{dt}=-0.6\ rad/\sec$

$$\tan\theta=\frac{x}{1} \quad\Rightarrow\quad \sec^2\theta\frac{d\theta}{dt}=\frac{dx}{dt}$$

a) At $x=0 \quad\Rightarrow\quad \theta=0$

$$\frac{dx}{dt}=\sec^2(0)(-0.6)=-0.6$$

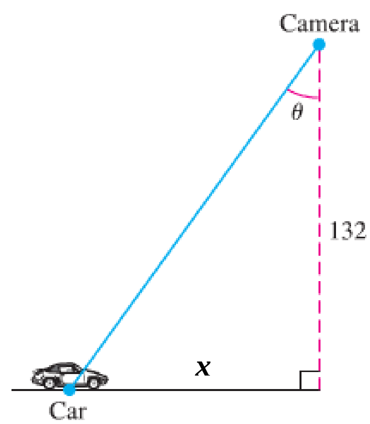
$\therefore$ The speed of the light is 0.6 *km/sec* when it reaches point *A*.

b) $0.6\frac{rad}{sec}\frac{1\ rev}{2\pi\ rad}\frac{60\ sec}{1\ min}\ \underline{=\frac{18}{\pi}\ \frac{rev}{min}|}$

Exercise

You are videotaping a race from a stand 132 *ft* from the track, following a car that is moving at 180 *mi/h* (264 *ft/sec*). How fast will your camera angle θ be changing when the car is right in front of you? A half second later?

Solution



$$\tan\theta=\frac{x}{132} \quad\Rightarrow\quad \sec^2\theta\frac{d\theta}{dt}=\frac{1}{132}\frac{dx}{dt}$$

$$\frac{d\theta}{dt}=\frac{1}{132\sec^2\theta}\frac{dx}{dt}$$

$$=\frac{1}{132\sec^2(0)}(-264)$$

$$\underline{=-2\ \ rad/\sec|}$$

At half second later the car has traveled 132 *ft* right to the perpendicular

$|\theta|=\frac{\pi}{4} \quad\to\quad \cos^2\theta=\frac{1}{2}$, and $\frac{dx}{dt}=264$ (since *x* increases)

$$\frac{d\theta}{dt}=\frac{1}{132(2)}(264)\ \underline{=1\ \ rad/\sec|}$$

Exercise

The coordinates of a particle in the metric xy–plane are differentiable functions of time t with $\frac{dx}{dt}=-1\ m/\sec$ and $\frac{dy}{dt}=-5\ m/\sec$. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)?

Solution

Given: $\frac{dx}{dt}=-1\ \ m/\sec \quad \frac{dy}{dt}=-5\ \ m/\sec$

$s^2=x^2+y^2 \qquad\qquad \Rightarrow \ \underline{s=\sqrt{x^2+y^2}=\sqrt{5^2+12^2}=13}$

$2s\frac{ds}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt}$

$\frac{ds}{dt}=\frac{1}{s}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)$

$\left.\frac{ds}{dt}\right|_{(5,12)}=\frac{1}{13}\left(5(-1)+12(-5)\right)=\underline{-5\ \ m/\sec}$

Exercise

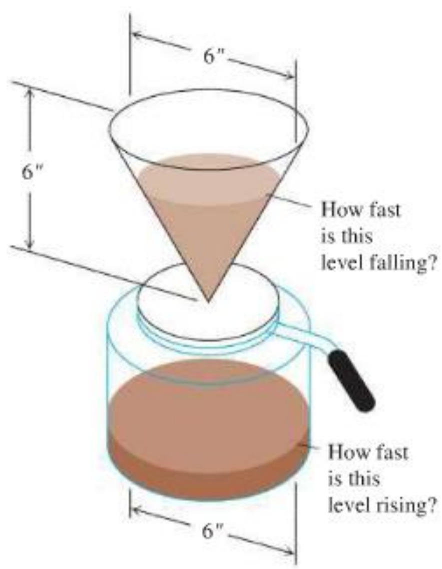
Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 $in^3/\min$.

a) How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?

b) How fast is the level in the cone falling then?

Solution

$r_{pot}=3 \quad \frac{dV}{dt}=10\ \ in^3/\min$



a) Let h be the height of the coffee in the pot.

Volume of the coffee: $V=\pi r^2h=9\pi h$

$\frac{dV}{dt}=9\pi\frac{dh}{dt}$

$\frac{dh}{dt}=\frac{1}{9\pi}\frac{dV}{dt}=\frac{1}{9\pi}(10)=\underline{\frac{10}{9\pi}\ \ in/\min}$

b) Radius of the filter: $r=\frac{h}{2}$

Volume of the filter: $V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2h=\frac{\pi h^3}{12}$

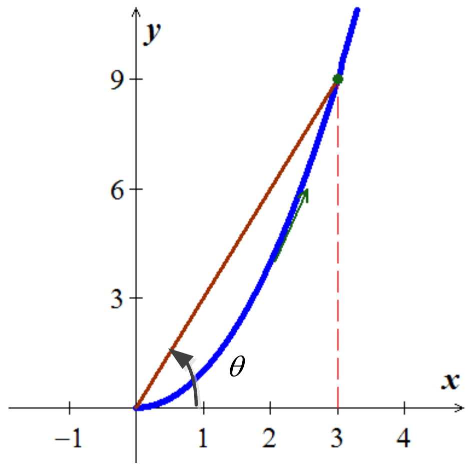
$\frac{dV}{dt}=\frac{\pi}{4}h^2\frac{dh}{dt}$

$\frac{dh}{dt}=\frac{4}{\pi h^2}\frac{dV}{dt}=\frac{4}{\pi(5)^2}(-10)=\underline{-\frac{8}{5\pi}\ \ in/\min}$

Exercise

A particle moves along the parabola $y = x^2$ *in* the first quadrant in such a way that its *x*-coordinate (measure in meters) increases at a steady 10 *m/sec*. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3$ *m*?

Solution



Given: $y = x^2 \quad v = \dfrac{dx}{dt} = 10 \ m/\sec \quad x = 3\,m$

$$\tan\theta = \frac{y}{x} = \frac{x^2}{x} = x$$

$$\frac{d}{dt}\tan\theta = \frac{d}{dt}x$$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{dx}{dt}$$

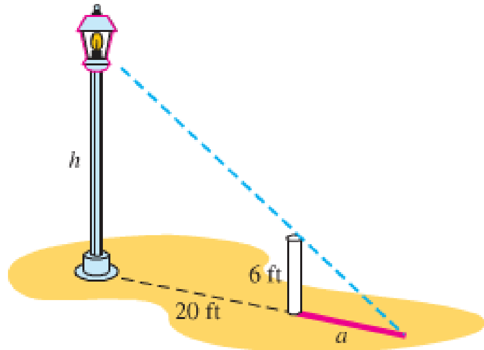
$$\frac{d\theta}{dt} = \frac{1}{\sec^2\theta}\frac{dx}{dt} = \cos^2\theta\frac{dx}{dt}$$

$$= \left(\frac{3}{\sqrt{9^2+3^2}}\right)^2(10)$$

$$\underline{= 1 \ \ rad/\sec|}$$

Exercise

To find the height of a lamppost, you stand a 6 *ft* pole 20 *ft* from the lamp and measure the length *a* of its shadow, finding it to be 15 *ft*, give or take an inch. Calculate the height of the lamppost using the value of *a* = 15 and estimate the possible error in the result.



Solution

$$\frac{h}{6} = \frac{20+a}{a} = \frac{35}{15} = \frac{7}{3} \quad \Rightarrow \quad \underline{h = 14 \ ft|}$$

$$\frac{h}{6} = \frac{20+a}{a} \quad \Rightarrow \ ah = 120 + 6a \ \rightarrow \ h = 6 + 120a^{-1}$$

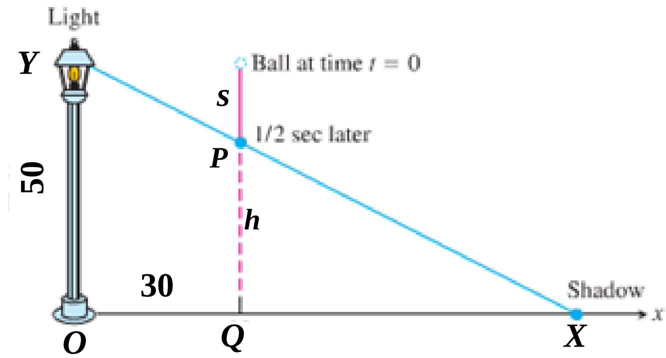
$$\frac{dh}{dt} = -120a^{-2}\frac{da}{dt} \quad \Rightarrow \quad dh = -\frac{120}{a^2}da = -\frac{120}{15^2}\left(\pm\frac{1}{12}\right)$$

$$dh = -\frac{120}{a^2}da = -\frac{120}{15^2}\left(\pm\frac{1}{12}\right) \ \underline{\approx \pm 0.044 \quad ft|}$$

Exercise

A light shines from the top of a pole 50 *ft* high. A ball is dropped from the same height from a point 30 *ft* away from the light. How fast is the shadow of the ball moving along the ground $\frac{1}{2}$ *sec* later? (Assume the ball falls a distance $s = 16t^2$ *ft* in *t sec*.)

Solution



$$s = 16t^2$$

$$s + h = 50$$

Triangles *XOY* and *XQP* are similar:

$$\therefore \frac{XQ}{h} = \frac{OX}{50} = \frac{30 + XQ}{50}$$

$$50|XQ| = 30h + h|XQ|$$

$$(50 - h)|XQ| = 30h$$

$$\begin{aligned}
|XQ| &= \frac{30h}{50 - h} \\
&= \frac{30(50 - s)}{50 - (50 - s)} \\
&= \frac{30\left(50 - 16t^2\right)}{50 - 50 + 16t^2} \\
&= \frac{1500 - 480t^2}{16t^2} \\
&= \frac{1500}{16t^2} - \frac{480t^2}{16t^2} \\
&= \frac{1500}{16t^2} - 30
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}|XQ| &= 1500\frac{-32t}{\left(16t^2\right)^2} \\
&= 1500\frac{-32t}{256t^4} \\
&= -\frac{375}{2t^3}
\end{aligned}$$

$$\left.\frac{d}{dt}|XQ|\right|_{t=\frac{1}{2}} = -\frac{375}{2\left(\frac{1}{2}\right)^3} = \underline{-1500 \;\; ft/\sec|}$$

Exercise

A spherical iron ball 8 *in*. in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of 10 $in^3/\min$, how fast is the thickness of the ice decreasing when it is 2 *in*. thick? How fast is the outer surface area of ice decreasing?

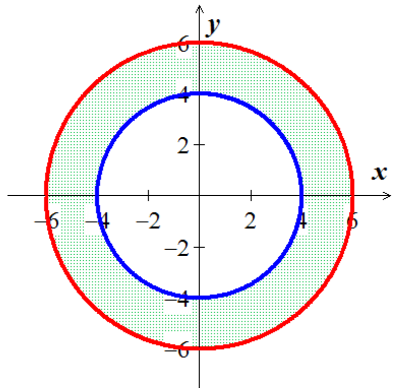
Solution

Given: $D = 8\ in \to r_1 = 4\ in \quad \frac{dV}{dt} = -10\ \text{ in}^3/\min \quad think = 2\ \text{in}$

$$V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

$$\left.\frac{dr}{dt}\right|_{r=6} = \frac{1}{4\pi(6)^2}(-10) = -\frac{5}{72\pi}\ in/\min$$



$$S = 4\pi r^2$$

$$\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$$

$$\left.\frac{dS}{dt}\right|_{r=6} = 8\pi(6)\left(-\frac{5}{72\pi}\right) = \boxed{-\frac{10}{3}\ in^2/\min}$$

The outer surface are of the ice is decreasing at $-\frac{10}{3}\ in^2/\min$

Exercise

On a morning of a day when the sun will pass directly overhead, the shadow of an 80–*ft* building on level ground is 60 *ft* long. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of 0.27 °/ *min*. At what rate is the shadow decreasing?

Solution

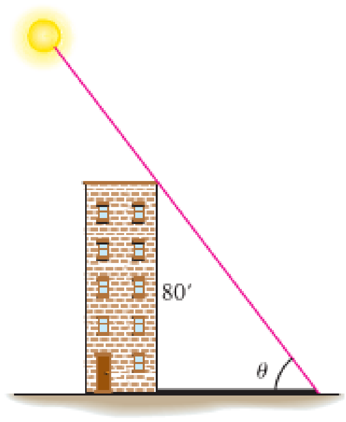
Given: $x = 60\ ft \quad h = 80\ ft$

$$\boxed{\frac{d\theta}{dt} = 0.27^\circ \min = 0.27^\circ \frac{\pi rad}{180^\circ}\frac{1}{\min} = \frac{3\pi}{2000} rad/\min}$$

$$\tan\theta = \frac{80}{x} \quad \Rightarrow \quad \frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{80}{x}$$

$$\sec^2\theta\frac{d\theta}{dt} = -\frac{80}{x^2}\frac{dx}{dt}$$

$$\frac{dx}{dt} = \left|-\frac{x^2\sec^2\theta}{80}\frac{d\theta}{dt}\right| \qquad \cos\theta = \frac{60}{\sqrt{60^2+80^2}} = \frac{60}{100} = \frac{3}{5}$$



$$= \frac{60^2\left(\frac{5}{3}\right)^2}{80}\left(\frac{3\pi}{2000}\right)$$

$$\boxed{= 0.589\ ft/\min}$$

Exercise

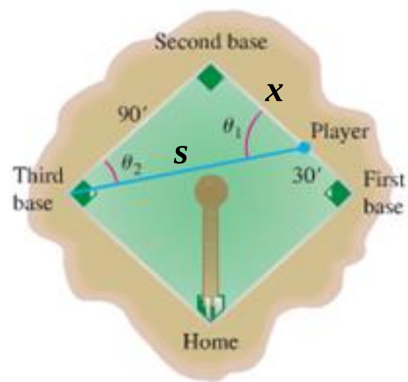
A baseball diamond is a square 90 *ft* on a side. A player runs from first base to second at a rate of 16 *ft/sec*.

a) At what rate is the player's distance from third base changing when the player is 30 *ft* from first base?

b) At what rates are angles θ_1 *and* θ_2 changing at that time?

c) The player slides into second base at the rate of 15 *ft/sec*. At what rates are angles θ_1 *and* θ_2 changing as the player touches base?

Solution



Given: $d_1 = 90\ ft \quad d_2 = 30\ ft \quad \frac{dx}{dt} = -16\ \ ft/\sec$

x: Distance between player and 2^{nd} base

s: Distance between player and 3^{rd} base

a) $x = 90 - 30 = 60\ ft$

$$s^2 = x^2 + 90^2 \quad \to \quad s = \sqrt{60^2 + 90^2} = \sqrt{11700} = 30\sqrt{13}$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt}$$

$$\frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}$$

$$= \frac{60}{30\sqrt{13}}(-16)$$

$$\underline{\approx -8.875\ ft/\sec}$$

b) $\sin\theta_1 = \frac{90}{s} \quad \to \quad \cos\theta_1 \frac{d\theta_1}{dt} = -\frac{90}{s^2}\frac{ds}{dt}$

$$\frac{d\theta_1}{dt} = -\frac{90}{s^2\cos\theta_1}\frac{ds}{dt} \qquad\qquad \cos\theta_1 = \frac{x}{s}$$

$$\frac{d\theta_1}{dt} = -\frac{90}{s^2\frac{x}{s}}\frac{ds}{dt}$$

$$= -\frac{90}{s\cdot x}\frac{ds}{dt}$$

$$= -\frac{90}{30\sqrt{13}(60)}(-8.875)$$

$$\underline{\approx 0.123\ rad/\sec}$$

$$\cos\theta_2 = \frac{90}{s} \quad \to \quad -\sin\theta_2\frac{d\theta_2}{dt} = -\frac{90}{s^2}\frac{ds}{dt}$$

$$\frac{d\theta_2}{dt} = \frac{90}{s^2 \sin\theta_2}\frac{ds}{dt} = \frac{90}{s\cdot x}\frac{ds}{dt} \qquad \sin\theta_2 = \frac{x}{s}$$

$$= \frac{90}{30\sqrt{13}(60)}(-8.875)$$

$$\boxed{\approx -0.123\ rad/\sec}$$

c) $$\frac{d\theta_1}{dt} = -\frac{90}{s^2\cos\theta_1}\frac{ds}{dt} \qquad \frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}$$

$$= -\frac{90}{s^2\frac{x}{s}}\frac{x}{s}\frac{dx}{dt}$$

$$= -\frac{90}{s^2}\frac{dx}{dt}$$

$$= -\frac{90}{x^2+8100}\frac{dx}{dt}$$

Player slides into second base $\Rightarrow x = 0$

$$\left.\frac{d\theta_1}{dt}\right|_{x=0} = -\frac{90}{0^2+8100}(-15) = \boxed{\frac{1}{6}\ rad/\sec}$$

$$\frac{d\theta_2}{dt} = \frac{90}{s^2\sin\theta_2}\frac{ds}{dt} = \frac{90}{s^2\frac{x}{s}}\frac{x}{s}\frac{dx}{dt} = \frac{90}{s^2}\frac{dx}{dt}$$

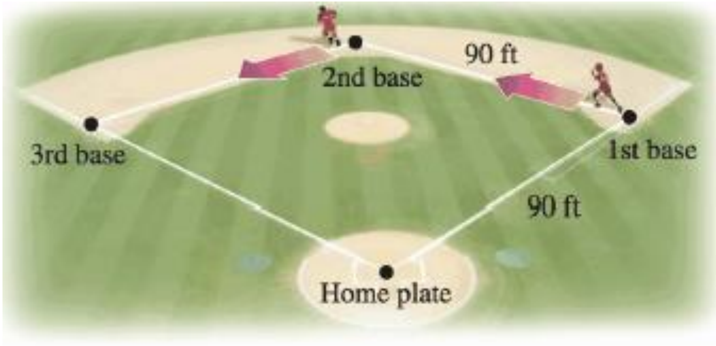
$$= \frac{90}{x^2+8100}\frac{dx}{dt}$$

Player slides into second base $\Rightarrow x = 0$

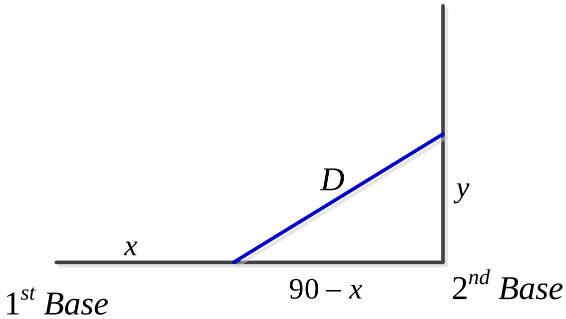
$$\left.\frac{d\theta_2}{dt}\right|_{x=0} = \frac{90}{0^2+8100}(-15) = \boxed{-\frac{1}{6}\ rad/\sec}$$

Exercise

Runners stand at first and second base in a baseball game. At the moment a ball is hit the runner at first base runs to second base at 18 *ft/s*; simultaneously the runner on second runs to third base at 20 *ft/s*. How fast is the distance between the runners changing 1 *sec* after the ball is hit?
(*Hint*: The distance between consecutive bases I 90 *ft* and the bases lie at the corners of a square.)



Solution



Given: $\frac{dx}{dt}=18\ ft/s,\quad \frac{dy}{dt}=20\ ft/s$

After 1 *sec*, $x=18$ *and* $y=20$

$$D^2=(90-x)^2+y^2$$

$$D=\sqrt{(90-18)^2+20^2}=\sqrt{5584}\approx 74.726$$

$$2D\frac{dD}{dt}=-2(90-x)\frac{dx}{dt}+2y\frac{dy}{dt}$$

$$D\frac{dD}{dt}=-(90-x)\frac{dx}{dt}+y\frac{dy}{dt}$$

$$\frac{dD}{dt}=\frac{-(90-18)(18)+20(20)}{74.726}\approx -11.99\ ft/\sec$$

So the distance between the runners is decreasing at a rate about 11.99 feet per second.