Section 3.4 – Using Laplace Transform to Solve Differential Equations

Homogeneous Equations

Example

Use Laplace transform to find the solution to the initial value problem

$$y'' - 2y' - 3y = 0 \qquad y(0) = 1 \quad \text{and} \quad y'(0) = 0$$

Solution

$$\begin{aligned}\mathcal{L}\left(y'' - 2y' - 3y\right) &= s^2 Y(s) - sy(0) - y'(0) - 2\left(sY(s) - y(0)\right) - 3Y(s) \\ &= \left(s^2 - 2s - 3\right)Y(s) - s + 2 \\ &= 0\end{aligned}$$

$$\begin{aligned}Y(s) &= \frac{s-2}{s^2 - 2s - 3} \\ &= \frac{A}{s-3} + \frac{B}{s+1} \\ &= \frac{(A+B)s + A - 3B}{(s-3)(s+1)} \qquad \begin{cases} A + B = 1 \\ A - 3B = -2 \end{cases} \Rightarrow A = \frac{1}{4}, \ \ B = \frac{3}{4} \\ &= \frac{1}{4}\frac{1}{s-3} + \frac{3}{4}\frac{1}{s+1}\end{aligned}$$

$$\begin{aligned}y(t) &= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ &= \frac{1}{4}e^{3t} + \frac{3}{4}e^{-t}\end{aligned}$$

Inhomogeneous **Equations**

Example

Use Laplace transform to find the solution to the initial value problem

$$y'' + 2y' + 2y = \cos 2t \qquad y(0) = 0 \quad \textit{and} \quad y'(0) = 1$$

Solution

$$\mathcal{L}\left(y'' + 2y' + 2y\right) = s^2 Y(s) - sy(0) - y'(0) + 2\left(sY(s) - y(0)\right) + 2Y(s)$$

$$= \left(s^2 + 2s + 2\right)Y(s) - 1$$

$$\mathcal{L}\left(\cos 2t\right) = \frac{s}{s^2 + 4}$$

$$\left(s^2 + 2s + 2\right)Y(s) - 1 = \frac{s}{s^2 + 4}$$

$$\left(s^2 + 2s + 2\right)Y(s) = \frac{s}{s^2 + 4} + 1$$

$$= \frac{s^2 + s + 4}{s^2 + 4}$$

$$Y(s) = \frac{s^2 + s + 4}{\left(s^2 + 4\right)\left(s^2 + 2s + 2\right)}$$

$$= \frac{As + B}{\left(s^2 + 2s + 2\right)} + \frac{Cs + D}{\left(s^2 + 4\right)}$$

$$= \frac{(A + C)s^3 + \left(2C + B + D\right)s^2 + \left(4A + 2C + 2D\right)s + 4B + 2D}{\left(s^2 + 2s + 2\right)\left(s^2 + 4\right)}$$

$$\begin{cases} A + C = 0 \\ B + 2C + D = 1 \\ 4A + 2C + 2D = 1 \\ 4B + 2D = 4 \end{cases} \rightarrow \begin{matrix} A = \frac{1}{10} & B = \frac{4}{5} = \frac{8}{10} \\ C = -\frac{1}{10} & D = \frac{2}{5} = \frac{4}{10} \end{matrix}$$

$$= \frac{1}{10}\frac{s + 8}{\left((s+1)^2 + 1\right)} - \frac{1}{10}\frac{s - 4}{\left(s^2 + 4\right)}$$

$$= \frac{1}{10}\frac{s + 1 + 7}{\left((s+1)^2 + 1\right)} - \frac{1}{10}\frac{s - 4}{\left(s^2 + 4\right)}$$

$$= \frac{1}{10}\frac{s + 1}{\left((s+1)^2 + 1\right)} + \frac{7}{10}\frac{1}{\left((s+1)^2 + 1\right)} - \frac{1}{10}\frac{s}{\left(s^2 + 4\right)} + \frac{1}{10}\frac{4}{\left(s^2 + 4\right)}$$

$$y(t) = \frac{1}{10}e^{-t}\cos t + \frac{7}{10}e^{-t}\sin t - \frac{1}{10}\cos 2t + \frac{1}{10}2\sin 2t$$

$$\underline{y(t) = \frac{1}{10}\left(e^{-t}\left(\cos t + 7\sin t\right) + 2\sin 2t - \cos 2t\right)}$$

Higher-Order Equations

Example

Find the solution to the initial value problem

$$y^{(4)} - y = 0 \qquad y(0) = 0,\ y'(0) = 1,\ y''(0) = 0,\ and\ y'''(0) = 0$$

Solution

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - Y(s) = 0$$

$$\left(s^4 - 1\right)Y(s) - s^2 = 0$$

$$\left(s^4 - 1\right)Y(s) = s^2$$

$$Y(s) = \frac{s^2}{s^4 - 1}$$

$$= \frac{s^2}{(s-1)(s+1)\left(s^2+1\right)}$$

$$= \frac{A}{(s-1)} + \frac{B}{(s+1)} + \frac{Cs+D}{\left(s^2+1\right)}$$

$$= \frac{As^3+As^2+As+A+Bs^3-Bs^2+Bs-B+Cs^3-Cs+Ds^2-D}{(s-1)(s+1)\left(s^2+1\right)}$$

$$= \frac{(A+B+C)s^3+(A-B+D)s^2+(A+B-C)s+A-B-D}{(s-1)(s+1)\left(s^2+1\right)}$$

$$\begin{cases} A+B+C=0 \\ A-B+D=1 \\ A+B-C=0 \\ A-B-D=0 \end{cases} \Rightarrow \begin{cases} 2A+2B=1 \\ 2A-2B=0 \end{cases} \Rightarrow \begin{cases} \boxed{A=\frac{1}{4}} \\ \boxed{B=-\frac{1}{4}} \end{cases} \rightarrow \begin{matrix} \boxed{C=0} \\ \boxed{D=\frac{1}{2}} \end{matrix}$$

$$Y(s) = \frac{1}{4}\frac{1}{s-1} - \frac{1}{4}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s^2+1}$$

$$\boxed{y(t) = \frac{1}{4}e^{t} - \frac{1}{4}e^{-t} + \frac{1}{2}\sin t}$$

Electrical Circuit

Resistor

Laplace Transform

$$I_R(s).R = V_R(s)$$

$$V_R = RI$$

The block diagram:

System

$$\textit{Input Signal} \longrightarrow \boxed{R} \longrightarrow \textit{Ouput Signal}\; V_R(s) = I_R(s).R$$

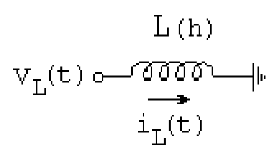
$$\Updownarrow$$

$$V_R(s) \longrightarrow \boxed{\frac{1}{R}} \longrightarrow I_R(s) = \frac{V_R(s)}{R}$$

Inductor

Laplace Transform

$$v_L(t) = L\frac{di_L(t)}{dt} \overset{\pounds}{\leftrightarrow} V_L(s) = L\left(sI_L(s) - I_L(0)\right)$$



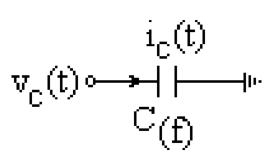
The block diagram:

$$I_L(s) \longrightarrow \boxed{sL} \longrightarrow V_L(s) = sL.I_L(s)$$

Capacitance

Laplace Transform

$$i_c(t) = C\frac{dV_c(t)}{dt} \overset{\pounds}{\leftrightarrow} I_c(s) = Cs.V_c(s)$$



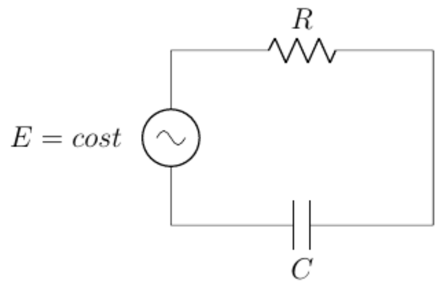
$$v_C(0) = E_0(0)$$

$$V_C(s) \longrightarrow \boxed{Cs} \longrightarrow I_C(s) \quad \Longleftrightarrow \quad I_C(s) \longrightarrow \boxed{\frac{1}{Cs}} \longrightarrow V_C(s)$$

Example

Suppose the electrical circuit has a resistor of $R = 2\Omega$ and a capacitor of $C = \frac{1}{5}F$. Assume the voltage source is $E = \cos t\ (V)$. If the initial current is 0 *A*, find the resulting current.

Solution



$$R\frac{dQ}{dt} + \frac{1}{C}Q = E$$

$$2Q' + 5Q = \cos t$$

$$\mathcal{L}(2Q' + 5Q) = \mathcal{L}(\cos t)$$

$$2sQ(s) - 2Q(0) + 5Q(s) = \frac{s}{s^2+1}$$

$$(2s+5)Q(s) = \frac{s}{s^2+1}$$

$$2\left(s+\tfrac{5}{2}\right)Q(s) = \frac{s}{s^2+1}$$

$$Q(s) = \frac{1}{2}\frac{s}{\left(s+\frac{5}{2}\right)\left(s^2+1\right)}$$

$$\frac{s}{\left(s+\frac{5}{2}\right)\left(s^2+1\right)} = \frac{A}{s+\frac{5}{2}} + \frac{Bs+C}{s^2+1}$$

$$s = As^2 + A + Bs^2 + \tfrac{5}{2}Bs + Cs + \tfrac{5}{2}C$$

$$s = (A+B)s^2 + \left(\tfrac{5}{2}B + C\right)s + A + \tfrac{5}{2}C$$

$$\begin{cases} A + B = 0 \\ \frac{5}{2}B + C = 1 \\ A + \frac{5}{2}C = 0 \end{cases} \Rightarrow A = -\frac{10}{29} \quad B = \frac{10}{29} \quad C = \frac{4}{29}$$

$$Q(s) = \frac{1}{2}\left(-\frac{10}{29}\frac{1}{s+\frac{5}{2}} + \frac{10}{29}\frac{s}{s^2+1} + \frac{4}{29}\frac{1}{s^2+1}\right)$$

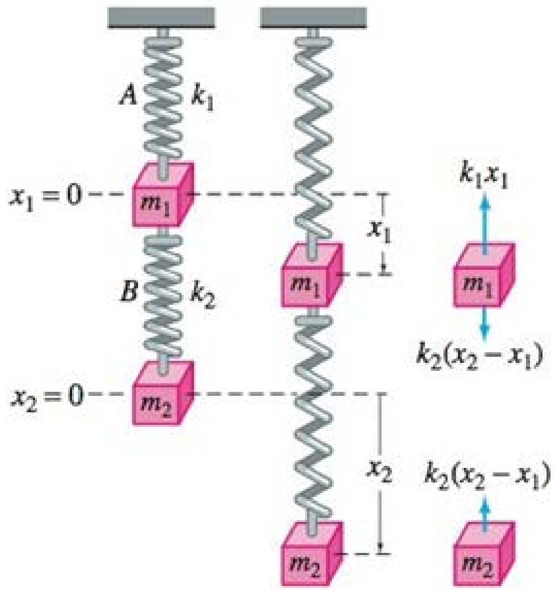
$$= \frac{1}{29}\left(-5\frac{1}{s+\frac{5}{2}} + 5\frac{s}{s^2+1} + 2\frac{1}{s^2+1}\right)$$

$$Q(t) = \frac{1}{29}\left(-5\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{5}{2}}\right\} + 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right)$$

$$\underline{= \frac{1}{29}\left(-5e^{-5t/2} + 5\cos t + 2\sin t\right)}$$

Springs-Masses

Two masses m_1 and m_2 are connected to two springs *A* and *B* of negligible mass having sprig constants k_1 and k_2, respectively.



Let $x_1(t)$ and $x_2(t)$ denote the vertical displacements of the masses from their equilibrium positions. When the system is in motion, spring *B* is subject to both an elongation and a compression; hence its net elongation is $x_2 - x_1$. Therefore, it follows from Hooke's law that springs *A* and *B* exert forces $-k_1x_1$ and $k_2(x_2 - x_1)$, respectively, on m_1.

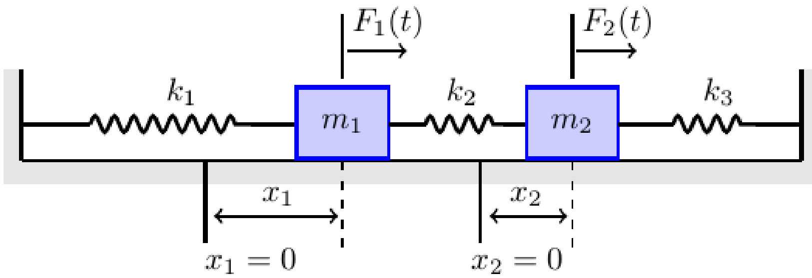
If no external force is impressed on the system and if no damping force is present, then the net force on m_1 is $-k_1x_1 + k_2(x_2 - x_1)$.

By Newton's second law we can write

$$m_1x_1'' = -k_1x_1 + k_2(x_2 - x_1)$$
$$m_2x_2'' = -k_2(x_2 - x_1)$$

Example

Write down the system of differential equations for the spring and mass system as shown below. Both masses moves to the right of their equilibrium points. The mass m_1 moves farther than m_2.



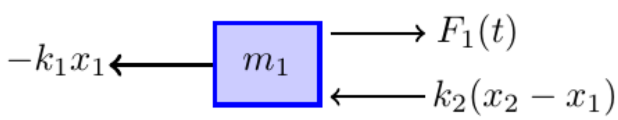
Solution

Let $x_1(t)$ and $x_2(t)$ denote the horizontal displacements of the masses from their equilibrium positions.

If both masses move the same amount in the same direction then the middle spring will not have changed length and it will result $x_2 - x_1 = 0$.

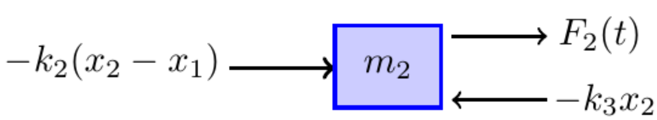
If both masses move in the positive direction then if m_1 moves more than m_2 then $x_2 - x_1 < 0$, the spring will be compressed. If m_2 moves more than m_1 then $x_2 - x_1 > 0$ (stretched).

At mass m_1 with $x_1 > 0$:



$$m_1x_1'' = -k_1x_1 + k_2\left(x_2 - x_1\right) + F_1(t)$$

At mass m_2:



$$m_2x_2'' = -k_3x_2 - k_2\left(x_2 - x_1\right) + F_2(t)$$

Exercises *Section* 3.4 - Using Laplace Transform to Solve Differential Equations

Solve using the Laplace transform:

1. $y' + y = te^t, \quad y(0) = -2$
2. $y' - y = 2\cos 5t, \quad y(0) = 0$
3. $y' - y = 1 + te^t, \quad y(0) = 0$
4. $y' + 3y = e^{2t}, \quad y(0) = -1$
5. $y' + 4y = \cos t, \quad y(0) = 0$
6. $y' + 4y = e^{-4t}, \quad y(0) = 2$
7. $y' - 4y = t^2 e^{-2t}, \quad y(0) = 1$
8. $y' + 9y = e^{-t}, \quad y(0) = 0$
9. $y' + 16y = \sin 3t, \quad y(0) = 1$

10. $y'' - y = e^{2t}; \quad y(0) = 0, \quad y'(0) = 1$
11. $y'' - y = 2t; \quad y(0) = 0, \quad y'(0) = -1$
12. $y'' - y = t - 2\ ; \quad y(2) = 3, \quad y'(2) = 0$
13. $y'' + y = t\ ; \quad y(\pi) = y'(\pi) = 0$
14. $y'' - 2y' + 5y = -8e^{\pi - t}\ ; \quad y(\pi) = 2, \ y'(\pi) = 12$
15. $y'' + y = t^2 + 2\ ; \quad y(0) = 1, \quad y'(0) = -1$
16. $y'' + y = \sqrt{2}\sin\sqrt{2}t; \quad y(0) = 10, \quad y'(0) = 0$
17. $y'' + y = -2\cos 2t\ ; \quad y(0) = 1, \quad y'(0) = -1$
18. $y'' - y' = e^t \cos t\ ; \quad y(0) = 0, \quad y'(0) = 0$
19. $y'' + y' - y = t^3\ ; \quad y(0) = 1, \quad y'(0) = 0$
20. $y'' - y' - 2y = 4t^2, \quad y(0) = 1, \quad y'(0) = 4$
21. $y'' - y' - 2y = e^{2t}\ ; \quad y(0) = -1, \quad y'(0) = 0$
22. $y'' - y' - 2y = 0, \quad y(0) = -2, \quad y'(0) = 5$
23. $y'' - y' - 2y = -8\cos t - 2\sin t\ ; \quad y\left(\frac{\pi}{2}\right) = 1, \quad y'\left(\frac{\pi}{2}\right) = 0$
24. $x'' - x' - 6x = 0\ ; \quad x(0) = 2, \quad x'(0) = -1$
25. $y'' + 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 1$
26. $y'' + 2y' + y = t\ , \quad y(0) = -3, \quad y(1) = -1$
27. $y'' - 2y' - y = e^{2t} - e^t\ ; \quad y(0) = 1, \quad y'(0) = 3$
28. $y'' - 2y' + y = 6t - 2\ ; \quad y(-1) = 3, \quad y'(-1) = 7$
29. $y'' - 2y' + y = \cos t - \sin t\ ; \quad y(0) = 1, \quad y'(0) = 3$
30. $y'' - 2y' + 5y = 0, \quad y(0) = 2, \quad y'(0) = 4$
31. $y'' - 2y' + 5y = 1 + t, \quad y(0) = 0, \quad y'(0) = 0$

32. $y'' + 3y' = -3t$; $y(0) = -1, \quad y'(0) = 1$

33. $y'' + 3y' = t^3$; $y(0) = 0, \quad y'(0) = 0$

34. $y'' - 3y' + 2y = e^{-t}$, $y(1) = 0, \quad y'(1) = 0$

35. $y'' - 3y' + 2y = \cos t$; $y(0) = 0, \quad y'(0) = -1$

36. $y'' - 4y' + 4y = t^3 e^{2t}$, $y(0) = 0, \quad y'(0) = 0$

37. $y'' - 4y' + 4y = t^3$, $y(0) = 1, \quad y'(0) = 0$

38. $y'' - 4y = e^{-t}$; $y(0) = -1, \quad y'(0) = 0$

39. $y'' - 4y' = 6e^{3t} - 3e^{-t}$; $y(0) = 1, \quad y'(0) = -1$

40. $x'' + 4x' + 4x = t^2$; $x(0) = x'(0) = 0$

41. $y'' + 4y = 4t^2 - 4t + 10$; $y(0) = 0, \quad y'(0) = 3$

42. $y'' - 4y = 4t - 8e^{-2t}$; $y(0) = 0, \quad y'(0) = 5$

43. $y'' + 4y' = \cos(t-3) + 4t$, $y(3) = 0, \quad y'(3) = 7$

44. $y'' + 4y' + 8y = \sin t$, $y(0) = 1, \quad y'(0) = 0$

45. $y'' + 5y' - y = e^t - 1$; $y(0) = 1, \quad y'(0) = 1$

46. $y'' + 5y' - 6y = 21e^{t-1}$ $y(1) = -1, \quad y'(1) = 9$

47. $y'' + 5y' + 4y = 0$; $y(0) = 1, \quad y'(0) = 0$

48. $y'' + 6y = t^2 - 1$; $y(0) = 0, \quad y'(0) = -1$

49. $y'' - 6y' + 9y = t$, $y(0) = 0, \quad y'(0) = 1$

50. $y'' - 6y' + 13y = 0$, $y(0) = 0, \quad y'(0) = -3$

51. $y'' - 6y' + 15y = 2\sin 3t$, $y(0) = -1, \quad y'(0) = -4$

52. $y'' + 6y' + 9y = 0$, $y(0) = -1, \quad y'(0) = 6$

53. $y'' + 6y' + 5y = 12e^t$, $y(0) = -1, \quad y'(0) = 7$

54. $y'' - 7y' + 10y = 9\cos t + 7\sin t$; $y(0) = 5, \quad y'(0) = -4$

55. $y'' + 8y' + 25y = 0$, $y(\pi) = 0, \quad y'(\pi) = 6$

56. $y'' + 9y = 2\sin 2t$; $y(0) = 0, \quad y'(0) = -1$

57. $y'' + 9y = 3\sin 2t$; $y(0) = 0, \quad y'(0) = -1$

58. $y'' - 10y' + 9y = 5t$; $y(0) = -1, \quad y'(0) = 2$

59. $y'' + 16y = 2\sin 4t$; $y(0) = -\frac{1}{2}, \quad y'(0) = 0$

60. $2y'' + 3y' - 2y = te^{-2t}$, $y(0) = 0, \quad y'(0) = -2$

61. $2y'' + 20y' + 51y = 0$, $y(0) = 2, \quad y'(0) = 0$

62. $y^{(3)} + y' = e^t$, $y(0) = y'(0) = y''(0) = 0$

63. $2y^{(3)}+3y''-3y'-2y=e^{-t}$; $y(0)=0$, $y'(0)=0$, $y''(0)=1$

64. $y^{(3)}+2y''-y'-2y=\sin 3t$; $y(0)=0$, $y'(0)=0$, $y''(0)=1$

65. $y^{(3)}-y''+y'-y=0$; $y(0)=1$, $y'(0)=1$, $y''(0)=3$

66. $y^{(3)}+4y''+y'-6y=-12$; $y(0)=1$, $y'(0)=4$, $y''(0)=-2$

67. $y^{(3)}+3y''+3y'+y=0$; $y(0)=-4$, $y'(0)=4$, $y''(0)=-2$

68. $y^{(3)}-3y''+3y'-y=t^2e^t$, $y(0)=1$, $y'(0)=2$, $y''(0)=3$

69. $y^{(3)}+y''+3y'-5y=16e^{-t}$; $y(0)=0$, $y'(0)=2$, $y''(0)=-4$

70. $y'''+4y''+5y'+2y=10\cos t$, $y(0)=y'(0)=0$, $y''(0)=3$

71. $y^{(4)}+2y''+y=4te^t$; $y(0)=y'(0)=y''(0)=y^{(3)}(0)=0$

72. $y^{(4)}-y=0$; $y(0)=1$, $y'(0)=0$, $y''(0)=0$, $y^{(3)}(0)=0$

73. $y^{(4)}-4y=0$; $y(0)=1$, $y'(0)=0$, $y''(0)=-2$, $y^{(3)}(0)=0$

74. $y^{(4)}-4y^{(3)}+6y''-4y'+y=0$; $y(0)=0$, $y'(0)=1$, $y''(0)=0$, $y^{(3)}(0)=1$

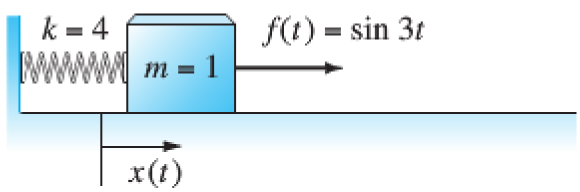
75. Given: $y''-4y'+3y=0$, $y(0)=1$ $y'(0)=-1$

a) Show that the general solution is: $y(t)=C_1e^{3t}+C_2e^t$ and find C_1 *and* C_2

b) Use Laplace transform to solve the system

76. Solve the initial value problem $x''+4x=\sin 3t$; $x(0)=x'(0)=0$.

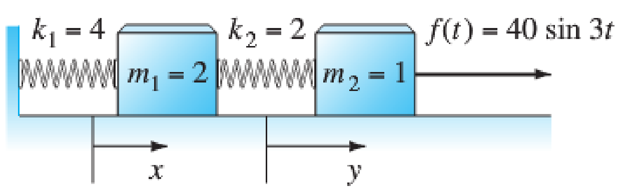
Such problem arises in the motion of a mass-and-spring system with external force as shown below.



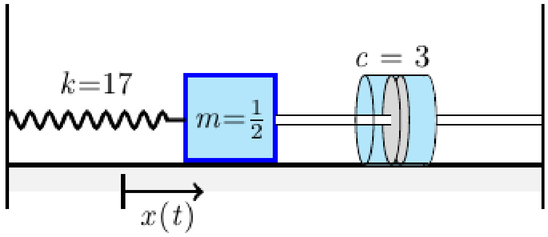
77. Solve the system
$$\begin{cases} 2x''=-6x+2y \\ y''=2x-2y+40\sin 3t \end{cases}$$

Subject to the initial conditions $x(0)=x'(0)=y(0)=y'(0)=0$

Thus the force $f(t)=40\sin 3t$ is applied to the second mass as shown below, beginning at time $t=0$ when the system is at rest in its equilibrium position.



78. Consider a mass-spring system with $m = \frac{1}{2}, \quad k = 17, \quad and \ c = 3$.



Let $x(t)$ be the displacement of the mass *m* from its equilibrium position. If the mass is set in motion with $x(0) = 3$ *and* $x'(0) = 1$, find $x(t)$ for the resulting damped free oscillations.

79. A $4-lb$ weight stretches a spring 2 *feet*. The weight is released from rest 18 *inches* above the equilibrium position, and the resulting motion takes place in a medium offering a damping force numerically equal to $\frac{7}{8}$ times the instantaneous velocity. Use the Laplace transform to find the equation of motion $x(t)$.

80. Consider a mass-spring-dashpot system with $m = \frac{1}{2}, \quad k = 17, \quad c = 3, \quad and \ f(t) = 15\sin 2t$ with initial conditions $x(0) = x'(0) = 0$. Let $x(t)$ be the displacement of the mass *m* from its equilibrium position. Find the resulting transient motion and steady periodic motion of the mass..

81. A 8-*kg* mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by 3 *N-sec/m* and the spring constant is 40 *N/m*. If the mass is driven by an external force equal to $f(t) = 2\sin 2t \cos 2t \ N$. Find the solution.

82. A 2-*kg* mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $c = 8$ *kg/sec* and the spring constant is $k = 80$ *N/m*. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20e^{-t} \ N$. (*t* in *seconds*). Find the equation of motion.

83. A 10-*kg* mass is attached to a spring having a spring constant of 140 *N/m*. The mass is started in motion initially from the equilibrium position with an initial velocity 1 *m/sec* in the upward direction and with an applied external force $F(t) = 5\sin t$. If the force due to air resistance is $-90y' \ N$. Find the equation motion of the mass.

84. A 128-*lb* weight is attached to a spring having a spring constant of 64 *lb/ft*. The weight is started in motion initially by displacing it 6 *in* above the equilibrium position with no initial velocity and with an applied external force $F(t) = 8\sin 4t$. Assume no air resistance. Find the equation motion of the mass.

85. Find the motion of a damped mass-and-spring system with $m=1$, $c=2$, and $k=26$ under the influence of an external force $F(t)=82\cos 4t$ with $x(0)=6$ and $x'(0)=0$.

86. A spring with a mass of 2-*kg* has natural length 0.5 *m*. A force of 25.6 *N* is required to maintain it stretched to a length of 0.7 *m*. The spring is immersed in a fluid with damping constant $c=40$. If the spring is started from the equilibrium position and is given a push to start it with initial velocity 0.6 *m/s*. Find the position of the mass at any time *t*.

87. A spring with a mass of 3-*kg* is held stretched 0.6 *m* beyond its natural length by a force of 20 *N*. If the spring begins at its equilibrium and with initial velocity 1.2 *m/s*. Find the position of the mass.

88. A spring with a mass of 2-*kg* is held stretched 0.5 *m*, has damping constant 14, and a force of 6 *N*. If the spring is stretched 1 *m* beyond at its equilibrium and with no initial velocity. Find the position of the mass at any time *t*.

89. Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L=0.25\ H$, $R=10\ \Omega$, $C=0.001\ F$, $E(t)=0$, $q(0)=q_0\ C$, and $i(0)=0$.

90. Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit at $t=0.01\ sec$ when $L=0.05\ h$, $R=2\ \Omega$, $C=0.01\ f$, $E(t)=0$, $q(0)=5\ C$, and $i(0)=0\ A$.

91. Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L=\frac{5}{3}\ h$, $R=10\ \Omega$, $C=\frac{1}{30}\ f$, $E(t)=0$, $q(0)=4\ C$, and $i(0)=0\ A$.

92. Find the current $i(t)$ in an *LRC*–series circuit when $L=1\ h$, $R=20\ \Omega$, $C=0.005\ f$, $E(t)=150\ V$, $q(0)=0\ C$, and $i(0)=0\ A$.

A resistor $R=20\ \Omega$ and a capacitor of $C=0.1\ F$ are joined in series with an electronic force (*emf*) $E=E(t)$ and no charge on the capacitor at $t=0$. Find the ensuing charge on the capacitor at time *t* for the given:

93. $E(t)=100\sin 2t$ **94.** $E(t)=100e^{-0.1t}$ **95.** $E(t)=100\left(1-e^{-0.1t}\right)$ **96.** $E(t)=100\cos 3t$

An inductor $(L=1\ H)$ and a resistor $(R=0.1\ \Omega)$ are joined in series with an electronic force (*emf*) $E=E(t)$ and no charge on the capacitor at $t=0$. Find the ensuing current in the current at time *t* for the given:

97. $E(t)=10-2t$ **98.** $E(t)=4\cos 3t$ **99.** $E(t)=4\sin 2\pi t$

100. Solve the general initial value problem modeling the *RC* circuit

$$R\frac{dQ}{dt}+\frac{1}{C}Q=E,\quad Q(0)=0$$

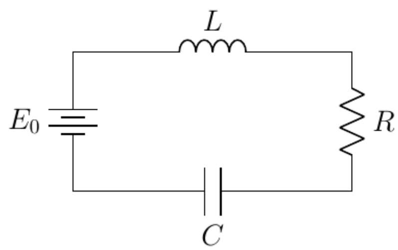
Where E is a constant source of *emf*

101. Solve the general initial value problem modeling the *LR* circuit

$$L\frac{dI}{dt}+RI=E,\quad I(0)=I_0$$

Where E is a constant source of *emf*

102. Consider a battery of constant voltage E_0 that charges the capacitor. $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E(t)$



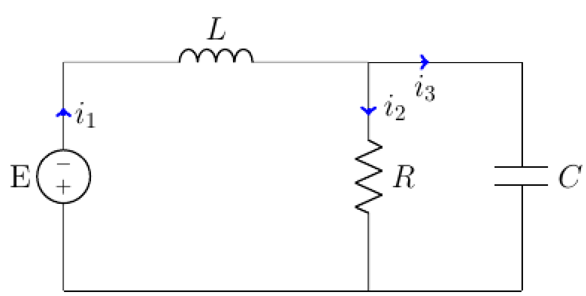
Divide the given equation by L and define $2\lambda=\frac{R}{L}$ and $\omega^2=\frac{1}{LC}$.

a) Use the Laplace transform to show that the solution $q(t)$ of $q''+2\lambda q'+\omega^2 q=\frac{E_0}{L}$ subject to $q(0)=0$, $i(0)=0$ is

$$q(t)=\begin{cases}E_0C\left[1-e^{-\lambda t}\left(\cosh\sqrt{\lambda^2-\omega^2}\,t+\frac{\lambda}{\sqrt{\lambda^2-\omega^2}}\sinh\sqrt{\lambda^2-\omega^2}\,t\right)\right] & \lambda>\omega\\ E_0C\left[1-e^{-\lambda t}(1+\lambda t)\right] & \lambda=\omega\\ E_0C\left[1-e^{-\lambda t}\left(\cos\sqrt{\omega^2-\lambda^2}\,t+\frac{\lambda}{\sqrt{\omega^2-\lambda^2}}\sin\sqrt{\omega^2-\lambda^2}\,t\right)\right] & \lambda<\omega\end{cases}$$

b) Use the Laplace transform to find the charge $q(t)$ in an *RC* series when $q(0)=0$ and $E(t)=E_0e^{-kt}$, $k>0$. Consider *two* cases: $k\neq\frac{1}{RC}$ and $k=\frac{1}{RC}$

103. Solve the system under the conditions $E(t)=60\ V$, $L=1\ h$, $R=50\ \Omega$, $C=10^{-4}\ f$, and the currents i_1 and i_2 are initially zero.



104. Solve

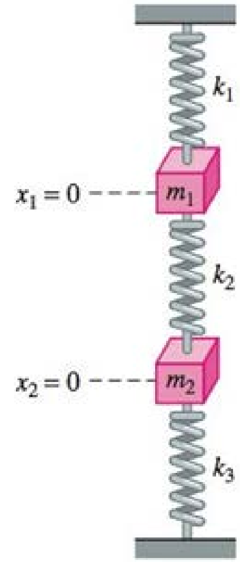
$$\begin{aligned} x_1'' + 10x_1 \quad - 4x_2 &= 0 \\ -4x_1 + x_2'' + 4x_2 &= 0 \end{aligned}$$

Subject to $x_1(0) = 0,\ x_1'(0) = 1,\ x_2(0) = 0,\ x_2'(0) = -1$

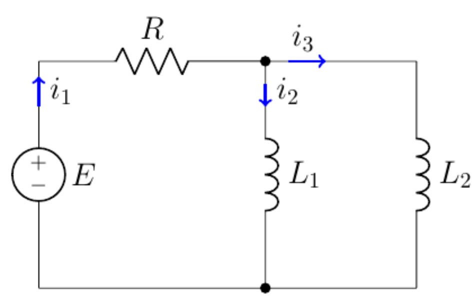
105. Derive the system of differential equations describing the straight-line vertical motion of the coupled springs. Use the Laplace transform to solve the system when

$k_1 = 1,\ k_2 = 1,\ k_3 = 1,\ m_1 = 1,\ m_2 = 1$ and

$x_1(0) = 0,\ x_1'(0) = -1,\ x_2(0) = 0,\ x_2'(0) = 1$

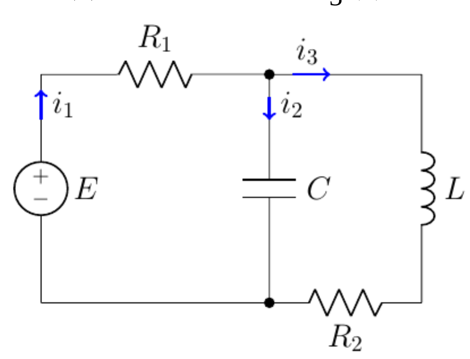


106. Solve the currents $i_1(t)$, $i_2(t)$ and $i_3(t)$ in the given electrical network.



Given $R = 5\,\Omega \quad L_1 = 0.01\,h, \quad L_2 = 0.0125\,h, \quad E = 100\ V$ and $i_2(0) = 0 \quad i_3(0) = 0$

107. Find the charge on the capacitor $q(t)$ and the current $i_3(t)$ in the given electrical network.

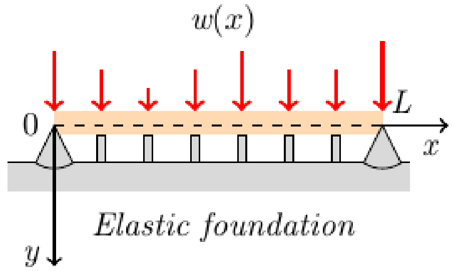


Given: $\quad R_1 = 1\,\Omega, \quad R_2 = 1\,\Omega, \quad L = 1\,h, \quad C = 1\,f$ & $q(0) = 0, \quad i_3(0) = 0$

$$E(t) = \begin{cases} 0, & 0 < t < 1 \\ 50e^{-t}, & t \ge 1 \end{cases}$$

108. When a uniform beam is supported by an elastic foundation, the differential equation for its deflection $y(x)$ is

$$EI\frac{d^4y}{dx^4}+ky=w(x)$$



Where k is the modulus of the foundation and $-ky$ is the restoring force of the foundation that acts in the direction opposite to that of the load $w(x)$. For algebraic convenience suppose that the differential equation is written as

$$\frac{d^4y}{dx^4}+4a^4y=\frac{w(x)}{EI}$$

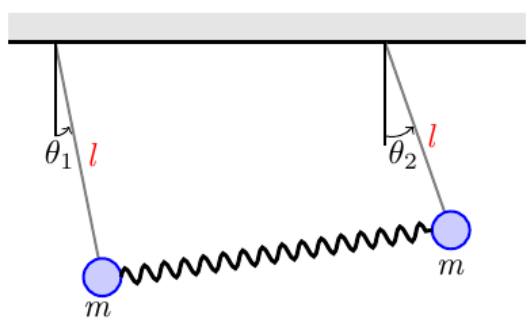
Where $a=\left(\frac{k}{4EI}\right)^{1/4}$. Assume $L=\pi$ and $a=1$. Find the deflection $y(x)$ of a beam that is supported on an elastic foundation when

a) The beam is simply supported at both ends and a constant load w_0 is uniformly distributed along its length,

b) The bean is embedded at both ends and $w(x)$ is concentrated load w_0 applied at $x=\frac{\pi}{2}$

109. Suppose two identical pendulums are coupled by means of a spring with constant k. when the displacement angles $\theta_1(t)$ and $\theta_2(t)$ are small, the system of linear differential equations describing the motion is

$$\begin{cases}\theta_1''+\frac{g}{l}\theta_1=-\frac{k}{m}(\theta_1-\theta_2)\\ \theta_2''+\frac{g}{l}\theta_2=\frac{k}{m}(\theta_1-\theta_2)\end{cases}$$



a) Use Laplace transform to solve the system when

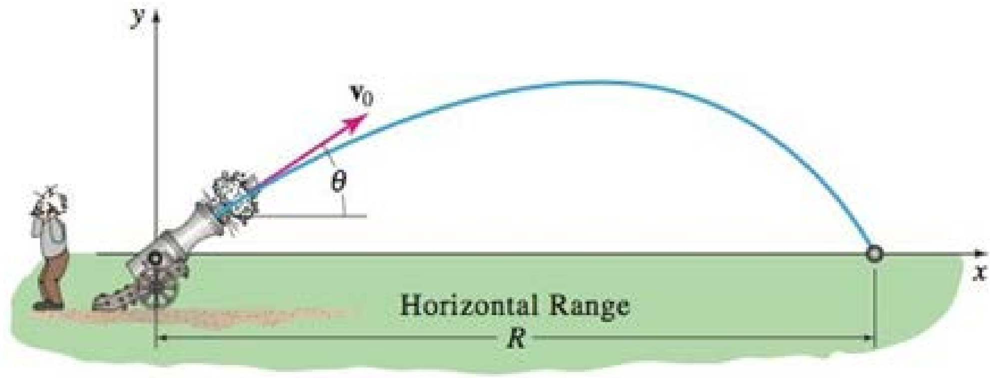
$$\theta_1'(0)=0 \quad \theta_1(0)=\theta_0 \quad \theta_2'(0)=0 \quad \theta_2(0)=\psi_0$$

Where θ_0 and ψ_0 constants. Let $\omega^2=\frac{g}{l}$, $K=\frac{k}{m}$

b) Use the solution in part (*a*) to discuss the motion of the coupled pendulums in the special case when the initial conditions are $\theta_1'(0)=0$, $\theta_1(0)=\theta_0$, $\theta_2'(0)=\theta_0$, $\theta_2(0)=0$

c) Use the solution in part (*a*) to discuss the motion of the coupled pendulums in the special case when the initial conditions are $\theta_1'(0)=0, \quad \theta_1(0)=\theta_0, \quad \theta_2'(0)=-\theta_0, \quad \theta_2(0)=0$

110. A projectile, such as the canon ball, has weight $w=mg$ and initial velocity $\boldsymbol{v}_0$ that is tangent to its path of motion.



If air resistance and all other forces except its weight are ignored, that motion of the projectile is describe by the system of linear differential equations:

$$\begin{cases} m\dfrac{d^2x}{dt^2}=0 \\ m\dfrac{d^2y}{dt^2}=-mg \end{cases}$$

a) Use Laplace transform to solve the system when

$$x(0)=0 \quad x'(0)=v_0\cos\theta \quad y(0)=0 \quad y'(0)=v_0\sin\theta$$

Where $v_0=|v|$ is constant and θ is the constant angle of elevation.

The solutions $x(t)$ and $y(t)$ are parametric equations of the trajectory of the projectile.

b) Use $x(t)$ in part (*a*) to eliminate the parameter *t* in $y(t)$. Use the resulting equation for *y* to show that the horizontal range *R* of the projectile is given by

$$R=\frac{v_0^2}{g}\sin 2\theta$$

c) From the formula in part (*b*), we see that *R* is a maximum when $\sin 2\theta=1$ or when $\theta=\frac{\pi}{4}$.

Show that the same range – less than the maximum– can be obtained by firing the gun at either of two complementary angles θ and $\frac{\pi}{2}-\theta$. The only difference is that the smaller angle results in a low trajectory whereas the larger angle fives a high trajectory.

d) Suppose $g=32\ ft/s^2$, $\theta=30°$, and $v_0=300\ ft/s$. Use part (*b*) to find the horizontal range of the projectile.

e) Find the time when the projectile hits the ground.

f) Use the parametric equations $x(t)$ and $y(t)$ in part (*a*) along with the numerical data in part (*d*) to plot the ballistic curve of the projectile.

g) Repeat with $\theta = 52°$ and $v_0 = 300$ *ft/s*.

h) Superimpose both curves part (*f*) & (*g*) on the same coordinate system.