Section 1.3 – Predicates and Quantifiers

Introduction

To express the meaning of a wide range of statements in mathematics and computer science in ways that permit us to reason and explore relationships between object, are called ***predicate logic***.

Predicates

Statements involving variables, such as

$$"x > 3," \quad "x = y + 3," \quad \textit{and} \ "\textit{computer x is under attack by an intruder.}"$$

are often found in mathematical assertions, in computer programs, and in system specifications

Example

Let $P(x)$ denote the statement $x > 3$. What are the truth values of $P(4)$ and $P(2)$?

Solution

We obtain the statement $P(4)$ by setting $x = 4$ in the statement $x > 3$. Hence, $P(4)$, which is the statement $4 > 3$ is true.

However, $P(2)$, which is the statement $2 > 3$ is false.

Example

Let $Q(x, y)$ denote the statement $x = y + 3$. What are the truth values of propositions $Q(1,\ 2)$ and $Q(3,\ 0)$?

Solution

To obtain $Q(1,\ 2)$, set $x = 1$ and $y = 2$ in the statement $Q(x, y)$. Hence, $Q(1,\ 2)$ is the statement $1 = 2 + 3$ which is false.

The statement $Q(3,\ 0)$ is the proposition $3 = 0 + 3$ which is true.

Example

Let $A(c,n)$ denote the statement "Computer *c* is connected to netwrok *n*," where *c* is a variable representing a computer and *n* is a variable representing a network. Suppose that the computer MATH1 is connected to network CAMPUS2, but not to network CAMPUS1. What are the values of $A(MATH1, CAMPUS1)$ and $A(MATH1, CAMPUS2)$?

Solution

Because MATH1 is not connected to the CAMPUS1 network, we see that $A(MATH1, CAMPUS1)$ is false.
However, because MATH1 is connected to the CAMPUS2 network, we see that $A(MATH1, CAMPUS2)$ is true.

- Consider the statement ***if*** $x > 0$ **then** $x := x + 1$
  When this statement is encountered in a program, the value of the variable *x* at the point in the execution of the program is inserted into $P(x)$, which is $x > 0$. If $P(x)$ is true for this value of *x*, the assignment statement $x := x + 1$ is executed. So the value of *x* is increased by 1. If $P(x)$ is false for this value of *x*, the assignment statement is not executed, so the value of *x* is not changed.

Preconditions and Postconditions

Predicates are also used to establish the correctness of computer programs, that is, to show that computer programs always produce desired output given valid input.
The statements that describe valid input are known as ***preconditions*** and the conditions that the output should satisfy when the program has run are known as ***postconditions***.

Quantifiers

To create a proposition from a propositional function is called ***quantification***. Quantification expresses the extent to which a predicate is true over a range of elements. The words ***all, some, many, none*** and ***few*** are used in quantifications.
The area of logic that deals with predicates and quantifiers is called the ***predicate calculus***.

There are two quantifiers

- Existential Quantifier "∃" reads "there exists"
- Universal Quantifier "∀" reads "for all"

Each is placed in front of a propositional function and ***binds*** it to obtain a proposition with semantic value.

Definition

The ***universal quantification*** of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain."

The notation $\forall x\, P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the universal quantifier. We read $\forall x\, P(x)$ as "for all $x\, P(x)$" or "for every $x\, P(x)$". An element for which $P(x)$ is false is called a counterexample of $\forall x\, P(x)$.

Statement	***When True?***	***When False?***
$\forall x\, P(x)$	$P(x)$ ***is true for every x.***	***There is an x for which*** $P(x)$ ***is false.***
$\forall x\, P(x)$	***There is an x for which*** $P(x)$ ***is true.***	$P(x)$ ***is false for every x.***

Example

Let $P(x)$ be the statement "$x+1>x$". What is the truth value of the quantification $\forall x\, P(x)$, where the domain consists of all real numbers?

Solution

Because $P(x)$ is true for all real numbers x, the quantification $\forall x\, P(x)$ is true.

Example

Let $Q(x)$ be the statement "$x<2$". What is the truth value of the quantification $\forall x\, Q(x)$, where the domain consists of all real numbers?

Solution

$Q(x)$ is not true for every real number x, because, $Q(3)$ is false.
That is, $x=3$ is a counterexample for the statement $\forall x\, Q(x)$. Thus $\forall x\, Q(x)$ is false.

Example

What is the truth value of $\forall x\, P(x)$, where $P(x)$ is the statement "$x^2<10$" and the domain consists of the positive integers not exceeding 4?

Solution

The domain consists of the integers 1, 2, 3, and 4. Since "$4^2<10$" is false, it follows that $\forall x\, P(x)$ is false.

Example

What is the truth value of $\forall x\left(x^2 \geq x\right)$, if the domain consists of real number? What is the truth value of this statement of the domain consists of all integers?

Solution

- $\left(\frac{1}{2}\right)^2 \not\geq \frac{1}{2}$, it follows that $\forall x\left(x^2 \geq x\right)$ is false.

  Note: $x^2 \geq x \quad iff \quad x^2 - x = x(x-1) \geq 0 \;\Rightarrow\; x \leq 0 \quad x \geq 1$

- If the domain consists of the integers, $\forall x\left(x^2 \geq x\right)$ is true, because there are no integers *x* with $0 < x < 1$

Definition

The ***existential quantification*** of $P(x)$ is the proposition

"There exists an element *x* in the domain such that $P(x)$"

We use the notation $\exists x\, P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the ***existential quantifier***.

Example

Let $P(x)$ denote the statement "$x > 3$". What is the truth value of the quantification $\exists x\, P(x)$, where the domain consists of all real numbers?

Solution

Because "$x > 3$" is sometimes true, for instance, when $x = 4 -$ the existence quantification of $P(x)$, which is $\exists x\, P(x)$, is true.

Example

Let $Q(x)$ denote the statement "$x = x + 1$". What is the truth value of the quantification $\exists x\, Q(x)$, where the domain consists of all real numbers?

Solution

$Q(x)$ is false for every real number *x*, the existential quantification of $Q(x)$, which is $\exists x\, Q(x)$, is false.

Example

What is the truth value of $\exists x\, P(x)$, where $P(x)$ is the statement "$x^2 > 10$" and the universe discourse of the positive integers not exceeding 4?

Solution

The domain consists of the integers 1, 2, 3, and 4. Since "$4^2 > 10$" is true, it follows that $\exists x\, P(x)$ is true.

Quantifiers with Restricted Domains

What do the statement $\forall x < 0 \; (x^2 > 0)$, $\forall y \neq 0 \; (y^3 \neq 0)$, and $\forall z > 0 \; (z^2 = 0)$ mean, where the domain in each case consists of the real numbers?

- The statement $\forall x < 0 \; (x^2 > 0)$ states that for every real numbers *x* with $x < 0$, $x^2 < 0$. That is, it states "the square of a negative real number is positive." This statement is the same as $\forall x \; (x < 0 \rightarrow x^2 > 0)$.
- The statement $\forall y \neq 0 \; (y^3 \neq 0)$ states that for every real numbers *y* with $y \neq 0$, $y^3 \neq 0$. That is, it states "the cube of every nonzero real number is nonzero." This statement is the equivalent to $\forall y \; (y \neq 0 \rightarrow y^3 \neq 0)$.
- The statement $\forall z > 0 \; (z^2 = 2)$ states that for every real numbers *z* with $z > 0$ such that $z^2 = 2$. That is, it states "There is a positive square root of 2." This statement is the equivalent to $\exists z \; (z > 0 \wedge z^2 = 2)$.

Binding Variables

When a quantifier is used on the variable x, we say that this occurrence of the variable is bound. An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be ***free***.

Logical Equivalences Involving Quantifiers

Definition

Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ involving predicates and quantifiers are logically equivalent.

Negating Quantified Expressions

We will often want to consider the negation of a quantified expression. For instance, consider the negation of the statemnt

"Every student in your class has taken a course in calculus"

This statement is a universal quantification, namely, $\forall x\, P(x)$

where $P(x)$ is the statement "*x* has taken a course in calculus" and the domain consists of the students in your class.

The negation of this statement is " It is not the case that every student in your class who has not taken a course in calculus". This is simply the existential quantification of the negation of the original proposition function, namely, $\exists x\, \neg P(x)$.

This example illustrates the following logical equivalence:

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$

$$\neg \exists x\, Q(x) \equiv \forall x\, \neg Q(x)$$

De Morgan's Laws for Quantifiers			
Negation	***Equivalent Statement***	***When Is Negation True?***	***When False?***
$\neg \exists x\, P(x)$	$\forall x\, \neg P(x)$	***For every x,*** $P(x)$ ***is false***	***There is an x for which*** $P(x)$ ***is true***
$\neg \forall x\, P(x)$	$\exists x\, \neg P(x)$	***There is an x for which*** $P(x)$ ***is false***	***For every x,*** $P(x)$ ***is true***

Example

What are the negations of the statements "There is an honest politician" and "All Americans eat cheeseburgers"?

Solution

Let $H(x)$ denote "x is honest."

Then the statement "there is an honest politician" is represented by $\exists x\, H(x)$

The negation statement is $\neg\exists x\, H(x)$, which is equivalent to $\forall x\, \neg H(x)$.

This negation can be expressed as "All politicians are not honest"

Let $G(x)$ denote "x eats cheeseburgers."

Then the statement "All Americans eat cheeseburgers" is represented by $\forall x\, G(x)$

The negation statement is $\neg\forall x\, G(x)$, which is equivalent to $\exists x\, \neg G(x)$.

This negation can be expressed as "There is an American who does not eat cheeseburgers."

Example

What are the negations of the statements $\forall x\left(x^2 > x\right)$ and $\exists x\left(x^2 = 2\right)$

Solution

The negation of $\forall x\left(x^2 > x\right)$ is the statement $\neg\forall x\left(x^2 > x\right) \equiv \exists x\, \neg\left(x^2 > x\right)$

Which can be written as $\exists x\left(x^2 \le x\right)$

The negation of $\exists x\left(x^2 = 2\right)$ is the statement $\neg\exists x\left(x^2 = 2\right) \equiv \forall x\, \neg\left(x^2 = 2\right)$

Which can be written as $\forall x\left(x^2 \ne 2\right)$

Example

Consider these statements, of which the first three are premises and the fourth is a valid conclusion

"All hummingbirds are richly colored"
"No large birds live on honey"
"Birds that do not live on honey are dull in color"
"Hummingbirds are small"

Let $P(x)$, $Q(x)$, $R(x)$ and $S(x)$ be the statements "x is a hummingbird, " "x is large," "x lives on honey," and " x is richly colored," respectively. Assuming that the domain consists of all birds, express the statements in the argument using quantifiers and $P(x)$, $Q(x)$, $R(x)$ and $S(x)$.

Solution

We can express the statements in the argument as

$$\forall x\left(P(x)\to S(x)\right)$$

$$\neg\exists x\left(Q(x)\wedge R(x)\right)$$

$$\forall x\left(\neg R(x)\to\neg S(x)\right)$$

$$\forall x\left(P(x)\to\neg Q(x)\right)$$

"***small***" is the same as "not large"

"***dull in color***" is the same as "not richly colored"

Exercises *Section* **1.3 – Predicates and Quantifiers**

1. Let $P(x)$ denote the statement "$x \le 4$". What are these truth values?

a) $P(0)$ *b)* $P(4)$ *c)* $P(6)$

2. Let $P(x)$ be the statement "*the word* ***x*** *contains the letter* ***a***". What are these truth values?

a) $P(orange)$ *b)* $P(lemon)$ *c)* $P(true)$ *d)* $P(false)$

3. State the value of *x* after the statement **if** $P(x)$ **then** $x := 1$ is executed, where $P(x)$ is the statement "$x > 1$", if the value of *x* when the statement is reached is

a) $x = 0$ *b)* $x = 1$ *c)* $x = 2$

4. Let $P(x)$ be the statement "***x*** *spends more than five hours every weekday in class,*" where the domain for *x* consists of all students. Express each of these quantifications in English.

a) $\exists x\, P(x)$ *b)* $\forall x\, P(x)$ *c)* $\exists x\, \neg P(x)$ *d)* $\forall x\, \neg P(x)$

5. Let $N(x)$ be the statement "***x*** *has visited North Dakota,*" where the domain consists of the students in your class. Express each of these quantifications in English.

a) $\exists x\, N(x)$ *b)* $\forall x\, N(x)$ *c)* $\neg\exists x\, N(x)$ *d)* $\exists x\, \neg N(x)$

e) $\neg\forall x\, N(x)$ *f)* $\forall x\, \neg N(x)$

6. Let $C(x)$ be the statement "***x*** *has a cat,*" let $D(x)$ be the statement "***x*** *has a dog,*", and let $F(x)$ be the statement "***x*** *has a ferret.*" Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

a) A student in your class has a cat, a dog, and a ferret.
b) All students in your class have a cat, a dog, or a ferret.
c) Some student in your class has a cat and a ferret, but not a dog.
d) No student in your class has a cat, a dog, and a ferret.
e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

7. Let $Q(x)$ be the statement "$x + 1 > 2x$". If the domain consists of all integers, what are these truth values?

a) $Q(0)$ *b)* $Q(-1)$ *c)* $Q(1)$ *d)* $\exists x\, Q(x)$

e) $\forall x\, Q(x)$ *f)* $\exists x\, \neg Q(x)$ *g)* $\forall x\, \neg Q(x)$

8. Determine the truth value of each of these statements if the domain consists of all integers

a) $\forall n(n + 1 > n)$ *b)* $\exists n(2n = 3n)$ *c)* $\exists n(n = -n)$ *d)* $\forall n(3n \le 4n)$

9. Determine the truth value of each of these statements if the domain consists of all real numbers *a*) $\exists x\left(x^3=-1\right)$ *b*) $\exists x\left(x^4<x^2\right)$ *c*) $\forall x\left((-x)^2=x^2\right)$ *d*) $\forall x(2x>x)$

10. Suppose that the domain of the propositional function $P(x)$ consists of the integers 1, 2, 3, 4, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

a) $\exists x\, P(x)$ *b*) $\forall x\, P(x)$ *c*) $\neg\exists x\, P(x)$ *d*) $\neg\forall x\, P(x)$

e) $\forall x\ \left((x\neq 3)\rightarrow P(x)\right)\ \vee\ \exists x\, \neg P(x)$

11. For each of these statements find a domain for which the statement is true and a domain for which the statement is false.

a) Everyone is studying discrete mathematics.
b) Everyone is older than 21 years.
c) Every two people have the same mother.
d) No Two different people have the same grandmother.

12. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) No one is perfect.
b) Not everyone is perfect.
c) All your friends are perfect.
d) At least one of your friends is perfect.
e) Everyone is your friend and is perfect.
f) Not everybody is your friend or someone is not perfect.

13. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) Something is not in the correct place.
b) All tools are in the correct place and are in excellent condition.
c) Everything is in the correct place and in excellent condition.
d) Nothing is in the correct place and is in excellent condition.
e) One of your tools is not in the correct place, but it is in excellent condition.