

Solution Section R.2 – Integration

Exercise

Find each indefinite integral. $\int \frac{x+2}{\sqrt{x}} dx$

Solution

$$\begin{aligned}\int \frac{x+2}{\sqrt{x}} dx &= \int \left[\frac{x}{x^{1/2}} + \frac{2}{x^{1/2}} \right] dx \\&= \int \frac{x}{x^{1/2}} dx + \int \frac{2}{x^{1/2}} dx \\&= \int x^{1/2} dx + 2 \int x^{-1/2} dx \\&= \frac{x^{3/2}}{3/2} + 2 \frac{x^{1/2}}{1/2} + C \\&= \underline{\frac{2}{3} x^{3/2} + 4x^{1/2} + C}\end{aligned}$$

Exercise

Find each indefinite integral $\int 4y^{-3} dy$

Solution

$$\begin{aligned}\int 4y^{-3} dy &= 4 \frac{y^{-2}}{-2} + C \\&= \underline{-\frac{2}{y^2} + C}\end{aligned}$$

Exercise

Find each indefinite integral $\int (x^3 - 4x + 2) dx$

Solution

$$\int (x^3 - 4x + 2) dx = \underline{\frac{1}{4} x^4 - 2x^2 + 2x + C}$$

Exercise

Find each indefinite integral $\int \left(\sqrt[4]{x^3} + 1 \right) dx$

Solution

$$\int \left(x^{3/4} + 1 \right) dx = \underline{\frac{4}{7} x^{7/4} + x + C}$$

Exercise

Find each indefinite integral $\int \sqrt{x}(x+1)dx$

Solution

$$\begin{aligned} \int x^{1/2}(x+1)dx &= \int \left(x^{3/2} + x^{1/2} \right) dx \\ &= \underline{\frac{2}{5} x^{5/2} + \frac{2}{3} x^{3/2} + C} = x + 5x^{-1} + C \end{aligned}$$

Exercise

Find each indefinite integral $\int (1+3t)t^2 dt$

Solution

$$\int \left(t^2 + 3t^3 \right) dt = \underline{\frac{1}{3} t^3 + \frac{3}{4} t^4 + C}$$

Exercise

Find each indefinite integral $\int \frac{x^2-5}{x^2} dx$

Solution

$$\begin{aligned} \int \frac{x^2-5}{x^2} dx &= \int \left(1 - \frac{5}{x^2} \right) dx \\ &= \int \left(1 - 5x^{-2} \right) dx \\ &= \underline{x + \frac{5}{x} + C} \end{aligned}$$

Exercise

Find each indefinite integral $\int (-40x + 250) dx$

Solution

$$\int (-40x + 250) dx = \underline{-20x^2 + 250x + C}$$

Exercise

Find each indefinite integral $\int (7 - 3x - 3x^2)(2x + 1) dx$

Solution

$$\begin{aligned}\int (7 - 3x - 3x^2)(2x + 1) dx &= \int (14x + 7 - 6x^2 - 3x - 6x^3 - 3x^2) dx \\ &= \int (-6x^3 - 9x^2 + 11x + 7) dx \\ &= \underline{-\frac{3}{2}x^4 - 3x^3 + \frac{11}{2}x^2 + 7x + C}\end{aligned}$$

Exercise

Evaluate the integral $\int xe^{2x} dx$

Solution

Let: $u = x \Rightarrow du = dx$

$$dv = e^{2x} dx \Rightarrow v = \int dv = \int e^{2x} dx = \frac{1}{2}e^{2x}$$

$$\int u dv = uv - \int v du$$

$$\begin{aligned}\int xe^{2x} dx &= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x} dx \\ &= \frac{1}{2}xe^{2x} - \frac{1}{2} \frac{1}{2}e^{2x} + C \\ &= \underline{\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C}\end{aligned}$$

Exercise

Evaluate the integral $\int x \ln x dx$

Solution

Let: $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$dv = x dx \Rightarrow v = \int dv = \int x dx = \frac{1}{2} x^2$

$$\begin{aligned} \int x \ln x dx &= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x^2 \frac{dx}{x} \\ &= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x dx \\ &= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C \end{aligned}$$

Exercise

Evaluate the integral $\int x^2 \sin x dx$

Solution

$$\int x^2 \sin x dx = \underline{-x^2 \cos x - 2x \sin x + 2 \cos x + C}$$

$\int \sin x$		
x^2	(+)	$-\cos x$
$2x$	(-)	$-\sin x$
2	(+)	$\cos x$

Exercise

Evaluate the integral $\int (x^2 - 2x + 1) e^{2x} dx$

Solution

$$\begin{aligned} \int (x^2 - 2x + 1) e^{2x} dx &= \frac{1}{2} (x^2 - 2x + 1) e^{2x} - \frac{1}{4} (2x - 2) e^{2x} + \frac{1}{8} (2) e^{2x} + C \\ &= \left(\frac{1}{2} x^2 - x + \frac{1}{2} - \frac{1}{2} x + \frac{1}{2} + \frac{1}{4} \right) e^{2x} + C \\ &= \underline{\left(\frac{1}{2} x^2 - \frac{3}{2} x + \frac{5}{4} \right) e^{2x} + C} \end{aligned}$$

$\int e^{2x}$		
+	$x^2 - 2x + 1$	$\frac{1}{2} e^{2x}$
-	$2x - 2$	$\frac{1}{4} e^{2x}$
+	2	$\frac{1}{8} e^{2x}$

Exercise

Evaluate the integral $\int e^{2x} \cos 3x dx$

Solution

		$\int \cos 3x$
+	e^{2x}	$\frac{1}{3} \sin 3x$
-	$\frac{1}{2} e^{2x}$	$-\frac{1}{9} \cos 3x$
+	$\frac{1}{4} e^{2x}$	

$$\int e^{2x} \cos 3x dx = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x - \frac{9}{4} \int e^{2x} \cos 3x dx$$

$$\int e^{2x} \cos 3x dx + \frac{9}{4} \int e^{2x} \cos 3x dx = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x + C_1$$

$$\frac{13}{4} \int e^{2x} \cos 3x dx = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x + C_1$$

$$\frac{4}{13} \frac{13}{4} \int e^{2x} \cos 3x dx = \frac{4}{13} \frac{1}{2} e^{2x} \cos 3x + \frac{4}{13} \frac{3}{4} e^{2x} \sin 3x + \frac{4}{13} C_1$$

$$\int e^{2x} \cos 3x dx = \underline{\underline{\frac{e^{2x}}{13} (2 \cos 3x + 3 \sin 3x) + C}}$$

Exercise

Find the general solution of the differential equation $y' = 2t + 3$

Solution

$$\int dy = \int (2t + 3) dt$$

$$\underline{\underline{y(t) = t^2 + 3t + C}}$$

Exercise

Find the general solution of the differential equation $y' = 3t^2 + 2t + 3$

Solution

$$\int dy = \int (3t^2 + 2t + 3) dt$$

$$\underline{\underline{y(t) = t^3 + t^2 + 3t + C}}$$

Exercise

Find the general solution of the differential equation $y' = \sin 2t + 2 \cos 3t$

Solution

$$\int dy = \int (\sin 2t + 2 \cos 3t) dt$$

$$\underline{y(t) = -\frac{1}{2} \cos 2t + \frac{2}{3} \sin 3t + C}$$

Exercise

Find the general solution of the differential equation: $y' = x^3(3x^4 + 1)^2$

Solution

$$\int x^3(3x^4 + 1)^2 dx$$

$$u = 3x^4 + 1 \Rightarrow du = 12x^3 dx$$

$$\begin{aligned} \int x^3(3x^4 + 1)^2 dx &= \int \frac{1}{12} u^2 du \\ &= \frac{1}{12} \frac{(3x^4 + 1)^3}{3} + C \\ &= \frac{1}{36} (3x^4 + 1)^3 + C \end{aligned}$$

$$\underline{y(x) = \frac{1}{36} (3x^4 + 1)^3 + C}$$

Exercise

Find the general solution of the differential equation: $y' = 5x\sqrt{x^2 - 1}$

Solution

$$\int 5x\sqrt{x^2 - 1} dx$$

$$u = x^2 - 1 \Rightarrow du = 2x dx$$

$$\int 5x(x^2 - 1)^{1/2} dx$$

$$= 5 \int u^{1/2} \frac{1}{2} du$$

Substitute for x and dx

$$= \frac{5}{2} \int u^{1/2} du$$

$$\begin{aligned}
&= \frac{5}{2} \frac{u^{3/2}}{3/2} + C \\
&= \frac{5}{3} u^{3/2} + C \\
&= \frac{5}{3} (x^2 - 1)^{3/2} + C
\end{aligned}$$

Exercise

Find the general solution of the differential equation: $y' = x\sqrt{x^2 + 4}$

Solution

$$\begin{aligned}
u = x^2 + 4 &\Rightarrow du = 2x dx \\
x dx &= \frac{1}{2} du \\
\int \sqrt{x^2 + 4} \, x dx &= \int u^{1/2} \frac{1}{2} du \\
&= \frac{1}{2} \frac{u^{3/2}}{3/2} + C \\
&= \frac{1}{3} u^{3/2} + C \\
&= \frac{1}{3} (x^2 + 4)^{3/2} + C
\end{aligned}$$

$$y(x) = \frac{1}{3} (x^2 + 4)^{3/2} + C$$

Exercise

Find the general solution of the differential equation: $y' = (2x + 1)e^{x^2 + x}$

Solution

$$\int dy = \int (2x + 1)e^{x^2 + x} dx \qquad u = x^2 + x \Rightarrow du = (2x + 1) dx$$

$$\int dy = \int e^u du$$

$$y = e^u + C$$

$$y(x) = e^{x^2 + x} + C$$

Exercise

Find the general solution of the differential equation: $y' = \frac{1}{6x-5}$

Solution

$$\int dy = \int \frac{1}{6x-5} dx$$

$$\int dy = \frac{1}{6} \int \frac{1}{6x-5} d(6x-5)$$

$$y(x) = \frac{1}{6} \ln|6x-5| + C$$

Exercise

Find the general solution of the differential equation: $y' = \frac{x^2+2x+3}{x^3+3x^2+9x+1}$

Solution

$$\int dy = \int \frac{x^2+2x+3}{x^3+3x^2+9x+1} dx$$

$$u = x^3 + 3x^2 + 9x + 1 \quad du = 3(x^2 + 2x + 3) dx$$

$$\int dy = \frac{1}{3} \int \frac{du}{u}$$

$$y(x) = \frac{1}{3} \ln|u| + C$$

$$y(x) = \frac{1}{3} \ln|x^3 + 3x^2 + 9x + 1| + C$$

Exercise

Find the general solution of the differential equation: $y' = \frac{1}{x(\ln x)^2}$

Solution

$$\int dy = \int \frac{1}{x(\ln x)^2} dx$$

$$u = \ln x \quad du = \frac{dx}{x}$$

$$\int dy = \int \frac{1}{u^2} du$$

$$y = -\frac{1}{u} + C$$

$$y(x) = -\frac{1}{\ln x} + C$$

Exercise

Evaluate the integrals $\int_{-2}^2 (x^3 - 2x + 3) dx$

Solution

$$\begin{aligned} \int_{-2}^2 (x^3 - 2x + 3) dx &= \left[\frac{x^4}{4} - x^2 + 3x \right]_{-2}^2 \\ &= \left(\frac{(2)^4}{4} - (2)^2 + 3(2) \right) - \left(\frac{(-2)^4}{4} - (-2)^2 + 3(-2) \right) \\ &= 12 \end{aligned}$$

Exercise

Evaluate the integrals $\int_0^1 (x^2 + \sqrt{x}) dx$

Solution

$$\begin{aligned} \int_0^1 (x^2 + \sqrt{x}) dx &= \left[\frac{x^3}{3} + \frac{2}{3} x^{3/2} \right]_0^1 \\ &= \left(\frac{(1)^3}{3} + \frac{2}{3} (1)^{3/2} \right) - 0 \\ &= 1 \end{aligned}$$

Exercise

Evaluate the integrals $\int_0^{\pi/3} 4 \sec u \tan u \, du$

Solution

$$\begin{aligned} \int_0^{\pi/3} 4 \sec u \tan u \, du &= 4 \sec u \Big|_0^{\pi/3} \\ &= 4 \left(\sec \frac{\pi}{3} - \sec 0 \right) \\ &= 4(2 - 1) \\ &= 4 \end{aligned}$$

Exercise

Evaluate the integrals $\int_{\pi/4}^{3\pi/4} \csc \theta \cot \theta d\theta$

Solution

$$\begin{aligned}\int_{\pi/4}^{3\pi/4} \csc \theta \cot \theta d\theta &= -\csc \theta \Big|_{\pi/4}^{3\pi/4} \\ &= -\left(\csc \frac{3\pi}{4} - \csc \frac{\pi}{4}\right) \\ &= -(\sqrt{2} - \sqrt{2}) \\ &= \underline{0}\end{aligned}$$

Exercise

Evaluate the integrals $\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt$

Solution

$$\begin{aligned}\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt &= \int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \pi t^{-2}\right) dt \\ &= \left[4\tan t - \pi t^{-1}\right]_{-\pi/3}^{-\pi/4} \\ &= \left(4\tan\left(-\frac{\pi}{4}\right) - \pi\left(-\frac{4}{\pi}\right)\right) - \left(4\tan\left(-\frac{\pi}{3}\right) - \pi\left(-\frac{3}{\pi}\right)\right) \\ &= (4(-1) + 4) - (4(-\sqrt{3}) + 3) \\ &= -(-4\sqrt{3} + 3) \\ &= \underline{4\sqrt{3} - 3}\end{aligned}$$

Exercise

Evaluate the integrals $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy$

Solution

$$\begin{aligned}\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy &= \int_{-3}^{-1} \left(\frac{y^5}{y^3} - \frac{2y}{y^3}\right) dy \\ &= \int_{-3}^{-1} (y^2 - 2y^{-2}) dy\end{aligned}$$

$$\begin{aligned}
&= \left[\frac{1}{3} y^3 + 2y^{-1} \right]_{-3}^{-1} \\
&= \left(\frac{1}{3}(-1)^3 + \frac{2}{-1} \right) - \left(\frac{1}{3}(-3)^3 + \frac{2}{-3} \right) \\
&= \underline{\underline{\frac{22}{3}}}
\end{aligned}$$

Exercise

Evaluate the integrals $\int_1^8 \frac{(x^{1/3} + 1)(2 - x^{2/3})}{x^{1/3}} dx$

Solution

$$\begin{aligned}
\int_1^8 \frac{(x^{1/3} + 1)(2 - x^{2/3})}{x^{1/3}} dx &= \int_1^8 \frac{2x^{1/3} - x + 2 - x^{2/3}}{x^{1/3}} dx \\
&= \int_1^8 \left(2 - x^{2/3} + 2x^{-1/3} - x^{1/3} \right) dx \\
&= \left[2x - \frac{3}{5}x^{5/3} + 3x^{2/3} - \frac{3}{4}x^{4/3} \right]_1^8 \\
&= \left(2(8) - \frac{3}{5}(8)^{5/3} + 3(8)^{2/3} - \frac{3}{4}(8)^{4/3} \right) - \left(2(1) - \frac{3}{5}(1)^{5/3} + 3(1)^{2/3} - \frac{3}{4}(1)^{4/3} \right) \\
&= \left(-\frac{16}{5} \right) - \left(\frac{73}{20} \right) \\
&= \underline{\underline{-\frac{137}{20}}}
\end{aligned}$$

Exercise

Evaluate: $\int_0^1 (2t + 3)^3 dt$

Solution

$$\begin{aligned}
\int_0^1 (2t + 3)^3 dt &= \int_0^1 u^3 \frac{1}{2} du & u = 2t + 3 \Rightarrow du = 2dt \rightarrow \frac{du}{2} = dt \\
&= \frac{1}{2} \int_0^1 u^3 du \\
&= \frac{1}{2} \frac{u^4}{4} \Big|_0^1 \\
&= \frac{1}{8} (2t + 3)^4 \Big|_0^1
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{8} \left[(2(1) + 3)^4 - (2(0) + 3)^4 \right] \\
&= \frac{1}{8} \left[5^4 - 3^4 \right] \\
&= \underline{68}
\end{aligned}$$

Exercise

Evaluate the integral $\int_{-1}^1 r\sqrt{1-r^2} \, dr$

Solution

$$\text{Let } u = 1 - r^2 \Rightarrow du = -2rdr \rightarrow -\frac{1}{2}du = rdr$$

$$\begin{aligned}
\int_{-1}^1 r\sqrt{1-r^2} \, dr &= \int_{-1}^1 u^{1/2} \left(-\frac{1}{2}du\right) \\
&= -\frac{1}{2} \frac{2}{3} u^{3/2} \Big|_{-1}^1 \\
&= -\frac{1}{3} \left[(1-r^2)^{3/2} \right]_{-1}^1 \\
&= -\frac{1}{3} \left[(1-(1)^2)^{3/2} - (1-(-1)^2)^{3/2} \right] \\
&= -\frac{1}{3} [0 - 0] \\
&= \underline{0}
\end{aligned}$$

Exercise

Find the general solution of $F'(x) = 4x + 2$, and find the particular solution that satisfies the initial condition $F(1) = 8$.

Solution

$$\begin{aligned}
F(x) &= \int (4x + 2) dx \\
&= 2x^2 + 2x + C
\end{aligned}$$

$$F(x) = 2(1)^2 + 2(1) + C = 8$$

$$2 + 2 + C = 8$$

$$C = 4$$

$$\Rightarrow \boxed{F(x) = 2x^2 + 2x + 4}$$

Exercise

Find the general solution of the differential equation: $y' = t \cos 3t$

Solution

$$u = t \rightarrow du = dt$$

$$dv = \cos 3t \rightarrow v = \frac{1}{3} \sin 3t$$

$$\begin{aligned} y &= \frac{1}{3} t \sin 3t - \frac{1}{3} \int \sin 3t dt \\ &= \frac{1}{3} t \sin 3t - \frac{1}{3} \frac{1}{3} \cos 3t + C \end{aligned}$$

$$y(t) = \frac{1}{3} t \sin 3t - \frac{1}{9} \cos 3t + C$$

Exercise

A ball is thrown into the air from an initial height of 6 m with an initial velocity of 120 m/s. What will be the maximum height of the ball and at what time will this event occur?

Solution

$$\frac{dv}{dt} = -g \Rightarrow dv = -g dt$$

$$v(t) = -gt + C_1$$

$$v(t = 0) = -g(0) + C_1 = 120$$

$$C_1 = 120$$

$$v(t) = -9.8t + 120$$

$$\frac{dx}{dt} = v \Rightarrow dx = v dt$$

$$x(t) = \int (-9.8t + 120) dt$$

$$= -4.9t^2 + 120t + C_2$$

$$x(0) = -4.9(0)^2 + 120(0) + C_2 = 6$$

$$C_2 = 6$$

$$x(t) = -4.9t^2 + 120t + 6$$

$$v(t) = -9.8t + 120 = 0 \rightarrow t = \frac{120}{9.8} = 12.24 \text{ sec}$$

$$x(t = 12.24) = -4.9(12.24)^2 + 120(12.24) + 6$$

$$x(t) = 740.69 \text{ m}$$

Exercise

Derive the position function if a ball is thrown upward with initial velocity of 32 *ft* per *second* from an initial height of 48 *ft*. When does the ball hit the ground? With what velocity does the ball hit the ground?

Solution

$$s(t) = -16t^2 + 32t + 48$$

$$s(0) = 48$$

$$s''(t) = -32$$

$$\begin{aligned} s'(t) &= \int -32 dt \quad s'(0) = 32 \\ &= -32t + C_1 \end{aligned}$$

$$\begin{aligned} s'(0) &= -32(0) + C_1 = 32 \\ \Rightarrow C_1 &= 32 \end{aligned}$$

$$s'(t) = -32t + 32$$

$$\begin{aligned} s(t) &= \int (-32t + 32) dt \\ &= -32 \frac{t^2}{2} + 32t + C_2 \end{aligned}$$

$$s(0) = -32 \frac{0^2}{2} + 32(0) + C_2 = 48 \quad \Rightarrow C_2 = 48$$

$$s(t) = -16t^2 + 32t + 48$$

$$s(t) = -16t^2 + 32t + 48 = 0$$

$$-t^2 + 2t + 3 = 0 \Rightarrow t = -1, t = 3$$

The ball hits the ground in **3** seconds

The velocity: $v(t) = s'(t) = -32t + 32$

$$v(t = 3) = -32(3) + 32 = \underline{-64 \text{ ft} / \text{sec}^2}$$