Solution Section 3.1 – Inner Products

Exercise

Let $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ be the Euclidean inner product on R^2, and let $\boldsymbol{u} = (1,\ 1)$, $\boldsymbol{v} = (3,\ 2)$, $\boldsymbol{w} = (0,\ -1)$, and $k = 3$. Compute the following.

a) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ *c)* $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle$ *e)* $d(\boldsymbol{u}, \boldsymbol{v})$

b) $\langle k\boldsymbol{v}, \boldsymbol{w} \rangle$ *d)* $\|\boldsymbol{v}\|$ *f)* $\|\boldsymbol{u} - k\boldsymbol{v}\|$

Solution

a) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 1(3) + 1(2) = \underline{5}$

b)
$$\begin{aligned}\langle k\boldsymbol{v}, \boldsymbol{w} \rangle &= \langle 3v,\ w \rangle \\ &= 9 \cdot 0 + 6 \cdot (-1) \\ &= \underline{-6}\end{aligned}$$

c)
$$\begin{aligned}\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle &= \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle \\ &= 1 \cdot 0 + 1 \cdot (-1) + 3 \cdot 0 + 2 \cdot (-1) \\ &\underline{= -3}\end{aligned}$$

d) $\|\boldsymbol{v}\| = \sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle} = \sqrt{3^2 + 2^2} = \underline{\sqrt{13}}$

e)
$$\begin{aligned}d(\boldsymbol{u}, \boldsymbol{v}) &= \|\boldsymbol{u} - \boldsymbol{v}\| \\ &= \|(-2,\ -1)\| \\ &= \sqrt{(-2)^2 + (-1)^2} \\ &= \underline{\sqrt{5}}\end{aligned}$$

f)
$$\begin{aligned}\|\boldsymbol{u} - k\boldsymbol{v}\| &= \|(1,1) - 3(3,2)\| \\ &= \|(-8,\ -5)\| \\ &= \sqrt{(-8)^2 + (-5)^2} \\ &= \underline{\sqrt{89}}\end{aligned}$$

Exercise

Let $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle$ be the Euclidean inner product on R^2, and let $\boldsymbol{u}=(1,\ 1)$, $\boldsymbol{v}=(3,\ 2)$, $\boldsymbol{w}=(0,\ -1)$, and $k=3$. Compute the following for the weighted Euclidean inner product $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = 2u_1v_1 + 3u_2v_2$.

a) $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle$ *c)* $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w}\rangle$ *e)* $d(\boldsymbol{u},\ \boldsymbol{v})$

b) $\langle k\boldsymbol{v},\ \boldsymbol{w}\rangle$ *d)* $\|\boldsymbol{v}\|$ *f)* $\|\boldsymbol{u}-k\boldsymbol{v}\|$

Solution

a) $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = 2(1)(3)+3(1)(2) = \underline{12|}$

b) $\langle k\boldsymbol{v},\ \boldsymbol{w}\rangle = 2(3\cdot 3)(0)+3(3\cdot 2)(-1) = \underline{-18|}$

c) $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w}\rangle = \langle \boldsymbol{u},\ \boldsymbol{w}\rangle + \langle \boldsymbol{v},\ \boldsymbol{w}\rangle$

$$= 1\cdot 0 + 1\cdot(-1) + 3\cdot 0 + 2\cdot(-1)$$

$$\underline{=-3|}$$

d) $\|\boldsymbol{v}\| = \sqrt{\langle \boldsymbol{v},\boldsymbol{v}\rangle} = \sqrt{2(3)(3)+3(2)(2)} = \underline{\sqrt{30}|}$

e) $d(\boldsymbol{u},\ \boldsymbol{v}) = \|\boldsymbol{u}-\boldsymbol{v}\|$

$$= \|\langle(-2,\ -1)\rangle\|$$

$$= \sqrt{2(-2)(-2)+3(-1)(-1)}$$

$$= \underline{\sqrt{11}|}$$

f) $\|\boldsymbol{u}-k\boldsymbol{v}\| = \|(1,1)-3(3,2)\|$

$$= \|\langle(-8,\ -5)\rangle\|$$

$$= \sqrt{2(-8)^2+3(-5)^2}$$

$$= \underline{\sqrt{203}|}$$

Exercise

Let $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ be the Euclidean inner product on R^2, and let $\boldsymbol{u} = (3, -2)$, $\boldsymbol{v} = (4, 5)$, $\boldsymbol{w} = (-1, 6)$, and $k = -4$. Verify the following.

a) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$

b) $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle$

c) $\langle \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{u}, \boldsymbol{w} \rangle$

d) $\langle k\boldsymbol{u}, \boldsymbol{v} \rangle = k\langle \boldsymbol{u}, \boldsymbol{v} \rangle$

e) $\langle \mathbf{0}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \mathbf{0} \rangle = 0$

Solution

a) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 3 \cdot 4 + (-2) \cdot (5) \underline{= 2}$

$\langle \boldsymbol{v}, \boldsymbol{u} \rangle = 4 \cdot 3 + (5) \cdot (-2) \underline{= 2}$

b) $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle (7,3), (-1,6) \rangle = 7(-1) + 3(6) = \underline{11}$

$\langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle = (3)(-1) + (-2)(6) + (4)(-1) + (5)(6) = \underline{11}$

c) $\langle \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w} \rangle = \langle (3,-2), (3,11) \rangle = 3(3) + (-2)(11) = \underline{-13}$

$\langle \boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{u}, \boldsymbol{w} \rangle = (3)(4) + (-2)(5) + (3)(-1) + (-2)(6) = \underline{-13}$

d) $\langle k\boldsymbol{u}, \boldsymbol{v} \rangle = (-4 \cdot 3) \cdot 4 + ((-4)(-2)) \cdot (5) = \underline{-8}$

$k\langle \boldsymbol{u}, \boldsymbol{v} \rangle = (-4)(3 \cdot 4 + (-2) \cdot (5)) = \underline{-8}$

e) $\langle \mathbf{0}, \boldsymbol{v} \rangle = 0 \cdot 4 + 0 \cdot (5) = \underline{0}$

$\langle \boldsymbol{v}, \mathbf{0} \rangle = 4 \cdot 0 + (5) \cdot (0) = \underline{0}$

Exercise

Let $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ be the Euclidean inner product on R^2, and let $\boldsymbol{u} = (3, -2)$, $\boldsymbol{v} = (4, 5)$, $\boldsymbol{w} = (-1, 6)$, and $k = -4$. Verify the following for the weighted Euclidean inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 4u_1v_1 + 5u_2v_2$.

a) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$

b) $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle$

c) $\langle \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{u}, \boldsymbol{w} \rangle$

d) $\langle k\boldsymbol{u}, \boldsymbol{v} \rangle = k\langle \boldsymbol{u}, \boldsymbol{v} \rangle$

e) $\langle \mathbf{0}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \mathbf{0} \rangle = 0$

Solution

a) $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = 4\cdot 3\cdot 4 + 5\cdot(-2)\cdot(5) = \underline{-2}|$

$\langle \boldsymbol{v},\ \boldsymbol{u}\rangle = 4\cdot 4\cdot 3 + 5\cdot(5)\cdot(-2) = \underline{-2}|$

b) $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w}\rangle = \langle (7,3),\ (-1,6)\rangle = 4\cdot 7(-1) + 5\cdot 3(6) = \underline{62}|$

$\langle \boldsymbol{u},\ \boldsymbol{w}\rangle + \langle \boldsymbol{v},\ \boldsymbol{w}\rangle = 4\cdot(3)(-1) + 5\cdot(-2)(6) + 4\cdot(4)(-1) + 5\cdot(5)(6) = \underline{62}|$

c) $\langle \boldsymbol{u},\ \boldsymbol{v}+\boldsymbol{w}\rangle = \langle (3,-2),\ (3,11)\rangle = 4\cdot 3(3) + 5\cdot(-2)(11) = \underline{-74}|$

$\langle \boldsymbol{u},\ \boldsymbol{v}\rangle + \langle \boldsymbol{u},\ \boldsymbol{w}\rangle = 4\cdot(3)(4) + 5\cdot(-2)(5) + 4\cdot(3)(-1) + 5\cdot(-2)(6) = \underline{-74}|$

d) $\langle k\boldsymbol{u},\ \boldsymbol{v}\rangle = 4\cdot(-4\cdot 3)\cdot 4 + 5\cdot((-4)(-2))\cdot(5) = \underline{8}|$

$k\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = (-4)(4\cdot 3\cdot 4 + 5\cdot(-2)\cdot(5)) = \underline{8}|$

e) $\langle \boldsymbol{0},\ \boldsymbol{v}\rangle = 4\cdot 0\cdot 4 + 5\cdot 0\cdot(5) = \underline{0}|$

$\langle \boldsymbol{v},\ \boldsymbol{0}\rangle = 4\cdot 4\cdot 0 + 5\cdot(5)\cdot(0) = \underline{0}|$

Exercise

Let $\boldsymbol{u} = (u_1,\ u_2)$ and $\boldsymbol{v} = (v_1,\ v_2)$. Show that the following are inner product on R^3 by verifying that the inner product axioms hold. $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = 3u_1v_1 + 5u_2v_2$

Solution

Axiom 1: $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = 3u_1v_1 + 5u_2v_2 = 3v_1u_1 + 5v_2u_2 = \langle \boldsymbol{v},\ \boldsymbol{u}\rangle$

Axiom 2:
$$\begin{aligned}\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w}\rangle &= 3(u_1+v_1)w_1 + 5(u_2+v_2)w_2 \\ &= 3(u_1w_1 + v_1w_1) + 5(u_2w_2 + v_2w_2) \\ &= 3u_1w_1 + 3v_1w_1 + 5u_2w_2 + 5v_2w_2 \\ &= (3u_1w_1 + 5u_2w_2) + (3v_1w_1 + 5v_2w_2) \\ &= \langle \boldsymbol{u},\ \boldsymbol{w}\rangle + \langle \boldsymbol{v},\ \boldsymbol{w}\rangle\end{aligned}$$

Axiom 3:
$$\begin{aligned}\langle k\boldsymbol{u},\ \boldsymbol{v}\rangle &= 3(ku_1)v_1 + 5(ku_2)v_2 \\ &= k(3u_1v_1 + 5u_2v_2) \\ &= k\langle \boldsymbol{u},\ \boldsymbol{v}\rangle\end{aligned}$$

Axiom 4:
$$\begin{aligned}\langle \boldsymbol{v},\ \boldsymbol{v}\rangle &= 3v_1v_1 + 5v_2v_2 \\ &= 3v_1^2 + 5v_2^2 \geq 0\end{aligned}$$

$v_1 = v_2 = 0 \quad iff \quad \boldsymbol{v} = \boldsymbol{0}$

Exercise

Show that the following identity holds for the vectors in any inner product space

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2+\|\boldsymbol{u}-\boldsymbol{v}\|^2=2\|\boldsymbol{u}\|^2+2\|\boldsymbol{v}\|^2$$

Solution

$$\begin{aligned}
\|\boldsymbol{u}+\boldsymbol{v}\|^2+\|\boldsymbol{u}-\boldsymbol{v}\|^2 &= \langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{u}+\boldsymbol{v}\rangle+\langle \boldsymbol{u}-\boldsymbol{v},\ \boldsymbol{u}-\boldsymbol{v}\rangle \\
&= \langle \boldsymbol{u},\boldsymbol{u}+\boldsymbol{v}\rangle+\langle \boldsymbol{v},\boldsymbol{u}+\boldsymbol{v}\rangle+\langle \boldsymbol{u},\boldsymbol{u}-\boldsymbol{v}\rangle-\langle \boldsymbol{v},\boldsymbol{u}-\boldsymbol{v}\rangle \\
&= \langle \boldsymbol{u},\boldsymbol{u}\rangle+\langle \boldsymbol{u},\boldsymbol{v}\rangle+\langle \boldsymbol{v},\boldsymbol{u}\rangle+\langle \boldsymbol{v},\boldsymbol{v}\rangle+\langle \boldsymbol{u},\boldsymbol{u}\rangle-\langle \boldsymbol{u},\boldsymbol{v}\rangle-\langle \boldsymbol{v},\boldsymbol{u}\rangle+\langle \boldsymbol{v},\boldsymbol{v}\rangle \\
&= 2\langle \boldsymbol{u},\boldsymbol{u}\rangle+2\langle \boldsymbol{v},\boldsymbol{v}\rangle \\
&= 2\|\boldsymbol{u}\|^2+2\|\boldsymbol{v}\|^2 \quad \checkmark
\end{aligned}$$

Solution   *Section* 3.2 – Angle and Orthogonality in Inner Product Spaces

Exercise

Which of the following form orthonormal sets?

a) $(1, 0), (0, 2)$ in $\boldsymbol{R}^2$

b) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\boldsymbol{R}^2$

c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\boldsymbol{R}^2$

d) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ in $\boldsymbol{R}^3$

e) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ in $\boldsymbol{R}^3$

f) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ in $\boldsymbol{R}^3$

Solution

a) $(1, 0)\cdot(0, 2) = 1(0) + 0(2) = 0$, they are ***orthonormal*** sets

b) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \bullet \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} = \frac{1}{2} - \frac{1}{2} = 0$, they are orthonormal sets

c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \bullet \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} = -\frac{1}{2} - \frac{1}{2}\ \underline{=-1}$

They are ***not orthonormal***

d) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \bullet \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \bullet \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{2}}\right) + 0 + \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{2}}$$

$$= -\frac{1}{2}\frac{1}{\sqrt{3}} - \frac{1}{2}\frac{1}{\sqrt{3}}$$

$$= -\frac{1}{\sqrt{3}} \neq 0$$

They are ***not orthonormal*** sets

e) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \bullet \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) \bullet \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

$$=\frac{2}{3}\frac{2}{3}\frac{1}{3}+\left(-\frac{2}{3}\right)\frac{1}{3}\frac{2}{3}+\frac{1}{3}\left(-\frac{2}{3}\right)\frac{2}{3}$$

$$=\frac{4}{27}-\frac{4}{27}-\frac{4}{27}$$

$$=-\frac{4}{27}\neq 0$$

They are ***not orthonormal*** sets

f) $\left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right)\bullet\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)=\frac{1}{\sqrt{6}}\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{2}}\right)+0\ \underline{=0}$

They are ***orthonormal*** sets

Exercise

Find the cosine of the angle between ***u*** and ***v***.

a) $\boldsymbol{u}=(1,-3),\quad \boldsymbol{v}=(2,4)$

b) $\boldsymbol{u}=(-1,0),\quad \boldsymbol{v}=(3,8)$

c) $\boldsymbol{u}=(-1,5,2),\quad \boldsymbol{v}=(2,4,-9)$

d) $\boldsymbol{u}=(4,1,8),\quad \boldsymbol{v}=(1,0,-3)$

e) $\boldsymbol{u}=(1,0,1,0),\quad \boldsymbol{v}=(-3,-3,-3,-3)$

f) $\boldsymbol{u}=(2,1,7,-1),\quad \boldsymbol{v}=(4,0,0,0)$

Solution

a) $\boldsymbol{u}=(1,-3),\quad \boldsymbol{v}=(2,4)$

$$\|\boldsymbol{u}\|=\sqrt{1^2+(-3)^2}=\underline{\sqrt{10}}$$

$$\|\boldsymbol{v}\|=\sqrt{2^2+4^2}=\underline{\sqrt{20}}$$

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle=1(2)+(-3)(4)=\underline{-10}$$

$$\cos\theta=\frac{\langle \boldsymbol{u},\boldsymbol{v}\rangle}{\|\boldsymbol{u}\|.\|\boldsymbol{v}\|}$$

$$=\frac{-10}{\sqrt{10}\sqrt{20}}$$

$$=-\frac{10}{\sqrt{200}}$$

$$\underline{=-\frac{1}{\sqrt{2}}}$$

b) $\boldsymbol{u} = (-1, 0), \quad \boldsymbol{v} = (3, 8)$

$$\|\boldsymbol{u}\| = \sqrt{(-1)^2 + 0^2} = 1$$

$$\|\boldsymbol{v}\| = \sqrt{3^2 + 8^2} = \sqrt{73}$$

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = (-1)(3) + (0)(8) = -3$$

$$\cos\theta = \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\|\boldsymbol{u}\| . \|\boldsymbol{v}\|} = \frac{-3}{1\sqrt{73}} = -\frac{3}{\sqrt{73}}$$

c) $\boldsymbol{u} = (-1, 5, 2), \quad \boldsymbol{v} = (2, 4, -9)$

$$\|\boldsymbol{u}\| = \sqrt{(-1)^2 + 5^2 + 2^2} = \sqrt{30}$$

$$\|\boldsymbol{v}\| = \sqrt{2^2 + 4^2 + (-9)^2} = \sqrt{101}$$

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = (-1)(2) + (5)(4) + (2)(-9) = 0$$

$$\cos\theta = \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\|\boldsymbol{u}\| . \|\boldsymbol{v}\|} = 0$$

d) $\boldsymbol{u} = (4, 1, 8), \quad \boldsymbol{v} = (1, 0, -3)$

$$\|\boldsymbol{u}\| = \sqrt{4^2 + 1^2 + 8^2} = 9$$

$$\|\boldsymbol{v}\| = \sqrt{1^2 + 0^2 + (-3)^2} = \sqrt{10}$$

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = (4)(1) + (1)(0) + (8)(-3) = -20$$

$$\cos\theta = \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\|\boldsymbol{u}\| . \|\boldsymbol{v}\|} = -\frac{20}{9\sqrt{10}}$$

e) $\boldsymbol{u} = (1, 0, 1, 0), \quad \boldsymbol{v} = (-3, -3, -3, -3)$

f) $\boldsymbol{u} = (2, 1, 7, -1), \quad \boldsymbol{v} = (4, 0, 0, 0)$

$$\|\boldsymbol{u}\| = \sqrt{2^2 + 1^2 + 7^2 + (-1)^2} = \sqrt{55}$$

$$\|\boldsymbol{v}\| = \sqrt{4^2 + 0} = 4$$

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = (2)(4) + (1)(0) + (7)(0) + (-1)(0) = 8$$

$$\cos\theta = \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\|\boldsymbol{u}\| . \|\boldsymbol{v}\|} = \frac{8}{4\sqrt{55}} = \frac{2}{\sqrt{55}}$$

Exercise

Find the cosine of the angle between ***A*** and ***B***.

a) $A=\begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix}$ $B=\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$

b) $A=\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}$ $B=\begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$

Solution

a)

$$\begin{aligned}\|A\| &= \sqrt{\langle A, A\rangle} \\ &= \sqrt{2^2+6^2+1^2+(-3)^2} \\ &= \sqrt{50} \\ &= 5\sqrt{2}\end{aligned}$$

$$\begin{aligned}\|B\| &= \sqrt{\langle B, B\rangle} \\ &= \sqrt{3^2+2^2+1^2+0^2} \\ &= \sqrt{14}\end{aligned}$$

$$\langle A, B\rangle = 2(3)+6(2)+1(1)+(-3)(0)=19$$

$$\cos\theta = \frac{\langle \boldsymbol{A}, \boldsymbol{B}\rangle}{\|\boldsymbol{A}\|.\|B\|} = \frac{19}{5\sqrt{2}\sqrt{14}} = \boxed{\frac{19}{10\sqrt{7}}}$$

b)

$$\begin{aligned}\|A\| &= \sqrt{\langle A, A\rangle} \\ &= \sqrt{2^2+4^2+(-1)^2+3^2} \\ &= \sqrt{30}\end{aligned}$$

$$\begin{aligned}\|B\| &= \sqrt{\langle B, B\rangle} \\ &= \sqrt{(-3)^2+1^2+4^2+2^2} \\ &= \sqrt{30}\end{aligned}$$

$$\langle A, B\rangle = 2(-3)+4(1)+(-1)(4)+3(2)=0$$

$$\cos\theta = \frac{\langle \boldsymbol{A}, \boldsymbol{B}\rangle}{\|\boldsymbol{A}\|.\|B\|} = \frac{0}{30} = \boxed{0}$$

Exercise

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

a) $\boldsymbol{u}=(-1,3,2), \quad \boldsymbol{v}=(4,2,-1)$

b) $\boldsymbol{u}=(a,b), \quad \boldsymbol{v}=(-b,a)$

c) $\boldsymbol{u}=(-2,-2,-2), \quad \boldsymbol{v}=(1,1,1)$

d) $\boldsymbol{u}=(-4,\ 6,\ -10,\ 1), \quad \boldsymbol{v}=(2,\ 1,\ -2,\ 9)$

e) $\boldsymbol{u}=(-4,\ 6,\ -10,\ 1), \quad \boldsymbol{v}=(2,\ 1,\ -2,\ 9)$

Solution

a) $\langle \boldsymbol{u},\boldsymbol{v}\rangle=(-1)(4)+3(2)+2(-1)=\underline{0|}$ Therefore the given vectors are orthogonal.

b) $\langle \boldsymbol{u},\boldsymbol{v}\rangle=a(-b)+b(a)=\underline{0|}$ Therefore the given vectors are orthogonal.

c) $\langle \boldsymbol{u},\boldsymbol{v}\rangle=(-2)(1)+(-2)(1)+(-2)(1)=\underline{-6|}$ Therefore the given vectors are ***not*** orthogonal.

d) $\langle \boldsymbol{u},\boldsymbol{v}\rangle=(-4)(2)+(6)(1)+(-10)(-2)+(1)(9)=\underline{27|}$ Therefore the given vectors are ***not*** orthogonal.

e) $\|\boldsymbol{u}\|=\sqrt{(-4)^2+6^2+(-10)^2+1^2}=\sqrt{153}=3\sqrt{17}$

$$\|\boldsymbol{v}\|=\sqrt{2^2+1^2+(-2)^2+9^2}=\sqrt{90}=3\sqrt{10}$$

$$\langle \boldsymbol{u},\ \boldsymbol{v}\rangle=(-4)(2)+6(1)-10(-2)+1(9)\underline{=27|}$$

$$\cos\theta=\frac{\langle \boldsymbol{u},\ \boldsymbol{v}\rangle}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}$$

$$=\frac{27}{3\sqrt{17}\left(3\sqrt{10}\right)}$$

$$\underline{=\frac{3}{\sqrt{170}}|}$$

The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are NOT orthogonal with respect to the Euclidean

Exercise

Do there exist scalars k and l such that the vectors $\boldsymbol{u}=(2,k,6), \quad \boldsymbol{v}=(l,5,3), \quad and \quad \boldsymbol{w}=(1,2,3)$ are mutually orthogonal with respect to the Euclidean inner product?

Solution

$$\langle \boldsymbol{u},\boldsymbol{w}\rangle=(2)(1)+(k)(2)+(6)(3)=20+2k=0 \quad \Rightarrow \quad \boxed{k=-10}$$

$$\langle \boldsymbol{v},\boldsymbol{w}\rangle=(l)(1)+(5)(2)+(3)(3)=l+19=0 \quad \Rightarrow \quad \boxed{l=-19}$$

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle=(2)(l)+(k)(5)+(6)(3)=2l+5k+18=0$$

$$2(-19)+5(-10)+18=-70\neq 0$$

Thus, there are no scalars such that the vectors are mutually orthogonal

Exercise

Let $\boldsymbol{R}^3$ have the Euclidean inner product. For which values of k are $\boldsymbol{u}$ and $\boldsymbol{v}$ orthogonal?

a) $\boldsymbol{u} = (2,1,3), \quad \boldsymbol{v} = (1,7,k)$

b) $\boldsymbol{u} = (k,k,1), \quad \boldsymbol{v} = (k,5,6)$

Solution

a) $\langle \boldsymbol{u},\boldsymbol{v} \rangle = (2)(1)+(1)(7)+(3)(k)$

$= 9 + 3k = 0$

$\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal for $\boxed{k = -3}$

b) $\langle \boldsymbol{u},\boldsymbol{v} \rangle = (k)(k)+(k)(5)+(1)(6)$

$= k^2 + 5k + 6 = 0$

$\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal for $\boxed{k = -2, \ -3}$

Exercise

Let V be an inner product space. Show that if $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal unit vectors in V, then $\|\boldsymbol{u}-\boldsymbol{v}\| = \sqrt{2}$

Solution

$$\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{v}\|^2 &= \langle \boldsymbol{u}-\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v} \rangle \\
&= \langle \boldsymbol{u}, \boldsymbol{u}-\boldsymbol{v} \rangle - \langle \boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v} \rangle \\
&= \langle \boldsymbol{u},\boldsymbol{u} \rangle - \langle \boldsymbol{u},\boldsymbol{v} \rangle - \langle \boldsymbol{v},\boldsymbol{u} \rangle + \langle \boldsymbol{v},\boldsymbol{v} \rangle \\
&= \|\boldsymbol{u}\|^2 - 0 - 0 + \|\boldsymbol{v}\|^2 \\
&= 2
\end{aligned}$$

Thus $\|\boldsymbol{u}-\boldsymbol{v}\| = \sqrt{2}$

Exercise

Let $\mathbf{S}$ be a subspace of $\mathbb{R}^n$. Explain what $\left(\mathbf{S}^{\perp}\right)^{\perp} = \mathbf{S}$ means and why it is true.

Solution

$\left(\mathbf{S}^{\perp}\right)^{\perp}$ is the orthogonal complement of , $\mathbf{S}^{\perp}$, which is itself the orthogonal complement of **S**, so $\left(\mathbf{S}^{\perp}\right)^{\perp} = \mathbf{S}$ means that **S** is the orthogonal of its orthogonal complement.

We need to show that **S** is contained in $\left(\mathbf{S}^{\perp}\right)^{\perp}$ and, conversely, that $\left(\mathbf{S}^{\perp}\right)^{\perp}$ is contained in **S** to be true.

i. Suppose $\vec{v} \in \mathbf{S}$ and $\vec{w} \in \mathbf{S}^{\perp}$. Then $\langle \vec{v},\ \vec{w} \rangle = 0$ by definition of $\mathbf{S}^{\perp}$. Thus **S** is certainly contained is $\left(\mathbf{S}^{\perp}\right)^{\perp}$ (which consists of all vectors in $\mathbb{R}^n$ which are orthogonal to $\mathbf{S}^{\perp}$).

ii. Suppose $\vec{v} \in \left(\mathbf{S}^{\perp}\right)^{\perp}$ (means $\vec{v}$ is orthogonal to all vectors in $\mathbf{S}^{\perp}$); then we need to show that $\vec{v} \in \mathbf{S}$.

Let assume $\left\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\right\}$ be a basis for **S** and let $\left\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_q\right\}$ be a basis for $\mathbf{S}^{\perp}$. If $\vec{v} \notin \mathbf{S}$, then $\left\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p, \vec{v}\right\}$ is linearly independent set. Since each vector ifs that set is orthogonal to all of $\mathbf{S}^{\perp}$, the set $\left\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p,\ \vec{v},\ \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_q\right\}$ is linearly independent. Since there are $p+q+1$ vectors in this set, this means that $p+q+1 \le n \quad \Leftrightarrow \quad p+q \le n-1$. On the other hand, If A is the matrix whose i^{th} row is $\vec{u}_i^{\,T}$, then the row space of A is **S** and the nullspace of A is $\mathbf{S}^{\perp}$. Since **S** is p-dimensional, the rank of A is p, meaning that the dimension of $\text{nul}(A) = \mathbf{S}^{\perp}$ is $q = n - p$. Therefore,

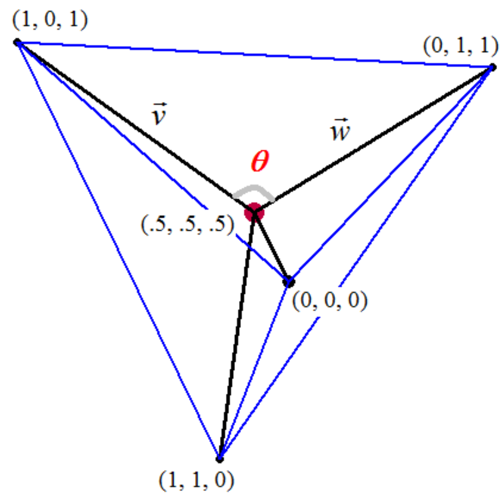
$$p + q = p + (n - p) = n$$

Which contradict the fact that $p + q \le n - 1$. From this, we see that, if $\vec{v} \in \left(\mathbf{S}^{\perp}\right)^{\perp}$, it must be the case that $\vec{v} \in \mathbf{S}$.

Exercise

The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0,\ 0,\ 0)$, $(1,\ 1,\ 0)$, $(1,\ 0,\ 1)$ and $(0,\ 1,\ 1)$ – (***note*** that all six edges have length $\sqrt{2}$, so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$ to the vertices?

Solution



Let $\vec{v}$ be the vector of the segment (1, 0, 1) and $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

Let GB-BXi5-4570R

be the vector of the segment (0, 1, 1) and $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$$\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

We have:

$$\begin{aligned} \cos\theta &= \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \cdot \|\vec{w}\|} \\ &= \frac{\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \cdot \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}{\sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}}\ \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}}} \\ &= \frac{-\frac{1}{4}}{\frac{3}{4}} \\ &= \boxed{-\frac{1}{3}} \end{aligned}$$

$$\boxed{\theta \approx 109.47°}$$

Exercise

Determine if the given vectors are orthogonal.

$$\boldsymbol{x}_1 = (1,\ 0,\ 1,\ 0), \quad \boldsymbol{x}_2 = (0,\ 1,\ 0,\ 1), \quad \boldsymbol{x}_3 = (1,\ 0,\ -1,\ 0), \quad \boldsymbol{x}_4 = (1,\ 1,\ -1,\ -1)$$

Solution

$$\boldsymbol{x}_1 \bullet \boldsymbol{x}_2 = (1,\ 0,\ 1,\ 0) \bullet (0,\ 1,\ 0,\ 1) = 0$$

$$\boldsymbol{x}_1 \bullet \boldsymbol{x}_3 = (1,\ 0,\ 1,\ 0) \bullet (1,\ 0,\ -1,\ 0) = 1 - 1 = 0$$

$$\boldsymbol{x}_1 \bullet \boldsymbol{x}_4 = (1,\ 0,\ 1,\ 0) \bullet (1,\ 1,\ -1,\ -1) = 1 - 1 = 0$$

$$\boldsymbol{x}_2 \bullet \boldsymbol{x}_3 = (0,\ 1,\ 0,\ 1) \bullet (1,\ 0,\ -1,\ 0) = 0$$

$$\boldsymbol{x}_2 \bullet \boldsymbol{x}_4 = (0,\ 1,\ 0,\ 1) \bullet (1,\ 1,\ -1,\ -1) = 1 - 1 = 0$$

$$\boldsymbol{x}_3 \bullet \boldsymbol{x}_4 = (1,\ 0,\ -1,\ 0) \bullet (1,\ 1,\ -1,\ -1) = 1 - 1 = 0$$

The given vectors are orthogonal

Exercise

Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

a) $\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\ \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)\ \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$

b) $\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right)\ \left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)\ \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)$

Solution

a)
$$\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) \bullet \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}}\ \underline{= 0}$$

$$\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) \bullet \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) = -\frac{1}{2} + 0 + \frac{1}{2}\ \underline{= 0}$$

$$\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right) \bullet \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} = -\frac{2}{\sqrt{6}}\ \underline{\neq 0}$$

Therefore the given vectors are ***not*** orthogonal.

b)
$$\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right) \bullet \left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right) = \frac{4}{9} - \frac{2}{9} - \frac{2}{9}\ \underline{= 0}$$

$$\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right) \bullet \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right) = \frac{2}{9} - \frac{4}{9} + \frac{2}{9}\ \underline{= 0}$$

$$\left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right) \bullet \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right) = \frac{2}{9} + \frac{2}{9} - \frac{4}{9}\ \underline{= 0}$$

Therefore the given vectors are orthogonal.

Solution Section 3.3 – Gram-Schmidt Process

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\boldsymbol{R}^m$.

a) $\boldsymbol{u}_1 = (1,\ -3),\quad \boldsymbol{u}_2 = (2,\ 2)$

b) $\boldsymbol{u}_1 = (1,\ 0),\quad \boldsymbol{u}_2 = (3,\ -5)$

c) $\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ (1,\ 2,\ 1)\}$

d) $\{(1,\ 1,\ 1),\ (0,\ 1,\ 1),\ (0,\ 0,\ 1)\}$

e) $\{(1,\ 1,\ 1,\ 1),\ (1,\ 2,\ 1,\ 0),\ (1,\ 3,\ 0,\ 0)\}$

f) $\{(0,\ 2,\ -1,\ 1),\ (0,\ 0,\ 1,\ 1),\ (-2,\ 1,\ 1,\ -1)\}$

g) $\boldsymbol{u}_1 = (1,\ 0,\ 0),\quad \boldsymbol{u}_2 = (3,\ 7,\ -2),\quad \boldsymbol{u}_3 = (0,\ 4,\ 1)$

h) $\boldsymbol{u}_1 = (0,2,1,0),\quad \boldsymbol{u}_2 = (1,-1,0,0),\quad \boldsymbol{u}_3 = (1,2,0,-1),\quad \boldsymbol{u}_4 = (1,0,0,1)$

Solution

a) $$v_1 = \frac{u_1}{\|u_1\|} = \frac{(1,\ -3)}{\sqrt{1^2+(-3)^2}} = \frac{(1,\ -3)}{\sqrt{10}} = \left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)$$

$$\begin{aligned}
w_2 &= u_2 - \left(u_2 . v_1\right) v_1 \\
&= (2,\ 2) - \left[(2,\ 2).\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)\right]\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right) \\
&= (2,\ 2) - \left[\frac{2}{\sqrt{10}} - \frac{6}{\sqrt{10}}\right]\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right) \\
&= (2,\ 2) - \left[-\frac{4}{\sqrt{10}}\right]\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right) \\
&= (2,\ 2) - \left(-\frac{4}{10},\ \frac{12}{10}\right) \\
&= (2,\ 2) - \left(-\frac{2}{5},\ \frac{6}{5}\right) \\
&= \left(\frac{12}{5},\ \frac{4}{5}\right)
\end{aligned}$$

$$\|w_2\| = \sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{144}{25} + \frac{16}{25}} = \sqrt{\frac{160}{25}} = \frac{\sqrt{16(10)}}{\sqrt{25}} = \frac{4\sqrt{10}}{5}$$

$$v_2 = \frac{w_2}{\|w_2\|} = \frac{4\sqrt{10}}{5}\left(\frac{12}{5},\ \frac{4}{5}\right) = \left(\frac{48\sqrt{10}}{25},\ \frac{16\sqrt{10}}{25}\right)$$

b) $u_1 = (1,\ 0),\ u_2 = (3,\ -5)$

$$v_1 = \frac{u_1}{\|u_1\|} = \frac{(1,\ 0)}{\sqrt{1^2+0^2}} = (1,\ 0)$$

$$w_2 = u_2 - (u_2 . v_1)v_1 = (0,\ -5)$$

$$= (3,\ -5) - [(3,\ -5).(1,\ 0)](1,\ 0)$$

$$= (3,\ -5) - [3](1,\ 0)$$

$$= (3,\ -5) - (3,\ 0)$$

$$\|w_2\| = \sqrt{0^2 + (-5)^2} = 5$$

$$v_2 = \frac{w_2}{\|w_2\|} = \frac{1}{5}(0,\ -5) = (0,\ -1)$$

c) $\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ (1,\ 2,\ 1)\}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1,1)}{\sqrt{1^2+1^2+1^2}} = \frac{(1,\ 1,\ 1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$w_2 = v_2 - (v_2 . u_1)u_1$$

$$= (-1,\ 1,\ 0) - \left[(-1,\ 1,\ 0)\bullet\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (-1,\ 1,\ 0) - \left[-\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 0\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (-1,\ 1,\ 0) - 0\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (-1,\ 1,\ 0)$$

$$\|w_2\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{(-1,\ 1,\ 0)}{\sqrt{2}} = \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$v_3 \bullet u_1 = (1,\ 2,\ 1)\bullet\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = \sqrt{3}$$

$$v_3 \bullet u_2 = (1,\ 2,\ 1)\bullet\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) = -\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} + 0 = \frac{1}{\sqrt{2}}$$

$$w_3 = v_3 - (v_3 \bullet u_1)u_1 - (v_3 \bullet u_2)u_2$$

$$= (1,\ 2,\ 1) - \sqrt{3}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) - \sqrt{2}\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= (1,\ 2,\ 1) - (1,\ 1,\ 1) - (-1,\ 1,\ 0)$$

$$= (1,\ 0,\ 0)$$

$$\boxed{u_3 = \frac{w_3}{\|w_3\|} = \frac{(1,\ 0,\ 0)}{\sqrt{1^2}} = (1,\ 0,\ 0)}$$

d) $\{(1,\ 1,\ 1),\ (0,\ 1,\ 1),\ (0,\ 0,\ 1)\}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1,1)}{\sqrt{1^2+1^2+1^2}} = \frac{(1,\ 1,\ 1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\begin{aligned}
w_2 &= v_2 - \left(v_2 . u_1\right) u_1 \\
&= (0,\ 1,\ 1) - \left[(0,\ 1,\ 1) \cdot \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) \\
&= (0,\ 1,\ 1) - \left[\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) \\
&= (0,\ 1,\ 1) - \left[\frac{2}{\sqrt{3}}\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) \\
&= (0,\ 1,\ 1) - \left(\frac{2}{3},\ \frac{2}{3},\ \frac{2}{3}\right) \\
&= \left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)
\end{aligned}$$

$$\|w_2\| = \sqrt{\left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}} = \sqrt{\frac{6}{9}} \boxed{= \frac{\sqrt{6}}{3}}$$

$$\boxed{u_2} = \frac{w_2}{\|w_2\|} = \frac{\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}{\frac{\sqrt{6}}{3}} = \frac{3}{\sqrt{6}}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right) = \left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$v_3 \cdot u_1 = (0,\ 0,\ 1) \cdot \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) \boxed{= \frac{1}{\sqrt{3}}}$$

$$v_3 \cdot u_2 = (0,\ 0,\ 1) \cdot \left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right) \boxed{= \frac{1}{\sqrt{6}}}$$

$$\begin{aligned}
w_3 &= v_3 - \left(v_3 \cdot u_1\right) u_1 - \left(v_3 \cdot u_2\right) u_2 \\
&= (0,\ 0,\ 1) - \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) - \frac{1}{\sqrt{6}}\left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right) \\
&= (0,\ 0,\ 1) - \left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right) - \left(-\frac{1}{3},\ \frac{1}{6},\ \frac{1}{6}\right) \\
&= \left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)
\end{aligned}$$

$$\boxed{u_3} = \frac{w_3}{\|w_3\|} = \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}}$$

$$= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\sqrt{\frac{1}{2}}}$$

$$= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\frac{1}{\sqrt{2}}}$$

$$= \sqrt{2}\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)$$

$$\boxed{= \left(0,\ -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)}$$

e) $\{(1,\ 1,\ 1,\ 1),\ (1,\ 2,\ 1,\ 0),\ (1,\ 3,\ 0,\ 0)\}$

$$\boldsymbol{v}_1 = \boldsymbol{u}_1 \ \boxed{= (1,\ 1,\ 1,\ 1)}$$

$$\boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \frac{(1,\ 1,\ 1,\ 1)}{\sqrt{4}} = \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$\boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$$

$$= (1,\ 2,\ 1,\ 0) - \frac{(1,\ 2,\ 1,\ 0)\bullet(1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}(1,\ 1,\ 1,\ 1)$$

$$= (1,\ 2,\ 1,\ 0) - \frac{1+2+1}{4}(1,\ 1,\ 1,\ 1)$$

$$= (1,\ 2,\ 1,\ 0) - \frac{4}{4}(1,\ 1,\ 1,\ 1)$$

$$= (1,\ 2,\ 1,\ 0) - (1,\ 1,\ 1,\ 1)$$

$$\boxed{= (0,\ 1,\ 0,\ -1)}$$

$$\boldsymbol{q}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} = \frac{(0,\ 1,\ 0,\ -1)}{\sqrt{1+1}} = \left(0,\ \frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right)$$

$$\boldsymbol{v}_3 = \boldsymbol{u}_3 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2$$

$$= (1,\ 3,\ 0,\ 0) - \frac{(1,\ 3,\ 0,\ 0)\bullet(1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}(1,\ 1,\ 1,\ 1) - \frac{(1,\ 3,\ 0,\ 0)\bullet(0,\ 1,\ 0,\ -1)}{\|(0,\ 1,\ 0,\ -1)\|^2}(0,\ 1,\ 0,\ -1)$$

$$= (1,\ 3,\ 0,\ 0) - \frac{4}{4}(1,\ 1,\ 1,\ 1) - \frac{3}{2}(0,\ 1,\ 0,\ -1)$$

$$= (1,\ 3,\ 0,\ 0) - (1,\ 1,\ 1,\ 1) - \left(0,\ \frac{3}{2},\ 0,\ -\frac{3}{2}\right)$$

$$\boxed{= \left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)}$$

$$q_3 = \frac{v_3}{\|v_3\|} = \frac{\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)}{\sqrt{\frac{1}{4}+1+\frac{1}{4}}} = \frac{\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)}{\frac{\sqrt{6}}{2}}$$

$$= \frac{2}{\sqrt{6}}\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)$$

$$\boxed{= \left(0,\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)}$$

f) $\{(0,\ 2,\ -1,\ 1),\ (0,\ 0,\ 1,\ 1),\ (-2,\ 1,\ 1,\ -1)\}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(0,\ 2,\ -1,\ 1)}{\sqrt{2^2+1^2+1^2}} = \frac{(0,\ 2,\ -1,\ 1)}{\sqrt{6}} = \left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$w_2 = v_2 - \left(v_2 . u_1\right)u_1$$

$$= (0,\ 0,\ 1,\ 1) - \left[(0,\ 0,\ 1,\ 1)\bullet\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)\right]\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$= (0,\ 0,\ 1,\ 1) - \left[-\frac{1}{\sqrt{6}}+\frac{1}{\sqrt{6}}\right]\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$= (0,\ 0,\ 1,\ 1) - [0]\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$= (0,\ 0,\ 1,\ 1)$$

$$\boxed{u_2} = \frac{w_2}{\|w_2\|} = \frac{(0,\ 0,\ 1,\ 1)}{\sqrt{2}} = \left(0,\ 0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$v_3\bullet u_1 = (-2,\ 1,\ 1,\ -1)\bullet\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right) = \frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}}\ \boxed{=0}$$

$$v_3\bullet u_2 = (-2,\ 1,\ 1,\ -1)\bullet\left(0,\ 0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\ \boxed{=0}$$

$$w_3 = v_3 - \left(v_3\bullet u_1\right)u_1 - \left(v_3\bullet u_2\right)u_2$$

$$= (-2,\ 1,\ 1,\ -1) - 0 - 0$$

$$= (-2,\ 1,\ 1,\ -1)$$

$$\boxed{u_3} = \frac{w_3}{\|w_3\|} = \frac{(-2,\ 1,\ 1,\ -1)}{\sqrt{(-2)^2+1^2+1^2+(-1)^2}} = \frac{(-2,\ 1,\ 1,\ -1)}{\sqrt{7}}$$

$$\boxed{= \left(-\frac{2}{\sqrt{7}},\ \frac{1}{\sqrt{7}},\ \frac{1}{\sqrt{7}},\ -\frac{1}{\sqrt{7}}\right)}$$

g) $\boldsymbol{u}_1 = (1,\ 0,\ 0),\quad \boldsymbol{u}_2 = (3,\ 7,\ -2),\quad \boldsymbol{u}_3 = (0,\ 4,\ 1)$

$$v_1 = \frac{u_1}{\|u_1\|} = \frac{(1,\ 0,\ 0)}{\sqrt{1^2+0^2+0^2}} = (1,\ 0,\ 0)$$

$$w_2 = u_2 - (u_2.v_1)v_1$$
$$= (3,7,-2) - \left[(3,7,-2)\bullet(1,0,0)\right](1,0,0)$$
$$= (3,7,-2) - 3(1,0,0)$$
$$= (0,\ 7,\ -2)$$

$$v_2 = \frac{w_2}{\|w_2\|} = \frac{(0,\ 7,\ -2)}{\sqrt{7^2+(-2)^2}} = \frac{1}{\sqrt{53}}(0,\ 7,\ -2) = \left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{53}}\right)$$

$$u_3\bullet v_1 = (0,\ 4,\ 1)\bullet(1,\ 0,\ 0) = 0$$

$$u_3\bullet v_2 = (0,\ 4,\ 1)\bullet\left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{53}}\right) = \frac{26}{\sqrt{53}}$$

$$w_3 = u_3 - (u_3\bullet v_1)v_1 - (u_3\bullet v_2)v_2$$
$$= (0,\ 4,\ 1) - 0 - \frac{26}{\sqrt{53}}\left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{6}}\right)$$
$$= (0,\ 4,\ 1) - \left(0,\ \frac{182}{53},\ -\frac{52}{53}\right)$$
$$= \left(0,\ \frac{30}{53},\ \frac{105}{53}\right)$$

$$v_3 = \frac{w_3}{\|w_3\|} = \frac{\left(0,\ \frac{30}{53},\ \frac{105}{53}\right)}{\sqrt{\left(\frac{30}{53}\right)^2 + \left(\frac{105}{53}\right)^2}}$$

$$= \frac{53}{\sqrt{11925}}\left(0,\ \frac{30}{53},\ \frac{105}{53}\right)$$

$$= \frac{53}{15\sqrt{53}}\left(0,\ \frac{30}{53},\ \frac{105}{53}\right)$$

$$= \left(0,\ \frac{2}{\sqrt{15}},\ \frac{7}{\sqrt{15}}\right)$$

h) $\boldsymbol{u}_1=(0,\ 2,\ 1,\ 0),\quad \boldsymbol{u}_2=(1,\ -1,\ 0,\ 0),\quad \boldsymbol{u}_3=(1,\ 2,\ 0,\ -1),\quad \boldsymbol{u}_4=(1,\ 0,\ 0,\ 1)$

$$v_1=\frac{u_1}{\|u_1\|}=\frac{(0,2,1,0)}{\sqrt{0^2+2^2+1^2+0^2}}=\frac{(0,\ 2,\ 1,\ 0)}{\sqrt{5}}=\left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)$$

$$w_2=u_2-(u_2.v_1)v_1$$

$$=(1,-1,\ 0,\ 0)-\left[(1,-1,\ 0,\ 0)\bullet\left(0,\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}},0\right)\right]\left(0,\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}},0\right)$$

$$=(1,-1,\ 0,\ 0)-\left(-\frac{2}{\sqrt{5}}\right)\left(0,\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}},0\right).$$

$$=\left(1,-\tfrac{1}{5},\ \tfrac{2}{5},\ 0\right)$$

$$v_2=\frac{w_2}{\|w_2\|}=\frac{\left(1,-\frac{1}{5},\ \frac{2}{5},\ 0\right)}{\sqrt{1+\frac{1}{25}+\frac{4}{25}+0}}=\frac{5}{\sqrt{30}}\left(1,-\frac{1}{5},\ \frac{2}{5},\ 0\right)=\left(\frac{5}{\sqrt{30}},-\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right)$$

$$u_3\bullet v_1=(1,\ 2,\ 0,\ -1)\bullet\left(0,\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)=\frac{4}{\sqrt{5}}$$

$$u_3\bullet v_2=(1,\ 2,\ 0,\ -1)\bullet\left(\frac{5}{\sqrt{30}},-\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right)=\frac{5}{\sqrt{30}}-\frac{2}{\sqrt{30}}=\frac{3}{\sqrt{30}}$$

$$w_3=u_3-(u_3\bullet v_1)v_1-(u_3\bullet v_2)v_2$$

$$=(1,\ 2,\ 0,\ -1)-\left(\frac{4}{\sqrt{5}}\right)\left(0,\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)-\left(\frac{3}{\sqrt{30}}\right)\left(\frac{5}{\sqrt{30}},-\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right)$$

$$=(1,\ 2,\ 0,\ -1)-\left(0,\frac{8}{5},\ \frac{4}{5},\ 0\right)-\left(\frac{1}{2},-\frac{1}{10},\ \frac{1}{5},\ 0\right)$$

$$=\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$$

$$v_3=\frac{w_3}{\|w_3\|}=\frac{\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)}{\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+(-1)^2+(-1)^2}}$$

$$=\frac{1}{\sqrt{\frac{5}{2}}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$$

$$=\frac{\sqrt{2}}{\sqrt{5}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)=\frac{2}{\sqrt{10}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$$

$$=\left(\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ -\frac{2}{\sqrt{10}},\ -\frac{2}{\sqrt{10}}\right)$$

$$u_4 \cdot v_1 = (1,0,0,1)\cdot\left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) = \underline{0}$$

$$u_4 \cdot v_2 = (1,0,0,1)\cdot\left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right) = \underline{\frac{5}{\sqrt{30}}}$$

$$u_4 \cdot v_3 = (1,0,0,1)\cdot\left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right) = \underline{-\frac{1}{\sqrt{10}}}$$

$$\begin{aligned}
w_4 &= u_4 - (u_4 \cdot v_1)v_1 - (u_4 \cdot v_2)v_2 - (u_4 \cdot v_3)v_3 \\
&= (1,2,0,-1) - (0) - \left(\frac{5}{\sqrt{30}}\right)\left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right) + \left(\frac{1}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right) \\
&= (1,2,0,-1) - \left(\frac{5}{6}, -\frac{1}{6}, \frac{1}{3}, 0\right) + \left(\frac{1}{10}, \frac{1}{10}, -\frac{1}{5}, -\frac{1}{5}\right) \\
&= \left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right)
\end{aligned}$$

$$\begin{aligned}
\underline{v_4} &= \frac{w_4}{\|w_4\|} = \frac{\left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right)}{\sqrt{\left(\frac{4}{15}\right)^2 + \left(\frac{4}{15}\right)^2 + \left(-\frac{8}{15}\right)^2 + \left(\frac{4}{5}\right)^2}} \\
&= \frac{1}{\sqrt{\frac{240}{225}}}\left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right) \\
&= \frac{1}{\frac{4}{\sqrt{15}}}\left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right) \\
&= \frac{15}{4\sqrt{15}}\left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right) \\
&= \underline{\left(\frac{1}{\sqrt{15}}, \frac{1}{\sqrt{15}}, -\frac{2}{\sqrt{15}}, \frac{3}{\sqrt{15}}\right)}
\end{aligned}$$

Exercise

Find the QR–decomposition of

a) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

Solution

a) Since $\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5 \neq 0$, The matrix is invertible

$$u_1(1,\ 2), \quad u_2 = (-1,\ 3)$$

$$v_1 = u_1 = (1,\ 2)$$

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{(1,\ 2)}{\sqrt{1^2 + 2^2}} = \frac{(1,\ 2)}{\sqrt{5}} \underline{= \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right)}$$

$$\begin{aligned} v_2 &= u_2 - \left(u_2 . v_1\right) v_1 \\ &= (-1,\ 3) - \left[(-1,\ 3).\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)\right]\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \\ &= (-1,\ 3) - \left(\frac{5}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \\ &= (-1,\ 3) - (1,\ 2) \\ &= (-2,\ 1) \end{aligned}$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{(-2,1)}{\sqrt{(-2)^2 + 1^2}} \underline{= \left(-\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right)}$$

$$\left\langle \boldsymbol{u}_1, \boldsymbol{q}_1 \right\rangle = (1,2) \bullet \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right) = \frac{5}{\sqrt{5}} = \sqrt{5}$$

$$\left\langle \boldsymbol{u}_2, \boldsymbol{q}_1 \right\rangle = (-1,\ 3) \bullet \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right) = \frac{5}{\sqrt{5}} = \sqrt{5}$$

$$\left\langle \boldsymbol{u}_2, \boldsymbol{q}_2 \right\rangle = (-1,\ 3) \bullet \left(-\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right) = \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{5}} = \sqrt{5}$$

$$R = \begin{bmatrix} \left\langle \boldsymbol{u}_1, \boldsymbol{q}_1 \right\rangle & \left\langle \boldsymbol{u}_2, \boldsymbol{q}_1 \right\rangle \\ 0 & \left\langle \boldsymbol{u}_2, \boldsymbol{q}_2 \right\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}$$

The *QR*-decomposition of the matrix is

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}$$

b) The column vectors of are: $\boldsymbol{u}_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$$\boldsymbol{v}_1 = \boldsymbol{u}_1 \ = (3,\ -4)$$

$$\boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \frac{(3,\ -4)}{\sqrt{9+16}} = \boxed{\left(\frac{3}{5},\ -\frac{4}{5}\right)}$$

$$\boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2} \boldsymbol{v}_1$$

$$= (5,\ 0) - \frac{(5,\ 0)\bullet(3,\ -4)}{25}(3,\ -4)$$

$$= (5,\ 0) - \frac{15}{25}(3,\ -4)$$

$$= (5,\ 0) - \frac{3}{5}(3,\ -4)$$

$$= (5,\ 0) - \left(\frac{9}{5},\ -\frac{12}{5}\right)$$

$$= \left(\frac{16}{5},\ \frac{12}{5}\right)$$

$$\boldsymbol{q}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} = \frac{\left(\frac{16}{5},\ \frac{12}{5}\right)}{\sqrt{\frac{256}{25}+\frac{144}{25}}} = \frac{1}{\sqrt{\frac{400}{25}}}\left(\frac{16}{5},\ \frac{12}{5}\right) = \frac{1}{\sqrt{16}}\left(\frac{16}{5},\ \frac{12}{5}\right) = \frac{1}{4}\left(\frac{16}{5},\ \frac{12}{5}\right) = \boxed{\left(\frac{4}{5},\ \frac{3}{5}\right)}$$

$$R = \begin{bmatrix} \langle \boldsymbol{u}_1, \boldsymbol{q}_1 \rangle & \langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle \\ 0 & \langle \boldsymbol{u}_2, \boldsymbol{q}_2 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} 3\left(\frac{3}{5}\right) - 4\left(-\frac{4}{5}\right) & 5\left(\frac{3}{5}\right) + 0\left(-\frac{4}{5}\right) \\ 0 & 5\left(\frac{4}{5}\right) - 0\left(\frac{3}{5}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \cdot \begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix}$$

$$\boldsymbol{A} \quad = \quad \boldsymbol{Q} \qquad \boldsymbol{R}$$

c) Since the column vectors $\boldsymbol{u}_1(1,\ 0,\ 1)$, $\boldsymbol{u}_2 = (2,\ 1,\ 4)$ are linearly independent, so has a QR–decomposition.

$$\boldsymbol{v}_1 = \boldsymbol{u}_1 = (1,\ 0,\ 1)$$

$$\boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \frac{(1,\ 0,\ 1)}{\sqrt{1^2+0+1^2}} = \frac{(1,\ 0,\ 1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$\begin{aligned} v_2 &= u_2 - (u_2.v_1)v_1 \\ &= (2,1,4) - \left[(2,1,4).\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\right]\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) \\ &= (2,1,4) - \left(\frac{6}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) \\ &= (2,1,4) - (3,\ 0,\ 3) \\ &= (-1,\ 1,\ 1) \end{aligned}$$

$$\boldsymbol{q}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} = \frac{(-1,\ 1,\ 1)}{\sqrt{(-1)^2+1^2+1^2}} = \left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\langle \boldsymbol{u}_1, \boldsymbol{q}_1 \rangle = (1,\ 0,\ 1) \bullet \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle = (2,\ 1,\ 4) \bullet \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}} = 3\sqrt{2}$$

$$\langle \boldsymbol{u}_2, \boldsymbol{q}_2 \rangle = (2,\ 1,\ 4) \bullet \left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) = -\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{4}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$\begin{aligned} R &= \begin{bmatrix} \langle \boldsymbol{u}_1, \boldsymbol{q}_1 \rangle & \langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle \\ 0 & \langle \boldsymbol{u}_2, \boldsymbol{q}_2 \rangle \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix} \end{aligned}$$

The QR–decomposition of the matrix is $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$

d) Since $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{vmatrix} = -4 \neq 0$, The matrix is invertible, so it has a *QR*-decomposition.

$u_1(1,\ 1,\ 0), \quad u_2 = (2,\ 1,\ 3), \quad u_3 = (1,\ 1,\ 1)$

$v_1 = u_1 = (1,\ 1,\ 0)$

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{(1,\ 1,\ 0)}{\sqrt{1^2 + 1^2 + 0}} = \frac{(1,\ 1,\ 0)}{\sqrt{2}} = \boxed{\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)}$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - (\vec{u}_2 \bullet \vec{v}_1)\vec{v}_1 \\
&= (2,\ 1,\ 3) - \left[(2,\ 1,\ 3).\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)\right]\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) \\
&= (2,\ 1,\ 3) - \frac{3}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) \\
&= (2,\ 1,\ 3) - \left(\frac{3}{2},\ \frac{3}{2},\ 0\right) \\
&= \left(\frac{1}{2},\ -\frac{1}{2},\ 3\right)
\end{aligned}$$

$$\begin{aligned}
q_2 = \frac{v_2}{\|v_2\|} &= \frac{\left(\frac{1}{2},\ -\frac{1}{2},\ 3\right)}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + 3^2}} \\
&= \frac{\left(\frac{1}{2},\ -\frac{1}{2},\ 3\right)}{\sqrt{\frac{19}{2}}} \\
&= \frac{\sqrt{2}}{\sqrt{19}}\left(\frac{1}{2},\ -\frac{1}{2},\ 3\right) \\
&= \boxed{\left(\frac{1}{\sqrt{38}},\ -\frac{1}{\sqrt{38}},\ \frac{6}{\sqrt{38}}\right)} \qquad = \boxed{\left(\frac{\sqrt{2}}{2\sqrt{19}},\ -\frac{\sqrt{2}}{2\sqrt{19}},\ \frac{3\sqrt{2}}{\sqrt{19}}\right)}
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - (\vec{u}_3 \bullet \vec{v}_1)\vec{v}_1 - (\vec{u}_3 \bullet \vec{v}_2)\vec{v}_2 \\
&= (1,1,1) - \left[(1,1,1).\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)\right]\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \\
&\qquad - \left[(1,1,1).\left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}}\right)\right]\left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}}\right) \\
&= (1,1,1) - \frac{2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) - \frac{6}{\sqrt{38}}\left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}}\right) \\
&= (1,1,1) - (1,1,0) - \left(\frac{3}{19}, -\frac{3}{19}, \frac{18}{19}\right) \\
&= \left(-\frac{3}{19},\ \frac{3}{19},\ \frac{1}{19}\right)
\end{aligned}$$

$$q_3 = \frac{v_3}{\|v_3\|} = \frac{\left(-\frac{3}{19},\ \frac{3}{19},\ \frac{1}{19}\right)}{\sqrt{\left(-\frac{3}{19}\right)^2 + \left(\frac{3}{19}\right)^2 + \left(\frac{1}{19}\right)^2}}$$

$$= \frac{19}{\sqrt{19}}\left(-\frac{3}{19},\ \frac{3}{19},\ \frac{1}{19}\right)$$

$$= \left(-\frac{3}{\sqrt{19}},\ \frac{3}{\sqrt{19}},\ \frac{1}{\sqrt{19}}\right)$$

$$\langle \boldsymbol{u}_1, \boldsymbol{q}_1 \rangle = (1,\ 1,\ 0) \cdot \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle = (2,\ 1,\ 3) \cdot \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) = \frac{3}{\sqrt{2}}$$

$$\langle \boldsymbol{u}_2, \boldsymbol{q}_2 \rangle = (2,\ 1,\ 3) \cdot \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}}\right) = \frac{2-1+18}{\sqrt{38}} = \frac{19}{\sqrt{38}} = \frac{19}{\sqrt{2}\sqrt{19}} = \frac{\sqrt{19}}{\sqrt{2}}$$

$$\langle \boldsymbol{u}_3, \boldsymbol{q}_1 \rangle = (1,\ 1,\ 1) \cdot \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\langle \boldsymbol{u}_3, \boldsymbol{q}_2 \rangle = (1,\ 1,\ 1) \cdot \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}}\right) = \frac{1-1+6}{\sqrt{38}} = \frac{6}{\sqrt{2}\sqrt{19}} \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{\sqrt{19}}$$

$$\langle \boldsymbol{u}_3, \boldsymbol{q}_3 \rangle = (1,\ 1,\ 1) \cdot \left(-\frac{3}{\sqrt{19}},\ \frac{3}{\sqrt{19}},\ \frac{1}{\sqrt{19}}\right) = \frac{-3+3+1}{\sqrt{19}} = \frac{1}{\sqrt{19}}$$

$$R = \begin{bmatrix} \langle \boldsymbol{u}_1, \boldsymbol{q}_1 \rangle & \langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle & \langle \boldsymbol{u}_3, \boldsymbol{q}_1 \rangle \\ 0 & \langle \boldsymbol{u}_2, \boldsymbol{q}_2 \rangle & \langle \boldsymbol{u}_3, \boldsymbol{q}_2 \rangle \\ 0 & 0 & \langle \boldsymbol{u}_3, \boldsymbol{q}_3 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}$$

The QR–decomposition of the matrix is $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The matrix is linearly dependent, so doesn't have a *QR*–decomposition.

Exercise

Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product

$$\boldsymbol{u} = (0,\ -2,\ 2,\ 1), \quad \boldsymbol{v} = (-1,\ -1,\ 1,\ 1)$$

Solution

$$\langle \boldsymbol{u},\ \boldsymbol{v} \rangle = 0 - 2(-1) + 2(1) + 1(1) = 5$$

$$\| \langle \boldsymbol{u}, \boldsymbol{v} \rangle \| = \sqrt{5}$$

$$\begin{aligned} \|\boldsymbol{u}\| . \|\boldsymbol{v}\| &= \sqrt{0+4+4+1}\sqrt{1+1+1+1} \\ &= \sqrt{9}\sqrt{4} \\ &= 6 \end{aligned}$$

$$\sqrt{5} < 6 \quad \Rightarrow \quad \| \langle \boldsymbol{u}, \boldsymbol{v} \rangle \| \le \|\boldsymbol{u}\| . \|\boldsymbol{v}\|$$

Solution   Section 3.4 – Orthogonal Matrices

Exercise

Show that the matrix is orthogonal

a) $A=\begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$

b) $A=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

Solution

a) $$AA^T=\begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}\begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=I$$

$$A^TA=\begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix}\begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=I$$

$\therefore$ ***A*** is an orthogonal

b) $$AA^T=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=I$$

$$AA^T=I$$

$\therefore$ ***A*** is an orthogonal

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b) $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

d) $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

f) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

e) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$

Solution

a) $AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

A is orthogonal with inverse $A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b) $AA^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore$ ***A*** is orthogonal with inverse $A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ (It is a standard matrix for a rotation of 45°)

c) $AA^T = \begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & & \\ & & \\ & & \end{bmatrix} \neq I$

Or $\|r_1\| = \sqrt{0 + 1^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{3}{2}} \neq 1$ $\therefore$ ***A*** is ***not*** orthogonal

d) $AA^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\therefore$ ***A*** is orthogonal with inverse $A^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

e) $AA^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\therefore$ ***A*** is orthogonal with inverse $A^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$

f) $\|\mathbf{r}_2\| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{3} + \frac{1}{4}} = \sqrt{\frac{7}{12}} \neq 1$

Or

$AA^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{5}{6} & & \\ & & & \\ & & & \end{pmatrix} \neq I$

$\therefore$ The matrix is ***not*** an orthogonal

Exercise

Prove that if A is orthogonal, then A^T is orthogonal.

Solution

If A is orthogonal then $A^T = A^{-1}$ and $A = \left(A^T\right)^T$

Then $\left(A^T\right)^T A^T = AA^T = I \Rightarrow A^T$ is orthogonal

Another word, since A is orthogonal, then both column and row vectors of A form an orthonormal set. A^T is just A with its row and column vectors are swapped. The column vectors of A^T (which are the row vectors of A) and row vectors of A^T (which are the column vectors of A) form orthonormal sets, therefore A^T is orthogonal

Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & \cdots \end{bmatrix}$$

Solution

$$\boldsymbol{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}^T \quad \rightarrow \quad \left\|\boldsymbol{q}_1\right\| = \sqrt{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 1$$

$$\boldsymbol{q}_2 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}^T \quad \rightarrow \quad \left\|\boldsymbol{q}_2\right\| = \sqrt{\frac{1}{6}+\frac{1}{6}+\frac{4}{6}} = 1$$

Let $\boldsymbol{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$

$$\boldsymbol{q}_1 \bullet \boldsymbol{q}_3 = \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y - \frac{1}{\sqrt{3}}z = 0 \quad \rightarrow \quad x + y - z = 0$$

$$\boldsymbol{q}_2 \bullet \boldsymbol{q}_3 = \frac{1}{\sqrt{6}}x + \frac{1}{\sqrt{6}}y - \frac{2}{\sqrt{6}}z = 0 \quad \rightarrow \quad x + y - 2z = 0$$

$$\begin{cases} x+y-z=0 \\ x+y-2z=0 \end{cases} \rightarrow \quad z = 0 \quad \text{and} \quad x + y = 0 \Rightarrow x = -y$$

$$\boxed{\boldsymbol{q}_3 = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T}$$

Exercise

Determine if the given matrix is orthogonal. If it is, find its inverse

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{5} & \frac{3}{7} \\ \frac{4}{9} & \frac{3}{5} & -\frac{2}{7} \\ \frac{8}{9} & -\frac{2}{5} & \frac{3}{7} \end{bmatrix}$$

Solution

$$\boldsymbol{q}_1 = \begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{8}{9} \end{bmatrix}^T \quad \boldsymbol{q}_2 = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} & -\frac{2}{5} \end{bmatrix}^T \quad \boldsymbol{q}_3 = \begin{bmatrix} \frac{3}{7} & -\frac{2}{7} & \frac{3}{7} \end{bmatrix}^T$$

$$\boldsymbol{q}_1 \bullet \boldsymbol{q}_2 = \frac{4}{45} + \frac{12}{45} - \frac{16}{45} = 0$$

$$\boldsymbol{q}_1 \bullet \boldsymbol{q}_3 = \frac{3}{63} - \frac{8}{63} + \frac{24}{63} = \frac{19}{63} \neq 0$$

$$\boldsymbol{q}_2 \bullet \boldsymbol{q}_3 = \frac{12}{35} - \frac{6}{35} + \frac{6}{35} = \frac{12}{35} \neq 0$$

The given matrix is ***not*** orthogonal

Solution   *Section* 3.5 – Least Squares Analysis

Exercise

Find the equation of the line that best fits the given points in the least-squares sense.

a) $\{(0,\ 2),\ (1,\ 2),\ (2,\ 0)\}$

b) $\{(1,\ 5),\ (2,\ 4),\ (3,\ 1),\ (4,\ 1),\ (5,-1)\}$

c) $\{(0,\ 1),\ (1,\ 3),\ (2,\ 4),\ (3,\ 4)\}$

d) $\{(-2,\ 0),\ (-1,\ 0),\ (0,\ 1),\ (1,\ 3),\ (2,\ 5)\}$

Solution

a) $\{(0,\ 2),\ (1,\ 2),\ (2,\ 0)\}$

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \quad \text{where } A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \quad x = \begin{bmatrix} m \\ b \end{bmatrix} \quad y = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

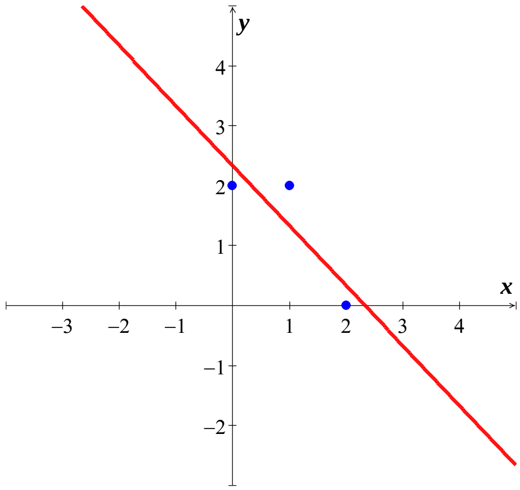
The normal equation formula: $A^T A\mathbf{x} = A^T \mathbf{y}$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

We have: $m = -1$ *and* $b = \frac{7}{3}$.

Thus, $\boxed{y = -x + \frac{7}{3}}$



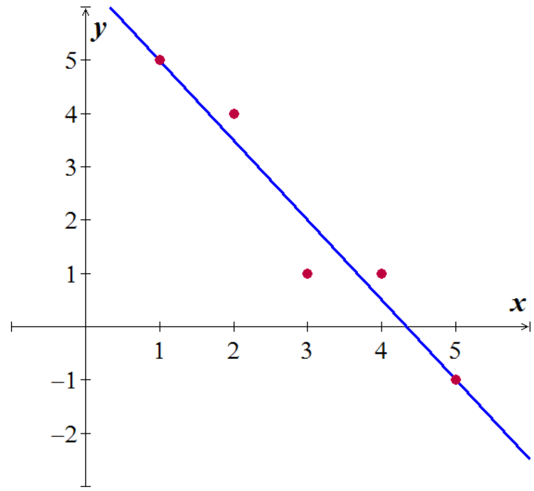
b) $\{(1,\ 5),\ (2,\ 4),\ (3,\ 1),\ (4,\ 1),\ (5,-1)\}$

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{where } A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \quad x = \begin{bmatrix} m \\ b \end{bmatrix} \quad y = \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

The normal equation: $A^T A\mathbf{x} = A^T \mathbf{y}$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$



$$\begin{bmatrix} 55 & 15 \\ 15 & 5 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}$$

We have: $m = -\frac{3}{2}$ *and* $b = \frac{13}{2}$.

Thus, $\boxed{y = -1.5x + 6.5}$

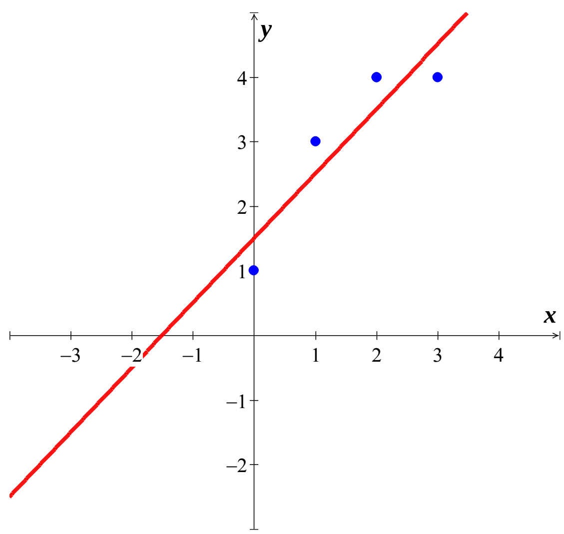
c) $\{(0, 1), (1, 3), (2, 4), (3, 4)\}$

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix} \qquad \text{where } A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \quad x = \begin{pmatrix} m \\ b \end{pmatrix} \quad y = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

The normal equation: $A^T A\mathbf{x} = A^T \mathbf{y}$

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$



$$\begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 23 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 4 & -6 \\ -6 & 14 \end{pmatrix}\begin{pmatrix} 23 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$$

We have: $m = 1$ *and* $b = \frac{3}{2}$.

Thus, $\boxed{y = x + 1.5}$

d) $\{(-2, 0), (-1, 0), (0, 1), (1, 3), (2, 5)\}$

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix} \qquad \text{where } A = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad x = \begin{pmatrix} m \\ b \end{pmatrix} \quad y = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

The normal equation: $A^T A\mathbf{x} = A^T \mathbf{y}$

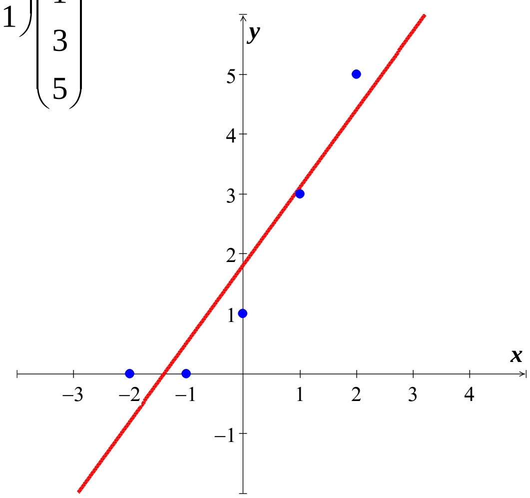
$$\begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 13 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{50}\begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}\begin{pmatrix} 13 \\ 9 \end{pmatrix} = \begin{pmatrix} \frac{13}{10} \\ \frac{9}{5} \end{pmatrix}$$

We have: $m = 1.3$ *and* $b = 1.8$.

Thus, $\boxed{y = 1.3x + 1.8}$



Exercise

Find the orthogonal projection of the vector $\mathbf{u}$ on the subspace of $\mathbf{R}^4$ spanned by the vectors

a) $\mathbf{u} = (-3,\ -3,\ 8,\ 9);\quad \mathbf{v}_1 = (3,\ 1,\ 0,\ 1),\quad \mathbf{v}_2 = (1,\ 2,\ 1,\ 1),\quad \mathbf{v}_3 = (-1,\ 0,\ 2,\ -1)$

b) $\mathbf{u} = (6,\ 3,\ 9,\ 6);\quad \mathbf{v}_1 = (2,\ 1,\ 1,\ 1),\quad \mathbf{v}_2 = (1,\ 0,\ 1,\ 1),\quad \mathbf{v}_3 = (-2,\ -1,\ 0,\ -1)$

c) $\mathbf{u} = (-2,\ 0,\ 2,\ 4);\quad \mathbf{v}_1 = (1,\ 1,\ 3,\ 0),\quad \mathbf{v}_2 = (-2,\ -1,\ -2,\ 1),\quad \mathbf{v}_3 = (-3,\ -1,\ 1,\ 3)$

Solution

a) Let $A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix} \Rightarrow A^T A = \begin{pmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{pmatrix}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{pmatrix}$

$$A^T\mathbf{u} = \begin{pmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{pmatrix}\begin{pmatrix} -3 \\ -3 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ 10 \end{pmatrix}$$

The normal solution is $A^T A\mathbf{x} = A^T\mathbf{u}$

$$\begin{pmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ 10 \end{pmatrix} \quad \Rightarrow \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

So $proj_W \boldsymbol{u} = A\boldsymbol{x} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 4 \\ 0 \end{pmatrix}$

$proj_W \boldsymbol{u} = (-2,\ 3,\ 4,\ 0)$

b) $\boldsymbol{u} = (6,\ 3,\ 9,\ 6)$; $\boldsymbol{v}_1 = (2,\ 1,\ 1,\ 1)$, $\boldsymbol{v}_2 = (1,\ 0,\ 1,\ 1)$, $\boldsymbol{v}_3 = (-2,\ -1,\ 0,\ -1)$

Let $A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \Rightarrow A^T A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix}$

$A^T \boldsymbol{u} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \\ 9 \\ 6 \end{pmatrix} = \begin{pmatrix} 30 \\ 21 \\ -21 \end{pmatrix}$

The normal solution is $A^T A\boldsymbol{x} = A^T \boldsymbol{u}$

$\begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 30 \\ 21 \\ -21 \end{pmatrix} \Rightarrow \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$

So $proj_W \boldsymbol{u} = A\boldsymbol{x} = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 9 \\ 5 \end{pmatrix}$

$proj_W \boldsymbol{u} = (7,\ 2,\ 9,\ 5)$

c) $\boldsymbol{u} = (-2,\ 0,\ 2,\ 4)$; $\boldsymbol{v}_1 = (1,\ 1,\ 3,\ 0)$, $\boldsymbol{v}_2 = (-2,\ -1,\ -2,\ 1)$, $\boldsymbol{v}_3 = (-3,\ -1,\ 1,\ 3)$

Let $A = \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix} \Rightarrow A^T A = \begin{pmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{pmatrix}$

$A^T \boldsymbol{u} = \begin{pmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix}\begin{pmatrix} -2 \\ 0 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix}$

The normal solution is $A^T A\boldsymbol{x} = A^T \boldsymbol{u}$

$$\begin{pmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} \\ -\frac{8}{5} \\ \frac{8}{5} \end{pmatrix}$$

So $proj_W \boldsymbol{u} = A\boldsymbol{x} = \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} -\frac{4}{5} \\ -\frac{8}{5} \\ \frac{8}{5} \end{pmatrix} = \begin{pmatrix} -\frac{12}{5} \\ -\frac{4}{5} \\ \frac{12}{5} \\ \frac{16}{5} \end{pmatrix}$

$$proj_W \boldsymbol{u} = \left(-\frac{12}{5},\ -\frac{4}{5},\ \frac{12}{5},\ \frac{16}{5}\right)$$

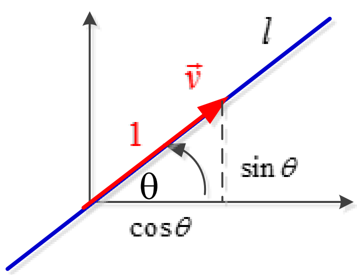
Exercise

Find the standard matrix for the orthogonal projection P of $\boldsymbol{R}^2$ on the line passes through the origin and makes an angle θ with the positive x-axis.

Solution

Since the line l in 2-dimensional, than we can take $\vec{v} = (\cos\theta,\ \sin\theta)$ as a basis for this subspace

$$A = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

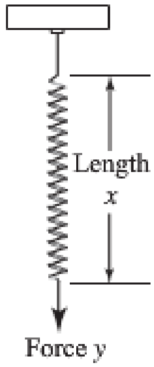


$$[P] = A^T A = [\cos\theta \quad \sin\theta]\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

Exercise

Hooke's law in physics states that the length x of a uniform spring is a linear function of the force y applied to it. If we express the relationship as $y = mx + b$, then the coefficient m is called the spring constant. Suppose a particular unstretched spring has a measured length of 6.1 *inches*.(i.e., x = 6.1 when y = 0). Forces of 2 pounds, 4 pounds, and 6 pounds are then applied to the spring, and the corresponding lengths are found to be 7.6 inches, 8.7 inches, and 10.4 inches. Find the spring constant.



Solution

$$M = \begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$

The normal equation: $A^T A\boldsymbol{x} = A^T \boldsymbol{y}$

$$\begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 278.82 & 32.8 \\ 32.8 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 112.4 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{39.44}\begin{pmatrix} 4 & -32.8 \\ -32.8 & 278.2 \end{pmatrix}\begin{pmatrix} 112.4 \\ 12 \end{pmatrix} = \begin{pmatrix} 1.4 \\ -8.8 \end{pmatrix}$$

Thus, the estimated value of the spring constant is $\approx 1.4\ \textit{pounds}$.

Exercise

Prove: If A has a linearly independent column vectors, and if $\boldsymbol{b}$ is orthogonal to the column space of A, then the least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x} = 0$.

Solution

If A has linearly independent column vectors, then $A^T A$ is invertible and the least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$ is the solution of $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$, but since $\boldsymbol{b}$ is orthogonal to the column space of A. $A^T\boldsymbol{b} = 0$, so $\boldsymbol{x}$ is a solution of $A^T A\boldsymbol{x} = 0$. Thus $\boldsymbol{x} = 0$ since $A^T A$ is invertible.

Exercise

Let A be an $m \times n$ matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of R^n onto the row space of A.

Solution

A^T will have linearly independent column vectors, and the column space A^T is the row space of A.

Thus, the standard matrix for the orthogonal projection of R^n onto the row space of A is

$$[P] = A^T \left[\left(A^T\right)^T A^T \right]^{-1} \left(A^T\right)^T = A^T \left(AA^T\right)^{-1} A$$

Exercise

Let W be the line with parametric equations $x = 2t, \quad t = -t, \quad z = 4t$

a) Find a basis for W.

b) Find the standard matrix for the orthogonal projection on W.

c) Use the matrix in part (b) to find the orthogonal projection of a point $P_0\left(x_0, y_0, z_0\right)$ on W.

d) Find the distance between the point $P_0\left(2, 1, -3\right)$ and the line W.

Solution

a) $W = span\{(2, -1, 4)\}$ so that the vector $(2, -1, 4)$ forms a basis for W (linear independence)

b) Let $A = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$

$$[P] = A\left(A^T A\right)^{-1} A^T$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \left(\begin{bmatrix} 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} [21]^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \frac{1}{21} \begin{bmatrix} 4 & -2 & 8 \\ -2 & 1 & -4 \\ 8 & -4 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix}$$

c) $$\begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \frac{4}{21}x_0 - \frac{2}{21}y_0 + \frac{8}{21}z_0 \\ -\frac{2}{21}x_0 + y_0 - \frac{4}{21}z_0 \\ \frac{8}{21}x_0 - \frac{4}{21}y_0 + \frac{16}{21}z_0 \end{bmatrix}$$

d) $$\begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{6}{7} \\ \frac{3}{7} \\ -\frac{12}{7} \end{bmatrix}$$

The distance between P_0 and W equals to the distance between P_0 and its projection on W.

The distance between $(2, 1, -3)$ and $\left(-\frac{6}{7}, \frac{3}{7}, -\frac{12}{7}\right)$ is

$$d = \sqrt{\left(2+\frac{6}{7}\right)^2 + \left(1-\frac{3}{7}\right)^2 + \left(-3+\frac{12}{7}\right)^2}$$

$$= \sqrt{\frac{400}{49} + \frac{16}{49} + \frac{81}{49}}$$

$$\boxed{= \frac{\sqrt{497}}{7}}$$

Exercise

In R^3, consider the line l given by the equations $\quad x = t, \quad t = t, \quad z = t$

And the line m given by the equations $\quad x = s, \quad t = 2s - 1, \quad z = 1$

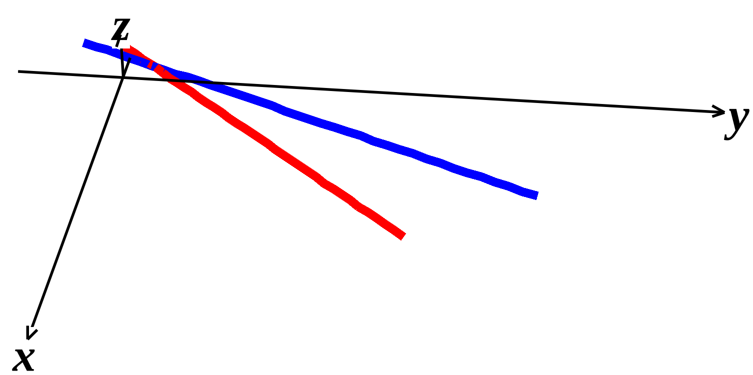
Let P be the point on l, and let Q be a point on m. Find the values of t and s that minimize the distance between the lines by minimizing the squared distance $\|P - Q\|^2$

Solution

When $t = 1 \;\Rightarrow\; Let\ P = (1, 1, 1)$ is on line l

When $s = 1 \;\Rightarrow\; Let\ Q = (1, 1, 1)$ is on line m

$$\|P - Q\| = \sqrt{(1-1)^2 + (1-1)^2 + (1-1)^2} = 0 \ge 0$$



Thus these are the values $P = (1,\ 1,\ 1)$ and $Q = (1,\ 1,\ 1)$ are the values for $s = t = 1$ that minimize the distance between the lines.

Exercise

Determine whether the statement is true or false,

a) If A is an $m \times n$ matrix, then $A^T A$ is a square matrix.

b) If $A^T A$ is invertible, then A is invertible.

c) If A is invertible, then $A^T A$ is invertible.

d) If $A\mathbf{x} = \mathbf{b}$ is a consistent linear system, then $A^T A\mathbf{x} = A^T\mathbf{b}$ is also consistent.

e) If $A\mathbf{x} = \mathbf{b}$ is an inconsistent linear system, then $A^T A\mathbf{x} = A^T\mathbf{b}$ is also inconsistent.

f) Every linear system has a least squares solution.

g) Every linear system has a unique least squares solution.

h) If A is an $m \times n$ matrix with linearly independent columns and $\mathbf{b}$ is in R^m, then $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution.

Solution

a) ***True***; $A^T A$ is an $n \times n$ matrix

b) ***False***; only square matrix has inverses, but $A^T A$ can be invertible when A is not square matrix.

c) ***True***; if A is invertible, so is A^T, so the product $A^T A$ is also invertible

d) ***True***

e) ***False***; the system $A^T A\mathbf{x} = A^T\mathbf{b}$ may be consistent

f) ***True***

g) ***False***; the least squares solution may involve a parameter

h) ***True***; if A has linearly independent column vectors; then $A^T A$ is invertible, so $A^T A\mathbf{x} = A^T\mathbf{b}$ has a unique solution