Exercise

The distribution of IQ scores is a nonstandard normal distribution with mean of 100 and standard deviation of 15. What are the values of the mean and standard deviation after all IQ scores have been standardized by converting them to *z*-scores using $z=\frac{x-\mu}{\sigma}$?

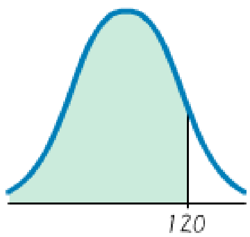
Solution

For any distribution, converting to *z* scores using the formula $z=\frac{x-\mu}{\sigma}$ produces a same-shaped distribution with mean 0 and standard deviation 1.

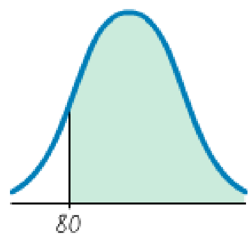
Exercise

Find the area of the shaded region. The graphs depict IQ scores adults, and those scores are normally distributed with mean of 100 and standard deviation of 15.

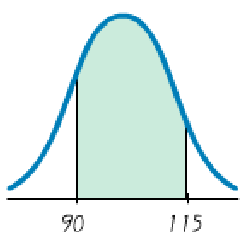
a)



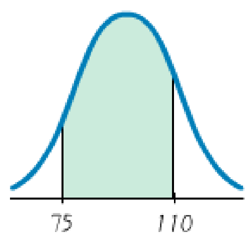
b)



c)



d)



Solution

a) $z=\frac{x-\mu}{\sigma}=\frac{120-100}{15}=\frac{20}{15}=1.33$

$P(x<120)=P(z=1.33)=0.9082$

b) $z=\frac{x-\mu}{\sigma}=\frac{80-100}{15}=-\frac{20}{15}=-1.33$

$$\begin{aligned}P(x>80)&=P(z>-1.33)\\&=1-P(z<-1.33)\\&=1-0.0918\\&=0.9082\end{aligned}$$

c) $x = 90 \quad \rightarrow \quad z = \dfrac{x-\mu}{\sigma} = \dfrac{90-100}{15} = -\dfrac{10}{15} \; \underline{= -0.67}$

$x = 115 \quad \rightarrow \quad z = \dfrac{x-\mu}{\sigma} = \dfrac{115-100}{15} \; \underline{= 1}$

$$\begin{aligned} P(90 < x < 115) &= P(-0.67 < z < 1.00) \\ &= P(z < 1.00) - P(z < -0.67) \\ &= 0.8413 - 0.0475 \\ &\underline{= 0.5899} \end{aligned}$$

d) $x = 75 \quad \rightarrow \quad z = \dfrac{x-\mu}{\sigma} = \dfrac{75-100}{15} = -\dfrac{25}{15} \; \underline{= -1.67}$

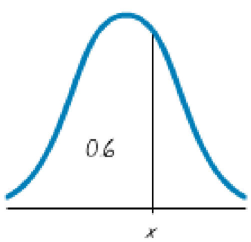
$x = 110 \quad \rightarrow \quad z = \dfrac{x-\mu}{\sigma} = \dfrac{110-100}{15} \; \underline{= 0.67}$

$$\begin{aligned} P(75 < x < 110) &= P(-1.67 < z < 0.67) \\ &= P(z < 067) - P(z < -1.67) \\ &= 0.7846 - 0.0475 \\ &\underline{= 0.7011} \end{aligned}$$

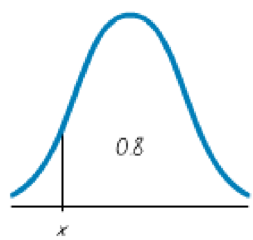
Exercise

Find the Indicated IQ scores. The graphs depict IQ scores adults, and those scores are normally distributed with mean of 100 and standard deviation of 15.

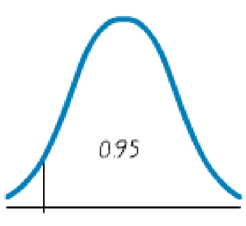
a)



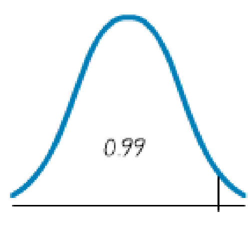
b)



c)



d)



Solution

a) The z score with $A = 0.6$ below it is: $z = 0.25$ (~0.5987)

$$\begin{aligned} x &= \mu + z\sigma \\ &= 100 + (0.25)(15) \\ &= 103.75 \\ &\underline{= 103.8} \end{aligned}$$

b) The z score with $A = 0.8$ above is the z score with $A = 0.2$ below; it is: $z = -0.84$

$$\begin{aligned} x &= \mu + z\sigma \\ &= 100 + (-0.84)(15) \\ &= 87.4 \end{aligned}$$

c) The z score with $A = 0.95$ above is the z score with $A = 0.05$ below; it is: $z = -1.645$

$$\begin{aligned} x &= \mu + z\sigma \\ &= 100 + (-1.645)(15) \\ &= 75.3 \end{aligned}$$

d) The z score with $A = 0.99$ below; it is: $z = 2.33$

$$\begin{aligned} x &= \mu + z\sigma \\ &= 100 + (2.33)(15) \\ &= 135.0 \end{aligned}$$

Exercise

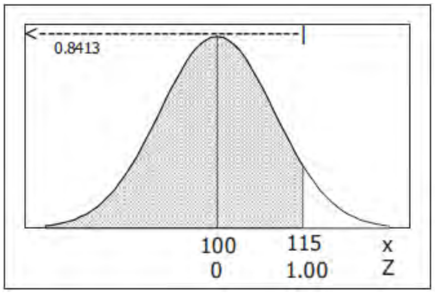
Assume that adults have IQ scores that are normally distributed with mean of 100 and standard deviation of 15

a) Find the probability that a randomly selected adult has an IQ that is less than 115.
b) Find the probability that a randomly selected adult has an IQ that is greater than 131.5.
c) Find the probability that a randomly selected adult has an IQ that is between 90 and 110.
d) Find the probability that a randomly selected adult has an IQ that is between 110 and 120.
e) Find P_{30} which is the IQ score separating the bottom 30% from the top 70%.
f) Find the first quartile Q_1 which is the IQ score separating the bottom 25% from the top 75%.

Solution

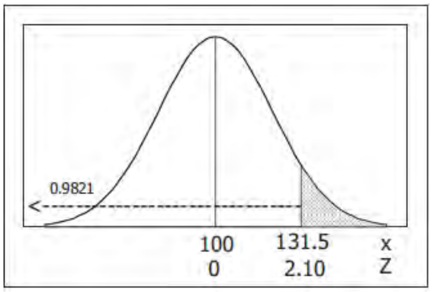
a) $x = 115 \quad \to \quad z = \dfrac{x-\mu}{\sigma} = \dfrac{115-100}{15} = \dfrac{15}{15} = 1$

$P(x < 115) = P(z < 1) = 0.8413$



b) $x = 131.5 \quad \to \quad z = \dfrac{x-\mu}{\sigma} = \dfrac{131.5-100}{15} = 2.10$

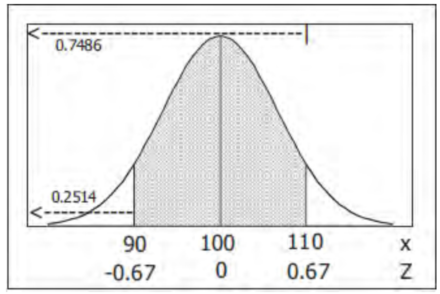
$$\begin{aligned} P(x > 131.5) &= P(z > 2.10) \\ &= 1 - P(z < 2.10) \\ &= 1 - 0.9821 \\ &= 0.0179 \end{aligned}$$



c) $x = 90 \quad \rightarrow \quad z = \dfrac{x-\mu}{\sigma} = \dfrac{90-100}{15} \; \underline{= -0.67}$

$x = 110 \quad \rightarrow \quad z = \dfrac{x-\mu}{\sigma} = \dfrac{110-100}{15} \; \underline{= 0.67}$

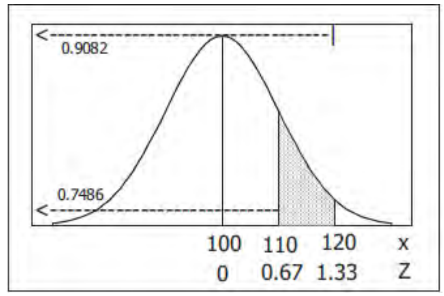
$$\begin{aligned} P(90 < x < 110) &= P(-0.67 < z < 0.67) \\ &= P(z < 0.67) - P(z < -0.67) \\ &= 0.7486 - 0.2514 \\ &\underline{= 0.4972} \end{aligned}$$



d) $x = 120 \quad \rightarrow \quad z = \dfrac{x-\mu}{\sigma} = \dfrac{120-100}{15} \; \underline{= 1.33}$

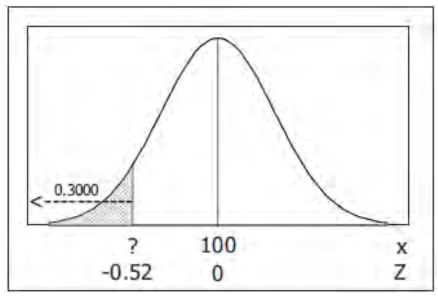
$x = 110 \quad \rightarrow \quad z = \dfrac{x-\mu}{\sigma} = \dfrac{110-100}{15} \; \underline{= 0.67}$

$$\begin{aligned} P(110 < x < 120) &= P(0.67 < z < 1.33) \\ &= P(z < 1.33) - P(z < 0.67) \\ &= 0.9082 - 0.7486 \\ &\underline{= 0.1596} \end{aligned}$$



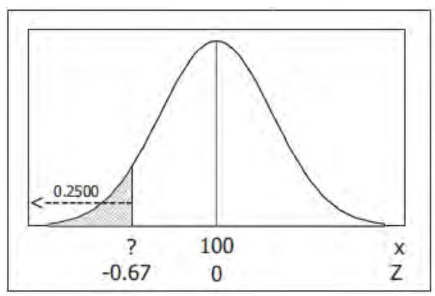
e) For P_{30} , $A = 0.300 \; [0.3015] \quad \rightarrow \quad z = -0.52$

$$\begin{aligned} x &= \mu + z\sigma \\ &= 100 + (-0.52)(15) \\ &\underline{= 92.2} \end{aligned}$$



f) For Q_1 , $A = 0.2500 \; [0.2514] \quad \rightarrow \quad z = -0.67$

$$\begin{aligned} x &= \mu + z\sigma \\ &= 100 + (-0.67)(15) \\ &\underline{\approx 90} \end{aligned}$$



Exercise

The Gulfstream 100 is an executive jet that seats six, and it has a doorway height of 51.6

- ***Men's*** heights are normally distributed with mean 69.0 in. and standard deviation 2.8 in.
- ***Women's*** heights are normally distributed with mean 63.6 in. and standard deviation 2.5 in.

a) What percentage of adult men can fit through the door without bending?

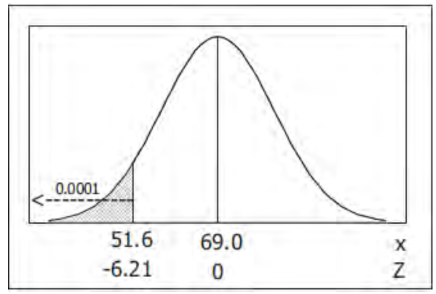
b) What percentage of adult women can fit through the door without bending?

c) Does the door design with a height of 51.6 in. appear to be adequate? Why didn't the engineers design larder door?

d) What doorway height would allow 60% of men to fit without bending?

Solution

a) Normal distribution with: $\mu = 69.0, \quad \sigma = 2.8$

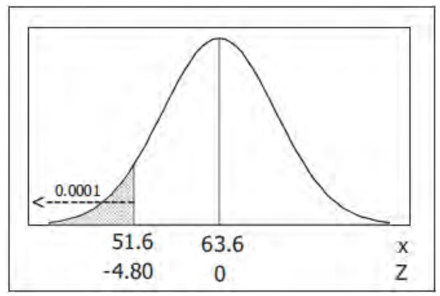


$$x = 51.6 \quad \rightarrow \quad z = \frac{x-\mu}{\sigma} = \frac{51.6 - 69.0}{2.8} = -6.21$$

$$P(x < 51.6) = P(z < -6.21)$$

$$= 0.0001 \quad or \quad 0.01\%$$

b) Normal distribution with: $\mu = 63.6, \quad \sigma = 2.5$



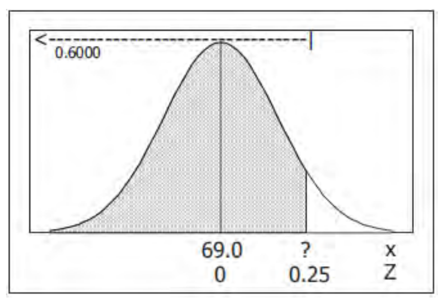
$$x = 51.6 \quad \rightarrow \quad z = \frac{x-\mu}{\sigma} = \frac{51.6 - 63.6}{2.5} = -4.80$$

$$P(x < 51.6) = P(z < -4.80)$$

$$= 0.0001 \quad or \quad 0.01\%$$

c) Maybe. While it may not be convenient, it presents no danger of injury because of the obvious need for everyone to bend. Considering the small size of the plane, the door is probably as large as possible.

d) For $A = 0.6000 \;\; [0.5987] \quad \rightarrow \quad z = 0.25$



$$x = \mu + z\sigma$$

$$= 69 + (0.25)(2.8)$$

$$= 69.7 \;\; inches$$

Exercise

Assume that human body temperatures are normally distributed with a mean of 98.20°F and a standard deviation of 0.62°F.

a) A Hospital uses 100.6°F as the lowest temperature considered to be a fever. What percentage of normal and healthy persons would be considered to have fever? Does this percentage suggest that a cutoff of 100.6°F is appropriate?

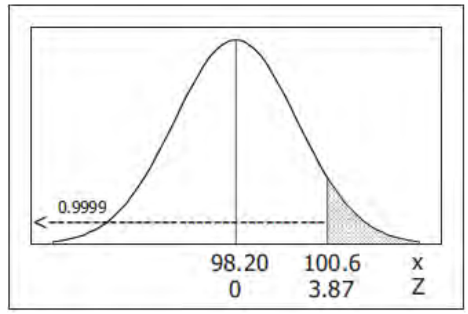
b) Physicians want to select a minimum temperature for requiring further medical tests. What should that temperature be, if we want only 5.0% of healthy people to exceed it? (Such a result is a false positive, meaning that the test result is positive, but the subject is not really sick.)

Solution

a) Normal distribution with: $\mu = 98.20, \quad \sigma = 0.62$

$$x = 100.6 \quad \rightarrow \quad z = \frac{x-\mu}{\sigma} = \frac{100.6 - 98.2}{0.62} \underline{= 3.87}$$

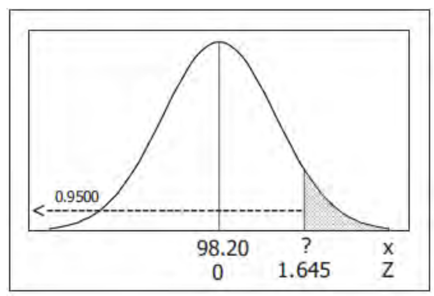
$$P(x > 100.6) = P(z > 3.87)$$
$$= 1 - P(z < 3.87)$$
$$= 1 - 0.9999$$
$$\underline{= 0.0001}$$



Yes. The cut-off is appropriate in that there is a small probability of saying that a healthy person has a fever, but many with low grade fevers may erroneously be labeled healthy.

b) For the highest 5%: $A = 0.9500 \quad \rightarrow \quad z = 1.645$

$$x = \mu + z\sigma$$
$$= 98.2 + (1.645)(0.62)$$
$$\underline{= 99.22\ ^\circ F}$$



Exercise

The lengths of pregnancies are normally distributed with a mean of 268 days and a standard deviation of 15 days.

a) One classical use of the normal distribution is inspired by a letter to "Dear Abby" in which a wife claimed to have given birth 308 days after a brief visit from her husband. Given this information, find the probability of a pregnancy lasting 308 days or longer. What does the result suggest?

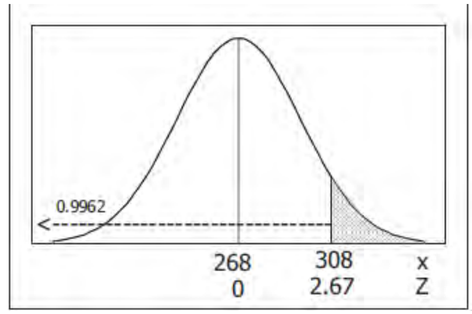
b) If we stipulate that a baby is premature if the length of pregnancy is the lowest 4%, find the length that separates premature babies from those who are not premature. Premature babies often require special care, and this result could be helpful to hospital administrators in planning for that care.

Solution

Normal distribution with: $\mu = 268, \quad \sigma = 15$

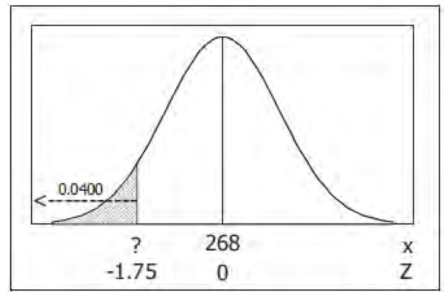
a) $P(x > 308) = P(z > 2.67)$

$$= 1 - P(z < 2.67)$$
$$= 1 - 0.9962$$
$$\underline{= 0.0038}$$



The result suggests that an unusual event has occurred – but certainly not an impossible one, as about 38 of every 10,000 pregnancies can be expected to last as long.

b) For the lowest 4%: $A = 0.0400 \quad [0.0401] \quad \to \quad z = -1.75$



$$x = \mu + z\sigma$$
$$= 268 + (-1.75)(15)$$
$$\underline{= 242 \;\; days}$$

Exercise

A statistics professor gives a test and finds that the scores are normally distributed with a mean of 25 and a standard deviation of 5. She plans to curve the scores.

a) If the curves by adding 50 to each grade, what is the new mean? What is the new standard deviation?

b) Is it fair to curve by adding 50 to each grade? Why or why not?

c) If the grades are curved according to the following scheme (instead of adding 50), find the numerical limits for each letter grade.

A: Top 10%
B: Scores above the bottom 70% and below the top 10%.
C: Scores above the bottom 30% and below the top 30%.
D: Scores above the bottom 10% and below the top 70%.
F: Bottom 10%.

d) Which method of curving the grades is fairer: Adding 50 to each grade or using the scheme given in part (*c*)? Explain.

Solution

Normal distribution with: $\mu = 25, \quad \sigma = 5$

a) For a population of size N, $\mu = \dfrac{\sum x}{N}, \quad \sigma^2 = \dfrac{\sum (x-\mu)^2}{N}$

Adding a constant to each score increases the mean by that amount but does not affect the standard deviation.

In non-statistical terms, shifting everything by k units does not affect the spread of the scores. This is true for any set of scores – regardless of the shape of the original distribution.

Let $y = x + k$

$$\begin{aligned}\mu_y &= \frac{\sum(x+k)}{N} \\ &= \frac{\sum x + \sum k}{N} \\ &= \frac{\sum x}{N} + \frac{\sum k}{N} \\ &= \frac{\sum x}{N} + \frac{Nk}{N} \\ &= \mu_x + k\end{aligned}$$

$$\begin{aligned}\sigma_y^2 &= \frac{\sum\left(y-\mu_y\right)^2}{N} \\ &= \frac{\sum\left((x+k)-\left(\mu_x+k\right)\right)^2}{N} \\ &= \frac{\sum\left(x-\mu_x\right)^2}{N} \\ &= \sigma_x^2\end{aligned}$$

If the teacher adds 50 to each grade,

New mean = 25 + 50 = 75

New standard deviation = 5

b) No; curving should consider the variation. Had the test been more appropriately constructed, it is not likely that every student would score exactly 50 points higher, If the typical student score increased by 50, we would expect the better students to increase by more than 50 and the poorer students to increase by less than 50. This would make the scores spread out and would increase the standard deviation.

c) For the top 10%: $A = 1 - 0.1 = 0.9000 \quad [0.8997] \quad \to \quad z = 1.28$

$$\begin{aligned}x &= \mu + z\sigma \\ &= 25 + (1.28)(5) \\ &= 31.4\end{aligned}$$

For the bottom 70%: $A = 0.7000 \quad [0.6985] \quad \to \quad z = 0.52$

$$\begin{aligned}x &= \mu + z\sigma \\ &= 25 + (0.52)(5) \\ &= 27.6\end{aligned}$$

For the bottom 30%: $A = 0.3000 \quad [0.3015] \quad \to \quad z = -0.52$

$$\begin{aligned}x &= \mu + z\sigma \\ &= 25 + (-0.52)(5)\end{aligned}$$

$\underline{= 22.4}$

For the bottom 10%: $A = 0.1000 \quad [0.1003] \quad \rightarrow \quad z = -1.28$

$$\begin{aligned} x &= \mu + z\sigma \\ &= 25 + (-1.28)(5) \\ &= \underline{18.6} \end{aligned}$$

A	Higher than 31.4
B	27.6 to 31.4
C	22.4 to 27.6
D	18.6 to 22.4
E	Less than 18.6

d) The curving scheme in part (*c*) is fairer because it takes into account the variation as discussed in part (*b*). Assuming the usual 90-80-7060 letter grade cut-offs, for example, the percentage of *A*'s under the scheme in part (a) with $\mu = 25 \textit{ and } \sigma = 5$ is

$$\begin{aligned} P(x > 90) &= 1 - P(x < 90) \\ &= 1 - P(z < 3.00) \\ &= 1 - .9987 \\ &= \underline{0.0013} \quad \textit{or} \quad 0.13\% \end{aligned}$$

This is considerably less than the 10% *A* under the scheme in part (*c*) and reflects the fact that the variation in part (a) in unrealistically small.