Lecture Four

Section 4.1 – Relations and Their Properties

Definition

Let A and B be sets. A binary relation from A to B is a subset of $A \times B$

A binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.

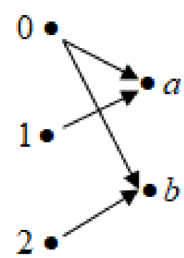
We use the notation $a\ R\ b$ to denote that $(a,\ b) \in R$ and $a\ \not R\ b$ to denote that $(a,\ b) \notin R$. Moreover, when $(a,\ b)$ belongs to R, a is said to be related to b by R.

Example

Let $A = \{0,\ 1,\ 2\}$ and $B = \{a,\ b\}$. Then $\{(0,\ a),\ (0,\ b),\ (1,\ a),\ (2,\ b)\}$ is a relation from A to B.

This means, for instance, that $0Ra$ but the $1\not R b$.

Relations can be represented graphically, as shown below, using arrows to represent ordered pairs.



Another way to represent this relation is to use a table.

R	***a***	***b***
0	x	x
1	x	
2		x

Relations on a Set

Definition

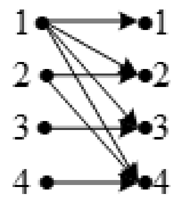
A ***relation*** on a set A is a relation from A to A. and it's a subset of $A \times A$

Example

Let $A = \{1, 2, 3, 4\}$ which ordered pairs are the relation $R = \{(a,\ b) | \ a \ divides\ b\}$?

Solution

$$R = \{(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (1,\ 4),\ (2,\ 2),\ (2,\ 4),\ (3,\ 3),\ (4,\ 4)\}$$



Example

Consider these relations on the set of integers:

$$R_1 = \{(a,\ b)\ |\ a\ \le\ b\}$$

$$R_2 = \{(a,\ b)\ |\ a\ >\ b\}$$

$$R_3 = \{(a,\ b)\ |\ a\ =\ b\ \text{ or }\ a\ =\ -b\}$$

$$R_4 = \{(a,\ b)\ |\ a\ =\ b\}$$

$$R_5 = \{(a,\ b)\ |\ a\ =\ b+1\}$$

$$R_6 = \{(a,\ b)\ |\ a+b\ \le 3\}$$

Which of these relations contain each of the pairs (1, 1), (1, 2), (2, 1), (1, −1), and (2, 2)?

Solution

(1, 1) $\rightarrow R_1, R_3, R_4, and\ R_6$

(1, 2) $\rightarrow R_1\ and\ R_6$

(2, 1) $\rightarrow R_2, R_5, and\ R_6$

(1, −1) $\rightarrow R_2, R_3, and\ R_6$

(2, 2) $\rightarrow R_1, R_3, and\ R_4$

Example

How many relations are there on a set with *n* elements?

Solution

A relation on a set *A* is a subset of $A \times A$. Because $A \times A$ has n^2 elements when *A* has *n* elements, and a set with *m* elements has 2^m subsets, there are 2^{n^2} subset of $A \times A$.

Thus there are 2^{n^2} relations on a set with *n* elements.

Properties of Relations

Reflexive

Definition

A relation *R* on a set *A* is called ***reflexive*** if $(a,\ a) \in R$ for every element $a \in A$

Example

Consider the following relations on $\{1,\ 2,\ 3,\ 4\}$:

$R_1 = \{(1,\ 1),\ (1,\ 2),\ (2,\ 1),\ (2,\ 2),\ (3,\ 4),\ (4,\ 1),\ (4,\ 4)\}$

$R_2 = \{(1,\ 1),\ (1,\ 2),\ (2,\ 1)\}$

$R_3 = \{(1,\ 1),\ (1,\ 2),\ (1,\ 4),\ (2,\ 1),\ (2,\ 2),\ (3,\ 3),\ (4,\ 1),\ (4,\ 4)\}$

$R_4 = \{(2,\ 1),\ (3,\ 1),\ (3,\ 2),\ (4,\ 1),\ (4,\ 2),\ (4,\ 3)\}$

$R_5 = \{(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (1,\ 4),\ (2,\ 2),\ (2,\ 3),\ (2,\ 4),\ (3,\ 3),\ (3,\ 4),\ (4,\ 4)\}$

$R_6 = \{(3,\ 4)\}$

Which of these relations are ***reflexive***?

Solution

The relations R_3 and R_5 are reflexive because they contain all pairs of the form (*a, a*), namely, $(1,\ 1),\ (2,\ 2),\ (3,\ 3),\ \textit{and}\ (4,\ 4)$.

$R_1,\ R_2,\ R_4,\ \textit{and}\ R_6$ are not reflexive because (3, 3) is not in any of these relations.

Example

Is the "divides" relation on the set of positive integers reflexive?

Solution

Because $a|a$ whenever *a* is a positive integer, the "divides" relation is reflexive.

(0 is doesn't divide 0)

Symmetric

Definition

A relation *R* on a set *A* is called ***symmetric*** if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.

$$\forall a \, \forall b \left((a, b) \in R \rightarrow (b, a) \in R \right)$$

A relation *R* on a set *A* such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called ***antisymmetric***.

$$\forall a \, \forall b \left(\left((a, b) \in R \wedge (b, a) \in R \right) \rightarrow (a = b) \right)$$

Example

Is the "divides" relation on the set of positive integers symmetric? Is it antisymmetric?

Solution

It is antisymmetric because $1|2$ but $2 \nmid 1$

Example

Consider the following relations on $\{1,2\ ,3\ ,4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$$

$$R_2 = \{(1, 1), (1, 2), (2, 1)\}$$

$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$$

$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$$

$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$$

$$R_6 = \{(3, 4)\}$$

Which of these relations are symmetric and which are antisymmetric?

Solution

The relations R_2 and R_3 are symmetric because in each case (*b, a*) belongs to the relation whenever (*a, b*) does. $(1,\ 2)$ *and* $(2,\ 1)$ in R_2 $(1,\ 2),\ (2,\ 1),\ (1,\ 4)$ *and* $(4,\ 1)$ in R_3.

The relations R_1, R_4, R_5 and R_6 are antisymmetric because for each relations there is no pair of elements *a* and *b* with $a \neq b$ such that both (*a, b*) and (*b, a*) belong to the relation.

Transitive

Definition

A relation ***R*** on a set ***A*** is called ***transitive*** if whenever $(a,\ b) \in R$ and $(b,\ c) \in R$ then $(a,\ c) \in R$, for all $a,\ b,\ c \in A$

$$\forall a \forall b \forall c \left(\left((a,\ b) \in R \wedge (b,\ c) \in R \right) \rightarrow (a,\ c) \in R \right)$$

Example

Consider the following relations on $\{1,2\ ,3\ ,4\}$:

$R_1 = \{(1,\ 1),\ (1,\ 2),\ (2,\ 1),\ (2,\ 2),\ (3,\ 4),\ (4,\ 1),\ (4,\ 4)\}$

$R_2 = \{(1,\ 1),\ (1,\ 2),\ (2,\ 1)\}$

$R_3 = \{(1,\ 1),\ (1,\ 2),\ (1,\ 4),\ (2,\ 1),\ (2,\ 2),\ (3,\ 3),\ (4,\ 1),\ (4,\ 4)\}$

$R_4 = \{(2,\ 1),\ (3,\ 1),\ (3,\ 2),\ (4,\ 1),\ (4,\ 2),\ (4,\ 3)\}$

$R_5 = \{(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (1,\ 4),\ (2,\ 2),\ (2,\ 3),\ (2,\ 4),\ (3,\ 3),\ (3,\ 4),\ (4,\ 4)\}$

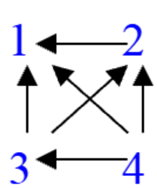
$R_6 = \{(3,\ 4)\}$

Which of these relations are transitive?

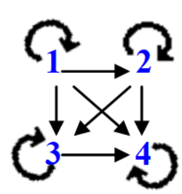
Solution

The relations R_4 and R_5 are transitive because in each of these relations case that is $(a,\ b)$ and $(b,\ c)$ belong to this relation then $(a,\ c)$ also does.

For R_4



For R_5



The relation R_1 is not transitive because $(3,\ 4)$ *and* $(4,\ 1)$ belong to R_1 but not $(3,\ 1)$

The relation R_2 is not transitive because $(2,\ 1)$ *and* $(1,\ 2)$ belong to R_2 but not $(2,\ 2)$

The relation R_3 is not transitive because $(4,\ 1)$ *and* $(1,\ 2)$ belong to R_3 but not $(4,\ 2)$

Example

Consider these relations on the set of integers:

$R_1 = \{(a,\ b) \mid a \le b\}$ $\qquad$ $R_2 = \{(a,\ b) \mid a > b\}$

$R_3 = \{(a,\ b) \mid a = b \text{ or } a = -b\}$ $\qquad$ $R_4 = \{(a,\ b) \mid a = b\}$

$R_5 = \{(a,\ b) \mid a = b+1\}$ $\qquad$ $R_6 = \{(a,\ b) \mid a+b \le 3\}$

Which of these relations contain each of the pairs (1, 1), (1, 2), (2, 1), (1, −1), and (2, 2)?

Solution

The relations R_1, R_2, R_3 and R_4 are transitive.

R_1 is transitive because $a \le b$ and $b \le c$ imply that $a \le c$

R_2 is transitive because $a > b$ and $b > c$ imply that $a > c$

R_3 is transitive because $a = \pm b$ and $b = \pm c$ imply that $a = \pm c$

R_4 is transitive because $a = b$ and $b = c$ imply that $a = c$

The relations R_5 and R_6 are not transitive.

R_5 is not transitive because $a = b+1$ and $b = c+1$ imply that

$$\begin{aligned} a &= (c+1)+1 \\ &= c+2 \ne c+1 \end{aligned}$$

R_6 is not transitive because $2+1 \le 3$ and $1+2 \le 3$ imply that $2+2 \not\le 3$

Example

Is the "divides" relation on the set of positive integers transitive?

Solution

Suppose a divides b and b divides c. Then there are positive integers m and n such that $b = ma$ and $c = nb$. Hence $c = n(ma) = (nm)a$, so a divides c.

Therefore this relation is transitive.

Combining Relations

Let $A = \{1,\ 2,\ 3\}$ and $B = \{1,\ 2,\ 3,\ 4\}$.

The relations

$R_1 = \{(1,\ 1),\ (2,\ 2),\ (3,\ 3)\}$ and $R_2 = \{(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (1,\ 4)\}$

$R_1 \cup R_2 = \{(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (1,\ 4),\ (2,\ 2),\ (3,\ 3)\}$

$R_1 \cap R_2 = \{(1,\ 1)\}$

$R_1 - R_2 = \{(2,\ 2),\ (3,\ 3)\}$

$R_2 - R_1 = \{(1,\ 2),\ (1,\ 3),\ (1,\ 4)\}$

Example

Let R_1 be the "less than" relation on the set of real numbers and let R_2 be the "greater than" relation on the set of real numbers, that is $R_1 = \{(x,y) \mid x < y\}$ and $R_2 = \{(x,y) \mid x > y\}$.
What are $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$, and $R_1 \oplus R_2$?

Solution

$(x,y) \in R_1 \cup R_2$ if and only if $(x,y) \in R_1$ *or* $(x,y) \in R_2$. That implies $(x,\ y) \in R_1 \cup R_2$ *iff* $x < y$ or $x > y$. Since $x < y$ or $x > y$ means that, that follows that $R_1 \cup R_2 = \{(x,\ y) \mid x \neq y\}$.

$R_1 \cap R_2 = \varnothing$, since it is impossible for a pair $(x,\ y)$ to belong to both R_1 and R_2 because $x < y$ and $x > y$.

$R_1 - R_2 = R_1$, since $R_1 \cap R_2 = \varnothing$

$R_2 - R_1 = R_2$, since $R_1 \cap R_2 = \varnothing$

$R_1 \oplus R_2 = R_1 \cup R_2 - R_1 \cap R_2 = \{(x,\ y) \mid x \neq y\}$

Definition

Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and A is the relation consisting of ordered pairs (a, c), where $a \in A,\ c \in C$, and for which there exists an element $b \in B$ such that $(a,\ b) \in R$ and $(b,\ c) \in S$. We denote the composite of R and S by $S \circ R$.

Example

What is the composite of the relation R and S, where

R is the relation from {1, 2, 3} to {1, 2, 3, 4} with $R = \{(1,\ 1),\ (1,\ 4),\ (2,\ 3),\ (3,\ 1),\ (3,\ 4)\}$.

S is the relation from {1, 2, 3, 4} to {0, 1, 2} with $S = \{(1,\ 0),\ (2,\ 0),\ (3,\ 1),\ (3,\ 2),\ (4,\ 1)\}$.

Solution

R	S		$S \circ R$
$(1,\ 1)$	$(1,\ 0)$	$\to$	$(1,\ 0)$
$(1,\ 4)$	$(4,\ 1)$	$\to$	$(1,\ 1)$
$(2,\ 3)$	$(3,\ 1)$	$\to$	$(2,\ 1)$
$(2,\ 3)$	$(3,\ 2)$	$\to$	$(2,\ 2)$
$(3,\ 1)$	$(1,\ 0)$	$\to$	$(3,\ 0)$
$(3,\ 4)$	$(4,\ 1)$	$\to$	$(3,\ 1)$

$$S \circ R = \{(1,\ 0),\ (1,\ 1),\ (2,\ 1),\ (2,\ 2),\ (3,\ 0),\ (3,\ 1)\}$$

Definition

Let R be a relation on the set A. Then powers $R^n,\ n = 1,\ 2,\ 3,\ \dots$ are defined recursively by

$$R^1 = R \quad and \quad R^{n+1} = R^n \circ R$$

Example

Let $R = \{(1,\ 1),\ (2,\ 1),\ (3,\ 2),\ (4,\ 3)\}$. Find the powers $R^n,\ n = 2,\ 3,\ 4,\ \dots$

Solution

$$R^2 = R \circ R = \{(1,\ 1),\ (2,\ 1),\ (3,\ 1),\ (4,\ 2)\}$$

$$R^3 = R^2 \circ R = \{(1,\ 1),\ (2,\ 1),\ (3,\ 1),\ (4,\ 1)\}$$

$$R^4 = R^3 \circ R = \{(1,\ 1),\ (2,\ 1),\ (3,\ 1),\ (4,\ 1)\}$$

From that, it follows that $R^n = R^3$ for $n = 5, 6, 7, \dots$.

Theorem

The relation on a set A is transitive ***iff*** $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

Proof

Suppose that $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$. In particular, $R^2 \subseteq R$. If $(a, b) \in R$ and $(b, c) \in R$, then by definition of composite, $(a, c) \in R^2$. Because $R^2 \subseteq R$, this means that $(a, c) \in R$. Hence, R is transitive.

Using mathematical induction to prove the only if part of the theorem

Assume that $R^n \subseteq R$ where n is a positive integer. This is the inductive hypothesis.

To complete the inductive step we must show that this implies that R^{n+1} is also a subset of R.

Assume that $(a, b) \in R^{n+1}$, then because $R^{n+1} = R^n \circ R$, there is an element x with $x \in A$ such that $(a, x) \in R$ and $(x, b) \in R^n$. The inductive hypothesis, namely, that $R^n \subseteq R$, implies that $(x, b) \in R$

Furthermore, because R is transitive, and $(a, x) \in R$ and $(x, b) \in R$, it follows that $(a, b) \in R$.

This shows that $R^n \subseteq R$.

Exercises *Section* 4.1 – Relations and Their Properties

1. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$ where $(a, b) \in R$ if and only if

a) $a = b$ *b*) $a + b = 4$ *c*) $a > b$

d) $a \mid b$ *e*) $\gcd(a,b) = 1$ *f*) $lcm(a,b) = 2$

2. *a*) List all the ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set $\{1, 2, 3, 4, 5, 6\}$

b) Display this relation graphically.

c) Display this relation in tabular form.

3. For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, symmetric, antisymmetric and transitive

a) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

b) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

c) $\{(2, 4), (4, 2)\}$

d) $\{(1, 2), (2, 3), (3, 4)\}$

e) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$

f) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

4. Determine whether the relation R on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if

a) a is taller than b.

b) a and b were born on the same day

c) a has the same first name as b.

d) a and b have a common grandparent.

5. Determine whether the relation ***R*** on the set of all ***real numbers*** is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in \boldsymbol{R}$ if and only if

a) $x + y = 0$

b) $x = \pm y$

c) $x - y$ *is a rational number*

d) $x = 2y$

e) $xy \ge 0$

f) $xy = 0$

g) $x = 1$

h) $x = 1$ *or* $y = 1$

6. Determine whether the relation ***R*** on the set of all ***integers numbers*** is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

a) $x \neq y$

b) $xy \geq 1$

c) $x = y + 1 \quad or \quad x = y - 1$

d) $x \equiv y (mod\ 7)$

e) x *is a multiple of* y

f) $x = y^2$

g) $x \geq y^2$

7. Show that the relation $R = \varnothing$ on nonempty set S is symmetric and transitive, but not reflexive.

8. Show that the relation $R = \varnothing$ on nonempty set $S = \varnothing$ is reflexive, symmetric and transitive.

9. Give an example of a relation on a set that is

a) both symmetric and antisymmetric

b) neither symmetric nor antisymmetric

10. A relation R is called ***asymmetric*** if $(a, b) \in R$ implies that $(b, a) \notin R$. Explore the notion of an asymmetric relation to the following

a) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

b) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

c) $\{(2, 4), (4, 2)\}$

d) $\{(1, 2), (2, 3), (3, 4)\}$

e) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$

f) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

g) *a* is taller than *b*.

h) *a* and *b* were born on the same day

i) *a* has the same first name as *b*.

j) *a* and *b* have a common grandparent.

11. Let R be the relation $R = \{(a, b) \mid a < b\}$ on the set of integers. Find

a) R^{-1} *b)* $\overline{R}$

12. Let R be the relation $R = \{(a, b) \mid a\ divides\ b\}$ on the set of positive integers. Find

a) R^{-1} *b)* $\overline{R}$

13. Let R be the relation on the set of all states in the U.S. consisting of pairs (a, b) where state a borders state b. Find

a) R^{-1} *b)* $\overline{R}$

14. Let $R_1 = \{(1, 2), (2, 3), (3, 4)\}$ and

$R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4)\}$ be relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$. Find

$a)$ $R_1 \cup R_2$ $b)$ $R_1 \cap R_2$ $c)$ $R_1 - R_2$ $d)$ $R_2 - R_1$

15. Let the relation $R = \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$ and the relation $S = \{(2, 1), (3, 1), (3, 2), (4, 2)\}$. Find $S \circ R$

16. $R_1 = \{(a, b) \in \boldsymbol{R}^2 \mid a > b\}$ $R_4 = \{(a, b) \in \boldsymbol{R}^2 \mid a \leq b\}$

$R_2 = \{(a, b) \in \boldsymbol{R}^2 \mid a \geq b\}$ $R_5 = \{(a, b) \in \boldsymbol{R}^2 \mid a = b\}$

$R_3 = \{(a, b) \in \boldsymbol{R}^2 \mid a < b\}$ $R_6 = \{(a, b) \in \boldsymbol{R}^2 \mid a \neq b\}$

Find the following:

$a)$ $R_1 \cup R_3$ $f)$ $R_2 - R_1$ $k)$ $R_1 \circ R_3$

$b)$ $R_1 \cup R_5$ $g)$ $R_1 \oplus R_3$ $l)$ $R_1 \circ R_4$

$c)$ $R_2 \cap R_4$ $h)$ $R_2 \oplus R_4$ $m)$ $R_1 \circ R_5$

$d)$ $R_3 \cap R_5$ $i)$ $R_1 \circ R_1$ $n)$ $R_1 \circ R_6$

$e)$ $R_1 - R_2$ $j)$ $R_1 \circ R_2$ $o)$ $R_2 \circ R_3$

17. Let R_1 *and* R_2 be the "*divides*" and "*is a multiple of*" relations on the set of all positive integers, respectively. That is $R_1 = \{(a, b) \mid a \text{ divides } b\}$ and $R_2 = \{(a, b) \mid a \text{ is a multiple of } b\}$

Find the following:

$a)$ $R_1 \cup R_2$ $c)$ $R_1 - R_2$ $e)$ $R_1 \oplus R_2$

$b)$ $R_1 \cap R_2$ $d)$ $R_2 - R_1$

Section 4.2 – Representing Relations

Representing Relations Using Matrices

A relation between finite sets can be represented using a zero-one matrix. Suppose that R is a relation from $A = \{a_1, a_2, a_3, \ldots, a_m\}$ to $B = \{b_1, b_2, b_3, \ldots, b_n\}$. The relation R can be represented by the matrix $M_a = \{m_{ij}\}$ where

$$m_{ij} = \begin{cases} 1 & if \ (a_i, b_j) \in R \\ 0 & if \ (a_i, b_j) \notin R \end{cases}$$

Example

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R the relation from A to B containing (a, b) if $a \in A, \ b \in B$, and $a > b$. What is the matrix representing R is $a_1 = 1, \ a_2 = 2, \ a_3 = 3,$ and $b_1 = 1, \ b_2 = 2$?

Solution

$$R = \{(2, 1), (3, 1), (3, 2)\}$$

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Example

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

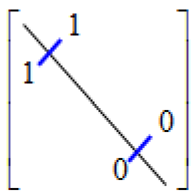
Solution

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$$

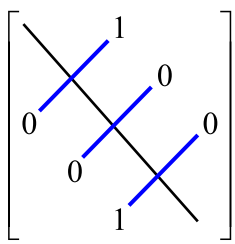
A relation R on A is ***reflexive*** if $(a, a) \in R$ whenever $a \in A$

$$M_R = (M_R)^t \qquad \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

A relation R on A is ***symmetric***



A relation R on A is ***antisymmetric*** iff $(a,\ b) \in R$ and $(b,\ a) \in R \ \Rightarrow\ a = b$



Example

Suppose that the relation R on the set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution

Because the diagonal elements are equal to 1, R is reflexive.

M_R is symmetric and it is not antisymmetric.

Relations Using Diagraphs

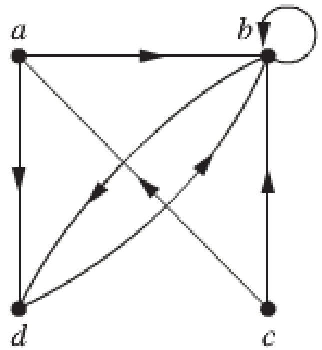
Definition

A directed ***graph***, or ***diagraph***, consists of a set V of ***vertices*** (or *nodes*) together with a set E ordered pairs of elements of V called ***edges*** (or *arcs*). The vertex a is called the ***initial*** vertex of the edge $(a,\ b)$, and the vertex b is called the ***terminal*** vertex of this edge.

Example

Draw the directed graph with vertices *a, b, c*, and *d*, and edges (*a, b*), (*a, d*), (*b, b*), (*b, d*), (*c, a*), (*c, b*), and (*d, b*)

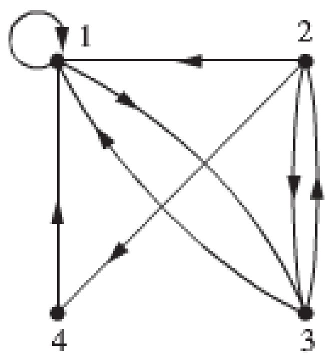
Solution



Example

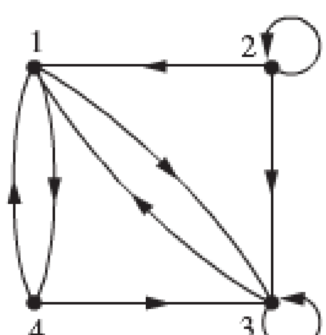
Draw the directed graph of the relation $R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$ on the set $\{1, 2, 3, 4\}$

Solution



Example

What are the ordered pairs in the relation *R* represented by the directed graph shown below



Solution

$$R = \{(1,\ 3),\ (1,\ 4),\ (2,\ 1),\ (2,\ 2),\ (2,\ 3),\ (3,\ 1),\ (3,\ 3),\ (4,\ 1),\ (4,\ 3)\}$$

Exercises Section 4.2 – Representing Relations

1. Represent each of these relations on $\{1, 2, 3\}$ with a matrix (with the elements of this set listed in increasing order). Then draw the directed graphs representing each relation

a) $\{(1,\ 1),\ (1,\ 2),\ (1,\ 3)\}$

b) $\{(1,\ 2),\ (2,\ 1),\ (2,\ 2),\ (3,\ 3)\}$

c) $\{(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (2,\ 2),\ (2,\ 3),\ (3,\ 3)\}$

d) $\{(1,\ 3),\ (3,\ 1)\}$

2. Represent each of these relations on $\{1, 2, 3, 4\}$ with a matrix (with the elements of this set listed in increasing order). Then draw the directed graphs representing each relation

a) $\{(1,\ 2),\ (1,\ 3),\ (1,\ 4),\ (2,\ 3),\ (2,\ 4),\ (3,\ 4)\}$

b) $\{(1,\ 1),\ (1,\ 4),\ (2,\ 2),\ (3,\ 3),\ (4,\ 1)\}$

c) $\{(1,\ 2),(1,\ 3),(1,\ 4),(2,\ 1),(2,\ 3),(2,\ 4),(3,\ 1),(3,\ 2),(3,\ 4),(4,\ 1),(4,\ 2),(4,\ 3)\}$

d) $\{(2,\ 4),\ (3,\ 1),\ (3,\ 2),\ (3,\ 4)\}$

3. List the ordered pairs in the relations on $\{1, 2, 3\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order). Then draw the directed graphs representing each relation

$a)\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad b)\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad c)\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

4. List the ordered pairs in the relations on $\{1, 2, 3, 4\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order). Then draw the directed graphs representing each relation

$a)\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad c)\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

$b)\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

5. Let R be the relation represented by the matrix

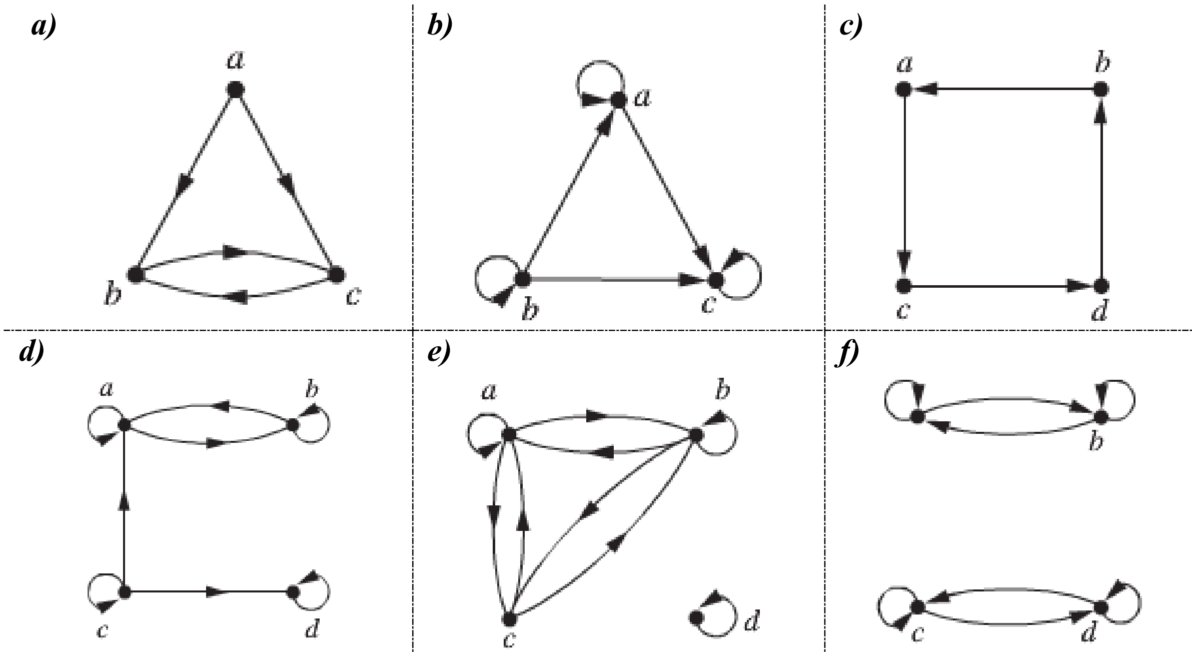
$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Find: $a)\ R^2 \quad b)\ R^3 \quad c)\ R^4$

6. Draw the directed graph that represents the relation

$\{(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)\}$

7. Determine whether the relations represented by the directed graphs are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive



Section 4.3 – Closures of Relations

Closures

The ***reflexive closure*** of R can be formed by adding to R all pairs of the form (a, a) with $a \in A$, not already in R.

The reflexive closure of R equals $R \cup \Delta$ where

$\Delta = \{(a,a) \mid a \in A\}$ is the ***diagonal relation*** on A.

Example

What is the reflexive closure of the relation $R = \{(a, b) \mid a < b\}$ on the set of integers?

Solution

The reflexive closure of R is the relation

$$\begin{aligned} R \cup \Delta &= \{(a, b) \mid a < b\} \cup \{(a, a) \mid a \in \mathbb{Z}\} \\ &= \{(a, b) \mid a \le b\} \end{aligned}$$

Example

What is the symmetric closure of the relation $R = \{(a, b) \mid a > b\}$ on the set of positive integers?

Solution

The symmetric closure of R is the relation

$$\begin{aligned} R \cup R^{-1} &= \{(a,b) \mid a > b\} \cup \{(b,a) \mid a < b\} \\ &= \{(a, b) \mid a \ne b\} \end{aligned}$$

Path in Directed Graphs

Definition

A path from a to b in the directed graph G is a sequence of edges (x_0, x_1), (x_1, x_2), (x_2, x_3), …, (x_{n-1}, x_n) in G, where n is nonnegative integer, and $x_0 = a$ and $x_n = b$, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path. The path is denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ and has length n. We view the empty set of edges as a path of length zero from a to a. A path of length $n \ge 1$ that begins and ends at the same vertex is called a ***circuit*** or ***cycle***.

Example

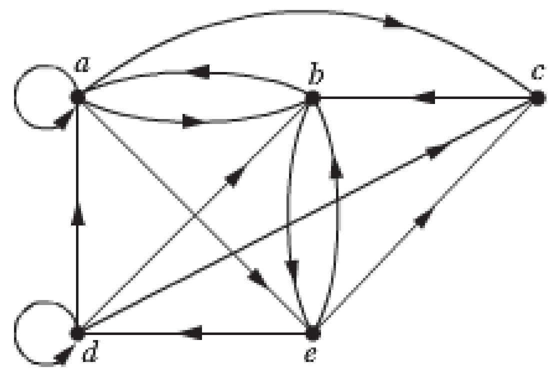
Which of the following are paths in the directed graph:

a, b, e, d; a, e, c, d, b; b, a, c, b, a, a, b; d, c; c, b, a; e, b, a, b, a, b, e ?

What are the lengths of those that are paths?

Which of the paths in this list are circuits?

Solution



Each of (a, b), (b, e), and (e, d) is an edge a, b, e, d is a path of length 3

(c, d) is not an edge, therefore a, e, c, d, b is not a path

b, a, c, b, a, a, b is a path of length 6

d, c is a path of length 1

c, b, a is a path of length 2

e, b, a, b, a, b, e is a path of length 6

The 2 paths b, a, c, b, a, a, b and e, b, a, b, a, b, e are circuits because they begin and end the same vertex.

The paths a, b, e, d; c, b, a; and d, c are not circuits

Theorem

Let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b if and only if $(a,b) \in R^n$

Proof

Using mathematical induction

There is a path from a to b of length one if and only if $(a,b) \in R$, which is true when $n = 1$.

Assume that the theorem is true for a positive integer n.

We need to prove that there is a path of length $n + 1$ from a to b if and only if $c \in A$ such there is a path of length 1 from a to c, so $(a,c) \in R$, and path of length n from c to b $(c,b) \in R^n$.

Consequently, by the inductive hypothesis, there is a path of length $n + 1$ from a to b if and only if there is an element c with $(a,c) \in R$ and $(c,b) \in R^n$. But there is such an element iff $(a,b) \in R^{n+1}$.

Therefore, there is a path of length $n + 1$ from a to b iff $(a,b) \in R^{n+1}$. This completes the proof.

Transitive Closures

Definition

Let R be a relation on a set A. The ***connectivity relation*** R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.

Example

Let R be the relation on the set of all people in the world that contains (a, b) if a has met b. What is R^n, where n is a positive integer greater than one? What is R^*?

Solution

The relation R^* contain (a, b) if there is a person c such that $(a, c) \in R$ and , that is, if there is a person c such that a has met c and c has met b.

Similarly, R^n consists of those pairs (a, b) such that there are people $x_1, x_2, \ldots, x_{n-1}$ such that a has met x_1. x_1 has met $x_2, \ldots, x_{n-1}$ has met b.

The relation R^* contains (a, b) if there is a sequence of people, starting with a and ending with b, such that each person in the sequence has met next person in the sequence.

Example

Let R be the relation on the set of all states in U.S. that contains (a, b) if state a and state b have a common border. What is R^n, where n is a positive integer? What is R^*?

Solution

The relation R^n contain (a, b), where it is possible to go from state a to state b by crossing exactly n state borders. The relation R^* consists of the ordered pairs (a, b), where it is possible to go from state a to state b crossing as many borders as necessary.

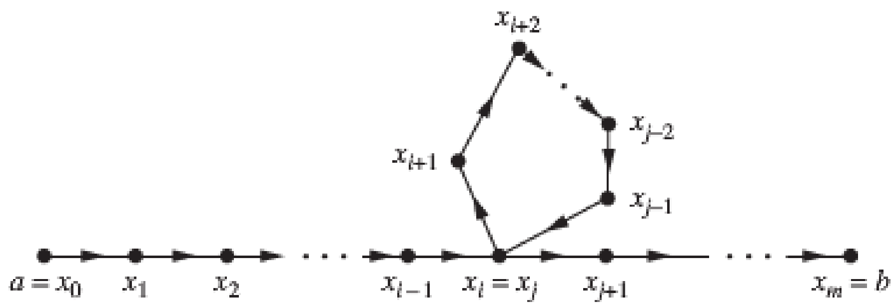
The only ordered pairs not in R^* are those containing sates that are not connected to the continental U.S.

Theorem

The transitive closure of a relation R equals the connectivity relation R^*.

Lemma

Let A be a set with n elements, and let R be the relation on A. If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n. Moreover, when $a \neq b$, if there is a path of length at least one in R from a to b, then there is such a path with length not exceeding $n - 1$.



Proof

Suppose there is a path from a to b in R. Let m be the length of the shortest such path.
Suppose that $x_0, x_1, x_2, \ldots, x_{m-1}, x_m$, where $x_0 = a$ and $x_m = b$, is such a path.
Suppose that $a = b$ and that $m > n$, so that $m \geq n+1$.
By the pigeonhole principle, because there are n vertices in A, among m vertices $x_0, x_1, \ldots, x_m$, at least two are equal.
Suppose that $x_i = x_j$ with $0 \leq i < j \leq m-1$. Then the path contains a circuit from x_i to itself. This circuit can be deleted from the path from a to b, leaving a path, namely, $x_0, x_1, \ldots, x_i, x_{j+1}, \ldots, x_m$, from a to b of shorter length. Hence, the path of shortest length must have less than or equal to n.

Exercises *Section* 4.3 – Closures of Relations

1. Let R be the relation on the set $\{0,\ 1,\ 2,\ 3\}$ containing the ordered pairs $(0,\ 1),\ (1,\ 1),\ (1,\ 2),\ (2,\ 0),(2,\ 2),\ and\ (3,\ 0)$. Find the

a) Reflexive closure of R.
b) Symmetric closure of R.

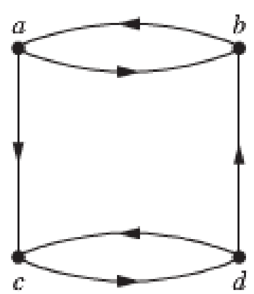
2. Let R be the relation $\{(a,\ b)\ \mid a \neq b\}$ on the set of integers. What is the reflexive closure of R?

3. Let R be the relation $\{(a,\ b)\ \mid\ a\ \ divides\ \ b\}$ on the set of integers. What is the symmetric closure of R?

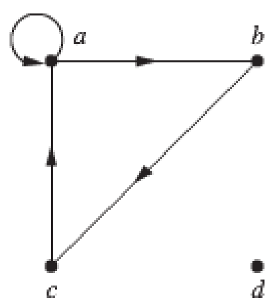
4. How can the directed graph representing the reflexive closure of a relation on a finite set be constructed from the directed graph of the relation?

5. Draw the directed graph of the *reflexive*, *symmetric*, and *both reflexive and symmetric* closure of the relations with the directed graph shown

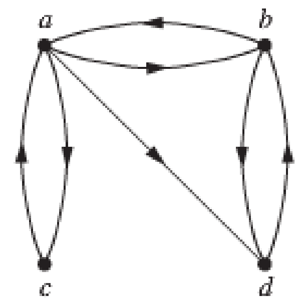
a)



b)

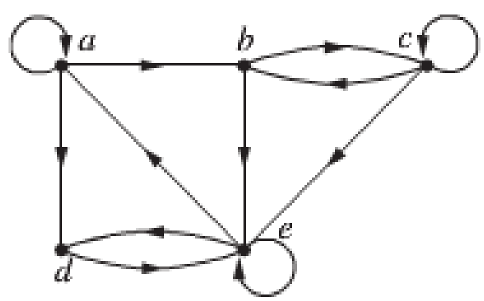


c)



6. **1**. Determine whether these sequences of vertices are paths in this directed graph

a) *a, b, c, e*
b) *b, e, c, b, e*
c) *a, a, b, e, d, e*
d) *b, c, e, d, a, a, b*
e) *b, c, c, b, e, d, e, d*
f) *a, a, b, b, c, ,c, b, e, d*



2. Find all circuits of length three in the directed graph

7. Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1,\ 3),\ (2,\ 4),\ (3,\ 1),\ (3,\ 5),\ (4,\ 3),\ (5,\ 1),\ and\ (5,\ 2)$. Find

$a)\ \ R^2$ $\qquad b)\ \ R^3$ $\qquad c)\ \ R^4$ $\qquad d)\ \ R^5$ $\qquad e)\ \ R^6$ $\qquad f)\ \ R^*$

8. Let R be the relation on the pair (a, b) if a and b are cities such that there is a direct non-stop airline flight from a to b. When is (a, b) in

a) R^2 *b*) R^3 *c*) R^*

9. Let R be the relation on the set of all students containing the ordered pair (a, b) if a and b are in at least one common class and $a \neq b$. When is (a, b) in

a) R^2 *b*) R^3 *c*) R^*

10. Suppose that the relation R is reflexive. Show that R^* is reflexive.

11. Suppose that the relation R is symmetric. Show that R^* is symmetric.

12. Suppose that the relation R is irreflexive. Is the relation R^2 necessarily irreflexive.

Section 4.4 – Equivalence Relations

Definition

A relation on a set A is called an ***equivalence relation*** if it is reflexive, symmetric, and transitive.

Definition

Two elements a and b that related by an equivalence relation are called ***equivalent***. The notation $a \sim b$ is often used to denotes that a and b are equivalent elements with respect to a particular equivalence relation.

Example

Let R be the relation on the set of integers such that aRb if and only if $a = b$ or $a = -b$. It follows that R is an equivalence relation.

Example

Let R be the relation on the set of real numbers such that aRb if and only if $a - b$ is an integer.
Is R an equivalence relation?

Solution

Because $a - a = 0$ is an integer for all real numbers a, aRa for all real numbers a. Hence, R is reflexive

Suppose that aRb, then $a - b$ is an integer, so $b - a$ and $b - c$ are integers.

Therefore, $a - c = (a - b) + (b - c)$ is also an integer. Hence, aRc.

Thus, R is transitive. Consequently, R is an equivalence relation.

Example

Let m be an integer with $m > 1$.

Show that the relation $R = \{(a,\ b)\ \mid\ a \equiv b\ (\boldsymbol{mod}\ m)\}$ is an equivalence relation on the set of integers.

Solution

$a \equiv b\ (\boldsymbol{mod}\ m)$ *iff* m divides $a - b$.

Since $0 = 0 \cdot m$ then $a - a = 0$ is divisible by m. Hence, $a \equiv a\ (\boldsymbol{mod}\ m)$, so congruence modulo m is reflexive.

Suppose that $a \equiv b\ (\boldsymbol{mod}\ m)$, then $a - b$ is divisible by m, so $a - b = km$, where k is an integer.

It follows that $b - a = (-k)m$, so $b \equiv a\ (\boldsymbol{mod}\ m)$. Hence, congruence modulo m is symmetric.

Suppose that $a \equiv b \ (\boldsymbol{mod}\ m)$ and $b \equiv c \ (\boldsymbol{mod}\ m)$, then m divides both $a - b$ and $b - c$. so $a - b = km$ and $b - c = lm$, where k and l are integers.

It follows that $a - c = (a-b) + (b-c) = km + lm = (k+l)m$, so $a \equiv c \ (\boldsymbol{mod}\ m)$.

Hence, congruence modulo m is transitive.

The congruence modulo m is an equivalence relation.

Example

Let n be a positive integer and S a set of strings. Suppose that R_n is the relation on S such that $s\ R_n t$ if and only if $s = t$, or both s and t have at least n characters and the first n characters of s and t are the same.

That is, a string of fewer than n characters is related only to itself; a string s with at least n characters is related to a string t if and only if t has at least n characters and t begins with the n characters at the start of s.

For example, let $n = 3$ and let S be the set of all bit strings.

Then $s\ R_3 t$ either when $s = t$ or both s and t are bit strings of length 3 or more that begin with the same three bits. For instant, $01\ R_3\ 01$ and $00111\ R_3\ 00101$ but $01 \not R_3\ 010$ and $01011 \not R_3\ 01110$

Show that every set S of strings and every positive integer n, R_n is an equivalence relation on S.

Solution

The relation R_n is reflexive because $s = s$, so that $s\ R_n s$ whenever s is a string in S.

If $s\ R_n t$, then either $s = t$ or s and t are both at least n characters long that begin with same n characters. This means that $t\ R_n s$. Therefore, R_n is symmetric.

Suppose that $s\ R_n t$ and $t\ R_n u$. Then either $s = t$ or s and t are both at least n characters long s and t begin with same n characters, and either $t = u$ or t and u are both at least n characters long t and u begin with same n characters. From this, we can deduce that either $s = u$ or s and u are both at least n characters long s and u begin with same n characters.

Because s, t and u are all at least n characters long s and u begin with same n characters as t does.

Therefore, R_n is transitive.

It follows that R_n is an equivalence relation.

Example

Let R be the relation on the set of real numbers such that $x\ R\ y$ if and only if x and y are real numbers that differ by less than 1, that is $|x - y| < 1$. Show that R is not an equivalence relation.

Solution

Let $x = 2.5$, $y = 1.8$, and $z = 1.1$, so that

$|x-y| = |2.5-1.8| = .7 < 1$ and $|y-z| = |1.8-1.1| = .7 < 1$

But $|x-z| = |2.5-1.1| = 1.4 > 1$.

That is $2.5 R\ 1.8$, $1.8 R\ 1.1$, but $2.5 \not{R}\ 1.1$

Equivalence Classes

Definition

Let R be an equivalent relation on a set A. The set of all elements that are related to an element a of A is called the ***equivalence class*** of a. The equivalence class of a with respect to R is denoted by $[a]_R$.

When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

$$[a]_R = \{s \mid (a, s) \in R\}$$

$b \in [a]_R$, then b called a ***representative*** of this equivalence class.

Example

Let R be the relation on the set of integers such that aRb if and only if $a = b$ or $a = -b$.
What is the equivalence class for this relation?

Solution

Because an integer is equivalent to itself and its negative in this equivalence relation, it follows that $[a] = \{-a, a\}$.

This set contains two distinct integers unless $a = 0$.

For instance, $[7] = \{-7, 7\}$, $[5] = \{-5, 5\}$, and $[0] = \{0\}$

Example

What is the equivalence class of 0 and 1 for congruence modulo 4?

Solution

The equivalence class of 0 contains all integers a such that $a \equiv 0(\textbf{mod } 4)$. The integers in this class are those divisible by 4, Hence, the equivalence class of 0 for this relation is

$$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$$

The equivalence class of 1 contains all integers a such that $a \equiv 1(\textbf{mod } 4)$. The integers in this class are those that have a remainder of 1when divided by 4, Hence, the equivalence class of 1 for this relation is

$$[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$$

Example

What is the equivalence class of the string 0111 with respect to the equivalence relation R_3 on the set of all bit strings?

Recall that $s\ R_3 t$ if and only if *s* and *t* are bit strings with *s* = *t* or *s* and *t* are strings of at least three bits that start with the same three bits.

Solution

The bit strings equivalent to 0111 are the bit strings with at least three bits that begin with 011. These are the bit strings 011, 0110, 0111, 01100, 01101, 01110, 01111, and so on …

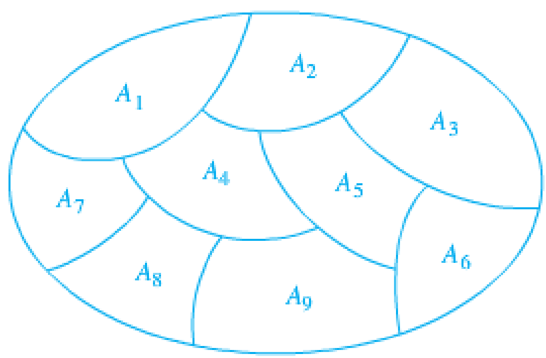
$$[011]_{R_3} = \{011,\ 0110,\ 0111,\ 01100,\ 01101,\ 01110,\ 01111,\ \ldots\}$$

Equivalence Classes and Partitions

Theorem

Let *R* be an equivalence relation on a set *A*. These statements for elements *a* and *b* of *A* are equivalent:

$(\boldsymbol{i})\ aRb \qquad (\boldsymbol{ii})\ [a]=[b] \qquad (\boldsymbol{iii})\ [a]\cap[b]\neq\varnothing$



$$\bigcup_{i\in I} A_i = S$$

Example

Suppose that *S* = {1, 2, 3, 4, 5, 6}.

The collection of sets $A_1 = \{1,\ 2,\ 3\}$, $A_2 = \{4,\ 5\}$, and $A_3 = \{6\}$ forms a partition of *S*.

Because these sets are disjoint and their union is *S*.

Theorem

Let R be an equivalent relation on a set S. Then the equivalence classes of R form a partition of S, Conversely, given a partition $\left\{A_i \mid i \in I\right\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Example

List the ordered pairs in the equivalence relation R produced by the partition $A_1 = \{1,\ 2,\ 3\}$, $A_2 = \{4,\ 5\}$, and $A_3 = \{6\}$ of $S = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$.

Solution

The subsets in the partition are the equivalence classes of R. The pair $(a,\ b) \in R$ if and only if a and b are in the same subset of the partition.
The pairs (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), and (3, 3) belong to R because $A_1 = \{1,\ 2,\ 3\}$ is an equivalence class.

The pairs (4, 4), (4, 5), (5, 4), and (5, 5) belong to R because $A_2 = \{4,\ 5\}$ is an equivalence class.

The pair (6, 6) belong to R because $A_3 = \{6\}$ is an equivalence class

Example

What are the sets in the partition of the integers arising from congruence modulo 4?

Solution

$[0]_4 = \{\ldots,\ -8,\ -4,\ 0,\ 4,\ 8,\ \ldots\}$

$[1]_4 = \{\ldots,\ -7,\ -3,\ 1,\ 5,\ 9,\ \ldots\}$

$[2]_4 = \{\ldots,\ -6,\ -2,\ 2,\ 6,\ 10,\ \ldots\}$

$[3]_4 = \{\ldots,\ -5,\ -1,\ 3,\ 7,\ 11,\ \ldots\}$

Exercises *Section* 4.4 – Equivalence Relations

1. Which of these relations on $\{0, 1, 2, 3\}$ are equivalence relations?
Determine the properties of an equivalence relation that the others lack.
What are the equivalence classes of the equivalence relations?

a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$

b) $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$

c) $\{(0, 0), (1, 1), (1, 2), (2, 1), (3, 2), (3, 3)\}$

d) $\{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

e) $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

2. Which of these relations on the set of all people are equivalence relations?
Determine the properties of an equivalence relation that the others lack.
What are the equivalence classes of the equivalence relations?

a) $\{(a, b) \mid a \text{ and } b \text{ are the same age}\}$

b) $\{(a, b) \mid a \text{ and } b \text{ have the same parents}\}$

c) $\{(a, b) \mid a \text{ and } b \text{ share a common parent}\}$

d) $\{(a, b) \mid a \text{ and } b \text{ have met}\}$

e) $\{(a, b) \mid a \text{ and } b \text{ speak a common language}\}$

3. Which of these relations on the set of all functions from $\mathbb{Z}$ to $\mathbb{Z}$ are equivalence relations?
Determine the properties of an equivalence relation that the others lack.
What are the equivalence classes of the equivalence relations?

a) $\{(f, g) \mid f(1) = g(1)\}$

b) $\{(f, g) \mid f(0) = g(0) \text{ or } f(1) = g(1)\}$

c) $\{(f, g) \mid f(x) - g(x) = 1 \text{ for all } x \in \mathbb{Z}\}$

d) $\{(f, g) \mid \text{for some } C \in \mathbb{Z}, \text{ for all } x \in \mathbb{Z}, f(x) - g(x) = C\}$

e) $\{(f, g) \mid f(0) = g(1) \text{ or } f(1) = g(0)\}$

4. Define three equivalence relations on the set of students in your discrete mathematics class different from the relations discussed in the text. Determine the equivalence classes for each of these equivalence relations.

5. Define three equivalence relations on the set of buildings on a college campus. Determine the equivalence classes for each of these equivalence relations.

6. Let R be the relation on the set of all sets of real numbers such that $S\ R\ T$ if and only if S and T have the same cardinality. Show that R is an equivalence relation. What are the equivalence classes of the sets $\{0, 1, 2\}$ and $\boldsymbol{Z}$?

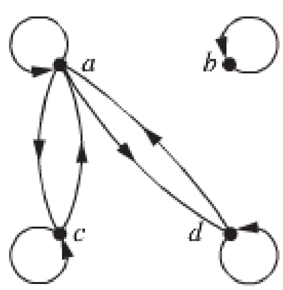
7. Suppose that A is a nonempty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y) such that $f(x) = f(y)$

a) Show that R is an equivalence relation on A.
b) What are the equivalence classes of R?

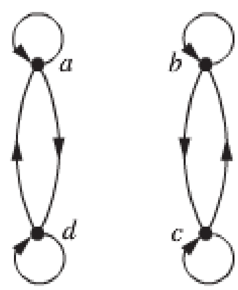
8. Suppose that A is a nonempty set, and R is an equivalence relation on A. Show that there is a function f with A as its domain such that $(x, y) \in R$ if and only if $f(x) = f(y)$

9. Determine whether the relation with the directed graph shown is an equivalence relation

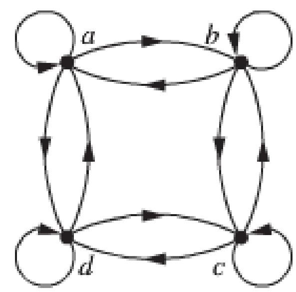
a)



b)



c)



10. Which of these collections of subsets are partitions of $\{1, 2, 3, 4, 5, 6\}$

a) $\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}$
b) $\{1\}, \{2, 3, 6\}, \{4\}, \{5\}$
c) $\{2, 4, 6\}, \{1, 3, 5\}$
d) $\{1, 4, 5\}, \{2, 6\}$

11. Which of these collections of subsets are partitions of $\{-3, -2, -1, 0, 1, 2, 3\}$

a) $\{-3, -1, 1, 3\}, \{-2, 0, 2\}$
b) $\{-3, -2, -1, 0\}, \{0, 1, 2, 3\}$
c) $\{-3, 3\}, \{-2, 2\}, \{-1, 1\}, \{0\}$
d) $\{-3, -2, 2, 3\}, \{-1, 1\}$

Section 4.5 – Partial Orderings

Definition

A relation R on set S is called a partial ordering or partial order if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a partially ordered set, or poset, and is denoted by (S, R). Members of S are called elements of the poset.

Example

Show that the "greater than or equal" relation $(\geq)$ is a partial ordering on the set of integers

Solution

Because $a \geq a$ for every integer a, $\geq$ is reflexive.
If $a \geq b$ and $b \geq a$, then $a = b$. Hence, $\geq$ is symmetric.
If $a \geq b$ and $b \geq c$ imply that $a \geq c$. Hence, $\geq$ is transitive.
It follows that $(\geq)$ is a partial ordering on the set of integers and $(\mathbb{Z}, \geq)$ is a poset.

Example

Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

Solution

Because $A \subseteq A$ whenever is a subset of S. $\subseteq$ is reflexive.
It is antisymmetric because $A \subseteq B$ and $B \subseteq A$ imply that $A = B$
$A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$, Hence $\subseteq$ is transitive.
Hence, $\subseteq$ is a partial ordering on $P(S)$ and $(P(S), \subseteq)$ is a poset.

Example

Let R be the relation on the set of people such that xRy if x and y are people and x is older than y. Show that is not a partial ordering,

Solution

R is not reflexive, because no person is older than herself or himself $x \not R x$
R is antisymmetric because if a person x is older than y, then y is not older than x. That is xRy, then $y \not R x$.
The relation is transitive because a person x is older than y, then y is older than z, then x is older than z.
R is not a partial ordering.

Definition

The elements *a* and *b* of poset $(S, \preceq)$ are called comparable if either $a \preceq b$ or $b \preceq a$. When *a* and *b* are elements of *S* such that neither $a \preceq b$ nor $b \preceq a$, *a* and *b* are called incomparable.

Example

In the poset $(\mathbb{Z}, |)$ are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Solution

The integers 3 and 9 are comparable, because 3 | 9.

The integers 5 and 7 are ***incomparable***, because $5 \nmid 7$ and $7 \nmid 5$

Definition

If $(S, \preceq)$ is a poset and every two elements of *S* are comparable, *S* is called a ***totally ordered*** or ***linearly ordered*** set, and $\rightleftarrows$ is called a ***total order*** or a ***linear order***. A totally ordered set is also called a ***chain***.

Example

The poset $(\mathbb{Z}, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever *a* and *b* are integers.

Example

The poset $(\mathbb{Z}^+, |)$ is not totally ordered, because it contains elements that are incomparable, such as 5 and 7.

Definition

If $(S, \preceq)$ is well-ordered set if it is a poset such that $\rightleftarrows$ is a total ordering and every nonempty subset of *S* has a least element.

Example

The set of ordered pairs of positive integers, $\mathbb{Z}^+ \times \mathbb{Z}^+$, with $(a_1, a_2) \preceq (b_1, b_2)$ if $a_1 < b_1$, or if $a_1 = b_1$ and $a_2 < b_2$ (Lexicographic ordering), is a well-ordered set.

The set ***Z***, with the usual ≤ ordering, is not well-ordered because the set of negative integers, which is a subset of ***Z***, has no least element.

Theorem – The Principle of Well-Ordered Induction

Suppose that S is a well-ordered set. Then $P(x)$ is true for all $x \in S$, if

Inductive Step: For every $y \in S$, if $P(x)$ is true for all $x \in S$ with $x \prec y$, then $P(y)$ is true.

Proof

Suppose it is not the case that $P(x)$ is true for all $x \in S$. Then there is an element $y \in S$ such that, $P(y)$ is false.

Consequently, the set $A = \{x \in S \mid P(x) \text{ } \textit{is false}\}$ is nonempty.

Because S is well ordered, A has a least element a. By the choice of a as a least element of A, we know that $P(x)$ is true for all with $x \prec a$.

This implies by the inductive step $P(a)$ is true. This contradiction shows that $P(x)$ must be true for all $x \in S$.

Example

Determine whether $(3,\ 5) \prec (4,\ 8)$, whether $(3,\ 8) \prec (4,\ 5)$, and whether $(4,\ 9) \prec (4,\ 11)$ in the poset $(\mathbb{Z} \times \mathbb{Z},\ \preceq)$, where $\preceq$ is the lexicographic ordering constructed from the usual $\leq$ relation on $\mathbb{Z}$.

Solution

Because $3 < 4$, it follows that $(3,\ 5) \prec (4,\ 8)$ and that $(3,\ 8) \prec (4,\ 5)$.

We have $(4,\ 9) \prec (4,\ 11)$, because the first entries of $(4,\ 9)$ and $(4,\ 11)$ are the same but $9 < 11$.

Maximal and Minimal Elements

An element of a poset is called maximal if it is not less than any element of the poset. That is, a is ***maximal*** in the poset $(S,\ \preceq)$ if there is no element $b \in S$ such that $a \prec b$.

Similarly, an element of a poset is called minimal if it is not greater than any element of the poset. That is, a is ***minimal*** in the poset $(S,\ \preceq)$ if there is no element $b \in S$ such that $b \prec a$.

Maximal and minimal elements are easy to spot using a ***Hasse*** diagram. They are the "top" and "bottom" elements in the diagram.

Sometimes there is an element in a poset that is greater than every other element. Such that an element is called the greatest element. That is, a s the ***greatest element*** of the poset $(S,\ \preceq)$

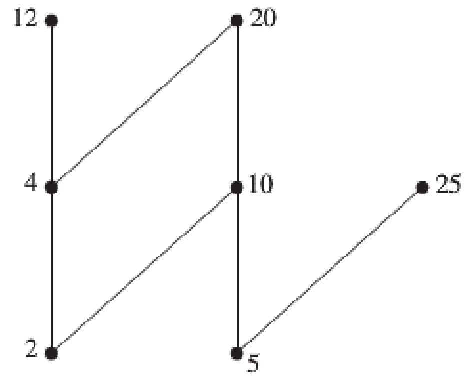
Example

Which elements of the poset $\left(\{2,\ 4,\ 5,\ 10,\ 12,\ 20,\ 25\} \cdots \mid\right)$ are maximal, and which are minimal?

Solution

From the Hasse diagram, the poset shows that the maximal elements are 12, 20, and 25.

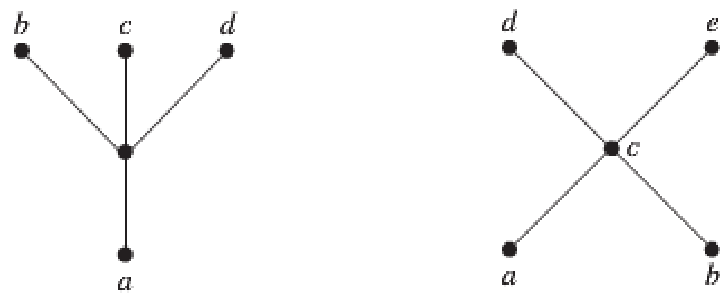
The minimal elements are 2 and 5.



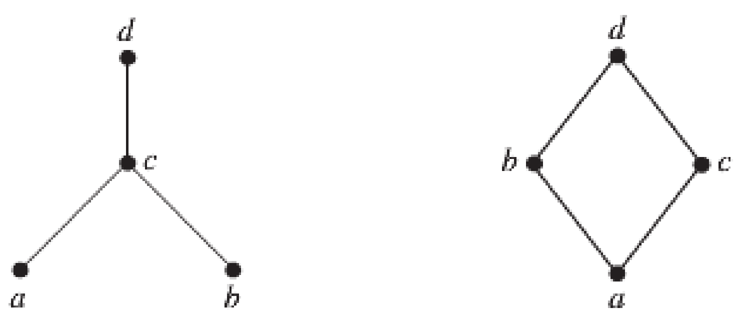
Hasse Diagram

Example

Determine whether the posets represented by each of the Hasse diagrams in figure below have greatest element and a least element.



(a) (b)



(c) (d)

Solution

The least element of the poset with Hasse diagram (a) is a. This poset has no greatest element.

The poset with Hasse diagram (b) has neither a least nor a greatest element.

The poset with Hasse diagram (c) has no least element. Its greatest element is d.

The poset with Hasse diagram (d) has least element a and greatest element d.

Example

Let S be a set. Determine whether there is a greatest element and a least element in the poset $\left(P(S),\subseteq\right)$

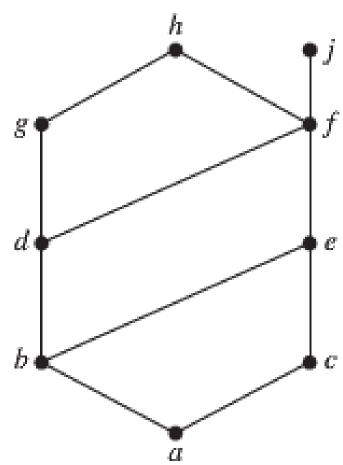
Solution

The least element is the empty set, because $\varnothing \subseteq T$ for any subset T of S.

The greatest element in this poset, because $T \subseteq S$ whenever T is a subset of S.

Example

Find the lower and upper bounds of the subsets $\{a,\ b,\ c\}$, $\{j,\ h\}$, and $\{a,\ c,\ d,\ f\}$ in the poset with the Hasse diagram shown in the figure.



Solution

The upper bounds of $\{a, b, c\}$ are e, f, j and h and its only lower bound is a.

There is no upper bounds of $\{j, h\}$, and its lower bounds are a, b, c, d, e, and f.

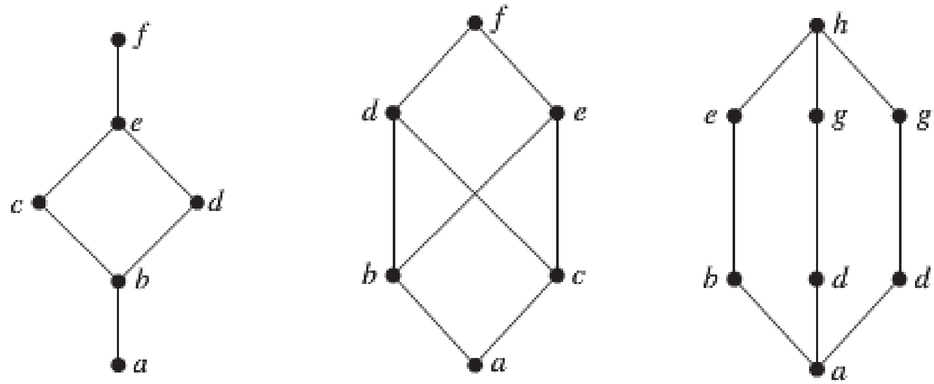
The upper bounds of $\{a, c, d, f\}$ are f, h, and j, and its lower bound is a.

Lattices

A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a ***lattice***. Lattices have many special properties.

Example

Determine whether the posets represented by each of the Hasse diagrams are lattices



(a) (b) (c)

Solution

The posets represented by the Hasse diagrams in (***a***) and (***c***) are both lattices because in each poset every pair of elements has both a least upper bound and a greatest lower bound.

On the other hand, the poset with the Hasse diagram shown in (***b***) is not a lattice, because the elements b and c have no least upper bound.

Each of the elements d, e, and f is an upper bound, but none of these 3 elements precedes the other two with respect to the ordering of this poset.

Example

Is the poset $\left(\mathbb{Z}^{+},\ |\right)$ a lattice?

Solution

Let a and b be two positive integers, The least upper bound and greatest lower bound of these 2 integers are the least common multiple and the greatest common divisor of these integers, respectively, as the reader should verify. It follows that this is a lattice.

Example

Determine whether the posets $(\{1,\ 2,\ 3,\ 4,\ 5\},\ |)$ and $(\{1,\ 2,\ 4,\ 8,\ 16\},\ |)$ are lattices

Solution

Because 2 and 3 have no upper bound in $(\{1,\ 2,\ 3,\ 4,\ 5\},\ |)$, they are certainly do not have a least upper bound. Hence, the first poset is not a lattice.

Every elements of the second poset have both a least upper bound and a greatest lower bound. The least upper bound of 2 elements in this poset is the larger of the elements and the greatest lower bound of 2 elements is the smaller of the elements. Hence, the second poset is a lattice.

Example

Determine whether $(P(S),\ \subseteq)$ is a lattice where S is a set.

Solution

Let A and B be 2 subsets of S. The least upper bound and the greatest lower bound of A and B are $A \cup B$ and $A \cap B$, respectively.

Hence, $(P(S),\ \subseteq)$ is a lattice.

Exercises *Section* 4.5 – Partial Orderings

1. Which of these relations on {0, 1, 2, 3} are partial orderings? Determine the properties of a partial ordering that the others lack.

a) $\{(0,\ 0),\ (1,\ 1),\ (2,\ 2),\ (3,\ 3)\}$

b) $\{(0,\ 0),\ (1,\ 1),\ (2,\ 0),\ (2,\ 2),\ (2,\ 3),\ (3,\ 2),\ (3,\ 3)\}$

c) $\{(0,\ 0),\ (1,\ 1),\ (1,\ 2),\ (2,\ 2),\ (3,\ 3)\}$

d) $\{(0,\ 0),\ (1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (2,\ 2),\ (2,\ 3),\ (3,\ 3)\}$

e) $\{(0,\ 0),\ (0,\ 1),\ (0,\ 2),\ (1,\ 0),\ (1,\ 1),\ (1,\ 2),\ (2,\ 0),\ (2,\ 2),\ (3,\ 3)\}$

f) $\{(0,\ 0),\ (2,\ 2),\ (3,\ 3)\}$

g) $\{(0,\ 0),\ (1,\ 1),\ (2,\ 0),\ (2,\ 2),\ (2,\ 3),\ (3,\ 3)\}$

h) $\{(0,\ 0),\ (1,\ 1),\ (1,\ 2),\ (2,\ 2),\ (3,\ 1),\ (3,\ 3)\}$

i) $\{(0,\ 0),\ (1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (2,\ 0),\ (2,\ 2),\ (2,\ 3),\ (3,\ 0),\ (3,\ 3)\}$

j) $\{(0,\ 0),\ (0,\ 1),\ (0,\ 2),\ (0,\ 3),\ (1,\ 0),\ (1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (2,\ 0),\ (2,\ 2),\ (3,\ 3)\}$

2. Is (*S*, *R*) a poset If *S* is the set of all people in the world and $(a,\ b) \in R$, where *a* and *b* are people, if

a) *a* is a taller than *b*?

b) *a* is not taller than *b*?

c) $a = b$ or *a* is an ancestor of *b*?

d) *a* and *b* have a common friend?

e) *a* is a shorter than *b*?

f) *a* weighs more than *b*?

g) $a = b$ or *a* is a descendant of *b*?

h) *a* and *b* do not have a common friend?

3. Which of these are posets?

a) (Z, =) *b)* $(Z,\ \neq)$ *c)* $(Z,\ \geq)$ *d)* $(Z,\ \nmid)$

e) $(R,\ =)$ *f)* $(R,\ <)$ *g)* $(R,\ \leq)$ *h)* $(R,\ \neq)$

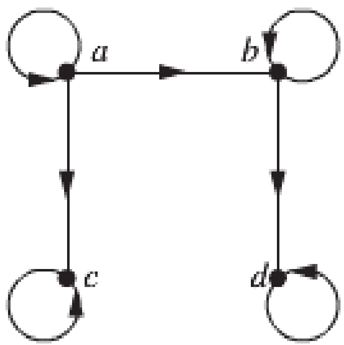
4. Determine whether the relations represented by these zero-one matrices are partial orders

a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *b)* $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *c)* $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *d)* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

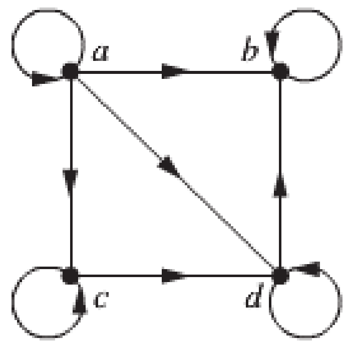
$$e)\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad f)\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

5. Determine whether the relation with the directed graph shown is a partial order.

a)



b)



c)

6. Let (S, R) be a poset. Show that $\left(S, R^{-1}\right)$ is also a poset, where R^{-1} is the inverse of R. The poset $\left(S, R^{-1}\right)$ is called the dual of (S, R).

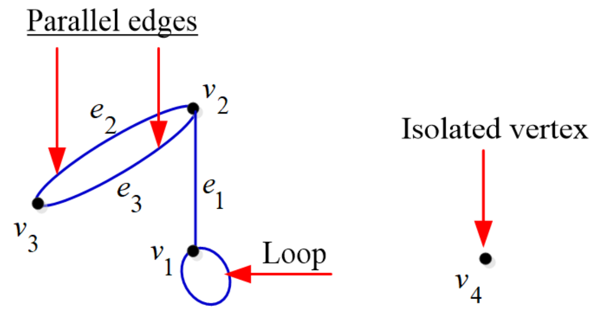
7. Draw the Hasse diagram for the "greater than or equal to" relation on $\{0, 1, 2, 3, 4, 5\}$

Section 4.6 – Graphs: Definitions and Basic Properties

Definition

A graph $G=(V,\ E)$ consists of V, a nonempty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two ***vertices*** (plural of ***vertex***) associated with it, called its ***endpoints***. An edge is said to connect its endpoints.

Visualize the graphs by using points to represent vertices and line segments, possibly curved, to represent edges, where the endpoints of a line segment representing an edge are the points representing the ***edge-endpoints***.

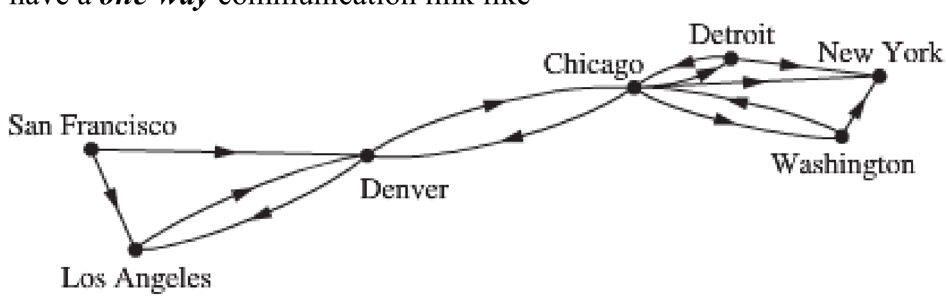


To model a computer network, we need graphs that have more than one edge connecting the same pair of vertices. Graphs that may have ***multiple edges*** connecting the same vertices are called ***multigraphs***.

Sometimes a communications link connects a data center with itself, a feedback loop for diagnostic purposes. Such edges are called ***loops***.

Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices or a vertex to itself, are sometimes called ***pseudographs***.

Sometimes we have a ***one-way*** communication link like



Basic Terminology

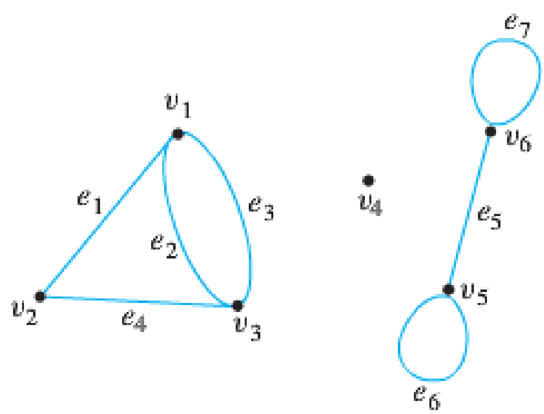
Definition

Two vertices u and v in an undirected graph G are called adjacent (or neighbors) in G if u and v are endpoints of an edge e of G. Such an edge e is called incident with the vertices u and v and e is said to connect u and v.

Graph Terminology			
Type	***Edges***	***Multiple Edges Allowed?***	***Loops Allowed?***
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

Example

Consider the following graph:



a) Write the vertex set and the edge set, and give a table showing the edge-point function.

b) Find all edges that are incident on v_1, all vertices that are adjacent to v_1, all edges that are adjacent to e_1, all loops, all parallel edges, all vertices that are adjacent to themselves, and all isolated vertices.

Solution

a) Vertex set $= \left\{v_1, v_2, v_3, v_4, v_5, v_6\right\}$

Edge set $= \left\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\right\}$

Edge-point function:

Edge	***Endpoints***
e_1	$\left\{v_1, v_2\right\}$
e_2	$\left\{v_1, v_3\right\}$
e_3	$\left\{v_1, v_3\right\}$
e_4	$\left\{v_2, v_3\right\}$
e_5	$\left\{v_5, v_6\right\}$
e_6	$\left\{v_5\right\}$
e_7	$\left\{v_6\right\}$

b) e_1, e_2, and e_3 are incident on v_1

v_2 and v_3 are adjacent to v_1

e_2, e_3, and e_4 are adjacent to e_1

e_6 and e_7 are loops.

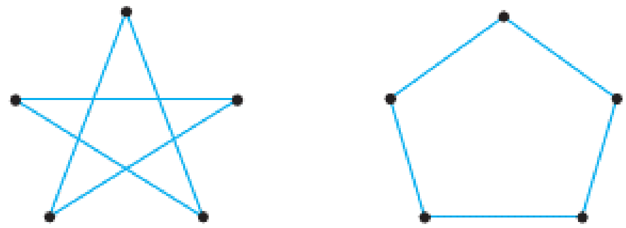
e_2 and e_3 are parallel.

v_5 and v_6 are adjacent to themselves.

v_4 is an isolated vertex.

Example

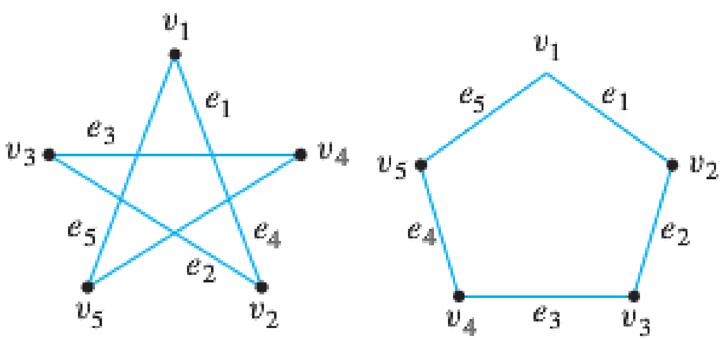
Consider the two drawing shown below.



(a) (b)

Label vertices and edges in such a way that both drawings represent the same graph

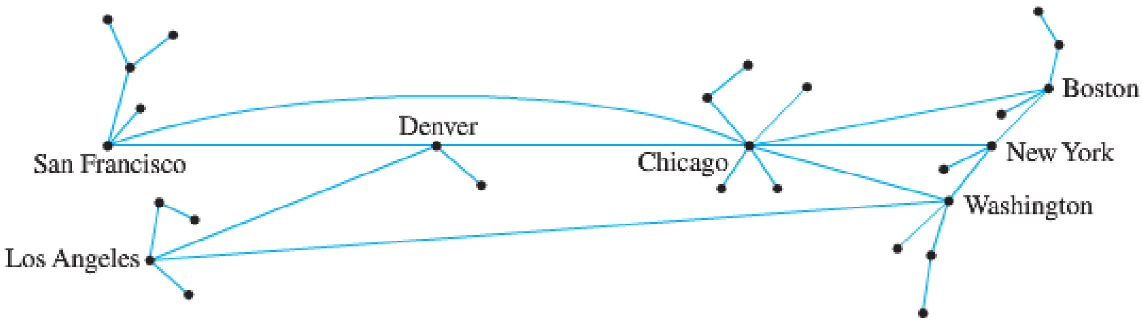
Solution



Edge	*Endpoints*
e_1	$\{v_1, v_2\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_3, v_4\}$
e_4	$\{v_4, v_5\}$
e_5	$\{v_5, v_1\}$

Definition

A ***directed graph*** (or ***digraph***) (V, E) consists of a nonempty set of vertices V and a set of ***directed edges*** (or ***arcs***) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to *start* at u and *end* at v.



Special Graphs

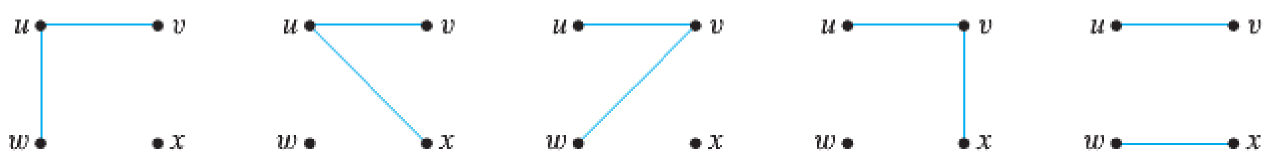
Definition

A ***simple graph*** is a graph that does not have any loops or parallel edges. In a simple graph, an edge with endpoints v and w is denoted $\{u, v\}$

Example

Draw all simple graphs with the four vertices $\{u, v, w, x\}$ and two edges, one of which is $\{u, v\}$

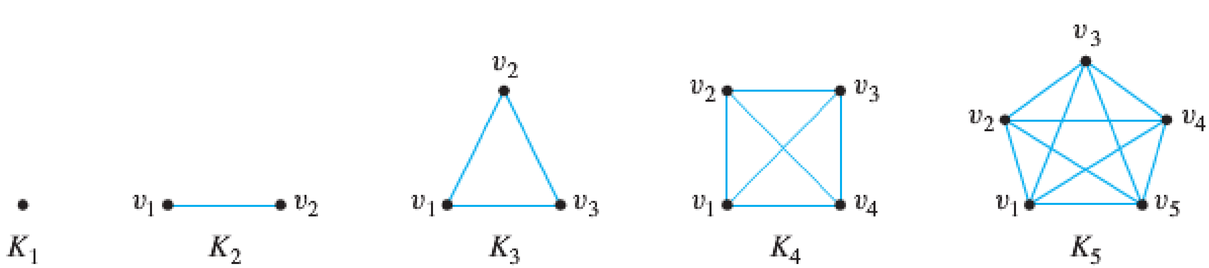
Solution



Definition

Let n be a positive integer. A ***complete graph*** on ***n vertices***, denoted $\boldsymbol{K}_n$ is a simple graph with n vertices and exactly one edge connecting each pair of distinct vertices.

The complete graphs $K_1, K_2, K_3, K_4,$ *and* K_5 can be drawn as follows



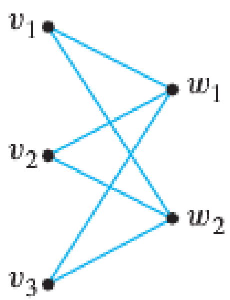
Definition

Let m and n be a positive integers. A ***complete bipartite graph*** on **(*m, n*)** ***vertices***, denoted $K_{m,n}$ is a simple graph with distinct vertices $v_1, v_2, \ldots, v_m$ *and* $w_1, w_2, \ldots, w_n$ that satisfies the following properties:

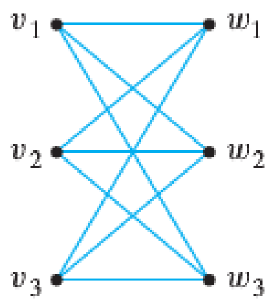
For all $i, k = 1, 2, \ldots, m$ and for all $j, l = 1, 2, \ldots, n,$

1. There is an edge from each vertex v_i to each vertex w_j
2. There is no edge from each vertex v_i to any other vertex v_k
3. There is no edge from each vertex w_j to any other vertex w_l

Example



$K_{3,2}$



$K_{3,3}$

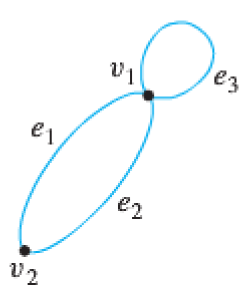
Definition

A graph H is said to be a subgraph of a graph if, and only if, every vertex in H is also a vertex in G, every edge in H is also an edge in G, and every edge in H has the same endpoints as it has in G.

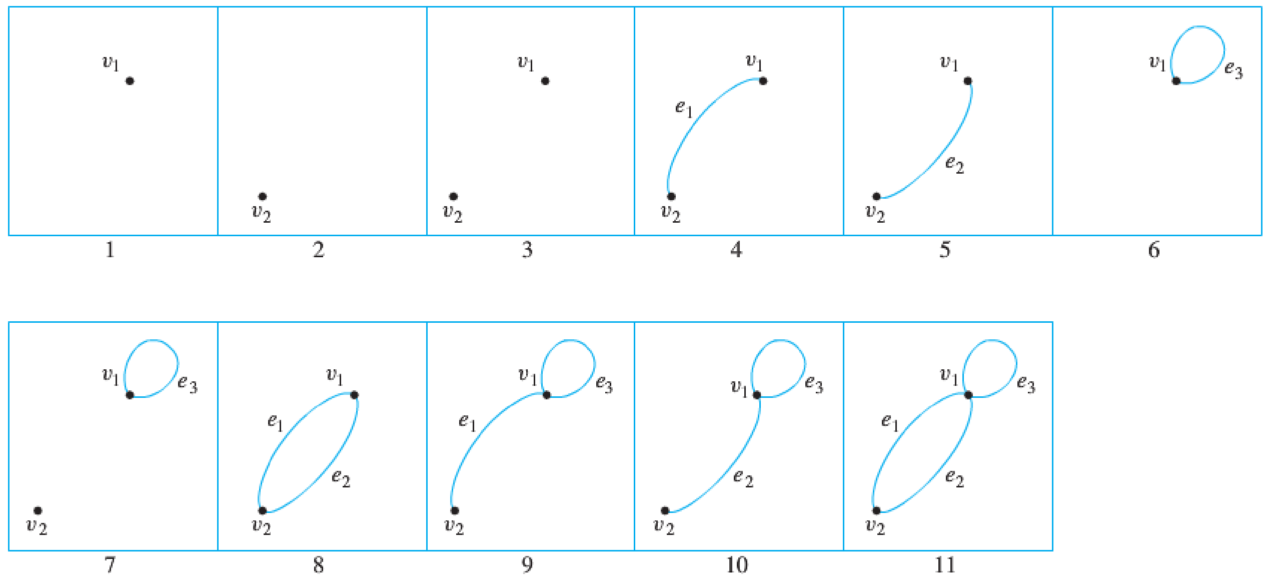
Example

List all subgraphs of the graph G with vertex set $\{v_1, v_2\}$ and edge set $\{e_1, e_2, e_3\}$ where the endpoints of e_1 are v_1 and v_2, the endpoints of e_2 are v_1 and v_2 and e_3 is a loop at v_1.

Solution



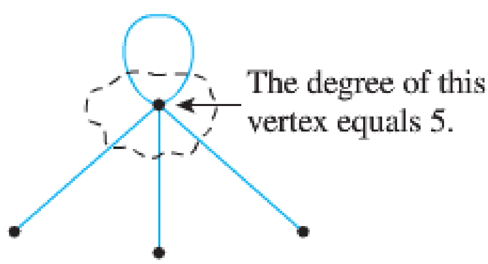
There are 11 subgraphs of G, which can be grouped according to those that do not have any edges, those that have one edge, those that have 2 edges, and those that have 3 edges.



The concept of Degree

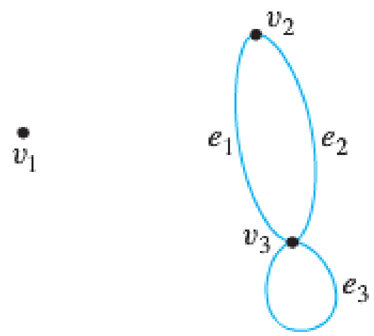
Definition

Let G be a graph and v a vertex of G. The ***degree of v***, denoted $\mathbf{deg}(v)$, equals the number of edges that are incident on v, with an edge that is a loop counted twice. The *total degree* of G is the sum of the degrees of all vertices of G.



Example

Find the degree of each vertex of the graph *G* shown below.



Then find the total degree of *G*.

Solution

$\deg(v_1) = 0$ since no edge is incident on v_1 $(v_1 \textit{ is isolated})$

$\deg(v_2) = 2$ since both e_1 and e_2 are incident on v_2

$\deg(v_3) = 4$ since both e_1 and e_2 are incident on v_3 and the loop e_3 is also incident on v_3 (contributes 2 to the degree of v_3)

$$\begin{aligned}\text{Total degree of } G &= \deg(v_1) + \deg(v_2) + \deg(v_3)\\ &= 0 + 2 + 4\\ &= 6\end{aligned}$$

The Handshake *Theorem*

If *G* is any graph, then the sum of the degrees of all the vertices of *G* equals twice the number of edges of *G*. Specially, if the vertices of *G* are $v_1, v_2, \ldots, v_n$, where *n* is a nonegative integer, then

$$\begin{aligned}\text{The total degree of G } &= \deg(v_1) + \deg(v_2) + \ldots + \deg(v_n)\\ &= 2 \cdot (\textit{the number of edges of } G)\end{aligned}$$

Corollary

The total degree of a graph is ***even***.

Example

Draw a graph with the specified properties or show that no such graph exists.

a) A graph with four vertices of degrees 1, 1, 2, and 3

b) A graph with four vertices of degrees 1, 1, 3, and 3

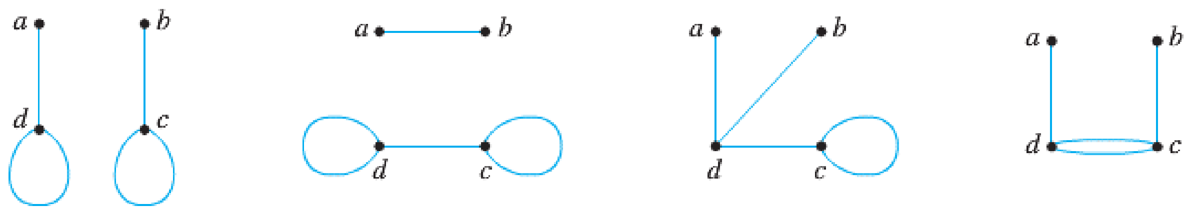
c) A simple graph with four vertices of degrees 1, 1, 2, and 3

Solution

a) No such graph is possible. By Corollary, the total degree of a graph is even.

But a graph with four vertices of degrees 1, 1, 2, and 3 would have a total degree of $1+1+2+3=7$ which is odd.

b) Let G be any of the graphs shown below



In each case, no matter how the edges are labeled, $\deg(a)=\deg(b)=1$ and $\deg(c)=\deg(d)=3$

c) There is no simple graph with four vertices of degrees 1, 1, 3, and 3.

Example

Is it possible in a group of 9 people for each to be friends with exactly five others?

Solution

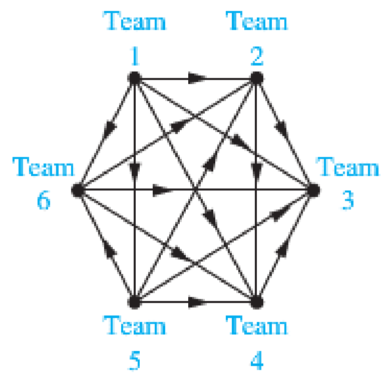
Imagine constructing an "acquaintance graph" in which each of the nine people represented by a vertex and 2 vertices are joined by an edge if, and only if, the people they represent are friends.

Suppose each of the people were friends with exactly five others. Then the degree of each of the 9 vertices of the graph would be 5, and so the total degree of the graph would be **45** (*odd*). Contradicts Corollary, which says that the total degree of a graph is even.

Therefore, the answer is ***no***.

Example

A tournament where each team plays every other team exactly once and no ties are allowed is called a round-robin tournament. Such tournaments can be modeled using directed graphs where each team is represented by a vertex, Note that (a, b) is an edge if team a beats team b. This graph is a simple directed graph, containing no loops or multiple directed edges (because no 2 teams play each other more than once. Such a directed graph model is presented

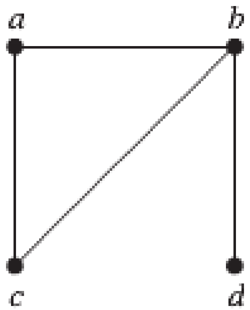


We see that team 1 is undefeated in this tournament, and Team 3 is winless.

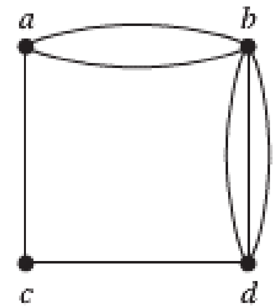
Exercises Section 4.6 – Graphs: Definitions and Basic Properties

1. Determine whether the graph shown has directed or undirected edges, whether it has multiple edges, and whether it has one or more loops.

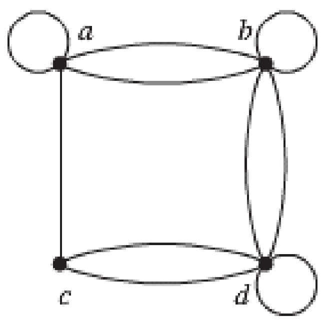
a)



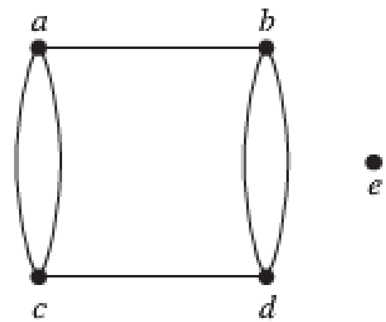
b)



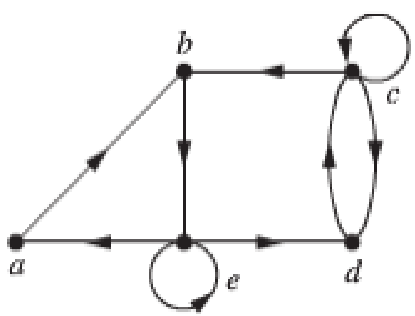
c)



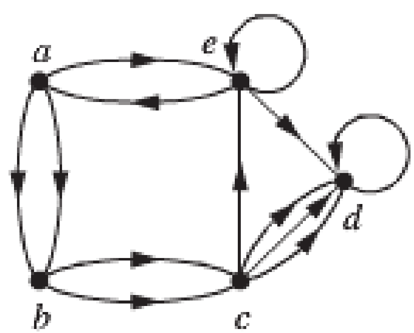
d)



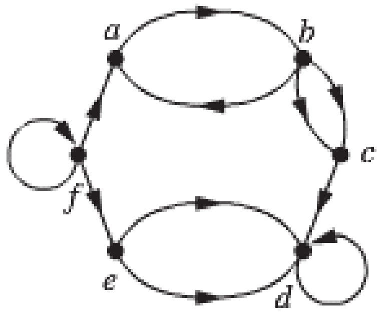
e)



f)

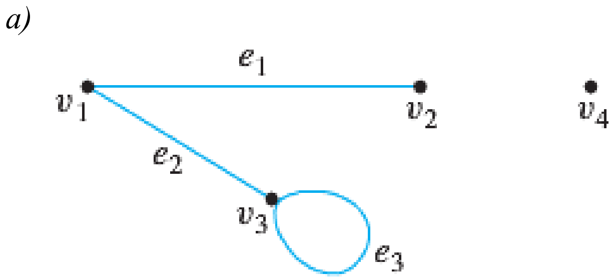


g)

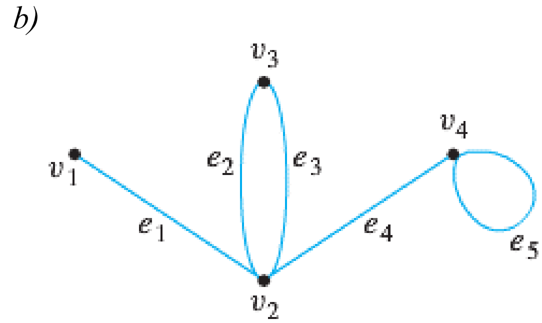


2. Define each graph formally by specifying its vertex set, its edge set, and a table giving the edge-endpoint function

a)



b)



3. Graph G has vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{e_1, e_2, e_3, e_4\}$, with edge-endpoint function as follow

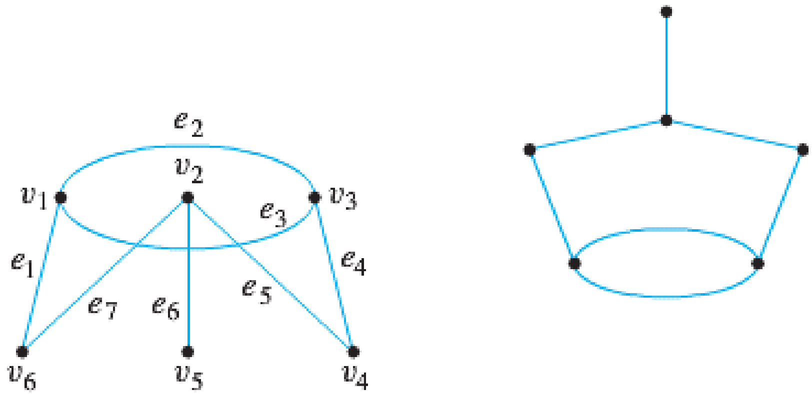
Edge	***Endpoints***
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_2\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_2\}$

4. Graph H has vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{e_1, e_2, e_3, e_4\}$, with edge-endpoint function as follow

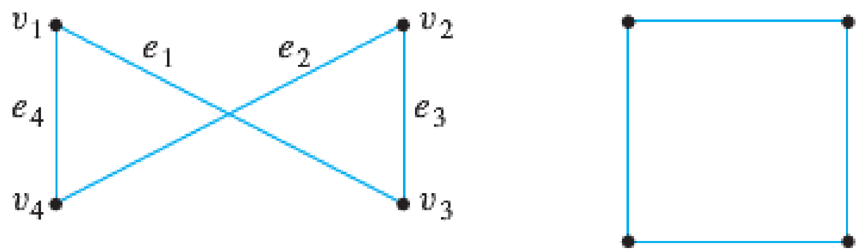
Edge	***Endpoints***
e_1	$\{v_1\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_1, v_5\}$

5. Show that the 2 drawings represent the same graph by labeling the vertices and edges of the right-hand drawing to correspond to those of the left-hand drawing.

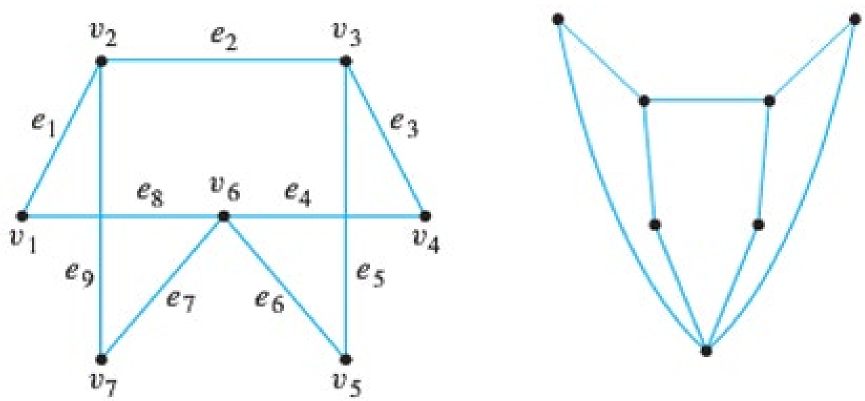
a)



b)



c)



6. For each of the graphs

i. Find all edges that are incident on v_1

ii. Find all vertices that are adjacent to v_3

iii. Find all edges that are adjacent to e_1

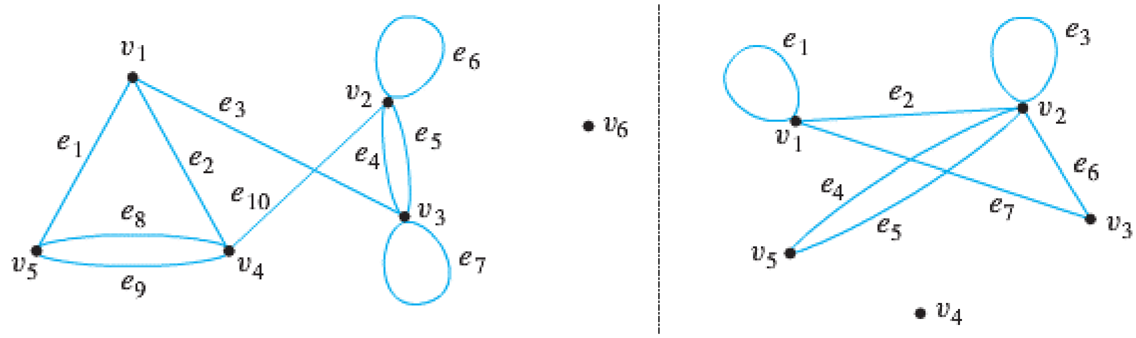
iv. Find all loops

v. Find all parallel edges

vi. Find all isolated vertices

vii. Find the degree of v_3

viii. Find the total degree of the graph



7. Let G be a simple graph. Show that the relation R on the set of vertices of G such that uRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, irreflexive relation on G.

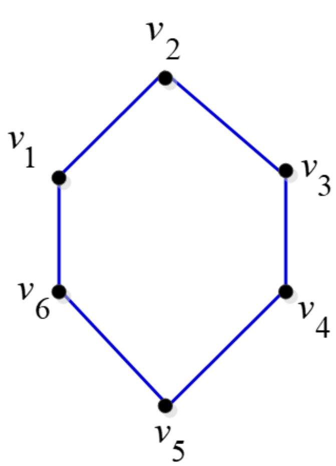
8. Let G be an undirected graph with a loop at every vertex. Show that the relation R on the set of vertices of G such that uRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, reflexive relation on G.

9. Explain how graphs can be used to model electronic mail messages in a network. Should the edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed? Describe a graph that models the electronic mail sent in a network in a particular week.

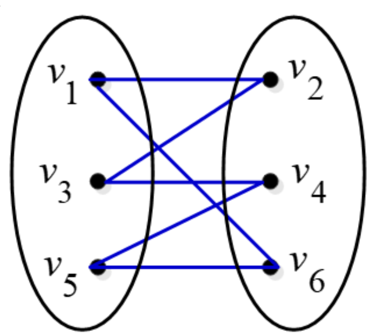
10. A bipartite graph G is a simple graph whose vertex set can be portioned into two disjoint nonempty subsets V_1 and V_2 such that vertices in V_1 may be connected to vertices in V_2, but no vertices in V_1 are connected to other vertices in V_1 and no vertices in V_2 are connected to other vertices in V_2.

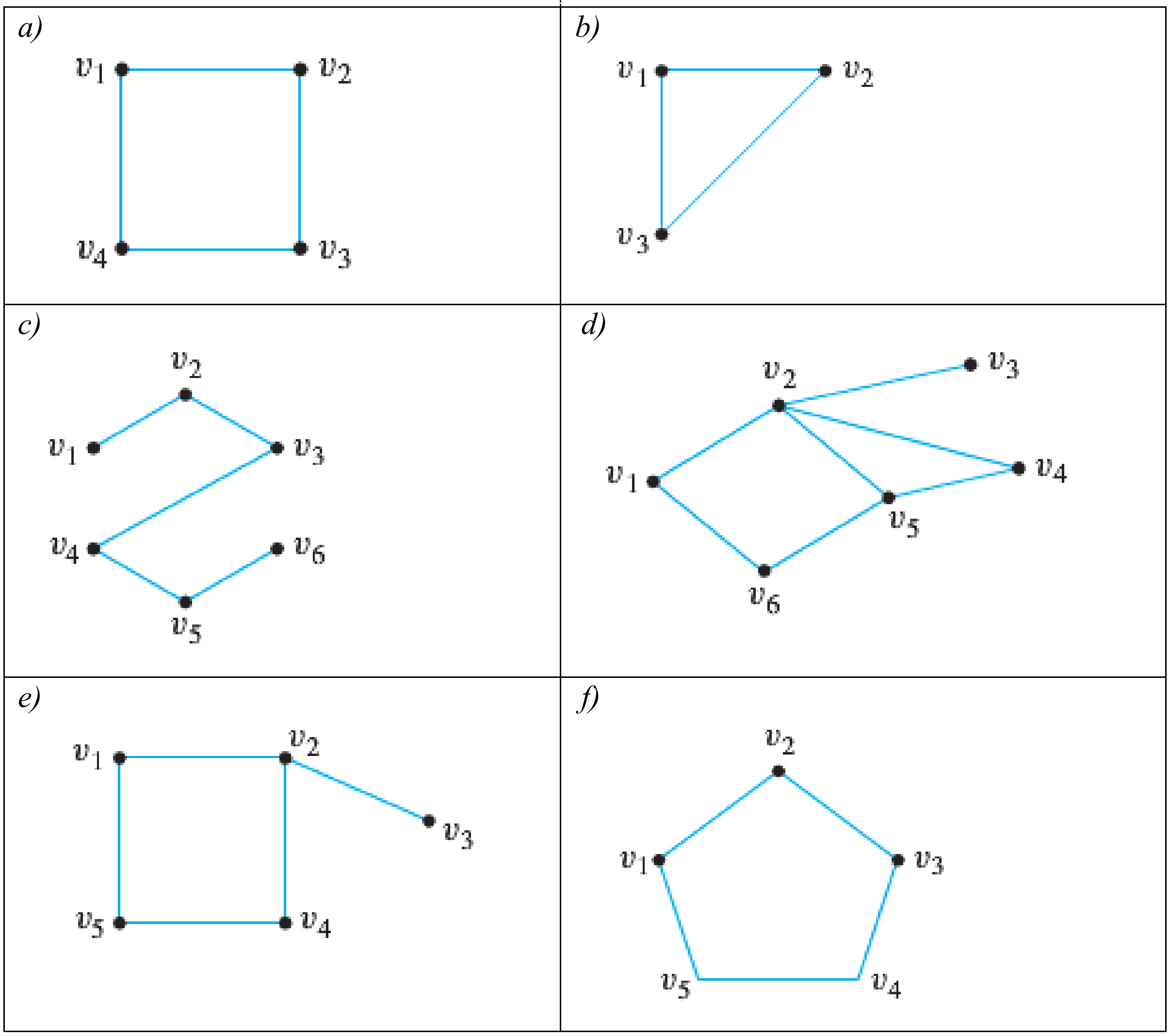
For example, the graph G illustrated in (i) can be redrawn as shown in (ii). From the drawing in (ii), you can see that G is bipartite with mutually disjoint vertex set $V_1 = \{v_1, v_3, v_5\}$ and $V_1 = \{v_2, v_4, v_6\}$

(i)



(ii)



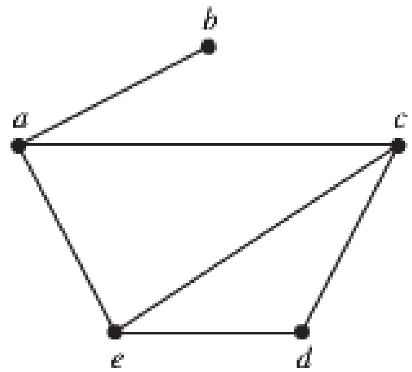

a)
v_1 v_2 v_4 v_3
b)
v_1 v_2 v_3
c)
v_2 v_1 v_3 v_4 v_6 v_5
d)
v_2 v_3 v_1 v_4 v_5 v_6
e)
v_1 v_2 v_3 v_5 v_4
f)
v_2 v_1 v_3 v_5 v_4

Section 4.7 – Representing Graphs and Graph Isomorphism

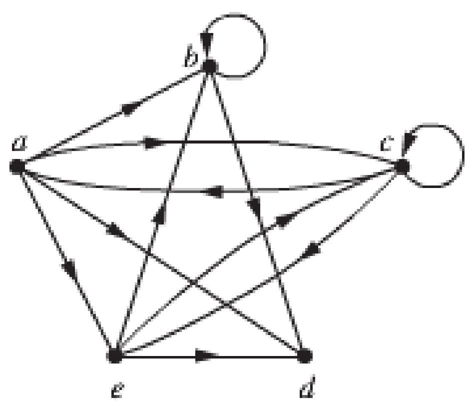
Sometimes, two graphs have exactly the same form, in the sense that there is a one-to-one correspondence between their vertex sets that preserves edges. In such case, we say that the two graphs are isomorphic.

Adjacency

Use the adjacency lists to describe the given graph



Adjacency List for a Simple Graph	
Vertex	***Adjacent Vertices***
a	*b, c , e*
b	*a*
c	*a, d, e*
d	*c, e*
e	*a, c , d*



Adjacency List for a Directed Graph	
Initial Vertex	***Terminal Vertices***
a	*b, c, d, e*
b	*b, d*
c	*a, c, e*
d	
e	*b, c , d*

Adjacency Matrices

Suppose that $G=(V,\ E)$ is a simple graph where $|V|=n$. Suppose that the vertices of *G* are listed arbitrary as $v_1,\ v_2,\ldots,\ v_n$. The adjacency matrix $\boldsymbol{A}\ \left(or\ \boldsymbol{A}_G\right)$ of *G*, with respect to this listing of the vertices, is the $n\times n$ zero-one matrix with 1 as its (*i, j*)*th* entry when v_i and v_j are adjacent, and 0 as its (*i, j*)*th* entry when they are not adjacent.

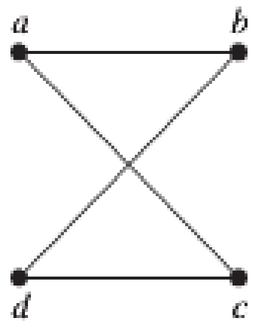
$$a_{ij}=\begin{cases}1 & if\ \left\{v_i,v_j\right\}\ \ is\ an\ edge\ of\ G\\ 0 & otherwise\end{cases}$$

Example

Draw a graph with the adjacency matrix

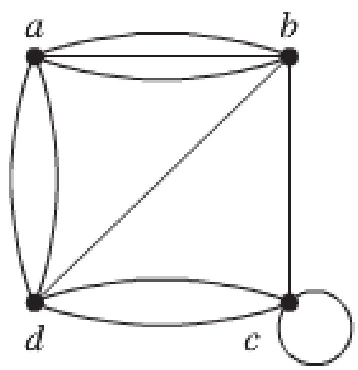
$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ with respect to the ordering of vertices *a, b, c, d*.

Solution



Example

Use an adjacency matrix to represent the pseudograph shown below.



Solution

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Incidence Matrices

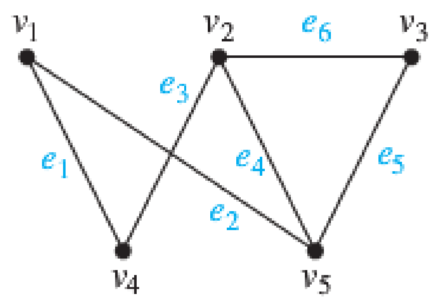
Let $G=(V,\ E)$ be undirected graph. Suppose that $v_1,\ v_2,\ldots,\ v_n$ are the vertices and $e_1,\ e_2,\ldots,\ e_n$ are the edges of G. Then the incident matrix with respect to this ordering of V and E is the $n\times m$ matrix $\boldsymbol{M}=\left[m_{ij}\right]$, where

$$m_{ij}=\begin{cases}1 & \textit{when edge } e_j \textit{ is incident with } v_i\\ 0 & \textit{otherwise}\end{cases}$$

Example

Represent the graph shown with an incidence matrix.

Solution

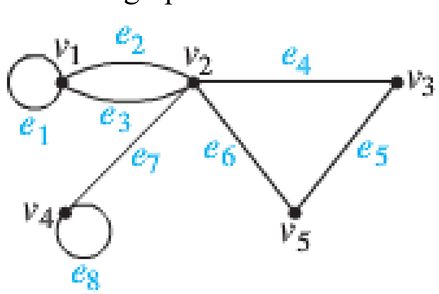


The incidence matrix is

$$\begin{array}{c} \\ v_1\\ v_2\\ v_3\\ v_4\\ v_5\end{array}\begin{array}{c}\begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6\end{matrix}\\ \begin{bmatrix}1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & 1\\ 1 & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1 & 1 & 0\end{bmatrix}\end{array}$$

Example

Represent the graph shown below with an incidence matrix.



Solution

The incidence matrix is

$$\begin{array}{c} \\ v_1\\ v_2\\ v_3\\ v_4\\ v_5\end{array}\begin{array}{c}\begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8\end{matrix}\\ \begin{bmatrix}1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\end{bmatrix}\end{array}$$

Isomorphism of Graphs

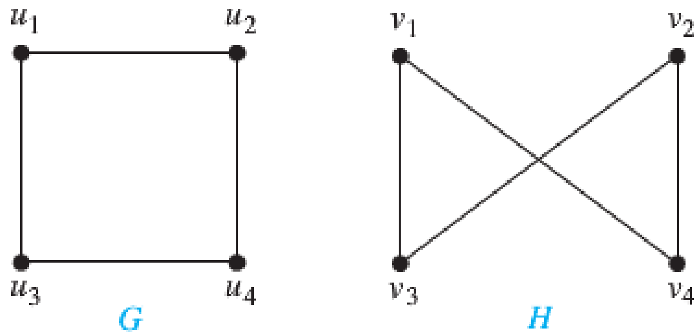
Definition

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a one-to-one and onto function f from V_1 to V_2 with the property that *a* and *b* are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all *a* and *b* in V_1. Such a function f is called an ***isomorphism***.

Two simple graphs that are not isomorphic are called ***nonisomorphic***.

Example

Show that the graphs $G = (V, E)$ and $H = (V, E)$, displayed below are isomorphic



Solution

The function f with $f(u_1) = v_1$, $f(u_2) = v_2$, $f(u_3) = v_3$, and $f(u_4) = v_4$ is a one-to-one correspondence between *V* and *W*.

To see that this correspondence preserves adjacency, note that adjacent vertices in *G* are u_1 *and* u_2, u_1 *and* u_3, u_2 *and* u_4, and u_3 *and* u_4, and each of the pairs $f(u_1) = v_1$ and $f(u_2) = v_4$, $f(u_1) = v_1$ and $f(u_3) = v_3$, $f(u_2) = v_4$ and $f(u_4) = v_2$, $f(u_3) = v_3$ and $f(u_4) = v_2$ consists of two adjacent vertices in *H*.

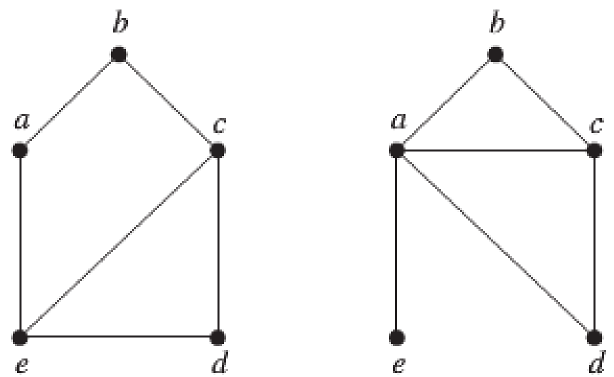
$\therefore$ If we flip v_2 & v_4, we end up the same graph.

Determining whether Two Simple Graphs are Isomorphic

Sometimes it is not hard to show that two graphs are not isomorphic. In Particular, we can show that two graphs are not isomorphic if we can find a property only one of the two graphs has, but that is preserved by isomorphism. A property preserved by isomorphism of graphs is called a ***graph invariant***.

Example

Show that the graphs shown below are not isomorphic



G ***H***

Solution

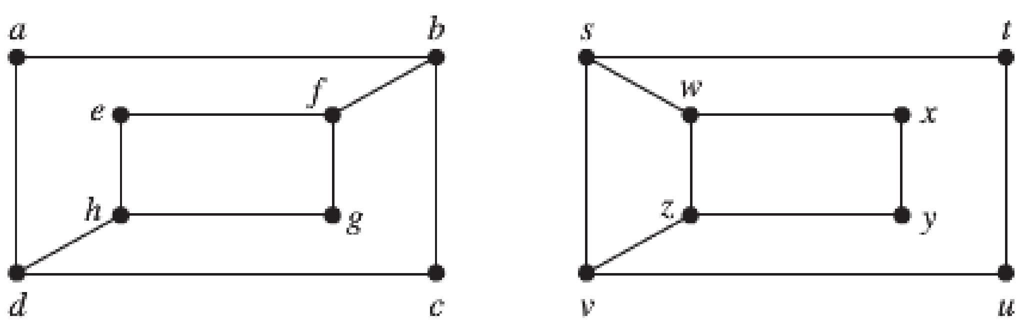
Both graphs G & H have 5 vertices and 6 edges.

H has a vertex of degree one, @ e, whereas G has no vertices of degree one.

It follows that G & H are not isomorphic.

Example

Determine whether the graphs shown below are isomorphic



G ***H***

Solution

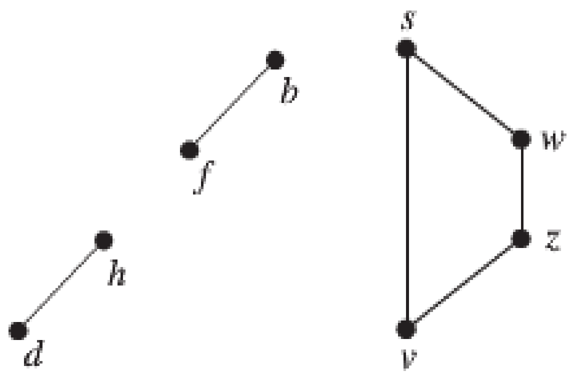
Both graphs G & H have 8 vertices and 10 edges.

Also both have 4 vertices of degree 2 and 4 vertices of degree 3.

$\deg(a) = 2$ in *G*, *a* must correspond to either *t, u ,x*, or *y* in *H*, because these are the vertices of degree 2 in *H*.

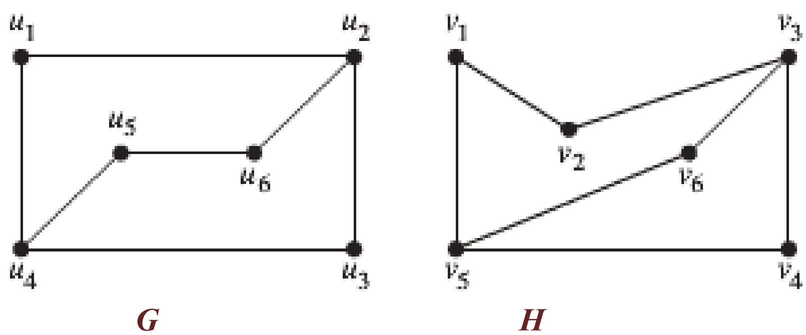
However, each of these four vertices in *H* is adjacent to another vertex of degree 2 in *H*, which is not true for *a* in *G*. Therefore, *G* & *H* are not isomorphic.

Another way to see that *G* & *H* are not isomorphic is by checking the subgraphs of *G* & *H* shown below, they have a different shape.



Example

Determine whether the graphs shown below are isomorphic



Solution

Both graphs *G* & *H* have 6 vertices and 7 edges.

Also both have 4 vertices of degree 2 and 2 vertices of degree 3.

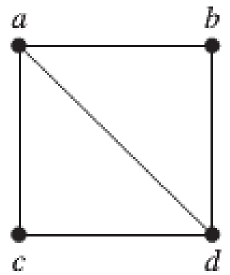
From the subgraphs of *G* & *H*, all vertices of degree 2 and the edges connecting them are isomorphic.

$u_1 \leftrightarrow v_6 \qquad u_4u_5u_6u_2 \leftrightarrow v_5v_1v_2v_3$

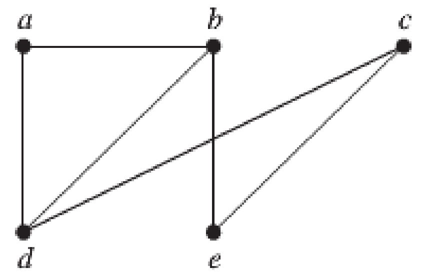
Exercises Section 4.7 – Representing Graphs and Graph Isomorphism

Use the adjacency list to represent the given graph, then represent with an adjacency matrix

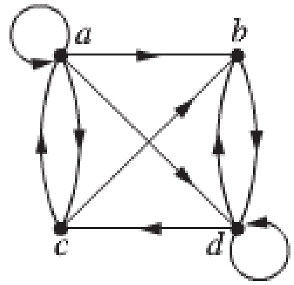
1.



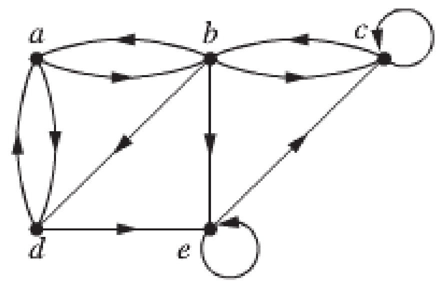
2.



3.



4.



5. Draw a graph with the given adjacency

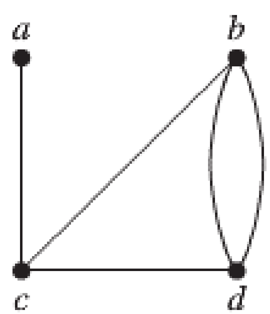
a) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

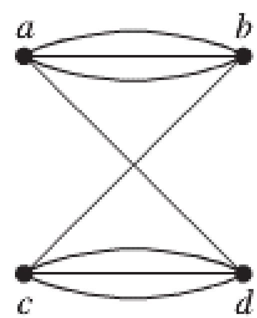
c) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

6. Represent the given graph using adjacency matrix

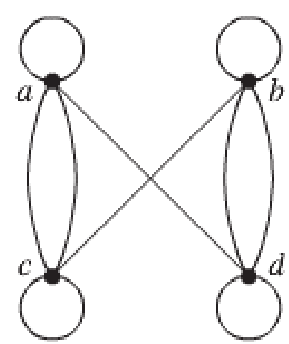
a)



b)



c)

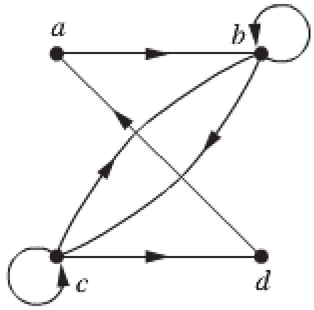


7. Draw an undirected graph represented by the given adjacency

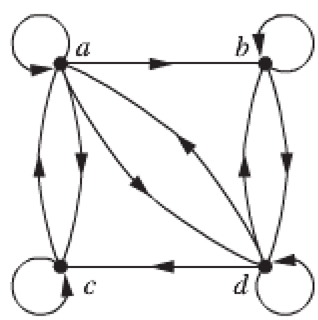
a) $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$ *b*) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ *c*) $\begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$

8. Find the adjacency matrix of the given directed multigraph with respect to the vertices listed in alphabetic order.

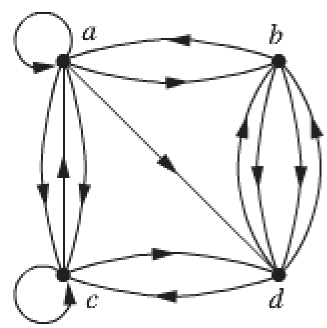
a)



b)

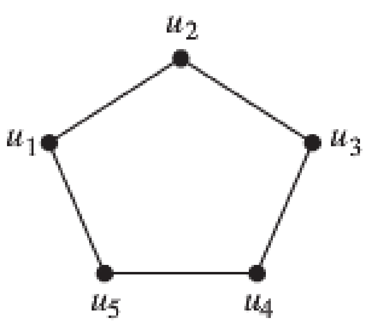


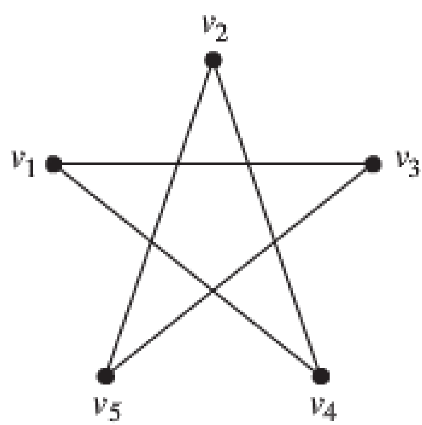
c)



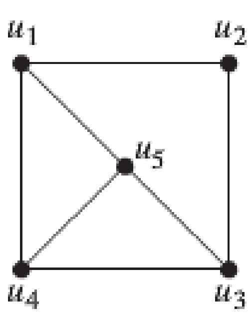
(**9 – 12**) Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

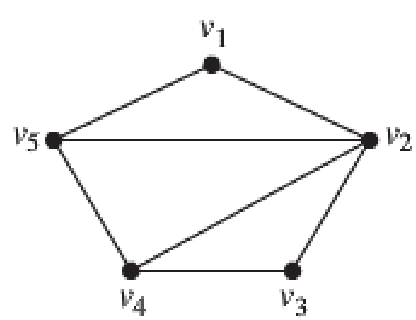
9.



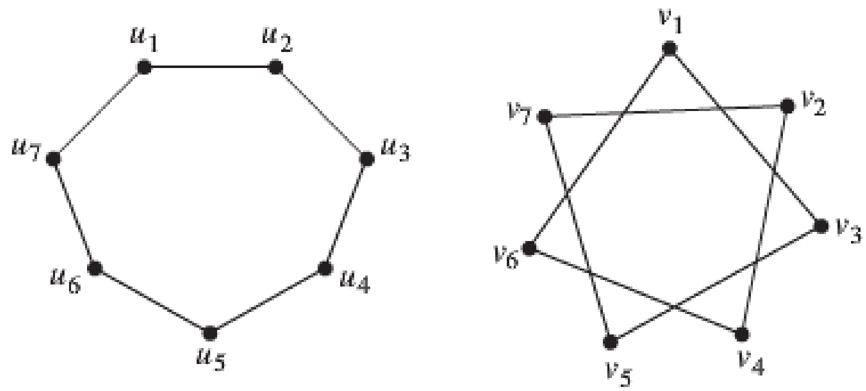


10.

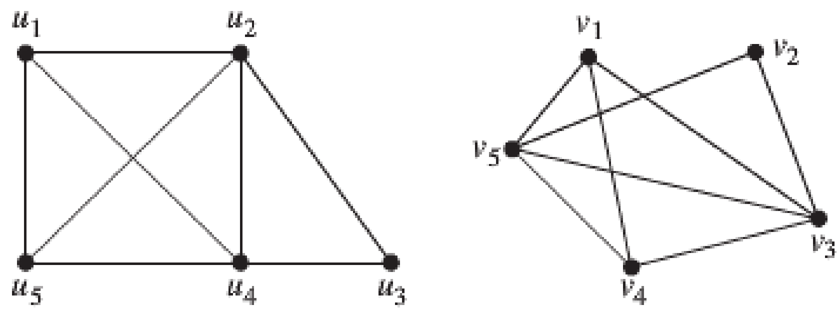




11.



12.



Section 4.8 – Connectivity

Paths

A path is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph.

Definitions

Let G be a graph, and let v and w be vertices in G.

A ***walk*** from v to w is a finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form

$$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n$$

Where the v's represent vertices, the e's represents edges, $v_0 = v,\ v_n = w$ and for all $i = 1, 2, \ldots, n,\ v_{i-1}\ and\ v_i$ are the endpoints of e_i.

The trivial walk from v to v consists of the single vertex v.

A ***trail*** from v to w is a walk from v to w that does not contain a repeated edge.

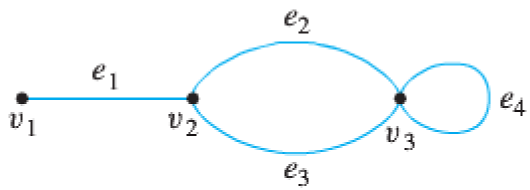
A ***closed walk*** is a walk that starts and ends at the same vertex.

A ***circuit*** is a closed walk that contains at least one edge and does not contain a repeated edge.

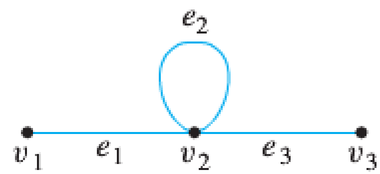
A ***simple circuit*** is a circuit that does not any other repeated vertex except first and last.

	Repeated Edge?	**Repeated Vertex**	**Starts & Ends at Same Point?**	**Must Contain at Least One Edge?**
Walk	Allowed	Allowed	Allowed	No
Trail	No	Allowed	Allowed	No
Path	No	No	No	No
Closed Walk	Allowed	Allowed	Yes	Yes
Circuit	No	Allowed	Yes	Yes
Simple Circuit	No	First & last only	Yes	Yes

Notation for Walks



The notation $e_1e_2e_4e_3$ refers unambiguous to the following walk: $v_1e_1v_2e_2e_4v_3e_3v_2$. On the other hand, the notation e_1 is ambiguous if used to refer to a walk. It could mean either $v_1e_1v_1$ or $v_2e_1v_1$. The notation v_2v_3 is ambiguous if used to refer to a walk. It could mean $v_2e_2v_3$ or $v_2e_3v_3$. On the other hand,



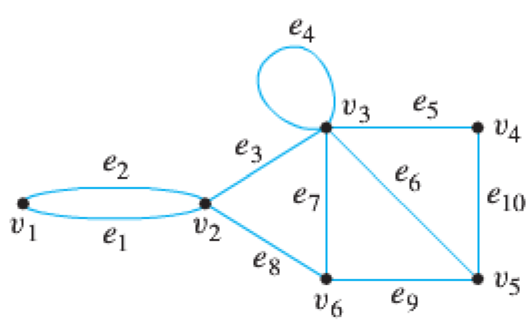
The notation $v_1v_2v_2v_3$ refers unambiguously to the walk $v_1e_1v_2e_2v_2e_3v_3$

Example

Determine which of the following walks are trails, paths, circuits, or simple circuits to the graph below.

a) $v_1e_1v_2e_3v_3e_4v_3e_5v_4$ *b)* $e_1e_3e_5e_5e_6$ *c)* $v_2v_3v_4v_5v_3v_6v_2$

d) $v_2v_3v_4v_5v_6v_2$ *e)* $v_1e_1v_2e_1v_1$ *f)* v_1



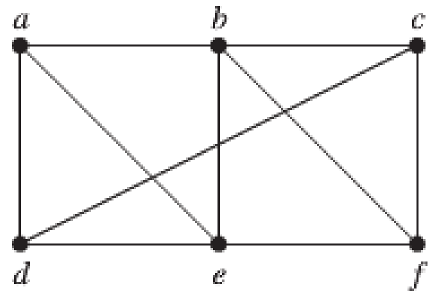
Solution

- ***a)*** This walk has a repeated vertex but does not have a repeated edge, so it is a trail from v_1 *to* v_4 but not a path.
- ***b)*** This is just a walk from v_1 *to* v_5. It is not a trail because it has a repeated edge.
- ***c)*** This walk starts and ends at v_2, contains at least one edge, and does not have a repeated edge, so it is a circuit. Since the vertex v_3 is repeated in the middle, it is not a simple circuit.
- ***d)*** This walk starts and ends at v_2, contains at least one edge, and does not have a repeated edge, and does not have a repeated vertex. Thus it is a simple circuit.

e) This is just a closed walk starting and ending at v_1. It is not a circuit because edge e_1 is repeated.

f) The first vertex of this walk is the same as its last vertex, but it does not contain an edge, and so it is not a circuit, It is a closed walk from v_1 *to* v_1. (It is also a trail from v_1 *to* v_1)

Example



The given graph, *a, d, c, f, e* is a simple path of length 4, because {*a, d*}, {*d, c*}, {*c, f*}, and {*f, e*} are all edges.

However, *d, e, c, a* is not a path, because {*e, c*} is not an edge.

Note that *b, c, f, e, b* is a circuit of length 4 because {*b, c*}, {*c, f*}, {*f, e*}, and {*e, b*} are edges, and this path begins and ends at *b*.

The path *a, b, e, d, a, b*, which is of length 5, is not simple because it contains the edge {*a, b*} twice.

Connectedness

Definition

Let *G* be a graph. Two vertices *v* and *w* of *G* are ***connected*** if, and only if, there is a walk from *v* to *w*.

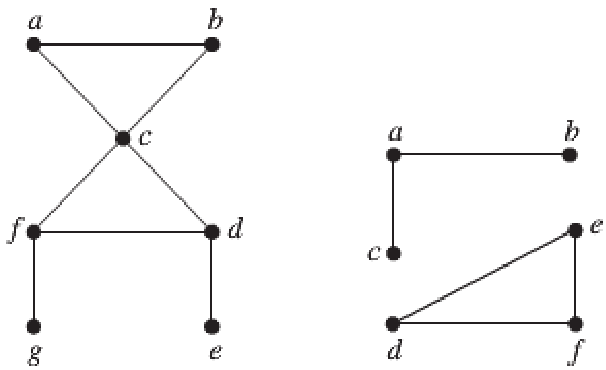
The graph *G* is connected if, and only if, given any two vertices *v* and *w* in *G*, there is a walk from *v* to *w*. Symbolically,

$$G \text{ is connected} \quad \Leftrightarrow \quad \forall \text{ vertices } v, w \in V(G), \exists \text{ a walk from } v \text{ to } w.$$

Definition

An undirected graph is called ***connected*** if there is a path between every pair of distinct vertices of the graph. An undirected graph that is not *connected* is called ***disconnected***. We say that we *disconnect* a graph when we remove vertices or edges, or both, to produce a disconnected subgraph.

Example



G_1 G_2

The graph G_1 is connected, because for every pair of distinct vertices there is a path between them.

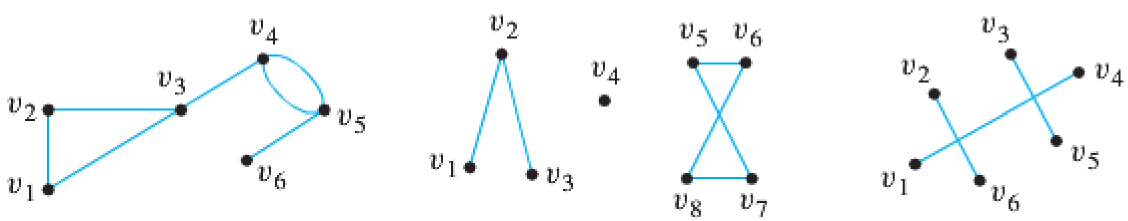
However, the graph G_2 is not connected.

For instance, there is no path in G_2 between vertices *a* and *b*.

Connected and Disconnected Graphs

Example

Which of the following graphs are connected?



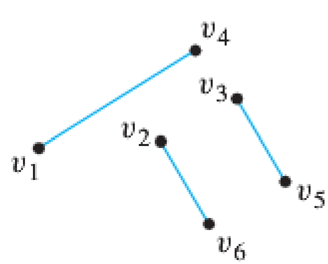
(*a*) **(*b*)** **(*c*)**

Solution

The graph represented in (*a*) is connected, whereas those of (*b*) and (*c*) are not.

To understand why (*c*) is not connected, two edges may cross at a point that is not a vertex.

Thus the graph in (*c*) can be drawn as follows:



Theorem

There is a simple path between every pair distinct vertices of a connected undirected graph.

Proof

Let u and v be two distinct vertices of the connected undirected graph $G=(V,\ E)$. Because G is connected, there is at least one path between u and v. Let $x_0, x_1, \ldots, x_n$ where $x_0 = u$ and $x_n = v$, be the vertex sequence of a path of least length.

This path of least length is simple.

To see this, suppose it is not simple. Then $x_i = x_j$ for some i and j with $0 \le i < j$.

This means that there is a path from u to v of shorter length with vertex sequence $x_0, x_1, \ldots, x_{i-1}\ x_j, \ldots, x_n$ obtained by deleting the edges corresponding to the vertex sequence $x_i, \ldots, x_{j-1}$

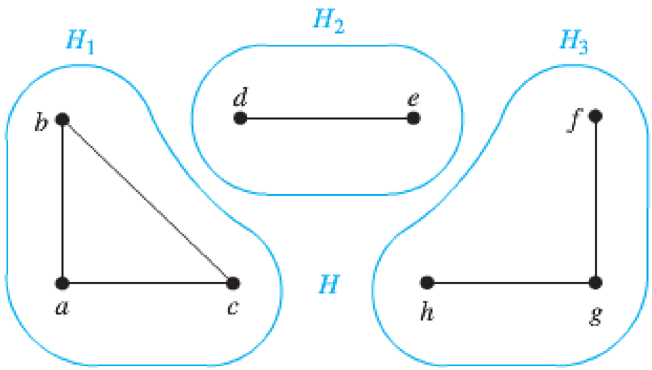
Definition

A graph H is a connected component of a graph G if, and only if,

- H is subgraph of G;
- H is connected; and
- No connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H.

Example

What are the connected components of the graph H shown below?



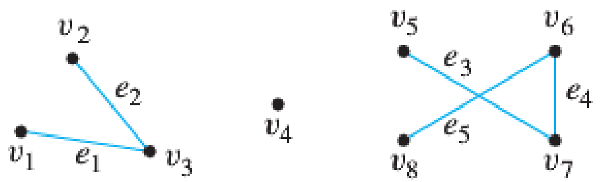
Solution

The graph H is the union of the three disjoint connected subgraphs $H_1, H_2, \textit{and } H_3$.

These three subgraphs are the connected components of H.

Example

Find all connected components of the following graph *G*.



Solution

G has three connected components: $H_1, H_2, and\ H_3$ with vertex sets $V_1, V_2, and\ V_3$ and edges $E_1, E_2, and\ E_3$, where

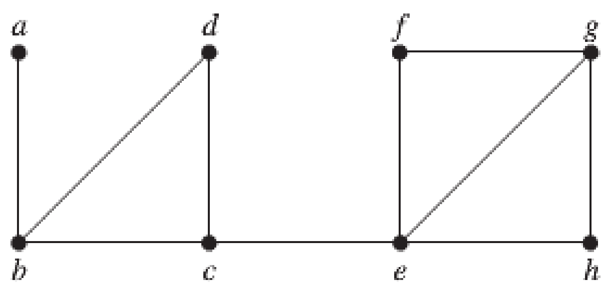
$$V_1 = \{v_1, v_2, v_3\} \qquad E_1 = \{e_1, e_2\}$$

$$V_2 = \{v_4\} \qquad E_2 = \varnothing$$

$$V_3 = \{v_5, v_6, v_7, v_8\} \qquad E_3 = \{e_3, e_4, e_5\}$$

Example

Find the cut vertices and cut edges in the graph G.



G

Solution

The cut vertices of *G* are *b, c*, and *e*.

The removal of one of these vertices (and its adjacent edges) disconnects the graph. The cut edges are $\{a, b\}$ and $\{c, e\}$.

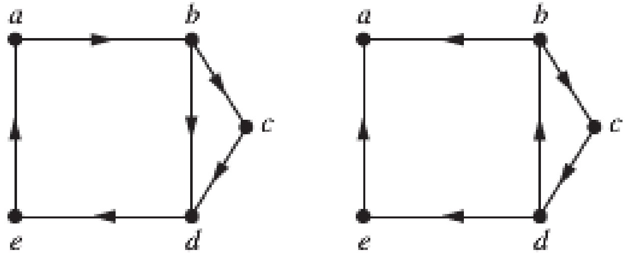
Removing either one of these edges disconnects *G*.

Definition

A directed graph is weakly connected of there is a path between every two vertices in the underlying undirected graph.

Example

Are the directed graphs G and H shown below strongly connected? Are they weakly connected?



G H

Solution

G is strongly connected because there is a path between any two vertices in this directed graph. Hence, G is also weakly connected.

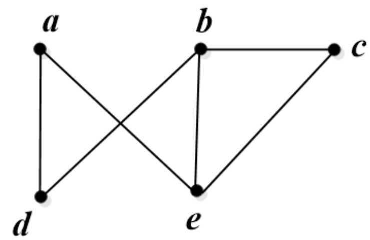
The graph H is not strongly connected. There is no directed path from a to b in this graph. However, H is weakly connected, because there is a path between any 2 vertices in the underlying undirected graph of H.

Exercises *Section* 4.8 – Connectivity

1. Does each of these lists of vertices form a path in the following graph?

Which paths are simple? Which are circuits?

Which are the lengths of those that are paths?



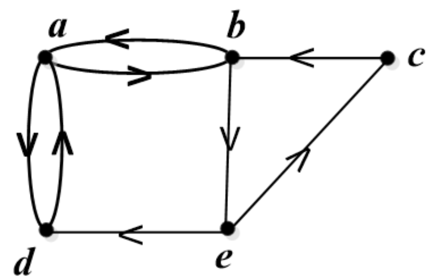
a) *a, e, b, c, b* ***b)*** *a, e, a, d, b, c, a* ***c)*** *e, b, a, d, b, e* ***d)*** *c, b , d, a, e, c*

2. Does each of these lists of vertices form a path in the following graph?

Which paths are simple?

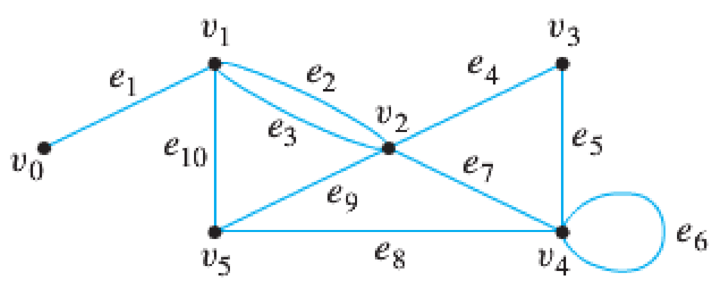
Which are circuits?

Which are the lengths of those that are paths?



a) *a, b, e, c, b* ***b)*** *a, d, a, d, a* ***c)*** *a, d, b, e, a* ***d)*** *a, b , e, c, b, d, a*

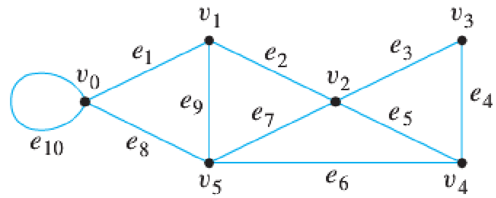
3. Determine whether of the following walks are trails, paths, circuits, or simple circuits or just walk to the graph below.



a) $v_0e_1v_1e_{10}v_5e_9v_2e_2v_1$ *b)* $v_4e_7v_2e_9v_5e_{10}v_1e_3v_2e_9v_5$ *c)* v_2

d) $v_5v_2v_3v_4v_4v_5$ *e)* $v_2v_3v_4v_5v_2v_4v_3v_2$ *f)* $e_5e_8e_{10}e_3$

4. Determine whether of the following walks are trails, paths, circuits, or simple circuits or just walk to the graph below.

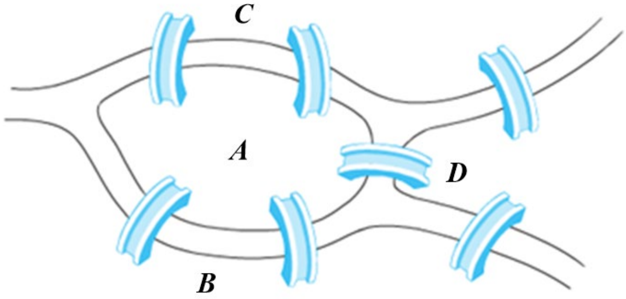


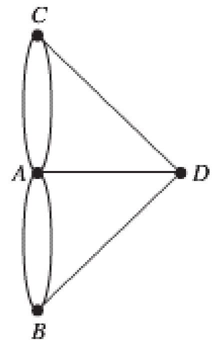
a) $v_1e_2v_2e_3v_3e_4v_4e_5v_2e_2v_1e_1v_0$ *b)* $v_2v_3v_4v_5v_2$ *c)* $v_4v_2v_3v_4v_5v_2v_4$

d) $v_2v_1v_5v_2v_3v_4v_2$ *e)* $v_0v_5v_2v_3v_4v_2v_1$ *f)* $v_5v_4v_2v_1$

Section 4.9 – Euler and Hamilton Paths

The town of *Königsberg* was divided into 4 sections by the branches of the Pregel River. The town people took long walks through town. They wondered whether it was possible to start at some location in the town, travel across all the bridges once without crossing and bridge twice, and return to the starting point.





Königsberg Town *Multigraph*

The Swiss mathematician Leonhard Euler solved this problem.

His solution, published in 1736, may be the first use of graph theory.

Euler studied this problem using the multigraph obtained when the four regions are represented by vertices and the bridges by edges.

Definition

Let G be a graph. An Euler Circuit for G is a circuit that contains every vertex and every edge of G. That is, an Euler circuit for G is a sequence of adjacent vertices and edges in G that has at least one edge, starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.

Theorem

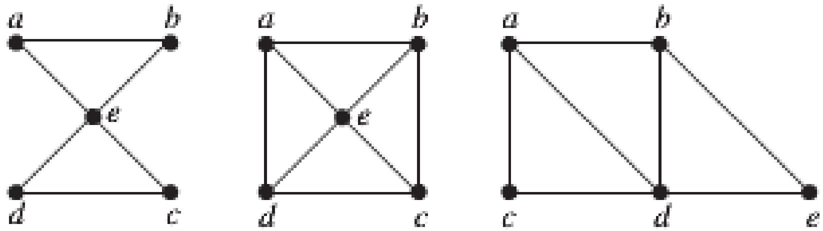
A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

Theorem

A connected multigraph has Euler path but not an Euler circuit if and only if it has exactly 2 vertices of odd degree.

Example

Which of the undirected graph have Euler circuit? Of those that do not, which have an Euler path?



G_1 G_2 G_3

Solution

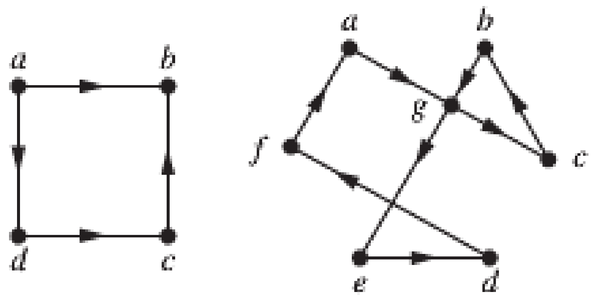
The graph G_1 has Euler circuit, for example, *a, e, c, d, e, b, a*.

G_2 & G_3 do not have Euler circuit. Because vertices *a, b, c, d* of G_2 & *a, b* of G_3 have degree 3

G_3 has an Euler path, *a, c, d, e, b, d, a, b*. G_2 does not have Euler path.

Example

Which of the undirected graph have Euler circuit? Of those that do not, which have an Euler path?



G_1 G_2 G_3

Solution

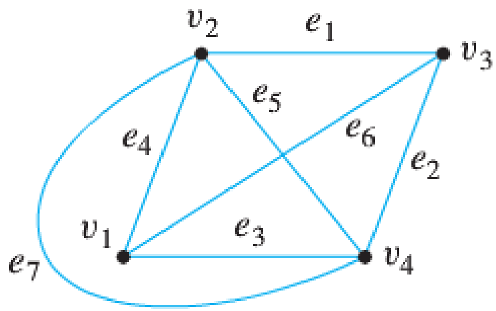
The graph G_2 has Euler circuit, for example, *a, g, c, b, g, e, d, f, a*.

G_1 & G_3 do not have Euler circuit. Because vertices *a, b, c, d* of G_1 have degree 1 (*odd*) & *c, b* of G_3 have degree 3

G_3 has an Euler path, *c, a, b, c, d, b*. but G_1 does not have Euler path.

Example

Show that the graph below does not have an Euler circuit



Solution

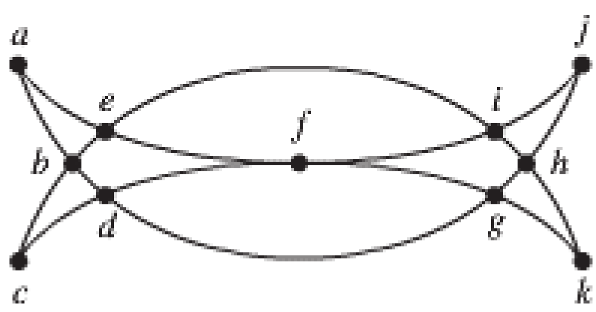
Vertices v_1 & v_3 both have degree 3, which is odd. Hence by the contrapositive form, this graph does not have an Euler circuit.

Definition

Let G be a graph, and let v and w be two distinct vertices of G. An Euler trail from v to w is a sequence of adjacent edges and vertices that starts at v, ends at w, passes through every vertex of G at least once, and traverses every edge of G exactly once.

Example

Many puzzles ask you to draw a picture in a continuous motion without lifting a pencil so that no part of the picture is retraced. We can solve such puzzles using Euler circuits and paths. For example, can Mohammed's scimitars, shown in Figure below, be drawn in this way, where the drawing begins and ends at the same point?



Solution

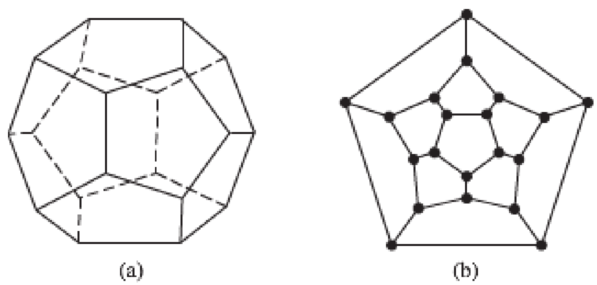
Let denote G for the graph. G has an Euler circuit, because all its vertices have even degree.

1. Form the circuit: *a, b, d, c, b, e, i, f, e, a*. We obtain the subgraph *H* by deleting the edges in this circuit and all vertices that become isolated when these edges are removed.
2. Form: *d, g, h, j, i, h, k, g, f, d* in H. After forming this circuit we have used all edges in G. Splicing this new circuit into the first circuit at the appropriate place produces the Euler circuit *a, b, d, g, h, j, i, h, k, g, f, d, c, b, e, i, f, e, a*. This circuit gives a way to draw the scimitars without lifting the pencil or retracting part of the picture.

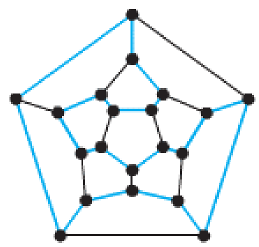
Hamilton Path and Circuits

A simple path in a graph G that passes through every vertex exactly once is called a ***Hamilton path***, and a simple circuit G in a graph G that passes through every vertex exactly once is called a *Hamilton circuit*.

That is, the simple path $x_0, x_1, \ldots, x_{n-1}, x_n$ in the graph $G = (V, E)$ is a Hamilton path if $V = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$, and the simple circuit $x_0, x_1, \ldots, x_{n-1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, \ldots, x_{n-1}, x_n$ is a Hamilton path.



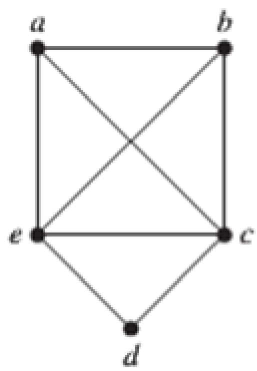
Hamilton's "A Voyage Round the World" Puzzle



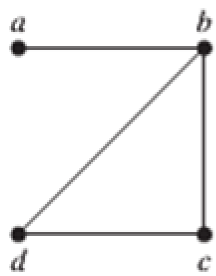
Solution to the "A Voyage Round the World" Puzzle

Example

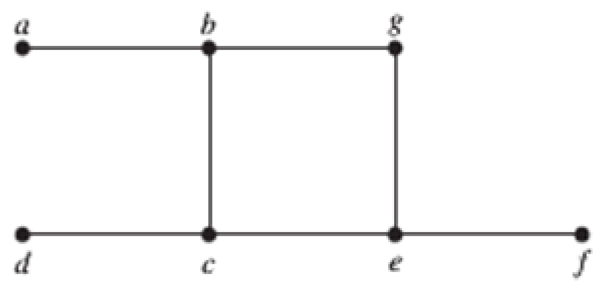
Which of the simple given graphs have a Hamilton circuit or, if not, a Hamilton path?



G_1



G_2



G_3

Solution

G_1 has a Hamilton circuit: *a, b, c, d, e, a*.

There is no Hamilton circuit in G_2, every vertex contain the edge $\{a, b\}$ twice, but G_2 does have a Hamilton path, namely *a, b, c, d*.

G_3 has neither a Hamilton circuit nor a Hamilton path, because any path containing all vertices must contain one of the edges $\{a, b\}$, $\{e, f\}$, and $\{c, d\}$ more than once.

Example

Show that K_n has a Hamilton circuit whenever $n \geq 3$.

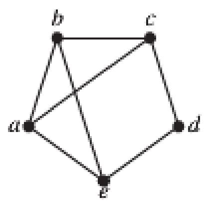
Solution

We can form a Hamilton circuit in K_n beginning at any vertex. Such a circuit can be built by visiting vertices in any order we choose, as long as the path begins and ends at the same vertex and visits each other vertex exactly once. This is possible because there are edges in K_n between any two vertices.

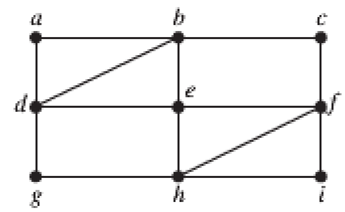
Exercises Section 4.9 – Euler and Hamilton Paths

(1 – 7) Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

1.



2.



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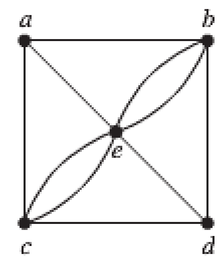

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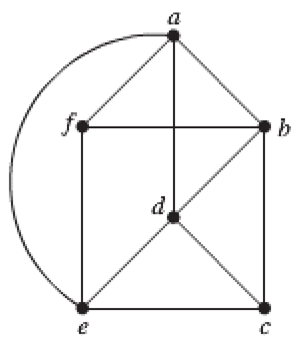

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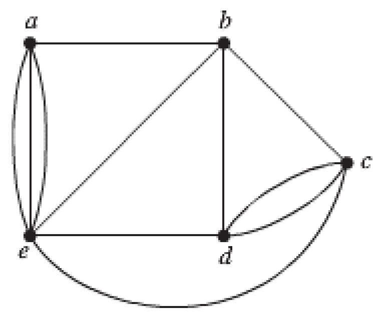
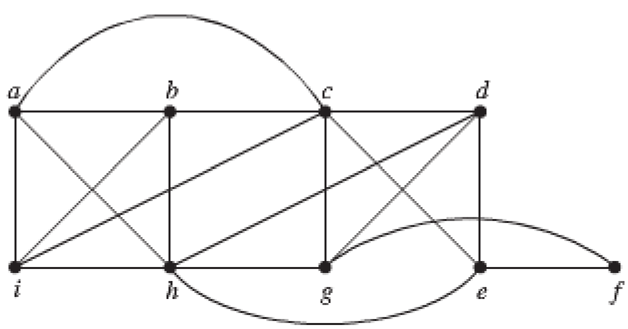
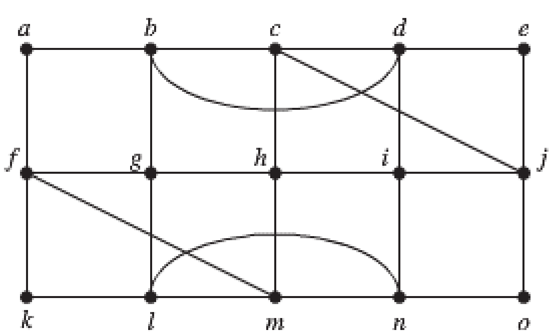


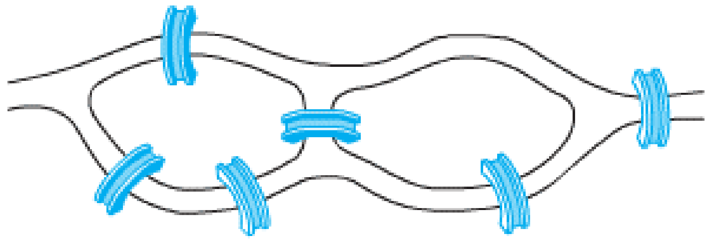
6.



7.

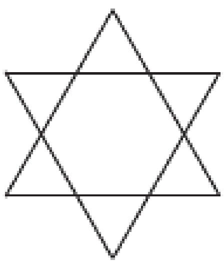


8. Can someone cross all the bridges shown in this map exactly once and return to the starting point?

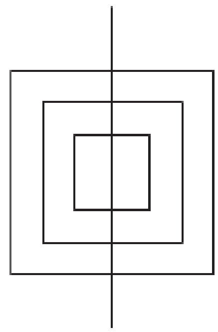


(9 – 11) Determine whether the picture shown can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture

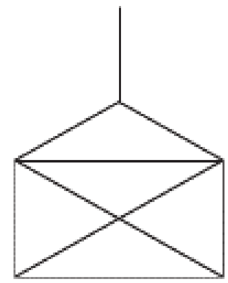
9.



10.

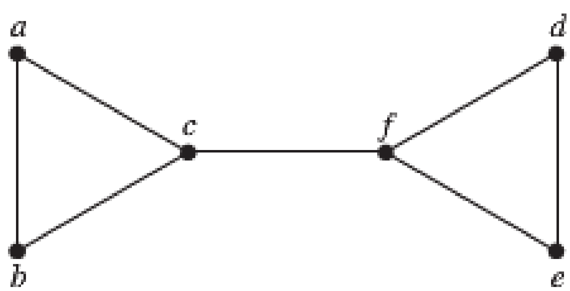


11.

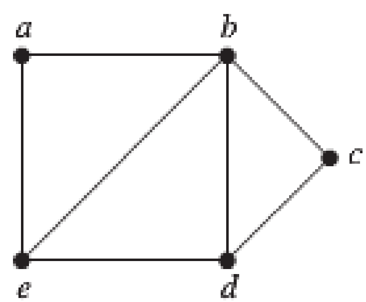


(12 – 17) Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

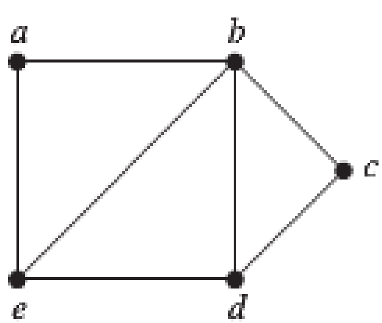
12.



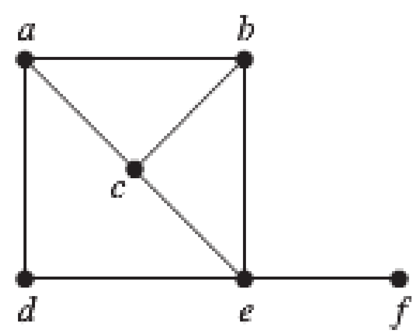
13.



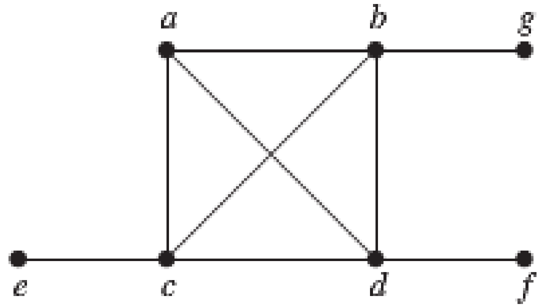
14.



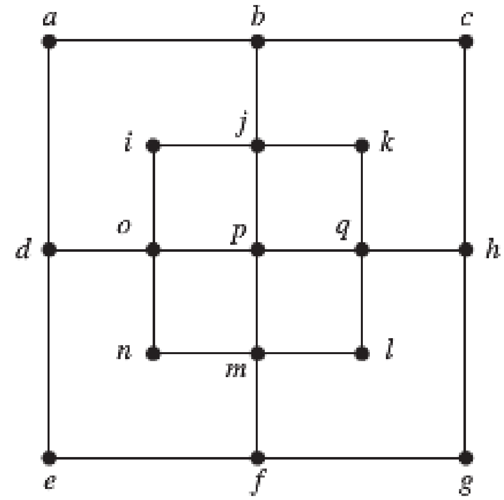
15.



16.



17.



18. Imagine that the drawing below is a map showing 4 cities and the distances in kilometers between them. Suppose that a salesman must travel to each city exactly once, starting and ending in city *A*. Which route from city to city will minimize the total distance that must be traveled?