# Solution    *Section* 2.5 – Numerical Integration

### *Exercise*

Find the Midpoint Rule approximations to: $\int_0^1 \sin \pi x \, dx$ *using* $n = 6$ *subintervals*

**Solution**

$$\Delta x = \frac{1-0}{6} \qquad\qquad \Delta x = \frac{b-a}{n}$$

$$= \frac{1}{6}$$

$$x_k = a + k\Delta x$$

$$x_0 = 0$$

$$x_1 = 0 + \frac{1}{6}$$

$$= \frac{1}{6}$$

$$x_2 = 0 + 2\left(\frac{1}{6}\right)$$

$$= \frac{1}{3}$$

$$x_3 = 0 + 3\left(\frac{1}{6}\right)$$

$$= \frac{1}{2}$$

$$x_4 = 0 + 4\left(\frac{1}{6}\right)$$

$$= \frac{2}{3}$$

$$x_5 = 0 + 5\left(\frac{1}{6}\right)$$

$$= \frac{5}{6}$$

$$x_6 = 0 + 6\left(\frac{1}{6}\right)$$

$$= 1$$

$$m_1 = \frac{1}{2}\left(0 + \frac{1}{6}\right) \qquad\qquad m_k = \frac{x_{k-1} + x_k}{2}$$

$$= \frac{1}{12}$$

$$m_2 = \frac{1}{2}\left(\frac{1}{6} + \frac{1}{3}\right)$$

$$= \boxed{\frac{1}{4}}$$

$$m_3 = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{2}\right)$$

$$= \boxed{\frac{5}{12}}$$

$$m_4 = \frac{1}{2}\left(\frac{1}{2} + \frac{2}{3}\right)$$

$$= \boxed{\frac{7}{12}}$$

$$m_5 = \frac{1}{2}\left(\frac{2}{3} + \frac{5}{6}\right)$$

$$= \boxed{\frac{3}{4}}$$

$$m_6 = \frac{1}{2}\left(\frac{5}{6} + 1\right)$$

$$= \boxed{\frac{11}{12}}$$

$$M(n) = f(m_1)\Delta x + f(m_2)\Delta x + \cdots + f(m_n)\Delta x$$

$$M(6) = \left(\sin\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{5\pi}{12}\right) + \sin\left(\frac{7\pi}{12}\right) + \sin\left(\frac{9\pi}{12}\right) + \sin\left(\frac{11\pi}{12}\right)\right)\left(\frac{1}{6}\right)$$

$$\boxed{\approx 0.6439505509}$$

## *Exercise*

Find the Midpoint Rule approximations to: $\displaystyle\int_0^{\pi} x^2 \sin x \, dx \quad n = 8 \;\; subintervals$

***Solution***

$$\Delta x = \frac{\pi - 0}{8} = \boxed{\frac{\pi}{8}} \qquad \Delta x = \frac{b-a}{n}$$

$$\boxed{x_0 = 0}$$

$$x_1 = 0 + \frac{\pi}{8} = \boxed{\frac{\pi}{8}} \qquad x_k = x_0 + k\Delta x$$

$$x_2 = 0 + 2\left(\frac{\pi}{8}\right) = \boxed{\frac{\pi}{4}}$$

$$x_3 = 0 + 3\left(\frac{\pi}{8}\right) = \boxed{\frac{3\pi}{8}}$$

$$x_4 = 0 + 4\left(\frac{\pi}{8}\right) = \boxed{\frac{\pi}{2}}$$

$$x_5 = 0 + 5\left(\frac{\pi}{8}\right) = \boxed{\frac{5\pi}{8}}$$

$$x_6 = 0 + 6\left(\frac{\pi}{8}\right) = \boxed{\frac{3\pi}{4}}$$

$$x_7 = 0 + 7\left(\frac{\pi}{8}\right) = \boxed{\frac{7\pi}{8}}$$

$$x_8 = 0 + 8\left(\frac{\pi}{8}\right) = \boxed{\pi}$$

$$m_1 = \frac{1}{2}\left(0 + \frac{\pi}{8}\right) = \boxed{\frac{\pi}{16}}$$

$$m_k = \frac{x_{k-1} + x_k}{2}$$

$$m_2 = \frac{1}{2}\left(\frac{\pi}{8} + \frac{\pi}{4}\right) = \boxed{\frac{3\pi}{16}}$$

$$m_3 = \frac{1}{2}\left(\frac{\pi}{4} + \frac{3\pi}{8}\right) = \boxed{\frac{5\pi}{16}}$$

$$m_4 = \frac{1}{2}\left(\frac{3\pi}{8} + \frac{\pi}{2}\right) = \boxed{\frac{7\pi}{16}}$$

$$m_5 = \frac{1}{2}\left(\frac{\pi}{2} + \frac{5\pi}{8}\right) = \boxed{\frac{9\pi}{16}}$$

$$m_6 = \frac{1}{2}\left(\frac{5\pi}{8} + \frac{3\pi}{4}\right) = \boxed{\frac{11\pi}{16}}$$

$$m_7 = \frac{1}{2}\left(\frac{3\pi}{4} + \frac{7\pi}{8}\right) = \boxed{\frac{13\pi}{16}}$$

$$m_8 = \frac{1}{2}\left(\frac{7\pi}{8} + \pi\right) = \boxed{\frac{15\pi}{16}}$$

$$M(n) = f(m_1)\Delta x + f(m_2)\Delta x + \cdots + f(m_n)\Delta x$$

$$M(8) = \left(\begin{array}{l} \left(\frac{\pi}{16}\right)^2 \sin\frac{\pi}{16} + \left(\frac{3\pi}{16}\right)^2 \sin\frac{3\pi}{16} + \left(\frac{5\pi}{16}\right)^2 \sin\frac{5\pi}{16} + \left(\frac{7\pi}{16}\right)^2 \sin\frac{7\pi}{16} + \left(\frac{9\pi}{16}\right)^2 \sin\frac{9\pi}{16} \\ + \left(\frac{11\pi}{16}\right)^2 \sin\frac{11\pi}{16} + \left(\frac{13\pi}{16}\right)^2 \sin\frac{13\pi}{16} + \left(\frac{15\pi}{16}\right)^2 \sin\frac{15\pi}{16} \end{array}\right)\left(\frac{\pi}{8}\right)$$

$$= \left(\begin{array}{l} \sin\frac{\pi}{16} + 9\sin\frac{3\pi}{16} + 25\sin\frac{5\pi}{16} + 49\sin\frac{7\pi}{16} + 81\sin\frac{9\pi}{16} + 121\sin\frac{11\pi}{16} \\ +169\sin\frac{13\pi}{16} + 225\sin\frac{15\pi}{16} \end{array}\right)\frac{\pi^3}{2,048}$$

$$\approx \left(\begin{array}{l} 0.19509 + 5.000132 + 20.7867403 + 67.3478925 + 79.4436077 \\ 100.607823 + 93.89136938 + 43.895322453 \end{array}\right)\frac{\pi^3}{2,048}$$

$$\boxed{\approx 6.22414635}$$

***Exercise***

Find the Midpoint Rule approximations to: $\displaystyle\int_0^1 e^{-\sqrt{x}}dx \quad n=6 \;\; subintervals$

***Solution***

$\Delta x = \frac{1-0}{6} = \boxed{\frac{1}{6}}$ $\qquad \Delta x = \frac{b-a}{n}$

$\boxed{x_0 = 0}$

$x_1 = 0+\frac{1}{6} = \boxed{\frac{1}{6}}$ $\qquad x_k = x_0 + k\Delta x$

$x_2 = 0+2\left(\frac{1}{6}\right) = \boxed{\frac{1}{3}}$

$x_3 = 0+3\left(\frac{1}{6}\right) = \boxed{\frac{1}{2}}$

$x_4 = 0+4\left(\frac{1}{6}\right) = \boxed{\frac{2}{3}}$

$x_5 = 0+5\left(\frac{1}{6}\right) = \boxed{\frac{5}{6}}$

$x_6 = 0+6\left(\frac{1}{6}\right) = \boxed{1}$

$m_1 = \frac{1}{2}\left(0+\frac{1}{6}\right) = \boxed{\frac{1}{12}}$ $\qquad m_k = \frac{x_{k-1}+x_k}{2}$

$m_2 = \frac{1}{2}\left(\frac{1}{6}+\frac{1}{3}\right) = \boxed{\frac{1}{4}}$

$m_3 = \frac{1}{2}\left(\frac{1}{3}+\frac{1}{2}\right) = \boxed{\frac{5}{12}}$

$m_4 = \frac{1}{2}\left(\frac{1}{2}+\frac{2}{3}\right) = \boxed{\frac{7}{12}}$

$m_5 = \frac{1}{2}\left(\frac{2}{3}+\frac{5}{6}\right) = \boxed{\frac{3}{4}}$

$m_6 = \frac{1}{2}\left(\frac{5}{6}+1\right) = \boxed{\frac{11}{12}}$

$M(n) = f\left(m_1\right)\Delta x + f\left(m_2\right)\Delta x + \cdots + f\left(m_n\right)\Delta x$

$M(6) = \left(e^{-\sqrt{1/12}} + e^{-\sqrt{1/4}} + e^{-\sqrt{5/12}} + e^{-\sqrt{7/12}} + e^{-\sqrt{3/4}} + e^{-\sqrt{11/12}}\right)\left(\frac{1}{6}\right)$

$\approx (.74925557 + .6065306597 + .52440173 + .46591 + .42062 + .383879)\left(\frac{1}{6}\right)$

$\boxed{\approx 0.67787732}$

### *Exercise*

Find the Midpoint Rule approximations to: $\int_0^1 e^{-x}dx \quad using \quad n=8 \ \ subintervals$

***Solution***

$$\Delta x = \frac{1-0}{8} = \frac{1}{8}$$

$$x_0 = 0$$

$$x_1 = 0+\frac{1}{8} = \frac{1}{8} \qquad\qquad x_k = x_0 + k\Delta x$$

$$x_2 = 0+2\left(\frac{1}{8}\right) = \frac{1}{4}$$

$$x_3 = 0+3\left(\frac{1}{8}\right) = \frac{3}{8}$$

$$x_4 = 0+4\left(\frac{1}{8}\right) = \frac{1}{2}$$

$$x_5 = 0+5\left(\frac{1}{8}\right) = \frac{5}{8}$$

$$x_6 = 0+6\left(\frac{1}{8}\right) = \frac{3}{4}$$

$$x_7 = 0+7\left(\frac{1}{8}\right) = \frac{7}{8}$$

$$x_8 = 0+8\left(\frac{1}{8}\right) = 1$$

$$m_1 = \frac{1}{2}\left(0+\frac{1}{8}\right) = \frac{1}{16} \qquad\qquad m_k = \frac{x_{k-1}+x_k}{2}$$

$$m_2 = \frac{1}{2}\left(\frac{1}{8}+\frac{1}{4}\right) = \frac{3}{16}$$

$$m_3 = \frac{1}{2}\left(\frac{1}{4}+\frac{3}{8}\right) = \frac{5}{16}$$

$$m_4 = \frac{1}{2}\left(\frac{3}{8}+\frac{1}{2}\right) = \frac{7}{16}$$

$$m_5 = \frac{1}{2}\left(\frac{1}{2}+\frac{5}{8}\right) = \frac{9}{16}$$

$$m_6 = \frac{1}{2}\left(\frac{5}{8}+\frac{3}{4}\right) = \frac{11}{16}$$

$$m_7 = \frac{1}{2}\left(\frac{3}{4}+\frac{7}{8}\right) = \frac{13}{16}$$

$$m_8 = \frac{1}{2}\left(\frac{7}{8}+1\right) = \frac{15}{16}$$

$$M(8) = \frac{1}{8}\left(e^{-1/16}+e^{-3/16}+e^{-5/16}+e^{-7/16}+e^{-9/16}+e^{-11/16}+e^{-13/16}+e^{-15/16}\right)$$

$$\approx 0.6317092095$$

## *Exercise*

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of $10^{-4}$ by (***a***) the Trapezoid Rule and (***b***) Simpson's Rule. $\int_1^3 (2x-1)\,dx$

***Solution***

***a)*** ***i***) $\Delta x = \frac{3-1}{4}$ $\qquad \Delta x = \frac{b-a}{n}$

$= \frac{1}{2}$

$T = \frac{1}{2}\Delta x\left(m\, f\left(x_i\right)\right)$

$= \frac{1}{2}\frac{1}{2}(24)$

$= 6$

| | $x_i$ | $f\left(x_i\right) = 2x_i - 1$ | $m$ | $mf\left(x_i\right)$ |
|---|---|---|---|---|
| $x_0$ | 1 | 1 | 1 | 1 |
| $x_1$ | $1+\frac{1}{2}=\frac{3}{2}$ | 2 | 2 | 4 |
| $x_2$ | 2 | 3 | 2 | 6 |
| $x_3$ | $\frac{5}{2}$ | 4 | 2 | 8 |
| $x_4$ | 3 | 5 | 1 | 5 |
| | | | | **24** |

$f(x) = 2x - 1 \quad \Rightarrow \quad f'(x) = 2$

$\Rightarrow \quad f''(x) = 0 = M$

$\Rightarrow Error = 0$

***ii***) $\int_1^3 (2x-1)\,dx = \left(x^2 - x\right|_1^3$

$= \left(3^2 - 3\right) - \left(1^2 - 1\right)$

$= 6$

***iii***) $Error = \frac{\left|E_T\right|}{True\ \ Value} \times 100$

$= 0\%$

***b)*** ***i***) $\Delta x = \frac{3-1}{4}$ $\qquad \Delta x = \frac{b-a}{n}$

$= \frac{1}{2}$

$S = \frac{1}{3}\Delta x\left(\sum m\, f\left(x_i\right)\right)$

$= \frac{1}{3}\frac{1}{2}(36)$

$= 6$

| | $x_i$ | $f\left(x_i\right) = 2x_i - 1$ | $m$ | $mf\left(x_i\right)$ |
|---|---|---|---|---|
| $x_0$ | 1 | 1 | 1 | 1 |
| $x_1$ | $\frac{3}{2}$ | 2 | 4 | 8 |
| $x_2$ | 2 | 3 | 2 | 6 |
| $x_3$ | $\frac{5}{2}$ | 4 | 4 | 16 |
| $x_4$ | 3 | 5 | 1 | 5 |
| | | | | **36** |

$f(x) = 2x - 1$

$\Rightarrow \quad f^{(4)}(x) = 0 = M$

$$\Rightarrow \left|E_s\right| = 0$$

$$ii)\quad \int_1^3 (2x-1)\,dx = 6$$

$$\left|E_s\right| = \int_1^3 (2x-1)\,dx - S$$

$$= 6 - 6$$

$$\underline{= 0}$$

$$iii)\quad Error = \frac{\left|E_T\right|}{True\ \ Value} \times 100$$

$$\underline{= 0\%}$$

## Exercise

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of $10^{-4}$ by (***a***) the Trapezoid Rule and (***b***) Simpson's Rule. $\int_{-1}^{1}\left(x^2+1\right)dx$

***Solution***

***a)*** ***i***) $\Delta x = \frac{b-a}{n}$

$$= \frac{1+1}{4}$$

$$= \frac{1}{2}$$

$$T = \frac{1}{2}\Delta x\left(m\, f\left(x_i\right)\right)$$

$$= \frac{1}{2}\frac{1}{2}(11)$$

$$= \frac{11}{4}$$

| | $x_i$ | $f\left(x_i\right)$ | $m$ | $m\,f\left(x_i\right)$ |
|---|---|---|---|---|
| $x_0$ | $-1$ | 2 | 1 | 2 |
| $x_1$ | $-\frac{1}{2}$ | $\frac{5}{4}$ | 2 | $\frac{5}{2}$ |
| $x_2$ | 0 | 1 | 2 | 2 |
| $x_3$ | $\frac{1}{2}$ | $\frac{5}{4}$ | 2 | $\frac{5}{2}$ |
| $x_4$ | 1 | 2 | 1 | 2 |
| | | | | **11** |

$$f(x) = x^2 + 1$$

$$f'(x) = 2x$$

$$f''(x) = 2 = M$$

$$\left|E_T\right| = \frac{1-(-1)}{12}\left(\frac{1}{2}\right)^2(2)$$

$$= \frac{1}{12}$$

$$= 0.0833\ldots$$

$$\boldsymbol{ii)}\ \int_{-1}^{1}\left(x^2+1\right)dx=\left(\tfrac{1}{3}x^3+x\right|_{-1}^{1}$$
$$=\left(\tfrac{1}{3}+1\right)-\left(-\tfrac{1}{3}-1\right)$$
$$\underline{=\tfrac{8}{3}}\Big|$$

$$E_T=\int_{-1}^{1}\left(x^2+1\right)dx-T$$
$$=\frac{8}{3}-\frac{11}{4}$$
$$\underline{=-\frac{1}{12}}\Big|$$

$$\boldsymbol{iii)}\ Error=\frac{|E_T|}{True\ \ Value}\times 100$$
$$=\frac{\frac{1}{12}}{\frac{8}{3}}$$
$$\underline{=\frac{100}{32}}\Big|\qquad \underline{\approx 3\%}\Big|$$

$$\boldsymbol{b)}\quad \boldsymbol{i)}\ \Delta x=\frac{b-a}{n}$$
$$=\frac{-1-(-1)}{4}$$
$$\underline{=\frac{1}{2}}\Big|$$

| | $x_i$ | $f\left(x_i\right)$ | $m$ | $m\,f\left(x_i\right)$ |
|---|---|---|---|---|
| $x_0$ | $-1$ | 2 | 1 | 2 |
| $x_1$ | $-\frac{1}{2}$ | $\frac{5}{4}$ | 4 | 5 |
| $x_2$ | 0 | 1 | 2 | 2 |
| $x_3$ | $\frac{1}{2}$ | $\frac{5}{4}$ | 4 | 5 |
| $x_4$ | 1 | 2 | 1 | 2 |
| | | | | **16** |

$$S=\frac{1}{3}\Delta x\left(\sum m\,f\left(x_i\right)\right)$$
$$=\frac{1}{3}\frac{1}{2}(16)$$
$$\underline{=\frac{8}{3}}\Big|$$

$$f(x)=x^2+1$$
$$f^{(4)}(x)=0=M$$
$$\Rightarrow\left|E_s\right|=0$$

$$\boldsymbol{ii)}\ \int_{-1}^{1}\left(x^2+1\right)dx\underline{=\frac{8}{3}}\Big|$$

$$E_S = \int_{-1}^{1}\left(x^2+1\right)dx - S$$

$$= \frac{8}{3} - \frac{8}{3}$$

$$\underline{= 0}$$

***iii***) $Error = \dfrac{|E_T|}{True\ \ Value}\times 100$

$$\underline{= 0\%}$$

## *Exercise*

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of $10^{-4}$ by (***a***) the Trapezoid Rule and (***b***) Simpson's Rule. $\displaystyle\int_2^4 \frac{1}{(s-1)^2}ds$

***Solution***

***a)*** $\Delta x = \dfrac{4-2}{4}$ $\qquad\qquad\qquad \Delta x = \dfrac{b-a}{n}$

$$= \frac{1}{2}$$

$x_0 = 2$

$x_1 = 2 + \frac{1}{2} = \frac{5}{2}$

$x_2 = 2 + 2\left(\frac{1}{2}\right) = 3$

$x_3 = 2 + 3\left(\frac{1}{2}\right) = \frac{7}{2}$

$x_4 = 4$

$$T = \frac{1}{2}\Delta x\left(m\, f\left(x_i\right)\right)$$

$$= \frac{1}{2}\frac{1}{2}\left(\frac{1}{(2-1)^2} + 2\frac{1}{\left(\frac{5}{2}-1\right)^2} + 2\frac{1}{(3-1)^2} + 2\frac{1}{\left(\frac{7}{2}-1\right)^2} + \frac{1}{(4-1)^2}\right)$$

$$= \frac{1}{4}\left(\frac{1}{1} + \frac{8}{9} + \frac{1}{2} + \frac{8}{25} + \frac{1}{9}\right)$$

$$= \frac{1269}{1800}$$

$$\underline{\approx 0.705}$$

$$f(s)=\frac{1}{(s-1)^2}$$

$$f'(s)=-\frac{2}{(s-1)^3}$$

$$f''(s)=\frac{6}{(s-1)^4} \quad \Rightarrow \quad \boxed{M=6}$$

$$\int_2^4 \frac{1}{(s-1)^2}\,ds=-\frac{1}{s-1}\bigg|_2^4$$

$$=-\left(\frac{1}{3}-1\right)$$

$$\boxed{=\frac{2}{3}}$$

The percentage error: $\approx \dfrac{|0.705-.6667|}{.6667}$

$$\boxed{\approx 0.0575} \quad \boxed{5.75\%}$$

***b)*** $\Delta x=\dfrac{4-2}{4}$ $\qquad\qquad \Delta x=\dfrac{b-a}{n}$

$$\boxed{=\frac{1}{2}}$$

$x_0=2$

$x_1=2+\frac{1}{2}=\frac{5}{2}$

$x_2=2+2\left(\frac{1}{2}\right)=3$

$x_3=2+3\left(\frac{1}{2}\right)=\frac{7}{2}$

$x_4=4$

$$S=\frac{1}{3}\Delta x\left(m\,f\left(x_i\right)\right)$$

$$=\frac{1}{3}\frac{1}{2}\left(\frac{1}{(2-1)^2}+4\frac{1}{\left(\frac{5}{2}-1\right)^2}+2\frac{1}{(3-1)^2}+4\frac{1}{\left(\frac{7}{2}-1\right)^2}+\frac{1}{(4-1)^2}\right)$$

$$=\frac{1}{6}\left(\frac{1}{1}+\frac{16}{9}+\frac{1}{2}+\frac{16}{25}+\frac{1}{9}\right)$$

$$\boxed{=\frac{1813}{450}}$$

$$\boxed{\approx 0.67148}$$

$$\int_2^4 \frac{1}{(s-1)^2}\,ds = \frac{2}{3}$$

The percentage error: $= \dfrac{|0.67148 - .6667|}{.6667}$

$\approx 0.0072$   $0.72\%$

## Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^1 \sin \pi x \, dx \quad n = 6 \;\; subintervals$

**Solution**

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f\left(x_n\right)$ | $m \cdot f\left(x_n\right)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.1666666667 | 0.5000000000 | 1.0000000000 |
| 2 | 0.3333333333 | 0.8660254000 | 1.7320508000 |
| 3 | 0.5000000000 | 1.0000000000 | 2.0000000000 |
| 4 | 0.6666666667 | 0.8660254000 | 1.7320508000 |
| 5 | 0.8333333333 | 0.5000000000 | 1.0000000000 |
| 6 | 1.0000000000 | 0.0000000000 | 0.0000000000 |

*Trapezoid Rule* approximation $\approx 0.62200847$

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f\left(x_n\right)$ | $m \cdot f\left(x_n\right)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.1666666667 | 0.5000000000 | 2.0000000000 |
| 2 | 0.3333333333 | 0.8660254000 | 1.7320508000 |
| 3 | 0.5000000000 | 1.0000000000 | 2.0000000000 |
| 4 | 0.6666666667 | 0.8660254000 | 1.7320508000 |
| 5 | 0.8333333333 | 0.5000000000 | 1.0000000000 |
| 6 | 1.0000000000 | 0.0000000000 | 0.0000000000 |

***Simpson's Rule*** approximation $\approx 0.63689453$

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 0.63661977 | 0.62200847 | 0.63689453 |
| Error: | | 2.2951 % | 0.0432 % |

## *Exercise*

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations to and error to $\int_0^1 e^{-x}dx \quad n=8 \;\; subintervals$

***Solution***

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 1.0000000000 | 1.0000000000 |
| 1 | 0.1250000000 | 0.8824969000 | 1.7649938000 |
| 2 | 0.2500000000 | 0.7788007800 | 1.5576015600 |
| 3 | 0.3750000000 | 0.6872892800 | 1.3745785600 |
| 4 | 0.5000000000 | 0.6065306600 | 1.2130613200 |
| 5 | 0.6250000000 | 0.5352614300 | 1.0705228600 |
| 6 | 0.7500000000 | 0.4723665500 | 0.9447331000 |
| 7 | 0.8750000000 | 0.4168620200 | 0.8337240400 |
| 8 | 1.0000000000 | 0.3678794400 | 0.3678794400 |

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*Trapezoid Rule* approximation ≈ 0.63294342

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 1.0000000000 | 1.0000000000 |
| 1 | 0.1250000000 | 0.8824969000 | 3.5299876000 |
| 2 | 0.2500000000 | 0.7788007800 | 1.5576015600 |
| 3 | 0.3750000000 | 0.6872892800 | 2.7491571200 |
| 4 | 0.5000000000 | 0.6065306600 | 1.2130613200 |
| 5 | 0.6250000000 | 0.5352614300 | 2.1410457200 |
| 6 | 0.7500000000 | 0.4723665500 | 0.9447331000 |
| 7 | 0.8750000000 | 0.4168620200 | 1.6674480800 |
| 8 | 1.0000000000 | 0.3678794400 | 0.3678794400 |

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***Simpson's Rule*** approximation ≈ 0.63212141

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 0.63212056 | 0.63294342 | 0.63212141 |
| Error: | | 0.1302 % | 0.0001 % |

## *Exercise*

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error to:

$$\int_1^5 \left(3x^2 - 2x\right) dx \quad n = 8 \;\; subintervals$$

***Solution***

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f\left(x_n\right)$ | $m \cdot f\left(x_n\right)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 1.0000000000 | 1.0000000000 |
| 1 | 1.5000000000 | 3.7500000000 | 7.5000000000 |
| 2 | 2.0000000000 | 8.0000000000 | 16.0000000000 |
| 3 | 2.5000000000 | 13.7500000000 | 27.5000000000 |
| 4 | 3.0000000000 | 21.0000000000 | 42.0000000000 |
| 5 | 3.5000000000 | 29.7500000000 | 59.5000000000 |
| 6 | 4.0000000000 | 40.0000000000 | 80.0000000000 |
| 7 | 4.5000000000 | 51.7500000000 | 103.500000000 |
| 8 | 5.0000000000 | 65.0000000000 | 65.0000000000 |

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*Trapezoid Rule* approximation ≈ 100.50000000

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f\left(x_n\right)$ | $m \cdot f\left(x_n\right)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 1.0000000000 | 1.0000000000 |
| 1 | 1.5000000000 | 3.7500000000 | 15.0000000000 |
| 2 | 2.0000000000 | 8.0000000000 | 16.0000000000 |
| 3 | 2.5000000000 | 13.7500000000 | 55.0000000000 |
| 4 | 3.0000000000 | 21.0000000000 | 42.0000000000 |
| 5 | 3.5000000000 | 29.7500000000 | 119.000000000 |
| 6 | 4.0000000000 | 40.0000000000 | 80.0000000000 |
| 7 | 4.5000000000 | 51.7500000000 | 207.000000000 |
| 8 | 5.0000000000 | 65.0000000000 | 65.0000000000 |

-------------------------------------------------------------------

***Simpson's Rule*** approximation ≈ 100.00000000

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 100.000000 | 100.500000 | 100.00000000 |
| Error: | | 0.5000% | 0.0000 % |

## *Exercise*

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^{\pi/4} 3\sin 2x\, dx \quad n = 8 \;\; subintervals$

***Solution***

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.0981747704 | 0.5852709700 | 1.1705419400 |
| 2 | 0.1963495408 | 1.1480503000 | 2.2961006000 |
| 3 | 0.2945243113 | 1.6667107000 | 3.3334214000 |
| 4 | 0.3926990817 | 2.1213203400 | 4.2426406800 |
| 5 | 0.4908738521 | 2.4944088400 | 4.9888176800 |
| 6 | 0.5890486225 | 2.7716386000 | 5.5432772000 |
| 7 | 0.6872233930 | 2.9423558400 | 5.8847116800 |
| 8 | 0.7853981634 | 3.0000000000 | 3.0000000000 |

*Trapezoid Rule* approximation ≈ 1.49517776

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.0981747704 | 0.5852709700 | 2.3410838800 |
| 2 | 0.1963495408 | 1.1480503000 | 2.2961006000 |
| 3 | 0.2945243113 | 1.6667107000 | 6.6668428000 |
| 4 | 0.3926990817 | 2.1213203400 | 4.2426406800 |
| 5 | 0.4908738521 | 2.4944088400 | 9.9776353600 |
| 6 | 0.5890486225 | 2.7716386000 | 5.5432772000 |
| 7 | 0.6872233930 | 2.9423558400 | 11.7694233600 |
| 8 | 0.7853981634 | 3.0000000000 | 3.0000000000 |

***Simpson's Rule*** approximation ≈ 1.50001244

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 1.500000 | 1.49517776 | 1.50001244 |
| Error: | | 0.3215 % | 0.0008 % |

## *Exercise*

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^8 e^{-2x}\,dx \quad n = 8 \;\; subintervals$

## Solution

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 1.0000000000 | 1.0000000000 |
| 1 | 1.0000000000 | 0.1353352800 | 0.2706705600 |
| 2 | 2.0000000000 | 0.0183156400 | 0.0366312800 |
| 3 | 3.0000000000 | 0.0024787500 | 0.0049575000 |
| 4 | 4.0000000000 | 0.0003354600 | 0.0006709200 |
| 5 | 5.0000000000 | 0.0000454000 | 0.0000908000 |
| 6 | 6.0000000000 | 0.0000061400 | 0.0000122800 |
| 7 | 7.0000000000 | 0.0000008300 | 0.0000016600 |
| 8 | 8.0000000000 | 0.0000001100 | 0.0000001100 |

----------------------------------------------------------------

*Trapezoid Rule* approximation ≈ 0.65651755

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 1.0000000000 | 1.0000000000 |
| 1 | 1.0000000000 | 0.1353352800 | 0.5413411200 |
| 2 | 2.0000000000 | 0.0183156400 | 0.0366312800 |
| 3 | 3.0000000000 | 0.0024787500 | 0.0099150000 |
| 4 | 4.0000000000 | 0.0003354600 | 0.0006709200 |
| 5 | 5.0000000000 | 0.0000454000 | 0.0001816000 |
| 6 | 6.0000000000 | 0.0000061400 | 0.0000122800 |
| 7 | 7.0000000000 | 0.0000008300 | 0.0000033200 |
| 8 | 8.0000000000 | 0.0000001100 | 0.0000001100 |

----------------------------------------------------------------

***Simpson's Rule*** approximation ≈ 0.52958521

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 0.49999994 | 0.65651755 | 0.52958521 |
| Error: | | 31.3035 % | 5.9171 % |

## *Exercise*

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_{-1}^{1} \sqrt{x^2+1}\, dx \quad n=8 \;\; subintervals$

## Solution

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f\left(x_n\right)$ | $m \cdot f\left(x_n\right)$ |
|---|---|---|---|
| 0 | –1.0000000000 | 1.4142135600 | 1.4142135600 |
| 1 | –0.7500000000 | 1.2500000000 | 2.5000000000 |
| 2 | –0.5000000000 | 1.1180339900 | 2.2360679800 |
| 3 | –0.2500000000 | 1.0307764100 | 2.0615528200 |
| 4 | 0.0000000000 | 1.0000000000 | 2.0000000000 |
| 5 | 0.2500000000 | 1.0307764100 | 2.0615528200 |
| 6 | 0.5000000000 | 1.1180339900 | 2.2360679800 |
| 7 | 0.7500000000 | 1.2500000000 | 2.5000000000 |
| 8 | 1.0000000000 | 1.4142135600 | 1.4142135600 |

*Trapezoid Rule* approximation ≈ 2.30295859

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f\left(x_n\right)$ | $m \cdot f\left(x_n\right)$ |
|---|---|---|---|
| 0 | –1.0000000000 | 1.4142135600 | 1.4142135600 |
| 1 | –0.7500000000 | 1.2500000000 | 5.0000000000 |
| 2 | –0.5000000000 | 1.1180339900 | 2.2360679800 |
| 3 | –0.2500000000 | 1.0307764100 | 4.1231056400 |
| 4 | 0.0000000000 | 1.0000000000 | 2.0000000000 |
| 5 | 0.2500000000 | 1.0307764100 | 4.1231056400 |
| 6 | 0.5000000000 | 1.1180339900 | 2.2360679800 |
| 7 | 0.7500000000 | 1.2500000000 | 5.0000000000 |
| 8 | 1.0000000000 | 1.4142135600 | 1.4142135600 |

*Simpson's Rule* approximation ≈ 2.29556453

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 2.29558715 | 2.30295859 | 2.29556453 |
| Error: | | 0.3211 % | 0.0010 % |

## *Exercise*

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^{1/2} \sin\left(x^2\right)\, dx \quad n = 4 \;\; subintervals$

***Solution***

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f\left(x_n\right)$ | $m \cdot f\left(x_n\right)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.1250000000 | 0.0156243642 | 0.0312487284 |
| 2 | 0.2500000000 | 0.0624593178 | 0.1249186357 |
| 3 | 0.3750000000 | 0.1401619723 | 0.2803239447 |
| 4 | 0.5000000000 | 0.2474039593 | 0.2474039593 |

*Trapezoid Rule* approximation ≈ 0.0427434543

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f\left(x_n\right)$ | $m \cdot f\left(x_n\right)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.1250000000 | 0.0156243642 | 0.0624974569 |
| 2 | 0.2500000000 | 0.0624593178 | 0.1249186357 |
| 3 | 0.3750000000 | 0.1401619723 | 0.5606478894 |
| 4 | 0.5000000000 | 0.2474039593 | 0.2474039593 |

*Simpson Rule* approximation ≈ 0.0414778309

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 0.0414810243 | 0.0427434543 | 0.0414778309 |
| Error: | | 3.04339% | 0.00770 % |

## *Exercise*

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\displaystyle\int_{\pi/2}^{\pi} \frac{\sin x}{x}\,dx \quad n=6 \;\; subintervals$

**Solution**

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 1.5708000000 | 0.6366182800 | 0.6366182800 |
| 1 | 1.8323333333 | 0.5271932200 | 1.0543864400 |
| 2 | 2.0938666667 | 0.4137271600 | 0.8274543200 |
| 3 | 2.3554000000 | 0.3004450800 | 0.6008901600 |
| 4 | 2.6169333333 | 0.1914141900 | 0.3828283800 |
| 5 | 2.8784666667 | 0.0903606800 | 0.1807213600 |
| 6 | 3.1400000000 | 0.0005072100 | 0.0005072100 |

*Trapezoid Rule* approximation ≈ 0.48166674

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 1.5708000000 | 0.6366182800 | 0.6366182800 |
| 1 | 1.8323333333 | 0.5271932200 | 2.1087728800 |
| 2 | 2.0938666667 | 0.4137271600 | 0.8274543200 |
| 3 | 2.3554000000 | 0.3004450800 | 1.2017803200 |
| 4 | 2.6169333333 | 0.1914141900 | 0.3828283800 |
| 5 | 2.8784666667 | 0.0903606800 | 0.3614427200 |
| 6 | 3.1400000000 | 0.0005072100 | 0.0005072100 |

*Simpson's Rule* approximation ≈ 0.48116938

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 0.48117214 | 0.48166674 | 0.48116938 |
| Error: | | 0.1028 % | 0.0006 % |

## *Exercise*

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^{\pi/4} x \tan x \, dx \quad n = 6 \;\; subintervals$

***Solution***

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.1308996939 | 0.0172332716 | 0.0344665433 |
| 2 | 0.2617993878 | 0.0701489345 | 0.1402978691 |
| 3 | 0.3926990817 | 0.1626612856 | 0.3253225711 |
| 4 | 0.5235987756 | 0.3022998940 | 0.6045997881 |
| 5 | 0.6544984695 | 0.5022143392 | 1.0044286785 |
| 6 | 0.7853981634 | 0.7853981634 | 0.7853981634 |

*Trapezoid Rule* approximation ≈ 0.1894454730

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.1308996939 | 0.0172332716 | 0.0689330865 |
| 2 | 0.2617993878 | 0.0701489345 | 0.1402978691 |
| 3 | 0.3926990817 | 0.1626612856 | 0.6506451423 |
| 4 | 0.5235987756 | 0.3022998940 | 0.6045997881 |
| 5 | 0.6544984695 | 0.5022143392 | 2.0088573569 |
| 6 | 0.7853981634 | 0.7853981634 | 0.7853981634 |

*Simpson Rule* approximation ≈ 0.1858222125

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 0.1857845357 | 0.1894454730 | 0.1858222125 |
| Error: | | 1.97053% | 0.02028 % |

## Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^1 e^{-x^2}\,dx \quad n=10 \ \ subintervals$

**Solution**

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 1.0000000000 | 1.0000000000 |
| 1 | 0.1000000000 | 0.9900498337 | 1.9800996675 |
| 2 | 0.2000000000 | 0.9607894392 | 1.9215788783 |
| 3 | 0.3000000000 | 0.9139311853 | 1.8278623705 |
| 4 | 0.4000000000 | 0.8521437890 | 1.7042875779 |
| 5 | 0.5000000000 | 0.7788007831 | 1.5576015661 |
| 6 | 0.6000000000 | 0.6976763261 | 1.3953526521 |
| 7 | 0.7000000000 | 0.6126263942 | 1.2252527884 |
| 8 | 0.8000000000 | 0.5272924240 | 1.0545848481 |
| 9 | 0.9000000000 | 0.4448580662 | 0.8897161324 |
| 10 | 1.0000000000 | 0.3678794412 | 0.3678794412 |

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*Trapezoid Rule* approximation ≈ 0.7462107961

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 1.0000000000 | 1.0000000000 |
| 1 | 0.1000000000 | 0.9900498337 | 3.9601993350 |
| 2 | 0.2000000000 | 0.9607894392 | 1.9215788783 |
| 3 | 0.3000000000 | 0.9139311853 | 3.6557247411 |
| 4 | 0.4000000000 | 0.8521437890 | 1.7042875779 |
| 5 | 0.5000000000 | 0.7788007831 | 3.1152031323 |
| 6 | 0.6000000000 | 0.6976763261 | 1.3953526521 |
| 7 | 0.7000000000 | 0.6126263942 | 2.4505055767 |
| 8 | 0.8000000000 | 0.5272924240 | 1.0545848481 |
| 9 | 0.9000000000 | 0.4448580662 | 1.7794322649 |
| 10 | 1.0000000000 | 0.3678794412 | 0.3678794412 |

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*Simpson Rule* approximation ≈ 0.7468249483

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 0.7468241328 | 0.7462107961 | 0.7468249483 |
| Error: | | 0.08213% | 0.00011 % |

### *Exercise*

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^2 \frac{1}{\sqrt{1+x^2}}\,dx \quad n=10 \;\; subintervals$

**Solution**

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m\cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 1.0000000000 | 1.0000000000 |
| 1 | 0.2000000000 | 0.9805806800 | 1.9611613600 |
| 2 | 0.4000000000 | 0.9284766900 | 1.8569533800 |
| 3 | 0.6000000000 | 0.8574929300 | 1.7149858600 |
| 4 | 0.8000000000 | 0.7808688100 | 1.5617376200 |
| 5 | 1.0000000000 | 0.7071067800 | 1.4142135600 |
| 6 | 1.2000000000 | 0.6401844000 | 1.2803688000 |
| 7 | 1.4000000000 | 0.5812381900 | 1.1624763800 |
| 8 | 1.6000000000 | 0.5299989400 | 1.0599978800 |
| 9 | 1.8000000000 | 0.4856429300 | 0.9712858600 |
| 10 | 2.0000000000 | 0.4472136000 | 0.4472136000 |

*Trapezoid Rule* approximation ≈ 1.44303943

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m\cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 1.0000000000 | 1.0000000000 |
| 1 | 0.2000000000 | 0.9805806800 | 3.9223227200 |
| 2 | 0.4000000000 | 0.9284766900 | 1.8569533800 |
| 3 | 0.6000000000 | 0.8574929300 | 3.4299717200 |
| 4 | 0.8000000000 | 0.7808688100 | 1.5617376200 |
| 5 | 1.0000000000 | 0.7071067800 | 2.8284271200 |
| 6 | 1.2000000000 | 0.6401844000 | 1.2803688000 |
| 7 | 1.4000000000 | 0.5812381900 | 2.3249527600 |
| 8 | 1.6000000000 | 0.5299989400 | 1.0599978800 |
| 9 | 1.8000000000 | 0.4856429300 | 1.9425717200 |
| 10 | 2.0000000000 | 0.4472136000 | 0.4472136000 |

*Simpson's Rule* approximation ≈ 1.44363449

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 1.44363548 | 1.44303943 | 1.44363449 |
| Error: | | 0.0413 % | 0.0001 % |

## *Exercise*

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^{1/2} \sin\left(e^{x/2}\right)\, dx \quad n = 8 \ \ subintervals$

***Solution***

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f\left(x_n\right)$ | $m \cdot f\left(x_n\right)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 0.8414709848 | 0.8414709848 |
| 1 | 0.0625000000 | 0.8581952249 | 1.7163904498 |
| 2 | 0.1250000000 | 0.8745438796 | 1.7490877592 |
| 3 | 0.1875000000 | 0.8904281963 | 1.7808563927 |
| 4 | 0.2500000000 | 0.9057510229 | 1.8115020459 |
| 5 | 0.3125000000 | 0.9204063003 | 1.8408126006 |
| 6 | 0.3750000000 | 0.9342785616 | 1.8685571232 |
| 7 | 0.4375000000 | 0.9472424468 | 1.8944848937 |
| 8 | 0.5000000000 | 0.9591622435 | 0.9591622435 |

*Trapezoid Rule* approximation ≈ 0.4519476404

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f\left(x_n\right)$ | $m \cdot f\left(x_n\right)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 0.8414709848 | 0.8414709848 |
| 1 | 0.0625000000 | 0.8581952249 | 3.4327808996 |
| 2 | 0.1250000000 | 0.8745438796 | 1.7490877592 |
| 3 | 0.1875000000 | 0.8904281963 | 3.5617127853 |
| 4 | 0.2500000000 | 0.9057510229 | 1.8115020459 |
| 5 | 0.3125000000 | 0.9204063003 | 3.6816252012 |
| 6 | 0.3750000000 | 0.9342785616 | 1.8685571232 |
| 7 | 0.4375000000 | 0.9472424468 | 3.7889697874 |
| 8 | 0.5000000000 | 0.9591622435 | 0.9591622435 |

*Simpson Rule* approximation ≈ 0.4519764340

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 0.4519764600 | 0.4519476404 | 0.4519764340 |
| Error: | | 0.00638% | 0.00001% |

## *Exercise*

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_2^3 \frac{1}{\ln x}\,dx \quad n=10 \;\; subintervals$

***Solution***

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 2.0000000000 | 1.4426950400 | 1.4426950400 |
| 1 | 2.1000000000 | 1.3478227100 | 2.6956454200 |
| 2 | 2.2000000000 | 1.2682994000 | 2.5365988000 |
| 3 | 2.3000000000 | 1.2006111700 | 2.4012223400 |
| 4 | 2.4000000000 | 1.1422452400 | 2.2844904800 |
| 5 | 2.5000000000 | 1.0913566700 | 2.1827133400 |
| 6 | 2.6000000000 | 1.0465599400 | 2.0931198800 |
| 7 | 2.7000000000 | 1.0067940700 | 2.0135881400 |
| 8 | 2.8000000000 | 0.9712326500 | 1.9424653000 |
| 9 | 2.9000000000 | 0.9392222400 | 1.8784444800 |
| 10 | 3.0000000000 | 0.9102392300 | 0.9102392300 |

*Trapezoid Rule* approximation ≈ 1.11906112

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 2.0000000000 | 1.4426950400 | 1.4426950400 |
| 1 | 2.1000000000 | 1.3478227100 | 5.3912908400 |
| 2 | 2.2000000000 | 1.2682994000 | 2.5365988000 |
| 3 | 2.3000000000 | 1.2006111700 | 4.8024446800 |
| 4 | 2.4000000000 | 1.1422452400 | 2.2844904800 |
| 5 | 2.5000000000 | 1.0913566700 | 4.3654266800 |
| 6 | 2.6000000000 | 1.0465599400 | 2.0931198800 |
| 7 | 2.7000000000 | 1.0067940700 | 4.0271762800 |
| 8 | 2.8000000000 | 0.9712326500 | 1.9424653000 |
| 9 | 2.9000000000 | 0.9392222400 | 3.7568889600 |
| 10 | 3.0000000000 | 0.9102392300 | 0.9102392300 |

*Simpson's Rule* approximation ≈ 1.11842787

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 1.11842481 | 1.11906112 | 1.11842787 |
| Error: | | 0.0569 % | 0.0003 % |

### Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_1^2 e^{1/x}\, dx \quad n = 4\ \ subintervals$

**Solution**

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 1.0000000000 | 2.7182818300 | 2.7182818300 |
| 1 | 1.2500000000 | 2.2255409300 | 4.4510818600 |
| 2 | 1.5000000000 | 1.9477340400 | 3.8954680800 |
| 3 | 1.7500000000 | 1.7707949500 | 3.5415899000 |
| 4 | 2.0000000000 | 1.6487212700 | 1.6487212700 |

*Trapezoid Rule* approximation ≈ 2.03189287

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 1.0000000000 | 2.7182818300 | 2.7182818300 |
| 1 | 1.2500000000 | 2.2255409300 | 8.9021637200 |
| 2 | 1.5000000000 | 1.9477340400 | 3.8954680800 |
| 3 | 1.7500000000 | 1.7707949500 | 7.0831798000 |
| 4 | 2.0000000000 | 1.6487212700 | 1.6487212700 |

*Simpson's Rule* approximation ≈ 2.02065122

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 2.02005862 | 2.03189287 | 2.02065122 |
| Error: | | 0.5858 % | 0.0293 % |

### *Exercise*

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^1 \ln\left(1+e^x\right)\,dx \quad n=8$ *subintervals*

***Solution***

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f\left(x_n\right)$ | $m\cdot f\left(x_n\right)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 0.6931471800 | 0.6931471800 |
| 1 | 0.1250000000 | 0.7575990400 | 1.5151980800 |
| 2 | 0.2500000000 | 0.8259394200 | 1.6518788400 |
| 3 | 0.3750000000 | 0.8981232600 | 1.7962465200 |
| 4 | 0.5000000000 | 0.9740769800 | 1.9481539600 |
| 5 | 0.6250000000 | 1.0537006800 | 2.1074013600 |
| 6 | 0.7500000000 | 1.1368710100 | 2.2737420200 |
| 7 | 0.8750000000 | 1.2234445800 | 2.4468891600 |
| 8 | 1.0000000000 | 1.3132616900 | 1.3132616900 |

*Trapezoid Rule* approximation ≈ 0.98411993

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f\left(x_n\right)$ | $m\cdot f\left(x_n\right)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 0.6931471800 | 0.6931471800 |
| 1 | 0.1250000000 | 0.7575990400 | 3.0303961600 |
| 2 | 0.2500000000 | 0.8259394200 | 1.6518788400 |
| 3 | 0.3750000000 | 0.8981232600 | 3.5924930400 |
| 4 | 0.5000000000 | 0.9740769800 | 1.9481539600 |
| 5 | 0.6250000000 | 1.0537006800 | 4.2148027200 |
| 6 | 0.7500000000 | 1.1368710100 | 2.2737420200 |
| 7 | 0.8750000000 | 1.2234445800 | 4.8937783200 |
| 8 | 1.0000000000 | 1.3132616900 | 1.3132616900 |

*Simpson's Rule* approximation ≈ 0.98381891

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 0.98381904 | 0.98411993 | 0.98381891 |
| Error: | | 0.0306 % | 0.0000 % |

## Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^1 x^5 e^x \, dx \quad n = 10 \;\; subintervals$

**Solution**

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.1000000000 | 0.0000110500 | 0.0000221000 |
| 2 | 0.2000000000 | 0.0003908500 | 0.0007817000 |
| 3 | 0.3000000000 | 0.0032801600 | 0.0065603200 |
| 4 | 0.4000000000 | 0.0152762800 | 0.0305525600 |
| 5 | 0.5000000000 | 0.0515225400 | 0.1030450800 |
| 6 | 0.6000000000 | 0.1416879600 | 0.2833759200 |
| 7 | 0.7000000000 | 0.3384514200 | 0.6769028400 |
| 8 | 0.8000000000 | 0.7292652500 | 1.4585305000 |
| 9 | 0.9000000000 | 1.4523710400 | 2.9047420800 |
| 10 | 1.0000000000 | 2.7182818300 | 2.7182818300 |

*Trapezoid Rule* approximation ≈ 0.40913975

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.1000000000 | 0.0000110500 | 0.0000442000 |
| 2 | 0.2000000000 | 0.0003908500 | 0.0007817000 |
| 3 | 0.3000000000 | 0.0032801600 | 0.0131206400 |
| 4 | 0.4000000000 | 0.0152762800 | 0.0305525600 |
| 5 | 0.5000000000 | 0.0515225400 | 0.2060901600 |
| 6 | 0.6000000000 | 0.1416879600 | 0.2833759200 |
| 7 | 0.7000000000 | 0.3384514200 | 1.3538056800 |
| 8 | 0.8000000000 | 0.7292652500 | 1.4585305000 |
| 9 | 0.9000000000 | 1.4523710400 | 5.8094841600 |
| 10 | 1.0000000000 | 2.7182818300 | 2.7182818300 |

*Simpson's Rule* approximation ≈ 0.39580225

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 0.39559955 | 0.40913975 | 0.39580225 |
| Error: | | 3.4227 % | 0.0512 % |

## *Exercise*

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^4 \sqrt{x}\ \sin x\, dx \quad n = 8\ \ subintervals$

***Solution***

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.5000000000 | 0.3390050500 | 0.6780101000 |
| 2 | 1.0000000000 | 0.8414709800 | 1.6829419600 |
| 3 | 1.5000000000 | 1.2216768700 | 2.4433537400 |
| 4 | 2.0000000000 | 1.2859407500 | 2.5718815000 |
| 5 | 2.5000000000 | 0.9462675500 | 1.8925351000 |
| 6 | 3.0000000000 | 0.2444270200 | 0.4888540400 |
| 7 | 3.5000000000 | –0.6562553300 | –1.3125106600 |
| 8 | 4.0000000000 | –1.5136049900 | –1.5136049900 |

*Trapezoid Rule* approximation ≈ 1.73286520

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 1 | 0.5000000000 | 0.3390050500 | 1.3560202000 |
| 2 | 1.0000000000 | 0.8414709800 | 1.6829419600 |
| 3 | 1.5000000000 | 1.2216768700 | 4.8867074800 |
| 4 | 2.0000000000 | 1.2859407500 | 2.5718815000 |
| 5 | 2.5000000000 | 0.9462675500 | 3.7850702000 |
| 6 | 3.0000000000 | 0.2444270200 | 0.4888540400 |
| 7 | 3.5000000000 | –0.6562553300 | –2.6250213200 |
| 8 | 4.0000000000 | –1.5136049900 | –1.5136049900 |

*Simpson's Rule* approximation ≈ 1.77214151

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 1.76874870 | 1.73286520 | 1.77214151 |
| Error: | | 2.0288 % | 0.1918 % |

## *Exercise*

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^3 \frac{1}{1+x^5}\,dx \quad n=6 \ \ subintervals$

***Solution***

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 1.0000000000 | 1.0000000000 |
| 1 | 0.5000000000 | 0.9696969700 | 1.9393939400 |
| 2 | 1.0000000000 | 0.5000000000 | 1.0000000000 |
| 3 | 1.5000000000 | 0.1163636400 | 0.2327272800 |
| 4 | 2.0000000000 | 0.0303030300 | 0.0606060600 |
| 5 | 2.5000000000 | 0.0101362100 | 0.0202724200 |
| 6 | 3.0000000000 | 0.0040983600 | 0.0040983600 |

*Trapezoid Rule* approximation ≈ 1.06427452

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 1.0000000000 | 1.0000000000 |
| 1 | 0.5000000000 | 0.9696969700 | 3.8787878800 |
| 2 | 1.0000000000 | 0.5000000000 | 1.0000000000 |
| 3 | 1.5000000000 | 0.1163636400 | 0.4654545600 |
| 4 | 2.0000000000 | 0.0303030300 | 0.0606060600 |
| 5 | 2.5000000000 | 0.0101362100 | 0.0405448400 |
| 6 | 3.0000000000 | 0.0040983600 | 0.0040983600 |

*Simpson's Rule* approximation ≈ 1.07491528

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 1.06587854 | 1.06427452 | 1.07491528 |
| Error: | | 0.150488% | .84782% |

## *Exercise*

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_1^4 \frac{e^x}{x}\,dx \quad n=10 \;\; subintervals$

***Solution***

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 1.0000000000 | 2.7182818300 | 2.7182818300 |
| 1 | 1.3000000000 | 2.8225359000 | 5.6450718000 |
| 2 | 1.6000000000 | 3.0956452700 | 6.1912905400 |
| 3 | 1.9000000000 | 3.5188918100 | 7.0377836200 |
| 4 | 2.2000000000 | 4.1022788600 | 8.2045577200 |
| 5 | 2.5000000000 | 4.8729975800 | 9.7459951600 |
| 6 | 2.8000000000 | 5.8730881300 | 11.7461762600 |
| 7 | 3.1000000000 | 7.1606294500 | 14.3212589000 |
| 8 | 3.4000000000 | 8.8129706000 | 17.6259412000 |
| 9 | 3.7000000000 | 10.9317038800 | 21.8634077600 |
| 10 | 4.0000000000 | 13.6495375100 | 13.6495375100 |

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*Trapezoid Rule* approximation ≈ 17.81239534

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 1.0000000000 | 2.7182818300 | 2.7182818300 |
| 1 | 1.3000000000 | 2.8225359000 | 11.2901436000 |
| 2 | 1.6000000000 | 3.0956452700 | 6.1912905400 |
| 3 | 1.9000000000 | 3.5188918100 | 14.0755672400 |
| 4 | 2.2000000000 | 4.1022788600 | 8.2045577200 |
| 5 | 2.5000000000 | 4.8729975800 | 19.4919903200 |
| 6 | 2.8000000000 | 5.8730881300 | 11.7461762600 |
| 7 | 3.1000000000 | 7.1606294500 | 28.6425178000 |
| 8 | 3.4000000000 | 8.8129706000 | 17.6259412000 |
| 9 | 3.7000000000 | 10.9317038800 | 43.7268155200 |
| 10 | 4.0000000000 | 13.6495375100 | 13.6495375100 |

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*Simpson's Rule* approximation ≈ 17.73628195

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 17.73575665 | 17.81239534 | 17.73628195 |
| Error: | | 0.4321 % | 0.0030 % |

## *Exercise*

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_1^2 \frac{dx}{x} \quad n = 10 \ \ subintervals$

## Solution

***Trapezoid Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 1.0000000000 | 1.0000000000 | 1.0000000000 |
| 1 | 1.1000000000 | 0.9090909100 | 1.8181818200 |
| 2 | 1.2000000000 | 0.8333333300 | 1.6666666600 |
| 3 | 1.3000000000 | 0.7692307700 | 1.5384615400 |
| 4 | 1.4000000000 | 0.7142857100 | 1.4285714200 |
| 5 | 1.5000000000 | 0.6666666700 | 1.3333333400 |
| 6 | 1.6000000000 | 0.6250000000 | 1.2500000000 |
| 7 | 1.7000000000 | 0.5882352900 | 1.1764705800 |
| 8 | 1.8000000000 | 0.5555555600 | 1.1111111200 |
| 9 | 1.9000000000 | 0.5263157900 | 1.0526315800 |
| 10 | 2.0000000000 | 0.5000000000 | 0.5000000000 |

Trapezoid Rule approximation ≈ 0.69377140

***Simpson's Rule*** Method

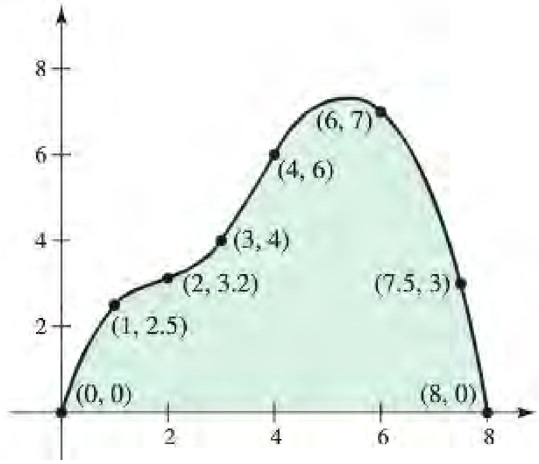
| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 1.0000000000 | 1.0000000000 | 1.0000000000 |
| 1 | 1.1000000000 | 0.9090909100 | 3.6363636400 |
| 2 | 1.2000000000 | 0.8333333300 | 1.6666666600 |
| 3 | 1.3000000000 | 0.7692307700 | 3.0769230800 |
| 4 | 1.4000000000 | 0.7142857100 | 1.4285714200 |
| 5 | 1.5000000000 | 0.6666666700 | 2.6666666800 |
| 6 | 1.6000000000 | 0.6250000000 | 1.2500000000 |
| 7 | 1.7000000000 | 0.5882352900 | 2.3529411600 |
| 8 | 1.8000000000 | 0.5555555600 | 1.1111111200 |
| 9 | 1.9000000000 | 0.5263157900 | 2.1052631600 |
| 10 | 2.0000000000 | 0.5000000000 | 0.5000000000 |

Simpson's Rule approximation ≈ 0.69315023

| | ***Exact*** | ***Trapezoid*** | ***Simpson*** |
|---|---|---|---|
| Value: | 0.69314718 | 0.69377140 | 0.69315023 |
| Error: | | 0.0901% | 0.0004% |

### *Exercise*

A piece of wood paneling must be cut in the shape shown below. The coordinates of several point on its curved surface are also shown (with units of inches).




*a)* Estimate the surface area of the paneling using the Trapezoid Rule

*b)* Estimate the surface area of the paneling using a left Riemann sum.

*c)* Could two identical pieces be cut from a 9-in by 9-in piece of wood?

***Solution***

***a)*** The *trapezoid* Rule gives

$$\frac{(0+.25)\cdot 1}{2}+\frac{(2.5+3.2)\cdot 1}{2}+\frac{(3.2+4)\cdot 1}{2}+\frac{(4+6)\cdot 1}{2}+\frac{(6+7)\cdot 2}{2}+\frac{(7+5.3)\cdot 1.5}{2}+\frac{(3+0)\cdot 0.5}{2}$$

$$\underline{=35.675}$$

***b)*** The left *Riemann* sum gives

$$0\cdot 1+2.5\cdot 1+3.2\cdot 1+4\cdot 1+6\cdot 2+7\cdot 1.5+5.3\cdot 0.5 \underline{=34.85}$$

***c)*** Although the surface area of the piece appears to be less than half of $81=9^2$ (area of $9\times 9$ piece of wood), the shape prohibits the creation of two identical pieces.

### *Exercise*

The region bounded by the curves $y=\frac{1}{1+e^{-x}}$, $x=0$ and $x=10$ is rotated about $x-axis$. Use Simpson's Rule with $n=10$ to estimate the volume of the resulting solid.

***Solution***

Using *Disk* method:

$$V=\pi\int_0^{10}\frac{1}{\left(1+e^{-x}\right)^2}\,dx$$

***Simpson's Rule*** Method

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 0.2500000000 | 0.2500000000 |
| 1 | 1.0000000000 | 0.5344466454 | 2.1377865816 |
| 2 | 2.0000000000 | 0.7758034926 | 1.5516069851 |
| 3 | 3.0000000000 | 0.9073974671 | 3.6295898684 |
| 4 | 4.0000000000 | 0.9643510838 | 1.9287021676 |
| 5 | 5.0000000000 | 0.9866590924 | 3.9466363696 |
| 6 | 6.0000000000 | 0.9950608676 | 1.9901217351 |
| 7 | 7.0000000000 | 0.9981787276 | 3.9927149105 |
| 8 | 8.0000000000 | 0.9993294122 | 1.9986588244 |
| 9 | 9.0000000000 | 0.9997532261 | 3.9990129043 |
| 10 | 10.000000000 | 0.9999092063 | 0.9999092063 |

---------------------------------------------------------------

*Simpson Rule* approximation $\approx 8.8082465177$

$$V = \pi \int_0^{10} \frac{1}{\left(1+e^{-x}\right)^2}\, dx$$

$$\approx \pi\left(8.8082465177\right)$$

$$\approx 27.6719226 \quad unit^3$$

## *Exercise*

A pendulum with length $L$ that makes a maximum angle $\theta_0$ with the vertical. Using Newton's Second Law it can be shown that the period $T$ (the time for one complete swing) is given by

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2 \sin^2 x}}$$

Where $k = \sin\left(\frac{1}{2}\theta_0\right)$ and $g$ is the acceleration due to gravity. If $L = 1\ m$ and $\theta_0 = 42°$, use Simpson's Rule with $n = 10$ to find the period.

***Solution***

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2 \sin^2 x}}$$

$$= 4\sqrt{\frac{1}{9.8}} \int_0^{\pi/2} \frac{dx}{\sqrt{1-\sin^2\left(\frac{1}{2}42°\right)\sin^2 x}}$$

$$= \frac{4}{\sqrt{9.8}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - \sin^2(21°)\sin^2 x}}$$

***Simpson's Rule*** Method $\displaystyle\int_0^{\pi/2} \frac{dx}{\sqrt{1 - \sin^2(21°)\sin^2 x}}$

| $n$ | $x_n$ | $f(x_n)$ | $m \cdot f(x_n)$ |
|---|---|---|---|
| 0 | 0.0000000000 | 1.0000000000 | 1.0000000000 |
| 1 | 0.1570796327 | 1.0031527554 | 4.0126110216 |
| 2 | 0.3141592654 | 1.0124160101 | 2.0248320201 |
| 3 | 0.4712388980 | 1.0271895774 | 4.1087583096 |
| 4 | 0.6283185307 | 1.0464308046 | 2.0928616093 |
| 5 | 0.7853981634 | 1.0686201540 | 4.2744806162 |
| 6 | 0.9424777961 | 1.0917709315 | 2.1835418629 |
| 7 | 1.0995574288 | 1.1135333115 | 4.4541332459 |
| 8 | 1.2566370614 | 1.1314314233 | 2.2628628466 |
| 9 | 1.4137166941 | 1.1432291699 | 4.5729166795 |
| 10 | 1.5707963268 | 1.1473515974 | 1.1473515974 |

*Simpson Rule* approximation $\approx 1.6825506215$

$$T = \frac{4}{\sqrt{9.8}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - \sin^2(21°)\sin^2 x}}$$

$$\approx \frac{4}{\sqrt{9.8}}(1.6825506215)$$

$$\approx 2.149884$$