Section 1.3 – Linear Differential Equations

Basic Assumption

The equation can be solved for y'; that is, the equation can be written in the form $y' = f(x, y)$

A linear differential equation of order ***n*** has the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x)$$

A ***first order*** linear equation is given by the form:

$$y' + p(x)y = f(x)$$

If $f(x) = 0 \rightarrow y' = p(x)y$. This linear equation is said to be ***homogeneous***. (Otherwise it is ***nonhomogeneous or inhomogeneous***).

$p(x)$ & $f(x)$ are called the coefficients and continuous function on some interval *I*.

Linear	***Non-linear***
$x' = \sin(t)x$	$x' = t\sin x$
$y' = e^{2t}y + \cos t$	$y' = 1 - y^2$
$x' = (3t + 2)x + t^2 - 1$	

Solution of the homogenous equation

$$\frac{dx}{dt} = a(t)x \quad \Rightarrow \quad \frac{dx}{x} = a(t)dt$$

$$\int \frac{dx}{x} = \int a(t)dt$$

$$\ln|x| = \int a(t)dt + C$$

Convert to exponential form

$$|x| = e^{\int a(t)dt + C} = e^C e^{\int a(t)dt}$$

Let $A = e^C$

$$\boxed{x(t) = Ae^{\int a(t)dt}}$$

Example

Solve: $x' = \sin(t)\ x$

Solution

$$\frac{dx}{dt} = \sin(t)\ x$$

$$x(t) = A.e^{\int \sin(t)dt}$$

$$\boxed{= A.e^{-\cos t}}$$

$$\frac{dx}{x} = \sin(t)\ dt$$

$$\int \frac{dx}{x} = \int \sin(t)\ dt$$

$$\ln|x| = \int \sin(t)dt + C$$

$$\ln|x| = -\cos(t) + C$$

$$\boxed{x = e^{-\cos(t)+C}}$$

Solving a linear first-order Equation (*Properties*)

1. Put a linear equation into a standard form $y' + p(x)y = f(x)$
2. Identify $p(x)$ then find $y_h = e^{-\int pdx}$
3. Multiply the standard form by y_h
4. Integrate both sides

Solution of the Inhomogeneous Equation

$$x' = p(t)x + f(t)$$

$$x' - px = f$$

$$u(t) = e^{-\int p(t)dt}$$

$$(ux)' = u\left(x' - px\right) = uf$$

$$u(t)x(t) = \int u(t)f(t)dt + C$$

1st Method

Example

Find the general solution to: $x' = x + e^{-t}$

Solution

$$x' - x = e^{-t} \qquad\qquad x' - p(t)x = f(t)$$

$$e^{-\int 1 dt} = e^{-t} \qquad\qquad e^{\int p(t)dt}$$

$$e^{-t}(x' - x) = e^{-t}e^{-t} \qquad\qquad e^{\int p(t)dt}x' - e^{\int p(t)dt}p(t)x = e^{\int p(t)dt}f(t)$$

$$\left(e^{-t}x\right)' = e^{-2t} \qquad\qquad \left(e^{\int p(t)dt}x\right)' = f(t)\, e^{\int p(t)dt}$$

$$e^{-t}x(t) = \int e^{-2t}dt \qquad\qquad e^{\int p(t)dt}x = \int f(t)\, e^{\int p(t)dt}$$

$$e^{-t}x(t) = -\frac{1}{2}e^{-2t} + C$$

$$\boxed{x(t) = -\frac{1}{2}e^{-t} + Ce^{t}}$$

Solution of the Nonhomogeneous Equation $y' + p(x)y = f(x)$

Let assume: $y = y_h + y_p$ *where* $\begin{cases} y_h & \textit{Homogeneous Solution} \\ y_p & \textit{Paticular Solution} \end{cases}$

The homogeneous equation is given by $y'_h + p(x)y_h = 0$

$$y'_h = -p(x)y_h$$

$$\boxed{y_h = e^{-\int pdx}}$$

$$y_p = u(x)y_h = u.e^{-\int pdx}$$

$$y'_p + p(x)y_p = f(x)$$

$$\left(uy_h\right)' + puy_h = f$$

$$u'y_h + uy'_h + puy_h = f$$

$$u'y_h + u\left(y'_h + py_h\right) = f \qquad \textit{Since } y'_h + py_h = 0 \qquad \textit{homogeneous}$$

$$u'y_h = f$$

$$\frac{du}{dx} = \frac{f}{y_h}$$

$$du = \frac{f}{e^{-\int pdx}}dx$$

$$= f.e^{\int pdx}dx$$

$$u = \int f.e^{\int pdx}dx$$

$$y_p = u.e^{-\int pdx}$$

$$u = \left(\int f.e^{\int pdx}dx\right)e^{-\int pdx}$$

$$\boxed{y_p = e^{-\int pdx}\int f.e^{\int pdx}dx}$$

$$\boxed{y = Ce^{-\int pdx} + e^{-\int pdx}\int f.e^{\int pdx}dx} \qquad y = y_h + y_p \qquad \boxed{y = e^{-\int pdx}\left(C + \int f.e^{\int pdx}dx\right)}$$

Example

Find the general solution of $x' = x\sin t + 2te^{-\cos t}$ and the particular solution that satisfies $x(0) = 1$.

Solution

$$x' - x\sin t = 2te^{-\cos t} \qquad P(t) = \sin t, \quad Q(t) = 2te^{-\cos t}$$

$$x_h = e^{-\int \sin t\,dt} = e^{\cos t}$$

$$\int Q(t)x_h\,dt = \int 2te^{-\cos t}e^{\cos t}dt = \int 2t\,dt = t^2$$

$$x(t) = e^{-\cos t}\left(t^2 + C\right) \qquad x = \frac{1}{e^{\int P dt}}\left(\int Q.e^{\int P dt}dt + C\right)$$

$$x(0) = \left((0)^2 + C\right)e^{-\cos 0} = 1$$

$$Ce^{-1} = 1$$

$$C = e$$

$$\boxed{x(t) = \left(t^2 + e\right)e^{-\cos t}}$$

Example

Find the general solution of $x' = x\tan t + \sin t$ and the particular solution that satisfies $x(0) = 2$..

Solution

$$x' - (\tan t)x = \sin t \qquad P(t) = -\tan t, \quad Q(t) = \sin t$$

$$e^{-\int \tan t\,dt} = e^{\ln(\cos t)} \underline{= \cos t}$$

$$\int (\sin t)(\cos t)\,dt = -\int \cos t\; d(\cos t) = -\frac{1}{2}\cos^2 t$$

$$x(t) = \frac{1}{\cos t}\left(-\frac{1}{2}\cos^2 t + C\right) = -\frac{1}{2}\cos t + \frac{1}{\cos t}C$$

$$\boxed{= -\frac{1}{2}\cos t + \frac{1}{\cos t}C}$$

$$x(0) = -\frac{1}{2}\cos(0) + \frac{C}{\cos(0)} = 2$$

$$-\frac{1}{2} + C = 2 \quad \Rightarrow \quad C = \frac{5}{2}$$

$$\boxed{x(t) = -\frac{1}{2}\cos t + \frac{5}{2\cos t}}$$

Linear Differential Operators

$L[y] = y' + p(x)y$ is a linear operator.

- $L[f+g] = L[f] + L[g]$

  Proof

$$\begin{aligned} L[f]+L[g] &= f' + p(x)f + g' + p(x)g \\ &= (f'+g') + p(x)(f+g) \\ &= (f+g)' + p(x)(f+g) \\ &= L[f+g] \end{aligned}$$

- $L[cf] = cL[f]$

  Proof

$$\begin{aligned} L[cf] &= (cf)' + p(x)(cf) \\ &= cf' + cp(x)f \\ &= c\left(f' + p(x)f\right) \\ &= cL[f] \end{aligned}$$

Any operation L that has the two properties

$$\begin{cases} L\left[y_1 + y_2\right] = L\left[y_1\right] + L\left[y_2\right] \\ L[cy] = cL[y], & c \text{ is constant} \end{cases}$$

is a ***linear operation***.

Differential is a linear operation; ***integration*** is a linear operation.

Notes

1. Integrating an expression that is not the differential of any elementary function is called non-elementary.

$$\int e^{x^2} dx \qquad \int x \tan x dx \qquad \int \frac{e^{-x}}{x} dx$$

$$\int \sin x^2 dx \qquad \int \cos x^2 dx \qquad \int \frac{\sin x}{x} dx \qquad \int \frac{\cos x}{x} dx$$

2. In math some important functions are defined in terms of non-elementary integrals. Two such functions are the error function and the complementary error function.

$$erf\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \qquad erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

Exercises *Section* 1.3 – Linear Differential Equations

Find the general solution of the first-order, linear equation.

1. $y' - y = 3e^t$
2. $y' + y = \sin t$
3. $y' + y = \dfrac{1}{1+e^t}$
4. $y' - y = e^{2t} - 1$
5. $y' + y = te^{-t} + 1$
6. $y' + y = 1 + e^{-x}\cos 2x$
7. $y' + y\cot x = \cos x$
8. $y' + y\sin t = \sin t$
9. $y' = \cos x - y\sec x$
10. $y' + (\tan x)y = \cos^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$
11. $y' + (\cot t)y = 2t\csc t$
12. $y' + (1+\sin t)y = 0$
13. $y' + \left(\frac{1}{2}\cos x\right)y = -\frac{3}{2}\cos x$
14. $\dfrac{dy}{dx} + y = e^{3x}$
15. $y' - ty = t$
16. $y' = 2y + x^2 + 5$
17. $xy' + 2y = 3$
18. $\dfrac{dy}{dt} - 2y = 4 - t$
19. $y' + 2y = 1$
20. $y' + 2y = e^{-t}$
21. $y' + 2y = e^{-2t}$
22. $y' - 2y = e^{3t}$
23. $y' + 2y = e^{-x} + x + 1$
24. $y' + 2xy = x$
25. $y' - 2ty = t$
26. $y' + 2ty = 5t$
27. $y' - 2xy = e^{x^2}$
28. $y' + 2xy = x^3$
29. $y' - 2y = t^2e^{2t}$
30. $x' - 2\dfrac{x}{t+1} = (t+1)^2$
31. $y' + \dfrac{2}{t}y = \dfrac{\cos t}{t^2}$
32. $y' - 2(\cos 2t)y = 0$
33. $y' + 2y = \cos 3t$
34. $y' - 3y = 5$
35. $y' + 3y = 2xe^{-3x}$
36. $y' + 3t^2y = t^2$
37. $y' + 3x^2y = x^2$
38. $y' + \dfrac{3}{t}y = \dfrac{\sin t}{t^3}, \quad (t \neq 0)$
39. $y' + \dfrac{3}{x}y = 1 + \dfrac{1}{x}$
40. $y' + \dfrac{3}{2}y = \dfrac{1}{2}e^x$
41. $y' + 5y = t + 1$
42. $xy' - y = x^2\sin x$
43. $x\dfrac{dy}{dx} + y = e^x, \quad x > 0$
44. $x\dfrac{dy}{dx} + 2y = 1 - \dfrac{1}{x}, \quad x > 0$
45. $y\dfrac{dx}{dy} + 2x = 5y^3$
46. $ty' + y = \cos t$
47. $xy' + 2y = x^2$
48. $xy' = 2y + x^3\cos x$
49. $xy' + 2y = x^{-3}$
50. $ty' + 2y = t^2$
51. $xy' + 2\left(y + x^2\right) = \dfrac{\sin x}{x}$
52. $xy' + 4y = x^3 - x$
53. $xy' + (x+1)y = e^{-x}\sin 2x$

54. $xy' + (3x+1)y = e^{3x}$

55. $xy' + (2x-3)y = 4x^4$

56. $2xy'' - 3y = 9x^3$

57. $2y' + 3y = e^{-t}$

58. $2y' + 2ty = t$

59. $3xy' + y = 10\sqrt{x}$

60. $3xy' + y = 12x$

61. $x^2y' + xy = 1$

62. $x^2y' + x(x+2)y = e^x$

63. $y^2 + (y')^2 = 1$

64. $(1+x)y' + y = \sqrt{x}$

65. $(1+x)y' + y = \cos x$

66. $(x+1)y' + (x+2)y = 2xe^{-x}$

67. $(x+1)y' - xy = x + x^2$

68. $\left(1+x^3\right)y' = 3x^2y + x^2 + x^5$

69. $(t+1)\dfrac{ds}{dt} + 2s = 3(t+1) + \dfrac{1}{(t+1)^2}, \quad t > -1$

70. $(x+2)^2 y' = 5 - 8y - 4xy$

71. $\left(x^2-1\right)y' + 2y = (x+1)^2$

72. $\left(x^2+4\right)y' + 2xy = x^2\left(x^2+4\right)$

73. $\left(1+e^t\right)y' + e^t y = 0$

74. $\left(t^2+9\right)y' + ty = 0$

75. $e^{2x}y' + 2e^{2x}y = 2x$

76. $\tan\theta \dfrac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \frac{\pi}{2}$

77. $(\cos t)y' + (\sin t)y = 1$

78. $\cos x \dfrac{dy}{dx} + (\sin x)y = 1$

79. $\cos^2 x \sin x \dfrac{dy}{dx} + \left(\cos^3 x\right)y = 1$

80. $\dfrac{dr}{d\theta} + r\sec\theta = \cos\theta$

81. $\dfrac{dr}{d\theta} + r\tan\theta = \sec\theta$

82. $\dfrac{dP}{dt} + 2tP = P + 4t - 2$

83. $ydx - 4\left(x + y^6\right)dy = 0$

84. $ydx = \left(ye^y - 2x\right)dy$

85. $(x+y+1)dx - dy = 0$

86. $\dfrac{dy}{dx} = x^2e^{-4x} - 4y$

87. $\left(x^2+1\right)y' + xy - x = 0$

88. $\dfrac{dx}{dt} = 9.8 - 0.196x$

89. $\dfrac{di}{dt} + 500i = 10\sin\omega t$

90. $2\dfrac{dQ}{dt} + 100Q = 10\sin 60t$

Find the solution of the initial value problem

91. $y' - 3y = 4$; $y(0) = 2$

92. $y' = y + 2xe^{2x}$; $y(0) = 3$

93. $\left(x^2+1\right)y' + 3xy = 6x$; $y(0) = -1$

94. $t\dfrac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(2) = 1$

95. $\theta\dfrac{dy}{d\theta} + y = \sin\theta, \quad \theta > 0, \quad y\left(\frac{\pi}{2}\right) = 1$

96. $\dfrac{dy}{dx} + xy = x, \quad y(0) = -6$

97. $ty' + 2y = 4t^2, \quad y(1) = 2$

98. $\left(1+t^2\right)y' + 4ty = \left(1+t^2\right)^{-2}, \quad y(1) = 0$

99. $y' + y = e^t, \quad y(0) = 1$

100. $y' + \frac{1}{2}y = t, \quad y(0) = 1$

101. $y' = x + 5y, \quad y(0) = 3$

102. $y' = 2x - 3y, \quad y(0) = \frac{1}{3}$

103. $xy' + y = e^x, \quad y(1) = 2$

104. $y\frac{dx}{dy} - x = 2y^2, \quad y(1) = 5$

105. $xy' + y = 4x + 1, \quad y(1) = 8$

106. $y' + 4xy = x^3 e^{x^2}, \quad y(0) = -1$

107. $(x+1)y' + y = \ln x, \quad y(1) = 10$

108. $x(x+1)y' + xy = 1, \quad y(e) = 1$

109. $y' - (\sin x)y = 2\sin x, \quad y\left(\frac{\pi}{2}\right) = 1$

110. $y' + (\tan x)y = \cos^2 x, \quad y(0) = -1$

111. $L\frac{di}{dt} + RI = E, \quad i(0) = i_0$

112. $\frac{dT}{dt} = k\left(T - T_m\right) \quad T(0) = T_0$

113. $y' + y = 2, \quad y(0) = 0$

114. $xy' + 2y = 3x, \quad y(1) = 5$

115. $y' - 2y = 3e^{2x}, \quad y(0) = 0$

116. $xy' + 5y = 7x^2, \quad y(2) = 5$

117. $xy' - y = x, \quad y(1) = 7$

118. $xy' + y = 3xy, \quad y(1) = 0$

119. $xy' + 3y = 2x^5, \quad y(2) = 1$

120. $y' + y = e^x, \quad y(0) = 1$

121. $xy' - 3y = x^3, \quad y(1) = 10$

122. $y' + 2xy = x, \quad y(0) = -2$

123. $y' = (1 - y)\cos x, \quad y(\pi) = 2$

124. $(1+x)y' + y = \cos x, \quad y(0) = 1$

125. $y' = 1 + x + y + xy, \quad y(0) = 0$

126. $xy' = 3y + x^4 \cos x, \quad y(2\pi) = 0$

127. $y' = 2xy + 3x^2 e^{x^2}, \quad y(0) = 5$

128. $\left(x^2 + 4\right)y' + 3xy = x, \quad y(0) = 1$

129. $y' - 2y = e^{3x}; \quad y(0) = 3$

130. $y' - 3y = 6; \quad y(0) = 1$

131. $2y' + 3y = e^x; \quad y(0) = 0$

132. $\left(x^2 + 1\right)y' + 3x^3 y = 6xe^{-3x^2/2}, \quad y(0) = 1$

133. $y' + y = 1 + e^{-x}\cos 2x; \quad y\left(\frac{\pi}{2}\right) = 0$

134. $2y' + (\cos x)y = -3\cos x; \quad y(0) = -4$

135. $y' + 2y = e^{-x} + x + 1; \quad y(-1) = e$

136. $y' + \frac{y}{x} = xe^{-x}; \quad y(1) = e - 1$

137. $y' + 4y = e^{-x}; \quad y(1) = \frac{4}{3}$

138. $x^2 y' + 3xy = x^4 \ln x + 1; \quad y(1) = 0$

139. $y' + \frac{3}{x}y = 3x - 2 \quad y(1) = 1$

140. $(\cos x)y' + y\sin x = 2x\cos^2 x \quad y\left(\frac{\pi}{4}\right) = \frac{-15\sqrt{2}\pi^2}{32}$

141. $(\cos x)y' + (\sin x)y = 2\cos^3 x \sin x - 1 \quad y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$

142. $t\,y' + 2y = t^2 - t + 1 \quad y(1) = \frac{1}{2}$

143. $t\,y' - 2y = t^5 \sin 2t - t^3 + 4t^4 \quad y(\pi) = \frac{3}{2}\pi^4$

144. $2y' - y = 4\sin 3t \quad y(0) = y_0$

145. $y' + 2y = 2 - e^{-4t} \quad y(0) = 1$

146. $y' - y = -\frac{1}{2}e^{t/2}\sin 5t + 5e^{t/2}\cos 5t \quad y(0) = 0$

147. $y' + 2y = 3\ ; \quad y(0) = -1$

148. $y' + (\cos t)y = \cos t\ ; \quad y(\pi) = 2$

149. $y' + 2ty = 2t\ ; \quad y(0) = 1$

150. $y' + y = \frac{e^{-t}}{t^2}\ ; \quad y(1) = 0$

151. $ty' + 2y = \sin t\ ; \quad y(\pi) = \frac{1}{\pi}$

152. $y' + \frac{2}{t}y = \frac{\cos t}{t^2}\ ; \quad y(\pi) = 0$

153. $(\sin t)y' + (\cos t)y = 0\ ; \quad y\left(\frac{3\pi}{4}\right) = 2$

154. $y' + 3t^2 y = t^2$; $y(0) = 2$

155. $ty' + y = t\sin t$; $y(\pi) = -1$

156. $y' + y = \sin t$; $y(\pi) = 1$

157. $y' + y = \cos 2t$; $y(0) = 5$

158. $y' + 3y = \cos 2t$; $y(0) = -1$

159. $y' - 2y = 7e^{2t}$; $y(0) = 3$

160. $y' - 2y = 3e^{-2t}$; $y(0) = 10$

161. $y' + 2y = t^2 + 2t + 1 + e^{4t}$; $y(0) = 0$

162. $y' - 3y = 2t - e^{4t}$; $y(0) = 0$

163. $y' + y = t^3 + \sin 3t$; $y(0) = 0$

164. $y' + 2y = \cos 2t + 3\sin 2t + e^{-t}$; $y(0) = 0$

165. $y' + y = e^{3t}$; $y(0) = y_0$

166. $t^2 y' - ty = 1$; $y(1) = y_0$

167. $y' + ay = e^{at}$; $y(0) = y_0$, $a \neq 0$

168. $3y' + 12y = 4$; $y(0) = y_0$

Find a solution to the initial value problem that is continuous on the given interval $[a,\ b]$

169. $y' + \frac{1}{x}y = f(x)$, $y(1) = 1$ $\quad f(x) = \begin{cases} 3x, & 1 \le x \le 2 \\ 0, & 2 < x \le 3 \end{cases}$ $\quad [a,\ b] = [1,\ 3]$

170. $y' + (\sin x)y = f(x)$, $y(0) = 3$ $\quad f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ -\sin x, & \pi < x \le 2\pi \end{cases}$ $\quad [a,\ b] = [0,\ 2\pi]$

171. $y' + p(t)y = 2$, $y(0) = 1$ $\quad p(t) = \begin{cases} 0, & 0 \le t \le 1 \\ \frac{1}{t}, & 1 < t \le 2 \end{cases}$ $\quad [a,\ b] = [0,\ 2]$

172. $y' + p(t)y = 0$, $y(0) = 3$ $\quad p(t) = \begin{cases} 2t - 1, & 0 \le t \le 1 \\ 0, & 1 < t \le 3 \\ -\frac{1}{t}, & 3 < t \le 4 \end{cases}$ $\quad [a,\ b] = [0,\ 4]$

Find the solution of the initial value problem. Discuss the interval of existence and provide a sketch of your solution

173. $xy' + 2y = \sin x$; $y\left(\frac{\pi}{2}\right) = 0$

174. $(2x+3)y' = y + (2x+3)^{1/2}$; $y(-1) = 0$

175. The following system of differential equations is encountered in the study of the decay of a special type of radioactive series of elements

$$\frac{dx}{dt} = -\lambda_1 x \qquad \frac{dy}{dt} = \lambda_1 x - \lambda_2 y$$

Where λ_1 *and* λ_2 are constants.

Discuss how to solve this system subject to $x(0) = x_0$, $y(0) = y_0$