

## ***Solution***

## **Section 4.1 – Matrix Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$**

### ***Exercise***

Find the standard matrix for the transformation defined by the equations

$$a) \begin{cases} w_1 = 2x_1 - 3x_2 + 4x_4 \\ w_2 = 3x_1 + 5x_2 - x_4 \end{cases}$$

$$b) \begin{cases} w_1 = 7x_1 + 2x_2 - 8x_3 \\ w_2 = -x_2 + 5x_3 \\ w_3 = 4x_1 + 7x_2 - x_3 \end{cases}$$

$$c) \begin{cases} w_1 = x_1 \\ w_2 = x_1 + x_2 \\ w_3 = x_1 + x_2 + x_3 \\ w_4 = x_1 + x_2 + x_3 + x_4 \end{cases}$$

### **Solution**

$$a) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 & 4 \\ 3 & 5 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The standard matrix is  $\begin{bmatrix} 2 & -3 & 0 & 4 \\ 3 & 5 & 0 & -1 \end{bmatrix}$

$$b) \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The standard matrix is  $\begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix}$

$$c) \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The standard matrix is  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

### ***Exercise***

Find the standard matrix for the operator  $T$  defined by the formula

$$T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$$

### **Solution**

$$\begin{aligned} T(x_1, x_2) &= \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

The standard matrix is  $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

### ***Exercise***

Find the standard matrix for the operator  $T$  defined by the formula

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3, x_1 + 5x_2, x_3)$$

### **Solution**

$$\begin{aligned} T(x_1, x_2, x_3) &= \begin{bmatrix} x_1 + 2x_2 + x_3 \\ x_1 + 5x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

The standard matrix is  $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

### ***Exercise***

Find the standard matrix for the operator  $T$  defined by the formula

$$T(x_1, x_2, x_3) = (4x_1, 7x_2, -8x_3)$$

### **Solution**

$$\begin{aligned} T(x_1, x_2, x_3) &= \begin{pmatrix} 4x_1 \\ 7x_2 \\ -8x_3 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned}$$

The standard matrix is  $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -8 \end{pmatrix}$

### ***Exercise***

Find the standard matrix for the operator  $T$  defined by the formula

$$T(x_1, x_2) = (x_2, -x_1, x_1 + 3x_2, x_1 - x_2)$$

### **Solution**

$$\begin{aligned} T(x_1, x_2) &= \begin{pmatrix} x_2 \\ -x_1 \\ x_1 + 3x_2 \\ x_1 - x_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

The matrix is  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 3 \\ 1 & -1 \end{pmatrix}$

### Exercise

Find the standard matrix for the operator  $T$  defined by the formula

$$T(x_1, x_2, x_3) = (0, 0, 0, 0, 0)$$

### Solution

$$T(x_1, x_2, x_3) = (0, 0, 0, 0, 0)$$

The matrix is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

### Exercise

Find the standard matrix for the operator  $T$  defined by the formula

$$T(x_1, x_2, x_3, x_4) = (x_4, x_1, x_3, x_2, x_1 - x_3)$$

### Solution

$$T(x_1, x_2, x_3, x_4) = (x_4, x_1, x_3, x_2, x_1 - x_3)$$

$$T(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_4 \\ x_1 \\ x_3 \\ x_2 \\ x_1 - x_3 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

The matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

### Exercise

Find the standard matrix for the operator  $T$  defined by the formula

$$T(x_1, x_2, x_3, x_4) = (7x_1 + 2x_2 - x_3 + x_4, x_2 + x_3, -x_1)$$

### Solution

$$T(x_1, x_2, x_3, x_4) = \begin{pmatrix} 7x_1 + 2x_2 - x_3 + x_4 \\ x_2 + x_3 \\ -x_1 \end{pmatrix}$$
$$= \begin{pmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

The matrix is  $\begin{pmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$

### Exercise

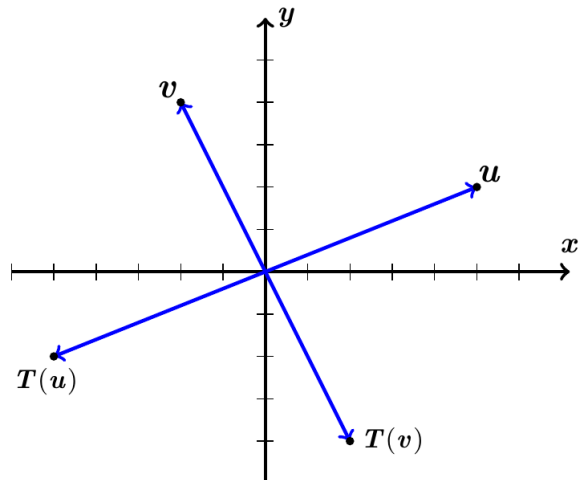
Plot  $\vec{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ ,  $\vec{v} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$  and their images under the given transformation  $T$

$$T(\vec{x}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

### Solution

$$T(\vec{u}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} -5 \\ -2 \end{pmatrix}$$

$$T(\vec{v}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$



$\therefore$  Reflection through the origin ( $180^\circ$ )

### Exercise

Plot  $\vec{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ ,  $\vec{v} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$  and their images under the given transformation  $T$

$$T(\vec{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

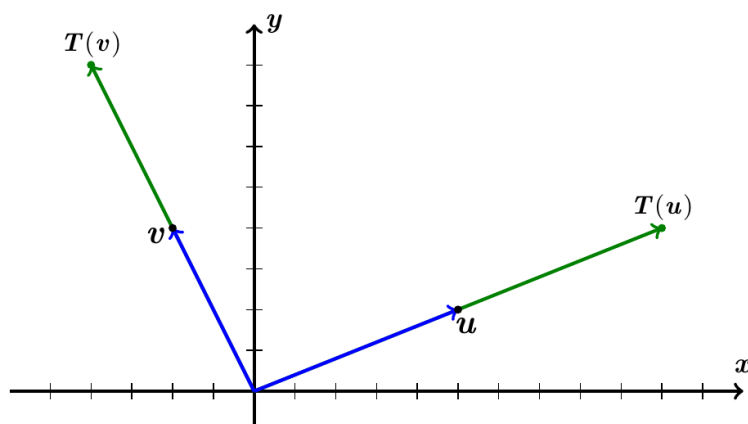
### Solution

$$\begin{aligned} T(\vec{u}) &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 10 \\ 4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} T(\vec{v}) &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ 8 \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} 10 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$T(\vec{u}) = \vec{u}$$



$\therefore$  Dilation with factor ( $k=2$ ) on  $\mathbb{R}^2$

### Exercise

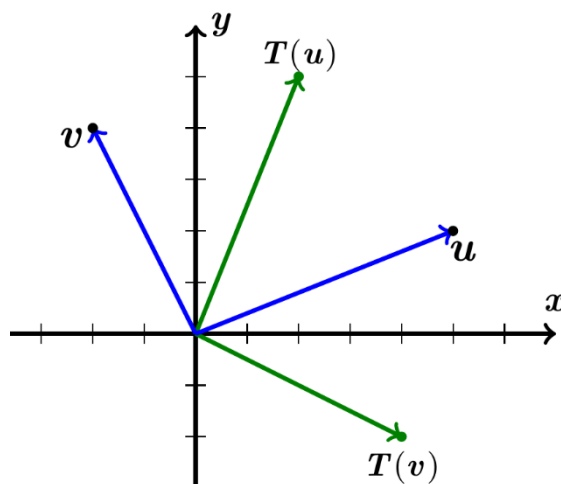
Plot  $\vec{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ ,  $\vec{v} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$  and their images under the given transformation  $T$

$$T(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

### Solution

$$\begin{aligned} T(\vec{u}) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 5 \end{pmatrix} \end{aligned}$$

$$T(\vec{v}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$



$$= \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 5 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\vec{x}_1 = \vec{x}_2$$

$\therefore$  reflection about the line  $\vec{x}_1 = \vec{x}_2$

### Exercise

Plot  $\vec{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ ,  $\vec{v} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$  and their images under the given transformation  $T$

$$T(\vec{x}) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

### Solution

$$T(\vec{u}) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$T(\vec{v}) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 8 \end{pmatrix}$$

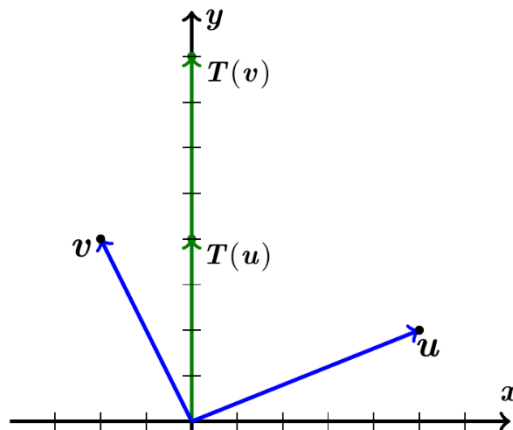
$$\begin{pmatrix} 0 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} * \\ 2 \end{pmatrix}$$

$$T(\vec{u}) \quad \vec{u}$$

$$\begin{pmatrix} 0 \\ 8 \end{pmatrix} = 2 \begin{pmatrix} * \\ 4 \end{pmatrix}$$

$$T(\vec{v}) \quad \vec{v}$$

$\therefore$  Orthogonal projection on the  $y$ -axis and with *Dilation* with factor ( $k = 2$ )



### Exercise

Plot  $\vec{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ ,  $\vec{v} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$  and their images under the given transformation  $T$

$$T(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

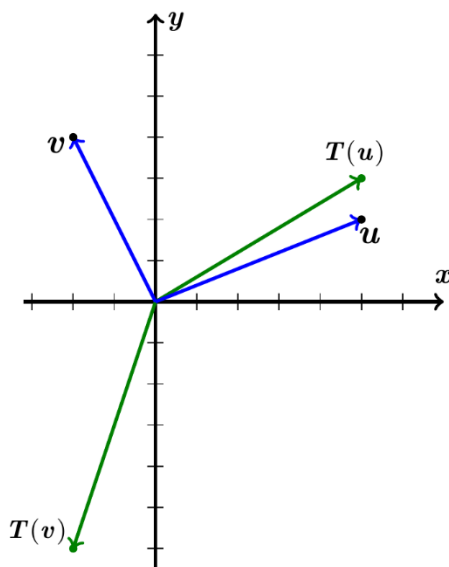
### Solution

$$\begin{aligned} T(\vec{u}) &= \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} T(\vec{v}) &= \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ -6 \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$
$$T(\vec{u}) = \vec{u}$$

$$\begin{pmatrix} -2 \\ -6 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$
$$T(\vec{v}) = \vec{v}$$



$\therefore$  Expansion of  $\vec{u}$  in the  $y$ -direction with factor  $k = \frac{3}{2}$

$\therefore$  Expansion of  $\vec{v}$  in the  $y$ -direction with factor  $k = \frac{3}{2}$  and reflection about  $x$ -axis.