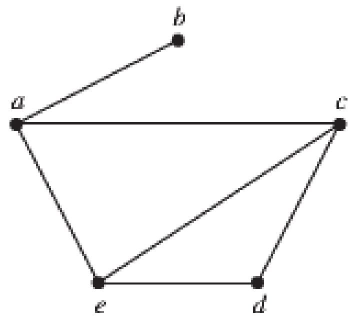
Section 4.7 – Representing Graphs and Graph Isomorphism

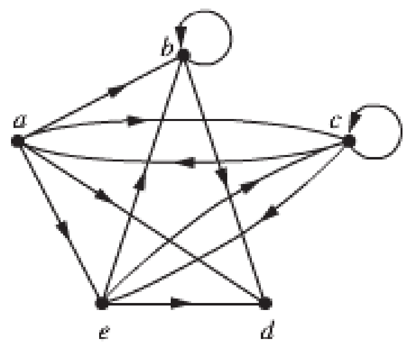
Sometimes, two graphs have exactly the same form, in the sense that there is a one-to-one correspondence between their vertex sets that preserves edges. In such case, we say that the two graphs are isomorphic.

Adjacency

Use the adjacency lists to describe the given graph



Adjacency List for a Simple Graph	
Vertex	***Adjacent Vertices***
a	*b, c , e*
b	*a*
c	*a, d, e*
d	*c, e*
e	*a, c , d*



Adjacency List for a Directed Graph	
Initial Vertex	***Terminal Vertices***
a	*b, c, d, e*
b	*b, d*
c	*a, c, e*
d	
e	*b, c , d*

Adjacency Matrices

Suppose that $G=(V,\ E)$ is a simple graph where $|V|=n$. Suppose that the vertices of *G* are listed arbitrary as $v_1,\ v_2,\ldots,\ v_n$. The adjacency matrix $\boldsymbol{A}$ $\left(or\ \boldsymbol{A}_G\right)$ of *G*, with respect to this listing of the vertices, is the $n\times n$ zero-one matrix with 1 as its $(i,\ j)th$ entry when v_i and v_j are adjacent, and 0 as its $(i,\ j)th$ entry when they are not adjacent.

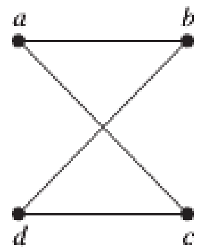
$$a_{ij}=\begin{cases}1 & if\ \left\{v_i,v_j\right\}\ is\ an\ edage\ of\ G\\ 0 & otherwise\end{cases}$$

Example

Draw a graph with the adjacency matrix

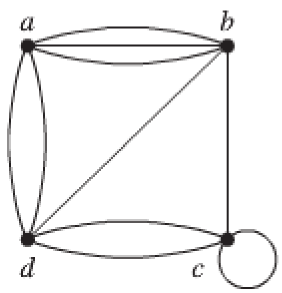
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$ with respect to the ordering of vertices *a*, *b*, *c*, *d*.

Solution



Example

Use an adjacency matrix to represent the pseudograph shown below.



Solution

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

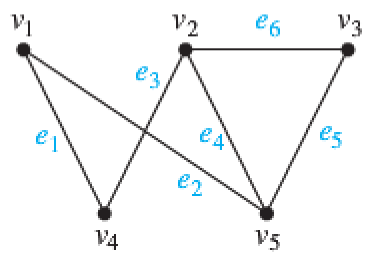
Incidence Matrices

Let $G=(V,\ E)$ be undirected graph. Suppose that $v_1,\ v_2,\ldots,\ v_n$ are the vertices and $e_1,\ e_2,\ldots,\ e_n$ are the edges of G. Then the incident matrix with respect to this ordering of V and E is the $n \times m$ matrix $\boldsymbol{M}=\left[m_{ij}\right]$, where

$$m_{ij}=\begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$

Example

Represent the graph shown with an incidence matrix.



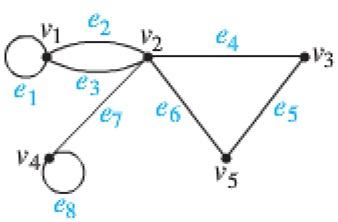
Solution

The incidence matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Example

Represent the graph shown below with an incidence matrix.



Solution

The incidence matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

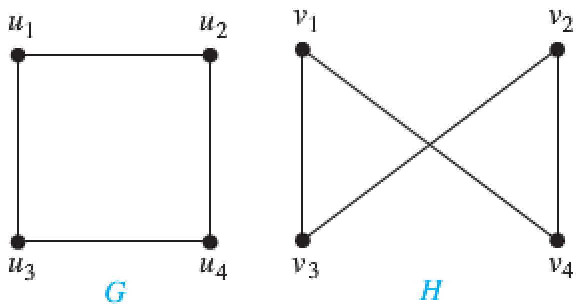
Isomorphism of Graphs

Definition

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a one-to-one and onto function f from V_1 to V_2 with the property that *a* and *b* are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all *a* and *b* in V_1. Such a function f is called an ***isomorphism***. Two simple graphs that are not isomorphic are called ***nonisomorphic***.

Example

Show that the graphs $G = (V, E)$ and $H = (V, E)$, displayed below are isomorphic



<u>Solution</u>

The function f with $f(u_1) = v_1$, $f(u_2) = v_2$, $f(u_3) = v_3$, and $f(u_4) = v_4$ is a one-to-one correspondence between *V* and *W*. To see that this correspondence preserves adjacency, note that adjacent vertices in *G* are u_1 *and* u_2, u_1 *and* u_3, u_2 *and* u_4, and u_3 *and* u_4, and each of the pairs $f(u_1) = v_1$ and $f(u_2) = v_4$, $f(u_1) = v_1$ and $f(u_3) = v_3$, $f(u_2) = v_4$ and $f(u_4) = v_2$, $f(u_3) = v_3$ and $f(u_4) = v_2$ consists of two adjacent vertices in *H*.

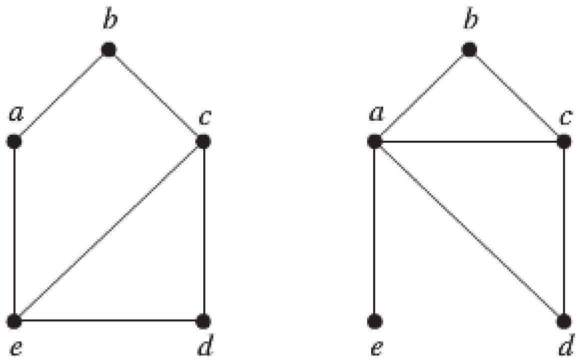
$\therefore$ If we flip v_2 & v_4, we end up the same graph.

Determining whether Two Simple Graphs are Isomorphic

Sometimes it is not hard to show that two graphs are not isomorphic. In Particular, we can show that two graphs are not isomorphic if we can find a property only one of the two graphs has, but that is preserved by isomorphism. A property preserved by isomorphism of graphs is called a ***graph invariant***.

Example

Show that the graphs shown below are not isomorphic



G ***H***

Solution

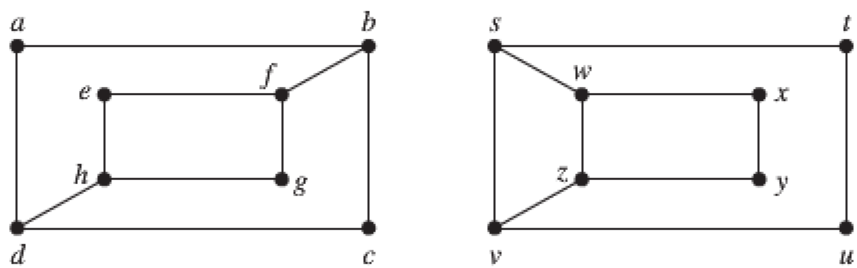
Both graphs *G* & *H* have 5 vertices and 6 edges.

H has a vertex of degree one, @ *e*, whereas *G* has no vertices of degree one.

It follows that *G* & *H* are not isomorphic.

Example

Determine whether the graphs shown below are isomorphic



G ***H***

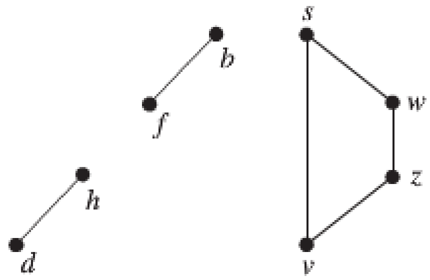
Solution

Both graphs *G* & *H* have 8 vertices and 10 edges.

Also both have 4 vertices of degree 2 and 4 vertices of degree 3.

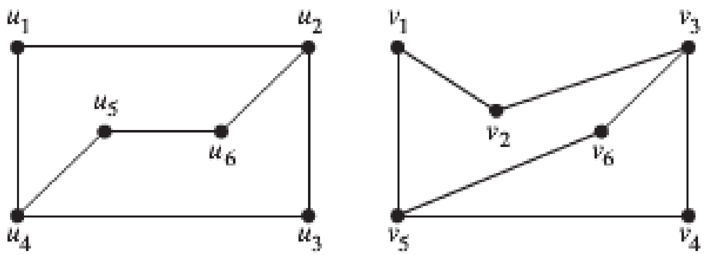
$\deg(a) = 2$ in *G*, *a* must correspond to either *t*, *u* ,*x*, or *y* in *H*, because these are the vertices of degree 2 in *H*. However, each of these four vertices in *H* is adjacent to another vertex of degree 2 in *H*, which is not true for *a* in *G*. Therefore, *G* & *H* are not isomorphic.

Another way to see that G & H are not isomorphic is by checking the subgraphs of G & H shown below, they have a different shape.



Example

Determine whether the graphs shown below are isomorphic



G ***H***

Solution

Both graphs G & H have 6 vertices and 7 edges.

Also both have 4 vertices of degree 2 and 2 vertices of degree 3.

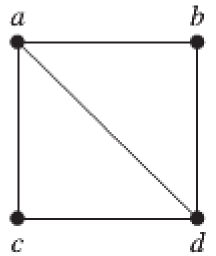
From the subgraphs of G & H, all vertices of degree 2 and the edges connecting them are isomorphic.

$u_1 \leftrightarrow v_6 \quad u_4u_5u_6u_2 \leftrightarrow v_5v_1v_2v_3$

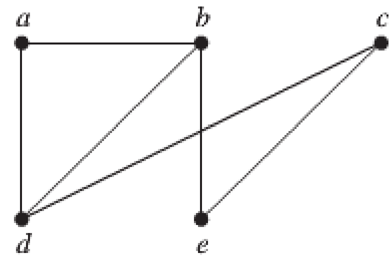
Exercises *Section* 4.7 – Representing Graphs and Graph Isomorphism

Use the adjacency list to represent the given graph, then represent with an adjacency matrix

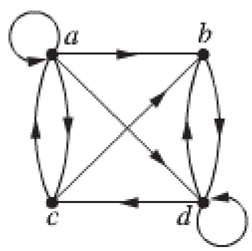
1.



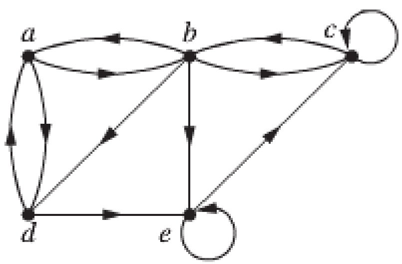
2.



3.



4.

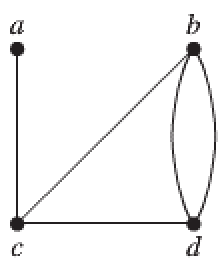


5. Draw a graph with the given adjacency

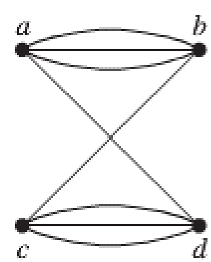
a) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ *b)* $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ *c)* $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

6. Represent the given graph using adjacency matrix

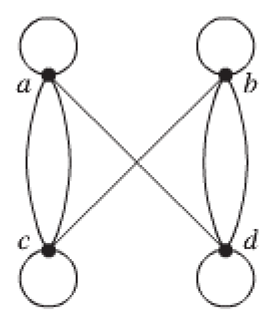
a)



b)



c)

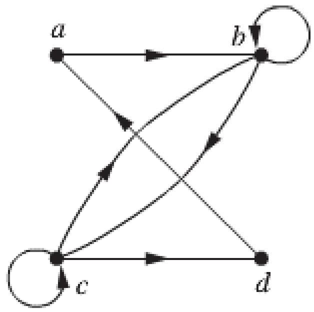


7. Draw an undirected graph represented by the given adjacency

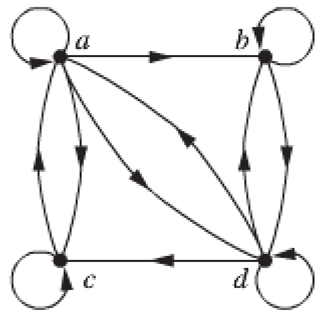
a) $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$ *b)* $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ *c)* $\begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$

8. Find the adjacency matrix of the given directed multigraph with respect to the vertices listed in alphabetic order.

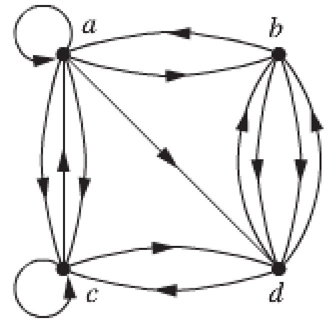
a)



b)

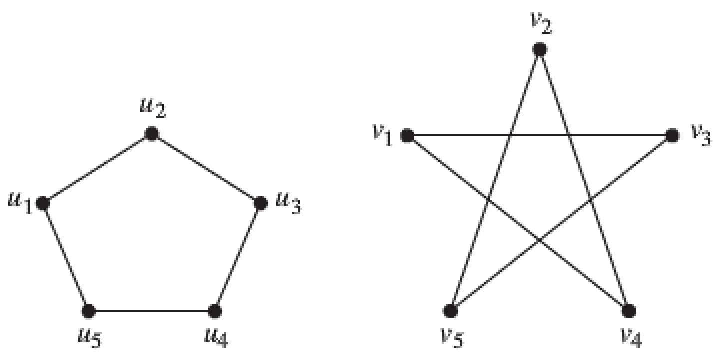


c)

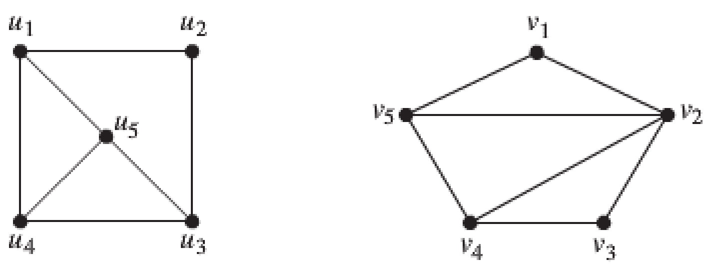


Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

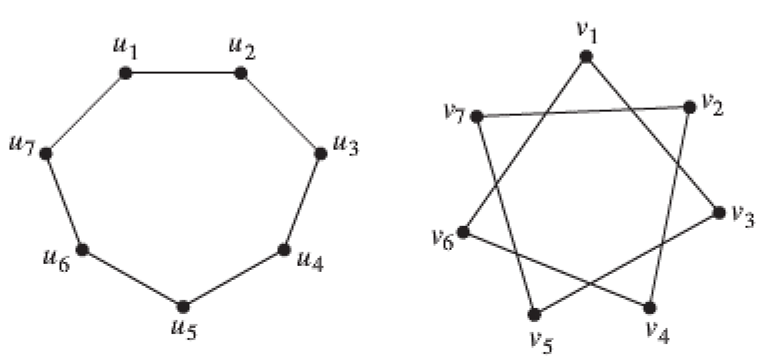
9.



10.



11.



12.