Prove that if A is an $m \times n$ matrix, there is an invertible matrix C such that CA is in reduced row-echelon form

The reduced row-echelon form of A can be written in the form $E_n \cdots E_1 A$, $E_1, E_2, \ldots, E_n$ are elementary matrices.

let $C = E_n \cdots E_1$, then C is invertibles since $E_1, \ldots, E_n$ are invertible. Hence, $\exists$ such a matrix C.

Prove that 2 $m \times n$ matrices A and B are row equivalent iff there exists a nonsingular matrix P such that $B = PA$

Suppose that $A \sim B$. Then there exist elementary matrices $E_1, E_2, \ldots, E_n$ such that $B = E_n \cdots E_1 A$.

Let $P = E_n \cdots E_1 \Rightarrow$ by the theorem, P is nonsingular.
Suppose that $B = PA$, for some nonsingular matrix P. By theorem P is row equivalent to I_k. That is, $I_k = E_n \cdots E_1 P$. Thus, $B = E_1^{-1} E_2^{-1} \cdots E_n^{-1} A$ and this implies that A is row equivalent to B.

Let A and B be 2 $n \times n$ matrices. Suppose A is row equivalent to B. Prove that A is nonsingular iff B is nonsingular.

Suppose that A is row equivalent to B. Then $B = PA$ (Prove Above) w/ P nonsingular. If A is nonsingular then B is nonsingular. Conversely, if B is nonsingular then $A = P^{-1}B$ is nonsingular.

Show that a 2×2 lower triangular matrix is invertible iff $a_{11}a_{22} \neq 0$ and in this case the inverse is also lower triangular

The lower triangular matrix $A = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}$ is invertible iff in $a_{11}a_{22} \neq 0$ and then

$$A^{-1} = \begin{pmatrix} \frac{1}{a_{11}} & 0 \\ \frac{-a_{21}}{a_{11}a_{22}} & \frac{1}{a_{22}} \end{pmatrix}$$

Show that if A and B are two $n \times n$ invertible matrices then A is row equivalent to B

Since A is invertible, then (by theorem) A is row equivalent to I_n. That is, there exist elementary matrices $E_1, \dots, E_k$ such that $I_n = E_k E_{k-1} \cdots E_1 A$.

Similarly, there exist elementary matrices $F_1, F_2, \dots, F_i$ such that $I_n = F_i F_{i-1} \cdots F_1 B$.

Hence $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} F_i F_{i-1} \cdots F_1 B$. That is, A is row equivalent to B.

Prove that a square matrix A is nonsingular iff A is a product of elementary matrices.

Suppose that A is nonsingular. Then A is row equivalent to I_n. That is, there exist elementary matrices $E_1, E_2, \dots, E_k$ such that $I_n = E_k E_{k-1} \cdots E_1 A$. Then $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$.

But each E_i^{-1} is an elementary matrix.

Conversely, suppose that $A = E_1 E_2 \cdots E_k$. Then $(E_1 E_2 \cdots E_k)^{-1} A = I_n$. That is, A is nonsingular.

Show that if $A \sim B$ (that is, if they are row equivalent), then $EA = B$ for some matrix E which is a product of elementary matrices.

If $A \sim B$, there is some sequence of elementary row operations which, when performed on A, produce B. Further, multiplying on the left by the corresponding elementary matrix is the same as performing that row operation. So we have

$$A \sim E_1 A \sim E_2 E_1 A \sim \cdots \sim E_k E_{k-1} \cdots E_1 A = B$$

Thus, if $E = E_k \cdots E_1$ we have $EA = B$

Show that if $EA = B$ for some matrix E which is a product of elementary matrices, then $AC \sim BC$ for every $n \times n$ matrix C

Let $E = E_k E_{k-1} \cdots E_2 E_1$ where each E_i is an elementary matrix.

$$AC \sim E_1 AC \sim E_2 E_1 AC \sim \cdots \sim E_k \cdots E_2 E_1 AC = EAC$$

since $EA = B \Rightarrow CEA = CB$

$ECA = CB \Rightarrow \therefore AC \sim BC$

Let $A\vec{x}=0$ be a homogeneous system of n linear equations in n unknowns that has only the trivial solution. Show that if k is any positive integer, then the system $A^k x=0$ also has only trivial solution

Since A is a square matrix, thus A has only the trivial soln $\Rightarrow$ A is invertible. But A^k is also invertible so $A^k x=0$ has only trivial soln.

Let $A\vec{x}=0$ be a homogeneous system of n linear eqns. in n unknowns, and let Q be an invertible $n\times n$ matrix. Show that $Ax=0$ has just trivial solution if and only if $(QA)\vec{x}=0$ has just trivial solution.

A is an $n\times n$ matrix. If $A\vec{x}=0$ has just trivial soln, then A is invertible
Since Q is an invertible $n\times n$ matrix $\Rightarrow QA$ is also invertible
Thus $(QA)\vec{x}=0$ has trivial soln.
On the other hand, if $(QA)x=0$ has trivial soln $\Rightarrow QA$ is invertible
Since Q is invertible $\Rightarrow Q^{-1}$ is also invertible.
Thus $A=Q^{-1}QA$ is invertible ie $Ax=0$ has just trivial soln.
$\Leftrightarrow A\vec{x}=0$ has just trivial soln iff $(QA)\vec{x}=0$ has just trivial soln.

Let $A\vec{x}=b$ be any consistent system of linear eqns, and let x_1 be a fixed soln. Show that every soln. to the system can be written in the form $x=x_1+x_0$ where x_0 is a solution to $A\vec{x}=0$. Show ~~also~~ also that every matrix of this form is a solution.

Since x_0 is a solution to $A\vec{x}=0$, we have $Ax_0=0$
Adding $Ax_0=0$ to $Ax=b \Rightarrow Ax+Ax_0=b+0$

$$A(x+x_0)=b$$

As adding an eqn to the original eqn does not affect the soln, if we let x_1 be a fixed solution, then every soln to $Ax=b$ is $x=x_1+x_0$

Besides

$$\begin{aligned} A(x_1+x_0) &= Ax_1+Ax_0 \\ &= b+0 \\ &= b \end{aligned}$$

So every matrix (vector) in the form x_1+x_0 is a solution to $Ax=b$