Section 3.5 – Additional Identities

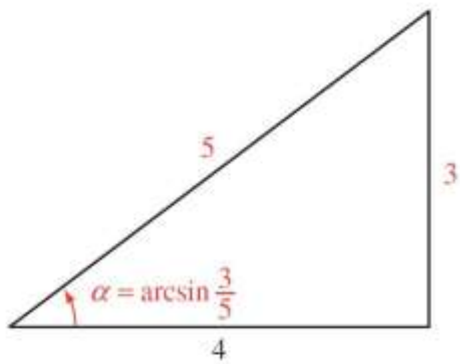
Identities and Formulas Involving Inverse Functions

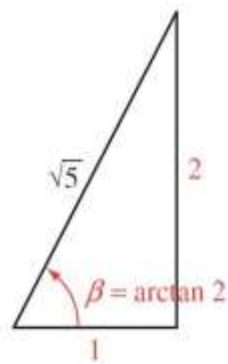
Example

Evaluate $\sin\left(\arcsin\frac{3}{5}+\arctan 2\right)$ without using a calculator.

<u>*Solution*</u>

$$\sin\left(\arcsin\frac{3}{5}+\arctan 2\right)=\sin(\alpha+\beta)$$
$$=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$





$\sin\alpha=\frac{3}{5}$ $\qquad$ $\cos\alpha=\frac{4}{5}$ $\qquad$ $\sin\beta=\frac{2}{\sqrt{5}}$ $\qquad$ $\cos\alpha=\frac{1}{\sqrt{5}}$

$$\begin{aligned}\sin\left(\arcsin\frac{3}{5}+\arctan 2\right) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta \\ &= \frac{3}{5}\frac{1}{\sqrt{5}}+\frac{4}{5}\frac{2}{\sqrt{5}} \\ &= \frac{3}{5\sqrt{5}}+\frac{8}{5\sqrt{5}} \\ &= \frac{11}{5\sqrt{5}}\end{aligned}$$

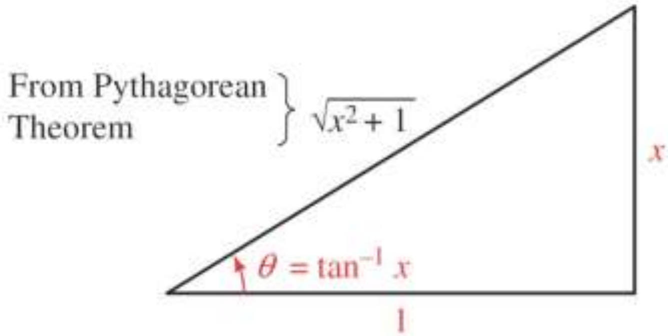
Example

Write $\sin(2\tan^{-1} x)$ as an equivalent expression involving only x. (Assume x is positive)

<u>*Solution*</u>

Let $\theta = \tan^{-1} x$

$\Rightarrow \tan\theta = x = \frac{x}{1}$



$\sin\theta = \frac{x}{\sqrt{x^2+1}} \qquad \cos\theta = \frac{1}{\sqrt{x^2+1}}$

$$\begin{aligned}\sin(2\tan^{-1} x) &= \sin(2\theta)\\ &= 2\sin\theta\cos\theta\\ &= 2\frac{x}{\sqrt{x^2+1}}\frac{1}{\sqrt{x^2+1}}\\ &= \frac{2x}{x^2+1}\end{aligned}$$

Product to Sum Formulas

$$\begin{array}{llll} \sin A\cos B & +\cos A\sin B & =\sin(A+B) & \\ \sin A\cos B & -\cos A\sin B & =\sin(A-B) & \\ \hline 2\sin A\cos B & & =\sin(A+B)+\sin(A-B) & \end{array}$$

$$\boxed{\begin{aligned} &\sin A\cos B=\tfrac{1}{2}\left[\sin(A+B)+\sin(A-B)\right] \\ &\cos A\sin B=\tfrac{1}{2}\left[\sin(A+B)-\sin(A-B)\right] \\ &\cos A\cos B=\tfrac{1}{2}\left[\cos(A+B)+\cos(A-B)\right] \\ &\sin A\sin B=\tfrac{1}{2}\left[\cos(A-B)-\cos(A+B)\right] \end{aligned}}$$

Example

Verify product formula $\cos A\cos B=\frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right]$ for $A=30°$ and $B=120°$

Solution

$$\cos 30°\cos 120°=\frac{1}{2}\left[\cos(30°+120°)+\cos(30°-120°)\right]$$

$$\cos 30°\cos 120°=\frac{1}{2}\left[\cos(150°)+\cos(-90°)\right]$$

$$\frac{\sqrt{3}}{2}\cdot\left(-\frac{1}{2}\right)=\frac{1}{2}\left[-\frac{\sqrt{3}}{2}+0\right]$$

$$-\frac{\sqrt{3}}{4}=-\frac{\sqrt{3}}{4}$$

Example

Write $4\cos 75°\sin 25°$ as a sum or difference

Solution

$$\begin{aligned} 4\cos 75°\sin 25° &= 4\frac{1}{2}\left[\sin\left(75°+25°\right)-\sin\left(75°-25°\right)\right] \\ &= 2\left[\sin\left(100°\right)-\sin\left(50°\right)\right] \end{aligned}$$

Sum to Product Formulas

$\sin A\cos B = \frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right]$

$2\sin A\cos B = \sin(A+B)+\sin(A-B)$

$\sin(A+B)+\sin(A-B) = 2\sin A\cos B$

$\cos A\sin B = \frac{1}{2}\left[\sin(A+B)-\sin(A-B)\right]$

$\sin(A+B)-\sin(A-B) = 2\cos A\sin B$

Let $\alpha = A+B$

$$\begin{array}{l} \alpha = A+B \\ \underline{\beta = A-B} \\ \alpha+\beta = 2A \quad \Rightarrow A = \dfrac{\alpha+\beta}{2} \\ \alpha-\beta = 2B \quad \Rightarrow B = \dfrac{\alpha-\beta}{2} \end{array}$$

$$\boxed{\begin{array}{l} \sin\alpha+\sin\beta = 2\sin\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right) \\ \sin\alpha-\sin\beta = 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right) \\ \cos\alpha+\cos\beta = 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right) \\ \cos\alpha-\cos\beta = -2\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right) \end{array}}$$

Example

Verify sum formula $\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ for $\alpha = 30°$ and $\beta = 90°$

<u>*Solution*</u>

$$\cos 30° + \cos 90° = 2\cos\left(\frac{30°+90°}{2}\right)\cos\left(\frac{30°-90°}{2}\right)$$

$$\cos 30° + \cos 90° = 2\cos\left(\frac{120°}{2}\right)\cos\left(\frac{-60°}{2}\right)$$

$$\cos 30° + \cos 90° = 2\cos(60°)\cos(-30°)$$

$$\frac{\sqrt{3}}{2}+0 = 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

Example

Verify the identity $-\tan x = \dfrac{\cos 3x - \cos x}{\sin 3x + \sin x}$

Solution

$$\frac{\cos 3x - \cos x}{\sin 3x + \sin x} = \frac{-2\sin\frac{3x+x}{2}\sin\frac{3x-x}{2}}{2\sin\frac{3x+x}{2}\cos\frac{3x-x}{2}}$$

$$= -\frac{2\sin 2x\sin x}{2\sin 2x\cos x}$$

$$= -\frac{\sin x}{\cos x}$$

$$= -\tan x$$

Example

Write $\sin 2\theta - \sin 4\theta$ as product of two functions.

Solution

$$\sin 2\theta - \sin 4\theta = 2\cos\left(\tfrac{2\theta+4\theta}{2}\right)\sin\left(\tfrac{2\theta-4\theta}{2}\right)$$

$$= 2\cos\left(\tfrac{6\theta}{2}\right)\sin\left(-\tfrac{2\theta}{2}\right)$$

$$= 2\cos 3\theta\sin(-\theta)$$

$$= -2\cos 3\theta\sin\theta$$

Exercises Section 3.5 – Additional Identities

1. Evaluate without using the calculator $\cos\left(\arctan\sqrt{3}+\arcsin\frac{1}{3}\right)$

2. Evaluate without using the calculator $\cos(\arcsin\frac{3}{5}-\arctan 2)$

3. Evaluate without using the calculator $\sin\left(2\cos^{-1}\frac{1}{\sqrt{5}}\right)$

4. Evaluate without using the calculator $\tan\left(2\arcsin\frac{2}{5}\right)$

5. Evaluate without using the calculator $\sin\left(\tan^{-1}u\right)$

6. Write $\sin\left(2\cos^{-1}x\right)$ as an equivalent expression involving only x.

7. Write $\cos\left(2\sin^{-1}u\right)$ as an equivalent expression involving only x.

8. Write $\sec\left(\tan^{-1}\frac{x-2}{2}\right)$ as an equivalent expression involving only x.

9. Write $10\cos 5x\sin 3x$ as a sum or difference

10. Prove the identity: $\frac{\sin 3x-\sin x}{\cos 3x-\cos x}=-\cot 2x$

11. Prove the following equation is an identity: $\sin(x+y)\cos(x-y)=\sin x\cos x+\cos y\sin y$

12. Prove the following equation is an identity: $2\sin(x+y)\cos(x-y)=\sin 2x+\sin 2y$

13. Prove the following equation is an identity: $\frac{\sin(26k)+\sin(8k)}{\cos(26k)-\cos(8k)}=-\cot(9k)$

14. Prove the following equation is an identity: $\frac{\sin(26k)-\sin(12k)}{\sin(26k)+\sin(12k)}=\cot(19k)\tan(7k)$

15. Prove the following equation is an identity: $\sin(x+y)\cos(x-y)=\sin x\cos x+\cos y\sin y$

16. Prove the following equation is an identity: $(\sin\alpha+\cos\alpha)(\sin\beta+\cos\beta)=\sin(\alpha+\beta)+\cos(\alpha-\beta)$

17. Prove the following equation is an identity: $\frac{\cos x-\cos 3x}{\cos x+\cos 3x}=\tan 2x\tan x$

18. Prove the following equation is an identity: $\frac{\cos 5x+\cos 3x}{\cos 5x-\cos 3x}=-\cot 4x\cot x$

19. Prove the following equation is an identity: $\frac{\sin 3t-\sin t}{\cos 3t+\cos t}=\tan t$

20. Prove the following equation is an identity: $\frac{\sin 3x+\sin 5x}{\sin 3x-\sin 5x}=-\frac{\tan 4x}{\tan x}$

21. Prove the following equation is an identity: $\cos^2 x - \cos^2 y = -\sin(x+y)\sin(x-y)$

22. Prove the following equation is an identity: $\dfrac{\sin 6x + \sin 2x}{2\sin 4x} = \cos 2x$

23. Prove the following equation is an identity: $\dfrac{\cos 8x - \cos 2x}{2\sin 5x} = -\sin 3x$

24. Prove the following equation is an identity: $\dfrac{\sin 9x + \sin 3x}{\cos 9x + \cos 3x} = \tan 6x$

25. Prove the following equation is an identity: $\dfrac{\cos 2x - \cos 6x}{\sin 2x + \sin 6x} = \tan 2x$

26. Prove the following equation is an identity: $\dfrac{\sin 8x + \sin 2x}{\sin 8x - \sin 2x} = \dfrac{\tan 5x}{\tan 3x}$

27. Prove the following equation is an identity: $\dfrac{\cos 6x - \cos 2x}{\cos 6x + \cos 2x} = -\tan 4x \tan 2x$

28. Prove the following equation is an identity: $\sin x\left(\sin x + \sin 5x\right) = \cos 2x\left(\cos 2x - \cos 4x\right)$

29. Prove the following equation is an identity: $\dfrac{\cos x + \cos y}{\sin x - \sin y} = \cot\dfrac{x-y}{2}$