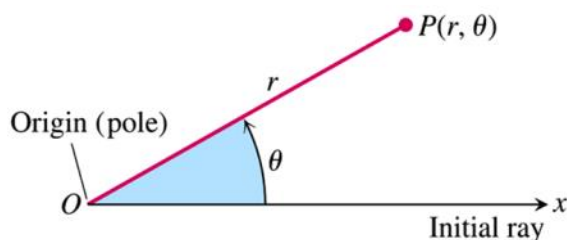


Section 3.6 – Polar Coordinates

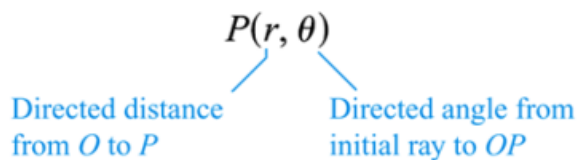
To reach the point whose address is $(2, 1)$, we start from origin and travel 2 units right and then 1 unit up. Another way to get to that point, we can travel $\sqrt{5}$ units on the terminal side of an angle in standard position and this type is called *Polar Coordinates*.

Definition of Polar Coordinates

To define polar coordinates, let an **origin** O (called the **pole**) and an **initial ray** from O . Then each point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives the directed from O to P and θ gives the directed angle from the initial ray to ray OP .



Polar Coordinates



Definition – Relationships between Rectangular and Polar Coordinates

The rectangular coordinates (x, y) and polar coordinates (r, θ) of a point P are related as follows:

1. $x = r \cos \theta, \quad y = r \sin \theta$
2. $r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x} \quad \text{if } x \neq 0$

Example

If $(r, \theta) = \left(4, \frac{7\pi}{6}\right)$ are polar coordinates of a point P , find the rectangular coordinates of P .

Solution

$$x = r \cos \theta = 4 \cos \frac{7\pi}{6} = 4 \left(-\frac{\sqrt{3}}{2} \right) = -2\sqrt{3}$$

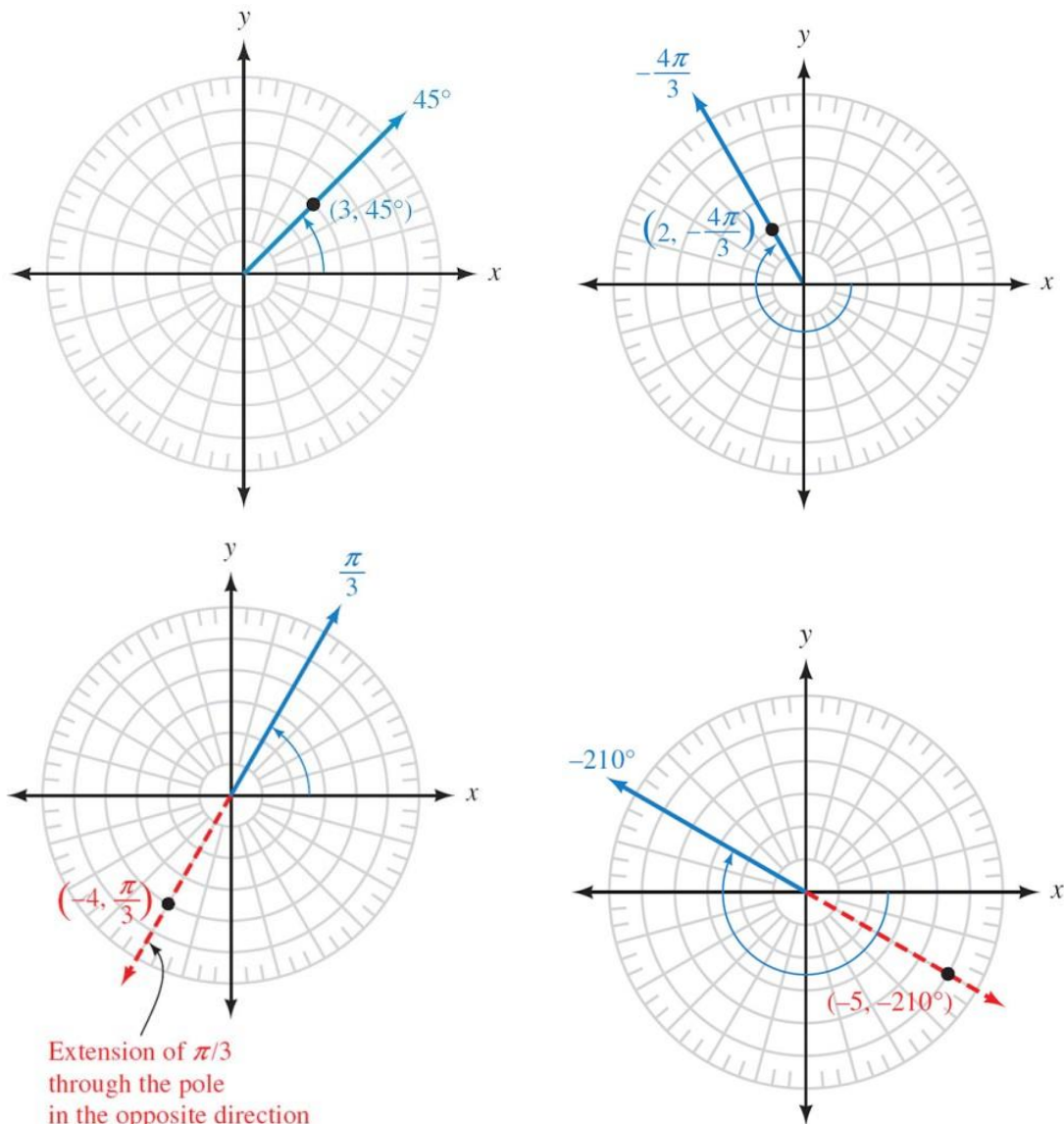
$$y = r \sin \theta = 4 \sin \frac{7\pi}{6} = 4 \left(-\frac{1}{2} \right) = -2$$

The rectangular coordinates of P are $(x, y) = (-2\sqrt{3}, -2)$

Example

Graph the points $(3, 45^\circ)$, $\left(2, -\frac{4\pi}{3}\right)$, $\left(-4, \frac{\pi}{3}\right)$, and $(-5, -210^\circ)$ on a polar coordinate system

Solution



Example

If $(x, y) = (-1, \sqrt{3})$ are rectangular coordinates of a point P , find three different pairs the polar coordinates of P .

Solution

$$\begin{aligned}r &= \pm\sqrt{x^2 + y^2} \\&= \pm\sqrt{(-1)^2 + (\sqrt{3})^2} \\&= \pm\sqrt{1+3} \\&= \pm\sqrt{4} \\&= \pm 2\end{aligned}$$

$$\tan \theta = \frac{y}{x} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

$$\hat{\theta} = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$$\theta_1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\theta_2 = \frac{2\pi}{3} + 2\pi = \frac{8\pi}{3}$$

$$\theta_3 = -\frac{\pi}{3}$$

The polar coordinates of P are: $\left(2, \frac{2\pi}{3}\right)$, $\left(2, \frac{8\pi}{3}\right)$, and $\left(-2, -\frac{\pi}{3}\right)$

Example

Find a polar equation of an arbitrary line.

Solution

An equation of a line can be written in the form: $ax + by = c$.

$$ax + by = c$$

$$ar \cos \theta + br \sin \theta = c$$

$$r(a \cos \theta + b \sin \theta) = c$$

$$r = \frac{c}{a \cos \theta + b \sin \theta}$$

Example

Find a polar equation of the hyperbola $x^2 - y^2 = 16$.

Solution

$$(r \cos \theta)^2 - (r \sin \theta)^2 = 16$$

$$r^2 \cos^2 \theta - r^2 \sin^2 \theta = 16$$

$$r^2 (\cos^2 \theta - \sin^2 \theta) = 16$$

$$r^2 (\cos 2\theta) = 16$$

$$\boxed{r^2 = \frac{16}{\cos 2\theta}} \quad \cos 2\theta \neq 0$$

$$\text{or } r^2 = 16 \sec 2\theta$$

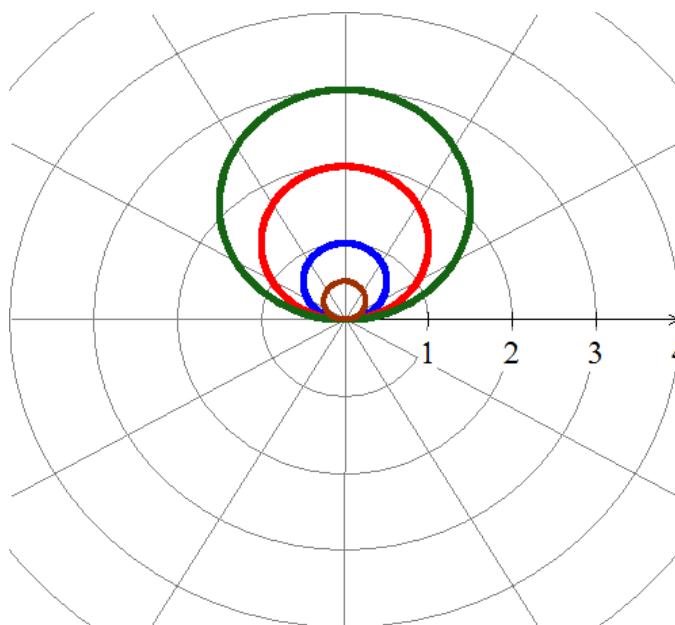
Example

Find an equation in x and y that has the same graph as the polar equation $r = a \sin \theta$, $a \neq 0$. Sketch the graph.

Solution

$$r^2 = ar \sin \theta$$

$$x^2 + y^2 = ay$$

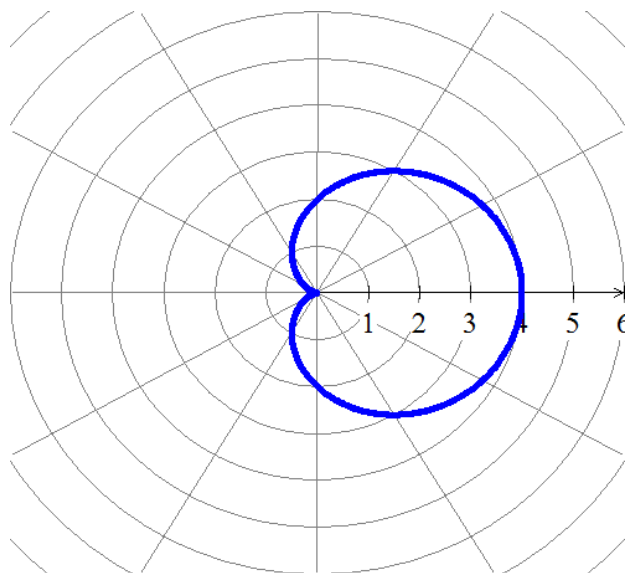


Example

Sketch the graph of the polar equation $r = 2 + 2 \cos \theta$.

Solution

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π
r	4	$2 + \sqrt{2}$	2	$2 - \sqrt{2}$	0	2	4



Exercises Section 3.6 – Polar Coordinates

Convert to rectangular coordinates

1. $(4, 30^\circ)$
2. $(-\sqrt{2}, \frac{3\pi}{4})$
3. $(3, 270^\circ)$
4. $(2, 60^\circ)$
5. $(\sqrt{2}, -225^\circ)$
6. $(4\sqrt{3}, -\frac{\pi}{6})$
7. Change the polar coordinates to rectangular coordinates $(-2, \frac{7\pi}{6})$
8. Change the polar coordinates to rectangular coordinates $(6, \arctan \frac{3}{4})$
9. Change the polar coordinates to rectangular coordinates $(10, \arccos(-\frac{1}{3}))$

Convert to polar coordinates

10. $(3, 3)$
11. $(-2, 0)$
12. $(-1, \sqrt{3})$
13. $(-3, -3) \quad r \geq 0 \quad 0^\circ \leq \theta < 360^\circ$
14. $(2, -2\sqrt{3}) \quad r \geq 0 \quad 0^\circ \leq \theta < 360^\circ$
15. $(-2, 0) \quad r \geq 0 \quad 0 \leq \theta < 2\pi$
16. $(-1, -\sqrt{3}) \quad r \geq 0 \quad 0 \leq \theta < 2\pi$
17. Change the rectangular coordinates to polar coordinates $(7, -7\sqrt{3}) \quad r > 0 \quad 0 \leq \theta < 2\pi$
18. Change the rectangular coordinates to polar coordinates $(-2\sqrt{2}, -2\sqrt{2}) \quad r > 0 \quad 0 \leq \theta < 2\pi$
19. The point $(0, -3)$ in rectangular coordinates is equivalent to $(3, 270^\circ)$ in polar coordinates.
20. The point $(1, -1)$ in rectangular coordinates is equivalent to $(-\sqrt{2}, \frac{3\pi}{4})$ in polar coordinates.
21. A point lies at $(4, 4)$ on a rectangular coordinate system. Give its address in polar coordinates (r, θ)

Write the equation in rectangular coordinates

22. $r^2 = 4$
23. $r = 6\cos\theta$
24. $r^2 = 4\cos 2\theta$
25. $r(\cos\theta - \sin\theta) = 2$
26. $r^2 = 4\sin 2\theta$
27. $r \sin\theta = -2$
28. $\theta = \frac{\pi}{4}$
29. $r^2(4\sin^2\theta - 9\cos^2\theta) = 36$
30. $r^2(\cos^2\theta + 4\sin^2\theta) = 16$
31. $r(\sin\theta - 2\cos\theta) = 6$
32. $r = 8\sin\theta - 2\cos\theta$
33. $r = \tan\theta$
34. $r(\sin\theta + r\cos^2\theta) = 1$

Find a polar equation that has the same graph as the equation in x and y

35. $y^2 = 6x$
36. $xy = 8$
37. $(x+2)^2 + (y-3)^2 = 13$
38. $y^2 - x^2 = 4$

Write the equation in polar coordinates

39. $x + y = 5$

41. $x^2 + y^2 = 4x$

43. $x + y = 4$

40. $x^2 + y^2 = 9$

42. $y = -x$

Sketch the graph of the polar equation

44. $r = 5$

48. $r = 2 - \cos \theta$

52. $r = e^{2\theta} \quad \theta \geq 0$

45. $\theta = \frac{\pi}{4}$

49. $r = 4 \csc \theta$

53. $r\theta = 1 \quad \theta > 0$

46. $r = 4 \cos \theta + 2 \sin \theta$

50. $r^2 = 4 \cos 2\theta$

54. $r = 2 + 2 \sec \theta$

47. $r = 2 + 4 \sin \theta$

51. $r = 2^\theta \quad \theta \geq 0$