Solution *Section* **1.3 – Volumes by Slicing**

Exercise

Find the volume of the solid formed by revolving the region about the x-axis : $\quad y = -x+1$

Solution

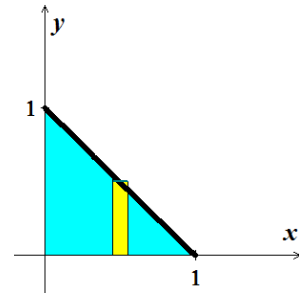$$V = \pi \int_0^1 (-x+1)^2 \, dx \qquad\qquad V = \pi \int_a^b \left(f(x)^2 - g(x)^2 \right) dx$$

$$= \pi \int_0^1 \left(x^2 - 2x + 1 \right) dx$$

$$= \pi \left(\frac{1}{3} x^3 - x^2 + x \right) \Big|_0^1$$

$$= \pi \left(\frac{1}{3} - 1 + 1 \right)$$

$$= \underline{\frac{\pi}{3}}$$

Exercise

Find the volume of the solid formed by revolving the region about the x-axis : $\quad y = 4 - x^2$

Solution
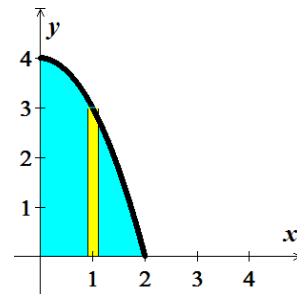
$$V = \pi \int_0^2 \left(4 - x^2 \right)^2 \, dx \qquad\qquad V = \pi \int_a^b \left(f(x)^2 - g(x)^2 \right) dx$$

$$= \pi \int_0^2 \left(16 - 8x^2 + x^4 \right) dx$$

$$= \pi \left(16x - \frac{8}{3} x^3 + \frac{1}{5} x^5 \right) \Big|_0^2$$

$$= \pi \left(32 - \frac{64}{3} + \frac{32}{5} \right)$$

$$= \underline{\frac{256\pi}{15}}$$
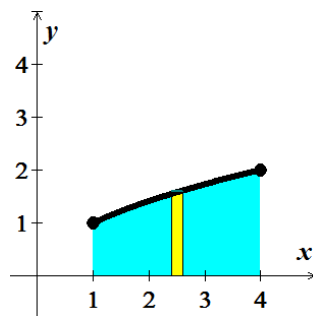
Exercise

Find the volume of the solid formed by revolving the region about the x-axis : $\quad y = \sqrt{x}$

Solution

$$V = \pi \int_1^4 \left(\sqrt{x} \right)^2 \, dx \qquad\qquad V = \pi \int_a^b \left(f(x)^2 - g(x)^2 \right) dx$$

$$= \pi \int_1^4 x\, dx$$

$$= \pi \left(\frac{1}{2} x^2 \right) \Big|_1^4$$

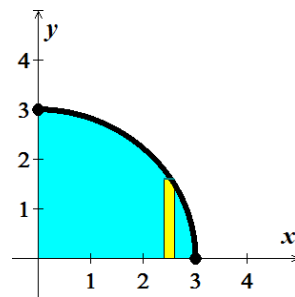$$= \pi \left(8 - \frac{1}{2} \right)$$

$$= \underline{\frac{15\pi}{2}}$$

Exercise

Find the volume of the solid formed by revolving the region about the *x-axis* : $y = \sqrt{9 - x^2}$

Solution

$$V = \pi \int_0^3 \left(\sqrt{9 - x^2} \right)^2 dx \qquad\qquad V = \pi \int_a^b \left(f(x)^2 - g(x)^2 \right) dx$$

$$= \pi \int_0^3 \left(9 - x^2 \right) dx$$

$$= \pi \left(9x - \frac{1}{3} x^3 \right) \Big|_0^3$$
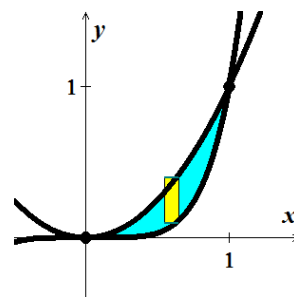
$$= \pi \left(27 - 9 \right)$$

$$= \underline{18\pi}$$

Exercise

Find the volume of the solid formed by revolving the region about the *x-axis* : $y = x^2, \quad y = x^5$

Solution

$$V = \pi \int_0^1 \left(x^4 - x^{10} \right) dx \qquad\qquad V = \pi \int_a^b \left(f(x)^2 - g(x)^2 \right) dx$$

$$= \pi \left(\frac{1}{5} x^5 - \frac{1}{11} x^{11} \right) \Big|_0^1$$

$$= \pi \left(\frac{1}{5} - \frac{1}{11} \right)$$

$$= \underline{\frac{6\pi}{55}}$$

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Exercise

Find the volume of the solid formed by revolving the region about the x-axis : $\quad y = 2, \quad y = 4 - \dfrac{x^2}{4}$

Solution

$$y = 4 - \frac{x^2}{4} = 2 \Rightarrow \frac{x^2}{4} = 2$$

$$x^2 = 8 \;\rightarrow\; \underline{x = \pm 2\sqrt{2}}$$

$$V = \pi \int_{-2\sqrt{2}}^{2\sqrt{2}} \left(\left(4 - \frac{x^2}{4} \right)^2 - 4 \right) dx \qquad V = \pi \int_a^b \left(f(x)^2 - g(x)^2 \right) dx$$

$$= 2\pi \int_0^{2\sqrt{2}} \left(16 - 2x^2 + \frac{1}{16}x^4 - 4 \right) dx$$

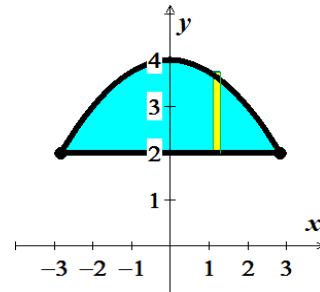$$= 2\pi \int_0^{2\sqrt{2}} \left(12 - 2x^2 + \frac{1}{16}x^4 \right) dx$$

$$= 2\pi \left(12x - \frac{2}{3}x^3 + \frac{1}{80}x^5 \right) \Big|_0^{2\sqrt{2}}$$

$$= 2\pi \left(24\sqrt{2} - \frac{32}{3}\sqrt{2} + \frac{128}{80}\sqrt{2} \right)$$

$$= 16\pi\sqrt{2} \left(3 - \frac{4}{3} + \frac{16}{80} \right)$$

$$= 16\pi\sqrt{2} \left(\frac{448}{240} \right)$$

$$= \underline{\frac{448\sqrt{2}}{15}\pi}$$

Exercise

Find the volume of the solid formed by revolving the region about the y-axis : $\quad y = x^2$

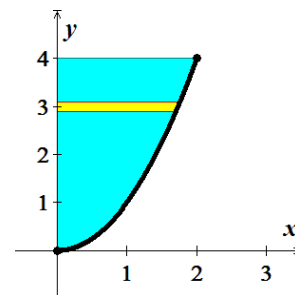### Solution

$$y = x^2 \;\rightarrow\; x = \sqrt{y}$$

$$V = \pi \int_0^4 \left(\sqrt{y} \right)^2 dy \qquad V = \pi \int_c^d \left(p(y)^2 - q(y)^2 \right) dy$$

$$= \pi \int_0^4 y\, dy$$

$$= \pi \left(\frac{1}{2}y^2 \right) \Big|_0^4$$

$$= \underline{8\pi}$$

Exercise

Find the volume of the solid formed by revolving the region about the y-axis : $\quad y = \sqrt{16 - x^2}$

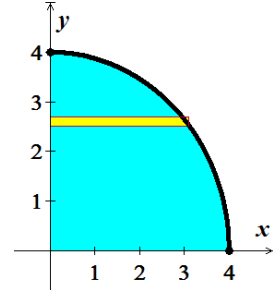$$y = \sqrt{16 - x^2} \;\rightarrow\; x = \sqrt{16 - y^2}$$

$$V = \pi \int_0^4 \left(16 - y^2\right) dy \qquad\qquad V = \pi \int_c^d \left(p(y)^2 - q(y)^2\right) dy$$

$$= \pi \left(16y - \frac{1}{3}y^3\right)\Big|_0^4$$

$$= \pi \left(64 - \frac{64}{3}\right)$$

$$= \underline{\frac{128\pi}{3}}$$

Exercise

Find the volume of the solid formed by revolving the region about the y-axis : $\quad y = x^{2/3}$
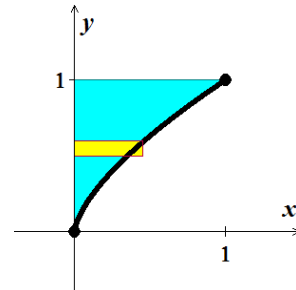
$$y = x^{2/3} \;\rightarrow\; x = y^{3/2}$$

$$V = \pi \int_0^1 \left(y^{3/2}\right)^2 dy \qquad\qquad V = \pi \int_c^d \left(p(y)^2 - q(y)^2\right) dy$$

$$= \pi \int_0^1 y^3 \, dy$$

$$= \pi \left(\frac{1}{4}y^4\right)\Big|_0^1$$

$$= \underline{\frac{\pi}{4}}$$

Exercise

Find the volume of the solid formed by revolving the region about the y-axis : $\quad x = -y^2 + 4y$

$$V = \pi \int_1^4 \left(-y^2 + 4y\right)^2 dy \qquad\qquad V = \pi \int_c^d \left(p(y)^2 - q(y)^2\right) dy$$

$$= \pi \int_1^4 \left(y^4 - 8y^3 + 16y^2\right) dy$$

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$$= \pi \left(\frac{1}{5}y^5 - 2y^4 + \frac{16}{3}y^3 \right) \Big|_1^4$$

$$= \pi \left(\frac{1024}{5} - 512 + \frac{1024}{3} - \frac{1}{5} + 2 - \frac{16}{3} \right)$$

$$= \pi \left(\frac{1023}{5} + \frac{1008}{3} - 510 \right)$$

$$= \pi \left(\frac{459}{15} \right)$$

$$= \frac{153\pi}{5}$$
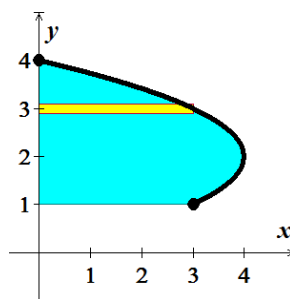
Exercise

Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $y = \sqrt{x}, \quad y = 0, \quad x = 3$

    a) the *x-axis*   *b)* the *y-axis*   *c)* the line $x = 3$ *d)* the line $x = 6$
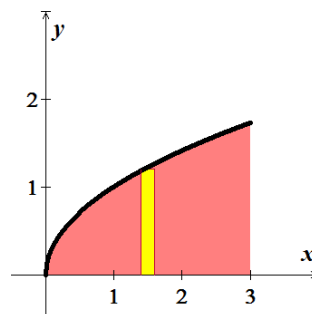
Solution

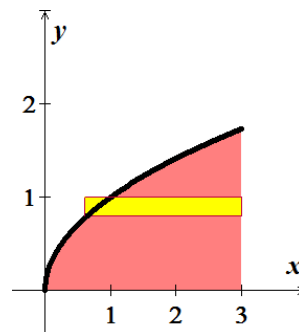*a)* $\sqrt{x} = 0 \rightarrow x = 0$

$$V = \pi \int_0^3 x\, dx \qquad V = \pi \int_a^b f(x)^2\, dx$$

$$= \pi \frac{1}{2}x^2 \Big|_0^3$$

$$= \frac{9\pi}{2}$$

b) $y = \sqrt{x} = \sqrt{3} \quad x = y^2$

$$V = \pi \int_0^{\sqrt{3}} \left(9 - y^4 \right) dy \qquad V = \pi \int_c^d \left(p(y)^2 - q(y)^2 \right) dy$$

$$= \pi \left(9x - \frac{1}{5}x^5 \right) \Big|_0^{\sqrt{3}}$$

$$= \pi \left(9\sqrt{3} - \frac{9}{5}\sqrt{3} \right)$$

$$= \frac{36\pi\sqrt{3}}{5}$$
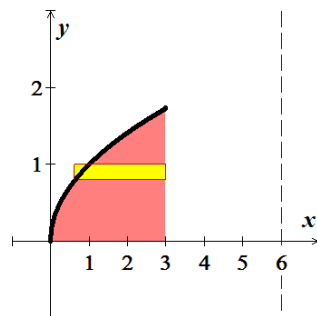

c) $V = \pi \int_0^{\sqrt{3}} \left(3 - y^2 \right)^2 dy$

$$= \pi \int_0^{\sqrt{3}} \left(9 - 6y^2 + y^4 \right) dy$$

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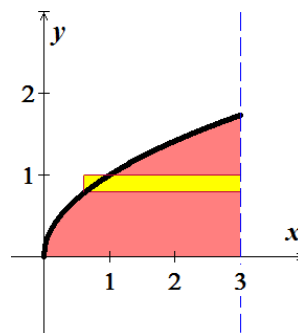
$$= \pi \left(9y - 2y^3 + \frac{1}{5}y^5\right)\Big|_0^{\sqrt{3}}$$

$$= \pi \left(9\sqrt{3} - 6\sqrt{3} + \frac{9}{5}\sqrt{3}\right)$$

$$= \underline{\underline{\frac{24\sqrt{3}}{5}\pi}}$$

d) $R(y) = 3 + \left(3 - y^2\right) = 6 - y^2 \quad r(y) = 3$

$$V = \pi \int_0^{\sqrt{3}} \left(\left(6 - y^2\right)^2 - 9\right) dy \qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2\right) dy$$

$$= \pi \int_0^{\sqrt{3}} \left(36 - 12y^2 + y^4 - 9\right) dy$$

$$= \pi \left(27y - 4y^3 + \frac{1}{5}y^5\right)\Big|_0^{\sqrt{3}}$$

$$= \pi \left(27\sqrt{3} - 12\sqrt{3} + \frac{9}{5}\sqrt{3}\right)$$

$$= \underline{\underline{\frac{84\sqrt{3}}{5}\pi}}$$

Exercise

Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $y = 2x^2, \quad y = 0, \quad x = 2$

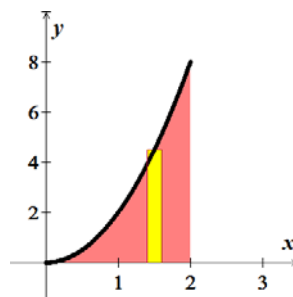 *a)* the *x-axis*   *b)* the *y-axis*   *c)* the line $y = 8$ *d)* the line $x = 2$

Solution

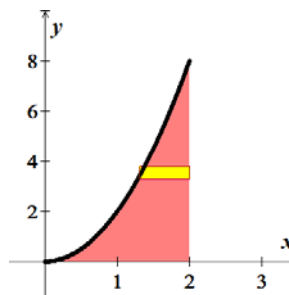**a)** $V = \pi \int_0^2 4x^4 \, dx \qquad V = \pi \int_a^b f(x)^2 \, dx$

$$= \frac{4\pi}{5}x^5 \Big|_0^2$$

$$= \underline{\underline{\frac{128\pi}{5}}}$$

b) $R(y) = 2 \quad x^2 = \frac{y}{2} \rightarrow r(y) = \sqrt{\frac{y}{2}}$
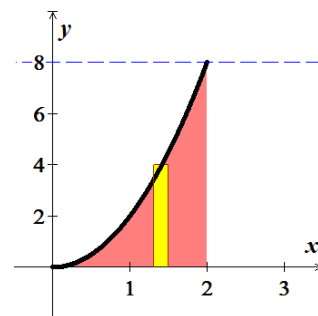
$x = 2 \rightarrow \underline{y = 2(2)^2 = 8}$

$$V = \pi \int_0^8 \left(4 - \frac{1}{2}y\right) dy \qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2\right) dy$$

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$$= \pi \left(4y - \frac{1}{4} y^2 \right) \Big|_0^8$$

$$= \pi (32 - 16)$$

$$\underline{= 16\pi}$$

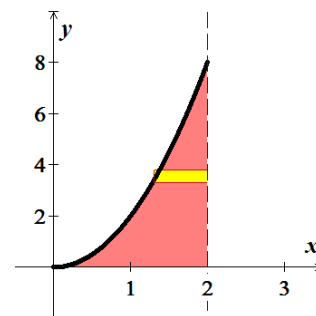**c)** $R(x) = 8 \quad r(x) = 8 - 2x^2$

$$V = \pi \int_0^2 \left(64 - \left(64 - 32x^2 + 4x^4 \right) \right) dx \qquad V = \pi \int_a^b \left(R(x)^2 - r(x)^2 \right) dx$$

$$= 4\pi \int_0^2 \left(8x^2 - x^4 \right) dx$$

$$= 4\pi \left(\frac{8}{3} x^3 - \frac{1}{5} x^5 \right) \Big|_0^2$$

$$= 4\pi \left(\frac{64}{3} - \frac{32}{5} \right)$$

$$\underline{= \frac{896\pi}{15}}$$

d) $R(y) = 2 - \sqrt{\frac{1}{2} y} \quad r(y) = 0$

$$V = \pi \int_0^8 \left(4 - 4\sqrt{\frac{1}{2}} y^{1/2} + \frac{1}{2} y \right) dy \qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2 \right) dy$$

$$= \pi \left(4y - \frac{4\sqrt{2}}{3} y^{3/2} + \frac{1}{4} y^2 \right) \Big|_0^8$$

$$= \pi \left(32 - \frac{4\sqrt{2}}{3} \left(2^{9/2} \right) + 16 \right)$$

$$= \pi \left(32 - \frac{128}{3} + 16 \right)$$

$$\underline{= \frac{16\pi}{3}}$$

Exercise

Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $y = x^2, \quad y = 4x - x^2$

    a)   the *x-axis*   *b)*   the line $y = 6$

Solution

$$x^2 = 4x - x^2 \;\rightarrow\; 2x^2 - 4x = 0 \quad \underline{x = 0, \; 2}$$

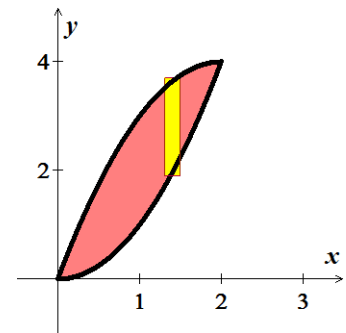*a)* $R(x) = 4x - x^2 \quad r(x) = x^2$

$$V = \pi \int_0^2 \left(16x^2 - 8x^3 + x^4 - x^4\right) dx \qquad V = \pi \int_a^b \left(R(x)^2 - r(x)^2\right) dx$$

$$= \pi \int_0^2 \left(16x^2 - 8x^3\right) dx$$

$$= \pi \left(\frac{16}{3}x^3 - 2x^4\right)\Big|_0^2$$

$$= \pi \left(\frac{128}{3} - 32\right)$$

$$= \underline{\frac{32\pi}{3}}$$

b) $R(x) = 6 - x^2 \quad r(x) = 6 - 4x + x^2$

$$V = \pi \int_0^2 \left(36 - 12x^2 + x^4 - \left(36 - 48x + 28x^2 - 8x^3 + x^4\right)\right) dx$$

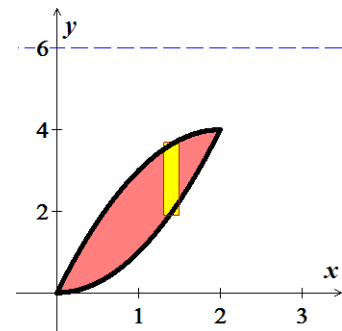$$= \pi \int_0^2 \left(48x - 40x^2 + 8x^3\right) dx$$

$$= 8\pi \int_0^2 \left(6x - 5x^2 + x^3\right) dx$$

$$= 8\pi \left(3x^2 - \frac{5}{3}x^3 + \frac{1}{4}x^4\right)\Big|_0^2$$

$$= 8\pi \left(12 - \frac{40}{3} + 4\right)$$

$$= 8\pi \left(\frac{8}{3}\right)$$

$$= \underline{\frac{64\pi}{3}}$$

Exercise

Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $y = -x^2 + 2x + 4, \quad y = 4 - x$
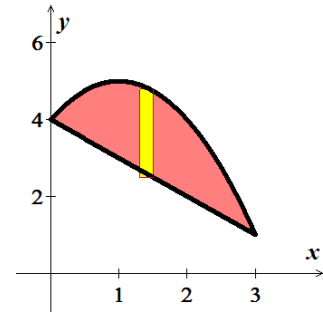
   a) the *x-axis*   *b)* the line $y = 1$

Solution

$$y = -x^2 + 2x + 4 = 4 - x \implies -x^2 + 3x = 0 \quad \rightarrow \quad \underline{x = 0,\ 3}$$

a) $R(x) = -x^2 + 2x + 4 \quad r(x) = 4 - x$

$$V = \pi \int_0^3 \left(x^4 - 4x^3 - 4x^2 + 16x + 16 - \left(16 - 8x + x^2 \right) \right) dx$$

$$= \pi \int_0^3 \left(x^4 - 4x^3 - 5x^2 + 24x \right) dx$$

$$= \pi \left(\tfrac{1}{5}x^5 - x^4 - \tfrac{5}{3}x^3 + 12x^2 \right) \Big|_0^3$$

$$= \pi \left(\tfrac{243}{5} - 81 - 45 + 108 \right)$$

$$= \underline{\dfrac{153\pi}{5}}$$

b) $R(x) = \left(-x^2 + 2x + 4 \right) - 1 \quad r(x) = (4 - x) - 1$

$$V = \pi \int_0^3 \left(\left(-x^2 + 2x + 3 \right)^2 - (3 - x)^2 \right) dx$$

$$= \pi \int_0^3 \left(x^4 - 4x^3 - 2x^2 + 12x + 9 - 9 + 6x - x^2 \right) dx$$

$$= \pi \int_0^3 \left(x^4 - 4x^3 - 3x^2 + 18x \right) dx$$

$$= \pi \left(\tfrac{1}{5}x^5 - x^4 - x^3 + 9x^2 \right) \Big|_0^3$$

$$= \pi \left(\tfrac{243}{5} - 81 - 27 + 81 \right)$$
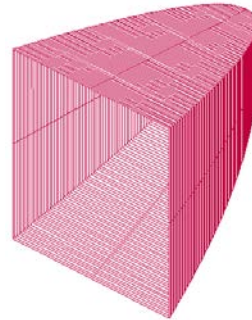
$$= \underline{\dfrac{108\pi}{5}}$$
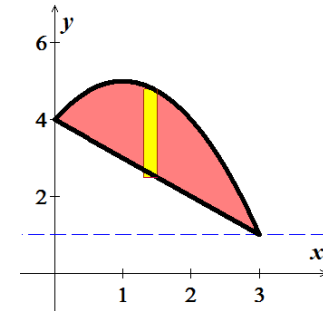
Exercise

The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$. Find the volume of the solid.

Solution

$$A(x) = \tfrac{1}{2}(diagonal)^2$$

$$= \tfrac{1}{2}\left(\sqrt{x} - \left(-\sqrt{x}\right)\right)^2$$

$$= \tfrac{1}{2}\left(2\sqrt{x}\right)^2$$

$$= \tfrac{1}{2}(4x)$$

$$\underline{= 2x \ unit^2} \qquad a = 0, \quad b = 4;$$

$$V = \int_a^b A(x)\,dx$$

$$= \int_0^4 2x\,dx$$

$$= \left[x^2\right]_0^4$$

$$= 4^2 - 0$$

$$\underline{= 16 \ unit^3}$$

Exercise

The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are circular whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$. Find the volume of the solid.

Solution

$$y = 2 - x^2 = x^2 \quad \Rightarrow \quad 2x^2 = 2 \to x^2 = 1 \ \Rightarrow \boxed{x = \pm 1}$$

$$A(x) = \tfrac{\pi}{4}(diameter)^2 = \tfrac{\pi}{4}\left(2 - x^2 - x^2\right)^2$$

$$= \tfrac{\pi}{4}\left(2\left(1 - x^2\right)\right)^2$$

$$= \tfrac{\pi}{4}4\left(1 - 2x^2 + x^4\right)$$

$$\underline{= \pi\left(1 - 2x^2 + x^4\right) \ unit^2} \qquad a = -1, \quad b = 1;$$

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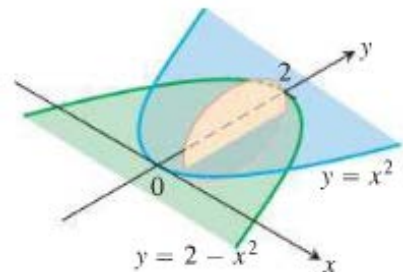
$$V = \int_a^b A(x)\,dx$$

$$= \int_{-1}^1 \pi\left(1 - 2x^2 + x^4\right)dx$$

$$= \pi\left[x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\right]_{-1}^1 \qquad\qquad = \pi\left[\left(1 - \tfrac{2}{3}(1)^3 + \tfrac{1}{5}(1)^5\right) - \left(-1 - \tfrac{2}{3}(-1)^3 + \tfrac{1}{5}(-1)^5\right)\right]$$

$$= \pi\left[\left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right) - \left(-1 + \tfrac{2}{3} - \tfrac{1}{5}\right)\right]$$

$$= 2\pi\left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right)$$

$$= \underline{\frac{16\pi}{15}\ unit^3}$$

Exercise

The solid lies between planes perpendicular to the *x*-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the axis between these planes are squares whose bases run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$. Find the volume of the solid.

Solution

$$A(x) = \tfrac{1}{2}(diagonal)^2$$

$$= \tfrac{1}{2}\left(\sqrt{1 - x^2} - \left(-\sqrt{1 - x^2}\right)\right)^2$$

$$= \tfrac{1}{2}\left(2\sqrt{1 - x^2}\right)^2$$

$$= \underline{2\left(1 - x^2\right)\ unit^2} \qquad a = -1,\quad b = 1;$$

$$V = \int_a^b A(x)\,dx$$

$$= \int_{-1}^1 2\left(1 - x^2\right)dx$$

$$= 2\left[x - \tfrac{1}{3}x^3\right]_{-1}^1$$

$$= 2\left[\left(1 - \tfrac{1}{3}1^3\right) - \left((-1) - \tfrac{1}{3}(-1)^3\right)\right] \qquad = 4\left(1 - \tfrac{1}{3}\right)$$

$$= \underline{\tfrac{8}{3}\ unit^3}$$

Exercise

The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis. The cross-sections perpendicular to the x-axis are
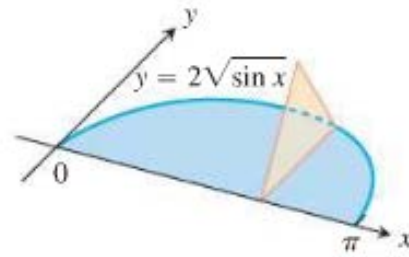Find the volume of the solid.

   a) Equilateral triangles with bases running from the x-axis to the curve as shown
   b) Squares with bases running from the x-axis to the curve.

Solution

  a) $A(x) = \frac{1}{2}(side)(side) \cdot \sin\frac{\pi}{3}$ *Equilateral triangle* $\theta = \frac{\pi}{3}$

$$= \frac{1}{2}\left(2\sqrt{\sin x}\right)\left(2\sqrt{\sin x}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$\underline{= \sqrt{3}\sin x \ \ unit^2} \qquad a = 0, \quad b = \pi;$$

$$V = \sqrt{3}\int_0^\pi \sin x\, dx$$

$$= \sqrt{3}\left[-\cos x\right]_0^\pi$$

$$= -\sqrt{3}\left[\cos\pi - \cos 0\right]$$

$$\underline{= 2\sqrt{3} \ \ unit^3}$$

  b) $A(x) = (side)^2 = \left(2\sqrt{\sin x}\right)^2 = \underline{4\sin x \ \ unit^2} \qquad a = 0, \quad b = \pi;$

$$V = 4\int_0^\pi \sin x\, dx$$

$$= 4\left[-\cos x\right]_0^\pi$$

$$= -4\left[\cos\pi - \cos 0\right]$$

$$\underline{= 8 \ \ unit^3}$$

Exercise

The base of the solid is the disk $x^2 + y^2 \leq 1$. The cross-sections by planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.

Solution

$$x^2 + y^2 = 1 \ \rightarrow \ x^2 = 1 - y^2 \Rightarrow \boxed{x = \pm\sqrt{1 - y^2}}$$

$$A(y) = \frac{1}{2}(leg)(leg) = \frac{1}{2}\left[\sqrt{1 - y^2} - \left(-\sqrt{1 - y^2}\right)\right]^2$$

<div align="center">48</div>

$$= \frac{1}{2}\left[2\sqrt{1-y^2}\right]^2$$

$$\underline{= 2\left(1-y^2\right) \; unit^2} \qquad c = -1, \quad d = 1;$$

$$V = \int_{-1}^{1} 2\left(1-y^2\right) dy \qquad\qquad V = \int_{c}^{d} A(y)\,dy$$

$$= 2\left[y - \frac{1}{3}y^3\right]_{-1}^{1}$$

$$= 2\left[\left(1 - \frac{1}{3}1^3\right) - \left((-1) - \frac{1}{3}(-1)^3\right)\right]$$

$$= 4\left(1 - \frac{1}{3}\right)$$

$$\underline{= \frac{8}{3} \; unit^3}$$

$x^2 + y^2 = 1$
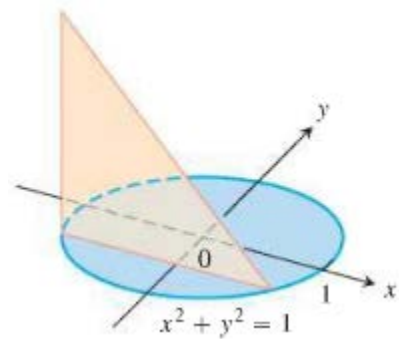
Exercise

Find the volume of the given tetrahedron. (***Hint***: Consider slices perpendicular to one of the labeled edges)

Solution

Let consider the slices perpendicular to edge labeled 5 are triangles.

By similar triangles, we have: $\dfrac{height}{base} = \dfrac{h}{b} = \dfrac{3}{4} \quad \Rightarrow \quad h = \dfrac{3}{4}b$

The equation of the line through $(5, 0)$ and $(0, 4)$ is: $\quad y = \dfrac{4-0}{0-5}(x-0) + 4 \quad \rightarrow \quad y = -\dfrac{4}{5}x + 4$
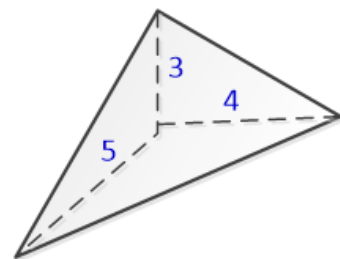
Therefore, the length of the base: $b = -\dfrac{4}{5}x + 4$

$$h = \frac{3}{4}b = \frac{3}{4}\left(-\frac{4}{5}x + 4\right) = -\frac{3}{5}x + 3$$

$$A(x) = \frac{1}{2}(base)\cdot(height) = \frac{1}{2}\left(-\frac{4}{5}x + 4\right)\left(-\frac{3}{5}x + 3\right)$$

$$= \frac{1}{2}\left(\frac{12}{25}x^2 - \frac{24}{5}x + 12\right)$$

$$= \frac{6}{25}x^2 - \frac{12}{5}x + 5 \quad unit^2$$

$$V = \int_{0}^{5}\left(\frac{6}{25}x^2 - \frac{12}{5}x + 5\right)dx \qquad\qquad V = \int_{a}^{b} A(x)\,dx$$

$$= \left[\frac{2}{25}x^3 - \frac{6}{5}x^2 + 5x\right]_{0}^{5}$$

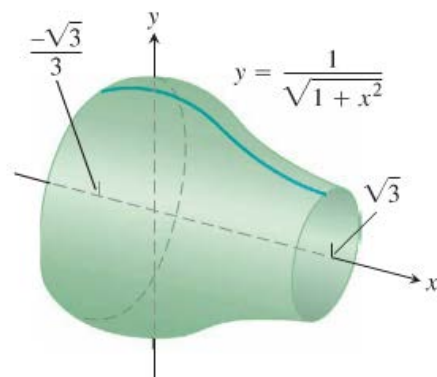$$= \left[\frac{2}{25}(5)^3 - \frac{6}{5}(5)^2 + 5(5)\right]$$

$$\underline{= 10 \; unit^3}$$

Exercise

Find the volume of the solid of revolution

Solution

$$V = \pi \int_{-\sqrt{3}/3}^{\sqrt{3}} \left(\frac{1}{\sqrt{1+x^2}} \right)^2 dx$$

$$= \pi \int_{-\sqrt{3}/3}^{\sqrt{3}} \frac{1}{1+x^2} dx$$

$$= \pi \left[\tan^{-1} x \right]_{-\sqrt{3}/3}^{\sqrt{3}}$$

$$= \pi \left(\tan^{-1} \sqrt{3} - \tan^{-1}\left(-\frac{\sqrt{3}}{3} \right) \right)$$

$$= \pi \left(\frac{\pi}{3} - \left(-\frac{\pi}{6} \right) \right)$$

$$= \pi \left(\frac{\pi}{3} + \frac{\pi}{6} \right)$$

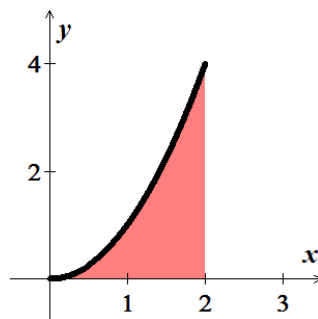$$= \pi \left(\frac{\pi}{2} \right)$$

$$= \frac{\pi^2}{2} \ unit^3$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y = x^2$ and the lines $y = 0$, $x = 2$ about the x-axis.

Solution

$$R(x) = x^2$$

$$V = \int_0^2 \pi \left[R(x) \right]^2 dx$$

$$= \pi \int_0^2 \left(x^2 \right)^2 dx$$

$$= \pi \int_0^2 x^4 dx$$

$$= \pi \left[\tfrac{1}{5}x^5 \right]_0^2$$

$$= \pi \left(\tfrac{1}{5}(2)^5 - 0 \right)$$
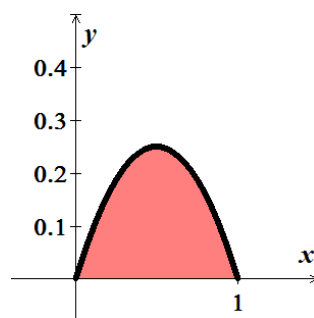
$$= \underline{\frac{32\pi}{5} \ unit^3}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y = x - x^2$ and the line $y = 0$ about the *x*-axis.

Solution

$R(x) = x - x^2$ $x - x^2 = 0 \rightarrow x = 0, \ 1$

$$V = \int_0^1 \pi \left[x - x^2 \right]^2 dx$$

$$= \pi \int_0^1 \left(x^2 - 2x^3 + x^4 \right) dx$$

$$= \pi \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^4 + \tfrac{1}{5}x^5 \right]_0^1$$

$$= \pi \left(\tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} \right)$$
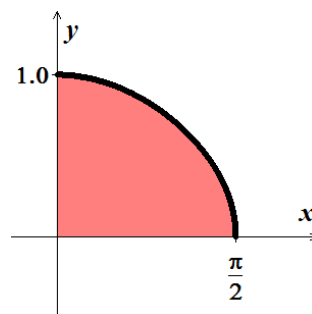
$$= \underline{\frac{\pi}{30} \ unit^3}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{\cos x}$ and the lines $0 \le x \le \frac{\pi}{2}, \ \ y = 0, \ \ x = 0$ about the *x*-axis.

Solution

$R(x) = \sqrt{\cos x}$

$$V = \int_0^{\pi/2} \pi \left[\sqrt{\cos x} \right]^2 dx$$

$$= \pi \int_0^{\pi/2} \cos x \, dx$$

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$$= \pi \left[\sin x \right]_{0}^{\pi/2}$$

$$= \pi \left(1 - 0 \right)$$
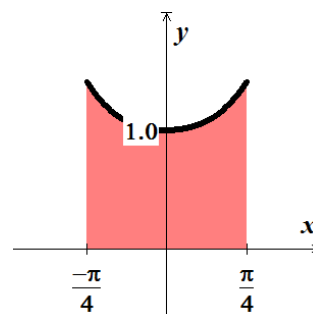
$$\underline{= \pi \ unit^{3}}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y = \sec x$ and the lines

$y = 0, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$ about the x-axis.

Solution

$$R(x) = \sec x$$

$$V = \pi \int_{-\pi/4}^{\pi/4} \sec^{2} x \, dx$$

$$= \pi \left[\tan x \right]_{-\pi/4}^{\pi/4}$$

$$= \pi \left(\tan\left(\frac{\pi}{4} \right) - \tan\left(-\frac{\pi}{4} \right) \right)$$

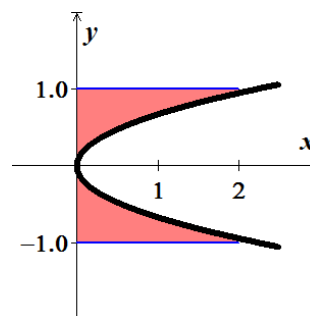$$= \pi \left(1 - (-1) \right)$$

$$\underline{= 2\pi \ unit^{3}}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $x = \sqrt{5} \, y^{2}$ and the lines

$x = 0, \quad y = -1, \quad y = 1$ about the y-axis.

Solution

$$R(y) = \sqrt{5}y^{2}$$

$$V = \pi \int_{-1}^{1} \left[\sqrt{5}y^{2} \right]^{2} dy$$

$$= \pi \int_{-1}^{1} 5y^{4} dy$$

$$= \pi \left(y^{5} \right)_{-1}^{1}$$

$$= \pi \left(1 - (-1) \right)$$

$$\underline{= 2\pi \ unit^{3}}$$

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Exercise

Find the volume of the solid generated by revolving the region bounded by $y = 2\sqrt{x}$ and the lines $y = 2, \quad x = 0$ about the x-axis.

Solution

$r(x) = 2\sqrt{x} \quad and \quad R(x) = 2$

$$V = \pi \int_0^1 \left((2)^2 - \left(2\sqrt{x}\right)^2 \right) dx \qquad V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2 \right) dx$$

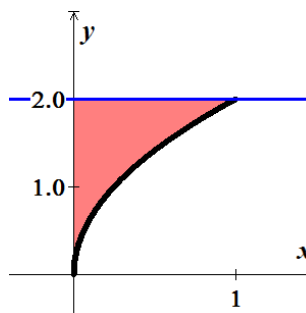$$= \pi \int_0^1 (4 - 4x) dx$$

$$= 4\pi \left(x - \frac{1}{2}x^2 \right) \Big|_0^1$$

$$= 4\pi \left[\left(1 - \frac{1}{2}\right) - 0 \right]$$

$$\underline{= 2\pi \ unit^3}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y = \sec x, \quad y = \tan x$ and the lines $x = 0, \quad x = 1$ about the x-axis.

Solution

$r(x) = \tan x \quad and \quad R(x) = \sec x$

$$V = \pi \int_0^1 \left(\sec^2 x - \tan^2 x \right) dx \qquad V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2 \right) dx$$

$$= \pi \int_0^1 (1) dx$$

$$= \pi (x) \Big|_0^1$$

$$\underline{= \pi \ unit^3}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $x = \sqrt{2\sin 2y}$ and the lines $0 \le y \le \frac{\pi}{2}, \quad x = 0$ about the y-axis.
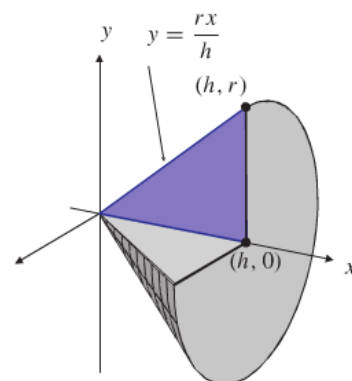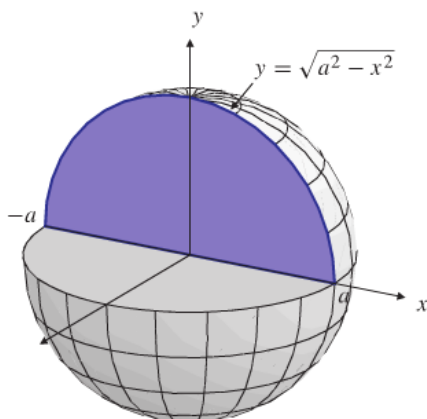
Solution

53

$$V = \pi \int_0^{\pi/2} \left(\sqrt{2\sin 2y}\right)^2 dy$$

$$= 2\pi \int_0^{\pi/2} (\sin 2y) \, dy$$

$$= -\pi \cos 2y \Big|_0^{\pi/2}$$

$$= -\pi(-1-1)$$

$$\underline{= 2\pi \ unit^3}$$

Exercise

Find the volume of a solid ball having radius a.

Solution

$$V = \pi \int_{-a}^{a} \left(\sqrt{a^2 - x^2}\right)^2 dx$$

$$= 2\pi \int_0^a \left(a^2 - x^2\right) dx$$

$$= 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= 2\pi \left(a^3 - \frac{a^3}{3} \right)$$

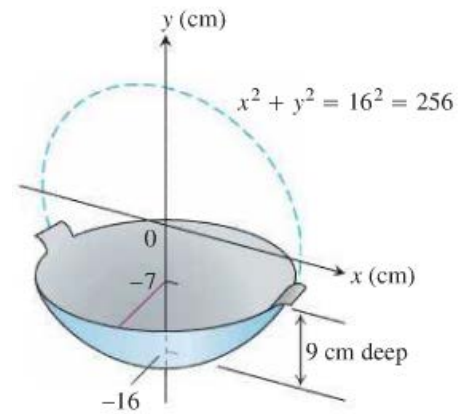$$\underline{= \frac{4}{3}\pi a^3 \ unit^3}$$

Exercise

You are designing a wok frying pan that will be shaped like a spherical bowl with handles. A bit of experimentation at home persuades you that you can get one that hold about 3 L if you make it 9 cm deep and give the sphere a radius of 16 cm. To be sure, you picture the wok as a solid of revolution, and calculate its volume with an integral. To the nearest cubic centimeter, what volume do you really get? $\left(1\,L = 1,000\ cm^3\right)$

Solution

$$x^2 + y^2 = 256 \quad \Rightarrow \quad x^2 = 256 - y^2$$

$$R(y) = \sqrt{256 - y^2}$$

$$V = \pi \int_{-16}^{-7} \left[\sqrt{256 - y^2} \right]^2 dy$$

$$= \pi \int_{-16}^{-7} \left(256 - y^2 \right) dy$$

$$= \pi \left[256y - \frac{1}{3} y^3 \right]_{-16}^{-7}$$

$$= \pi \left[\left(256(-7) - \frac{1}{3}(-7)^3 \right) - \left(256(-16) - \frac{1}{3}(-16)^3 \right) \right]$$

$$\underline{= 1053\pi \ cm^3} \approx 3308 \ cm^3$$

y (cm)

$x^2 + y^2 = 16^2 = 256$

0

−7

x (cm)

9 cm deep

−16

Exercise

A curved wedge is cut from a circular cylinder of radius 3 by two planes. On plane is perpendicular to the axis of the cylinder. The second place crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

Solution

The base of the cylinder is a circle $x^2 + y^2 = 9$.

Since the second plane cut the base at the cylinder at the center, therefore, the base of the wedge is
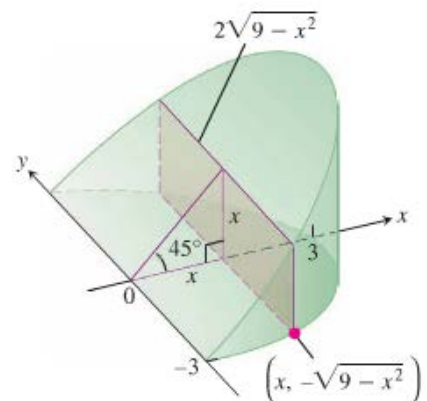
semi-circle. $y = \pm\sqrt{9 - x^2} = radius$.

When we slice the wedge by a plane perpendicular to the axis of the cylinder, we obtained a cross-section at x which is a rectangle of height x.

The area of this cross-section is: $A(x) = height \times width$

$$= x \left(2\sqrt{9 - x^2} \right)$$

$$= 2x\sqrt{9 - x^2}$$

The rectangles run from $x = 0$ to $x = 3$, so

$$V = \int_0^3 2x\sqrt{9 - x^2} dx \qquad or \ u = 9 - x^2 \ \rightarrow \ du = -2xdx$$

$$= -\int_0^3 \left(9 - x^2 \right)^{1/2} d\left(9 - x^2 \right)$$

$$= -\frac{2}{3} \left[\left(9 - x^2 \right)^{3/2} \right]_0^3$$

$$= -\frac{2}{3} \left[\left(9 - 3^2 \right)^{3/2} - \left(9 - 0^2 \right)^{3/2} \right]$$

$$\underline{= 18 \ unit^3}$$

$2\sqrt{9 - x^2}$

y

x
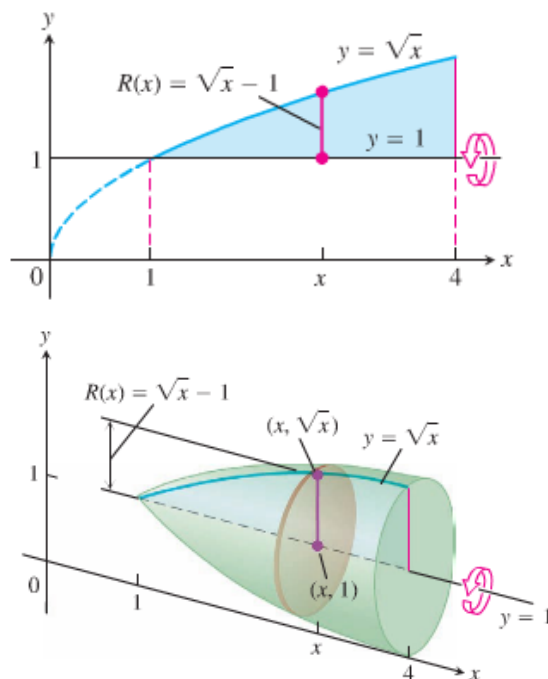
45°

x

3

0

−3

$\left(x, -\sqrt{9 - x^2} \right)$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

Solution

$$V = \int_1^4 \pi \left[R(x)\right]^2 dx$$

$$= \pi \int_1^4 \left[\sqrt{x} - 1\right]^2 dx$$

$$= \pi \int_1^4 \left(x - 2\sqrt{x} + 1\right) dx$$

$$= \pi \left[\frac{x^2}{2} - 2\frac{2}{3}x^{3/2} + x\right]_1^4$$

$$= \pi \left[\left(\frac{4^2}{2} - \frac{4}{3}4^{3/2} + 4\right) - \left(\frac{1^2}{2} - \frac{4}{3}1^{3/2} + 1\right)\right]$$
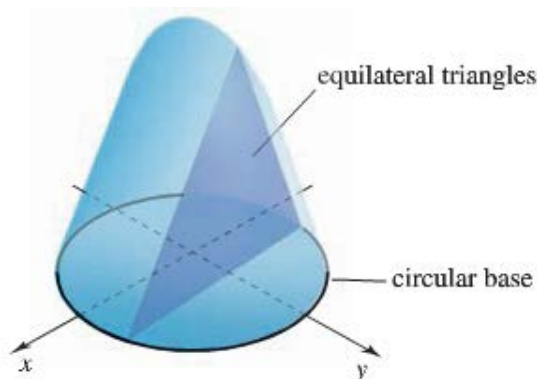
$$= \frac{7\pi}{6} \ unit^3$$

Exercise

The solid with a circular base of radius 5 whose cross sections perpendicular to the base and parallel to the x-axis are equilateral triangles. Use the general slicing method to find the volume of the solid.

Solution

$$x^2 + y^2 = 5^2 \quad \rightarrow \quad y = \pm\sqrt{25 - x^2}$$

$$A(x) = \left(2\sqrt{25 - x^2}\right)^2 = 100 - 4x^2$$

$$V = \int_0^5 \left(100 - 4x^2\right) dx$$

$$= \left(100x - \frac{4}{3}x^3\right)\Big|_0^5$$

$$= 500 - \frac{500}{3}$$

$$= \frac{1000}{3} \ unit^3$$

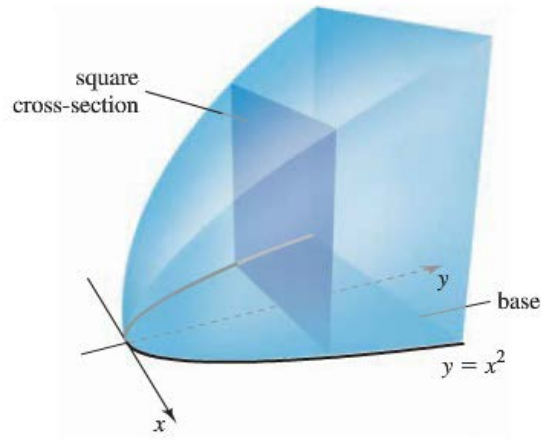equilateral triangles

circular base

Exercise

The solid whose base is the region bounded by $y = x^2$ and the line $y = 1$ and whose cross sections perpendicular to the base and parallel to the *x*-axis squares. Use the general slicing method to find the volume of the solid.

Solution

$$y = x^2 \quad \Rightarrow \quad x = \sqrt{y}$$

$$V = \int_0^1 A(y)\,dy$$

$$= \int_0^1 \left(2\sqrt{y}\right)^2 dy$$

$$= \int_0^1 (4y)\,dy$$

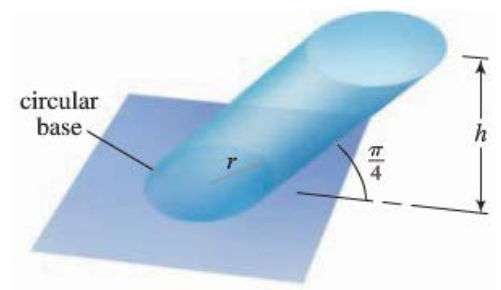$$= 2y^2 \Big|_0^1$$

$$= 2 \ unit^3$$

Exercise

A circular cylinder of radius *r* and height *h* whose curved surface is at an angle of $\frac{\pi}{4} \ rad$. Use the general slicing method to find the volume of the solid

Solution

The cross sections are all circles with area πr^2

$$V = \int_0^h \left(\pi r^2\right) dz$$

$$= \pi r^2 z \Big|_0^h$$

$$= \pi r^2 h \ unit^3$$
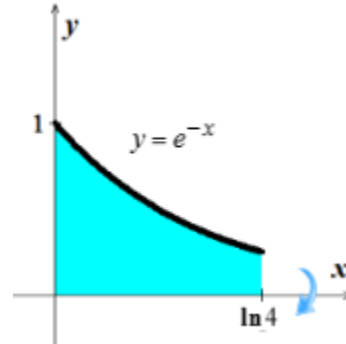
∴ The 45° angle does not affect the volume.

Exercise

Let R be the region bounded by $y = e^{-x}$, $y = 0$, $x = 0$, $x = \ln 4$. Use the disk method to find the volume of the solid generated when R is revolved about x-axis.

Solution

$$V = \pi \int_0^{\ln 4} e^{-2x} dx$$

$$= -\frac{\pi}{2} e^{-2x} \Big|_0^{\ln 4}$$

$$= -\frac{\pi}{2} \left(e^{-2\ln 4} - 1 \right)$$

$$= -\frac{\pi}{2} \left(\frac{1}{16} - 1 \right)$$

$$= \underline{\frac{15\pi}{32} \; unit^3}$$
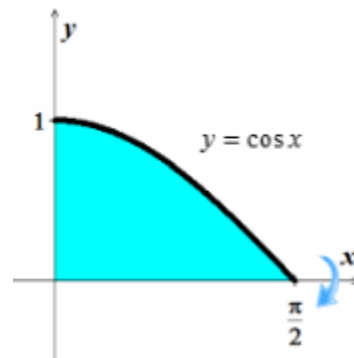

Exercise

Let R be the region bounded by $y = \cos x$, $y = 0$, $x = 0$. Use the disk method to find the volume of the solid generated when R is revolved about x-axis,

Solution

$$V = \pi \int_0^{\pi/2} \cos^2 x \, dx$$

$$= \frac{\pi}{2} \int_0^{\pi/2} (1 + \cos 2x) \, dx$$

$$= \frac{\pi}{2} \left(x + \frac{1}{2} \sin 2x \right) \Big|_0^{\pi/2}$$

$$= \frac{\pi}{2} \left(\frac{\pi}{2} \right)$$
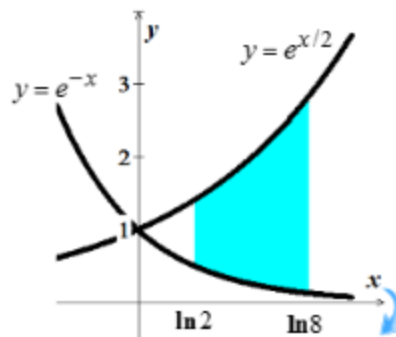
$$= \underline{\frac{\pi^2}{4} \; unit^3}$$

Exercise

Let R be the region bounded by $y = e^{x/2}$, $y = e^{-x}$, $x = \ln 2$, $x = \ln 8$. Use the disk method to find the volume of the solid generated when R is revolved about x-axis.

Solution

$$V = \pi \int_{\ln 2}^{\ln 8} \left(\left(e^{x/2} \right)^2 - \left(e^{-x} \right)^2 \right) dx$$

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$$= \pi \int_{\ln 2}^{\ln 8} \left(e^x - e^{-2x} \right) dx$$

$$= \pi \left(e^x + \frac{1}{2} e^{-2x} \right) \Big|_{\ln 2}^{\ln 8}$$

$$= \pi \left(8 + \frac{1}{2} \frac{1}{64} - 2 - \frac{1}{8} \right)$$

$$= \underline{\frac{753\pi}{128} \ unit^3}$$

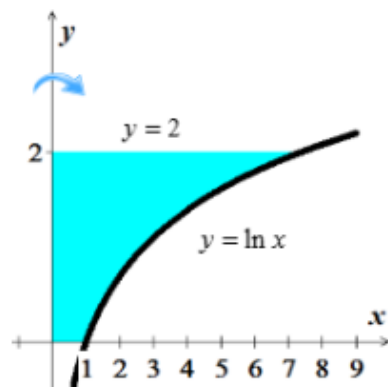## Exercise

Let R be the region bounded by $y = 0$, $y = \ln x$, $y = 2$, $x = 0$. Use the disk method to find the volume of the solid generated when R is revolved about y-axis.

Solution

$$y = \ln x \quad \rightarrow \quad x = e^y$$

$$V = \pi \int_0^2 e^{2y} dy$$

$$= \frac{\pi}{2} e^{2y} \Big|_0^2$$

$$= \underline{\frac{\pi}{2} \left(e^4 - 1 \right) \ unit^3}$$

Exercise

Let R be the region bounded by $y = \sin^{-1} x$, $x = 0$, $y = \frac{\pi}{4}$. Use the disk method to find the volume of the solid generated when R is revolved about y-axis.

Solution

$$y = \sin^{-1} x \quad \rightarrow \quad x = \sin y$$

$$V = \pi \int_0^{\pi/4} \sin^2 y \ dy$$

$$= \frac{\pi}{2} \int_0^{\pi/4} (1 - \cos 2y) \ dy$$

$$= \frac{\pi}{2} \left(y - \frac{1}{2} \sin 2y \right) \Big|_0^{\pi/4}$$

$$= \frac{\pi}{2} \left(\frac{\pi}{4} - \frac{1}{2} \right)$$

$$= \underline{\frac{\pi(\pi - 2)}{8} \ unit^3}$$

Exercise

Find the volume of the solid of revolution bounded by $y = \dfrac{\ln x}{\sqrt{x}}$, $y = 0$, *and* $x = 2$ revolved about the

x-axis. Sketch the region

Solution

$$y = \frac{\ln x}{\sqrt{x}} = 0 \;\rightarrow\; \underline{x = 1|}$$
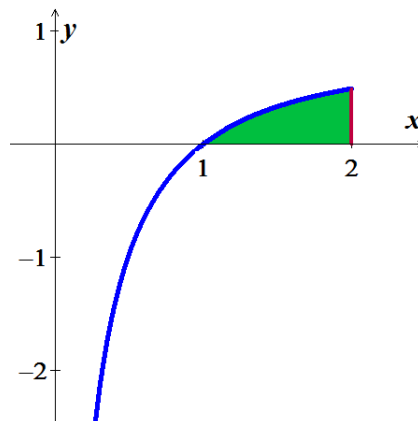
$$V = \pi \int_1^2 \left(\frac{\ln x}{\sqrt{x}} \right)^2 dx$$

$$= \pi \int_1^2 \frac{\ln^2 x}{x} dx$$

$$= \pi \int_1^2 \ln^2 x \, d(\ln x)$$

$$= \frac{\pi}{3} \left[\ln^3 x \right]_1^2$$

$$= \underline{\frac{\pi \ln^3 2}{3} \; unit^3}$$

Exercise

Find the volume of the solid of revolution bounded by $y = e^{-x}$, $y = e^{x}$, $x = 0$, $x = \ln 4$ revolved about the x-axis. Sketch the region
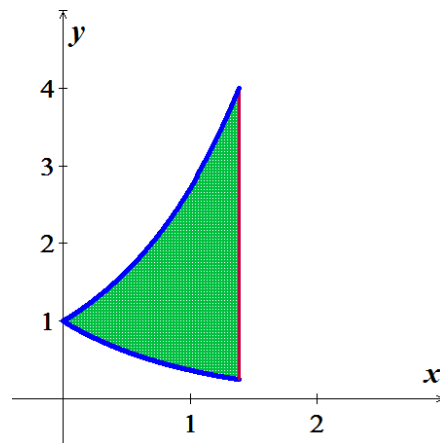
Solution

$$V = \pi \int_0^{\ln 4} \left(e^{2x} - e^{-2x} \right) dx$$

$$= \frac{\pi}{2} \left(e^{2x} + e^{-2x} \right) \Big|_0^{\ln 4}$$

$$= \frac{\pi}{2} \left(16 + \frac{1}{16} \right) \qquad e^{2\ln 4} = e^{\ln 4^2} = 16$$

$$= \underline{\frac{225\pi}{32} \; unit^3}$$

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Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $y = 4$

$$y = x, \quad y = 3, \quad x = 0$$

Solution

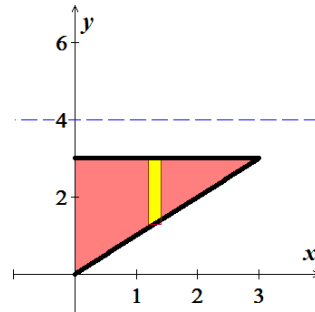$y = x = 3|$

$R(x) = 4 - x \quad r(x) = 4 - 3 = 1$

$$V = \pi \int_0^3 \left((4-x)^2 - 1 \right) dx \qquad V = \pi \int_a^b \left(R(x)^2 - r(x)^2 \right) dx$$

$$= \pi \int_0^3 \left(16 - 8x + x^2 - 1 \right) dx$$

$$= \pi \left(15x - 4x^2 + \frac{1}{3}x^3 \right) \Big|_0^3$$

$$= \pi (45 - 36 + 9)$$

$$= 18\pi|$$

Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $y = 4$

$$y = \frac{1}{2}x^3, \quad y = 4, \quad x = 0$$

Solution

$y = \frac{1}{2}x^3 = 4 \Rightarrow x^3 = 8 \quad \rightarrow \quad x = 2|$
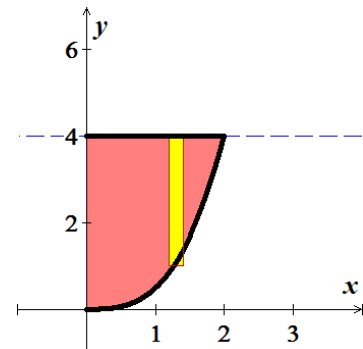
$R(x) = 4 - \frac{1}{2}x^3 \quad r(x) = 0$

$$V = \pi \int_0^2 \left(4 - \frac{1}{2}x^3 \right)^2 dx \qquad V = \pi \int_a^b \left(R(x)^2 - r(x)^2 \right) dx$$

$$= \pi \int_0^2 \left(16 - 4x^3 + \frac{1}{4}x^6 \right) dx$$

$$= \pi \left(16x - x^4 + \frac{1}{28}x^7 \right) \Big|_0^2$$

$$= \pi \left(32 - 16 + \frac{128}{28} \right)$$

$$= \frac{144\pi}{7}$$

Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $y = 4$

$$y = \frac{3}{1+x}, \quad y = 0, \quad x = 0, \quad x = 3$$

Solution

$R(x) = 4 \quad r(x) = 4 - \frac{3}{1+x}$

$$V = \pi \int_0^3 \left(16 - \left(4 - \frac{3}{1+x} \right)^2 \right) dx \qquad V = \pi \int_a^b \left(R(x)^2 - r(x)^2 \right) dx$$
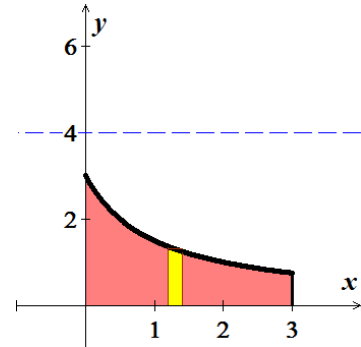
$$= \pi \int_0^3 \left(\frac{24}{1+x} - \frac{9}{(1+x)^2} \right) dx \qquad \int \frac{d(x+1)}{(x+1)^2} = \frac{-1}{x+1}$$

$$= \pi \left(24 \ln(1+x) + \frac{9}{1+x} \right) \Big|_0^3$$

$$= \pi \left(24 \ln 4 + \frac{9}{4} - 9 \right)$$

$$= \left(48 \ln 2 - \frac{27}{4} \right) \pi$$

Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $y = 4$

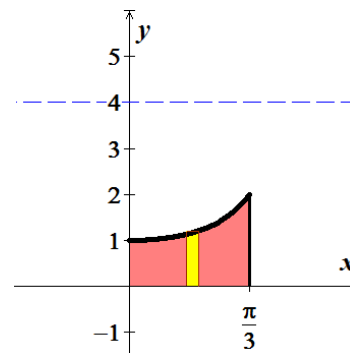$$y = \sec x, \quad y = 0, \quad 0 \le x \le \frac{\pi}{3}$$

Solution

$R(x) = 4 \quad r(x) = 4 - \sec x$

$$V = \pi \int_0^{\pi/3} \left(16 - (4 - \sec x)^2 \right) dx$$

$$= \pi \int_0^{\pi/3} \left(16 - 16 + 8\sec x - \sec^2 x \right) dx$$

$$= \pi \left(8 \ln |\sec x + \tan x| - \tan x \right) \Big|_0^{\pi/3}$$

$$= \pi \left(8 \ln \left(2 + \sqrt{3} \right) - \sqrt{3} \right)$$

Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $x = 5$

$$y = x, \quad y = 0, \quad y = 4, \quad x = 5$$

Solution
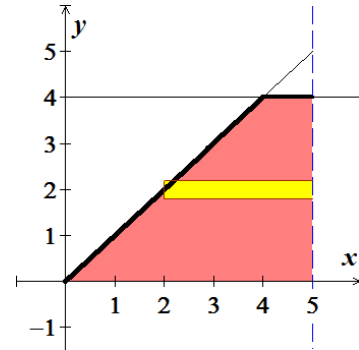
$$R(y) = 5 - y \quad r(y) = 0$$

$$V = \pi \int_0^4 (5 - y)^2 \, dy \qquad\qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2 \right) dy$$

$$= -\pi \frac{1}{3}(5 - y)^3 \Big|_0^4$$

$$= -\pi \frac{1}{3}\left(1 - 5^3 \right)$$

$$= \underline{\frac{124\pi}{3}}$$

Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $x = 5$

$$y = 3 - x, \quad y = 0, \quad y = 2, \quad x = 0$$

Solution

$$R(y) = 5 \quad r(y) = 5 - (3 - y) = 2 + y$$

$$V = \pi \int_0^2 \left(25 - (2 + y)^2 \right) dy \qquad\qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2 \right) dy$$

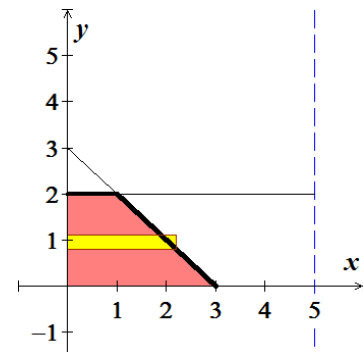$$= \pi \int_0^2 \left(25 - 4 - 4y - y^2 \right) dy$$

$$= \pi \int_0^2 \left(21 - 4y - y^2 \right) dy$$

$$= \pi \left(21y - 2y^2 - \frac{1}{3}y^3 \right) \Big|_0^2$$

$$= \pi \left(42 - 8 - \frac{8}{3} \right)$$

$$= \underline{\frac{94\pi}{3}}$$

Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $x = 5$

$$x = y^2, \quad x = 4$$

Solution

$x = y^2 = 4 \rightarrow \underline{y = \pm 2}$

$R(y) = 5 - y^2 \quad r(y) = 5 - 4 = 1$

$$V = \pi \int_{-2}^{2} \left(\left(5 - y^2\right)^2 - 1 \right) dy \qquad\qquad V = \pi \int_{c}^{d} \left(R(y)^2 - r(y)^2 \right) dy$$

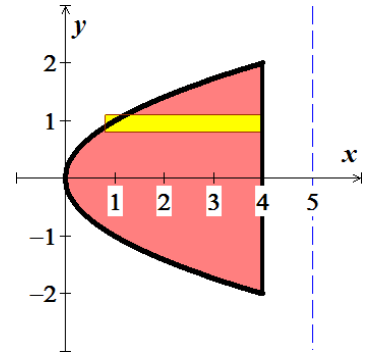$$= 2\pi \int_{0}^{2} \left(24 - 10y^2 + y^4 \right) dy$$

$$= 2\pi \left(24y - \frac{10}{3} y^3 + \frac{1}{5} y^5 \right) \Big|_{0}^{2}$$

$$= 2\pi \left(48 - \frac{80}{3} + \frac{32}{5} \right)$$

$$= \underline{\frac{832\pi}{15}}$$

Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $x = 5$

$$xy = 3, \quad y = 1, \quad y = 4, \quad x = 5$$

Solution

$xy = 3 \rightarrow x = \dfrac{3}{y}$

$R(y) = 5 - \dfrac{3}{y} \quad r(y) = 0$

$$V = \pi \int_{1}^{4} \left(5 - \frac{3}{y} \right)^2 dy \qquad\qquad V = \pi \int_{c}^{d} \left(R(y)^2 - r(y)^2 \right) dy$$

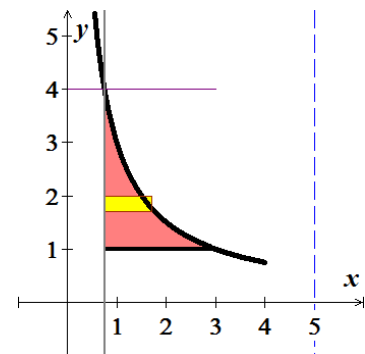$$= \pi \int_{1}^{4} \left(25 - \frac{30}{y} + \frac{9}{y^2} \right) dy$$

$$= \pi \left(25y - 30\ln y - \frac{9}{y} \right) \Big|_{1}^{4}$$

$$= \pi \left(100 - 30\ln 4 - \frac{9}{4} - 25 + 9 \right)$$

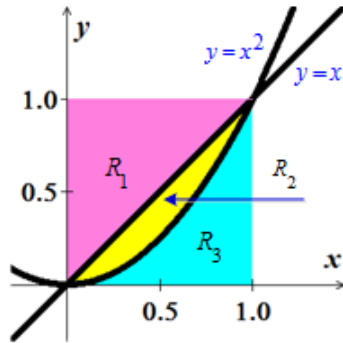$$= \pi \left(84 - \frac{9}{4} - 30\ln 2^2 \right)$$

$$= \underline{\pi \left(\frac{327}{4} - 60\ln 2 \right)}$$

Exercise

Find the volume generated by rotating the given region $y = x^2$ *and* $y = x$ about the specified line.

a) R_1 *about* $x = 0$ d) R_2 *about* $y = 1$ g) R_2 *about* $x = 0$

b) R_1 *about* $x = 1$ e) R_3 *about* $x = 0$ h) R_2 *about* $x = 1$

c) R_2 *about* $y = 0$ f) R_3 *about* $x = 1$

Solution

a) $V = \pi \displaystyle\int_0^1 y^2 \, dy$ $V = \pi \displaystyle\int_c^d \left(R(y)^2 - r(y)^2 \right) dy$

$= \dfrac{\pi}{3} y^3 \Big|_0^1$

$= \underline{\dfrac{\pi}{3}}$

b) $R(y) = 1$ $r(y) = 1 - y$

$V = \pi \displaystyle\int_0^1 \left(1 - (1-y)^2 \right) dy$ $V = \pi \displaystyle\int_c^d \left(R(y)^2 - r(y)^2 \right) dy$

$= \pi \displaystyle\int_0^1 \left(2y - y^2 \right) dy$

$= \pi \left(y^2 - \dfrac{1}{3} y^3 \right) \Big|_0^1$

$= \pi \left(1 - \dfrac{1}{3} \right)$

$= \underline{\dfrac{2\pi}{3}}$

c) R_2 *about* $y = 0$

$R(x) = x$ $r(x) = x^2$

$V = \pi \displaystyle\int_0^1 \left(x^2 - x^4 \right) dx$ $V = \pi \displaystyle\int_a^b \left(R(x)^2 - r(x)^2 \right) dx$

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$$= \pi\left(\frac{1}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^1$$

$$= \pi\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$\underline{= \frac{2\pi}{15}}$$

d) R_2 about $y = 1$

$$R(x) = 1 - x^2 \quad r(x) = 1 - x$$

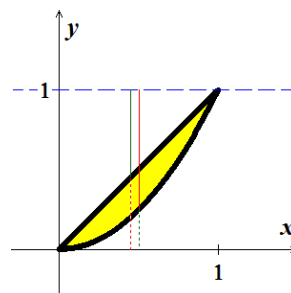$$V = \pi\int_0^1\left(\left(1 - x^2\right)^2 - (1-x)^2\right)dx \qquad\qquad V = \pi\int_a^b\left(R(x)^2 - r(x)^2\right)dx$$

$$= \pi\int_0^1\left(1 - 2x^2 + x^4 - 1 + 2x - x^2\right)dx$$

$$= \pi\int_0^1\left(x^4 - 3x^2 + 2x\right)dx$$

$$= \pi\left(\frac{1}{5}x^4 - x^3 + x^2\right)\Big|_0^1$$

$$= \pi\left(\frac{1}{5} - 1 + 1\right)$$

$$\underline{= \frac{\pi}{5}}$$

e) R_3 about $x = 0$

$$y = x^2 \quad\rightarrow\quad x = \sqrt{y}$$
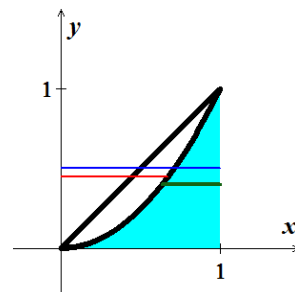$$R(y) = 1 \quad r(y) = \sqrt{y}$$

$$V = \pi\int_0^1(1 - y)\,dy \qquad\qquad V = \pi\int_c^d\left(R(y)^2 - r(y)^2\right)dy$$

$$= \pi\left(y - \frac{1}{2}y^2\right)\Big|_0^1$$

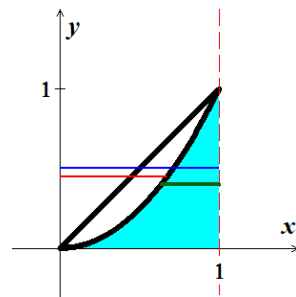$$= \pi\left(1 - \frac{1}{2}\right)$$

$$\underline{= \frac{\pi}{2}}$$

f) R_3 about $x = 1$

$$R(y) = 1 - \sqrt{y} \quad r(y) = 1 - 1 = 0$$

$$V = \pi\int_0^1\left(1 - \sqrt{y}\right)^2 dy \qquad\qquad V = \pi\int_c^d\left(R(y)^2 - r(y)^2\right)dy$$
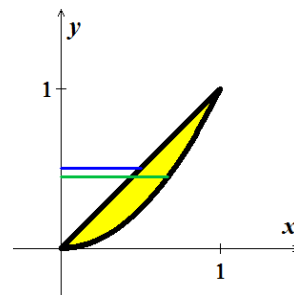
$$= \pi \int_0^1 \left(1 - 2\sqrt{y} + y\right)dy$$

$$= \pi \left(y - \frac{4}{3}y^{3/2} + \frac{1}{2}y^2\right)\Big|_0^1$$

$$= \pi \left(1 - \frac{4}{3} + \frac{1}{2}\right)$$

$$= \frac{\pi}{6}$$

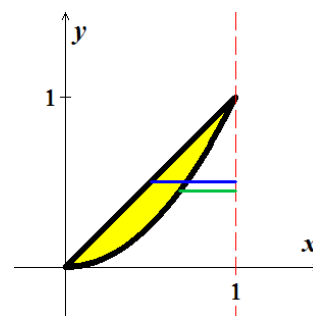**g)** R_2 *about* $x = 0$

$$R(y) = \sqrt{y} \quad r(y) = y$$

$$V = \pi \int_0^1 \left(y - y^2\right)dy \qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2\right)dy$$

$$= \pi \left(\frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Big|_0^1$$

$$= \pi \left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{\pi}{6}$$

h) R_2 *about* $x = 1$

$$R(y) = 1 - y \quad r(y) = 1 - \sqrt{y}$$

$$V = \pi \int_0^1 \left(\left(1-y\right)^2 - \left(1 - \sqrt{y}\right)^2\right)dy \qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2\right)dy$$

$$= \pi \int_0^1 \left(1 - 2y + y^2 - 1 + 2\sqrt{y} - y\right)dy$$

$$= \pi \int_0^1 \left(-3y + y^2 + 2y^{1/2}\right)dy$$

$$= \pi \left(-\frac{3}{2}y^2 + \frac{1}{3}y^3 + \frac{4}{3}y^{3/2}\right)\Big|_0^1$$

$$= \pi \left(-\frac{3}{2} + \frac{1}{3} + \frac{4}{3}\right)$$

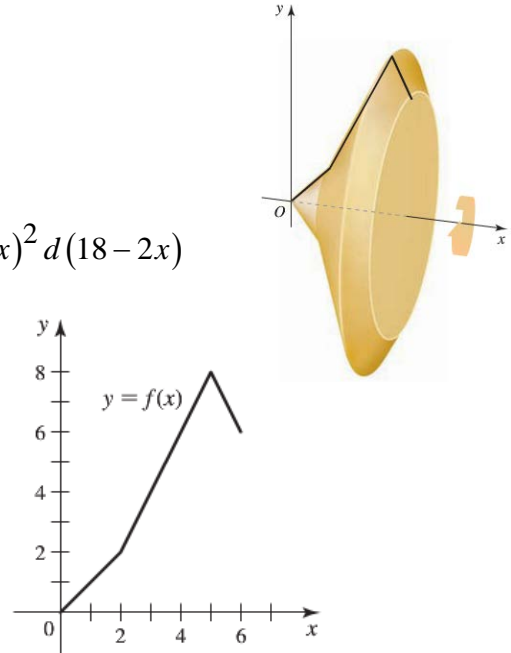$$= \pi \left(\frac{5}{3} - \frac{3}{2}\right)$$

$$= \frac{\pi}{6}$$

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Exercise

Let $f(x) = \begin{cases} x & if \quad 0 \le x \le 2 \\ 2x-2 & if \quad 2 < x \le 5 \\ -2x+18 & if \quad 5 < x \le 6 \end{cases}$

Find the volume of the solid formed when the region bounded by the graph of f, the x-axis, and the line $x = 6$ is revolved about the x-axis

Solution

$V = \pi \int_0^2 x^2 dx + \pi \int_2^5 (2x-2)^2 dx + \pi \int_5^6 (18-2x)^2 dx$

$= \pi \int_0^2 x^2 dx + \frac{\pi}{2} \int_2^5 (2x-2)^2 d(2x-2) - \frac{\pi}{2} \int_5^6 (18-2x)^2 d(18-2x)$

$= \frac{\pi}{3} x^3 \Big|_0^2 + \frac{\pi}{6}(2x-2)^3 \Big|_2^5 - \frac{\pi}{6}(18-2x)^3 \Big|_5^6$

$= \pi \left[\frac{8}{3} + \frac{1}{6}(512-8) - \frac{1}{6}(216-512) \right]$

$= \pi \left(\frac{8}{3} + \frac{252}{3} + \frac{148}{3} \right)$

$= \underline{136\pi \quad unit^3}$

Exercise

Consider the solid formed by revolving the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$ about the x-axis

    a) Find the value of x in the interval $[0, 4]$ that divides the solids into two parts of equal volume.

    b) Find the values of x in the interval $[0, 4]$ that divide the solids into three parts of equal volume.

Solution

$V = \pi \int_0^4 (\sqrt{x})^2 dx$

$= \pi \int_0^4 x \, dx$

$= \frac{\pi}{2} x^2 \Big|_0^4$

$= \underline{8\pi}$

a) Let $0 < c < 4$

$V = \pi \int_0^c x \, dx$

$$= \frac{\pi}{2}x^2 \Big|_0^c$$

$$= \frac{\pi}{2}c^2 = \frac{1}{2}(8\pi)$$

$$\rightarrow c^2 = 8 \ \Rightarrow \ \underline{c = 2\sqrt{2}}$$

$\therefore \ x = 2\sqrt{2}$, the solid is divided into 2 equal volume parts.

b) The first one third part:

$$\frac{\pi}{2}c^2 = \frac{1}{3}(8\pi) \ \rightarrow \ c^2 = \frac{16}{3} \ \Rightarrow \ \underline{c = \frac{4\sqrt{3}}{3}}$$

The second part $\left(\frac{2}{3}\right)$:

$$\frac{\pi}{2}d^2 = \frac{2}{3}(8\pi) \ \rightarrow \ d^2 = \frac{32}{3} \ \Rightarrow \ \underline{d = \frac{4\sqrt{2}}{\sqrt{3}} = \frac{4\sqrt{6}}{3}}$$

$\therefore \ x = \frac{4\sqrt{3}}{3}, \ \frac{4\sqrt{6}}{3}$, the solid is divided into 3 equal volume parts.

Exercise

The arc of $y = 4 - \frac{1}{4}x^2$ on the interval $[0, \ 4]$ is revolved about the line $y = b$

a) Find the volume of the resulting solid as a function of b.

b) Graph the function in part (a), and approximate the value of b that minimizes the volume of the solid.

c) Find the value of b that minimizes the volume of the solid, and compare the result with the answer in part (b).

Solution

a) $y = 4 - \frac{1}{4}x^2 = b \ \Rightarrow \ x^2 = 4(4-b) \ \rightarrow \ \underline{x = 2\sqrt{4-b}}$

$$V = \pi \int_0^{2\sqrt{4-b}} \left(4 - \frac{1}{4}x^2 - b\right)^2 dx + \pi \int_{2\sqrt{4-b}}^{4} \left(b - 4 + \frac{1}{4}x^2\right)^2 dx$$
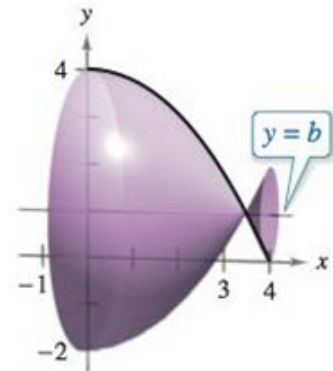
$$= \pi \int_0^4 \left(4 - \frac{1}{4}x^2 - b\right)^2 dx$$

$$= \pi \int_0^4 \left(16 - 2x^2 + \frac{1}{2}bx^2 + \frac{1}{16}x^4 - 8b + b^2\right)dx$$

$$= \pi \left(16x - \frac{2}{3}x^3 + \frac{1}{6}bx^3 + \frac{1}{80}x^5 - 8bx + b^2 x\right)\Big|_0^4$$

$$= \pi \left(64 - \frac{16}{3} + \frac{64}{6}b + \frac{64}{5} - 32b + 4b^2\right)$$

$$= \pi \left(4b^2 - \frac{64}{3}b + \frac{512}{15}\right)$$

$$V(b) = 4\pi \left(b^2 - \frac{16}{3}b + \frac{128}{15} \right)$$
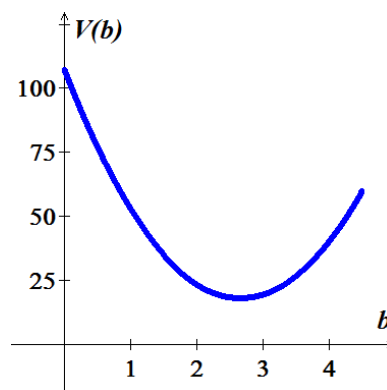
b) From the graph (*using software*)

2.660 17.873
2.670 17.872
2.680 17.874

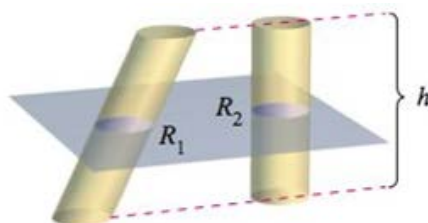The minumium volume is 17.872 for $b = 2.67$

c) $V'(b) = 4\pi \left(2b - \frac{16}{3} \right) = 0 \quad \rightarrow \quad \underline{b = \frac{16}{6} = \frac{8}{3}}$

$$V\left(b = \frac{8}{3} \right) = 4\pi \left(\left(\frac{8}{3} \right)^2 - \frac{16}{3}\frac{8}{3} + \frac{128}{15} \right)$$

$$= 4\pi \left(\frac{64}{9} - \frac{128}{9} + \frac{128}{15} \right)$$

$$= 4\pi \left(\frac{128}{15} - \frac{64}{9} \right)$$

$$= 4\pi \left(\frac{64}{45} \right)$$

$$= \underline{\frac{256\pi}{45}} \quad \approx 17.872$$

Exercise

Prove that if two solids have equal altitudes and all plane sections parallel to their bases and at equal distances from their bases have equal areas, then the solids hace the same volume.

Area of R_1 = Area of R_2

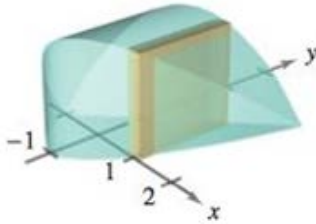### *Solution*

Since $A_1(x) = A_2(x)$ when $a \le x \le b$, then

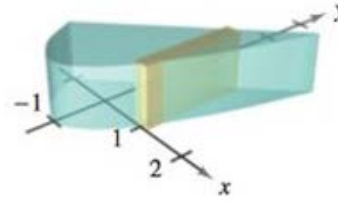$$V_1 = \int_a^b A_1(x)\,dx = \int_a^b A_2(x)\,dx = V_2$$

∴ The volume are the same.

Exercise

Find the volumes of the solids whose bases are bounded by the graph of $y = x + 1$ and $y = x^2 - 1$, with the indicated cross sections taken perpendicular to the *x-axis*

a) Squares

b) Rectangles of height 1

Solution

a) Base of cross section $= (x + 1) - \left(x^2 - 1\right) = x - x^2 + 2$

$A = b^2 = \left(2 + x - x^2\right)^2$

$\qquad = 4 + 4x - 3x^2 - 2x^3 + x^4$

$V = \displaystyle\int_{-1}^{2}\left(4 + 4x - 3x^2 - 2x^3 + x^4\right)dx$

$\qquad = \left(4x + 2x^2 - x^3 - \dfrac{1}{2}x^4 + \dfrac{1}{5}x^5\right)\Big|_{-1}^{2}$

$\qquad = 8 + 8 - 8 - 8 + \dfrac{32}{5} + 4 - 2 - 1 + \dfrac{1}{2} + \dfrac{1}{5}$

$\qquad = \dfrac{32}{5} + 1 + \dfrac{7}{10}$

$\qquad = \underline{\dfrac{81}{10}}$

$x - x^2 + 2$

$x - x^2 + 2$
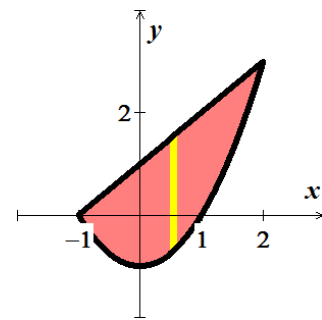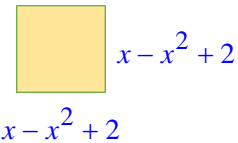
b) $A = bh = \left(2 + x - x^2\right)(1)$

$V = \displaystyle\int_{-1}^{2}\left(2 + x - x^2\right)dx$

$\qquad = 2x + \dfrac{1}{2}x^2 - \dfrac{1}{3}x^3\Big|_{-1}^{2}$

$\qquad = 4 + 2 - \dfrac{8}{3} + 2 - \dfrac{1}{2} - \dfrac{1}{3}$

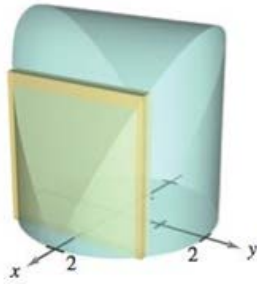$\qquad = 8 - 3 - \dfrac{1}{2}$
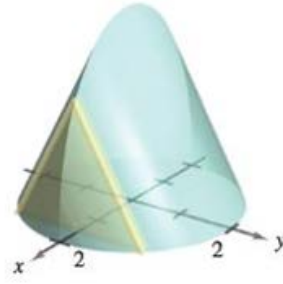
$\qquad = \underline{\dfrac{9}{2}}$

1

$x - x^2 + 2$
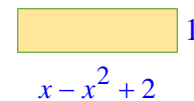
Exercise

Find the volumes of the solids whose bases are bounded by the circle $x^2 + y^2 = 4$, with the indicated cross sections taken perpendicular to the *x-axis*

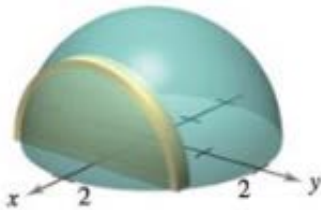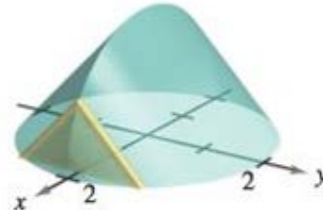   *a)* Squares

    b) Equilateral triangles

    c) Semicircles

    d) Iscosceles right triangles

Solution

$$y = \pm\sqrt{4 - x^2}$$
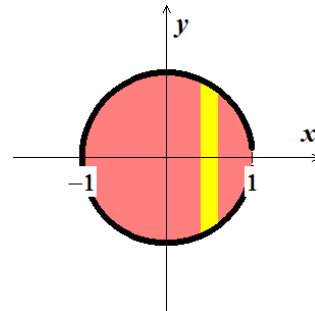
Base cross section $y = 2\sqrt{4 - x^2}$

a) $A = b^2 = 4\left(4 - x^2\right)$

$$V = 4\int_{-2}^{2}\left(4 - x^2\right)dx$$

$$= 8\left(4x - \frac{1}{3}x^3\right)\Big|_{0}^{2}$$

$$= 8\left(8 - \frac{8}{3}\right)$$

$$= \underline{\frac{128}{3}}$$

b) $\sin 60° = \dfrac{h}{2\sqrt{4 - x^2}} \;\rightarrow\; \underline{h = \sqrt{3}\sqrt{4 - x^2}}$

$$A = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{4 - x^2}\right)\sqrt{3}\sqrt{4 - x^2}$$

$$= \sqrt{3}\left(4 - x^2\right)$$

$$V = \sqrt{3}\int_{-2}^{2}\left(4 - x^2\right)dx$$

$$= 2\sqrt{3}\left(4x - \frac{1}{3}x^3\right)\Big|_{0}^{2}$$

$2\sqrt{4 - y^2}$ $2\sqrt{4 - y^2}$

$2\sqrt{4 - y^2}$

72
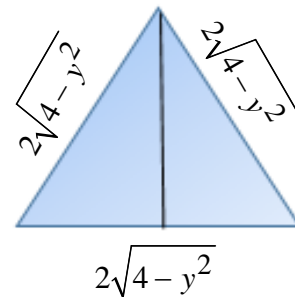
$$= 2\sqrt{3}\left(8 - \frac{8}{3}\right)$$
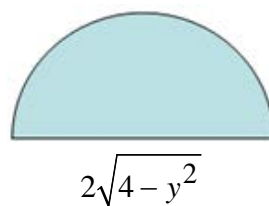
$$\underline{= \frac{32\sqrt{3}}{3}}$$

c) $\quad A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\sqrt{4-x^2}\right)^2 = \frac{\pi}{2}\left(4-x^2\right)$

$$V = \frac{\pi}{2}\int_{-2}^{2}\left(4-x^2\right)dx$$

$$= \pi\left(4x - \frac{1}{3}x^3\right)\bigg|_0^2$$

$$= \pi\left(8 - \frac{8}{3}\right)$$

$$\underline{= \frac{16\pi}{3}}$$

$2\sqrt{4-y^2}$

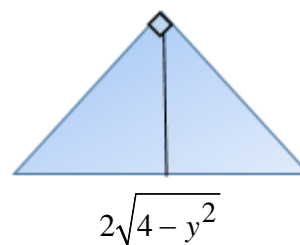**d)** $\quad \tan 45° = \dfrac{h}{\sqrt{4-x^2}} \quad \rightarrow \quad h = \sqrt{4-x^2}$

$$A = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{4-x^2}\right)\sqrt{4-x^2} = 4 - x^2$$

$$V = \int_{-2}^{2}\left(4-x^2\right)dx$$
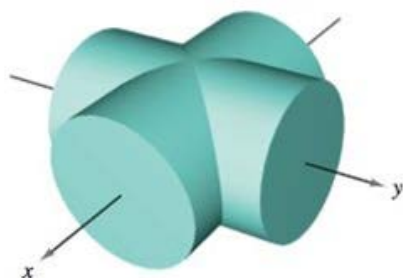
$$= 2\left(4x - \frac{1}{3}x^3\right)\bigg|_0^2$$

$$= 2\left(8 - \frac{8}{3}\right)$$

$$\underline{= \frac{32}{3}}$$

$2\sqrt{4-y^2}$

Exercise

Find the volume of the solid of intersection (the solid common to both) of the two right circular cylinders of radius r whose axes meet at right angles.

Two intersecting cylinders

Solid of intersection

Solution

The cross sections are squares.

By symmetry, we can divide the volume to 8 equal sections.

$$x^2 + y^2 = r^2 \rightarrow x = \sqrt{r^2 - y^2}$$

$$A = b^2 = \left(\sqrt{r^2 - y^2}\right)^2 = r^2 - y^2$$

$$V = 8\int_0^r \left(r^2 - y^2\right) dy$$

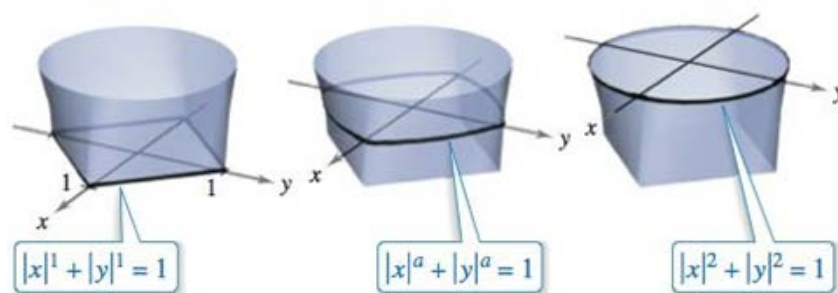$$= 8\left(r^2 y - \frac{1}{3}y^3\right)\Bigg|_0^r$$

$$= 8\left(r^3 - \frac{1}{3}r^3\right)$$

$$= \underline{\frac{16}{3}r^3}$$

Exercise

The solid shown in the figure has cross sections bounded by the graph $|x|^a + |y|^a = 1$ where $1 \le a \le 2$.

    a) Describe the cross section when $a = 1$ and $a = 2$.

    b) Describe a procedure for approximating the volume of the solid.

$$|x|^1 + |y|^1 = 1 \qquad |x|^a + |y|^a = 1 \qquad |x|^2 + |y|^2 = 1$$
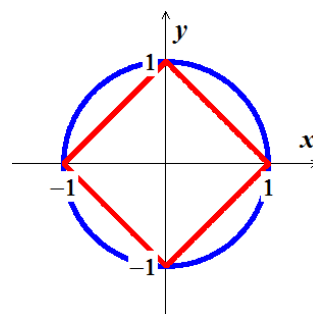
Solution

    a) When $a = 1 \Rightarrow |x| + |y| = 1$ represents a square.

        When $a = 2 \Rightarrow |x|^2 + |y|^2 = 1$ represents a circle.

    b) $|x|^a + |y|^a = 1 \rightarrow |y| = \left(1 - |x|^a\right)^{1/a}$

$$A = 2\int_{-1}^1 \left(1 - |x|^a\right)^{1/a} dx$$

$$= 4\int_0^1 \left(1 - |x|^a\right)^{1/a} dx$$

    To approximate the volume of the solid, from n slices, each of whose area is approximated by the integral above,

    Then sum the volumes of these n slices.

Exercise

Two planes cut a right circular cylinder to form a wedge. One plane is perpendicular to the axis of the cylinder and the second makes an angle of θ degrees with the first.

   a) Find the volume of the wedge if $\theta = 45°$.

   b) Find the volume of the wedge for an arbitrary angle θ. Assuming that the cylinder has sufficient length, how does the volume of the wedge change as θ increases from $0°$ to $90°$?
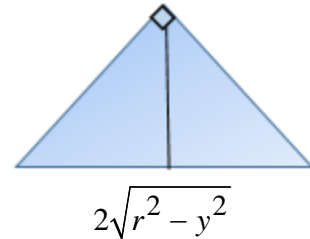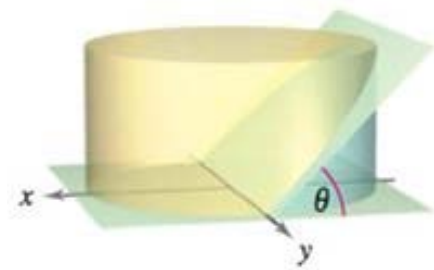
Solution

   a) Since $\theta = 45°$, then the cross sections are isosceles right triangles.

$$x^2 + y^2 = r^2 \rightarrow x = \pm\sqrt{r^2 - y^2}$$

$$\tan 45° = \frac{h}{\sqrt{r^2 - y^2}} = 1 \rightarrow h = \sqrt{r^2 - y^2}$$

$$A(y) = \frac{1}{2}bh = \frac{1}{2}\sqrt{r^2 - y^2}\sqrt{r^2 - y^2}$$

$$= \frac{1}{2}\left(r^2 - y^2\right)$$

$$V = \frac{1}{2}\int_{-r}^{r}\left(r^2 - y^2\right)dy$$

$$= \left.\left(r^2 y - \frac{1}{3}y^3\right)\right|_0^r$$

$$\underline{= \frac{2}{3}r^3}$$

   b) $\tan\theta = \dfrac{h}{\sqrt{r^2 - y^2}} \rightarrow h = (\tan\theta)\sqrt{r^2 - y^2}$

$$A(y) = \frac{1}{2}bh = \frac{1}{2}\left(r^2 - y^2\right)\tan\theta$$

$$V = \frac{1}{2}\tan\theta\int_{-r}^{r}\left(r^2 - y^2\right)dy$$

$$= \left.\tan\theta\left(r^2 y - \frac{1}{3}y^3\right)\right|_0^r$$

$$\underline{= \frac{2}{3}r^3 \tan\theta}$$

As $\theta \to 90° \Rightarrow V \to \infty$

75

Exercise

For the given torus (donut).

  a) Show that the volume of the torus is given by the integral

$$8\pi R \int_0^r \sqrt{r^2 - y^2}\, dy \quad \text{where} \quad R > r > 0$$

  b) Find the volume of the torus

Solution

a) $(x - R)^2 + y^2 = r^2 \quad \rightarrow \quad x = R \pm \sqrt{r^2 - y^2}$

$$V = \pi \int_{-r}^r \left[\left(R + \sqrt{r^2 - y^2} \right)^2 - \left(R - \sqrt{r^2 - y^2} \right)^2 \right] dy$$

$$= 2\pi \int_0^r \left[R^2 + 2R\sqrt{r^2 - y^2} + r^2 - y^2 - R^2 + 2R\sqrt{r^2 - y^2} - r^2 + y^2 \right] dy$$

$$= 8\pi R \int_0^r \sqrt{r^2 - y^2}\, dy \quad \checkmark$$

b) $\int_0^r \sqrt{r^2 - y^2}\, dy$ is $\frac{1}{4}$ of the area of a circle of radius r.

$$A = \frac{1}{4} \pi r^2$$

$$V = 8\pi R \int_0^r \sqrt{r^2 - y^2}\, dy$$

$$= 8\pi R \left(\frac{1}{4} \pi r^2 \right)$$

$$\underline{= 2\pi^2 r^2 R}$$