Solution *Section* 2.1 – Functions and Graphs

Exercise

$$f(x)=\begin{cases} 2+x & if\ \ x<-4 \\ -x & if\ \ -4\le x\le 2 \\ 3x & if\ \ x>2 \end{cases}$$ **Find**: $f(-5), f(-1), f(0),\ and\ f(3)$

Solution

a) $f(-5)=2-5=-3$

b) $f(-1)=-(-1)=1$

c) $f(0)=-0=0$

d) $f(3)=3(3)=9$

Exercise

$$f(x)=\begin{cases} -2x & if\ \ x<-3 \\ 3x-1 & if\ \ -3\le x\le 2 \\ -4x & if\ \ x>2 \end{cases}$$ **Find**: $f(-5), f(-1), f(0),\ and\ f(3)$

Solution

a) $f(-5)=-2(-5)=10$

b) $f(-1)=3(-1)-1=-4$

c) $f(0)=3(0)-1=-1$

d) $f(3)=-4(3)=-12$

Exercise

$$f(x)=\begin{cases} x^3+3 & if\ \ -2\le x\le 0 \\ x+3 & if\ \ 0<x<1 \\ 4+x-x^2 & if\ \ 1\le x\le 3 \end{cases}$$ **Find**: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

Solution

a) $f(-5)=\ \ doesn't\ exist$

b) $f(-1)=(-1)^3+3$

$\quad\quad =2 \rfloor$

c) $f(0)=(0)^3+3$

$$\underline{=3}\,\rvert$$

d) $f(3)=4+(3)-(3)^2$

$$\underline{=-2}\,\rvert$$

Exercise

$$h(x)=\begin{cases}\dfrac{x^2-9}{x-3} & if \quad x\neq 3\\ 6 & if \quad x=3\end{cases}$$

Find: $h(5)$, $h(0)$, *and* $h(3)$

Solution

a) $h(5)=\dfrac{5^2-9}{5-3}$

$$\underline{=8}\,\rvert$$

b) $h(0)=\dfrac{0^2-9}{0-3}$

$$\underline{=3}\,\rvert$$

c) $\underline{h(3)=6}\,\rvert$

Exercise

$$f(x)=\begin{cases}3x+5 & if \quad x<0\\ 4x+7 & if \quad x\geq 0\end{cases}$$

Find

a) $f(0)$ *b)* $f(-2)$ *c)* $f(1)$ *d)* $f(3)+f(-3)$ *e)* Graph $f(x)$

Solution

a) $f(0)=4(0)+7$

$$\underline{=7}\,\rvert$$

b) $f(-2)=3(-2)+5$

$$\underline{=-1}\,\rvert$$

c) $f(1)=4(1)+7$

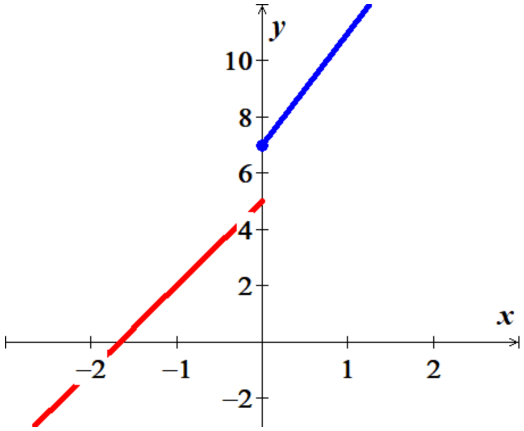
$$\underline{=11}\,\rvert$$

d) $f(3)+f(-3)=4(3)+7+3(-3)+5$

$$=12+7-9+5$$

$$\underline{=15}\,\rvert$$

e)



Exercise

$$f(x)=\begin{cases} 6x-1 & \text{if} \quad x<0 \\ 7x+3 & \text{if} \quad x\ge 0 \end{cases} \quad \text{Find}$$

a) $f(0)$ *b)* $f(-1)$ *c)* $f(4)$ *d)* $f(2)+f(-2)$ *e)* Graph $f(x)$

Solution

a) $f(0)=7(0)+3$

$\underline{=3}$

b) $f(-2)=6(-1)-1$

$\underline{=-7}$

c) $f(4)=7(4)+3$

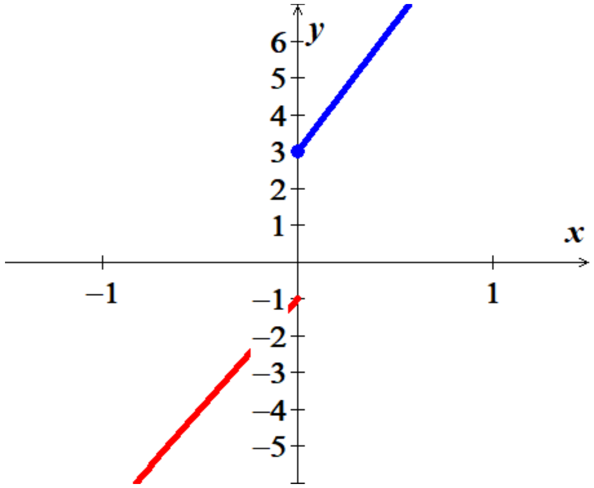
$\underline{=31}$

d) $f(2)+f(-2)=7(2)+3+6(-2)-1$

$=14+3-12-1$

$\underline{=4}$

e)



Exercise

$f(x)=\begin{cases} 2x+1 & if \quad x\le 1 \\ 3x-2 & if \quad x>1 \end{cases}$ Find

a) $f(0)$ *b)* $f(2)$ *c)* $f(-2)$ *d)* $f(1)+f(-1)$ *e)* Graph $f(x)$

Solution

a) $f(0)=2(0)+1$

$\underline{=1}$

b) $f(2)=3(2)-2$

$\underline{=4}$

c) $f(-2)=2(-2)+1$

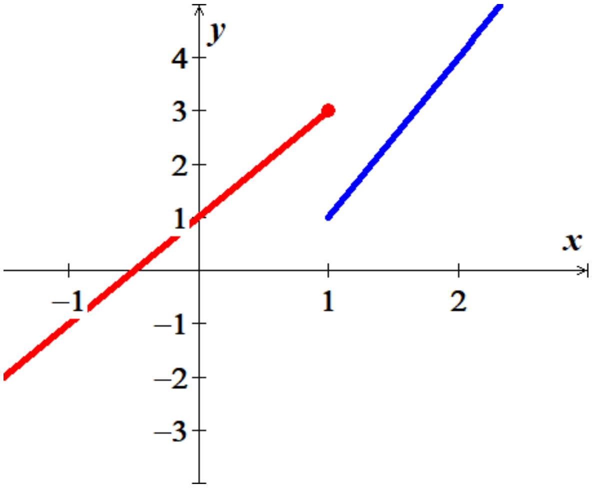
$\underline{=-3}$

d) $f(1)+f(-1)=(2(1)+1)+(2(-1)+1)$

$=2+1-2+1$

$\underline{=2}$

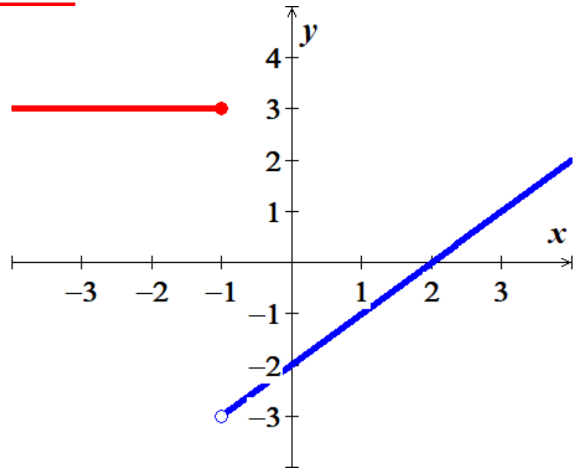
e)



Exercise

Graph the piecewise function defined by $f(x)=\begin{cases} 3 & if \quad x\le -1 \\ x-2 & if \quad x>-1 \end{cases}$

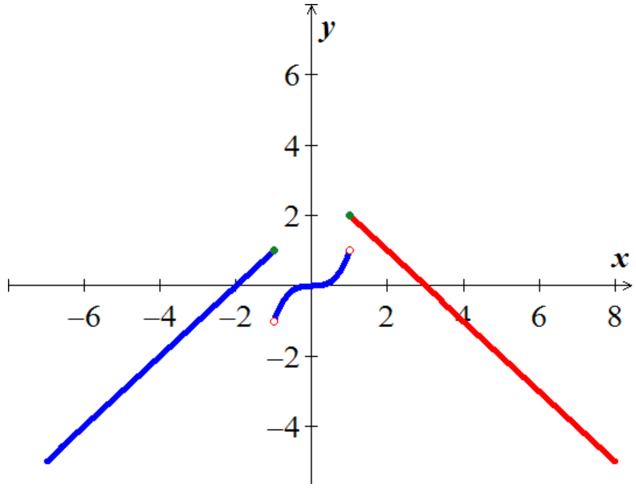
Solution



Exercise

Sketch the graph $f(x)=\begin{cases} x+2 & if \quad x\le -1 \\ x^3 & if \quad -1<x<1 \\ -x+3 & if \quad x\ge 1 \end{cases}$

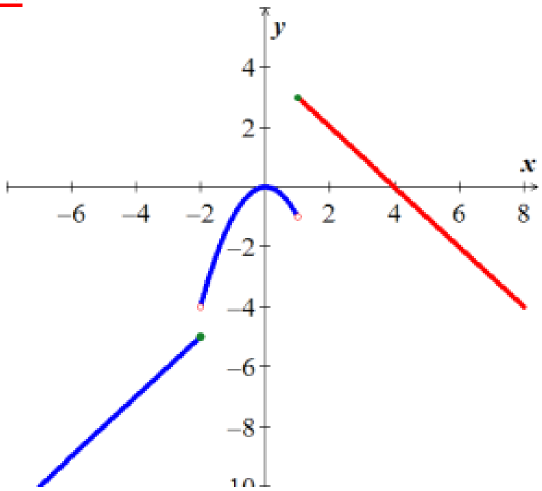
Solution



Exercise

Sketch the graph $f(x)=\begin{cases} x-3 & if \quad x\le -2 \\ -x^2 & if \quad -2<x<1 \\ -x+4 & if \quad x\ge 1 \end{cases}$

Solution



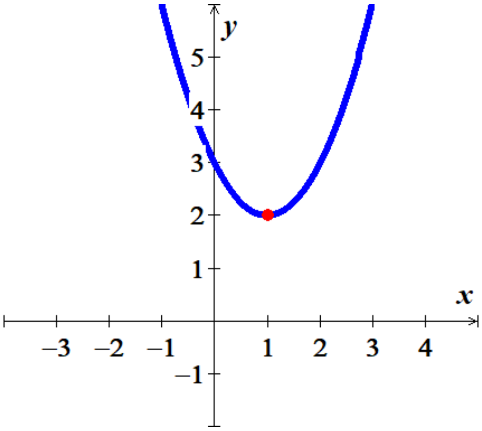
Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$f(x)=x^2-2x+3$

Solution

Relative Maximum:	*None*
Minimum Point:	$(1,\ 2)$
Increasing:	$(1,\ \infty)$
Decreasing:	$(-\infty,\ 1)$
Domain:	$\mathbb{R}$
Range:	$[2,\ \infty)$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = -x^2 - 2x + 3$$

Solution

Maximum Point: $(-1,\ 4)$

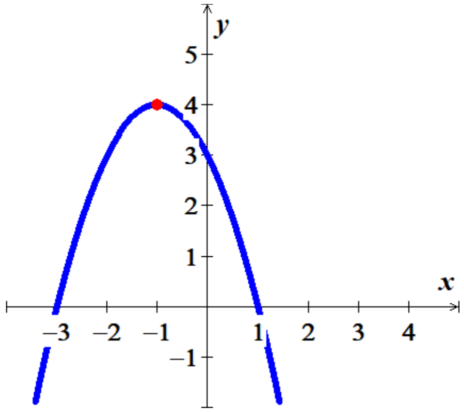
Relative Minimum: *None*

Increasing: $(-\infty,\ -1)$

Decreasing: $(-1,\ \infty)$

Domain: $\mathbb{R}$

Range: $(-\infty,\ 4]$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = -x^3 + 3x^2$$

Solution

Relative Maximum: $(2,\ 4)$

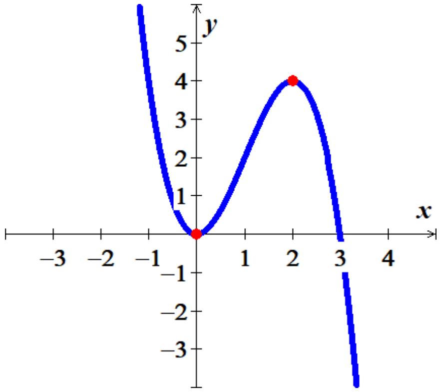
Relative Minimum: $(0,\ 0)$

Increasing: $(0,\ 2)$

Decreasing: $(-\infty,\ 0)\quad (2,\ \infty)$

Domain: $\mathbb{R}$

Range: $\mathbb{R}$



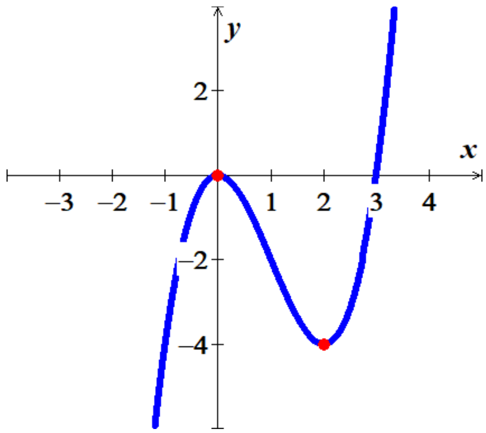
Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = x^3 - 3x^2$$

Solution

Relative Maximum:	$(0,\ 0)$
Relative Minimum:	$(2,\ -4)$
Increasing:	$(-\infty,\ 0)\ \ (2,\ \infty)$
Decreasing:	$(0,\ 2)$
Domain:	$\mathbb{R}$
Range:	$\mathbb{R}$



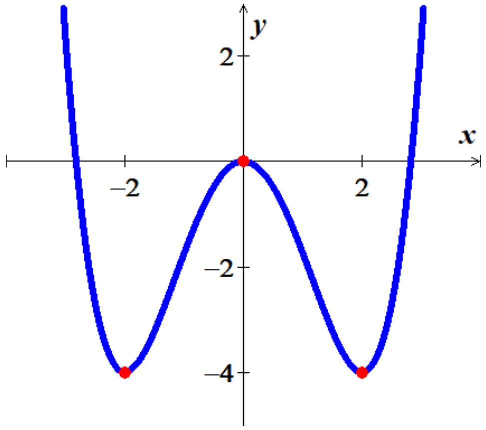
Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = \frac{1}{4}x^4 - 2x^2$$

Solution

Relative Maximum:	$(0,\ 0)$
Minimum Points:	$(-2,\ -4)\ \&\ (2,\ -4)$
Increasing:	$(-2,\ 0) \cup (2,\ \infty)$
Decreasing:	$(-\infty,\ -2) \cup (0,\ 2)$
Domain:	$\mathbb{R}$
Range:	$[-4,\ \infty)$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x)=\frac{4}{81}x^4-\frac{8}{9}x^2+4$$

Solution

Relative Maximum: $(0,\ 4)$

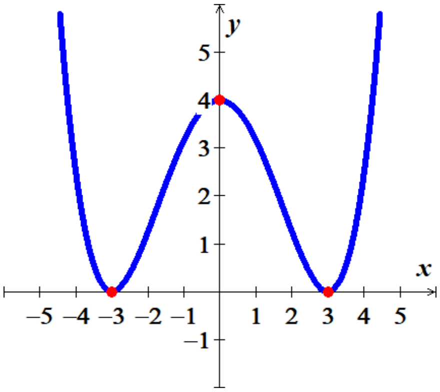
Minimum Points: $(-3,\ 0)\ \&\ (3,\ 0)$

Increasing: $(-3,\ 0)\cup(3,\ \infty)$

Decreasing: $(-\infty,\ -3)\cup(0,\ 3)$

Domain: $\mathbb{R}$

Range: $[0,\ \infty)$



Exercise

The elevation H, in *meters*, above sea level at which the boiling point of water is in t degrees Celsius is given by the function

$$H(t)=1000(100-t)+580(100-t)^2$$

At what elevation is the boiling point 99.5°.

Solution

$$H(99.5)=1000(100-99.5)+580(100-99.5)^2$$

$$=645\ m$$

Exercise

A hot-air balloon rises straight up from the ground at a rate of 120 *ft./min*. The balloon is tracked from a rangefinder on the ground at point P, which is 400 *ft*. from the release point Q of the balloon. Let d = the distance from the balloon to the rangefinder and t – the time, in minutes, since the balloon was released. Express d as a function of t.

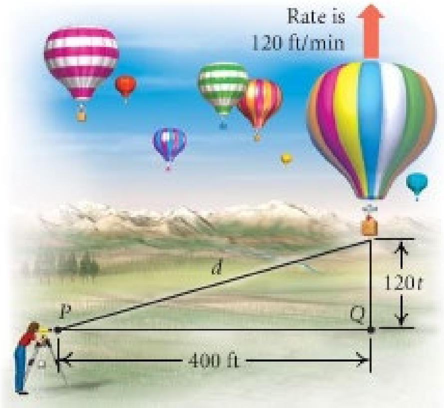
Solution

$$d^2=(120t)^2+400^2$$

$$d=\sqrt{14400t^2+160000}$$

$$d=\sqrt{1600\left(9t^2+100\right)}$$

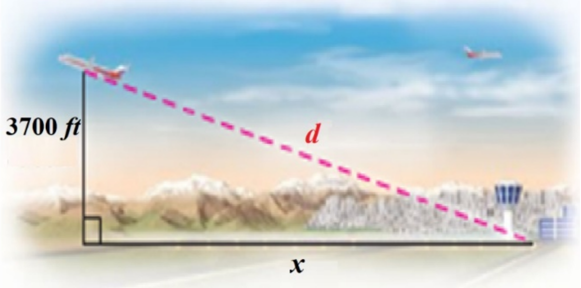
$$d(t)=40\sqrt{9t^2+100}$$



Exercise

An airplane is flying at an altitude of 3700 *feet*. The slanted distance directly to the airport is d *feet*. Express the horizontal distance x as a function of d.

Solution



$$d^2 = (3,700)^2 + x^2$$

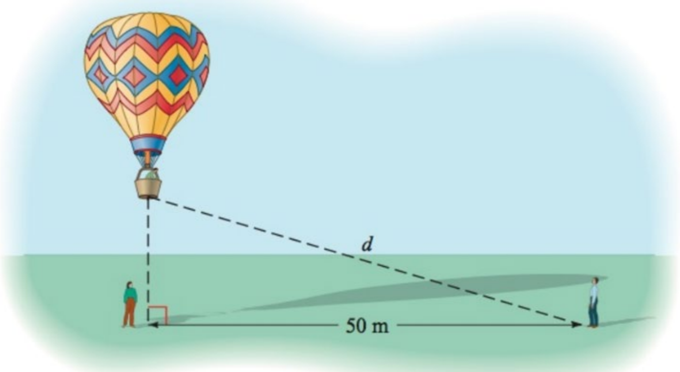
$$h^2 = d^2 - (3700)^2$$

$$\boxed{h(t) = \sqrt{d^2 - (3,700)^2}}$$

Exercise

For the first minute of flight, a hot air balloon rises vertically at a rate of 3 *m/sec*. If t is the time in *seconds* that the balloon has been airborne, write the distance d between the balloon and a point on the ground 50 *meters* from the point to lift off as a function of t.

Solution



$$\boxed{h = 3t} \qquad v = \frac{h}{t}$$

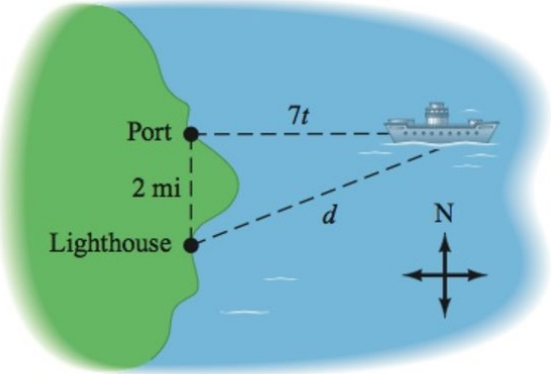
$$d^2 = h^2 + 50^2$$

$$\boxed{d(t) = \sqrt{9t^2 + 2,500}}$$

Exercise

A light house is 2 *miles* south of a port. A ship leaves port and sails east at a rate of 7 *miles* per *hour*. Express the distance d between the ship and the lighthouse as a function of time, given that the ship has been sailing for t hours.

Solution

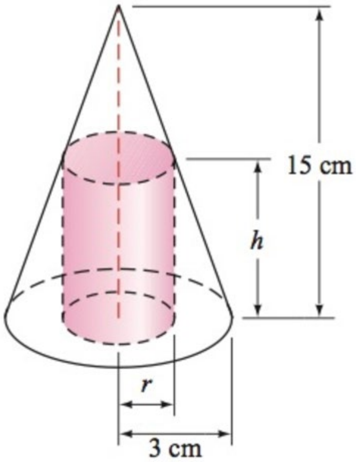


$$d^2 = 4^2 + (7t)^2$$

$$\boxed{d(t) = \sqrt{16 + 49t^2}}$$

Exercise

A cone has an altitude of 15 *cm* and a radius of 3 *cm*. A right circular cylinder of radius *r* and height *h* is inscribed in the cone. Use similar triangles to write *h* as a function of *r*.



Solution

$$\frac{15-h}{15}=\frac{r}{3}$$

$$15-h=5r$$

$$\boxed{h(r)=15-5r}$$

Exercise

Water is flowing into a conical drinking cup with an altitude of 4 *inches* am a radius of 2 *inches*.

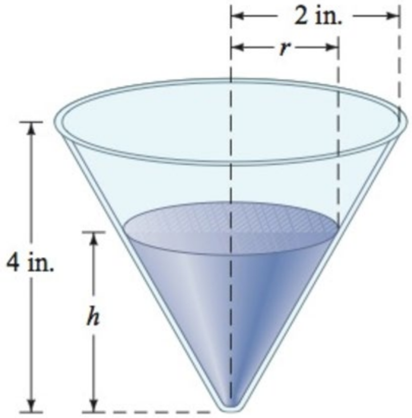
a) Write the radius *r* of the surface of the water as a function of its depth *h*.

b) Write the volume *V* of the water as a function of its depth *h*.

Solution

a) $\frac{h}{4}=\frac{r}{2}$

$$\boxed{r(h)=\frac{1}{2}h}$$



b) $Area=\pi r^2$

$$V=\frac{1}{3}\pi r^2 h$$

$$=\frac{1}{3}\pi\left(\frac{h^2}{4}\right)h$$

$$\boxed{=\frac{1}{12}\pi h^3}$$

Exercise

A water tank has the shape of a right circular cone with height 16 *feet* and radius 8 *feet*. Water is running into the tank so that the radius *r* (in *feet*) of the surface of the water is given by $r=1.5t$, where *t* is the time (in *minutes*) that the water has been running.

a) The area *A* of the surface of the water is $A=\pi r^2$. Find $A(t)$ and use it to determine the area of the surface of the water when $t=2\ \ minutes$.

b) The volume *V* of the water is given by $V=\frac{1}{3}\pi r^2 h$. Find $V(t)$ and use it to determine the volume of the water when $t=3\ \ minutes$

Solution

c) $Area = \pi r^2$

$$A(t) = \pi\left(\frac{3}{2}t\right)^2$$

$$\boxed{= \frac{9\pi}{4}t^2}$$

d) $\frac{h}{16} = \frac{r}{8}$

$$\boxed{h = 2r}$$

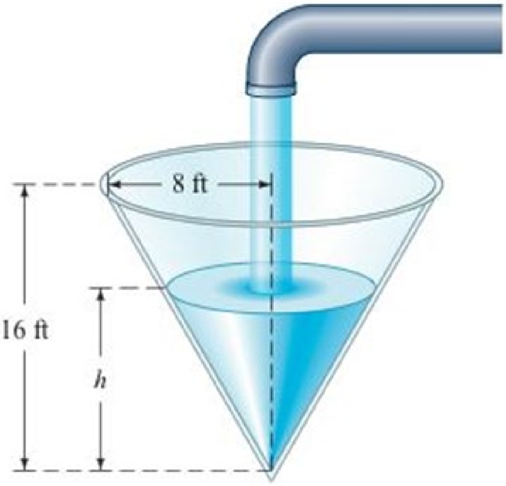
$$V(t) = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi r^2 (2r)$$

$$= \frac{2}{3}\pi r^3$$

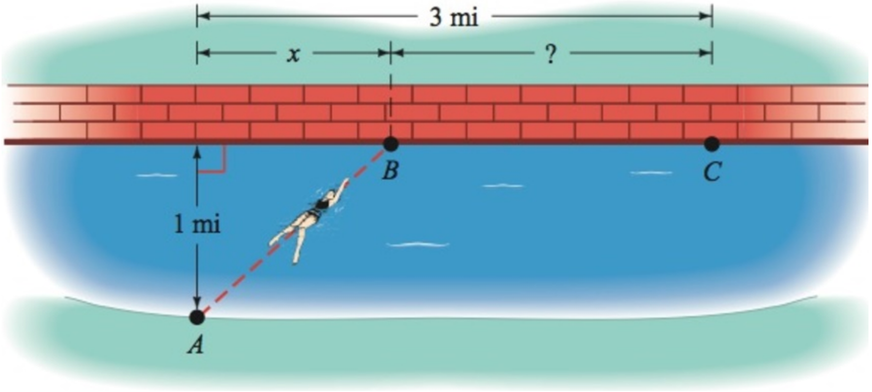
$$= \frac{2}{3}\pi\left(\frac{3}{2}t\right)^3$$

$$\boxed{= \frac{9}{4}\pi t^3}$$



Exercise

An athlete swims from point ***A*** to point ***B*** at a rate of 2 *miles* per *hour* and runs from point ***B*** to point ***C*** at a rate of 8 *miles* per *hour*. Use the dimensions in the figure to write the time *t* required to reach point ***C*** as a function of *x*.



Solution

$$Swimming\ distance = \sqrt{x^2 + 1}$$

$$t_{swim} = \frac{\sqrt{x^2+1}}{2} \qquad t = \frac{d}{v}$$

$$Running\ distance = 3 - x$$

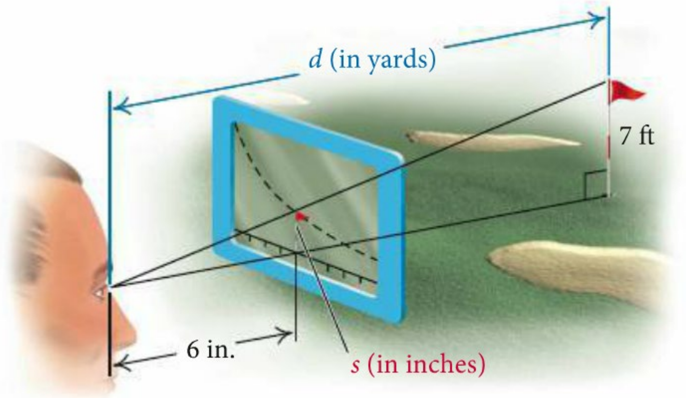
$$t_{run} = \frac{3-x}{8} \qquad t = \frac{d}{v}$$

$$\boxed{t_{total} = \frac{\sqrt{x^2+1}}{2} + \frac{3-x}{8}}$$

Exercise

A device used in golf to estimate the distance d, in *yards*, to a hole measures the size $\mathbf{s}$, in *inches*, that the 7-*foot* pin appears to be in a viewfinder. Express the distance d as a function of $\mathbf{s}$.

Solution



$$\frac{d}{6} = \frac{7}{s}\frac{ft}{in}$$

$$d = \frac{7}{s}\frac{ft}{in}6in$$

$$d = \frac{42}{s}ft\frac{1yd}{3ft}$$

$$\boxed{d(s) = \frac{14}{s}}$$

Exercise

A *rhombus* is inscribed in a rectangle that is w *meters* wide with a perimeter of 40 m. Each vertex of the rhombus is a midpoint of a side of the rectangle. Express the area of the *rhombus* as a function of the rectangle's width.

Solution

The area of the rhombus = $\frac{1}{2}$ area of the rectangle, since each vertex of the rhombus is a midpoint of a side of the rectangle.

$$Perimter: \quad 2l + 2w = 40 \qquad \textbf{Divide both sides by 2}$$

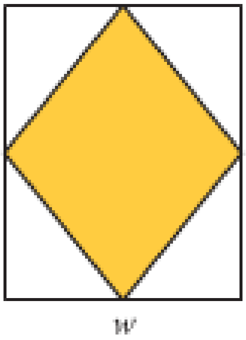
$$l + w = 20$$

$$l = 20 - w$$

$$Area\ of\ the\ rectangle = lw = (20-w)w$$

$$Area\ of\ the\ rhombus = \frac{1}{2}\left(20w - w^2\right)$$

$$\boxed{= -\frac{1}{2}w^2 + 10w}$$



Exercise

The surface area *S* of a right circular cylinder is given by the formula $S = 2\pi rh + 2\pi r^2$. if the height is twice the radius, find each of the following.

a) A function $S(r)$ for the surface area as a function of *r*.

b) A function $S(h)$ for the surface area as a function of *h*.

Solution

Given: $h = 2r$

a) $S = 2\pi rh + 2\pi r^2$

$$S(r) = 2\pi r(2r) + 2\pi r^2$$

$$= 4\pi r^2 + 2\pi r^2$$

$$\underline{= 6\pi r^2}\,\big|$$

b) $r = \frac{1}{2}h$

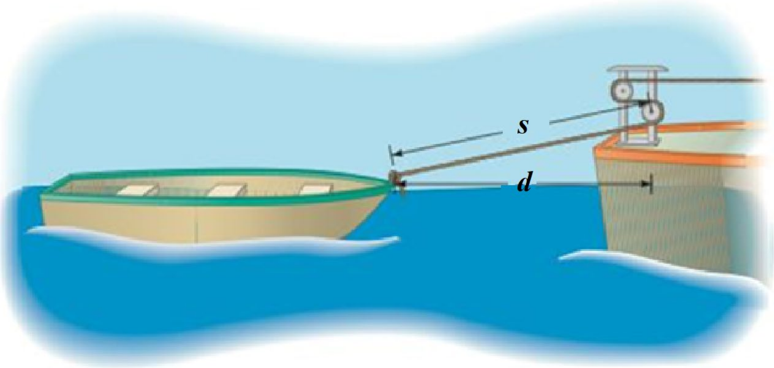
$$S(h) = 2\pi\left(\frac{1}{2}h\right)h + 2\pi\left(\frac{1}{2}h\right)^2$$

$$= \pi h^2 + \frac{1}{2}\pi h^2$$

$$\underline{= \frac{3}{2}\pi h^2}\,\big|$$

Exercise

A boat is towed by a rope that runs through a pulley that is 4 *feet* above the point where the rope is tied to the boat. The length (in *feet*) of the rope from the boat to the pulley is given by $s = 48 - t$, where *t* is the time in *seconds* that the boat has been in tow. The horizontal distance from the pulley to the boat is *d*.



a) Find $d(t)$

b) Evaluate $s(35)$ and $d(35)$

Solution

a) $s^2 = d^2 + 4^2$

$$d^2 = (48-t)^2 - 16$$

$$d(t) = \sqrt{2,304 - 96t + t^2 - 16}$$

$$\boxed{= \sqrt{t^2 - 96t + 2,288}}$$

b) $s(35) = 48 - 35$

$$\boxed{= 13 \ \ feet}$$

$$d(35) = \sqrt{(48-35)^2 - 16}$$

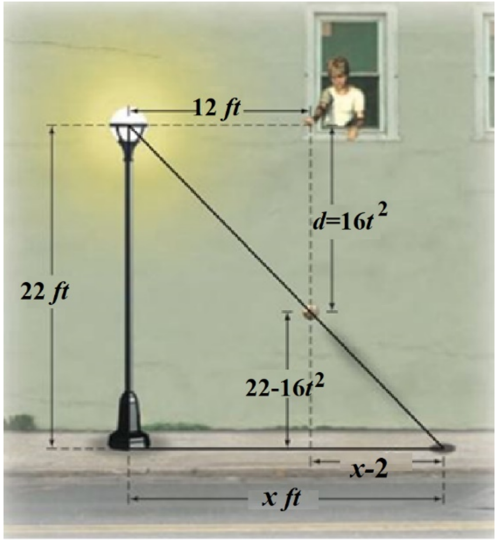
$$= \sqrt{13^2 - 16}$$

$$\boxed{= \sqrt{153} \ \ feet}$$

Exercise

The light from a lamppost casts a shadow from a ball that was dropped from a height of 22 *feet* above the ground. The distance *d*, in *feet*, the ball has dropped *t seconds* after it is released is given by $d(t) = 16t^2$. Find the distance *x*, in *feet*, of the shadow from the base of the lamppost as a function of time *t*.

Solution



$$\frac{22-16t^2}{22} = \frac{x-12}{x}$$

$$\left(22-16t^2\right)x = 22(x-12)$$

$$\left(22-16t^2\right)x = 22x - 264$$

$$\left(22-16t^2-22\right)x = -264$$

$$-16t^2 x = -264$$

$$\boxed{x(t) = \frac{33}{2t^2}}$$

Exercise

A right circular cylinder of height *h* and a radius *r* is inscribed in a right circular cone with a height of 10 *feet* and a base with radius 6 *feet*.

a) Express the height *h* of the cylinder as a function of *r*.

b) Express the volume *V* of the cylinder as a function of *r*.

c) Express the volume *V* of the cylinder as a function of *h*.

Solution

a) $\frac{h}{10} = \frac{6-r}{6}$

$$\boxed{h(r) = \frac{5}{3}(6-r)}$$

b) $V = \pi r^2 h$

$$V(r) = \frac{5}{3}\pi r^2 (6-r)$$

$$\boxed{= \frac{5}{3}\pi\left(6r^2 - r^3\right)}$$

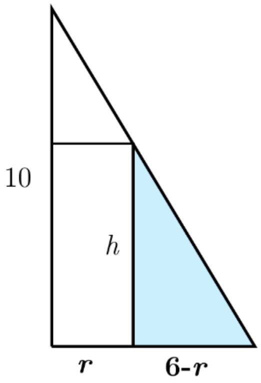
c) $\frac{3}{5}h = 6-r$

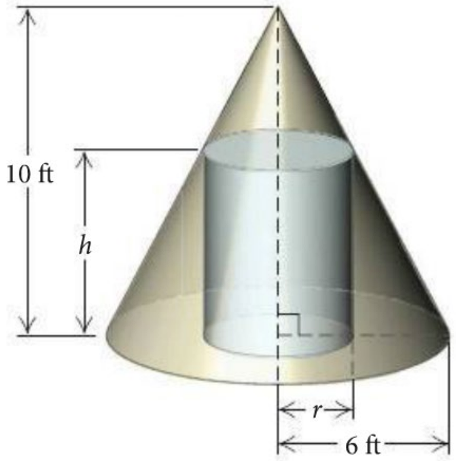
$$r = 6 - \frac{3}{5}h$$

$$V = \pi r^2 h$$

$$V(h) = \pi\left(\frac{30-3h}{5}\right)^2 h$$

$$\boxed{= \frac{1}{25}\pi h(30-3h)^2}$$





Solution *Section* 2.2 – Function Operations

Exercise

Find the domain: $f(x) = 7x + 4$

Solution

Domain: $(-\infty, \infty)$

Exercise

Find the domain: $f(x) = |3x - 2|$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x) = 3x + \pi$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x) = \sqrt{7}x + \frac{1}{2}$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x) = -2x^2 + 3x - 5$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x) = x^3 - 2x^2 + x - 3$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x) = x^2 - 2x - 15$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain $f(x) = 4 - \frac{2}{x}$

Solution

Domain: $x \neq 0$

Exercise

Find the domain $f(x) = \frac{1}{x^4}$

Solution

Domain: $x \neq 0$

Exercise

Find the domain: $g(x) = \frac{3}{x-4}$

Solution

Domain: $x \neq 4$

Exercise

Find the domain $y = \frac{2}{x-3}$

Solution

Domain: $x \neq 3$

Exercise

Find the domain $y = \dfrac{-7}{x-5}$

Solution

Domain: $\boxed{x \neq 5}$

Exercise

Find the domain $f(x) = \dfrac{x+5}{2-x}$

Solution

$2 - x \neq 0$

Domain: $\boxed{x \neq 2}$

Exercise

Find the domain $f(x) = \dfrac{8}{x+4}$

Solution

$x + 4 \neq 0$

Domain: $\boxed{x \neq -4}$

Exercise

Find the domain $f(x) = \dfrac{1}{x+4}$

Solution

Domain: $\boxed{x \neq -4}$

Exercise

Find the domain $f(x) = \dfrac{1}{x-4}$

Solution

Domain: $\boxed{x \neq 4}$

Exercise

Find the domain $f(x) = \dfrac{3x}{x+2}$

Solution

Domain: $\underline{x \neq -2}$

Exercise

Find the domain $f(x) = x - \dfrac{2}{x-3}$

Solution

Domain: $\underline{x \neq 3}$

Exercise

Find the domain $f(x) = x + \dfrac{3}{x-5}$

Solution

Domain: $\underline{x \neq 5}$

Exercise

Find the domain $f(x) = \dfrac{1}{2}x - \dfrac{8}{x+7}$

Solution

Domain: $\underline{x \neq -7}$

Exercise

Find the domain $f(x) = \dfrac{1}{x-3} - \dfrac{8}{x+7}$

Solution

Domain: $\underline{x \neq -7,\ 3}$

Exercise

Find the domain $f(x) = \dfrac{1}{x+4} - \dfrac{2x}{x-4}$

Solution

Domain: $\underline{x \neq \pm 4}$

Exercise

Fib+cnd the domain $f(x) = \frac{3x^2}{x+3} - \frac{4x}{x-2}$

Solution

Domain: $x \neq -3,\ 2$

Exercise

Find the domain $f(x) = \frac{1}{x^2 - 2x + 1}$

Solution

$x^2 - 2x + 1 \neq 0 \qquad a + b + c = 0 \ \rightarrow \ x = 1,\ \frac{c}{a}$

Domain: $x \neq 1$

Exercise

Find the domain $f(x) = \frac{x}{x^2 + 3x + 2}$

Solution

$x^2 + 3x + 2 \neq 0 \qquad a - b + c = 0 \ \rightarrow \ x = -1,\ -\frac{c}{a}$

Domain: $x \neq -1,\ -2$

Exercise

Find the domain $f(x) = \frac{x^2}{x^2 - 5x + 4}$

Solution

$x^2 - 5x + 4 \neq 0 \qquad a + b + c = 0 \ \rightarrow \ x = 1,\ \frac{c}{a}$

Domain: $x \neq 1,\ 4$

Exercise

Find the domain $f(x) = \frac{1}{x^2 - 4x - 5}$

Solution

$x^2 - 4x - 5 \neq 0 \qquad a - b + c = 0 \ \rightarrow \ x = -1,\ -\frac{c}{a}$

Domain: $x \neq -1,\ 5$

Exercise

Find the domain $g(x)=\dfrac{2}{x^2+x-12}$

Solution

$$x^2+x-12\neq 0$$

$$(x+4)(x-3)\neq 0$$

Domain: $\boxed{x\neq -4,\ 3}$ $\boxed{(-\infty,-4)\cup(-4,3)\cup(3,\infty)}$

Exercise

Find the domain $h(x)=\dfrac{5}{\frac{4}{x}-1}$

Solution

$x\neq 0$

$$\frac{4}{x}-1\neq 0$$

$$\frac{4-x}{x}\neq 0$$

$$4-x\neq 0$$

$$x\neq 4$$

Domain: $\boxed{x\neq 0,\ 4}$ $\boxed{(-\infty,0)\cup(0,4)\cup(4,\infty)}$

Exercise

Find the domain $y=\sqrt{x}$

Solution

$$x\geq 0$$

Domain: $\boxed{x\geq 0}$ $\boxed{[0,\ \infty)}$

Exercise

Find the domain $f(x)=\sqrt{8-3x}$

Solution

$$8-3x\geq 0$$

$$8\geq 3x$$

Domain: $\boxed{x\leq \frac{8}{3}}$ $\boxed{\left(-\infty,\ \frac{8}{3}\right]}$

Exercise

Find the domain $y=\sqrt{4x+1}$

Solution

$$4x+1\ge 0\Rightarrow x\ge -\frac{1}{4}$$

Domain: $\boxed{x\ge -\frac{1}{4}}$ $\left[-\frac{1}{4},\infty\right)$

Exercise

Find the domain $y=\sqrt{7-2x}$

Solution

$$7-2x\ge 0$$

$$-2x\ge -7$$

Domain: $\boxed{x\le \frac{7}{2}}$ $\left(-\infty,\frac{7}{2}\right]$

Exercise

Find the domain $f(x)=\sqrt{8-x}$

Solution

$$8-x\ge 0$$

Domain: $\boxed{x\le -8}$ $(-\infty,\ 8]$

Exercise

Find the domain $f(x)=\sqrt{3-2x}$

Solution

Domain: $\boxed{x\le \frac{3}{2}}$

Exercise

Find the domain $f(x)=\sqrt{3+2x}$

Solution

Domain: $\boxed{x\ge -\frac{3}{2}}$

Exercise

Find the domain $f(x) = \sqrt{5-x}$

Solution

Domain: $\boxed{x \le 5}$

Exercise

Find the domain $f(x) = \sqrt{x-5}$

Solution

Domain: $\boxed{x \ge 5}$

Exercise

Find the domain $f(x) = \sqrt{6-3x}$

Solution

Domain: $\boxed{x \le 2}$

Exercise

Find the domain $f(x) = \sqrt{3x-6}$

Solution

Domain: $\boxed{x \ge 2}$

Exercise

Find the domain $f(x) = \sqrt{2x+7}$

Solution

Domain: $\boxed{x \ge -\frac{7}{2}}$

Exercise

Find the domain $f(x) = \sqrt{x^2 - 16}$

Solution

$x^2 - 16 = 0$

$x^2 = 16$

$x = \pm 4$

Domain: $\underline{x \le -4 \quad x \ge 4}$

Exercise

Find the domain $\quad f(x) = \sqrt{16 - x^2}$

Solution

$x = \pm 4$

Domain: $\underline{-4 \le x \le 4}$

Exercise

Find the domain $\quad f(x) = \sqrt{9 - x^2}$

Solution

$x = \pm 3$

Domain: $\underline{-3 \le x \le 3}$

Exercise

Find the domain $\quad f(x) = \sqrt{x^2 - 25}$

Solution

$x = \pm 5$

Domain: $\underline{-5 \le x \le 5}$

Exercise

Find the domain $\quad f(x) = \sqrt{x^2 - 5x + 4}$

Solution

$x^2 - 5x + 4 \qquad a + b + c = 0 \;\rightarrow\; x = 1, \ \frac{c}{a}$

$x = 1, \ 4$

Domain: $\underline{x \le 1 \quad x \ge 4}$

Exercise

Find the domain $f(x)=\sqrt{x^2+5x+4}$

Solution

x^2+5x+4 $\qquad a-b+c=0 \;\rightarrow\; x=-1,\ -\frac{c}{a}$

$x=-1,\ -4$

Domain: $\boxed{x\le -4 \quad x\ge -1}$

Exercise

Find the domain $f(x)=\sqrt{x^2+3x+2}$

Solution

x^2+3x+2 $\qquad a-b+c=0 \;\rightarrow\; x=-1,\ -\frac{c}{a}$

$x=-1,\ -2$

Domain: $\boxed{x\le -2 \quad x\ge -1}$

Exercise

Find the domain $f(x)=\sqrt{x^2-3x+2}$

Solution

x^2-3x+2 $\qquad a+b+c=0 \;\rightarrow\; x=1,\ \frac{c}{a}$

$x=1,\ 2$

Domain: $\boxed{x\le 1 \quad x\ge 2}$

Exercise

Find the domain $f(x)=\sqrt{x-4}+\sqrt{x+1}$

Solution

$x\ge 4 \quad x\ge -1$

Domain: $\boxed{x\ge 4}$

Exercise

Find the domain $f(x) = \sqrt{3-x} + \sqrt{x-2}$

Solution

$x \le 3 \quad x \ge 2$

Domain: $\underline{2 \le x \le 3}$

Exercise

Find the domain $f(x) = \sqrt{1-x} + \sqrt{4-x}$

Solution

$x \le 1 \quad x \le 4$

Domain: $\underline{x \le 1}$

Exercise

Find the domain $f(x) = \sqrt{1-x} - \sqrt{x-3}$

Solution

$x \le 1 \quad x \ge 3$

Domain: $\underline{\varnothing}$

Exercise

Find the domain $f(x) = \sqrt{x+4} - \sqrt{x-1}$

Solution

$x \ge -4 \quad x \ge 1$

Domain: $\underline{x \ge 1}$

Exercise

Find the domain $f(x) = \dfrac{\sqrt{x+1}}{x}$

Solution

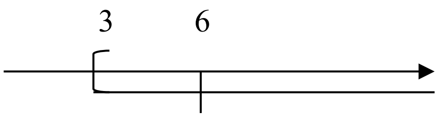
$x + 1 \ge 0 \qquad x \ne 0$

$x \ge -1$

Domain: $\underline{x \ge -1 \quad x \ne 0}$ $\qquad \underline{[-1,\ 0) \cup (0,\ \infty)}$

Exercise

Find the domain $g(x)=\dfrac{\sqrt{x-3}}{x-6}$



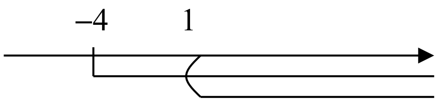
Solution

$$\to\begin{cases} x\ge 3 \\ x\ne 6 \end{cases}$$

Domain: $x\ge 3 \quad x\ne 6$ $[3,\ 6)\cup(6,\ \infty)$

Exercise

Find the domain $f(x)=\dfrac{\sqrt{x+4}}{\sqrt{x-1}}$



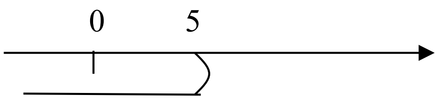
Solution

$$\to\begin{cases} x\ge -4 \\ x>1 \end{cases}$$

Domain: $x>1$ $(1,\ \infty)$

Exercise

Find the domain $f(x)=\dfrac{\sqrt{5-x}}{x}$



Solution

$x\le 5 \quad x\ne 0$

Domain: $x\le 5 \quad x\ne 0$ $(-\infty,\ 0)\cup(0,\ 5]$

Exercise

Find the domain $f(x)=\dfrac{x}{\sqrt{5-x}}$

Solution

Domain: $x<5$ $(-\infty,\ 5)$

Exercise

Find the domain $f(x) = \dfrac{1}{x\sqrt{5-x}}$

Solution

$$x < 5 \quad x \neq 0$$

Domain: $\boxed{x < 5 \quad x \neq 0}$ $\boxed{(-\infty,\ 0) \cup (0,\ 5)}$

Exercise

Find the domain $f(x) = \dfrac{x+1}{x^3 - 4x}$

Solution

$$x^3 - 4x \neq 0$$

$$x\left(x^2 - 4\right) \neq 0$$

Domain: $\boxed{x \neq 0,\ \ \pm 2}$ $(-\infty,-2) \cup (-2,0) \cup (0,2) \cup (2,\infty)$

Exercise

Find the domain $f(x) = \dfrac{\sqrt{x+5}}{x}$

Solution

$$x \geq -5 \quad x \neq 0$$

Domain: $\boxed{x \geq -5 \quad x \neq 0}$

Exercise

Find the domain $f(x) = \dfrac{x}{\sqrt{x+5}}$

Solution

$$x > -5$$

Domain: $\boxed{x > -5}$

Exercise

Find the domain $f(x)=\dfrac{1}{x\sqrt{x+5}}$

Solution

$x>-5 \quad x\neq 0$

Domain: $\underline{x>-5 \quad x\neq 0}\Big|$

Exercise

Find the domain $f(x)=\dfrac{x+3}{\sqrt{x-3}}$

Solution

Domain: $\underline{x>3}\Big|$

Exercise

Find the domain $f(x)=\dfrac{\sqrt{x+3}}{\sqrt{x-3}}$

Solution

$x\geq -3 \quad x>3$

Domain: $\underline{x>3}\Big|$

Exercise

Find the domain $f(x)=\dfrac{\sqrt{x-2}}{\sqrt{x+2}}$

Solution

$x\geq 2 \quad x>-2$

Domain: $\underline{x\geq 2}\Big|$

Exercise

Find the domain $f(x)=\dfrac{\sqrt{2-x}}{\sqrt{x+2}}$

Solution

$x\leq 2 \quad x>-2$

Domain: $\underline{-2<x\leq 2}\Big|$

Exercise

Find the domain $f(x) = \dfrac{x-4}{\sqrt{x-2}}$

Solution

Domain: $x > 2$

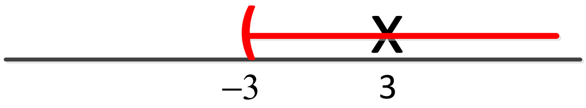
Exercise

Find the domain of $f(x) = \dfrac{1}{(x-3)\sqrt{x+3}}$

Solution

$x - 3 \neq 0 \qquad x + 3 > 0$

$x \neq 3 \qquad x > -3$



Domain: $\{x \mid x > -3 \ \ and \ \ x \neq 3\}$

$(-3,\ 3) \cup (3,\ \infty)$

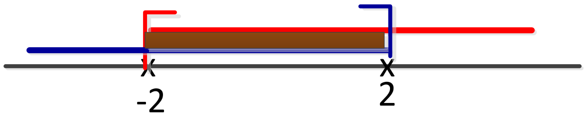
Exercise

Find the domain of $f(x) = \sqrt{x+2} + \sqrt{2-x}$

Solution

$x + 2 \geq 0 \qquad 2 - x \geq 0$

$x \geq -2 \qquad -x \geq -2 \to x \leq 2$



Domain: $\{x \mid -2 \leq x \leq 2\}$

Exercise

Find the domain of $f(x) = \sqrt{(x-2)(x-6)}$

Solution

$x - 2 \geq 0 \quad x - 6 \geq 0$

$x \geq 2 \quad x \geq 6$

Domain: $\{x \mid x \leq 2,\ x \geq 6\}$

2 6

−	+	+
−	−	+
+	−	+

Exercise

Find the domain of $f(x)=\sqrt{x+3}-\sqrt{4-x}$

Solution

$x \ge -3 \quad x \le 4$

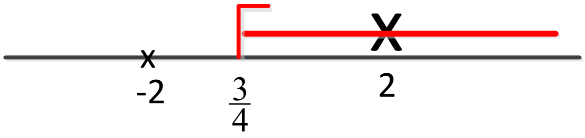
Domain: $-3 \le x \le 4$

Exercise

Find the domain of $f(x)=\dfrac{\sqrt{4x-3}}{x^2-4}$

Solution

$$4x-3 \ge 0 \qquad x^2-4 \ne 0$$
$$4x \ge 3 \qquad x \ne \pm 2$$
$$x \ge \frac{3}{4}$$



Domain: $\left[\frac{3}{4},\ 2\right)\cup(2,\ \infty)$

Exercise

Find the domain of $f(x)=\dfrac{4x}{6x^2+13x-5}$

Solution

$$6x^2+13x-5 \ne 0$$

$$x=\frac{-13\pm\sqrt{169+120}}{12}$$

$$=\begin{cases}\dfrac{-13-17}{12}=-\dfrac{5}{2}\\[2mm] \dfrac{-13+17}{12}=\dfrac{1}{3}\end{cases}$$

Domain: $x \ne -\frac{5}{2},\ \frac{1}{3}$

Exercise

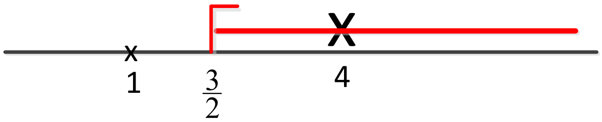
Find the domain of $f(x)=\dfrac{\sqrt{2x-3}}{x^2-5x+4}$

Solution

$$2x-3 \ge 0 \qquad x^2-5x+4 \ne 0$$

$2x \ge 3 \qquad\qquad x \ne 1,\ 4$

$x \ge \frac{3}{2}$



Domain: $x \ge \frac{3}{2}, \ \ x \ne 4$ $\qquad \left[\frac{3}{2},\ 4\right) \cup (4,\ \infty)$

Exercise

Find the domain of $\quad f(x) = \dfrac{x^2}{\sqrt{x^2 - 5x + 4}}$

Solution

$x^2 - 5x + 4 \qquad\qquad a + b + c = 0 \ \rightarrow \ x = 1,\ \frac{c}{a}$

$x = 1,\ \ 4$

Domain: $x < 1 \quad x > 4$

Exercise

Find the domain of $\quad f(x) = \dfrac{x+2}{\sqrt{x^2 + 5x + 4}}$

Solution

$x^2 + 5x + 4 \qquad\qquad a - b + c = 0 \ \rightarrow \ x = -1,\ -\frac{c}{a}$

$x = -1, \ \ -4$

Domain: $x < -4 \quad x > -1$

Exercise

Find the domain of $\quad f(x) = \dfrac{\sqrt{x+2}}{\sqrt{x^2 + 3x + 2}}$

Solution

$x^2 + 3x + 2 \qquad\qquad a - b + c = 0 \ \rightarrow \ x = -1,\ -\frac{c}{a}$

$x < -2 \quad x > -1$

$\sqrt{x+2} \ \rightarrow \ x \ge -2$

Domain: $x > -1$

Exercise

Find the domain of $f(x)=\dfrac{\sqrt{2x+3}}{x^2-6x+5}$

Solution

$x^2-6x+5 \qquad\qquad a+b+c=0 \;\rightarrow\; x=1,\; \frac{c}{a}$

$x\neq 1,\ 5$

$\sqrt{2x+3} \;\rightarrow\; x\geq -\frac{3}{2}$

Domain: $\boxed{x\geq -\frac{3}{2}\quad x\neq 1,\ 5}$

Exercise

Let $f(x)=4x-3$ and $g(x)=5x+7$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

Solution

a) $(f+g)(x)=4x-3+5x+7$

$\boxed{=9x+4}$

Domain: $\boxed{\mathbb{R}}$

b) $(f-g)(x)=4x-3-(5x+7)$

$=4x-3-5x-7$

$\boxed{=-x-10}$

Domain: $\boxed{\mathbb{R}}$

c) $(fg)(x)=(4x-3)(5x+7)$

$\boxed{=20x^2+13x-21}$

Domain: $\boxed{\mathbb{R}}$

d) $\boxed{\left(\frac{f}{g}\right)(x)=\frac{4x-3}{5x+7}}$

Domain: $\boxed{x\neq -\frac{7}{5}}$

Exercise

Let $f(x) = 2x^2 + 3$ and $g(x) = 3x - 4$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

Solution

a) $(f+g)(x) = 2x^2 + 3 + 3x - 4$

$$\boxed{= 2x^2 + 3x - 1}$$

Domain: $\boxed{\mathbb{R}}$

b) $(f-g)(x) = 2x^2 + 3 - (3x - 4)$

$$= 2x^2 + 3 - 3x + 4$$

$$\boxed{= 2x^2 - x + 7}$$

Domain: $\boxed{\mathbb{R}}$

c) $(fg)(x) = \left(2x^2 + 3\right)(3x - 4)$

$$\boxed{= 6x^2 + x - 12}$$

Domain: $\boxed{\mathbb{R}}$

d) $\boxed{\left(\frac{f}{g}\right)(x) = \frac{2x^2 + 3}{3x - 4}}$

Domain: $\boxed{x \neq -\frac{4}{3}}$

Exercise

Let $f(x) = x^2 - 2x - 3$ and $g(x) = x^2 + 3x - 2$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

Solution

a) $(f+g)(x) = x^2 - 2x - 3 + x^2 + 3x - 2$

$$\boxed{= 2x^2 + x - 5}$$

Domain: $\boxed{\mathbb{R}}$

b) $(f-g)(x) = x^2 - 2x - 3 - x^2 - 3x + 2$

$$\boxed{= -5x - 1}$$

Domain: $\boxed{\mathbb{R}}$

c) $(fg)(x) = \left(x^2 - 2x - 3\right)\left(x^2 + 3x - 2\right)$

$$= x^4 + 3x^3 - 2x^2 - 2x^3 - 6x^2 + 4x - 3x^2 - 9x + 6$$

$$\boxed{= x^4 + x^3 - 11x^2 - 5x + 6}$$

Domain: $\boxed{\mathbb{R}}$

d) $\boxed{\left(\frac{f}{g}\right)(x) = \frac{x^2 - 2x - 3}{x^2 + 3x - 2}}$

Domain: $\boxed{x \ne \frac{-3 \pm \sqrt{17}}{2}}$

Exercise

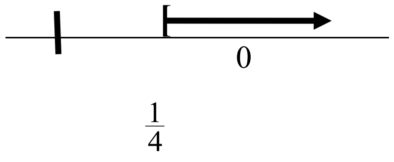
Let $f(x) = \sqrt{4x-1}$ and $g(x) = \frac{1}{x}$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

Solution

a) $(f+g)(x)$

$(f+g)(x) = \sqrt{4x-1} + \frac{1}{x}$



$4x - 1 \ge 0$ $x \ne 0$

$x \ge \frac{1}{4}$

Domain: $\left[\frac{1}{4}, \infty\right)$

b) $(f-g)(x)$

$(f-g)(x) = \sqrt{4x-1} - \frac{1}{x}$

$4x - 1 \ge 0$ $x \ne 0$

$x \ge \frac{1}{4}$

Domain: $\left[\frac{1}{4}, \infty\right)$

c) $(fg)(x) = \sqrt{4x-1}\left(\frac{1}{x}\right)$

$$= \frac{\sqrt{4x-1}}{x}$$

$4x-1\ge 0 \qquad\qquad x\ne 0$

$x\ge \frac{1}{4}$

Domain: $\left[\frac{1}{4},\infty\right)$

d) $\left(\frac{f}{g}\right)(x)=\dfrac{\sqrt{4x-1}}{\frac{1}{x}}$ *Domain:* $x\ne 0$

$=x\sqrt{4x-1}$

$4x-1\ge 0$

$x\ge \frac{1}{4}$

Domain: $\left[\frac{1}{4},\infty\right)$

Exercise

Given that $f(x)=x+1$ and $g(x)=\sqrt{x+3}$

a) Find $(f+g)(x)$

b) Find the domain of $(f+g)(x)$

c) Find: $(f+g)(6)$

Solution

a) $(f+g)(x)=f(x)+g(x)$

$=x+1+\sqrt{x+3}$

b) $x+3\ge 0\to x\ge -3$

Domain $=[-3,\ \infty)$

c) $(f+g)(6)=6+1+\sqrt{6+3}$

$=10$

Exercise

Given that $f(x)=x^2-4$ and $g(x)=x+2$

a) Find $(f+g)(x)$ and its domain

b) Find $(f/g)(x)$ and its domain

Solution

a) $(f+g)(x) = x^2 - 4 + x + 2$

$= x^2 + x - 2$

Domain: $\boxed{\mathbb{R}}$

b) $\dfrac{f(x)}{g(x)} = \dfrac{x^2-4}{x+2}$

$x \neq -2$

Domain: $\boxed{(-\infty,-2)\cup(-2,\infty)}$

Exercise

Let $f(x) = x^2 + 1$ and $g(x) = 3x + 5$. Find $(f+g)(1)$, $(f-g)(-3)$, $(fg)(5)$, and $\left(\frac{f}{g}\right)(0)$

Solution

a) $(f+g)(1) = f(1) + g(1)$

$= 1^2 + 1 + 3(1) + 5$

$\boxed{= 10}$

b) $(f-g)(-3) = f(-3) - g(-3)$

$= (-3)^2 + 1 - (3(-3) + 5)$

$\boxed{= 10}$

c) $(fg)(5) = f(5) \cdot g(5)$

$= (5^2 + 1) \cdot (3(5) + 5)$

$= (26) \cdot (20)$

$\boxed{= 520}$

d) $\left(\frac{f}{g}\right)(0) = \dfrac{f(0)}{g(0)}$

$= \dfrac{0^2 + 1}{3(0) + 5}$

$\boxed{= \frac{1}{5}}$

Exercise

Find $(f+g)(x),\ \ (f-g)(x),\ \ (f\cdot g)(x),\ \ and\ \ (f/g)(x)$ and the domain of

$f(x)=\sqrt{3-2x},\ \ \ g(x)=\sqrt{x+4}$

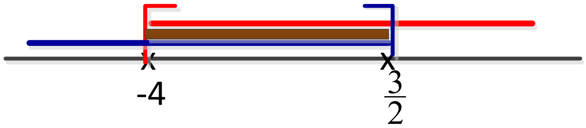
Solution

$$f(x)+g(x)=\sqrt{3-2x}+\sqrt{x+4}$$

$3-2x\ge 0 \qquad x+4\ge 0$

$-2x\ge -3 \qquad x\ge -4$

$x\le \frac{3}{2}$



Domain: $\left\{x\,|\,-4\le x\le \frac{3}{2}\right\}$

$$f(x)-g(x)=\sqrt{3-2x}-\sqrt{x+4}$$

$3-2x\ge 0 \qquad x+4\ge 0$

$-2x\ge -3 \qquad x\ge -4$

$x\le \frac{3}{2}$

Domain: $\left\{x\,|\,-4\le x\le \frac{3}{2}\right\}$

$$\begin{aligned}(f\cdot g)(x)&=\left(\sqrt{3-2x}\right)\left(\sqrt{x+4}\right)\\&=\sqrt{(3-2x)(x+4)}\\&=\sqrt{-2x^2-5x+12}\end{aligned}$$

$3-2x\ge 0 \qquad x+4\ge 0$

$-2x\ge -3 \qquad x\ge -4$

$x\le \frac{3}{2}$

Domain: $\left\{x\,|\,-4\le x\le \frac{3}{2}\right\}$

$$\begin{aligned}(f/g)(x)&=\frac{\sqrt{3-2x}}{\sqrt{x+4}}\,\frac{\sqrt{x+4}}{\sqrt{x+4}}\\&=\frac{\sqrt{-2x^2-5x+12}}{x+4}\end{aligned}$$

$3-2x\ge 0 \qquad x+4> 0$

$-2x\ge -3 \qquad x> -4$

$x\le \frac{3}{2}$

Domain: $\left\{x\,|\,-4< x\le \frac{3}{2}\right\}$ $\qquad \left(-4,\ \frac{3}{2}\right]$

Exercise

Find $(f+g)(x)$, $(f-g)(x)$, $(f\cdot g)(x)$, *and* $(f/g)(x)$ and the domain of

$f(x)=\dfrac{2x}{x-4}$, $g(x)=\dfrac{x}{x+5}$

Solution

$$
\begin{aligned}
(f+g)(x) &= \frac{2x}{x-4}+\frac{x}{x+5}\\
&= \frac{2x(x+5)+x(x-4)}{(x-4)(x+5)}\\
&= \frac{2x^2+10x+x^2-4x}{(x-4)(x+5)}\\
&= \frac{3x^2+6x}{(x-4)(x+5)}
\end{aligned}
$$

$x-4\neq 0 \qquad x+5\neq 0$

$x\neq 4 \qquad x\neq -5$

Domain: $\{x \mid x\neq -5,4\}$ $\qquad (-\infty,\ -5)\cup(-5,\ 4)\cup(4,\ \infty)$

$$
\begin{aligned}
(f-g)(x) &= \frac{2x}{x-4}-\frac{x}{x+5}\\
&= \frac{2x(x+5)-x(x-4)}{(x-4)(x+5)}\\
&= \frac{2x^2+10x-x^2+4x}{(x-4)(x+5)}\\
&= \frac{x^2+14x}{(x-4)(x+5)}
\end{aligned}
$$

$x\neq 4 \qquad x\neq -5$

Domain: $\{x \mid x\neq -5,4\}$

$$
\begin{aligned}
(f\bullet g)(x) &= \frac{2x}{x-4}\frac{x}{x+5}\\
&= \frac{2x^2}{(x-4)(x+5)}
\end{aligned}
$$

$x\neq 4 \qquad x\neq -5$

Domain: $\{x \mid x\neq -5,4\}$

$$
\begin{aligned}
(f/g)(x) &= \frac{2x}{x-4}\div\frac{x}{x+5}\\
&= \frac{2x}{x-4}\,\frac{x+5}{x}
\end{aligned}
$$

$$= 2\frac{x+5}{x-4}$$

$$x \neq 4 \qquad x \neq -5$$

Domain: $\{x \mid x \neq -5, 4\}$

Exercise

Find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, *and* $(f/g)(x)$ of $f(x) = x-5$ and $g(x) = x^2 - 1$

Solution

a) $(f+g)(x) = f(x) + g(x)$

$$= x - 5 + x^2 - 1$$

$$= x^2 + x - 6$$

b) $(f-g)(x) = f(x) - g(x)$

$$= x - 5 - (x^2 - 1)$$

$$= x - 5 - x^2 + 1$$

$$= -x^2 + x - 4$$

c) $(fg)(x) = f(x)g(x)$

$$= (x-5)(x^2-1)$$

$$= x^3 - x - 5x^2 + 5$$

$$= x^3 - 5x^2 - x + 5$$

d) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$$= \frac{x-5}{x^2-1}$$

Exercise

For the function *f* given by $f(x) = 9x + 5$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$f(x+h) = 9(x+h) + 5 = 9x + 9h + 5$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\overset{f(x+h)}{9x + 9h + 5} \; - \; \overset{f(x)}{(9x+5)}}{h}$$

$$= \frac{9x + 9h + 5 \; - \; 9x - 5}{h}$$

$$= \frac{9h}{h}$$

$$\underline{= 9}$$

Exercise

For the function *f* given by $f(x) = 6x + 2$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\frac{f(x+h) - f(x)}{h} = \frac{6(x+h) + 2 - (6x+2)}{h}$$

$$= \frac{6x + 6h + 2 - 6x - 2}{h}$$

$$= \frac{6h}{h}$$

$$\underline{= 6}$$

Exercise

For the function *f* given by $f(x) = 4x + 11$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\frac{f(x+h) - f(x)}{h} = \frac{4(x+h) + 11 - (4x+11)}{h}$$

$$= \frac{4x + 4h + 11 - 4x - 11}{h}$$

$$= \frac{4h}{h}$$

$$\underline{= 4}$$

Exercise

For the function *f* given by $f(x) = 3x - 5$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\frac{f(x+h) - f(x)}{h} = \frac{3(x+h) - 5 - 3x + 5}{h}$$

$$= \frac{3x + 3h - 5 - 3x + 5}{h}$$

$$= \frac{3h}{h}$$

$$\underline{= 3}$$

Exercise

For the function *f* given by $f(x) = -2x - 3$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{-2(x+h)-3+2x+3}{h}\\ &= \frac{-2x-2h-3+2x+3}{h}\\ &= \frac{-2h}{h}\\ &\underline{= -2}\end{aligned}$$

Exercise

For the function *f* given by $f(x) = -4x + 3$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{-4(x+h)+3+4x-3}{h}\\ &= \frac{-4x-4h+3+4x-3}{h}\\ &= \frac{-4h}{h}\\ &\underline{= -4}\end{aligned}$$

Exercise

For the function *f* given by $f(x) = 3x - 6$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{3(x+h)-6-3x+6}{h}\\ &= \frac{3x+3h-6-3x+6}{h}\\ &= \frac{3h}{h}\\ &\underline{= 3}\end{aligned}$$

Exercise

For the function *f* given by $f(x) = -5x - 7$, find the difference quotient $\frac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{-5(x+h)-7+5x+7}{h} \\ &= \frac{-5x-5h-7+5x+7}{h} \\ &= \frac{-5h}{h} \\ &= -5\end{aligned}$$

Exercise

Given the function: $f(x) = 2x^2$. Find and simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned}f(x+h) &= 2(x+h)^2 \\ &= 2\left(x^2+2hx+h^2\right) \\ &= 2x^2+4hx+2h^2\end{aligned}$$

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{2x^2+4hx+2h^2-2x^2}{h} \\ &= \frac{4hx+2h^2}{h} \\ &= \frac{4hx}{h}+\frac{2h^2}{h} \\ &= 4x+2h\end{aligned}$$

Exercise

For the function *f* given by $f(x) = 5x^2$, find the difference quotient $\frac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{5(x+h)^2-5x^2}{h} \\ &= \frac{5\left(x^2+2hx+h^2\right)-5x^2}{h}\end{aligned}$$

$$= \frac{5x^2 + 10hx + 5h^2 - 5x^2}{h}$$

$$= \frac{10hx + 5h^2}{h}$$

$$\boxed{= 10x + 5h}$$

Exercise

For the function *f* given by $f(x) = 3x^2 - 4x$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\frac{f(x+h) - f(x)}{h} = \frac{3(x+h)^2 - 4x - 3x^2 + 4x}{h}$$

$$= \frac{3\left(x^2 + 2hx + h^2\right) - 4(x+h) - 3x^2 + 4x}{h}$$

$$= \frac{3x^2 + 6hx + 3h^2 - 4x - 4h - 3x^2 + 4x}{h}$$

$$= \frac{6hx + 3h^2 - 4h}{h}$$

$$\boxed{= 6x + 3h - 4}$$

Exercise

For the function *f* given by $f(x) = 2x^2 - 3x$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$f(x+h) = 2(\text{——})^2 - 3(\text{——})$$

$$= 2(x+h)^2 - 3(x+h) \qquad\qquad (a+b)^2 = a^2 + 2ab + b^2$$

$$= 2\left(x^2 + 2xh + h^2\right) - 3x - 3h$$

$$= 2x^2 + 4xh + 2h^2 - 3x - 3h$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\overset{f(x+h)}{\overline{2x^2 + 4xh + 2h^2 - 3x - 3h}} \; - \; \overset{f(x)}{\overline{(2x^2 - 3x)}}}{h}$$

$$= \frac{2x^2 + 4xh + 2h^2 - 3x - 3h - 2x^2 + 3x}{h}$$

$$= \frac{4xh + 2h^2 - 3h}{h}$$

$$=\frac{4xh}{h}+\frac{2h^2}{h}-\frac{3h}{h}$$

$$\underline{=4x+2h-3|}$$

Exercise

For the function f given by $f(x)=2x^2-x-3$, find the difference quotient $\frac{f(x+h)-f(x)}{h}$

Solution

$$f(x+h)=2(x+h)^2-(x+h)-3$$

$$=2\left(x^2+2hx+h^2\right)-x-h-3$$

$$=2x^2+4hx+2h^2-x-h-3$$

$$\frac{f(x+h)-f(x)}{h}=\frac{2x^2+2h^2+4hx-x-h-3-\left(2x^2-x-3\right)}{h}$$

$$=\frac{2x^2+2h^2+4hx-x-h-3-2x^2+x+3}{h}$$

$$=\frac{2h^2+4hx-h}{h}$$

$$=\frac{2h^2}{h}+\frac{4hx}{h}-\frac{h}{h}$$

$$\underline{=2h+4x-1|}$$

Exercise

For the given function $f(x)=2x^2-x-3$, find the difference quotient $\frac{f(x+h)-f(x)}{h}$

Solution

$$\frac{f(x+h)-f(x)}{h}=\frac{2(x+h)^2-(x+h)-3-2x^2+x+3}{h}$$

$$=\frac{2\left(x^2+2hx+h^2\right)-x-h-2x^2+x}{h}$$

$$=\frac{2x^2+4hx+2h^2-h-2x^2}{h}$$

$$=\frac{4hx+2h^2-h}{h}$$

$$\underline{=4x+2h-1\,|}$$

Exercise

For the given function $f(x) = x^2 - 2x + 5$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^2 - 2(x+h) + 5 - x^2 + 2x - 5}{h} \\
&= \frac{x^2 + 2hx + h^2 - 2x - 2h - x^2 + 2x}{h} \\
&= \frac{2hx + h^2 - 2h}{h} \\
&= 2x + h - 2
\end{aligned}$$

Exercise

For the given function $f(x) = 3x^2 - 2x + 5$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{3(x+h)^2 - 2(x+h) + 5 - 3x^2 + 2x - 5}{h} \\
&= \frac{3\left(x^2 + 2hx + h^2\right) - 2x - 2h - 3x^2 + 2x}{h} \\
&= \frac{3x^2 + 6hx + 3h^2 - 2h - 3x^2}{h} \\
&= \frac{6hx + 3h^2 - 2h}{h} \\
&= 6x + 3h - 2
\end{aligned}$$

Exercise

For the given function $f(x) = -2x^2 - 3x + 7$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{-2(x+h)^2 - 3(x+h) + 7 + 2x^2 + 3x - 7}{h} \\
&= \frac{-2\left(x^2 + 2hx + h^2\right) - 3x - 3h + 2x^2 + 3x}{h} \\
&= \frac{-2x^2 - 4hx - 2h^2 - 3h + 2x^2}{h}
\end{aligned}$$

$$= \frac{-4hx - 2h^2 - 3h}{h}$$

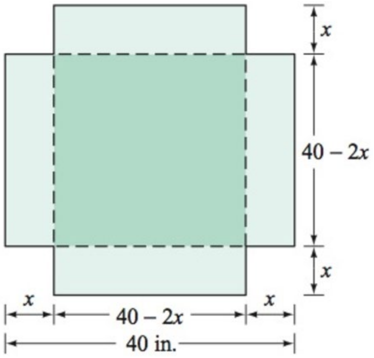
$$= -4x - 2h - 3$$

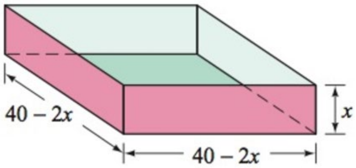
Exercise

An open box is to be made from a square piece of cardboard that measures 40 *inches* on each side, to construct the box, squares that measure x *inches* on each side are cut from each corner of the cardboard.

a) Express the volume V of the box as a function of x.

b) Determine the domain of V.





Solution

a) $V(x) = x(40 - 2x)^2$

$$= x\left(1600 - 160x + 4x^2\right)$$

$$= 4x^3 - 160x^2 + 1600x$$

b) $40 - 2x = 0$

$$x = 20$$

Domain: $\{x \mid 0 < x < 20\}$

Exercise

A child 4 *feet* tall is standing near a street lamp that is 12 *feet* high. The light from the lamp casts a shadow.

a) Find the length l of the shadow as a function of the distance d of the child from the lamppost.

b) What is the domain of the function?

c) What is the length of the shadow when the child is 8 *feet* from the base of the lamppost?

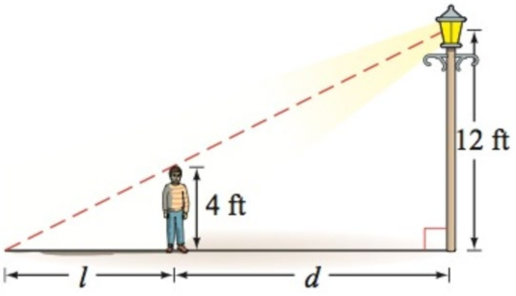
Solution

a) $\frac{l + d}{12} = \frac{l}{4}$

$$l + d = 3l$$

$$2l = d$$

$$l(d) = \frac{1}{2}d$$



b) ***Domain***: $\{x \mid 0 \le d < \infty\}$

c) ***Given***: $d = 8$

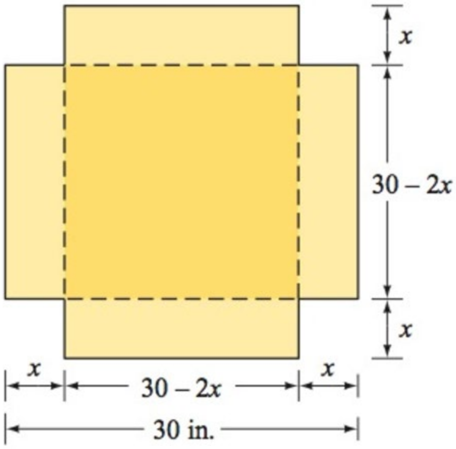
$l = 4$ *feet*

Exercise

An open box is to be made from a square piece of cardboard with the dimensions 30 *inches* by 30 *inches* by cutting out squares of area x^2 from each corner.

a) Express the volume V of the box as a function of x.

b) Determine the domain of V.



Solution

a) $V(x) = x(30-2x)^2$

$$= x\left(900 - 120x + 4x^2\right)$$

$$= 4x^3 - 120x^2 + 900x$$

b) $30 - 2x = 0$

$x = 15$

Domain: $\{x \mid 0 < x < 15\}$

Exercise

Two guy wires are attached to utility poles that are 40 *feet* apart.

a) Find the total length of the two guy wires as a function of x.

b) What is the domain of this function?

Solution

a) $\ell_1 = \sqrt{(40-x)^2 + 20^2}$

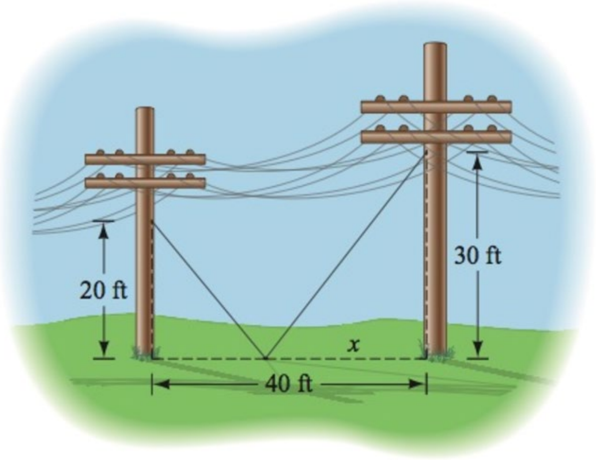
$$= \sqrt{1,600 - 80x + x^2 + 400}$$

$$= \sqrt{2,000 - 80x + x^2}$$

$\ell_2 = \sqrt{x^2 + 30^2}$

$$= \sqrt{x^2 + 900}$$

$$\ell(x) = \sqrt{2,000 - 80x + x^2} + \sqrt{x^2 + 900}$$



b) ***Domain***: $[0,\ 40]$

Exercise

A rancher has 360 *yards* of fencing with which to enclose two adjacent rectangular corrals, one for sheep and one for cattle. A river forms one side of the corrals. Suppose the width of each corral is *x* yards.

a) Express the total area of the two corrals as a function of *x*.

b) Find the domain of the function.

Solution

a) $P = 3x + l = 360$

$l = 360 - 3x$

$A = xl$

$= x(360 - 3x)$

$\boxed{A(x) = 360x - 3x^2}$

b) $x(360 - 3x) = 0$

$\boxed{x = 0}$

$360 - 3x = 0$

$3x = 360$

$\Rightarrow \boxed{x = 120}$

Domain: $\boxed{0 < x < 120}$

Exercise

A rectangle is bounded by the *x*- and *y-axis* of $y = -\frac{1}{2}x + 4$

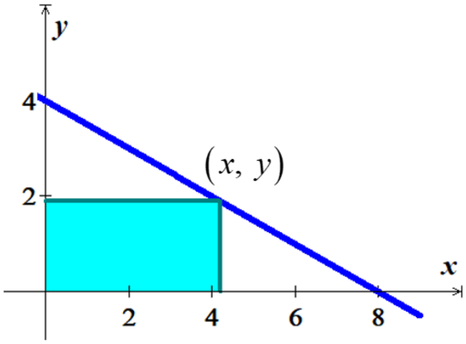
a) Find the area of the rectangle as a function of *x*.
b) What is the domain of this function.

Solution

a) $Area = xy$

$A(x) = x\left(-\frac{1}{2}x + 4\right)$

$\boxed{= -\frac{1}{2}x^2 + 4x}$



b) $x\left(-\frac{1}{2}x + 4\right) = 0$

$x = 0 \quad x = 8$

Domain: $\boxed{0 < x < 8}$

Solution *Section* 2.3 – Composition Functions

Exercise

Given that $f(x)=2x-5$ and $g(x)=x^2-3x+8$, find $(f\circ g)(x)$, $(g\circ f)(x)$ and their domain then find $(f\circ g)(7)$

Solution

$$f\left(g\left(x\right)\right)=f\left(x^2-3x+8\right) \qquad \textit{Domain: } \left(-\infty,\infty\right)$$

$$=2(\text{------------})-5$$

$$=2\left(2x^2-3x+8\right)-5$$

$$=2x^2-6x+16-5$$

$$=2x^2-6x+11 \qquad \textit{Domain: } \left(-\infty,\infty\right)$$

***Domain**:* $\mathbb{R}$

$$g\left(f\left(x\right)\right)=g\left(2x-5\right) \qquad \textit{Domain: } \left(-\infty,\infty\right)$$

$$=\left(---\right)^2-3\left(---\right)+8$$

$$=\left(2x-5\right)^2-3\left(2x-5\right)+8$$

$$=4x^2-20x+25-6x+15+8$$

$$=4x^2-26x+48 \qquad \textit{Domain: } \left(-\infty,\infty\right)$$

***Domain**:* $\mathbb{R}$

$$f\left(g\left(7\right)\right)=2(7)^2-6(7)+11=67$$

Exercise

Given that $f(x)=\sqrt{x}$ and $g(x)=x-1$, find

a) $(f\circ g)(x)=f(g(x))$

b) $(g\circ f)(x)=g(f(x))$

c) $(f\circ g)(2)=f(g(2))$

Solution

a) $(f\circ g)(x)=f(g(x))$

$$=f\left(x-1\right)$$

$$=\sqrt{x-1}$$

b) $$\begin{aligned}(g\circ f)(x) &= g(f(x))\\ &= g\left(\sqrt{x}\right)\\ &\boxed{=\sqrt{x}-1}\end{aligned}$$

c) $$\begin{aligned}(f\circ g)(2) &= f(g(2))\\ &= \sqrt{x-1}\\ &= \sqrt{2-1}\\ &\boxed{=1}\end{aligned}$$

Exercise

Given that $f(x)=\dfrac{x}{x+5}$ and $g(x)=\dfrac{6}{x}$, find

a) $(f\circ g)(x)=f(g(x))$

b) $(g\circ f)(x)=g(f(x))$

c) $(f\circ g)(2)=f(g(2))$

Solution

a) $$\begin{aligned}(f\circ g)(x) &= f(g(x))\\ &= f\left(\frac{6}{x}\right)\\ &= \frac{\frac{6}{x}}{\frac{6}{x}+5}\\ &= \frac{\frac{6}{x}}{\frac{6+5x}{x}}\\ &\boxed{=\frac{6}{6+5x}}\end{aligned}$$

b) $$\begin{aligned}(g\circ f)(x) &= g(f(x))\\ &= g\left(\frac{x}{x+5}\right)\\ &= \frac{6}{\frac{x}{x+5}}\\ &= \frac{6(x+5)}{x}\\ &\boxed{=\frac{6x+30}{x}}\end{aligned}$$

c) $(f\circ g)(2)=f(g(2))$

$$= \frac{6}{6+5(2)}$$

$$= \frac{6}{16}$$

$$\boxed{= \frac{3}{8}}$$

Exercise

Find $(f \circ g)(x)$, $(g \circ f)(x)$, $f(g(-2))$ *and* $g(f(3))$: $f(x) = 2x^2 + 3x - 4$, $g(x) = 2x - 1$

Solution

$$f(g(x)) = f(2x-1)$$

$$= 2(2x-1)^2 + 3(2x-1) - 4$$

$$= 2\left(4x^2 - 4x + 1\right) + 6x - 3 - 4$$

$$= 8x^2 - 8x + 2 + 6x - 7$$

$$\boxed{= 8x^2 - 2x - 5}$$

$$g(f(x)) = g\left(2x^2 + 3x - 4\right)$$

$$= 2\left(2x^2 + 3x - 4\right) - 1$$

$$= 4x^2 + 6x - 8 - 1$$

$$\boxed{= 4x^2 + 6x - 9}$$

$$f(g(-2)) = 8(-2)^2 - 2(-2) - 5$$

$$\boxed{= 31}$$

$$g(f(3)) = 4(3)^2 + 6(3) - 9$$

$$\boxed{= 45}$$

Exercise

Find $(f \circ g)(x)$, $(g \circ f)(x)$, $f(g(-2))$ *and* $g(f(3))$: $f(x) = x^3 + 2x^2$, $g(x) = 3x$

Solution

$$f(g(x)) = f(3x)$$

$$= (3x)^3 + 2(3x)^2$$

$$= 27x^3 + 18x^2$$

$$g(f(x)) = g\left(x^3 + 2x^2\right)$$

$$= 3\left(x^3 + 2x^2\right)$$

$$= 3x^3 + 6x^2$$

$$f(g(-2)) = 27(-2)^3 + 18(-2)^2$$

$$= 288$$

$$g(f(3)) = 3(3)^3 + 6(3)^2$$

$$= 135$$

Exercise

Find $(f \circ g)(x)$, $(g \circ f)(x)$, $f(g(-2))$ *and* $g(f(3))$: $f(x) = |x|$, $g(x) = -7$

Solution

$$f(g(x)) = f(-7)$$

$$= |-7|$$

$$= 7$$

$$g(f(x)) = g(|x|)$$

$$= -7$$

$$f(g(-2)) = 7$$

$$g(f(3)) = -7$$

Exercise

Given $f(x) = x - 3$ *and* $g(x) = x + 3$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(x+3)$ ***Domain***: $\mathbb{R}$

$$= (x-3) + 3$$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g\left(f(x)\right) = g(x-3)$ ***Domain***: $\mathbb{R}$

$= (x+3) - 3$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = \frac{2}{3}x$ *and* $g(x) = \frac{3}{2}x$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(\frac{3}{2}x\right)$ ***Domain***: $\mathbb{R}$

$= \frac{2}{3}\left(\frac{3}{2}x\right)$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g\left(f(x)\right) = g\left(\frac{2}{3}x\right)$ ***Domain***: $\mathbb{R}$

$= \frac{3}{2}\left(\frac{2}{3}x\right)$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = x - 1$ *and* $g(x) = 3x^2 - 2x - 1$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(3x^2 - 2x - 1\right)$ ***Domain***: $\mathbb{R}$

$= 3(x-1)^2 - 2(x-1) - 1$

$= 3\left(x^2 - 2x + 1\right) - 2x + 2 - 1$

$= 3x^2 - 6x + 3 - 2x + 1$

$\boxed{= 3x^2 - 8x + 4}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

b) $g\left(f(x)\right) = g(x-1)$ ***Domain***: $\mathbb{R}$

$= 3x^2 - 2x - 1 - 1$

$\boxed{= 3x^2 - 2x - 2}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x) = 3x - 2$ *and* $g(x) = x^2 - 5$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(x^2 - 5\right)$ ***Domain***: $\mathbb{R}$

$= 3\left(x^2 - 5\right) - 2$

$= 3x^2 - 15 - 2$

$\boxed{= 3x^2 - 17}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

b) $g\left(f(x)\right) = g(3x - 2)$ ***Domain***: $\mathbb{R}$

$= (3x - 2)^2 - 5$

$= 9x^2 - 12x + 4 - 5$

$\boxed{= 9x^2 - 12x - 1}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x) = x^2 - 2$ *and* $g(x) = 4x - 3$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(4x-3)$ **Domain**: $\mathbb{R}$

$$= (4x-3)^2 - 2$$

$$= 16x^2 - 24x + 9 - 2$$

$$\boxed{= 16x^2 - 24x + 7}$$ **Domain**: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

b) $g(f(x)) = g\left(x^2 - 2\right)$ **Domain**: $\mathbb{R}$

$$= 4\left(x^2 - 2\right) - 3$$

$$= 4x^2 - 8 - 3$$

$$\boxed{= 4x^2 - 11}$$ **Domain**: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x) = 4x^2 - x + 10$ *and* $g(x) = 2x - 7$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(2x-7)$ **Domain**: $\mathbb{R}$

$$= 4(2x-7)^2 - (2x-7) + 10$$

$$= 4\left(4x^2 - 28x + 49\right) - 2x + 7 + 10$$

$$= 16x^2 - 112x + 196 - 2x + 17$$

$$\boxed{= 16x^2 - 114x + 213}$$ **Domain**: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

b) $g(f(x)) = g\left(4x^2 - x + 10\right)$ **Domain**: $\mathbb{R}$

$$= 2\left(4x^2 - x + 10\right) - 7$$

$$= 8x^2 - 2x + 20 - 7$$

$$\boxed{= 8x^2 - 2x + 13} \qquad \textbf{Domain}: \mathbb{R}$$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x) = \sqrt{x}$ *and* $g(x) = x + 3$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(x+3)$ ***Domain***: $\mathbb{R}$

$\boxed{= \sqrt{x+3}}$ ***Domain***: $x \ge -3$

Domain: $\boxed{x \ge -3}$

b) $g(f(x)) = g\left(\sqrt{x}\right)$ ***Domain***: $x \ge 0$

$\boxed{= \sqrt{x} + 3}$ ***Domain***: $x \ge 0$

Domain: $\boxed{x \ge 0}$

Exercise

Given $f(x) = \sqrt{x}$ *and* $g(x) = 2 - 3x$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(2-3x)$ ***Domain***: $\mathbb{R}$

$\boxed{= \sqrt{2-3x}}$ ***Domain***: $x \le \frac{2}{3}$

Domain: $\boxed{x \le \frac{2}{3}}$

b) $g(f(x)) = g\left(\sqrt{x}\right)$ ***Domain***: $x \ge 0$

$\boxed{= 2 - 3\sqrt{x}}$ ***Domain***: $x \ge 0$

Domain: $\boxed{x \ge 0}$

Exercise

Given $f(x) = 3x + 2$ *and* $g(x) = \sqrt{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(\sqrt{x})$ ***Domain***: $x \geq 0$

$\boxed{= 3\sqrt{x}+2}$ ***Domain***: $x \geq 0$

Domain: $\boxed{x \geq 0}$

b) $g(f(x)) = g(3x + 2)$ ***Domain***: $\mathbb{R}$

$\boxed{= \sqrt{3x+2}}$ ***Domain***: $x \geq -\frac{2}{3}$

Domain: $\boxed{x \geq -\frac{2}{3}}$

Exercise

Given $f(x) = x^4$ *and* $g(x) = \sqrt[4]{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(\sqrt[4]{x})$ ***Domain***: $x \geq 0$

$= \left(\sqrt[4]{x}\right)^4$

$\boxed{= x}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{x \geq 0}$

b) $g(f(x)) = g(x^4)$ ***Domain***: $\mathbb{R}$

$= \sqrt[4]{x^4}$

$\boxed{= x}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x) = x^n$ *and* $g(x) = \sqrt[n]{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\sqrt[n]{x}\right)$

Domain: $\begin{cases} \text{If } n \text{ is even} & x \ge 0 \\ \text{If } n \text{ is odd} & \mathbb{R} \end{cases}$

$= \left(\sqrt[n]{x}\right)^n$

$= x$

Domain: $\mathbb{R}$

Domain: $\begin{cases} \text{If } n \text{ is even} & x \ge 0 \\ \text{If } n \text{ is odd} & \mathbb{R} \end{cases}$

b) $g(f(x)) = g\left(x^n\right)$

Domain: $\mathbb{R}$

$= \sqrt[n]{x^n}$

$= x$

Domain: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = x^2 - 3x$ *and* $g(x) = \sqrt{x+2}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\sqrt{x+2}\right)$

$x + 2 \ge 0 \Rightarrow x \ge -2$

$= \left(\sqrt{x+2}\right)^2 - 3\sqrt{x+2}$

$= x + 2 - 3\sqrt{x+2}$

$x + 2 \ge 0 \Rightarrow x \ge -2$

Domain: $\{x \mid x \ge -2\}$

b) $g(f(x)) = g\left(x^2 - 3x\right)$

$\mathbb{R}$

$= \sqrt{x^2 - 3x + 2}$

$x^2 - 3x + 2 \ge 0 \Rightarrow (x = 1,\ 2) \leftrightarrow x \le 1,\ x \ge 2$

Domain: $\{x \mid x \le 1,\ x \ge 2\}$

Exercise

Given $f(x)=\sqrt{x-2}$ *and* $g(x)=\sqrt{x+5}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\sqrt{x+5}\right)$ $\qquad x+5\ge 0\Rightarrow x\ge -5$

$=\sqrt{\sqrt{x+5}-2}$ $\qquad \sqrt{x+5}-2\ge 0\Rightarrow \sqrt{x+5}\ge 2$

$x+5\ge 4$

$x\ge -1$

Domain: $\{x\mid x\ge -1\}$

b) $g(f(x))=g\left(\sqrt{x-2}\right)$ $\qquad x-2\ge 0\Rightarrow x\ge 2$

$=\sqrt{\sqrt{x-2}+5}$ $\qquad \sqrt{x-2}+5\ge 0\Rightarrow \sqrt{x-2}\ge -5$ *Always true when* $x\ge 2$

Domain: $\{x\mid x\ge 2\}$

Exercise

Given $f(x)=x^2+2$ *and* $g(x)=\sqrt{3-x}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\sqrt{3-x}\right)$ $\qquad$ **Domain**: $x\le 3$

$=\left(\sqrt{3-x}\right)^2+2$

$=3-x+2$

$\underline{=5-x}$ $\qquad$ **Domain**: $\mathbb{R}$

Domain: $\underline{x\le 3}$

b) $g(f(x))=g\left(x^2+2\right)$ $\qquad$ **Domain**: $\mathbb{R}$

$=\sqrt{3-x^2-2}$

$\underline{=\sqrt{1-x^2}}$ $\qquad$ **Domain**: $-1\le x\le 1$

Domain: $\underline{-1\le x\le 1}$

Exercise

Given $f(x) = x^5 - 2$ *and* $g(x) = \sqrt[5]{x+2}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\sqrt[5]{x+2}\right)$ **Domain**: $\mathbb{R}$

$$= \left(\sqrt[5]{x+2}\right)^5 - 2$$

$$= x + 2 - 2$$

$$\boxed{= x}$$ **Domain**: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

b) $g(f(x)) = g\left(x^5 - 2\right)$ **Domain**: $\mathbb{R}$

$$= \sqrt[5]{x^5 - 2 + 2}$$

$$= \sqrt[5]{x^5}$$

$$\boxed{= x}$$ **Domain**: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x) = 1 - x^2$ *and* $g(x) = \sqrt{x^2 - 25}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\sqrt{x^2 - 25}\right)$ **Domain**: $x \le -5 \quad x \ge 5$

$$= 1 - \left(\sqrt{x^2 - 25}\right)^2$$

$$= 1 - \left(x^2 - 25\right)$$

$$= 1 - x^2 + 25$$

$$\boxed{= 26 - x^2}$$ **Domain**: $\mathbb{R}$

Domain: $\boxed{x \le -5 \quad x \ge 5}$

b) $g\left(f(x)\right)=g\left(1-x^2\right)$ ***Domain***: $\mathbb{R}$

$$=\sqrt{\left(1-x^2\right)^2-25}$$

$$=\sqrt{1-2x^2+x^4-25}$$

$$\boxed{=\sqrt{x^4-2x^2-24}}$$

$$x^2=\frac{2\pm\sqrt{4+96}}{2}$$

$$=\begin{cases}\frac{2-10}{2}=-4\ \times\\ \frac{2+10}{2}=6\end{cases}$$

$x^2=6 \quad\to\quad \boxed{x=\pm\sqrt{6}}$ ***Domain***: $x\le-\sqrt{6} \quad x\ge\sqrt{6}$

Domain: $\boxed{x\le-\sqrt{6} \quad x\ge\sqrt{6}}$

Exercise

Given $f(x)=2x+3$ *and* $g(x)=\frac{x-3}{2}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f\left(g(x)\right)=f\left(\frac{x-3}{2}\right)$ ***Domain***: $\mathbb{R}$

$$=2\left(\frac{x-3}{2}\right)+3$$

$$=x-3+3$$

$\boxed{=x}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

b) $g\left(f(x)\right)=g(2x+3)$ ***Domain***: $\mathbb{R}$

$$=\frac{1}{2}(2x+3-3)$$

$\boxed{=x}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x) = 4x - 5$ *and* $g(x) = \frac{x+5}{4}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(\frac{x+5}{4}\right)$ ***Domain***: $\mathbb{R}$

$= 4\left(\frac{x+5}{4}\right) - 5$

$= x + 5 - 5$

$\boxed{= x}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

b) $g\left(f(x)\right) = g(4x - 5)$ ***Domain***: $\mathbb{R}$

$= \frac{1}{4}(4x - 5 + 5)$

$\boxed{= x}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x) = \frac{4}{1-5x}$ *and* $g(x) = \frac{1}{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(\frac{1}{x}\right)$ ***Domain***: $x \neq 0$

$= \frac{4}{1 - 5\frac{1}{x}}$

$\boxed{= \frac{4x}{x-5}}$ ***Domain***: $x \neq 5$

Domain: $\boxed{x \neq 0,\ 5}$

b) $g\left(f(x)\right) = g\left(\frac{4}{1-5x}\right)$ ***Domain***: $x \neq \frac{1}{5}$

$\boxed{= \frac{1-5x}{4}}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{x \neq \frac{1}{5}}$

Exercise

Given $f(x) = \dfrac{1}{x-2}$ *and* $g(x) = \dfrac{x+2}{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\big(g(x)\big) = f\left(\dfrac{x+2}{x}\right)$ **Domain**: $x \neq 0$

$$= \frac{1}{\frac{x+2}{x} - 2}$$

$$= \frac{1}{\frac{x+2-2x}{x}}$$

$$\boxed{= \frac{x}{2-x}}$$ **Domain**: $x \neq 2$

Domain: $\boxed{x \neq 0,\ 2}$ $(-\infty,\ 0) \cup (0,\ 2) \cup (2,\ \infty)$

b) $g\big(f(x)\big) = g\left(\dfrac{1}{x-2}\right)$ **Domain**: $x \neq 2$

$$= \frac{\frac{1}{x-2} + 2}{\frac{1}{x-2}}$$

$$= \frac{\frac{1+2x-4}{x-2}}{\frac{1}{x-2}}$$

$$\boxed{= 2x-3}$$ **Domain**: $\mathbb{R}$

Domain: $\boxed{x \neq 2}$ $(-\infty,\ 2) \cup (2,\ \infty)$

Exercise

Given $f(x) = \dfrac{3x+5}{2}$ *and* $g(x) = \dfrac{2x-5}{3}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\big(g(x)\big) = f\left(\dfrac{2x-5}{3}\right)$ **Domain**: $\mathbb{R}$

$$= \frac{3\,\frac{2x-5}{3} + 5}{2}$$

$$= \frac{2x-5+5}{2}$$

$$= \frac{2x}{2}$$

$$\underline{= x}\rfloor$$

Domain: $\underline{\mathbb{R}}\rfloor$

b) $g\left(f(x)\right) = g\left(\frac{3x+5}{2}\right)$ ***Domain***: $\mathbb{R}$

$$= \frac{2\frac{3x+5}{2} - 5}{3}$$

$$= \frac{3x+5-5}{3}$$

$$= \frac{3x}{3}$$

$$\underline{= x}\rfloor$$

Domain: $\underline{\mathbb{R}}\rfloor$

Exercise

Given $f(x) = \frac{1}{1+x}$ *and* $g(x) = \frac{1-x}{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(\frac{1-x}{x}\right)$ ***Domain***: $x \neq 0$

$$= \frac{1}{1+\frac{1-x}{x}}$$

$$= \frac{x}{x+1-x}$$

$\underline{= x}\rfloor$ ***Domain***: $\mathbb{R}$

Domain: $\underline{x \neq 0}\rfloor$

b) $g\left(f(x)\right) = g\left(\frac{1}{x+1}\right)$ ***Domain***: $x \neq -1$

$$= \frac{1-\frac{1}{x+1}}{\frac{1}{x+1}}$$

$$= x+1-1$$

$\underline{= x}\rfloor$ ***Domain***: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}\rfloor$

Exercise

Given $f(x)=\dfrac{x-1}{x-2}$ *and* $g(x)=\dfrac{x-3}{x-4}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f\big(g(x)\big)=f\left(\dfrac{x-3}{x-4}\right)$

Domain: $x\neq 4$

$$=\frac{\frac{x-3}{x-4}-1}{\frac{x-3}{x-4}-2}$$

$$=\frac{\frac{x-3-(x-4)}{x-4}}{\frac{x-3-2(x-4)}{x-4}}$$

$$=\frac{x-3+x+4}{x-3-2x+8}$$

$$=\frac{2x+1}{-x+5}$$

Domain: $x\neq 5$

Domain: $\{x \mid x\neq 4,\ 5\}$

b) $g\big(f(x)\big)=g\left(\dfrac{x-1}{x-2}\right)$

Domain: $x\neq 2$

$$=\frac{\frac{x-1}{x-2}-3}{\frac{x-1}{x-2}-4}$$

$$=\frac{x-1-3(x-2)}{x-1-4(x-2)}$$

$$=\frac{x-1-3x+6}{x-1-4x+8}$$

$$=\frac{-2x+5}{-3x+7}$$

Domain: $x\neq \frac{7}{3}$

Domain: $\left\{x \mid x\neq 2,\ \frac{7}{3}\right\}$

Exercise

Given $f(x)=\dfrac{6}{x-3}$ *and* $g(x)=\dfrac{1}{x}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $(f\circ g)(x)$

$$f\left(g(x)\right)=f\left(\frac{1}{x}\right) \qquad \textbf{\textit{Domain}}: x\neq 0$$

$$=\frac{6}{\frac{1}{x}-3}$$

$$=\frac{6}{\frac{1-3x}{x}}$$

$$=\frac{6x}{1-3x} \qquad \textbf{\textit{Domain}}: x\neq \frac{1}{3}$$

Domain: $\underline{x\neq 0,\ \frac{1}{3}}\,\big|$ $\qquad$ $\underline{(-\infty,0)\cup\left(0,\frac{1}{3}\right)\cup\left(\frac{1}{3},\infty\right)}\,\big|$

b) $(g\circ f)(x)$

$$g\left(f(x)\right)=g\left(\frac{6}{x-3}\right) \qquad \textbf{\textit{Domain}}: x\neq 3$$

$$=\frac{1}{\frac{6}{x-3}}$$

$$=\frac{x-3}{6} \qquad \textbf{\textit{Domain}}: (-\infty,\infty)$$

Domain: $\underline{x\neq 3}\,\big|$ $\qquad$ $(-\infty,3)\cup(3,\infty)$

Exercise

Given $f(x)=\dfrac{6}{x}$ *and* $g(x)=\dfrac{1}{2x+1}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f\left(g(x)\right)=f\left(\dfrac{1}{2x+1}\right)$ $\qquad$ ***Domain***: $x\neq -\dfrac{1}{2}$

$$=\frac{6}{\frac{1}{2x+1}}$$

$$\underline{=12x+6}\,\big| \qquad \textbf{\textit{Domain}}: \mathbb{R}$$

Domain: $\boxed{x \neq -\frac{1}{2}}$

b) $g\left(f(x)\right) = g\left(\frac{6}{x}\right)$ ***Domain***: $x \neq 0$

$$= \frac{1}{2\frac{6}{x}+1}$$

$$\boxed{= \frac{x}{12+x}}$$ ***Domain***: $x \neq -12$

Domain: $\boxed{x \neq -12,\ 0}$

Exercise

Given $f(x) = 3x - 7$ *and* $g(x) = \frac{x+7}{3}$

a) Find $\left(f \circ g\right)(x)$ and the domain of $f \circ g$

b) Find $\left(g \circ f\right)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(\frac{x+7}{3}\right)$ ***Domain***: $\mathbb{R}$

$$= 3\frac{x+7}{3} - 7$$

$$= x + 7 - 7$$

$$\boxed{= x}$$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

b) $g\left(f(x)\right) = g\left(3x - 7\right)$ ***Domain***: $\mathbb{R}$

$$= \frac{3x - 7 + 7}{3}$$

$$\boxed{= x}$$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x) = \frac{2x+3}{x-4}$ *and* $g(x) = \frac{4x+3}{x-2}$

a) Find $\left(f \circ g\right)(x)$ and the domain of $f \circ g$

b) Find $\left(g \circ f\right)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(\frac{4x+3}{x-2}\right)$ ***Domain***: $x \neq 2$

$$= \frac{2\,\frac{4x+3}{x-2}+3}{\frac{4x+3}{x-2}-4}$$

$$= \frac{8x+6+3x-6}{4x+3-4x+8}$$

$$= \frac{11x}{11}$$

$\boxed{= x}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{x \neq 2}$

b) $g\left(f(x)\right) = g\left(\frac{2x+3}{x-4}\right)$ ***Domain***: $x \neq 4$

$$= \frac{4\,\frac{2x+3}{x-4}+3}{\frac{2x+3}{x-4}-2}$$

$$= \frac{8x+12+3x-4}{2x+3-2x+8}$$

$$= \frac{11x}{11}$$

$\boxed{= x}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{x \neq 4}$

Exercise

Given $f(x) = \frac{2x+3}{x+4}$ *and* $g(x) = \frac{-4x+3}{x-2}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(\frac{-4x+3}{x-2}\right)$ ***Domain***: $x \neq 2$

$$= \frac{2\,\frac{-4x+3}{x-2}+3}{\frac{4x+3}{x-2}+4}$$

$$= \frac{-8x+6+3x-6}{4x+3+4x-8}$$

$$= \frac{-5x}{-5}$$

$\boxed{= x}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{x \neq 2}$

b) $g\left(f(x)\right) = g\left(\frac{2x+3}{x+4}\right)$ **Domain**: $x \neq -4$

$$= \frac{-4\frac{2x+3}{x+4}+3}{\frac{2x+3}{x+4}-2}$$

$$= \frac{-8x-12+3x+12}{2x+3-2x-8}$$

$$= \frac{-5x}{-5}$$

$\boxed{= x}$ **Domain**: $\mathbb{R}$

Domain: $\boxed{x \neq -4}$

Exercise

Given $f(x) = x+1$ *and* $g(x) = x^3 - 5x^2 + 3x + 7$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(x^3 - 5x^2 + 3x + 7\right)$ **Domain**: $\mathbb{R}$

$$= x^3 - 5x^2 + 3x + 7 + 1$$

$\boxed{= x^3 - 5x^2 + 3x + 8}$ **Domain**: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

b) $g\left(f(x)\right) = g(x+1)$ **Domain**: $\mathbb{R}$

$$= (x+1)^3 - 5(x+1)^2 + 3(x+1) + 7$$

$$= x^3 + 3x^2 + 3x + 1 - 5\left(x^2 + 2x + 1\right) + 3x + 3 + 7$$

$$= x^3 + 3x^2 + 6x + 11 - 5x^2 - 10x - 5$$

$\boxed{= x^3 - 2x^2 - 4x + 6}$ **Domain**: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x)=x-1$ *and* $g(x)=x^3+2x^2-3x-9$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(x^3+2x^2-3x-9\right)$ ***Domain***: $\mathbb{R}$

$$=x^3+2x^2-3x-9-1$$

$$=x^3+2x^2-3x-10$$

Domain: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g(f(x))=g(x-1)$ ***Domain***: $\mathbb{R}$

$$=(x-1)^3+2(x-1)^2-(x-1)-9$$

$$=x^3-3x^2+3x-1+2\left(x^2-2x+1\right)-3x+3-9$$

$$=x^3-3x^2-7+2x^2-4x+2$$

$$=x^3-x^2-4x-5$$

Domain: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$: $(f\circ g)(4)$

Solution

$$(f\circ g)(4)=f(g(4))$$

$$=f(16-20)$$

$$=f(-4)$$

$$=-8-3$$

$$=-11$$

Exercise

Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$: $(g\circ f)(4)$

Solution

$$(g\circ f)(4)=g(f(4))$$

$$= g(8-3)$$

$$= g(5)$$

$$= 25 - 25$$

$$\underline{= 0}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ *and* $g(x) = x^2 - 5x$: $(f \circ g)(-2)$

Solution

$$\begin{aligned}(f \circ g)(-2) &= f(g(-2)) \\ &= f(4+10) \\ &= f(14) \\ &= 28 - 3 \\ &\underline{= 25}\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ *and* $g(x) = x^2 - 5x$: $(g \circ f)(-2)$

Solution

$$\begin{aligned}(g \circ f)(-2) &= g(f(-2)) \\ &= g(-4-3) \\ &= g(-7) \\ &= 49 + 35 \\ &\underline{= 84}\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ *and* $g(x) = x^2 - 5x$: $(f \circ f)(-3)$

Solution

$$\begin{aligned}(f \circ f)(-3) &= f(f(-3)) \\ &= f(-6-3) \\ &= f(-9) \\ &= -18 - 3 \\ &\underline{= -21}\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ *and* $g(x) = x^2 - 5x$: $(g \circ g)(7)$

Solution

$$\begin{aligned}(g \circ g)(7) &= g(g(7)) \\ &= g(49 - 35) \\ &= g(14) \\ &= 196 - 70 \\ &\underline{= 126 \,|}\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ *and* $g(x) = x^2 - 5x$: $(f \circ g)(\sqrt{2})$

Solution

$$\begin{aligned}(f \circ g)(\sqrt{2}) &= f\left(g\left(\sqrt{2}\right)\right) \\ &= f\left(2 - 5\sqrt{2}\right) \\ &= 2\left(2 - 5\sqrt{2}\right) - 3 \\ &= 4 - 10\sqrt{2} - 3 \\ &\underline{= 1 - 10\sqrt{2} \,|}\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ *and* $g(x) = x^2 - 5x$: $(g \circ f)(\sqrt{3})$

Solution

$$\begin{aligned}(g \circ f)(\sqrt{3}) &= g\left(f\left(\sqrt{3}\right)\right) \\ &= g\left(2\sqrt{3} - 3\right) \\ &= \left(2\sqrt{3} - 3\right)^2 - 5\left(2\sqrt{3} - 3\right) \\ &= 12 - 12\sqrt{3} + 9 - 10\sqrt{3} + 15 \\ &\underline{= 36 - 22\sqrt{3} \,|}\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$: $(f\circ g)(2a)$

Solution

$$\begin{aligned}(f\circ g)(2a)&=f(g(2a))\\&=f\left(4a^2-10a\right)\\&=2\left(4a^2-10a\right)-3\\&=8a^2-20a-3\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$: $(g\circ f)(3b)$

Solution

$$\begin{aligned}(g\circ f)(3b)&=g(f(3b))\\&=g(6b-3)\\&=(6b-3)^2-5(6b-3)\\&=36b^2-36b+9-30b+15\\&=36b^2-66b+24\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$: $(f\circ g)(k+1)$

Solution

$$\begin{aligned}(f\circ g)(k+1)&=f(g(k+1))\\&=f\left((k+1)^2-5k-5\right)\\&=2\left((k+1)^2-5k-5\right)-3\\&=2\left(k^2+2k+1\right)-10k-10-3\\&=2k^2+4k+2-10k-13\\&=2k^2-6k-11\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3 \quad and \quad g(x) = x^2 - 5x$: $(g \circ f)(k-1)$

Solution

$$\begin{aligned}
(g \circ f)(k-1) &= g\left(f(k-1)\right) \\
&= g(2k - 2 - 3) \\
&= g(2k - 5) \\
&= (2k-5)^2 - 5(2k-5) \\
&= 4k^2 - 20k + 25 - 10k + 25 \\
&= 4k^2 - 30k + 50
\end{aligned}$$

Solution Section 2.4 – Properties of Division

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=2x^4-x^3+7x-12$; $p(x)=x^2-3$

Solution

$$\begin{array}{r}
2x^2-x+6 \\
x^2-3\overline{\big)2x^4-x^3+0x^2+7x-12} \\
\underline{2x^4 \qquad -6x^2 \qquad\qquad} \\
-x^3+6x^2+7x \\
\underline{-x^3 \qquad\quad +3x} \\
6x^2+4x-12 \\
\underline{6x^2 \qquad -18} \\
4x+6
\end{array}$$

$$Q(x)=2x^2-x+6; \quad R(x)=4x+6$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=3x^3+2x-4$; $p(x)=2x^2+1$

Solution

$$\begin{array}{r}
\frac{3}{2}x \\
2x^2+1\overline{\big)3x^3+0x^2+2x-4} \\
\underline{3x^3 \qquad +\frac{3}{2}x } \\
\frac{1}{2}x-4
\end{array}$$

$$Q(x)=\frac{3}{2}x; \quad R(x)=\frac{1}{2}x-4$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=7x+2;\quad p(x)=2x^2-x-4$

Solution

$$
\begin{array}{r|l}
 & \frac{2}{7}x-\frac{11}{49} \\ \hline
7x+2 & 2x^2-x-4 \\
 & \underline{2x^2+\frac{4}{7}x} \\
 & -\frac{11}{7}x-4 \\
 & \underline{-\frac{11}{7}x-\frac{22}{49}} \\
 & -\frac{174}{49}
\end{array}
$$

$$\boxed{Q(x)=\frac{2}{7}x-\frac{11}{49}}\quad \boxed{R(x)=-\frac{174}{49}}$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=9x+4;\quad p(x)=2x-5$

Solution

$$
\begin{array}{r|l}
 & \frac{9}{2} \\ \hline
2x-5 & 9x+4 \\
 & \underline{9x-\frac{45}{2}} \\
 & -\frac{37}{2}
\end{array}
$$

$$\boxed{Q(x)=\frac{9}{2};\quad R(x)=-\frac{37}{2}}$$

Exercise

Use the remainder theorem to find $f(c)$: $f(x)=x^4-6x^2+4x-8;\quad c=-3$

Solution

$$f(-3)=(-3)^4-6(-3)^2+4(-3)-8$$

$$\boxed{=7}$$

Exercise

Use the remainder theorem to find $f(c)$: $f(x) = x^4 + 3x^2 - 12; \quad c = -2$

Solution

$$f(-2) = (-2)^4 + 3(-2)^2 - 12$$

$$= 16$$

Exercise

Use the factor theorem to show that $x - c$ is a factor of $f(x)$: $f(x) = x^3 + x^2 - 2x + 12; \quad c = -3$

Solution

$$f(-3) = (-3)^3 + (-3)^2 - 2(-3) + 12$$

$$= 0$$

From the factor theorem; $x + 3$ is a factor of $f(x)$.

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $2x^3 - 3x^2 + 4x - 5; \quad x - 2$

Solution

$$\begin{array}{r|rrrr} 2 & 2 & -3 & 4 & -5 \\ & & 4 & 2 & 12 \\ \hline & 2 & 1 & 6 & \boxed{7} \end{array}$$

$$Q(x) = 2x^2 + x + 6 \qquad R(x) = 7$$

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $5x^3 - 6x^2 + 15; \quad x - 4$

Solution

$$\begin{array}{r|rrrr} 4 & 5 & -6 & 0 & 15 \\ & & 20 & 56 & 224 \\ \hline & 5 & 14 & 56 & \boxed{239} \end{array}$$

$$Q(x) = 5x^2 + 14x + 56 \qquad R(x) = 239$$

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $9x^3-6x^2+3x-4; \quad x-\frac{1}{3}$

Solution

$$\begin{array}{r|rrrr}
\frac{1}{3} & 9 & -6 & 3 & -4 \\
 & & 3 & -1 & \frac{2}{3} \\
\hline
 & 9 & -3 & 2 & \boxed{-\frac{10}{3}}
\end{array}$$

$$\boxed{Q(x)=9x^2-3x+2 \qquad R(x)=-\frac{10}{3}}$$

Exercise

Use the synthetic division to find $f(c)$: $f(x)=2x^3+3x^2-4x+4; \quad c=3$

Solution

$$\begin{array}{r|rrrr}
3 & 2 & 3 & -4 & 4 \\
 & & 9 & 36 & 93 \\
\hline
 & 3 & 12 & 32 & \boxed{97}
\end{array}$$

$$\boxed{f(3)=97}$$

Exercise

Use the synthetic division to find $f(c)$: $f(x)=8x^5-3x^2+7; \quad c=\frac{1}{2}$

Solution

$$\begin{array}{r|rrrrrr}
\frac{1}{2} & 8 & 0 & 0 & -3 & 0 & 7 \\
 & & 4 & 2 & 1 & -1 & -\frac{1}{2} \\
\hline
 & 8 & 4 & 2 & -2 & -1 & \boxed{\frac{13}{2}}
\end{array}$$

$$\boxed{f\left(\frac{1}{2}\right)=\frac{13}{2}}$$

Exercise

Use the synthetic division to find $f(c)$: $f(x) = x^3 - 3x^2 - 8; \quad c = 1 + \sqrt{2}$

Solution

$$\begin{array}{r|rrrr} 1+\sqrt{2} & 3 & -3 & 0 & -8 \\ & & 3+3\sqrt{2} & 6+3\sqrt{2} & 12+9\sqrt{2} \\ \hline & 3 & 3\sqrt{2} & 6+3\sqrt{2} & \boxed{4+9\sqrt{2}} \end{array}$$

$f\left(1+\sqrt{2}\right) = 4 + 9\sqrt{2}$

Exercise

Use the synthetic division to show that c is a zero of $f(x)$: $f(x) = 3x^4 + 8x^3 - 2x^2 - 10x + 4; \quad c = -2$

Solution

$$\begin{array}{r|rrrrr} -2 & 3 & 8 & -2 & -10 & 4 \\ & & -6 & -4 & 12 & -4 \\ \hline & 3 & 2 & -6 & 2 & \boxed{0} \end{array}$$

$f(-2) = 0$

Exercise

Use the synthetic division to show that c is a zero of $f(x)$: $f(x) = 27x^4 - 9x^3 + 3x^2 + 6x + 1; \quad c = -\frac{1}{3}$

Solution

$$\begin{array}{r|rrrrr} -\frac{1}{3} & 27 & -9 & 3 & 6 & 1 \\ & & -9 & 6 & -3 & -1 \\ \hline & 27 & -18 & 9 & 3 & \boxed{0} \end{array}$$

$f\left(-\frac{1}{3}\right) = 0$

Exercise

Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

$f(x) = kx^3 + x^2 + k^2x + 3k^2 + 11; \quad x + 2$

Solution

$$\begin{array}{r|cccc} -2 & k & 1 & k^2 & 3k^2+11 \\ & & -2k & 4k-2 & -2k^2-8k+4 \\ \hline & k & 1-2k & k^2+4k-2 & k^2-8k+15 \end{array}$$

$$k^2-8k+15=0 \Rightarrow \boxed{k=3,\ 5}$$

Exercise

Find all solutions of the equation: $x^3-x^2-10x-8=0$

Solution

$$\textit{possibilities for } \tfrac{c}{d}: \pm\left\{\frac{8}{1}\right\} = \pm\{1,\ 2,\ 4,\ 8\}$$

$$\begin{array}{r|cccc} -1 & 1 & -1 & -10 & -8 \\ & & -1 & 2 & 8 \\ \hline & 1 & -2 & -8 & 0 \end{array} \to \quad x^2-2x-8=0$$

The solutions are: $x=-1,\ -2,\ 4$

Exercise

Find all solutions of the equation: $x^3+x^2-14x-24=0$

Solution

$$\textit{possibilities for } \tfrac{c}{d}: \pm\left\{\frac{24}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24\}$$

$$\begin{array}{r|cccc} -2 & 1 & 1 & -14 & -24 \\ & & -2 & 2 & 24 \\ \hline & 1 & -1 & -12 & 0 \end{array} \to \quad x^2-x-12=0$$

The solutions are: $x=-2,\ -3,\ 4$

Exercise

Find all solutions of the equation: $2x^3-3x^2-17x+30=0$

Solution

$$\textit{possibilities for } \tfrac{c}{d}: \pm\left\{\frac{30}{2}\right\} = \pm\left\{1,\ 2,\ 3,\ 5,\ 6,\ 10,\ 15,\ 30,\ \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \frac{15}{2}\right\}$$

$$\begin{array}{r|rrrr} 2 & 2 & -3 & -17 & 30 \\ & & 4 & 2 & -30 \\ \hline & 2 & 1 & -15 & \boxed{0} \end{array} \rightarrow \quad 2x^2+x-15=0$$

The solutions are: $x=2,\ -3,\ \frac{5}{2}$

Exercise

Find all solutions of the equation: $12x^3+8x^2-3x-2=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{2}{12}\right\}=\pm\left\{1,\ 2,\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{2},\ \frac{1}{6},\ \frac{1}{12}\right\}$

$$\begin{array}{r|rrrr} \frac{1}{2} & 12 & 8 & -3 & -2 \\ & & 6 & 7 & 2 \\ \hline & 12 & 14 & 4 & \boxed{0} \end{array} \rightarrow \quad 12x^2+14x+4=0$$

The solutions are: $x=\frac{1}{2},\ -\frac{1}{2},\ -\frac{2}{3}$

Exercise

Find all solutions of the equation: $x^3+x^2-6x-8=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{8}{1}\right\}=\pm\{1,\ 2,\ 4,\ 8\}$

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -6 & -8 \\ & & -2 & 2 & 8 \\ \hline & 1 & -1 & -4 & \boxed{0} \end{array} \rightarrow \quad x^2-x-4=0$$

$$x=\frac{1\pm\sqrt{1+16}}{2}$$

The solutions are: $x=-2,\ \frac{1\pm\sqrt{17}}{2}$

Exercise

Find all solutions of the equation: $x^3-19x-30=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{30}{1}\right\}=\pm\{1,\ 2,\ 3,\ 5,\ 6,\ 15,\ 30\}$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -19 & -30 \\ & & -2 & 4 & 30 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array} \rightarrow \quad x^2 - 2x - 15$$

$$x = \frac{2 \pm \sqrt{4+60}}{2}$$

$$= \begin{cases} \frac{2-8}{2} = -3 \\ \frac{2+8}{2} = 5 \end{cases}$$

The solutions are: $\underline{x = -2, \ -3, \ 5}$

Exercise

Find all solutions of the equation: $2x^3 + x^2 - 25x + 12 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{12}{2} \right\} = \pm \left\{ 1, \ 2, \ 3, \ 4, \ 6, \ 12, \ \frac{1}{2}, \ \frac{3}{2} \right\}$

$$\begin{array}{r|rrrr} 3 & 2 & 1 & -25 & 12 \\ & & 6 & 21 & -12 \\ \hline & 2 & 7 & -4 & \boxed{0} \end{array} \rightarrow \quad 2x^2 + 7x - 4$$

$$x = \frac{-7 \pm \sqrt{49+32}}{4}$$

$$= \begin{cases} \frac{-7-9}{4} = -4 \\ \frac{-7+9}{4} = \frac{1}{2} \end{cases}$$

The solutions are: $\underline{x = -4, \ \frac{1}{2}, \ 3}$

Exercise

Find all solutions of the equation: $3x^3 + 11x^2 - 6x - 8 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{8}{3} \right\} = \pm \left\{ 1, \ 2, \ 4, \ 8, \ \frac{1}{3}, \ \frac{2}{3}, \ \frac{4}{3}, \ \frac{8}{3} \right\}$

$$\begin{array}{r|rrrr} 1 & 3 & 11 & -6 & -8 \\ & & 3 & 14 & 8 \\ \hline & 3 & 14 & 8 & \boxed{0} \end{array} \rightarrow \quad 3x^2 + 14x + 8$$

$$x = \frac{-14 \pm \sqrt{196-96}}{6}$$

$$= \begin{cases} \frac{-14-10}{6} = -4 \\ \frac{-14+10}{6} = -\frac{2}{3} \end{cases}$$

The solutions are: $x = -4,\ -\frac{2}{3},\ 1$

Exercise

Find all solutions of the equation: $2x^3 + 9x^2 - 2x - 9 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{9}{2}\right\} = \pm\left\{1,\ 3,\ 9,\ \frac{1}{2},\ \frac{3}{2},\ \frac{9}{2}\right\}$

$$\begin{array}{r|rrrr} 1 & 2 & 9 & -2 & -9 \\ & & 2 & 11 & 9 \\ \hline & 2 & 11 & 9 & \boxed{0} \end{array} \rightarrow \ 2x^2 + 11x + 9$$

$x = -1,\ -\frac{9}{2}$ $\qquad a - b + c = 0 \ \rightarrow\ x = -1,\ -\frac{c}{a}$

The solutions are: $x = -\frac{9}{2},\ -1,\ 1$

Exercise

Find all solutions of the equation: $x^3 + 3x^2 - 6x - 8 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{8}{1}\right\} = \pm\{1,\ 2,\ 4,\ 8\}$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & -6 & -8 \\ & & -1 & -2 & 8 \\ \hline & 1 & 2 & -8 & \boxed{0} \end{array} \rightarrow \ x^2 + 2x - 8 = 0$$

$$x = \frac{-2 \pm \sqrt{4+32}}{2}$$

$$= \begin{cases} \frac{-2-6}{2} = -4 \\ \frac{-2+6}{2} = 2 \end{cases}$$

The solutions are: $x = -4,\ -1,\ 2$

Exercise

Find all solutions of the equation: $3x^3 - x^2 - 6x + 2 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{2}{3}\right\} = \pm\left\{1,\ 2,\ \frac{1}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -1 & -6 & 2 \\ & & 1 & 0 & -2 \\ \hline & 3 & 0 & -6 & \boxed{0} \end{array} \quad \rightarrow \quad 3x^2 - 6 = 0$$

$x^2 = 2$

The solutions are: $x = \frac{1}{3},\ \pm\sqrt{2}$

Exercise

Find all solutions of the equation: $x^3 - 8x^2 + 8x + 24 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{24}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24\}$

$$\begin{array}{r|rrrr} 6 & 1 & -8 & 8 & 24 \\ & & 6 & -12 & -24 \\ \hline & 1 & -2 & -4 & \boxed{0} \end{array} \quad \rightarrow \quad x^2 - 2x - 4 = 0$$

$$x = \frac{2 \pm \sqrt{4+16}}{2}$$

$$= \frac{2 \pm 2\sqrt{5}}{2}$$

The solutions are: $x = 6,\ 1 \pm \sqrt{5}$

Exercise

Find all solutions of the equation: $x^3 - 7x^2 - 7x + 69 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{69}{1}\right\} = \pm\{1,\ 3,\ 23,\ 69\}$

$$\begin{array}{r|rrrr} -3 & 1 & -7 & -7 & 69 \\ & & -3 & 30 & -69 \\ \hline & 1 & -10 & 23 & \boxed{0} \end{array} \quad \rightarrow \quad x^2 - 10x + 23 = 0$$

$$x = \frac{10 \pm \sqrt{100-92}}{2}$$

$$= \frac{10 \pm 2\sqrt{2}}{2}$$

The solutions are: $\left.\underline{x = -3,\ 5 \pm \sqrt{2}}\,\right|$

Exercise

Find all solutions of the equation: $x^3 - 3x - 2 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & -2 \\ & & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array} \quad \to \quad x^2 - x - 2 = 0$$

$\left.\underline{x = -1,\ 2}\right|$ $\qquad a - b + c = 0 \quad \to \quad x = -1,\ -\frac{c}{a}$

The solutions are: $\left.\underline{x = -1,\ \ -1,\ 2}\,\right|$

Exercise

Find all solutions of the equation: $x^3 - 2x + 1 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\{1\}$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -2 & 1 \\ & & 1 & 1 & 1 \\ \hline & 1 & 1 & -1 & \boxed{0} \end{array} \quad \to \quad x^2 + x - 1 = 0$$

$$x = \frac{-1 \pm \sqrt{5}}{2}$$

The solutions are: $\left.\underline{x = 1,\ \frac{-1 \pm \sqrt{5}}{2}}\,\right|$

Exercise

Find all solutions of the equation: $x^3-2x^2-11x+12=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{12}{1}\right\}=\pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -11 & 12 \\ & & 1 & -1 & 12 \\ \hline & 1 & -1 & -12 & \boxed{0} \end{array} \quad \rightarrow \quad x^2-x-12=0$$

$$x=\frac{1\pm\sqrt{1+48}}{2}$$

$$=\begin{cases}\frac{1-7}{2}=-3\\ \frac{1+7}{2}=4\end{cases}$$

The solutions are: $\underline{x=-3,\ 1,\ 4}|$

Exercise

Find all solutions of the equation: $x^3-2x^2-7x-4=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{4}{1}\right\}=\pm\{1,\ 2,\ 4\}$

$$\begin{array}{r|rrrr} -1 & 1 & -2 & -7 & -4 \\ & & -1 & 3 & 4 \\ \hline & 1 & -3 & -4 & \boxed{0} \end{array} \quad \rightarrow \quad x^2-3x-4=0$$

$\underline{x=-1,\ 4}|$ $\qquad a-b+c=0 \ \rightarrow \ x=-1,\ -\frac{c}{a}$

The solutions are: $\underline{x=-1,\ -1,\ 4}|$

Exercise

Find all solutions of the equation: $x^3-10x-12=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{12}{1}\right\}=\pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -10 & -12 \\ & & -2 & 4 & 12 \\ \hline & 1 & -2 & -6 & \boxed{0} \end{array} \quad \to \quad x^2-2x-6=0$$

$$\begin{aligned} x &= \frac{2 \pm \sqrt{4+24}}{2} \\ &= \frac{2 \pm 2\sqrt{7}}{2} \end{aligned}$$

The solutions are: $\underline{x = -2,\ 1 \pm \sqrt{7}}\big|$

Exercise

Find all solutions of the equation: $x^3 - 5x^2 + 17x - 13 = 0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{13}{1}\right\} = \pm\{1,\ 13\}$

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 17 & -13 \\ & & 1 & -4 & 13 \\ \hline & 1 & -4 & 13 & \boxed{0} \end{array} \quad \to \quad x^2-4x+13=0$$

$$\begin{aligned} x &= \frac{4 \pm \sqrt{16-52}}{2} \\ &= \frac{4 \pm 6i}{2} \end{aligned}$$

The solutions are: $\underline{x = 1,\ 2 \pm 3i}\big|$

Exercise

Find all solutions of the equation: $6x^3 + 25x^2 - 24x + 5 = 0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{5}{6}\right\} = \pm\left\{1,\ 5,\ \frac{1}{2},\ \frac{5}{2},\ \frac{1}{3},\ \frac{5}{3},\ \frac{1}{6},\ \frac{5}{6}\right\}$

$$\begin{array}{r|rrrr} -5 & 6 & 25 & -24 & 5 \\ & & -30 & 25 & -5 \\ \hline & 6 & -5 & 1 & \boxed{0} \end{array} \quad \to \quad 6x^2-5x+1=0$$

$$x = \frac{5 \pm \sqrt{25-24}}{12}$$

$$= \begin{cases} \frac{5-1}{12} = \frac{1}{3} \\ \frac{5+1}{12} = \frac{1}{2} \end{cases}$$

The solutions are: $x = -5,\ \frac{1}{3},\ \frac{1}{2}$

Exercise

Find all solutions of the equation: $8x^3 + 18x^2 + 45x + 27 = 0$

Solution

$$possibilities: \ \pm\left\{\frac{27}{8}\right\} = \pm\left\{\frac{1,\ 3,\ 9,\ 27}{1,\ 2,\ 4,\ 8}\right\}$$

$$= \pm\left\{1,\ 3,\ 9,\ 27,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{3}{2},\ \frac{3}{4},\ \frac{3}{8},\ \frac{9}{2},\ \frac{9}{4},\ \frac{9}{8},\ \frac{27}{2},\ \frac{27}{4},\ \frac{27}{8}\right\}$$

$$\begin{array}{r|rrrr} -\frac{3}{4} & 8 & 18 & 45 & 27 \\ & & -6 & -9 & -27 \\ \hline & 8 & 12 & 36 & \boxed{0} \end{array} \quad \rightarrow \ 8x^2 + 12x + 36 = 0$$

The solutions are: $x = -\frac{3}{4},\ -\frac{3}{4} \pm i\frac{3\sqrt{7}}{4}$

Exercise

Find all solutions of the equation: $3x^3 - x^2 + 11x - 20 = 0$

Solution

$$possibilities: \ \pm\left\{\frac{20}{3}\right\} = \pm\left\{\frac{1,\ 2,\ 4,\ 5,\ 10,\ 20}{1,\ 3}\right\}$$

$$= \pm\left\{1,\ 2,\ 4,\ 5,\ 10,\ 20,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3},\ \frac{5}{3},\ \frac{10}{3},\ \frac{20}{3}\right\}$$

$$\begin{array}{r|rrrr} \frac{4}{3} & 3 & -1 & 11 & -20 \\ & & 4 & 4 & 20 \\ \hline & 3 & 3 & 15 & \boxed{0} \end{array} \quad \rightarrow \ 3x^2 + 3x + 15 = 0$$

$$x = \frac{-3 \pm \sqrt{9-180}}{6}$$

$$= \frac{-3 \pm 3\sqrt{19}}{6}$$

The solutions are: $x = \frac{4}{3},\ -\frac{1}{2} \pm i\frac{\sqrt{19}}{2}$

Exercise

Find all solutions of the equation: $x^4 - x^3 - 9x^2 + 3x + 18 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{18}{1}\right\} = \pm\{1,\ 2,\ 3,\ 6,\ 9,\ 18\}$

$$\begin{array}{r|rrrrr}
-2 & 1 & -1 & -9 & 3 & 18 \\
 & & -2 & 6 & 6 & -18 \\
\hline
3 & 1 & -3 & -3 & 9 & 0 \\
 & & 3 & 0 & -9 & \\
\hline
 & 1 & 0 & -3 & 0 &
\end{array}$$

$\to x^3 - 3x^2 - 3x + 9 = 0 \to \pm\left\{\frac{9}{1}\right\} = \pm\{1,\ 3,\ 9\}$

$\to x^2 - 3 = 0 \Rightarrow \underline{x = \pm\sqrt{3}}$

The solutions are: $x = -2,\ 3,\ \pm\sqrt{3}$

Exercise

Find all solutions of the equation: $2x^4 - 9x^3 + 9x^2 + x - 3 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$$\begin{array}{r|rrrrr}
1 & 2 & -9 & 9 & 1 & -3 \\
 & & 2 & -7 & 2 & 3 \\
\hline
1 & 2 & -7 & 2 & 3 & 0 \\
 & & 2 & -5 & -3 & \\
\hline
 & 2 & -5 & -3 & 0 &
\end{array}$$

$\to 2x^3 - 7x^2 + 2x + 3 = 0 \to \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$\to 2x^2 - 5x - 3 = 0$

$$x = \frac{5 \pm \sqrt{25 + 24}}{4}$$

$$= \begin{cases} \frac{5-7}{4} = -\frac{1}{2} \\ \frac{5+7}{4} = 3 \end{cases}$$

The solutions are: $x = 1,\ 1,\ -\frac{1}{2},\ 3$

Exercise

Find all solutions of the equation: $6x^4+5x^3-17x^2-6x=0$

Solution

$$x\left(6x^3+5x^2-17x-6\right)=0 \quad \rightarrow \quad \underline{x=0}$$

$$\textit{possibilities}: \ \pm\left\{\frac{6}{6}\right\}=\pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{3}\right\}$$

$$\begin{array}{r|rrrr} -2 & 6 & 5 & -17 & -6 \\ & & -12 & 14 & 6 \\ \hline & 6 & -7 & -3 & \boxed{0} \end{array} \quad \rightarrow \ 6x^2-7x-3=0$$

$$x=\frac{7\pm\sqrt{49+72}}{12}$$

$$=\begin{cases} \dfrac{7-11}{12}=-\dfrac{1}{3} \\ \dfrac{7+11}{12}=\dfrac{3}{2} \end{cases}$$

The solutions are: $\underline{x=0,\ -2,\ -\frac{1}{3},\ \frac{3}{2}}$

Exercise

Find all solutions of the equation: $x^4-2x^2-16x-15=0$

Solution

$$\textit{possibilities}: \ \pm\left\{\frac{15}{1}\right\}=\pm\{1,\ 3,\ 5,\ 15\}$$

$$\begin{array}{r|rrrrr} -1 & 1 & 0 & -2 & -16 & -15 \\ & & -1 & 1 & 1 & 15 \\ \hline 3 & 1 & -1 & -1 & -15 & 0 \\ & & 3 & 6 & 15 & \\ \hline & 1 & 2 & 5 & 0 & \end{array}$$

$$\rightarrow \ x^3-x^2-x-15=0 \ \rightarrow \ \pm\left\{\frac{15}{1}\right\}=\pm\{1,\ 3,\ 5,\ 15\}$$

$$\rightarrow \ x^2+2x+5=0$$

$$x=\frac{-2\pm\sqrt{-16}}{2}$$

$$=-1\pm 2i$$

The solutions are: $\underline{x=-1,\ 3,\ -1\pm 2i}$

Exercise

Find all solutions of the equation: $x^4 - 2x^3 - 5x^2 + 8x + 4 = 0$

Solution

$\text{possibilities}: \ \pm\left\{\frac{4}{1}\right\} = \pm\{1,\ 2,\ 4\}$

$$\begin{array}{r|rrrrr} 2 & 1 & -2 & -5 & 8 & 4 \\ & & 2 & 0 & -10 & -4 \\ \hline -2 & 1 & 0 & -5 & -2 & \color{red}{0} \\ & & -2 & 4 & 2 & \\ \hline & 1 & -2 & -1 & \color{red}{0} & \end{array}$$

$\rightarrow x^3 - 5x - 2 = 0 \rightarrow \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$\rightarrow x^2 - 2x - 1 = 0$

$$x = \frac{2 \pm \sqrt{8}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2}$$

The solutions are: $\underline{x = -2,\ 2,\ 1 \pm \sqrt{2}}$

Exercise

Find all solutions of the equation: $2x^4 - 17x^3 + 4x^2 + 35x - 24 = 0$

Solution

$\text{possibilities}: \ \pm\left\{\frac{24}{2}\right\} = \pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$$\begin{array}{r|rrrrr} 1 & 2 & -17 & 4 & 35 & -24 \\ & & 2 & -15 & -11 & 24 \\ \hline 1 & 2 & -15 & -11 & 24 & \color{red}{0} \\ & & 2 & -13 & 24 & \\ \hline & 2 & -13 & -24 & \color{red}{0} & \end{array}$$

$\rightarrow 2x^3 - 15x^2 - 11x + 24 = 0 \rightarrow \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$\rightarrow 2x^2 - 13x - 24 = 0$

$$x = \frac{13 \pm \sqrt{169 + 192}}{4}$$

$$= \begin{cases} \frac{13 - 19}{4} = -\frac{3}{2} \\ \frac{13 + 19}{4} = 8 \end{cases}$$

The solutions are: $\underline{x = -\frac{3}{2},\ 1,\ 1,\ 8}$

Exercise

Find all solutions of the equation: $x^4 + x^3 - 3x^2 - 5x - 2 = 0$

Solution

possibilities : $\pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$$\begin{array}{r|rrrrr}
-1 & 1 & 1 & -3 & -5 & -2 \\
 & & -1 & 0 & 3 & 2 \\
\hline
-1 & 1 & 0 & -3 & -2 & 0 \\
 & & -1 & 1 & 2 & \\
\hline
 & 1 & -1 & -2 & 0 &
\end{array}
\begin{array}{l}
\\ \\
\to x^3 - 3x - 2 = 0 \to \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\} \\
\\
\to x^2 - x - 2 = 0
\end{array}$$

$$x = \frac{1 \pm \sqrt{9}}{2}$$

$$= \begin{cases} \frac{1-3}{2} = -1 \\ \frac{1+3}{2} = 2 \end{cases}$$

The solutions are: $\boxed{x = -1,\ -1,\ -1,\ 2}$

Exercise

Find all solutions of the equation: $6x^4 - 17x^3 - 11x^2 + 42x = 0$

Solution

$$x\left(6x^3 - 17x^2 - 11x + 42\right) = 0$$

$$x = 0 \quad 6x^3 - 17x^2 - 11x + 42 = 0$$

possibilities : $\pm\left\{\frac{42}{6}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ 7,\ 14,\ 21,\ 42,\ \frac{1}{2},\ \frac{3}{2},\ \frac{7}{2},\ \frac{21}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{7}{3},\ \frac{14}{3},\ \frac{1}{6},\ \frac{7}{6},\ \frac{21}{6}\right\}$

$$\begin{array}{r|rrrr}
2 & 6 & -17 & -11 & 42 \\
 & & 12 & -10 & -42 \\
\hline
 & 6 & -5 & -21 & \boxed{0}
\end{array}
\to 6x^2 - 5x - 21 = 0$$

$$x = \frac{5 \pm \sqrt{25 + 504}}{12}$$

$$= \begin{cases} \frac{5-23}{12} = -\frac{3}{2} \\ \frac{5+23}{12} = \frac{7}{3} \end{cases}$$

The solutions are: $\boxed{x = -\frac{3}{2},\ 0,\ 2,\ \frac{7}{3}}$

Exercise

Find all solutions of the equation: $x^4 - 5x^2 - 2x = 0$

Solution

$$x\left(x^3 - 5x - 2\right) = 0$$

$$x = 0 \quad x^3 - 5x - 2 = 0$$

$$\textit{possibilities}: \ \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -5 & -2 \\ & & -2 & 4 & 2 \\ \hline & 1 & -2 & -1 & \boxed{0} \end{array} \rightarrow x^2 - 2x - 1 = 0$$

$$x = \frac{2 \pm \sqrt{4+4}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2}$$

The solutions are: $x = -2,\ 0,\ 1 \pm \sqrt{2}$

Exercise

Find all solutions of the equation: $3x^4 - 4x^3 - 11x^2 + 16x - 4 = 0$

Solution

$$\textit{possibilities}: \ \pm\left\{\frac{4}{3}\right\} = \pm\left\{1,\ 2,\ 4,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3}\right\}$$

$$\begin{array}{r|rrrrr} 1 & 3 & -4 & -11 & 16 & -4 \\ & & 3 & -1 & -12 & 4 \\ \hline 2 & 3 & -1 & -12 & 4 & 0 \\ & & 6 & 10 & -4 & \\ \hline & 3 & 5 & -2 & 0 & \end{array} \rightarrow 3x^3 - x^2 - 12x + 4 = 0 \rightarrow \pm\left\{\frac{4}{3}\right\} = \pm\left\{1,\ 2,\ 4,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3}\right\}$$

$$\rightarrow 3x^2 + 5x - 2 = 0$$

$$x = \frac{-5 \pm \sqrt{25+24}}{6}$$

$$= \begin{cases} \dfrac{-5-7}{6} = -2 \\ \dfrac{-5+7}{6} = \dfrac{1}{3} \end{cases}$$

The solutions are: $x = -2,\ \frac{1}{3},\ 1,\ 2$

Exercise

Find all solutions of the equation: $6x^4+23x^3+19x^2-8x-4=0$

Solution

$$\textit{possibilities}: \ \pm\left\{\frac{4}{6}\right\}=\pm\left\{1,\ 2,\ 4,\ \frac{1}{6},\ \frac{2}{3},\ \frac{2}{3}\right\}$$

$$\begin{array}{r|rrrrr}
-2 & 6 & 23 & 19 & -8 & -4 \\
 & & -12 & -22 & 6 & 4 \\
\hline
-2 & 6 & 11 & -3 & -2 & 0 \\
 & & -12 & 2 & 2 & \\
\hline
 & 6 & -1 & -1 & 0 &
\end{array}$$

$$\to 6x^3+11x^2-3x-2=0 \ \to \ \pm\left\{\frac{2}{6}\right\}=\pm\left\{1,\ 2,\ \frac{1}{6},\ \frac{1}{3}\right\}$$

$$\to 6x^2-x-1=0$$

$$x=\frac{1\pm\sqrt{25}}{12}$$

$$=\begin{cases}\dfrac{1-5}{12}=-\dfrac{1}{3}\\[2mm] \dfrac{1+5}{12}=\dfrac{1}{2}\end{cases}$$

The solutions are: $\underline{x=-2,\ -2,\ -\frac{1}{3},\ \frac{1}{2}}$

Exercise

Find all solutions of the equation: $4x^4-12x^3+3x^2+12x-7=0$

Solution

$$\textit{possibilities}: \ \pm\left\{\frac{7}{4}\right\}=\pm\left\{1,\ 7,\ \frac{1}{2},\ \frac{7}{2},\ \frac{1}{4},\ \frac{7}{4}\right\}$$

$$\begin{array}{r|rrrrr}
1 & 4 & -12 & 3 & 12 & -7 \\
 & & 4 & -8 & -5 & 7 \\
\hline
-1 & 4 & -8 & -5 & 7 & 0 \\
 & & -4 & 12 & -7 & \\
\hline
 & 4 & -12 & 7 & 0 &
\end{array}$$

$$\to 4x^3-8x^2-5x+7=0 \ \to \ \pm\left\{\frac{7}{4}\right\}=\pm\left\{1,\ 7,\ \frac{1}{2},\ \frac{7}{2},\ \frac{1}{4},\ \frac{7}{4}\right\}$$

$$\to 4x^2-12x+7=0$$

$$x=\frac{12\pm\sqrt{144-112}}{8}$$

$$=\frac{12\pm4\sqrt{2}}{8}$$

The solutions are: $\underline{x=-1,\ 1,\ \frac{3\pm\sqrt{2}}{2}}$

Exercise

Find all solutions of the equation: $2x^4 - 9x^3 - 2x^2 + 27x - 12 = 0$

Solution

$$\textit{possibilities}: \ \pm\left\{\frac{12}{2}\right\} = \pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 12,\ \frac{1}{2},\ \frac{3}{2}\right\}$$

$$\begin{array}{r|rrrrr} 4 & 2 & -9 & -2 & 27 & -12 \\ & & 8 & -4 & -24 & 12 \\ \hline \frac{1}{2} & 2 & -1 & -6 & 3 & 0 \\ & & 1 & 0 & -3 & \\ \hline & 2 & 0 & -6 & 0 & \end{array}$$

$$\rightarrow 2x^3 - x^2 - 6x + 3 = 0 \ \rightarrow \ \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$$

$$\rightarrow 2x^2 - 6 = 0$$

$x^2 = 3$

The solutions are: $x = \frac{1}{2},\ 4,\ \pm\sqrt{3}$

Exercise

Find all solutions of the equation: $2x^4 - 19x^3 + 51x^2 - 31x + 5 = 0$

Solution

$$\textit{possibilities}: \ \pm\left\{\frac{5}{2}\right\} = \pm\left\{1,\ 5,\ \frac{1}{2},\ \frac{5}{2}\right\}$$

$$\begin{array}{r|rrrrr} 5 & 2 & -19 & 51 & -31 & 5 \\ & & 10 & -45 & 30 & -5 \\ \hline \frac{1}{2} & 2 & -9 & 6 & -1 & 0 \\ & & 1 & -4 & 1 & \\ \hline & 2 & -8 & 2 & 0 & \end{array}$$

$$\rightarrow 2x^3 - 9x^2 + 6x - 1 = 0 \ \rightarrow \ \pm\left\{\frac{1}{2}\right\}$$

$$\rightarrow 2x^2 - 8x + 2 = 0$$

$$x = \frac{8 \pm \sqrt{64 - 16}}{4}$$

$$= \frac{8 \pm 4\sqrt{3}}{4}$$

The solutions are: $x = \frac{1}{2},\ 5,\ 2 \pm \sqrt{3}$

Exercise

Find all solutions of the equation: $4x^4 - 35x^3 + 71x^2 - 4x - 6 = 0$

Solution

$$\textit{possibilities}: \ \pm\left\{\frac{6}{4}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{4},\ \frac{3}{4}\right\}$$

$$\begin{array}{r|rrrrr} 3 & 4 & -35 & 71 & -4 & -6 \\ & & 12 & -69 & 6 & 6 \\ \hline -\frac{1}{4} & 4 & -23 & 2 & 2 & 0 \\ & & -1 & 6 & -2 & \\ \hline & 4 & -24 & 8 & 0 & \end{array}$$

$$\rightarrow\ 4x^3 - 23x^2 + 2x + 2 = 0 \ \rightarrow\ \pm\left\{\frac{2}{4}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2},\ \frac{1}{4}\right\}$$

$$\rightarrow\ 4x^2 - 24x + 8 = 0$$

$$x^2 - 6x + 2 = 0$$

$$x = \frac{6 \pm \sqrt{36-8}}{2}$$

$$= \frac{6 \pm 2\sqrt{7}}{4}$$

The solutions are: $\underline{x = -\frac{1}{4},\ 3,\ 3 \pm \sqrt{7}}$

Exercise

Find all solutions of the equation: $2x^4 + 3x^3 - 4x^2 - 3x + 2 = 0$

Solution

possibilities : $\pm\left\{\frac{2}{2}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2}\right\}$

$$\begin{array}{r|rrrrr} 1 & 2 & 3 & -4 & -3 & 2 \\ & & 2 & 5 & 1 & -2 \\ \hline -1 & 2 & 5 & 1 & -2 & 0 \\ & & -2 & -3 & 2 & \\ \hline & 2 & 3 & -2 & 0 & \end{array}$$

$$\rightarrow\ 2x^3 - 23x^2 + 2x - 2 = 0 \ \rightarrow\ \pm\left\{\frac{2}{2}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2}\right\}$$

$$\rightarrow\ 2x^2 + 3x - 2 = 0$$

$$x = \frac{-3 \pm \sqrt{9+16}}{4}$$

$$= \begin{cases} \frac{-3-5}{4} = -2 \\ \frac{-3+5}{4} = \frac{1}{2} \end{cases}$$

The solutions are: $\underline{x = -2,\ -1,\ \frac{1}{2},\ 1}$

Exercise

Find all solutions of the equation: $x^4 + 3x^3 - 30x^2 - 6x + 56 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{56}{1}\right\} = \pm\{1,\ 2,\ 4,\ 7,\ 8,\ 14,\ 28,\ 56\}$

$$\begin{array}{r|rrrrr}
4 & 1 & 3 & -3 & -6 & 56 \\
 & & 4 & 28 & -8 & -56 \\ \hline
-7 & 1 & 7 & -2 & -14 & 0 \\
 & & -7 & 0 & 14 & \\ \hline
 & 1 & 0 & -2 & &
\end{array}$$

$$\to \quad x^3+7x^2-2x-14=0 \;\Rightarrow\; \frac{c}{d}=\pm\left\{\frac{14}{1}\right\}=\pm\{1,\ 2,\ 7,\ 14\}$$

$$\to \quad x^2-2=0$$

$$\Rightarrow \boxed{x=\pm\sqrt{2}}$$

The solutions are: $\boxed{x=4,\ -7,\ \pm\sqrt{2}}$

Exercise

Find all solutions of the equation: $3x^5-10x^4-6x^3+24x^2+11x-6=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{6}{3}\right\}=\pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrrrr}
-1 & 3 & -10 & -6 & 24 & 11 & -6 \\
 & & -3 & 13 & -7 & -17 & 6 \\ \hline
-1 & 3 & -13 & 7 & 17 & -6 & 0 \\
 & & -3 & 16 & -23 & 6 & \\ \hline
2 & 3 & -16 & 23 & -6 & 0 & \\
 & & 6 & 20 & 6 & & \\ \hline
 & 3 & -10 & 3 & 0 & &
\end{array}$$

$$x^4-13x^3+7x^2+17x-6=0 \to \pm\left\{\frac{6}{3}\right\}=\pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$$

$$3x^3-16x^2+26x-6=0 \to \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$$

$$3x^2-10x+3=0$$

$$x=\frac{10\pm\sqrt{100-36}}{6}$$

$$=\begin{cases}\frac{10-8}{6}=\frac{1}{3}\\[4pt] \frac{10+8}{6}=3\end{cases}$$

The solutions are: $\boxed{x=-1,\ -1,\ \frac{1}{3},\ 2,\ 3}$

Exercise

Find all solutions of the equation: $6x^5+19x^4+x^3-6x^2=0$

Solution

$$x^2\left(6x^3+19x^2+x-6\right)=0 \quad\to\quad \boxed{x=0,\ 0}$$

$$6x^3+19x^2+x-6=0$$

possibilities for $\frac{c}{d}:\pm\left\{\frac{6}{6}\right\}=\pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrr} -3 & 6 & 19 & 1 & -6 \\ & & -18 & -3 & 6 \\ \hline & 6 & 1 & -2 & \boxed{0} \end{array} \qquad 6x^2+x-2=0$$

$$x=\frac{-1\pm\sqrt{1+48}}{12}$$

$$=\begin{cases} \frac{-1-7}{12}=-\frac{2}{3} \\ \frac{-1+7}{12}=\frac{1}{2} \end{cases}$$

The solutions are: $\underline{x=0,\ 0,\ -\frac{2}{3},\ -3,\ \frac{1}{2}}$

Exercise

Find all solutions of the equation: $x^5+5x^4+10x^3+10x^2+5x+1=0$

Solution

$$x^5+5x^4+10x^3+10x^2+5x+1=(x+1)^5=0$$

possibilities for $\frac{c}{d}:\pm\{1\}$

$$\begin{array}{r|rrrrrr} -1 & 1 & 5 & 10 & 10 & 5 & 1 \\ & & -1 & -4 & -6 & -4 & -1 \\ \hline -1 & 1 & 4 & 6 & 4 & 1 & 0 \\ & & -1 & -3 & -3 & -1 & \\ \hline -1 & 1 & 3 & 3 & 1 & 0 & \\ & & -1 & -2 & -1 & & \\ \hline & 1 & 2 & 1 & 0 & & \end{array}$$

$\to\ x^4+4x^3+6x^2+4x+1=0\to\pm\{1\}$

$\to\ x^3+3x^2+3x+1=0\to\pm\{1\}$

$\to\ x^2+2x+1=0$

$$x^2+2x+1=(x+1)^2$$

The solutions are: $\underline{x=-1,\ -1,\ -1,\ -1,\ -1}$

Exercise

Find all solutions of the equation: $x^5 - x^4 - 7x^3 + 7x^2 + 12x - 12 = 0$

Solution

possibilities for $\frac{c}{d}: \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$$\begin{array}{r|rrrrrr} 1 & 1 & -1 & -7 & 7 & 12 & -12 \\ & & 1 & 0 & -7 & 0 & 12 \\ \hline 2 & 1 & 0 & -7 & 0 & 12 & 0 \\ & & 2 & 4 & -6 & -12 & \\ \hline -2 & 1 & 2 & -3 & -6 & 0 & \\ & & -2 & 0 & 6 & & \\ \hline & 1 & 0 & -3 & 0 & & \end{array}$$

$\rightarrow x^4 - 7x^2 - 12 = 0 \rightarrow \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$\rightarrow x^3 + 2x^2 - 3x - 6 = 0 \rightarrow \pm\{1,\ 2,\ 3,\ 6\}$

$\rightarrow x^2 - 3 = 0$

$x^2 = 3$

The solutions are: $\underline{x = -2,\ 1,\ 2,\ \pm\sqrt{3}}$

Exercise

Find all solutions of the equation: $x^5 - 2x^3 - 8x = 0$

Solution

$$x\left(x^4 - 2x^2 - 8\right) = 0$$

$\underline{x = 0}$

$$x^4 - 2x^2 - 8 = 0$$

$$x^2 = \frac{2 \pm \sqrt{4+32}}{2}$$

$$= \begin{cases} \frac{2-6}{2} = -2 \\ \frac{2+6}{2} = 4 \end{cases}$$

$$\begin{cases} x^2 = -2 & \rightarrow & x = \pm i\sqrt{2} \\ x^2 = 4 & \rightarrow & x = \pm 2 \end{cases}$$

The solutions are: $\underline{x = 0,\ \pm 2,\ \pm i\sqrt{2}}$

Exercise

Find all solutions of the equation: $x^5 - 32 = 0$

Solution

possibilities for $\frac{c}{d}: \pm\{1,\ 2,\ 4,\ 8,\ 16,\ 32\}$

$$\begin{array}{r|rrrrrr}
2 & 1 & 0 & 0 & 0 & 0 & -32 \\
 & & 2 & 4 & 8 & 16 & 32 \\
\hline
 & 1 & 2 & 4 & 8 & 16 & 0
\end{array} \quad \to\ x^4 + 2x^3 + 4x^2 + 8x + 16 = 0 \to \pm\{1,\ 2,\ 4,\ 8,\ 16\}$$

$x^4 + 2x^3 + 4x^2 + 8x + 16 = 0$

Cannot be solved using rational zero theorem.
Therefore; using program

The solutions are: $\underline{x = 2,\ \frac{-1 - \sqrt{5} \pm i\sqrt{2}\sqrt{5 - \sqrt{5}}}{2},\ \frac{-1 + \sqrt{5} \pm i\sqrt{2}\sqrt{5 - \sqrt{5}}}{2}}\big|$

Exercise

Find all solutions of the equation: $3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24 = 0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{24}{3}\right\} = \pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3},\ \frac{8}{3}\right\}$

$$\begin{array}{r|rrrrrrrl}
1 & 3 & -10 & -29 & 34 & 50 & -24 & -24 & \\
 & & 3 & -7 & -36 & -2 & 48 & 24 & \\
\hline
-1 & 3 & -7 & -36 & -2 & 48 & 24 & 0 & \to\ 3x^5 - 7x^4 - 36x^3 - 2x^2 + 48x + 24 = 0 \\
 & & -3 & 10 & 26 & -24 & -24 & & \\
\hline
-2 & 3 & -10 & -26 & 24 & 24 & 0 & & \to\ 3x^4 - 10x^3 - 26x^2 + 24x + 24 = 0 \\
 & & -6 & 32 & -12 & -24 & & & \\
\hline
-\frac{2}{3} & 3 & -16 & 6 & 12 & 0 & & & \to\ 3x^3 - 16x^2 + 12x - 12 = 0 \\
 & & -2 & 12 & -12 & & & & \\
\hline
 & 3 & -18 & 18 & 0 & & & & \to\ 3x^2 - 18x + 18 = 0
\end{array}$$

$x^2 - 6x + 6 = 0$

$$\begin{aligned}
x &= \frac{6 \pm \sqrt{36 - 24}}{2} \\
&= \frac{6 \pm 2\sqrt{3}}{2}
\end{aligned}$$

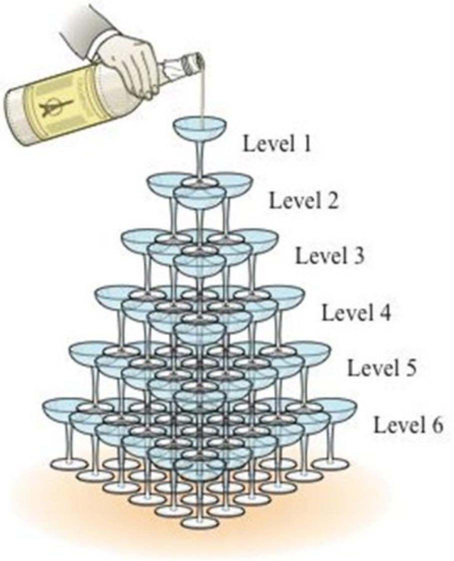
The solutions are: $\underline{x = -2,\ -1,\ 1,\ -\frac{2}{3},\ 3 \pm \sqrt{3}}\big|$

Exercise

Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k)=\frac{1}{6}\left(k^3+3k^2+2k\right)$$

Where k is the number of levels in the pyramid. If 220 glasses are used to form a triangle pyramid, how many levels are in the pyramid?



Solution

$$\frac{1}{6}\left(k^3+3k^2+2k\right)=220$$

$$k^3+3k^2+2k-1,320=0$$

$$\begin{array}{r|rrrr} 10 & 1 & 3 & 2 & -1320 \\ & & 10 & 130 & 1320 \\ \hline & 1 & 13 & 132 & 0 \end{array} \quad \rightarrow \quad k^2+13k+132=0$$

$$k=\frac{-13\pm\sqrt{-359}}{2} \qquad \mathbb{C}$$

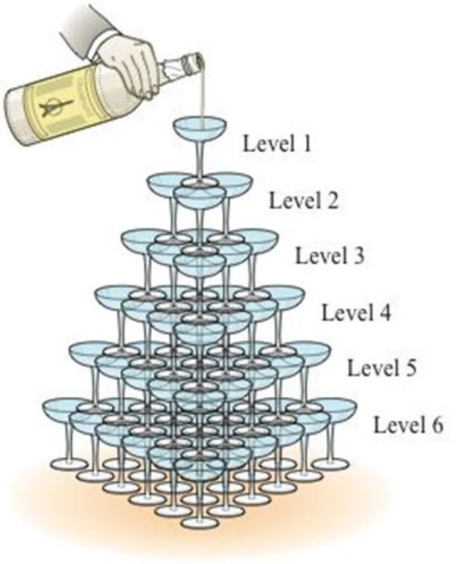
The are **10** levels in the pyramid.

Exercise

Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k)=\frac{1}{6}\left(2k^3+3k^2+k\right)$$

Where k is the number of levels in the pyramid. If 140 glasses are used to form a triangle pyramid, how many levels are in the pyramid?



Solution

$$\frac{1}{6}\left(2k^3+3k^2+k\right)=150$$

$$2k^3+3k^2+k-840=0$$

$$\begin{array}{r|rrrr} 7 & 2 & 3 & 1 & -840 \\ & & 14 & 119 & 840 \\ \hline & 2 & 17 & 120 & 0 \end{array} \quad \rightarrow \quad 2k^2+17k+120=0$$

$$k=\frac{-17\pm\sqrt{-671}}{4} \qquad \mathbb{C}$$

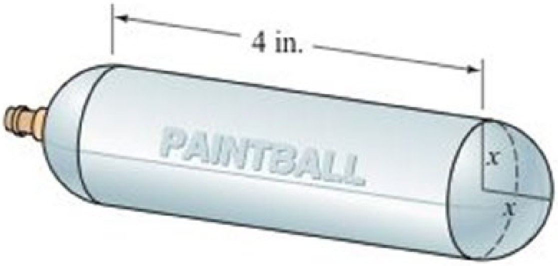
The are **7** levels in the pyramid.

Exercise

A carbon dioxide cartridge for a paintball rifle has the shape of a right circular cylinder with a hemisphere at each end. The cylinder is 4 *inches* long, and the volume of the cartridge is $2\pi \ \ in^3$.
The common interior radius of the cylinder and the hemispheres is denoted by x. Estimate the length of the radius x.

Solution

Volume of the Cartridge = 2 × (Volume of Hemisphere) + Volume of Cylinder



$$Volume\ of\ \ Sphere = \frac{4}{3}\pi x^3$$

$$Volume\ of\ \ Cylinder = 4\pi x^2$$

$$Volume\ of\ \ Cartridge = \frac{4}{3}\pi x^3 + 4\pi x^2$$

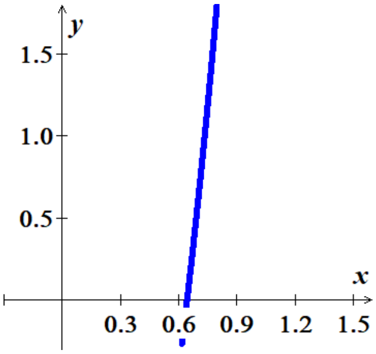
$$2\pi = \frac{4}{3}\pi x^3 + 4\pi x^2$$

$$2x^3 + 6x^2 = 3$$

$$2x^3 + 6x^2 - 3 = 0$$

Using Graph:

$x \approx 0.64 \ \ in.$

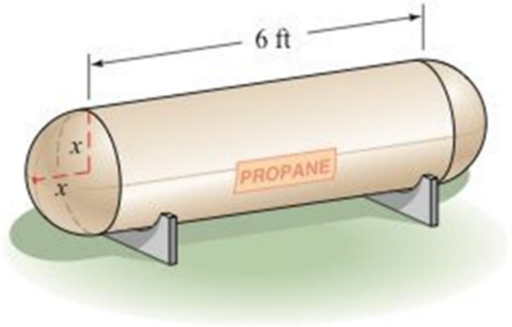


Exercise

A propane tank has the shape of a circular cylinder with a hemisphere at each end. The cylinder is 6 *feet* long and the volume of the tank is $9\pi \ \ ft^3$. Find the length of the radius x.

Solution

Volume of the Cartridge = 2 × (Volume of Hemisphere) + Volume of Cylinder



$$Volume\ of\ \ Sphere = \frac{4}{3}\pi x^3$$

$$Volume\ of\ \ Cylinder = 6\pi x^2$$

$$Volume\ of\ \ Cartridge = \frac{4}{3}\pi x^3 + 6\pi x^2$$

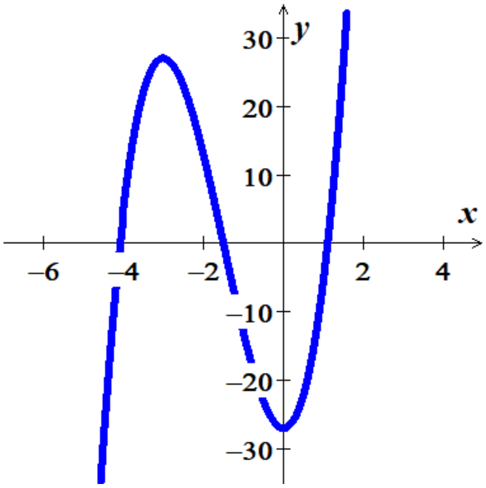
$$9\pi = \frac{4}{3}\pi x^3 + 6\pi x^2$$

$$27 = 4x^3 + 18x^2$$

$$4x^3 + 18x^2 - 27 = 0$$

$$possibilities\ for\ \ \frac{c}{d} : \pm\left\{\frac{27}{4}\right\} = \pm\left\{1,\ 3,\ 9,\ 27,\ \frac{1}{2},\ \frac{3}{2},\ \frac{9}{2},\ \frac{27}{2},\ \frac{1}{4},\ \frac{3}{4},\ \frac{9}{4},\ \frac{27}{4}\right\}$$

$$\begin{array}{r|rrrr} -\frac{3}{2} & 4 & 18 & 0 & -27 \\ & & -6 & -18 & 27 \\ \hline & 4 & 12 & -18 & 0 \end{array} \quad \rightarrow \quad 4x^2+12x-18=0$$



$$2x^2+6x-9=0$$

$$x=\frac{-6\pm\sqrt{36+72}}{4}$$

$$=\frac{-6\pm 6\sqrt{3}}{4}$$

$$=\frac{-3\pm 3\sqrt{3}}{2}$$

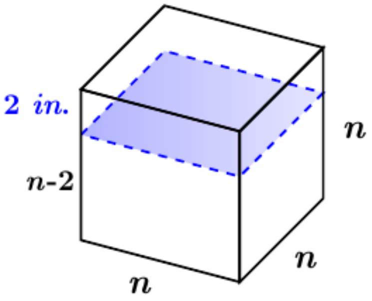
$$\underline{x=-\frac{3}{2},\ \frac{-3-3\sqrt{3}}{2},\ \frac{-3+3\sqrt{3}}{2}}\Big|$$

∴ the length of the radius x is $\frac{-3+3\sqrt{3}}{2}\approx 1.1\ \ foot$

Exercise

A cube measures n inches on each edge. If a slice 2 *inches* thick is cut from one face of the cube, the resulting solid has a volume of $567\ \ in^3$. Find n.

Solution



$$Volume=n^2(n-2)$$

$$n^3-2n^2=567$$

$$n^3-2n^2-567=0$$

$$possibilities\ for\ \frac{c}{d}:=\pm\{1,\ 3,\ 7,\ 9,\ 21,\ 27,\ 63,\ 81,\ 189,\ 567\}$$

$$\begin{array}{r|rrrr} 9 & 1 & -2 & 0 & -567 \\ & & 9 & 63 & 567 \\ \hline & 1 & 7 & 63 & 0 \end{array} \quad \rightarrow \quad n^2+7n+63=0$$

$$n=\frac{-7\pm\sqrt{49-252}}{2}$$

$$=\frac{-7\pm i\sqrt{203}}{2}\quad \times$$

$$\therefore\ \ \underline{n=9}\,\Big|$$

Exercise

A cube measures n inches on each edge. If a slice 1 *inch* thick is cut from one face of the cube and then a slice 3 inches thick is cut from another face of the cube, the resulting solid has a volume of $1560\ in^3$. Find the dimensions of the original cube.

Solution

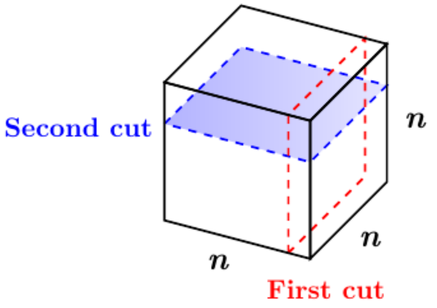
$$Volume = n(n-1)(n-3)$$

$$n^3 - 4n^2 + 3n = 1560$$

$$n^3 - 4n^2 + 3n - 1560 = 0$$

$$possibilities\ for\ \frac{c}{d} := \pm\left\{\begin{matrix} 1,\ 2,\ 4,\ 5,\ 6,\ 8,\ 10,\ 12,\ 13,\ 15,\ 20,\ 24,\ 26,\ 30,\ 39, \\ 40,\ 52,\ 60,\ 65,\ 78,\ 104,\ 120,\ 130,\ 156,\ 195,\ 260,\ 312,\ 390,\ 780,\ 1560 \end{matrix}\right\}$$

$$\begin{array}{r|rrrr} 13 & 1 & -4 & 3 & -1560 \\ & & 13 & 117 & 1560 \\ \hline & 1 & 9 & 120 & 0 \end{array} \quad \rightarrow \quad n^2 + 9n + 120 = 0$$



$$n = \frac{-9 \pm \sqrt{81-480}}{2}$$

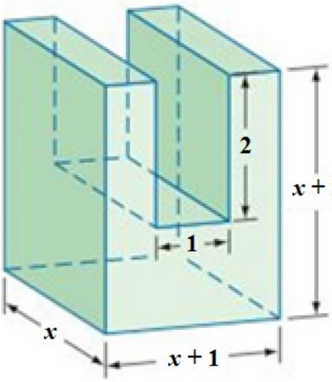
$$= \frac{-9 \pm i\sqrt{399}}{2} \quad \times$$

$$\therefore\ \ \underline{n = 13\ |}$$

Exercise

For what value of x will the volume of the following solid be $112\ in^3$

Solution



$$\text{Volume of the bottom portion } = x^2(x+1)$$

$$\text{Volume of one side portion } = 2x\left(\frac{1}{2}x\right)$$

$$= x^2$$

$$\text{Total Volume } = x^2(x+1) + 2x^2$$

$$112 = x^3 + 3x^2$$

$$x^3 + 3x^2 - 112 = 0$$

$$possibilities\ for\ \frac{c}{d} := \pm\{1,\ 2,\ 4,\ 8,\ 14,\ 28,\ 56,\ 112\}$$

$$\begin{array}{r|rrrr} 4 & 1 & 3 & 0 & -112 \\ & & 4 & 28 & 112 \\ \hline & 1 & 7 & 28 & 0 \end{array} \rightarrow \quad x^2+7x+28=0$$

$$x=\frac{-7\pm\sqrt{49-112}}{2}$$

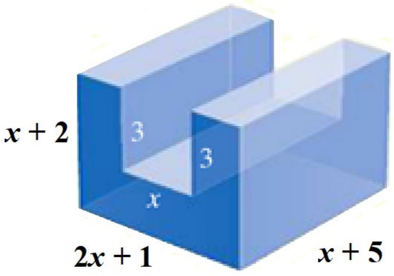
$$=\frac{-7\pm 3i\sqrt{7}}{2} \quad \times$$

$$\therefore \quad \underline{x=4}$$

Exercise

For what value of x will the volume of the following solid be $208 \ in^3$

Solution



$$\text{Volume of the bottom portion } =(2x+1)(x+5)(x+2-3)$$

$$=\left(2x^2+11x+5\right)(x-1)$$

$$=2x^3+11x^2+5x-2x^2-11x-5$$

$$=2x^3+9x^2-6x-5$$

$$\text{Volume of one side portion } =(3)\tfrac{1}{2}(2x+1-x)(x+5)$$

$$=\frac{3}{2}(x+1)(x+5)$$

$$=\frac{3}{2}\left(x^2+6x+5\right)$$

$$\text{Total Volume } =2x^3+9x^2-6x-5+2\left(\frac{3}{2}\right)\left(x^2+6x+5\right)$$

$$208=2x^3+9x^2-6x-5+3x^2+18x+15$$

$$2x^3+12x^2+12x-198=0$$

$$x^3+6x^2+6x-99=0$$

$$\text{possibilities for } \tfrac{c}{d}:=\pm\{1,\ 3,\ 9,\ 11,\ 33,\ 99\}$$

$$\begin{array}{r|rrrr} 3 & 1 & 6 & 6 & -99 \\ & & 3 & 27 & 99 \\ \hline & 1 & 9 & 33 & 0 \end{array} \rightarrow \quad x^2+9x+33=0$$

$$x=\frac{-9\pm\sqrt{81-132}}{2}$$

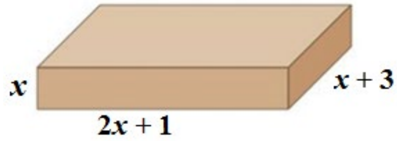
$$=\frac{-9\pm i\sqrt{51}}{2} \quad \times$$

$$\therefore \quad \underline{x=3}$$

Exercise

The length of rectangular box is 1 *inch* more than twice the height of the box, and the width is 3 *inches* more than the height. If the volume of the box is $126\ in^3$, find the dimensions of the box.

Solution



$$Volume = x(2x+1)(x+3)$$

$$2x^3+7x^2+3x=126$$

$$2x^3+7x^2+3x-126=0$$

$$possibilities\ for\ \frac{c}{d} := \pm\left\{\frac{126}{2}\right\}$$

$$=\pm\left\{1,\ 2,\ 3,\ 6,\ 9,\ 14,\ 21,\ 42,\ 63,\ 126,\ \frac{1}{2},\ \frac{3}{2},\ \frac{9}{2},\ \frac{21}{2},\ \frac{63}{2}\right\}$$

$$\begin{array}{r|rrrr} 3 & 2 & 7 & 3 & -126 \\ & & 6 & 39 & 126 \\ \hline & 2 & 13 & 42 & 0 \end{array} \quad \rightarrow \quad 2x^2+13x+42=0$$

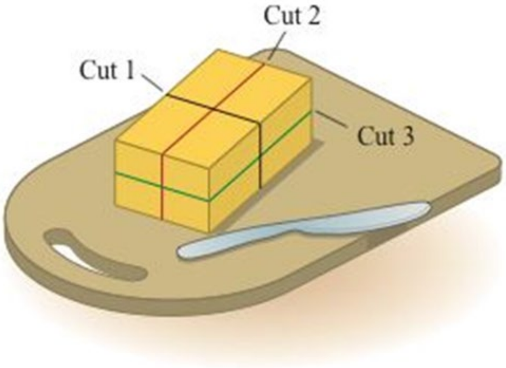
$$x=\frac{-13\pm\sqrt{169-336}}{4}$$

$$=\frac{-13\pm i\sqrt{167}}{4} \quad \times$$

$$\therefore \quad \underline{x=3}\,|$$

Exercise

One straight cut through a thick piece of cheese produces two pieces. Two straight cuts can produce a maximum of four pieces. Two straight cuts can produce a maximum of four pieces. Three straight cuts can produce a maximum of eight pieces.



You might be inclined to think that every additional cut double number of pieces. However, for four straight cuts, you get a maximum of 15 pieces. The maximum number of pieces P that can be produced by n straight cuts is given by

$$P(n)=\frac{n^3+5n+6}{6}$$

a) Determine number of pieces that can be produces by five straight cuts.

b) What is the fewest number of straight cuts that are needed to produce 64 pieces?

Solution

a) $P(5) = \dfrac{5^3 + 25 + 6}{6}$

$$\underline{= 26\,}|$$

b) $\dfrac{n^3 + 5n + 6}{6} = 64$

$$n^3 + 5n + 6 = 384$$

$$n^3 + 5n - 378 = 0$$

$$\textit{possibilities for } \tfrac{c}{d} := \pm\{378\}$$

$$= \pm\{1,\ 2,\ 3,\ 6,\ 7,\ 9,\ 14,\ 18,\ 21,\ 27,\ 42,\ 54,\ 63,\ 126,\ 189,\ 378\}$$

$$\begin{array}{r|rrrr} 7 & 1 & 0 & 5 & -378 \\ & & 7 & 49 & 378 \\ \hline & 1 & 7 & 54 & 0 \end{array} \quad \rightarrow \quad n^2 + 7n + 54 = 0$$

$$n = \frac{-7 \pm \sqrt{49 - 216}}{2}$$

$$= \frac{-7 \pm i\sqrt{167}}{2} \quad \times$$

$$\therefore \quad \underline{n = 7\,}|$$

Exercise

The number of ways one can select three cards from a group of n cards (the order of the selection matters), where $n \geq 3$, is given by $P(n) = n^3 - 3n^2 + 2n$. For a certain card trick, a magician has determined that there are exactly 504 *ways* to choose three cards from a given group. How many cards are in the group?

Solution

$$P(n) = n^3 - 3n^2 + 2n = 504$$

$$n^3 - 3n^2 + 2n - 504 = 0$$

$$\textit{possibilities for } \tfrac{c}{d} := \pm\{504\}$$

$$= \pm\left\{\begin{array}{l} 1,\ 2,\ 3,\ 4,\ 6,\ 7,\ 8,\ 9,\ 12,\ 14,\ 18,\ 21, \\ \ 24,\ 28,\ 36,\ 42,\ 56,\ 63,\ 72,\ 84,\ 126,\ 168,\ 252,\ 504 \end{array}\right\}$$

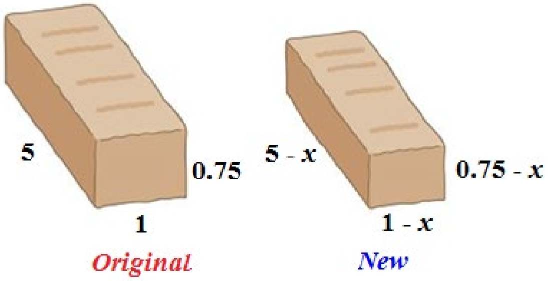
$$\begin{array}{r|rrrr} 9 & 1 & -3 & 2 & -504 \\ & & 9 & 54 & 504 \\ \hline & 1 & 6 & 56 & 0 \end{array} \quad \rightarrow \quad n^2+6n+56=0$$

$$\begin{aligned} n &= \frac{-6 \pm \sqrt{36-224}}{2} \\ &= -3 \pm i\sqrt{47} \quad \times \end{aligned}$$

$\therefore \quad \underline{n=9}|$

Exercise

A nutrition bar in the shape of a rectangular solid measure 0.75 *in*. by 1 *in*. by 5 *inches*.



To reduce costs, the manufacturer has decided to decrease each of the dimensions of the nutrition bar by x *inches*, what value of x will produce a new bar with a volume that is $0.75 \ in^3$ less than the present bar's volume.

Solution

$$\begin{aligned} V_{original} &= (5)(1)\left(\frac{3}{4}\right) \\ &= \frac{15}{4} \end{aligned}$$

$$V_{new} = (5-x)(1-x)\left(\frac{3}{4}-x\right) \qquad \left(x<\frac{3}{4}\right)$$

$$\left(5-6x+x^2\right)\left(\frac{3-4x}{4}\right) = \frac{15}{4}-\frac{3}{4}$$

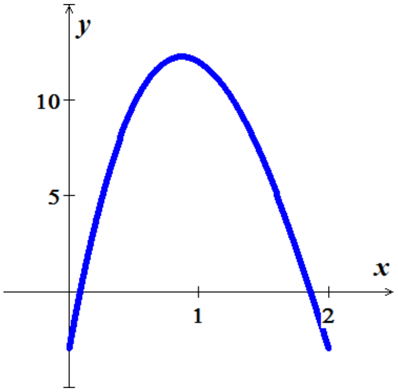
$$15-20x-18x+24x^2+3x^2-4x^3 = 4(3)$$

$$4x^3-27x^2+38x-3=0$$

From graph table:

```
0.08200   -0.06334
0.08400    0.00386
```

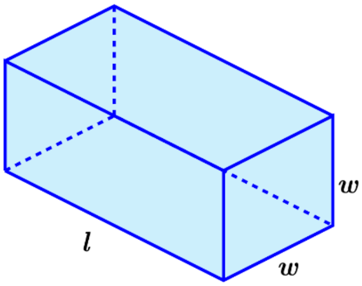
$\underline{x \approx 0.083 \ \ in.}|$



Exercise

A rectangular box is square on two ends and has length plus girth of 81 *inches*. (Girth: distance *around* the box). Determine the possible lengths $l\ (l > w)$ of the box if its volume is $4900\ in^3$.

Solution



$$81 = l + 4w$$
$$l = 81 - 4w$$
$$V = lw^2$$
$$= (81-4w)w^2$$

$$-4w^3 + 81w^2 = 4900$$
$$4w^3 - 81w^2 + 4900 = 0$$

$$\text{possibilities for } \frac{c}{d} := \pm\left\{\frac{4900}{4}\right\} = \pm\left\{\begin{array}{l}1,\ 2,\ 4,\ 7,\ 10,\ 14,\ 20,\ 28,\ 49, \\ \ \ 100,\ 175,\ 245,\ 350,\ 490,\ 700,\ 1225,\ 2450,\ 4900,\cdots\end{array}\right\}$$

$$\begin{array}{r|rrrr} 14 & 4 & -81 & 0 & 4900 \\ & & 56 & -350 & -4900 \\ \hline & 4 & -25 & -350 & 0 \end{array} \quad \rightarrow \quad 4w^2 - 25w - 350 = 0$$

$$w = \frac{25 \pm \sqrt{625+5600}}{8}$$
$$= \frac{25 \pm 5\sqrt{249}}{8}$$
$$= \begin{cases} \dfrac{25-5\sqrt{249}}{8} < 0 \\ \dfrac{25+5\sqrt{249}}{8} \approx 13 \end{cases}$$

$$l = 81 - 4(14) = 25$$
$$l = 81 - 4(13) = 29$$

∴ the possible lengths l are around **25** *in*. ***or*** **29** *in*.

Solution *Section* 2.5 – Graphing Polynomial Functions

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = 5x^3 + 7x^2 - x + 9$

Solution

Leading term: $5x^3$ with 3^{rd} degree (n is ***odd***)

$x \to -\infty \Rightarrow f(x) \to -\infty$ $f(x)$ falls left

$x \to \infty \Rightarrow f(x) \to \infty$ $f(x)$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = 11x^3 - 6x^2 + x + 3$

Solution

Leading term: $11x^3$ with 3^{rd} degree (n is ***odd***)

$x \to -\infty \Rightarrow f(x) \to -\infty$ $f(x)$ falls left

$x \to \infty \Rightarrow f(x) \to \infty$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = -11x^3 - 6x^2 + x + 3$

Solution

Leading term: $-11x^3$ with 3^{rd} degree (n is ***odd***)

$x \to -\infty \Rightarrow f(x) \to \infty$ $f(x)$ rises left

$x \to \infty \Rightarrow f(x) \to -\infty$ $f(x)$ falls right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = 2x^3 + 3x^2 - 23x - 42$

Solution

Leading term: $2x^3$ with 3^{rd} degree (n is ***odd***)

$x \to -\infty \Rightarrow f(x) \to -\infty$ $f(x)$ falls left

$x \to \infty \Rightarrow f(x) \to \infty$ $f(x)$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=5x^4+7x^2-x+9$

Solution

Leading term: $5x^4$ with 4^{rd} degree (n is ***even***)

$x \to -\infty \Rightarrow f(x) \to \infty$ $\quad f(x)$ rises left

$x \to \infty \Rightarrow f(x) \to \infty$ $\quad f(x)$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=11x^4-6x^2+x+3$

Solution

Leading term: $11x^4$ with 4^{rd} degree (n is ***even***)

$x \to -\infty \Rightarrow f(x) \to \infty$ $\quad f(x)$ rises left

$x \to \infty \Rightarrow f(x) \to \infty$ $\quad f(x)$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=-5x^4+7x^2-x+9$

Solution

Leading term: $-5x^4$ with 4^{rd} degree (n is ***even***)

$x \to -\infty \Rightarrow f(x) \to -\infty$ $\quad f(x)$ falls left

$x \to \infty \Rightarrow f(x) \to -\infty$ $\quad f(x)$ falls right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=-11x^4-6x^2+x+3$

Solution

Leading term: $-11x^4$ with 4^{rd} degree (n is ***even***)

$x \to -\infty \Rightarrow f(x) \to -\infty$ $\quad f(x)$ falls left

$x \to \infty \Rightarrow f(x) \to -\infty$ $\quad f(x)$ falls right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = 5x^5 - 16x^2 - 20x + 64$

Solution

Leading term: $5x^5$ with 5^{th} degree (n is ***odd***)

$x \to -\infty \quad \Rightarrow \quad f(x) \to -\infty$ $\qquad f(x)$ falls left

$x \to \infty \quad \Rightarrow \quad f(x) \to \infty$ $\qquad f(x)$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = -5x^5 - 16x^2 - 20x + 64$

Solution

Leading term: $-5x^5$ with 5^{th} degree (n is ***odd***)

$x \to -\infty \quad \Rightarrow \quad f(x) \to \infty$ $\qquad f(x)$ rises left

$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty$ $\qquad f(x)$ falls right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = -3x^6 - 16x^3 + 64$

Solution

Leading term: $-3x^6$ with 6^{th} degree (n is ***even***)

$x \to -\infty \quad \Rightarrow \quad f(x) \to -\infty$ $\qquad f(x)$ falls left

$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty$ $\qquad f(x)$ falls right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = 3x^6 - 16x^3 + 4$

Solution

Leading term: $3x^6$ with 6^{th} degree (n is ***even***)

$x \to -\infty \quad \Rightarrow \quad f(x) \to \infty$ $\qquad f(x)$ rises left

$x \to \infty \quad \Rightarrow \quad f(x) \to \infty$ $\qquad f(x)$ rises right

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x)=x^3-x-1;\quad between\ \ 1\ and\ 2$

Solution

$$f(1)=(1)^3-(1)-1$$
$$\underline{=-1\,|}$$
$$f(2)=(2)^3-(2)-1$$
$$\underline{=5\,|}$$

Since $f(1)\ and\ f(2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 1 and 2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x)=x^3-4x^2+2;\quad between\ \ 0\ and\ 1$

Solution

$$f(0)=(0)^3-4(0)^2+2$$
$$\underline{=2\,|}$$
$$f(1)=(1)^3-4(1)^2+2$$
$$\underline{=-1\,|}$$

Since $f(0)\ and\ f(1)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 0 and 1.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x)=2x^4-4x^2+1;\quad between\ \ -1\ and\ 0$

Solution

$$f(-1)=2(-1)^4-4(-1)^2+1$$
$$\underline{=-1\,|}$$
$$f(0)=2(0)^4-4(0)^2+1$$
$$\underline{=1\,|}$$

Since $f(0)$ *and* $f(-1)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between -1 and 0.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x)=x^4+6x^3-18x^2;\quad between\ \ 2\ and\ 3$

Solution

$$f(2)=(2)^4+6(2)^3-18(2)^2$$

$$\underline{=-8}\,|$$

$$f(3)=(3)^4+6(3)^3-18(3)^2$$

$$\underline{=81}\,|$$

Since $f(2)$ *and* $f(3)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 2 and 3.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x)=x^3+x^2-2x+1;\quad between\ \ -3\ and\ -2$

Solution

$$f(-3)=(-3)^3+(-3)^2-2(-3)+1$$

$$\underline{=-11}\,|$$

$$f(-2)=(-2)^3+(-2)^2-2(-2)+1$$

$$\underline{=1}\,|$$

Since $f(-3)$ *and* $f(-2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between -2 and -3.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x)=x^5-x^3-1;\quad between\ \ 1\ and\ 2$

Solution

$$f(1) = (1)^5 - (1)^3 - 1$$

$$\underline{= -1}\,\rfloor$$

$$f(2) = (2)^5 - (2)^3 - 1$$

$$\underline{= 23}\,\rfloor$$

Since $f(1)$ *and* $f(2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 1 and 2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = 3x^3 - 10x + 9;$ *between* -3 *and* -2

Solution

$$f(-3) = 3(-3)^3 - 10(-3) + 9$$

$$\underline{= -42}\,\rfloor$$

$$f(-2) = 3(-2)^3 - 10(-2) + 9$$

$$\underline{= 5}\,\rfloor$$

Since $f(-3)$ *and* $f(-2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between –3 and –2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = 3x^3 - 8x^2 + x + 2;$ *between* 2 *and* 3

Solution

$$f(2) = 3(2)^3 - 8(2)^2 + (2) + 2$$

$$\underline{= -4}\,\rfloor$$

$$f(3) = 3(3)^3 - 8(3)^2 + (3) + 2$$

$$\underline{= 14}\,\rfloor$$

Since $f(2)$ *and* $f(3)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 2 and 3.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = 3x^3 - 8x^2 + x + 2; \quad \textit{between} \ \ 1 \ \textit{and} \ 2$

Solution

$$f(1) = 3(1)^3 - 8(1)^2 + (1) + 2$$

$$\underline{= -2\,}\big|$$

$$f(2) = 3(2)^3 - 8(2)^2 + (2) + 2$$

$$\underline{= -4\,}\big|$$

Since $f(1)$ *and* $f(2)$ have same signs.

Therefore, ***cannot be determined***.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3; \quad \textit{between} \ \ 0 \ \textit{and} \ 1$

Solution

$$f(0) = (0)^5 + 2(0)^4 - 6(0)^3 + 2(0) - 3$$

$$\underline{= -3\,}\big|$$

$$f(1) = (1)^5 + 2(1)^4 - 6(1)^3 + 2(1) - 3$$

$$\underline{= -4\,}\big|$$

Since $f(0)$ *and* $f(1)$ have same signs.

Therefore, ***cannot be determined***.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 2x^3 + 3x^2 - 23x - 42, \quad a = 3, \quad b = 4$

Solution

$$P(3) = 54 + 27 - 69 - 42$$

$$\underline{= -30\,}\big|$$

$$P(4) = 128 + 48 - 92 - 42$$

$$\underline{= 90\,}\big|$$

Since $P(3)$ *and* $P(4)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 3 and 4.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 4x^3 - x^2 - 6x + 1, \quad a = 0, \quad b = 1$

Solution

$P(0) \underline{= 1}\rfloor$

$P(1) = 4 - 1 - 6 + 1$

$\underline{= -2}\rfloor$

Since $P(0)$ *and* $P(1)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 0 and 1.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 3x^3 + 7x^2 + 3x + 7, \quad a = -3, \quad b = -2$

Solution

$P(-3) = -81 + 63 - 9 + 7$

$\underline{= -20}\rfloor$

$P(-2) = -24 + 28 - 6 + 7$

$\underline{= 5}\rfloor$

Since $P(-3)$ *and* $P(-2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between −3 and −2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 2x^3 - 21x^2 - 2x + 25, \quad a = 1, \quad b = 2$

Solution

$P(1) = 2 - 21 - 2 + 25$

$\underline{= 4}\rfloor$

$$P(2) = 16 - 84 - 4 + 25$$

$$\underline{= -47}$$

Since $P(1)$ *and* $P(2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 1 and 2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 4x^4 + 7x^3 - 11x^2 + 7x - 15, \quad a = 1, \quad b = \frac{3}{2}$

Solution

$$P(1) = 4 + 7 - 11 + 7 - 15$$

$$\underline{= -8}$$

$$P\left(\frac{3}{2}\right) = 81 + \frac{189}{8} - \frac{99}{4} + \frac{21}{2} - 15$$

$$= 66 + \frac{189 - 198 + 84}{8}$$

$$= 66 + \frac{75}{8}$$

$$\underline{= \frac{603}{8}}$$

Since $P(1)$ *and* $P\left(\frac{3}{2}\right)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 1 and $\frac{3}{2}$.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 5x^3 - 16x^2 - 20x + 64, \quad a = 3, \quad b = \frac{7}{2}$

Solution

$$P(3) = 135 - 144 - 60 + 64$$

$$\underline{= -5}$$

$$P\left(\frac{7}{2}\right) = \frac{1715}{8} - 196 - 70 + 64$$

$$= \frac{1715}{8} - 202$$

$$\underline{= \frac{99}{8}}$$

Since $P(3)$ *and* $P\left(\frac{7}{2}\right)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 3 and $\frac{7}{2}$.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x)=x^4-x^2-x-4, \quad a=1, \quad b=2$

Solution

$$P(1)=1-1-1-4$$

$$\underline{=-5}\,|$$

$$P(2)=16-4-2-4$$

$$\underline{=6}\,|$$

Since $P(1)$ *and* $P(2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 1 and 2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x)=x^3-x-8, \quad a=2, \quad b=3$

Solution

$$P(2)=8-2-8$$

$$\underline{=-2}\,|$$

$$P(3)=27-3-8$$

$$\underline{=16}\,|$$

Since $P(2)$ *and* $P(3)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 2 and 3.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x)=x^3-x-8, \quad a=0, \quad b=1$

Solution

$$\underline{P(0)=-8}\,|$$

$$P(1) = 1 - 1 - 8$$

$$\underline{= -8}$$

Since $P(0)$ *and* $P(1)$ have same sign.

Therefore, ***cannot be determined***.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = x^3 - x - 8, \quad a = 2.1, \quad b = 2.2$

Solution

$$P(2.1) = P\left(\frac{21}{10}\right)$$

$$= \frac{9261}{1000} - \frac{21}{10} - 8$$

$$= \frac{9261 - 2100 - 8000}{1000}$$

$$\underline{= -\frac{839}{1,000}}$$

$$P(2.2) = P\left(\frac{2.2}{10}\right)$$

$$= \frac{10,648}{1000} - \frac{22}{10} - 8$$

$$= \frac{10,648 - 2,200 - 8,000}{1000}$$

$$= \frac{448}{1,000}$$

$$\underline{= 0.448}$$

Since $P(2.1)$ *and* $P(2.2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 2.1 and 2.2.

Exercise

Let $f(x) = x^4 - 4x^2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$f(x) = x^2\left(x^2 - 4\right)$$

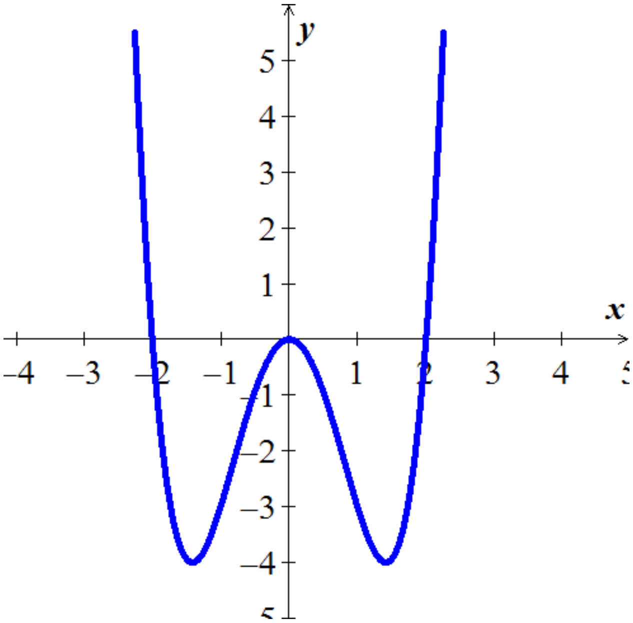
$$= x^2(x-2)(x+2)$$

The zeros are: 0, 0, 2, −2.

$-\infty$	-2	**0,0**	2	∞
+		−		+

$f(x) < 0 \quad (-2,\ 0) \cup (0,\ 2)$

$f(x) > 0 \quad (-\infty,\ -2) \cup (2,\ \infty)$



Exercise

Let $f(x) = x^4 + 3x^3 - 4x^2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

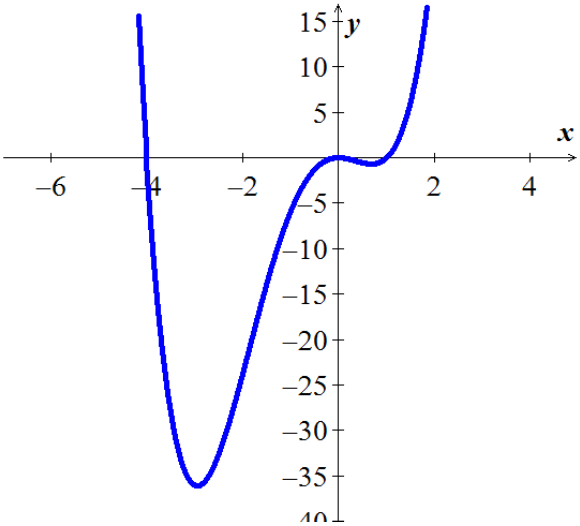
$$f(x) = x^2\left(x^2 + 3x - 4\right)$$

The zeros are: 0, 0, 1, −4.

$-\infty$	-4	**0,0**	1	∞
+		−		+

$f(x) > 0 \quad (-\infty,\ -4) \cup (1,\ \infty)$

$f(x) < 0 \quad (-4,\ 0) \cup (0,\ 1)$

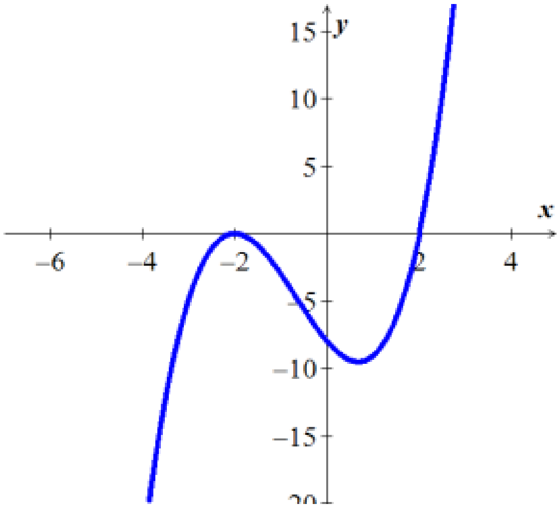


Exercise

Let $f(x)=x^3+2x^2-4x-8$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

Solution

$$\begin{aligned} f(x) &= x^2(x+2)-4(x+2) \\ &= (x+2)\left(x^2-4\right) \\ &= (x+2)(x+2)(x-2) = 0 \end{aligned}$$



The zeros are: 2, −2, −2

$-\infty$ -2 0 2 ∞

−	−	+

$f(x)>0$ $(2,\ \infty)$

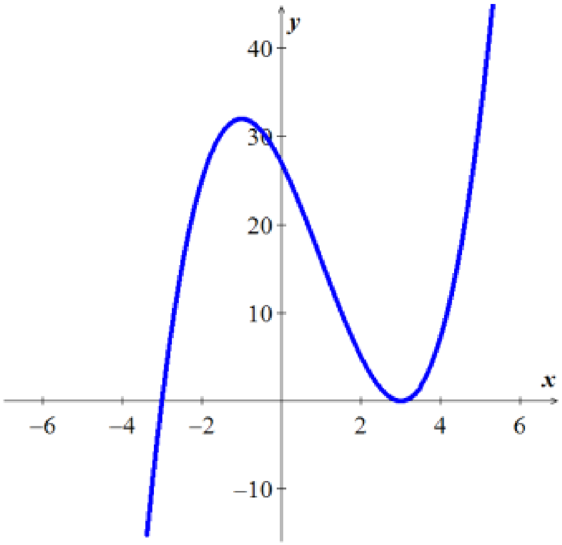
$f(x)<0$ $(-\infty,\ -2)\cup(-2,\ 2)$

Exercise

Let $f(x)=x^3-3x^2-9x+27$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

Solution

$$\begin{aligned} f(x) &= x^2(x-3)-9(x-3) \\ &= (x-3)\left(x^2-9\right) \\ &= (x-3)(x-3)(x+3) \end{aligned}$$



The zeros are: −3, 3 (multiplicity)

$-\infty$ -3 0 3 ∞

−	+	+

$f(x)>0$ $(-3,\ 3)\cup(3,\ \infty)$

$f(x)<0$ $(-\infty,\ -3)$

Exercise

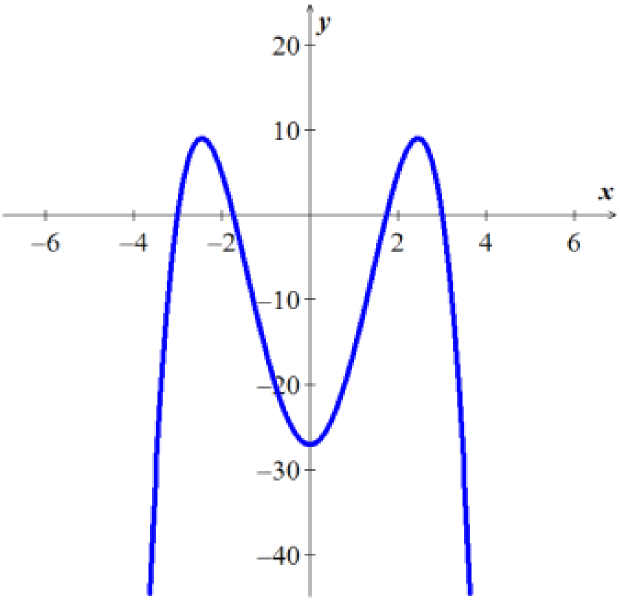
Let $f(x) = -x^4 + 12x^2 - 27$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$x^2 = \frac{-12 \pm \sqrt{36}}{-2}$$

$$= \begin{cases} \frac{-12-6}{-2} = 9 \\ \frac{-12+6}{-2} = 3 \end{cases}$$

$$\rightarrow \begin{cases} x^2 = 9 \\ x^2 = 3 \end{cases} \Rightarrow \begin{matrix} x = \pm 3 \\ x = \pm\sqrt{3} \end{matrix}$$



	-3	$-\sqrt{3}$	$\sqrt{3}$	3
−	+	−	+	−

$f(x) > 0$ $\quad \underline{\left(-3,\ -\sqrt{3}\right) \cup \left(\sqrt{3},\ 3\right)}$

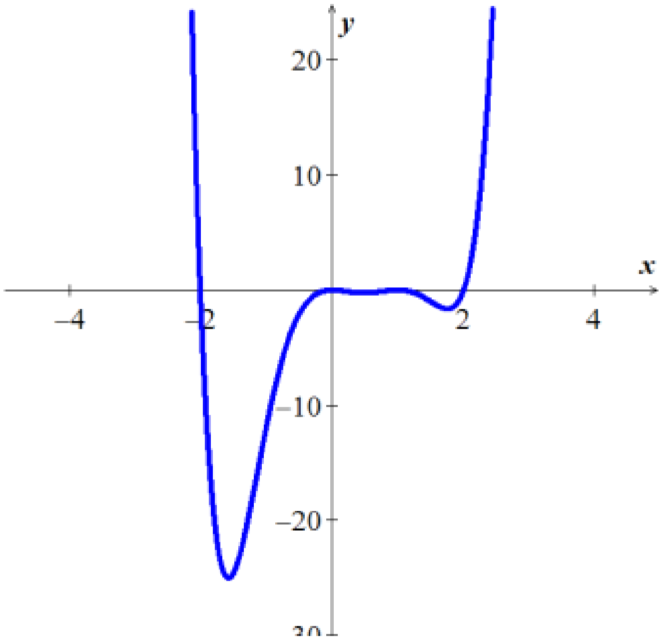
$f(x) < 0$ $\quad \underline{(-\infty,\ -3) \cup \left(-\sqrt{3},\ \sqrt{3}\right) \cup (3,\ \infty)}$

Exercise

Let $f(x) = x^2(x+2)(x-1)^2(x-2)$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

The zeros are: −2, 2, 0, 0, 1, 1



-2	**0,0** **1,1**	2
+	−	+

$f(x) > 0$ $\quad \underline{(-\infty,\ -2) \cup (2,\ \infty)}$

$f(x) < 0$ $\quad \underline{(-2,\ 0) \cup (0,\ 1) \cup (1,\ 2)}$

Exercise

Let $f(x) = 2x^3 + 11x^2 - 7x - 6$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$possibilities: \pm\left\{\frac{6}{2}\right\} = \pm\left\{\frac{1,\ 2,\ 3,\ 6}{1,\ 2}\right\}$$

$$= \pm\left\{1,\ 2,\ 3,\ 5,\ \frac{1}{2},\ \frac{3}{2}\right\}$$

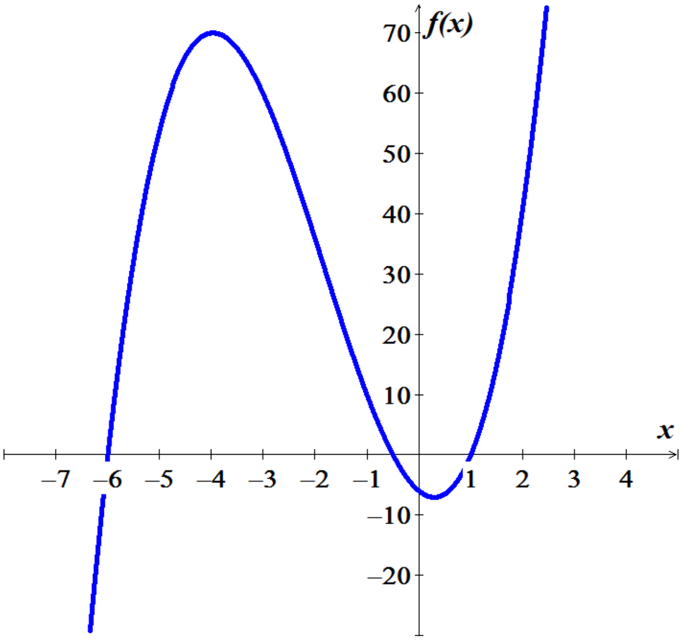
$$\begin{array}{r|rrrr} 1 & 2 & 11 & -7 & -6 \\ & & 2 & 13 & 6 \\ \hline & 2 & 13 & 6 & \boxed{0} \end{array} \rightarrow 2x^2 + 13x + 6 = 0$$

The zeros are: $x = 1,\ -\frac{1}{2},\ -6$

$$\begin{array}{ccccccc} & -6 & & -\frac{1}{2} & & 1 & \\ - & & + & & - & & + \end{array}$$

$f(x) > 0$ $\left(-6,\ -\frac{1}{2}\right) \cup (1,\ \infty)$

$f(x) < 0$ $(-\infty,\ -6) \cup \left(-\frac{1}{2},\ 1\right)$



Exercise

Let $f(x) = x^3 + 2x^2 - 5x - 6$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$possibilities\ :\ \pm\left\{\frac{6}{1}\right\}$$

$$= \pm\{1,\ 2,\ 3,\ 6\}$$

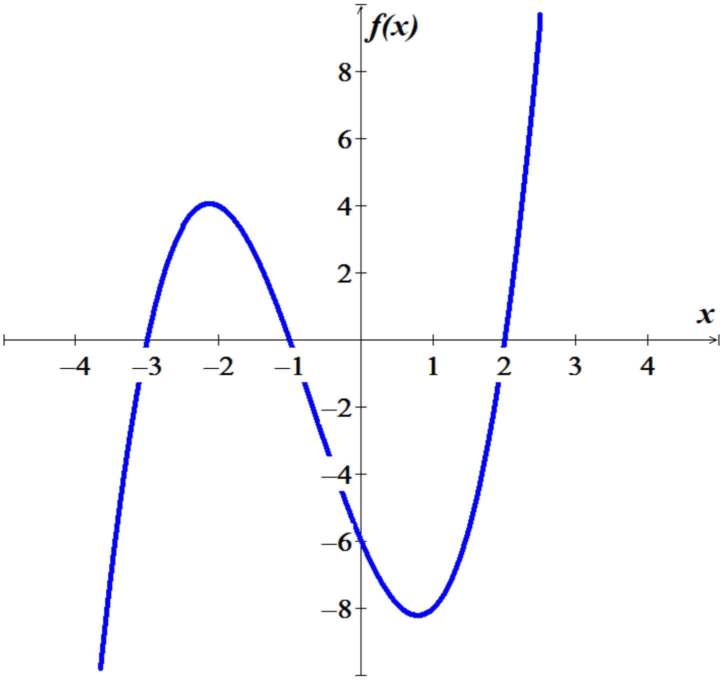
$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & & -1 & -1 & 6 \\ \hline & 1 & 1 & -6 & \boxed{0} \end{array} \rightarrow x^2 + x - 6 = 0$$

The zeros are: $x = -1,\ -3,\ 2$

$$\begin{array}{ccccccc} & -3 & & -1 & & 2 & \\ - & & + & & - & & + \end{array}$$

$f(x) > 0 \quad \boxed{(-3,\ -1)\cup(2,\ \infty)}$

$f(x) < 0 \quad \boxed{(-\infty,\ -3)\cup(-1,\ 2)}$



Exercise

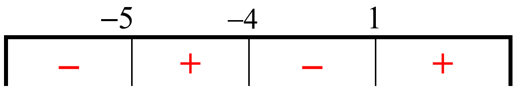
Let $f(x) = x^3 + 8x^2 + 11x - 20$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$\text{possibilities } : \pm\left\{\frac{20}{1}\right\} = \pm\{1,\ 2,\ 4,\ 5,\ 20,\ 20\}$$

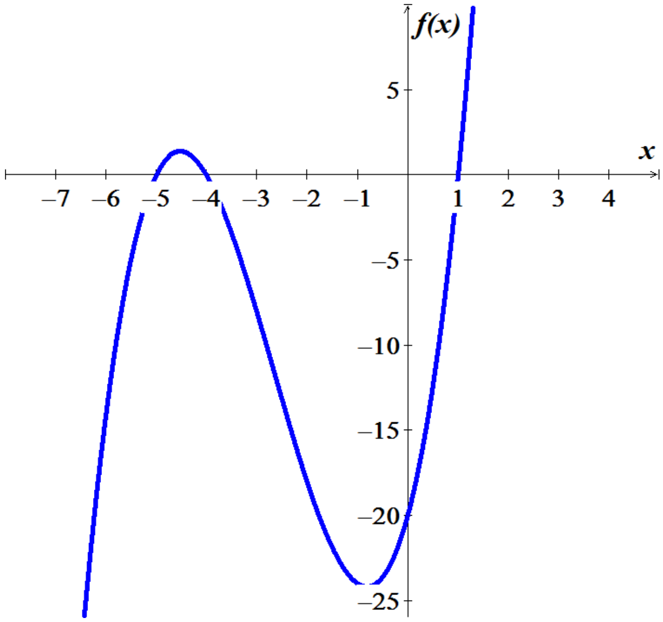
$$\begin{array}{c|cccc} 1 & 1 & 8 & 11 & -20 \\ & & 1 & 9 & 20 \\ \hline & 1 & 9 & 20 & \boxed{0} \end{array} \quad \rightarrow\ x^2 + 9x + 20 = 0$$

The zeros are: $\boxed{x = -5,\ -4,\ 1}$



$f(x) > 0 \quad \boxed{(-5,\ -1)\cup(1,\ \infty)}$

$f(x) < 0 \quad \boxed{(-\infty,\ -5)\cup(-4,\ 1)}$



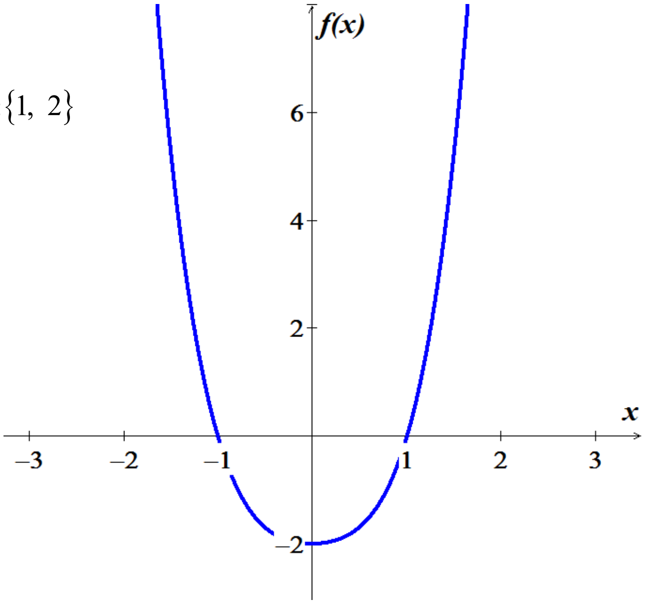
Exercise

Let $f(x) = x^4 + x^2 - 2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

possibilities: $\pm\{1,\ 2\}$

$$\begin{array}{r|rrrrr} 1 & 1 & 0 & 1 & 0 & -2 \\ & & 1 & 1 & 2 & 1 \\ \hline -1 & 1 & 1 & 2 & 2 & 0 \\ & & -1 & 0 & -2 & \\ \hline & 1 & 0 & 2 & 0 & \end{array} \to x^3 + x^2 + 2x + 1 = 0 \to \pm\{1,\ 2\}$$



$\to x^2 + 2 = 0 \quad \Rightarrow \quad \underline{x = \pm i\sqrt{2}}$

The zeros are: $\underline{x = \pm 1}$

$$\begin{array}{c|c|c} & -1 \quad\quad 1 & \\ \hline + & - & + \end{array}$$

$f(x) > 0 \quad \underline{(-\infty,\ -1) \cup (1,\ \infty)}$

$f(x) < 0 \quad \underline{(-1,\ 1)}$

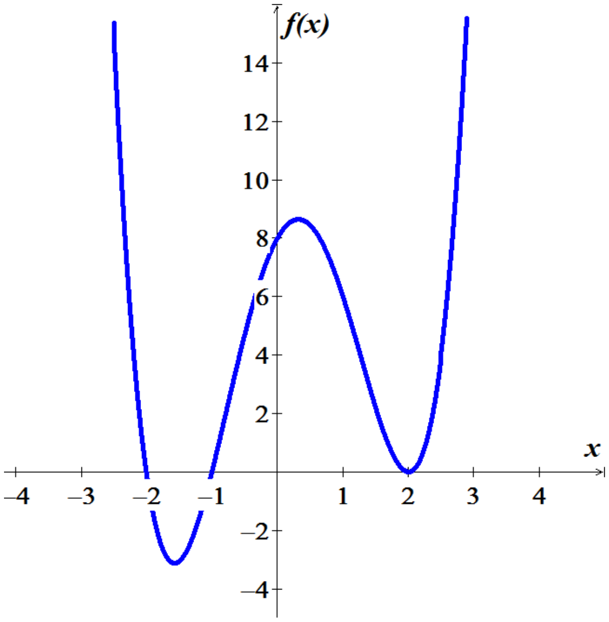
Exercise

Let $f(x) = x^4 - x^3 - 6x^2 + 4x + 8$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

possibilities: $\pm\{1,\ 2,\ 4,\ 8\}$

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & -6 & 4 & 8 \\ & & -1 & 2 & 4 & -8 \\ \hline -2 & 1 & -2 & -4 & 8 & 0 \\ & & -2 & 8 & -8 & \\ \hline & 1 & -4 & 4 & 0 & \end{array} \begin{array}{l} \to x^3 - 2x^2 - 4x + 8 = 0 \\ \to \pm\{1,\ 2,\ 4,\ 8\} \end{array}$$



$\to x^2 - 4x + 4 = 0 \quad \Rightarrow \quad \underline{x = 2,\ 2}$

The zeros are: $\underline{x = -2,\ \ -1,\ 2,\ 2}$

$$\begin{array}{c|c|c|c} & -2 \quad\quad -1 & \quad\quad 2 & \\ \hline + & - & + & + \end{array}$$

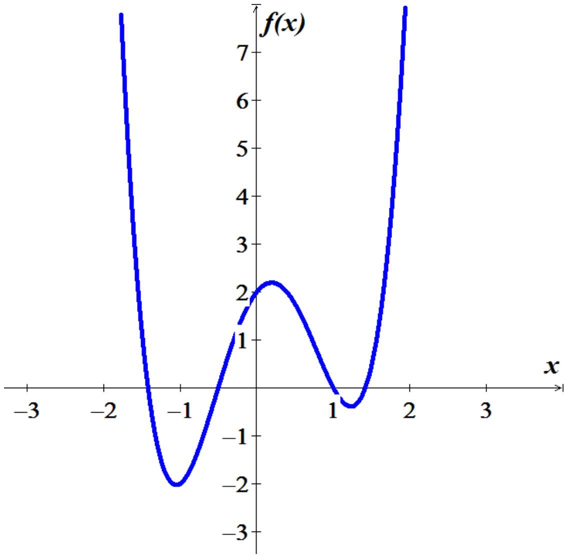
$f(x) > 0 \quad \underline{\left.(-\infty,\ -1)\cup(1,\ \infty)\right|}$

$f(x) < 0 \quad \underline{\left.(-1,\ 1)\right|}$

Exercise

Let $f(x) = 2x^4 - x^3 - 5x^2 + 2x + 2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution



$$\textit{possibilities}: \ \pm\left\{\frac{2}{2}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2}\right\}$$

$$\begin{array}{r|rrrrr}
1 & 2 & -1 & -5 & 2 & 2 \\
 & & 2 & 1 & -4 & -2 \\
\hline
-\frac{1}{2} & 2 & 1 & -4 & -2 & 0 \\
 & & -1 & 0 & 2 & \\
\hline
 & 2 & 0 & -4 & 0 &
\end{array}
\quad \begin{array}{l} \to 2x^3+x^2-4x-2=0 \\ \to\pm\left\{1,\ 2,\ \frac{1}{2}\right\} \end{array}$$

$$\to\ 2x^2 - 4 = 0 \Rightarrow \underline{\left. x = \pm\sqrt{2}\right|}$$

The zeros are: $\underline{\left. x = -\frac{1}{2},\ 1,\ -\sqrt{2},\ \sqrt{2}\right|}$

$$\begin{array}{ccccccccc}
 & -\sqrt{2} & & -\frac{1}{2} & & 1 & & \sqrt{2} & \\
+ & | & - & | & + & | & - & | & +
\end{array}$$

$f(x) > 0 \quad \underline{\left.(-\infty,\ -\sqrt{2})\cup\left(-\frac{1}{2},\ 1\right)\cup(\sqrt{2},\ \infty)\right|}$

$f(x) < 0 \quad \underline{\left.\left(-\sqrt{2},\ -\frac{1}{2}\right)\cup(1,\ \sqrt{2})\right|}$

Exercise

Let $f(x) = 4x^5 - 8x^4 - x + 2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$\begin{aligned} f(x) &= 4x^4(x-2) - (x-2) \\ &= (x-2)\left(4x^4 - 1\right) = 0 \end{aligned}$$

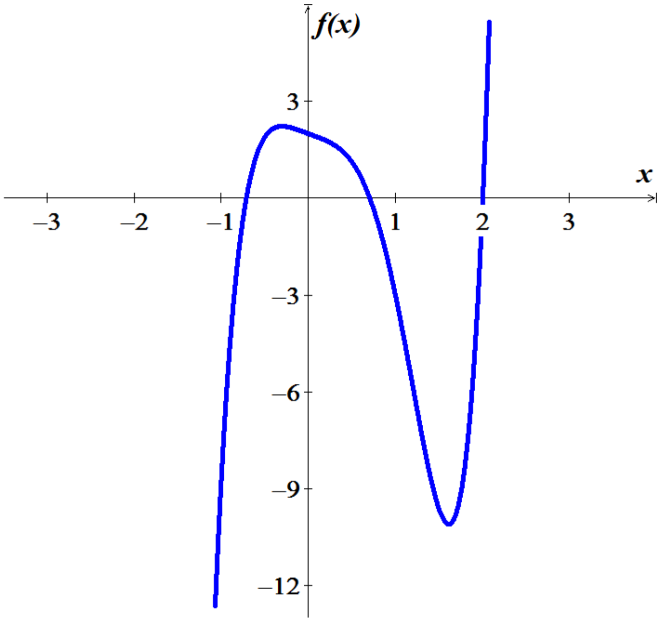
$$4x^4-1=0 \quad \Rightarrow \quad \begin{cases} x^2=-\frac{1}{2} & \mathbb{C} \\ x^2=\frac{1}{2} & x=\pm\frac{\sqrt{2}}{2} \end{cases}$$

The zeros are: $\boxed{x=2,\ -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}}$

	$-\frac{\sqrt{2}}{2}$		$\frac{\sqrt{2}}{2}$		2	
−		+		−		+

$f(x)>0$ $\boxed{\left(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)\cup(2,\ \infty)}$

$f(x)<0$ $\boxed{\left(-\infty,\ -\frac{\sqrt{2}}{2}\right)\cup\left(\frac{\sqrt{2}}{2},\ 2\right)}$



Exercise

Let $f(x)=x^5-7x^4+19x^3-37x^2+60x-36$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

Solution

possibilities: $\pm\left\{\frac{36}{1}\right\}=\pm\{1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12,\ 18,\ 36\}$

$$\begin{array}{c|cccccc}
1 & 1 & -7 & 19 & -37 & 60 & -36 \\
 & & 1 & -6 & 13 & -24 & 36 \\
\hline
3 & 1 & -6 & 13 & -24 & 36 & 0 \\
 & & 3 & -9 & 12 & -36 & \\
\hline
3 & 1 & -3 & 4 & -12 & 0 & \\
 & & 3 & 0 & 12 & & \\
\hline
 & 1 & 0 & 4 & 0 & &
\end{array}$$

$x^4-6x^3+13x^2-24x+36=0 \rightarrow \pm\{1,2,3,4,6,9,12,18,36\}$

$x^3-3x^2+4x-12=0 \rightarrow \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

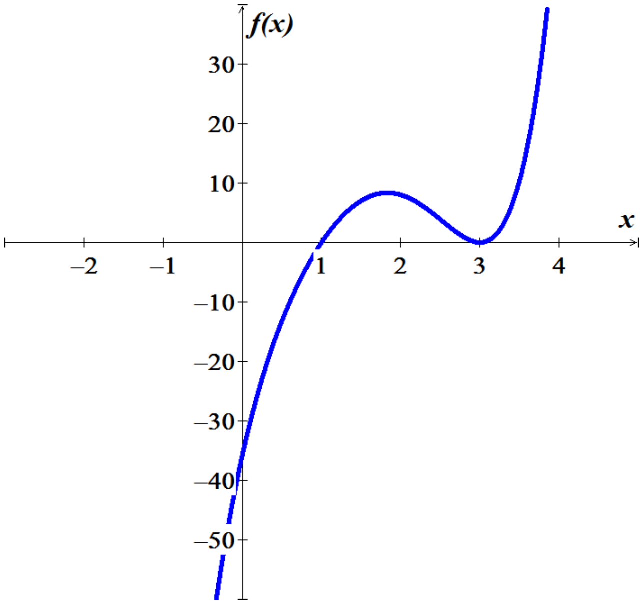
$x^2+4=0 \quad \Rightarrow \quad x=\pm 2i$

The zeros are: $\boxed{x=1,\ 3,\ 3}$

	1		3	
−		+		+

$f(x)>0$ $\boxed{(1,\ 3)\cup(3,\ \infty)}$

$f(x)<0$ $\boxed{(-\infty,\ 1)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - x^2 - 10x - 8$$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{8}{1}\right\} = \pm\{1,\ 2,\ 4,\ 8\}$

$$\begin{array}{r|rrrr} -1 & 1 & -1 & -10 & -8 \\ & & -1 & 2 & 8 \\ \hline & 1 & -2 & -8 & 0 \end{array} \rightarrow \quad x^2 - 2x - 8 = 0$$

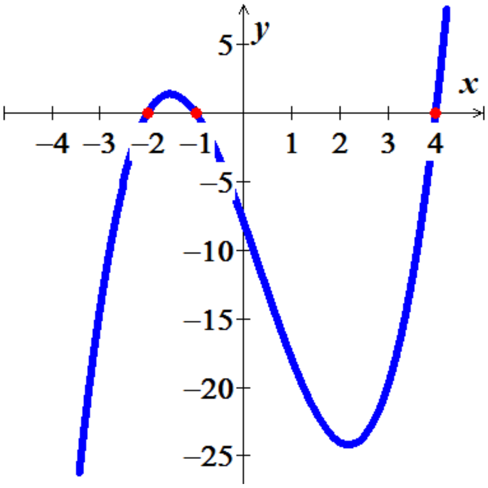
$x = -1,\ -2,\ 4$

$-2 \qquad -1 \qquad 4$

−	+	−	+

$f(x) > 0$ $(-2,\ -1) \cup (4,\ \infty)$

$f(x) < 0$ $(-\infty,\ -2) \cup (-1,\ 4)$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 + x^2 - 14x - 24$$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{24}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24\}$

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -14 & -24 \\ & & -2 & 2 & 24 \\ \hline & 1 & -1 & -12 & 0 \end{array} \rightarrow \quad x^2 - x - 12 = 0$$

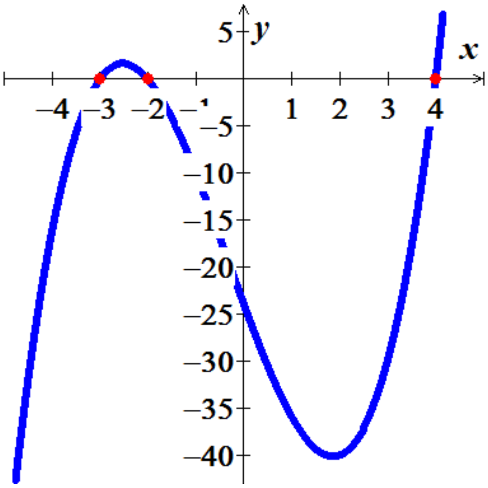
$x = -2,\ -3,\ 4$

$-3 \qquad -2 \qquad 4$

−	+	−	+

$f(x) > 0$ $(-3,\ -2) \cup (4,\ \infty)$

$f(x) < 0$ $(-\infty,\ -3) \cup (-2,\ 4)$



Exercise

Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of $f(x)$

$$f(x)=2x^3-3x^2-17x+30$$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{30}{2}\right\}=\pm\left\{1,\ 2,\ 3,\ 5,\ 6,\ 10,\ 15,\ 30,\ \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \frac{15}{2}\right\}$

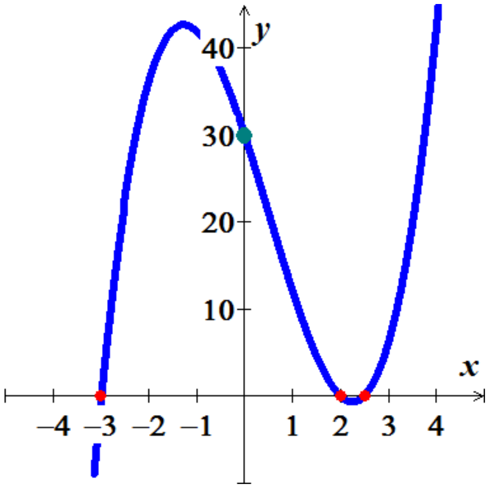
$$\begin{array}{r|rrrr} 2 & 2 & -3 & -17 & 30 \\ & & 4 & 2 & -30 \\ \hline & 2 & 1 & -15 & \boxed{0} \end{array} \rightarrow\ 2x^2+x-15=0$$

$x=2,\ -3,\ \frac{5}{2}$

$$\begin{array}{c|c|c|c} & -3 & 2 & \frac{5}{2} \\ - & + & - & + \end{array}$$

$f(x)>0\quad (-3,\ 2)\cup\left(\frac{5}{2},\ \infty\right)$

$f(x)<0\quad (-\infty,\ -3)\cup\left(2,\ \frac{5}{2}\right)$



Exercise

Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of $f(x)$

$$f(x)=12x^3+8x^2-3x-2$$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{2}{12}\right\}=\pm\left\{1,\ 2,\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{2},\ \frac{1}{6},\ \frac{1}{12}\right\}$

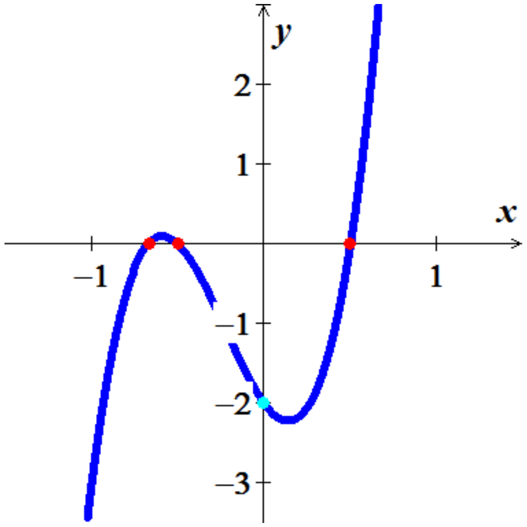
$$\begin{array}{r|rrrr} \frac{1}{2} & 12 & 8 & -3 & -2 \\ & & 6 & 7 & 2 \\ \hline & 12 & 14 & 4 & \boxed{0} \end{array} \rightarrow\ 12x^2+14x+4=0$$

$x=\frac{1}{2},\ -\frac{1}{2},\ -\frac{2}{3}$

$$\begin{array}{c|c|c|c} & -\frac{2}{3} & -\frac{1}{2} & \frac{1}{2} \\ - & + & - & + \end{array}$$

$f(x)>0\quad \left(-\frac{2}{3},\ -\frac{1}{2}\right)\cup\left(\frac{1}{2},\ \infty\right)$

$f(x) < 0 \quad \boxed{\left(-\infty,\ -\frac{2}{3}\right) \cup \left(-\frac{1}{2},\ \frac{1}{2}\right)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 + x^2 - 6x - 8$$

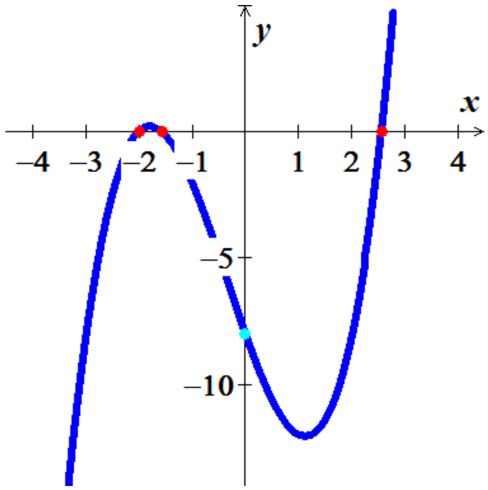
Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{8}{1}\right\} = \pm\{1,\ 2,\ 4,\ 8\}$

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -6 & -8 \\ & & -2 & 2 & 8 \\ \hline & 1 & -1 & -4 & \boxed{0} \end{array} \rightarrow \quad x^2 - x - 4 = 0$$

$$x = \frac{1 \pm \sqrt{1+16}}{2}$$

$$\boxed{x = -2,\ \frac{1 \pm \sqrt{17}}{2}}$$



	-2		$\frac{1-\sqrt{17}}{2}$		$\frac{1+\sqrt{17}}{2}$	
−		+		−		+

$f(x) > 0 \quad \boxed{\left(-2,\ \frac{1-\sqrt{17}}{2}\right) \cup \left(\frac{1+\sqrt{17}}{2},\ \infty\right)}$

$f(x) < 0 \quad \boxed{(-\infty,\ -2) \cup \left(\frac{1-\sqrt{17}}{2},\ \frac{1+\sqrt{17}}{2}\right)}$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 19x - 30$$

Solution

$$\textit{possibilities for } \frac{c}{d} : \pm\left\{\frac{30}{1}\right\} = \pm\{1,\ 2,\ 3,\ 5,\ 6,\ 15,\ 30\}$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -19 & -30 \\ & & -2 & 4 & 30 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array} \rightarrow\ x^2 - 2x - 15$$

$$x = \frac{2 \pm \sqrt{4+60}}{2}$$

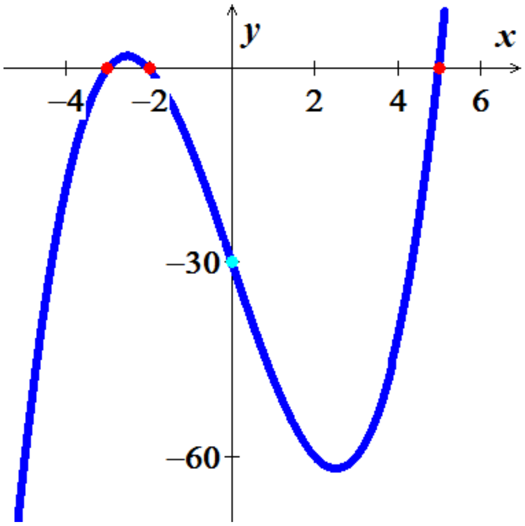
$$= \begin{cases} \frac{2-8}{2} = -3 \\ \frac{2+8}{2} = 5 \end{cases}$$

$$\underline{x = -2,\ -3,\ 5}$$

	−3		−2		5	
−		+		−		+

$f(x) > 0 \quad \underline{(-3,\ -2) \cup (5,\ \infty)}$

$f(x) < 0 \quad \underline{(-\infty,\ -3) \cup (-2,\ 5)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 2x^3 + x^2 - 25x + 12$$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{12}{2}\right\} = \pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 12,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$$\begin{array}{r|rrrr} 3 & 2 & 1 & -25 & 12 \\ & & 6 & 21 & -12 \\ \hline & 2 & 7 & -4 & \boxed{0} \end{array} \quad \to \quad 2x^2 + 7x - 4$$

$$x = \frac{-7 \pm \sqrt{49 + 32}}{4}$$

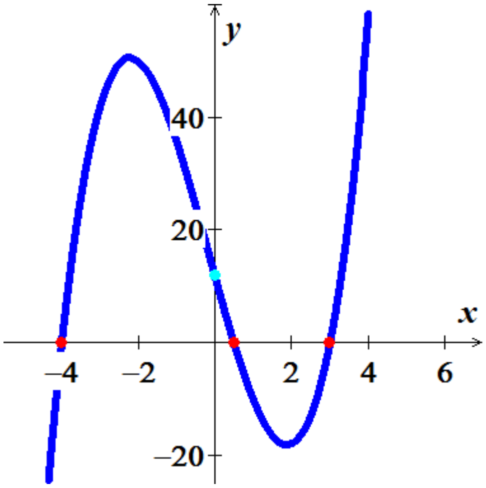
$$= \begin{cases} \frac{-7-9}{4} = -4 \\ \frac{-7+9}{4} = \frac{1}{2} \end{cases}$$

$x = -4,\ \frac{1}{2},\ 3$

-4	$\frac{1}{2}$	3	
$-$	$+$	$-$	$+$

$f(x) > 0$ $\quad \left(-4,\ \frac{1}{2}\right) \cup (3,\ \infty)$

$f(x) < 0$ $\quad (-\infty,\ -1) \cup \left(\frac{1}{2},\ 3\right)$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 3x^3 + 11x^2 - 6x - 8$$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{8}{3}\right\} = \pm\left\{1,\ 2,\ 4,\ 8,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3},\ \frac{8}{3}\right\}$

$$\begin{array}{r|rrrr} 1 & 3 & 11 & -6 & -8 \\ & & 3 & 14 & 8 \\ \hline & 3 & 14 & 8 & \boxed{0} \end{array} \quad \rightarrow \quad 3x^2 + 14x + 8$$

$$x = \frac{-14 \pm \sqrt{196 - 96}}{6}$$

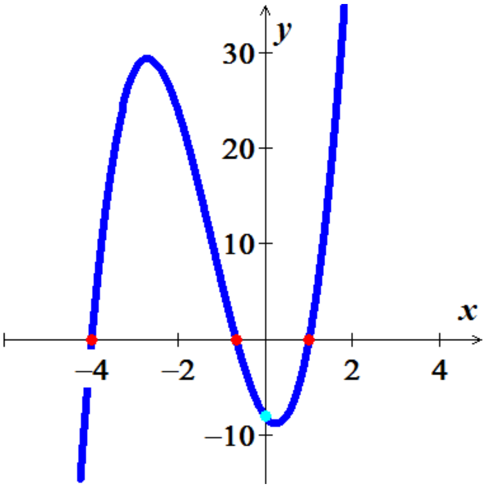
$$= \begin{cases} \frac{-14-10}{6} = -4 \\ \frac{-14+10}{6} = -\frac{2}{3} \end{cases}$$

$$x = -4,\ -\frac{2}{3},\ 1$$

$$\begin{array}{ccccccc} & -4 & & -\frac{2}{3} & & 1 & \\ \hline - & | & + & | & - & | & + \end{array}$$

$f(x) > 0 \quad \left(-4,\ -\frac{2}{3}\right) \cup (1,\ \infty)$

$f(x) < 0 \quad (-\infty,\ -4) \cup \left(-\frac{2}{3},\ 1\right)$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 2x^3 + 9x^2 - 2x - 9$$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{9}{2}\right\} = \pm\left\{1,\ 3,\ 9,\ \frac{1}{2},\ \frac{3}{2},\ \frac{9}{2}\right\}$

$$\begin{array}{r|rrrr} 1 & 2 & 9 & -2 & -9 \\ & & 2 & 11 & 9 \\ \hline & 2 & 11 & 9 & \boxed{0} \end{array} \quad \to \quad 2x^2 + 11x + 9$$

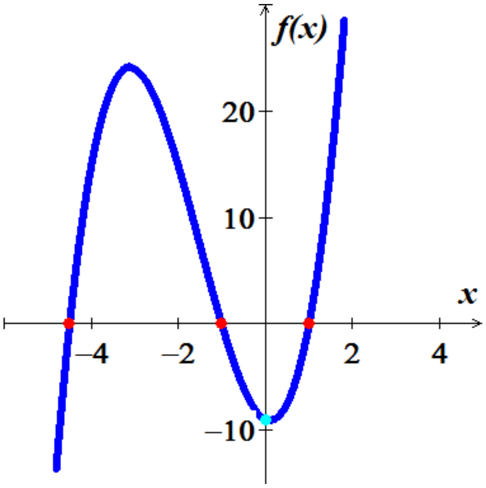
$x = -1,\ -\frac{9}{2}$ $\qquad a - b + c = 0 \quad \to \quad x = -1,\ -\frac{c}{a}$

$x = -\frac{9}{2},\ -1,\ 1$

	$-\frac{9}{2}$		-1		1	
$-$		$+$		$-$		$+$

$f(x) > 0 \quad \left(-\frac{9}{2},\ -1\right) \cup (1,\ \infty)$

$f(x) < 0 \quad \left(-\infty,\ -\frac{9}{2}\right) \cup (-1,\ 1)$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 + 3x^2 - 6x - 8$$

Solution

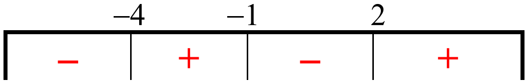
$$\textit{possibilities for } \frac{c}{d} : \pm\left\{\frac{8}{1}\right\} = \pm\{1,\ 2,\ 4,\ 8\}$$

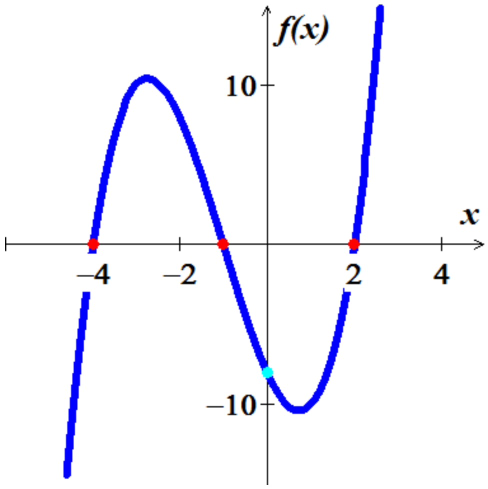
$$\begin{array}{r|rrrr} -1 & 1 & 3 & -6 & -8 \\ & & -1 & -2 & 8 \\ \hline & 1 & 2 & -8 & \boxed{0} \end{array} \rightarrow \ x^2 + 2x - 8 = 0$$

$$x = \frac{-2 \pm \sqrt{4+32}}{2}$$

$$= \begin{cases} \frac{-2-6}{2} = -4 \\ \frac{-2+6}{2} = 2 \end{cases}$$

$$\underline{x = -4,\ \ -1,\ \ 2\ \big|}$$





$f(x) > 0 \quad \underline{(-4,\ -1) \cup (2,\ \infty)\ \big|}$

$f(x) < 0 \quad \underline{(-\infty,\ -4) \cup (-1,\ 2)\ \big|}$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

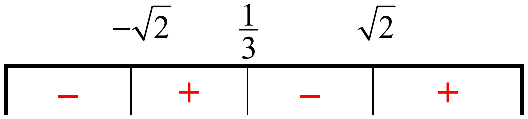
$$f(x) = 3x^3 - x^2 - 6x + 2$$

Solution

$$\textit{possibilities for } \frac{c}{d} : \pm\left\{\frac{2}{3}\right\} = \pm\left\{1,\ 2,\ \frac{1}{3},\ \frac{2}{3}\right\}$$

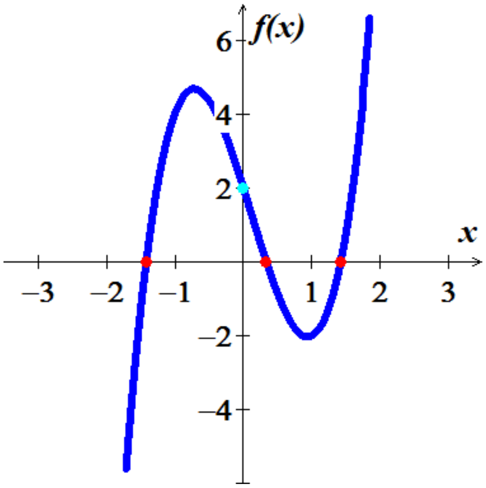
$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -1 & -6 & 2 \\ & & 1 & 0 & -2 \\ \hline & 3 & 0 & -6 & \boxed{0} \end{array} \ \rightarrow \ 3x^2 - 6 = 0$$

$$\underline{x = \frac{1}{3},\ \pm\sqrt{2}\ \big|}$$



$f(x) > 0$ $\quad \underline{\left(-\sqrt{2},\ \frac{1}{3}\right) \cup \left(\sqrt{2},\ \infty\right)}$

$f(x) < 0$ $\quad \underline{\left(-\infty,\ -\sqrt{2}\right) \cup \left(\frac{1}{3},\ \sqrt{2}\right)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 8x^2 + 8x + 24$$

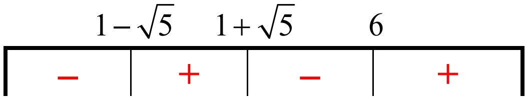
Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{24}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24\}$

$$\begin{array}{r|rrrr} 6 & 1 & -8 & 8 & 24 \\ & & 6 & -12 & -24 \\ \hline & 1 & -2 & -4 & \boxed{0} \end{array} \quad \to \quad x^2 - 2x - 4 = 0$$

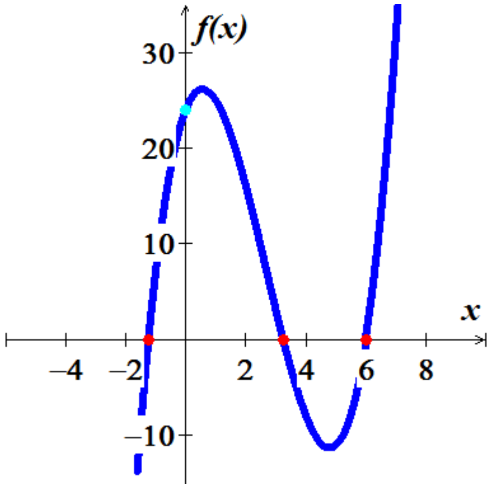
$$x = \frac{2 \pm \sqrt{4+16}}{2}$$

$\underline{x = 6,\ 1 \pm \sqrt{5}}$



$f(x) > 0$ $\quad \underline{\left(1-\sqrt{5},\ 1+\sqrt{5}\right) \cup (6,\ \infty)}$

$f(x) < 0 \quad \boxed{\left(-\infty,\ 1-\sqrt{5}\right)\cup\left(1+\sqrt{5},\ 6\right)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 7x^2 - 7x + 69$$

Solution

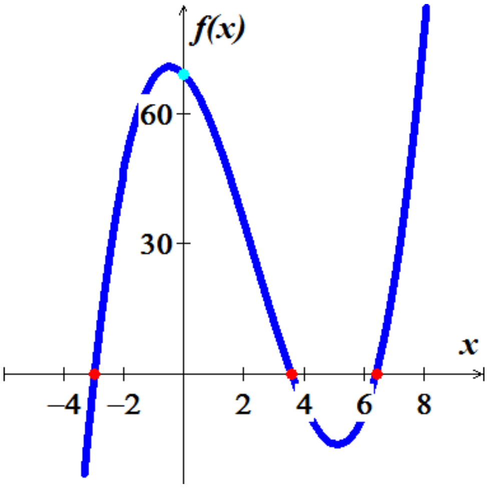
possibilities for $\frac{c}{d} : \pm\left\{\frac{69}{1}\right\} = \pm\{1,\ 3,\ 23,\ 69\}$

$$\begin{array}{r|rrrr} -3 & 1 & -7 & -7 & 69 \\ & & -3 & 30 & -69 \\ \hline & 1 & -10 & 23 & \boxed{0} \end{array} \quad \to \quad x^2 - 10x + 23 = 0$$

$$x = \frac{10 \pm \sqrt{100-92}}{2}$$

$$= \frac{10 \pm 2\sqrt{2}}{2}$$

$$\boxed{x = -3,\ 5 \pm \sqrt{2}}$$



	-3		$5-\sqrt{2}$		$5+\sqrt{2}$	
−		+		−		+

$f(x) > 0 \quad \boxed{\left(-3,\ 5-\sqrt{2}\right)\cup\left(5+\sqrt{2},\ \infty\right)}$

$f(x) < 0 \quad \boxed{\left(-\infty,\ -3\right)\cup\left(5-\sqrt{2},\ 5+\sqrt{2}\right)}$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 3x - 2$$

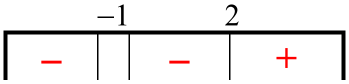
Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & -2 \\ & & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array} \quad \rightarrow \quad x^2 - x - 2 = 0$$

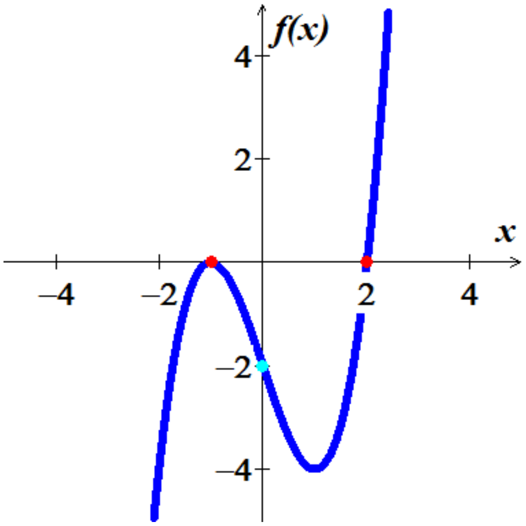
$x = -1,\ 2$ $\qquad a - b + c = 0 \quad \rightarrow \quad x = -1,\ -\frac{c}{a}$

$\underline{x = -1,\ -1,\ 2}$



$f(x) > 0 \quad \underline{(2,\ \infty)}$

$f(x) < 0 \quad \underline{(-\infty,\ -1) \cup (-1,\ 2)}$



Exercise

Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of $f(x)$

$$f(x)=x^3-2x+1$$

Solution

possibilities for $\frac{c}{d}:\pm\{1\}$

$$\begin{array}{c|cccc} 1 & 1 & 0 & -2 & 1 \\ & & 1 & 1 & 1 \\ \hline & 1 & 1 & -1 & \boxed{0} \end{array} \quad \rightarrow \quad x^2+x-1=0$$

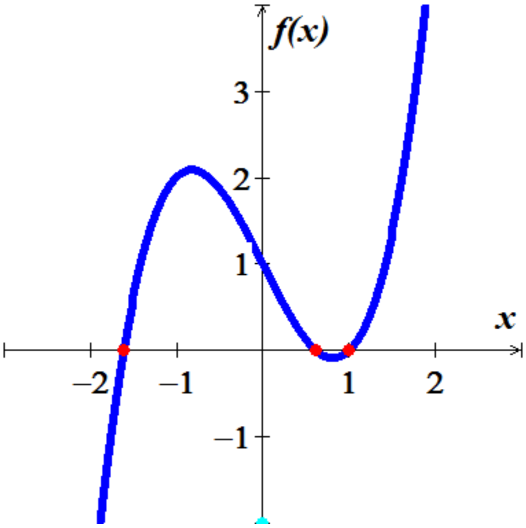
$$x=\frac{-1\pm\sqrt{5}}{2}$$

$$\boxed{x=1,\ \frac{-1\pm\sqrt{5}}{2}}$$

$$\begin{array}{ccccccc} & \frac{-1-\sqrt{5}}{2} & & \frac{-1+\sqrt{5}}{2} & & 1 & \\ - & | & + & | & - & | & + \end{array}$$

$$f(x)>0 \quad \boxed{\left(\frac{-1-\sqrt{5}}{2},\ \frac{-1+\sqrt{5}}{2}\right)\cup(1,\ \infty)}$$

$$f(x)<0 \quad \boxed{\left(-\infty,\ \frac{-1-\sqrt{5}}{2}\right)\cup\left(\frac{-1+\sqrt{5}}{2},\ 1\right)}$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 2x^2 - 11x + 12$$

Solution

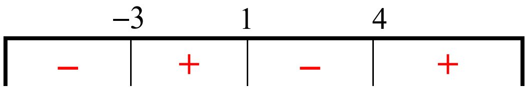
possibilities for $\frac{c}{d} : \pm\left\{\frac{12}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -11 & 12 \\ & & 1 & -1 & 12 \\ \hline & 1 & -1 & -12 & \boxed{0} \end{array} \quad \to \quad x^2 - x - 12 = 0$$

$$x = \frac{1 \pm \sqrt{1+48}}{2}$$

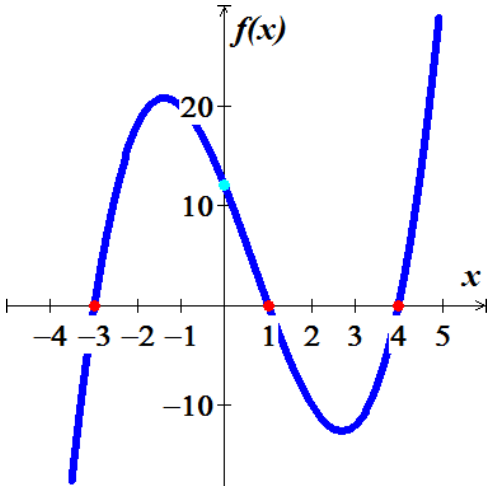
$$= \begin{cases} \frac{1-7}{2} = -3 \\ \frac{1+7}{2} = 4 \end{cases}$$

$$\underline{x = -3,\ 1,\ 4}$$



$f(x) > 0 \quad \underline{(-3,\ 1) \cup (4,\ \infty)}$

$f(x) < 0 \quad \underline{(-\infty,\ -3) \cup (1,\ 4)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 2x^2 - 7x - 4$$

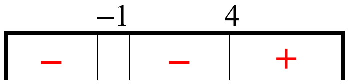
Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{4}{1}\right\} = \pm\{1,\ 2,\ 4\}$

$$\begin{array}{r|rrrr} -1 & 1 & -2 & -7 & -4 \\ & & -1 & 3 & 4 \\ \hline & 1 & -3 & -4 & \boxed{0} \end{array} \quad \to \quad x^2 - 3x - 4 = 0$$

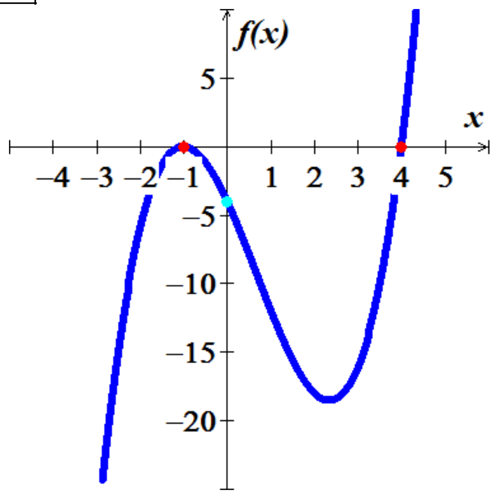
$\underline{x = -1,\ 4}$ $\qquad a - b + c = 0 \quad \to \quad x = -1,\ -\frac{c}{a}$

$\underline{x = -1,\ -1,\ 4}$



$f(x) > 0 \quad \underline{(4,\ \infty)|}$

$f(x) < 0 \quad \underline{(-\infty,\ -1) \cup (-1,\ 4)|}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 10x - 12$$

Solution

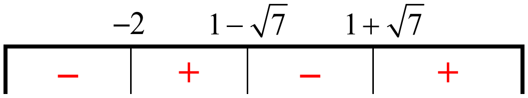
possibilities for $\frac{c}{d} : \pm\left\{\frac{12}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -10 & -12 \\ & & -2 & 4 & 12 \\ \hline & 1 & -2 & -6 & \boxed{0} \end{array} \quad \to \quad x^2 - 2x - 6 = 0$$

$$x = \frac{2 \pm \sqrt{4+24}}{2}$$

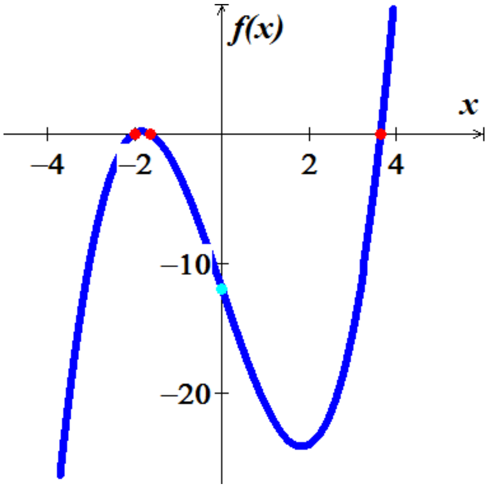
$$= \frac{2 \pm 2\sqrt{7}}{2}$$

$$\underline{x = -2,\ 1 \pm \sqrt{7}\ |}$$



$f(x) > 0 \quad \underline{\left(-2,\ 1-\sqrt{7}\right) \cup \left(1+\sqrt{7},\ \infty\right)|}$

$f(x) < 0 \quad \underline{\left(-\infty,\ -2\right) \cup \left(1-\sqrt{7},\ 1+\sqrt{7}\right)}\big|$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 5x^2 + 17x - 13$$

Solution

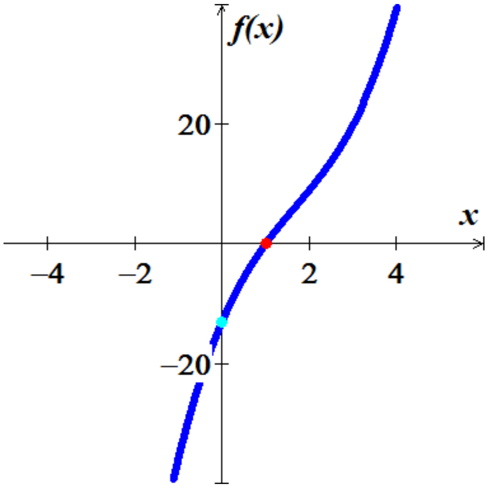
possibilities for $\frac{c}{d} : \pm\left\{\frac{13}{1}\right\} = \pm\{1,\ 13\}$

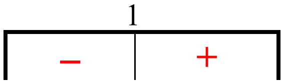
$$\begin{array}{r|rrrr} 1 & 1 & -5 & 17 & -13 \\ & & 1 & -4 & 13 \\ \hline & 1 & -4 & 13 & \boxed{0} \end{array} \quad \to \quad x^2 - 4x + 13 = 0$$

$$x = \frac{4 \pm \sqrt{16-52}}{2}$$

$$= \frac{4 \pm 6i}{2}$$

$\underline{x = 1,\ 2 \pm 3i}\big|$





$f(x) > 0 \quad \underline{(1,\ \infty)}\big|$

$f(x) < 0 \quad \underline{(-\infty,\ 1)}\big|$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 6x^3 + 25x^2 - 24x + 5$$

Solution

$$\textit{possibilities for } \frac{c}{d} : \pm\left\{\frac{5}{6}\right\} = \pm\left\{1,\ 5,\ \frac{1}{2},\ \frac{5}{2},\ \frac{1}{3},\ \frac{5}{3},\ \frac{1}{6},\ \frac{5}{6}\right\}$$

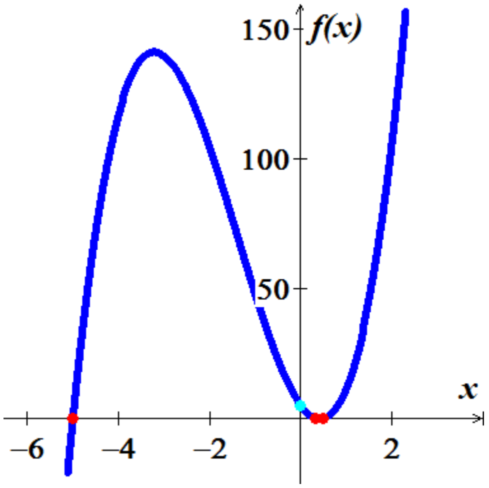
$$\begin{array}{r|rrrr} -5 & 6 & 25 & -24 & 5 \\ & & -30 & 25 & -5 \\ \hline & 6 & -5 & 1 & \boxed{0} \end{array} \quad \to \quad 6x^2 - 5x + 1 = 0$$

$$x = \frac{5 \pm \sqrt{25-24}}{12}$$

$$= \begin{cases} \frac{5-1}{12} = \frac{1}{3} \\ \frac{5+1}{12} = \frac{1}{2} \end{cases}$$

$$x = -5,\ \frac{1}{3},\ \frac{1}{2}$$

$$\begin{array}{c|c|c|c|c} & -5 & \frac{1}{3} & \frac{1}{2} & \\ \hline - & + & - & + \end{array}$$



$f(x) > 0 \quad \left(-5,\ \frac{1}{3}\right) \cup \left(\frac{1}{2},\ \infty\right)$

$f(x) < 0 \quad (-\infty,\ -5) \cup \left(\frac{1}{3},\ \frac{1}{2}\right)$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 8x^3 + 18x^2 + 45x + 27$$

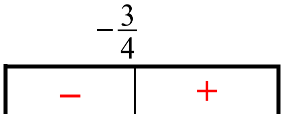
Solution

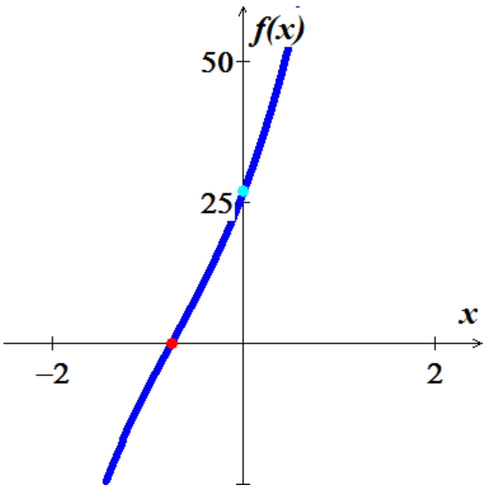
$$\textit{possibilities}: \ \pm\left\{\frac{27}{8}\right\} = \pm\left\{\frac{1,\ 3,\ 9,\ 27}{1,\ 2,\ 4,\ 8}\right\}$$

$$= \pm\left\{1,\ 3,\ 9,\ 27,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{3}{2},\ \frac{3}{4},\ \frac{3}{8},\ \frac{9}{2},\ \frac{9}{4},\ \frac{9}{8},\ \frac{27}{2},\ \frac{27}{4},\ \frac{27}{8}\right\}$$

$$\begin{array}{r|rrrr} -\frac{3}{4} & 8 & 18 & 45 & 27 \\ & & -6 & -9 & -27 \\ \hline & 8 & 12 & 36 & \boxed{0} \end{array} \quad \to \quad 8x^2 + 12x + 36 = 0$$

$$\boxed{x = -\frac{3}{4},\ -\frac{3}{4} \pm i\frac{3\sqrt{7}}{4}}$$





$f(x) > 0 \quad \boxed{\left(-\frac{3}{4},\ \infty\right)}$

$f(x) < 0 \quad \boxed{\left(-\infty,\ -\frac{3}{4}\right)}$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 3x^3 - x^2 + 11x - 20$$

Solution

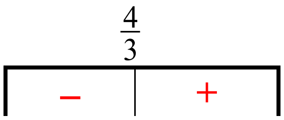
$$possibilities: \ \pm\left\{\frac{20}{3}\right\} = \pm\left\{\frac{1,\ 2,\ 4,\ 5,\ 10,\ 20}{1,\ 3}\right\}$$

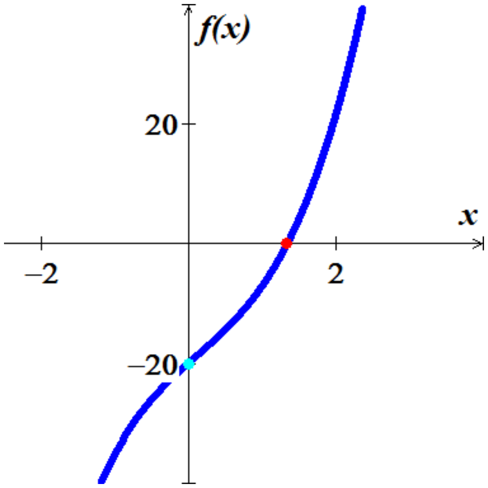
$$= \pm\left\{1,\ 2,\ 4,\ 5,\ 10,\ 20,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3},\ \frac{5}{3},\ \frac{10}{3},\ \frac{20}{3}\right\}$$

$$\begin{array}{r|rrrr} \frac{4}{3} & 3 & -1 & 11 & -20 \\ & & 4 & 4 & 20 \\ \hline & 3 & 3 & 15 & \boxed{0} \end{array} \quad \to \ 3x^2 + 3x + 15 = 0$$

$$x = \frac{-3 \pm \sqrt{9 - 180}}{6}$$

$$\boxed{x = \frac{4}{3},\ -\frac{1}{2} \pm i\frac{\sqrt{19}}{2}}$$





$f(x) > 0 \quad \boxed{\left(\frac{4}{3},\ \infty\right)}$

$f(x) < 0 \quad \boxed{\left(-\infty,\ \frac{4}{3}\right)}$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^4 - x^3 - 9x^2 + 3x + 18$$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{18}{1}\right\} = \pm\{1,\ 2,\ 3,\ 6,\ 9,\ 18\}$

$$\begin{array}{r|rrrrr} -2 & 1 & -1 & -9 & 3 & 18 \\ & & -2 & 6 & 6 & -18 \\ \hline 3 & 1 & -3 & -3 & 9 & 0 \\ & & 3 & 0 & -9 & \\ \hline & 1 & 0 & -3 & 0 & \end{array}$$

$\rightarrow x^3 - 3x^2 - 3x + 9 = 0 \rightarrow \pm\left\{\frac{9}{1}\right\} = \pm\{1,\ 3,\ 9\}$

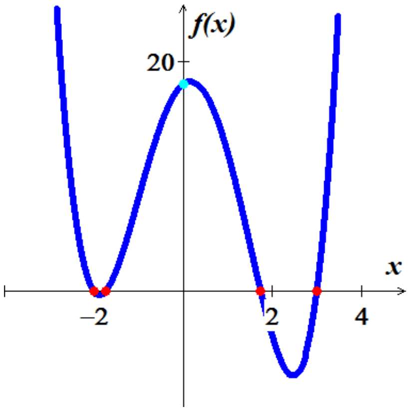
$\rightarrow x^2 - 3 = 0 \Rightarrow x = \pm\sqrt{3}$

$x = -2,\ 3,\ \pm\sqrt{3}$

	-2		$-\sqrt{3}$		$\sqrt{3}$		3	
+		−		+		−		+

$f(x) > 0 \quad (-\infty,\ -2) \cup (-\sqrt{3},\ \sqrt{3}) \cup (3,\ \infty)$

$f(x) < 0 \quad (-2,\ -\sqrt{3}) \cup (\sqrt{3},\ 3)$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 2x^4 - 9x^3 + 9x^2 + x - 3$$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$$\begin{array}{r|rrrrr} 1 & 2 & -9 & 9 & 1 & -3 \\ & & 2 & -7 & 2 & 3 \\ \hline 1 & 2 & -7 & 2 & 3 & 0 \\ & & 2 & -5 & -3 & \\ \hline & 2 & -5 & -3 & 0 & \end{array}$$

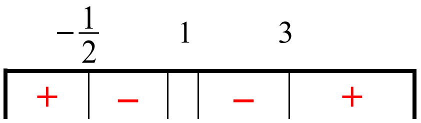
$\rightarrow 2x^3 - 7x^2 + 2x + 3 = 0 \rightarrow \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$

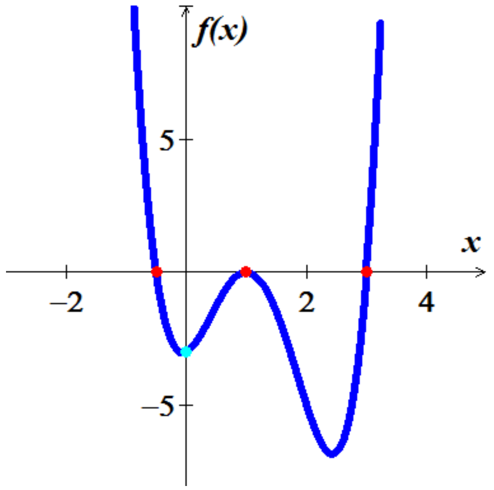
$\rightarrow 2x^2 - 5x - 3 = 0$

$$x = \frac{5 \pm \sqrt{25 + 24}}{4}$$

$$= \begin{cases} \frac{5-7}{4} = -\frac{1}{2} \\ \frac{5+7}{4} = 3 \end{cases}$$

$x = 1,\ 1,\ -\frac{1}{2},\ 3$





$f(x) > 0 \quad \left(-\infty,\ -\frac{1}{2}\right) \cup (3,\ \infty)$

$f(x) < 0 \quad \left(-\frac{1}{2},\ 1\right) \cup (1,\ 3)$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 6x^4 + 5x^3 - 17x^2 - 6x$$

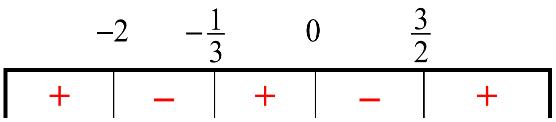
Solution

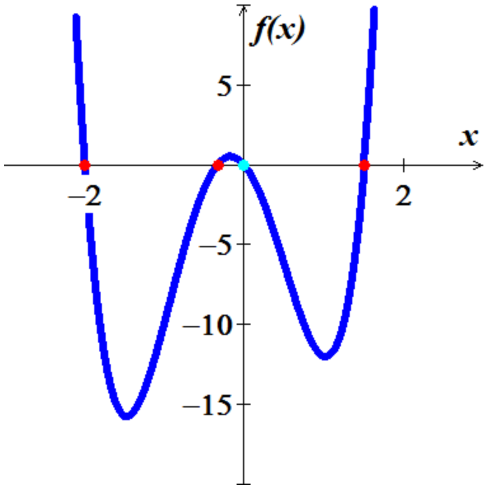
$x\left(6x^3 + 5x^2 - 17x - 6\right) = 0 \quad \rightarrow \quad x = 0$

possibilities : $\pm\left\{\frac{6}{6}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrr} -2 & 6 & 5 & -17 & -6 \\ & & -12 & 14 & 6 \\ \hline & 6 & -7 & -3 & \boxed{0} \end{array} \rightarrow 6x^2 - 7x - 3 = 0$$

$x = 0,\ -2,\ -\frac{1}{3},\ \frac{3}{2}$





$f(x) > 0 \quad (-\infty,\ -2) \cup \left(-\frac{1}{3},\ 0\right) \cup \left(\frac{3}{2},\ \infty\right)$

$f(x) < 0 \quad \left(-2,\ -\frac{1}{3}\right) \cup \left(0,\ \frac{3}{2}\right)$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^4 - 2x^2 - 16x - 15$$

Solution

$$\textit{possibilities}: \ \pm\left\{\frac{15}{1}\right\} = \pm\{1,\ 3,\ 5,\ 15\}$$

$$\begin{array}{r|rrrrr}
-1 & 1 & 0 & -2 & -16 & -15 \\
 & & -1 & 1 & 1 & 15 \\
\hline
3 & 1 & -1 & -1 & -15 & 0 \\
 & & 3 & 6 & 15 & \\
\hline
 & 1 & 2 & 5 & 0 &
\end{array}
\quad \begin{array}{l} \to x^3 - x^2 - x - 15 = 0 \ \to \ \pm\left\{\frac{15}{1}\right\} = \pm\{1,\ 3,\ 5,\ 15\} \\ \\ \to x^2 + 2x + 5 = 0 \end{array}$$

$$x = \frac{-2 \pm \sqrt{-16}}{2}$$

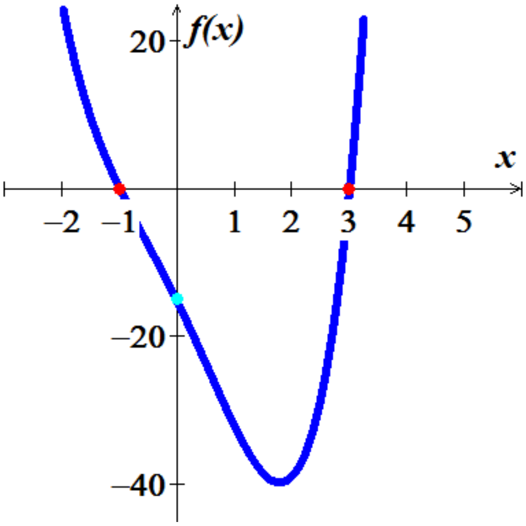
$$= -1 \pm 2i$$

$$\underline{x = -1,\ 3,\ -1 \pm 2i}\,|$$

	−1		3	
+		−		+

$f(x) > 0$ $\quad \underline{(-\infty,\ -1) \cup (3,\ \infty)}\,|$

$f(x) < 0$ $\quad \underline{(-1,\ 3)}\,|$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^4 - 2x^3 - 5x^2 + 8x + 4$$

Solution

possibilities : $\pm\left\{\frac{4}{1}\right\} = \pm\{1,\ 2,\ 4\}$

$$\begin{array}{r|rrrrr}
2 & 1 & -2 & -5 & 8 & 4 \\
 & & 2 & 0 & -10 & -4 \\ \hline
-2 & 1 & 0 & -5 & -2 & 0 \\
 & & -2 & 4 & 2 & \\ \hline
 & 1 & -2 & -1 & 0 &
\end{array}$$

$\rightarrow x^3 - 5x - 2 = 0 \rightarrow \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$\rightarrow x^2 - 2x - 1 = 0$

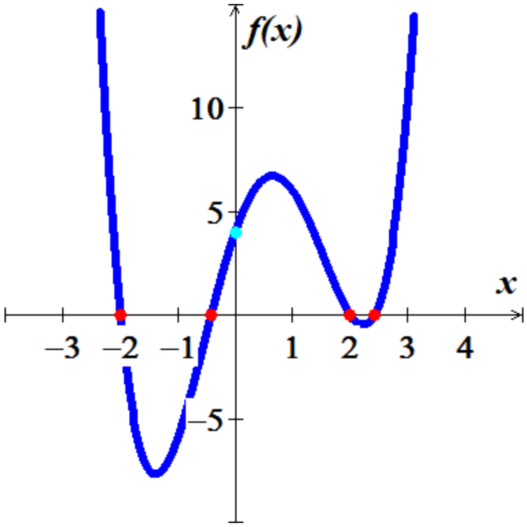
$$x = \frac{2 \pm \sqrt{8}}{2}$$

$$x = -2,\ 2,\ 1 \pm \sqrt{2}$$

	-2		$1-\sqrt{2}$		2		$1+\sqrt{2}$	
+		−		+		−		+

$f(x) > 0$ $\quad (-\infty,\ -2) \cup (1-\sqrt{2},\ 2) \cup (1+\sqrt{2},\ \infty)$

$f(x) < 0$ $\quad (-2,\ 1-\sqrt{2}) \cup (2,\ 1+\sqrt{2})$



Exercise

Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of $f(x)$

$$f(x)=2x^4-17x^3+4x^2+35x-24$$

Solution

$$possibilities: \ \pm\left\{\frac{24}{2}\right\}=\pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24,\ \frac{1}{2},\ \frac{3}{2}\right\}$$

$$\begin{array}{r|rrrrr} 1 & 2 & -17 & 4 & 35 & -24 \\ & & 2 & -15 & -11 & 24 \\ \hline \overline{1} & 2 & -15 & -11 & 24 & 0 \\ & & 2 & -13 & 24 & \\ \hline & 2 & -13 & -24 & 0 & \end{array}$$

$\to\ 2x^3-15x^2-11x+24=0\ \to\ \pm\left\{\frac{2}{1}\right\}=\pm\{1,\ 2\}$

$\to\ 2x^2-13x-24=0$

$$x=\frac{13\pm\sqrt{169+192}}{4}$$

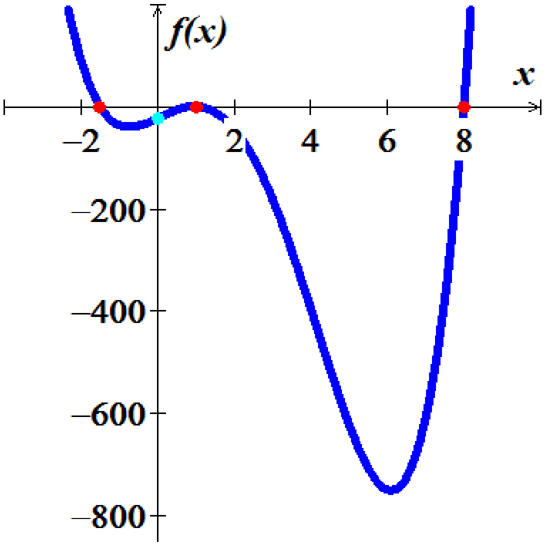
$$=\begin{cases} \frac{13-19}{4}=-\frac{3}{2} \\ \frac{13+19}{4}=8 \end{cases}$$

$$\boxed{x=-\frac{3}{2},\ 1,\ 1,\ 8}$$

$$\begin{array}{ccccccc} & -\frac{3}{2} & & 1 & & 8 & \\ + & | & - & || & - & | & + \end{array}$$

$f(x)>0$ $\boxed{\left(-\infty,\ -\frac{3}{2}\right)\cup(8,\ \infty)}$

$f(x)<0$ $\boxed{\left(-\frac{3}{2},\ 1\right)\cup(1,\ 8)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^4 + x^3 - 3x^2 - 5x - 2$$

Solution

possibilities : $\pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$$\begin{array}{r|rrrrr} -1 & 1 & 1 & -3 & -5 & -2 \\ & & -1 & 0 & 3 & 2 \\ \hline -1 & 1 & 0 & -3 & -2 & 0 \\ & & -1 & 1 & 2 & \\ \hline & 1 & -1 & -2 & 0 & \end{array}$$

$\rightarrow x^3 - 3x - 2 = 0 \rightarrow \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$\rightarrow x^2 - x - 2 = 0$

$$x = \frac{1 \pm \sqrt{1+8}}{2}$$

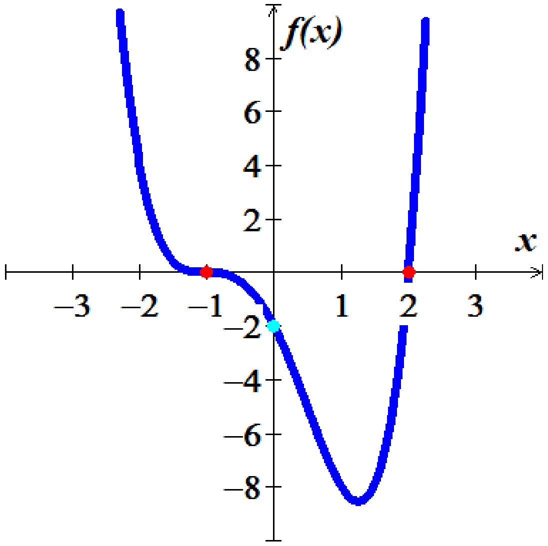
$$= \begin{cases} \frac{1-3}{2} = -1 \\ \frac{1+3}{2} = 2 \end{cases}$$

$$\underline{x = -1,\ -1,\ -1,\ 2 \,|}$$

$$\begin{array}{|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{-1} & \multicolumn{1}{c}{2} \\ \hline + & - & + \end{array}$$

$f(x) > 0$ $\quad \underline{(-\infty,\ -1) \cup (2,\ \infty)\,|}$

$f(x) < 0$ $\quad \underline{(-2,\ 2)\,|}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 6x^4 - 17x^3 - 11x^2 + 42x$$

Solution

$$x\left(6x^3 - 17x^2 - 11x + 42\right) = 0$$

$$x = 0 \quad 6x^3 - 17x^2 - 11x + 42 = 0$$

$$possibilities: \ \pm\left\{\frac{42}{6}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ 7,\ 14,\ 21,\ 42,\ \frac{1}{2},\ \frac{3}{2},\ \frac{7}{2},\ \frac{21}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{7}{3},\ \frac{14}{3},\ \frac{1}{6},\ \frac{7}{6},\ \frac{21}{6}\right\}$$

$$\begin{array}{r|rrrr} 2 & 6 & -17 & -11 & 42 \\ & & 12 & -10 & -42 \\ \hline & 6 & -5 & -21 & \boxed{0} \end{array} \quad \to \ 6x^2 - 5x - 21 = 0$$

$$x = \frac{5 \pm \sqrt{25 + 504}}{12}$$

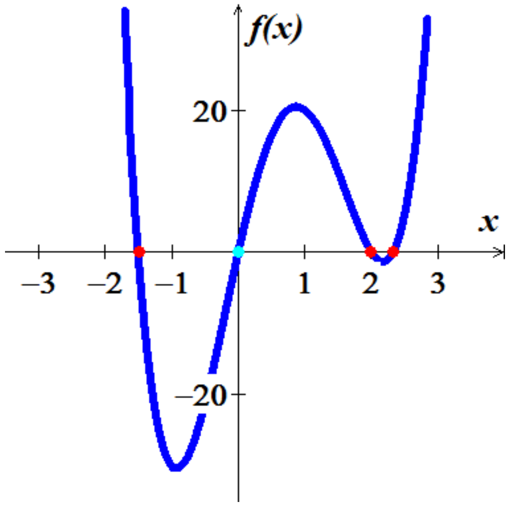
$$= \begin{cases} \dfrac{5-23}{12} = -\dfrac{3}{2} \\ \dfrac{5+23}{12} = \dfrac{7}{3} \end{cases}$$

$$\boxed{x = -\frac{3}{2},\ 0,\ 2,\ \frac{7}{3}}$$

$$\begin{array}{ccccccccc} & -\frac{3}{2} & & 0 & & 2 & & \frac{7}{3} & \\ + & | & - & | & + & | & - & | & + \end{array}$$

$$f(x) > 0 \quad \boxed{\left(-\infty,\ -\tfrac{3}{2}\right) \cup (0,\ 2) \cup \left(\tfrac{7}{3},\ \infty\right)}$$

$$f(x) < 0 \quad \boxed{\left(-\tfrac{3}{2},\ 0\right) \cup \left(2,\ \tfrac{7}{3}\right)}$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^4 - 5x^2 - 2x$$

Solution

$$x\left(x^3 - 5x - 2\right) = 0$$

$$x = 0 \quad x^3 - 5x - 2 = 0$$

$$\textit{possibilities}: \ \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -5 & -2 \\ & & -2 & 4 & 2 \\ \hline & 1 & -2 & -1 & \boxed{0} \end{array} \rightarrow x^2 - 2x - 1 = 0$$

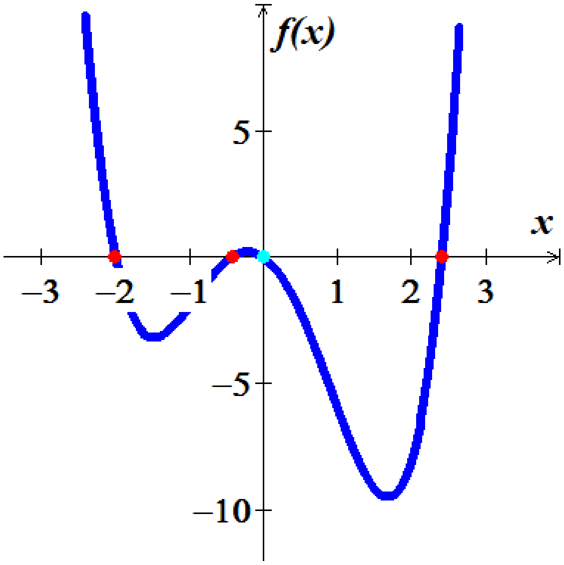
$$x = \frac{2 \pm \sqrt{8}}{2}$$

$$\underline{x = -2,\ 0,\ 1 \pm \sqrt{2}\ |}$$

$$\begin{array}{ccccccccc} & -2 & & 1-\sqrt{2} & & 2 & & 1+\sqrt{2} & \\ \hline + & | & - & | & + & | & - & | & + \end{array}$$

$$f(x) > 0 \quad \underline{(-\infty,\ -2) \cup \left(1-\sqrt{2},\ 2\right) \cup \left(1+\sqrt{2},\ \infty\right)\ |}$$

$$f(x) < 0 \quad \underline{\left(-2,\ 1-\sqrt{2}\right) \cup \left(2,\ 1+\sqrt{2}\right)\ |}$$



Exercise

Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of $f(x)$

$$f(x)=3x^4-4x^3-11x^2+16x-4$$

Solution

possibilities : $\pm\left\{\frac{4}{3}\right\}=\pm\left\{1,\ 2,\ 4,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3}\right\}$

$$\begin{array}{r|rrrrr}
1 & 3 & -4 & -11 & 16 & -4 \\
 & & 3 & -1 & -12 & 4 \\
\hline
\overline{2} & 3 & -1 & -12 & 4 & 0 \\
 & & 6 & 10 & -4 & \\
\hline
 & 3 & 5 & -2 & 0 &
\end{array}$$

$\rightarrow 3x^3-x^2-12x+4=0 \rightarrow \pm\left\{\frac{4}{3}\right\}=\pm\left\{1,\ 2,\ 4,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3}\right\}$

$\rightarrow 3x^2+5x-2=0$

$$x=\frac{-5\pm\sqrt{25+24}}{6}$$

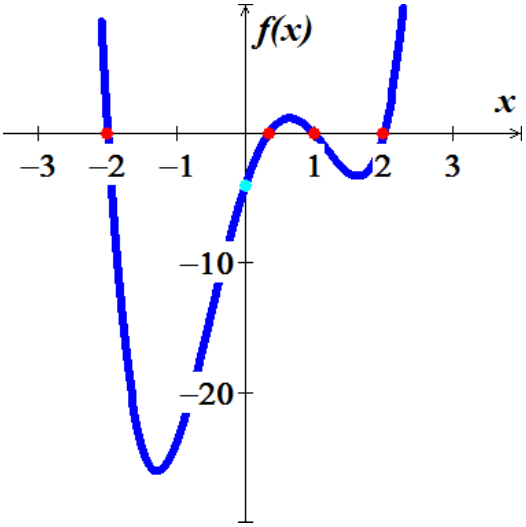
$$=\begin{cases}\frac{-5-7}{6}=-2\\ \frac{-5+7}{6}=\frac{1}{3}\end{cases}$$

$$\underline{x=-2,\ \frac{1}{3},\ 1,\ 2\Big|}$$

$$\begin{array}{ccccccccc}
 & -2 & & \frac{1}{3} & & 1 & & 2 & \\
\hline
+ & | & - & | & + & | & - & | & +
\end{array}$$

$f(x)>0$ $\quad \underline{\left(-\infty,\ -2\right)\cup\left(\frac{1}{3},\ 1\right)\cup\left(2,\ \infty\right)\Big|}$

$f(x)<0$ $\quad \underline{\left(-2,\ \frac{1}{3}\right)\cup\left(1,\ 2\right)\Big|}$



Exercise

Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of $f(x)$

$$f(x)=6x^4+23x^3+19x^2-8x-4$$

Solution

$$\text{possibilities}: \ \pm\left\{\frac{4}{6}\right\}=\pm\left\{1,\ 2,\ 4,\ \frac{1}{6},\ \frac{2}{3},\ \frac{2}{3}\right\}$$

$$\begin{array}{r|rrrrr} -2 & 6 & 23 & 19 & -8 & -4 \\ & & -12 & -22 & 6 & 4 \\ \hline -2 & 6 & 11 & -3 & -2 & 0 \\ & & -12 & 2 & 2 & \\ \hline & 6 & -1 & -1 & 0 & \end{array}$$

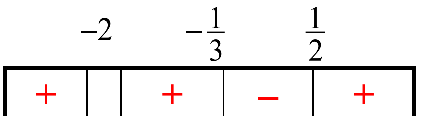
$$\rightarrow 6x^3+11x^2-3x-2=0 \ \rightarrow \ \pm\left\{\frac{2}{6}\right\}=\pm\left\{1,\ 2,\ \frac{1}{6},\ \frac{1}{3}\right\}$$

$$\rightarrow 6x^2-x-1=0$$

$$x=\frac{1\pm\sqrt{25}}{12}$$

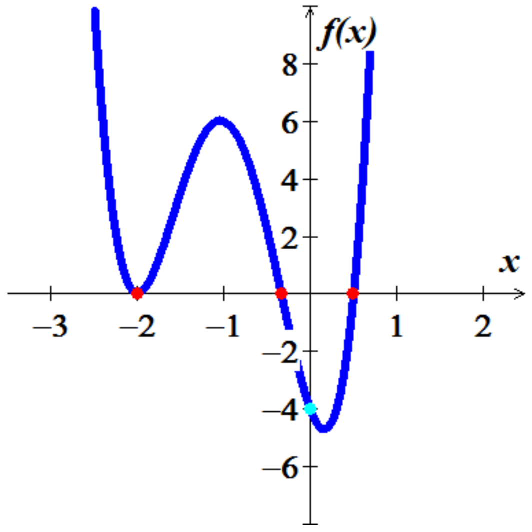
$$=\begin{cases}\frac{1-5}{12}=-\frac{1}{3}\\ \frac{1+5}{12}=\frac{1}{2}\end{cases}$$

$$\boxed{x=-2,\ -2,\ -\frac{1}{3},\ \frac{1}{2}}$$



$$f(x)>0 \quad \boxed{(-\infty,\ -2)\cup\left(-2,\ -\frac{1}{3}\right)\cup\left(\frac{1}{2},\ \infty\right)}$$

$$f(x)<0 \quad \boxed{\left(-\frac{1}{3},\ \frac{1}{2}\right)}$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 4x^4 - 12x^3 + 3x^2 + 12x - 7$$

Solution

$$possibilities: \ \pm\left\{\frac{7}{4}\right\} = \pm\left\{1,\ 7,\ \frac{1}{2},\ \frac{7}{2},\ \frac{1}{4},\ \frac{7}{4}\right\}$$

$$\begin{array}{r|rrrrr} 1 & 4 & -12 & 3 & 12 & -7 \\ & & 4 & -8 & -5 & 7 \\ \hline -1 & 4 & -8 & -5 & 7 & 0 \\ & & -4 & 12 & -7 & \\ \hline & 4 & -12 & 7 & 0 & \end{array}$$

$$\to 4x^3 - 8x^2 - 5x + 7 = 0 \ \to \ \pm\left\{\frac{7}{4}\right\} = \pm\left\{1,\ 7,\ \frac{1}{2},\ \frac{7}{2},\ \frac{1}{4},\ \frac{7}{4}\right\}$$

$$\to 4x^2 - 12x + 7 = 0$$

$$x = \frac{12 \pm \sqrt{144 - 112}}{8}$$

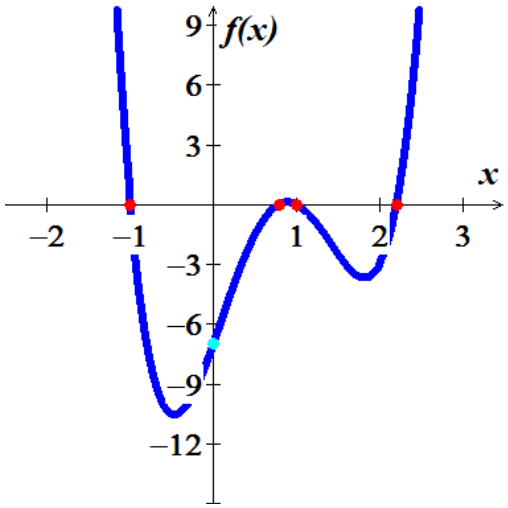
$$= \frac{12 \pm 4\sqrt{2}}{8}$$

$$\boxed{x = -1,\ 1,\ \frac{3 \pm \sqrt{2}}{2}}$$

$$\begin{array}{ccccccccc} & -1 & & \frac{3-\sqrt{2}}{2} & & 1 & & \frac{3+\sqrt{2}}{2} & \\ + & | & - & | & + & | & - & | & + \end{array}$$

$$f(x) > 0 \quad \boxed{(-\infty,\ -1) \cup \left(\frac{3-\sqrt{2}}{2},\ 1\right) \cup \left(\frac{3+\sqrt{2}}{2},\ \infty\right)}$$

$$f(x) < 0 \quad \boxed{\left(-1,\ \frac{3-\sqrt{2}}{2}\right) \cup \left(1,\ \frac{3+\sqrt{2}}{2}\right)}$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 2x^4 - 9x^3 - 2x^2 + 27x - 12$$

Solution

$$\textit{possibilities}: \ \pm\left\{\frac{12}{2}\right\} = \pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 12,\ \frac{1}{2},\ \frac{3}{2}\right\}$$

$$\begin{array}{r|rrrrr}
4 & 2 & -9 & -2 & 27 & -12 \\
 & & 8 & -4 & -24 & 12 \\
\hline
\frac{1}{2} & 2 & -1 & -6 & 3 & 0 \\
 & & 1 & 0 & -3 & \\
\hline
 & 2 & 0 & -6 & 0 &
\end{array}$$

$$\to 2x^3 - x^2 - 6x + 3 = 0 \ \to \ \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$$

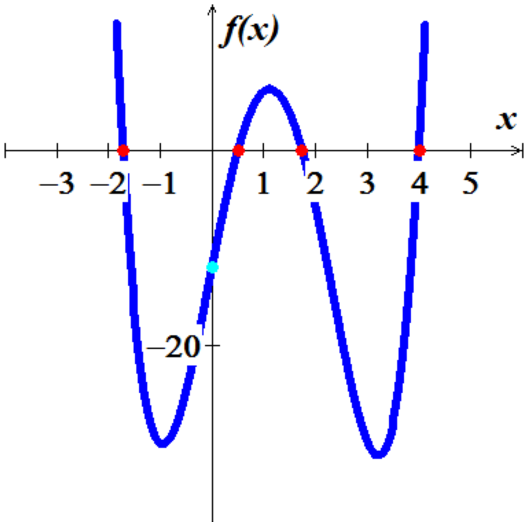
$$\to 2x^2 - 6 = 0$$

$$\underline{x = \frac{1}{2},\ 4,\ \pm\sqrt{3}\,}|$$

$$\begin{array}{cccccccccc}
 & -\sqrt{3} & & \frac{1}{2} & & \sqrt{3} & & 4 & \\
+ & | & - & | & + & | & - & | & +
\end{array}$$

$f(x) > 0$ $\quad \left(-\infty,\ -\sqrt{3}\right) \cup \left(\frac{1}{2},\ \sqrt{3}\right) \cup (4,\ \infty)$

$f(x) < 0$ $\quad \left(-\sqrt{3},\ \frac{1}{2}\right) \cup \left(\sqrt{3},\ 4\right)$



Exercise

Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of $f(x)$

$$f(x)=2x^4-19x^3+51x^2-31x+5$$

Solution

$$\text{possibilities}: \ \pm\left\{\frac{5}{2}\right\}=\pm\left\{1,\ 5,\ \frac{1}{2},\ \frac{5}{2}\right\}$$

$$\begin{array}{c|rrrrr} 5 & 2 & -19 & 51 & -31 & 5 \\ & & 10 & -45 & 30 & -5 \\ \hline \frac{1}{2} & 2 & -9 & 6 & -1 & 0 \\ & & 1 & -4 & 1 & \\ \hline & 2 & -8 & 2 & 0 & \end{array}$$

$\to 2x^3-9x^2+6x-1=0 \to \pm\left\{\frac{1}{2}\right\}$

$\to 2x^2-8x+2=0$

$$x=\frac{8\pm\sqrt{64-16}}{4}$$

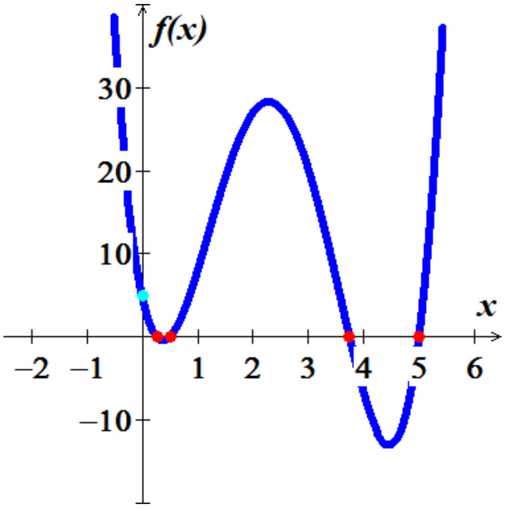
$$=\frac{8\pm4\sqrt{3}}{4}$$

$$x=\frac{1}{2},\ 5,\ 2\pm\sqrt{3}$$

	$2-\sqrt{3}$		$\frac{1}{2}$		$2+\sqrt{3}$		5	
$+$		$-$		$+$		$-$		$+$

$f(x)>0$ $\quad \left(-\infty,\ 2-\sqrt{3}\right)\cup\left(\frac{1}{2},\ 2+\sqrt{3}\right)\cup\left(5,\ \infty\right)$

$f(x)<0$ $\quad \left(2-\sqrt{3},\ \frac{1}{2}\right)\cup\left(2+\sqrt{3},\ 5\right)$



Exercise

Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of $f(x)$

$$f(x)=4x^4-35x^3+71x^2-4x-6$$

Solution

$$\textit{possibilities}: \ \pm\left\{\frac{6}{4}\right\}=\pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{4},\ \frac{3}{4}\right\}$$

$$\begin{array}{r|rrrrr}
3 & 4 & -35 & 71 & -4 & -6 \\
 & & 12 & -69 & 6 & 6 \\ \hline
-\frac{1}{4} & 4 & -23 & 2 & 2 & 0 \\
 & & -1 & 6 & -2 & \\ \hline
 & 4 & -24 & 8 & 0 &
\end{array}$$

$$\to 4x^3-23x^2+2x+2=0 \ \to \ \pm\left\{\frac{2}{4}\right\}=\pm\left\{1,\ 2,\ \frac{1}{2},\ \frac{1}{4}\right\}$$

$$\to 4x^2-24x+8=0$$

$$x^2-6x+2=0$$

$$x=\frac{6\pm\sqrt{36-8}}{2}$$

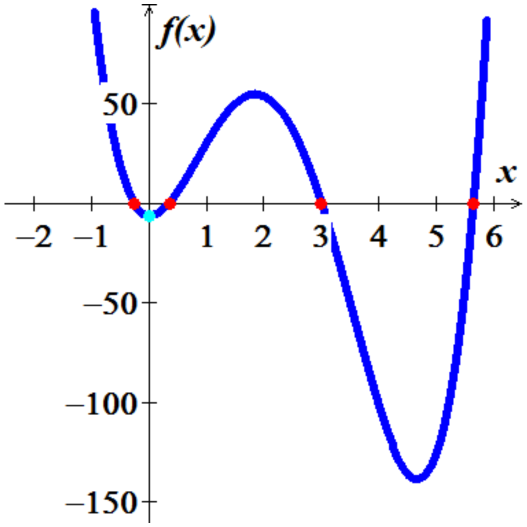
$$=\frac{6\pm2\sqrt{7}}{4}$$

$$\boxed{x=-\frac{1}{4},\ 3,\ 3\pm\sqrt{7}}$$

	$-\frac{1}{4}$		$3-\sqrt{7}$		3		$3+\sqrt{7}$	
$+$		$-$		$+$		$-$		$+$

$f(x)>0$ $\boxed{\left(-\infty,\ -\frac{1}{4}\right)\cup\left(3-\sqrt{7},\ 3\right)\cup\left(3+\sqrt{7},\ \infty\right)}$

$f(x)<0$ $\boxed{\left(-\frac{1}{4},\ 3-\sqrt{7}\right)\cup\left(3,\ 3+\sqrt{7}\right)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 2x^4 + 3x^3 - 4x^2 - 3x + 2$$

Solution

possibilities : $\pm\left\{\frac{2}{2}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2}\right\}$

$$\begin{array}{r|rrrrr}
1 & 2 & 3 & -4 & -3 & 2 \\
 & & 2 & 5 & 1 & -2 \\
\hline
-1 & 2 & 5 & 1 & -2 & 0 \\
 & & -2 & -3 & 2 & \\
\hline
 & 2 & 3 & -2 & 0 &
\end{array}$$

$\to 2x^3 - 23x^2 + 2x - 2 = 0 \to \pm\left\{\frac{2}{2}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2}\right\}$

$\to 2x^2 + 3x - 2 = 0$

$$x = \frac{-3 \pm \sqrt{9+16}}{4}$$

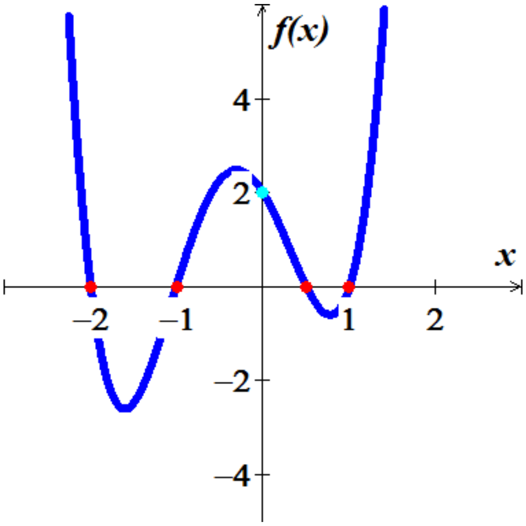
$$= \begin{cases} \frac{-3-5}{4} = -2 \\ \frac{-3+5}{4} = \frac{1}{2} \end{cases}$$

$$\boxed{x = -2,\ -1,\ \frac{1}{2},\ 1}$$

	-2		-1		$\frac{1}{2}$		1	
+		−		+		−		+

$f(x) > 0 \quad \boxed{(-\infty,\ -2) \cup \left(-1,\ \frac{1}{2}\right) \cup (1,\ \infty)}$

$f(x) < 0 \quad \boxed{(-2,\ -1) \cup \left(\frac{1}{2},\ 1\right)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^4 + 3x^3 - 30x^2 - 6x + 56$$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{56}{1}\right\} = \pm\{1,\ 2,\ 4,\ 7,\ 8,\ 14,\ 28,\ 56\}$

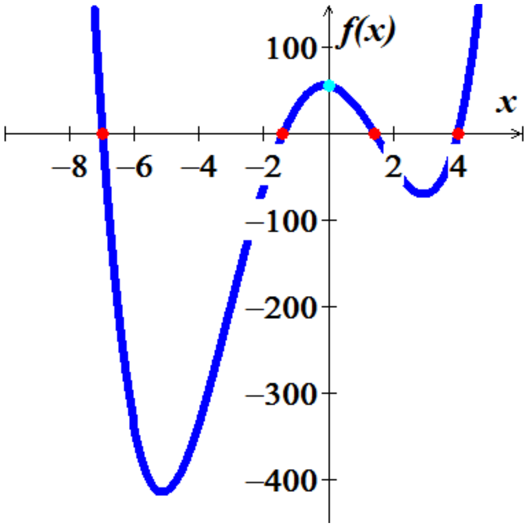
$$\begin{array}{r|rrrrr}
4 & 1 & 3 & -3 & -6 & 56 \\
 & & 4 & 28 & -8 & -56 \\
\hline
\overline{-7} & 1 & 7 & -2 & -14 & 0 \\
 & & -7 & 0 & 14 & \\
\hline
 & 1 & 0 & -2 & &
\end{array}
\quad
\begin{array}{l}
\to\ x^3 + 7x^2 - 2x - 14 = 0 \ \Rightarrow\ \frac{c}{d} = \pm\left\{\frac{14}{1}\right\} = \pm\{1,\ 2,\ 7,\ 14\} \\
\\
\to\ x^2 - 2 = 0 \Rightarrow \underline{x = \pm\sqrt{2}|}
\end{array}$$

$\underline{x = 4,\ -7,\ \pm\sqrt{2}|}$

$$\begin{array}{c|c|c|c|c}
\multicolumn{1}{c}{} & -7 & -\sqrt{2} & \sqrt{2} & 4 \\
\hline
+ & - & + & - & +
\end{array}$$

$f(x) > 0 \quad \underline{(-\infty,\ -7) \cup \left(-\sqrt{2},\ \sqrt{2}\right) \cup (4,\ \infty)\ |}$

$f(x) < 0 \quad \underline{\left(-7,\ -\sqrt{2}\right) \cup \left(\sqrt{2},\ 4\right)\ |}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6$$

Solution

$$\text{possibilities for } \frac{c}{d} : \pm\left\{\frac{6}{3}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$$

$$\begin{array}{r|rrrrrr}
-1 & 3 & -10 & -6 & 24 & 11 & -6 \\
 & & -3 & 13 & -7 & -17 & 6 \\ \hline
-1 & 3 & -13 & 7 & 17 & -6 & 0 \\
 & & -3 & 16 & -23 & 6 & \\ \hline
2 & 3 & -16 & 23 & -6 & 0 & \\
 & & 6 & 20 & 6 & & \\ \hline
 & 3 & -10 & 3 & 0 & &
\end{array}$$

$$x^4 - 13x^3 + 7x^2 + 17x - 6 = 0 \rightarrow \pm\left\{\frac{6}{3}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$$

$$3x^3 - 16x^2 + 26x - 6 = 0 \rightarrow \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$$

$$3x^2 - 10x + 3 = 0$$

$$x = \frac{10 \pm \sqrt{100 - 36}}{6}$$

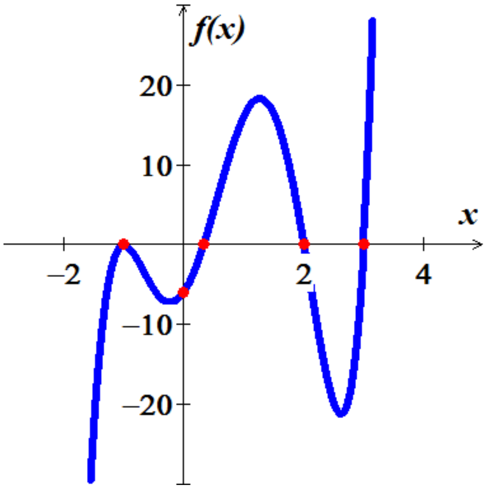
$$= \begin{cases} \dfrac{10-8}{6} = \dfrac{1}{3} \\ \dfrac{10+8}{4} = 3 \end{cases}$$

$$\boxed{x = -1,\ -1,\ \frac{1}{3},\ 2,\ 3}$$

$$\begin{array}{ccccccccc}
 & -1 & & \frac{1}{3} & & 2 & & 3 & \\
- & | & - & | & + & | & - & | & +
\end{array}$$

$f(x) > 0$ $\boxed{\left(\frac{1}{3},\ 2\right) \cup (3,\ \infty)}$

$f(x) < 0$ $\boxed{(-\infty,\ -1) \cup \left(-1,\ \frac{1}{3}\right) \cup (2,\ 3)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 6x^5 + 19x^4 + x^3 - 6x^2$$

Solution

$x^2\left(6x^3 + 19x^2 + x - 6\right) = 0 \quad \rightarrow \quad \underline{x = 0,\ 0|}$

$6x^3 + 19x^2 + x - 6 = 0$

possibilities for $\frac{c}{d} : \pm\left\{\frac{6}{6}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrr} -3 & 6 & 19 & 1 & -6 \\ & & -18 & -3 & 6 \\ \hline & 6 & 1 & -2 & \boxed{0} \end{array} \qquad 6x^2 + x - 2 = 0$$

$$x = \frac{-1 \pm \sqrt{1+48}}{12}$$

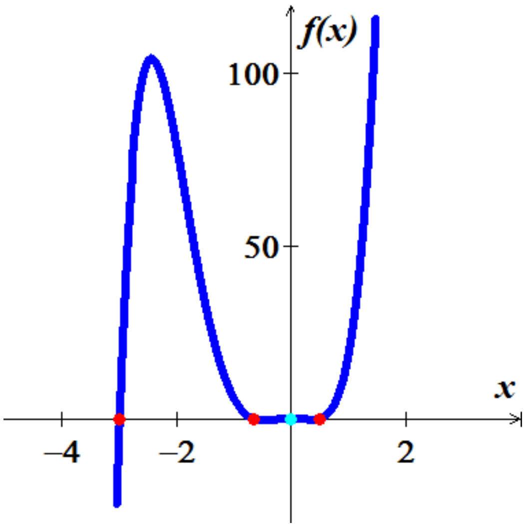
$$= \begin{cases} \frac{-1-7}{12} = -\frac{2}{3} \\ \frac{-1+7}{12} = \frac{1}{2} \end{cases}$$

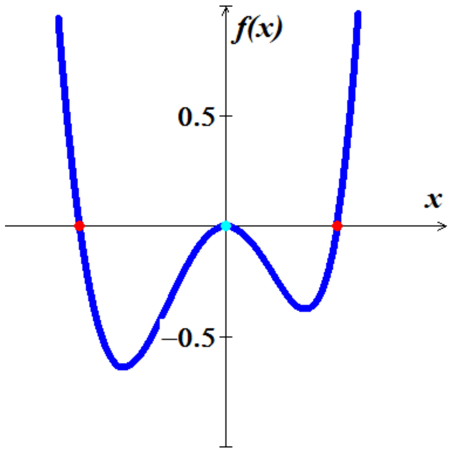
$\underline{x = 0,\ 0,\ -\frac{2}{3},\ -3,\ \frac{1}{2}\ |}$

	-3		$-\frac{2}{3}$		0		$\frac{1}{2}$	
−		+		−		−		+

$f(x) > 0 \quad \underline{\left(-3,\ -\frac{2}{3}\right) \cup \left(\frac{1}{2},\ \infty\right)\ |}$

$f(x) < 0 \quad \underline{(-\infty,\ -3) \cup \left(-\frac{2}{3},\ 0\right) \cup \left(0,\ \frac{1}{2}\right)\ |}$





Exercise

Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of $f(x)$

$$f(x)=x^5+5x^4+10x^3+10x^2+5x+1$$

Solution

$$x^5+5x^4+10x^3+10x^2+5x+1=(x+1)^5=0$$

possibilities for $\frac{c}{d}:\pm\{1\}$

$$\begin{array}{r|rrrrrr}
-1 & 1 & 5 & 10 & 10 & 5 & 1 \\
 & & -1 & -4 & -6 & -4 & -1 \\
\hline
-1 & 1 & 4 & 6 & 4 & 1 & 0 \\
 & & -1 & -3 & -3 & -1 & \\
\hline
-1 & 1 & 3 & 3 & 1 & 0 & \\
 & & -1 & -2 & -1 & & \\
\hline
 & 1 & 2 & 1 & 0 & &
\end{array}
\qquad
\begin{array}{l}
\\
\\
\to\ x^4+4x^3+6x^2+4x+1=0\to\pm\{1\} \\
\\
\to\ x^3+3x^2+3x+1=0\to\pm\{1\} \\
\\
\to\ x^2+2x+1=0
\end{array}$$

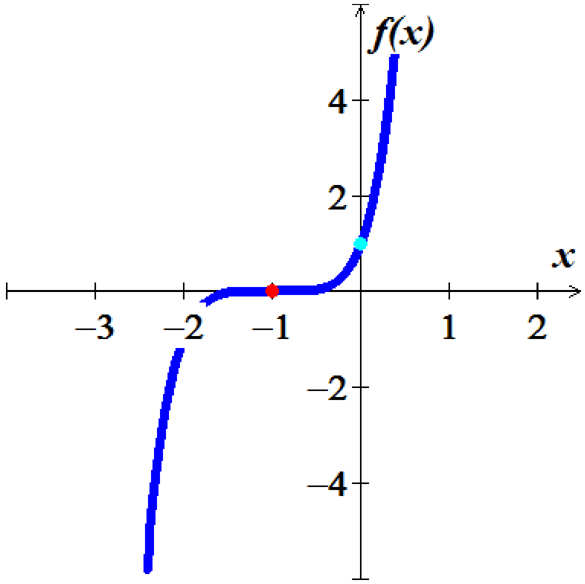
$$x^2+2x+1=(x+1)^2$$

$\underline{x=-1}$ (*multiplicity* of 5)

$$\begin{array}{c}
\qquad -1 \\
\begin{array}{|c|c|c|}
\hline
\quad - \quad & & \quad + \quad
\end{array}
\end{array}$$

$f(x)>0$ $\quad \underline{(-1,\ \infty)}$

$f(x)<0$ $\quad \underline{(-\infty,\ -1)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$f(x) = x^5 - x^4 - 7x^3 + 7x^2 + 12x - 12$

Solution

possibilities for $\frac{c}{d} : \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$$\begin{array}{r|rrrrrr}
1 & 1 & -1 & -7 & 7 & 12 & -12 \\
 & & 1 & 0 & -7 & 0 & 12 \\
\hline
2 & 1 & 0 & -7 & 0 & 12 & 0 \\
 & & 2 & 4 & -6 & -12 & \\
\hline
-2 & 1 & 2 & -3 & -6 & 0 & \\
 & & -2 & 0 & 6 & & \\
\hline
 & 1 & 0 & -3 & 0 & &
\end{array}$$

$\to x^4 - 7x^2 - 12 = 0 \to \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$\to x^3 + 2x^2 - 3x - 6 = 0 \to \pm\{1,\ 2,\ 3,\ 6\}$

$\to x^2 - 3 = 0$

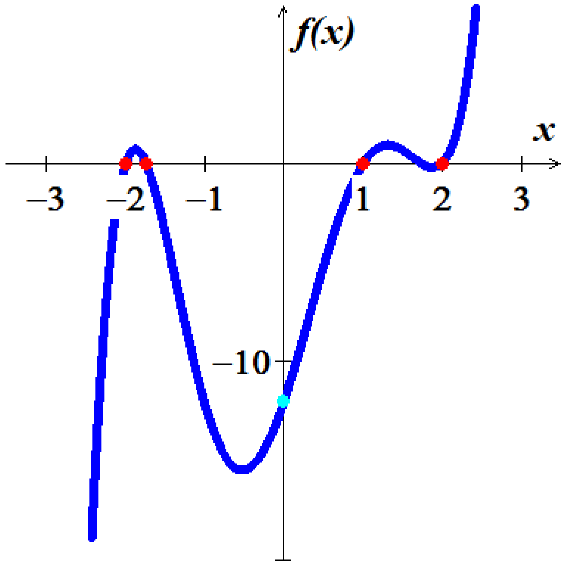
$x^2 = 3$

$x = -2,\ 1,\ 2,\ \pm\sqrt{3}$

	-2		$-\sqrt{3}$		1		$\sqrt{3}$		2	
$-$		$+$		$-$		$+$		$-$		$+$

$f(x) > 0 \quad (-2,\ -\sqrt{3}) \cup (1,\ \sqrt{3}) \cup (2,\ \infty)$

$f(x) < 0 \quad (-\infty,\ -2) \cup (-\sqrt{3},\ 1) \cup (\sqrt{3},\ 2)$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^5 - 2x^3 - 8x$$

Solution

$$x\left(x^4 - 2x^2 - 8\right) = 0$$

$$\boxed{x = 0}$$

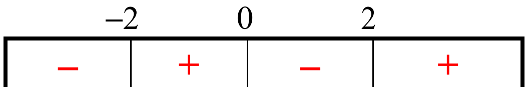
$$x^4 - 2x^2 - 8 = 0.$$

$$x^2 = \frac{2 \pm \sqrt{4+32}}{2}$$

$$= \begin{cases} \frac{2-6}{2} = -2 \\ \frac{2+6}{2} = 4 \end{cases}$$

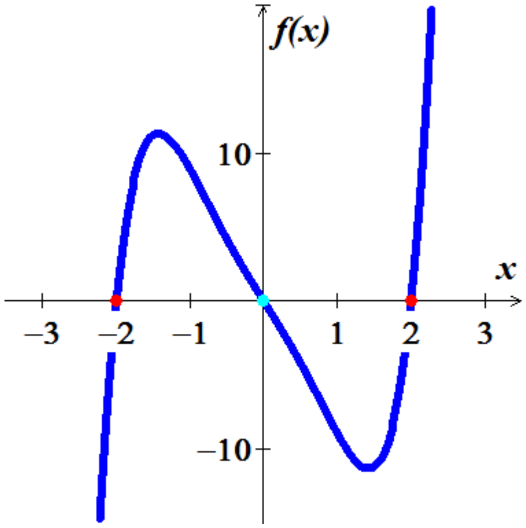
$$\begin{cases} x^2 = -2 & \rightarrow & x = \pm i\sqrt{2} \\ x^2 = 4 & \rightarrow & x = \pm 2 \end{cases}$$

$$\boxed{x = 0,\ \pm 2,\ \pm i\sqrt{2}}$$



$f(x) > 0$ $\boxed{(-2,\ 0) \cup (2,\ \infty)}$

$f(x) < 0$ $\boxed{(-\infty,\ -2) \cup (0,\ 2)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24$$

Solution

$$\textit{possibilities for } \frac{c}{d} : \pm\left\{\frac{24}{3}\right\}$$

$$= \pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3},\ \frac{8}{3}\right\}$$

1	3	−10	−29	34	50	−24	−24	
		3	−7	−36	−2	48	24	
−1	3	−7	−36	−2	48	24	0	$\rightarrow\ 3x^5 - 7x^4 - 36x^3 - 2x^2 + 48x + 24 = 0$
		−3	10	26	−24	−24		
−2	3	−10	−26	24	24	0		
		−6	32	−12	−24			$\rightarrow\ 3x^4 - 10x^3 - 26x^2 + 24x + 24 = 0$
$-\frac{2}{3}$	3	−16	6	12	0			$\rightarrow\ 3x^3 - 16x^2 + 12x - 12 = 0$
		−2	12	−12				
	3	−18	18	0				$\rightarrow\ 3x^2 - 18x + 18 = 0$

$$x^2 - 6x + 6 = 0$$

$$x = \frac{6 \pm \sqrt{36 - 24}}{2}$$

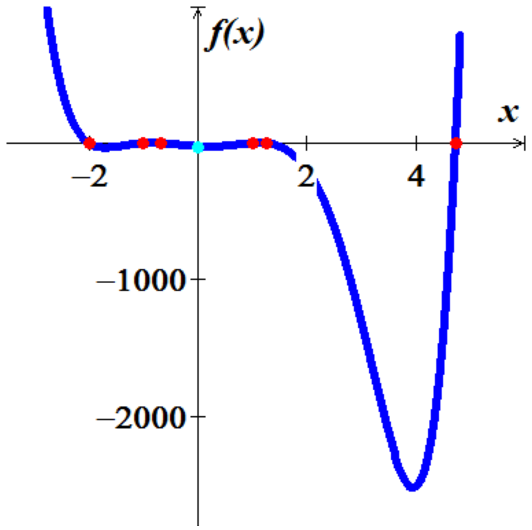
$$= \frac{6 \pm 2\sqrt{3}}{2}$$

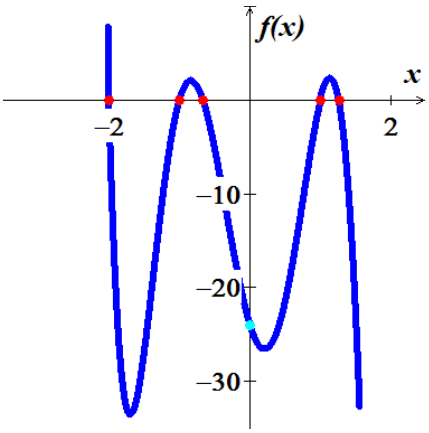
$$\boxed{x = -2,\ -1,\ 1,\ -\frac{2}{3},\ 3 \pm \sqrt{3}}$$

	-2		-1		$-\frac{2}{3}$		1		$3-\sqrt{3}$		$3+\sqrt{3}$	
+		–		+		–		+		–		+

$f(x) > 0$ $\quad \boxed{(-\infty,\ -2) \cup \left(-1,\ -\frac{2}{3}\right) \cup \left(1,\ 3-\sqrt{3}\right) \cup \left(3+\sqrt{3},\ \infty\right)}$

$f(x) < 0$ $\quad \boxed{(-2,\ -1) \cup \left(-\frac{2}{3},\ 1\right) \cup \left(3-\sqrt{3},\ 3+\sqrt{3}\right)}$


f(x)
x
−2
2
4
−1000
−2000


f(x)
x
−2
2
−10
−20
−30

Solution *Section* 2.6 – Graphing Rational Functions

Exercise

Determine all asymptotes of the function: $y = \frac{3x}{1-x}$

Solution

VA: $x = 1$ ***HA***: $y = -3$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \frac{x^2}{x^2+9}$

Solution

VA: n/a $x^2 + 9 \neq 0$ ***HA***: $y = 1$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \frac{x-2}{x^2-4x+3}$

Solution

$$x^2 - 4x + 3 = 0 \Rightarrow x = 1,\ 3$$

$$y = \frac{x}{x^2} \to 0$$

VA: $x = 1,\ x = 3$ ***HA***: $y = 0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \frac{3}{x-5}$

Solution

VA: $x = 5$ ***HA***: $y = 0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y=\dfrac{x^3-1}{x^2+1}$

Solution

VA: *none*

HA: *none*

Hole: n/a

Oblique asymptote: $y=x$

$$\begin{array}{r} x \\ x^2+1\overline{)\,x^3-1} \\ \underline{-x^3-x} \\ -x-1 \end{array}$$

$$y=x-\frac{x+1}{x^2+1}$$

Exercise

Determine all asymptotes of the function: $y=\dfrac{x^3+3x^2-2}{x^2-4}$

Solution

VA: $x=\pm 2$

HA: n/a

Hole: n/a

Oblique asymptote: $y=x+3$

$$\begin{array}{r} x+3 \\ x^2-4\overline{)\,x^3+3x^2-2} \\ \underline{-x^3+4x} \\ 3x^2+4x-2 \\ \underline{-3x^2+12} \\ 4x+10 \end{array}$$

$$y=x+3+\frac{4x+10}{x^2-4}$$

Exercise

Determine all asymptotes of the function: $y=\dfrac{3x^2-27}{(x+3)(2x+1)}$

Solution

$$y=\frac{3x^2-27}{(x+3)(2x+1)}=\frac{3\left(x^2-9\right)}{(x+3)(2x+1)}=\frac{3(x+3)(x-3)}{(x+3)(2x+1)}=\frac{3(x-3)}{(2x+1)}$$

VA: $x=-3,\ -\frac{1}{2}$ ***HA***: $y=\frac{3}{2}$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{x-3}{x^2-9}$

Solution

$$x^2 - 9 = 0 \to \boxed{x = \pm 3}$$

$$y = \frac{x-3}{(x-3)(x+3)}$$

$$= \frac{1}{x+3}$$

VA: $x = 3$ ***HA***: $y = 0$

Hole: $x = 3 \to y = \frac{1}{6}$ ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{6}{\sqrt{x^2 - 4x}}$

Solution

$$x^2 - 4x = 0$$

$$\Rightarrow x(x-4) = 0 \to \boxed{x = 0, 4}$$

VA: $x = 0,\ x = 4$ ***HA***: $y = 0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{5x-1}{1-3x}$

Solution

VA: $x = \frac{1}{3}$ ***HA***: $y = -\frac{5}{3}$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $f(x) = \dfrac{2x-11}{x^2+2x-8}$

Solution

VA: $x = 2,\ x = -4$ ***HA***: $y = 0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $f(x)=\dfrac{x^2-4x}{x^3-x}$

Solution

$$f(x)=\frac{x(x-4)}{x\left(x^2-1\right)}$$

$$=\frac{x-4}{x^2-1}$$

VA: $x=-1,\ x=1$ ***HA***: $y=0$

Hole: $x=0 \rightarrow y=4$ ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $f(x)=\dfrac{x-2}{x^3-5x}$

Solution

VA: $x=0,\ x=\pm\sqrt{5}$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{4x}{x^2+10x}$

Solution

$$x^2+10x=0 \quad \rightarrow \quad x=0,\ -10$$

Domain: $(-\infty,\ -10)\cup(-10,\ 0)\cup(0,\ \infty)$

$$f(x)=\frac{4x}{x(x+10)}$$

$$=\frac{4}{x+10}$$

VA: $x=-10$ ***HA***: $y=0$

Hole: $x=0 \quad \rightarrow \quad y=\frac{4}{10} \Rightarrow$ *hole* $\left(0,\ \frac{2}{5}\right)$ ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{3-x}{(x-4)(x+6)}$

Solution

VA: $x=-6 \quad and \quad x=4$ **HA**: $y=0$

Hole: n/a **Oblique asymptote**: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^3}{2x^3-x^2-3x}$

Solution

$$2x^3-x^2-3x=x\left(2x^2-x-3\right)=0 \quad \to \quad x=0,\ -1,\ \frac{3}{2}$$

$$\begin{aligned} f(x) &= \frac{x^3}{2x^3-x^2-3x} \\ &= \frac{x^3}{x\left(2x^2-x-3\right)} \\ &= \frac{x^2}{2x^2-x-3} \end{aligned}$$

VA: $x=-1 \quad and \quad x=\frac{3}{2}$ **HA**: $y=\frac{1}{2}$

Hole: $x=0 \quad \to \quad y=0 \Rightarrow$ hole $(0,\ 0)$ **Oblique asymptote**: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{3x^2+5}{4x^2-3}$

Solution

$$4x^2-3=0 \quad \to \quad x=\pm\frac{\sqrt{3}}{2}$$

Domain: $\left(-\infty,\ -\frac{\sqrt{3}}{2}\right)\cup\left(-\frac{\sqrt{3}}{2},\ \frac{\sqrt{3}}{2}\right)\cup\left(\frac{\sqrt{3}}{2},\ \infty\right)$

VA: $x=-\frac{\sqrt{3}}{2} \quad and \quad x=\frac{\sqrt{3}}{2}$ **HA**: $y=\frac{3}{4}$

Hole: n/a **Oblique asymptote**: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x+6}{x^3+2x^2}$

Solution

$$x^3+2x^2=x^2(x+2)=0 \quad \to \quad x=0,-2$$

Domain: $(-\infty, \ -2)\cup(-2, \ 0)\cup(0, \ \infty)$

VA: $x=0$ *and* $x=2$ **HA**: $y=0$

Hole: n/a **Oblique asymptote**: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2+4x-1}{x+3}$

Solution

VA: $x=-3$

HA: n/a

Hole: n/a

Oblique asymptote: $y=x+1$

$$\begin{array}{r}
x+1 \\
x+3\overline{)x^2+4x-1} \\
\underline{-x^2-3x} \\
x-1 \\
\underline{-x-3} \\
-4
\end{array}$$

$$f(x)=\frac{x^2+4x-1}{x+3}=x+1-\frac{4}{x+3}$$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2-6x}{x-5}$

Solution

$x-5=0 \quad \rightarrow \quad x=5$

Domain: $(-\infty, \ 5)\cup(5, \ \infty)$

$$\begin{aligned} f(x)&=\frac{x^2-6x}{x-5} \\ &=x-1-\frac{5}{x-5} \end{aligned}$$

$$\begin{array}{r}
x-1 \\
x-5\overline{)x^2-6x} \\
\underline{-x^2+5x} \\
-x \\
\underline{x-5} \\
-5
\end{array}$$

VA: $x=5$

HA: N/A

Hole: N/A

Oblique asymptote: $y=x-1$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^3-x^2+x-4}{x^2+2x-1}$

Solution

$$x^2+2x-1=0 \quad \rightarrow \quad x=-1\pm\sqrt{2}$$

Domain: $\left(-\infty,\ -1-\sqrt{2}\right)\cup\left(-1-\sqrt{2},\ -1+\sqrt{2}\right)\cup\left(-1+\sqrt{2},\ \infty\right)$

$$\begin{aligned} f(x)&=\frac{x^3-x^2+x-4}{x^2+2x-1} \\ &=x-3+\frac{8x-7}{x^2+2x-1} \end{aligned}$$

VA: $x=-1\pm\sqrt{2}$

HA: n/a

Hole: n/a

Oblique asymptote: $y=x-3$

$$\begin{array}{rl} & \phantom{x^3-{}}x-3 \\ x^2+2x-1 \big) & \overline{x^3-x^2+x-4} \\ & \underline{-x^3-2x^2+x} \\ & -3x^2+2x-4 \\ & \phantom{x^3-{}}\underline{3x^2+6x-3} \\ & \phantom{x^3-3x^2+{}}8x-7 \end{array}$$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{4x}{x^2+10x}$

Solution

$x^2+10x=0 \quad \rightarrow \quad x=0,\ -10$ ***Domain***: $(-\infty,\ -10)\cup(-10,\ 0)\cup(0,\ \infty)$

$$f(x)=\frac{4x}{x(x+10)}=\frac{4}{x+10}$$

VA: $x=-10$ ***HA***: $y=0$

Hole: $x=0 \quad \rightarrow \quad y=\frac{4}{10} \Rightarrow$ *hole* $\left(0,\ \frac{2}{5}\right)$

Oblique asymptote: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{3-x}{(x-4)(x+6)}$

Solution

Domain: $(-\infty,\ -6)\cup(-6,\ 4)\cup(4,\ \infty)$

VA: $x=-6$ *and* $x=4$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^3}{2x^3-x^2-3x}$

Solution

$$2x^3-x^2-3x=x\left(2x^2-x-3\right)=0 \quad\to\quad x=0,\ -1,\ \frac{3}{2}$$

Domain: $(-\infty,\ -1)\cup(-1,\ 0)\cup\left(0,\ \frac{3}{2}\right)\cup\left(\frac{3}{2},\ \infty\right)$

$$f(x)=\frac{x^3}{2x^3-x^2-3x}=\frac{x^3}{x\left(2x^2-x-3\right)}=\frac{x^2}{2x^2-x-3}$$

VA: $x=-1$ *and* $x=\frac{3}{2}$ ***HA***: $y=\frac{1}{2}$

Hole: $x=0 \quad\to\quad y=0\Rightarrow$ *hole* $(0,\ 0)$

Oblique asymptote: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{3x^2+5}{4x^2-3}$

Solution

$4x^2-3=0 \quad\to\quad x=\pm\frac{\sqrt{3}}{2}$ ***Domain***: $\left(-\infty,\ -\frac{\sqrt{3}}{2}\right)\cup\left(-\frac{\sqrt{3}}{2},\ \frac{\sqrt{3}}{2}\right)\cup\left(\frac{\sqrt{3}}{2},\ \infty\right)$

VA: $x=-\frac{\sqrt{3}}{2}$ *and* $x=\frac{\sqrt{3}}{2}$ ***HA***: $y=\frac{3}{4}$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x+6}{x^3+2x^2}$

Solution

$x^3+2x^2=x^2(x+2)=0 \quad\to\quad x=0,-2$ ***Domain***: $(-\infty,\ -2)\cup(-2,\ 0)\cup(0,\ \infty)$

VA: $x=0$ *and* $x=2$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2+4x-1}{x+3}$

Solution

$x+3=0 \quad \rightarrow \quad x=-3$ **Domain**: $(-\infty,\ -3)\cup(-3,\ \infty)$

$$\begin{array}{r|l} & x+1 \\ \hline x+3 & x^2+4x-1 \\ & \underline{-x^2-3x} \\ & \quad\ x-1 \\ & \quad\ \underline{-x-3} \\ & \qquad -4 \end{array}$$

$$f(x)=\frac{x^2+4x-1}{x+3}=x+1-\frac{4}{x+3}$$

VA: $x=-3$ ***HA***: n/a

Hole: n/a ***Oblique asymptote***: $y=x+1$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2-6x}{x-5}$

Solution

$x-5=0 \quad \rightarrow \quad x=5$ **Domain**: $(-\infty,\ 5)\cup(5,\ \infty)$

$$\begin{array}{r|l} & x-1 \\ \hline x-5 & x^2-6x \\ & \underline{-x^2+5x} \\ & \quad -x \\ & \quad\ \underline{x-5} \\ & \quad\ -5 \end{array}$$

VA: $x=5$

HA: N/A

Hole: N/A

Oblique asymptote: $y=x-1$

$$f(x)=\frac{x^2-6x}{x-5}=x-1-\frac{5}{x-5}$$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^3-x^2+x-4}{x^2+2x-1}$

Solution

$x^2+2x-1=0 \quad \rightarrow \quad x=-1\pm\sqrt{2}$

Domain: $\left(-\infty,\ -1-\sqrt{2}\right)\cup\left(-1-\sqrt{2},\ -1+\sqrt{2}\right)\cup\left(-1+\sqrt{2},\ \infty\right)$

$$\begin{array}{r}
x-3 \\
x^2+2x-1\overline{\smash{)}\,x^3-x^2+x-4} \\
\underline{-x^3-2x^2+x} \\
-3x^2+2x-4 \\
\underline{3x^2+6x-3} \\
8x-7
\end{array}$$

VA: $x=-1\pm\sqrt{2}$

HA: N/A

Hole: N/A

Oblique asymptote: $y=x-3$

$$\begin{aligned} f(x) &= \frac{x^3-x^2+x-4}{x^2+2x-1} \\ &= x-3+\frac{8x-7}{x^2+2x-1} \end{aligned}$$

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

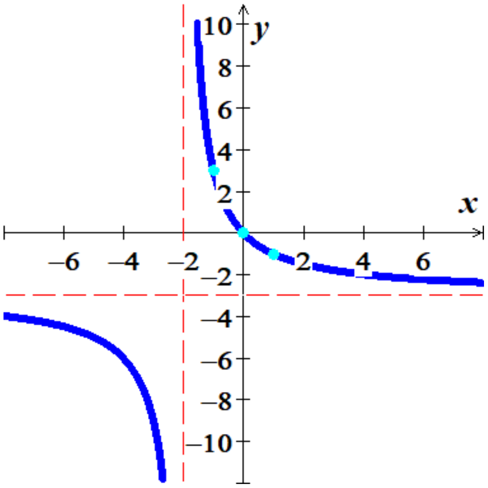
$$f(x)=\frac{-3x}{x+2}$$

Solution

VA: $x=-2$ ***HA***: $y=-3$

Hole: n/a ***OA***: n/a

x	***y***
0	0
1	–1
–1	3



Exercise

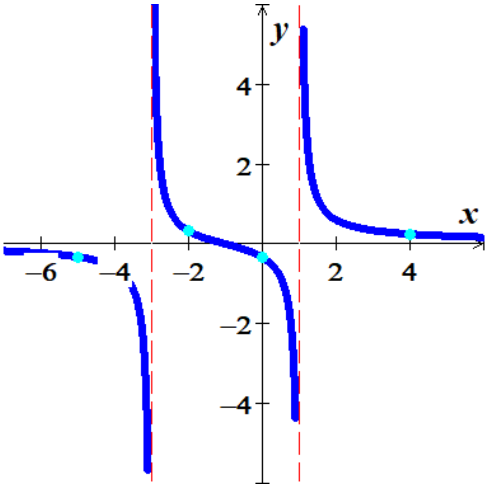
Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x+1}{x^2+2x-3}$$

Solution

VA: $x=1,\ x=-3$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a



x	y
−5	−0.33
−2	0.33
0	−1/3
4	0.24

Exercise

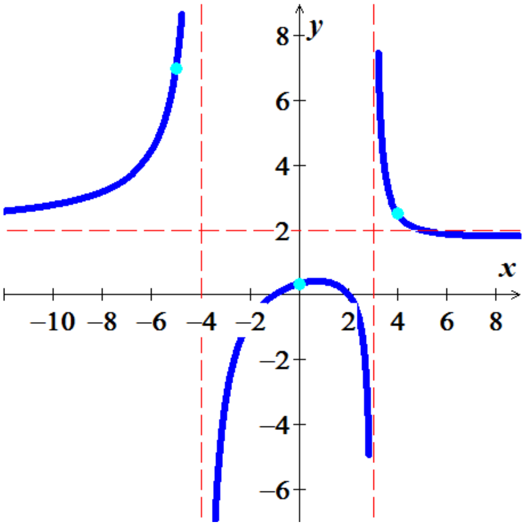
Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{2x^2-2x-4}{x^2+x-12}$$

Solution

VA: $x=-4,\ 3$ ***HA***: $y=2$

Hole: n/a ***OA***: n/a



x	y
−5	7
−2	−0.8
0	1/3
4	2.5
5	2

Exercise

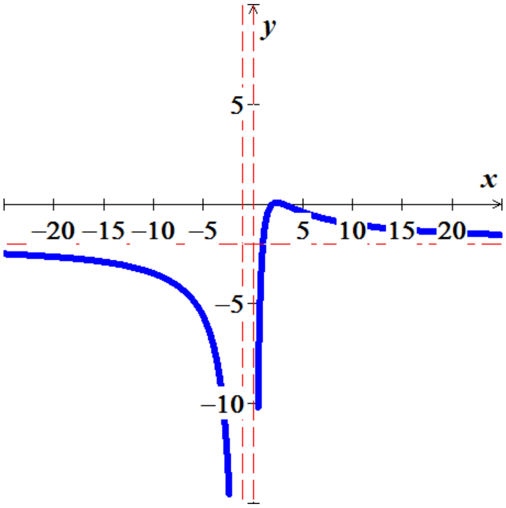
Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

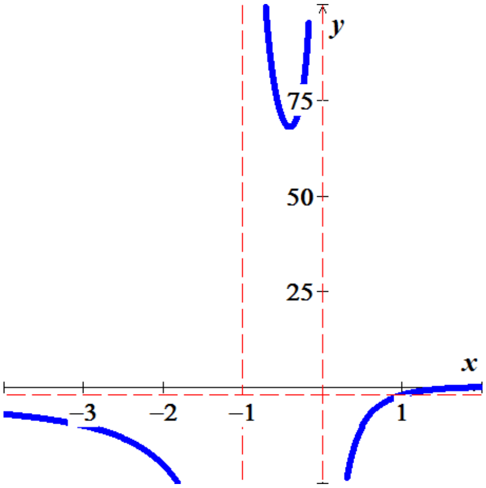
$$f(x)=\frac{-2x^2+10x-12}{x^2+x}$$

Solution

VA: $x=-1,\ 0$ ***HA***: $y=-2$

Hole: n/a ***OA***: n/a





Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x^2-x-6}{x+1}$$

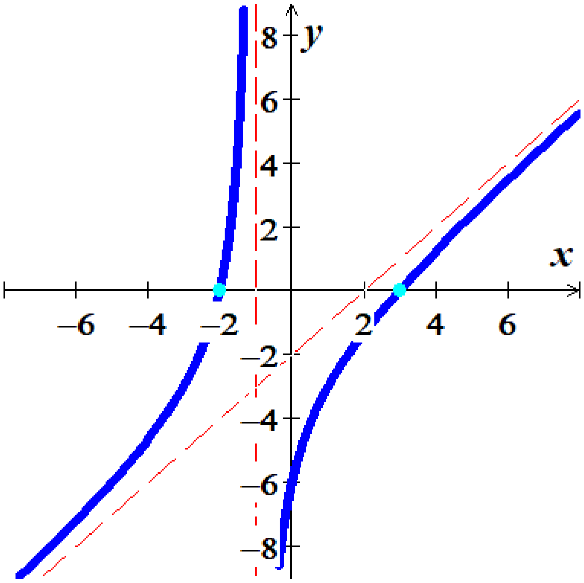
Solution

$$\begin{array}{r} x-2 \\ x+1\overline{)\,x^2-x-6} \\ \underline{x^2+x} \\ -2x-6 \\ \underline{-2x-2} \\ \boxed{-4} \end{array}$$

VA: $x=-1$ ***HA***: n/a

Hole: n/a ***OA***: $y=x-2$

x	y
2	0
–2	0
0	–6



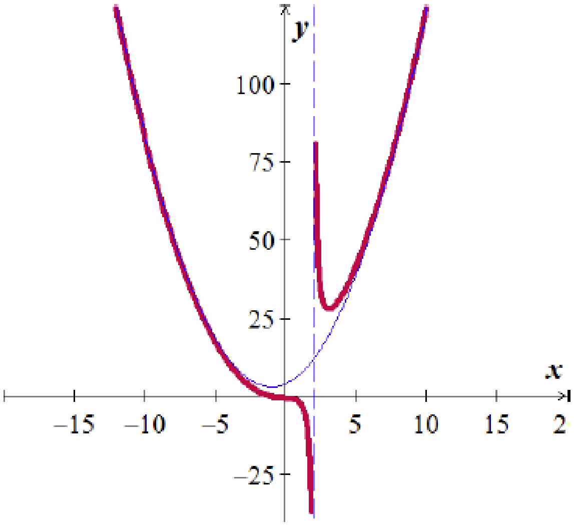
Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x^3+1}{x-2}$$

Solution

$$\begin{array}{r l}
 & x^2+2x+4 \\
x-2 & \overline{)\,x^3-1} \\
 & \underline{x^3-2x^2} \\
 & \qquad 2x^2 \\
 & \qquad \underline{2x^2-4x} \\
 & \qquad\qquad 4x-1 \\
 & \qquad\qquad \underline{4x-8} \\
 & \qquad\qquad\qquad 7
\end{array}$$



VA: $x=2$ ***HA***: n/a

Hole: n/a ***OA***: $y=x^2+2x+4$

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{2x^2+x-6}{x^2+3x+2}$$

Solution

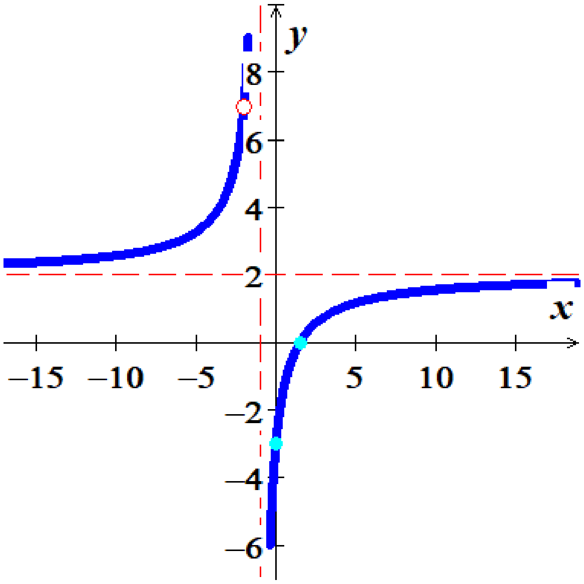
$$f(x)=\frac{(2x-3)(x+2)}{(x+1)(x+2)}$$

$$=\frac{2x-3}{x+1}$$

VA: $x=-1$ ***HA***: $y=2$

Hole: $(-2,\ 7)$ ***OA***: n/a

x	y
0	-3
$-\frac{3}{2}$	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x-1}{1-x^2}$$

Solution

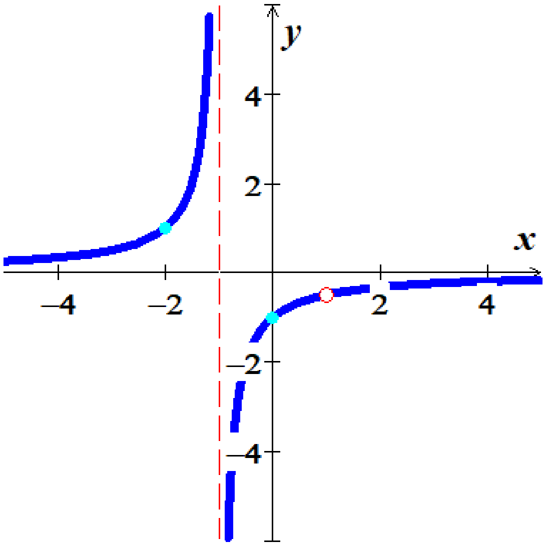
$$f(x)=\frac{x-1}{(x+1)(1-x)}$$

$$=-\frac{1}{x+1}$$

VA: $x=-1$ ***HA***: $y=0$

Hole: $\left(1,\ -\frac{1}{2}\right)$ ***OA***: n/a

x	y
0	−1
−2	1



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x^2+x-2}{x+2}$$

Solution

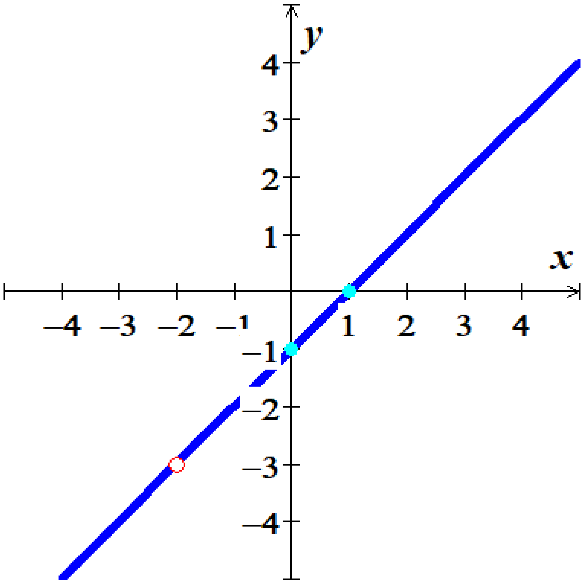
$$f(x)=\frac{(x+2)(x-1)}{x+2}$$

$$=x-1$$

VA: n/a ***HA***: n/a

Hole: $(-2,\ -3)$ ***OA***: n/a

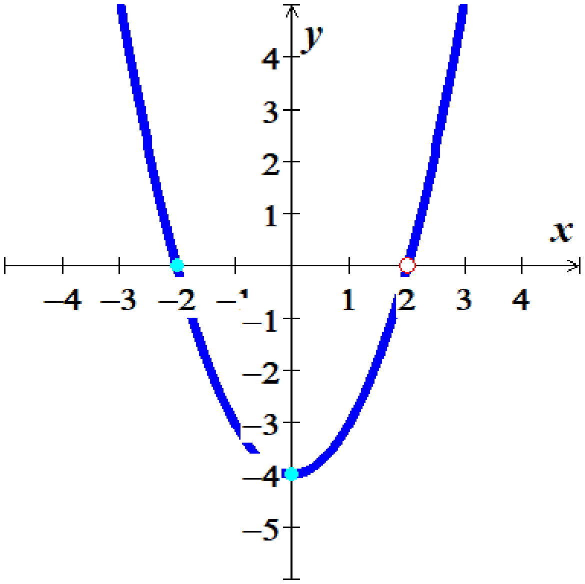
x	y
0	−1
1	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x^3-2x^2-4x+8}{x-2}$$



Solution

$$f(x)=\frac{\left(x^2-4\right)(x-2)}{x-2}$$

$$=x^2-4$$

VA: n/a ***HA***: n/a

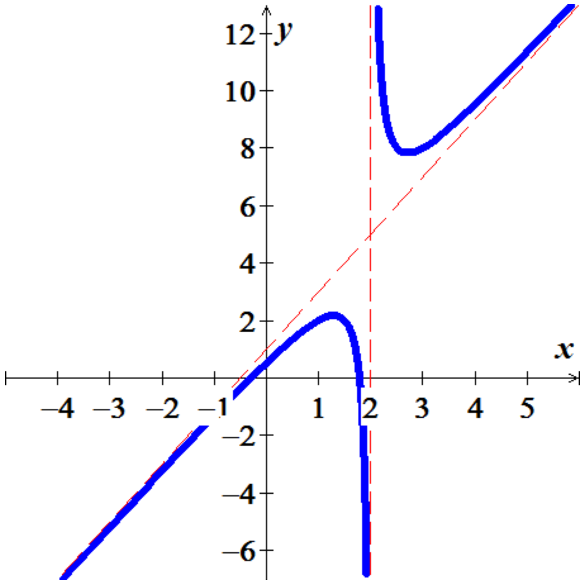
Hole: $(2,\ 0)$ ***OA***: n/a

x	***y***
0	−4
−2	0

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{2x^2-3x-1}{x-2}$$



Solution

$$\begin{array}{r} 2x+1 \\ x-2\overline{\smash{\big)}2x^2-3x-1} \\ \underline{-2x^2+4x} \\ x-1 \\ \underline{-x+2} \\ 1 \end{array}$$

$$f(x)=\frac{2x^2-3x-1}{x-2}$$

$$=(2x+1)+\frac{1}{x-2}$$

VA: $x=2$ ***HA***: $y=1$

Hole: n/a ***OA***: $y=2x+1$

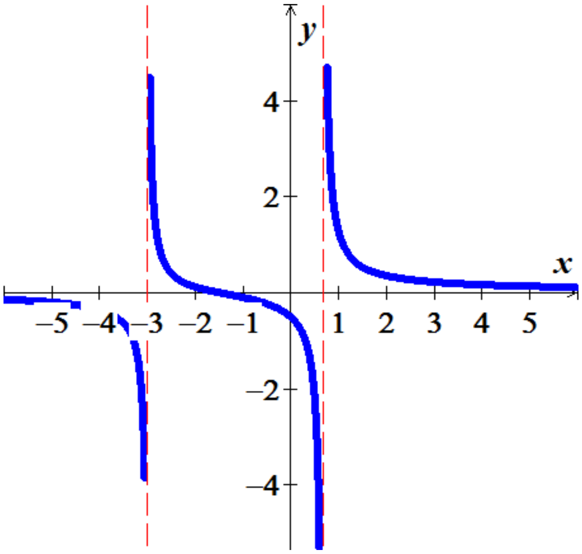
Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{2x+3}{3x^2+7x-6}$$

Solution

$$3x^2+7x-6=0 \quad \Rightarrow \quad x=-3,\ \frac{2}{3}$$



VA: $x=-3$ *and* $x=\frac{2}{3}$

HA: $y=0$

Hole: n/a

OA: n/a

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x^2-1}{x^2+x-6}$$

Solution

$$x^2+x-6=0 \quad \Rightarrow \quad \underline{x=-3,\ 2|}$$

VA: $x=-3$ *and* $x=2$ **HA**: $y=1$

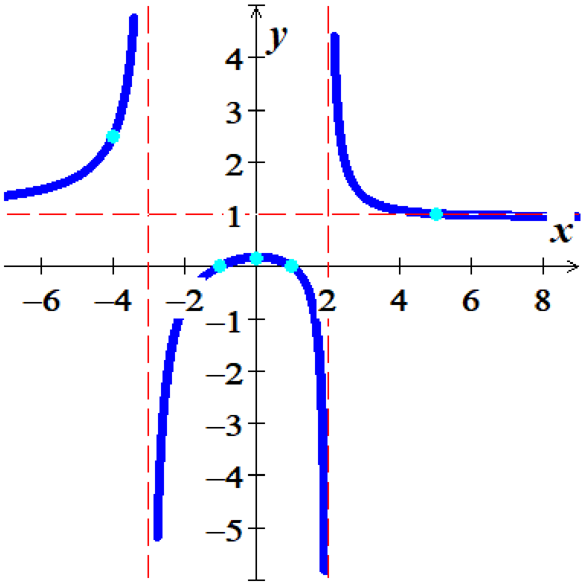
Hole: n/a **OA:** n/a

$$1=\frac{x^2-1}{x^2+x-6}$$

$$x^2+x-6=x^2-1$$

$$\underline{x=5|}$$

x	**y**
0	$\frac{1}{6}$
5	1
±1	0
−4	$\frac{5}{2}$

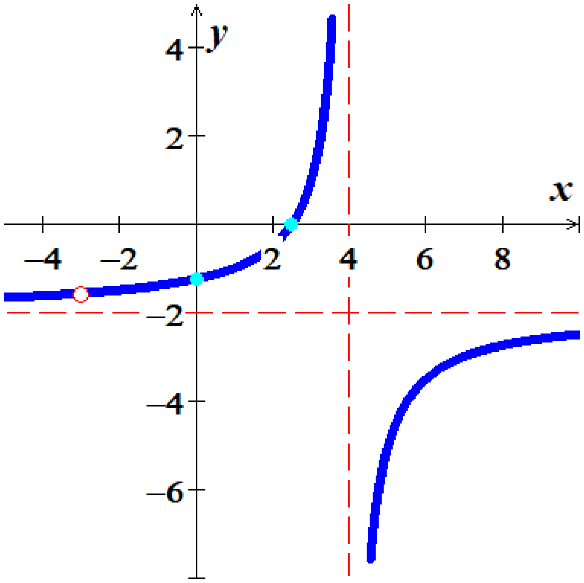


Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{-2x^2-x+15}{x^2-x-12}$$

Solution



$x^2-x-12=0 \quad \Rightarrow \quad \underline{x=-3,\ 4}|$

Domain: $(-\infty,-3)\cup(-3,4)\cup(4,\infty)$

$$f(x)=\frac{(-2x+5)(x+3)}{(x-4)(x+3)}$$

$$=\frac{-2x+5}{x-4}$$

VA: $x=4$ ***HA***: $y=-2$

Hole: $\left(-3,\ -\frac{11}{7}\right)$ ***OA***: n/a

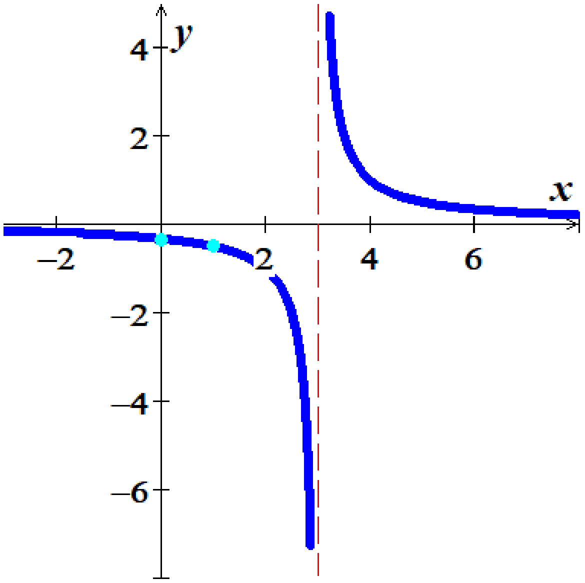
x	y
0	$-\frac{5}{4}$
$\frac{5}{2}$	0

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{1}{x-3}$$

Solution



VA: $x=3$ ***HA***: $y=0$

Hole: n/a ***OA***: n/a

x	y
0	$-\frac{1}{3}$
1	$-\frac{1}{2}$

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

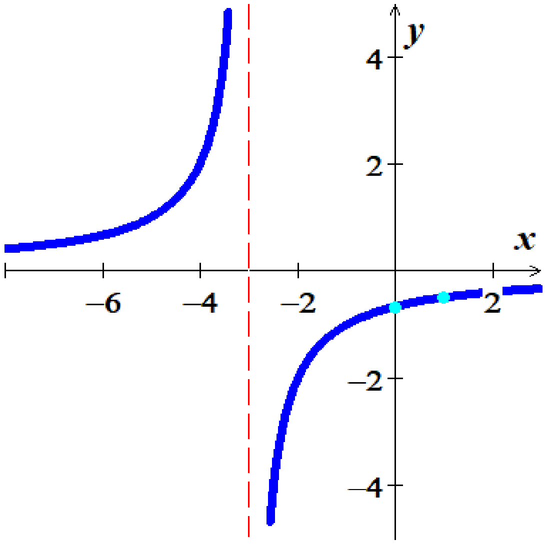
$$f(x)=\frac{-2}{x+3}$$

Solution

VA: $x=-3$ ***HA***: $y=0$

Hole: n/a ***OA:*** n/a

x	y
0	$-\frac{2}{3}$
1	$-\frac{1}{2}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

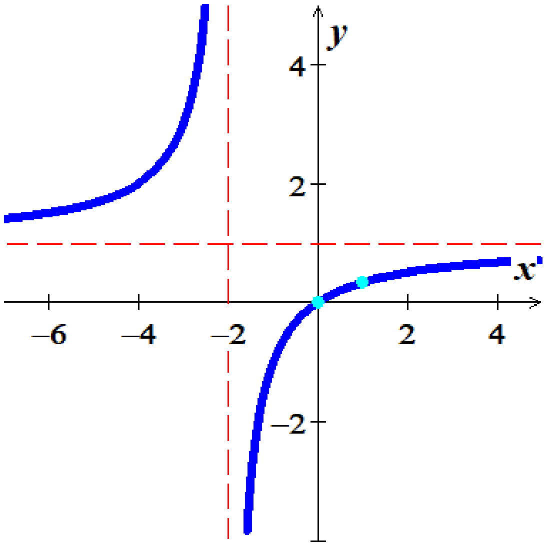
$$f(x)=\frac{x}{x+2}$$

Solution

VA: $x=-2$ ***HA***: $y=1$

Hole: n/a ***OA:*** n/a

x	y
0	0
1	$\frac{1}{3}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

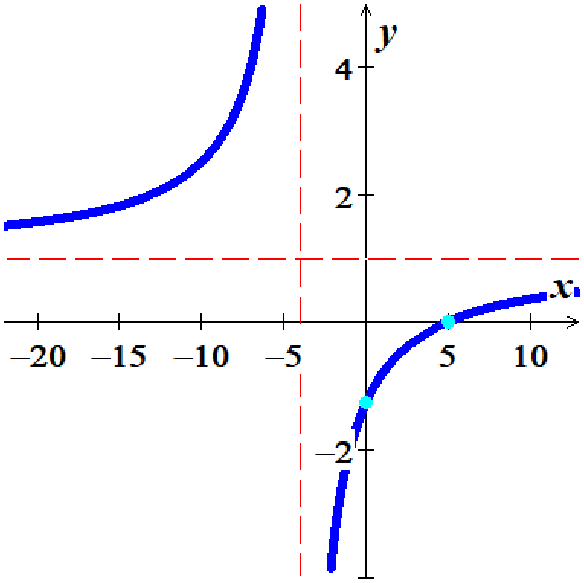
$$f(x)=\frac{x-5}{x+4}$$

Solution

VA: $x=-4$ *HA*: $y=1$

Hole: n/a *OA*: n/a

x	y
0	$-\frac{5}{4}$
5	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{2x^2-2}{x^2-9}$$

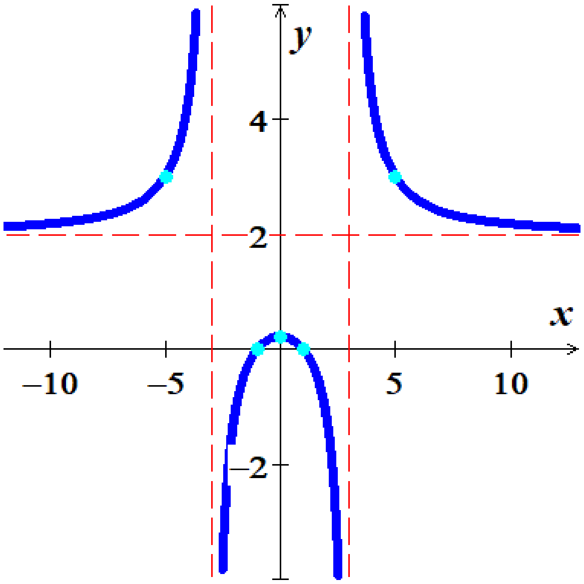
Solution

$x^2=9 \rightarrow \underline{x=\pm 3}|$

VA: $x=\pm 3$ *HA*: $y=2$

Hole: n/a *OA*: n/a

x	y
0	$\frac{2}{9}$
±1	0
±5	3



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

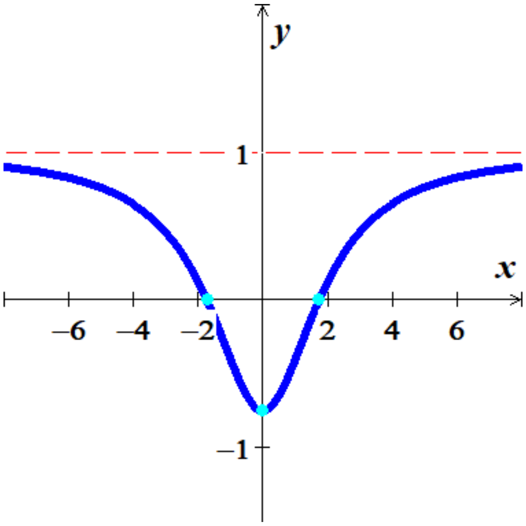
$$f(x)=\frac{x^2-3}{x^2+4}$$

Solution

VA: n/a HA: $y=1$

$Hole$: n/a **OA:** n/a

x	y
0	$-\frac{3}{4}$
$\pm\sqrt{3}$	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2+4}{x^2-3}$$

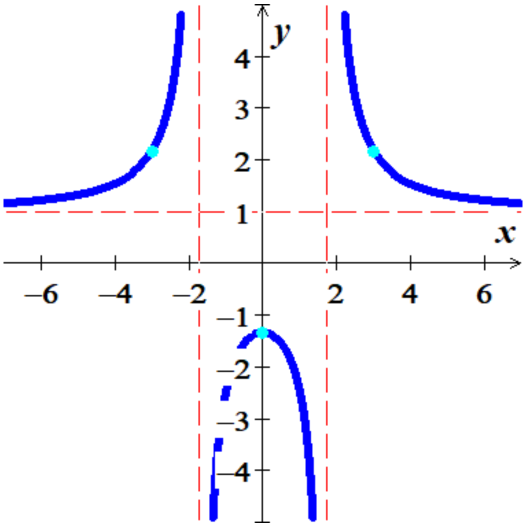
Solution

$x^2-3=0 \quad \rightarrow \quad x=\pm\sqrt{3}$

VA: $x=\pm\sqrt{3}$ HA: $y=1$

$Hole$: n/a **OA:** n/a

x	y
0	$-\frac{4}{3}$
±3	$\frac{13}{6}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x^2}{x^2 - 6x + 9}$$

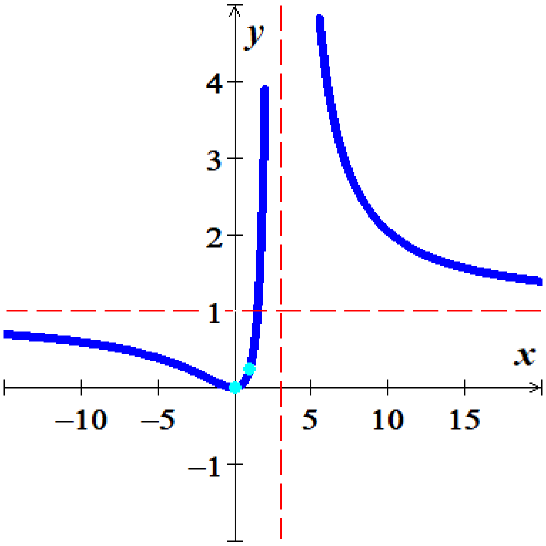
Solution

$x^2 - 6x + 9 = 0 \quad \rightarrow \quad x = 3$

VA: $x = 3$ **HA**: $y = 1$

Hole: n/a **OA:** n/a

x	y
0	0
1	$\frac{1}{4}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x^2 + x + 4}{x^2 + 2x - 1}$$

Solution

$x^2 + 2x - 1 = 0$

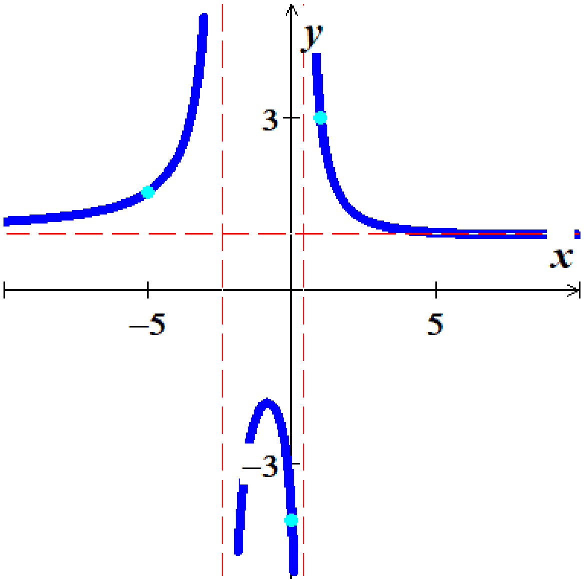
$x = \frac{-2 \pm \sqrt{8}}{2}$

$= -1 \pm \sqrt{2}$

VA: $x = -1 \pm \sqrt{2}$ **HA**: $y = 1$

Hole: n/a **OA:** n/a

x	y
0	–4
1	3
–5	$\frac{12}{7}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

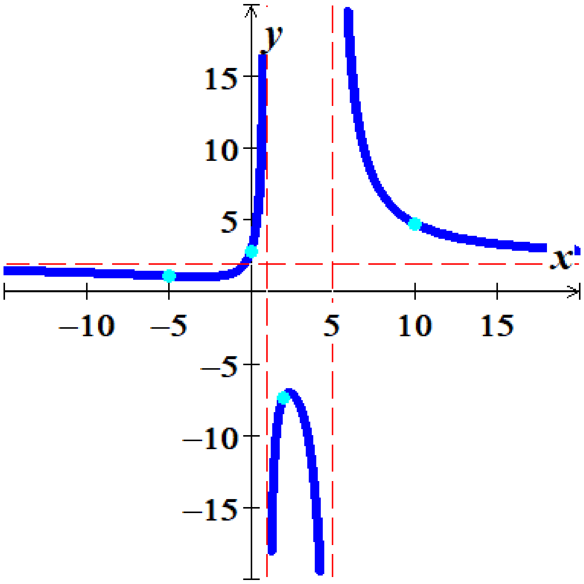
$$f(x)=\frac{2x^2+14}{x^2-6x+5}$$

Solution

VA: $x=1,\ 5$ **HA**: $y=2$

Hole: n/a **OA:** n/a

x	y
0	$\frac{14}{5}$
2	$-\frac{22}{3}$
–5	$\frac{16}{15}$
10	$\frac{214}{45}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2-4x-5}{2x+5}$$

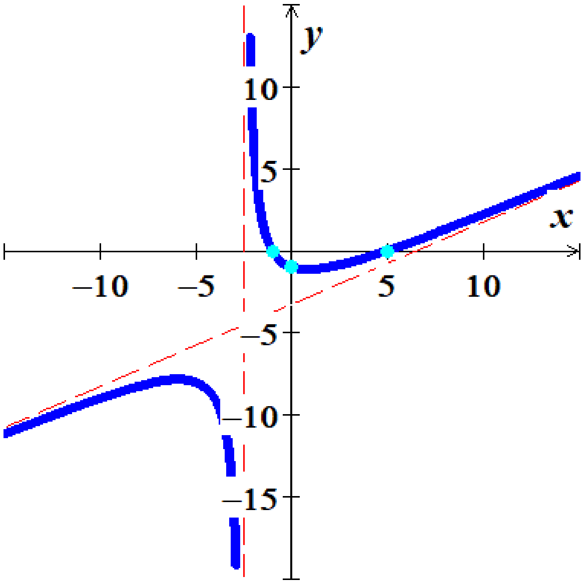
Solution

$$\begin{array}{r} \frac{1}{2}x-\frac{13}{4} \\ 2x+5\overline{\big)\, x^2-4x-5} \\ \underline{x^2+\frac{5}{2}x} \\ -\frac{13}{2}x-5 \end{array}$$

VA: $x=-\frac{5}{2}$ **HA**: n/a

Hole: n/a **OA:** $y=\frac{1}{2}x-\frac{13}{2}$

x	y
0	–1
–1, 5	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x-3}{x^2-3x+2}$$

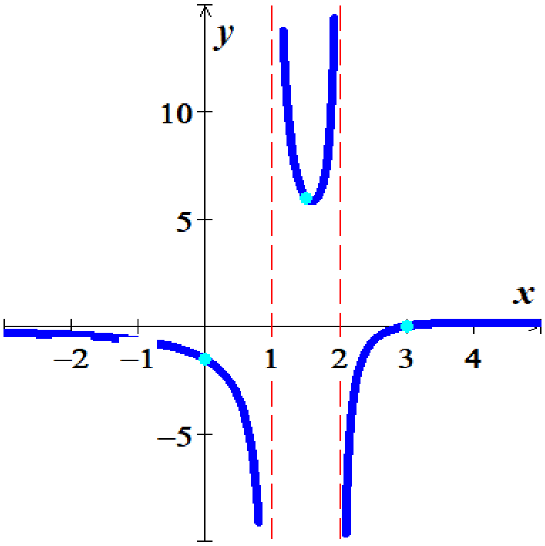
Solution

$x^2-3x+2 \quad \to \quad \underline{x=1,\ 2}|$

VA: $x=1,\ 2$ ***HA***: $y=0$

Hole: n/a ***OA:*** n/a

x	y
0	$-\frac{3}{2}$
3	0
$\frac{3}{2}$	6



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2+2}{x^2+3x+2}$$

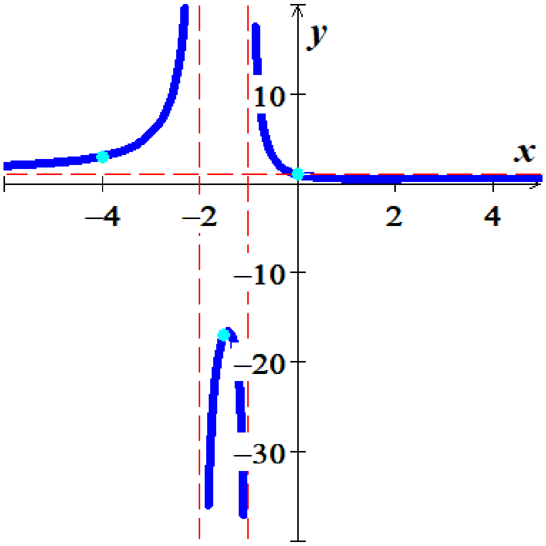
Solution

$x^2+3x+2 \quad \to \quad \underline{x=-1,\ -2}|$

VA: $x=-1,\ -2$ ***HA***: $y=1$

Hole: n/a ***OA:*** n/a

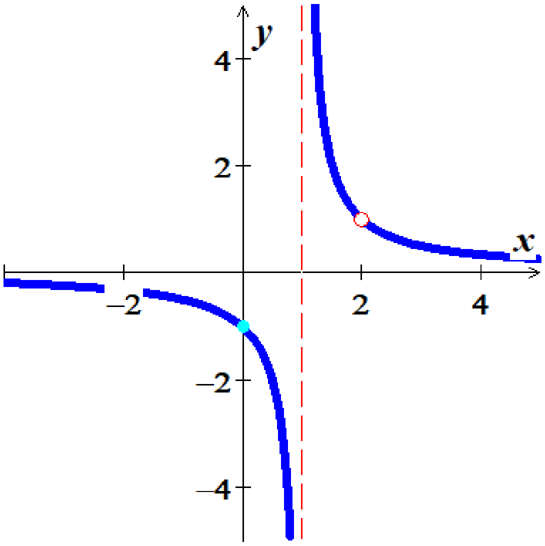
x	y
0	1
$-\frac{3}{2}$	-17
-4	3



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x-2}{x^2-3x+2}$$



Solution

$$x^2-3x+2 \quad \to \quad \underline{x=1,\ 2|}$$

$$f(x)=\frac{x-2}{(x-2)(x-1)}$$

$$=\frac{1}{x-1}$$

VA: $x=1$ ***HA***: $y=0$

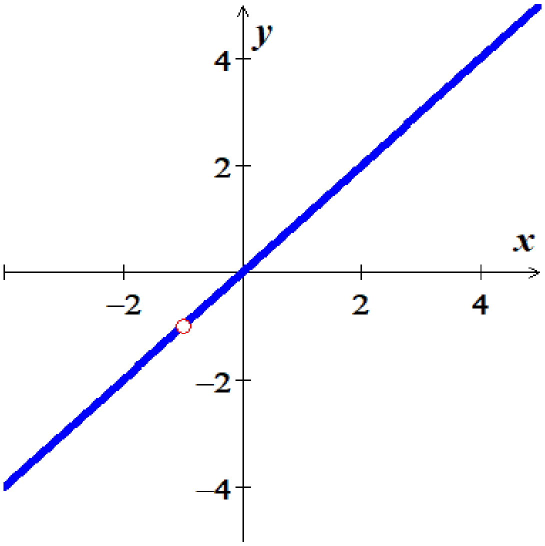
Hole: $(2,\ 1)$ ***OA:*** n/a

x	y
0	-1

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2+x}{x+1}$$



Solution

$$f(x)=\frac{x(x+1)}{x+1}$$

$$\underline{=x|}$$

VA: n/a ***HA***: n/a

Hole: $(-1,\ -1)$ ***OA:*** n/a

Hole: $\left(-3,\ -\frac{11}{7}\right)$ ***OA:*** n/a

x	y
0	0

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2-2x}{x-2}$$

Solution

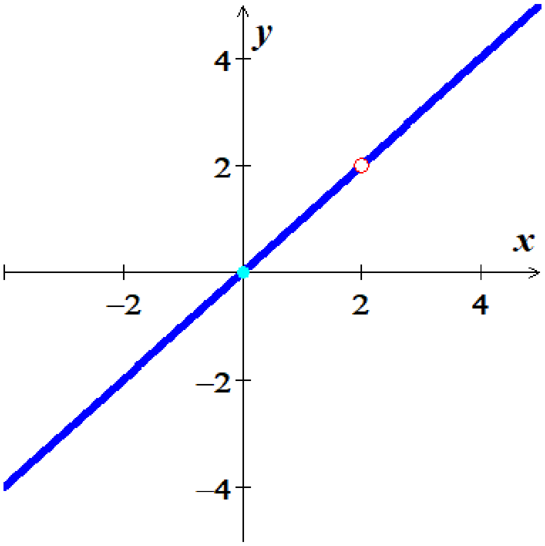
$$f(x)=\frac{x(x-2)}{x-2}$$
$$\underline{=x\,|}$$

VA: n/a ***HA***: n/a

Hole: $(2,\ 2)$ ***OA:*** n/a

Hole: $\left(-3,\ -\frac{11}{7}\right)$ ***OA:*** n/a

x	***y***
0	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2-3x}{x+3}$$

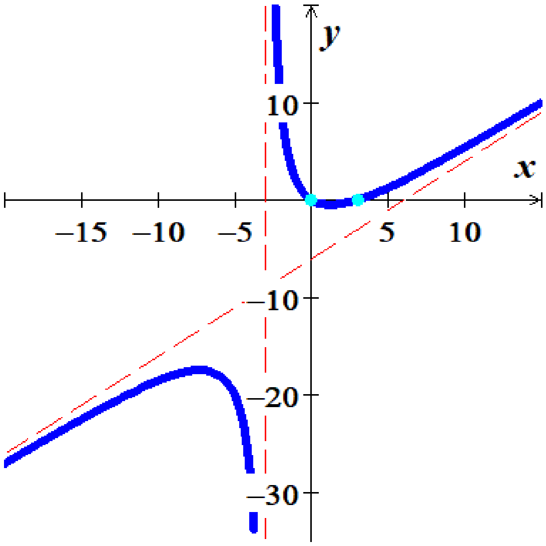
Solution

$$\begin{array}{r} x-6 \\ x+3\ \overline{\big)\,x^2-3x} \\ \underline{x^2+3x} \\ -6x-5 \end{array}$$

VA: $x=-3$ ***HA***: n/a

Hole: n/a ***OA:*** $y=x-6$

x	***y***
0	0
3	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^3+3x^2-4x+6}{x+2}$$

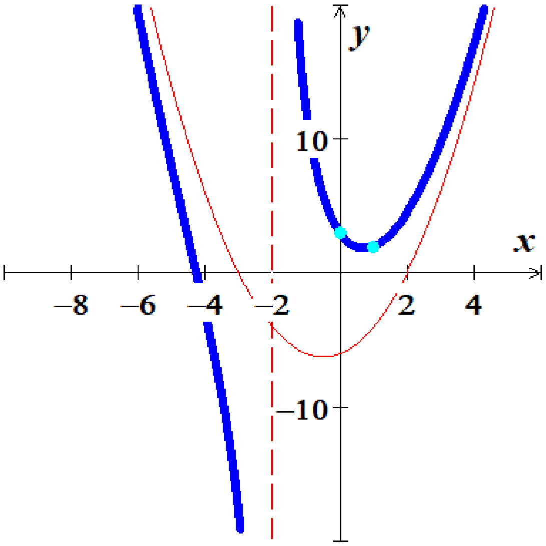
Solution

$$\begin{array}{rl} & \quad x^2+x-6 \\ x+2 & \overline{)\,x^3+3x^2-4x+6} \\ & \quad \underline{x^3+2x^2} \\ & \quad x^2-4x \\ & \quad \underline{x^2+2x} \\ & \quad -6x+6 \end{array}$$

VA: $x=-2$ ***HA***: n/a

Hole: n/a ***OA:*** $y=x^2+x-6$

x	y
0	3
1	2



Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote: \ x=4 \\ horizontal\ asymptote: \ y=-1 \\ x-intercept: \ 3 \end{cases}$$

Solution

Vertical Asymptote: $f(x)=\frac{\quad}{x-4}$

Horizontal Asymptote: $f(x)=\frac{-x+a}{x-4}$

x-intercept: $f(x=3)=\frac{-3+a}{3-4}=0 \quad\Rightarrow\quad \underline{a=3}$

$$f(x)=\frac{-x+3}{x-4}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote:\ x=-4, x=5 \\ horizontal\ asymptote:\ y=\frac{3}{2} \\ x-intercept:\ -2 \end{cases}$$

Solution

Vertical Asymptote: $\quad f(x)=\frac{}{(x+4)(x-5)}$

Horizontal Asymptote: $\quad f(x)=\frac{3}{2}\frac{(x+a)(x+b)}{(x+4)(x-5)}$

x-intercept: $\quad f(x=-2)=\frac{3}{2}\frac{(-2+a)(-2+b)}{}$

$$0=(-2+a)(-2+b)$$

$$\boxed{a=b=2}$$

$$f(x)=\frac{3}{2}\frac{(x-2)^2}{x^2-x-20}$$

$$\boxed{=\frac{3x^2-12x+12}{2x^2-2x-40}}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote:\ x=5 \\ horizontal\ asymptote:\ y=-1 \\ x-intercept:\ 2 \end{cases}$$

Solution

Vertical Asymptote: $\quad f(x)=\frac{}{x-5}$

x-intercept : $\quad f(x)=\frac{x-2}{x-5}$

Horizontal Asymptote: $\quad f(x)=-\frac{x-2}{x-5}$

$$\boxed{f(x)=-\frac{x-2}{x-5}}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote:\ x=-2,\ x=0 \\ horizontal\ asymptote:\ y=0 \\ x-intercept:\ 2, \quad f(3)=1 \end{cases}$$

Solution

Vertical Asymptote: $f(x)=\dfrac{}{x(x+2)}$

x-intercept : $f(x)=\dfrac{x-2}{x(x+2)}$

Horizontal Asymptote: $f(x)=\dfrac{a(x-2)}{x(x+2)}$

$f(3)=1 \quad \rightarrow \dfrac{a(1)}{(3)(5)}=1 \quad \Rightarrow \underline{a=15|}$

$$\boxed{f(x)=\dfrac{15x-30}{x^2+2x}}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote:\ x=-3,\ x=1 \\ horizontal\ asymptote:\ y=0 \\ x-intercept:\ -1, \quad f(0)=-2 \\ hole:\ x=2 \end{cases}$$

Solution

Vertical Asymptote: $f(x)=\dfrac{}{(x+3)(x-1)}$

x-intercept : $f(x)=\dfrac{(x+1)}{(x+3)(x-1)}$

Horizontal Asymptote: $f(x)=\dfrac{a(x+1)}{(x+3)(x-1)}$

$f(0)=-2 \quad \rightarrow \dfrac{a}{-3}=-2 \quad \Rightarrow \underline{a=6|}$

Hole at $x=2$**:** $f(x)=\dfrac{6(x+1)(x-2)}{\left(x^2+2x-3\right)(x-2)}$

$$\boxed{f(x)=\dfrac{6x^2-6x-12}{x^3-7x+6}}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote:\ x=-1,\ \ x=3 \\ horizontal\ asymptote:\ y=2 \\ x-intercept:\ \ -2,\ \ 1 \\ \ \ hole:\ \ x=0 \end{cases}$$

Solution

Vertical Asymptote: $f(x)=\dfrac{}{(x+1)(x-3)}$

Horizontal Asymptote: $f(x)=\dfrac{2}{(x+1)(x-3)}$

x-intercept : $f(x)=\dfrac{2(x+2)(x-1)}{(x+1)(x-3)}$

Hole at $x=0$: $f(x)=\dfrac{2x(x+2)(x-1)}{x(x+1)(x-3)}$

$$\underline{f(x)=\dfrac{2x^3+2x^2-4x}{x^3-2x^2-3x}}\Big|$$