# Solution Section 3.4 – L'Hôpital's Rule

### Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to -2} \dfrac{x+2}{x^2-4}$

**Solution**

$$\lim\limits_{x\to -2} \frac{x+2}{x^2-4} = \left.\frac{1}{2x}\right|_{x=-2} \qquad \frac{(x+2)'}{\left(x^2-4\right)'} = \frac{1}{2x}$$

$$= -\frac{1}{4}$$

### Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 1} \dfrac{x^3-1}{4x^3-x-3}$

**Solution**

$$\lim\limits_{x\to 1} \frac{x^3-1}{4x^3-x-3} = \lim\limits_{x\to 1} \frac{3x^2}{12x^2-1}$$

$$= \frac{3}{11}$$

### Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to -5} \dfrac{x^2-25}{x+5}$

**Solution**

$$\lim\limits_{x\to -5} \frac{x^2-25}{x+5} = \lim\limits_{x\to -5} \frac{2x}{1}$$

$$= -10$$

### Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{t\to 0} \dfrac{\sin 5t}{2t}$

**Solution**

$$\lim\limits_{t\to 0} \frac{\sin 5t}{2t} = \lim\limits_{t\to 0} \frac{5\cos 5t}{2}$$

$$= \frac{5}{2}$$

### Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{\theta \to -\pi/3} \dfrac{3\theta + \pi}{\sin\left(\theta + \frac{\pi}{3}\right)}$

***Solution***

$$\lim_{\theta \to -\pi/3} \frac{3\theta + \pi}{\sin\left(\theta + \frac{\pi}{3}\right)} = \lim_{\theta \to -\pi/3} \frac{3}{\cos\left(\theta + \frac{\pi}{3}\right)}$$

$$= 3$$

### Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x \to 0} \dfrac{x^2}{\ln(\sec x)}$

***Solution***

$$\lim_{x \to 0} \frac{x^2}{\ln(\sec x)} = \lim_{x \to 0} \frac{2x}{\frac{\sec x \tan x}{\sec x}}$$

$$= \lim_{x \to 0} \frac{2x}{\tan x} = \frac{0}{0}$$

$$\frac{(2x)'}{(\tan x)'} = \frac{2}{\sec^2 x}$$

$$= \lim_{x \to 0} \frac{2}{\sec^2 x}$$

$$= \frac{2}{1}$$

$$= 2$$

### Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{\theta \to 0} \dfrac{3^{\sin\theta} - 1}{\theta}$

***Solution***

$$\lim_{\theta \to 0} \frac{3^{\sin\theta} - 1}{\theta} = \lim_{\theta \to 0} \frac{3^{\sin\theta} \ln(3)(\cos\theta)}{1}$$

$$= 3^{\sin 0} \ln(3)(\cos 0)$$

$$= \ln(3)$$

## Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 0} \frac{3^x - 1}{2^x - 1}$

***Solution***

$$\lim_{x\to 0} \frac{3^x - 1}{2^x - 1} = \lim_{x\to 0} \frac{3^x \ln(3)}{2^x \ln(2)}$$

$$= \frac{3^0 \ln 3}{2^0 \ln 2}$$

$$\boxed{= \frac{\ln 3}{\ln 2}} \quad \boxed{= \log_2^3}$$

## Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 0^+} (\ln x - \ln \sin x)$

***Solution***

$$\lim_{x\to 0^+} (\ln x - \ln \sin x) = \lim_{x\to 0^+} \left(\ln \frac{x}{\sin x}\right)$$

$$= \ln \lim_{x\to 0^+} \left(\frac{x}{\sin x}\right) = \ln \frac{0}{0}$$

$$= \ln \lim_{x\to 0^+} \left(\frac{1}{\cos x}\right)$$

$$= \ln(1)$$

$$\boxed{= 0}$$

## Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 0} \frac{\left(e^x - 1\right)^2}{x \sin x}$

***Solution***

$$\lim_{x\to 0} \frac{\left(e^x - 1\right)^2}{x \sin x} = \lim_{x\to 0} \frac{2\left(e^x - 1\right)e^x}{\sin x + x\cos x}$$

$$= \frac{2(1-1)}{0+0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{2e^{2x} - 2e^x}{\sin x + x\cos x}$$

$$= \lim_{x\to 0} \frac{4e^{2x} - 2e^{x}}{\cos x + \cos x - x\sin x}$$

$$= \frac{4-2}{1+1-0}$$

$$\underline{=1}$$

***Exercise***

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\pi/2^-} \dfrac{1+\tan x}{\sec x}$

***Solution***

$$\lim_{x\to\pi/2^-} \frac{1+\tan x}{\sec x} = \frac{\infty}{\infty} = \lim_{x\to\pi/2^-} \frac{\sec^2 x}{\sec x \tan x}$$

$$= \lim_{x\to\pi/2^-} \frac{\sec x}{\tan x}$$

$$= \lim_{x\to\pi/2^-} \frac{1}{\sin x}$$

$$\underline{=1}$$

***Exercise***

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\infty} \dfrac{4x^3 - 6x^2 + 1}{2x^3 - 10x + 3}$

***Solution***

$$\lim_{x\to\infty} \frac{4x^3 - 6x^2 + 1}{2x^3 - 10x + 3} \underline{= 2}$$

***Exercise***

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 0} \dfrac{3\sin 4x}{5x}$

***Solution***

$$\lim_{x\to 0} \frac{3\sin 4x}{5x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{12\cos 4x}{5}$$

$$\underline{= \frac{12}{5}}$$

### *Exercise*

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 2\pi} \dfrac{x\sin x + x^2 - 4\pi^2}{x - 2\pi}$

***Solution***

$$\lim\limits_{x\to 2\pi} \frac{x\sin x + x^2 - 4\pi^2}{x - 2\pi} = \frac{0}{0}$$

$$= \lim\limits_{x\to 2\pi} \frac{\sin x + x\cos x + 2x}{1}$$

$$= 2\pi + 4\pi$$

$$= 6\pi$$

### *Exercise*

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 0} \dfrac{\tan 4x}{\tan 7x}$

***Solution***

$$\lim\limits_{x\to 0} \frac{\tan 4x}{\tan 7x} = \frac{0}{0}$$

$$= \lim\limits_{x\to 0} \frac{4\sec^2 4x}{7\sec^2 7x}$$

$$= \frac{4}{7}$$

### *Exercise*

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 0} \dfrac{\sin^2 3x}{x^2}$

***Solution***

$$\lim\limits_{x\to 0} \frac{\sin^2 3x}{x^2} = \frac{0}{0}$$

$$= \lim\limits_{x\to 0} \left(\frac{\sin 3x}{x}\right)^2 \left(\frac{3}{3}\right)^2$$

$$= 9 \lim\limits_{x\to 0} \left(\frac{\sin 3x}{3x}\right)^2 \qquad \lim\limits_{x\to 0} \frac{\sin ax}{ax} = 1$$

$$= 9$$

## *Exercise*

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to -1} \dfrac{x^3 - x^2 - 5x - 3}{x^4 + 2x^3 - x^2 - 4x - 2}$

**Solution**

$$\lim_{x\to -1} \frac{x^3 - x^2 - 5x - 3}{x^4 + 2x^3 - x^2 - 4x - 2} = \frac{0}{0}$$

$$= \lim_{x\to -1} \frac{3x^2 - 2x - 5}{4x^3 + 6x^2 - 2x - 4}$$

$$= \lim_{x\to -1} \frac{3x^2 - 2x - 5}{4x^3 + 6x^2 - 2x - 4} = \frac{0}{0}$$

$$= \lim_{x\to -1} \frac{6x - 2}{12x^2 + 12x - 2}$$

$$= 4$$

## *Exercise*

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 1} \dfrac{x^n - 1}{x - 1} \quad (n > 0)$

**Solution**

$$\lim_{x\to 1} \frac{x^n - 1}{x - 1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{nx^{n-1}}{1}$$

$$= n$$

## *Exercise*

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 1^-} (1 - x)\tan\left(\dfrac{\pi x}{2}\right)$

**Solution**

$$\lim_{x\to 1^-} (1 - x)\tan\left(\frac{\pi x}{2}\right) = 0 \cdot \infty$$

$$= \lim_{x\to 1^-} \frac{1 - x}{\cot\left(\frac{\pi x}{2}\right)}$$

$$= \lim_{x\to 1^-} \frac{-1}{-\frac{\pi}{2}\csc^2\left(\frac{\pi x}{2}\right)}$$

$$= \frac{2}{\pi}$$

### Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\infty} \frac{3}{x}\csc\frac{5}{x}$

**Solution**

$$\lim_{x\to\infty} \frac{3}{x}\csc\frac{5}{x} = 0\cdot\infty$$

$$= 3\lim_{y\to 0} \frac{y}{\sin 5y} \qquad\qquad y = \frac{1}{x}$$

$$= 3\lim_{y\to 0} \frac{1}{5\cos 5y}$$

$$= \frac{3}{5}$$

### Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\pi/4} \frac{\tan x - \cot x}{x - \frac{\pi}{4}}$

**Solution**

$$\lim_{x\to\pi/4} \frac{\tan x - \cot x}{x - \frac{\pi}{4}} = \frac{0}{0}$$

$$= \lim_{x\to\pi/4} \frac{\sec^2 x + \csc^2 x}{1}$$

$$= 2 + 2$$

$$= 4$$

### Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 0} \frac{1-\cos 3x}{8x^2}$

**Solution**

$$\lim_{x\to 0} \frac{1-\cos 3x}{8x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{3\sin 3x}{16x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{9\cos 3x}{16}$$

$$= \frac{9}{16}$$

### *Exercise*

Apply l'Hôpital Rule to evaluate $\lim\limits_{x \to 3} \dfrac{x-1-\sqrt{x^2-5}}{x-3}$

***Solution***

$$\lim_{x \to 3} \frac{x-1-\sqrt{x^2-5}}{x-3} = \frac{0}{0}$$

$$= \lim_{x \to 3} \frac{1-\dfrac{x}{\sqrt{x^2-5}}}{1}$$

$$= 1-\frac{3}{2}$$

$$= -\frac{1}{2}$$

### *Exercise*

Apply l'Hôpital Rule to evaluate $\lim\limits_{x \to 2} \dfrac{x^2+x-6}{\sqrt{8-x^2}-x}$

***Solution***

$$\lim_{x \to 2} \frac{x^2+x-6}{\sqrt{8-x^2}-x} = \frac{0}{0}$$

$$= \lim_{x \to 2} \frac{2x+1}{\dfrac{-x}{\sqrt{8-x^2}}-1}$$

$$= \frac{5}{-1-1}$$

$$= -\frac{5}{2}$$

### *Exercise*

Apply l'Hôpital Rule to evaluate $\lim\limits_{h \to 0} \dfrac{\sin(x+h)-\sin x}{h}$ *x is a real number*

***Solution***

$$\lim_{h \to 0} \frac{\sin(x+h)-\sin x}{h} = \frac{0}{0}$$

$$= \lim_{h \to 0} \frac{\cos(x+h)}{1}$$

$$= \cos x$$

### Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 2} \dfrac{\sqrt[3]{3x+2}-2}{x-2}$

**Solution**

$$\lim_{x\to 2} \frac{\sqrt[3]{3x+2}-2}{x-2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(3x+2)^{-2/3}}{1}$$

$$= \frac{1}{4}$$

### Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\infty} \dfrac{3x^4-x^2}{6x^4+12}$

**Solution**

$$\lim_{x\to\infty} \frac{3x^4-x^2}{6x^4+12} = \frac{1}{2}$$

### Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\infty} \dfrac{4x^3-2x^2+6}{\pi x^3+4}$

**Solution**

$$\lim_{x\to\infty} \frac{4x^3-2x^2+6}{\pi x^3+4} = \frac{4}{\pi}$$

### Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\infty} \dfrac{8-4x^2}{3x^3+x-1}$

**Solution**

$$\lim_{x\to\infty} \frac{8-4x^2}{3x^3+x-1} = 0$$

### Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\pi/2} \dfrac{2\tan x}{\sec^2 x}$

**Solution**

$$\lim_{x\to\pi/2} \frac{2\tan x}{\sec^2 x} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\pi/2} \frac{2\sec^2 x}{2\sec^2 x \tan x}$$

$$= \lim_{x\to\pi/2} \cot x$$

$$\boxed{= 0}$$

## *Exercise*

Apply l'Hôpital Rule to evaluate $\lim_{x\to 0} \frac{e^x - x - 1}{5x^2}$

***Solution***

$$\lim_{x\to 0} \frac{e^x - x - 1}{5x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{e^x - 1}{10x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{e^x}{10}$$

$$\boxed{= \frac{1}{10}}$$

## *Exercise*

Apply l'Hôpital Rule to evaluate $\lim_{x\to 0} \frac{e^x - \sin x - 1}{x^4 + 8x^3 + 12x^2}$

***Solution***

$$\lim_{x\to 0} \frac{e^x - \sin x - 1}{x^4 + 8x^3 + 12x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{e^x - \cos x}{4x^3 + 24x^2 + 24x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{e^x + \sin x}{12x^2 + 48x + 24}$$

$$\boxed{= \frac{1}{24}}$$

## Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\infty} \dfrac{e^{1/x}-1}{1/x}$

**Solution**

$$\lim\limits_{x\to\infty} \frac{e^{1/x}-1}{1/x} = \frac{0}{0}$$

$$= \lim\limits_{x\to\infty} \frac{\frac{-1}{x^2}e^{1/x}}{-1/x^2}$$

$$= \lim\limits_{x\to\infty} e^{1/x}$$

$$\boxed{=1}$$

## Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\infty} \dfrac{e^{3x}}{3e^{3x}+5}$

**Solution**

$$\lim\limits_{x\to\infty} \frac{e^{3x}}{3e^{3x}+5} = \frac{\infty}{\infty}$$

$$= \lim\limits_{x\to\infty} \frac{3e^{3x}}{9e^{3x}}$$

$$\boxed{=\frac{1}{3}}$$

## Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\infty} \dfrac{\ln\left(3x+5e^{x}\right)}{\ln\left(7x+3e^{2x}\right)}$

**Solution**

$$\lim\limits_{x\to\infty} \frac{\ln\left(3x+5e^{x}\right)}{\ln\left(7x+3e^{2x}\right)} = \frac{\infty}{\infty}$$

$$= \lim\limits_{x\to\infty} \frac{3+5e^{x}}{3x+5e^{x}} \cdot \lim\limits_{x\to\infty} \frac{7x+3e^{2x}}{7+6e^{2x}}$$

$$= \lim\limits_{x\to\infty} \frac{5e^{x}}{3+5e^{x}} \cdot \lim\limits_{x\to\infty} \frac{7+6e^{2x}}{12e^{2x}}$$

$$= \lim\limits_{x\to\infty} \frac{5e^{x}}{5e^{x}} \cdot \lim\limits_{x\to\infty} \frac{6e^{2x}}{12e^{2x}}$$

$$=1\cdot\frac{1}{2}$$

$$\boxed{=\frac{1}{2}}$$

### Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\infty}\dfrac{x^2-\ln\left(\frac{2}{x}\right)}{3x^2+2x}$

**Solution**

$$\lim_{x\to\infty}\frac{x^2-\ln\left(\frac{2}{x}\right)}{3x^2+2x}=\lim_{x\to\infty}\frac{2x-\left(\frac{x}{2}\right)\left(\frac{-1}{x^2}\right)}{6x+2}$$

$$=\lim_{x\to\infty}\frac{2x+\frac{1}{x}}{6x+2}$$

$$\boxed{=\frac{1}{3}}$$

### Exercise

Find $\lim\limits_{x\to1^+} x^{1/(x-1)}$

**Solution**

$$\lim_{x\to1^+} x^{1/(x-1)}=1^\infty$$

$$f(x)=x^{1/(x-1)} \quad\Rightarrow \ln f(x)=\ln x^{1/(x-1)}=\boxed{\frac{1}{x-1}\ln x}$$

$$\lim_{x\to1^+}\ln f(x)=\lim_{x\to1^+}\frac{\ln x}{x-1}=\lim_{x\to1^+}\frac{\frac{1}{x}}{1}=1$$

$$\lim_{x\to1^+} x^{1/(x-1)}=\lim_{x\to1^+} f(x)$$

$$=\lim_{x\to1^+} e^{\ln f(x)}$$

$$=\lim_{x\to1^+} e^{1}$$

$$\boxed{=e}$$

### *Exercise*

Find $\lim\limits_{x\to e^+} (\ln x)^{1/(x-e)}$

***Solution***

$$\lim_{x\to e^+} (\ln x)^{1/(x-e)} = 1^\infty$$

$$f(x) = (\ln x)^{1/(x-e)}$$

$$\ln f(x) = \ln\left( (\ln x)^{1/(x-e)} \right)$$

$$= \frac{1}{x-e}\ln(\ln x)$$

$$\lim_{x\to e^+} \ln f(x) = \lim_{x\to e^+} \frac{\ln(\ln x)}{x-e}$$

$$= \lim_{x\to e^+} \frac{\frac{1}{x\ln x}}{1}$$

$$= \lim_{x\to e^+} \frac{1}{x\ln x}$$

$$= \frac{1}{e\ln e}$$

$$= \frac{1}{e}$$

$$\lim_{x\to e^+} (\ln x)^{1/(x-e)} = \lim_{x\to e^+} f(x)$$

$$= \lim_{x\to e^+} e^{\ln f(x)}$$

$$= e^{1/e}$$

### *Exercise*

Find $\lim\limits_{x\to\infty} (1+2x)^{1/(2\ln x)}$

***Solution***

$$\lim_{x\to\infty} (1+2x)^{1/(2\ln x)} = \infty^0$$

$$f(x) = (1+2x)^{1/(2\ln x)} \quad \Rightarrow \ln f(x) = \ln\left( (1+2x)^{1/(2\ln x)} \right) = \frac{1}{2\ln x}\ln(1+2x)$$

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln(1+2x)}{2\ln x}$$

$$= \lim_{x\to\infty} \frac{\frac{2}{1+2x}}{\frac{2}{x}}$$

$$= \lim_{x\to\infty} \frac{x}{1+2x}$$

$$= \frac{1}{2}$$

$$\lim_{x\to\infty} (1+2x)^{1/(2\ln x)} = \lim_{x\to\infty} e^{\ln f(x)}$$

$$= e^{1/2}$$

## *Exercise*

Find $\lim_{x\to\infty} \left(\frac{x^2+1}{x+2}\right)^{1/x}$

***Solution***

$$\lim_{x\to\infty} \left(\frac{x^2+1}{x+2}\right)^{1/x} = \infty^0$$

$$f(x) = \left(\frac{x^2+1}{x+2}\right)^{1/x} \quad \Rightarrow \ln f(x) = \ln\left(\left(\frac{x^2+1}{x+2}\right)^{1/x}\right) = \frac{1}{x}\ln\left(\frac{x^2+1}{x+2}\right)$$

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{1}{x}\ln\left(\frac{x^2+1}{x+2}\right)$$

$$= \lim_{x\to\infty} \frac{\ln\left(x^2+1\right) - \ln(x+2)}{x}$$

$$= \lim_{x\to\infty} \frac{\frac{2x}{x^2+1} - \frac{1}{x+2}}{1}$$

$$= \lim_{x\to\infty} \frac{2x^2+4x-x^2-1}{\left(x^2+1\right)(x+2)}$$

$$= \lim_{x\to\infty} \frac{x^2+4x-1}{x^3+2x^2+x+2} \qquad \lim_{x\to\infty} \frac{x^2+4x-1}{x^3+2x^2+x+2} = 0$$

$$= \lim_{x\to\infty} \frac{2x+4}{3x^2+4x+1}$$

$$= \lim_{x\to\infty} \frac{2}{6x+4}$$

$$= 0$$

$$\lim_{x\to\infty} \left(\frac{x^2+1}{x+2}\right)^{1/x} = \lim_{x\to\infty} e^{\ln f(x)}$$

$$= e^0$$

$$= 1$$

## *Exercise*

Evaluate the following limit $\displaystyle\lim_{t\to 2} \frac{t^3-t^2-2t}{t^2-4}$

***Solution***

$$\lim_{t\to 2} \frac{t^3-t^2-2t}{t^2-4} = \frac{0}{0}$$

$$= \lim_{t\to 2} \frac{3t^2-2t-2}{2t}$$

$$= \frac{3}{2}$$

## *Exercise*

Evaluate the following limit $\displaystyle\lim_{x\to 0} \frac{1-\cos 6x}{2x}$

***Solution***

$$\lim_{x\to 0} \frac{1-\cos 6x}{2x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{6\sin 6x}{2}$$

$$= 0$$

## *Exercise*

Evaluate the following limit $\displaystyle\lim_{x\to\infty} \frac{5x^2+2x-5}{\sqrt{x^4-1}}$

***Solution***

$$\lim_{x\to\infty} \frac{5x^2+2x-5}{\sqrt{x^4-1}} = \lim_{x\to\infty} \frac{5x^2}{x^2}$$

$$= 5$$

## *Exercise*

Evaluate the following limit $\lim\limits_{\theta\to 0} \dfrac{3\sin^2 2\theta}{\theta^2}$

***Solution***

$$\lim_{\theta\to 0} \frac{3\sin^2 2\theta}{\theta^2} = \frac{0}{0}$$

$$= \lim_{\theta\to 0} \frac{12\sin 2\theta\cos 2\theta}{2\theta}$$

$$= 12 \lim_{2\theta\to 0} \frac{\sin 2\theta}{2\theta} \lim_{\theta\to 0} \cos 2\theta \qquad\qquad \lim_{2\theta\to 0} \frac{\sin 2\theta}{2\theta} = 1$$

$$\underline{= 12}$$

## *Exercise*

Evaluate the following limit $\lim\limits_{x\to\infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2-x}\right)$

***Solution***

$$\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2-x}\right) = \infty - \infty$$

$$= \lim_{x\to\infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2-x}\right)\frac{\sqrt{x^2+x+1} + \sqrt{x^2-x}}{\sqrt{x^2+x+1} + \sqrt{x^2-x}}$$

$$= \lim_{x\to\infty} \frac{x^2+x+1-x^2+x}{\sqrt{x^2+x+1} + \sqrt{x^2-x}}$$

$$= \lim_{x\to\infty} \frac{2x}{\sqrt{x^2+x+1} + \sqrt{x^2-x}}$$

$$= \lim_{x\to\infty} \frac{2x}{x+x}$$

$$\underline{= 1}$$

## *Exercise*

Evaluate the following limit $\lim\limits_{\theta\to 0} 2\theta\cot 3\theta$

***Solution***

$$\lim_{\theta\to 0} 2\theta\cot 3\theta = 0\cdot\infty = \frac{0}{0}$$

$$= \lim_{\theta\to 0} 2\theta\frac{\cos 3\theta}{\sin 3\theta}$$

$$= \lim_{\theta\to 0} \frac{2\theta}{\sin 3\theta} \lim_{\theta\to 0} \cos 3\theta$$

$$\underline{= \frac{2}{3}}$$

## Exercise

Evaluate the following limit $\lim\limits_{x\to 0} \dfrac{e^{-2x}-1+2x}{x^2}$

***Solution***

$$\lim_{x\to 0} \frac{e^{-2x}-1+2x}{x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{-2e^{-2x}+2}{2x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{4e^{-2x}}{2}$$

$$= 2$$

## Exercise

Evaluate the following limit $\lim\limits_{x\to 1} \dfrac{x^4-x^3-3x^2+5x-2}{x^3+x^2-5x+3}$

***Solution***

$$\lim_{x\to 1} \frac{x^4-x^3-3x^2+5x-2}{x^3+x^2-5x+3} = \frac{1-1-3+5-2}{1+1-5+3} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{4x^3-3x^2-6x+5}{3x^2+2x-5} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{12x^2-6x-6}{6x+2}$$

$$= 0$$

## Exercise

Evaluate the following limit $\lim\limits_{y\to 0^+} \dfrac{\ln^{10} y}{\sqrt{y}}$

***Solution***

$$\lim_{y\to 0^+} \frac{\ln^{10} y}{\sqrt{y}} = \frac{\infty}{0}$$

$$= \lim_{x\to \infty} \frac{\left(\ln \frac{1}{x}\right)^{10}}{1/\sqrt{x}} \qquad \text{Let } y = \frac{1}{x}$$

$$= \lim_{x\to \infty} \frac{\left(\ln \frac{1}{x}\right)^{10}}{1/\sqrt{x}}$$

$$= \lim_{x\to\infty} \sqrt{x}\,(-\ln x)^{10}$$

$$\underline{= \infty}$$

### *Exercise*

Evaluate the following limit $\quad \lim_{\theta\to 0} \dfrac{3\sin 8\theta}{8\sin 3\theta}$

***Solution***

$$\lim_{\theta\to 0} \frac{3\sin 8\theta}{8\sin 3\theta} = \frac{0}{0}$$

$$= \frac{3}{8}\frac{8}{3}$$

$$\underline{=1}$$

### *Exercise*

Evaluate the following limit $\quad \lim_{x\to\infty} \dfrac{\ln x^{100}}{\sqrt{x}}$

***Solution***

$$\lim_{x\to\infty} \frac{\ln x^{100}}{\sqrt{x}} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{100(\ln x)}{\sqrt{x}}$$

$$= 100 \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}$$

$$= 200 \lim_{x\to\infty} \frac{1}{\sqrt{x}}$$

$$\underline{=0}$$

### *Exercise*

Evaluate the following limit $\quad \lim_{x\to 0} \csc x \sin^{-1} x$

***Solution***

$$\lim_{x\to 0} \csc x \sin^{-1} x = \lim_{x\to 0} \frac{\sin^{-1} x}{\sin x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\cos x}$$

$$\underline{=1}$$

### *Exercise*

Evaluate the following limit $\displaystyle\lim_{x\to\infty} \frac{\ln^3 x}{\sqrt{x}}$

***Solution***

$$\lim_{x\to\infty} \frac{\ln^3 x}{\sqrt{x}} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{3\ln^2 x}{x}\left(2\sqrt{x}\right) \qquad \left(\ln^3 x\right)' = \frac{3\ln^2 x}{x}$$

$$= 6\lim_{x\to\infty} \frac{\ln^2 x}{\sqrt{x}} = \frac{\infty}{\infty}$$

$$= 6\lim_{x\to\infty} \frac{2\ln x}{x}\left(2\sqrt{x}\right)$$

$$= 24\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$$

$$= 24\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}$$

$$= 48\lim_{x\to\infty} \frac{1}{\sqrt{x}}$$

$$\underline{= 0}$$

### *Exercise*

Evaluate the following limit $\displaystyle\lim_{x\to\infty} \ln\left(\frac{x+1}{x-1}\right)$

***Solution***

$$\lim_{x\to\infty} \ln\left(\frac{x+1}{x-1}\right) = \lim_{x\to\infty} \ln(1)$$

$$\underline{= 0}$$

### *Exercise*

Evaluate the following limit $\displaystyle\lim_{x\to 0^+} (1+x)^{\cot x}$

***Solution***

$$\lim_{x\to 0^+} (1+x)^{\cot x} = 1^{\infty}$$

$$\lim_{x\to 0^+} \ln(1+x)^{\cot x} = \lim_{x\to 0^+} (\cot x)\ln(1+x)$$

$$= \lim_{x\to 0^+} \frac{\ln(1+x)}{\tan x} = \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{\sec^2 x}$$

$$\underline{=1}\rfloor$$

$$\lim_{x\to 0^+} (1+x)^{\cot x} = e^1 \ \underline{= e}\rfloor$$

## *Exercise*

Evaluate the following limit $\lim\limits_{x\to \frac{\pi}{2}^+} (\sin x)^{\tan x}$

***Solution***

$$\lim_{x\to \frac{\pi}{2}^+} (\sin x)^{\tan x} = 1^\infty$$

$$\lim_{x\to \frac{\pi}{2}^+} \ln(\sin x)^{\tan x} = \lim_{x\to \frac{\pi}{2}^+} (\tan x)\ln(\sin x)$$

$$= \lim_{x\to \frac{\pi}{2}^+} \frac{\ln(\sin x)}{\cot x}$$

$$= \lim_{x\to \frac{\pi}{2}^+} \frac{\frac{\cos x}{\sin x}}{-\csc^2 x}$$

$$= -\lim_{x\to \frac{\pi}{2}^+} \frac{\cos x}{\sin x}\sin^2 x$$

$$= -\lim_{x\to \frac{\pi}{2}^+} (\cos x \sin x)$$

$$\underline{=0}\rfloor$$

$$\lim_{x\to \frac{\pi}{2}^+} (\sin x)^{\tan x} = e^0 \ \underline{=1}\rfloor$$

### Exercise

Evaluate the following limit $\lim\limits_{x\to\infty} \left(\sqrt{x}+1\right)^{1/x}$

***Solution***

$$\lim_{x\to\infty} \left(\sqrt{x}+1\right)^{1/x} = \infty^0$$

$$\begin{aligned}\lim_{x\to\infty} \ln\left(\sqrt{x}+1\right)^{1/x} &= \lim_{x\to\infty} \frac{1}{x}\ln\left(\sqrt{x}+1\right)\\ &= \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}\frac{1}{\sqrt{x}+1}}{1}\\ &= 0\end{aligned}$$

$$\lim_{x\to\infty} \left(\sqrt{x}+1\right)^{1/x} = e^0 = 1$$

### Exercise

Evaluate the following limit $\lim\limits_{x\to 0^+} |\ln x|^x$

***Solution***

$$\lim_{x\to 0^+} |\ln x|^x = -\infty^0$$

$$\begin{aligned}\lim_{x\to 0^+} \ln|\ln x|^x &= \lim_{x\to 0^+} x\ln|\ln x|\\ &= \lim_{x\to 0^+} \frac{\ln|\ln x|}{\frac{1}{x}} = \frac{\infty}{\infty}\\ &= \lim_{x\to 0^+} \frac{\frac{1}{x|\ln x|}}{-\frac{1}{x^2}}\\ &= -\lim_{x\to 0^+} \frac{x}{|\ln x|}\\ &= 0\end{aligned}$$

$$\lim_{x\to 0^+} |\ln x|^x = e^0 = 1$$

## *Exercise*

Evaluate the following limit $\lim\limits_{x\to\infty} x^{1/x}$

***Solution***

$$\lim_{x\to\infty} x^{1/x} = \infty^0$$

$$\lim_{x\to\infty} \ln x^{1/x} = \lim_{x\to\infty} \frac{1}{x}\ln x$$

$$= \lim_{x\to\infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$\boxed{= 0}$$

$$\lim_{x\to\infty} x^{1/x} = e^0 \; \boxed{= 1}$$

## *Exercise*

Evaluate the following limit $\lim\limits_{x\to\infty} \left(1-\frac{3}{x}\right)^x$

***Solution***

$$\lim_{x\to\infty} \left(1-\frac{3}{x}\right)^x = 1^\infty$$

$$\lim_{x\to\infty} \ln\left(1-\frac{3}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1-\frac{3}{x}\right)$$

$$= \lim_{x\to\infty} \frac{1}{\frac{1}{x}}\ln\left(1-\frac{3}{x}\right) = \frac{0}{0}$$

$$= \lim_{x\to\infty} \frac{3}{x^2}\frac{1}{1-\frac{3}{x}} \bullet \frac{1}{-\frac{1}{x^2}}$$

$$= -3\lim_{x\to\infty} \frac{1}{1-\frac{3}{x}}$$

$$\boxed{= -3}$$

$$\lim_{x\to\infty} \left(1-\frac{3}{x}\right)^x \; \boxed{= e^{-3}}$$

## *Exercise*

Evaluate the following limit $\lim\limits_{x\to\infty}\left(\frac{2}{\pi}\tan^{-1}x\right)^x$

***Solution***

$$\lim_{x\to\infty}\left(\frac{2}{\pi}\tan^{-1}x\right)^x = 1^\infty$$

$$\begin{aligned}
\lim_{x\to\infty}\ln\left(\frac{2}{\pi}\tan^{-1}x\right)^x &= \lim_{x\to\infty} x\ln\left(\frac{2}{\pi}\tan^{-1}x\right) \\
&= \lim_{x\to\infty}\frac{1}{\frac{1}{x}}\ln\left(\frac{2}{\pi}\tan^{-1}x\right) \\
&= \lim_{x\to\infty}\frac{1}{-\frac{1}{x^2}}\cdot\frac{\frac{2}{\pi}\frac{1}{1+x^2}}{\frac{2}{\pi}\tan^{-1}x} \\
&= -\lim_{x\to\infty}\frac{x^2}{\left(1+x^2\right)\tan^{-1}x} \qquad \lim_{x\to\infty}\frac{x^2}{1+x^2}=1 \\
&= -\frac{1}{\frac{\pi}{2}} \\
&= \boxed{-\frac{2}{\pi}}
\end{aligned}$$

$$\lim_{x\to\infty}\left(\frac{2}{\pi}\tan^{-1}x\right)^x = \boxed{e^{-\frac{2}{\pi}}}$$

## *Exercise*

Evaluate the following limit $\lim\limits_{x\to 1}(x-1)^{\sin\pi x}$

***Solution***

$$\lim_{x\to 1}(x-1)^{\sin\pi x} = 0^0$$

$$\begin{aligned}
\lim_{x\to 1}\ln(x-1)^{\sin\pi x} &= \lim_{x\to 1}(\sin\pi x)\ln(x-1) \\
&= \lim_{x\to 1}\frac{\ln(x-1)}{\csc\pi x} \\
&= \lim_{x\to 1}\frac{\frac{1}{x-1}}{-\pi\csc\pi x\ \cot\pi x} \\
&= -\frac{1}{\pi}\lim_{x\to 1}\frac{1}{x-1}\frac{\sin^2\pi x}{\cos\pi x}
\end{aligned}$$

$$= -\frac{1}{\pi}\lim_{x\to 1}\frac{1}{\cos\pi x}\bullet\lim_{x\to 1}\frac{\sin^2\pi x}{x-1}$$

$$= \frac{1}{\pi}\lim_{x\to 1}\frac{2\pi\sin\pi x\cos\pi x}{1}$$

$$\underline{= 0}$$

$$\lim_{x\to 1}(x-1)^{\sin\pi x} = e^0 \ \underline{=1}$$

## Exercise

Evaluate the following limit $\displaystyle\lim_{x\to\infty}\frac{2x^5-x+1}{5x^6+x}$

**Solution**

$$\lim_{x\to\infty}\frac{2x^5-x+1}{5x^6+x} = \lim_{x\to\infty}\frac{2x^5}{5x^6}$$

$$\underline{= 0}$$

## Exercise

Evaluate the following limit $\displaystyle\lim_{x\to\infty}\frac{4x^4-\sqrt{x}}{2x^4+x^{-1}}$

**Solution**

$$\lim_{x\to\infty}\frac{4x^4-\sqrt{x}}{2x^4+x^{-1}} \ \underline{= 2}$$

## Exercise

Evaluate the following limit $\displaystyle\lim_{x\to 0}\frac{1-\cos x^n}{x^{2n}}$

**Solution**

$$\lim_{x\to 0}\frac{1-\cos x^n}{x^{2n}} = \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{nx^{n-1}\sin x^n}{2nx^{2n-1}}$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{\sin x^n}{x^n}$$

$$= \frac{1}{2}\lim_{x^n\to 0}\frac{\sin x^n}{x^n} \qquad \lim_{u\to 0}\frac{\sin u}{u} = 1$$

$$\underline{= \frac{1}{2}}$$

## Exercise

Evaluate the following limit $\lim_{x\to 0} \frac{1-\cos^n x}{x^2}$

**Solution**

$$\lim_{x\to 0} \frac{1-\cos^n x}{x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{n(\sin x)\cos^{n-1} x}{2x} = \frac{0}{0}$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{n(\cos x)\cos^{n-1} x - n(n-1)\sin^2 x \cos^{n-2} x}{1}$$

$$= \frac{n}{2}$$

## Exercise

Evaluate the following limit $\lim_{x\to 0} \frac{1-\cos x^n}{x^2}$

**Solution**

$$\lim_{x\to 0} \frac{1-\cos x^n}{x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{nx^{n-1}\sin x^n}{2x} = \frac{0}{0}$$

$$= \frac{1}{2} \lim_{x\to 0} x^{n-2} \sin x^n$$

$$= 0$$

## Exercise

Evaluate the following limit $\lim_{x\to 0} \frac{3x}{\tan 4x}$

**Solution**

$$\lim_{x\to 0} \frac{3x}{\tan 4x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{3}{4\sec^2 4x}$$

$$= \frac{3}{4}$$

### *Exercise*

Evaluate the following limit $\lim\limits_{x\to 0}\dfrac{\sin ax}{\sin bx}$

***Solution***

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{a\cos ax}{b\cos bx}$$

$$=\frac{a}{b}$$

### *Exercise*

Evaluate the following limit $\lim\limits_{x\to 2}\dfrac{\ln(2x-3)}{x^2-4}$

***Solution***

$$\lim_{x\to 2}\frac{\ln(2x-3)}{x^2-4}=\frac{0}{0}$$

$$=\lim_{x\to 2}\frac{\frac{2}{2x-3}}{2x}$$

$$=\lim_{x\to 2}\frac{1}{x(2x-3)}$$

$$=\frac{1}{2}$$

### *Exercise*

Evaluate the following limit $\lim\limits_{x\to 0}\dfrac{1-\cos ax}{1-\cos bx}$

***Solution***

$$\lim_{x\to 0}\frac{1-\cos ax}{1-\cos bx}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{a\sin ax}{b\sin bx}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{a^2\cos ax}{b^2\cos bx}$$

$$=\frac{a^2}{b^2}$$

## Exercise

Evaluate the following limit $\lim_{x\to 0} \frac{\sin^{-1} x}{\tan^{-1} x}$

***Solution***

$$\lim_{x\to 0} \frac{\sin^{-1} x}{\tan^{-1} x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{1}{1+x^2}}$$

$$= \lim_{x\to 0} \frac{1+x^2}{\sqrt{1-x^2}}$$

$$= 1$$

## Exercise

Evaluate the following limit $\lim_{x\to 1} \frac{x^{1/3}-1}{x^{2/3}-1}$

***Solution***

$$\lim_{x\to 1} \frac{x^{1/3}-1}{x^{2/3}-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{\frac{1}{3}x^{-2/3}}{\frac{2}{3}x^{-1/3}}$$

$$= \frac{1}{2}$$

## Exercise

Evaluate the following limit $\lim_{x\to 0} x\cot x$

***Solution***

$$\lim_{x\to 0} x\cot x = 0\cdot\infty$$

$$= \lim_{x\to 0} \frac{x\cos x}{\sin x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\cos x - x\sin x}{\cos x}$$

$$= 1$$

### Exercise

Evaluate the following limit $\lim_{x\to 0} \frac{1-\cos x}{\ln\left(1+x^2\right)}$

**Solution**

$$\lim_{x\to 0} \frac{1-\cos x}{\ln\left(1+x^2\right)} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{\frac{2x}{1+x^2}}$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x}\left(1+x^2\right) \qquad \lim_{u\to 0} \frac{\sin u}{u} = 1$$

$$= \frac{1}{2}$$

### Exercise

Evaluate the following limit $\lim_{x\to \pi} \frac{\sin^2 x}{x-\pi}$

**Solution**

$$\lim_{x\to \pi} \frac{\sin^2 x}{x-\pi} = \frac{0}{0}$$

$$= \lim_{x\to \pi} \frac{2\sin x \cos x}{1}$$

$$= 0$$

### Exercise

Evaluate the following limit $\lim_{x\to 0} \frac{10^x - e^x}{x}$

**Solution**

$$\lim_{x\to 0} \frac{10^x - e^x}{x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{10^x \ln 10 - e^x}{1}$$

$$= \ln(10) - 1$$

### Exercise

Evaluate the following limit $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cos 3x}{\pi - 2x}$

***Solution***

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos 3x}{\pi - 2x} = \frac{0}{0}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{-3\sin 3x}{-2}$$

$$= \boxed{\frac{3}{2}}$$

### Exercise

Evaluate the following limit $\lim\limits_{x \to 1} \dfrac{\ln(ex) - 1}{\sin \pi x}$

***Solution***

$$\lim_{x \to 1} \frac{\ln(ex) - 1}{\sin \pi x} = \frac{0}{0}$$

$$= \lim_{x \to 1} \frac{\frac{1}{x}}{\cos \pi x}$$

$$= \boxed{\frac{1}{\pi}}$$

### Exercise

Evaluate the following limit $\lim\limits_{x \to \infty} x \sin \dfrac{1}{x}$

***Solution***

$$\lim_{x \to \infty} x \sin \frac{1}{x} = \lim_{x \to \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = \frac{0}{0} \qquad y = \frac{1}{x} \xrightarrow[x \to \infty]{} 0$$

$$= \lim_{y \to 0} \frac{\sin y}{y}$$

$$= \boxed{1}$$

### Exercise

Evaluate the following limit $\lim\limits_{x \to 0} \dfrac{x - \sin x}{x^3}$

***Solution***

$$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{6x} = \frac{0}{0}$$

$$\boxed{= \frac{1}{6}}$$

## *Exercise*

Evaluate the following limit $\lim_{x\to 0} \frac{x-\sin x}{x-\tan x}$

**Solution**

$$\lim_{x\to 0} \frac{x-\sin x}{x-\tan x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1-\cos x}{1-\sec^2 x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{-2\sec^2 x \tan x} = \frac{0}{0}$$

$$= -\frac{1}{2} \lim_{x\to 0} \frac{\sin x}{\frac{1}{\cos^2 x}\frac{\sin x}{\cos x}}$$

$$= -\frac{1}{2} \lim_{x\to 0} \cos x$$

$$\boxed{= -\frac{1}{2}}$$

## *Exercise*

Evaluate the following limit $\lim_{x\to 0} \frac{2-x^2-2\cos x}{x^4}$

**Solution**

$$\lim_{x\to 0} \frac{2-x^2-2\cos x}{x^4} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{-2x+2\sin x}{4x^3} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{-2+2\cos x}{12x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{-2\sin x}{24x} = \frac{0}{0}$$

$$\boxed{= -\frac{1}{12}}$$

## Exercise

Evaluate the following limit $\lim\limits_{x\to 0^+} \dfrac{\sin^2 x}{\tan x - x}$

**Solution**

$$\lim_{x\to 0^+} \frac{\sin^2 x}{\tan x - x} = \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{2\sin x \cos x}{\sec^2 x - 1} = \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\sin 2x}{\tan^2 x} = \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{2\cos 2x}{2\tan x \sec^2 x}$$

$$= \frac{1}{0^+}$$

$$\underline{= \infty}$$

## Exercise

Evaluate the following limit $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\ln \sin x}{\cos x}$

**Solution**

$$\lim_{x\to \frac{\pi}{2}} \frac{\ln \sin x}{\cos x} = \frac{0}{0}$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{\frac{\cos x}{\sin x}}{-\sin x}$$

$$= -\lim_{x\to \frac{\pi}{2}} \frac{\cos x}{\sin^2 x}$$

$$\underline{= 0}$$

## Exercise

Evaluate the following limit $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\sin x}{x}$

**Solution**

$$\lim_{x\to \frac{\pi}{2}} \frac{\sin x}{x} = \frac{1}{\frac{\pi}{2}}$$

$$\underline{= \frac{2}{\pi}}$$

### *Exercise*

Evaluate the following limit $\lim\limits_{x\to 1^-} \dfrac{\arccos x}{x-1}$

***Solution***

$$\lim_{x\to 1^-} \frac{\arccos x}{x-1} = \frac{0}{0}$$

$$= \lim_{x\to 1^-} \frac{-\frac{1}{\sqrt{1-x^2}}}{1}$$

$$\underline{= -\infty}$$

### *Exercise*

Evaluate the following limit $\lim\limits_{x\to\infty} x\left(2\tan^{-1} x - \pi\right)$

***Solution***

$$\lim_{x\to\infty} x\left(2\tan^{-1} x - \pi\right) = \infty \cdot 0$$

$$= \lim_{x\to\infty} \left(\frac{2\tan^{-1} x}{x} - \frac{\pi}{x}\right)$$

$$= \lim_{x\to\infty} \left(\frac{2\tan^{-1} x}{x}\right) - \lim_{x\to\infty} \left(\frac{\pi}{x}\right)$$

$$= 2\lim_{x\to\infty} \left(\frac{\tan^{-1} x}{x}\right)$$

$$= 2\lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{1}$$

$$= 2\lim_{x\to\infty} \frac{1}{x^2}$$

$$\underline{= 0}$$

### *Exercise*

Evaluate the following limit $\lim\limits_{x\to\frac{\pi}{2}^+} x(\sec x - \tan x)$

***Solution***

$$\lim_{x\to\frac{\pi}{2}^+} x(\sec x - \tan x) = \infty - \infty$$

$$= \lim_{x\to\frac{\pi}{2}^+} x\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

$$= \lim_{x\to\frac{\pi}{2}^+} x\left(\frac{1-\sin x}{\cos x}\right) = \frac{0}{0}$$

$$= \lim_{x\to\frac{\pi}{2}^+} x\left(\frac{\cos x}{-\sin x}\right)$$

$$\underline{= 0}$$

## Exercise

Evaluate the following limit $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{xe^{ax}}\right)$

**Solution**

$$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{xe^{ax}}\right) = \infty - \infty$$

$$= \lim_{x\to 0} \frac{e^{ax}-1}{xe^{ax}} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{ae^{ax}}{(1+ax)e^{ax}}$$

$$\underline{= a}$$

## Exercise

Evaluate the following limit $\lim_{x\to 0^+} x^{\sqrt{x}}$

**Solution**

$$\lim_{x\to 0^+} x^{\sqrt{x}} = 0^0$$

$$\lim_{x\to 0^+} \ln x^{\sqrt{x}} = \lim_{x\to 0^+} \sqrt{x}\ln x$$

$$= \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}}$$

$$= -2\lim_{x\to 0^+} \sqrt{x}$$

$$\underline{= 0}$$

$$\lim_{x\to 0^+} x^{\sqrt{x}} = e^0 \ \underline{= 1}$$

## *Exercise*

Evaluate the following limit $\lim\limits_{x\to\pi} \frac{\cos x+1}{(x-\pi)^2}$

**Solution**

$$\lim\limits_{x\to\pi} \frac{\cos x+1}{(x-\pi)^2} = \frac{0}{0}$$

$$= \lim\limits_{x\to\pi} \frac{-\sin x}{2(x-\pi)} = \frac{0}{0}$$

$$= \lim\limits_{x\to\pi} \frac{-\cos x}{2}$$

$$= \frac{1}{2}$$

## *Exercise*

Evaluate the following limit $\lim\limits_{x\to 0} \frac{\sin x - x}{7x^3}$

**Solution**

$$\lim\limits_{x\to 0} \frac{\sin x - x}{7x^3} = \frac{0}{0}$$

$$= \lim\limits_{x\to 0} \frac{\cos x - 1}{21x^2} = \frac{0}{0}$$

$$= \lim\limits_{x\to 0} \frac{-\sin x}{42x} \qquad \lim\limits_{x\to 0} \frac{\sin x}{x} = 1$$

$$= -\frac{1}{42}$$

## *Exercise*

Evaluate the following limit $\lim\limits_{x\to\infty} \frac{\tan^{-1} x - \frac{\pi}{2}}{\frac{1}{x}}$

**Solution**

$$\lim\limits_{x\to\infty} \frac{\tan^{-1} x - \frac{\pi}{2}}{\frac{1}{x}} = \frac{\frac{\pi}{2} - \frac{\pi}{2}}{0} = \frac{0}{0}$$

$$= \lim\limits_{x\to\infty} \frac{\frac{1}{1+x^2}}{-\frac{1}{x^2}}$$

$$= -\lim\limits_{x\to\infty} \frac{x^2}{1+x^2}$$

$$= -1$$

## *Exercise*

Evaluate the following limit $\lim\limits_{x\to 3} \dfrac{x-1-\sqrt{x^2-5}}{x-3}$

***Solution***

$$\lim\limits_{x\to 3} \frac{x-1-\sqrt{x^2-5}}{x-3} = \frac{2-2}{0} = \frac{0}{0}$$

$$= \lim\limits_{x\to 3} \frac{1-\frac{x}{\sqrt{x^2-5}}}{1}$$

$$= 1-\frac{3}{2}$$

$$\boxed{= -\frac{1}{2}}$$

## *Exercise*

Evaluate the following limit $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{\sqrt{8-x^2}-x}$

***Solution***

$$\lim\limits_{x\to 2} \frac{x^2+x-6}{\sqrt{8-x^2}-x} = \frac{0}{0}$$

$$= \lim\limits_{x\to 2} \frac{2x+1}{\frac{-x}{\sqrt{8-x^2}}-1}$$

$$\boxed{= -\frac{5}{2}}$$

## *Exercise*

Evaluate the following limit $\lim\limits_{x\to 2} \dfrac{x^2-4x+4}{\sin^2 \pi x}$

***Solution***

$$\lim\limits_{x\to 2} \frac{x^2-4x+4}{\sin^2 \pi x} = \frac{0}{0}$$

$$= \lim\limits_{x\to 2} \frac{2x-4}{2\pi \sin \pi x \cos \pi x}$$

$$= \lim\limits_{x\to 2} \frac{2x-4}{\pi \sin 2\pi x} = \frac{0}{0}$$

$$= \lim\limits_{x\to 2} \frac{2}{2\pi^2 \cos 2\pi x}$$

$$\boxed{= \frac{1}{\pi^2}}$$

### *Exercise*

Evaluate the following limit $\lim\limits_{x\to 2} \dfrac{(3x+2)^{1/3}-2}{x-2}$

***Solution***

$$\lim_{x\to 2} \frac{(3x+2)^{1/3}-2}{x-2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(3x+2)^{-2/3}}{1}$$

$$= 8^{-2/3}$$

$$\boxed{= \frac{1}{4}}$$

### *Exercise*

Evaluate the following limit $\lim\limits_{x\to \infty} \dfrac{3x^4-x^2}{6x^4+12}$

***Solution***

$$\lim_{x\to \infty} \frac{3x^4-x^2}{6x^4+12} = \frac{3}{6} \boxed{= \frac{1}{2}}$$

### *Exercise*

Evaluate the following limit $\lim\limits_{x\to \infty} \dfrac{4x^3-2x^2+6}{\pi x^3+4}$

***Solution***

$$\lim_{x\to \infty} \frac{4x^3-2x^2+6}{\pi x^3+4} \boxed{= \frac{4}{\pi}}$$

### *Exercise*

Evaluate the following limit $\lim\limits_{x\to \frac{\pi}{2}^-} \dfrac{1}{3}(2x-\pi)\tan x$

***Solution***

$$\lim_{x\to \frac{\pi}{2}^-} \frac{1}{3}(2x-\pi)\tan x = 0\cdot\infty$$

$$= \frac{1}{3} \lim_{x\to \frac{\pi}{2}^-} \frac{\tan x}{\frac{1}{2x-\pi}} = \frac{0}{0}$$

$$= \frac{1}{3} \lim_{x \to \frac{\pi}{2}^-} \frac{\sec^2 x}{\frac{-2}{(2x-\pi)^2}}$$

$$= -\frac{1}{6} \lim_{x \to \frac{\pi}{2}^-} \frac{(2x-\pi)^2}{\cos^2 x} = \frac{0}{0}$$

$$= -\frac{1}{6} \lim_{x \to \frac{\pi}{2}^-} \frac{4(2x-\pi)}{-2\sin x \cos x}$$

$$= \frac{2}{3} \lim_{x \to \frac{\pi}{2}^-} \frac{2x-\pi}{\sin 2x} = \frac{0}{0}$$

$$= \frac{2}{3} \lim_{x \to \frac{\pi}{2}^-} \frac{2}{2\cos 2x}$$

$$= -\frac{2}{3}$$

## *Exercise*

Evaluate the following limit $\lim_{x \to \infty} x \ln\left(1 + \frac{1}{x}\right)$

***Solution***

$$\lim_{x \to \infty} x \ln\left(1 + \frac{1}{x}\right) = \lim_{x \to \infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \frac{0}{0}$$

$$= \lim_{x \to \infty} \frac{-\frac{1}{x^2}}{1 + \frac{1}{x}} \bullet \frac{1}{-\frac{1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{1}{1 + \frac{1}{x}} \qquad \lim_{x \to \infty} \frac{1}{x} = 0$$

$$= 1$$

## *Exercise*

Evaluate the following limit $\lim_{x \to \frac{\pi}{2}^-} \left(\frac{\pi}{2} - x\right) \sec x$

***Solution***

$$\lim_{x \to \frac{\pi}{2}^-} \left(\frac{\pi}{2} - x\right) \sec x = \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{\pi}{2} - x}{\cos x} = \frac{0}{0}$$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{-1}{-\sin x}$$

$$\underline{=1}$$

***Exercise***

Evaluate the following limit $\lim\limits_{x\to\infty} \dfrac{e^{1/x}-1}{\sin\frac{1}{x}}$

***Solution***

$$\lim_{x\to\infty} \frac{e^{1/x}-1}{\sin\frac{1}{x}} = \frac{0}{0}$$

$$= \lim_{x\to\infty} \frac{-\frac{1}{x^2}e^{1/x}}{-\frac{1}{x^2}\cos\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{e^{1/x}}{\cos\frac{1}{x}}$$

$$\underline{=1}$$

***Exercise***

Evaluate the following limit $\lim\limits_{x\to 0^+} \sin x\sqrt{\dfrac{1-x}{x}}$

***Solution***

$$\lim_{x\to 0^+} \sin x\sqrt{\frac{1-x}{x}} = \lim_{x\to 0^+} \sin x\sqrt{\frac{x(1-x)}{x^2}}$$

$$= \lim_{x\to 0^+} \frac{\sin x}{x}\sqrt{x(1-x)} \qquad \lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

$$= 1 \bullet 0$$

$$\underline{= 0}$$

***Exercise***

Evaluate the following limit $\lim\limits_{x\to 0} \left(\cot x - \dfrac{1}{x}\right)$

***Solution***

$$\lim_{x\to 0} \left(\cot x - \frac{1}{x}\right) = \infty - \infty$$

$$= \lim_{x\to 0} \left(\frac{\cos x}{\sin x} - \frac{1}{x}\right)$$

$$= \lim_{x\to 0} \left(\frac{x\cos x - \sin x}{x\sin x}\right) = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x}$$

$$= \lim_{x\to 0} \frac{-x\sin x}{\sin x + x\cos x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x}$$

$$= \frac{0}{2}$$

$$\underline{= 0}$$

## *Exercise*

Evaluate the following limit $\lim_{x\to\infty} \left(x - \sqrt{x^2+1}\right)$

***Solution***

$$\lim_{x\to\infty} \left(x - \sqrt{x^2+1}\right) = \infty - \infty$$

$$= \lim_{x\to\infty} \left(x - x\sqrt{1+\frac{1}{x^2}}\right)$$

$$= \lim_{x\to\infty} x\left(1 - \sqrt{1+\frac{1}{x^2}}\right) \qquad t = \frac{1}{x} \xrightarrow[x\to\infty]{} 0$$

$$= \lim_{t\to 0} \frac{1}{t}\left(1 - \sqrt{1+t^2}\right)$$

$$= \lim_{t\to 0} \frac{1-\sqrt{1+t^2}}{t} = \frac{0}{0}$$

$$= \lim_{t\to 0} \frac{\frac{t}{\sqrt{1+t^2}}}{1}$$

$$\underline{= 0}$$

## *Exercise*

Evaluate the following limit $\lim_{\theta\to\frac{\pi}{2}^-} \left(\tan\theta - \sec\theta\right)$

***Solution***

$$\lim_{\theta\to\frac{\pi}{2}^-}\left(\tan\theta-\sec\theta\right)=\infty-\infty$$

$$=\lim_{\theta\to\frac{\pi}{2}^-}\left(\frac{\sin\theta}{\cos\theta}-\frac{1}{\cos\theta}\right)$$

$$=\lim_{\theta\to\frac{\pi}{2}^-}\frac{\sin\theta-1}{\cos\theta}=\frac{0}{0}$$

$$=\lim_{\theta\to\frac{\pi}{2}^-}\frac{\cos\theta}{-\sin\theta}$$

$$=0$$

## *Exercise*

Evaluate the following limit $\lim_{x\to 0^+}\ln x^{2x}$

***Solution***

$$\lim_{x\to 0^+}\ln x^{2x}=\lim_{x\to 0^+}2x\ln x$$

$$=2\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}$$

$$=2\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$=-2\lim_{x\to 0^+}x$$

$$=0$$

## *Exercise*

Evaluate the following limit $\lim_{x\to 0}\ln(1+4x)^{3/x}$

***Solution***

$$\lim_{x\to 0}\ln(1+4x)^{3/x}=3\lim_{x\to 0}\frac{\ln(1+4x)}{x}=\frac{0}{0}$$

$$=3\lim_{x\to 0}\frac{\frac{4}{1+4x}}{1}$$

$$=12$$

### *Exercise*

Evaluate the following limit $\lim\limits_{\theta\to\frac{\pi}{2}^-} \ln(\tan\theta)^{\cos\theta}$

***Solution***

$$\lim_{\theta\to\frac{\pi}{2}^-} \ln(\tan\theta)^{\cos\theta} = \lim_{\theta\to\frac{\pi}{2}^-} \cos\theta\ln(\tan\theta)$$

$$= \lim_{\theta\to\frac{\pi}{2}^-} \frac{\ln(\tan\theta)}{\sec\theta}$$

$$= \lim_{\theta\to\frac{\pi}{2}^-} \frac{\frac{\sec^2\theta}{\tan\theta}}{\sec\theta\tan\theta}$$

$$= \lim_{\theta\to\frac{\pi}{2}^-} \frac{\sec\theta}{\tan^2\theta}$$

$$= \frac{0}{\infty}$$

$$\underline{= 0}$$

### *Exercise*

Evaluate the following limit $\lim\limits_{x\to0^+} (1+x)^{\cot x}$

***Solution***

$$\lim_{x\to0^+} (1+x)^{\cot x} = 1^\infty$$

$$\lim_{x\to0^+} \ln(1+x)^{\cot x} = \lim_{x\to0^+} \cot x\ln(1+x)$$

$$= \lim_{x\to0^+} \frac{\ln(1+x)}{\tan x} = \frac{0}{0}$$

$$= \lim_{x\to0^+} \frac{\frac{1}{1+x}}{\sec^2 x}$$

$$= \lim_{x\to0^+} \frac{1}{(1+x)\sec^2 x}$$

$$\underline{= 1}$$

$$\lim_{x\to0^+} (1+x)^{\cot x} = e^1 \underline{= e}$$

## *Exercise*

Evaluate the following limit $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{\ln x}$

***Solution***

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\ln x}=e^L$$

$$L=\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^{\ln x}$$

$$=\lim_{x\to\infty}(\ln x)\ln\left(1+\frac{1}{x}\right)$$

$$=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{\ln x}}$$

$$=\lim_{x\to\infty}\frac{-\frac{1}{x^2}}{1+\frac{1}{x}}\frac{1}{\frac{-1}{(\ln x)^2}\cdot\frac{1}{x}}$$

$$=\lim_{x\to\infty}\frac{1}{x^2}\frac{1}{\frac{x+1}{x}}x(\ln x)^2$$

$$=\lim_{x\to\infty}\frac{(\ln x)^2}{x+1}=\frac{\infty}{\infty}$$

$$=\lim_{x\to\infty}\frac{\frac{2\ln x}{x}}{1}$$

$$=2\lim_{x\to\infty}\frac{\ln x}{x}$$

$$=2\lim_{x\to\infty}\frac{\frac{1}{x}}{1}$$

$$=0$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\ln x}=e^0=1$$

## *Exercise*

Evaluate the following limit $\lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^{x}$

***Solution***

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{x}=e^L$$

$$L = \lim_{x\to\infty} \ln\left(1+\frac{a}{x}\right)^x$$

$$= \lim_{x\to\infty} x\ln\left(1+\frac{a}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{1}{-\frac{1}{x^2}}\frac{-\frac{a}{x^2}}{1+\frac{a}{x}}$$

$$= \lim_{x\to\infty} \frac{a}{1+\frac{a}{x}}$$

$$\underline{= a}$$

$$\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x \underline{= e^a}$$

## Exercise

Evaluate the following limit $\lim_{x\to 0} \left(e^{5x}+x\right)^{1/x}$

***Solution***

$$\lim_{x\to 0} \left(e^{5x}+x\right)^{1/x} = e^L$$

$$L = \lim_{x\to 0} \ln\left(e^{5x}+x\right)^{1/x}$$

$$= \lim_{x\to 0} \frac{\ln\left(e^{5x}+x\right)}{x}$$

$$= \lim_{x\to 0} \frac{5e^{5x}+1}{e^{5x}+x}\cdot\frac{1}{1}$$

$$\underline{= 6}$$

$$\lim_{x\to 0} \left(e^{5x}+x\right)^{1/x} \underline{= e^6}$$

### *Exercise*

Evaluate the following limit $\lim_{x\to 0}\left(e^{ax}+x\right)^{1/x}$

***Solution***

$$\lim_{x\to 0}\left(e^{ax}+x\right)^{1/x}=e^{L}$$

$$L=\lim_{x\to 0}\ln\left(e^{ax}+x\right)^{1/x}$$

$$=\lim_{x\to 0}\frac{\ln\left(e^{ax}+x\right)}{x}$$

$$=\lim_{x\to 0}\frac{ae^{ax}+1}{e^{ax}+x}\bullet\frac{1}{1}$$

$$\underline{=a+1}$$

$$\lim_{x\to 0}\left(e^{ax}+x\right)^{1/x}\ \underline{=e^{a+1}}$$

### *Exercise*

Evaluate the following limit $\lim_{x\to 0}\left(2^{ax}+x\right)^{1/x}$

***Solution***

$$\lim_{x\to 0}\left(2^{ax}+x\right)^{1/x}=e^{L}$$

$$L=\lim_{x\to 0}\ln\left(2^{ax}+x\right)^{1/x}$$

$$=\lim_{x\to 0}\frac{\ln\left(2^{ax}+x\right)}{x}$$

$$=\lim_{x\to 0}\frac{a2^{ax}\ln 2+1}{2^{ax}+x}\bullet\frac{1}{1}$$

$$\underline{=a\ln 2+1}$$

$$\lim_{x\to 0}\left(2^{ax}+x\right)^{1/x}=e^{a\ln 2+1}$$

$$=e\ e^{a\ln 2}$$

$$\underline{=e\cdot 2^{a}}$$

## ***Exercise***

Evaluate the following limit $\lim\limits_{x\to 0^+} (\tan x)^x$

***Solution***

$$\lim_{x\to 0^+} (\tan x)^x = e^L$$

$$\begin{aligned}
L &= \lim_{x\to 0^+} \ln(\tan x)^x \\
&= \lim_{x\to 0^+} x\ln(\tan x) \\
&= \lim_{x\to 0^+} \frac{\ln(\tan x)}{\frac{1}{x}} \\
&= \lim_{x\to 0^+} \frac{\sec^2 x}{\tan x} \bullet \frac{1}{-\frac{1}{x^2}} \\
&= -\lim_{x\to 0^+} \frac{x^2 \cos x}{\sin x \cos^2 x} \\
&= -\lim_{x\to 0^+} \frac{x^2}{\sin x \cos x} \\
&= -\lim_{x\to 0^+} \frac{x}{\sin x} \bullet \lim_{x\to 0^+} \frac{x}{\cos x} \\
&= -(1)\bullet(0) \\
&= 0
\end{aligned}$$

$$\lim_{x\to 0^+} (\tan x)^x = e^0 = 1$$

## ***Exercise***

The functions $f(x) = \left(x^x\right)^x$ *and* $g(x) = x^{\left(x^x\right)}$ are different functions. For example, $f(3) = 19{,}683$ and $g(3) \approx 7.6\times 10^{12}$. Determine whether $\lim\limits_{x\to 0^+} f(x)$ and $\lim\limits_{x\to 0^+} g(x)$ are intermediate forms and evaluate the limits.

***Solution***

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(x^x\right)^x$$

$$\lim_{x\to 0^+} \ln\left(x^x\right)^x = \lim_{x\to 0^+} x\ln\left(x^x\right)$$

$$= \lim_{x\to 0^+} x^2 \ln x$$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{2}{x^3}}$$

$$= -\lim_{x\to 0^+} \frac{x^2}{2}$$

$$\boxed{= 0}$$

$$\lim_{x\to 0^+} \left(x^x\right)^x = e^0 \ \boxed{= 1}$$

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} x^{\left(x^x\right)}$$

$$\lim_{x\to 0^+} \left(x^x\right) = e^L$$

$$L = \lim_{x\to 0^+} \ln\left(x^x\right)$$

$$= \lim_{x\to 0^+} x \ln x$$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} (-x)$$

$$\boxed{= 0}$$

$$\lim_{x\to 0^+} \left(x^x\right) = e^0 \ \boxed{= 1}$$

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} x^{\left(x^x\right)}$$

$$= \lim_{x\to 0^+} 0^1$$

$$\boxed{= 0}$$

## *Exercise*

Consider the function $g(x)=\left(1+\frac{1}{x}\right)^{x+a}$. show that if $0\le a<\frac{1}{2}$, then $g(x)\to e$ from below as $x\to\infty$; if $\frac{1}{2}\le a<1$, then $g(x)\to e$ from above as $x\to\infty$

***Solution***

$$\begin{aligned}\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+a} &= \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{a}\\ &= e\bullet 1\\ &= e\end{aligned}$$

$$\lim_{x\to\infty}\ln g(x)=\ln 2=1$$

It suffices to determine whether $\ln g(x)-1$ is $(+)$ or $(-)$ as $x\to\infty$

Consider

$$\lim_{x\to\infty}x\left(\ln g(x)-1\right)=\lim_{x\to\infty}x\left[(x+a)\ln\left(1+\frac{1}{x}\right)-1\right]$$

Let $t=\frac{1}{x}\to 0 \quad\Rightarrow x=\frac{1}{t}$

$$\begin{aligned}\lim_{x\to\infty}x\left[(x+a)\ln\left(1+\frac{1}{x}\right)-1\right] &= \lim_{t\to 0}\frac{1}{t}\left[\left(\frac{1}{t}+a\right)\ln(1+t)-1\right]\\ &= \lim_{t\to 0}\frac{(1+at)\ln(1+t)-t}{t^2}=\frac{0}{0}\\ &= \lim_{t\to 0}\frac{a\ln(1+t)+\frac{1+at}{1+t}-1}{2t}\\ &= \lim_{t\to 0}\frac{a(1+t)\ln(1+t)+1+at-1-t}{2t(1+t)}\\ &= \lim_{t\to 0}\frac{a(1+t)\ln(1+t)+(a-1)t}{2t+2t^2}=\frac{0}{0}\\ &= \lim_{t\to 0}\frac{a\ln(1+t)+a+a-1}{2+4t}\\ &= \lim_{t\to 0}\frac{a\ln(1+t)+2a-1}{2+4t}\\ &= \frac{2a-1}{2}\\ &= a-\frac{1}{2}\end{aligned}$$

When $a>\frac{1}{2} \quad\Rightarrow\quad g(x)>e$ *as* $x\to\infty$

If $0\le a<\frac{1}{2} \quad\Rightarrow\quad g(x)<e$ *as* $x\to\infty$

## *Exercise*

Let $f(x)=(a+x)^x$, where $a>0$

*a)* What is the domain of *f* (in terms of *a*)?

*b)* Describe the end behavior of *f* (near the left boundary of its domain and as $x\to\infty$).

*c)* Compute $f'$.

*d)* Show that *f* has a single local minimum at the point *z* that satisfies $(z+a)\ln(z+a)+z=0$

*e)* Describe how $f(z)$ varies as *a* increases.

## **Solution**

***a)*** $a+x\ge 0 \quad\Rightarrow\quad$ *Domain*: $[-a,\ \infty)$

***b)*** $\lim\limits_{x\to -a^+}(a+x)^x=e^L$

$$\begin{aligned}
L &= \lim_{x\to -a^+}\ln(a+x)^x\\
&= \lim_{x\to -a^+} x\ln(a+x)\\
&= \lim_{x\to -a^+}\frac{\ln(a+x)}{\frac{1}{x}}\\
&= \lim_{x\to -a^+}\frac{\frac{1}{a+x}}{-\frac{1}{x^2}}\\
&= \lim_{x\to -a^+}\frac{-x^2}{a+x}\\
&= \frac{-a^2}{0^-}\\
&= \infty
\end{aligned}$$

$$\lim_{x\to -a^+}(a+x)^x=e^{\infty}=\infty$$

***c)*** $\ln f(x)=\ln(a+x)^x$

$$(\ln f(x))'=(x\ln(a+x))'$$

$$\frac{f'(x)}{f(x)}=\ln(a+x)+\frac{x}{a+x}$$

$$\begin{aligned}
f'(x)&=\left(\ln(a+x)+\frac{x}{a+x}\right)(a+x)^x\\
&=(a+x)^x\ln(a+x)+x(a+x)^{x-1}
\end{aligned}$$

***d)*** $f'(x) = \left(\ln(a+x) + \frac{x}{a+x}\right)(a+x)^x = 0$

$\ln(a+x) + \frac{x}{a+x} = 0$

$(a+x)\ln(a+x) + x = 0$

Let $z = a$

$(z+a)\ln(z+a) + z = 0$

$\ln(a+z) + \frac{z}{a+z} = 0$

$\ln(a+z) + \frac{z+a-a}{a+z} = 0$

$\ln(a+z) + 1 - \frac{a}{a+z} = 0$

$\ln(a+z) + -\frac{a}{a+z} = -1$

As $z$ increases left side increases.

***e)*** As $a \to \infty \;\; \Rightarrow z \to -\infty \;\; \Rightarrow f(x) \to 0$