SOLUTION Section 3.1 – Sequences

Exercise

Find the values of $a_1, a_2, a_3, \text{ and } a_4$ for $a_n = \dfrac{1-n}{n^2}$

Solution

$$a_1 = \frac{1-1}{1^2} \underline{= 0}|$$

$$a_2 = \frac{1-2}{2^2} \underline{= -\frac{1}{4}}|$$

$$a_3 = \frac{1-3}{3^2} \underline{= -\frac{2}{9}}|$$

$$a_4 = \frac{1-4}{4^2} \underline{= -\frac{3}{16}}|$$

Exercise

Find the values of $a_1, a_2, a_3, \text{ and } a_4$ for $a_n = \dfrac{1}{n!}$

Solution

$$a_1 = \frac{1}{1!} \underline{= 1}|$$

$$a_2 = \frac{1}{2!} \underline{= \frac{1}{4}}|$$

$$a_3 = \frac{1}{3!} \underline{= \frac{1}{6}}|$$

$$a_4 = \frac{1}{4!} \underline{= \frac{1}{24}}|$$

Exercise

Find the values of $a_1, a_2, a_3, \text{ and } a_4$ for $a_n = \dfrac{(-1)^{n+1}}{2n-1}$

Solution

$$a_1 = \frac{(-1)^{1+1}}{2(1)-1} \underline{= 1}|$$

$$a_2 = \frac{(-1)^{2+1}}{2(2)-1} \underline{= -\frac{1}{3}}|$$

$$a_3 = \frac{(-1)^{3+1}}{2(3)-1} \underline{= \frac{1}{5}}|$$

$$a_4 = \frac{(-1)^{4+1}}{2(4)-1} = -\frac{1}{7}$$

Exercise

Find the values of $a_1,\ a_2,\ a_3,\ and\ a_4$ for $a_n = 2 + (-1)^n$

Solution

$$a_1 = 2 + (-1)^1 = 1$$

$$a_2 = 2 + (-1)^2 = 3$$

$$a_3 = 2 + (-1)^3 = 1$$

$$a_4 = 2 + (-1)^4 = 3$$

Exercise

Find the values of $a_1,\ a_2,\ a_3,\ and\ a_4$ for $a_n = \dfrac{2^n - 1}{2^n}$

Solution

$$a_1 = \frac{2^1 - 1}{2^1} = \frac{1}{2}$$

$$a_2 = \frac{2^2 - 1}{2^2} = \frac{3}{4}$$

$$a_3 = \frac{2^3 - 1}{2^3} = \frac{7}{8}$$

$$a_4 = \frac{2^4 - 1}{2^5} = \frac{15}{32}$$

Exercise

Write the first ten terms of the sequence $a_1 = 1, \quad a_{n+1} = a_n + \dfrac{1}{2^n}$

Solution

$$a_2 = a_1 + \frac{1}{2^1}$$

$$= 1 + \frac{1}{2}$$

$$= \frac{3}{2}$$

$$a_3 = a_2 + \frac{1}{2^2}$$

$$= \frac{3}{2} + \frac{1}{4}$$

$$\boxed{= \frac{7}{4}}$$

$$a_4 = a_3 + \frac{1}{2^3}$$

$$= \frac{7}{4} + \frac{1}{8}$$

$$\boxed{= \frac{15}{8}}$$

$$a_5 = a_4 + \frac{1}{2^4}$$

$$= \frac{15}{8} + \frac{1}{16}$$

$$\boxed{= \frac{31}{16}}$$

$$a_6 = a_5 + \frac{1}{2^5}$$

$$= \frac{31}{16} + \frac{1}{32}$$

$$\boxed{= \frac{63}{32}}$$

$$a_7 = a_6 + \frac{1}{2^6}$$

$$= \frac{63}{32} + \frac{1}{64}$$

$$\boxed{= \frac{127}{64}}$$

$$a_8 = a_7 + \frac{1}{2^7}$$

$$= \frac{127}{64} + \frac{1}{128}$$

$$\boxed{= \frac{255}{128}}$$

$$a_9 = a_8 + \frac{1}{2^8}$$

$$= \frac{255}{128} + \frac{1}{256}$$

$$\boxed{= \frac{511}{256}}$$

$$a_{10} = a_9 + \frac{1}{2^9}$$

$= \frac{511}{256} + \frac{1}{512}$

$= \frac{1023}{512}$

Exercise

Write the first ten terms of the sequence $a_1 = 1, \quad a_{n+1} = \frac{a_n}{n+1}$

Solution

$a_1 = 1$

$a_2 = \frac{1}{1+1} = \frac{1}{2}$

$a_3 = \frac{\frac{1}{2}}{2+1} = \frac{1}{6}$

$a_4 = \frac{\frac{1}{6}}{3+1} = \frac{1}{24}$

$a_5 = \frac{\frac{1}{24}}{4+1} = \frac{1}{120}$

$a_6 = \frac{\frac{1}{120}}{5+1}$

$= \frac{1}{720}$

$a_7 = \frac{\frac{1}{720}}{6+1}$

$= \frac{1}{5040}$

$a_8 = \frac{\frac{1}{5040}}{7+1}$

$= \frac{1}{40,320}$

$a_9 = \frac{\frac{1}{40,320}}{8+1}$

$= \frac{1}{362,880}$

$a_{10} = \frac{\frac{1}{362,880}}{9+1}$

$= \frac{1}{3,628,800}$

Exercise

Write the first ten terms of the sequence $a_1 = 2, \quad a_2 = -1, \quad a_{n+2} = \dfrac{a_{n+1}}{a_n}$

Solution

$a_1 = 2, \quad a_2 = -1$

$\boxed{a_3 = -\frac{1}{2}}$

$$a_4 = \frac{-\frac{1}{2}}{-1}$$

$$\boxed{= \frac{1}{2}}$$

$$a_5 = \frac{\frac{1}{2}}{-\frac{1}{2}}$$

$$\boxed{= -1}$$

$$a_6 = \frac{-1}{\frac{1}{2}}$$

$$\boxed{= -2}$$

$$a_7 = \frac{-2}{-1}$$

$$\boxed{= 2}$$

$$a_8 = \frac{2}{-2}$$

$$\boxed{= -1}$$

$$a_9 = \frac{-1}{2}$$

$$\boxed{= -\frac{1}{2}}$$

$$a_{10} = \frac{-\frac{1}{2}}{-1}$$

$$\boxed{= \frac{1}{2}}$$

Exercise

Find a formula for the *n*th term of the sequence $-1, \ 1, \ -1, \ 1, \ -1, \cdots$

Solution

$\boxed{a_n = (-1)^n} \quad n \in \mathbb{N}$

Exercise

Find a formula for the nth term of the sequence $1,\ -\frac{1}{4},\ \frac{1}{9},\ -\frac{1}{16},\ \frac{1}{25},\cdots$

Solution

$$a_1 = 1 \quad r = -\frac{1}{4}$$

$$\begin{aligned} a_n &= a_1 r \\ &= -\frac{1}{4} \\ &= \frac{(-1)^{n+1}}{n^2} \end{aligned}$$

$$a_n = \frac{(-1)^{n+1}}{n^2} \quad n \in \mathbb{N}$$

Exercise

Find a formula for the nth term of the sequence $\frac{1}{9},\ \frac{2}{12},\ \frac{2^2}{15},\ \frac{2^3}{18},\ \frac{2^4}{21},\cdots$

Solution

$$a_n = \frac{2^{n-1}}{3(n+2)} \quad n \in \mathbb{N}$$

Exercise

Find a formula for the nth term of the sequence $-3,\ -2,\ -1,\ 0,\ 1,\cdots$

Solution

$$d = -2 - (-3) = 1$$

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= -3 + (n-1)(1) \\ &= -3 + n - 1 \\ &= n - 4 \quad n \in \mathbb{N} \end{aligned}$$

Exercise

Find a formula for the nth term of the sequence $\frac{1}{25},\ \frac{8}{125},\ \frac{27}{625},\ \frac{64}{3125},\ \frac{125}{15{,}625},\cdots$

Solution

$$\frac{1}{5^2}, \frac{2^3}{5^3}, \frac{3^3}{5^4}, \frac{4^3}{5^5}, \frac{5^3}{5^6}, \cdots$$

$$\boxed{a_n = \frac{n^3}{5^{n+1}}} \quad n \in \mathbb{N}$$

Exercise

Find a formula for the nth term of the sequence $0, 1, 1, 2, 2, 3, 3, 4, \cdots$

Solution

$$\boxed{a_n = \frac{n - \frac{1}{2} + (-1)^n\left(\frac{1}{2}\right)}{2}} \quad n \in \mathbb{N}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{n + (-1)^n}{n}$$

Solution

$$\lim_{n\to\infty} \frac{n + (-1)^n}{n} = \lim_{n\to\infty}\left(1 + \frac{(-1)^n}{n}\right)$$

$$\boxed{=1} \quad \Rightarrow \quad \textit{converges}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{1-2n}{1+2n}$$

Solution

$$\lim_{n\to\infty} \frac{1-2n}{1+2n} = \lim_{n\to\infty}\left(\frac{\frac{1}{n} - 2}{\frac{1}{n} + 2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{-2}{2}\right)$$

$$\boxed{=-1} \qquad \text{The limit } \textbf{\textit{converges}}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{1-n^3}{70-4n^2}$$

Solution

$$\lim_{n\to\infty} \frac{1-n^3}{70-4n^2} = \lim_{n\to\infty} \frac{\frac{1}{n^2}-n}{\frac{70}{n^2}-4}$$

$$\lim_{n\to\infty} \frac{0-n}{0-4}$$

$$\underline{=\infty} \quad \Rightarrow \quad \textit{diverges}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \left(2-\frac{1}{2^n}\right)\left(3+\frac{1}{2^n}\right)$$

Solution

$$\lim_{n\to\infty} \left(2-\frac{1}{2^n}\right)\left(3+\frac{1}{2^n}\right) = (2)(3)$$

$$\underline{=6} \quad \Rightarrow \quad \textit{converges}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = n\pi\cos(n\pi)$$

Solution

$$\lim_{n\to\infty} n\pi\cos(n\pi) = \lim_{n\to\infty} n\pi(-1)^n$$

$$\underline{=\infty} \quad \Rightarrow \quad \textit{diverges}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = n-\sqrt{n^2-n}$$

Solution

$$\lim_{n\to\infty} n-\sqrt{n^2-n} = \lim_{n\to\infty}\left(n-\sqrt{n^2-n}\right)\frac{n+\sqrt{n^2-n}}{n+\sqrt{n^2-n}}$$

$$= \lim_{n\to\infty}\frac{n^2-\left(n^2-n\right)}{n+\sqrt{n^2-n}}$$

$$= \lim_{n\to\infty}\frac{n}{n+\sqrt{n^2-n}}$$

$$= \lim_{n\to\infty}\frac{1}{1+\sqrt{1-\frac{1}{n}}}$$

$$\underline{=\frac{1}{2}}$$

The given series *converges*.

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \sqrt{\frac{2n}{n+1}}$$

Solution

$$\lim_{n\to\infty}\sqrt{\frac{2n}{n+1}} = \sqrt{\lim_{n\to\infty}\frac{2}{1+\frac{1}{n}}}$$

$$\underline{=\sqrt{2}}$$

The given series converges

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\sin^2 n}{2^n}$$

Solution

$$0 \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n}$$ By the Sandwich Theorem for sequences

$$\lim_{n\to\infty}\frac{\sin^2 n}{2^n}\ \underline{=0}$$

The given series converges

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\ln n}{\ln 2n}$$

Solution

$$\lim_{n\to\infty} \frac{\ln n}{\ln 2n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{2}{2n}}$$

$$\underline{=1}$$

The given series converges

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{3^n \cdot 6^n}{2^{-n} \cdot n!}$$

Solution

$$\lim_{n\to\infty} \frac{3^n \cdot 6^n}{2^{-n} \cdot n!} = \lim_{n\to\infty} \frac{2^n \cdot 3^n \cdot 6^n}{n!}$$

$$= \lim_{n\to\infty} \frac{36^n}{n!}$$

$$\underline{=0}$$

The given series converges

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \sqrt{n}\ \sin\frac{1}{\sqrt{n}}$$

Solution

$$\lim_{n\to\infty} \frac{\sin\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\left(-\frac{1}{2n^{3/2}}\right)\cos\frac{1}{\sqrt{n}}}{-\frac{1}{2n^{3/2}}}$$

$$= \lim_{n\to\infty} \cos\frac{1}{\sqrt{n}}$$

$$= \cos 0$$

$$\underline{=1}$$

The given series converges

or $\lim\limits_{n\to\infty} \dfrac{\sin\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim\limits_{u\to 0} \dfrac{\sin u}{u} = 1$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{n^2}{2^n - 1}$$

Solution

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{2^n - 1} \qquad \text{L'Hôpital Rule}$$

$$= \lim_{x\to\infty} \frac{2x}{(\ln 2)\cdot 2^x}$$

$$= \lim_{x\to\infty} \frac{2}{(\ln 2)^2 \cdot 2^x}$$

$$\underline{= 0}|$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$\{c_n\} = \left\{(-1)^n \frac{1}{n!}\right\}$$

Solution

$$n! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdots n = 24\cdot \underbrace{5\cdot 6\cdots n}_{n-4}$$

$$2^2 = 2\cdot 2\cdot 2\cdot 2\cdot 2\cdots 2 = 16\cdot \underbrace{2\cdot 2\cdots 2}_{n-4}$$

$$\frac{-1}{2^n} \le (-1)^n \frac{1}{n!} \le \frac{1}{2^n} \qquad n \ge 4$$

By the Squeeze Theorem

$$\lim_{n\to\infty} (-1)^n \frac{1}{n!} \underline{= 0}|$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{5}{n+2}$$

Solution

$$\boxed{\lim_{n\to\infty} \frac{5}{n+2} = 0}$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = 8 + \frac{5}{n}$$

Solution

$$\boxed{\lim_{n\to\infty} \left(8 + \frac{5}{n}\right) = 8}$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = (-1)^n \left(\frac{n}{n+1}\right)$$

Solution

$$\lim_{n\to\infty} (-1)^n \left(\frac{n}{n+1}\right) \quad \text{does not exist (oscillates between } -1 \text{ and } 1\text{)}$$

The sequence ***diverges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{1+(-1)^n}{n^2}$$

Solution

$$\boxed{\lim_{n\to\infty} \frac{1+(-1)^n}{n^2} = 0}$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{10n^2 + 3n + 7}{2n^2 - 6}$$

Solution

$$\lim_{n\to\infty} \frac{10n^2 + 3n + 7}{2n^2 - 6} = \lim_{n\to\infty} \frac{10n^2}{2n^2}$$

$$= 5$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\sqrt[3]{n}}{\sqrt[3]{n} + 1}$$

Solution

$$\lim_{n\to\infty} \frac{\sqrt[3]{n}}{\sqrt[3]{n} + 1} = 1$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\ln\left(n^3\right)}{2n}$$

Solution

$$\lim_{n\to\infty} \frac{\ln\left(n^3\right)}{2n} = \lim_{n\to\infty} \frac{3\ln(n)}{2n}$$

$$= \lim_{n\to\infty} \frac{3}{2}\frac{\frac{1}{n}}{1}$$

$$= 0$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{5^n}{3^n}$$

Solution

$$\lim_{n\to\infty} \frac{5^n}{3^n} = \lim_{n\to\infty} \left(\frac{5}{3}\right)^n$$

$$= \infty$$

The sequence ***diverges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{(n+1)!}{n!}$$

Solution

$$\lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1)$$

$$= \infty$$

The sequence ***diverges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{(n-2)!}{n!}$$

Solution

$$\lim_{n\to\infty} \frac{(n-2)!}{n!} = \lim_{n\to\infty} \frac{1}{n(n-1)}$$

$$= 0$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{n^p}{e^n}, \quad p > 0$$

Solution

$$\lim_{n\to\infty} \frac{n^p}{e^n} \underline{\boxed{= 0}}$$

The sequence ***converges*** $(p > 0, \quad n \ge 2)$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = n \sin\frac{1}{n}$$

Solution

$$\lim_{n\to\infty} n\sin\frac{1}{n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} \qquad \text{Let } x = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \qquad \text{Since } \lim_{x\to 0} \frac{\sin x}{x} = 1 \quad \lim_{x\to 0} \frac{\cos x}{1} = 1$$

$$\underline{\boxed{= 1}}$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = 2^{1/n}$$

Solution

$$\lim_{n\to\infty} 2^{1/n} = 2^0$$

$$\underline{\boxed{= 1}}$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = -3^{-n}$$

Solution

$$\lim_{n\to\infty} -3^{-n} = \lim_{n\to\infty}\left(-\frac{1}{3^n}\right)$$

$$\underline{\boxed{= 0}}$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\sin n}{n}$$

Solution

$$\lim_{n\to\infty} \frac{\sin n}{n} = \lim_{n\to\infty} \frac{1}{n}(\sin n)$$

$$= 0 \qquad \text{since } \frac{1}{n} \to 0$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\cos \pi n}{n^2}$$

Solution

$$\lim_{n\to\infty} \frac{\cos \pi n}{n^2} = \lim_{n\to\infty} \frac{1}{n^2}(\cos \pi n)$$

$$= 0$$

The sequence ***converges***