Mathematica Manual

Notebook 16: Multiple Integrals

Double Integrals

Example: Integrate $\int_1^3 \int_1^x \frac{1}{x\,y}\, dy\, dx$ using *Mathematica*.

The easiest way to input double integrals is to open the **Basic Math Assistant** (Calculator - Advanced tab) or the **Basic Math Input** palette (under **Other**) and then click on the button containing $\int_{\square}^{\square} \square\, d\square$. Enter this twice and insert the appropriate values or functions in the positions indicated, deleting the integrand for the outer integral. You can also use the Integrate command as follows. Note the difference in the order in which you position the bounds in the **Integrate** example.

```
In[1]:= Clear[x, y, f]
        f[x_, y_] := 1 / (x y)
        ∫_1^3 ∫_1^x f[x, y] ⅆy ⅆx
        Integrate[f[x, y], {x, 1, 3}, {y, 1, x}]
```

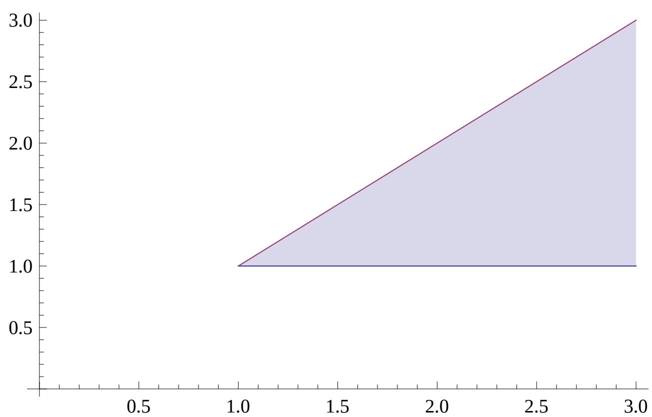
Out[3]= $\frac{\text{Log}[3]^2}{2}$

Out[4]= $\frac{\text{Log}[3]^2}{2}$

To reverse the order of integration, it is best to first plot the region over which the integration extends. To plot this region we will use with the **Filling** option in the **Plot** command. This will shade the region between $y = 1$ and $y = x$ between $x = 1$ and $x = 3$.

```
In[5]:= Plot[{1, x}, {x, 1, 3}, Filling → {1 → {2}}, AxesOrigin → {0, 0}]
```

Out[5]=



To reverse the order of integration, we see that x will go from y to 3, while y goes from 1 to 3.

In[6]:= $\int_1^3 \int_y^3 f[x, y]\, dx\, dy$

Out[6]= $\frac{\text{Log}[3]^2}{2}$

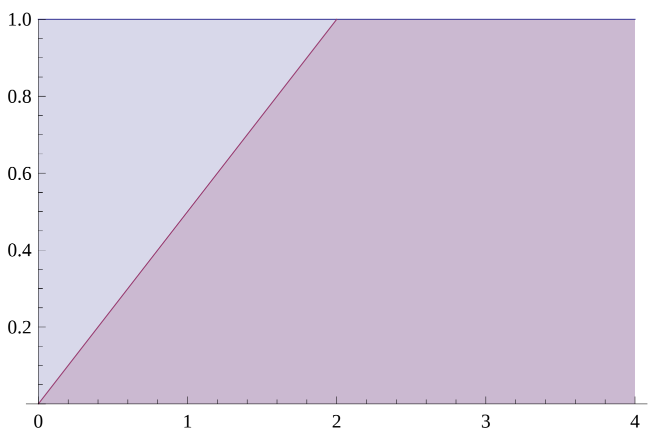
Sometimes the reversal of the order of integration requires two integrals. Here, we will use both the **Integrate** command and the numerical **NIntegrate** command and also draw the region (darker shade) over which the integral extends.

In[7]:=
```
Clear[x, y, f]
f[x_, y_] := ⅇ^(x^2)
```
$\int_0^1 \int_{2y}^4 f[x, y]\, dx\, dy$
```
NIntegrate[f[x, y], {y, 0, 1}, {x, 2 y, 4}]
Plot[{1, x / 2}, {x, 0, 4}, Filling → Axis, PlotRange → {0, 1}]
```

Out[9]= $\frac{1}{4}\left(-1 + e^4 - 2\sqrt{\pi}\ (\text{Erfi}[2] - \text{Erfi}[4])\right)$

Out[10]= 1.1494×10^6

Out[11]=



This integral does not have a closed form; the result is the Gaussian error function (from the normal distribution To reverse the order of integration for this function, two separate integrals must be used.

In[12]:=
```
Integrate[f[x, y], {x, 0, 2}, {y, 0, x / 2}] + Integrate[f[x, y], {x, 2, 4}, {y, 0, 1}]
NIntegrate[f[x, y], {x, 0, 2}, {y, 0, x / 2}] + NIntegrate[f[x, y], {x, 2, 4}, {y, 0, 1}]
```

Out[12]= $\frac{1}{4}\left(-1 + e^4\right) + \frac{1}{2}\sqrt{\pi}\ (-\text{Erfi}[2] + \text{Erfi}[4])$

Out[13]= 1.1494×10^6

We can see that the results are identical to those arrived at using the single integral above.

■ Double Integrals in Polar Form

Example: Given the double integral $\int_0^1 \int_{-y/5}^{y/5} x \sqrt{x^2 + y^2}\ dx\, dy$, complete each of the following.

(a) Plot the region of integration in the xy-plane.

(b) Change each boundary curve of the Cartesian region in part (a) to its polar representation by solving its Cartesian equation for r and θ.

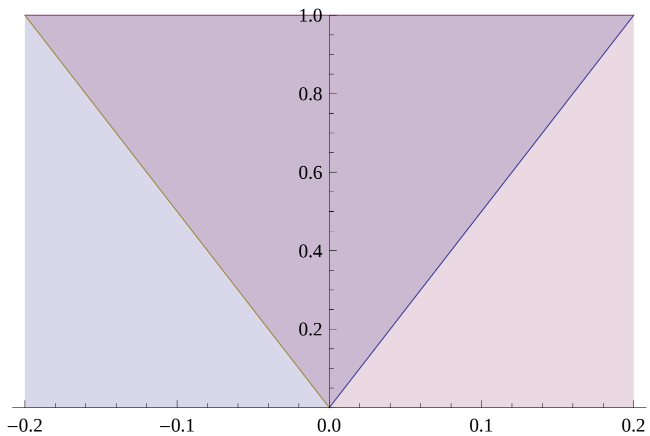
(c) Using part (b), plot the polar region of integration in the rθ-plane.

(d) Change the integrand from Cartesian to polar coordinates. Determine the limits of integration from your plot in part (c) and evaluate the polar integral using *Mathematica*.

Part (a) Since $x = \pm y/5$ is equivalent to $y = \pm 5x$, the following **Plot** command can be used to plot the cartesian region (darker shade) of integration.

```
In[14]:= Clear[x, y, r, θ]
         pxy = Plot[{5 x, 1, -5 x}, {x, -1 / 5, 1 / 5}, PlotRange → {0, 1}, Filling → {{2 → {1}}, {2 → {3}}}]
```

Out[15]=



Part (b) Replacing y with $r \sin\theta$ and x with $r \cos\theta$, the `Solve` command can then be used to find the values of r and θ. (Ignore the warning messages.)

```
In[16]:= Solve[r Cos[θ] == -r Sin[θ] / 5, θ]
         Solve[r Cos[θ] == r Sin[θ] / 5, θ]
         Solve[r Sin[θ] == 1, r]
```

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Out[16]= $\left\{\left\{\theta \to \text{ArcCos}\left[-\frac{1}{\sqrt{26}}\right]\right\}, \left\{\theta \to -\text{ArcCos}\left[\frac{1}{\sqrt{26}}\right]\right\}\right\}$

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

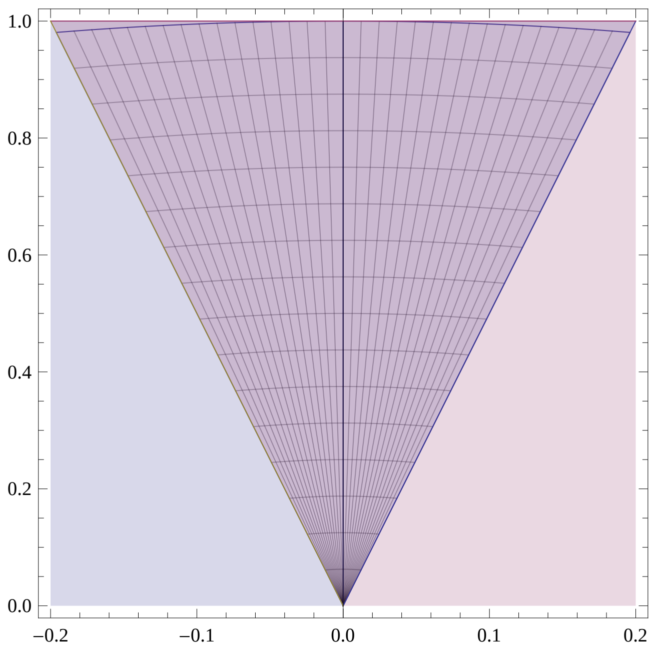
Out[17]= $\left\{\left\{\theta \to -\text{ArcCos}\left[-\frac{1}{\sqrt{26}}\right]\right\}, \left\{\theta \to \text{ArcCos}\left[\frac{1}{\sqrt{26}}\right]\right\}\right\}$

Out[18]= $\{\{r \to \text{Csc}[\theta]\}\}$

Part (c): The following command will plot the polar coordinate system for $r = 0$ to 1 in the region bounded by the θ values specified above. We use the **ParametricPlot** command and the polar definition of complex numbers using the Euler Identity: $x + i\,y = r\,(\cos\theta + i\sin\theta)$.
We will show this together with the shaded plot created in the *xy*-plane

In[19]:=
```
pp1 = ParametricPlot[Through[{Re, Im}[r Exp[I θ]]],
    {r, 0, 1}, {θ, ArcCos[1/√26], π/2}, PlotStyle → None];
pp2 = ParametricPlot[Through[{Re, Im}[-r Exp[I θ]]], {r, 0, 1},
    {θ, -π/2, -ArcCos[1/√26]}, PlotStyle → None];
Show[pp1, pp2, pxy, PlotRange → {0, 1}, AspectRatio → 1]
```

Out[21]=



The region of integration is now displayed in the polar coordinate system. Note that our region of integration is bounded by the lines $\theta = \text{ArcCos}\left[\frac{1}{\sqrt{26}}\right]$, $\theta = \text{ArcCos}\left[\frac{-1}{\sqrt{26}}\right]$, and
r = csc θ as the upper bound.

Part (d): The integral is now converted to polar form and evaluated. The integrand $x\sqrt{x^2 + y^2}$ in polar coordinates becomes $r^3\,\text{Cos}[\theta]^2$. Recall that the element of area in polar coordinates is $r\,dr\,d\theta$. This computation may take longer in some cases than in others.

In[22]:= $$\int_{\text{ArcCos}\left[\frac{1}{\sqrt{26}}\right]}^{\text{ArcCos}\left[\frac{-1}{\sqrt{26}}\right]}\int_{0}^{\frac{1}{\text{Sin}[\theta]}} r^{4}\,\text{Cos}[\theta]^{2}\,dr\,d\theta$$

Out[22]= $$\frac{1}{5}\Bigg(-\frac{13}{200}\,\text{Csc}\left[\frac{1}{2}\text{ArcCos}\left[-\frac{1}{\sqrt{26}}\right]\right]^{2}+\frac{1}{665600}$$
$$279\,\text{Csc}\left[\frac{1}{2}\text{ArcCos}\left[-\frac{1}{\sqrt{26}}\right]\right]^{4}\left(\sqrt{26}-25\,\text{Log}\left[\frac{5\,\text{Csc}\left[\frac{1}{2}\text{ArcCos}\left[-\frac{1}{\sqrt{26}}\right]\right]^{2}}{2\sqrt{26}}\right]\right)-$$
$$\left(25\,\text{Csc}\left[\frac{1}{2}\text{ArcCos}\left[-\frac{1}{\sqrt{26}}\right]\right]^{8}\left(\sqrt{26}-25\,\text{Log}\left[\frac{5\,\text{Csc}\left[\frac{1}{2}\text{ArcCos}\left[-\frac{1}{\sqrt{26}}\right]\right]^{2}}{2\sqrt{26}}\right]\right)\right)\Big/\,11075584+$$
$$\Bigg(-69299\sqrt{26}+2163200\,\text{Sin}\left[\frac{1}{2}\text{ArcCos}\left[-\frac{1}{\sqrt{26}}\right]\right]^{2}+58032\sqrt{26}\,\text{Sin}\left[\frac{1}{2}\text{ArcCos}\left[-\frac{1}{\sqrt{26}}\right]\right]^{4}-$$
$$5408\sqrt{26}\,\text{Sin}\left[\frac{1}{2}\text{ArcCos}\left[-\frac{1}{\sqrt{26}}\right]\right]^{8}+25\,\text{Log}\left[\frac{5\,\text{Csc}\left[\frac{1}{2}\text{ArcCos}\left[-\frac{1}{\sqrt{26}}\right]\right]^{2}}{2\sqrt{26}}\right]$$
$$\left(69299-58032\,\text{Sin}\left[\frac{1}{2}\text{ArcCos}\left[-\frac{1}{\sqrt{26}}\right]\right]^{4}+5408\,\text{Sin}\left[\frac{1}{2}\text{ArcCos}\left[-\frac{1}{\sqrt{26}}\right]\right]^{8}\right)\Bigg)\Big/\,8000000+$$
$$\Bigg(-270400\left(-353+32\sqrt{26}\right)\text{Csc}\left[\frac{1}{2}\text{ArcCos}\left[\frac{1}{\sqrt{26}}\right]\right]^{2}+1625000\,\text{Csc}\left[\frac{1}{2}\text{ArcCos}\left[\frac{1}{\sqrt{26}}\right]\right]^{6}-$$
$$390625\,\text{Csc}\left[\frac{1}{2}\text{ArcCos}\left[\frac{1}{\sqrt{26}}\right]\right]^{8}\text{Log}\left[\frac{5\,\text{Csc}\left[\frac{1}{2}\text{ArcCos}\left[\frac{1}{\sqrt{26}}\right]\right]^{2}}{2\sqrt{26}}\right]+$$
$$260000\,\text{Csc}\left[\frac{1}{2}\text{ArcCos}\left[\frac{1}{\sqrt{26}}\right]\right]^{4}\left(2\left(52+\sqrt{26}\right)+279\,\text{Log}\left[\frac{5\,\text{Csc}\left[\frac{1}{2}\text{ArcCos}\left[\frac{1}{\sqrt{26}}\right]\right]^{2}}{2\sqrt{26}}\right]\right)+$$
$$21632\left(4\left(-3731+401\sqrt{26}\right)+\text{Log}\left[\frac{5\,\text{Csc}\left[\frac{1}{2}\text{ArcCos}\left[\frac{1}{\sqrt{26}}\right]\right]^{2}}{2\sqrt{26}}\right]\right.$$
$$\left.\left(-69299+58032\,\text{Sin}\left[\frac{1}{2}\text{ArcCos}\left[\frac{1}{\sqrt{26}}\right]\right]^{4}-5408\,\text{Sin}\left[\frac{1}{2}\text{ArcCos}\left[\frac{1}{\sqrt{26}}\right]\right]^{8}\right)\right)\Bigg)\Big/\,6922240000\Bigg)$$

Now the numerical approximation of the integral is obtained.

In[23]:= `N[%]`

Out[23]= 0.00107938

In order to check the answer, suppose you attempt to integrate in Cartesian coordinates.

In[24]:= $\int_0^1 \int_{-\frac{y}{5}}^{\frac{y}{5}} x^2 \sqrt{x^2 + y^2}\, dx\, dy$

Out[24]= $\frac{27\sqrt{26} - 625\,\text{ArcCsch}[5]}{12500}$

Despite the difference in form, we will see that this result agrees with that arrived at with polar coordinates.

In[25]:= `N[%]`

Out[25]= 0.00107938

■ Triple Integrals in Rectangular and Cylindrical Coordinates

Evaluating triple integrals with *Mathematica* is completely analogous to computing double integrals. For example, the triple integral to find the volume of a region is evaluated below.

In[26]:= $\int_0^1 \int_x^1 \int_0^{y-x} 1\, dz\, dy\, dx$

Out[26]= $\frac{1}{6}$

At times, some of the bounds might be easier to consider using polar coordinates, for example, if the side bounds are simply $r = 1$. The following code defines the functions, specifies the switch to polar coordinates for the function, then evaluates the triple integral in two forms. Here, the order of integration is immaterial, since all the bounds are constants using polar coordinates.

In[27]:=
```
f := x^2 y^2 z
topolar = {x → r Cos[t], y → r Sin[t]};
fp = f /. topolar // Simplify
```
$\int_0^1 \int_0^{2\pi} \int_0^1 r\, fp\, dr\, dt\, dz$
```
Integrate[r fp, {t, 0, 2 π}, {r, 0, 1}, {z, 0, 1}]
N[%]
```

Out[29]= $r^4\, z\, \text{Cos}[t]^2\, \text{Sin}[t]^2$

Out[30]= $\frac{\pi}{48}$

Out[31]= $\frac{\pi}{48}$

Out[32]= 0.0654498

We can compare this result to what we would have gotten without the polar coordinate switch.

In[33]:=
```
NIntegrate[f, {x, -1, 1}, {y, -Sqrt[1 - x^2], Sqrt[1 - x^2]}, {z, 0, 1}]
```

Out[33]= 0.0654498