Chapter 12. Fourier Series

Section 12.1. Computation of Fourier Series

1. The function $f(x) = |\sin x|$ is even on the interval $[-\pi, \pi]$. Hence, the Fourier expansion will contain only cosine terms. Its Fourier coefficients are

$$a_n = \frac{2}{\pi}\int_0^{\pi} |\sin x| \cos nx\, dx.$$

However, $\sin x \geq 0$ on $[0, \pi]$, so

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\, dx.$$

Now, for $n = 0$,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x dx = \frac{2}{\pi}[-\cos x]_0^{\pi} = \frac{4}{\pi}.$$

For $n = 1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x dx = 0.$$

Now, for $n > 1$,

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\, dx \\
&= \frac{1}{\pi}\int_0^{\pi} [\sin(1+n)x + \sin(1-n)x]\, dx \\
&= \frac{1}{\pi}\left[\frac{-1}{1+n}\cos(1+n)x - \frac{1}{1-n}\cos(1-n)x\right]_0^{\pi} \\
&= \frac{1}{\pi}\left[\frac{-1}{1+n}(-1)^{1+n} - \frac{1}{1-n}(-1)^{1-n}\right. \\
&\qquad \left. + \frac{1}{1+n} + \frac{1}{1-n}\right] \\
&= \frac{1}{\pi}\left\{\frac{1}{1+n}\left[1-(-1)^{1+n}\right]\right. \\
&\qquad \left. + \frac{1}{1-n}\left[1-(-1)^{1-n}\right]\right\}
\end{aligned}$$

Thus, if n is odd, $a_n = 0$. On the other hand, if n is even,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\left[\frac{2}{1+n} + \frac{2}{1-n}\right] \\
&= \frac{1}{\pi}\left[\frac{2-2n+2+2n}{(1+n)(1-n)}\right] \\
&= \frac{4}{\pi(1-n^2)}.
\end{aligned}$$

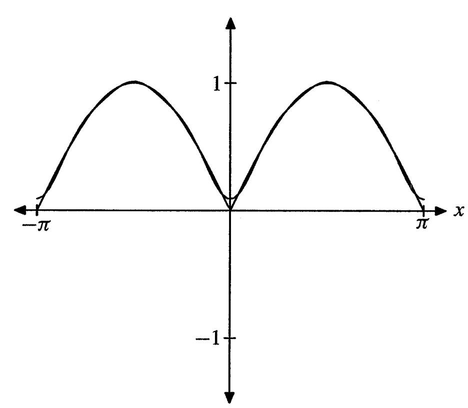
Let us now write

$$a_{2n} = \frac{4}{\pi(1-4n^2)},$$

for $n \geq 1$. Thus, the Fourier representation is

$$f(x) \sim \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)} \cos 2nx.$$

The partial sum S_6 on $[-\pi, \pi]$ and $[-3\pi, 3\pi]$ follow.



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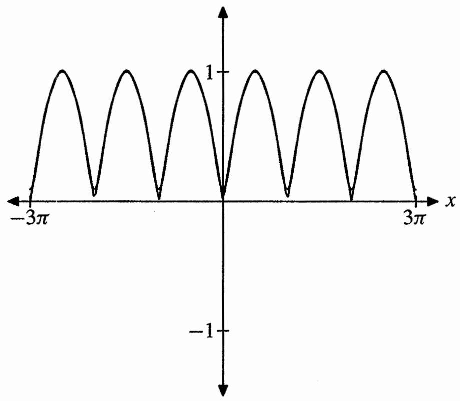


2. On the interval $-\pi \le x \le \pi$, the function $f(x) = |x|$ is even, so only cosine terms will appear in the Fourier series expansion and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx \, dx = \frac{2}{\pi}\int_{0}^{\pi} |x| \cos nx \, dx.$$

But $|x| = x$ on $[0, \pi]$. Thus,

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx,$$

and integration by parts provides

$$\begin{aligned} a_n &= \frac{2}{\pi}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] \\ &= \frac{2}{n^2\pi}\left[(-1)^n - 1\right], \end{aligned}$$

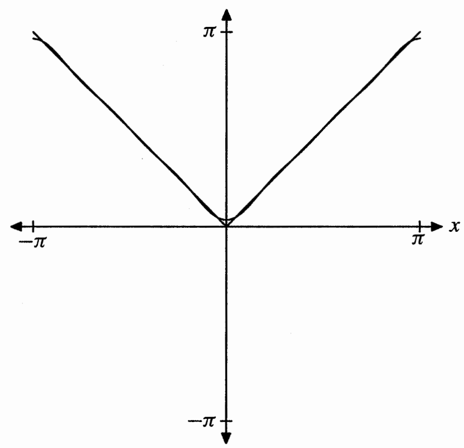
provided $n \neq 0$. In the case that $n = 0$,

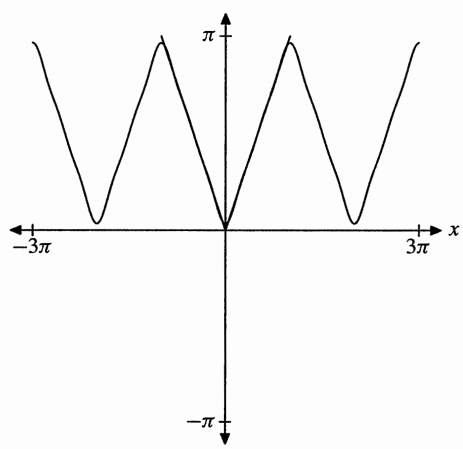
$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi} x \, dx \\ &= \frac{2}{\pi}\left[\frac{1}{2}x^2\right]_0^{\pi} \\ &= \frac{1}{\pi}(\pi^2) \\ &= \pi. \end{aligned}$$

Thus, $f(x) = |x|$ has Fourier Series

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ &= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}[(-1)^n - 1] \cos nx. \end{aligned}$$

The partial sum S_6 on $[-\pi, \pi]$ and $[-3\pi, 3\pi]$ follow.





3. On the interval $[-\pi, \pi]$, the function

$$f(x) = \begin{cases} 0, & -\pi \le x < 0, \\ x, & 0 \le x \le \pi, \end{cases}$$

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is neither even nor odd, so the Fourier series may have both sine and cosine terms. However, f is identically zero on $[-\pi, 0]$, so

$$a_n = \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx.$$

Integration by parts provides

$$\begin{aligned} a_n &= \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} \\ &= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] \\ &= \frac{1}{\pi n^2}\left[(-1)^n - 1\right], \end{aligned}$$

provided $n \neq 0$. In the case that $n = 0$,

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\cdot\frac{1}{2}x^2\Big|_0^{\pi} = \frac{\pi}{2}.$$

Similarly,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx. \end{aligned}$$

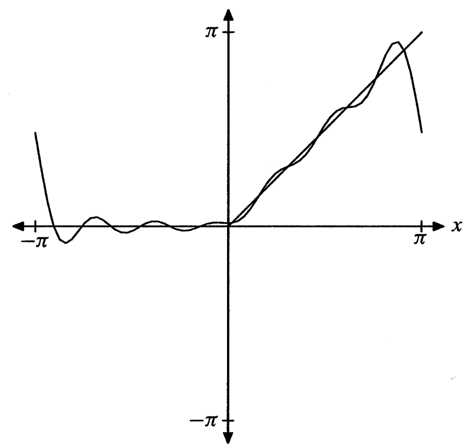
Integration by parts provides

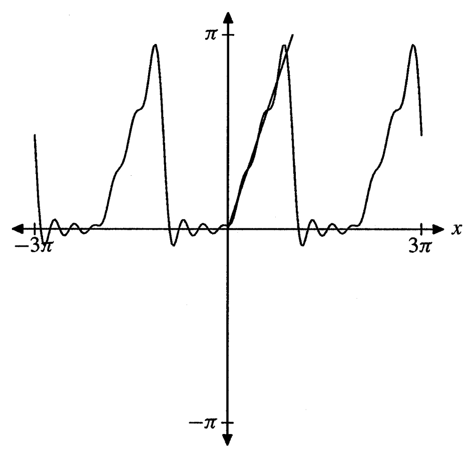
$$\begin{aligned} b_n &= \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} \\ &= \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] \\ &= \frac{(-1)^{n+1}}{n}. \end{aligned}$$

Hence, $f(x)$ has Fourier series expansion

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx \\ &= \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{[(-1)^n - 1]}{\pi n^2}\cos nx \right. \\ &\qquad \left. + \frac{(-1)^{n+1}}{n}\sin nx\right] \end{aligned}$$

The partial sum S_6 on $[-\pi, \pi]$ and $[-3\pi, 3\pi]$ follow.





4. On the interval $[-\pi, \pi]$, the function

$$f(x) = \begin{cases} 0, & -\pi \le x < 0, \\ \sin x, & 0 \le x \le \pi, \end{cases}$$

is neither even nor odd, so the fourier series may

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have both sine and cosine terms.

$$\begin{aligned}a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx \\ &\quad + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx\end{aligned}$$

However, f is identically zero on $[-\pi, 0)$, so

$$\begin{aligned}a_n &= \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} \sin x\cos nx\,dx.\end{aligned}$$

However, two integration by parts provides

$$\begin{aligned}a_n &= \frac{n^2}{\pi(n^2-1)}\left[\frac{\sin x\sin nx}{n} + \frac{\cos x\cos nx}{n^2}\right]_0^{\pi} \\ &= \frac{n^2}{\pi(n^2-1)}\left[\frac{-\cos n\pi}{n^2} - \frac{1}{n^2}\right] \\ &= \frac{n^2}{\pi(n^2-1)}\left[\frac{(-1)^{n+1}}{n^2} - \frac{1}{n^2}\right] \\ &= \frac{1}{\pi(n^2-1)}\left[(-1)^{n+1} - 1\right],\end{aligned}$$

provided $n \neq 1, 0$. In the case where $n = 1$,

$$\begin{aligned}a_1 &= \frac{1}{\pi}\int_{0}^{\pi} \sin x\cos x\,dx \\ &= \frac{1}{2\pi}\int_{0}^{\pi} \sin 2x\,dx = 0.\end{aligned}$$

In the case where $n = 0$,

$$a_0 = \frac{1}{\pi}\int_{0}^{\pi} \sin x\,dx = \frac{2}{\pi}.$$

Similarly,

$$\begin{aligned}b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} \sin x\sin nx\,dx.\end{aligned}$$

This time we use a product to sum identity to write

$$\begin{aligned}b_n &= \frac{1}{2\pi}\int_{0}^{\pi} [\cos(1-n)x - \cos(1+n)x]\,dx \\ &= \frac{1}{2\pi}\left[\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right]_0^{\pi} \\ &= 0,\end{aligned}$$

provided $n \neq 1$. In the case that $n = 1$,

$$\begin{aligned}b_1 &= \frac{1}{\pi}\int_{0}^{\pi} \sin^2 x\,dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} \frac{1-\cos 2x}{2}\,dx \\ &= \frac{1}{2\pi}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} \\ &= \frac{1}{2}.\end{aligned}$$

Hence, $f(x)$ has Fourier series expansion

$$\begin{aligned}f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx \\ &= \frac{2/\pi}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}-1}{\pi(n^2-1)}\cos nx + \frac{1}{2}\sin x \\ &= \frac{1}{\pi} + \frac{1}{\pi}\sum_{n=2}^{\infty} \frac{(-1)^{n+1}-1}{n^2-1}\cos nx + \frac{1}{2}\sin x.\end{aligned}$$

The partial sum S_6 on $[-\pi, \pi]$ and $[-3\pi, 3\pi]$ fol-

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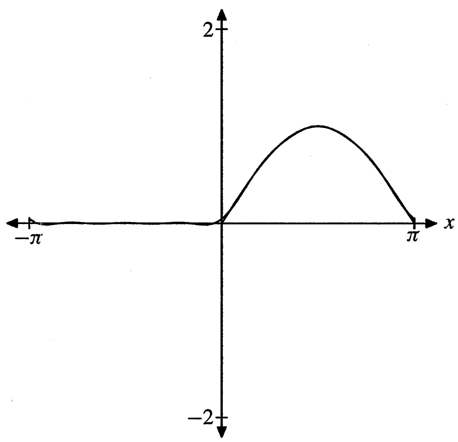
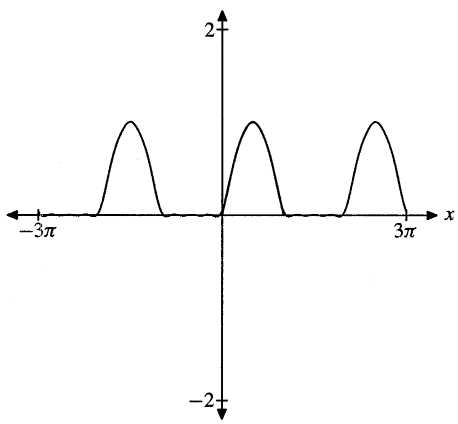
low.





5. The function $f(x) = x\cos x$ is odd on the interval $[-\pi, \pi]$. Thus, the Fourier expansion will contain only sine terms.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} x\cos x\sin nx\,dx. \end{aligned}$$

Using a product to sum identity,

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi}\frac{x}{2}\left[\sin(n-1)x + \sin(n+1)x\right]dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\sin(n-1)x\,dx \\ &\quad + \frac{1}{\pi}\int_0^{\pi} x\sin(n+1)x\,dx. \end{aligned}$$

Using integration by parts, the first integral is

$$\begin{aligned} &\frac{1}{\pi}\int_0^{\pi} x\sin(n-1)x\,dx \\ &= \frac{1}{\pi}\left[\frac{-x\cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{\pi} \\ &= \frac{1}{\pi}\left[\frac{-\pi\cos(n-1)\pi}{n-1}\right] \\ &= \frac{-\cos(n-1)\pi}{n-1} \\ &= \frac{(-1)^n}{n-1}, \end{aligned}$$

provided $n \neq 1$. Similarly, the second integral is

$$\frac{1}{\pi}\int_0^{\pi} x\sin(n+1)x\,dx = \frac{(-1)^n}{n+1}.$$

Thus,

$$b_n = (-1)^n\left[\frac{1}{n-1} + \frac{1}{n+1}\right] = (-1)^n\frac{2n}{n^2-1},$$

provided $n \neq 1$. In the case that $n = 1$,

$$\begin{aligned} b_1 &= \frac{1}{\pi}\int_0^{\pi} x\cos x\sin x\,dx \\ &= \frac{1}{2\pi}\int_0^{\pi} x\sin 2x\,dx. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} b_1 &= \frac{1}{2\pi}\left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi} \\ &= \frac{1}{2\pi}\left[\frac{-\pi}{2}\right] \\ &= -\frac{1}{4}. \end{aligned}$$

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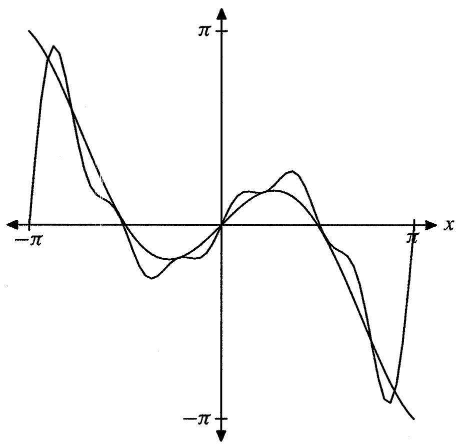
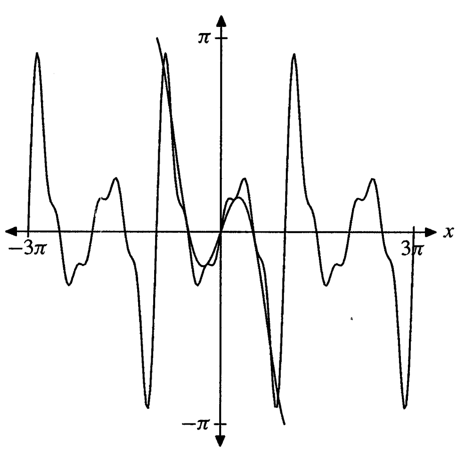
Thus, the Fourier series expansion is

$$f(x) = -\frac{1}{4}\sin x + \sum_{n=2}^{\infty}(-1)^n \frac{2n}{n^2-1}\sin nx.$$

The partial sum S_6 on $[-\pi, \pi]$ and $[-3\pi, 3\pi]$ follow.





6. The function $f(x) = x\sin x$ is even in the interval $[-\pi, \pi]$.

$$\begin{aligned} f(-x) &= -x\sin(-x) = -x(-\sin x) = x\sin x \\ &= f(x) \end{aligned}$$

Thus, the Fourier series contains only cosine terms and

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx. \end{aligned}$$

Using a product to sum identity,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi}\frac{x}{2}[\sin(1-n)x + \sin(1+n)x]\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\sin(1-n)x\,dx \\ &\quad + \frac{1}{\pi}\int_0^{\pi} x\sin(1+n)x\,dx \end{aligned}$$

Integrating by parts, the first integral becomes

$$\begin{aligned} &\frac{1}{\pi}\int_0^{\pi} x\sin(1-n)x\,dx \\ &= -\frac{1}{\pi}\int_0^{\pi} x\sin(n-1)x\,dx \\ &= -\frac{1}{\pi}\left[\frac{-x\cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{\pi} \\ &= -\frac{1}{\pi}\left[\frac{-\pi\cos(n-1)\pi}{n-1}\right] \\ &= \frac{(-1)^{n-1}}{n-1}, \end{aligned}$$

provided $n \neq 1$. Similarly, the second integral is

$$\frac{1}{\pi}\int_0^{\pi} x\sin(1+n)x\,dx = \frac{(-1)^n}{n+1}.$$

Thus,

$$\begin{aligned} a_n &= \frac{(-1)^{n-1}}{n-1} + \frac{(-1)^n}{n+1} \\ &= (-1)^{n-1}\left[\frac{1}{n-1} - \frac{1}{n+1}\right] \\ &= (-1)^{n-1}\frac{2}{n^2-1} \end{aligned}$$

Note that this gives us $a_0 = (-1)(-2) = 2$. In the

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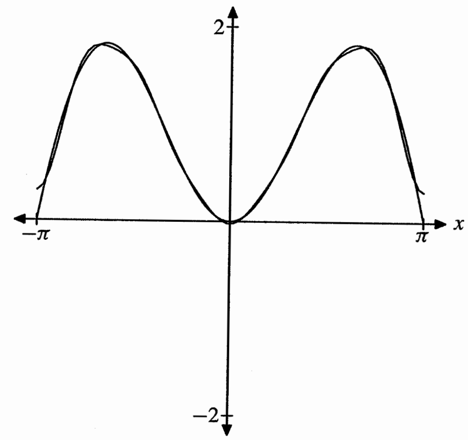
case where $n = 1$,

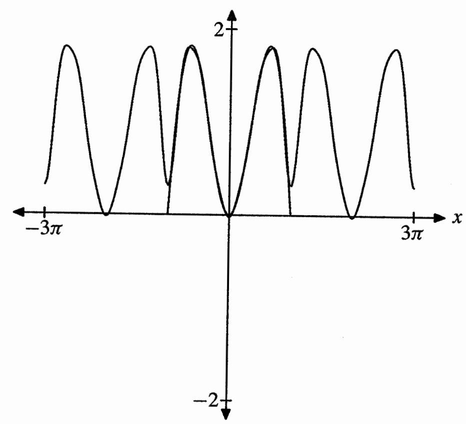
$$
\begin{aligned}
a_1 &= \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx \\
&= \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx \\
&= \frac{1}{\pi}\left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi} \\
&= \frac{1}{\pi}\left[\frac{-\pi}{2}\right] \\
&= -\frac{1}{2}.
\end{aligned}
$$

Hence the Fourier series expansion is

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\
&= 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 2\cos nx}{n^2 - 1}.
\end{aligned}
$$

The partial sum S_6 on $[-\pi, \pi]$ and $[-3\pi, 3\pi]$ follow.





7. If

$$
\begin{cases} 1 + x, & -1 \le x \le 0, \\ 1, & 0 < x \le 1, \end{cases}
$$

then on the interval $[-1, 1]$

$$
\begin{aligned}
a_n &= \int_{-1}^{1} f(x) \cos n\pi x dx \\
&= \int_{-1}^{0} (1 + x) \cos n\pi x dx + \int_0^1 \cos n\pi x dx.
\end{aligned}
$$

The second integral is straight forward.

$$
\int_0^1 \cos n\pi x dx = \frac{1}{n\pi} \sin n\pi x \Big|_0^1 = \frac{1}{n\pi}(0 - 0) = 0.
$$

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The first integral requires integration by parts.

$$\int_{-1}^{0}(1+x)\cos n\pi x dx$$
$$= \left[\frac{(1+x)\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2}\right]_{-1}^{0}$$
$$= \frac{1}{n^2\pi^2}[1-\cos n\pi]$$

Hence, combining integrals, for $n \geq 1$,

$$a_n = \frac{1}{n^2\pi^2}[1-(-1)^n] = \begin{cases} 0, & n \text{ even,} \\ \frac{2}{n^2\pi^2}, & n \text{ odd.} \end{cases}$$

For $n = 0$,

$$a_0 = \int_{-1}^{1} f(x)dx$$
$$= \int_{-1}^{0}(1+x)dx + \int_{0}^{1} dx$$
$$= \left(x + \frac{1}{2}x^2\right)\Big|_{-1}^{0} + 1$$
$$= \frac{3}{2}.$$

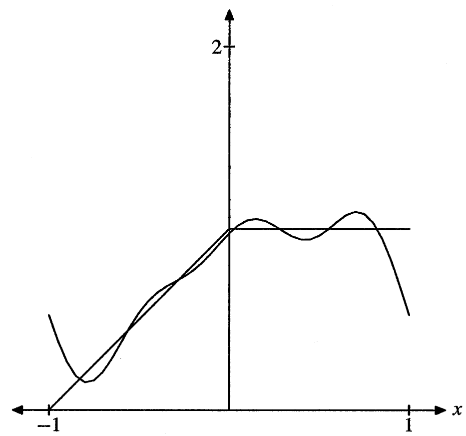
Similarly,

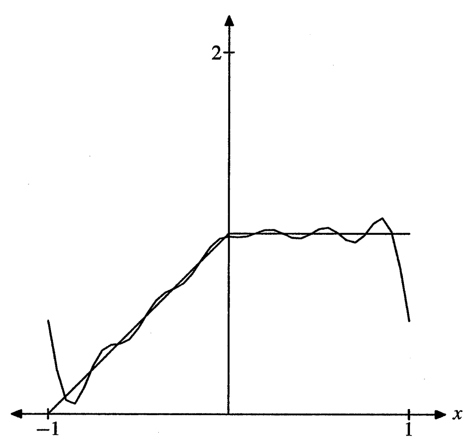
$$b_n = \int_{-1}^{1} f(x)\sin n\pi x dx$$
$$= \int_{-1}^{0}(1+x)\sin n\pi x dx + \int_{0}^{1}\sin n\pi x dx$$
$$= -\frac{1}{n\pi} - \frac{1}{n\pi}[\cos n\pi - 1]$$
$$= \frac{(-1)^{n+1}}{n\pi}.$$

Thus,

$$f(x) \sim \frac{3}{4} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos((2n+1)\pi x)}{(2n+1)^2}$$
$$-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi x.$$

In the following figures, the first displays S_3, the second S_6.





8. Note that $f(x) = 4 - x^2$ is even on the interval $[-2, 2]$, so the Fourier expansion has only cosine terms. Indeed,

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx$$
$$= \int_{0}^{2}(4-x^2)\cos\frac{n\pi x}{2}dx.$$

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Integrating by parts,

$$a_n = \left[\frac{2(4-x^2)\sin(n\pi x/2)}{n\pi} - \frac{8x\cos(n\pi x/2)}{n^2\pi^2} + \frac{16\sin(n\pi x/2)}{n^3\pi^3}\right]_0^2 = -\frac{16\cos n\pi}{n^2\pi^2} = \frac{16(-1)^{n+1}}{n^2\pi^2},$$

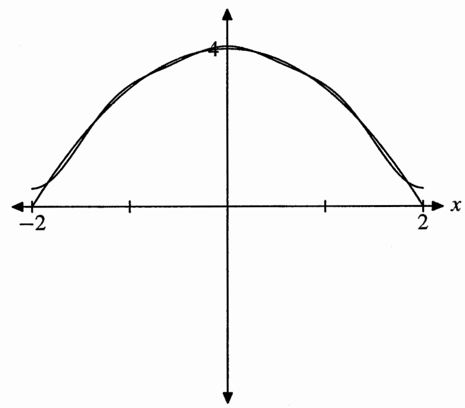
provided $n \neq 0$. In the case that $n = 0$,

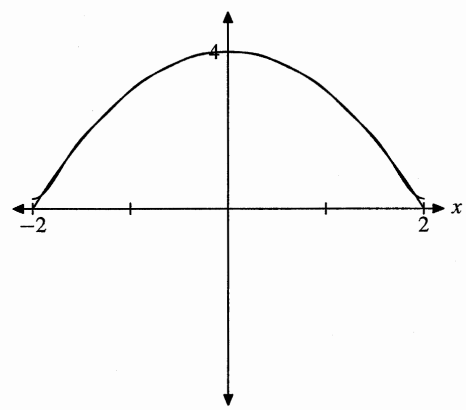
$$a_0 = \int_0^2 (4-x^2)dx = 4x - \frac{1}{3}x^3 = \frac{16}{3}.$$

Hence, the Fourier representation is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} = \frac{8}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{2}.$$

In the following figures, the fifirst displays S_3, the fisecond S_6.





9. Note that $f(x) = x^3$ is odd on the interval $[-1, 1]$. Thus, the Fourier series representation will only contain sine terms with coefficient

$$b_n = 2\int_0^1 f(x)\sin n\pi x dx = 2\int_0^1 x^3 \sin n\pi x dx.$$

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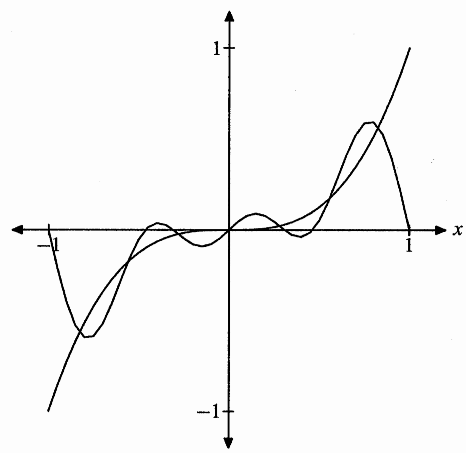
Several applications of integration by parts provides

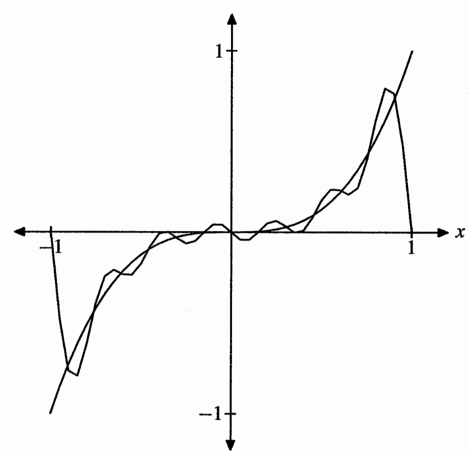
$$\begin{aligned} b_n &= -\frac{2x^3 \cos n\pi x}{n\pi} + \frac{6x^2 \sin n\pi x}{n^2\pi^2} \\ &\quad + \frac{12x \cos n\pi x}{n^3\pi^3} - \frac{12 \sin n\pi x}{n^4\pi^4}\Bigg|_0^1 \\ &= -\frac{2\cos n\pi}{n\pi} + \frac{12\cos n\pi}{n^3\pi^3} \\ &= (-1)^n \frac{2(6 - n^2\pi^2)}{n^3\pi^3}. \end{aligned}$$

Therefore, the Fourier representation is

$$f(x) \sim \sum_{n=1}^{\infty} (-1)^n \frac{2(6 - n^2\pi^2)}{n^3\pi^3} \sin n\pi x.$$

In the following figures, the first displays S_3, the second right S_6.





10. The function $f(x) = \sin x \cos^2 x$ is odd on the interval $[-\pi, \pi]$, so the Fourier representation should contain only sine terms. Indeed,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos^2 x \sin nx\, dx.$$

Using the fact that $\sin 2x = 2 \sin x \cos x$,

$$b_n = \frac{1}{\pi}\int_0^{\pi} \cos x \sin 2x \sin nx\, dx.$$

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Using a product to sum identity,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{\pi} (\cos x)\frac{1}{2}\big[\cos(n-2)x \\ &\qquad - \cos(n+2)x\big]dx \\ &= \frac{1}{2\pi}\int_0^{\pi} \cos x \cos(n-2)x dx \\ &\qquad - \frac{1}{2\pi}\int_0^{\pi} \cos x\cos(n+2)x dx. \end{aligned}$$

Using a product to sum identity, the first integral on the right becomes

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{\pi} \frac{1}{2}[\cos((n-2)+1)x \\ &\qquad + \cos((n-2)-1)x]dx \\ &= \frac{1}{4\pi}\left[\frac{\sin(n-1)x}{n-1} + \frac{\sin(n-3)x}{n-3}\right]_0^{\pi} \\ &= 0, \end{aligned}$$

provided $n \neq 1, 3$. Similarly, the second integral is also zero, provided $n \neq -3, -1$ (which of course cannot happen). It remains to find what happens at $n = 1$ and $n = 3$.

$$\begin{aligned} b_1 &= \frac{2}{\pi}\int_0^{\pi} \sin^2 x \cos^2 x dx \\ &= \frac{2}{\pi}\cdot\frac{1}{4}\int_0^{\pi} \sin^2 2x dx \\ &= \frac{1}{2\pi}\int_0^{\pi} \frac{1-\cos 4x}{2} dx \\ &= \frac{1}{4\pi}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi} \\ &= \frac{1}{4}. \end{aligned}$$

Next, for $n = 3$,

$$\begin{aligned} b_3 &= \frac{2}{\pi}\int_0^{\pi} \sin x \cos^2 x \sin 3x dx \\ &= \frac{1}{\pi}\int_0^{\pi} \cos x \sin 2x \sin 3x dx \\ &= \frac{1}{\pi}\int_0^{\pi} (\cos x)\frac{1}{2}[\cos(3x-2x) \\ &\qquad - \cos(3x+2x)]dx \\ &= \frac{1}{2\pi}\int_0^{\pi} \cos^2 x dx \\ &\qquad - \frac{1}{2\pi}\int_0^{\pi} \cos x \cos 5x. \end{aligned}$$

The first integral is

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{\pi} \frac{1+\cos 2x}{2} dx \\ &\qquad = \frac{1}{4\pi}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{1}{4}. \end{aligned}$$

The second integral is

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{\pi} \frac{1}{2}\left[\cos(5x+x) + \cos(5x-x)\right] dx \\ &= \frac{1}{4\pi}\int_0^{\pi} (\cos 6x + \cos 4x) dx \\ &= \frac{1}{4\pi}\left[\frac{1}{6}\sin 6x + \frac{1}{4}\sin 4x\right]_0^{\pi} \\ &= 0. \end{aligned}$$

Thus, $b_3 = 1/4$ and the Fourier representation is

$$\begin{aligned} f(x) &\sim b_1 \sin x + b_3 \sin 3x \\ &\sim \frac{1}{4}\sin x + \frac{1}{4}\sin 3x. \end{aligned}$$

It is interesting to attack this problem with trig identities, writing

$$\begin{aligned} \sin x \cos^2 x &= \frac{1}{2}\sin 2x \cos x \\ &= \frac{1}{2}\left[\frac{1}{2}(\sin(2x+x) + \sin(2x-x))\right] \\ &= \frac{1}{4}\sin 3x + \frac{1}{4}\sin x. \end{aligned}$$

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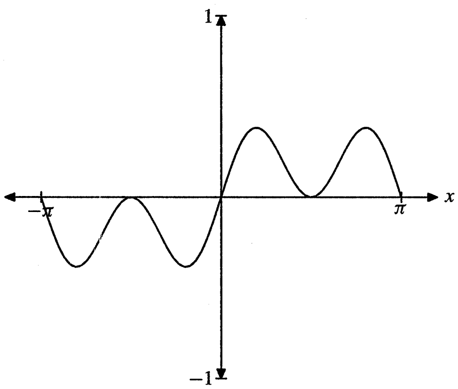
This agrees nicely with the calculus solution, but with much less effort. Note that the partial sum S_3 agrees exactly with the graph of f in the following image.



11. If

$$f(x) = \begin{cases} 0, & -1 \le x \le 0, \\ x^2, & 0 < x \le 1, \end{cases}$$

then on the interval $[-1, 1]$

$$\begin{aligned} a_n &= \int_{-1}^{1} f(x) \cos n\pi x dx \\ &= \int_{0}^{1} x^2 \cos n\pi x dx. \end{aligned}$$

With $n = 0$,

$$a_0 = \int_0^1 x^2 dx = \frac{1}{3} x^3 \bigg|_0^1 = \frac{1}{3}.$$

For $n \ge 1$, several applications of integration by parts provides

$$\begin{aligned} a_n &= \frac{x^2 \sin n\pi x}{n\pi} + \frac{2x \cos n\pi x}{n^2\pi^2} - \frac{2 \sin n\pi x}{n^3\pi^3}\bigg|_0^1 \\ &= \frac{2 \cos n\pi}{n^2\pi^2} \\ &= (-1)^n \frac{2}{n^2\pi^2}. \end{aligned}$$

Next, for $n \ge 1$

$$\begin{aligned} b_n &= \int_{-1}^{1} f(x) \sin n\pi x dx \\ &= \int_{0}^{1} x^2 \sin n\pi x dx. \end{aligned}$$

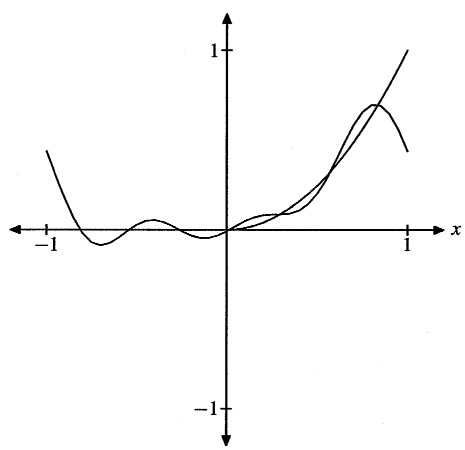
Several applications of integration by parts provides

$$\begin{aligned} b_n &= -\frac{x^2 \cos n\pi x}{n\pi} + \frac{2x \sin n\pi x}{n^2\pi^2} + \frac{2 \cos n\pi x}{n^3\pi^3}\bigg|_0^1 \\ &= -\frac{\cos n\pi}{n\pi} + \frac{2 \cos n\pi}{n^3\pi^3} - \frac{2}{n^3\pi^3} \\ &= \frac{1}{n^3\pi^3}\left[2(-1)^n - n^2\pi^2(-1)^n - 2\right]. \end{aligned}$$

Thus, the Fourier representation for the function is

$$\begin{aligned} f(x) \sim \frac{1}{6} + \sum_{n=1}^{\infty} \bigg[& (-1)^n \frac{2}{n^2\pi^2} \cos n\pi x \\ & + \frac{2(-1)^n - n^2\pi^2(-1)^n - 2}{n^3\pi^3} \sin n\pi x \bigg]. \end{aligned}$$

In the following figures, the first displays S_3, the second S_6.



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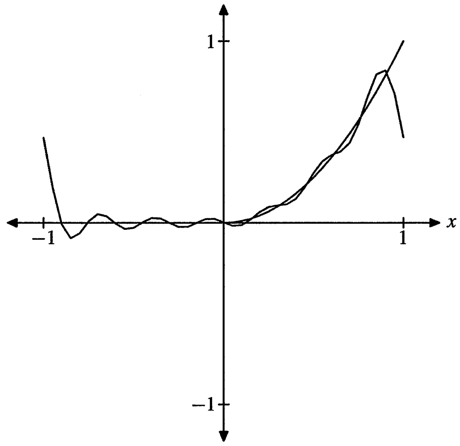

12. If

$$f(x) = \begin{cases} \sin(\pi x/2), & -2 \le x \le 0, \\ 0, & 0 < x \le 2, \end{cases}$$

then on the interval $[-2, 2]$,

$$\begin{aligned} a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx \\ &= \frac{1}{2}\int_{-2}^{0} \sin\frac{\pi x}{2}\cos\frac{n\pi x}{2}dx \\ &= \frac{1}{2}\int_{-2}^{0} \frac{1}{2}\left[\sin\frac{(1+n)\pi x}{2} + \sin\frac{(1-n)\pi x}{2}\right]dx \\ &= \frac{1}{4}\left[-\frac{2}{\pi(1+n)}\cos\frac{(1+n)\pi x}{2} - \frac{2}{\pi(1-n)}\cos\frac{(1-n)\pi x}{2}\right]_{-2}^{0} \\ &= -\frac{1}{4}\left\{\frac{4}{\pi(1+n)(1-n)} + (-1)^n\frac{4}{\pi(1+n)(1-n)}\right\} \\ &= \frac{(-1)^{n+1}-1}{\pi(1+n)(1-n)}, \end{aligned}$$

provided $n \ne 1$. Note that this implies that $a_0 = -2/\pi$. In the case that $n = 1$,

$$a_1 = \frac{1}{2}\int_{-2}^{0} \sin\frac{\pi x}{2}\cos\frac{\pi x}{2}dx = \frac{1}{4}\int_{-2}^{0}\sin\pi x dx = 0.$$

Next,

$$\begin{aligned} b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx \\ &= \frac{1}{2}\int_{-2}^{0} \sin\frac{\pi x}{2}\sin\frac{n\pi x}{2}dx \\ &= \frac{1}{2}\int_{-2}^{0}\frac{1}{2}\left[\cos\frac{(1-n)\pi x}{2} - \cos\frac{(1+n)\pi x}{2}\right]dx \\ &= \frac{1}{4}\left[\frac{2}{\pi(1-n)}\sin\frac{(1-n)\pi x}{2} - \frac{2}{\pi(1+n)}\sin\frac{(1+n)\pi x}{2}\right]_{-2}^{0} \\ &= 0, \end{aligned}$$

provided $n \ne 1$. In the case that $n = 1$,

$$\begin{aligned} b_1 &= \frac{1}{2}\int_{-2}^{0}\sin^2\frac{\pi x}{2}dx \\ &= \frac{1}{2}\int_{-2}^{0}\frac{1-\cos\pi x}{2}dx \\ &= \frac{1}{4}\left[x - \frac{1}{\pi}\sin\pi x\right]_{-2}^{0} \\ &= \frac{1}{2}. \end{aligned}$$

Thus, the Fourier representation is

$$f(x) \sim -\frac{1}{\pi} + \frac{1}{2}\sin\frac{\pi x}{2} + \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^{n+1}-1}{1-n^2}\cos\frac{n\pi x}{2}.$$

In the following figures, the first displays S_3, the second S_6.

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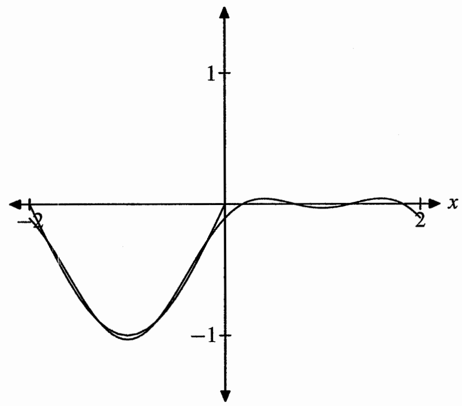

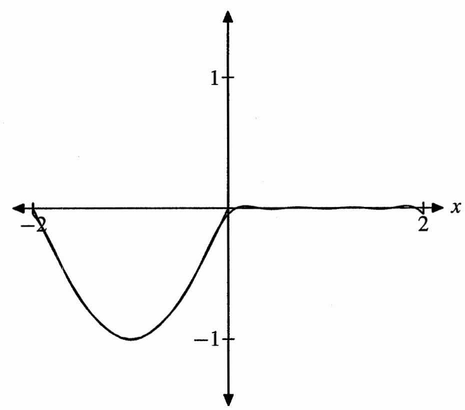

13. If
$$f(x) = \begin{cases} \cos \pi x, & -1 \le x \le 0, \\ 1, & 0 < x \le 1, \end{cases}$$
then on the interval $[-1, 1]$,
$$\begin{aligned} a_n &= \int_{-1}^{1} f(x) \cos n\pi x dx \\ &= \int_{-1}^{0} \cos \pi x \cos n\pi x dx + \int_{0}^{1} \cos n\pi x dx. \end{aligned}$$
For $n = 0$,
$$a_0 = \int_{-1}^{0} \cos \pi x dx + \int_{0}^{1} dx = 1.$$

For $n = 1$, the first integral becomes
$$\begin{aligned} \int_{-1}^{0} \cos^2 \pi x dx &= \int_{-1}^{0} \frac{1 + \cos 2\pi x}{2} dx \\ &= \frac{1}{2} \left[x + \frac{1}{2\pi} \sin 2\pi x \right]_{-1}^{0} \\ &= \frac{1}{2}. \end{aligned}$$

With $n > 1$,
$$\begin{aligned} &\int_{-1}^{0} \cos \pi x \cos n\pi x dx \\ &= \frac{1}{2} \int_{-1}^{0} [\cos(n-1)\pi x + \cos(n+1)\pi x]\, dx \\ &= \frac{1}{2} \left[\frac{\sin(n-1)\pi x}{(n-1)\pi} + \frac{\sin(n+1)\pi x}{(n+1)\pi} \right]_{-1}^{0} \\ &= 0. \end{aligned}$$

With $n \ge 1$, the second integral becomes
$$\int_{0}^{1} \cos n\pi x dx = \left. \frac{\sin n\pi x}{n\pi} \right|_{0}^{1} = 0.$$

In summary, $a_0 = 1$, $a_1 = 1/2$, and $a_n = 0$ for all $n > 1$. Next, with $n \ge 1$,
$$\begin{aligned} b_n &= \int_{-1}^{1} f(x) \sin n\pi x dx \\ &= \int_{-1}^{0} \cos \pi x \sin n\pi x dx + \int_{0}^{1} \sin n\pi x dx. \end{aligned}$$

With $n = 1$, the first integral becomes
$$\int_{-1}^{0} \cos \pi x \sin \pi x dx = \frac{1}{2} \int_{-1}^{0} \sin 2\pi x dx = 0.$$

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With $n > 1$, the first integral becomes

$$\begin{aligned}
&\int_{-1}^{0} \cos \pi x \sin n\pi x dx \\
&= \frac{1}{2}\int_{-1}^{0} [\sin(n-1)\pi x + \sin(n+1)\pi x] dx \\
&= -\frac{1}{2}\left[\frac{\cos(n-1)\pi x}{(n-1)\pi} + \frac{\cos(n+1)\pi x}{(n+1)\pi}\right]_{-1}^{0} \\
&= -\frac{1}{2}\left\{\left[\frac{1}{(n-1)\pi} + \frac{1}{(n+1)\pi}\right]\right. \\
&\qquad \left. - \left[\frac{(-1)^{n-1}}{(n-1)\pi} + \frac{(-1)^{n+1}}{(n+1)\pi}\right]\right\},
\end{aligned}$$

which, after some simplification, equals

$$\int_{-1}^{0} \cos \pi x \sin n\pi x dx = -n\left[\frac{1+(-1)^n}{(n-1)(n+1)\pi}\right].$$

The second integral is straight forward.

$$\int_0^1 \sin n\pi x dx = -\frac{\cos n\pi x}{n\pi}\bigg|_0^1 = \frac{1}{n\pi}[1-(-1)^n].$$

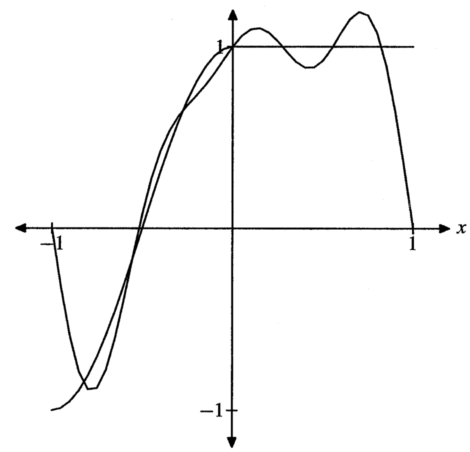
In summary,

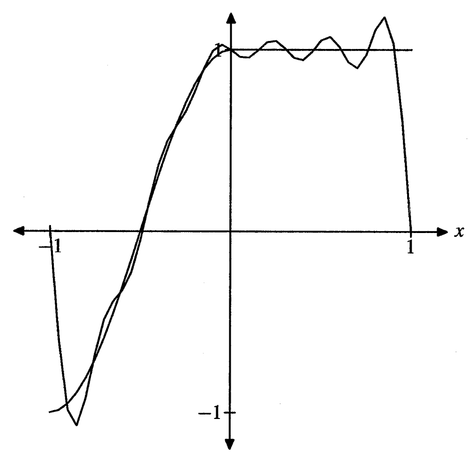
$$b_n = \begin{cases} \frac{2}{\pi}, & n = 1 \\ -n\left[\frac{1+(-1)^n}{(n-1)(n+1)\pi}\right] + \frac{1}{n\pi}[1-(-1)^n], & n > 1. \end{cases}$$

Thus, the Fourier representation of f is

$$\begin{aligned}
f(x) \sim \frac{1}{2} + \frac{1}{2}\cos \pi x + \frac{2}{\pi}\sin \pi x \\
+ \sum_{n=2}^{\infty}\left\{ -n\left[\frac{1+(-1)^n}{(n-1)(n+1)\pi}\right]\right. \\
\left. + \frac{1}{n\pi}[1-(-1)^n]\right\} \sin n\pi x.
\end{aligned}$$

In the following figures, the first S_3, the second S_6.





14. The function

$$f(x) = \begin{cases} 1+x, & -1 \le x \le 0, \\ 1-x, & 0 < x \le 1, \end{cases}$$

is even on the interval $[-1, 1]$, so the Fourier repre-

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sentation contains only cosine terms.

$$
\begin{aligned}
a_n &= \int_{-1}^{1} f(x)\cos n\pi x \\
&= \int_{-1}^{0} (1+x)\cos n\pi x dx \\
&\quad + \int_{0}^{1} (1-x)\cos n\pi x dx.
\end{aligned}
$$

Integrating by parts, the first integral becomes

$$
\begin{aligned}
&\int_{-1}^{0} (1+x)\cos n\pi x dx \\
&= \left[\frac{(1+x)\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2}\right]_{-1}^{0} \\
&= \frac{1}{n^2\pi^2} - \frac{(-1)^n}{n^2\pi^2} \\
&= \frac{1-(-1)^n}{n^2\pi^2}.
\end{aligned}
$$

Similarly, the second integral becomes

$$
\int_0^1 (1-x)\cos n\pi x dx = \frac{1-(-1)^n}{n^2\pi^2}.
$$

Thus,

$$
a_n = \frac{1-(-1)^n}{n^2\pi^2} + \frac{1-(-1)^n}{n^2\pi^2} = \frac{2}{n^2\pi^2}(1-(-1)^n),
$$

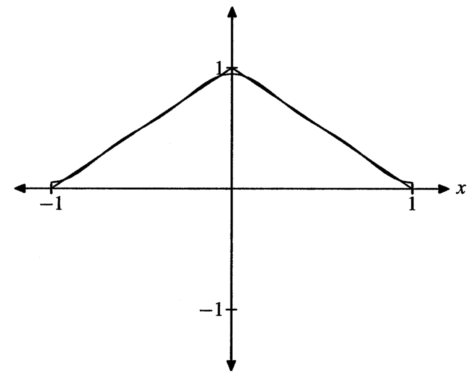
provided $n \neq 0$. In the case that $n = 0$,

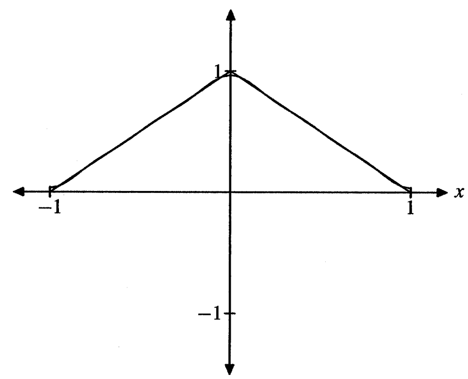
$$
a_0 = \int_{-1}^{0} (1+x)dx + \int_0^1 (1-x)dx = 1.
$$

Thus, the Fourier representation is

$$
f(x) = \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2}\cos n\pi x.
$$

In the following figures, the first displays S_3, the second S_6.





15. If

$$
f(x) = \begin{cases} 2+x, & -2 \le x \le 0, \\ -2+x, & 0 < x \le 2, \end{cases}
$$

then f is odd on the interval $[-2, 2]$ and we will need only calculate sine terms.

$$
\begin{aligned}
b_n &= \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}dx \\
&= \int_0^2 (-2+x)\sin\frac{n\pi x}{2}dx
\end{aligned}
$$

Integration by parts provides

$$
\begin{aligned}
b_n &= \left[-\frac{2(x-2)\cos(n\pi x/2)}{n\pi} + \frac{4\sin(n\pi x/2)}{n^2\pi^2}\right]_0^2 \\
&= -\frac{4}{n\pi}.
\end{aligned}
$$

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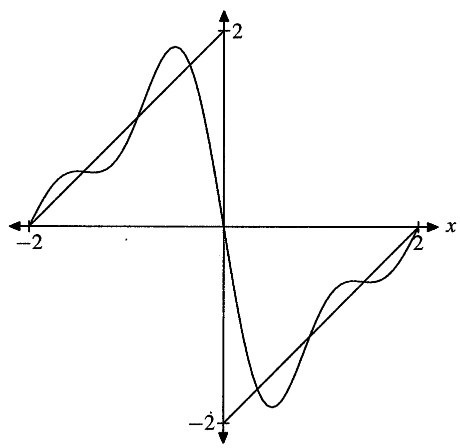
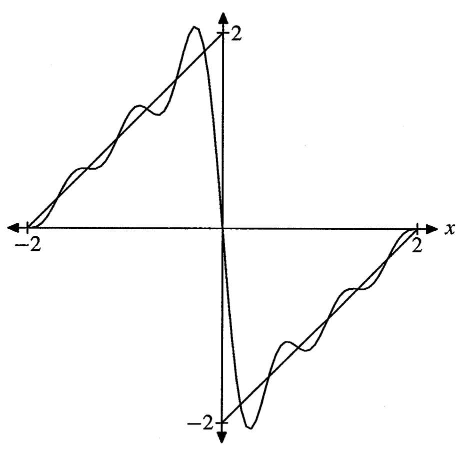
Hence, the Fourier representation of the function f is

$$f(x) \sim \sum_{n=1}^{\infty} \frac{-4}{n\pi} \sin \frac{n\pi x}{2}.$$

In the following figures, the first displays S_3, the second S_6.





16. If

$$f(x) = \begin{cases} 2, & -2 \le x \le 0 \\ 2 - x, & 0 < x \le 2, \end{cases}$$

then on the interval $[-2, 2]$,

$$\begin{aligned} a_n &= \frac{1}{2} \int_{-2}^{2} f(x) \cos \frac{n\pi x}{2} dx \\ &= \frac{1}{2} \int_{-2}^{0} 2 \cos \frac{n\pi x}{2} dx \\ &\quad + \frac{1}{2} \int_{0}^{2} (2 - x) \cos \frac{n\pi x}{2} dx. \end{aligned}$$

The first integral is straightforward,

$$\int_{-2}^{0} \cos \frac{n\pi x}{2} dx = \frac{2}{\pi n} \sin \frac{n\pi x}{2} \Big|_{-2}^{0} = 0,$$

provided $n \neq 0$. Using integration by parts, the second integral becomes

$$\begin{aligned} &\frac{1}{2} \int_{0}^{2} (2 - x) \cos \frac{n\pi x}{2} dx \\ &= \frac{1}{2} \left[\frac{2(2 - x) \sin(n\pi x/2)}{n\pi} - \frac{4 \cos(n\pi x/2)}{n^2\pi^2} \right]_0^2 \\ &= \frac{1}{2} \left[-\frac{4(-1)^n}{n^2\pi^2} + \frac{4}{n^2\pi^2} \right] \\ &= \frac{2}{n^2\pi^2} [1 - (-1)^n], \end{aligned}$$

provided $n \neq 0$. Thus,

$$a_n = \frac{2}{n^2\pi^2} [1 - (-1)^n],$$

provided $n \neq 0$. In the case that $n = 0$,

$$a_0 = \frac{1}{2} \int_{-2}^{0} 2 dx + \frac{1}{2} \int_{0}^{2} (2 - x) dx = 3.$$

Next,

$$\begin{aligned} b_n &= \frac{1}{2} \int_{-2}^{2} f(x) \sin \frac{n\pi x}{2} dx \\ &= \frac{1}{2} \int_{-2}^{0} 2 \sin \frac{n\pi x}{2} dx \\ &\quad + \frac{1}{2} \int_{0}^{2} (2 - x) \sin \frac{n\pi x}{2} dx. \end{aligned}$$

The first integral is straightforward,

$$\begin{aligned} \int_{-2}^{0} \sin \frac{n\pi x}{2} dx &= \frac{-2}{n\pi} \cos \frac{n\pi x}{2} \Big|_{-2}^{0} \\ &= -\frac{2}{n\pi} [1 - (-1)^n]. \end{aligned}$$

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Integrating by parts, the second integral becomes

$$\begin{aligned}
&\frac{1}{2}\int_0^2 (2-x)\sin\frac{n\pi x}{2}dx\\
&=\frac{1}{2}\left[-\frac{2(2-x)\cos(n\pi x/2)}{n\pi}-\frac{4\sin(n\pi x/2)}{n^2\pi^2}\right]_0^2\\
&=-\frac{1}{2}\left[-\frac{4}{n\pi}\right]\\
&=\frac{2}{n\pi}.
\end{aligned}$$

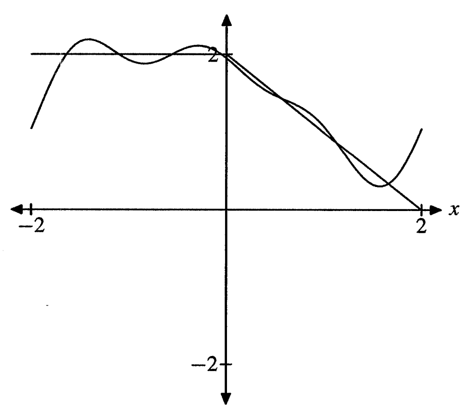
Thus,

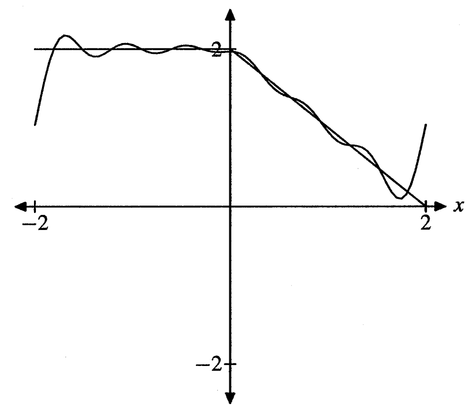
$$b_n = -\frac{2}{n\pi}[1-(-1)^n]+\frac{2}{n\pi}=\frac{2}{n\pi}(-1)^n.$$

Therefore, the Fourier representation is

$$\begin{aligned}
f(x) \sim \frac{3}{2}+\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\cos\frac{n\pi x}{2}\\
+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{2}.
\end{aligned}$$

In the following figures, the first displays S_3, the second S_6.





17. Note that $f(x) = x^2$ is even, so the Fourier series will have only cosine terms and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx.$$

Two applications of integration by parts provides

$$\begin{aligned}
a_n &= \frac{2}{\pi}\left[\frac{x^2\sin nx}{n}+\frac{2x\cos nx}{n^2}-\frac{2\sin nx}{n^3}\right]_0^{\pi}\\
&=\frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n^2}\right]\\
&=\frac{4\cos n\pi}{n^2}\\
&=(-1)^n\frac{4}{n^2},
\end{aligned}$$

provided $n \neq 0$. In the case that $n = 0$,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\cdot\frac{x^3}{3}\bigg|_0^{\pi} = \frac{2}{3}\pi^2.$$

Hence, $f(x) = x^2$ has Fourier series

$$\begin{aligned}
f(x) &= \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx\\
&=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}.
\end{aligned}$$

The partial sum S_7 is displayed on $[-\pi,\pi]$ and $[-2\pi,2\pi]$.

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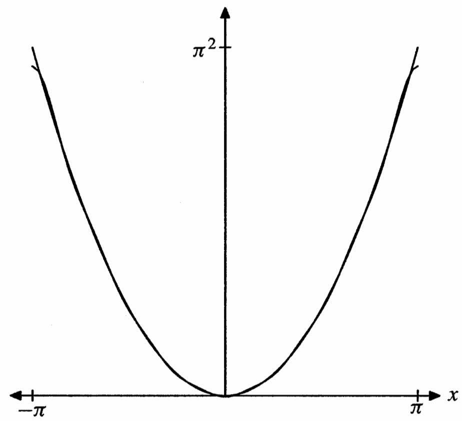
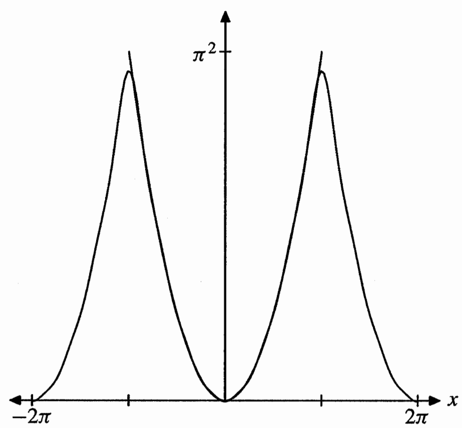




18. Note that $f(x) = x^2$ is even, so the Fourier series will have only cosine terms and

$$a_n = \frac{1}{1}\int_{-1}^{1} x^2 \cos n\pi x\, dx = 2\int_0^1 x^2 \cos n\pi x\, dx.$$

Two applications of integration by parts provides

$$\begin{aligned} a_n &= 2\left[\frac{x^2 \sin n\pi x}{n\pi} + \frac{2x \cos n\pi x}{n^2\pi^2} - \frac{2 \sin n\pi x}{n^3\pi^3}\right]_0^1 \\ &= 2\left[\frac{2\cos n\pi}{n^2\pi^2}\right] \\ &= \frac{4}{n^2\pi^2}(-1)^n, \end{aligned}$$

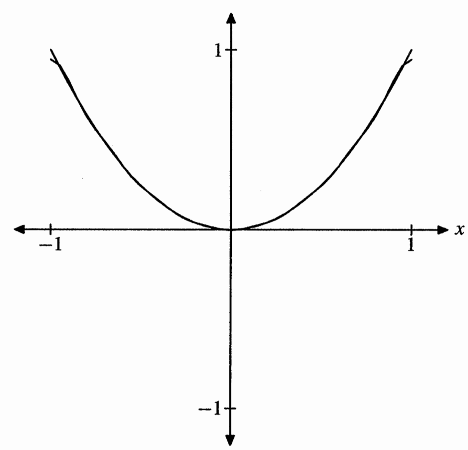
provided $n \neq 0$. In the case that $n = 0$,

$$a_0 = 2\int_0^1 x^2\, dx = \frac{2}{3}x^3\bigg|_0^1 = \frac{2}{3}.$$

Hence, $f(x) = x^2$ has Fourier series

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \\ &= \frac{2/3}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2\pi^2} \cos n\pi x \\ &= \frac{1}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n 4 \cos n\pi x}{n^2\pi^2}. \end{aligned}$$

The partial sum S_7 is displayed on $[-1, 1]$ and $[-2, 2]$.



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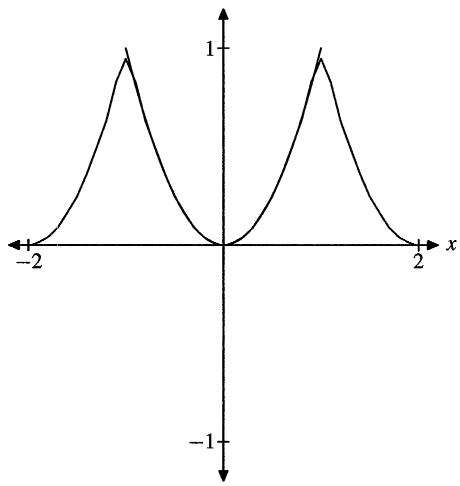

19. Since $\sin(-x) = -\sin(x)$, we have $|\sin(-x)| = |\sin(x)|$. Hence $f(x) = |\sin x|$ is even.

20. Since $f(-x) = -x + 3(-x)^3 = -(x + 3x^3) = -f(x)$, f is odd.

21. Since $f(-1) = e^{-1} = 1/e$, and $f(1) = e$, we see that f is neither even nor odd.

22. Notice that $f(-1) = 0$, while $f(1) = 2$. Hence f is neither even nor odd.

23. Using the cosine expansion,
$$\begin{aligned}&\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]\\&=\frac{1}{2}[\cos\alpha\cos\beta+\sin\alpha\sin\beta\\&\qquad+\cos\alpha\cos\beta-\sin\alpha\sin\beta]\\&=\cos\alpha\cos\beta.\end{aligned}$$
Similarly,
$$\begin{aligned}&\frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]\\&=\frac{1}{2}[\cos\alpha\cos\beta+\sin\alpha\sin\beta\\&\qquad-\cos\alpha\cos\beta+\sin\alpha\sin\beta]\\&=\sin\alpha\sin\beta.\end{aligned}$$
Finally, the sine expansion delivers
$$\begin{aligned}&\frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]\\&=\frac{1}{2}[\sin\alpha\cos\beta-\cos\alpha\sin\beta\\&\qquad+\sin\alpha\cos\beta+\cos\alpha\sin\beta]\\&=\sin\alpha\cos\beta.\end{aligned}$$

24. The function
$$\sin\frac{p\pi x}{L}\cos\frac{q\pi x}{L}\,dx,$$
being the product of an odd and even function, is odd on the interval $[-L, L]$. Thus,
$$\int_{-L}^{L}\sin\frac{p\pi x}{L}\cos\frac{q\pi x}{L}\,dx=0$$
for all nonnegative integers p and q.
Secondly, $\cos(p\pi x/L)\cos(q\pi x/L)$ is even on $[-L, L]$, so
$$\begin{aligned}&\int_{-L}^{L}\cos\frac{p\pi x}{L}\cos\frac{q\pi x}{L}\,dx\\&=2\int_{0}^{L}\cos\frac{p\pi x}{L}\cos\frac{q\pi x}{L}\,dx.\end{aligned}$$
Using the product to sum identity,
$$\begin{aligned}&2\int_{0}^{L}\cos\frac{p\pi x}{L}\cos\frac{q\pi x}{L}\,dx\\&=2\int_{0}^{L}\frac{1}{2}\left[\cos\frac{\pi(p-q)x}{L}\right.\\&\qquad\left.+\cos\frac{\pi(p+q)x}{L}\right]dx.\end{aligned}$$
But
$$\begin{aligned}&\int_{0}^{L}\cos\frac{\pi(p-q)x}{L}\,dx\\&=\frac{L}{\pi(p-q)}\sin\frac{\pi(p-q)x}{L}\Big|_0^L\\&=\frac{L}{\pi(p-q)}[\sin(p-q)\pi-\sin 0]\\&=0,\end{aligned}$$

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provided $p \neq q$. Also,

$$\begin{aligned}
&\int_0^L \cos\frac{\pi(p+q)x}{L}\,dx \\
&= \frac{L}{\pi(p+q)}\sin\frac{\pi(p+q)x}{L}\Big|_0^L \\
&= \frac{L}{\pi(p+q)}[\sin(p+q)\pi - 0] \\
&= 0,
\end{aligned}$$

provided it is *not* the case that $p = q = 0$. Hence, if it is not the case the $p = q = 0$ and it is not the case that $p = q$,

$$\int_{-L}^L \cos\frac{p\pi x}{L}\cos\frac{q\pi x}{L}\,dx = 0.$$

Consider the case where $p = q = 0$. Then

$$\int_{-L}^L \cos\frac{p\pi x}{L}\cos\frac{q\pi x}{L}\,dx = \int_{-L}^L dx = 2L.$$

Consider the case where $p = q$. Then

$$\begin{aligned}
&\int_{-L}^L \cos\frac{p\pi x}{L}\cos\frac{q\pi x}{L}\,dx \\
&= 2\int_0^L \cos^2\frac{p\pi x}{L}\,dx \\
&= 2\int_0^L \frac{1+\cos\frac{2p\pi x}{L}}{2}\,dx \\
&= \left[x + \frac{L}{2p\pi}\sin\frac{2p\pi x}{L}\right]_0^L \\
&= [L+0]-[0+0] \\
&= L.
\end{aligned}$$

Next, $\sin(p\pi x/L)\sin(q\pi x/L)$ is even on $[-L, L]$, so

$$\begin{aligned}
&\int_{-L}^L \sin\frac{p\pi x}{L}\sin\frac{q\pi x}{L}\,dx \\
&= 2\int_0^L \sin\frac{p\pi x}{L}\sin\frac{q\pi x}{L}\,dx.
\end{aligned}$$

Using a product to sum identity,

$$\begin{aligned}
&2\int_0^L \sin\frac{p\pi x}{L}\sin\frac{q\pi x}{L}\,dx \\
&= 2\int_0^L \frac{1}{2}\left[\cos\frac{\pi(p-q)x}{L} - \cos\frac{\pi(p+q)x}{L}\right]dx.
\end{aligned}$$

But,

$$\begin{aligned}
&\int_0^L \cos\frac{\pi(p-q)x}{L}\,dx \\
&= \frac{L}{\pi(p-q)}\sin\frac{\pi(p-q)x}{L}\Big|_0^L \\
&= \frac{L}{\pi(p-q)}[\sin\pi(p-q) - \sin 0] \\
&= 0,
\end{aligned}$$

provided $p \neq q$. Similarly,

$$\int_0^L \cos\frac{\pi(p+q)x}{L}\,dx = 0,$$

provided it is not the case that $p = q = 0$. Thus,

$$\int_{-L}^L \sin\frac{p\pi x}{L}\sin\frac{q\pi x}{L}\,dx = 0,$$

provided $p \neq q$ and it is not the case that $p = q = 0$. In the case that $p = q = 0$,

$$\int_{-L}^L \sin\frac{p\pi x}{L}\sin\frac{q\pi x}{L}\,dx = \int_{-L}^L 0\,dx = 0.$$

In the case that $p = q$,

$$\begin{aligned}
&\int_{-L}^L \sin\frac{p\pi x}{L}\sin\frac{q\pi x}{L}\,dx \\
&= \int_{-L}^L \sin^2\frac{p\pi x}{L}\,dx \\
&= \int_{-L}^L \frac{1-\cos\frac{2p\pi x}{L}}{2}\,dx \\
&= \frac{1}{2}\left[x - \frac{L}{2p\pi}\sin\frac{2p\pi x}{L}\right]_{-L}^L \\
&= \frac{1}{2}[(L-0)-(-L-0)] \\
&= L.
\end{aligned}$$

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25. We multiply both sides of

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\pi x + b_k \sin k\pi x$$

by $\sin nx$ and integrate. By the orthogonality relations in Lemma 1.2, all terms will equal zero, save one.

$$\int_{-\pi}^{\pi} f(x) \sin nx \, dx = b_k \int_{-\pi}^{\pi} \sin nx \sin nx \, dx$$
$$\int_{-\pi}^{\pi} f(x) \sin nx \, dx = b_k \pi$$

Hence,

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

26. If f and g are even, then

$$(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x).$$

Hence, fg is even. Next, if f and g are odd, then

$$\begin{aligned}(fg)(-x) &= f(-x)g(-x) = [-f(x)][-g(x)] \\ &= f(x)g(x) = (fg)(x).\end{aligned}$$

Hence, fg is even. Finally, if f is even and g is odd, then

$$\begin{aligned}(fg)(-x) = f(-x)g(-x) = f(x)[-g(x)] = \\ - f(x)g(x) = -(fg)(x).\end{aligned}$$

Thus, fg is odd.

27. Whether f is even or odd, we can write

$$\int_{-L}^{L} f(x)\, dx = \int_{-L}^{0} f(x)\, dx + \int_{0}^{L} f(x)\, dx.$$

However, if we let $u = -x$, the $du = -dx$ and the first integral on the right becomes

$$\int_{-L}^{0} f(x)\, dx = \int_{L}^{0} f(-u)(-du).$$

If we reverse the bounds, then

$$= \int_{0}^{L} f(-u)\, du.$$

But f is even, so

$$\int_{0}^{L} f(u)\, du.$$

But u is just a dummy variable of integration, so

$$\begin{aligned}\int_{-L}^{L} f(x)\, dx &= \int_{-L}^{0} f(x)\, dx + \int_{0}^{L} f(x)\, dx \\ &= \int_{0}^{L} f(x)\, dx + \int_{0}^{L} f(x)\, dx \\ &= 2\int_{0}^{L} f(x)\, dx.\end{aligned}$$

If f is odd, then with $u = -x$ and $du = -dx$,

$$\int_{-L}^{0} f(x)\, dx = \int_{L}^{0} f(-u)\,(-du).$$

Reversing the bounds,

$$= \int_{0}^{L} f(-u)\, du.$$

But f is odd, so

$$-\int_{0}^{L} f(u)\, du.$$

But u is just a dummy variable of integration, so

$$\begin{aligned}\int_{-L}^{L} f(x)\, dx &= \int_{-L}^{0} f(x)\, dx + \int_{0}^{L} f(x)\, dx \\ &= -\int_{0}^{L} f(x)\, dx + \int_{0}^{L} f(x)\, dx \\ &= 0.\end{aligned}$$

28. If $f(x)$ is odd, then $f(x)\cos(n\pi x/L)$ is odd and

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{k\pi x}{L}\, dx = 0$$

for $k \geq 0$. Therefore, all cosine terms vanish and

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\frac{k\pi x}{L},$$

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where

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{k\pi x}{L}\,dx.$$

Further, $f(x)\sin(k\pi x/L)$ is even and

$$b_k = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx.$$

29. We start with f being odd on the interval $[-L, L]$. Then the Fourier representation contains only sine terms. Thus,

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

Now, suppose we have the additional symmetry that f is symmetrical with respect to the line $x = L/2$. That is, $f(L-x) = f(x)$ for all $0 \le x \le L$. We can write

$$b_n = \frac{2}{L}\left[\int_0^{L/2} f(x)\sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\right].$$

Now, consider the second integral with the substitution

$$u = L - x \quad \text{and} \quad du = -dx.$$

Then,

$$\begin{aligned}\int_{L/2}^{L} f(x)\sin\frac{n\pi x}{L}dx &= \int_{L/2}^{0} f(L-u)\sin\frac{n\pi(L-u)}{L}(-du)\\ &= \int_0^{L/2} f(L-u)\sin\left(n\pi - \frac{n\pi u}{L}\right)du.\end{aligned}$$

But $f(L-u) = f(u)$, and the sine expansion provides

$$\begin{aligned}&\int_0^{L/2} f(u)\left[\sin n\pi\cos\frac{n\pi u}{L} - \sin\frac{n\pi u}{L}\cos n\pi\right]du\\ &= -(-1)^n\int_0^{L/2} f(u)\sin\frac{n\pi u}{L}\,du.\end{aligned}$$

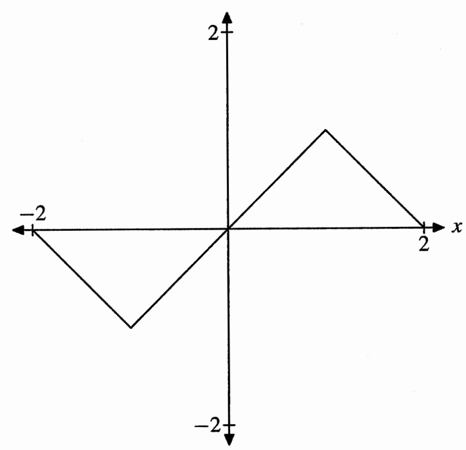
Hence, if n is even,

$$\begin{aligned}b_n &= \frac{2}{L}\left[\int_0^{L/2} f(x)\sin\frac{n\pi x}{L}\,dx - \int_0^{L/2} f(u)\sin\frac{n\pi u}{L}\,du\right]\\ &= 0.\end{aligned}$$

As an example, consider

$$f(x) = \begin{cases} -x-2, & -2 \le x < -1 \\ x, & -1 \le x < 1 \\ -x+2, & 1 \le x \le 2 \end{cases}.$$

Note that f is symmetric about $x = 1$ on $[0, 2]$. That is, $f(2-x) = f(x)$ for $0 \le x \le 2$.



30. By the definition of derivative and the periodicity of f we have

$$\begin{aligned}f'(x+T) &= \lim_{h\to 0}\frac{f(x+T+h) - f(x+T)}{h}\\ &= \frac{f(x+h) - f(x)}{h}\\ &= f'(x).\end{aligned}$$

Therefore f' is periodic with period T.

31. Set $f_{\text{odd}}(x) = [f(x) - f(-x)]/2$ and $f_{\text{even}}(x) = [f(x) + f(-x)]/2$. Then f_{odd} is off and f_{even} is even, and $f(x) = f_{\text{odd}}(x) + f_{\text{even}}(x)$ for all x.

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Section 12.2. Convergence of Fourier Series

1. Since $\sin(x+\pi) = -\sin x$, $|\sin(x+\pi)| = |\sin x|$. Hence f is periodic with period π. For any $T < \pi$, we have $\sin T > 0 = \sin 0$. Hence f cannot be periodic with period T. Therefore the smallest period is π.

2. Since $\cos y$ is periodic with period 2π, we have
$$\begin{aligned} f(x+2/3) &= \cos(3\pi(x+2/3)) \\ &= \cos(3\pi x + 2\pi) \\ &= \cos(3\pi x) = f(x). \end{aligned}$$
Hence f is periodic with period 2/3.

3. Since f is strictly increasing, $f(x) \neq f(y)$ as long as $x \neq y$. Hence f cannot be periodic.

4. The function $\sin x$ is periodic with period 2π, while $\cos(x/2$ is periodic with period 4π. Hence f is periodic with period 4π.

5. Since $f(0) = 0$ and $f(x) \neq 0$ for $x \neq 0$, f cannot be periodic.

6. Since f is strictly increasing, $f(x) \neq f(y)$ as long as $x \neq y$. Hence f cannot be periodic.

7. f_p has one-sided derivatives everywhere, and is continuous everywhere except at odd multiples of π. Therefore, the Fourier series converges to $f_p(x)$ except at the odd multiples of π. At the odd multiples of π the series converges to $[\pi + 0]/2 = \pi/2$.

8. f_p has one-sided derivatives everywhere, and is continuous everywhere, so the Fourier series converges to $f_p(x)$ everywhere.

9. f_p has one-sided derivatives everywhere, and is continuous everywhere except at the odd integers. Therefore, the Fourier series converges to $f_p(x)$ except at the odd integers. At the odd integers, the series converges to $[1+0]/2 = 1/2$.

10. f_p has one-sided derivatives everywhere, and is continuous everywhere, so the Fourier series converges to $f_p(x)$ everywhere.

11. f_p has one-sided derivatives everywhere, and is continuous everywhere except at the odd integers. Therefore, the Fourier series converges to $f_p(x)$ except at the odd integers. At the odd integers, the series converges to $[1-1]/2 = 0$.

12. f_p has one-sided derivatives everywhere, and is continuous everywhere except at the odd integers. Therefore, the Fourier series converges to $f_p(x)$ except at the odd integers. At the odd integers, the series converges to $[1+0]/2 = 1/2$.

13. f_p has one-sided derivatives everywhere, and is continuous everywhere, so the Fourier series converges to $f_p(x)$ everywhere.

14. f_p has one-sided derivatives everywhere, and is continuous everywhere except at the odd multiples of 2. Therefore, the Fourier series converges to $f_p(x)$ except at the odd multiples of 2. At the odd multiples of 2, the series converges to $[2+0]/2 = 1$.

15. The function $f(x) = |x|$ is even on the interval $[-\pi, \pi]$. Hence, the Fourier representation contains only cosine terms. Hence,
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx,$$
since $|x| = x$ on $[0, \pi]$. For $n = 0$,
$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}x^2\Big|_0^{\pi} = \pi.$$
For $n \geq 1$, integration by parts provides
$$\begin{aligned} a_n &= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} \\ &= \frac{2}{\pi}\left\{\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right\} \\ &= \frac{2}{\pi n^2}((-1)^n - 1). \end{aligned}$$
Thus, if n is even, $a_n = 0$. If n is odd,
$$a_n = -\frac{4}{\pi n^2}.$$
We can write
$$a_{2n+1} = -\frac{4}{\pi(2n+1)^2},$$

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which is valid for $n \geq 1$. Thus, the Fourier representation of f is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\cos((2n+1)x).$$

Using Corollary 2.4, the function $f(x) = |x|$ is continuous at $x = 0$ so the series converges to $f(0) = 0$ at $x = 0$. Thus,

$$0 = f(0) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2},$$

and

$$\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

16. The function $f(x) = x^2$ is even on the interval $[-\pi, \pi]$ so its Fourier representation has only cosine terms.

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx.$$

With $n = 0$,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{3\pi}x^3\bigg|_0^{\pi} = \frac{2\pi^2}{3}.$$

With $n \geq 1$, several applications of integration by parts provides

$$\begin{aligned}
a_n &= \frac{2}{\pi}\left[\frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n^2}\right] \\
&= (-1)^n\frac{4}{n^2}.
\end{aligned}$$

Thus, the Fourier representation of $f(x) = x^2$ is

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

Because f is continuous at $x = 0$, Corollary 2.4 implies that the series converges at $x = 0$ to $f(0) = 0$.

$$0 = f(0) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

Thus,

$$\begin{aligned}
-4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} &= \frac{\pi^2}{3} \\
\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} &= \frac{\pi^2}{12}.
\end{aligned}$$

Again, Corollary 2.4 implies that the series converges to $f(\pi)$ at $x = \pi$. Thus,

$$\pi^2 = f(\pi) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi.$$

Since $\cos n\pi = (-1)^n$,

$$\begin{aligned}
\frac{2\pi^2}{3} &= 4\sum_{n=1}^{\infty}\frac{1}{n^2} \\
\sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{\pi^2}{6}.
\end{aligned}$$

17. The function $f(x) = x^4$ is even on the interval $[-\pi, \pi]$, so the Fourier representation has only cosine terms.

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^4 \cos nx\, dx$$

For $n = 0$,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^4\, dx = \frac{2}{5\pi}x^5\bigg|_0^{\pi} = \frac{2\pi^4}{5}.$$

For $n \geq 1$, several applications of integration by parts provides

$$\begin{aligned}
a_n &= \frac{2}{\pi}\left[\frac{x^4\sin nx}{n} + \frac{4x^3\cos nx}{n^2} - \frac{12x^2\sin nx}{n^3}\right. \\
&\qquad \left. - \frac{24x\cos nx}{n^4} + \frac{24\sin nx}{n^5}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[\frac{4\pi^3\cos n\pi}{n^2} - \frac{24\pi\cos n\pi}{n^4}\right].
\end{aligned}$$

Since $\cos n\pi = (-1)^n$,

$$a_n = 8(-1)^n\left[\frac{\pi^2}{n^2} - \frac{6}{n^4}\right].$$

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Hence, the Fourier representation of $f(x) = x^4$ is

$$f(x) = \frac{\pi^4}{5} + 8\sum_{n=1}^{\infty}(-1)^n\left[\frac{\pi^2}{n^2} - \frac{6}{n^4}\right]\cos nx.$$

Because f is continuous at $x = 0$, the Fourier series converges to $f(0)$ at $x = 0$.

$$0 = f(0) = \frac{\pi^4}{5} + 8\sum_{n=1}^{\infty}(-1)^n\left[\frac{\pi^2}{n^2} - \frac{6}{n^4}\right]$$

$$-\frac{\pi^4}{40} = \pi^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} - 6\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$$

In Exercise 16, we found that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Substituting,

$$-\frac{\pi^4}{40} = \pi^2\left(-\frac{\pi^2}{12}\right) + 6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}$$

$$6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4} = \frac{\pi^4}{12} - \frac{\pi^4}{40}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}.$$

In similar fashion, $f(x) = x^4$ is continuous at $x = \pi$, so the series converges to $f(\pi)$ at $x = \pi$.

$$\pi^4 = f(\pi) = \frac{\pi^4}{5} + 8\sum_{n=1}^{\infty}(-1)^n\left[\frac{\pi^2}{n^2} - \frac{6}{n^4}\right]\cos n\pi$$

Since $\cos n\pi = (-1)^n$, we have $(-1)^n(-1)^n = 1$ and

$$\frac{4\pi^4}{5} = 8\sum_{n=1}^{\infty}\left[\frac{\pi^2}{n^2} - \frac{6}{n^4}\right]$$

$$\frac{\pi^4}{10} = \pi^2\sum_{n=1}^{\infty}\frac{1}{n^2} - 6\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

From Exercise 16,

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6},$$

so

$$\frac{\pi^4}{10} = \pi^2\left(\frac{\pi^2}{6}\right) - 6\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$6\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{6} - \frac{\pi^4}{10} = \frac{5\pi^4 - 3\pi^4}{30} = \frac{\pi^4}{15}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

18. If

$$f(x) = \begin{cases} 0, & -1 < x \le -1/2, \\ 1, & -1/2 < x \le 1/2, \\ 0, & 1/2 < x \le 1, \end{cases}$$

then f is even on the interval $[-1, 1]$ (except at points of discontinuity) and its Fourier representation contains only cosine terms.

$$\begin{aligned} a_n &= \int_{-1}^{1} f(x)\cos n\pi x\,dx \\ &= \int_{-1/2}^{1/2} \cos n\pi x\,dx \\ &= \frac{1}{n\pi}\sin n\pi x\Big|_{-1/2}^{1/2} \\ &= \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} - \sin\left(-\frac{n\pi}{2}\right)\right] \\ &= \frac{2}{n\pi}\sin\frac{n\pi}{2}, \end{aligned}$$

provided $n \ne 0$. In the case that $n = 0$,

$$a_0 = \int_{-1/2}^{1/2} dx = 1.$$

Thus, the Fourier representation is

$$f(x) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\cos n\pi x.$$

The plot of S_7 on $[-1, 1]$ and $[-2, 2]$ follows.

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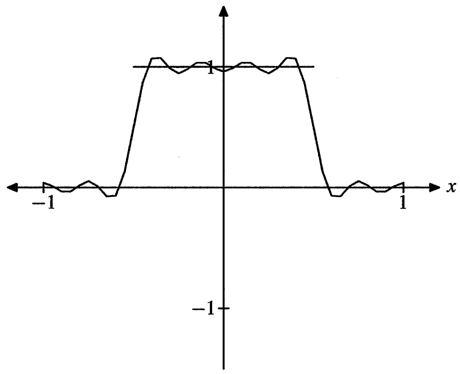

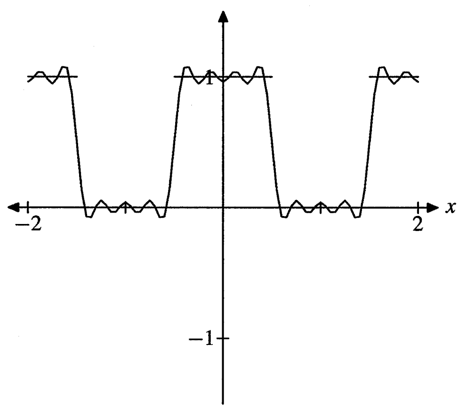

Note the slower convergence due to the discontinuities at the half integers.

19. Consider $f(x) = e^{rx}$ on the interval $[-\pi, \pi]$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{rx}dx = \frac{1}{\pi r}e^{rx}\Big|_{-\pi}^{\pi} = \frac{2}{\pi r}\sinh \pi r.$$

For $n \geq 1$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{rx}\cos nx\, dx.$$

Two applications of integration by parts provides

$$\begin{aligned}
a_n &= \frac{1}{\pi}\cdot\frac{n^2}{n^2+r^2}\left[\frac{e^{rx}\sin nx}{n} + \frac{re^{rx}\cos nx}{n^2}\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi(n^2+r^2)}\left[e^{rx}(n\sin nx + r\cos nx)\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi(n^2+r^2)}[e^{r\pi} r\cos n\pi \\
&\qquad - e^{-r\pi} r\cos n\pi] \\
&= \frac{r(-1)^n}{\pi(n^2+r^2)}[e^{r\pi} - e^{-r\pi}] \\
&= \frac{2r(-1)^n}{\pi(n^2+r^2)}\sinh \pi r.
\end{aligned}$$

Next,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{rx}\sin nx\, dx.$$

Again, two applications of integration by parts provides the answer.

$$\begin{aligned}
b_n &= \frac{1}{\pi}\cdot\frac{n^2}{n^2+r^2}\left[-\frac{e^{rx}\cos nx}{n} + \frac{re^{rx}\sin nx}{n^2}\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi(n^2+r^2)}\left[e^{rx}(-n\cos nx + r\sin nx)\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi(n^2+r^2)}\big[e^{r\pi}(-n\cos n\pi) \\
&\qquad - e^{-r\pi}(-n\cos n\pi)\big] \\
&= \frac{n(-1)^{n+1}}{\pi(n^2+r^2)}\left[e^{r\pi} - e^{-r\pi}\right] \\
&= \frac{2n(-1)^{n+1}}{\pi(n^2+r^2)}\sinh r\pi.
\end{aligned}$$

Thus, the Fourier representation is

$$\begin{aligned}
e^{rx} &= \frac{\sinh \pi r}{\pi r} + \sum_{n=1}^{\infty}\frac{2r(-1)^n\sinh \pi r}{\pi(n^2+r^2)}\cos nx \\
&\quad - \sum_{n=1}^{\infty}\frac{2n(-1)^n\sinh \pi r}{\pi(n^2+r^2)}\sin nx.
\end{aligned}$$

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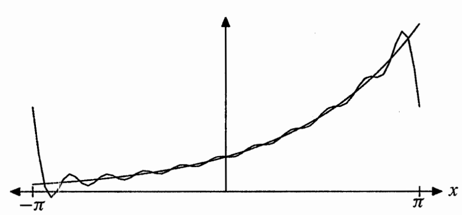
In the case that $r = 1/2$,

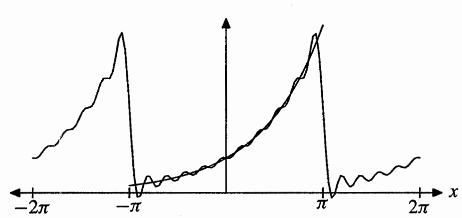
$$f(x) = \frac{2}{\pi}\sinh\frac{\pi}{2} + \frac{\sinh(\pi/2)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1/4}\cos nx$$
$$-\frac{2\sinh(\pi/2)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1/4}\sin nx.$$

This is perhaps better written as

$$f(x) = \frac{1}{\pi}\sinh\frac{\pi}{2}\left[2 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1/4}\cos nx\right.$$
$$\left.-2\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1/4}\sin nx\right].$$

The following plots use $N = 10$ terms.





20. In Exercise 19, we found that on the interval $[-\pi, \pi]$,

$$e^{rx} = \frac{\sinh \pi r}{\pi r} + \sum_{n=1}^{\infty}\frac{2r(-1)^n \sinh \pi r}{\pi(n^2+r^2)}\cos nx$$
$$-\sum_{n=1}^{\infty}\frac{2n(-1)^n \sinh \pi r}{\pi(n^2+r^2)}\sin nx.$$

Thus, with $r = 1$,

$$e^x = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\cos nx$$
$$-\frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\sin nx.$$

Similarly, with $r = -1$,

$$e^{-x} = \frac{\sinh(-\pi)}{-\pi} - \frac{2\sinh(-\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\cos nx$$
$$-\frac{2\sinh(-\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1}\sin nx.$$

However, $\sinh x$ is an odd function, since

$$\sinh(-\pi) = \frac{e^{-\pi} - e^{\pi}}{2}$$
$$= -\frac{e^{\pi} - e^{-\pi}}{2} = -\sinh(\pi).$$

Thus,

$$e^{-x} = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\cos nx$$
$$+\frac{2\sin h\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1}\sin nx.$$

Subtracting,

$$e^x - e^{-x} = -\frac{4\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1}\sin nx.$$

Thus,

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$
$$= \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1} n}{n^2+1}\sin nx.$$

Note that the answer contains only sine terms, which is consistent with the fact that $f(x) = \sinh x$ is odd on the interval $[-\pi, \pi]$. In similar fashion,

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$
$$= \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\cos nx.$$

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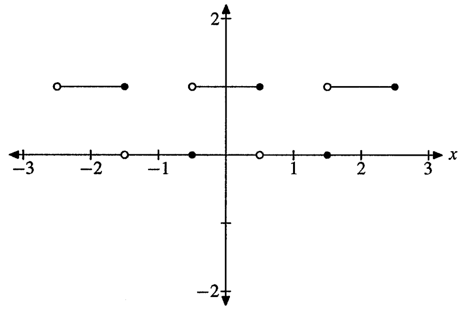
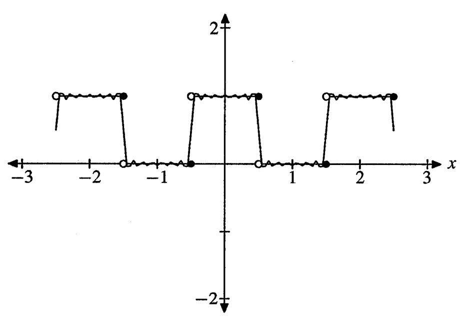
21. Consider

$$f(x) = \begin{cases} 0, & -1 < x \le -1/2, \\ 1, & -1/2 < x \le 1/2, \\ 0, & 1/2 < x \le 1. \end{cases}$$

The plot of $f_p(x)$ is shown in blue.



The Fourier series will converge to $f_p(x)$ whenever f_p is continuous at x. At each point of discontinuity we have a limit from one side equalling 1, while the limit coming in from the other side is 0. Thus, the Fourier series should converge to 1/2 at these points, as is evident in the following image depicting $S_{13}(x)$.



22. Using the limit quotient definition of the derivative and the periodicity of f we have

$$\begin{aligned} f''(x+T) &= \lim_{h\to 0} \frac{f(x+T+h) - f(x+T)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= f'(x). \end{aligned}$$

Hence f' is periodic with period T.

23. First notice that

$$\int_0^T f(x)\,dx = \int_0^a f(x)\,dx + \int_a^T f(x)\,dx.$$

For the first integral, we use the periodicity of f and the substitution $x = u + T$ to get

$$\int_0^a f(x)\,dx = \int_T^{a+T} f(u+T)\,du = \int_T^{a+T} f(u)\,du.$$

Since u is a dummy variable in the last integral, we can replace it by x. Then we get

$$\begin{aligned} \int_0^T f(x)\,dx &= \int_T^{a+T} f(x)\,dx + \int_a^T f(x)\,dx \\ &= \int_a^{a+T} f(x)\,dx. \end{aligned}$$

Using this with a replaced by b, we get the final result

$$\int_b^{b+T} f(x)\,dx = \int_0^T f(x)\,dx = \int_a^{a+T} f(x)\,dx$$

If f is periodic with period L, then $F(x) = f(x)\cos(n\pi x/L)$ is also periodic with period L. Hence, for any c,

$$a_n = \frac{1}{L}\int_{-L}^{L} F(x)\,dx = \frac{1}{L}\int_c^{c+2L} F(x)\,dx.$$

The same argument works for b_n.

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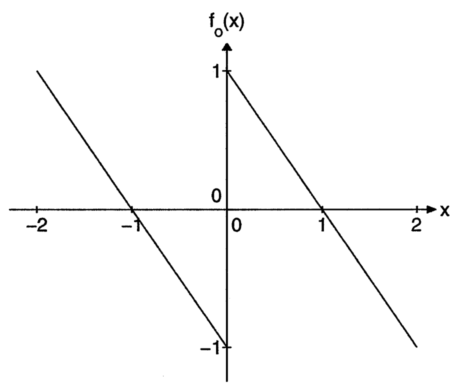
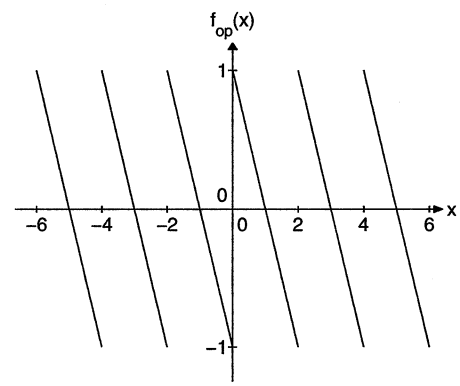
Section 12.3. Fourier Cosine and Sine Series

1. If $f(x) = 1 - x$ on $[0, 2]$, then

$$f_o(x) = \begin{cases} -(1+x), & -2 \leq x < 0, \\ 0, & x = 0, \\ 1 - x, & 0 < x \leq 2. \end{cases}$$

The graphs of the odd extension f_o on $[-2, 2]$ and the odd periodic extension f_{op} on $[-6, 6]$ follow.

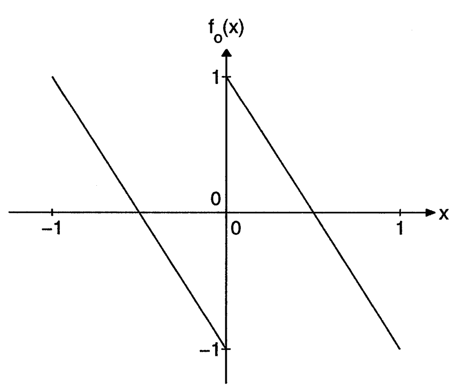


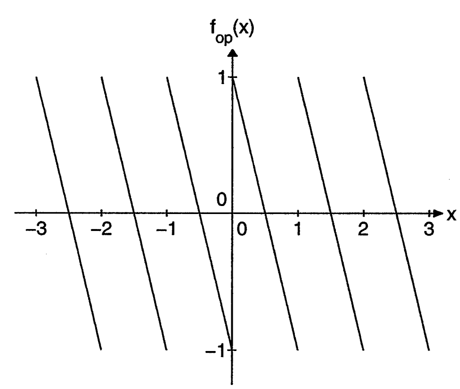


2. If $f(x) = 1 - 2x$ on $[0, 1]$, then

$$f_o(x) = \begin{cases} -(1+2x), & -1 \leq x < 0, \\ 0, & x = 0, \\ 1 - 2x, & 0 < x \leq 1. \end{cases}$$

The graphs of the odd extension f_o on $[-1, 1]$ and the odd periodic extension f_{op} on $[-3, 3]$ follow.





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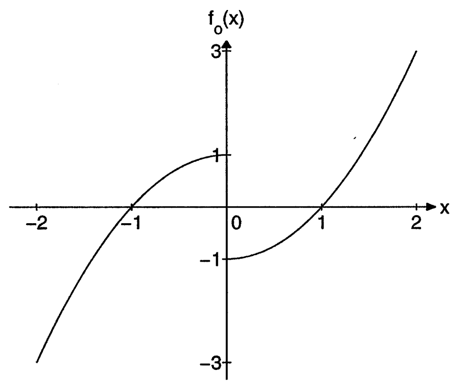
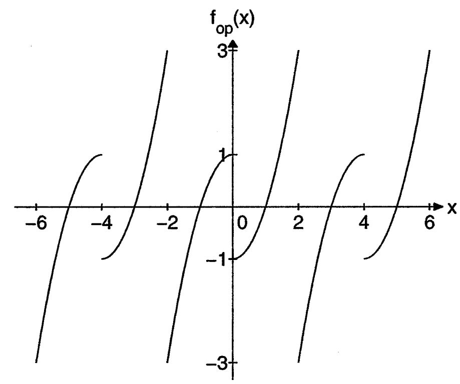
3. If $f(x) = x^2 - 1$ on $[0, 2]$, then

$$f_o(x) = \begin{cases} -(x^2 - 1), & -2 \le x < 0, \\ 0, & x = 0, \\ x^2 - 1, & 0 < x \le 2. \end{cases}$$

The graphs of the odd extension f_o on $[-2, 2]$ and the odd periodic extension f_{op} on $[-6, 6]$ follow.

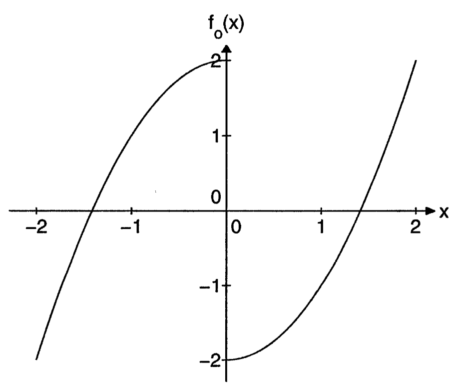


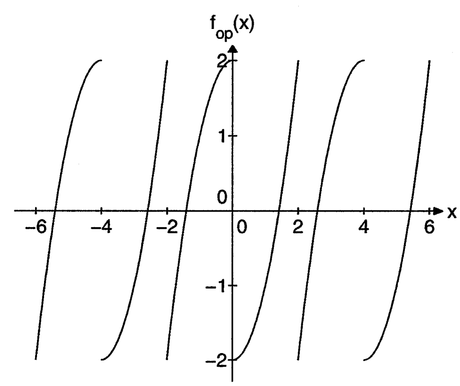


4. If $f(x) = x^2 - 2$ on $[0, 2]$, then

$$f_o(x) = \begin{cases} -(x^2 - 2), & -2 \le x < 0, \\ 0, & x = 0, \\ x^2 - 2, & 0 < x \le 2. \end{cases}$$

The graphs of the odd extension f_o on $[-2, 2]$ and the odd periodic extension f_{op} on $[-6, 6]$ follow.





5. If $f(x) = 1 - x$ on $[0, 2]$, then

$$f_e(x) = \begin{cases} 1 + x, & -2 \le x < 0, \\ 1 - x, & 0 \le x \le 2. \end{cases}$$

The graphs of the even extension f_e on $[-2, 2]$ and the even periodic extension f_{ep} on $[-6, 6]$ follow.

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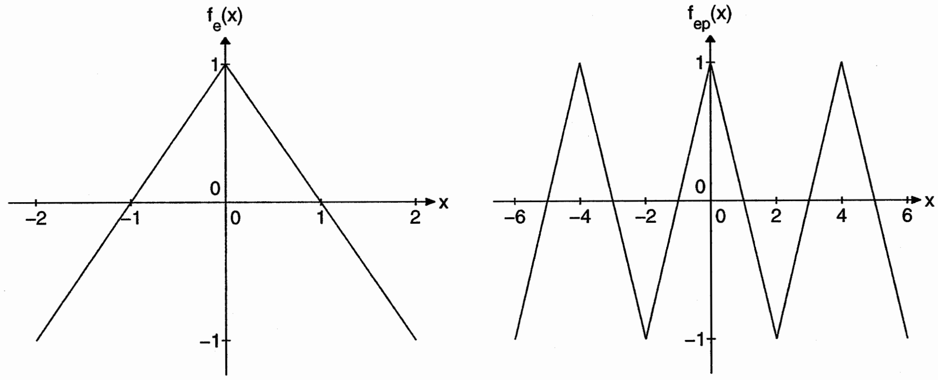
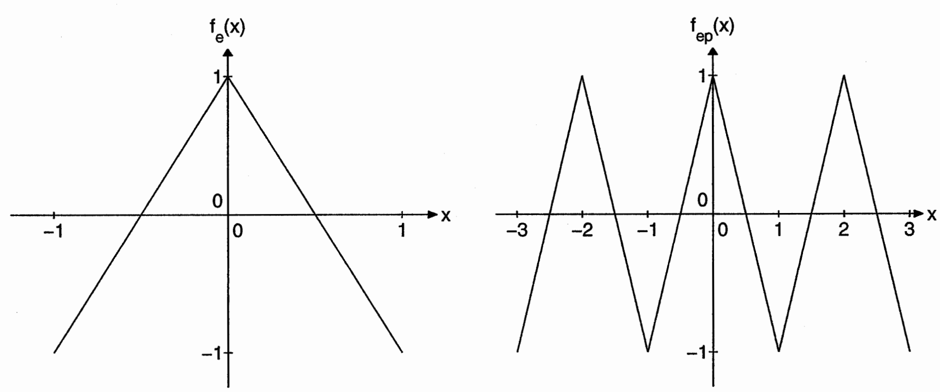


6. If $f(x) = 1 - 2x$ on $[0, 1]$, then

$$f_e(x) = \begin{cases} 1 + 2x, & -1 \le x < 0, \\ 1 - 2x, & 0 \le x \le 1. \end{cases}$$

The graphs of the even extension f_e on $[-1, 1]$ and the even periodic extension f_{ep} on $[-3, 3]$ follow.



7. If $f(x) = x^2 - 1$ on $[0, 2]$, then

$$f_e(x) = \begin{cases} x^2 - 1, & -2 \le x < 0, \\ x^2 - 1, & 0 \le x \le 2. \end{cases}$$

The graphs of the even extension f_e on $[-2, 2]$ and the even periodic extension f_{ep} on $[-6, 6]$ follow.

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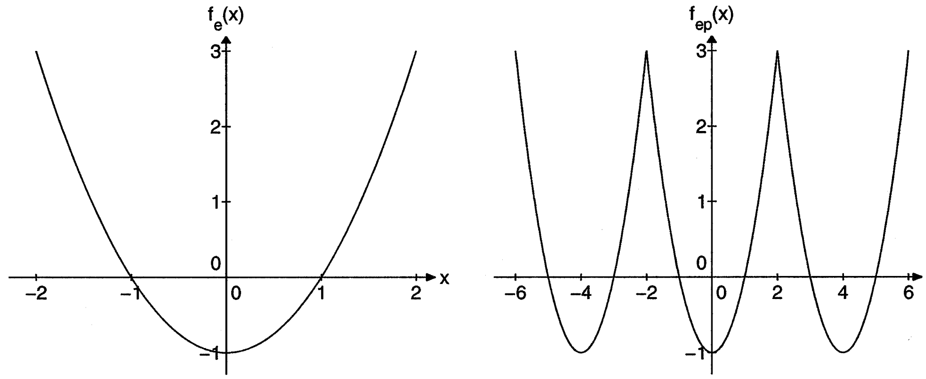
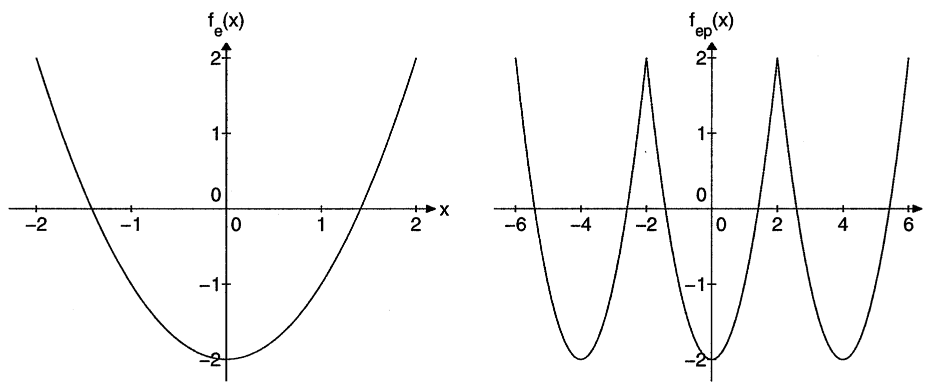


8. If $f(x) = x^2 - 2$ on $[0, 2]$, then

$$f_e(x) = \begin{cases} x^2 - 2, & -2 \le x < 0, \\ x^2 - 2, & 0 \le x \le 2. \end{cases}$$

The graphs of the even extension f_e on $[-2, 2]$ and the even periodic extension f_{ep} on $[-6, 6]$ follow.



9. The function $f(x) = x$ on $[0, \pi]$ has even extension

$$f_e(x) = \begin{cases} -x, & -\pi \le x < 0, \\ x, & 0 \le x \le \pi, \end{cases}$$

on the interval $[-\pi, \pi]$. The Fourier expansion of $f_e(x)$ has only cosine terms and

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_e(x)\cos nx\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} f_e(x)\cos nx\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} a_n &= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] \\ &= \frac{2}{\pi n^2}[(-1)^n - 1], \end{aligned}$$

provided $n \neq 0$. In the case $n = 0$,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{1}{2}x^2\right]_0^{\pi} = \pi.$$

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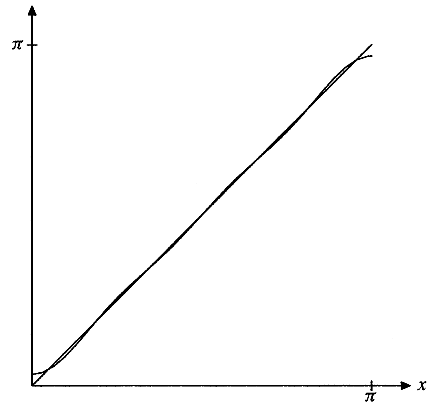
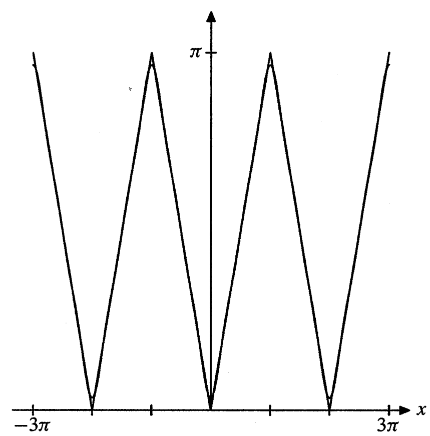
Thus, the Fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$
$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}[(-1)^n - 1] \cos nx.$$

The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.





10. The function $f(x) = \sin x$ has even extension

$$f_e(x) = \begin{cases} -\sin x, & -\pi \le x < 0, \\ \sin x, & 0 \le x \le \pi, \end{cases}$$

on the interval $[-\pi, \pi]$. Hence, the Fourier series expansion has only cosine terms and

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_e(x) \cos nx \, dx$$
$$= \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx \, dx.$$

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Using a product to sum identity,

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}[\sin(1-n)x + \sin(1+n)x]\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\,dx \\
&= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} \\
&= \frac{1}{\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} - \left(\frac{-1}{n+1} + \frac{1}{n-1}\right)\right] \\
&= \frac{1}{\pi}\left[\frac{-n(-1)^{n+1} + (-1)^{n+1} + n(-1)^{n-1} + (-1)^{n-1} + n - 1 - n - 1}{(n+1)(n-1)}\right] \\
&= \frac{1}{\pi}\left[\frac{n(-1)^{n-1}[1-(-1)^2] + (-1)^{n-1}[1+(-1)^2] - 2}{n^2-1}\right] \\
&= \frac{1}{\pi}\left[\frac{2(-1)^{n-1} - 2}{n^2-1}\right] \\
&= \frac{2}{\pi}\left[\frac{(-1)^{n-1} - 1}{n^2-1}\right],
\end{aligned}$$

provided $n \neq 1$. In the case that $n = 1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\,dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x\,dx = 0.$$

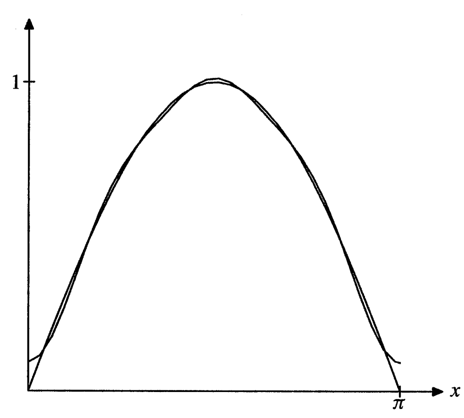
Note that

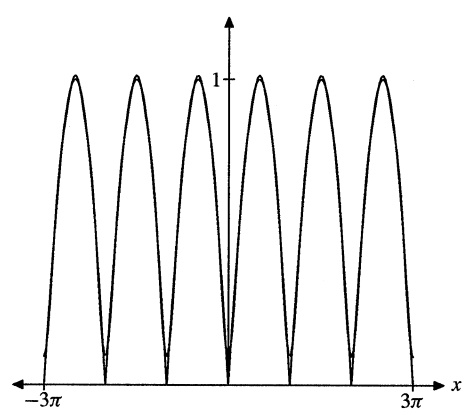
$$a_0 = \frac{2}{\pi}\left[\frac{(-1)^{-1} - 1}{0^2 - 1}\right] = \frac{2}{\pi}\left[\frac{-2}{-1}\right] = \frac{4}{\pi}.$$

Hence, the Fourier series expansion for f is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{2}{\pi}\left[\frac{(-1)^{n-1} - 1}{n^2-1}\right]\cos nx.$$

The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.





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11. As $f(x) = \cos x$ is already a cosine series, any analysis we do should agree with this fact. The function has even extension

$$f_e(x) = \begin{cases} \cos(-x), & -\pi \le x < 0, \\ \cos x, & 0 \le x \le \pi, \end{cases}$$
$$= \begin{cases} \cos x, & -\pi \le x < 0, \\ \cos x, & 0 \le x \le \pi, \end{cases}$$

on the interval $[-\pi, \pi]$. Hence, the Fourier series expansion of $f_e(x)$ has only cosine terms and

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{\pi}^{\pi} f_e(x)\cos nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} \cos x \cos nx\,dx. \end{aligned}$$

Using a product to sum identity,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}[\cos(1-n)x + \cos(1+n)x]\,dx \\ &= \frac{1}{\pi}\left[\frac{\sin(n-1)x}{n-1} + \frac{\sin(n+1)x}{n+1}\right]_0^{\pi} \\ &= 0, \end{aligned}$$

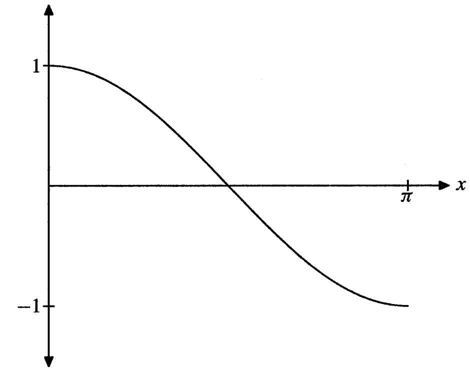
provided $n \neq 1$. In the case that $n = 1$,

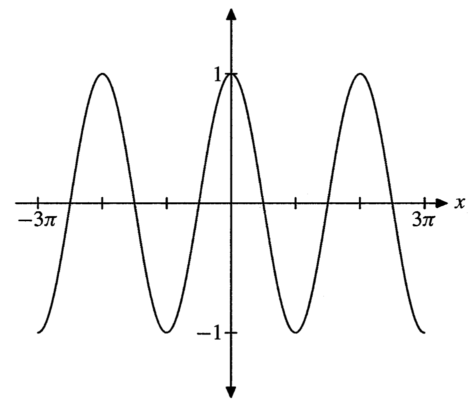
$$\begin{aligned} a_1 &= \frac{2}{\pi}\int_0^{\pi} \cos^2 x\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{1+\cos 2x}{2}\,dx \\ &= \frac{1}{\pi}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi} \\ &= \frac{1}{\pi}[\pi] \\ &= 1. \end{aligned}$$

Hence, the Fourier series expansion is

$$f(x) = \cos x.$$

No surprises here! The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.





12. The function $f(x) = 1$, being already even on $[0, \pi]$, has even extension $f_e(x) = 1$. Moreover, the function is already a Fourier series with $a_0 = 1$ and $a_n = b_n = 0$, $n \ge 1$. So any analysis should agree with this fact. Thus, $f_e(x)$, being even on $[-\pi, \pi]$ will only contain cosine terms and

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_e(x)\cos nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} \cos nx\,dx \\ &= \frac{2}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} \\ &= 0, \end{aligned}$$

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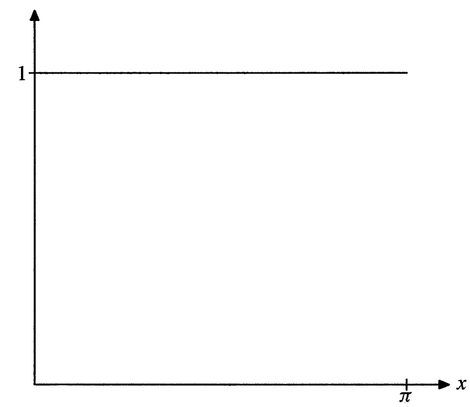
provided $n \neq 0$. For the case $n = 0$,

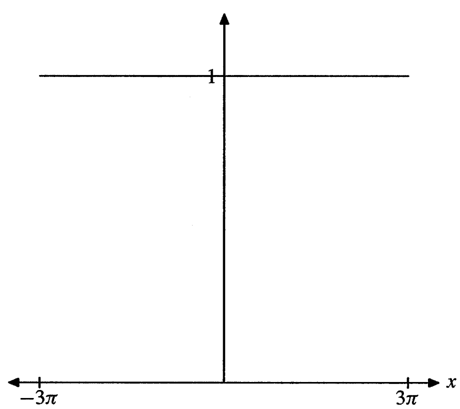
$$a_0 = \frac{2}{\pi} \int_0^{\pi} dx = 2.$$

Hence, the Fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = 1.$$

No surprises here! The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.





13. The function $f(x) = \pi - x$ on $[0, \pi]$ has even extension

$$f_e(x) = \begin{cases} \pi + x, & -\pi \leq x < 0, \\ \pi - x, & 0 \leq x \leq \pi, \end{cases}$$

on the interval $[-\pi, \pi]$. The Fourier series expansion of $f_e(x)$ has only cosine terms and

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_e(x) \cos nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} (\pi - x) \cos nx \, dx. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} a_n &= \frac{2}{\pi} \left[\frac{(\pi - x) \sin nx}{n} - \frac{\cos nx}{n^2} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[-\frac{\cos n\pi}{n^2} + \frac{1}{n^2} \right] \\ &= \frac{2}{\pi n^2} [1 - (-1)^n], \end{aligned}$$

provided $n \neq 0$. In the case that $n = 0$,

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} (\pi - x) dx \\ &= \frac{2}{\pi} \left[\pi x - \frac{1}{2} x^2 \right]_0^{\pi} \\ &= \pi. \end{aligned}$$

Thus, the Fourier series expansion is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ &= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} [1 - (-1)^n] \cos nx. \end{aligned}$$

The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.

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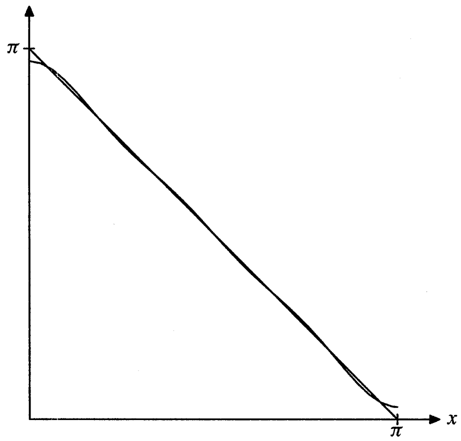
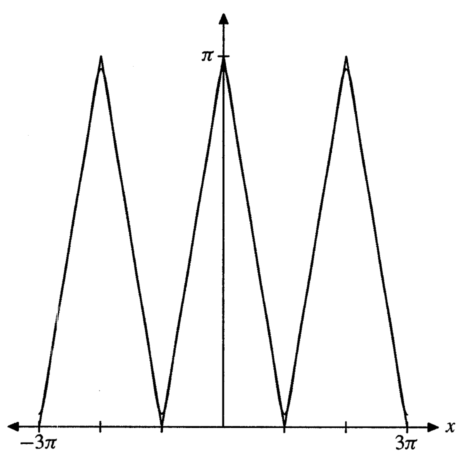




14. The function $f(x) = x^2$ on $[0, \pi]$ has even extension $f_e(x) = x^2$ on $[-\pi, \pi]$. The Fourier expansion will have cosine terms only and

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_e(x)\cos nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx \end{aligned}$$

Integrating by parts,

$$\begin{aligned} a_n &= \frac{2}{\pi}\left[\frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n^2}\right] \\ &= (-1)^n\frac{4}{n^2}, \end{aligned}$$

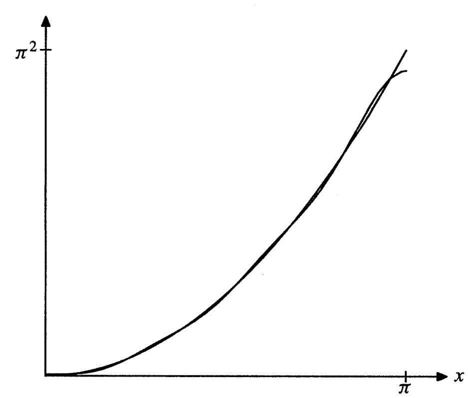
provided $n \neq 0$. In the case that $n = 0$,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{1}{3}x^3\right]_0^{\pi} = \frac{2\pi^2}{3}.$$

Hence, the Fourier expansion is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos nx. \end{aligned}$$

The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.



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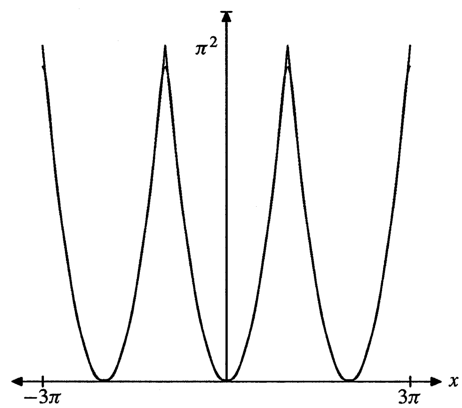


15. The function $f(x) = x^3$ on $[0, \pi]$ has even extension

$$f_e(x) = \begin{cases} -x^3, & -\pi \le x < 0, \\ x^3, & 0 \le x \le \pi, \end{cases}$$

on the interval $[-\pi, \pi]$. The Fourier expansion of $f_e(x)$ has only cosine terms and

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_e(x)\cos nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} x^3 \cos nx\,dx. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} a_n &= \frac{2}{\pi}\left[\frac{x^3\sin nx}{n} + \frac{3x^2\cos nx}{n^2}\right. \\ &\qquad \left. - \frac{6x\sin nx}{n^3} - \frac{6\cos nx}{n^4}\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[\frac{3\pi^2\cos n\pi}{n^2} - \frac{6\cos n\pi}{n^4} + \frac{6}{n^4}\right] \\ &= \frac{6\pi(-1)^n}{n^2} - \frac{12}{\pi n^4}\left((-1)^n - 1\right), \end{aligned}$$

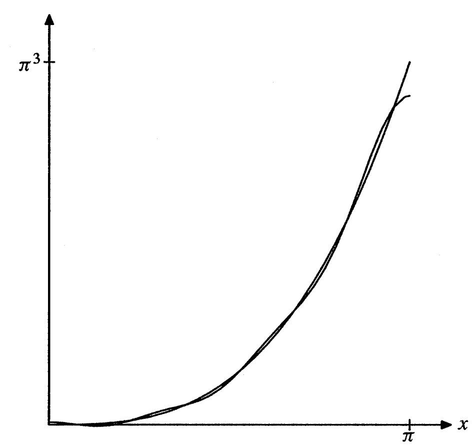
provided $n \neq 0$. In the case that $n = 0$,

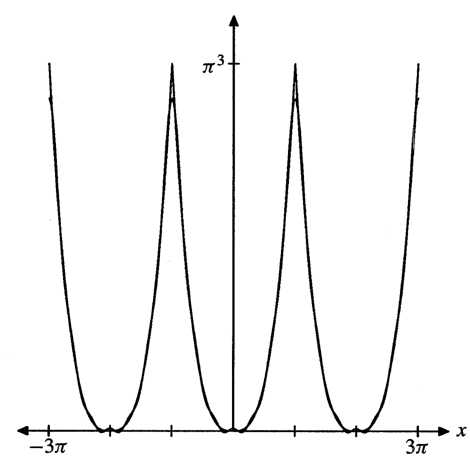
$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^3\,dx = \frac{2}{\pi}\left[\frac{1}{4}x^4\right]_0^{\pi} = \frac{\pi^3}{2}.$$

Hence, the Fourier series expansion is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx \\ &= \frac{\pi^3}{4} + \sum_{n=1}^{\infty}\left[\frac{6\pi(-1)^n}{n^2}\right. \\ &\qquad \left. - \frac{12}{\pi n^4}\left((-1)^n - 1\right)\right]\cos nx. \end{aligned}$$

The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.





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16. The function $f(x) = x^4$ on $[0, \pi]$ has even extension $f_e(x) = x^4$ on $[-\pi, \pi]$. The Fourier expansion of $f_e(x)$ has only cosine terms and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_e(x)\cos nx$$
$$= \frac{2}{\pi}\int_0^{\pi} x^4 \cos nx\,dx.$$

Integrating by parts,

$$a_n = \frac{2}{\pi}\left[\frac{x^4 \sin nx}{n} + \frac{4x^3\cos nx}{n^2} - \frac{12x^2\sin nx}{n^3} - \frac{24x\cos nx}{n^4} + \frac{24\sin nx}{n^5}\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\frac{4\pi^3\cos n\pi}{n^2} - \frac{24\pi\cos n\pi}{n^4}\right]$$
$$= 8(-1)^n\left[\frac{\pi^2}{n^2} - \frac{6}{n^4}\right]$$
$$= 8(-1)^n\frac{\pi^2 n^2 - 6}{n^4},$$

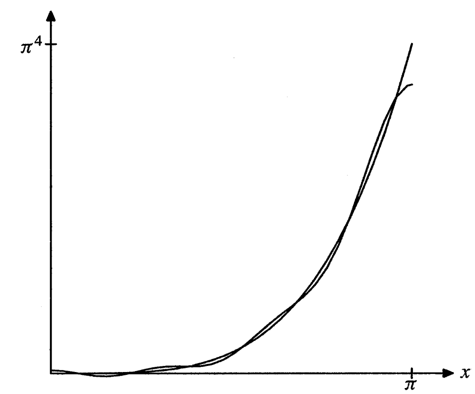
provided $n \neq 0$. In the case that $n = 0$,

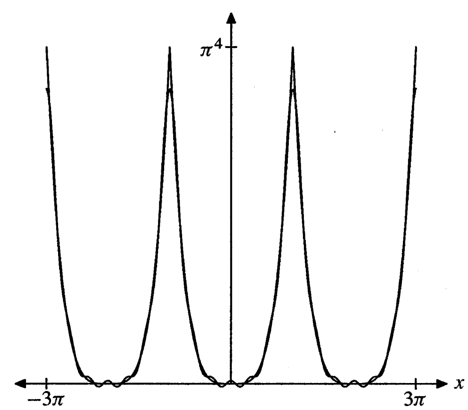
$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^4\,dx = \frac{2}{\pi}\left[\frac{1}{5}x^5\right]_0^{\pi} = \frac{2\pi^4}{5}.$$

Thus, the Fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx$$
$$= \frac{\pi^4}{5} + \sum_{n=1}^{\infty} 8(-1)^n\frac{\pi^2 n^2 - 6}{n^4}\cos nx.$$

The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.





17. The function

$$f(x) = \begin{cases} 1, & 0 \le x < \pi/2, \\ 0, & \pi/2 \le x \le \pi, \end{cases}$$

has even extension

$$f_e(x) = \begin{cases} 0, & -\pi \le x < -\pi/2, \\ 1, & -\pi/2 \le x < \pi/2, \\ 0, & \pi/2 \le x < \pi, \end{cases}$$

on the interval $[-\pi, \pi]$. The Fourier expansion of $f_e(x)$ has only cosine terms and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_e(x)\cos nx\,dx$$
$$= \frac{2}{\pi}\int_0^{\pi} f_e(x)\cos nx\,dx.$$

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However, $f_e(x) = 0$ on $[\pi/2, \pi]$, so this last integral becomes

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi/2} \cos nx\,dx = \frac{2}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi/2} \\ &= \frac{2}{n\pi}\sin\frac{n\pi}{2}, \end{aligned}$$

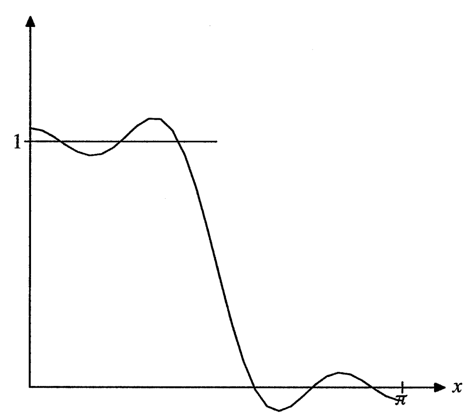
provided $n \neq 0$. In the case that $n = 0$,

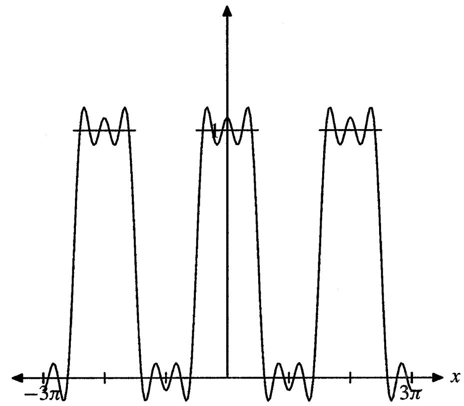
$$a_0 = \frac{2}{\pi}\int_0^{\pi/2} dx = 1.$$

Thus, the Fourier series expansion is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{2}\cos nx. \end{aligned}$$

The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.





18. The function

$$f(x) = \begin{cases} x, & 0 \le x < \pi/2, \\ \pi/2, & \pi/2 \le x \le \pi, \end{cases}$$

has even extension

$$f_e(x) = \begin{cases} \pi/2, & -\pi \le x < -\pi/2, \\ -x, & -\pi/2 \le x < 0, \\ x, & 0 \le x < \pi/2, \\ \pi/2, & \pi/2 \le x \le \pi, \end{cases}$$

on the interval $[-\pi, \pi]$. The Fourier expansion of $f_e(x)$ has only cosine terms and

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_e(x)\cos nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} f_e(x)\cos nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi/2} x\cos nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}\frac{\pi}{2}\cos nx\,dx \end{aligned}$$

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Integrating by parts, the first integral is

$$\begin{aligned}\frac{2}{\pi}\int_0^{\pi/2} x\cos nx\,dx &= \frac{2}{\pi}\left[\frac{x\sin nx}{n}+\frac{\cos nx}{n^2}\right]_0^{\pi/2}\\ &= \frac{2}{\pi}\left[\frac{\frac{\pi}{2}\sin\frac{n\pi}{2}}{n}+\frac{\cos\frac{n\pi}{2}}{n^2}-\frac{1}{n^2}\right]\\ &= \frac{\sin\frac{n\pi}{2}}{n}+\frac{2\cos\frac{n\pi}{2}}{\pi n^2}-\frac{2}{\pi n^2}.\end{aligned}$$

The second integral is

$$\begin{aligned}\frac{2}{\pi}\int_{\pi/2}^{\pi}\frac{\pi}{2}\cos nx\,dx = \int_{\pi/2}^{\pi}\cos nx\,dx &= \frac{1}{n}\sin nx\Big|_{\pi/2}^{\pi}\\ &= -\frac{1}{n}\sin\frac{n\pi}{2}\end{aligned}$$

Adding these last two results,

$$\begin{aligned}a_n &= \frac{2}{\pi n^2}\cos\frac{n\pi}{2}-\frac{2}{\pi n^2}\\ &= \frac{2}{\pi n^2}\left[\cos\frac{n\pi}{2}-1\right],\end{aligned}$$

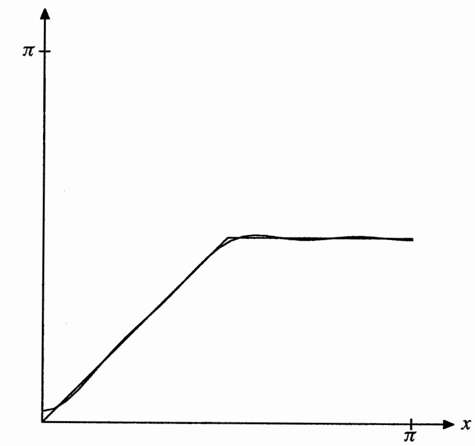
provided $n \neq 0$. In the case that $n = 0$,

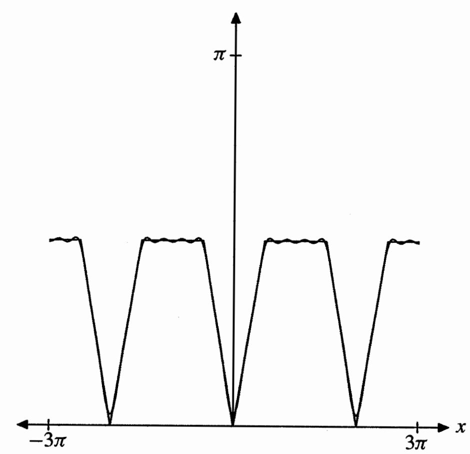
$$\begin{aligned}a_0 &= \frac{2}{\pi}\int_0^{\pi/2} x\,dx+\int_{\pi/2}^{\pi} dx\\ &= \frac{2}{\pi}\left[\frac{1}{2}x^2\right]_0^{\pi/2}+\left(\pi-\frac{\pi}{2}\right)\\ &= \frac{3\pi}{4}.\end{aligned}$$

Hence, the Fourier expansion is

$$\begin{aligned}f(x) &= \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx\\ &= \frac{3\pi}{8}+\sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left[\cos\frac{n\pi}{2}-1\right]\cos nx.\end{aligned}$$

The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.





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19. The function $f(x) = x\cos x$ on $[0, \pi]$ has even extension

$$f_e(x) = \begin{cases} -x\cos x, & -\pi \le x < 0, \\ x\cos x, & 0 \le x < \pi, \end{cases}$$

on the interval $[-\pi, \pi]$. The Fourier expansion of $f_e(x)$ has only cosine terms and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_e(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos x\cos nx\,dx.$$

Using a product to sum identity,

$$a_n = \frac{2}{\pi}\int_0^{\pi} \frac{x}{2}[\cos(1-n)x + \cos(1+n)x]\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos(n-1)x\,dx + \frac{1}{\pi}\int_0^{\pi} x\cos(n+1)x\,dx.$$

Integrating by parts, the first integral becomes

$$\begin{aligned}
\frac{1}{\pi}\int_0^{\pi} x\cos(n-1)x\,dx &= \frac{1}{\pi}\left[\frac{x\sin(n-1)x}{n-1} + \frac{\cos(n-1)x}{(n-1)^2}\right]_0^{\pi} \\
&= \frac{1}{\pi}\left[\frac{\cos(n-1)\pi}{(n-1)^2} - \frac{1}{(n-1)^2}\right] \\
&= \frac{1}{\pi(n-1)^2}\left[(-1)^{n-1} - 1\right],
\end{aligned}$$

provided $n \neq 1$. Similarly, the second integral is

$$\frac{1}{\pi}\int_0^{\pi} x\cos(n+1)x\,dx = \frac{1}{\pi(n+1)^2}\left[(-1)^{n+1} - 1\right].$$

Adding,

$$\begin{aligned}
a_n &= \frac{1}{\pi(n-1)^2}\left[(-1)^{n-1} - 1\right] + \frac{1}{\pi(n+1)^2}\left[(-1)^{n+1} - 1\right] \\
&= \frac{(n+1)^2\left((-1)^{n-1} - 1\right) + (n-1)^2\left((-1)^{n+1} - 1\right)}{\pi(n-1)^2(n+1)^2} \\
&= \frac{\left((-1)^{n-1} - 1\right)\left((n+1)^2 + (n-1)^2\right)}{\pi(n-1)^2(n+1)^2} \\
&= \frac{2\left((-1)^{n-1} - 1\right)(n^2+1)}{\pi(n-1)^2(n+1)^2},
\end{aligned}$$

provided $n \neq 1$. In the case that $n = 1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\cos^2 x\,dx.$$

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Using the half angle identity,

$$\begin{aligned} a_1 &= \frac{2}{\pi}\int_0^{\pi} x\left(\frac{1+\cos 2x}{2}\right)dx \\ &= \frac{1}{\pi}\int_0^{\pi}(x + x\cos 2x)\,dx \\ &= \frac{1}{\pi}\left[\frac{1}{2}x^2 + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right]_0^{\pi} \\ &= \frac{1}{\pi}\left[\frac{\pi^2}{2}\right] \\ &= \frac{\pi}{2}. \end{aligned}$$

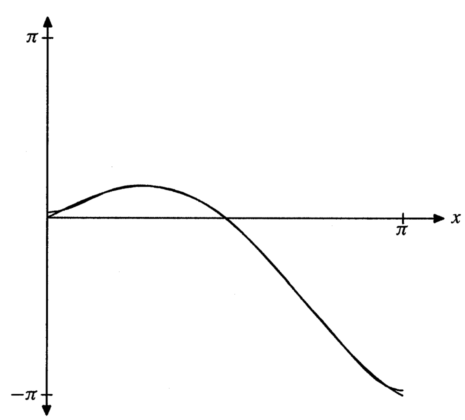
Also, note that with computation above,

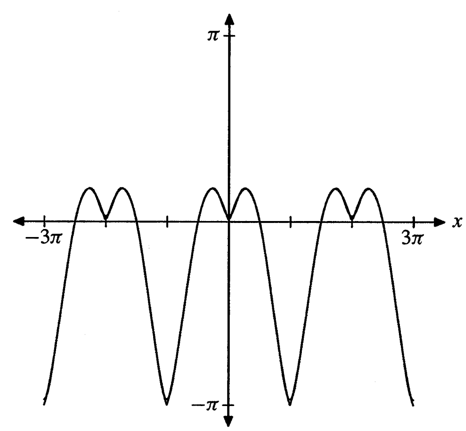
$$a_0 = \frac{2(-1-1)(1)}{\pi\cdot 1\cdot 1} = -\frac{4}{\pi}.$$

Hence, the Fourier expansion is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n\cos nx \\ &= -\frac{2}{\pi} + \frac{\pi}{2}\cos x + \sum_{n=2}^{\infty}\frac{2\left((-1)^{n-1}-1\right)(n^2+1)}{\pi(n-1)^2(n+1)^2}\cos nx. \end{aligned}$$

The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.





20. The function $f(x) = x\sin x$ on $[0, \pi]$ has even extension $f_e(x) = x\sin x$ on the interval $[-\pi, \pi]$. Hence, the Fourier expansion of $f_e(x)$ has only cosine terms and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_e(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx.$$

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Using a product to sum identity,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} \frac{x}{2}\,[\sin(1-n)x + \sin(1+n)x]\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\sin(n+1)x\,dx \\ &\quad - \frac{1}{\pi}\int_0^{\pi} x\sin(n-1)x\,dx. \end{aligned}$$

Using integration by parts, the first integral becomes

$$\begin{aligned} &\frac{1}{\pi}\int_0^{\pi} x\sin(n+1)x\,dx \\ &= \frac{1}{\pi}\left[\frac{-x\cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2}\right]_0^{\pi} \\ &= \frac{1}{\pi}\left[\frac{-\pi\cos(n+1)\pi}{n+1}\right] \\ &= \frac{(-1)^{n+2}}{n+1}. \end{aligned}$$

Similarly, the second integral is

$$\frac{1}{\pi}\int_0^{\pi} x\sin(n-1)x\,dx = \frac{(-1)^n}{n-1}.$$

Subtracting,

$$\begin{aligned} a_n &= \frac{(-1)^{n+2}}{n+1} - \frac{(-1)^n}{n-1} \\ &= \frac{(-1)^{n+2}(n-1) - (-1)^n(n+1)}{(n+1)(n-1)} \\ &= \frac{(-1)^n[n-1-n-1]}{(n+1)(n-1)} \\ &= \frac{(-1)^n(-2)}{n^2-1} \\ &= \frac{2(-1)^{n+1}}{n^2-1}, \end{aligned}$$

provided $n \neq 1$. Note that $a_0 = 2$. In the event that $n = 1$.

$$\begin{aligned} a_1 &= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx. \end{aligned}$$

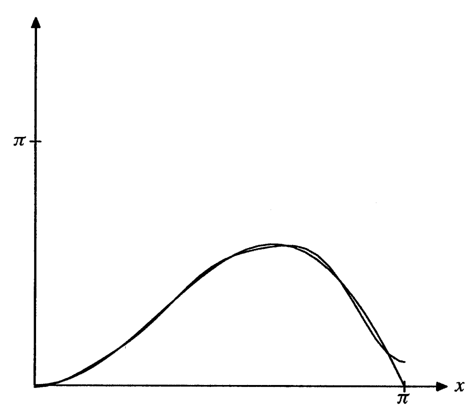
Integrating by parts,

$$\begin{aligned} a_1 &= \frac{1}{\pi}\left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi} \\ &= \frac{1}{\pi}\left[\frac{-\pi}{2}\right] \\ &= -\frac{1}{2}. \end{aligned}$$

Hence, the Fourier expansion is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx \\ &= 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1}\cos nx. \end{aligned}$$

The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.



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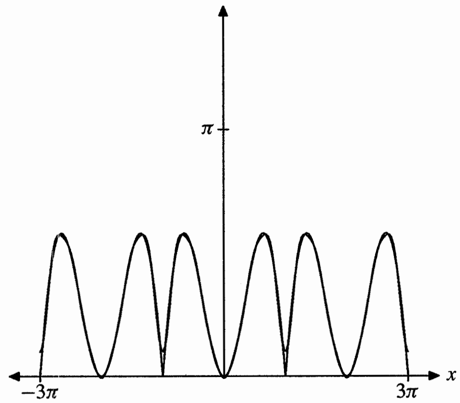
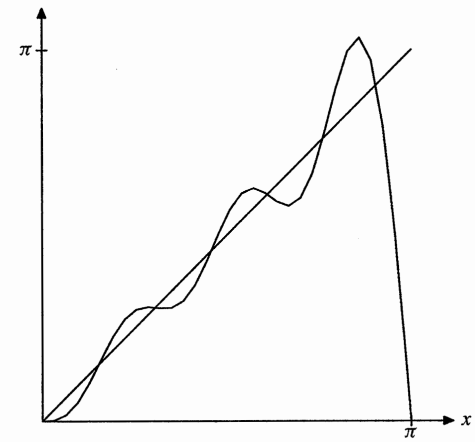




21. The function $f(x) = x$ on $[0, \pi]$ is already odd on the interval $[-\pi, \pi]$, so it is the same as its odd extension. Thus the Fourier expansion has only sine terms and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx.$$

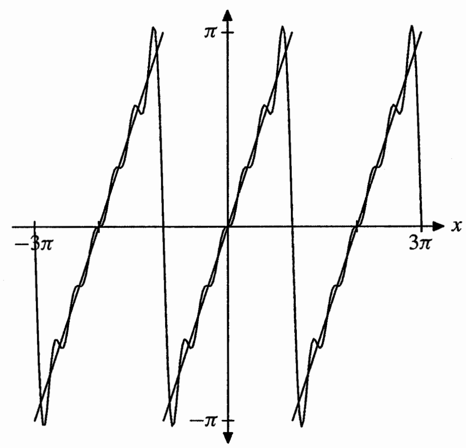
Integrating by parts,

$$\begin{aligned} b_n &= \frac{2}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] \\ &= 2\left[\frac{-(-1)^n}{n}\right] \\ &= (-1)^{n+1}\frac{2}{n}, \end{aligned}$$

for $n \geq 1$. Thus, the Fourier series expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx.$$

The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.



22. The function $f(x) = \sin x$ on $[0, \pi]$ is already odd on the interval $[-\pi, \pi]$, so it is the same as its odd extension. Thus, the Fourier expansion has only sine terms and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\sin x\sin nx\,dx.$$

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Using a product to sum identity,

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi \frac{1}{2}[\cos(x-nx)-\cos(x+nx)]\,dx \\ &= \frac{1}{\pi}\int_0^\pi [\cos(n-1)x-\cos(n+1)x]\,dx \\ &= \frac{1}{\pi}\left[\frac{\sin(n-1)x}{n-1}-\frac{\sin(n+1)x}{n+1}\right]_0^\pi \\ &= 0, \end{aligned}$$

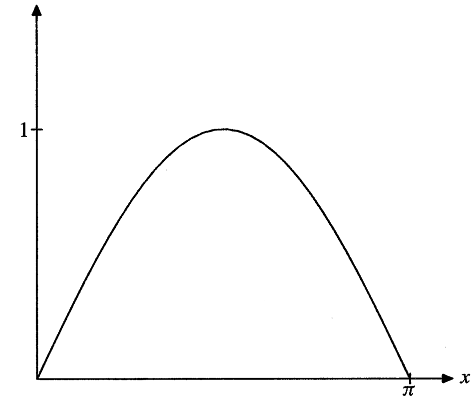
provided $n \neq 1$. In the case that $n = 1$,

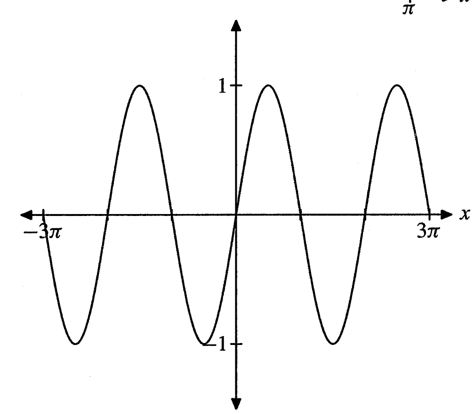
$$\begin{aligned} b_1 &= \frac{2}{\pi}\int_0^\pi \sin^2 x\,dx \\ &= \frac{2}{\pi}\int_0^\pi \frac{1-\cos 2x}{2}\,dx \\ &= \frac{1}{\pi}\left[x-\frac{1}{2}\sin 2x\right]_0^\pi \\ &= \frac{1}{\pi}[\pi] \\ &= 1. \end{aligned}$$

Thus the Fourier expansion for $f(x) = \sin x$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sin x,$$

as it should be. The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.





23. An odd extension for $f(x) = \cos x$ on the interval $[-\pi, \pi]$ is

$$f_o(x) = \begin{cases} -\cos x, & -\pi \le x < 0, \\ \cos x, & 0 \le x \le \pi. \end{cases}$$

The Fourier expansion of $f_o(x)$ contains only sine terms and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_o(x)\sin nx\,dx = \frac{2}{\pi}\int_0^\pi \cos x \sin nx\,dx.$$

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Using a product to sum identity,

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left[\sin(n-1)x + \sin(n+1)x\right] dx \\ &= \frac{1}{\pi}\left[-\frac{\cos(n-1)x}{n-1} - \frac{\cos(n+1)x}{n+1}\right]_0^{\pi} \\ &= -\frac{1}{\pi}\left\{\frac{(-1)^{n-1}}{n-1} + \frac{(-1)^{n+1}}{n+1} - \frac{1}{n-1} - \frac{1}{n+1}\right\} \\ &= -\frac{1}{\pi}\left\{\frac{(-1)^{n-1}(n+1) + (-1)^{n+1}(n-1) - (n+1) - (n-1)}{n^2-1}\right\}. \end{aligned}$$

Because $(-1)^{n+1} = (-1)^{n-1}$,

$$b_n = -\frac{1}{\pi}\left\{\frac{(-1)^{n-1}[(n+1)+(n-1)] - 2n}{n^2-1}\right\} = \frac{2n}{\pi}\cdot\frac{(-1)^n + 1}{n^2-1},$$

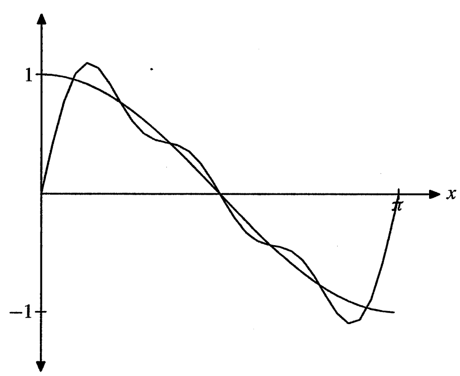
provided $n \neq 1$. In the case that $n = 1$,

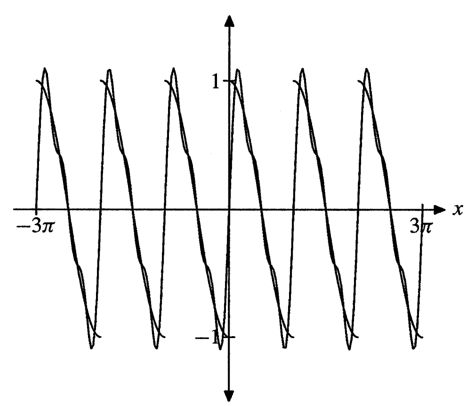
$$b_1 = \frac{2}{\pi}\int_0^{\pi} \cos x \sin x\, dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x\, dx = \frac{1}{\pi}\left[-\frac{1}{2}\cos 2x\right]_0^{\pi} = 0.$$

Thus, the Fourier expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=2}^{\infty} \frac{2\left((-1)^n + 1\right)n}{\pi(n^2-1)} \sin nx.$$

The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.





24. An odd extension for $f(x) = 1$ on the interval $[-\pi, \pi]$ is

$$f_o = \begin{cases} -1, & -\pi \le x < 0, \\ 1, & 0 \le x \le \pi. \end{cases}$$

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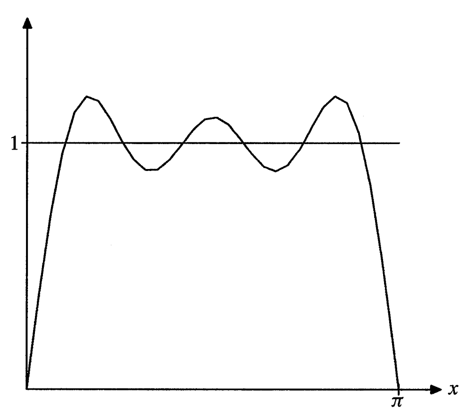
The Fourier expansion of $f_o(x)$ contains only sine terms with

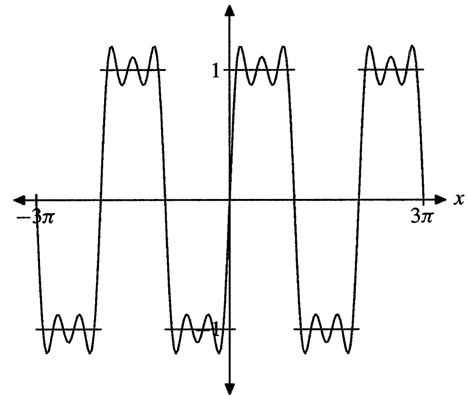
$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_o(x)\sin nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi}\sin nx\,dx \\ &= \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} \\ &= -\frac{2}{\pi n}[\cos n\pi - 1] \\ &= -\frac{2}{\pi n}\left[(-1)^n - 1\right] \\ &= \frac{2}{\pi n}\left[(-1)^{n+1} + 1\right]. \end{aligned}$$

Hence, the Fourier expansion is

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin nx \\ &= \sum_{n=1}^{\infty} \frac{2\left((-1)^{n+1}+1\right)}{\pi n}\sin nx. \end{aligned}$$

The partial sum S_6 is shown on the intervals $[0,\pi]$ and $[-3\pi, 3\pi]$.





25. An odd extension of $f(x) = \pi - x$ on the interval $[-\pi, \pi]$ is

$$f_o(x) = \begin{cases} -\pi - x, & -\pi \le x < 0, \\ \pi - x, & 0 \le x \le \pi. \end{cases}$$

The Fourier expansion contains only sine terms with

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_o(x)\sin nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,dx. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} b_n &= \frac{2}{\pi}\left[\frac{-(\pi - x)\cos nx}{n} - \frac{\sin nx}{n^2}\right]_0^{\pi} \\ &= -\frac{2}{\pi n^2}\left[n(\pi - x)\cos nx + \sin nx\right]_0^{\pi} \\ &= -\frac{2}{\pi n^2}(-n\pi) \\ &= \frac{2}{n}. \end{aligned}$$

Hence, the Fourier expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} \frac{2}{n}\sin nx.$$

The partial sum S_6 is shown on the interval $[0,\pi]$ and $[-3\pi, 3\pi]$.

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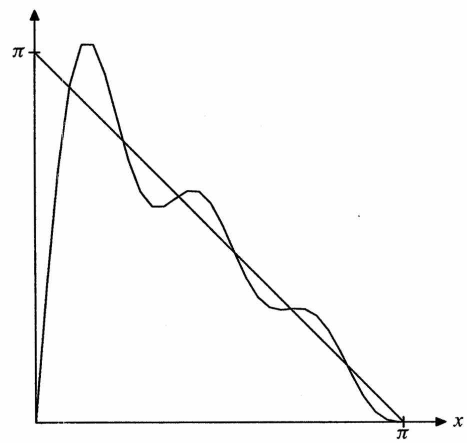
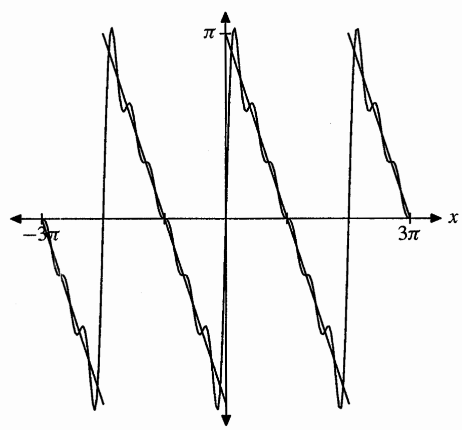




26. An odd extension of $f(x) = x^2$ on the interval $[-\pi, \pi]$ is

$$f_o(x) = \begin{cases} -x^2, & -\pi \le x < 0, \\ x^2, & 0 \le x \le \pi. \end{cases}$$

The Fourier expansion contains only sine terms and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_o(x) \sin nx\, dx \\ &= \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\, dx. \end{aligned}$$

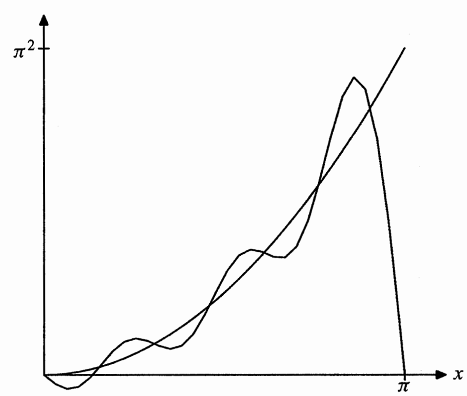
Integrating by parts,

$$\begin{aligned} b_n &= \frac{2}{\pi}\left[-\frac{x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[\left(-\frac{\pi^2\cos n\pi}{n} + \frac{2\cos n\pi}{n^3}\right) - \frac{2}{n^3}\right] \\ &= \frac{2}{\pi n^3}\left(-\pi^2 n^2(-1)^n + 2(-1)^n - 2\right) \\ &= \frac{2\left((-1)^n(2 - \pi^2 n^2) - 2\right)}{\pi n^3}. \end{aligned}$$

Hence, the Fourier expansion is

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin nx \\ &= \sum_{n=1}^{\infty} \frac{2\left((-1)^n(2 - \pi^2 n^2) - 2\right)}{\pi n^3} \sin nx. \end{aligned}$$

The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.



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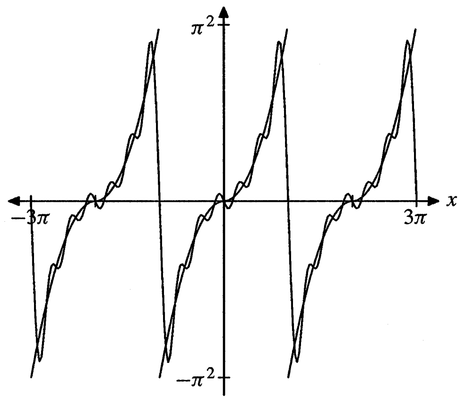


27. The function $f(x) = x^3$ is already odd on the interval $[-\pi, \pi]$, so its odd extension is itself and its Fourier expansion contains only sine terms.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx\,dx.$$

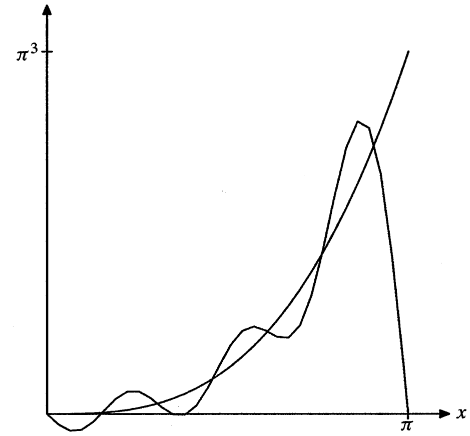
Integrating by parts,

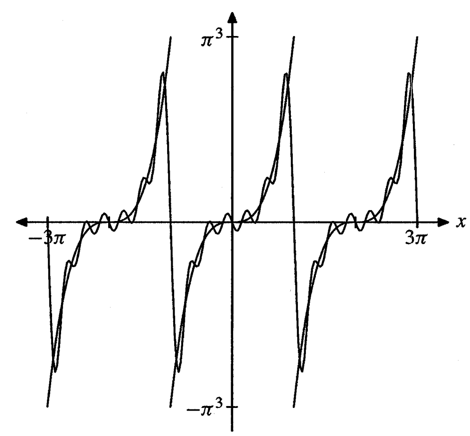
$$\begin{aligned} b_n &= \frac{2}{\pi}\Bigg[-\frac{x^3\cos nx}{n} + \frac{3x^2\sin nx}{n^2} \\ &\qquad + \frac{6x\cos nx}{n^3} - \frac{6\sin nx}{n^4}\Bigg]_0^{\pi} \\ &= \frac{2}{\pi}\left[-\frac{\pi^3\cos n\pi}{n} + \frac{6\pi\cos n\pi}{n^3}\right] \\ &= \frac{2}{\pi}\left[\frac{-\pi^3 n^2(-1)^n + 6\pi(-1)^n}{n^3}\right] \\ &= \frac{2(-1)^n(6 - n^2\pi^2)}{n^3} \end{aligned}$$

Thus, the Fourier expansion is

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin nx \\ &= \sum_{n=1}^{\infty} \frac{2(-1)^n(6 - n^2\pi^2)}{n^3}\sin nx. \end{aligned}$$

The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.





28. The function $f(x) = x^4$ has odd extension

$$f_o(x) = \begin{cases} -x^4, & -\pi \le x < 0, \\ x^4, & 0 \le x \le \pi. \end{cases}$$

The Fourier expansion contains only sine terms and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} x^4 \sin nx\,dx. \end{aligned}$$

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Integrating by parts,

$$b_n = \frac{2}{\pi}\left[-\frac{x^4 \cos nx}{n} + \frac{4x^3 \sin nx}{n^2} + \frac{12x^2 \cos nx}{n^3} - \frac{24x \sin nx}{n^4} - \frac{24 \cos nx}{n^5}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi^4 \cos n\pi}{n} + \frac{12\pi^2 \cos n\pi}{n^3} - \frac{24 \cos n\pi}{n^5} + \frac{24}{n^5}\right]$$

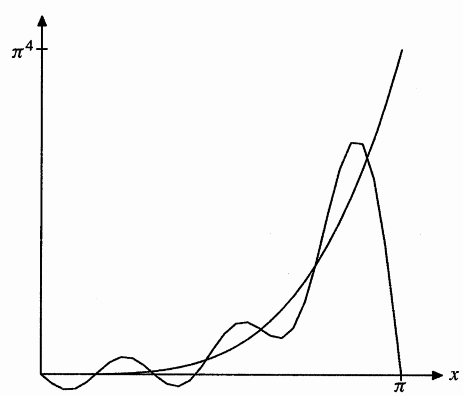
$$= \frac{2}{\pi n^5}\left[(-1)^{n+1}(\pi^4 n^4 - 12\pi^2 n^2 + 24) + 24\right]$$

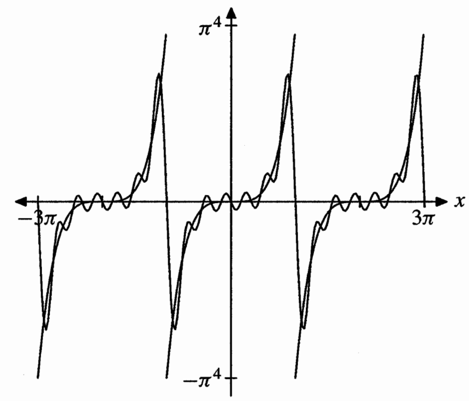
Thus, the Fourier series expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{2}{\pi n^5}\left[(-1)^{n+1}(\pi^4 n^4 - 12\pi^2 n^2 + 24) + 24\right] \sin nx.$$

The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.





29. The function

$$f(x) = \begin{cases} 1, & 0 \le x < \pi/2, \\ 0, & \pi/2 \le x \le \pi, \end{cases}$$

has odd extension

$$f_o(x) = \begin{cases} 0, & -\pi \le x < -\pi/2, \\ -1, & -\pi/2 \le x < 0, \\ 1, & 0 \le x < \pi/2, \\ 0, & \pi/2 \le x \le \pi, \end{cases}$$

on the interval $[-\pi, \pi]$. The Fourier expansion of $f_o(x)$ has only sine terms with

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_o(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f_o(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi/2} \sin nx \, dx$$

$$= \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi n}\left[\cos\frac{n\pi}{2} - 1\right].$$

Hence, the Fourier expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{2}{\pi n}\left[1 - \cos\frac{n\pi}{2}\right] \sin nx.$$

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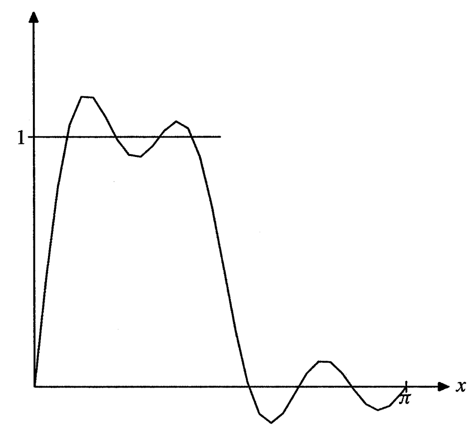
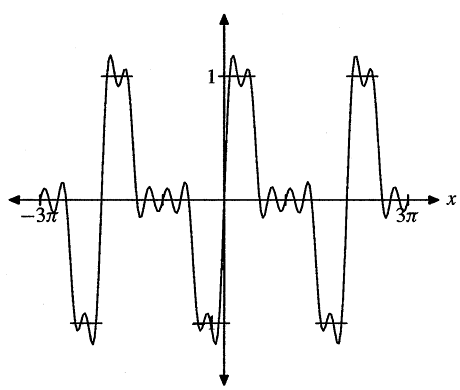
The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.





30. The function

$$f(x) = \begin{cases} x, & 0 \le x < \pi/2, \\ \pi/2, & \pi/2 \le x \le \pi, \end{cases}$$

has odd extension

$$f_o(x) = \begin{cases} -\pi/2, & -\pi \le x < -\pi/2, \\ x, & -\pi/2 \le x < 0, \\ x, & 0 \le x < \pi/2, \\ \pi/2, & \pi/2 \le x \le \pi, \end{cases}$$

on the interval $[-\pi, \pi]$. Hence, the Fourier expansion has only sine terms and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_o(x)\sin nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} f_o(x)\sin nx\,dx \\ &= \frac{2}{\pi}\left\{\int_0^{\pi/2} x\sin nx\,dx \right. \\ &\qquad \left. + \int_{\pi/2}^{\pi} \frac{\pi}{2}\sin nx\,dx\right\}. \end{aligned}$$

The first integral is found by integrating by parts.

$$\begin{aligned} \int_0^{\pi/2} x\sin nx\,dx &= \left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi/2} \\ &= -\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} \end{aligned}$$

The second integral is

$$\begin{aligned} \frac{\pi}{2}\int_{\pi/2}^{\pi} \sin nx\,dx &= \frac{\pi}{2}\left[-\frac{\cos nx}{n}\right]_{\pi/2}^{\pi} \\ &= -\frac{\pi}{2n}\left[\cos n\pi - \cos\frac{n\pi}{2}\right]. \end{aligned}$$

Hence,

$$\begin{aligned} b_n &= \frac{2}{\pi}\left\{-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} \right. \\ &\qquad \left. -\frac{\pi}{2n}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right\} \\ &= \frac{2}{\pi}\left\{\frac{1}{n^2}\sin\frac{n\pi}{2} - \frac{\pi}{2n}\cos n\pi\right\}. \end{aligned}$$

Thus, the Fourier expansion is

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin nx \\ &= \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\sin\frac{n\pi}{2} - \frac{\pi}{2n}\cos n\pi\right)\sin nx. \end{aligned}$$

The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.

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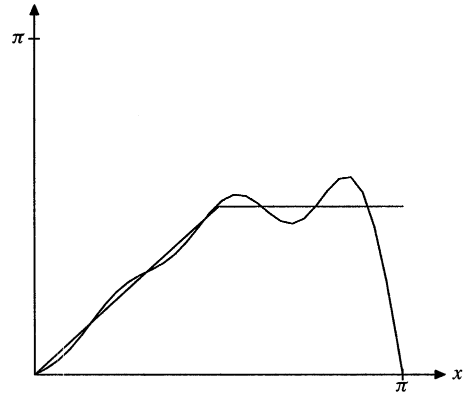

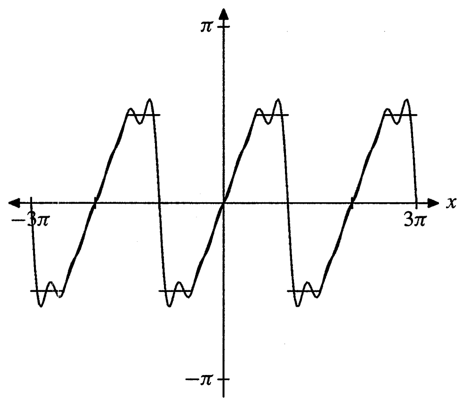

31. The function $f(x) = x\cos x$ is already odd on the interval $[-\pi, \pi]$, so the Fourier expansion contains only sine terms and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos x\sin nx\,dx.$$

Using a product to sum identity,

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\left\{\frac{1}{2}\left[\sin(n-1)x + \sin(n+1)x\right]\right\}dx = \frac{1}{\pi}\left\{\int_0^{\pi} x\sin(n-1)x\,dx + \int_0^{\pi} x\sin(n+1)x\,dx\right\}.$$

Integrating by parts, the first integral is

$$\int_0^{\pi} x\sin(n-1)x\,dx = \left[-\frac{x\cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{\pi} = \frac{-\pi\cos(n-1)\pi}{n-1}.$$

Integrating by parts, the second integral is

$$\int_0^{\pi} x\sin(n+1)x\,dx = \left[\frac{-x\cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2}\right]_0^{\pi} = \frac{-\pi\cos(n+1)\pi}{n+1}.$$

Hence,

$$\begin{aligned} b_n &= \frac{1}{\pi}\left\{\frac{-\pi\cos(n-1)\pi}{n-1} - \frac{\pi\cos(n+1)\pi}{n+1}\right\} \\ &= \frac{-(-1)^{n-1}}{n-1} - \frac{(-1)^{n+1}}{n+1} \\ &= \frac{(-1)^n}{n-1} + \frac{(-1)^{n+2}}{n+1} \\ &= (-1)^n\frac{2n}{n^2-1}, \end{aligned}$$

provided $n \neq 1$. In the case that $n = 1$,

$$b_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x\sin x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx.$$

Integrating by parts,

$$b_1 = \frac{1}{\pi}\left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi} = -\frac{1}{2}.$$

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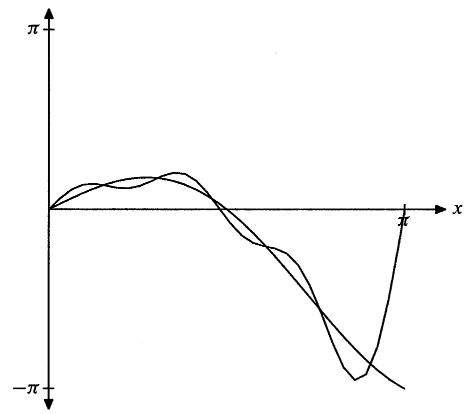
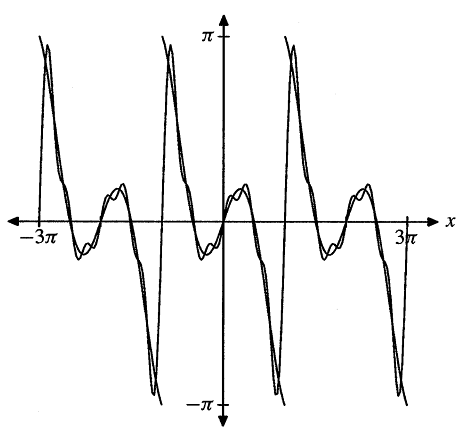
Hence, the Fourier expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$
$$= -\frac{1}{2}\sin x + \sum_{n=2}^{\infty} \frac{(-1)^n 2n}{n^2-1} \sin nx.$$

The partial sum S_6 is shown on the interval $[0, \pi]$ and $[-3\pi, 3\pi]$.





32. The function $f(x) = x \sin x$ has odd extension

$$f_o(x) = \begin{cases} -x \sin x, & -\pi \le x < 0, \\ x \sin x, & 0 \le x \le \pi. \end{cases}$$

Thus, the Fourier expansion has only sine terms and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_o(x) \sin nx\, dx$$
$$= \frac{2}{\pi}\int_0^{\pi} x \sin x \sin nx\, dx.$$

Using a product to sum identity,

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\left[\frac{1}{2}\Big(\cos(x-nx) - \cos(x+nx)\Big)\right]dx$$
$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cos(n-1)x\, dx - \int_0^{\pi} x\cos(n+1)x\, dx\right].$$

Integrating by parts, the first integral is

$$\int_0^{\pi} x\cos(n-1)x\, dx$$
$$= \left[\frac{x\sin(n-1)x}{n-1} + \frac{\cos(n-1)x}{(n-1)^2}\right]_0^{\pi}$$
$$= \frac{\cos(n-1)\pi}{(n-1)^2} - \frac{1}{(n-1)^2}$$
$$= \frac{(-1)^{n-1}}{(n-1)^2} - \frac{1}{(n-1)^2}.$$

Similarly, the second integral is

$$\int_0^{\pi} x\cos(n+1)x\, dx$$
$$= \left[\frac{x\sin(n+1)x}{n+1} + \frac{\cos(n+1)x}{(n+1)^2}\right]_0^{\pi}$$
$$= \frac{\cos(n+1)\pi}{(n+1)^2} - \frac{1}{(n+1)^2}$$
$$= \frac{(-1)^{n+1}}{(n+1)^2} - \frac{1}{(n+1)^2}.$$

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Hence,

$$\begin{aligned} b_n &= \frac{1}{\pi}\left[\frac{(-1)^{n-1}}{(n-1)^2} - \frac{1}{(n-1)^2} \right. \\ &\qquad \left. - \frac{(-1)^{n+1}}{(n+1)^2} + \frac{1}{(n+1)^2}\right] \\ &= \frac{1}{\pi}\left[\frac{1}{(n+1)^2} - \frac{1}{(n-1)^2}\right]\left((-1)^n + 1\right) \\ &= \frac{1}{\pi}\left[\frac{(n^2-2n+1)-(n^2+2n+1)}{(n+1)^2(n-1)^2}\right] \\ &\qquad \times \left((-1)^n + 1\right) \\ &= \frac{4n\left((-1)^{n+1}-1\right)}{\pi(n+1)^2(n-1)^2}, \end{aligned}$$

provided $n \neq 1$. In the case that $n = 1$,

$$\begin{aligned} b_1 &= \frac{2}{\pi}\int_0^{\pi} x \sin^2 x \, dx \\ &= \frac{1}{\pi}\int_0^{\pi} (x - x\cos 2x)\, dx. \end{aligned}$$

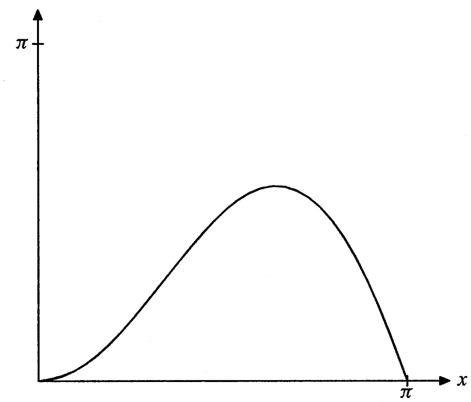
Integrating by parts,

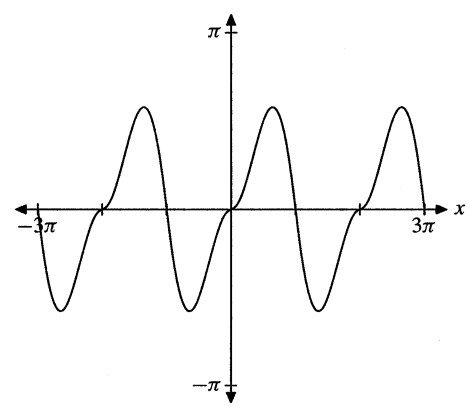
$$\begin{aligned} b_1 &= \frac{1}{\pi}\left[\frac{x^2}{2} - \left(\frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right)\right]_0^{\pi} \\ &= \frac{1}{\pi}\left[\frac{\pi^2}{2}\right] \\ &= \frac{\pi}{2}. \end{aligned}$$

Hence, the Fourier expansion is

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin nx \\ &= \frac{\pi}{2}\sin x + \sum_{n=2}^{\infty} \frac{4n\left((-1)^{n+1}-1\right)}{\pi(n+1)^2(n-1)^2}\sin nx. \end{aligned}$$

The partial sum S_6 is shown on the intervals $[0, \pi]$ and $[-3\pi, 3\pi]$.





33. Use the trig identity $\cos\alpha\cos\beta = 1/2[\cos(\alpha-\beta)+$

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$\cos(\alpha+\beta)]$ to write (if $n \neq p$),

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L}dx$$
$$=\frac{1}{2}\int_0^L\left[\cos\frac{(n-p)\pi x}{L}+\cos\frac{(n+p)\pi x}{L}\right]dx$$
$$=\frac{1}{2}\left[\frac{L}{(n-p)\pi}\sin\frac{(n-p)\pi x}{L}+\frac{L}{(n+p)\pi}\sin\frac{(n+p)\pi x}{L}\right]_0^L$$
$$=\frac{1}{2}\left[\frac{L}{(n-p)\pi}\sin(n-p)\pi+\frac{L}{(n+p)\pi}\sin(n+p)\pi\right]$$

Because $n-p$ and $n+p$ are integers, $\sin(n-p)\pi=0$ and $\sin(n+p)\pi=0$. Thus, if $n\neq p$, then

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\,dx=0.$$

34. Using a product to sum trig identity,

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}dx$$
$$=\int_0^L\frac{1}{2}\left[\cos\left(\frac{n\pi x}{L}-\frac{p\pi x}{L}\right)-\cos\left(\frac{n\pi x}{L}+\frac{p\pi x}{L}\right)\right]dx$$
$$=\frac{1}{2}\int_0^L\left[\cos\frac{(n-p)\pi x}{L}-\cos\frac{(n+p)\pi x}{L}\right]dx$$
$$=\frac{1}{2}\left[\frac{L}{(n-p)\pi}\sin\frac{(n-p)\pi x}{L}-\frac{L}{(n+p)\pi}\sin\frac{(n+p)\pi}{L}\right]_0^L$$
$$=0,$$

provided n and p are integers and $n\neq p$.

35. Use the trig identity $\sin\alpha\cos\beta=1/2[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$ to write

$$\int_0^1\cos(2n\pi x)\sin(2k\pi x)dx$$
$$=\frac{1}{2}\int_0^1[\sin 2(k-n)\pi x+\sin 2(k+n)\pi x]\,dx$$
$$=\frac{1}{2}\left[-\frac{1}{2(k-n)\pi}\cos 2(k-n)\pi x-\frac{1}{2(k+n)\pi}\cos 2(k+n)\pi x\right]_0^1$$
$$=\frac{1}{2}\left[-\frac{1}{2(k-n)\pi}(\cos 2(k-n)\pi-1)-\frac{1}{2(k+n)\pi}(\cos 2(k+n)\pi-1)\right]$$

If k and n are different integers, $\cos 2(k-n)\pi=1$ and $\cos 2(k+n)\pi=1$. Thus, if $k\neq n$,

$$\int_0^1\cos(2n\pi x)\sin(2k\pi x)dx=0.$$

Now, if $k=n$,

$$\int_0^1\cos(2n\pi x)\sin(2k\pi x)dx$$
$$=\int_0^1\cos(2n\pi x)\sin(2n\pi x)dx$$
$$=\frac{1}{2}\int_0^1\sin(4n\pi x)dx$$
$$=\frac{1}{2}\cdot\frac{-1}{4n\pi}\cos(4n\pi x)\Big|_0^1$$
$$=0.$$

36. The point here is that starting with a continuous function f on $[0,L]$, then its even extension

$$f_e(x)=\begin{cases}f(-x), & -L\le x<0\\ f(x), & 0\le x\le L,\end{cases}$$

will hook up when extended periodically. For example start with f and its even extension, as shown in

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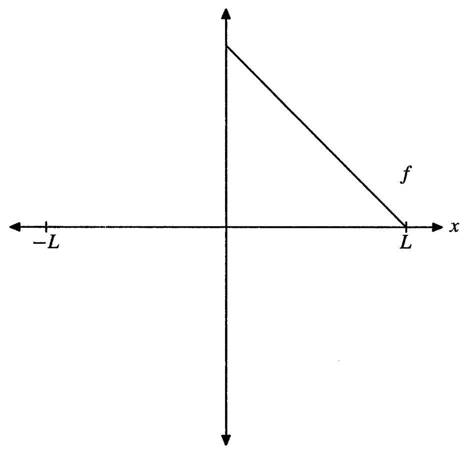
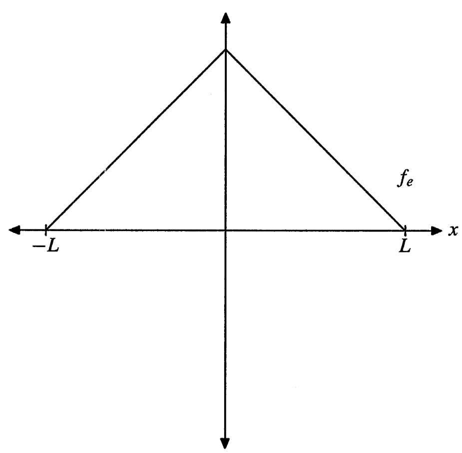
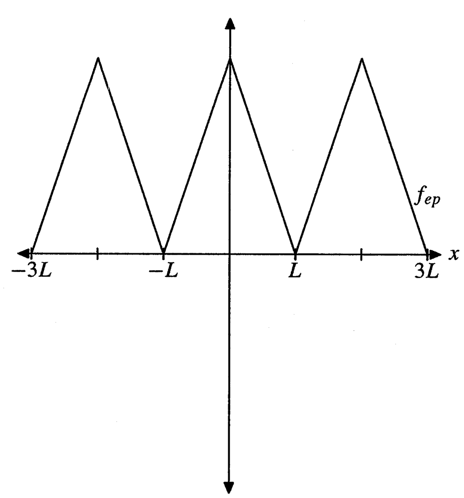
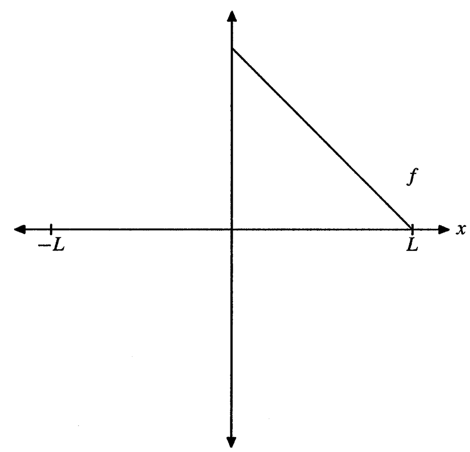
the first two images.

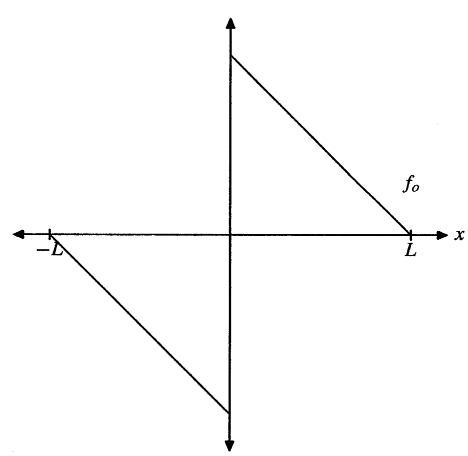






Note that when f_e is extended periodically, the result is continuous everywhere, as shown in the third image above. However this is not the case for an odd extension. Consider, for example, this graph of f and its odd extension f_o, shown in the first and second images that follow.





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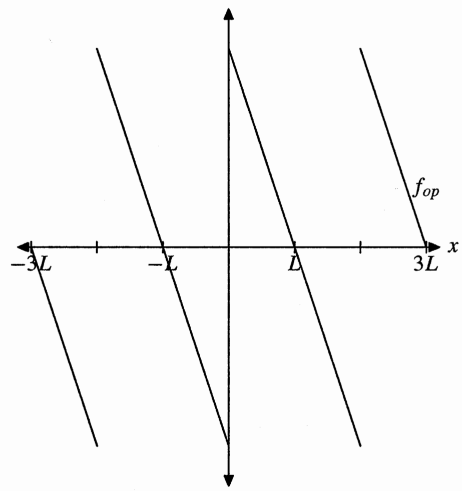
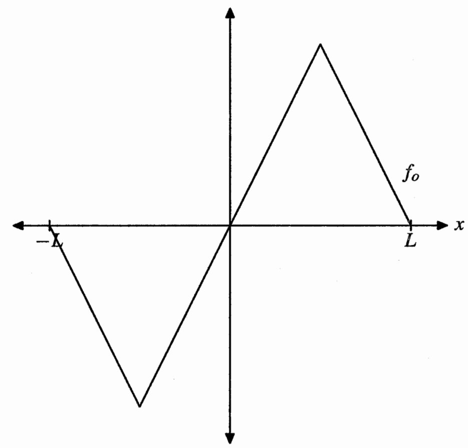
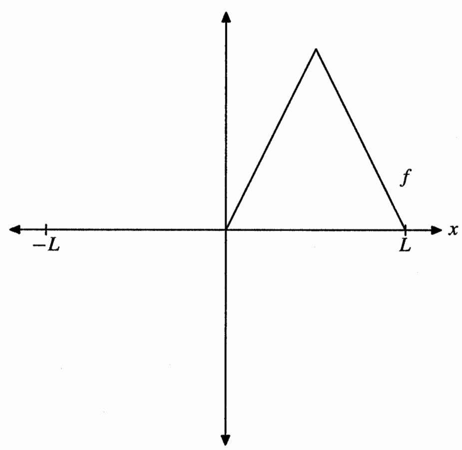
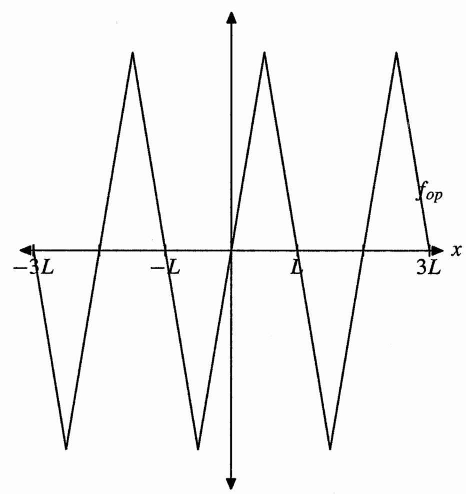




When f_o is extended periodically, note that it is not continuous at even multiples of L, as shown in the third image above. One thing we can do is require that $f(0) = f(L) = 0$. Then the periodic extension is continuous everywhere, as shown in the images that follow.





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Section 12.4. The Complex Form of a Fourier Series

1. If f is real, we know that

$$\overline{\alpha_n} = \alpha_{-n}, \qquad \text{for all } n.$$

However, if

$$\alpha_n = \frac{1}{L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx,$$

then

$$\alpha_{-n} = \frac{1}{L}\int_{-L}^{L} f(x)e^{in\pi x/L}dx.$$

Since f is even, $f(-x) = f(x)$ and

$$\alpha_{-n} = \frac{1}{L}\int_{-L}^{L} f(-x)e^{in\pi x/L}dx.$$

Now, let

$$u = -x \quad \text{and} \quad du = -dx,$$

and write

$$\begin{aligned}\alpha_{-n} &= \frac{1}{L}\int_{L}^{-L} f(u)e^{-in\pi u/L}(-du)\\ &= \frac{1}{L}\int_{-L}^{L} f(u)e^{-in\pi u/L}du.\end{aligned}$$

But this is precisely the definition for α_n (ignore the dummy variable u). Thus,

$$\overline{\alpha_n} = \alpha_{-n} = \alpha_n.$$

Because $\overline{\alpha_n} = \alpha_n$, it must be the case that α_n is real.

Next, suppose that f is real and odd. Again, because f is real, we know that

$$\overline{\alpha_n} = \alpha_{-n}, \qquad \text{for all } n.$$

The coefficient α_n is purely imaginary if and only if we can show that $\overline{\alpha_n} = -\alpha_n$ for all n. However,

$$\alpha_n = \frac{1}{L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx,$$

so again,

$$\alpha_{-n} = \frac{1}{L}\int_{-L}^{L} f(x)e^{in\pi x/L}dx.$$

Because f is odd, $f(-x) = -f(x)$ for all $x\epsilon[-L, L]$. We can write

$$\alpha_{-n} = \frac{1}{L}\int_{-L}^{L} -f(-x)e^{in\pi x/L}dx$$

and again make the change of variable

$$u = -x \quad \text{and} \quad du = -dx.$$

Thus,

$$\begin{aligned}\alpha_{-n} &= \frac{1}{L}\int_{L}^{-L} -f(u)e^{-in\pi u/L}(-du)\\ &= -\frac{1}{L}\int_{-L}^{L} f(u)e^{-in\pi u/L}du\\ &= -\alpha_n,\end{aligned}$$

the last statement being true because u is just a dummy variable. Thus,

$$\overline{\alpha_n} = \alpha_{-n} = -\alpha_n,$$

and thus α_n is purely imaginary for all n.

2. With $L = \pi$, the nth coefficient becomes

$$\begin{aligned}\alpha_n &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\,dx\end{aligned}$$

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Integrating by parts,

$$\begin{aligned}\alpha_n &= \frac{1}{2\pi}\left[-\frac{xe^{-inx}}{in}+\frac{e^{-inx}}{n^2}\right]_{-\pi}^{\pi}\\
&= \frac{1}{2\pi}\left[e^{-inx}\left(\frac{1}{n^2}-\frac{x}{in}\right)\right]_{-\pi}^{\pi}\\
&= \frac{1}{2\pi}\left[e^{-in\pi}\left(\frac{1}{n^2}-\frac{\pi}{in}\right)\right.\\
&\qquad\left.-e^{in\pi}\left(\frac{1}{n^2}+\frac{\pi}{in}\right)\right]\\
&= \frac{1}{2\pi}\left[(-1)^n\left(\frac{1}{n^2}-\frac{\pi}{in}\right)\right.\\
&\qquad\left.-(-1)^n\left(\frac{1}{n^2}+\frac{\pi}{in}\right)\right]\\
&= \frac{1}{2\pi}\left[(-1)^n\left(-\frac{2\pi}{in}\right)\right]\\
&= \frac{(-1)^{n+1}}{in},\end{aligned}$$

provided $n \neq 0$. In the case that $n = 0$,

$$\alpha_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0.$$

Thus,

$$f(x) \sim \sum_{n=-\infty}^{\infty}\alpha_n e^{inx} = \sum_{n\neq 0}\frac{(-1)^{n+1}}{in}e^{inx}.$$

3. With $L = \pi$, the nth coefficient becomes

$$\begin{aligned}\alpha_n &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|e^{-inx}\,dx\\
&= \frac{1}{2\pi}\int_{-\pi}^{0}(-x)e^{-inx}\,dx\\
&\qquad+\frac{1}{2\pi}\int_{0}^{\pi}xe^{-inx}\,dx.\end{aligned}$$

Integrating by parts, the first integral becomes

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{0}(-x)e^{-inx}\,dx\\
&= \frac{1}{2\pi}\left[\frac{xe^{-inx}}{in}-\frac{e^{-inx}}{n^2}\right]_{-\pi}^{0}\\
&= \frac{1}{2\pi}\left[e^{-inx}\left(\frac{x}{in}-\frac{1}{n^2}\right)\right]_{-\pi}^{0}\\
&= \frac{1}{2\pi}\left[-\frac{1}{n^2}-e^{in\pi}\left(\frac{-\pi}{in}-\frac{1}{n^2}\right)\right]\\
&= \frac{1}{2\pi}\left[-\frac{1}{n^2}-(-1)^n\left(-\frac{\pi}{in}-\frac{1}{n^2}\right)\right]\\
&= \begin{cases}\frac{1}{2\pi}\left(\frac{\pi}{in}\right), & n \text{ even},\\ \frac{1}{2\pi}\left(-\frac{2}{n^2}-\frac{\pi}{in}\right), & n \text{ odd}\end{cases}\\
&= \begin{cases}\frac{1}{2in}, & n \text{ even}\\ -\frac{1}{\pi n^2}-\frac{1}{2in}, & n \text{ odd}\end{cases}\end{aligned}$$

The second integral is

$$\begin{aligned}&\frac{1}{2\pi}\int_{0}^{\pi}xe^{-inx}\,dx\\
&= \frac{1}{2\pi}\left[\frac{-xe^{-inx}}{in}+\frac{e^{-inx}}{n^2}\right]_{0}^{\pi}\\
&= \frac{1}{2\pi}\left[e^{-inx}\left(\frac{1}{n^2}-\frac{x}{in}\right)\right]_{0}^{\pi}\\
&= \frac{1}{2\pi}\left[e^{-in\pi}\left(\frac{1}{n^2}-\frac{\pi}{in}\right)-\frac{1}{n^2}\right]\\
&= \frac{1}{2\pi}\left[(-1)^n\left(\frac{1}{n^2}-\frac{\pi}{in}\right)-\frac{1}{n^2}\right]\\
&= \begin{cases}\frac{1}{2\pi}\left(-\frac{\pi}{in}\right), & n \text{ even},\\ \frac{1}{2\pi}\left(\frac{\pi}{in}-\frac{2}{n^2}\right), & n \text{ odd}\end{cases}\\
&= \begin{cases}-\frac{1}{2in}, & n \text{ even},\\ \frac{1}{2in}-\frac{1}{\pi n^2}, & n \text{ odd}\end{cases}\end{aligned}$$

Thus,

$$\alpha_n = \begin{cases}0, & n \text{ even},\\ -\frac{2}{\pi n^2}, & n \text{ odd},\end{cases}$$

provided $n \neq 0$. In the case that $n = 0$,

$$\alpha_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|\,dx = \frac{1}{2\pi}(\pi^2) = \frac{\pi}{2}.$$

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Hence,

$$\begin{aligned} f(x) &\sim \sum_{n=-\infty}^{\infty} \alpha_n e^{inx} \\ &= \frac{\pi}{2} + \sum_{n \text{ odd}} \left(-\frac{2}{\pi n^2}\right) e^{inx} \\ &= \frac{\pi}{2} - \frac{2}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2} e^{inx}. \end{aligned}$$

4. With $L = \pi$, the nth coefficient becomes

$$\begin{aligned} \alpha_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{0} (-1) e^{-inx}\, dx \\ &\quad + \frac{1}{2\pi} \int_{0}^{\pi} (1) e^{-inx}\, dx \end{aligned}$$

The first integral is

$$\begin{aligned} -\frac{1}{2\pi} \int_{-\pi}^{0} e^{-inx}\, dx &= -\frac{1}{2\pi} \frac{e^{-inx}}{-in}\bigg|_{-\pi}^{0} \\ &= \frac{e^{-inx}}{2\pi in}\bigg|_{-\pi}^{0} \\ &= \frac{1 - e^{in\pi}}{2\pi in}. \end{aligned}$$

The second integral is

$$\begin{aligned} \frac{1}{2\pi} \int_{0}^{\pi} e^{-inx}\, dx &= \frac{1}{2\pi} \frac{e^{-inx}}{-in}\bigg|_{0}^{\pi} \\ &= -\frac{e^{-inx}}{2\pi in}\bigg|_{0}^{\pi} \\ &= -\frac{e^{-in\pi} - 1}{2\pi in}. \end{aligned}$$

Thus,

$$\begin{aligned} \alpha_n &= \frac{1 - (-1)^n}{2\pi in} - \frac{(-1)^n - 1}{2\pi in} \\ &= \frac{2 - 2(-1)^n}{2\pi in} = \frac{1 - (-1)^n}{\pi in}, \\ &= \begin{cases} 0, & n \text{ even}, \\ 2/(\pi in), & n \text{ odd}, \end{cases} \end{aligned}$$

provided $n \neq 0$. In the case that $n = 0$,

$$\alpha_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = 0,$$

because f is odd on $[-\pi, \pi]$. Consequently,

$$f(x) \sim \sum_{n=-\infty}^{\infty} \alpha_n e^{inx} = \sum_{n \text{ odd}} \frac{2}{\pi in} e^{inx}.$$

5. With $L = \pi$, the nth coefficient becomes

$$\begin{aligned} \alpha_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx \\ &= \frac{1}{2\pi} \int_{0}^{\pi} e^{-inx}\, dx \\ &= \frac{1}{2\pi} \left[\frac{e^{-inx}}{-in}\right]_0^{\pi} \\ &= -\frac{1}{2\pi in} \left[e^{-in\pi} - 1\right] \\ &= \frac{1 - (-1)^n}{2\pi in} \\ &= \begin{cases} 0, & n \text{ even}, \\ \frac{1}{\pi in}, & n \text{ odd}, \end{cases} \end{aligned}$$

provided $n \neq 0$. In the case that $n = 0$,

$$\alpha_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi} \int_{0}^{\pi} dx = \frac{1}{2}.$$

Thus,

$$f(x) \sim \sum_{n=-\infty}^{\infty} \alpha_n e^{inx} = \frac{1}{2} + \sum_{n \text{ odd}} \frac{1}{\pi in} e^{inx}.$$

6. With $L = \pi$, the nth coefficient becomes

$$\alpha_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{-inx}\, dx.$$

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Integrating by parts,

$$\begin{aligned}\alpha_n &= \frac{1}{2\pi}\left[\frac{-x^2e^{-inx}}{in}+\frac{2xe^{-inx}}{n^2}+\frac{2e^{-inx}}{in^3}\right]_{-\pi}^{\pi}\\ &= \frac{1}{2\pi}\left[e^{-inx}\left(-\frac{x^2}{in}+\frac{2x}{n^2}+\frac{2}{in^3}\right)\right]_{-\pi}^{\pi}\\ &= \frac{1}{2\pi}\left[(-1)^n\left(-\frac{\pi^2}{in}+\frac{2\pi}{n^2}+\frac{2}{in^3}\right)\right.\\ &\qquad\left.-(-1)^n\left(-\frac{\pi^2}{in}-\frac{2\pi}{n^2}+\frac{2}{in^3}\right)\right]\\ &= \frac{1}{2\pi}\left[(-1)^n\left(\frac{4\pi}{n^2}\right)\right]\\ &= (-1)^n\frac{2}{n^2},\end{aligned}$$

provided $n \neq 0$. In the case that $n = 0$,

$$\alpha_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^2}{3}.$$

Hence,

$$\begin{aligned}f(x) &\sim \sum_{n=-\infty}^{\infty}\alpha_n e^{inx}\\ &= \frac{\pi^2}{3}+\sum_{n\neq 0}(-1)^n\frac{2}{n^2}e^{inx}.\end{aligned}$$

7. With $L = \pi$, the nth coefficient becomes

$$\begin{aligned}\alpha_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{bx}e^{-inx}\,dx\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(b-in)x}\,dx\\ &= \frac{1}{2\pi}\frac{e^{(b-in)x}}{b-in}\Big|_{-\pi}^{\pi}\\ &= \frac{1}{2\pi(b-in)}\left[e^{(b-in)\pi}-e^{(b-in)(-\pi)}\right]\\ &= \frac{1}{2\pi(b-in)}\left[e^{b\pi}e^{-in\pi}-e^{-b\pi}e^{in\pi}\right]\\ &= \frac{(-1)^n}{\pi(b-in)}\left[\frac{e^{b\pi}-e^{-b\pi}}{2}\right]\\ &= \frac{(-1)^n\sinh b\pi}{\pi(b-in)},\end{aligned}$$

provided, of course, that $b \neq in$. Hence,

$$f(x) \sim \sum_{n=-\infty}^{\infty}\alpha_n e^{inx} = \frac{\sinh b\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{b-in}e^{inx}$$

8. With $L = \pi$, the nth coefficient becomes

$$\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^3e^{-inx}dx.$$

Several integrations by parts provides

$$\begin{aligned}\alpha_n &= \frac{1}{2\pi}\left[\frac{ix^3e^{-inx}}{n}+\frac{3x^2e^{-inx}}{n^2}\right.\\ &\qquad\left.-\frac{i6xe^{-inx}}{n^3}-\frac{6e^{-inx}}{n^4}\right]_{-\pi}^{\pi}\\ &= \frac{1}{2\pi n^4}\left[e^{-inx}(in^3x^3+3n^2x^2-i6nx-6)\right]_{-\pi}^{\pi}\\ &= \frac{1}{2\pi n^4}\left[(-1)^n(in^3\pi^3+3n^2\pi^2-i6n\pi-6)\right.\\ &\qquad\left.-(-1)^n(-in^3\pi^3+3n^2\pi^2+i6n\pi-6)\right]\\ &= \frac{(-1)^n}{2\pi n^4}\left[2in^3\pi^3-12in\pi\right]\\ &= \frac{(-1)^n i}{n^3}(n^2\pi^2-6),\end{aligned}$$

provided $n \neq 0$. In the case that $n = 0$,

$$\alpha_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^3dx = 0,$$

since x^3 is odd on $[-\pi, \pi]$. Hence,

$$f(x) = \sum_{n=-\infty}^{\infty}\alpha_n e^{inx} = \sum_{n\neq 0}\frac{(-1)^n i(n^2\pi^2-6)}{n^3}e^{inx}.$$

9. If $f(x) = \pi - x$ on the interval $[-\pi, \pi]$, then

$$\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi - x)e^{-inx}dx.$$

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Integration by parts provides

$$\begin{aligned}\alpha_n &= \frac{1}{2\pi}\left[-\frac{(\pi - x)e^{-inx}}{in} - \frac{e^{-inx}}{n^2}\right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi}\left[-e^{-inx}\left(\frac{\pi - x}{in} + \frac{1}{n^2}\right)\right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi}\left[-(-1)^n \cdot \frac{1}{n^2} + (-1)^n\left(\frac{2\pi}{in} + \frac{1}{n^2}\right)\right] \\ &= (-1)^{n+1}\frac{i}{n},\end{aligned}$$

provided $n \neq 0$. In the case that $n = 0$,

$$\begin{aligned}\alpha_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi - x)dx \\ &= \frac{1}{2\pi}\left[\pi x - \frac{1}{2}x^2\right]_{-\pi}^{\pi} \\ &= \pi.\end{aligned}$$

Hence,

$$f(x) \sim \pi + i\sum_{n\neq 0}\frac{(-1)^{n+1}}{n}e^{inx}.$$

10. With $L = \pi$,

$$\begin{aligned}\alpha_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|\cos x|e^{-inx}dx \\ &= -\frac{1}{2\pi}\int_{-\pi}^{-\pi/2}(\cos x)e^{-inx}dx \\ &\quad + \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(\cos x)e^{-inx}dx \\ &\quad - \frac{1}{2\pi}\int_{\pi/2}^{\pi}(\cos x)e^{-inx}dx.\end{aligned}$$

Integrating by parts,

$$\begin{aligned}&-\frac{1}{2\pi}\int_{-\pi}^{-\pi/2}(\cos x)e^{-inx}dx \\ &= \frac{-1}{2\pi(1-n^2)}\left[e^{-inx}(\sin x - in\cos x)\right]_{-\pi}^{-\pi/2} \\ &= \frac{1}{2\pi(1-n^2)}[i^n + (-1)^n in]\end{aligned}$$

Similarly,

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(\cos x)e^{-inx}dx = \frac{(-1)^n i^n + i^n}{2\pi(1-n^2)},$$

and

$$-\frac{1}{2\pi}\int_{\pi/2}^{\pi}(\cos x)e^{-inx}dx = \frac{(-1)^n i^n - (-1)^n in}{2\pi(1-n^2)}.$$

Adding,

$$\begin{aligned}\alpha_n &= \frac{1}{2\pi(1-n^2)}[2i^n + 2(-1)^n i^n] \\ &= \frac{1+(-1)^n}{\pi(1-n^2)}i^n,\end{aligned}$$

providing $n \neq \pm 1$. In the case that $n = -1$,

$$\begin{aligned}\alpha_1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|\cos x|e^{ix}dx \\ &= -\frac{1}{2\pi}\int_{-\pi}^{-\pi/2}(\cos x)e^{ix}dx \\ &\quad + \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(\cos x)e^{ix}dx \\ &\quad - \frac{1}{2\pi}\int_{\pi/2}^{\pi}(\cos x)e^{ix}dx.\end{aligned}$$

Note that

$$\begin{aligned}&\int(\cos x)e^{ix}dx \\ &= \int\cos x(\cos x + i\sin x)dx \\ &= \int\left[\frac{1+\cos 2x}{2} + i\frac{1}{2}\sin 2x\right]dx \\ &= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right] - \frac{1}{4}i\cos 2x.\end{aligned}$$

Thus,

$$\begin{aligned}&-\frac{1}{2\pi}\int_{-\pi}^{-\pi/2}(\cos x)e^{ix}dx \\ &= -\frac{1}{2\pi}\left[\frac{2x+\sin 2x}{4} - \frac{i\cos 2x}{4}\right]_{-\pi}^{-\pi/2} \\ &= -\frac{1}{2\pi}\left\{\frac{\pi}{4} + \frac{2i}{4}\right\} \\ &= -\frac{\pi + 2i}{8\pi}.\end{aligned}$$

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Similarly,

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(\cos x)e^{ix}dx = \frac{1}{4},$$

and

$$-\frac{1}{2\pi}\int_{\pi/2}^{\pi}(\cos x)e^{ix}dx = -\frac{\pi - 2i}{8\pi}.$$

Adding,

$$\begin{aligned}\alpha_{-1} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|\cos x|e^{ix}dx \\ &= -\frac{\pi + 2i}{8\pi} + \frac{2\pi}{8\pi} - \frac{\pi - 2i}{8\pi} \\ &= 0.\end{aligned}$$

Similarly,

$$\alpha_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|\cos x|e^{ix}dx = 0.$$

Hence,

$$f(x) = \sum_{n\neq 1,\,-1}\frac{1+(-1)^n}{\pi(1-n^2)}i^n e^{inx}.$$

11. With $L = \pi$, the nth coefficient becomes

$$\begin{aligned}\alpha_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|\sin x|e^{-inx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{0}(-\sin x)e^{-inx}\,dx \\ &\qquad + \frac{1}{2\pi}\int_{0}^{\pi}(\sin x)e^{-inx}\,dx.\end{aligned}$$

Integrating by parts,

$$\begin{aligned}&-\frac{1}{2\pi}\int_{-\pi}^{0}(\sin x)e^{-inx}\,dx \\ &= -\frac{1}{2\pi}\cdot\frac{n^2}{n^2-1}\left[\frac{-e^{-inx}\sin x}{in}\right. \\ &\qquad\left. + \frac{e^{-inx}\cos x}{n^2}\right]_{-\pi}^{0} \\ &= \frac{-n^2}{2\pi(n^2-1)}\left[e^{-inx}\left(\frac{\cos x}{n^2} - \frac{\sin x}{in}\right)\right]_{-\pi}^{0} \\ &= \frac{-n^2}{2\pi(n^2-1)}\left[\frac{1}{n^2} - e^{in\pi}\left(\frac{-1}{n^2}\right)\right] \\ &= \frac{-(-1)^n - 1}{2\pi(n^2-1)}.\end{aligned}$$

Similarly,

$$\frac{1}{2\pi}\int_{0}^{\pi}(\sin x)e^{-inx}\,dx = \frac{-1-(-1)^n}{2\pi(n^2-1)}.$$

Adding,

$$\begin{aligned}\alpha_n &= \frac{-2-2(-1)^n}{2\pi(n^2-1)} = \frac{-1-(-1)^n}{\pi(1-n^2)} \\ &= \begin{cases}2/(\pi(n^2-1)), & n \text{ even},\\ 0, & n \text{ odd},\end{cases}\end{aligned}$$

providing $n \neq \pm 1, 0$. In the case that $n = -1$,

$$\begin{aligned}\alpha_{-1} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|\sin x|e^{ix}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{0}(-\sin x)e^{ix}\,dx \\ &\qquad + \frac{1}{2\pi}\int_{0}^{\pi}(\sin x)e^{ix}\,dx.\end{aligned}$$

Note that

$$\begin{aligned}&\int e^{ix}\sin x dx \\ &= \int(\cos x + i\sin x)\sin x dx \\ &= \int\left(\frac{1}{2}\sin 2x + i\left(\frac{1-\cos 2x}{2}\right)\right)dx \\ &= -\frac{1}{4}\cos 2x + i\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right).\end{aligned}$$

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Thus

$$-\frac{1}{2\pi}\int_{-\pi}^{0} e^{ix}\sin x\,dx$$
$$= -\frac{1}{2\pi}\left[-\frac{1}{4}\cos 2x + i\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right)\right]_{-\pi}^{0}$$
$$= -\frac{1}{2\pi}\left[-\frac{1}{4} - \left(-\frac{1}{4} - \frac{\pi}{2}i\right)\right]$$
$$= -\frac{1}{4}i.$$

Similarly,

$$\frac{1}{2\pi}\int_{0}^{\pi} e^{ix}\sin x\,dx = \frac{1}{4}i.$$

Thus, $\alpha_1 = 0$. In a similar manner, $\alpha_{-1} = 0$. In the case that $n = 0$,

$$\alpha_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin x|\,dx = \frac{1}{\pi}\int_{0}^{\pi}\sin x\,dx = \frac{2}{\pi}.$$

Thus,

$$f(x) \sim \sum_{n=-\infty}^{\infty}\alpha_n e^{inx} = \frac{2}{\pi} + \sum_{\substack{n \text{ even}\\ n\neq 0}} \frac{2}{\pi(n^2-1)}e^{inx}$$

12. Suppose that p and q are integers, $p \neq q$. Then

$$\int_{-\pi}^{\pi} e^{ipx}\overline{e^{iqx}}dx = \int_{-\pi}^{\pi} e^{ipx}e^{-iqx}dx$$
$$= \int_{-\pi}^{\pi} e^{i(p-q)x}dx$$
$$= \frac{1}{i(p-q)}e^{i(p-q)x}\Big|_{-\pi}^{\pi}$$
$$= \frac{1}{i(p-q)}\left[e^{i(p-q)\pi} - e^{-i(p-q)\pi}\right].$$

Now, the key idea is the fact that $e^{i\pi} = e^{-i\pi} = -1$. Thus,

$$\int_{-\pi}^{\pi} e^{ipx}\overline{e^{iqx}}dx = \frac{1}{i(p-q)}\left[(-1)^{p-q} - (-1)^{p-q}\right]$$
$$= 0.$$

Thus, e^{ipx} and e^{iqx} are orthogonal, provided $p \neq q$.

13. Begin with the assumption that

$$f(x) = \sum_{n=-\infty}^{\infty}\alpha_n e^{inx}, \quad -\pi \le x \le \pi.$$

Multiply both sides by $\overline{e^{ikx}} = e^{-ikx}$ and integrate.

$$f(x)e^{-ikx} = \sum_{n=-\infty}^{\infty}\alpha_n e^{inx}e^{-ikx}$$
$$\int_{-\pi}^{\pi} f(x)e^{-ikx}dx = \sum_{n=-\infty}^{\infty}\alpha_n\int_{-\pi}^{\pi} e^{inx}e^{-ikx}dx.$$

In Exercise 12, we showed that e^{inx} and e^{ikx} are orthogonal, provided n and k are integers, $n \neq k$. Thus, the right side simplifies to

$$\int_{-\pi}^{\pi} f(x)e^{-ikx}dx = \alpha_k\int_{-\pi}^{\pi} e^{ikx}e^{-ikx}dx = \alpha_k(2\pi)$$

Therefore,

$$\alpha_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx.$$

Now, k is a dummy variable, so just substitute n for k.

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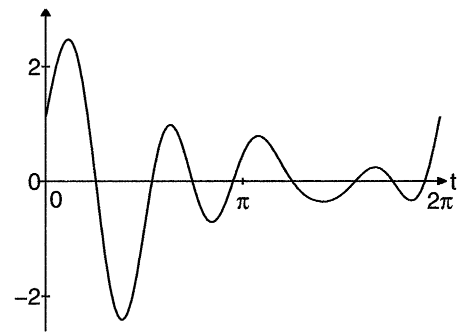
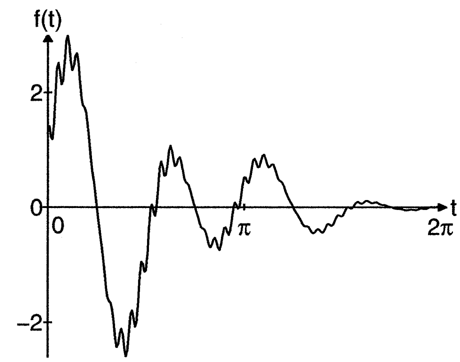
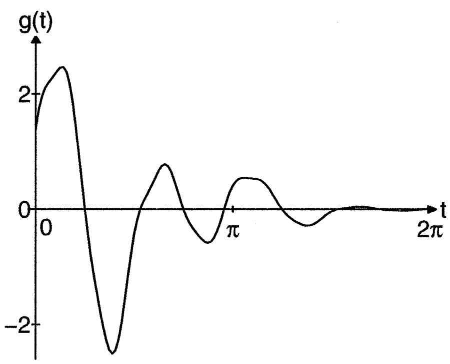
Section 12.5. The Discrete Fourier Transform and the FFT

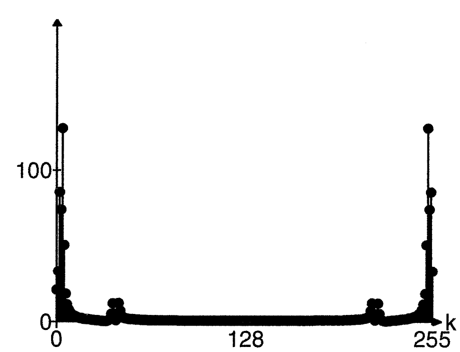
1. The function f is plotted on the left below. Because of the terms involving $\cos 2x$ and $\sin 4x$, we would expect the coefficients for $n = 4$ to be the largest, followed by $n = 2$. The middle figure plots the discrete Fourier transform with N = 256, and it verifies the conjecture. The third figure is a plot of the partial sum of the Fourier series of order $n = 6$. The partial sum keeps the main features of f, but misses the high frequency vibrations in f.

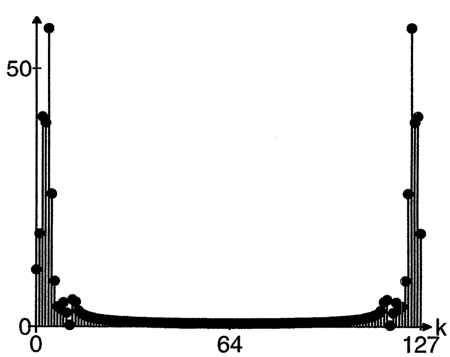


2. The function g is plotted on the left below. The middle figure plots the discrete Fourier transform with N = 128. The third figure is a plot of the Fourier series with the coefficients of small magnitude filtered out. There were 13 nonzero coefficients.









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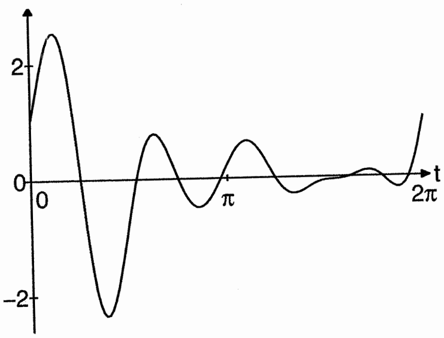
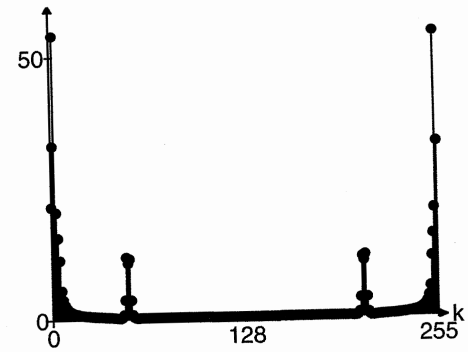
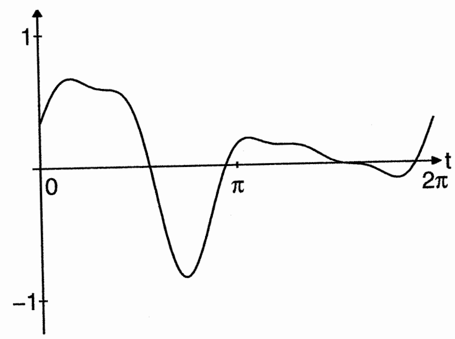




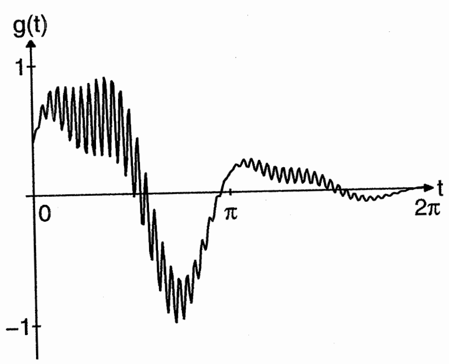
3. By 5.5, $\widehat{y}_m = \sum_{j=0}^{N-1} y_j e^{(-2\pi imj/N)} = \sum_{j=0}^{N-1} y_j \overline{w}^{jm}$. Hence, using the facts that $\overline{y_j} = y_j$ and $w = \overline{w}^{-1}$, we have

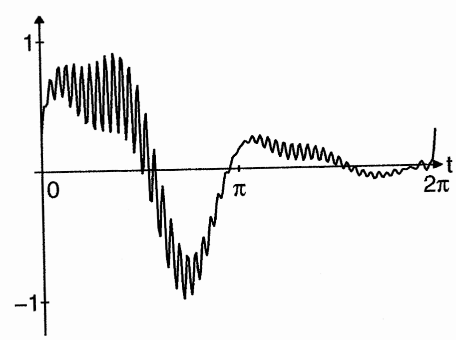
$$\overline{\widehat{y}_m} = \overline{\sum_{j=0}^{N-1} y_j \overline{w}^{jm}} = \sum_{j=0}^{N-1} y_j w^{jm} = \sum_{j=0}^{N-1} y_j \overline{w}^{-jm}$$
$$= \widehat{y}_{-m} = \widehat{y}_{N-m}.$$



4. The results are shown in the following three figures. The first shows the graph of f. The second shows the graph of the FFT, which shows that the low frequencies dominate except for a few large coefficients that account for the high frequency noise in the graph of f. The third figure shows the result of using only the low frequency components.

5. The graph of f and the FFT are shown in the previous exercise. The three figures shown below show the results of filtering with different tolerances. The first used tol $= 0.01$, and needed 71 nonzero coefficients. The second used tol $= 0.2$, and needed 15 nonzero coefficients. The second used tol $= 0.05$, and needed 27 nonzero coefficients.





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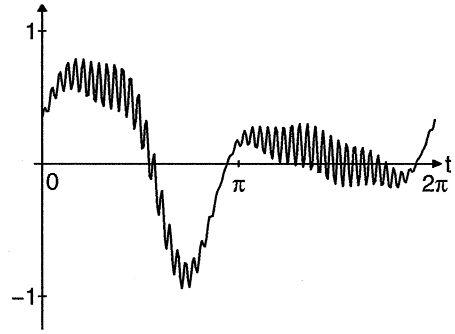
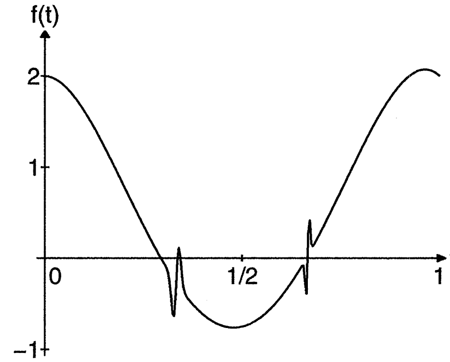
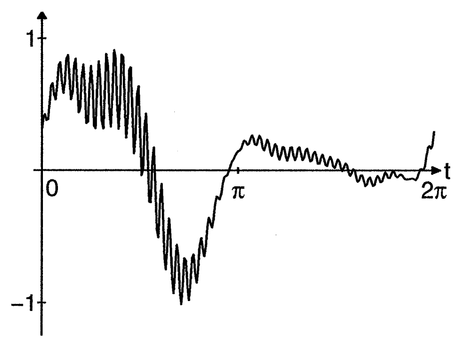
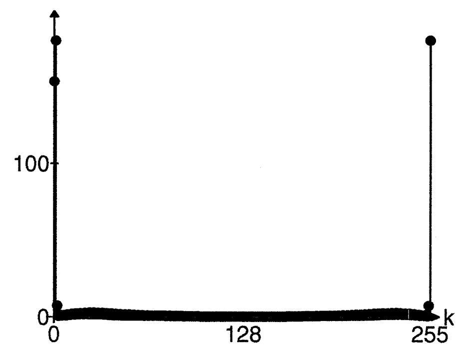
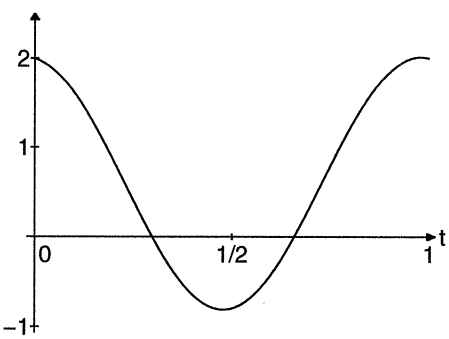








6. The results are shown in the following four figures. The first shows the graph of f showing a relatively smooth function with the exception of two spikes. The second is the graph of the FFT, which shows that the low frequencies dominate. The third figure shows the result of using only the low frequency components. In fact we use $m = 2$, so there are only 3 nonzero coefficients. The spikes are filtered out. In the fourth figure we zero out the coefficients with small magnitude, and we wanted to see the spikes. This required the use of a very small tolerance. For the figure we used tol $= 0.0025$, and the result was 179 nonzero coefficients.



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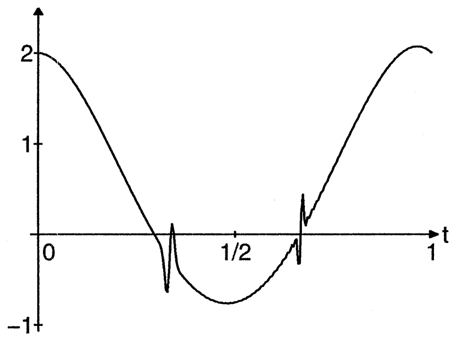

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