Solution Section 2.4 – Integration of Rational Functions by Partial Fractions

Exercise

Evaluate $\displaystyle\int \frac{dx}{x^2+2x}$

Solution

$$\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2} = \frac{Ax+2A+Bx}{x^2+2x}$$

$$1 = (A+B)x + 2A \quad \Rightarrow \begin{cases} 2A = 1 & \to A = \frac{1}{2} \\ A+B=0 & \to B = -\frac{1}{2} \end{cases}$$

$$\int \frac{1}{x^2+2x}dx = \frac{1}{2}\int \frac{1}{x}dx - \frac{1}{2}\int \frac{1}{x+2}dx$$

$$= \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + C$$

Exercise

Evaluate $\displaystyle\int \frac{2x+1}{x^2-7x+12}dx$

Solution

$$\frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3} = \frac{(A+B)x-3A-4B}{(x-4)(x-3)}$$

$$\to \begin{cases} A+B=2 \\ -3A-4B=1 \end{cases} \Rightarrow \boxed{A=9} \quad \boxed{B=-7}$$

$$\int \frac{2x+1}{x^2-7x+12}dx = 9\int \frac{dx}{x-4} - 7\int \frac{dx}{x-3}$$

$$= 9\ln|x-4| - 7\ln|x-3| + C$$

$$= \ln\left|\frac{(x-4)^9}{(x-3)^7}\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{x+3}{2x^3-8x}\,dx$

Solution

$$\frac{x+3}{2x^3-8x} = \frac{1}{2}\frac{x+3}{x\left(x^2-4\right)} = \frac{1}{2}\left(\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-2}\right)$$

$$= \frac{1}{2}\frac{A(x+2)(x-2)+Bx(x-2)+Cx(x+2)}{x(x+2)(x-2)}$$

$$(A+B+C)x^2+(2C-2B)x-4A = x+3$$

$$\begin{cases} A+B+C=0 \\ 2C-2B=1 \\ -4A=3 \end{cases} \rightarrow \boxed{A=-\frac{3}{4}} \quad \boxed{B=\frac{1}{8}} \quad \boxed{C=\frac{5}{8}}$$

$$\int \frac{x+3}{2x^3-8x}\,dx = \frac{1}{2}\int -\frac{3}{4}\frac{dx}{x} + \frac{1}{2}\int \frac{1}{8}\frac{dx}{x+2} + \frac{1}{2}\int \frac{5}{8}\frac{dx}{x-2}$$

$$= -\frac{3}{8}\ln|x| + \frac{1}{16}\ln|x+2| + \frac{5}{16}\ln|x-2| + K$$

$$= \frac{1}{16}\left(\ln|x+2| + 5\ln|x-2| - 6\ln|x|\right) + K$$

$$\boxed{= \frac{1}{16}\ln\left|\frac{(x+2)(x-2)^5}{x^6}\right| + K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^2}{(x-1)\left(x^2+2x+1\right)}dx$

Solution

$$\frac{x^2}{(x-1)\left(x^2+2x+1\right)} = \frac{x^2}{(x-1)(x+1)^2} = \frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$$x^2 = (A+B)x^2 + (2A+C)x + A - B - C$$

$$\begin{cases} A+B=1 \\ 2A+C=0 \\ A-B-C=0 \end{cases} \rightarrow \boxed{A=\frac{1}{4}} \quad \boxed{B=\frac{3}{4}} \quad \boxed{C=-\frac{1}{2}}$$

$$\int \frac{x^2}{(x-1)\left(x^2+2x+1\right)}dx = \frac{1}{4}\int \frac{dx}{x-1} + \frac{3}{4}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{(x+1)^2}$$

$$= \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2}\frac{1}{(x+1)} + K$$

$$= \frac{1}{4}\left(\ln|x-1| + \ln|x+1|^3\right) + \frac{1}{2(x+1)} + K$$

$$\boxed{= \frac{1}{4}\ln\left|(x-1)(x+1)^3\right| + \frac{1}{2(x+1)} + K}$$

Exercise

Evaluate $\displaystyle\int \frac{8x^2+8x+2}{\left(4x^2+1\right)^2}dx$

Solution

$$\frac{8x^2+8x+2}{\left(4x^2+1\right)^2} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{\left(4x^2+1\right)^2} = \frac{(Ax+B)\left(4x^2+1\right)+Cx+D}{\left(4x^2+1\right)^2}$$

$$8x^2+8x+2 = 4Ax^3+4Bx^2+(A+C)x+B+D$$

$$\begin{cases} A=0 \\ 4B=8 \\ A+C=8 \\ B+D=2 \end{cases} \to \boxed{A=0} \quad \boxed{B=2} \quad \boxed{C=8} \quad \boxed{D=0}$$

$$\int \frac{8x^2+8x+2}{\left(4x^2+1\right)^2}dx = \int \frac{2}{4x^2+1}dx + \int \frac{8x}{\left(4x^2+1\right)^2}dx \qquad d\left(4x^2+1\right) = 8xdx$$

$$= \int \frac{2}{4x^2+1}dx + \int \frac{d\left(4x^2+1\right)}{\left(4x^2+1\right)^2} \qquad \int \frac{du}{u^2} = -\frac{1}{u} \quad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\boxed{= \tan^{-1}2x - \frac{1}{4x^2+1} + K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^2+x}{x^4-3x^2-4}dx$

Solution

$$\frac{x^2+x}{x^4-3x^2-4} = \frac{x^2+x}{\left(x^2-4\right)\left(x^2+1\right)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1}$$

$$x^2+x = A(x+2)\left(x^2+1\right) + B(x-2)\left(x^2+1\right) + (Cx+D)\left(x^2-4\right)$$

$$= Ax^3 + Ax + 2Ax^2 + 2A + Bx^3 + Bx - 2Bx^2 - 2B + Cx^3 - 4Cx + Dx^2 - 4D$$

$$= (A+B+C)x^3 + (2A-2B+D)x^2 + (A+B-4C)x + 2A - 2B - 4D$$

$$\begin{cases} A+B+C=0 \\ 2A-2B+D=1 \\ A+B-4C=1 \\ 2A-2B-4D=0 \end{cases} \Rightarrow \boxed{A=\frac{3}{10}} \quad \boxed{B=-\frac{1}{10}} \quad \boxed{C=-\frac{1}{5}} \quad \boxed{D=\frac{1}{5}}$$

$$\int \frac{x^2+x}{x^4-3x^2-4}dx = \frac{3}{10}\int \frac{1}{x-2}dx - \frac{1}{10}\int \frac{1}{x+2}dx + \frac{1}{5}\int \frac{-x+1}{x^2+1}dx$$

$$= \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{5}\int \frac{x}{x^2+1}dx + \frac{1}{5}\int \frac{1}{x^2+1}dx \qquad d\left(x^2+1\right) = 2xdx$$

$$= \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{10}\int \frac{d\left(x^2+1\right)}{x^2+1} + \frac{1}{5}\int \frac{1}{x^2+1}dx \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{10}\ln\left(x^2+1\right) + \frac{1}{5}\tan^{-1}x + K$$

Exercise

Evaluate $\displaystyle\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{\left(\theta^2+1\right)^3}d\theta$

Solution

$$\frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{\left(\theta^2+1\right)^3} = \frac{A\theta+B}{\theta^2+1} + \frac{C\theta+D}{\left(\theta^2+1\right)^2} + \frac{E\theta+F}{\left(\theta^2+1\right)^3}$$

$$\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1 = (A\theta+B)\left(\theta^2+1\right)^2 + (C\theta+D)\left(\theta^2+1\right) + E\theta + F$$

$$= (A\theta+B)\left(\theta^4+2\theta^2+1\right) + C\theta^3 + C\theta + D\theta^2 + D + E\theta + F$$

$$= A\theta^5 + B\theta^4 + (2A+C)\theta^3 + (2B+D)\theta^2 + (A+C+E)\theta + B + D + F$$

$$\begin{cases} \boxed{A=0} \\ \boxed{B=1} \\ 2A+C=-4 \\ 2B+D=2 \\ A+C+E=-3 \\ B+D+F=1 \end{cases} \rightarrow \boxed{C=-4} \quad \boxed{D=0} \quad \boxed{E=1} \quad \boxed{F=0}$$

$$\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{\left(\theta^2+1\right)^3}d\theta = \int \frac{1}{\theta^2+1}d\theta - 4\int \frac{\theta}{\left(\theta^2+1\right)^2}d\theta + \int \frac{\theta}{\left(\theta^2+1\right)^3}d\theta$$

$$= \int \frac{1}{\theta^2+1}\,d\theta - 2\int \frac{d\left(\theta^2+1\right)}{\left(\theta^2+1\right)^2} + \frac{1}{2}\int \frac{d\left(\theta^2+1\right)}{\left(\theta^2+1\right)^3} \qquad d\left(\theta^2+1\right) = 2\theta d\theta$$

$$\boxed{= \tan^{-1}\theta + 2\frac{1}{\theta^2+1} - \frac{1}{4}\frac{1}{\left(\theta^2+1\right)^2} + K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^4}{x^2-1}\,dx$

Solution

$$\frac{x^4}{x^2-1} = x^2+1+\frac{1}{(x-1)(x+1)}$$

$$\begin{array}{r} x^2+1 \\ x^2-1 \overline{\big)\, x^4 \qquad} \\ \underline{x^4-x^2} \\ x^2 \\ \underline{x^2-1} \\ 1 \end{array}$$

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1}+\frac{B}{x+1} = \frac{(A+B)x+A-B}{(x-1)(x+1)}$$

$$\begin{cases} A+B=0 \\ A-B=1 \end{cases} \rightarrow \boxed{A=\frac{1}{2}} \quad \boxed{B=-\frac{1}{2}}$$

$$\int \frac{x^4}{x^2-1}\,dx = \int \left(x^2+1\right)dx + \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx$$

$$= \frac{1}{3}x^3 + x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

$$= \frac{1}{3}x^3 + x + \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right) + C$$

$$\boxed{= \frac{1}{3}x^3 + x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{16x^3}{4x^2-4x+1}\,dx$

Solution

$$\begin{array}{r} 4x+4 \\ 4x^2-4x+1 \overline{\big)\, 16x^3 \qquad\qquad} \\ \underline{16x^3-16x^2+4x} \\ 16x^2-4x \\ \underline{16x^2-16x+4} \\ 12x-4 \end{array}$$

$$\frac{16x^3}{4x^2-4x+1} = 4x+4+\frac{12x-4}{(2x-1)^2}$$

$$= 4x+4+\frac{A}{2x-1}+\frac{B}{(2x-1)^2}$$

$$12x-4 = 2Ax - A + B$$

$$\begin{cases} 2A = 12 \\ -A + B = -4 \end{cases} \quad \rightarrow \quad \boxed{A=6} \quad \boxed{B=2}$$

$$\int \frac{16x^3}{4x^2-4x+1}dx = \int (4x+4)dx + 6\int \frac{dx}{2x-1} + 2\int \frac{dx}{(2x-1)^2}$$

$$= 2x^2 + 4x + 6\left(\frac{1}{2}\right)\ln|2x-1| + 2\left(-\frac{1}{2}\right)\frac{1}{2x-1} + C$$

$$= 2x^2 + 4x + 3\ln|2x-1| - \frac{1}{2x-1} + C$$

Exercise

Evaluate $\displaystyle\int \frac{e^{4x}+2e^{2x}-e^x}{e^{2x}+1}dx$

Solution

$$\int \frac{e^{4x}+2e^{2x}-e^x}{e^{2x}+1}dx = \int \frac{e^x\left(e^{3x}+2e^x-1\right)}{e^{2x}+1}dx \qquad y = e^x \ \Rightarrow \ dy = e^x dx$$

$$= \int \frac{y^3+2y-1}{y^2+1}dy$$

$$= \int \left(y + \frac{y-1}{y^2+1}\right)dy$$

$$= \int ydy + \int \frac{y}{y^2+1}dy - \int \frac{1}{y^2+1}dy$$

$$= \int ydy + \frac{1}{2}\int \frac{1}{y^2+1}d\left(y^2+1\right) - \int \frac{1}{y^2+1}dy \qquad d\left(y^2+1\right) = 2ydy$$

$$= \frac{1}{2}y^2 + \frac{1}{2}\ln\left(y^2+1\right) - \tan^{-1}y + C$$

$$= \frac{1}{2}e^{2x} + \frac{1}{2}\ln\left(e^{2x}+1\right) - \tan^{-1}e^x + C$$

Exercise

Evaluate $\displaystyle\int \frac{\sin\theta d\theta}{\cos^2\theta + \cos\theta - 2}$

Solution

Let $y = \cos\theta \ \Rightarrow \ dy = -\sin\theta d\theta$

$$\int \frac{\sin\theta d\theta}{\cos^2\theta+\cos\theta-2} = -\int \frac{dy}{y^2+y-2}$$

$$\frac{1}{y^2+y-2}=\frac{1}{(y+2)(y-1)}=\frac{A}{y+2}+\frac{B}{y-1}$$

$$1=(A+B)y-A+2B$$

$$\begin{cases} A+B=0 \\ -A+2B=1 \end{cases} \rightarrow \boxed{A=-\frac{1}{3}} \quad \boxed{B=\frac{1}{3}}$$

$$\int \frac{\sin\theta d\theta}{\cos^2\theta+\cos\theta-2}=-\left(-\frac{1}{3}\int\frac{dy}{y+2}+\frac{1}{3}\int\frac{dy}{y-1}\right)$$

$$=\frac{1}{3}\ln|y+2|-\frac{1}{3}\ln|y-1|+C$$

$$=\frac{1}{3}\left(\ln|y+2|-\ln|y-1|\right)+C$$

$$=\frac{1}{3}\ln\left|\frac{y+2}{y-1}\right|+C$$

$$=\frac{1}{3}\ln\left|\frac{\cos\theta+2}{\cos\theta-1}\right|+C$$

Exercise

Evaluate $\displaystyle\int \frac{(x-2)^2\tan^{-1}(2x)-12x^3-3x}{\left(4x^2+1\right)(x-2)^2}\,dx$

Solution

$$\int \frac{(x-2)^2\tan^{-1}(2x)-12x^3-3x}{\left(4x^2+1\right)(x-2)^2}\,dx=\int\frac{(x-2)^2\tan^{-1}(2x)}{\left(4x^2+1\right)(x-2)^2}\,dx-\int\frac{12x^3+3x}{\left(4x^2+1\right)(x-2)^2}\,dx$$

$$=\int\frac{\tan^{-1}(2x)}{4x^2+1}\,dx-\int\frac{3x\left(4x^2+1\right)}{\left(4x^2+1\right)(x-2)^2}\,dx$$

$$=\int\frac{\tan^{-1}(2x)}{4x^2+1}\,dx-\int\frac{3x}{(x-2)^2}\,dx$$

$$d\left(\tan^{-1}2x\right)=\frac{dx}{(2x)^2+1}=\frac{dx}{4x^2+1}$$

$$\frac{3x}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}=\frac{Ax-2A+B}{(x-2)^2}$$

$$\begin{cases} \boxed{A=3} \\ -2A+B=0 \end{cases} \rightarrow \boxed{B=6}$$

$$\int \frac{(x-2)^2 \tan^{-1}(2x) - 12x^3 - 3x}{\left(4x^2+1\right)(x-2)^2}\,dx = \frac{1}{2}\int \tan^{-1}(2x)\,d\left(\tan^{-1}(2x)\right) - 3\int \frac{dx}{x-2} - 6\int \frac{dx}{(x-2)^2}$$

$$= \frac{1}{4}\left(\tan^{-1}(2x)\right)^2 - 3\int \frac{d(x-2)}{x-2} - 6\int \frac{d(x-2)}{(x-2)^2}$$

$$= \frac{1}{4}\left(\tan^{-1}(2x)\right)^2 - 3\ln|x-2| - \frac{6}{x-2} + C$$

Exercise

Evaluate $\displaystyle\int \frac{\sqrt{x+1}}{x}\,dx$

Solution

Let $x+1=u^2 \Rightarrow dx = 2udu$

$$\int \frac{\sqrt{x+1}}{x}\,dx = \int \frac{u}{u^2-1}\,2udu$$

$$= 2\int \frac{u^2}{u^2-1}\,du$$

$$\begin{array}{r|l} & 1 \\ \hline u^2-1 & u^2 \\ & \underline{u^2-1} \\ & 1 \end{array}$$

$$= 2\int \left(1+\frac{1}{u^2-1}\right)du$$

$$= 2\int du + 2\int \frac{1}{u^2-1}\,du$$

$$\frac{1}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1} = \frac{(A+B)u + A - B}{(u-1)(u+1)}$$

$$\begin{cases} A+B=0 \\ A-B=1 \end{cases} \Rightarrow \boxed{A=\frac{1}{2}} \quad \boxed{B=-\frac{1}{2}}$$

$$= 2\int du + 2\int \left(\frac{1}{2}\frac{1}{u-1} - \frac{1}{2}\frac{1}{u+1}\right)du$$

$$= 2u + \int \frac{1}{u-1}\,du - \int \frac{1}{u+1}\,du$$

$$= 2u + \ln|u-1| - \ln|u+1| + C$$

$$= 2\sqrt{x+1} + \ln\left|\sqrt{x+1}-1\right| - \ln\left|\sqrt{x+1}+1\right| + C$$

$$= 2\sqrt{x+1} + \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{x^3-2x^2+3x-4}{x^2+1}dx$

Solution

$$\int \frac{x^3-2x^2+3x-4}{x^2+1}dx = \int \left(x-2+\frac{2x-2}{x^2+1}\right)dx$$

$$= \int (x-2)\,dx + \int \frac{2x}{x^2+1}dx - 2\int \frac{1}{x^2+1}dx$$

$$= \int (x-2)\,dx + \int \frac{d\left(x^2+1\right)}{x^2+1} - 2\int \frac{1}{x^2+1}dx$$

$$= \frac{1}{2}x^2 - 2x + \ln\left(x^2+1\right) - 2\tan^{-1}(x) + C$$

$$\begin{array}{r|l} & x-2 \\ \hline x^2+1 & x^3-2x^2+3x-4 \\ & \underline{x^3 \qquad\quad +x} \\ & -2x^2+2x-4 \\ & \underline{-2x^2 \qquad\quad -2} \\ & 2x-2 \end{array}$$

Exercise

Evaluate $\displaystyle\int \frac{4x^2+2x+4}{x+1}dx$

Solution

$$\int \frac{4x^2+2x+4}{x+1}dx = \int \left(4x+2+\frac{6}{x+1}\right)dx$$

$$= \int (4x-2)dx + \int \frac{6}{x+1}dx$$

$$= \int (4x-2)dx + 6\int \frac{d(x+1)}{x+1}$$

$$\int \frac{d(U)}{U} = \ln|U|$$

$$= 2x^2 - 2x + 6\ln|x+1| + C$$

Exercise

Evaluate $\displaystyle\int \frac{3x^2+7x-2}{x^3-x^2-2x}dx$

Solution

$$\frac{3x^2+7x-2}{x^3-x^2-2x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$$

$$3x^2+7x-2 = A(x+1)(x-2) + Bx(x-2) + Cx(x+1)$$

$$= Ax^2 - Ax - 2A$$

$$Bx^2 - 2Bx$$

$$Cx^2 + Cx$$

$$\begin{cases} A+B+C=3 \\ -A-2B+C=7 \\ -2A=-2 \end{cases} \rightarrow \boxed{A=1} \quad \begin{cases} B+C=2 \\ -2B+C=8 \end{cases} \rightarrow \underline{B=-2} \quad \underline{C=4}$$

$$\int \frac{3x^2+7x-2}{x^3-x^2-2x}dx = \int \left(\frac{1}{x} - \frac{2}{x+1} + \frac{4}{x-2}\right)dx$$

$$= \ln|x| - 2\ln|x+1| + 4\ln|x-2| + K$$

$$= \ln \frac{|x|(x-2)^4}{(x+1)^2} + K$$

Exercise

Evaluate $\displaystyle \int \frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)} dx$

Solution

$$\frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)} = \frac{A}{x-1} + \frac{B}{x-5} + \frac{C}{x+4}$$

$$3x^2+2x+5 = (A+B+C)x^2 + (-A+3B-6C)x - 20A - 4B + 5C$$

$$\begin{cases} A+B+C=3 \\ -A+3B-6C=2 \\ -20A-4B+5C=5 \end{cases} \rightarrow A=\frac{1}{2}, \quad B=\frac{5}{2}, \quad C=1$$

$$\int \frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)} dx = \int \left(\frac{1}{2}\frac{1}{x-1} + \frac{5}{2}\frac{1}{x-5} + \frac{1}{x+4}\right)dx$$

$$= \frac{1}{2}\ln|x-1| + \frac{5}{2}\ln|x-5| + \ln|x+4| + K$$

Exercise

Evaluate $\displaystyle \int \frac{5x^2-3x+2}{x^3-2x^2} dx$

Solution

$$\frac{5x^2-3x+2}{x^3-2x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$5x^2-3x+2 = Ax^2 - 2Ax + Bx - 2B + Cx^2$$

$$\begin{cases} A+C=5 \\ -2A+B=-3 \\ -2B=2 \end{cases} \rightarrow \underline{B=-1; \ \ A=1; \ \ C=4}$$

$$\int \frac{5x^2-3x+2}{x^3-2x^2}dx = \int \frac{dx}{x} - \int \frac{dx}{x^2} + 4\int \frac{dx}{x-2}$$

$$\boxed{= \ln|x| + \frac{1}{x} + 4\ln|x-2| + K}$$

Exercise

Evaluate $\displaystyle\int \frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}dx$

Solution

$$\frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)} = \frac{A}{x-2} + \frac{Bx+C}{x^2-2x+3}$$

$$7x^2-13x+13 = Ax^2-2Ax+3A+Bx^2-2Bx+Cx-2C$$

$$\begin{cases} A+B=7 \\ -2A-2B+C=-13 \\ 3A-2C=13 \end{cases} \rightarrow \boxed{A=5;\ \ B=2;\ C=1}$$

$$\begin{aligned}
\int \frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}dx &= \int \frac{5dx}{x-2} + \int \frac{2x+1}{x^2-2x+3}dx \\
&= 5\ln|x-2| + \int \frac{2x-2+3}{x^2-2x+3}dx \\
&= 5\ln|x-2| + \int \frac{2x-2}{x^2-2x+3}dx + \int \frac{3}{(x-1)^2+3}dx \\
&= \boxed{5\ln|x-2| + \ln\left(x^2-2x+3\right) + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + K}
\end{aligned}$$

Exercise

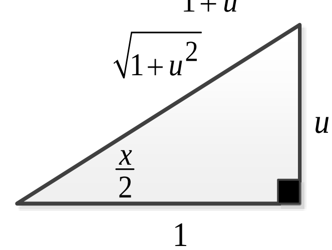
Evaluate $\displaystyle\int \frac{dx}{1+\sin x}$

Solution

$$\begin{aligned}
\int \frac{dx}{1+\sin x} &= \int \frac{1}{1+\frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2}du \\
&= \int \frac{2}{u^2+2u+1}du \\
&= \int \frac{2}{(u+1)^2}d(u+1)
\end{aligned}$$

Let $u = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\tan^{-1}u \quad \rightarrow dx = \frac{2du}{1+u^2}$

$$\begin{aligned}
\sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2} \\
&= 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}} \\
&= \frac{2u}{1+u^2}
\end{aligned}$$



$$= -\frac{2}{u+1} + C$$

$$\boxed{= -\frac{2}{\tan\left(\frac{x}{2}\right)+1} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{2+\cos x}$

Solution

$$\int \frac{dx}{2+\cos x} = \int \frac{1}{2+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}du$$

$$= 2\int \frac{1}{u^2+3}\, du$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C$$

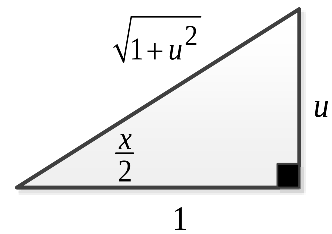
$$\boxed{= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}} \tan\frac{x}{2}\right) + C}$$

Let $u = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\tan^{-1} u \quad \to dx = \frac{2du}{1+u^2}$

$$\cos x = 2\cos^2\frac{x}{2} - 1$$

$$= 2\frac{1}{1+u^2} - 1$$

$$= \frac{1-u^2}{1+u^2}$$



$$\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{1-\cos x}$

Solution

$$\int \frac{dx}{1-\cos x} = \int \frac{1}{1-\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}du$$

$$= \int \frac{1}{u^2}\, du$$

$$= -\frac{1}{u} + C$$

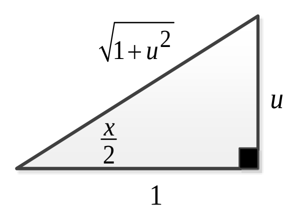
$$\boxed{= -\frac{1}{\tan\frac{x}{2}} + C} \quad \boxed{= -\cot\frac{x}{2} + C}$$

Let $u = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\tan^{-1} u \quad \to dx = \frac{2du}{1+u^2}$

$$\cos x = 2\cos^2\frac{x}{2} - 1$$

$$= 2\frac{1}{1+u^2} - 1$$

$$= \frac{1-u^2}{1+u^2}$$



Exercise

Evaluate $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$

Solution

$$\int \frac{dx}{1+\sin x+\cos x}=\int \frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\cdot\frac{2}{1+u^2}du$$

$$=2\int \frac{1}{2+2u}\,du$$

$$=\int \frac{1}{1+u}\,d(1+u)$$

$$=\ln|1+u|+C$$

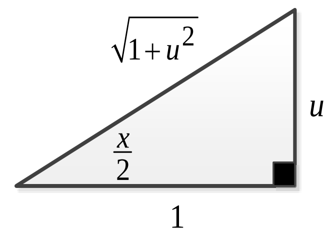
$$\boxed{=\ln\left|1+\tan\frac{x}{2}\right|+C}$$

$u=\tan\left(\frac{x}{2}\right)\Rightarrow x=2\tan^{-1}u \quad \to dx=\frac{2du}{1+u^2}$

$$\cos x=2\cos^2\frac{x}{2}-1$$

$$=2\frac{1}{1+u^2}-1$$

$$=\frac{1-u^2}{1+u^2}$$



$$\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$=2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}}=\frac{2u}{1+u^2}$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x^2-5x+6}\,dx$

Solution

$$\frac{1}{x^2-5x+6}=\frac{A}{x-2}+\frac{B}{x-3}$$

$$Ax-3A+Bx-2B=1 \quad \to \begin{cases} A+B=0 \\ -3A-2B=1 \end{cases} \quad \to \quad A=-1 \quad B=1$$

$$\int \frac{1}{x^2-5x+6}\,dx=\int\left(\frac{-1}{x-2}+\frac{1}{x-3}\right)dx$$

$$=\ln|x-3|-\ln|x-2|+C$$

$$\boxed{=\ln\left|\frac{x-3}{x-2}\right|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x^2-5x+5}\,dx$

Solution

$$\frac{1}{x^2-5x+5}=\frac{A}{x-\frac{5+\sqrt{5}}{2}}+\frac{B}{x-\frac{5-\sqrt{5}}{2}} \qquad x=\frac{5\pm\sqrt{5}}{2}$$

$$Ax-\left(\frac{5-\sqrt{5}}{2}\right)A+Bx-\left(\frac{5+\sqrt{5}}{2}\right)B=1$$

$$\begin{cases} A+B=0 \\ -\frac{5-\sqrt{5}}{2}A-\frac{5+\sqrt{5}}{2}B=1 \end{cases} \to \begin{matrix} \frac{5-\sqrt{5}}{2}A+\frac{5-\sqrt{5}}{2}B=0 \\ -\frac{5-\sqrt{5}}{2}A-\frac{5+\sqrt{5}}{2}B=1 \end{matrix}$$

$$-\sqrt{5}B=1 \to B=-\frac{1}{\sqrt{5}} \quad \Rightarrow \quad A=\frac{1}{\sqrt{5}}$$

$$\int \frac{1}{x^2-5x+5}\,dx = \int \left(\frac{\sqrt{5}}{5}\frac{2}{2x-5-\sqrt{5}} - \frac{\sqrt{5}}{5}\frac{2}{2x-5+\sqrt{5}} \right) dx$$

$$= \frac{\sqrt{5}}{5}\ln\left|2x-5-\sqrt{5}\right| - \frac{\sqrt{5}}{5}\ln\left|2x-5+\sqrt{5}\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$

Solution

$$\frac{5x^2+20x+6}{x^3+2x^2+x} = \frac{5x^2+20x+6}{x(x+1)^2} = \frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$Ax^2+2Ax+A+Bx^2+Bx+Cx = 5x^2+20x+6$$

$$\begin{cases} A+B=5 \\ 2A+B+C=20 \\ \underline{A=6} \end{cases} \to \quad \underline{B=-1} \quad \underline{C=9}$$

$$\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx = \int \left(\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2} \right) dx$$

$$= 6\ln|x| - \ln|x+1| - \frac{9}{x+1} + C$$

$$= \ln\frac{x^6}{|x+1|} - \frac{9}{x+1} + C$$

Exercise

Evaluate $\displaystyle\int \frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}\,dx$

Solution

$$\frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)} = \frac{2x^3-4x-8}{x(x-1)\left(x^2+4\right)} = \frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+4}$$

$$Ax^3-Ax^2+4Ax-4A+Bx^3+4Bx+Cx^3-Cx^2+Dx^2-Dx = 2x^3-4x-8$$

$$\begin{cases} x^3 & A+B+C=2 \\ x^2 & -A-C+D=0 \\ x^1 & 4A+4B-D=-4 \\ x^0 & -4A=-8 \end{cases} \rightarrow \begin{cases} B+C=0 \\ -C+D=2 \\ 4B-D=-12 \\ \underline{A=2} \end{cases} \Rightarrow \begin{cases} B+D=2 \\ 4B-D=-12 \end{cases} \rightarrow \begin{cases} A=2 \\ B=-2 \\ C=2 \\ D=4 \end{cases}$$

$$\int \frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}\,dx = \int \left(\frac{2}{x} - \frac{2}{x-1} + \frac{2x}{x^2+4} + \frac{4}{x^2+4}\right) dx \qquad \int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a}$$

$$\boxed{= 2\ln|x| - 2\ln|x-1| + \ln\left(x^2+4\right) + 2\tan^{-1}\frac{x}{2} + C}$$

Exercise

Evaluate $\displaystyle \int \frac{8x^3+13x}{\left(x^2+2\right)^2}\,dx$

Solution

$$\frac{8x^3+13x}{\left(x^2+2\right)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{\left(x^2+2\right)^2}$$

$$Ax^3 + 2Ax + Bx^2 + 2B + Cx + D = 8x^3 + 13x \qquad \begin{cases} x^3 & A=8 \\ x^2 & B=0 \\ x^1 & 2A+C=13 \\ x^0 & D=0 \end{cases} \rightarrow C=-3$$

$$\int \frac{8x^3+13x}{\left(x^2+2\right)^2}\,dx = \int \frac{8x}{x^2+2}\,dx - \int \frac{3x}{\left(x^2+2\right)^2}\,dx$$

$$= 2\int \frac{1}{x^2+2}\,d\left(x^2+2\right) - \frac{3}{2}\int \frac{1}{\left(x^2+2\right)^2}\,d\left(x^2+2\right)$$

$$\boxed{= 2\ln\left(x^2+2\right) + \frac{3}{2}\frac{1}{x^2+2} + C}$$

Exercise

Evaluate $\displaystyle \int \frac{\sin x}{\cos x + \cos^2 x}\,dx$

Solution

$$\frac{\sin x}{\cos x + \cos^2 x} = \frac{A}{\cos x} + \frac{B}{1+\cos x}$$

$$A + A\cos x + B\cos x = \sin x \qquad \begin{cases} \underline{A = \sin x|} \\ A + B = 0 \end{cases} \to \underline{B = -\sin x|}$$

$$\int \frac{\sin x}{\cos x + \cos^2 x}\, dx = \int \frac{\sin x}{\cos x} dx - \int \frac{\sin x}{1+\cos x} dx$$

$$= -\int \frac{1}{\cos x}\, d(\cos x) + \int \frac{1}{1+\cos x}\, d(1+\cos x)$$

$$= -\ln|\cos x| + \ln|1+\cos x| + C$$

$$\underline{= \ln\left|\frac{1+\cos x}{\cos x}\right| + C|} \qquad \underline{= \ln|\sec x + 1| + C|}$$

Exercise

Evaluate $\displaystyle\int \frac{5\cos x}{\sin^2 x + 3\sin x - 4}\, dx$

Solution

$$\frac{5\cos x}{\sin^2 x + 3\sin x - 4} = \frac{A}{\sin x - 1} + \frac{B}{\sin x + 4}$$

$$A\sin x + 4A + B\sin x - B = 5\cos x \qquad \begin{cases} 4A - B = 5\cos x \\ A + B = 0 \end{cases} \quad \underline{A = \cos x|} \quad \underline{B = -\cos x|}$$

$$\int \frac{5\cos x}{\sin^2 x + 3\sin x - 4}\, dx = \int \frac{\cos x}{\sin x - 1} dx - \int \frac{\cos x}{\sin x + 4} dx$$

$$= \int \frac{1}{\sin x - 1} d(\sin x - 1) - \int \frac{1}{\sin x + 4} d(\sin x + 4)$$

$$= \ln|\sin x - 1| - \ln|\sin x + 4| + C$$

$$\underline{= \ln\left|\frac{\sin x - 1}{\sin x + 4}\right| + C|}$$

Exercise

Evaluate $\displaystyle\int \frac{e^x}{\left(e^x - 1\right)\left(e^x + 4\right)}\, dx$

Solution

Let $u = e^x \quad \to \quad du = e^x dx$

$$\int \frac{e^x}{\left(e^x - 1\right)\left(e^x + 4\right)}\, dx = \int \frac{du}{(u-1)(u+4)}$$

$$\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4}$$

$$Au + 4A + Bu - B = 1 \quad \Rightarrow \begin{cases} A + B = 0 \\ 4A - B = 1 \end{cases} \to \quad \underline{A = \frac{1}{5},\ B = -\frac{1}{5}|}$$

$$\int \frac{du}{(u-1)(u+4)} = \frac{1}{5}\int \frac{1}{u-1}\, du + \frac{4}{5}\int \frac{1}{u+4}\, du$$

$$= \frac{1}{5}\int \frac{1}{u-1}\, d(u-1) + \frac{4}{5}\int \frac{1}{u+4}\, d(u+4)$$

$$= \frac{1}{5}\ln\left|e^x - 1\right| - \frac{1}{5}\ln\left(e^x + 4\right) + C$$

$$\boxed{= \frac{1}{5}\ln\left|\frac{e^x - 1}{e^x + 4}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{e^x}{\left(e^{2x}+1\right)\left(e^x-1\right)}\, dx$

Solution

Let $u = e^x \quad \to \quad du = e^x dx$

$$\int \frac{e^x}{\left(e^{2x}+1\right)\left(e^x-1\right)}\, dx = \int \frac{du}{\left(u^2+1\right)(u-1)}$$

$$\frac{1}{\left(u^2+1\right)(u-1)} = \frac{Au+B}{u^2+1} + \frac{C}{u-1}$$

$$Au^2 - Au + Bu - B + Cu^2 + C = 1$$

$$\begin{cases} u^2 & A+C=0 \\ u^1 & -A+B=0 \\ u^0 & -B+C=1 \end{cases} \to \begin{cases} B+C=0 \\ -B+C=1 \end{cases} \qquad \boxed{C = \frac{1}{2} \quad B = -\frac{1}{2} \quad A = -\frac{1}{2}}$$

$$\int \frac{du}{\left(u^2+1\right)(u-1)} = -\frac{1}{2}\int \frac{u}{u^2+1} du - \frac{1}{2}\int \frac{du}{u^2+1} + \frac{1}{2}\int \frac{du}{u-1}$$

$$= -\frac{1}{4}\int \frac{1}{u^2+1} d\left(u^2+1\right) - \frac{1}{2}\arctan u + \frac{1}{2}\ln|u-1|$$

$$\boxed{= -\frac{1}{4}\ln\left(e^{2x}+1\right) - \frac{1}{2}\arctan e^x + \frac{1}{2}\ln\left|e^x - 1\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{\sqrt{x}}{x-4}\, dx$

Solution

Let $u = \sqrt{x} \quad \to \quad du = \dfrac{1}{2\sqrt{x}} dx \;\Rightarrow\; 2udu = dx$

$$\int \frac{\sqrt{x}}{x-4}\,dx = \int \frac{u}{u^2-4}\,2u\,du$$

$$= \int \frac{2u^2}{u^2-4}\,du$$

$$= \int \left(2 + \frac{8}{u^2-4}\right) du$$

$$\frac{8}{u^2-4} = \frac{A}{u-2} + \frac{B}{u+2}$$

$$Au + 2A + Bu - 2B = 8 \qquad \to \begin{cases} A+B=0 \\ 2A-2B=8 \end{cases} \Rightarrow \boxed{A=2 \quad B=-2}$$

$$= \int \left(2 + \frac{2}{u-2} - \frac{2}{u+2}\right) du$$

$$= 2\sqrt{x} + 2\ln\left|\sqrt{x}-2\right| - 2\ln\left|\sqrt{x}+2\right| + C$$

$$\boxed{= 2\sqrt{x} + 2\ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right| + C}$$

Exercise

Evaluate $\displaystyle \int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$

Solution

Let $u = x^{1/6} \to u^6 = x \quad \to \quad 6u^5 du = dx$

$u^2 = x^{1/3} \quad u^3 = x^{1/2}$

$$\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx = \int \frac{6u^5}{u^3-u^2}\,du$$

$$= \int \frac{6u^3}{u-1}\,du$$

$$\begin{array}{r|l} & 6u^2+6u+6 \\ u-1 & 6u^3 \\ & \underline{-6u^3+6u^2} \\ & 6u^2 \\ & \underline{-6u^2+6u} \\ & 6u \\ & \underline{-6u+6} \\ & 6 \end{array}$$

$$= \int \left(6u^2 + 6u + 6 + \frac{6}{u-1}\right) du$$

$$= 2u^3 + 3u^2 + 6u + 6\ln|u-1| + C$$

$$\boxed{= 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln\left|\sqrt[6]{x}-1\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x^2-9}\,dx$

Solution

$$\frac{1}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3}$$

$$Ax + 3A + Bx - 3B = 1 \qquad \Rightarrow \begin{cases} A+B=0 \\ 3A-3B=1 \end{cases} \rightarrow \boxed{A = \tfrac{1}{6} \quad B = -\tfrac{1}{6}}$$

$$\int \frac{1}{x^2-9}\,dx = \frac{1}{6}\int \frac{1}{x-3}\,dx - \frac{1}{6}\int \frac{1}{x+3}\,dx$$

$$= \frac{1}{6}\ln|x-3| - \frac{1}{6}\ln|x+3| + C$$

$$\boxed{= \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{2}{9x^2-1}\,dx$

Solution

$$\frac{2}{9x^2-1} = \frac{A}{3x-1} + \frac{B}{3x+1}$$

$$3Ax + A + 3Bx - B = 2 \qquad \Rightarrow \begin{cases} 3A+3B=0 \\ A-B=2 \end{cases} \rightarrow \boxed{A = 1 \quad B = -1}$$

$$\int \frac{2}{9x^2-1}\,dx = \int \frac{1}{3x-1}\,dx - \int \frac{1}{3x+1}\,dx$$

$$= \frac{1}{3}\ln|3x-1| - \frac{1}{3}\ln|3x+1| + C$$

$$\boxed{= \frac{1}{3}\ln\left|\frac{3x-1}{3x+1}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{5}{x^2+3x-4}\,dx$

Solution

$$\frac{5}{x^2+3x-4} = \frac{A}{x-1} + \frac{B}{x+4}$$

$$Ax + 4A + Bx - B = 5 \qquad \Rightarrow \begin{cases} A+B=0 \\ 4A-B=5 \end{cases} \rightarrow \boxed{A = 1 \quad B = -1}$$

$$\int \frac{5}{x^2+3x-4}\,dx = \int \frac{1}{x-1}\,dx - \int \frac{1}{x+4}\,dx$$

$$= \ln|x-1| - \ln|x+4| + C$$

$$\boxed{= \ln\left|\frac{x-1}{x+4}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{3-x}{3x^2-2x-1}\,dx$

Solution

$$\frac{3-x}{3x^2-2x-1} = \frac{A}{x-1} + \frac{B}{3x+1}$$

$$3Ax + A + Bx - B = 3 - x \qquad \Rightarrow \begin{cases} 3A + B = -1 \\ A - B = 3 \end{cases} \rightarrow \boxed{A = \frac{1}{2} \quad B = -\frac{5}{2}}$$

$$\int \frac{3-x}{3x^2-2x-1}\,dx = \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{5}{2}\int \frac{1}{3x+1}\,dx$$

$$\boxed{= \frac{1}{2}\ln|x-1| - \frac{5}{6}\ln|3x+1| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x^2+12x+12}{x^3-4x}\,dx$

Solution

$$\frac{x^2+12x+12}{x^3-4x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

$$Ax^2 - 4A + Bx^2 + 2Bx + Cx^2 - 2Cx = x^2 + 12x + 12$$

$$\begin{cases} x^2 & A + B + C = 1 \\ x^1 & 2B - 2C = 12 \quad \rightarrow A = -3 \quad B = 5 \quad C = -1 \\ x^0 & -4A = 12 \end{cases}$$

$$\int \frac{x^2+12x+12}{x^3-4x}\,dx = -\frac{3}{x} + \frac{5}{x-2} - \frac{1}{x+2}$$

$$\boxed{= -3\ln|x| + 5\ln|x-2| - \ln|x+2| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x^3-x+3}{x^2+x-2}\,dx$

Solution

$$\frac{x^3 - x + 3}{x^2 + x - 2} = x - 1 + \frac{2x+1}{x^2+x-2}$$

$$\begin{array}{r}
x-1 \\
x^2+x-2\,\big)\overline{x^3 - x + 3} \\
\underline{-x^3 - x^2 + 2x} \\
-x^2 + x + 3 \\
\underline{x^2 + x - 2} \\
2x - 1
\end{array}$$

$$\frac{2x+1}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$Ax + 2A + Bx - B = 2x + 1 \quad \Rightarrow \begin{cases} A + B = 2 \\ 2A - B = 1 \end{cases} \rightarrow A = 1 \quad B = 1$$

$$\int \frac{x^3 - x + 3}{x^2 + x - 2}\, dx = \int \left(x - 1 + \frac{1}{x-1} + \frac{1}{x+2} \right) dx$$

$$= \frac{1}{2}x^2 - x + \ln|x-1| + \ln|x+2| + C$$

Exercise

Evaluate $\displaystyle\int \frac{5x-2}{(x-2)^2}\, dx$

Solution

$$\frac{5x-2}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$$

$$Ax - 2A + B = 5x - 2 \quad \Rightarrow \begin{cases} A = 5 \\ -2A + B = -2 \end{cases} \rightarrow B = 8$$

$$\int \frac{5x-2}{(x-2)^2}\, dx = \frac{5}{x-2} + \frac{8}{(x-2)^2}$$

$$= 5\ln|x-2| - \frac{8}{x-2} + C$$

Exercise

Evaluate $\displaystyle\int \frac{2x^3 - 4x^2 - 15x + 4}{x^2 - 2x - 8}\, dx$

Solution

$$\begin{array}{r}
2x \\
x^2-2x-8\,\big)\overline{2x^3 - 4x^2 - 15x + 4} \\
\underline{2x^3 - 4x^2 - 16x} \\
x + 4
\end{array}$$

$$\int \frac{2x^3 - 4x^2 - 15x + 4}{x^2 - 2x - 8}\, dx = \int 2x\, dx + \int \frac{x+4}{x^2 - 2x - 8}\, dx$$

$$\frac{x+4}{x^2-2x-8} = \frac{A}{x-4} + \frac{B}{x+2}$$

$$Ax + 2A + Bx - 4B = x + 4 \quad \Rightarrow \begin{cases} A + B = 1 \\ 2A - 4B = 4 \end{cases} \rightarrow A = \frac{4}{3} \quad B = -\frac{1}{3}$$

$$\int \frac{2x^3 - 4x^2 - 15x + 4}{x^2 - 2x - 8}\, dx = x^2 + \frac{4}{3}\int \frac{1}{x-4}\, dx - \frac{1}{3}\int \frac{1}{x+2}\, dx$$

$$= x^2 + \frac{4}{3}\ln|x-4| - \frac{1}{3}\ln|x+2| + C$$

Exercise

Evaluate $\displaystyle\int \frac{x+2}{x^2+5x}\,dx$

Solution

$$\frac{x+2}{x^2+5x} = \frac{A}{x} + \frac{B}{x+5}$$

$$Ax + 5A + Bx = x + 2 \qquad \Rightarrow \begin{cases} A+B=1 \\ 5A=2 \end{cases} \rightarrow \underline{A=\tfrac{2}{5} \quad B=\tfrac{3}{5}}$$

$$\int \frac{x+2}{x^2+5x}\,dx = \frac{2}{5}\int \frac{1}{x}\,dx + \frac{3}{5}\int \frac{1}{x+5}\,dx$$

$$= \frac{2}{5}\ln|x| + \frac{3}{5}\ln|x+5| + C$$

Exercise

Evaluate $\displaystyle\int \frac{\sec^2 x}{\tan^2 x + 5\tan x + 6}\,dx$

Solution

Let $u = \tan x \quad du = \sec^2 x\,dx$

$$\int \frac{\sec^2 x}{\tan^2 x + 5\tan x + 6}\,dx = \int \frac{1}{u^2+5u+6}\,du$$

$$\frac{1}{u^2+5u+6} = \frac{1}{(u+2)(u+3)} = \frac{A}{u+2} + \frac{B}{u+3}$$

$$1 = Au + 3A + Bu + 2B$$

$$\begin{cases} A+B=0 \\ 3A+2B=1 \end{cases} \rightarrow A=1 \quad B=-1$$

$$\int \frac{\sec^2 x}{\tan^2 x + 5\tan x + 6}\,dx = \int \frac{1}{u+2}\,du - \int \frac{1}{u+3}\,du$$

$$= \ln|\tan x + 2| - \ln|\tan x + 3| + C$$

$$= \ln\left|\frac{\tan x + 2}{\tan x + 3}\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{\sec^2 x}{\tan x(\tan x + 1)}\,dx$

Solution

Let $u = \tan x \quad du = \sec^2 x\,dx$

$$\int \frac{\sec^2 x}{\tan x(\tan x + 1)}\,dx = \int \frac{du}{u(u+1)}$$

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

$$1 = Au + A + Bu$$

$$\begin{cases} A + B = 0 \\ \underline{A = 1} \end{cases} \rightarrow \underline{B = -1}$$

$$\int \frac{\sec^2 x}{\tan x(\tan x + 1)}\, dx = \int \frac{du}{u} - \int \frac{du}{u+1}$$

$$= \ln|\tan x| - \ln|\tan x + 1| + C$$

$$\boxed{= \ln\left|\frac{\tan x}{\tan x + 1}\right| + C}$$

Exercise

Evaluate $\displaystyle\int_0^2 \frac{3}{4x^2 + 5x + 1}\, dx$

Solution

$$\frac{3}{4x^2 + 5x + 1} = \frac{A}{x+1} + \frac{B}{4x+1}$$

$$4Ax + A + Bx + B = 3 \qquad \Rightarrow \begin{cases} 4A + B = 0 \\ A + B = 3 \end{cases} \rightarrow \underline{A = -1 \quad B = 4}$$

$$\int_0^2 \frac{3}{4x^2 + 5x + 1}\, dx = -\int_0^2 \frac{1}{x+1}\, dx + \int_0^2 \frac{4}{4x+1}\, dx$$

$$= -\ln(x+1) + \ln(4x+1)\Big|_0^2$$

$$= \ln \frac{4x+1}{x+1}\Big|_0^2$$

$$\underline{= \ln 3}$$

Exercise

Evaluate $\displaystyle\int_1^5 \frac{x-1}{x^2(x+1)}\, dx$

Solution

$$\frac{x-1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$Ax^2 + Ax + Bx + B + Cx^2 = x - 1$$

$$\begin{cases} x^2 & A+C=0 \\ x^1 & A+B=1 \quad \rightarrow A=2 \quad C=-2 \\ x^0 & \underline{B=-1} \end{cases}$$

$$\int_1^5 \frac{x-1}{x^2(x+1)}\,dx = \int_1^5 \left(\frac{2}{x} - \frac{1}{x^2} - \frac{2}{x+1}\right) dx$$

$$= 2\ln x + \frac{1}{x} - 2\ln(x+1)\Big|_1^5$$

$$= 2\ln 5 + \frac{1}{5} - 2\ln 6 - 1 + 2\ln 2$$

$$\boxed{= 2\ln\frac{5}{3} - \frac{4}{5}}$$

Exercise

Evaluate $\displaystyle\int_1^2 \frac{x+1}{x\left(x^2+1\right)}\,dx$

Solution

$$\frac{x+1}{x\left(x^2+1\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$Ax^2 + A + Bx^2 + Cx = x+1$$

$$\begin{cases} x^2 & A+B=0 \\ x^1 & \underline{C=1} \quad \rightarrow \underline{B=-1} \\ x^0 & \underline{A=1} \end{cases}$$

$$\int_1^2 \frac{x+1}{x\left(x^2+1\right)}\,dx = \int_1^2 \frac{1}{x}\,dx - \int_1^2 \frac{x}{x^2+1}\,dx + \int_1^2 \frac{1}{x^2+1}\,dx$$

$$= \int_1^2 \frac{1}{x}\,dx - \frac{1}{2}\int_1^2 \frac{1}{x^2+1}\,d\left(x^2+1\right) + \int_1^2 \frac{1}{x^2+1}\,dx$$

$$= \ln x - \frac{1}{2}\ln\left(x^2+1\right) + \arctan x\Big|_1^2$$

$$= \ln 2 - \frac{1}{2}\ln 5 + \arctan 2 + \frac{1}{2}\ln 2 - \frac{\pi}{4}$$

$$= \frac{1}{2}(3\ln 2 - \ln 5) - \frac{\pi}{4} + \arctan 2$$

$$\boxed{= \frac{1}{2}\ln\frac{8}{5} - \frac{\pi}{4} + \arctan 2}$$

Exercise

Evaluate $\displaystyle\int_0^1 \frac{x^2 - x}{x^2 + x + 1}\, dx$

Solution

$$\int_0^1 \frac{x^2 - x}{x^2 + x + 1}\, dx = \int_0^1 \left(1 - \frac{2x+1}{x^2 + x + 1}\right) dx$$

$$= \int_0^1 dx - \int_0^1 \frac{1}{x^2 + x + 1}\, d\left(x^2 + x + 1\right)$$

$$= x - \ln\left(x^2 + x + 1\right)\Big|_0^1$$

$$= 1 - \ln 3$$

Exercise

Evaluate $\displaystyle\int_4^8 \frac{y\,dy}{y^2 - 2y - 3}$

Solution

$$\frac{y}{y^2 - 2y - 3} = \frac{A}{y-3} + \frac{B}{y+1} = \frac{(A+B)y + A - 3B}{(y-3)(y+1)} \qquad \rightarrow \begin{cases} A + B = 1 \\ A - 3B = 0 \end{cases} \Rightarrow \boxed{A = \frac{3}{4}} \quad \boxed{B = \frac{1}{4}}$$

$$\int_4^8 \frac{y\,dy}{y^2 - 2y - 3} = \frac{3}{4}\int_4^8 \frac{dy}{y-3} + \frac{1}{4}\int_4^8 \frac{dy}{y+1}$$

$$= \left[\tfrac{3}{4}\ln|y-3| + \tfrac{1}{4}\ln|y+1|\right]_4^8$$

$$= \tfrac{3}{4}\ln|5| + \tfrac{1}{4}\ln|9| - \left(\tfrac{3}{4}\ln|1| + \tfrac{1}{4}\ln|5|\right)$$

$$= \frac{3}{4}\ln 5 + \frac{1}{4}\ln 9 - \frac{1}{4}\ln 5$$

$$= \frac{1}{2}\ln 5 + \frac{1}{4}\ln 3^2 \qquad \textit{Power Rule}$$

$$= \frac{1}{2}\ln 5 + \frac{1}{2}\ln 3$$

$$= \frac{1}{2}(\ln 5 + \ln 3) \qquad \textit{Product Rule}$$

$$= \frac{1}{2}\ln 15$$

Exercise

Evaluate $\displaystyle\int_1^{\sqrt{3}} \frac{3x^2+x+4}{x^3+x}\,dx$

Solution

$$\frac{3x^2+x+4}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{(A+B)x^2+Cx+A}{x\left(x^2+1\right)}$$

$$\begin{cases} A+B=3 \\ C=1 \\ A=4 \end{cases} \to \boxed{A=4} \quad \boxed{B=-1} \quad \boxed{C=1}$$

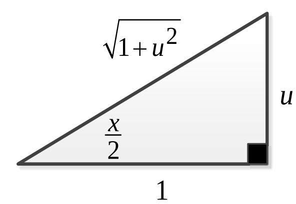
$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{3x^2+x+4}{x^3+x}\,dx &= \int_1^{\sqrt{3}} \frac{4}{x}\,dx + \int_1^{\sqrt{3}} \frac{-x+1}{x^2+1}\,dx \\
&= 4\int_1^{\sqrt{3}} \frac{1}{x}\,dx - \int_1^{\sqrt{3}} \frac{x}{x^2+1}\,dx + \int_1^{\sqrt{3}} \frac{1}{x^2+1}\,dx \qquad d\left(x^2+1\right) = 2xdx \\
&= 4\int_1^{\sqrt{3}} \frac{1}{x}\,dx - \frac{1}{2}\int_1^{\sqrt{3}} \frac{d\left(x^2+1\right)}{x^2+1} + \int_1^{\sqrt{3}} \frac{1}{x^2+1}\,dx \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} \\
&= \left[4\ln|x| - \tfrac{1}{2}\ln\left(x^2+1\right) + \tan^{-1}x\right]_1^{\sqrt{3}} \\
&= 4\ln\sqrt{3} - \tfrac{1}{2}\ln 4 + \tan^{-1}\sqrt{3} - \left(4\ln 1 - \tfrac{1}{2}\ln 2 + \tan^{-1}1\right) \\
&= 4\ln 3^{1/2} - \frac{1}{2}\ln 2^2 + \frac{\pi}{3} + \frac{1}{2}\ln 2 - \frac{\pi}{4} \\
&= 2\ln 3 - \ln 2 + \frac{\pi}{12} + \frac{1}{2}\ln 2 \\
&= 2\ln 3 - \frac{1}{2}\ln 2 + \frac{\pi}{12} \\
&= \ln\left(\frac{9}{\sqrt{2}}\right) + \frac{\pi}{12}
\end{aligned}$$

Exercise

Evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$

Solution

$$\begin{aligned}
\int_0^{\pi/2} \frac{dx}{\sin x + \cos x} &= \int_0^{\pi/2} \frac{1}{\frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}\,du \\
&= 2\int_0^{\pi/2} \frac{du}{2u+1-u^2}
\end{aligned}$$



$$u = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\tan^{-1}u \quad \to dx = \frac{2du}{1+u^2}$$

$$= -2\int_0^{\pi/2} \frac{du}{u^2 - 2u - 1}$$

$$\cos x = 2\cos^2 \frac{x}{2} - 1 = 2\frac{1}{1+u^2} - 1 = \frac{1-u^2}{1+u^2}$$

$$\sin x = 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2}$$

$$= -\frac{1}{\sqrt{2}}\int_0^{\pi/2} \left(\frac{1}{u-1-\sqrt{2}} - \frac{1}{u-1+\sqrt{2}}\right) du$$

$$\frac{2}{u^2 - 2u - 1} = \frac{A}{u-1-\sqrt{2}} + \frac{B}{u-1+\sqrt{2}}$$

$$2 = Au + \left(-1+\sqrt{2}\right)A + Bu + \left(-1-\sqrt{2}\right)B$$

$$\begin{cases} A + B = 0 \\ \left(-1+\sqrt{2}\right)A - \left(1+\sqrt{2}\right)B = 2 \end{cases} \to \begin{cases} B = -A = -\frac{1}{\sqrt{2}} \\ 2\sqrt{2}A = 2 \end{cases}$$

$$= -\frac{1}{\sqrt{2}}\left(\ln\left|\frac{1}{u-1-\sqrt{2}}\right| - \ln\left|\frac{1}{u-1+\sqrt{2}}\right|\right)\Bigg|_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}}\left(\ln\left|\frac{u-1+\sqrt{2}}{u-1-\sqrt{2}}\right|\right)\Bigg|_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}}\left(\ln\left|\frac{\tan\frac{x}{2} - 1 + \sqrt{2}}{\tan\frac{x}{2} - 1 - \sqrt{2}}\right|\right)\Bigg|_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}}\left(\ln|-1| - \ln\left|\frac{-1+\sqrt{2}}{-1-\sqrt{2}}\right|\right)$$

$$= \frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

Exercise

Evaluate $\displaystyle\int_0^{\pi/3} \frac{\sin\theta}{1-\sin\theta}\,d\theta$

Solution

$$\int_0^{\pi/3} \frac{\sin\theta}{1-\sin\theta}\,d\theta = \int_0^{\pi/3} \frac{1}{\csc\theta - 1}\,d\theta$$

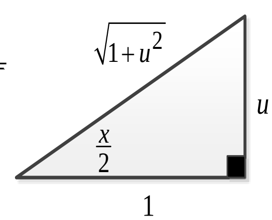
$$u = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\tan^{-1} u \quad \to dx = \frac{2du}{1+u^2}$$

$$= \int_0^{\pi/3} \frac{1}{\frac{1+u^2}{2u} - 1} \cdot \frac{2}{1+u^2}\,du$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}}$$

$$= \frac{2u}{1+u^2}$$



$$= \int_0^{\pi/3} \frac{4u}{\left(1+u^2-2u\right)\left(1+u^2\right)} du$$

$$= \int_0^{\pi/3} \frac{4u}{(u-1)^2\left(1+u^2\right)} du$$

$$\frac{4u}{(u-1)^2\left(1+u^2\right)} = \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{Cu+D}{1+u^2}$$

$$4u = Au + Au^3 - A - Au^2 + B + Bu^2 + Cu^3 - 2Cu^2 + Cu + Du^2 - 2Du + D$$

$$\begin{cases} A+C=0 \\ -A+B-2C+D=0 \\ C-2D=4 \\ -A+B+D=0 \end{cases} \rightarrow \begin{cases} A=0; & B=2 \\ C=0; & D=-2 \end{cases}$$

$$= \int_0^{\pi/3} \left(\frac{2}{(u-1)^2} - \frac{2}{1+u^2} \right) du$$

$$= \frac{-2}{u-1} - 2\tan^{-1} u \Big|_0^{\pi/3}$$

$$= \frac{-2}{\tan\frac{x}{2}-1} - 2\tan^{-1}\left(\tan\frac{x}{2}\right) \Big|_0^{\pi/3}$$

$$= \frac{-2}{\tan\frac{x}{2}-1} - x \Big|_0^{\pi/3}$$

$$= \frac{-2}{\frac{1}{\sqrt{3}}-1} - \frac{\pi}{3} - 2$$

$$= \frac{-2\sqrt{3}}{1-\sqrt{3}} - \frac{\pi}{3} - 2$$

$$= \frac{-2}{1-\sqrt{3}} - \frac{\pi}{3} \qquad\qquad = \frac{-2}{1-\sqrt{3}} \frac{1+\sqrt{3}}{1+\sqrt{3}} - \frac{\pi}{3}$$

$$\boxed{= 1 + \sqrt{3} - \frac{\pi}{3}}$$

Exercise

Find the volume of the solid generated by the revolving the shaded region about *x*-axis

Solution

$$V = \pi \int_{0.5}^{2.5} y^2 dx$$

$$= \pi \int_{0.5}^{2.5} \frac{9}{3x - x^2}\,dx$$

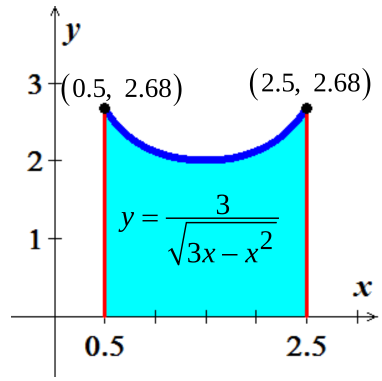
$$= 9\pi \int_{0.5}^{2.5} \frac{1}{3x - x^2}\,dx$$

$$\frac{1}{3x - x^2} = \frac{1}{x(3-x)} = \frac{A}{x} + \frac{B}{3-x} = \frac{(B-A)x + 3A}{x(3-x)}$$

$$= 9\pi \int_{0.5}^{2.5} \frac{1}{3}\left(\frac{1}{x} + \frac{1}{3-x}\right)dx$$

$$\begin{cases} B - A = 0 \\ 3A = 1 \end{cases} \Rightarrow \boxed{A = \frac{1}{3}} \quad \boxed{B = \frac{1}{3}}$$

$$= 3\pi \int_{0.5}^{2.5} \left(\frac{1}{x} - \frac{1}{x-3}\right)dx$$



$$= 3\pi \left[\int_{0.5}^{2.5} \frac{1}{x}\,dx - \int_{0.5}^{2.5} \frac{1}{x-3}\,dx \right]$$

$$= 3\pi \left[\ln|x| - \ln|x-3| \right]_{0.5}^{2.5}$$

$$= 3\pi \left[\ln\left|\frac{x}{x-3}\right| \right]_{0.5}^{2.5}$$

$$= 3\pi \left[\ln\left|\frac{2.5}{-.5}\right| - \ln\left|\frac{0.5}{-2.5}\right| \right]$$

$$= 3\pi \left[\ln 5 - \ln\frac{1}{5} \right]$$

$$= 3\pi \left[\ln 5 + \ln 5 \right]$$

$$= 3\pi \left[2\ln 5 \right]$$

$$= 3\pi \ln 25$$

Exercise

Find the area of the region bounded by the graphs of $y = \dfrac{12}{x^2 + 5x + 6}, \quad y = 0, \quad x = 0, \quad and \quad x = 1$

Solution

$$A = \int_0^1 \frac{12}{x^2 + 5x + 6}\,dx$$

$$\frac{12}{x^2 + 5x + 6} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$12 = Ax + 3A + Bx + 2B$$

$$\begin{cases} A + B = 0 \\ 3A + 2B = 12 \end{cases} \to A = 12 \quad B = -12$$

$$A=\int_0^1 \frac{12}{x+2}dx-\int_0^1 \frac{12}{x+3}dx$$

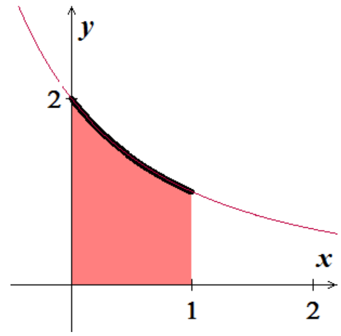
$$=12\left(\ln|x+2|-\ln|x+3|\right)\Big|_0^1$$

$$=12\left(\ln 3-\ln 4-\ln 2+\ln 3\right)$$

$$=12\left(2\ln 3-3\ln 2\right)$$

$$=12\left(\ln 9-\ln 8\right)$$

$$=12\ln\frac{9}{8}$$



Exercise

Find the area of the region bounded by the graphs of $y=\dfrac{7}{16-x^2}$ *and* $y=1$

Solution

$$A=2\int_0^3\left(1-\frac{7}{16-x^2}\right)dx$$

$$=2\int_0^3 dx-2\int_0^3\frac{7}{16-x^2}\,dx$$

$x=4\sin\theta \qquad 16-x^2=16\cos^2\theta$

$dx=4\cos\theta d\theta$

$$=2x\Big|_0^3-14\int_0^3\frac{1}{4\cos\theta}\,d\theta$$

$$=6-\frac{7}{2}\int_0^3\sec\theta\,d\theta$$

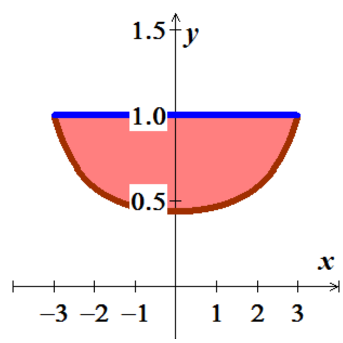
$$=6-\frac{7}{2}\ln|\sec\theta+\tan\theta|\Big|_0^3$$

$$=6-\frac{7}{2}\ln\left|\frac{4+x}{\sqrt{16-x^2}}\right|\Bigg|_0^3$$

$$=6-\frac{7}{2}\ln\left|\frac{7}{\sqrt{7}}\right|$$

$$=6-\frac{7}{2}\ln\sqrt{7}$$

$$=6-\frac{7}{4}\ln 7 \quad\approx 2.595$$



Exercise

Consider the region bounded by the graphs $y=\dfrac{2x}{x^2+1}, \quad y=0, \quad x=0, \quad and\ x=3.$

a) Find the volume of the solid generated by revolving the region about the *x-axis*

b) Find the centroid of the region.

Solution

a) $V=\pi\displaystyle\int_0^3 \left(\frac{2x}{x^2+1}\right)^2 dx$

$$=4\pi\int_0^3 \frac{x^2}{\left(x^2+1\right)^2}\, dx$$

$$\frac{x^2}{\left(x^2+1\right)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{\left(x^2+1\right)^2}$$

$$x^2=Ax^3+Ax+Bx^2+B+Cx+D$$

$$\begin{cases} x^3 & A=0 \\ x^2 & B=1 \\ x & A+C=0 \to C=0 \\ x^0 & B+D=0 \to D=-1 \end{cases}$$

$$=4\pi\int_0^3 \frac{1}{x^2+1]}\, dx-4\pi\int_0^3 \frac{1}{\left(x^2+1\right)^2}\, dx$$

$x=\tan\theta \qquad x^2+1=\sec^2\theta$

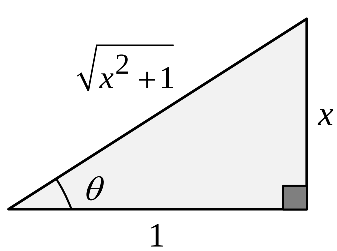
$dx=\sec^2\theta\, d\theta$

$$=4\pi\arctan x\Big|_0^3-4\pi\int_0^3 \frac{1}{\sec^2\theta}\, d\theta$$

$$=4\pi\arctan 3-4\pi\int_0^3 \cos^2\theta\, d\theta$$

$$=4\pi\arctan 3-2\pi\int_0^3 (1+\cos 2\theta)\, d\theta$$

$$=4\pi\arctan 3-2\pi(\theta+\sin\theta\cos\theta)\Big|_0^3$$



$$=4\pi\arctan 3-2\pi\left(\arctan x+\frac{x}{x^2+1}\right)\Big|_0^3$$

$$=4\pi\arctan 3-2\pi\left(\arctan 3+\frac{3}{10}\right)$$

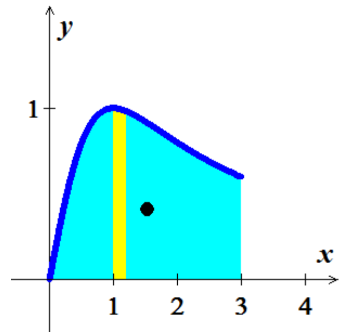
$$=2\pi\arctan 3-\frac{3\pi}{5} \qquad \approx 5.963$$

b) $A = \int_0^3 \frac{2x}{x^2+1}\,dx$

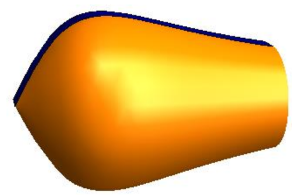
$= \int_0^3 \frac{1}{x^2+1}\,d\left(x^2+1\right)$

$= \ln\left(x^2+1\right)\Big|_0^3$

$= \ln 10$



$\bar{x} = \frac{1}{\ln 10}\int_0^3 \frac{2x^2}{x^2+1}\,dx$ $\qquad$ $\bar{x} = \frac{1}{A}\int_a^b x \cdot f(x)\,dx$



$= \frac{1}{\ln 10}\int_0^3 \left(2 - \frac{2}{x^2+1}\right)dx$

$= \frac{1}{\ln 10}\left(2x - 2\arctan x\right)\Big|_0^3$

$= \frac{2}{\ln 10}\left(3 - \arctan 3\right)$ $\qquad$ ≈ 1.521

$\bar{y} = \frac{1}{2}\frac{1}{\ln 10}\int_0^3 \left(\frac{2x}{x^2+1}\right)^2 dx$ $\qquad$ $\bar{x} = \frac{1}{A}\int_a^b x \cdot f(x)\,dx$

$= \frac{2}{\ln 10}\int_0^3 \frac{x^2}{\left(x^2+1\right)^2}\,dx$

$= \frac{2}{\ln 10}\int_0^3 \frac{1}{x^2+1}\,dx - \frac{2}{\ln 10}\int_0^3 \frac{1}{\left(x^2+1\right)^2}\,dx$

$= \frac{2}{\ln 10}\left(\arctan x - \frac{1}{2}\arctan x - \frac{1}{2}\frac{x}{x^2+1}\right)\Big|_0^3$

$= \frac{2}{\ln 10}\left(\frac{1}{2}\arctan 3 - \frac{3}{20}\right)$

$= \frac{1}{\ln 10}\left(\arctan 3 - \frac{3}{10}\right)$ $\qquad$ ≈ 0.412

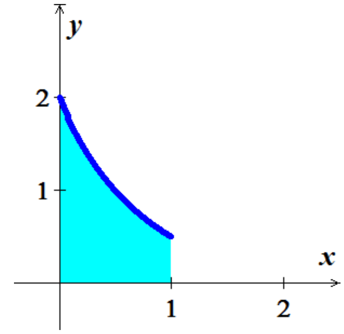
$(\bar{x}, \bar{y}) = \left(\frac{2}{\ln 10}\left(3 - \arctan 3\right),\ \frac{1}{\ln 10}\left(\arctan 3 - \frac{3}{10}\right)\right)$ $\qquad$ $\approx (1.521,\ 0.412)$

Exercise

Consider the region bounded by the graph $y^2 = \dfrac{(2-x)^2}{(1+x)^2} \quad 0 \le x \le 1$.

Find the volume of the solid generated by revolving this region about the *x-axis*.

Solution



$$V = \pi \int_0^1 \frac{(2-x)^2}{(1+x)^2}\, dx$$

$$= 4\pi \int_0^1 \frac{1}{(1+x)^2}\, dx - 4\pi \int_0^1 \frac{x}{(1+x)^2}\, dx + \pi \int_0^1 \frac{x^2}{(1+x)^2}\, dx$$

$$\frac{x}{(1+x)^2} = \frac{A}{x+1} + \frac{B}{(1+x)^2}$$

$$\frac{x^2}{x^2+2x+1} = 1 - \frac{2x+1}{(1+x)^2} = 1 - \left(\frac{C}{x+1} + \frac{D}{(1+x)^2}\right)$$

$Ax + A + B = x$ $\qquad$ $Cx + C + D = 2x + 1$

$A = 1, \quad B = -1$ $\qquad$ $C = 2, \quad D = -1$

$$= -4\pi \frac{1}{1+x}\bigg|_0^1 - 4\pi \int_0^1 \frac{1}{x+1}\, dx + 4\pi \int_0^1 \frac{1}{(1+x)^2}\, dx + \pi \int_0^1 dx - 2\pi \int_0^1 \frac{1}{x+1}\, dx + \pi \int_0^1 \frac{1}{(1+x)^2}\, dx$$

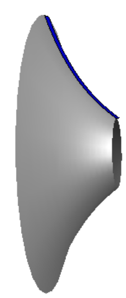
$$= 2\pi + \left(-4\pi \ln(x+1) - 4\pi \frac{1}{x+1} + \pi x - 2\pi \ln(x+1) - \pi \frac{1}{x+1}\right)\bigg|_0^1$$

$$= 2\pi - \left(6\pi \ln(x+1) + 5\pi \frac{1}{x+1} - \pi x\right)\bigg|_0^1$$

$$= 2\pi - \left(6\pi \ln(2) + \frac{5}{2}\pi - \pi - 5\pi\right)$$

$$= 2\pi - 6\pi \ln 2 + \frac{7}{2}\pi$$

$$= \frac{\pi}{2}(11 - 12\ln 2)$$



Exercise

A single infected individual enters a community of n susceptible individuals. Let x be the number of newly infected individuals at time t. The common epidemic model assumes that the disease spreads at a rate proportional to the product of the total number infected and the number not yet infected. So,

$\dfrac{dx}{dt} = k(x+1)(n-x)$ and you obtain

$\displaystyle\int \frac{1}{(x+1)(n-x)}\, dx = \int k\, dt$ $\qquad$ Solve for x as a function of t.

Solution

$$\frac{1}{(x+1)(n-x)} = \frac{A}{x+1} + \frac{B}{n-x}$$

$$1 = An - Ax + Bx + B \quad \to \quad \begin{cases} -A + B = 0 \\ nA + B = 1 \end{cases}$$

$$(n+1)A = 1 \Rightarrow \boxed{A = \frac{1}{n+1} = B}$$

$$\int \frac{1}{(x+1)(n-x)}\, dx = \frac{1}{n+1}\int \frac{1}{x+1}\, dx + \frac{1}{n+1}\int \frac{1}{n-x}\, dx$$

$$= \frac{1}{n+1}\left(\ln|x+1| - \ln|n-x|\right)$$

$$= \frac{1}{n+1}\ln\left|\frac{x+1}{n-x}\right|$$

$$\int k\; dt = kt + C$$

$$\frac{1}{n+1}\ln\left|\frac{x+1}{n-x}\right| = kt + C$$

$$x(t=0) = 0 \qquad \to \boxed{\frac{1}{n+1}\ln\left|\frac{1}{n}\right| = C}$$

$$\frac{1}{n+1}\ln\left|\frac{x+1}{n-x}\right| = kt + \frac{1}{n+1}\ln\left|\frac{1}{n}\right|$$

$$\ln\left|\frac{x+1}{n-x}\right| - \ln\left|\frac{1}{n}\right| = (n+1)kt$$

$$\ln\left|\frac{nx+n}{n-x}\right| = (n+1)kt$$

$$\frac{nx+n}{n-x} = e^{(n+1)kt}$$

$$nx + n = ne^{(n+1)kt} - xe^{(n+1)kt}$$

$$\left(n + e^{(n+1)kt}\right)x = ne^{(n+1)kt} - n$$

$$\boxed{x = \frac{ne^{(n+1)kt} - n}{n + e^{(n+1)kt}}} \qquad\qquad \lim_{t\to\infty} x = n$$

Exercise

Evaluate $\int_0^1 \frac{x}{1+x^4}\, dx$ in ***two*** different ways.

Solution

1- Partial method

$$\frac{x}{1+x^4} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}$$

$$x = Ax^3 - \sqrt{2}Ax^2 + Ax + Bx^2 - \sqrt{2}Bx + B + Cx^3 + \sqrt{2}Cx^2 + Cx + Dx^2 + \sqrt{2}Dx + D$$

$$x^3 \qquad A + C = 0 \to C = -A$$

$$x^2 \qquad -\sqrt{2}A + B + \sqrt{2}C + D = 0$$

$$x \qquad A - \sqrt{2}B + C + \sqrt{2}D = 1$$

$$x^0 \qquad B + D = 0 \to D = -B$$

$$\begin{cases} -2\sqrt{2}A = 0 \to \underline{A = 0 = C} \\ -2\sqrt{2}B = 1 \Rightarrow \underline{B = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4}} \end{cases} \to \underline{D = \frac{\sqrt{2}}{4}}$$

$$\int_0^1 \frac{x}{1+x^4}\,dx = -\frac{\sqrt{2}}{4}\int_0^1 \frac{1}{x^2+\sqrt{2}x+1}dx + \frac{\sqrt{2}}{4}\int_0^1 \frac{1}{x^2-\sqrt{2}x+1}dx$$

$$= -\frac{\sqrt{2}}{4}\int_0^1 \frac{1}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}dx + \frac{\sqrt{2}}{4}\int_0^1 \frac{1}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}dx$$

$$= \frac{\sqrt{2}}{4}\left(-\sqrt{2}\arctan\sqrt{2}\left(x+\frac{\sqrt{2}}{2}\right) + \sqrt{2}\arctan\sqrt{2}\left(x-\frac{\sqrt{2}}{2}\right)\right)\Bigg|_0^1$$

$$= \frac{1}{2}\left(-\arctan\left(\sqrt{2}x+1\right) + \arctan\left(\sqrt{2}x-1\right)\right)\Big|_0^1$$

$$= \frac{1}{2}\left(-\arctan\left(\sqrt{2}+1\right) + \arctan\left(\sqrt{2}-1\right) + \arctan 1 - \arctan(-1)\right)$$

$$\underline{= \frac{1}{2}\left(-\arctan\left(\sqrt{2}+1\right) + \arctan\left(\sqrt{2}-1\right) + \frac{\pi}{2}\right)}$$

2- Let $u = x^2 \to du = 2xdx$

$$\int_0^1 \frac{x}{1+x^4}\,dx = \frac{1}{2}\int_0^1 \frac{1}{1+u^2}du$$

$$= \frac{1}{2}\arctan x^2\Big|_0^1$$

$$\underline{= \frac{\pi}{8}}$$

$$\frac{1}{2}\left(\arctan\left(\sqrt{2}-1\right) - \arctan\left(\sqrt{2}+1\right) + \frac{\pi}{2}\right) = \frac{1}{2}\left(\arctan\left(\frac{\sqrt{2}-1-\sqrt{2}-1}{1+\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}\right) + \frac{\pi}{2}\right)$$

$$= \frac{1}{2}\left(\arctan(-1) + \frac{\pi}{2}\right)$$

$$= \frac{1}{2}\left(-\frac{\pi}{4} + \frac{\pi}{2}\right)$$

$$\underline{= \frac{\pi}{8}}$$