

Solution

Section 1.8 – Curvature and Normal Vectors

Exercise

Find T , N , and κ for the plane curves: $\vec{r}(t) = t\hat{i} + (\ln \cos t)\hat{j}$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

Solution

$$\begin{aligned}\vec{v}(t) &= \hat{i} - \frac{\sin t}{\cos t} \hat{j} \\ &= \hat{i} - \tan t \hat{j}\end{aligned}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned}|\vec{v}| &= \sqrt{1 + \tan^2 t} \\ &= \sqrt{\sec^2 t} \\ &= |\sec t|\end{aligned}$$

$$\vec{T} = \frac{1}{\sec t} \hat{i} - \frac{\tan t}{\sec t} \hat{j}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\begin{aligned}&= \cos t \hat{i} - \cos t \frac{\sin t}{\cos t} \hat{j} \\ &= \cos t \hat{i} - \sin t \hat{j}\end{aligned}$$

$$\frac{d\vec{T}}{dt} = -(\sin t)\hat{i} - (\cos t)\hat{j}$$

$$\left| \frac{d\vec{T}}{dt} \right| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\begin{aligned}\vec{N} &= \frac{-(\sin t)\hat{i} - (\cos t)\hat{j}}{1} \\ &= -(\sin t)\hat{i} - (\cos t)\hat{j}\end{aligned}$$

$$\vec{N} = \frac{d\vec{T} / dt}{|d\vec{T} / dt|}$$

$$\begin{aligned}\kappa &= \frac{1}{\sec t} (1) \\ &= \cos t\end{aligned}$$

$$\kappa = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$$

Exercise

Find T , N , and κ for the plane curves: $\vec{r}(t) = (\ln \sec t)\hat{i} + t\hat{j}$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

Solution

$$\begin{aligned}\vec{v}(t) &= \frac{\sec t \tan t}{\sec t} \hat{i} + \hat{j} \\ &= \tan t \hat{i} + \hat{j}\end{aligned}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned}
 |\vec{v}| &= \sqrt{\tan^2 t + 1} \\
 &= \sqrt{\sec^2 t} \\
 &= \sec t
 \end{aligned}$$

$$\begin{aligned}
 \vec{T} &= \frac{\tan t}{\sec t} \hat{i} + \frac{1}{\sec t} \hat{j} \\
 &= (\sin t) \hat{i} + (\cos t) \hat{j}
 \end{aligned}$$

$$\frac{d\vec{T}}{dt} = (\cos t) \hat{i} - (\sin t) \hat{j}$$

$$\begin{aligned}
 \left| \frac{d\vec{T}}{dt} \right| &= \sqrt{\cos^2 t + \sin^2 t} \\
 &= 1
 \end{aligned}$$

$$\vec{N} = (\sin t) \hat{i} + (\cos t) \hat{j}$$

$$\begin{aligned}
 \kappa &= \frac{1}{\sec t} (1) \\
 &= \cos t
 \end{aligned}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{N} = \frac{d\vec{T} / dt}{|d\vec{T} / dt|}$$

$$\kappa = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$$

Exercise

Find T , N , and κ for the plane curves: $\vec{r}(t) = (2t + 3)\hat{i} + (5 - t^2)\hat{j}$

Solution

$$\vec{v}(t) = 2\hat{i} - 2t\hat{j}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned}
 |\vec{v}| &= \sqrt{4 + 4t^2} \\
 &= 2\sqrt{1 + t^2}
 \end{aligned}$$

$$\begin{aligned}
 \vec{T} &= \frac{2}{2\sqrt{1+t^2}} \hat{i} - \frac{2t}{2\sqrt{1+t^2}} \hat{j} \\
 &= \frac{1}{\sqrt{1+t^2}} \hat{i} - \frac{t}{\sqrt{1+t^2}} \hat{j}
 \end{aligned}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\frac{d\vec{T}}{dt} = \frac{-2t}{2(1+t^2)^{3/2}} \hat{i} - \frac{(1+t^2)^{1/2} - \frac{1}{2}(1+t^2)^{-1/2}(2t)t}{(1+t^2)} \hat{j}$$

$$\begin{aligned}
&= \frac{-t}{(1+t^2)^{3/2}} \hat{i} - \frac{1+t^2-t^2}{(1+t^2)^{3/2}} \hat{j} \\
&= \frac{-t}{(1+t^2)^{3/2}} \hat{i} - \frac{1}{(1+t^2)^{3/2}} \hat{j}
\end{aligned}$$

$$\begin{aligned}
\left| \frac{d\vec{T}}{dt} \right| &= \sqrt{\frac{t^2}{(1+t^2)^3} + \frac{1}{(1+t^2)^3}} \\
&= \sqrt{\frac{t^2+1}{(1+t^2)^3}} \\
&= \sqrt{\frac{1}{(1+t^2)^2}} \\
&= \frac{1}{(1+t^2)}
\end{aligned}$$

$$\begin{aligned}
\vec{N} &= (1+t^2) \left(\frac{-t}{(1+t^2)^{3/2}} \hat{i} - \frac{1}{(1+t^2)^{3/2}} \hat{j} \right) \\
&= \frac{-t}{\sqrt{1+t^2}} \hat{i} - \frac{1}{\sqrt{1+t^2}} \hat{j}
\end{aligned}$$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\begin{aligned}
\kappa &= \frac{1}{2\sqrt{1+t^2}} \frac{1}{(1+t^2)} \\
&= \frac{1}{2(1+t^2)^{3/2}}
\end{aligned}$$

$$\kappa = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$$

Exercise

Find \mathbf{T} , \mathbf{N} , and κ for the plane curves: $\vec{r}(t) = (\cos t + t \sin t) \hat{i} + (\sin t - t \cos t) \hat{j}$, $t > 0$

Solution

$$\begin{aligned}
\vec{v} &= (-\sin t + \sin t + t \cos t) \hat{i} + (\cos t - \cos t + t \sin t) \hat{j} \\
&= (t \cos t) \hat{i} + (t \sin t) \hat{j}
\end{aligned}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{t^2 \cos^2 t + t^2 \sin^2 t}$$

$$\begin{aligned}
&= \sqrt{t^2 (\cos^2 t + \sin^2 t)} \\
&= \sqrt{t^2} \\
&= |t| \\
&= t
\end{aligned}$$

$$\begin{aligned}
\vec{T} &= \left(\frac{t \cos t}{t} \right) \hat{i} + \left(\frac{t \sin t}{t} \right) \hat{j} \\
&= \underline{(\cos t) \hat{i} + (\sin t) \hat{j}}
\end{aligned}
\qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\begin{aligned}
\frac{d\vec{T}}{dt} &= (-\sin t) \hat{i} + (\cos t) \hat{j} \\
\left| \frac{d\vec{T}}{dt} \right| &= \sqrt{\sin^2 t + \cos^2 t} = \underline{1}
\end{aligned}$$

$$\vec{N} = \underline{(-\sin t) \hat{i} + (\cos t) \hat{j}} \qquad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\underline{\kappa = \frac{1}{t}} \qquad \kappa = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$$

Exercise

Find \vec{T} , \vec{N} , and κ for the space curves: $\vec{r}(t) = (3 \sin t) \hat{i} + (3 \cos t) \hat{j} + 4t \hat{k}$

Solution

$$\vec{v} = (3 \cos t) \hat{i} - (3 \sin t) \hat{j} + 4 \hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned}
|\vec{v}| &= \sqrt{9 \cos^2 t + 9 \sin^2 t + 16} \\
&= \sqrt{9 + 16} \\
&= \underline{5}
\end{aligned}$$

$$\vec{T} = \underline{\frac{3}{5} \cos t \hat{i} - \frac{3}{5} \sin t \hat{j} + \frac{4}{5} \hat{k}} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\begin{aligned}
\frac{d\vec{T}}{dt} &= -\frac{3}{5} \sin t \hat{i} - \frac{3}{5} \cos t \hat{j} \\
\left| \frac{d\vec{T}}{dt} \right| &= \sqrt{\frac{9}{25} \sin^2 t + \frac{9}{25} \cos^2 t} \\
&= \sqrt{\frac{9}{25}} \\
&= \underline{\frac{3}{5}}
\end{aligned}$$

$$\vec{N} = \frac{5}{3} \left(-\frac{3}{5} \sin t \hat{i} - \frac{3}{5} \cos t \hat{j} \right) \quad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$= \left(-\sin t \hat{i} - (\cos t) \hat{j} \right)$$

$$\kappa = \frac{1}{5} \frac{3}{5}$$

$$\kappa = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$$

$$= \frac{3}{25}$$

Exercise

Find T , N , and κ for the space curves: $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + 2t\mathbf{k}$

Solution

$$\vec{v}(t) = (e^t \cos t - e^t \sin t)\hat{i} + (e^t \sin t + e^t \cos t)\hat{j} \quad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned} |\vec{v}| &= \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2} \\ &= \sqrt{e^{2t} \cos^2 t - e^{2t} \cos t \sin t + e^{2t} \sin^2 t + e^{2t} \sin^2 t + e^{2t} \cos t \sin t + e^{2t} \cos^2 t} \\ &= \sqrt{2e^{2t} (\cos^2 t + \sin^2 t)} \\ &= e^t \sqrt{2} \end{aligned}$$

$$\begin{aligned} \vec{T} &= \left(\frac{e^t \cos t - e^t \sin t}{\sqrt{2}e^t} \right) \hat{i} + \left(\frac{e^t \sin t + e^t \cos t}{\sqrt{2}e^t} \right) \hat{j} \quad \vec{T} = \frac{\vec{v}}{|\vec{v}|} \\ &= \left(\frac{\cos t - \sin t}{\sqrt{2}} \right) \hat{i} + \left(\frac{\sin t + \cos t}{\sqrt{2}} \right) \hat{j} \end{aligned}$$

$$\frac{d\vec{T}}{dt} = \left(\frac{-\sin t - \cos t}{\sqrt{2}} \right) \hat{i} + \left(\frac{\cos t - \sin t}{\sqrt{2}} \right) \hat{j}$$

$$\begin{aligned} \left| \frac{d\vec{T}}{dt} \right| &= \sqrt{\frac{(-\sin t - \cos t)^2}{2} + \frac{(\cos t - \sin t)^2}{2}} \\ &= \frac{1}{\sqrt{2}} \sqrt{\sin^2 t + 2 \sin t \cos t + \cos^2 t + \sin^2 t - 2 \sin t \cos t + \cos^2 t} \\ &= \frac{1}{\sqrt{2}} \sqrt{2 \sin^2 t + 2 \cos^2 t} \\ &= \frac{1}{\sqrt{2}} \sqrt{2} \end{aligned}$$

$$=1]$$

$$\vec{N} = \left(\frac{-\sin t - \cos t}{\sqrt{2}} \right) \hat{i} + \left(\frac{\cos t - \sin t}{\sqrt{2}} \right) \hat{j} \quad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\kappa = \frac{1}{\sqrt{2}e^t} \quad \kappa = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$$

Exercise

Find T , N , and κ for the space curves: $\vec{r}(t) = \frac{t^3}{3} \hat{i} + \frac{t^2}{2} \hat{j}$, $t > 0$

Solution

$$\vec{v} = (t^2) \hat{i} + t \hat{j} \quad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned} |\vec{v}| &= \sqrt{t^4 + t^2} \\ &= |t| \sqrt{t^2 + 1} \\ &= t \sqrt{t^2 + 1} \quad (t > 0) \end{aligned}$$

$$\begin{aligned} \vec{T} &= \left(\frac{t^2}{t \sqrt{t^2 + 1}} \right) \hat{i} + \left(\frac{t}{t \sqrt{t^2 + 1}} \right) \hat{j} \quad \vec{T} = \frac{\vec{v}}{|\vec{v}|} \\ &= \left(\frac{t}{\sqrt{t^2 + 1}} \right) \hat{i} + \left(\frac{1}{\sqrt{t^2 + 1}} \right) \hat{j} \end{aligned}$$

$$\frac{d\vec{T}}{dt} = \frac{(1+t^2)^{1/2} - \frac{1}{2}(1+t^2)^{-1/2}(2t)t}{(1+t^2)} \hat{i} + \frac{-2t}{2(1+t^2)^{3/2}} \hat{j}$$

$$= \frac{1+t^2-t^2}{(1+t^2)^{3/2}} \hat{i} - \frac{t}{(1+t^2)^{3/2}} \hat{j}$$

$$= \frac{1}{(1+t^2)^{3/2}} \hat{i} - \frac{t}{(1+t^2)^{3/2}} \hat{j}$$

$$\left| \frac{d\vec{T}}{dt} \right| = \sqrt{\frac{1}{(1+t^2)^3} + \frac{t^2}{(1+t^2)^3}}$$

$$\begin{aligned}
&= \sqrt{\frac{t^2+1}{(1+t^2)^3}} \\
&= \sqrt{\frac{1}{(1+t^2)^2}} \\
&= \frac{1}{1+t^2} \\
\vec{N} &= (1+t^2) \left(\frac{1}{(1+t^2)^{3/2}} \hat{i} - \frac{t}{(1+t^2)^{3/2}} \hat{j} \right) & \vec{N} &= \frac{d\vec{T}/dt}{|d\vec{T}/dt|} \\
&= \frac{1}{\sqrt{1+t^2}} \hat{i} - \frac{t}{\sqrt{1+t^2}} \hat{j} \\
\kappa &= \frac{1}{t\sqrt{t^2+1}} \frac{1}{1+t^2} & \kappa &= \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right| \\
&= \frac{t}{t(t^2+1)^{3/2}}
\end{aligned}$$

Exercise

Find T , N , and κ for the space curves: $\vec{r}(t) = (\cos^3 t) \hat{i} + (\sin^3 t) \hat{j}$, $0 < t < \frac{\pi}{2}$

Solution

$$\begin{aligned}
\vec{v} &= -(3\cos^2 t \sin t) \hat{i} + (3\sin^2 t \cos t) \hat{j} & \vec{v}(t) &= \frac{d\vec{r}}{dt} \\
|\vec{v}| &= \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} \\
&= 3\sqrt{\cos^2 t \sin^2 t (\cos^2 t + \sin^2 t)} \\
&= 3\sqrt{\cos^2 t \sin^2 t} \\
&= 3|\cos t \sin t| \\
&= 3\cos t \sin t \\
\vec{T} &= -\left(\frac{3\cos^2 t \sin t}{3|\cos t \sin t|} \right) \hat{i} + \left(\frac{3\sin^2 t \cos t}{3|\cos t \sin t|} \right) \hat{j} & \vec{T} &= \frac{\vec{v}}{|\vec{v}|} \\
&= -(\cos t) \hat{i} + (\sin t) \hat{j}
\end{aligned}$$

$$\frac{d\vec{T}}{dt} = (\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\left| \frac{d\vec{T}}{dt} \right| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\vec{N} = \frac{(\sin t)\hat{i} + (\cos t)\hat{j}}{\left| (\sin t)\hat{i} + (\cos t)\hat{j} \right|}$$

$$\vec{N} = \frac{d\vec{T}/dt}{\left| d\vec{T}/dt \right|}$$

$$\kappa = \frac{1}{3 \cos t \sin t} (1)$$

$$= \frac{t}{3 \cos t \sin t}$$

$$\kappa = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$$

Exercise

Find T , N , and κ for the space curves: $\vec{r}(t) = (\cosh t)\hat{i} - (\sinh t)\hat{j} + t\hat{k}$

Solution

$$\vec{v} = (\sinh t)\hat{j} - (\cosh t)\hat{j} + \hat{k}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{\sinh^2 t + \cosh^2 t + 1}$$

$$\cosh^2 t - \sinh^2 t = 1 \Rightarrow \cosh^2 t = 1 + \sinh^2 t$$

$$= \sqrt{\cosh^2 t + \cosh^2 t}$$

$$= \sqrt{2 \cosh^2 t}$$

$$\vec{T} = \left(\frac{1}{\sqrt{2} \cosh t} \sinh t \right) \hat{i} - \left(\frac{1}{\sqrt{2} \cosh t} \cosh t \right) \hat{j} + \frac{1}{\sqrt{2} \cosh t} \hat{k}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \left(\frac{1}{\sqrt{2}} \tanh t \right) \hat{i} - \frac{1}{\sqrt{2}} \hat{j} + \left(\frac{1}{\sqrt{2}} \operatorname{sech} t \right) \hat{k}$$

$$\frac{d\vec{T}}{dt} = \left(\frac{1}{\sqrt{2}} \operatorname{sech}^2 t \right) \hat{i} - \left(\frac{1}{\sqrt{2}} \operatorname{sech} t \tanh t \right) \hat{k}$$

$$\left| \frac{d\vec{T}}{dt} \right| = \sqrt{\frac{1}{2} \operatorname{sech}^4 t + \frac{1}{2} \operatorname{sech}^2 t \tanh^2 t}$$

$$= \frac{1}{\sqrt{2}} \operatorname{sech} t \sqrt{\operatorname{sech}^2 t + \tanh^2 t}$$

$$= \frac{1}{\sqrt{2}} \operatorname{sech} t \sqrt{1}$$

$$= \frac{1}{\sqrt{2}} \operatorname{sech} t$$

$$\begin{aligned}\vec{N} &= \frac{\sqrt{2}}{\operatorname{sech} t} \left(\left(\frac{1}{\sqrt{2}} \operatorname{sech}^2 t \right) \hat{i} - \left(\frac{1}{\sqrt{2}} \operatorname{sech} t \tanh t \right) \hat{k} \right) & \vec{N} &= \frac{d\vec{T}/dt}{|d\vec{T}/dt|} \\ &= \underline{(\operatorname{sech} t) \hat{i} - (\tanh t) \hat{k}}\end{aligned}$$

$$\begin{aligned}\kappa &= \frac{1}{\sqrt{2} \cosh t} \left(\frac{1}{\sqrt{2}} \operatorname{sech} t \right) & \kappa &= \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right| \\ &= \underline{\frac{1}{2} \operatorname{sech}^2 t}\end{aligned}$$

Exercise

Find an equation for the circle of curvature of the curve $\vec{r}(t) = t \hat{i} + (\sin t) \hat{j}$, at the point $\left(\frac{\pi}{2}, 1\right)$. (The curve parametrizes the graph $y = \sin x$ in the xy -plane.)

Solution

$$\vec{v} = \hat{j} + (\cos t) \hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{1 + \cos^2 t}$$

$$\begin{aligned}\left| \vec{v} \left(\frac{\pi}{2} \right) \right| &= \sqrt{1 + \cos^2 \left(\frac{\pi}{2} \right)} \\ &= \sqrt{1 + 0} \\ &= \underline{1}\end{aligned}$$

$$\vec{T} = \frac{1}{\sqrt{1 + \cos^2 t}} \hat{i} + \left(\frac{\cos t}{\sqrt{1 + \cos^2 t}} \right) \hat{j} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\begin{aligned}\frac{d\vec{T}}{dt} &= -\frac{1}{2} \frac{2 \cos t (-\sin t)}{(1 + \cos^2 t)^{3/2}} \hat{i} + \frac{-\sin t (1 + \cos^2 t)^{1/2} - \cos t \left(\left(\frac{1}{2} \right) 2 \cos t (-\sin t) \right) (1 + \cos^2 t)^{-1/2}}{(1 + \cos^2 t)} \hat{j} \\ &= \frac{\cos t \sin t}{(1 + \cos^2 t)^{3/2}} \hat{i} + \frac{-\sin t (1 + \cos^2 t) + \sin t \cos^2 t}{(1 + \cos^2 t)^{3/2}} \hat{j} \\ &= \frac{\cos t \sin t}{(1 + \cos^2 t)^{3/2}} \hat{i} + \frac{-\sin t (1 + \cos^2 t - \cos^2 t)}{(1 + \cos^2 t)^{3/2}} \hat{j} \\ &= \frac{\sin t \cos t}{(1 + \cos^2 t)^{3/2}} \hat{i} - \frac{\sin t}{(1 + \cos^2 t)^{3/2}} \hat{j}\end{aligned}$$

$$\begin{aligned}
\left| \frac{d\vec{T}}{dt} \right| &= \left| \frac{(\sin t \cos t) \hat{i} - (\sin t) \hat{j}}{(1 + \cos^2 t)^{3/2}} \right| \\
&= \frac{\sqrt{\sin^2 t \cos^2 t + \sin^2 t}}{(1 + \cos^2 t)^{3/2}} \\
&= \frac{|\sin t| \sqrt{\cos^2 t + 1}}{(1 + \cos^2 t)^{3/2}} \\
&= \frac{|\sin t|}{1 + \cos^2 t}
\end{aligned}$$

$$\begin{aligned}
\left| \frac{d\vec{T}}{dt} \right|_{t=\frac{\pi}{2}} &= \frac{\left| \sin \frac{\pi}{2} \right|}{1 + \cos^2 \frac{\pi}{2}} \\
&= 1
\end{aligned}$$

$$\kappa = 1$$

$$\kappa = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$$

The radius of curvature is: $\rho = \frac{1}{\kappa} = \frac{1}{1} = 1$

The center of the circle is $\left(\frac{\pi}{2}, 0 \right)$

The equation of the osculating circle is: $\left(x - \frac{\pi}{2} \right)^2 + y^2 = 1$

Exercise

Write \vec{a} of the motion $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$ without finding \mathbf{T} and \mathbf{N} . $\vec{r}(t) = (a \cos t) \hat{i} + (a \sin t) \hat{j} + bt \hat{k}$

Solution

$$\vec{v} = (-a \sin t) \hat{i} + (a \cos t) \hat{j} + (b) \hat{k}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned}
|\vec{v}| &= \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} \\
&= \sqrt{a^2 + b^2}
\end{aligned}$$

$$a_T = \frac{d}{dt} |\vec{v}| = 0$$

$$\vec{a} = \vec{v}' = (-a \cos t) \hat{i} - (a \sin t) \hat{j}$$

$$|\vec{a}| = \sqrt{a^2 \cos^2 t + a^2 \sin^2 t}$$

$$=|a|$$

$$a_N = \sqrt{a^2 + 0}$$

$$=|a|$$

$$\vec{a} = (0)\vec{T} + |a|\vec{N}$$

$$\underline{=|a|\vec{N}}|$$

$$a_N = \sqrt{|a|^2 - a_T^2}$$

$$\vec{a} = a_T\vec{T} + a_N\vec{N}$$

Exercise

Write \vec{a} of the motion $\vec{a} = a_T\vec{T} + a_N\vec{N}$ without finding \mathbf{T} and \mathbf{N} . $\vec{r}(t) = (1+3t)\hat{i} + (t-2)\hat{j} - 3t\hat{k}$

Solution

$$\vec{v} = 3\hat{i} + \hat{j} - 3\hat{k}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{9+1+9}$$

$$= \sqrt{19}$$

$$a_T = \frac{d}{dt}|\vec{v}| = 0$$

$$\vec{a} = \vec{v}' = 0$$

$$a_N = 0$$

$$a_N = \sqrt{|a|^2 - a_T^2}$$

$$\vec{a} = (0)\vec{T} + 0\vec{N}$$

$$\vec{a} = a_T\vec{T} + a_N\vec{N}$$

$$\underline{= \vec{0}}|$$

Exercise

Write \vec{a} of the motion $\vec{a} = a_T\vec{T} + a_N\vec{N}$ at the given value of t without finding \mathbf{T} and \mathbf{N} .

$$\vec{r}(t) = (t+1)\hat{i} + 2t\hat{j} + t^2\hat{k}, \quad t = 1$$

Solution

$$\vec{v} = \hat{i} + 2\hat{j} + 2t\hat{k}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{1+4+4t^2}$$

$$= \sqrt{5+4t^2}$$

$$a_T = \frac{1}{2}(8t)(5+4t^2)^{-1/2}$$

$$a_T = \frac{d}{dt}|\vec{v}|$$

$$= 4t(5 + 4t^2)^{-1/2}$$

$$\begin{aligned} a_T \Big|_{t=1} &= 4(5 + 4)^{-1/2} \\ &= 4(9)^{-1/2} \\ &= \frac{4}{3} \end{aligned}$$

$$\begin{aligned} \vec{a} &= \vec{v}' = 2\hat{k} \\ |\vec{a}| &= \sqrt{4} = 2 \end{aligned}$$

$$\begin{aligned} a_N &= \sqrt{4 - \frac{16}{9}} & a_N &= \sqrt{|\mathbf{a}|^2 - a_T^2} \\ &= \sqrt{\frac{20}{9}} \\ &= \frac{2\sqrt{5}}{3} \end{aligned}$$

$$\vec{a} = \frac{4}{3}\vec{T} + \frac{2\sqrt{5}}{3}\vec{N} \quad \vec{a} = a_T\vec{T} + a_N\vec{N}$$

Exercise

Write \vec{a} of the motion $\vec{a} = a_T\vec{T} + a_N\vec{N}$ at the given value of t without finding T and N .

$$\vec{r}(t) = (t \cos t)\hat{i} + (t \sin t)\hat{j} + t^2\hat{k}, \quad t = 0$$

Solution

$$\vec{v} = (\cos t - t \sin t)\hat{i} + (\sin t + t \cos t)\hat{j} + 2t\hat{k} \quad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned} |\vec{v}| &= \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + 4t^2} \\ &= \sqrt{\cos^2 t - 2t \sin t \cos t + t^2 \sin^2 t + \sin^2 t + 2t \sin t \cos t + t^2 \cos^2 t + 4t^2} \\ &= \sqrt{\cos^2 t + \sin^2 t + t^2(\sin^2 t + \cos^2 t) + 4t^2} \\ &= \sqrt{1 + 5t^2} \end{aligned}$$

$$\begin{aligned} a_T &= \frac{d}{dt}|\vec{v}| \\ &= \frac{1}{2}(10t)(1 + 5t^2)^{-1/2} \\ &= 5t(1 + 5t^2)^{-1/2} \end{aligned}$$

$$a_T \Big|_{t=0} = 0$$

$$\begin{aligned}\vec{a} = \vec{v}' &= (-\sin t - \sin t - t \cos t) \hat{i} + (\cos t + \cos t - t \sin t) \hat{j} + 2\hat{k} \\ &= (-2\sin t - t \cos t) \hat{i} + (2\cos t - t \sin t) \hat{j} + 2\hat{k}\end{aligned}$$

$$\vec{a} \Big|_{t=0} = 2\hat{j} + 2\hat{k}$$

$$\begin{aligned}|\vec{a}| \Big|_{t=0} &= \sqrt{4+4} \\ &= 2\sqrt{2}\end{aligned}$$

$$\begin{aligned}a_N &= \sqrt{8-0} \\ &= 2\sqrt{2}\end{aligned}$$

$$a_N = \sqrt{|\vec{a}|^2 - a_T^2}$$

$$\vec{a} = 2\sqrt{2}\vec{N}$$

$$\vec{a} = a_T \vec{T} + a_N \vec{N}$$

Exercise

Write \vec{a} of the motion $\vec{a} = a_T \vec{T} + a_N \vec{N}$ at the given value of t without finding \vec{T} and \vec{N} .

$$\vec{r}(t) = (e^t \cos t) \hat{i} + (e^t \sin t) \hat{j} + \sqrt{2}e^t \hat{k}, \quad t = 0$$

Solution

$$\vec{v} = (e^t \cos t - e^t \sin t) \hat{i} + (e^t \sin t + e^t \cos t) \hat{j} + \sqrt{2}e^t \hat{k} \quad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned}|\vec{v}| &= \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + 2e^{2t}} \\ &= \sqrt{e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t) + e^{2t}(\cos^2 t + 2\cos t \sin t + \sin^2 t) + 2e^{2t}} \\ &= e^t \sqrt{1 - 2\cos t \sin t + 1 + 2\cos t \sin t + 2} \\ &= e^t \sqrt{4} \\ &= 2e^t\end{aligned}$$

$$a_T = \frac{d}{dt}|\vec{v}| = 2e^t$$

$$a_T \Big|_{t=0} = 2$$

$$\begin{aligned}\vec{a} = \vec{v}' &= (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t) \hat{i} + (e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t) \hat{j} + \sqrt{2}e^t \hat{k} \\ &= (-2e^t \sin t) \hat{i} + (2e^t \cos t) \hat{j} + \sqrt{2}e^t \hat{k}\end{aligned}$$

$$\vec{a}\Big|_{t=0} = 2\hat{j} + \sqrt{2}\hat{k}$$

$$\begin{aligned} |\vec{a}\Big|_{t=0} &= \sqrt{4+2} \\ &= \sqrt{6} \end{aligned}$$

$$\begin{aligned} a_N &= \sqrt{6-4} \\ &= \sqrt{2} \end{aligned}$$

$$a_N = \sqrt{|\vec{a}|^2 - a_T^2}$$

$$\vec{a} = 2\vec{T} + \sqrt{2}\vec{N}$$

$$\vec{a} = a_T \vec{T} + a_N \vec{N}$$

Exercise

Write \vec{a} of the motion $\vec{a} = a_T \vec{T} + a_N \vec{N}$ without finding \vec{T} and \vec{N} .

$$\vec{r}(t) = (2 + 3t + 3t^2)\hat{i} + (4t + 4t^2)\hat{j} - (6\cos t)\hat{k} \quad t = 0$$

Solution

$$\vec{v} = (3 + 6t)\hat{i} + (4 + 8t)\hat{j} + 6\sin t\hat{k}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned} |\vec{v}| &= \sqrt{(3 + 6t)^2 + (4 + 8t)^2 + 36\sin^2 t} \\ &= \sqrt{9(1 + 2t)^2 + 16(1 + 2t)^2 + 36\sin^2 t} \\ &= \sqrt{(9 + 16)(1 + 2t)^2 + 36\sin^2 t} \\ &= \sqrt{25(1 + 2t)^2 + 36\sin^2 t} \end{aligned}$$

$$\begin{aligned} a_T &= \frac{1}{2} (100(1 + 2t) + 72\sin t \cos t) \left(25(1 + 2t)^2 + 36\sin^2 t \right)^{-1/2} \\ &= \frac{1}{2} \frac{100(1 + 2t) + 72\sin t \cos t}{\sqrt{25(1 + 2t)^2 + 36\sin^2 t}} \end{aligned}$$

$$a_T = \frac{d}{dt} |\vec{v}|$$

$$\begin{aligned} a_T \Big|_{t=0} &= \frac{1}{2} \frac{100}{\sqrt{25}} \\ &= 10 \end{aligned}$$

$$\vec{a} = 6\hat{i} + 8\hat{j} + 6\cos t\hat{k}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$\vec{a}\Big|_{t=0} = 6\hat{i} + 8\hat{j} + 6\hat{k}$$

$$\begin{aligned} |\vec{a}\Big|_{t=0} &= \sqrt{36 + 64 + 36} \\ &= \sqrt{136} \end{aligned}$$

$$a_N = \sqrt{133-100}$$

$$= 6$$

$$\vec{a} = 10\vec{T} + 6\vec{N}$$

$$a_N = \sqrt{|\vec{a}|^2 - a_T^2}$$

$$\vec{a} = a_T \vec{T} + a_N \vec{N}$$

Exercise

Write \vec{a} of the motion $\vec{a} = a_T \vec{T} + a_N \vec{N}$ without finding \vec{T} and \vec{N} .

$$\vec{r}(t) = (2+t)\hat{i} + (t+2t^2)\hat{j} + (1+t^2)\hat{k} \quad t=0$$

Solution

$$\vec{v} = \hat{i} + (1+4t)\hat{j} + 2t\hat{k}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{1 + (1+4t)^2 + 4t^2}$$

$$= \sqrt{1 + 1 + 8t + 16t^2 + 4t^2}$$

$$= \sqrt{2 + 8t + 20t^2}$$

$$a_T = \frac{8 + 40t}{2\sqrt{2 + 8t + 20t^2}}$$

$$a_T = \frac{d}{dt} |\vec{v}|$$

$$= \frac{4 + 20t}{\sqrt{2 + 8t + 20t^2}}$$

$$a_T \Big|_{t=0} = \frac{4}{\sqrt{2}}$$

$$= 2\sqrt{2}$$

$$\vec{a} = 4\hat{j} + 2\hat{k}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$|\vec{a}| \Big|_{t=0} = \sqrt{16 + 4}$$

$$= 2\sqrt{5}$$

$$a_N = \sqrt{20 - 8}$$

$$= 2\sqrt{3}$$

$$\vec{a} = 2\sqrt{2}\vec{T} + 2\sqrt{3}\vec{N}$$

$$a_N = \sqrt{|\vec{a}|^2 - a_T^2}$$

$$\vec{a} = a_T \vec{T} + a_N \vec{N}$$

Exercise

Graph the curves and sketch their velocity and acceleration vectors at the given values of t . Then write \mathbf{a} of the motion $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$ without finding \mathbf{T} and \mathbf{N} , and find the value of κ at the given values of t .

$$\vec{r}(t) = (4 \cos t) \hat{i} + (\sqrt{2} \sin t) \hat{j}, \quad t = 0 \text{ and } \frac{\pi}{4}$$

Solution

$$\begin{cases} x = 4 \cos t & \rightarrow \cos t = \frac{x}{4} \\ y = \sqrt{2} \sin t & \rightarrow \sin t = \frac{y}{\sqrt{2}} \end{cases}$$

$$\cos^2 t + \sin^2 t = 1$$

$$\frac{x^2}{16} + \frac{y^2}{2} = 1 \rightarrow \text{Ellipse}$$

$$\vec{v} = -4 \sin t \hat{i} + \sqrt{2} \cos t \hat{j}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\vec{a} = -4 \cos t \hat{i} - \sqrt{2} \sin t \hat{j}$$

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$t = 0$$

$$\vec{r}(0) = 4\hat{i}$$

$$\vec{v}(0) = \sqrt{2}\hat{j}$$

$$\vec{a}(0) = -4\hat{i}$$

$$t = \frac{\pi}{4}$$

$$\vec{r}\left(\frac{\pi}{4}\right) = 2\sqrt{2}\hat{i} + \hat{j}$$

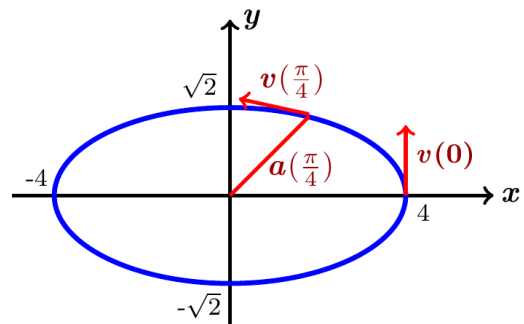
$$\vec{v}\left(\frac{\pi}{4}\right) = -2\sqrt{2}\hat{i} + \hat{j}$$

$$\vec{a}\left(\frac{\pi}{4}\right) = -2\sqrt{2}\hat{i} - \hat{j}$$

$$|\vec{v}| = \sqrt{16 \sin^2 t + 2 \cos^2 t}$$

$$\begin{aligned} a_T &= \frac{32 \sin t \cos t - 4 \cos t \sin t}{2\sqrt{16 \sin^2 t + 2 \cos^2 t}} \\ &= \frac{14 \cos t \sin t}{\sqrt{16 \sin^2 t + 2 \cos^2 t}} \end{aligned}$$

$$a_T = \frac{d}{dt} |\vec{v}|$$



$$t = 0$$

$$a_T = 0$$

$$|\vec{a}| = \sqrt{16 + 0} = 4$$

$$a_N = \sqrt{4 - 0} = 4$$

$$\vec{a} = 0\vec{T} + 4\vec{N}$$

$$|\vec{v}(0)| = \sqrt{2}$$

$$\kappa = \frac{4}{2} = 2$$

$$\kappa = \frac{a_N}{|\vec{v}|^2}$$

$$t = \frac{\pi}{4}$$

$$a_T = \frac{14 \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}}}{\sqrt{8+1}} = \frac{7}{3}$$

$$|\vec{a}| = \sqrt{8+1} = 3$$

$$a_N = \sqrt{9 - \frac{49}{9}} \\ = \frac{4\sqrt{2}}{3}$$

$$\vec{a} = \frac{7}{3} \vec{T} + \frac{4\sqrt{2}}{3} \vec{N}$$

$$\left| \vec{v} \left(\frac{\pi}{4} \right) \right| = \sqrt{8+1} = 3$$

$$\kappa = \frac{4\sqrt{2}}{3} \frac{1}{9} \quad \kappa = \frac{a_N}{|\vec{v}|^2} \\ = \frac{4\sqrt{2}}{27}$$

Exercise

Graph the curves and sketch their velocity and acceleration vectors at the given values of t . Then write \mathbf{a} of the motion $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$ without finding \mathbf{T} and \mathbf{N} , and find the value of κ at the given values of t .

$$\vec{r}(t) = (\sqrt{3} \sec t) \hat{i} + (\sqrt{3} \tan t) \hat{j}, \quad t = 0$$

Solution

$$\begin{cases} x = \sqrt{3} \sec t \rightarrow \sec t = \frac{x}{\sqrt{3}} \\ y = \sqrt{3} \tan t \rightarrow \tan t = \frac{y}{\sqrt{3}} \end{cases}$$

$$\sec^2 t - \tan^2 t = 1$$

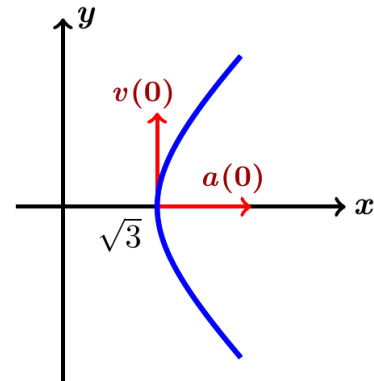
$$\frac{x^2}{3} - \frac{y^2}{3} = 1 \rightarrow \text{Hyperbolic}$$

$$\vec{v} = \sqrt{3} \sec t \tan t \hat{i} + \sqrt{3} \sec^2 t \hat{j}$$

$$\vec{a} = \sqrt{3} (\sec t \tan^2 t + \sec^3 t) \hat{i} + 2\sqrt{3} \sec^2 t \tan t \hat{j}$$

At $t = 0$

$$\vec{r}(0) = \sqrt{3} \hat{i}, \quad \vec{v}(0) = \sqrt{3} \hat{j}, \quad \vec{a}(0) = \sqrt{3} \hat{i}$$



$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$|\vec{v}(0)| = \sqrt{3}$$

$$\begin{aligned} |\vec{v}| &= \sqrt{3 \sec^2 t \tan^2 t + 3 \sec^4 t} \\ &= \sqrt{3} \sqrt{\sec^2 t (\sec^2 t - 1) + \sec^4 t} \\ &= \sqrt{3} \sqrt{2 \sec^4 t - \sec^2 t} \end{aligned}$$

$$\begin{aligned} a_T &= \frac{\sqrt{3}}{2} \frac{8 \sec^4 t \tan t + 2 \sec^2 t \tan t}{\sqrt{2 \sec^4 t - \sec^2 t}} & a_T &= \frac{d}{dt} |\vec{v}| \\ &= \sqrt{3} \frac{\sec^2 t \tan t (4 \sec^2 t + 1)}{\sec t \sqrt{2 \sec^2 t - 1}} \\ &= \frac{\sqrt{3} \sec t \tan t (4 \sec^2 t + 1)}{\sqrt{2 \sec^2 t - 1}}. \end{aligned}$$

$$a_T \Big|_{t=0} = 0$$

$$\begin{aligned} a_N &= \sqrt{3-0} & a_N &= \sqrt{|\mathbf{a}|^2 - a_T^2} \\ &= \sqrt{3} \end{aligned}$$

$$\vec{a} = 0 \vec{T} + \sqrt{3} \vec{N} \quad \vec{a} = a_T \vec{T} + a_N \vec{N}$$

$$\vec{a} = \sqrt{3} \vec{N}$$

$$\kappa = \frac{\sqrt{3}}{3} \quad \kappa = \frac{a_N}{|\vec{v}|^2}$$

Exercise

Find \mathbf{T} , \mathbf{N} , \mathbf{B} , τ , and κ at the given value of t for the plane curves

$$\vec{r}(t) = \frac{4}{9}(1+t)^{3/2} \hat{i} + \frac{4}{9}(1-t)^{3/2} \hat{j} + \frac{1}{3}t \hat{k}; \quad t = 0$$

Solution

$$\vec{v} = \frac{2}{3}(1+t)^{1/2} \hat{i} - \frac{2}{3}(1-t)^{1/2} \hat{j} + \frac{1}{3} \hat{k} \quad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned} |\vec{v}| &= \sqrt{\frac{4}{9}(1+t) + \frac{4}{9}(1-t) + \frac{1}{9}} \\ &= \sqrt{\frac{4}{9}(1+t+1-t) + \frac{1}{9}} \\ &= \sqrt{\frac{8}{9} + \frac{1}{9}} \end{aligned}$$

$$=1]$$

$$\vec{T} = \frac{2}{3}(1+t)^{1/2} \hat{i} - \frac{2}{3}(1-t)^{1/2} \hat{j} + \frac{1}{3} \hat{k}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{T}(\mathbf{0}) = \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$$

$$\frac{d\vec{T}}{dt} = \frac{1}{3}(1+t)^{-1/2} \hat{i} + \frac{1}{3}(1-t)^{-1/2} \hat{j}$$

$$\left. \frac{d\vec{T}}{dt} \right|_{t=0} = \frac{1}{3}\hat{i} + \frac{1}{3}\hat{j}$$

$$\left| \frac{d\vec{T}}{dt}(0) \right| = \sqrt{\frac{1}{9} + \frac{1}{9}}$$

$$= \frac{\sqrt{2}}{3}$$

$$\vec{N}(\mathbf{0}) = \frac{3}{\sqrt{2}} \left(\frac{1}{3}\hat{i} + \frac{1}{3}\hat{j} \right)$$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$= \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

$$\vec{B}(0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{vmatrix}$$

$$\vec{B} = \vec{T} \times \vec{N}$$

$$= -\frac{1}{3\sqrt{2}}\hat{i} + \frac{1}{3\sqrt{2}}\hat{j} + \frac{4}{3\sqrt{2}}\hat{k}$$

$$\vec{a}(t) = \frac{1}{3}(1+t)^{-1/2} \hat{i} + \frac{1}{3}(1-t)^{-1/2} \hat{j}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$\vec{a}(0) = \frac{1}{3}\hat{i} + \frac{1}{3}\hat{j}$$

$$\vec{v}(0) \times \vec{a}(0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{vmatrix}$$

$$= -\frac{1}{9}\hat{i} + \frac{1}{9}\hat{j} + \frac{4}{9}\hat{k}$$

$$|\vec{v}(0) \times \vec{a}(0)| = \sqrt{\frac{1}{81} + \frac{1}{81} + \frac{16}{81}}$$

$$= \sqrt{\frac{18}{81}}$$

$$= \frac{3\sqrt{2}}{9}$$

$$= \frac{\sqrt{2}}{3}$$

$$\kappa(0) = \frac{\sqrt{2}}{3} \quad \kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$$

$$\vec{a}' = -\frac{1}{6}(1+t)^{-3/2} \hat{i} + \frac{1}{6}(1-t)^{-3/2} \hat{j}$$

$$\vec{a}'(0) = -\frac{1}{6} \hat{i} + \frac{1}{6} \hat{j}$$

$$\tau(0) = \frac{1}{\left(\frac{\sqrt{2}}{3}\right)^2} \begin{vmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 \end{vmatrix}$$

$$\tau = \frac{1}{|\vec{v} \times \vec{a}|^2} \begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix}$$

$$= \frac{\frac{1}{3} \left(2 \frac{1}{18} \right)}{\frac{2}{9}}$$

$$= \frac{1}{6}$$

Exercise

Find \mathbf{T} , \mathbf{N} , \mathbf{B} , τ , and κ at the given value of t for the plane curves

$$\vec{r}(t) = (e^t \sin 2t) \hat{i} + (e^t \cos 2t) \hat{j} + 2e^t \hat{k}; \quad t = 0$$

Solution

$$\vec{v} = e^t (\sin 2t + 2 \cos 2t) \hat{i} + e^t (\cos 2t - 2 \sin 2t) \hat{j} + 2e^t \hat{k} \quad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned} |\vec{v}| &= e^t \sqrt{(\sin 2t + 2 \cos 2t)^2 + (\cos 2t - 2 \sin 2t)^2 + 4} \\ &= e^t \sqrt{\sin^2 2t + 4 \sin 2t \cos 2t + 4 \cos^2 2t + \cos^2 2t - 4 \sin 2t \cos 2t + 4 \sin^2 2t + 4} \\ &= e^t \sqrt{5 \cos^2 2t + 5 \sin^2 2t + 4} \\ &= e^t \sqrt{5 + 4} \\ &= 3e^t \end{aligned}$$

$$\vec{T} = \frac{e^t (\sin 2t + 2 \cos 2t) \hat{i} + e^t (\cos 2t - 2 \sin 2t) \hat{j} + 2e^t \hat{k}}{3e^t} \quad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{1}{3} ((\sin 2t + 2 \cos 2t) \hat{i} + (\cos 2t - 2 \sin 2t) \hat{j} + 2 \hat{k})$$

$$\vec{T}(0) = \frac{2}{3} \hat{i} + \frac{1}{3} \hat{j} + \frac{2}{3} \hat{k}$$

$$\frac{d\vec{T}}{dt} = \frac{1}{3}(2\cos 2t - 4\sin 2t)\hat{i} + \frac{1}{3}(-2\sin 2t - 4\cos 2t)\hat{j}$$

$$\left. \frac{d\vec{T}}{dt} \right|_{t=0} = \frac{2}{3}\hat{i} - \frac{4}{3}\hat{j}$$

$$\left| \frac{d\vec{T}}{dt}(0) \right| = \sqrt{\frac{4}{9} + \frac{16}{9}}$$

$$= \frac{2}{3}\sqrt{5}$$

$$\vec{N}(0) = \frac{3}{2\sqrt{5}}\left(\frac{2}{3}\hat{i} - \frac{4}{3}\hat{j}\right)$$

$$= \frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}$$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\vec{B}(0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \end{vmatrix}$$

$$= \frac{4}{3\sqrt{5}}\hat{i} + \frac{2}{3\sqrt{5}}\hat{j} - \frac{5}{3\sqrt{5}}\hat{k}$$

$$\vec{B} = \vec{T} \times \vec{N}$$

$$\vec{a}(t) = e^t(4\cos 2t - 3\sin 2t)\hat{i} + e^t(-4\sin 2t - 3\cos 2t)\hat{j} + 2e^t\hat{k}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$\vec{a}(0) = 4\hat{i} - 3\hat{j} + 2\hat{k}$$

$$\vec{v}(0) \times \vec{a}(0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 2 \\ 4 & -3 & 2 \end{vmatrix}$$

$$= 8\hat{i} + 4\hat{j} - 10\hat{k}$$

$$|\vec{v}(0) \times \vec{a}(0)| = \sqrt{64 + 16 + 100}$$

$$= \sqrt{180}$$

$$= 6\sqrt{5}$$

$$\kappa(0) = \frac{6\sqrt{5}}{3^2}$$

$$= \frac{2\sqrt{5}}{3}$$

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$$

$$\vec{a}' = e^t(4\cos 2t - 3\sin 2t - 8\sin 2t - 6\cos 2t)\hat{i}$$

$$+ e^t(-4\sin 2t - 3\cos 2t - 8\cos 2t + 6\sin 2t)\hat{j} + 2e^t\hat{k}$$

$$= e^t(-2\cos 2t - 11\sin 2t)\hat{i} + e^t(2\sin 2t - 11\cos 2t)\hat{j} + 2e^t\hat{k}$$

$$\vec{a}'(0) = -2\hat{i} - 11\hat{j} + 2\hat{k}$$

$$\begin{aligned}\tau(0) &= \frac{1}{180} \begin{vmatrix} 2 & 1 & 2 \\ 4 & -3 & 2 \\ 2 & -11 & 2 \end{vmatrix} \\ &= \frac{-80}{180} \\ &= \underline{-\frac{4}{9}}\end{aligned}$$

$$\tau = \frac{1}{|\vec{v} \times \vec{a}|^2} \begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix}$$

Exercise

Find T , N , B , τ , and κ at the given value of t for the plane curves $\vec{r}(t) = t\hat{i} + \left(\frac{1}{2}e^{2t}\right)\hat{j}$; $t = \ln 2$

Solution

$$\vec{v} = \hat{i} + e^{2t}\hat{j}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned}\vec{v}(\ln 2) &= \hat{i} + e^{2\ln 2}\hat{j} \\ &= \hat{i} + e^{\ln 4}\hat{j} \\ &= \underline{\hat{i} + 4\hat{j}}\end{aligned}$$

$$\begin{aligned}|\vec{v}| &= \sqrt{1 + e^{4t}} \Big|_{t=\ln 2} \\ &= \sqrt{1 + e^{4\ln 2}} \\ &= \sqrt{1 + e^{\ln 2^4}} \\ &= \sqrt{1 + 16} \\ &= \underline{\sqrt{17}}\end{aligned}$$

$$\vec{T} = \frac{1}{\sqrt{1 + e^{4t}}}\hat{i} + \frac{e^{2t}}{\sqrt{1 + e^{4t}}}\hat{j}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\underline{\vec{T}(\ln 2) = \frac{1}{\sqrt{17}}\hat{i} + \frac{4}{\sqrt{17}}\hat{j}}$$

$$\begin{aligned}\frac{d\vec{T}}{dt} &= \frac{-2e^{4t}}{(1 + e^{4t})^{3/2}}\hat{i} + \frac{2e^{2t}(1 + e^{4t}) - (2e^{4t})e^{2t}}{(1 + e^{4t})^{3/2}}\hat{j} \\ &= \frac{-2e^{4t}}{(1 + e^{4t})^{3/2}}\hat{i} + \frac{2e^{2t}}{(1 + e^{4t})^{3/2}}\hat{j}\end{aligned}$$

$$\begin{aligned}\left.\frac{d\vec{T}}{dt}\right|_{t=\ln 2} &= -\frac{2e^{4\ln 2}}{(1+16)^{3/2}}\hat{i} + \frac{2(4)}{17\sqrt{17}}\hat{j} \\ &= -\frac{32}{17\sqrt{17}}\hat{i} + \frac{8}{17\sqrt{17}}\hat{j}\end{aligned}$$

$$\begin{aligned}\left|\frac{d\vec{T}}{dt}(0)\right| &= \sqrt{\frac{32^2}{17^3} + \frac{64}{17^3}} \\ &= \frac{8\sqrt{17}}{17\sqrt{17}} \\ &= \frac{8}{17}\end{aligned}$$

$$\begin{aligned}\vec{N}(\ln 2) &= \frac{17}{8}\left(-\frac{32}{17\sqrt{17}}\hat{i} + \frac{8}{17\sqrt{17}}\hat{j}\right) & \vec{N} &= \frac{d\vec{T}/dt}{|d\vec{T}/dt|} \\ &= -\frac{4}{\sqrt{17}}\hat{i} + \frac{1}{\sqrt{17}}\hat{j}\end{aligned}$$

$$\begin{aligned}\vec{B}(\ln 2) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} & 0 \\ \frac{-4}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0 \end{vmatrix} & \vec{B} &= \vec{T} \times \vec{N} \\ &= \hat{k}\end{aligned}$$

$$\vec{a}(t) = 2e^{2t}\hat{j} \qquad \vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$\vec{a}(\ln 2) = 8\hat{j}$$

$$\begin{aligned}(\vec{v} \times \vec{a})(\ln 2) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 0 \\ 0 & 8 & 0 \end{vmatrix} \\ &= 8\hat{k}\end{aligned}$$

$$|(\vec{v} \times \vec{a})(\ln 2)| = 8$$

$$\kappa(\ln 2) = \frac{8}{\sqrt{17}} \qquad \kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$$

$$\vec{a}' = 4e^{2t}\hat{j}$$

$$\vec{a}'(\ln 2) = 16\hat{j}$$

$$\tau(\ln 2) = \frac{1}{180} \begin{vmatrix} 1 & 4 & 0 \\ 0 & 8 & 0 \\ 0 & 16 & 0 \end{vmatrix}$$

$$= 0$$

$$\tau = \frac{1}{|\vec{v} \times \vec{a}|^2} \begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix}$$

Exercise

Find \mathbf{r} , \mathbf{T} , \mathbf{N} , and \mathbf{B} at the given value of t . Then find equations for the osculating, normal, and rectifying planes at that value of t . $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - \mathbf{k}$, $t = \frac{\pi}{4}$

Solution

$$\vec{r}\left(t = \frac{\pi}{4}\right) = \left(\cos \frac{\pi}{4}\right)\hat{i} + \left(\sin \frac{\pi}{4}\right)\hat{j} - \hat{k}$$

$$= \frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j} - \hat{k}$$

$$\vec{v} = (-\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\vec{T} = (-\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{T}\left(t = \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j}$$

$$\frac{d\vec{T}}{dt} = (-\cos t)\hat{i} + (-\sin t)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

$$\vec{N} = (-\cos t)\hat{i} + (-\sin t)\hat{j}$$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\vec{N}\left(t = \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\hat{i} - \frac{\sqrt{2}}{2}\hat{j}$$

$$\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \hat{k}$$

$$\vec{B}\left(t = \frac{\pi}{4}\right) = \hat{k}$$

The normal to the osculating plane $\mathbf{r} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} - \mathbf{k} \Rightarrow P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1\right)$ lies on the

osculating plane (using \mathbf{B}):

$$0\left(x - \frac{\sqrt{2}}{2}\right) + 0\left(y - \frac{\sqrt{2}}{2}\right) + (z - (-1)) = 0$$

$z = -1$ is the osculating plane.

T is normal to the normal plane

$$-\frac{\sqrt{2}}{2}\left(x - \frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}y\left(x - \frac{\sqrt{2}}{2}\right) + 0(z - (-1)) = 0$$

$$-\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y = 0$$

$-x + y = 0$ is the normal plane

N is normal to the rectifying plane:

$$-\frac{\sqrt{2}}{2}\left(x - \frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}\left(y - \frac{\sqrt{2}}{2}\right) + 0(z - (-1)) = 0$$

$$-\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y + \frac{1}{2} + \frac{1}{2} = 0$$

$x + y = \sqrt{2}$ is the rectifying plane.

Exercise

Find \mathbf{r} , \mathbf{T} , \mathbf{N} , and \mathbf{B} at the given value of t . Then find equations for the osculating, normal, and rectifying planes at that value of t . $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}$, $t = 0$

Solution

$$\vec{r}(t=0) = (\cos 0)\hat{i} + (\sin 0)\hat{j} + 0\hat{k}$$

$$= \hat{i}$$

$$\vec{v} = (-\sin t)\hat{i} + (\cos t)\hat{j} + \hat{k}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{\sin^2 t + \cos^2 t + 1}$$

$$= \sqrt{2}$$

$$\vec{T} = -\left(\frac{\sin t}{\sqrt{2}}\right)\hat{i} + \left(\frac{\cos t}{\sqrt{2}}\right)\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{T}(t=0) = \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$$

$$\frac{d\vec{T}}{dt} = \left(-\frac{1}{\sqrt{2}}\cos t\right)\hat{i} - \left(\frac{1}{\sqrt{2}}\sin t\right)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{1}{2}\cos^2 t + \frac{1}{2}\sin^2 t}$$

$$= \frac{1}{\sqrt{2}}$$

$$\begin{aligned}\vec{N} &= \sqrt{2} \left(\left(-\frac{1}{\sqrt{2}} \cos t \right) \hat{i} - \left(\frac{1}{\sqrt{2}} \sin t \right) \hat{j} \right) \\ &= (-\cos t) \hat{i} - (\sin t) \hat{j}\end{aligned}$$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\vec{N}(t=0) = -\hat{i}$$

$$\begin{aligned}\vec{B}(t=0) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \end{vmatrix} \\ &= -\frac{1}{\sqrt{2}} \hat{j} + \frac{1}{\sqrt{2}} \hat{k}\end{aligned}$$

$$\vec{B} = \vec{T} \times \vec{N}$$

The normal to the osculating plane $\vec{r}(t) = \hat{i} \Rightarrow P(1, 0, 0)$ lies on the osculating plane (using \vec{B}):

$$\begin{aligned}0(x-1) - \frac{1}{\sqrt{2}}(y-0) + \frac{1}{\sqrt{2}}(z-0) &= 0 \\ -\frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}z &= 0\end{aligned}$$

$y - z = 0$ is the osculating plane.

\vec{T} is normal to the normal plane

$$\begin{aligned}0(x-1) + \frac{1}{\sqrt{2}}(y-0) + \frac{1}{\sqrt{2}}(z-0) &= 0 \\ \frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}z &= 0\end{aligned}$$

$y + z = 0$ is the normal plane

\vec{N} is normal to the rectifying plane:

$$\begin{aligned}-(x-1) + 0(y-0) + 0(z-0) &= 0 \\ -x + 1 &= 0\end{aligned}$$

$x = 1$ is the rectifying plane.

Exercise

Find \vec{B} and τ for: $\vec{r}(t) = (3\sin t)\hat{i} + (3\cos t)\hat{j} + 4t\hat{k}$

Solution

$$\vec{v} = (3\cos t)\hat{i} - (3\sin t)\hat{j} + 4\hat{k}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned}
|\vec{v}| &= \sqrt{9\cos^2 t + 9\sin^2 t + 16} \\
&= \sqrt{9+16} \\
&= 5
\end{aligned}$$

$$\vec{T} = \frac{3}{5}\cos t \hat{i} - \frac{3}{5}\sin t \hat{j} + \frac{4}{5} \hat{k}$$

$$\frac{d\vec{T}}{dt} = -\frac{3}{5}\sin t \hat{i} - \frac{3}{5}\cos t \hat{j}$$

$$\begin{aligned}
\left| \frac{d\vec{T}}{dt} \right| &= \sqrt{\frac{9}{25}\sin^2 t + \frac{9}{25}\cos^2 t} \\
&= \sqrt{\frac{9}{25}} \\
&= \frac{3}{5}
\end{aligned}$$

$$\begin{aligned}
\vec{N} &= \frac{5}{3} \left(-\frac{3}{5}\sin t \hat{i} - \frac{3}{5}\cos t \hat{j} \right) \\
&= (-\sin t) \hat{i} - (\cos t) \hat{j}
\end{aligned}$$

$$\begin{aligned}
\vec{B} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{3}{5}\cos t & -\frac{3}{5}\sin t & \frac{4}{5} \\ -\sin t & -\cos t & 0 \end{vmatrix} \\
&= \left(\frac{4}{5}\cos t \right) \hat{i} - \left(\frac{4}{5}\sin t \right) \hat{j} - \frac{3}{5} \hat{k}
\end{aligned}$$

$$\vec{a} = (-3\sin t) \hat{i} - (3\cos t) \hat{j}$$

$$\begin{aligned}
\vec{v} \times \vec{a} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \end{vmatrix} \\
&= (12\cos t) \hat{i} - (12\sin t) \hat{j} - 9\hat{k}
\end{aligned}$$

$$\begin{aligned}
|\vec{v} \times \vec{a}|^2 &= 144\cos^2 t + 144\sin^2 t + 81 \\
&= 144 + 81 \\
&= 225
\end{aligned}$$

$$\tau = \frac{\begin{vmatrix} 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \\ -3\cos t & 3\sin t & 0 \end{vmatrix}}{225}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\vec{B} = \vec{T} \times \vec{N}$$

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2}$$

$$\begin{aligned}
&= \frac{4(-9\sin^2 t - 9\cos^2 t)}{225} \\
&= -\frac{36}{225} \\
&= -\frac{4}{25} \quad \Big|
\end{aligned}$$

Exercise

Find \mathbf{B} and τ for: $\vec{r}(t) = (\cos t + t \sin t)\hat{i} + (\sin t - t \cos t)\hat{j} + 3\hat{k}$

Solution

$$\vec{v} = (-\sin t + \sin t + t \cos t)\hat{i} + (\cos t - \cos t + t \sin t)\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$= (t \cos t)\hat{i} + (t \sin t)\hat{j}$$

$$|\vec{v}| = \sqrt{t^2 \cos^2 t + t^2 \sin^2 t}$$

$$= \sqrt{t^2 (\cos^2 t + \sin^2 t)}$$

$$= \sqrt{t^2}$$

$$= |t|$$

$$= t$$

$$\vec{T} = \left(\frac{t \cos t}{t} \right) \hat{i} + \left(\frac{t \sin t}{t} \right) \hat{j}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$= (\cos t)\hat{i} + (\sin t)\hat{j} \quad \Big|$$

$$\frac{d\vec{T}}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\left| \frac{d\vec{T}}{dt} \right| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\vec{N} = (-\sin t)\hat{j} + (\cos t)\hat{i} \quad \Big|$$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\mathbf{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & \sin t & 0 \\ -\sin t & -\cos t & 0 \end{vmatrix}$$

$$\vec{B} = \vec{T} \times \vec{N}$$

$$= (\cos^2 t + \sin^2 t)\hat{k}$$

$$= \hat{k} \quad \Big|$$

$$\vec{a}(\cos t - t \sin t)\hat{i} + (\sin t + t \cos t)\hat{j} \quad \vec{a} = \frac{d\vec{v}}{dt}$$

$$\begin{aligned} \vec{v} \times \vec{a} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t \cos t & t \sin t & 0 \\ \cos t - t \sin t & \sin t + t \cos t & 0 \end{vmatrix} \\ &= (t \cos t \sin t + t^2 \cos^2 t - t \sin t \cos t + t^2 \sin^2 t)\hat{k} \\ &= t^2 \hat{k} \end{aligned}$$

$$|\vec{v} \times \vec{a}|^2 = (t^2)^2 = t^4$$

$$\tau = \frac{\begin{vmatrix} t \cos t & t \sin t & 0 \\ \cos t - t \sin t & \sin t + t \cos t & 0 \\ -2 \sin t - t \cos t & 2 \cos t - t \sin t & 0 \end{vmatrix}}{t^4}$$

$$= \frac{0}{t^4}$$

$$= 0$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \ddot{\ddot{x}} & \ddot{\ddot{y}} & \ddot{\ddot{z}} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2}$$

Exercise

Find \mathbf{B} and τ for: $\vec{r}(t) = (6 \sin 2t)\hat{i} + (6 \cos 2t)\hat{j} + 5t\hat{k}$

Solution

$$\vec{v} = (12 \cos 2t)\hat{i} - (12 \sin 2t)\hat{j} + 5\hat{k} \quad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned} |\vec{v}| &= \sqrt{144 \cos^2 t + 144 \sin^2 t + 25} \\ &= \sqrt{144 + 25} \\ &= 13 \end{aligned}$$

$$\vec{T} = \frac{12 \cos 2t \hat{i} - 12 \sin 2t \hat{j} + 5 \hat{k}}{13}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\frac{d\vec{T}}{dt} = -\frac{24}{13} \sin 2t \hat{i} - \frac{24}{13} \cos 2t \hat{j}$$

$$\begin{aligned} \left| \frac{d\vec{T}}{dt} \right| &= \sqrt{\frac{576}{169} \sin^2 t + \frac{576}{169} \cos^2 t} \\ &= \sqrt{\frac{576}{169}} \\ &= \frac{24}{13} \end{aligned}$$

$$\vec{N} = \frac{13}{24} \left(-\frac{24}{13} \sin 2t \hat{i} - \frac{24}{13} \cos 2t \hat{j} \right)$$

$$= \left(-\sin 2t \right) \hat{i} - \left(\cos 2t \right) \hat{j}$$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{12}{13} \cos 2t & -\frac{12}{13} \sin 2t & \frac{5}{13} \\ -\sin 2t & -\cos 2t & 0 \end{vmatrix}$$

$$= \left(\frac{5}{13} \cos 2t \right) \hat{i} - \left(\frac{5}{13} \sin 2t \right) \hat{j} - \frac{12}{13} \hat{k}$$

$$\vec{a} = (-24 \sin 2t) \hat{i} - (24 \cos 2t) \hat{j}$$

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 12 \cos 2t & -12 \sin 2t & 5 \\ -24 \sin 2t & -24 \cos 2t & 0 \end{vmatrix}$$

$$= (120 \cos 2t) \hat{i} - (120 \sin 2t) \hat{j} - 288 \hat{k}$$

$$|\vec{v} \times \vec{a}|^2 = 14400 \cos^2 2t + 14400 \sin^2 2t + 288^2$$

$$= 14400 + 82944 = 97344$$

$$\tau = \frac{\begin{vmatrix} 12 \cos 2t & -12 \sin 2t & 5 \\ -24 \sin 2t & -24 \cos 2t & 0 \\ -48 \cos 2t & 48 \sin 2t & 0 \end{vmatrix}}{97344}$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2}$$

$$= \frac{5(-1152 \sin^2 2t - 1152 \cos^2 2t)}{97344}$$

$$= -\frac{5760}{97344}$$

$$= -\frac{10}{169}$$

Exercise

The speedometer on your car reads a steady 35 *mph*. Could you be accelerating? Explain.

Solution

Yes.

If the car is moving along a curved path, then $\kappa \neq 0$ and $a_N = \kappa |\vec{v}|^2 \neq 0$

$$\Rightarrow \vec{a} = a_T \vec{T} + a_N \vec{N} \neq \mathbf{0}$$

Exercise

Can anything be said about the acceleration of a particle that is moving at a constant speed? Give reasons for your answer.

Solution

$$|v| \text{ is constant } \Rightarrow \vec{a}_T = \frac{d\vec{v}}{dt} = 0$$

$$\vec{a} = a_T \vec{T} + a_N \vec{N} = a_N \vec{N} \text{ is orthogonal to } \vec{T}.$$

\therefore The acceleration is normal to the path.

Exercise

Find T , N , B , τ and κ as functions of t for the plane curves: $\vec{r}(t) = (\sin t)\hat{i} + (\sqrt{2}\cos t)\hat{j} + (\sin t)\hat{k}$, then write \vec{a} of the motion $\vec{a} = a_T \vec{T} + a_N \vec{N}$

Solution

$$\vec{v} = (\cos t)\hat{i} - (\sqrt{2}\sin t)\hat{j} + (\cos t)\hat{k}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned} |\vec{v}| &= \sqrt{\cos^2 t + 2\sin^2 t + \cos^2 t} \\ &= \sqrt{2\cos^2 t + 2\sin^2 t} \\ &= \sqrt{2} \end{aligned}$$

$$\vec{T} = \left(\frac{\cos t}{\sqrt{2}} \right) \hat{i} - (\sin t) \hat{j} + \left(\frac{\cos t}{\sqrt{2}} \right) \hat{k}$$

$$\frac{d\vec{T}}{dt} = -\frac{\sin t}{\sqrt{2}} \hat{i} - (\cos t) \hat{j} - \frac{\sin t}{\sqrt{2}} \hat{k}$$

$$\begin{aligned} \left| \frac{d\vec{T}}{dt} \right| &= \sqrt{\frac{1}{2}\sin^2 t + \cos^2 t + \frac{1}{2}\sin^2 t} \\ &= \sqrt{\sin^2 t + \cos^2 t} \\ &= 1 \end{aligned}$$

$$\vec{N} = -\frac{\sin t}{\sqrt{2}} \hat{i} - (\cos t) \hat{j} - \frac{\sin t}{\sqrt{2}} \hat{k}$$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\begin{aligned}
\mathbf{B} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\cos t}{\sqrt{2}} & -\sin t & \frac{\cos t}{\sqrt{2}} \\ -\frac{\sin t}{\sqrt{2}} & -\cos t & -\frac{\sin t}{\sqrt{2}} \end{vmatrix} \\
&= \left(\frac{\sin^2 t}{\sqrt{2}} + \frac{\cos^2 t}{\sqrt{2}} \right) \hat{i} - \left(-\frac{\sin t \cos t}{\sqrt{2}} + \frac{\sin t \cos t}{\sqrt{2}} \right) \hat{j} + \left(-\frac{\cos^2 t}{\sqrt{2}} - \frac{\sin^2 t}{\sqrt{2}} \right) \hat{k} \\
&= \frac{1}{\sqrt{2}} \hat{i} - \frac{1}{\sqrt{2}} \hat{k}
\end{aligned}$$

$$\kappa = \frac{1}{\sqrt{2}} \quad \kappa = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$$

$$\vec{a} = (-\sin t) \hat{i} - (\sqrt{2} \cos t) \hat{j} - (\sin t) \hat{k}$$

$$\begin{aligned}
\vec{v} \times \vec{a} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & -\sqrt{2} \sin t & \cos t \\ -\sin t & -\sqrt{2} \cos t & -\sin t \end{vmatrix} \\
&= (\sqrt{2} \sin^2 t + \sqrt{2} \cos^2 t) \hat{i} + (-\sqrt{2} \cos^2 t - \sqrt{2} \sin^2 t) \hat{k} \\
&= (\sqrt{2}) \hat{i} - (\sqrt{2}) \hat{k}
\end{aligned}$$

$$|\vec{v} \times \vec{a}| = \sqrt{2+2} = 2$$

$$\begin{aligned}
\tau &= \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2} = \frac{\begin{vmatrix} \cos t & -\sqrt{2} \sin t & \cos t \\ -\sin t & -\sqrt{2} \cos t & -\sin t \\ -\cos t & \sqrt{2} \sin t & -\cos t \end{vmatrix}}{4} \\
&= \frac{\sqrt{2} \cos^3 t - \sqrt{2} \sin^2 t \cos t - \sqrt{2} \sin^2 t \cos t - \sqrt{2} \cos^3 t + \sqrt{2} \sin^2 t \cos t + \sqrt{2} \sin^2 t \cos t}{4} \\
&= 0
\end{aligned}$$

Exercise

Consider the ellipse $\vec{r}(t) = \langle 3 \cos t, 4 \sin t \rangle$ for $0 \leq t \leq 2\pi$

- Find the tangent vector \vec{r}' , the unit vector \vec{T} , and the principal unit normal vector \vec{N} at all points on the curve.
- At what points does $|\vec{r}'|$ have maximum and minimum values?

- c) At what points does the curvature have maximum and minimum values? Interpret this result in light of part (b).
- d) Find the points (if any) at which \vec{r} and \vec{N} are parallel.

Solution

a) $\vec{r}'(t) = \langle -3\sin t, 4\cos t \rangle$

$$\vec{T} = \frac{\langle -3\sin t, 4\cos t \rangle}{\sqrt{9\sin^2 t + 16\cos^2 t}} \quad \vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$= \frac{1}{\sqrt{9\sin^2 t + 9\cos^2 t + 7\cos^2 t}} \langle -3\sin t, 4\cos t \rangle$$

$$= \left\langle -\frac{3\sin t}{\sqrt{9+7\cos^2 t}}, \frac{4\cos t}{\sqrt{9+7\cos^2 t}} \right\rangle$$

$$\frac{d\vec{T}}{dt} = \left\langle -3 \frac{\cos t(9+7\cos^2 t) + 7\cos t \sin^2 t}{(9+7\cos^2 t)^{3/2}}, 4 \frac{-\sin t(9+7\cos^2 t) + 7\cos^2 t \sin t}{(9+7\cos^2 t)^{3/2}} \right\rangle$$

$$= \left\langle -3 \frac{9\cos t + 7\cos^3 t + 7\cos t \sin^2 t}{(9+7\cos^2 t)^{3/2}}, 4 \frac{-9\sin t - 7\cos^2 t \sin t + 7\cos^2 t \sin t}{(9+7\cos^2 t)^{3/2}} \right\rangle$$

$$= \left\langle -3\cos t \frac{9+7\cos^2 t + 7\sin^2 t}{(9+7\cos^2 t)^{3/2}}, -\frac{36\sin t}{(9+7\cos^2 t)^{3/2}} \right\rangle$$

$$= \left\langle -\frac{48\cos t}{(9+7\cos^2 t)^{3/2}}, -\frac{36\sin t}{(9+7\cos^2 t)^{3/2}} \right\rangle$$

$$= -\frac{12}{(9+7\cos^2 t)^{3/2}} \langle 4\cos t, 3\sin t \rangle$$

$$\left| \frac{d\vec{T}}{dt} \right| = \frac{12}{(9+7\cos^2 t)^{3/2}} \sqrt{16\cos^2 t + 9\sin^2 t}$$

$$= \frac{12}{(9+7\cos^2 t)^{3/2}} \sqrt{9+7\cos^2 t}$$

$$= \frac{12}{\sqrt{9+7\cos^2 t}}$$

$$\vec{N} = \frac{\sqrt{9+7\cos^2 t}}{12} \frac{-12}{(9+7\cos^2 t)^{3/2}} \langle 4\cos t, 3\sin t \rangle \quad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$= \left\langle -\frac{4\cos t}{\sqrt{9+7\cos^2 t}}, -\frac{3\sin t}{\sqrt{9+7\cos^2 t}} \right\rangle$$

b) $|\vec{r}'(t)| = \sqrt{9\sin^2 t + 16\cos^2 t}$

$$\frac{d}{dt}|\vec{r}'(t)| = \frac{18\sin t \cos t - 32\cos t \sin t}{2\sqrt{9\sin^2 t + 16\cos^2 t}}$$

$$= \frac{-7\cos t \sin t}{\sqrt{9\sin^2 t + 16\cos^2 t}} = 0$$

$$\begin{cases} \cos t = 0 & \rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2} & |\vec{r}'(t)| = 3 & \text{Minimum} \\ \sin t = 0 & \rightarrow t = 0, \pi & |\vec{r}'(t)| = 4 & \text{Maximum} \end{cases}$$

c) $\vec{r}''(t) \times \vec{r}'(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\cos t & -4\sin t & 0 \\ -3\sin t & 4\cos t & 0 \end{vmatrix}$

$$= \langle 0, 0, -12\cos^2 t - 12\sin^2 t \rangle$$

$$= \langle 0, 0, -12 \rangle$$

$$\tau = \frac{|\langle 0, 0, -12 \rangle|}{(9+7\cos^2 t)^{3/2}} \quad \kappa = \frac{|\vec{r}''(t) \times \vec{r}'(t)|}{(|\vec{r}'(t)|)^3}$$

$$= \frac{12}{(9+7\cos^2 t)^{3/2}}$$

For τ to be maximum the denominator has to be the smallest

$$\cos^2 t = 0 \rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$$

From part (b) result of t -values

$$\begin{cases} t = \frac{\pi}{2}, \frac{3\pi}{2} & \tau = \frac{12}{27} = \frac{4}{9} & \text{Maximum} \\ t = 0, \pi & \tau = \frac{12}{64} = \frac{3}{16} & \text{Minimum} \end{cases}$$

Velocity is maximized where curvature is minimal.

Velocity is maximized where curvature is minimal.

$$d) \quad \vec{r}(t) = \langle 3\cos t, 4\sin t \rangle \quad // \quad \vec{N} = \left\langle -\frac{4\cos t}{\sqrt{9+7\cos^2 t}}, -\frac{3\sin t}{\sqrt{9+7\cos^2 t}} \right\rangle$$

$$3\cos t = -\frac{4\cos t}{\sqrt{9+7\cos^2 t}} \cdot m \rightarrow \cos t = 0$$

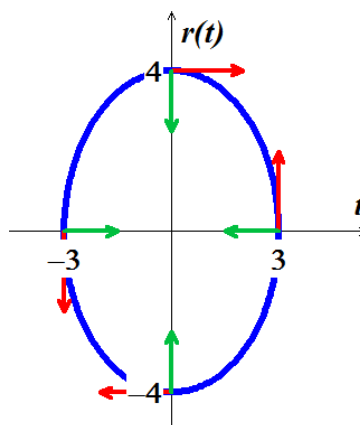
$$4\sin t = -\frac{3\sin t}{\sqrt{9+7\cos^2 t}} \cdot m \rightarrow \sin t = 0$$

$$\cos t = 0 \rightarrow \vec{r}(t) = \langle 0, 4\sin t \rangle$$

$$\text{Points are: } \underline{(0, 4) \quad (0, -4)}$$

$$\sin t = 0 \rightarrow \vec{r}(t) = \langle 3\cos t, 0 \rangle$$

$$\text{Points are: } \underline{(3, 0) \quad (-3, 0)}$$



Exercise

Find the following for all values of t for which the given curve is defined by

$$\vec{r}(t) = \langle 6\cos t, 3\sin t \rangle, \quad 0 \leq t \leq 2\pi$$

- Find the tangent vector and the unit tangent vector
- Find the curvature.
- Find the principal unit normal vector.
- Verify that $|\vec{N}| = 1$ and $\vec{T} \cdot \vec{N} = 0$
- Graph the curve and sketch \vec{T} and \vec{N} at two points.

Solution

$$a) \quad \vec{v}(t) = \langle -6\sin t, 3\cos t \rangle \quad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}(t)| = \sqrt{36\sin^2 t + 9\cos^2 t}$$

$$= 3\sqrt{4\sin^2 t + \cos^2 t}$$

$$= 3\sqrt{1 + 3\sin^2 t}$$

$$\vec{T} = \frac{\langle -6\sin t, 3\cos t \rangle}{3\sqrt{1 + 3\sin^2 t}}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{1}{\sqrt{1 + 3\sin^2 t}} \langle -2\sin t, \cos t \rangle$$

$$b) \quad \vec{r}''(t) \times \vec{r}'(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6\cos t & -3\sin t & 0 \\ -6\sin t & 3\cos t & 0 \end{vmatrix}$$

$$\begin{aligned}
&= \langle 0, 0, -18\cos^2 t - 18\sin^2 t \rangle \\
&= \langle 0, 0, -18 \rangle
\end{aligned}$$

$$\begin{aligned}
\tau &= \frac{|\langle 0, 0, -18 \rangle|}{\left(3\sqrt{1+3\sin^2 t}\right)^3} & \kappa &= \frac{|r''(t) \times \vec{r}'(t)|}{\left(|r'(t)|\right)^3} \\
&= \frac{18}{27\left(1+3\sin^2 t\right)^{3/2}} \\
&= \frac{2}{3\left(1+3\sin^2 t\right)^{3/2}}
\end{aligned}$$

$$\begin{aligned}
c) \quad \frac{d\vec{T}}{dt} &= \left\langle -2 \frac{\cos t(1+3\sin^2 t) - 3\cos t \sin^2 t}{\left(1+3\sin^2 t\right)^{3/2}}, \frac{-\sin t(1+3\sin^2 t) - 3\cos^2 t \sin t}{\left(1+3\sin^2 t\right)^{3/2}} \right\rangle \\
&= \left\langle \frac{-2\cos t}{\left(1+3\sin^2 t\right)^{3/2}}, -\frac{\sin t + 3\sin^3 t - 3\cos^2 t \sin t}{\left(1+3\sin^2 t\right)^{3/2}} \right\rangle \\
&= \left\langle \frac{-2\cos t}{\left(1+3\sin^2 t\right)^{3/2}}, -\sin t \frac{1+3(\sin^2 t + \cos^2 t)}{\left(1+3\sin^2 t\right)^{3/2}} \right\rangle \\
&= \left\langle \frac{-2\cos t}{\left(1+3\sin^2 t\right)^{3/2}}, \frac{-4\sin t}{\left(1+3\sin^2 t\right)^{3/2}} \right\rangle \\
&= \frac{-2}{\left(1+3\sin^2 t\right)^{3/2}} \langle \cos t, 2\sin t \rangle \\
\left| \frac{d\vec{T}}{dt} \right| &= \frac{2}{\left(1+3\sin^2 t\right)^{3/2}} \sqrt{\cos^2 t + 4\sin^2 t} \\
&= \frac{2}{\left(1+3\sin^2 t\right)^{3/2}} \sqrt{1+3\sin^2 t} \\
&= \frac{2}{\sqrt{1+3\sin^2 t}}
\end{aligned}$$

$$\vec{N} = \frac{\sqrt{1+3\sin^2 t}}{2} = \frac{-2}{(1+3\sin^2 t)^{3/2}} \langle \cos t, 2\sin t \rangle \quad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$= \frac{1}{\sqrt{1+3\sin^2 t}} \langle -\cos t, -2\sin t \rangle$$

d) $|\vec{N}| = \frac{1}{\sqrt{1+3\sin^2 t}} \sqrt{\cos^2 t + 4\sin^2 t}$

$$= \frac{1}{\sqrt{1+3\sin^2 t}} \sqrt{1+3\sin^2 t}$$

$$= 1 \quad \checkmark$$

$$\vec{T} \cdot \vec{N} = \frac{1}{\sqrt{1+3\sin^2 t}} \langle -2\sin t, \cos t \rangle \cdot \frac{1}{\sqrt{1+3\sin^2 t}} \langle -\cos t, -2\sin t \rangle$$

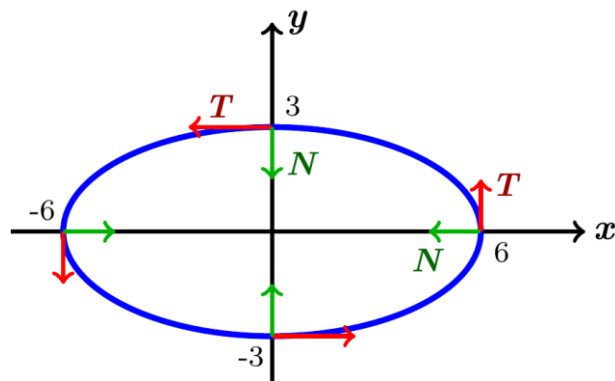
$$= \frac{2\sin t \cos t - 2\sin t \cos t}{1+3\sin^2 t}$$

$$= 0 \quad \checkmark$$

e) $x = 6\cos t \quad y = 3\sin t$

$$\cos^2 t + \sin^2 t = 1$$

$$\frac{x^2}{36} + \frac{y^2}{9} = 1$$



Exercise

Find the following for all values of t for which the given curve is defined by

$$\vec{r}(t) = (\cos t)\hat{i} + (2\sin t)\hat{j} + \hat{k}, \quad 0 \leq t \leq 2\pi$$

- Find the tangent vector and the unit tangent vector
- Find the curvature.
- Find the principal unit normal vector.
- Verify that $|\vec{N}| = 1$ and $\vec{T} \cdot \vec{N} = 0$
- Graph the curve and sketch \vec{T} and \vec{N} at two points.

Solution

$$a) \quad \vec{v}(t) = \langle -\sin t, 2\cos t, 0 \rangle$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}(t)| = \sqrt{\sin^2 t + 4\cos^2 t}$$

$$= \sqrt{1 + 3\cos^2 t}$$

$$\vec{T} = \frac{\langle -\sin t, 2\cos t, 0 \rangle}{\sqrt{1 + 3\cos^2 t}}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$b) \quad \vec{r}''(t) \times \vec{r}'(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\cos t & -2\sin t & 0 \\ -\sin t & 2\cos t & 0 \end{vmatrix}$$

$$= \langle 0, 0, -2\cos^2 t - 2\sin^2 t \rangle$$

$$= \langle 0, 0, -2 \rangle$$

$$\kappa = \frac{|\langle 0, 0, -2 \rangle|}{\left(\sqrt{1 + 3\cos^2 t}\right)^3}$$

$$= \frac{2}{\left(1 + 3\cos^2 t\right)^{3/2}}$$

$$\kappa = \frac{|\vec{r}''(t) \times \vec{r}'(t)|}{\left(|\vec{r}'(t)|\right)^3}$$

$$c) \quad \frac{d\vec{T}}{dt} = \left\langle -\frac{\cos t(1 + 3\cos^2 t) + 3\cos t \sin^2 t}{(1 + 3\cos^2 t)^{3/2}}, 2\frac{-\sin t(1 + 3\cos^2 t) + 3\cos^2 t \sin t}{(1 + 3\cos^2 t)^{3/2}}, 0 \right\rangle$$

$$= \frac{-1}{(1 + 3\cos^2 t)^{3/2}} \langle \cos t + 3\cos^3 t + 3\cos t \sin^2 t, 2\sin t, 0 \rangle$$

$$= \frac{-2}{(1+3\cos^2 t)^{3/2}} \langle 2\cos t, \sin t, 0 \rangle$$

$$\left| \frac{d\vec{T}}{dt} \right| = \frac{2}{(1+3\cos^2 t)^{3/2}} \sqrt{4\cos^2 t + \sin^2 t}$$

$$= \frac{2}{(1+3\cos^2 t)^{3/2}} \sqrt{3\cos^2 t + 1}$$

$$= \frac{2}{\sqrt{1+3\cos^2 t}}$$

$$\vec{N} = \frac{\sqrt{1+3\cos^2 t}}{2} \cdot \frac{-2}{(1+3\cos^2 t)^{3/2}} \langle 2\cos t, \sin t, 0 \rangle \quad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$= \frac{1}{\sqrt{1+3\cos^2 t}} \langle -2\cos t, -\sin t, 0 \rangle$$

$$d) \quad |\vec{N}| = \frac{1}{\sqrt{1+3\cos^2 t}} \sqrt{4\cos^2 t + \sin^2 t}$$

$$= \frac{1}{\sqrt{1+3\cos^2 t}} \sqrt{1+3\cos^2 t}$$

$$= 1 \quad \checkmark$$

$$\vec{T} \cdot \vec{N} = \frac{\langle -\sin t, 2\cos t, 0 \rangle}{\sqrt{1+3\cos^2 t}} \cdot \frac{\langle -2\cos t, -\sin t, 0 \rangle}{\sqrt{1+3\cos^2 t}}$$

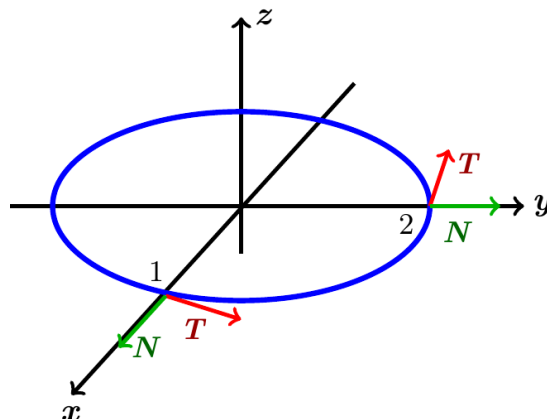
$$= \frac{2\sin t \cos t - 2\sin t \cos t}{1+3\cos^2 t}$$

$$= 0 \quad \checkmark$$

$$e) \quad x = \cos t \quad y = 2\sin t$$

$$\cos^2 t + \sin^2 t = 1$$

$$x^2 + \frac{y^2}{4} = 1$$



Exercise

Find the following for all values of t for which the given curve is defined by

$$\vec{r}(t) = t\hat{i} + (2\cos t)\hat{j} + (2\sin t)\hat{k}, \quad 0 \leq t \leq 2\pi$$

- Find the tangent vector and the unit tangent vector
- Find the curvature.
- Find the principal unit normal vector.
- Verify that $|\vec{N}| = 1$ and $\vec{T} \cdot \vec{N} = 0$
- Graph the curve and sketch \vec{T} and \vec{N} at two points.

Solution

$$a) \quad \vec{v}(t) = \langle 1, -2\sin t, 2\cos t \rangle \quad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}(t)| = \sqrt{1 + 4\sin^2 t + 4\cos^2 t} \\ = \sqrt{5}$$

$$\vec{T} = \frac{1}{\sqrt{5}} \langle 1, -2\sin t, 2\cos t \rangle \quad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$b) \quad \vec{r}''(t) \times \vec{r}'(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2\cos t & -2\sin t \\ 1 & -2\sin t & 2\cos t \end{vmatrix} \\ = \langle -4, -2\sin t, 2\cos t \rangle$$

$$\tau = \frac{|\langle -4, -2\sin t, 2\cos t \rangle|}{(\sqrt{5})^3} \quad \kappa = \frac{|\vec{r}''(t) \times \vec{r}'(t)|}{(|\vec{r}'(t)|)^3} \\ = \frac{\sqrt{16 + 4\sin^2 t + 4\cos^2 t}}{5\sqrt{5}} \\ = \frac{\sqrt{20}}{5\sqrt{5}} \\ = \frac{2}{5}$$

$$c) \quad \frac{d\vec{T}}{dt} = \frac{1}{\sqrt{5}} \langle 0, -2\cos t, -2\sin t \rangle$$

$$\left| \frac{d\vec{T}}{dt} \right| = \frac{1}{\sqrt{5}} \sqrt{4\cos^2 t + 4\sin^2 t} \\ = \frac{2}{\sqrt{5}}$$

$$\vec{N} = \frac{\sqrt{5}}{2} \cdot \frac{1}{\sqrt{5}} \langle 0, -2\cos t, -2\sin t \rangle$$

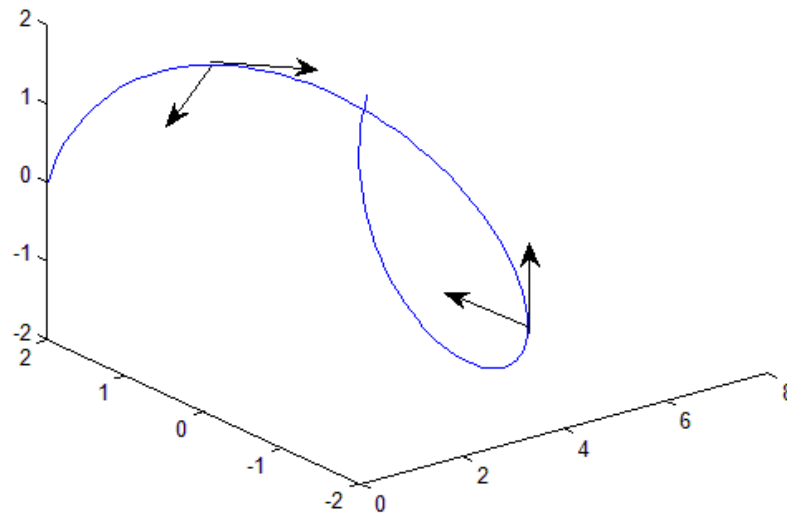
$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$= \langle 0, -\cos t, -\sin t \rangle$$

$$d) \quad |\vec{N}| = \sqrt{\cos^2 t + \sin^2 t} \\ = 1 \quad \checkmark$$

$$\begin{aligned} \vec{T} \cdot \vec{N} &= \frac{1}{\sqrt{5}} \langle 1, -2\sin t, 2\cos t \rangle \cdot \langle 0, -\cos t, -\sin t \rangle \\ &= \frac{1}{\sqrt{5}} (2\sin t \cos t - 2\sin t \cos t) \\ &= 0 \quad \checkmark \end{aligned}$$

e)



Exercise

Find the following for all values of t for which the given curve is defined by

$$\vec{r}(t) = (\cos t)\hat{i} + (2\cos t)\hat{j} + (\sqrt{5}\sin t)\hat{k}, \quad 0 \leq t \leq 2\pi$$

- Find the tangent vector and the unit tangent vector
- Find the curvature.
- Find the principal unit normal vector.
- Verify that $|\vec{N}| = 1$ and $\vec{T} \cdot \vec{N} = 0$
- Graph the curve and sketch \vec{T} and \vec{N} at two points.

Solution

$$a) \quad \vec{v}(t) = \langle -\sin t, -2\sin t, \sqrt{5}\cos t \rangle \quad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned} |\vec{v}(t)| &= \sqrt{\sin^2 t + 4\sin^2 t + 5\cos^2 t} \\ &= \sqrt{5\sin^2 t + 5\cos^2 t} \end{aligned}$$

$$= \sqrt{5}$$

$$\vec{T} = \frac{1}{\sqrt{5}} \langle -\sin t, -2 \sin t, \sqrt{5} \cos t \rangle \quad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$b) \quad \vec{r}''(t) = \langle -\cos t, -2 \cos t, -\sqrt{5} \sin t \rangle$$

$$\begin{aligned} \vec{r}''(t) \times \vec{r}'(t) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\cos t & -2 \cos t & -\sqrt{5} \sin t \\ -\sin t & -2 \sin t & \sqrt{5} \cos t \end{vmatrix} \\ &= \left(-2\sqrt{5} \cos^2 t - 2\sqrt{5} \sin^2 t \right) \hat{i} - \left(-\sqrt{5} \cos^2 t - \sqrt{5} \sin^2 t \right) \hat{j} \\ &\quad + \left(2 \sin t \cos t - 2 \sin t \cos t \right) \hat{k} \\ &= -2\sqrt{5} \hat{i} + \sqrt{5} \hat{j} \end{aligned}$$

$$\begin{aligned} |\vec{r}''(t) \times \vec{r}'(t)| &= \sqrt{20 + 5} \\ &= 5 \end{aligned}$$

$$\begin{aligned} \kappa &= \frac{5}{(\sqrt{5})^3} \\ &= \frac{1}{\sqrt{5}} \end{aligned} \quad \kappa = \frac{|\vec{r}''(t) \times \vec{r}'(t)|}{(|\vec{r}'(t)|)^3}$$

$$c) \quad \frac{d\vec{T}}{dt} = \frac{1}{\sqrt{5}} \langle -\cos t, -2 \cos t, -\sqrt{5} \sin t \rangle$$

$$\begin{aligned} \left| \frac{d\vec{T}}{dt} \right| &= \frac{1}{\sqrt{5}} \sqrt{\cos^2 t + 4 \cos^2 t + 5 \sin^2 t} \\ &= \frac{1}{\sqrt{5}} \sqrt{5 \cos^2 t + 5 \sin^2 t} \\ &= \frac{\sqrt{5}}{\sqrt{5}} \\ &= 1 \end{aligned}$$

$$\vec{N} = \frac{1}{\sqrt{5}} \langle -\cos t, -2 \cos t, -\sqrt{5} \sin t \rangle \quad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\begin{aligned} d) \quad |\vec{N}| &= \frac{1}{\sqrt{5}} \sqrt{\cos^2 t + 4 \cos^2 t + 5 \sin^2 t} \\ &= \frac{1}{\sqrt{5}} \sqrt{5} \\ &= 1 \end{aligned} \quad \checkmark$$

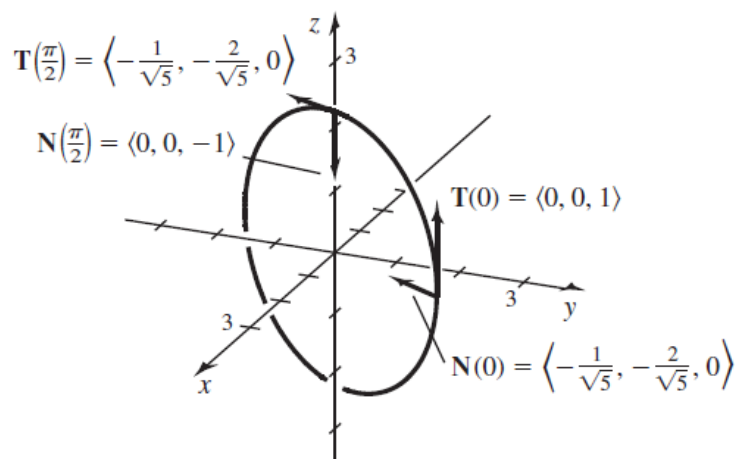
$$\begin{aligned}\vec{T} \cdot \vec{N} &= \frac{\langle -\sin t, -2\sin t, \sqrt{5}\cos t \rangle}{\sqrt{5}} \cdot \frac{\langle -\cos t, -2\cos t, -\sqrt{5}\sin t \rangle}{\sqrt{5}} \\ &= \frac{\sin t \cos t + 4\sin t \cos t - 5\cos t \sin t}{5} \\ &= 0 \quad \checkmark\end{aligned}$$

$$e) \quad \vec{r}(t) = (\cos t)\hat{i} + (2\cos t)\hat{j} + (\sqrt{5}\sin t)\hat{k}$$

$$\vec{T} = \frac{1}{\sqrt{5}} \langle -\sin t, -2\sin t, \sqrt{5}\cos t \rangle$$

$$\vec{N} = \frac{1}{\sqrt{5}} \langle -\cos t, -2\cos t, -\sqrt{5}\sin t \rangle$$

t	\vec{r}	\vec{T}	\vec{N}
0	(1, 1, 0)	(0, 0, 1)	$\left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0\right)$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, \sqrt{2}, \sqrt{\frac{5}{2}}\right)$		
$\frac{\pi}{2}$	(0, 0, $\sqrt{5}$)	$\left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0\right)$	(0, 0, -1)
π	(-1, -1, 0)		
$\frac{3\pi}{2}$	(0, 0, $-\sqrt{5}$)		
2π	(1, 1, 0)		



Exercise

Find equations for the osculating, normal and rectifying planes of the curve $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$ at the point (1, 1, 1).

Solution

$$\vec{v}(t) = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}(t)| = \sqrt{1 + 4t^2 + 9t^4}$$

$$|\vec{v}(1)| = \sqrt{1 + 4 + 9} \\ = \sqrt{14}$$

$$\vec{T}(t) = \frac{1}{\sqrt{14}}(\hat{i} + 2t\hat{j} + 3t^2\hat{k})$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{T}(1) = \frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k} \quad (\text{Normal to the normal plane}).$$

$$\frac{1}{\sqrt{14}}(x-1) + \frac{2}{\sqrt{14}}(y-1) + \frac{3}{\sqrt{14}}(z-1) = 0$$

$$x-1 + 2(y-1) + 3(z-1) = 0$$

$$\underline{x + 2y + 3z = 6} \quad (\text{equation of the normal plane}).$$

$$\vec{a}(t) = 2\hat{j} + 6t\hat{k}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$\vec{a}(1) = 2\hat{j} + 6\hat{k}$$

$$(\vec{v} \times \vec{a})(1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} \\ = 6\hat{i} - 6\hat{j} + 2\hat{k}$$

$$|\vec{v} \times \vec{a}| = \sqrt{36 + 36 + 4} \\ = \sqrt{76}$$

$$\kappa = \frac{\sqrt{76}}{(\sqrt{14})^3} \\ = \frac{2\sqrt{19}}{14\sqrt{14}} \\ = \frac{\sqrt{19}}{7\sqrt{14}} \quad \Bigg|$$

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{(|\vec{v}|)^3}$$

$$\frac{ds}{dt} = |\vec{v}(t)| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\frac{ds}{dt}(1) = \sqrt{14}$$

$$\frac{d^2s}{dt^2} = \frac{4t + 18t^3}{\sqrt{1 + 4t^2 + 9t^4}} \Bigg|_{t=1} \\ = \frac{22}{\sqrt{14}}$$

$$\vec{a} = \frac{d^2 s}{dt^2} \vec{T} + \kappa \left(\frac{ds}{dt} \right)^2 \vec{N}$$

$$2\hat{j} + 6\hat{k} = \frac{22}{\sqrt{14}} \left(\frac{1}{\sqrt{14}} \hat{i} + \frac{2}{\sqrt{14}} \hat{j} + \frac{3}{\sqrt{14}} \hat{k} \right) + \frac{\sqrt{19}}{7\sqrt{14}} (\sqrt{14})^2 \vec{N}$$

$$2\hat{j} + 6\hat{k} = \frac{11}{7} (\hat{i} + 2\hat{j} + 3\hat{k}) + \frac{2\sqrt{19}}{\sqrt{14}} \vec{N}$$

$$\frac{2\sqrt{19}}{\sqrt{14}} \vec{N} = 2\hat{j} + 6\hat{k} - \frac{11}{7} \hat{i} - \frac{22}{7} \hat{j} - \frac{33}{7} \hat{k}$$

$$\frac{2\sqrt{19}}{\sqrt{14}} \vec{N} = -\frac{11}{7} \hat{i} - \frac{8}{7} \hat{j} + \frac{9}{7} \hat{k}$$

$$\vec{N} = \frac{\sqrt{14}}{2\sqrt{19}} \left(-\frac{11}{7} \hat{i} - \frac{8}{7} \hat{j} + \frac{9}{7} \hat{k} \right)$$

$$-\frac{11}{7}(x-1) - \frac{8}{7}(y-1) + \frac{9}{7}(z-1) = 0$$

$$-11x + 11 - 8y + 8 + 9z - 9 = 0$$

$$\underline{11x + 8y - 9z = 10}$$

$$\vec{B}(1) = \vec{T}(1) \times \vec{N}(1)$$

$$= \frac{1}{\sqrt{14}} \cdot \frac{\sqrt{14}}{2\sqrt{19}} \cdot \frac{1}{7} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -11 & -8 & 9 \end{vmatrix}$$

$$= \frac{1}{14\sqrt{19}} (42\hat{i} - 42\hat{j} + 14\hat{k})$$

$$= \frac{1}{\sqrt{19}} (3\hat{i} - 3\hat{j} + \hat{k})$$

$$3(x-1) - 3(y-1) + (z-1) = 0$$

$$3x - 3 - 3y + 3 + z - 1 = 0$$

$$\underline{3x - 3y + z = 1}$$

Exercise

Consider the position vector $\vec{r}(t) = (t^2 + 1)\hat{i} + (2t)\hat{j}$, $t \geq 0$ of the moving objects

- Find the normal and tangential components of the acceleration.
- Graph the trajectory and sketch the normal and tangential components of the acceleration at two points on the trajectory. Show that their sum gives the total acceleration.

Solution

$$a) \quad \vec{v}(t) = 2t\hat{i} + 2\hat{j}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned}
|\vec{v}(t)| &= \sqrt{4t^2 + 4} \\
&= 2\sqrt{t^2 + 1}
\end{aligned}$$

$$\vec{T}(t) = \frac{2t\hat{i} + 2\hat{j}}{2\sqrt{t^2 + 1}}$$

$$= \frac{t}{\sqrt{t^2 + 1}} \hat{i} + \frac{1}{\sqrt{t^2 + 1}} \hat{j}$$

$$\frac{d\vec{T}}{dt} = \frac{t^2 + 1 - t^2}{(t^2 + 1)^{3/2}} \hat{i} - \frac{t}{(t^2 + 1)^{3/2}} \hat{j}$$

$$= \frac{1}{(t^2 + 1)^{3/2}} \hat{i} - \frac{t}{(t^2 + 1)^{3/2}} \hat{j}$$

$$\left| \frac{d\vec{T}}{dt} \right| = \sqrt{\frac{1}{(t^2 + 1)^3} + \frac{t^2}{(t^2 + 1)^3}}$$

$$= \sqrt{\frac{1 + t^2}{(t^2 + 1)^3}}$$

$$= \frac{1}{t^2 + 1}$$

$$\vec{N} = (t^2 + 1) \left(\frac{1}{(t^2 + 1)^{3/2}} \hat{i} - \frac{t}{(t^2 + 1)^{3/2}} \hat{j} \right)$$

$$= \frac{1}{\sqrt{t^2 + 1}} \hat{i} - \frac{t}{\sqrt{t^2 + 1}} \hat{j}$$

$$\vec{a}(t) = 2\hat{i}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$a_T = \frac{2t}{\sqrt{t^2 + 1}}$$

$$a_T = \frac{d}{dt} |\vec{v}|$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t & 2 & 0 \\ 2 & 0 & 0 \end{vmatrix}$$

$$= -4\hat{k}$$

$$|\vec{v} \times \vec{a}| = 4$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\kappa = \frac{4}{8\left(\sqrt{t^2+1}\right)^3}$$

$$= \frac{1}{2\left(t^2+1\right)^{3/2}}$$

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{(|\vec{v}|)^3}$$

$$a_N = \frac{4}{2\sqrt{t^2+1}}$$

$$= \frac{2}{\sqrt{t^2+1}}$$

$$a_N = \kappa |\vec{v}|^2 = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$$

$$\vec{a} = \frac{2t}{\sqrt{t^2+1}} \vec{T} + \frac{2}{\sqrt{t^2+1}} \vec{N}$$

b) $x = t^2 + 1 \quad y = 2t$

$$t = \frac{1}{2}y \rightarrow x = \frac{1}{4}y^2 + 1$$

At $t = 1$

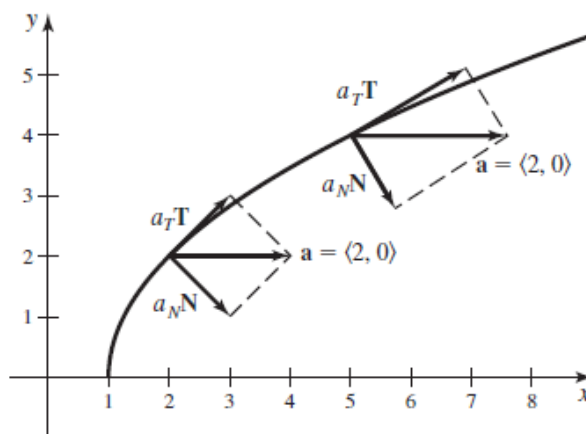
$$\begin{aligned} \vec{a} &= \frac{2}{\sqrt{2}} \vec{T} + \frac{2}{\sqrt{2}} \vec{N} \\ &= \frac{2}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \hat{i} + \frac{1}{\sqrt{2}} \hat{j} \right) + \frac{2}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \hat{i} - \frac{1}{\sqrt{2}} \hat{j} \right) \\ &= \hat{i} + \hat{j} + \hat{i} - \hat{j} \\ &= 2\hat{i} \\ &= \langle 2, 0 \rangle \end{aligned}$$

At $t = 2$

$$\begin{aligned} \vec{a} &= \frac{4}{\sqrt{5}} \left(\frac{2}{\sqrt{5}} \hat{i} + \frac{1}{\sqrt{5}} \hat{j} \right) + \frac{2}{\sqrt{5}} \left(\frac{1}{\sqrt{5}} \hat{i} - \frac{2}{\sqrt{5}} \hat{j} \right) \\ &= \frac{8}{5} \hat{i} + \frac{4}{5} \hat{j} + \frac{2}{5} \hat{i} - \frac{4}{5} \hat{j} \\ &= 2\hat{i} = \langle 2, 0 \rangle \end{aligned}$$

At $t = 5$

$$\begin{aligned} \vec{a} &= \frac{10}{\sqrt{26}} \left(\frac{5}{\sqrt{26}} \hat{i} + \frac{1}{\sqrt{26}} \hat{j} \right) + \frac{2}{\sqrt{26}} \left(\frac{1}{\sqrt{26}} \hat{i} - \frac{5}{\sqrt{26}} \hat{j} \right) \\ &= \frac{50}{26} \hat{i} + \frac{10}{26} \hat{j} + \frac{2}{26} \hat{i} - \frac{10}{26} \hat{j} \\ &= \frac{52}{26} \hat{i} \\ &= 2\hat{i} = \langle 2, 0 \rangle \end{aligned}$$



Exercise

Consider the position vector $\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j}$, $0 \leq t \leq 2\pi$ of the moving objects

- Find the normal and tangential components of the acceleration.
- Graph the trajectory and sketch the normal and tangential components of the acceleration at two points on the trajectory. Show that their sum gives the total acceleration.

Solution

$$a) \quad \vec{v}(t) = -2\sin t \hat{i} + 2\cos t \hat{j}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\vec{a}(t) = -2\cos t \hat{i} - 2\sin t \hat{j}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$|\vec{v}| = \sqrt{4\cos^2 t + 4\sin^2 t}$$

$$= \sqrt{2}$$

$$a_T = \frac{d|\vec{v}|}{dt} = 0$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin t & 2\cos t & 0 \\ -2\cos t & -2\sin t & 0 \end{vmatrix}$$

$$= (4\sin^2 t + 4\cos^2 t)\hat{k}$$

$$= 4\hat{k}$$

$$a_N = \frac{4}{2} = 2$$

$$a_N = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$$

$$\vec{a} = 0\vec{T} + 2\vec{N}$$

$$b) \quad t = 0 \rightarrow \vec{a} = -2\hat{i}$$

$$= 2\langle -1, 0 \rangle$$

$$= 2\vec{N}$$

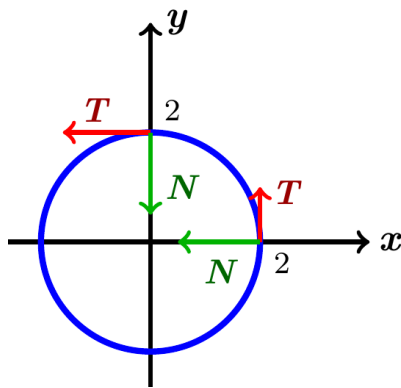
$$t = \frac{\pi}{2} \rightarrow \vec{a} = \langle 0, -2 \rangle$$

$$= 2\langle 0, -1 \rangle$$

$$= 2\vec{N}$$

$$x = 2\cos t \quad y = 2\sin t$$

$$x^2 + y^2 = 4$$



Exercise

Consider the position vector $\vec{r}(t) = 3t\hat{i} + (4-t)\hat{j} + t\hat{k}$, $t \geq 0$ of the moving objects

Find the normal and tangential components of the acceleration.

Solution

$$\begin{aligned} a) \quad \vec{v}(t) &= 3\hat{i} - \hat{j} + \hat{k} & \vec{v}(t) &= \frac{d\vec{r}}{dt} \\ \vec{a}(t) &= \vec{0} & \vec{a}(t) &= \frac{d\vec{v}}{dt} \\ |\vec{v}| &= \sqrt{9+1+1} \\ &= \sqrt{11} \\ a_T &= \frac{d|\vec{v}|}{dt} = 0 \\ a_N &= 0 \\ \vec{a} &= 0\vec{T} + 0\vec{N} \end{aligned}$$

Exercise

Consider the position vector $\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j} + (10t)\hat{k}$, $0 \leq t \leq 2\pi$ of the moving objects

- Find the normal and tangential components of the acceleration.
- Graph the trajectory and sketch the normal and tangential components of the acceleration at two points on the trajectory. Show that their sum gives the total acceleration.

Solution

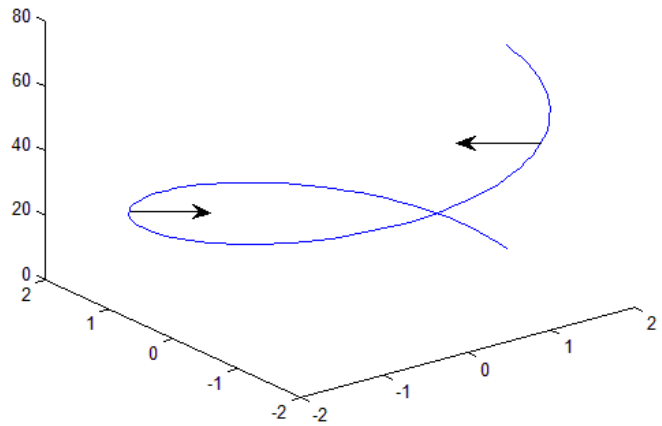
$$\begin{aligned} a) \quad \vec{v}(t) &= -2\sin t\hat{i} + 2\cos t\hat{j} + 10\hat{k} & \vec{v}(t) &= \frac{d\vec{r}}{dt} \\ \vec{a}(t) &= -2\cos t\hat{i} - 2\sin t\hat{j} & \vec{a}(t) &= \frac{d\vec{v}}{dt} \\ |\vec{v}| &= \sqrt{4\sin^2 t + 4\cos^2 t + 100} \\ &= 2\sqrt{26} \\ a_T &= \frac{d|\vec{v}|}{dt} = 0 \\ \vec{v} \times \vec{a} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin t & 2\cos t & 10 \\ -2\cos t & -2\sin t & 0 \end{vmatrix} \\ &= -20\sin t\hat{i} + 20\cos t\hat{j} - 4\hat{k} \\ a_N &= \frac{\sqrt{400\sin^2 t + 400\cos^2 t + 16}}{2\sqrt{26}} & a_N &= \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|} \end{aligned}$$

$$\begin{aligned}
 &= \frac{\sqrt{416}}{2\sqrt{26}} \\
 &= \frac{4\sqrt{26}}{2\sqrt{26}} \\
 &= 2
 \end{aligned}$$

$$\vec{a} = 0\vec{T} + 2\vec{N}$$

$$\begin{aligned}
 b) \quad t=0 &\rightarrow \vec{a} = \langle -2, 0, 0 \rangle \\
 &= 2\langle -1, 0, 0 \rangle \\
 &2\vec{N}
 \end{aligned}$$

$$\begin{aligned}
 t = \frac{\pi}{2} &\rightarrow \vec{a} = \langle 0, -2, 0 \rangle \\
 &= 2\langle 0, -1, 0 \rangle \\
 &2\vec{N}
 \end{aligned}$$



Exercise

Compute the unit binormal vector \mathbf{B} and the torsion of the curve $\vec{r}(t) = \langle t, t^2, t^3 \rangle$, at $t = 1$

Solution

$$\vec{v}(t) = \langle 1, 2t, 3t^2 \rangle \quad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\vec{v}(t=1) = \langle 1, 2, 3 \rangle$$

$$|\vec{v}(t)| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\begin{aligned}
 |\vec{v}(t=1)| &= \sqrt{1 + 4 + 9} \\
 &= \sqrt{14}
 \end{aligned}$$

$$\vec{T} = \frac{1}{\sqrt{1 + 4t^2 + 9t^4}} \langle 1, 2t, 3t^2 \rangle \quad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{T}(t=1) = \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle$$

$$\begin{aligned}
 \frac{d\vec{T}}{dt} &= \frac{1}{(1 + 4t^2 + 9t^4)^{3/2}} \langle -4t - 18t^3, 2 + 8t^2 + 18t^4 - 8t^2 - 36t^4, 3t(2 + 8t^2 + 18t^4 - 4t^2 - 18t^4) \rangle \\
 &= \frac{1}{(1 + 4t^2 + 9t^4)^{3/2}} \langle -4t - 18t^3, 2 - 18t^4, 6t + 12t^3 \rangle
 \end{aligned}$$

$$= \frac{2}{(1+4t^2+9t^4)^{3/2}} \langle -2t-9t^3, 1-9t^4, 3t+6t^3 \rangle$$

$$\begin{aligned} \left| \frac{d\vec{T}}{dt} \right| &= \frac{2}{(1+4t^2+9t^4)^{3/2}} \sqrt{(-2t-9t^3)^2 + (1-9t^4)^2 + (3t+6t^3)^2} \\ &= \frac{2}{(1+4t^2+9t^4)^{3/2}} \sqrt{4t^2+36t^4+81t^6+1-18t^4+81t^8+9t^2+36t^4+36t^6} \\ &= \frac{2}{(1+4t^2+9t^4)^{3/2}} \sqrt{1+13t^2+54t^4+117t^6+81t^8} \end{aligned}$$

$$\begin{aligned} \vec{N} &= \frac{(1+4t^2+9t^4)^{3/2}}{2\sqrt{1+13t^2+54t^4+117t^6+81t^8}} \frac{2}{(1+4t^2+9t^4)^{3/2}} \langle -2t-9t^3, 1-9t^4, 3t+6t^3 \rangle \quad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} \\ &= \frac{1}{\sqrt{1+13t^2+54t^4+117t^6+81t^8}} \langle -2t-9t^3, 1-9t^4, 3t+6t^3 \rangle \end{aligned}$$

$$\begin{aligned} \vec{N}(t=1) &= \frac{1}{\sqrt{1+13+54+117+81}} \langle -11, -8, 9 \rangle \\ &= \frac{1}{\sqrt{266}} \langle -11, -8, 9 \rangle \end{aligned}$$

$$\begin{aligned} \vec{B} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ \frac{-11}{\sqrt{266}} & \frac{-8}{\sqrt{266}} & \frac{9}{\sqrt{266}} \end{vmatrix} \quad \vec{B} = \vec{T} \times \vec{N} \\ &= \left\langle \frac{42}{14\sqrt{19}}, \frac{-42}{14\sqrt{19}}, \frac{14}{14\sqrt{19}} \right\rangle \\ &= \left\langle \frac{3}{\sqrt{19}}, \frac{-3}{\sqrt{19}}, \frac{1}{\sqrt{19}} \right\rangle \end{aligned}$$

$$\vec{r}''(t) = \langle 0, 2, 6t \rangle$$

$$\begin{aligned} \vec{r}'' \times \vec{r}' &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 6t \\ 1 & 2t & 3t^2 \end{vmatrix} \\ &= \langle -6t^2, 6t, -2 \rangle \end{aligned}$$

$$|\vec{r}'' \times \vec{r}'| = \sqrt{36t^4 + 36t^2 + 4} \Big|_{t=1}$$

$$= \sqrt{76}$$

$$\vec{r}'''(t) = \langle 0, 0, 6 \rangle$$

$$\tau = \frac{1}{76} \begin{vmatrix} 1 & 2t & 3t^2 \\ 0 & 2 & 6t \\ 0 & 0 & 6 \end{vmatrix}$$

$$= \frac{12}{76}$$

$$= \frac{3}{19}$$

$$\tau = \frac{1}{|\vec{r}' \times \vec{r}''|^2} \begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix}$$

Exercise

At point P , the velocity and acceleration of a particle moving in the plane are $\vec{v} = 3\hat{i} + 4\hat{j}$ and $\vec{a} = 5\hat{i} + 15\hat{j}$. Find the curvature of the particle's path at P .

Solution

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ 5 & 15 & 0 \end{vmatrix}$$

$$= 25\hat{k}$$

$$|\vec{v} \times \vec{a}| = 25$$

$$|\vec{v}| = \sqrt{9+16}$$

$$= 5$$

$$\kappa = \frac{25}{5^3}$$

$$= \frac{1}{5}$$

Exercise

Consider the curve $C: \vec{r}(t) = \langle 3\sin t, 4\sin t, 5\cos t \rangle$, for $0 \leq t \leq 2\pi$

- Find $\mathbf{T}(t)$ at all points of C .
- Find $\mathbf{N}(t)$ and the curvature at all points of C .
- Sketch the curve and show $\mathbf{T}(t)$ and $\mathbf{N}(t)$ at the points of C corresponding to $t = 0$ and $t = \frac{\pi}{2}$.
- Are the results of parts (a) and (b) consistent with the graph?

- e) Find $\mathbf{B}(t)$ at all points of C .
- f) Describe three calculations that serve to check the accuracy of your results in part (a) – (f).
- g) Compute the torsion at all points of C . Interpret this result.

Solution

a) $\vec{v}(t) = \langle 3\cos t, 4\cos t, -5\sin t \rangle \quad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$$\begin{aligned} |\vec{v}(t)| &= \sqrt{9\cos^2 t + 16\cos^2 t + 25\sin^2 t} \\ &= \sqrt{25\cos^2 t + 25\sin^2 t} \\ &= 5 \end{aligned}$$

$$\vec{T} = \frac{1}{5} \langle 3\cos t, 4\cos t, -5\sin t \rangle \quad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

b) $\frac{d\vec{T}}{dt} = \frac{1}{5} \langle -3\sin t, -4\sin t, -5\cos t \rangle$

$$\begin{aligned} \left| \frac{d\vec{T}}{dt} \right| &= \frac{1}{5} \sqrt{9\sin^2 t + 16\sin^2 t + 25\cos^2 t} \\ &= 1 \end{aligned}$$

$$\vec{N} = \frac{1}{5} \langle -3\sin t, -4\sin t, -5\cos t \rangle \quad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\kappa = \frac{1}{25} \sqrt{9\sin^2 t + 16\sin^2 t + 25\cos^2 t} \quad \kappa = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$$

c) At $t = 0 \rightarrow \vec{T} = \langle \frac{3}{5}, \frac{4}{5}, 0 \rangle \quad \vec{N} = \langle 0, 0, -1 \rangle$

$$t = \frac{\pi}{2} \rightarrow \vec{T} = \langle 0, 0, -1 \rangle \quad \vec{N} = \langle -\frac{3}{5}, -\frac{4}{5}, 0 \rangle$$

$$9\sin^2 t + 16\sin^2 t + 25\cos^2 t = 25$$

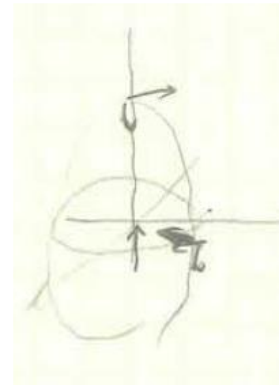
$$x^2 + y^2 + z^2 = 25$$

- d) Yes; the results of parts (a) and (b) consistent with the graph

e) $\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{3}{5}\cos t & \frac{4}{5}\cos t & -\sin t \\ -\frac{3}{5}\sin t & -\frac{4}{5}\sin t & -\cos t \end{vmatrix} \quad \vec{B} = \vec{T} \times \vec{N}$

$$\begin{aligned} &= \left\langle -\frac{4}{5}\cos^2 t - \frac{4}{5}\sin^2 t, \frac{3}{5}\sin^2 t + \frac{3}{5}\cos^2 t, 0 \right\rangle \\ &= \left\langle -\frac{4}{5}, \frac{3}{5}, 0 \right\rangle \end{aligned}$$

f) $|\vec{T}| = \frac{1}{5} \sqrt{9\cos^2 t + 16\cos^2 t + 25\sin^2 t}$



$$= \frac{1}{5} \sqrt{25 \cos^2 t + 25 \sin^2 t}$$

$$= \frac{1}{5} \sqrt{25}$$

$$= \underline{1} \quad \checkmark$$

$$|\vec{N}| = \frac{1}{5} \sqrt{9 \sin^2 t + 16 \sin^2 t + 25 \cos^2 t}$$

$$= \frac{1}{5} \sqrt{25 \cos^2 t + 25 \sin^2 t}$$

$$= \underline{1} \quad \checkmark$$

$$|\vec{B}| = \sqrt{\frac{16}{25} + \frac{9}{25}}$$

$$= \sqrt{\frac{25}{25}}$$

$$= \underline{1} \quad \checkmark$$

$$\vec{T} \cdot \vec{N} = \frac{1}{5} \langle 3 \cos t, 4 \cos t, -5 \sin t \rangle \cdot \frac{1}{5} \langle -3 \sin t, -4 \sin t, -5 \cos t \rangle$$

$$= \frac{1}{25} (-9 \cos t \sin t - 16 \cos t \sin t + 25 \cos t \sin t)$$

$$= \underline{0} \quad \checkmark$$

$$\vec{T} \cdot \vec{B} = \frac{1}{5} \langle 3 \cos t, 4 \cos t, -5 \sin t \rangle \cdot \left\langle -\frac{4}{5}, \frac{3}{5}, 0 \right\rangle$$

$$= \frac{1}{25} (-12 \cos t + 12 \cos t)$$

$$= \underline{0} \quad \checkmark$$

$$\vec{B} \cdot \vec{N} = \left\langle -\frac{4}{5}, \frac{3}{5}, 0 \right\rangle \cdot \frac{1}{5} \langle -3 \sin t, -4 \sin t, -5 \cos t \rangle$$

$$= \frac{1}{25} (12 \sin t - 12 \sin t + 0)$$

$$= \underline{0} \quad \checkmark$$

g) Since \vec{B} is constant, then $\tau = \underline{0}$

Exercise

Consider the curve $C: \vec{r}(t) = \langle 3 \sin t, 3 \cos t, 4t \rangle$, for $0 \leq t \leq 2\pi$

a) Find $\vec{T}(t)$ at all points of C .

b) Find $\vec{N}(t)$ and the curvature at all points of C .

c) Sketch the curve and show $\vec{T}(t)$ and $\vec{N}(t)$ at the points of C corresponding to $t = 0$ and $t = \frac{\pi}{2}$.

d) Are the results of parts (a) and (b) consistent with the graph?

- e) Find $\mathbf{B}(t)$ at all points of C .
- f) Describe three calculations that serve to check the accuracy of your results in part (a) – (f).
- g) Compute the torsion at all points of C . Interpret this result.

Solution

a) $\vec{v}(t) = \langle 3\cos t, -3\sin t, 4 \rangle \quad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$$\begin{aligned} |\vec{v}(t)| &= \sqrt{9\cos^2 t + 9\sin^2 t + 16} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

$$\vec{T} = \frac{1}{5} \langle 3\cos t, -3\sin t, 4 \rangle$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

b) $\frac{d\vec{T}}{dt} = \frac{1}{5} \langle -3\sin t, -3\cos t, 0 \rangle$

$$\begin{aligned} \left| \frac{d\vec{T}}{dt} \right| &= \frac{1}{5} \sqrt{9\sin^2 t + 9\cos^2 t} \\ &= \frac{3}{5} \end{aligned}$$

$$\vec{N} = \langle -\sin t, -\cos t, 0 \rangle$$

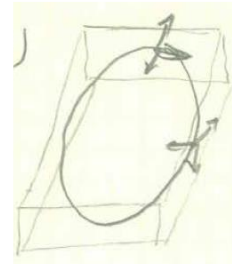
$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\kappa = \frac{3}{25}$$

$$\kappa = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$$

c) At $t = 0 \rightarrow \vec{T} = \langle \frac{3}{5}, 0, \frac{4}{5} \rangle \quad \vec{N} = \langle 0, -1, 0 \rangle$

$$t = \frac{\pi}{2} \rightarrow \vec{T} = \langle 0, -\frac{3}{5}, \frac{4}{5} \rangle \quad \vec{N} = \langle -1, 0, 0 \rangle$$



d) Yes; the results of parts (a) and (b) consistent with the graph

e) $\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{3}{5}\cos t & -\frac{3}{5}\sin t & \frac{4}{5} \\ -\sin t & -\cos t & 0 \end{vmatrix}$

$$\vec{B} = \vec{T} \times \vec{N}$$

$$= \left\langle \frac{4}{5}\cos t, -\frac{4}{5}\sin t, -\frac{3}{5} \right\rangle$$

f) $|\vec{T}| = \frac{1}{5} \sqrt{9\cos^2 t + 9\sin^2 t + 16}$

$$= \frac{1}{5} \sqrt{9 + 16}$$

$$= \frac{1}{5} \sqrt{25}$$

$$\underline{=1} \quad \checkmark$$

$$|\vec{N}| = \sqrt{\sin^2 t + \cos^2 t}$$

$$\underline{=1} \quad \checkmark$$

$$|\vec{B}| = \sqrt{\frac{16}{25}\cos^2 t + \frac{16}{25}\sin^2 t + \frac{9}{25}}$$

$$= \sqrt{\frac{16}{25} + \frac{9}{25}}$$

$$\underline{=1} \quad \checkmark$$

$$\vec{T} \cdot \vec{N} = \frac{1}{5} \langle 3\cos t, -3\sin t, 4 \rangle \cdot \langle -\sin t, -\cos t, 0 \rangle$$

$$= \frac{1}{5} (-3\cos t \sin t + 3\cos t \sin t + 0)$$

$$\underline{=0} \quad \checkmark$$

$$\vec{T} \cdot \vec{B} = \frac{1}{25} \langle 3\cos t, -3\sin t, 4 \rangle \cdot \langle 4\cos t, -4\sin t, -3 \rangle$$

$$= \frac{1}{25} (12\cos^2 t + 12\sin^2 t - 12)$$

$$= \frac{1}{25} (12 - 12)$$

$$\underline{=0} \quad \checkmark$$

$$\vec{B} \cdot \vec{N} = \frac{1}{5} \langle 4\cos t, -4\sin t, -3 \rangle \cdot \langle -\sin t, -\cos t, 0 \rangle$$

$$= \frac{1}{5} (-4\sin t \cos t + 4\sin t \cos t + 0)$$

$$\underline{=0} \quad \checkmark$$

$$g) \quad \frac{d\vec{B}}{dt} = \frac{1}{5} \langle -4\sin t, -4\cos t, 0 \rangle$$

$$\tau = \frac{1}{5} \langle -4\sin t, -4\cos t, 0 \rangle \cdot \langle -\sin t, -\cos t, 0 \rangle$$

$$= \frac{1}{5} (4\sin^2 t + 4\cos^2 t + 0)$$

$$\underline{= \frac{4}{5}}$$

$$\tau = \frac{d\vec{B}}{dt} \cdot \vec{N}$$

Exercise

Suppose $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f, g , and h are the quadratic functions $f(t) = a_1 t^2 + b_1 t + c_1$, $g(t) = a_2 t^2 + b_2 t + c_2$, and $h(t) = a_3 t^2 + b_3 t + c_3$, and where at least one of the leading coefficients a_1, a_2 , or a_3 is nonzero. Apart from a set of degenerate cases (for example $\mathbf{r}(t) = \langle t^2, t^2, t^2 \rangle$, whose graph is a line), it can be shown that the graph of $\mathbf{r}(t)$ is a parabola that lies in a plane

- Show by direct computation that $\mathbf{v} \times \mathbf{a}$ is constant. Then explain why the unit binormal vector is constant at all points on the curve. What does this result say about the torsion of the curve?
- Compute $\mathbf{a}'(t)$ and explain why the torsion is zero at all points on the curve for which the torsion is defined.

Solution

$$\begin{aligned} \mathbf{a)} \quad \vec{r}(t) &= \langle a_1 t^2 + b_1 t + c_1, a_2 t^2 + b_2 t + c_2, a_3 t^2 + b_3 t + c_3 \rangle \\ \vec{v}(t) &= \langle 2a_1 t + b_1, 2a_2 t + b_2, 2a_3 t + b_3 \rangle & \vec{v}(t) &= \frac{d\vec{r}}{dt} \\ \vec{a}(t) &= \langle 2a_1, 2a_2, 2a_3 \rangle & \vec{a}(t) &= \frac{d\vec{v}}{dt} \end{aligned}$$

$$\begin{aligned} \vec{v} \times \vec{a} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2a_1 t + b_1 & 2a_2 t + b_2 & 2a_3 t + b_3 \\ 2a_1 & 2a_2 & 2a_3 \end{vmatrix} \\ &= 2 \langle 2a_2 a_3 t + a_3 b_2 - 2a_2 a_3 t - a_2 b_3, 2a_1 a_3 t + a_1 b_3 - 2a_1 a_3 t - a_3 b_1, \\ &\quad 2a_1 a_2 t + a_2 b_1 - 2a_1 a_2 t - a_1 b_2 \rangle \\ &= 2 \langle a_3 b_2 - a_2 b_3, a_1 b_3 - a_3 b_1, a_2 b_1 - a_1 b_2 \rangle \\ &= \text{Constant} \end{aligned}$$

$$|\vec{v} \times \vec{a}| = 2\sqrt{(a_3 b_2 - a_2 b_3)^2 + (a_1 b_3 - a_3 b_1)^2 + (a_2 b_1 - a_1 b_2)^2}$$

$$\vec{B} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|} = \text{constant}$$

$$\Rightarrow \tau = 0$$

$$\mathbf{b)} \quad \vec{a}' = \langle 0, 0, 0 \rangle$$

$$\tau = \frac{(\vec{v} \times \vec{a}) \times \vec{a}'}{|\vec{v} \times \vec{a}|^2}$$

$$= 0$$

Exercise

Let f and g be continuous on an interval I . consider the curve

$$C: \mathbf{r}(t) = \langle a_1 f(t) + a_2 g(t) + a_3, b_1 f(t) + b_2 g(t) + b_3, c_1 f(t) + c_2 g(t) + c_3 \rangle$$

For t in I , and where a_i , b_i , and c_i , for $i = 1, 2$, and 3 , are real numbers

- a) Show that, in general, C lies in a plane.
- b) Explain why the torsion is zero at all points of C for which the torsion is defined.

Solution

$$\begin{aligned} \text{a) } \vec{r}(t) &= \langle a_1 f(t) + a_2 g(t) + a_3, b_1 f(t) + b_2 g(t) + b_3, c_1 f(t) + c_2 g(t) + c_3 \rangle \\ \vec{r}(s) &= \langle a_1 f(s) + a_2 g(s) + a_3, b_1 f(s) + b_2 g(s) + b_3, c_1 f(s) + c_2 g(s) + c_3 \rangle \\ \vec{r}(t) \times \vec{r}(s) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 f(t) + a_2 g(t) + a_3 & b_1 f(t) + b_2 g(t) + b_3 & c_1 f(t) + c_2 g(t) + c_3 \\ a_1 f(s) + a_2 g(s) + a_3 & b_1 f(s) + b_2 g(s) + b_3 & c_1 f(s) + c_2 g(s) + c_3 \end{vmatrix} \\ &= \left[(b_1 f(t) + b_2 g(t) + b_3)(c_1 f(s) + c_2 g(s) + c_3) - \right. \\ &\quad \left. (c_1 f(t) + c_2 g(t) + c_3)(b_1 f(s) + b_2 g(s) + b_3) \right] \hat{i} \\ &\quad + \left[(c_1 f(t) + c_2 g(t) + c_3)(a_1 f(s) + a_2 g(s) + a_3) - \right. \\ &\quad \left. (a_1 f(t) + a_2 g(t) + a_3)(c_1 f(s) + c_2 g(s) + c_3) \right] \hat{j} \\ &\quad + \left[(a_1 f(t) + a_2 g(t) + a_3)(b_1 f(s) + b_2 g(s) + b_3) - \right. \\ &\quad \left. (b_1 f(t) + b_2 g(t) + b_3)(a_1 f(s) + a_2 g(s) + a_3) \right] \hat{k} \end{aligned}$$

$$\begin{aligned}
&= \left[b_1 c_1 f(t) f(s) + b_1 c_2 f(t) g(s) + b_1 c_3 f(t) + b_2 c_1 g(t) f(s) \right. \\
&\quad + b_2 c_2 g(t) g(s) + b_2 c_3 g(t) + b_3 c_1 f(s) + b_3 c_2 g(s) + b_3 c_3 \\
&\quad - b_1 c_1 f(t) f(s) - b_2 c_1 f(t) g(s) - b_3 c_1 f(t) - b_1 c_2 g(t) f(s) \\
&\quad \left. - b_2 c_2 g(t) g(s) - b_3 c_2 g(t) - b_1 c_3 f(s) - b_2 c_3 g(s) - b_3 c_3 \right] \hat{i} \\
&+ \left[a_1 c_1 f(t) f(s) + a_2 c_1 f(t) g(s) + a_3 c_1 f(t) + a_1 c_2 g(t) f(s) \right. \\
&\quad + a_2 c_2 g(t) g(s) + a_3 c_2 g(t) + a_1 c_3 f(s) + a_2 c_3 g(s) + a_3 c_3 \\
&\quad - a_1 c_1 f(s) f(t) - a_2 c_1 f(s) g(t) - a_3 c_1 f(s) - a_1 c_2 g(s) f(t) \\
&\quad \left. - a_2 c_2 g(s) g(t) - a_3 c_2 g(s) - a_1 c_3 f(t) - a_2 c_3 g(t) - a_3 c_3 \right] \hat{j} \\
&+ \left[a_1 b_1 f(s) f(t) + a_2 b_1 f(s) g(t) + a_3 b_1 f(s) + a_1 b_2 g(s) f(t) \right. \\
&\quad + a_2 b_2 g(s) g(t) + a_3 b_2 g(s) + a_1 b_3 f(t) + a_2 b_3 g(t) + a_3 b_3 \\
&\quad - a_1 b_1 f(t) f(s) - a_2 b_1 f(t) g(s) - a_3 b_1 f(t) - a_1 b_2 g(t) f(s) \\
&\quad \left. - a_2 b_2 g(t) g(s) - a_3 b_2 g(t) - a_1 b_3 f(s) - a_2 b_3 g(s) - a_3 b_3 \right] \hat{k} \\
&= \left[(b_1 c_2 - b_2 c_1) f(t) g(s) + (b_2 c_1 - b_1 c_2) f(s) g(t) + (b_1 c_3 - b_3 c_1) f(t) \right. \\
&\quad \left. + (b_2 c_3 - b_3 c_2) g(t) + (b_3 c_1 - b_1 c_3) f(s) + (b_3 c_2 - b_2 c_3) g(s) \right] \hat{i} \\
&+ \left[(a_2 c_1 - a_1 c_2) f(t) g(s) + (a_1 c_2 - a_2 c_1) f(s) g(t) + (a_3 c_1 - a_1 c_3) f(t) \right. \\
&\quad \left. + (a_3 c_2 - a_2 c_3) g(t) + (a_1 c_3 - a_3 c_1) f(s) + (a_2 c_3 - a_3 c_2) g(s) \right] \hat{j} \\
&+ \left[(a_2 b_1 - a_1 b_2) f(s) g(t) + (a_1 b_2 - a_2 b_1) f(t) g(s) + (a_1 b_3 - a_3 b_1) f(t) \right. \\
&\quad \left. + (a_2 b_3 - a_3 b_2) g(t) + (a_3 b_1 - a_1 b_3) f(s) + (a_3 b_2 - a_2 b_3) g(s) \right] \hat{k}
\end{aligned}$$

If $a_3 = b_3 = c_3 = 0$

$$\begin{aligned}
\vec{r}(t) \times \vec{r}(s) &= \left[(b_1 c_2 - b_2 c_1) f(t) g(s) + (b_2 c_1 - b_1 c_2) f(s) g(t) \right] \hat{i} \\
&+ \left[(a_2 c_1 - a_1 c_2) f(t) g(s) + (a_1 c_2 - a_2 c_1) f(s) g(t) \right] \hat{j} \\
&+ \left[(a_1 b_2 - a_2 b_1) f(t) g(s) + (a_2 b_1 - a_1 b_2) f(s) g(t) \right] \hat{k} \\
&= \left[(b_1 c_2 - b_2 c_1) (f(t) g(s) - f(s) g(t)) \right] \hat{i} \\
&+ \left[(a_2 c_1 - a_1 c_2) (f(t) g(s) - f(s) g(t)) \right] \hat{j} \\
&+ \left[(a_1 b_2 - a_2 b_1) (f(t) g(s) - f(s) g(t)) \right] \hat{k} \\
&= (f(t) g(s) - f(s) g(t)) \langle b_1 c_2 - b_2 c_1, a_2 c_1 - a_1 c_2, a_1 b_2 - a_2 b_1 \rangle
\end{aligned}$$

Which is orthogonal to the same vector, the vectors $\vec{r}(t)$ must all be in the same plane.

If $a_3, b_3, \& c_3 \neq 0$

Consider $\vec{p}(t) = \vec{r}(t) - \langle a_3, b_3, c_3 \rangle$

$\vec{p}(t)$ is the form of $\vec{r}(t) \times \vec{r}(s)$ with $a_3 = b_3 = c_3 = 0$.

$\therefore \vec{p}(t)$ lies in a plane where $\vec{r}(t) = \vec{p}(t) + \langle a_3, b_3, c_3 \rangle$ lies in a plane too.

b) If the curve lies in a plane, \vec{B} is always normal to the plane with $|\vec{B}| = 1$.

Hence, \vec{B} is constant, so $\tau = \frac{d\vec{B}}{dt} \cdot \vec{N} = 0$