Solution *Section* 4.4 – Fundamental Theorem of Calculus

Exercise

Find the integral $\int_0^3 (2x+1)dx$

Solution

$$\int_0^3 (2x+1)dx = x^2 + x\Big|_0^3$$
$$= 3^2 + 3 - (0+0)$$
$$= 12$$

Exercise

Find the integral $\int_{-1}^4 |x-2|dx$

Solution

$$\int_{-1}^4 |x-2|dx = \int_{-1}^2 -(x-2)dx + \int_2^4 (x-2)dx$$
$$= -\frac{1}{2}x^2 + 2x\Big|_{-1}^2 + \left[\frac{1}{2}x^2 - 2x\right]_2^4$$
$$= -\frac{1}{2}2^2 + 2(2) - \left(-\frac{1}{2}(-1)^2 + 2(-1)\right) + \frac{1}{2}4^2 - 2(4) - \left(\frac{1}{2}(2)^2 - 2(2)\right)$$
$$= \frac{13}{2}$$

Exercise

Find the integral $\int_0^2 \sqrt{4-x^2}\,dx$

Solution

$$\int_0^2 \sqrt{4-x^2}\,dx = \frac{1}{2}x\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}$$

$\sqrt{4-x^2}$ is a semi-circle with center (0, 0) and radius = 2

Since x from 0 to 2

$\Rightarrow$ Area = $\frac{1}{4}$(Area of this circle) = $\frac{1}{4}2\pi 2^2 = 2\pi$

Exercise

Evaluate $\int_0^1 x^2 e^x dx$

Solution

$$\int_0^1 x^2 e^x dx = x^2 e^x - 2\int xe^x dx$$

$$= x^2 e^x \Big|_0^1 - 2\left[xe^x \Big|_0^1 - \int_0^1 e^x dx \right]$$

$$= \left[x^2 e^x - 2\left(xe^x - e^x \right) \right]_0^1$$

$$= x^2 e^x - 2xe^x + 2e^x \Big|_0^1$$

$$= e^1 - 2e^1 + 2e^1 - (0 - 0 + 2e^0)$$

$$= e - 2$$

Exercise

Evaluate the integrals $\int_0^2 x(x-3)\,dx$

Solution

$$\int_0^2 x(x-3)\,dx = \int_0^2 \left(x^2 - 3x\right) dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2} \right]_0^2$$

$$= \left(\frac{2^3}{3} - \frac{3(2)^2}{2} \right) - \left(\frac{0^3}{3} - \frac{3(2)^2}{2} \right)$$

$$= -\frac{10}{3}$$

Exercise

Evaluate the integrals $\int_0^4 \left(3x - \frac{x^3}{4}\right) dx$

Solution

$$\int_0^4 \left(3x - \frac{x^3}{4}\right) dx = \left[3\frac{x^2}{2} - \frac{x^4}{16}\right]_0^4$$

$$= \left(3\frac{(4)^2}{2} - \frac{(4)^4}{16}\right) - 0$$

$$= 8$$

Exercise

Evaluate the integrals $\int_{-2}^2 \left(x^3 - 2x + 3\right) dx$

Solution

$$\int_{-2}^2 \left(x^3 - 2x + 3\right) dx = \left[\frac{x^4}{4} - x^2 + 3x\right]_{-2}^2$$

$$= \left(\frac{(2)^4}{4} - (2)^2 + 3(2)\right) - \left(\frac{(-2)^4}{4} - (-2)^2 + 3(-2)\right)$$

$$= 12$$

Exercise

Evaluate the integrals $\int_0^1 \left(x^2 + \sqrt{x}\right) dx$

Solution

$$\int_0^1 \left(x^2 + \sqrt{x}\right) dx = \left[\frac{x^3}{3} + \frac{2}{3}x^{3/2}\right]_0^1$$

$$= \left(\frac{(1)^3}{3} + \frac{2}{3}(1)^{3/2}\right) - 0$$

$$= 1$$

Exercise

Evaluate the integrals $\displaystyle\int_{-3}^{-1} \frac{y^5 - 2y}{y^3}\, dy$

Solution

$$\int_{-3}^{-1} \frac{y^5 - 2y}{y^3}\, dy = \int_{-3}^{-1} \left(\frac{y^5}{y^3} - \frac{2y}{y^3} \right) dy$$

$$= \int_{-3}^{-1} \left(y^2 - 2y^{-2} \right) dy$$

$$= \left[\frac{1}{3} y^3 + 2y^{-1} \right]_{-3}^{-1}$$

$$= \left(\frac{1}{3}(-1)^3 + \frac{2}{-1} \right) - \left(\frac{1}{3}(-3)^3 + \frac{2}{-3} \right)$$

$$= \frac{22}{3}$$

Exercise

Evaluate the integrals $\displaystyle\int_{1}^{8} \frac{\left(x^{1/3} + 1\right)\left(2 - x^{2/3}\right)}{x^{1/3}}\, dx$

Solution

$$\int_{1}^{8} \frac{\left(x^{1/3} + 1\right)\left(2 - x^{2/3}\right)}{x^{1/3}}\, dx = \int_{1}^{8} \frac{2x^{1/3} - x + 2 - x^{2/3}}{x^{1/3}}\, dx$$

$$= \int_{1}^{8} \left(2 - x^{2/3} + 2x^{-1/3} - x^{1/3} \right) dx$$

$$= \left[2x - \frac{3}{5} x^{5/3} + 3x^{2/3} - \frac{3}{4} x^{4/3} \right]_{1}^{8}$$

$$= \left(2(8) - \frac{3}{5}(8)^{5/3} + 3(8)^{2/3} - \frac{3}{4}(8)^{4/3} \right) - \left(2(1) - \frac{3}{5}(1)^{5/3} + 3(1)^{2/3} - \frac{3}{4}(1)^{4/3} \right)$$

$$= \left(-\frac{16}{5} \right) - \left(\frac{73}{20} \right)$$

$$= -\frac{137}{20}$$

Exercise

Evaluate the integral $\displaystyle\int_0^3 \sqrt{y+1}\, dy$

Solution

$u = y+1 \Rightarrow du = dy$

$$\begin{aligned}\int_0^3 \sqrt{y+1}\, dy &= \int_0^3 u^{1/2}\, du \\ &= \frac{2}{3}u^{3/2}\Big|_0^3 \\ &= \frac{2}{3}(y+1)^{3/2}\Big|_0^3 \\ &= \frac{2}{3}\left[(3+1)^{3/2} - (0+1)^{3/2}\right] \\ &= \frac{2}{3}[8-1] \\ &= \frac{14}{3}\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^1 r\sqrt{1-r^2}\, dr$

Solution

Let $u = 1-r^2 \quad \Rightarrow \quad du = -2rdr \quad \rightarrow \quad -\frac{1}{2}du = rdr$

$$\begin{aligned}\int_{-1}^1 r\sqrt{1-r^2}\, dr &= \int_{-1}^1 u^{1/2}\left(-\frac{1}{2}du\right) \\ &= -\frac{1}{2}\frac{2}{3}u^{3/2}\Big|_{-1}^1 \\ &= -\frac{1}{3}\left[\left(1-r^2\right)^{3/2}\right]_{-1}^1 \\ &= -\frac{1}{3}\left[\left(1-(1)^2\right)^{3/2} - \left(1-(-1)^2\right)^{3/2}\right] \\ &= -\frac{1}{3}[0-0] \\ &= 0\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 t^3\left(1+t^4\right)^3\, dt$

Solution

Let $u=1+t^4 \quad\Rightarrow\quad du=4t^3dt \quad\to\quad \frac{1}{4}du=t^3dt \quad \begin{cases} t=1 & \to u=2 \\ t=0 & \to u=1 \end{cases}$

$$\int_0^1 t^3\left(1+t^4\right)^3\, dt=\int_1^2 \frac{1}{4}u^3du$$

$$=\frac{1}{4}\left(\frac{u^4}{4}\right)_1^2$$

$$=\frac{1}{16}\left(u^4\right)_1^2$$

$$=\frac{1}{16}\left(2^4-1^4\right)$$

$$=\frac{15}{16}$$

Exercise

A company manufactures *x* HDTVs per month. The monthly marginal profit (in dollars) is given by

$$P'(x)=165-0.1x \qquad 0\le x\le 4{,}000$$

The company is currently manufacturing 1,500 HDTVs per month, but is planning to increase production. Find the change in the monthly profit if monthly production is increased to 1,600 HDTVs.

Solution

$$P=\int_{1,500}^{1,600}(165-0.1x)\,dx$$

$$=\left[165x-0.05x^2\right]_{1,500}^{1,600}$$

$$=\left(165(1{,}600)-0.05(1{,}600)^2\right)-\left(165(1{,}500)-0.05(1{,}500)^2\right)$$

$$=136{,}000-135{,}000$$

$$=1{,}000$$

Increasing monthly production from 1,500 units to 1,600 units will increase the monthly profit by \$1,000.

Exercise

An amusement company maintains records for each video game installed in an arcade. Suppose that $C(t)$ and $R(t)$ represent the total accumulated costs and revenues (in thousands of dollars), respectively, t years after a particular game has been installed. Suppose also that

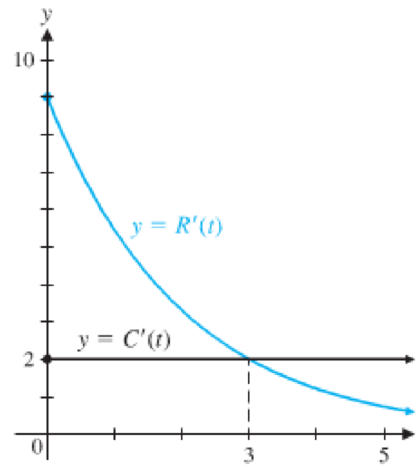
$$C'(t)=2 \quad R'(t)=9e^{-0.5t}$$

The value of t for which $C'(t)=R'(t)$ is called the ***useful life*** of the game.

a) Find the useful life of the game, to the nearest year.

b) Find the total profit accumulated during the useful life of the game.

Solution



a) $R'(t)=C'(t)$

$$9e^{-0.5t}=2$$

$$e^{-0.5t}=\frac{2}{9}$$

$$-0.5t=\ln\left(\frac{2}{9}\right) \quad\rightarrow\quad \underline{\left| t=\frac{-1}{0.5}\ln\left(\frac{2}{9}\right) \approx 3\,years \right|}$$

The game has a useful life of 3 years.

b) The total profit during the useful life of the game is

$$\begin{aligned}
P(t) &= \int_0^3 P'(t)\,dt \\
&= \int_0^3 \left(R'(t)-C'(t)\right)dt \\
&= \int_0^3 \left(9e^{-0.5t}-2\right)dt \\
&= \left[\frac{9}{-.5}e^{-0.5t}-2t\right]_0^3 \\
&= \left(-18e^{-0.5(3)}-2(3)\right)-\left(-18e^{-0.5(0)}-2(0)\right) \\
&= -18e^{-1.5}-6+18 \\
&= 12-18e^{-1.5} \\
&\approx 7.984 \\
&\underline{\left| = \$7,984 \right|}
\end{aligned}$$

Exercise

The total cost (in dollars) of printing x dictionaries is $C(x)=20,000+10x$

a) Find the average cost per unit if 1,000 dictionaries are produced.

b) Find the average value of the cost function over the interval [0, 1,000}

c) Discuss the difference between parts (***a***) and (***b***)

Solution

a) Average cost per unit: $\overline{C}(x)=\dfrac{C(x)}{x}$

$$\overline{C}(x)=\frac{20,000+10x}{x}$$

$$\overline{C}(1,000)=\frac{20,000+10(1,000)}{1,000}\ \underline{=\$30|}$$

b) Average $C(x)=\dfrac{1}{1,000}\displaystyle\int_0^{1,000}(20,000+10x)\,dx$

$$=\frac{1}{1,000}\left[20,000x+5x^2\right]_0^{1,000}$$

$$\underline{=\$25,000|}$$

c) $\overline{C}(1,000)$ is the average cost per unit at a production level of 1,000 units.

Ave $C(x)$ is the average value of the total coast as production increases from 0 unit to 1,000 units.

Exercise

If the rate of labor is $g(x)=2,000\,x^{-1/3}$, then approximately how many labor–hours will be required to assemble the 9th through the 27th.

Solution

The number labor–hours to assemble the 9th through the 27th control units is

$$\int_8^{27} g(x)\,dx=\int_8^{27} 2,000\,x^{-1/3}\,dx$$

$$=2,000\left[\frac{3}{2}x^{2/3}\right]_8^{27}$$

$$=3,000\left(27^{2/3}-8^{2/3}\right)$$

$$=3,000(9-4)$$

$$\underline{=15,000\ \ \textit{labor hrs}|}$$