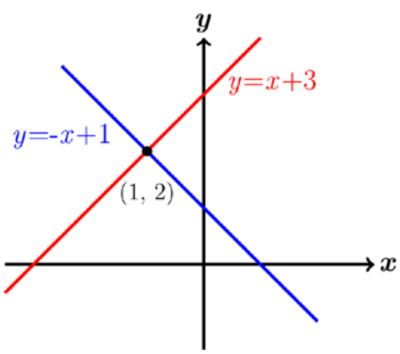
Lecture Four – Matrices

Section 4.1 – System of linear Equations

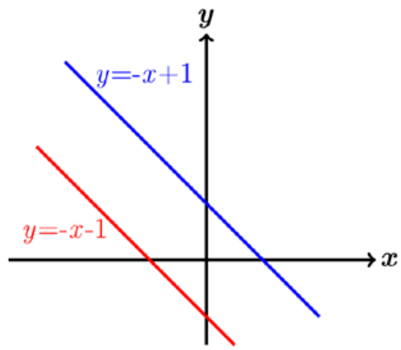
Solving Systems of Equations

1. Graphically
2. Substitution Method
3. Elimination Method

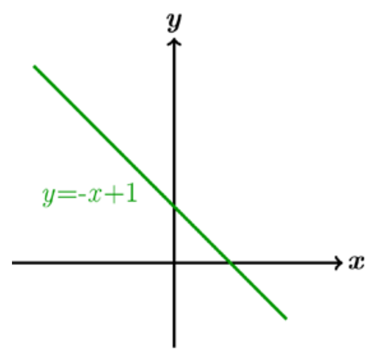
Graphically



One solution (lines intersect)
Consistent
Independent



No Solution (lines //)
Inconsistent
Independent



Infinite solution
Consistent
Dependent

Substitution Method

Solve: $\begin{cases} 3x + 2y = 11 & (1) \\ -x + y = 3 & (2) \end{cases}$

Solution

$From\ (2) \to y = x + 3 \quad (3)$

$(1) \Rightarrow 3x + 2(x+3) = 11$

$3x + 2x + 6 = 11$

$5x + 6 = 11$

$5x + 6 - 6 = 11 - 6$

$5x = 5$

$x = 1$

$From\ (3) \to y = 1 + 3 = 4$

$Solution: \ \underline{(1,\ 4)}|$

Elimination Method

Solve: $\begin{cases} 3x - 4y = 1 & (1) \\ 2x + 3y = 12 & (2) \end{cases}$

Solution

$$\begin{aligned} -2\times) \quad & 3x - 4y = 1 \\ 3\times) \quad & 2x + 3y = 12 \end{aligned}$$

$$\begin{array}{r} -6x + 8y = -2 \\ 6x + 9y = 36 \\ \hline 17y = 34 \end{array}$$

$y = \frac{34}{17} = 2$

$$\begin{aligned} From\ (1) \Rightarrow 3x &= 1 + 4y \\ 3x &= 1 + 4(2) \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

$Solution: \ \underline{(3,\ 2)}|$

Matrices

$$\begin{array}{cc}
 & \textit{Column} \\
 & \begin{array}{ccc} C_1 & C_2 & C_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\
\begin{array}{r} Row\ 1 \rightarrow R_1 \\ Row\ 2 \rightarrow R_2 \\ Row\ 3 \rightarrow R_3 \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
\end{array}$$

This is called Matrix (*Matrices*)

Each number in the array is an ***element*** or ***entry***

The matrix is said to be of order $m \times n$

m: numbers of rows,

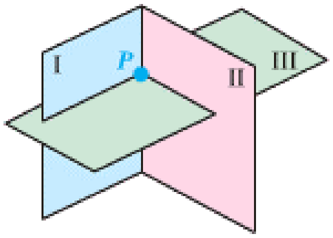
n: number of columns

When $m = n$, then matrix is said to be ***square***.

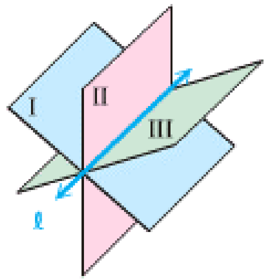
Given the system equations

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

The ***augmented matrix*** form is: $\left[\begin{array}{ccc|c} 3 & 1 & 2 & 31 \\ 1 & 1 & 2 & 19 \\ 1 & 3 & 2 & 25 \end{array}\right]$

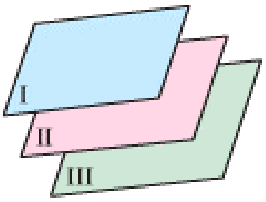


A single solution

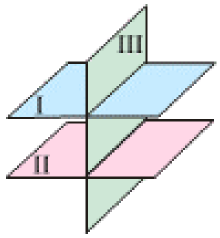


Points of a line in common

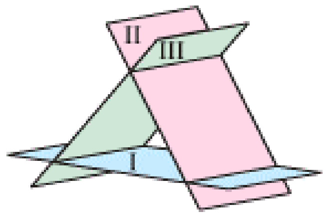
All points in common



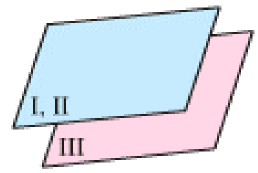
No points in common



No points in common



No points in common



No points in common

Gaussian Elimination

Example

Use the Gaussian elimination method to solve the system

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 3 & 1 & 2 & 31 \\ 1 & 3 & 2 & 25 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array} \qquad \begin{array}{cccc} 3 & 1 & 2 & 31 \\ -3 & -3 & -6 & -57 \\ \hline 0 & -2 & -4 & -26 \end{array} \qquad \begin{array}{cccc} 1 & 3 & 2 & 25 \\ -1 & -1 & -2 & -19 \\ \hline 0 & 2 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & -2 & -4 & -26 \\ 0 & 2 & 0 & 6 \end{array}\right] -\tfrac{1}{2}R_2 \qquad \begin{array}{cccc} 0 & 1 & 2 & 13 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 2 & 0 & 6 \end{array}\right] R_3 - 2R_2 \qquad \begin{array}{cccc} 0 & 2 & 0 & 6 \\ 0 & -2 & -4 & -26 \\ \hline 0 & 0 & -4 & -20 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -4 & -20 \end{array}\right] -\tfrac{1}{4}R_3 \qquad \begin{array}{cccc} 0 & 0 & 1 & 5 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array}\right] \Rightarrow \begin{array}{rl} x + y + 2z = 19 & (3) \\ y + 2z = 13 & (2) \\ z = 5 & (1) \end{array}$$

$(2) \Rightarrow y = 13 - 2z = 13 - 2(5) = 3$

$(3) \Rightarrow\ x = 19 - y - 2z = 19 - 3 - 10 = 6$

$\Rightarrow$ **(6, 3, 5)**

Gauss-Jordan Elimination

Example

Use the Gauss-Jordan method to solve the system

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 3 & 1 & 2 & 31 \\ 1 & 3 & 2 & 25 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 1 & 2 & 31 \\ -3 & -3 & -6 & -57 \\ \hline 0 & -2 & -4 & -26 \end{array} \qquad \begin{array}{cccc} 1 & 3 & 2 & 25 \\ -1 & -1 & -2 & -19 \\ \hline 0 & 2 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & -2 & -4 & -26 \\ 0 & 2 & 0 & 6 \end{array}\right] -\tfrac{1}{2}R_2 \qquad 0 \quad 1 \quad 2 \quad 13$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 2 & 0 & 6 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 2R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 2 & 0 & 6 \\ 0 & -2 & -4 & -26 \\ \hline 0 & 0 & -4 & -20 \end{array} \qquad \begin{array}{cccc} 1 & 1 & 2 & 19 \\ 0 & -1 & -2 & -13 \\ \hline 1 & 0 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -4 & -20 \end{array}\right] -\tfrac{1}{4}R_3 \qquad 0 \quad 0 \quad 1 \quad 5$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array}\right] R_2 - 2R_3 \qquad \begin{array}{cccc} 0 & 1 & 2 & 13 \\ 0 & 0 & -2 & -10 \\ \hline 0 & 1 & 0 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

***Solution*: (6, 3, 5)**

Example

Use the Gaussian elimination method to solve the system

$$\begin{aligned} 2x + \ \ y + 2z &= 4 \\ 2x + 2y \qquad &= 5 \\ 2x - \ \ y + 6z &= 2 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \ \tfrac{1}{2}R_1 \qquad\qquad 1 \quad \tfrac{1}{2} \quad 1 \quad 2$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array} \qquad \begin{array}{cccc} 2 & 2 & 0 & 5 \\ -2 & -1 & -2 & -4 \\ \hline 0 & 1 & -2 & 1 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 6 & 2 \\ -2 & -1 & -2 & -4 \\ \hline 0 & -2 & 4 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & -2 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 2R_2 \end{array} \qquad \begin{array}{cccc} 0 & -2 & 4 & -2 \\ 0 & 2 & -4 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{rl} x + \frac{1}{2}y + z = 2 & (3) \\ y - 2z = 1 & (2) \\ 0 = 0 & (1) \end{array}$$

From (1): 0 = 0 is a true statement. Let ***z*** be the variable.

From (2): $y = 1 + 2z$

From (3): $x = -\frac{1}{2}y - z + 2$

$$x = -\frac{1}{2}(1 + 2z) - z + 2$$

$$x = -\frac{1}{2} - z - z + 2$$

$$x = -2z + \frac{3}{2}$$

Solution: $\left(-2z + \frac{3}{2},\ 2z + 1,\ z\right)$

Example

Use the Gaussian elimination method to solve the system

$$\begin{aligned} x+2y-5z&=-1\\ 2x+3y-2z&=2\\ 3x+5y-7z&=4 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & -1 \\ 2 & 3 & -2 & 2 \\ 3 & 5 & -7 & 4 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-3R_1 \end{array}$$

$$\begin{array}{rrrr} 2 & 3 & -2 & 2 \\ -2 & -4 & 10 & 2 \\ \hline 0 & -1 & 8 & 4 \end{array} \qquad \begin{array}{rrrr} 3 & 5 & -7 & 4 \\ -3 & -6 & 15 & 3 \\ \hline 0 & -1 & 8 & 7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & -1 \\ 0 & -1 & 8 & 4 \\ 0 & -1 & 8 & 7 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ \end{array} \qquad 0 \quad 1 \quad -8 \quad -4$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & -1 \\ 0 & 1 & -8 & -4 \\ 0 & -1 & 8 & 7 \end{array}\right] \begin{array}{l} \\ \\ R_3+R_2 \end{array}$$

$$\begin{array}{rrrr} 0 & -1 & 8 & 7 \\ 0 & 1 & -8 & -4 \\ \hline 0 & 0 & 0 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & -1 \\ 0 & 1 & -8 & -4 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

From Row 3: 0 = 3 is a False statement.

No Solution or ***Inconsistent***

Exercises Section 4.1 – System of linear Equations

(1 – 15) Use any method to solve the system equation (***elimination*** or ***substitution*** method)

1. $\begin{cases} 3x+2y=-4 \\ 2x-y=-5 \end{cases}$

2. $\begin{cases} 2x+5y=7 \\ 5x-2y=-3 \end{cases}$

3. $\begin{cases} 4x-7y=-16 \\ 2x+5y=9 \end{cases}$

4. $\begin{cases} 3x+2y=4 \\ 2x+y=1 \end{cases}$

5. $\begin{cases} 3x+4y=2 \\ 2x+5y=-1 \end{cases}$

6. $\begin{cases} 5x-2y=4 \\ -10x+4y=7 \end{cases}$

7. $\begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$

8. $\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

9. $\begin{cases} 2x+10y=-14 \\ 7x-2y=-16 \end{cases}$

10. $\begin{cases} 4x-3y=24 \\ -3x+9y=-1 \end{cases}$

11. $\begin{cases} 4x+2y=12 \\ 3x-2y=16 \end{cases}$

12. $\begin{cases} x+2y=-1 \\ 4x-2y=6 \end{cases}$

13. $\begin{cases} x-2y=5 \\ -10x+2y=4 \end{cases}$

14. $\begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$

15. $\begin{cases} 4x-4y=-12 \\ 4x+4y=-20 \end{cases}$

(16 – 27) Perform the matrix row operation (or operations) and write the new matrix.

16. $\left[\begin{array}{cc|c} 1 & 4 & 7 \\ 3 & 5 & 0 \end{array}\right]$ R_2-3R_1

17. $\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right]$ R_2-2R_1

18. $\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 5 & 2 & 19 \end{array}\right]$ R_2-5R_1

19. $\left[\begin{array}{cc|c} 2 & -3 & 8 \\ -6 & 9 & 4 \end{array}\right]$ R_2+3R_1

20. $\left[\begin{array}{cc|c} 2 & 3 & 11 \\ 1 & 2 & 8 \end{array}\right]$ $2R_2-R_1$

21. $\left[\begin{array}{cc|c} 3 & 5 & -13 \\ 2 & 3 & -9 \end{array}\right]$ $3R_2-2R_1$

22. $\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 5 & 4 & 1 \end{array}\right]$ R_3-5R_2

23. $\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right]$ $\begin{array}{l} R_2-3R_1 \\ R_3-R_1 \end{array}$

24. $\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 2 & 4 & 4 & 22 \\ -1 & -2 & 3 & 15 \end{array}\right]$ $\begin{array}{l} 3R_2-2R_1 \\ 3R_3+R_1 \end{array}$

25. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & -4 & 2 & -7 \end{array}\right]$ $\begin{array}{l} R_2-2R_1 \\ R_3-3R_1 \end{array}$

26. $\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ 2 & -3 & 5 & -1 & 0 \\ 1 & 0 & 3 & 1 & -4 \\ -4 & 3 & 2 & -1 & 3 \end{array}\right]$ $\begin{array}{l} R_2-2R_1 \\ R_3-R_1 \\ R_4+4R_1 \end{array}$

27. $\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ -3 & 6 & -3 & -9 & 6 \\ 2 & 1 & 2 & 3 & 4 \\ 5 & 3 & 2 & -1 & -7 \end{array}\right]$ $\begin{array}{l} R_2+3R_1 \\ R_3-2R_1 \\ R_4-5R_1 \end{array}$

(**28 – 34**) Use the Gauss-Jordan method to solve the system

28. $\begin{cases} x-y+5z=-6 \\ 3x+3y-z=10 \\ x+3y+2z=5 \end{cases}$

29. $\begin{cases} 2x-y+4z=-3 \\ x-2y-10z=-6 \\ 3x \qquad +4z=7 \end{cases}$

30. $\begin{cases} 4x+3y-5z=-29 \\ 3x-7y-z=-19 \\ 2x+5y+2z=-10 \end{cases}$

31. $\begin{cases} x+2y-3z=-15 \\ 2x-3y+4z=18 \\ -3x+y+z=1 \end{cases}$

32. $\begin{cases} x+2y+3z=10 \\ 4x+5y+6z=11 \\ 7x+8y+9z=12 \end{cases}$

33. $\begin{cases} 2x+y+2z=4 \\ 2x+2y \qquad =5 \\ 2x-y+6z=2 \end{cases}$

34. $\begin{cases} x_1+x_2+2x_3=8 \\ -x_1-2x_2+3x_3=1 \\ 3x_1-7x_2+4x_3=10 \end{cases}$

(**35 – 69**) Use augmented elimination to solve linear system

35. $\begin{cases} 2x-5y+3z=1 \\ x-2y-2z=8 \end{cases}$

36. $\begin{cases} x+y+z=2 \\ 2x+y-z=5 \\ x-y+z=-2 \end{cases}$

37. $\begin{cases} 2x+y+z=9 \\ -x-y+z=1 \\ 3x-y+z=9 \end{cases}$

38. $\begin{cases} 3y-z=-1 \\ x+5y-z=-4 \\ -3x+6y+2z=11 \end{cases}$

39. $\begin{cases} x+3y+4z=14 \\ 2x-3y+2z=10 \\ 3x-y+z=9 \end{cases}$

40. $\begin{cases} x+4y-z=20 \\ 3x+2y+z=8 \\ 2x-3y+2z=-16 \end{cases}$

41. $\begin{cases} 2y-z=7 \\ x+2y+z=17 \\ 2x-3y+2z=-1 \end{cases}$

42. $\begin{cases} -2x+6y+7z=3 \\ -4x+5y+3z=7 \\ -6x+3y+5z=-4 \end{cases}$

43. $\begin{cases} 2x-y+z=1 \\ 3x-3y+4z=5 \\ 4x-2y+3z=4 \end{cases}$

44. $\begin{cases} 3x-4y+4z=7 \\ x-y-2z=2 \\ 2x-3y+6z=5 \end{cases}$

45. $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

46. $\begin{cases} x+y+z=3 \\ -y+2z=1 \\ -x+z=0 \end{cases}$

47. $\begin{cases} 3x+y+3z=14 \\ 7x+5y+8z=37 \\ x+3y+2z=9 \end{cases}$

48. $\begin{cases} 4x-2y+z=7 \\ x+y+z=-2 \\ 4x+2y+z=3 \end{cases}$

49. $\begin{cases} 2x-2y+z=-4 \\ 6x+4y-3z=-24 \\ x-2y+2z=1 \end{cases}$

50. $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

51. $\begin{cases} 2x-y+2z=-8 \\ x+2y-3z=9 \\ 3x-y-4z=3 \end{cases}$

52. $\begin{cases} x \qquad -3z=-5 \\ 2x-y+2z=16 \\ 7x-3y-5z=19 \end{cases}$

53. $\begin{cases} x+2y-z=5 \\ 2x-y+3z=0 \\ 2y+z=1 \end{cases}$

54. $\begin{cases} x+y+z=6 \\ 3x+4y-7z=1 \\ 2x-y+3z=5 \end{cases}$

55. $\begin{cases} 3x+2y+3z=3 \\ 4x-5y+7z=1 \\ 2x+3y-2z=6 \end{cases}$

56. $\begin{cases} x-3y+z=2 \\ 4x-12y+4z=8 \\ -2x+6y-2z=-4 \end{cases}$

57. $\begin{cases} 2x-2y+z=-1 \\ x+2y-z=2 \\ 6x+4y+3z=5 \end{cases}$

58. $\begin{cases} x_1-5x_2+2x_3-2x_4=4 \\ x_2-3x_3-x_4=0 \\ 3x_1+2x_3-x_4=6 \\ -4x_1+x_2+4x_3+2x_4=-3 \end{cases}$

59. $\begin{cases} x_1+x_2+x_3+x_4=5 \\ x_1+2x_2-x_3-2x_4=-1 \\ x_1-3x_2-3x_3-x_4=-1 \\ 2x_1-x_2+2x_3-x_4=-2 \end{cases}$

60. $\begin{cases} 2x+8y-z+w=0 \\ 4x+16y-3z-w=-10 \\ -2x+4y-z+3w=-6 \\ -6x+2y+5z+w=3 \end{cases}$

61. $\begin{cases} 2x_1+x_2+3x_3=0 \\ x_1+2x_2=0 \\ x_2+x_3=0 \end{cases}$

62. $\begin{cases} 2x+2y+4z=0 \\ -y-3z+w=0 \\ 3x+y+z+2w=0 \\ x+3y-2z-2w=0 \end{cases}$

63. $\begin{cases} 2x+z+w=5 \\ y-w=-1 \\ 3x-z-w=0 \\ 4x+y+2z+w=9 \end{cases}$

64. $\begin{cases} 4y+z=20 \\ 2x-2y+z=0 \\ x+z=5 \\ x+y-z=10 \end{cases}$

65. $\begin{cases} x-y+2z-w=-1 \\ 2x+y-2z-2w=-2 \\ -x+2y-4z+w=1 \\ 3x-3w=-3 \end{cases}$

66. $\begin{cases} 2u-3v+w-x+y=0 \\ 4u-6v+2w-3x-y=-5 \\ -2u+3v-2w+2x-y=3 \end{cases}$

67. $\begin{cases} 6x_3+2x_4-4x_5-8x_6=8 \\ 3x_3+x_4-2x_5-4x_6=4 \\ 2x_1-3x_2+x_3+4x_4-7x_5+x_6=2 \\ 6x_1-9x_2+11x_4-19x_5+3x_6=1 \end{cases}$

68. $\begin{cases} 3x_1+2x_2-x_3=-15 \\ 5x_1+3x_2+2x_3=0 \\ 3x_1+x_2+3x_3=11 \\ -6x_1-4x_2+2x_3=30 \end{cases}$

69. $\begin{cases} x_1+3x_2-2x_3+2x_5=0 \\ 2x_1+6x_2-5x_3-2x_4+4x_5-3x_6=-1 \\ 5x_3+10x_4+15x_6=5 \\ 2x_1+6x_2+8x_4+4x_5+18x_6=6 \end{cases}$

70. At Snack Mix, caramel corn worth \$2.50 per *pound* is mixed with honey roasted missed nuts worth \$7.50 per *pound* in order to get 20 *lbs*. of a mixture worth \$4.50 per *pound*. How much of each snack is used?

Section 4.2 – Matrix operations and Their Applications

Matrix Notation

The Matrix:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix} \text{ is called the coefficient matrix of the system}$$

The matrix is said to be of order *m x n*

m: numbers of rows,

n: number of columns

A matrix A with m rows and n columns can be written in a general form

$$A = \left[a_{\text{ij}}\right] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

The matrix A above has 3 rows and 3 columns; therefore, the order of the matrix A is (3 *x* 3)

When $m = n$, then matrix is said to be ***square***.

The numbers in a matrix are called ***entries.***

Example

$$Let \quad A = \begin{bmatrix} 5 & -2 \\ -3 & \pi \\ 1 & 6 \end{bmatrix}$$

a. What is the order of A?

3 rows and 2 columns $\Rightarrow A$ is 3 *x* 2

b. $a_{12} = -2 \qquad a_{31} = 1$

Equality of Matrices

Definition of Equality of Matrices

Two matrices ***A*** and ***B*** are equal if and only if they have the same order (size) $m \times n$ and if each pair corresponding elements is equal

$$a_{ij} = b_{ij} \text{ for } i = 1, 2, \ldots, m \textbf{ and } j = 1, 2, \ldots, n$$

Example

Find the values of the variables for which each statement is true, if possible.

a) $\begin{bmatrix} 2 & 1 \\ p & q \end{bmatrix} = \begin{bmatrix} x & y \\ -1 & 0 \end{bmatrix}$

$x = 2,\ y = 1,\ \ p = -1,\ \ q = 0$

b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$

can't be true

c) $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

$\Rightarrow \begin{bmatrix} w = 9 & x = 17 \\ 8 = y & -12 = z \end{bmatrix}$

Matrix Addition and Subtraction

Given two *m x n* matrices $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ their sum is $A + B = \left[a_{ij} + b_{ij}\right]$

And their difference is $A - B = \left[a_{ij} - b_{ij}\right]$

The matrices have to be the ***same order***

Example

Find $\begin{bmatrix} -4 & 3 \\ 7 & -6 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & -4 \end{bmatrix}$

Solution

$$\begin{aligned}
\begin{bmatrix} -4 & 3 \\ 7 & -6 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & -4 \end{bmatrix} &= \begin{bmatrix} -4+6 & 3+(-3) \\ 7+2 & -6+(-4) \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 \\ 9 & -10 \end{bmatrix}
\end{aligned}$$

Example

Find $\begin{bmatrix} 5 & 4 \\ -3 & 7 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 8 \\ 6 & 0 \\ -5 & 3 \end{bmatrix}$

Solution

$$\begin{aligned}
\begin{bmatrix} 5 & 4 \\ -3 & 7 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 8 \\ 6 & 0 \\ -5 & 3 \end{bmatrix} &= \begin{bmatrix} 5-(-4) & 4-8 \\ -3-6 & 7-0 \\ 0-(-5) & 1-3 \end{bmatrix} \\
&= \begin{bmatrix} 9 & -4 \\ -9 & 7 \\ 5 & -2 \end{bmatrix}
\end{aligned}$$

Example

Find $\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix}$

Solution

$$\begin{aligned}
\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix} &= \begin{bmatrix} 5-4 & -6+6 \\ 8+8 & 9-3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 16 & 6 \end{bmatrix}
\end{aligned}$$

Scalar Multiplication

The scalar product of a number k and a matrix A is denoted by $\boldsymbol{kA}$.

$$kA = k\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$$

Example

Find $5\begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix}$

Solution

$$5\begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2(5) & -3(5) \\ 0(5) & 4(5) \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -15 \\ 0 & 20 \end{bmatrix}$$

Example

Find $\frac{3}{4}\begin{bmatrix} 20 & 36 \\ 12 & -16 \end{bmatrix}$

Solution

$$\frac{3}{4}\begin{bmatrix} 20 & 36 \\ 12 & -16 \end{bmatrix} = \begin{bmatrix} 15 & 27 \\ 9 & -12 \end{bmatrix}$$

Example

Given: $A = \begin{bmatrix} -4 & 1 \\ 3 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix}$

Find:

a) $-6B$

b) $3A + 2B$

Solution

a) $-6B = -6\begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix}$

$$= \begin{bmatrix} -1(-6) & -2(-6) \\ 8(-6) & 5(-6) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 12 \\ -48 & -30 \end{bmatrix}$$

b) $3A + 2B = 3\begin{bmatrix} -4 & 1 \\ 3 & 0 \end{bmatrix} + 2\begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix}$

$$= \begin{bmatrix} -4(3) & 1(3) \\ 3(3) & 0(3) \end{bmatrix} + \begin{bmatrix} -1(2) & -2(2) \\ 8(2) & 5(2) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 3 \\ 9 & 0 \end{bmatrix} + \begin{bmatrix} -2 & -4 \\ 16 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} -12-2 & 3-4 \\ 9+16 & 0+10 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & -1 \\ 25 & 10 \end{bmatrix}$$

Matrix Multiplication

Product of Two Matrices

Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix. To find the element in the i^{th} row and j^{th} column of the product matrix AB, multiply each element in the i^{th} row of A by the corresponding element in the j^{th} column of B, and then add these products. The product matrix AB is an $m \times k$ matrix.

Matrix ***A*** *Matrix B*

$m \times n$ $n \times k$

Inner *must be equal*

Outer: Order of AB is $m \times k$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$2x2 \qquad 2x2 \quad \rightarrow \quad 2x2$

$$a_{11} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & \_ \\ \_ & \_ \end{bmatrix}$$

$$a_{12} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & af + bh \\ \_ & \_ \end{bmatrix}$$

$$a_{21} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ ce + dg & \_ \end{bmatrix}$$

$$a_{22} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ \_ & cf + dh \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Example

Given: $A = \begin{bmatrix} 1 & -3 \\ 7 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 4 & -1 \end{bmatrix}$

Find ***AB*** and ***BA***.

Solution

$$\boldsymbol{AB} = \begin{bmatrix} 1 & -3 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)+(-3)3 & 1(0)+(-3)1 & 1(-1)+(-3)4 & 1(2)+(-3)(-1) \\ 7(1)+2(3) & 7(0)+2(1) & 7(-1)+2(4) & 7(2)+2(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -8 & -3 & -13 & 5 \\ 13 & 2 & 1 & 12 \end{bmatrix}$$

BA can be found since: B: $2x4$ and A: $2x2$

Note: $AB \neq BA$

Example

Given: $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix}$

Find ***AB***.

Solution

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1(4)+3(1) & 1(6)+3(0) \\ 2(4)+5(1) & 2(6)+5(0) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 6 \\ 13 & 12 \end{bmatrix}$$

Example

Given: $A=\begin{bmatrix}3 & 1 & -1\\ 2 & 0 & 3\end{bmatrix}$ $B=\begin{bmatrix}1 & 6\\ 3 & -5\\ -2 & 4\end{bmatrix}$ Find $\boldsymbol{AB}$.

Solution

$$AB=\begin{bmatrix}3 & 1 & -1\\ 2 & 0 & 3\end{bmatrix}\begin{bmatrix}1 & 6\\ 3 & -5\\ -2 & 4\end{bmatrix}$$

$$=\begin{bmatrix}3(1)+1(3)-1(-2) & 3(6)+1(-5)-1(4)\\ 2(1)+0(3)+3(-2) & 2(6)+0(-5)+3(4)\end{bmatrix}$$

$$=\begin{bmatrix}8 & 9\\ -4 & 24\end{bmatrix}$$

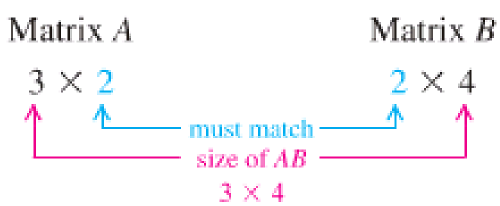
Example

Suppose A is a 3*x*2 matrix, while B is a 2*x*4 matrix.

a) Can the product *AB* be calculated?
b) If *AB* can be calculated, what size is it?
c) Can *BA* be calculated?
d) If *BA* can be calculated, what size is it?

Solution

a)



b) The product *AB* size is 3 *x* 4

c)

d) Can't be calculated

Example

Given: $A_{1x3} = \begin{bmatrix} 2 & 0 & 4 \end{bmatrix}$ $B_{3x1} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$ Find ***AB*** and ***BA***.

Solution

A B

1 *x* 3 3 *x* 1

Inner: Are equal

Outer: Order of *AB* is 1x1

AB = [2(1) + 0(3) + 4(7)] = [30]

BA : 3*x*1 ---- 1*x*3

$$BA = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1(2) & 1(0) & 1(4) \\ 3(2) & 3(0) & 3(4) \\ 7(2) & 7(0) & 7(4) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 4 \\ 6 & 0 & 12 \\ 14 & 0 & 28 \end{bmatrix}$$

Example

Given: $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 & -1 & 6 \\ 0 & 5 & 4 & 1 \end{bmatrix}$ Find ***AB*** and ***BA***.

Solution

a) $AB = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \bullet \begin{bmatrix} 2 & 3 & -1 & 6 \\ 0 & 5 & 4 & 1 \end{bmatrix}$ 2*x*2 --- 2*x*4

$$= \begin{bmatrix} 1(2)+3(0) & 1(3)+3(5) & 1(-1)+3(4) & 1(6)+3(1) \\ 0(2)+2(0) & 0(3)+2(5) & 0(-1)+2(4) & 0(6)+2(1) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 18 & 11 & 9 \\ 0 & 10 & 8 & 2 \end{bmatrix}$$

b) $BA = \begin{bmatrix} 2 & 3 & -1 & 6 \\ 0 & 5 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ + Undefined 2*x*4 --- 2*x*2 (*Inner order are not equal* 2, 4)

Properties of Matrix

Addition and Scalar Multiplication

$A+B=B+A$	*Commutative Property of Addition*
$A+(B+C)=(A+B)+C$	*Associative Property of Addition*
$(kl)A=k(lA)$	*Associative Property of Scalar Multiplication*
$k(A+B)=kA+kB$	*Distributive Property*
$(k+l)A=kA+lA$	*Distributive Property*
$A+0=0+A=A$	*Additive Identity Property*
$A+(-A)=(-A)+A=0$	*Additive Inverse Property*

Multiplication

$A(BC)=(AB)C$	*Associative Property of Multiplication*
$A(B+C)=AB+AC$	*Distributive Property*
$(B+C)A=BA+CA$	*Distributive Property*

Exercises Section 4.2 – Matrix operations and Their Applications

(1 – 7) Find values for the variables so that the matrices are equal.

1. $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

2. $\begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$

3. $\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix} = \begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$

4. $\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$

5. $\begin{bmatrix} a+11 & 12z+1 & 5m \\ 11k & 3 & 1 \end{bmatrix} + \begin{bmatrix} 9a & 9z & 4m \\ 12k & 5 & 3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$

6. $\begin{bmatrix} x+2 & 3y+1 & 5z \\ 8w & 2 & 3 \end{bmatrix} + \begin{bmatrix} 3x & 2y & 5z \\ 2w & 5 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$

7. $\begin{bmatrix} 2x-3 & y-2 & 2z+1 \\ 5 & 2w & 7 \end{bmatrix} + \begin{bmatrix} 3x-3 & y+2 & z-1 \\ -5 & 5w+1 & 3 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$

8. Given $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix}$ Find: $A-B,\ \ 3A+2B$

9. Given $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ $F = \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}$ Find: $3F+2A$

(10 – 22) Evaluate

10. $\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \\ 12 \end{bmatrix}$

11. $\begin{bmatrix} 5 & 8 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 9 & 1 \\ 4 & 2 & 5 \end{bmatrix}$

12. $\begin{bmatrix} -5 & 0 \\ 4 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix}$

13. $\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix}$

14. $\begin{bmatrix} -5 & 6 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ 5 & -8 \end{bmatrix}$

15. $\begin{bmatrix} 8 & 6 & -4 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -8 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix}$

17. $\begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix}$

18. $\begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix} . \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix} . \begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}$

20. $\begin{bmatrix} -2 & -3 & -4 \\ 2 & -1 & 0 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$

21. $\begin{bmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{18} \\ \sqrt{3} & \sqrt{27} & 0 \end{bmatrix} \begin{bmatrix} 8 & -10 \\ 9 & 12 \\ 0 & 2 \end{bmatrix}$

22. $\begin{bmatrix} x & 2x+1 & 4 \\ 5 & x-1 & 8 \\ -2 & 3x & 2x+1 \end{bmatrix} + \begin{bmatrix} 2x-1 & -2x-1 & 4x \\ -5 & 6 & x+1 \\ -5 & 2 & -2x \end{bmatrix}$

(**23 – 33**) Find AB and BA, if possible

23. $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}$

24. $A = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} \quad B = \begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}$

25. $A = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}$

26. $A = \begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix}$

27. $A = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$

28. $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$

29. $A = \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$

30. $A = \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix}$

31. $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$

32. $A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

33. $A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}$

34. Given $A = \begin{bmatrix} -3 & 4 \\ 2 & -3 \\ -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 1 \\ 1 & -2 \\ 3 & -4 \end{bmatrix}$, Find

a) $A+B$
b) $A-B$
c) $3A$
d) $-2B$
e) $2A+3B$
f) A^2
g) AB
h) BA

35. Given $A=\begin{bmatrix}2 & -2\\ 3 & 4\\ 1 & 0\end{bmatrix}$ $B=\begin{bmatrix}-1 & 8\\ 2 & -2\\ -4 & 3\end{bmatrix}$, Find

a) $A+B$ *b)* $A-B$ *c)* $3A$ *d)* $-2B$ *e)* $2A+3B$ *f)* A^2 *g)* AB *h)* BA

36. Given $A=\begin{bmatrix}-2 & 3 & -1\\ 0 & -1 & 2\\ -4 & 3 & 3\end{bmatrix}$ $B=\begin{bmatrix}1 & -2 & 0\\ 2 & 3 & -1\\ 3 & -1 & 2\end{bmatrix}$, Find

a) $A+B$ *b)* $A-B$ *c)* $3A$ *d)* $-2B$ *e)* $2A+3B$ *f)* A^2 *g)* AB *h)* BA

37. Given $A=\begin{bmatrix}0 & 2 & 0\\ 1 & -3 & 3\\ 5 & 4 & -2\end{bmatrix}$ $B=\begin{bmatrix}-1 & 2 & 4\\ 3 & 3 & -2\\ -4 & 4 & 3\end{bmatrix}$, Find

a) $A+B$ *b)* $A-B$ *c)* $3A$ *d)* $-2B$ *e)* $2A+3B$ *f)* A^2 *g)* AB *h)* BA

38. Given $A=\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}$ $B=\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}$ $C=\begin{pmatrix}4 & 3 & 2\\ -1 & 2 & 1\end{pmatrix}$ $D=\begin{pmatrix}-2 & 3\\ 2 & -1\\ 3 & 2\end{pmatrix}$, Find

a) $4A-2B$ *b)* $3A+C$ *c)* $3A+B$ *d)* $2A-3B$ *e)* AB *f)* BA *g)* A^2 *h)* B^3 *i)* AC *j)* CA *k)* CD *l)* DC

39. Given $A=\begin{pmatrix}2 & 4\\ 3 & -1\end{pmatrix}$ $B=\begin{pmatrix}-1 & 3\\ 2 & -1\end{pmatrix}$ $C=\begin{pmatrix}1 & 4 & 5\\ -2 & 3 & 4\\ -1 & 0 & -2\end{pmatrix}$ $D=\begin{pmatrix}2 & 4 & -2\\ 0 & 3 & 5\\ -3 & 1 & 1\end{pmatrix}$, Find

a) $4A-2B$ *b)* $3A+C$ *c)* $3A+B$ *d)* $2A-3B$ *e)* AB *f)* BA *g)* A^2 *h)* B^3 *i)* AC *j)* CB *k)* CD *l)* DC

40. A contractor builds three kinds of houses, models *A*, *B*, and *C*, with a choice of two styles, Spanish and contemporary. Matrix *P* shows the number of each kind of house planned for a new 100-home subdivision. The amounts for each of the exterior materials depend primarily on the style of the house. These amounts are shown in matrix *Q*. (concrete is in *cubic yards*, lumber in units of 1000 board *feet*, brick in 1000*s*, and shingles in units of 100 ft^2.) Matrix *R* gives the cost in dollars for each kind of material.

a) What is the total cost of these materials for each model?

b) How much of each of four kinds of material must be ordered
c) What is the total cost for exterior materials?

41. Mitchell Fabricators manufactures three styles of bicycle frames in its two plants. The following table shows the number of each style produced at each plant

	Mountain Bike	***Racing Bike***	***Touring Bike***
North Plant	150	120	100
South Plant	180	90	130

a) Write a 2 *x* 3 matrix ***A*** that represents the information in the table
b) The manufacturer increased production of each style by 20%. Find a Matrix ***M*** that represents the increased production figures.
c) Find the matrix ***A + M*** and tell what it represents

42. Sal's Shoes and Fred's Footwear both have outlets in California and Arizona. Sal's sells shoes for $80, sandals for $40, and boots for $120. Fred's prices are $60, $30, and $150 for shoes, sandals and boots, respectively. Half of all sales in California stores are shoes, 1/4 are *sandals*, and 1/4 are *boots*. In Arizona, the fractions are 1/5 *shoes*, 1/5 are *sandals*, and 3/5 are *boots*.
a) Write a 2 x 3 matrix called *P* representing prices for the two stores and three types of footwear.
b) Write a 2 x 3 matrix called *F* representing fraction of each type of footwear sold in each state.
c) Only one of the two products *PF* and *FP* is meaningful. Determine which one it is, calculate the product, and describe what the entries represent.

Section 4.3 – Multiplicative Inverses of Matrices

Identity Matrix

The n *x* n identity matrix with 1's on the main diagonal and 0's elsewhere and is denoted by

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2x2} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3x3} \qquad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{I} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

The Multiplicative Identity Matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then $AI = IA = A$

Example

$$A = \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution

$$AI = \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4(1) - 7(0) & 4(0) - 7(1) \\ -3(1) + 2(0) & -3(0) + 2(1) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix} = A$$

$$= A$$

Multiplicative inverse of a matrix

Multiplicative inverse of a matrix $A_{n\times n}$ and $A^{-1}_{n\times n}$ if exists, then:

$$A\cdot A^{-1}=A^{-1}\cdot A=I$$

Example

Show that B is Multiplicative inverse of A.

$A=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$ $\quad$ $B=\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}$

Solution

$$\boldsymbol{A.B}=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}2(1)-1(1) & 2(-1)+1(2)\\ 1(1)+1(-1) & 1(-1)+1(2)\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\underline{=I}$$

$$BA=\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\underline{=I}$$

$\therefore$ B is multiplicative inverse of a matrix A: $B=A^{-1}$

Finding Inverse matrix

To find inverse matrix using Gauss-Jordan method:

$$\left[A \middle| I\right] \rightarrow \left[I \middle| A^{-1}\right] \qquad \textit{where } A^{-1} \textit{ read as "A inverse"}$$

For 2 by 2 matrices (*only*)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix}$$

If $ad - bc = 0$, then A^{-1} doesn't exist

Example

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix} \Rightarrow A^{-1} = ?$$

Solution

$$A^{-1} = \frac{1}{(-1)(4) - (-2)(3)}\begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Example

$$A=\begin{bmatrix}3 & -2\\ -1 & 1\end{bmatrix} \Rightarrow A^{-1}=?$$

Solution

$$A^{-1}=\frac{1}{(3)(1)-(-2)(-1)}\begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix}$$

To find inverse matrix using Gauss-Jordan method:

$$\left[A \middle| I\right] \rightarrow \left[I \middle| A^{-1}\right] \qquad \textit{where } A^{-1} \textit{ read as "A inverse"}$$

Example

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \qquad \text{Find } A^{-1}$$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccccc} -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 \\ \hline 0 & 2 & 5 & 1 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & -1 & 0 & 0 \\ \hline 0 & -1 & -2 & -1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right] \tfrac{1}{2}R_2$$

$$\begin{array}{cccccc} 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right] R_3 + R_2$$

$$\begin{array}{cccccc} 0 & -1 & -2 & -1 & 0 & 1 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \hline 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right] 2R_3$$

$$\begin{array}{cccccc} 0 & 0 & 1 & -1 & 1 & 2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 - 2R_3 \\ R_2 - \frac{5}{2}R_3 \\ \end{array}$$

$$\begin{array}{cccccc} 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{5}{2} & \frac{5}{2} & -\frac{5}{2} & -5 \\ \hline 0 & 1 & 0 & 3 & -2 & -5 \end{array} \qquad \begin{array}{cccccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 2 & -2 & -4 \\ \hline 1 & 0 & 0 & 3 & -2 & -4 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right]$$

$$\Rightarrow \qquad A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

Solving a System Using A^{-1}

To solve the matrix equation $AX = B$.

- **X**: matrix of the variables
- **A**: Coefficient matrix
- **B**: Constant matrix

$$
\begin{aligned}
AX &= B \\
A^{-1}(AX) &= A^{-1}B && \textit{Multiply both side by } A^{-1} \\
\left(A^{-1}A\right)X &= A^{-1}B && \textit{Associate property} \\
IX &= A^{-1}B && \textit{Multiplicative inverse property} \\
X &= A^{-1}B && \textit{Identity property}
\end{aligned}
$$

Example

Solve the system using A^{-1}

$$
\begin{cases} x + 2z = 6 \\ -x + 2y + 3z = -5 \\ x - y = 6 \end{cases}
\qquad \text{Given } A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}
$$

Solution

$$
\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X}
\underset{=}{=}
\underbrace{\begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix}}_{B}
$$

$X = A^{-1}B$

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} . \begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix}
= \begin{bmatrix} 3(6)-2(-5)-4(6) \\ 3(6)-2(-5)-5(6) \\ -1(6)+1(-5)+2(6) \end{bmatrix}
= \begin{bmatrix} 18+10-24 \\ 18+10-30 \\ -6-5+12 \end{bmatrix}
$$

$$
= \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}
$$

Solution: $\{(4, -2, 1)\}$

Example

Use the inverse of the coefficient matrix to solve the linear system

$$\begin{aligned} 2x - 3y &= 4 \\ x + 5y &= 2 \end{aligned}$$

Solution

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix} \qquad B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

The solution of the system is $(2, 0)$

Exercise Section 4.3 – Multiplicative Inverses of Matrices

Show that B is Multiplicative inverse of A

1. $A=\begin{bmatrix}-2 & 4\\ 1 & -2\end{bmatrix}$ $B=\begin{bmatrix}1 & 2\\ -1 & -2\end{bmatrix}$

2. $A=\begin{pmatrix}3 & 1\\ 2 & 1\end{pmatrix}$ & $B=\begin{pmatrix}1 & -1\\ -2 & 3\end{pmatrix}$

Find the inverse, if exists, of

3. $A=\begin{bmatrix}2 & -6\\ 1 & -2\end{bmatrix}$

4. $A=\begin{bmatrix}10 & -2\\ -5 & 1\end{bmatrix}$

5. $A=\begin{bmatrix}-2 & 3\\ -3 & 4\end{bmatrix}$

6. $A=\begin{bmatrix}a & b\\ 3 & 3\end{bmatrix}$

7. $A=\begin{bmatrix}-2 & a\\ 4 & a\end{bmatrix}$

8. $A=\begin{bmatrix}4 & 4\\ b & a\end{bmatrix}$

9. $A=\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}$

10. $A=\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}$

11. $A=\begin{pmatrix}2 & 4\\ 3 & -1\end{pmatrix}$

12. $A=\begin{pmatrix}-1 & 3\\ 2 & -1\end{pmatrix}$

13. $A=\begin{pmatrix}1 & 3\\ -2 & 5\end{pmatrix}$

14. $A=\begin{pmatrix}4 & 6\\ 2 & 3\end{pmatrix}$

15. $A=\begin{pmatrix}-6 & 9\\ 2 & -3\end{pmatrix}$

16. $A=\begin{pmatrix}-2 & 7\\ 0 & 2\end{pmatrix}$

17. $A=\begin{pmatrix}4 & -16\\ 1 & -4\end{pmatrix}$

18. $A=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$

19. $A=\begin{pmatrix}2 & 1\\ a & a\end{pmatrix}$

20. $A=\begin{pmatrix}b & 3\\ b & 2\end{pmatrix}$

21. $A=\begin{pmatrix}1 & a\\ 3 & a\end{pmatrix}$

22. $A=\begin{pmatrix}a & 2\\ 2 & a\end{pmatrix}$

23. $A=\begin{pmatrix}4 & -2\\ 2 & -1\end{pmatrix}$

24. $A=\begin{pmatrix}-3 & \frac{1}{2}\\ 6 & -1\end{pmatrix}$

25. $A=\begin{bmatrix}1 & 0 & 1\\ 2 & -2 & -1\\ 3 & 0 & 0\end{bmatrix}$

26. $A=\begin{bmatrix}1 & 2 & -1\\ 3 & 5 & 3\\ 2 & 4 & 3\end{bmatrix}$

27. $A=\begin{bmatrix}1 & 2 & -1\\ -2 & 0 & 1\\ 1 & -1 & 0\end{bmatrix}$

28. $A=\begin{bmatrix}-2 & 5 & 3\\ 4 & -1 & 3\\ 7 & -2 & 5\end{bmatrix}$

29. $A=\begin{pmatrix}1 & 1 & 0\\ -1 & 3 & 4\\ 0 & 4 & 3\end{pmatrix}$

30. $A=\begin{pmatrix}1 & -1 & 1\\ 0 & -2 & 1\\ -2 & -3 & 0\end{pmatrix}$

31. $A=\begin{pmatrix}1 & 0 & 2\\ -1 & 2 & 3\\ 1 & -1 & 0\end{pmatrix}$

32. $A=\begin{pmatrix}1 & 1 & 1\\ 3 & 2 & -1\\ 3 & 1 & 2\end{pmatrix}$

33. $A=\begin{pmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$

34. $A=\begin{pmatrix} -3 & 1 & -1 \\ 1 & -4 & -7 \\ 1 & 2 & 5 \end{pmatrix}$

35. $A=\begin{pmatrix} 1 & 1 & -3 \\ 2 & -4 & 1 \\ -5 & 7 & 1 \end{pmatrix}$

36. $A=\begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{pmatrix}$

37. $A=\begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -6 & 1 \\ -2 & -2 & 5 & 1 \end{bmatrix}$

38. $A=\begin{bmatrix} 1 & -14 & 7 & 38 \\ -1 & 2 & 1 & -2 \\ 1 & 2 & -1 & -6 \\ 1 & -2 & 3 & 6 \end{bmatrix}$

39. $A=\begin{bmatrix} 10 & 20 & -30 & 15 \\ 3 & -7 & 14 & -8 \\ -7 & -2 & -1 & 2 \\ 4 & 4 & -3 & 1 \end{bmatrix}$

State the conditions under which A^{-1} exists. Then find a formula for A^{-1}

40. $A=[x]$

41. $A=\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$

42. $A=\begin{bmatrix} 0 & 0 & x \\ 0 & y & 0 \\ z & 0 & 0 \end{bmatrix}$

43. $A=\begin{bmatrix} x & 1 & 1 & 1 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{bmatrix}$

44. Solve the system using A^{-1} $\begin{cases} x \qquad\;\; +2z=6 \\ -x+2y+3z=-5 \\ x-\;y \qquad\;\;\; =6 \end{cases}$ Given $A^{-1}=\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

45. Solve the system using A^{-1}

$$\begin{cases} x+2y+5z=2 \\ 2x+3y+8z=3 \\ -x+\;\;y+2z=3 \end{cases}$$

a) Write the linear system as a matrix equation in the form *AX*= *B*

b) Solve the system using the inverse that is given for the coefficient matrix

$$\textit{the inverse of } \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix} \textit{ is } \begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}$$

46. Solve the system using A^{-1}

$$\begin{cases} x - y + z = 8 \\ 2y - z = -7 \\ 2x + 3y = 1 \end{cases}$$

a) Write the linear system as a matrix equation in the form $AX = B$

b) Solve the system using the inverse that is given for the coefficient matrix

$$\textit{the inverse is } \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

(47– 75) Use the ***inverse*** of the coefficient matrix to solve the linear system

47. $\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$

48. $\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$

49. $\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$

50. $\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$

51. $\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$

52. $\begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$

53. $\begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$

54. $\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$

55. $\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$

56. $\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$

57. $\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$

58. $\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$

59. $\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$

60. $\begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases}$

61. $\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$

62. $\begin{cases} -2x + 3y = 4 \\ -3x + 4y = 5 \end{cases}$

63. $\begin{cases} x - 2y = 6 \\ 4x + 3y = 2 \end{cases}$

64. $\begin{cases} 2x - 3y = 7 \\ 4x + y = -7 \end{cases}$

65. $\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$

66. $\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$

67. $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

68. $\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$

69. $\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$

70. $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

71. $\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$

72. $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

73. $\begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$

Section 4.4 – Determinants and Cramer's Rule

Determinant of a 2 *x* 2 Matrix

Determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and is define as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example

Let $A = \begin{bmatrix} -3 & 4 \\ 6 & 8 \end{bmatrix}$. Find $|A|$

Solution

$$|A| = \begin{vmatrix} -3 & 4 \\ 6 & 8 \end{vmatrix}$$

$$= -3(8) - 4(6)$$

$$\underline{= -48}$$

Example

Evaluate: $\begin{vmatrix} 2 & -3 \\ -4 & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & -3 \\ -4 & 1 \end{vmatrix} = 2(1) - (-3)(-4)$$

$$= 2 - 12$$

$$\underline{= -10}$$

$$\boldsymbol{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor

For a square matrix $A = \left[a_{ij}\right]$, the minor M_{ij}. Of an element a_{ij} is the determinant of the matrix formed by deleting the i^{th} row and the j^{th} column of A.

Cofactor: $A_{ij} = (-1)^{i+j} M_{ij}$

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example

$A = \begin{pmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{pmatrix}$ Find the determinant of A.

Solution

$$|A| = \begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix}$$

$$= -8\begin{vmatrix} -6 & 7 \\ -3 & 5 \end{vmatrix} - 0\begin{vmatrix} 4 & 7 \\ -1 & 5 \end{vmatrix} + 6\begin{vmatrix} 4 & -6 \\ -1 & -3 \end{vmatrix}$$

$= -8(-30 - (-21)) - 0 + 6(-12 - 6)$

$= -8(-9) + 6(-18)$

$\underline{= -36}$

Determinant Using Diagonal Method

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix} \qquad a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \quad (1)$$

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix} \qquad a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} \quad (2)$$

Determinant: $\boldsymbol{D = (1) - (2)}$

Example

Evaluate $\begin{vmatrix} 2 & -3 & -2 \\ -1 & -4 & -3 \\ -1 & 0 & 2 \end{vmatrix}$

Solution

$$\left|\begin{matrix} 2 & -3 & -2 \\ -1 & -4 & -3 \\ -1 & 0 & 2 \end{matrix}\right| \begin{matrix} 2 & -3 \\ -1 & -4 \\ -1 & 0 \end{matrix} = 2(-4)(2) + (-3)(-3)(-1) + (-2)(-1)(0) - (-2)(-4)(-1) - (2)(-3)(0) - (-3)(-1)(2)$$

$$\underline{= -23}$$

Example

Evaluate $\begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix}$

Solution

$$\left|\begin{matrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{matrix}\right| \begin{matrix} -8 & 0 \\ 4 & -6 \\ -1 & -3 \end{matrix} = (-8)(-6)(-5) + 0(7)(-1) + 6(4)(-3) - 6(-6)(-1) - (-8)(7)(-3) - 0(4)(5)$$

$$\underline{= -36}$$

Example

Evaluate $\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix}\begin{matrix} x & 0 \\ 2 & x \\ -3 & x \end{matrix} = x^2 + 0 - 2x - (3x) - x^4 - 0$$

$$\underline{= -x^4 + x^2 - 5x}$$

Cramer's Rule

Given:

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

$$\text{If} \quad D \ne 0 \qquad x = \frac{D_x}{D} \qquad y = \frac{D_y}{D}$$

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Example

Use Cramer's rule to solve the system

$$5x + 7y = -1$$
$$6x + 8y = 1$$

Solution

$$D = \begin{vmatrix} 5 & 7 \\ 6 & 8 \end{vmatrix} = -2$$

$$D_x = \begin{vmatrix} -1 & 7 \\ 1 & 8 \end{vmatrix} = -15$$

$$D_y = \begin{vmatrix} 5 & -1 \\ 6 & 1 \end{vmatrix} = 11$$

$$x = \frac{D_x}{D} = \frac{-15}{-2} = \frac{15}{2}$$

$$y = \frac{D_y}{D} = \frac{11}{-2} = -\frac{11}{2}$$

Solution: $\left(\frac{15}{2}, \; -\frac{11}{2} \right)$

$$D_x = \begin{matrix} b_1 & a_{12} & a_{13} & b_1 & a_{12} \\ b_2 & a_{22} & a_{23} & b_2 & a_{22} \\ b_3 & a_{32} & a_{33} & b_3 & a_{32} \end{matrix}$$

$$D_x = b_1 a_{22} a_{33} + a_{12} a_{23} b_3 + a_{13} b_2 a_{32} - a_{13} a_{22} b_3 - b_1 a_{23} a_{32} - a_{12} b_2 a_{33}$$

$$D_y = \begin{matrix} a_{11} & b_1 & a_{13} & a_{11} & b_1 \\ a_{21} & b_2 & a_{23} & a_{21} & b_2 \\ a_{31} & b_3 & a_{33} & a_{31} & b_3 \end{matrix}$$

$$D_z = \begin{matrix} a_{11} & a_{12} & b_1 & a_{11} & a_{12} \\ a_{21} & a_{22} & b_2 & a_{21} & a_{22} \\ a_{31} & a_{32} & b_3 & a_{31} & a_{32} \end{matrix}$$

$$x = \frac{D_x}{D} \qquad y = \frac{D_y}{D} \qquad z = \frac{D_z}{D}$$

Example

Use Cramer's rule to solve the system

$x - 3y + 7z = 13$
$x + y + z = 1$
$x - 2y + 3z = 4$

Solution

$$D = \begin{vmatrix} 1 & -3 & 7 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = -10 \qquad D_x = \begin{vmatrix} 13 & -3 & 7 \\ 1 & 1 & 1 \\ 4 & -2 & 3 \end{vmatrix} = 20$$

$$D_y = \begin{vmatrix} 1 & 13 & 7 \\ 1 & 1 & 1 \\ 1 & 4 & 3 \end{vmatrix} = -6 \qquad D_z = \begin{vmatrix} 1 & -3 & 13 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{vmatrix} = -24$$

$$x = \frac{20}{-10} = -2 \qquad y = \frac{-6}{-10} = \frac{3}{5} \qquad z = \frac{-24}{-10} = \frac{12}{5}$$

Solution: $\left(-2, \frac{3}{5}, \frac{12}{5}\right)$

Exercises   *Section* 4.4 – Determinants and Cramer's Rule

(1 – 34) Evaluate

1. $\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix}$

2. $\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix}$

3. $\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix}$

4. $\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix}$

5. $\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix}$

6. $\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix}$

7. $\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}$

8. $\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix}$

9. $\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix}$

10. $\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix}$

11. $\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix}$

12. $\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix}$

13. $\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix}$

14. $\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix}$

15. $\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix}$

16. $\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix}$

17. $\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix}$

18. $\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix}$

19. $\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix}$

20. $\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix}$

21. $\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix}$

22. $\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix}$

23. $\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix}$

24. $\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix}$

25. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix}$

26. $\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix}$

27. $\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix}$

28. $\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix}$

29. $\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix}$

30. $\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix}$

31. $\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix}$

32. $\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix}$

33. $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix}$

34. $\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix}$

(**35 – 89**) Use Cramer's rule to solve the system

35. $\begin{cases} 3x+2y=-4 \\ 2x-y=-5 \end{cases}$

36. $\begin{cases} 2x+5y=7 \\ 5x-2y=-3 \end{cases}$

37. $\begin{cases} 4x-7y=-16 \\ 2x+5y=9 \end{cases}$

38. $\begin{cases} 3x+2y=4 \\ 2x+y=1 \end{cases}$

39. $\begin{cases} 3x+4y=2 \\ 2x+5y=-1 \end{cases}$

40. $\begin{cases} 5x-2y=4 \\ -10x+4y=7 \end{cases}$

41. $\begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$

42. $\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

43. $\begin{cases} 2x+10y=-14 \\ 7x-2y=-16 \end{cases}$

44. $\begin{cases} 4x-3y=24 \\ -3x+9y=-1 \end{cases}$

45. $\begin{cases} 4x+2y=12 \\ 3x-2y=16 \end{cases}$

46. $\begin{cases} x+2y=-1 \\ 4x-2y=6 \end{cases}$

47. $\begin{cases} x-2y=5 \\ -10x+2y=4 \end{cases}$

48. $\begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$

49. $\begin{cases} 4x-4y=-12 \\ 4x+4y=-20 \end{cases}$

50. $\begin{cases} x+y=7 \\ x-y=3 \end{cases}$

51. $\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

52. $\begin{cases} 12x+3y=15 \\ 2x-3y=13 \end{cases}$

53. $\begin{cases} x-2y=5 \\ 5x-y=-2 \end{cases}$

54. $\begin{cases} 3x+2y=2 \\ 2x+2y=3 \end{cases}$

55. $\begin{cases} 4x-5y=17 \\ 2x+3y=3 \end{cases}$

56. $\begin{cases} x-3y=4 \\ 3x-4y=12 \end{cases}$

57. $\begin{cases} 2x-9y=5 \\ 3x-3y=11 \end{cases}$

58. $\begin{cases} 3x-4y=4 \\ x+y=6 \end{cases}$

59. $\begin{cases} 3x=7y+1 \\ 2x=3y-1 \end{cases}$

60. $\begin{cases} 2x=3y+2 \\ 5x=51-4y \end{cases}$

61. $\begin{cases} y=-4x+2 \\ 2x=3y-1 \end{cases}$

62. $\begin{cases} 3x=2-3y \\ 2y=3-2x \end{cases}$

63. $\begin{cases} x+2y-3=0 \\ 12=8y+4x \end{cases}$

64. $\begin{cases} 7x-2y=3 \\ 3x+y=5 \end{cases}$

65. $\begin{cases} 3x+2y-z=4 \\ 3x-2y+z=5 \\ 4x-5y-z=-1 \end{cases}$

66. $\begin{cases} x+y+z=2 \\ 2x+y-z=5 \\ x-y+z=-2 \end{cases}$

67. $\begin{cases} 2x+y+z=9 \\ -x-y+z=1 \\ 3x-y+z=9 \end{cases}$

68. $\begin{cases} 3y-z=-1 \\ x+5y-z=-4 \\ -3x+6y+2z=11 \end{cases}$

69. $\begin{cases} x+3y+4z=14 \\ 2x-3y+2z=10 \\ 3x-y+z=9 \end{cases}$

70. $\begin{cases} x+4y-z=20 \\ 3x+2y+z=8 \\ 2x-3y+2z=-16 \end{cases}$

71. $\begin{cases} -2x+6y+7z=3 \\ -4x+5y+3z=7 \\ -6x+3y+5z=-4 \end{cases}$

72. $\begin{cases} 2x-y+z=1 \\ 3x-3y+4z=5 \\ 4x-2y+3z=4 \end{cases}$

73. $\begin{cases} 3x-4y+4z=7 \\ x-y-2z=2 \\ 2x-3y+6z=5 \end{cases}$

74. $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

75. $\begin{cases} x+y+z=3 \\ -y+2z=1 \\ -x+z=0 \end{cases}$

76. $\begin{cases} 3x+y+3z=14 \\ 7x+5y+8z=37 \\ x+3y+2z=9 \end{cases}$

77. $\begin{cases} 4x-2y+z=7 \\ x+y+z=-2 \\ 4x+2y+z=3 \end{cases}$

78. $\begin{cases} 2y-z=7 \\ x+2y+z=17 \\ 2x-3y+2z=-1 \end{cases}$

79. $\begin{cases} 2x-2y+z=-4 \\ 6x+4y-3z=-24 \\ x-2y+2z=1 \end{cases}$

80. $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

81. $\begin{cases} 2x-y+2z=-8 \\ x+2y-3z=9 \\ 3x-y-4z=3 \end{cases}$

82. $\begin{cases} x \quad -3z=-5 \\ 2x-y+2z=16 \\ 7x-3y-5z=19 \end{cases}$

83. $\begin{cases} x+2y-z=5 \\ 2x-y+3z=0 \\ 2y+z=1 \end{cases}$

84. $\begin{cases} x+y+z=6 \\ 3x+4y-7z=1 \\ 2x-y+3z=5 \end{cases}$

85. $\begin{cases} 3x+2y+3z=3 \\ 4x-5y+7z=1 \\ 2x+3y-2z=6 \end{cases}$

86. $\begin{cases} 4x+5y \quad =2 \\ 11x+y+2z=3 \\ x+5y+2z=1 \end{cases}$

87. $\begin{cases} x-4y+z=6 \\ 4x-y+2z=-1 \\ 2x+2y-3z=-20 \end{cases}$

88. $\begin{cases} 2x-y+z=-1 \\ 3x+4y-z=-1 \\ 4x-y+2z=-1 \end{cases}$

89. $\begin{cases} -x_1-4x_2+2x_3+x_4=-32 \\ 2x_1-x_2+7x_3+9x_4=14 \\ -x_1+x_2+3x_3+x_4=11 \\ -x_1-2x_2+x_3-4x_4=-4 \end{cases}$

(**90 – 101**) Solve for x

90. $\begin{vmatrix} x & 3 \\ 2 & 1 \end{vmatrix}=12$

91. $\begin{vmatrix} x & 1 \\ 2 & x \end{vmatrix}=-1$

92. $\begin{vmatrix} 3 & x \\ x & 4 \end{vmatrix}=-13$

93. $\begin{vmatrix} x & 2 \\ 3 & x \end{vmatrix}=x$

94. $\begin{vmatrix} 4 & 6 \\ -2 & x \end{vmatrix}=32$

95. $\begin{vmatrix} x+2 & -3 \\ x+5 & -4 \end{vmatrix}=3x-5$

96. $\begin{vmatrix} x+3 & -6 \\ x-2 & -4 \end{vmatrix}=28$

97. $\begin{vmatrix} x & -3 \\ -1 & x \end{vmatrix}\geq 0$

98. $\begin{vmatrix} 2 & x & 1 \\ 1 & 2 & -1 \\ 3 & 4 & -2 \end{vmatrix}=-6$

99. $\begin{vmatrix} 1 & x & -3 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix}=8$

100. $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix}=39$

101. $\begin{vmatrix} x & 0 & 0 \\ 7 & x & 1 \\ 7 & 2 & 1 \end{vmatrix}=-1$

102. Find the quadratic function $f(x)=ax^2+bx+c$ for which $f(1)=-10, \quad f(-2)=-31, \quad f(2)=-19$. What is the function?

103. you wish to mix candy worth $3.44 per pound with candy worth $9.96 per pound to form 24 pounds of a mixture worth $8.33 per pound.

a) Write the system equations?

b) How many pounds of each candy should you use?

104. Anne and Nancy use a metal alloy that is 17.76% copper to make jewelry. How many ounces of a 15% alloy must be mixed with a 19% alloy to form 100 ounces of the desired alloy?

105. A company makes 3 types of cable. Cable ***A*** requires 3 black, 3 white, and 2 red wires. ***B*** requires 1 black, 2 white, and 1 red. ***C*** requires 2 black, 1 white, and 2 red. They used 95 black, 100 white and 80 red wires.

a) Write the system equations?

b) How many of each cable were made?

106. A basketball fieldhouse seats 15,000. Courtside seats sell for $8.00, end zone for $6.00, and balcony for $5.00. Total for a sell-out is $86,000. If half the courtside and balcony and all end zone seats are sold, ticket sales total $49,000.

a) Write the system equations?

b) How many of each type of seat are there?

107. A movie theater charges $9.00 for adults and $7.00 for senior citizens. On a day when 325 people paid admission, the total receipts were $2,495.

a) Write the system equations?

b) How many who paid were adults? How many were seniors?

108. A Broadway theater has 500 seats, divided into orchestra, main, and balcony seating. Orchestra seats sell for $150, main seats for $135, and balcony seats for $110. If all the seats are sold, the gross revenue to the theater is $64,250. If all the main and balcony seats are sold, but only half the orchestra seats are sold, the gross revenue is $56,750.

a) Write the system equations?

b) How many of each kind of seat are there?

109. A movie theater charges $11 for adults, $6.50 for children, and $9 for senior citizens. One day the theater sold 405 tickets and collected $3,315 in receipts. Twice as many children's tickets were sold as adult tickets.

a) Write the system equations?

b) How many adults, children, and senior citizens went to the theater that day?

110. Emma has $20,000 to invest. As her financial planner, you recommend that she diversify into three investements: Treasure bills that yield 5% simple interest. Treasury bonds tht yield 7% simple interest, and corporate bonds that yield 10% simple interest. Emma wishes to earn $1,390 per year in income. Also, Emma wants her investment in Treasury bills to be $3,000 more than her investment in corporate bonds. How much money should Emma place in each investement?

111. A person invested $17,000 for one year, part at 10%, part at 12%, and the remainder at 15%. The total annual income from these investements was $2,110. The amount of money invested at 12% was $1,000 less than the amounts invested at 10% and 15% combined. Find the amount invested at each rate.

112. At a production, 400 tickets were sold. The ticket prices were \$8, \$10, and \$12, and the total income from ticket sales was \$3,700. How many tickets of each type were sold if the combined number of \$8 and \$10 tickets sold was 7 times the number of \$12 tickets sold?

113. A certain brand of razor blades comes in packages if 6, 12, and 24 blades, costing \$2, \$3, and \$4 per package, respectively. A store sold 12 packages containing a total of 162 razor blades and took in \$35. How many packages of each type were sold?

114. A store sells cashews for \$5.00 per pound and peanuts for \$1.50 per pound. The manager decides to mix 30 pounds of peanuts with some cashews and sell the mixture for \$3.00 per pound.

a) Write the system equations?

b) How many pounds of cashews should be mixed with peanuts so that the mixture will produce the same revenue as selling the nuts separately?

115. A wireless store takes presale orders for a new smartphone and tablet. He gets 340 preorders for the smartphone and 250 preorders for the tablet. The combined value of the preorders is \$270,500.00. If the price of a smartphone and tablet together is \$965, how much does each device cost?

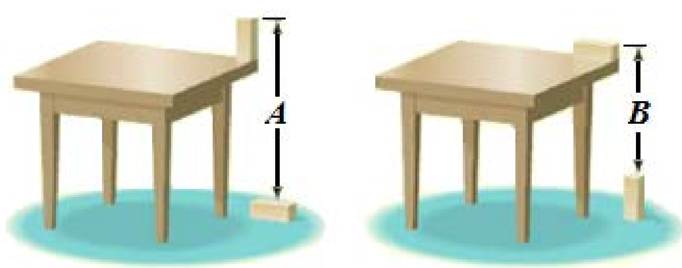
116. A restaurant manager wants to purchase 200 sets of dishes. One design costs \$25 per set, and another costs \$45 per set. If she has only \$7400 to spend, how many sets of each design should be order?

117. One group of people purchased 10 hot dogs and 5 soft drinks at a cost of \$35.00. A second bought 7 hot dogs and 4 soft drinks at a cost of \$25.25. What is the cost of a single hot dog and a single soft drink?

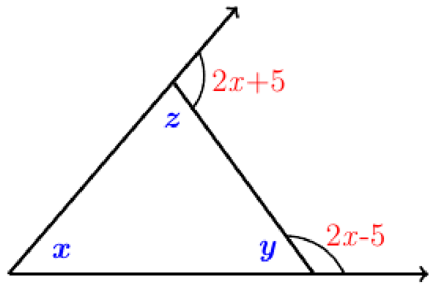
118. The sum of three times the first number, plus the second number, and twice the third number is 5. If 3 times the second number is subtracted from the sum of the first number and 3 times the third number, the result is 2. If the third number is subtracted from the sum of 2 times the first number and 3 times the second number, the result is 1. Find the three numbers.

119. The sum of three numbers is 16. The sum of twice the first number, 3 times the second number, and 4 times the third number is 46. The difference between 5 times the first number and the second number is 31. Find the three numbers.

120. Two blocks of wood having the same length and width are placed on the top and bottom of a table. Length *A* measure 32 *cm*. The blocks are rearranged. Length *B* measures 28 *cm*. Determine the height of the table.

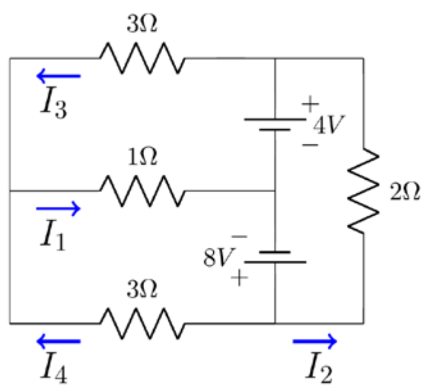


121. In the following triangle, the degree measures of the three interior angles and two of the exterior angles are represented with variables. Find the measure of each interior angle.



122. Three painters (Beth, Bill, and Edie), working together, can paint the exterior of a home in 10 *hours*. Bill and Edie together have painted similar house in 15 *hours*. One day, all three worked on this same kind of house for 4 *hours*, after which Edie left. Beth and Bill required 8 more *hours* to finish. Assuming no gain or loss in efficiency, how long should it take each person to complete such a job alone?

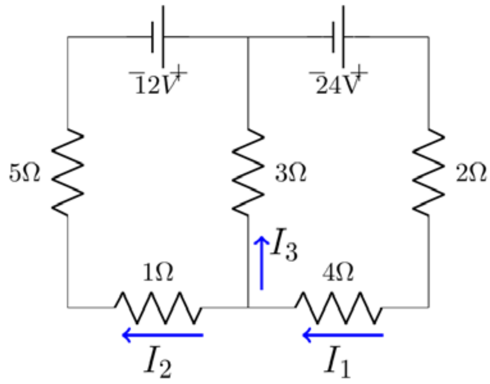
123. An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:



$$\begin{cases} I_1 = I_3 + I_4 \\ I_1 + 5I_4 = 8 \\ I_1 + 3I_3 = 4 \\ 8 - 4 - 2I_2 = 0 \end{cases}$$

Find the currents I_1, I_2, I_3, and I_4

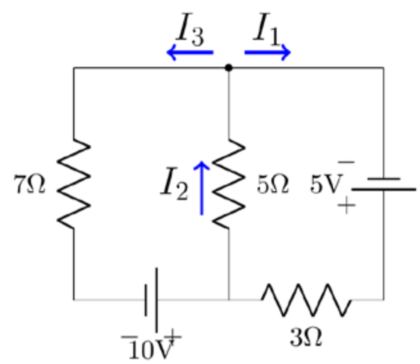
124. An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:



$$\begin{cases} I_1 = I_2 + I_3 \\ 24 - 6I_1 - 3I_3 = 0 \\ 12 + 24 - 6I_1 - 6I_2 = 0 \end{cases}$$

Find the currents I_1, I_2, and I_3

125. An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:



$$\begin{cases} I_2 = I_1 + I_3 \\ 5 - 3I_1 - 5I_2 = 0 \\ 10 - 5I_2 - 7I_3 = 0 \end{cases}$$

Find the currents I_1, I_2, and I_3

126. An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:

$$\begin{cases} I_3 = I_1 + I_2 \\ 6I_2 + 4I_3 = 8 \\ 8I_1 = 4 + 6I_2 \end{cases}$$

Find the currents I_1, I_2, and I_3