Chapter 3. Modeling and Applications

Section 3.1. Modeling Population Growth

1. The equation of the Malthusian model is $P(t) = Ce^{rt}$. Apply the initial condition $P(0) = 100$. Then $100 = Ce^0$, or $C = 100$. Next apply the condition $P(1) = 300$. Then $300 = 100e^r$. Solving gives the reproductive rate $r = \ln 3 \approx 1.0986$. The population after five days is $P(5) = 100e^{5\ln 3} = 24300$.

2. The equation of the Malthusian model is $P(t) = Ce^{rt}$. Apply the cell counts to solve for r and C. $P(1) = 1000$, so $Ce^r = 1000$, i.e. $r = \ln(1000/C)$. Also, $P(2) = 3000$, so $Ce^{2r} = 3000$, i.e. $r = \frac{1}{2}\ln(3000/C)$. Setting these equal and solving, one obtains $C = 1000/3$. Substituting C into $r = \ln(1000/C)$ gives that $r = \ln 3 \approx 1.0986$. In addition $P(0) = C = 1000/3$.

3. Recall the equation of the Malthusian model $P(t) = Ce^{rt}$. The constant C is the initial population $P(0)$, so the tripling condition is stated as $P(10) = P(0)e^{10r} = 3P(0)$. Thus, $r = (1/10)\ln 3 \approx 0.1099$. The population doubles precisely when $P(t) = 2P(0) = P(0)e^{(t\ln 3)/10}$, that is, when $(t\ln 3)/10 = \ln 2$. Solving, one obtains $t = (10\ln 2)/\ln 3 \approx 6.3093$.

4. The solution of the Malthusian model is $P(t) = Ce^{rt}$. The constant C is the initial population $P(0)$, so the doubling condition is stated as $P(10) = P(0)e^{10r} = 2P(0)$. Thus, $r = (1/10)\ln 2$. To find the value of t so that $P(t) = 10000$, when $P(0) = 1000$, solve $10000 = 1000e^{(t\ln 2)/10}$ for t. This gives $t = (10\ln 10)/\ln 2 \approx 33.2193$.

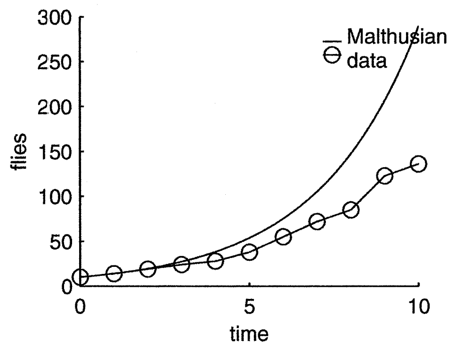
5. The modified Malthusian model will take the form $P' = rP - h$. Multiplying by the integrating factor e^{-rt} and putting the terms involving P on the left gives the equation $P'e^{-rt} - re^{-rt}P = -he^{-rt}$. Solve to obtain $Pe^{-rt} = (h/r)e^{-rt} + C$, where C is a constant. This constant has the value $P(0) - (h/r)$, obtained by substituting $t = 0$. Multiply by e^{rt} and get the solution $P(t) = (h/r) + (P(0) - (h/r))e^{rt}$. The long-time activity of the population depends entirely on the sign of $P(0) - (h/r)$. If it is negative, then the population will die out. If it is positive, the population will grow exponentially. If it is zero, then the population will remain at a constant h/r.

6. First, calculate the reproduction rate r. Under normal Malthusian conditions, the population doubles every eight hours, so $P(8) = 2P(0) = P(0)e^{8r}$. Thus, $r = (1/8)\ln 2 \approx 0.0866$. Now, using the solution to the modified Malthusian model $P' = rP - h$, obtain $P(t) = (h/r) + (P(0) - (h/r))e^{rt}$. Here, $P(0) = 20000$ and $h = 2000$, so $P(t) = (16000/\ln 2) + (20000 - (16000/\ln 2))e^{(t\ln 2)/8} \approx 23083 - 3083e^{0.0866t}$. The culture will be depleted when $(16000/\ln 2) + (20000 - (16000/\ln 2))e^{(t\ln 2)/8} = 0$. Solving for t gives $t = (8/\ln 2)\ln(-16000/(20000\ln 2 - 16000)) \approx 23.2350$. Now, if $P(0) = 25000$, then the solution becomes $P(t) = (16000/\ln 2) + (25000 - (16000/\ln 2))e^{(t\ln 2)/8} \approx 23083 + 1917e^{0.0866t}$. When t is as above, $P(t) \approx 37421$. Recalling the solution to problem 3.1.5, one will note that this indeed should be the case, as the initial population $P(0)$ is now greater than the critical value h/r.

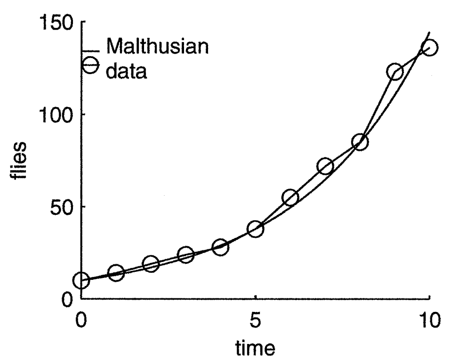
7. The reproduction rate r is calculated as before by solving $P(4) = 2P(0) = P(0)e^{4r}$. Thus, $r = (1/4)\ln 2 \approx 0.1733$. Using the result from problem 3.1.5, the population is given by $P(t) = (h/r) + (P(0) - (h/r))e^{rt}$. In order for the population to not explode, the coefficient $P(0) - (h/r)$ must not be positive. Hence the harvesting rate must be at least $rP(0)$, which here is $(10000)(1/4)\ln 2 = 2500\ln 2 \approx 1732.9$.

8. (a) The first data point is $P(0) = 10$, and the second is $P(1) = 14$. The equation of the Malthusian model is $P(t) = Ce^{rt}$. Thus, $C = P(0) =$

10, and $r = \ln(P(1)/10) = \ln(7/5)$. See the following figure for plots.



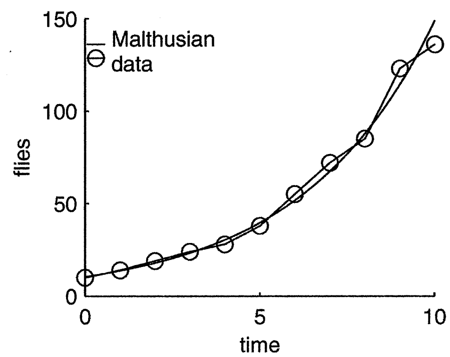
(b) Now use the data points $P(0) = 10$ and $P(5) = 38$. Using the equation of the Malthusian model, we get $C = P(0) = 10$ and $r = (1/5)\ln(P(5)/10)$ or $r = (1/5)\ln(3.8)$. See the following figure for plots.



(c) Using linear regression on the function $\ln P = \ln C + rt$ here gives a system

$$\begin{pmatrix} 11 & 55 \\ 55 & 385 \end{pmatrix} \begin{pmatrix} \ln C \\ r \end{pmatrix} = \begin{pmatrix} 40.4854 \\ 231.5369 \end{pmatrix}$$

The solution to this is $C \approx 10.5626$ and $r \approx 0.2646$. See the following figure for plots.



(d) The reproductive rate found using only the first two data points (part a) was the least accurate. The rate found using an early data point and a later data point (part b) improved the accuracy. The best estimate of r came from the linear regression in part c, because it used all of the data.

(e) As long as an accurate value of r is used, the Malthusian model is a good approximation of the data in the table.

9. (a) For simplicity of notation, we will let $\sum$ represent $\sum_{i=1}^{n}$. Start with

$$S = \sum [y_i - (mx_i + b)]^2 \qquad (1.1)$$

and differentiate with respect to m to obtain

$$\frac{\partial S}{\partial m} = 2\sum [y_i - (mx_i + b)](-x_i).$$

Set this result equal to zero and divide both sides of the resulting equation by -2 to obtain the result

$$0 = \sum \left[x_i y_i - mx_i^2 - bx_i\right].$$

Familiar properties of summation allow us to write this in the form

$$m\sum x_i^2 + b\sum x_i = \sum x_i y_i. \qquad (1.2)$$

Next, differentiate S, as defined in equation (1.1), with respect to b to obtain

$$\frac{\partial S}{\partial b} = 2\sum [y_i - (mx_i + b)](-1).$$

Set this result equal to zero then divide both sides of the resulting equation to obtain

$$0 = \sum [y_i - mx_i - b].$$

Familiar properties of summation (recall $\sum_{i=1}^{n} b = bn$) allows to write this in the form

$$m \sum x_i + bn = \sum y_i. \qquad (1.3)$$

Let's eliminate b from equations (1.2) and (1.3). Multiply equation (1.2) by n, then multiply equation (1.3) by $-\sum x_i$ to obtain

$$mn \sum x_i^2 + bn \sum x_i = n \sum x_i y_i$$

$$-m \left(\sum x_i\right)^2 - bn \sum x_i = -\sum x_i \sum y_i.$$

Adding,

$$mn \sum x_i^2 - m \left(\sum x_i\right)^2 = n \sum x_i y_i - \sum x_i \sum y_i.$$

Solving this for m,

$$m = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - \left(\sum x_i\right)^2}. \qquad (1.4)$$

Finally, solve equation (1.3) for b to obtain

$$b = \frac{\sum y_i - m \sum x_i}{n}. \qquad (1.5)$$

(b) Now, compare

$$y = mx + b$$

with equation

$$\ln P = \ln C + rt$$

and you will note that we need to associate t_i with x_i, $\ln P_i$ with y_i, r with m, and $\ln C$ with b. With these changes, equation (1.4) becomes

$$r = \frac{n \sum t_i \ln P_i - \sum t_i \sum \ln P_i}{n \sum t_i^2 - \left(\sum t_i\right)^2}, \qquad (1.6)$$

and equation (1.5) becomes

$$\ln C = \frac{\sum \ln P_i - r \sum t_i}{n}. \qquad (1.7)$$

(c) It's helpful to arrange the data in tabular form.

t_i	P_i	$\ln P_i$	t_i^2	$t_i \ln P_i$
0	10	2.3026	0	0.0000
1	25	3.2189	1	3.2189
2	61	4.1109	4	8.2217
3	144	4.9698	9	14.9094
4	360	5.8861	16	23.5444
10	600	20.4883	30	49.8944

Substitute the appropriate totals from the table into equation (1.6).

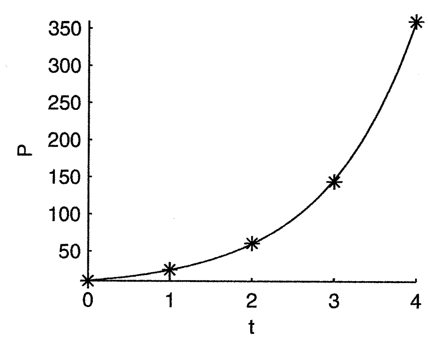
$$r = \frac{5(49.8944) - 10(20.4883)}{5(30) - 10^2} \approx 0.8918$$

Substitute this value of r and the appropriate totals from the table into equation (1.7).

$$\ln C = \frac{20.4883 - 0.8918(10)}{5} \approx 2.3141$$

Thus, $C \approx 10.1154$.

(d) Substituting the values of C and r in $P = Ce^{rt}$, $P = 10.1154e^{0.8918t}$. The plot of the data and the exponential fit follows.



10. To find the minimum and maximum growth rate, use the first derivative test on the growth rate dP/dt. That is, set $d^2P/dt^2 = 0$. Obtain

$$\frac{d^2P}{dt^2} = rP'\left(1 - \frac{2P}{k}\right).$$

Thus, dP/dt reaches an extremum when $P' = 0$ and when $P = k/2$. One wants to show that dP/dt is a maximum when $P = k/2$. So, one checks that d^2P/dt^2 changes sign from positive to negative at $P = k/2$. When $P = k/2$, $P' = r(k/2)(1-(k/2)/k) = rk/4$, which is positive. So any possible change in sign of d^2P/dt^2 is in the third factor $(1-2P/k)$. This factor does change sign from positive to negative at $P = k/2$, so d^2P/dt^2 changes sign from positive to negative at $P = k/2$. Therefore dP/dt reaches a maximum at $P = k/2$.

11. (a) Let $\omega = \alpha P$ and $s = \beta t$. Then solving for P and t and substituting into the logistic equation, one obtains

$$\frac{\beta}{\alpha}\frac{d\omega}{ds} = r\frac{\omega}{\alpha}\left(1-\frac{\omega}{\alpha K}\right).$$

Isolating $d\omega/ds$ and simplifying, one obtains

$$\frac{d\omega}{ds} = \frac{r}{\beta}\omega - \frac{r}{\alpha\beta K}\omega^2.$$

This is as desired.

(b) If $\beta = r$ and $\alpha = 1/K$, then we have $d\omega/ds = \omega - \omega^2$.

(c) Separating variables and using partial fractions, one obtains

$$\left(\frac{1}{\omega} - \frac{1}{1-\omega}\right) d\omega = ds.$$

Thus, $\ln|\omega| - \ln|1-\omega| = s + c$. Exponentiating and combining all constants into a constant a gives

$$\frac{\omega}{1-\omega} = ae^s.$$

Solving for ω gives the desired form

$$\omega = \frac{1}{1-(-a)e^{-s}}.$$

(d) Note that $P = \omega K$ and $s = rt$. Thus,

$$P = \frac{K}{1+ae^{-rt}}.$$

The initial condition $P(t_0) = P_0$ gives $P_0 + P_0ae^{-rt_0} = K$, so

$$a = \frac{K-P_0}{P_0}e^{rt_0}.$$

Using this in the equation for P, one obtains

$$P = \frac{KP_0}{P_0 + (K-P_0)e^{-r(t-t_0)}}.$$

This is the solution (1.13) given in the text.

12. The carrying capacity $K = 20,000$, and the initial condition $P_0 = 1000$, and it is given that $P(10) = 2000$. Using equation (1.13), one obtains

$$2000 = \frac{(20000)(1000)}{1000 + (19000)e^{-10r}}.$$

Solving gives $r = -(\ln(9/19))/10 \approx 0.0747$. After 25 hours, the population is

$$P(25) = \frac{(20000)(1000)}{1000 + (19000)e^{2.5\ln(9/19)}} \approx 5084.$$

Now, find t so that $P(t) = K/2 = 10000$. This gives

$$e^{\frac{1}{10}\ln(\frac{9}{19})t} = \frac{2000-1000}{19000}.$$

That is, $t = 10(-\ln(19))/(\ln(9/19)) \approx 39.4055$.

13. The carrying capacity $K = 20,000$, and the initial condition $P_0 = 1000$, and it is given that $P(8) = 1200$. Using equation (1.13), one obtains

$$1200 = \frac{(20000)(1000)}{1000 + (19000)e^{-8r}}.$$

Solving we get

$$r = (-1/8)\ln(188000/((12)(19000)) \approx 0.0241.$$

Now, find t so that $P(t) = 3K/4 = 15000$. This gives $e^{-rt} = 1/57$. That is, $t = (\ln 57)/r \approx 167.6713$.

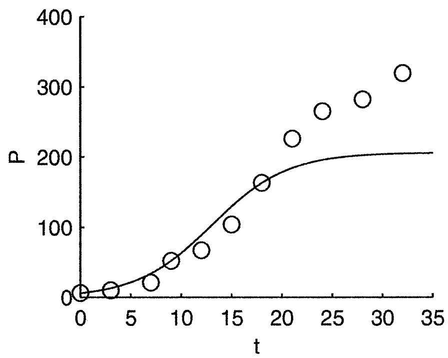
14. (a) Use the following three data points $P_0 = P(0) = 6$, $P_1 = P(9) = 52$, and $P_3 = P(18) = 163$ to estimate r and k: Set $h = 9$. Equation (1.17) gives

$$r = \frac{1}{9}\ln\left(\frac{163(52-6)}{6(163-52)}\right) = 0.269.$$

Then using this in equation (1.16), one obtains

$$K = \frac{(6)(52)(1-e^{-0.269(9)})}{6-52e^{-0.269(9)}} = 205.90.$$

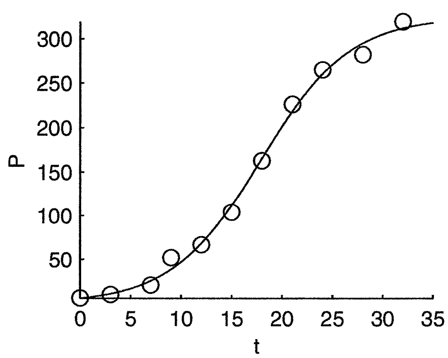
The plot of the logistic solution with these parameters is shown in the following figure.



(b) To compute the parameters we want to find values of the parameters P_0, r, and K which minimize

$$\sum_{j=1}^{11}\left[\log\left(\frac{P(t_j)}{P_j}\right)\right]^2,$$

where t_j and P_j are the data from Table 1, and $P(t)$ is the logistic function in equation 3.1.13 with parameters P_0, r, and K. Using a nonlinear least squares minimization program we find that $K = 324.1241$, $P_0 = 5.6047$, and $r = 0.2268$. The plot of the logistic solution with these parameters is shown in the following figure.



(c) If we use only three points to compute the parameters, the fit of the logistic model is very poor. However, if we use all of the data the fit is quite good.

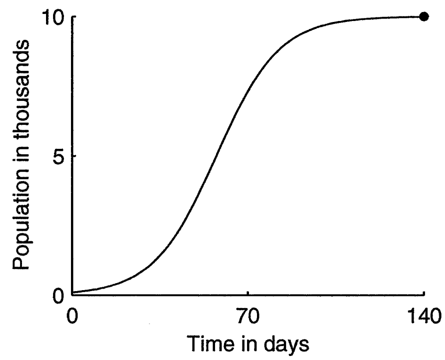
15. The data is shown in the following figure plotted as points. Using this we estimate $K = 9990$. To find r we use the solution to the logistic model

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}},$$

and solve for

$$r = \frac{1}{t}\ln\left(\frac{P(t)(K - P_0)}{P_0(K - P(t))}\right).$$

If we use $t = 60$, with $P(60) = 5510$, we get $r = 0.08$. The logistic solution with these parameters is the solid curve in the figure.



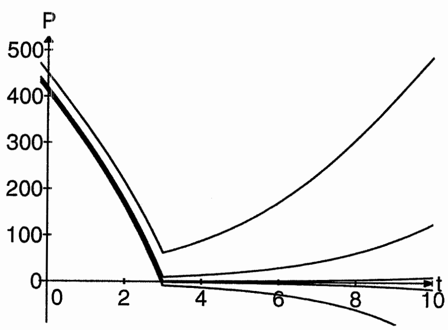
The fit seems to be quite good. However, this method depends on the choice of middle point. If that is an outlier the fit might not be so good as it was in this case.

16. (a) Since the population is measured in thousands, the fishing rate is $h = 0.1$ thousand fish per day. The modified model is $P' = 0.1P(1 - P/10) - 0.1$.

(b) We look for roots of $0.1P(1-P/10)-0.1 = 0$. This leads to the quadratic equation $P^2-10P+10 = 0$, which has roots $P = 5 \pm \sqrt{15}$. Using the graph of $0.1P(1 - P/10) - 0.1$ we see that $P_1 = 5 - \sqrt{15} \approx 1.127$ is an unstable equilibrium point, and $P_2 = 5 + \sqrt{15} \approx 8.873$ is an asymptotically stable equilibrium point.

(c) We have $P' = 0.1P(1 - P/10) - 0.1 > 0$ for $P_1 < P < P_2$ and negative elsewhere. It follows using qualitative analysis that for any starting population greater than P_1, the population tends to the stable equilibrium at P_2. For any starting population smaller than P_1, the population decreases until it dies out. In particular a population of 1000 is doomed while a population of 2000 tends to P_2.

17. (a) In the following figure, the five curves are given by initial conditions $P(0) = 410$, $P(0) = 414$, $P(0) = 415$, $P(0) = 420$, and $P(0) = 450$. Notice that the solution curve with initial condition $P(0) = 415$ never reaches zero, although the curve with initial condition $P(0) = 414$ does. Thus, the critical population is between 414 and 415.



(b) The critical initial population will be the initial condition for a solution curve which has $P = 0$ when $t = 3$. Writing the differential equation for $t < 3$ gives

$$\frac{dP}{dt} = 0.38P - 0.00038P^2 - 200.$$

Separate variables.

$$\frac{dP}{-0.00038P^2 + 0.38P - 200} = dt.$$

Integrating this, one obtains

$$t+C = \frac{2}{\sqrt{.1596}}\tan^{-1}\left(\frac{-0.00076P + 0.38}{\sqrt{.1596}}\right).$$

Using conditions $P(3) = 0$, one obtains

$$3 + C = \frac{2}{\sqrt{.1596}}\tan^{-1}\left(\frac{0.38}{\sqrt{.1596}}\right)$$

or $C = 0.8067$. Now setting $t = 0$ and solving for $P = P_0$, one obtains $P_0 = 414.5557$.

18. We will use days and 1000 individuals as our units. We are given that the carrying capacity is $K = 10$, $P(0) = 1$, and with $t_1 = 2.3/24$,

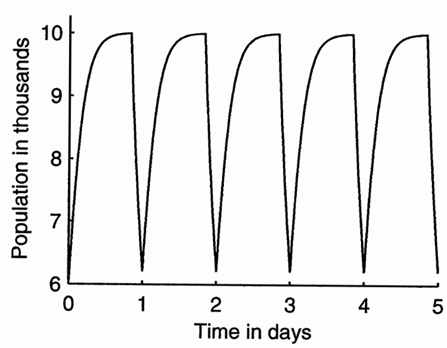
$$2 = P(t_1) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt_1}}.$$

Solving for r, we get $r = 8.4619$.

The harvesting rate during the last four hours of each day is 1500/hr = 36,000/day. Hence the harvesting rate is

$$h(t) = \begin{cases} 0, & \text{in the first 20 hours} \\ 36, & \text{in the last 4 hours} \end{cases}$$

of each day. Using a square wave function to model this in our solver we get the following graph of the solution.



The population at the end of each day is approximately 10,000.

19. (a) The harvesting rate is $0.01P$, so the modified model is

$$P' = 0.1P(1 - P/10) - 0.01P.$$

(b) The model equation simplifies to $P' = 0.01P(9-P)$. This is another logistic equation. Thus $P = 0$ is an unstable equilibrium point, and $P = 9$ is an asymptotically stable equilibrium point, representing the carrying capacity.

(c) The population tends to the carrying capacity $P = 9$.

20. The harvesting rate is γP, so the modified model is $P' = rP(1 - P/K) - \gamma P$. This can be written as

$$P' = (r - \gamma)P(1 - P/K_\gamma),$$

where the new carrying capacity is $K_\gamma = (r - \gamma)K/r$. If $\gamma < r$, this is again a logistic equation with carrying capacity K_γ, so all populations tend to this value. If $\gamma > r$, then $K_\gamma < 0$, and we see that $P' < 0$ whenever $P > 0$. Thus any real population tends to 0. The population dies out.

21. Refer to the answer to Exercise 19. The units of γP are individuals/ unit time. It represents the number of individuals harvested in a unit of time. For this reason it is called the yield. Notice that for values of $\gamma < r$ the model is another logistic equation with carrying capacity $K_\gamma = (r - \gamma)K/r$. For populations at the carrying capacity the yield is $\gamma K_\gamma = \gamma(r - \gamma)K/r$. This function of γ is maximized when $\gamma = r/2$. The carrying capacity for this value of γ is $K/2$ and the yield is $rK/4$.

Section 3.2. Models and the Real World

1. We have $P_0 = P(1790) = 3929827$, and $P(1800) = 5305925 = P_0 e^{10r}$. Solving the last equation for r we get $r = \ln(5305925/3929827)/10 \approx 0.03$.

2. We have $h = 60$, and the three data points $P_0 = P(1790) = 3929$, $P_1 = P(1850) = 23192$, and $P_2 = P(1910) = 91972$. Thus using (1.19) we get

$$r = \frac{1}{h}\ln\left(\frac{P_2(P_1 - P_0)}{P_0(P_2 - P_1)}\right) = 0.0313.$$

With this value we use the first equation in (1.18) to find the carrying capacity

$$K = \frac{P_0 P_1(1 - e^{-rh})}{P_0 - P_1 e^{-rh}} = 197274.$$

3. Answers will vary depending on the data chosen.

Section 3.3. Personal Finance

1. (a) Let $P(t)$ represent the balance in the account after t years. Because the initial deposit at time $t = 0$ is $P = \$1200$,

$$P' = 0.05P, \quad P(0) = 1200.$$

Separate the variables and solve for P, then use the initial condition to produce

$$P(t) = 1200e^{0.05t}.$$

Thus, the amount in the account after 10 years is

$$P(10) = 1200e^{0.05(10)} \approx \$1978.50.$$

(b) Solve $P(t) = 5000$, or

$$\begin{aligned} 1200e^{0.05t} &= 5000 \\ e^{(1/20)t} &= \frac{25}{6} \\ t &= 20\ln\frac{25}{6} \\ t &\approx 28.5\,\text{yr} \end{aligned}$$

2. Let $P(t)$ represent the balance in the account t years after the initial investment. Let P_0 represent the initial investment, i.e., $P(0) = P_0$. Consequently,

$$P' = rP, \quad P(0) = P_0.$$

Separate variables and use the initial condition to produce

$$P = P_0 e^{rt}.$$

Because the initial investment doubles in five years,

$$\begin{aligned} 2P_0 &= P_0 e^{r(5)}, \\ 2 &= e^{5r}, \\ r &= \frac{\ln 2}{5}, \\ r &\approx 0.1386, \end{aligned}$$

or, 13.86%.

3. Let $P(t)$ represent the balance in the account t years after the initial investment. Let r represent the annual rate, d the yearly deposit, and P_0 the initial investment. Then,

$$P' = rP + d, \quad P(0) = P_0.$$

This equation is linear, with integrating factor e^{-rt}. Consequently,

$$\begin{aligned} \left(e^{-rt}P\right)' &= de^{-rt}, \\ e^{-rt}P &= -\frac{d}{r}e^{-rt} + C, \\ P &= -\frac{d}{r} + Ce^{rt}. \end{aligned}$$

Use $P(0) = P_0$ to produce $C = P_0 + d/r$ and

$$P(t) = -\frac{d}{r} + \left(P_0 + \frac{d}{r}\right)e^{rt}.$$

Thus, the amount in the account after 10 years is

$$\begin{aligned} P(10) &= -\frac{1200}{0.06} + \left(5000 + \frac{1200}{0.06}\right)e^{0.06(10)} \\ &\approx \$25,553. \end{aligned}$$

4. Let $P(t)$ represent the balance in the account t years after Jason's day of birth. Let r represent the annual rate, d the annual deposit. Since no initial investment is required,

$$P' = rP + d, \quad P(0) = 0.$$

This equation is linear, with integrating factor e^{-rt}. Consequently,

$$\begin{aligned} \left(e^{-rt}P\right)' &= de^{-rt}, \\ e^{-rt}P &= -\frac{d}{r}e^{-rt} + C, \\ P &= -\frac{d}{r} + Ce^{rt}. \end{aligned}$$

Use $P(0) = 0$ to produce $C = d/r$ and

$$P(t) = \frac{d}{r}\left(e^{rt} - 1\right).$$

Because $P(18) = 50000$,

$$\begin{aligned} 50000 &= \frac{d}{r}\left(e^{r(18)} - 1\right), \\ d &= \frac{50000r}{e^{18r} - 1}, \\ d &= \frac{50000(0.0625)}{e^{18(0.0625)} - 1}, \\ d &\approx \$1,502.25 \end{aligned}$$

5. Let $P(t)$ represent the balance in the account after t years. Let r represent the annual rate, w the yearly withdrawal, and P_0 the amount of the inheritance. Then

$$P' = rP - w \quad P(0) = P_0.$$

The equation is linear with integrating factor e^{-rt}. Consequently,

$$\begin{aligned} \left(e^{-rt}P\right)' &= -we^{-rt}, \\ e^{-rt}P &= \frac{w}{r}e^{-rt} + C, \\ P &= \frac{w}{r} + Ce^{rt}. \end{aligned}$$

Use $P(0) = P_0$ to produce $C = P_0 - w/r$ and

$$P(t) = \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{rt}.$$

Now, to find when the funds are depleted, set $P(t) = 0$.

$$\begin{aligned} 0 &= \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{rt}, \\ e^{rt} &= \frac{w/r}{w/r - P_0}, \\ t &= \frac{1}{r}\ln\frac{w/r}{w/r - P_0}. \end{aligned}$$

Thus, the account will be depleted in

$$t = \frac{1}{0.05}\ln\frac{8000/0.05}{8000/0.05 - 50000} \approx 7.5\text{ years}.$$

6. Let $P(t)$ represent the loan balance after t years. Let r represent the annual rate, w the annual payment, and P_0 the amount of the loan. Then

$$P' = rP - w \quad P(0) = P_0.$$

The equation is linear with integrating factor e^{-rt}. Consequently,

$$\begin{aligned} \left(e^{-rt}P\right)' &= -we^{-rt}, \\ e^{-rt}P &= \frac{w}{r}e^{-rt} + C, \\ P &= \frac{w}{r} + Ce^{rt}. \end{aligned}$$

Use $P(0) = P_0$ to produce $C = P_0 - w/r$ and

$$P(t) = \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{rt}.$$

Now, the loan is exhausted at the end of four years. Consequently, $P(4) = 0$, so

$$\begin{aligned} 0 &= \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{r(4)}, \\ \frac{w}{r}\left(e^{4r} - 1\right) &= P_0e^{4r}, \\ P_0 &= \frac{w}{r}\left(1 - e^{-4r}\right) \\ P_0 &= \frac{(225)(12)}{0.08}\left(1 - e^{-4(0.08)}\right), \\ P_0 &\approx \$9,242,47 \end{aligned}$$

7. Let $P(t)$ represent the loan balance after t years. Let r represent the annual rate, w the annual payment, and P_0 the amount of the loan. Then

$$P' = rP - w \quad P(0) = P_0.$$

The equation is linear with integrating factor e^{-rt}. Consequently,

$$\begin{aligned} \left(e^{-rt}P\right)' &= -we^{-rt}, \\ e^{-rt}P &= \frac{w}{r}e^{-rt} + C, \\ P &= \frac{w}{r} + Ce^{rt}. \end{aligned}$$

Use $P(0) = P_0$ to produce $C = P_0 - w/r$ and

$$P(t) = \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{rt}.$$

Now, the loan is exhausted at the end of 30 years. Consequently, $P(30) = 0$, so

$$\begin{aligned} 0 &= \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{r(30)}, \\ \frac{w}{r}\left(e^{30r} - 1\right) &= P_0e^{30r}, \\ w &= \frac{rP_0}{1 - e^{-30r}}, \\ w &= \frac{0.08(100000)}{1 - e^{-30(0.08)}} \\ w &\approx \$8,798.15 \end{aligned}$$

8. Let $P(t)$ represent the loan balance after t years. Let r represent the annual interest rate, w the annual payment, and P_0 the amount of the loan. Then

$$P' = rP - w, \quad P(0) = P_0.$$

The equation is linear with integrating factor e^{-rt}. Consequently,

$$\begin{aligned} \left(e^{-rt}P\right)' &= -we^{-rt}, \\ e^{-rt}P &= \frac{w}{r}e^{-rt} + C, \\ P &= \frac{w}{r} + Ce^{rt}. \end{aligned}$$

Use $P(0) = P_0$ to produce $C = P_0 - w/r$ and

$$P(t) = \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{rt}.$$

Now, \$1,000 per month makes \$12,000 per year, so $w = 12000$. Furthermore, the term of the loan is 30 years, so $P(30) = 0$ and

$$\begin{aligned}
0 &= \frac{w}{r} + \left(P_0 - \frac{w}{r}\right) e^{r(30)}, \\
we^{30r} - w &= P_0 r e^{30r}, \\
P_0 &= \frac{w\left(e^{30r} - 1\right)}{re^{30r}}, \\
P_0 &= \frac{12000\left(e^{30(0.0725)} - 1\right)}{0.0725e^{30(0.0725)}}, \\
P_0 &\approx \$146,713.
\end{aligned}$$

9. Let $S(t)$ represent José's salary after t years. Consequently,

$$S' = 0.01S, \quad S(0) = 28,$$

where S is measured in *thousands* of dollars. Therefore, José's salary is given by

$$S(t) = 28e^{0.01t}.$$

Let $P(t)$ represent the balance of the account after t years. Let ρ represent the fixed percentage of José's salary that is deposited in the account on an annual basis. Thus,

$$P' = 0.06P + 28\rho e^{0.01t}, \quad P(0) = 2.5,$$

where $P(t)$ is also measured in *thousands* of dollars. Multiply by the integrating factor, $e^{-0.06t}$, and integrate.

$$\begin{aligned}
\left(e^{-0.06t}P\right)' &= 28\rho e^{-0.05t} \\
e^{-0.06t}P &= \frac{28}{-0.05}\rho e^{-0.05t} + C \\
P &= -560\rho e^{0.01t} + Ce^{0.06t}
\end{aligned}$$

The initial condition $P(0) = 2.5$ produces $C = 2.5 + 560\rho$ and

$$P = -560\rho e^{0.01t} + (2.5 + 560\rho)e^{0.06t}.$$

Now, José wants \$50,000 in the account at the end of 20 years, so $P(20) = 50$ and

$$\begin{aligned}
50 &= -560\rho e^{0.01(20)} + (2.5 + 560\rho)e^{0.06(20)}, \\
\rho &= \frac{50 - 2.5e^{1.2}}{560(e^{1.2} - e^{0.2})}, \\
\rho &\approx 0.035,
\end{aligned}$$

or 3.5%.

10. Let $P(t)$ represent the balance after t years, r the annual rate, and d the annual deposit. Thus,

$$P' = rP + d, \quad P(0) = 1000.$$

However, in this case the annual deposit is a function of time. The initial deposit is \$1,000 and this is increased by \$500 each year. Consequently, $d(t) = 1000 + 500t$ and

$$P' = rP + (1000 + 500t), \quad P(0) = 1000.$$

Multiply by the integrating factor, e^{-rt}, and integrate.

$$\begin{aligned}
\left(e^{-rt}P\right)' &= (1000 + 500t)e^{-rt} \\
e^{-rt}P &= \left(-\frac{1000 + 500t}{r} - \frac{500}{r^2}\right)e^{-rt} + C \\
P &= -\frac{1000 + 500t}{r} - \frac{500}{r^2} + Ce^{rt}
\end{aligned}$$

The initial condition $P(0) = 1000$ gives $C = 1000 + 1000/r + 500/r^2$ and

$$\begin{aligned}
P = &-\frac{1000 + 500t}{r} - \frac{500}{r^2} \\
&+ \left(1000 + \frac{1000}{r} + \frac{500}{r^2}\right)e^{rt}.
\end{aligned}$$

At ten years,

$$\begin{aligned}
P(10) = &-\frac{1000 + 500(10)}{0.06} - \frac{500}{(0.06)^2} \\
&+ \left(1000 + \frac{1000}{0.06} + \frac{500}{(0.06)^2}\right)e^{0.06(10)}, \\
P(10) \approx &\ \$46,373.93.
\end{aligned}$$

11. (a) Let $P(n)$ represent the balance at the end of n compounding periods, I the annual interest rate, m the number of compounding periods per year, and P_0 the initial investment. Thus,

$$P(n+1) = \left(1 + \frac{I}{m}\right)P(n), \quad P(0) = P_0.$$

(b) Compare

$$P(n+1) = \left(1 + \frac{I}{m}\right) P(n), \quad P(0) = P_0.$$

with

$$a(n+1) = ra(n), \quad a(0) = a_0,$$

and note that $r = 1 + I/m$ and $a_0 = P_0$. Consequently,

$$a(n) = a_0 r^n$$

becomes

$$P(n) = P_0 \left(1 + \frac{I}{m}\right)^n.$$

12. (a) Let $P(t)$ represent the balance at the end of t years. Let r represent the annual rate and P_0 the initial investment. Thus,

$$P' = rP, \quad P(0) = P_0.$$

Consequently,

$$\begin{aligned} P(t) &= P_0 e^{rt}, \\ P(10) &= 2000e^{0.06(10)}, \\ P(10) &\approx \$3,644.24 \end{aligned}$$

(b) In semiannual case, $m = 2$. Furthermore, there are 20 compounding periods in 10 years, so

$$P(20) = 2000\left(1 + \frac{0.06}{2}\right)^{20} \approx \$3,612.22.$$

In the monthly case, $m = 12$. There are 120 compounding periods in 10 years, so

$$P(120) = 2000\left(1 + \frac{0.06}{12}\right)^{120} \approx \$3,638.79.$$

In the daily case, $m = 365$. There are 3650 compounding periods in 10 years, so

$$P(3650) = 2000\left(1 + \frac{0.06}{365}\right)^{3650} \approx \$3,644.06.$$

13. (a) The series $1 + r + r^2 + \cdots + r^{n-1}$ is geometric, finite, with sum $(1-r^n)/(1-r)$. Consequently,

$$\begin{aligned} a(n) &= a_0 r^n + b\left(1 + r + r^2 + \cdots + r^{n-1}\right), \\ a(n) &= a_0 r^n + b\left(\frac{1-r^n}{1-r}\right), \\ a(n) &= a_0 r^n + \frac{b}{1-r} - \frac{b}{1-r} r^n, \\ a(n) &= \left(a_0 - \frac{b}{1-r}\right) r^n + \frac{b}{1-r}. \end{aligned}$$

(b) Comparing

$$P(n+1) = \left(1 + \frac{I}{m}\right) P(n) + d, \quad P(0) = P_0.$$

with

$$a(n+1) = ra(n) + b, \quad a(0) = a_0$$

shows us that $r = 1 + I/m$, $b = d$, and $a_0 = P_0$. Consequently, the solution

$$a(n) = \left(a_0 - \frac{b}{1-r}\right) r^n + \frac{b}{1-r}$$

becomes

$$\begin{aligned} P(n) &= \left(P_0 - \frac{d}{1-(1+I/m)}\right)\left(1 + \frac{I}{m}\right)^n \\ &\quad + \frac{d}{1-(1+I/m)}, \\ P(n) &= \left(P_0 + \frac{md}{I}\right)\left(1 + \frac{I}{m}\right)^n - \frac{md}{I}. \end{aligned}$$

14. (a) Let $P(t)$ represent the balance after t years, r the annual interest rate, d the yearly deposit, and P_0 the initial investment, Thus,

$$P' = rP + d, \quad P(0) = P_0.$$

Multiply by the integrating factor, e^{-rt}, and integrate.

$$\begin{aligned} \left(e^{-rt} P\right)' &= de^{-rt} \\ e^{-rt} P &= -\frac{d}{r} e^{-rt} + C \\ P(t) &= -\frac{d}{r} + Ce^{rt} \end{aligned}$$

The initial condition $P(0) = P_0$ gives $C = P_0 + d/r$ and

$$P(t) = -\frac{d}{r} + \left(P_0 + \frac{d}{r}\right)e^{rt}.$$

After 10 years,

$$\begin{aligned} P(10) &= -\frac{1000}{0.05} + \left(2000 + \frac{1000}{0.05}\right)e^{0.05(10)} \\ &\approx \$16,271.87. \end{aligned}$$

(b) Let $P(n)$ represent the balance after n compounding periods, I the annual interest rate, m the number of compounding periods per year, and P_0 the initial investment. Then,

$$P(n+1) = \left(1 + \frac{I}{m}\right)P(n) + d, \quad P(0) = P_0,$$

with solution (from exercise 13)

$$P(n) = \left(P_0 + \frac{md}{I}\right)\left(1 + \frac{I}{m}\right)^n - \frac{md}{I}.$$

In ten years, there are 40 compounding periods. Thus,

$$\begin{aligned} P(40) &= \left(2000 + \frac{(4)(250)}{0.05}\right)\left(1 + \frac{0.05}{4}\right)^{40} \\ &\quad - \frac{(4)(250)}{0.05} \\ &\approx \$16,159.63. \end{aligned}$$

(c) One significant change is the deposit at the end of each compounding period (at the end of each day). Because Demetrios is depositing \$1,000 per year, his daily deposit is $d = \$1000/365$. Next, there are 3,650 compounding periods in the ten-year period. Thus,

$$\begin{aligned} P(3650) &= \left(2000 + \frac{(365)(1000/365)}{0.05}\right) \\ &\quad \times \left(1 + \frac{0.05}{365}\right)^{3650} \\ &\quad - \frac{(365)(1000/365)}{0.05}, \\ &\approx \$16,270.63. \end{aligned}$$

15. (a) Let $P(t)$ represent the balance of the loan after t years, I the annual interest rate, w the yearly payment, and P_0 the amount of the loan. Then,

$$P' = rP - w, \quad P(0) = P_0.$$

Multiply by the integrating factor, e^{-rt}, and integrate.

$$\begin{aligned} \left(e^{-rt}P\right)' &= -we^{-rt} \\ e^{-rt}P &= \frac{w}{r}e^{-rt} + C \\ P &= \frac{w}{r} + Ce^{-rt} \end{aligned}$$

The initial condition $P(0) = P_0$ gives $C = P_0 - w/r$ and

$$P = \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{-rt}.$$

Because the loan expires in 5 years, $P(5) = 0$ and

$$\begin{aligned} 0 &= \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{r(5)}, \\ w &= \frac{P_0 r}{1 - e^{-5r}}, \\ w &= \frac{(12000)(0.08)}{1 - e^{-5(0.08)}}, \\ w &\approx \$2,911.91, \end{aligned}$$

or \$242.66 per month.

(b) Let $P(n)$ represent the balance on the loan after n compounding periods, I the annual rate, m the number of compounding periods per year, w the monthly payment, and P_0 the initial amount of the loan. Then, in a manner similar to that in Exercise 13, the progress of the loan is modeled by the first order difference equation

$$P(n+1) = \left(1 + \frac{I}{m}\right)P(n) - w, \quad P(0) = P_0,$$

with solution

$$P(n) = \left(P_0 - \frac{mw}{I}\right)\left(1 + \frac{I}{m}\right)^n + \frac{mw}{I}.$$

There are 60 compounding periods in 5 years, so $P(60) = 0$ and

$$0 = \left(P_0 - \frac{mw}{I}\right)\left(1 + \frac{I}{m}\right)^{60} + \frac{mw}{I},$$
$$w = \frac{P_0 I (1 + I/m)^{60}}{m((1 + I/m)^{60} - 1)},$$
$$w = \frac{P_0 I}{m(1 - (1 + I/m)^{-60})},$$
$$w = \frac{(12000)(0.08)}{12(1 - (1 + 0.08/12)^{-60})},$$
$$w \approx \$243.32.$$

Section 3.4. Electrical Circuits

1. The model equation is $Q' + Q/2 = 5$. The solution is $Q(t) = 10 - 10e^{-t/2}$.

2. The model equation is $Q' + Q/2 = 5e^{-t/10}$. The solution is
$$Q(t) = 25(e^{-t/10} - e^{-t/2})/2.$$

3. The model equation is $Q' + Q/2 = 5\sin 2t$. The solution is
$$Q(t) = (-40\cos 2t + 10\sin 2t + 40e^{-t/2})/17.$$

4. The model equation is $Q' + Q/2 = 5\cos 3t$. The solution is
$$Q(t) = (10\cos 3t + 60\sin 3t - 10e^{-t/2})/37.$$

5. The model equation is $Q' + Q/2 = 5 - t/20$. The solution is
$$Q(t) = 51(1 - e^{-t/2})/5 - t/10.$$

6. The model equation is $Q' + Q/2 = 5(1 - e^{-t/10})$. The solution is
$$Q(t) = 10 - (25e^{-t/10} - 5e^{-t/2})/2.$$

7. The model equation is $I' + I/10 = 1$. The solution is $I(t) = 10(1 - e^{-t/10})$.

8. The model equation is $I' = e^{-t/10} - I/10$. The solution is $I(t) = te^{-t/10}$.

9. The model equation is $I' + I/10 = 5\sin 2\pi t$. The solution is
$$I(t) = \frac{50(-20\pi\cos 2\pi t + 20\pi e^{-t/10} + \sin 2\pi t)}{1 + 400\pi^2}.$$

10. The model equation is $I' + I/10 = 4\cos 3t$. The solution is
$$I(t) = (40\cos 3t + 1200\sin 3t - 40e^{-t/10})/901.$$

11. The model equation is $I' + I/10 = 10 - 2t$. The solution is
$$I(t) = 300 - 20t - 300e^{-t/10}.$$

12. The model equation is $I' + I/10 = 10(1 - e^{-t/20})$. The solution is
$$I(t) = 100(1 - 2e^{-t/20} + e^{-t/10}).$$

13. $Q(t) = EC(1 - e^{-t/RC})$

14. $I(t) = (E + (RI_0 - E)e^{-Rt/L})/R$

15. The model equation, $Q' + Q/20 = (1/2)e^{-t/100}$ with $Q(0) = 0$, has solution

$$Q(t) = 25(e^{-t/100} - e^{-t/20})/2.$$

We find the maximum by setting the derivative

$$Q'(t) = 25\left[\frac{1}{20}e^{-t/20} - \frac{1}{100}e^{-t/100}\right]\Big/2 = 0,$$

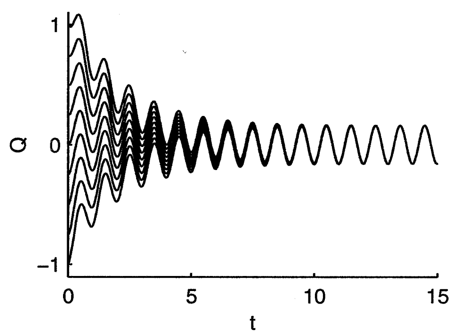
and solving for t. It has a maximum at $t = 25\ln 5 \approx 40.24$. The maximum value is $Q(25\ln 5) = 6.69$.

16. The model equation, $I' + I/10 = (1/2)e^{-t/20}$ with $I(0) = 0$, has solution $I = 10(e^{-t/20} - e^{-t/10})$. The solution is maximized at $t = 20\ln 2 \approx 13.86$ with maximum value $I(20\ln 2) = 2.5$

17. (a) 23.87

(b) The model equation, $Q' + Q/10 = 10$ with $Q(0) = 0$, has solution $Q = 100(1 - e^{-t/10})$ for $0 \le t \le 5$. Note that $Q(5) = Q_5 = 100(1 - e^{-1/2})$. Then for the next 5 seconds, the model equation is $Q' = -Q/10$ with $Q(5) = Q_5$. The solution to this equation is $Q = Q_5 e^{-(t-5)/10}$. At $t = 10$ seconds, the amount of charge is $Q = Q_5 e^{-1/2} \approx 23.87$.

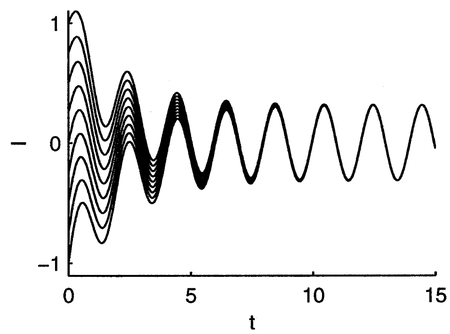
18. (a) The nine solution are plotted below.



The steady state solution is a particular solution to the inhomogeneous equation, which all solutions converge to regardless of their initial condition. Note that this is the case for all nine solutions corresponding to the nine different initial conditions.

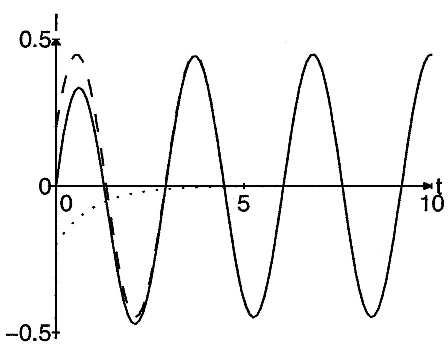
(b) The frequency of the force is $2\pi/2\pi = 1$ Hz. This also appears to be the period of the steady-state response.

19. The nine solutions are plotted below.

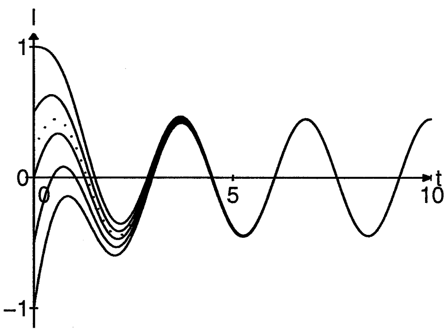


The frequency is of the force is $2\pi/2 = \pi$ Hz. This also appears to be the frequency of the steady-state response.

20. (a) In the following plot, the solution is the solid curve, the transient response is the dotted curve, and the steady-state solution is dashed.



(b) In the following figure, the steady-state solution is the dotted curve.



All solutions converge to the steady state solution as time gets large.

21. If we use $I = Q'$, we get the differential equation

$$Q' + \frac{1}{RC}Q = \frac{E}{C}\sin\omega t.$$

This linear equation can be solved using the integrating factor $e^{t/RC}$. We get the solution

$$Q(t) = \frac{EC}{1+R^2C^2\omega^2}(\sin\omega t - RC\omega\cos\omega t) + Ae^{-t/RC},$$

where A is an arbitrary constant. The term $Ae^{-t/RC}$ dies out as t increases, so it is the transient term. The first term does not die out and is therefore the steady-state solution.

22. By the capacitance law, the voltage drop across the capacitor is $V = Q/C$. Thus $Q = CV$. If we substitute this into the given equation we get the result. The equation is linear, and it can be solved using the integrating factor $e^{t/RC}$. The general solution is

$$V(t) = \frac{E}{1+R^2C^2\omega^2}(RC\omega\sin\omega t + \cos\omega t) + Ae^{-t/RC},$$

where A is an arbitrary constant. The term $Ae^{-t/RC}$ dies out as t increases, so it is the transient term. The first term does not die out and is therefore the steady-state solution.

23. By the chain rule

$$\frac{dI}{dt} = \frac{dI}{dQ}\frac{dQ}{dt} = I\frac{dI}{dt}.$$

Hence the equation becomes

$$LI\frac{dI}{dQ} + Q/C = 0.$$

This separable equation has the solution $I = \sqrt{k - Q^2/LC}$ where k is a constant. Since $I = dQ/dt$, the equation now becomes

$$\frac{dQ}{dt} = \sqrt{k - Q^2/LC}.$$

This is another separable equation, and

$$\frac{dQ}{\sqrt{k - Q^2/LC}} = dt$$

The integral on the left involves an arcsine and the solution is

$$Q(t) = k_1\sqrt{LC}\sin\left(\frac{t}{\sqrt{LC}} + k_2\right).$$

Since $I = dQ/dt$, we also obtain

$$I(t) = k_1\cos\left(\frac{t}{\sqrt{LC}} + k_2\right),$$

where k_1 and k_2 are constants.