Section 2.4 –Derivatives of Trigonometric Functions

Derivative of the *Sine* Function

If $f(x)=\sin x$, then

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$
$$=\lim_{h\to 0}\frac{(\sin x\cos h+\cos x\sin h)-\sin x}{h}$$
$$=\lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$
$$=\lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}$$
$$=\lim_{h\to 0}\frac{\sin x(\cos h-1)}{h}+\lim_{h\to 0}\frac{\cos x\sin h}{h}$$
$$=\sin x\underbrace{\lim_{h\to 0}\frac{\cos h-1}{h}}_{0}+\cos x\underbrace{\lim_{h\to 0}\frac{\sin h}{h}}_{1}$$
$$=\sin x\bullet(0)+\cos x\bullet(1)$$
$$=\cos x$$

$$\cos h=1-2\sin^2\left(\frac{h}{2}\right)$$

$$\lim_{h\to 0}\frac{\cosh-1}{h}=\lim_{h\to 0}\frac{1-2\sin^2\left(\frac{h}{2}\right)-1}{h}$$
$$=\lim_{h\to 0}\frac{-2\sin^2\left(\frac{h}{2}\right)}{h}\qquad \textit{Let } \theta=\frac{h}{2}$$

Let $\theta=\frac{h}{2}$

$$=-\lim_{\theta\to 0}\frac{2\sin^2(\theta)}{2\theta}$$
$$=-\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\sin\theta$$
$$=-(1)(0)$$
$$=0$$

$$\boxed{\frac{d}{dx}(\sin x)=\cos x}$$

Example

Find the derivative of $y=x^2-\sin x$

Solution

$$y'=2x-(\sin x)'$$
$$=2x-\cos x$$

Example

Find the derivative of $y=x^2\sin x$

Solution

$$y'=2x\sin x+x^2\cos x$$

Example

Find the derivative of $y = \dfrac{\sin x}{x}$

Solution

$$y' = \frac{x\cos x - \sin x \cdot (1)}{x^2}$$

$$= \frac{x\cos x - \sin x}{x^2}$$

Derivative of the *Cosine* Function

If $f(x) = \cos x$, then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{(\cos x\cos h - \sin x\sin h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x\cos h - \sin x\sin h - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x\sin h}{h} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1)}{h} - \lim_{h\to 0} \frac{\sin x\sin h}{h} \\
&= \cos x \underbrace{\lim_{h\to 0} \frac{\cos h - 1}{h}}_{0} - \sin x \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{1} \\
&= \cos x \bullet (0) - \sin x \bullet (1) \\
&= -\sin x
\end{aligned}$$

$$\lim_{h\to 0} \frac{\cosh - 1}{h} = 0$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

Example

Find the derivative of $y = 5x + \cos x$

Solution

$$y' = 5 - \sin x$$

Example

Find the derivative of $y = \sin x \cos x$

Solution

$$\begin{aligned} y' &= (\sin x)' \cos x + (\cos x)' \sin x \\ &= (\cos x)\cos x + (-\sin x)\sin x \\ &= \cos^2 x - \sin^2 x \end{aligned}$$

Example

Find the derivative of $y = \dfrac{\cos x}{1 - \sin x}$

Solution

$$\begin{aligned} y' &= \frac{(1-\sin x)(\cos x)' - \cos x(1-\sin x)'}{(1-\sin x)^2} \\ &= \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2} \\ &= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \qquad \sin^2 x + \cos^2 x = 1 \\ &= \frac{1-\sin x}{(1-\sin x)^2} \\ &= \frac{1}{1-\sin x} \end{aligned}$$

Derivatives of the Other Trigonometric Functions

$$\begin{cases} (\tan x)' = \sec^2 x & (\cot x)' = -\csc^2 x \\ (\sec x)' = \sec x \tan x & (\csc x)' = -\csc x \cot x \end{cases}$$

Prove

Example

Find $\frac{d}{dx}(\tan x)$

Solution

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\ &= \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x} \\ &= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x \end{aligned}$$

Quotient Rule

$\frac{1}{\cos x} = \sec x$

Example

Find y'' *if* $y = \sec x$

Solution

$$y' = \sec x \tan x$$

$$\begin{aligned} y'' &= (\sec x)' \tan x + \sec x(\tan x)' \\ &= (\sec x \tan x)\tan x + \sec x\left(\sec^2 x\right) \\ &= \sec x \tan^2 x + \sec^3 x \end{aligned}$$

Exercises Section 2.4 –Derivatives of Trigonometric Functions

Find the derivative of

1. $y=-10x+3\cos x$

2. $y=\csc x-4\sqrt{x}+7$

3. $y=x^2\cos x$

4. $y=\csc x\cot x$

5. $y=(\sin x+\cos x)\sec x$

6. $y=(\sec x+\tan x)(\sec x-\tan x)$

7. $y=\dfrac{\cos x}{x}+\dfrac{x}{\cos x}$

8. $y=x^2\cos x-2x\sin x-2\cos x$

9. $y=(2-x)\tan^2 x$

10. $y=t^2-\sec t+1$

11. $y=\dfrac{1+\csc t}{1-\csc t}$

12. $r=\theta\sin\theta+\cos\theta$

13. $y=\dfrac{3x+\tan x}{x\sec x}$

14. $p=\dfrac{\sin q+\cos q}{\cos q}$

15. $p=\dfrac{3q+\tan q}{q\sec q}$

16. $f(x)=\dfrac{\sin x+2x}{x}$

17. $f(x)=\dfrac{\sin x}{x^2}$

18. $f(x)=x^3\cos x$

19. $f(x)=\dfrac{1}{x}-12\sec x$

20. $f(\theta)=5\theta\sec\theta+\theta\tan\theta$

21. $y=\sec\pi x$

22. $y=\cos 5x$

23. $y=\cos(4-3x)$

24. $f(x)=\sin(4-3x)$

25. $f(\theta)=\dfrac{\sin a\theta}{\cos b\theta}$

26. $f(\theta)=\sin 2\theta-\cos 2\theta$

27. $f(\theta)=\tan\theta-\cot\theta$

28. $\dfrac{d}{dx}\left(5x^2\sin x\right)$

29. $\dfrac{d}{dx}\left(2x(\sin x)\sqrt{3x-1}\right)$

30. Find $y^{(4)}$ *if* $y=9\cos x$

31. Find $y',\ y'',\ y'''$: $y=(x-3)\sqrt{x+2}$

32. Find $\dfrac{d^{999}}{dx^{999}}(\cos x)$

33. Find $\lim\limits_{x\to-\frac{\pi}{6}}\sqrt{1+\cos(\pi\csc x)}$

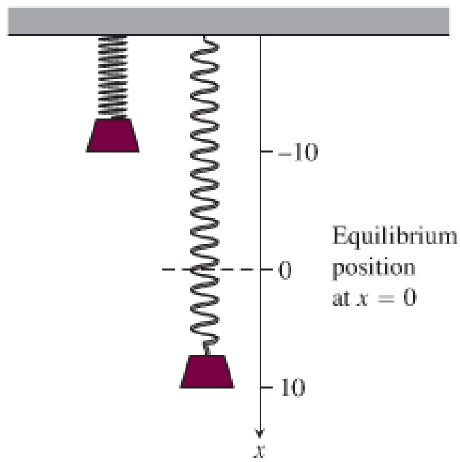
34. Assume that a particle's position on the *x*-axis is given by

$$x = 3\cos t + 4\sin t$$

a) Find the particle's position when $t = 0, \quad t = \frac{\pi}{2}, \quad and \quad t = \pi$

b) Find the particle's velocity when $t = 0, \quad t = \frac{\pi}{2}, \quad and \quad t = \pi$

35. A weight is attached to a spring and reaches its equilibrium position $(x = 0)$. It is then set in motion resulting in a displacement of $x = 10\cos t$



Where *x* is measured in centimeters and *t* is measured in seconds.

a) Find the spring's displacement when $t = 0, \quad t = \frac{\pi}{3}, \quad and \quad t = \frac{3\pi}{4}$

b) Find the spring's velocity when $t = 0, \quad t = \frac{\pi}{3}, \quad and \quad t = \frac{3\pi}{4}$