SOLUTION   *Section* **3.3 – Integral Test**

Exercise

Use the integral Test to determine if the series converge or diverge. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{0.2}}$

Solution

$$f(x)=\frac{1}{x^{0.2}} \quad\rightarrow\quad \int_{1}^{\infty}\frac{dx}{x^{0.2}}=\lim_{b\to\infty}\int_{1}^{\infty}\frac{dx}{x^{0.2}}$$

$$=\lim_{b\to\infty}\left[\frac{1}{0.8}x^{0.8}\right]_{1}^{b}$$

$$=\frac{1}{0.8}\lim_{b\to\infty}\left(b^{0.8}-1\right)$$

$$=\infty \quad\rightarrow\quad \int_{1}^{\infty}\frac{dx}{x^{0.2}}\ \ diverges$$

By the integral test, the given series *diverges.*

Exercise

Use the integral Test to determine if the series converge or diverge. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{2}+4}$

Solution

$$f(x)=\frac{1}{x^{2}+4} \quad\rightarrow\quad \int_{1}^{\infty}\frac{dx}{x^{2}+4}=\lim_{b\to\infty}\int_{1}^{\infty}\frac{dx}{x^{2}+4}$$

$$=\lim_{b\to\infty}\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{1}^{b}$$

$$=\frac{1}{2}\lim_{b\to\infty}\left(\tan^{-1}\frac{b}{2}-\tan^{-1}\frac{1}{2}\right)$$

$$=\frac{1}{2}\left(\frac{\pi}{2}-\tan^{-1}\frac{1}{2}\right) \quad\rightarrow\quad \int_{1}^{\infty}\frac{dx}{x^{2}+4}\ \ converges$$

By the integral test, the given series *converges.*

Exercise

Use the integral Test to determine if the series converge or diverge. $\displaystyle\sum_{n=1}^{\infty} e^{-2n}$

Solution

$$f(x) = e^{-2x} \quad \to \quad \int_1^{\infty} e^{-2x}\,dx = \lim_{b\to\infty}\int_1^{\infty} e^{-2x}\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{1}{2}e^{-2x}\right]_1^b$$

$$= -\frac{1}{2}\lim_{b\to\infty}\left(e^{-2b} - e^{-2}\right)$$

$$= -\frac{1}{2}\lim_{b\to\infty}\left(\frac{1}{e^{2b}} - \frac{1}{e^2}\right)$$

$$= -\frac{1}{2}\left(\frac{1}{e^2}\right) \quad \to \quad \int_1^{\infty} e^{-2x}\,dx \;\; converges$$

By the integral test, the given series *converges*.

Exercise

Use the integral Test to determine if the series converge or diverge. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$

Solution

Function is positive, continuous, and decreasing for $x \geq 2$.

$$f(x) = \frac{1}{x(\ln x)^2} \quad \to \quad \int_2^{\infty}\frac{dx}{x(\ln x)^2} = \lim_{b\to\infty}\int_2^{\infty}\frac{dx}{x(\ln x)^2} \qquad d(\ln x) = \frac{1}{x}dx$$

$$= \lim_{b\to\infty}\int_2^{\infty}\frac{d(\ln x)}{(\ln x)^2}$$

$$= \lim_{b\to\infty}\left[-\frac{1}{\ln x}\right]_2^b$$

$$= -\lim_{b\to\infty}\left(\frac{1}{\ln b} - \frac{1}{\ln 2}\right)$$

$$= \frac{1}{\ln 2} \quad \to \quad \int_1^{\infty}\frac{dx}{x(\ln x)^2} \;\; converges$$

By the integral test, the given series *converges*.

Exercise

Use the integral Test to determine if the series converge or diverge. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$

Solution

$f(x) = \dfrac{x^2}{e^{x/3}}$ is positive, continuous for $x \geq 1$.

$$f'(x) = \frac{2xe^{x/3} - \frac{1}{3}x^2 e^{x/3}}{\left(e^{x/3}\right)^2} = \frac{6x - x^2}{3e^{x/3}} = \frac{-x(x-6)}{3e^{x/3}} < 0 \quad for \quad x > 6$$

$$\int_7^{\infty} \frac{x^2}{e^{x/3}} dx = \lim_{b\to\infty} \int_7^{\infty} \frac{x^2}{e^{x/3}} dx$$

$$= \lim_{b\to\infty} \left[-\frac{3x^2}{e^{x/3}} - \frac{18x}{e^{x/3}} - \frac{54}{e^{x/3}} \right]_7^b$$

$$= -\lim_{b\to\infty} \left[\frac{3x^2 + 18x + 54}{e^{x/3}} \right]_7^b$$

$$= -\lim_{b\to\infty} \left[\frac{3b^2 + 18b + 54}{e^{b/3}} - \frac{327}{e^{7/3}} \right]$$

$$= \frac{327}{e^{7/3}} \quad \to \quad \int_1^{\infty} \frac{x^2}{e^{x/3}} \quad converges$$

By the integral test, the given series *converges*.

Exercise

Use the integral Test to determine if the series converge or diverge. $\displaystyle\sum_{n=1}^{\infty} \frac{n-4}{n^2 - 2n + 1}$

Solution

$$f(x) = \frac{x-4}{x^2 - 2x + 1}$$

$$= \frac{x-4}{(x-1)^2} \qquad \text{is continuous for } x \geq 2, \text{ and positive } x > 4.$$

$$f'(x) = \frac{(x-1)^2 - 2(x-1)(x-4)}{(x-1)^4}$$

$$= \frac{(x-1)(x-1-2x+8)}{(x-1)^4}$$

$$= \frac{-x+7}{(x-1)^3} < 0 \quad for \quad x > 7$$

$$\int_8^\infty \frac{x-4}{(x-1)^2} dx = \lim_{b\to\infty} \left[\int_8^b \frac{x-1}{(x-1)^2} dx - \int_b^\infty \frac{3}{(x-1)^2} dx \right] \qquad d(x-1) = dx$$

$$= \lim_{b\to\infty} \left[\int_8^b \frac{1}{x-1} dx - \int_b^\infty \frac{3}{(x-1)^2} d(x-1) \right]$$

$$= \lim_{b\to\infty} \left[\ln|x-1| \right]_8^b - \lim_{b\to\infty} \left[\frac{3}{(x-1)} \right]_8^b$$

$$= \lim_{b\to\infty} \left[\ln|b-1| - \ln 7 - \frac{3}{b-1} + \frac{3}{7} \right]$$

$$= \infty$$

$$\int_8^\infty \frac{x-4}{(x-1)^2} dx \; diverges. \qquad \sum_{n=1}^\infty \frac{n-4}{n^2 - 2n + 1} \; diverges.$$

Exercise

Use the integral Test to determine if the series converge or diverge. $\displaystyle\sum_{n=2}^\infty \frac{1}{n \ln n}$

Solution

Let $f(x) = \frac{1}{x \ln x}$

$$\int_1^\infty f(x) dx = \int_1^\infty \frac{1}{x \ln x} dx \qquad\qquad \int \frac{dx}{x \ln ax} = \ln|\ln ax|$$

$$= \ln|\ln x| \Big|_1^\infty$$

$$= \infty - 0$$

$$= \infty$$

Therefore, by the integral test, the given series *diverges*.

Exercise

Use the integral Test to determine if the series converge or diverge. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^2 + 4}}$

Solution

Let $f(x) = \dfrac{x}{\sqrt{x^2 + 4}}$, $f(x)$ is continuous for $x \geq 1$.

$$f'(x) = \frac{\sqrt{x^2 + 4} - x^2\left(x^2 + 4\right)^{-1/2}}{\left(\sqrt{x^2 + 4}\right)^2}$$

$$= \frac{4}{\left(\sqrt{x^2 + 4}\right)^3} > 0$$

Thus $f(x)$ is increasing, and the conditions of the Integral Test are not satisfied.

Therefore, by the integral test, the given series ***diverges***.

Exercise

Use the integral Test to determine if the series converge or diverge. $\displaystyle\sum_{k=1}^{\infty} ke^{-2k^2}$

Solution

Let $f(x) = x \cdot e^{-2x^2}$, $f(x)$ is continuous for $x \geq 1$.

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} xe^{-2x^2}\,dx$$

$$= -\frac{1}{4}\int_1^{\infty} e^{-2x^2}\,d\left(-2x^2\right)$$

$$= -\frac{1}{4} e^{-2x^2}\Big|_1^{\infty}$$

$$= -\frac{1}{4}\left(0 - e^{-2}\right)$$

$$= \underline{\frac{1}{4e^2}}$$

Therefore, by the integral test, the given series ***converges***.

Exercise

Use the integral Test to determine if the series converge or diverge. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{\sqrt{k+8}}$

Solution

Let $f(x) = \dfrac{1}{\sqrt{x+8}}$, $f(x)$ is continuous and decreasing for $x \geq 1$.

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{\sqrt{x+8}}dx$$

$$= 2\sqrt{x+8}\Big|_1^{\infty}$$

$$= \infty \qquad\qquad \text{Therefore, by the integral test, the given series \textbf{\textit{diverges}}.}$$

Exercise

Use the integral Test to determine if the series converge or diverge. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k \ln(\ln k)}$

Solution

Let $f(x) = \dfrac{1}{x \ln x \ln(\ln x)}$, $f(x)$ is **not continuous** at $x = e$.

Therefore, the integral test does not apply.

Exercise

Use the integral Test to determine if the series converge or diverge. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Solution

$$\int_1^{\infty} \frac{x}{x^2+1}dx = \int_1^{\infty} \frac{1}{x^2+1}d\left(x^2+1\right)$$

$$= \ln\left(x^2+1\right)\Big|_1^{\infty}$$

$$= \infty$$

Therefore, by the integral test, the given series **_diverges_**.

Exercise

Use the integral Test to determine if the series converge or diverge. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \arctan x \Big|_1^{\infty}$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Therefore, by the integral test, the given series **converges**.

Exercise

Use the integral Test to determine if the series converge or diverge. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

$$\int_1^{\infty} \frac{1}{x^3}\,dx = -\frac{1}{2}\frac{1}{x^2}\Big|_1^{\infty}$$

$$= -\frac{1}{2}(0-1)$$

$$= \frac{1}{2}$$

Therefore, by the integral test, the given series **converges**.

Exercise

Use the integral Test to determine if the series converge or diverge. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

$$\int_1^{\infty} x^{-1/2}\,dx = \frac{1}{2}\sqrt{x}\Big|_1^{\infty}$$

$$= \infty$$

Therefore, by the integral test, the given series **diverges**.

Exercise

Use the integral Test to determine if the series converge or diverge. $\sum_{n=1}^{\infty} \dfrac{1}{n^{1/4}}$

$$\int_1^{\infty} x^{-1/4}\,dx = \dfrac{4}{3}x^{3/4}\bigg|_1^{\infty}$$

$$= \infty$$

Therefore, by the integral test, the given series **diverges**.

Exercise

Use the integral Test to determine if the series converge or diverge. $\sum_{n=1}^{\infty} \dfrac{1}{n^5}$

Solution

$$\int_1^{\infty} x^{-5}\,dx = -\dfrac{1}{4}\dfrac{1}{x^4}\bigg|_1^{\infty}$$

$$= -\dfrac{1}{4}(0-1)$$

$$= \dfrac{1}{4}$$

Therefore, by the integral test, the given series **converges**.

Exercise

Determine if the series converge or diverge $\sum_{n=1}^{\infty} e^{-n}$

Solution

The series is a geometric series with $r = \dfrac{1}{e} < 1$, therefore it **converges**.

Exercise

Determine if the series converge or diverge $\sum_{n=1}^{\infty} \dfrac{n}{n+1}$

Solution

$\lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1 \neq 0$, therefore the given series **diverges**.

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Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$

Solution

$$\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} = 3\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}, \text{ which is a divergent } p\text{-series } \left(p = \frac{1}{2} \right).$$

Therefore, the given series *diverges*.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{-8}{n}$

Solution

$$\sum_{n=1}^{\infty} \frac{-8}{n} = -8\sum_{n=1}^{\infty} \frac{1}{n},$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

Therefore, the given series *diverges*.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=2}^{\infty} \frac{\ln}{n}$

Solution

By the Integral Test: $\displaystyle\int_{2}^{\infty} \frac{\ln x}{x}\, dx = \frac{1}{2}\left(\ln^2 n - \ln 2 \right) \to \infty .$

Therefore, the given series *diverges*.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{4^n + 3}$

Solution

Using L'Hôpital rule:

$$\sum_{n=1}^{\infty} \frac{5^n}{4^n + 3} = \sum_{n=1}^{\infty} \frac{5^n \ln 5}{4^n \ln 4} = \frac{\ln 5}{\ln 4} \sum_{n=1}^{\infty} \frac{5^n}{4^n} \to \infty \,.$$

Therefore, the given series **diverges**.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} n \sin \frac{1}{n}$

Solution

$$\lim_{n \to \infty} n \sin \frac{1}{n} = \lim_{n \to \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} = 1 \neq 0$$

Therefore, the given series **diverges**.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{10}}$

Solution

This is a p-series with $p = 10$.

Therefore, by the p-series, the given series **converges**.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=2}^{\infty} \frac{n^e}{n^\pi}$

Solution

$$\sum_{n=2}^{\infty} \frac{n^e}{n^\pi} = \sum_{n=2}^{\infty} \frac{1}{n^{\pi-e}}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n^{3.1416-2.71828}}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n^{0.42331}}$$

$0.42331 < 1$

Therefore, the given series *diverges*.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n-2)^4}$

Solution

$$\sum_{n=3}^{\infty} \frac{1}{(n-2)^4} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

This is a *p*-series with $p = 4$.

Therefore, by the *p*-series, the given series *converges*.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} 2n^{-3/2}$

Solution

$$\sum_{n=1}^{\infty} 2n^{-3/2} = 2\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

This is a p-series with $p = \dfrac{3}{2}$

Therefore, by the p-series, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{k=0}^{\infty} \dfrac{k}{\sqrt{k^2 + 4}}$

Solution

$$\lim_{k \to \infty} \dfrac{k}{\sqrt{k^2 + 4}} = \lim_{k \to \infty} \dfrac{k}{\sqrt{k^2}}$$

$$= \lim_{k \to \infty} \dfrac{k}{k}$$

$$= \underline{1} \neq 0$$

Therefore, the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{k=1}^{\infty} \dfrac{2^k + 3^k}{4^k}$

Solution

$$\int_1^{\infty} \left(\dfrac{2^x}{4^x} + \dfrac{3^x}{4^x} \right) dx = \int_1^{\infty} \left(\left(\dfrac{1}{2}\right)^x + \left(\dfrac{3}{4}\right)^x \right) dx$$

$$= \left[-\dfrac{1}{\ln(2)} \left(\dfrac{1}{2}\right)^x - \dfrac{1}{\ln(3/4)} \left(\dfrac{3}{4}\right)^x \right]_1^{\infty}$$

$$= 0 - \left(-\dfrac{1}{2\ln(2)} - \dfrac{3}{4\ln(3/4)} \right)$$

$$\approx \underline{3.3284 < \infty}$$

Therefore, by the integral test, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{k=2}^{\infty} \dfrac{4}{k \ln^2 k}$

Solution

$$\int_2^{\infty} \dfrac{4}{x \ln^2 x} dx = 4 \int_2^{\infty} \dfrac{1}{\ln^2 x} d(\ln x)$$

$$= -4 \frac{1}{\ln x}\Big|_2^\infty$$

$$\underline{= 4\frac{1}{\ln 2} < \infty}$$

Therefore, by the integral test, the given series **_converges_**.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

Solution

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/5}}$$

This is a *p*-series with $p = \dfrac{1}{5} < 1$

Therefore, by the *p*-series, the given series **_diverges_**.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$

Solution

This is a *p*-series with $p = \dfrac{5}{3} > 1$

Therefore, by the *p*-series, the given series **_converges_**.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

Solution

This is a *p*-series with $p = 1.04 > 1$

Therefore, by the *p*-series, the given series **_converges_**.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{\pi}}$

Solution

This is a *p*-series with $p = \pi > 1$

Therefore, by the *p*-series, the given series ***converges***.

Exercise

Determine if the series converge or diverge $1 + \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} + \dfrac{1}{4\sqrt{4}} + \dfrac{1}{5\sqrt{5}} + \cdots$

Solution

$$1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots = \sum_{1}^{\infty}\frac{1}{n^{3/2}}$$

This is a *p*-series with $p = \dfrac{3}{2} > 1$

Therefore, by the *p*-series, the given series ***converges***.

Exercise

Determine if the series converge or diverge $1 + \dfrac{1}{\sqrt[3]{4}} + \dfrac{1}{\sqrt[3]{9}} + \dfrac{1}{\sqrt[3]{16}} + \dfrac{1}{\sqrt[3]{25}} + \cdots$

Solution

$$1 + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{9}} + \frac{1}{\sqrt[3]{16}} + \frac{1}{\sqrt[3]{25}} + \cdots = \sum_{1}^{\infty}\frac{1}{n^{2/3}}$$

This is a *p*-series with $p = \dfrac{2}{3} < 1$

Therefore, by the *p*-series, the given series ***diverges***.

Exercise

Consider a wedding cake of infinite height, each layer of which is a right circular cylinder of height 1. The bottom layer of the cake has a radius of 1, the second layer has a radius of $\dfrac{1}{2}$, the third layer has a radius of $\dfrac{1}{3}$, and the nth layer has a radius of $\dfrac{1}{n}$.

a) To determine how much frosting is needed to cover the cake, find the area of the lateral (vertical sides of the wedding cake. What is the area of the horizontal surfaces of the cake?

b) Determine the volume of the cake.

c) Comment on your answer to parts (*a*) and (*b*)

a) The circumference of the kth layer is: $2\pi\frac{1}{k}$, so its area $\frac{2\pi}{k}$

The total vertical surface area:

$$\sum_{k=1}^{\infty}\frac{2\pi}{k}=2\pi\sum_{k=1}^{\infty}\frac{1}{k}$$

$$=\infty$$

Looking at the cake from above, the horizontal area

$$Area=\pi r^2=\pi\cdot 1^2$$

$$=\pi$$

b) The volume of a cylinder $=\pi r^2 h$

Volume of the kth layer $=\pi\frac{1}{k^2}\cdot 1=\frac{\pi}{k^2}$

Thus the volume of the cake is:

$$\sum_{k=1}^{\infty}\frac{\pi}{k^2}=\pi\sum_{k=1}^{\infty}\frac{1}{k^2}$$

$$=\frac{\pi^3}{6}$$

$$\approx 5.168$$

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}\quad(Leonhard\ Euler)$$

c) This cake has infinite area, it has finite volume.

Exercise

The Riemann zeta function is the subject of extensive research and is associated with several renowned

unsolved problems. Is its defined by $\zeta(x)=\displaystyle\sum_{k=1}^{\neq}\frac{1}{k^x}$, when x is a real number, the zeta funcion becomes

a p-series. For even positive integers ρ, the value of $\zeta(\rho)$ is known exactly. For example,

$$\sum_{k=1}^{\neq}\frac{1}{k^2}=\frac{\pi^2}{6},\quad \sum_{k=1}^{\neq}\frac{1}{k^4}=\frac{\pi^4}{90},\quad and\quad \sum_{k=1}^{\neq}\frac{1}{k^6}=\frac{\pi^6}{945},\quad\ldots$$

a) Use the estimation techniques to approximate $\zeta(3)$ and $\zeta(5)$ (whose values are not known exactly) with a remainder less than 10^{-3}.

b) Determine the sum of the reciprocals of the squares of the odd positive integers by rearranging the terms of the series $(x=2)$ without changing the value of the series.

Solution

a) $\zeta(m) = \int_n^{\infty} \frac{1}{x^m}\,dx = \frac{1}{m-1}x^{1-m}\Big|_n^{\infty} = \underline{\frac{1}{m-1}n^{1-m}}$

For $\zeta(3) = \frac{1}{2}n^{-2} < 10^{-3}$

$$\frac{1}{2n^2} < \frac{1}{10^3} \quad \rightarrow \quad 2n^2 > 10^3$$

$$\underline{n > \sqrt{500} \approx 23}$$

$$\sum_{k=1}^{23} \frac{1}{k^3} \approx \underline{1.201151955}$$

The true value is $\approx \underline{1.202056903}$

For $\zeta(5) = \frac{1}{4}n^{-4} < 10^{-3}$

$$\frac{1}{4n^4} < \frac{1}{10^3} \quad \rightarrow \quad 4n^4 > 10^3$$

$$\underline{n > (250)^{1/4} \approx 4}$$

$$\sum_{k=1}^{4} \frac{1}{k^5} \approx \underline{1.0363417888}$$

The true value is $\approx \underline{1.036927755}$

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for i = 1:n
    kk = 1 / (i^x);
    k = k + kk;
end
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b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$

$$= \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

$$\frac{3}{4}\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{3}{4}\frac{\pi^2}{6} = \frac{\pi^2}{8}$$

Exercise

Consider a set of identical dominoes that are 2 inches long. The dominoes are stacked on top of each other with their long edges aligned so that each domino overhangs the one beneath is as far as possible

a) If there are n dominoes in the stack, what is the greatest distance that the top domino can be made to overhang the bottom domino? (*Hint*: Put the nth domino beneath the previous $n-1$ dominoes.)

b) If we allow for infinitely many dominoes in the stack, what is the greatest distance that the top domino can be made to overhang the bottom domino?

Solution

a) The center of gravity of any stack of dominoes is the average of the locations of their centers. Define the midpoint of the zeroth (top) domino to be $x = 0$, and stack additional dominoes down and to its right (to increasingly positive x-coordinates).

Let $m(n)$ be the x-coordinate of the midpoint of the nth domino. Then in order for the stack not to fall over, the left edge of the nth domino must be placed directly under the center of gravity of dominos 0 through $n-1$, which is $\dfrac{1}{n}\displaystyle\sum_{i=0}^{n-1} m(i)$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$$

So $m(n) = 1 + \dfrac{1}{n}\displaystyle\sum_{i=0}^{n-1} m(i) = \sum_{k=1}^{n}\frac{1}{k}$
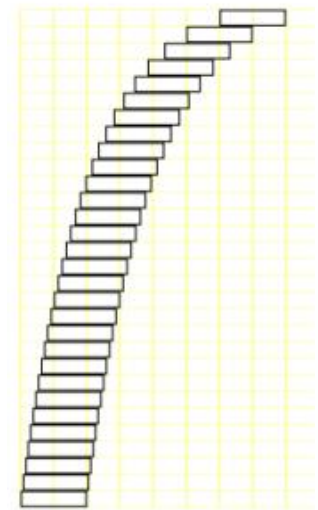
Proof by induction;

For $n = 1 \Rightarrow m(1) = 1$ ✓ P_1 is true

Let P_j is true $m(j) = \displaystyle\sum_{k=1}^{j}\frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{j}$;

we need to prove it is also true for P_{j+1}

$$m(j+1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{j} + \frac{1}{j+1}$$

$$= \sum_{k=1}^{j}\frac{1}{k} + \frac{1}{j+1}$$

$$= \sum_{k=1}^{j+1} \frac{1}{k}$$

Therefore, the formula is clearly true by mathematical induction.

b) For infinite number of dominos, because the overhang is the harmonic series, the distance is potentially infinite. This series diverges so with enough dominoes.