Section R.2 – Integration

Definition of Antiderivative

A Function F is an Antiderivative of a function f if for every x in the domain of f, it follows that $F'(x) = f(x)$

Notation for Antiderivatives and indefinite integrals

The notation

$$\int f(x)dx = F(x) + C$$

where C is an arbitrary constant, means that F is an Antiderivative of f.
That is $F'(x) = f(x)$ for all x in the domain of f.

$\int f(x)dx$ Indefinite integral

Integrand Antiderivative

Integral sign

$$\int f(x)dx = \overbrace{F(x)+C}$$

Differential

Basic Integration Rules

$$\int kdx = kx + C$$

$$\int kf(x)dx = k\int f(x)dx$$

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

The General Power Rule

The Simple Power Rule is given by:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \mathbf{n \neq -1}$$

$$\int \overbrace{\left(x^2+1\right)}^{u^3}{}^{3} \underbrace{2x dx}_{du} = \int u^3 du$$

$$= \frac{u^4}{4} + C$$

General Power Rule for Integration

If u is a differentiable function of x, then

$$\int u^n \frac{du}{dx} dx = \int u^n du = \frac{u^{n+1}}{n+1} + C \qquad \mathbf{n \neq -1}$$

Example

Find each indefinite integral.

$$\int 5x dx = \int 5x^1 dx$$

$$= 5\frac{x^{1+1}}{1+1} + C$$

$$= \frac{5}{2}x^2 + C$$

$$\int \sqrt[3]{x} dx = \int x^{1/3} dx$$

$$= \frac{x^{1/3+1}}{1/3+1} + C$$

$$= \frac{x^{4/3}}{4/3} + C$$

$$\boxed{= \frac{3}{4}x^{4/3} + C} \qquad \textit{or} \qquad = \frac{3}{4}x\sqrt[3]{x} + C$$

Using the Exponential Rule

Let u be a differentiable function of u

$$\int e^{u}du = e^{u} + C \qquad \textit{General Exponential Rule}$$

Example

Find the indefinite integral $\int e^{2x+3}dx$

Solution

Let $u = 2x + 3 \to du = 2dx$

$$\int e^{2x+3}dx = \int e^{u}\,\tfrac{1}{2}du$$

$$= \tfrac{1}{2}\int e^{u}du$$

$$= \tfrac{1}{2}e^{u} + C$$

$$\boxed{= \tfrac{1}{2}e^{2x+3} + C}$$

Using the Log Rule

Let u be a differentiable function of x.

$$\int \frac{du/dx}{u}dx = \int \frac{1}{u}du = \ln|u| + C \qquad \textit{General Logarithmic Rule}$$

Example

Find the indefinite integral $\int \frac{1}{4x+1}dx$

Solution

Let $u = 4x + 1 \to du = 4dx \to \frac{1}{4}du = dx$

$$\int \frac{1}{4x+1}dx = \int \frac{1}{u}\frac{1}{4}du$$

$$= \frac{1}{4}\int \frac{1}{u}du$$

$$= \frac{1}{4}\ln|u| + C$$

$$\boxed{= \frac{1}{4}\ln|4x+1| + C}$$

Integration by Parts

Let u and v be differentiable functions of x.

$$\int u\,dv = uv - \int v\,du$$

Guidelines for integration by Parts

1. Let dv be the most complicated portion of the integrand that fits a basic integration formula. Let u be the remaining factor.
2. Let u be the portion of the integrand whose derivative is a function simpler than u. Let dv be the remaining factor.

Example

Find $\int x\cos x\,dx$

Solution

Let:

$$u = x \qquad dv = \cos x\,dx$$

$$du = dx \quad v = \int dv = \int \cos x\,dx = \sin x$$

$$\int u\,dv = uv - \int v\,du$$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + C$$

Example

Evaluate $\int x^2 e^x\,dx$

Solution

$$f(x) = x^2 \quad and \quad g(x) = e^x$$

$$\int x^2 e^x\,dx = x^2e^x - 2xe^x + 2e^x + C$$

	derivative	$\int e^x dx$
$(+)$	x^2	e^x
$(-)$	$2x$	e^x
$(+)$	2	e^x
	0	

Particular Solutions

In many applications of integrations, we are given enough information to determine a particular solution. To do this, we need to know the value of *F(x)* for one value of *x*. This information is called an initial condition.

Example

Solve the differential equation: $y' = te^t$ that satisfies $y(0) = 2$

Solution

$$y = \int te^t dt$$

Integration by part: $\int udv = uv - \int vdu$

$$\begin{cases} u = t \Rightarrow du = dt \\ dv = e^t dt \Rightarrow v = e^t \end{cases}$$

$$y = te^t - \int e^t dt$$

$$= te^t - e^t + C$$

$$y(0) = (0)e^0 - e^0 + C = 2$$

$$-1 + C = 2$$

$$C = 3$$

$$\underline{y(t) = e^t (t-1) + 3}$$

Example

Solve the differential equation: $y' = \frac{1}{x}$ that satisfies $y(1) = 3$

Solution

$$y = \int \frac{1}{x} dx$$

$$= \ln|x| + C$$

$$y(1) = \ln|1| + C = 3$$

$$\boxed{C = 3}$$

$$\underline{y(x) = \ln x + 3} \qquad with\ x > 0$$

Example

Suppose a ball thrown into the air with initial velocity $v_0 = 20\ ft/\sec$. Assuming the ball thrown from a height of $x_0 = 6\ ft$, how long does it take for the ball to hit the ground?

Solution

$$\frac{dv}{dt} = -g \Rightarrow dv = -gdt$$

$$v(t) = -gt + C_1$$

$$v(t=0) = -g(0) + C_1 = 20$$

$$C_1 = 20$$

$$v(t) = -32t + 20$$

$$\frac{dx}{dt} = v \Rightarrow dx = vdt$$

$$\int dx = \int vdt$$

$$x(t) = \int (-32t + 20)dt$$

$$= -16t^2 + 20t + C_2$$

$$x(t=0) = -16(0)^2 + 20(0) + C_2 = 6$$

$$C_2 = 6$$

$$\underline{x(t) = -16t^2 + 20t + 6}$$

Theorem – The Fundamental Theorem of Calculus, P-2

If f is continuous at every point in $[a, b]$, then F is any antiderivative of f on $[a, b]$, then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Example

a) $\displaystyle\int_0^{\pi} \cos x\, dx = \sin x \Big|_0^{\pi}$

$= \sin \pi - \sin 0$

$= \underline{0}$

b) $\displaystyle\int_{-\frac{\pi}{4}}^{0} \sec x \tan x\, dx = \sec x \Big|_{-\frac{\pi}{4}}^{0}$

$= \sec 0 - \sec\left(-\frac{\pi}{4}\right)$

$= \underline{1 - \sqrt{2}}$

c) $\displaystyle\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \left[x^{3/2} + \frac{4}{x}\right]_1^4$

$= \left((4)^{3/2} + \frac{4}{4}\right) - \left((1)^{3/2} + \frac{4}{1}\right)$

$= (9) - (5)$

$= \underline{4}$

Exercises *Section* R.2 – Integration

Find each indefinite integral.

1. $\int \frac{x+2}{\sqrt{x}}dx$

2. $\int 4y^{-3}dy$

3. $\int (x^3-4x+2)dx$

4. $\int \left(\sqrt[4]{x^3}+1\right)dx$

5. $\int \sqrt{x}(x+1)dx$

6. $\int (1+3t)t^2dt$

7. $\int \frac{x^2-5}{x^2}dx$

8. $\int (-40x+250)dx$

9. $\int \left(7-3x-3x^2\right)(2x+1)\,dx$

10. $\int xe^{2x}dx$

11. $\int x\ln x\,dx$

12. $\int \left(x^2-2x+1\right)e^{2x}dx$

13. $\int e^{2x}\cos 3x\,dx$

Find the general solution of the differential equation

14. $y'=2t+3$

15. $y'=3t^2+2t+3$

16. $y'=\sin 2t+2\cos 3t$

17. $y'=x^3(3x^4+1)^2$

18. $y'=5x\sqrt{x^2-1}$

19. $y'=x\sqrt{x^2+4}$

20. $y'=(2x+1)e^{x^2+x}$

21. $y'=\frac{1}{6x-5}$

22. $y'=\frac{x^2+2x+3}{x^3+3x^2+9x+1}$

23. $y'=\frac{1}{x(\ln x)^2}$

Evaluate the integrals

24. $\int_{-2}^{2}\left(x^3-2x+3\right)dx$

25. $\int_{0}^{1}\left(x^2+\sqrt{x}\right)dx$

26. $\int_{0}^{\pi/3} 4\sec u\tan u\;du$

27. $\int_{\pi/4}^{3\pi/4} \csc\theta\cot\theta\,d\theta$

28. $\int_{-\pi/3}^{-\pi/4}\left(4\sec^2 t+\frac{\pi}{t^2}\right)dt$

29. $\int_{-3}^{-1}\frac{y^5-2y}{y^3}dy$

30. $\int_{1}^{8}\frac{\left(x^{1/3}+1\right)\left(2-x^{2/3}\right)}{x^{1/3}}dx$

31. $\int_{0}^{1}(2t+3)^3dt$

32. $\int_{-1}^{1} r\sqrt{1-r^2}\;dr$

33. Find the general solution of $F'(x) = 4x + 2$, and find the particular solution that satisfies the initial condition $F(1) = 8$.

34. Find the general solution of the differential equation: $y' = t\cos 3t$

35. A ball is thrown into the air from an initial height of 6 *m* with an initial velocity of 120 *m/s*. What will be the maximum height of the ball and at what time will this event occur?

36. Derive the position function if a ball is thrown upward with initial velocity of 32 *ft* per *second* from an initial height of 48 *ft*. When does the ball hit the ground? With what velocity does the ball hit the ground?