Solution   *Section* 3.1 – Introduction to Linear Systems

Exercise

Find a solution for x, y, z to the system of equations

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3e+2\sqrt{2}+\pi \\ 6e+5\sqrt{2}+4\pi \\ 9e+8\sqrt{2}+7\pi \end{pmatrix}$$

Solution

$$\begin{pmatrix} x+2y+3z \\ 4x+5y+6z \\ 7x+8y+9z \end{pmatrix} = \begin{pmatrix} 3e+2\sqrt{2}+\pi \\ 6e+5\sqrt{2}+4\pi \\ 9e+8\sqrt{2}+7\pi \end{pmatrix}$$

$$\Rightarrow \begin{cases} x+2y+3z = \pi+2\sqrt{2}+3e \\ 4x+5y+6z = 4\pi+5\sqrt{2}+6e \\ 7x+8y+9z = 7\pi+8\sqrt{2}+9e \end{cases}$$

Solution: $\boxed{x=\pi \quad y=\sqrt{2} \quad z=e}$

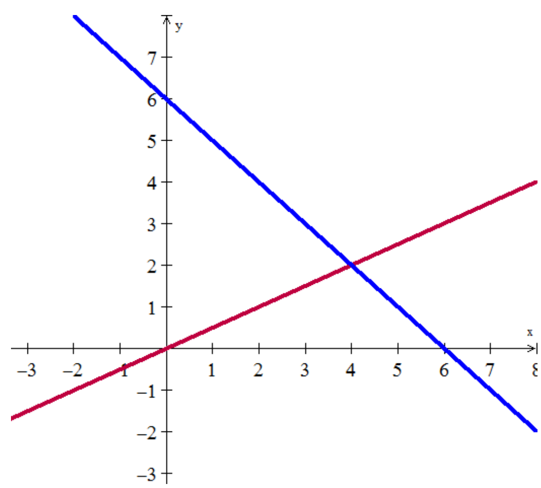
Exercise

Draw the two pictures in two planes for the equations: $x-2y=0, \quad x+y=6$

Solution

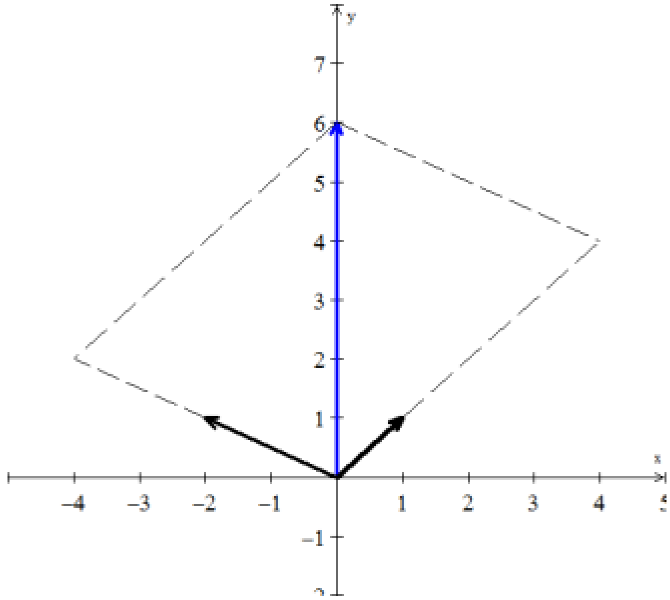
The matrix form of the 2 equations:

$$\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$



Row picture is the 2 lines from the given equations and their intersection is the point (4, 2) which is the solution for the system.

Column Picture is the column vectors $\begin{pmatrix}1 & 1\end{pmatrix}$ and $\begin{pmatrix}-2 & 1\end{pmatrix}$



$$x\begin{pmatrix}1\\1\end{pmatrix}+y\begin{pmatrix}-2\\1\end{pmatrix}=4\begin{pmatrix}1\\1\end{pmatrix}+2\begin{pmatrix}-2\\1\end{pmatrix}=\begin{pmatrix}0\\6\end{pmatrix}$$

The parallelogram show how the solution vector $\begin{pmatrix}0 & 6\end{pmatrix}$ can be written as the linear combination of the column vectors.

Exercise

Normally 4 planes in 4-dimensional space meet at a ________. Normally 4 column vectors in 4-deimensional space can combine to produce b. what combinations of $(1,\ 0,\ 0,\ 0),\quad (1,\ 1,\ 0,\ 0),\quad (1,\ 1,\ 1,\ 0),\quad (1,\ 1,\ 1,\ 1)$ produces $b=(3,\ 3,\ 3,\ 2)$?
What 4 equations for $x,\ y,\ z,\ w$ are you solving?

Solution

Normally 4 planes in 4-dimensional space meet at a ***point***.

The combination of the vectors producing b is:

$$0\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+0\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+1\begin{pmatrix}1\\1\\1\\0\end{pmatrix}+2\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}3\\3\\3\\2\end{pmatrix}$$

$$1\begin{pmatrix}1\\1\\1\\0\end{pmatrix}+2\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}3\\3\\3\\2\end{pmatrix}$$

The system of equations that satisfies the given vectors is:

$$\begin{cases} x+y+z+w=3 \\ y+z+w=3 \\ z+w=3 \\ w=2 \end{cases}$$

Exercise

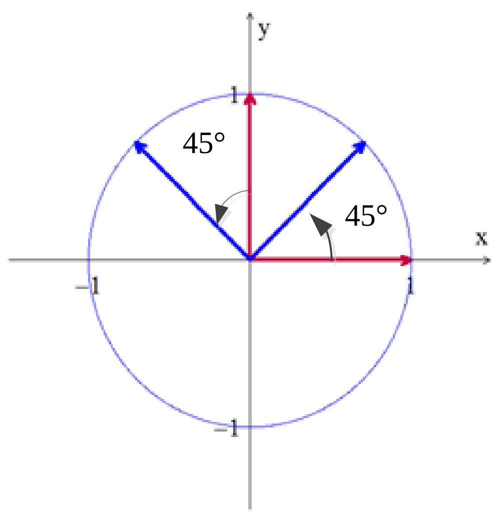
What 2 by 2 matrix A rotates every vector through 45° ?

The vector (1, 0) goes to $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. The vector (0, 1) goes to $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Those determine the matrix. Draw these particular vectors is the xy-plane and find A.

Solution

$$A=\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$



Exercise

What two vectors are obtained by rotating the plane vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ by 30° (*cw*) ?

Write a matrix A such that for every vector v in the plane, Av is the vector obtained by rotating v clockwise by 30°.

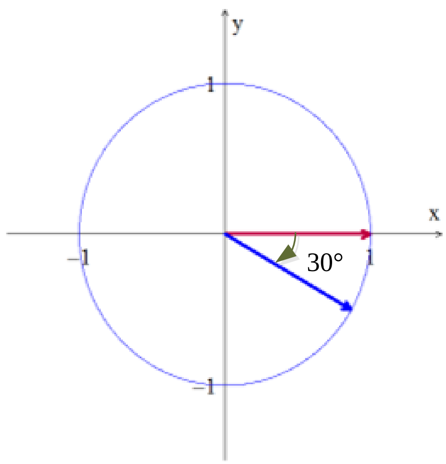
Find a matrix B such that for every 3-dimensional vector v, the vector Bv is the reflection of v through the plane $x+y+z=0$. $Hint: v=(1,\ 0,\ 0)$

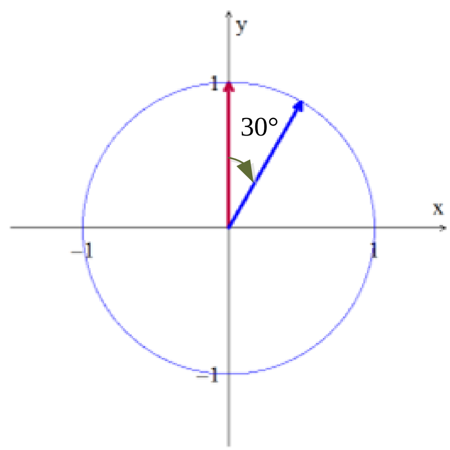
Solution

Rotating the vectors by 30° (*cw*) yields:

For the vector $\begin{bmatrix}1\\0\end{bmatrix}$ yields to $\begin{pmatrix}\cos(-30°)\\ \sin(-30°)\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{2}\\ -\frac{1}{2}\end{pmatrix}$

And for the vector $\begin{bmatrix}0\\1\end{bmatrix}$ yields to $\begin{pmatrix}\sin(30°)\\ \cos(30°)\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}$





The desired matrix is: $A=\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}$

To get 1 from $\frac{\sqrt{3}}{2}$ is to multiply by $\frac{2}{\sqrt{3}}=2\frac{1}{\sqrt{3}}$

The unit vector to the plane $x+y+z=0$ is $\hat{u}=\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$

$$Bv=B\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}-\frac{2}{\sqrt{3}}\hat{u}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{2}{\sqrt{3}} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$B\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{\sqrt{3}}\hat{u} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$$

$$B\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{2}{\sqrt{3}}\hat{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

The solution: $\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$

Exercise

Find a system of linear equation corresponding to the given augmented matrix

$$\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$$

Solution

$$\begin{cases} \quad 3x_2 - x_3 - x_4 = -1 \\ 5x_1 + 2x_2 \quad - 3x_4 = -6 \end{cases}$$

Exercise

Find a system of linear equation corresponding to the given augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \\ 5 & -6 & 1 & 1 \\ -8 & 0 & 0 & 3 \end{bmatrix}$$

Solution

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ -4x_1 - 3x_2 - 2x_3 = -1 \\ 5x_1 - 6x_2 + x_3 = 1 \\ -8x_1 = 3 \end{cases}$$

Exercise

Find the augmented matrix for the given system of linear equations.

$$\begin{cases} -2x_1 = 6 \\ 3x_1 = 8 \\ 9x_1 = -3 \end{cases}$$

Solution

$$\left[\begin{array}{c|c} -2 & 6 \\ 3 & 8 \\ 9 & -3 \end{array}\right]$$

Exercise

Find the augmented matrix for the given system of linear equations.

$$\begin{cases} 3x_1 - 2x_2 = -1 \\ 4x_1 + 5x_2 = 3 \\ 7x_1 + 3x_2 = 2 \end{cases}$$

Solution

$$\left[\begin{array}{cc|c} 3 & -2 & -1 \\ 4 & 5 & 3 \\ 7 & 3 & 2 \end{array}\right]$$

Exercise

Find the augmented matrix for the given system of linear equations.

$$\begin{cases} 2x_1 \qquad\quad + 2x_3 = 1 \\ 3x_1 - x_2 + 4x_3 = 7 \\ 6x_1 + x_2 - x_3 = 0 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 1 \\ 3 & -1 & 4 & 7 \\ 6 & 1 & -1 & 0 \end{array}\right]$$

Solution *Section* 3.2 – Gaussian Elimination

Exercise

When elimination is applied to the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix}$

a) What are the first and second pivots?

b) What is the multiplier l_{21} in the first step (l_{21} times row 1 is subtracted from row 2)?

c) What entry in the 2, 2 position (instead of 9) would force an exchange of rows 2 and 3?

d) What is the multiplier $l_{31} = 0$, subtracting 0 times row 1 from row 3?

Solution

a) The first pivot is 3 and when 2 times row 1 is subtracted from row 2, the second pivot is revealed as 7.

$$\begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix} \quad \begin{matrix} \textit{subtract } 2 \textit{ times row.1} \\ \textit{from row.2} \end{matrix} \quad \begin{bmatrix} 3 & 1 & 0 \\ 0 & \boxed{7} & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

b) The multiplier l_{21} in the first step is $\frac{6}{3} = 2$.

c) If we reduce the entry 9 to 2, that drop of 7 in the a_{22} position would force a row exchange.

$$\begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix} \quad \begin{matrix} \textit{subtract } 7 \textit{ times row.1} \\ \textit{from row.2} \end{matrix} \quad \begin{bmatrix} 3 & 1 & 0 \\ -15 & \boxed{2} & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

d) The multiplier l_{31} is already zero because $a_{31} = 0$ and no needs row elimination.

Exercise

Use elimination to reach upper triangular matrices ***U***. Solve by back substitution or explain why this impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the $-x$ in equation (3).

$$\begin{cases} x+y+z=7 \\ x+y-z=5 \\ x-y+z=3 \end{cases} \qquad \begin{cases} x+y+z=7 \\ x+y-z=5 \\ -x-y+z=3 \end{cases}$$

Solution

For the *first* system:

$$\begin{matrix} x+y+z=7 & \textit{subtract eqn.1} & x+y+z=7 \\ x+y-z=5 & \textit{from eqn.2} & 0y-2z=-2 \\ x-y+z=3 & \textit{from eqn.3} & -2y-0z=-4 \end{matrix}$$

$$\begin{matrix} x+y+z=7 & & 1x+\ y+\ z=7 \\ x+y-z=5 & \textit{Exchange eqn.2} & -2y-0z=-4 \\ x-y+z=3 & \textit{and eqn.3} & -2z=-2 \end{matrix}$$

The solutions are: $z=1 \quad y=2 \quad x=4$ and the pivots are 1, -2, -2.

For the *second* system:

$$\begin{matrix} x+y+z=7 & \textit{Subtract eqn.1} & x+y+z=7 \\ x+y-z=5 & \textit{from eqn.2} & 0y-2z=-2 \\ -x-y+z=3 & \textit{Add eqn.1 to eqn.3} & 0y+2z=10 \end{matrix}$$

$$\begin{matrix} x+y+z=7 & & x+y+z=7 \\ 0y-2z=-2 & \textit{Add eqn.2} & 0y-2z=-2 \\ 0y+2z=10 & \textit{to eqn.3} & \boxed{0z=8} \end{matrix}$$

The three planes don't meet. But if we change '3' in the last equation to '-5'

$$\begin{matrix} x+y+z=7 & \textit{Subtract eqn.1} & x+y+z=7 \\ x+y-z=5 & \textit{from eqn.2} & 0y-2z=-2 \\ -x-y+z=-5 & \textit{Add eqn.1 to eqn.3} & 0y+2z=2 \end{matrix}$$

$$\begin{matrix} x+y+z=7 & x+y=6 \\ 0y-2z=-2 & \\ 0y+2z=10 & z=1 \end{matrix}$$

There are unique infinite many solutions!

The three planes now meet along a whole line.

Exercise

For which numbers a does the elimination break down (1) permanently (2) temporarily

$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= 6 \end{aligned}$$

Solve for x and y after fixing the second breakdown by a row change.

Solution

The matrix form is: $\begin{pmatrix} a & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$

If $a = 0$, the elimination brakes down temporarily.

$$\begin{pmatrix} 4 & 6 \\ 0 & \boxed{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$$

The system is in upper triangular form and entry row 2 column 2 is not equal to zero, therefore the system has a solution.

If $a \ne 0$,

$$\begin{pmatrix} a & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix} \qquad R_2 - \frac{4}{a}R_1$$

$$\begin{pmatrix} a & 3 \\ 0 & 6 - \frac{12}{a} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 + \frac{12}{a} \end{pmatrix}$$

$$6 - \frac{12}{a} = 0 \Rightarrow \frac{12}{a} = 6$$

$$\to \underline{\left| a = \frac{12}{6} = 2 \right|}$$

If $a = 2$,

$\begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \end{pmatrix}$, the system will fail and has no solution.

If $a \ne 2$;

$\begin{pmatrix} a & 3 \\ 0 & 6 - \frac{12}{a} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 + \frac{12}{a} \end{pmatrix}$, the system has a unique solution.

Exercise

Find the pivots and the solution for these four equations:

$$\begin{aligned} 2x + y \qquad &= 0 \\ x + 2y + z \quad &= 0 \\ y + 2z + t &= 0 \\ z + 2t &= 5 \end{aligned}$$

Solution

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} R_2 - \tfrac{1}{2}R_1$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 1.5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} R_3 - \tfrac{2}{3}R_2$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} R_4 - \tfrac{3}{4}R_3$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 5 \end{pmatrix} \begin{aligned} 2x = -y \Rightarrow \lfloor x = -2\tfrac{1}{2} = -1 \rfloor \\ \tfrac{3}{2}y + z = 0 \Rightarrow y = -z\tfrac{2}{3} = -(-3)\tfrac{2}{3} \to \boxed{y = 2} \\ \tfrac{4}{3}z + t = 0 \to \tfrac{4}{3}z = -t \to \lfloor z = -4\tfrac{3}{4} = -3 \rfloor \\ \tfrac{5}{4}t = 5 \to \boxed{t = 4} \end{aligned}$$

The pivots are diagonal entries and the solution is: $(-1,\ 2,\ -3,\ 4)$

Exercise

Look for a matrix that has row sums 4 and 8, and column sums 2 and s.

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{matrix} a+b=4 & a+c=2 \\ c+d=8 & b+d=s \end{matrix}$$

The four equations are solvable only if s = ____. Then find two different matrices that have the correct row and column sums.

Solution

$$\begin{array}{rr} & a+b=4 \\ + & c+d=8 \\ \hline & a+c+b+d=12 \end{array}$$

$2+s=12$

$\boxed{s=10}$

Exercise

Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of A is a _______ of the first two rows. Find a third equation that can't be solved together with $x+y+z=0$ and $x-2y-z=1$

Solution

The system is singular if row 3 of A is a ***linear combination*** of the first two rows.

There are many possible of a third equation that can't be solved together with $x+y+z=0$ and $x-2y-z=1$.

$$\begin{array}{rr} 3 \text{ times } 1^{st} \text{ equation} & 3x+3y+3z \\ \text{minus } 2nd & -x+2y+z \\ \hline & 2x+5y+4z=1 \end{array}$$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} -1 & -2 & 3 & 1 \\ 1 & 1 & 2 & 8 \\ \hline 0 & -1 & 5 & 9 \end{array} \qquad \begin{array}{cccc} 3 & -7 & 4 & 10 \\ -3 & -3 & -6 & -24 \\ \hline 0 & -10 & -2 & -14 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right] -R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 10R_2 \end{array}$$

$$\begin{array}{cccc} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ \hline 1 & 0 & 7 & 17 \end{array} \qquad \begin{array}{cccc} 0 & -10 & -2 & -14 \\ 0 & 10 & -50 & -90 \\ \hline 0 & 0 & -52 & -104 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right] -\frac{1}{52}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 - 7R_3 \\ R_2 + 5R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & 7 & 17 \\ 0 & 0 & -7 & -14 \\ \hline 1 & 0 & 0 & 3 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -5 & -9 \\ 0 & 0 & 5 & 10 \\ \hline 0 & 1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: $\boxed{(3,\ 1,\ 2)}$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \qquad\qquad - 3w = -3 \end{cases}$$

Solution

Solution: $\boxed{(w-1,\ 2z,\ z,\ w)}$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \qquad\qquad - 3w = -3 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ -1 & 3 & -2 & 1 \\ 3 & 4 & -7 & 10 \end{array}\right] \quad \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & -2 & -10 & -14 \end{array}\right] \quad \begin{array}{l} \\ \\ 5R_3 + 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & 0 & -52 & -52 \end{array}\right] \quad \begin{array}{rl} x + 2y + z = 8 & (3) \\ 5y - z = 9 & (2) \\ -52z = -52 & (1) \end{array}$$

$(1) \quad \Rightarrow \quad z = 1$

$(2) \quad \Rightarrow \quad 5y = 9 + 1 = 10 \rightarrow y = 2$

$(3) \quad \Rightarrow \quad x = 8 - 4 - 1 = 3$

$\therefore$ Solution: $\boxed{(3,\ 2,\ 1)}$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} 2u-3v+\ w-\ \ x+y=\ \ 0 \\ 4u-6v+2w-3x-y=-5 \\ -2u+3v-2w+2x-y=\ \ 3 \end{cases}$$

Solution

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 4 & -6 & 2 & -3 & -1 & -5 \\ -2 & 3 & -2 & 2 & -1 & 3 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3+R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -3 & -5 \\ 0 & 0 & -1 & 1 & 0 & 3 \end{array}\right] \qquad \begin{array}{cr} 2u-3v+w-x+y=0 & (3) \\ -x-3y=-5 & (2) \\ -w+x=3 & (1) \end{array}$$

$(2) \Rightarrow \ x=5-3y$

$(1) \Rightarrow \ w=x-3=2-3y$

$(3) \Rightarrow \ 2u=3v-2+3y+5-3y-y=3v-y+3$

$$u=\frac{3}{2}v-\frac{1}{2}y+\frac{3}{2}$$

$\therefore$ Solution: $\left(\frac{3}{2}v-\frac{1}{2}y+\frac{3}{2}, v, 2-3y, 5-3y, y\right)$

Exercise

Solve the given linear system by any method

$$\begin{cases} 2x_1+x_2+3x_3=0 \\ x_1+2x_2\qquad\ =0 \\ \qquad x_2+x_3=0 \end{cases}$$

Solution

$$\begin{cases} x_1=-2x_2 \\ x_3=-x_2 \end{cases} \ \rightarrow\ -4x_2+x_2-3x_2=0 \ \Rightarrow\ x_2=0$$

Solution: $\boxed{(0,\ 0,\ 0)}$

Exercise

Solve the given linear system by any method $\begin{cases} 2x+2y+4z \quad\ =0 \\ \quad -y-3z+w=0 \\ 3x+y+z+2w=0 \\ x+3y-2z-2w=0 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & -2 & 0 \\ 0 & -1 & -3 & 1 & 0 \\ 3 & 1 & 1 & 2 & 0 \\ 2 & 2 & 4 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3-3R_1 \\ R_4-2R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & -2 & 0 \\ 0 & -1 & -3 & 1 & 0 \\ 0 & -8 & 7 & 8 & 0 \\ 0 & -4 & 8 & 4 & 0 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & -2 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & -8 & 7 & 8 & 0 \\ 0 & -4 & 8 & 4 & 0 \end{array}\right] \begin{array}{l} R_1-3R_2 \\ \\ R_3+8R_2 \\ R_4+4R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -11 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 31 & 0 & 0 \\ 0 & 0 & 20 & 0 & 0 \end{array}\right] \begin{array}{l} x+w=0 \\ y-w=0 \\ \to z=0 \\ \\ \end{array}$$

Solution: $\boxed{(-w,\ w,\ 0,\ w)}$

Exercise

Add 3 times the second row to the first of $\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$

Solution

$$E=\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$EA=\begin{bmatrix}1&3&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}5&-1&5\\7&3&-2\\8&1&2\\6&0&-1\end{bmatrix}=\begin{bmatrix}27&8&-1\\7&3&-2\\8&1&2\\6&0&-1\end{bmatrix}$$

$$\begin{bmatrix}5&-1&5\\7&3&-2\\8&1&2\\6&0&-1\end{bmatrix}\quad \begin{matrix}R_1+3R_2\\ \\ \\ \end{matrix} =\begin{bmatrix}27&8&-1\\7&3&-2\\8&1&2\\6&0&-1\end{bmatrix}$$

Exercise

Solve the system using Gaussian elimination $\begin{cases}3x_1+2x_2-\ x_3=-15\\ 5x_1+3x_2+2x_3=\ 0\\ 3x_1+\ \ x_2+3x_3=\ 11\\ -6x_1-4x_2+2x_3=\ 30\end{cases}$

Solution

$$\left[\begin{array}{ccc|c}3&2&-1&-15\\5&3&2&0\\3&1&3&11\\-6&-4&2&30\end{array}\right]\quad \begin{array}{c}\\3R_2-5R_1\\R_3-R_1\\R_4+2R_1\end{array}$$

$$\left[\begin{array}{ccc|c}3&2&-1&-15\\0&-1&11&75\\0&-1&4&26\\0&0&0&0\end{array}\right]\quad \begin{array}{c}\\ \\R_3-R_2\\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c}3&2&-1&-15\\0&-1&11&75\\0&0&-7&-49\\0&0&0&0\end{array}\right]\quad \begin{array}{rl}3x_1+2x_2-x_3=-15 & (3)\\ -x_2+11x_3=75 & (2)\\ -7x_3=-49 & (1)\end{array}$$

$(1)\ \rightarrow\ x_3=7$

$(2)\ \rightarrow\ x_2=77-75=2$

$(1)\ \rightarrow\ 3x_1=-15-4+7=12\ \Rightarrow\ x_1=-4$

$\therefore$ Solution: $\underline{(-4,2,7)|}$

Exercise

For what value(s) of k, if any, does the system $\begin{cases} x+\ y-\ z=1 \\ 2x+3y+kz=3 \\ x+ky+3z=2 \end{cases}$ have

a) A unique solution?

b) Infinitely many solutions?

c) No solution?

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & k & 3 \\ 1 & k & 3 & 2 \end{array}\right] \quad \begin{array}{l} R_2-2R_1 \\ R_3-R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & k-1 & 4 & 1 \end{array}\right] \quad R_3-(k-1)R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & 0 & 4-(k-1)(k+2) & 2-k \end{array}\right] \qquad \begin{array}{r} x=1-y+z \\ y=1-(k+2)z \\ \to\left(6-k^2-k\right)z=-(k-2) \end{array}$$

$$\begin{cases} z=-\dfrac{k-2}{-(k-2)(k+3)}=\dfrac{1}{k+3} \quad {\color{red}(k\neq 2,-3)} \\ y=1-\dfrac{k+2}{k+3}=\dfrac{1}{k+3} \\ x=\dfrac{k+2}{k+3}+\dfrac{1}{k+3}=1 \end{cases}$$

a) Unique solution if $k\neq 2,-3$

b) Infinitely solution if ${\color{blue}k=2}$

c) No solution if ${\color{blue}k=-3}$

Solution Section 3.3 – Algebra of Matrices

Exercise

For the matrices: $A=\begin{bmatrix} p & 0 \\ q & r \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, when does $AB=BA$

Solution

$$AB=\begin{pmatrix} p & 0 \\ q & r \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} p & p \\ q & q+r \end{pmatrix}$$

$$BA=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & 0 \\ q & r \end{pmatrix}=\begin{pmatrix} p+q & r \\ q & r \end{pmatrix}$$

$$AB=BA$$

$$\begin{pmatrix} p & p \\ q & q+r \end{pmatrix}=\begin{pmatrix} p+q & r \\ q & r \end{pmatrix}$$

$$\begin{cases} p=p+q \\ \boxed{p=r} \\ q+r=r \end{cases} \Rightarrow \begin{cases} \boxed{q=0} \\ \\ q=0 \end{cases}$$

Exercise

A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

a) AB *c)* ABD *e)* ABC *g)* $A(B+C)$

b) BA *d)* DBA *f)* $ABCD$

Solution

$$A=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad B=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad C=\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad D=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

a) $AB:(3\times 5)(5\times 3)=(3\times 3)$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}=\begin{pmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{pmatrix}$$

b) $BA:(5\times3)(3\times5)=(5\times5)$

$$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{pmatrix}=\begin{pmatrix}3&3&3&3&3\\3&3&3&3&3\\3&3&3&3&3\\3&3&3&3&3\\3&3&3&3&3\end{pmatrix}$$

c) $ABD:(3\times5)(5\times3)(3\times1)=(3\times1)$

$$\begin{pmatrix}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}5&5&5\\5&5&5\\5&5&5\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}15\\15\\15\end{pmatrix}$$

d) $DBA:(3\times1)(5\times3)(3\times5)=NA$

e) $ABC:(3\times5)(5\times3)(5\times1)=NA$

f) $ABCD:(3\times5)(5\times3)(5\times1)(3\times1)=NA$

g) $A(B+C):(3\times5)((5\times3)+(5\times1))=NA$

Matrices B and C are not the same size.

Exercise

What rows or columns or matrices do you multiply to find.

a) The third column of AB?

b) The second column of AB?

c) The first row of AB?

d) The second row of AB?

e) The entry in row 3, column 4 of AB?

f) The entry in row 2, column 3 of AB?

Solution

a) A (column 3 of B)

b) A (column 2 of B)

c) (Row 1 of A) B

d) (Row 2 of A) B

e) (Row 3 of A) (Column 4 of B)

f) (Row 2 of A) (Column 3 of B)

Exercise

Add AB to AC and compare with $A(B+C)$:

$$A=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix} \quad and \quad B=\begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix} \quad and \quad C=\begin{bmatrix}3 & 1\\ 0 & 0\end{bmatrix}$$

Solution

$$AB=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix}\begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 7\\ 0 & 7\end{bmatrix}$$

$$AC=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix}\begin{bmatrix}3 & 1\\ 0 & 0\end{bmatrix}=\begin{bmatrix}3 & 1\\ 6 & 2\end{bmatrix}$$

$$\begin{aligned}A(B+C)&=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix}\left(\begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix}+\begin{bmatrix}3 & 1\\ 0 & 0\end{bmatrix}\right)\\ &=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix}\begin{bmatrix}3 & 3\\ 0 & 1\end{bmatrix}\\ &=\begin{bmatrix}3 & 8\\ 6 & 9\end{bmatrix}\end{aligned}$$

$$\begin{aligned}AB+AC&=\begin{bmatrix}0 & 7\\ 0 & 7\end{bmatrix}+\begin{bmatrix}3 & 1\\ 6 & 2\end{bmatrix}\\ &=\begin{bmatrix}3 & 8\\ 6 & 9\end{bmatrix}\end{aligned}$$

$$\boxed{A(B+C)=AB+AC}$$

Exercise

True or False

a) If A^2 is defined then A is necessarily square.

b) If AB and BA are defined then A and B are square.

c) If AB and BA are defined then AB and BA are square.

d) If $AB=B$, then $A=I$

Solution

a) True

b) False, if A has an order m by n and B n by m: $AB: m\times m \qquad BA: n\times n$

c) True; $AB: m\times m \qquad BA: n\times n$

d) False, if B is the matrix of all zeros.

Exercise

a) Find a nonzero matrix A such that $A^2 = 0$

b) Find a matrix that has $A^2 \neq 0$ but $A^3 = 0$

Solution

a) A nonzero matrix A such that $A^2 = 0$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

b) A matrix that has $A^2 \neq 0$ but $A^3 = 0$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = A^2 A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercise

Suppose you solve $Ax = b$ for three special right sides b:

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If the three solutions x_1, x_2, x_3 are the columns of a matrix X, what is A times X?

Solution

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Therefore, $Ax = I$

Exercise

Show that $(A+B)^2$ is different from $A^2+2AB+B^2$, when

$$A=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix} \quad \textit{and} \quad B=\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}$$

Write down the correct rule for $(A+B)(A+B)=A^2+$ __________ $+B^2$

Solution

$$A+B=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}+\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}=\begin{bmatrix}2 & 2\\ 3 & 0\end{bmatrix}$$

$$(A+B)^2=\begin{bmatrix}2 & 2\\ 3 & 0\end{bmatrix}\begin{bmatrix}2 & 2\\ 3 & 0\end{bmatrix}=\begin{bmatrix}10 & 4\\ 6 & 6\end{bmatrix}$$

$$A^2=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}$$

$$B^2=\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}$$

$$AB=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}=\begin{bmatrix}7 & 0\\ 0 & 0\end{bmatrix}$$

$$2AB=2\begin{bmatrix}7 & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}14 & 0\\ 0 & 0\end{bmatrix}$$

$$\begin{aligned}A^2+2AB+B^2&=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}+\begin{bmatrix}14 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}\\ &=\begin{bmatrix}16 & 2\\ 3 & 0\end{bmatrix}\end{aligned}$$

$$\begin{bmatrix}10 & 4\\ 6 & 6\end{bmatrix}\neq\begin{bmatrix}16 & 2\\ 3 & 0\end{bmatrix} \quad \Rightarrow \quad \boxed{(A+B)^2\neq A^2+2AB+B^2}$$

$$BA=\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}$$

$$\begin{aligned}A^2+AB+BA+B^2&=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}+\begin{bmatrix}7 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}+\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}\\ &=\begin{bmatrix}10 & 4\\ 5 & 6\end{bmatrix}\end{aligned}$$

$$\boxed{(A+B)(A+B)=A^2+\_AB+BA\_+B^2}$$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

Solution

By rows: $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 5 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 14 \\ 22 \end{pmatrix}$

By columns: $\begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 4\begin{pmatrix} 2 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 \\ 22 \end{pmatrix}$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Solution

By rows: $\begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 3 & 6 \end{pmatrix}\begin{pmatrix} 2 & -1 \end{pmatrix} \\ \begin{pmatrix} 6 & 12 \end{pmatrix}\begin{pmatrix} 2 & -1 \end{pmatrix} \end{pmatrix} \quad = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

By columns: $\begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 6 \end{pmatrix} - 1\begin{pmatrix} 6 \\ 12 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

Solution

By rows: $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 & 1 \end{pmatrix} \end{pmatrix}$

$$= \begin{pmatrix} 1(3) + 2(1) + 4(1) \\ 2(3) + 0(1) + 1(1) \end{pmatrix}$$

$$= \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

By columns: $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

Solution

$$\text{By rows: } \begin{pmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} (1 \quad 2 \quad 4)(2 \quad 2 \quad 3) \\ (-2 \quad 3 \quad 1)(2 \quad 2 \quad 3) \\ (-4 \quad 1 \quad 2)(2 \quad 2 \quad 3) \end{pmatrix}$$

$$= \begin{pmatrix} 18 \\ 5 \\ 0 \end{pmatrix}$$

$$\text{By columns: } \begin{pmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 18 \\ 5 \\ 0 \end{pmatrix}$$

Exercise

Given $A = \begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix}$ $\quad$ Find $A+B, \quad 2A, \quad and \quad -B$

Solution

$$A+B = \begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix} + \begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -2 \\ 8 & -2 & 0 \end{bmatrix}$$

$$2A = 2\begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 6 \\ 6 & -2 & -4 \\ 0 & 0 & 8 \end{bmatrix}$$

$$-B = -\begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 3 \\ 1 & 0 & 0 \\ -8 & 2 & 4 \end{bmatrix}$$

Exercise

Given $A = \begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad$ Find AB *and* BA *if possible*

Solution

$$AB = \begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -10 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3(3)+2(0)-3(1) & 3(-4)+2(1)-3(0) \\ 0(3)+1(0)+0(1) & 0(-4)+1(1)+0(0) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -10 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 2 & -9 \\ 0 & 1 & 0 \\ 3 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3(3)-4(0) & 3(2)-4(1) & 3(-3)-4(0) \\ 0(3)+1(0) & 0(2)+1(1) & 0(-3)+1(0) \\ 1(3)+0(0) & 1(2)+0(1) & 1(-3)+0(0) \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 2 & -9 \\ 0 & 1 & 0 \\ 3 & 2 & -3 \end{bmatrix}$$

Exercise

Given $A = \begin{bmatrix} 5 & 3 \\ -1 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 4 & -2 \\ -2 & 0 \\ 9 & 1 \end{bmatrix}$ $\quad$ Find AB *and* BA *if possible*

Solution

$AB = \textbf{\textit{Undefined}}$

$$BA = \begin{bmatrix} 4 & -2 \\ -2 & 0 \\ 9 & 1 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 22 & 12 \\ -10 & -6 \\ 44 & 27 \end{bmatrix}$$

Exercise

Given $A=\begin{bmatrix}3 & 0\\ -1 & 2\\ 1 & 1\end{bmatrix}$ $\quad B=\begin{bmatrix}4 & -1\\ 0 & 2\end{bmatrix}$ $\quad$ Find AB *and* BA *if possible*

Solution

a) $AB=\begin{bmatrix}3 & 0\\ -1 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}4 & -1\\ 0 & 2\end{bmatrix}=\begin{bmatrix}12 & -3\\ -4 & 5\\ 4 & 1\end{bmatrix}$

b) $BA=Undefined$

Exercise

Consider the matrices

$$A=\begin{bmatrix}3 & 0\\ -1 & 2\\ 1 & 1\end{bmatrix}\quad B=\begin{bmatrix}4 & -1\\ 0 & 2\end{bmatrix}\quad C=\begin{bmatrix}1 & 4 & 2\\ 3 & 1 & 5\end{bmatrix}\quad D=\begin{bmatrix}1 & 5 & 2\\ -1 & 0 & 1\\ 3 & 2 & 4\end{bmatrix}\quad E=\begin{bmatrix}6 & 1 & 3\\ -1 & 1 & 2\\ 4 & 1 & 3\end{bmatrix}$$

Compute the following (where possible):

a) $D+E$ ***b***) $D-E$ ***c***) $5A$ ***d***) $-7C$ ***e***) $2B-C$ ***g***) $-3(D+2E)$

Solution

a) $D+E=\begin{bmatrix}1 & 5 & 2\\ -1 & 0 & 1\\ 3 & 2 & 4\end{bmatrix}+\begin{bmatrix}6 & 1 & 3\\ -1 & 1 & 2\\ 4 & 1 & 3\end{bmatrix}=\begin{bmatrix}7 & 6 & 5\\ -2 & 1 & 3\\ 7 & 3 & 7\end{bmatrix}$

b) $D-E=\begin{bmatrix}1 & 5 & 2\\ -1 & 0 & 1\\ 3 & 2 & 4\end{bmatrix}-\begin{bmatrix}6 & 1 & 3\\ -1 & 1 & 2\\ 4 & 1 & 3\end{bmatrix}=\begin{bmatrix}-5 & 4 & -1\\ 0 & -1 & -1\\ -1 & 1 & 1\end{bmatrix}$

c) $5A=5\begin{bmatrix}3 & 0\\ -1 & 2\\ 1 & 1\end{bmatrix}=\begin{bmatrix}15 & 0\\ -5 & 10\\ 5 & 5\end{bmatrix}$

d) $-7C=-7\begin{bmatrix}1 & 4 & 2\\ 3 & 1 & 5\end{bmatrix}=\begin{bmatrix}-7 & -28 & -14\\ -21 & -7 & -35\end{bmatrix}$

e) $2B-C=can't\ be\ calculated$

g) $-3(D+2E)=-3\left(\begin{bmatrix}1 & 5 & 2\\ -1 & 0 & 1\\ 3 & 2 & 4\end{bmatrix}+\begin{bmatrix}12 & 2 & 6\\ -2 & 2 & 4\\ 8 & 2 & 6\end{bmatrix}\right)=-3\begin{bmatrix}13 & 7 & 8\\ -3 & 2 & 5\\ 11 & 4 & 10\end{bmatrix}=\begin{bmatrix}-39 & -21 & -24\\ 9 & -6 & -15\\ -33 & -12 & -30\end{bmatrix}$

Solution   *Section* 3.4 – Inverse Matrices

Exercise

Apply Gauss-Jordan method to find the inverse of this triangular "Pascal matrix"

Triangular Pascal matrix $A=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\1&1&0&0&0&1&0&0\\1&2&1&0&0&0&1&0\\1&3&3&1&0&0&0&1\end{array}\right]\begin{array}{l}\\R_2-R_1\\R_3-R_1\\R_4-R_1\end{array}$$

$$\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&-1&1&0&0\\0&2&1&0&-1&0&1&0\\0&3&3&1&-1&0&0&1\end{array}\right]\begin{array}{l}\\ \\R_3-2R_2\\R_4-3R_2\end{array}$$

$$\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&1&-2&1&0\\0&0&3&1&2&-3&0&1\end{array}\right]\begin{array}{l}\\ \\ \\R_4-3R_3\end{array}$$

$$\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&1&-2&1&0\\0&0&0&1&-1&3&-3&1\end{array}\right]$$

$$A^{-1}=\begin{pmatrix}1&0&0&0\\-1&1&0&0\\1&-2&1&0\\-1&3&-3&1\end{pmatrix}$$

- The inverse matrix A^{-1} looks like A, except odd-numbered diagonals are multiplied by -1.

Exercise

If A is invertible and $AB = AC$, prove that $B = C$

Solution

$AB = AC$ ***Multiply by A^{-1} both sides.***

$A^{-1}(AB) = A^{-1}(AC)$ ***Multiplication is associative***

$\left(A^{-1}A\right)B = \left(A^{-1}A\right)C$ $A^{-1}A = I$

$IB = IC$

$\boxed{B = C}$

Exercise

If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find two matrices $B \ne C$ such that $AB = AC$

Solution

Let $B = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ *and* $C = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$\boxed{B \ne C \Rightarrow AB = AC}$

Exercise

If A has ***row*** 1 + ***row*** 2 = ***row*** 3, show that A is not invertible

a) Explain why $Ax=(1,\ 0,\ 0)$ can't have a solution.

b) Which right sides $\left(b_1,\ b_2,\ b_3\right)$ might allow a solution to $Ax=b$

c) What happens to ***row*** 3 in elimination?

Solution

a) Let $A_1,\ A_2,\ A_3$ be the row vectors of A and x is a solution to $Ax=(1,\ 0,\ 0)$.

Then $A_1.x=1,\ A_2.x=0,\ A_3.x=0$.

Since $\quad A_1+A_2=A_3$

Means $\quad A_1.x+A_2.x=A_3.x$

Implies $\quad 1+0=0$ a contradiction

b) If $Ax=\left(b_1,\ b_2,\ b_3\right)\Rightarrow A_1.x=b_1,\ A_2.x=b_2,\ A_3.x=b_3$

Since $\quad A_1+A_2=A_3$

$A_1.x+A_2.x=A_3.x$

$\Rightarrow b_1+b_2=b_3$

c) In the elimination matrix, the third row will be zero.

Exercise

True or false (with a counterexample if false and a reason if true):

a) A 4 by 4 matrix with a row of zeros is not invertible.

b) A matrix with 1's down the main diagonal is invertible.

c) If A is invertible then A^{-1} is invertible.

d) If A is invertible then A^2 is invertible.

Solution

a) True, because it can have at most 3 pivots.

b) False, if the matrix: $A=\begin{pmatrix}1&0&0\\0&1&1\\0&1&1\end{pmatrix}$ and only has 2 pivots, thus is not invertible.

c) True, If A is invertible then necessarily A^{-1} is invertible.

d) True, $A^2x=0$ where x is nonzero matrix.

$$A^{-1}A^2x=\left(A^{-1}A\right)Ax=IAx=Ax=0$$

Since A is invertible, this can only be true if x was zero to begin with. Thus A^2 must also be invertible.

Exercise

Do there exist 2 by 2 matrices A and B with real entries such that $AB-BA=I$, where I is the identity matrix?

Solution

Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$$AB=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}=\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$BA=\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}$$

$$\begin{aligned} AB-BA&=\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}-\begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix} \\ &=\begin{pmatrix} bg-cf & af+bh-be-df \\ ce+dg-ag-ch & cf-bg \end{pmatrix} \\ &=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{cases} bg-cf=1 \\ af+bh-be-df=0 \\ ce+dg-ag-ch=0 \\ cf-bg=1 \end{cases} \rightarrow \begin{cases} bg-cf=1 \\ \underline{cf-bg=1} \\ 0=2 \end{cases}$$

Therefore, $AB-BA\neq I$ for any 2 by 2 matrices.

Exercise

If B is the inverse of A^2, show that AB is the inverse of A.

Solution

Since B is the inverse of A^2 that implies: $\underline{B=\left(A^2\right)^{-1}}=(AA)^{-1}\underline{=A^{-1}A^{-1}}$

Show that AB is the inverse of A

$$\begin{aligned}(AB)A &= \left(A\left(A^{-1}A^{-1}\right)\right)A \\ &= \left(\left(AA^{-1}\right)A^{-1}\right)A \\ &= \left(IA^{-1}\right)A \\ &= A^{-1}A \\ &\underline{= I}\end{aligned}$$

Therefore, AB is the inverse of A.

Exercise

Find and check the inverses (assuming they exist) of these block matrices.

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}$$

Solution

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix}\begin{bmatrix} I & 0 \\ A & I \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ C+A & I \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \Rightarrow C+A=0 \to A=-C$$

$$\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}^{-1}=\begin{pmatrix} I & 0 \\ -C & I \end{pmatrix}$$

$$\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}\begin{bmatrix} E & 0 \\ F & G \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} AE & 0 \\ CE+DF & DG \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\Rightarrow \begin{cases} AE = I \\ CE + DF = 0 \\ DG = I \end{cases} \rightarrow \begin{cases} E = A^{-1} \\ \\ G = D^{-1} \end{cases}$$

$$\begin{aligned} CE + DF = 0 \rightarrow CA^{-1} + DF &= 0 \\ DF &= -CA^{-1} \\ D^{-1}DF &= -D^{-1}CA^{-1} \\ IF &= -D^{-1}CA^{-1} \\ F &= -D^{-1}CA^{-1} \end{aligned}$$

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{pmatrix}$$

$$\begin{bmatrix} 0 & I \\ I & D \end{bmatrix}\begin{bmatrix} A & I \\ I & B \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & B \\ A+D & I+DB \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\rightarrow \begin{cases} B = 0 \\ A + D = 0 \\ I + DB = I \end{cases} \Rightarrow \begin{cases} A = -D \\ DB = 0 \end{cases}$$

$$\begin{pmatrix} 0 & I \\ I & D \end{pmatrix}^{-1} = \begin{pmatrix} -D & I \\ I & 0 \end{pmatrix}$$

Exercise

For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

Solution

$c = 0$, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 7 & 0 \end{bmatrix}$ (zero column 2 / row 2)

$$c=2,\ A=\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 8 & 7 & 2 \end{bmatrix} \text{ (equal rows)}$$

$$c=7,\ A=\begin{bmatrix} 2 & 7 & 7 \\ 7 & 7 & 7 \\ 8 & 7 & 7 \end{bmatrix} \text{ (equal columns)}$$

Exercise

Find A^{-1} and B^{-1} (if they exist) by elimination.

$$A=\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad B=\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Solution

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right)\begin{array}{l} \frac{1}{2}R_1 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right)\begin{array}{l} \\ R_2-R_1 \\ R_3-R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right)\begin{array}{l} \\ \frac{2}{3}R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right)\begin{array}{l} R_1-\frac{1}{2}R_2 \\ \\ R_3-\frac{1}{2}R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array}\right)\begin{array}{l} \\ \\ \frac{3}{4}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right) \begin{array}{l} R_1 - \frac{1}{3}R_3 \\ R_2 - \frac{1}{3}R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \frac{1}{2}R_1 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 + R_1 \\ R_3 + R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ R_3 + R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

B^{-1} doesn’t exist, and if we add the columns in B, the result is zero.

Exercise

Find A^{-1} using the Gauss-Jordan method, which has a remarkable inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 + R_2 \\ \\ \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 + R_3 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ R_3 + R_4 \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise

Find the inverse.

a) $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$

c) $\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

e) $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$

f) $\begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

g) $\begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}$

Solution

a) $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{12} & \frac{1}{6} \\ \frac{1}{8} & \frac{1}{4} \end{bmatrix}$

b)
$$A^{-1} = \frac{1}{7-8}\begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix}$$
$$= -\begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$$

c)
$$A^{-1} = \frac{1}{-15-24}\begin{bmatrix} 5 & -6 \\ -4 & -3 \end{bmatrix}$$
$$= -\frac{1}{39}\begin{bmatrix} 5 & -6 \\ -4 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{5}{39} & \frac{2}{13} \\ \frac{4}{39} & \frac{1}{13} \end{bmatrix}$$

d)
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_3 - R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \quad R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] \quad -\tfrac{1}{2}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \quad \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

e) $$\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}^{-1} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{pmatrix}$$

f) $$\begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{pmatrix} \frac{\sqrt{2}}{26} & -\frac{3\sqrt{2}}{26} & 0 \\ \frac{2\sqrt{2}}{13} & \frac{\sqrt{2}}{26} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

g) $$\begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}^{-1} = \textit{doesn't exist}$$ This matrix is ***singular***

Exercise

Show that A is not invertible for any values of the entries

$$A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

Solution

Since the matrix A had zero's on its diagonals, therefore A is not invertible.

Exercise

Prove that if A is an invertible matrix and B is row equivalent to A, then B is also invertible.

Solution

Since B is row equivalent to A, there exist some elementary matrices $E_1, E_2, \ldots, E_n$ such that $B = E_n \ \cdots E_1 A$. Because $E_1, E_2, \ldots, E_n$ and A are invertible, then B is also invertible.

Exercise

Determine if the given matrix has an inverse, and find the inverse if it exists. Check your answer by multiplying $A \cdot A^{-1} = I$

a) $\begin{bmatrix} 2 & 3 \\ -3 & -5 \end{bmatrix}$ *b)* $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}$

Solution

a) $2(-5) - 3(-3) = -10 + 9 = -1$

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} -5 & -3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ -3 & -2 \end{bmatrix}$$

$$AA^{-1} = \begin{pmatrix} 2 & 3 \\ -3 & -5 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

b) $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \quad R_3 - 2R_1$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 3 & 3 & -2 & 0 & 1 \end{array}\right] \quad R_3 - 3R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & * & * & * \end{array}\right]$$

The inverse matrix doesn't exist

Exercise

Show that the inverse of $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is $\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$

Solution

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$=\begin{bmatrix} (\cos\theta)\cos(-\theta)-(\sin\theta)\sin(-\theta) & (\cos\theta)\sin(-\theta)-(\sin\theta)\cos(-\theta) \\ (-\sin\theta)\cos(-\theta)-(\cos\theta)\sin(-\theta) & (-\sin\theta)\sin(-\theta)+(\cos\theta)\cos(-\theta) \end{bmatrix}$$

$$=\begin{bmatrix} \cos\theta\cos\theta+\sin\theta\sin\theta & -\cos\theta\sin\theta-\sin\theta\cos\theta \\ -\sin\theta\cos\theta+\cos\theta\sin\theta & \sin\theta\sin\theta+\cos\theta\cos\theta \end{bmatrix} \qquad \begin{cases} \cos(-\theta)=\cos\theta & (even) \\ \sin(-\theta)=-\sin\theta & (odd) \end{cases}$$

$$=\begin{bmatrix} \cos^2\theta+\sin^2\theta & 0 \\ 0 & \sin^2\theta+\cos^2\theta \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=I$$

Solution Section 3.5 – Determinants and Cramer's Rule

Exercise

Verify that $\det(AB) = \det(A)\det(B)$ when: $A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{pmatrix}$

Solution

$$AB = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{pmatrix}$$

$$\det(AB) = \left|\begin{matrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{matrix}\right| \begin{matrix} 9 & -1 \\ 31 & 1 \\ 10 & 0 \end{matrix} = -170$$

$$\det(A) = \begin{vmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 10$$

$$\det(B) = \begin{vmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{vmatrix} = -17$$

$$\det(AB) = \det(A)\det(B) \quad = -170 \quad \checkmark$$

Exercise

For which value(s) of $\boldsymbol{k}$ does A fail to be invertible? $A = \begin{bmatrix} k-3 & -2 \\ -2 & k-2 \end{bmatrix}$

Solution

For $\boldsymbol{A}$ to have an invertible the determinant cannot be equal to zero. To ***fail*** det(A) = 0.

$$|A| = \begin{vmatrix} k-3 & -2 \\ -2 & k-2 \end{vmatrix} = 0$$

$$(k-3)(k-2) - 4 = 0$$

$$k^2 - 5k + 6 - 4 = 0$$

$$k^2 - 5k + 2 = 0 \Rightarrow \boxed{k = \frac{5 \pm \sqrt{17}}{2}}$$

Exercise

Without directly evaluating, show that $\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$

Solution

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \overset{R_2+R_1}{=} \begin{vmatrix} a+b+c & b+c+a & c+b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$$

It is equal to zero, since first row and third row are proportional.

$$\begin{vmatrix} a+b+c & b+c+a & c+b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \underset{R_3-\frac{1}{a+b+c}R_1}{=} \begin{vmatrix} a+b+c & b+c+a & c+b+a \\ a & b & c \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Exercise

If the entries in every row of A add to zero, solve $\boldsymbol{Ax} = 0$ to prove det $A = 0$. If those entries add to one, show that det $(A - I) = 0$. Does this mean det $A = I$?

Solution

If $\boldsymbol{x} = (1, 1, \dots, 1)$, then $\boldsymbol{Ax}$ = the sums of the rows of $\boldsymbol{A}$. Since every row of A add to zero, that implies $\boldsymbol{Ax} = 0$. Since A has non-zero nullspace, it is not invertible and det $A = 0$. If the entries in every row of A sum to one, then the entries in every row of $A - \text{I}$ sum to zero. A – I has a non-zero nullspace and det $(A - \text{I}) = 0$. This does not mean that det $A = \text{I}$.

Example: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ every row of A add to zero $\Rightarrow \det A = -1 \neq 1 = \det I$

Exercise

Does $\det(AB) = \det(BA)$ in general?

a) True or false if $\boldsymbol{A}$ and $\boldsymbol{B}$ are square n x n matrices?

b) True or false if $\boldsymbol{A}$ is m x n and B is n x m with $m \neq n$?

Solution

a) Matrices A and B are square matrices, then by the property:

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$$

Therefore it is true for any $\boldsymbol{A}$ and $\boldsymbol{B}$ square matrices.

b) False, example if $A=\begin{pmatrix}1\\1\end{pmatrix}$ $\quad B=\begin{pmatrix}1 & 1\end{pmatrix}$

$$\det AB=\det\left[\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix}\right]=\det\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}=0$$

$$\det AB=\det\left[\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right]=\det(2)=2$$

Exercise

True or false, with a reason if true or a counterexample if false:

a) The determinant of $I+A$ is 1 + det $\boldsymbol{A}$.

b) The determinant of ABC is $|A||B||C|$.

c) The determinant of $4A$ is $4|A|$

d) The determinant of $AB-BA$ is zero. (try an example)

e) If A is not invertible then AB is not invertible.

f) The determinant of $A-B$ equals to det A – det B.

Solution

a) ***False***, if $A=\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\Rightarrow\det(I+A)=\det\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}=0$

$$\det A=1\Rightarrow 1+\det A=1+1=2\ \underline{\neq\det(I+A)}$$

b) ***True***, $\det(ABC)=\det(A)\det(BC)=\det(A)\det(B)\det(C)$.

c) ***False***, in general $\det(4A)=4^n\det(A)$ if A is n x n.

d) ***False***, $A=\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$, $\quad B=\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$

$$\begin{aligned}AB-BA&=\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}-\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\\&=\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}-\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\\&=\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\end{aligned}$$

$$\det(AB-BA)=\begin{vmatrix}0 & 1\\-1 & 0\end{vmatrix}=1\neq 0$$

e) False, any matrix is invertible, iff its determinant is nonzero. So det A = 0 which

$\det(AB) = \det(A)\det(B) = 0$. Therefore, AB can't be invertible.

f) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow |A| = 0, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow |B| = -1$

$\det(A) - \det(B) = 0 - (-1) \ = 1$

$\det(A - B) = \det\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \ = -1 \qquad \Rightarrow \det(A - B) \neq \det(A) - \det(B)$

Exercise

Use row operations to show the 3 by 3 "Vandermonde determinant" is

$$\det\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$$

Solution

$$\det\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = \det\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$= \det\begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix} \quad factor\,(b-a)$$

$$= (b-a)\det\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & c-a & c^2-a^2 \end{bmatrix} \begin{matrix} \\ \\ R_2 - (c-a)R_2 \end{matrix}$$

$$(c-a)(c+a) - (b+a)(c-a) = (c-a)(c+a-b-a)$$

$$= (b-a)\det\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & (c-a)(c-b) \end{bmatrix} \quad \textbf{\textit{Multiply the main diagonal by (b - a)}}$$

$$= (b-a)(c-a)(c-b)$$

Exercise

The inverse of a 2 by 2 matrix seems to have determinant = 1:

$$\det A^{-1} = \det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{ad-bc}{ad-bc} = 1$$

What is wrong with this calculation? What is the correct $\det A^{-1}$

Solution

The $\det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $(ad - bc)$ it is part of the determinant and it is not the solution.

$$\det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\det\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad-bc}\frac{1}{ad-bc}(ad-bc)$$

$$= \frac{1}{ad-bc}$$

Exercise

A *Hessenberg* matrix is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci's rule $|H_4| = |H_3| + |H_2|$. The same rule will continue for all sizes $|H_n| = |H_{n-1}| + |H_{n-2}|$. Which Fibonacci number is $|H_n|$?

$$H_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad H_3 = \begin{bmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad H_4 = \begin{bmatrix} 2 & 1 & & \\ \mathbf{1} & 2 & \mathbf{1} & \\ \mathbf{1} & 1 & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

Solution

$$|H_4| = 2C_{11} + 1C_{12}$$

The cofactor C_{11} for H_4 is the determinant $|H_3|$.

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

The cofactor $C_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$

$$= -\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= -\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} \qquad \left|H_2\right| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= -\left|H_3\right| + \left|H_2\right|$$

$$\left|H_4\right| = 2C_{11} + 1C_{12}$$

$$= 2\left|H_3\right| - \left|H_3\right| + \left|H_2\right|$$

$$= \left|H_3\right| + \left|H_2\right|$$

The actual number: $\left|H_2\right| = 3, \quad \left|H_3\right| = 5, \quad H_4 = 8$.

Since $\left|H_n\right|$ follows Fibonacci's rule $\left|H_{n-1}\right| + \left|H_{n-2}\right|$, it must be $\left|H_n\right| = F_{n+2}$.

Exercise

Evaluate the determinant:

a) $\begin{vmatrix} -1 & 7 \\ -8 & -3 \end{vmatrix}$

b) $\begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix}$

c) $\begin{vmatrix} k-1 & 2 \\ 4 & k-3 \end{vmatrix}$

d) $\begin{vmatrix} c & -4 & 3 \\ 2 & 1 & c^2 \\ 4 & c-1 & 2 \end{vmatrix}$

e) $\begin{vmatrix} 1 & 0 & 3 \\ 4 & 0 & -1 \\ 2 & 8 & 6 \end{vmatrix}$

g) $\begin{vmatrix} -3 & 1 & 2 \\ 6 & 2 & 1 \\ -9 & 1 & 2 \end{vmatrix}$

h) $\begin{vmatrix} 1 & 4 & -3 & 1 \\ 2 & 0 & 6 & 3 \\ 4 & -1 & 2 & 5 \\ 1 & 0 & -2 & 4 \end{vmatrix}$

i) $\begin{vmatrix} 1 & 3 & 2 & -1 \\ 4 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 5 & 1 & 3 & -2 \end{vmatrix}$

j) $\begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & 1 & -2 & 0 \\ 1 & 0 & 3 & -3 \end{vmatrix}$

f) $\begin{vmatrix} x & -3 & 9 \\ 2 & 4 & x+1 \\ 1 & x^2 & 3 \end{vmatrix}$

Solution

a) $\begin{vmatrix} -1 & 7 \\ -8 & -3 \end{vmatrix} = (-1)(-3)-(7)(-8) \underline{= 59}$

b) $\begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix} = (a-3)(a-2)+15$

$= a^2-5a+6+15$

$\underline{= a^2-5a+21}$

c) $\begin{vmatrix} k-1 & 2 \\ 4 & k-3 \end{vmatrix} = (k-1)(k-3)-8$

$= k^2-4k+3-8$

$\underline{= k^2-4k-5}$

d) $\begin{vmatrix} c & -4 & 3 \\ 2 & 1 & c^2 \\ 4 & c-1 & 2 \end{vmatrix} = 2c-16c^2+6c-6-12-c^4+c^3+16 \ \underline{= -c^4+c^3-16c^2+8c-2}$

e) $\begin{vmatrix} 1 & 0 & 3 \\ 4 & 0 & -1 \\ 2 & 8 & 6 \end{vmatrix} = 0+0+96-0+8-0 \ \underline{= 104}$

f) $\begin{vmatrix} x & -3 & 9 \\ 2 & 4 & x+1 \\ 1 & x^2 & 3 \end{vmatrix} = 12x-3(x+1)+18x^2-36-x^3(x+1)+18$

$= 12x-3x-3+18x^2-36-x^4-x^3+18$

$\underline{= -x^4-x^3+18x^2+9x-21}$

g) $\begin{vmatrix} -3 & 1 & 2 \\ 6 & 2 & 1 \\ -9 & 1 & 2 \end{vmatrix} = -12-9+12+36+3-12 \ \underline{= 18}$

h) $\begin{vmatrix} 1 & 4 & -3 & 1 \\ 2 & 0 & 6 & 3 \\ 4 & -1 & 2 & 5 \\ 1 & 0 & -2 & 4 \end{vmatrix} \underline{= 275}$

i) $\begin{vmatrix} 1 & 3 & 2 & -1 \\ 4 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 5 & 1 & 3 & -2 \end{vmatrix} = 0$ Since row 3 has zero.

j) $\begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & 1 & -2 & 0 \\ 1 & 0 & 3 & -3 \end{vmatrix} = (2)(-1)(-2)(-3) \ \boxed{=-12}$

Exercise

Find all the values of λ for which det($\boldsymbol{A}$) = 0: $A = \begin{bmatrix} \lambda - 1 & -2 \\ 1 & \lambda - 4 \end{bmatrix}$

Solution

$$\begin{vmatrix} \lambda - 1 & -2 \\ 1 & \lambda - 4 \end{vmatrix} = (\lambda - 1)(\lambda - 4) + 2$$

$$= \lambda^2 - 5\lambda + 4 + 2$$

$$= \lambda^2 - 5\lambda + 6 = 0 \qquad \textit{Solve for } \lambda.$$

$$\boxed{\lambda = -1, \ 6}$$

Exercise

Find all the values of λ for which det($\boldsymbol{A}$) = 0: $A = \begin{bmatrix} \lambda - 6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda - 4 \end{bmatrix}$

Solution

$$\begin{vmatrix} \lambda - 6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda - 4 \end{vmatrix} = \lambda(\lambda - 6)(\lambda - 4) + 4(\lambda - 6)$$

$$= \lambda\left(\lambda^2 - 10\lambda + 24\right) + 4\lambda - 24$$

$$= \lambda^3 - 10\lambda^2 + 24\lambda + 4\lambda - 24$$

$$= \lambda^3 - 10\lambda^2 + 28\lambda - 24$$

$$\lambda^3 - 10\lambda^2 + 28\lambda - 24 = 0 \to \boxed{\lambda = 2, \ 2, \ 6}$$

Exercise

Prove that if a square matrix $\boldsymbol{A}$ has a column of zeros, then det($\boldsymbol{A}$) = 0

Solution

Consider a 3 by 3 matrix with a zero column, however to find the determinant we can interchange any column of that matrix; therefore:

$$\boldsymbol{A} = [a_{ij}] = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$$

By definition, the determinant of $\boldsymbol{A}$ using the cofactor:

$$\begin{aligned} |A| &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= 0\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} 0 & a_{23} \\ 0 & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} 0 & a_{22} \\ 0 & a_{32} \end{vmatrix} \\ &= 0 \end{aligned}$$

Exercise

With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D| \quad \textit{but} \quad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|$$

a) Why is the first statement true? Somehow B doesn't enter.

b) Show by example that equality fails (as shown) when C enters.

c) Show by example that the answer $\det(AD - CB)$ is also wrong.

Solution

a) If we don't pick any 0 entries, then the first two columns are picked from $\boldsymbol{A}$ and the last two rows are from D. We can't pick any columns or rows from B, because there aren't any left.

b) $$\left|\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right| = 1.$$

and $A = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0, \quad B = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0, \quad C = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0, \quad D = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$

c) Use the example from part (*b*): $1 \neq 0$ $\quad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|$

Exercise

Show that the value of the following determinant is independent of θ.

$$\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta-\cos\theta & \sin\theta+\cos\theta & 1 \end{vmatrix}$$

Solution

$$\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta-\cos\theta & \sin\theta+\cos\theta & 1 \end{vmatrix} = \begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix}$$

$$= \sin^2\theta + \cos^2\theta$$

$$= \sin^2\theta - \left(-\cos^2\theta\right)$$

$$\underline{=1}$$

Therefore, the determinant is independent of θ.

Exercise

Show that the matrices $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ *and* $B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$ commute if and only if $\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$

Solution

$$AB = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} d & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix}$$

$$BA = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} da & db+ec \\ 0 & fc \end{pmatrix}$$

$$AB = BA \quad \Rightarrow \quad \begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix} = \begin{pmatrix} da & db+ec \\ 0 & fc \end{pmatrix}$$

Iff $ae+bf = db+ec$

$$\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = b(d-f) - e(a-c) = bd - bf - ea + ec = 0$$

$$\boxed{bd+ec = bf+ae} \quad \checkmark$$

Exercise

Show that $\det(A) = \frac{1}{2}\begin{vmatrix} tr(A) & 1 \\ tr(A^2) & tr(A) \end{vmatrix}$ for every 2×2 matrix A.

Solution

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow tr(A) = a + d$

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} \Rightarrow tr(A^2) = a^2+bc+bc+d^2$$

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{aligned}\frac{1}{2}\begin{vmatrix} tr(A) & 1 \\ tr(A^2) & tr(A) \end{vmatrix} &= \frac{1}{2}\begin{vmatrix} a+d & 1 \\ a^2+bc+bc+d^2 & a+d \end{vmatrix} \\ &= \frac{1}{2}\left[(a+d)^2 - \left(a^2+bc+bc+d^2\right)\right] \\ &= \frac{1}{2}\left(a^2+2ad+d^2-a^2-bc-bc-d^2\right) \\ &= \frac{1}{2}(2ad-2bc) \\ &= ad-bc \\ &= \det(A)\end{aligned}$$

Exercise

What is the maximum number of zeros that a 4×4 matrix can have without a zero determinant? Explain your reasoning.

Solution

The maximum number of zeros that a 4×4 matrix can have without a zero determinant is 12 zeros. If the main diagonal has nonzero entries and the rest are zero, then the determinant of the matrix is equal to the product of the main diagonal entries.

Exercise

Evaluate *det* ***A***, *det* ***E***, and *det* (***AE***). Then verify that (*det* ***A***)(*det* ***E***) = *det*(***AE***)

$$A = \begin{bmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & & \\ & 3 & \\ & & 1 \end{bmatrix}$$

Solution

$$\det(A)=\begin{vmatrix}4 & 1 & 3\\ 0 & -2 & 0\\ 3 & 1 & 5\end{vmatrix}=-40+18\ \underline{=-22}$$

$$\det(E)=\begin{vmatrix}1 & & \\ & 3 & \\ & & 1\end{vmatrix}\ \underline{=3}$$

$$AE=\begin{bmatrix}4 & 1 & 3\\ 0 & -2 & 0\\ 3 & 1 & 5\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}4 & 3 & 3\\ 0 & -6 & 0\\ 3 & 3 & 5\end{bmatrix}$$

$$\det(AE)=\begin{vmatrix}4 & 3 & 3\\ 0 & -6 & 0\\ 3 & 3 & 5\end{vmatrix}=-120+54\ \underline{=-66}$$

$$\det(A)\det(E)=(-22)(3)\ \underline{=-66}$$

$$\det(A)\det(E)=\det(AE) \qquad \checkmark$$

Exercise

Show that $\begin{bmatrix}\sin^2\alpha & \sin^2\beta & \sin^2\gamma\\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma\\ 1 & 1 & 1\end{bmatrix}$ is not invertible for any values of α, β, γ

Solution

$$\begin{vmatrix}\sin^2\alpha & \sin^2\beta & \sin^2\gamma\\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma\\ 1 & 1 & 1\end{vmatrix}=\sin^2\alpha\cos^2\beta+\sin^2\beta\cos^2\gamma+\sin^2\gamma\cos^2\alpha$$
$$-\sin^2\gamma\cos^2\beta-\sin^2\alpha\cos^2\gamma-\cos^2\alpha\sin^2\beta$$

$$=\sin^2\alpha\left(\cos^2\beta-\cos^2\gamma\right)+\cos^2\alpha\left(\sin^2\gamma-\sin^2\beta\right)+\sin^2\beta\cos^2\gamma-\sin^2\gamma\cos^2\beta$$

$$=\sin^2\alpha\left(\cos^2\beta-\cos^2\gamma\right)+\cos^2\alpha\left(1-\cos^2\gamma-1+\cos^2\beta\right)+\left(1-\cos^2\beta\right)\cos^2\gamma-\left(1-\cos^2\gamma\right)\cos^2\beta$$

$$=\sin^2\alpha\left(\cos^2\beta-\cos^2\gamma\right)+\cos^2\alpha\left(\cos^2\beta-\cos^2\gamma\right)+\cos^2\gamma-\cos^2\gamma\cos^2\beta-\cos^2\beta+\cos^2\gamma\cos^2\beta$$

$$=\left(\sin^2\alpha++\cos^2\alpha\right)\left(\cos^2\beta-\cos^2\gamma\right)+\cos^2\gamma-\cos^2\beta$$

$$=\cos^2\beta-\cos^2\gamma+\cos^2\gamma-\cos^2\beta$$

$\underline{=0}$ Therefore, this matrix in not invertible.

Exercise

Use Cramer's Rule with ratios $\frac{\det B_j}{\det A}$ to solve $A\boldsymbol{x} = b$. Also find the inverse matrix $A^{-1} = \frac{C^T}{\det A}$. Why is the solution $\boldsymbol{x}$ is the first part the same as column 3 of A^{-1}? Which cofactors are involved in computing that column $\boldsymbol{x}$?

$$Ax = b \quad is \quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Find the volumes of the boxes whose edges are columns of $\boldsymbol{A}$ and then rows of A^{-1}.

Solution

$$|A| = \begin{vmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{vmatrix} = 2 \qquad |B_1| = \begin{vmatrix} 0 & 6 & 2 \\ 0 & 4 & 2 \\ 1 & 9 & 0 \end{vmatrix} = 4 \qquad |B_2| = \begin{vmatrix} 2 & 0 & 2 \\ 1 & 0 & 2 \\ 5 & 1 & 0 \end{vmatrix} = -2 \qquad |B_1| = \begin{vmatrix} 2 & 6 & 0 \\ 1 & 4 & 0 \\ 5 & 9 & 1 \end{vmatrix} = 2$$

$$x = \frac{4}{2} = 2; \quad y = \frac{-2}{2} = -1; \quad z = \frac{2}{2} = 1$$

The solution is: $(2, \ -1, \ 1)$

$$C_{11} = \begin{vmatrix} 4 & 2 \\ 9 & 0 \end{vmatrix} = -18 \quad C_{12} = -\begin{vmatrix} 1 & 2 \\ 5 & 0 \end{vmatrix} = 10 \quad C_{13} = \begin{vmatrix} 1 & 4 \\ 5 & 9 \end{vmatrix} = -11$$

$$C_{21} = -\begin{vmatrix} 6 & 2 \\ 9 & 0 \end{vmatrix} = 18 \quad C_{22} = \begin{vmatrix} 2 & 2 \\ 5 & 0 \end{vmatrix} = -10 \quad C_{23} = -\begin{vmatrix} 2 & 6 \\ 5 & 9 \end{vmatrix} = 12$$

$$C_{31} = \begin{vmatrix} 6 & 2 \\ 4 & 2 \end{vmatrix} = 4 \quad C_{32} = -\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = -2 \quad C_{33} = \begin{vmatrix} 2 & 6 \\ 1 & 4 \end{vmatrix} = 2$$

$$C = \begin{pmatrix} -18 & 10 & -11 \\ 18 & -10 & 12 \\ 4 & -2 & 2 \end{pmatrix} \Rightarrow C^T = \begin{pmatrix} -18 & 18 & 4 \\ 10 & -10 & -2 \\ -11 & 12 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{C^T}{\det A} = \frac{1}{2}\begin{pmatrix} -18 & 18 & 4 \\ 10 & -10 & -2 \\ -11 & 12 & 2 \end{pmatrix} = \begin{pmatrix} -9 & 9 & 2 \\ 5 & -5 & -1 \\ -\frac{11}{2} & 6 & 1 \end{pmatrix}$$

The solution $\boldsymbol{x}$ is the third column of A^{-1} because $\boldsymbol{b}$ = (0, 0, 1) is the third column of I.

The volume of the boxes whose edges are columns of $\boldsymbol{A}$ = det($\boldsymbol{A}$) = 2.

Since $\left|A^T\right| = |A|$. The box from rows of A^{-1} has volume $\left|A^{-1}\right| = \frac{1}{|A|} \ \boxed{= \frac{1}{2}}$

Exercise

Verify that $\det(AB)=\det(BA)$ and determine whether the equality $\det(A+B)=\det(A)+\det(B)$ holds

$$A=\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \textit{and} \quad B=\begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

Solution

$$AB=\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{bmatrix} \qquad \det(AB)=\begin{vmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{vmatrix}=-170$$

$$BA=\begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}=\begin{bmatrix} -1 & -3 & 6 \\ 17 & 11 & 4 \\ 10 & 5 & 2 \end{bmatrix} \qquad \det(BA)=\begin{vmatrix} -1 & -3 & 6 \\ 17 & 11 & 4 \\ 10 & 5 & 2 \end{vmatrix}=-170$$

Thus, $\boxed{\det(AB)=\det(BA)}$

$$\det(A)=\begin{vmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{vmatrix}=10$$

$$\det(B)=\begin{vmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{vmatrix}=-17$$

$$A+B=\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}+\begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 3 & 0 & 3 \\ 10 & 5 & 2 \\ 5 & 0 & 3 \end{bmatrix} \qquad \det(A+B)=\begin{vmatrix} 3 & 0 & 3 \\ 10 & 5 & 2 \\ 5 & 0 & 3 \end{vmatrix}=-30$$

$$\det(A)+\det(B)=10-17=-7$$

$$\neq \det(A+B)$$

Exercise

Verify that $\det(kA) = k^n \det(A) \quad A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}, \quad k = 2$

Solution

$$\det(A) = \begin{vmatrix} -1 & 2 \\ 3 & 4 \end{vmatrix} = -10$$

$$\begin{aligned} \det(2A) &= \begin{vmatrix} -2 & 4 \\ 6 & 8 \end{vmatrix} \\ &= -40 \\ &= 4(-10) \\ &= 2^2(-10) \\ &= k^2 \det(A) \end{aligned}$$

Exercise

Verify that $\det(kA) = k^n \det(A) \quad A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}, \quad k = -2$

Solution

$$\det(A) = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{vmatrix} = 56$$

$$\begin{aligned} \det(-2A) &= \begin{vmatrix} -4 & 2 & -6 \\ -6 & -4 & -2 \\ -2 & -8 & -0 \end{vmatrix} \\ &= -448 \\ &= (-2)^3(56) \\ &= k^3 \det(A) \end{aligned}$$

Exercise

Verify that $\det(kA) = k^n \det(A)$ $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}, \quad k = 3$

Solution

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{vmatrix} = -7$$

$$\begin{aligned} \det(3A) &= \begin{vmatrix} 3 & 3 & 3 \\ 0 & 6 & 9 \\ 0 & 3 & -6 \end{vmatrix} \\ &= -189 \\ &= 3^3(-7) \\ &= k^3 \det(A) \end{aligned}$$

Exercise

Solve by using Cramer's rule

a) $\begin{cases} 7x - 2y = 3 \\ 3x + y = 5 \end{cases}$

b) $\begin{cases} 4x + 5y \qquad = 2 \\ 11x + y + 2z = 3 \\ x + 5y + 2z = 1 \end{cases}$

c) $\begin{cases} x - 4y + z = 6 \\ 4x - y + 2z = -1 \\ 2x + 2y - 3z = -20 \end{cases}$

d) $\begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 = -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 = 14 \\ -x_1 + x_2 + 3x_3 + x_4 = 11 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -4 \end{cases}$

e) $\begin{cases} 2x - y + z = -1 \\ 3x + 4y - z = -1 \\ 4x - y + 2z = -1 \end{cases}$

Solution

a) $\begin{cases} 7x - 2y = 3 \\ 3x + y = 5 \end{cases}$

$$D = \begin{vmatrix} 7 & -2 \\ 3 & 1 \end{vmatrix} = 13 \qquad D_x = \begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 13 \qquad D_y = \begin{vmatrix} 7 & 3 \\ 3 & 5 \end{vmatrix} = 26$$

$$x = \frac{D_x}{D} = \frac{13}{13} = 1 \qquad y = \frac{D_y}{D} = \frac{26}{13} = 2$$

Solution: $\boxed{(1,\ 2)}$

b) $\begin{cases} 4x+5y \quad\quad = 2 \\ 11x+y+2z=3 \\ x+5y+2z=1 \end{cases}$

$$D = \begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} = -132 \qquad D_x = \begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} = -36$$

$$D_y = \begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} = -24 \qquad D_z = \begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} = 12$$

$$x = \frac{D_x}{D} = \frac{-36}{-132} = \frac{3}{11} \qquad y = \frac{D_y}{D} = \frac{-24}{-132} = \frac{2}{11} \qquad z = \frac{D_z}{D} = \frac{12}{-132} = -\frac{1}{11}$$

Solution: $\boxed{\left(\frac{3}{11},\ \frac{2}{11},\ -\frac{1}{11}\right)}$

c) $\begin{cases} x-4y+z=6 \\ 4x-y+2z=-1 \\ 2x+2y-3z=-20 \end{cases}$

$$D = \begin{vmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix} = 3-16+8+2-4-48 = -55$$

$$D_x = \begin{vmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{vmatrix} = 18+160-2-20-24+12 = 144$$

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{vmatrix} = 3+24-80+2+40+72 = 61$$

$$D_z = \begin{vmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{vmatrix} = 20+8+48+12+2-320 = -230$$

$$x = \frac{D_x}{D} = -\frac{144}{55}, \quad y = \frac{D_y}{D} = -\frac{61}{55}, \quad z = \frac{D_z}{D} = \frac{-230}{-55} = \frac{46}{11}$$

Solution: $\left(-\frac{144}{55},\ -\frac{61}{55},\ \frac{46}{11}\right)$

d) $\begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 = -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 = 14 \\ -x_1 + x_2 + 3x_3 + x_4 = 11 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -4 \end{cases}$

$D = -423 \quad D_{x_1} = -2115 \quad D_{x_2} = -3384 \quad D_{x_3} = -1269 \quad D_{x_4} = 423$

$x_1 = \dfrac{D_{x_1}}{D} = \dfrac{-2115}{-423} = 5 \qquad x_2 = \dfrac{D_{x_2}}{D} = \dfrac{-3384}{-423} = 8$

$x_3 = \dfrac{D_{x_3}}{D} = \dfrac{-1269}{-423} = 3 \qquad x_4 = \dfrac{D_{x_4}}{D} = \dfrac{423}{-423} = -1$

Solution: $\boxed{(5,\ 8,\ 3,\ -1)}$

e) $\begin{cases} 2x - y + z = -1 \\ 3x + 4y - z = -1 \\ 4x - y + 2z = -1 \end{cases}$

$$D = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 4 & -1 \\ 4 & -1 & 2 \end{vmatrix} = 16 + 4 - 3 - 16 - 2 + 6 = 5$$

$$D_x = \begin{vmatrix} -1 & -1 & 1 \\ -1 & 4 & -1 \\ -1 & -1 & 2 \end{vmatrix} = -8 - 1 + 1 + 4 + 1 - 2 = -5$$

$$D_y = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -1 & -1 \\ 4 & -1 & 2 \end{vmatrix} = -4 + 4 - 3 + 4 - 2 + 6 = 5$$

$$D_z = \begin{vmatrix} 2 & -1 & -1 \\ 3 & 4 & -1 \\ 4 & -1 & -1 \end{vmatrix} = -8 + 4 + 3 + 16 - 2 - 3 = 10$$

$x = \dfrac{D_x}{D} = \dfrac{-5}{5} = -1, \quad y = \dfrac{D_y}{D} = \dfrac{5}{5} = 1, \quad z = \dfrac{D_z}{D} = \dfrac{10}{5} = 2$

$\therefore$ Solution: $(-1,\ 1,\ 2)$

Exercise

Show that the matrix A is invertible for all values of θ, then find A^{-1} using $A^{-1}=\frac{1}{\det(A)}adj(A)$

$$A=\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>***Solution***</u>

$$\det(A)=\begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}=\cos^2\theta+\sin^2\theta=1 \qquad \Rightarrow \boldsymbol{A} \text{ is invertible}$$

$$C_{11}=\begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix}=\cos\theta; \quad C_{12}=-\begin{vmatrix} -\sin\theta & 0 \\ 0 & 1 \end{vmatrix}=\sin\theta; \quad C_{13}=\begin{vmatrix} -\sin\theta & \cos\theta \\ 0 & 0 \end{vmatrix}=0$$

$$C_{21}=-\begin{vmatrix} \sin\theta & 0 \\ 0 & 1 \end{vmatrix}=-\sin\theta; \quad C_{22}=\begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix}=\cos\theta; \quad C_{23}=-\begin{vmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{vmatrix}=0$$

$$C_{31}=\begin{vmatrix} \sin\theta & 0 \\ \cos\theta & 0 \end{vmatrix}=0; \quad C_{32}=-\begin{vmatrix} \cos\theta & 0 \\ -\sin\theta & 0 \end{vmatrix}=0; \quad C_{33}=\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}=1$$

$$adj(A)=\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}=\frac{1}{\det(A)}adj(A)$$

$$=\frac{1}{1}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution *Section* 3.6 – Vectors in 2-Space, 3-Space, and *n*-Space

Exercise

Sketch the following vectors with initial points located at the origin

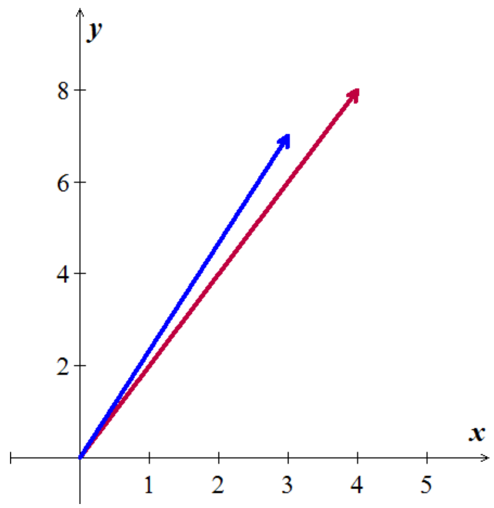
a) $P_1(4,8)\quad P_2(3,7)$

b) $P_1(-1,0,2)\quad P_2(0,-1,0)$

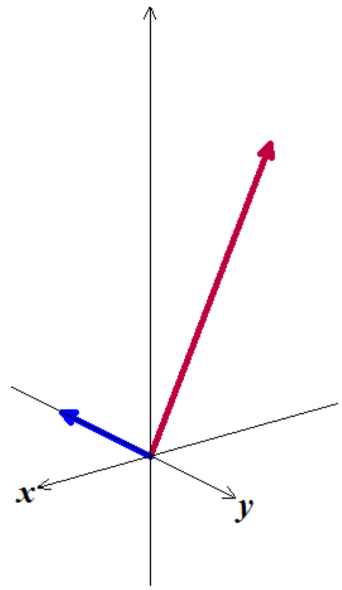
c) $P_1(3,-7,2)\quad P_2(-2,5,-4)$

Solution

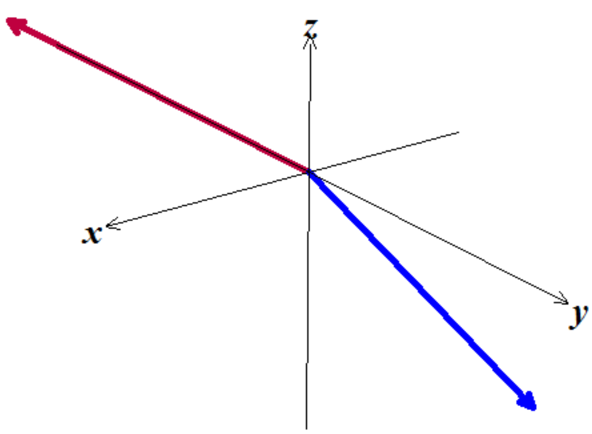
a)



b)



c)



Exercise

Find the components of the vector $\overrightarrow{P_1P_2}$

a) $P_1(3,5)$ $P_2(2,8)$

b) $P_1(5,-2,1)$ $P_2(2,4,2)$

c) $P_1(0,0,0)$ $P_2(-1,6,1)$

Solution

a) $\overrightarrow{P_1P_2} = (2-3,\ 8-5)\ \underline{=(-1,\ 3)}$

b) $\overrightarrow{P_1P_2} = (2-5,\ 4-(-2),\ 2-1)\ \underline{=(-3,\ 6,\ 1)}$

c) $\overrightarrow{P_1P_2} = (-1-0,\ 6-0,\ 1-0)\ \underline{=(-1,\ 6,\ 1)}$

Exercise

Find the terminal point of the vector that is equivalent to $\boldsymbol{u} = (1, 2)$ and whose initial point is $A(1,1)$

Solution

The terminal point: $B(b_1,\ b_2)$

$(b_1 - 1,\ b_2 - 1) = (1,2)$

$$\begin{cases} b_1 - 1 = 1 & \Rightarrow b_1 = 2 \\ b_2 - 1 = 2 & \Rightarrow b_2 = 3 \end{cases}$$

The terminal point: $\underline{B(2,\ 3)}$

Exercise

Find the initial point of the vector that is equivalent to $\boldsymbol{u} = (1, 1, 3)$ and whose terminal point is $B(-1,-1,2)$

Solution

The initial point: $A(x,y,z)$

$(-1-x,-1-y,2-z) = (1,1,3)$

$$\begin{cases} -1-x=1 & \Rightarrow x=-2 \\ -1-y=1 & \Rightarrow y=-2 \\ 2-z=3 & \Rightarrow z=-1 \end{cases}$$

The initial point: $\underline{A(-2,-2,-1)}$

Exercise

Find a nonzero vector $\boldsymbol{u}$ with initial point $P(-1, 3, -5)$ such that

a) $\boldsymbol{u}$ has the same direction as $\boldsymbol{v} = (6, 7, -3)$

b) $\boldsymbol{u}$ is oppositely directed as $\boldsymbol{v} = (6, 7, -3)$

Solution

a) $\boldsymbol{u}$ has the same direction as $\boldsymbol{v} \Rightarrow \boldsymbol{u} = \boldsymbol{v} = (6, 7, -3)$

The initial point $P(-1, 3, -5)$ then the terminal point : $(-1+6, 3+7, -5-3) = \boxed{(5, 10, -8)}$

b) $\boldsymbol{u}$ is oppositely as $\boldsymbol{v} \Rightarrow \boldsymbol{u} = -\boldsymbol{v} = (-6, -7, 3)$

The initial point $P(-1, 3, -5)$ then the terminal point : $(-1-6, 3-7, -5+3) = \boxed{(-7, -4, -2)}$

Exercise

Let $\boldsymbol{u} = (-3, 1, 2)$, $\boldsymbol{v} = (4, 0, -8)$, and $\boldsymbol{w} = (6, -1, -4)$. Find the components

a) $v - w$

b) $6u + 2v$

c) $5(v - 4u)$

d) $-3(v - 8w)$

e) $(2u - 7w) - (8v + u)$

f) $-u + (v - 4w)$

Solution

a) $v - w = (4-6,\ 0-(-1),\ -8-(-4)) = \boxed{(-2,\ 1,\ -4)}$

b) $6u + 2v = (-18, 6, 12) + (8, 0, -16) = \boxed{(-10,\ 6,\ -4)}$

c) $5(v - 4u) = 5(4-(-12), 0-4, -8-8) = 5(16, -4, -16) = \boxed{(80,\ -20,\ -80)}$

d) $-3(v - 8w) = -3(4-48, 0-(-8), -8-(-32)) = -3(-44, 8, 24) = \boxed{(32,\ -24,\ -72)}$

e)
$$\begin{aligned}(2u - 7w) - (8v + u) &= \left[(-6, 2, 4) - (42, -7, -28)\right] - \left[(32, 0, -64) + (-3, 1, 2)\right] \\ &= (-48, 9, 32) - (29, 1, -62) \\ &= \boxed{(-77,\ 8,\ 94)}\end{aligned}$$

f)
$$\begin{aligned}-u + (v - 4w) &= (3, -1, -2) + \left[(4, 0, -8) - (24, -4, -16)\right] \\ &= (3, -1, -2) + (-20, 4, 8) \\ &= \boxed{(-17,\ 3,\ 6)}\end{aligned}$$

Exercise

Let $\boldsymbol{u}$ = (2, 1, 0, 1, –1) and $\boldsymbol{v}$ = (–2, 3, 1, 0, 2). Find scalars a and b so that $a\boldsymbol{u} + b\boldsymbol{v}$ = (–8, 8, 3, –1, 7)

Solution

$$\begin{aligned} au+bv &= a(2,1,0,1,-1)+b(-2,3,1,0,2) \\ &= (a-2b,\ a+3b,\ b,\ a,\ -a+2b) \\ &= (-8,8,3,-1,7) \end{aligned}$$

$$\begin{cases} a-2b=-8 \\ a+3b=8 \\ b=3 \\ a \quad =-1 \\ -a+2b=7 \end{cases} \quad \rightarrow \quad a=-1 \quad b=3 \textit{ Unique solution}$$

Exercise

Find all scalars $c_1,\ c_2,\ and\ c_3$ such that $c_1(1,2,0)+c_2(2,1,1,)+c_3(0,3,1)=(0,0,0)$

Solution

$$c_1(1,2,0)+c_2(2,1,1,)+c_3(0,3,1)=(c_1+2c_2, 2c_1+c_2+3c_3, c_2+c_3)=(0,0,0)$$

$$\begin{cases} c_1+2c_2 \qquad =0 \\ 2c_1+c_2+3c_3=0 \\ \qquad c_2+c_3=0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\boxed{c_1=c_2=c_3=0}$$

Exercise

Find the distance between the given points $\begin{bmatrix}5 & 1 & 8 & -1 & 2 & 9\end{bmatrix}$, $\begin{bmatrix}4 & 1 & 4 & 3 & 2 & 8\end{bmatrix}$

Solution

$$\begin{aligned} d &= \sqrt{(4-5)^2+(1-1)^2+(4-8)^2+(3+1)^2+(2-2)^2+(8-9)^2} \\ &= \sqrt{1+0+16+16+0+1} \\ &= \sqrt{34} \end{aligned}$$

Exercise

Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operation on $\boldsymbol{u}=(u_1,\ u_2)\quad \boldsymbol{v}=(v_1,\ v_2)$

$$\boldsymbol{u}+\boldsymbol{v}=(u_1+v_1+1,\ u_2+v_2+1)\qquad k\boldsymbol{u}=(ku_1,\ ku_2)$$

a) Compute $\boldsymbol{u}+\boldsymbol{v}$ and $k\boldsymbol{u}$ for $\boldsymbol{u}=(0, 4)$, $\boldsymbol{v}=(1, -3)$, and $k=2$.

b) Show that $(0, 0) \neq \mathbf{0}$.

c) Show that $(-1, -1) = 0$.

d) Show that $\boldsymbol{u}+(-\boldsymbol{u})=0$ for $\boldsymbol{u}=(u_1,\ u_2)$

e) Find two vector space axioms that fail to hold.

Solution

a) $\boldsymbol{u}+\boldsymbol{v}=(0+1+1,\ 4-3+1)=\underline{(2,\ 2)|}$

$k\boldsymbol{u}=(ku_1,\ ku_2)=(2(0),\ 2(4))=\underline{(0,\ 8)|}$

b)
$$\begin{aligned}(0,0)+(u_1,u_2)&=(0+u_1+1,\ 0+u_2+1)\\&=(u_1+1,\ u_2+1)\\&\neq(u_1,\ u_2)\end{aligned}$$

Therefore (0, 0) is not the zero vector **0** required (by Axiom).

c)
$$\begin{aligned}(-1,-1)+(u_1,u_2)&=(-1+u_1+1,\ -1+u_2+1)\\&=(u_1,\ u_2)\end{aligned}$$
$$\begin{aligned}(u_1,u_2)+(-1,-1)&=(u_1-1+1,\ u_2-1+1)\\&=(u_1,\ u_2)\end{aligned}$$

Therefore $(-1, -1) = \mathbf{0}$ holds.

d) Let $\boldsymbol{u}=(u_1,\ u_2)\ -\boldsymbol{u}=(-2-u_1,\ -2-u_2)$

$$\begin{aligned}\boldsymbol{u}+(-\boldsymbol{u})&=(u_1+(-2-u_1)+1,\ u_2+(-2-u_2)+1)\\&=(-1,\ -1)\\&\underline{=\mathbf{0}|}\end{aligned}$$

$\boldsymbol{u}+(-\boldsymbol{u})=0$ holds

e) Axiom 7: $k(\boldsymbol{u}+\boldsymbol{v})\overset{?}{=}k\boldsymbol{u}+k\boldsymbol{v}$

$$k(\boldsymbol{u}+\boldsymbol{v}) = k\left(u_1 + v_1 + 1,\ u_2 + v_2 + 1\right) = \left(ku_1 + kv_1 + k,\ ku_2 + kv_2 + k\right)$$

$$k\boldsymbol{u} + k\boldsymbol{v} = \left(ku_1,\ ku_2\right) + \left(kv_1,\ kv_2\right) = \left(ku_1 + kv_1 + 1,\ ku_2 + kv_2 + 1\right)$$

Therefore, $k(\boldsymbol{u}+\boldsymbol{v}) \neq k\boldsymbol{u} + k\boldsymbol{v}$; Axiom 7 fails to hold

Axiom 8: $(k+m)\boldsymbol{u} \overset{?}{=} k\boldsymbol{u} + m\boldsymbol{u}$

$$(k+m)\boldsymbol{u} = \left((k+m)u_1,\ (k+m)u_2\right) = \left(ku_1 + mu_1,\ ku_2 + mu_2\right)$$

$$k\boldsymbol{u} + m\boldsymbol{u} = \left(ku_1,\ ku_2\right) + \left(mu_1,\ mu_2\right) = \left(ku_1 + mu_1 + 1,\ ku_2 + mu_2 + 1\right)$$

Therefore, $(k+m)\boldsymbol{u} \neq k\boldsymbol{u} + m\boldsymbol{u}$; Axiom 8 fails to hold

Solution Section 3.7 – Linear Dependence and Independence

Exercise

Given three independent vectors w_1, w_2, w_3. Take combinations of those vectors to produce v_1, v_2, v_3. Write the combinations in a matrix form as $V = WM$.

$$\begin{aligned} v_1 &= w_1 + w_2 \\ v_2 &= w_1 + 2w_2 + w_3 \\ v_1 &= w_2 + cw_3 \end{aligned} \quad \text{which is } \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix}$$

What is the test on a matrix **V** to see if its columns are linearly independent?

If $c \neq 1$ show that v_1, v_2, v_3 are linearly independent.

If $c = 1$ show that $v's$ are linearly *dependent*.

Solution

The nullspace of **V** must contain only the *zero* vector. Then $x = (0,\ 0,\ 0)$ is the only combination of the columns that gives **V**x = zero vector.

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & c \end{bmatrix} \xrightarrow[R_3 - R_2]{R_1 - R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & \boxed{c-1} \end{bmatrix}$$

If $c \neq 1$, then the matrix M is invertible. So if x is any nonzero vector we know that Mx is nonzero. Since $\boldsymbol{w}$'s are given as independent and WMx is nonzero. Since $V = WM$, this says that x is not in the nullspace of **V**. therefore; v_1, v_2, v_3 are independent.

If $c = 1$, that implies $\begin{aligned} v_1 &= w_1 + w_2 \\ v_2 &= w_1 + w_2 + w_2 + w_3 \\ v_3 &= w_2 + w_3 \end{aligned} \Rightarrow \begin{aligned} v_1 &= w_1 + w_2 \\ &\boxed{v_2 = v_1 + v_3} \\ v_3 &= w_2 + w_3 \end{aligned}$

$-v_1 + v_2 - v_3 = 0$, which means that $v's$ are linearly *dependent*.

The other way, the vector $x = (1,\ -1,\ 1)$ is in that nullspace, and $Mx = 0$. Then certainly $WMx = 0$ which is the same as $Vx = 0$. So the $v's$ are dependent.

Exercise

Find the largest possible number of independent vectors among

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad v_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad v_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad v_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

Solution

Since $v_4 = v_2 - v_1,\ v_5 = v_3 - v_1,\ \textit{and}\ \ v_6 = v_3 - v_2$, there are at most three independent vectors among these: furthermore, applying row reduction to the matrix $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ gives three pivots, showing that v_1, v_2, v_3 are independent.

Exercise

Show that $v_1,\ v_2,\ v_3$ are independent but $v_1,\ v_2,\ v_3,\ v_4$ are dependent:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad v_4 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

Solve either $c_1v_1 + c_2v_2 + c_3v_3 = 0$ *or* $Ax = 0$. The v's go in the columns of $\boldsymbol{A}$.

Solution

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

This matrix has 3 pivots with rank of 3 equals to rows that implies the $v_1,\ v_2,\ v_3$ are independent.

$v_4 = v_1 + v_2 - 4v_3$ *or* $v_1 + v_2 - 4v_3 - v_4 = 0$ that shows that $v_1,\ v_2,\ v_3,\ v_4$ are dependent.

Exercise

Decide the dependence or independence of

a) The vectors (1, 3, 2) and (2, 1, 3) and (3, 2, 1).

b) The vectors $(1,\ -3,\ 2)$ and $(2,\ 1,\ -3)$ and $(-3,\ 2,\ 1)$.

Solution

a) These are linearly independent. $x_1(1,\ 3,\ 2)+x_2(2,\ 1,\ 3)+x_3(3,\ 2,\ 1)=(0,\ 0,\ 0)$ only if $x_1=x_2=x_3=0$

b) These are linearly dependent: $1(1,\ -3,\ 2)+1(2,\ 1,\ -3)+1(-3,\ 2,\ 1)=(0,\ 0,\ 0)$

Exercise

Find two independent vectors on the plane $x+2y-3z-t=0$ in $\boldsymbol{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

Solution

This plane is the nullspace of the matrix

$$A=\begin{pmatrix}1 & 2 & -3 & -1\end{pmatrix}$$

$x_1+2x_2-3x_3-x_4=0$

The pivot is 1st column, and the rest are 3 variables.

If $x_2=-1\quad x_3=x_4=0\ \Rightarrow x_1=2$. The vector is $(2,\ -1,\ 0,\ 0)$

If $x_3=1\quad x_1=x_4=0\ \Rightarrow x_1=3$. The vector is $(3,\ 0,\ 1,\ 0)$

If $x_4=1\quad x_1=x_3=0\ \Rightarrow x_1=1$. The vector is $(1,\ 0,\ 0,\ 1)$

The 3 vectors $(2,\ -1,\ 0,\ 0)$, $(3,\ 0,\ 1,\ 0)$, $(1,\ 0,\ 0,\ 1)$ are linearly independent.

We can't find 4 independent vectors because the nullspace only has dimension 3 (have 3 variables).

Exercise

Determine whether the vectors are linearly dependent or linearly independent in $\boldsymbol{R}^3$

a) (4, –1, 2), (–4, 10, 2)

b) (8, –1, 3), (4, 0, 1)

c) (–3, 0, 4), (5, –1, 2), (1, 1, 3)

d) (–2, 0, 1), (3, 2, 5), (6, –1, 1), (7, 0, –2)

Solution

a) The vector equation $a(4, -1, 2) + b(-4, 10, 2) = (0, 0, 0)$

$$\left[\begin{array}{cc|c} 4 & -4 & 0 \\ -1 & 10 & 0 \\ 2 & 2 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system has only the trivial solution $a = b = 0$.
We conclude that the given set of vectors is linearly independent.

b) $a(8, -1, 3) + b(4, 0, 1) = (0, 0, 0)$

$$\left[\begin{array}{cc|c} 8 & 4 & 0 \\ -1 & 0 & 0 \\ 3 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, the system has only one trivial solution $a = b = 0$.
We conclude that the given set of vectors is linearly independent

c) The vector equation $a(-3, 0, 4) + b(5, -1, 2) + c(1, 1, 3) = (0, 0, 0)$

$$\left[\begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore the system has only the trivial solution $a = b = c = 0$.
We conclude that the given set of vectors is linearly independent.

d) The vector equation $a(-2, 0, 1) + b(3, 2, 5) + c(6, -1, 1) + d(7, 0, -2) = (0, 0, 0)$

$$\left[\begin{array}{cccc|c} -2 & 3 & 6 & 7 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 1 & 5 & 1 & -2 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{79}{29} & 0 \\ 0 & 1 & 0 & \frac{3}{29} & 0 \\ 0 & 0 & 1 & \frac{6}{29} & 0 \end{array}\right]$$

Therefore the system has nontrivial solutions $a = \frac{79}{29}t, \quad b = -\frac{3}{29}t, \quad c = -\frac{6}{29}t, \quad d = t$

We conclude that the given set of vectors is linearly dependent.

Exercise

Determine whether the vectors are linearly dependent or linearly independent in $\boldsymbol{R}^4$

a) (3, 8, 7, –3), (1, 5, 3, –1), (2, –1, 2, 6), (1, 4, 0, 3)

b) (0, 0, 2, 2), (3, 3, 0, 0), (1, 1, 0, –1)

c) (0, 3, –3, –6), (–2, 0, 0, –6), (0, –4, –2, –2), (0, –8, 4, –4)

d) (3, 0, –3, 6), (0, 2, 3, 1), (0, –2, –2, 0), (–2, 1, 2, 1)

Solution

a) $\det\begin{pmatrix} 3 & 1 & 2 & 1 \\ 8 & 5 & -1 & 4 \\ 7 & 3 & 2 & 0 \\ -3 & -1 & 6 & 3 \end{pmatrix} = 128 \neq 0$

The system has only the trivial solution and the vectors are linearly independent.

b) $k_1(0,0,2,2) + k_2(3,3,0,0) + k_3(1,1,0,-1) = (0,0,0,0)$

$$\left[\begin{array}{ccc|c} 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 = k_2 = k_3 = 0$

The system has only the trivial solution and the vectors are linearly independent.

c) $\det\begin{pmatrix} 0 & -2 & 0 & 0 \\ 3 & 0 & -4 & -8 \\ -3 & 0 & -2 & 4 \\ -6 & -6 & -2 & -4 \end{pmatrix} = 480 \neq 0$

The system has only the trivial solution and the vectors are linearly independent.

d) $a(3,\ 0,\ -3,\ 6) + b(0,\ 2,\ 3,\ 1) + c(0,\ -2,\ -2,\ 0) + d(-2,\ 1,\ 2,\ 1) = (0,\ 0,\ 0,\ 0)$

$$\left[\begin{array}{cccc|c} 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & 1 & 0 \\ -3 & 3 & -2 & 2 & 0 \\ 6 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore, the system has only one trivial solution $a = b = c = d = 0$.

The given set of vectors is linearly independent

Exercise

a) Show that the three vectors $v_1 = (1,2,3,4)$ $v_2 = (0,1,0,-1)$ $v_3 = (1,3,3,3)$ form a linearly dependent set in $\boldsymbol{R}^4$.

b) Express each vector in part (*a*) as a linear combination of the other two.

Solution

a) The vector equation $k_1(1,2,3,4) + k_2 = (0,1,0,-1) + k_3(1,3,3,3) = (0,0,0,0)$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ 3 & 0 & 3 & 0 \\ 4 & -1 & 3 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The solution: $k_1 = -t, \ k_2 = -t, \ k_3 = t$

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

b) Since $k_1 = -t, \ k_2 = -t, \ k_3 = t$ and if we let $t = 1$, then $-v_1 - v_2 + v_3 = 0$

$$v_1 = -v_2 + v_3, \quad v_2 = -v_1 + v_3, \quad v_3 = v_1 + v_2$$

Exercise

For which real values of λ do the following vectors form a linearly dependent set in $\boldsymbol{R}^3$

$$v_1 = \left(\lambda, -\tfrac{1}{2}, -\tfrac{1}{2}\right) \quad v_2 = \left(-\tfrac{1}{2}, \lambda, -\tfrac{1}{2}\right) \quad v_3 = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, \lambda\right)$$

Solution

$$k_1\left(\lambda, -\tfrac{1}{2}, -\tfrac{1}{2}\right) + k_2 = \left(-\tfrac{1}{2}, \lambda, -\tfrac{1}{2}\right) + k_3\left(-\tfrac{1}{2}, -\tfrac{1}{2}, \lambda\right) = (0,0,0,0)$$

$$\det\begin{pmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{pmatrix} = \frac{1}{4}\left(4\lambda^3 - 3\lambda - 1\right)$$

For $\lambda = 1 \quad \lambda = -\frac{1}{2}$, the determinant is zero and the vectors form a linearly dependent set.

Exercise

Show that if $S = \{v_1, v_2, \ldots, v_n\}$ is a linearly independent set of vectors, then so is every nonempty subset of S.

Solution

Let $\{v_a, v_b, \ldots, v_r\}$ be a nonempty subset of *S*.

If this set is linearly dependent, then there would be a nonzero solution $(k_a, k_b, \ldots, k_r)$ to $k_a v_a + k_b v_b + \ldots + k_r v_r = 0$. This can be expanded to a nonzero solution of $k_1 v_1 + k_2 v_2 + \ldots + k_n v_n = 0$ by taking all other coefficients as 0. This contradicts the linear independence of S, so the subset must be linearly independent.

Exercise

Show that if $S = \{v_1, v_2, \ldots, v_r\}$ is a linearly dependent set of vectors in a vector space *V*, and if $v_{r+1}, \ldots, v_n$ are vectors in *V* that are not in *S*, then $\{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ is also linearly dependent.

Solution

If *S* is linearly dependent, then there is a nonzero solution $(k_1, k_2, \ldots, k_r)$ to $k_1 v_1 + k_2 v_2 + \ldots + k_r v_r = 0$. Thus $(k_1, k_2, \ldots, k_r, 0, 0, \ldots, 0)$ is a nonzero solution to $k_1 v_1 + k_2 v_2 + \ldots + k_r v_r + k_{r+1} v_{r+1} \ldots + k_n v_n = 0$ so the set $\{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ is linearly dependent.

Exercise

Show that $\{v_1, v_2\}$ is linearly independent and v_3 does not lie in span $\{v_1, v_2\}$, then $\{v_1, v_2, v_3\}$ is a linearly independent.

Solution

If $\{v_1, v_2, v_3\}$ are linearly dependent, there exist a nonzero solution to $k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$ with $k_3 \neq 0$ (since v_1 and v_2 are linearly independent).

$k_3 v_3 = -k_1 v_1 - k_2 v_2 \quad \Rightarrow \quad v_3 = -\dfrac{k_1}{k_3} v_1 - \dfrac{k_2}{k_3} v_2$ which contradicts that v_3 is not in span $\{v_1, v_2\}$.

Thus $\{v_1, v_2, v_3\}$ is a linearly independent.

Exercise

By using the appropriate identities, where required, determine $F(-\infty, \infty)$ are linearly dependent.

a) $6,\ 3\sin^2 x,\ 2\cos^2 x$

b) $x,\ \cos x$

c) $1,\ \sin x,\ \sin 2x$

d) $(3-x)^2,\ x^2-6x,\ 5$

e) $\cos 2x,\ \sin^2 x,\ \cos^2 x$

Solution

a) From the identity $\sin^2 x + \cos^2 x = 1$

$$(-1)(6)+(2)\left(3\sin^2 x\right)+(3)\left(2\cos^2 x\right) = -6+6\left(\sin^2 x+\cos^2 x\right) \underline{=0}$$

Therefore, the set is linearly dependent.

b) $ax + b\cos x = 0$

$$x=0 \quad\Rightarrow\quad b=0$$

$$x=\tfrac{\pi}{2} \quad\Rightarrow\quad a=0$$

Therefore, the set is linearly independent.

c) $a(1)+b\sin x + c\sin 2x = 0$

$$x=0 \quad\Rightarrow\quad a=0$$

$$x=\tfrac{\pi}{2} \quad\Rightarrow\quad b=0$$

$$x=\tfrac{\pi}{4} \quad\Rightarrow\quad c=0$$

Therefore, the set is linearly independent.

d) $(3-x)^2 = 9-6x+x^2$

$$(3-x)^2-\left(9-6x+x^2\right)=0$$

$$(3-x)^2-\left(x^2-6x\right)-9=0$$

$$(1)(3-x)^2+(-1)\left(x^2-6x\right)+\left(-\tfrac{9}{5}\right)5=0$$

Therefore, the set is linearly dependent.

e) By using the double angle:

$\cos 2x = \cos^2 x - \sin^2 x$ are linearly dependent.

Exercise

$f_1(x) = \sin x, \quad f_2(x) = \cos x$ are linearly independent in $F(-\infty, \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using Wroński's test.

Solution

The Wronskian: $W(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$

$$= -\sin^2 x - \cos^2 x$$

$$= -\left(\sin^2 x + \cos^2 x\right)$$

$$\underline{= -1 \neq 0}$$

$\sin x$ *and* $\cos x$ are linearly independent

Exercise

Use the Wronskian to show that $f_1(x) = \sin x, \quad f_2(x) = \cos x, \quad f_3(x) = x\cos x$ span a three-dimensional subspace of $F(-\infty, \infty)$

Solution

The Wronskian: $W(x) = \begin{vmatrix} \sin x & \cos x & x\cos x \\ \cos x & -\sin x & \cos x - x\sin x \\ -\sin x & -\cos x & -2\sin x - x\cos x \end{vmatrix}$

$$= 2\sin^3 x + x\sin^2 x\cos x - \sin x\cos^2 x + x\sin^2 x\cos x - x\cos^3 x$$

$$- x\sin^2 x\cos x + \sin x\cos^2 x - x\sin^2 x\cos x + 2\sin x\cos^2 x + x\cos^3 x$$

$$= 2\sin^3 x + 2\sin x\cos^2 x$$

$$= 2\sin x\left(\sin^2 x + \cos^2 x\right)$$

$$\underline{= 2\sin x}$$

Since $\sin x \neq 0$ for all real x values, the vectors are linearly independent.

Exercise

Show by inspection that the vectors are linearly dependent.

$\mathbf{v}_1(4,\ -1,\ 3), \quad \mathbf{v}_2(2,\ 3,\ -1), \quad \mathbf{v}_3(-1,\ 2,\ -1), \quad \mathbf{v}_4(5,\ 2,\ 3), \quad \textit{in } \mathbb{R}^3$

Solution

$$\begin{bmatrix} 4 & 2 & -1 & 5 \\ -1 & 3 & 2 & 2 \\ 3 & -1 & -1 & 3 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & \frac{11}{7} \\ 0 & 1 & 0 & \frac{1}{7} \\ 0 & 0 & 1 & \frac{11}{7} \end{bmatrix}$$

$7\boldsymbol{v}_4 = 11\boldsymbol{v}_1 + \boldsymbol{v}_2 + 11\boldsymbol{v}_2$

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

a) $(2,\ -1,\ 3),\ \ (3,\ 4,\ 1),\ \ (2,\ -3,\ 4),\ \ in\ \mathbb{R}^3.$

b) $(1,\ 0,\ 0,\ 0),\ \ (1,\ 1,\ 0,\ 0),\ \ (1,\ 1,\ 1,\ 0),\ \ (1,\ 1,\ 1,\ 1),\ \ in\ \mathbb{R}^4.$

c) $A_1 \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix},\ \ A_2 \begin{bmatrix} -2 & 4 \\ 1 & 0 \end{bmatrix},\ \ A_3 \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix},\ \ in\ M_{22}$

Solution

a) $a(2,\ -1,\ 3) + b(3,\ 4,\ 1) + c(2,\ -3,\ 4) = (0,\ 0,\ 0)$

$$\begin{bmatrix} 2 & 3 & 2 & 0 \\ -1 & 4 & -3 & 0 \\ 3 & 1 & 4 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The system has only he trivial solution $a = b = c = 0$.

$$\begin{vmatrix} 2 & 3 & 2 \\ -1 & 4 & -3 \\ 3 & 1 & 4 \end{vmatrix} = 32 - 27 - 2 - 24 + 6 + 12 \neq 0$$

The system has only the trivial solution and the vectors are linearly independent

b) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$

The system has only the trivial solution and the vectors are linearly independent

c) $\begin{bmatrix} 1 & -2 & 3 & 0 \\ 2 & 4 & -1 & 0 \\ 3 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ The vectors are linearly independent

Exercise

Suppose that the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \textit{and } \boldsymbol{u}_3$ are linearly dependent. Are the vectors $\boldsymbol{v}_1 = \boldsymbol{u}_1 + \boldsymbol{u}_2$, $\boldsymbol{v}_2 = \boldsymbol{u}_1 + \boldsymbol{u}_3$, and $\boldsymbol{v}_3 = \boldsymbol{u}_2 + \boldsymbol{u}_3$ also linearly dependent?

(***Hint***: Assume that $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 = 0$, and see what the $a_i\text{'}s$ can be.)

Solution

Given: $\boldsymbol{u}_1, \boldsymbol{u}_2, \textit{and } \boldsymbol{u}_3$ are linearly dependent, then there are scalar $b_1, b_2, \textit{and } b_3$ such that $b_1\boldsymbol{u}_1 + b_2\boldsymbol{u}_2 + b_3\boldsymbol{u}_3 = 0$.

Assume that $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 = 0$

$$a_1\left(\boldsymbol{u}_1 + \boldsymbol{u}_2\right) + a_2\left(\boldsymbol{u}_1 + \boldsymbol{u}_3\right) + a_3\left(\boldsymbol{u}_2 + \boldsymbol{u}_3\right) = 0$$

$$a_1\boldsymbol{u}_1 + a_1\boldsymbol{u}_2 + a_2\boldsymbol{u}_1 + a_2\boldsymbol{u}_3 + a_3\boldsymbol{u}_2 + a_3\boldsymbol{u}_3 = 0$$

$$\left(a_1 + a_2\right)\boldsymbol{u}_1 + \left(a_1 + a_3\right)\boldsymbol{u}_2 + \left(a_2 + a_3\right)\boldsymbol{u}_3 = 0$$

If $a_1 + a_2 = b_1 \quad a_1 + a_3 = b_2 \quad a_2 + a_3 = b_3$ and since $\boldsymbol{u}_1, \boldsymbol{u}_2, \textit{and } \boldsymbol{u}_3$ are linearly dependent, therefore, $\boldsymbol{v}_1, \boldsymbol{v}_2, \textit{and } \boldsymbol{v}_3$ are linearly dependent

Solution *Section* 3.8 – Dot product and Orthogonality

Exercise

If $\|\vec{v}\| = 5$ and $\|\vec{w}\| = 3$, what are the smallest and largest possible values of $\|\vec{v} - \vec{w}\|$ and $\vec{v}.\vec{w}$?

Solution

$$\|\vec{v} - \vec{w}\| \le \|\vec{v}\| + \|\vec{w}\| = 5 + 3 = 8$$

$$\|\vec{v} - \vec{w}\| \ge \|\vec{v}\| - \|\vec{w}\| = 5 - 3 = 2$$

$$|\vec{v}.\vec{w}| = \|\vec{v}\|.\|\vec{w}\|.\cos\theta \le \|\vec{v}\|.\|\vec{w}\|$$

$$-\|\vec{v}\|.\|\vec{w}\| \le |\vec{v}.\vec{w}| \le \|\vec{v}\|.\|\vec{w}\|$$

$$-(3)(5) \le |\vec{v}.\vec{w}| \le (3)(5)$$

$$-15 \le |\vec{v}.\vec{w}| \le 15$$

The minimum value occurs when the dot product is a small as possible, v and w are parallel, but point in opposite directions. Thus the smallest value is -15.

The maximum value occurs when the dot product is a large as possible, v and w are parallel and point in same direction. Thus the largest value is 15.

Exercise

If $\|\vec{v}\| = 7$ and $\|\vec{w}\| = 3$, what are the smallest and largest possible values of $\|\vec{v} + \vec{w}\|$ and $\vec{v}.\vec{w}$?

Solution

$$\|\vec{v} + \vec{w}\| \le \|\vec{v}\| + \|\vec{w}\| = 7 + 3 = 10$$

$$\|\vec{v} + \vec{w}\| \ge \|\vec{v}\| - \|\vec{w}\| = 7 - 3 = 4$$

$$|\vec{v}.\vec{w}| \le \|\vec{v}\|.\|\vec{w}\|$$

$$-\|\vec{v}\|.\|\vec{w}\| \le |\vec{v}.\vec{w}| \le \|\vec{v}\|.\|\vec{w}\|$$

$$-(7)(3) \le |\vec{v}.\vec{w}| \le (7)(3)$$

$$-21 \le |\vec{v}.\vec{w}| \le 21$$

The minimum value occurs when the dot product is a small as possible, v and w are parallel, but point in opposite directions. Thus the smallest value is -21. $\vec{v} = (7,\ 0,\ 0,\ \cdots)$ and $\vec{w} = (-3,\ 0,\ 0,\ \cdots)$

The maximum value occurs when the dot product is a large as possible, v and w are parallel and point in same direction. Thus the largest value is 21. $\vec{v} = (7,\ 0,\ 0,\ \cdots)$ and $\vec{w} = (3,\ 0,\ 0,\ \cdots)$

Exercise

Given that $cos(\alpha) = \frac{v_1}{\|v\|}$ and $sin(\alpha) = \frac{v_2}{\|v\|}$. Similarly, $cos(\beta) =$ ______ and $sin(\beta) =$ ______. The angle θ is $\beta - \alpha$. Substitute into the trigonometry formula $cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta)$ for $cos(\beta - \alpha)$ to find $cos(\theta) = \frac{v.w}{\|v\|.\|w\|}$

Solution

$$cos(\beta) = \frac{w_1}{\|w\|}$$

$$sin(\beta) = \frac{w_2}{\|w\|}$$

$$\begin{aligned} cos(\beta - \alpha) &= cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta) \\ &= \frac{v_1}{\|v\|}\frac{w_1}{\|w\|} + \frac{v_2}{\|v\|}\frac{w_2}{\|w\|} \\ &= \frac{v_1 w_1 + v_2 w_2}{\|v\|.\|w\|} \\ &= \frac{v.w}{\|v\|.\|w\|} \end{aligned}$$

Exercise

Can three vectors in the xy plane have $u.v < 0$ and $v.w < 0$ and $u.w < 0$?

Solution

Let consider: $u = (1,\ 0)$, $v = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $w = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

$$u.v = (1)\left(-\frac{1}{2}\right) + 0 = -\frac{1}{2}$$

$$\begin{aligned} v.w &= \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) \\ &= \frac{1}{4} - \frac{3}{4} \\ &= -\frac{1}{2} \end{aligned}$$

$$u.w = (1)\left(-\frac{1}{2}\right) + (0)\left(-\frac{\sqrt{3}}{2}\right) = -\frac{1}{2}$$

Yes, it is.

Exercise

Find the norm of v, a unit vector that has the same direction as v, and a unit vector that is oppositely directed.

a) $v = (4, -3)$

b) $v = (1, -1, 2)$

c) $v = (-2, 3, 3, -1)$

Solution

a) $\|v\| = \sqrt{4^2 + (-3)^2} = 5$

Same direction unit vector: $u_1 = \frac{v}{\|v\|} = \frac{1}{5}(4, -3) = \left(\frac{4}{5}, -\frac{3}{5}\right)$

Opposite direction unit vector: $u_2 = -\frac{v}{\|v\|} = -\frac{1}{5}(4, -3) = \left(-\frac{4}{5}, \frac{3}{5}\right)$

b) $\|v\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$

Same direction unit vector:

$$u_1 = \frac{v}{\|v\|} = \frac{1}{\sqrt{6}}(1, -1, 2) = \left(\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)$$

Opposite direction unit vector:

$$u_2 = -\frac{v}{\|v\|} = -\frac{1}{\sqrt{6}}(1, -1, 2) = \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right)$$

c) $\|v\| = \sqrt{(-2)^2 + (3)^2 + (3)^2 + (-1)^2} = \sqrt{23}$

Same direction unit vector:

$$u_1 = \frac{v}{\|v\|} = \frac{1}{\sqrt{23}}(-2, 3, 3, -1) = \left(\frac{-2}{\sqrt{23}},\ \frac{3}{\sqrt{23}},\ \frac{3}{\sqrt{23}},\ -\frac{1}{\sqrt{23}}\right)$$

Opposite direction unit vector:

$$u_2 = -\frac{v}{\|v\|} = -\frac{1}{\sqrt{23}}(-2, 3, 3, -1) = \left(\frac{2}{\sqrt{23}},\ -\frac{3}{\sqrt{23}},\ -\frac{3}{\sqrt{23}},\ \frac{1}{\sqrt{23}}\right)$$

Exercise

Evaluate the given expression with $\boldsymbol{u} = (2, -2, 3)$, $\boldsymbol{v} = (1, -3, 4)$, and $\boldsymbol{w} = (3, 6, -4)$

a) $\|u+v\|$ *b)* $\|-2u+2v\|$

c) $\|3u-5v+w\|$ *d)* $\|3v\|-3\|v\|$

e) $\|u\|+\|-2v\|+\|-3w\|$

Solution

a)
$$\begin{aligned}\|u+v\| &= \|(2,-2,3)+(1,-3,4)\| \\ &= \|(3,-5,7)\| \\ &= \sqrt{3^2+(-5)^2+7^2} \\ &= \sqrt{83}\end{aligned}$$

b)
$$\begin{aligned}\|-2u+2v\| &= \|(-4,4,-6)+(2,-6,8)\| \\ &= \|(-2,-2,2)\| \\ &= \sqrt{(-2)^2+(-2)^2+2^2} \\ &= \sqrt{12} \\ &= 2\sqrt{3}\end{aligned}$$

c)
$$\begin{aligned}\|3u-5v+w\| &= \|(6,-6,9)-(5,-15,20)+(3,6,-4)\| \\ &= \|(4,15,-15)\| \\ &= \sqrt{(4)^2+(15)^2+(-15)^2} \\ &= \sqrt{466}\end{aligned}$$

d)
$$\begin{aligned}\|3v\|-3\|v\| &= \|(3,-9,12)\|-3\|(1,-3,4)\| \\ &= \sqrt{3^2+(-9)^2+12^2}-3\sqrt{1^2+(-3)^2+4^2} \\ &= \sqrt{234}-3\sqrt{26} \\ &= 3\sqrt{26}-3\sqrt{26} \\ &= 0\end{aligned}$$

$$\|3v\|-3\|v\| = 3\|v\|-3\|v\| = 0$$

e)
$$\begin{aligned}\|u\|+\|-2v\|+\|-3w\| &= \|u\|-2\|v\|-3\|w\| \\ &= \sqrt{2^2+(-2)^2+3^2}-2\sqrt{1^2+(-3)^2+4^2}-3\sqrt{3^2+6^2+(-4)^2} \\ &= \sqrt{17}-2\sqrt{26}-3\sqrt{61}\end{aligned}$$

Exercise

Let $\boldsymbol{v}$ = (1, 1, 2, –3, 1). Find all scalars k such that $\|kv\| = 5$

Solution

$$\begin{aligned}\|kv\| &= |k|\|v\| \\ &= |k|\ \|(1,1,2,-3,1)\| \\ &= |k|\ \sqrt{1^2+1^2+2^2+(-3)^2+1^2} \\ &= |k|\sqrt{49} \\ &= 7|k|\end{aligned}$$

$$7|k| = 5 \quad \to \quad |k| = \frac{5}{7} \Rightarrow \boxed{k = \pm\frac{5}{7}}$$

Exercise

Find $u \cdot v, \quad u \cdot u, \quad and \quad v \cdot v$

a) u = (3, 1, 4), v = (2, 2, –4)

b) u = (1, 1, 4, 6), v = (2, –2, 3, –2)

c) u = (2, –1, 1, 0, –2), v = (1, 2, 2, 2, 1)

Solution

a) $u \cdot v = (3,1,4)\cdot(2,2,-4) = 3(2)+1(2)+4(-4) \underline{=-8|}$

$u \cdot u = \|u\|^2 = 3^2+1^2+4^2 \underline{=26|}$

$v \cdot v = \|v\|^2 = 2^2+2^2+(-4)^2 \underline{=24|}$

b) $u \cdot v = (1,1,4,6)\cdot(2,-2,3,-2) = 1(2)+1(-2)+4(3)+6(-2) \underline{=0|}$

$u \cdot u = \|u\|^2 = 1^2+1^2+4^2+6^2 \underline{=54|}$

$v \cdot v = \|v\|^2 = 2^2+(-2)^2+3^2+(-2)^2 \underline{=21|}$

c) $u \cdot v = (2,-1,1,0,-2)\cdot(1,2,2,2,1) = 2(1)-1(2)+1(2)+0(2)-2(1) \underline{=0|}$

$u \cdot u = \|u\|^2 = 2^2+(-1)^2+1^2+0+(-2)^2 \underline{=10|}$

$v \cdot v = \|v\|^2 = 1^2+2^2+2^2+2^2+1^2 \underline{=14|}$

Exercise

Find the Euclidean distance between $\boldsymbol{u}$ and $\boldsymbol{v}$, then find the angle between them

a) $u = (3, 3, 3)$, $v = (1, 0, 4)$

b) $u = (1, 2, -3, 0)$, $v = (5, 1, 2, -2)$

c) $u = (0, 1, 1, 1, 2)$, $v = (2, 1, 0, -1, 3)$

Solution

a)
$$d = \|u - v\| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$$
$$= \sqrt{(-2)^2 + (-3)^2 + (1)^2}$$
$$= \sqrt{14}$$

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\ \|\boldsymbol{v}\|}$$
$$= \frac{3(1) + 3(0) + 3(4)}{\sqrt{3^2 + 3^2 + 3^2}\sqrt{1^2 + 0^2 + 4^2}}$$
$$= \frac{15}{\sqrt{27}\sqrt{17}}$$

$$\theta = \cos^{-1}\left(\frac{15}{\sqrt{27}\sqrt{17}}\right) = 45.56°$$

b)
$$d = \|u - v\| = \sqrt{(1-5)^2 + (-2-1)^2 + (-3-2)^2 + (-2-0)^2}$$
$$= \sqrt{(-4)^2 + (-3)^2 + (-5)^2 + (-2)^2}$$
$$= \sqrt{46}$$

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\ \|\boldsymbol{v}\|}$$
$$= \frac{1(5) + 2(1) - 3(2) + 0(-2)}{\sqrt{1^2 + 2^2 + (-3)^2 + 0}\ \sqrt{5^2 + 1^2 + 2^2 + (-2)^2}}$$
$$= \frac{1}{\sqrt{14}\sqrt{34}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{14}\sqrt{34}}\right) = 87.37°$$

c) $d=\|u-v\|=\sqrt{(0-2)^2+(1-1)^2+(1-0)^2+(1-(-1))^2+(2-3)^2}$

$$\boxed{=\sqrt{10}}$$

$$\cos\theta=\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|}$$

$$=\frac{0(2)+1(1)+1(0)+1(-1)+2(3)}{\sqrt{0+1^2+1^2+1^2+2^2}\,\sqrt{2^2+1^2+0+(-1)^2+(3)^2}}$$

$$=\frac{6}{\sqrt{7}\sqrt{15}}$$

$$\theta=\cos^{-1}\left(\frac{6}{\sqrt{7}\sqrt{15}}\right)=\boxed{54.16°}$$

Exercise

Find a unit vector that has the same direction as the given vector

a) $(-4,-3)$ *b)* $\left(-3,\ 2,\ \sqrt{3}\right)$ *c)* $(1, 2, 3, 4, 5)$

Solution

a) $u=\frac{u}{\|u\|}=\frac{(-4,-3)}{\sqrt{(-4)^2+(-3)^2}}$

$$=\frac{(-4,-3)}{\sqrt{25}}$$

$$\boxed{=\left(-\frac{4}{5},-\frac{3}{5}\right)}$$

b) $u=\frac{1}{\sqrt{(-3)^2+(2)^2+\left(\sqrt{3}\right)^2}}\left(-3,2,\sqrt{3}\right)$

$$=\frac{1}{\sqrt{17}}\left(-3,\ 2,\ \sqrt{3}\right)$$

$$\boxed{=\left(-\frac{3}{\sqrt{17}},\ \frac{2}{\sqrt{17}},\ \frac{\sqrt{3}}{\sqrt{17}}\right)}$$

c) $u=\frac{1}{\sqrt{1^2+2^2+3^2+4^2+5^2}}(1,2,3,4,5)$

$$=\frac{1}{\sqrt{55}}(1,2,3,4,5)$$

$$\boxed{=\left(\frac{1}{\sqrt{55}},\ \frac{2}{\sqrt{55}},\ \frac{3}{\sqrt{55}},\ \frac{4}{\sqrt{55}},\ \frac{5}{\sqrt{55}}\right)}$$

Exercise

Find a unit vector that is oppositely to the given vector

a) $(-12, -5)$

b) $(3, -3, 3)$

c) $\left(-3,\ 1,\ \sqrt{6},\ 3\right)$

Solution

a) $u = -\dfrac{1}{\sqrt{(-12)^2 + (-5)^2}}(-12, -5)$

$= -\dfrac{1}{\sqrt{169}}(-12, -5)$

$\boxed{= \left(\dfrac{12}{13},\ \dfrac{5}{13}\right)}$

b) $u = -\dfrac{1}{\sqrt{(3)^2 + (-3)^2 + (3)^2}}(3, -3, 3)$

$= -\dfrac{1}{\sqrt{27}}(3, -3, 3)$

$= -\dfrac{1}{3\sqrt{3}}(3, -3, 3)$

$\boxed{= \left(-\dfrac{1}{\sqrt{3}},\ \dfrac{1}{\sqrt{3}},\ -\dfrac{1}{\sqrt{3}}\right)}$

c) $u = -\dfrac{1}{\sqrt{(-3)^2 + 1^2 + \left(\sqrt{6}\right)^2 + 3^2}}\left(-3,\ 1,\ \sqrt{6},\ 3\right)$

$= -\dfrac{1}{\sqrt{25}}\left(-3,\ 1,\ \sqrt{6},\ 3\right)$

$\boxed{= \left(\dfrac{3}{5},\ -\dfrac{1}{5},\ -\dfrac{\sqrt{6}}{5},\ -\dfrac{3}{5}\right)}$

Exercise

Verify that the Cauchy-Schwarz inequality holds

a) $u = (-3, 1, 0)$, $v = (2, -1, 3)$

b) $u = (0, 2, 2, 1)$, $v = (1, 1, 1, 1)$

c) $u = (1, 3, 5, 2, 0, 1)$, $v = (0, 2, 4, 1, 3, 5)$

Solution

a) $|u \cdot v| = |(-3,1,0) \bullet (2,-1,3)|$

$= |-3(2) + 1(-1) + 0(3)|$

$= |-7|$

$\underline{= 7|}$

$\|u\|\|v\| = \sqrt{(-3)^2 + 1^2 + 0}\ \sqrt{(2)^2 + (-1)^2 + 3^2}$

$= \sqrt{10}\sqrt{14}$

$\underline{\approx 11.83|}$

$|u \cdot v| \le \|u\|\|v\|$ Cauchy-Schwarz inequality holds

b) $|u \cdot v| = |(0,2,2,1) \bullet (1,1,1,1)|$

$= |0 + 2 + 2 + 1|$

$\underline{= 5|}$

$\|u\|\|v\| = \sqrt{0 + 2^2 + 2^2 + 1^2}\ \sqrt{1^2 + 1^2 + 1^2 + 1^2}$

$= \sqrt{9}\sqrt{4}$

$\underline{= 6|}$

$|u \cdot v| \le \|u\|\|v\|$ Cauchy-Schwarz inequality holds

c) $|u \cdot v| = |(1,3,5,2,0,1) \bullet (0,2,4,1,3,5)|$

$= |0 + 6 + 20 + 2 + 0 + 5|$

$\underline{= 23|}$

$\|u\|\|v\| = \sqrt{1^2 + 3^2 + 5^2 + 2^2 + 0 + 1^2}\ \sqrt{0 + 2^2 + 4^2 + 1^2 + 3^2 + 5^2}$

$= \sqrt{40}\sqrt{55}$

$\underline{\approx 46|}$

$|u \cdot v| \le \|u\|\|v\|$ Cauchy-Schwarz inequality holds

Exercise

Find $\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v}$ and then the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$ $\quad \boldsymbol{u}=\begin{bmatrix}3\\-1\\2\\1\end{bmatrix} \quad \boldsymbol{v}=\begin{bmatrix}1\\0\\-1\\-1\end{bmatrix}$

Solution

$$\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v}=3+0-2-1\underline{=0|}$$

$$\theta=\cos^{-1}\frac{0}{\sqrt{15}\sqrt{3}}=\cos^{-1}(0)=\underline{90^\circ|}$$

Exercise

Find the norm: $\|\boldsymbol{u}\|+\|\boldsymbol{v}\|$, $\|\boldsymbol{u}+\boldsymbol{v}\|$ for $\boldsymbol{u}=(3,\ -1,\ -2,\ 1,\ 4)$ $\quad \boldsymbol{v}=(1,\ 1,\ 1,\ 1,\ 1)$

Solution

$$\|\boldsymbol{u}\|+\|\boldsymbol{v}\|=\sqrt{3^2+(-1)^2+(-2)^2+1^2+4^2}+\sqrt{1+1+1+1+1}\ \underline{=\sqrt{31}+\sqrt{5}|}$$

$$\|\boldsymbol{u}+\boldsymbol{v}\|=\|(4,\ 0,\ -1,\ 2,\ 5)\|\ =\sqrt{16+0+1+4+25}\ \ \underline{=\sqrt{46}|}$$

Exercise

Find all numbers r such that: $\|r(1,\ 0,\ -3,\ -1,\ 4,\ 1)\|=1$

Solution

$$r\sqrt{1+9+1+16+1}=\pm1$$

$$r\sqrt{28}=\pm1$$

$$r=\pm\frac{1}{2\sqrt{7}}\underline{=\pm\frac{\sqrt{7}}{14}|}$$

Exercise

Find the distance between $P_1(7,-5,1)$ and $P_2(-7,-2,-1)$

Solution

$$\begin{aligned}\|P_1P_2\|&=\sqrt{(-7-7)^2+(-2+5)^2+(-1-1)^2}\\&=\sqrt{14^2+3^2+(-2)^2}\\&=\sqrt{196+9+4}\\&\underline{=\sqrt{209}|}\end{aligned}$$

Exercise

Given $\boldsymbol{u}$ = (1, –5, 4), $\boldsymbol{v}$ = (3, 3, 3)

a) Find $\boldsymbol{u}\cdot\boldsymbol{v}$

b) Find the cosine of the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$.

Solution

a) $\boldsymbol{u}\cdot\boldsymbol{v} = 3-15+12 = \underline{0}$

b) $\cos\theta = \dfrac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\ \|\boldsymbol{v}\|}\ \underline{=0}$

Exercise

Determine whether $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

a) $\boldsymbol{u}=(-6,\ -2),\quad \boldsymbol{v}=(5,\ -7)$

b) $\boldsymbol{u}=(6,\ 1,\ 4),\quad \boldsymbol{v}=(2,\ 0,\ -3)$

c) $\boldsymbol{u}=(1,\ -5,\ 4),\quad \boldsymbol{v}=(3,\ 3,\ 3)$

d) $\boldsymbol{u}=(-2,\ 2,\ 3),\quad \boldsymbol{v}=(1,\ 7,\ -4)$

Solution

a) $\boldsymbol{u}\cdot\boldsymbol{v}=(-6)(5)+(-2)(-7)$

$=-30+14$

$\underline{=-16\neq 0}$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are not orthogonal

b) $\boldsymbol{u}\cdot\boldsymbol{v}=6(2)+1(0)+4(-3)\ \underline{=0}$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

c) $\boldsymbol{u}\cdot\boldsymbol{v}=1(3)-5(3)+4(3)\ \underline{=0}$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

d) $\boldsymbol{u}\cdot\boldsymbol{v}=-2(1)+2(7)+3(-4)\ \underline{=0}$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

Exercise

Determine whether the vectors form an orthogonal set

a) $\boldsymbol{v}_1=(2,\ 3),\quad \boldsymbol{v}_2=(3,\ 2)$

b) $\boldsymbol{v}_1=(1,\ -2),\quad \boldsymbol{v}_2=(-2,\ 1)$

c) $\boldsymbol{u}=(-4,\ 6,\ -10,\ 1)\quad \boldsymbol{v}=(2,\ 1,\ -2,\ 9)$

d) $\boldsymbol{u}=(a,\ b)\quad \boldsymbol{v}=(-b,\ a)$

e) $\boldsymbol{v}_1=(-2,\ 1,\ 1),\quad \boldsymbol{v}_2=(1,\ 0,\ 2),\quad \boldsymbol{v}_3=(-2,\ -5,\ 1)$

f) $\boldsymbol{v}_1 = (1,\ 0,\ 1),\quad \boldsymbol{v}_2 = (1,\ 1,\ 1),\quad \boldsymbol{v}_3 = (-1,\ 0,\ 1)$

g) $\boldsymbol{v}_1 = (2,-2,1),\quad \boldsymbol{v}_2 = (2,1,-2),\quad \boldsymbol{v}_3 = (1,2,2)$

Solution

a) $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 2(3)+3(2)\ \underline{=12 \neq 0}$

$\therefore$ Vectors don't form an orthogonal set

b) $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 1(-2)-2(1)\ \underline{=-4 \neq 0}$

$\therefore$ Vectors don't form an orthogonal set

c) $\boldsymbol{u}\bullet\boldsymbol{v} = -8+6+20+9\ \underline{=27 \neq 0}$; These vectors are not orthogonal

d) $\boldsymbol{u}\bullet\boldsymbol{v} = -ab+ab\ \underline{=0}$; These vectors are orthogonal

e) $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = -2(1)+1(0)+1(2)\underline{=0}$

$\boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = -2(-2)+1(-5)+1(1)\underline{=0}$

$\boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = 1(-2)+0(-5)+2(1)\underline{=0}$

$\therefore$ Vectors form an orthogonal set

f) $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 1(1)+0(1)+1(1)\ \underline{=2 \neq 0}$

$\therefore$ Vectors don't form an orthogonal set

g) $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 2(2)-2(1)+1(-2)\underline{=0}$

$\boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = 2(1)-2(2)+1(2)\underline{=0}$

$\boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = 2(1)+1(2)-2(2)\underline{=0}$

$\therefore$ Vectors form an orthogonal set

Exercise

Find a unit vector that is orthogonal to both $\boldsymbol{u} = (1, 0, 1)$ and $\boldsymbol{v} = (0, 1, 1)$

Solution

Let $\boldsymbol{w} = (w_1, w_2, w_3)$ be the unit vector that is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$.

$$\boldsymbol{u}\cdot\boldsymbol{w} = 1(w_1)+0(w_2)+1(w_3) = \underline{w_1+w_3=0}$$

$$\boxed{w_3=-w_1}$$

$$\boldsymbol{v}\cdot\boldsymbol{w} = 0(w_1)+1(w_2)+1(w_3) = \underline{w_2+w_3=0}$$

$$\boxed{w_3=-w_2}$$

$w_1 = w_2 = -w_3$

The orthogonal vector to both $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{w} = (1,\ 1,\ -1)$, therefore the unit vector is

$$\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = \frac{1}{\sqrt{1^2 + 1^2 + (-1)^2}}(1,\ 1,\ -1)$$

$$= \frac{1}{\sqrt{3}}(1,\ 1,\ -1)$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)$$

The possible vectors are: $\boxed{\pm\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)}$

Exercise

a) Show that $\boldsymbol{v} = (a, b)$ and $\boldsymbol{w} = (-b, a)$ are orthogonal vectors.

b) Use the result to find two vectors that are orthogonal to $\boldsymbol{v} = (2, -3)$.

c) Find two unit vectors that are orthogonal to $(-3, 4)$

Solution

a) $\boldsymbol{v} \cdot \boldsymbol{w} = a(-b) + b(a) = -ab + ab = 0$ are orthogonal vectors.

b) (2, 3) and (–2, 3).

c) $u_1 = \frac{1}{\sqrt{4^2 + 3^2}}(4,3) = \left(\frac{4}{5},\ \frac{3}{5}\right)$

$u_2 = -\frac{1}{\sqrt{4^2 + 3^2}}(4,3) = \left(-\frac{4}{5},\ -\frac{3}{5}\right)$

Exercise

Show that if $\boldsymbol{v}$ is orthogonal to both $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, then $\boldsymbol{v}$ is orthogonal to $k_1\boldsymbol{w}_1 + k_2\boldsymbol{w}_2$ for all scalars k_1 and k_2.

Solution

$$\boldsymbol{v} \cdot (k_1\boldsymbol{w}_1 + k_2\boldsymbol{w}_2) = \boldsymbol{v} \cdot (k_1\boldsymbol{w}_1) + \boldsymbol{v} \cdot (k_2\boldsymbol{w}_2)$$

$$= k_1(\boldsymbol{v} \cdot \boldsymbol{w}_1) + k_2(\boldsymbol{v} \cdot \boldsymbol{w}_2) \qquad \textit{If v is orthogonal to } \boldsymbol{w}_1 \ \& \ \boldsymbol{w}_2 \rightarrow \boldsymbol{v} \cdot \boldsymbol{w}_1 = \boldsymbol{v} \cdot \boldsymbol{w}_2 = 0$$

$$= k_1(0) + k_2(0)$$

$$= 0$$

Exercise

Show that $\vec{u}-\vec{v}$ is orthogonal to $\vec{u}+\vec{v}$ if and only if $\|\vec{u}\|=\|\vec{v}\|$

Solution

Suppose that $\vec{u}-\vec{v}$ is orthogonal to $\vec{u}+\vec{v}$. Then

$$\begin{aligned}
0 &= \langle \vec{u}-\vec{v},\ \vec{u}+\vec{v}\rangle \\
&= (\vec{u}-\vec{v})^T(\vec{u}+\vec{v}) \\
&= \left(\vec{u}^T-\vec{v}^T\right)(\vec{u}+\vec{v}) \\
&= \vec{u}^T\vec{u}+\vec{u}^T\vec{v}-\vec{v}^T\vec{u}-\vec{v}^T\vec{v} \\
&= \langle \vec{u},\ \vec{u}\rangle+\langle \vec{u},\ \vec{v}\rangle-\langle \vec{v},\ \vec{u}\rangle-\langle \vec{v},\ \vec{v}\rangle \qquad \langle \vec{u},\ \vec{v}\rangle=\langle \vec{v},\ \vec{u}\rangle \\
&= \langle \vec{u},\ \vec{u}\rangle-\langle \vec{v},\ \vec{v}\rangle
\end{aligned}$$

So $\langle \vec{u},\ \vec{u}\rangle=\langle \vec{v},\ \vec{v}\rangle$. Therefore, $\|\vec{u}\|^2=\|\vec{v}\|^2 \quad\Rightarrow\quad \|\vec{u}\|=\|\vec{v}\|$.

Suppose $\|\vec{u}\|=\|\vec{v}\|$. Then

$$\begin{aligned}
\langle \vec{u}-\vec{v},\ \vec{u}+\vec{v}\rangle &= (\vec{u}-\vec{v})^T(\vec{u}+\vec{v}) \\
&= \left(\vec{u}^T-\vec{v}^T\right)(\vec{u}+\vec{v}) \\
&= \vec{u}^T\vec{u}+\vec{u}^T\vec{v}-\vec{v}^T\vec{u}-\vec{v}^T\vec{v} \\
&= \langle \vec{u},\ \vec{u}\rangle+\langle \vec{u},\ \vec{v}\rangle-\langle \vec{v},\ \vec{u}\rangle-\langle \vec{v},\ \vec{v}\rangle \qquad \langle \vec{u},\ \vec{v}\rangle=\langle \vec{v},\ \vec{u}\rangle \\
&= \langle \vec{u},\ \vec{u}\rangle-\langle \vec{v},\ \vec{v}\rangle \\
&= \|\vec{u}\|^2-\|\vec{v}\|^2 \\
&= 0
\end{aligned}$$

So we can see that $\vec{u}-\vec{v}$ is orthogonal to $\vec{u}+\vec{v}$

We conclude that $\vec{u}-\vec{v}$ is orthogonal to $\vec{u}+\vec{v}$ if and only if $\|\vec{u}\|=\|\vec{v}\|$, as desired.

Exercise

Given $\boldsymbol{u}=(3,\ -1,\ 2)\quad \boldsymbol{v}=(4,\ -1,\ 5)\quad and\quad \boldsymbol{w}=(8,\ -7,\ -6)$

a) Find $3\boldsymbol{v}-4(5\boldsymbol{u}-6\boldsymbol{w})$

b) Find $\boldsymbol{u}\bullet\boldsymbol{v}$ and then the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$.

Solution

a)

$$\begin{aligned}
3\boldsymbol{v}-4(5\boldsymbol{u}-6\boldsymbol{w}) &= 3(4,\ -1,\ 5)-4\big(5(3,\ -1,\ 2)-6(8,\ -7,\ -6)\big) \\
&= (12,\ -3,\ 15)-4\big((15,\ -5,\ 10)-(48,\ -42,\ -36)\big) \\
&= (12,\ -3,\ 15)-4(-33,\ 37,\ 46) \\
&= (12,\ -3,\ 15)-(-132,\ 148,\ 184)
\end{aligned}$$

$$\boxed{=(144,\ -151,\ -169)}$$

b) $\boldsymbol{u}\bullet\boldsymbol{v}=(3,\ -1,\ 2)\bullet(1,\ 1,\ -1)$

$=3-1-2$

$\boxed{=0}$

$\boxed{\theta=90^\circ}$

Exercise

a) Show that $\boldsymbol{v}=(a,\ b)$ and $\boldsymbol{w}=(-b,\ a)$ are orthogonal vectors

b) Use the result in part (*a*) to find two vectors that are orthogonal to $\boldsymbol{v}=(2,\ -3)$

c) Find two unit vectors that are orthogonal to $(-3,\ 4)$

Solution

a) $\boldsymbol{u}\cdot\boldsymbol{v}=-ab+ba=0$; 2 vectors are orthogonal vectors.

b) $\boldsymbol{v}=(2,\ -3)\ \Rightarrow \boldsymbol{w}=(-3,\ -2)$ and $\boldsymbol{w}=(3,\ 2)$

c) $(-3,\ 4)\ \Rightarrow\ \boldsymbol{u}=\dfrac{(-3,\ 4)}{\sqrt{9+16}}=\left(-\frac{3}{5},\ \frac{4}{5}\right)$

$\boldsymbol{u}_1=\left(\frac{4}{5},\frac{3}{5}\right)\quad and\quad \boldsymbol{u}_2=\left(-\frac{4}{5},-\frac{3}{5}\right)$

Exercise

Show that $A(3,\ 0,\ 2)$, .$B(4,\ 3,\ 0)$, and $C(8,\ 1,\ -1)$ are vertices of a right triangle. At which vertex is the right angle?

Solution

$AB=(4-3,3-0,0-2)=(1,3,-2)\qquad AC=(5,1,-3)\qquad BC=(4,-2,-1)$

$AB\bullet AC=5+3+6=14$

$AB\bullet BC=4-6+2=0$

$AC\bullet BC=20-2+3=21$

The right triangle at point *B*

Exercise

Establish the identity: $\quad \boldsymbol{u}\bullet\boldsymbol{v}=\frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2-\frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2$

Solution

Let $\boldsymbol{u}(u_1,\ u_2,\ \dots,\ u_n)\quad and\quad \boldsymbol{v}=(v_1,\ v_2,\ \dots,\ v_n)$

$\boxed{\boldsymbol{u}\bullet\boldsymbol{v}=u_1v_1+\ u_2v_2+\ \dots+u_nv_n}$

$$\boldsymbol{u}+\boldsymbol{v}=\left(u_1+v_1,\ u_2+v_2,\ \ldots,u_n+v_n\right)$$

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2=\left(u_1+v_1\right)^2+\left(u_2+v_2\right)^2+\cdots+\left(u_n+v_n\right)^2$$

$$=u_1^2+v_1^2+2u_1v_1+u_2^2+v_2^2+2u_2v_2+\cdots\ +u_2^2+v_n^2+2u_nv_n$$

$$\boldsymbol{u}-\boldsymbol{v}=\left(u_1-v_1,\ u_2-v_2,\ \ldots,u_n-v_n\right)$$

$$\|\boldsymbol{u}-\boldsymbol{v}\|^2=\left(u_1-v_1\right)^2+\left(u_2-v_2\right)^2+\cdots+\left(u_n-v_n\right)^2$$

$$=u_1^2+v_1^2-2u_1v_1+u_2^2+v_2^2-2u_2v_2+\cdots\ +u_2^2+v_n^2-2u_nv_n$$

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2-\|\boldsymbol{u}-\boldsymbol{v}\|^2=u_1^2+v_1^2+2u_1v_1+u_2^2+v_2^2+2u_2v_2+\cdots\ +u_2^2+v_n^2+2u_nv_n$$

$$-\left(u_1^2+v_1^2-2u_1v_1+u_2^2+v_2^2-2u_2v_2+\cdots\ +u_2^2+v_n^2-2u_nv_n\right)$$

$$=u_1^2+v_1^2+2u_1v_1+u_2^2+v_2^2+2u_2v_2+\cdots\ +u_2^2+v_n^2+2u_nv_n$$

$$-u_1^2-v_1^2+2u_1v_1-u_2^2-v_2^2+2u_2v_2-\cdots\ -u_2^2-v_n^2+2u_nv_n$$

$$=4u_1v_1+4u_2v_2+\cdots\ +4u_nv_n$$

$$\frac{1}{4}\left(\|\boldsymbol{u}+\boldsymbol{v}\|^2-\|\boldsymbol{u}-\boldsymbol{v}\|^2\right)=u_1v_1+u_2v_2+\cdots\ +u_nv_n$$

Therefore; $\boldsymbol{u}\bullet\boldsymbol{v}=\frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2-\frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2$ is true.

2nd method:

$$\frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2-\frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2=\frac{1}{4}\left[(\boldsymbol{u}+\boldsymbol{v})(\boldsymbol{u}+\boldsymbol{v})-(\boldsymbol{u}-\boldsymbol{v})(\boldsymbol{u}-\boldsymbol{v})\right]$$

$$=\frac{1}{4}\left[\boldsymbol{uu}+2\boldsymbol{uv}+\boldsymbol{vv}-(\boldsymbol{uu}-2\boldsymbol{uv}+\boldsymbol{vv})\right]$$

$$=\frac{1}{4}\left[\boldsymbol{uu}+2\boldsymbol{uv}+\boldsymbol{vv}-\boldsymbol{uu}+2\boldsymbol{uv}-\boldsymbol{vv}\right]$$

$$=\frac{1}{4}(4\boldsymbol{uv})$$

$$=\boldsymbol{u}\bullet\boldsymbol{v}$$

Solution *Section* 3.9 – Eigenvalues and Eigenvectors

Exercise

Find the eigenvalues and eigenvectors of $A,\ A^2,\ A^{-1},\ and\ A+4I$:

$$A=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix} \quad and \quad A^2=\begin{bmatrix}5 & -4\\ -4 & 5\end{bmatrix}$$

Check the trace $\lambda_1+\lambda_2$ and the determinant $\lambda_1\lambda_2$ for A and also A^2.

Solution

For A:

$$\det(A-\lambda I)=\begin{vmatrix}2-\lambda & -1\\ -1 & 2-\lambda\end{vmatrix}$$

$$=(2-\lambda)^2-1$$

$$=\lambda^2-4\lambda+3=0$$

The eigenvalues of $\boldsymbol{A}$ are $\lambda_1=1\quad and\quad \lambda_2=3$.

The trace of a square matrix A is the sum of the elements on the main diagonal: 2 + 2 agrees with 1+ 3. The det($\boldsymbol{A}$) = 3 agrees with the product $\lambda_1\lambda_2$.

The eigenvectors for $\boldsymbol{A}$ are:

$\lambda_1=1: \left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix}2-1 & -1\\ -1 & 2-1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\to\begin{cases}x-y=0\\ -x+y=0\end{cases}\Rightarrow x=y$$

Therefore the eigenvector $V_1=\begin{pmatrix}1\\ 1\end{pmatrix}$

$\lambda_2=3: \left(A-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\to\begin{cases}-x-y=0\\ -x-y=0\end{cases}\Rightarrow x=-y$$

Therefore the eigenvector $V_2=\begin{pmatrix}1\\ -1\end{pmatrix}$

For A^2:

$$\det\left(A^2-\lambda I\right)=\begin{vmatrix}5-\lambda & -4\\ -4 & 5-\lambda\end{vmatrix}=(5-\lambda)^2-16=\lambda^2-10\lambda+9=0$$

The eigenvalues of A^2 are $\lambda_1=1 \quad and \quad \lambda_2=9$. ***Or*** $\lambda_1=1^2=1 \quad and \quad \lambda_2=3^2=9$

$$\begin{cases} tr(A)=5+5=10\\ \lambda_1+\lambda_2=1+9=10\end{cases} \Rightarrow tr(A)=\lambda_1+\lambda_2$$

$$\begin{cases} \left|A^2\right|=\begin{vmatrix}5 & -4\\ -4 & 5\end{vmatrix}=9\\ \lambda_1\lambda_2=1(9)=9\end{cases} \Rightarrow \left|A^2\right|=\lambda_1\lambda_2$$

$\lambda_1=1:\left(A^2-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix}4 & -4\\ -4 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\rightarrow\begin{cases}4x-4y=0\\ -4x+4y=0\end{cases}\Rightarrow x=y$$

Therefore the eigenvector $V_1=\begin{pmatrix}1\\ 1\end{pmatrix}$

$\lambda_2=9:\left(A^2-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix}-4 & -4\\ -4 & -4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\rightarrow\begin{cases}-4x-4y=0\\ -4x-4y=0\end{cases}\Rightarrow x=y$$

Therefore the eigenvector $V_2=\begin{pmatrix}1\\ -1\end{pmatrix}$

For A^{-1}:

$$A^{-1}=\frac{1}{3}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}=\begin{pmatrix}\frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}$$

$$\det\left(A^{-1}-\lambda I\right)=\begin{vmatrix}\frac{2}{3}-\lambda & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}-\lambda\end{vmatrix}=\left(\frac{2}{3}-\lambda\right)^2-\frac{1}{9}=\lambda^2-\frac{4}{3}\lambda+\frac{1}{3}=0$$

The eigenvalues of A^{-1} are $\lambda_1=1 \quad and \quad \lambda_2=\frac{1}{3}$.

$\lambda_1=1:\left(A^{-1}-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix} \frac{2}{3}-1 & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3}-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -\frac{1}{3}x+\frac{1}{3}y=0 \\ \frac{1}{3}x-\frac{1}{3}y=0 \end{cases} \to \boxed{x=y}$$

Therefore the eigenvector $V_1=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\lambda_2=\frac{1}{3}:\left(A^{-1}-\lambda_2 I\right)V_2=0$$

$$\begin{pmatrix} \frac{2}{3}-\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3}-\frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \frac{1}{3}x+\frac{1}{3}y=0 \\ \frac{1}{3}x+\frac{1}{3}y=0 \end{cases} \to \boxed{x=-y}$$

Therefore the eigenvector $V_2=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

For $A+4I$:

$$A+4I=\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}+\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}=\begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix}$$

$$\det\left(A^{-1}-\lambda I\right)=\begin{vmatrix} 6-\lambda & 1 \\ 1 & 6-\lambda \end{vmatrix}=(6-\lambda)^2-1=\lambda^2-12\lambda+35=0$$

The eigenvalues of A^{-1} are $\lambda_1=5$ *and* $\lambda_2=7$.

$$\lambda_1=5:\left(A+4I-\lambda_1 I\right)V_1=0$$

$$\begin{pmatrix} 6-5 & 1 \\ 1 & 6-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x+y=0 \\ x+y=0 \end{cases} \to \boxed{x=-y}$$

Therefore the eigenvector $V_1=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\lambda_2=7:\left(A+4I-\lambda_2 I\right)V_2=0$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \begin{cases} -x + \frac{1}{3}y = 0 \\ x - y = 0 \end{cases} \rightarrow \boxed{x = y}$$

Therefore the eigenvector $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The eigenvalues $(A) = \lambda$

The eigenvalues $\left(A^2\right) = \lambda^2$

The eigenvalues $\left(A^{-1}\right) = \frac{1}{\lambda}$

Exercise

Show directly that the given vectors are eigenvectors of the given matrix. What are the corresponding eigenvalues

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$$

Solution

$$A\boldsymbol{v}_1 = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ -21 \end{bmatrix} = 7\begin{bmatrix} 1 \\ -3 \end{bmatrix} = 7\boldsymbol{v}_1$$

$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is an eigenvectors corresponding to the eigenvalue 7.

$$A\boldsymbol{v}_2 = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0\boldsymbol{v}_2$$

$\boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvectors corresponding to the eigenvalue 0.

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & -2 \\ -3 & 6-\lambda \end{vmatrix} \\ &= (1-\lambda)(6-\lambda) - 6 \\ &= 6 - 7\lambda + \lambda^2 - 6 \\ &= \lambda^2 - 7\lambda = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = 0 \quad \textit{and} \quad \lambda_2 = 7$

Exercise

For which real numbers c does this matrix A have

$$A=\begin{pmatrix} 2 & -c \\ -1 & 2 \end{pmatrix}$$

a) Two real eigenvalues and eigenvectors.

b) A repeated eigenvalue with only one eigenvector

c) Two complex eigenvalues and eigenvectors.

Solution

$$\begin{aligned}\left|A-\lambda I\right| &= \begin{vmatrix} 2-\lambda & -c \\ -1 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)^2 - c \\ &= \lambda^2 - 4\lambda + 4 - c = 0\end{aligned}$$

$$\lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-c)}}{2(1)}$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16+4c}}{2}$$

a) Two real eigenvalues and eigenvectors, when $16+4c>0 \ \rightarrow \ 4c>-16 \ \Rightarrow \ \boxed{c>-4}$

b) A repeated eigenvalue with only one eigenvector, when $16+4c=0 \ \Rightarrow \ \boxed{c=-4}$

c) Two complex eigenvalues and eigenvectors, when $16+4c<0 \ \Rightarrow \ \boxed{c<-4}$

Exercise

Find the eigenvalues of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{AB}$, and $\boldsymbol{BA}$:

$$A=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad AB=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad BA=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

a) The eigenvalues of $\boldsymbol{AB}$ (are equal to) (are not equal to) eigenvalues of $\boldsymbol{A}$ times eigenvalues of $\boldsymbol{B}$.

b) The eigenvalues of $\boldsymbol{AB}$ (are equal to) (are not equal to) eigenvalues of $\boldsymbol{BA}$.

Solution

Since $\boldsymbol{A}$ is a lower triangular, then $\lambda_1 = \lambda_2 = 1$

Since $\boldsymbol{B}$ is a upper triangular, then $\lambda_1 = \lambda_2 = 1$

$$\det(AB-I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)-1 = \lambda^2 - 3\lambda + 1 = 0 \qquad \lambda_1 = \frac{3-\sqrt{5}}{2} \quad \lambda_2 = \frac{3+\sqrt{5}}{2}$$

$$\det(BA-I)=\begin{vmatrix}2-\lambda & 1\\ 1 & 1-\lambda\end{vmatrix}=(2-\lambda)(1-\lambda)-1=\lambda^2-3\lambda+1=0 \qquad \lambda_1=\frac{3-\sqrt{5}}{2} \quad \lambda_2=\frac{3+\sqrt{5}}{2}$$

a) The eigenvalues of $\boldsymbol{AB}$ are not equal to eigenvalues of $\boldsymbol{A}$ times eigenvalues of $\boldsymbol{B}$.

b) The eigenvalues of $\boldsymbol{AB}$ are equal to the eigenvalues of $\boldsymbol{BA}$.

Exercise

When $a+b=c+d$ show that (1, 1) is an eigenvector and find both eigenvalues of

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

Solution

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}=\begin{pmatrix}a+b\\ c+d\end{pmatrix}=\begin{pmatrix}a+b\\ a+b\end{pmatrix}=(a+b)\begin{bmatrix}1\\ 1\end{bmatrix}$$

If $a+b=c+d=\lambda_1$

$tr(A)=a+d=\lambda_1+\lambda_2$

$$\begin{aligned}\lambda_2&=(a+d)-\lambda_1\\ &=a+d-(a+b)\\ &=a+d-a-b\\ &=d-b \qquad \text{or} \quad =a-c\end{aligned}$$

The eigenvalues for λ_2:

$$\begin{pmatrix}a-\lambda_2 & b\\ c & d-\lambda_2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}a-(a-c) & b\\ c & d-(d-b)\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}c & b\\ c & b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\rightarrow\{cx+by=0 \quad \Rightarrow \boxed{cx=-by}$$

The eigenvector: $V_2=\begin{pmatrix}b\\ -c\end{pmatrix}$

Exercise

The eigenvalues of A equal to the eigenvalues of A^T. This is because $\det(A-\lambda I)$ equals $\det\left(A^T-\lambda I\right)$.

That is true because ______. Show by an example that the eigenvectors of A and A^T are not the same.

Solution

$$\det(A-\lambda I)=\det(A-\lambda I)^T=\det\left(A^T-(\lambda I)^T\right)=\det\left(A^T-\lambda I\right)$$

Therefore, A and A^T have the same eigenvalues.

Let consider the matrix: $A=\begin{pmatrix}0&1\\4&0\end{pmatrix} \Rightarrow A^T=\begin{pmatrix}0&4\\1&0\end{pmatrix}$

$$|A-\lambda I|=\begin{vmatrix}-\lambda&1\\4&-\lambda\end{vmatrix}=\lambda^2-4=0$$

The eigenvalues are: $\lambda=\pm 2$

For $\lambda=2$

$$(A-\lambda_1 I)V_1=\begin{pmatrix}-2&1\\4&-2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\rightarrow\begin{cases}-2x+y=0\\4x-2y=0\end{cases}\Rightarrow y=2x$$

$$V_1=\begin{pmatrix}1\\2\end{pmatrix}$$

$$\left(A^T-\lambda_1 I\right)V_1=\begin{pmatrix}-2&4\\1&-2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\rightarrow\begin{cases}-2x+4y=0\\x-2y=0\end{cases}\Rightarrow x=2y$$

$$V_1=\begin{pmatrix}2\\1\end{pmatrix}$$

Exercise

Let $A=\begin{bmatrix}2&-1\\1&2\end{bmatrix}$. Compute the eigenvalues and eigenvectors of A.

Solution

$$\det(A-\lambda I)=\begin{vmatrix}2-\lambda&-1\\1&2-\lambda\end{vmatrix}=(2-\lambda)^2+1=0$$

$$(2-\lambda)^2=-1$$

$$2-\lambda=\pm\sqrt{-1}=\pm i$$

The eigenvalues of $\boldsymbol{A}$ are: $\lambda=2\pm i$

For $\lambda_1=2-i\Rightarrow\left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix} 2-(2-i) & -1 \\ 1 & 2-(2-i) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} ix - y = 0 \\ x + iy = 0 \end{cases} \Rightarrow x = -iy$$

The eigenvector is: $V_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

For $\lambda_2 = 2 + i \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} -ix - y = 0 \\ x - iy = 0 \end{cases} \Rightarrow x = iy$$

The eigenvector is: $V_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

Exercise

Let $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

a) What is the characteristic polynomial for A (i.e. compute $\det\left(A - \lambda I\right)$?

b) Verify that 1 is an eigenvalue of A. What is a corresponding eigenvector?

c) What are the other eigenvalues of A?

Solution

a) $\det\left(A - \lambda I\right) = \begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix}$

$$= (2-\lambda)(1-\lambda)(-1-\lambda) - 2 + 9 - 3(1-\lambda) - 3(2-\lambda) + 2(-1-\lambda)$$

$$= \left(2 - 3\lambda + \lambda^2\right)(-1-\lambda) + 7 - 3 + 3\lambda - 6 + 3\lambda - 2 - 2\lambda$$

$$= -2 + 3\lambda - \lambda^2 - 2\lambda + 3\lambda^2 - \lambda^3 + 4\lambda - 4$$

$$= -\lambda^3 + 2\lambda^2 + 5\lambda - 6$$

b) *If* $\lambda = 1 \quad \to -\lambda^3 + 2\lambda^2 + 5\lambda - 6 \overset{?}{=} 0$

$$-1^3 + 2(1)^2 + 5(1) - 6 \overset{?}{=} 0$$

$$-1 + 2 + 5 - 6 \overset{?}{=} 0$$

$$\boxed{0 = 0}$$

1 is an eigenvalue of A.

$$\begin{pmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \to \begin{cases} x - 2y + 3z = 0 \\ x \qquad + z = 0 \\ x + 3y - 2z = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \qquad \boxed{x = -z} \\ 3y = 2z - x = 2z + z = 3z \quad \Rightarrow \boxed{y = z} \end{cases}$$

The eigenvector for $\lambda = 1$ is $V = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

c) $-\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0 \;\Rightarrow\; \underline{\lambda_1 = 1 \quad \lambda_2 = -2 \quad \lambda_3 = 3}$

Exercise

For the matrix:

$a)\ \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ $\quad b)\ \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$ $\quad c)\ \begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}$

$d)\ \begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$ $\quad e)\ \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$ $\quad f)\ \begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$

$g)\ \begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}$ $\quad h)\ \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\quad i)\ \begin{pmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

a)

i. $\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix}$

$$= (3-\lambda)(-1-\lambda) - 0$$

$$= \lambda^2 - 2\lambda - 3$$

The characteristic equation: $\lambda^2 - 2\lambda - 3$

ii. $\lambda^2 - 2\lambda - 3 = 0$

The eigenvalues are $\lambda_1 = -1$ *and* $\lambda_2 = 3$

iii. $\lambda_1 = -1 \rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & 0 \\ 8 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 4x = 0 \\ 8x = 0 \end{cases} \Rightarrow x = 0$$

Therefore the eigenvector $V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda_2 = 3 \rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 0 = 0 \\ 8x - 4y = 0 \end{cases} \Rightarrow 8x = 4y \rightarrow \boxed{2x = y}$$

Therefore the eigenvector $V_2 = \begin{pmatrix} x \\ 2x \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

b) For the matrix: $\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

i. $\det(A-\lambda I) = \begin{vmatrix} 10-\lambda & -9 \\ 4 & -2-\lambda \end{vmatrix}$

$= (10-\lambda)(-2-\lambda) + 36$

$= \lambda^2 - 8\lambda + 16$

$\Rightarrow$ The characteristic equation: $\lambda^2 - 8\lambda + 16$

ii. $\lambda^2 - 8\lambda + 16 = 0$

$\Rightarrow$ The eigenvalues are $\lambda_{1,2} = 4$

iii. $\lambda_1 = 4 \quad \rightarrow \quad (A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 6x - 9y = 0 \\ 4x - 6y = 0 \end{cases} \Rightarrow \boxed{2x = 3y}$$

Therefore the eigenvector $V_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$

c) For the matrix: $\begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}$

i. $\det(A-\lambda I) = \begin{vmatrix} -\lambda & 3 \\ 4 & -\lambda \end{vmatrix}$

$= \lambda^2 - 12$

$\Rightarrow$ The characteristic equation: $\lambda^2 - 12$

ii. $\lambda^2 - 12 = 0 \quad \Rightarrow \quad \lambda = \pm\sqrt{12}$

The eigenvalues are $\lambda_{1,2} = 4$

iii. For $\lambda_1 = \sqrt{12} \quad \rightarrow \quad \begin{pmatrix} -\sqrt{12} & 3 \\ 4 & -\sqrt{12} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -\sqrt{12} & 3 \\ 4 & -\sqrt{12} \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & -\frac{3}{\sqrt{12}} \\ 0 & 0 \end{pmatrix} \Rightarrow x - \frac{3}{\sqrt{12}}y = 0$$

Therefore the eigenvector $V_1 = \begin{pmatrix} \frac{3}{\sqrt{12}} \\ 1 \end{pmatrix}$

For $\lambda_2 = -\sqrt{12} \rightarrow \begin{pmatrix} \sqrt{12} & 3 \\ 4 & \sqrt{12} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} \sqrt{12} & 3 \\ 4 & \sqrt{12} \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & \frac{3}{\sqrt{12}} \\ 0 & 0 \end{pmatrix} \Rightarrow x + \frac{3}{\sqrt{12}} y = 0$$

Therefore the eigenvector $V_2 = \begin{pmatrix} -\frac{3}{\sqrt{12}} \\ 1 \end{pmatrix}$

d) For the matrix $\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$

i. $\begin{vmatrix} -2-\lambda & -7 \\ 1 & 2-\lambda \end{vmatrix} = (-2-\lambda)(2-\lambda)+7$

$$= -4+\lambda^2+7$$

$$= \lambda^2+3$$

The characteristic equation: $\boxed{\lambda^2+3=0}$

ii. $\lambda^2 = -3 \rightarrow$ *The eigenvalues* $\boxed{\lambda_{1,2} = \pm i\sqrt{3}}$

iii. For $\lambda_1 = -i\sqrt{3}$, we have: $(A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -2+i\sqrt{3} & -7 \\ 1 & 2+i\sqrt{3} \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \left(-2+i\sqrt{3}\right)x_1 - 7y_1 = 0 \\ x_1 + \left(2+i\sqrt{3}\right)y_1 = 0 \end{cases}$$

$$x_1 = -\left(2+i\sqrt{3}\right)y_1$$

Therefore the eigenvector $V_1 = \begin{pmatrix} 2+i\sqrt{3} \\ -1 \end{pmatrix}$

For $\lambda_2 = i\sqrt{3}$, we have: $(A-\lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -2-i\sqrt{3} & -7 \\ 1 & 2-i\sqrt{3} \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \left(-2-i\sqrt{3}\right)x_2 - 7y_2 = 0 \\ x_2 + \left(2-i\sqrt{3}\right)y_2 = 0 \end{cases}$$

$$x_2 = -\left(2-i\sqrt{3}\right)y_2$$

Therefore the eigenvector $V_2 = \begin{pmatrix} 2-i\sqrt{3} \\ -1 \end{pmatrix}$

e) For the matrix: $\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

i. $\det(A-\lambda I)=\begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda)(1-\lambda)+2(1-\lambda)$

$$=(1-\lambda)\left[(4-\lambda)(1-\lambda)+2\right]$$

$$=(1-\lambda)\left(\lambda^2-5\lambda+6\right) \quad \Rightarrow \text{The characteristic equation: } \underline{-\lambda^3+6\lambda^2-11\lambda+6}$$

ii. $-\lambda^3+6\lambda^2-11\lambda+6=0 \quad \Rightarrow \quad$ The eigenvalues are $\boxed{\lambda=1,\ 2,\ 3}$

iii. $\lambda_1=1 \rightarrow \begin{pmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 3x_1+x_3=0 \\ x_1=0 \end{cases} \Rightarrow x_1=x_3=0$

Therefore the eigenvector $V_1=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_2=2 \rightarrow \begin{pmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x_1+x_3=0 \\ -2x_1-x_2=0 \\ -2x_1-x_3=0 \end{cases} \Rightarrow \begin{cases} x_3=-2x_1 \\ x_2=-2x_1 \end{cases}$

Therefore the eigenvector $V_2=\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$

$\lambda_3=3 \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x_1+x_3=0 \\ -2x_1-2x_2=0 \\ -2x_1-2x_3=0 \end{cases} \Rightarrow \begin{cases} x_3=-x_1 \\ x_2=-x_1 \end{cases}$

Therefore the eigenvector $V_3=\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

f) For the matrix: $\begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$

i. $\det(A-\lambda I)=\begin{vmatrix} 3-\lambda & 0 & -5 \\ \frac{1}{5} & -1-\lambda & 0 \\ 1 & 1 & -2-\lambda \end{vmatrix}$

$$=(3-\lambda)(-1-\lambda)(-2-\lambda)-1+5(-1-\lambda)$$
$$=(3-\lambda)\left(\lambda^2+3\lambda+2\right)-1-5-5\lambda$$
$$=3\lambda^2+9\lambda+6-\lambda^3-3\lambda^2-2\lambda-6-5\lambda$$
$$=-\lambda^3+2\lambda$$

$\Rightarrow$ The characteristic equation: $\underline{-\lambda^3+2\lambda}|$

ii. $-\lambda^3+2\lambda=0 \quad \Rightarrow \quad$ The eigenvalues are $\boxed{\lambda=0,\ \pm\sqrt{2}}$

iii. $\lambda_1=-\sqrt{2} \quad \to \quad \begin{pmatrix} 3+\sqrt{2} & 0 & -5 \\ \frac{1}{5} & -1+\sqrt{2} & 0 \\ 1 & 1 & -2+\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} \left(3+\sqrt{2}\right)x_1-5x_3=0 \\ \frac{1}{5}x_1+\left(-1+\sqrt{2}\right)x_2=0 \\ x_1+x_2+\left(-2+\sqrt{2}\right)x_3=0 \end{cases}$

$$\to \begin{cases} x_3=\frac{3+\sqrt{2}}{5}x_1 \\ \left(-1+\sqrt{2}\right)x_2=-\frac{1}{5}x_1 \end{cases} \quad \Rightarrow x_2=-\frac{1}{5\left(-1+\sqrt{2}\right)}x_1$$

Therefore the eigenvector $V_1=\begin{pmatrix} 1 \\ \frac{1}{5\left(1-\sqrt{2}\right)} \\ \frac{3+\sqrt{2}}{5} \end{pmatrix}$

$$\lambda_2=0 \quad \to \quad \begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} 3x_1-5x_3=0 \\ \frac{1}{5}x_1-x_2=0 \\ x_1+x_2-2x_3=0 \end{cases} \Rightarrow \begin{cases} x_3=\frac{3}{5}x_1 \\ x_2=\frac{1}{5}x_1 \end{cases}$$

Therefore the eigenvector $V_2=\begin{pmatrix} 5 \\ \frac{1}{5} \\ \frac{3}{5} \end{pmatrix}$

$$\lambda_3=\sqrt{2} \;\rightarrow\; \begin{pmatrix} 3-\sqrt{2} & 0 & -5 \\ \frac{1}{5} & -1-\sqrt{2} & 0 \\ 1 & 1 & -2-\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \left(3-\sqrt{2}\right)x_1-5x_3=0 \\ \frac{1}{5}x_1+\left(-1-\sqrt{2}\right)x_2=0 \\ x_1+x_2+\left(-2-\sqrt{2}\right)x_3=0 \end{cases}$$

$$\rightarrow \begin{cases} x_3=\frac{3-\sqrt{2}}{5}x_1 0 \\ \left(-1-\sqrt{2}\right)x_2=-\frac{1}{5}x_1 \end{cases} \Rightarrow x_2=\frac{1}{5\left(1+\sqrt{2}\right)}x_1$$

Therefore the eigenvector $V_3=\begin{pmatrix} 1 \\ \frac{1}{5\left(1+\sqrt{2}\right)} \\ \frac{3-\sqrt{2}}{5} \end{pmatrix}$

g) For the matrix: $\begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}$

i. $\det(A-\lambda I)=\begin{vmatrix} -2-\lambda & 0 & 1 \\ -6 & -2-\lambda & 0 \\ 19 & 5 & -4-\lambda \end{vmatrix}$

$$=(-2-\lambda)^2(-4-\lambda)-30-19(-2-\lambda)$$
$$=\left(4+4\lambda+\lambda^2\right)(-4-\lambda)-30+38+19\lambda$$
$$=-16-16\lambda-4\lambda^2-4\lambda-4\lambda^2-\lambda^3+8+19\lambda$$
$$=-\lambda^3-8\lambda^2-\lambda-8$$

$\Rightarrow$ The characteristic equation: $\underline{-\lambda^3-8\lambda^2-\lambda-8}$

ii. $\lambda^3+8\lambda^2+\lambda+8=0 \;\Rightarrow\; (\lambda+8)\left(\lambda^2+1\right)=0$

$\lambda^3+8\lambda^2+\lambda+8=0 \;\Rightarrow\;$ The eigenvalues are $\boxed{\lambda_{1,2,3}=-8,\ \pm i}$

iii. $\lambda_1=-8 \;\rightarrow\; \begin{pmatrix} 6 & 0 & 1 \\ -6 & 6 & 0 \\ 19 & 5 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 6 & 0 & 1 \\ -6 & 6 & 0 \\ 19 & 5 & 4 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & \frac{1}{6} \\ 0 & 1 & \frac{1}{6} \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + \frac{1}{6} z_1 = 0 \\ y_1 + \frac{1}{6} z_1 = 0 \end{cases}$$

Therefore the eigenvector $V_1 = \begin{pmatrix} -\frac{1}{6} \\ -\frac{1}{6} \\ 1 \end{pmatrix}$

For $\lambda_2 = -i \quad \rightarrow \quad \begin{pmatrix} -2+i & 0 & 1 \\ -6 & -2+i & 0 \\ 19 & 5 & -4+i \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\rightarrow \begin{cases} (-2+i)x_2 + z_2 = 0 \\ -6x_2 + (-2+i)y_2 = 0 \\ 19x_2 + 5y_2 + (-4+i)z_2 = 0 \end{cases}$$

Therefore the eigenvector $V_2 = \begin{pmatrix} 1 \\ -\frac{12}{5} - i\frac{6}{5} \\ 2-i \end{pmatrix}$

For $\lambda_3 = i \quad \rightarrow \quad \begin{pmatrix} -2-i & 0 & 1 \\ -6 & -2-i & 0 \\ 19 & 5 & -4-i \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{cases} (-2-i)x_2 + z_2 = 0 \\ -6x_2 + (-2-i)y_2 = 0 \\ 19x_2 + 5y_2 + (-4-i)z_2 = 0 \end{cases}$$

Therefore the eigenvector $V_2 = \begin{pmatrix} 1 \\ -\frac{12}{5} + i\frac{6}{5} \\ 2+i \end{pmatrix}$

h) For the matrix: $\begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

i. $\det(A-\lambda I)=\begin{pmatrix}-\lambda & 0 & 2 & 0\\ 1 & -\lambda & 1 & 0\\ 0 & 1 & -2-\lambda & 0\\ 0 & 0 & 0 & 1-\lambda\end{pmatrix}=(1-\lambda)\begin{vmatrix}-\lambda & 0 & 2\\ 1 & -\lambda & 1\\ 0 & 1 & -2-\lambda\end{vmatrix}$

$$=(1-\lambda)\left(\lambda^2(-2-\lambda)+2+\lambda\right)$$

$$=(1-\lambda)\left(-\lambda^3-2\lambda^2+\lambda+2\right)$$

$$=\lambda^4+\lambda^3-3\lambda^2-\lambda+2$$

$\Rightarrow$ The characteristic equation: $\lambda^4+\lambda^3-3\lambda^2-\lambda+2$

ii. $\lambda^4+\lambda^3-3\lambda^2-\lambda+2=0 \quad\Rightarrow\quad$ The eigenvalues are $\boxed{\lambda=-2,-1,1,1}$

iii. $\lambda_1=-2 \quad\rightarrow\quad \begin{pmatrix}2 & 0 & 2 & 0\\ 1 & 2 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\rightarrow\begin{cases}2x_1+2x_3=0\\ x_1+2x_2+x_3=0\\ x_2=0\\ x_4=0\end{cases}$

$$\rightarrow\begin{cases}x_1=-x_3\\ x_1=-x_3\\ x_2=0\\ x_4=0\end{cases}$$

Therefore the eigenvector $V_1=\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}$

$\lambda_2=-1 \quad\rightarrow\quad \begin{pmatrix}1 & 0 & 2 & 0\\ 1 & 1 & 1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\rightarrow\begin{cases}x_1+2x_3=0\\ x_1+x_2+x_3=0\\ x_2-x_3=0\\ x_4=0\end{cases}$

$$\rightarrow\begin{cases}x_1=-2x_3\\ x_1=-x_2-x_3\\ x_2=x_3\end{cases}$$

Therefore the eigenvector $V_2=\begin{pmatrix}-2\\1\\1\\0\end{pmatrix}$

$$\lambda_3 = 1 \quad \rightarrow \quad \begin{pmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -x_1 + 2x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_2 - 3x_3 = 0 \end{cases}$$

$$\rightarrow \begin{cases} x_1 = 2x_3 \\ x_1 = x_2 - x_3 \\ x_2 = 3x_3 \end{cases}$$

Therefore the eigenvector $V_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_4 = 1 \quad \rightarrow$ Therefore the eigenvector $V_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

i) For the matrix: $\begin{pmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

i. $\det(A - \lambda I) = \begin{pmatrix} 10-\lambda & -9 & 0 & 0 \\ 4 & -2-\lambda & 0 & 0 \\ 0 & 0 & -2-\lambda & -7 \\ 0 & 0 & 1 & 2-\lambda \end{pmatrix}$

$$= (10-\lambda)\begin{vmatrix} -2-\lambda & 0 & 0 \\ 0 & -2-\lambda & -7 \\ 0 & 1 & 2-\lambda \end{vmatrix} + 9\begin{vmatrix} 4 & 0 & 0 \\ 0 & -2-\lambda & -7 \\ 0 & 1 & 2-\lambda \end{vmatrix}$$

$$= (10-\lambda)\left[(-2-\lambda)^2(2-\lambda) + 7(-2-\lambda)\right] + 9\left[(4)(-2-\lambda)(2-\lambda) + 28\right]$$

$$= (10-\lambda)(-2-\lambda)\left(3+\lambda^2\right) + 9\left(4\lambda^2 + 12\right)$$

$$= \left(3+\lambda^2\right)\left(-8\lambda + \lambda^2 + 16\right)$$

$$= \left(3+\lambda^2\right)(\lambda - 4)^2$$

$\Rightarrow$ The characteristic equation: $\left(3+\lambda^2\right)(\lambda-4)^2$

ii. $\left(3+\lambda^2\right)(\lambda-4)^2 = 0 \quad \Rightarrow$ The eigenvalues are $\boxed{\lambda = 4,\ 4,\ \pm i\sqrt{3}}$

iii. $\lambda_1 = 4 \rightarrow \begin{pmatrix} 6 & -9 & 0 & 0 \\ 4 & -6 & 0 & 0 \\ 0 & 0 & -6 & -7 \\ 0 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 6x_1 - 9x_2 = 0 \\ 4x_1 - 6x_2 = 0 \\ -6x_3 - 7x_4 = 0 \\ x_3 - 2x_4 = 0 \end{cases}$

$$\begin{cases} 6x_1 = 9x_2 \\ 6x_3 = -7x_4 \\ x_3 = 2x_4 \end{cases} \Rightarrow \begin{matrix} x_1 = \frac{3}{2}x_2 \\ x_3 = x_4 = 0 \end{matrix}$$

Therefore the eigenvector $V_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda_2 = 4 \rightarrow$ Therefore the eigenvector $V_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\lambda_3 = -i\sqrt{3} \rightarrow \begin{pmatrix} 10+i\sqrt{3} & -9 & 0 & 0 \\ 4 & -2+i\sqrt{3} & 0 & 0 \\ 0 & 0 & -2+i\sqrt{3} & -7 \\ 0 & 0 & 1 & 2+i\sqrt{3} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} \left(10+i\sqrt{3}\right)x_1 - 9x_2 = 0 \\ 4x_1 + \left(-2+i\sqrt{3}\right)x_2 = 0 \\ \left(-2+i\sqrt{3}\right)x_3 - 7x_4 = 0 \\ x_3 + \left(2+i\sqrt{3}\right)x_4 = 0 \end{cases}$$

$$\rightarrow \begin{cases} \left(10+i\sqrt{3}\right)x_1 = 9x_2 \\ 4x_1 = -\left(-2+i\sqrt{3}\right)x_2 \\ \left(-2+i\sqrt{3}\right)x_3 = 7x_4 \\ x_3 = -\left(2+i\sqrt{3}\right)x_4 \end{cases} \Rightarrow \begin{cases} x_1 = x_2 = 0 \\ x_3 = \frac{7}{-2+i\sqrt{3}}x_4\left(\frac{-2-i\sqrt{3}}{-2-i\sqrt{3}}\right) = -\left(2+i\sqrt{3}\right)x_4 \\ x_3 = -\left(2+i\sqrt{3}\right)x_4 \end{cases}$$

Therefore the eigenvector $V_3 = \begin{pmatrix} 0 \\ 0 \\ -\left(2+i\sqrt{3}\right) \\ 1 \end{pmatrix}$

$$\lambda_4 = i\sqrt{3} \quad \rightarrow \quad \begin{pmatrix} 10-i\sqrt{3} & -9 & 0 & 0 \\ 4 & -2-i\sqrt{3} & 0 & 0 \\ 0 & 0 & -2-i\sqrt{3} & -7 \\ 0 & 0 & 1 & 2-i\sqrt{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} \left(10-i\sqrt{3}\right)x_1 - 9x_2 = 0 \\ 4x_1 + \left(-2-i\sqrt{3}\right)x_2 = 0 \\ \left(-2-i\sqrt{3}\right)x_3 - 7x_4 = 0 \\ x_3 + \left(2-i\sqrt{3}\right)x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_2 = 0 \\ x_3 = \dfrac{7}{-2-i\sqrt{3}}x_4 \left(\dfrac{-2+i\sqrt{3}}{-2+i\sqrt{3}}\right) = \left(-2+i\sqrt{3}\right)x_4 \\ x_3 = -\left(2-i\sqrt{3}\right)x_4 \end{cases}$$

Therefore the eigenvector $V_4 = \begin{pmatrix} 0 \\ 0 \\ -2+i\sqrt{3} \\ 1 \end{pmatrix}$

j) For the matrix $\begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$

i. $\begin{vmatrix} -1-\lambda & 0 & 1 \\ -1 & 3-\lambda & 0 \\ -4 & 13 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2(3-\lambda) - 13 + 4(3-\lambda)$

$$= \left(\lambda^2 + 2\lambda + 1\right)(3-\lambda) - 13 + 12 - 4\lambda$$

$$= 3\lambda^2 + 6\lambda + 3 - \lambda^3 - 2\lambda^2 - \lambda - 1 - 4\lambda$$

$$= -\lambda^3 + \lambda^2 + \lambda + 2$$

The characteristic equation: $\underline{-\lambda^3 + \lambda^2 + \lambda + 2 = 0}$

ii. $\rightarrow$ *The eigenvalues* $\underline{\lambda_{1,2,3} = 2,\ -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}}$

iii. For $\lambda_1 = 2$, we have: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -3 & 0 & 1 \\ -1 & 1 & 0 \\ -4 & 13 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} -3x_1 + z_1 = 0 \\ -x_1 + y_1 = 0 \\ -4x_1 + 13y_1 - 3z_1 = 0 \end{cases} \quad \Rightarrow \begin{cases} z_1 = 3x_1 \\ y_1 = x_1 \end{cases}$$

If we let $x_1 = 1$; therefore the eigenvector $V_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

For $\lambda_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, we have: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -\frac{1}{2}+i\frac{\sqrt{3}}{2} & 0 & 1 \\ -1 & \frac{7}{2}+i\frac{\sqrt{3}}{2} & 0 \\ -4 & 13 & -\frac{1}{2}+i\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x_2 + z_2 = 0 \\ -x_2 + \left(\frac{7}{2}+i\frac{\sqrt{3}}{2}\right)y_2 = 0 \\ -4x_2 + 13y_2 + \left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)z_2 = 0 \end{cases}$$

$$\rightarrow \begin{cases} z_2 = -\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x_2 \\ y_2 = \left(\frac{2}{7+i\sqrt{3}}\right)x_2 \end{cases}$$

If we let $x_2 = 1$; therefore the eigenvector $V_2 = \begin{pmatrix} 1 \\ \frac{1-i\sqrt{3}}{2} \\ \frac{2}{7+i\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1-i\sqrt{3}}{2} \\ \frac{7-i\sqrt{3}}{26} \end{pmatrix}$

For $\lambda_3 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, we have: $\left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -\frac{1}{2}-i\frac{\sqrt{3}}{2} & 0 & 1 \\ -1 & \frac{7}{2}-i\frac{\sqrt{3}}{2} & 0 \\ -4 & 13 & -\frac{1}{2}-i\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)x_3 + z_3 = 0 \\ -x_3 + \left(\frac{7}{2}-i\frac{\sqrt{3}}{2}\right)y_3 = 0 \\ -4x_3 + 13y_3 + \left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)z_3 = 0 \end{cases}$$

$$\rightarrow \begin{cases} z_3 = -\left(\frac{1+i\sqrt{3}}{2}\right)x_3 \\ y_3 = \left(\frac{2}{7-i\sqrt{3}}\right)x_3 \end{cases}$$

If we let $x_3 = 1$; therefore the eigenvector $V_3 = \begin{pmatrix} 1 \\ \frac{1+i\sqrt{3}}{2} \\ \frac{2}{7-i\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1+i\sqrt{3}}{2} \\ \frac{7+i\sqrt{3}}{26} \end{pmatrix}$

Exercise

Find the eigenvalues of A^9 for $A = \begin{pmatrix} 1 & 3 & 7 & 11 \\ 0 & \frac{1}{2} & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

Solution

The eigenvalues are: $\lambda = 1,\ \frac{1}{2},\ 0,\ 2$

The eigenvalues of A^9 are: $1^9 = 1 \quad \left(\frac{1}{2}\right)^9 = \frac{1}{512} \quad 0^9 = 0 \quad 2^9 = 512$

Exercise

Find the eigenvalues of the matrices

$$A = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0.61 & 0.52 \\ 0.39 & 0.48 \end{pmatrix}, \quad A^\infty = \begin{pmatrix} 0.57143 & 0.57143 \\ 0.42857 & 0.42857 \end{pmatrix}, \quad and \quad B = \begin{pmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{pmatrix}$$

Solution

The eigenvalues for A:

$$\begin{vmatrix} 0.7-\lambda & 0.4 \\ 0.3 & 0.6-\lambda \end{vmatrix} = (0.7-\lambda)(0.6-\lambda) - .12$$

$$= \lambda^2 - 1.3\lambda + .3 = 0 \qquad \lambda_{1,2} = 0.65 \pm 0.35$$

The eigenvalues are: $\lambda_1 = 1 \quad \lambda_2 = 0.3$

The eigenvalues for A^2: $\lambda_1 = 1^2 = 1 \quad \lambda_2 = 0.3^2 = 0.09$

The eigenvalues for A^∞: $\lambda^2 - \lambda = 0 \quad \lambda_1 = 1 \quad \lambda_2 = 0.3^\infty = 0$

The eigenvalues for B:

$$\begin{vmatrix} 0.3-\lambda & 0.6 \\ 0.7 & 0.4-\lambda \end{vmatrix} = \lambda^2 - .7\lambda - .3 = 0 \qquad \lambda_{1,2} = 0.35 \pm 0.65$$

The eigenvalues are: $\lambda_1 = 1 \quad \lambda_2 = -0.3$

Exercise

Given the matrix $\begin{bmatrix} -1 & -3 \\ -3 & 7 \end{bmatrix}$

a) Find the characteristic polynomial.
b) Find the eigenvalues
c) Find the bases for its eigenspaces
d) Graph the eigenspaces
e) Verify directly that $A\mathbf{v} = \lambda\mathbf{v}$, for all associated eigenvectors and eigenvalues.

Solution

a) $\begin{vmatrix} -1-\lambda & -3 \\ -3 & 7-\lambda \end{vmatrix} = (-1-\lambda)(7-\lambda)-9$

$$= -7-6\lambda+\lambda^2-9$$
$$= \lambda^2-6\lambda-16$$

The characteristic polynomial is $\lambda^2-6\lambda-16=0$

b) $\lambda^2-6\lambda-16=0 \quad \Rightarrow \quad \lambda_1=-2 \quad \textit{and} \quad \lambda_2=8$

c) For $\lambda_1=-2$, we have: $(A+2I)V_1=0$

$$\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} x_1-3y_1=0 \\ -3x_1+9y_1=0 \end{cases} \quad \Rightarrow x_1=3y_1$$

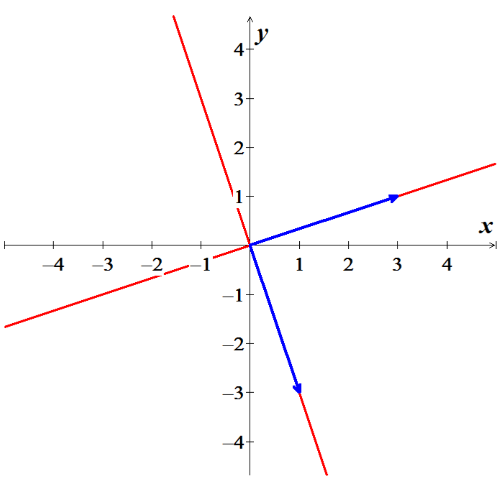
Therefore the eigenvector $V_1=\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

For $\lambda_2=8$, we have: $(A+8I)V_2=0$

$$\begin{pmatrix} -9 & -3 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} -9x_2-3y_2=0 \\ -3x_2-y_2=0 \end{cases} \quad \Rightarrow y_2=-3x_2$$

Therefore the eigenvector $V_2=\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

d)



e) $AV_1 = \lambda V_1 \quad \to \quad \begin{pmatrix} -1 & -3 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = -2\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\begin{pmatrix} -6 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \end{pmatrix} \checkmark$

$AV_2 = \lambda V_2 \quad \to \quad \begin{pmatrix} -1 & -3 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = 8\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$\begin{pmatrix} 8 \\ -24 \end{pmatrix} = \begin{pmatrix} -6 \\ -24 \end{pmatrix} \checkmark$

Exercise

Given the matrix $\begin{bmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{bmatrix}$

a) Find the characteristic polynomial.
b) Find the eigenvalues
c) Find the bases for its eigenspaces
d) Graph the eigenspaces
e) Verify directly that $A\mathbf{v} = \lambda\mathbf{v}$, for all associated eigenvectors and eigenvalues.

Solution

a)
$$\begin{vmatrix} 5-\lambda & 0 & -4 \\ 0 & -3-\lambda & 0 \\ -4 & 0 & -1-\lambda \end{vmatrix} = (5-\lambda)(-3-\lambda)(-1-\lambda) - 16(-3-\lambda)$$
$$= (5-\lambda)\left(3+4\lambda+\lambda^2\right) + 48 + 16\lambda$$
$$= 15 + 20\lambda + 5\lambda^2 - 3\lambda - 4\lambda^2 - \lambda^3 + 48 + 16\lambda$$
$$= -\lambda^3 + \lambda^2 + 33\lambda + 63$$

The characteristic polynomial is $\underline{-\lambda^3 + \lambda^2 + 33\lambda + 63 = 0}$

b) $-\lambda^3 + \lambda^2 + 33\lambda + 63 = 0 \quad \Rightarrow \quad \lambda = -3,\ -3,\ 7$

c) For $\lambda_{1,2} = -3$, we have: $(A+3I)V_1 = 0$

$$\begin{pmatrix} 8 & 0 & -4 \\ 0 & 0 & 0 \\ -4 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} 8x_1 - 4z_1 = 0 \\ -4x_1 + 2z_1 = 0 \end{cases} \quad \Rightarrow z_1 = 2x_1$$

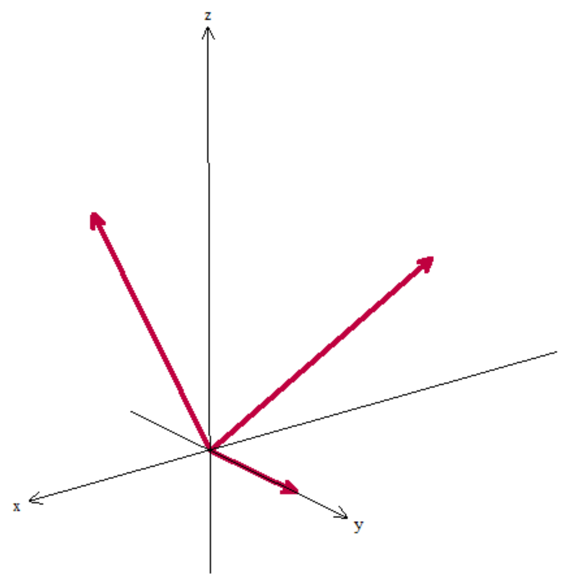
Therefore the eigenvector $V_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ *and* $V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_3 = 7$, we have: $(A - 7I)V_3 = 0$

$$\begin{pmatrix} -2 & 0 & -4 \\ 0 & -10 & 0 \\ -4 & 0 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -2x_1 - 4z_1 = 0 \\ -10y_1 = 0 \\ -4x_1 - 8z_1 = 0 \end{cases} \Rightarrow x_1 = -2z_1 \text{ and } y_1 = 0$$

Therefore the eigenvector $V_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

d)



e) $AV_1 = \lambda V_1 \quad \rightarrow \quad \begin{pmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} -3 \\ 0 \\ -6 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ -6 \end{pmatrix} \checkmark$$

$AV_2 = \lambda V_2 \quad \rightarrow \quad \begin{pmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} \checkmark$$

$$AV_3 = \lambda V_3 \quad \to \quad \begin{pmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 7\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -14 \\ 0 \\ 7 \end{pmatrix} = \begin{pmatrix} -14 \\ 0 \\ 7 \end{pmatrix} \quad \surd$$

Exercise

Given: $A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$. Compute A^{11}

Solution

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 7 & -1 \\ 0 & 1-\lambda & 0 \\ 0 & 15 & -2-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(1-\lambda)(-2-\lambda)$$

The eigenvalues are: $-1,\ 1,\ -2$

For $\lambda_1 = -1$, we have: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 7 & -1 \\ 0 & 2 & 0 \\ 0 & 15 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} 7y_1 - z_1 = 0 \\ 2y_1 = 0 \\ 15y_1 - z_1 = 0 \end{cases} \quad \Rightarrow \begin{cases} z_1 = 7y_1 \\ y_1 = 0 \end{cases}$$

The eigenvector $V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

For $\lambda_2 = 1$, we have: $(A - I)V_2 = 0$

$$\begin{pmatrix} -2 & 7 & -1 \\ 0 & 0 & 0 \\ 0 & 15 & -3 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} -2x_2 + 7y_2 - z_2 = 0 \\ \\ 15y_2 - 3z_2 = 0 \end{cases} \quad \Rightarrow \begin{cases} 2x_2 = 7y_2 - z_2 \\ \\ 5y_2 = z_2 \end{cases}$$

If we let $y_2 = 1 \ \to\ z_2 = 5$ and $x_2 = \frac{7-5}{2} = 1;$

The eigenvector $V_2 = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$

For $\lambda_3 = -2$, we have: $(A+2I)V_3 = 0$

$$\begin{pmatrix} 1 & 7 & -1 \\ 0 & 3 & 0 \\ 0 & 15 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x_3 + 7y_3 - z_3 = 0 \\ 3y_3 = 0 \\ 15y_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = -7y_3 + z_3 \\ y_3 = 0 \end{cases}$$

The eigenvector $V_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \Rightarrow D^{11} = \begin{pmatrix} (-1)^{11} & 0 & 0 \\ 0 & 1^{11} & 0 \\ 0 & 0 & (-2)^{11} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2048 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \Rightarrow P^{-1} = \begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

$$A^{11} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2048 \end{pmatrix}\begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & -2048 \\ 0 & 1 & 0 \\ 0 & 5 & -2048 \end{pmatrix}\begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 10237 & 2047 \\ 0 & 1 & 0 \\ 0 & 10245 & -2048 \end{pmatrix}$$