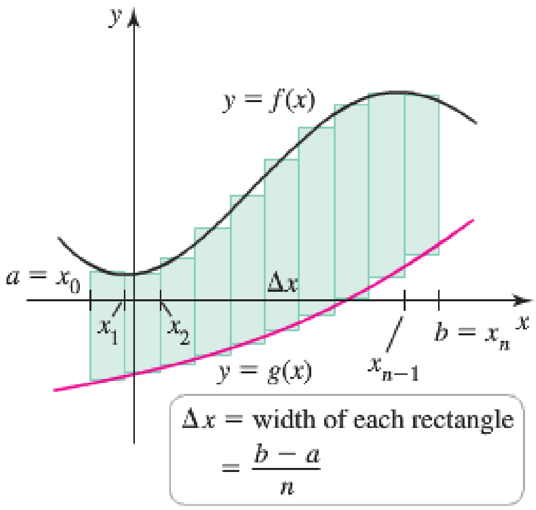
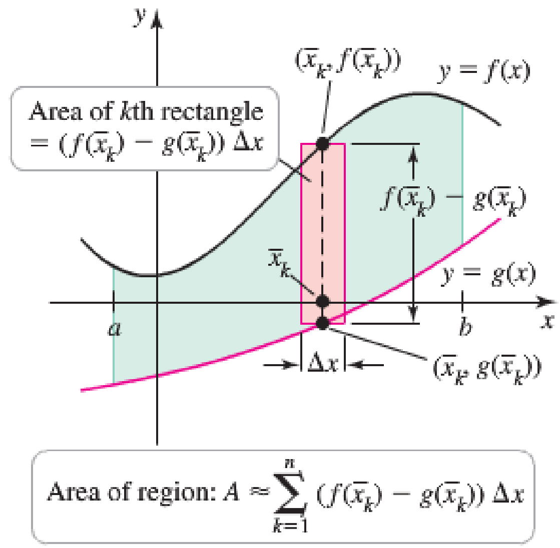
Section 1.2 – Region between Curves

Areas between Curves





Definition

If f and g are continuous with $f(x) \ge g(x)$ throughout $[a, b]$, then the ***area of the region between the curves*** $y = f(x)$ *and* $y = g(x)$ **from *a* to *b*** is:

$$A = \int_a^b \left[f(x) - g(x) \right] dx$$

Example

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution

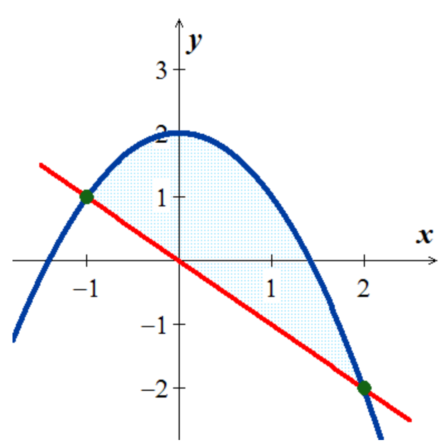
The limits of integrations are found by letting:

$2 - x^2 = -x \qquad \Rightarrow \; x^2 - x - 2 = 0 \quad \rightarrow \quad \underline{x = -1,\ 2}$

$$A = \int_{-1}^{2} \left[f(x) - g(x) \right] dx$$

$$= \int_{-1}^{2} \left[2 - x^2 - (-x) \right] dx$$

$$= \int_{-1}^{2} \left(2 - x^2 + x \right) dx$$



$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^{2}$$

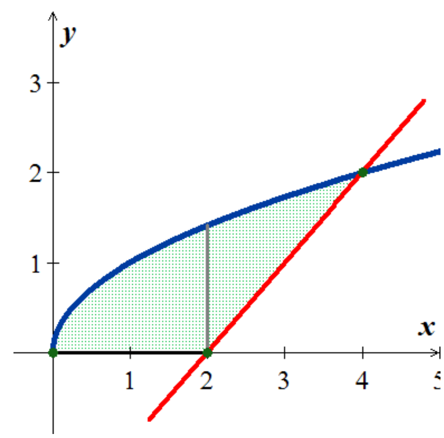
$$= \left(4 - \frac{8}{3} + \frac{4}{2}\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right)$$

$$= \frac{9}{2} \ \ unit^2$$

Example

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below the x-axis and the line $y = x - 2$

Solution



$$\left(y = \sqrt{x}\right) \cap \left(y = 0\right) \quad \to \quad (0,\ 0)$$

$$\left(y = \sqrt{x}\right) \cap \left(y = x - 2\right) \quad \to \quad \sqrt{x} = x - 2$$

$$\left(\sqrt{x}\right)^2 = (x-2)^2$$

$$x = x^2 - 4x + 4$$

$$x^2 - 5x + 4 = 0$$

$$\to \quad x = \cancel{1},\ 4$$

$$\left(y = 0\right) \cap \left(y = x - 2\right) \quad \to \quad x = 2$$

$$\text{Total Area } = \int_0^2 \left[\sqrt{x} - 0\right] dx + \int_2^4 \left[\sqrt{x} - (-x+2)\right] dx$$

$$= \left[\frac{2}{3} x^{3/2}\right]_0^2 + \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

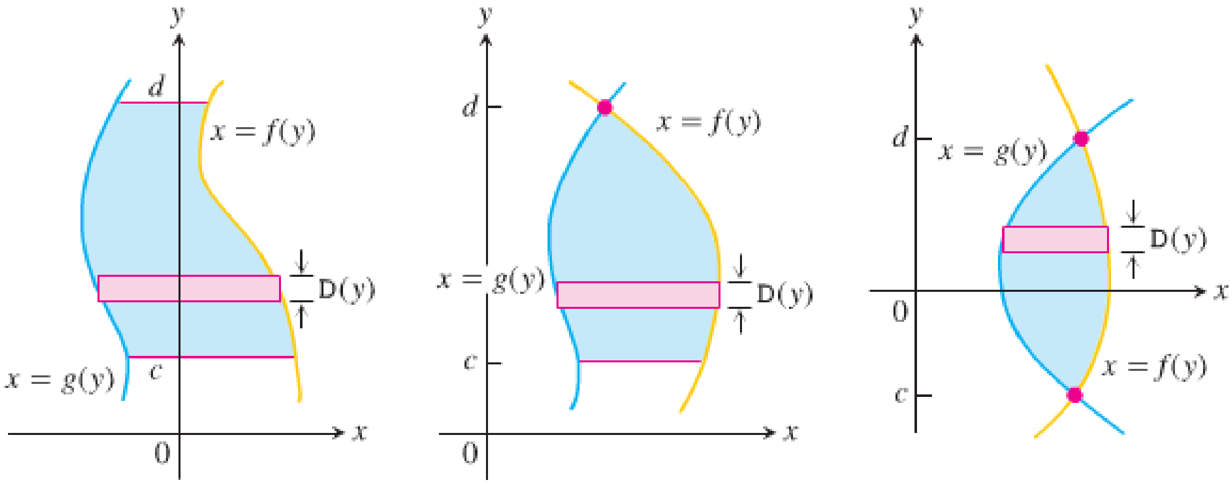
$$= \left[\frac{2}{3}\left(2^{3/2}\right) - 0\right] + \left(\frac{2}{3} 4^{3/2} - \frac{4^2}{2} + 2(4)\right) - \left(\frac{2}{3} 2^{3/2} - \frac{2^2}{2} + 2(2)\right)$$

$$= \frac{2}{3}\left(2^{3/2}\right) + \frac{2}{3} 4^{3/2} - \frac{16}{2} + 8 - \frac{2}{3} 2^{3/2} + \frac{4}{2} - 4$$

$$= \frac{2}{3}(8) - 2$$

$$= \frac{10}{3} \ \ unit^2$$

Integration with Respect to *y*



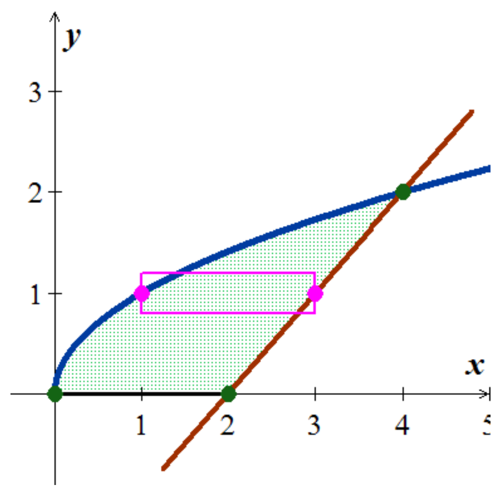
$$A=\int_{c}^{d}\left[f(y)-g(y)\right]dy$$

(*From right hand to left* hand)

Example

Find the area of the region by integrating with respect to y, in the first quadrant that is bounded above by $y=\sqrt{x}$ and below the *x*-axis and the line $y=x-2$.

Solution



$$y=\sqrt{x} \quad\to\quad x=y^2$$
$$y=x-2 \quad\to\quad x=y+2$$

$$\left(x=y^2\right)\cap\left(y=0\right) \quad\to\quad (0,\ 0)$$
$$\left(x=y^2\right)\cap\left(x=y+2\right) \quad\to\quad y^2=y+2$$
$$y^2-y-2=0 \quad\to\quad y=-1,\ 2$$
$$\left(y=0\right)\cap\left(x=y+2\right) \quad\to\quad y=0$$

$$A=\int_{0}^{2}\left[y+2-y^2\right]dy$$
$$=\left[\frac{y^2}{2}+2y-\frac{y^3}{3}\right]_0^2$$
$$=\frac{2^2}{2}+2(2)-\frac{2^3}{3}-0$$
$$=\frac{10}{3}\ \ unit^2$$

Exercises *Section* 1.2 – Region between Curves

Find the area of the region bounded by the graphs of

1. $y = 2x - x^2$ and $y = -3$

2. $y = 7 - 2x^2$ and $y = x^2 + 4$

3. $y = x^4 - 4x^2 + 4$ and $y = x^2$

4. $x = 2y^2, \quad x = 0, \quad and \quad y = 3$

5. $x = y^3 - y^2 \quad and \quad x = 2y$

6. $4x^2 + y = 4 \quad and \quad x^4 - y = 1$

7. $y = \sin\frac{\pi x}{2} \quad and \quad y = x$

8. $y = 3 - x^2$ *and* $y = 2x$

9. $y = x^2 - x - 2$ *and* x-axis

10. $y = \sqrt{x}, \quad y = x\sqrt{x}$

11. $y = x^{1/2} \quad and \quad y = x^3$

12. $x + 4y^2 = 4, \quad x + y^4 = 1, \quad x \ge 0$

13. $y = 2\sin x, \quad y = \sin 2x, \quad 0 \le x \le \pi$

14. $y = x^2 + 1$ *and* $y = x$ for $0 \le x \le 2$

15. $y = x^2 - 2x$ *and* $y = x$ on [0, 4]

16. $x = 1, \quad x = 2, \quad y = x^3 + 2, \quad y = 0$

17. $y = x^2 - 18, \quad y = x - 6$

18. $y = -x^2 + 3x + 1, \quad y = -x + 1$

19. $y = x, \quad y = 2 - x, \quad y = 0$

20. $y = \frac{4}{x^2}, \quad y = 0, \quad x = 1, \quad x = 4$

21. $f(y) = y^2, \quad g(y) = y + 2$

22. $f(x) = 2^x, \quad g(x) = \frac{3}{2}x + 1$

23. $x = \sqrt[3]{y} \quad and \quad x = \sqrt[5]{y}$

24. $f(x) = x^3 + 2x^2 - 3x, \quad g(x) = x^2 + 3x$

25. $y = \sec^2 x, \quad y = \tan^2 x, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$

26. $f(x) = -x^2 + 1, \quad g(x) = 2x + 4, \quad x = -1, \ x = 2$

27. $f(x) = \sqrt{x} + 3, \quad g(x) = \frac{1}{2}x + 3$

28. $f(x) = \sqrt[3]{x-1}, \quad g(x) = x - 1$

29. $f(y) = y(2 - y), \quad g(y) = -y$

30. $f(y) = \dfrac{y}{\sqrt{16 - y^2}}, \quad g(y) = 0, \quad y = 3$

31. $f(y) = y^2 + 1, \quad g(y) = 0, \quad y = -1, \quad y = 2$

32. $f(x) = \frac{10}{x}, \quad x = 0, \quad y = 2, \quad y = 10$

33. $g(x) = \frac{4}{2 - x}, \quad y = 4, \quad x = 0$

34. $f(x) = \cos x, \quad g(x) = 2 - \cos x, \quad 0 \le x \le 2\pi$

35. $f(x) = \sin x, \quad g(x) = \cos 2x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{6}$

36. $f(x) = 2\sin x, \quad g(x) = \tan x, \quad -\frac{\pi}{3} \le x \le \frac{\pi}{3}$

37. $f(x) = \sec\frac{\pi x}{4}\tan\frac{\pi x}{4}, \quad g(x) = \left(\sqrt{2} - 4\right)x + 4, \quad x = 0$

38. $f(x) = xe^{-x^2}, \quad y = 0, \quad 0 \le x \le 1$

39. $y = \sin x$ *and* $y = x \quad 0 \le x \le 2\pi$

40. $y = x^2, \ y = 2x^2 - 4x$ and $y = 0$

41. $y = 8\cos x, \quad y = \sec^2 x, \quad -\frac{\pi}{3} \le x \le \frac{\pi}{3}$

42. $y^2 = 4x + 4, \quad y = 4x - 16$

43. $x = 2y^2, \quad x = 0, \quad y = 3$

44. $x = y^3 \quad and \quad y = x$

45. Find the area of the region in the first quadrant bounded by $y = 4x$ *and* $y = x\sqrt{25 - x^2}$

46. Find the area of the region in the first quadrant bounded by the curve $\sqrt{x} + \sqrt{y} = 1$

47. Find the area of the region in the first quadrant bounded by $y = \frac{x}{6}$ and $y = 1 - \left|\frac{x}{2} - 1\right|$

48. Find the area of the region in the first quadrant bounded by $y = x^p$ and $y = \sqrt[p]{x}$ where $p = 100$ and $p = 1000$

49. Consider the functions $y = \frac{x^2}{a}$ and $y = \sqrt{\frac{x}{a}}$, where $a > 0$. Find $A(a)$, the area of the region between the curves.

50. Find the area between the curves $y = \ln x$ and $y = \ln 2x$ from $x = 1$ to $x = 5$.

51. Find the total area of the region enclosed by the curve $x = y^{2/3}$ and lines $x = y$ and $y = -1$.

52. Find the area of the "triangular region in the first quadrant bounded on the left by the *y-axis* and on the right by the curves $\sin x$ *and* $\cos x$.

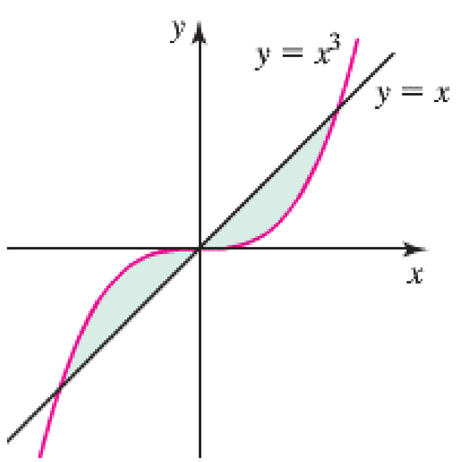
53. Find the area of the "triangular region in the first quadrant bounded above by the curve $y = e^{2x}$, below by the curve $y = e^x$, and on the right by the line $x = \ln 3$.

54. Find the area of the triangular region bounded on the left by $x + y = 2$, on the right by $y = x^2$, and above by $y = 2$

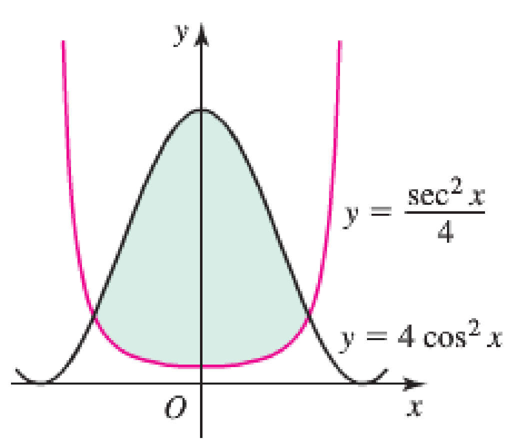
55. Find the extreme values of $f(x) = x^3 - 3x^2$ and find the area of the region enclosed by the graph of *f* and the *x*-axis.

(56 – 59) Determine the area of the shaded region in the following

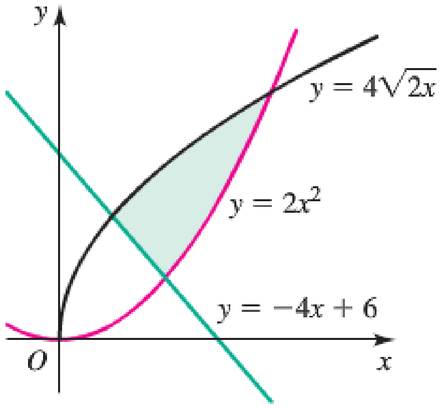
56.



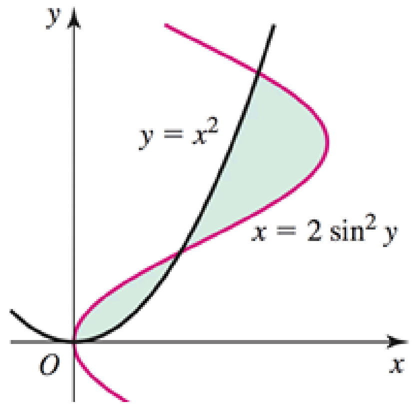
57.



58.

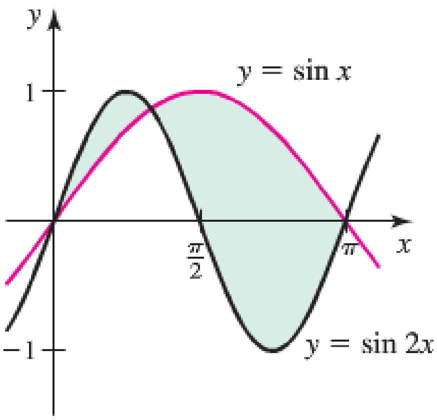


59.

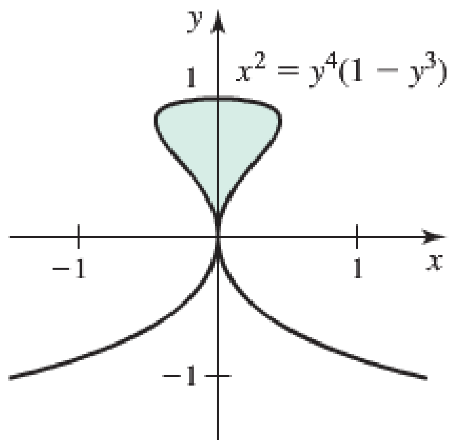


(**60 – 71**) Determine the area of the shaded regions

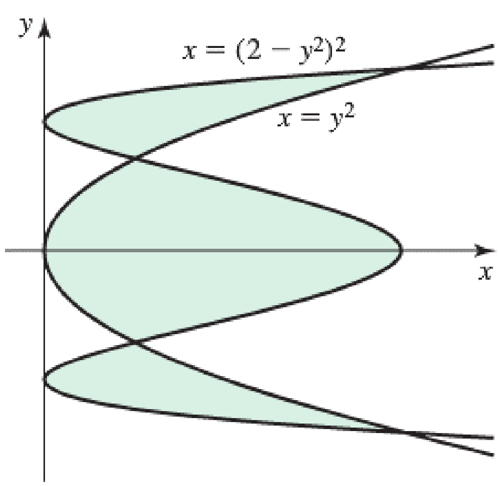
60. $y = \sin x$ *and* $y = \sin 2x$, *for* $0 \le x \le \pi$



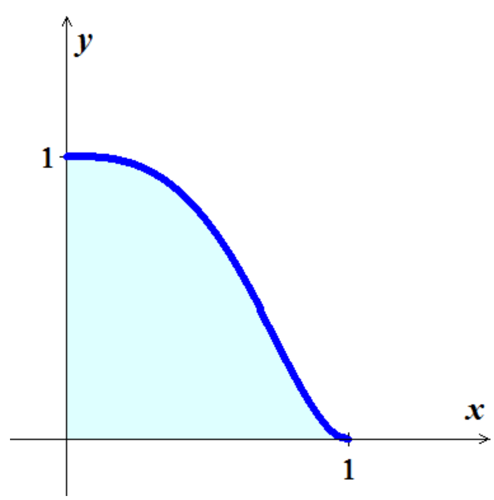
61. Bounded by $x^2 = y^4\left(1 - y^3\right)$



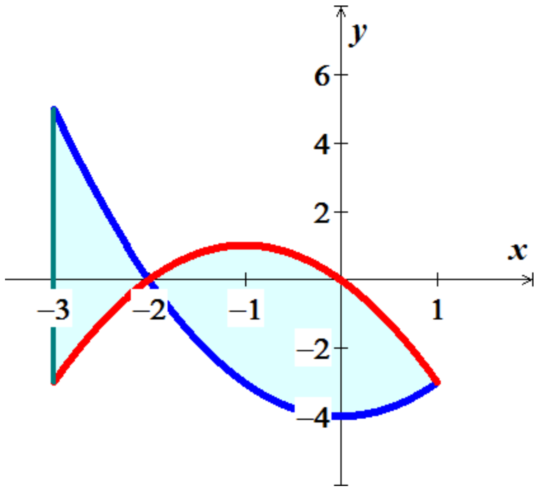
62. bounded by $x = y^2$ and $x = \left(2 - y^2\right)^2$



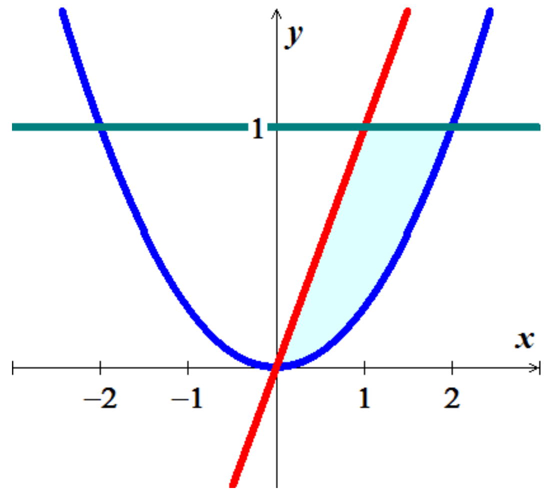
63. $x^3 + \sqrt{y} = 1, \quad x = 0, \quad y = 0, \quad 0 \le x \le 1$



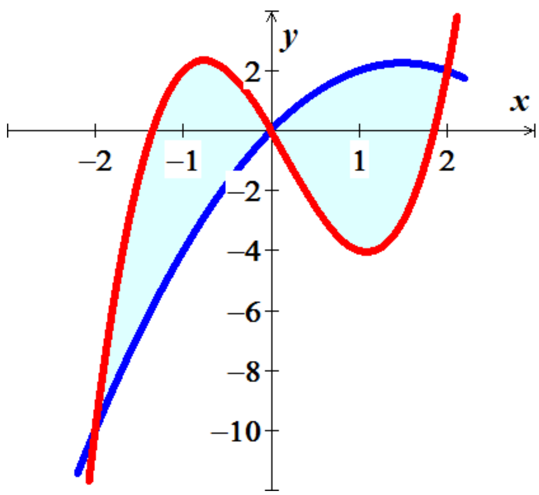
64. $y = x^2 - 4, \quad y = -x^2 - 2x, \quad -3 \le x \le 1$



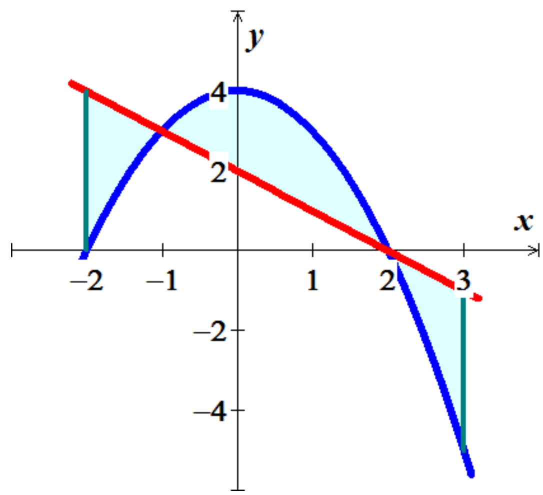
65. $y = \frac{1}{4}x^2, \quad y = x, \quad y = 1$



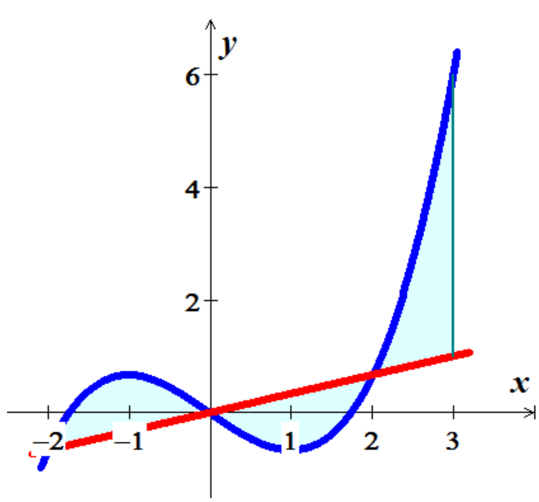
66. $y = -x^2 + 3x, \quad y = 2x^3 - x^2 - 5x, \quad -2 \le x \le 2$



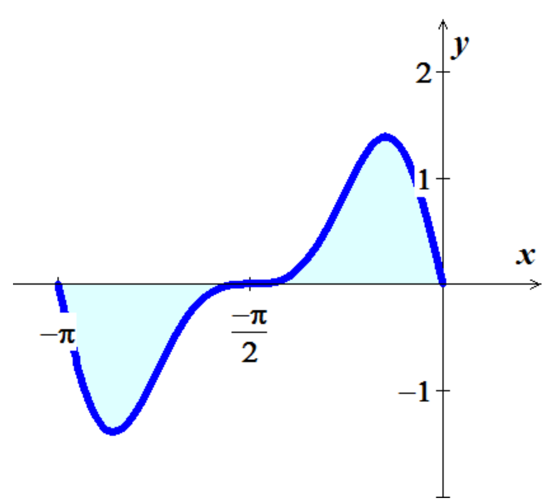
67. $y = 4 - x^2, \quad y = -x + 2, \quad -2 \le x \le 3$



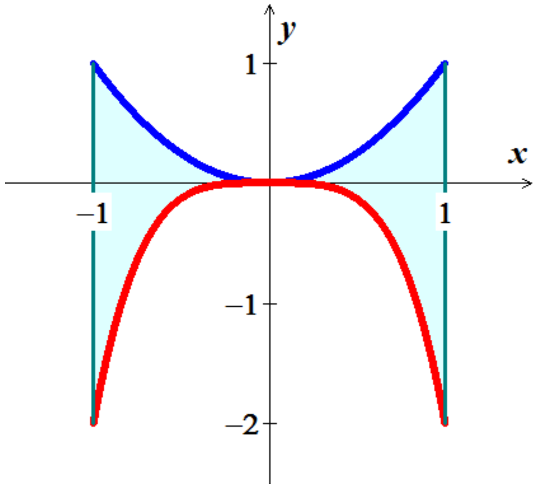
68. $y = \frac{1}{3}x^3 - x, \quad y = \frac{1}{3}x, \quad -2 \le x \le 3$



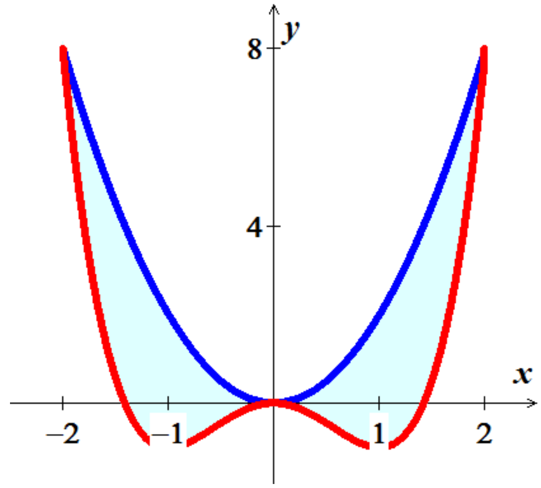
69. $y = \frac{\pi}{2}\cos x \ \sin\left(\pi + \pi \sin x\right) \quad -\pi \le x \le 0$



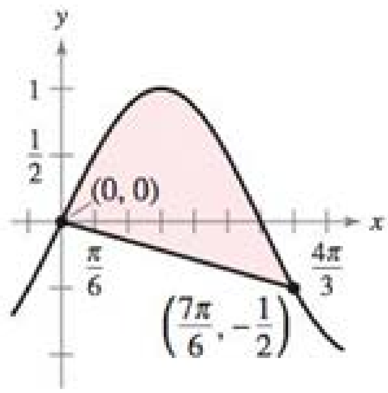
70. $y = x^2, \quad y = -2x^4, \quad -1 \le x \le 1$



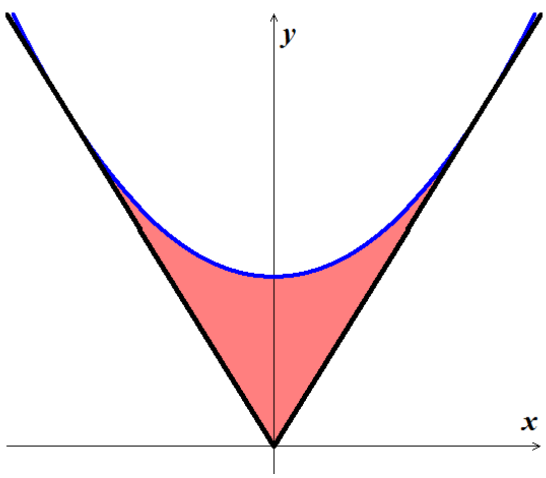
71. $y = 2x^2, \quad y = x^4 - 2x^2, \quad -2 \le x \le 2$



72. Find the area between the graph of $y = \sin x$ and the line segment joining the points $(0,\ 0)$ and $\left(\frac{7\pi}{6},\ -\frac{1}{2}\right)$.



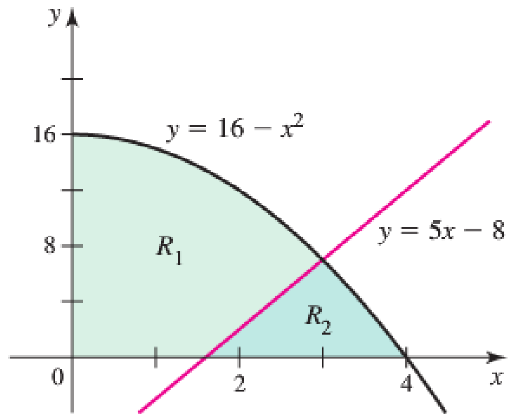
73. The surface of a machine part is the region between the graphs of $y_1 = |x|$ and $y_2 = 0.08x^2 + k$



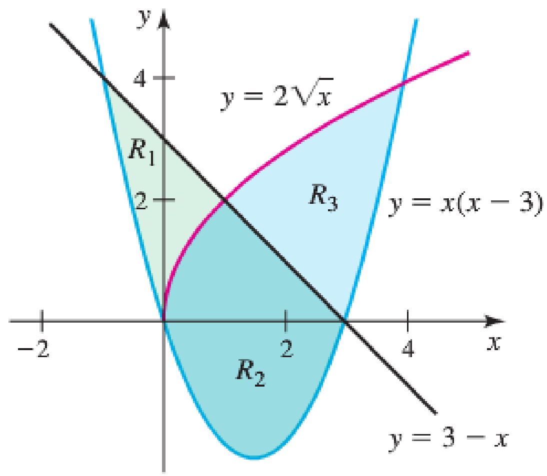
a) Find k where the parabola is tangent to the graph of y_1

b) Find the area of the surface of the machine part.

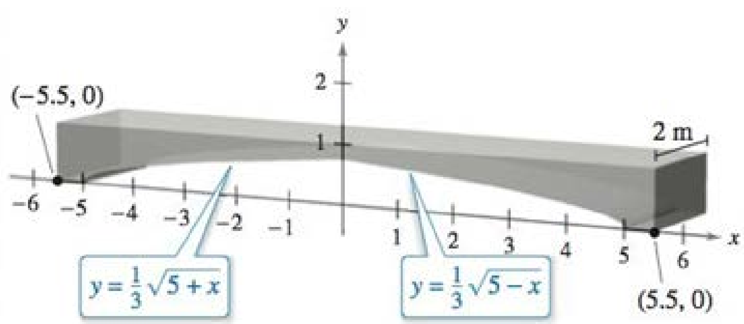
74. Find the area of the regions R_1 and R_2 (separately) shown in the figure, which are formed by the graphs of $y = 16 - x^2$ and $y = 5x - 8$



75. Find the area of the regions R_1, R_2 and R_3 (separately) shown in the figure, which are formed by the graphs of $y = 2\sqrt{x}$, $y = 3 - x$, and $y = x(x-3)$



76. Concrete sections for a new building have the dimensions (in meters) and shape shown in figure



a) Find the area of the face of the section superimposed on the rectangularcoordinate system.

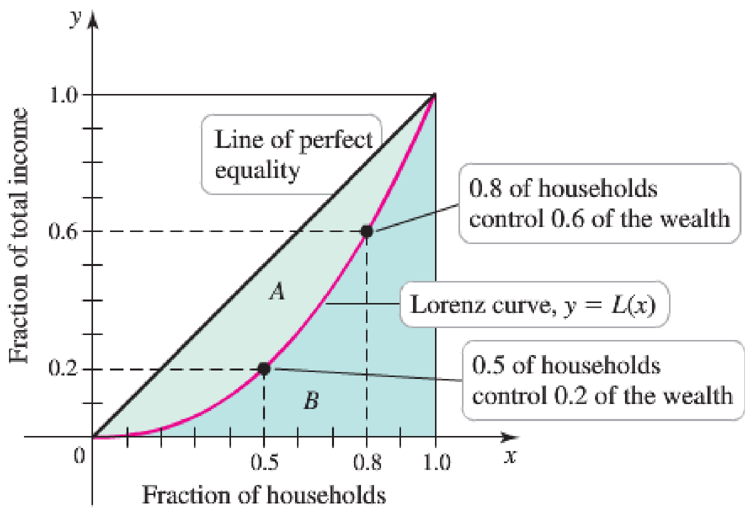
b) Find the volume of concrete in one of the sections by multiplying the area in part (*a*) by 2 *meters*.

c) One cubic meter of concrete weighs 5,000 *pounds*. Find the weight of the section.

77. A Lorenz curve is given by $y = L(x)$, where $0 \le x \le 1$ represents the lowest fraction of the population of a society in terms of wealth and $0 \le y \le 1$ represents the fraction of the total wealth that is owned by that fraction of the society. For example, the Lorenz curve in the figure shows that $L(0.5)=0.2$, which means that the lowest 0.5 (50%) of the society owns 0.2 (20%) of the wealth.

a) A Lorenz curve $y=L(x)$ is accompanied by the line $y=x$, called the ***line of perfect equality***. Explain why this line is given the name.

b) Explain why a Lorenz curve satisfies the conditions $L(0)=0,\ L(1)=1,\ and\ L'(x)\geq 0 \quad on\ [0,\ 1]$



c) Graph the Lorenz curves $L(x)=x^p$ corresponding to *p* = 1.1, 1.5, 2, 3, 4. Which value of *p* corresponds to the ***most*** equitable distribution of wealth (closest to the line of perfect equality)? Which value of *p* corresponds to the ***least*** equitable distribution of wealth? Explain.

d) The information in the Lorenz curve is often summarized in a single measure called the ***Gini index***, which is defined as follows. Let *A* be the area of the region between $y=x$ and $y=L(x)$ and Let *B* be the area of the region between $y=L(x)$ and the *x*-axis. Then the Gini index is $G=\dfrac{A}{A+B}$.

Show that $G=2A=1-2\int_0^1 L(x)\,dx$.

e) Compute the Gini index for the cases $L(x)=x^p$ and *p* = 1.1, 1.5, 2, 3, 4.

f) What is the smallest interval [*a*, *b*] on which values of the Gini index lie, for $L(x)=x^p$ with $p\geq 1$? Which endpoints of [*a*, *b*] correspond to the least and most equitable distribution of wealth?

g) Consider the Lorenz curve described by $L(x)=\dfrac{5x^2}{6}+\dfrac{x}{6}$. Show that it satisfies the conditions $L(0)=0,\ L(1)=1,\ and\ L'(x)\geq 0 \quad on\ [0,\ 1]$. Find the Gini index for this function.