Sec 1.3

1)
$$\begin{bmatrix} 16 & 4 & 5 & 4 \\ -3 & 13 & 15 & 6 \\ 0 & 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 4 & 2x+1 & 4 \\ -3 & 13 & 15 & 3x \\ 0 & 2 & 3y-5 & 0 \end{bmatrix}$$

$2x+1=5 \rightarrow \boxed{x=2}$

$3x=6$

$3y-5=4 \Rightarrow \boxed{y=3}$

2) $A=\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $B=\begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix}$

a) $A+B=\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}+\begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix}$
$=\begin{pmatrix} -2 & 0 \\ 6 & 3 \end{pmatrix}$

b) $A-B=\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}-\begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix}$
$=\begin{pmatrix} 4 & 4 \\ -2 & -1 \end{pmatrix}$

c) $2A=2\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$
$=\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$

d) $2A-B=\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}-\begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix}$
$=\begin{pmatrix} 5 & 6 \\ 0 & 0 \end{pmatrix}$

e) $B+\frac{1}{2}A=\begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix}+\frac{1}{2}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$
$=\begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix}+\begin{pmatrix} \frac{1}{2} & 1 \\ 1 & 1/2 \end{pmatrix}$
$=\begin{pmatrix} -\frac{5}{2} & -1 \\ 6 & \frac{5}{2} \end{pmatrix}$

3/ $A = \begin{pmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 8 & -1 \\ 4 & -3 \end{pmatrix}$

2×3 $\quad$ 2×2

a) $A + B$ not possible (A & B are not same size)

b) $A - B$ " " " same "

c) $2A = 2\begin{pmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{pmatrix}$

$= \begin{pmatrix} 12 & 0 & 6 \\ -2 & -8 & 0 \end{pmatrix}$

d) $2A - B$ not possible

e) $B + \frac{1}{2}A$ " "

4/ $A = \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 2 & -1 \\ -1 & 8 \end{pmatrix}$

a) $AB = \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 15 \\ 6 & 12 \end{pmatrix}$

b) $BA = \begin{pmatrix} 2 & -1 \\ -1 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 31 & 14 \end{pmatrix}$

5/ $A = \begin{pmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{pmatrix}$ $B = \begin{pmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{pmatrix}$

3×2 $\quad$ 3×3

a) $AB =$ not possible

b) $\underbrace{BA}_{3\times 3 \ 3\times 2} = \begin{pmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{pmatrix}$

$= \begin{pmatrix} 3 & -4 \\ 10 & 16 \\ 26 & 46 \end{pmatrix}$

6/ $A: 3\times 4$ $\quad E: 4\times 3$

$E-2A$ not possible, E + A are not the same size

7/ $D: 4\times 2$ $\quad C: 4\times 2$

$2D+C : \underline{4\times 2}$.

8/ $A=\begin{pmatrix} 2 & -1 & -1 \\ 1 & -2 & 2 \end{pmatrix}$ $\quad x=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $\quad 0=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$Ax : 2\times 3 - 3\times 1$

$Ax = 0$

$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x_1 - x_2 - x_3 = 0 \\ x_1 - 2x_2 + 2x_3 = 0 \end{cases}$$

let $x_3 = t$

$$\Rightarrow \begin{cases} 2x_1 - x_2 = t \\ x_1 - 2x_2 = -2t \end{cases} \quad \underline{\times -2}$$

$$\begin{array}{r} 2x_1 - x_2 = t \\ \underline{-2x_1 + 4x_2 = 4t} \\ 3x_2 = 5t \end{array} \Rightarrow x_2 = \frac{5}{3}t$$

$x_1 = -2t + \frac{10}{3}t$

$= \frac{4}{3}t$

$\therefore \ x = \left(\frac{4}{3}t, \ \frac{5}{3}t, \ t\right)$

9/

$$\begin{cases} -x_1 + x_2 = 4 \\ -2x_1 + x_2 = 0 \end{cases}$$

$$\underset{A}{\begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}}\underset{x}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \underset{b}{\begin{pmatrix} 4 \\ 0 \end{pmatrix}}$$

9-cont $A=\begin{bmatrix}1 & -1 & 2\\ 3 & -3 & 1\end{bmatrix} \quad b=\begin{bmatrix}-1\\ 7\end{bmatrix}$

$x_1\begin{pmatrix}1\\3\end{pmatrix}+x_2\begin{pmatrix}-1\\-3\end{pmatrix}+x_3\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}-1\\7\end{pmatrix}$

$\begin{cases} x_1 - x_2 + 2x_3 = -1 \\ 3x_1 - 3x_2 + x_3 = 7 \end{cases}$

$\left[\begin{array}{ccc|c}1 & -1 & 2 & -1\\ 3 & -3 & 1 & 7\end{array}\right] R_2-3R_1$

$\left[\begin{array}{ccc|c}1 & -1 & 2 & -1\\ 0 & 0 & -5 & +10\end{array}\right] \begin{array}{l}\to ①\\ \to \underline{x_3=-2}\end{array}$

① $x_1 - x_2 = -1 - 2(-2)$
$= 3$

let $x_2=0 \Rightarrow x_1=3$

$3\begin{pmatrix}1\\3\end{pmatrix}+0\begin{pmatrix}-1\\-3\end{pmatrix}-2\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}-1\\7\end{pmatrix}$

check: $3+0-4=-1$ ✓
$9 \quad -2 = 7$ ✓

if we let $x_1=0 \Rightarrow x_2=3$

$0\begin{pmatrix}1\\3\end{pmatrix}+3\begin{pmatrix}-1\\-3\end{pmatrix}-2\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}-1\\7\end{pmatrix}$

$-3-4\neq -1$ #

$\{ x_1=3,\ x_2=0,\ x_3=-2$

(not unique) $\{ x_1=2,\ x_2=-1,\ x_3=-2$ ✓

11/ $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} a_{11} + 2a_{21} = 1 \\ a_{12} + 2a_{22} = 0 \\ 3a_{11} + 5a_{21} = 0 \\ 3a_{12} + 5a_{22} = 1 \end{cases}$$

$$-3 \begin{cases} a_{11} + 2a_{21} = 1 \\ 3a_{11} + 5a_{21} = 0 \end{cases}$$

$$\begin{aligned} -3a_{11} - 6a_{21} &= -3 \\ 3a_{11} + 5a_{21} &= 0 \\ \hline a_{21} &= 3 \\ a_{11} &= -5 \end{aligned}$$

$$-3 \begin{cases} a_{12} + 2a_{22} = 0 \\ 3a_{12} + 5a_{22} = 1 \end{cases}$$

$\Rightarrow a_{22} = -1, \; a_{12} = 2$

$$\boxed{A = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}}$$

12/ $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 19 & 2 \end{pmatrix}$

$$-3 \quad \begin{aligned} a + 2c &= 6 \\ 3a + 4c &= 19 \\ \hline -2c &= 1 \\ c &= -\tfrac{1}{2} \\ a &= 7 \end{aligned}$$

$$-3 \quad \begin{aligned} b + 2d &= 3 \\ 3b + 4d &= 2 \\ \hline -2d &= -7 \\ d &= \tfrac{7}{2} \\ b &= -4 \end{aligned}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 7 & -4 \\ -\frac{1}{2} & \frac{7}{2} \end{pmatrix}$$

13/ $A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$ $B = \begin{pmatrix} -5 & 0 \\ 0 & 4 \end{pmatrix}$

$$AB = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} -5 & 0 \\ 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -10 & 0 \\ 0 & -12 \end{pmatrix}$$

$$BA = \begin{pmatrix} -5 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -10 & 0 \\ 0 & -12 \end{pmatrix}$$

$AB = BA$

14/ a) $Tr(A+B) = Tr(A) + Tr(B)$

$Tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$

$Tr(B) = b_{11} + b_{22} + \cdots + b_{nn}$

$A + B = [a_{ij} + b_{ij}]$

$$
\begin{aligned}
tr(A+B) &= (a_{11}+b_{11}) + (a_{22}+b_{22}) + \cdots + (a_{nn}+b_{nn}) \\
&= (a_{11}+a_{22}+\cdots+a_{nn}) + (b_{11}+b_{22}+\cdots+b_{nn}) \\
&= Tr(A) + Tr(B) \leftarrow
\end{aligned}
$$

b) $Tr(cA) = c\,Tr(A)$

$$
\begin{aligned}
c\,Tr(A) &= c(a_{11}+a_{22}+\cdots+a_{nn}) \\
&= ca_{11} + ca_{22} + \cdots + ca_{nn} \\
&= Tr(cA) \leftarrow
\end{aligned}
$$

15/ $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ $B = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$

$$
\begin{aligned}
AB &= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \\
&= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix} \\
&= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
BA &= \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \\
&= \begin{pmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha & -\sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \cos\alpha\sin\beta + \sin\alpha\cos\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix} \\
&= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}
\end{aligned}
$$

$\Rightarrow AB = BA.$

16/ $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$

$$aA + bB = a\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} a & 2a+b \\ 3a-b & 4a+2b \end{pmatrix}$$

17/ $A = \begin{pmatrix} -4 & 0 \\ 1 & -5 \\ -3 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 4 \end{pmatrix}$

a) $3X + 2A = B$

$$3\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} + 2\begin{pmatrix} -4 & 0 \\ 1 & -5 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 3x_1 - 8 & 3x_2 \\ 3x_3 + 2 & 3x_4 - 10 \\ 3x_5 - 6 & 3x_6 + 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 4 \end{pmatrix}$$

$3x_1 - 8 = 1 \Rightarrow \underline{x_1 = 3}$

$3x_2 = 2 \rightarrow x_2 = 2/3$

$3x_3 + 2 = -2 \rightarrow x_3 = -4/3$

$3x_4 - 10 = 1 \rightarrow x_4 = 11/3$

$3x_5 - 6 = 4 \rightarrow x_5 = 10/3$

$3x_6 + 4 = 4 \rightarrow x_6 = 0$

$$X = \begin{pmatrix} 3 & \frac{2}{3} \\ \frac{-4}{3} & \frac{11}{3} \\ \frac{10}{3} & 0 \end{pmatrix}$$