Section 1.4 – Linear Equations

A first order linear equation is given by the form:

$$y' + p(x)y = f(x)$$

If $f(x) = 0 \to y' = p(x)y$. This linear equation is said to be ***homogeneous***. (Otherwise it is ***nonhomogeneous or inhomogeneous***).

$p(x)$ & $f(x)$ are called the coefficients

Linear	***Non-linear***
$x' = \sin(t)x$	$x' = t\sin x$
$y' = e^{2t}y + \cos t$	$y' = 1 - y^2$
$x' = (3t+2)x + t^2 - 1$	

Solution of the homogenous equation

$$\frac{dx}{dt} = a(t)x \quad \Rightarrow \quad \frac{dx}{x} = a(t)dt$$

$$\int \frac{dx}{x} = \int a(t)dt$$

$$\ln|x| = \int a(t)dt + C$$

Convert to exponential form

$$|x| = e^{\int a(t)dt + C} = e^C e^{\int a(t)dt}$$

Let $A = e^C$

$$\boxed{x(t) = A.e^{\int a(t)dt}}$$

Example

Solve: $x' = \sin(t)\ x$

Solution

$$\frac{dx}{dt} = \sin(t)\ x$$

$$\frac{dx}{x} = \sin(t)\ dt$$

$$\int \frac{dx}{x} = \int \sin(t)\ dt$$

$$\ln|x| = \int \sin(t)dt + C$$

$$\ln|x| = -\cos(t) + C$$

$$x = e^{-\cos(t)+C}$$

$$x(t) = A.e^{\int \sin(t)dt}$$

$$\boxed{= A.e^{-\cos t}}$$

Solving a linear first-order Equation (*Properties*)

1. Put a linear equation into a standard form $y' + p(x)y = f(x)$
2. Identify $p(x)$ then find $y_h = e^{-\int p\,dx}$
3. Multiply the standard form by y_h
4. Integrate both sides

Solution of the Inhomogeneous Equation $u(t) = e^{-\int a(t)dt}$

$$x' = a(t)x + f(t)$$

$$x' - ax = f$$

$$(ux)' = u\left(x' - ax\right) = uf$$

$$u(t)x(t) = \int u(t)f(t)dt + C$$

Example

Find the general solution to: $x' = x + e^{-t}$

Solution

$$x' - x = e^{-t}$$

$$e^{-\int 1dt} = e^{-t}$$

$$e^{-t}\left(x' - x\right) = e^{-t}e^{-t}$$

$$\left(e^{-t}x\right)' = e^{-2t}$$

$$e^{-t}x(t) = \int e^{-2t}dt$$

$$e^{-t}x(t) = -\tfrac{1}{2}e^{-2t} + C$$

$$\boxed{x(t) = -\tfrac{1}{2}e^{-t} + Ce^{t}}$$

Solution of the Nonhomogeneous Equation $y' + p(x)y = f(x)$

Let assume: $y = y_h + y_p$ where $\begin{cases} y_h & \textit{Homogeneous Solution} \\ y_p & \textit{Paticular Solution} \end{cases}$

The homogeneous equation is given by $y_h' + p(x)y_h = 0$

$$y_h' = -p(x)y_h$$

$$\boxed{y_h = e^{-\int p\,dx}}$$

$$y_p = u(x)y_h = u.e^{-\int p\,dx}$$

$$y_p' + p(x)y_p = f(x)$$

$$\left(uy_h\right)' + puy_h = f$$

$$u'y_h + uy_h' + puy_h = f$$

$$u'y_h + u\left(y_h' + py_h\right) = f \qquad \textit{Since } y_h' + py_h = 0 \qquad \textit{homogeneous}$$

$$u'y_h = f$$

$$\frac{du}{dx} = \frac{f}{y_h}$$

$$du = \frac{f}{e^{-\int p\,dx}}dx$$

$$= f.e^{\int p\,dx}dx$$

$$u = \int f.e^{\int p\,dx}dx$$

$$y_p = u.e^{-\int p\,dx}$$

$$u = \left(\int f.e^{\int p\,dx}dx\right)e^{-\int p\,dx}$$

$$\boxed{y_p = e^{-\int p\,dx}\int f.e^{\int p\,dx}dx}$$

$$y = y_h + y_p$$

$$\boxed{y = Ce^{-\int p\,dx} + e^{-\int p\,dx}\int f.e^{\int p\,dx}dx} \qquad \boxed{y = e^{-\int p\,dx}\left(C + \int f.e^{\int p\,dx}dx\right)}$$

Example

Find the general solution of $x' = x\sin t + 2te^{-\cos t}$ and the particular solution that satisfies $x(0) = 1$.

Solution

$$x' - x\sin t = 2te^{-\cos t} \qquad\qquad P(t) = \sin t, \quad Q(t) = 2te^{-\cos t}$$

$$x_h = e^{-\int \sin t dt} = e^{\cos t}$$

$$\int Q(t)x_h dt = \int 2te^{-\cos t}e^{\cos t} dt = \int 2tdt = t^2$$

$$x(t) = e^{-\cos t}\left(t^2 + C\right) \qquad\qquad x = \frac{1}{e^{\int P dt}}\left(\int Q.e^{\int P dt}dt + C\right)$$

$$x(0) = \left((0)^2 + C\right)e^{-\cos 0} = 1$$

$$Ce^{-1} = 1$$

$$C = e$$

$$\boxed{x(t) = \left(t^2 + e\right)e^{-\cos t}}$$

Example

Find the general solution of $x' = x\tan t + \sin t$ and the particular solution that satisfies $x(0) = 2.$.

Solution

$$x' - (\tan t)x = \sin t \qquad\qquad P(t) = -\tan t, \quad Q(t) = \sin t$$

$$e^{-\int \tan t dt} = e^{\ln(\cos t)} \boxed{= \cos t}$$

$$\int (\sin t)(\cos t)\ dt = -\int \cos t\ d(\cos t) = -\frac{1}{2}\cos^2 t$$

$$x(t) = \frac{1}{\cos t}\left(-\frac{1}{2}\cos^2 t + C\right) = -\frac{1}{2}\cos t + \frac{1}{\cos t}C$$

$$\boxed{= -\frac{1}{2}\cos t + \frac{1}{\cos t}C}$$

$$x(0) = -\frac{1}{2}\cos(0) + \frac{C}{\cos(0)} = 2$$

$$-\frac{1}{2} + C = 2 \quad \Rightarrow \quad C = \frac{5}{2}$$

$$\boxed{x(t) = -\frac{1}{2}\cos t + \frac{5}{2\cos t}}$$

Notes

1. Integrating an expression that is not the differential of any elementary function is called non-elementary.

$$\int e^{x^2}dx \qquad \int x\tan x dx \qquad \int \frac{e^{-x}}{x}dx$$

$$\int \sin x^2 dx \qquad \int \cos x^2 dx \qquad \int \frac{\sin x}{x}dx \qquad \int \frac{\cos x}{x}dx$$

2. In math some important functions are defined in terms of non-elementary integrals. Two such functions are the error function and the complementary error function.

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \qquad erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}dt$$

Exercises *Section* 1.4 – Linear Equations

Find the general solution of the first-order, linear equation.

1. $y' - y = 3e^{t}$
2. $y' + y = \sin t$
3. $y' + y = \dfrac{1}{1+e^{t}}$
4. $y' - y = e^{2t} - 1$
5. $y' + y = te^{-t} + 1$
6. $y' + y = 1 + e^{-x}\cos 2x$
7. $y' + y\cot x = \cos x$
8. $y' + y\sin t = \sin t$
9. $y' = \cos x - y\sec x$
10. $y' + (\tan x)y = \cos^{2} x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$
11. $y' + (\cot t)y = 2t\csc t$
12. $y' + (1+\sin t)y = 0$
13. $y' + \left(\frac{1}{2}\cos x\right)y = -\frac{3}{2}\cos x$
14. $\dfrac{dy}{dx} + y = e^{3x}$
15. $y' - ty = t$
16. $y' = 2y + x^{2} + 5$
17. $xy' + 2y = 3$
18. $\dfrac{dy}{dt} - 2y = 4 - t$
19. $y' + 2y = 1$
20. $y' + 2y = e^{-t}$
21. $y' + 2y = e^{-2t}$
22. $y' - 2y = e^{3t}$
23. $y' + 2y = e^{-x} + x + 1$
24. $y' + 2xy = x$
25. $y' - 2ty = t$
26. $y' + 2ty = 5t$
27. $y' - 2xy = e^{x^{2}}$
28. $y' + 2xy = x^{3}$
29. $y' - 2y = t^{2}e^{2t}$
30. $x' - 2\dfrac{x}{t+1} = (t+1)^{2}$
31. $y' + \dfrac{2}{t}y = \dfrac{\cos t}{t^{2}}$
32. $y' - 2(\cos 2t)y = 0$
33. $y' + 2y = \cos 3t$
34. $y' - 3y = 5$
35. $y' + 3y = 2xe^{-3x}$
36. $y' + 3t^{2}y = t^{2}$
37. $y' + 3x^{2}y = x^{2}$
38. $y' + \dfrac{3}{t}y = \dfrac{\sin t}{t^{3}}, \quad (t \neq 0)$
39. $y' + \dfrac{3}{x}y = 1 + \dfrac{1}{x}$
40. $y' + \dfrac{3}{2}y = \dfrac{1}{2}e^{x}$
41. $y' + 5y = t + 1$
42. $xy' - y = x^{2}\sin x$
43. $x\dfrac{dy}{dx} + y = e^{x}, \quad x > 0$
44. $x\dfrac{dy}{dx} + 2y = 1 - \dfrac{1}{x}, \quad x > 0$
45. $y\dfrac{dx}{dy} + 2x = 5y^{3}$
46. $ty' + y = \cos t$
47. $xy' + 2y = x^{2}$
48. $xy' = 2y + x^{3}\cos x$
49. $xy' + 2y = x^{-3}$
50. $ty' + 2y = t^{2}$
51. $xy' + 2\left(y + x^{2}\right) = \dfrac{\sin x}{x}$
52. $xy' + 4y = x^{3} - x$
53. $xy' + (x+1)y = e^{-x}\sin 2x$

54. $xy' + (3x+1)y = e^{3x}$

55. $xy' + (2x-3)y = 4x^4$

56. $2xy'' - 3y = 9x^3$

57. $2y' + 3y = e^{-t}$

58. $2y' + 2ty = t$

59. $3xy' + y = 10\sqrt{x}$

60. $3xy' + y = 12x$

61. $x^2 y' + xy = 1$

62. $x^2 y' + x(x+2)y = e^x$

63. $y^2 + (y')^2 = 1$

64. $(1+x)y' + y = \sqrt{x}$

65. $(1+x)y' + y = \cos x$

66. $(x+1)y' + (x+2)y = 2xe^{-x}$

67. $(x+1)y' - xy = x + x^2$

68. $\left(1+x^3\right)y' = 3x^2 y + x^2 + x^5$

69. $(t+1)\dfrac{ds}{dt} + 2s = 3(t+1) + \dfrac{1}{(t+1)^2}, \quad t > -1$

70. $(x+2)^2 y' = 5 - 8y - 4xy$

71. $\left(x^2-1\right)y' + 2y = (x+1)^2$

72. $\left(x^2+4\right)y' + 2xy = x^2\left(x^2+4\right)$

73. $\left(1+e^t\right)y' + e^t y = 0$

74. $\left(t^2+9\right)y' + ty = 0$

75. $e^{2x}y' + 2e^{2x}y = 2x$

76. $\tan\theta \dfrac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \frac{\pi}{2}$

77. $(\cos t)y' + (\sin t)y = 1$

78. $\cos x \dfrac{dy}{dx} + (\sin x)y = 1$

79. $\cos^2 x \sin x \dfrac{dy}{dx} + \left(\cos^3 x\right)y = 1$

80. $\dfrac{dr}{d\theta} + r\sec\theta = \cos\theta$

81. $\dfrac{dr}{d\theta} + r\tan\theta = \sec\theta$

82. $\dfrac{dP}{dt} + 2tP = P + 4t - 2$

83. $ydx - 4\left(x + y^6\right)dy = 0$

84. $ydx = \left(ye^y - 2x\right)dy$

85. $(x+y+1)dx - dy = 0$

86. $\dfrac{dy}{dx} = x^2 e^{-4x} - 4y$

87. $\left(x^2+1\right)y' + xy - x = 0$

88. $\dfrac{dx}{dt} = 9.8 - 0.196x$

89. $\dfrac{di}{dt} + 500i = 10\sin\omega t$

90. $2\dfrac{dQ}{dt} + 100Q = 10\sin 60t$

Find the solution of the initial value problem

91. $y' - 3y = 4\,; \quad y(0) = 2$

92. $y' = y + 2xe^{2x}; \quad y(0) = 3$

93. $\left(x^2+1\right)y' + 3xy = 6x; \quad y(0) = -1$

94. $t\dfrac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(2) = 1$

95. $\theta\dfrac{dy}{d\theta} + y = \sin\theta, \quad \theta > 0, \quad y\left(\frac{\pi}{2}\right) = 1$

96. $\dfrac{dy}{dx} + xy = x, \quad y(0) = -6$

97. $ty' + 2y = 4t^2, \quad y(1) = 2$

98. $\left(1+t^2\right)y' + 4ty = \left(1+t^2\right)^{-2}, \quad y(1) = 0$

99. $y' + y = e^t, \quad y(0) = 1$

100. $y' + \frac{1}{2}y = t, \quad y(0) = 1$

101. $y' = x + 5y, \quad y(0) = 3$

102. $y' = 2x - 3y, \quad y(0) = \frac{1}{3}$

103. $xy' + y = e^x, \quad y(1) = 2$

104. $y\frac{dx}{dy} - x = 2y^2, \quad y(1) = 5$

105. $xy' + y = 4x + 1, \quad y(1) = 8$

106. $y' + 4xy = x^3 e^{x^2}, \quad y(0) = -1$

107. $(x+1)y' + y = \ln x, \quad y(1) = 10$

108. $x(x+1)y' + xy = 1, \quad y(e) = 1$

109. $y' - (\sin x)y = 2\sin x, \quad y\left(\frac{\pi}{2}\right) = 1$

110. $y' + (\tan x)y = \cos^2 x, \quad y(0) = -1$

111. $L\frac{di}{dt} + RI = E, \quad i(0) = i_0$

112. $\frac{dT}{dt} = k\left(T - T_m\right) \quad T(0) = T_0$

113. $y' + y = 2, \quad y(0) = 0$

114. $xy' + 2y = 3x, \quad y(1) = 5$

115. $y' - 2y = 3e^{2x}, \quad y(0) = 0$

116. $xy' + 5y = 7x^2, \quad y(2) = 5$

117. $xy' - y = x, \quad y(1) = 7$

118. $xy' + y = 3xy, \quad y(1) = 0$

119. $xy' + 3y = 2x^5, \quad y(2) = 1$

120. $y' + y = e^x, \quad y(0) = 1$

121. $xy' - 3y = x^3, \quad y(1) = 10$

122. $y' + 2xy = x, \quad y(0) = -2$

123. $y' = (1 - y)\cos x, \quad y(\pi) = 2$

124. $(1+x)y' + y = \cos x, \quad y(0) = 1$

125. $y' = 1 + x + y + xy, \quad y(0) = 0$

126. $xy' = 3y + x^4 \cos x, \quad y(2\pi) = 0$

127. $y' = 2xy + 3x^2 e^{x^2}, \quad y(0) = 5$

128. $\left(x^2 + 4\right)y' + 3xy = x, \quad y(0) = 1$

129. $y' - 2y = e^{3x}; \quad y(0) = 3$

130. $y' - 3y = 6; \quad y(0) = 1$

131. $2y' + 3y = e^x; \quad y(0) = 0$

132. $\left(x^2 + 1\right)y' + 3x^3 y = 6xe^{-3x^2/2}, \quad y(0) = 1$

133. $y' + y = 1 + e^{-x}\cos 2x; \quad y\left(\frac{\pi}{2}\right) = 0$

134. $2y' + (\cos x)y = -3\cos x; \quad y(0) = -4$

135. $y' + 2y = e^{-x} + x + 1; \quad y(-1) = e$

136. $y' + \frac{y}{x} = xe^{-x}; \quad y(1) = e - 1$

137. $y' + 4y = e^{-x}; \quad y(1) = \frac{4}{3}$

138. $x^2 y' + 3xy = x^4 \ln x + 1; \quad y(1) = 0$

139. $y' + \frac{3}{x}y = 3x - 2 \quad y(1) = 1$

140. $(\cos x)y' + y\sin x = 2x\cos^2 x \quad y\left(\frac{\pi}{4}\right) = \frac{-15\sqrt{2}\pi^2}{32}$

141. $(\cos x)y' + (\sin x)y = 2\cos^3 x \sin x - 1 \quad y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$

142. $ty' + 2y = t^2 - t + 1 \quad y(1) = \frac{1}{2}$

143. $ty' - 2y = t^5 \sin 2t - t^3 + 4t^4 \quad y(\pi) = \frac{3}{2}\pi^4$

144. $2y' - y = 4\sin 3t \quad y(0) = y_0$

145. $y' + 2y = 2 - e^{-4t} \quad y(0) = 1$

146. $y' - y = -\frac{1}{2}e^{t/2}\sin 5t + 5e^{t/2}\cos 5t \quad y(0) = 0$

147. $y' + 2y = 3; \quad y(0) = -1$

148. $y' + (\cos t)y = \cos t; \quad y(\pi) = 2$

149. $y' + 2ty = 2t; \quad y(0) = 1$

150. $y' + y = \frac{e^{-t}}{t^2}; \quad y(1) = 0$

151. $ty' + 2y = \sin t; \quad y(\pi) = \frac{1}{\pi}$

152. $y' + \frac{2}{t}y = \frac{\cos t}{t^2}; \quad y(\pi) = 0$

153. $(\sin t)y' + (\cos t)y = 0; \quad y\left(\frac{3\pi}{4}\right) = 2$

154. $y' + 3t^2 y = t^2$; $y(0) = 2$

155. $ty' + y = t\sin t$; $y(\pi) = -1$

156. $y' + y = \sin t$; $y(\pi) = 1$

157. $y' + y = \cos 2t$; $y(0) = 5$

158. $y' + 3y = \cos 2t$; $y(0) = -1$

159. $y' - 2y = 7e^{2t}$; $y(0) = 3$

160. $y' - 2y = 3e^{-2t}$; $y(0) = 10$

161. $y' + 2y = t^2 + 2t + 1 + e^{4t}$; $y(0) = 0$

162. $y' - 3y = 2t - e^{4t}$; $y(0) = 0$

163. $y' + y = t^3 + \sin 3t$; $y(0) = 0$

164. $y' + 2y = \cos 2t + 3\sin 2t + e^{-t}$; $y(0) = 0$

165. $y' + y = e^{3t}$; $y(0) = y_0$

166. $t^2 y' - ty = 1$; $y(1) = y_0$

167. $y' + ay = e^{at}$; $y(0) = y_0$, $a \neq 0$

168. $3y' + 12y = 4$; $y(0) = y_0$

Find a solution to the initial value problem that is continuous on the given interval $[a,\ b]$

169. $y' + \frac{1}{x}y = f(x),\quad y(1) = 1$ $\qquad f(x) = \begin{cases} 3x, & 1 \le x \le 2 \\ 0, & 2 < x \le 3 \end{cases}$ $\qquad [a,\ b] = [1,\ 3]$

170. $y' + (\sin x)y = f(x),\quad y(0) = 3$ $\qquad f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ -\sin x, & \pi < x \le 2\pi \end{cases}$ $\qquad [a,\ b] = [0,\ 2\pi]$

171. $y' + p(t)y = 2,\quad y(0) = 1$ $\qquad p(t) = \begin{cases} 0, & 0 \le t \le 1 \\ \frac{1}{t}, & 1 < t \le 2 \end{cases}$ $\qquad [a,\ b] = [0,\ 2]$

172. $y' + p(t)y = 0,\quad y(0) = 3$ $\qquad p(t) = \begin{cases} 2t-1, & 0 \le t \le 1 \\ 0, & 1 < t \le 3 \\ -\frac{1}{t}, & 3 < t \le 4 \end{cases}$ $\qquad [a,\ b] = [0,\ 4]$

Find the solution of the initial value problem. Discuss the interval of existence and provide a sketch of your solution

173. $xy' + 2y = \sin x$; $y\left(\frac{\pi}{2}\right) = 0$

174. $(2x+3)y' = y + (2x+3)^{1/2}$; $y(-1) = 0$

175. The following system of differential equations is encountered in the study of the decay of a special type of radioactive series of elements

$$\frac{dx}{dt} = -\lambda_1 x \qquad \frac{dy}{dt} = \lambda_1 x - \lambda_2 y$$

Where λ_1 *and* λ_2 are constants.

Discuss how to solve this system subject to $x(0) = x_0$, $y(0) = y_0$

176. Let $P(t)$ be the performance level of someone learning a skill as a function of the training time *t*. The graph of *P* is called a ***learning curve***. We proposed the differential equation

$$\frac{dP}{dt} = k\left(M - P(t)\right)$$

As a reasonable model for learning, where *k* is a positive constant. Solve it as a linear differential equation and use your solution to graph the learning curve.

177. A differential equation describing the velocity *v* of a falling mass subject to air resistance proportional to the instantaneous velocity is

$$m\frac{dv}{dt} = mg - kv$$

Where $k > 0$ is a constant of proportionality. The positive direction is downward.

a) Solve the equation subject to the initial condition $v(0) = v_0$

b) Use the solution in part (*a*) to determine the limiting, or terminal, velocity of the mass.

c) If the distance *s*, measured from the point where the mass was released above ground, is related to velocity *v* by $\frac{ds}{dt} = v(t)$, find an explicit expression for $s(t)$ if $s(0) = 0$

178. As a raindrop falls, it evaporates while retaining its spherical shape. If we make the further assumptions that the rate at which the raindrop evaporates is proportional to its surface and that air resistance is negligible, then a model for the velocity $v(t)$ of the raindrop is

$$\frac{dv}{dt} + \frac{3(k/\rho)}{\frac{k}{\rho}t + r_0}v = g$$

Here ρ is the density of water, r_0 is the radius of the raindrop at $t = 0$, $k < 0$ is the constant of proportionality, and downward direction is taken to be the positive direction.

a) Solve for $v(t)$ if the raindrop falls from rest.

b) Show that the radius of the raindrop at time *t* is $r(t) = \frac{k}{\rho}t + r_0$.

c) If $r_0 = 0.01\,ft$ and $r = 0.007\,ft$ 10 *seconds* after the raindrop falls from a cloud, determine the time at which the raindrop has evaporated completely.

179. A model that describes the population of a fishery in which harvesting takes place at a constant rate is given by

$$\frac{dP}{dt} = kP - h$$

Where *k* and *h* are positive constants.

a) Solve $P(t)$ given the initial value $P(0) = P_0$

b) Describe the behavior of the population $P(t)$ for increasing time in three cases $P_0 > \frac{h}{k}$, $P_0 = \frac{h}{k}$, and $P_0 < \frac{h}{k}$

c) Use the results from part (*b*) to determine whether the fish population will ever go extinct in finite time, that is, whether there exists a time $T > 0$ such that $P(T) = 0$. If the population goes extinct then find *T*.

180. A certain body weighing $45\ lb$, is heated to a temperature of $300°$. Then at $t = 0$ it is plunged into $100\ lb$ of water at a temperature of $50°$. Given that the specific heat of the body is $\frac{1}{9}$, find the formula for the temperature *T* of the body during its cooling.