Section 2.8 – Derivatives of Logarithmic & Exponential Functions

Derivative of $y = \ln x$

$$\boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}} \qquad x \neq 0$$

The chain rule extends: $\boxed{\frac{d}{dx}\ln u = \frac{1}{u}\cdot\frac{du}{dx} = \frac{u'}{u}}$ $\quad u > 0$

Example

Find $\quad \frac{d}{dx}\ln 2x$

Solution

$$\begin{aligned}\frac{d}{dx}\ln 2x &= \frac{(2x)'}{2x}\\ &= \frac{2}{2x}\\ &= \frac{1}{x}\end{aligned}$$

Example

Find the derivative of $\ln\left(x^2+3\right)$

Solution

$$\frac{d}{dx}\ln\left(x^2+3\right) = \frac{2x}{x^2+3}$$

Properties of the Natural logarithm

Product Rule $\quad \ln bx = \ln b + \ln x$

Quotient Rule $\quad \ln\frac{b}{x} = \ln b - \ln x$

Reciprocal Rule $\quad \ln\frac{1}{x} = -\ln x$

Power Rule $\quad \ln x^r = r\ln x$

Example

a) $\ln(4\sin x) = \ln 4 + \ln\sin x$

b) $\ln\frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$

c) $\ln\frac{1}{8} = -\ln 8 = -\ln 2^3 = -3\ln 2$

Example

Find $\frac{dy}{dx}$ if $y = \frac{\left(x^2+1\right)(x+3)^{1/2}}{x-1}, \quad x > 1$

Solution

$$\ln y = \ln\frac{\left(x^2+1\right)(x+3)^{1/2}}{x-1}$$

$$= \ln\left(x^2+1\right)(x+3)^{1/2} - \ln(x-1) \qquad \textit{Quotient Rule}$$

$$= \ln\left(x^2+1\right) + \ln(x+3)^{1/2} - \ln(x-1) \qquad \textit{Product Rule}$$

$$= \ln\left(x^2+1\right) + \frac{1}{2}\ln(x+3) - \ln(x-1) \qquad \textit{Power Rule}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+1} + \frac{1}{2}\frac{1}{x+3} - \frac{1}{x-1}$$

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right)$$

$$\frac{dy}{dx} = \frac{\left(x^2+1\right)(x+3)^{1/2}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right)$$

The Derivative and Integral of e^x

The natural exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero.

$$\ln\left(e^x\right) = x \qquad \textit{Inverse relationship}$$

$$\frac{d}{dx}\ln\left(e^x\right) = 1 \qquad \textit{Differentiate both sides}.$$

$$\frac{1}{e^x}\frac{d}{dx}\left(e^x\right) = 1 \qquad \frac{d}{dx}\ln u = \frac{1}{u}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}e^x = e^x$$

If u is any differentiable function of x, then $\boxed{\frac{d}{dx}e^u = e^u\frac{du}{dx}}$ $\boxed{\left(e^u\right)' = u'e^u}$

Example

Find the derivative of $\frac{d}{dx}\left(5e^x\right)$

Solution

$$\frac{d}{dx}\left(5e^x\right) = 5\frac{d}{dx}e^x$$

$$= 5e^x$$

Example

Find the derivative of $\frac{d}{dx}\left(e^{\sin x}\right)$

Solution

$$\frac{d}{dx}\left(e^{\sin x}\right) = e^{\sin x}\frac{d}{dx}(\sin x)$$

$$= e^{\sin x}\cdot\cos x$$

Example

Find the derivative of $\frac{d}{dx}\left(e^{\sqrt{3x+1}}\right)$

Solution

$$\begin{aligned}\frac{d}{dx}\left(e^{\sqrt{3x+1}}\right) &= e^{\sqrt{3x+1}}\frac{d}{dx}\left(\sqrt{3x+1}\right)\\ &= e^{\sqrt{3x+1}}\cdot\tfrac{1}{2}(3x+1)^{-1/2}\cdot 3\\ &= \boxed{\frac{3}{2\sqrt{3x+1}}e^{\sqrt{3x+1}}}\end{aligned}$$

Definition

For any numbers $a > 0$ and x, the ***exponential function with base a*** is

$$a^x = e^{x\ln a}$$

When $a = e$, the function gives $a^x = e^{x\ln a} = e^{x\ln e} = e^x$

Power Rule – *Definition*

For any $x > 0$ and for any real number n, $\quad x^n = e^{n\ln x}$

General Power Rule for Derivatives

For any $x > 0$ and for any real number n, $\quad \frac{d}{dx}x^n = nx^{n-1}$

Proof

$$\begin{aligned}\frac{d}{dx}x^n &= \frac{d}{dx}e^{n\ln x}\\ &= e^{n\ln x}\frac{d}{dx}(n\ln x)\\ &= x^n\cdot\frac{n}{x}\\ &= nx^{n-1}\end{aligned}$$

Example

Differentiate $f(x) = x^x, \quad x > 0$

Solution

$$\begin{aligned}f'(x) &= \frac{d}{dx}\left(e^{x\ln x}\right)\\ &= e^{x\ln x}\frac{d}{dx}(x\ln x)\end{aligned}$$

$$= e^{x\ln x}\frac{d}{dx}(x\ln x)$$

$$= e^{x\ln x}\left(\ln x + x\cdot\frac{1}{x}\right)$$

$$\boxed{= x^x(\ln x+1)} \quad x>0$$

Theorem – The Number *e* as a Limit

The number e can be calculated as the limit $\quad e = \lim\limits_{x\to 0}(1+x)^{1/x}$

Proof

If $f(x)=\ln x \quad \to f'(x)=\frac{1}{x} \quad so \ \ f'(1)=1$

$$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{x\to 0}\frac{f(1+x)-f(1)}{x}$$

$$= \lim_{x\to 0}\frac{\ln(1+x)-\ln(1)}{x}$$

$$= \lim_{x\to 0}\left[\frac{1}{x}\ln(1+x)\right]$$

$$= \lim_{x\to 0}\ln(1+x)^{1/x}$$

$$= \ln\left[\lim_{x\to 0}(1+x)^{1/x}\right] \qquad f'(1)=1$$

$$\boxed{=1}$$

$$\lim_{x\to 0}(1+x)^{1/x} = e$$

Definition

If $a>0$ and u is a differentiable of x, then a^u is a differentiable function of x and

$$\boxed{\frac{d}{dx}a^u = a^u\ln a\frac{du}{dx}}$$

Example

- $\frac{d}{dx}3^x = 3^x\ln 3$

- $\frac{d}{dx}3^{-x} = 3^{-x}\ln 3\frac{d}{dx}(-x) = -3^{-x}\ln 3$
- $\frac{d}{dx}3^{\sin x} = 3^{\sin x}\ln 3\frac{d}{dx}(\sin x) = 3^{\sin x}\ln 3(\cos x)$

Logarithms with base *a*

For any positive number $a \neq 1$, $\log_a x$ is the inverse function of a^x

Inverse Equations for $\log_a x$ and a^x

$$a^{\log_a x} = x \quad (\forall x > 0)$$

$$\log_a\left(a^x\right) = x \quad (all\ x)$$

$$\log_a x = \frac{\ln x}{\ln a}$$

Derivative $\boxed{\frac{d}{dx}\left(\log_a u\right) = \frac{1}{u}\cdot\frac{1}{\ln a}\frac{du}{dx}}$

Example

- $\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{(3x+1)\cdot\ln 10}\frac{d}{dx}(3x+1) = \frac{3}{(3x+1)\cdot\ln 10}$

Exercises Section 2.8 – Derivatives of Logarithmic & Exponential Functions

Find the derivative

1. $y = \ln\sqrt{x+5}$
2. $y = (3x+7)\ln(2x-1)$
3. $f(x) = \ln\sqrt[3]{x+1}$
4. $f(x) = \ln\left[x^2\sqrt{x^2+1}\right]$
5. $y = \ln\dfrac{x^2}{x^2+1}$
6. $y = \ln\left[\dfrac{x^2(x+1)^3}{(x+3)^{1/2}}\right]$
7. $y = \ln(x^2+1)$
8. $f(x) = \ln\left(x^2-4\right)$
9. $f(x) = 2\ln\left(x^2-3x+4\right)$
10. $f(x) = 3\ln\left(1+x^2\right)$
11. $f(x) = (1+\ln x)^3$
12. $f(x) = (x-2\ln x)^4$
13. $f(x) = x^2\ln x$
14. $f(x) = -\dfrac{\ln x}{x^2}$
15. $y = \ln\left(t^2\right)$
16. $y = \ln(2\theta+2)$
17. $y = (\ln x)^3$
18. $y = x(\ln x)^2$
19. $y = \dfrac{x^4}{4}\ln x - \dfrac{x^4}{16}$
20. $y = \dfrac{1+\ln t}{t}$
21. $f(x) = \dfrac{\ln x}{1+x}$
22. $f(x) = \dfrac{2x}{1+\ln x}$
23. $f(x) = x^3\ln x$
24. $f(x) = 6x^4\ln x$
25. $f(x) = \ln x^8$
26. $f(x) = 5x - \ln x^5$
27. $f(x) = \ln x^{10} + 2\ln x$
28. $f(x) = \dfrac{\ln x}{2x+5}$
29. $f(x) = -2\ln x + x^2 - 4$
30. $y = \ln\left(\dfrac{1}{x\sqrt{x+1}}\right)$
31. $y = \ln(\ln(\ln x))$
32. $y = \ln(\sec(\ln x))$
33. $y = \ln\left(\dfrac{\left(x^2+1\right)^5}{\sqrt{1-x}}\right)$
34. $y = \ln\sqrt{\dfrac{(x+1)^5}{(x+2)^{20}}}$
35. $f(x) = e^{3x}$
36. $f(x) = e^{-2x^3}$
37. $f(x) = 4e^{x^2}$
38. $f(x) = 2x^3e^x$
39. $f(x) = \dfrac{3e^x}{1+e^x}$
40. $f(x) = 5e^x + 3x + 1$
41. $f(x) = x^2e^x$
42. $f(x) = \dfrac{e^x+e^{-x}}{2}$
43. $f(x) = \dfrac{e^x}{x^2}$
44. $f(x) = x^2e^x - e^x$
45. $f(x) = (1+2x)e^{4x}$
46. $y = x^2e^{5x}$
47. $y = x^2e^{-2x}$
48. $f(x) = \dfrac{e^x}{x^2+1}$
49. $f(x) = \dfrac{1-e^x}{1+e^x}$
50. $y = \dfrac{e^x+e^{-x}}{x}$
51. $y = \sqrt{e^{2x^2}+e^{-2x^2}}$
52. $y = \dfrac{x}{e^{2x}}$
53. $y = 3e^{5x^3+1}$

54. $f(x)=\left(x^2-2x+2\right)e^x$

55. $f(\theta)=e^{\theta}(\sin\theta+\cos\theta)$

56. $f(\theta)=\ln\left(3\theta e^{-\theta}\right)$

57. $f(\theta)=\theta^3 e^{-2\theta}\cos 5\theta$

58. $f(\theta)=\ln\left(\dfrac{\sqrt{\theta}}{1+\sqrt{\theta}}\right)$

59. $f(t)=e^{(\cos t+\ln t)}$

60. $y=e^{\sin t}\left(\ln t^2+1\right)$

61. $y=e^{x^2}\ln x$

62. $f(x)=e^x+x-\ln x$

63. $f(x)=\ln x+2e^x-3x^2$

64. $f(x)=\ln x^2+4e^x$

65. $y=\ln\dfrac{1+e^x}{1-e^x}$

66. $y=\dfrac{\ln x}{e^{2x}}$

67. $f(x)=e^{2x}\ln\left(xe^x+1\right)$

68. $f(x)=\dfrac{xe^x}{\ln\left(x^2+1\right)}$

69. $f(x)=xe^{-10x}$

70. $f(x)=x\ln^2 x$

71. $f(x)=e^{-x}\ln x$

72. $f(x)=2^{x^2-x}$

Use logarithmic differentiation to find the derivative of

73. $y=\sqrt{x(x+1)}$

74. $y=\sqrt{\left(x^2+1\right)(x-1)^2}$

75. $y=\sqrt{\dfrac{1}{t(t+1)}}$

76. $y=\dfrac{\theta+5}{\theta\cos\theta}$

77. $y=\sqrt[3]{\dfrac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}}$

Find the derivative of

78. $y=t^{1-e}$

79. $y=2^{\sin 3t}$

80. $y=\log_3(1+\theta\ln 3)$

81. $y=\log_{25}e^x-\log_5\sqrt{x}$

82. $y=\log_3 r\cdot\log_9 r$

83. $y=\log_7\left(\dfrac{\sin\theta\cos\theta}{e^{\theta}2^{\theta}}\right)$

84. $y=3\log_8\left(\log_2 t\right)$

85. $y=t\log_3\left(e^{(\sin t)(\ln 3)}\right)$

86. $f(x)=2^{x^2-x}$

87. $f(x)=\log_3(x+8)$

Use logarithmic differentiation to find the derivative of

88. $y=(x+1)^x$

89. $y=x^2+x^{2x}$

90. $y=(\sin x)^x$

91. $y=x^{\sin x}$

92. $y=(\ln x)^{\ln x}$

93. Find the second derivative of $y = 3\,e^{5x^3+1}$

94. Find the equation of the tangent line to $f(x) = e^x$ at the point (0, 1)

95. Find the equation of the tangent line to $f(x) = e^x$ at the point (1, e)

96. Find the equation of the tangent lines to $f(x) = 4e^{-8x}$ at the points (0, 4)

97. Find the equation of the tangent line to $y = 4xe^{-x} + 5$ at $x = 1$

98. The following formula accurately models the relationship between the size of a certain type of tumor and the amount of time that it has been growing:

$$V(t) = 450\left(1 - e - 0.0022t\right)^3$$

where *t* is in months and $V(t)$ is measured in cubic centimeters. Calculate the rate of change of tumor volume at 80 *months*.

99. A yeast culture at room temperature (68° F) is placed in a refrigerator set at a constant temperature of 38° F. After *t* hours, the temperature *T* of the culture is given approximately by

$$T = 30e^{-0.58t} + 38 \quad t \geq 0$$

What is the rate of change of temperature of the culture at the end of 1 *hour*? At the end of 4 hours?

100. A mathematical model for the average age of a group of people learning to type is given by

$$N(t) = 10 + 6\ln t \quad t \geq 1$$

Where $N(t)$ is the number of words per minute typed after *t hours* of instruction and practice (2 hours per day, 5 days per week). What is the rate of learning after 10 *hours* of instruction and practice? After 100 *hours*?

101. The population of coyotes in the northwestern portion of Alabama is given by the formula $P(t) = \left(t^2 + 100\right)\ln\left(t + 2\right)$, where *t* represents the time in years since 2000 (the year 2000 corresponds to $(t = 0)$ Find the rate of change of the coyote population in 2013 $(t = 13)$.