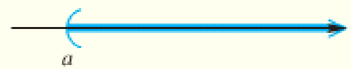
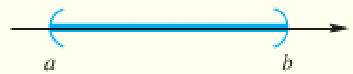
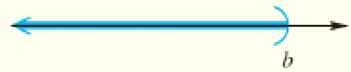
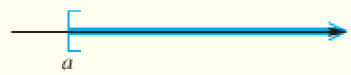
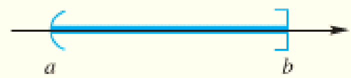
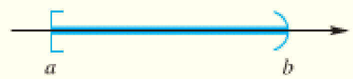
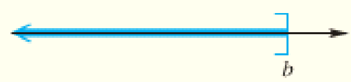
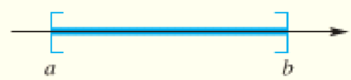
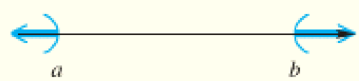
Section 1.7 – Inequalities

Notation

Type of Interval	Set	Interval Notation	Graph
Open interval	$\{x \mid x > a\}$	(a, ∞)	
	$\{x \mid a < x < b\}$	(a, b)	
	$\{x \mid x < b\}$	$(-\infty, b)$	
Other intervals	$\{x \mid x \ge a\}$	$[a, \infty)$	
	$\{x \mid a < x \le b\}$	$(a, b]$	
	$\{x \mid a \le x < b\}$	$[a, b)$	
	$\{x \mid x \le b\}$	$(-\infty, b]$	
Closed interval	$\{x \mid a \le x \le b\}$	$[a, b]$	
Disjoint interval	$\{x \mid x < a \text{ or } x > b\}$	$(-\infty, a) \cup (b, \infty)$	
All real numbers	$\{x \mid x \text{ is a real number}\}$	$(-\infty, \infty)$	

Properties of inequality

1. ***If a < b, then a + c < b + c***
2. ***If a < b and if c > 0, then ac < bc***
3. ***If a < b and if c < 0, then ac > bc***

Example

Solve $3x + 1 > 7x - 15$

Solution

$3x - 7x > -1 - 15$

$-4x > -16$ *Divide by – 4 both sides*

$\underline{x < 4}$ *or* $(-\infty\ ,\ 4)$ *or* $\{x \mid x < 4\}$

Example

Solve $\quad \dfrac{x-4}{2} \ge \dfrac{x-2}{3} + \dfrac{5}{6}$ $\qquad$ *LCD*: 2, 3, 6

Solution

$$(6)\frac{x-4}{2} \ge (6)\frac{x-2}{3} + (6)\frac{5}{6}$$

$$3(x-4) \ge 2(x-2) + 5$$

$$3x - 12 \ge 2x - 4 + 5$$

$$3x - 12 \ge 2x + 1$$

$$3x - 2x \ge 12 + 1$$

$$\underline{x \ge 13}\,|$$

Example

a) $3(x+1) > 3x + 2$

$$3x + 3 > 3x + 2$$

$$3x - 3x > -3 + 1$$

$$0 > -1 \ (\textit{True statement})$$

Sol.: $\mathbb{R}$ ***or*** $\{x|\ \textit{All Real numbers}\}$ ***or*** $(-\infty, \infty)$

b) $x + 1 \le x - 1$

$$x - x \le -1 - 1$$

$$0 \le -2$$

Sol.: $\varnothing$

Example

Solve $\ -2 < 5 + 3x < 20\ $ Give the solution set in interval notation and graph it.

Solution

$$-2 - 5 < 5 + 3x - 5 < 20 - 5$$

$$-7 < 3x < 15$$

$$-\frac{7}{3} < \frac{3}{3}x < \frac{15}{3}$$

$$-\frac{7}{3} < x < 5$$

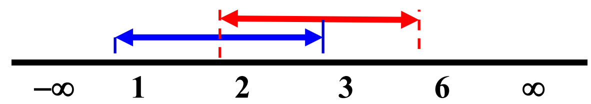
Solution: $\left(-\frac{7}{3},\ 5\right)$

Intersections of Interval $\cap$

To find the intersection, take the portion of the number line that the two graphs have in ***common***

Example

$[1, 3] \cap (2, 6) = (2, 3]$

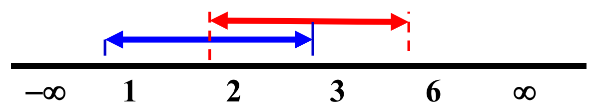


Unions of Interval $\cup$

To find the union, take the portion of the number line representing the total ***collection*** of numbers in the two graphs.

Example

$[1, 3] \cup (2, 6) = [1, 6)$



Solving an *Absolute Value* Inequality:

If X is an algebraic expression and c is a positive number,

1. The solutions of $|X| < c$ are the numbers that satisfy $-c < X < c$.
2. The solutions of $|X| > c$ are the numbers that satisfy $X < -c$ or $X > c$.

Example

Solve: $-3|5x-2| + 20 \geq -19$

Solution

$-3|5x-2| \geq -39$

$-|5x-2| \geq -13$

$|5x-2| \leq 13$

$-13 \leq 5x-2 \leq 13$

$-11 \leq 5x \leq 15$

$-\frac{11}{5} \leq x \leq 3$ or $\left[-\frac{11}{5},\ 3\right]$

Example

Solve: $18 < |6-3x|$

Solution

$|6-3x| > 18$

$6-3x < -18$	$6-3x > 18$
$-3x < -18-6$	$-3x > 18-6$
$-3x < -24$	$-3x > 12$
$\frac{-3}{-3}x > -\frac{24}{-3}$	$\frac{-3}{-3}x < \frac{12}{-3}$
$x > 8$	$x < -4$

Solution: $(-\infty,\ -4) \cup (8,\ \infty)$

Special Cases

Example

Solve the inequality $|2-5x| \geq -4$

Solution

$|2-5x| \geq -4$

It is always ***true***

$\therefore$ The solution set is: $\mathbb{R}$ All real numbers $(-\infty, \infty)$

Example

Solve the inequality $|4x-7| < -3$

Solution

$|4x-7| < -3$

Any absolute value can't be less than any negative number.

$\therefore$ No solution or $\varnothing$

Example

Solve the inequality $|5x+15| = 0$

Solution

$|5x+15| = 0$

$$5x + 15 = 0$$
$$5x = -15$$

$\therefore$ Solution: $\underline{x = -3}$

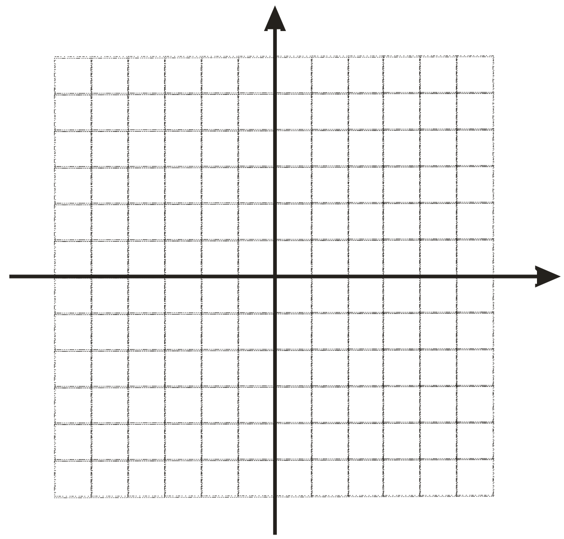
Definition of a Polynomial Inequality

A polynomial inequality is any inequality that can be put into one of the forms

$$f(x) < 0 \qquad f(x) > 0 \qquad f(x) \le 0 \qquad f(x) \ge 0$$

Where f is a polynomial function.

$f(x) = x^2 - 5x + 4 \quad (x = 1, 4)$



Procedure for Solving Polynomial Inequalities **_Example_**

1. Express the inequality in the form $f(x)$? 0	$x^2 - x < 12$ $x^2 - x - 12 < 0$
2. Solve $f(x) = 0$	$x^2 - x - 12 = 0$ $x = -3,\ 4$
3. Locate the boundary 4. Choose one test value	$-3 \quad 0 \quad 4$ $+ \quad - \quad +$
5. Write the solution set	$(-3,\ 4)$

$$\checkmark \quad ax^2 + bx + c \ge 0 \ \rightarrow \ \textit{if } a > 0 \ \Rightarrow \ x \le x_1,\ x \ge x_2$$

$$\checkmark \quad ax^2 + bx + c \le 0 \ \rightarrow \ \textit{if } a > 0 \ \Rightarrow \ x_1 \le x \le x_2$$

Example

Solve $2x^2 + 5x - 12 \ge 0$

Solution

$$2x^2 + 5x - 12 = 0$$

$$(2x-3)(x+4) = 0$$

$$x = -4,\ \frac{3}{2}$$

	-4	0	$\frac{3}{2}$	
$+$		$-$		$+$

$$\left(-\infty,\ -4\right] \cup \left[\frac{3}{2},\ \infty\right)$$

Solution: $x \le -4 \quad x \ge \frac{3}{2}$

Example

Solve: $x^3 + 3x^2 \le x + 3$

Solution

$$x^3 + 3x^2 - x - 3 = 0$$

$$x^2(x+3) - (x+3) = 0$$

$$(x+3)(x^2-1) = 0$$

$$x + 3 = 0 \qquad x^2 - 1 = 0$$

$$x = -3 \qquad x^2 = 1$$

$$x = -3 \qquad x = \pm 1$$

	-3		-1	0	1	
$-$		$+$		$-$		$+$

Solution: $x \le -3 \quad -1 \le x \le 1$ $\qquad$ $\left(-\infty,\ -3\right] \cup \left[-1,\ 1\right]$

Rational Inequality

Example

Solve: $\dfrac{2x}{x+1} \ge 1$

Solution

$\dfrac{2x}{x+1} = 1 \qquad \rightarrow \ Cond.: x+1 \ne 0 \Rightarrow \ x \ne -1$

$(x+1)\dfrac{2x}{x+1} - 1(x+1) = 0$

$2x - x - 1 = 0$

$x - 1 = 0$

$x = 1$

	-1	0	1	
$+$		$-$		$+$

Solution: $x \le -1 \quad x \ge 1$ $\qquad (-\infty,\ -1) \cup [1,\ \infty)$

Example

Solve $\dfrac{5}{x+4} \ge 1$

Solution

$\dfrac{5}{x+4} - 1 = 0 \qquad$ *Exception:* $x+4 \ne 0 \ \Rightarrow x \ne -4$

$(x+4)\dfrac{5}{x+4} - 1(x+4) = 0$

$5 - x - 4 = 0$

$x = 1$

	-4	0	1	
$-$		$+$		$-$

Solution: $-4 < x \le 1$ $\qquad (-4,\ 1]$

Example

Solve $\dfrac{2x-1}{3x+4} < 5$

Solution

$\dfrac{2x-1}{3x+4} - 5 = 0 \qquad$ *Restriction:* $3x+4 \ne 0 \ \Rightarrow \ x \ne -\dfrac{4}{3}$

$(3x+4)\dfrac{2x-1}{3x+4} - 5(3x+4) = 0$

$2x - 1 - 15x - 20 = 0$

$-13x - 21 = 0$

$x = -\dfrac{21}{13}$

	$-\frac{21}{13}$		$-\frac{4}{3}$	0	
$-$		$+$			$-$

Solution: $x < -\dfrac{21}{13} \quad x > -\dfrac{4}{3}$ $\qquad \left(-\infty,\ -\dfrac{21}{13}\right) \cup \left(-\dfrac{4}{3},\ \infty\right)$

Exercises *Section* 1.7 – Inequalities

(1 – 6) Find:

1. $(-3,0)\cap[-1,2]$

2. $(-3,0)\cup[-1,2]$

3. $(-4,0)\cap[-2,1]$

4. $(-4,0)\cup[-2,1]$

5. $(-\infty,5)\cap[1,8)$

6. $(-\infty,5)\cup[1,8)$

(7 – 45) Solve the inequality equation

7. $-3x+5>-7$

8. $2-3x\le 5$

9. $4-3x\le 7+2x$

10. $5x+11<26$

11. $3x-8\ge 13$

12. $-9x\ge 36$

13. $-4x\le 64$

14. $8x-11\le 3x-13$

15. $18x+45\le 12x-8$

16. $4(x+1)+2\ge 3x+6$

17. $8x+3>3(2x+1)+x+5$

18. $2x-11<-3(x+2)$

19. $-4(x+2)>3x+20$

20. $1-(x+3)\ge 4-2x$

21. $5(3-x)\le 3x-1$

22. $\frac{x}{4}-\frac{1}{2}\le\frac{x}{2}+1$

23. $\frac{3x}{10}+1\ge\frac{1}{5}-\frac{x}{10}$

24. $6x-(2x+3)\ge 4x-5$

25. $\frac{2x-5}{-8}\le 1-x$

26. $1-\frac{x}{2}>4$

27. $7-\frac{4}{5}x<\frac{3}{5}$

28. $\frac{x-4}{6}\ge\frac{x-2}{9}+\frac{5}{18}$

29. $\frac{4x-3}{6}+2\ge\frac{2x-1}{12}$

30. $4(3x-2)-3x<3(1+3x)-7$

31. $3(x-8)-2(10-x)<5(x-1)$

32. $8(x+1)\le 7(x+5)+x$

33. $4(x-1)\ge 3(x-2)+x$

34. $7(x+4)-13>12+13(3+x)$

35. $-2[7x-(2x-3)]<-2(x+1)$

36. $6-\frac{2}{3}(3x-12)\le\frac{2}{5}(10x+50)$

37. $\frac{2}{7}(7-21x)-4<10-\frac{3}{11}(11x-11)$

38. $3[3(x+5)+8x+7]+5[3(x-6)-2(3x-5)]<2(4x+3)$

39. $5[3(2-3x)-2(5-x)]-6[5(x-2)-2(4x-3)]<3x+19$

40. $0\le 3x-1\le 10$

41. $0\le 1-3x\le 10$

42. $0\le 2x+6\le 54$

43. $-3\le\frac{2}{3}x-5\le -1$

44. $-6\le 6x+3\le 21$

45. $1\le 2x+3\le 11$

(**46 – 85**) Solve the inequality equation

46. $|x| < 2$

47. $|x| \ge 2$

48. $|x-2| < 1$

49. $|x-1| < 4$

50. $|x+2| \ge 1$

51. $|x+1| \ge 4$

52. $|3x+5| < 17$

53. $|5x-2| < 13$

54. $|5x-2| \ge 13$

55. $|2(x-1)+4| \le 8$

56. $|3(x-1)+2| \le 20$

57. $\left|\frac{2x+6}{3}\right| > 2$

58. $\left|\frac{3x-3}{4}\right| < 6$

59. $\left|\frac{2x+2}{4}\right| \ge 2$

60. $\left|\frac{3x-3}{9}\right| \le 1$

61. $\left|3-\frac{2x}{3}\right| > 5$

62. $\left|3-\frac{3x}{4}\right| < 9$

63. $|x-2| < -1$

64. $|x+2| < -3$

65. $|x+6| > -10$

66. $|x+2| > -8$

67. $|x+2| + 9 \le 16$

68. $|x-2| + 4 \ge 5$

69. $2|2x-3| + 10 > 12$

70. $3|2x-1| + 2 < 8$

71. $-4|1-x| < -16$

72. $-2|5-x| < -6$

73. $3 \le |2x-1|$

74. $9 \le |4x+7|$

75. $12 < \left|-2x+\frac{6}{7}\right| + \frac{3}{7}$

76. $4 + \left|3-\frac{x}{3}\right| \ge 9$

77. $|x-2| < 5$

78. $|2x+1| < 7$

79. $|5x+2| - 2 < 3$

80. $|2-7x| - 1 > 4$

81. $|3x-4| < 2$

82. $|2x+5| \ge 3$

83. $|12-9x| \ge -12$

84. $|6-3x| < -11$

85. $|7+2x| < 0$

(**86 – 107**) Solve the inequality equation

86. $x^2 - 7x + 10 > 0$

87. $2x^2 - 9x \le 18$

88. $x^2 - 5x + 4 > 0$

89. $x^2 + x - 2 > 0$

90. $x^2 - 4x + 12 < 0$

91. $x^2 + 7x > 0$

92. $x^2 - 49 < 0$

93. $x^2 - 5x \ge 0$

94. $x^2 - 16 \le 0$

95. $x^2 + 7x + 10 < 0$

96. $x^2 - 3x \ge 28$

97. $x^2 + 5x + 6 < 0$

98. $x^2 < -x + 30$

99. $x^3 - 3x^2 - 9x + 27 < 0$

100. $x^3 - x > 0$

101. $x^3 + 3x^2 \le x + 3$

102. $x^3 + x^2 \ge 48x$

103. $x^3 - x^2 - 16x + 16 < 0$

104. $x^3 + x^2 - 9x - 9 > 0$

105. $x^3 + 3x^2 - 4x - 12 \ge 0$

106. $x^4 - 20x^2 + 64 \le 0$

107. $x^4 - 10x^2 + 9 \ge 0$

(**108 – 130**) Solve the inequality equation

108. $\frac{x+4}{x-1}<0$

109. $\frac{x-2}{x+3}>0$

110. $\frac{x-5}{x+8}\geq 3$

111. $\frac{x-4}{x+6}\leq 1$

112. $\frac{x}{2x+7}\geq 4$

113. $\frac{x}{3x-5}\leq -5$

114. $\frac{x+2}{x-5}\leq 2$

115. $\frac{3x+1}{x-2}\geq 4$

116. $\frac{x}{x-3}>0$

117. $\frac{x-3}{x+2}\geq 0$

118. $\frac{x-2}{x+2}\leq 2$

119. $\frac{x+2}{x-2}\geq 2$

120. $\frac{x+2}{3+2x}\leq 5$

121. $\frac{x+6}{x-14}\geq 1$

122. $\frac{x-3}{x+4}\geq \frac{x+2}{x-5}$

123. $\frac{x-4}{x+3}-\frac{x+2}{x-1}\leq 0$

124. $\frac{2x-1}{x+3}\geq \frac{x+1}{3x+1}$

125. $\frac{(x+1)(x-4)}{x-2}<0$

126. $\frac{x(x-4)}{x+5}>0$

127. $\frac{6x^2-11x-10}{x}>0$

128. $\frac{3x^2-2x-8}{x-1}\geq 0$

129. $\frac{x^2-6x+9}{x-5}\leq 0$

130. $\frac{x^2+10x+25}{x+1}\leq 0$