Solution   *Section* **2.7 – Derivatives of Exponential and Logarithmic Functions**

Exercise

Find the derivative of $f(x) = e^{3x}$

Solution

$$f'(x) = 3e^{3x}$$

Exercise

Find the derivative of $f(x) = e^{-2x^3}$

Solution

$$f'(x) = e^{-2x^3} \frac{d}{dx}[-2x^3]$$

$$= e^{-2x^3}\left[-6x^2\right]$$

$$= -\frac{6x^2}{e^{2x^3}}$$

Exercise

Find the derivative of $f(x) = 4e^{x^2}$

Solution

$$f'(x) = 4e^{x^2} \frac{d}{dx}[x^2]$$

$$= 4e^{x^2}(2x)$$

$$= 8xe^{x^2}$$

Exercise

Find the derivative of $f(x) = e^{-2x}$

Solution

$$f'(x) = -2e^{-2x}$$

$$= -\frac{2}{e^{2x}}$$

Exercise

Find the derivative of $f(x) = x^2 e^x$

Solution

$$f'(x) = e^x \frac{d}{dx}[x^2] + x^2 \frac{d}{dx}[e^x]$$

$$= e^x(2x) + x^2 e^x$$

$$= xe^x(2+x)$$

Exercise

Find the derivative of $f(x) = \frac{e^x + e^{-x}}{2}$

Solution

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

$$f'(x) = \frac{1}{2}\left(\frac{d}{dx}[e^x] + \frac{d}{dx}[e^{-x}]\right)$$

$$= \frac{1}{2}(e^x - e^{-x})$$

Exercise

Find the derivative of $f(x) = \dfrac{e^x}{x^2}$

$$f'(x) = \frac{x^2 e^x - e^x(2x)}{x^4}$$

$$= \frac{x^2 e^x - 2xe^x}{x^4}$$

$$= \frac{xe^x(x-2)}{x^4}$$

$$= \underline{\left.\frac{e^x(x-2)}{x^3}\right|}$$

Exercise

Find the derivative of $f(x) = x^2 e^x - e^x$

$$f'(x) = e^x \frac{d}{dx}[x^2] + x^2 \frac{d}{dx}[e^x] - \frac{d}{dx}[e^x]$$

$$= e^x(2x) + x^2 e^x - e^x$$

$$= \underline{\left.e^x(x^2 + 2x - 1)\right|}$$

Exercise

Find the derivative of $f(x) = (1 + 2x)e^{4x}$

$$f'(x) = (2)e^{4x} + (1 + 2x)(4e^{4x})$$

$$= 2e^{4x} + (1 + 2x)(4e^{4x})$$

$$= 2e^{4x}(1 + 2(1 + 2x))$$

$$= 2e^{4x}(1 + 2 + 4x)$$

$$= \underline{\left.2e^{4x}(3 + 4x)\right|}$$

Exercise

Find the derivative of $y = x^2 e^{5x}$

Solution

$$y' = x^2 \left(5e^{5x} \right) + 2x \left(e^{5x} \right)$$

$$\underline{= xe^{5x} \left(5x + 2 \right)}$$

Exercise

Find the derivative of $f(x) = \dfrac{100,000}{1 + 100e^{-0.3x}}$

Solution

$$f'(x) = \frac{(0)\left(1 + 100e^{-0.3x} \right) - 100,000 \left(0 + (-0.3)100e^{-0.3x} \right)}{\left(1 + 100e^{-0.3x} \right)^2}$$

$$= \frac{-100,000 \left(-30e^{-0.3x} \right)}{\left(1 + 100e^{-0.3x} \right)^2}$$

$$\underline{= \frac{3,000,000e^{-0.3x}}{\left(1 + 100e^{-0.3x} \right)^2}}$$

Exercise

Find the derivative of $y = x^2 e^{-2x}$

Solution

$$y' = 2xe^{-2x} - 2x^3 e^{-2x}$$

$$\underline{= 2xe^{-2x} \left(1 - x^2 \right)}$$

Exercise

Find the derivative of $y = \dfrac{e^x + e^{-x}}{x}$

Solution

$$f = e^x + e^{-x} \qquad g = x$$
$$f' = e^x - e^{-x} \qquad g' = 1$$

$$y = \frac{\left(e^x - e^{-x}\right)x - \left(e^x + e^{-x}\right)}{x^2}$$

$$= \frac{xe^x - xe^{-x} - e^x - e^{-x}}{x^2}$$

$$= \frac{(x-1)e^x - (x+1)e^{-x}}{x^2}$$

Exercise

Find the derivative of $y = \sqrt{e^{2x^2} + e^{-2x^2}}$

Solution

$$y = \sqrt{e^{2x^2} + e^{-2x^2}} = \left(e^{2x^2} + e^{-2x^2}\right)^{1/2} = U^{1/2}$$

$$U = e^{2x^2} + e^{-2x^2} \qquad\qquad \left(e^{2x^2}\right)' = \left(2x^2\right)' e^{2x^2} = 4xe^{2x^2}$$

$$U' = 4xe^{2x^2} - 4xe^{-2x^2}$$

$$y' = \frac{1}{2}\left(4xe^{2x^2} - 4xe^{-2x^2}\right)\left(e^{2x^2} + e^{-2x^2}\right)^{-1/2}$$

$$= \frac{1}{2} \frac{4x\left(e^{2x^2} - e^{-2x^2}\right)}{\left(e^{2x^2} + e^{-2x^2}\right)^{1/2}}$$

$$= \frac{2x\left(e^{2x^2} - e^{-2x^2}\right)}{\sqrt{e^{2x^2} + e^{-2x^2}}}$$

Exercise

Find the derivative of $y = \dfrac{x}{e^{2x}}$

$$f = x \qquad\qquad g = e^{2x}$$

$$f' = 1 \qquad\qquad g' = 2e^{2x}$$

$$y' = \frac{1\left(e^{2x}\right) - x\left(2e^{2x}\right)}{\left(e^{2x}\right)^2}$$

$$= \frac{e^{2x}(1 - 2x)}{\left(e^{2x}\right)^2}$$

$$= \underline{\frac{1 - 2x}{e^{2x}}}$$

Exercise

Find the second derivative of $y = 3\,e^{5x^3 + 1}$

$$y' = 3(15x^2)e^{5x^3 + 1}$$

$$y' = 45x^2 e^{5x^3 + 1}$$

$$f = x^2 \qquad\qquad g = e^{5x^3 + 1}$$

$$f' = 2x \qquad\qquad g' = 15x^2 e^{5x^3 + 1}$$

$$y'' = 45\left(2xe^{5x^3 + 1} + \left(x^2\right)15x^2 e^{5x^3 + 1}\right)$$

$$= 45e^{5x^3 + 1}\left(2x + 15x^4\right)$$

$$= \underline{45xe^{5x^3 + 1}\left(2 + 15x^3\right)}$$

Exercise

Find the derivative of $y = \ln \sqrt{x+5}$

Solution

$$y = \ln(x+5)^{1/2}$$

$$= \frac{1}{2}\ln(x+5)$$

$$y' = \frac{1}{2(x+5)}$$

Exercise

Find the Derivatives of $y = (3x+7)\ln(2x-1)$

Solution

$$f = 3x+7 \quad f' = 3$$

$$g = \ln(2x-1) \quad g' = \frac{2}{2x-1}$$

$$y' = 3x\ln(2x-1) + \frac{2(3x+7)}{2x-1}$$

Exercise

Find the Derivatives of $y = e^{x^2}\ln x$

Solution

$$f = e^{x^2} \quad f' = 2xe^{x^2}$$

$$g = \ln x \quad g' = \frac{1}{x}$$

$$y' = 2xe^{x^2}\ln x + \frac{e^{x^2}}{x}$$

73

Exercise

Find the Derivatives of $y = \log_7 \sqrt{4x-3}$

Solution

$$y' = \frac{1}{\ln 7} \frac{\left(\sqrt{4x-3}\right)'}{\sqrt{4x-3}}$$

$$\frac{d}{dx}\left[\log_a |g(x)|\right] = \frac{1}{(\ln a)} \cdot \frac{g'(x)}{g(x)}$$

$$\left(\sqrt{4x-3}\right)' = \left((4x-3)^{1/2}\right)'$$

$$= \frac{1}{2}(4)(4x-3)^{-1/2} \qquad \left(U^n\right)' = nU'U^{n-1}$$

$$= 2(4x-3)^{-1/2}$$

$$y' = \frac{1}{\ln 7} \frac{2(4x-3)^{-1/2}}{\sqrt{4x-3}}$$

$$= \frac{1}{\ln 7} \frac{2}{(4x-3)^{1/2}(4x-3)^{1/2}}$$

$$= \frac{1}{\ln 7} \frac{2}{(4x-3)}$$

Exercise

Find the Derivatives of $f(x) = \ln \sqrt[3]{x+1}$

Solution

$$f(x) = \ln(x+1)^{1/3}$$

$$= \frac{1}{3}\ln(x+1)$$

$$u = x+1 \Rightarrow \frac{du}{dx} = 1$$

$$f'(x) = \frac{1}{3}\frac{1}{x+1}$$

$$= \frac{1}{3(x+1)}$$

Exercise

Find the Derivatives of $f(x) = \ln\left[x^2\sqrt{x^2+1}\right]$

Solution

$f(x) = \ln(x^2) + \ln\sqrt{x^2+1}$ *Product Property*

$f(x) = \ln(x^2) + \ln(x^2+1)^{1/2}$

$f(x) = 2\ln x + \frac{1}{2}\ln\left(x^2+1\right)$ *Power Property*

$f'(x) = 2\frac{1}{x} + \frac{1}{2}\frac{2x}{x^2+1}$ *Differentiate*

$\qquad = \frac{2}{x} + \frac{x}{x^2+1}$

Exercise

Find the Derivatives of $y = \ln\dfrac{1+e^x}{1-e^x}$

Solution

$y = \ln\left(1+e^x\right) - \ln\left(1-e^x\right)$

$y' = \dfrac{e^x}{1+e^x} - \dfrac{-e^x}{1-e^x}$

$\quad = \dfrac{e^x}{1+e^x} + \dfrac{e^x}{1-e^x}$

$\quad = \dfrac{e^x - e^{2x} + e^x + e^{2x}}{\left(1+e^x\right)\left(1-e^x\right)}$

$\quad = \dfrac{2e^x}{\left(1+e^x\right)\left(1-e^x\right)}$

Exercise

Find the Derivatives of $y = \ln \dfrac{x^2}{x^2 + 1}$

Solution

$$y = \ln x^2 - \ln x^2 + 1$$

$$y' = \frac{2x}{x^2} - \frac{2x}{x^2 + 1}$$

$$= \frac{2}{x} - \frac{2x}{x^2 + 1}$$

Exercise

Find the Derivatives of $y = \ln\left[\dfrac{x^2(x+1)^3}{(x+3)^{1/2}} \right]$

Solution

$$y = \ln\left[x^2(x+1)^3 \right] - \ln(x+3)^{1/2} \qquad \textit{Quotient Rule}$$

$$= \ln x^2 + \ln(x+1)^3 - \ln(x+3)^{1/2} \qquad \textit{Product Rule}$$

$$= 2\ln x + 3\ln(x+1) - \frac{1}{2}\ln(x+3) \qquad \textit{Power Rule}$$

$$y' = \frac{2}{x} + \frac{3}{x+1} - \frac{1}{2(x+3)}$$

Exercise

Find the Derivatives of $y = \ln(x^2 + 1)$

Solution

$$y' = \frac{2x}{x^2 + 1} \qquad\qquad (\ln U)' = \frac{U'}{U}$$

76

Exercise

Find the Derivatives of $y = \dfrac{\ln x}{e^{2x}}$

$$y' = \frac{e^{2x}(1/x) - \ln x(2e^{2x})}{e^{4x}}$$

$$= \frac{e^{2x} - 2x\ln x(e^{2x})}{e^{4x}}$$

$$= \frac{e^{2x}(1 - 2x\ln x)}{e^{4x}}$$

Exercise

Find the Derivatives of $f(x) = \ln(x^2 - 4)$

$$\text{Let } u = x^2 - 4 \Rightarrow \frac{du}{dx} = 2x$$

$$f'(x) = \frac{1}{u}\frac{du}{dx}$$

$$= \frac{1}{x^2 - 4}(2x)$$

$$= \frac{2x}{x^2 - 4}$$

Exercise

Find the Derivatives of $f(x) = x^2 \ln x$

$$f' = x^2 \frac{d}{dx}[\ln x] + \ln x \frac{d}{dx}[x^2] \qquad (fg)' = f'g + fg'$$

$$= x^2\left(\frac{1}{x}\right) + 2x\ln x$$

$$= x + 2x\ln x$$

$$= x(1 + 2\ln x)$$

77

Exercise

Find the Derivatives of $f(x) = -\dfrac{\ln x}{x^2}$

Solution

$$f' = -\frac{x^2 \frac{d}{dx}[\ln x] - \ln x \frac{d}{dx}\left[x^2\right]}{\left(x^2\right)^2}$$

$$= -\frac{x^2 \frac{1}{x} - 2x \ln x}{x^4}$$

$$= -\frac{x - 2x \ln x}{x^4}$$

$$= -\frac{x(1 - 2\ln x)}{x^4}$$

$$= -\frac{1 - 2\ln x}{x^3}$$

Exercise

Find the Derivative of $f(x) = \dfrac{e^{\sqrt{x}}}{\ln\left(\sqrt{x}+1\right)}$

Solution

$$f = e^{\sqrt{x}} \qquad U = \sqrt{x} = x^{1/2} \Rightarrow U' = \tfrac{1}{2}x^{-1/2} \qquad f' = \tfrac{1}{2}x^{-1/2}e^{\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

$$g = \ln\left(\sqrt{x}+1\right) \qquad U = x^{1/2}+1 \Rightarrow U' = \tfrac{1}{2}x^{-1/2} \qquad g' = \frac{\tfrac{1}{2}x^{-1/2}}{\sqrt{x}+1} = \frac{1}{2x^{1/2}\left(\sqrt{x}+1\right)}$$

$$f'(x) = \frac{\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}\ln\left(\sqrt{x}+1\right) - \dfrac{1}{2\sqrt{x}\left(\sqrt{x}+1\right)}e^{\sqrt{x}}}{\left(\sqrt{x}+1\right)^2}$$

$$= \frac{\dfrac{\left(\sqrt{x}+1\right)e^{\sqrt{x}}\ln\left(\sqrt{x}+1\right) - e^{\sqrt{x}}}{2\sqrt{x}\left(\sqrt{x}+1\right)}}{\left(\ln\left(\sqrt{x}+1\right)\right)^2}$$

$$= \frac{e^{\sqrt{x}}\left[\left(\sqrt{x}+1\right)\ln\left(\sqrt{x}+1\right) - 1\right]}{2\sqrt{x}\left(\sqrt{x}+1\right)\left(\ln\left(\sqrt{x}+1\right)\right)^2}$$

78

Exercise

Find the Derivative of $f(x) = e^{2x}\ln\left(xe^x + 1\right)$

Solution

$$f = e^{2x} \qquad\qquad U = 2x \rightarrow U' = 2 \qquad\qquad f' = 2e^{2x}$$

$$g = \ln\left(xe^x + 1\right) \quad U = xe^x + 1 \rightarrow U' = e^x + xe^x \quad g' = \frac{e^x + xe^x}{xe^x + 1}$$

$$f'(x) = 2e^{2x}\ln\left(xe^x + 1\right) + e^{2x}\frac{e^x + xe^x}{xe^x + 1}$$

$$= e^{2x}\left[2\ln\left(xe^x + 1\right) + \frac{e^x(1+x)}{xe^x + 1}\right]$$

Exercise

Find the Derivative of $f(x) = \dfrac{xe^x}{\ln\left(x^2 + 1\right)}$

Solution

$$f = xe^x \qquad\qquad f' = e^x + xe^x$$

$$g = \ln\left(x^2 + 1\right) \qquad\qquad g' = \frac{2x}{x^2 + 1}$$

$$f'(x) = \frac{e^x(1+x)\ln\left(x^2 + 1\right) - \frac{2x}{x^2+1}xe^x}{\left[\ln\left(x^2 + 1\right)\right]^2}$$

$$= \frac{e^x\left[(1+x)\ln\left(x^2 + 1\right) - \frac{2x^2}{x^2+1}\right]}{\left[\ln\left(x^2 + 1\right)\right]^2}$$

$$= \frac{e^x\left[\frac{\left(x^2+1\right)(1+x)\ln\left(x^2+1\right) - 2x^2}{x^2+1}\right]}{\left[\ln\left(x^2 + 1\right)\right]^2}$$

$$= \frac{e^x\left[\left(x^2 + 1\right)(1+x)\ln\left(x^2 + 1\right) - 2x^2\right]}{\left(x^2 + 1\right)\left[\ln\left(x^2 + 1\right)\right]^2}$$

79

Exercise

Find the derivative $\quad f(x) = 2\ln\left(x^2 - 3x + 4\right)$

Solution

$$f'(x) = 2\frac{2x - 3}{x^2 - 3x + 4}$$
$$= \underline{\frac{4x - 6}{x^2 - 3x + 4}}$$

Exercise

Find the derivative $\quad f(x) = e^{x^2 + 3x + 1}$

Solution

$$f'(x) = \underline{(2x + 3)e^{x^2 + 3x + 1}}$$

Exercise

Find the derivative $\quad f(x) = 3\ln\left(1 + x^2\right)$

Solution

$$f'(x) = 3\frac{2x}{1 + x^2}$$
$$= \underline{\frac{6x}{1 + x^2}}$$

Exercise

Find the derivative $\quad f(x) = (1 + \ln x)^3$

Solution

$$f'(x) = 3(1 + \ln x)^2 (1 + \ln x)'$$
$$= 3(1 + \ln x)^2 \left(\frac{1}{x}\right)$$
$$= \frac{3}{x}(1 + \ln x)^2$$

Exercise

Find the derivative $\quad f(x)=(x-2\ln x)^4$

Solution

$$f'(x)=4(x-2\ln x)^3(x-2\ln x)'$$

$$=4(x-2\ln x)^3\left(1-\frac{2}{x}\right)$$

$$=4(x-2\ln x)^3\left(\frac{x-2}{x}\right)$$

$$=\frac{4x-8}{x}(x-2\ln x)^3$$

Exercise

Find the derivative $\quad f(x)=\dfrac{e^x}{x^2+1}$

Solution

$$u=e^x \quad v=x^2+1$$

$$u'=e^x \quad v'=2x$$

$$f'(x)=\frac{e^x\left(x^2+1\right)-2xe^x}{\left(x^2+1\right)^2}$$

$$=\frac{\left(x^2+1-2x\right)e^x}{\left(x^2+1\right)^2}$$

Exercise

Find the derivative $\quad f(x)=\dfrac{1-e^x}{1+e^x}$

Solution

$$u=1-e^x \quad v=1+e^x$$

$$u'=-e^x \quad v'=e^x$$

$$f'(x)=\frac{-e^x\left(1+e^x\right)-e^x\left(1-e^x\right)}{\left(1+e^x\right)^2}$$

81

$$= \frac{-e^x - e^{2x} - e^x + e^{2x}}{\left(1 + e^x\right)^2}$$

$$= -\frac{2e^x}{\left(1 + e^x\right)^2} \Bigg|$$

Exercise

Find the derivative $\quad f(x) = \dfrac{\ln x}{1 + x}$

Solution

$$u = \ln x \quad v = 1 + x$$

$$u' = \frac{1}{x} \qquad v' = 1$$

$$f'(x) = \frac{\left(\frac{1}{x}\right)(1 + x) - \ln x}{(1 + x)^2}$$

$$= \frac{1}{x} \frac{1 + x - x \ln x}{(1 + x)^2}$$

$$= \frac{1 + x - x \ln x}{x(1 + x)^2} \Bigg|$$

Exercise

Find the derivative $\quad f(x) = \dfrac{2x}{1 + \ln x}$

Solution

$$u = 2x \quad v = 1 + \ln x$$

$$u' = 2 \qquad v' = \frac{1}{x}$$

$$f'(x) = \frac{2(1 + \ln x) - (2x)\frac{1}{x}}{(1 + \ln x)^2}$$

$$= \frac{2 + 2\ln x - 2}{(1 + \ln x)^2}$$

$$= \frac{2\ln x}{(1 + \ln x)^2} \Bigg|$$

Exercise

Find the derivative $f(x) = x^2 e^x$

Solution

$$u = x^2 \quad v = e^x$$
$$u' = 2x \quad v' = e^x$$

$$f'(x) = 2xe^x + x^2 e^x$$
$$= \left(2x + x^2\right)e^x$$

Exercise

Find the derivative $f(x) = x^3 \ln x$

Solution

$$u = x^3 \quad v = \ln x$$
$$u' = 3x^2 \quad v' = \frac{1}{x}$$

$$f'(x) = 3x^2 \ln x + x^3 \frac{1}{x}$$
$$= 3x^2 \ln x + x^2$$
$$= \left(3\ln x + 1\right)x^2$$

Exercise

Find the derivative $f(x) = 6x^4 \ln x$

Solution

$$u = 6x^4 \quad v = \ln x$$
$$u' = 24x^3 \quad v' = \frac{1}{x}$$

$$f'(x) = 24x^3 \ln x + 6x^4 \frac{1}{x}$$
$$= 24x^3 \ln x + 6x^3$$
$$= 6x^3 \left(4\ln x + 1\right)$$

Exercise

Find the derivative $\quad f(x) = 2x^3 e^x$

$$u = 2x^3 \quad v = e^x$$
$$u' = 6x^2 \quad v' = e^x$$
$$f'(x) = 6x^2 e^x + 2x^3 e^x$$
$$= 2x^2 e^x (3+x)$$

Exercise

Find the derivative $\quad f(x) = \dfrac{3e^x}{1+e^x}$

$$u = 3e^x \quad v = 1 + e^x$$
$$u' = 3e^x \quad v' = e^x$$

$$f'(x) = \frac{3e^x \left(1 + e^x\right) - 3e^x e^x}{\left(1 + e^x\right)^2}$$

$$= \frac{3e^x + 3e^{2x} - 3e^{2x}}{\left(1 + e^x\right)^2}$$

$$= \frac{3e^x}{\left(1 + e^x\right)^2}$$

Exercise

Find the derivative $\quad f(x) = 5e^x + 3x + 1$

$$f'(x) = 5e^x + 3$$

Exercise

Find the derivative $f(x) = \dfrac{\ln x}{2x + 5}$

Solution

$$u = \ln x \quad v = 2x + 5$$

$$u' = \frac{1}{x} \quad v' = 2$$

$$f'(x) = \frac{\frac{1}{x}(2x+5) - (2)\ln x}{(2x+5)^2} \cdot \frac{x}{x}$$

$$= \frac{2x + 5 - 2x\ln x}{x(2x+5)^2}$$

Exercise

Find the derivative $f(x) = -2\ln x + x^2 - 4$

Solution

$$f'(x) = -\frac{2}{x} + 2x$$

Exercise

Find the derivative $f(x) = e^x + x - \ln x$

Solution

$$f'(x) = e^x + 1 - \frac{1}{x}$$

Exercise

Find the derivative $f(x) = \ln x + 2e^x - 3x^2$

Solution

$$f'(x) = \frac{1}{x} + 2e^x - 6x$$

Exercise

Find the derivative $\quad f(x) = \ln x^8$

Solution

$f(x) = \ln x^8 = 8\ln x$

$\underline{f'(x) = \dfrac{8}{x}}$

Power Rule

$(\ln x)' = \dfrac{1}{x}$

Exercise

Find the derivative $\quad f(x) = 5x - \ln x^5$

Solution

$f(x) = 5x - \ln x^5$

$\qquad = 5x - 5\ln x$

$\underline{f'(x) = 5 - \dfrac{5}{x}}$

Power Rule

$(\ln x)' = \dfrac{1}{x}$

Exercise

Find the derivative $\quad f(x) = \ln x^2 + 4e^x$

Solution

$f(x) = 2\ln x + 4e^x$

$\underline{f'(x) = \dfrac{2}{x} + 4e^x}$

Power Rule

$(\ln x)' = \dfrac{1}{x}$

Exercise

Find the derivative $\quad f(x) = \ln x^{10} + 2\ln x$

Solution

$f(x) = 10\ln x + 2\ln x$

$\qquad = 12\ln x$

$\underline{f'(x) = \dfrac{12}{x}}$

Power Rule

$(\ln x)' = \dfrac{1}{x}$

86

Exercise

The percentage of people of any particular age group that will die in a given year may be approximated by the formula

$$P(t) = 0.00239e^{0.0957t}$$

Where t is the age of the person in years

Solution

a) Find $P(25)$

$$P(25) = 0.00239e^{0.0957(25)} = \underline{0.02615}$$

b) Find $P'(25)$

$$P'(t) = 0.000228723e^{0.0957t}$$

$$P'(25) = 0.000228723e^{0.0957(25)} = \underline{0.0025}$$

Exercise

Find the equations of the tangent lines to $f(x) = e^x$ at the points (0, 1)

Solution

$$f'(x) = e^x$$

$$(0, 1) \Rightarrow m = f'(x = 0)$$

$$= e^0$$

$$= 1$$

$$y - y_1 = m\left(x - x_1\right)$$

$$y - 1 = 1(x - 0)$$

$$y - 1 = x$$

$$\underline{y = x + 1}$$

Exercise

Find the equations of the tangent lines to $f(x) = e^x$ at the points (1, e)

Solution

$$f'(x) = e^x$$

$(1, e) \Rightarrow m = f'(x = 1) = e^1 = e$

$$y - e = e(x - 1)$$

$$y - e = ex - e$$

$$\underline{y = ex}$$

Exercise

Find the equations of the tangent lines to $y = 4xe^{-x} + 5$ at $x = 1$

Solution

$$y' = 4e^{-x} - 4xe^{-x} = 4e^{-x}(1 - x)$$

$$= 4e^{-x}(1 - x)$$

$$m = y'(x = 1)$$

$$= 4e^{-1}(1 - 1) = 0$$

$$\Rightarrow x = 1 \rightarrow y = 4e^{-1} + 5$$

$$\left(1, 4e^{-1} + 5\right)$$

$$y - (4e^{-1} + 5) = 0(x - 1)$$

$$\underline{y = 4e^{-1} + 5}$$

Exercise

Find the equation of the tangent lines to $f(x) = 4e^{-8x}$ at the points $(0, 4)$

Solution

$$f'(x) = -32e^{-8x}$$

$$m = f'(0) = -32e^{-8(0)} = -32$$

$$y - 4 = -32(x - 0) \qquad\qquad y - y_1 = m\left(x - x_1\right)$$

$$y - 4 = -32x$$

$$\boxed{y = -32x + 4}$$

Exercise

Assume the cost of a gallon of milk is $2.90. With continuous compounding, find the time it would take the cost to be 5 times as much (to the nearest tenth of a year), at an annual inflation rate of 6 %.

Solution

$$A = Pe^{rt}$$

$$5(2.90) = 2.9e^{.06t} \qquad \textit{Divide both sides by 2.9}$$

$$5 = e^{.06t} \qquad \textit{"ln" both sides}$$

$$\ln 5 = \ln e^{.06t} \qquad \textit{use inverse property}$$

$$.06t = \ln 5$$

$$\left\lfloor t = \frac{\ln 5}{.06} = 26.8 \text{ years} \right\rfloor$$

Exercise

The sales in thousands of a new type of product are given by $S(t) = 30 - 90e^{-0.5t}$, where t represents time in years. Find the rate of change of sales at the time when $t = 3$

Solution

$$S' = -90(-.5)e^{-0.5t}$$

$$= 45e^{-0.5t}$$

$$S'(t = 3) = 45e^{-0.5(3)}$$

$$= 10.04$$

The rate of change of sales at the time when $t = 3$ is 10,040.

Exercise

A company's total cost, in millions of dollars, is given by $C(t) = 300 - 70e^{-t}$ where t = time in years. Find the marginal cost when $t = 3$.

Solution

$$C'(t) = -70(-1)e^{-t}$$

$$= 70e^{-t}$$

$$C'(t = 3) = 70e^{-3} = 3.485$$

The marginal cost is $3,485,000.

Exercise

A company's total cost, in millions of dollars, is given by $C(t) = 280 - 30e^{-t}$ where t = time in years. Find the marginal cost when $t = 2$.

Solution

$$C'(t) = 30e^{-t}$$

$$C'(t = 2) = 30e^{-2} = 4.06$$

The marginal cost is $4,060,000.

Exercise

The demand function for a certain book is given by the function $x = D(p) = 70e^{-0.006p}$. Find the marginal demand $D'(p)$

Solution

$$D'(p) = 70(-.006)e^{-0.006p}$$

$$= -.42e^{-0.006p}$$

Exercise

Suppose that the amount in grams of a radioactive substance present at time t (in years) is given by $A(t) = 840e^{-0.63t}$. Find the rate of change of the quantity present at the time when $t = 2$.

Solution

$$A'(t) = 840(-0.63)e^{-0.63t}$$

$$= -529.2e^{-0.63t}$$

$$A'(t = 2) = -529.2e^{-0.63(2)}$$

$$= -150.11$$

Exercise

Researchers have found that the maximum number of successful trials that a laboratory rat can complete in a week is given by

$$P(t) = 53\left(1 - e^{-0.4t}\right)$$

where t is the number of weeks the rat has been trained. Find the rate of change $P'(t)$.

$$P'(t) = 53\left(-(-.4)e^{-0.4t}\right)$$

$$= 21.2e^{-0.4t}$$

Exercise

When a telephone wire is hung between two poles, the wire forms a U-shape curve called a Catenary. For instance, the function $y = 30\left(e^{x/60} + e^{-x/60}\right)$ $-30 \le x \le 30$ models the shape of the telephone wire strung between two poles that are 60 ft. apart (x & y are measured in ft.). Show that the lowest point on the wire is midway between two poles. How much does the wire sag between the two poles?

Solution

$$y' = 30\left(\frac{1}{60}e^{x/60} - \frac{1}{60}e^{-x/60}\right)$$

$$= \frac{1}{2}\left(e^{x/60} - e^{-x/60}\right)$$

Find the critical number(s)

$$y' = 0$$

$$\frac{1}{2}\left(e^{x/60} - e^{-x/60}\right) = 0$$

$$e^{x/60} - e^{-x/60} = 0$$

$$e^{x/60} = e^{-x/60}$$

$$\frac{x}{60} = -\frac{x}{60}$$

$$\Rightarrow x = 0$$

$$y(x = -30) = 30\left(e^{-30/60} + e^{-(-30)/60}\right) \approx 67.7\,ft$$

$$y(x = 0) = 30\left(e^{0} + e^{0}\right) = 30(2) = 60\,ft$$

$$y(x = 30) = 30\left(e^{30/60} + e^{-(30)/60}\right) \approx 67.7\,ft$$

Sag 7.7 *ft*

Exercise

Find $f''(x)$ for $f(x) = \dfrac{\ln x}{7x}$, then find $f''(0)$ and $f''(2)$

Solution

$$f(x) = \frac{\ln x}{7x}$$

$$f = \ln x \qquad f' = \frac{1}{x}$$
$$g = 7x \qquad g' = 7$$

$$f'(x) = \frac{\frac{1}{x}(7x) - 7\ln x}{(7x)^2}$$

$$= \frac{7 - 7\ln x}{49x^2}$$

$$= \frac{7(1 - \ln x)}{49x^2}$$

$$= \frac{1 - \ln x}{7x^2}$$

$$f = 1 - \ln x \qquad f' = -\frac{1}{x}$$
$$g = 7x^2 \qquad g' = 14x$$

$$f''(x) = \frac{-\frac{1}{x}(7x^2) - 14x(1 - \ln x)}{(7x^2)^2}$$

$$= \frac{-7x - 14x + 14x\ln x}{49x^4}$$

$$= \frac{-21x + 14x\ln x}{49x^4}$$

$$= \frac{7x(-3 + 2\ln x)}{49x^4}$$

$$= \frac{-3 + 2\ln x}{7x^3}$$

$$\boxed{f''(x) = \frac{2\ln x - 3}{7x^3}}$$

$$f''(x = 0) = \frac{2\boxed{\ln(0)} - 3}{7(0)^3} : undefined \qquad \text{Inside log has to be} > 0 \text{ Therefore is undefined for } \ln(0)$$

$$f''(x = 7) = \frac{2\ln(7) - 3}{7(7)^3} \approx 0.0004$$

92

Exercise

Suppose the average test score p and was modeled by $p = 92.3 - 16.9\ln(t+1)$, where t is the time in months. How would the rate at which the average test score changed after 1 year?

Solution

$$\frac{dp}{dt} = -\frac{16.9}{t+1}$$

$$t = 1 \ yr = 12 \ mths$$

$$\Rightarrow \frac{dp}{dt} = -\frac{16.9}{12+1}$$

$$= -\frac{16.9}{13}$$

$$\underline{= -1.3}$$

Exercise

Suppose that the population of a certain collection of rare ants is given by

$$P(t) = (t+100)\ln(t+2)$$

Where t represents the time in days. Find the rates of change of the population on the second day and on the eighth day.

Solution

$$P'(t) = \ln(t+2) + (t+100)\frac{1}{t+2}$$

$$= \ln(t+2) + \frac{t+100}{t+2}$$

$$P'(2) = \ln(2+2) + \frac{2+100}{2+2} = \underline{27.89}$$

$$P'(8) = \ln(8+2) + \frac{8+100}{8+2} = \underline{13.10}$$

Exercise

Suppose that the demand function for x units of a certain item is $P(x) = 100 + \dfrac{180\ln(x+5)}{x}$ where P is the price per unit, in dollars. Find the marginal revenue.

Solution

$$R = x.P(x)$$

$$= x\left(100 + \frac{180\ln(x+5)}{x}\right)$$

$$= 100x + 180\ln(x+5)$$

$$R'(x) = 100 + 180\frac{1}{x+5}$$

$$= \frac{100(x+5) + 180}{x+5}$$

$$= \frac{100x + 500 + 180}{x+5}$$

$$= \frac{100x + 680}{x+5}$$

Exercise

The population of coyotes in the northwestern portion of Alabama is given by the formula $P(t) = \left(t^2 + 100\right)\ln(t+2)$, where t represents the time in years since 2000 (the year 2000 corresponds to $(t = 0)$ Find the rate of change of the coyote population in 2013 $(t = 13)$.

Solution

$$f = t^2 + 100 \quad \rightarrow f' = 2t$$

$$g = \ln(t+2) \quad \rightarrow g' = \frac{1}{t+2}$$

$$P' = f'g + g'f$$

$$P'(t) = 2t\ln(t+2) + \frac{1}{t+2}\left(t^2 + 100\right)$$

$$= 2t\ln(t+2) + \frac{t^2 + 100}{t+2}$$

$$P'(t = 13) = 2(13)\ln(13+2) + \frac{13^2 + 100}{13+2} \qquad 2*13\ln(13+2) + (13^2+100)/(13+2)$$

$$= 88.34$$

94

Exercise

Students in a math class took a final exam. They took equivalent forms of the exam in monthly intervals thereafter. The average score $S(t)$, in percent, after t months was found to be given by

$$S(t) = 73 - 17\ln(t+1), \quad t \geq 0$$

Find $S'(t)$.

Solution

$$S'(t) = -17\frac{1}{t+1}$$

$$\underline{= -\frac{17}{t+1}}$$

Exercise

Suppose that the population of a town is given by $P(t) = 8\ln\sqrt{8t+7}$ where t is the time in years after 1980 and P is the population of the town in thousands. Find $P'(t)$.

Solution

$$U = 8t + 7 \quad \rightarrow U' = 8$$

$$V = \sqrt{8t+7} = (8t+7)^{1/2} = U^{1/2}$$

$$\rightarrow V' = nU'U^{n-1} = \frac{1}{2}8(8t+7)^{-1/2}$$

$$= \frac{4}{(8t+7)^{1/2}}$$

$$P'(t) = 8\frac{V'}{V}$$

$$= 8\underbrace{\frac{4}{(8t+7)^{1/2}}}_{V'}\underbrace{\frac{1}{(8t+7)^{1/2}}}_{V}$$

$$\underline{= \frac{32}{8t+7}}$$

Exercise

The following formula accurately models the relationship between the size of a certain type of tumor and the amount of time that it has been growing:

$$V(t) = 450(1 - e - 0.0022t)^3$$

where t is in months and $V(t)$ is measured in cubic centimeters. Calculate the rate of change of tumor volume at 80 months.

Solution

$$U = 1 - e - 0.0022t \qquad V = 450U^3$$

$$U' = -.0022 \qquad V' = 450(3)U^2U'$$

$$V'(t) = 450(3)(1 - e - 0.0022t)^2(-.0022) \underline{= 2.97(1 - e - 0.0022t)^2}$$

$$V'(t = 80) = 2.97(1 - e - 0.0022(80))^2 \underline{= 10.66}$$

Exercise

A yeast culture at room temperature (68° F) is placed in a refrigerator set at a constant temperature of 38° F. After t hours, the temperature T of the culture is given approximately by

$$T = 30e^{-0.58t} + 38 \quad t \geq 0$$

What is the rate of change of temperature of the culture at the end of 1 hour? At the end of 4 hours?

Solution

$$T' = 30(-0.58)e^{-0.58t} \underline{= -17.4e^{-0.58t}}$$

$$T'(1) = -17.4e^{-0.58(1)} \underline{= -9.74° \ F/hr}$$

$$T'(4) = -17.4e^{-0.58(4)} \underline{= -1.71° \ F/hr}$$

Exercise

A mathematical model for the average age of a group of people learning to type is given by

$$N(t) = 10 + 6\ln t \quad t \geq 1$$

Where $N(t)$ is the number of words per minute typed after t hours of instruction and practice (2 hours per day, 5 days per week). What is the rate of learning after 10 hours of instruction and practice? After 100 hours?

Solution

$$N'(t) = \frac{6}{t}$$

$$N'(10) = \frac{6}{10} \underline{= 0.6}$$

After 10 hours of instruction and practice, the rate of learning is 0.6 words/minute per hour of instruction and practice.

$$N'(100) = \frac{6}{100} = 0.06$$

After 100 hours of instruction and practice, the rate of learning is 0.06 words/minute per hour of instruction and practice.