Solution *Section* 2.1 – Definition of the Derivative

Exercise

Find the derivative of y with the respect to t for the function $y = \frac{4}{t}$

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta t \to 0} \frac{\frac{4}{t+\Delta t} - \frac{4}{t}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\frac{4t - 4(t+\Delta t)}{t(t+\Delta t)}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{4t - 4(t+\Delta t)}{t(t+\Delta t)}$$

$$= \lim_{\Delta t \to 0} \frac{-4\Delta t}{t(t+\Delta t)\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{-4}{t(t+\Delta t)}$$

$$= -\frac{4}{t^2}$$

Exercise

Find the equation of the tangent line to $f(x) = x^2 + 1$ that is parallel to $2x + y = 0$

Solution

$2x + y = 0 \Rightarrow y = -2x \Rightarrow$ slope = -2

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + 1 - (x^2 + 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + \Delta x^2 + 2x\Delta x + 1 - x^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x^2 + 2x\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \Delta x + 2x = 2x$$

$$f' = 2x = -2 \ \Rightarrow x = -1 \Rightarrow f(-1) = (-1)^2 + 1 = 2 \quad \to \text{(-1, 2)}$$

The line equation is given by

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -2(x + 1)$$

$$y - 2 = -2x - 2$$

$$y = -2x$$

Exercise

Use the definition of limits to find the derivative: $f(x) = \dfrac{3}{\sqrt{x}}$

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(\dfrac{3}{\sqrt{x+\Delta x}}\right) - \left(\dfrac{3}{\sqrt{x}}\right)}{\Delta x} \cdot \frac{\sqrt{x}\cdot\sqrt{x+\Delta x}}{\sqrt{x}\cdot\sqrt{x+\Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{3\sqrt{x} - 3\sqrt{x+\Delta x}}{\Delta x\left(\sqrt{x}\cdot\sqrt{x+\Delta x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{3\left(\sqrt{x} - \sqrt{x+\Delta x}\right)}{\Delta x\left(\sqrt{x}\cdot\sqrt{x+\Delta x}\right)} \cdot \frac{\sqrt{x} + \sqrt{x+\Delta x}}{\sqrt{x} + \sqrt{x+\Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{3\left(x - (x+\Delta x)\right)}{\Delta x\left(\sqrt{x}\cdot\sqrt{x+\Delta x}\right)\left(\sqrt{x} + \sqrt{x+\Delta x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{-3\Delta x}{\Delta x\left(\sqrt{x}\cdot\sqrt{x+\Delta x}\right)\left(\sqrt{x} + \sqrt{x+\Delta x}\right)}$$

$$= \frac{-3}{x\left(2\sqrt{x}\right)} = \frac{-3}{2x^{3/2}}$$

Exercise

Use the definition of limits to find the derivative: $f(x) = \sqrt{x+2}$

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+2} - \sqrt{x+2}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x+2} + \sqrt{x+2}}{\sqrt{x+\Delta x+2} + \sqrt{x+2}}$$

$$= \lim_{\Delta x \to 0} \frac{x+\Delta x+2-(x+2)}{\Delta x\left(\sqrt{x+\Delta x+2} + \sqrt{x+2}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{x+\Delta x+2} + \sqrt{x+2}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x+2} + \sqrt{x+2}}$$

$$= \frac{1}{2\sqrt{x+2}}$$