Solution Section 3.3 – Double-angle Formulas

Exercise

Let $\sin A = -\frac{3}{5}$ with A in QIII and find $\cos 2A$

Solution

$$\cos 2A = 1 - 2\sin^2 A$$

$$= 1 - 2\left(-\frac{3}{5}\right)^2$$

$$= 1 - 2\left(\frac{9}{25}\right)$$

$$= \frac{25-18}{25}$$

$$= \frac{7}{25}$$

Exercise

Let $\cos x = \frac{1}{\sqrt{10}}$ with x in QIV and find $\cot 2x$

Solution

x in QIV $\Rightarrow \sin x < 0$

$$\sin x = -\sqrt{1-\cos^2 x}$$

$$= -\sqrt{1-\frac{1}{10}}$$

$$= -\sqrt{\frac{9}{10}}$$

$$= -\frac{3}{\sqrt{10}}$$

$$\cot 2x = \frac{\cos 2x}{\sin 2x}$$

$$= \frac{2\cos^2 x - 1}{2\sin x\cos x}$$

$$=\frac{2\left(\frac{1}{\sqrt{10}}\right)^2-1}{2\frac{1}{\sqrt{10}}\left(-\frac{3}{\sqrt{10}}\right)}$$

$$=\frac{2\frac{1}{10}-1}{-\frac{6}{10}}$$

$$=\frac{\frac{2-10}{10}}{-\frac{6}{10}}$$

$$=\frac{-8}{-6}$$

$$=\frac{4}{3}$$

Exercise

Verify: $(\cos x-\sin x)(\cos x+\sin x)=\cos 2x$

Solution

$$\begin{aligned}(\cos x-\sin x)(\cos x+\sin x)&=\cos^2 x-\sin^2 x\\&=\cos 2x\end{aligned}$$

$(a+b)(a-b)=a^2-b^2$

Exercise

Prove: $\cot x\sin 2x=1+\cos 2x$

Solution

$$\begin{aligned}\cot x\sin 2x&=\frac{\cos x}{\sin x}(2\sin x\cos x)\\&=2\cos^2 x\\&=\cos 2x+1\end{aligned}$$

$\cos 2x=2\cos^2 x-1\Rightarrow 2\cos^2 x=\cos 2x+1$

Exercise

Prove: $\cot\theta = \dfrac{\sin 2\theta}{1-\cos 2\theta}$

Solution

$$\frac{\sin 2\theta}{1-\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1-\left(1-2\sin^2\theta\right)}$$

$$= \frac{2\sin\theta\cos\theta}{1-1+2\sin^2\theta}$$

$$= \frac{2\sin\theta\cos\theta}{2\sin^2\theta}$$

$$= \frac{\cos\theta}{\sin\theta}$$

$$= \cot\theta$$

Exercise

Simplify $\cos^2 7x - \sin^2 7x$

Solution

$$\cos^2 7x - \sin^2 7x = \cos\left(2(7x)\right) \qquad \cos 2x = \cos^2 x - \sin^2 x$$

$$= \cos 14x$$

Exercise

Write $\sin 3x$ in terms of $\sin x$

Solution

$$\sin 3x = \sin(2x + x)$$

$$= \sin 2x\cos x + \cos 2x\sin x$$

$$= (2\sin x\cos x)\cos x + \left(1-2\sin^2 x\right)\sin x$$

$$= 2\sin x\cos^2 x + \sin x - 2\sin^3 x \qquad \cos^2 x = 1-\sin^2 x$$

$$= 2\sin x\left(1-\sin^2 x\right) + \sin x - 2\sin^3 x$$

$$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$$

$$= 3\sin x - 4\sin^3 x$$

Exercise

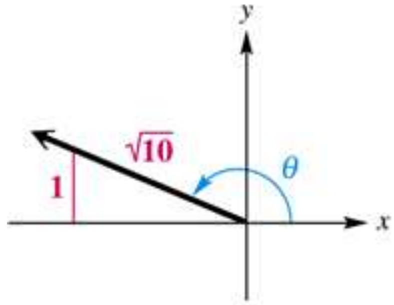
Find the values of the six trigonometric functions of θ if $\cos 2\theta = \frac{4}{5}$ *and* $90^\circ < \theta < 180^\circ$

Solution

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$ $= \frac{1+\frac{4}{5}}{2}$ $= \frac{\frac{9}{5}}{2}$ $= \frac{9}{10}$ $\cos\theta = -\sqrt{\frac{9}{10}}$ $= -\frac{3}{\sqrt{10}}\frac{\sqrt{10}}{\sqrt{10}}$ $= -\frac{3\sqrt{10}}{10}$	$\sin^2\theta = \frac{1-\cos 2\theta}{2}$ $= \frac{1-\frac{4}{5}}{2}$ $= \frac{\frac{1}{5}}{2}$ $= \frac{1}{10}$ $\sin\theta = \sqrt{\frac{1}{10}}$ $= \frac{1}{\sqrt{10}}\frac{\sqrt{10}}{\sqrt{10}}$ $= \frac{\sqrt{10}}{10}$
$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $= \frac{\frac{\sqrt{10}}{10}}{-\frac{3\sqrt{10}}{10}}$ $= -\frac{\sqrt{10}}{10}\frac{10}{3\sqrt{10}}$ $= -\frac{1}{3}$	$\cot\theta = \frac{1}{\tan\theta}$ $= \frac{1}{-\frac{1}{3}}$ $= -3$
$\csc\theta = \frac{1}{\sin\theta}$ $= \frac{1}{\frac{1}{\sqrt{10}}}$ $= \sqrt{10}$	$\sec\theta = \frac{1}{\cos\theta}$ $= \frac{1}{-\frac{3}{\sqrt{10}}}$ $= -\frac{\sqrt{10}}{3}$

Exercise

Use a right triangle in QII to find the value of $\cos\theta$ *and* $\tan\theta$



Solution

$$r=\sqrt{10},\ y=1$$

$$\begin{aligned}x&=-\sqrt{r^2-y^2}\\&=-\sqrt{\left(\sqrt{10}\right)^2-1^2}\\&=-\sqrt{10-1}\\&=-\sqrt{9}\\&=-3\end{aligned}$$

$$\cos\theta=\frac{-3}{\sqrt{10}}\frac{\sqrt{10}}{\sqrt{10}}=-\frac{3\sqrt{10}}{10}$$

$$\tan\theta=-\frac{1}{3}$$

Exercise

Prove the following equation is an identity: $\sin 3x=\sin x\left(3\cos^2 x-\sin^2 x\right)$

Solution

$$\begin{aligned}\sin 3x&=\sin(x+2x)\\&=\sin x\cos 2x+\sin 2x\cos x\\&=\sin x\left(\cos^2 x-\sin^2 x\right)+\left(2\sin x\cos x\right)\cos x\\&=\sin x\cos^2 x-\sin^3 x+2\sin x\cos^2 x\\&=3\sin x\cos^2 x-\sin^3 x\\&=\sin x\left(3\cos^2 x-\sin^2 x\right)\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cos 3x = \cos^3 x - 3\cos x \sin^2 x$

Solution

$$
\begin{aligned}
\cos 3x &= \cos(x+2x) \\
&= \cos x \cos 2x - \sin x \sin 2x \\
&= \cos x\left(\cos^2 x - \sin^2 x\right) - \sin x\left(2\sin x \cos x\right) \\
&= \cos^3 x - \sin^2 x \cos x - 2\sin^2 x \cos x \\
&= \cos^3 x - 3\sin^2 x \cos x
\end{aligned}
$$

Exercise

Prove the following equation is an identity: $\cos^4 x - \sin^4 x = \cos 2x$

Solution

$$
\begin{aligned}
\cos^4 x - \sin^4 x &= \left(\cos^2 x - \sin^2 x\right)\left(\cos^2 x + \sin^2 x\right) \qquad (a-b)(a+b) = a^2 + b^2 \\
&= (\cos 2x)(1) \\
&= \cos 2x
\end{aligned}
$$

Exercise

Prove: $\cot\theta = \dfrac{\sin 2\theta}{1-\cos 2\theta}$

Solution

$$
\begin{aligned}
\frac{\sin 2\theta}{1-\cos 2\theta} &= \frac{2\sin\theta\cos\theta}{1-\left(1-2\sin^2\theta\right)} \\
&= \frac{2\sin\theta\cos\theta}{1-1+2\sin^2\theta} \\
&= \frac{2\sin\theta\cos\theta}{2\sin^2\theta} \\
&= \frac{\cos\theta}{\sin\theta} \\
&= \cot\theta
\end{aligned}
$$

Exercise

Prove the following equation is an identity: $\sin 2x = -2\sin x \sin\left(x - \frac{\pi}{2}\right)$

Solution

$$\sin 2x = 2\sin x \cos x \qquad \cos x = \sin\left(\tfrac{\pi}{2} - x\right)$$

$$= 2\sin x \sin\left(\frac{\pi}{2} - x\right) \qquad \sin(-x) = -\sin x$$

$$= -2\sin x \sin\left(x - \frac{\pi}{2}\right)$$

Exercise

Prove the following equation is an identity: $\frac{\sin 4t}{4} = \cos^3 t \sin t - \sin^3 t \cos t$

Solution

$$\frac{\sin 4t}{4} = \frac{1}{4}(2\sin 2t \cos 2t)$$

$$= \frac{1}{2}(2\sin t \cos t)\left(\cos^2 t - \sin^2 t\right)$$

$$= \sin t \cos t\left(\cos^2 t - \sin^2 t\right)$$

$$= \sin t \cos^3 t - \cos t \sin^3 t$$

Exercise

Prove the following equation is an identity: $\frac{\cos 2x}{\sin^2 x} = \csc^2 x - 2$

Solution

$$\frac{\cos 2x}{\sin^2 x} = \frac{1 - 2\sin^2 x}{\sin^2 x}$$

$$= \frac{1}{\sin^2 x} - \frac{2\sin^2 x}{\sin^2 x}$$

$$= \csc^2 x - 2$$

Exercise

Prove the following equation is an identity: $\frac{\cos 2x+\cos 2y}{\sin x+\cos y}=2\cos y-2\sin x$

Solution

$$\begin{aligned}
\frac{\cos 2x+\cos 2y}{\sin x+\cos y} &= \frac{2\cos\left(\frac{2x+2y}{2}\right)\cos\left(\frac{2x-2y}{2}\right)}{\sin x+\cos y} \\
&= \frac{2\cos(x+y)\cos(x-y)}{\sin x+\cos y} \\
&= \frac{2(\cos x\cos y-\sin x\sin y)(\cos x\cos y+\sin x\sin y)}{\sin x+\cos y} \\
&= 2\frac{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}{\sin x+\cos y} \\
&= 2\frac{\left(1-\sin^2 x\right)\cos^2 y-\sin^2 x\left(1-\cos^2 y\right)}{\sin x+\cos y} \\
&= 2\frac{\cos^2 y-\sin^2 x\cos^2 y-\sin^2 x+\sin^2 x\cos^2 y}{\sin x+\cos y} \\
&= 2\frac{\cos^2 y-\sin^2 x}{\sin x+\cos y} \\
&= 2\frac{(\cos y-\sin x)(\cos y+\sin x)}{\sin x+\cos y} \\
&= 2(\cos y-\sin x) \\
&= 2\cos y-2\sin x
\end{aligned}$$

$$\begin{aligned}
\frac{\cos 2x+\cos 2y}{\sin x+\cos y} &= \frac{\cos^2 x-\sin^2 x+\cos^2 y-\sin^2 y}{\sin x+\cos y} \\
&= \frac{1-\sin^2 x-\sin^2 x+\cos^2 y-\left(1-\cos^2 y\right)}{\sin x+\cos y} \\
&= \frac{1-2\sin^2 x+\cos^2 y-1+\cos^2 y}{\sin x+\cos y} \\
&= \frac{2\cos^2 y-2\sin^2 x}{\sin x+\cos y} \\
&= 2\frac{\cos^2 y-\sin^2 x}{\sin x+\cos y} \\
&= 2\frac{(\cos y-\sin x)(\cos y+\sin x)}{\sin x+\cos y} \\
&= 2(\cos y-\sin x) \\
&= 2\cos y-2\sin x
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cos 2x}{\cos^2 x} = \sec^2 x - 2\tan^2 x$

Solution

$$\frac{\cos 2x}{\cos^2 x} = \frac{1-2\sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} - \frac{2\sin^2 x}{\cos^2 x}$$

$$= \sec^2 x - 2$$

Exercise

Prove the following equation is an identity: $\sin 4x = (4\sin x\cos x)\left(2\cos^2 x - 1\right)$

Solution

$$\sin 4x = \sin(2(2x))$$

$$= 2\sin 2x\cos 2x$$

$$= 2(2\sin x\cos x)\left(2\cos^2 x - 1\right)$$

$$= (4\sin x\cos x)\left(2\cos^2 x - 1\right)$$

Exercise

Prove the following equation is an identity: $\cos 4x = \cos^4 x - 6\sin^2 x\cos^2 x + \sin^4 x$

Solution

$$\cos 4x = \cos(2(2x))$$

$$= \cos^2 2x - \sin^2 2x$$

$$= (\cos 2x)^2 - (\sin 2x)^2$$

$$= \left(\cos^2 x - \sin^2 x\right)^2 - (2\sin x\cos x)^2$$

$$= \cos^4 x - 2\sin^2 x\cos^2 x - \sin^4 x - 4\sin^2 x\cos^2 x$$

$$= \cos^4 x - 6\sin^2 x\cos^2 x - \sin^4 x$$

Exercise

Prove the following equation is an identity: $\cos 2y = \dfrac{1-\tan^2 y}{1+\tan^2 y}$

Solution

$$\cos 2y = \cos^2 y - \sin^2 y$$

$$= \frac{\cos^2 y - \sin^2 y}{1}$$

$$= \frac{\cos^2 y - \sin^2 y}{\cos^2 y + \sin^2 y}$$

$$= \frac{\frac{\cos^2 y}{\cos^2 y} - \frac{\sin^2 y}{\cos^2 y}}{\frac{\cos^2 y}{\cos^2 y} + \frac{\sin^2 y}{\cos^2 y}}$$

$$= \frac{1-\tan^2 y}{1+\tan^2 y}$$

$$\frac{1-\tan^2 y}{1+\tan^2 y} = \frac{1-\frac{\sin^2 y}{\cos^2 y}}{1+\frac{\sin^2 y}{\cos^2 y}}$$

$$= \frac{\frac{\cos^2 y - \sin^2 y}{\cos^2 y}}{\frac{\cos^2 y + \sin^2 y}{\cos^2 y}}$$

$$= \frac{\cos^2 y - \sin^2 y}{\cos^2 y + \sin^2 y}$$

$$= \frac{\cos^2 y - \sin^2 y}{1}$$

$$= \cos^2 y - \sin^2 y$$

Exercise

Prove the following equation is an identity: $\tan^2 x(1+\cos 2x) = 1 - \cos 2x$

Solution

$$\tan^2 x(1+\cos 2x) = \frac{\sin^2 x}{\cos^2 x}\left(1+2\cos^2 x - 1\right)$$

$$= \frac{\sin^2 x}{\cos^2 x}\left(2\cos^2 x\right)$$

$$= 2\sin^2 x$$

$$= 1 - 1 + 2\sin^2 x$$

$$= 1 - \left(1 - 2\sin^2 x\right)$$

$$= 1 - \cos 2x$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos 2x}{\sin^2 x} = 2\cot^2 x - \csc^2 x$

Solution

$$\frac{\cos 2x}{\sin^2 x} = \frac{\cos^2 x - \sin^2 x}{\sin^2 x}$$

$$= \frac{\cos^2 x}{\sin^2 x} - \frac{\sin^2 x}{\sin^2 x}$$

$$= \cot^2 x - 1 \qquad\qquad \cot^2 x + 1 = \csc^2 x$$

$$= \cot^2 x + \cot^2 x - \csc^2 x$$

$$= 2\cot^2 x - \csc^2 x$$

Exercise

Prove the following equation is an identity: $\tan x + \cot x = 2\csc 2x$

Solution

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$

$$= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x}$$

$$= \frac{1}{\cos x \sin x}$$

$$= \frac{1}{\frac{1}{2}\sin 2x}$$

$$= 2\frac{1}{\sin 2x}$$

$$= 2\csc 2x$$

Exercise

Prove the following equation is an identity: $\tan 2x = \dfrac{2}{\cot x - \tan x}$

Solution

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$= \frac{2\frac{\tan x}{\tan x}}{\frac{1}{\tan x} - \frac{\tan^2 x}{\tan x}}$$

$$= \frac{2}{\cot x - \tan x}$$

Exercise

Prove the following equation is an identity: $\dfrac{1-\tan x}{1+\tan x} = \dfrac{1-\sin 2x}{\cos 2x}$

Solution

$$\frac{1-\tan x}{1+\tan x} = \frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}$$

$$= \frac{\frac{\cos x - \sin x}{\cos x}}{\frac{\cos x + \sin x}{\cos x}}$$

$$= \frac{\cos x - \sin x}{\cos x + \sin x}\frac{\cos x - \sin x}{\cos x - \sin x}$$

$$= \frac{\cos^2 x - 2\cos x \sin x + \sin^2 x}{\cos^2 x + \sin^2 x}$$

$$= \frac{1-\sin 2x}{\cos 2x}$$

Exercise

Prove the following equation is an identity: $\sin 2\alpha \sin 2\beta = \sin^2(\alpha+\beta) - \sin^2(\alpha-\beta)$

Solution

$$\begin{aligned}
\sin 2\alpha \sin 2\beta &= (2\sin\alpha\cos\alpha)(2\sin\beta\cos\beta) \\
&= (2\sin\alpha\cos\beta)(2\sin\beta\cos\alpha) \\
&= \left(2\tfrac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]\right)\left(2\tfrac{1}{2}\left[\sin(\beta+\alpha)+\sin(\beta-\alpha)\right]\right) \\
&= \left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right)\left(\sin(\alpha+\beta)-\sin(\alpha-\beta)\right) \\
&= \sin^2(\alpha+\beta) - \sin^2(\alpha-\beta)
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cos^2(A-B) - \cos^2(A+B) = \sin 2A\ \sin 2B$

Solution

$$\begin{aligned}
\cos^2(A-B) - \cos^2(A+B) &= \left(\cos(A-B)-\cos(A+B)\right)\left(\cos(A-B)+\cos(A+B)\right) \\
&= (2\sin A\sin B)(2\cos A\cos B) \\
&= (2\sin A\cos A)(2\sin B\cos B) \\
&= \sin 2A\ \sin 2B
\end{aligned}$$