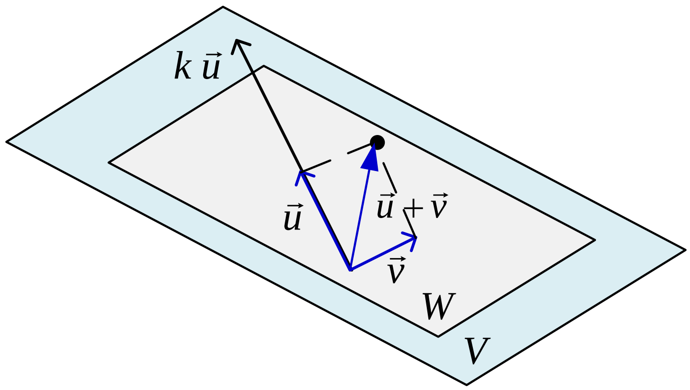
Section 2.5 – Subspaces, Span and Null Spaces

Subspaces

Definition

A subset *W* of a vector space *V* is called a ***subspace*** of *V* if *W* itself a vector space under the addition and scalar multiplication defined in *V*.



Theorem

If *W* is a set of one or more vectors in a vector space *V*, then *W* is a subspace of *V* iff the following conditions holds

1. If $\vec{u}$ and $\vec{v}$ are vectors in *W*, then $\vec{u}+\vec{v}$ is in *W*.
2. If *k* is any scalar and $\vec{v}$ is any vector in *W*, the $k\vec{v}$ is in the subspace in *W*.

- The most fundamental ideas in linear algebra are that the plane is a subspace of the full vector space $\mathbb{R}^n$.

- Every subspace contains the zero vector. The plane vector in $\mathbb{R}^3$ has to go through (0, 0, 0). From rule (**2**), if we choose $k=0$ and the rule requires $0v$ to be in the subspace.

The ***axioms*** that are ***not*** inherited by *W* are

Axiom 1 – Closure of *W* under addition

Axiom 4 – Existence of a zero vector in *W*

Axiom 5 – Existence of a negative in *W* for every vector in *W*

Axiom 6 – Closure of *W* under scalar multiplication

Example

Keep only the vectors $(x,\ y)$ whose components are positive or zero (first quadrant ***"quarter-plane"***).

The vector $(2,\ 3)$ is included but $(-2,\ -3)$ is not. So, rule (**2**) is violated when we try $k=-1$. ***The quarter-plane is not a subspace***.

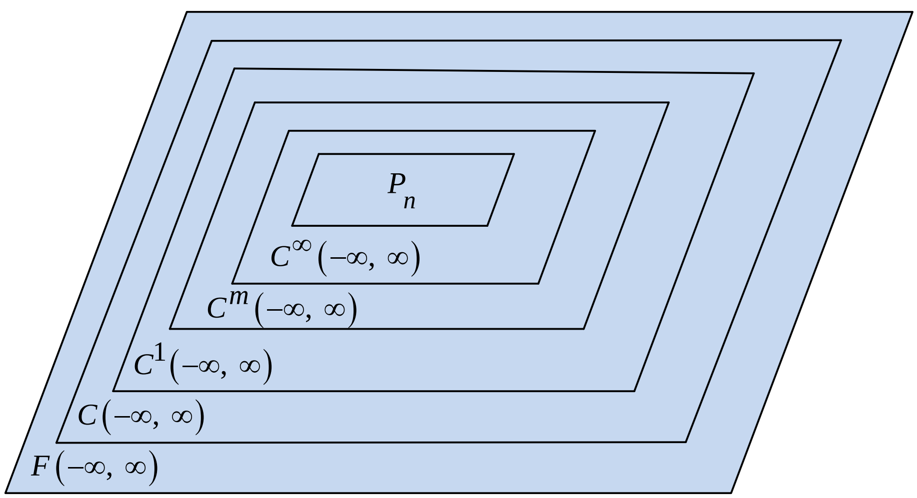
Example

Include also the vectors whose components are both negative. Now we have two quarter-planes. Rule (***ii***) satisfies when we multiply by any c. But rule (***i***) fails. The sum of $v=(2,\ 3)$ and $w=(-3,\ -2)$ is $(-1,\ 1)$ which is outside the quarter-plane. ***Two quarter-planes don't make a subspace***.

Example

The **Subspace** $C(-\infty,\ \infty)$

There is a theorem in calculus which states that a sum of continuous functions is continuous and than a constant times a continuous function is continuous. In vector word, the set of continuous functions on $(-\infty,\ \infty)$ is a subspace of $F(-\infty,\ \infty)$. We dente this subspace by $C(-\infty,\ \infty)$



Theorem

If $W_1,\ W_2,\ \ldots,\ W_n$ are subspaces of a vector space *V*, then intersection of these subspaces is also a subspace of *V*.

- ***A subspace containing*** $\vec{v}$ ***and*** $\vec{w}$ ***must contain all linear combination*** $c\vec{v}+d\vec{w}$.

Example

Inside the vector space M of all 2 by 2 matrices, given two subspaces:

U all upper triangular matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

D all diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$

Solution

If we add 2 matrices in **U**: $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 0 & 2d \end{bmatrix}$ is in **U**.

If we add 2 matrices in **D**: $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 0 & 2d \end{bmatrix}$ is in **D**.

In this case **D** is also a subspace of **U**!. The zero matrix is in these subspaces, when a, b, and d all equal zero.

Span

Definition

The subspace of a vector space V that is formed from all possible linear combinations of the vectors in a nonempty set S is called the ***span of S***, and we say that the vectors in S *span* that subspace. If $S = \left\{w_1,\ w_2,\ \ldots,\ w_r\right\}$, then we denoted the span of S by

$$span\left\{w_1,\ w_2,\ \ldots,\ w_r\right\} \quad or \quad span(S)$$

Theorem

Let $\vec{v}_1,\ \ldots,\ \vec{v}_n$ be vectors in vector space V and S be their span. Then,

a) S is a subspace of V.

Proof: $\forall\ \vec{u},\ \vec{v} \in S$, $\vec{u} = a_1\vec{v}_1 + \ldots + a_n\vec{v}_n$ and $\vec{v} = b_1\vec{v}_1 + \ldots + b_n\vec{v}_n$

$$\vec{u} + \vec{v} = \left(a_1 + b_1\right)\vec{v}_1 + \ldots + \left(a_n + b_n\right)\vec{v}_n \quad \in \boldsymbol{S}$$

$$k\vec{u} = ka_1\vec{v}_1 + \ldots + ka_n\vec{v}_n \quad \in \boldsymbol{S}$$

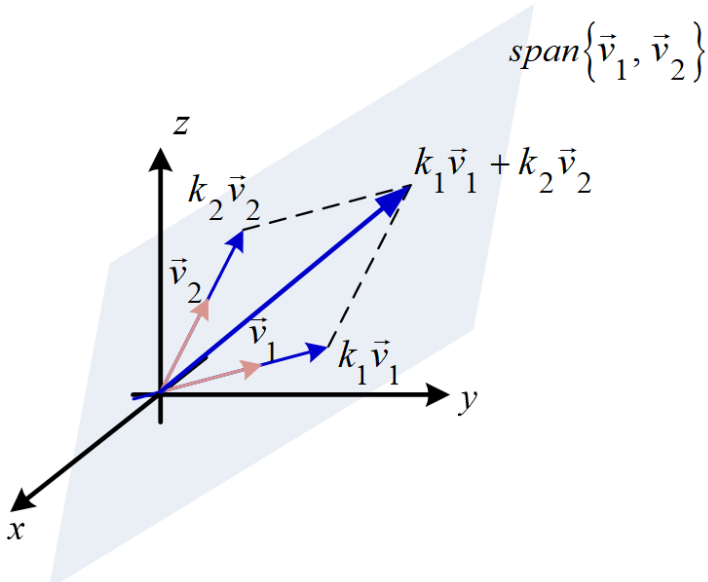
b) S is the smallest subspace of V that contains $\vec{v}_1,\ \ldots,\ \vec{v}_k$. i.e. any other subspace $\vec{w}$ containing $\vec{v}_1,\ \ldots,\ \vec{v}_n$ also contains S.

Proof: *let* $\vec{u} \in S$, $\vec{u} = a_1\vec{v}_1 + \ldots + a_n\vec{v}_n$

But $a_1\vec{v}_1, \ldots, a_n\vec{v}_n \in \vec{w}$ $\therefore$ $\vec{w}$ *closed under scalar multiplication.*

$a_1\vec{v}_1, \ldots, a_n\vec{v}_n \in \vec{w}$ $\therefore$ $\vec{w}$ *closed under addition.*

$\therefore$ $\underline{\vec{u} \in \vec{w}}$



Example

a) $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ span the full two-dimensional space $\mathbb{R}^2$.

b) $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, *and* $\vec{v}_3 = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$ span the full space $\mathbb{R}^2$.

c) $\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *and* $\vec{w}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ only span a line in $\mathbb{R}^2$.

Definition

The ***row space*** of a matrix is the subspace of $\mathbb{R}^n$ spanned by the rows.

Example

Determine whether $\vec{v}_1 = (1,\ 1,\ 2)$, $\vec{v}_2 = (1,\ 0,\ 1)$, *and* $\vec{v}_3 = (2,\ 1,\ 3)$ span the vector space $\mathbb{R}^3$

Solution

Let $b = (b_1,\ b_2,\ b_3)$ be the arbitrary vector in $\mathbb{R}^3$ can be expressed as a linear combination

$$\vec{b} = k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3$$

$$(b_1,\ b_2,\ b_3) = k_1(1,\ 1,\ 2) + k_2(1,\ 0,\ 1) + k_3(2,\ 1,\ 3)$$

$$(b_1,\ b_2,\ b_3) = (k_1 + k_2 + 2k_3,\ k_1 + k_3,\ 2k_1 + k_2 + 3k_3)$$

$$\rightarrow \begin{cases} k_1 + k_2 + 2k_3 = b_1 \\ k_1 \qquad + k_3 = b_2 \\ 2k_1 + k_2 + 3k_3 = b_3 \end{cases}$$

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$

$$\underline{= 0|}$$

Since the determinant is zero, the $\vec{v}_1$, $\vec{v}_2$, *and* $\vec{v}_3$ ***do not*** span space $\mathbb{R}^3$

Solution Spaces of *Homogeneous* (*Null Space*) Systems

Theorem

The solution set of a homogeneous linear system $A\vec{x} = \vec{0}$ in *n* unknowns is a subspace of $\mathbb{R}^n$

Proof

Let *W* be the solution set for the system. The set *W* is not empty because it contains at least the trivial solution $\vec{x} = \vec{0}$.

To show that *W* is a subspace of $\mathbb{R}^n$, we must show that it is closed under addition and scalar multiplication.

Let $\vec{x}_1$ and $\vec{x}_2$ be vectors in *W* and these vectors are solution of $A\vec{x} = \vec{0}$.

$$A\vec{x}_1 = \vec{0} \quad and \quad A\vec{x}_2 = \vec{0}$$

Therefore,

$$
\begin{aligned}
A\left(\vec{x}_1+\vec{x}_2\right) &= A\vec{x}_1+A\vec{x}_2 \\
&= \vec{0}+\vec{0} \\
&= \vec{0}
\end{aligned}
$$

So, W is closed under addition.

$$A\left(k\vec{x}_1\right)=kA\vec{x}_1=k0=0$$

So, W is closed under scalar multiplication.

Null Spaces

Definition

The nullspace of A consists of all solutins to $A\vec{x}=\vec{0}$. These solution vectors $\vec{x}$ are in $\mathbb{R}^n$. The Nullspace containing all solutions is denoted by $N(A)$ *or* $NS(A)$.

$\left\{\vec{x}\in\mathbb{R}^n \mid A\vec{x}=\vec{0}\right\}$ is the nullspace of A, $NS(A)$

(Can also be called ***Kernel*** of A: $Ker(A)$)

Theorem

Suppose $NS(A)$ is a subspace of $\mathbb{R}^n$ for $A_{m\times n}$

- ✓ Let $\vec{x}$ and $\vec{y}$ are in the nullspace $(\vec{x},\ \vec{y}\in NS(A))$ then

$$\begin{aligned} A(\vec{x}+\vec{y}) &= A\vec{x}+A\vec{y} \\ &= \vec{0}+\vec{0} \\ &= \vec{0} \end{aligned}$$

- ✓ Let $\vec{x}\in NS(A)$ then $c\vec{x}\in NS(A)$

$$\begin{aligned} \therefore A(c\vec{x}) &= cA\vec{x} \\ &= c\vec{0} \\ &= \vec{0} \end{aligned}$$

Since we can add and multiply without leaving the Nullspace, it is a subspace.

Example

The equation $x+2y+3z=0$ comes from the 1 by 3 matrix $A=\begin{bmatrix}1 & 2 & 3\end{bmatrix}$. This equation produces a plane through the origin. The plane is a subspace of $\mathbb{R}^3$. *It is the Nullspace* of A.

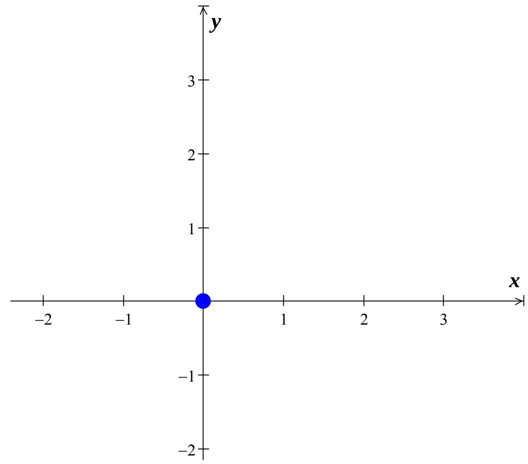
Solution

The solution to $x+2y+3z=6$ also form a plane, but not a subspace.

Example

Find the null space of

$$a)\quad A=\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix} \qquad b)\ B=\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}$$



Solution

a) $\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\to \begin{matrix}x_1+x_2=0\\ 3x_2=0\end{matrix}$

$\Rightarrow x_1=x_2=0$

So $NS(A)=\{\vec{0}\}$

b) $\begin{bmatrix}1 & 2 & 0\\ 3 & 6 & 0\end{bmatrix}\ R_2-3R_1$

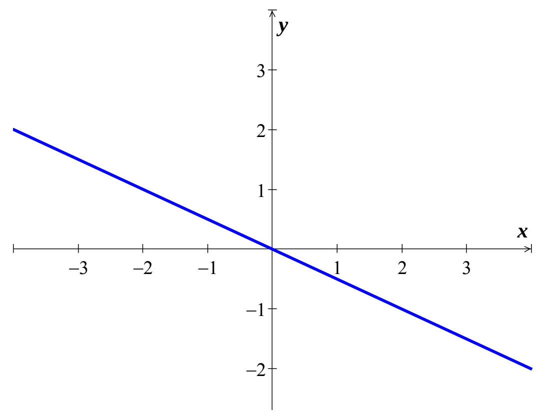
$\begin{bmatrix}1 & 2 & 0\\ 0 & 0 & 0\end{bmatrix}$

$\Rightarrow x_1=-2x_2$

If we let $x_2=s$, then

$\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ is in $NS(B)$ if and only if

$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=s\begin{bmatrix}-2\\ 1\end{bmatrix}$



Example

Describe the nullspace of $A=\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}$

Solution

Apply the elimination to the linear equations $Ax=0$:

$$\begin{bmatrix}x_1+2x_2=0\\ 3x_1+6x_2=0\end{bmatrix}\xrightarrow[R_2-3R_1]{}\begin{bmatrix}x_1+2x_2=0\\ 0=0\end{bmatrix}$$

There is only one equation $\left(x_1+2x_2=0\right)$, this line is the Nullspace $N(A)$.

Example

Consider the linear system $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution

$z = t, \quad y = s, \quad x = 2s - 3t$

$\Rightarrow x - 2y + 3z = 0$

This is the equation of a plane through the origin that has $\vec{n} = (1, -2, 3)$ as a normal.

Example

Consider the linear system $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution

$x = 0, \quad y = 0, \quad z = 0$

The solution space is $\{\vec{0}\}$

Exercises *Section* 2.5 – Subspaces, Span and Null Spaces

1. Suppose S and T are two subspaces of a vector space **V**.

a) The sum $S+T$ contains all sums $\vec{s}+\vec{t}$ of a vector $\vec{s}$ in S and a vector $\vec{t}$ in T. Show that $S+T$ satisfies the requirements (addition and scalar multiplication) for a vector space.

b) If S and T are lines in $\mathbb{R}^m$, what is the difference between $S+T$ and $S\cup T$? That union contains all vectors from S and T or both. Explain this statement: The span of $S\cup T$ is $S+T$.

2. Determine which of the following are subspaces of $\mathbb{R}^3$?

a) All vectors of the form $(a, 0, 0)$

b) All vectors of the form $(a, 1, 1)$

c) All vectors of the form (a, b, c), where $b = a + c$

d) All vectors of the form (a, b, c), where $b = a + c + 1$

e) All vectors of the form $(a, b, 0)$

3. Determine which of the following are subspaces of $\mathbb{R}^\infty$?

a) All sequences $\vec{v}$ in $\mathbb{R}^\infty$ of the form $\vec{v} = (v, 0, v, 0, \ldots)$

b) All sequences $\vec{v}$ in $\mathbb{R}^\infty$ of the form $\vec{v} = (v, 1, v, 1, \ldots)$

c) All sequences $\vec{v}$ in $\mathbb{R}^\infty$ of the form $\vec{v} = \left(v,\ 2v,\ 4v,\ 8v,\ 16v,\ \ldots\right)$

4. Which of the following are linear combinations of $\vec{u} = (0, -2, 2)$ and $\vec{v} = (1, 3, -1)$?

a) $(2, 2, 2)$ *b)* $(3, 1, 5)$ *c)* $(0, 4, 5)$ *d)* $(0, 0 ,0)$

5. Which of the following are linear combinations of $\vec{u} = (2, 1, 4)$, $\vec{v} = (1, -1, 3)$ and $\vec{w} = (3, 2, 5)$?

a) $(-9, -7, -15)$ *b)* $(6, 11, 6)$ *c)* $(0, 0 ,0)$

6. Determine whether the given vectors span $\mathbb{R}^3$

a) $\vec{v}_1 = (2,\ 2,\ 2),\quad \vec{v}_2 = (0,\ 0,\ 3),\quad \vec{v}_3 = (0,\ 1,\ 1)$

b) $\vec{v}_1 = (2,\ -1,\ 3),\quad \vec{v}_2 = (4,\ 1,\ 2),\quad \vec{v}_3 = (8,\ -1,\ 8)$

c) $\vec{v}_1 = (3,\ 1,\ 4),\quad \vec{v}_2 = (2,\ -3,\ 5),\quad \vec{v}_3 = (5,\ -2,\ 9),\quad \vec{v}_4 = (1,\ 4,\ -1)$

7. Which of the following are linear combinations of $A = \begin{pmatrix} 4 & 0 \\ -2 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}$

a) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ *b)* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *c)* $\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$

8. Suppose that $\vec{v}_1 = (2,\ 1,\ 0,\ 3)$, $\vec{v}_2 = (3,\ -1,\ 5,\ 2)$, $\vec{v}_3 = (-1,\ 0,\ 2,\ 1)$. Which of the following vectors are in span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

a) (2, 3, −7, 3) *b)* (0, 0, 0, 0) *c)* (1, 1, 1, 1) *d)* (−4, 6, −13, 4)

9. Let $\boldsymbol{f} = \cos^2 x$ and $\boldsymbol{g} = \sin^2 x$. Which of the following lie in the space spanned by $\boldsymbol{f}$ and $\boldsymbol{g}$

a) $\cos 2x$ *b)* $3 + x^2$ *c)* $\sin x$ *d)* 0

10. Let $S = \left\{(x,\ y) \middle| \ x^2 + y^2 = 0;\ \ x,\ y \in \mathbb{R}\right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^2$?

11. Let $S = \left\{(x,\ y) \middle| \ x^2 + y^2 = 0;\ \ x,\ y \in \mathbb{C}\right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{C}^2$?

12. Let $S = \left\{(x,\ y) \middle| \ x^2 - y^2 = 0;\ \ x,\ y \in \mathbb{R}\right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^2$?

13. Let $S = \left\{(x,\ y) \middle| \ x - y = 0;\ \ x,\ y \in \mathbb{R}\right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^2$?

14. Let $S = \left\{(x,\ y) \middle| \ x - y = 1;\ \ x,\ y \in \mathbb{R}\right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^2$?

15. $V=\mathbb{R}^3,\quad S=\{(0,\ s,\ t)|\ s,\ t\ are\ real\ numbers\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V?

16. $V=\mathbb{R}^3,\quad S=\{(x,\ y,\ z)|\ x,\ y\ ,\ z\ \geq 0\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V?

17. $V=\mathbb{R}^3,\quad S=\{(x,\ y,\ z)|\ z=x+y+1\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V?

18. Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ a_1=3a_2\ and\ a_3=-a_2\right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^3$?

19. Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ a_1=a_3+2\right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^3$?

20. Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ 2a_1-7a_2+a_3=0\right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^3$?

21. Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ a_1-4a_2-a_3=0\right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^3$?

22. Let $S=\{(a_1, a_2, a_3)\in\mathbb{R}^3 : a_1+2a_2-3a_3=0\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

23. Let $S=\{(a_1, a_2, a_3)\in\mathbb{R}^3 : a_1+2a_2-3a_3=1\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

24. Let $S=\{(a_1, a_2, a_3)\in\mathbb{R}^3 : 5a_1^2-3a_2^2+6a_3^2=0\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

25. Let $S=\{(a_1, a_2, a_3)\in\mathbb{R}^3 : a_3=a_1+a_2\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

26. Let $S=\{(a_1, a_2, a_3)\in\mathbb{R}^3 : a_1+a_2+a_3=0\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

27. $S=\{(x_1, x_2, 1): x_1 \textit{ and } x_2 \textit{ are real numbers}\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

28. $S=\{(x_1, x_2, x_3)\in\mathbb{R}^3 : x_2=x_1+2x_3\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

29. $S=\left\{\begin{pmatrix} a & 1 \\ c & d \end{pmatrix}\in M_{2\times 2} \mid a,\ b,\ c\in\mathbb{R}\right\}$ and $V=M_{2,2}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V ?

30. $S=\left\{\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\in M_{2\times 2} \mid a,\ b,\ c\in\mathbb{R}\right\}$ and $V=M_{2,2}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V ?

31. Let $S=\left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\in M_{2\times 2} \mid a,\ d\in\mathbb{R}\ \&\ ad\geq 0\right\}$ and $V=M_{2,2}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V ?

32. $V=M_{33}$, $S=\{A \mid A \text{ is invertible}\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V?

33. Let $S=\left\{p(t)=a+2at+3at^3 \mid a\in\mathbb{R}\ \&\ p(t)\in\mathbb{P}_2\right\}$ and $V=\mathbb{P}_2$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V ?

34. Let $S=\{p(t) \mid p(t)\in\mathbb{P}[t] \text{ has degree } 3\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{P}[t]$?

35. Let $S=\{p(t) \mid p(0)=0,\ p(t)\in\mathbb{P}[t]\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{P}[t]$?

36. Given: $A=\begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix}$

a) Find $NS(A)$

b) For which n is $NS(A)$ a subspace of $\mathbb{R}^n$

c) Sketch $NS(A)$ in $\mathbb{R}^2$ or $\mathbb{R}^3$

37. Determine which of the following are subspaces of M_{22}

a) All 2×2 matrices with integer entries

b) All matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a+b+c+d=0$

38. Let $V=\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}: \ ad-bc=1\right\}$. Is V a vector space?

39. Let $V=\{(x,0,y): \ x \,\&\, y \textit{ are arbitrary } \mathbb{R}\}$. Define addition and scalar multiplication as follows:

$$\begin{cases} (x_1,\ 0,\ y_1)+(x_2,\ 0,\ y_2)=(x_1+x_2,\ y_1+y_2) \\ \quad c(x,\ 0,\ y)=(cx,\ cy) \end{cases}$$

Is V a vector space?

40. Construct a matrix whose column space contains $(1,\ 1,\ 0)$ and $(0,\ 1,\ 1)$ and whose nullspace contains $(1,\ 0,\ 1)$ and $(0,\ 0,\ 1)$

41. How is the nullspace $N(C)$ related to the spaces $N(A)$ and $N(B)$, is $C=\begin{bmatrix} A \\ B \end{bmatrix}$?

42. True or False (check addition or give a counterexample)

a) If V is a vector space and W is a subset of V that is a vector space, then W is s subspace of V.
b) The empty set is a subspace of every vector space.
c) If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$.
d) The intersection of any two subsets of V is a subspace of V.
e) Let W be the *xy*-plane in $\mathbb{R}^3$; that is, $W=\{(a_1,\ a_2,\ 0): \ a_1,\ a_2 \in \mathbb{R}\}$. Then $W=\mathbb{R}^2$

43. Let $A\vec{x}=\vec{0}$ be a homogeneous system of n linear equations in n unknowns that has only the trivial solution. Show that of k is any positive integer, then the system $A^k\vec{x}=\vec{0}$ also has only trivial solution.

44. Let $A\vec{x}=\vec{0}$ be a homogeneous system of n linear equations in n unknowns and let Q be an invertible $n\times n$ matrix. Show that of $A\vec{x}=\vec{0}$ has just trivial solution if and only if $(QA)\vec{x}=\vec{0}$ has just trivial solution.

45. Let $A\vec{x}=\vec{b}$ be a consistent system of linear equations and let $\vec{x}_1$ be a fixed solution. Show that every solution to the system can be written in the form $\vec{x}=\vec{x}_1+\vec{x}_0$ where $\vec{x}_0$ is a solution to $A\vec{x}=\vec{0}$. Show also that every matrix of this form is a solution.