Solution Section 3.3 – Inverse Laplace Transform

Exercise

Find the inverse Laplace Transform of $Y(s) = \frac{1}{3s+2}$

Solution

$$Y(s) = \frac{1}{3}\frac{1}{s+2/3} \qquad \textit{Factor}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{3}\frac{1}{s+2/3}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2/3}\right\}$$

$$= \frac{1}{3}e^{-(2/3)t}$$

Exercise

Find the inverse Laplace Transform of $Y(s) = \frac{2}{3-5s}$

Solution

$$Y(s) = -2\frac{1}{5s-3}$$

$$= -\frac{2}{5}\frac{1}{s-\frac{3}{5}}$$

Thus, by linearity;

$$y(t) = -\frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-\frac{3}{5}}\right\}$$

$$= -\frac{2}{5}e^{(3/5)t}$$

Exercise

Find the inverse Laplace Transform of $Y(s) = \frac{1}{s^2+4}$

Solution

$$Y(s) = \frac{1}{2}\frac{2}{s^2+4}$$

$$y(t) = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= \frac{1}{2}\sin 2t$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{3}{s^2}$

Solution

$$y(t)=3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$\boxed{=3t}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{3s+2}{s^2+25}$

Solution

$$Y(s)=\frac{3s}{s^2+25}+\frac{2}{s^2+25}$$

$$=3\frac{s}{s^2+25}+\frac{2}{5}\frac{5}{s^2+25}$$

$$y(t)=3\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\}+\frac{2}{5}\,\mathcal{L}^{-1}\left\{\frac{5}{s^2+25}\right\}$$

$$\boxed{=3\cos 5t+\frac{2}{5}\sin 5t}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{2-5s}{s^2+9}$

Solution

$$Y(s)=\frac{2}{s^2+9}-\frac{5s}{s^2+25}$$

$$=\frac{2}{3}\frac{3}{s^2+9}-5\frac{s}{s^2+9}$$

$$y(t)=\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}-5\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}$$

$$\boxed{=\frac{2}{3}\sin 3t-5\cos 3t}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{5}{(s+2)^3}$

Solution

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s+a)^{n+1}}\right\}=t^n e^{-at} \qquad n=2 \quad a=2$$

$$Y(s) = \frac{5}{2!}\frac{2!}{(s+2)^3}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{5}{(s+2)^3}\right\}$$

$$= \frac{5}{2}t^2 e^{-2t}$$

Exercise

Find the inverse Laplace Transform of $Y(s) = \dfrac{1}{(s-1)^6}$

Solution

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s+a)^{n+1}}\right\} = t^n e^{-at} \qquad n = 5 \quad a = -1$$

$$Y(s) = \frac{1}{5!}\frac{5!}{(s-1)^6}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{5!}\frac{5!}{(s-1)^6}\right\}$$

$$= \frac{1}{120}t^5 e^t$$

Exercise

Find the inverse Laplace Transform of $\qquad Y(s) = \dfrac{4(s-1)}{(s-1)^2+4}$

Solution

$$\mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+\omega^2}\right\} = e^{-at}\cos\omega t \qquad a = -1 \quad \omega = 2$$

$$Y(s) = 4\frac{s-1}{(s-1)^2+4}$$

$$y(t) = \mathcal{L}^{-1}\left\{4\frac{s-1}{(s-1)^2+4}\right\}$$

$$= 4e^t\cos 2t$$

Exercise

Find the inverse Laplace Transform of $\qquad Y(s)=\dfrac{2s-3}{(s-1)^2+5}$

Solution

$$y(t)=\mathcal{L}^{-1}\left\{\frac{2s-3}{(s-1)^2+5}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{2s-2-1}{(s-1)^2+5}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{2(s-1)}{(s-1)^2+5}-\frac{1}{(s-1)^2+5}\right\}$$

$$=\mathcal{L}^{-1}\left\{2\frac{s-1}{(s-1)^2+5}-\frac{1}{\sqrt{5}}\frac{\sqrt{5}}{(s-1)^2+5}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+\omega^2}\right\}=e^{-at}\cos\omega t$$

$$\mathcal{L}^{-1}\left\{\frac{\omega}{(s+a)^2+\omega^2}\right\}=e^{-at}\sin\omega t$$

$$=2e^{t}\cos\sqrt{5}t-\frac{1}{\sqrt{5}}e^{t}\sin\sqrt{5}t$$

$$\underline{=e^{t}\left(2\cos\sqrt{5}t-\frac{\sqrt{5}}{5}\sin\sqrt{5}t\right)}$$

Exercise

Find the inverse Laplace Transform of $\qquad Y(s)=\dfrac{2s-1}{(s+1)(s-2)}$

Solution

Use partial fraction $\qquad \dfrac{2s-1}{(s+1)(s-2)}=\dfrac{A}{s+1}+\dfrac{B}{s-2}$

$$=\frac{As-2A+Bs+B}{(s+1)(s-2)}$$

$$2s-1=(A+B)s-2A+B$$

$$\begin{cases} A+B=2 \\ -2A+B=-1 \end{cases} \Rightarrow A=B=1$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{2s-1}{(s+1)(s-2)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$\underline{=e^{-t}+e^{2t}}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{2s-2}{(s-4)(s+2)}$

Solution

$$\frac{2s-2}{(s-4)(s+2)}=\frac{A}{s-4}+\frac{B}{s+2}$$

$$=\frac{As+2A+Bs-4B}{(s-4)(s+2)}$$

$$2s-2=(A+B)s+2A-4B$$

$$\begin{cases} A+B=2 \\ 2A-4B=-2 \end{cases} \Rightarrow A=B=1$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{2s-2}{(s-4)(s+2)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s-4}+\frac{1}{s+2}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\underline{=e^{4t}+e^{-2t}}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{7s^2+3s+16}{(s+1)\left(s^2+4\right)}$

Solution

$$\frac{7s^2+3s+16}{(s+1)\left(s^2+4\right)}=\frac{A}{s+1}+\frac{Bs+C}{s^2+4}$$

$$=\frac{As^2+4A+Bs^2+Bs+Cs+C}{(s+1)\left(s^2+4\right)}$$

$$=\frac{(A+B)s^2+(B+C)s+4A+C}{(s+1)\left(s^2+4\right)}$$

$$7s^2+3s+16=(A+B)s^2+(B+C)s+4A+C$$

$$\begin{cases} A+B=7 \\ B+C=3 \\ 4A+C=16 \end{cases} \quad \begin{matrix} \rightarrow A-C=4 \\ 4A+C=16 \end{matrix} \quad \Rightarrow 5A=20 \rightarrow A=4 \quad B=3 \quad C=0$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{7s^2+3s+16}{(s+1)\left(s^2+4\right)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{4}{s+1}+\frac{3s}{s^2+4}\right\}$$

$$= 4\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}+3\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$\underline{= 4e^{-t}+3\cos 2t}$$

Exercise

Find the inverse Laplace Transform of $\quad Y(s)=\dfrac{1}{(s+2)^2\left(s^2+9\right)}$

Solution

$$\frac{1}{(s+2)^2\left(s^2+9\right)} = \frac{A}{s+2}+\frac{B}{(s+2)^2}+\frac{Cs+D}{s^2+9}$$

$$= \frac{A(s+2)\left(s^2+9\right)+Bs^2+9B+(Cs+D)(s^2+4s+4)}{(s+2)^2\left(s^2+9\right)}$$

$$= \frac{As^3+2As^2+9As+18A+Bs^2+9B+Cs^3+4Cs^2+4Cs+Ds^2+4Ds+4D}{(s+2)^2\left(s^2+9\right)}$$

$$1=(A+C)s^3+(2A+B+4C+D)s^2+(9A+4C+4D)s+18A+9B+4D$$

$$\begin{cases} A+C=0 \\ 2A+B+4C+D=0 \\ 9A+4C+4D=0 \\ 18A+9B+4D=1 \end{cases} \Rightarrow A=\frac{4}{169} \quad B=\frac{1}{13} \quad C=-\frac{4}{169} \quad D=-\frac{5}{169}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2\left(s^2+9\right)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{4}{169}\frac{1}{s+2}+\frac{1}{13}\frac{1}{(s+2)^2}-\frac{1}{169}\frac{4s+5}{s^2+9}\right\}$$

$$= \frac{4}{169}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}+\frac{1}{13}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}-\frac{4}{169}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}-\frac{5}{169}\mathcal{L}^{-1}\left\{\frac{1}{3}\frac{3}{s^2+9}\right\}$$

$$\underline{= \frac{4}{169}e^{-2t}+\frac{1}{13}te^{-2t}-\frac{4}{169}\cos 3t-\frac{5}{507}\sin 3t}$$

Exercise

Find the inverse Laplace Transform of $\qquad Y(s)=\dfrac{s}{(s+2)^2\left(s^2+9\right)}$

Solution

$$\frac{s}{(s+2)^2\left(s^2+9\right)}=\frac{A}{s+2}+\frac{B}{(s+2)^2}+\frac{Cs+D}{s^2+9}$$

$$=\frac{As^3+2As^2+9As+18A+Bs^2+9B+Cs^3+4Cs^2+4Cs+Ds^2+4Ds+4D}{(s+1)^2\left(s^2+9\right)}$$

$$s=(A+C)s^3+(2A+B+4C+D)s^2+(9A+4C+4D)s+18A+9B+4D$$

$$\begin{cases} A+C=0 \\ 2A+B+4C+D=0 \\ 9A+4C+4D=1 \\ 18A+9B+4D=0 \end{cases} \Rightarrow A=\frac{5}{169} \quad B=-\frac{2}{13} \quad C=-\frac{5}{169} \quad D=\frac{36}{169}$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2\left(s^2+9\right)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{5}{169}\frac{1}{s+2}-\frac{2}{13}\frac{1}{(s+2)^2}-\frac{1}{169}\frac{5s+36}{s^2+9}\right\}$$

$$=\frac{5}{169}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}-\frac{2}{13}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}-\frac{5}{169}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}-\frac{36}{169}\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$\underline{=\frac{5}{169}e^{-2t}-\frac{2}{13}te^{-2t}-\frac{5}{169}\cos 3t+\frac{12}{169}\sin 3t}$$

Exercise

Find the inverse Laplace Transform of $\qquad Y(s)=\dfrac{1}{(s+1)^2\left(s^2-4\right)}$

Solution

$$\frac{1}{(s+1)^2\left(s^2-4\right)}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{Cs+D}{s^2-4}$$

$$=\frac{A(s+1)\left(s^2-4\right)+B\left(s^2-4\right)+(Cs+D)(s+1)^2}{(s+1)^2\left(s^2-4\right)}$$

$$1=As^3-4As+As^2-4A+Bs^2-4B+Cs^3+2Cs^2+Cs+Ds^2+2Ds+D$$

$$=(A+C)s^3+(A+B+2C+D)s^2+(-4A+C+2D)s-4A-4B+D$$

$$\begin{array}{l} s^3 \\ s^2 \\ s^1 \\ s^0 \end{array}\begin{cases} A+C=0 \\ A+B+2C+D=0 \\ -4A+C+2D=0 \\ -4A-4B+D=1 \end{cases} \qquad A=-\frac{2}{15} \quad B=\frac{1}{5} \quad C=\frac{2}{15} \quad D=-\frac{1}{3}$$

$$Y(s)=-\frac{2}{15}\frac{1}{s+1}+\frac{1}{5}\frac{1}{(s+1)^2}+\frac{\frac{2}{15}s-\frac{1}{3}}{s^2-4}$$

$$=-\frac{2}{15}\frac{1}{s+1}+\frac{1}{5}\frac{1}{(s+1)^2}+\frac{2}{15}\frac{s}{s^2-4}-\frac{1}{3}\frac{1}{s^2-4}$$

$$y(t)=-\frac{2}{15}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}+\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}+\frac{2}{15}\mathcal{L}^{-1}\left\{\frac{s}{s^2-4}\right\}-\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2-4}\right\}$$

$$=-\frac{2}{15}e^{-t}+\frac{1}{5}te^{-t}+\frac{2}{15}\cosh 2t-\frac{1}{6}\sinh 2t$$

$$=-\frac{2}{15}e^{-t}+\frac{1}{5}te^{-t}+\frac{2}{15}\frac{e^{2t}+e^{-2t}}{2}-\frac{1}{6}\frac{e^{2t}-e^{-2t}}{2}$$

$$=-\frac{2}{15}e^{-t}+\frac{1}{5}te^{-t}+\frac{1}{15}e^{2t}+\frac{1}{15}e^{-2t}-\frac{1}{12}e^{2t}+\frac{1}{12}e^{-2t}$$

$$=-\frac{2}{15}e^{-t}+\frac{1}{5}te^{-t}-\frac{1}{60}e^{2t}+\frac{3}{20}e^{-2t}$$

Exercise

Find the inverse Laplace Transform of $\quad Y(s)=\dfrac{7s^2+20s+53}{(s-1)\left(s^2+2s+5\right)}$

Solution

$$\frac{7s^2+20s+53}{(s-1)\left(s^2+2s+5\right)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+2s+5}$$

$$7s^2+20s+53=As^2+2As+5A+Bs^2-Bs+Cs-C$$

$$\begin{array}{l} s^2 \\ s^1 \\ s^0 \end{array}\begin{cases} A+B=7 \\ 2A-B+C=20 \\ 5A-C=53 \end{cases} \Rightarrow \begin{cases} A=10 \\ B=-3 \\ C=-3 \end{cases}$$

$$Y(x)=\frac{10}{s-1}+\frac{-3s-3}{s^2+2s+5}$$

$$=\frac{10}{s-1}-3\frac{s+1}{s^2+2s+5}$$

$$y(t)=10\mathcal{L}^{-1}\left\{\frac{10}{s-1}\right\}-3\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\}$$

$$=10e^{t}-3e^{-t}\cos 2t$$

Exercise

Find the inverse Laplace Transform of $F(s)=\frac{1}{s^3}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$=\frac{1}{2}t^2$$

Exercise

Find the inverse Laplace Transform of $F(s)=\frac{1}{s^4}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$$

$$=\frac{1}{6}t^3$$

Exercise

Find the inverse Laplace Transform of $F(s)=\frac{1}{s^2}-\frac{48}{s^5}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^2}-\frac{48}{s^5}\right\}$$

$$=t-2t^4$$

Exercise

Find the inverse Laplace Transform of $F(s)=\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s-2}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s-2}\right\}$$

$$=t-1+e^{2t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\frac{4}{s}+\frac{4}{s^5}+\frac{1}{s-8}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{4}{s}+\frac{4}{s^5}+\frac{1}{s-8}\right\}$$

$$= 4 + \frac{1}{6}t^4 + e^{8t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \frac{1}{4s+1}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s+\frac{1}{4}}\right\}$$

$$= \frac{1}{4}e^{-t/2}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \frac{1}{5s-2}$

Solution

$$f(t) = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-\frac{2}{5}}\right\}$$

$$= \frac{1}{5}e^{-2t/5}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \frac{s+1}{s^2+2}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2} + \frac{1}{s^2+2}\right\}$$

$$= \cos\sqrt{2}t + \frac{1}{2}\sin\sqrt{2}t$$

Exercise

Find the inverse Laplace Transform of $F(s) = \frac{2s-6}{s^2+9}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{2s}{s^2+9} - \frac{6}{s^2+9}\right\}$$

$$= 2\cos 3t - 2\sin 3t$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{10s}{s^2+16}$

Solution

$$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{10s}{s^2+16}\right\}$$

$$\underline{f(t)=10\cos 4t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\left(\dfrac{2}{s}-\dfrac{1}{s^3}\right)^2$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{4}{s^2}-\frac{4}{s^4}+\frac{1}{s^6}\right\}$$

$$\underline{=4t-\frac{2}{3}t^3+\frac{1}{5!}t^5}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{(s+1)^3}{s^4}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{s^3+3s^2+3s+1}{s^4}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{3}{s^2}+\frac{3}{s^3}+\frac{1}{s^4}\right\}$$

$$\underline{=1+3t+\frac{3}{2}t^2+\frac{1}{6}t^3}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{(s+2)^2}{s^3}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{s^2+4s+4}{s^3}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{4}{s^2}+\frac{4}{s^3}\right\}$$

$$\underline{=1+4t+2t^2}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{1}{s^4-9}$

Solution

$$F(s)=\frac{1}{s^4-9}=\frac{A}{s-\sqrt{3}}+\frac{B}{s+\sqrt{3}}+\frac{Cs+D}{s^2+3} \qquad s^4-9=\left(s^2-3\right)\left(s^2+3\right)=\left(s-\sqrt{3}\right)\left(s+\sqrt{3}\right)\left(s^2+3\right)$$

$$A\left(s+\sqrt{3}\right)\left(s^2+3\right)+B\left(s-\sqrt{3}\right)\left(s^2+3\right)+Cs^3-3Cs+Ds^2-3D=1$$

$$As^3+3As+As^2\sqrt{3}+3A\sqrt{3}+Bs^3+3Bs-Bs^2\sqrt{3}-3B\sqrt{3}+Cs^3-3Cs+Ds^2-3D=1$$

$$\begin{array}{ll} s^3 & A+B+C=0 \\ s^2 & \sqrt{3}A-\sqrt{3}B+D=0 \\ s^1 & 3A+3B-3C=0 \\ s^0 & 3\sqrt{3}A-3\sqrt{3}B-3D=1 \end{array} \rightarrow \begin{cases} A+B+C=0 \\ A+B-C=0 \end{cases} \quad \begin{cases} \sqrt{3}A-\sqrt{3}B+D=0 \\ \sqrt{3}A-\sqrt{3}B-D=1 \end{cases} \rightarrow \begin{cases} A+B=0 \\ 2\sqrt{3}A-2\sqrt{3}B=1 \end{cases}$$

$$\underline{A=\frac{1}{4\sqrt{3}}=\frac{\sqrt{3}}{12}} \qquad \underline{B=-\frac{\sqrt{3}}{12}}$$

$$\underline{C=A+B=0} \qquad \underline{D=\sqrt{3}B-\sqrt{3}A=-\frac{1}{4}}$$

$$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{12}\frac{1}{s-\sqrt{3}}-\frac{\sqrt{3}}{12}\frac{1}{s+\sqrt{3}}-\frac{1}{4}\frac{\sqrt{3}}{\sqrt{3}}\frac{1}{s^2+3}\right\}$$

$$\boxed{f(t)=\frac{\sqrt{3}}{12}\left(e^{\sqrt{3}t}-e^{-\sqrt{3}t}-\sin\sqrt{3}t\right)}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{1}{s^3+5s}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s\left(s^2+5\right)}\right\}$$

$$\frac{1}{s\left(s^2+5\right)}=\frac{A}{s}+\frac{Bs+C}{s^2+5}$$

$$1=As^2+5A+Bs^2+Cs$$

$$\begin{array}{lll} s^2 & A+B=0 & \underline{B=-\frac{1}{5}} \\ s & \underline{C=0} & \\ s^0 & 5A=1 & \underline{A=\frac{1}{5}} \end{array}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{5}\frac{1}{s} - \frac{1}{5}\frac{s}{s^2+5}\right\}$$

$$\boxed{= \frac{t}{5} + \frac{1}{5}\cos\sqrt{5}t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{5}{s^2+36}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{5}{s^2+36}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \sin\omega t$$

$$\boxed{= \frac{5}{6}\sin 6t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{10s}{s^2+16}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{10s}{s^2+16}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos\omega t$$

$$\boxed{= 10\cos 4t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{4s}{4s^2+1}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{4s}{4s^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+\frac{1}{4}}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos\omega t$$

$$\boxed{= \cos\frac{1}{2}t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{1}{4s^2+1}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{4s^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{s}{s^2+\frac{1}{4}}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos\omega t$$

$$= \frac{1}{4}\cos\frac{1}{2}t$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{1}{s^2+3s}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s+3)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{3}\frac{1}{s} - \frac{1}{3}\frac{1}{s+3}\right\}$$

$$= \frac{1}{3}t - \frac{1}{3}e^{-3t}$$

$$\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$

$$1 = As + 3A + Bs$$

$$s \quad A + B = 0 \quad B = -\frac{1}{3}$$

$$s^0 \quad 3A = 1 \quad A = \frac{1}{3}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{s+1}{s^2-4s}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s+1}{s(s-4)}\right\}$$

$$= \mathcal{L}^{-1}\left\{-\frac{1}{4}\frac{1}{s} + \frac{5}{4}\frac{1}{s-4}\right\}$$

$$= -\frac{1}{4}t + \frac{5}{4}e^{4t}$$

$$\frac{s+1}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4}$$

$$s + 1 = As - 4A + Bs$$

$$s \quad A + B = 1 \quad B = \frac{5}{4}$$

$$s^0 \quad -4A = 1 \quad A = -\frac{1}{4}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{1}{s^3+5s}$

Solution

$$F(s) = \frac{1}{s^3+5s} = \frac{A}{s} + \frac{Bs+C}{s^2+5}$$

$$As^2 + 5A + Bs^2 + Cs = 1$$

$$s^2 \quad A + B = 0 \quad B = -\frac{1}{5}$$

$$s^1 \quad C = 0$$

$$s^0 \quad 5A = 1 \quad A = \frac{1}{5}$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{5}\frac{1}{s}-\frac{1}{5}\frac{s}{s^2+5}\right\}$$

$$=\frac{1}{5}\left(t-\cos\sqrt{5}t\right)$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{3}{s^2+9}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\}=\sin\omega t$$

$$=\sin 3t$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{2}{s^2+4}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\}=\sin\omega t$$

$$=\sin 2t$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{3}{(2s+5)^3}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{3}{(2s+5)^3}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{3}{2^3\left(s+\frac{5}{8}\right)^3}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{n!}{(s+a)^{n+1}}\right\}=t^ne^{-at}$$

$$=\frac{3}{16}t^2e^{-5t/8}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{6}{(s-1)^4}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{6}{(s-1)^4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s+a)^{n+1}}\right\}=t^n e^{-at}$$

$$\underline{=t^3e^t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{5}{(s+2)^4}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{5}{(s+2)^4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s+a)^{n+1}}\right\}=t^n e^{-at}$$

$$=\frac{5}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{(s+2)^4}\right\}$$

$$\underline{=\frac{5}{6}t^3e^{-2t}}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{s-1}{s^2-2s+5}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-2s+5}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+\omega^2}\right\}=e^{-at}\cos\omega t$$

$$\underline{=e^t\cos 2t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{3s+2}{s^2+2s+10}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{3s+2}{s^2+2s+10}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3s+3-1}{(s+1)^2+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3(s+1)}{(s+1)^2+9} - \frac{1}{(s+1)^2+9}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+\omega^2}\right\} = e^{-at}\cos\omega t$$

$$= 3e^{t}\cos 3t - \frac{1}{3}e^{-t}\sin 3t$$

$$\mathcal{L}^{-1}\left\{\frac{\omega}{(s+a)^2+\omega^2}\right\} = e^{-at}\sin\omega t$$

Exercise

Find the inverse Laplace Transform of $F(s) = \frac{s}{s^2+2s-3}$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s-1)(s+3)}\right\}$$

$$\frac{s}{(s-1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+3}$$

$$s = As + 3A + Bs - B$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s-1} + \frac{3}{4}\frac{1}{s+3}\right\}$$

$s \qquad A + B = 1 \qquad B = \frac{3}{4}$

$s^0 \qquad 3A - B = 0 \qquad A = \frac{1}{4}$

$$= -\frac{1}{4}e^{t} + \frac{3}{4}e^{-3t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \frac{1}{s^2+2s-20}$

Solution

$$s^2 + 2s - 20 = 0 \quad \rightarrow \quad s_{1,2} = -1 \pm 2\sqrt{21}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{\left(s+1+2\sqrt{21}\right)\left(s+1-2\sqrt{21}\right)}\right\}$$

$$\frac{1}{\left(s+1+2\sqrt{21}\right)\left(s+1-2\sqrt{21}\right)} = \frac{A}{s+1+2\sqrt{21}} + \frac{B}{s+1-2\sqrt{21}}$$

$$1 = sA + \left(1-2\sqrt{21}\right)A + sB + \left(1+2\sqrt{21}\right)B$$

$s \qquad A + B = 0 \qquad A = -B$

$s^0 \qquad \left(1-2\sqrt{21}\right)A + \left(1+2\sqrt{21}\right)B = 1 \quad \rightarrow \left(1-2\sqrt{21}-1-2\sqrt{21}\right)A = 1$

$$A = -\frac{1}{4\sqrt{21}} \qquad B = \frac{1}{4\sqrt{21}}$$

$$= \mathcal{L}^{-1}\left\{-\frac{1}{4\sqrt{21}}\frac{1}{s+1+2\sqrt{21}}+\frac{1}{4\sqrt{21}}\frac{1}{s+1-2\sqrt{21}}\right\}$$

$$\boxed{= -\frac{1}{4\sqrt{21}}e^{\left(-1-2\sqrt{21}\right)t}+\frac{1}{4\sqrt{21}}e^{\left(-1+2\sqrt{21}\right)t}}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{s+1}{s^2+2s+10}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+10}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+\omega^2}\right\}=e^{-at}\cos\omega t$$

$$\boxed{= e^{-t}\cos 3t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{1}{s^2+4s+8}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+8}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{\omega}{(s+a)^2+\omega^2}\right\}=e^{-at}\sin\omega t$$

$$\boxed{= \frac{1}{2}e^{-2t}\sin 2t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{2s+16}{s^2+4s+13}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{2s+16}{s^2+4s+13}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2(s+2)+12}{(s+2)^2+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2(s+2)}{(s+2)^2+3^2}+\frac{4(3)}{(s+2)^2+3^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{\omega}{(s+a)^2+\omega^2}\right\} = e^{-at}\sin\omega t \qquad \mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+\omega^2}\right\} = e^{-at}\cos\omega t$$

$$\boxed{= \frac{1}{2}e^{-2t}\sin 2t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{s-1}{2s^2+s+6}$

Solution

$$\frac{s-1}{2s^2+s+6} = \frac{s-1}{2\left(s^2+\frac{1}{2}s+3\right)}$$

$$= \frac{1}{2}\frac{s-1}{\left(s+\frac{1}{4}\right)^2+\frac{47}{16}}$$

$$= \frac{1}{2}\frac{s+\frac{1}{4}-\frac{5}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{47}{16}}$$

$$= \frac{1}{2}\left[\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{47}{16}}-\frac{5}{4}\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{47}{16}}\right]$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s-1}{2s^2+s+6}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\left(\frac{\sqrt{47}}{4}\right)^2}-\frac{5}{4}\frac{4}{\sqrt{47}}\frac{\frac{\sqrt{47}}{4}}{\left(s+\frac{1}{4}\right)^2+\left(\frac{\sqrt{47}}{4}\right)^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{\omega}{(s+a)^2+\omega^2}\right\} = e^{-at}\sin\omega t \qquad \mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+\omega^2}\right\} = e^{-at}\cos\omega t$$

$$\boxed{= \frac{1}{2}e^{-t/4}\cos\left(\frac{\sqrt{47}}{4}t\right)-\frac{5}{2\sqrt{47}}e^{-t/4}\sin\left(\frac{\sqrt{47}}{4}t\right)}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{s^2+1}{s^3-2s^2-8s}$

Solution

$$\frac{s^2+1}{s^3-2s^2-8s}=\frac{A}{s}+\frac{B}{s-4}+\frac{C}{s+2}$$

$$s^2+1=As^2-2As-8A+Bs^2+2Bs+Cs^2-4Cs$$

$$\begin{array}{l} s^2 \\ s^1 \\ s^0 \end{array}\begin{cases} A+B+C=1 \\ -2A+2B-4C=0 \\ -8A=1 \end{cases} \Rightarrow A=-\frac{1}{8} \quad B=\frac{17}{24} \quad C=\frac{5}{12}$$

$$F(s)=-\frac{1}{8}\frac{1}{s}+\frac{17}{24}\frac{1}{s-4}+\frac{5}{12}\frac{1}{s+2}$$

$$f(t)=\mathcal{L}^{-1}\left\{-\frac{1}{8}\frac{1}{s}+\frac{17}{24}\frac{1}{s-4}+\frac{5}{12}\frac{1}{s+2}\right\}$$

$$=-\frac{1}{8}t+\frac{17}{24}e^{4t}+\frac{5}{12}e^{-2t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{6s+3}{s^4+5s^2+4}$

Solution

$$F(s)=\frac{6s+3}{s^4+5s^2+4}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+4}$$

$$6s+3=As^3+4As+Bs^2+4B+Cs^3+Cs+Ds^2+D$$

$$\begin{array}{ll} s^3 & A+C=0 \\ s^2 & B+D=0 \\ s^1 & 4A+C=6 \\ s^0 & 4B+D=3 \end{array} \qquad \begin{array}{ll} A=2 & B=1 \\ C=-2 & D=-1 \end{array}$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1}+\frac{1}{s^2+1}-\frac{2s}{s^2+4}-\frac{1}{s^2+4}\right\}$$

$$=2\cos t+\sin t-2\cos 2t-\frac{1}{2}\sin 2t$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{s-3}{\left(s-\sqrt{3}\right)\left(s+\sqrt{3}\right)}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{s-3}{\left(s-\sqrt{3}\right)\left(s+\sqrt{3}\right)}\right\}$$

$$\frac{s-3}{\left(s-\sqrt{3}\right)\left(s+\sqrt{3}\right)}=\frac{A}{s-\sqrt{3}}+\frac{B}{s+\sqrt{3}}$$

$$s-3=sA+\sqrt{3}A+sB-\sqrt{3}B$$

$$s \qquad A+B=1$$

$$s^0 \qquad \sqrt{3}A-\sqrt{3}B=-3$$

$$\Delta=\begin{vmatrix}1 & 1\\ \sqrt{3} & -\sqrt{3}\end{vmatrix}=-2\sqrt{3} \quad \Delta_A=\begin{vmatrix}1 & 1\\ -3 & -\sqrt{3}\end{vmatrix}=3-\sqrt{3} \quad \Delta_B=\begin{vmatrix}1 & 1\\ \sqrt{3} & -3\end{vmatrix}=-3-\sqrt{3}$$

$$\boxed{A=\frac{-3+\sqrt{3}}{2\sqrt{3}}=\frac{1-\sqrt{3}}{2} \qquad B=\frac{3+\sqrt{3}}{2\sqrt{3}}=\frac{\sqrt{3}+1}{2}}$$

$$=\mathcal{L}^{-1}\left\{\frac{1-\sqrt{3}}{2}\frac{1}{s-\sqrt{3}}+\frac{1+\sqrt{3}}{2}\frac{1}{s+\sqrt{3}}\right\}$$

$$\boxed{=\frac{1-\sqrt{3}}{2}e^{\sqrt{3}t}+\frac{1+\sqrt{3}}{2}e^{\sqrt{3}t}}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{1}{\left(s^2+1\right)\left(s^2+4\right)}$

Solution

$$F(s)=\frac{1}{\left(s^2+1\right)\left(s^2+4\right)}=\frac{A}{s^2+1}+\frac{B}{s^2+4}$$

$$(A+B)s^2+4A+B=1$$

$$\begin{cases}A+B=0\\ 4A+B=1\end{cases} \rightarrow \boxed{A=\frac{1}{3};\ B=-\frac{1}{3}}$$

$$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{3}\frac{1}{s^2+1}-\frac{1}{3}\frac{2}{2}\frac{1}{s^2+4}\right\}$$

$$\boxed{f(t)=\frac{1}{3}\sin t-\frac{1}{6}\sin 2t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{2s-4}{\left(s^2+s\right)\left(s^2+1\right)}$

Solution

$$F(s)=\frac{2s-4}{\left(s^2+s\right)\left(s^2+1\right)}=\frac{A}{s}+\frac{B}{s+1}+\frac{Cs+D}{s^2+1}$$

$$As^3+4As+As^2+4A+Bs^3+Bs+Cs^3+Cs^2+Ds^2+Ds=2s-4$$

$$\begin{array}{ll} s^3 & A+B+C=0 \\ s^2 & A+C+D=0 \\ s^1 & 4A+B+D=2 \\ s^0 & 4A=-4\to \boxed{A=-1} \end{array} \quad \to \begin{cases} B+C=1 & \to \quad 6-D+1-D=1\Rightarrow \boxed{D=3} \\ C+D=1 \to C=1-D & \to \boxed{C=-2} \\ B+D=6 \to B=6-D & \to \boxed{B=3} \end{cases}$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{-1}{s}+\frac{3}{s+1}-\frac{2s}{s^2+1}+\frac{3}{s^2+1}\right\}$$

$$\boxed{=-t+3e^{-t}-2\cos t+3\sin t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{s}{(s+2)\left(s^2+4\right)}$

Solution

$$F(s)=\frac{s}{(s+2)\left(s^2+4\right)}=\frac{A}{s+2}+\frac{Bs+C}{s^2+4}$$

$$As^2+4A+Bs^2+2Bs+Cs+2C=s$$

$$\begin{array}{llll} s^2 & A+B=0 & \to A=-B & \boxed{A=-\frac{1}{4}} \\ s^1 & 2B+C=1 & \to C=1-2B & \boxed{C=\frac{1}{2}} \\ s^0 & 4A+2C=0 & \Rightarrow & -4B+2-4B=0\Rightarrow \boxed{B=\frac{1}{4}} \end{array}$$

$$f(t)=\mathcal{L}^{-1}\left\{-\frac{1}{4}\frac{1}{s+2}+\frac{1}{4}\frac{s}{s^2+4}+\frac{1}{2}\frac{2}{2}\frac{1}{s^2+4}\right\}$$

$$\boxed{=-\frac{1}{4}e^{-2t}+\frac{1}{4}\cos 2t+\frac{1}{4}\sin 2t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{s^2+1}{s(s-1)(s+1)(s-2)}$

Solution

$$F(s)=\frac{s^2+1}{s(s-1)(s+1)(s-2)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+1}+\frac{D}{s-2}$$

$$A\left(s^2-1\right)(s-2)+Bs(s+1)(s-2)+Cs(s-1)(s-2)+Ds\left(s^2-1\right)=s^2+1$$

$$As^3-2As^2-As+2A+Bs^3-Bs^2-2Bs+Cs^3-3Cs^2+2Cs+Ds^3-Ds=s^2+1$$

$$\begin{array}{ll} s^3 & A+B+C+D=0 \\ s^2 & -2A-B-3C=1 \\ s^1 & -A-2B+2C-D=0 \\ s^0 & 2A=1 \to A=\frac{1}{2} \end{array} \to \begin{cases} B+C+D=-\frac{1}{2} \\ B+3C=-2 \\ -2B+2C-D=\frac{1}{2} \end{cases} \to \begin{array}{l} B=-1 \\ C=-\frac{1}{3} \\ D=\frac{5}{6} \end{array}$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s}-\frac{1}{s-1}-\frac{1}{3}\frac{1}{s+1}+\frac{5}{6}\frac{1}{s-2}\right\}$$

$$=\frac{1}{2}t-e^t-\frac{1}{3}e^{-t}+\frac{5}{6}e^{2t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{s}{(s-2)(s-3)(s-6)}$

Solution

$$F(s)=\frac{s}{(s-2)(s-3)(s-6)}=\frac{A}{s-2}+\frac{B}{s-3}+\frac{C}{s-6}$$

$$As^2-9As+18A+Bs^2-8Bs+12B+Cs^2-5Cs+6C=s$$

$$\begin{array}{ll} s^2 & A+B+C=0 \\ s^1 & -9A-8B-5C=1 \\ s^0 & 18A+12B+6C=0 \end{array} \to A=\frac{1}{2} \quad B=-1 \quad C=\frac{1}{2}$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s-2}-\frac{1}{s-3}+\frac{1}{2}\frac{1}{s-6}\right\}$$

$$=\frac{1}{2}e^{2t}-e^{3t}+\frac{1}{2}e^{6t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{7s-1}{(s+1)(s+2)(s-3)}$

Solution

$$f(t)=\mathcal{L}^{-1}\left\{\frac{7s-1}{(s+1)(s+2)(s-3)}\right\}$$

$$\frac{7s-1}{(s+1)(s+2)(s-3)}=\frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{s-3}$$

$$7s-1=As^2-As-6A+Bs^2-2Bs-3B+Cs^2+3Cs+2C$$

$$s^2 \quad A+B+C=0$$
$$s \quad -A-2B+3C=7$$
$$s^0 \quad -6A-3B+2C=-1$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ -1 & -2 & 3 \\ -6 & -3 & 2 \end{vmatrix} = -20 \quad \Delta_A = \begin{vmatrix} 0 & 1 & 1 \\ 7 & -2 & 3 \\ -1 & -3 & 2 \end{vmatrix} = -40 \quad \Delta_B = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 7 & 3 \\ -6 & -1 & 2 \end{vmatrix} = 60$$

$$\underline{A=2, \quad B=-3, \quad C=1}$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s+1} - \frac{3}{s+2} + \frac{1}{s-3}\right\}$$

$$\underline{= 2e^{-t} - 3e^{-2t} + e^{3t}}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{s^2+9s+2}{(s-1)^2(s+3)}$

Solution

$$\frac{s^2+9s+2}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}$$

$$s^2+9s+2 = As^2+2As-3A+Bs+3B+Cs^2-2Cs+C$$

$$s^2 \quad A+C=1$$
$$s \quad 2A+B-2C=9$$
$$s^0 \quad -3A+3B+C=2$$

$$\Delta = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & -2 \\ -3 & 3 & 1 \end{vmatrix} = 16 \quad \Delta_A = \begin{vmatrix} 1 & 0 & 1 \\ 9 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 32 \quad \Delta_B = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 9 & -2 \\ -3 & 2 & 1 \end{vmatrix} = 48$$

$$\underline{A=2, \quad B=3, \quad C=-1}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s-1} + \frac{3}{(s-1)^2} - \frac{1}{s+3}\right\}$$

$$\underline{= 2e^{t} + 3te^{t} - e^{-3t}}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{2s^2+10s}{\left(s^2-2s+5\right)(s+1)}$

Solution

$$\frac{2s^2+10s}{\left(s^2-2s+5\right)(s+1)}=\frac{2s^2+10s}{\left((s-1)^2+4\right)(s+1)}=\frac{A(s-1)+B}{(s-1)^2+4}+\frac{C}{s+1}$$

$$2s^2+10s=As^2-A+Bs+B+Cs^2-2Cs+5C$$

$$\begin{aligned} s^2 \quad & A+C=2 \\ s \quad & B-2C=10 \\ s^0 \quad & -A+B+5C=0 \end{aligned}$$

$$\Delta=\begin{vmatrix}1&0&1\\0&1&-2\\-1&1&5\end{vmatrix}=8 \quad \Delta_A=\begin{vmatrix}2&0&1\\10&1&-2\\0&1&5\end{vmatrix}=24 \quad \Delta_B=\begin{vmatrix}1&2&1\\0&10&-2\\-1&0&5\end{vmatrix}=64$$

$$\underline{A=3, \quad B=8, \quad C=-1}$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{2s^2+10s}{\left(s^2-2s+5\right)(s+1)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{3(s-1)}{(s-1)^2+4}+\frac{4(2)}{(s-1)^2+4}-\frac{1}{s+1}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+\omega^2}\right\}=e^{-at}\cos\omega t \qquad \mathcal{L}^{-1}\left\{\frac{\omega}{(s+a)^2+\omega^2}\right\}=e^{-at}\sin\omega t$$

$$\underline{=3e^t\cos 2t+4e^t\sin 2t-e^{-t}}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{s^2-26s-47}{(s-1)(s+2)(s+5)}$

Solution

$$\frac{s^2-26s-47}{(s-1)(s+2)(s+5)}=\frac{A}{s-1}+\frac{B}{s+2}+\frac{C}{s+5}$$

$$s^2-26s-47=As^2+7As+10A+Bs^2+4Bs-5B+Cs^2+Cs-2C$$

$$\begin{aligned} s^2 \quad & A+B+C=1 \\ s \quad & 7A+4B+C=-26 \\ s^0 \quad & 10A-5B-2C=-47 \end{aligned}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 7 & 4 & 1 \\ 10 & -5 & -2 \end{vmatrix} = -54 \quad \Delta_A = \begin{vmatrix} 1 & 1 & 1 \\ -26 & 4 & 1 \\ -47 & -5 & -2 \end{vmatrix} = 216 \quad \Delta_B = \begin{vmatrix} 1 & 1 & 1 \\ 7 & -26 & 1 \\ 10 & -47 & -2 \end{vmatrix} = 54$$

$$\underline{A = -4, \quad B = -1, \quad C = 6}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s^2 - 26s - 47}{(s-1)(s+2)(s+5)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{-4}{s-1} - \frac{1}{s+2} + \frac{6}{s+5}\right\}$$

$$\underline{= -4e^{t} - e^{-2t} + 6e^{-5t}}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{-s-7}{(s-1)(s+2)}$

Solution

$$\frac{-s-7}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}$$

$$-s - 7 = As + 2A + Bs - B$$

$$s \qquad A + B = -1$$

$$s^0 \qquad 2A - B = -7$$

$$\underline{A = -\frac{8}{3}, \quad B = \frac{5}{3}}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{-s-7}{(s-1)(s+2)}\right\}$$

$$= \mathcal{L}^{-1}\left\{-\frac{8}{3}\frac{1}{s-1} + \frac{5}{3}\frac{1}{s+2}\right\}$$

$$\underline{= -\frac{8}{3}e^{t} + \frac{5}{3}e^{-2t}}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{-8s^2 - 5s + 9}{\left(s^2 - 3s + 2\right)(s+1)}$

Solution

$$\frac{-8s^2 - 5s + 9}{\left(s^2 - 3s + 2\right)(s+1)} = \frac{-8s^2 - 5s + 9}{(s-1)(s-2)(s+1)}$$

$$= \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+1}$$

$$-8s^2 - 5s + 9 = As^2 - As - 2A + Bs^2 - B + Cs^2 - 3Cs + 2C$$

$s^2 \quad A + B + C = -8$

$s \quad -A - 3C = -5$

$s^0 \quad -2A - B + 2C = 9$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & -3 \\ -2 & -1 & 2 \end{vmatrix} = 6 \quad \Delta_A = \begin{vmatrix} -8 & 1 & 1 \\ -5 & 0 & -3 \\ 9 & -1 & 2 \end{vmatrix} = 12 \quad \Delta_B = \begin{vmatrix} 1 & -8 & 1 \\ -1 & -5 & -3 \\ -2 & 9 & 2 \end{vmatrix} = -66$$

$A = 2, \quad B = -11, \quad C = 1$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{-8s^2 - 5s + 9}{\left(s^2 - 3s + 2\right)(s+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s-1} - \frac{11}{s-2} + \frac{1}{s+1}\right\}$$

$$= 2e^t - 11e^{2t} + e^{-t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{-2s^2 + 8s - 14}{(s+1)\left(s^2 - 2s + 5\right)}$

Solution

$$\frac{-2s^2 + 8s - 14}{(s+1)\left(s^2 - 2s + 5\right)} = \frac{-2s^2 + 8s - 14}{(s-1)\left((s-1)^2 + 4\right)}$$

$$= \frac{A}{s-1} + \frac{B(s-1) + C}{(s-1)^2 + 4}$$

$$-2s^2 + 8s - 14 = As^2 - 2As + 5A + Bs^2 - 2Bs + B + Cs - C$$

$s^2 \quad A + B = -2$

$s \quad -2A - 2B + C = 8$

$s^0 \quad 5A + B - C = -14$

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ -2 & -2 & 1 \\ 5 & 1 & -1 \end{vmatrix} = 4 \quad \Delta_A = \begin{vmatrix} -2 & 1 & 0 \\ 8 & -2 & 1 \\ -14 & 1 & -1 \end{vmatrix} = -8 \quad \Delta_B = \begin{vmatrix} 1 & -2 & 0 \\ -2 & 8 & 1 \\ 5 & -14 & -1 \end{vmatrix} = 0 \quad \Delta_C = \begin{vmatrix} 1 & 1 & -2 \\ -2 & -2 & 8 \\ 5 & 1 & -14 \end{vmatrix} = 16$$

$A = -2, \quad B = 0, \quad C = 4$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{-2s^2 + 8s - 14}{(s+1)\left(s^2 - 2s + 5\right)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{-2}{s-1}+\frac{4}{(s-1)^2+2^2}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{\omega}{(s+a)^2+\omega^2}\right\}=e^{-at}\sin\omega t$$

$$\underline{= -2e^{t}+2e^{t}\sin 2t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{-5s-36}{(s+2)\left(s^2+9\right)}$

Solution

$$\frac{-5s-36}{(s+2)\left(s^2+9\right)}=\frac{A}{s+2}+\frac{Bs+C}{s^2+9}$$

$$-5s-36=As^2+9A+Bs^2+2Bs+Cs+2C$$

$s^2 \quad A+B=0$

$s \quad 2B+C=-5$

$s^0 \quad 9A+2C=-6$

$$\Delta=\begin{vmatrix}1&1&0\\0&2&1\\9&0&2\end{vmatrix}=13 \qquad \Delta_A=\begin{vmatrix}0&1&0\\-5&2&1\\-36&0&2\end{vmatrix}=-26$$

$$\underline{A=-2, \quad B=2, \quad C=-9}$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{-2}{s+2}+\frac{2s}{s^2+3^2}-\frac{9}{s^2+3^2}\right\}$$

$$\underline{= -2e^{-2t}+2\cos 3t-3\sin 3t}$$

Exercise

Find the inverse Laplace Transform of $F(s)=\dfrac{3s^2+5s+3}{s^4+s^3}$

Solution

$$\frac{3s^2+5s+3}{s^4+s^3}=\frac{3s^2+5s+3}{s^3(s+1)}$$

$$=\frac{A}{s^3}+\frac{B}{s^2}+\frac{C}{s}+\frac{D}{s+1}$$

$$3s^2+5s+3=As+A+Bs^2+Bs+Cs^3+Cs^2+Ds^3$$

$$s^3 \quad C + D = 0 \quad \underline{D = -1}$$

$$s^2 \quad B + C = 3 \quad \underline{C = 1}$$

$$s \quad A + B = 5 \quad \underline{B = 2}$$

$$s^0 \quad \underline{A = 3}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{3s^2 + 5s + 3}{s^4 + s^3}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3}{s^3} + \frac{2}{s^2} + \frac{1}{s} - \frac{1}{s+1}\right\}$$

$$= \frac{3}{2}t^2 + 2t + 1 - e^{-t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{7s^3 - 2s^2 - 3s + 6}{s^3(s-2)}$

Solution

$$\frac{7s^3 - 2s^2 - 3s + 6}{s^3(s-2)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s-2}$$

$$7s^3 - 2s^2 - 3s + 6 = As - 2A + Bs^2 - 2Bs + Cs^3 - 2Cs^2 + Ds^3$$

$$s^3 \quad C + D = 7 \quad \underline{D = 6}$$

$$s^2 \quad B - 2C = -2 \quad \underline{C = 1}$$

$$s \quad A - 2B = -3 \quad \underline{B = 0}$$

$$s^0 \quad -2A = 6 \quad \underline{A = -3}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{7s^3 - 2s^2 - 3s + 6}{s^3(s-2)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{-3}{s^3} + \frac{1}{s} + \frac{6}{s-2}\right\}$$

$$= -\frac{3}{2}t^2 + 1 + 6e^{2t}$$

Exercise

Find the inverse Laplace Transform of $F(s) = \dfrac{7s^2 - 41s + 84}{(s-1)(s^2 - 4s + 13)}$

Solution

$$\frac{7s^2-41s+84}{(s-1)\left(s^2-4s+13\right)}=\frac{A}{s-1}+\frac{B(s-2)+C}{(s-2)^2+9}$$

$$7s^2-41s+84=As^2-4As+13A+Bs^2-3Bs+2B+Cs-C$$

$$\begin{array}{ll} s^2 & A+B=7 \\ s & -4A-3B+C=-41 \\ s^0 & 13A+2B-C=84 \end{array}$$

$$\Delta=\begin{vmatrix}1 & 1 & 0\\ -4 & -3 & 1\\ 13 & 2 & -1\end{vmatrix}=10 \quad \Delta_A=\begin{vmatrix}7 & 1 & 0\\ -41 & -3 & 1\\ 84 & 2 & -1\end{vmatrix}=50 \quad \Delta_B=\begin{vmatrix}1 & 7 & 0\\ -4 & -41 & 1\\ 13 & 84 & -1\end{vmatrix}=20$$

$$A=5 \quad B=2 \quad C=-15$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{7s^2-41s+84}{(s-1)\left(s^2-4s+13\right)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{5}{s-1}+\frac{2(s-2)}{(s-2)^2+3^2}-\frac{5(3)}{(s-2)^2+3^2}\right\}$$

$$=5e^t+2e^{2t}\cos 3t-5e^{2t}\sin 3t$$

Exercise

Find the inverse Laplace transform of $F(s)=\dfrac{6s-5}{s^2+7}$

Solution

$$F(s)=\frac{6s}{s^2+7}-\frac{5}{s^2+7}$$

$$=\frac{6s}{s^2+7}-\frac{\sqrt{7}}{\sqrt{7}}\frac{5}{s^2+7}$$

$$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{6s}{s^2+7}-\frac{5}{\sqrt{7}}\frac{\sqrt{7}}{s^2+7}\right\}$$

$$f(t)=6\cos\sqrt{7}t-\frac{5}{\sqrt{7}}\sin\sqrt{7}t$$

Exercise

Find the inverse Laplace transform of $F(s)=\dfrac{1-3s}{s^2+8s+21}$

Solution

$$s^2 + 8s + 21 = s^2 + 8s + 16 - 16 + 21$$
$$= (s+4)^2 + 5$$

$$F(s) = \frac{1-3s}{s^2 + 8s + 21}$$
$$= \frac{1 - 3(s+4) + 12}{(s+4)^2 + 5}$$
$$= \frac{13}{(s+4)^2 + 5} - 3\frac{s+4}{(s+4)^2 + 5}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{13}{(s+4)^2 + 5} - 3\frac{s+4}{(s+4)^2 + 5}\right\}$$

$$\boxed{f(t) = \frac{13}{\sqrt{5}} e^{-4t} \sin\sqrt{5}t - 3e^{-4t} \cos\sqrt{5}t}$$

Exercise

Find the inverse Laplace transform of $F(s) = \dfrac{3s-2}{2s^2 - 6s - 2}$

Solution

$$2s^2 - 6s - 2 = 2\left(s^2 - 3s - 1\right)$$
$$= 2\left(s - \frac{3-\sqrt{13}}{2}\right)\left(s - \frac{3+\sqrt{13}}{2}\right)$$
$$= 2\left[\left(s - \frac{3}{2}\right)^2 - \frac{9}{4} - 1\right]$$
$$= 2\left(\left(s - \frac{3}{2}\right)^2 - \frac{13}{4}\right)$$

$$F(s) = \frac{1}{2}\frac{3\left(s - \frac{3}{2}\right) + \frac{9}{2} - 2}{\left(s - \frac{3}{2}\right)^2 - \frac{13}{4}}$$
$$= \frac{3}{2}\frac{s - \frac{3}{2}}{\left(s - \frac{3}{2}\right)^2 - \frac{13}{4}} + \frac{5}{4}\frac{1}{\left(s - \frac{3}{2}\right)^2 - \frac{13}{4}}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{2}\frac{s - \frac{3}{2}}{\left(s - \frac{3}{2}\right)^2 - \frac{13}{4}} + \frac{5}{2\sqrt{13}}\frac{\frac{\sqrt{13}}{2}}{\left(s - \frac{3}{2}\right)^2 - \frac{13}{4}}\right\}$$

$$f(t)=\frac{3}{2}e^{3t/2}\cosh\left(\frac{\sqrt{13}}{2}t\right)+\frac{5}{2\sqrt{13}}e^{3t/2}\sinh\left(\frac{\sqrt{13}}{2}t\right)$$

Exercise

Find the inverse Laplace transform of $F(s)=\dfrac{s+7}{s^2-3s-10}$

Solution

$$F(s)=\frac{s+7}{(s+2)(s-5)}=\frac{A}{s+2}+\frac{B}{s-5}$$

$$s+7=As-5A+Bs+2B$$

$$\begin{cases} A+B=1 \\ -5A+2B=7 \end{cases} \rightarrow \ A=-\frac{5}{7},\ \ B=\frac{12}{7}$$

$$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{-\frac{5}{7}\frac{1}{s+2}+\frac{12}{7}\frac{1}{s-5}\right\}$$

$$f(t)=-\frac{5}{7}e^{-2t}+\frac{12}{7}e^{5t}$$

Exercise

Find the inverse Laplace transform of $F(s)=\dfrac{86s-78}{(s+3)(s-4)(5s-1)}$

Solution

$$F(s)=\frac{86s-78}{(s+3)(s-4)(5s-1)}=\frac{A}{s+3}+\frac{B}{s-4}+\frac{C}{5s-1}$$

$$86s-78=A(s-4)(5s-1)+B(s+3)(5s-1)+C(s+3)(s-4)$$

$$\begin{cases} s^2 & 5A+5B+C=0 \\ s & -21A+14B-C=86 \\ s^0 & 4A-3B-12C=-78 \end{cases}$$

$$\Delta=\begin{vmatrix} 5 & 5 & 1 \\ -21 & 14 & -1 \\ 4 & -3 & -12 \end{vmatrix}=-2128 \quad \Delta_A=\begin{vmatrix} 0 & 5 & 1 \\ 86 & 14 & -1 \\ -78 & -3 & -12 \end{vmatrix}=6384 \quad \Delta_B=\begin{vmatrix} 5 & 0 & 1 \\ -21 & 86 & -1 \\ 4 & -78 & -12 \end{vmatrix}=-4256$$

$$A=-\frac{6384}{2128}=-3,\quad B=\frac{4256}{2128}=2,\quad C=5$$

$$F(s)=-\frac{3}{s+3}+\frac{2}{s-4}+\frac{5}{5s-1}$$

$$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{-\frac{3}{s+3}+\frac{2}{s-4}+\frac{1}{s-\frac{1}{5}}\right\}$$

$$\underline{f(t) = -3e^{-3t} + 2e^{4t} + e^{t/5}}$$

Exercise

Find the inverse Laplace transform of $F(s) = \dfrac{2-5s}{(s-6)\left(s^2+11\right)}$

Solution

$$F(s) = \frac{2-5s}{(s-6)\left(s^2+11\right)} = \frac{A}{s-6} + \frac{Bs+C}{s^2+11}$$

$$2-5s = As^2 + 11A + Bs^2 - 6Bs + Cs - 6C$$

$$\begin{cases} s^2 & A+B=0 \\ s & -6B+C=-5 \\ s^0 & 11A-6C=2 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -6 & 1 & -5 \\ 11 & 0 & -6 & 2 \end{array}\right)$$

$$\underline{A = -\frac{28}{47}, \quad B = \frac{28}{47}, \quad C = -\frac{67}{47}}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{-\frac{28}{47}\frac{1}{s-6} + \frac{28}{47}\frac{s}{s^2+11} - \frac{67}{47}\frac{1}{s^2+11}\frac{\sqrt{11}}{\sqrt{11}}\right\}$$

$$\underline{f(t) = -\frac{28}{47}e^{6t} + \frac{28}{47}\cos\sqrt{11}t - \frac{67}{47\sqrt{11}}\sin\sqrt{11}t}$$

Exercise

Find the inverse Laplace transform of $F(s) = \dfrac{25}{s^3\left(s^2+4s+5\right)}$

Solution

$$F(s) = \frac{25}{s^3\left(s^2+4s+5\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{s^2+4s+5}$$

$$25 = As^4 + 4As^3 + 5As^2 + Bs^3 + 4Bs^2 + 5Bs + Cs^2 + 4Cs + 5C + Ds^4 + Es^3$$

$$\begin{cases} s^4 & A+D=0 & \rightarrow \underline{D = -\frac{11}{5}} \\ s^3 & 4A+B+E=0 & \rightarrow \underline{E = -\frac{24}{5}} \\ s^2 & 5A+4B+C=0 & \rightarrow \underline{A = \frac{11}{5}} \\ s & 5B+4C=0 & \rightarrow \underline{B = -4} \\ s^0 & 5C=25 & \rightarrow \underline{C = 5} \end{cases}$$

$$F(s) = \frac{11}{5}\frac{1}{s} - \frac{4}{s^2} + \frac{5}{s^3} - \frac{1}{5}\frac{11s+24}{(s+2)^2-4+5}$$

$$= \frac{11}{5}\frac{1}{s} - \frac{4}{s^2} + \frac{5}{s^3} - \frac{1}{5}\frac{11(s+2)-22+24}{(s+2)^2+1}$$

$$= \frac{11}{5}\frac{1}{s} - \frac{4}{s^2} + \frac{5}{s^3} - \frac{1}{5}\frac{11(s+2)+2}{(s+2)^2+1}$$

$$= \frac{11}{5}\frac{1}{s} - \frac{4}{s^2} + \frac{5}{s^3} - \frac{11}{5}\frac{s+2}{(s+2)^2+1} - \frac{2}{5}\frac{1}{(s+2)^2+1}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{11}{5}\frac{1}{s} - \frac{4}{s^2} + \frac{5}{s^3} - \frac{11}{5}\frac{s+2}{(s+2)^2+1} - \frac{2}{5}\frac{1}{(s+2)^2+1}\right\}$$

$$f(t) = \frac{11}{5} - 4t + \frac{5}{2}t^2 - \frac{11}{5}e^{-2t}\cos t - \frac{2}{5}\sin t$$

Exercise

Find the inverse Laplace transform of $F(s) = \dfrac{5e^{-6s} - 3e^{-11s}}{(s+2)(s^2+9)}$

Solution

$$F(s) = \left(5e^{-6s} - 3e^{-11s}\right)\frac{1}{(s+2)(s^2+9)}$$

$$= \left(5e^{-6s} - 3e^{-11s}\right)G(s)$$

$$G(s) = \frac{1}{(s+2)(s^2+9)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+9}$$

$$1 = As^2 + 9A + Bs^2 + 2Bs + Cs + 2C$$

$$\begin{cases} s^2 & A+B=0 \\ s & 2B+C=0 \\ s^0 & 9A+2C=1 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 9 & 0 & 2 & 1 \end{array}\right)$$

$$A = \frac{1}{13}, \quad B = -\frac{1}{13}, \quad C = \frac{2}{13}$$

$$\mathcal{L}^{-1}\{G(s)\} = \frac{1}{13}\mathcal{L}^{-1}\left\{\frac{1}{s+2} - \frac{s}{s^2+9} + \frac{2}{s^2+9}\right\}$$

$$g(t) = \frac{1}{13}\left(e^{-2t} - \cos 3t + \frac{2}{3}\sin 3t\right)$$

$$f(t) = 5u_6(t)g(t-6) - 3u_{11}(t)g(t-11)$$