Section 2.5 – Graphing Polynomial Functions

Polynomial Function

A *Polynomial function* $P(x)$ in x is a sum of the form is given by:

$$P(x)=a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$$

Where the coefficients a_n, a_{n-1}, $\ldots, a_2$, a_1, a_0 are real numbers and the exponents are whole numbers.

$$\underline{a_n} x^{n}$$

Degree

Leading Term

Leading Coefficient

Non-polynomial Functions: $\frac{1}{x}+2x$; $\sqrt{x^2-3}+x$; $\frac{x-5}{x^2+2}$

Degree of f	***Form of f(x)***	***Graph of f(x)***
0	$f(x)=a_0$	A horizontal line
1	$f(x)=a_1 x + a_0$	A line with slope a_1
2	$f(x)=a_2 x^2 + a_1 x + a_0$	A parabola with a vertical axis

All polynomial functions are ***continuous functions***.

End Behavior $\left(a_n x^n\right)$

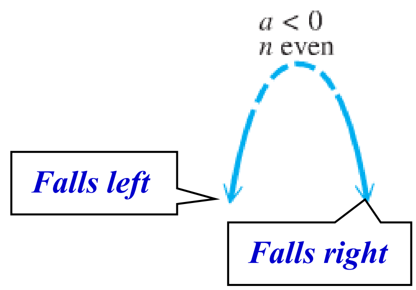
If n (degree) is ***even***:

If $a_n < 0$ (in front x^n is negative).

Then the function falls from the left and right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to -\infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty$$

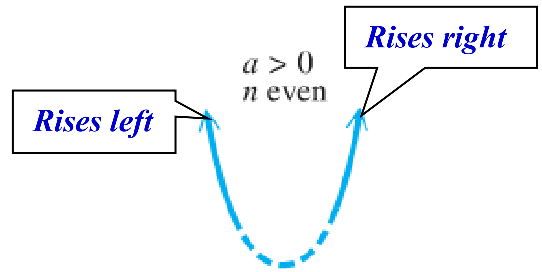


If $a_n > 0$ (in front x^n is positive).

Then the function rises from the left and right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to \infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to \infty$$



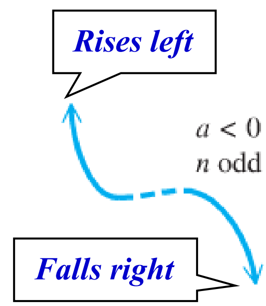
If n (degree) is ***odd***:

If $a_n < 0$ (negative).

Then the function rises from the left side and falls from the right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to \infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty$$

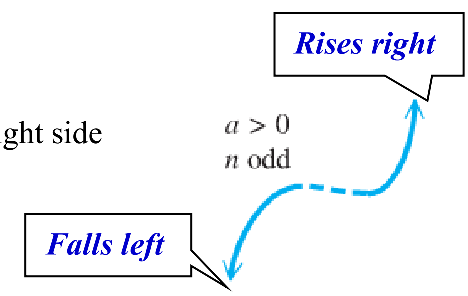


If $a_n > 0$ (positive).

Then the function falls from the left side and rises from the right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to -\infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to \infty$$



Example

Determine the end behavior of the graph of the polynomial function $\quad f(x) = -4x^5 + 7x^2 - x + 9$

Solution

Leading term: $-4x^5$ with 5th degree (n is odd)

$x \to -\infty \quad \Rightarrow \quad f(x) = -(-)^5 = (-)(-) = + \to \infty \qquad f(x)$ rises left

$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty \qquad f(x)$ falls right

The Intermediate Value *Theorem*

For any polynomial function $f(x)$ with real coefficients and $f(a) \neq f(b)$ for $a < b$, then f takes on every value between $f(a)$ and $f(b)$ in the interval $[a, b]$.

$\therefore$ $f(a)$ and $f(b)$ are the ***opposite signs***. Then the function has a real zero between a and b.

Example

Using the intermediate value theorem, determine, if possible, whether the function has a real zero between a and b.

a) $f(x) = x^3 + x^2 - 6x; \quad a = -4, \ b = -2$

b) $f(x) = x^3 + x^2 - 6x; \quad a = -1, \ b = 3$

Solution

a) $f(x) = x^3 + x^2 - 6x; \quad a = -4, \ b = -2$

$f(-4) = (-4)^3 + (-4)^2 - 6(-4)$

$= -24$

$f(-2) = (-2)^3 + (-2)^2 - 6(-2)$

$= 8$

$\therefore$ $f(x)$ has a zero between -4 and -2

b) $f(x) = x^3 + x^2 - 6x; \quad a = -1, \ b = 3$

$f(-1) = (-1)^3 + (-1)^2 - 6(-1)$

$= 6$

$f(3) = (3)^3 + (3)^2 - 6(3) = 18$

$= 18$

$\therefore$ $f(x)$ *zeros* ***can't be determined***

Example

Show that $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3$ has a zero between 1 and 2.

Solution

$f(1) = 1 + 2 - 6 + 2 - 3$

$= -4$

$$f(2) = (2)^5 + 2(2)^4 - 6(2)^3 + 2(2) - 3$$

$$\underline{= 17}$$

Since $f(1)$ *and* $f(2)$ have opposite signs.

Therefore, $f(c) = 0$ for at least one real number c between 1 and 2.

Sketching

Example

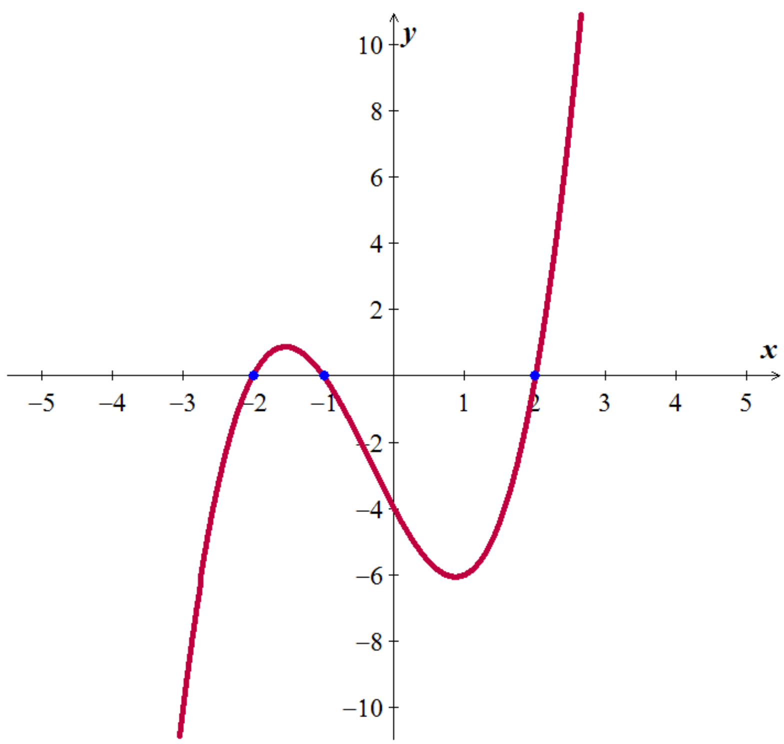
Let $f(x)=x^3+x^2-4x-4$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

Solution

$$
\begin{aligned}
f(x) &= x^3+x^2-4x-4 \\
&= x^2(x+1)-4(x+1) \\
&= (x+1)\left(x^2-4\right) \\
&= (x+1)(x+2)(x-2)
\end{aligned}
$$

The zeros of $f(x)$ (x-intercepts) are: $-2,\ -1,\ and\ 2$

Interval	$-\infty$ $\quad$ -2	-1	0 $\quad$ 2	∞
Sign of $f(x)$	$-$	$+$	$-$	$+$
Position	**Below *x*-axis**	**Above *x*-axis**	**Below *x*-axis**	**Above *x*-axis**



We can conclude from the chart and the graph that:

$$f(x)>0 \quad if\ x\ is\ in\ (-2,\ -1)\cup(2,\ \infty)$$

$$f(x)<0 \quad if\ x\ is\ in\ (-\infty,\ -2)\cup(-1,\ 2)$$

Example

Let $f(x) = x^4 - 4x^3 + 3x^2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

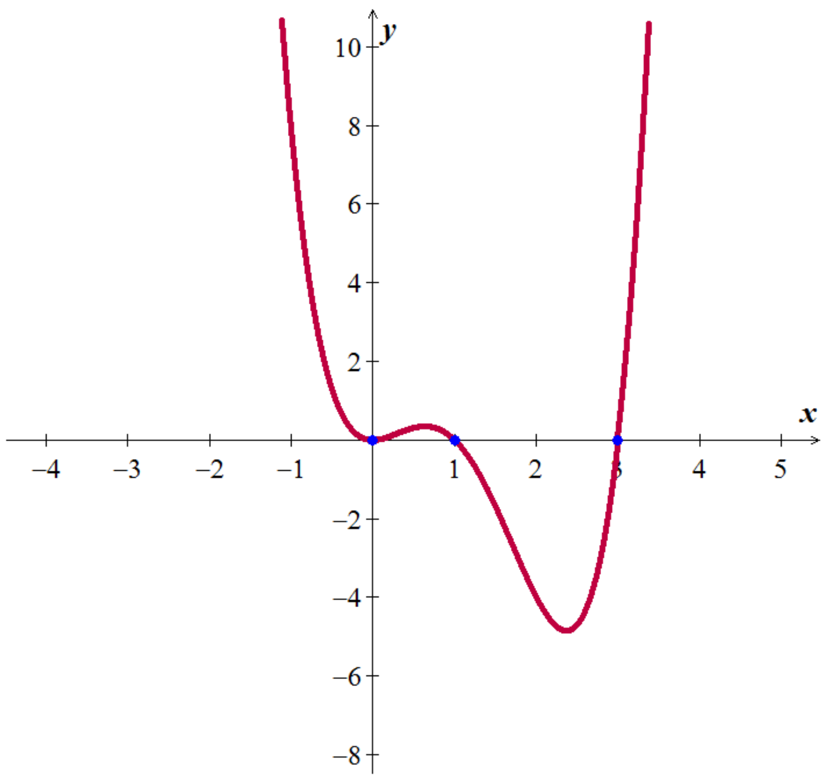
Solution

$$f(x) = x^2\left(x^2 - 4x + 3\right)$$
$$= x^2(x-1)(x-3)$$

The zeros are: 0, 1, 3.

Since the factor x^2 is always positive, it has no factor

$-\infty$ 1	2	3 ∞
+	–	+



$$f(x) > 0 \quad \Rightarrow \quad x \text{ is in } (-\infty,\ 0) \cup (0,\ 1) \cup (3,\ \infty)$$
$$f(x) < 0 \quad \Rightarrow \quad x \text{ is in } (1,\ 3)$$

Exercises *Section* 2.5 – Polynomial Functions

(**1 – 12**) Determine the end behavior of the graph of the polynomial function

1. $f(x) = 5x^3 + 7x^2 - x + 9$

2. $f(x) = 11x^3 - 6x^2 + x + 3$

3. $f(x) = -11x^3 - 6x^2 + x + 3$

4. $f(x) = 2x^3 + 3x^2 - 23x - 42$

5. $f(x) = 5x^4 + 7x^2 - x + 9$

6. $f(x) = 11x^4 - 6x^2 + x + 3$

7. $f(x) = -5x^4 + 7x^2 - x + 9$

8. $f(x) = -11x^4 - 6x^2 + x + 3$

9. $f(x) = 5x^5 - 16x^2 - 20x + 64$

10. $f(x) = -5x^5 - 16x^2 - 20x + 64$

11. $f(x) = -3x^6 - 16x^3 + 64$

12. $f(x) = 3x^6 - 16x^3 + 4$

(**13 – 32**) Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers.

13. $f(x) = x^3 - x - 1;\quad between\ \ 1\ and\ 2$

14. $f(x) = x^3 - 4x^2 + 2;\quad between\ \ 0\ and\ 1$

15. $f(x) = 2x^4 - 4x^2 + 1;\quad between\ \ -1\ and\ 0$

16. $f(x) = x^4 + 6x^3 - 18x^2;\quad between\ \ 2\ and\ 3$

17. $f(x) = x^3 + x^2 - 2x + 1;\quad between\ \ -3\ and\ -2$

18. $f(x) = x^5 - x^3 - 1;\quad between\ \ 1\ and\ 2$

19. $f(x) = 3x^3 - 10x + 9;\quad between\ \ -3\ and\ -2$

20. $f(x) = 3x^3 - 8x^2 + x + 2;\quad between\ \ 2\ and\ 3$

21. $f(x) = 3x^3 - 8x^2 + x + 2;\quad between\ \ 1\ and\ 2$

22. $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3;\quad between\ \ 0\ and\ 1$

23. $P(x) = 2x^3 + 3x^2 - 23x - 42,\quad a = 3,\quad b = 4$

24. $P(x) = 4x^3 - x^2 - 6x + 1,\quad a = 0,\quad b = 1$

25. $P(x) = 3x^3 + 7x^2 + 3x + 7,\quad a = -3,\quad b = -2$

26. $P(x) = 2x^3 - 21x^2 - 2x + 25,\quad a = 1,\quad b = 2$

27. $P(x) = 4x^4 + 7x^3 - 11x^2 + 7x - 15,\quad a = 1,\quad b = \frac{3}{2}$

28. $P(x) = 5x^3 - 16x^2 - 20x + 64,\quad a = 3,\quad b = \frac{7}{2}$

29. $P(x) = x^4 - x^2 - x - 4, \quad a = 1, \quad b = 2$

30. $P(x) = x^3 - x - 8, \quad a = 2, \quad b = 3$

31. $P(x) = x^3 - x - 8, \quad a = 0, \quad b = 1$

32. $P(x) = x^3 - x - 8, \quad a = 2.1, \quad b = 2.2$

(33 – 91) Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

33. $f(x) = x^4 - 4x^2$

34. $f(x) = x^4 + 3x^3 - 4x^2$

35. $f(x) = x^3 + 2x^2 - 4x - 8$

36. $f(x) = x^3 - 3x^2 - 9x + 27$

37. $f(x) = -x^4 + 12x^2 - 27$

38. $f(x) = x^2(x+2)(x-1)^2(x-2)$

39. $f(x) = 2x^3 + 11x^2 - 7x - 6$

40. $f(x) = x^3 + 2x^2 - 5x - 6$

41. $f(x) = x^3 + 8x^2 + 11x - 20$

42. $f(x) = x^4 + x^2 - 2$

43. $f(x) = x^4 - x^3 - 6x^2 + 4x + 8$

44. $f(x) = 4x^5 - 8x^4 - x + 2$

45. $f(x) = 2x^4 - x^3 - 5x^2 + 2x + 2$

46. $f(x) = x^3 - x^2 - 10x - 8$

47. $f(x) = x^3 + x^2 - 14x - 24$

48. $f(x) = 2x^3 - 3x^2 - 17x + 30$

49. $f(x) = 12x^3 + 8x^2 - 3x - 2$

50. $f(x) = x^3 + x^2 - 6x - 8$

51. $f(x) = x^3 - 19x - 30$

53. $f(x) = 3x^3 + 11x^2 - 6x - 8$

54. $f(x) = 2x^3 + 9x^2 - 2x - 9$

55. $f(x) = x^3 + 3x^2 - 6x - 8$

56. $f(x) = 3x^3 - x^2 - 6x + 2$

57. $f(x) = x^3 - 8x^2 + 8x + 24$

58. $f(x) = x^3 - 7x^2 - 7x + 69$

59. $f(x) = x^3 - 3x - 2$

60. $f(x) = x^3 - 2x + 1$

61. $f(x) = x^3 - 2x^2 - 11x + 12$

62. $f(x) = x^3 - 2x^2 - 7x - 4$

63. $f(x) = x^3 - 10x - 12$

64. $f(x) = x^3 - 5x^2 + 17x - 13$

65. $f(x) = 6x^3 + 25x^2 - 24x + 5$

66. $f(x) = 8x^3 + 18x^2 + 45x + 27$

67. $f(x) = 3x^3 - x^2 + 11x - 20$

68. $f(x) = x^4 - x^3 - 9x^2 + 3x + 18$

69. $f(x) = 2x^4 - 9x^3 + 9x^2 + x - 3$

70. $f(x) = 6x^4 + 5x^3 - 17x^2 - 6x$

71. $f(x) = x^4 - 2x^2 - 16x - 15$

72. $f(x) = x^4 - 2x^3 - 5x^2 + 8x + 4$

73. $f(x) = 2x^4 - 17x^3 + 4x^2 + 35x - 24$

74. $f(x) = x^4 + x^3 - 3x^2 - 5x - 2$

52. $f(x)=2x^3+x^2-25x+12$

75. $f(x)=6x^4-17x^3-11x^2+42x$

76. $f(x)=x^4-5x^2-2x$

77. $f(x)=3x^4-4x^3-11x^2+16x-4$

78. $f(x)=6x^4+23x^3+19x^2-8x-4$

79. $f(x)=4x^4-12x^3+3x^2+12x-7$

80. $f(x)=2x^4-9x^3-2x^2+27x-12$

81. $f(x)=2x^4-19x^3+51x^2-31x+5$

82. $f(x)=4x^4-35x^3+71x^2-4x-6$

83. $f(x)=2x^4+3x^3-4x^2-3x+2$

84. $f(x)=x^4+3x^3-30x^2-6x+56$

85. $f(x)=3x^5-10x^4-6x^3+24x^2+11x-6$

86. $f(x)=6x^5+19x^4+x^3-6x^2$

87. $f(x)=x^5+5x^4+10x^3+10x^2+5x+1$

88. $f(x)=x^5-x^4-7x^3+7x^2+12x-12$

89. $f(x)=x^5-2x^3-8x$

90. $f(x)=x^5-7x^4+19x^3-37x^2+60x-36$

91. $f(x)=3x^6-10x^5-29x^4+34x^3+50x^2-24x-24$