

Solution **Section 2.1 – Introducing the Derivative**

Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P ; then graph the curve and the tangent line.

$$y = 4 - x^2; \quad P(-1, 3)$$

Solution

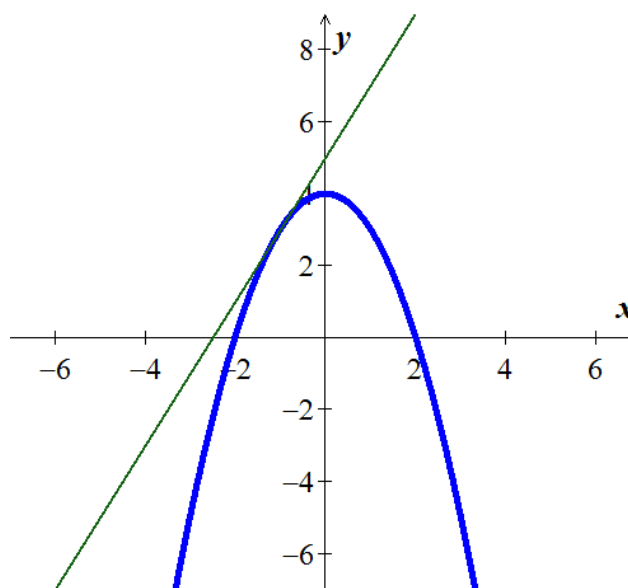
$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{4 - (x+h)^2 - (4 - x^2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{4 - (-1+h)^2 - (4 - (-1)^2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{4 - (1 - 2h + h^2) - (4 - 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{4 - 1 + 2h - h^2 - 3}{h} \\ &= \lim_{h \rightarrow 0} \frac{2h - h^2}{h} \\ &= \lim_{h \rightarrow 0} (2 - h) \\ &= \underline{2} \end{aligned}$$

$$y - y_1 = m(x - x_1)$$

$$\text{At } (-1, 3) \Rightarrow y - 3 = 2(x - (-1))$$

$$y - 3 = 2x + 2$$

$$\underline{y = 2x + 5}$$



Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P ; then graph the curve and the tangent line.

$$y = \frac{1}{x^2}; \quad P(-1, 1)$$

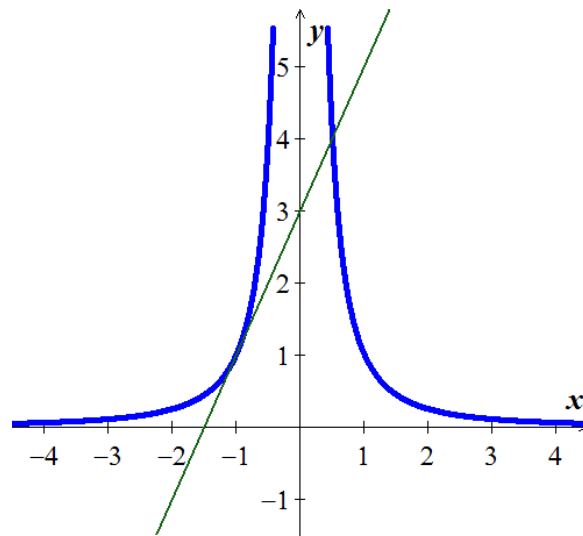
Solution

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{(-1+h)^2} - \frac{1}{1} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1 - (1 - 2h + h^2)}{(-1+h)^2} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1 - 1 + 2h - h^2}{(-1+h)^2} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{2h - h^2}{(-1+h)^2} \right) \\ &= \lim_{h \rightarrow 0} \frac{h}{h} \left(\frac{2 - h}{(-1+h)^2} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{2 - h}{(-1+h)^2} \right) \\ &= \frac{2 + 0}{(-1 + 0)^2} \\ &= 2 \end{aligned}$$

$$\text{At } (-1, 3) \Rightarrow y - 1 = 2(x - (-1))$$

$$y - 1 = 2x + 2$$

$$\underline{y = 2x + 3}$$



Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P ; then graph the curve and the tangent line.

$$f(x) = 2\sqrt{x}; \quad P(1, 2)$$

Solution

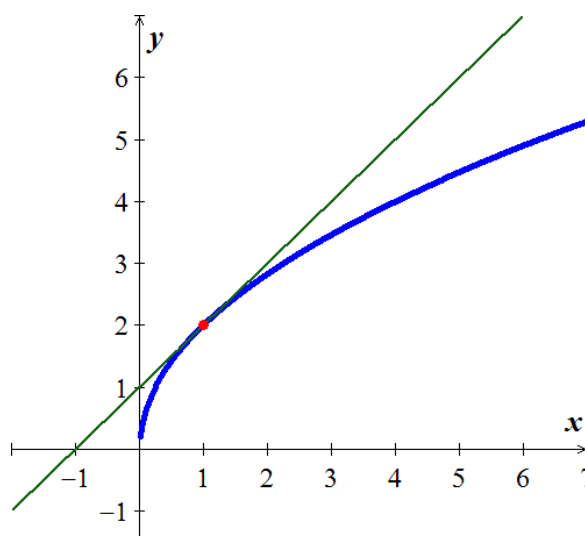
$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{2\sqrt{x+h} - 2\sqrt{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{2\sqrt{1+h} - 2\sqrt{x}}{h} \cdot \frac{2\sqrt{1+h} + 2}{2\sqrt{1+h} + 2} \\ &= \lim_{h \rightarrow 0} \frac{4(1+h) - 4}{h(2\sqrt{1+h} + 2)} \\ &= \lim_{h \rightarrow 0} \frac{4 + 4h - 4}{h(2\sqrt{1+h} + 2)} \\ &= \lim_{h \rightarrow 0} \frac{4h}{h(2\sqrt{1+h} + 2)} \\ &= \lim_{h \rightarrow 0} \frac{4}{2\sqrt{1+h} + 2} \\ &= \frac{4}{2+2} \\ &= 1 \end{aligned}$$

$$\text{At } (1, 2) \Rightarrow y - 2 = (x - 1)$$

$$y - y_1 = m(x - x_1)$$

$$y - 2 = x - 1$$

$$\underline{y = x + 1}$$



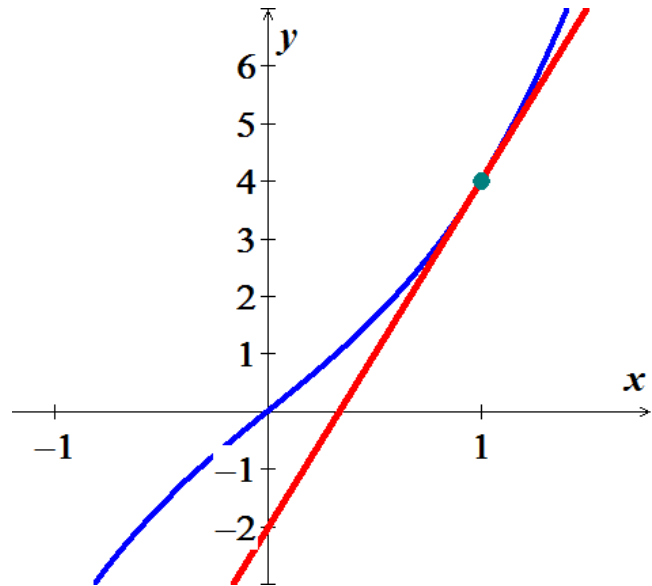
Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P ; then graph the curve and the tangent line.

$$f(x) = x^3 + 3x; \quad P(1, 4)$$

Solution

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{(x+h)^3 + 3(x+h) - (x^3 + 3x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 + 3h + 3h^2 + h^3 + 3 + 3h - (4)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3h + 3h^2 + h^3 + 3h}{h} \\ &= \lim_{h \rightarrow 0} \frac{3h^2 + h^3 + 6h}{h} \\ &= 6 \end{aligned}$$



At $(1, 4)$

$$y - 4 = 6(x - 1) \qquad y - y_1 = m(x - x_1)$$

$$y - 4 = 6x - 6$$

$$y = 6x - 2$$

Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P ; then graph the curve and the tangent line.

$$f(x) = 4x^2 - 7x + 5; \quad P(2, 7)$$

Solution

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{4(x+h)^2 - 7(x+h) + 5 - 4x^2 + 7x - 5}{h} \\ &= \lim_{h \rightarrow 0} \frac{4(x^2 + 2xh + h^2) - 7x - 7h - 4x^2 + 7x}{h} \\ &= \lim_{h \rightarrow 0} \frac{4x^2 + 8xh + 4h^2 - 7h - 4x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{8xh + 4h^2 - 7h}{h} \end{aligned}$$

$$= \lim_{h \rightarrow 0} (8xh + 4h - 7)$$

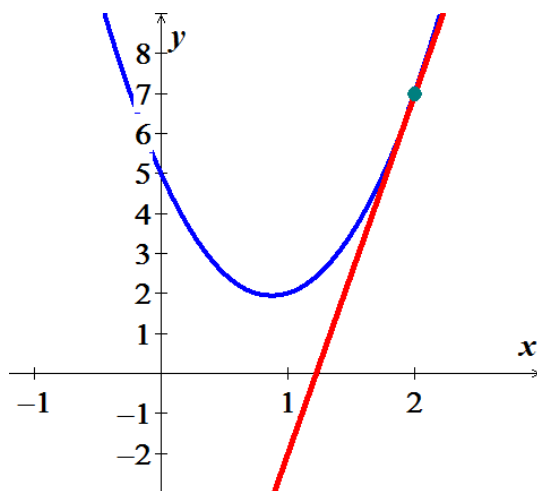
$$= \underline{8x - 7}$$

At $(2, 7) \rightarrow m = 9$

$$y = 9(x - 2) + 7$$

$$y = m(x - x_1) + y_1$$

$$= \underline{9x - 11}$$



Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P ; then graph the curve and the tangent line.

$$f(x) = 5x^3 + x; \quad P(1, 6)$$

Solution

$$m = \lim_{h \rightarrow 0} \frac{5(x+h)^3 + (x+h) - 5x^3 - x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{5(x^3 + 3x^2h + 3xh^2 + h^3) + h - 5x^3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{15x^2h + 15xh^2 + 5h^3 + h}{h}$$

$$= \lim_{h \rightarrow 0} (15x^2 + 15xh + 5h^2 + 1)$$

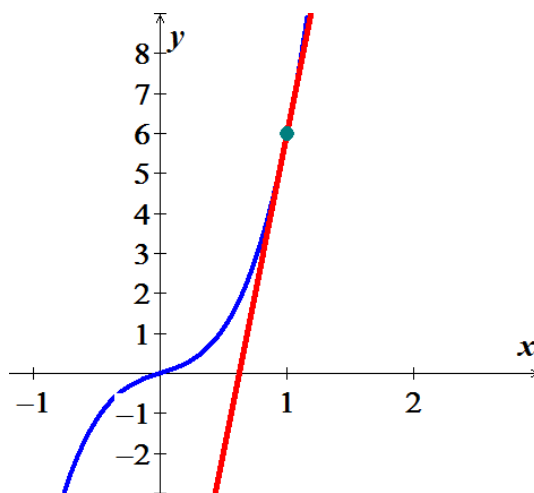
$$= 15x^2 + 1 \Big|_{(1, 6)}$$

$$= \underline{16}$$

$$y = 16(x - 1) + 6$$

$$y = m(x - x_1) + y_1$$

$$= \underline{16x - 10}$$



Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P ; then graph the curve and the tangent line.

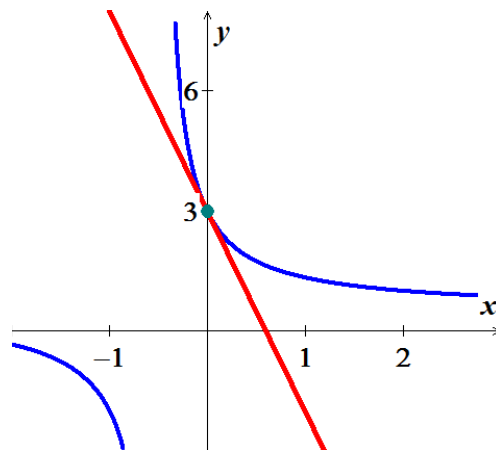
$$f(x) = \frac{x+3}{2x+1}; \quad P(0, 3)$$

Solution

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x+h+3}{2x+2h+1} - \frac{x+3}{2x+1} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{2x^2 + 2hx + 6x + x + h + 3 - 2x^2 - 2hx - x - 6x - 6h - 3}{(2x+2h+1)(2x+1)} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-5h}{(2x+2h+1)(2x+1)} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{-5}{(2x+2h+1)(2x+1)} \right) \\ &= \frac{-5}{(2x+1)^2} \Big|_{(0, 3)} \\ &= -5 \end{aligned}$$

$$\underline{y = -5x + 3}$$

$$y = m(x - x_1) + y_1$$



Exercise

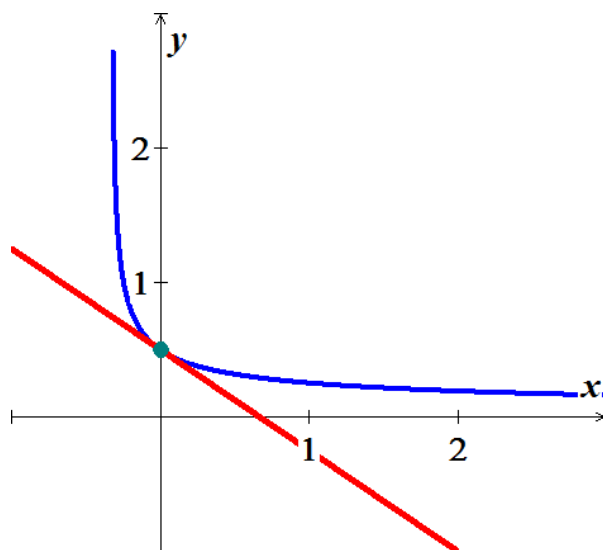
Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P ; then graph the curve and the tangent line.

$$f(x) = \frac{1}{2\sqrt{3x+1}}; \quad P\left(0, \frac{1}{2}\right)$$

Solution

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{2\sqrt{3x+3h+1}} - \frac{1}{2\sqrt{3x+1}} \right] \\ &= \frac{1}{2} \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\sqrt{3x+1} - \sqrt{3x+3h+1}}{\sqrt{3x+3h+1} \sqrt{3x+1}} \right) = \frac{0}{0} \\ &= \frac{1}{2} \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\sqrt{3x+1} - \sqrt{3x+3h+1}}{\sqrt{3x+3h+1} \sqrt{3x+1}} \right) \frac{\sqrt{3x+1} + \sqrt{3x+3h+1}}{\sqrt{3x+1} + \sqrt{3x+3h+1}} \\ &= \frac{1}{2} \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{3x+1-3x-3h-1}{\sqrt{3x+3h+1} \sqrt{3x+1} (\sqrt{3x+1} + \sqrt{3x+3h+1})} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-3h}{\sqrt{3x+3h+1} \sqrt{3x+1} (\sqrt{3x+1} + \sqrt{3x+3h+1})} \right) \\
&= -\frac{3}{2} \lim_{h \rightarrow 0} \left(\frac{1}{\sqrt{3x+3h+1} \sqrt{3x+1} (\sqrt{3x+1} + \sqrt{3x+3h+1})} \right) \\
&= -\frac{3}{2} \frac{1}{(3x+1)(2\sqrt{3x+1})} \\
&= -\frac{3}{4} \frac{1}{(3x+1)^{3/2}} \left| \left(0, \frac{1}{2}\right) \right. \\
&= -\frac{3}{4} \left| \right. \\
\underline{y = -\frac{3}{4}x + \frac{1}{2}} & \quad y = m(x - x_1) + y_1
\end{aligned}$$



Exercise

Find the slope of the curve $y = 1 - x^2$ at the point $x = 2$

Solution

$$\begin{aligned}
m &= \lim_{h \rightarrow 0} \frac{1 - (x+h)^2 - (1 - x^2)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1 - (2+h)^2 - (1 - 2^2)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1 - (4 + 4h + h^2) - (-3)}{h}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{1 - 4 - 4h - h^2 + 3}{h} \\
&= \lim_{h \rightarrow 0} \frac{-4h - h^2}{h} \\
&= \lim_{h \rightarrow 0} (-4 - h) \\
&= -4
\end{aligned}$$

Exercise

Find the slope of the curve $y = \frac{1}{x-1}$ at the point $x = 3$

Solution

$$\begin{aligned}
m &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\frac{1}{3+h-1} - \frac{1}{3-1}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{2-2-h}{2+h} \right) \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-h}{2+h} \right) \\
&= \lim_{h \rightarrow 0} \left(\frac{-1}{2+h} \right) \\
&= -\frac{1}{2}
\end{aligned}$$

Exercise

Find the slope of the curve $y = \frac{x-1}{x+1}$ at the point $x = 0$

Solution

$$\begin{aligned}
m &= \lim_{h \rightarrow 0} \frac{\frac{x+h-1}{x+h+1} - \frac{x-1}{x+1}}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{0+h-1}{0+h+1} - \frac{0-1}{0+1} \right) \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{h-1}{h+1} + 1 \right)
\end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{h-1+h+1}{h+1} \right) \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{2h}{h+1} \right) \\
&= \lim_{h \rightarrow 0} \left(\frac{2}{h+1} \right) \\
&= 2
\end{aligned}$$

Exercise

Find equations of all lines having slope -1 that are tangent to the curve $y = \frac{1}{x-1}$

Solution

$$\begin{aligned}
m &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h} \\
-1 &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h} \\
-1 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{x-1-(x+h-1)}{x+h-1} \right) \\
-1 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{x-1-x-h+1}{x+h-1} \right) \\
-1 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-h}{x+h-1} \right) \\
-1 &= \lim_{h \rightarrow 0} \left(\frac{-1}{x+h-1} \right) \\
-1 &= \frac{-1}{x-1} \\
-x+1 &= -1 \\
x &= 2
\end{aligned}$$

Cross multiplication

$$\begin{aligned}
y &= \frac{1}{x-1} \\
&= \frac{1}{2-1} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\text{At } (2, 1) &\Rightarrow y-1 = -1(x-2) \\
&y-1 = -x+2 \\
&y = -x+3
\end{aligned}$$

Exercise

What is the rate of change of the area of a circle $(A = \pi r^2)$ with respect to the radius when the radius is $r = 3$?

Solution

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{\pi(3+h)^2 - \pi(3)^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\pi(9 + 6h + h^2) - 9\pi}{h} \\ &= \lim_{h \rightarrow 0} \frac{9\pi + 6\pi h + \pi h^2 - 9\pi}{h} \\ &= \lim_{h \rightarrow 0} \frac{6\pi h + \pi h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\pi h(6 + h)}{h} \\ &= \lim_{h \rightarrow 0} \pi(6 + h) \\ &= 6\pi \end{aligned}$$

Exercise

Find the slope of the tangent to the curve $y = \frac{1}{\sqrt{x}}$ at the point where $x = 4$

Solution

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\sqrt{4} - \sqrt{4+h}}{2\sqrt{4+h}} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{2 - \sqrt{4+h}}{2\sqrt{4+h}} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{2 - \sqrt{4+h}}{2\sqrt{4+h}} \cdot \frac{2 + \sqrt{4+h}}{2 + \sqrt{4+h}} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{4 - (4+h)}{2\sqrt{4+h}(2 + \sqrt{4+h})} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-h}{2\sqrt{4+h}(2 + \sqrt{4+h})} \right) \end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \left(\frac{-1}{2\sqrt{4+h}(2+\sqrt{4+h})} \right) \\
&= \frac{-1}{2\sqrt{4}(2+\sqrt{4})} \\
&= \frac{-1}{2(2)(2+2)} \\
&= \underline{\underline{\frac{-1}{16}}}
\end{aligned}$$

Exercise

Find the values of the derivatives of the function $f(x) = 4 - x^2$. Then find the values of $f'(-3)$, $f'(0)$, $f'(1)$

Solution

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{4 - (x+h)^2 - (4 - x^2)}{h} \\
&= \frac{4 - (x^2 + 2xh + h^2) - (4 - x^2)}{h} \\
&= \frac{4 - x^2 - 2xh - h^2 - 4 + x^2}{h} \\
&= \frac{-2xh - h^2}{h} \\
&= \underline{\underline{-2x - h}}
\end{aligned}$$

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} (-2x - h) \\
&= \underline{\underline{-2x}}
\end{aligned}$$

$$f'(-3) = \underline{\underline{6}}$$

$$f'(0) = \underline{\underline{0}}$$

$$f'(1) = \underline{\underline{-2}}$$

Exercise

Find the values of the derivatives of the function $r(s) = \sqrt{2s+1}$. Then find the values of $r'(0)$, $r'\left(\frac{1}{2}\right)$, $r'(1)$

Solution

$$\begin{aligned}
r'(s) &= \lim_{h \rightarrow 0} \frac{r(s+h) - r(s)}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sqrt{2(s+h)+1} - \sqrt{2s+1}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sqrt{2s+2h+1} - \sqrt{2s+1}}{h} \cdot \frac{\sqrt{2s+2h+1} + \sqrt{2s+1}}{\sqrt{2s+2h+1} + \sqrt{2s+1}} \\
&= \lim_{h \rightarrow 0} \frac{2s+2h+1 - (2s+1)}{h(\sqrt{2s+2h+1} + \sqrt{2s+1})} \\
&= \lim_{h \rightarrow 0} \frac{2s+2h+1 - 2s-1}{h(\sqrt{2s+2h+1} + \sqrt{2s+1})} \\
&= \lim_{h \rightarrow 0} \frac{2h}{h(\sqrt{2s+2h+1} + \sqrt{2s+1})} \\
&= \lim_{h \rightarrow 0} \frac{2}{\sqrt{2s+2h+1} + \sqrt{2s+1}} \\
&= \frac{2}{\sqrt{2s+1} + \sqrt{2s+1}} \\
&= \frac{2}{2\sqrt{2s+1}} \\
&= \frac{1}{\sqrt{2s+1}}
\end{aligned}$$

$$r'(0) = \frac{1}{\sqrt{2(0)+1}} = \underline{1}$$

$$r'\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2\frac{1}{2}+1}} = \underline{\frac{1}{\sqrt{2}}}$$

$$r'(1) = \frac{1}{\sqrt{2(1)+1}} = \underline{\frac{1}{\sqrt{3}}}$$

Exercise

Find the derivative of $f(x) = 3x^2 - 2x$

Solution

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{3(x+\Delta x)^2 - 2(x+\Delta x) - (3x^2 - 2x)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{3(x^2 + \Delta x^2 + 2x\Delta x) - 2x - 2\Delta x - 3x^2 + 2x}{\Delta x}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\Delta x \rightarrow 0} \frac{3x^2 + 3\Delta x^2 + 6x\Delta x - 2x - 2\Delta x - 3x^2 + 2x}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{3\Delta x^2 + 6x\Delta x - 2\Delta x}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} 3\Delta x + 6x - 2 \\
&= \underline{6x - 2}
\end{aligned}$$

Exercise

Find the derivative of y with the respect to t for the function $y = \frac{4}{t}$

Solution

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta t \rightarrow 0} \frac{\frac{4}{t + \Delta t} - \frac{4}{t}}{\Delta t} \\
&= \lim_{\Delta t \rightarrow 0} \frac{\frac{4t - 4(t + \Delta t)}{t(t + \Delta t)}}{\Delta t} \\
&= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \frac{4t - 4(t + \Delta t)}{t(t + \Delta t)} \\
&= \lim_{\Delta t \rightarrow 0} \frac{-4\Delta t}{t(t + \Delta t)\Delta t} \\
&= \lim_{\Delta t \rightarrow 0} \frac{-4}{t(t + \Delta t)} \\
&= \underline{-\frac{4}{t^2}}
\end{aligned}$$

Exercise

Find the derivative of $\frac{dy}{dx}$ if $y = 2x^3$

Solution

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{2(x + \Delta x)^3 - 2x^3}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{2\left(x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3\right) - 2x^3}{\Delta x}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\Delta x \rightarrow 0} \frac{2x^3 + 6x^2\Delta x + 6x(\Delta x)^2 + 3(\Delta x)^3 - 2x^3}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(6x^2 + 6x(\Delta x) + 3(\Delta x)^2)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} (6x^2 + 6x(\Delta x) + 3(\Delta x)^2) \\
&= \underline{6x^2}
\end{aligned}$$

Exercise

Differentiate the function $y = \frac{x+3}{1-x}$ and find the slope of the tangent line at the given value of the independent variable.

Solution

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\frac{x+\Delta x+3}{1-x-\Delta x} - \frac{x+3}{1-x}}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \left(\frac{1}{\Delta x} \right) \left(\frac{(x+\Delta x+3)(1-x) - (x+3)(1-x-\Delta x)}{(1-x-\Delta x)(1-x)} \right) \\
&= \lim_{\Delta x \rightarrow 0} \left(\frac{1}{\Delta x} \right) \left(\frac{x+\Delta x+3-x^2-x\Delta x-3x - (x-x^2-x\Delta x+3-3x-3\Delta x)}{(1-x-\Delta x)(1-x)} \right) \\
&= \lim_{\Delta x \rightarrow 0} \left(\frac{1}{\Delta x} \right) \left(\frac{x+\Delta x+3-x^2-x\Delta x-3x-x+x^2+x\Delta x-3+3x+3\Delta x}{(1-x-\Delta x)(1-x)} \right) \\
&= \lim_{\Delta x \rightarrow 0} \left(\frac{1}{\Delta x} \right) \left(\frac{4\Delta x}{(1-x-\Delta x)(1-x)} \right) \\
&= \lim_{\Delta x \rightarrow 0} \frac{4}{(1-x-\Delta x)(1-x)} \\
&= \frac{4}{(1-x)(1-x)} \\
&= \underline{\frac{4}{(1-x)^2}}
\end{aligned}$$

Exercise

Find the equation of the tangent line to $f(x) = x^2 + 1$ that is parallel to $2x + y = 0$

Solution

$$2x + y = 0 \Rightarrow y = -2x \Rightarrow \text{slope} = -2$$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 + 1 - (x^2 + 1)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + \Delta x^2 + 2x\Delta x + 1 - x^2 - 1}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x^2 + 2x\Delta x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \Delta x + 2x = 2x \end{aligned}$$

$$f' = 2x = -2$$

$$\Rightarrow x = -1$$

$$f(-1) = (-1)^2 + 1 = 2$$

$$\rightarrow (-1, 2)$$

The line equation is given by $y = m(x - x_1) + y_1$

$$y = -2(x + 1) + 2$$

$$\underline{y = -2x}$$

Exercise

Use the definition of limits to find the derivative: $f(x) = \frac{3}{\sqrt{x}} y - 2 = -2x - 2$

Solution

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\left(\frac{3}{\sqrt{x + \Delta x}}\right) - \left(\frac{3}{\sqrt{x}}\right)}{\Delta x} \cdot \frac{\sqrt{x} \cdot \sqrt{x + \Delta x}}{\sqrt{x} \cdot \sqrt{x + \Delta x}} \end{aligned}$$

$$\begin{aligned}
&= \lim_{\Delta x \rightarrow 0} \frac{3\sqrt{x} - 3\sqrt{x + \Delta x}}{\Delta x (\sqrt{x} \cdot \sqrt{x + \Delta x})} \\
&= \lim_{\Delta x \rightarrow 0} \frac{3(\sqrt{x} - \sqrt{x + \Delta x})}{\Delta x (\sqrt{x} \cdot \sqrt{x + \Delta x})} \cdot \frac{\sqrt{x} + \sqrt{x + \Delta x}}{\sqrt{x} + \sqrt{x + \Delta x}} \\
&= \lim_{\Delta x \rightarrow 0} \frac{3(x - (x + \Delta x))}{\Delta x (\sqrt{x} \cdot \sqrt{x + \Delta x}) (\sqrt{x} + \sqrt{x + \Delta x})} \\
&= \lim_{\Delta x \rightarrow 0} \frac{-3\Delta x}{\Delta x (\sqrt{x} \cdot \sqrt{x + \Delta x}) (\sqrt{x} + \sqrt{x + \Delta x})} \\
&= \frac{-3}{x(2\sqrt{x})} \\
&= \frac{-3}{2x^{3/2}} \Bigg|
\end{aligned}$$

Exercise

Use the definition of limits to find the derivative: $f(x) = \sqrt{x + 2}$

Solution

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x + 2} - \sqrt{x + 2}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x + 2} + \sqrt{x + 2}}{\sqrt{x + \Delta x + 2} + \sqrt{x + 2}} \\
&= \lim_{\Delta x \rightarrow 0} \frac{x + \Delta x + 2 - (x + 2)}{\Delta x (\sqrt{x + \Delta x + 2} + \sqrt{x + 2})} \\
&= \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x (\sqrt{x + \Delta x + 2} + \sqrt{x + 2})} \\
&= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x + 2} + \sqrt{x + 2}} \\
&= \frac{1}{2\sqrt{x + 2}} \Bigg|
\end{aligned}$$

Exercise

Suppose the height s of an object (in m) above the ground after t seconds is approximated by the function

$$s(t) = -4.9t^2 + 25t + 1$$

- a) Make a table showing the average velocities of the object from time $t = 1$ to $t = 1 + h$, for $h = 0.01, 0.001, 0.0001$, and 0.00001 .
- b) Use the table in part (a) to estimate the instantaneous velocity of the object at $t = 1$.
- c) Use limits to verify your estimate in part (b).

Solution

$$\begin{aligned} a) \quad \frac{f(1+h) - f(1)}{h} &= \frac{1}{h} \left(-4.9(1+h)^2 + 25(1+h) + 1 + 4.9 - 25 - 1 \right) \\ &= \frac{1}{h} \left(-4.9 - 9.8h - 4.9h^2 + 25h + 4.9 \right) \\ &= \frac{1}{h} \left(-4.9h^2 + 15.2h \right) \\ &= \underline{15.2 - 4.9h} \end{aligned}$$

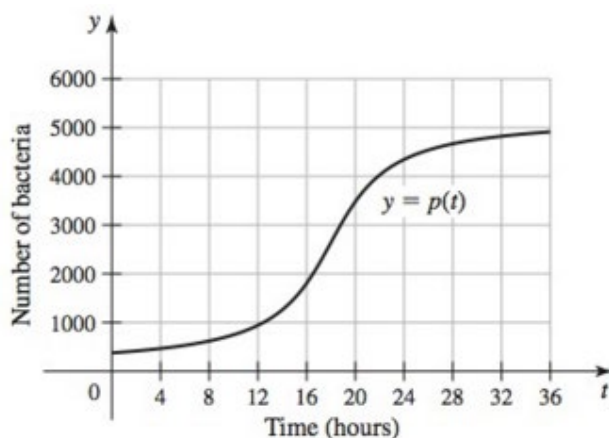
h	$\frac{f(1+h) - f(1)}{h}$
0.01	15.151
0.001	15.1951
0.0001	15.1995
0.00001	15.2
0.000001	15.2

$$\begin{aligned} b) \quad f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\ &\approx \underline{15.2 \text{ m/sec}} \end{aligned}$$

$$\begin{aligned} c) \quad f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\ &= \underline{15.2 \text{ m/sec}} \end{aligned}$$

Exercise

Suppose the following graph represents the number of bacteria in a culture t hours after the start of an experiment.



- a) At approximately what time is the instantaneous growth rate the greatest, for $0 \leq t \leq 36$? Estimate the growth rate at this time.
- b) At approximately what time is the instantaneous growth rate the least, for $0 \leq t \leq 36$? Estimate the growth rate at this time.
- c) What is the average growth rate over the interval $0 \leq t \leq 36$?

Solution

a) $t = \frac{36}{2} = 18$

$$\begin{aligned}\text{Point the rate} &= \frac{N(20) - N(16)}{20 - 16} \\ &= \frac{2500 - 1900}{4} \\ &= 400 \text{ bacteria/hr}\end{aligned}$$

b) It is smallest at $t = 0$ or $t = 36$

$$\begin{aligned}\frac{N(36) - N(32)}{4} &= \frac{4900 - 4800}{4} \\ &= 25 \text{ bacteria/hr}\end{aligned}$$

c) Growth rate $= \frac{N(36) - N(0)}{36}$

$$\begin{aligned}&\approx \frac{4900 - 400}{36} \\ &= 125 \text{ bacteria/hr}\end{aligned}$$

Solution **Section 2.2 – Differentiation Rules**

Exercise

Find the derivative of $y = \frac{1}{x^3}$

Solution

$$y = x^{-3}$$

$$y' = -3x^{-3-1}$$

$$\underline{= -3x^{-4}} \quad \text{or} \quad -\frac{3}{x^4}$$

Exercise

Find the derivative of $D_x \left(x^{4/3} \right)$

Solution

$$\underline{D_x \left(x^{4/3} \right) = \frac{4}{3} x^{1/3}}$$

Exercise

Find the derivative of $y = \sqrt{z}$

Solution

$$\frac{dy}{dz} = \frac{d}{dz} \left(z^{1/2} \right)$$

$$= \frac{1}{2} z^{1/2-1}$$

$$\underline{= \frac{1}{2} z^{-1/2}} \quad \frac{1}{2z^{1/2}} \quad \frac{1}{2\sqrt{z}}$$

Exercise

Find the derivative of $D_t (-8t)$

Solution

$$\underline{D_t (-8t) = -8}$$

Exercise

Find the derivative of $y = \frac{9}{4x^2}$

Solution

$$y = \frac{9}{4}x^{-2}$$

$$y' = \frac{9}{4}(-2)x^{-3}$$

$$= -\frac{9}{2x^3}$$

Exercise

Find the derivative of $y = 6x^3 + 15x^2$

Solution

$$y' = 18x^2 + 30x$$

Exercise

Find the first derivative of $y = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$

Solution

$$y' = 3(4)x^3 - 6(3)x^2 + \frac{2}{8}x + 0$$

$$= 12x^3 - 18x^2 + \frac{1}{4}x$$

Exercise

Find the derivative of $p(t) = 12t^4 - 6\sqrt{t} + \frac{5}{t}$

Solution

$$p(t) = 12t^4 - 6t^{1/2} + 5t^{-1}$$

$$p' = 48t^3 - 3t^{-1/2} - 5t^{-2}$$

$$= 48t^3 - \frac{3}{t^{1/2}} - \frac{5}{t^2}$$

Exercise

Find the derivative of $f(x) = \frac{x^3 + 3\sqrt{x}}{x}$

Solution

$$\begin{aligned} f(x) &= \frac{x^3}{x} + 3 \frac{x^{1/2}}{x} \\ &= x^2 + 3x^{-1/2} \end{aligned}$$

$$\begin{aligned} f'(x) &= 2x - \frac{3}{2}x^{-3/2} \\ &= 2x - \frac{3}{2x^{3/2}} \\ &= 2x - \frac{3}{2\sqrt{x^3}} \end{aligned}$$

Exercise

Find the derivative of $y = \frac{x^3 - 4x}{\sqrt{x}}$

Solution

$$y = \frac{x^3}{x^{1/2}} - 4 \frac{x}{x^{1/2}} = x^{5/2} - 4x^{1/2}$$

$$\begin{aligned} y' &= \frac{5}{2}x^{3/2} - 4 \frac{1}{2}x^{-1/2} \\ &= \frac{5}{2}x\sqrt{x} - 2 \frac{2}{\sqrt{x}} \end{aligned}$$

Exercise

Find the derivative of $f(x) = (4x^2 - 3x)^2$

Solution

$$\begin{aligned} f(x) &= (4x^2 - 3x)^2 \\ &= 16x^4 - 24x^3 + 9x^2 \end{aligned}$$

$$f'(x) = 64x^3 - 72x^2 + 18x$$

$$(a + b)^2 = a^2 + 2ab + b^2$$

Exercise

Find the derivative of $y = 3x(2x^2 + 5x)$

Solution

$$y = 6x^3 + 15x^2$$

$$\underline{y' = 18x^2 + 30x}$$

Exercise

Find the derivative of $y = 3(2x^2 + 5x)$

Solution

$$y = 6x^2 + 15x$$

$$\underline{y' = 12x + 15}$$

Exercise

Find the derivative of $y = \frac{x^2 + 4x}{5}$

Solution

$$\underline{y' = \frac{1}{5}(2x + 4)}$$

Exercise

Find the derivative of $y = \frac{3x^4}{5}$

Solution

$$\underline{y' = \frac{12}{5}x^3}$$

Exercise

Find the derivative of $g(s) = \frac{s^2 - 2s + 5}{\sqrt{s}}$

Solution

$$g(s) = \frac{s^2}{s^{1/2}} - 2\frac{s}{s^{1/2}} + \frac{5}{s^{1/2}}$$

$$= s^{3/2} - 2s^{1/2} + 5s^{-1/2}$$

$$g'(s) = \frac{3}{2}s^{1/2} - 2\frac{1}{2}s^{-1/2} + 5\left(-\frac{1}{2}\right)s^{-3/2}$$

$$= \frac{3}{2}s^{1/2} - s^{-1/2} - \frac{5}{2}s^{-3/2}$$

$$= \frac{3}{2}\sqrt{s} - \frac{1}{\sqrt{s}} - \frac{5}{2s^{3/2}}$$

$$= \frac{3}{2}\sqrt{s} - \frac{1}{\sqrt{s}} - \frac{5}{2s\sqrt{s}} \quad \Bigg|$$

Exercise

Find the derivative of $f(x) = \frac{x+1}{\sqrt{x}}$

Solution

$$f(x) = \frac{x}{x^{1/2}} + \frac{1}{x^{1/2}}$$

$$= x^{1/2} + x^{-1/2}$$

$$f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}$$

$$= \frac{1}{2x^{1/2}} - \frac{1}{2x^{3/2}} \quad \Bigg|$$

Exercise

Find the derivative of $f(x) = 4x^{5/3} + 6x^{-3/2} - 11x$

Solution

$$f'(x) = \frac{20}{3}x^{2/3} - 9x^{-5/2} - 11 \quad \Bigg|$$

Exercise

Find the derivative of $f(x) = \frac{2}{3}x^3 + \pi x^2 + 7x + 1$

Solution

$$f'(x) = 2x^2 + 2\pi x + 7 \quad \Bigg|$$

Exercise

Find the derivative of $f(x) = \frac{x^5 - x^3}{15}$

Solution

$$f(x) = \frac{1}{15}x^5 - \frac{1}{15}x^3$$

$$f'(x) = \frac{1}{3}x^4 - \frac{1}{5}x^2 \quad \Bigg|$$

Exercise

Find the derivative of $f(x) = x^{1/3} + 2x^{1/4} - 3x^{1/5}$

Solution

$$\underline{f'(x) = \frac{1}{3}x^{-2/3} + \frac{1}{2}x^{-3/4} - \frac{3}{5}x^{-4/5} \quad |}$$

Exercise

Find the derivative of $f(t) = 3\sqrt[3]{t^2} - \frac{2}{\sqrt{t^3}}$

Solution

$$f(t) = 3t^{2/3} - 2t^{-1/3}$$

$$\underline{f'(t) = 2t^{-1/3} + \frac{2}{3}2t^{-4/3} \quad |}$$

Exercise

Find the derivative of $f(t) = \sqrt{t}\left(5 - t - \frac{1}{3}t^2\right)$

Solution

$$f(t) = 5t^{1/2} - t^{3/2} - \frac{1}{3}t$$

$$\underline{f'(t) = \frac{5}{2}t^{-1/2} - \frac{3}{2}t^{1/2} - \frac{1}{3} \quad |}$$

Exercise

Find the derivative of $f(x) = \frac{3}{5}x^{5/3} + \frac{5}{3}x^{-3/5}$

Solution

$$\underline{f'(x) = x^{2/3} - x^{-8/5} \quad |}$$

Exercise

Find the derivative of $f(x) = x^{23} - x^{-23}$

Solution

$$\underline{f'(x) = 23x^{22} + 23x^{-24} \quad |}$$

Exercise

Find the **first** and **second** derivatives $y = -x^3 + 3$

Solution

$$\underline{y' = -3x^2}$$

$$\underline{y'' = -6x}$$

Exercise

Find the **first** and **second** derivatives $y = 3x^7 - 7x^3 + 21x^2$

Solution

$$\underline{y' = 21x^6 - 21x^2 + 42x}$$

$$\underline{y'' = 126x^5 - 42x + 42}$$

Exercise

Find the **first** and **second** derivatives $y = 6x^2 - 10x - \frac{1}{x}$

Solution

$$\underline{y' = 12x - 10 + \frac{1}{x^2}}$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$y'' = 12 + \frac{-2x}{x^4}$$

$$\underline{= 12 - \frac{2}{x^3}}$$

Exercise

Find the **first** and **second** derivatives $f(x) = \frac{1}{2}x^4 + \pi x^3 - 7x + 1$

Solution

$$\underline{f'(x) = 2x^3 + 3\pi x^2 - 7}$$

$$\underline{f''(x) = 6x^2 + 6\pi x}$$

Exercise

Find the *first* and *second* derivatives $y = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$

Solution

$$\underline{y' = 12x^3 - 18x^2 + \frac{x}{4}}$$

$$\underline{y'' = 36x^2 - 36x + \frac{1}{4}}$$

Exercise

Find the *first* and *second* derivatives $y = (2x - 3)(1 - 5x)$

Solution

$$y = -10x^2 + 17x - 3$$

$$\underline{y' = -20x + 17}$$

$$\underline{y'' = -20}$$

Exercise

Find the derivative $f(x) = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$, $f^{(4)}(x)$

Solution

$$\underline{f^{(4)}(x) = 3(4!) = 72}$$

Exercise

Find the derivative $f(x) = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$, $f^{(5)}(x)$

Solution

$$\underline{f^{(5)}(x) = 0}$$

Exercise

Find the derivative $f(x) = 2x^6 + 4x^4 - x + 2$, $f^{(6)}(x)$

Solution

$$f^{(6)}(x) = 2(6!)$$

$$\underline{= 1,440}$$

Exercise

Find the derivative $f(x) = 4x^5 + 4x^4 + x^2 - 2$, $f^{(5)}(x)$

Solution

$$f^{(5)}(x) = 4(5!) \\ \underline{= 480}$$

Exercise

Find the derivative $f(x) = 4x^5 + 4x^4 + x^2 - 2$, $f^{(6)}(x)$

Solution

$$f^{(6)}(x) = \underline{0}$$

Exercise

Find the derivative $f(x) = 4x^4 - 2x^3 + x + 2$, $f^{(4)}(x)$

Solution

$$f^{(4)}(x) = 4(4!) \\ \underline{= 96}$$

Exercise

Find an equation for the line perpendicular to the tangent to the curve $y = x^3 - 4x + 1$ at the point (2, 1).

Solution

$$y' = 3x^2 - 4$$

$$m = y'|_{x=2} = 3(2)^2 - 4 = 8$$

$$\underline{m_1 = -\frac{1}{8}}$$

$$y = -\frac{1}{8}(x - 2) + 1$$

$$\underline{y = -\frac{1}{8}x - \frac{3}{4}}$$

$$y = m(x - x_1) + y_1$$

Exercise

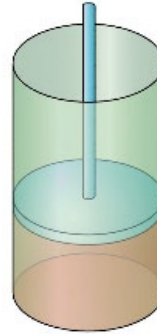
If gas in a cylinder is maintained at a constant temperature T , the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2}$$

In which a , b , n , and R are constants. Find $\frac{dP}{dV}$

Solution

$$\begin{aligned}\frac{dP}{dV} &= \frac{d}{dV} \left(\frac{nRT}{V - nb} \right) - \frac{d}{dV} \left(\frac{an^2}{V^2} \right) \\ &= -nRT \frac{(V - nb)'}{(V - nb)^2} - an^2 \left(-\frac{2V}{V^4} \right) \\ &= -nRT \frac{1}{(V - nb)^2} + an^2 \left(\frac{2}{V^3} \right) \\ &= -\frac{nRT}{(V - nb)^2} + \frac{2an^2}{V^3} \quad | \end{aligned}$$



Exercise

Show that if $(a, f(a))$ is any point on the graph of $f(x) = x^2$, then the slope of the tangent line at that point is $m = 2a$

Solution

$$\begin{aligned}m = f'(a) &= \lim_{x \rightarrow a} \frac{x^2 - a^2}{x - a} \\ &= \lim_{x \rightarrow a} \frac{(x - a)(x + a)}{x - a} \\ &= \lim_{x \rightarrow a} (x + a) \\ &= 2a \quad | \end{aligned}$$

Exercise

Show that if $(a, f(a))$ is any point on the graph of $f(x) = bx^2 + cx + d$, then the slope of the tangent line at that point is $m = 2ab + c$

Solution

$$\begin{aligned}
m = f'(a) &= \lim_{h \rightarrow 0} \frac{b(a+h)^2 + c(a+h) + d - ba^2 - ca - d}{h} \\
&= \lim_{h \rightarrow 0} \frac{ba^2 + 2abh + bh^2 + ch - ba^2}{h} \\
&= \lim_{h \rightarrow 0} \frac{2abh + bh^2 + ch}{h} \\
&= \lim_{h \rightarrow 0} (2ab + bh + c) \\
&= \underline{2ab + c}
\end{aligned}$$

Exercise

Let $f(x) = x^2$

- Show that $\frac{f(x) - f(y)}{x - y} = f'\left(\frac{x + y}{2}\right)$, for all $x \neq y$
- Is this property true for $f(x) = ax^2$, where a is a nonzero real number?
- Give a geometrical interpretation of this property.
- Is this property true for $f(x) = ax^3$?

Solution

a) $f'(x) = 2x$

$$\begin{aligned}
\frac{f(x) - f(y)}{x - y} &= \frac{x^2 - y^2}{x - y} \\
&= \frac{(x - y)(x + y)}{x - y} \\
&= \underline{x + y}
\end{aligned}$$

$$\begin{aligned}
f'\left(\frac{x + y}{2}\right) &= 2\left(\frac{x + y}{2}\right) \\
&= \underline{x + y}
\end{aligned}$$

$$\frac{f(x) - f(y)}{x - y} = f'\left(\frac{x + y}{2}\right), \text{ for all } x \neq y$$

b) $f(x) = ax^2 \rightarrow f'(x) = 2ax$

$$\begin{aligned}
f'\left(\frac{x + y}{2}\right) &= 2a\left(\frac{x + y}{2}\right) \\
&= \underline{a(x + y)}
\end{aligned}$$

$$\begin{aligned}\frac{f(x) - f(y)}{x - y} &= \frac{ax^2 - ay^2}{x - y} \\ &= \frac{a(x - y)(x + y)}{x - y} \\ &= a(x + y) \quad | \end{aligned}$$

$$\frac{f(x) - f(y)}{x - y} = f'\left(\frac{x + y}{2}\right), \text{ for all } x \neq y$$

c) Line thru $(x, f(x))$ and $(y, f(y))$ is parallel to the tangent line and midpoint is between x and y .

d) $f(x) = ax^3 \rightarrow f'(x) = 3ax^2$

$$\begin{aligned}f'\left(\frac{x + y}{2}\right) &= 3a\left(\frac{x + y}{2}\right)^2 \\ &= \frac{3}{4}a(x + y)^2 \quad | \end{aligned}$$

$$\begin{aligned}\frac{f(x) - f(y)}{x - y} &= \frac{ax^3 - ay^3}{x - y} \\ &= \frac{a(x - y)(x^2 + xy + y^2)}{x - y} \\ &= a(x^2 + xy + y^2) \quad | \end{aligned}$$

$$x^2 + xy + y^2 \neq (x + y)^2$$

$$\frac{f(x) - f(y)}{x - y} \neq f'\left(\frac{x + y}{2}\right) \quad (\text{No})$$

Solution **Section 2.3 – Product and Quotient Rules**

Exercise

Find the derivative of $y = (x+1)(\sqrt{x} + 2)$

Solution

$$\begin{aligned} y' &= (1)\left(x^{1/2} + 2\right) + (x+1)\left(\frac{1}{2}x^{-1/2}\right) \\ &= x^{1/2} + 2 + \frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2} \\ &= \frac{3}{2}x^{1/2} + \frac{1}{2}x^{-1/2} + 2 \end{aligned}$$

Exercise

Find the derivative of $y = (4x + 3x^2)(6 - 3x)$

Solution

$$\begin{aligned} y' &= (4x + 3x^2)\frac{d}{dx}(6 - 3x) + (6 - 3x)\frac{d}{dx}(4x + 3x^2) \\ &= (4x + 3x^2)(-3) + (6 - 3x)(4 + 6x) \\ &= -12x - 9x^2 + 24 + 36x - 12x - 18x^2 \\ &= -27x^2 + 12x + 24 \end{aligned}$$

$$y = 24x + 6x^2 - 9x^3$$

Exercise

Find the derivative of $y = \left(\frac{1}{x} + 1\right)(2x + 1)$

Solution

$$\begin{aligned} y' &= \left(x^{-1} + 1\right)\frac{d}{dx}(2x + 1) + (2x + 1)\frac{d}{dx}\left(x^{-1} + 1\right) \\ &= \left(x^{-1} + 1\right)(2) + (2x + 1)\left(-x^{-2}\right) \\ &= \frac{2}{x} + 2 + (2x + 1)\left(-\frac{1}{x^2}\right) \\ &= \frac{2}{x} + 2 - \frac{2}{x} - \frac{1}{x^2} \\ &= 2 - \frac{1}{x^2} \\ &= \frac{2x^2 - 1}{x^2} \end{aligned}$$

Exercise

Find the derivative of $y = \frac{3x-2}{x+4}$

Solution

$$\begin{aligned}y &= \frac{3x-2}{x+4} \\&= \frac{3x-2}{x} \cdot \frac{1}{x+4} \\&= \frac{3x-2}{x^2+4x}\end{aligned}$$

$$\begin{aligned}y' &= \frac{\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} x^2 + 2 \begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} x + \begin{vmatrix} 3 & -2 \\ 4 & 0 \end{vmatrix}}{(x^2+4x)^2} \\&= \frac{-3x^2+4x+8}{x^2(x+4)^2}\end{aligned}$$

OR

$$\begin{aligned}y' &= \frac{(x^2+4x)(3) - (3x-2)(2x+4)}{[x(x+4)]^2} \\&= \frac{3x^2+12x-6x^2-12x+4x+8}{x^2(x+4)^2} \\&= \frac{-3x^2+4x+8}{x^2(x+4)^2}\end{aligned}$$

Exercise

Find the derivative of $g(x) = \frac{x^2-4x+2}{x^2+3}$

Solution

$$g'(x) = \frac{\begin{vmatrix} 1 & -4 \\ 1 & 0 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} x + \begin{vmatrix} -4 & 2 \\ 0 & 3 \end{vmatrix}}{(x^2+3)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2+bx+c}{dx^2+ex+f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2+ex+f)^2}$$

$$= \frac{4x^2 + 2x - 12}{(x^2 + 3)^2}$$

Or

$$g' = \frac{(2x-4)(x^2+3) - (x^2-4x+2)(2x)}{(x^2+3)^2}$$

$$= \frac{2x^3 + 6x - 4x^2 - 12 - 2x^3 + 8x^2 - 4x}{(x^2+3)^2}$$

$$= \frac{4x^2 + 2x - 12}{(x^2 + 3)^2}$$

Exercise

Find the derivative of $f(x) = \frac{(3-4x)(5x+1)}{7x-9}$

Solution

$$f(x) = \frac{-20x^2 + 11x + 3}{7x-9}$$

$$f'(x) = \frac{\begin{vmatrix} -20 & 11 \\ 0 & 7 \end{vmatrix} x^2 + 2 \begin{vmatrix} -20 & 3 \\ 0 & -9 \end{vmatrix} x + \begin{vmatrix} 11 & 3 \\ 7 & -9 \end{vmatrix}}{(7x-9)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

$$= \frac{-140x^2 + 360x - 120}{(7x-9)^2}$$

Or

$$D_x \left[\frac{(3-4x)(5x+1)}{7x-9} \right] = \frac{[(-4)(5x+1) + (3-4x)(5)](7x-9) - (3-4x)(5x+1)(7)}{(7x-9)^2}$$

$$= \frac{[-20x - 4 + 15 - 20x](7x-9) - (15x + 3 - 20x^2 - 4x)(7)}{(7x-9)^2}$$

$$= \frac{(-40x + 11)(7x-9) - 7(-20x^2 + 11x + 3)}{(7x-9)^2}$$

$$= \frac{-280x^2 + 360x + 77x - 99 + 140x^2 - 77x - 21}{(7x-9)^2}$$

$$\boxed{= \frac{-140x^2 + 360x - 120}{(7x - 9)^2}}$$

Exercise

Find the derivative of $f(x) = x\left(1 - \frac{2}{x+1}\right)$

Solution

$$f(x) = x - \frac{2x}{x+1}$$

$$\boxed{f'(x) = 1 - \frac{2}{(x+1)^2}}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

OR

$$\left(\frac{2x}{x+1}\right)' \Rightarrow \begin{array}{ll} f = 2x & f' = 2 \\ g = x+1 & g' = 1 \end{array}$$

$$\begin{aligned} f'(x) &= 1 - \frac{2(x+1) - 2x}{(x+1)^2} \\ &= 1 - \frac{2x+2-2x}{(x+1)^2} \\ &= 1 - \frac{2}{(x+1)^2} \end{aligned} \boxed{}$$

Exercise

Find the derivative of $f(x) = (\sqrt{x} + 3)(x^2 - 5x)$

Solution

$$\begin{aligned} f' &= \left(\frac{1}{2}x^{-1/2}\right)(x^2 - 5x) + (\sqrt{x} + 3)(2x - 5) \\ &= \frac{1}{2}x^{3/2} - \frac{5}{2}x^{1/2} + 2x^{3/2} - 5x^{1/2} + 6x - 15 \\ &= \frac{5}{2}x^{3/2} - \frac{15}{2}x^{1/2} + 6x - 15 \\ &= \frac{5}{2}x^{3/2} + 6x - \frac{15}{2}x^{1/2} - 15 \end{aligned} \boxed{}$$

Exercise

Find the derivative of $y = (2x + 3)(5x^2 - 4x)$

Solution

$$\begin{aligned}y &= (2x + 3)(5x^2 - 4x) = 10x^3 - 8x^2 + 15x^2 - 12x \\&= 10x^3 + 7x^2 - 12x \\y' &= \underline{30x^2 + 14x - 12} \quad | \end{aligned}$$

Exercise

Find the derivative of $y = (x^2 + 1)\left(x + 5 + \frac{1}{x}\right)$

Solution

$$\begin{aligned}y &= x^3 + 5x^2 + x + x + 5 + \frac{1}{x} \\&= x^3 + 5x^2 + 2x + 5 + x^{-1} \\y' &= 3x^2 + 10x + 2 - x^{-2} \\&= \underline{3x^2 + 10x + 2 - \frac{1}{x^2}} \quad | \end{aligned}$$

Exercise

Find the derivative of $y = \frac{x + 4}{5x - 2}$

Solution

$$y' = \underline{-\frac{22}{(5x - 2)^2}} \quad |$$

$$\left(\frac{ax + b}{cx + d}\right)' = \frac{ad - bc}{(cx + d)^2}$$

OR

$$\begin{aligned}y' &= \frac{(5x - 2) \frac{d}{dx}[(x + 4)] - (x + 4) \frac{d}{dx}[(5x - 2)]}{(5x - 2)^2} \\&= \frac{(5x - 2)(1) - (x + 4)(5)}{(5x - 2)^2} \\&= \frac{5x - 2 - 5x - 20}{(5x - 2)^2} \\&= \underline{-\frac{22}{(5x - 2)^2}} \quad | \end{aligned}$$

Exercise

Find the derivative of $z = \frac{4-3x}{3x^2+x}$

Solution

$$\underline{z' = \frac{9x^2 - 24x - 4}{(3x^2 + x)^2}} \quad \begin{array}{ccc} 0 & -3 & 4 \\ 3 & 1 & 0 \end{array} \quad \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right)' = \frac{(ae - bd)x^2 + 2(af - cd)x + bf - ce}{(dx^2 + ex + f)^2}$$

OR

$$u = 4 - 3x \quad v = 3x^2 + x$$

$$u' = -3 \quad v' = 6x + 1$$

$$\begin{aligned} z' &= \frac{-3(3x^2 + x) - (6x + 1)(4 - 3x)}{(3x^2 + x)^2} \\ &= \frac{-9x^2 - 3x - (24x - 18x^2 + 4 - 3x)}{(3x^2 + x)^2} \\ &= \frac{-9x^2 - 3x - 21x + 18x^2 - 4}{(3x^2 + x)^2} \\ &= \frac{9x^2 - 24x - 4}{(3x^2 + x)^2} \end{aligned}$$

$$z' = \frac{u'v - v'u}{u^2}$$

Exercise

Find the derivative of $y = (2x - 7)^{-1}(x + 5)$

Solution

$$y = \frac{x+5}{2x-7}$$

$$\underline{y' = -\frac{17}{(2x-7)^2}}$$

$$\left(\frac{ax+b}{cx+d} \right)' = \frac{ad-bc}{(cx+d)^2}$$

OR

$$\begin{aligned} y' &= -(2x-7)^{-2}(2)(x+5) + (2x-7)^{-1} \\ &= -(2x-7)^{-2}(2x+10) + (2x-7)^{-1} \end{aligned}$$

$$\begin{aligned}
&= \left[-(2x-7)^{-2}(2x+10) + (2x-7)^{-1} \right] \frac{(2x-7)^2}{(2x-7)^2} \\
&= \frac{-2x-10+2x-7}{(2x-7)^2} \\
&= \frac{-17}{(2x-7)^2}
\end{aligned}$$

Exercise

Find the derivative of $f(x) = \frac{\sqrt{x}-1}{\sqrt{x}+1}$

Solution

$$\begin{aligned}
f'(x) &= \frac{\frac{1}{2}(1+1)x^{-1/2}}{(\sqrt{x}+1)^2} & \left(\frac{ax^n+b}{cx^n+d} \right)' &= \frac{n(ad-bc)x^{n-1}}{(cx+d)^2} \\
&= \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}
\end{aligned}$$

OR

$$\begin{aligned}
f'(x) &= \frac{\frac{1}{2}x^{-1/2}(x^{1/2}+1) - \frac{1}{2}x^{-1/2}(x^{1/2}-1)}{(\sqrt{x}+1)^2} & u &= x^{1/2}-1 & v &= x^{1/2}+1 \\
&= \frac{\frac{1}{2}1+x^{-1/2}-1+x^{-1/2}}{(\sqrt{x}+1)^2} & u' &= \frac{1}{2}x^{-1/2} & v' &= \frac{1}{2}x^{-1/2} \\
&= \frac{\frac{1}{2}2x^{-1/2}}{(\sqrt{x}+1)^2} \\
&= \frac{1}{x^{1/2}(\sqrt{x}+1)^2} \\
&= \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}
\end{aligned}$$

Exercise

Find the derivative of $y = \frac{1}{(x^2-1)(x^2+x+1)}$

Solution

$$y = \frac{1}{x^4 + x^3 + x^2 - x^2 - x - 1}$$

$$= \frac{1}{x^4 + x^3 - x - 1}$$

$$y' = \frac{-(4x^3 + 3x^2 - 1)}{(x^4 + x^3 - x - 1)^2} \quad \left(\frac{1}{v}\right)' = -\frac{v'}{v^2}$$

$$= \frac{-4x^3 - 3x^2 + 1}{(x^4 + x^3 - x - 1)^2}$$

Exercise

Find the derivative of $f(x) = \frac{x^{3/2}(x^2 + 1)}{x + 1}$

Solution

$$f(x) = \frac{x^{7/2} + x^{3/2}}{x + 1}$$

$$u = x^{7/2} + x^{3/2} \quad v = x + 1$$

$$u' = \frac{7}{2}x^{5/2} + \frac{3}{2}x^{1/2} \quad v' = 1$$

$$f'(x) = \frac{\frac{7}{2}x^{7/2} + \frac{3}{2}x^{3/2} + \frac{7}{2}x^{5/2} + \frac{3}{2}x^{1/2} - x^{7/2} - x^{3/2}}{(x + 1)^2}$$

$$= \frac{\frac{1}{2}5x^{7/2} + x^{3/2} + 7x^{5/2} + 3x^{1/2}}{(x + 1)^2}$$

Exercise

Find the derivative of $f(x) = \frac{x^3 - 4x^2 + x}{x - 2}$

Solution

$$f'(x) = \frac{3x^3 - 8x^2 + x - 6x^2 + 16x - 2 - x^3 + 4x^2 - x}{(x - 2)^2}$$

$$u = x^3 - 4x^2 + x \quad v = x - 2$$

$$u' = 3x^2 - 8x + 1 \quad v' = 1$$

$$= \frac{2x^3 - 10x^2 + 16x - 2}{(x - 2)^2}$$

Exercise

Find the derivative of $g(x) = \frac{x(3-x)}{2x^2}$

Solution

$$g(x) = \frac{1}{2} \frac{3-x}{x}$$

$$g' = -\frac{3}{2x^2}$$

$$\left(\frac{ax+b}{cx+d} \right)' = \frac{ad-bc}{(cx+d)^2}$$

OR

$$u = 3-x \quad v = x$$

$$u' = -1 \quad v' = 1$$

$$g'(x) = \frac{1}{2} \frac{-x-3+x}{x^2}$$

$$= -\frac{3}{2x^2}$$

Exercise

Find the derivative of $y = \frac{2x^2}{3x+1}$

Solution

$$y' = \frac{6x^2+4x}{(3x+1)^2}$$

$$\begin{array}{ccc} 2 & 0 & 0 \\ 0 & 3 & 1 \end{array}$$

$$\left(\frac{ax^2+bx+c}{dx^2+ex+f} \right)' = \frac{(ae-bd)x^2 + 2(af-cd)x + bf-ce}{(dx^2+ex+f)^2}$$

OR

$$u = x^2 \quad v = 3x+1$$

$$u' = 2x \quad v' = 3$$

$$y' = 2 \frac{6x^2+2x-3x^2}{(3x+1)^2}$$

$$= \frac{6x^2+4x}{(3x+1)^2}$$

Exercise

Find the derivative of $f(x) = \frac{x^9 + x^8 + 4x^5 - 7x}{x^4 - 3x^2 + 2x + 1}$

Solution

$$u = x^9 + x^8 + 4x^5 - 7x \quad v = x^4 - 3x^2 + 2x + 1$$

$$u' = 9x^8 + 8x^7 + 20x^4 - 7 \quad v' = 4x^3 - 6x + 2$$

x^{12}	x^{11}	x^{10}	x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x	x^0
				9								
			18	16				20				
9	8	-27	-24	20	8	-60	40	-7		21	-14	-7
-12	-4	6	6	-16		24	-8	-28		42	14	
			-2	-2								

$$f'(x) = \frac{-3x^{12} + 4x^{11} - 21x^{10} - 2x^9 + 27x^8 + 8x^7 - 36x^6 + 32x^5 - 15x^4 + 63x^2 - 7}{(x^4 - 3x^2 + 2x + 1)^2}$$

Exercise

Find the derivative of $f(x) = \frac{x}{1+x^2}$

Solution

$f' = \frac{1-x^2}{(x^2+1)^2}$	$\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}$	$\left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right)' = \frac{(ae - bd)x^2 + 2(af - cd)x + bf - ce}{(dx^2 + ex + f)^2}$
--------------------------------	---	--

OR

$$u = x \quad v = 1 + x^2$$

$$u' = 1 \quad v' = 2x$$

$$f'(x) = \frac{1+x^2 - 2x^2}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

Exercise

Find the derivative of $y = \frac{x^2 - 2ax + a^2}{x - a}$

Solution

$$y = \frac{(x-a)^2}{x-a}$$

$$= x - a$$

$$\underline{y' = 1}$$

Exercise

Find the derivative of $f(x) = \frac{x^2 + 4x^{1/2}}{x^2}$

Solution

$$f(x) = 1 + 4x^{-3/2}$$

$$\underline{f'(x) = -6x^{-5/2}}$$

Exercise

Find the derivative of $f(x) = (2x+1)(3x^2+2)$

Solution

$$f'(x) = 2(3x^2+2) + (6x)(2x+1)$$

$$= 6x^2 + 4 + 12x^2 + 6x$$

$$\underline{= 18x^2 + 6x + 4}$$

Exercise

Find the derivative of $f(x) = \frac{x^2-1}{x^2+1}$

Solution

$$\underline{f' = \frac{-2x^3 + 2x^2 + 4x}{(x^2+1)^2}}$$

$$\begin{array}{ccc} 1 & 0 & -1 \\ 1 & 0 & 1 \end{array} \quad \left(\frac{ax^2+bx+c}{dx^2+ex+f} \right)' = \frac{(ae-bd)x^2 + 2(af-cd)x + bf-ce}{(dx^2+ex+f)^2}$$

OR

$$u = x^2 - 1 \quad v = x^2 + 1$$

$$u' = 2x \quad v' = 2x$$

$$f'(x) = \frac{2x^2 + 2x - 2x^3 + 2x}{(x^2 + 1)^2}$$

$$= \frac{-2x^3 + 2x^2 + 4x}{(x^2 + 1)^2}$$

Exercise

Find the derivative of $y = \frac{4x^3 + 3x + 1}{2x^5}$

Solution

$$y = 2x^{-2} + \frac{3}{2}x^{-4} + \frac{1}{2}x^{-5}$$

$$y' = -4x^{-3} - 6x^{-5} - \frac{5}{2}x^{-6}$$

$$= -\frac{1}{2}x^{-6}(8x^3 + 12x + 5)$$

$$= -\frac{8x^3 + 12x + 5}{2x^6}$$

Exercise

Find the derivative of $y = \frac{4}{3-x}$

Solution

$$y' = \frac{4}{(3-x)^2}$$

$$\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$$

Exercise

Find the derivative of $y = \frac{2}{1-x^2}$

Solution

$$y' = \frac{4x}{(1-x^2)^2}$$

$$\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$$

Exercise

Find the derivative of $f(x) = \frac{\pi}{2 - \pi x}$

Solution

$$\underline{f'(x) = \frac{\pi^2}{(2 - \pi x)^2}}$$

$$\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$$

Exercise

Find the derivative of $y = \frac{x-4}{5x-2}$

Solution

$$\underline{y' = \frac{1(-2) - (-4)(5)}{(5x-2)^2} = \frac{18}{(5x-2)^2}}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{3x-4}{2x-1}$

Solution

$$\underline{y' = \frac{-3+8}{(2x-1)^2} = \frac{5}{(2x-1)^2}}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{3x+4}{2x+1}$

Solution

$$\underline{y' = \frac{3-8}{(2x+1)^2} = \frac{-5}{(2x+1)^2}}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{-3x+4}{2x+1}$

Solution

$$y' = \frac{-3-8}{(2x+1)^2}$$
$$= -\frac{11}{(2x+1)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{-3x-4}{2x-1}$

Solution

$$y' = \frac{3+8}{(2x-1)^2}$$
$$= \frac{11}{(2x-1)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{2x-3}{x+1}$

Solution

$$y' = \frac{2+3}{(x+1)^2}$$
$$= \frac{5}{(x+1)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{3x}{3x-2}$

Solution

$$y' = \frac{-6}{(3x-2)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{x-3}{2x+5}$

Solution

$$y' = \frac{11}{(2x+5)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{5x-3}{2x+5}$

Solution

$$y' = \frac{31}{(2x+5)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{6x-8}{2x-3}$

Solution

$$y' = -\frac{2}{(2x-3)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{x^2-4}{5x^2-2}$

Solution

$$y' = \frac{2(-2+20)x}{(5x^2-2)^2} \\ = \frac{36x}{(5x^2-2)^2}$$

$$\left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{(cx^n+d)^2}$$

Exercise

Find the derivative of $y = \frac{3x^2-4}{2x^2-1}$

Solution

$$y' = \frac{2(-3+8)x}{(2x^2-1)^2}$$

$$= \frac{10x}{(2x^2-1)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{3x^2 + 4}{2x^2 + 1}$

Solution

$$y' = \frac{2(3-8)x}{(2x^2+1)^2}$$

$$= -\frac{10x}{(2x^2+1)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{2x^2 - 3}{x^2 + 1}$

Solution

$$y' = \frac{2(2+3)x}{(x^2+1)^2}$$

$$= \frac{10x}{(x^2+1)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{3x^2}{3x^2 - 2}$

Solution

$$y' = -\frac{12x}{(3x^2-2)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{5x^2 - 3}{2x^2 + 5}$

Solution

$$y' = \frac{2(25 + 6)x}{(2x^2 + 5)^2}$$
$$= \frac{62x}{(2x^2 + 5)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{6x^2 - 8}{2x^2 + 1}$

Solution

$$y' = \frac{2(6 + 16)x}{(2x^2 + 1)^2}$$
$$= \frac{44x}{(2x^2 + 1)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{6x^3 + 8}{2x^3 + 1}$

Solution

$$y' = \frac{3(6 - 16)x^2}{(2x^3 + 1)^2}$$
$$= -\frac{30x^2}{(2x^3 + 1)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{5x^3 - 3}{2x^3 + 5}$

Solution

$$y' = \frac{3(25+6)x^2}{(2x^3+5)^2}$$

$$= \frac{93x^2}{(2x^3+1)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{x^3}{3x^3 - 2}$

Solution

$$y' = -\frac{6x^2}{(3x^3 - 2)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{2x^3 - 3}{2x^3 + 1}$

Solution

$$y' = \frac{24x^2}{(2x^3 + 1)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{2x^4 - 3}{2x^4 + 1}$

Solution

$$y' = \frac{4(2+6)x^3}{(2x^4+1)^2}$$

$$= \frac{32x^3}{(2x^4+1)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{x^2 - 4x + 1}{5x^2 - 2x - 1}$

Solution

$$y' = \frac{\begin{vmatrix} 1 & -4 \\ 5 & -2 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} x + \begin{vmatrix} -4 & 1 \\ -2 & -1 \end{vmatrix}}{(5x^2 - 2x - 1)^2}$$
$$= \frac{18x^2 - 12x + 6}{(5x^2 - 2x - 1)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

Exercise

Find the derivative of $y = \frac{3x^2 - 4x + 2}{2x^2 + x - 1}$

Solution

$$y' = \frac{\begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} x^2 + 2 \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} x + \begin{vmatrix} -4 & 2 \\ 1 & -1 \end{vmatrix}}{(2x^2 + x - 1)^2}$$
$$= \frac{11x^2 - 14x + 6}{(2x^2 + x - 1)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

Exercise

Find the derivative of $y = \frac{3x^2 + x - 4}{2x^2 + 1}$

Solution

$$y' = \frac{\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} x^2 + 2 \begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} x + \begin{vmatrix} 1 & -4 \\ 0 & 1 \end{vmatrix}}{(2x^2 + 1)^2}$$
$$= \frac{2x^2 + 22x + 1}{(2x^2 + 1)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

Exercise

Find the derivative of $y = \frac{2x^2 - 3}{x^2 + 5x + 1}$

Solution

$$y' = \frac{\begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix} x^2 + 2 \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} x + \begin{vmatrix} 0 & -3 \\ 5 & 1 \end{vmatrix}}{(x^2 + 5x + 1)^2}$$
$$= \frac{10x^2 + 10x + 15}{(x^2 + 5x + 1)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

Exercise

Find the derivative of $y = \frac{3x^2}{3x^2 + 6x - 8}$

Solution

$$y' = \frac{\begin{vmatrix} 3 & 0 \\ 3 & 6 \end{vmatrix} x^2 + 2 \begin{vmatrix} 3 & 0 \\ 3 & -8 \end{vmatrix} x + \begin{vmatrix} 0 & 0 \\ 6 & -8 \end{vmatrix}}{(3x^2 + 6x - 8)^2}$$
$$= \frac{18x^2 - 48x}{(3x^2 + 6x - 8)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

Exercise

Find the derivative of $y = \frac{x^2 + 2x}{2x^2 + x - 5}$

Solution

$$y' = \frac{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 0 \\ 2 & -5 \end{vmatrix} x + \begin{vmatrix} 2 & 0 \\ 1 & -5 \end{vmatrix}}{(2x^2 + x - 5)^2}$$
$$= \frac{-3x^2 - 10x - 10}{(2x^2 + x - 5)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

Exercise

Find the derivative of $y = \frac{x^2 + 5x + 1}{x^2}$

Solution

$$y' = \frac{\begin{vmatrix} 1 & 5 \\ 1 & 0 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} x + \begin{vmatrix} 5 & 1 \\ 0 & 0 \end{vmatrix}}{x^4}$$

$$= \frac{-5x^2 - 4x}{x^4}$$

$$= \frac{-5x - 4}{x^3}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

Exercise

Find the derivative of $y = \frac{x^2 - 3x + 1}{x^2 - 8x + 5}$

Solution

$$y' = \frac{\begin{vmatrix} 1 & -3 \\ 1 & -8 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} x + \begin{vmatrix} -3 & 1 \\ -8 & 5 \end{vmatrix}}{(x^2 - 8x + 5)^2}$$

$$= \frac{-5x^2 + 8x - 7}{(x^2 - 8x + 5)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

Exercise

Find the **first** and **second** derivative $y = \frac{x^2 + 5x - 1}{x^2}$

Solution

$$y' = \frac{(2x + 5)x^2 - 2x(x^2 + 5x - 1)}{x^4}$$

$$= \frac{(2x + 5)x^2 - 2x(x^2 + 5x - 1)}{x^4}$$

$$= x \frac{(2x + 5)x - 2(x^2 + 5x - 1)}{x^4}$$

$$\left(\frac{u}{v} \right)' = \frac{u'v - v'u}{v^2}$$

$$\begin{aligned} u &= x^2 + 5x - 1 & v &= x^2 \\ u' &= 2x + 5 & v' &= 2x \end{aligned}$$

$$= \frac{2x^2 + 5x - 2x^2 - 10x + 2}{x^3}$$

$$= \frac{-5x + 2}{x^3}$$

$$u = -5x + 2 \quad v = x^3$$

$$u' = -5 \quad v' = 3x^2$$

$$y'' = \frac{(-5)x^3 - 3x^2(-5x + 2)}{x^6}$$

$$= x^2 \frac{-5x^3 + 15x - 6}{x^6}$$

$$= \frac{-5x^3 + 15x - 6}{x^4}$$

Exercise

Find y' , y'' , y''' : $y = (x-3)\sqrt{x+2}$

Solution

$$y' = \sqrt{x+2} + \frac{1}{2}(x-3)(x+2)^{-1/2}$$

$$y'' = \frac{1}{2}(x+2)^{-1/2} + \frac{1}{2}(x+2)^{-1/2} - \frac{1}{4}(x-3)(x+2)^{-3/2}$$

$$= (x+2)^{-1/2} - \frac{1}{4}(x-3)(x+2)^{-3/2}$$

$$y''' = -\frac{1}{2}(x+2)^{-3/2} - \frac{1}{4}(x+2)^{-3/2} + \frac{3}{8}(x-3)(x+2)^{-5/2}$$

$$= -\frac{3}{4}(x+2)^{-3/2} + \frac{3}{8}(x-3)(x+2)^{-5/2}$$

Exercise

For what value(s) of x is the line tangent to the curve $y = x\sqrt{6-x}$ horizontal? Vertical?

Solution

$$y' = \sqrt{6-x} - \frac{x}{2\sqrt{6-x}}$$

$$= \frac{12-3x}{2\sqrt{6-x}} = 0$$

$$12-3x=0$$

$$\underline{x = 4, y = 4\sqrt{2} \mid}$$

∴ Point $(4, 4\sqrt{2})$ is a horizontal tangent line.

∴ The vertical tangent line inside the square root of y . $\Rightarrow \underline{x = 6 \mid}$

$$\begin{aligned} \lim_{x \rightarrow 6} y' &= \lim_{x \rightarrow 6} \frac{12 - 3x}{2\sqrt{6 - x}} \\ &= \frac{-6}{0} \\ &= \underline{-\infty \mid} \end{aligned}$$

Exercise

Find an equation of the tangent line to the graph of $y = \frac{x^2 - 4}{2x + 5}$ when $x = 0$

Solution

$$\underline{f' = \frac{2x^2 + 10x + 8}{(2x + 5)^2} \mid} \quad \begin{matrix} 1 & 0 & -4 \\ 0 & 2 & 5 \end{matrix} \quad \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right)' = \frac{(ae - bd)x^2 + 2(af - cd)x + bf - ce}{(dx^2 + ex + f)^2}$$

OR

$$\begin{aligned} y' &= \frac{(2x + 5)(2x) - (x^2 - 4)(2)}{(2x + 5)^2} \\ &= \frac{4x^2 + 10x - 2x^2 + 8}{(2x + 5)^2} \\ &= \underline{\frac{2x^2 + 10x + 8}{(2x + 5)^2} \mid} \end{aligned}$$

$$\left(\frac{u}{v} \right)' = \frac{u'v - v'u}{v^2}$$

$$\Rightarrow x = 0 \rightarrow y' = \frac{8}{25} = m$$

$$x = 0 \rightarrow y = \frac{x^2 - 4}{2x + 5} = -\frac{4}{5}$$

$$y = \frac{8}{25}(x - 0) - \frac{4}{5}$$

$$y = m(x - x_1) + y_1$$

$$\underline{y = \frac{8}{25}x - \frac{4}{5} \mid}$$

Solution **Section 2.4 –Derivatives of Trigonometric Functions**

Exercise

Find the derivative of $y = -10x + 3\cos x$

Solution

$$\underline{y' = -10 - 3\sin x}$$

Exercise

Find the derivative of $y = \csc x - 4\sqrt{x} + 7$

Solution

$$\begin{aligned} y' &= -\csc x \cot x - 4\left(\frac{1}{2}x^{-1/2}\right) \\ &= -\csc x \cot x - \frac{2}{\sqrt{x}} \end{aligned}$$

Exercise

Find the derivative of $y = x^2 \cos x$

Solution

$$\begin{aligned} y &= 2x \cos x + x^2(-\sin x) \\ &= 2x \cos x - x^2 \sin x \end{aligned}$$

$$(uv)' = u'v + v'u$$

Exercise

Find the derivative of $y = \csc x \cot x$

Solution

$$\begin{aligned} y' &= (-\csc x \cot x) \cot x + \csc x(-\csc^2 x) \\ &= -\csc x \cot^2 x - \csc^3 x \\ &= -\csc x(\cot^2 x + \csc^2 x) \end{aligned}$$

$$(uv)' = u'v + uv'$$

Exercise

Find the derivative of $y = (\sin x + \cos x)\sec x$

Solution

$$u = \sin x + \cos x \quad v = \sec x$$

$$u' = \cos x - \sin x \quad v' = \sec x \tan x$$

$$y' = (\cos x - \sin x)\sec x + (\sin x + \cos x)(\sec x \tan x)$$

$$= \sec x \left[\cos x - \sin x + (\sin x + \cos x) \frac{\sin x}{\cos x} \right]$$

$$= \sec x \left(\cos x - \sin x + \frac{\sin^2 x}{\cos x} + \sin x \right)$$

$$= \sec x \left(\cos x + \frac{\sin^2 x}{\cos x} \right)$$

$$= \sec x \left(\frac{\cos^2 x + \sin^2 x}{\cos x} \right)$$

$$= \sec x \left(\frac{1}{\cos x} \right)$$

$$= \sec x \sec x$$

$$= \sec^2 x$$

$$y = (\sin x + \cos x) \frac{1}{\cos x}$$

$$= \tan x + 1$$

$$y' = \sec^2 x$$

Exercise

Find the derivative of $y = (\sec x + \tan x)(\sec x - \tan x)$

Solution

$$y = (\sec x + \tan x)(\sec x - \tan x)$$

$$= \sec^2 x - \tan^2 x$$

$$= 1 + \tan^2 x - \tan^2 x$$

$$= 1$$

$$y' = 0$$

$$y' = (\sec x + \tan x)'(\sec x - \tan x) + (\sec x + \tan x)(\sec x - \tan x)'$$

$$= (\sec x \tan x + \sec^2 x)(\sec x - \tan x)$$

$$+ (\sec x + \tan x)(\sec x \tan x - \sec^2 x)$$

$$= \sec^2 x \tan x - \sec x \tan^2 x + \sec^3 x - \sec^2 x \tan x$$

$$+ \sec^2 x \tan x - \sec^3 x + \sec x \tan^2 x - \sec^2 x \tan x$$

$$= 0$$

Exercise

Find the derivative of $y = \frac{\cos x}{x} + \frac{x}{\cos x}$

Solution

$$y = \frac{\cos^2 x + x^2}{x \cos x}$$

$$u = \cos^2 x + x^2 \quad v = x \cos x$$

$$u' = 2 \cos x (-\sin x) + 2x \quad v' = \cos x - x \sin x$$

$$\begin{aligned} y' &= \frac{(-2 \cos x \sin x + 2x)x \cos x - (\cos x - x \sin x)(\cos^2 x + x^2)}{(x \cos x)^2} & \left(\frac{u}{v}\right)' &= \frac{u'v - v'u}{v^2} \\ &= \frac{-2x \sin x \cos^2 x + 2x^2 \cos x - \cos^3 x - x^2 \cos x + x \sin x \cos^2 x + x^3 \sin x}{(x \cos x)^2} \\ &= \frac{-x \sin x \cos^2 x - x^2 \cos x - \cos^3 x + x^3 \sin x}{(x \cos x)^2} \end{aligned}$$

Exercise

Find the derivative of $y = x^2 \cos x - 2x \sin x - 2 \cos x$

Solution

$$\begin{aligned} y' &= 2x \cos x - x^2 \sin x - 2(\sin x + x \cos x) - 2(-\sin x) \\ &= 2x \cos x - x^2 \sin x - 2 \sin x - 2x \cos x + 2 \sin x \\ &= -x^2 \sin x \end{aligned}$$

Exercise

Find the derivative of $y = (2 - x) \tan^2 x$

Solution

$$\begin{aligned} y' &= -\tan^2 x + (2 - x)(2 \tan x \sec^2 x) \\ &= \tan x(-\tan x + 2(2 - x) \sec^2 x) \\ &= \tan x(2(2 - x) \sec^2 x - \tan x) \end{aligned}$$

Exercise

Find the derivative of $y = t^2 - \sec t + 1$

Solution

$$\underline{y' = 2t - \sec t \tan t}$$

Exercise

Find the derivative of $y = \frac{1 + \csc t}{1 - \csc t}$

Solution

$$\begin{aligned} u &= 1 + \csc t & v &= 1 - \csc t \\ u' &= -\csc x \cot x & v' &= \csc x \cot x \\ y' &= \frac{(-\csc x \cot x)(1 - \csc t) - (1 + \csc t)(\csc x \cot x)}{(1 - \csc t)^2} \\ &= \frac{-\csc x \cot x + \csc^2 x \cot x - \csc x \cot x - \csc^2 x \cot x}{(1 - \csc t)^2} \\ &= \underline{-\frac{2 \csc x \cot x}{(1 - \csc t)^2}} \end{aligned}$$

Exercise

Find the derivative of $r = \theta \sin \theta + \cos \theta$

Solution

$$\begin{aligned} r' &= \sin \theta + \theta \cos \theta - \sin \theta \\ &= \underline{\theta \cos \theta} \end{aligned}$$

Exercise

Find the derivative of $p = \frac{\sin q + \cos q}{\cos q}$

Solution

$$\begin{aligned} u &= \sin q + \cos q & v &= \cos q \\ u' &= \cos q - \sin q & v' &= -\sin q \\ p' &= \frac{(\cos q - \sin q)\cos q - (-\sin q)(\sin q + \cos q)}{\cos^2 q} \\ &= \frac{\cos^2 q - \sin q \cos q + \sin^2 q + \sin q \cos q}{\cos^2 q} \end{aligned}$$

$$\begin{aligned}
&= \frac{\cos^2 q + \sin^2 q}{\cos^2 q} \\
&= \frac{1}{\cos^2 q} \\
&= \sec^2 q \quad |
\end{aligned}$$

Exercise

Find the derivative of $p = \frac{3q + \tan q}{q \sec q}$

Solution

$$u = 3q + \tan q \quad v = q \sec q$$

$$u' = 3 + \sec^2 q \quad v' = \sec q + q \sec q \tan q$$

$$\begin{aligned}
p' &= \frac{(3 + \sec^2 q)(q \sec q) - (3q + \tan q)(\sec q + q \sec q \tan q)}{(q \sec q)^2} & \left(\frac{u}{v}\right)' &= \frac{u'v - v'u}{v^2} \\
&= \frac{3q \sec q + q \sec^3 q - 3q \sec q - 3q^2 \sec q \tan q - \tan q \sec q - q \sec q \tan^2 q}{q^2 \sec^2 q} \\
&= \frac{q \sec^3 q - 3q^2 \sec q \tan q - \tan q \sec q - q \sec q \tan^2 q}{q^2 \sec^2 q} \quad |
\end{aligned}$$

Exercise

Find the derivative of $f(x) = \frac{\sin x + 2x}{x}$

Solution

$$\begin{aligned}
f'(x) &= \frac{x \cos x + 2x - \sin x - 2x}{x^2} \\
&= \frac{x \cos x - \sin x}{x^2} \quad |
\end{aligned}$$

Exercise

Find the derivative of $f(x) = \frac{\sin x}{x^2}$

Solution

$$f'(x) = \frac{x^2 \cos x - 2x \sin x}{x^4}$$

$$= \frac{x \cos x - 2 \sin x}{x^3}$$

Exercise

Find the derivative of $f(x) = x^3 \cos x$

Solution

$$f'(x) = 3x^2 \cos x - x^3 \sin x$$

Exercise

Find the derivative of $f(x) = \frac{1}{x} - 12 \sec x$

Solution

$$f'(x) = -\frac{1}{x^2} - 12 \sec x \tan x$$

Exercise

Find the derivative of $f(\theta) = 5\theta \sec \theta + \theta \tan \theta$

Solution

$$f'(\theta) = 5 \sec \theta + 5\theta \sec \theta \tan \theta + \tan \theta + \theta \sec^2 \theta$$

Exercise

Find the derivative of $y = \sec \pi x$

Solution

$$y' = \pi \sec \pi x \tan \pi x$$

Exercise

Find the derivative of $y = \cos 5x$

Solution

$$y' = -5 \sin 5x$$

Exercise

Find the derivative of $y = \cos(4 - 3x)$

Solution

$$\underline{y' = 3 \sin(4 - 3x) \quad |}$$

Exercise

Find the derivative of $f(x) = \sin(4 - 3x)$

Solution

$$\underline{f'(x) = -3 \cos(4 - 3x) \quad |}$$

Exercise

Find the derivative of $f(\theta) = \frac{\sin a\theta}{\cos b\theta}$

Solution

$$u = \sin a\theta \quad v = \cos b\theta$$

$$u' = a \cos a\theta \quad v' = -b \sin b\theta$$

$$\underline{f'(\theta) = \frac{a \cos a\theta \cos b\theta + b \sin a\theta \sin b\theta}{\cos^2 b\theta} \quad |}$$

Exercise

Find the derivative of $f(\theta) = \sin 2\theta - \cos 2\theta$

Solution

$$\underline{f'(\theta) = 2 \cos 2\theta + 2 \sin 2\theta \quad |}$$

Exercise

Find the derivative of $f(\theta) = \tan \theta - \cot \theta$

Solution

$$\underline{f'(\theta) = \sec^2 \theta + \csc^2 \theta \quad |}$$

Exercise

Find the derivative of $\frac{d}{dx}(5x^2 \sin x)$

Solution

$$\underline{\frac{d}{dx}(5x^2 \sin x) = 10x \sin x + 5x^2 \cos x}$$

Exercise

Find the derivative of $\frac{d}{dx}(2x(\sin x)\sqrt{3x-1})$

Solution

$$\begin{aligned} \frac{d}{dx}(2x(\sin x)\sqrt{3x-1}) &= \underbrace{2}_{u'}(\sin x)\sqrt{3x-1} + 2x(\underbrace{\cos x}_{v'})\sqrt{3x-1} + 2x(\sin x)\underbrace{\frac{1}{2}(3)(3x-1)^{-1/2}}_{w'} \\ &= 2(\sin x)\sqrt{3x-1} + 2x(\cos x)\sqrt{3x-1} + \frac{3x \sin x}{\sqrt{3x-1}} \end{aligned}$$

Exercise

Find $y^{(4)}$ if $y = 9 \cos x$

Solution

$$\underline{y' = -9 \sin x}$$

$$\underline{y'' = -9 \cos x}$$

$$\underline{y''' = 9 \sin x}$$

$$\underline{y^{(4)} = 9 \cos x}$$

Exercise

Find $\frac{d^{999}}{dx^{999}}(\cos x)$

Solution

$$\underline{y' = -\sin x}$$

$$\underline{y'' = -\cos x}$$

$$\underline{y''' = \sin x}$$

$$\underline{y^{(4)} = \cos x}$$

$$999 = 249 \times 4 + 3$$

$$\begin{aligned} \frac{d^{999}}{dx^{999}}(\cos x) &= \frac{d^3}{dx^3}(\cos x) \\ &= \sin x \end{aligned}$$

Exercise

Find y', y'', y''' $y = \sin \sqrt{x}$

Solution

$$\underline{y' = \frac{1}{2\sqrt{x}} \cos \sqrt{x}}$$

$$\underline{y'' = -\frac{1}{4x^{3/2}} \cos \sqrt{x} - \frac{1}{4x} \sin \sqrt{x}}$$

$$\begin{aligned} y''' &= \frac{3}{8x^{5/2}} \cos \sqrt{x} + \frac{1}{8x^2} \sin \sqrt{x} + \frac{1}{4x^2} \sin \sqrt{x} - \frac{1}{8x^{3/2}} \cos \sqrt{x} \\ &= \frac{3}{8x^2} \sin \sqrt{x} + \frac{3-x}{8x^{5/2}} \cos \sqrt{x} \end{aligned}$$

Exercise

Find $\lim_{x \rightarrow -\frac{\pi}{6}} \sqrt{1 + \cos(\pi \csc x)}$

Solution

$$\begin{aligned} \lim_{x \rightarrow -\frac{\pi}{6}} \sqrt{1 + \cos(\pi \csc x)} &= \sqrt{1 + \cos\left(\pi \csc\left(-\frac{\pi}{6}\right)\right)} \\ &= \sqrt{1 + \cos(\pi(-2))} \\ &= \sqrt{1 + \cos(-2\pi)} \\ &= \sqrt{1+1} \\ &= \underline{\sqrt{2}} \end{aligned}$$

Exercise

Assume that a particle's position on the x -axis is given by $x = 3 \cos t + 4 \sin t$; ft

a) Find the particle's position when $t = 0$, $t = \frac{\pi}{2}$, and $t = \pi$

b) Find the particle's velocity when $t = 0$, $t = \frac{\pi}{2}$, and $t = \pi$

Solution

a) $t = 0$

$$x = 3 \cos 0 + 4 \sin 0$$

$$= 3 \text{ ft}$$

$$t = \frac{\pi}{2}$$

$$x = 3 \cos \frac{\pi}{2} + 4 \sin \frac{\pi}{2}$$

$$= 0 + 4$$

$$= 4 \text{ ft}$$

$$t = \pi$$

$$x = 3 \cos \pi + 4 \sin \pi$$

$$= 3(-1) + 0$$

$$= -3 \text{ ft}$$

b) $v = x' = -3 \sin t + 4 \cos t$

$$t = 0$$

$$x = -3 \sin 0 + 4 \cos 0$$

$$= 4 \text{ ft/sec}$$

$$t = \frac{\pi}{2}$$

$$x = -3 \sin \frac{\pi}{2} + 4 \cos \frac{\pi}{2}$$

$$= -3 + 0$$

$$= -3 \text{ ft/sec}$$

$$t = \pi$$

$$x = -3 \sin \pi + 4 \cos \pi$$

$$= 0 - 4$$

$$= -4 \text{ ft/sec}$$

Exercise

A weight is attached to a spring and reaches its equilibrium position ($x = 0$). It is then set in motion resulting in a displacement of $x = 10 \cos t$

Where x is measured in centimeters and t is measured in seconds.

a) Find the spring's displacement when $t = 0$, $t = \frac{\pi}{3}$, and $t = \frac{3\pi}{4}$

b) Find the spring's velocity when $t = 0$, $t = \frac{\pi}{3}$, and $t = \frac{3\pi}{4}$

Solution

a) $t = 0$

$$x = 10 \cos 0$$

$$= 10 \text{ cm}$$

$$t = \frac{\pi}{3}$$

$$x = 10 \cos \frac{\pi}{3}$$

$$= 10 \left(\frac{1}{2} \right)$$

$$= 5 \text{ cm}$$

$$t = \frac{3\pi}{4}$$

$$x = 10 \cos \frac{3\pi}{4}$$

$$= 10 \frac{\sqrt{2}}{2}$$

$$= 5\sqrt{2} \text{ cm}$$

b) $v = x' = -10 \sin t$

$$t = 0$$

$$x = -10 \sin 0$$

$$= 0 \text{ cm/sec}$$

$$t = \frac{\pi}{3}$$

$$x = -10 \sin \frac{\pi}{3}$$

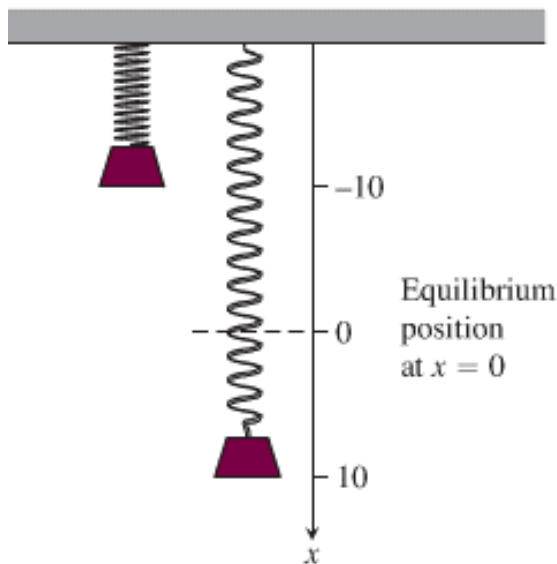
$$= 10 \left(\frac{\sqrt{3}}{2} \right)$$

$$= 5\sqrt{3} \text{ cm/sec}$$

$$t = \frac{3\pi}{4}$$

$$x = -10 \sin \frac{3\pi}{4}$$

$$= -5\sqrt{2} \text{ cm/sec}$$



Solution **Section 2.5 – Derivative as Rates of Change**

Exercise

The position $s(t) = t^2 - 3t + 2$, $0 \leq t \leq 2$ of a body moving on a coordinate line, with s in meters and t in seconds.

- a) Find the body's displacement and average velocity for the given time interval.
- b) Find the body's speed and acceleration at the endpoints of the interval.
- c) When, if ever, during the interval does the body change direction?

Solution

a) Displacement: $\Delta s = s(2) - s(0)$

$$= 2^2 - 3(2) + 2 - (0^2 - 3(0) + 2)$$
$$= -2 \text{ m}$$

Average velocity = $\frac{\Delta s}{\Delta t}$

$$= \frac{-2}{2-0}$$
$$= -1 \text{ m/sec}$$

b) $v = \frac{ds}{dt} = 2t - 3$

$$\Rightarrow \begin{cases} |v(0)| = |-3| = 3 \text{ m/sec} \\ |v(2)| = 1 \text{ m/sec} \end{cases}$$

$$a = \frac{dv}{dt} = 2$$

$$a(0) = a(2) = 2 \text{ m/sec}^2$$

c) $v = 0$

$$2t - 3 = 0$$
$$t = \frac{3}{2}$$

v is negative in the interval $0 < t < \frac{3}{2}$

v is positive in the interval $\frac{3}{2} < t < 2$

The body changes direction at $t = \frac{3}{2}$

Exercise

The position $s(t) = \frac{25}{t+5}$, $-4 \leq t \leq 0$ of a body moving on a coordinate line, with s in meters and t in seconds.

- a) Find the body's displacement and average velocity for the given time interval.
- b) Find the body's speed and acceleration at the endpoints of the interval.
- c) When, if ever, during the interval does the body change direction?

Solution

$$\begin{aligned} \text{a) Displacement: } \Delta s &= s(0) - s(-4) \\ &= \frac{25}{0+5} - \frac{25}{-4+5} \\ &= 5 - 25 \\ &= -20 \text{ m} \end{aligned}$$

$$\text{Average velocity} = \frac{\Delta s}{\Delta t} = \frac{-20}{0 - (-4)} = -5 \text{ m/sec}$$

$$\begin{aligned} \text{b) } v &= \frac{ds}{dt} = \frac{25(-1)}{(t+5)^2} = -\frac{25}{(t+5)^2} \\ \Rightarrow \begin{cases} |v(-4)| = \left| -\frac{25}{(-4+5)^2} \right| = 25 \text{ m/sec} \\ |v(0)| = \left| -\frac{25}{(0+5)^2} \right| = 1 \text{ m/sec} \end{cases} \end{aligned}$$

$$\begin{aligned} a &= \frac{dv}{dt} = -\frac{-25[2(t+5)(1)]}{(t+5)^4} \\ &= \frac{50}{(t+5)^3} \end{aligned}$$

$$a(-4) = \frac{50}{(-4+5)^3} = 50 \text{ m/sec}^2$$

$$a(0) = \frac{50}{(0+5)^3} = \frac{2}{5} \text{ m/sec}^2$$

$$\begin{aligned} \text{c) } v &= 0 \\ -\frac{25}{(t+5)^2} &= 0 \rightarrow v < 0 \end{aligned}$$

v is never equal to zero \Rightarrow The body never changes direction.

Exercise

At time t , the position of a body moving along the s -axis is $s = t^3 - 6t^2 + 9t$ m.

- a) Find the body's acceleration each time the velocity is zero.
- b) Find the body's speed each time the acceleration is zero.
- c) Find the total distance traveled by the body from $t = 0$ to $t = 2$.

Solution

$$a) \quad v = s' = 3t^2 - 12t + 9 = 0$$

$$\underline{t_1 = 1 \quad \& \quad t_2 = 3}$$

$$a = v' = 6t - 12$$

$$\begin{cases} a(1) = 6 - 12 = -6 \text{ m/sec}^2 \\ a(3) = 6(3) - 12 = 6 \text{ m/sec}^2 \end{cases}$$

The body is motionless but being accelerated left when $t = 1$, and motionless but being accelerated right when $t = 3$.

$$b) \quad a = 0 = 6t - 12$$

$$\Rightarrow \underline{t = 2}$$

$$\begin{aligned} |v(2)| &= |3(2)^2 - 12(2) + 9| \\ &= 3 \text{ m/sec} \end{aligned}$$

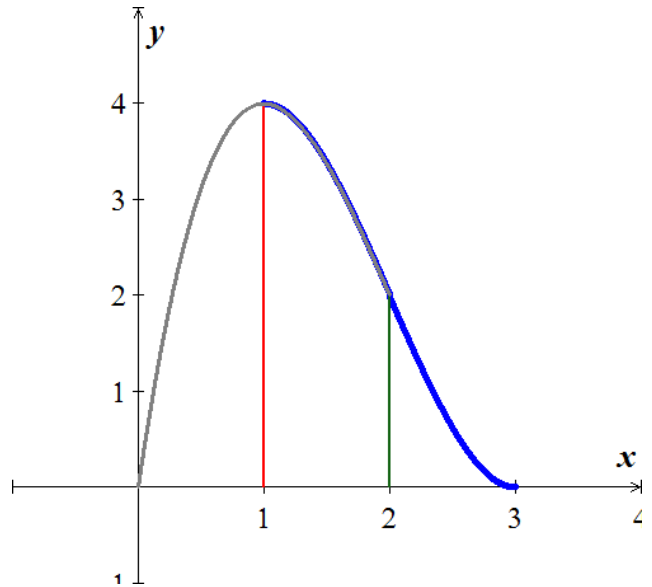
$$c) \quad \text{The body moves forward on } 0 \leq t < 1$$

$$\begin{aligned} d_1 &= s(1) - s(0) \\ &= 1 - 6 + 9 \\ &= 4 \end{aligned}$$

The body moves backward on $1 \leq t < 2$

$$\begin{aligned} d_2 &= |s(2) - s(1)| \\ &= |2 - 4| \\ &= 2 \end{aligned}$$

$$\text{Total distance} = d_1 + d_2 = 4 + 2 = 6 \text{ m}$$



Exercise

A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of $s(t) = 24t - 0.8t^2 \text{ m}$ in $t \text{ sec}$.

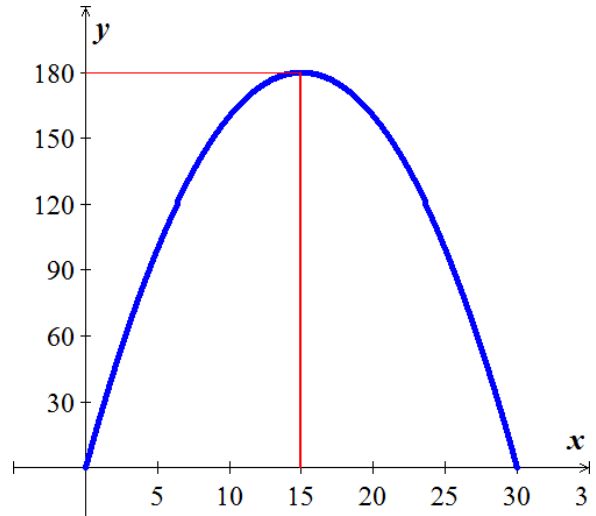
- Find the rock's velocity and acceleration at time t . (The acceleration in this case is the acceleration of gravity on the moon.)
- How long does it take the rock to reach its highest point?
- How high does the rock go?
- How long does it take the rock to reach half its maximum height?
- How long is the rock aloft?

Solution

$$\begin{aligned} a) \quad v(t) &= s' = 24 - 1.6t \text{ m/sec} \\ a(t) &= v' = s'' = -1.6 \text{ m/sec}^2 \end{aligned}$$

$$\begin{aligned} b) \quad v(t) &= 0 = 24 - 1.6t \\ t &= \frac{24}{1.6} \\ &= 15 \text{ sec} \end{aligned}$$

$$\begin{aligned} c) \quad s(15) &= 24(15) - 0.8(15)^2 \\ &= 180 \text{ m} \end{aligned}$$



- d) Since the maximum high is 180 m , then half is 90 m :

$$s(t) = 24t - 0.8t^2 = 90$$

$$-0.8t^2 + 24t - 90 = 0 \Rightarrow t = 4.39 \quad t = 25.61$$

$$2t^2 - 60t + 225 = 0$$

$$\begin{aligned} t_{1,2} &= \frac{60 \pm \sqrt{3,600 - 1,800}}{4} \\ &= 15 \pm \frac{15\sqrt{2}}{2} \end{aligned}$$

$$t_1 = 4.39 \quad t_2 = 25.61$$

It took 4.39 sec going up and 25.6 sec going down.

- e) The rock took 30 sec to reach its highest point.

Exercise

Had Galileo dropped a cannonball from the Tower of Pisa, 179 *ft* above the ground, the ball's height above the ground t *sec* into the fall would have been $s = 179 - 16t^2$.

- a) What would have been the ball's velocity, speed, and acceleration at time t ?
- b) About how long would it have taken the ball to hit the ground?
- c) What would have been the ball's velocity at the moment of impact?

Solution

a) $v = s' = -32t$ |

$$\text{speed} = |v| = 32t \text{ ft/sec}$$

$$a = -32 \text{ ft/sec}^2$$

b) $s = 0 = 179 - 16t^2$

$$16t^2 = 179$$

$$t = \sqrt{\frac{179}{16}}$$

$$= \frac{\sqrt{179}}{4} \text{ sec}$$

$$\approx 3.3 \text{ sec}$$

c) When $t = 3.3 \text{ sec}$

$$\Rightarrow v = -32t$$

$$= -32(3.3)$$

$$= -107 \text{ ft/sec}$$

Exercise

A toy rocket fired straight up into the air has height $s(t) = 160t - 16t^2$ *feet* after t seconds.

- a) What is the rocket's initial velocity (when $t = 0$)?
- b) What is the acceleration when $t = 3$?
- c) At what time will the rocket hit the ground?
- d) At what velocity will the rocket be traveling just as it smashes into the ground?

Solution

a) $v(t) = s'(t) = 160 - 32t$

$$v(0) = 160$$

b) $a(t) = v'(t) = -32$

$$a(t=3) = -32 \text{ ft/sec}^2$$

$$c) \quad s(t) = 160t - 16t^2 = 0$$

The rocket hit the ground at $t = 0$

$$t = \frac{160}{16}$$

$$= 10 \text{ sec}$$

Exercise

A helicopter is rising straight up in the air. Its distance from the ground t seconds after takeoff is

$$s(t) = t^2 + t \text{ feet}$$

- How long will it take for the helicopter to rise 20 feet ?
- Find the velocity and the acceleration of the helicopter when it is 20 feet above the ground.

Solution

$$a) \quad s(t) = t^2 + t = 20$$

$$t^2 + t - 20 = 0$$

$$t = -5, \quad t = 4$$

It will take 10 sec. for the helicopter to rise 20 feet.

$$b) \quad v(t) = s'(t) = 2t + 1$$

$$v(t = 10) = 21 \text{ ft / sec}$$

$$a(t) = v'(t) = 2$$

$$a(t = 10) = 2 \text{ ft}^2 / \text{sec}$$

Exercise

The position of a particle moving on a line is given by $s(t) = 2t^3 - 21t^2 + 60t$, $t \geq 0$, where t is measured in seconds and s in feet.

- What is the velocity after 3 seconds and after 6 seconds?
- When the particle moving in the positive direction?
- Find the total distance traveled by the particle during the first 7 seconds.

Solution

$$a) \quad v(t) = s'(t) = 6t^2 - 42t + 60$$

$$v(t = 3) = 6(9) - 42(3) + 60$$

$$= -12 \text{ ft / sec}$$

$$v(t=6) = 6(36) - 42(6) + 60$$

$$= 24 \text{ ft/sec}$$

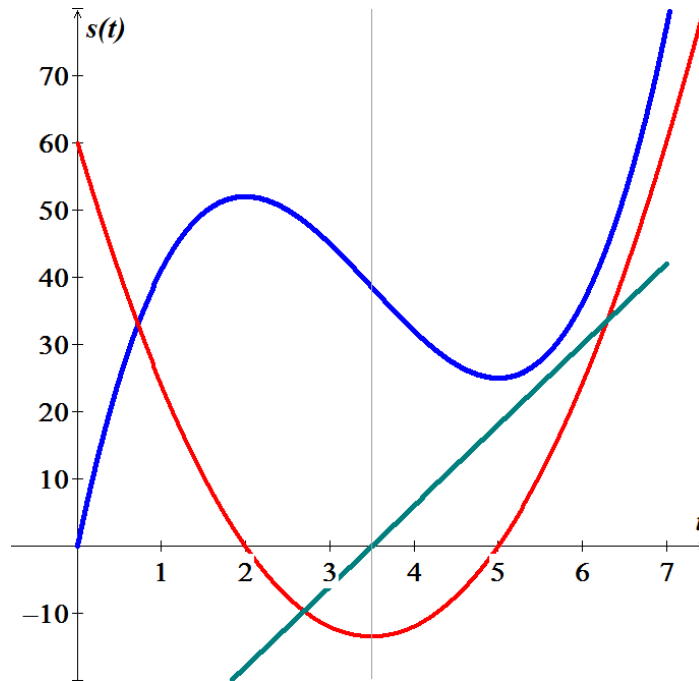
b) $a(t) = v'(t) = 12t - 42 = 0$

$$t = \frac{7}{2} \text{ sec}$$

The particle is moving in the positive direction at 3.5 sec

c) $s(t=7) = 2(7)^3 - 21(7)^2 + 60(7)$

$$= 77 \text{ ft}$$



Exercise

A small probe is launched vertically from the ground. After it reaches its high point, a parachute deploys, and the probe descends to Earth. The height of the probe the ground is

$$s(t) = \frac{300t - 50t^2}{t^3 + 2} \quad \text{for } 0 \leq t \leq 6$$

- Graph the height function and describe the motion of the probe.
- Find the velocity of the probe.
- Graph the velocity function and determine the approximate time at which the velocity is a maximum.

Solution

a) $s'(t) = \frac{(300 - 100t)(t^3 + 2) - 3t^2(300t - 50t^2)}{(t^3 + 2)^2}$

$$= \frac{300t^3 - 100t^4 + 600 - 200t - 900t^3 + 150t^4}{(t^3 + 2)^2}$$

$$= \frac{50t^4 - 600t^3 - 200t + 600}{(t^3 + 2)^2}$$

$$50t^4 - 600t^3 - 200t + 600 = 0$$

$$t^4 - 12t^3 - 4t + 12 = 0$$

$$t = 0.91, \quad \cancel{12.02} > 6 \quad |$$

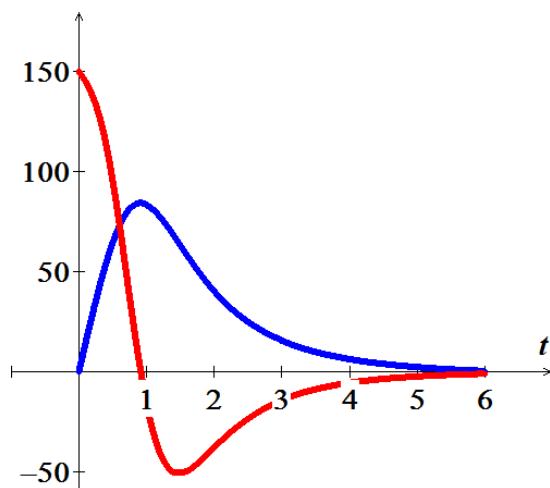
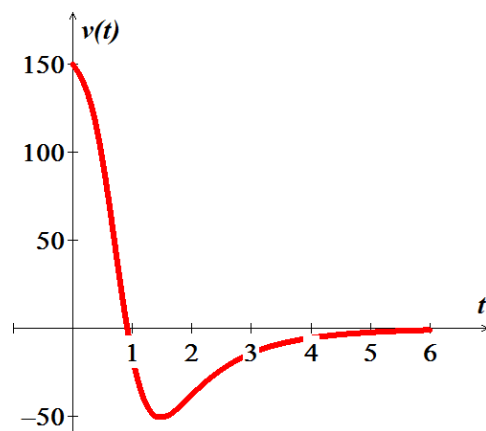
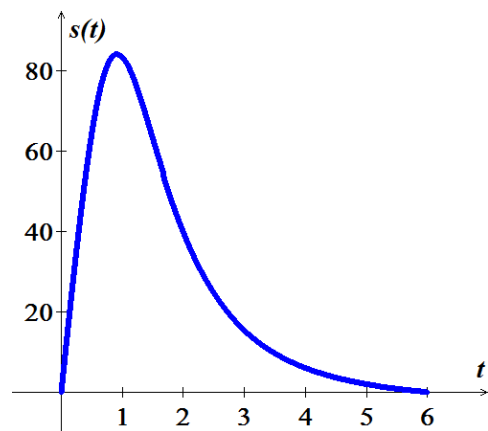
$$s(t = .91) = 84.107 \quad |$$

The maximum height is 84.107 at $t = 0.91$

$$b) \quad v(t) = s'(t) = \frac{50t^4 - 600t^3 - 200t + 600}{(t^3 + 2)^2}$$

$$c) \quad v(t = 0) = \frac{600}{4} \\ = 150 \quad |$$

The maximum velocity is 150



Exercise

Suppose the cost of producing x lawn mowers is $C(x) = -0.02x^2 + 400x + 5000$

- a) Determine the average and marginal costs for $x = 3000$ lawn mowers.
- b) Interpret the meaning of your results in part (a)

Solution

$$\begin{aligned} \text{a) Average Cost} &= \frac{C(3,000)}{3,000} \\ &= \frac{-0.02(9 \times 10^6) + 1,200,000 + 5,000}{3,000} \\ &= \frac{1,025,000}{3,000} \\ &= \underline{\$341.67} \end{aligned}$$

$$\text{Marginal Cost} = C'(x) = -0.04x + 400$$

$$\begin{aligned} C'(3,000) &= -0.04(3,000) + 400 \\ &= \underline{\$280.00} \end{aligned}$$

- b) The average cost of producing 3,000 lawnmowers is \$341.67 per mower.
The cost of producing the 3,001st lawnmower is about \$280.00

Exercise

Suppose a company produces fly rods. Assume $C(x) = -0.0001x^3 + 0.05x^2 + 60x + 800$ represents the cost of making x fly rods.

- a) Determine the average and marginal costs for $x = 400$ fly rods.
- b) Interpret the meaning of your results in part (a)

Solution

$$\begin{aligned} \text{a) Average Cost} &= \frac{C(400)}{400} \\ &= \frac{-0.0001(400)^3 + 0.05(400)^2 + 24,000 + 800}{400} \\ &= \frac{26,400}{400} \\ &= \underline{\$66.00} \end{aligned}$$

$$\text{Marginal Cost} = C'(x) = -0.0003x^2 + 0.1x + 60$$

$$\begin{aligned} C'(400) &= -0.0003(160,000) + 40 + 60 \\ &= \underline{\$52.00} \end{aligned}$$

- c) The average cost of producing 400 fly rods is \$66.00 per fly rod.
The cost of producing the 401st flying rod is about \$52.00

Exercise

Suppose $p(t) = -1.7t^3 + 72t^2 + 7200t + 80,000$ is the population of a city t years after 1950.

- a) Determine the average rate of growth of the city from 1950 to 2000.
- b) What was the rate of growth of the city in 1990?

Solution

From 1950 to 2000 $\rightarrow 0 \leq t \leq 50$

$$\begin{aligned} \text{a) Average growth rate} &= \frac{P(50) - P(0)}{50 - 0} \\ &= \frac{407,500 - 80,000}{50} \\ &= \underline{6,550 \text{ ppl/yr}} \end{aligned}$$

$$\text{b) } p'(t) = -5.1t^2 + 144t + 7200$$

$$\begin{aligned} p'(40) &= -5.1(1,600) + 144(40) + 7200 \\ &= \underline{4,800 \text{ ppl/yr}} \end{aligned}$$

Solution ***Section 2.6 – Chain Rule***

Exercise

Find the derivative of $y = (3x^4 + 1)^4 (x^3 + 4)$

Solution

$$\begin{aligned} y' &= (3x^4 + 1)^3 \left(4(12x^3)(x^3 + 4) + 3x^2(3x^4 + 1) \right) \\ &= (3x^4 + 1)^3 (48x^6 + 192x^3 + 9x^6 + 3x^2) \\ &= \underline{(3x^4 + 1)^3 (57x^6 + 192x^3 + 3x^2)} \end{aligned}$$

$$(U^n V^m)' = U^{n-1} V^{m-1} (nU'V + mUV')$$

OR

$$\begin{aligned} y' &= 4(12x^3)(3x^4 + 1)^3 (x^3 + 4) + 3x^2(3x^4 + 1)^4 \\ &= 48x^3(3x^4 + 1)^3 (x^3 + 4) + 3x^2(3x^4 + 1)^4 \\ &= 3x^2(3x^4 + 1)^3 [16x(x^3 + 4) + 3x^4 + 1] \\ &= 3x^2(3x^4 + 1)^3 (16x^4 + 64x + 3x^4 + 1) \\ &= \underline{3x^2(3x^4 + 1)^3 (19x^4 + 64x + 1)} \end{aligned}$$

$$(uv)' = u'v + uv'$$

Exercise

Find the derivative of $p(t) = \frac{(2t+3)^3}{4t^2-1}$

Solution

$$\begin{aligned} p'(t) &= \frac{(2t+3)^2}{(4t^2-1)^2} (6(4t^2-1) - 8t(2t+3)) \\ &= \underline{\frac{(2t+3)^2 (8t^2 - 24t - 6)}{(4t^2-1)^2}} \end{aligned}$$

$$(U^n V^m)' = U^{n-1} V^{m-1} (nU'V + mUV')$$

OR

$$\begin{aligned}
 P'(t) &= \frac{2(3)(2t+3)^2(4t^2-1) - 8t(2t+3)^3}{(4t^2-1)^2} \\
 &= \frac{(2t+3)^2 [6(4t^2-1) - 8t(2t+3)]}{(4t^2-1)^2} \\
 &= \frac{(2t+3)^2 [24t^2 - 6 - 16t^2 - 24t]}{(4t^2-1)^2} \\
 &= \frac{(2t+3)^2 (8t^2 - 24t - 6)}{(4t^2-1)^2} \\
 &= \frac{2(2t+3)^2 (4t^2 - 12t - 3)}{(4t^2-1)^2} \Bigg|
 \end{aligned}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

Exercise

Find the derivative of $y = (x^3 + 1)^2$

Solution

$$u = x^3 + 1 \rightarrow y = u^2$$

$$\frac{d}{dx} y = \frac{dy}{du} \frac{du}{dx}$$

$$= 2u(3x^2)$$

$$\begin{aligned}
 y' &= 2(x^3 + 1)(3x^2) \\
 &= 6x^2(x^3 + 1) \Bigg|
 \end{aligned}$$

Exercise

Find the derivative of $y = (x^2 + 3x)^4$

Solution

$$u = x^2 + 3x$$

$$\begin{aligned} y' &= n \quad (u)^{n-1} \quad \frac{d}{dx}[u] \\ &= 4(x^2 + 3x)^3 \frac{d}{dx}[x^2 + 3x] \\ &= 4(2x + 3)(x^2 + 3x)^3 \end{aligned}$$

Exercise

Find the derivative of $y = \frac{4}{2x+1}$

Solution

$$y' = \frac{-8}{(2x+1)^2}$$

$$\left(\frac{ax+b}{cx+d} \right)' = \frac{ad-bc}{(cx+d)^2}$$

OR

$$y = 4(2x+1)^{-1}$$

$$\begin{aligned} y' &= -4(2x+1)^{-2} (2) \\ &= -8(2x+1)^{-2} \\ &= -\frac{8}{(2x+1)^2} \end{aligned}$$

Exercise

Find the derivative of $y = \frac{2}{(x-1)^3}$

Solution

$$y' = -\frac{6}{(x-1)^4}$$

$$\left(\frac{1}{U^n} \right)' = -\frac{nU'}{U^{n+1}}$$

Exercise

Find the derivative of $y = x^2 \sqrt{x^2 + 1}$

Solution

$$y' = \frac{x}{\sqrt{x^2+1}} \left(2(x^2+1) + \frac{1}{2}(2x)x \right)$$

$$= \frac{x(3x^2+2)}{\sqrt{x^2+1}} \Bigg|$$

$$\left(U^n V^m \right)' = U^{n-1} V^{m-1} (nUV' + mUV')$$

OR

$$y = x^2 (x^2 + 1)^{1/2}$$

$$y' = x^2 \frac{d}{dx} \left[(x^2 + 1)^{1/2} \right] + (x^2 + 1)^{1/2} \frac{d}{dx} \left[x^2 \right]$$

$$= x^2 \left[\frac{1}{2} (x^2 + 1)^{-1/2} (2x) \right] + (x^2 + 1)^{1/2} [2x]$$

$$= x^3 (x^2 + 1)^{-1/2} + 2x (x^2 + 1)^{1/2}$$

$$= \frac{(x^2+1)^{1/2}}{(x^2+1)^{1/2}} \left[x^3 (x^2 + 1)^{-1/2} + 2x (x^2 + 1)^{1/2} \right]$$

$$= \frac{x^3 (x^2+1)^{-1/2} (x^2+1)^{1/2} + 2x (x^2+1)^{1/2} (x^2+1)^{1/2}}{(x^2+1)^{1/2}}$$

$$= \frac{x^3 + 2x(x^2+1)}{(x^2+1)^{1/2}}$$

$$= \frac{x^3 + 2x^3 + 2x}{\sqrt{x^2+1}}$$

$$= \frac{3x^3 + 2x}{\sqrt{x^2+1}}$$

$$= \frac{x(3x^2+2)}{\sqrt{x^2+1}} \Bigg|$$

Exercise

Find the derivative of $y = \left(\frac{x+1}{x-5}\right)^2$

Solution

$$y' = 2\left(\frac{x+1}{x-5}\right) \frac{-5-1}{(x-5)^2}$$
$$= -\frac{12(x+1)}{(x-5)^3}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

OR

$$y' = 2\left(\frac{x+1}{x-5}\right) \frac{d}{dx} \left[\frac{x+1}{x-5} \right]$$
$$= 2\left(\frac{x+1}{x-5}\right) \left[\frac{(1)(x-5) - (1)(x+1)}{(x-5)^2} \right]$$
$$= 2\left(\frac{x+1}{x-5}\right) \left(\frac{x-5-x-1}{(x-5)^2} \right)$$
$$= 2\left(\frac{x+1}{x-5}\right) \left(\frac{-6}{(x-5)^2} \right)$$
$$= -\frac{12(x+1)}{(x-5)^3}$$

Exercise

Find the derivative of $s(t) = \sqrt{2t^2 + 5t + 2}$

Solution

$$s(t) = (2t^2 + 5t + 2)^{1/2}$$

$$s'(t) = \frac{1}{2}(4t+5)(2t^2 + 5t + 2)^{-1/2}$$
$$= \frac{1}{2} \frac{4t+5}{\sqrt{2t^2 + 5t + 2}}$$

$$U = 2t^2 + 5t + 2 \rightarrow U' = 4t + 5$$

$$(U^n)' = nU'U^{n-1}$$

Exercise

Find the derivative of $f(x) = \frac{1}{(x^2 - 3x)^2}$

Solution

$$\underline{f'(x) = -\frac{2(2x-3)}{(x^2-3x)^3}}$$

$$\left(\frac{1}{U^n}\right)' = -\frac{nU'}{U^{n+1}}$$

Exercise

Find the derivative of $y = t^2 \sqrt{t-2}$

Solution

$$\begin{aligned} y' &= \frac{t}{\sqrt{t-2}} \left(2(t-2) + \frac{1}{2}t \right) \\ &= \frac{1}{2} \frac{5t-4}{\sqrt{t-2}} \end{aligned}$$

$$(U^n V^m)' = U^{n-1} V^{m-1} (nU'V + mUV')$$

OR

$$\begin{aligned} f &= t^2 & f' &= 2t \\ g &= (t-2)^{1/2} & g' &= \frac{1}{2}(t-2)^{-1/2} \end{aligned}$$

$$\begin{aligned} y' &= 2t\sqrt{t-2} + t^2 \frac{1}{2}(t-2)^{-1/2} \\ &= \left[2t(t-2)^{1/2} + t^2 \frac{1}{2}(t-2)^{-1/2} \right] \frac{2(t-2)^{1/2}}{2(t-2)^{1/2}} \\ &= \frac{4t(t-2) + t^2}{2(t-2)^{1/2}} \\ &= \frac{4t^2 - 8t + t^2}{2\sqrt{t-2}} \\ &= \frac{5t^2 - 8t}{2\sqrt{t-2}} \end{aligned}$$

Exercise

Find the derivative of $y = \left(\frac{6-5x}{x^2-1} \right)^2$

Solution

$$f' = 2 \frac{5x^2 - 12x + 5}{(x^2 - 1)^2} \left(\frac{6-5x}{x^2-1} \right) \quad \begin{matrix} 0 & -5 & 6 \\ 1 & 0 & -1 \end{matrix} \quad \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right)' = \frac{(ae - bd)x^2 + 2(af - cd)x + bf - ce}{(dx^2 + ex + f)^2}$$
$$= \frac{2(5x^2 - 12x + 5)(6 - 5x)}{(x^2 - 1)^3}$$

OR

$$f = 6 - 5x \quad f' = -5$$

$$g = x^2 - 1 \quad g' = 2x$$

$$y' = 2 \frac{-5(x^2 - 1) - (2x)(6 - 5x)}{(x^2 - 1)^2} \left(\frac{6-5x}{x^2-1} \right) \quad (U^n)' = nU'U^{n-1}$$
$$= 2 \frac{-5x^2 + 5 - 12x + 10x^2}{(x^2 - 1)^3} (6 - 5x)$$
$$= \frac{2(5x^2 - 12x + 5)(6 - 5x)}{(x^2 - 1)^3}$$

Exercise

Find the derivative of $y = 4x(3x + 5)^5$

Solution

$$y' = 4(3x + 5)^5 + 5(3)(3x + 5)^4(4x)$$
$$= 4(3x + 5)^5 + 60x(3x + 5)^4$$
$$= 4(3x + 5)^4(3x + 5 + 15x)$$
$$= 4(3x + 5)^4(18x + 5)$$

Exercise

Find the derivative of $y = (3x^2 - 5x)^{1/2}$

Solution

$$u = 3x^2 - 5x \quad \& \quad y = u^{1/2}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{2} u^{-1/2} (6x - 5) \\ &= \frac{1}{2} (6x - 5) (3x^2 - 5x)^{-1/2} \\ &= \frac{6x - 5}{2(3x^2 - 5x)^{1/2}} \end{aligned}$$

Exercise

Find the derivative of $D_x (x^2 + 5x)^8$

Solution

$$\begin{aligned} D_x (x^2 + 5x)^8 &= 8(x^2 + 5x)^7 (x^2 + 5x)' \\ &= 8(x^2 + 5x)^7 (2x + 5) \\ &= 8(2x + 5)(x^2 + 5x)^7 \end{aligned}$$

Exercise

Find the derivative of $y = \frac{(3x + 2)^7}{x - 1}$

Solution

$$\begin{aligned} y' &= \frac{7(3)(3x + 2)^6 (x - 1) - (1)(3x + 2)^7}{(x - 1)^2} \\ &= \frac{(3x + 2)^6 (21x - 21 - 3x - 2)}{(x - 1)^2} \\ &= \frac{(3x + 2)^6 (18x - 23)}{(x - 1)^2} \end{aligned}$$

Exercise

Find the derivative of $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$

Solution

$$\begin{aligned} y' &= 4 \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^3 \left(\frac{2x}{8} + 1 - \frac{-1}{x^2}\right) \\ &= 4 \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^3 \left(\frac{x}{4} + 1 + \frac{1}{x^2}\right) \\ &= \left(x + 4 + \frac{4}{x^2}\right) \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^3 \end{aligned}$$

Exercise

Find the derivative of $y = \sqrt{3x^2 - 4x + 6}$

Solution

$$\begin{aligned} y &= \left(3x^2 - 4x + 6\right)^{1/2} = u^{1/2} & u &= 3x^2 - 4x + 6 \Rightarrow u' = 6x - 4 \\ y' &= \frac{1}{2} u^{1/2} u' \\ &= \frac{1}{2} \left(3x^2 - 4x + 6\right)^{-1/2} 2(3x - 4) \\ &= \frac{3x - 4}{\sqrt{3x^2 - 4x + 6}} \end{aligned}$$

Exercise

Find the derivative of $y = \cot\left(\pi - \frac{1}{x}\right)$

Solution

$$\begin{aligned} u &= \pi - \frac{1}{x} \rightarrow u' = \frac{1}{x^2} \\ y' &= -\csc^2\left(\pi - \frac{1}{x}\right) \left(\frac{1}{x^2}\right) \\ &= -\frac{1}{x^2} \csc^2\left(\pi - \frac{1}{x}\right) \end{aligned}$$

Exercise

Find the derivative of $y = 5 \cos^{-4} x$

Solution

$$y = 5 \cos^{-4} x \quad u = \cos x \rightarrow u' = -\sin x$$

$$\begin{aligned} y' &= 5u^{-5}u' \\ &= 5(-4)\cos^{-5} x(-\sin x) \\ &= \underline{20 \sin x \cos^{-5} x} \end{aligned}$$

Exercise

Find the derivative of $y = \sin\left(\frac{3\pi t}{2}\right) + \cos\left(\frac{3\pi t}{2}\right)$

Solution

$$\begin{aligned} y' &= \frac{3\pi}{2} \cos\left(\frac{3\pi t}{2}\right) + \frac{3\pi}{2} \left(-\cos\left(\frac{3\pi t}{2}\right)\right) \\ &= \frac{3\pi}{2} \cos\left(\frac{3\pi t}{2}\right) - \frac{3\pi}{2} \cos\left(\frac{3\pi t}{2}\right) \\ &= \underline{\frac{3\pi}{2} \left(\cos\left(\frac{3\pi t}{2}\right) - \cos\left(\frac{3\pi t}{2}\right)\right)} \end{aligned}$$

Exercise

Find the derivative of $r = 6(\sec \theta - \tan \theta)^{3/2}$

Solution

$$\begin{aligned} r &= 6(\sec \theta - \tan \theta)^{3/2} = 6u^{3/2} \Rightarrow u = \sec \theta - \tan \theta \rightarrow u' = \sec \theta \tan \theta - \sec^2 \theta \\ &\Rightarrow u = \sec \theta - \tan \theta \rightarrow u' = \sec \theta \tan \theta - \sec^2 \theta \end{aligned}$$

$$\begin{aligned} r' &= 6\left(\frac{3}{2}\right)(\sec \theta - \tan \theta)^{3/2-1}(\sec \theta \tan \theta - \sec^2 \theta) \\ &= 9(\sec \theta - \tan \theta)^{1/2}(\sec \theta \tan \theta - \sec^2 \theta) \\ &= \underline{9(\sec \theta \tan \theta - \sec^2 \theta)\sqrt{\sec \theta - \tan \theta}} \end{aligned}$$

Exercise

Find the derivative of $g(x) = \frac{\tan 3x}{(x+7)^4}$

Solution

$$\begin{aligned} g'(x) &= \frac{(3\sec^2 3x)(x+7)^4 - 4(x+7)^3 \tan 3x}{(x+7)^8} & \left(\frac{u}{v}\right)' &= \frac{u'v - v'u}{v^2} & u &= \tan 3x & v &= (x+7)^4 \\ &= \frac{(x+7)^3 [3(x+7)\sec^2 3x - 4\tan 3x]}{(x+7)^8} & u' &= 3\sec^2 3x & v' &= 4(x+7)^3 \\ &= \frac{3(x+7)\sec^2 3x - 4\tan 3x}{(x+7)^5} \end{aligned}$$

Exercise

Find the derivative of $f(\theta) = \left(\frac{\sin \theta}{1 + \cos \theta}\right)^2$

Solution

$$\begin{aligned} f'(\theta) &= 2\left(\frac{\sin \theta}{1 + \cos \theta}\right)\left(\frac{\sin \theta}{1 + \cos \theta}\right)' \\ &= \frac{2\sin \theta}{1 + \cos \theta} \left(\frac{\cos \theta(1 + \cos \theta) - (-\sin \theta)\sin \theta}{(1 + \cos \theta)^2} \right) \\ &= \frac{2\sin \theta}{1 + \cos \theta} \left(\frac{\cos \theta + \cos^2 \theta + \sin^2 \theta}{(1 + \cos \theta)^2} \right) \\ &= \frac{2\sin \theta}{1 + \cos \theta} \left(\frac{\cos \theta + 1}{(1 + \cos \theta)^2} \right) \\ &= \frac{2\sin \theta}{(1 + \cos \theta)^2} \end{aligned}$$

Exercise

Find the derivative of $y = \sin^2(\pi t - 2)$

Solution

$$y' = 2\sin(\pi t - 2)(\sin(\pi t - 2))'$$

$$\begin{aligned}
 &= 2 \sin(\pi t - 2) (\pi \cos(\pi t - 2)) \\
 &= \underline{2\pi \sin(\pi t - 2) \cos(\pi t - 2)}
 \end{aligned}$$

Exercise

Find the derivative of $y = (t \tan t)^{10}$

Solution

$$\begin{aligned}
 y' &= 10(t \tan t)^9 (t \tan t)' \\
 &= 10(t \tan t)^9 (\tan t + t \sec^2 t) \\
 &= 10(t \tan t)^9 \tan t + 10t(t \tan t)^9 \sec^2 t \\
 &= \underline{10t^9 \tan^{10} t + 10t^{10} \tan^9 t \sec^2 t}
 \end{aligned}$$

Exercise

Find the derivative of $y = \cos\left(5 \sin\left(\frac{t}{3}\right)\right)$

Solution

$$\begin{aligned}
 y' &= -\sin\left(5 \sin\left(\frac{t}{3}\right)\right) \left(5 \sin\left(\frac{t}{3}\right)\right)' \\
 &= -\sin\left(5 \sin\left(\frac{t}{3}\right)\right) \left(5 \frac{1}{3} \cos\left(\frac{t}{3}\right)\right) \\
 &= \underline{-\frac{5}{3} \sin\left(5 \sin\left(\frac{t}{3}\right)\right) \cos\left(\frac{t}{3}\right)}
 \end{aligned}$$

Exercise

Find the derivative of $y = 4 \sin\left(\sqrt{1 + \sqrt{t}}\right)$

Solution

$$\begin{aligned}
 y' &= 4 \cos\left(\sqrt{1 + \sqrt{t}}\right) \left(\sqrt{1 + \sqrt{t}}\right)' \\
 &\quad \left(\left(1 + \sqrt{t}\right)^{1/2}\right)' = \frac{1}{2} \left(1 + \sqrt{t}\right)^{-1/2} \left(t^{1/2}\right)' \\
 &= \frac{1}{2} \left(1 + \sqrt{t}\right)^{-1/2} \left(\frac{1}{2} t^{-1/2}\right) \\
 &= \frac{1}{4} \frac{1}{\sqrt{t} \sqrt{1 + \sqrt{t}}}
 \end{aligned}$$

$$= \frac{1}{4} \frac{1}{\sqrt{t(1+\sqrt{t})}}$$

$$y' = 4 \cos(\sqrt{1+\sqrt{t}}) \left(\frac{1}{4} \frac{1}{\sqrt{t+t\sqrt{t}}} \right)$$

$$= \frac{\cos(\sqrt{1+\sqrt{t}})}{\sqrt{t+t\sqrt{t}}}$$

Exercise

Find the derivative of $y = \tan^2(\sin^3 x)$

Solution

$$u = \sin^3 x \Rightarrow u' = 3 \sin^2 x (\sin x)' = 3 \sin^2 x (\cos x)$$

$$\begin{aligned} y' &= 2 \tan(\sin^3 x) \cdot (\tan(\sin^3 x))' \\ &= 2 \tan(\sin^3 x) \cdot \sec^2(\sin^3 x) \cdot (\sin^3 x)' \\ &= 2 \tan(\sin^3 x) \cdot \sec^2(\sin^3 x) \cdot (3 \sin^2 x \cos x) \\ &= 6 \cos x \sin^2 x \cdot \tan(\sin^3 x) \cdot \sec^2(\sin^3 x) \end{aligned}$$

Exercise

Find the derivative of $f(x) = \left((x^2 + 3)^5 + x \right)^2$

Solution

$$f'(x) = 2 \left((x^2 + 3)^5 + x \right) \left(10x(x^2 + 3)^4 + 1 \right)$$

Exercise

Find the derivative of $y = \left(\frac{3x-1}{x^2+3} \right)^2$

Solution

$$y = (3x-1)^2 (x^2+3)^{-2}$$

$$(U^m V^n)' = U^{m-1} V^{n-1} (mU'V + nUV')$$

$$\begin{aligned}
 y' &= (3x-1)(x^2+3)^{-3} \left(6x(x^2+3) - 4x(3x-1) \right) \\
 &= \frac{3x-1}{(x^2+3)^3} (6x^3 + 18x - 12x^2 + 4x) \\
 &= \frac{(3x-1)(6x^3 - 12x^2 + 22x)}{(x^2+3)^3}
 \end{aligned}$$

Exercise

Find the derivative of $y = \cos \sqrt{\sin(\tan \pi x)}$

Solution

$$\begin{aligned}
 y' &= -\left(\sin \sqrt{\sin(\tan \pi x)} \right) \left(\frac{1}{2} \frac{\pi \cos(\tan \pi x) \sec^2 \pi x}{\sqrt{\sin(\tan \pi x)}} \right) \\
 &= -\frac{\pi \sec^2 \pi x \cos(\tan \pi x) \sin \sqrt{\sin(\tan \pi x)}}{2 \sqrt{\sin(\tan \pi x)}}
 \end{aligned}$$

Exercise

Find the derivative of $f(x) = \frac{x}{\sqrt{x^2+1}}$

Solution

$$f(x) = x(x^2+1)^{-1/2}$$

$$\begin{aligned}
 f'(x) &= \frac{x^2+1 - \frac{1}{2}(2x^2)}{x^2+1} \\
 &= \frac{1}{x^2+1}
 \end{aligned}$$

$$(U^m V^n)' = U^{m-1} V^{n-1} (mU'V + nUV')$$

Exercise

Find the derivative of $y = \cos(1-2x)^2$

Solution

$$\begin{aligned}
 y' &= -(2(-2)(1-2x)) \sin(1-2x)^2 \\
 &= 4(1-2x) \sin(1-2x)^2
 \end{aligned}$$

Exercise

Find the derivative of $f(x) = (4x - 3)^2$

Solution

$$\underline{f'(x) = 8(4x - 3)}$$

$$(U^n)' = nU' U^{n-1}$$

Exercise

Find the derivative of $f(x) = \frac{x}{\sqrt[3]{x^2 + 4}}$

Solution

$$f(x) = x(x^2 + 4)^{-1/3}$$

$$\begin{aligned} f'(x) &= (x^2 + 4)^{-4/3} \left(x^2 + 4 - \frac{1}{3}(2x^2) \right) \\ &= \frac{1}{3} \frac{x^2 + 12}{(x^2 + 4)^{4/3}} \end{aligned}$$

$$(U^m V^n)' = U^{m-1} V^{n-1} (mU'V + nUV')$$

Exercise

Find the derivative of $f(x) = \left(\frac{x^2}{x^3 + 2} \right)^2$

Solution

$$f(x) = x^4 (x^3 + 2)^{-2}$$

$$\begin{aligned} f'(x) &= x^3 (x^3 + 2)^{-3} (4x^3 + 8 - 2x^3) \\ &= \frac{x^3 (2x^3 + 8)}{(x^3 + 2)^3} \end{aligned}$$

$$(U^m V^n)' = U^{m-1} V^{n-1} (mU'V + nUV')$$

Exercise

Find the derivative of $y = \sin \sqrt[3]{x} + \sqrt[3]{\sin x}$

Solution

$$\underline{y' = \frac{1}{3} x^{-2/3} \cos \sqrt[3]{x} + \frac{1}{3} \cos x (\sin x)^{-2/3}}$$

Exercise

Find the derivative of $f(\theta) = 4 \tan(\theta^2 + 3\theta + 2)$

Solution

$$\underline{f'(\theta) = 4(2\theta + 3) \sec^2(\theta^2 + 3\theta + 2)}$$

Exercise

Find the derivative of $f(\theta) = \tan(\sin \theta)$

Solution

$$\underline{f'(\theta) = \cos \theta \sec^2(\sin \theta)}$$

Exercise

Find the derivative of $y = 5x + \sin^3 x + \sin x^3$

Solution

$$\underline{y' = 5 + 3 \cos x \sin^2 x + 3x^2 \cos x^3}$$

Exercise

Find the derivative of $y = \csc^5 3x$

Solution

$$\begin{aligned} y' &= 15 \csc^4 3x (-\csc 3x \cot 3x) \\ &= \underline{-15 \cot 3x \csc^5 3x} \end{aligned}$$

Exercise

Find the derivative of $y = 2x\sqrt{x^2 - 2x + 2}$

Solution

$$\begin{aligned} y' &= 2\sqrt{x^2 - 2x + 2} + 2x(2x - 2)(x^2 - 2x + 2)^{-1/2} \\ &= \underline{2\sqrt{x^2 - 2x + 2} + \frac{4x^2 - 4x}{\sqrt{x^2 - 2x + 2}}} \end{aligned}$$

Exercise

Find the derivative of $\frac{d}{du} \left(\frac{4u^2 + u}{8u + 1} \right)^3$

Solution

$$(U^n)' = nU' U^{n-1} \quad \frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

$$\begin{aligned} \frac{d}{du} \left(\frac{4u^2 + u}{8u + 1} \right)^3 &= 3 \left(\frac{4u^2 + u}{8u + 1} \right)^2 \frac{\begin{vmatrix} 4 & 1 \\ 0 & 8 \end{vmatrix} u^2 + \begin{vmatrix} 4 & 0 \\ 0 & 1 \end{vmatrix} u + \begin{vmatrix} 1 & 0 \\ 8 & 1 \end{vmatrix}}{(8u + 1)^2} \\ &= 3 \left(32u^2 + 4u + 1 \right) \frac{(4u^2 + u)^2}{(8u + 1)^4} \end{aligned}$$

Exercise

Find the derivative of $y = \frac{1}{2}x^2 \sqrt{16 - x^2}$

Solution

$$y = \frac{1}{2}x^2 (16 - x^2)^{1/2}$$

$$y' = \frac{1}{2}x (16 - x^2)^{-1/2} (32 - 2x^2 - x^2)$$

$$= \frac{1}{2} \frac{32x - 3x^3}{\sqrt{16 - x^2}}$$

$$(U^m V^n)' = U^{m-1} V^{n-1} (mU'V + nUV')$$

Exercise

Find the derivative of $y = \left(\frac{x-3}{2x+5} \right)^4$

Solution

$$y' = 4 \frac{5+6}{(2x+5)^2} \left(\frac{x-3}{2x+5} \right)^3$$

$$= \frac{44(x-3)^3}{(2x+5)^5}$$

$$(U^n)' = nU' U^{n-1} \quad \left(\frac{ax+b}{cx+d} \right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \left(\frac{5x-3}{2x+5}\right)^5$

Solution

$$y' = 5 \frac{25+6}{(2x+5)^2} \left(\frac{5x-3}{2x+5}\right)^4 \quad \left(U^n\right)' = nU' U^{n-1} \quad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$
$$= \frac{155(5x-3)^4}{(2x+5)^6}$$

Exercise

Find the derivative of $y = \left(\frac{6x-8}{2x-3}\right)^6$

Solution

$$y' = 6 \frac{-18+16}{(2x-3)^2} \left(\frac{6x-8}{2x-3}\right)^5 \quad \left(U^n\right)' = nU' U^{n-1} \quad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$
$$= -\frac{12(6x-8)^5}{(2x-3)^7}$$

Exercise

Find the derivative of $y = \left(\frac{3x^2-4}{2x^2-1}\right)^3$

Solution

$$y' = 3 \frac{2(-3+8)x}{(2x^2-1)^2} \left(\frac{3x^2-4}{2x^2-1}\right)^2 \quad \left(U^n\right)' = nU' U^{n-1} \quad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{(cx^n+d)^2}$$
$$= \frac{30x(3x^2-4)^2}{(2x^2-1)^4}$$

Exercise

Find the derivative of $y = \left(\frac{3x^2+4}{2x^2+1}\right)^{-3}$

Solution

$$\begin{aligned}
 y' &= (-3) \frac{2(3-8)x}{(2x^2+1)^2} \left(\frac{3x^2+4}{2x^2+1} \right)^{-4} & (U^n)' &= nU' U^{n-1} \quad \left(\frac{ax^n+b}{cx^n+d} \right)' = \frac{n(ad-bc)x^{n-1}}{(cx^n+d)^2} \\
 &= \frac{15x}{(2x^2+1)^2} \left(\frac{2x^2+1}{3x^2+4} \right)^4 \\
 &= \frac{15x(2x^2+1)^2}{(3x^2+4)^4} \quad \Bigg|
 \end{aligned}$$

Exercise

Find the derivative of $y = \left(\frac{2x^2-3}{x^2+1} \right)^{1/3}$

Solution

$$\begin{aligned}
 y' &= \frac{1}{3} \frac{2(2+3)x}{(x^2+1)^2} \left(\frac{2x^2-3}{x^2+1} \right)^{-2/3} & (U^n)' &= nU' U^{n-1} \quad \left(\frac{ax^n+b}{cx^n+d} \right)' = \frac{n(ad-bc)x^{n-1}}{(cx^n+d)^2} \\
 &= \frac{10}{3} \frac{x}{(x^2+1)^2} \left(\frac{x^2+1}{2x^2-3} \right)^{2/3} \\
 &= \frac{10}{3} \frac{x}{(x^2+1)^{4/3} (2x^2-3)^{2/3}} \quad \Bigg|
 \end{aligned}$$

Exercise

Find the derivative of $y = \sqrt{\frac{2x^3-3}{2x^3+1}}$

Solution

$$\begin{aligned}
 y' &= \frac{1}{2} \frac{3(2+6)x^2}{(x^3+1)^2} \left(\frac{2x^3-3}{x^3+1} \right)^{-1/2} & (U^n)' &= nU' U^{n-1} \quad \left(\frac{ax^n+b}{cx^n+d} \right)' = \frac{n(ad-bc)x^{n-1}}{(cx^n+d)^2} \\
 &= \frac{12x^2}{(x^3+1)^2} \left(\frac{x^3+1}{2x^3-3} \right)^{1/2} \\
 &= \frac{12x^2}{(x^3+1)^{3/2} \sqrt{2x^3-3}} \quad \Bigg|
 \end{aligned}$$

Exercise

Find the derivative of $y = \left(\frac{2x^4 - 3}{2x^4 + 1} \right)^5$

Solution

$$y' = 5 \frac{4(2+6)x^3}{(2x^4 + 1)^2} \left(\frac{2x^4 - 3}{2x^4 + 1} \right)^4$$

$$(U^n)' = nU' U^{n-1} \quad \left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

$$= \frac{160x^3(2x^4 - 3)^4}{(2x^4 + 1)^6}$$

Exercise

Find the derivative of $y = \left(\frac{x^2 - 4x + 1}{5x^2 - 2x - 1} \right)^3$

Solution

$$(U^n)' = nU' U^{n-1} \quad \frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

$$y' = (3) \frac{\begin{vmatrix} 1 & -4 \\ 5 & -2 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} x + \begin{vmatrix} -4 & 1 \\ -2 & -1 \end{vmatrix}}{(5x^2 - 2x - 1)^2} \left(\frac{x^2 - 4x + 1}{5x^2 - 2x - 1} \right)^2$$

$$= \frac{(18x^2 - 12x + 6)(x^2 - 4x + 1)^2}{(5x^2 - 2x - 1)^4}$$

Exercise

Find the derivative of $y = \left(\frac{3x^2 - 4x + 2}{2x^2 + x - 1} \right)^{2/3}$

Solution

$$(U^n)' = nU' U^{n-1} \quad \frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

$$\begin{aligned}
 y' &= \frac{\frac{2}{3} \begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} x^2 + 2 \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} x + \begin{vmatrix} -4 & 2 \\ 1 & -1 \end{vmatrix}}{\left(2x^2 + x - 1\right)^2} \left(\frac{3x^2 - 4x + 2}{2x^2 + x - 1}\right)^{-1/3} \\
 &= \frac{2}{3} \frac{11x^2 - 14x + 6}{\left(2x^2 + x - 1\right)^2} \left(\frac{2x^2 + x - 1}{3x^2 - 4x + 2}\right)^{1/3} \\
 &= \frac{2}{3} \frac{11x^2 - 14x + 6}{\left(2x^2 + x - 1\right)^{5/3} \left(3x^2 - 4x + 2\right)^{1/3}}
 \end{aligned}$$

Exercise

Find the derivative of $f(x) = \left(\frac{3t^2 - 1}{3t^2 + 1}\right)^{-3}$

Solution

$$\begin{aligned}
 f(x) &= \left(\frac{3t^2 + 1}{3t^2 - 1}\right)^3 \\
 f'(x) &= 3 \frac{3(-3-3)t}{\left(3t^2 - 1\right)^2} \left(\frac{3t^2 + 1}{3t^2 - 1}\right)^2 \\
 &= -\frac{6t(3t^2 + 1)^2}{\left(3t^2 - 1\right)^4}
 \end{aligned}
 \qquad
 \begin{aligned}
 \left(U^n\right)' &= nU' U^{n-1} \quad \left(\frac{ax^n + b}{cx^n + d}\right)' = \frac{n(ad - bc)x^{n-1}}{\left(cx^n + d\right)^2}
 \end{aligned}$$

Exercise

Find the derivative of $f(x) = \left(\frac{x}{3x^2 + 2x + 1}\right)^{1/3}$

Solution

$$\begin{aligned}
 \left(U^n\right)' &= nU' U^{n-1} \quad \frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f}\right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{\left(dx^2 + ex + f\right)^2} \\
 f'(x) &= \frac{1}{3} \frac{\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} x^2 + 2 \begin{vmatrix} 0 & 0 \\ 3 & 1 \end{vmatrix} x + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}}{\left(3x^2 + 2x + 1\right)^2} \left(\frac{x}{3x^2 + 2x + 1}\right)^{-2/3}
 \end{aligned}$$

$$= \frac{1}{3} \frac{-3x^2 + 1}{(3x^2 + 2x + 1)^2} \left(\frac{3x^2 + 2x + 1}{x} \right)^{2/3}$$

$$= \frac{-3x^2 + 1}{3x^{2/3} (3x^2 + 2x + 1)^{4/3}} \Bigg|$$

Exercise

Find the derivative of $f(x) = (x^2 + 2x - 3)^5 (2x + 3)^6$

Solution

$$(U^m V^n)' = U^{m-1} V^{n-1} (mU'V + nUV')$$

$$f'(x) = (x^2 + 2x - 3)^4 (2x + 3)^5 \left[5(2x + 2)(2x + 3) + 12(x^2 + 2x - 3) \right]$$

$$= (x^2 + 2x - 3)^4 (2x + 3)^5 (20x^2 + 50x + 30 + 12x^2 + 24x - 36)$$

$$= (x^2 + 2x - 3)^4 (2x + 3)^5 (32x^2 + 74x - 6) \Bigg|$$

Exercise

Find the derivative of $f(x) = (2x^2 - 4x + 3)^4 (3x - 5)^5$

Solution

$$(U^m V^n)' = U^{m-1} V^{n-1} (mU'V + nUV')$$

$$f'(x) = (2x^2 - 4x + 3)^3 (3x - 5)^4 \left[4(4x - 4)(3x - 5) + 15(2x^2 - 4x + 3) \right]$$

$$= (2x^2 - 4x + 3)^3 (3x - 5)^4 (48x^2 - 128x + 80 + 30x^2 - 60x + 45)$$

$$= (2x^2 - 4x + 3)^3 (3x - 5)^4 (88x^2 - 188x + 135) \Bigg|$$

Exercise

Find the derivative of $f(x) = (x^2 + 2x - 3)^4 (x^2 + 3x + 5)^6$

Solution

$$(U^m V^n)' = U^{m-1} V^{n-1} (mU'V + nUV')$$

$$\begin{aligned} f'(x) &= (x^2 + 2x - 3)^3 (x^2 + 3x + 5)^5 \left[4(2x + 2)(x^2 + 3x + 5) + 6(2x + 3)(x^2 + 2x - 3) \right] \\ &= (x^2 + 2x - 3)^3 (x^2 + 3x + 5)^5 (8x^3 + 32x^2 + 64x + 40 + 12x^3 + 42x^2 - 54) \\ &= \underline{(x^2 + 2x - 3)^3 (x^2 + 3x + 5)^5 (20x^3 + 74x^2 + 64x - 14)} \end{aligned}$$

Exercise

Find the derivative of $f(x) = (2x^3 - 5x)^3 (x^2 + 2x + 1)^4 (2x - 3)^5$

Solution

$$\begin{aligned} (U^m V^n W^p)' &= U^{m-1} V^{n-1} W^{p-1} (mU'VW + nUV'W + pUVW') \\ f'(x) &= (2x^3 - 5x)^2 (x^2 + 2x + 1)^3 (2x - 3)^4 \left[3(6x^2 - 5)(x^2 + 2x + 1)(2x - 3) \right. \\ &\quad \left. + 4(2x + 2)(2x^3 - 5x)(2x - 3) + 5(2)(2x^3 - 5x)(x^2 + 2x + 1) \right] \\ &= (2x^3 - 5x)^2 (x^2 + 2x + 1)^3 (2x - 3)^4 \left[(18x^2 - 15)(2x^3 + x^2 - 4x - 3) \right. \\ &\quad \left. + (8x + 8)(4x^4 - 6x^3 - 10x^2 + 15x) + (20x^5 + 40x^4 - 20x^3 - 100x^2 - 50x) \right] \\ &= (2x^3 - 5x)^2 (x^2 + 2x + 1)^3 (2x - 3)^4 \\ &\quad \begin{array}{rcccccc} x^5 & x^4 & x^3 & x^2 & x & x^0 \\ 36 & 18 & -72 & -54 & -60 & 45 \\ 32 & -48 & -30 & -15 & 120 & \\ 20 & 32 & -80 & 120 & 50 & \\ & 40 & -48 & -80 & & \\ & & -20 & -100 & & \end{array} \\ &= \underline{(2x^3 - 5x)^2 (x^2 + 2x + 1)^3 (2x - 3)^4 (88x^5 + 42x^4 - 250x^3 - 129x^2 + 110x + 45)} \end{aligned}$$

Exercise

Find the derivative of $f(x) = (x^4 + 3x)^4 (x^3 + 2x)^5 (2x - 3)^6$

Solution

$$(U^m V^n W^p)' = U^{m-1} V^{n-1} W^{p-1} (mU'VW + nUV'W + pUVW')$$

$$f'(x) = (x^4 + 3x)^3 (x^3 + 2x)^4 (2x - 3)^5 \left[4(4x^3 + 3)(x^3 + 2x)(2x - 3) + 5(x^4 + 3x)(3x^2 + 2)(2x - 3) + 12(x^4 + 3x)(x^3 + 2x) \right]$$

$$f'(x) = (x^4 + 3x)^3 (x^3 + 2x)^4 (2x - 3)^5 \left[(16x^3 + 9)(8x^4 - 9x^3 + 16x^2 - 18x) + (5x^4 + 15x)(6x^3 - 9x^2 + 4x - 6) + (12x^4 + 36x)(x^3 + 2x) \right]$$

x^7	128 + 30 + 12
x^6	-144 - 45
x^5	256 + 20 + 24
x^4	-288 + 72 - 30 + 90 + 36
x^3	-81 - 135
x^2	144 + 60 + 72
x^1	-162 - 90

$$f'(x) = (x^4 + 3x)^3 (x^3 + 2x)^4 (2x - 3)^5 (170x^7 - 189x^6 + 300x^5 - 120x^4 - 216x^3 + 206x^2 - 252x)$$

Exercise

Find the derivative of $f(x) = \frac{(x^2 - 6x)^5}{(3x^2 + 5x - 2)^4}$

Solution

$$f(x) = (x^2 - 6x)^5 (3x^2 + 5x - 2)^{-4} \quad (U^m V^n)' = U^{m-1} V^{n-1} (mU'V + nUV')$$

$$f'(x) = (x^2 - 6x)^4 (3x^2 + 5x - 2)^{-5} \left[5(2x - 6)(3x^2 + 5x - 2) - 4(x^2 - 6x)(6x + 5) \right]$$

$$= (x^2 - 6x)^4 (3x^2 + 5x - 2)^{-5} \left[(10x - 30)(3x^2 + 5x - 2) - 4(6x^3 - 31x^2 - 30x) \right]$$

$$\begin{aligned}
&= (x^2 - 6x)^4 (3x^2 + 5x - 2)^{-5} \\
&\quad \begin{array}{r} x^3 \quad 30 - 24 \\ x^2 \quad 50 - 90 + 124 \\ x \quad -20 - 150 + 120 \\ x^0 \quad 60 \end{array} \\
&= \frac{(x^2 - 6x)^4 (6x^3 + 84x^2 - 50x + 60)}{(3x^2 + 5x - 2)^5}
\end{aligned}$$

Exercise

Find the derivative of $f(x) = \frac{(2x^2 + 3x + 1)^4}{(x^2 + 5x - 6)^5}$

Solution

$$f(x) = (2x^2 + 3x + 1)^4 (x^2 + 5x - 6)^{-5} \quad \left(U^m V^n \right)' = U^{m-1} V^{n-1} (mU'V + nUV')$$

$$f'(x) = (2x^2 + 3x + 1)^3 (x^2 + 5x - 6)^{-6} \left[4(4x + 3)(x^2 + 5x - 6) - 5(2x^2 + 3x + 1)(2x + 5) \right].$$

$$= \frac{(2x^2 + 3x + 1)^3}{(x^2 + 5x - 6)^6} \left[(16x + 12)(x^2 + 5x - 6) - (2x^2 + 3x + 1)(10x + 25) \right]$$

$$\begin{array}{r} x^3 \quad 16 - 20 \\ x^2 \quad 80 + 12 - 50 - 30 \\ x \quad -96 + 60 - 75 \\ x^0 \quad -7 - 25 \end{array}$$

$$f'(x) = \frac{(2x^2 + 3x + 1)^3}{(x^2 + 5x - 6)^6} (-4x^3 + 12x^2 - 11x - 32)$$

Exercise

Find the derivative of $f(x) = \frac{(x^3 - 3x)^3 (x^2 + 4x)^4}{(x^2 + 4x + 1)^2}$

Solution

$$(U^m V^n W^p)' = U^{m-1} V^{n-1} W^{p-1} (mU'VW + nUV'W + pUVW')$$

$$f'(x) = \frac{(x^3 - 3x)^2 (x^2 + 4x)^3}{(x^2 + 4x + 1)^3} \left[\begin{aligned} &3(3x^2 - 3)(x^2 + 4x)(x^2 + 4x + 1) \\ &+ 3(x^3 - 3x)(2x + 4)(x^2 + 4x + 1) - 2(2x + 4)(x^3 - 3x)(x^2 + 4x) \end{aligned} \right]$$

$$= \frac{(x^3 - 3x)^2 (x^2 + 4x)^3}{(x^2 + 4x + 1)^3} \left[\begin{aligned} &(9x^2 - 9)(x^4 + 8x^3 + 17x^2 + 4x) \\ &+ (3x^3 - 9x)(2x^3 + 12x^2 + 18x + 4) \\ &- (4x + 8)(x^5 + 4x^4 - 3x^3 - 12x^2) \end{aligned} \right]$$

$$\begin{array}{ll} x^6 & 9 + 6 - 4 \\ x^5 & 72 + 36 - 16 - 16 - 8 \\ x^4 & 153 - 9 + 54 - 18 + 12 - 32 \\ x^3 & 36 - 72 + 12 - 108 + 48 + 24 \\ x^2 & -153 - 162 + 96 \\ x^1 & -36 - 36 \end{array}$$

$$f'(x) = \frac{(x^3 - 3x)^2 (x^2 + 4x)^3}{(x^2 + 4x + 1)^3} (11x^6 + 68x^5 + 160x^4 - 60x^3 - 219x^2 - 72x)$$

Exercise

Find the derivative of $f(x) = \frac{x^2 + 3}{(2x - 1)^3 (3x + 1)^4}$

Solution

$$f(x) = (x^2 + 3)(2x - 1)^{-3} (3x + 1)^{-4} \quad (U^m V^n W^p)' = U^{m-1} V^{n-1} W^{p-1} (mU'VW + nUV'W + pUVW')$$

$$f'(x) = (2x - 1)^{-4} (3x + 1)^{-5} \left[2x(2x - 1)(3x + 1) - 6(x^2 + 3)(3x + 1) - 12(x^2 + 3)(2x - 1) \right]$$

$$= \frac{1}{(2x-1)^4(3x+1)^5} \left((4x^2-2x)(3x+1) - 6(3x^3+x^2+9x+3) - 12(2x^3-x^2+6x-3) \right)$$

$$x^3 \quad 12-18-24$$

$$x^2 \quad 4-6-6+12$$

$$x \quad -2-54-72$$

$$x^0 \quad -18+36$$

$$f'(x) = \frac{-30x^3 + 4x^2 - 128x + 18}{(2x-1)^4(3x+1)^5}$$

Exercise

Find the derivative of $f(x) = \frac{(x^3-3x)^3(x^2+4x)^4}{(x^2+4x+1)^2}$

Solution

$$f(x) = (x^3-3x)^3(x^2+4x)^4(x^2+4x+1)^{-2}$$

$$(U^m V^n W^p)' = U^{m-1} V^{n-1} W^{p-1} (mU'VW + nUV'W + pUVW')$$

$$f'(x) = (x^3-3x)^2(x^2+4x)^3(x^2+4x+1)^{-3} \left[3(3x^2-3)(x^2+4x)(x^2+4x+1) \right. \\ \left. + 4(x^3-3x)(2x+4)(x^2+4x+1) - 2(x^3-3x)(x^2+4x)(2x+4) \right]$$

$$f'(x) = (x^3-3x)^2(x^2+4x)^3(x^2+4x+1)^{-3} \left[(9x^2-9)(x^4+8x^3+9x^2+4x) \right. \\ \left. + (4x^3-12x)(2x^3+12x^2+18x+4) + (-2x^3+6x)(2x^3+12x^2+16x) \right]$$

$$x^6 \quad 9+8-4$$

$$x^5 \quad 72+48-24$$

$$x^4 \quad 81-9+72-24-32+12$$

$$x^3 \quad 36-72+16-144+72$$

$$x^2 \quad -81-216+96$$

$$x^1 \quad -36-48$$

$$f'(x) = \frac{(13x^6 + 96x^5 + 100x^4 - 92x^3 - 201x^2 - 84x)(x^3-3x)^2(x^2+4x)^3}{(x^2+4x+1)^3}$$

Exercise

Find the **second** derivative $y = \frac{x^2 + 3}{(x-1)^3 + (x+1)^3}$

Solution

$$\begin{aligned}(x-1)^3 + (x+1)^3 &= x^3 - 3x^2 + 3x - 1 + x^3 + 3x^2 + 3x + 1 \\ &= 2x^3 + 6x\end{aligned}$$

$$y = \frac{x^2 + 3}{2x^3 + 6x} \qquad \begin{aligned} u &= x^2 + 3 & v &= 2x^3 + 6x \\ u' &= 2x & v' &= 6x^2 + 6 \end{aligned}$$

$$y' = \frac{4x^4 + 12x^2 - 6x^4 - 18x^2 - 6x^2 - 18}{(2x^3 + 6x)^2} \qquad \left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$= \frac{-2x^4 - 12x^2 - 18}{(2x^3 + 6x)^2}$$

$$= -2 \frac{x^4 + 6x^2 + 9}{(2x^3 + 6x)^2}$$

$$u = x^4 + 6x^2 + 9 \qquad v = (2x^3 + 6x)^2$$

$$\begin{aligned} u' &= 4x^3 + 12x & v' &= 2(2x^3 + 6x)(6x^2 + 6) \\ &= 4x(x^2 + 3) \end{aligned}$$

$$y'' = -2 \frac{4x(x^2 + 3)(2x^3 + 6x)^2 - 2(2x^3 + 6x)(6x^2 + 6)(x^4 + 6x^2 + 9)}{(2x^3 + 6x)^4}$$

$$= -4(2x^3 + 6x) \frac{2x(2x^5 + 6x^3 + 6x^3 + 18x) - (6x^6 + 36x^4 + 54x^2 + x^4 + 36x^2 + 54)}{(2x^3 + 6x)^4}$$

$$= -4 \frac{4x^5 + 24x^3 + 36x^2 - 6x^6 - 37x^4 - 90x^2 - 54}{(2x^3 + 6x)^3}$$

$$= -4 \frac{-6x^6 + 4x^5 - 37x^4 + 24x^3 - 54x^2 - 54}{(2x^3 + 6x)^3}$$

Exercise

Find the **second** derivative of $y = \left(1 + \frac{1}{x}\right)^3$

Solution

$$\begin{aligned} y' &= 3\left(1 + \frac{1}{x}\right)^2 \left(1 + \frac{1}{x}\right)' & \left(\frac{1}{x}\right)' &= -\frac{1}{x^2} \\ &= 3\left(1 + \frac{1}{x}\right)^2 \left(-\frac{1}{x^2}\right) \\ &= -\frac{3}{x^2} \left(1 + \frac{1}{x}\right)^2 \end{aligned}$$

$$\begin{aligned} y'' &= \left(-\frac{3}{x^2}\right)' \left(1 + \frac{1}{x}\right)^2 + \left(-\frac{3}{x^2}\right) \left(1 + \frac{1}{x}\right)^2' \\ &= \left(-\frac{3(2x)}{x^4}\right) \left(1 + \frac{1}{x}\right)^2 + \left(-\frac{3}{x^2}\right) \left(2\left(1 + \frac{1}{x}\right) \left(-\frac{1}{x^2}\right)\right) \\ &= \frac{6}{x^3} \left(1 + \frac{1}{x}\right)^2 + \frac{6}{x^4} \left(1 + \frac{1}{x}\right) \\ &= \frac{6}{x^3} \left(1 + \frac{1}{x}\right) \left(1 + \frac{1}{x} + \frac{1}{x}\right) \\ &= \frac{6}{x^3} \left(1 + \frac{1}{x}\right) \left(1 + \frac{2}{x}\right) \end{aligned}$$

Exercise

Find the **second** derivative of $y = 9 \tan\left(\frac{x}{3}\right)$

Solution

$$\begin{aligned} y' &= 9 \sec^2\left(\frac{x}{3}\right) \cdot \left(\frac{x}{3}\right)' \\ &= 9 \sec^2\left(\frac{x}{3}\right) \cdot \left(\frac{1}{3}\right) \\ &= 3 \sec^2\left(\frac{x}{3}\right) \end{aligned}$$

$$\begin{aligned} y'' &= 6 \sec\left(\frac{x}{3}\right) \cdot \left(\sec\left(\frac{x}{3}\right)\right)' \\ &= 6 \sec\left(\frac{x}{3}\right) \cdot \frac{1}{3} \sec\left(\frac{x}{3}\right) \cdot \tan\left(\frac{x}{3}\right) \\ &= 2 \sec^2\left(\frac{x}{3}\right) \cdot \tan\left(\frac{x}{3}\right) \end{aligned}$$

Exercise

Find the tangent line to the graph of $y = \sqrt[3]{(x+4)^2}$ when $x = 4$.

Solution

$$y = (x+4)^{2/3}$$

$$y' = \frac{2}{3}(x+4)^{-1/3}$$

$$= \frac{2}{3} \frac{1}{(x+4)^{1/3}}$$

$$= \frac{2}{3 \sqrt[3]{x+4}} \Bigg|$$

$$x = 4 \rightarrow m = y' = \frac{2}{3 \sqrt[3]{4+4}}$$

$$= \frac{2}{3 \sqrt[3]{2^3}}$$

$$= \frac{2}{3(2)}$$

$$= \frac{1}{3} \Bigg|$$

$$x = 4 \rightarrow y = \sqrt[3]{(4+4)^2} = 4$$

$$y = \frac{1}{3}(x-4) + 4$$

$$y = \frac{1}{3}x - \frac{4}{3} + 4$$

$$\underline{y = \frac{1}{3}x + \frac{8}{3}} \Bigg|$$

Exercise

Evaluate the limit $\lim_{h \rightarrow 0} \frac{\sin^2\left(\frac{\pi}{4} + h\right) - \frac{1}{2}}{h}$

Solution

$$\lim_{h \rightarrow 0} \frac{\sin^2\left(\frac{\pi}{4} + h\right) - \frac{1}{2}}{h} = \frac{\frac{1}{2} - \frac{1}{2}}{0} = \frac{0}{0}$$

$$f\left(\frac{\pi}{4}\right) = \sin^2 \frac{\pi}{4} = \frac{1}{2}$$

$$\lim_{h \rightarrow 0} \frac{\sin^2\left(\frac{\pi}{4} + h\right) - \frac{1}{2}}{h} = f'\left(\frac{\pi}{4}\right)$$

$$\begin{aligned}
 &= 2 \sin \frac{\pi}{4} \cos \frac{\pi}{4} \\
 &= 2 \frac{\sqrt{2}}{2} \frac{\sqrt{2}}{2} \\
 &= 1
 \end{aligned}$$

Exercise

Evaluate the limit $\lim_{x \rightarrow 5} \frac{\tan(\pi\sqrt{3x-11})}{x-5}$

Solution

$$\lim_{x \rightarrow 5} \frac{\tan(\pi\sqrt{3x-11})}{x-5} = \frac{\tan 2\pi}{0} = \frac{0}{0}$$

$$f(x) = \tan(\pi\sqrt{3x-11})$$

$$\lim_{x \rightarrow 5} \frac{f(x) - f(5)}{x-5} = f'(5)$$

$$= \frac{3\pi}{2\sqrt{3x-11}} \sec^2(\pi\sqrt{3x-11}) \Big|_{x=5}$$

$$= \frac{3\pi}{4} \sec^2(2\pi)$$

$$= \frac{3\pi}{4}$$

Solution **Section 2.7 – Implicit Differentiation**

Exercise

Find $\frac{dy}{dx}$: $y^2 + x^2 - 2y - 4x = 4$

Solution

$$\frac{d}{dx}(y^2 + x^2 - 2y - 4x) = \frac{d}{dx}(4)$$

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(x^2) - \frac{d}{dx}(2y) - \frac{d}{dx}(4x) = \frac{d}{dx}(4)$$

$$2y \frac{dy}{dx} + 2x - 2 \frac{dy}{dx} - 4 = 0$$

$$2(y-1) \frac{dy}{dx} = 4 - 2x$$

$$(y-1) \frac{dy}{dx} = 2 - x$$

$$\boxed{\frac{dy}{dx} = \frac{2-x}{y-1}}$$

Exercise

Find $\frac{dy}{dx}$: $x^2 y^2 - 2x = 3$

Solution

$$2xy^2 + 2x^2 yy' - 2 = 0$$

$$2x^2 yy' = 2 - 2xy^2$$

$$y' = \frac{2(1 - xy^2)}{2x^2 y}$$

$$\boxed{\frac{dy}{dx} = \frac{1 - xy^2}{x^2 y}}$$

Exercise

Find $\frac{dy}{dx}$: $x + \sqrt{x}\sqrt{y} = y^2$

Solution

$$\frac{d}{dx}(x + x^{1/2}y^{1/2}) = \frac{d}{dx}y^2$$

$$1 + \frac{d}{dx}(x^{1/2})y^{1/2} + x^{1/2}\frac{d}{dx}(y^{1/2}) = 2y\frac{dy}{dx}$$

$$1 + \frac{1}{2}x^{-1/2}y^{1/2} + x^{1/2}\frac{1}{2}y^{-1/2}\frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$1 + \frac{y^{1/2}}{2x^{1/2}} + \frac{x^{1/2}}{2y^{1/2}}\frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$1 + \frac{y^{1/2}}{2x^{1/2}} = 2y\frac{dy}{dx} - \frac{x^{1/2}}{2y^{1/2}}\frac{dy}{dx}$$

$$\left(\frac{4y^{3/2} - x^{1/2}}{2y^{1/2}}\right)\frac{dy}{dx} = \frac{2x^{1/2} + y^{1/2}}{2x^{1/2}}$$

$$\frac{dy}{dx} = \frac{2x^{1/2} + y^{1/2}}{2x^{1/2}} \cdot \frac{2y^{1/2}}{4y^{3/2} - x^{1/2}}$$

$$= \frac{4x^{1/2}y^{1/2} + 2y}{8x^{1/2}y^{3/2} - 2x}$$

Divide every term by 2

$$= \frac{2x^{1/2}y^{1/2} + y}{4x^{1/2}y^{3/2} - x}$$

Exercise

Find $\frac{dy}{dx}$: $x^2y + xy^2 = 6$

Solution

$$\left(2xy + x^2\frac{dy}{dx}\right) + \left(y^2 + 2xy\frac{dy}{dx}\right) = 0$$

$$x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -2xy - y^2$$

$$(x^2 + 2xy)\frac{dy}{dx} = -2xy - y^2$$

$$\frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy}$$

Exercise

Find $\frac{dy}{dx}$: $x^3 - xy + y^3 = 1$

Solution

$$3x^2 - \left(y + x \frac{dy}{dx}\right) + 3y^2 \frac{dy}{dx} = 0$$

$$3x^2 - y - x \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 0$$

$$(3y^2 - x) \frac{dy}{dx} = y - 3x^2$$

$$\frac{dy}{dx} = \frac{y - 3x^2}{3y^2 - x}$$

Exercise

Find $\frac{dy}{dx}$: $y^2 = \frac{x-1}{x+1}$

Solution

$$2yy' = \frac{1(x+1) - (1)(x-1)}{(x+1)^2}$$

$$2yy' = \frac{x+1-x+1}{(x+1)^2}$$

$$y' = \frac{2}{2y(x+1)^2}$$

$$\frac{dy}{dx} = \frac{1}{y(x+1)^2}$$

Exercise

Find $\frac{dy}{dx}$: $(3xy + 7)^2 = 6y$

Solution

$$2(3xy + 7)(3y + 3xy') = 6y'$$

$$6(3xy + 7)(y + xy') = 6y'$$

$$(3xy + 7)(y + xy') = y'$$

$$3xy^2 + 3x^2yy' + 7y + 7xy' = y'$$

Divide by 6 both sides

$$\begin{aligned}
 3x^2 yy' + 7xy' - y' &= -3xy^2 - 7y \\
 (3x^2 y + 7x - 1)y' &= -(3xy^2 + 7y) \\
 \frac{dy}{dx} &= -\frac{3xy^2 + 7y}{3x^2 y + 7x - 1}
 \end{aligned}$$

Exercise

Find $\frac{dy}{dx}$: $xy = \cot(xy)$

Solution

$$\begin{aligned}
 y + xy' &= -\csc^2(xy) (y + xy') \\
 y + xy' &= -y \csc^2(xy) - x \csc^2(xy) y' \\
 x \csc^2(xy) y' + xy' &= -y \csc^2(xy) - y \\
 x(\csc^2(xy) + 1)y' &= -y(\csc^2(xy) + 1) \\
 y' &= -\frac{y(\csc^2(xy) + 1)}{x(\csc^2(xy) + 1)} \\
 \frac{dy}{dx} &= -\frac{y}{x}
 \end{aligned}$$

Exercise

Find $\frac{dy}{dx}$: $x + \tan(xy) = 0$

Solution

$$\begin{aligned}
 1 + \sec^2(xy)(y + xy') &= 0 \\
 1 + y \sec^2(xy) + x \sec^2(xy) y' &= 0 \\
 x \sec^2(xy) y' &= -y \sec^2(xy) - 1 \\
 y' &= -\frac{y \sec^2(xy)}{x \sec^2(xy)} - \frac{1}{x \sec^2(xy)} \\
 \frac{dy}{dx} &= -\frac{y}{x} - \frac{\cos^2 x}{x} \\
 &= \frac{-y - \cos^2 x}{x}
 \end{aligned}$$

Exercise

Find $\frac{dy}{dx}$: $x \cos(2x + 3y) = y \sin x$

Solution

$$\cos(2x + 3y) - \sin(2x + 3y)(2x + 3y') = y' \sin x + y \cos x$$

$$\cos(2x + 3y) - 2x \sin(2x + 3y) - 3 \sin(2x + 3y) y' = y' \sin x + y \cos x$$

$$\cos(2x + 3y) - 2x \sin(2x + 3y) - y \cos x = y' \sin x + 3 \sin(2x + 3y) y'$$

$$\cos(2x + 3y) - 2x \sin(2x + 3y) - y \cos x = y'(\sin x + 3 \sin(2x + 3y))$$

$$y' = \frac{\cos(2x + 3y) - 2x \sin(2x + 3y) - y \cos x}{\sin x + 3 \sin(2x + 3y)}$$

Exercise

Find $\frac{dy}{dx}$: $y = \frac{e^y}{1 + \sin x}$

Solution

$$y(1 + \sin x) = e^y$$

$$y'(1 + \sin x) + y \cos x = y' e^y$$

$$y'(e^y - 1 - \sin x) = y \cos x$$

$$\frac{dy}{dx} = \frac{y \cos x}{e^y - 1 - \sin x}$$

Exercise

Find $\frac{dy}{dx}$: $\sin x \cos(y - 1) = \frac{1}{2}$

Solution

$$\cos x \cos(y - 1) - y' \sin x \sin(y - 1) = 0$$

$$y' \sin x \sin(y - 1) = \cos x \cos(y - 1)$$

$$y' = \frac{\cos x \cos(y - 1)}{\sin x \sin(y - 1)}$$

$$\frac{dy}{dx} = \cot x \cot(y - 1)$$

Exercise

Find $\frac{dy}{dx}$: $y\sqrt{x^2 + y^2} = 15$

Solution

$$y'\sqrt{x^2 + y^2} + \frac{1}{2}y(2x + 2yy')\left(x^2 + y^2\right)^{-1/2} = 0 \quad \times \sqrt{x^2 + y^2}$$

$$y'(x^2 + y^2) + y(x + yy') = 0$$

$$y'(x^2 + y^2) + y^2y' = -xy$$

$$y'(x^2 + 2y^2) = -xy$$

$$\frac{dy}{dx} = -\frac{xy}{x^2 + 2y^2}$$

Exercise

Find $\frac{dr}{d\theta}$ $r - 2\sqrt{\theta} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4}$

Solution

$$r - 2\theta^{1/2} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4}$$

$$\frac{dr}{d\theta} - 2\frac{1}{2}\theta^{-1/2} = \frac{3}{2}\frac{2}{3}\theta^{-1/3} + \frac{4}{3}\frac{3}{4}\theta^{-1/4}$$

$$\frac{dr}{d\theta} = \theta^{-1/3} + \theta^{-1/4} + \theta^{-1/2}$$

Exercise

Find $\frac{dr}{d\theta}$ $\sin(r\theta) = \frac{1}{2}$

Solution

$$\cos(r\theta)\left(\theta\frac{dr}{d\theta} + r\right) = 0$$

$$\theta\frac{dr}{d\theta} + r = 0 \quad \cos(r\theta) \neq 0$$

$$\frac{dr}{d\theta} = -\frac{r}{\theta} \quad \cos(r\theta) \neq 0$$

Exercise

Find $\frac{d^2y}{dx^2}$ $x^{2/3} + y^{2/3} = 1$

Solution

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

Multiply all terms by $\frac{3}{2}$

$$x^{-1/3} + y^{-1/3}y' = 0$$

$$y^{-1/3}y' = -x^{-1/3}$$

$$y' = -\frac{x^{-1/3}}{y^{-1/3}}$$

$$= -\frac{y^{1/3}}{x^{1/3}} = -\left(\frac{y}{x}\right)^{1/3}$$

$$y'' = -\frac{1}{3}\left(\frac{y}{x}\right)^{-2/3}\left(\frac{xy' - y}{x^2}\right)$$

$$= -\frac{1}{3}\left(\frac{x}{y}\right)^{2/3}\left(\frac{-x\left(\frac{y}{x}\right)^{1/3} - y}{x^2}\right) = \frac{1}{3}\left(\frac{x^{4/3}y^{1/3}}{y^{2/3}x^2} + \frac{x^{2/3}y}{y^{2/3}x^2}\right)$$

$$= \frac{1}{3}\left(\frac{x}{y}\right)^{2/3}\left(\frac{x\frac{y^{1/3}}{x^{1/3}} + y}{x^2}\right)$$

$$= \frac{1}{3}\frac{x^{2/3}}{y^{2/3}}\frac{x^{2/3}y^{1/3} + y}{x^2}$$

$$= \frac{1}{3}\left(\frac{1}{y^{1/3}x^{2/3}} + \frac{y^{1/3}}{x^{4/3}}\right)$$

Exercise

Find $\frac{d^2y}{dx^2}$ $2\sqrt{y} = x - y$

Solution

$$2\frac{1}{2}y^{-1/2}y' = 1 - y'$$

$$2\frac{1}{2}y^{-1/2}y' + y' = 1$$

$$\left(y^{-1/2} + 1\right)y' = 1$$

$$y' = \frac{1}{y^{-1/2} + 1} = \frac{1}{\frac{1}{\sqrt{y}} + 1} = \frac{\sqrt{y}}{1 + \sqrt{y}}$$

$$\left(y^{-1/2} + 1\right)y'' + \left(-\frac{1}{2}y^{-3/2}y'\right)y' = 0$$

$$\left(y^{-1/2} + 1\right)y'' - \frac{1}{2}y^{-3/2}(y')^2 = 0$$

$$\left(y^{-1/2} + 1\right)y'' = \frac{1}{2}y^{-3/2}\left(\frac{1}{y^{-1/2} + 1}\right)^2$$

$$y'' = \frac{1}{2}y^{-3/2} \frac{1}{\left(y^{-1/2} + 1\right)^2} \frac{1}{y^{-1/2} + 1}$$

$$= \frac{1}{2}y^{-3/2} \frac{1}{\left(\frac{1}{\sqrt{y}} + 1\right)^3}$$

$$= \frac{1}{2}y^{-3/2} \frac{1}{\left(\frac{1 + \sqrt{y}}{\sqrt{y}}\right)^3}$$

$$= \frac{1}{2}y^{-3/2} \frac{1}{\frac{(1 + \sqrt{y})^3}{(y^{1/2})^3}}$$

$$= \frac{1}{2}y^{-3/2} \frac{y^{3/2}}{(1 + \sqrt{y})^3}$$

$$= \frac{1}{2(1 + \sqrt{y})^3}$$

Exercise

If $x^3 + y^3 = 16$, find the value of $\frac{d^2y}{dx^2}$ at the point (2, 2).

Solution

$$3x^2 + 3y^2y' = 0$$

$$3y^2y' = -3x^2$$

$$y^2 y' = -x^2$$

$$2yy'y' + y^2 y'' = -2x$$

$$y^2 y'' = -2x - 2y(y')^2$$

$$y^2 y'' = -2x - 2y \left(\frac{-x^2}{y^2} \right)^2$$

$$y^2 y'' = -2x - 2 \frac{x^4}{y^3}$$

$$y'' = -2 \frac{x}{y^2} - 2 \frac{x^4}{y^5}$$

$$= \frac{-2xy^3 - 2x^4}{y^5}$$

$$y'' \Big|_{(2,2)} = \frac{-2(2)2^3 - 2(2)^4}{2^5}$$

$$= \frac{-2^5 - 2^5}{2^5}$$

$$\underline{\underline{= -2}}$$

Exercise

Find dy/dx : $x^2 - xy + y^2 = 4$ and evaluate the derivative at the given point $(0, -2)$

Solution

$$2x - (y + xy') + 2yy' = 0$$

$$-y - xy' + 2yy' = -2x$$

$$(2y - x)y' = y - 2x$$

$$\underline{\underline{\frac{dy}{dx} = \frac{y - 2x}{2y - x}}}$$

$$@ (0, -2) \rightarrow \frac{dy}{dx} = \frac{-2 - 2(0)}{2(-2) - (0)}$$

$$= \frac{-2}{-4}$$

$$\underline{\underline{= \frac{1}{2}}}$$

Exercise

Find the slope of the curve $(x^2 + y^2)^2 = (x - y)^2$ at the point $(-2, 1)$ and $(-2, -1)$

Solution

1 and -1

Exercise

Find the slope of the tangent line to the circle $x^2 - 9y^2 = 16$ at the point $(5, 1)$

Solution

$$2x - 18y \frac{dy}{dx} = 0$$

$$-18y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{-18y} = \frac{x}{9y}$$

$$\begin{aligned} @ (5, 1) \rightarrow \frac{dy}{dx} &= \frac{5}{9(1)} \\ &= \frac{5}{9} \end{aligned}$$

Exercise

Find the slope of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$

Solution

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$2y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\begin{aligned} \text{Slope: } \frac{dy}{dx} &= -\frac{3}{-4} \\ &= \frac{3}{4} \end{aligned}$$

Exercise

Find an equation of the line tangent to the following curves at the given point

$$y = 3x^3 + \sin x; \quad (0, 0)$$

Solution

$$m = y' = 9x^2 + \cos x \Big|_{(0, 0)}$$

$$= 1 \Big|$$

$$\underline{y = x}$$

$$y = m(x - x_1) + y_1$$

Exercise

Find an equation of the line tangent to the following curves at the given point

$$y = \frac{4x}{x^2 + 3}; \quad (3, 1)$$

Solution

$$m = y' = \frac{4x^2 + 12 - 8x^2}{(x^2 + 3)^2}$$

$$= \frac{12 - 4x^2}{(x^2 + 3)^2} \Big|_{(3, 1)}$$

$$= \frac{-24}{144}$$

$$= -\frac{1}{6} \Big|$$

$$y = -\frac{1}{6}(x - 3) + 1$$

$$\underline{= -\frac{1}{6}x + \frac{3}{2}}$$

$$y = m(x - x_1) + y_1$$

Exercise

Find an equation of the line tangent to the following curves at the given point

$$y + \sqrt{xy} = 6; \quad (1, 4)$$

Solution

$$y' + \frac{1}{2}(y + xy') \frac{1}{\sqrt{xy}} = 0 \Big|_{(1, 4)}$$

$$y' + \frac{1}{2}(4 + y') \frac{1}{2} = 0$$

$$y' + \frac{1}{4}y' = -1$$

$$\frac{5}{4}y' = -1$$

$$m = y' = -\frac{4}{5} \quad \Bigg|$$

$$y = -\frac{4}{5}(x-1) + 4$$

$$= -\frac{4}{5}x + \frac{24}{5} \quad \Bigg|$$

$$y = m(x - x_1) + y_1$$

Exercise

Find an equation of the line tangent to the following curves at the given point

$$x^2y + y^3 = 75; \quad (4, 3)$$

Solution

$$2xy + x^2y' + 3y^2y' = 0 \quad \Bigg| \quad (4, 3)$$

$$(16 + 27)y' = -24$$

$$y' = -\frac{24}{43} = m \quad \Bigg|$$

$$y = -\frac{24}{43}(x-4) + 3$$

$$= -\frac{24}{43}x + \frac{225}{43} \quad \Bigg|$$

$$y = m(x - x_1) + y_1$$

Exercise

Find the equation of the tangent line to the circle $x^3 + y^3 = 9xy$ at the point (2, 4)

Solution

$$3x^2 + 3y^2y' = 9y + 9xy'$$

$$3y^2y' - 9xy' = 9y - 3x^2$$

$$(3y^2 - 9x)y' = 9y - 3x^2$$

$$y' = \frac{3(3y - x^2)}{3(y^2 - 3x)}$$

$$= \frac{3y - x^2}{y^2 - 3x}$$

$$\begin{aligned}
 m \Big|_{(2,4)} &= \frac{3(4)-2^2}{4^2-3(2)} \\
 &= \frac{8}{10} \\
 &= \frac{4}{5} \Big|
 \end{aligned}$$

$$\begin{aligned}
 y &= \frac{4}{5}(x-2) + 4 \\
 &= \frac{4}{5}x - \frac{8}{5} + 4 \\
 y &= \frac{4}{5}x + \frac{12}{5} \Big|
 \end{aligned}$$

$$y = m(x - x_1) + y_1$$

Exercise

Find the lines that are **(a)** tangent and **(b)** normal to the curve $x^2 + xy - y^2 = 1$ at the point $(2, 3)$.

Solution

$$2x + y + xy' - 2yy' = 0$$

$$(x - 2y)y' = -2x - y$$

$$y' = \frac{-2x-y}{x-2y} = \frac{2x+y}{2y-x}$$

$$\begin{aligned}
 \text{a) tangent slope} &= y' \Big|_{(2,3)} \\
 &= \frac{2(2)+3}{2(3)-2} \\
 &= \frac{7}{4} \Big|
 \end{aligned}$$

$$\begin{aligned}
 y &= \frac{7}{4}(x-2) + 3 \\
 &= \frac{7}{4}x - \frac{7}{2} + 3 \\
 &= \frac{7}{4}x - \frac{1}{2} \Big|
 \end{aligned}$$

$$y = m(x - x_1) + y_1$$

$$\text{b) normal slope} = -\frac{4}{7}$$

$$y = -\frac{4}{7}(x-2) + 3$$

$$y = \frac{4}{7}x - \frac{8}{7} + 3$$

$$y = -\frac{4}{7}x + \frac{29}{7} \Big|$$

$$y = m(x - x_1) + y_1$$

Exercise

Find the lines that are **(a)** tangent and **(b)** normal to the curve $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$ at the point $(-1, 0)$.

Solution

$$12x + 3y + 3xy' + 4yy' + 17y' = 0$$

$$(3x + 4y + 17)y' = -12x - 3y$$

$$y' = \frac{-12x - 3y}{3x + 4y + 17}$$

$$\begin{aligned} \text{a) tangent slope} &= y' \Big|_{(-1, 0)} \\ &= \frac{-12(-1) - 3(0)}{3(-1) + 4(0) + 17} \\ &= \frac{6}{7} \end{aligned}$$

$$\begin{aligned} y &= \frac{6}{7}(x + 1) \\ &= \frac{6}{7}x + \frac{6}{7} \end{aligned}$$

$$y = m(x - x_1) + y_1$$

$$\begin{aligned} \text{b) normal slope} &= -\frac{7}{6} \\ y &= -\frac{7}{6}(x + 1) \Rightarrow y = -\frac{7}{6}x - \frac{7}{6} \end{aligned}$$

$$y = m(x - x_1) + y_1$$

Exercise

Find the lines that are **(a)** tangent and **(b)** normal to the curve $x^2 \cos^2 y - \sin y = 0$ at the point $(0, \pi)$.

Solution

$$2x \cos^2 y + x^2 (2 \cos y (-\sin y) y') - (\cos y) y' = 0$$

$$(-2x^2 \cos y \sin y - \cos y) y' = -2x \cos^2 y$$

$$y' = \frac{-2x \cos^2 y}{-(2x^2 \sin y + 1) \cos y} = \frac{2x \cos y}{2x^2 \sin y + 1}$$

$$\begin{aligned} \text{a) tangent slope} &= y' \Big|_{(0, \pi)} \\ &= \frac{2(0) \cos(\pi)}{2(0)^2 \sin(\pi) + 1} \end{aligned}$$

$$= 0$$

$$y - \pi = 0(x - 0)$$

$$y = \pi$$

b) normal slope = 0

$$\Rightarrow x = 0$$

Exercise

Suppose that x and y are both functions of t , which can be considered to represent time, and that x and y are related by the equation $xy^2 + y = x^2 + 17$

Suppose further that when $x = 2$ and $y = 3$, then $\frac{dx}{dt} = 13$. Find the value of the $\frac{dy}{dt}$ at that moment.

Solution

$$y^2 \frac{dx}{dt} + 2xy \frac{dy}{dt} + \frac{dy}{dt} = 2x \frac{dx}{dt}$$

$$3^2(13) + 2(2)(3) \frac{dy}{dt} + \frac{dy}{dt} = 2(2)(13)$$

$$117 + 12 \frac{dy}{dt} + \frac{dy}{dt} = 52$$

$$13 \frac{dy}{dt} = -65$$

$$\frac{dy}{dt} = \frac{-65}{13}$$

$$= -5$$

Exercise

A cone-shaped icicle is dripping from the roof. The radius of the icicle is decreasing at a rate of 0.2 cm per hour, while the length is increasing at a rate of 0.8 cm per hour. If the icicle is currently 4 cm in radius and 20 cm long, is the volume of the icicle increasing or decreasing and at what rate?

Solution

The volume of the cone is given by the formula: $V = \frac{1}{3} \pi r^2 h$.

$$\frac{dV}{dt} = \frac{1}{3} \pi \left(2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right)$$

Given the values:

$$\frac{dr}{dt} = -0.2 \quad \frac{dh}{dt} = 0.8 \quad r = 4 \quad h = 20$$

$$\frac{dV}{dt} = \frac{1}{3} \pi \left(2(4)(20)(-0.2) + 4^2(0.8) \right)$$

$$\underline{=-20}$$

The volume is decreasing at a rate of 20 cm^3 per *hour*.

Solution **Section 2.8 – Derivatives of Logarithmic & Exponential Functions**

Exercise

Find the derivative of $y = \ln \sqrt{x+5}$

Solution

$$y = \ln(x+5)^{1/2}$$

$$= \frac{1}{2} \ln(x+5)$$

$$\underline{y' = \frac{1}{2(x+5)}} \quad \Bigg|$$

Exercise

Find the Derivatives of $y = (3x+7)\ln(2x-1)$

Solution

$$f = 3x+7 \quad f' = 3$$

$$g = \ln(2x-1) \quad g' = \frac{2}{2x-1}$$

$$\underline{y' = 3\ln(2x-1) + \frac{2(3x+7)}{2x-1}} \quad \Bigg|$$

Exercise

Find the Derivatives of $f(x) = \ln \sqrt[3]{x+1}$

Solution

$$f(x) = \ln(x+1)^{1/3}$$

$$= \frac{1}{3} \ln(x+1)$$

$$u = x+1 \Rightarrow \frac{du}{dx} = 1$$

$$f'(x) = \frac{1}{3} \frac{1}{x+1}$$

$$\underline{= \frac{1}{3(x+1)}} \quad \Bigg|$$

Exercise

Find the Derivatives of $f(x) = \ln\left(x^2 \sqrt{x^2 + 1}\right)$

Solution

$$f(x) = \ln\left(x^2\right) + \ln\sqrt{x^2 + 1} \quad \text{Product Property}$$

$$f(x) = \ln\left(x^2\right) + \ln\left(x^2 + 1\right)^{1/2}$$

$$f(x) = 2 \ln x + \frac{1}{2} \ln\left(x^2 + 1\right) \quad \text{Power Property}$$

$$\begin{aligned} f'(x) &= 2 \frac{1}{x} + \frac{1}{2} \frac{2x}{x^2 + 1} \\ &= \frac{\frac{2}{x} + \frac{x}{x^2 + 1}}{\quad} \quad \text{Differentiate} \end{aligned}$$

Exercise

Find the Derivatives of $y = \ln \frac{x^2}{x^2 + 1}$

Solution

$$y = \ln x^2 - \ln x^2 + 1$$

$$\begin{aligned} y' &= \frac{2x}{x^2} - \frac{2x}{x^2 + 1} \\ &= \frac{\frac{2}{x} - \frac{2x}{x^2 + 1}}{\quad} \end{aligned}$$

Exercise

Find the Derivatives of $y = \ln \left[\frac{x^2(x+1)^3}{(x+3)^{1/2}} \right]$

Solution

$$y = \ln \left[x^2 (x+1)^3 \right] - \ln (x+3)^{1/2} \quad \text{Quotient Rule}$$

$$= \ln x^2 + \ln (x+1)^3 - \ln (x+3)^{1/2} \quad \text{Product Rule}$$

$$= 2 \ln x + 3 \ln (x+1) - \frac{1}{2} \ln (x+3) \quad \text{Power Rule}$$

$$\underline{y' = \frac{2}{x} + \frac{3}{x+1} - \frac{1}{2(x+3)}}$$

Exercise

Find the Derivatives of $y = \ln(x^2 + 1)$

Solution

$$\underline{y' = \frac{2x}{x^2 + 1}} \qquad (\ln U)' = \frac{U'}{U}$$

Exercise

Find the Derivatives of $f(x) = \ln(x^2 - 4)$

Solution

$$\text{Let } u = x^2 - 4 \Rightarrow \frac{du}{dx} = 2x$$

$$\begin{aligned} f'(x) &= \frac{1}{x^2 - 4} (2x) \\ &= \frac{2x}{x^2 - 4} \end{aligned}$$

Exercise

Find the derivative $f(x) = 2 \ln(x^2 - 3x + 4)$

Solution

$$\begin{aligned} f'(x) &= 2 \frac{2x - 3}{x^2 - 3x + 4} \\ &= \frac{4x - 6}{x^2 - 3x + 4} \end{aligned}$$

Exercise

Find the derivative $f(x) = 3 \ln(1 + x^2)$

Solution

$$f'(x) = 3 \frac{2x}{1+x^2}$$

$$= \frac{6x}{1+x^2}$$

Exercise

Find the derivative $f(x) = (1 + \ln x)^3$

Solution

$$f'(x) = 3(1 + \ln x)^2 (1 + \ln x)'$$

$$= 3(1 + \ln x)^2 \left(\frac{1}{x}\right)$$

$$= \frac{3}{x}(1 + \ln x)^2$$

Exercise

Find the derivative $f(x) = (x - 2 \ln x)^4$

Solution

$$f'(x) = 4(x - 2 \ln x)^3 (x - 2 \ln x)'$$

$$= 4(x - 2 \ln x)^3 \left(1 - \frac{2}{x}\right)$$

$$= 4(x - 2 \ln x)^3 \left(\frac{x-2}{x}\right)$$

$$= \frac{4x-8}{x}(x - 2 \ln x)^3$$

Exercise

Find the Derivatives of $f(x) = x^2 \ln x$

Solution

$$f' = x^2 \left(\frac{1}{x}\right) + 2x \ln x$$

$$= x + 2x \ln x$$

$$= x(1 + 2 \ln x)$$

$$(fg)' = f'g + fg'$$

Exercise

Find the Derivatives of $f(x) = -\frac{\ln x}{x^2}$

Solution

$$\begin{aligned} f' &= -\frac{x^2 \frac{d}{dx}(\ln x) - \ln x \frac{d}{dx}(x^2)}{(x^2)^2} \\ &= -\frac{x^2 \frac{1}{x} - 2x \ln x}{x^4} \\ &= -\frac{x - 2x \ln x}{x^4} \\ &= -\frac{x(1 - 2 \ln x)}{x^4} \\ &= -\frac{1 - 2 \ln x}{x^3} \end{aligned}$$

Exercise

Find the derivative of $y = \ln(t^2)$

Solution

$$\begin{aligned} y' &= \frac{(t^2)'}{t^2} \\ &= \frac{2t}{t^2} \\ &= \frac{2}{t} \end{aligned}$$

Exercise

Find the derivative of $y = \ln(2\theta + 2)$

Solution

$$\begin{aligned} y' &= \frac{2}{2\theta + 2} \\ &= \frac{1}{\theta + 1} \end{aligned}$$

Exercise

Find the derivative of $y = (\ln x)^3$

Solution

$$y' = 3(\ln x)^2 \frac{1}{x}$$

$$= \frac{3(\ln x)^2}{x}$$

Exercise

Find the derivative of $y = x(\ln x)^2$

Solution

$$y' = (\ln x)^2 + x \left(2(\ln x) \frac{1}{x} \right)$$

$$= (\ln x)^2 + 2 \ln x$$

Exercise

Find the derivative of $y = \frac{x^4}{4} \ln x - \frac{x^4}{16}$

Solution

$$y' = \frac{4x^3}{4} \ln x + \frac{x^4}{4} \frac{1}{x} - \frac{4x^3}{16}$$

$$= x^3 \ln x + \frac{1}{4}x^3 - \frac{1}{4}x^3$$

$$= x^3 \ln x$$

Exercise

Find the derivative of $y = \frac{1 + \ln t}{t}$

Solution

$$y' = \frac{\frac{1}{t} - (1 + \ln t)}{t^2}$$

$$= \frac{1 - 1 - \ln t}{t^2}$$

$$= -\frac{\ln t}{t^2}$$

Exercise

Find the derivative $f(x) = \frac{\ln x}{1+x}$

Solution

$$\begin{aligned} f'(x) &= \frac{\left(\frac{1}{x}\right)(1+x) - \ln x}{(1+x)^2} \\ &= \frac{\frac{1}{x} 1 + x - x \ln x}{(1+x)^2} \\ &= \frac{1+x-x \ln x}{x(1+x)^2} \end{aligned}$$

$$u = \ln x \quad v = 1+x$$

$$u' = \frac{1}{x} \quad v' = 1$$

Exercise

Find the derivative $f(x) = \frac{2x}{1+\ln x}$

Solution

$$\begin{aligned} f'(x) &= \frac{2(1+\ln x) - (2x)\frac{1}{x}}{(1+\ln x)^2} \\ &= \frac{2+2\ln x-2}{(1+\ln x)^2} \\ &= \frac{2\ln x}{(1+\ln x)^2} \end{aligned}$$

$$u = 2x \quad v = 1+\ln x$$

$$u' = 2 \quad v' = \frac{1}{x}$$

Exercise

Find the derivative $f(x) = x^3 \ln x$

Solution

$$u = x^3 \quad v = \ln x$$

$$u' = 3x^2 \quad v' = \frac{1}{x}$$

$$\begin{aligned} f'(x) &= 3x^2 \ln x + x^3 \frac{1}{x} \\ &= 3x^2 \ln x + x^2 \\ &= (3\ln x + 1)x^2 \end{aligned}$$

Exercise

Find the derivative $f(x) = 6x^4 \ln x$

Solution

$$u = 6x^4 \quad v = \ln x$$

$$u' = 24x^3 \quad v' = \frac{1}{x}$$

$$\begin{aligned} f'(x) &= 24x^3 \ln x + 6x^4 \frac{1}{x} \\ &= 24x^3 \ln x + 6x^3 \\ &= \underline{6x^3(4 \ln x + 1)} \end{aligned}$$

Exercise

Find the derivative $f(x) = \ln x^8$

Solution

$$f(x) = \ln x^8 = 8 \ln x$$

$$\underline{f'(x) = \frac{8}{x}}$$

Power Rule

$$(\ln x)' = \frac{1}{x}$$

Exercise

Find the derivative $f(x) = 5x - \ln x^5$

Solution

$$\begin{aligned} f(x) &= 5x - \ln x^5 \\ &= 5x - 5 \ln x \end{aligned}$$

$$\underline{f'(x) = 5 - \frac{5}{x}}$$

Power Rule

$$(\ln x)' = \frac{1}{x}$$

Exercise

Find the derivative $f(x) = \ln x^{10} + 2 \ln x$

Solution

$$f(x) = 10 \ln x + 2 \ln x$$

Power Rule

$$= 12 \ln x$$

$$\underline{f'(x) = \frac{12}{x} \quad \Big|}$$

$$(\ln x)' = \frac{1}{x}$$

Exercise

Find the derivative $f(x) = \frac{\ln x}{2x+5}$

Solution

$$u = \ln x \quad v = 2x + 5$$

$$u' = \frac{1}{x} \quad v' = 2$$

$$\begin{aligned} f'(x) &= \frac{\frac{1}{x}(2x+5) - (2)\ln x}{(2x+5)^2} \cdot \frac{x}{x} \\ &= \frac{2x+5-2x\ln x}{x(2x+5)^2} \quad \Big| \end{aligned}$$

Exercise

Find the derivative $f(x) = -2 \ln x + x^2 - 4$

Solution

$$\underline{f'(x) = -\frac{2}{x} + 2x \quad \Big|}$$

Exercise

Find the derivative of $y = \ln\left(\frac{1}{x\sqrt{x+1}}\right)$

Solution

$$\begin{aligned} y &= \ln(1) - \ln(x\sqrt{x+1}) \\ &= -\ln x - \ln(x+1)^{1/2} \\ &= -\ln x - \frac{1}{2} \ln(x+1) \\ y' &= -\frac{1}{x} - \frac{1}{2} \frac{1}{x+1} \\ &= -\frac{3x+2}{2x(x+1)} \quad \Big| \end{aligned}$$

Exercise

Find the derivative of $y = \ln(\ln(\ln x))$

Solution

$$\begin{aligned} y' &= \frac{1}{\ln(\ln x)} \cdot (\ln(\ln x))' \\ &= \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot (\ln x)' \\ &= \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{1}{x} \\ &= \frac{1}{x(\ln x)(\ln(\ln x))} \end{aligned}$$

Exercise

Find the derivative of $y = \ln(\sec(\ln x))$

Solution

$$\begin{aligned} y' &= \frac{1}{\sec(\ln x)} \cdot (\sec(\ln x))' \\ &= \frac{1}{\sec(\ln x)} \cdot (\sec(\ln x) \tan(\ln x)) \cdot (\ln x)' \\ &= \frac{\sec(\ln x)}{\sec(\ln x)} \tan(\ln x) \cdot \frac{1}{x} \\ &= \frac{\tan(\ln x)}{x} \end{aligned}$$

Exercise

Find the derivative of $y = \ln \left(\frac{(x^2 + 1)^5}{\sqrt{1 - x}} \right)$

Solution

$$\begin{aligned} y &= \ln(x^2 + 1)^5 - \ln(1 - x)^{1/2} \\ &= 5 \ln(x^2 + 1) - \frac{1}{2} \ln(1 - x) \\ y' &= 5 \frac{2x}{x^2 + 1} - \frac{1}{2} \frac{-1}{1 - x} \end{aligned}$$

$$= \frac{10x}{x^2+1} + \frac{1}{2(1-x)} \quad \Bigg|$$

Exercise

Find the derivative of $y = \ln \sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$

Solution

$$\begin{aligned} y &= \frac{1}{2} \ln \left(\frac{(x+1)^5}{(x+2)^{20}} \right) \\ &= \frac{1}{2} \left[\ln(x+1)^5 - \ln(x+2)^{20} \right] \\ &= \frac{1}{2} (5 \ln(x+1) - 20 \ln(x+2)) \end{aligned}$$

$$\begin{aligned} y' &= \frac{1}{2} \left(5 \frac{1}{x+1} - 20 \frac{1}{x+2} \right) \\ &= \frac{5}{2} \left(\frac{1}{x+1} - \frac{4}{x+2} \right) \\ &= \frac{5}{2} \left(\frac{x+2-4x-4}{(x+1)(x+2)} \right) \\ &= \frac{5}{2} \frac{-3x-2}{(x+1)(x+2)} \\ &= -\frac{5}{2} \frac{3x+2}{(x+1)(x+2)} \quad \Bigg| \end{aligned}$$

Exercise

Find the derivative of $f(x) = e^{3x}$

Solution

$$f'(x) = 3e^{3x} \quad \Bigg|$$

Exercise

Find the derivative of $f(x) = e^{-2x^3}$

Solution

$$f'(x) = e^{-2x^3} (-6x^2)$$

$$\underline{= -\frac{6x^2}{e^{2x^3}}}$$

Exercise

Find the derivative of $f(x) = 4e^{x^2}$

Solution

$$\begin{aligned} f'(x) &= 4e^{x^2} (2x) \\ &\underline{= 8xe^{x^2}} \end{aligned}$$

Exercise

Find the derivative of $f(x) = x^2 e^x$

Solution

$$\begin{aligned} f'(x) &= e^x \frac{d}{dx}[x^2] + x^2 \frac{d}{dx}[e^x] \\ &= e^x(2x) + x^2 e^x \\ &\underline{= xe^x(2+x)} \end{aligned}$$

Exercise

Find the derivative $f(x) = 2x^3 e^x$

Solution

$$\begin{aligned} f'(x) &= 6x^2 e^x + 2x^3 e^x \\ &\underline{= 2x^2 e^x (3+x)} \end{aligned}$$

$$\begin{aligned} u &= 2x^3 & v &= e^x \\ u' &= 6x^2 & v' &= e^x \end{aligned}$$

Exercise

Find the derivative $f(x) = \frac{3e^x}{1+e^x}$

Solution

$$f'(x) = \frac{3e^x(1+e^x) - 3e^x e^x}{(1+e^x)^2}$$

$$\begin{aligned} u &= 3e^x & v &= 1+e^x \\ u' &= 3e^x & v' &= e^x \end{aligned}$$

$$= \frac{3e^x + 3e^{2x} - 3e^{2x}}{(1+e^x)^2}$$

$$= \frac{3e^x}{(1+e^x)^2}$$

Exercise

Find the derivative $f(x) = 5e^x + 3x + 1$

Solution

$$f'(x) = 5e^x + 3$$

Exercise

Find the derivative of $f(x) = \frac{e^x + e^{-x}}{2}$

Solution

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

$$f'(x) = \frac{1}{2} \left(\frac{d}{dx} [e^x] + \frac{d}{dx} [e^{-x}] \right)$$

$$= \frac{1}{2}(e^x - e^{-x})$$

Exercise

Find the derivative of $f(x) = \frac{e^x}{x^2}$

Solution

$$f'(x) = \frac{x^2 e^x - e^x (2x)}{x^4}$$

$$= \frac{x^2 e^x - 2x e^x}{x^4}$$

$$= \frac{x e^x (x-2)}{x^4}$$

$$= \frac{e^x (x-2)}{x^3}$$

Exercise

Find the derivative of $f(x) = x^2 e^x - e^x$

Solution

$$\begin{aligned} f'(x) &= e^x \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(e^x) - \frac{d}{dx}(e^x) \\ &= e^x(2x) + x^2 e^x - e^x \\ &= \underline{e^x(x^2 + 2x - 1)} \end{aligned}$$

Exercise

Find the derivative of $f(x) = (1 + 2x)e^{4x}$

Solution

$$\begin{aligned} f'(x) &= (2)e^{4x} + (1 + 2x)(4e^{4x}) \\ &= 2e^{4x} + (1 + 2x)(4e^{4x}) \\ &= 2e^{4x}(1 + 2(1 + 2x)) \\ &= 2e^{4x}(1 + 2 + 4x) \\ &= \underline{2e^{4x}(3 + 4x)} \end{aligned}$$

Exercise

Find the derivative of $y = x^2 e^{5x}$

Solution

$$\begin{aligned} y' &= x^2(5e^{5x}) + 2x(e^{5x}) \\ &= \underline{xe^{5x}(5x + 2)} \end{aligned}$$

Exercise

Find the derivative of $y = x^2 e^{-2x}$

Solution

$$\begin{aligned} y' &= 2xe^{-2x} - 2x^2 e^{-2x} \\ &= \underline{2xe^{-2x}(1 - x)} \end{aligned}$$

Exercise

Find the derivative $f(x) = \frac{e^x}{x^2 + 1}$

Solution

$$\begin{aligned} f'(x) &= \frac{e^x(x^2 + 1) - 2xe^x}{(x^2 + 1)^2} \\ &= \frac{(x^2 + 1 - 2x)e^x}{(x^2 + 1)^2} \end{aligned}$$

$$u = e^x \quad v = x^2 + 1$$

$$u' = e^x \quad v' = 2x$$

Exercise

Find the derivative $f(x) = \frac{1 - e^x}{1 + e^x}$

Solution

$$\begin{aligned} f'(x) &= \frac{-e^x(1 + e^x) - e^x(1 - e^x)}{(1 + e^x)^2} \\ &= \frac{-e^x - e^{2x} - e^x + e^{2x}}{(1 + e^x)^2} \\ &= -\frac{2e^x}{(1 + e^x)^2} \end{aligned}$$

$$u = 1 - e^x \quad v = 1 + e^x$$

$$u' = -e^x \quad v' = e^x$$

Exercise

Find the derivative of $y = \frac{e^x + e^{-x}}{x}$

Solution

$$\begin{aligned} y &= \frac{(e^x - e^{-x})x - (e^x + e^{-x})}{x^2} \\ &= \frac{xe^x - xe^{-x} - e^x - e^{-x}}{x^2} \\ &= \frac{(x-1)e^x - (x+1)e^{-x}}{x^2} \end{aligned}$$

$$f = e^x + e^{-x} \quad g = x$$

$$f' = e^x - e^{-x} \quad g' = 1$$

Exercise

Find the derivative of $y = \sqrt{e^{2x^2} + e^{-2x^2}}$

Solution

$$y = \sqrt{e^{2x^2} + e^{-2x^2}} = \left(e^{2x^2} + e^{-2x^2} \right)^{1/2} = U^{1/2}$$

$$U = e^{2x^2} + e^{-2x^2} \quad \left(e^{2x^2} \right)' = \left(2x^2 \right)' e^{2x^2} = 4xe^{2x^2}$$

$$U' = 4xe^{2x^2} - 4xe^{-2x^2}$$

$$y' = \frac{1}{2} \left(4xe^{2x^2} - 4xe^{-2x^2} \right) \left(e^{2x^2} + e^{-2x^2} \right)^{-1/2}$$

$$= \frac{1}{2} \frac{4x \left(e^{2x^2} - e^{-2x^2} \right)}{\left(e^{2x^2} + e^{-2x^2} \right)^{1/2}}$$

$$= \frac{2x \left(e^{2x^2} - e^{-2x^2} \right)}{\sqrt{e^{2x^2} + e^{-2x^2}}}$$

Exercise

Find the derivative of $y = \frac{x}{e^{2x}}$

Solution

$$y' = \frac{1 \left(e^{2x} \right) - x \left(2e^{2x} \right)}{\left(e^{2x} \right)^2}$$

$$= \frac{e^{2x} (1 - 2x)}{\left(e^{2x} \right)^2}$$

$$= \frac{1 - 2x}{e^{2x}}$$

$$f = x \quad g = e^{2x}$$

$$f' = 1 \quad g' = 2e^{2x}$$

Exercise

Find the derivative of $y = 3e^{5x^3+1}$

Solution

$$y' = 3(15x^2)e^{5x^3+1}$$

$$y' = 45x^2e^{5x^3+1}$$

$$f = x^2 \quad g = e^{5x^3+1}$$

$$f' = 2x \quad g' = 15x^2e^{5x^3+1}$$

$$y'' = 45 \left(2xe^{5x^3+1} + (x^2)15x^2e^{5x^3+1} \right)$$

$$= 45e^{5x^3+1} (2x + 15x^4)$$

$$= 45xe^{5x^3+1} (2 + 15x^3) \quad |$$

Exercise

Find the derivative of $(x^2 - 2x + 2)e^x$

Solution

$$y = (x^2 - 2x + 2)e^x$$

$$y' = (2x - 2)e^x + (x^2 - 2x + 2)e^x$$

$$= (2x - 2 + x^2 - 2x + 2)e^x$$

$$= x^2e^x \quad |$$

Exercise

Find the derivative of $e^\theta (\sin \theta + \cos \theta)$

Solution

$$\frac{d}{d\theta} e^\theta (\sin \theta + \cos \theta) = e^\theta (\sin \theta + \cos \theta) + e^\theta (\cos \theta - \sin \theta)$$

$$= e^\theta (\sin \theta + \cos \theta + \cos \theta - \sin \theta)$$

$$= 2e^\theta \cos \theta \quad |$$

Exercise

Find the derivative of $\ln(3\theta e^{-\theta})$

Solution

$$\begin{aligned}\frac{d}{d\theta} \ln(3\theta e^{-\theta}) &= \frac{(3\theta e^{-\theta})'}{3\theta e^{-\theta}} \\ &= 3 \frac{e^{-\theta} - \theta e^{-\theta}}{\theta e^{-\theta}} \\ &= \frac{e^{-\theta}(1 - \theta)}{\theta e^{-\theta}} \\ &= \underline{\underline{\frac{1 - \theta}{\theta}}}\end{aligned}$$

$$\begin{aligned}\ln(3\theta e^{-\theta}) &= \ln(3) + \ln(\theta) + \ln(e^{-\theta}) \\ &= \ln 3 + \ln \theta - \theta \\ \frac{d}{d\theta} \ln(3\theta e^{-\theta}) &= \underline{\underline{\frac{1}{\theta} - 1}}\end{aligned}$$

Exercise

Find the derivative of $\theta^3 e^{-2\theta} \cos 5\theta$

Solution

$$\begin{aligned}\frac{dy}{d\theta} &= (\theta^3)' e^{-2\theta} \cos 5\theta + \theta^3 (e^{-2\theta})' \cos 5\theta + \theta^3 e^{-2\theta} (\cos 5\theta)' \\ &= 3\theta^2 e^{-2\theta} \cos 5\theta - 2\theta^3 e^{-2\theta} \cos 5\theta - 5\theta^3 e^{-2\theta} \sin 5\theta \\ &= \underline{\underline{\theta^3 e^{-2\theta} (3 \cos 5\theta - 2\theta \cos 5\theta - 5\theta \sin 5\theta)}}$$

Exercise

Find the derivative of $\ln\left(\frac{\sqrt{\theta}}{1 + \sqrt{\theta}}\right)$

Solution

$$\begin{aligned}\frac{d}{d\theta} \ln\left(\frac{\sqrt{\theta}}{1 + \sqrt{\theta}}\right) &= \frac{d}{d\theta} \left[\ln \theta^{1/2} - \ln(1 + \sqrt{\theta}) \right] \\ &= \frac{d}{d\theta} \left[\frac{1}{2} \ln \theta - \ln(1 + \sqrt{\theta}) \right] \\ &= \frac{1}{2} \frac{1}{\theta} - \frac{\frac{1}{2} \theta^{-1/2}}{1 + \sqrt{\theta}} \\ &= \frac{1}{2\theta} - \frac{1}{2\sqrt{\theta}(1 + \sqrt{\theta})}\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left(\frac{1}{\theta} - \frac{1}{\sqrt{\theta}(1+\sqrt{\theta})} \right) \\
&= \frac{1}{2} \frac{\sqrt{\theta}(1+\sqrt{\theta}) - \theta}{\theta\sqrt{\theta}(1+\sqrt{\theta})} \\
&= \frac{1}{2} \frac{\sqrt{\theta} + \theta - \theta}{\theta\sqrt{\theta}(1+\sqrt{\theta})} \\
&= \frac{1}{2} \frac{\sqrt{\theta}}{\theta\sqrt{\theta}(1+\sqrt{\theta})} \\
&= \frac{1}{2\theta(1+\sqrt{\theta})} \Bigg|
\end{aligned}$$

Exercise

Find the derivative of $e^{(\cos t + \ln t)}$

Solution

$$\begin{aligned}
e^{(\cos t + \ln t)} &= e^{\cos t} e^{\ln t} \\
&= t e^{\cos t}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} e^{(\cos t + \ln t)} &= \frac{d}{dt} (t e^{\cos t}) \\
&= e^{\cos t} + t e^{\cos t} (-\sin t) \\
&= \underline{(1 - t \sin t) e^{\cos t}} \Bigg|
\end{aligned}$$

Exercise

Find the derivative of $e^{\sin t} (\ln t^2 + 1)$

Solution

$$\begin{aligned}
\frac{d}{dt} e^{\overset{u}{\sin t}} (\overset{v}{\ln t^2 + 1}) &= e^{\overset{u'}{\sin t}} \overset{v'}{\cos t} (\ln t^2 + 1) + \frac{v'}{t} e^{\overset{u}{\sin t}} \\
&= \underline{e^{\sin t} \left[(\ln t^2 + 1) \cos t + \frac{2}{t} \right]} \Bigg|
\end{aligned}$$

Exercise

Find the Derivatives of $y = e^{x^2} \ln x$

Solution

$$\underline{y' = 2xe^{x^2} \ln x + \frac{e^{x^2}}{x}}$$

$$\begin{aligned} f &= e^{x^2} & g &= \ln x \\ f' &= 2xe^{x^2} & g' &= \frac{1}{x} \end{aligned}$$

Exercise

Find the derivative $f(x) = e^x + x - \ln x$

Solution

$$\underline{f'(x) = e^x + 1 - \frac{1}{x}}$$

Exercise

Find the derivative $f(x) = \ln x + 2e^x - 3x^2$

Solution

$$\underline{f'(x) = \frac{1}{x} + 2e^x - 6x}$$

Exercise

Find the derivative $f(x) = \ln x^2 + 4e^x$

Solution

$$f(x) = 2 \ln x + 4e^x$$

Power Rule

$$\underline{f'(x) = \frac{2}{x} + 4e^x}$$

$$(\ln x)' = \frac{1}{x}$$

Exercise

Find the Derivatives of $y = \ln \frac{1+e^x}{1-e^x}$

Solution

$$y = \ln(1+e^x) - \ln(1-e^x)$$

$$y' = \frac{e^x}{1+e^x} - \frac{-e^x}{1-e^x}$$

$$\begin{aligned}
&= \frac{e^x}{1+e^x} + \frac{e^x}{1-e^x} \\
&= \frac{e^x - e^{2x} + e^x + e^{2x}}{(1+e^x)(1-e^x)} \\
&= \frac{2e^x}{(1+e^x)(1-e^x)} \quad \Bigg|
\end{aligned}$$

Exercise

Find the Derivatives of $y = \frac{\ln x}{e^{2x}}$

Solution

$$\begin{aligned}
y' &= \frac{e^{2x}(1/x) - \ln x(2e^{2x})}{e^{4x}} \\
&= \frac{e^{2x} - 2x \ln x(e^{2x})}{e^{4x}} \\
&= \frac{e^{2x}(1 - 2x \ln x)}{e^{4x}} \quad \Bigg|
\end{aligned}$$

Exercise

Find the Derivatives of $f(x) = e^{2x} \ln(xe^x + 1)$

Solution

$$f = e^{2x} \quad U = 2x \rightarrow U' = 2 \quad f' = 2e^{2x}$$

$$g = \ln(xe^x + 1) \quad U = xe^x + 1 \rightarrow U' = e^x + xe^x \quad g' = \frac{e^x + xe^x}{xe^x + 1}$$

$$\begin{aligned}
f'(x) &= 2e^{2x} \ln(xe^x + 1) + e^{2x} \frac{e^x + xe^x}{xe^x + 1} \\
&= e^{2x} \left[2 \ln(xe^x + 1) + \frac{e^x(1+x)}{xe^x + 1} \right] \quad \Bigg|
\end{aligned}$$

Exercise

Find the Derivatives of $f(x) = \frac{xe^x}{\ln(x^2 + 1)}$

Solution

$$u = xe^x$$

$$u' = e^x + xe^x$$

$$v = \ln(x^2 + 1)$$

$$v' = \frac{2x}{x^2 + 1}$$

$$\begin{aligned} f'(x) &= \frac{e^x(1+x)\ln(x^2 + 1) - \frac{2x}{x^2 + 1}xe^x}{\left[\ln(x^2 + 1)\right]^2} \\ &= \frac{e^x \left[(1+x)\ln(x^2 + 1) - \frac{2x^2}{x^2 + 1} \right]}{\left[\ln(x^2 + 1)\right]^2} \\ &= \frac{e^x \left[\frac{(x^2 + 1)(1+x)\ln(x^2 + 1) - 2x^2}{x^2 + 1} \right]}{\left[\ln(x^2 + 1)\right]^2} \\ &= \frac{e^x \left[(x^2 + 1)(1+x)\ln(x^2 + 1) - 2x^2 \right]}{(x^2 + 1)\left[\ln(x^2 + 1)\right]^2} \end{aligned}$$

Exercise

Find the Derivatives of $f(x) = xe^{-10x}$

Solution

$$f'(x) = e^{-10x} - 10xe^{-10x}$$

Exercise

Find the Derivatives of $f(x) = x \ln^2 x$

Solution

$$\begin{aligned} f'(x) &= \ln^2 x + x \left(2 \frac{1}{x} \ln x \right) \\ &= \ln^2 x + 2 \ln x \end{aligned}$$

Exercise

Find the Derivatives of $f(x) = e^{-x} \ln x$

Solution

$$\underline{f'(x) = -e^{-x} \ln x + \frac{e^{-x}}{x}}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt{x(x+1)}$

Solution

$$\ln y = \ln(x(x+1))^{1/2} = -\ln x - \frac{1}{2} \ln(x+1)$$

$$\frac{y'}{y} = \frac{1}{2} \left(\frac{1}{x} + \frac{1}{x+1} \right)$$

$$\frac{y'}{y} = \frac{1}{2} \left(\frac{2x+1}{x(x+1)} \right)$$

$$y' = \frac{1}{2} \left(\frac{2x+1}{x(x+1)} \right) \cdot y$$

$$\underline{= \frac{1}{2} \left(\frac{2x+1}{x(x+1)} \right) \sqrt{x(x+1)}}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt{(x^2+1)(x-1)^2}$

Solution

$$\ln y = \ln \left((x^2+1)(x-1)^2 \right)^{1/2}$$

$$= \frac{1}{2} \ln \left((x^2+1)(x-1)^2 \right)$$

$$= \frac{1}{2} \left[\ln(x^2+1) + \ln(x-1)^2 \right]$$

$$= \frac{1}{2} \left[\ln(x^2+1) + 2 \ln(x-1) \right]$$

$$= \frac{1}{2} \ln(x^2+1) + \ln(x-1)$$

$$\frac{y'}{y} = \frac{1}{2} \frac{2x}{x^2+1} + \frac{1}{x-1}$$

$$\begin{aligned}
&= \frac{x}{x^2+1} + \frac{1}{x-1} \\
&= \frac{x(x-1) + (x^2+1)}{(x^2+1)(x-1)} \\
&= \frac{x^2 - x + x^2 + 1}{(x^2+1)(x-1)} \\
&= \frac{2x^2 - x + 1}{(x^2+1)(x-1)}
\end{aligned}$$

$$\begin{aligned}
y' &= \frac{2x^2 - x + 1}{(x^2+1)(x-1)} \cdot y \\
&= \frac{2x^2 - x + 1}{(x^2+1)(x-1)} \sqrt{(x^2+1)(x-1)^2} \\
&= \frac{2x^2 - x + 1}{(x^2+1)(x-1)} |x-1| \sqrt{x^2+1} \\
&= \frac{(2x^2 - x + 1)|x-1|}{(x^2+1)(x-1)} (x^2+1)^{1/2} \\
&= \frac{(2x^2 - x + 1)|x-1|}{\sqrt{x^2+1}(x-1)}
\end{aligned}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt{\frac{1}{t(t+1)}}$

Solution

$$y = \left(\frac{1}{t(t+1)} \right)^{1/2}$$

$$\ln y = \ln \left(\frac{1}{t(t+1)} \right)^{1/2}$$

$$\begin{aligned}
\ln y &= \frac{1}{2} \ln \left(\frac{1}{t(t+1)} \right) \\
&= -\frac{1}{2} \ln(t(t+1)) \\
&= -\frac{1}{2} [\ln t + \ln(t+1)]
\end{aligned}$$

$$\begin{aligned}
\frac{y'}{y} &= -\frac{1}{2}\left(\frac{1}{t} + \frac{1}{t+1}\right) \\
y' &= -\frac{1}{2}\left(\frac{1}{t} + \frac{1}{t+1}\right)y \\
&= -\frac{1}{2}\left(\frac{t+1+t}{t(t+1)}\right)\frac{1}{(t(t+1))^{1/2}} \\
&= -\frac{1}{2}\frac{2t+1}{(t(t+1))^{3/2}} \\
&= -\frac{2t+1}{2(t^2+t)^{3/2}}
\end{aligned}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \frac{\theta+5}{\theta \cos \theta}$

Solution

$$\begin{aligned}
\ln y &= \ln\left(\frac{\theta+5}{\theta \cos \theta}\right) \\
\ln y &= \ln(\theta+5) - \ln(\theta \cos \theta) \\
\ln y &= \ln(\theta+5) - \ln \theta - \ln(\cos \theta) \\
\frac{y'}{y} &= \frac{1}{\theta+5} - \frac{1}{\theta} + \frac{\sin \theta}{\cos \theta} \\
y' &= \left(\frac{1}{\theta+5} - \frac{1}{\theta} + \frac{\sin \theta}{\cos \theta}\right)y \\
y' &= \left(\frac{\theta+5}{\theta \cos \theta}\right)\left(\frac{1}{\theta+5} - \frac{1}{\theta} + \tan \theta\right)
\end{aligned}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$

Solution

$$\begin{aligned}
\ln y &= \ln\left(\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}\right)^{1/3} \\
&= \frac{1}{3}\left[\ln x + \ln(x+1) + \ln(x-2) - \ln(x^2+1) - \ln(2x+3)\right] \\
\frac{y'}{y} &= \frac{1}{3}\left(\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3}\right)
\end{aligned}$$

$$y' = \frac{1}{3} \cdot \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}} \cdot \left(\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3} \right)$$

Exercise

Find the derivative of $y = t^{1-e}$

Solution

$$y' = (1-e)t^{1-e-1}$$

$$= (1-e)t^{-e}$$

Exercise

Find the derivative of $y = 2^{\sin 3t}$

Solution

$$y = a^u \Rightarrow y' = a^u \ln a \cdot (u')$$

$$y' = (2^{\sin 3t} \ln 2)(\cos 3t)(3)$$

$$= 3(\ln 2) \cos 3t (2^{\sin 3t})$$

Exercise

Find the derivative of $y = \log_3 (1 + \theta \ln 3)$

Solution

$$y = \frac{\ln(1 + \theta \ln 3)}{\ln 3}$$

$$y' = \frac{1}{\ln 3} \cdot \frac{\ln 3}{1 + \theta \ln 3}$$

$$= \frac{1}{1 + \theta \ln 3}$$

$$y = \ln u \Rightarrow y' = \frac{u'}{u}$$

Exercise

Find the derivative of $y = \log_{25} e^x - \log_5 \sqrt{x}$

Solution

$$\begin{aligned}
 y &= \frac{\ln e^x}{\ln 25} - \frac{\ln x^{1/2}}{\ln 5} \\
 &= \frac{x}{2 \ln 5} - \frac{1}{2} \frac{\ln x}{\ln 5} \\
 &= \frac{1}{2 \ln 5} (x - \ln x)
 \end{aligned}$$

$$\begin{aligned}
 y' &= \frac{1}{2 \ln 5} \left(1 - \frac{1}{x} \right) \\
 &= \frac{x-1}{2x \ln 5}
 \end{aligned}$$

Exercise

Find the derivative of $y = \log_3 r \cdot \log_9 r$

Solution

$$\begin{aligned}
 y &= \frac{\ln r}{\ln 3} \cdot \frac{\ln r}{\ln 9} \\
 &= \frac{1}{\ln 3 \cdot \ln 9} \cdot \ln^2 r
 \end{aligned}$$

$$\begin{aligned}
 y' &= \frac{1}{\ln 3 \cdot \ln 9} \cdot (2 \ln r) \left(\frac{1}{r} \right) \\
 &= \frac{2 \ln r}{r \cdot \ln 3 \cdot \ln 9}
 \end{aligned}$$

Exercise

Find the derivative of $y = \log_7 \left(\frac{\sin \theta \cos \theta}{e^\theta 2^\theta} \right)$

Solution

$$\begin{aligned}
 y &= \frac{\ln(\sin \theta) + \ln(\cos \theta) - \ln(e^\theta) - \ln(2^\theta)}{\ln 7} \\
 &= \frac{1}{\ln 7} [\ln(\sin \theta) + \ln(\cos \theta) - \theta - \theta \ln(2)]
 \end{aligned}$$

$$\begin{aligned}
 y' &= \frac{1}{\ln 7} \left[\frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} - 1 - \ln(2) \right] \\
 &= \frac{1}{\ln 7} (\cot \theta - \tan \theta - 1 - \ln 2)
 \end{aligned}$$

Exercise

Find the derivative of $y = 3 \log_8 \left(\log_2 t \right)$

Solution

$$y = 3 \frac{\ln \left(\log_2 t \right)}{\ln 8} = \frac{3}{\ln 8} \ln \left(\frac{\ln t}{\ln 2} \right)$$

$$\begin{aligned} y' &= \frac{3}{\ln 8} \left(\frac{1}{\frac{\ln t}{\ln 2}} \right) \left(\frac{1}{\ln 2} \cdot \frac{1}{t} \right) \\ &= \frac{3}{2 \ln 2} \left(\frac{\ln 2}{\ln t} \right) \left(\frac{1}{t \ln 2} \right) \\ &= \frac{1}{t (\ln t) (\ln 2)} \end{aligned}$$

Exercise

Find the derivative of $y = t \log_3 \left(e^{(\sin t)(\ln 3)} \right)$

Solution

$$\begin{aligned} y &= t \frac{\ln e^{(\sin t)(\ln 3)}}{\ln 3} \\ &= \frac{1}{\ln 3} t (\sin t) (\ln 3) \\ &= t \sin t \end{aligned}$$

$$y' = \sin t + t \cos t$$

Exercise

Find the derivative of $f(x) = \log_3 (x + 8)$

Solution

$$f'(x) = \frac{1}{\ln 3} \left(\frac{1}{x + 8} \right)$$

$$\frac{d}{dx} \left[\log_a u \right] = \left(\frac{1}{\ln a} \right) \left(\frac{1}{u} \right) \frac{du}{dx}$$

Exercise

Find the derivative of $f(x) = 2^{x^2 - x}$

Solution

$$f'(x) = (2x - 1)(\ln 2) 2^{x^2 - x}$$

$$\frac{d}{dx} [a^u] = a^u \ln(a) \frac{du}{dx}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = (x+1)^x$

Solution

$$\ln y = \ln(x+1)^x = x \cdot \ln(x+1)$$

$$\frac{y'}{y} = \ln(x+1) + x \cdot \frac{1}{x+1}$$

$$\underline{y' = (x+1)^x \left(\ln(x+1) + \frac{x}{x+1} \right)}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = x^2 + x^{2x}$

Solution

$$y - x^2 = x^{2x}$$

$$\ln(y - x^2) = \ln x^{2x} = 2x \ln x$$

$$\frac{1}{y - x^2} (y' - 2x) = 2 \ln x + 2x \frac{1}{x}$$

$$y' - 2x = (y - x^2)(2 \ln x + 2)$$

$$y' - 2x = (x^2 + x^{2x} - x^2)(2 \ln x + 2)$$

$$y' = 2x^{2x}(\ln x + 1) + 2x$$

$$\underline{= 2(x^{2x} \ln x + x^{2x} + x)}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = (\sin x)^x$

Solution

$$\ln y = \ln(\sin x)^x$$

$$\ln y = x \ln(\sin x)$$

$$u = x \quad v = \ln(\sin x)$$

$$u' = 1 \quad v' = \frac{\cos x}{\sin x}$$

$$\frac{y'}{y} = \ln(\sin x) + x \frac{\cos x}{\sin x}$$

$$y' = y(\ln(\sin x) + x \cot x)$$

$$\underline{= (\sin x)^x [\ln(\sin x) + x \cot x]}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = x^{\sin x}$

Solution

$$\ln y = \ln x^{\sin x}$$

$$\ln y = \sin x \ln x$$

$$\frac{y'}{y} = \cos x \ln x + \frac{\sin x}{x}$$

$$\frac{y'}{y} = \frac{x \cos x \ln x + \sin x}{x}$$

$$y' = y \frac{x \cos x \ln x + \sin x}{x}$$
$$= x^{\sin x} \frac{\sin x + x(\ln x)(\cos x)}{x}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = (\ln x)^{\ln x}$

Solution

$$\ln y = \ln (\ln x)^{\ln x}$$

$$\ln y = (\ln x) \ln (\ln x)$$

$$\frac{y'}{y} = \frac{1}{x} \ln (\ln x) + \ln x \frac{1}{\ln x}$$

$$y' = \left(\frac{1}{x} \ln (\ln x) + \frac{1}{x} \right) y$$

$$= \left(\frac{\ln (\ln x) + 1}{x} \right) (\ln x)^{\ln x}$$

Exercise

Find the second derivative of $y = 3 e^{5x^3+1}$

Solution

$$y' = 45x^2 e^{5x^3+1}$$

$$y'' = (90x + 675x^5) e^{5x^3+1}$$

Exercise

Find the equations of the tangent lines to $f(x) = e^x$ at the points (0, 1)

Solution

$$f'(x) = e^x$$

$$\begin{aligned}(0, 1) \Rightarrow m &= f'(x=0) \\ &= e^0 \\ &= 1\end{aligned}$$

$$y - 1 = 1(x - 0) + 1$$

$$\underline{y = x + 1}$$

$$y = m(x - x_1) + y_1$$

Exercise

Find the equations of the tangent lines to $f(x) = e^x$ at the points (1, e)

Solution

$$f'(x) = e^x$$

$$\begin{aligned}(1, e) \Rightarrow m &= f'(x=1) = e^1 = e \\ y &= e(x - 1) + e\end{aligned}$$

$$\underline{y = ex}$$

$$y = m(x - x_1) + y_1$$

Exercise

Find the equations of the tangent lines to $y = 4xe^{-x} + 5$ at $x = 1$

Solution

$$y' = 4e^{-x} - 4xe^{-x} = 4e^{-x}(1 - x)$$

$$= 4e^{-x}(1 - x)$$

$$m = y'(x=1)$$

$$= 4e^{-1}(1 - 1) = 0$$

$$\Rightarrow x = 1 \rightarrow y = 4e^{-1} + 5 \quad \left(1, 4e^{-1} + 5\right)$$

$$y = 0(x - 1) + 4e^{-1} + 5$$

$$\underline{y = 4e^{-1} + 5}$$

$$y = m(x - x_1) + y_1$$

Exercise

Find the equation of the tangent lines to $f(x) = 4e^{-8x}$ at the points (0, 4)

Solution

$$f'(x) = -32e^{-8x}$$

$$m = f'(0) = -32e^{-8(0)} = -32$$

$$y = -32(x - 0) + 4$$

$$\underline{y = -32x + 4}$$

$$y = m(x - x_1) + y_1$$

Exercise

The following formula accurately models the relationship between the size of a certain type of tumor and the amount of time that it has been growing:

$$V(t) = 450(1 - e - 0.0022t)^3$$

where t is in months and $V(t)$ is measured in cubic centimeters. Calculate the rate of change of tumor volume at 80 months.

Solution

$$U = 1 - e - 0.0022t \quad V = 450U^3$$

$$U' = -.0022 \quad V' = 450(3)U^2U'$$

$$V'(t) = 450(3)(1 - e - 0.0022t)^2(-.0022)$$

$$\underline{= 2.97(1 - e - 0.0022t)^2}$$

$$V'(t = 80) = 2.97(1 - e - 0.0022(80))^2$$

$$\underline{\approx 10.66}$$

Exercise

A yeast culture at room temperature ($68^\circ F$) is placed in a refrigerator set at a constant temperature of $38^\circ F$. After t hours, the temperature T of the culture is given approximately by

$$T = 30e^{-0.58t} + 38 \quad t \geq 0$$

What is the rate of change of temperature of the culture at the end of 1 hour? At the end of 4 hours?

Solution

$$T' = 30(-0.58)e^{-0.58t} \underline{= -17.4e^{-0.58t}}$$

$$T'(1) = -17.4e^{-0.58(1)} \approx -9.74^\circ F / hr$$

$$T'(4) = -17.4e^{-0.58(4)} \approx -1.71^\circ F / hr$$

Exercise

A mathematical model for the average age of a group of people learning to type is given by

$$N(t) = 10 + 6 \ln t \quad t \geq 1$$

Where $N(t)$ is the number of words per minute typed after t hours of instruction and practice (2 hours per day, 5 days per week). What is the rate of learning after 10 hours of instruction and practice? After 100 hours?

Solution

$$N'(t) = \frac{6}{t}$$

$$N'(10) = \frac{6}{10} = 0.6$$

After 10 hours of instruction and practice, the rate of learning is 0.6 words/minute per hour of instruction and practice.

$$N'(100) = \frac{6}{100} = 0.06$$

After 100 hours of instruction and practice, the rate of learning is 0.06 words/minute per hour of instruction and practice.

Exercise

The population of coyotes in the northwestern portion of Alabama is given by the formula

$P(t) = (t^2 + 100) \ln(t + 2)$, where t represents the time in years since 2000 (the year 2000 corresponds to $t = 0$). Find the rate of change of the coyote population in 2013 ($t = 13$).

Solution

$$P'(t) = 2t \ln(t + 2) + \frac{1}{t + 2} (t^2 + 100)$$

$$P' = f'g + g'f$$

$$f = t^2 + 100 \quad g = \ln(t + 2)$$

$$f' = 2t \quad g' = \frac{1}{t + 2}$$

$$= 2t \ln(t + 2) + \frac{t^2 + 100}{t + 2}$$

$$P'(t = 13) = 2(13) \ln(13 + 2) + \frac{13^2 + 100}{13 + 2}$$

$$\approx 88.34$$

$$2 * 13 \ln(13 + 2) + (13^2 + 100) / (13 + 2)$$

Solution **Section 2.9 – Derivatives of Inverse Trigonometric Functions**

Exercise

Find the value of $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

Solution

$$\begin{aligned}\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) &= \sin\left(\frac{\pi}{4}\right) \\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

Exercise

Find the value of $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

Solution

$$\begin{aligned}\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) &= \cot\left(-\frac{\pi}{3}\right) \\ &= -\frac{1}{\sqrt{3}}\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow -1^+} \cos^{-1} x$

Solution

$$\begin{aligned}\lim_{x \rightarrow -1^+} \cos^{-1} x &= \cos^{-1}(-1) \\ &= \pi\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow -\infty} \tan^{-1} x$

Solution

$$\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

Exercise

Find the limit: $\lim_{x \rightarrow \infty} \csc^{-1} x$

Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \csc^{-1} x &= \lim_{x \rightarrow \infty} \sin^{-1} \left(\frac{1}{x} \right) \\ &= \sin^{-1} \left(\frac{1}{\infty} \right) \\ &= \underline{0} \quad | \end{aligned}$$

Exercise

Find the derivative $y = \cos^{-1} \left(\frac{1}{x} \right)$

Solution

$$\begin{aligned}y &= \cos^{-1} \left(\frac{1}{x} \right) \\ &= \sec^{-1}(x) \\ y' &= \underline{\frac{1}{|x| \cdot \sqrt{x^2 - 1}}} \quad | \end{aligned}$$

Exercise

Find the derivative $y = \sin^{-1} \sqrt{2}t$

Solution

$$\begin{aligned}y' &= \frac{\sqrt{2}}{\sqrt{1 - (\sqrt{2}t)^2}} \\ &= \underline{\frac{\sqrt{2}}{\sqrt{1 - 2t^2}}} \quad | \end{aligned}$$

Exercise

Find the derivative $y = \sec^{-1}(5s)$

Solution

$$y' = \frac{5s}{|5s| \sqrt{(5s)^2 - 1}}$$

$$= \frac{s}{|s|\sqrt{25s^2 - 1}} \Bigg|$$

Exercise

Find the derivative $y = \cot^{-1} \sqrt{t-1}$

Solution

$$\begin{aligned} y' &= -\frac{\frac{1}{2}(t-1)^{-1/2}}{1 + \left[(t-1)^{1/2}\right]^2} \\ &= -\frac{1}{2(t-1)^{1/2}(1+t-1)} \\ &= -\frac{1}{2t\sqrt{t-1}} \Bigg| \end{aligned}$$

Exercise

Find the derivative $y = \ln(\tan^{-1} x)$

Solution

$$\begin{aligned} y' &= \frac{\frac{1}{1+x^2}}{\tan^{-1} x} \\ &= \frac{1}{(1+x^2)\tan^{-1} x} \Bigg| \end{aligned}$$

Exercise

Find the derivative $y = \tan^{-1}(\ln x)$

Solution

$$\begin{aligned} y' &= \frac{\frac{1}{x}}{1 + (\ln x)^2} \\ &= \frac{1}{x[1 + (\ln x)^2]} \Bigg| \end{aligned}$$

$$\left(\tan^{-1} u\right)' = \frac{u'}{1+u^2}$$

Exercise

Find the derivative $y = \csc^{-1}(e^t)$

Solution

$$\begin{aligned} y' &= -\frac{e^t}{|e^t|\sqrt{(e^t)^2 - 1}} \\ &= -\frac{1}{\sqrt{e^{2t} - 1}} \end{aligned}$$

Exercise

Find the derivative $y = x\sqrt{1-x^2} + \cos^{-1}x$

Solution

$$\begin{aligned} y' &= \sqrt{1-x^2} + x\left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x) - \frac{1}{\sqrt{1-x^2}} \\ &= \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \\ &= \frac{1-x^2-x^2-1}{\sqrt{1-x^2}} \\ &= \frac{-2x^2}{\sqrt{1-x^2}} \end{aligned}$$

Exercise

Find the derivative $y = \ln(x^2 + 4) - x \tan^{-1}\left(\frac{x}{2}\right)$

Solution

$$\begin{aligned} y' &= \frac{2x}{x^2 + 4} - \tan^{-1}\left(\frac{x}{2}\right) - x \frac{\frac{1}{2}}{1 + \left(\frac{x}{2}\right)^2} \\ &= \frac{2x}{x^2 + 4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{x}{2} \cdot \frac{1}{1 + \frac{x^2}{4}} \\ &= \frac{2x}{x^2 + 4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{x}{2} \cdot \frac{4}{4 + x^2} \\ &= \frac{2x}{x^2 + 4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{2x}{4 + x^2} \end{aligned}$$

$$\left. = -\tan^{-1}\left(\frac{x}{2}\right) \right|$$

Exercise

Find the derivative $f(x) = \sin^{-1} \frac{1}{x}$

Solution

$$\begin{aligned} f'(x) &= -\frac{1}{x^2} \frac{1}{\sqrt{1 - \left(\frac{1}{x}\right)^2}} \\ &= \frac{-1}{|x|\sqrt{x^2 - 1}} \end{aligned}$$

Exercise

Find the derivative $\left. \frac{d}{dx}(x \sec^{-1} x) \right|_{x=\frac{2}{\sqrt{3}}}$

Solution

$$\begin{aligned} \frac{d}{dx}(x \sec^{-1} x) &= \sec^{-1} x + \frac{x}{x\sqrt{x^2 - 1}} \bigg|_{x=\frac{2}{\sqrt{3}}} \\ &= \sec^{-1} \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{\frac{4}{3} - 1}} \\ &= \frac{\pi}{6} + \sqrt{3} \end{aligned}$$

Exercise

Find the derivative $\left. \frac{d}{dx}(\tan^{-1} e^{-x}) \right|_{x=0}$

Solution

$$\begin{aligned} \frac{d}{dx}(\tan^{-1} e^{-x}) &= \frac{-e^{-x}}{1 + e^{-2x}} \bigg|_{x=0} \\ &= -\frac{1}{2} \end{aligned}$$

Exercise

Find the angle α

Solution

$$65^\circ + (90^\circ - \beta) + (90^\circ - \alpha) = 180^\circ$$

$$65^\circ + 180^\circ - \beta - \alpha = 180^\circ$$

$$\beta + \alpha = 65^\circ$$

$$\underline{\alpha = 65^\circ - \beta}$$

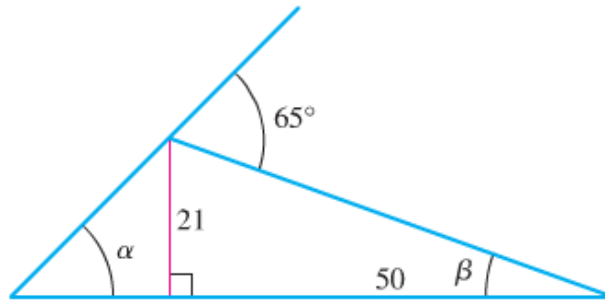
$$\tan \beta = \frac{21}{50}$$

$$\beta = \tan^{-1}\left(\frac{21}{50}\right)$$

$$\underline{\approx 22.78^\circ}$$

$$\underline{\alpha \approx 65^\circ - 22.78^\circ}$$

$$\underline{\approx 42.22^\circ}$$



Solution **Section 2.10 – Related Rates**

Exercise

If $y = x^2$ and $\frac{dx}{dt} = 3$, then what is $\frac{dy}{dt}$ when $x = -1$

Solution

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dx} \frac{dx}{dt} \\ &= 2x(3) \\ &= \underline{6x}\end{aligned}$$

$$\begin{aligned}\frac{dy}{dt} \Big|_{x=-1} &= 6(-1) \\ &= \underline{-6}\end{aligned}$$

Exercise

If $x = y^3 - y$ and $\frac{dy}{dt} = 5$, then what is $\frac{dx}{dt}$ when $y = 2$

Solution

$$\begin{aligned}\frac{dx}{dt} &= \frac{dx}{dy} \frac{dy}{dt} \\ &= (3y^2 - 1)(5) \\ &= 5(3y^2 - 1)\end{aligned}$$

$$\begin{aligned}\frac{dx}{dt} \Big|_{y=2} &= 5(3(2)^2 - 1) \\ &= \underline{55}\end{aligned}$$

Exercise

A cube's surface area increases at the rate of $72 \text{ in}^2 / \text{sec}$. At what rate is the cube's volume changing when the edge length is $x = 3 \text{ in}$?

Solution

Cube's surface: $S = 6x^2$

$$\frac{dS}{dt} = 12x \frac{dx}{dt}$$

$$72 = 12x(3)$$

$$x = \frac{72}{26}$$

$$\begin{aligned}
 &= 2 \Big| \\
 V &= x^3 \\
 \frac{dV}{dt} &= 3x^2 \frac{dx}{dt} \\
 \frac{dV}{dt} \Big|_{x=3} &= 3(3)^2(2) \\
 &= 54 \text{ in}^2 / \text{sec} \Big|
 \end{aligned}$$

Exercise

The radius r and height h of a right circular cone are related to the cone's volume V by the equation

$$V = \frac{1}{3}\pi r^2 h.$$

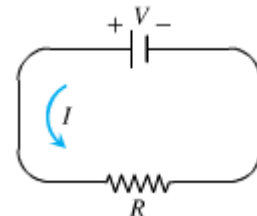
- How is $\frac{dV}{dt}$ related to $\frac{dh}{dt}$ if r is constant?
- How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ if h is constant?
- How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ and $\frac{dh}{dt}$ if neither r nor h is constant?

Solution

- $\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt}$
- $\frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt}$
- $\frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$

Exercise

The voltage V (volts), current I (amperes), and resistance R (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of $\frac{1}{3}$ amp / sec. Let t denote time in seconds.



- What is the value of $\frac{dV}{dt}$?
- What is the value of $\frac{dI}{dt}$?
- What equation relates $\frac{dR}{dt}$ to $\frac{dV}{dt}$ and $\frac{dI}{dt}$?
- Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amp. Is R increasing or decreasing?

Solution

$$a) \quad \frac{dV}{dt} = 1 \text{ volt / sec} \Big|$$

$$b) \left. \frac{dI}{dt} = \frac{1}{3} \text{ amp / sec} \right|$$

$$c) \frac{dV}{dt} = R \frac{dI}{dt} + I \frac{dR}{dt}$$

$$I \frac{dR}{dt} = \frac{dV}{dt} - R \frac{dI}{dt}$$

$$V = IR \Rightarrow R = \frac{V}{I}$$

$$\frac{dR}{dt} = \frac{1}{I} \left(\frac{dV}{dt} - \frac{V}{I} \frac{dI}{dt} \right)$$

$$d) \left. \begin{aligned} \frac{dR}{dt} &= \frac{1}{2} \left((1) - \frac{12}{2} \left(-\frac{1}{3} \right) \right) \\ &= \frac{3}{2} \text{ ohms / sec} \end{aligned} \right|$$

R is increasing

Exercise

Let x and y be differentiable functions of t and let $s = \sqrt{x^2 + y^2}$ be the distance between the points $(x, 0)$ and $(0, y)$ in the xy -plane.

a) How is $\frac{ds}{dt}$ related to $\frac{dx}{dt}$ if y is constant?

b) How is $\frac{ds}{dt}$ related to $\frac{dx}{dt}$ and $\frac{dy}{dt}$ if neither x nor y is constant?

c) How is $\frac{dx}{dt}$ related to $\frac{dy}{dt}$ if s is constant?

Solution

$$s = \sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2}$$

$$a) \frac{ds}{dt} = \frac{1}{2} (x^2 + y^2)^{-1/2} \left(2x \frac{dx}{dt} \right)$$

$$= \frac{x}{\sqrt{x^2 + y^2}} \frac{dx}{dt} \left| \right.$$

$$b) \frac{ds}{dt} = \frac{1}{2} (x^2 + y^2)^{-1/2} \left(2x \frac{dx}{dt} + 2y \frac{dy}{dt} \right)$$

$$= \frac{1}{\sqrt{x^2 + y^2}} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right)$$

$$= \frac{x}{\sqrt{x^2 + y^2}} \frac{dx}{dt} + \frac{y}{\sqrt{x^2 + y^2}} \frac{dy}{dt} \left| \right.$$

$$c) s = \sqrt{x^2 + y^2}$$

$$s^2 = x^2 + y^2$$

$$0 = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$2x \frac{dx}{dt} = -2y \frac{dy}{dt}$$

$$\boxed{\frac{dx}{dt} = -\frac{y}{x} \frac{dy}{dt}}$$

Exercise

A 13-foot ladder is leaning against a house when its base starts to slide away. By the time the base is 12 feet from the house, the base is moving at the rate of 5 ft/sec.

- How fast is the top of the ladder sliding down the wall then?
- At what rate is the area of the triangle formed by the ladder, wall, and the ground changing then?
- At what rate is the angle θ between the ladder and the ground changing then?

Solution

Given: $L = 13 \text{ ft}$ $x = 12$ $\frac{dx}{dt} = 5 \text{ ft/sec}$

$$y = \sqrt{13^2 - 12^2} = 5$$

a) $x^2 + y^2 = 13^2$

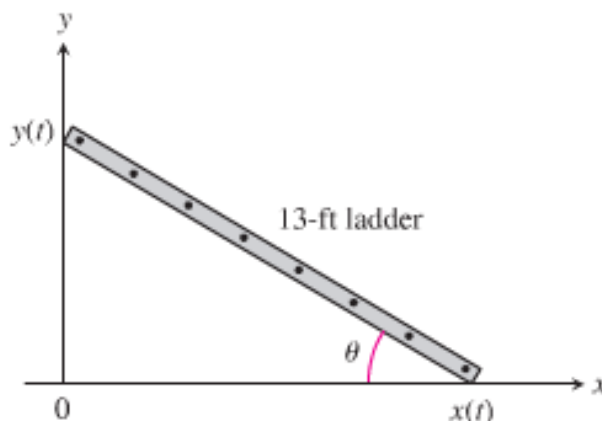
$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$y \frac{dy}{dt} = -x \frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$$

$$= -\frac{12}{5}(5)$$

$$\boxed{= -12 \text{ ft/sec}}$$



The ladder is sliding down the wall

- b) Area of the triangle formed by the ladder and the walls is: $A = \frac{1}{2}xy$

$$\frac{dA}{dt} = \frac{1}{2} \left(y \frac{dx}{dt} + x \frac{dy}{dt} \right)$$

$$= \frac{1}{2} ((5)(5) + (12)(-12))$$

$$\boxed{= -19.5 \text{ ft}^2/\text{sec}}$$

c) $\cos \theta = \frac{x}{13}$

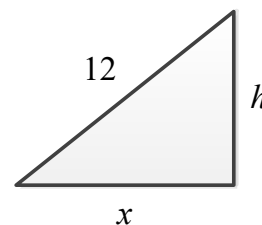
$$-\sin \theta \frac{d\theta}{dt} = \frac{1}{13} \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = -\frac{1}{13 \sin \theta} \frac{dx}{dt}$$

$$= -\frac{1}{13 \sin \theta} (5) \quad \sin \theta = \frac{5}{13}$$

$$= -\frac{1}{13 \left(\frac{5}{13} \right)} (5)$$

$$= -1 \text{ rad / sec}$$



Exercise

A 13-foot ladder is leaning against a vertical wall when he begins pulling the foot of the ladder away from the wall at a rate of 0.5 ft/s. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 5 feet from the wall?

Solution

$$x^2 + h^2 = 13^2$$

$$x^2 + h^2 = 169$$

$$h = \sqrt{169 - 25}$$

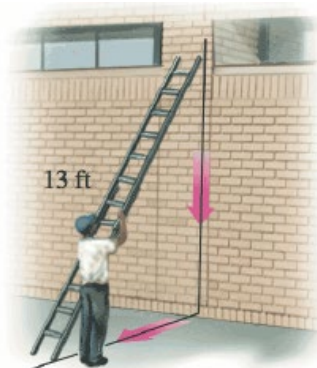
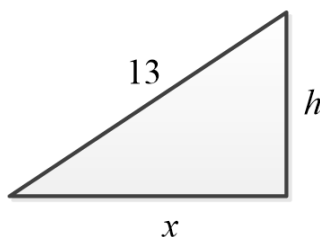
$$= 12$$

$$2x \frac{dx}{dt} + 2h \frac{dh}{dt} = 0$$

$$\frac{dh}{dt} = -\frac{x}{h} \frac{dx}{dt}$$

$$= -\frac{5}{12} (0.5)$$

$$= -\frac{5}{24} \text{ ft / sec}$$



So, the top of the ladder slides down the wall at $\frac{5}{24}$ ft / sec

Exercise

A 12-foot ladder is leaning against a vertical wall when he begins pulling the foot of the ladder away from the wall at a rate of 0.2 ft/s. What is the configuration of the ladder at the instant that the vertical speed of the top of the ladder equals the horizontal speed of the foot of the ladder?

Solution

$$x^2 + h^2 = 144$$

$$2x \frac{dx}{dt} + 2h \frac{dh}{dt} = 0$$

$$x \frac{dx}{dt} + h \frac{dh}{dt} = 0$$

The vertical speed of the top of the ladder equals the horizontal speed of the foot of the ladder.

$$\frac{dx}{dt} = 0.2$$

$$\frac{dh}{dt} = -0.2$$

$$0.2x - 0.2h = 0$$

$$x = h$$

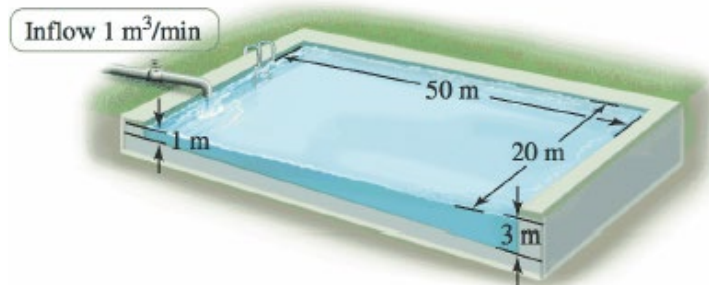
Since $x = h$, the triangle is forming a $(45^\circ - 45^\circ - 90^\circ)$ with

$$\begin{aligned} |x = h| &= 12 \cos 45^\circ \\ &= 6\sqrt{2} \end{aligned}$$

Exercise

A swimming pool is 50 m long and 20 m wide. Its length decreases linearly along the length from 3 m to 1 m. It is initially empty and is filled at a rate of $1 \text{ m}^3 / \text{min}$.

- How fast is the water level rising 250 min after the filling begins?
- How long will it take to fill the pool?



Solution

$$\frac{h}{2} = \frac{b}{50}$$

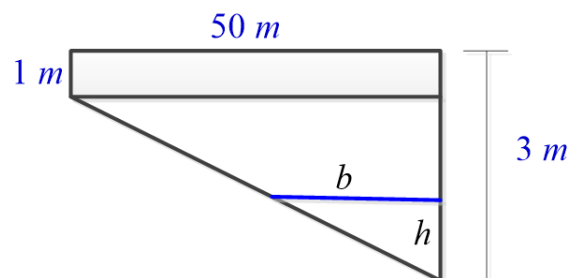
$$b = 25h$$

The area of the side:

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{25}{2}h^2 \end{aligned}$$

For $0 \leq h \leq 2$

$$\begin{aligned} V(h) &= 12.5h^2(20) \\ &= 250h^2 \end{aligned}$$



For $2 < h \leq 3$

$$V(h) = 250 \times 2^2 + 50 \times 20 \times (h - 2)$$

$$= 1000h - 1000$$

a) When $t = 250$ min

$$V = 250 \text{ min} \times 1 \frac{m^3}{\text{min}}$$

$$= 250 \text{ m}^3$$

$$\text{So } V(h) = 250h^2 = 250$$

$$\rightarrow h = 1$$

$$\frac{dV}{dt} = 500h \frac{dh}{dt}$$

$$= 1 \frac{m^3}{\text{min}}$$

$$\frac{dh}{dt} = \frac{1}{500} \frac{m}{\text{min}}$$

$$= .002 \frac{m}{\text{min}}$$

b) $V(h) = 1000(3) - 1000$

$$= 2000 \text{ m}^3$$

$$\text{Since } \frac{dV}{dt} = 1 \frac{m^3}{\text{min}}$$

Then it will take 2,000 minutes.

Exercise

An inverted conical water tank with a height of 12 feet and a radius of 6 feet is drained through a hole in the vertex at a rate of $2 \text{ ft}^3 / \text{sec}$. What is the rate of change of the water depth when the water depth is 3 feet?

Solution

Given: $\frac{dV}{dt} = -2 \frac{\text{ft}^3}{\text{min}}$

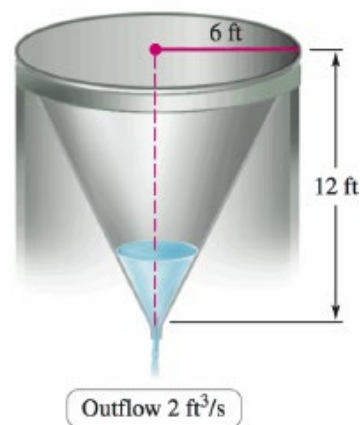
The water forms a cone with volume:

$$V = \frac{1}{3} \pi r^2 h$$

From the triangles:

$$\frac{r}{6} = \frac{h}{12} \Rightarrow r = \frac{1}{2} h$$

$$V = \frac{1}{3} \pi \left(\frac{h}{2} \right)^2 h$$



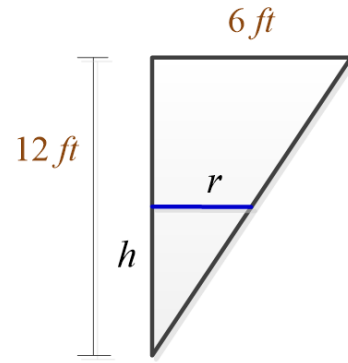
$$= \frac{1}{12} \pi h^3 \Big|$$

$$\frac{dV}{dt} = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$$

$$-2 = \frac{\pi 3^2}{4} \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{8}{9\pi} \text{ ft/s} \Big|$$

So, the depth of the water is decreasing at a rate of $\frac{8}{9\pi} \text{ ft/s}$



Exercise

Water runs into a conical tank at the rate of $6 \text{ ft}^3/\text{min}$. The tank stands point down and has a height of 20 feet and a base radius of 8 feet. How fast is the water level rising when the water is 6 feet deep?

Solution

Given: $\frac{dV}{dt} = 6 \frac{\text{ft}^3}{\text{min}}, \quad y = 20 \text{ ft}, \quad x = 8 \text{ ft}, \quad h = 6 \text{ ft}$

The water forms a cone with volume:

$$V = \frac{1}{3} \pi r^2 h$$

From the triangles:

$$\frac{r}{h} = \frac{8}{20} \Rightarrow r = \frac{2}{5} h$$

$$V = \frac{1}{3} \pi \left(\frac{2}{5} h \right)^2 h$$

$$= \frac{4\pi}{75} h^3 \Big|$$

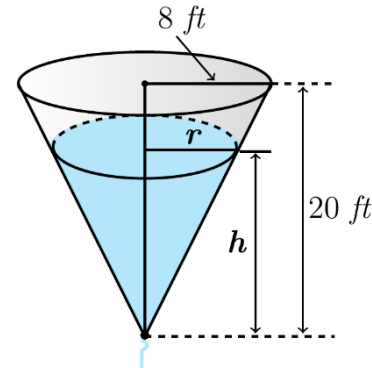
$$\frac{dV}{dt} = \frac{4\pi}{75} \left(3h^2 \frac{dh}{dt} \right)$$

$$\frac{dV}{dt} = \frac{4\pi}{25} h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{25}{4\pi h^2} \frac{dV}{dt}$$

$$= \frac{25}{4\pi (36)} (6)$$

$$= \frac{25}{24\pi} \text{ ft/min} \Big|$$



The water level is rising at about 0.33157 ft/min .

Exercise

Water runs into a conical tank at the rate of $5 \text{ ft}^3/\text{min}$. The tank stands point down and has a height of 10 feet and a base radius of 4 feet. How fast is the water level rising when the water is 6 feet deep?

Solution

Given: $\frac{dV}{dt} = 5 \frac{\text{ft}^3}{\text{min}}, \quad y = 10 \text{ ft}, \quad x = 4 \text{ ft}, \quad h = 6 \text{ ft}$

The water forms a cone with volume:

$$V = \frac{1}{3}\pi r^2 h$$

From the triangles:

$$\frac{r}{h} = \frac{4}{10} \Rightarrow r = \frac{2}{5}h$$

$$\begin{aligned} V &= \frac{1}{3}\pi \left(\frac{2}{5}h\right)^2 h \\ &= \frac{4\pi}{75}h^3 \end{aligned}$$

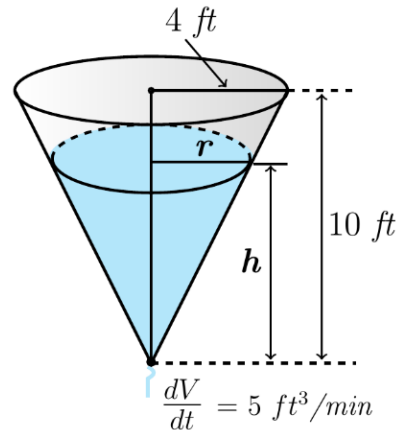
$$\frac{dV}{dt} = \frac{4\pi}{75} \left(3h^2 \frac{dh}{dt}\right)$$

$$\frac{dV}{dt} = \frac{4\pi}{25} h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{25}{4\pi h^2} \frac{dV}{dt}$$

$$= \frac{25}{4\pi(36)}(5)$$

$$= \frac{125}{144\pi} \text{ ft/min}$$



The water level is rising at about 0.2763 ft/min .

Exercise

Water runs into a conical tank at the rate of $5 \text{ ft}^3/\text{min}$. The tank stands point down and has a height of 20 feet and a base radius of 5 feet. How fast is the water level rising when the water is 4 feet deep?

Solution

Given: $\frac{dV}{dt} = 5 \frac{\text{ft}^3}{\text{min}}, \quad y = 20 \text{ ft}, \quad x = 5 \text{ ft}, \quad h = 4 \text{ ft}$

The water forms a cone with volume:

$$V = \frac{1}{3}\pi r^2 h$$

From the triangles:

$$\frac{r}{h} = \frac{5}{20} \Rightarrow r = \frac{1}{4}h$$

$$V = \frac{1}{3}\pi\left(\frac{1}{4}h\right)^2 h$$

$$= \frac{\pi}{48}h^3 \quad \Big|$$

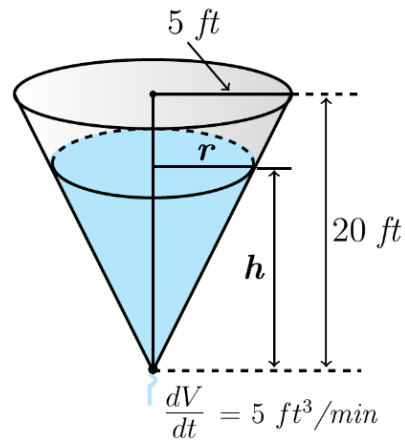
$$\frac{dV}{dt} = \frac{\pi}{48}\left(3h^2 \frac{dh}{dt}\right)$$

$$\frac{dV}{dt} = \frac{\pi}{16}h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{16}{\pi h^2} \frac{dV}{dt}$$

$$= \frac{16}{\pi(16)}(5)$$

$$= \frac{5}{\pi} \text{ ft/min} \quad \Big|$$



The water level is rising at about 1.59155 *ft/min*.

Exercise

Water runs into a conical tank at the rate of $4 \text{ ft}^3/\text{min}$. The tank stands point down and has a height of 20 feet and a base radius of 5 feet. How fast is the water level rising when the water is 5 feet deep?

Solution

Given: $\frac{dV}{dt} = 4 \frac{\text{ft}^3}{\text{min}}$, $y = 20 \text{ ft}$, $x = 5 \text{ ft}$, $h = 5 \text{ ft}$

The water forms a cone with volume:

$$V = \frac{1}{3}\pi r^2 h$$

From the triangles:

$$\frac{r}{h} = \frac{5}{20} \Rightarrow r = \frac{1}{4}h$$

$$V = \frac{1}{3}\pi\left(\frac{1}{4}h\right)^2 h$$

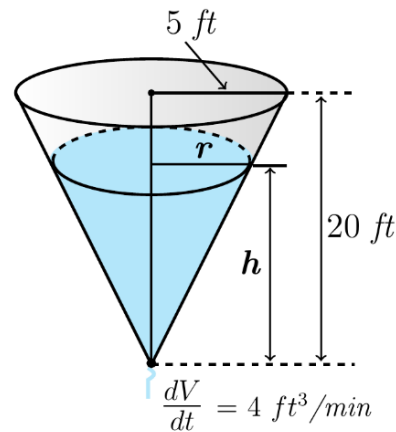
$$= \frac{\pi}{48}h^3 \quad \Big|$$

$$\frac{dV}{dt} = \frac{\pi}{48}\left(3h^2 \frac{dh}{dt}\right)$$

$$\frac{dV}{dt} = \frac{\pi}{16}h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{16}{\pi h^2} \frac{dV}{dt}$$

$$= \frac{16}{\pi(25)}(4)$$



$$= \frac{64}{25\pi} \text{ ft/min} \Big|$$

The water level is rising at about 0.814873 *ft/min*.

Exercise

Water runs into a conical tank at the rate of $2 \text{ ft}^3/\text{min}$. The tank stands point down and has a height of 10 feet and a base radius of 5 feet. How fast is the water level rising when the water is 4 feet deep?

Solution

Given: $\frac{dV}{dt} = 2 \frac{\text{ft}^3}{\text{min}}, \quad y = 10 \text{ ft}, \quad x = 5 \text{ ft}, \quad h = 4 \text{ ft}$

The water forms a cone with volume:

$$V = \frac{1}{3} \pi r^2 h$$

From the triangles:

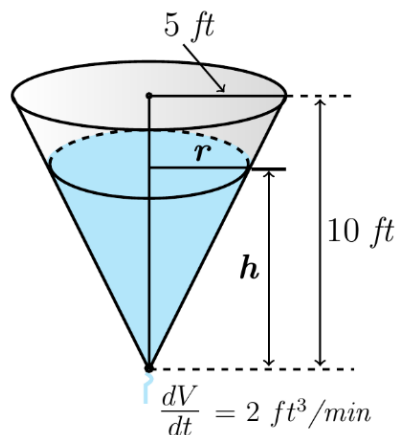
$$\frac{r}{h} = \frac{5}{10} \Rightarrow r = \frac{1}{2} h$$

$$\begin{aligned} V &= \frac{1}{3} \pi \left(\frac{1}{2} h \right)^2 h \\ &= \frac{\pi}{12} h^3 \Big| \end{aligned}$$

$$\begin{aligned} \frac{dV}{dt} &= \frac{\pi}{12} \left(3h^2 \frac{dh}{dt} \right) \\ &= \frac{\pi}{4} h^2 \frac{dh}{dt} \end{aligned}$$

$$\begin{aligned} \frac{dh}{dt} &= \frac{4}{\pi h^2} \frac{dV}{dt} \\ &= \frac{4}{\pi (16)} (2) \\ &= \frac{1}{2\pi} \text{ ft/min} \Big| \end{aligned}$$

The water level is rising at about 0.159155 *ft/min*.



Exercise

Water runs into a conical tank at the rate of $3 \text{ ft}^3/\text{min}$. The tank stands point down and has a height of 18 feet and a base radius of 6 feet. How fast is the water level rising when the water is 6 feet deep?

Solution

Given: $\frac{dV}{dt} = 3 \frac{\text{ft}^3}{\text{min}}, \quad y = 18 \text{ ft}, \quad x = 6 \text{ ft}, \quad h = 6 \text{ ft}$

The water forms a cone with volume:

$$V = \frac{1}{3}\pi r^2 h$$

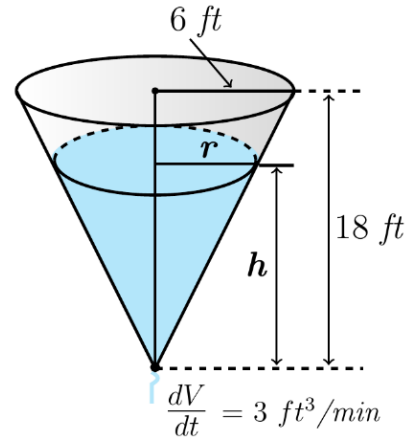
From the triangles:

$$\frac{r}{h} = \frac{6}{18} \Rightarrow r = \frac{1}{3}h$$

$$\begin{aligned} V &= \frac{1}{3}\pi \left(\frac{1}{3}h\right)^2 h \\ &= \frac{\pi}{27}h^3 \end{aligned}$$

$$\begin{aligned} \frac{dV}{dt} &= \frac{\pi}{27} \left(3h^2 \frac{dh}{dt}\right) \\ &= \frac{\pi}{9}h^2 \frac{dh}{dt} \end{aligned}$$

$$\begin{aligned} \frac{dh}{dt} &= \frac{9}{\pi h^2} \frac{dV}{dt} \\ &= \frac{9}{\pi(36)}(3) \\ &= \frac{3}{4\pi} \text{ ft/min} \end{aligned}$$



The water level is rising at about 0.23873 ft/min .

Exercise

Water runs into a conical tank at the rate of $6 \text{ ft}^3/\text{min}$. The tank stands point down and has a height of 18 feet and a base radius of 6 feet. How fast is the water level rising when the water is 12 feet deep?

Solution

Given: $\frac{dV}{dt} = 6 \frac{\text{ft}^3}{\text{min}}, \quad y = 18 \text{ ft}, \quad x = 6 \text{ ft}, \quad h = 12 \text{ ft}$

The water forms a cone with volume:

$$V = \frac{1}{3}\pi r^2 h$$

From the triangles:

$$\frac{r}{h} = \frac{6}{18} \Rightarrow r = \frac{1}{3}h$$

$$V = \frac{1}{3} \pi \left(\frac{1}{3} h \right)^2 h$$

$$= \frac{\pi}{27} h^3$$

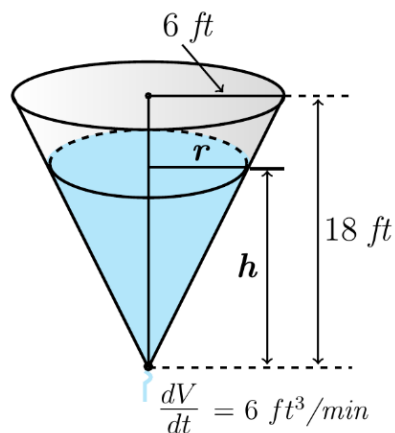
$$\frac{dV}{dt} = \frac{\pi}{27} \left(3h^2 \frac{dh}{dt} \right)$$

$$= \frac{\pi}{9} h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{9}{\pi h^2} \frac{dV}{dt}$$

$$= \frac{9}{\pi (144)} (6)$$

$$= \frac{3}{8\pi} \text{ ft/min}$$



The water level is rising at about 0.119366 *ft/min*.

Exercise

A hemispherical tank with a radius of 10 *m* is filled from an inflow pipe at a rate of 3 *m³/min*. (Hint:

The volume of a cap of thickness *h* sliced from a sphere of radius *r* is $\frac{\pi h^2 (3r - h)}{3}$).

- How fast is the water level rising when the water level is 5 *m* from the bottom of the tank?
- What is the rate of change of the surface area of the water when the water is 5 *m* deep?

Solution

Given: $\frac{dV}{dt} = 3 \frac{\text{m}^3}{\text{min}}$, $r = 10 \text{ m}$

$$a) \quad V(h) = \frac{1}{3} \pi h^2 (3r - h)$$

$$= 10\pi h^2 - \frac{1}{3} \pi h^3$$

$$\frac{dV}{dt} = (20\pi h - \pi h^2) \frac{dh}{dt}$$

When $h = 5 \text{ m}$

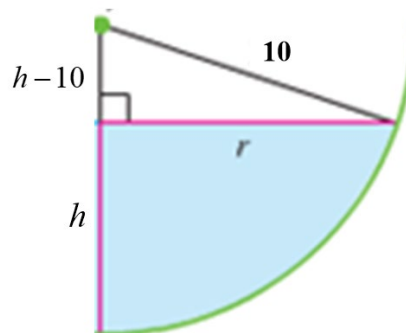
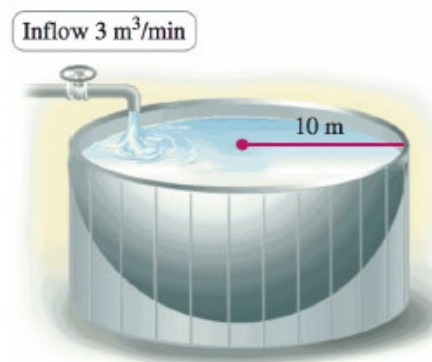
$$3 = (100 - 25) \pi \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{3}{75\pi} \text{ m/min}$$

$$b) \quad S = \pi r^2 \Rightarrow \frac{dS}{dt} = 2\pi r \frac{dr}{dt}$$

From the right triangle:

$$10^2 = r^2 + (10 - h)^2$$



$$100 = r^2 + 100 - 20h + h^2$$

$$20h = h^2 + r^2$$

$$h = 5, \quad r = \sqrt{100 - 25} = 5\sqrt{3}$$

$$20 \frac{dh}{dt} = 2h \frac{dh}{dt} + 2r \frac{dr}{dt}$$

$$10 \frac{dh}{dt} = h \frac{dh}{dt} + r \frac{dr}{dt}$$

$$5\sqrt{3} \frac{dr}{dt} = 10 \frac{3}{75\pi} - 5 \frac{3}{75\pi}$$

$$\frac{dr}{dt} = \frac{1}{5\sqrt{3}} \frac{15}{75\pi} = \frac{\sqrt{3}}{75\pi}$$

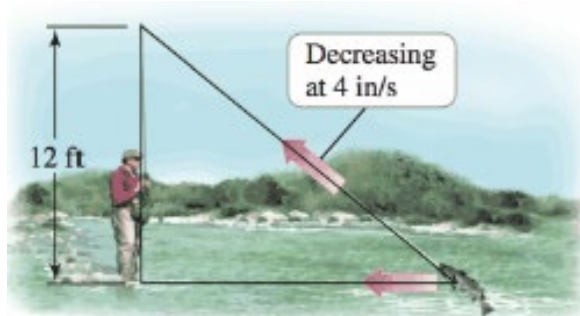
$$\frac{dS}{dt} = 2\pi r \frac{dr}{dt}$$

$$= 2\pi 5\sqrt{3} \frac{\sqrt{3}}{75\pi}$$

$$= \frac{2}{5} \frac{m^2}{min}$$

Exercise

A fisherman hooks a trout and reels in his line at 4 in/sec. Assume the trip of the fishing rod is 12 feet above the water directly above the fisherman and the fish is pulled horizontally directly towards the fisherman. Find the horizontal speed of the fish when it is 20 feet from the fisherman.



Solution

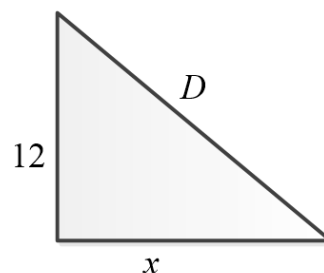
Let x be the distance between the fisherman's feet & fish.

Let D be the distance between the fisherman's head & the fish.

Given: $\frac{dD}{dt} = -4 \text{ in / s}, \quad x = 20 \text{ ft}$

$$D^2 = x^2 + 144$$

$$2D \frac{dD}{dt} = 2x \frac{dx}{dt}$$



$$\begin{aligned}\frac{dx}{dt} &= \frac{D}{x} \frac{dD}{dt} \\ &= \frac{\sqrt{20^2 + 144}}{20} (-4) \\ &\approx -4.66 \frac{\text{in}}{\text{s}}\end{aligned}$$

The fish is moving toward the fisherman at about 4.66 inches per second.

Exercise

Water is flowing at the rate of 6 from a reservoir shaped like a hemispherical bowl of radius 13 m. Answer the following questions, given that the volume of water in a hemispherical bowl of radius R is $V = \frac{\pi}{3} y^2 (3R - y)$ when the water is y meters deep.

- At what rate the water level changing when the water is 8 m deep?
- What is the radius r of the water's surface when the water is y m deep?
- At what rate is the radius r changing when the water is 8 m deep?

Solution

Given: $\frac{dV}{dt} = 6 \text{ m}^3 / \text{min}$ $R = 13 \text{ m}$

$$\begin{aligned}a) \quad V &= \frac{\pi}{3} y^2 (3R - y) \\ &= \pi R y^2 - \frac{\pi}{3} y^3\end{aligned}$$

$$\frac{dV}{dt} = \left(2\pi R y - \pi y^2 \right) \frac{dy}{dt}$$

Factor πy

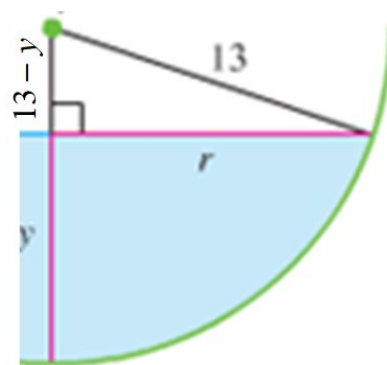
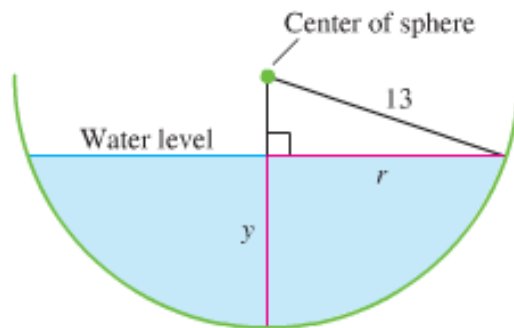
$$\frac{dV}{dt} = \pi y (2R - y) \frac{dy}{dt}$$

$$\begin{aligned}\frac{dy}{dt} &= \frac{1}{\pi (8) (2(13) - (8))} (-6) \\ &= -\frac{1}{24\pi} \text{ m/min}\end{aligned}$$

- b) The hemispherical is on the circle:

$$r^2 + (13 - y)^2 = 13^2$$

$$\begin{aligned}r^2 &= 169 - (169 - 26y + y^2) \\ &= 169 - 169 + 26y - y^2 \\ &= 26y - y^2\end{aligned}$$



$$r = \sqrt{26y - y^2}$$

$$c) \quad r = (26y - y^2)^{1/2}$$

$$\begin{aligned} \frac{dr}{dt} &= \frac{1}{2} (26y - y^2)^{-1/2} (26 - 2y) \frac{dy}{dt} \\ &= \frac{1}{2} \frac{26 - 2y}{\sqrt{26y - y^2}} \frac{dy}{dt} \end{aligned}$$

$$\begin{aligned} \left. \frac{dr}{dt} \right|_{y=8} &= \frac{1}{2} \frac{26 - 2(8)}{\sqrt{26(8) - (8)^2}} \left(-\frac{1}{24\pi} \right) \\ &= -\frac{5}{288\pi} \\ &\approx \underline{0.005526 \text{ m/s}} \quad \text{or} \quad \underline{5.526 \times 10^{-3} \text{ m/s}} \end{aligned}$$

Exercise

A spherical balloon is inflated with helium at the rate of $100\pi \text{ ft}^3 / \text{min}$. How fast is the balloon's radius increasing at the instant the radius is 5 feet? How fast the surface area increasing?

Solution

$$\text{Given:} \quad \frac{dV}{dt} = 100\pi \text{ ft}^3 / \text{min} \quad r = 5 \text{ ft}$$

$$\text{If } V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \frac{dV}{dt}$$

$$= \frac{1}{4\pi(5)^2} (100\pi)$$

$$= \underline{1 \text{ ft/min}}$$

$$S = 4\pi r^2$$

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

$$= 8\pi(5)(1)$$

$$= \underline{40\pi \text{ ft}^2 / \text{min}}$$

The rate of the surface area is increasing.

Exercise

A hot air balloon rising straight up from a level field is tracked by a range finder 500 feet from the liftoff point. At the moment, the range finder's elevation angle is $\frac{\pi}{3}$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Solution

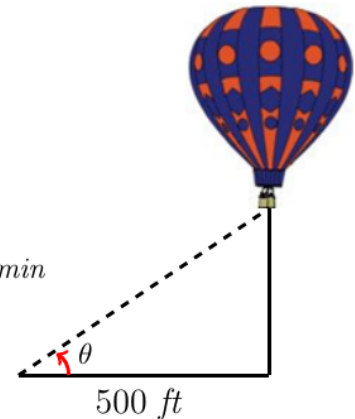
Given: $\frac{d\theta}{dt} = 0.14 \text{ rad/min}$ when $\theta = \frac{\pi}{3}$
distance = 500 ft

$$\tan \theta = \frac{y}{500}$$

$$y = 500 \tan \theta$$

$$\begin{aligned}\frac{dy}{dt} &= 500 \left(\sec^2 \theta \right) \frac{d\theta}{dt} \\ &= 500 \left(\sec^2 \frac{\pi}{3} \right) (0.14) \\ &= 280 \text{ ft/min} \end{aligned}$$

$$\frac{d\theta}{dt} = 0.14 \text{ rad/min}$$



The balloon is rising at the rate of 280 ft/min.

Exercise

A hot air balloon rising straight up from a level field is tracked by a range finder 1,000 feet from the liftoff point. At the moment, the range finder's elevation angle is $\frac{\pi}{3}$, the angle is increasing at the rate of 0.2 rad/min. How fast is the balloon rising at that moment?

Solution

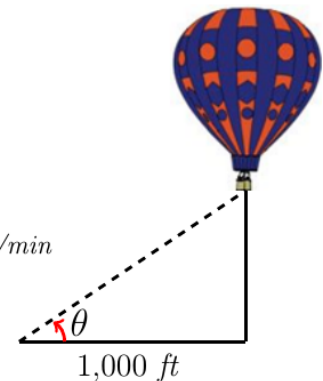
Given: $\frac{d\theta}{dt} = 0.2 \text{ rad/min}$ when $\theta = \frac{\pi}{3}$
distance = 1,000 ft

$$\tan \theta = \frac{y}{1,000}$$

$$y = 1,000 \tan \theta$$

$$\begin{aligned}\frac{dy}{dt} &= 1,000 \left(\sec^2 \theta \right) \frac{d\theta}{dt} \\ &= 1,000 \left(\sec^2 \frac{\pi}{3} \right) (0.2) \\ &= 800 \text{ ft/min} \end{aligned}$$

$$\frac{d\theta}{dt} = 0.2 \text{ rad/min}$$



The balloon is rising at the rate of 800 ft/min.

Exercise

A hot air balloon rising straight up from a level field is tracked by a range finder 1,000 *feet* from the liftoff point. At the moment, the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.2 *rad/min*. How fast is the balloon rising at that moment?

Solution

Given: $\frac{d\theta}{dt} = 0.2 \text{ rad/min}$ when $\theta = \frac{\pi}{4}$

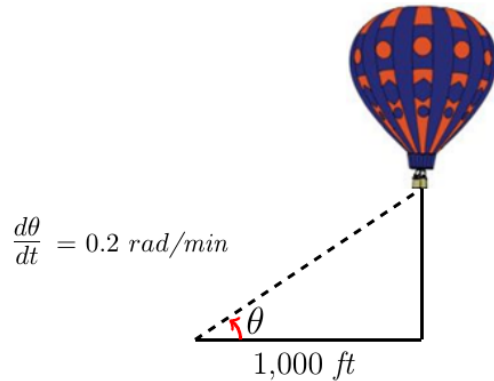
distance = 1,000 ft

$$\tan \theta = \frac{y}{1,000}$$

$$y = 1,000 \tan \theta$$

$$\begin{aligned} \frac{dy}{dt} &= 1,000 \left(\sec^2 \theta \right) \frac{d\theta}{dt} \\ &= 1,000 \left(\sec^2 \frac{\pi}{4} \right) (0.2) \\ &= \underline{400 \text{ ft/min}} \end{aligned}$$

The balloon is rising at the rate of 400 *ft/min*.



Exercise

A balloon rising vertically above a level, straight road at a constant rate of 1 *ft/sec*. Just when the balloon is 65 *feet* above the ground, a bicycle moving at a constant rate of 17 *ft/sec* passes under it. How fast is the distance $s(t)$ between the bicycle and the balloon increasing 3 *sec* later?

Solution

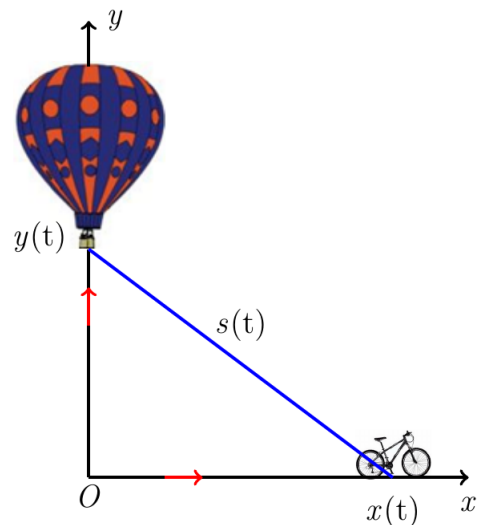
Given: $\frac{dy}{dt} = 1 \text{ ft/sec}$ $y = 65 \text{ ft}$ $\frac{dx}{dt} = 17 \text{ ft/sec}$

Bicycle increasing 3 *sec*:

$$\begin{aligned} x &= vt = 17(3) \\ &= \underline{51 \text{ ft}} \end{aligned}$$

$$\begin{aligned} s^2 &= x^2 + y^2 \\ s &= \sqrt{51^2 + 65^2} \\ &= \underline{\approx 83 \text{ ft}} \end{aligned}$$

$$\begin{aligned} 2s \frac{ds}{dt} &= 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \\ \frac{ds}{dt} &= \frac{1}{s} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right) \end{aligned}$$



$$\begin{aligned}
 &= \frac{1}{83}(51(17) + 65(1)) \\
 &= \frac{932}{83} \text{ ft/sec} \\
 &\approx 11 \text{ ft/sec}
 \end{aligned}$$

Exercise

An observer stands 300 feet from the launch site of a hot-air balloon. The balloon is launched vertically and maintains a constant upward velocity of 20 ft/sec. what is the rate of change of the angle of elevation of the balloon when it is 400 feet from the ground? The angle of elevation is the angle θ between the observer's line of sight to the balloon and the ground.

Solution

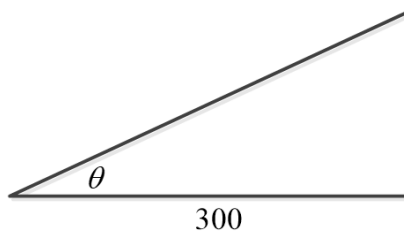
Given: $\frac{dh}{dt} = 20 \text{ ft/s}$, $h = 400 \text{ ft}$

$$\tan \theta = \frac{h}{300}$$

$$\theta = \tan^{-1}\left(\frac{h}{300}\right)$$

$$\begin{aligned}
 \frac{d\theta}{dt} &= \frac{1}{300\left(1 + \left(\frac{h}{300}\right)^2\right)} \frac{dh}{dt} \\
 &= \frac{1}{300\left(1 + \left(\frac{400}{300}\right)^2\right)} (20)
 \end{aligned}$$

$$\approx .024 \text{ rad/sec}$$



Exercise

A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow. The rope is hauled in at rate of 2 ft/sec.

- How fast is the boat approaching the dock when 10 ft of rope are out?
- At what rate is the angle θ changing at this instant?

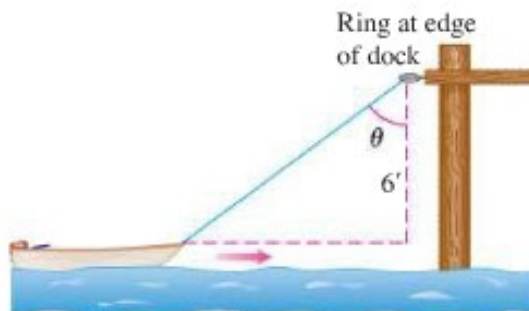
Solution

Given: $h = 6 \text{ ft}$ $\frac{ds}{dt} = -2 \text{ ft/sec}$

a) $s = 10 \text{ ft}$

$$s^2 = x^2 + 6^2 \Rightarrow x = \sqrt{s^2 - 36}$$

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt}$$



$$\frac{dx}{dt} = \frac{s}{\sqrt{s^2 - 36}} \frac{ds}{dt}$$

$$\left. \frac{dx}{dt} \right|_{s=10} = \frac{10}{\sqrt{10^2 - 36}} (-2)$$

$$= \underline{-2.5 \text{ ft/sec}}$$

$$b) \cos \theta = \frac{6}{s}$$

$$-\sin \theta \frac{d\theta}{dt} = -\frac{6}{s^2} \frac{ds}{dt}$$

$$\frac{d\theta}{dt} = \frac{6}{\sin \theta s^2} \frac{ds}{dt}$$

$$\sin \theta = \frac{x}{s} = \frac{\sqrt{10^2 - 36}}{10} = \frac{8}{10}$$

$$\frac{d\theta}{dt} = \frac{6}{(.8)10^2} (-2)$$

$$= \underline{-0.15 \text{ rad/sec}}$$

Exercise

The figure shows a boat 1 km offshore, sweeping the shore with a searchlight. The light turns at a constant rate $\frac{d\theta}{dt} = -0.6 \text{ rad/sec}$.

- How fast is the light moving along the shore when it reaches point A?
- How many revolutions per minute is 0.6 rad/sec ?

Solution

Given: $\frac{d\theta}{dt} = -0.6 \text{ rad/sec}$

$$\tan \theta = \frac{x}{1}$$

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{dx}{dt}$$

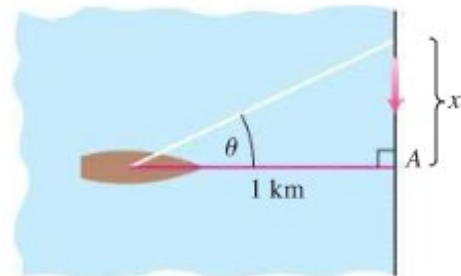
a) At $x = 0 \Rightarrow \theta = 0$

$$\frac{dx}{dt} = \sec^2(0)(-0.6)$$

$$= \underline{-0.6}$$

\therefore The speed of the light is 0.6 km/sec when it reaches point A.

b) $0.6 \frac{\text{rad}}{\text{sec}} \frac{1 \text{ rev}}{2\pi \text{ rad}} \frac{60 \text{ sec}}{1 \text{ min}} = \underline{\frac{18}{\pi} \frac{\text{rev}}{\text{min}}}$



Exercise

You are videotaping a race from a stand 132 feet from the track, following a car that is moving at 180 mi/h (264 ft/sec). How fast will your camera angle θ be changing when the car is right in front of you? A half second later?

Solution

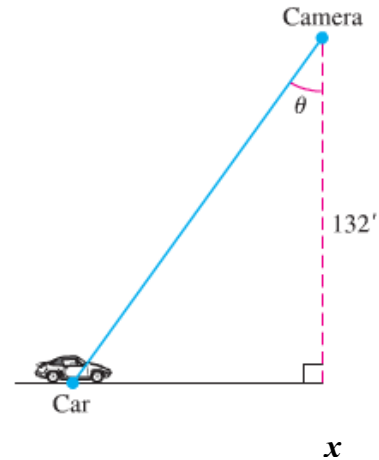
$$\tan \theta = \frac{x}{132}$$

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{132} \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{132 \sec^2 \theta} \frac{dx}{dt}$$

$$= \frac{1}{132 \sec^2(0)} (-264)$$

$$= -2 \text{ rad/sec}$$



At half second later the car has traveled 132 feet right to the perpendicular

$$|\theta| = \frac{\pi}{4} \rightarrow \cos^2 \theta = \frac{1}{2}, \text{ and } \frac{dx}{dt} = 264 \text{ (since } x \text{ increases)}$$

$$\frac{d\theta}{dt} = \frac{1}{132(2)} (264)$$

$$= 1 \text{ rad/sec}$$

Exercise

The coordinates of a particle in the metric xy -plane are differentiable functions of time t with

$\frac{dx}{dt} = -1 \text{ m/sec}$ and $\frac{dy}{dt} = -5 \text{ m/sec}$. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)?

Solution

Given: $\frac{dx}{dt} = -1 \text{ m/sec}$ $\frac{dy}{dt} = -5 \text{ m/sec}$

$$s^2 = x^2 + y^2$$

$$s = \sqrt{x^2 + y^2} = \sqrt{5^2 + 12^2} = 13$$

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{1}{s} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right)$$

$$\left. \frac{ds}{dt} \right|_{(5, 12)} = \frac{1}{13} (5(-1) + 12(-5))$$

$$= -5 \text{ m/sec}$$

Exercise

Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of $10 \text{ in}^3 / \text{min}$.

- a) How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?
- b) How fast is the level in the cone falling then?

Solution

$$r_{\text{pot}} = 3 \quad \frac{dV}{dt} = 10 \text{ in}^3 / \text{min}$$

- a) Let h be the height of the coffee in the pot.

Volume of the coffee:

$$\begin{aligned} V &= \pi r^2 h \\ &= 9\pi h \end{aligned}$$

$$\frac{dV}{dt} = 9\pi \frac{dh}{dt}$$

$$\begin{aligned} \frac{dh}{dt} &= \frac{1}{9\pi} \frac{dV}{dt} \\ &= \frac{1}{9\pi} (10) \\ &= \frac{10}{9\pi} \text{ in/min} \end{aligned}$$

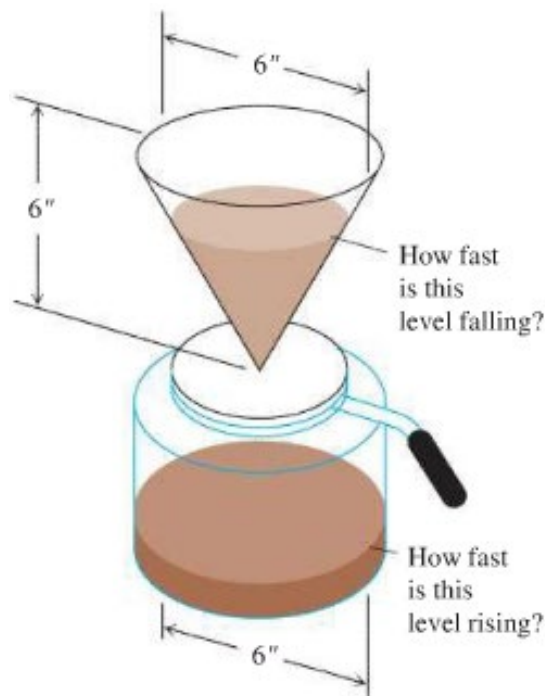
- b) Radius of the filter: $r = \frac{h}{2}$

Volume of the filter:

$$\begin{aligned} V &= \frac{1}{3} \pi r^2 h \\ &= \frac{1}{3} \pi \left(\frac{h}{2} \right)^2 h \\ &= \frac{\pi h^3}{12} \end{aligned}$$

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$$

$$\begin{aligned} \frac{dh}{dt} &= \frac{4}{\pi h^2} \frac{dV}{dt} \\ &= \frac{4}{\pi (5)^2} (-10) \\ &= -\frac{8}{5\pi} \text{ in/min} \end{aligned}$$



Exercise

A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x -coordinate (measure in meters) increases at a steady 10 m/sec . How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3 \text{ m}$?

Solution

Given: $y = x^2$ $v = \frac{dx}{dt} = 10 \text{ m/sec}$ $x = 3 \text{ m}$

$$\tan \theta = \frac{y}{x} = \frac{x^2}{x} = x$$

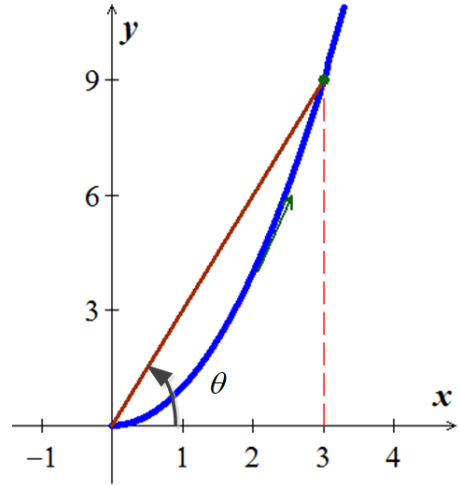
$$\frac{d}{dt} \tan \theta = \frac{d}{dt} x$$

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{\sec^2 \theta} \frac{dx}{dt} = \cos^2 \theta \frac{dx}{dt}$$

$$= \left(\frac{3}{\sqrt{9^2 + 3^2}} \right)^2 (10)$$

$$= 1 \text{ rad/sec}$$



Exercise

To find the height of a lamppost, you stand a 6 feet pole 20 feet from the lamp and measure the length a of its shadow, finding it to be 15 feet , give or take an inch. Calculate the height of the lamppost using the value of $a = 15$ and estimate the possible error in the result.

Solution

$$\frac{h}{6} = \frac{20+a}{a} = \frac{35}{15}$$

$$= \frac{7}{3}$$

$$h = 14 \text{ ft}$$

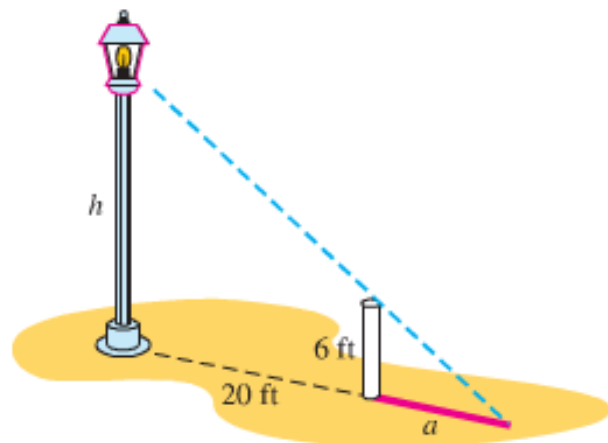
$$\frac{h}{6} = \frac{20+a}{a}$$

$$ah = 120 + 6a$$

$$h = 6 + 120a^{-1}$$

$$\frac{dh}{dt} = -120a^{-2} \frac{da}{dt}$$

$$dh = -\frac{120}{a^2} da = -\frac{120}{15^2} \left(\pm \frac{1}{12} \right)$$



$$\begin{aligned}
 dh &= -\frac{120}{a^2} da \\
 &= -\frac{120}{15^2} \left(\pm \frac{1}{12} \right) \\
 &= -\frac{120}{15^2} \left(\pm \frac{1}{12} \right) \\
 &= \pm \frac{2}{45} \text{ ft} \quad \left| \quad \approx \pm 0.044 \text{ ft} \right|
 \end{aligned}$$

Exercise

A light shines from the top of a pole 50 feet high. A ball is dropped from the same height from a point 30 feet away from the light. How fast is the shadow of the ball moving along the ground $\frac{1}{2}$ sec later?

(Assume the ball falls a distance $s = 16t^2$ ft in t sec.)

Solution

$$s = 16t^2$$

$$s + h = 50$$

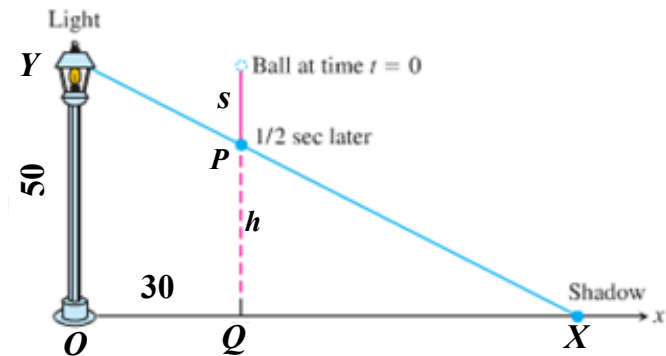
Triangles XOY and XQP are similar:

$$\begin{aligned}
 \therefore \frac{XQ}{h} &= \frac{OX}{50} \\
 &= \frac{30 + XQ}{50}
 \end{aligned}$$

$$50|XQ| = 30h + h|XQ|$$

$$(50 - h)|XQ| = 30h$$

$$\begin{aligned}
 |XQ| &= \frac{30h}{50 - h} \\
 &= \frac{30(50 - s)}{50 - (50 - s)} \\
 &= \frac{30(50 - 16t^2)}{50 - 50 + 16t^2} \\
 &= \frac{1500 - 480t^2}{16t^2} \\
 &= \frac{1500}{16t^2} - \frac{480t^2}{16t^2} \\
 &= \frac{1500}{16t^2} - 30
 \end{aligned}$$



$$\begin{aligned}
 \frac{d}{dt}|XQ| &= 1500 \frac{-32t}{(16t^2)^2} \\
 &= 1500 \frac{-32t}{256t^4} \\
 &= -\frac{375}{2t^3} \\
 \left. \frac{d}{dt}|XQ| \right|_{t=\frac{1}{2}} &= -\frac{375}{2\left(\frac{1}{2}\right)^3} \\
 &= \underline{-1500 \text{ ft/sec}}
 \end{aligned}$$

Exercise

A spherical iron ball 8 in. in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of $10 \text{ in}^3 / \text{min}$, how fast is the thickness of the ice decreasing when it is 2 in. thick? How fast is the outer surface area of ice decreasing?

Solution

Given: $D = 8 \text{ in} \rightarrow r_1 = 4 \text{ in}$ $\frac{dV}{dt} = -10 \text{ in}^3 / \text{min}$ $\text{think} = 2 \text{ in}$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

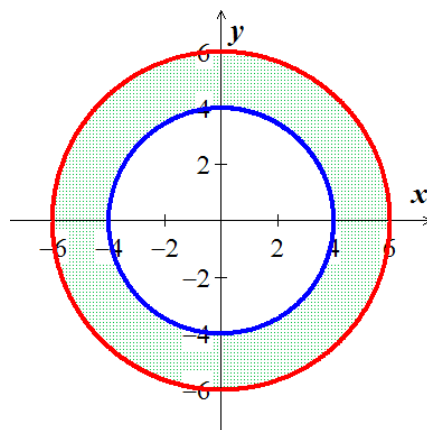
$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \frac{dV}{dt}$$

$$\begin{aligned}
 \left. \frac{dr}{dt} \right|_{r=6} &= \frac{1}{4\pi(6)^2}(-10) \\
 &= \underline{-\frac{5}{72\pi} \text{ in/min}}
 \end{aligned}$$

$$S = 4\pi r^2$$

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

$$\begin{aligned}
 \left. \frac{dS}{dt} \right|_{r=6} &= 8\pi(6)\left(-\frac{5}{72\pi}\right) \\
 &= \underline{-\frac{10}{3} \text{ in}^2 / \text{min}}
 \end{aligned}$$



The outer surface of the ice is decreasing at $-\frac{10}{3} \text{ in}^2 / \text{min}$

Exercise

On a morning of a day when the sun will pass directly overhead, the shadow of an 80-foot building on level ground is 60 feet long. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of $0.27^\circ/\text{min}$. At what rate is the shadow decreasing?

Solution

Given: $x = 60 \text{ ft}$ $h = 80 \text{ ft}$

$$\begin{aligned}\frac{d\theta}{dt} &= 0.27^\circ/\text{min} \\ &= 0.27^\circ \frac{\pi \text{ rad}}{180^\circ} \frac{1}{\text{min}} \\ &= \frac{3\pi}{2000} \text{ rad/min} \end{aligned}$$

$$\tan \theta = \frac{80}{x}$$

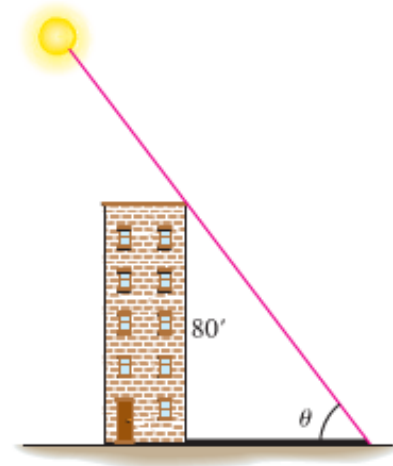
$$\frac{d}{dt} \tan \theta = \frac{d}{dt} \frac{80}{x}$$

$$\sec^2 \theta \frac{d\theta}{dt} = -\frac{80}{x^2} \frac{dx}{dt}$$

$$\frac{dx}{dt} = \left| -\frac{x^2 \sec^2 \theta}{80} \frac{d\theta}{dt} \right|$$

$$\begin{aligned} &= \frac{60^2 \left(\frac{5}{3}\right)^2}{80} \left(\frac{3\pi}{2000}\right) \\ &= 0.589 \text{ ft/min} \end{aligned}$$

$$\cos \theta = \frac{60}{\sqrt{60^2 + 80^2}} = \frac{60}{100} = \frac{3}{5}$$



Exercise

A baseball diamond is a square 90 feet on a side. A player runs from first base to second at a rate of 16 ft/sec.

- At what rate is the player's distance from third base changing when the player is 30 feet from first base?
- At what rates are angles θ_1 and θ_2 changing at that time?
- The player slides into second base at the rate of 15 ft/sec. At what rates are angles θ_1 and θ_2 changing as the player touches base?

Solution

Given: $d_1 = 90 \text{ ft}$ $d_2 = 30 \text{ ft}$ $\frac{dx}{dt} = -16 \text{ ft/sec}$

x : Distance between player and 2nd base

s : Distance between player and 3rd base

$$a) \quad x = 90 - 30 = 60 \text{ ft}$$

$$s^2 = x^2 + 90^2$$

$$s = \sqrt{60^2 + 90^2}$$

$$= \sqrt{11700}$$

$$= 30\sqrt{13}$$

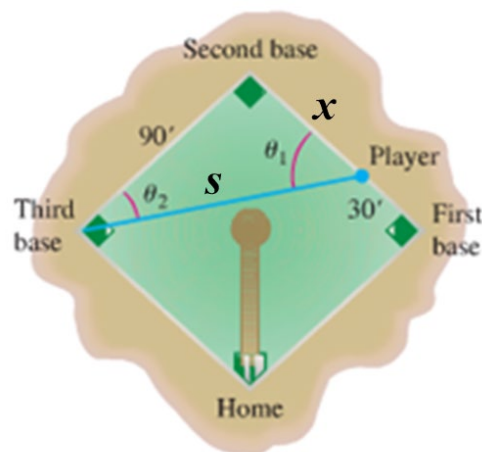
$$2s \frac{ds}{dt} = 2x \frac{dx}{dt}$$

$$\frac{ds}{dt} = \frac{x}{s} \frac{dx}{dt}$$

$$= \frac{60}{30\sqrt{13}}(-16)$$

$$= -\frac{32}{\sqrt{13}} \text{ ft/sec}$$

$$\approx -8.875 \text{ ft/sec}$$



$$b) \quad \sin \theta_1 = \frac{90}{s}$$

$$\cos \theta_1 \frac{d\theta_1}{dt} = -\frac{90}{s^2} \frac{ds}{dt}$$

$$\frac{d\theta_1}{dt} = -\frac{90}{s^2 \cos \theta_1} \frac{ds}{dt}$$

$$\frac{d\theta_1}{dt} = -\frac{90}{s^2 \frac{x}{s}} \frac{ds}{dt}$$

$$= -\frac{90}{s \cdot x} \frac{ds}{dt}$$

$$= -\frac{90}{30\sqrt{13}(60)}(-8.875)$$

$$\approx 0.123 \text{ rad/sec}$$

$$\cos \theta_1 = \frac{x}{s}$$

$$\cos \theta_2 = \frac{90}{s}$$

$$-\sin \theta_2 \frac{d\theta_2}{dt} = -\frac{90}{s^2} \frac{ds}{dt}$$

$$\frac{d\theta_2}{dt} = \frac{90}{s^2 \sin \theta_2} \frac{ds}{dt} \quad \sin \theta_2 = \frac{x}{s}$$

$$= \frac{90}{s \cdot x} \frac{ds}{dt}$$

$$= \frac{90}{30\sqrt{13}(60)}(-8.875)$$

$$\approx -0.123 \text{ rad/sec} \quad |$$

$$\begin{aligned} c) \quad \frac{d\theta_1}{dt} &= -\frac{90}{s^2 \cos \theta_1} \frac{ds}{dt} & \frac{ds}{dt} &= \frac{x}{s} \frac{dx}{dt} \\ &= -\frac{90}{s^2} \frac{x}{s} \frac{dx}{dt} \\ &= -\frac{90}{s^2} \frac{dx}{dt} \\ &= -\frac{90}{x^2 + 8100} \frac{dx}{dt} \end{aligned}$$

Player slides into second base $\Rightarrow x = 0$

$$\begin{aligned} \left. \frac{d\theta_1}{dt} \right|_{x=0} &= -\frac{90}{0^2 + 8100} (-15) \\ &= \frac{1}{6} \text{ rad/sec} \quad | \end{aligned}$$

$$\begin{aligned} \frac{d\theta_2}{dt} &= \frac{90}{s^2 \sin \theta_2} \frac{ds}{dt} \\ &= \frac{90}{s^2} \frac{x}{s} \frac{dx}{dt} \\ &= \frac{90}{s^2} \frac{dx}{dt} \\ &= \frac{90}{x^2 + 8100} \frac{dx}{dt} \end{aligned}$$

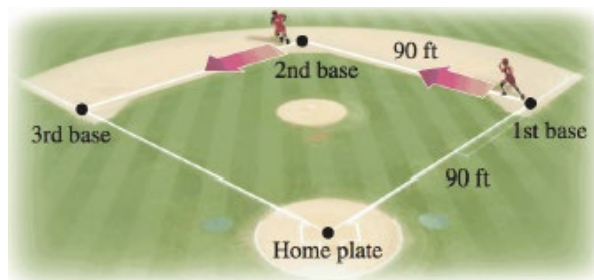
Player slides into second base $\Rightarrow x = 0$

$$\begin{aligned} \left. \frac{d\theta_2}{dt} \right|_{x=0} &= \frac{90}{0^2 + 8100} (-15) \\ &= -\frac{1}{6} \text{ rad/sec} \quad | \end{aligned}$$

Exercise

Runners stand at first and second base in a baseball game. At the moment, a ball is hit the runner at first base runs to second base at 18 ft/s ; simultaneously the runner on second runs to third base at 20 ft/s . How fast is the distance between the runners changing 1 sec after the ball is hit?

(Hint: The distance between consecutive bases is 90 feet and the bases lie at the corners of a square.)



Solution

Given: $\frac{dx}{dt} = 18 \text{ ft/s}$, $\frac{dy}{dt} = 20 \text{ ft/s}$

After 1 sec , $x = 18$ and $y = 20$

$$D^2 = (90 - x)^2 + y^2$$

$$D = \sqrt{(90 - 18)^2 + 20^2}$$

$$= \sqrt{5584}$$

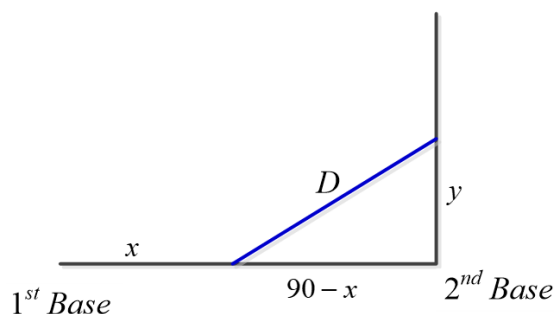
$$\approx 74.726$$

$$2D \frac{dD}{dt} = -2(90 - x) \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$D \frac{dD}{dt} = -(90 - x) \frac{dx}{dt} + y \frac{dy}{dt}$$

$$\frac{dD}{dt} = \frac{-(90 - 18)(18) + 20(20)}{74.726}$$

$$\approx -11.99 \text{ ft/sec}$$



So, the distance between the runners is decreasing at a rate about $11.99 \text{ feet per second}$.

Exercise

The variables x and y are both differentiable functions of t and are related by the equation $y = x^2 + 3$.

Find $\frac{dy}{dt}$ when $x = 1$, given $\frac{dx}{dt} = 2$ when $x = 1$.

Solution

$$\frac{dy}{dt} = \frac{d}{dt}(x^2 + 3)$$

$$= 2x \frac{dx}{dt}$$

$$= 2(1)(2)$$

$$= 4$$

Exercise

A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second.

When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

Solution

Given: $\frac{dr}{dt} = 1$

Find: $\frac{dA}{dt}$ when $r = 4$

$$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2)$$

$$= 2\pi r \frac{dr}{dt}$$

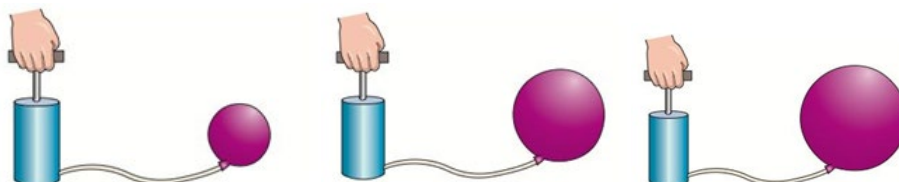
$$= 2\pi(4)(1)$$

$$= 8\pi \text{ ft}^2/\text{sec}$$



Exercise

Air is being pumped into a spherical balloon at a rate of $4.5 \text{ ft}^3/\text{min}$.



Find the rate of change of the radius when the radius is 2 feet.

Solution

Given: $\frac{dV}{dt} = \frac{9}{2}$

Find: $\frac{dr}{dt}$ when $r = 2$

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$= 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \frac{dV}{dt}$$

$$= \frac{1}{16\pi} \frac{9}{2}$$

$$= \frac{9}{32\pi} \text{ ft/min}$$

Exercise

An Airplane is flying on a flight path that will take it directly over a radar tracking station. The distance s is decreasing at a rate of 400 *mph* when $s = 10$ *mi*. what is the speed of the plane?

Solution

Given: $\frac{ds}{dt} = -400$ when $s = 10$

Find: $\frac{dx}{dt}$ when $s = 10$ & $x = 8$

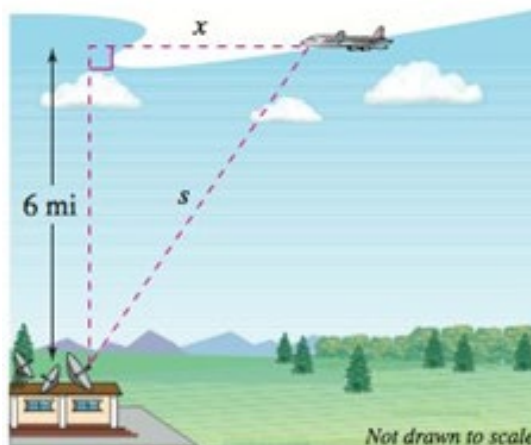
$$s^2 = x^2 + 6^2$$

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{s}{x} \frac{ds}{dt}$$

$$= \frac{10}{8}(-400)$$

$$= -500 \text{ mph}$$



Exercise

Find the rate of change in the angle of elevation of the camera at 10 *seconds* after lift-off.

Solution

Given: $\frac{ds}{dt} = 100t$

Find: $\frac{d\theta}{dt}$ when $t = 10$ & $s = 5,000$

$$\tan \theta = \frac{s}{2,000}$$

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{2,000} \frac{ds}{dt}$$

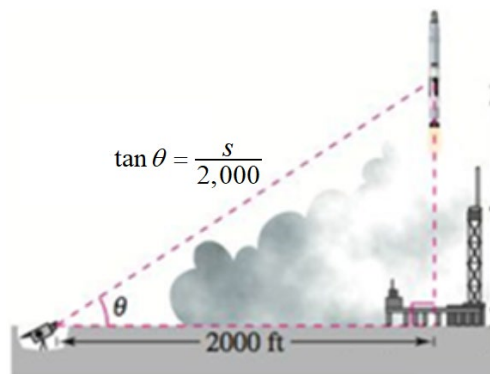
$$\text{hyp} = \sqrt{s^2 + 2,000^2} \quad \cos \theta = \frac{2,000}{\sqrt{s^2 + 2,000^2}}$$

$$\frac{d\theta}{dt} = \frac{\cos^2 \theta}{2,000} \frac{ds}{dt}$$

$$= \frac{1}{2,000} \left(\frac{2,000}{\sqrt{5,000^2 + 2,000^2}} \right)^2 (100(10))$$

$$= \frac{1}{2} \left(\frac{2,000}{\sqrt{25 \times 10^6 + 4 \times 10^6}} \right)^2$$

$$= \frac{2}{29} \text{ rad/sec}$$



Exercise

In the engine, a 7-inch connecting rod is fastened to a crank of radius 3 inches, the crankshaft rotates counterclockwise at a constant rate of 200 revolutions per minute.

Find the velocity of the piston when $\theta = \frac{\pi}{3}$.

Solution

Given: $\frac{d\theta}{dt} = 400\pi$

Find: $\frac{dx}{dt}$ when $\theta = \frac{\pi}{3}$

$$7^2 = x^2 + 3^2 - 6x \cos \theta \quad a^2 = b^2 + c^2 - 2bc \cos A \quad (\text{Law of cosine})$$

$$x^2 - 6x \cos \theta = 40$$

$$\Rightarrow \left(\theta = \frac{\pi}{3} \right)$$

$$x^2 - 3x - 40 = 0 \rightarrow x = 8, \quad \cancel{x = -5}$$

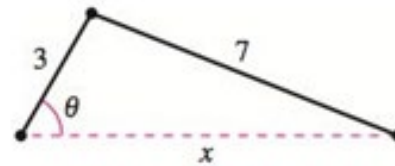
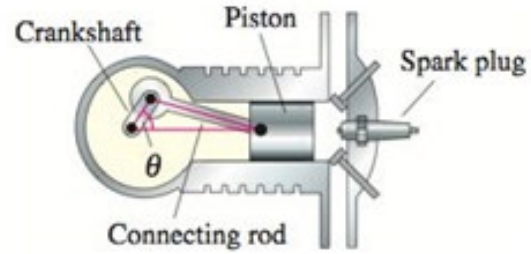
$$2x \frac{dx}{dt} - 6 \cos \theta \frac{dx}{dt} + 6x \sin \theta \frac{d\theta}{dt} = 0$$

$$(6 \cos \theta - 2x) \frac{dx}{dt} = 6x \sin \theta \frac{d\theta}{dt}$$

$$\frac{dx}{dt} = \frac{6x \sin \theta}{6 \cos \theta - 2x} \frac{d\theta}{dt}$$

$$= \frac{6(8) \sin \frac{\pi}{3}}{6 \cos \frac{\pi}{3} - 2(8)} (400\pi)$$

$$\approx -4018 \text{ in / min}$$



Exercise

A swimming pool is 12 meters long, 6 meters wide, 1 meter deep at the shallow end, and 3 meters deep at the deep end. Water is being pumped into the pool at $\frac{1}{4} \text{ m}^3/\text{min}$, and there is 1 meter of water at the deep end.

- What percent of the pool is filled?
- At what rate is the water level rising?

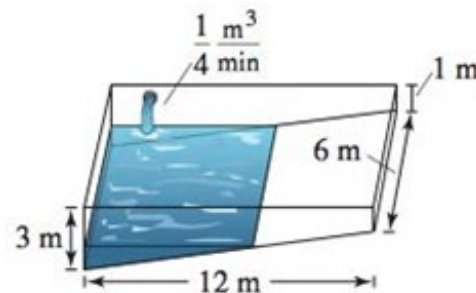
Solution

Given: $\frac{dV}{dt} = \frac{1}{4} \text{ m}^3/\text{min}$

a) Total Volume:

$$V = \frac{1}{2}(2)(12)(6) + (1)(12)(6)$$

$$= 144 \text{ m}^3$$



Volume of 1 m of water:

$$V = \frac{1}{2}(1)(6)(6)$$

$$= 18 \text{ m}^3$$

$$\% \text{ pool filled} = \frac{18}{144}(100\%)$$

$$= 12.5\%$$

b) Since there is 1 m of water in the pool, then $0 \leq h \leq 2$

$$\frac{b}{6} = \frac{h}{1}$$

$$b = 6h$$

$$V = \frac{1}{2}bh(6)$$

$$= 3(6h)h$$

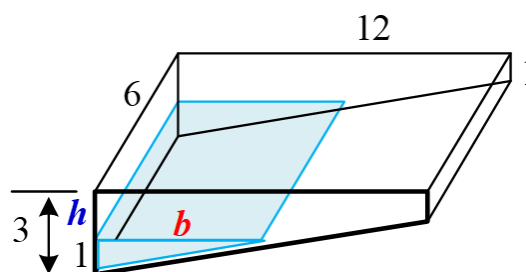
$$= 18h^2$$

$$\frac{dV}{dt} = 36h \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{36h} \frac{dV}{dt}$$

$$= \frac{1}{36(1)} \frac{1}{4}$$

$$= \frac{1}{144} \text{ m/min}$$



Exercise

A trough is 12 feet long and 3 feet across the top. Its ends are isosceles triangles with altitudes of 3 feet.

- a) Water is being pumped into the trough at $2 \text{ ft}^3 / \text{min}$. How fast is the water level rising when the depth h is 1 foot?
- b) The water is rising at a rate of $\frac{3}{8} \text{ in/min}$ when $h = 2$. Determine the rate at which water is being pumped into the trough.

Solution

Given: $\frac{dV}{dt} = 2 \text{ ft}^3 / \text{min}$

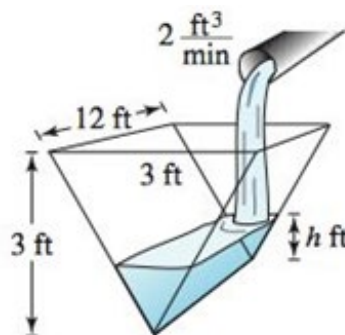
a) $V = \frac{1}{2}bh(12) = 6bh$

$V = 6h^2$ since $b = h$

$$\frac{dV}{dt} = 12h \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{12h} \frac{dV}{dt}$$

$h = 1$



$$= \frac{1}{12(1)}(2)$$

$$= \frac{1}{6} \text{ ft/min} \quad \Big|$$

b) **Given:** $\frac{dh}{dt} = \frac{3}{8} \text{ in/min}$, $h = 2$

$$\frac{dV}{dt} = 12(2) \left(\frac{3}{8} \frac{\text{in}}{\text{min}} \frac{1 \text{ ft}}{12 \text{ in}} \right)$$

$$= \frac{3}{4} \text{ ft}^3 / \text{min} \quad \Big|$$

Exercise

A ladder 25 feet long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 feet per second.

- How fast is the top of the ladder moving down the wall when its base is 7 feet, 15 feet, and 24 feet from the wall?
- Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 feet from the wall.
- Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 7 feet from the wall.

Solution

Given: $\frac{dx}{dt} = 2 \text{ ft/sec}$

$$x^2 + y^2 = 25^2$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$$

$$= -\frac{2x}{y} \quad \Big|$$

a) $x = 7$

$$y = \sqrt{25^2 - 7^2}$$

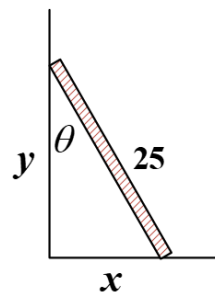
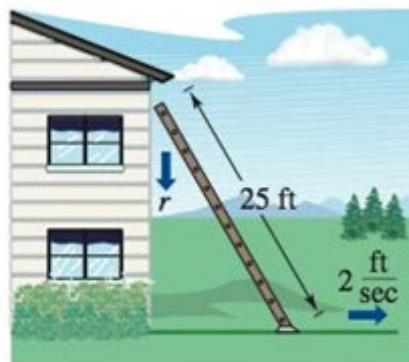
$$= \sqrt{576}$$

$$= 24 \quad \Big|$$

$$\frac{dy}{dt} = -\frac{2(7)}{24}$$

$$= -\frac{7}{12} \text{ ft/sec} \quad \Big|$$

$x = 15$



$$y = \sqrt{25^2 - 15^2}$$

$$= 20$$

$$\frac{dy}{dt} = -\frac{2(15)}{20}$$

$$= -\frac{3}{2} \text{ ft / sec}$$

$$x = 24$$

$$y = \sqrt{25^2 - 24^2}$$

$$= 7$$

$$\frac{dy}{dt} = -\frac{2(24)}{7}$$

$$= -\frac{48}{7} \text{ ft / sec}$$

$$b) A = \frac{1}{2}xy$$

$$\frac{dA}{dt} = \frac{1}{2} \left(y \frac{dx}{dt} + x \frac{dy}{dt} \right)$$

$$\text{We have: } x = 7; \quad y = 24; \quad \frac{dy}{dt} = -\frac{7}{12}; \quad \frac{dx}{dt} = 2$$

$$\frac{dA}{dt} = \frac{1}{2} \left(24(2) + 7 \left(-\frac{7}{12} \right) \right)$$

$$= \frac{527}{24} \text{ ft}^2 / \text{sec}$$

$$c) \tan \theta = \frac{x}{y}$$

$$\sec^2 \theta \, d\theta = \frac{1}{y} \frac{dx}{dt} - \frac{x}{y^2} \frac{dy}{dt}$$

$$d\theta = \cos^2 \theta \left(\frac{1}{y} \frac{dx}{dt} - \frac{x}{y^2} \frac{dy}{dt} \right)$$

$$\text{We have: } x = 7; \quad y = 24; \quad \frac{dy}{dt} = -\frac{7}{12}; \quad \frac{dx}{dt} = 2; \quad \cos \theta = \frac{y}{25} = \frac{24}{25}$$

$$d\theta = \left(\frac{24}{25} \right)^2 \left(\frac{1}{24}(2) - \frac{7}{24^2} \left(-\frac{7}{12} \right) \right)$$

$$= \left(\frac{24}{25} \right)^2 \left(\frac{48 \times 12 + 49}{12 \times 24^2} \right)$$

$$= \frac{625}{12 \times 25^2}$$

$$= \frac{1}{12} \text{ rad / sec}$$

Exercise

A construction worker pulls a five-meter plank up the side of a building under construction by means of a rope tied to one end of the plank. Assume the opposite end of the plank follows a path perpendicular to the wall of the building and the worker pulls the rope at a rate of 0.15 m/sec . How fast is the end of the plank sliding along the ground when it is 2.5 meters from the wall of the building?

Solution

$$x^2 + y^2 = 25$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

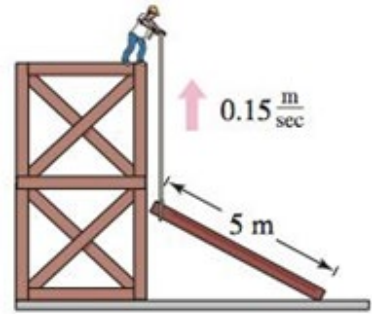
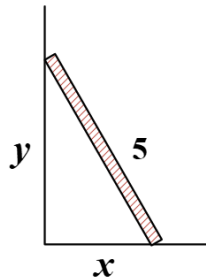
$$\frac{dx}{dt} = -\frac{y}{x} \frac{dy}{dt}$$

Given: $\frac{dy}{dt} = 0.15 \text{ m/sec}$ $x = 2.5$

$$y = \sqrt{25 - 2.5^2} = \sqrt{18.75}$$

$$\frac{dx}{dt} = -\frac{\sqrt{18.75}}{2.5} (0.15)$$

$$\approx \underline{-0.26 \text{ m/sec}}$$



Exercise

A winch at the top of a 12-meter building pulls a pipe of the same length to a vertical position. The winch pulls in rope at a rate of -0.2 m/sec . Find the rate of vertical change and the rate of horizontal change at the end of the pipe when $y = 6$

Solution

When $y = 6 \Rightarrow x = \sqrt{12^2 - 6^2} = \sqrt{108} = 6\sqrt{3}$

$$s = \sqrt{x^2 + (12 - y)^2}$$

$$= \sqrt{108^2 + (12 - 6)^2}$$

$$= \sqrt{144}$$

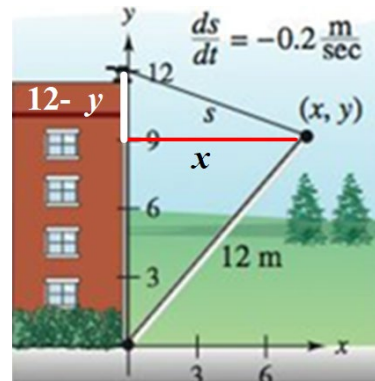
$$= \underline{12}$$

$$x^2 + (12 - y)^2 = s^2$$

$$2x \frac{dx}{dt} + 2(12 - y)(-1) \frac{dy}{dt} = 2s \frac{ds}{dt}$$

$$x \frac{dx}{dt} - (12 - y) \frac{dy}{dt} = s \frac{ds}{dt}$$

$$x^2 + y^2 = 12^2$$



$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dx}{dt} = -\frac{y}{x} \frac{dy}{dt}$$

$$x \left(-\frac{y}{x} \frac{dy}{dt} \right) - (12 - y) \frac{dy}{dt} = s \frac{ds}{dt}$$

$$-y \frac{dy}{dt} - (12 - y) \frac{dy}{dt} = s \frac{ds}{dt}$$

$$-12 \frac{dy}{dt} = s \frac{ds}{dt}$$

$$\frac{dy}{dt} = -\frac{s}{12} \frac{ds}{dt} \qquad \frac{ds}{dt} = -0.2; \quad y = 6; \quad x = 6\sqrt{3}$$

$$= -\frac{12}{12}(-0.2)$$

$$= \frac{1}{5} \text{ m / sec}$$

$$\frac{dx}{dt} = -\frac{y}{x} \frac{dy}{dt}$$

$$= -\frac{6}{6\sqrt{3}} \frac{1}{5}$$

$$= -\frac{\sqrt{3}}{15} \text{ m / sec}$$

Exercise

A boat is pulled into a dock by means of a winch 12 *feet* above the deck of the boat.

- The winch pulls in rope at a rate of 4 *feet per second*. Determine the speed of the boat when there is 13 *feet* of rope out. What happens to the speed of the boat as it gets closer to the dock?
- Suppose the boat is moving at a constant rate of 4 *feet per second*. Determine the speed at which the winch pulls in rope when there is a total of 13 *feet* of rope out. What happens to the speed at which the winch pulls in rope as the boat gets closer to the dock?

Solution

Let L be the length of the rope.

$$a) \quad L^2 = 12^2 + x^2$$

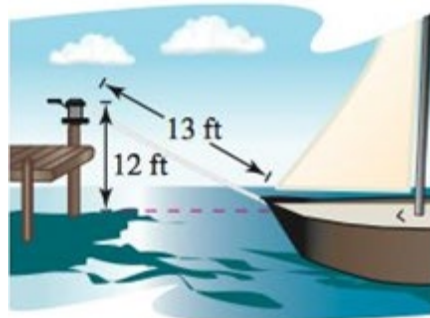
$$2L \frac{dL}{dt} = 2x \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{L}{x} \frac{dL}{dt}$$

$$\text{Given: } \frac{dL}{dt} = -4 \text{ ft / sec}$$

When $L = 13$

$$x = \sqrt{129 - 144}$$



$$= 5 \text{ |}$$

$$\begin{aligned} \frac{dx}{dt} &= \frac{13}{5}(-4) \\ &= -\frac{52}{5} \text{ ft / sec} \end{aligned} \text{ |}$$

b) $\frac{dx}{dt} = -4 \text{ ft / sec}; L = 13$

$$\begin{aligned} \frac{dL}{dt} &= \frac{x}{L} \frac{dx}{dt} \\ &= \frac{5}{13}(-4) \\ &= -\frac{20}{13} \text{ ft / sec} \end{aligned} \text{ |}$$

$$\begin{aligned} \frac{dL}{dt} &= \frac{\sqrt{L^2 - 144}}{L}(-4) \\ \lim_{L \rightarrow 12^+} \frac{dL}{dt} &= -4 \lim_{L \rightarrow 12^+} \frac{\sqrt{L^2 - 144}}{L} \\ &= 0 \end{aligned} \text{ |}$$

Exercise

An air traffic controller spots two planes at the same altitude converging on a point as they fly at right angles to each other. One plane is 225 miles from the point moving at 450 mph. The other plane is 300 miles from the point moving at 600 mph.

- At what rate is the distance between the planes decreasing?
- How much time does the air traffic controller have to get one of the planes on a different flight path?

Solution

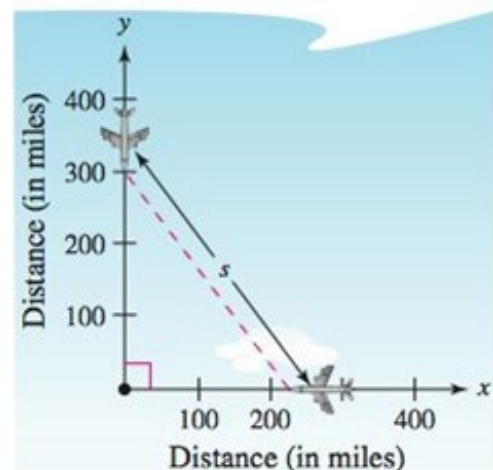
Given: $x = 225$, $\frac{dx}{dt} = -450$, $y = 300$, $\frac{dy}{dt} = -600$

a) $s^2 = x^2 + y^2$

$$\begin{aligned} s &= \sqrt{225^2 + 300^2} \\ &= 375 \end{aligned} \text{ |}$$

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\begin{aligned} \frac{ds}{dt} &= \frac{1}{s} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right) \\ &= \frac{1}{375} (225(-450) + 300(-600)) \\ &= -\frac{281,250}{375} \end{aligned}$$



$$= -750 \text{ mph} \quad |$$

$$b) \quad t = \frac{375}{750} = \frac{1}{2} \text{ hr} \quad | \quad = 30 \text{ min} \quad |$$

Exercise

An airplane is flying at an altitude of 5 miles and passes directly over a radar antenna. When the plane is 10 miles away ($s = 10$), the radar detects that the distance s is changing at a rate of 240 mph. What is the speed of the plane?

Solution

Given: $s = 10$, $\frac{ds}{dt} = 240$ $\frac{dy}{dt} = 0$

$$c) \quad s^2 = x^2 + y^2$$

$$x = \sqrt{10^2 - 5^2}$$

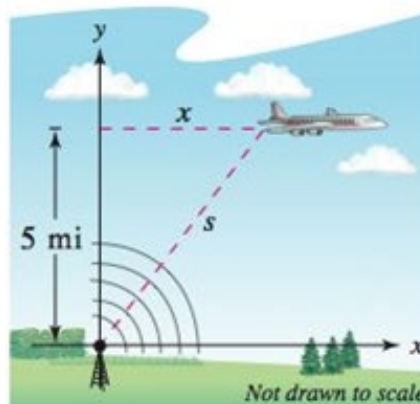
$$= 5\sqrt{3} \quad |$$

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{s}{x} \frac{ds}{dt}$$

$$= \frac{10}{5\sqrt{3}} (240)$$

$$= 160\sqrt{3} \text{ mph} \quad |$$



Exercise

A man 6 feet tall walks at a rate of 5 feet per second away from a light that is 15 feet above the ground.

- When he is 10 feet from the base of the light, at what rate is the tip of his shadow moving?
- When he is 10 feet from the base of the light, at what rate is the length of his shadow changing?

Solution

$$a) \quad \frac{y}{y-x} = \frac{15}{6}$$

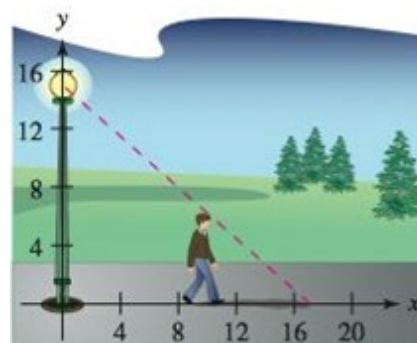
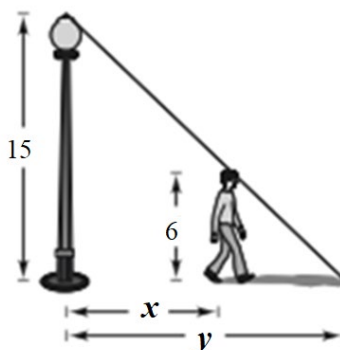
$$6y = 15y - 15x$$

$$9y = 15x$$

$$y = \frac{5}{3}x$$

$$\frac{dy}{dt} = \frac{5}{3} \frac{dx}{dt}$$

$$= \frac{5}{3}(5)$$



$$= \frac{25}{3} \text{ ft / sec}$$

$$\begin{aligned} b) \quad \frac{d(y-x)}{dt} &= \frac{dy}{dt} - \frac{dx}{dt} \\ &= \frac{25}{3} - 5 \\ &= \frac{10}{3} \text{ ft / sec} \end{aligned}$$

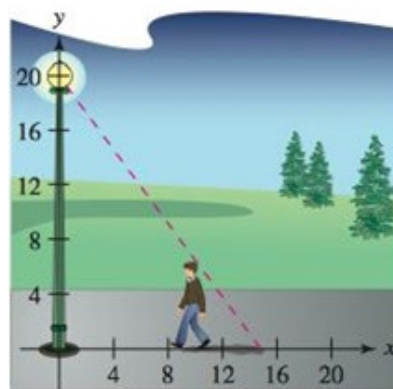
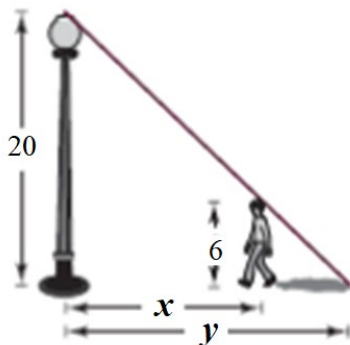
Exercise

A man 6 feet tall walks at a rate of 5 feet per second toward a light that is 20 feet above the ground.

- When he is 10 feet from the base of the light, at what rate is the tip of his shadow moving?
- When he is 10 feet from the base of the light, at what rate is the length of his shadow changing?

Solution

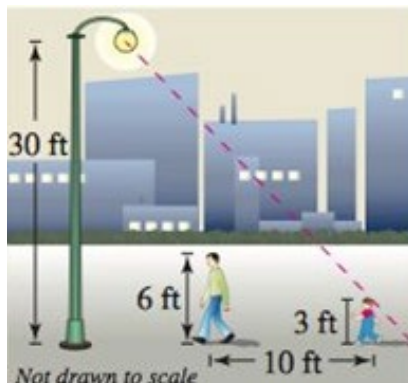
$$\begin{aligned} a) \quad \frac{y}{y-x} &= \frac{20}{6} \\ 6y &= 20y - 20x \\ 14y &= 20x \\ y &= \frac{10}{7}x \\ \frac{dy}{dt} &= \frac{10}{7} \frac{dx}{dt} \\ &= \frac{10}{7}(-5) \\ &= -\frac{50}{7} \text{ ft / sec} \end{aligned}$$



$$\begin{aligned} b) \quad \frac{d(y-x)}{dt} &= \frac{dy}{dt} - \frac{dx}{dt} \\ &= -\frac{50}{7} - (-5) \\ &= -\frac{15}{7} \text{ ft / sec} \end{aligned}$$

Exercise

A man 6 feet tall walks at a rate of 5 ft/sec toward a streetlight that is 30 feet high. The man's 3-foot-tall child follows at the same speed, but 10 feet behind the man. At times, the shadow behind the child is caused by the man, and at other times, by the child.



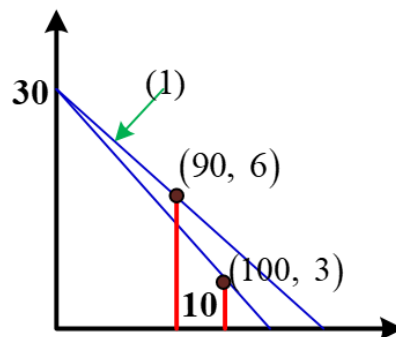
- Suppose the man is 90 feet from the streetlight. Show that the man's shadow extends beyond the child's shadow.
- Suppose the man is 60 feet from the streetlight. Show that the child's shadow extends beyond the man's shadow.
- Determine the distance d from the man to the streetlight at which the tips of the two shadows are exactly the same distance from the streetlight.
- Determine how fast the tip of the man's shadow is moving as a function of x , the distance between the man and the streetlight. Discuss the continuity of this shadow speed function.

Solution

$$\begin{aligned} a) \text{ Line (1): } y &= \frac{6-30}{90-0}(x-0) + 30 & y &= m(x-x_0) + y_0 \\ &= -\frac{4}{15}x + 30 \end{aligned}$$

When $x = 100$

$$\begin{aligned} y &= -\frac{400}{15} + 30 \\ &= \frac{10}{3} > 3 \end{aligned}$$

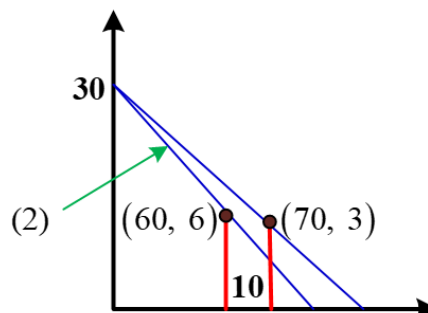


The shadow determined by the man extends beyond the shadow by the child.

$$\begin{aligned} b) \text{ Line (2): } y &= \frac{6-30}{60-0}(x-0) + 30 \\ &= -\frac{24}{60}x + 30 \\ &= -\frac{2}{5}x + 30 \end{aligned}$$

When $x = 70$

$$\begin{aligned} y &= -\frac{140}{5} + 30 \\ &= 2 < 3 \end{aligned}$$



The shadow determined by the child extends beyond the shadow by the man.

c) The points: $(0, 30)$, $(d, 6)$, and $(d + 10, 3)$

$$\frac{30 - 6}{0 - d} = \frac{30 - 3}{0 - d - 10}$$

$$-\frac{24}{d} = -\frac{27}{d + 10}$$

$$24d + 240 = 27d$$

$$3d = 240$$

$$d = 80 \text{ ft}$$

d) **Given:** $\frac{dx}{dt} = -5$

The shadow is determined by the man

$$\frac{y}{30} = \frac{y - x}{6}$$

$$24y = 30x$$

$$y = \frac{5}{4}x$$

$$\frac{dy}{dt} = \frac{5}{4} \frac{dx}{dt}$$

$$= -\frac{25}{4}$$

The shadow is determined by the child

$$\frac{y}{30} = \frac{y - x - 10}{3}$$

$$27y = 30x + 300$$

$$y = \frac{10}{9}x + \frac{100}{9}$$

$$\frac{dy}{dt} = \frac{10}{9} \frac{dx}{dt}$$

$$= -\frac{50}{9}$$

$$\therefore \frac{dy}{dt} = \begin{cases} -\frac{50}{9} & 0 < x < 80 \\ -\frac{25}{4} & x > 80 \end{cases}$$

$\frac{dy}{dt}$ is not continuous at $x = 80$.

Exercise

A ball is dropped from a height of 20 m, 12 m away from the top of a 20-meter lamppost. The ball's shadow, caused by the light at the top of the lamppost, is moving along the level ground. How fast is the shadow moving 1 second after the ball is released?

Solution

$$y(t) = -4.9t^2 + 20$$

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0 \quad g = 9.8 \text{ m/s}^2$$

$$\frac{dy}{dt} = -9.8t$$

$$\text{At } t = 1 \rightarrow y(1) = 15.1 \text{ m} \quad y'(1) = -9.8 \text{ m/s}$$

$$\frac{y}{20} = \frac{x-12}{x}$$

$$xy = 20x - 240$$

$$y(1) = 15.1$$

$$20x - 15.1x = 240$$

$$x = \frac{240}{4.9}$$

$$x \frac{dy}{dt} + y \frac{dx}{dt} = 20 \frac{dx}{dt}$$

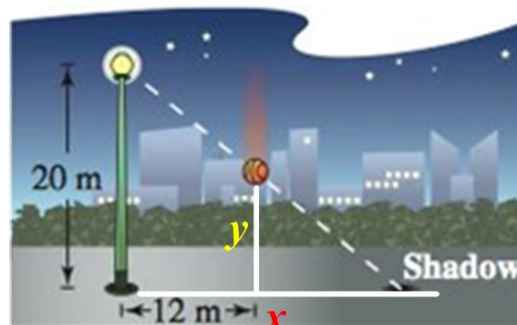
$$(20 - y) \frac{dx}{dt} = x \frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{x}{20 - y} \frac{dy}{dt}$$

$$= \frac{240}{4.9} \frac{1}{20 - 15.1} (-9.8)$$

$$= -\frac{480}{4.9}$$

$$= -\frac{4800}{49} \text{ m/s}$$



Exercise

The endpoints of a movable rod of length 1 meter have coordinates $(x, 0)$ and $(0, y)$. The position of the end on the x -axis is

$$x(t) = \frac{1}{2} \sin \frac{\pi t}{6} \quad \text{where } t \text{ is the time in seconds.}$$

- Find the time of one complete cycle of the rod.
- What is the lowest point reached by the end of the rod on the y -axis?
- Find the speed of the y -axis endpoint when the x -axis endpoints is $\left(\frac{1}{4}, 0\right)$

Solution

Given: $x(t) = \frac{1}{2} \sin \frac{\pi t}{6}$ $x^2 + y^2 = 1$

a) Period: $\frac{2\pi}{\pi/6} = 12 \text{ sec}$

b) When $x = \frac{1}{2}$

$$y = \sqrt{1 - \frac{1}{4}}$$

$$= \frac{\sqrt{3}}{2}$$

\therefore The lowest point $\left(0, \frac{\sqrt{3}}{2}\right)$

c) When $x = \frac{1}{4}$

$$y = \sqrt{1 - \frac{1}{16}}$$

$$= \frac{\sqrt{15}}{4}$$

$$x(t) = \frac{1}{2} \sin \frac{\pi t}{6}$$

$$\frac{dx}{dt} = \frac{\pi}{12} \cos \frac{\pi t}{6}$$

$$x^2 + y^2 = 1$$

$$x \frac{dx}{dt} + y \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$$

$$= -\left(\frac{1}{4} \cdot \frac{4}{\sqrt{15}}\right) \frac{\pi}{12} \cos \frac{\pi t}{6} \Bigg|_{t=1}$$

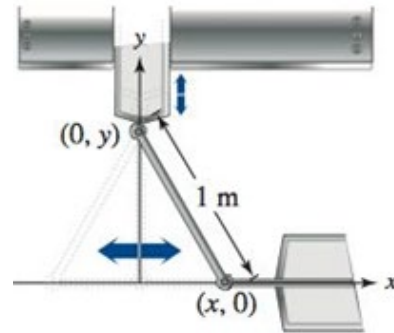
$$= -\frac{\pi}{12\sqrt{15}} \cos \frac{\pi}{6}$$

$$= -\frac{\pi}{12\sqrt{15}} \frac{\sqrt{3}}{2}$$

$$= -\frac{\pi}{24\sqrt{5}}$$

$$\text{Speed} = \left| -\frac{\pi\sqrt{5}}{120} \right|$$

$$= \frac{\pi\sqrt{5}}{120} \text{ m / sec}$$



Exercise

Cars on a certain roadway travel on a circular arc of radius r . In order not to rely on friction alone to overcome the centrifugal force, the road is banked at an angle of magnitude θ from the horizontal. The banking angle must satisfy the equation $rg \tan \theta = v^2$, where v is the velocity of the cars and

$g = 32 \text{ ft/sec}^2$ is the acceleration due to gravity. Find the relationship between the related rates $\frac{dv}{dt}$ and $\frac{d\theta}{dt}$.

Solution

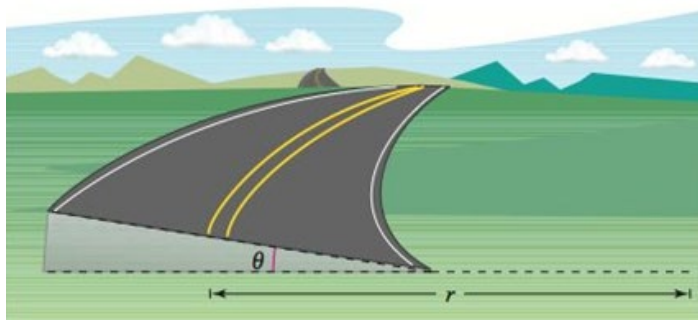
$$rg \tan \theta = v^2$$

$$32r \tan \theta = v^2$$

$$32r \sec^2 \theta \frac{d\theta}{dt} = 2v \frac{dv}{dt}$$

$$\frac{dv}{dt} = \frac{16r}{v} \sec^2 \theta \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{v}{16r} \cos^2 \theta \frac{dv}{dt}$$



Exercise

A fish is reeled in at a rate of 1 ft/sec from a point 10 feet above the water. At what rate is the angle θ between the line and the water changing when there is a total of 25 feet of line from the end of the rod to the water?

Solution

Given: $\frac{dx}{dt} = -1 \text{ ft/sec}$

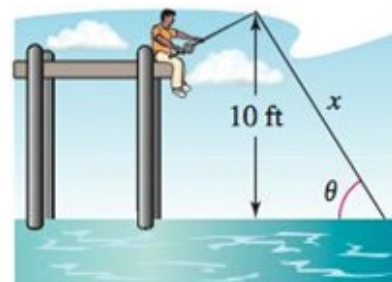
$$\sin \theta = \frac{10}{x}$$

$$\cos \theta \frac{d\theta}{dt} = -\frac{10}{x^2} \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = -\frac{10 \sec \theta}{x^2} \frac{dx}{dt}$$

$$= -\frac{10}{25^2} \frac{5}{\sqrt{21}} (-1)$$

$$= \frac{2}{25\sqrt{21}} \text{ rad/sec}$$



$$\sec \theta = \frac{x}{\sqrt{x^2 - 10^2}} = \frac{25}{\sqrt{25^2 - 10^2}} = \frac{25}{\sqrt{525}} = \frac{5}{\sqrt{21}}$$

Exercise

An airplane flies at an altitude of 5 miles toward a point directly over an observer. The speed of the plane is 600 mph. Find the rates at which the angle of elevation θ is changing when the angle is

a) $\theta = 30^\circ$ b) $\theta = 60^\circ$ c) $\theta = 75^\circ$

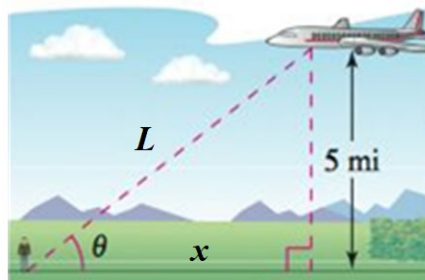
Solution

Given: $y = 5$; $\frac{dx}{dt} = -600$ mph

$$\tan \theta = \frac{y}{x} = \frac{5}{x}$$

$$\sec^2 \theta \frac{d\theta}{dt} = -\frac{5}{x^2} \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = -\frac{5 \cos^2 \theta}{x^2} \frac{dx}{dt}$$



Let L be the distance from observer to the plane. $\cos \theta = \frac{x}{L}$

$$\frac{d\theta}{dt} = -\frac{5}{x^2} \frac{x^2}{L^2} \frac{dx}{dt}$$

$$= -\frac{5}{L^2} \frac{dx}{dt}$$

$$\sin \theta = \frac{5}{L} \rightarrow L = \frac{5}{\sin \theta}$$

$$= -5 \frac{\sin^2 \theta}{5^2} (-600)$$

$$= 120 \sin^2 \theta$$

a) $\theta = 30^\circ$

$$\frac{d\theta}{dt} = 120 \sin^2 30^\circ$$

$$= \frac{120}{4}$$

$$= 30 \text{ rad / hr}$$

b) $\theta = 60^\circ$

$$\frac{d\theta}{dt} = 120 \sin^2 60^\circ$$

$$= 120 \left(\frac{3}{4} \right)$$

$$= 90 \text{ rad / hr}$$

c) $\theta = 75^\circ$

$$\frac{d\theta}{dt} = 120 \sin^2 75^\circ$$

$$\approx 111.96 \text{ rad / hr}$$

Exercise

A patrol car is parked 50 feet from a long warehouse. The revolving light on top of the car turns at a rate of 30 revolutions per minute. How fast is the light beam moving along the wall when the beam makes angles of

a) $\theta = 30^\circ$ b) $\theta = 60^\circ$ c) $\theta = 70^\circ$

With the perpendicular line from the light to the wall?

Solution

Given: $y = 50$; $\frac{d\theta}{dt} = 30 \frac{\text{rev}}{\text{min}} \cdot \frac{2\pi \text{ rad}}{1 \text{ rev}} \cdot \frac{1 \text{ min}}{60 \text{ sec}}$
 $= \pi \text{ rad / sec}$

$$\tan \theta = \frac{x}{50}$$

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{50} \frac{dx}{dt}$$

$$\frac{dx}{dt} = 50 \sec^2 \theta \frac{d\theta}{dt}$$

$$\frac{dx}{dt} = 50\pi \sec^2 \theta$$

a) $\theta = 30^\circ$

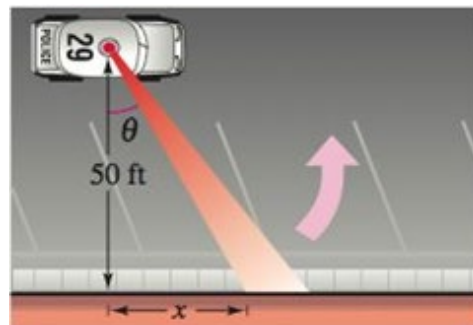
$$\begin{aligned} \frac{dx}{dt} &= 50\pi \sec^2 30^\circ \\ &= 50\pi \left(\frac{4}{3}\right) \\ &= \frac{200\pi}{3} \text{ ft / sec} \end{aligned}$$

b) $\theta = 60^\circ$

$$\begin{aligned} \frac{dx}{dt} &= 50\pi \sec^2 60^\circ \\ &= 50\pi (4) \\ &= 200\pi \text{ ft / sec} \end{aligned}$$

c) $\theta = 70^\circ$

$$\begin{aligned} \frac{dx}{dt} &= 50\pi \sec^2 70^\circ \\ &= 427.43\pi \text{ ft / sec} \end{aligned}$$



Exercise

A wheel of radius 30 cm revolves at a rate of 10 revolutions per second. A dot is painted at a point P on the rim of the wheel.

- Find $\frac{dx}{dt}$ as a function of θ .
- Graph the function.
- When is the absolute value of the rate of change of x greatest?
- When is it least?
- Find $\frac{dx}{dt}$ when $\theta = 30^\circ$ and $\theta = 60^\circ$

Solution

Given: $r = 30$; $\frac{d\theta}{dt} = 10 \frac{\text{rev}}{\text{sec}} \frac{2\pi \text{ rad}}{1 \text{ rev}} = 20\pi \text{ rad/sec}$



$$\begin{aligned} a) \quad \cos \theta &= \frac{x}{30} \\ -\sin \theta \frac{d\theta}{dt} &= \frac{1}{30} \frac{dx}{dt} \\ \frac{dx}{dt} &= -30 \sin \theta \frac{d\theta}{dt} \\ &= -600\pi \sin \theta \end{aligned}$$

b)

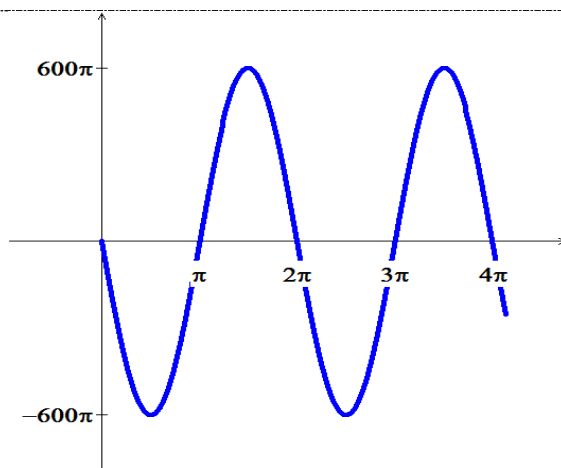
Amplitude: $|A| = 600\pi$

Period: $P = 2\pi$

Phase Shift: $\phi = 0$

VT: $y = 0$

x	y
0	0
$\frac{\pi}{2}$	-600π
π	0
$\frac{3\pi}{2}$	600π
2π	0



c) $|\sin \theta| = 1$ is the greatest value, therefore $\theta = \frac{\pi}{2} + n\pi$

\therefore The greatest: $\frac{dx}{dt} = |-600\pi \sin \theta|$
 $= 600\pi$

d) The least when $|\sin \theta| = 0 \rightarrow \theta = n\pi$:

$\frac{dx}{dt} = 0$

e) $\theta = 30^\circ$

$$\begin{aligned}\frac{dx}{dt} &= -600\pi \sin 30^\circ \\ &= \underline{-300\pi \text{ cm/sec}} \\ \theta &= 60^\circ \\ \frac{dx}{dt} &= -600\pi \sin 60^\circ \\ &= \underline{-300\pi\sqrt{3} \text{ cm/sec}}\end{aligned}$$

Exercise

A security camera is centered 50 feet above a 100-foot hallway. It is easiest to design the camera with a constant angular rate of rotation, but this results in recording the images of the surveillance area at a variable rate. So, it is desirable to design a system with a variable rate of rotation and a constant rate of movement of the scanning beam along the hallway. Find a model for the variable rate of rotation when

$$\left| \frac{dx}{dt} \right| = 2 \text{ ft/sec}$$

Solution

Given: $y = 50$; $\left| \frac{dx}{dt} \right| = 2 \text{ ft/sec}$

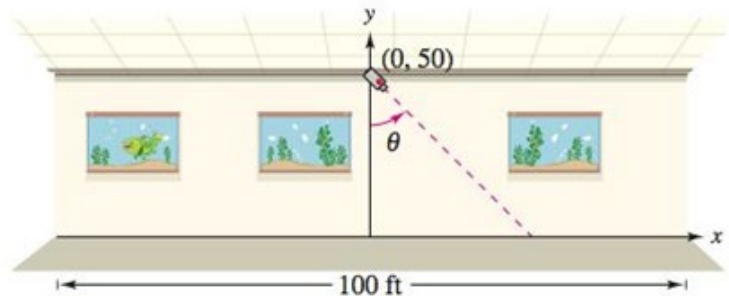
$$\tan \theta = \frac{x}{50}$$

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{50} \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{\cos^2 \theta}{50} \frac{dx}{dt}$$

$$= \frac{\cos^2 \theta}{50} (2)$$

$$= \underline{\frac{1}{25} \cos^2 \theta} \quad -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$$



Exercise

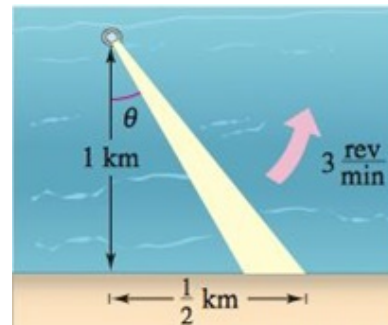
A rotating beacon is located 1 km off a straight shoreline. The beacon rotates at a rate of 3 rev/min . How fast (in km/hr) does the beam of light appear to be moving to a viewer who is $\frac{1}{2} \text{ km}$ down the shoreline?

Solution

Given: $y = 1$; $\frac{d\theta}{dt} = 3 \frac{\text{rev}}{\text{min}} \frac{2\pi \text{ rad}}{1 \text{ rev}} = \underline{6\pi \text{ rad/min}}$

$$\tan \theta = x$$

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{dx}{dt}$$



$$\begin{aligned}
 \frac{dx}{dt} &= \left(1 + \tan^2 \theta\right) \frac{d\theta}{dt} \\
 &= \left(1 + \frac{1}{4}\right) \left(6\pi \frac{\text{rad}}{\text{min}} \frac{60 \text{ min}}{1 \text{ hr}}\right) \\
 &= \underline{450\pi \text{ km / hr}}
 \end{aligned}$$

Exercise

A sandbag is dropped from a balloon at a height of 60 m when the angle of elevation to the sun is 30° . The position of the sandbag is

$$s(t) = 60 - 4.9t^2$$

Find the rate at which the shadow of the sandbag is traveling along the ground when the sandbag is at height of 35 m.

Solution

$$s(t) = 60 - 4.9t^2$$

$$\frac{d}{dt}s(t) = -9.8$$

$$s(t) = 60 - 4.9t^2 = 35$$

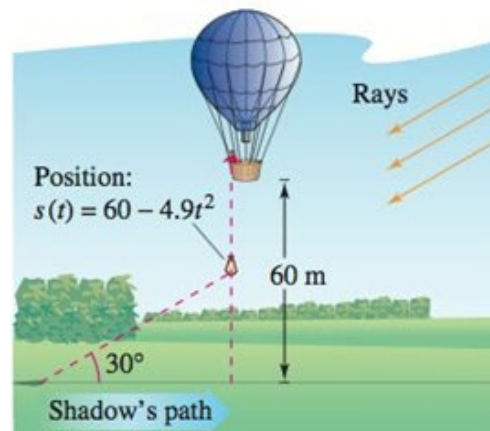
$$t^2 = \frac{25}{4.9}$$

$$t = \frac{5}{\sqrt{4.9}}$$

$$\tan 30^\circ = \frac{s(t)}{x(t)} = \frac{1}{\sqrt{3}}$$

$$\begin{aligned}
 x(t) &= \sqrt{3} s(t) \\
 &= 60\sqrt{3} - 4.9\sqrt{3} t^2
 \end{aligned}$$

$$\begin{aligned}
 \frac{dx}{dt} &= -9.8t\sqrt{3} \\
 &= -\frac{5(9.8)\sqrt{3}}{\sqrt{4.9}} \\
 &= \underline{-38.37 \text{ m / sec}}
 \end{aligned}$$



Exercise

The distance between the head of a piston and the end of a cylindrical chamber is given by $x(t) = \frac{8t}{t+1}$ cm, for $t \geq 0$ (measured in seconds). The radius of the cylinder is 4 cm.

- Find the volume of the chamber, for $t \geq 0$.
- Find the rate of change of the volume $V'(t)$ for $t \geq 0$.
- Graph the derivative of the volume function. On what intervals is the volume increasing? Decreasing?

Solution

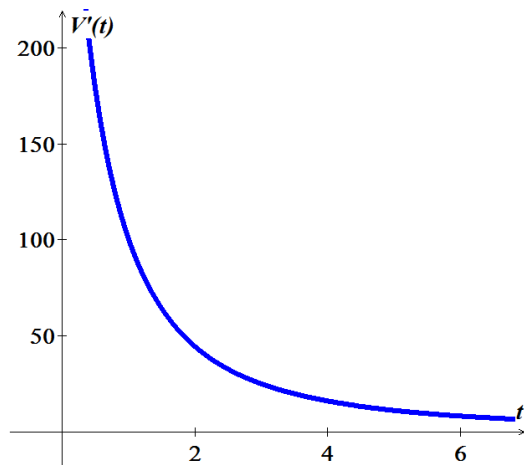
$$\begin{aligned} a) \quad V(t) &= \pi r^2 h \\ &= \pi (4)^2 \frac{8t}{t+1} \\ &= \frac{128\pi t}{t+1} \text{ cm}^3 \end{aligned}$$

$$b) \quad V'(t) = \frac{128\pi}{(t+1)^2}$$

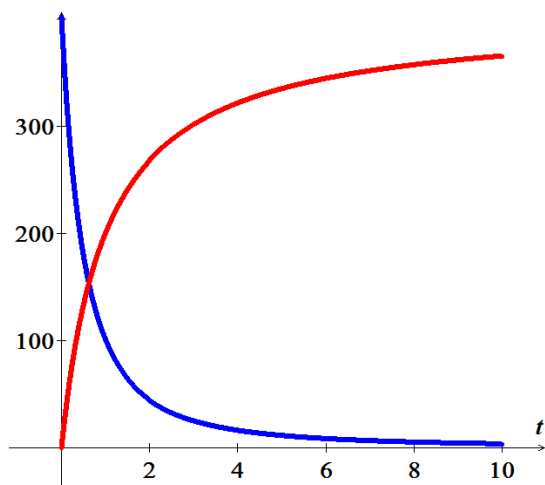
$$\left(\frac{ax+b}{cx+d} \right)' = \frac{ad-bc}{(cx+d)^2}$$

c)

t	$V'(t)$
0	128π
1	32π



Since the rate of change of the volume is strictly positive, the volume function must be increasing for $t > 0$.



Exercise

Two boats leave a dock at the same time. One boat travels south at 30 *mi/hr* and the other travels east at 40 *mi/hr*. after half an hour, how fast is the distance between the boats increasing?

Solution

Given: $\frac{dx}{dt} = 40 \text{ mi/hr}$, $\frac{dy}{dt} = 30 \text{ mi/hr}$, $t = 0.5 \text{ hr}$

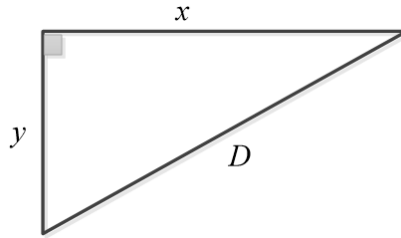
$$t = 0.5 \rightarrow x = 20 \quad y = 15$$

$$D^2 = x^2 + y^2$$

$$D = \sqrt{20^2 + 15^2}$$
$$= 25 \text{ |}$$

$$2D \frac{dD}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\frac{dD}{dt} = \frac{1}{D} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right) \bigg|_{\frac{dx}{dt}=40, \frac{dy}{dt}=30, D=25}$$
$$= \frac{20(40) + 15(30)}{25}$$
$$= 50 \text{ mph |}$$



Exercise

A spherical balloon is inflated at a rate of 10 cm^3 / min . At what rate is the diameter of the balloon increasing when the balloon has a diameter of 5 *cm*.

Solution

Given: $\frac{dV}{dt} = 10 \text{ cm}^3 / \text{min}$, $D = 5 \text{ cm}$

$$V = \frac{4}{3} \pi r^3$$

$$= \frac{4}{3} \pi \left(\frac{D}{2} \right)^3$$

$$= \frac{\pi}{6} D^3$$

$$\frac{dV}{dt} = \frac{\pi}{2} D^2 \frac{dD}{dt}$$

$$\frac{dD}{dt} = \frac{2}{\pi D^2} \frac{dV}{dt} \bigg|_{\frac{dV}{dt}=10, D=5}$$
$$= \frac{2}{\pi 5^2} (10)$$
$$= \frac{4}{5\pi} \text{ cm/min |}$$

Exercise

A rope is attached to the bottom of a hot-air balloon that is floating above the flat field. If the angle of the rope to the ground remains 65° and the rope is pulled in at 5 ft/s , how quickly is the elevation of the balloon changing?

Solution

Given: $\frac{ds}{dt} = 5 \text{ ft/sec}, \quad \theta = 65^\circ$

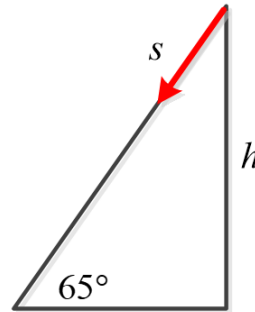
$$\sin 65^\circ = \frac{h}{s}$$

$$h = s \sin 65^\circ$$

$$\frac{dh}{dt} = \sin 65^\circ \frac{ds}{dt}$$

$$= -5 \sin 65^\circ$$

$$\approx -4.53 \text{ ft/sec}$$



Exercise

Water flows into a conical tank at a rate of $2 \text{ ft}^3 / \text{min}$. If the radius of the top of the tank is 4 feet and the height is 6 feet , determine how quickly the water level is rising when the water is 2 feet deep in the tank.

Solution

Given: $\frac{dV}{dt} = 2 \frac{\text{ft}^3}{\text{min}}, \quad y = 6 \text{ ft}, \quad x = 4 \text{ ft}, \quad h = 2 \text{ ft}$

The water forms a cone with volume: $V = \frac{1}{3} \pi r^2 h$

From the triangles: $\frac{r}{h} = \frac{4}{6}$

$$r = \frac{2}{3} h$$

$$V = \frac{1}{3} \pi \left(\frac{2}{3} h \right)^2 h$$

$$= \frac{4\pi}{27} h^3$$

$$\frac{dV}{dt} = \frac{4\pi}{27} \left(3h^2 \frac{dh}{dt} \right)$$

$$= \frac{4\pi}{9} h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{9}{4\pi h^2} \frac{dV}{dt}$$

$$= \frac{9}{4\pi(4)} (2)$$

$$= \frac{9}{8\pi} \text{ ft/min}$$

Exercise

A jet flies horizontally 500 feet directly above a spectator at an air show at 450 mi/hr. How quickly is the angle of elevation (between the ground and the line from the spectator to the jet) changing 2 seconds later?

Solution

$$\begin{aligned} \text{Given: } \frac{dx}{dt} &= 450 \frac{\text{mi}}{\text{hr}} \frac{1 \text{ hr}}{3600 \text{ sec}} \frac{5280 \text{ ft}}{1 \text{ mi}} \\ &= 660 \text{ ft/sec} \end{aligned}$$

$$\cot \theta = \frac{x}{500}$$

$$\theta = \cot^{-1} \frac{x}{500}$$

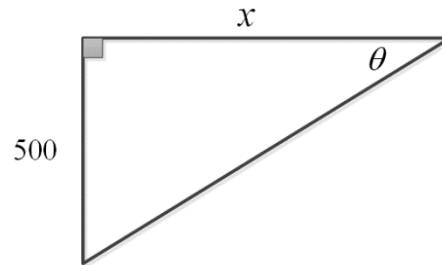
$$\theta' = - \frac{\frac{d}{dt} \left(\frac{x}{500} \right)}{1 + \left(\frac{x}{500} \right)^2}$$

$$= - \frac{\frac{1}{500} \frac{dx}{dt}}{\frac{500^2 + x^2}{500^2}} \quad t = 2 \rightarrow x = 2(660) = 1320$$

$$= - \frac{500}{500^2 + x^2} \frac{dx}{dt} \bigg|_{\frac{dx}{dt}=660, x=1320}$$

$$= - \frac{500}{500^2 + 1320^2} (660)$$

$$\approx -0.166 \text{ rad/sec}$$



Exercise

A man whose eyelevel is 6 feet above the ground walks toward a billboard at a rate of 2 ft/s. The bottom of the billboard is 10 feet above the ground, and it is 15 feet high. The man's viewing angle is the angle formed by the lines between the man's eyes and the top and bottom of the billboard. At what rate is the viewing angle changing when the man is 30 feet from the billboard?

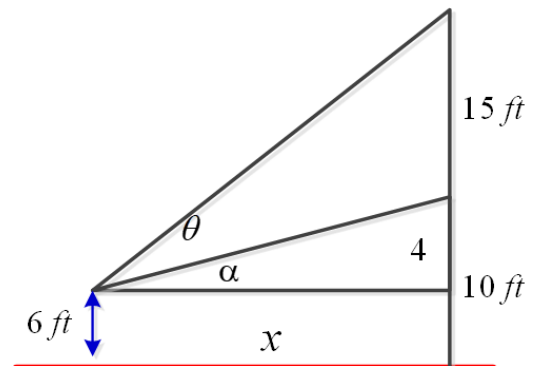
Solution

$$\text{Given: } \frac{dx}{dt} = -2 \text{ ft/s} \quad x_0 = 30 \text{ ft}$$

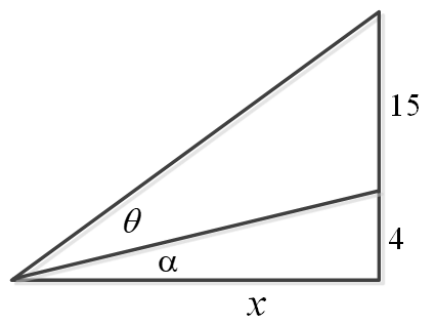
$$\cot \alpha = \frac{x}{4} \rightarrow \alpha = \cot^{-1} \left(\frac{x}{4} \right)$$

$$\cot(\alpha + \theta) = \frac{x}{19}$$

$$\theta = \cot^{-1} \left(\frac{x}{19} \right) - \alpha$$



$$\begin{aligned}
&= \cot^{-1}\left(\frac{x}{19}\right) - \cot^{-1}\left(\frac{x}{4}\right) \\
\theta' &= -\frac{\frac{d}{dt}\left(\frac{x}{19}\right)}{1 + \left(\frac{x}{19}\right)^2} + \frac{\frac{d}{dt}\left(\frac{x}{4}\right)}{1 + \left(\frac{x}{4}\right)^2} \\
&= -\frac{\frac{1}{19} \frac{dx}{dt}}{1 + \frac{x^2}{361}} + \frac{\frac{1}{4} \frac{dx}{dt}}{1 + \frac{x^2}{16}} \\
&= \left(\frac{4}{x^2 + 16} - \frac{19}{x^2 + 361} \right) \frac{dx}{dt} \bigg|_{\frac{dx}{dt} = -2, x=30} \\
&= (-2) \left(\frac{4}{900 + 16} - \frac{19}{900 + 361} \right) \\
&= (-2) \left(\frac{1}{229} - \frac{19}{1,261} \right) \\
&= (-2) \frac{-3090}{288,769} \\
&= \frac{6,180}{288,769} \text{ rad/sec} \approx 0.0214 \text{ rad/sec}
\end{aligned}$$



Exercise

A trough is shaped like a half cylinder with length 5 m and radius 1 m. The trough is full of water when a valve is opened and water flows out of the bottom of the trough at a rate of $1.5 \text{ m}^3 / \text{hr}$.

(Hint: Area of the sector = $\frac{1}{2} r^2 \theta$, r is the radius of a sector of the circle subtended by an angle of θ)

- How fast is the water level changing when the water level is 0.5 m from the bottom of the trough?
- What is the rate of change of the surface area of the water when the water is 0.5 m deep?

Solution

$$\cos \frac{\theta}{2} = \frac{r-h}{h} \Rightarrow \frac{\theta}{2} = \cos^{-1} \frac{r-h}{h}$$

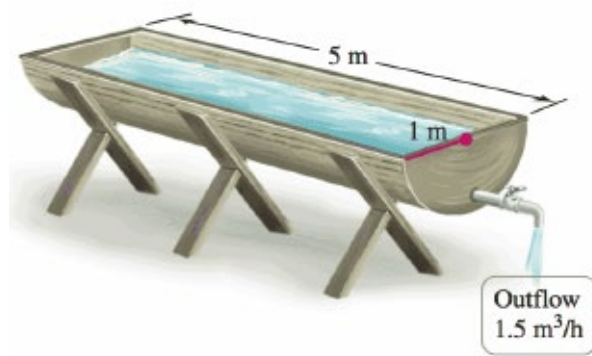
$$\theta = 2 \cos^{-1} \left(\frac{r-h}{h} \right)$$

$$x^2 = r^2 - (r-h)^2$$

$$x = \sqrt{r^2 - r^2 + 2rh - h^2}$$

$$= \sqrt{2rh - h^2}$$

ΔAOB area is A_1



$$A_1 = 2 \frac{1}{2} (r-h) \sqrt{2rh-h^2}$$

$$= (r-h) \sqrt{2rh-h^2}$$

Area of sector AOB is A_2

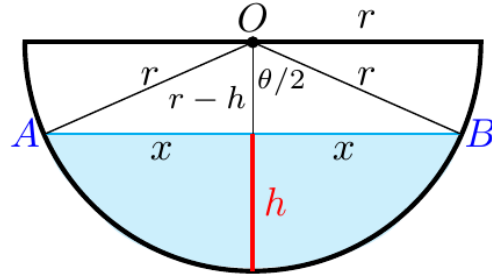
$$A_2 = \frac{1}{2} r^2 \theta$$

$$= \frac{1}{2} r^2 2 \cos^{-1} \left(\frac{r-h}{r} \right)$$

$$= r^2 \cos^{-1} \left(\frac{r-h}{r} \right)$$

Area of cross-sectional:

$$A = r^2 \cos^{-1} \left(\frac{r-h}{r} \right) - (r-h) \sqrt{2rh-h^2}$$



a) Given: $r=1$ $\frac{dV}{dt} = -1.5$ $= 0.5$

$$A = \cos^{-1}(1-h) - (1-h) \sqrt{2h-h^2}$$

$$V = 5A$$

$$= 5 \cos^{-1}(1-h) - 5(1-h) \sqrt{2h-h^2}$$

$$\frac{dV}{dt} = 5 \frac{1}{\sqrt{1-(1-h)^2}} \frac{dh}{dt} - 5 \frac{1}{\sqrt{2h-h^2}} \left(-2h+h^2 + (1-h)^2 \right) \frac{dh}{dt}$$

$$= 5 \frac{1}{\sqrt{1-(1-h)^2}} \frac{dh}{dt} - 5 \frac{1}{\sqrt{2h-h^2}} \left(1-4h+2h^2 \right) \frac{dh}{dt}$$

$$-\frac{3}{2} = \left(5 \frac{1}{\sqrt{1-\frac{1}{4}}} - 5 \frac{1}{\sqrt{1-\frac{1}{4}}} \left(1-2+\frac{1}{2} \right) \right) \frac{dh}{dt}$$

$$-\frac{3}{10} = \left(\frac{2}{\sqrt{3}} - \frac{2}{\sqrt{3}} \left(-\frac{1}{2} \right) \right) \frac{dh}{dt}$$

$$-\frac{3}{10} = \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) \frac{dh}{dt}$$

$$-\frac{3}{10} = \sqrt{3} \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{\sqrt{3}}{10} \text{ m/hr}$$

b) Surface: $S = 5(2x)$

$$S = 10\sqrt{2h-h^2}$$

$$\begin{aligned}
 \frac{dS}{dt} &= 10 \frac{1-h}{\sqrt{2h-h^2}} \frac{dh}{dt} \\
 &= 10 \frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} \left(-\frac{\sqrt{3}}{10} \right) \\
 &= -1 \text{ m}^2/\text{hr}
 \end{aligned}$$

Exercise

A conical tank with an upper radius of 4 m and a height of 5 m drains into a cylindrical tank with a radius of 4 m and a height of 5 m. If the water level in the conical tank drops at a rate of 0.5 m/min, at what rate does the water in the cylindrical tank rise when the water level in the conical tank is 3 m? 1 m?

Solution

$$\begin{aligned}
 V_1 &= \frac{\pi}{3} r^2 h_1 \\
 \frac{r}{4} &= \frac{h_1}{5} \rightarrow r = \frac{4h_1}{5}
 \end{aligned}$$

$$\begin{aligned}
 V_1 &= \frac{\pi}{3} \left(\frac{4h_1}{5} \right)^2 h_1 \\
 &= \frac{16\pi}{75} h_1^3
 \end{aligned}$$

$$\frac{dV_1}{dt} = \frac{16\pi}{25} h_1^2 \frac{dh_1}{dt}$$

a) **For** $h_1 = 3$

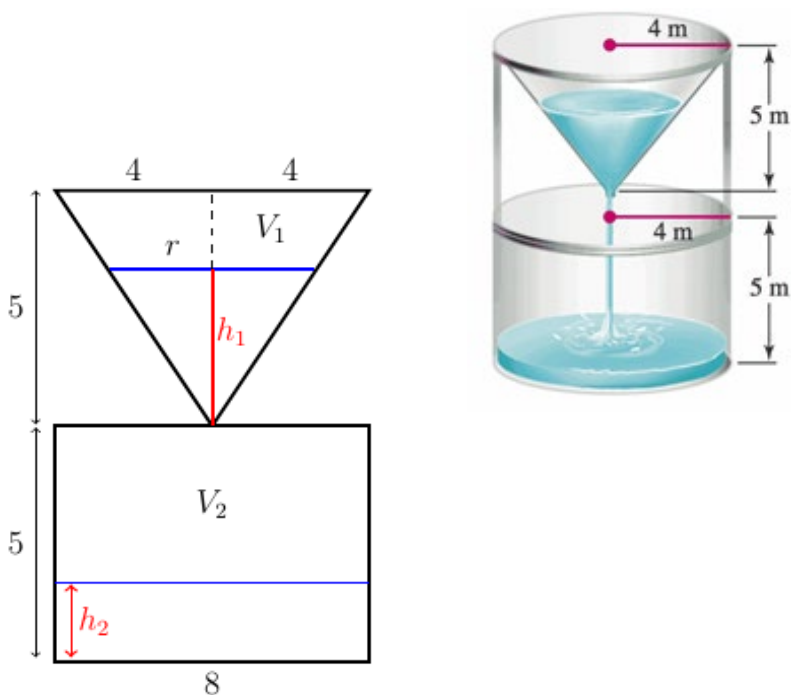
Given: $\frac{dh_1}{dt} = -0.5$

$$\begin{aligned}
 \frac{dV_1}{dt} &= \frac{16\pi}{25} h_1^2 \frac{dh_1}{dt} \\
 &= \frac{16\pi}{25} 9 \left(-\frac{1}{2} \right) \\
 &= -\frac{72\pi}{25} \text{ m}^3/\text{min}
 \end{aligned}$$

Volume of the lower tank:

$$\begin{aligned}
 V_2 &= \pi r^2 h_2 \\
 &= 16\pi h_2
 \end{aligned}$$

$$\frac{dV_2}{dt} = 16\pi \frac{dh_2}{dt}$$



$$\frac{72\pi}{25} = 16\pi \frac{dh_2}{dt}$$

$$\frac{dh_2}{dt} = \frac{9}{50} \text{ m/min}$$

b) **For** $h_1 = 1$

$$\frac{dV_1}{dt} = \frac{16\pi}{25} h_1^2 \frac{dh_1}{dt}$$

$$= \frac{16\pi}{25} \left(-\frac{1}{2} \right)$$

$$= -\frac{8\pi}{25} \text{ m}^3/\text{min}$$

$$\frac{dV_2}{dt} = 16\pi \frac{dh_2}{dt}$$

$$\frac{8\pi}{25} = 16\pi \frac{dh_2}{dt}$$

$$\frac{dh_2}{dt} = \frac{1}{50} \text{ m/min}$$

Exercise

Two cylindrical swimming pools are being filled simultaneously at the same rate (in m^3/min). The smaller pool has a radius of 5 m, and the water level rises at a rate of 0.5 m/min. The larger pool has a radius of 8 m. How fast is the water level rising in the larger pool?

Solution

Small Pool:

$$V_s = \pi r^2 h_s$$

$$= 25\pi h_s$$

$$\frac{dV_s}{dt} = 25\pi \frac{dh_s}{dt}$$

$$= 25\pi \left(\frac{1}{2} \right)$$

$$= 12.5\pi \text{ m}^3/\text{min}$$

Large Pool:

$$V_L = \pi r^2 h_L$$

$$= 64\pi h_L$$

$$\frac{dV_L}{dt} = 64\pi \frac{dh_L}{dt}$$



$$\text{Since } \frac{dV_L}{dt} = \frac{dV_s}{dt}$$

$$\frac{25\pi}{2} = 64\pi \frac{dh_L}{dt}$$

$$\frac{dh_L}{dt} = \frac{25}{128} \text{ m/min}$$

Exercise

An observer is 20 m above the ground floor of a large hotel atrium looking at a glass enclosed elevator shaft that is 20 m horizontally from the observer. The angle of elevation of the elevator is the angle that the observer's line of sight makes with the horizontal (it may be positive or negative).

- Assuming that the elevator rises at a rate of 5 m/s, what is the rate of change of the angle of the angle of elevation when the elevator is 10 m above the ground?
- When the elevator is 40 m above the ground?

Solution

$$\tan \theta = \frac{h}{20} \rightarrow h = 20 \tan \theta$$

$$\frac{dh}{dt} = 20 \sec^2 \theta \frac{d\theta}{dt} \quad (1)$$

$$a) \text{ Given: } \frac{dh}{dt} = 5$$

$$\text{At } h = -10 \rightarrow \tan \theta = -\frac{10}{20} = -\frac{1}{2}$$

$$\sec^2 \theta = 1 + \frac{1}{4} = \frac{5}{4}$$

$$\sec^2 \theta = 1 + \tan^2 \theta$$

$$(1) \rightarrow \frac{d\theta}{dt} = \frac{1}{20 \sec^2 \theta} \frac{dh}{dt}$$

$$= \frac{1}{20 \cdot \frac{5}{4}} (5)$$

$$= \frac{1}{5} \text{ rad/sec}$$

$$b) 40 \text{ m above the ground} \Rightarrow h = 20$$

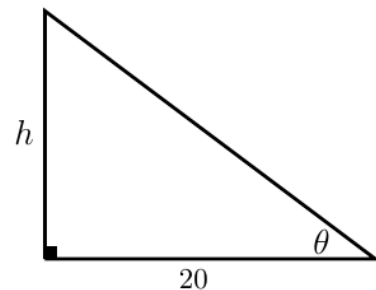
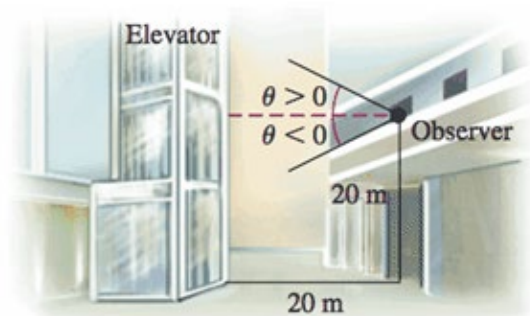
$$\tan \theta = \frac{20}{20} = 1$$

$$\sec^2 \theta = 1 + 1 = 2$$

$$\sec^2 \theta = 1 + \tan^2 \theta$$

$$\frac{d\theta}{dt} = \frac{1}{20(2)} (5)$$

$$= \frac{1}{8} \text{ rad/sec}$$



Exercise

A camera is set up at the starting line of a drag race 50 *ft.* from a dragster at the starting line (camera 1). Two seconds after the start race, the dragster has traveled 100 *ft.* and the camera is turning at 0.75 *rad/s* while filming the dragster.

- What is the speed of the dragster at this point?
- A second camera (camera 2) filming the dragster is located on the starting line 100 *ft.* away from the dragster at the start of the race. How fast is this camera turning 2 *sec* after the start of the race?

Solution

$$a) \tan \theta = \frac{y}{50}$$

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{50} \frac{dy}{dt} \quad (1)$$

$$\text{Given: } \frac{d\theta}{dt} = .75 = \frac{3}{4}$$

$$t = 2 \rightarrow y = 100$$

$$\tan \theta = \frac{100}{50} = 2$$

$$\sec^2 \theta = 1 + 4$$

$$= 5$$

$$(1) \rightarrow 5 \left(\frac{3}{4} \right) = \frac{1}{50} \frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{375}{2} \text{ ft/sec}$$

\therefore Speed at $t = 2$ *sec* is 187.5 *ft/sec*

$$b) \tan \alpha = \frac{y}{100}$$

$$\sec^2 \alpha \frac{d\alpha}{dt} = \frac{1}{100} \frac{dy}{dt}$$

$$\frac{d\alpha}{dt} = \frac{1}{100 \sec^2 \alpha} \frac{dy}{dt}$$

$$\tan \alpha = \frac{100}{100} = 1$$

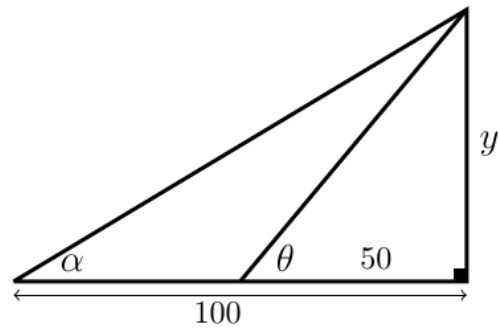
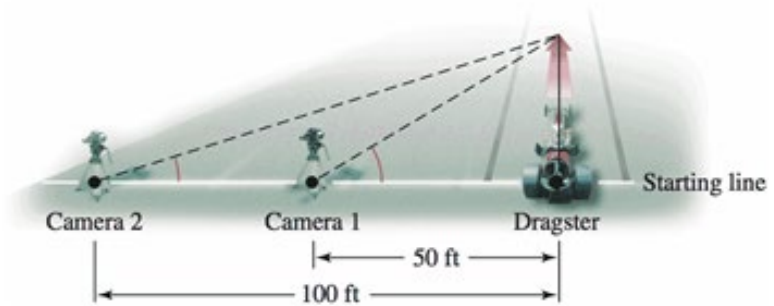
$$\sec^2 \theta = 1 + 1$$

$$= 2$$

$$\frac{d\alpha}{dt} = \frac{1}{200} \frac{375}{2}$$

$$= \frac{15}{16} \text{ rad/sec}$$

Speed of the camera 2.



Exercise

A port and a radar station are 2 *mi* apart on a straight shore running east and west. A ship leaves the port at noon traveling northeast at a rate of 15 *mi/hr*. If the ship maintains its speed and course, what is the rate of change of the tracking angle θ between the shore and the line between the radar station and the ship at 12:30 PM?

Solution

$$\begin{aligned} C &= 180^\circ - 45^\circ - \theta \\ &= 135^\circ - \theta \\ &= \frac{3\pi}{4} - \theta \end{aligned}$$

$$\frac{\sin \theta}{s} = \frac{\sin\left(\frac{3\pi}{4} - \theta\right)}{2}$$

$$\begin{aligned} 2 \sin \theta &= s \left(\sin \frac{3\pi}{4} \cos \theta - \cos \frac{3\pi}{4} \sin \theta \right) \\ &= s \left(\frac{\sqrt{2}}{2} \cos \theta + \frac{\sqrt{2}}{2} \sin \theta \right) \\ &= \frac{\sqrt{2}}{2} s (\cos \theta + \sin \theta) \end{aligned}$$

$$4 \sin \theta = \sqrt{2} s (\cos \theta + \sin \theta) \quad \times \frac{1}{\cos \theta}$$

$$4 \tan \theta = \sqrt{2} s + \sqrt{2} s \tan \theta$$

$$(4 - \sqrt{2} s) \tan \theta = \sqrt{2} s$$

$$\tan \theta = \frac{\sqrt{2} s}{4 - \sqrt{2} s} \rightarrow \theta = \tan^{-1} \frac{\sqrt{2} s}{4 - \sqrt{2} s}$$

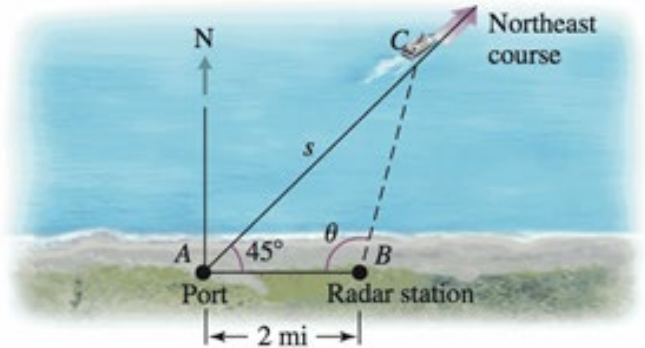
$$\frac{d\theta}{dt} = \frac{4\sqrt{2}}{(4 - \sqrt{2} s)^2} \cdot \frac{1}{1 + \left(\frac{\sqrt{2} s}{4 - \sqrt{2} s}\right)^2} \frac{ds}{dt}$$

$$= \frac{4\sqrt{2}}{16 - 8\sqrt{2} s + 4s^2} \frac{ds}{dt}$$

$$= \frac{\sqrt{2}}{4 - 2\sqrt{2} s + s^2} \frac{ds}{dt}$$

$$= \frac{\sqrt{2}}{4 - 15\sqrt{2} + \frac{225}{4}} (15)$$

$$= \frac{60\sqrt{2}}{241 - 60\sqrt{2}} \text{ rad/hr} \Bigg| \approx 0.54 \text{ rad/hr}$$



$$\left(\frac{ax+b}{cx+d} \right)' = \frac{ad-bc}{(cx+d)^2}$$

$$s = 7.5 = \frac{15}{2} \quad \frac{ds}{dt} = 15$$

Exercise

A ship leaves port traveling southwest at a rate of 12 *mi/hr*. At noon, the ship reaches its closest approach to a radar station, which is on the shore 1.5 *mi* from the port. If the ship maintains its speed and course, what is the rate of change of the tracking angle θ between the radar station and the ship at 1:30 PM?

Solution

Let x be the distance the ship has traveled.

$$\begin{aligned} P &= 180^\circ - 45^\circ - \theta \\ &= 135^\circ - \theta \\ &= \frac{3\pi}{4} - \theta \end{aligned}$$

$$\frac{\sin \theta}{x} = \frac{\sin\left(\frac{3\pi}{4} - \theta\right)}{\frac{3}{2}}$$

$$\begin{aligned} 3 \sin \theta &= 2x \left(\sin \frac{3\pi}{4} \cos \theta - \cos \frac{3\pi}{4} \sin \theta \right) \\ &= 2x \left(\frac{\sqrt{2}}{2} \cos \theta + \frac{\sqrt{2}}{2} \sin \theta \right) \\ &= \sqrt{2}x (\cos \theta + \sin \theta) \end{aligned}$$

$$3 \sin \theta = \sqrt{2}x (\cos \theta + \sin \theta) \quad \times \frac{1}{\cos \theta}$$

$$3 \tan \theta = \sqrt{2}x + \sqrt{2}x \tan \theta$$

$$(3 - \sqrt{2}x) \tan \theta = \sqrt{2}x$$

$$\tan \theta = \frac{\sqrt{2}x}{3 - \sqrt{2}x} \rightarrow \theta = \tan^{-1} \frac{\sqrt{2}x}{3 - \sqrt{2}x}$$

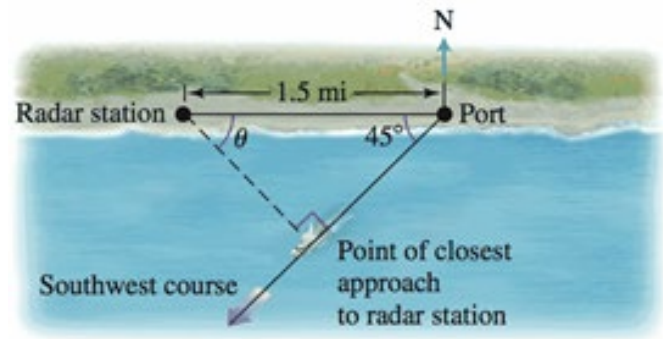
$$\frac{d\theta}{dt} = \frac{3\sqrt{2}}{(3 - \sqrt{2}x)^2} \cdot \frac{1}{1 + \left(\frac{\sqrt{2}x}{3 - \sqrt{2}x}\right)^2} \frac{dx}{dt}$$

$$= \frac{3\sqrt{2}}{9 - 6\sqrt{2}x + 4x^2} \frac{dx}{dt}$$

$$\text{At 12:00: } 2x^2 = 1.5^2 = \frac{9}{4} \rightarrow x = \frac{3}{2\sqrt{2}}$$

$$\begin{aligned} \text{At 1:30: } x &= \left(12 + \frac{12}{2}\right) + \frac{3}{2\sqrt{2}} \\ &= 18 + \frac{3}{2\sqrt{2}} \text{ mi} \\ &\approx 19.06 \text{ mi} \end{aligned}$$

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{3\sqrt{2}}{9 - 6\sqrt{2}(19.06) + 4(19.06)^2} (12) \\ &\approx 0.04 \text{ rad/hr} \end{aligned}$$



$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

A lighthouse stands 500 m off of a straight shore, the focused beam of its light revolving four times each minute. P is the point on shore closest to the lighthouse and Q is a point on the shore 200 m from P .

- What is the speed of the beam along the shore when it strikes the point Q ?
- Describe how the speed of the beam along the shore varies with the distance between P and Q . (neglect the height of the lighthouse)

Solution

$$\begin{aligned} a) \quad \tan \alpha &= \frac{x}{500} \\ \sec^2 \alpha \frac{d\alpha}{dt} &= \frac{1}{500} \frac{dx}{dt} \quad (1) \end{aligned}$$

At Point Q :

$$\tan \alpha = \frac{200}{500} = \frac{2}{5}$$

$$\sec^2 \theta = 1 + \frac{4}{25} = \frac{29}{25}$$

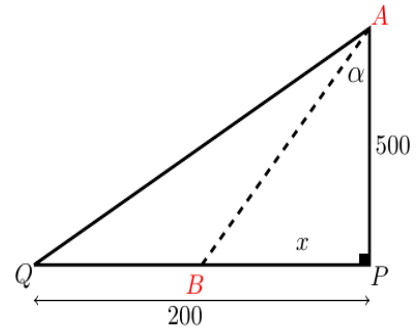
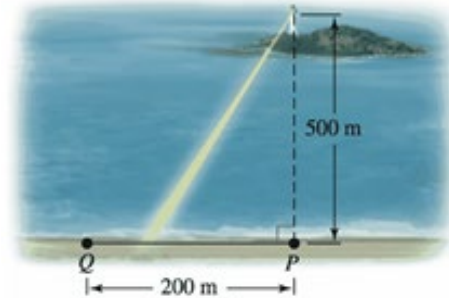
$$\sec^2 \theta = 1 + \tan^2 \theta$$

4 times each minute:

$$\begin{aligned} \frac{d\alpha}{dt} &= 4 \frac{2\pi}{1 \text{ min } 60 \text{ sec}} \\ &= \frac{2\pi}{15} \text{ rad/sec} \end{aligned}$$

$$(1) \rightarrow \frac{29}{25} \frac{2\pi}{15} = \frac{1}{500} \frac{dx}{dt}$$

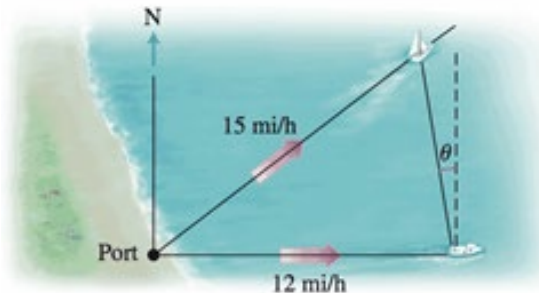
$$\frac{dx}{dt} = \frac{232\pi}{3} \text{ m/sec}$$



- The beam moves slower when B near P , and faster as it approaches Q (or further away from P)

Exercise

A boat leaves a port traveling due east at 12 mi/hr. At the same time, another boat leaves the same port traveling northeast at 15 mi/hr. The angle θ of the line between the boats is measured relative to due north.



What is the rate of change of this angle 30 min. after the boats leave the port? 2 hr. after the boats leave the port?

Solution

Given: $\frac{dx}{dt} = 12$ $\frac{dy}{dt} = 15$

$$\frac{y}{\sin\left(\frac{\pi}{2} - \theta\right)} = \frac{x}{\sin\left(\frac{\pi}{4} + \theta\right)} \quad (\text{Law of Sines})$$

$$y \sin\left(\frac{\pi}{4} + \theta\right) = x \sin\left(\frac{\pi}{2} - \theta\right)$$

$$y \left(\sin \frac{\pi}{4} \cos \theta + \cos \frac{\pi}{4} \sin \theta \right) = x \left(\sin \frac{\pi}{2} \cos \theta - \cos \frac{\pi}{2} \sin \theta \right)$$

$$\frac{1}{\sqrt{2}} y (\cos \theta + \sin \theta) = x \cos \theta$$

$$\cos \theta + \sin \theta = \frac{x}{y} \sqrt{2} \cos \theta$$

$$1 + \tan \theta = \frac{x}{y} \sqrt{2}$$

$$\tan \theta = \frac{x\sqrt{2} - y}{y}$$

$$\theta = \tan^{-1} \frac{x\sqrt{2} - y}{y}$$

$$\frac{d\theta}{dt} = \frac{\sqrt{2}y \frac{dx}{dt} - y \frac{dy}{dt} - \sqrt{2}x \frac{dy}{dt} + y \frac{dx}{dt}}{y^2} \cdot \frac{1}{1 + \left(\frac{x\sqrt{2} - y}{y} \right)^2}$$

$$= \frac{\sqrt{2} \left(y \frac{dx}{dt} - x \frac{dy}{dt} \right)}{y^2 + (x\sqrt{2} - y)^2}$$

After **30 min** $\left(= \frac{1}{2} \text{ hr} \right)$

$$\rightarrow x = \frac{12}{2} = 6 \text{ mi} \quad y = \frac{15}{2} = 7.5 \text{ mi}$$

$$\frac{d\theta}{dt} = \frac{\sqrt{2} \left(\frac{15}{2}(12) - 6(15) \right)}{\frac{225}{4} + \left(6\sqrt{2} - \frac{15}{2} \right)^2}$$

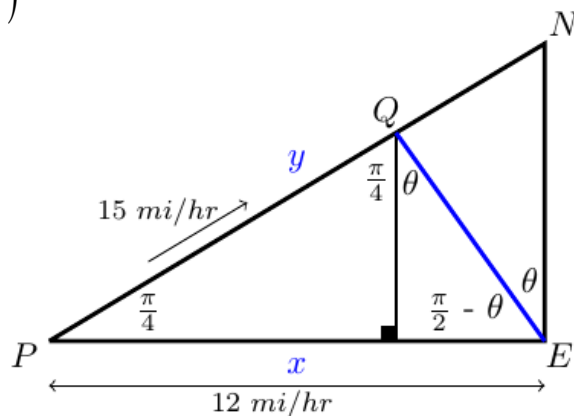
$$= 0$$

After **2 hrs.**

$$\rightarrow x = 12(2) = 24 \text{ mi} \quad y = 15(2) = 30 \text{ mi}$$

$$\frac{d\theta}{dt} = \frac{\sqrt{2} (30(12) - 24(15))}{900 + (24\sqrt{2} - 30)^2}$$

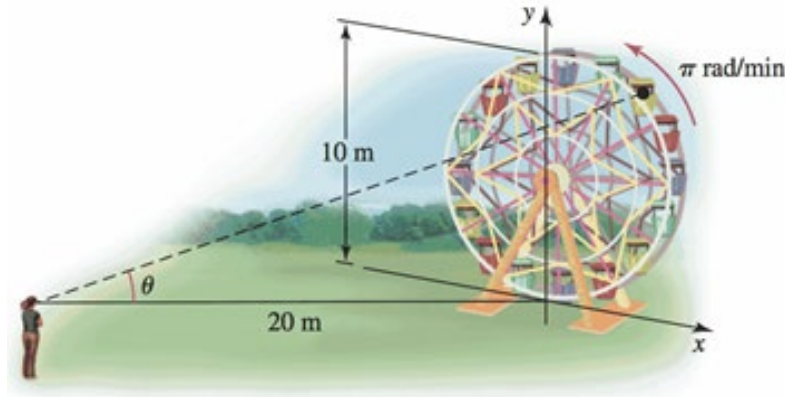
$$= 0$$



$$\left(\tan^{-1} u \right)' = \frac{u'}{1 + u^2}$$

Exercise

An observer stands 20 m from the bottom of a 10-m tall Ferris wheel on a line that is perpendicular to the face of the Ferris wheel. The wheel revolves at a rate of $\pi \text{ rad/min}$ and the observer's line of sight with a specific seat on the wheel makes an angle θ with the ground. 40 seconds after that seat leaves the lowest point on the wheel, what is the rate of change of θ ?



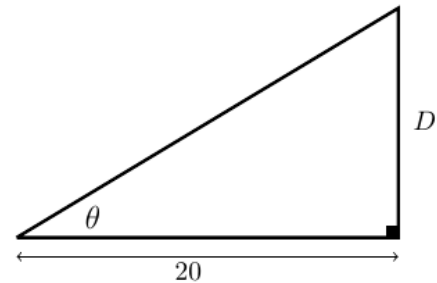
Assume the observer's eyes are level with the bottom of the wheel.

Solution

$$\tan \theta = \frac{D}{20}$$

$$\theta = \tan^{-1} \frac{D}{20}$$

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{1}{20} \frac{1}{1 + \frac{D^2}{400}} \frac{dD}{dt} \\ &= \frac{20}{400 + D^2} \frac{dD}{dt} \quad (1) \end{aligned}$$



$$D^2 = 5^2 + 5^2 - 2(5)(5)\cos \alpha \quad (\text{Law of Cosines})$$

$$D^2 = 50 - 50\cos \alpha$$

$$2D \frac{dD}{dt} = 50 \sin \alpha \frac{d\alpha}{dt}$$

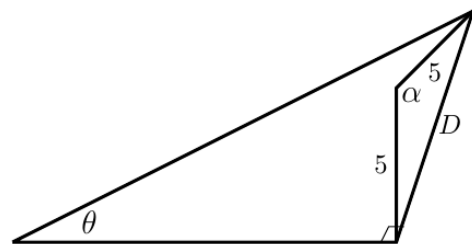
$$\frac{dD}{dt} = \frac{25}{D} \sin \alpha \frac{d\alpha}{dt} \quad (2)$$

$$\text{At } t = 40 \text{ sec } \frac{1 \text{ min}}{60 \text{ sec}} = \frac{2}{3} \text{ min}$$

$$\text{Given: } \frac{d\alpha}{dt} = \pi \frac{\text{rad}}{\text{min}} \rightarrow \alpha = \frac{2\pi}{3} \text{ rad}$$

$$\begin{aligned} D &= \sqrt{50 - 50\cos \frac{2\pi}{3}} \\ &= \sqrt{50 + 25} \\ &= 5\sqrt{3} \end{aligned}$$

$$(2) \rightarrow \frac{dD}{dt} = \frac{25}{5\sqrt{3}} \sin \left(\frac{2\pi}{3} \right) (\pi)$$



$$\frac{dD}{dt} = \frac{5\pi}{\sqrt{3}} \frac{\sqrt{3}}{2}$$

$$= \frac{5\pi}{2} \text{ rad/min}$$

$$(1) \quad \frac{d\theta}{dt} = \frac{20}{400 + 75} \left(\frac{5\pi}{2} \right)$$

$$\frac{d\theta}{dt} = \frac{2\pi}{19} \text{ rad/min}$$

Exercise

The bottom of a large theater screen is 3 ft. above your eye level and the top of the screen is 10 ft. above your eye level. Assume you walk away from the screen (perpendicular to the screen) at a rate of 3 ft/s while looking at the screen. What is the rate of change of the viewing angle θ when you are 30 ft. from the wall on which the screen hangs, assuming the floor is flat?

Solution

$$\tan \alpha = \frac{3}{x}$$

$$\alpha = \tan^{-1} \frac{3}{x}$$

$$\tan(\alpha + \theta) = \frac{10}{x}$$

$$\alpha + \theta = \tan^{-1} \frac{10}{x}$$

$$\theta = \tan^{-1} \frac{10}{x} - \alpha$$

$$= \tan^{-1} \frac{10}{x} - \tan^{-1} \frac{3}{x}$$

$$\frac{d\theta}{dt} = \left(-\frac{10}{x^2} \right) \frac{1}{1 + \frac{100}{x^2}} \frac{dx}{dt} - \left(-\frac{3}{x^2} \right) \frac{1}{1 + \frac{9}{x^2}} \frac{dx}{dt}$$

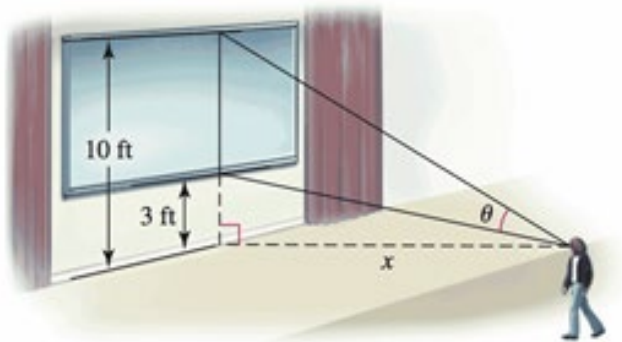
$$= \left(\frac{3}{x^2 + 9} - \frac{10}{x^2 + 100} \right) \frac{dx}{dt} \quad \frac{dx}{dt} = 3, \quad x = 30$$

$$= \left(\frac{3}{900 + 9} - \frac{10}{900 + 100} \right) (3)$$

$$= 3 \left(\frac{3}{909} - \frac{1}{100} \right)$$

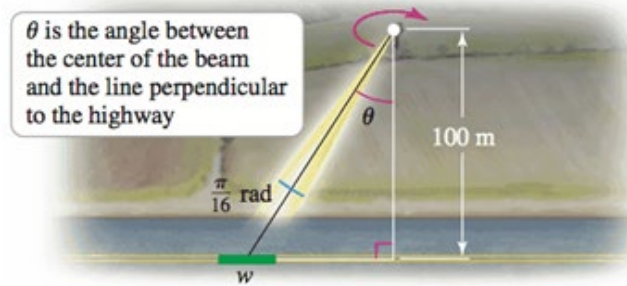
$$= 3 \frac{-609}{90900}$$

$$= -\frac{203}{10100} \text{ rad/sec}$$



Exercise

A revolving searchlight, 100 m from the nearest point on the center line of a straight highway, casts a horizontal beam along a highway. The beam leaves the spotlight at an angle of $\frac{\pi}{16}$ rad and revolves at a rate $\frac{\pi}{16}$ rad/s. Let w be the width of the beam as it sweeps along the highway and θ be the angle that the center of the beam makes with the perpendicular to the highway. What is the rate of change of w when $\theta = \frac{\pi}{3}$? Neglect the height of the lighthouse.



Solution

$$\tan\left(\theta - \frac{\pi}{32}\right) = \frac{r}{100} \Rightarrow r = 100 \tan\left(\theta - \frac{\pi}{32}\right)$$

$$\ell = AC$$

$$\tan\left(\theta + \frac{\pi}{32}\right) = \frac{\ell}{100} \Rightarrow \ell = 100 \tan\left(\theta + \frac{\pi}{32}\right)$$

$$w = \ell - r$$

$$\frac{dw}{dt} = \frac{d\ell}{dt} - \frac{dr}{dt}$$

$$= \frac{d}{dt}\left(100 \tan\left(\theta + \frac{\pi}{32}\right)\right) - \frac{d}{dt}\left(100 \tan\left(\theta - \frac{\pi}{32}\right)\right)$$

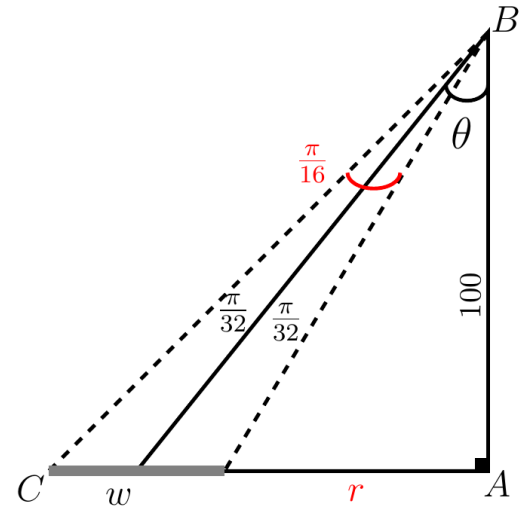
$$= 100 \left[\frac{1}{1 + \left(\theta + \frac{\pi}{32}\right)^2} - \frac{1}{1 + \left(\theta - \frac{\pi}{32}\right)^2} \right] \frac{d\theta}{dt} \quad \left| \frac{d\theta}{dt} = \frac{\pi}{16}, \theta = \frac{\pi}{3} \right.$$

$$= 100 \left[\frac{1}{1 + \left(\frac{\pi}{3} + \frac{\pi}{32}\right)^2} - \frac{1}{1 + \left(\frac{\pi}{3} - \frac{\pi}{32}\right)^2} \right] \frac{\pi}{16}$$

$$= \frac{25\pi}{4} \left(\frac{96^2}{96^2 + (35\pi)^2} - \frac{96^2}{96^2 + (29\pi)^2} \right)$$

$$= 57,600\pi \left(\frac{1}{96^2 + (35\pi)^2} - \frac{1}{96^2 + (29\pi)^2} \right)$$

$$\approx -4.9 \text{ m/sec}$$



Exercise

A piston is seated at the top of a cylindrical chamber with radius 5 cm when it starts moving into the chamber at a constant speed of 3 cm/sec . What is the rate of change of the volume of the cylinder when the piston is 2 cm from the base of the chamber?

Solution

$$V = 25\pi h$$

$$V = \pi r^2 h$$

$$\frac{dV}{dt} = 25\pi \frac{dh}{dt}$$

$$= 25\pi(-3)$$

$$= \underline{-75\pi\text{ cm/s}}$$

