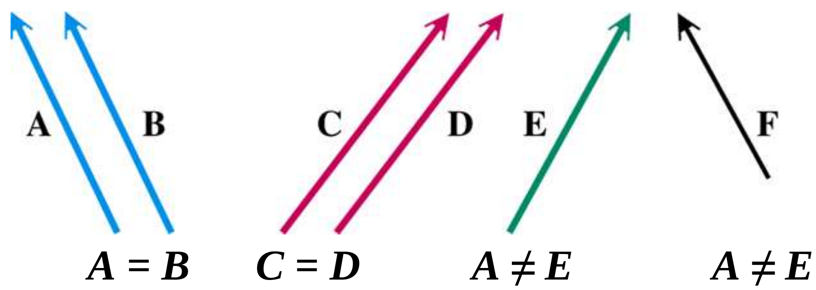
Section 4.3 – Vectors and Dot Product

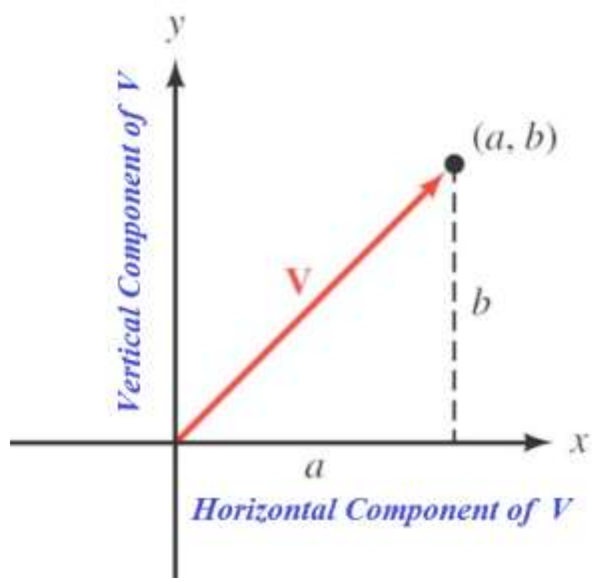
Notation	***The quantity is***
$\mathbf{V}$	*a vector*
$\vec{V}$	*a vector*
$\bar{V}$	*a vector*
$\overrightarrow{AB}$	*a vector*
x	*Scalar*
$\|V\|$	*Magnitude of vector V, a scalar*

Equality for Vectors

The vectors are equivalent if they have the same magnitude and the same direction. $V_1 = V_2$



Standard Position



A vector with its initial point at the origin is called a ***position vector***.

Magnitude of a Vector

The length or ***magnitude of a vector*** can be written:

$$|V| = \sqrt{a^2 + b^2} \qquad\qquad |V| = \sqrt{|V_x|^2 + |V_y|^2}$$

Direction *Angle* of a Vector

The direction angle θ satisfies $\tan\theta = \dfrac{b}{a}$, where $a \neq 0$.

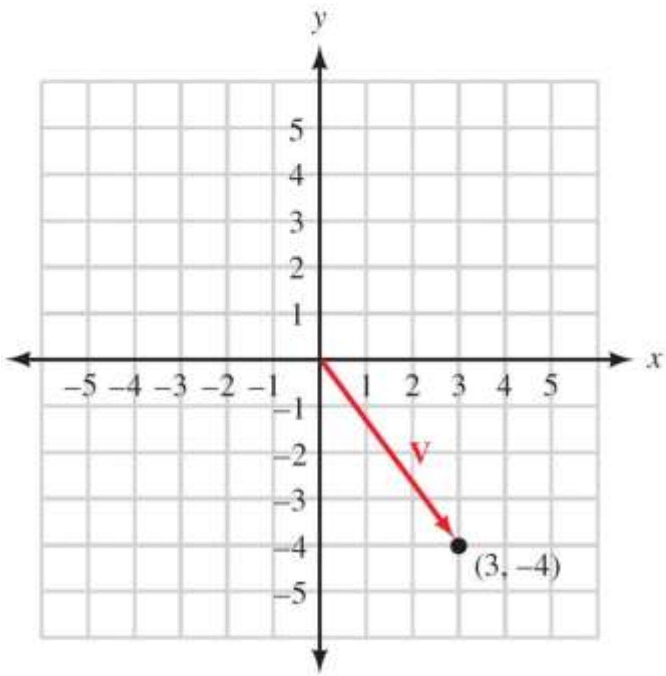
Zero Vector

A vector has a magnitude of zero $|V| = 0$ and has no defined direction.

Example

Draw the vector $V = (3, -4)$ in standard position and find its magnitude.

<u>*Solution*</u>



$$\begin{aligned} |V| &= \sqrt{a^2 + b^2} \\ &= \sqrt{3^2 + (-4)^2} \\ &= 5 \end{aligned}$$

Example

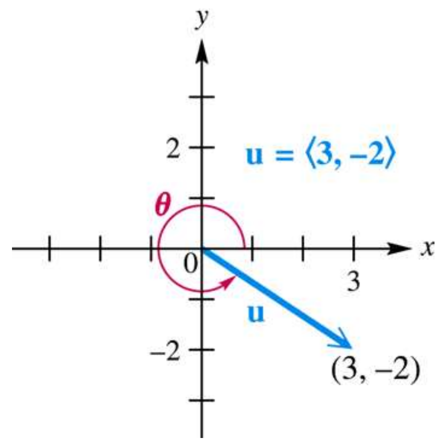
Find the magnitude and direction angle for **u** = ⟨3, –2⟩.

<u>*Solution*</u>

$$|u| = \sqrt{3^2 + (-2)^2}$$
$$= \sqrt{13}$$

$$\hat{\theta} = \tan^{-1}\frac{y}{x}$$
$$= \tan^{-1}\left(-\frac{2}{3}\right)$$
$$\approx -33.7^\circ$$

$$\theta \approx 360^\circ - 33.7^\circ$$
$$\approx 326.3^\circ$$



Horizontal & Vertical Vector Components

The horizontal and vertical components, respectively, of a vector **V** having magnitude |**V**| and direction angle θ are given by:

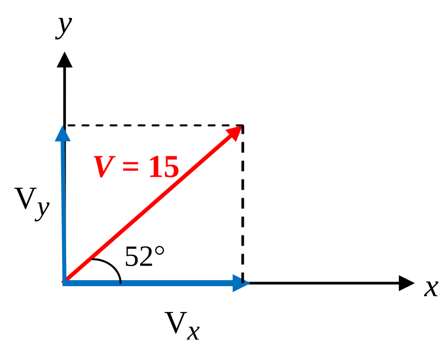
$$V_x = |V|\cos\theta \quad \textit{and} \quad V_y = |V|\sin\theta$$

V_x is the ***horizontal vector component*** of V

$$|V_x| = |V|\cos 52^\circ$$
$$= 15\cos 52^\circ$$
$$= 9.2$$

V_y is the ***vertical vector component*** of V

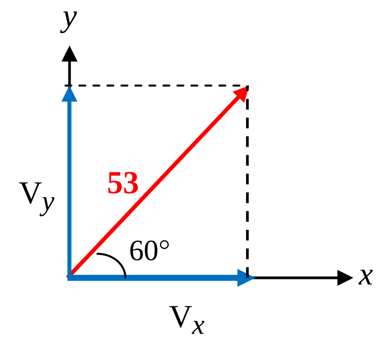
$$|V_y| = |V|\sin 52^\circ$$
$$= 15\sin 52^\circ$$
$$= 12$$



Example

The human cannonball is shot from cannon with an initial velocity of 53 miles per hour at an angle of 60° from the horizontal. Find the magnitude of the horizontal and vertical vector components of the velocity vector.

<u>*Solution*</u>

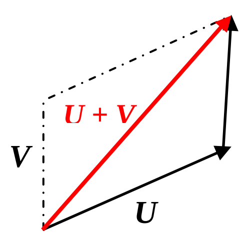


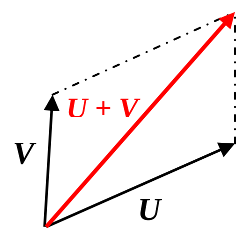
$$\left|V_x\right| = 53\cos 60^\circ$$
$$= 27\ mi/hr$$

$$\left|V_y\right| = 53\sin 60^\circ$$
$$= 46\ mi/hr$$

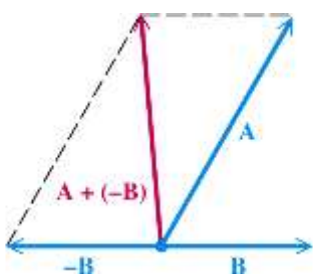
Addition and Subtraction of Vectors

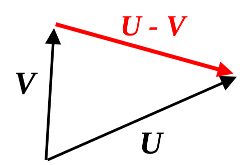
The sum of the vectors ***U*** and ***V*** (**U + V**) is called the ***resultant vector***.





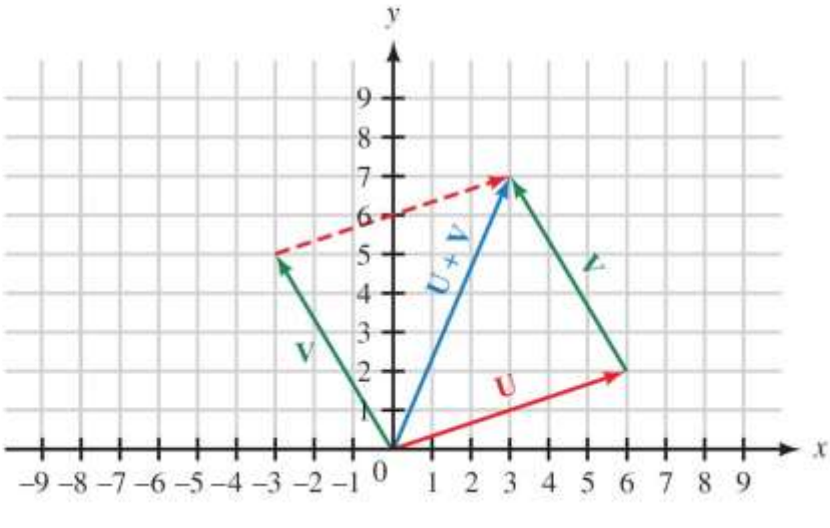
U – V = U + (-V)





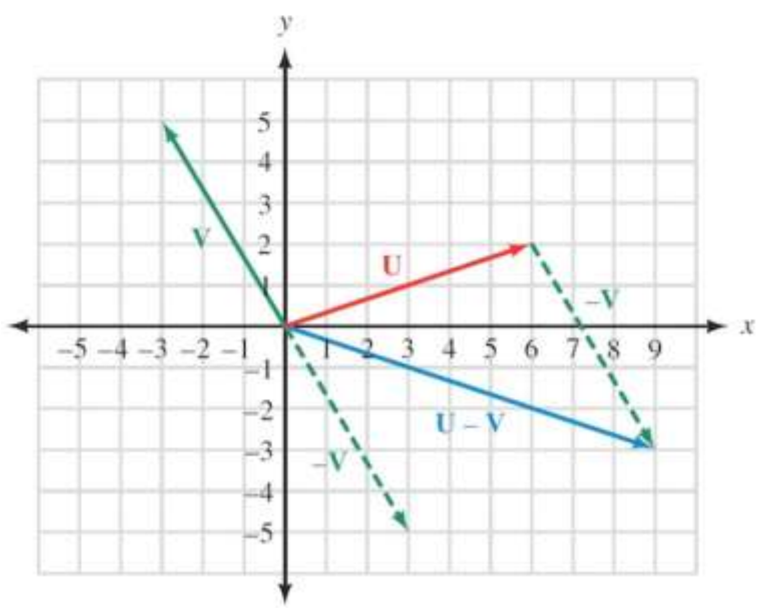
The sum of a vector ***V*** and its opposite –***V*** has magnitude 0 and is called the ***zero vector***.

Addition and subtraction with ***Algebraic*** Vectors



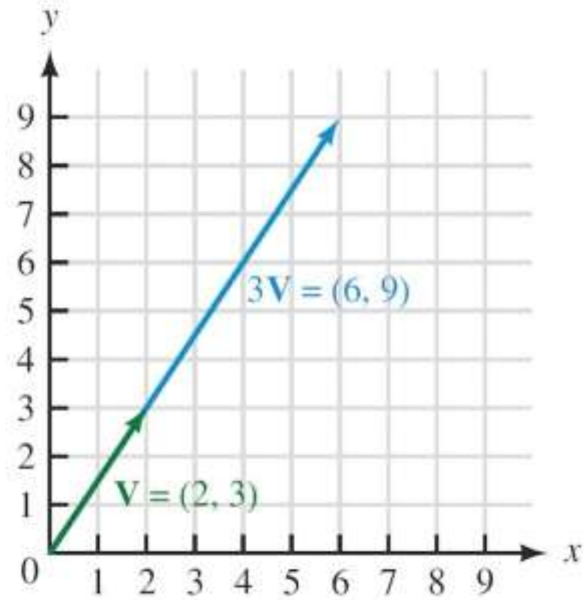
$$
\begin{aligned}
U + V &= \langle 6,2 \rangle + \langle -3,5 \rangle \\
&= \langle 6-3,\ 2+5 \rangle \\
&= \langle 3,\ 7 \rangle
\end{aligned}
$$

$$
\begin{aligned}
U - V &= \langle 6,2 \rangle - \langle -3,5 \rangle \\
&= \langle 6-(-3),\ 2-5 \rangle \\
&= \langle 9,\ -3 \rangle
\end{aligned}
$$



Scalar Multiplication

Example



$$3V = 3\langle 2, 3\rangle$$
$$= \langle 6,\ 9\rangle$$

Example

If $U = \langle 5, -3\rangle$ and $V = \langle -6, 4\rangle$, find:

a. $U + V$

b. $4U - 5V$

<u>*Solution*</u>

a. $U + V = < 5,-3 > + < -6,4 >$

$$= < 5-6, -3+4 >$$

$$= < -1, 1 >$$

b. $4U - 5V = 4 < 5,-3 > -5 < -6,4 >$

$$= < 20,-12 > - < -30,20 >$$

$$= < 20-(-30), -12-20 >$$

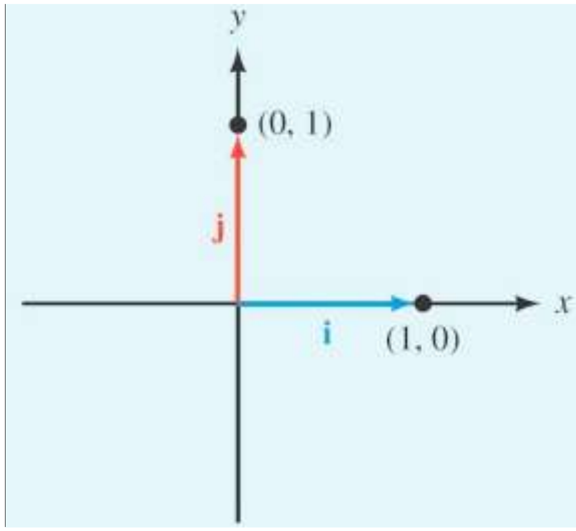
$$= < 20+30, -32 >$$

$$= \ < 50, -32 >$$

Component Vector Form

The vector that extends from the origin to the point (1, 0) is called the unit horizontal vector and is denoted by ***i***.

The vector that extends from the origin to the point (0, 1) is called the unit vertical vector and is denoted by ***j***.



Example

Write the vector $V = \langle 3,4 \rangle$ in terms of the unit vectors ***i*** and ***j***.

<u>*Solution*</u>

$V = 3i + 4j$

Algebraic Vectors

If ***i*** is the unit vector from (0, 0) to (1, 0), and ***j*** is the unit vector from (0, 0) to (0, 1), then any vector V can be written as

$$V = a\mathbf{i} + b\mathbf{j} = \langle a, b \rangle$$

Where a and b are real numbers. The magnitude of V is

$$|V| = \sqrt{a^2 + b^2}$$

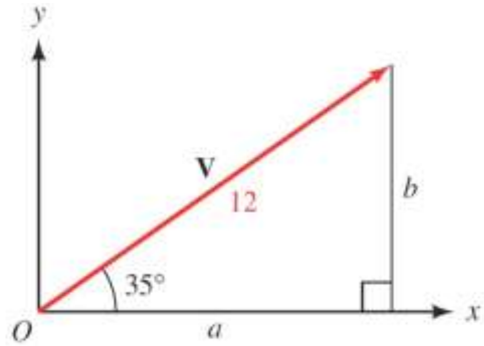
$$a = |V|\cos\theta \quad \textit{and} \quad b = |V|\sin\theta$$

$$V = \langle a,\ b \rangle = \langle |V|\cos\theta,\ |V|\sin\theta \rangle$$

Example

Vector V has its tail at the origin, and makes an angle of 35° with the positive x-axis. Its magnitude is 12. Write V in terms of the unit vectors i and j.

<u>*Solution*</u>



$$a = 12\cos 35° = 9.8$$

$$b = 12\sin 35° = 6.9$$

$$V = 9.8\boldsymbol{i} + 6.9\boldsymbol{j}$$

Example

If $U = 5\boldsymbol{i} - 3\boldsymbol{j}$ and $V = -6\boldsymbol{i} + 4\boldsymbol{j}$

a. $U + V$

$$U + V = 5\boldsymbol{i} - 3\boldsymbol{j} - 6\boldsymbol{i} + 4\boldsymbol{j}$$

$$= -\boldsymbol{i} + \boldsymbol{j}$$

b. $4U - 5V$

$$4U - 5V = 4(5\boldsymbol{i} - 3\boldsymbol{j}) - 5(-6\boldsymbol{i} + 4\boldsymbol{j})$$

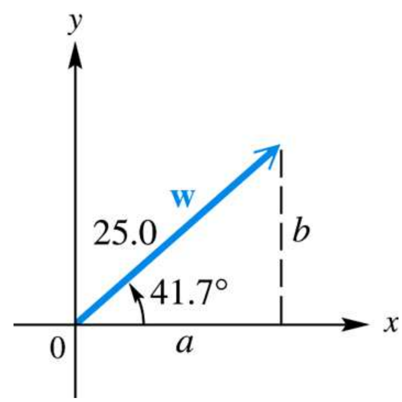
$$= 20\boldsymbol{i} - 12\boldsymbol{j} + 30\boldsymbol{i} - 20\boldsymbol{j}$$

$$= 50\boldsymbol{i} - 32\boldsymbol{j}$$

Example

Vector **w** has magnitude 25.0 and direction angle 41.7°. Find the horizontal and vertical components.

<u>*Solution*</u>



$$a = |w|\cos\theta$$
$$= 25\cos 41.7°$$
$$\approx 18.7$$

$$b = |w|\sin\theta$$
$$= 25\sin 41.7°$$
$$\approx 16.6$$

$$w = \langle 18.7,\ 16.6\rangle$$

Horizontal component: 18.7

Vertical component: 16.6

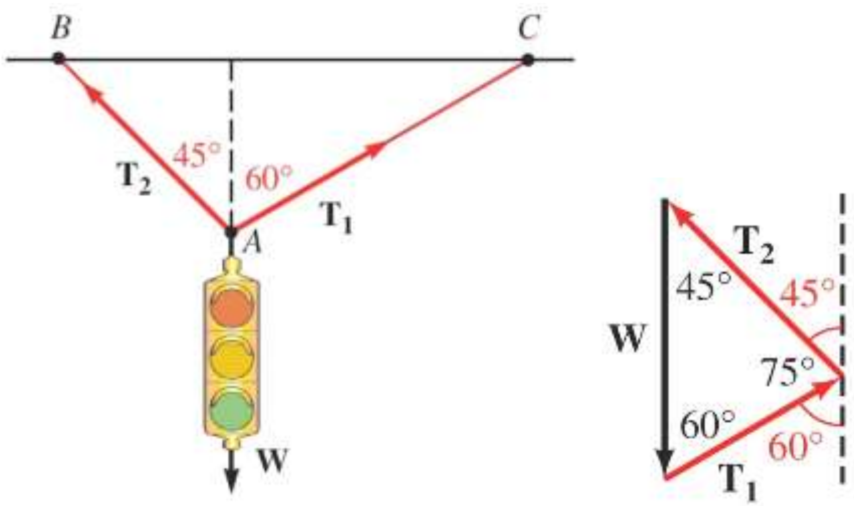
Force

When an object is stationary (at rest) we say it is in a state of static equilibrium.

When an object is in this state, the sum of the forces acting on the object must be equal to the zero vector **0**.

Example

A traffic light weighing 22 pounds is suspended by two wires. Find the magnitude of the tension in wire *AB*, and the magnitude of the tension in wire *AC*.



<u>*Solution*</u>

$$\frac{|T_1|}{\sin 45°} = \frac{22}{\sin 75°}$$

$$|T_1| = \frac{22 \sin 45°}{\sin 75°}$$

$$= 16\, lb$$

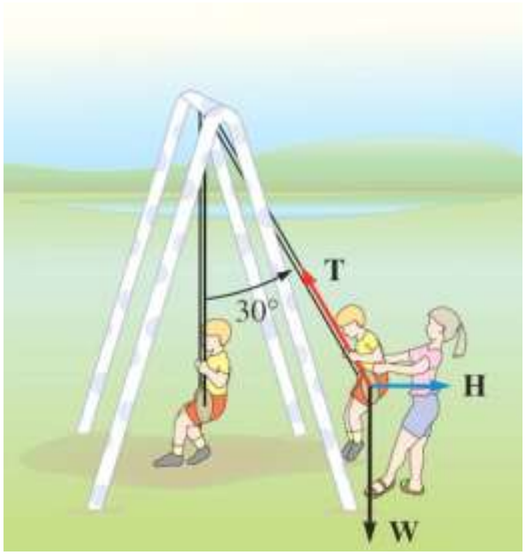
$$\frac{|T_2|}{\sin 60°} = \frac{22}{\sin 75°}$$

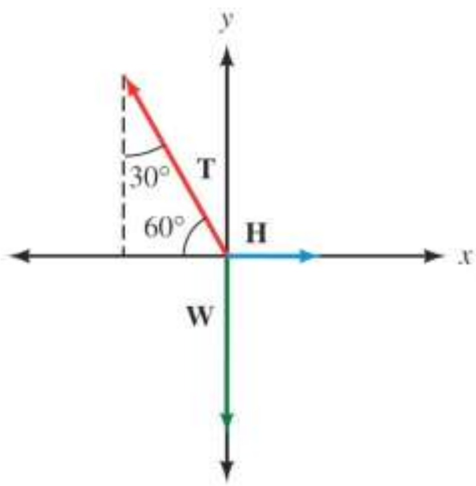
$$|T2| = \frac{22 \sin 60°}{\sin 75°}$$

$$\underline{= 20\ lb}$$

Example

Danny is 5 years old and weighs 42 pounds. He is sitting on a swing when his sister Stacey pulls him and the swing back horizontally through an angle of 30° and then stops. Find the tension in the ropes of the swing and the magnitude of the force exerted by Stacey.





<u>Solution</u>

$$H = |H|\,i$$

$$W = -|W|\,j = -42\,j$$

$$T = -|T|\cos 60°\,i + |T|\sin 60°\,j$$

$T + H + W = 0$ Static equilibrium $\Rightarrow \sum all\ forces = 0$

$$-|T|\cos 60°\,i + |T|\sin 60°\,j + |H|\,i - 42\,j = 0$$

$$-|T|\cos 60°\,i + |H|\,i + |T|\sin 60°\,j - 42\,j = 0$$

$$\left(-|T|\cos 60° + |H|\right)i + \left(|T|\sin 60° - 42\right)j = 0$$

$|T|\sin 60° - 42 = 0$ $\qquad -|T|\cos 60° + |H| = 0$

$$|T|\sin 60° = 42$$

$$|T| = \frac{42}{\sin 60°}$$

$$|T| = 48\ lb$$

$$\begin{aligned} |H| &= |T|\cos 60° \\ &= 48\cos 60° \\ &= 24\ lb \end{aligned}$$

The DOT *Product*

The *dot product* (or scalar product) of two vectors $U = ai + bj$ and $V = ci + dj$ is written $U \bullet V$ and is defined as follows:

$$U \bullet V = (ai + bj) \bullet (ci + dj)$$
$$= ac + bd$$

The dot product is a real number (scalar), not a vector.

- ✓ *It is helpful to find the angle between two vectors.*
- ✓ *Finding the work done by a force*

Dot Product	Angle Between Vectors
Positive	Acute
0	Right
Negative	Obtuse

Example

Find each of the following dot products

a. $U \cdot V$ *when* $U = \langle 3, 4 \rangle$ *and* $V = \langle 2, 5 \rangle$

$$U \cdot V = \langle 3,\ 4 \rangle \cdot \langle 2,\ 5 \rangle$$
$$= 3(2) + 4(5)$$
$$= 26$$

b. $\langle -1, 2 \rangle \cdot \langle 3, -5 \rangle$

$$\langle -1,\ 2 \rangle \cdot \langle 3,\ -5 \rangle = -3 - 10$$
$$= -13$$

c. $S \cdot W$ *when* $S = 6i + 3j$ *and* $W = 2i - 7j$

$$S \cdot W = 12 - 21$$
$$= -9$$

Finding the Angle Between Two Vectors

The dot product of two vectors is equal to the product of their magnitudes multiplies by the cosine of the angle between them. That is, when θ is the angle between two nonzero vectors U and V, then

$$U \cdot V = |U||V|\cos\theta$$

$$\cos\theta = \frac{U \cdot V}{|U||V|}$$

Example

Find the angle between the vectors U and V.

a. $U = \langle 2,3 \rangle$ *and* $V = \langle -3,2 \rangle$

b. $U = 6i - j$ *and* $V = i + 4j$

<u>Solution</u>

a) $U = \langle 2,3 \rangle$ *and* $V = \langle -3,2 \rangle$

$$\begin{aligned}\cos\theta &= \frac{U \cdot V}{|U||V|} \\ &= \frac{2(-3)+3(2)}{\sqrt{2^2+3^2}\sqrt{(-3)^2+2^2}} \\ &= \frac{-6+6}{\sqrt{13}\sqrt{13}} \\ &= \frac{0}{13} \\ &= 0\end{aligned}$$

$$\theta = \cos^{-1}(0) = 90°$$

b) $U = 6i - j$ *and* $V = i + 4j$

$$\begin{aligned}\cos\theta &= \frac{U \cdot V}{|U||V|} \\ &= \frac{6(1)+(-1)(4)}{\sqrt{6^2+(-1)^2}\sqrt{1^2+4^2}} \\ &= \frac{6-4}{\sqrt{37}\sqrt{17}} \\ &= \frac{2}{25.08} \\ &= 0.0797\end{aligned}$$

$$\theta = \cos^{-1}(0.0797) = 85.43°$$

Perpendicular Vectors

If U and V are two nonzero vectors, then

$$U \cdot V = 0 \iff U \perp V$$

Two vectors are perpendicular if and only if (*iff*) their dot product is 0.

Example

Which of the following vectors are perpendicular to each other?

$U = 8i + 6j \qquad V = 3i - 4j \qquad W = 4i + 3j$

<u>*Solution*</u>

$U \cdot V = (8i + 6j) \cdot (3i - 4j)$

$= 24 - 24$

$= 0 \qquad \therefore$ U and V are perpendicular

$U \cdot W = (8i + 6j) \cdot (4i + 3j)$

$= 32 + 18$

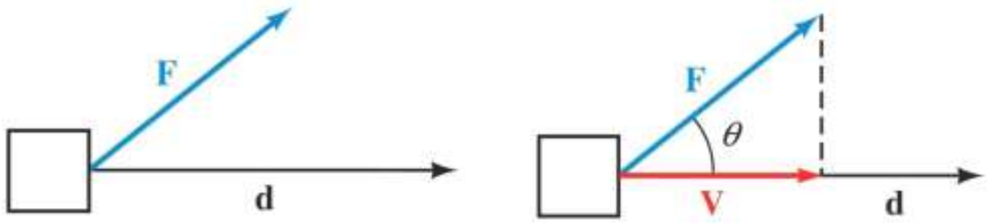
$= 50 \qquad \therefore$ U and W are not perpendicular

$V \cdot W = (3i - 4j) \cdot (4i + 3j)$

$= 12 - 12$

$= 0 \qquad \therefore$ V and W are perpendicular

Work



Work is performed when a force (constant) is used to move an object a certain distance.

d: displacement vector.

V: Represents the component of ***F*** that is the same direction of ***d,***
is sometimes called the ***projection*** of onto ***d***.

$$|V| = |F|\cos\theta$$

$$\begin{aligned} Work &= |V||d| \\ &= |F|\cos\theta|d| \\ &= |F||d|\cos\theta \\ &= F \cdot d \end{aligned}$$

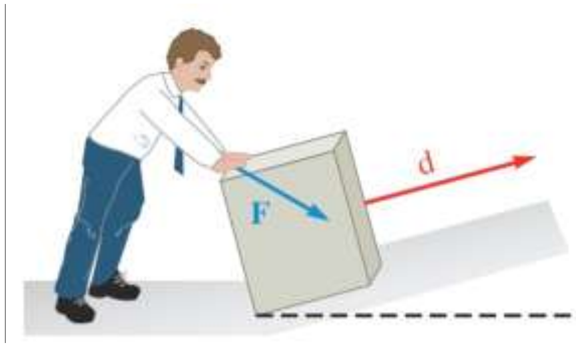
Definition

If a constant force ***F*** is applied, and the resulting movement of the object is represented by the displacement vector d, then the work performed by the force is

$$Work = F \cdot d$$

Example

A force $\boldsymbol{F} = 35i - 12j$ (in pounds) is used to push an object up a ramp. The resulting movement of the object is represented by the displacement vector $\boldsymbol{d} = 15i + 4j$ (in feet). Find the work done by the force.



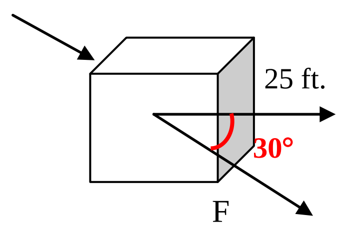
<u>*Solution*</u>

$$\begin{aligned} Work &= F \cdot d \\ &= (35)(15) + (-12)(4) \\ &= 480 \ \ ft - lb \end{aligned}$$

Example

A shipping clerk pushes a heavy package across the floor. He applies a force of 64 pounds in a downward direction, making an angle of 35° with the horizontal. If the package is moved 25 feet, how much work is done by the clerk?

<u>*Solution*</u>



$$\begin{aligned} \left|F_x\right| &= \left|F\right| \cos 30° \\ &= 64 \cos 30° \end{aligned}$$

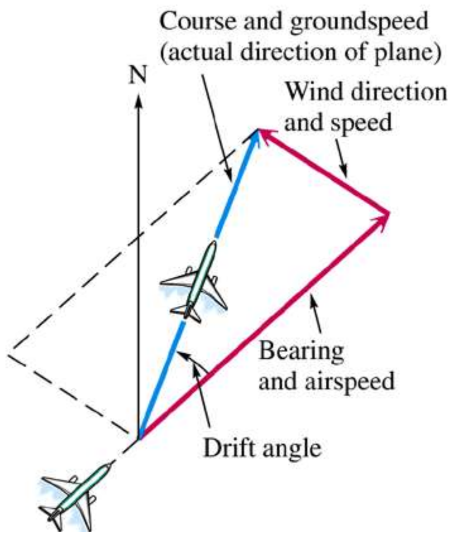
$$\begin{aligned} W &= \left|F_x\right| . d \\ &= (64 \cos 35°) .25 \\ &= 1300 \ \ ft - lb \end{aligned}$$

Airspeed and Groundspeed

The ***airspeed*** of a plane is its speed relative to the air

The ***groundspeed*** of a plane is its speed relative to the ground.

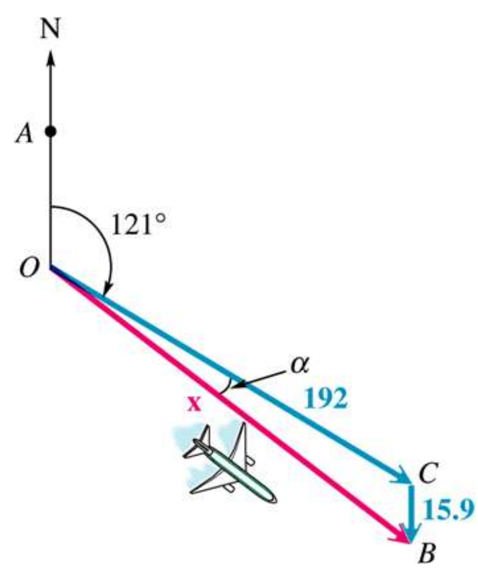
The ***groundspeed*** of a plane is represented by the vector sum of the airspeed and windspeed vectors.



Example

A plane with an airspeed of 192 mph is headed on a bearing of 121°. A north wind is blowing (from north to south) at 15.9 mph. Find the groundspeed and the actual bearing of the plane.

<u>Solution</u>



$\angle BCO = \angle AOC = 121°$

The groundspeed is represented by $|\mathbf{x}|$.

$$|x|^2 = 192^2 + 15.9^2 - 2(192)(15.9)\cos 121°$$
$$\approx 40{,}261$$

$$|x| \approx 200.7 \ mph$$

The plane's groundspeed is about 201 mph.

$$\frac{\sin\alpha}{15.9} = \frac{\sin 121°}{200.7}$$

$$\sin\alpha = \frac{15.9\sin 121°}{200.7}$$

$$\alpha = \sin^{-1}\left(\frac{15.9\sin 121°}{200.7}\right) \approx 3.89°$$

The plane's groundspeed is about 201 mph on a bearing of 121° + 4° = 125°.

Exercises *Section* 4.3 – Vectors and Dot Product

1. Let **u** = $\langle -2, 1\rangle$ and **v** = $\langle 4, 3\rangle$. Find the following.

a) **u** + **v**
b) –2**u**
c) 4**u** – 3**v**

2. Given: $|V| = 13.8,\ \theta = 24.2°$, find the magnitudes of the horizontal and vertical vector components of V, V_x *and* V_y, respectively

3. Find the angle θ between the two vectors **u** = $\langle 3, 4\rangle$ and **v** = $\langle 2, 1\rangle$.

4. A bullet is fired into the air with an initial velocity of 1,800 feet per second at an angle of 60° from the horizontal. Find the magnitude of the horizontal and vertical vector component as of the velocity vector.

5. A bullet is fired into the air with an initial velocity of 1,200 feet per second at an angle of 45° from the horizontal.

a) Find the magnitude of the horizontal and vertical vector component as of the velocity vector.
b) Find the horizontal distance traveled by the bullet in 3 seconds. (Neglect the resistance of air on the bullet).

6. A ship travels 130 km on a bearing of S 42° E. How far east and how far south has it traveled?

7. An arrow is shot into the air with so that its horizontal velocity is 15.0 ft./sec and its vertical velocity is 25.0 ft./sec. Find the velocity of the arrow?

8. An arrow is shot into the air so that its horizontal velocity is 25 feet per second and its vertical is 15 feet per second. Find the velocity of the arrow.

9. A plane travels 170 miles on a bearing of N 18 E and then changes its course to N 49 E and travels another 120 miles. Find the total distance traveled north and the total distance traveled east.

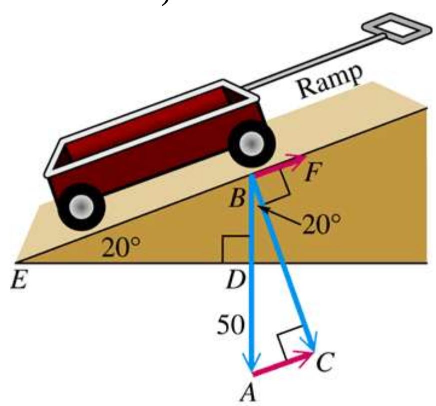
10. A boat travels 72 miles on a course of bearing N 27° E and then changes its course to travel 37 miles at N 55° E. How far north and how far east has the boat traveled on this 109-mile trip?

11. A boat is crossing a river that run due north. The boat is pointed due east and is moving through the water at 12 miles per hour. If the current of the river is a constant 5.1 miles per hour, find the actual course of the boat through the water to two significant digits.

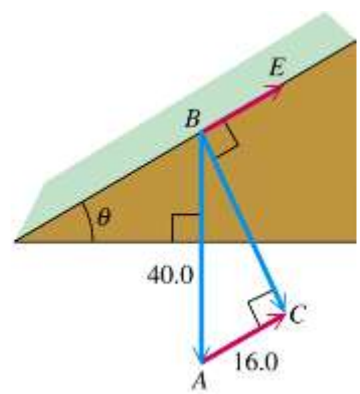
12. Two forces of 15 and 22 Newtons act on a point in the plane. (A **newton** is a unit of force that equals .225 lb.) If the angle between the forces is 100°, find the magnitude of the resultant vector.

13. Find the magnitude of the equilibrant of forces of 48 Newtons and 60 Newtons acting on a point *A*, if the angle between the forces is 50°. Then find the angle between the equilibrant and the 48-newton force.

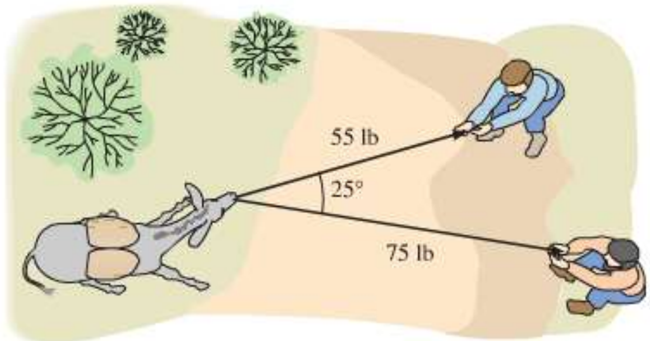
14. Find the force required to keep a 50-lb wagon from sliding down a ramp inclined at 20° to the horizontal. (Assume there is no friction.)



15. A force of 16.0 lb. is required to hold a 40.0 lb. lawn mower on an incline. What angle does the incline make with the horizontal?



16. Two prospectors are pulling on ropes attached around the neck of a donkey that does not want to move. One prospector pulls with a force of 55 lb, and the other pulls with a force of 75 lb. If the angle between the ropes is 25°, then how much force must the donkey use in order to stay put? (The donkey knows the proper direction in which to apply his force.)



17. A ship leaves port on a bearing of 28.0° and travels 8.20 mi. The ship then turns due east and travels 4.30 mi. How far is the ship from port? What is its bearing from port?

18. A solid steel ball is placed on a 10° incline. If a force of 3.2 lb in the direction of the incline is required to keep the ball in place, then what is the weight of the ball?

19. Find the amount of force required for a winch to pull a 3000-lb car up a ramp that is inclined at 20°.

20. If the amount of force required to push a block of ice up an ice-covered driveway that is inclined at 25° is 100lb, then what is the weight of the block?

21. If superman exerts 1000 lb of force to prevent a 5000-lb boulder from rolling down a hill and crushing a bus full of children, then what is the angle of inclination of the hill?

22. If Sisyphus exerts a 500-lb force in rolling his 4000-lb spherical boulder uphill, then what is the angle of inclination of the hill?

23. A plane is headed due east with an air speed of 240 mph. The wind is from the north at 30 mph. Find the bearing for the course and the ground speed of the plane.

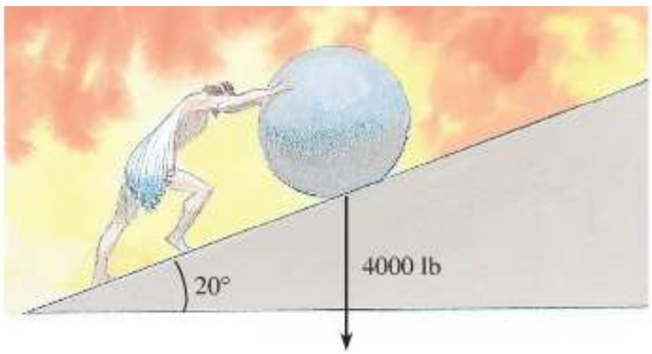
24. A plane is headed due west with an air speed of 300 mph. The wind is from the north at 80 mph. Find the bearing for the course and the ground speed of the plane.

25. An ultralight is flying northeast at 50 mph. The wind is from the north at 20 mph. Find the bearing for the course and the ground speed of the ultralight.

26. A superlight is flying northwest at 75 mph. The wind is from the south at 40 mph. Find the bearing for the course and the ground speed of the superlight.

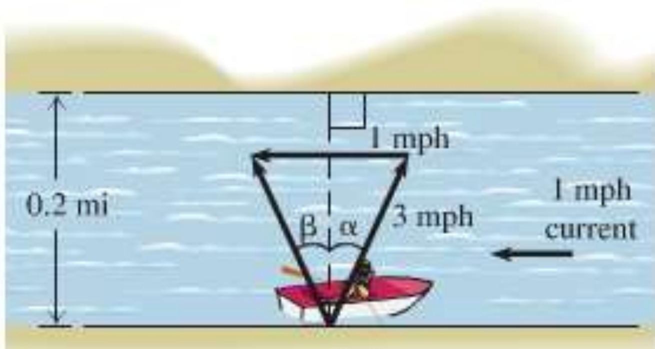
27. An airplane is heading on a bearing of 102° with an air speed of 480 mph. If the wind is out of the northeast (bearing 225°) at 58 mph, then what are the bearing of the course and the ground speed of the airplane?

28. In Roman mythology, Sisyphus revealed a secret of Zeus and thus incurred the god's wrath. As punishment, Zeus banished him to Hades, where he was doomed for eternity to roll a rock uphill, only to have it roll back on him. If Sisyphus stands in front of a 4000-lb spherical rock on a 20° incline, then what force applied in the direction of the incline would keep the rock from rolling down the incline?

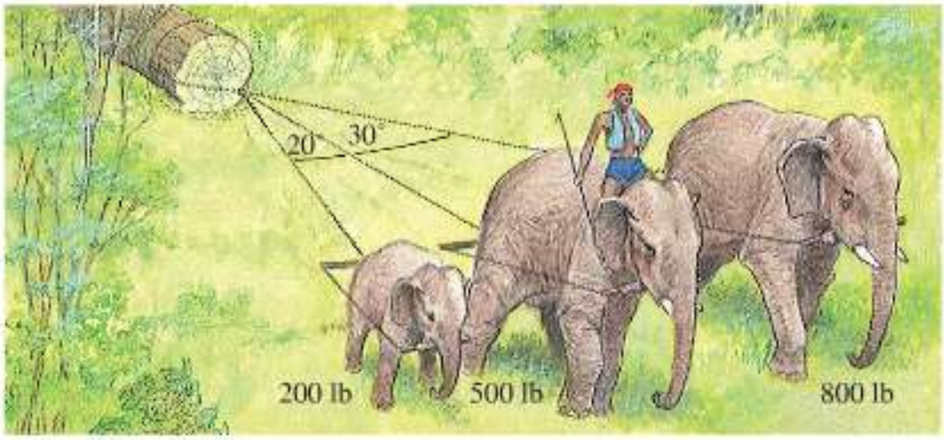


29. A trigonometry student wants to cross a river that is 0.2 mi wide and has a current of 1 mph. The boat goes 3 mph in still water.

a) Write the distance the boats travels as a function of the angle β.
b) Write the actual speed of the boat as a function of α and β.
c) Write the time for the trip as a function of α. Find the angle α for which the student will cross the river in the shortest amount of time.



30. Amal uses three elephants to pull a very large log out of the jungle. The papa elephant pulls with 800 lb. of force, the mama elephant pulls with 500 lb. of force, and the baby elephant pulls with 200 lb. force. The angles between the forces are shown in the figure. What is the magnitude of the resultant of all three forces? If mama is pulling due east, then in what direction will the log move?


20°
30°
200 lb
500 lb
800 lb