Solution   *Section* 2.6 – Improper Integrals

Exercise

Evaluate the integral $\displaystyle\int_0^{\infty} \frac{dx}{x^2+1}$

Solution

$$\begin{aligned}
\int_0^{\infty} \frac{dx}{x^2+1} &= \lim_{b\to\infty} \int_0^{b} \frac{dx}{x^2+1} \\
&= \lim_{b\to\infty} \left[\tan^{-1} x\right]_0^b \\
&= \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 0\right) \\
&= \frac{\pi}{2} - 0 \\
&= \frac{\pi}{2}
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{4} \frac{dx}{\sqrt{4-x}}$

Solution

$$\begin{aligned}
\int_0^{4} \frac{dx}{\sqrt{4-x}} &= \lim_{b\to 4^-} \int_0^{b} (4-x)^{-1/2}\, dx \\
&= \lim_{b\to 4^-} \int_0^{b} -(4-x)^{-1/2}\, d(4-x) \\
&= -2 \lim_{b\to 4^-} \left[(4-x)^{1/2}\right]_0^b \\
&= -2 \lim_{b\to 4^-} \left[(4-b)^{1/2} - (4)^{1/2}\right] \\
&= -2(0-2) \\
&= 4
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{2} \frac{2dx}{x^2+4}$

Solution

$$\begin{aligned}
\int_{-\infty}^{2} \frac{2dx}{x^2+4} &= 2\lim_{b\to-\infty}\int_{b}^{2} \frac{dx}{x^2+2^2} \\
&= 2\lim_{b\to-\infty}\frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_b^2 \\
&= \lim_{b\to-\infty}\left[\tan^{-1}1-\tan^{-1}\frac{b}{2}\right] \\
&= \frac{\pi}{4}-\left(-\frac{\pi}{2}\right) \\
&= \frac{3\pi}{4}
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{xdx}{\left(x^2+4\right)^{3/2}}$

Solution

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{xdx}{\left(x^2+4\right)^{3/2}} &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{d\left(x^2+4\right)}{\left(x^2+4\right)^{3/2}} \qquad u=x^2+4 \quad \to du=2xdx \\
&= \frac{1}{2}\left[-2\left(x^2+4\right)^{-1/2}\right]_{-\infty}^{\infty} \\
&= -\left[\frac{1}{\sqrt{x^2+4}}\right]_{-\infty}^{\infty} \\
&= -(0-0) \\
&= 0
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int_{1}^{\infty} \frac{dx}{x\sqrt{x^2-1}}$

Solution

$$\int_{1}^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \int_{1}^{2} \frac{dx}{x\sqrt{x^2-1}} + \int_{2}^{\infty} \frac{dx}{x\sqrt{x^2-1}}$$

$$= \lim_{b\to 1^+} \int_b^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{c\to\infty} \int_2^c \frac{dx}{x\sqrt{x^2-1}}$$

$$= \lim_{b\to 1^+} \left[\sec^{-1}|x|\right]_b^2 + \lim_{c\to\infty} \left[\sec^{-1}|x|\right]_2^c$$

$$= \lim_{b\to 1^+} \left(\sec^{-1} 2 - \sec^{-1} b\right) + \lim_{c\to\infty} \left(\sec^{-1} c - \sec^{-1} 2\right)$$

$$= \left(\frac{\pi}{3} - 0\right) + \left(\frac{\pi}{2} - \frac{\pi}{3}\right)$$

$$= \frac{\pi}{2}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx$

Solution

$$\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx = \int_{-\infty}^{0} 2xe^{-x^2}\,dx + \int_0^{\infty} 2xe^{-x^2}\,dx \qquad d\left(-x^2\right) = -2xdx$$

$$= -\lim_{b\to-\infty} \int_b^0 e^{-x^2}\,d\left(-x^2\right) - \lim_{c\to\infty} \int_0^c e^{-x^2}\,d\left(-x^2\right)$$

$$= -\lim_{b\to-\infty} \left[e^{-x^2}\right]_b^0 - \lim_{c\to\infty} \left[e^{-x^2}\right]_0^c$$

$$= -\lim_{b\to-\infty} \left(1 - e^{-b^2}\right) - \lim_{c\to\infty} \left(e^{-c^2} - 1\right) \qquad = -(1-0) - (0-1)$$

$$= 0$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 (-\ln x)\,dx$

Solution

$$\int_0^1 (-\ln x)\,dx = -\lim_{b\to 0^+} \int_b^1 (\ln x)\,dx$$

$$= -\lim_{b\to 0^+} \left[x\ln x - x\right]_b^1$$

$$= -\lim_{b\to 0^+} \left(\ln 1 - 1 - \left(b\ln b - b\right)\right)$$

$$= -\left(0 - 1 - 0 + 0\right)$$

$$\underline{= 1}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{4} \frac{dx}{\sqrt{|x|}}$

Solution

$$\int_{-1}^{4} \frac{dx}{\sqrt{|x|}} = \lim_{b \to 0^-} \int_{-1}^{b} \frac{dx}{\sqrt{-x}} + \lim_{c \to 0^+} \int_{c}^{4} \frac{dx}{\sqrt{x}}$$

$$= \lim_{b \to 0^-} \left[-2\sqrt{-x}\right]_{-1}^{b} + \lim_{c \to 0^+} \left[2\sqrt{x}\right]_{c}^{4}$$

$$= \lim_{b \to 0^-} \left(-2\sqrt{-b} + 2\right) + \lim_{c \to 0^+} \left(2\sqrt{4} - 2\sqrt{c}\right)$$

$$= 2 + 4$$

$$\underline{= 6}$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{\infty} e^{-3x}\, dx$

Solution

$$\int_{0}^{\infty} e^{-3x}\, dx = -\frac{1}{3} e^{-3x} \Big|_{0}^{\infty}$$

$$= -\frac{1}{3}\left(e^{-\infty} - 1\right)$$

$$\underline{= \frac{1}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$

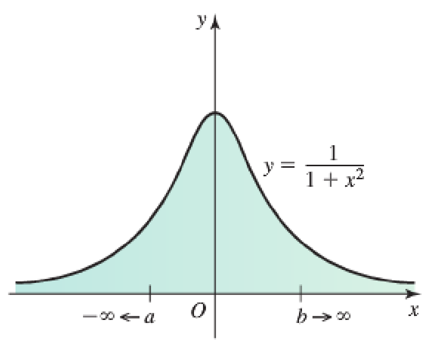
Solution

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \tan^{-1} x \Big|_{-\infty}^{\infty}$$

$$= \tan^{-1} \infty - \tan^{-1}(-\infty)$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$\underline{= \pi}$$



Area of region under the curve $y = \frac{1}{1 + x^2}$ on $(-\infty, \infty)$ has finite value π.

Exercise

Evaluate the integral $\displaystyle\int_1^{10} \frac{dx}{(x-2)^{1/3}}$

Solution

$$\int_1^{10} (x-2)^{-1/3}\,dx = \frac{3}{2}(x-2)^{2/3}\Big|_1^{10}$$

$$= \frac{3}{2}\left(8^{2/3} - (-1)^{2/3}\right)$$

$$= \frac{3}{2}(4-1)$$

$$= \frac{9}{2}$$

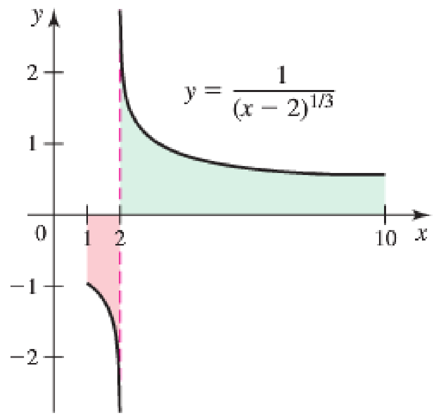
$$\int_1^{10} (x-2)^{-1/3}\,dx = \int_1^{2} (x-2)^{-1/3}\,dx + \int_2^{10} (x-2)^{-1/3}\,dx$$

$$= \frac{3}{2}(x-2)^{2/3}\Big|_1^{2} + (x-2)^{2/3}\Big|_2^{10}$$

$$= \frac{3}{2}\left(0 - (-1)^{2/3}\right) + \frac{3}{2}\left(8^{2/3} - 0\right)$$

$$= \frac{3}{2}(-1+4)$$

$$= \frac{9}{2}$$



Exercise

Evaluate the integral $\displaystyle\int_1^{\infty} \frac{dx}{x^2}$

Solution

$$\int_1^{\infty} \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^{\infty}$$

$$= -\left(\frac{1}{\infty} - 1\right)$$

$$= -(0-1)$$

$$= 1$$

Exercise

Evaluate the integral $\displaystyle\int_0^\infty \frac{dx}{(x+1)^3}$

Solution

$$\int_0^\infty (x+1)^{-3}dx = -\frac{2}{(x+1)^2}\Big|_0^\infty$$

$$= -2\left(\frac{1}{\infty}-1\right)$$

$$= -2(0-1)$$

$$\underline{= 2}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^0 e^x\, dx$

Solution

$$\int_{-\infty}^0 e^x\, dx = e^x\Big|_{-\infty}^0$$

$$= \left(1-e^{-\infty}\right)$$

$$\underline{= 1}$$

Exercise

Evaluate the integral $\displaystyle\int_1^\infty 2^{-x}\, dx$

Solution

$$\int_1^\infty 2^{-x}\, dx = -\int_1^\infty 2^{-x}\, d(-x)$$

$$\int a^x dx = \frac{a^x}{\ln a}$$

$$= -\frac{2^{-x}}{\ln 2}\Big|_1^\infty$$

$$= -\frac{1}{\ln 2}\left(0-\frac{1}{2}\right)$$

$$\underline{= \frac{1}{2\ln 2}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{0} \frac{dx}{\sqrt[3]{2-x}}$

Solution

$$\int_{-\infty}^{0} \frac{dx}{\sqrt[3]{2-x}} = -\int_{-\infty}^{0} (2-x)^{-1/3}\, d(2-x)$$

$$= -\frac{3}{2}(2-x)^{2/3}\Big|_{-\infty}^{0}$$

$$= -\frac{3}{2}\left(2^{2/3} - \infty\right)$$

$$\boxed{=\infty} \quad \textit{diverges}$$

Exercise

Evaluate the integral $\displaystyle\int_{4/\pi}^{\infty} \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right) dx$

Solution

$$\int_{4/\pi}^{\infty} \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right) dx = -\int_{4/\pi}^{\infty} \sec^2\left(\frac{1}{x}\right) d\left(\frac{1}{x}\right) \qquad d\left(\frac{1}{x}\right) = -\frac{1}{x^2}dx$$

$$= -\tan\left(\frac{1}{x}\right)\Big|_{4/\pi}^{\infty}$$

$$= -\left(\tan 0 - \tan\frac{\pi}{4}\right)$$

$$\boxed{=1}$$

Exercise

Evaluate the integral $\displaystyle\int_{e^2}^{\infty} \frac{dx}{x\ln^p x} \quad p > 1$

Solution

$$\int_{e^2}^{\infty} \frac{dx}{x\ln^p x} = \int_{e^2}^{\infty} (\ln x)^{-p}\, d(\ln x)$$

$$= \frac{1}{1-p}(\ln x)^{1-p}\Big|_{e^2}^{\infty}$$

$$= \frac{1}{1-p}\left((\ln x)^{-\infty} - \left(\ln e^2\right)^{1-p}\right)$$

$$= \frac{-1}{1-p} 2^{1-p}$$

$$\boxed{= \frac{1}{(p-1)2^{p-1}}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^\infty \frac{p}{\sqrt[5]{p^2+1}}\, dp$

Solution

$$\int_0^\infty \frac{p}{\sqrt[5]{p^2+1}}\, dp = \frac{1}{2}\int_0^\infty \left(p^2+1\right)^{-1/5}\, d\left(p^2+1\right) \qquad d\left(p^2+1\right) = 2pdp$$

$$= \frac{5}{8}\left(p^2+1\right)^{4/5}\Big|_0^\infty$$

$$\boxed{= \infty} \quad \textit{diverges}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{1} \ln y^2\, dy$

Solution

$$\int_{-1}^{1} \ln y^2\, dy = 2\int_0^1 \ln y^2\, dy$$

$$= 4\left(y\ln y - y\right)\Big|_0^1$$

$$= 4\left[-1-0\right]$$

$$\boxed{= -4}$$

$$\int \ln x^2 dx = 2\int \ln x dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx \qquad v = \int dx = x$$

$$= 2\left[x\ln x - \int dx\right]$$

$$\boxed{= 2(x\ln x - x) + C}$$

Exercise

Evaluate the integral $\displaystyle\int_{-2}^{6} \frac{dx}{\sqrt{|x-2|}}$

Solution

$$\int_{-2}^{6} \frac{dx}{\sqrt{|x-2|}} = \int_{-2}^{2} \frac{dx}{\sqrt{2-x}} + \int_{2}^{6} \frac{dx}{\sqrt{x-2}}$$

$$= -\int_{-2}^{2} (2-x)^{-1/2}\, d(2-x) + \int_{2}^{6} (x-2)^{-1/2}\, d(x-2)$$

$$= -2\sqrt{2-x}\Big|_{-2}^{2} + 2\sqrt{x-2}\Big|_{2}^{6}$$

$$= -2(0-2) + 2(2-0)$$

$$\underline{= 8}$$

Exercise

Evaluate $\displaystyle\int_0^{\infty} xe^{-x}dx$

Solution

$$\int_0^{\infty} xe^{-x}dx = -xe^{-x} - e^{-x}\Big|_0^{\infty}$$

$$= 0 - (-1)$$

$$\underline{= 1}$$

		$\int e^{-x}$
+	x	$-e^{-x}$
−	1	e^{-x}

Exercise

Evaluate $\displaystyle\int_0^{1} x\ln x\, dx$

Solution

$u = \ln x \qquad dv = x\, dx$

$du = \dfrac{dx}{x} \qquad v = \dfrac{1}{2}x^2$

$$\int x\ln x\, dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int xdx \;\; \underline{= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2}$$

$$\int_0^1 x\ln x\, dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2\Big|_0^1$$

$$\underline{= -\frac{1}{4}}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}dx$

Solution

$u = \ln x \qquad dv = \dfrac{1}{x^2}\, dx$

$du = \dfrac{dx}{x} \qquad v = -\dfrac{1}{x}$

$$\int \frac{\ln x}{x^2}\, dx = -\frac{1}{x}\ln x + \int \frac{1}{x^2}dx$$

$$= -\frac{1}{x}\ln x - \frac{1}{x}$$

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = -\frac{1}{x}(\ln x + 1)\Big|_1^\infty$$

$$\boxed{= 1}$$

Exercise

Evaluate $\displaystyle\int_1^\infty (1-x)e^{-x}\,dx$

Solution

$$\int_1^\infty (1-x)e^{-x}\,dx = \left[-e^{-x} - (-x-1)e^{-x}\right]_1^\infty$$

$$= \left[xe^{-x}\right]_1^\infty$$

$$= 0 - e^1$$

$$\boxed{= \frac{1}{e}}$$

Exercise

Evaluate $\displaystyle\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx$

Solution

$$\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx = \int_{-\infty}^\infty \frac{du}{1+u^2} \qquad u = e^x \;\to\; du = e^x dx$$

$$= \arctan e^x\Big|_{-\infty}^{\infty} \qquad = \arctan\infty - \arctan 0$$

$$\boxed{= \frac{\pi}{2}}$$

Exercise

Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}$

Solution

$$\int_0^1 x^{-1/3}dx = \frac{3}{2}x^{2/3}\Big|_0^1 \quad \boxed{= \frac{3}{2}}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} \frac{4}{\sqrt[4]{x}}\, dx$

Solution

$$\int_1^{\infty} 4x^{-1/4}dx = \frac{16}{3}x^{3/4}\Big|_1^{\infty}$$

$$= \infty \qquad \textit{Diverges}$$

Exercise

Evaluate $\displaystyle\int_0^{2} \frac{dx}{x^3}$

Solution

$$\int_0^{2} \frac{dx}{x^3} = -\frac{1}{2x^2}\Big|_0^{2}$$

$$= -\frac{1}{8} + \infty$$

$$= \infty \qquad \textit{Diverges}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} \frac{dx}{x^3}$

Solution

$$\int_1^{\infty} \frac{dx}{x^3} = -\frac{1}{2x^2}\Big|_1^{\infty} \quad = \frac{1}{2}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} \frac{6}{x^4}\, dx$

Solution

$$\int_1^{\infty} 6x^{-4}\, dx = -2\frac{1}{x^3}\Big|_1^{\infty} \quad = 2$$

Exercise

Evaluate $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x}\,(x+1)}$

Solution

$$u=\sqrt{x} \ \rightarrow \ u^2 = x \quad \Rightarrow dx = 2udu$$

$$\int_0^{\infty} \frac{dx}{\sqrt{x}\,(x+1)} = \int_0^{\infty} \frac{2u}{u\left(u^2+1\right)}du$$

$$= 2\int_0^{\infty} \frac{1}{u^2+1}du$$

$$= 2\arctan\sqrt{x}\,\Big|_0^{\infty}$$

$$= 2\left(\frac{\pi}{2}-0\right)$$

$$= \pi$$

Exercise

Evaluate $\displaystyle\int_{-\infty}^{0} xe^{-4x}dx$

Solution

$$\int_{-\infty}^{0} xe^{-4x}dx = \left(-\frac{x}{4}-\frac{1}{16}\right)e^{-4x}\Big|_{-\infty}^{0}$$

$$= -\frac{1}{16}-\infty$$

$$= -\infty \qquad \textit{Diverges}$$

Exercise

Evaluate $\displaystyle\int_0^{\infty} xe^{-x/3}dx$

Solution

$$\int_0^{\infty} xe^{-x/3}dx = (-3x-9)e^{-x/3}\Big|_0^{\infty}$$

$$= 9$$

Exercise

Evaluate $\displaystyle\int_0^\infty x^2 e^{-x} dx$

Solution

$$\int_0^\infty x^2 e^{-x} dx = \left(-x^2 - 2x - 2\right)e^{-x}\Big|_0^\infty \quad \underline{= 2}$$

Exercise

Evaluate $\displaystyle\int_0^\infty e^{-x}\cos x\, dx$

Solution

		$\int \cos x$
+	e^{-x}	$\sin x$
−	$-e^{-x}$	$-\cos x$
+	e^{-x}	$-\int \cos x$

$$\int e^{-x}\cos x\, dx = e^{-x}(\sin x - \cos x) - \int e^{-x}\cos x\, dx$$

$$2\int e^{-x}\cos x\, dx = e^{-x}(\sin x - \cos x)$$

$$\int_0^\infty e^{-x}\cos x\, dx = \frac{1}{2}e^{-x}(\sin x - \cos x)\Big|_0^\infty$$

$$= \frac{1}{2}(0 - (-1))$$

$$\underline{= \frac{1}{2}}$$

Exercise

Evaluate $\displaystyle\int_4^\infty \frac{1}{x(\ln x)^3} dx$

Solution

$$\int_4^\infty \frac{1}{x(\ln x)^3} dx = \int_4^\infty (\ln x)^{-3}\, d(\ln x)$$

$$= -\frac{1}{2}\frac{1}{(\ln x)^2}\Big|_4^\infty$$

$$= \frac{1}{2}\left(0 - \frac{1}{(\ln 4)^2}\right)$$

$$\underline{= \frac{1}{2(\ln 4)^2}}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} \frac{\ln x}{x}\,dx$

Solution

$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \int_1^{\infty} \ln x \; d(\ln x)$$

$$= \frac{1}{2}(\ln x)^2 \Big|_1^{\infty}$$

$$\underline{= \infty} \qquad \textit{Diverges}$$

Exercise

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{4}{16+x^2}\,dx$

Solution

$$\int_{-\infty}^{\infty} \frac{4}{16+x^2}\,dx = \arctan\left(\frac{x}{4}\right)\Big|_{-\infty}^{\infty}$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$

$$\underline{= \pi}$$

Exercise

Evaluate $\displaystyle\int_0^{\infty} \frac{x^3}{\left(x^2+1\right)^2}\,dx$

Solution

$$\frac{x^3}{\left(x^2+1\right)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{\left(x^2+1\right)^2}$$

$$x^3 = Ax^3 + Ax + Bx^2 + B + Cx + D$$

$$\begin{cases} x^3 & \underline{A=1} \\ x^2 & \underline{B=0} \\ x & A+C=0 \rightarrow \underline{C=-1} \\ x^0 & B+D=0 \rightarrow \underline{D=0} \end{cases}$$

$$\int_0^{\infty} \frac{x^3}{\left(x^2+1\right)^2}\,dx = \int_0^{\infty} \frac{x}{x^2+1}\,dx - \int_0^{\infty} \frac{x}{\left(x^2+1\right)^2}\,dx$$

$$=\frac{1}{2}\int_0^{\infty}\frac{1}{x^2+1}d\left(x^2+1\right)-\frac{1}{2}\int_0^{\infty}\frac{1}{\left(x^2+1\right)^2}d\left(x^2+1\right)$$

$$=\left[\frac{1}{2}\ln\left(x^2+1\right)+\frac{1}{2}\frac{1}{x^2+1}\right]_0^{\infty}$$

$$\boxed{=\infty} \quad \textit{diverges}$$

Exercise

Evaluate $\displaystyle\int_0^{\infty}\frac{1}{e^x+e^{-x}}dx$

Solution

$$\int_0^{\infty}\frac{1}{e^x+e^{-x}}dx=\int_0^{\infty}\frac{1}{e^x+e^{-x}}\frac{e^x}{e^x}dx$$

$$=\int_0^{\infty}\frac{e^x}{e^{2x}+1}dx$$

$$=\int_0^{\infty}\frac{1}{\left(e^x\right)^2+1}d\left(e^x\right)$$

$$=\arctan e^x\Big|_0^{\infty}$$

$$=\arctan(\infty)-\arctan(1)$$

$$=\frac{\pi}{2}-\frac{\pi}{4}$$

$$\boxed{=\frac{\pi}{4}}$$

Exercise

Evaluate $\displaystyle\int_0^{\infty}\frac{e^x}{1+e^x}dx$

Solution

$$\int_0^{\infty}\frac{e^x}{1+e^x}dx=\int_0^{\infty}\frac{1}{1+e^x}d\left(e^x\right)$$

$$=\ln\left(1+e^x\right)\Big|_0^{\infty}$$

$$\boxed{=\infty} \quad \textit{diverges}$$

Exercise

Evaluate $\displaystyle\int_0^{\infty} \cos \pi x \, dx$

Solution

$$\int_0^{\infty} \cos \pi x \, dx = \frac{1}{\pi} \sin \pi x \Big|_0^{\infty}$$

$$= \infty \quad \textit{diverges}$$

Exercise

Evaluate $\displaystyle\int_0^{\infty} \sin \frac{x}{2} \, dx$

Solution

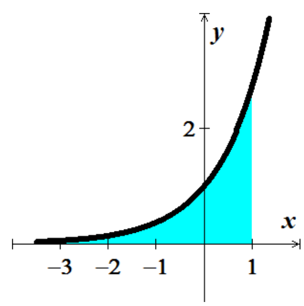
$$\int_0^{\infty} \sin \frac{x}{2} \, dx = -2 \cos \frac{x}{2} \Big|_0^{\infty}$$

$$= \infty \quad \textit{diverges}$$

Exercise

Find the area of the unbounded shaded region $\quad y = e^x, \quad -\infty < x \le 1$

Solution



$$A = \int_{-\infty}^{1} e^x dx$$

$$= e^x \Big|_{-\infty}^{1}$$

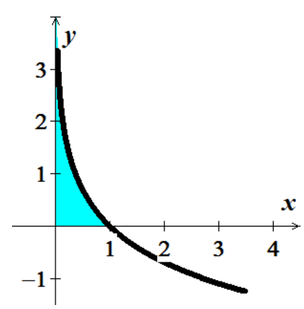
$$= e - 0$$

$$= e$$

Exercise

Find the area of the unbounded shaded region $\quad y = -\ln x$

Solution



$$A = -\int_0^1 \ln x \, dx$$

$$= -(x \ln x - x)\Big|_0^1$$

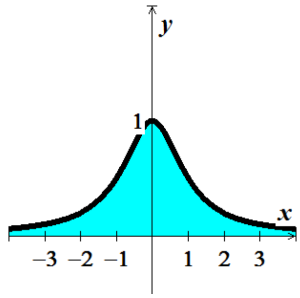
$$= 1$$

Exercise

Find the area of the unbounded shaded region $\quad y = \dfrac{1}{x^2+1}$

Solution

$$A = \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$$
$$= \arctan x \Big|_{-\infty}^{\infty}$$
$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$
$$\underline{= \pi}$$

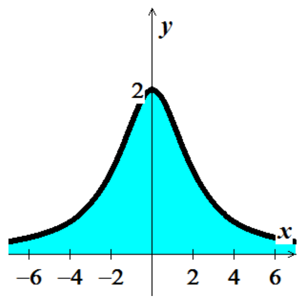


Exercise

Find the area of the unbounded shaded region $\quad y = \dfrac{8}{x^2+4}$

Solution

$$A = \int_{-\infty}^{\infty} \frac{8}{x^2+4}\,dx$$
$$= 4\arctan \frac{x}{2} \Big|_{-\infty}^{\infty}$$
$$= 4\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$
$$\underline{= 4\pi}$$

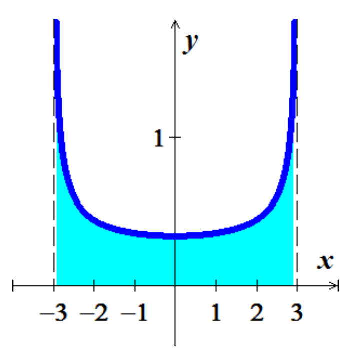


Exercise

Find the area of the region R between the graph of $f(x) = \dfrac{1}{\sqrt{9-x^2}}$ and the x-*axis* on the interval $(-3,\ 3)$ (if it exists)

Solution

$$A = \int_{-3}^{3} \frac{dx}{\sqrt{9-x^2}}$$
$$= 2\int_{0}^{3} \frac{dx}{\sqrt{9-x^2}}$$
$$= 2\sin^{-1}\frac{x}{3}\Big|_{0}^{3}$$



$$= 2\left(\sin^{-1}1 - \sin^{-1}0\right)$$

$$\boxed{= \pi \ \ unit^2}$$

Exercise

Find the volume of the region bounded by $f(x) = \left(x^2+1\right)^{-1/2}$ and the x-*axis* on the interval $[2, \infty)$ is revolved about the x-*axis*.

Solution

$$V = \pi \int_2^\infty \frac{1}{x^2+1}\,dx \qquad\qquad V = \pi \int_a^b \left(f(x)\right)^2 dx$$

$$= \pi \tan^{-1} x \Big|_2^\infty$$

$$= \pi\left(\tan^{-1}\infty - \tan^{-1}2\right)$$

$$\boxed{= \pi\left(\frac{\pi}{2} - \tan^{-1}2\right) \ \ unit^3}$$

Exercise

Find the volume of the region bounded by $f(x) = \sqrt{\frac{x+1}{x^3}}$ and the x-*axis* on the interval $[1, \infty)$ is revolved about the x-*axis*.

Solution

$$V = \pi \int_1^\infty \frac{x+1}{x^3}\,dx \qquad\qquad V = \pi \int_a^b \left(f(x)\right)^2 dx$$

$$= \pi \int_1^\infty \left(\frac{1}{x^2} + x^{-3}\right) dx$$

$$= \pi\left(-\frac{1}{x} - \frac{1}{2}\frac{1}{x^2}\right)\Big|_1^\infty$$

$$= \pi\left(1 + \frac{1}{2}\right)$$

$$\boxed{= \frac{3\pi}{2} \ \ unit^3}$$

Exercise

Find the volume of the region bounded by $f(x) = (x+1)^{-3}$ and the x-*axis* on the interval $[0, \infty)$ is revolved about the y-*axis*.

Solution

$$V=2\pi\int_0^{\infty} x\frac{1}{(x+1)^3}dx$$

$$=2\pi\int_0^{\infty}\left(\frac{1}{(x+1)^2}-\frac{1}{(x+1)^3}\right)d(x+1)$$

$$=2\pi\left(\frac{-1}{x+1}+\frac{1}{2}\frac{1}{(x+1)^2}\right)\Big|_0^{\infty}$$

$$=2\pi\left(1-\frac{1}{2}\right)$$

$$\boxed{=\pi\ \ unit^3}$$

$$V=2\pi\int_a^b x\cdot f(x)\,dx \quad (Shell\ method)$$

$$\frac{x}{(x+1)^3}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$

$$x=Ax^2+2Ax+A+Bx+B+C$$

$$\begin{cases} \boxed{A=0} \\ 2A+B=1 \ \rightarrow \boxed{B=1} \quad \boxed{C=-1} \\ B+C=0 \end{cases}$$

Exercise

Find the volume of the region bounded by $f(x)=\frac{1}{\sqrt{x}\ln x}$ and the x-*axis* on the interval $[2,\ \infty)$ is revolved about the x-*axis* .

Solution

$$V=\pi\int_2^{\infty}\frac{1}{x\ln^2 x}\ dx \qquad\qquad V=\pi\int_a^b (f(x))^2\,dx$$

$$=\pi\int_2^{\infty}\frac{1}{\ln^2 x}\ d(\ln x)$$

$$=\pi\left(-\frac{1}{\ln x}\right)\Big|_2^{\infty}$$

$$=\pi\left(-0+\frac{1}{\ln 2}\right)$$

$$\boxed{=\frac{\pi}{\ln 2}\ \ unit^3}$$

Exercise

Find the volume of the region bounded by $f(x)=\frac{\sqrt{x}}{\sqrt[3]{x^2+1}}$ and the x-*axis* on the interval $[0,\ \infty)$ is revolved about the x-*axis* .

Solution

$$V=\pi\int_0^{\infty}\frac{x}{\left(x^2+1\right)^{2/3}}\ dx \qquad\qquad V=\pi\int_a^b (f(x))^2\,dx$$

$$=\frac{\pi}{2}\int_0^{\infty}\left(x^2+1\right)^{-2/3}\ d\left(x^2+1\right)$$

$$= \frac{3\pi}{2}\left(x^2+1\right)^{1/3}\Big|_0^{\infty}$$

$$= \frac{3\pi}{2}(\infty - 1)$$

$$\boxed{= \infty \quad \textit{diverges}}$$ So the volume doesn't exist

Exercise

Find the volume of the region bounded by $f(x) = \left(x^2 - 1\right)^{-1/4}$ and the *x-axis* on the interval $(1,\ 2]$ is revolved about the *y-axis*.

Solution

$$V = 2\pi \int_1^2 x\left(x^2-1\right)^{-1/4} dx \qquad V = 2\pi \int_a^b x \cdot f(x)\, dx \quad (\textit{Shell method})$$

$$= \pi \int_1^2 \left(x^2-1\right)^{-1/4} d\left(x^2-1\right)$$

$$= \frac{4\pi}{3}\left(x^2-1\right)^{3/4}\Big|_1^2$$

$$= \frac{4\pi}{3}(3)^{3/4}$$

$$\boxed{= \frac{4\pi}{3^{1/4}} \ \ unit^3}$$

Exercise

Find the volume of the region bounded by $f(x) = \tan x$ and the *x-axis* on the interval $\left[0,\ \frac{\pi}{2}\right)$ is revolved about the *x-axis*.

Solution

$$V = \pi \int_0^{\pi/2} \tan^2 x\, dx \qquad V = \pi \int_a^b (f(x))^2\, dx$$

$$= \pi \int_0^{\pi/2} \left(\sec^2 x - 1\right) dx$$

$$= \pi(\tan x - x)\Big|_0^{\pi/2} \qquad \left(\tan\frac{\pi}{2} = \infty\right)$$

$$\boxed{= \infty \quad \textit{diverges}}$$ So the volume doesn't exist

Exercise

Find the volume of the region bounded by $f(x) = -\ln x$ and the *x-axis* on the interval $(0,\ 1]$ is revolved about the *x-axis*.

Solution

$$V = \pi\int_0^1 \ln^2 x\, dx \qquad\qquad V = \pi\int_a^b (f(x))^2\, dx$$

$$u = \ln x \qquad dv = \ln x\, dx$$

$$du = \frac{dx}{x} \qquad v = x\ln x - x$$

$$u = \ln x \quad dv = dx$$

$$du = \frac{dx}{x} \quad v = x \quad \to \int \ln x dx = x\ln x - \int dx = x\ln x - x$$

$$\int \ln^2 x\, dx = \ln x(x\ln x - x) - \int (\ln x - 1)dx$$

$$= x\ln^2 x - x\ln x - (x\ln x - x - x)$$

$$= x\ln^2 x - 2x\ln x + 2x$$

$$V = \pi\left(x\ln^2 x - 2x\ln x + 2x\right)\Big|_0^1$$

$$= 2\pi \ \ unit^3$$

Exercise

Find the volume of the solid generated by revolving the region bounded by the graphs of $y = xe^{-x}$, $y = 0$, and $x = 0$ about the *x-axis*.

Solution

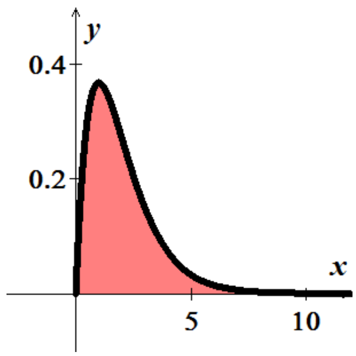
$$V = \pi\int_0^\infty \left(xe^{-x}\right)^2 dx$$

$$= \pi\int_0^\infty x^2 e^{-2x} dx$$

$$= \pi e^{-2x}\left(-\tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{1}{4}\right)\Big|_0^\infty$$

$$= \pi\left(0 + \tfrac{1}{4}\right)$$

$$= \frac{\pi}{4}$$



Exercise

Consider the region satisfying the inequalities $y \le e^{-x}, \quad y \ge 0, \quad x \ge 0$

a) Find the area of the region

b) Find the volume of the solid generated by revolving the region about the *x-axis*.

c) Find the volume of the solid generated by revolving the region about the *y-axis*.

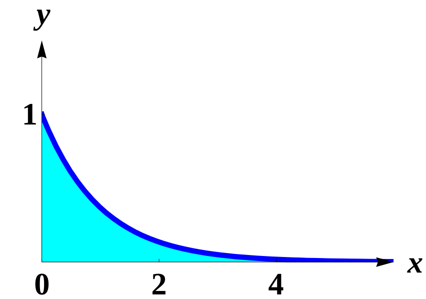
Solution

a) $A = \int_0^\infty e^{-x} dx$

$= -e^{-x}\Big|_0^\infty$

$= -(0-1)$

$= 1$



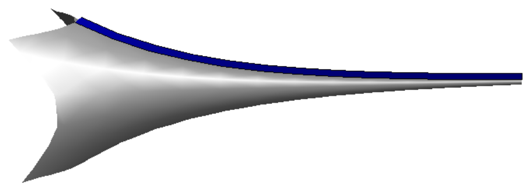
b) $V = \pi \int_0^\infty \left(e^{-x}\right)^2 dx$ $\qquad V = \pi \int_a^b \left(R(x)^2 - r(x)^2\right) dx \quad (Disk\ Method)$

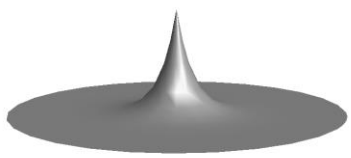
$= \pi \int_0^\infty e^{-2x}\, dx$

$= -\frac{\pi}{2} e^{-2x}\Big|_0^\infty$

$= -\frac{\pi}{2}(0-1)$

$= \frac{\pi}{2}$



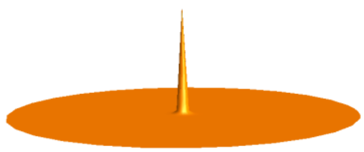


c) $V = 2\pi \int_0^\infty xe^{-x}\, dx$ $\qquad V = 2\pi \int_a^b xf(x)\, dx \quad (Shell\ Method)$

$= -2\pi e^{-x}(x+1)\Big|_0^\infty$

$= -2\pi(0-1)$

$= 2\pi$



Exercise

Consider the region satisfying the inequalities $y \le \frac{1}{x^2}, \quad y \ge 0, \quad x \ge 1$

a) Find the area of the region

b) Find the volume of the solid generated by revolving the region about the *x-axis*.

c) Find the volume of the solid generated by revolving the region about the *y-axis*.

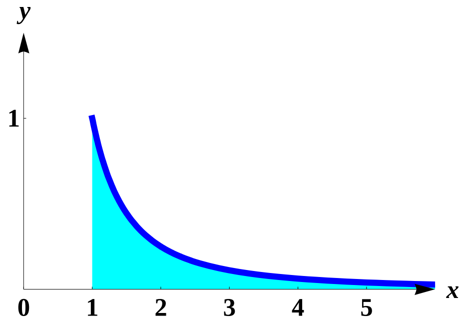
Solution

a) $A=\displaystyle\int_1^{\infty}\frac{1}{x^2}\,dx$

$=-\dfrac{1}{x}\Big|_1^{\infty}$

$=-(0-1)$

$\underline{=1}$



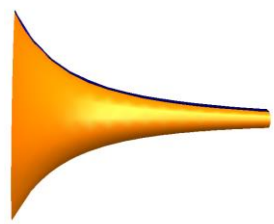
b) $V=\pi\displaystyle\int_0^{\infty}\left(\frac{1}{x^2}\right)^2\,dx$ $\qquad$ $V=\pi\displaystyle\int_a^b\left(R(x)^2-r(x)^2\right)dx$ $\quad$ (*Disk Method*)

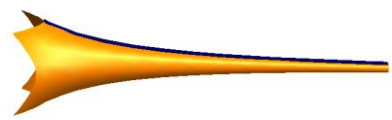
$=\pi\displaystyle\int_0^{\infty}x^{-4}dx$

$=-\dfrac{\pi}{3x^3}\Big|_1^{\infty}$

$=-\dfrac{\pi}{3}(0-1)$

$\underline{=\dfrac{\pi}{3}}$





c) $V=2\pi\displaystyle\int_0^{\infty}x\left(\frac{1}{x^2}\right)\,dx$ $\qquad$ $V=2\pi\displaystyle\int_a^b xf(x)\,dx$ $\quad$ (*Shell Method*)

$=2\pi\displaystyle\int_0^{\infty}\frac{1}{x}\,dx$

$=2\pi\ln x\Big|_1^{\infty}$

$\underline{=\infty}$ $\quad$ *Diverges*

Exercise

Find the perimeter of the hypocycloid of four cusps $x^{2/3}+y^{2/3}=4$

Solution

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0$$

$$y'=-\frac{x^{-1/3}}{y^{-1/3}}=-\frac{y^{1/3}}{x^{1/3}}$$

$$\sqrt{1+(y')^2}=\sqrt{1+\frac{y^{2/3}}{x^{2/3}}}$$

$$=\frac{\sqrt{x^{2/3}+y^{2/3}}}{x^{1/3}}$$

$$= \frac{\sqrt{4}}{x^{1/3}}$$

$$= 2x^{-1/3}$$

$$S = 4\int_0^8 2x^{-1/3}\, dx$$

$$= 12x^{2/3}\Big|_0^8$$

$$= 12(4-0)$$

$$\underline{= 48|}$$

Exercise

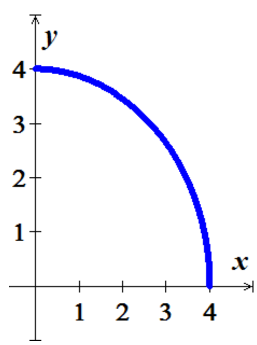
Find the arc length of the graph $y = \sqrt{16-x^2}$ over the interval $[0,\ 4]$

Solution

$$y' = -\frac{x}{\sqrt{16-x^2}}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{x^2}{16-x^2}}$$

$$= \frac{4}{\sqrt{16-x^2}}$$



$$L = \int_0^4 \frac{4}{\sqrt{16-x^2}}\, dx$$

$$= 4\arcsin\frac{x}{4}\Big|_0^4$$

$$= 4(\arcsin 1 - \arcsin 0)$$

$$= 4\left(\frac{\pi}{2}\right)$$

$$\underline{= 2\pi|}$$

Exercise

The region bounded by $(x-2)^2 + y^2 = 1$ is revolved about the *y-axis* to form a torus. Find the surface area of the torus.

Solution

$$2(x-2) + 2yy' = 0$$

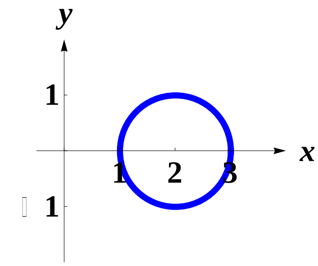
$$y' = -\frac{x-2}{y}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{(x-2)^2}{y^2}}$$

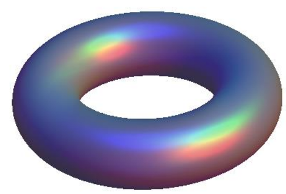
$$= \sqrt{\frac{y^2+(x-2)^2}{y^2}} \qquad (x-2)^2 + y^2 = 1$$

$$= \frac{1}{y}$$

$$= \frac{1}{\sqrt{1-(x-2)^2}}$$



$$S = 4\pi \int_1^3 \frac{x}{\sqrt{1-(x-2)^2}}\, dx$$



$$= 4\pi \int_1^3 \frac{x-2+2}{\sqrt{1-(x-2)^2}}\, dx$$

$$= 4\pi \int_1^3 \frac{x-2}{\sqrt{1-(x-2)^2}}\, dx + 4\pi \int_1^3 \frac{2}{\sqrt{1-(x-2)^2}}\, dx$$

$$= -2\pi \int_1^3 \left(1-(x-2)^2\right)^{-1/2} d\left(1-(x-2)^2\right) + 8\pi \arctan(x-2)\Big|_1^3$$

$$= -4\pi \sqrt{1-(x-2)^2}\Big|_1^3 + 8\pi\left(\arctan(1) - \arctan(-1)\right)$$

$$= -4\pi(0-0) + 8\pi\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$\boxed{= 8\pi^2}$$

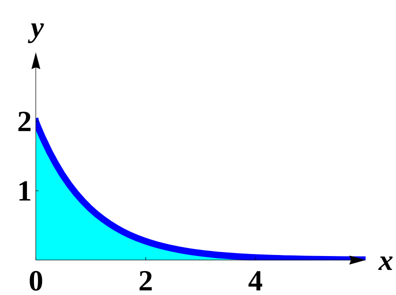
Exercise

Find the surface area formed by revolving the graph $y = 2e^{-x}$ on the interval $[0, \infty)$ about the *x-axis*

Solution

$$y' = -2e^{-x}$$

$$\sqrt{1+(y')^2} = \sqrt{1+4e^{-2x}}$$



$$S = 2\pi \int_0^\infty 2e^{-x}\sqrt{1+4e^{-2x}}\, dx$$

$$= -4\pi \int_0^\infty \sqrt{1+4\left(e^{-x}\right)^2}\, d\left(e^{-x}\right)$$

$$\int \sqrt{1+4u^2}\ du = \frac{1}{2}\int \sec^3\theta\ d\theta$$

$2u = \tan\theta \qquad \sqrt{4u^2+1} = \sec\theta$

$du = \frac{1}{2}\sec^2\theta\ d\theta$

$$= \frac{1}{4}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)$$

$$= -\pi\left(2e^{-x}\sqrt{1+4e^{-2x}} + \ln\left|2e^{-x} + \sqrt{1+4e^{-2x}}\right|\right)\Big|_0^{\infty}$$

$$= -\pi\left(-2\sqrt{5} + \ln\left(2+\sqrt{5}\right)\right)$$

$$= \pi\left(2\sqrt{5} - \ln\left(2+\sqrt{5}\right)\right)$$

Exercise

The magnetic potential P at a point on the axis of a circular coil is given by

$$P = \frac{2\pi NIr}{k}\int_c^{\infty} \frac{1}{\left(r^2+x^2\right)^{3/2}}\ dx$$

Where N, I, r, k, and c are constants. Find P.

Solution

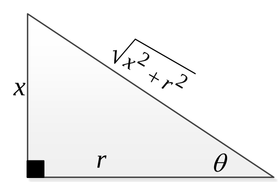
Let $\quad K = \dfrac{2\pi NIr}{k}$

$$P = K\int_c^{\infty} \frac{1}{\left(r^2+x^2\right)^{3/2}}\ dx$$

$x = r\tan\theta \qquad \sqrt{x^2+r^2} = r\sec\theta$

$dx = r\sec^2\theta\ d\theta$

$$= K\int_c^{\infty} \frac{r\sec^2\theta}{r^3\sec^3\theta}\ d\theta$$



$$= \frac{K}{r^2}\int_c^{\infty} \cos\theta\ d\theta$$

$$= \frac{K}{r^2}\sin\theta\Big|_c^{\infty}$$

$$= \frac{K}{r^2}\frac{x}{\sqrt{r^2+x^2}}\Bigg|_c^{\infty}$$

$$= \frac{K}{r^2}\left(1 - \frac{c}{\sqrt{r^2+c^2}}\right)$$

$$= \frac{2\pi NI\left(\sqrt{r^2+c^2} - c\right)}{kr\sqrt{r^2+c^2}}$$

Exercise

A "semi-infinite" uniform rod occupies the nonnegative $x\text{-}axis$. The rod has a linear density δ, which means that a segment of length dx has a mass of δdx. A particle of mass M is located at the point $(-a,\ 0)$. The gravitational force F that the rod exerts on the mass is given by

$$F = \int_0^\infty \frac{GM\delta}{(a+x)^2}\,dx$$

Where G is the gravitational constant. Find F.

Solution

$$\begin{aligned} F &= \int_0^\infty \frac{GM\delta}{(a+x)^2}\,dx \\ &= -\frac{GM\delta}{a+x}\Big|_0^\infty \\ &= \boxed{\frac{GM\delta}{a}} \end{aligned}$$

Exercise

Let R be the region bounded by the graph of $f(x) = x^{-p}$ and the $x\text{-}axis$

a) Let S be the solid generated when R is revolved about the $x\text{-}axis$. For what values of p is the volume of S finite for $0 < x \le 1$?

b) Let S be the solid generated when R is revolved about the $y\text{-}axis$. For what values of p is the volume of S finite for $0 < x \le 1$?

c) Let S be the solid generated when R is revolved about the $x\text{-}axis$. For what values of p is the volume of S finite for $x \ge 1$?

d) Let S be the solid generated when R is revolved about the $y\text{-}axis$. For what values of p is the volume of S finite for $x \ge 1$?

Solution

a) $V = \pi\int_0^1 \left(x^{-p}\right)^2 dx$ $\qquad\qquad V = \pi\int_a^b f(x)^2\,dx$

$$\begin{aligned} &= \pi\int_0^1 x^{-2p}dx \\ &= \pi\frac{x^{-2p+1}}{1-2p}\Big|_0^1 \\ &= \frac{\pi}{1-2p}\left(1-0^{-2p+1}\right) \end{aligned}$$

The volume of S finite when $1-2p > 0 \;\Rightarrow\; \boxed{p < \frac{1}{2}}$

b) $V = 2\pi \int_0^1 x \cdot x^{-p} dx$ $\qquad V = 2\pi \int_a^b xf(x)\,dx$

$$= 2\pi \int_0^1 x^{1-p} dx$$

$$= \frac{2\pi}{2-p} x^{2-p} \Big|_0^1$$

$$= \frac{2\pi}{2-p}\left(1 - 0^{2-p}\right)$$

The volume of S finite when $2 - p > 0 \;\Rightarrow\; \boxed{p < 2}$

c) $V = \pi \int_1^\infty \left(x^{-p}\right)^2 dx$ $\qquad V = \pi \int_a^b f(x)^2\,dx$

$$= \pi \int_1^\infty x^{-2p} dx$$

$$= \pi \frac{x^{-2p+1}}{1-2p} \Big|_1^\infty$$

$$= \frac{\pi}{1-2p}\left(\infty^{1-2p} - 1\right)$$

The volume of S finite when $1 - 2p < 0 \;\Rightarrow\; \boxed{p > \frac{1}{2}}$ $\left(\frac{1}{\infty} = 0\right)$

d) $V = 2\pi \int_0^1 x \cdot x^{-p} dx$ $\qquad V = 2\pi \int_a^b xf(x)\,dx$

$$= 2\pi \int_0^1 x^{1-p} dx$$

$$= \frac{2\pi}{2-p} x^{2-p} \Big|_0^1$$

$$= \frac{2\pi}{2-p}\left(1 - 0^{2-p}\right)$$

The volume of S finite when $2 - p > 0 \;\Rightarrow\; \boxed{p < 2}$

Exercise

The solid formed by revolving (about the *x-axis*) the unbounded region lying between the graph of $f(x) = \frac{1}{x}$ and the *x-axis* $(x \ge 1)$ is called ***Gabriel's Horn***.

Show that this solid has a finite volume and an infinite surface area

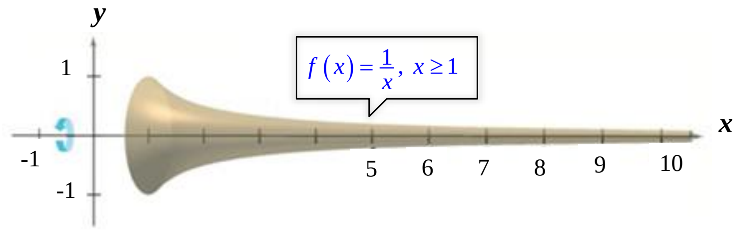
Solution

$$V = \pi\int_1^{\infty} \frac{1}{x^2}\,dx \qquad\qquad V = \pi\int_x^b \left(f(x)\right)^2 dx \ \ (disk\ method)$$

$$= -\pi \frac{1}{x}\Big|_1^{\infty}$$

$$= -\pi(0-1)$$

$$\boxed{= \pi \ \ unit^3}$$



$$f'(x) = -\frac{1}{x^2}$$

$$S = 2\pi\int_1^{\infty} \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx \qquad\qquad S = 2\pi\int_a^b f(x)\sqrt{1+\left(f'(x)\right)^2}\,dx$$

Since $\quad 1+\frac{1}{x^4} > 1$ and $\int_1^{\infty} \frac{1}{x}\,dx$ diverges

Therefore the surface area in infinite.

Exercise

Water is drained from a 3000-*gal* tank at a rate that starts at 100 *gal/hr*. and decreases continuously by 5% /*hr*. If the drain left open indefinitely, how much water drains from the tank? Can a full tank be emptied at this rate?

Solution

Rate of the drain water: $r(t) = 100(1-.05)^t$

$$= 100(0.95)^t$$

$$\boxed{= 100e^{(\ln 0.95)t}}$$

Total water amount drained:

$$D = \int_0^{\infty} 100e^{(\ln 0.95)t}\,dt$$

$$= \frac{100}{\ln 0.95} e^{(\ln 0.95)t}\Big|_0^{\infty}$$

$$= \frac{100}{\ln 0.95}(0-1) \qquad\qquad \ln 0.95 < 0 \xrightarrow[t\to\infty]{} e^{(\ln 0.95)t} = e^{-\infty} = 0$$

$$\boxed{= -\frac{100}{\ln 0.95} \approx 1950 \ \ gal}$$

Since 1950 *gal* < 3000 *gal* which it takes infinite time.

Therefore, the full 3,000–gallon tank cannot be emptied at this rate.