Lecture Three – Probability

Section 3.1 – Sets

Set Properties & Notation

- $\{b \in B\}$ b is an element of B.
- $\varnothing$: The empty set
- $A \subset B$: Subset
- $A \not\subset B$: Not a Subset
- $\varnothing \subset A$
- $A = B$: Equal To
- $A \neq B$: Not Equal To
- $A \bigcup B$ Union
- $A \bigcap B$ Intersection
- $\{A'\}$ Complement

Set Notations

- U is the universal set. The universal set contains all of the elements.
- $A \bigcup B$ is read "A union B". An element is in the set $A \bigcup B$ if the element lies in either set A or set B.
- $A \bigcap B$ is read "A intersect B". An element is in the set $A \bigcap B$ if the element lies (simultaneously) in both set A and set B.
- A' or $\overline{A}$ is read "A complement" or "not A". An element is in the set A′ if the element lies outside of set A.
- The capital letter notation A defines the set A, $n(A)$ is the notation used to give the number of elements in set A.
- Example: $A = \{2,4,6,8\}$, $n(A) = 4$
- $\varnothing$ represents the empty set. {a set with no elements, like, the number of pregnant men in class}

Example

Write the elements $\{x \mid x$ is a natural number less than $5\}$

<u>*Solution*</u>

$\{1, 2, 3, 4\}$

Subset

Set A is a subset of set B (written $A \subseteq B$) if every element of A is also an element of B. Set A is a proper subset (written $A \subset B$) if $A \subseteq B$ and $A \neq B$

Example

Decide whether the following statements are true or false

$a)$ $\{3,4,5,6\} = \{4,6,3,5\}$

$b)$ $\{5,6,9,12\} \subseteq \{5,6,7,8,9,10,11\}$

Solution

a) $\{3,4,5,6\} = \{4,6,3,5\}$

True, since both sets contain exactly the same element.

b) $\{5,6,9,12\} \subseteq \{5,6,7,8,9,10,11\}$

False, since first set contains 12 that doesn't belong to the second set.

Example

List all possible subsets for {7, 8}

Solution

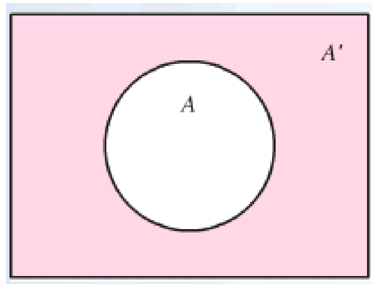
There are 4 subsets:

∅, {7}, {8}, {7, 8}

Complement of a Set

Let A be any set, with U representing the universal set, then the complement of A.

$$A' = \{x \mid x \notin A \text{ and } x \in U\}$$



Example

Let $U = \{1,2,3,4,5,6,7,8,9,10,11\}$, $A = \{1,2,3,4,5,7\}$, and $B = \{2,4,5,7,9,11\}$. Find each set.

a) A'
b) B'
c) $\phi' = U$ and $U' = \phi$
d) $(A')' = A$

Solution

a) $A' = \{6,8,9,10,11\}$

b) $B' = \{1,3,6,8,10\}$

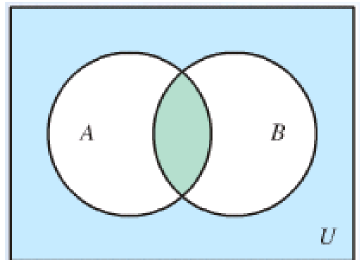
c) $\phi' = U$ and $U' = \phi$

d) $(A')' = A$

Intersection of Two Sets

The intersection of sets *A* and *B*, is:

$$A \bigcap B = \{e \in S \mid e \in A \textbf{ and } e \in B\}$$



Example

Let $A = \{3,6,9\}$, $B = \{2,4,6,8\}$, and $U = \{0,1,2,3,4,5,6,7,8,9,10\}$

a) $A \bigcap B$

b) $A \bigcap B'$

<u>*Solution*</u>

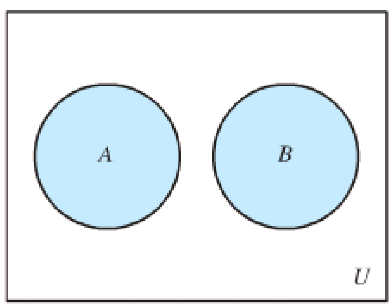
a) $A \bigcap B = \{6\}$

b) $A \bigcap B' = \{3,6,9\} \bigcap \{0,1,3,5,7,9,10\} = \{3,9\}$

$= \{3,9\}$

Disjoint Sets

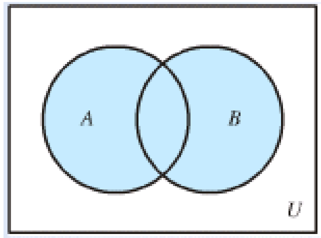
For any sets *A* and *B*, if *A* and *B* are disjoint sets, then $A \bigcap B = \phi$



Union of Two Sets

The union of sets A and B, is:

$$A \cup B = \{e \in S \mid e \in A \textbf{ or } e \in B\}$$



Example

Let $A = \{1,3,5,7,9,11\}$, $B = \{3,6,9,12\}$, $C = \{1,2,3,4,5\}$, and $U = \{0,1,2,\ldots\ldots,11,12\}$ Find each set.

a) $A \cup B$

b) $(A \cup B) \cap C'$

<u>*Solution*</u>

a) $A \cup B = \{1,3,5,6,7,9,11,12\}$

b) $(A \cup B) \cap C' = \{1,3,5,6,7,9,11,12\} \cap \{0,6,7,8,9,10,11,12\}$

$= \{6,7,9,11,12\}$

Example

A department store classifies credit applicants by gender, marital status, and employment status. Let the universal set be the set of all applicants, M be the set of male applicants, S be the set of single applicants, and E be the set of employed applicants. Describe each set in words.

a) $M \cap E$

b) $M' \cup S'$

c) $M' \cap S'$

<u>*Solution*</u>

a) $M \cap E$

This set includes all applicants who are both male and employed; that is, employed male applicants

b) $M' \cup S'$

This set includes all applicants who are female (not male) or married, all married female applicants are in this set

c) $M' \cap S'$

This set includes all applicants who are female and married, this is the set of all married female applicants

Exercises **Section 3.1 – Sets**

1. Let $U=\{1,2,3,4,5,6,7,8,9\}$, $X=\{2,4,6,8\}$, $Y=\{2,3,4,5,6\}$, *and* $Z=\{1,2,3,8,9\}$

a) $X\cap Y$ *b)* $X\cup Y$ *c)* Y' *d)* $X'\cap Z$

e) $Y\cap(X\cup Z)$ *f)* $X'\cap(Y'\cup Z)$ *g)* $(X\cap Y')\cup Z'$

2. Given $A = \{0, 2, 4, 6\}$, $B = \{0, 1, 2, 3, 4, 5, 6\}$, and $C = \{2, 6, 0, 4\}$, determine if the statement is true or false?

a) $A\subset B$ *b)* $A\subset C$ *c)* $A=B$ *d)* $C\subset B$

e) $\text{B}\not\subset\text{A}$ *f)* $\varnothing\subset B$

3. Given $R = \{1, 2, 3, 4\}$, $S = \{1, 3, 5, 7\}$, $T = \{2, 4\}$, and $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, find the following:

a) $R\cup S$ *b)* $R\cap S$ *c)* $S\cap T$ *d)* S'

4. Write true or false for each statement

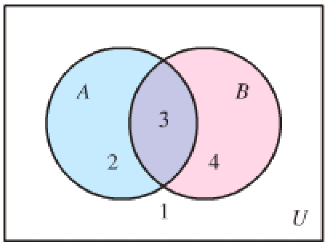
a) $3\in\{2,\ 5,\ 7,\ 9,\ 10\}$ *b)* $6\in\{-2,\ 5,\ 6,\ 9\}$

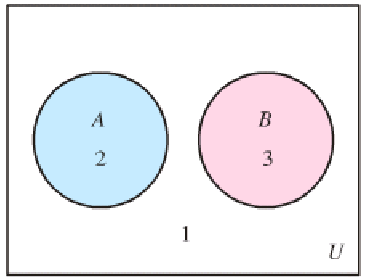
c) $9\notin\{2,\ 1,\ 5,\ 8\}$ *d)* $3\notin\{7,\ 6,\ 5,\ 4,\ 10\}$

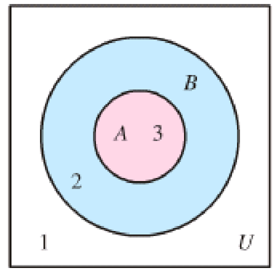
e) $\{2,\ 5,\ 8,\ 9\}=\{2,\ 5,\ 9,\ 8\}$ *f)* $\{3,\ 7,\ 12,\ 14\}=\{3,\ 7,\ 12,\ 14,\ 9\}$

Section 3.2 – Applications of Venn Diagrams

The event might be; winning the lottery, guessing the correct answer on a test question, selecting an Ace from a deck of cards ….

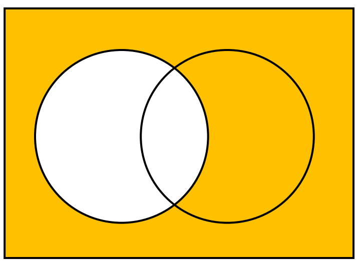


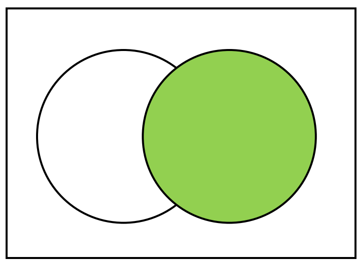


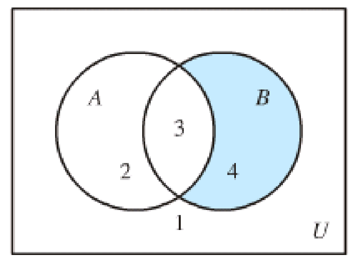


Example

Draw Venn diagrams and shade the regions representing $A' \cap B$







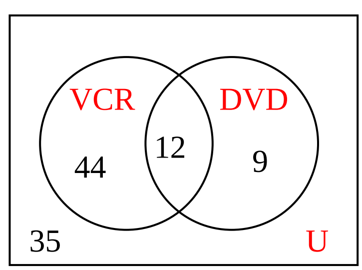
Example

A researcher collecting data on 100 households finds that

- ✓ 21 have a DVD player;
- ✓ 56 have a videocassette recorder (VCR); and
- ✓ 12 have both

a) How many do not have a VCR?
b) How many have neither a DVD player nor a VCR?
c) How many have a DVD player but not a VCR?

<u>*Solution*</u>



a) How many do not have a VCR?

35 + 9 = 44

b) How many have neither a DVD player nor a VCR?

35

c) How many have a DVD player but not a VCR?

9

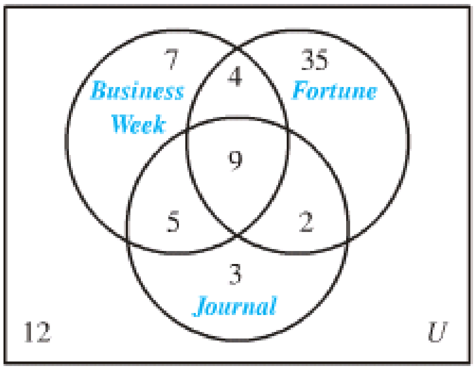
Example

A survey of 77 freshman business students at a large university produced the following results

- ✓ 25 of the students read Business Week;
- ✓ 19 read the Wall Street Journal;
- ✓ 27 do not read Fortune
  - 11 read Business Week but not Wall Street Journal
  - 11 read the Wall Street Journal and Fortune
  - 13 read Business Week and Fortune;
  - 9 read all three

a) How many students read none of the publications?
b) How many read only Fortune?
c) How many read Business Week and the Wall Street Journal, but not Fortune?

<u>Solution</u>



a) **12** students read none of the publications

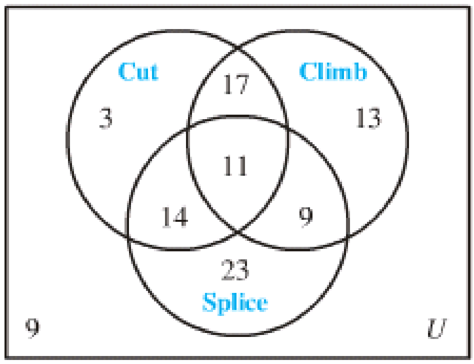
b) **35** read only Fortune

c) **5** read Business Week and the Wall Street Journal, but not Fortune $(B \cap J) \cap F'$

Example

Jeff is a section chief of an electric utility company. The employees in his section cut down trees, climb poles, and splice wire. Jeff reported the following information to the management of the utility

"Of the 100 employees in my section,
45 can cut trees;
50 can climb poles;
57 can splice wire;
22 can climb poles but can't cut trees;
20 can climb poles but splice wire;
25 can cut trees and splice wire;
14 can cut trees and splice wire but can't climb poles;
9 can't do any of the three (management trainees)."



<u>Solution</u>

Jeff claimed to have 100 employees, but his information indicate only 99.

Management decided that Jeff didn't qualify as a section chief.

***Moral**: He should have taken this course.*

***Counting Techniques*:** A method that gives us a way to count how many elements are in a set, without having to actually count them.

Union Rule for Sets - *Addition Principle* (*For Counting*)

$$n(A\cup B) = n(A) + n(B) - n(A\cap B)$$

$n(A\cap B) = 0 \quad \Rightarrow A \& B \;\; \textit{are disjoint}$

$$\begin{aligned} n(A\cup B\cup C) = {} & n(A) + n(B) + n(C) \\ & - n(A\cap B) - n(A\cap C) - n(B\cap B) \\ & - n(A\cap B\cap C) \end{aligned}$$

Example

A group of 10 students meet to plan a school function. All are majoring in accounting or economics or both. Five of the students are economics majors and 7 are majors in accounting. How many major in both subjects?

<u>*Solution*</u>

$n(A\cup B) = 10$

$n(A) = 5$

$n(B) = 7$

$n(A\cup B) = n(A) + n(B) - n(A\cap B)$

$10 = 5 + 7 - n(A\cap B)$

$10 - 12 = -n(A\cap B)$

$-2 = -n(A\cap B)$

$n(A\cap B) = 2$

Exercises *Section* 3.2 – Applications of Venn Diagrams

1. Use the union rule to answer the following

a) If $n(A)=5;\quad n(B)=12,\quad and\quad n(A\cap B)=4$ what is $n(A\cup B)$?

b) If $n(A)=15;\quad n(B)=30,\quad and\quad n(A\cup B)=33$ what is $n(A\cap B)$?

c) $n(B)=9;\quad n(A\cap B)=5,\quad and\quad n(A\cup B)=22$ what is $n(A)$?

2. Draw a Venn diagram and use the given information to fill in the number of elements

a) $n(U)=41;\quad n(A)=16,\quad n(A\cap B)=12,\quad n(B')=20$

b) $n(A)=28;\quad n(B)=12,\quad n(A\cup B)=32,\quad n(A')=19$

c) $n(A)=11;\quad n(A\cap B)=6,\quad n(A\cup B)=24,\quad n(A'\cup B')=25$

d) $n(A)=28,\quad n(B)=34,\quad n(C)=25,\quad n(A\cap B)=14,\quad n(B\cap C)=15$
$n(A\cap C)=11;\quad n(A\cap B\cap C)=9,\quad n(U)=59$

e) $n(A)=54,\quad n(B')=63,\quad n(C)=44,\quad n(A\cap B)=22,\quad n(B\cap C)=16,$
$n(A\cap C)=15;\quad n(A\cap B\cap C)=4,\quad n(A\cup B)=85$

f) $n(A\cap C')=11,\quad n(B\cap C')=8,\quad n(C)=15,\quad n(A\cap B)=6,\quad n(B\cap C)=4$
$n(A\cap C)=7;\quad n(A\cap B\cap C)=4,\quad n(A'\cap B'\cap C')=5$

3. Toward the middle of the harvesting season peaches for canning come in three types, early, late, and extra late, depending on the expected date of ripening. During a certain week, the following data were recorded at a fruit delivery station:

34 trucks went out carrying early peaches;
61 carried late peaches;
50 carried extra late;
25 carried early and late;
30 carried late and extra late;
8 carried early and extra late;
6 carried all three
9 carried only figs (no peaches at all).

a) How many trucks carried only variety peaches?
b) How many carried only extra late?
c) How many carried only one type of peach?
d) How many trucks (in all) went during the week?

4. In a survey of 100 randomly chosen students, a marketing questionnaire included the following:

- ✓ 75 own a TV
- ✓ 45 own a car
- ✓ 35 own a TV and a car

a) How many students owned a car but not a TV set?
b) How many students did not own both a car and a TV set?

5. A small town has two radio stations, an AM station and an FM station. A survey of 100 residents of the town produced the following results: In the last 30 days, 65 people have listened to the AM station, 45 have listened to the FM station, and 30 have listened to both stations.

6. In a class of 35 students, 19 are married and 20 are blondes. Given that there are 7 students that are both married and blonde, answer the following questions.

a) How many are married, but not blonde?
b) How many are blonde but not married?
c) How many are blonde or married?
d) How many are neither blonde nor married?
e) How many are not blonde?

7. In a survey of 500 businesses it was found that 250 had copiers and 300 had fax machines. It was also determined that 100 businesses had both copiers and fax machines.
a) How many had either a copier or a fax machine?
b) How many had neither a copier nor a fax machine?
c) How many had a copier, but no fax machine?
d) How many had a fax machine, but no copier?
e) How many had no fax machines?

8. Given: $n(U)=105,\quad n(A)=50,\quad n(B)=75,\quad n(A\cup B)=105$, find the following:

a) $n(A\cap B)$ *b)* $n(A'\cap B)$ *c)* $n(A'\cap B')$

d) $n(A\cup B')$ *e)* $n(B')$

9. Fred interviewed 140 people in a shopping center to discover some of their cooking habits. He obtained the following results:

58 use microwave ovens
63 use electric ranges
58 use gas ranges
19 use microwave ovens and electric ranges
17 use microwave ovens and gas ranges
4 use both gas and electric ranges
1 uses all three
2 use none of the three

Should he be reassigned one more time? Why or why not?

10. Toward the middle of the harvesting season, peaches for canning come in three types, early, late, and extra late, depending on the expected date of ripening. During a certain week, the following data were recorded at a fruit delivery station:

34 trucks went out carrying early peaches
61 carried late peaches
50 carried extra late
25 carried early and late
30 carried late and extra late
8 carried early and extra late
6 carried all three
9 carried only figs (no peaches at all)

a) How many trucks carried only late variety peaches?
b) How many carried only extra late?
c) How many carried only one type of peach?
d) How many trucks (in all) went out during the week?

11. Most mathematics professors love to invest their hard earned money. A recent survey of 150 math professors revealed that

111 invested in stocks
98 invested in bonds
100 invested in certificates of deposit
80 invested in stocks and bonds
83 invested in bonds and certificates of deposit
85 invested in stocks and certificates of deposit
9 did not invest in any of three

How many mathematics professors invested in stocks and bonds and certificates of deposit?

12. Suppose that a group of 150 students have joined at least one of three chat rooms; one on auto-racing, one on bicycling, and one for college students. For simplicity, we will call these rooms A, B, and C. In addition,

90 students joined room A;
50 students joined room B;
70 students joined room C;
15 students joined room A and C;
12 students joined room B and C;
10 students joined all three rooms;

Determine how many students joined both chat rooms A and B.

Section 3.3 – Counting; Multiplication Principle

Basic Counting Principle

The Product Rule

A procedure can be broken down into a sequence of two tasks. There are n_1 ways to do the first task and n_2 ways to do the second task. Then there are $n_1 \cdot n_2$ ways to do the procedure

Example

How many bit strings of length seven are there?

Solution

Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = 128$.

Example

A new company with just two employees rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution

The procedure of assigning offices to these 2 employees consists of assigning an office to one employee, which can be done in 12 ways, then assigning an office to the second different from the office assigned to the first, which can be done in 11 ways.

By the product rule, there are $12 \cdot 11 \boxed{= 132 \ \ \text{ways}}$ to assign offices to these 2 employees.

Example

There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a computer in the center are there?

Solution

$32 \cdot 24 \boxed{= 768 \ \ \text{ports}}$

Multiplication Principle

Sequence of operations $\Rightarrow$ set multiplication to count numbers

> Suppose n choices must be made, with m_1 ways to make choice 1,
>
> *and* for each of these ways, with m_2 ways to make choice 2,
>
> *and* so on, with m_n ways to make choice n.
>
> Then there are: $m_1.m_2 \ldots m_n$

Theorem

- If two operations O_1 O_2 are performed in O order with N_1 possible outcomes for 1st O_1 & N_2 possible outcomes for $O_2 \Rightarrow N_1.N_2$
- $O_1, O_2, \cdots, O_n \Rightarrow N_1.N_2$ ..... N_3

Example

A certain combination lock can be set to open to any 3-letter sequence.

a) How many sequences are possible?

b) How many sequences are possible if no letter is repeated?

Solution

a) Possible sequences: $26.26.26 = \boxed{17,576 \ \textit{different sequences}}$

b) Possibility if no letter is repeated: $26.25.24 = 15,600$ *possibilities*

Example

Each question on a multiple-choice test has 5 choices. If there are 5 such question on a test, how many different response sheets are possible if only 1 choice is marked for each question?

Solution

$$5\cdot5\cdot5\cdot5\cdot5 = 5^5 = \boxed{3125 \ \textit{different responses}}$$

Example

Morse code uses a sequence of dots and dashes to represent letters and words. How many sequences are possible with at most 3 symbols?

Solution

At most 3 means "1 *or* 2 *or* 3"
Sequence: *dot and dash*

Number of Symbols	Number of Sequences
1	**2**
2	$\mathbf{2} \cdot \mathbf{2} = 4$
3	$\mathbf{2} \cdot \mathbf{2} \cdot \mathbf{2} = 8$

Possibilities: 2 + 4 +8 = 14

Example

A teacher has 5 different books that he wishes to arrange side by side. How many different arrangements are possible?

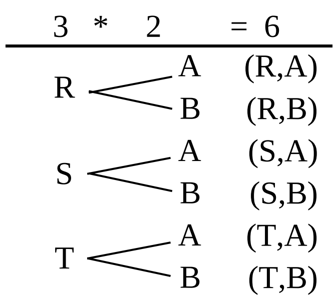
Solution

Possibilities: 5.4.3.2.1 = 120

Example

A company offers its employee's health plans from three different companies *R*, *S*, and *T*. Each company offers two levels of coverage, *A* and *B*, with one level requirement additional employee contributions. What are the combined choices, and how many choices are?

Solution



Count = 6

Tree Diagram

Factorial

$n! = n(n-1)(n-2)\cdots(3).(2).(1)$ (*n factorial*)

Calculators: **Math → Prob → 4**

5.4.3.2.1 = 5!

$\frac{7!}{6!} = 7$

0! = 1

$\frac{8!}{5!} = \frac{(8)(7)(6)(5!)}{5!} = 8(7)(6) = 336$

$4! = 4 \bullet 3 \bullet 2 \bullet 1 = 24$

4	**Math**	**PRB**	**Type 4**	
	MATH NUM CPX PRB 1:▸Frac 2:▸Dec	MATH NUM CPX PRB 1:rand 2:nPr 3:nCr 4:! 5:randInt(6:randNorm(7:randBin(	4! 24	4 MATH ◀ 4 ENTER

Example

During the summer, you are planning to visit these 6 national parks: Glacier, Yellowstone, Yosemite, Arches, Zion, and Grand Canyon. You would like to plan the most efficient route and you decide to list all of the possible routes. How many different routes are possible?

<u>*Solution*</u>

There 6 different parks can be arranged in order 6! different ways.

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ *ways*

Example

A teacher has 5 different books that he wishes to place only 3 of the 5 books on his desk. How many arrangements of 3 books are possible?

<u>*Solution*</u>

Possibilities: 5.4.3 = 60 ***arrangements***

Exercises *Section* 3.3 – Counting; Multiplication Principle

1. How many different types of homes are available if a builder offers a choice of 6 basic plans, 3 roof styles, and 2 exterior finishes?

2. A menu offers a choice of 3 salads, 8 main dishes, and 7 desserts. How many different meals consisting of one salad, one main dish, and one dessert are possible?

3. A couple has narrowed down the choice of a name for their new baby to 4 first names and 5 middle names. How many different first- and middle-name arrangements are possible?

4. An automobile manufacturer produces 8 models, each available in 7 different exterior colors, with 4 different upholstery fabrics and 5 interior colors. How many varieties of automobile are available?

5. A biologist is attempting to classify 52,000 species of insects by assigning 3 initials to each species. Is it possible to classify all the species in this way? If not, how many initials should be used?

6. How many 4-letter code words are possible using the first 10 letters of the alphabet under:

a) No letter can be repeated
b) Letters can be repeated
c) Adjacent can't be alike

7. How many 3 letters license plate code words are possible without repeats possible

8. How many ways can 2 coins turn up heads, H, or tails, T – if the combined outcome (H, T) is to be distinguished from the outcome (T, H)?

9. How many 2-letter code words can be formed from the first 3 letters of the alphabet if no letter can be used more than once?

10. A coin is tossed with possible outcomes heads, H, or tails, T. Then a single die is tossed with possible outcomes 1, 2, 3, 4, 5, or 6. How many combined outcomes are there?

11. In how many ways can 3 coins turn up heads, H, or tails, T – if combined outcomes such as (H,T,H), (H, H, T), and (T, H, H) are to be considered different?

12. An entertainment guide recommends 6 restaurants and 3 plays that appeal to a couple.
a) If the couple goes to dinner or to a play, how many selections are possible?
b) If the couple goes to dinner and then to a play, how many combined selections are possible?

13. There are 18 mathematics majors and 325 computer science majors at a college
a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?
b) In how many ways can one representative be picked who either a mathematics major or a computer science major?

14. An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

15. A multiple-choice test contains 10 questions. There are four possible answers for each question

a) In how many ways can a student answer the questions on the test if the student answers every question?

b) In how many ways can a student answer the questions on the test if the student can leave answers blank?

16. A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of the shirts are made?

17. How many different three-letter initials can people have?

18. How many different three-letter initials with none of the letters repeated can people have?

19. How many different three-letter initials are there that begin with an A?

20. How many strings are there of four lowercase letters that have the letter x in them?

21. How many license plates can be made using either three digits followed by three uppercase English letters or three uppercase English letters followed by three digits?

22. How many license plates can be made using either two uppercase English letters followed by four digits or two digits followed by four uppercase English letters?

23. How many license plates can be made using either three uppercase English letters followed by three digits or four uppercase English letters followed by two digits?

24. How many strings of eight English letter are there

a) That contains no vowels, if letters can be repeated?

b) That contains no vowels, if letters cannot be repeated?

c) That starts with a vowel, if letters can be repeated?

d) That starts with a vowel, if letters cannot be repeated?

e) That contains at least one vowel, if letters can be repeated?

f) That contains at least one vowel, if letters cannot be repeated?

25. In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, where the bride and the groom are among these 10 people, if

a) The bride must be in the picture?

b) Both the bride and groom must be in the picture?

c) Exactly one of the bride and the groom is in the picture?

Section 3.4 – Permutations and Combinations

Permutation

A permutation of a set of distinct objects is an arrangement of the objects is a ***specific Order Without*** repetition. An ordered arrangement of r elements of a set is called an ***r -permutation***.

$$P_{n,r} = \frac{n!}{(n-r)!}$$

n Math $\rightarrow$ Prob $\rightarrow$ 2(nPr) r

$$P_{n,n} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n! \qquad\qquad P_{7,7} = 7!$$

Example

In mid 2007, eight candidates sought the Democratic nomination for president. In how many ways could voters rank their first, second, and third choices?

Solution

$P_{8,3} = 336$ 8 *Math* $\rightarrow$ *Prob* $\rightarrow$ (nPr) 3

Requirements:

1. There are n items available, and some items are identical to others.
2. We select all of the n items (without replacement).
3. We consider rearrangements of distinct items to be different sequences.

If the preceding requirements are satisfied, and if there are n_1 alike, n_2 alike, . . . , n_k alike, the number of permutations (or sequences) of all items selected without replacement is

$$\frac{n!}{n_1! \cdot n_2! \cdots n_k!}$$

Example

In how many ways can the letters in the word *Mississippi* be arranged?

Solution

$$\frac{11!}{1!4!4!2!} \boxed{= 34,650 \ \ ways}$$

m	*i*	*s*	*p*
1	4	4	2

Example

A student buys 3 cherry yogurts, 2 raspberry yogurts, and 2 blueberry yogurts. She puts them in her dormitory refrigerator to eat one a day for the next week. Assuming yogurts of the same flavor are indistinguishable, in how many ways can she select yogurts to eat for the next week?

<u>Solution</u>

$$\frac{7!}{3!2!2!} \underline{=210 \ \ ways}|$$

Example

A televised show will include 4 women and 3 men as panelists

a) In how many ways can the panelists be seated in a row of 7 chairs?

b) In how many ways can the panelists be seated if the men and women are to be alternated?

c) In how many ways can the panelists be seated if the men must sit together, and the women must also sit together?

d) In how many ways can one woman and one man from the panel be selected?

<u>Solution</u>

a) $P_{7,7} \underline{=5040 \ \ ways}|$

b) 1^{st} seat: 4 **(4 *women and not enough men* 3)**

2^{nd} seat: 3

3^{rd} seat: 3

W	M	W	M	W	M	W
4	3	3	2	2	1	1

4.3.3.2.2.1.1 $\underline{=144 \ \ ways}|$

c) Arrange the 2 groups: 2!

Arranging women: 4!

Arranging men: 3!

Total: 2!.4!.3! $\underline{=288 \ \ ways}|$

d) 4.3 $\underline{=12 \ \ ways}|$

Example

Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

<u>Solution</u>

There are: $P(8,3)=8\cdot 7\cdot 6 \underline{=336 \ \ ways}|$

Combination

Combination of a set of n distinct objects taken r @ a time ***without*** repetition is an r element subset of the set of n objects.

The arrangement of the elements *doesn't matter*.

$$C_{n,r} = \binom{n}{r} = \frac{P_{n,r}}{r!} = \frac{n!}{r!(n-r)!}$$

n Math $\rightarrow$ Prob $\rightarrow$ 3(nCr) r

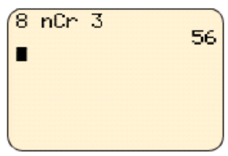
Example

How many committees of 3 people can be formed from a group of 8 people?

Solution

$$C_{8,3} = \binom{8}{3} = 56$$

8 Math $\rightarrow$ Prob $\rightarrow 3(nCr)$ 3



Example

Three lawyers are to be selected from a group of 30 to work on a special project.

a) In how many different ways can the lawyers be selected?

$$C_{30,3} = 4060 \ \ ways$$

b) In how many ways can the group of 3 be selected if a certain lawyer must work on the project?

1 already selected which left 29 to select from

$$C_{29,2} = \binom{29}{2} = 406 \ \ ways$$

c) In how many ways can a nonempty group of at most 3 lawyers be selected from these 30 lawyers?

At most 3 = "1 or 2 or 3"

$$C_{30,1} + C_{30,2} + C_{30,3} = 30 + 435 + 4060 = 4525 \ \ ways$$

Example

A salesman has 10 accounts in a certain city

a) In how many ways can he select 3 accounts to call on?

b) In how many ways can he select at least 8 of the accounts to use in preparing a report?

<u>*Solution*</u>

a) In how many ways can he select 3 accounts to call on?

$$C_{10,3} = \binom{10}{3} = 120 \ \ ways$$

b) In how many ways can he select at least 8 of the accounts to use in preparing a report?

At least 8 = "8 or 9 or 10"

$$C_{10,8} + C_{10,9} + C_{10,10} = 45 + 10 + 1 = 56 \ \ ways$$

Permutation: *order matter*.

Combination: *Order doesn't matter*.

Examples

For each problem, tell whether permutations or combinations should be used to solve the problem.

a) How many 4-digit code numbers are possible if no digits are repeated?

Permutation

b) A sample of 3 light bulbs is randomly selected from batch of 15. How many different samples are possible?

Combination

c) In a baseball conference with 8 teams, how many games must be played so that each team plays every other team exactly once?

Combination

d) In how many ways can 4 patients be assigned to 6 different hospital rooms so that each patient has a private room?

Permutation

Example

A manger must select 4 employees for promotion; 12 employees are eligible.

a) In how many ways can the 4 be chosen?

$$C_{12,4} \ \boxed{= 495 \ \ \textit{ways}}$$

b) In how many ways can 4 employees be chosen (from 12) to be placed in 4 different jobs?

$$P_{12,4} \ \boxed{= 11,880 \ \ \textit{ways}}$$

Example

In how many ways can a full house of aces and eights (3 aces and 2 eights) occur in 5-card poker?

Solution

$C_{4,3}$: get 3 aces out of 4

$C_{4,2}$: get 2 eights out of 4

$$C_{4,3}C_{4,2} = 24$$

Example

Five cards are dealt from a standard 52-card deck

a) How many hands have only face cards?

$$C_{12,5} \ \boxed{= 792 \ \ \textit{hands}}$$

b) How many such hands have exactly 2 hearts?

2 hearts will be selected from 13 cards, the other 3 cards will be selected from 39 remaining cards.

$$C_{13,2}C_{39,3} \ \boxed{= 712,842 \ \ \textit{hands}}$$

c) How many such hands have cards of a single suit?

There are 4 different suits

$$4.C_{13,5} \ \boxed{= 5148 \ \ \textit{hands}}$$

Exercises *Section* 3.4 – Permutations and Combinations

1. Decide whether the situation involves ***permutations*** or ***combinations***

a) A batting order for 9 players for a baseball game
b) An arrangement of 8 people for a picture
c) A committee of 7 delegates chosen from a class of 30 students to bring a petition to the administration
d) A selection of a chairman and a secretary from a committee of 14 people
e) A sample of 5 items taken from 71 items on an assembly line
f) A blend of 3 spices taken from 7 spices on a spice rack
g) From the 7 male and 10 female sales representatives for an insurance company, team of 8 will be selected to attend a national conference on insurance fraud.
h) Marbles are being drawn without replacement from a bag containing 15 marbles.
i) The new university president named 3 new officers a vice-president of finance, a vice-president of academic affairs, and a vice-president of student affairs.
j) A student checked out 4 novels from the library to read over the holiday.
k) A father ordered an ice cream cone (chocolate, vanilla, or strawberry) for each of his 4 children.

2. Wing has different books to arrange on a shelf: 4 blue, 3 green, and 2 red.

a) In how many ways can the books be arranged on a shelf?
b) If books of the same color are to be grouped together, how many arrangements are possible?
c) In how many distinguishable ways can the books be arranged if books of the same color are identical but need not be grouped together?
d) In how many ways can you select 3 books, one of each color, if the order in which the books are selected does not matter?
e) In how many ways can you select 3 books, one of each color, if the order in which the books are selected matters?

3. A child has a set of differently shaped plastic objects. There are 3 pyramids, 4 cubes, and 7 spheres.

a) In how many ways can she arrange the objects in a row if each is a different color?
b) How many arrangements are possible if objects of the same shape must be grouped together and each object is a different color?
c) In how many distinguishable ways can the objects be arranged in a row if objects of the same shape are also the same color, but need not be grouped together?
d) In how many ways can you select 3 objects, one of each shape, if the order in which the objects are selected does not matter and each object is a different color?
e) In how many ways can you select 3 objects, one of each shape, if the order in which the objects are selected matters and each object is a different color?

4. Twelve drugs have been found to be effective in the treatment of a disease. It is believed that the sequence in which the drugs are administered is important in the effectiveness of the treatment. In how many different sequences can 5 of the 12 drugs be administered?

5. In a club with 16 members, how many ways can a slate of 3 officers consisting of president, vice-president, and secretary/treasurer be chosen?

6. In how many ways can 7 of 11 monkeys be arranged in a row for a genetics experiment?

7. In an experiment on social interaction, 6 people will sit in 6 seats in a row. In how many ways can this be done?

8. In an election with 3 candidates for one office and 6 candidates for another office, how many different ballots may be printed?

9. A business school gives courses in typing, shorthand, transcription, business English, technical writing, and accounting. In how many ways can a student arrange a schedule if 3 courses are taken? Assume that the order in which courses are schedules matters.

10. If your college offers 400 courses, 25 of which are in mathematics, and your counselor arranges your schedule of 4 courses by random selection, how many schedules are possible that do not include a math course? Assume that the order in which courses are scheduled matters.

11. A baseball team has 19 players. How many 9-player batting orders are possible?

12. A chapter of union Local 715 has 35 members. In how many different ways can the chapter select a president, a vice-president, a treasurer, and a secretary?

13. A concert to raise money for an economics prize is to consist of 5 works; 2 overtures, 2 sonatas, and a piano concerto.
   a) In how many ways can the program be arranged?
   b) In how many ways can the program be arranged if an overture must come first?

14. A zydeco band from Louisiana will play 5 traditional and 3 original Cajun compositions at a concert. In how many ways can they arrange the program if
   a) it begins with a traditional piece?
   b) An original piece will be played last?

15. Given the set $\{A, B, C, D\}$, how many permutations are there of this set of 4 object taken 2 at a time?
   a) Using the multiplication principle
   b) Using the Permutation

16. Find the number of permutations of 30 objects taken 4 at a time.

17. Five cards are marked with the numbers 1, 2, 3, 4, and 5, then shuffled, and 2 cards are drawn.
   a) How many different 2-card combinations are possible?
   b) How many 2-card hands contain a number less than 3?

18. An economics club has 31 members.
   a) If a committee of 4 is to be selected, in how many ways can the selection be made?
   b) In how many ways can a committee of at least 1 and at most 3 be selected?

19. Use a tree diagram for the following

a) Find the number of ways 2 letters can be chosen from the set $\{L, M, N\}$ if order is important and repetition is allowed.

b) Reconsider part a if no repeats are allowed

c) Find the number of combinations of 3 elements taken 2 at a time. Does this answer differ from part *a* or *b*?

For each problem, decide whether permutations or combinations should be used to solve the problem.

20. In a club with 9 male and 11 female members, how many 5-member committees can be chosen that have

a) All men?

b) All women?

c) 3 men and 2 women?

21. In a club with 9 male and 11 female members, how many 5-member committees can be selected that have

a) At least 4 women?

b) No more than 2 men?

22. In a game of musical chairs, 12 children will sit in 11 chairs arranged in a row (one will be left out). In how many ways can this happen, if we count rearrangements of the children in the chairs as different outcomes?

23. A group of 3 students is to be selected from a group of 14 students to take part in a class in cell biology.

a) In how many ways can this be done?

b) In how many ways can the group who will not take part be chosen?

24. Marbles are being drawn without replacement from a bag containing 16 marbles.

a) How many samples of 2 marbles can be drawn?

b) How many samples of 2 marbles can be drawn?

c) If the bag contains 3 yellow, 4 white, and 9 blue marbles, how many samples of 2 marbles can be drawn in which both marbles are blue?

25. There are 7 rotten apples in a crate of 26 apples

a) How many samples of 3 apples can be drawn from the crate?

b) How many samples of 3 could be drawn in which all 3 are rotten?

c) How many samples of 3 could be drawn in which there are two good apples and one rotten one?

26. A bag contains 5 black, 1 red, and 3 yellow jelly beans; you take 3 at random. How many samples are possible in which the jelly beans are

a) All black?

b) All red?

c) All yellow?

d) 2 black and 1 red?
e) 2 black and 1 yellow?
f) 2 yellow and 1 black?
g) 2 red and 1 yellow?

27. In how many ways can 5 out of 9 plants be arranged in a row on a windowsill?

28. From a pool of 8 secretaries, 3 are selected to be assigned to 3 managers, one per manager. In how many ways can they be selected and assigned?

29. A salesperson has the names of 6 prospects.
a) In how many ways can she arrange her schedule if she calls on all 6?
b) In how many ways can she arrange her schedule if she can call on only 4 of the 6?

30. A group of 9 workers decides to send a delegation of 3 to their supervisor to discuss their grievances.
a) How many delegations are possible?
b) If it is decided that a particular worker must be in the delegation, how many different delegations are possible?
c) If there are 4 women and 5 men in the group, how many delegations would include at least 1 woman?

31. Hamburger Hut sells regular hamburgers as well as a larger burger. Either type can include cheese, relish, lettuce, tomato, mustard, or catsup.
a) How many different hamburgers can be ordered with exactly three extras?
b) How many different regular hamburgers can be ordered with exactly three extras?
c) How many different regular hamburgers can be ordered with at least five extras?

32. Five items are to be randomly selected from the first 50 items on an assembly line to determine the defect rate. How many different samples of 5 items can be chosen?

33. From a group of 16 smokers and 22 nonsmokers, a researcher wants to randomly select 8 smokers and 8 nonsmokers for a study. In how many ways can the study group be selected?

34. In an experiment on plant hardiness, a researcher gathers 6 wheat plants, 3 barley plants, and 2 rye plants. She wishes to select 4 plants at random.
a) In how many ways can this be done?
b) In how many ways can this be done if exactly 2 wheat plants must be included?

35. A legislative committee consists of 5 Democrats and 4 Republicans. A delegation of 3 is to be selected to visit a small Pacific island republic.
a) How many different delegations are possible?
b) How many delegations would have all Democrats?
c) How many delegations would have 2 Democrats and 1 Republican?
d) How many delegations would have at least 1 Republican?

36. From 10 names on a ballot, 4 will be elected to a political party committee. In how many ways can the committee of 4 be formed if each person will have a different responsibility, and different assignments of responsibility are considered different committees?

37. How many different 13-card bridge hands can be selected from an ordinary deck?

38. Five cards are chosen from an ordinary deck to form a hand in poker. In how many ways is it possible to get the following results?

a) 4 queens

b) No face card

c) Exactly 2 face cards

d) At least 2 face cards

e) 1 heart, 2 diamonds, and 2 clubs

39. In poker, a flush consists of 5 cards with the same suit, such as 5 diamonds.

a) Find the number of ways of getting a flush consisting of cards with values from 5 to 10 by listing all the possibilities.

b) Find the number of ways of getting a flush consisting of cards with values from 5 to 10 by using combinations

40. If a baseball coach has 5 good hitters and 4 poor hitters on the bench and chooses 3 players at random, in how many ways can be choose at least 2 good hitters?

41. The coach of a softball team has 6 good hitters and 8 poor hitters. He chooses 3 hitters at random.

a) In how many ways can he choose 2 good hitters and 1 poor hitter?

b) In how many ways can he choose 3 good hitters?

c) In how many ways can he choose at least 3 good hitters?

42. How many 5 card hands will have 3 aces and 2 kings?

43. How many 5 card hands will have 3 hearts and 2 spades?

44. 2 letters follow by 3 numbers; 2 letters out of 8 & 3 numbers out of 10

45. Serial numbers for a product are to be made using 3 letters follow by 2 digits (0 – 9 no repeats). If the letters are to be taken from the first 8 letters of the alphabet with no repeats, how many serial numbers are possible?

46. A company has 7 senior and 5 junior officers. An ad hoc legislative committee is to be formed.

a) How many 4-officer committees with 1 senior officer and 3 junior officers can be formed?

b) How many 4-officer committees with 4 junior officers can be formed?

c) How many 4-officer committees with at least 2 junior officers can be formed?

47. From a committee of 12 people,

a) In how many ways can we choose a chairperson, a vice-chairperson, a secretary, and a treasurer, assuming that one person can’t hold more than on position

b) In how many ways can we choose a subcommittee of 4 people?

48. Find the number of combinations of 30 objects taken 4 at a time.

49. How many different permutations are the of the set $\{a, b, c, d, e, f, g\}$?

50. How many permutations of $\{a, b, c, d, e, f, g\}$ end with a?

51. Find the number of 5-permutations of a set with nine elements

52. In how many different orders can five runners finish a race if no ties are allowed?

53. A coin flipped eight times where each flip comes up either heads or tails. How many possible outcomes

a) Are there in total?
b) Contain exactly three heads?
c) Contain at least three heads?
d) Contain the same number of heads and tails?

54. In how many ways can a set of two positive integers less than 100 be chosen?

55. In how many ways can a set of five letters be selected from the English alphabet?

Section 3.5 – Probability

The type of experiments on which probability studies are based called random experiments: flipping coins, dice.

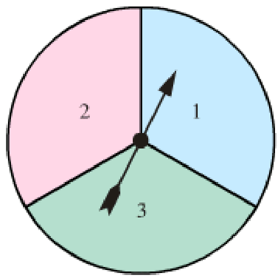
Experiment means a random experiment; an activity with a result.

Each possible result is called an ***outcome*** of the experiment (each trial).

Sample Spaces (*S*)

Sample Space is the set of all possible outcomes for an experiment.

Outcome: Event – subset of S: ***simple*** ***or*** ***compound***



Example

a) Give the sample space for a the spinner $\boxed{S = \{1, 2, 3\}}$

b) Give the sample space for an experiment consists of studying the numbers of boys and girls in families with exactly 3 children. Let *b* represent *boy* and *g* represent *girl*.

$$S = \{bbb,\ bbg,\ bgb,\ gbb,\ bgg,\ gbg,\ ggb,\ ggg\}$$

Events

Event is any subset of S including $\varnothing$

$$\begin{cases} Simple\ event & \text{Contains only one element} \\ Compound\ event & \text{Contains more than one element} \end{cases}$$

Example

Give the sample space for an experiment consists of studying the numbers of boys and girls in families with exactly 3 children. Write the event foe the family that has exactly two girls

<u>Solution</u>

$$E = \{bgg, gbg, ggb\}$$

Example

Suppose a coin is flipped until both a head and a tail appear, or until the coin has been flipped four times, whichever comes first.

a) Write the event that the coin flipped exactly three times.

b) Write the event that the coin flipped at least three times.

c) Write the event that the coin flipped at least two times.

<u>*Solution*</u>

a) $S = \{HHH, HHT, HTH, THH, HTT, HTH, TTH, TTT\}$

$E = \{HHT, TTH\}$

b) $E = \{TTH, HHT, HHHH, HHHT, TTTH, TTTT\}$

c) $E = \{HT, TH, TTH, HHT, HHHH, HHHT, TTTH, TTTT\}$

Probability

Let *S* be a sample space of equally likely outcomes, and let event *E* be a subset of *S*. Then the probability that event *E* occurs is

$$P(E) = \frac{\text{number of elements of } E}{\text{number of elements in } S} = \frac{n(E)}{n(S)}$$

Example

Suppose a single fair die is rolled. Use the sample space $S = \{1, 2, 3, 4, 5, 6\}$ and give the probability of each event.

a) E: the die shows an even number

b) F: the die show a number less than 10

c) G: the die shows an 8

<u>*Solution*</u>

a) Even number: $E = \{2, 4, 6\}$

$$P(E) = \frac{n(S)}{n(E)}$$

$$= \frac{3}{6}$$

$$\boxed{= \frac{1}{2}}$$

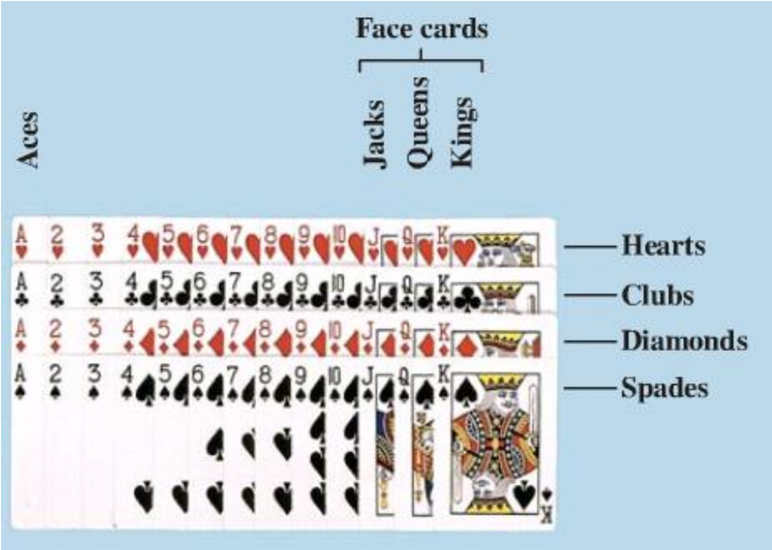
b) Number less than 10

$F = \{1, 2, 3, 4, 5, 6\}$

$P(F) = \frac{6}{6} \ \boxed{= 1}$

c) Die shows an 8

$\boxed{P(G) = 0}$ ***Impossible***



Example

If a single playing card is drawn at random from a standard 52-card deck, find the probability of each event.

a) Drawing an ace
b) Drawing a face card
c) Drawing a spade
d) Drawing a spade or a heart

<u>*Solution*</u>

a) $P(Ace)=\frac{4}{52}=\frac{1}{13}$

b) $P(\text{face card})=\frac{12}{52}=\frac{3}{13}$

c) $P(spade)=\frac{13}{52}=\frac{1}{4}$

d) $P(spade \text{ or } heart)=\frac{26}{52}=\frac{1}{2}$

Probability of an Event

$0 \le P(E) \le 1$

$S = \{e_1, \cdots, e_i\}$

Probability of the event e_i : $P(e)$ *or* $\Pr(e)$

$$\begin{cases} 0 \le P \le 1 \\ \sum P_i = 1 \end{cases} \quad \Rightarrow \textit{Acceptable probability assignment}$$

1) If ***E*** is an empty set $\Rightarrow$ P(E) = 0

2) If ***E*** is a simple event $\Rightarrow$ P(E) has already been assigned

3) If ***E*** is a compound event $\Rightarrow$ P(E) is the sum of probability of all sample E

4) If ***E*** is a sample space $\Rightarrow$ P(E) = P(S) = 1

Example

After flipping the nickel and dime 1,000 times, we find that HH turns up 273 times, HT turns up 206 times, TH turns up 312 times, TT turns up 209 times.

Simple Event e_i	*HH*	*HT*	*TH*	*TT*
$P(e_i)$	.273	.206	.312	.209

What are the probabilities of the following events?

a) E_1 = getting at least 1 tail

b) E_2 = getting 2 tails

c) E_3 = getting at least 1 tail or at least 1 head

<u>*Solution*</u>

a) $P(E_1) = P(HT) + P(TH) + P(TT)$

$= .206 + .312 + .209$

$\underline{= .727}$

b) $P(E_2) = P(TT) \ \underline{= .209}$

c) $P(E_3) = P(HH) + P(HT) + P(TH) + P(TT)$

$\underline{= 1}$

Set Operations for Events

Let E and F be events for a sample space S.

$E\bigcup F=\{e\in S\mid e\in E \text{ or } e\in F\}$ Occurs when E **or** F **or both** occur

$E\bigcap F=\{e\in S\mid e\in E \text{ and } e\in F\}$ Occurs when both E **and** F occur

E' Occurs when E ***does not*** occur

Event

The event E **or** $F=E\bigcup F$

The event E **and** $F=E\bigcap F$

Example

A study of workers earning the minimum wage grouped such workers into various categories, which can be interpreted as events when a worker is selected at random. Consider the following events:

E: worker is under 20;
F: worker is white;
G: worker is female.

a) E'
b) $F\bigcap G'$
c) $E\bigcap G$

<u>Solution</u>

a) E' is the event that the worker is 20 or over.

b) $F\bigcap G'$ is the event that the worker is white and not a female

c) $E\bigcap G$ is the event that the worker is under 20 or is female

Union Rule For Probability

For any events E and F from a sample space S,

$$\boxed{P(E\bigcup F)=P(E)+P(F)-P(E\bigcap F)}$$

Example

If a single card is drawn from an ordinary deck of cards, find the probability that it will be a red or a face card

<u>Solution</u>

$P(Red \text{ or } Face)=P\left(R\bigcup F\right)$

$$P(R) = \boxed{\frac{26}{52}}$$

$$P(F) = \boxed{\frac{12}{52}}$$

$$P(R \cap F) = \boxed{\frac{6}{52}}$$

$$\begin{aligned} P(R \cup F) &= P(R) + P(F) - P(R \cap F) \\ &= \frac{26}{52} + \frac{12}{52} - \frac{6}{52} \\ &= \frac{32}{52} \\ &= \boxed{\frac{8}{13}} \end{aligned}$$

Example

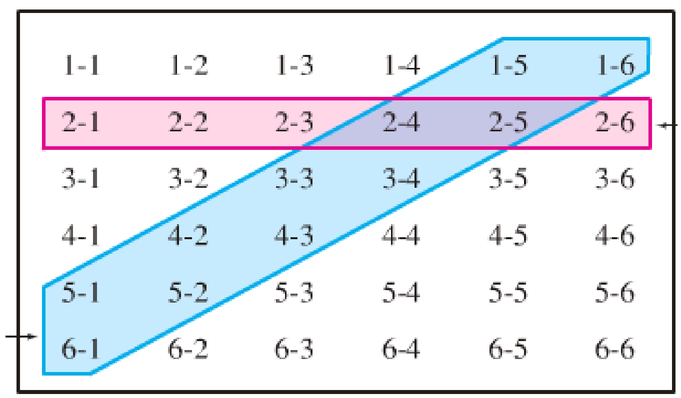
Suppose two fair dice are rolled. Find each probability

a) The first die shows a 2, or the sum of the results is 6 or 7.

b) The sum of the results is 11 or the second die shows a 5.

<u>*Solution*</u>

a) The first die shows a 2, or the sum of the results is 6 or 7.



$$P\left(2 \text{ or } \sum 6 \text{ or } 7\right) = \frac{15}{36} = \frac{5}{12}$$

b) $P\left(\sum 11 \text{ or second shows } 5\right) = \boxed{\frac{7}{36}}$

Complement Rule

E' is called the complement of E relative to S

$$E \cap E' = \varnothing, \qquad E \cup E' = S \qquad \textit{Mutually Exclusive}$$

$$\mathrm{P}(S) = \mathrm{P}(E \cup E') = \mathrm{P}(E) + \mathrm{P}(E') = 1$$

$$P(E) = 1 - P(E') \quad and \quad P(E') = 1 - P(E)$$

Example

If a fair die is rolled, what is the probability that any number but 5 will come up?

<u>*Solution*</u>

$$P(E=5) = \frac{1}{6}$$

$$\begin{aligned} P(E' = any\ but\ 5) &= 1 - P(E) \\ &= 1 - \frac{1}{6} \\ &= \frac{5}{6} \end{aligned}$$

Example

Suppose two fair dice are rolled. Find each probability that the sum of the numbers rolled is greater than 3.

<u>*Solution*</u>

$$P(sum > 3) = 1 - P(sum \le 3)$$

$$\begin{aligned} P(sum \le 3) &= P(sum\ is\ 2) + P(sum\ is\ 3) \\ &= \frac{1}{36} + \frac{2}{36} \\ &= \frac{3}{36} \\ &= \frac{1}{12} \end{aligned}$$

$$\begin{aligned} P(sum > 3) &= 1 - P(sum \le 3) \\ &= 1 - \frac{1}{12} \\ &= \frac{11}{12} \end{aligned}$$

Mutually Exclusive

Events E and F are ***mutually exclusive events*** if $E \cap F = \phi$

Mutually exclusive events are disjoint sets

If ***E*** and F are mutually exclusive (i.e. their intersection is empty), then

$$P(E \cup F) = P(E) + P(F)$$

In looking $P(E)$ always find $P(E')$ first!

Example

Let $E = \{4, 5, 6\}$ and $G = \{1, 2\}$

Then E and G are mutually exclusive events since they have no outcomes in common: $E \cap G = \phi$

Example

A shipment of 40 precision parts, including 8 that are defective, is sent to an assembly plant. The quality control division selects 10 at random for testing and rejects the entire shipment if 1 or more in the sample are found defective. What is the probability that the shipment will be rejected?

Solution

8 ***defective*** $\Rightarrow$ 32 ***non-defective***.

$n(S) = C_{40,10}$

$n(E') = C_{32,10}$

$P(E') = \dfrac{n(E')}{n(S)} \approx 0.08$

$P(E) = 1 - P(E') \underline{\approx 0.92}$

ODDS

Definition

The ***actual odds against*** event A occurring are the ratio $\frac{P(\bar{A})}{P(A)}$, usually expressed in the form of a:b (or "a to b"), where a and b are integers having no common factors.

The ***actual odds in favor*** of event A occurring are the ratio $\frac{P(A)}{P(\bar{A})}$, which is the reciprocal of the actual odds against the event. If the odds against A are a:b, then the odds in favor of A are b:a.

The ***payoff odds*** against event A occurring are the ratio of the net profit (if you win) to the amount bet.

$$payoff\ \ odds\ against\ event\ A = \frac{net\ profit}{amount\ bet}$$

$$\begin{cases} Odds\ of\ \ E = \dfrac{P(E)}{1-P(E)} = \dfrac{P(E)}{P(E')} & ,\ \ P(E) \neq 1\ or\ P(E') \neq 0 \\ Odds\ against\ E = \dfrac{P(E')}{P(E)} & ,\ \ P(E) \neq 0 \end{cases}$$

Note:
Odds are expressed as ratios of whole numbers

- If the odds favoring event ***E*** are ***m*** to ***n***, then

$$P(E) = \frac{m}{m+n} \qquad \textbf{and} \qquad P(E') = \frac{n}{m+n}$$

Example

Suppose the weather forecast says that the probability of rain tomorrow is $\frac{1}{3}$. Find the odds in favor of rain tomorrow.

Solution

$$P(E) = \frac{1}{3} \ \Rightarrow\ P(E') = 1 - \frac{1}{3} = \frac{2}{3}$$

$$Odds: \frac{1/3}{2/3} = \frac{1}{2} \quad \Rightarrow Written\text{: "1:2"} \ \ or \ \ \text{"1 } to \text{ 2"} \qquad\qquad Read\text{: 1 } to \text{ 2}$$

The odds that it will not rain:

$$Odds\ against: \frac{2/3}{1/3} = \frac{2}{1} \qquad \Rightarrow Written\text{: "2:1"} \ \ or \ \ \text{"2 } to \text{ 1"}$$

Example

The odds that a particular bid will be the low bid are 4 to 5

a) Find the probability that the bid will be the low bid.

$$P(bid\ will\ be\ low) = \frac{4}{4+5} = \frac{4}{9}$$

b) Find the odds against that bid being the low bid.

$$P(bid\ will\ \text{not}\ be\ low) = \frac{5}{4+5} = \frac{5}{9}$$

$$odds\ Against = \frac{P(bid\ will\ \text{not}\ be\ low)}{P(bid\ will\ be\ low)} = \frac{5/9}{4/9} = \frac{5}{4}$$

So, 5:4

Example

If the odds in favor of a particular horse's winning a race are 5 to 7, what is the probability that the horse will win the race?

<u>*Solution*</u>

$$P(winning) = \frac{5}{5+7} = \boxed{\frac{5}{12}}$$

Example

a) What are the odds for rolling a sum of 8 in a single roll of two fair dice?

$$P(E) = \frac{5}{36}$$

$$\Rightarrow\ P(E') = 1 - \frac{5}{36} = \frac{31}{36}$$

$$Odds\ of\ \ E = \frac{P(E)}{P(E')} = \frac{\frac{5}{36}}{\frac{31}{36}} = \boxed{\frac{5}{31}}$$

$$\Rightarrow read\ \ "\mathbf{5\ to\ 31}" \rightarrow written\ \ "5{:}31"$$

b) If you bet \$5 that sum of 8 will turn up, what should the house pay (plus returning your \$5 bet) if a sum of 8 does turn up for the game to be fair?

Odds: 5 *to* 31 $\rightarrow$ \$31

Empirical Probability

It is not possible to establish exact probabilities for events; instead, useful approximations are often found by drawing on past experience. This approach is called ***Empirical probabilities***.

If an experiment is conducted n times and event E occurs with frequency f(E), then the ratio f(E)/n is called the relative frequency of the occurrence of event E in n trials.

$$P(E) = \frac{frequency\ of\ occurrence\ of\ E}{total\ number\ of\ trials} = \frac{f(E)}{n} \rightarrow \text{Ratio called } \textbf{\textit{relative frequency}}$$

$$P(E) = \frac{number\ of\ elements\ of\ E}{number\ of\ elements\ in\ S} = \frac{n(E)}{n(S)}$$

The larger n is, the better the approximation.

Example

Let consider rolling 2 dice. Find the probabilities of the following events

a) E = Sum of 5 turns up

b) F = a sum that is a prime number greater than 7 turns up

<u>*Solution*</u>

a) $P(E) = \frac{4}{36} = \frac{1}{9}$

b) $P(F) = \frac{2}{36} = \frac{1}{18}$

$$\text{P}\left(E_1 = \sum 7\right) = \frac{n\left(E_1\right)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

$$\text{P}\left(E_2 = \sum 11\right) = \frac{n\left(E_2\right)}{n(S)} = \frac{2}{36} = \frac{1}{18}$$

Example

The following table lists U.S. advertising volume in millions of dollars by medium in 2004

Medium	Expenditures
Direct mail	52,191
Newspapers	46,614
Broadcast TV	46,264
Cable TV	21,527
Radio	19,581
Yellow pages	14,002
Magazines	12,247
Other	51,340

Find the empirical probability that a dollar of advertising is spent on each medium.

<u>*Solution*</u>

Medium	***Expenditures***	***Probabilities***
Direct mail	53,191	$\frac{53191}{263766} = 0.1979$
Newspapers	46,614	$\frac{46614}{263766} = 0.1767$
Broadcast TV	46,264	$\frac{46264}{263766} = 0.1754$
Cable TV	21,527	$\frac{21527}{263766} = 0.0816$
Radio	19,581	$\frac{19581}{263766} = 0.0742$
Yellow pages	14,002	$\frac{14002}{263766} = 0.0531$
Magazines	12,247	$\frac{12247}{263766} = 0.0464$
Other	51,340	$\frac{51340}{263766} = 0.1946$
$\sum$	263,766	$0.9999 \approx 1$

Properties of Probability

1. The probability of each outcome is a number between 0 and 1.

$$0 \le P_1 \le 1, \; 0 \le P_2 \le 1, \; \ldots, \; 0 \le P_n \le 1$$

2. The sum of the probabilities of all possible outcomes is 1.

$$p_1 + p_2 + \ldots + P_n = 1$$

Example

In a group of n people, what is the probability that at least 2 people have the same birthday (the same month and day, excluding February 29)? Evaluate *P(E)* for $n = 4$

<u>*Solution*</u>

Birthday: at least 2 same.

$$S = 365.365\cdots 365 = 365^n$$

E = at least 2 same birthday

E' no people have the same

$$n(E') = 365.364.363\cdots\big(365-(n-1)\big) = 366-n$$

$$n(E') = \frac{365!}{(365-n)!}$$

$$P(E') = \frac{n(E')}{n(S)}$$

$$= \frac{\frac{365!}{(365-n)!}}{365^n}$$

$$= \frac{365!}{365^n(365-n)!}$$

$$P(E) = 1 - P(E') = 1 - \frac{365!}{365^n(365-n)!}$$

Group of $n = 4$: $P(E) = 1 - \frac{365!}{365^4(365-4)!} \approx 0.016$

Exercises *Section* 3.5 – Probability

1. An experiment consists of recording the boy-girl composition of a two-child family.

a) What is an appropriate sample space if we are interested in the genders of the children in the order of their births? Draw a tree diagram.

b) What is an appropriate sample space if we are interested only in the *number* of the girls in a family?

2. Given $S = \{1, 2, 3, \cdots, 17, 18\}$

a) The outcome is a number divisible by 12

b) The outcome is an even number greater than 15

c) Is divisible by 4

d) Is divisible by 5

3. Consider rolling 2 Dice.

a) What is the event that a sum of 5 turns up

b) What is the event that a sum that is a prime number greater than 7 turns up

4. A single fair die is rolled. Find the probability of each event

a) Getting a 2

b) Getting an odd number

c) Getting a number less than 5

d) Getting a number greater than 2

e) Getting a 3 or a 4

f) Getting any number except 3

5. A card is drawn from a well-shuffled deck of 52 cards. Find the probability of drawing the following

a) A 9

b) A black card

c) A black 9

d) A heart

e) The 9 of hearts

f) A face card

g) A 2 or a queen

h) A black 7 or red 8

i) A red card or a 10

j) A spade or a king

6. The student sitting next to you in class concludes that the probability of the ceiling falling down on both of you before class ends is 1/2, because there are two possible outcomes - the ceiling will fall or not fall. What is wrong with this reasoning?

7. A jar contains 3 white, 4 orange, 5 yellow, and 8 black marbles. If a marble is drawn at random, find the probability that it is the following.

a) White

b) Orange

c) Yellow

d) Black

e) Not black

f) Orange or Yellow

8. Let consider rolling 2 dice. Find the probabilities of the following events

a) ***E*** = Sum of 5 turns up

b) ***F*** = a sum that is a prime number greater than 7 turns up

9. The board of regents of a university is made up of 12 men and 16 women. If a committee of 6 chosen at random, what is the probability that it will contain 4 men and 2 women?

10. In drawing 7 cards from a 52-card deck without replacement, what is the probability of getting 7 hearts.

11. A committee of 4 people is to be chosen from a group of 5 men and 6 women. What is the probability that the committee will consist of 2 men and 2 women?

E is the set of all possible ways to select 2 men and 2 women

S is the set of all possible ways to select 4 people from 11

12. A department store receives a shipment of 27 new portable radios. There are 4 defective radios in the shipment. If 6 radios are selected for display, what is the probability that 2 of them are defective?

E is the set of all possible ways to have 2 defective and 4 not defective.

S is the set of all possible ways to select 6 radios from 27.

13. Eight cards are drawn from a standard deck of cards. What is the probability that there are 4 face cards and 4 non-face cards?

E is the set of all possible ways to have 4 faces and 4 non-faces.

S is the set of all possible ways to select 8 cards from 52.

14. Five cards are drawn from a standard deck of cards. What is the probability that there are exactly 3 hearts?

15. There are 11 members on the board of directors for the Coca Cola Company.

a) If they must select a chairperson, first vice chairperson, second vice chairperson, and secretary, how many different slates of candidates are possible?

b) If they must form an ethics subcommittee of 4 members, how many different subcommittees are possible?

16. A poll was conducted preceding an election to determine the relationship between voter persuasion concerning a controversial issue and the area of the city in which the voter lives. Five hundred registered voters were interviewed from three areas of the city. The data are shown below. Compute the probability of having no opinion on the issue or living in the inner city.

Area of city	***Favor***	***Oppose***	***No Opinion***
East	30	40	55
North	25	45	50
Inner	95	65	85

17. When testing for current in a cable with five color-coded wires, the author used a meter to test two wires at a time. How many different tests are required for every possible pairing of two wires?

18. Identity theft often begins by someone discovering your 9-digit social security number. Answer each of the following. Express probabilities as fractions.

a) What is the probability of randomly generating 9 digits and getting your social security number?

b) In the past, many teachers posted grades along with the last 4 digits of your social security number, what is the probability that if they randomly generated the order digits, they would match yours? Is that something to worry about?

19. You become suspicious when a genetics researcher randomly selects groups of 20 newborn babies and seems to consistently get 10 girls and 10 boys. The researchers claims that it is common to get 10 girls and 10 boys in such cases,

a) If 20 newborn babies are randomly selected, how many different gender sequences are possible?

b) How many different ways can 10 girls and 10 boys be arranged in sequence?

c) What is the probability of getting 10 girls and 10 boys when 10 babies are born?

20. Two dice are rolled. Find the probabilities of the following events.

a) The first die is 3 or the sum is 8

b) The second die is 5 or the sum is 10.

21. One card is drawn from an ordinary of 52 cards. Find the probabilities of drawing the following cards

a) A 9 or 10

b) A red card or a 3

c) A 9 or a black 10

d) A heart or a black card

e) A face card or a diamond

22. One card is drawn from an ordinary of 52 cards. Find the probabilities of drawing the following cards

a) Less than a 4 (count aces as ones)

b) A diamond or a 7

c) A black card or an ace
d) A heart or a jack
e) A red card or a face card

23. Pam invites relatives to a party: her mother, 2 aunts, 3 uncles, 2 brothers, 1 male cousin, and 4 female cousins. If the chances of any one guest first equally likely, find the probabilities that the first guest to arrive is as follows.
a) A brother or an uncle
b) A brother or a cousin
c) A brother or her mother
d) An uncle or a cousin
e) A male or a cousin
f) A female or a cousin

24. The numbers $\{1, 2, 3, 4, and\ 5\}$ are written on slips of paper, and 2 slips are drawn at random one at a time without replacement. Find the probabilities:
a) The sum of the numbers is 9.
b) The sum of the numbers is 5 or less.
c) The first number is 2 or the sum is 6
d) Both numbers are even.
e) One of the numbers is even or greater than 3.
f) The sum is 5 or the second number is 2.

Use Venn diagrams

25. Suppose $P(E) = 0.26, \quad P(F) = 0.41, \quad P(E \cap F) = 0.16$. Find the following
a) $P(E \cup F) =$
b) $P(E' \cap F) =$
c) $P(E \cap F') =$
d) $P(E' \cup F') =$

26. Suppose $P(E) = 0.42, \quad P(F) = 0.35, \quad P(E \cap F) = 0.59$. Find the following
a) $P(E' \cap F') =$
b) $P(E' \cup F') =$
c) $P(E' \cup F) =$
d) $P(E \cap F') =$

27. A single fair die is rolled. Find the odds in favor of getting the results
a) 3
b) 4, 5, or 6
c) 2, 3, 4, or 5
d) Some number less than 6

28. If in repeated rolls of two fair dice the odds against rolling a 6 before rolling a 7 are 6 to 5, what is the probability of rolling a 6 before rolling 7?

29. From survey involving 1,000 people in the certain city, it was found that 500 people had tried a certain brand of diet cola, 600 had tried a certain brand of regular cola, and 200 had tried both types of cola. If a resident of the city is selected at random, what is the empirical probability that

a) The resident has not tried either cola? What are the empirical odds for this event?

b) The resident has tried the diet or has not tried the regular cola? What are the empirical odds against this event?

30. The odds in favor of a particular horse winning a race are 4:5.

a) Find the probability of the horse winning.

b) Find the odds against the horse winning.

31. Consider the sample space of equally likely events for the rolling of a single fair die.

a) What is the probability of rolling an odd number **and** a prime number?

b) What is the probability of rolling an odd number **or** a prime number?

32. Suppose that 2 fair Dice are rolled

a) What is the probability of that a sum of 2 or 3 turns up?

b) What is the probability of that both dice turn up the same or that a sum greater than 8 turns up?

33. A single card is drawn from an ordinary of 52 cards. Calculate the probabilities of and odds for each event

a) A face card or a club is drawn

b) A king or a heart is drawn

c) A black card or an ace is drawn

d) A heart or a number less than 7 (count an ace as 1) is drawn.

34. What is the probability of getting at least 1 black card in a 7-card hand dealt from a standard 52-card deck?

35. What is the probability that a number selected at random from the first 600 positive integers is (exactly) divisible by 6 or 9?

36. What is the probability that a number selected at random from the first 1,000 positive integers is (exactly) divisible by 6 or 8?

37. From a survey involving 1,000 students at a large university, a market research company found that 750 students owned stereos, 450 owned cars, and 350 owned cars and stereos. If a student at the university is selected at random, what is the (empirical) probability that

a) The student owns either a car or a stereo?

b) The student owns neither a car nor a stereo?

38. In order to test a new car, an automobile manufacturer wants to select 4 employees to test drive the car for 1 year. If 12 management and 8 union employees volunteer to be test drivers and the selection is made at random, what is the probability that at least 1 union employee is selected.

39. A shipment of 60 inexpensive digital watches, including 9 that are defective, is sent to a department store. The receiving department selects 10 at random for testing and rejects the whole shipment if 1 or more in the sample are found defective. What is the probability that the shipment will be rejected?

40. If you bet \$5 on the number 13 in roulette, your probability of winning is $\frac{1}{38}$ and the payoff odds are given by the casino as 35:1.

a) Find the actual odds against the outcome of 13.

b) How much net profit would you make if you win by betting on 13?

c) If the casino was not operating for profit, and the payoff odds were changed to match the actual odds against 13, how much would you win if the outcome were 13?

Section 3.6 – Conditional Probability, Independent Events

Conditional Probabilities

The probability of the occurrence of an event *A*, given the occurrence of another event B is called a ***conditional probability***.

"***Changes due the occurrence of another event***"

Example

Age > 21 $\Rightarrow$ the probability of having cancer would be too high

Class of 21 students $\Rightarrow$ passing and > 90 (*conditional*)

Conditional Probability $P(A\mid B)=\dfrac{n(A\cap B)}{n(B)}=\dfrac{\frac{n(A\cap B)}{n(S)}}{\frac{n(B)}{n(S)}}=\dfrac{P(A\cap B)}{P(B)};\quad P(B)\neq 0$

Example

A pointer is spun once, the probability assigned to the pointer landing on a given integer (1 to 6) as given in the table

e_i	1	2	3	4	5	6
$P(e_i)$	.1	.2	.1	.1	.3	.2

a) What is the probability of the pointer landing on a number greater than 4?

$$\begin{aligned}P(>4)&=P(E)\\&=P(5)+P(6)\\&=.3+.2\\&=.5\end{aligned}$$

b) What is the probability of the pointer landing on a number greater than 4 given that it landed on an even number?

E: > 4

F: even number

$$\Rightarrow P(F)=.2+.1+.2=.5$$

$$\begin{aligned}P(E\mid F)&=\frac{P(E\cap F)}{P(F)}=\frac{.2}{.5}=.4\\&=\frac{.2}{.5}=.4\\&=.4\end{aligned}$$

Intersection **of Events:** Product Rule

$$P(A|B)=\frac{P(A\cap B)}{P(B)} \qquad \Rightarrow P(A\cap B)=P(A|B)P(B)$$

$P(A)\neq 0$ & $P(B)\neq 0$

$$P(B|A)=\frac{P(A\cap B)}{P(A)} \qquad \Rightarrow P(A\cap B)=P(B|A)P(A)$$

Product Rule: $P(A\cap B)=P(A|B)P(B)=P(B|A)P(A)$

Example

If 40% 0f the department store's customers are male and 80% of the male customers of the department store have charge accounts, what is the probability that a customer selected at random is a male and has a charge account?

<u>*Solution*</u>

M: Male customer

C: Customers with a charge account

$P(M)=0.4$

$P(C|M)=0.8$

$$\begin{aligned} P(M\cap C) &= P(M)P(C|M) \\ &= (.4)(.8) \\ &= .32 \end{aligned}$$

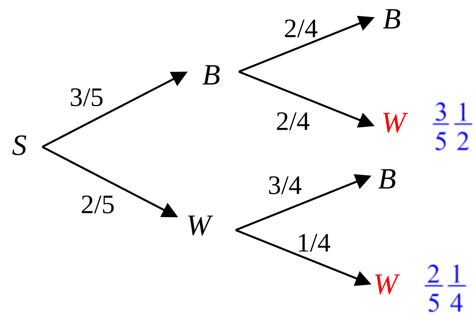
Probability Tree

Example

Two balls are drawn in succession, without replacement, from a box containing 3 blue and 2 white balls. What is the probability of drawing a white ball on the second draw?

<u>*Solution*</u>

$$P(2^{nd}\ White) = P(B \cap W) + P(W \cap W)$$
$$= \frac{1}{10} + \frac{3}{10}$$
$$= \frac{2}{5}$$

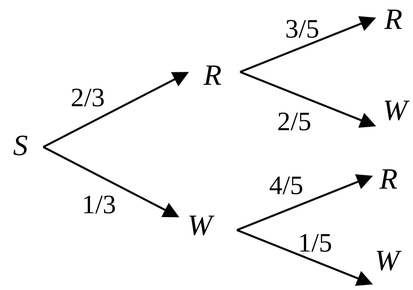


Example

Two balls are drawn in succession, without replacement, from a box containing 4 red and 2 white balls. What is the probability of drawing a red ball on the second draw?

<u>*Solution*</u>

$$P\left(2^{nd}\ Red\right) = \frac{2}{3}\frac{3}{5} + \frac{4}{5}\frac{1}{3}$$
$$= .67$$

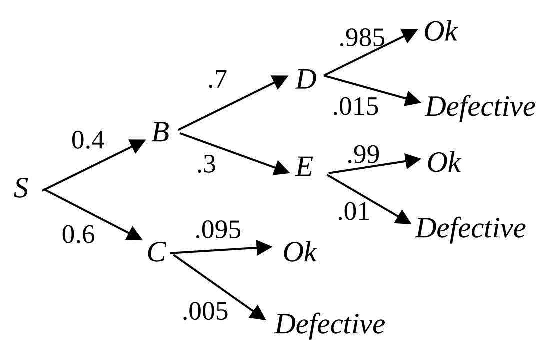


Example

A large computer company A subcontracts the manufacturing of it circuit boards to two companies, 40% to company B and 60% to company C. Company B in turn subcontracts 70% of the orders it receives from company A to company D and the remaining 30% to company E, both subsidiaries of company B. When the boards are completed by companies D, E, and C, they are shipped to company A to be used in various computer models. It has been found that 1.5%, 1%, and .5% of the boards from D, E, and C respectively, prove defective during the 90-day warranty period after a computer is first sold. What is the probability that a given board in a computer will be defective during the 90-day warranty period?

<u>*Solution*</u>

$$P\left(defective\right) = .4\left(.7\right)\left(.015\right) + .4\left(.3\right)\left(01\right) + .6\left(.005\right)$$
$$= 0.0084$$



Independent Events

A & *B* are independent if and only if $P(A \cap B) = P(A)P(B)$

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

Otherwise, *A* & *B* are said to be dependent

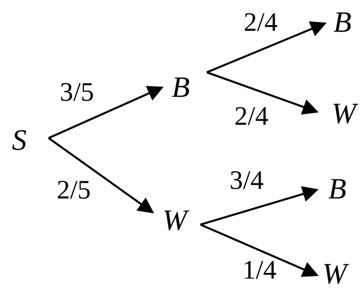
With ***or*** Without Replacement

Example

Two balls are drawn in succession, without replacement, from a box containing 3 blue and 2 white balls. What is the probability of drawing a white ball on the second draw?

Solution

Without



$$P(w) = \frac{2}{5}\frac{1}{4} + \frac{3}{5}\frac{2}{4} = 0.4$$

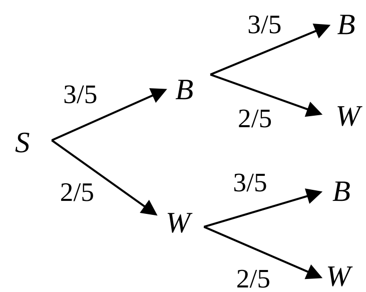
$$P\left(w \mid w_1\right) = \frac{P\left(w \cap w_1\right)}{P(w) + P\left(w_1\right)}$$

$$= \frac{\frac{2}{5}\frac{1}{4}}{\frac{2}{5}\frac{1}{4} + \frac{3}{5}\frac{2}{4}}$$

$$\underline{= 0.25|}$$

$$\Rightarrow P\left(w \mid w_1\right) \neq P(w) \ \rightarrow \ \textbf{\textit{Dependent}}$$

With



$$P(w) = \frac{2}{5}\frac{2}{5} + \frac{3}{5}\frac{2}{5} = 0.4$$

$$P(w \mid w_1) = \frac{\frac{2}{5}\frac{2}{5}}{\frac{2}{5}\frac{2}{5} + \frac{3}{5}\frac{2}{5}} = 0.4$$

$$\underline{= 0.4|}$$

$$\Rightarrow P\left(w \mid w_1\right) = P(w) \ \rightarrow \ \textbf{\textit{Independent}}$$

Example

A single card is drawn from a standard 52-card deck. Test the following events for independence:

a) E: Red card F: divisible by 5

b) G: Kings H: Queen

<u>*Solution*</u>

a) $P(E \cap F) = \frac{4}{52} = \frac{1}{13}$

$P(E) = \frac{26}{52} \quad P(F) = \frac{8}{52}$

$P(E)P(F) = \frac{26}{52}\frac{8}{52} = \frac{1}{13}$ ***Independent***

b) $P(G \cap H) = 0$

$P(G) = \frac{4}{52} \quad P(H) = \frac{4}{52}$

$P(G)P(H) = \frac{4}{52}\frac{4}{52} = \frac{1}{169}$ ***Dependent***

Independent Set of Events

$$\Rightarrow P\left(E_1 \cap E_2 \cap \cdots\right) = P\left(E_1\right) \bullet P\left(E_2\right) \bullet \cdots$$

Exercises *Section* 3.6 – Conditional Probability, Independent Events

1. In building the space shuttle, NASA contracts for certain guidance components to be supplied by three different companies: 41% by company A, 25% by company B, and 34% by company C. It has been found that 1%, 1.75%, and 2% of the components from companies A, B, and C, respectively, are defective. If one of these guidance components is selected at random, what is the probability that it is defective?

2. Suppose the probability of A is $P(A)=\frac{1}{4}$ and the probability of B is $P(B)=\frac{2}{3}$. What would the probability of A intersect B need to be for A and B to be independent events?

3. In 2 throws of a fair die, what is the probability that you will get at least 5 on each throw? At least 5 on the first or second throw?

4. 2 balls are drawn in succession out a box containing 2 red and 5 white balls. Find the probability that the second ball was red, given that the first ball was

a) Replaced before the second draw

b) Not replaced before the second draw

5. 2 balls are drawn in succession out a box containing 2 red and 5 white balls. Find the probability that at least 1 ball was red, given that the first ball was

a) Replaced before the second draw

b) Not replaced before the second draw

6. 2 balls are drawn in succession out a box containing 2 red and 5 white balls. Find the probability that both balls were the same color, given that the first ball was

a) Replaced before the second draw

b) Not replaced before the second draw

7. An automobile manufacturer produces 37% of its cars at plant A. If 5% of the cars manufactured at plant A have defective emission control devices, what is the probability that one of this manufacturer's cars was manufactured at plant A and has a defective emission control device?

8. To transfer into a particular department, a company requires an employee to pass a screening test. A maximum of 3 attempts are allowed at 6-month intervals between trials. From past records it is found that 40% pass on the first trial; of those that fail the first trial and take the test a second time, 60% pass; and of those that fail on the second trial and take the test a third time, 20% pass. For an employee wishing to transfer:

a) What is the probability of passing the test on the first or second try?

b) What is the probability of failing on the first 2 trials and passing on the third?

c) What is the probability of failing on all 3 attempts?

9. A survey of the residents of a precinct in a large city revealed that 55% of the residents where members of the Democratic party and that 60% of the Democratic party members voted in the last election. What is the probability that a person selected at random from the residents of this precinct is a member of the Democratic party and voted in the last election?

Section 3.7 – Probability Applications of Counting Principles

$$P(E) = \frac{n(E)}{n(S)}$$

Example

The Environment Protection Agency is considering inspecting 6 plants for environment compliance: 3 in Chicago, 2 in Los Angeles, and 1 NY. Due to a lack of inspectors, they decide to inspect 2 plants selected at random, 1 this month and 1 next month, with each plant equally likely to be selected, but no plant is selected twice. What is the probability that 1 Chicago plant and 1 Los Angeles plant are selected?

<u>*Solution*</u>

Chicago plant can be selected from 3 in $\binom{3}{1}$ ways.

Los Angeles plant can be selected from 2 in $\binom{2}{1}$ ways.

$n(S)$ = (# ways selected 1st month) (# ways selected 2nd month)

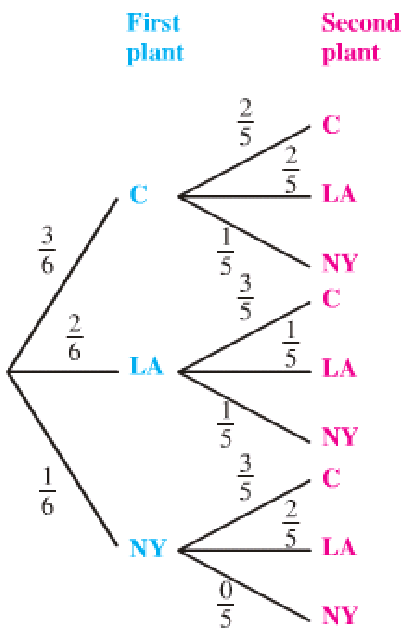
= 6 . 5

= 30 ways

$$P(1C \text{ and } 1LA) = \frac{\binom{3}{1}\binom{2}{1} + \binom{2}{1}\binom{3}{1}}{30} = 0.4$$

OR

$$P(1C \text{ and } 1LA) = \frac{3}{6}\frac{2}{5} + \frac{2}{6}\frac{3}{5} = 0.4$$



Example

From a group of 22 nurses, 4 are to be selected to present a list of grievances to management.

a) In how many ways can this be done?

b) One of the nurses is Julie. Find the probability that Julie will be among the 4 selected.

c) Find the probability that Julie will not be selected

<u>*Solution*</u>

a) $C_{22,4} \ \underline{= 7,315}$

b) $P(\text{Julie is chosen}) = \frac{n(E)}{n(S)} = \frac{\binom{1}{1}\binom{21}{3}}{\binom{22}{4}} \approx 0.1818$

c) $P(\text{Julie is not chosen}) = 1 - 0.1818 \ \underline{= 0.8182}$

Example

When shipping diesel engines abroad, it is common to pack 12 engines in one container that is then loaded on a rail car and sent to a port. Suppose that a company has received complaints from its customers that many of the engines arrive in nonworking condition. To help solve this problem, the company decides to make a spot check of containers after loading. The company will test 3 engines from a container at random; if any of the 3 is nonworking, the container will not be shipped until each engine in it is checked. Suppose a given container has 2 nonworking engines. Find the probability that the container will not be shipped.

<u>*Solution*</u>

$$\begin{aligned}\Pr(\textit{not shipping}) &= \Pr(1\ \textit{defective}) + \Pr(2\ \textit{defective}) \\ &= \frac{\binom{2}{1}\binom{10}{2}}{\binom{12}{3}} + \frac{\binom{2}{2}\binom{10}{1}}{\binom{12}{3}} \\ &= 0.4545\end{aligned}$$

OR

$$\begin{aligned}\Pr(\textit{not shipping}) &= 1 - \Pr(0\ \textit{defective}) \\ &= 1 - \frac{\binom{2}{0}\binom{10}{3}}{\binom{12}{3}} \\ &= 0.4545\end{aligned}$$

Example

In a common form of the card game poker, a hand of 5 cards is dealt to each player from a deck of 52 cards. Find the probability of getting each of the following hands.

a) A hand containing only hearts, called a heart flush

$$P(\textit{heart flush}) = \frac{\binom{13}{5}\binom{39}{0}}{\binom{52}{5}} \approx 0.0004952$$

b) A flush of any suit (5 cards of the same suit)

$$P(\textit{flush}) = 4(0.0004952) = 0.001981$$

c) A full house of aces and eights (3 aces & 2 eights)

$$P(3\ \textit{aces},\ 2\ \textit{eights}) = \frac{\binom{4}{3}\binom{4}{2}\binom{44}{0}}{\binom{52}{5}} \approx 0.000009234$$

Example

A music teacher has 3 violin pupils, Fred, Carl, and Helen. For a recital, the teacher selects a first violinist and a second violinist. The third pupil will play with the others, but not solo. If the teacher selects randomly, what is the probability that Helen is the first violinist, Carl is second violinist, and Fred does not solo?

<u>*Solution*</u>

$$P(3,3)=6$$

$$P=\frac{1}{6}$$

Example

Ray and Nate are arranging a row of fruit at random on a table. They have 5 apples, 6 oranges, and 7 lemons. What is the probability that all fruit of the same kind are together?

<u>*Solution*</u>

To arrange all 18 pieces of fruit $=18!$ *ways*

To arrange 3 kinds of fruit $=3!$ *ways*

$$P(\text{all fruit of the same kind are together})=\frac{3!5!6!7!}{18!}=0.4081\times 10^{-7}$$

Exercises *Section* 3.7 – Probability Applications of Counting Principles

1. A basket contains 7 red apples and 4 yellow apples. A sample of 3 apples is drawn. Find the probabilities that the sample contains the following.

a) All red apples

b) All yellow apples

c) 2 yellow and 1 red apple

d) More red than yellow apples

2. Two cards are drawn at random from an ordinary deck of 52. How many 2-card hands are possible?

3. Find the probability that the 2-card hand contains the following.

a) 2 aces

b) At least 1 ace

c) All spades

d) 2 cards of the same suit

e) Only face cards

f) No face cards

g) No card higher than 8 (count ace as 1)

4. A reader wrote to the "Ask Marilyn" column in a magazine. "You have six envelopes to pick from. Two-thirds (= 4) are empty. One-third (= 2) contain a \$100 bill. You're told to choose 2 envelopes at random. Which is more likely: **(1)** that you'll get at least one \$100 bill, or **(2)** that you'll get no \$100 bill at all?" Find the two probabilities.

5. After studying all night for a final exam, a bleary-eyed student randomly grabs 2 socks from a drawer containing 9 black, 6 brown, and 2 blue socks, all mixed together. What is the probability that she grabs a matched pair?

6. At a conference of writers, special-edition books were selected to be given away in contests. There were 9 books written by Hughes, 5 books by Baldwin, and 7 books by Morrison. The judge of one contest selected 6 books at random for prizes. Find the probabilities that he selection consisted of the following.

a) 3 Hughes and 3 Morrison books

b) Exactly 4 Baldwin books

c) 2 Hughes, 3 Baldwin, and 1 Morrison book

d) At least 4 Hughes books

e) Exactly 4 books written by males (Morrison is female)

f) No more than 2 books written by Baldwin

7. A school in Bangkok requires that students take an entrance examination. After the examination, there is a drawing in which 5 students are randomly selected from each group of 40 for automatic acceptance into the school, regardless of their performance on the examination. The drawing consists of pacing 35 red and 5 green pieces of paper into a box. Each student picks a piece of paper from the box and then does not return the piece of paper to the box. The 5 lucky students who pick the green pieces are automatically accepted into the school.

a) What is the probability that the first person wins automatic acceptance?
b) What is the probability that the last person wins automatic acceptance?
c) If the students are chosen by the order of their seating does this give the student who goes first a better chance of winning than the second, third… person?

(*Hint*: Imagine that the 40 pieces of paper have been mixed up and laid in a row so that the first student picks the first piece of paper, the second student picks the second piece of paper, and so on.)

8. A controversy arose in 1992 over the Teen Talk Barbie doll, each of which was programmed with four saying randomly picked from a set of 270 sayings. The controversy was over the saying, "Math class is tough," which some felt gave a negative message toward girls doing well in math. In an interview with Science, a spokeswoman for Mattel, the makers of Barbie, said that "There is a less than 1% chance you're going to get a doll that says math class is tough". Is this figure correct? If not, give the correct figure.

9. Bingo has become popular in the U.S., and it is an efficient way for many organizations to raise money. The bingo card has 5 rows and 5 columns of numbers from 1 to 75, with the center given as a free cell. Balls showing one of the 75 numbers are picked at random from a container. If the drawn number appears on a player's card, then the player covers the number. In general, the winner is the person who first has a card with an entire row, column, or diagonal covered.

a) Find the probability that a person will win bingo after just four numbers are called.
b) An L occurs when the first column and the bottom row are both covered. Find the probability that an L will occur in the fewest number of calls.
c) An X-out occurs when both diagonals are covered. Find the probability than an X-out occurs in the fewest number of calls.
d) If bingo cards are constructed so that column one has 5 of the numbers from 1 to 15, column two has 5 of the numbers from 16 to 30, column three has 4 of the numbers from 31 to 45, column four has 5 of the numbers from 46 to 60, column five has 5 of the numbers from 61 to 75, how many different bingo cards could be constructed? (*Hint*: Order matters!)

Section 3.8 – Bayes' Theorem

A continuation of Conditional Probability, try to find the probability of earlier event conditioned on the occurrence of a later event.

Example

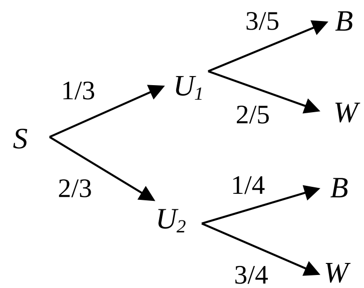
One urn has 3 blue and 2 white balls; a second urn has 1 blue and 3 white balls. A single fair die is rolled and if 1 or 2 comes up, a ball is drawn out of the first urn; otherwise, a ball is drawn out of the second urn. If the drawn ball is blue, what is the probability that it came out of the first urn? Out of the second urn?

<u>Solution</u>

$$\begin{cases} U_1 \to 3\ Blue,\ 2\ White \\ U_2 \to 1\ Blue,\ 3\ White \end{cases}$$

Fair Die

$$\begin{cases} if\ \ 1\ or\ 2 \to (1) & P = \frac{2}{6} = \frac{1}{3} \\ otherwise \to (2) & P = \frac{4}{6} = \frac{2}{3} \end{cases}$$



$$P(1^{st} Urn\ has\ tale\ B) = P\left(U_1 \mid B\right)$$

$$= \frac{P\left(U_1 \cap B\right)}{P(B)}$$

$$P\left(U_1 \mid B\right) = \frac{P\left(U_1 \cap B\right)}{P\left(U_1 \cap B\right) + P\left(U_2 \cap B\right)}$$

$$= \frac{\frac{3}{5}\frac{1}{3}}{\frac{3}{5}\frac{1}{3} + \frac{2}{3}\frac{1}{4}}$$

$$\underline{\approx .55|}$$

$$P\left(U_2 \mid B\right) = \frac{P\left(U_2 \cap B\right)}{P\left(U_2 \cap B\right) + P\left(U_1 \cap B\right)}$$

$$= \frac{\frac{1}{4}\frac{2}{3}}{\frac{3}{5}\frac{1}{3} + \frac{2}{3}\frac{1}{4}}$$

$$\underline{\approx .45|}$$

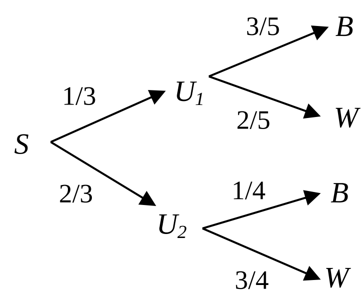
Example

One urn has 3 blue and 2 white balls; a second urn has 1 blue and 3 white balls. A single fair die is rolled and if 1 or 2 comes up, a ball is drawn out of the first urn; otherwise, a ball is drawn out of the second urn. If the drawn ball is white, what is the probability that it came out of the first urn? Out of the second urn?

<u>*Solution*</u>

Fair Die

$$\begin{cases} if\ 1\,or\ 2\ \ (\text{n} = 2) \to P = \frac{2}{6} = \frac{1}{3} \\ \quad otherwise \to P = \frac{4}{6} = \frac{2}{3} \end{cases}$$



$$P\left(U_1\,|W\right) = \frac{\frac{1}{3}\ \frac{2}{5}}{\frac{1}{3}\ \frac{2}{5} + \frac{2}{3}\ \frac{3}{4}}$$

$$= \frac{4}{19}$$

$$\underline{\approx .21}$$

$$P\left(U_2\,|W\right) = \frac{\frac{2}{3}\ \frac{3}{4}}{\frac{1}{3}\ \frac{2}{5} + \frac{2}{3}\ \frac{3}{4}}$$

$$= \frac{15}{19}$$

$$\underline{\approx .79}$$

Bayes' Formula

$$P\left(U_1 \mid E\right) = \frac{P\left(U_1 \cap E\right)}{P(E)}$$

$$= \frac{P\left(U_1 \cap E\right)}{P\left(U_1 \cap E\right) + P\left(U_2 \cap E\right) + \cdots}$$

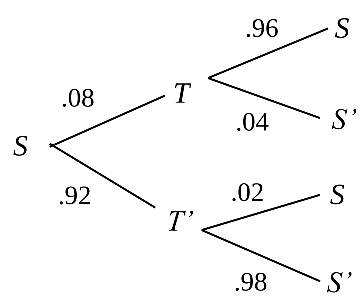
$$= \frac{P\left(E \mid U_1\right)P\left(U_1\right)}{P\left(E \mid U_1\right)P\left(U_1\right) + P\left(E \mid U_2\right)P\left(U_2\right) + \cdots}$$

Example

A new, inexpensive skin test is devised for detecting tuberculosis. To evaluate the test before it is put into use, a medical researcher randomly selects 1,000 people. Using the precise but more expensive methods already available, it is found that 8% of the 1,000 people have tested tuberculosis. Now each of the 1,000 subjects is given the new skin test and the following results are recorded: The test indicates tuberculosis in 96% of those who have it and in 2% of those who do not. Based on these results,

a) What is the probability of a randomly chosen person having tuberculosis given that the skin test indicates the disease?
b) What is the probability of a person not having tuberculosis given that the skin test indicates the disease?
c) What is the probability that a person has tuberculosis given that the test indicates no tuberculosis is present?
d) What is the probability that a person does not have tuberculosis given that the test indicates no tuberculosis is present?

<u>*Solution*</u>



a) $P(T \mid S) = \dfrac{(.08)(.96)}{(.08)(.96)+(.92)(.02)} \; \boxed{= .81}$

b) $P(T' \mid S) = 1 - P(T \mid S) \; \boxed{= .19}$

c) $P(T \mid S') = .004$

d) $P(T' \mid S') = .996$

Example

A company produces 1,000 refrigerators a week at three plants. Plant *A* produces 350 refrigerators a week, plant *B* produces 250 refrigerators a week, and plant *C* produces 400 refrigerators a week. Production records indicate that 5% of the refrigerators produced at plant *A* will be defective, 3% of the refrigerators produced at plant *B* will be defective, 7% of the refrigerators produced at plant *C* will be defective. All the refrigerators are shipped to a central warehouse. If a refrigerator at the warehouse is found to be defective,

a) What is the probability that it was produced at plant *A*?

b) What is the probability that it was produced at plant *B*?

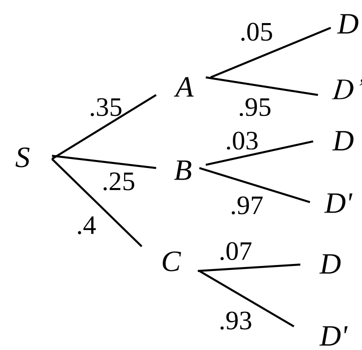
c) What is the probability that it was produced at plant *C*?

<u>*Solution*</u>

a) $$P(A\mid D)=\frac{(.35)(.05)}{(.35)(.05)+(.25)(.03)+(.4)(.07)}=.33$$

b) $$P(B\mid D)=\frac{(.25)(.03)}{(.35)(.05)+(.25)(.03)+(.4)(.07)}=.14$$

c) $$P(C\mid D)=\frac{(.4)(.07)}{(.35)(.05)+(.25)(.03)+(.4)(.07)}=.53$$



Exercises *Section* 3.8 – Bayes' Theorem

1. One urn has 4 red balls and 1 white ball; a second urn has 2 red balls and 3 white balls. A single card is randomly selected from a standard deck. If the card is less than 5 (aces count as 1), a ball is drawn out of the first urn; otherwise a ball is drawn out of the second urn. If the drawn ball is red, what is the probability that it came out of the second urn?

2. A small manufacturing company has rated 75% of its employees as satisfactory (S) and 25% as unsatisfactory (S'). Personnel records show that 80% of the satisfactory workers had previous work experience (E) in the job they are now doing, while 15% of the unsatisfactory workers had no work experience (E') in the job they are now doing. If a person who has had previous work experience is hired, what is the approximate empirical probability that this person will be an unsatisfactory employee?

3. A basketball team is to play two games in a tournament. The probability of winning the first game is .10. If the first game is won, the probability of winning the second game is .15. If the first game is lost, the probability of winning the second game is .25. What is the probability the first game was won if the second game is lost?

4. To evaluate a new test for detecting Hansen's disease, a group of people 5% of which are known to have Hansen's disease are tested. The test finds Hansen's disease among 98% of those with the disease and 3% of those who don't. What is the probability that someone testing positive for Hansen's disease under this new test actually has it?

5. An urn contains 4 red and 5 white balls. Two balls are drawn in succession without replacement. If the second ball is white, what is the probability that the first ball was white?

6. An urn contains 4 red and 5 white balls. Two balls are drawn in succession without replacement. If the second ball is red, what is the probability that the first ball was red?

7. Urn 1 contains 7 red and 3 white balls. Urn 2 contains 4 red and 5 white balls. A ball is drawn from urn 1 and placed in urn 2. Then a ball is drawn from urn2. If the ball drawn from urn 2 is red, what is the probability that the ball drawn from urn 1 was red?

8. Urn 1 contains 7 red and 3 white balls. Urn 2 contains 4 red and 5 white balls. A ball is drawn from urn 1 and placed in urn 2. Then a ball is drawn from urn2. If the ball drawn from urn 2 is white, what is the probability that the ball drawn from urn 1 was white?

9. A company has rated 75% of its employees as satisfactory and 25% as unsatisfactory. Personnel records indicate that 80% of the satisfactory workers had previous work experience, while only 40% of the unsatisfactory workers had any previous work experience. If a person with previous work experience is hired, what is the probability that this person will be a satisfactory employee? If a person with no previous work experience is hired, what is the probability that this person will be a satisfactory employee?

10. A manufacturer obtains clock-radios from three different subcontractors: 20% from A, 40% from B, and 40% from C. The defective rates for these subcontractors are 1%, 3%, and 2%, respectively. If a defective clock-radio is returned by a customer, what is the probability that it came from subcontractor A? From B? From C?

11. A computer store sells three types of microcomputer, brand A, brand B, brand C. Of the computers sell, 60% are brands A, 25% are brand B, 15% are brand C. They have found that 20% of the brand A computers, 15% of the brand B computers, and 5% of the brand C computers are returned for service during the warranty period. If a computer is returned for service during the warranty period, what is the probability that it is a brand A computer, A brand B computer? A brand C computer?

12. A new, simple test has been developed to detect a particular type of cancer. The test must be evaluated before it is put into use. A medical researcher selects a random sample of 1,000 adults and finds (by other means) that 2% have this type of cancer. Each of the 1,000 adults is given test, and it is found that the test indicates cancer in 98% of those who have it and in 1% of those who do not. Based on these results, what is the probability of a randomly chosen person having cancer given that the test indicates cancer? Of a person having cancer given that the test does not indicate cancer?

13. In a random sample of 200 women who suspect that they are pregnant, 100 turn out to be pregnant. A new pregnancy test given to these women indicated pregnancy in 92 of the 100 pregnant women and in 12 of the 100 non-pregnant women. If a woman suspects she is pregnant and this test indicates that she is pregnant, what is the probability that she is pregnant? If the test indicates that she is not pregnant, what is the probability that she is not pregnant?

14. One of two urns is chosen at random with one as likely to be chosen as the other. Then a ball is drawn from the chosen urn, Urn 1 contains 1 white and 4 red balls, and urn 2 has 3 white and 2 red balls.

a) If a white ball is drawn, what is the probability that it came from urn 1?
b) If a white ball is drawn, what is the probability that it came from urn 2?
c) If a red ball is drawn, what is the probability that it came from urn 2?
d) If a red ball is drawn, what is the probability that it came from urn 1?