Section 1.4 – Inverse Matrices - Finding A^{-1}

Definition

The matrix A is invertible if there exists a matrix A^{-1} such that:

$$A^{-1}A = AA^{-1} = I$$

where A^{-1} *read as* "A ***inverse***" and A has to be a ***square matrix***.

Not all matrices have inverses.

1. The inverse exists *iff* elimination produces n pivots (row exchanges allow).

2. The matrix A cannot have two different inverses.

3. If A is invertible, the one and only one solution to $Ax = B$ is $x = A^{-1}B$

$$AX = B$$

$$A^{-1}(AX) = A^{-1}B \qquad \textit{Multiply both side by } A^{-1}$$

$$\left(A^{-1}A\right)X = A^{-1}B \qquad \textit{Associate property}$$

$$IX = A^{-1}B \qquad \textit{Multiplicative inverse property}$$

$$X = A^{-1}B \qquad \textit{Identity property}$$

4. Suppose there is a ***nonzero*** vector x such that $Ax = 0$. Then A cannot have an inverse

5. A 2 by 2 matrix is invertible iff $ad - bc$ is not zero.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \textit{Only for 2 by 2 matrices}$$

If $ad - bc = 0$ is the determinant, then A^{-1} doesn't exist

The Inverse of a Product AB

Theorem

If an $n \times n$ matrix has an inverse, that inverse is unique.

Proof

Suppose that A has an inverse A^{-1} and B is a matrix such that $BA = I$

$$B = BI = B\left(AA^{-1}\right) = (BA)A^{-1} = IA^{-1} = A^{-1}$$

Theorem

If A and B are invertible then so is AB. The inverse of a product AB is $(AB)^{-1} = B^{-1}A^{-1}$

Proof

$$\begin{aligned}(AB)\left(B^{-1}A^{-1}\right) &= A\left(BB^{-1}\right)A^{-1} \\ &= (AI)A^{-1} \\ &= AA^{-1} \\ &= \underline{I}\end{aligned}$$

Reverse Order

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

Theorem

If A is invertible and n is a nonnegative integer, then:

a) A^{-1} is *invertible* and $\left(A^{-1}\right)^{-1} = A$

b) A^{n} is *invertible* and $\left(A^{n}\right)^{-1} = A^{-n} = \left(A^{-1}\right)^{n}$

c) kA is *invertible* for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$

Proof

$$(kA)\left(k^{-1}A^{-1}\right) = k^{-1}(kA)A^{-1} = \left(k^{-1}k\right)AA^{-1} = (1)I = I$$

$$\left(k^{-1}A^{-1}\right)(kA) = k^{-1}\left(kA^{-1}\right)A = \left(k^{-1}k\right)A^{-1}A = (1)I = I$$

Finding A^{-1} using Gauss-Jordan Elimination

$$\left[A \middle| I\right] \to \left[I \middle| A^{-1}\right]$$

Find A^{-1} *if* $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccccc} 2 & -2 & -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -2 & 0 & 0 \\ \hline 0 & -2 & -3 & -2 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 3 & 0 & 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & -3 & 0 & 0 \\ \hline 0 & 0 & -3 & -3 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \quad -\tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ -\tfrac{1}{3}R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - \frac{3}{2}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & \frac{1}{3} \\ \hline 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{array} \qquad \begin{array}{cccccc} 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2} \\ \hline 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

- ✓ Matrix ***A*** is ***symmetric*** across its main diagonal. So is A^{-1}
- ✓ Matrix ***A*** is ***tridiagonal*** (only three nonzero diagonals). But A^{-1} is a full matrix with no zeros. (another reason we don't compute A^{-1})

Singular *versus* Invertible

A^{-1} exists when A has a full set of n pivots. (Row exchanges allowed)

- With n pivots, elimination solves all the equations $Ax_i = b_i$. The columns x_i go into A^{-1}.

  Then $AA^{-1} = I$ is at least a ***right-inverse***.

- Elimination is really a sequence of multiplications.

Conclusion

- If A doesn't have n pivots, elimination will lead to a ***zero row***.
- Elimination steps are taken by an invertible M. So a row of MA is zero.
- If $AB = I$ then $MAB = M$. The zero row of MA, times B, gives a zero row of M.
- The invertible matrix M can't have a zero row! A must have n pivots if $AB = I$.

Elementary Matrices

Definition

Let ***e*** be an elementary row operation. Then the $n \times n$ ***elementary matrix*** E associated with ***e*** is the matrix obtained by applying ***e*** to the $n \times n$ identity matrix. Thus

$$E = eI$$

Example

a) $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \rightarrow$ *Multiply R_2 of I by -3*

b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow$ *Multiply the third row by -5*

c) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ *Interchange the first and second rows*

d) $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ *Add -3 times R_1 to R_2*

Theorem

Let e be an elementary operation and let E be the corresponding elementary matrix $E = e(I)$. Then for every $m \times n$ matrix A

$$e(A) = EA$$

That is, an elementary row operation can be performed on A by multiplying A on the left by the corresponding elementary matrix.

Example $m \times m$

Let $A = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$ $\quad M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

$$MA = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 6 & 2 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$

This result can be obtained from A by multiplying the first row by 2.

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 & 1 \\ -6 & 2 & 1 & 5 \\ 1 & 0 & 2 & 4 \end{bmatrix}$$

This result can be obtained from A by interchanging rows 2 and 3.

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ 0 & -4 & 10 & 8 \end{bmatrix}$$

This result can be obtained from A by adding 3 times row 1 to row 3.

Uniqueness of Echelon Form

Two matrices A and B are row-equivalent if and only if they have the same reduced echelon form.

Proof

If A and B have the same reduced echelon form E, then A is row-equivalent to E and E is row-equivalent to B. It follows that A is row-equivalent to B.

Now Suppose A and B are row-equivalent. Let E_1 be a reduced echelon form of A and E_2 be a reduced echelon form of B. Then E_1 and E_2 are row equivalent.

Suppose $E_1 = IF_1 \quad and \quad E_2 = IF_2$. Since E_1 and E_2 are row equivalent, $E_2 = CE_1$ for some matrix C. This means $I = CI$ and $F_2 = CF_1$. But then $C = I \quad and \quad F_2 = F_1$.

Example

Show that the two matrices are row equivalent

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

Solution

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{matrix} R_1 + R_2 \\ R_2 - 2R_1 \end{matrix}$$

$$= \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

$$= B$$

Definition

A relationship ~ (equivalent) between elements of a set is called an equivalence relation if

- ✓ $A \sim A$ is always true,
- ✓ $A \sim B$ always implies $B \sim A$,
- ✓ $A \sim B$ and $B \sim C$ always implies $A \sim C$.

Exercises Section 1.4 – Inverse Matrices - Finding A^{-1}

1. Apply Gauss-Jordan method to find the inverse of this triangular "Pascal matrix"

Triangular Pascal matrix $A=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix}$

2. If A is invertible and $AB=AC$, prove that $B=C$

3. If $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$, find two matrices $B\neq C$ such that $AB=AC$

4. If A has ***row*** 1 + ***row*** 2 = ***row*** 3, show that A is not invertible

a) Explain why $Ax=(1,\ 0,\ 0)$ can't have a solution.

b) Which right sides $(b_1,\ b_2,\ b_3)$ might allow a solution to $Ax=b$

c) What happens to ***row*** 3 in elimination?

5. True or false (with a counterexample if false and a reason if true):

a) A 4 by 4 matrix with a row of zeros is not invertible.
b) A matrix with 1's down the main diagonal is invertible.
c) If A is invertible then A^{-1} is invertible.
d) If A is invertible then A^2 is invertible.

6. Do there exist 2 by 2 matrices A and B with real entries such that $AB-BA=I$, where I is the identity matrix?

7. If B is the inverse of A^2, show that AB is the inverse of A.

8. Find and check the inverses (assuming they exist) of these block matrices.

$$\begin{bmatrix}I&0\\C&I\end{bmatrix}\quad\begin{bmatrix}A&0\\C&D\end{bmatrix}\quad\begin{bmatrix}0&I\\I&D\end{bmatrix}$$

9. For which three numbers c is this matrix not invertible, and why not?

$$A=\begin{bmatrix}2&c&c\\c&c&c\\8&7&c\end{bmatrix}$$

10. Find A^{-1} and B^{-1} (if they exist) by elimination.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

11. Find A^{-1} using the Gauss-Jordan method, which has a remarkable inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

12. Find the inverse.

a) $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$

c) $\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

e) $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$

f) $\begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

g) $\begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}$

13. Show that A is not invertible for any values of the entries

$$A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

14. Prove that if A is an invertible matrix and B is row equivalent to A, then B is also invertible.

15. Determine if the given matrix has an inverse, and find the inverse if it exists. Check your answer by multiplying $A \cdot A^{-1} = I$

a) $\begin{bmatrix} 2 & 3 \\ -3 & -5 \end{bmatrix}$ *b)* $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}$

16. Show that the inverse of $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is $\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$

17. If the product $C = AB$ is invertible (and *A* & *B* are square matrices), find a formula for A^{-1} that involves C^{-1} and *B*.
Hence, it is not possible to multiply a non-invertible matrix by another matric and obtain an invertible matrix as a result.

18. Prove that if *A* is an $m \times n$ matrix, there is an invertible matrix *C* such that CA is in reduced row-echelon form.

19. Prove that 2 $m \times n$ matrices *A* and *B* are row equivalent if and only if there exists a nonsingular matrix *P* such that $B = PA$

20. Let *A* and *B* be 2 $m \times n$ matrices. Suppose *A* is row equivalent to *B*. Prove that *A* is nonsingular if and only if *B* is nonsingular.

21. Show that if *A* and *B* are two $n \times n$ invertible matrices then *A* is row equivalent to *B*.

22. Prove that a square matrix *A* is nonsingular if and only if *A* is a product of elementary matrices.

23. Show that if $A \sim B$ (that is, if they are row equivalent), then $EA = B$ for some matrix *E* which is a product of elementary matrices.

24. Show that if $EA = B$ for some matrix *E* which is a product of elementary matrices, then $AC \sim BC$ for every $n \times n$ matrix *C*.

25. Let $A\vec{x} = 0$ be a homogeneous system of *n* linear equations in *n* unknowns that has only the trivial solution. Show that of *k* is any positive integer, then the system $A^k\vec{x} = 0$ also has only trivial solution.

26. Let $A\vec{x} = 0$ be a homogeneous system of *n* linear equations in *n* unknowns, and let *Q* be an invertible $n \times n$ matrix. Show that $A\vec{x} = 0$ has just trivial solution if and only if $(QA)\vec{x} = 0$ has just trivial solution.

27. Let $A\vec{x} = b$ be any consistent system of linear equations, and let $\vec{x}_1$ be a fixed solution. Show that every solution to the system can be written in the form $\vec{x} = \vec{x}_1 + \vec{x}_0$ where $\vec{x}_0$ is a solution to $A\vec{x} = 0$. Show also that every matrix of this form is a solution.

28. If A and B are $n \times n$ matrices satisfying $A^2 = B^2 = (AB)^2 = I_n$. Prove that $AB = BA$.

29. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix}$. Verify that $A^3 = 5I$, then find A^{-1} in term of A.