Section 2.4 – Chain Rule

Functions of Two Variables

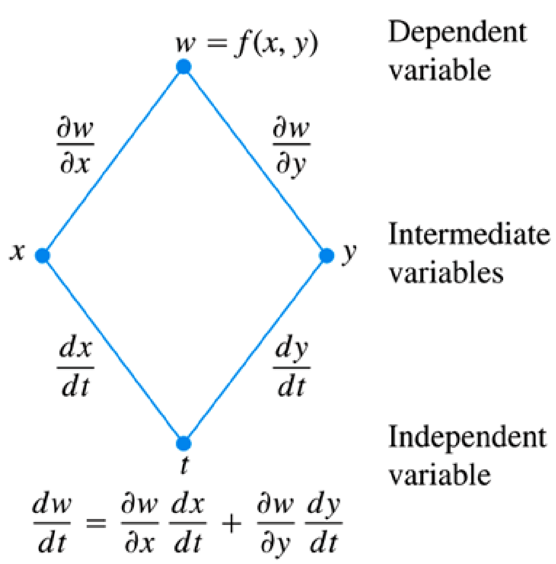
Theorem – Chain Rule for Functions of Two Independent Variables

If $w = f(x, y)$ is differentiable and if $x = x(t),\ y = y(t)$ are differentiable functions of *t*, then the composite $w = f(x(t), y(t))$ is a differentiable function of *t* and

$$\frac{dw}{dt} = f_x(x(t), y(t)) \bullet x'(t) + f_y(x(t), y(t)) \bullet y'(t)$$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Chain Rule



Example

Use the Chain Rule to find the derivative of $w = xy$ with respect to *t* along the path $x = \cos t, \quad y = \sin t$. What is the derivative's value at $t = \frac{\pi}{2}$?

Solution

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial(xy)}{\partial x}\frac{d}{dt}(\cos t) + \frac{\partial(xy)}{\partial y}\frac{d}{dt}(\sin t) \\
&= y(-\sin t) + x(\cos t) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) \\
&= -\sin^2 t + \cos^2 t
\end{aligned}$$

$= \cos 2t$

$$w = xy$$
$$= \cos t \sin t$$
$$= \frac{1}{2}\sin 2t$$

$$\frac{dw}{dt} = \frac{1}{2}(2\cos 2t)$$
$$= \cos 2t$$

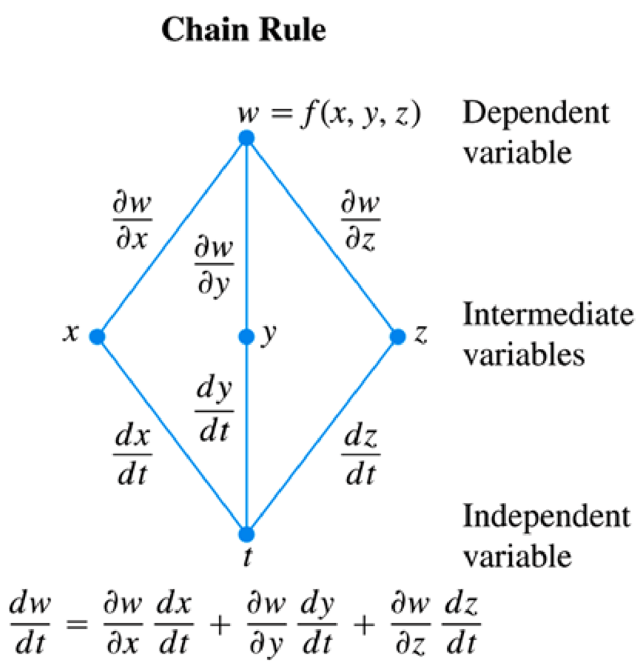
$$\left.\frac{dw}{dt}\right|_{t=\pi/2} = \cos 2\left(\frac{\pi}{2}\right)$$
$$= \cos(\pi)$$
$$= -1$$

Functions of Three Variables

Theorem – Chain Rule for Functions of Three Independent Variables

If $w = f(x, y, z)$ is differentiable and if $x, y, and\ z$ are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$



Example

Find $\frac{dw}{dt}$ if $w = xy + z, \quad x = \cos t, \quad y = \sin t, \quad z = t$

In this example the values of $w(t)$ are changing along the path of a helix as t changes. What is the derivative's value at $t = 0$?

Solution

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= (y)(-\sin t) + (x)(\cos t) + (1)(1)$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1$$

$$= -\sin^2 t + \cos^2 t + 1$$

$$= \cos 2t + 1$$

$$\left.\frac{dw}{dt}\right|_{t=0} = \cos(0) + 1 = 2$$

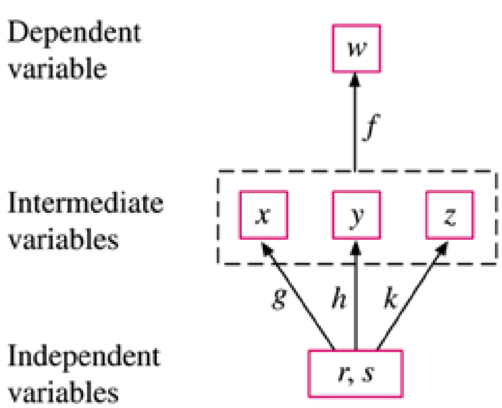
Functions Defined on Surfaces

Theorem **– Chain Rule for Two Independent Variables and Three Intermediate Variables**

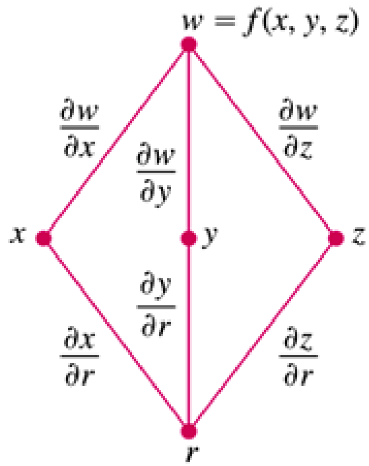
Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

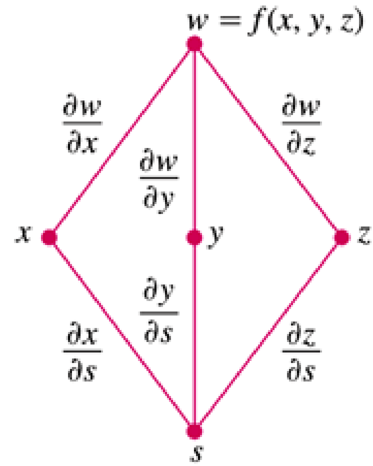
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$



$$w = f(g(r, s), h(r, s), k(r, s))$$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Example

Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if $\quad w = x + 2y + z^2, \quad x = \frac{r}{s}, \quad y = r^2 + \ln s, \quad z = 2r$

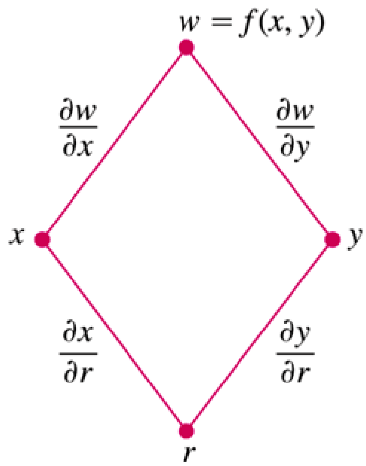
Solution

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}\\ &= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2)\\ &= \frac{1}{s} + 4r + (4r)(2)\\ &\boxed{= \frac{1}{s} + 12r}\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}\\ &= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0)\\ &\boxed{= -\frac{r}{s^2} + \frac{2}{s}}\end{aligned}$$

➢ If $w = f(x, y)$, $x = g(r, s)$ and $y = h(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

Example

Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if $\quad w = x^2 + y^2, \quad x = r - s, \quad y = r + s$

Solution

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

$$= (2x)(1) + (2y)(1)$$

$$= 2(r - s) + 2(r + s)$$

$$= 2r - 2s + 2r + 2s$$

$$\underline{= 4r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

$$= (2x)(-1) + (2y)(1)$$

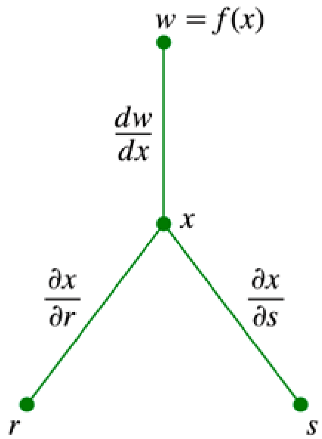
$$= -2(r - s) + 2(r + s)$$

$$= -2r + 2s + 2r + 2s$$

$$\underline{= 4s}$$

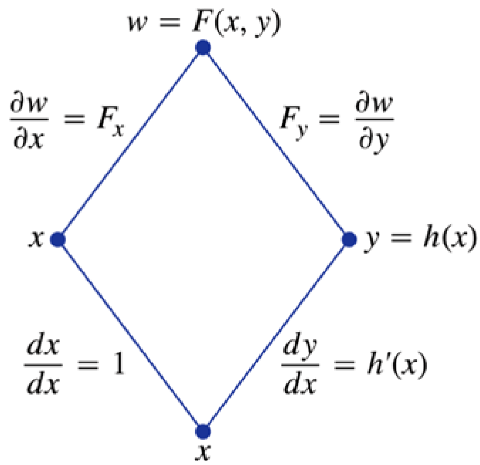
- If $w = f(x)$, $x = g(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} \quad and \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s}$$



$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$



$$\frac{dw}{dx} = F_x \cdot 1 + F_y \cdot \frac{dy}{dx}$$

Implicit Differentiation Revisited

Theorem – A Formula for Implicit Differentiation

Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y)=0$ defines *y* as a differentiable function of *x*. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

$$\frac{dz}{dx}=-\frac{F_x}{F_z} \qquad \frac{dz}{dy}=-\frac{F_y}{F_z}$$

Example

Find $\frac{dy}{dx}$ if $y^2-x^2-\sin xy=0$

Solution

$$F(x, y)=y^2-x^2-\sin xy$$

$$\begin{aligned}\frac{dy}{dx}&=-\frac{F_x}{F_y}\\ &=-\frac{-2x-y\cos xy}{2y-x\cos xy}\\ &=\boxed{\frac{2x+y\cos xy}{2y-x\cos xy}}\end{aligned}$$

Example

Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at (0, 0, 0) if $x^3+z^2+ye^{xz}+z\cos y=0$

Solution

$$F(x, y, z)=x^3+z^2+ye^{xz}+z\cos y$$

$$F_x=3x^2+yze^{xz}, \quad F_y=e^{xz}-z\sin y, \quad \text{and} \quad F_z=2z+xye^{xz}+\cos y$$

$$F(0, 0, 0)=0 \quad F_z=1\neq 0$$

$$\begin{aligned}\frac{dz}{dx}&=-\frac{F_x}{F_z}\\ &=-\left.\frac{3x^2+yze^{xz}}{2z+xye^{xz}+\cos y}\right|_{(0,0,0)}\end{aligned}$$

$$= -\frac{0}{1}$$

$$\underline{= 0}$$

$$\frac{dz}{dy} = -\frac{F_y}{F_z}$$

$$= -\frac{e^{xz} - z\sin y}{2z + xye^{xz} + \cos y}\Bigg|_{(0,0,0)}$$

$$= -\frac{1}{1}$$

$$\underline{= -1}$$

Exercises Section 2.4 – Chain Rule

(1 – 6) Express $\frac{dw}{dt}$ as a function of *t*, then evaluate $\frac{dw}{dt}$ at the given value of *t*.

1. $w = x^2 + y^2, \quad x = \cos t, \quad y = \sin t, \quad t = \pi$

2. $w = x^2 + y^2, \quad x = \cos t + \sin t, \quad y = \cos t - \sin t, \quad t = 0$

3. $w = \ln\left(x^2 + y^2 + z^2\right), \quad x = \cos t, \quad y = \sin t, \quad z = 4\sqrt{t}, \quad t = 3$

4. $w = z - \sin xy, \quad x = t, \quad y = \ln t, \quad z = e^{t-1}, \quad t = 1$

5. $w = \sin(xy + \pi), \quad x = e^t, \quad y = \ln(t+1) \quad t = 0$

6. $w = xe^y + y\sin z - \cos z, \quad x = 2\sqrt{t}, \quad y = t - 1 + \ln t, \quad z = \pi t, \quad t = 1$

7. Express $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as functions of *u* and *v* if $z = 4e^x \ln y, \quad x = \ln(u\cos v), \quad y = u\sin v$, then evaluate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at the point $(u,v) = \left(2, \frac{\pi}{4}\right)$.

8. Express $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ as functions of *u* and *v* if $w = xy + yz + xz, \quad x = u + v, \quad y = u - v, \quad z = uv$, then evaluate $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ at the point $(u,v) = \left(\frac{1}{2}, 1\right)$.

9. Express $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ as functions of *x*, *y* and *z* if $u = e^{qr}\sin^{-1} p, \quad p = \sin x, \quad q = z^2 \ln y, \quad r = \frac{1}{z}$, then evaluate $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ at the point $(x,y,z) = \left(\frac{\pi}{4}, \frac{1}{2}, -\frac{1}{2}\right)$.

10. Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z^3 - xy + yz + y^3 - 2 = 0$ at the point $(1,\ 1,\ 1)$

11. Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $\sin(x+y) + \sin(y+z) + \sin(x+z) = 0$ at the point $(\pi,\ \pi,\ \pi)$

12. Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0$ at the point $(1,\ \ln 2,\ \ln 3)$

13. Find $\frac{\partial w}{\partial r}$ when $r = 1,\ s = -1$ if $w = (x + y + z)^2, \quad x = r - s, \quad y = \cos(r+s), \quad z = \sin(r+s)$

14. Find $\frac{\partial z}{\partial u}$ when $u = 0,\ v = 1$ if $z = \sin xy + x\sin y, \quad x = u^2 + v^2, \quad y = uv$

15. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ when $u = \ln 2,\ v = 1$ if $z = 5\tan^{-1} x, \quad x = e^u + \ln v$

16. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ when $u=1,\ v=-2$ if $z=\ln q,\quad q=\sqrt{v+3}\ \tan^{-1}u$

17. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ when $r=\pi$ and $s=0$ if $w=\sin(2x-y),\quad x=r+\sin s,\quad y=rs$

18. Assume that $w=f\left(s^3+t^2\right)$ and $f'(x)=e^x$. Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$

(**19 – 22**) Evaluate the derivatives

19. $w'(t)$, where $w=xy\sin z,\quad x=t^2,\quad y=4t^3,\quad \textit{and}\quad z=t+1$

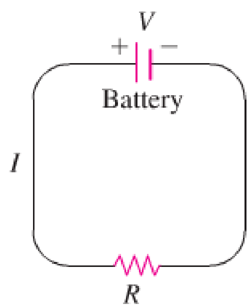
20. $w'(t)$, where $w=\sqrt{x^2+y^2+z^2},\quad x=\sin t,\quad y=\cos t,\quad \textit{and}\quad z=\cos t$

21. w_s and w_t, where $w=xyz,\quad x=2st,\quad y=st^2,\quad \textit{and}\quad z=s^2t$

22. w_r, w_s, and w_t, where $w=\ln\left(xy^2\right),\quad x=rst,\quad \textit{and}\quad y=r+s$

23. The voltage V in a circuit that satisfies the law $V = IR$ is slowly dropping as the battery wears out. At the same time, the resistance R is increasing as the resistor heats up. Use the equation

$$\frac{dV}{dt}=\frac{\partial V}{\partial I}\frac{dI}{dt}+\frac{\partial V}{\partial R}\frac{dR}{dt}$$



To find how the current is changing at the instant when $R=600\ \Omega$, $I=0.04A$, $\frac{dR}{dt}=0.5\ ohm/\sec$, and $\frac{dV}{dt}=-0.01\ volt/\sec$

24. The lengths *a*, *b*, and *c* of the edges of a rectangular box are changing with time. At the instant in question, $a=1\ m,\quad b=2\ m,\quad c=3\ m$, $\frac{da}{dt}=\frac{db}{dt}=1\ m/\sec$, and $\frac{dc}{dt}=-3\ m/\sec$. At what rates the box's volume V and surface area S changing at that instant? Are the box's interior diagonals increasing in length or decreasing?

25. Let $T = f(x,y)$ be the temperature at the point (x,y) on the circle $x=\cos t,\quad y=\sin t,\ 0\le t\le 2\pi$ and suppose that

$$\frac{\partial T}{\partial x}=8x-4y,\quad \frac{\partial T}{\partial y}=8y-4x$$

a) Find where the maximum and minimum temperatures on the circle occur by examining the derivatives $\frac{dT}{dt}$ and $\frac{d^2T}{dt^2}$.

b) Suppose that $T = 4x^2 - 4xy + 4y^2$. Find the maximum and minimum values of T on the circle.

(**26 – 33**) Evaluate $\frac{dy}{dx}$

26. $x^2 - 2y^2 - 1 = 0$

27. $x^3 + 3xy^2 - y^5 = 0$

28. $2\sin xy = 1$

29. $ye^{xy} - 2 = 0$

30. $\sqrt{x^2 + 2xy + y^4} = 3$

31. $y\ln\left(x^2 + y^2 + 4\right) = 3$

32. $y\ln\left(x^2 + y^2\right) = 4$

33. $2x^2 + 3xy - 3y^4 = 2$

(**34 – 37**) Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at the given point.

34. $z^3 - xy + yz + y^3 - 2 = 0; \quad (1,\ 1,\ 1)$

35. $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0; \quad (2,\ 3,\ 6)$

36. $\sin(x+y) + \sin(y+z) + \sin(x+z) = 0; \quad (\pi,\ \pi,\ \pi)$

37. $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0; \quad (1,\ \ln 2,\ \ln 3)$

38. Consider the surface and parameterized curves C in the xy-plane

$$z = 4x^2 + y^2 - 2; \quad C:\ x = \cos t, \quad y = \sin t, \quad \text{for } \ 0 \le t \le 2\pi$$

a) Find $z'(t)$ on C.

b) Imagine that you are walking on the surface directly above C consistent with the positive orientation of C. Find the values of ***t*** for which you are walking uphill.

39. Consider the surface and parameterized curves C in the xy-plane

$$z = 4x^2 - 2y^2 + 4; \quad C:\ x = 2\cos t, \quad y = 2\sin t, \quad \text{for } \ 0 \le t \le 2\pi$$

a) Find $z'(t)$ on C.

b) Imagine that you are walking on the surface directly above C consistent with the positive orientation of C. Find the values of ***t*** for which you are walking uphill.

40. Find the value of the derivative of $f(x,y,z) = xy + yz + xz$ with respect to t on the curve $x = \cos t, \quad y = \sin t, \quad z = \cos 2t$ at $t = 1$

41. Define y as a differentiable function of x for $2xy + e^{x+y} - 2 = 0$, find the values of $\frac{dy}{dx}$ at point $P(0,\ \ln 2)$