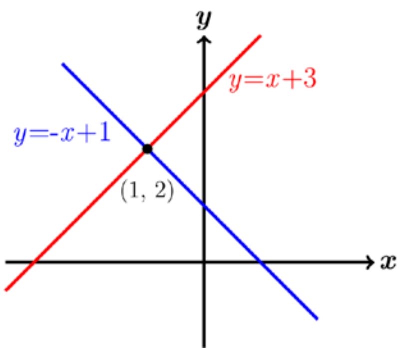
Lecture Four – Matrices

Section 4.1 – System of linear Equations

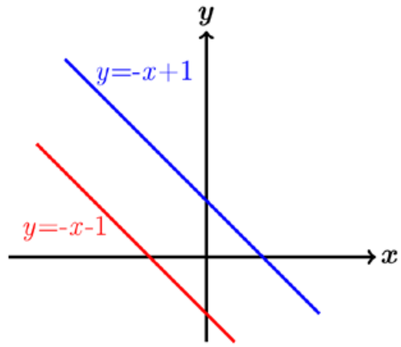
Solving Systems of Equations

1. Graphically
2. Substitution Method
3. Elimination Method

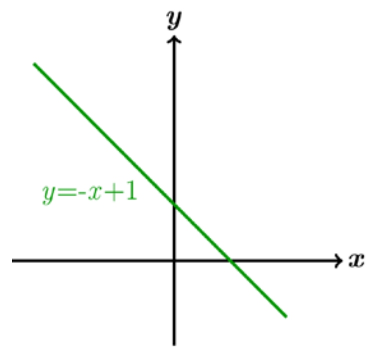
Graphically



One solution (lines intersect)
Consistent
Independent



No Solution (lines //)
Inconsistent
Independent



Infinite solution
Consistent
Dependent

Substitution Method

Solve: $\begin{cases} 3x+2y=11 & (1) \\ -x+y=3 & (2) \end{cases}$

Solution

$$\textit{From } (2) \to y = x+3 \qquad (3)$$

$$(1) \Rightarrow 3x+2(x+3)=11$$

$$3x+2x+6=11$$

$$5x+6=11$$

$$5x+6-6=11-6$$

$$5x=5$$

$$x=1$$

$From\ (3) \to y = 1 + 3 = 4$

$Solution: \underline{(1,\ 4)}|$

Elimination Method

Solve: $\begin{cases} 3x - 4y = 1 & (1) \\ 2x + 3y = 12 & (2) \end{cases}$

Solution

$$\begin{aligned} -2\times) \quad & 3x - 4y = 1 \\ 3\times) \quad & 2x + 3y = 12 \end{aligned}$$

$$\begin{array}{r} -6x + 8y = -2 \\ 6x + 9y = 36 \\ \hline 17y = 34 \end{array}$$

$y = \frac{34}{17} = 2$

$$\begin{aligned} From\ (1) \Rightarrow 3x &= 1 + 4y \\ 3x &= 1 + 4(2) \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

$Solution: \underline{(3,\ 2)}|$

Matrices

$$
\begin{array}{cc}
 & \mathit{Column} \\
 & \begin{array}{ccc} C_1 & C_2 & C_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\
\begin{array}{c} \mathit{Row}\ 1 \rightarrow R_1 \\ \mathit{Row}\ 2 \rightarrow R_2 \\ \mathit{Row}\ 3 \rightarrow R_3 \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
\end{array}
$$

This is called Matrix (*Matrices*)

Each number in the array is an ***element*** or ***entry***

The matrix is said to be of order m x n

m: numbers of rows,

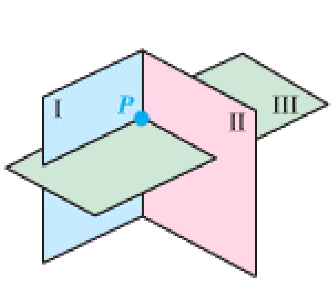
n: number of columns

When $m = n$, then matrix is said to be ***square***.

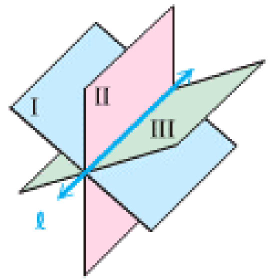
Given the system equations

$$
\begin{aligned}
3x + \ \ y + 2z &= 31 \\
x + \ \ y + 2z &= 19 \\
x + 3y + 2z &= 25
\end{aligned}
$$

The ***augmented matrix*** form is: $\left[\begin{array}{ccc|c} 3 & 1 & 2 & 31 \\ 1 & 1 & 2 & 19 \\ 1 & 3 & 2 & 25 \end{array}\right]$

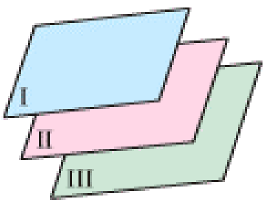


A single solution

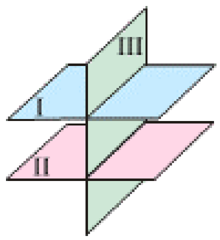


Points of a line in common

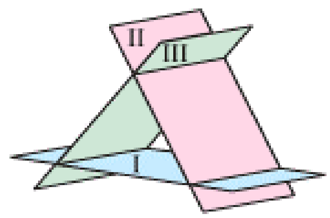
All points in common



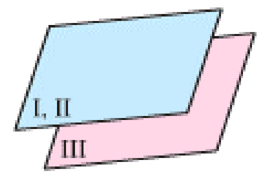
No points in common



No points in common



No points in common



No points in common

Gaussian Elimination

Example

Use the Gaussian elimination method to solve the system

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 3 & 1 & 2 & 31 \\ 1 & 3 & 2 & 25 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array} \qquad \begin{array}{cccc} 3 & 1 & 2 & 31 \\ -3 & -3 & -6 & -57 \\ \hline 0 & -2 & -4 & -26 \end{array} \qquad \begin{array}{cccc} 1 & 3 & 2 & 25 \\ -1 & -1 & -2 & -19 \\ \hline 0 & 2 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & -2 & -4 & -26 \\ 0 & 2 & 0 & 6 \end{array}\right] \ -\tfrac{1}{2}R_2 \qquad \begin{array}{cccc} 0 & 1 & 2 & 13 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 2 & 0 & 6 \end{array}\right] \ R_3 - 2R_2 \qquad \begin{array}{cccc} 0 & 2 & 0 & 6 \\ 0 & -2 & -4 & -26 \\ \hline 0 & 0 & -4 & -20 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -4 & -20 \end{array}\right] \ -\tfrac{1}{4}R_3 \qquad \begin{array}{cccc} 0 & 0 & 1 & 5 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array}\right] \Rightarrow \begin{aligned} x + y + 2z &= 19 \quad (3) \\ y + 2z &= 13 \quad (2) \\ z &= 5 \quad (1) \end{aligned}$$

(2) $\Rightarrow y = 13 - 2z = 13 - 2(5) = 3$

(3) $\Rightarrow x = 19 - y - 2z = 19 - 3 - 10 = 6$

$\Rightarrow$ **(6, 3, 5)**

Gauss-Jordan Elimination

Example

Use the Gauss-Jordan method to solve the system

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 3 & 1 & 2 & 31 \\ 1 & 3 & 2 & 25 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 1 & 2 & 31 \\ -3 & -3 & -6 & -57 \\ \hline 0 & -2 & -4 & -26 \end{array} \qquad \begin{array}{cccc} 1 & 3 & 2 & 25 \\ -1 & -1 & -2 & -19 \\ \hline 0 & 2 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & -2 & -4 & -26 \\ 0 & 2 & 0 & 6 \end{array}\right] \begin{array}{l} \\ -\frac{1}{2}R_2 \\ \\ \end{array}$$

$$\begin{array}{cccc} 0 & 1 & 2 & 13 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 2 & 0 & 6 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 2R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 2 & 0 & 6 \\ 0 & -2 & -4 & -26 \\ \hline 0 & 0 & -4 & -20 \end{array} \qquad \begin{array}{cccc} 1 & 1 & 2 & 19 \\ 0 & -1 & -2 & -13 \\ \hline 1 & 0 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -4 & -20 \end{array}\right] \begin{array}{l} \\ \\ -\frac{1}{4}R_3 \end{array}$$

$$\begin{array}{cccc} 0 & 0 & 1 & 5 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 0 & 1 & 2 & 13 \\ 0 & 0 & -2 & -10 \\ \hline 0 & 1 & 0 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

***Solution*: (6, 3, 5)**

Example

Use the Gaussian elimination method to solve the system

$$\begin{aligned} 2x + \ \ y + 2z &= 4 \\ 2x + 2y \qquad &= 5 \\ 2x - \ \ y + 6z &= 2 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \ \tfrac{1}{2}R_1 \qquad\qquad 1 \quad \tfrac{1}{2} \quad 1 \quad 2$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array} \qquad \begin{array}{cccc} 2 & 2 & 0 & 5 \\ -2 & -1 & -2 & -4 \\ \hline 0 & 1 & -2 & 1 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 6 & 2 \\ -2 & -1 & -2 & -4 \\ \hline 0 & -2 & 4 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & -2 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 2R_2 \end{array} \qquad \begin{array}{cccc} 0 & -2 & 4 & -2 \\ 0 & 2 & -4 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{rl} x + \frac{1}{2}y + z = 2 & (3) \\ y - 2z = 1 & (2) \\ 0 = 0 & (1) \end{array}$$

From (1): $\quad 0 = 0$ is a true statement. Let $\boldsymbol{z}$ be the variable.

From (2): $\quad y = 1 + 2z$

From (3): $\quad x = -\frac{1}{2}y - z + 2$

$$x = -\frac{1}{2}(1 + 2z) - z + 2$$

$$x = -\frac{1}{2} - z - z + 2$$

$$x = -2z + \frac{3}{2}$$

Solution: $\left(-2z + \frac{3}{2},\ 2z + 1,\ z\right)$

Example

Use the Gaussian elimination method to solve the system

$$\begin{aligned} x+2y-5z&=-1\\ 2x+3y-2z&=2\\ 3x+5y-7z&=4 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & -1\\ 2 & 3 & -2 & 2\\ 3 & 5 & -7 & 4 \end{array}\right] \begin{array}{l} \\ R_2-2R_1\\ R_3-3R_1 \end{array}$$

$$\begin{array}{cccc} 2 & 3 & -2 & 2\\ -2 & -4 & 10 & 2\\ \hline 0 & -1 & 8 & 4 \end{array} \qquad \begin{array}{cccc} 3 & 5 & -7 & 4\\ -3 & -6 & 15 & 3\\ \hline 0 & -1 & 8 & 7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & -1\\ 0 & -1 & 8 & 4\\ 0 & -1 & 8 & 7 \end{array}\right] \begin{array}{l} \\ -R_2\\ \\ \end{array}$$

$$0 \quad 1 \quad -8 \quad -4$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & -1\\ 0 & 1 & -8 & -4\\ 0 & -1 & 8 & 7 \end{array}\right] \begin{array}{l} \\ \\ R_3+R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -1 & 8 & 7\\ 0 & 1 & -8 & -4\\ \hline 0 & 0 & 0 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & -1\\ 0 & 1 & -8 & -4\\ 0 & 0 & 0 & 3 \end{array}\right]$$

From Row 3: 0 = 3 is a False statement.

No Solution or ***Inconsistent***

Exercises ***Section* 4.1 – System of linear Equations**

(1 – 15) Use any method to solve the system equation (***elimination*** or ***substitution*** method)

1. $\begin{cases} 3x+2y=-4 \\ 2x-y=-5 \end{cases}$

2. $\begin{cases} 2x+5y=7 \\ 5x-2y=-3 \end{cases}$

3. $\begin{cases} 4x-7y=-16 \\ 2x+5y=9 \end{cases}$

4. $\begin{cases} 3x+2y=4 \\ 2x+y=1 \end{cases}$

5. $\begin{cases} 3x+4y=2 \\ 2x+5y=-1 \end{cases}$

6. $\begin{cases} 5x-2y=4 \\ -10x+4y=7 \end{cases}$

7. $\begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$

8. $\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

9. $\begin{cases} 2x+10y=-14 \\ 7x-2y=-16 \end{cases}$

10. $\begin{cases} 4x-3y=24 \\ -3x+9y=-1 \end{cases}$

11. $\begin{cases} 4x+2y=12 \\ 3x-2y=16 \end{cases}$

12. $\begin{cases} x+2y=-1 \\ 4x-2y=6 \end{cases}$

13. $\begin{cases} x-2y=5 \\ -10x+2y=4 \end{cases}$

14. $\begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$

15. $\begin{cases} 4x-4y=-12 \\ 4x+4y=-20 \end{cases}$

(16 – 27) Perform the matrix row operation (or operations) and write the new matrix.

16. $\left[\begin{array}{cc|c} 1 & 4 & 7 \\ 3 & 5 & 0 \end{array}\right]$ R_2-3R_1

17. $\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right]$ R_2-2R_1

18. $\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 5 & 2 & 19 \end{array}\right]$ R_2-5R_1

19. $\left[\begin{array}{cc|c} 2 & -3 & 8 \\ -6 & 9 & 4 \end{array}\right]$ R_2+3R_1

20. $\left[\begin{array}{cc|c} 2 & 3 & 11 \\ 1 & 2 & 8 \end{array}\right]$ $2R_2-R_1$

21. $\left[\begin{array}{cc|c} 3 & 5 & -13 \\ 2 & 3 & -9 \end{array}\right]$ $3R_2-2R_1$

22. $\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 5 & 4 & 1 \end{array}\right]$ R_3-5R_2

23. $\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right]$ $\begin{array}{l} R_2-3R_1 \\ R_3-R_1 \end{array}$

24. $\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 2 & 4 & 4 & 22 \\ -1 & -2 & 3 & 15 \end{array}\right]$ $\begin{array}{l} 3R_2-2R_1 \\ 3R_3+R_1 \end{array}$

25. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & -4 & 2 & -7 \end{array}\right]$ $\begin{array}{l} R_2-2R_1 \\ R_3-3R_1 \end{array}$

26. $\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ 2 & -3 & 5 & -1 & 0 \\ 1 & 0 & 3 & 1 & -4 \\ -4 & 3 & 2 & -1 & 3 \end{array}\right]$ $\begin{array}{l} R_2-2R_1 \\ R_3-R_1 \\ R_4+4R_1 \end{array}$

27. $\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ -3 & 6 & -3 & -9 & 6 \\ 2 & 1 & 2 & 3 & 4 \\ 5 & 3 & 2 & -1 & -7 \end{array}\right]$ $\begin{array}{l} R_2+3R_1 \\ R_3-2R_1 \\ R_4-5R_1 \end{array}$

(**28 – 34**) Use the Gauss-Jordan method to solve the system

28. $\begin{cases} x-y+5z=-6 \\ 3x+3y-z=10 \\ x+3y+2z=5 \end{cases}$

29. $\begin{cases} 2x-y+4z=-3 \\ x-2y-10z=-6 \\ 3x \qquad +4z=7 \end{cases}$

30. $\begin{cases} 4x+3y-5z=-29 \\ 3x-7y-z=-19 \\ 2x+5y+2z=-10 \end{cases}$

31. $\begin{cases} x+2y-3z=-15 \\ 2x-3y+4z=18 \\ -3x+y+z=1 \end{cases}$

32. $\begin{cases} x+2y+3z=10 \\ 4x+5y+6z=11 \\ 7x+8y+9z=12 \end{cases}$

33. $\begin{cases} 2x+y+2z=4 \\ 2x+2y \qquad =5 \\ 2x-y+6z=2 \end{cases}$

34. $\begin{cases} x_1+x_2+2x_3=8 \\ -x_1-2x_2+3x_3=1 \\ 3x_1-7x_2+4x_3=10 \end{cases}$

(**35 – 69**) Use augmented elimination to solve linear system

35. $\begin{cases} 2x-5y+3z=1 \\ x-2y-2z=8 \end{cases}$

36. $\begin{cases} x+y+z=2 \\ 2x+y-z=5 \\ x-y+z=-2 \end{cases}$

37. $\begin{cases} 2x+y+z=9 \\ -x-y+z=1 \\ 3x-y+z=9 \end{cases}$

38. $\begin{cases} 3y-z=-1 \\ x+5y-z=-4 \\ -3x+6y+2z=11 \end{cases}$

39. $\begin{cases} x+3y+4z=14 \\ 2x-3y+2z=10 \\ 3x-y+z=9 \end{cases}$

40. $\begin{cases} x+4y-z=20 \\ 3x+2y+z=8 \\ 2x-3y+2z=-16 \end{cases}$

41. $\begin{cases} 2y-z=7 \\ x+2y+z=17 \\ 2x-3y+2z=-1 \end{cases}$

42. $\begin{cases} -2x+6y+7z=3 \\ -4x+5y+3z=7 \\ -6x+3y+5z=-4 \end{cases}$

43. $\begin{cases} 2x-y+z=1 \\ 3x-3y+4z=5 \\ 4x-2y+3z=4 \end{cases}$

44. $\begin{cases} 3x-4y+4z=7 \\ x-y-2z=2 \\ 2x-3y+6z=5 \end{cases}$

45. $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

46. $\begin{cases} x+y+z=3 \\ -y+2z=1 \\ -x+z=0 \end{cases}$

47. $\begin{cases} 3x+y+3z=14 \\ 7x+5y+8z=37 \\ x+3y+2z=9 \end{cases}$

48. $\begin{cases} 4x-2y+z=7 \\ x+y+z=-2 \\ 4x+2y+z=3 \end{cases}$

49. $\begin{cases} 2x-2y+z=-4 \\ 6x+4y-3z=-24 \\ x-2y+2z=1 \end{cases}$

50. $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

51. $\begin{cases} 2x-y+2z=-8 \\ x+2y-3z=9 \\ 3x-y-4z=3 \end{cases}$

52. $\begin{cases} x \qquad -3z=-5 \\ 2x-y+2z=16 \\ 7x-3y-5z=19 \end{cases}$

53. $\begin{cases} x+2y-z=5 \\ 2x-y+3z=0 \\ 2y+z=1 \end{cases}$

54. $\begin{cases} x+y+z=6 \\ 3x+4y-7z=1 \\ 2x-y+3z=5 \end{cases}$

55. $\begin{cases} 3x+2y+3z=3 \\ 4x-5y+7z=1 \\ 2x+3y-2z=6 \end{cases}$

56. $\begin{cases} x-3y+z=2 \\ 4x-12y+4z=8 \\ -2x+6y-2z=-4 \end{cases}$

57. $\begin{cases} 2x-2y+z=-1 \\ x+2y-z=2 \\ 6x+4y+3z=5 \end{cases}$

58. $\begin{cases} x_1-5x_2+2x_3-2x_4=4 \\ x_2-3x_3-x_4=0 \\ 3x_1+2x_3-x_4=6 \\ -4x_1+x_2+4x_3+2x_4=-3 \end{cases}$

59. $\begin{cases} x_1+x_2+x_3+x_4=5 \\ x_1+2x_2-x_3-2x_4=-1 \\ x_1-3x_2-3x_3-x_4=-1 \\ 2x_1-x_2+2x_3-x_4=-2 \end{cases}$

60. $\begin{cases} 2x+8y-z+w=0 \\ 4x+16y-3z-w=-10 \\ -2x+4y-z+3w=-6 \\ -6x+2y+5z+w=3 \end{cases}$

61. $\begin{cases} 2x_1+x_2+3x_3=0 \\ x_1+2x_2=0 \\ x_2+x_3=0 \end{cases}$

62. $\begin{cases} 2x+2y+4z=0 \\ -y-3z+w=0 \\ 3x+y+z+2w=0 \\ x+3y-2z-2w=0 \end{cases}$

63. $\begin{cases} 2x+z+w=5 \\ y-w=-1 \\ 3x-z-w=0 \\ 4x+y+2z+w=9 \end{cases}$

64. $\begin{cases} 4y+z=20 \\ 2x-2y+z=0 \\ x+z=5 \\ x+y-z=10 \end{cases}$

65. $\begin{cases} x-y+2z-w=-1 \\ 2x+y-2z-2w=-2 \\ -x+2y-4z+w=1 \\ 3x-3w=-3 \end{cases}$

66. $\begin{cases} 2u-3v+w-x+y=0 \\ 4u-6v+2w-3x-y=-5 \\ -2u+3v-2w+2x-y=3 \end{cases}$

67. $\begin{cases} 6x_3+2x_4-4x_5-8x_6=8 \\ 3x_3+x_4-2x_5-4x_6=4 \\ 2x_1-3x_2+x_3+4x_4-7x_5+x_6=2 \\ 6x_1-9x_2+11x_4-19x_5+3x_6=1 \end{cases}$

68. $\begin{cases} 3x_1+2x_2-x_3=-15 \\ 5x_1+3x_2+2x_3=0 \\ 3x_1+x_2+3x_3=11 \\ -6x_1-4x_2+2x_3=30 \end{cases}$

69. $\begin{cases} x_1+3x_2-2x_3+2x_5=0 \\ 2x_1+6x_2-5x_3-2x_4+4x_5-3x_6=-1 \\ 5x_3+10x_4+15x_6=5 \\ 2x_1+6x_2+8x_4+4x_5+18x_6=6 \end{cases}$

70. At Snack Mix, caramel corn worth \$2.50 per *pound* is mixed with honey roasted missed nuts worth \$7.50 per *pound* in order to get 20 *lbs*. of a mixture worth \$4.50 per *pound*. How much of each snack is used?