Solution Section 4.2 – Line Integrals

Exercise

Evaluate $\int_C (x+y)\,ds$ where C is the straight-line segment $x=t,\ y=(1-t),\ z=0$ from (0, 1, 0) to $(1,\ 0,\ 0)$.

Solution

$$\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= t\hat{i} + (1-t)\,\hat{j}$$

$$\frac{dr}{dt} = \boldsymbol{i} - \boldsymbol{j} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right| = \sqrt{1+1} = \sqrt{2}$$

$$\begin{matrix} x=t \\ y=1-t \end{matrix} \quad \to \quad x+y = t+1-t = 1$$

$$\int_C f(x,y,z) = \int_0^1 f(t,1-t,0)\left|\frac{d\vec{r}}{dt}\right|dt$$
$$= \int_0^1 (1)\sqrt{2}\,dt$$
$$= \sqrt{2}t\Big|_0^1$$
$$= \sqrt{2}$$

Exercise

Evaluate $\int_C (x-y+z-2)\,ds$ where C is the straight-line segment $x=t,\ y=(1-t),\ z=1$ from $(0,\ 1,\ 1)$ to $(1,\ 0,\ 1)$.

Solution

$$\vec{r}(t) = t\hat{i} + (1-t)\,\hat{j} + \hat{k} \qquad 0 \le t \le 1$$

$$\frac{d\vec{r}}{dt} = \hat{i} - \hat{j} \quad \Rightarrow \quad \left|\frac{d\vec{r}}{dt}\right| = \sqrt{1+1} = \sqrt{2}$$

$$\begin{cases} x = t \\ y = 1 - t \\ z = 1 \end{cases}$$

$$x - y + z - 2 = t - 1 + t + 1 - 2$$
$$= 2t - 2$$

$$\int_C f(x,y,z) = \int_0^1 (2t-2)\sqrt{2}\,dt$$
$$= \sqrt{2}\left[t^2 - 2t\right]_0^1$$
$$= \sqrt{2}(1-2)$$
$$= -\sqrt{2}$$

Exercise

Evaluate $\int_C (xy + y + z)\,ds$ along the curve $\vec{r}(t) = 2t\hat{i} + t\hat{j} + (2-2t)\hat{k}, \quad 0 \le t \le 1$

Solution

$$\vec{r}(t) = 2t\hat{i} + t\hat{j} + (2-2t)\hat{k}, \quad 0 \le t \le 1$$

$$\frac{d\vec{r}}{dt} = 2\hat{j} + \hat{j} - 2\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{4+1+4} = 3$$

$$\begin{matrix} x = 2t \\ y = t \\ z = 2 - 2t \end{matrix} \quad \rightarrow \quad xy + y + z = 2t^2 + t + 2 - 2t = 2t^2 - t + 2$$

$$\int_C (xy + y + z)\,ds = \int_0^1 \left(2t^2 - t + 2\right)(3)\,dt$$
$$= 3\left[\frac{2}{3}t^3 - \frac{1}{2}t^2 + 2t\right]_0^1$$
$$= 3\left(\frac{2}{3} - \frac{1}{2} + 2\right)$$
$$= 3\left(\frac{13}{6}\right)$$
$$= \frac{13}{2}$$

Exercise

Evaluate $\displaystyle\int_C \left(xz - y^2\right) ds$ C: is the line segment from $(0,\ 1,\ 2)$ to $(-3,\ 7,\ -1)$.

Solution

Equation of the line is:

$$\begin{cases} x = 0 + (-3-0)t \\ y = 1 + (7-1)t \\ z = 2 + (-1-2)t \end{cases} \rightarrow \langle -3t,\ 1+6t,\ 2-3t \rangle$$

$$\vec{r}(t) = \langle -3t,\ 1+6t,\ 2-3t \rangle \quad 0 \le t \le 1$$

$$\vec{r}'(t) = \langle -3,\ 6,\ -3 \rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{9+36+9}$$

$$= 3\sqrt{6}$$

$$\int_C \left(xz - y^2\right) ds = 3\sqrt{6}\int_0^1 \left((-3t)(2-3t) - (1+6t)^2\right) dt$$

$$= 3\sqrt{6}\int_0^1 \left(-6t + 9t^2 - 1 - 12t - 36t^2\right) dt$$

$$= 3\sqrt{6}\int_0^1 \left(-27t^2 - 18t - 1\right) dt$$

$$= 3\sqrt{6}\left(-9t^3 - 9t^2 - t\right)\Big|_0^1$$

$$= 3\sqrt{6}(-9-9-1)$$

$$\underline{= -57\sqrt{6}}$$

Exercise

Evaluate $\displaystyle\int_C xy\ ds$; C: is the unit circle $\vec{r}(t) = \langle \cos t,\ \sin t \rangle$; $\quad 0 \le t \le 2\pi$

Solution

$$\vec{r}'(t) = \langle -\sin t,\ \cos t \rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{\sin^2 t + \cos^2 t}$$

$$\underline{= 1}$$

$$\int_C xy\ ds = \int_0^{2\pi} \cos t \sin t\ dt$$

$$= \int_0^{2\pi} \sin t\ d(\sin t)$$

$$= \frac{1}{2}\sin^2 t \Big|_0^{2\pi}$$

$$\underline{= 0}$$

Exercise

Evaluate $\int_C (x+y)ds$ C: is the circle of radius 1 centered at $(0,\ 0)$

Solution

$$\vec{r}(t) = \langle \cos t,\ \sin t \rangle; \quad 0 \le t \le 2\pi$$

$$\vec{r}'(t) = \langle -\sin t,\ \cos t \rangle$$

$$|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t}$$

$$\underline{= 1}$$

$$\int_C (x+y)\ ds = \int_0^{2\pi} (\cos t + \sin t)\ dt$$

$$= (\sin t - \cos t)\Big|_0^{2\pi}$$

$$= -1 + 1$$

$$\underline{= 0}$$

Exercise

Evaluate $\int_C \left(x^2 - 2y^2\right)ds$ C: is the line $\vec{r}(t) = \left\langle \frac{t}{\sqrt{2}},\ \frac{t}{\sqrt{2}} \right\rangle; \quad 0 \le t \le 4$

Solution

$$\vec{r}'(t) = \left\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}} \right\rangle$$

$$|\vec{r}'(t)| = \sqrt{\frac{1}{2} + \frac{1}{2}}$$

$$\boxed{=1}$$

$$\int_C \left(x^2 - 2y^2\right) ds = \int_0^4 \left(\frac{1}{2}t^2 - t^2\right) dt$$

$$= -\frac{1}{2}\int_0^4 t^2 dt$$

$$= -\frac{1}{6}t^3 \Big|_0^4$$

$$\boxed{= -\frac{32}{3}}$$

Exercise

Evaluate $\int_C x^2 y \; ds$ C: is the line $\vec{r}(t) = \left\langle \frac{t}{\sqrt{2}},\ 1 - \frac{t}{\sqrt{2}} \right\rangle;\quad 0 \le t \le 4$

Solution

$$\vec{r}'(t) = \left\langle \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}} \right\rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{\frac{1}{2} + \frac{1}{2}}$$

$$\boxed{=1}$$

$$\int_C x^2 y \; ds = \int_0^4 \frac{1}{2}t^2 \left(1 - \frac{1}{\sqrt{2}}t\right) dt$$

$$= \int_0^4 \left(\frac{1}{2}t^2 - \frac{1}{2\sqrt{2}}t^3\right) dt$$

$$= \left(\frac{1}{6}t^3 - \frac{1}{8\sqrt{2}}t^4\right)\Big|_0^4$$

$$= \frac{32}{3} - \frac{32}{\sqrt{2}}$$

$$\boxed{= \frac{32 - 48\sqrt{2}}{3}}$$

Exercise

Evaluate $\int_C \left(x^2+y^2\right)ds$ C: is the circle of radius 4 centered at $(0,\ 0)$

Solution

$$\vec{r}(t)=\langle 4\cos t,\ 4\sin t\rangle;\quad 0\le t\le 2\pi$$

$$\vec{r}'(t)=\langle -4\sin t,\ 4\cos t\rangle$$

$$\left|\vec{r}'(t)\right|=\sqrt{16\sin^2 t+16\cos^2 t}$$

$$=4$$

$$\int_C \left(x^2+y^2\right)ds=4\int_0^{2\pi}\left(16\cos^2 t+16\sin^2 t\right)dt$$

$$=64\int_0^{2\pi} dt$$

$$=128\pi$$

Exercise

Evaluate $\int_C \left(x^2+y^2\right)ds$ C: is the line segment from $(0,\ 0)$ to $(5,\ 5)$

Solution

$$\vec{r}(t)=\langle 5t,\ 5t\rangle;\quad 0\le t\le 1$$

$$\vec{r}'(t)=\langle 5,\ 5\rangle$$

$$\left|\vec{r}'(t)\right|=\sqrt{25+25}$$

$$=5\sqrt{2}$$

$$\int_C \left(x^2+y^2\right)ds=5\sqrt{2}\int_0^1\left(25t^2+25t^2\right)dt$$

$$=250\sqrt{2}\int_0^1 t^2dt$$

$$=\frac{250}{3}\sqrt{2}\ t^3\Big|_0^1$$

$$=\frac{250}{3}\sqrt{2}$$

Exercise

Evaluate $\displaystyle\int_C \frac{x}{x^2+y^2}\,ds$ C: is the line segment from $(1,\ 1)$ to $(10,\ 10)$

Solution

$$\vec{r}(t)=\langle t,\ t\rangle;\quad 1\le t\le 10$$

$$\vec{r}\,'(t)=\langle 1,\ 1\rangle$$

$$\left|\vec{r}\,'(t)\right|=\sqrt{1+1}$$

$$=\sqrt{2}\,\rfloor$$

$$\int_C \frac{x}{x^2+y^2}\,ds=\sqrt{2}\int_1^{10}\frac{t}{t^2+t^2}\,dt$$

$$=\frac{\sqrt{2}}{2}\int_1^{10}\frac{1}{t}\,dt$$

$$=\frac{\sqrt{2}}{2}\ \ln t\,\Big|_1^{10}$$

$$=\frac{\sqrt{2}}{2}\ln 10\,\rfloor$$

Exercise

Evaluate $\displaystyle\int_C (xy)^{1/3}\,ds$ C: is the curve $y=x^2,\quad 0\le x\le 1$

Solution

$$\vec{r}(t)=\left\langle t,\ t^2\right\rangle;\quad 0\le t\le 1$$

$$\vec{r}\,'(t)=\langle 1,\ 2t\rangle$$

$$\left|\vec{r}\,'(t)\right|=\sqrt{1+4t^2}$$

$$\int_C (xy)^{1/3}\,ds=\int_0^1\left(t^3\right)^{1/3}\ \sqrt{1+4t^2}\ dt$$

$$=\int_0^1 t\left(1+4t^2\right)^{1/2}\ dt$$

$$=\frac{1}{8}\int_0^1 \left(1+4t^2\right)^{1/2}\ d\left(1+4t^2\right)$$

$$=\frac{1}{12}\left(1+4t^2\right)^{3/2}\Big|_0^1$$

$$=\frac{1}{12}\left(5\sqrt{5}-1\right)$$

Exercise

Evaluate $\int_C xy\ ds$ C: is a portion of the ellipse $\frac{x^2}{4}+\frac{y^2}{16}=1$ in the first quadrant, oriented counterclockwise.

Solution

$$\vec{r}(t)=\langle 2\cos t,\ 4\sin t\rangle; \quad 0\le t\le \pi$$

$$\vec{r}\,'(t)=\langle -2\sin t,\ 4\cos t\rangle$$

$$\left|\vec{r}\,'(t)\right|=\sqrt{4\sin^2 t+16\cos^2 t}$$

$$=2\sqrt{\sin^2 t+4\cos^2 t}$$

$$\int_C xy\ ds=2\int_0^{\pi}(8\cos t\sin t)\sqrt{1-\cos^2 t+4\cos^2 t}\,dt$$

$$=16\int_0^{\pi}(\cos t\sin t)\left(1+3\cos^2 t\right)^{1/2}\ dt$$

$$=-\frac{8}{3}\int_0^{\pi}\left(1+3\cos^2 t\right)^{1/2}\ d\left(1+3\cos^2 t\right)$$

$$=-\frac{16}{9}\left(1+3\cos^2 t\right)^{3/2}\Big|_0^{\pi}$$

$$=-\frac{16}{9}(8-8)$$

$$=0$$

Exercise

Evaluate $\int_C (2x-3y)\,ds$ C: is the line segment from $(-1,\ 0)$ to $(0,\ 1)$ followed by the line segment from $(0,\ 1)$ to $(1,\ 0)$

Solution

$(-1,\ 0)$ to $(0,\ 1)$

$\vec{r}_1(t) = \langle t-1,\ \ t\rangle \quad 0 \le t \le 1$

$\vec{r}_1'(t) = \langle 1,\ \ 1\rangle$

$\boxed{\left|\vec{r}_1'(t)\right| = \sqrt{2}}$

$(0,\ 1)$ to $(1,\ 0)$

$\vec{r}_2(t) = \langle t,\ \ 1-t\rangle$

$\vec{r}_2'(t) = \langle 1,\ \ -1\rangle$

$\boxed{\left|\vec{r}_2'(t)\right| = \sqrt{2}}$

$$\begin{aligned}
\int_C (2x-3y)\,ds &= \sqrt{2}\int_0^1 \big(2(t-1)-3t\big)\,dt + \sqrt{2}\int_0^1 (2t-3+3t)\,dt \\
&= \sqrt{2}\int_0^1 (-2-t)\,dt + \sqrt{2}\int_0^1 (5t-3)\,dt \\
&= \sqrt{2}\int_0^1 (-2-t+5t-3)\,dt \\
&= \sqrt{2}\int_0^1 (4t-5)\,dt \\
&= \sqrt{2}\left(2t^2-5t\right)\Big|_0^1 \\
&= \sqrt{2}(2-5) \\
&= \boxed{-3\sqrt{2}}
\end{aligned}$$

Exercise

Evaluate $\int_C (x+y+z)\,ds$; C is the circle $\vec{r}(t) = \langle 2\cos t,\ 0,\ 2\sin t\rangle \quad 0 \le t \le 2\pi$

Solution

$$\vec{r}'(t) = \langle -2\sin t,\ 0,\ 2\cos t\rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{4\sin^2 t + 4\cos^2 t}$$

$$= 2$$

$$\int_C (x+y+z)\,ds = \int_0^{2\pi} (2\cos t + 2\sin t)(2)\,dt$$

$$= 4(-\sin t + \cos t)\Big|_0^{2\pi}$$

$$= 0$$

Exercise

Evaluate $\int_C (x-y+2z)\ ds$; C is the circle $\vec{r}(t) = \langle 1,\ 3\cos t,\ 3\sin t\rangle \quad 0 \le t \le 2\pi$

Solution

$$\vec{r}'(t) = \langle 0,\ -3\sin t,\ 3\cos t\rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{9\sin^2 t + 9\cos^2 t}$$

$$= 3$$

$$\int_C (x-y+2z)\ ds = 3\int_0^{2\pi} (1 - 3\cos t + 6\sin t)\,dt$$

$$= 3(t - 3\sin t - 6\cos t)\Big|_0^{2\pi}$$

$$= 3(2\pi - 6 + 6)(t - 3\sin t - 6\cos t)$$

$$= 6\pi$$

Exercise

Evaluate $\int_C xyz\ ds$; C is the circle $\vec{r}(t) = \langle 1,\ 3\cos t,\ 3\sin t\rangle \quad 0 \le t \le 2\pi$

Solution

$$\vec{r}'(t) = \langle 0, \ -3\sin t, \ 3\cos t \rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{9\sin^2 t + 9\cos^2 t}$$

$$\underline{= 3 \,|}$$

$$\int_C xyz \ ds = 3\int_0^{2\pi} (9\cos t \sin t)\, dt$$

$$= 27\int_0^{2\pi} \sin t \ d(\sin t)$$

$$= \frac{27}{2}\sin^2 t \Big|_0^{2\pi}$$

$$\underline{= 0 \,|}$$

Exercise

Evaluate $\int_C xyz \ ds$; C is the line segment from $(0, \ 0, \ 0)$ to $(1, \ 2, \ 3)$

Solution

$$\vec{r}(t) = \langle t, \ 2t, \ 3t \rangle \qquad 0 \le t \le 1$$

$$\vec{r}'(t) = \langle 1, \ 2, \ 3 \rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{1+4+9}$$

$$\underline{= \sqrt{14} \,|}$$

$$\int_C xyz \ ds = \sqrt{14}\int_0^1 6t^3 dt$$

$$= \frac{3}{2}\sqrt{14}t^4 \Big|_0^1$$

$$\underline{= \frac{3}{2}\sqrt{14} \,|}$$

Exercise

Evaluate $\int_C \frac{xy}{z} \ ds$; C is the line segment from $(1, \ 4, \ 1)$ to $(3, \ 6, \ 3)$

Solution

$$\vec{r}(t) = \langle 2t+1, \ 2t+4, \ 2t+1 \rangle \qquad 0 \le t \le 1$$

$\vec{r}(t) = \langle 2,\ 2,\ 2 \rangle$

$\boxed{|\vec{r}'(t)| = 2\sqrt{3}}$

$$\int_C \frac{xy}{z}\, ds = 2\sqrt{3}\int_0^1 \frac{(2t+1)(2t+4)}{2t+1}\, dt$$

$$= 2\sqrt{3}\int_0^1 (2t+4)\, dt$$

$$= 2\sqrt{3}\left(t^2+4t\right)\Big|_0^1$$

$$\boxed{= 10\sqrt{3}}$$

Exercise

Evaluate $\int_C (y-z)\, ds$; C is the helix $\vec{r}(t) = \langle 3\cos t,\ 3\sin t,\ t \rangle \quad 0 \le t \le 2\pi$

Solution

$\vec{r}'(t) = \langle -3\sin t,\ 3\cos t,\ 1 \rangle$

$|\vec{r}'(t)| = \sqrt{9\sin^2 t + 9\cos^2 t + 1}$

$\boxed{= \sqrt{10}}$

$$\int_C (y-z)\, ds = \sqrt{10}\int_0^{2\pi} (3\sin t - t)\, dt$$

$$= \sqrt{10}\left(-3\cos t - \frac{1}{2}t^2\right)\Big|_0^{2\pi}$$

$$= \sqrt{10}\left(-3 - 2\pi^2 + 3\right)$$

$$\boxed{= -2\pi\sqrt{10}}$$

Exercise

Evaluate $\int_C xe^{yz}\, ds$; C is $\vec{r}(t) = \langle t,\ 2t,\ -4t \rangle \quad 1 \le t \le 2$

Solution

$\vec{r}'(t) = \langle 1,\ 2,\ -4 \rangle$

$\boxed{|\vec{r}'(t)| = \sqrt{21}}$

$$\int_C xe^{yz}\,ds = \sqrt{21}\int_1^2 te^{-8t^2}\,dt$$

$$= -\frac{\sqrt{21}}{16}\int_1^2 e^{-8t^2}\,d\left(-8t^2\right)$$

$$= -\frac{\sqrt{21}}{16}e^{-8t^2}\Big|_1^2$$

$$= -\frac{\sqrt{21}}{16}\left(e^{-32}-e^{-8}\right)$$

$$= -\frac{\sqrt{21}}{16e^8}\left(\frac{1}{e^{24}}-1\right)$$

$$= \frac{\sqrt{21}}{16e^{32}}\left(e^{24}-1\right)$$

Exercise

Find the integral of $f(x,y,z) = x + y + z$ over the straight–line segment from $(1,\ 2,\ 3)$ to $(0,\ -1,\ 1)$

Solution

$$\boldsymbol{r}(t) = (\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) + t\big((0-1)\boldsymbol{i} + (-1-2)\boldsymbol{j} + (1-3)\boldsymbol{k}\big)$$

$$= (\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) + t(-\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k})$$

$$= (1-t)\boldsymbol{i} + (2-3t)\boldsymbol{j} + (3-2t)\boldsymbol{k}, \quad 0 \le t \le 1$$

$$\frac{dr}{dt} = -\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right| = \sqrt{1+9+4} = \sqrt{14}$$

$$\begin{aligned} x &= 1-t \\ y &= 2-3t \quad \rightarrow \quad x+y+z = 1-t+2-3t+3-2t \\ z &= 3-2t \end{aligned}$$

$$x + y + z = 6 - 6t$$

$$\int_C (x+y+z)\,ds = \int_0^1 (6-6t)\left(\sqrt{14}\right)dt$$

$$= \sqrt{14}\left[6t - 3t^2\right]_0^1$$

$$= 3\sqrt{14}$$

Exercise

Find the integral of $f(x,y,z)=\dfrac{\sqrt{3}}{x^2+y^2+z^2}$ over the curve $\mathbf{r}(t)=t\mathbf{i}+t\mathbf{j}+t\mathbf{k}, \quad 1\le t\le\infty$

Solution

$$\mathbf{r}(t)=t\mathbf{i}+t\mathbf{j}+t\mathbf{k}, \quad 1\le t\le\infty$$

$$\frac{dr}{dt}=\mathbf{i}+\mathbf{j}+\mathbf{k} \quad\Rightarrow\quad \left|\frac{dr}{dt}\right|=\sqrt{1+1+1}=\sqrt{3}$$

$$x^2+y^2+z^2=t^2+t^2+t^2=3t^2$$

$$\int_C \frac{\sqrt{3}}{x^2+y^2+z^2}\,ds=\int_1^\infty \frac{\sqrt{3}}{3t^2}\left(\sqrt{3}\right)dt$$

$$=\left[-\frac{1}{t}\right]_1^\infty$$

$$=-\left(\frac{1}{\infty}-1\right)$$

$$=1$$

Exercise

Evaluate $\displaystyle\int_C x\,ds$ where C is

a) The straight-line segment $x=t, \ \ y=\frac{t}{2}$, from (0, 0) to (4, 2).

b) The parabolic curve $x=t, \ \ y=t^2$, from (0, 0) to (2, 4).

Solution

a) $x=t \ \rightarrow \ \begin{cases} x=0 & t=0 \\ x=4 & t=4 \end{cases} \qquad t=2y \ \rightarrow \ \begin{cases} y=0 & t=0 \\ y=2 & t=4 \end{cases}$

$$\mathbf{r}(t)=t\mathbf{i}+\frac{t}{2}\mathbf{j}, \quad 0\le t\le 4$$

$$\frac{dr}{dt}=\mathbf{i}+\frac{1}{2}\mathbf{j} \quad\Rightarrow\quad \left|\frac{dr}{dt}\right|=\sqrt{1+\frac{1}{4}}=\frac{\sqrt{5}}{2}$$

$$\int_C x\,ds=\int_0^4 t\frac{\sqrt{5}}{2}dt$$

$$=\frac{\sqrt{5}}{2}\left[\frac{1}{2}t^2\right]_0^4$$

$$=4\sqrt{5}$$

b) $x=t \quad \rightarrow \quad \begin{cases} x=0 & t=0 \\ x=2 & t=2 \end{cases} \qquad t=\sqrt{y} \quad \rightarrow \quad \begin{cases} y=0 & t=0 \\ y=4 & t=2 \end{cases}$

$\mathbf{r}(t)=t\mathbf{i}+t^2\mathbf{j}, \quad 0\le t\le 2$

$\frac{dr}{dt}=\mathbf{i}+2t\mathbf{j} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right|=\sqrt{1+4t^2}$

$$\int_C x\,ds=\int_0^2 t\sqrt{1+4t^2}\,dt \qquad\qquad d\left(1+4t^2\right)=8tdt$$

$$=\frac{1}{8}\int_0^2 \left(1+4t^2\right)^{1/2} d\left(1+4t^2\right)$$

$$=\frac{1}{8}\left[\frac{2}{3}\left(1+4t^2\right)^{3/2}\right]_0^2$$

$$=\frac{1}{12}\left[(17)^{3/2}-1\right]$$

$$=\frac{1}{12}\left(17\sqrt{17}-1\right)$$

Exercise

Evaluate $\int_C \sqrt{x+2y}\; ds$ where C is

a) The straight-line segment $x=t, \;\; y=4t$, from (0, 0) to (1, 4).

b) $C_1\cup C_2$: C_1 is the line segment (0, 0) to (1, 0) and C_2 is the line segment (1, 0) to (1, 2).

Solution

a) $x=t \quad \rightarrow \quad \begin{cases} x=0 & t=0 \\ x=1 & t=1 \end{cases} \qquad t=\frac{y}{4} \quad \rightarrow \quad \begin{cases} y=0 & t=0 \\ y=4 & t=1 \end{cases}$

$\mathbf{r}(t)=t\mathbf{i}+4t\mathbf{j}, \quad 0\le t\le 1$

$\frac{dr}{dt}=\mathbf{i}+4\mathbf{j} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right|=\sqrt{1+16}=\sqrt{17}$

$$\int_C \sqrt{x+2y}\; ds=\int_0^1 \sqrt{t+8t}\left(\sqrt{17}\right)dt$$

$$=\sqrt{17}\int_0^1 \sqrt{9t}\,dt$$

$$=3\sqrt{17}\left[\frac{2}{3}t^{3/2}\right]_0^1$$

$$=2\sqrt{17}$$

b) $C_1: \quad \boldsymbol{r}(t) = (0\boldsymbol{i} + 0\boldsymbol{j}) + t(\boldsymbol{i} + 0\boldsymbol{j}) = t\boldsymbol{i} \quad 0 \le t \le 1$

$$\frac{dr}{dt} = \boldsymbol{i} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right| = 1$$

$C_2: \quad \boldsymbol{r}(t) = (1\boldsymbol{i} + 0\boldsymbol{j}) + t((1-1)\boldsymbol{i} + (2-0)\boldsymbol{j})$

$$= \boldsymbol{i} + 2t\boldsymbol{j} \qquad\qquad 0 \le t \le 2$$

$$\frac{dr}{dt} = 2\boldsymbol{j} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right| = 2$$

$$\int_C \sqrt{x+2y}\ ds = \int_0^1 \sqrt{t}\,(1)\,dt + \int_0^2 \sqrt{1+4t}\,(2)\,dt$$

$$= \left[\frac{2}{3}t^{3/2}\right]_0^1 + \frac{1}{2}\int_0^2 (1+4t)^{1/2}\ d(1+4t)$$

$$= \frac{2}{3} + \frac{1}{3}\left[(1+4t)^{3/2}\right]_0^2$$

$$= \frac{2}{3} + \frac{1}{3}\left[(9)^{3/2} - 1\right]$$

$$= \frac{2}{3} + \frac{1}{3}(26)$$

$$\boxed{= \frac{28}{3}}$$

Exercise

Find the line integral of $f(x,y) = \frac{\sqrt{y}}{x}$ along the curve $r(t) = t^3\boldsymbol{i} + t^4\boldsymbol{j}, \quad \frac{1}{2} \le t \le 1$

Solution

$r(t) = t^3\boldsymbol{i} + t^4\boldsymbol{j}, \quad \frac{1}{2} \le t \le 1$

$$\frac{dr}{dt} = 3t^2\boldsymbol{i} + 4t^3\boldsymbol{j} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right| = \sqrt{9t^4 + 16t^6} = t^2\sqrt{9+16t^2}$$

$$\int_C \frac{\sqrt{y}}{x}\ ds = \int_{1/2}^1 \frac{\sqrt{t^4}}{t^3}\left(t^2\sqrt{9+16t^2}\right)dt$$

$$= \int_{1/2}^1 t\left(9+16t^2\right)^{1/2}\ dt \qquad\qquad d\left(9+16t^2\right) = 32t\,dt$$

$$= \frac{1}{32}\int_{1/2}^1 \left(9+16t^2\right)^{1/2}\ d\left(9+16t^2\right)$$

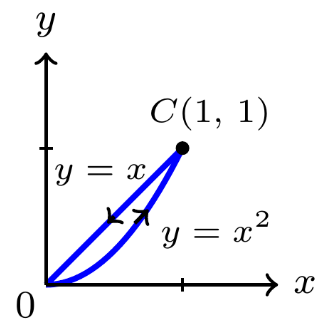
$$=\frac{1}{32}\left(\frac{2}{3}\right)\left[\left(9+16t^2\right)^{3/2}\right]_{1/2}^{1}$$

$$=\frac{1}{48}\left[(25)^{3/2}-(13)^{3/2}\right]$$

$$\boxed{=\frac{1}{48}\left(125-13\sqrt{13}\right)}$$

Exercise

Evaluate $\int_C \left(x+\sqrt{y}\right)\, ds$ where C is



Solution

$C_1: \quad \boldsymbol{r}(t)=t\boldsymbol{i}+t^2\boldsymbol{j} \quad 0\le t\le 1$

$$\frac{dr}{dt}=\boldsymbol{i}+2t\boldsymbol{j} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right|=\sqrt{1+4t^2}$$

$C_2: \quad \boldsymbol{r}(t)=(1\boldsymbol{i}+1\boldsymbol{j})+t(-\boldsymbol{i}-\boldsymbol{j})$

$$=(1-t)\boldsymbol{i}+(1-t)\boldsymbol{j} \qquad 0\le t\le 1$$

$$\frac{dr}{dt}=-\boldsymbol{i}-\boldsymbol{j} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right|=\sqrt{2}$$

$$\int_C\left(x+\sqrt{y}\right)\, ds=\int_0^1\left(t+\sqrt{t^2}\right)\left(\sqrt{1+4t^2}\right)dt+\int_0^1\left(1-t+\sqrt{1-t}\right)\left(\sqrt{2}\right)dt$$

$$=\int_0^1 2t\left(\sqrt{1+4t^2}\right)dt-\sqrt{2}\int_0^1\left((1-t)+\sqrt{1-t}\right)d(1-t)$$

$$=\frac{1}{4}\int_0^1\left(1+4t^2\right)^{1/2}d\left(1+4t^2\right)-\sqrt{2}\left[\frac{1}{2}(1-t)^2+\frac{2}{3}(1-t)^{3/2}\right]_0^1$$

$$=\frac{1}{6}\left[\left(1+4t^2\right)^{3/2}\right]_0^1-\sqrt{2}\left[-\frac{1}{2}-\frac{2}{3}\right]$$

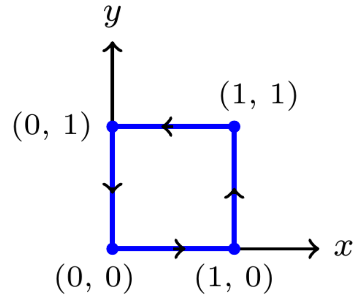
$$=\frac{1}{6}\left[(5)^{3/2}-1\right]+\frac{7\sqrt{2}}{6}$$

$$\boxed{=\frac{5\sqrt{5}-1+7\sqrt{2}}{6}}$$

Exercise

Evaluate $\displaystyle\int_C \frac{1}{x^2+y^2+1}\,ds$ where C is

Solution



$C_1:\ \ \boldsymbol{r}(t)=t\boldsymbol{i}\quad 0\le t\le 1$

$$\frac{dr}{dt}=\boldsymbol{i}\ \Rightarrow\ \left|\frac{dr}{dt}\right|=1$$

$C_2:\ \ \boldsymbol{r}(t)=\boldsymbol{i}+t\boldsymbol{j}\qquad 0\le t\le 1$

$$\frac{dr}{dt}=\boldsymbol{j}\ \Rightarrow\ \left|\frac{dr}{dt}\right|=1$$

$C_3:\ \ \boldsymbol{r}(t)=(1-t)\boldsymbol{i}+\boldsymbol{j}\quad 0\le t\le 1$

$$\frac{dr}{dt}=-\boldsymbol{i}\ \Rightarrow\ \left|\frac{dr}{dt}\right|=1$$

$C_4:\ \ \boldsymbol{r}(t)=(1-t)\boldsymbol{j}\qquad 0\le t\le 1$

$$\frac{dr}{dt}=-\boldsymbol{j}\ \Rightarrow\ \left|\frac{dr}{dt}\right|=1$$

$$\int_C \frac{1}{x^2+y^2+1}\,ds=\int_0^1\frac{1}{t^2+1}(1)\,dt+\int_0^1\frac{1}{1+t^2+1}(1)\,dt$$

$$+\int_0^1\frac{1}{(1-t)^2+1+1}(1)\,dt+\int_0^1\frac{1}{(1-t)^2+1}(1)\,dt$$

$$=\int_0^1\frac{1}{t^2+1}\,dt+\int_0^1\frac{1}{t^2+2}\,dt-\int_0^1\frac{1}{(1-t)^2+2}\,d(1-t)-\int_0^1\frac{1}{(1-t)^2+1}\,d(1-t)$$

$$=\Big[\tan^{-1}t\Big]_0^1+\frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{t}{\sqrt{2}}\right]_0^1-\frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{1-t}{\sqrt{2}}\right]_0^1-\Big[\tan^{-1}(1-t)\Big]_0^1$$

$$=\frac{\pi}{4}+\frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}+\frac{\pi}{4}$$

$$=\frac{\pi}{2}+\frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

Exercise

Find the line integral of $f(x,y)=\dfrac{x^3}{y}$ over the curve $C:\ \ y=\dfrac{x^2}{2},\quad 0\le x\le 2$

Solution

$$r(t) = x\boldsymbol{i} + y\boldsymbol{j} = x\boldsymbol{i} + \frac{1}{2}x^2\boldsymbol{j} \qquad 0 \le x \le 2$$

$$\frac{dr}{dt} = \boldsymbol{i} + xj \quad \Rightarrow \quad \left|\frac{dr}{dt}\right| = \sqrt{1+x^2}$$

$$\int_C f(x,y)\,ds = \int_C \frac{x^3}{y^2}\,ds$$

$$= \int_0^2 2x\sqrt{1+x^2}\,dx \qquad\qquad d\left(1+x^2\right) = 2xdx$$

$$= \int_0^2 \left(1+x^2\right)^{1/2} d\left(1+x^2\right)$$

$$= \frac{2}{3}\left[\left(1+x^2\right)^{3/2}\right]_0^2$$

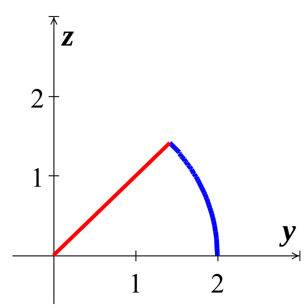
$$= \frac{2}{3}\left[(5)^{3/2} - 1\right]$$

$$= \frac{10\sqrt{5}-2}{3}$$

Exercise

Find the line integral of $f(x,y) = x^2 - y$ over the curve $C: \quad x^2 + y^2 = 4$ in the first quadrant from (0, 2) to $\left(\sqrt{2},\ \sqrt{2}\right)$

Solution



$$x = r\cos t \qquad y = r\sin t$$

$$r(t) = (2\sin t)\boldsymbol{i} + (2\cos t)\boldsymbol{j} \qquad 0 \le t \le \frac{\pi}{4}$$

$$\frac{dr}{dt} = (2\cos t)\boldsymbol{i} - (2\sin t)\boldsymbol{j} \quad \to \quad \left|\frac{dr}{dt}\right| = \sqrt{4\cos^2 t + 4\sin^2 t} = 2$$

$$f(x,y) = x^2 - y = 4\sin^2 t - 2\cos t$$

$$\int_C f(x,y)\,ds = \int_0^{\pi/4} \left(4\sin^2 t - 2\cos t\right)(2)\,dt \qquad\qquad \sin^2 t = \frac{1-\cos 2t}{2}$$

$$= 4\int_0^{\pi/4} (1 - \cos 2t - \cos t)\,dt$$

$$= 4\left[t - \frac{1}{2}\sin 2t - \sin t\right]_0^{\pi/4}$$

$$= 4\left(\frac{\pi}{4} - \frac{1}{2} - \frac{\sqrt{2}}{2}\right)$$

$$= 4\left(\frac{\pi}{4} - \frac{1+\sqrt{2}}{2}\right)$$

$$\underline{= \pi - 2\left(1+\sqrt{2}\right)}$$

Exercise

Evaluate the line integral $\int_C \left(x^2 - 2xy + y^2\right) ds$; C is the upper half of a circle $\vec{r}(t) = \langle 5\cos t,\ 5\sin t\rangle, \quad 0 \le t \le \pi \quad (ccw)$

Solution

$$\vec{r}' = \langle -5\sin t,\ 5\cos t\rangle$$

$$|\vec{r}'| = \sqrt{25\sin^2 t + 25\cos^2 t}$$

$$\underline{= 5}$$

$$\int_C \left(x^2 - 2xy + y^2\right) ds = 5\int_0^{\pi} \left(25\cos^2 t - 50\cos t \sin t + 25\sin^2 t\right) dt$$

$$= 125\int_0^{\pi} (1 - 2\cos t \sin t)\, dt$$

$$= 125\int_0^{\pi} (1 - \sin 2t)\, dt$$

$$= 125\left(t + \frac{1}{2}\cos 2t\right)\Big|_0^{\pi}$$

$$= 125\left(\pi + \frac{1}{2} - \frac{1}{2}\right)$$

$$\underline{= 125\pi}$$

Exercise

Evaluate the line integral $\int_C ye^{-xz} ds$; C is the path $\vec{r}(t) = \langle t,\ 3t,\ -6t\rangle, \quad 0 \le t \le \ln 8$

Solution

$$\vec{r}' = \langle 1,\ 3,\ 6\rangle$$

$$|\vec{r}'| = \sqrt{1 + 9 + 36}$$

$$= \boxed{\sqrt{46}}$$

$$\int_C ye^{-xz}ds = \sqrt{46}\int_0^{\ln 8} 3te^{6t^2}\,dt$$

$$= \frac{\sqrt{46}}{4}\int_0^{\ln 8} e^{6t^2}\,d\left(6t^2\right)$$

$$= \frac{\sqrt{46}}{4}e^{6t^2}\Big|_0^{3\ln 2}$$

$$= \boxed{\frac{\sqrt{46}}{4}\left(e^{54\ln 2} - 1\right)}$$

Exercise

Integrate $f(x, y, z) = \sqrt{x^2 + z^2}$ over the circle $\vec{r}(t) = (a\cos t)\,\hat{j} + (a\sin t)\,\hat{k}, \quad 0 \le t \le 2\pi$

Solution

$$\vec{r}' = \langle 0, \ -a\sin t, \ a\cos t\rangle$$

$$|\vec{r}'| = \sqrt{a^2\sin^2 t + a^2\cos^2 t}$$

$$= \boxed{a}$$

$$f(t) = \sqrt{0 + a^2\sin^2 t}$$

$$= \boxed{a|\sin t|}$$

$$\int_C f\ |r'|\,dt = a^2\int_0^{2\pi} |\sin t|\,dt$$

$$= a^2\int_0^{\pi} \sin t\ dt + a^2\int_{\pi}^{2\pi} \sin t\ dt$$

$$= -a^2\cos t\Big|_0^{\pi} - a^2\cos t\Big|_{\pi}^{2\pi}$$

$$= -a^2(-1-1) - a^2(1+1)$$

$$= 2a^2 + 2a^2$$

$$= \boxed{4a^2}$$

Exercise

Integrate $f(x,y,z)=\sqrt{x^2+y^2}$ over the involute curve

$\vec{r}(t)=\langle \cos t+t\sin t, \quad \sin t-t\cos t\rangle, \quad 0\le t\le \sqrt{3}$

Solution

$$\vec{r}\,'=\langle -\sin t+\sin t+t\cos t, \quad \cos t-\cos t+t\sin t\rangle$$

$$=\langle t\cos t, \quad t\sin t\rangle$$

$$|v|=\sqrt{t^2\cos^2 t+t^2\sin^2 t}$$

$$\boxed{=t}$$

$$f(t)=\sqrt{(\cos t+t\sin t)^2+(\sin t-t\cos t)^2}$$

$$=\sqrt{\cos^2 t+2t\cos t\sin t+t^2\sin^2 t+\sin^2 t-2t\cos t\sin t+t^2\cos^2 t}$$

$$=\sqrt{1+t^2}$$

$$\int_C f\ |v|\,dt=\int_0^{\sqrt{3}} t\sqrt{1+t^2}\ dt$$

$$=\frac{1}{2}\int_0^{\sqrt{3}}\left(1+t^2\right)^{1/2}\ dt$$

$$=\frac{1}{3}\left(1+t^2\right)^{3/2}\Big|_0^{\sqrt{3}}$$

$$=\frac{1}{3}(8-1)$$

$$\boxed{=\frac{7}{3}}$$

Exercise

Find the average of the function on the given curves

$f(x,\ y)=x+2y$ on the line segment from $(1,\ 1)$ to $(2,\ 5)$

Solution

$$\vec{r}(t)=\langle (2-1)t+1, \ (5-1)t+1\rangle$$

$$=\langle t+1, \ 4t+1\rangle$$

$$|\vec{r}\,'(t)|=\sqrt{1+16}\ \boxed{=\sqrt{17}}$$

$$\int_C (x+2y)\,ds=\int_0^1 (t+1+2(4t+1))\cdot\sqrt{17}\ dt$$

$$= \sqrt{17} \int_0^1 (9t + 3)\, dt$$

$$= \sqrt{17} \left(\frac{9}{2} t^2 + 3t \right) \Big|_0^1$$

$$= \sqrt{17} \left(\frac{9}{2} + 3 \right)$$

$$\boxed{= \frac{15}{2} \sqrt{17}}$$

The length of the line segment is $\sqrt{17}$

$\therefore$ The average value is $\frac{15}{2}$

Exercise

Find the average of the function on the given curves

$f(x, y) = x^2 + 4y^2$ on the circle of radius 9 centered at the origin.

Solution

$$\vec{r}(t) = \langle 9\cos t,\ 9\sin t \rangle \quad 0 \le t \le 2\pi$$

$$\vec{r}\,'(t) = \langle -9\sin t,\ 9\cos t \rangle$$

$$\left| \vec{r}\,'(t) \right| = \sqrt{81\sin^2 t + 81\cos^2 t}$$

$$\boxed{= 9}$$

$$\int_C \left(x^2 + 4y^2 \right) ds = 9 \int_0^{2\pi} \left(81\cos^2 t + 324\sin^2 t \right) dt$$

$$= 729 \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2}\cos 2t + 2 - 2\cos 2t \right) dt$$

$$= 729 \int_0^{2\pi} \left(\frac{5}{2} - \frac{3}{2}\cos 2t \right) dt$$

$$= 729 \left(\frac{5}{2} t - \frac{3}{4}\sin 2t \right) \Big|_0^{2\pi}$$

$$\boxed{= 3,645\pi}$$

The circumference of the circle is $9(2\pi) = 18\pi$

$\therefore$ The average value is $\frac{3645\pi}{18\pi} = \frac{405}{2}$

Exercise

Find the average of the function on the given curves

$f(x, y) = xe^y$ on the circle of radius 1 centered at the origin.

Solution

$$\vec{r}(t) = \langle \cos t, \ \sin t \rangle \quad 0 \le t \le 2\pi$$

$$\vec{r}'(t) = \langle -\sin t, \ \cos t \rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{\sin^2 t + \cos^2 t}$$

$$\underline{= 1|}$$

$$\int_C xe^y ds = \int_0^{2\pi} \cos t \ e^{\sin t} \ dt$$

$$= \int_0^{2\pi} e^{\sin t} \ d(\sin t)$$

$$= e^{\sin t} \Big|_0^{2\pi}$$

$$= 1 - 1$$

$$\underline{= 0 \,|}$$

$\therefore$ The average value is 0

Exercise

Find the average of the function on the given curves

$f(x, y) = \sqrt{4 + 9y^{2/3}}$ on the curve $y = x^{3/2}$, for $0 \le x \le 5$

Solution

$$\vec{r}(t) = \left\langle t, \ t^{3/2} \right\rangle$$

$$\vec{r}'(t) = \left\langle 1, \ \frac{3}{2} t^{1/2} \right\rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{1 + \frac{9}{4} t}$$

$$= \frac{1}{2}\sqrt{4 + 9t}$$

$$\int_C \sqrt{4 + 9y^{2/3}} ds = \frac{1}{2} \int_0^5 \sqrt{4 + 9\left(t^{3/2}\right)^{2/3}} \sqrt{4 + 9t} \ dt$$

$$= \frac{1}{2}\int_0^5 \sqrt{4+9t}\sqrt{4+9t}\ dt$$

$$= \frac{1}{2}\int_0^5 (4+9t)\ dt$$

$$= \frac{1}{2}\left(4t+\frac{9}{2}t^2\right)\Big|_0^5$$

$$= \frac{1}{2}\left(20+\frac{225}{2}\right)$$

$$\boxed{= \frac{265}{4}}$$

The length of the curve is

$$\int_0^5 \sqrt{4+9\left(x^{3/2}\right)^{2/3}}\ dx = \frac{1}{2}\int_0^5 \sqrt{4+9x}\ dx$$

$$= \frac{1}{18}\int_0^5 (4+9x)^{1/2}\ d(4+9x)$$

$$= \frac{1}{27}(4+9x)^{3/2}\Big|_0^5$$

$$= \frac{1}{27}(343-8)$$

$$= \frac{335}{27}$$

$\therefore$ The average value is $= \frac{265}{4}\times\frac{27}{335} = \frac{1431}{268}$

Exercise

Find the length of the curve $\quad \vec{r}(t) = \left\langle 20\sin\frac{t}{4},\ 20\cos\frac{t}{4},\ \frac{t}{2}\right\rangle \quad 0 \le t \le 2$

Solution

$$\vec{r}'(t) = \left\langle 5\cos\frac{t}{4},\ -5\sin\frac{t}{4},\ \frac{1}{2}\right\rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{25\cos^2\frac{t}{4}+25\sin^2\frac{t}{4}+\frac{1}{4}}$$

$$= \sqrt{25+\frac{1}{4}}$$

$$\boxed{= \frac{1}{2}\sqrt{101}}$$

$$L = \int_0^2 \frac{1}{2}\sqrt{101}dt$$

$$= \frac{1}{2}\sqrt{101}(2)$$

$$\underline{= \sqrt{101}}$$

Exercise

Find the length of the curve $\vec{r}(t) = \langle 30\sin t,\ 40\sin t,\ 50\cos t\rangle \quad 0 \le t \le 2\pi$

Solution

$$|\vec{r}'(t)| = \sqrt{900\cos^2 t + 1600\cos^2 t + 2500\sin^2 t}$$

$$= \sqrt{2500\cos^2 t + 2500\sin^2 t}$$

$$\underline{= 50}$$

$$L = \int_0^{2\pi} 50\ dt$$

$$\underline{= 100\pi}$$