Lecture One

Section 1.1 – Introduction to System of Linear Equations

Given the linear equations

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases}$$

The solution to this system is $(3,\ 1)$, which means that 2 lines meeting at a single point.

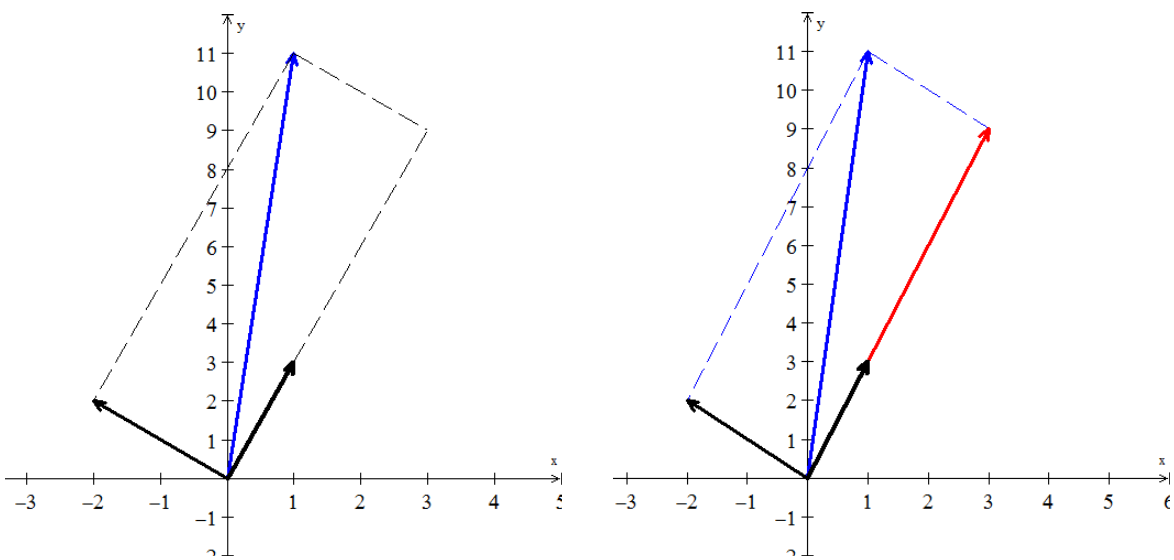
We can rewrite the system equation as linear combination:

$$x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

$$x.v_1 + y.v_2 = v$$

$$\begin{bmatrix} 1+x \\ 3+y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} \Rightarrow \begin{bmatrix} x=3 \\ y=9 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 9 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$



Therefore, the side vectors are $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$ *and* $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$

The diagonal sum is $\begin{bmatrix} 3-2 \\ 9+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$

The linear combination is given by:

$$3\begin{bmatrix}1\\3\end{bmatrix}+1\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$$

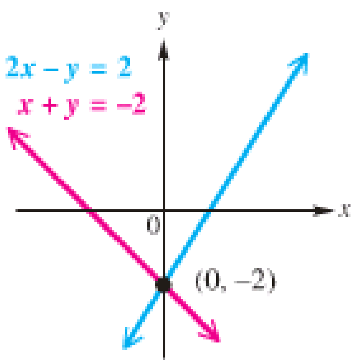
Thus, the solution is $x=3 \quad y=1$

<u>Note</u>

$\begin{bmatrix}1 & -2\\3 & 2\end{bmatrix}$ is called the coefficient matrix

The matrix form of the system is written as $\qquad Ax=b$

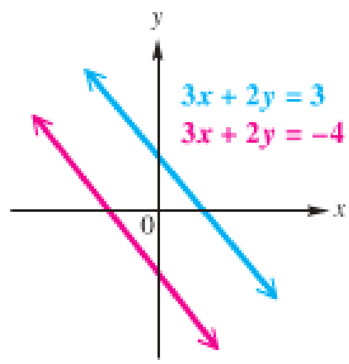
$$\begin{bmatrix}1 & -2\\3 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$$



One solution (lines intersect)

Consistent

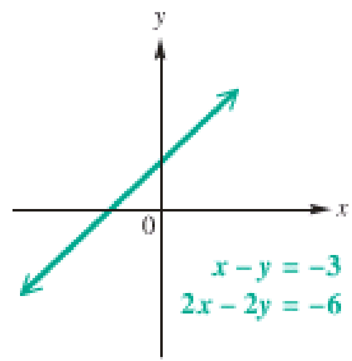
Independent



No Solution (lines //)

Inconsistent

Independent



Infinite solution

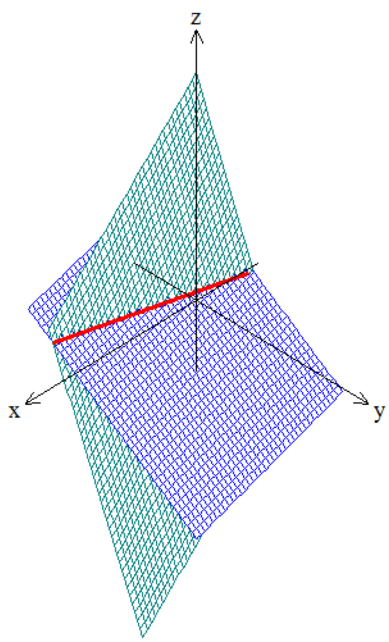
Consistent

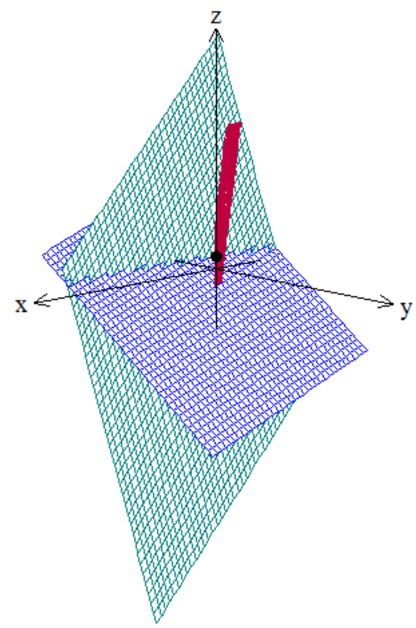
Dependent

Three Equations in 3 Unknowns

Given the system equations

$$x + 2y + 3z = 6$$
$$2x + 5y + 2z = 4$$
$$6x - 3y + z = 2$$

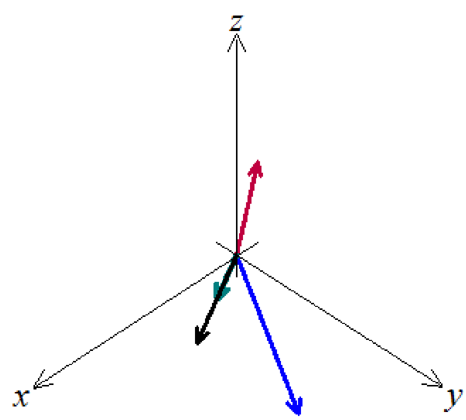




This system can be written as linear combination:

$$x\begin{bmatrix}1\\2\\6\end{bmatrix} + y\begin{bmatrix}2\\5\\-3\end{bmatrix} + z\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix}$$

Let $b = \begin{bmatrix}6\\4\\2\end{bmatrix}$



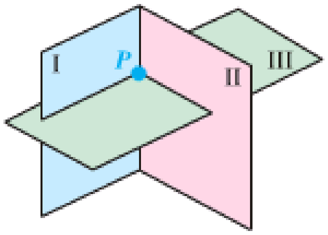
We want to multiply the three column vectors by x, y, z to produce ***b***.

The combination of the three vectors that produces vector b is 2 times the third vector.

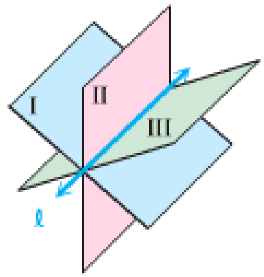
$$2(3, 2, 1) = (6, 4, 2) = b$$

Therefore the coefficients that we need are $x = 0, \quad y = 0, \quad and\ z = 2$.

$$0\begin{bmatrix}1\\2\\6\end{bmatrix} + 0\begin{bmatrix}2\\5\\-3\end{bmatrix} + 2\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix}$$

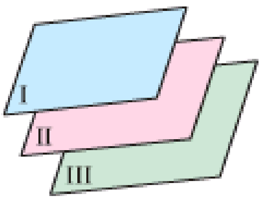


A single solution

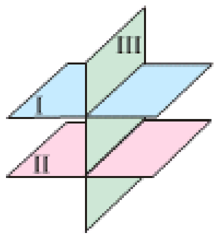


Points of a line in common

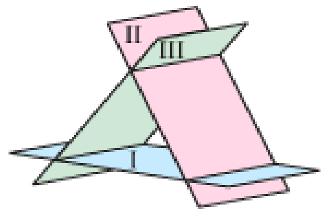
All points in common



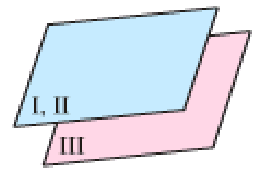
No points in common



No points in common



No points in common



No points in common

Exercises *Section* 1.1 – Introduction to System of Linear Equations

1. Find a solution for x, y, z to the system of equations

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3e+2\sqrt{2}+\pi \\ 6e+5\sqrt{2}+4\pi \\ 9e+8\sqrt{2}+7\pi \end{pmatrix}$$

2. Draw the two pictures in two planes for the equations: $x-2y=0, \quad x+y=6$

3. Normally 4 planes in 4-dimensional space meet at a ________. Normally 4 column vectors in 4-deimensional space can combine to produce b. what combinations of $(1,\ 0,\ 0,\ 0), \quad (1,\ 1,\ 0,\ 0), \quad (1,\ 1,\ 1,\ 0), \quad (1,\ 1,\ 1,\ 1)$ produces $b=(3,\ 3,\ 3,\ 2)$?
What 4 equations for $x,\ y,\ z,\ w$ are you solving?

4. What 2 by 2 matrix A rotates every vector through 45° ?
The vector (1, 0) goes to $\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$. The vector (0, 1) goes to $\left(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$.
Those determine the matrix. Draw these particular vectors is the xy-plane and find A.

5. What two vectors are obtained by rotating the plane vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by 30° (*cw*) ?
Write a matrix A such that for every vector v in the plane, Av is the vector obtained by rotating v clockwise by 30°.
Find a matrix B such that for every 3-dimensional vector v, the vector Bv is the reflection of v through the plane $x+y+z=0$. *Hint* : $v=(1,\ 0,\ 0)$

6. In each part, find a system of linear equation corresponding to the given augmented matrix

a) $\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \\ 5 & -6 & 1 & 1 \\ -8 & 0 & 0 & 3 \end{bmatrix}$

7. Find the augmented matrix for the given system of linear equations.

a) $\begin{cases} -2x_1 = 6 \\ 3x_1 = 8 \\ 9x_1 = -3 \end{cases}$

b) $\begin{cases} 3x_1 - 2x_2 = -1 \\ 4x_1 + 5x_2 = 3 \\ 7x_1 + 3x_2 = 2 \end{cases}$

c) $\begin{cases} 2x_1 \quad\quad + 2x_3 = 1 \\ 3x_1 - x_2 + 4x_3 = 7 \\ 6x_1 + x_2 - x_3 = 0 \end{cases}$

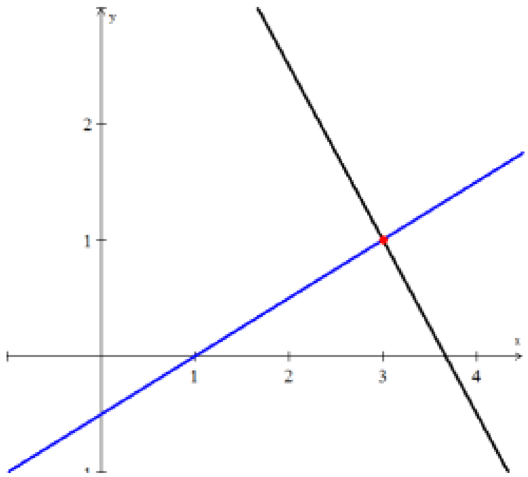
Section 1.2 – Gaussian Elimination

Elimination produces an ***upper triangular system***.

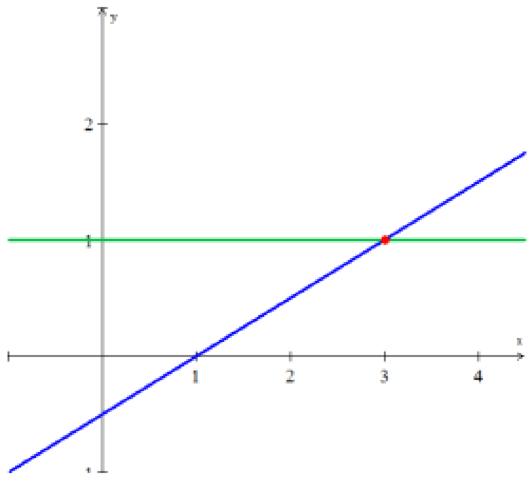
$$\begin{cases} x-2y=1 \\ 3x+2y=11 \end{cases} \Rightarrow \begin{cases} x-2y=1 & \textit{Mutliply by } 3 \\ \quad 8y=8 & \textit{and subtract} \end{cases}$$

The equation $8y=8$ *reveals* $y=1$

This process is called ***back substitution***.



Before elimination



After elimination

Definitions

Pivot: first nonzero in the row that does the elimination

Multiplier: (entry to eliminate) divide by pivot

$$\begin{array}{lll} 4x-8y=4 & \textit{Multiply equation } 1 \textit{ by } \frac{3}{4} & 4x-8y=4 \\ 3x+2y=11 & \textit{Subtract from equation } 2 & \quad 8y=8 \end{array}$$

The first pivot is 4 (the coefficient of x) and the multiplier is $l=\frac{3}{4}$

The pivots are on the diagonal of the triangle after elimination.

Reduced Row Echelon Form

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

Use the Gaussian elimination method to solve the system

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 3 & 1 & 2 & 31 \\ 1 & 3 & 2 & 25 \end{array}\right] \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{matrix} \qquad \begin{array}{cccc} 3 & 1 & 2 & 31 \\ -3 & -3 & -6 & -57 \\ \hline 0 & -2 & -4 & -26 \end{array} \qquad \begin{array}{cccc} 1 & 2 & 2 & 25 \\ -1 & -1 & -2 & -19 \\ \hline 0 & 2 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & -2 & -4 & -26 \\ 0 & 2 & 0 & 6 \end{array}\right] -\tfrac{1}{2}R_2 \qquad \begin{array}{cccc} 0 & 1 & 2 & 13 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 2 & 0 & 6 \end{array}\right] R_3 - 2R_2 \qquad \begin{array}{cccc} 0 & 2 & 0 & 6 \\ 0 & -2 & -4 & -26 \\ \hline 0 & 0 & -4 & -20 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -4 & -20 \end{array}\right] -\tfrac{1}{4}R_3 \qquad \begin{array}{cccc} 0 & 0 & 1 & 5 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array}\right] \Rightarrow \begin{aligned} x + y + 2z &= 19 \quad (3) \\ y + 2z &= 13 \quad (2) \\ z &= 5 \quad (1) \end{aligned}$$

$(2) \Rightarrow y = 13 - 2z = 13 - 2(5) = 3$

$(3) \Rightarrow x = 19 - y - 2z = 19 - 3 - 10 = 6$

$\Rightarrow$ **(6, 3, 5)**

Example

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \quad\quad + 2x_5 \quad\quad &= 0\\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1\\
5x_3 + 10x_4 \quad\quad + 15x_6 &= 5\\
2x_1 + 6x_2 \quad\quad + 8x_4 + 4x_5 + 18x_6 &= 6
\end{aligned}$$

Solution

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 2 & 6 & -5 & -2 & 4 & -3 & -1\\ 0 & 0 & 5 & 10 & 0 & 15 & 5\\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \quad \textit{Adding } (-2) \textit{ times the } 1st \textit{ row to the } 2nd \\ \\ R_4 - 2R_1 \quad \textit{Adding } (-2) \textit{ times the } 1st \textit{ row to the } 4th \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 0 & 0 & -1 & -2 & 0 & -3 & -1\\ 0 & 0 & 5 & 10 & 0 & 15 & 5\\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] -R_2$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 2 & 0 & 3 & 1\\ 0 & 0 & 5 & 10 & 0 & 15 & 5\\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 5R_2 \\ R_3 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 2 & 0 & 3 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right] \tfrac{1}{6}R_4 \quad \textit{then interchanging row3 and row4}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 2 & 0 & 3 & 1\\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3}\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] R_2 - 3R_3$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3}\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{cases} x_1 + 3x_2 \quad + 4x_4 + 2x_5 = 0\\ x_3 + 2x_4 = 0\\ +x_6 = \frac{1}{3} \end{cases}$$

The general solution of the system: $x_6 = \frac{1}{3}, \quad x_3 = -2x_4, \quad x_1 = -3x_2 - 4x_4 - 2x_5$

Example

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 4 & 16 & -3 & -1 & -10 \\ -2 & 4 & -1 & 3 & -6 \\ -6 & 2 & 5 & 1 & 3 \end{array}\right] \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 + 3R_1 \end{array} \qquad \left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 - \frac{13}{6}R_2 \end{array}$$

$$\begin{aligned} 2x + 8y - z + w &= 0 \\ 4x + 16y - 3z - w &= -10 \\ -2x + 4y - z + 3w &= -6 \\ -6x + 2y + 5z + w &= 3 \end{aligned}$$

Solution

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \qquad \textit{Interchange } R_2 \textit{ and } R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & \frac{19}{3} & -\frac{14}{3} & 16 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 + \frac{19}{3}R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & 0 & -\frac{71}{3} & -\frac{142}{3} \end{array}\right] \begin{array}{rcll} 2x + 8y - z + w = 0 & \rightarrow & 2x = -8y + z - w = 6 & \Rightarrow \boxed{x = 3} \\ 12y - 2z + 4w = -6 & \rightarrow & 12y = 2z - 4w - 6 = -6 & \Rightarrow \boxed{y = -\frac{1}{2}} \\ -z - 3w = -10 & \rightarrow & z = 10 - 3w = 4 & \\ -\frac{71}{3}w = -\frac{142}{3} & \rightarrow & \boxed{w = 2} & \end{array}$$

Solution: $\left(3,\ -\frac{1}{2},\ 4,\ 2\right)$

***Theorem*: Free Variable Theorem for Homogeneous Systems**

If a *homogeneous linear* system has ***n*** unknowns, and if the reduced row echelon form of its augmented matrix has ***r*** nonzero rows, then the system has $n - r$ free variables.

Theorem

A *homogeneous linear* system with more unknowns than equations has ***infinitely many*** *unknowns*.

Breakdown Elimination

Permanent failure with no solution

$$\begin{array}{rcr} x-2y=1 & \textit{Subtract 3 times} & x-2y=1 \\ 3x-6y=11 & \textit{eqn. 1 from eqn. 2} & 0y=8 \end{array}$$

The last equation $0y=8$; therefore there is *no* solution. This system has no second pivot, since no zero allowed as a pivot.

Permanent failure with infinitely many solutions

$$\begin{array}{rcr} x-2y=1 & \textit{Subtract 3 times} & x-2y=1 \\ 3x-6y=3 & \textit{eqn. 1 from eqn. 2} & 0y=0 \end{array}$$

Every *y* satisfies $0y=0$. There is only one equation $x-2y=1$.

There are ***unique infinitely*** many solutions.

Three Equations in Three Unknowns

To understand Gaussian elimination, you have to go beyond 2 by 2 systems.

Consider the system equations: $\begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases}$

$$\begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases} \quad \begin{array}{c} \textit{subtract 2 times eqn.1} \\ \textit{from eqn.2} \\ \ \end{array} \quad \begin{array}{r} 2x+4y-2z=2 \\ y+z=4 \\ -2x-3y+7z=10 \end{array}$$

$$\begin{cases} 2x+4y-2z=2 \\ y+z=4 \\ -2x-3y+7z=10 \end{cases} \quad \begin{array}{l} \textit{Add eqn.1} \\ \ \\ \textit{and eqn.3} \end{array} \quad \begin{array}{r} 2x+4y-2z=2 \\ y+z=4 \\ y+5z=12 \end{array}$$

$$\begin{cases} 2x+4y-2z=2 \\ y+z=4 \\ y+5z=12 \end{cases} \quad \begin{array}{c} \ \\ \textit{Subtract eqn.2} \\ \textit{from eqn.3} \end{array} \quad \begin{array}{r} 2x+4y-2z=2 \\ y+z=4 \\ 4z=8 \end{array} \quad \begin{array}{c} \Rightarrow \\ \Rightarrow \\ \Rightarrow \end{array} \quad \begin{array}{r} \boxed{x=1-2y+z=-1} \\ \boxed{y=4-z=2} \\ \boxed{z=2} \end{array}$$

The solution is $\boxed{(-1,\ 2,\ 2)}$

Exercises *Section* 1.2 – Gaussian Elimination

1. When elimination is applied to the matrix $A=\begin{bmatrix}3&1&0\\6&9&2\\0&1&5\end{bmatrix}$

a) What are the first and second pivots?

b) What is the multiplier l_{21} in the first step (l_{21} times row 1 is subtracted from row 2)?

c) What entry in the 2, 2 position (instead of 9) would force an exchange of rows 2 and 3?

d) What is the multiplier $l_{31}=0$, subtracting 0 times row 1 from row 3?

2. Use elimination to reach upper triangular matrices U. Solve by back substitution or explain why this impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the $-x$ in equation (3).

$$\begin{cases}x+y+z=7\\x+y-z=5\\x-y+z=3\end{cases}\qquad\qquad\begin{cases}x+y+z=7\\x+y-z=5\\-x-y+z=3\end{cases}$$

3. For which numbers a does the elimination break down (1) permanently (2) temporarily

$$ax+3y=-3$$
$$4x+6y=6$$

Solve for x and y after fixing the second breakdown by a row change.

4. Find the pivots and the solution for these four equations:

$$\begin{aligned}2x+y\qquad\quad&=0\\x+2y+z\quad\;&=0\\y+2z+t&=0\\z+2t&=5\end{aligned}$$

5. Look for a matrix that has row sums 4 and 8, and column sums 2 and s.

$$A=\begin{bmatrix}a&b\\c&d\end{bmatrix}\qquad\begin{matrix}a+b=4&a+c=2\\c+d=8&b+d=s\end{matrix}$$

The four equations are solvable only if s = ____. Then find two different matrices that have the correct row and column sums.

6. Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of A is a _______ of the first two rows. Find a third equation that can't be solved together with $x+y+z=0$ and $x-2y-z=1$

7. Solve the linear system by Gauss-Jordan elimination.

$a)\ \begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$ $\qquad b)\ \begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x - 3w = -3 \end{cases}$

$c)\ \begin{cases} x + 2y + z = 8 \\ -x + 3y - 2z = 1 \\ 3x + 4y - 7z = 10 \end{cases}$ $\qquad d)\ \begin{cases} 2u - 3v + w - x + y = 0 \\ 4u - 6v + 2w - 3x - y = -5 \\ -2u + 3v - 2w + 2x - y = 3 \end{cases}$

8. Solve the given linear system by any method

$a)\ \begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 = 0 \\ x_2 + x_3 = 0 \end{cases}$ $\qquad b)\ \begin{cases} 2x + 2y + 4z = 0 \\ -y - 3z + w = 0 \\ 3x + y + z + 2w = 0 \\ x + 3y - 2z - 2w = 0 \end{cases}$

9. Add 3 times the second row to the first of

$$\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

10. Solve the system using Gaussian elimination $\begin{cases} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{cases}$

Section 1.3 – The Algebra of Matrices

Matrices

$$\begin{array}{r c} & \text{Column} \\ & \begin{array}{ccc} C_1 & C_2 & C_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{r} Row\ 1 \to R_1 \\ Row\ 2 \to R_2 \\ Row\ 3 \to R_3 \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

This is called Matrix (*Matrices*)

Each number in the array is an ***element*** or ***entry***

The matrix is said to be of order $m \ x \ n$

m: numbers of rows,

n: number of columns

When $m = n$, then matrix is said to be ***square***.

Given the system equations

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Write into an ***augmented matrix*** form

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 31 \\ 1 & 1 & 2 & 19 \\ 1 & 3 & 2 & 25 \end{array}\right]$$

The Matrix: $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}$ is called the ***coefficient matrix*** of the system.

The matrix A above has 3 rows and 3 columns, therefore the order of the matrix A is $(3 \ x \ 3)$

$$A = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Equality of Matrices

Definition of Equality of Matrices

Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are equal if and only if they have the same order (size) $m \times n$ and if each pair corresponding elements is equal

$a_{ij} = b_{ij}$ for $i = 1, 2, \ldots, m$ ***and*** $j = 1, 2, \ldots, n$

Example

Find the values of the variables for which each statement is true, if possible.

a) $\begin{bmatrix} 2 & 1 \\ p & q \end{bmatrix} = \begin{bmatrix} x & y \\ -1 & 0 \end{bmatrix}$

$x = 2,\ y = 1,\ \ p = -1,\ \ q = 0$

b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$

can't be true

c) $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

$\Rightarrow \begin{bmatrix} w = 9 & x = 17 \\ 8 = y & -12 = z \end{bmatrix}$

Addition and *Subtraction* of Matrices

Definition

If $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ are $m\times n$ matrices, their sum $A+B$, is the $m\times n$ matrix obtained by adding the corresponding entries; that is

$$\left[a_{ij}\right]+\left[b_{ij}\right]=\left[a_{ij}+b_{ij}\right]$$

Matrices can be added if their shapes are the same, meaning have the same ***order***.

$$\begin{bmatrix}1 & 2\\ 3 & 4\\ 0 & 0\end{bmatrix}+\begin{bmatrix}2 & 2\\ 4 & 4\\ 9 & 9\end{bmatrix}=\begin{bmatrix}1+2 & 2+2\\ 3+4 & 4+4\\ 0+9 & 0+9\end{bmatrix}$$

$$=\begin{bmatrix}3 & 4\\ 7 & 8\\ 9 & 9\end{bmatrix}$$

Scalar Multiplication Matrices

Definition

If k is a scalar and $A=\left[a_{ij}\right]$ is an $m\times n$ matrices, then scalar product $\boldsymbol{kA}$ is the $m\times n$ matrix obtained by multiplying each entry of A by k; that is

$$k\left[a_{ij}\right]=\left[ka_{ij}\right]$$

$$kA=k\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$$

$$=\begin{bmatrix}ka_{11} & ka_{12}\\ ka_{21} & ka_{22}\end{bmatrix}$$

Example

$$2\begin{bmatrix}1 & 2\\ 3 & 4\\ 0 & 0\end{bmatrix}=\begin{bmatrix}(2)1 & (2)2\\ (2)3 & (2)4\\ (2)0 & (2)0\end{bmatrix}$$

$$=\begin{bmatrix}2 & 4\\ 6 & 8\\ 0 & 0\end{bmatrix}$$

Definition

If $A_1, A_2, \ldots, A_n$ are matrices of the same size, and if $c_1, c_2, \ldots, c_n$ are scalars, then expression of the form

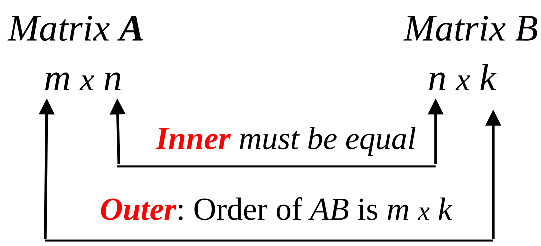
$$c_1A_1 + c_2A_2 + \ldots + c_nA_n$$

Is called a ***linear combination*** of $A_1, A_2, \ldots, A_n$ with *coefficients* $c_1, c_2, \ldots, c_n$.

Matrix Multiplication

Product of Two Matrices

Let ***A*** be an *m x n* matrix and let ***B*** be an *n x k* matrix. To find the element in the i^{th} row and j^{th} column of the product matrix ***AB***, multiply each element in the i^{th} row of ***A*** by the corresponding element in the j^{th} column of ***B***, and then add these products. The product matrix ***AB*** is an *m x k* matrix.



- ✓ To multiply ***AB*** or dot product, if ***A*** has ***n*** columns, ***B*** must have ***n*** rows.
- ✓ Squares matrices can be multiplied if and only if (***iff***) they have the same size.
- ✓ The entry in row *i* and column *j* of AB is $\boxed{(row\ i\ of\ A).(col\ j\ of\ B)}$

  The result: $\sum a_{ik}b_{kj}$

$$\begin{bmatrix} * & * & & & \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{i5} \\ * & & & & \\ * & & & & \end{bmatrix} \begin{bmatrix} * & * & b_{1j} & * & * & * \\ & & b_{2j} & & & \\ & & \vdots & & & \\ & & \vdots & & & \\ & & b_{5j} & & & \end{bmatrix} = \begin{bmatrix} & & * & & & \\ * & * & (AB)_{ij} & * & * & * \\ & & * & & & \\ & & * & & & \end{bmatrix}$$

$$4\ by\ 5 \qquad\qquad 5\ by\ 6 \qquad\qquad 4\ by\ 6$$

$$\boldsymbol{AB} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$2x2 \qquad 2x2 \quad \rightarrow \quad 2x2$$

$$a_{11} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & \_ \\ \_ & \_ \end{bmatrix}$$

$$a_{12} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & af + bh \\ \_ & \_ \end{bmatrix}$$

$$a_{21} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ ce + dg & \_ \end{bmatrix}$$

$$a_{22} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ \_ & cf + dh \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Example

Find: $\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}$

Solution

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} 1(5) + 1(1) & 1(6) + 1(0) \\ 2(5) - 1(1) & 2(6) - 1(0) \end{bmatrix} \\ &= \begin{bmatrix} 6 & 6 \\ 9 & 12 \end{bmatrix} \end{aligned}$$

Special Case

When A is a square matrix, then

$$A \quad times \quad A^2 = A^2 \quad times \quad A \;\; \underline{= A^3}$$

$$A^p = AA \cdots A \qquad (p \text{ factors})$$

$$\left(A^p\right)\left(A^q\right) = A^{p+q}$$

$$\left(A^p\right)^q = A^{pq}$$

Block Multiplication

If the cuts between columns of ***A*** match the cuts between rows of ***B***, then the block multiplication of ***AB*** allowed.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \left[\begin{array}{c|c} a_{11}b_{11} + a_{12}b_{21} & a_{12}b_{21} + a_{12}b_{22} \\ \hline a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{21} + a_{22}b_{22} \end{array}\right]$$

Important special case

$$\begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix}$$

Matrix Form of the Equations

The coefficient matrix is $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}$

The equivalent matrix equation is in the form $AX = b$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Multiplication by ***rows*** $\quad AX = \begin{bmatrix} (row\ 1).X \\ (row\ 2).X \\ (row\ 3).X \end{bmatrix}$

Multiplication by ***columns*** $\quad AX = x\ (column\ 1) + y\ (column\ 2) + z\ (column\ 3)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Identity Matrix

The identity matrix is given by the form: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \boxed{Ix = x}$

Properties of Matrix

Addition and Scalar Multiplication

$A+B=B+A$	*Commutative Property of Addition*
$A+(B+C)=(A+B)+C$	*Associative Property of Addition*
$(kl)A=k(lA)$	*Associative Property of Scalar Multiplication*
$k(A+B)=kA+kB$	*Distributive Property*
$k(A-B)=kA-kB$	*Distributive Property*
$(k+l)A=kA+lA$	*Distributive Property*
$(k-l)A=kA-lA$	*Distributive Property*
$A+0=0+A=A$	*Additive Identity Property*
$A+(-A)=(-A)+A=0$	*Additive Inverse Property*
$k(AB)=kA(B)=A(kB)$	

Multiplication

$AB\neq BA$	*Commutative "**law**" is usually broken*
$A(BC)=(AB)C$	*Associative Property of Multiplication* (***Parentheses not needed***)
$A(B+C)=AB+AC$	*Distributive Property*
$(B+C)A=BA+CA$	*Distributive Property*
$A(B-C)=AB-AC$	*Distributive Property*
$(B-C)A=BA-CA$	*Distributive Property*

****

Consider the three vectors $\boldsymbol{u}, \quad \boldsymbol{v}, \quad and\ \boldsymbol{w}$:

$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The linear combinations in three-dimensional space are $cu + dv + ew$

Combination $$c\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}$$

Combine the three vectors $u,\ v,\ and\ w$ into on matrix A.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Multiplies the matrix A by a vector x, where $c,\ d,\ e$ are the component of a vector x.

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}$$

We can rewrite the form, matrix A times the vector x, as the combination $cu + dv + ew$

$$Ax = \begin{bmatrix} u & v & w \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$$

Write the matrix in the form $Ax = b$

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b$$

Where the x is the input and b is the output.

Cyclic Difference

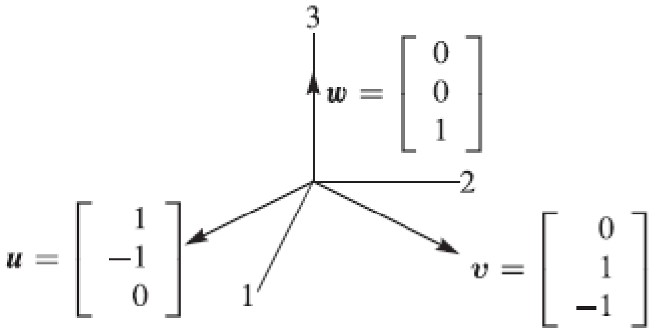
The linear combinations of three vectors $u,\ v,\ and\ w^*$ lead to a cyclic difference matrix C and is given by:

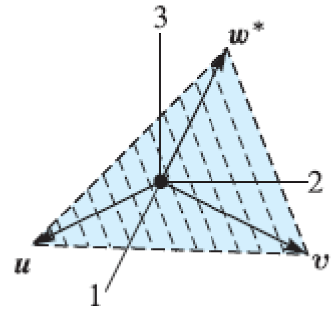
$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad w^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = b$$

The matrix C is not triangular. It is not easy to find the solution to $Cx = b$, because either we are going to have ***infinitely many solution*** or ***no solution***..

Let looks at these problems geometrically.





Exercises *Section* 1.3 – The Algebra of Matrices

1. For the matrices: $A=\begin{bmatrix} p & 0 \\ q & r \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, when does $AB=BA$

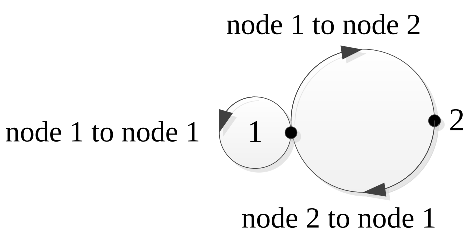
2. Find a combination $x_1w_1+x_2w_2+x_3w_3$ that gives the zero vector:

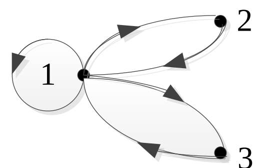
$$w_1=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \quad w_2=\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}; \quad w_3=\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are independent or dependent?
The vectors lie in a ______.
The matrix W with those columns is not invertible.

3. The very last words say that the 5 by 5 centered difference matrix is not invertible, Write down the 5 equations $Cx=b$. Find a combination of left sides that gives zero. What combination of b_1, b_2, b_3, b_4, b_5 must be zero?

4. A direct graph starts with *n* nodes. There are n^2 possible edges, each edge leaves one of the *n* nodes and enters one of the *n* nodes (possibly itself). The *n* by *n* adjacency matrix has $a_{ij}=1$ when edge leaves node *i* and enter node *j*; if no edge then $a_{ij}=0$. Here are directed graphs and their adjacency matrices:





$$A=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The *i*, *j* entry of A^2 is $a_{i1}a_{1j}+\ldots+a_{in}a_{nj}$.

Why does that sum count the two-step paths from *i* to any node to *j*?

The *i*, *j* entry of A^k counts *k*-steps paths:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{matrix} \textit{counts the paths} \\ \textit{with two edges} \end{matrix} \quad \begin{bmatrix} 1 \text{ to } 2 \text{ to } 1, 1 \text{ to } 1 \text{ to } 1 & 1 \text{ to } 1 \text{ to } 2 \\ 2 \text{ to } 1 \text{ to } 1 & 2 \text{ to } 1 \text{ to } 2 \end{bmatrix}$$

List all 3-step paths between each pair of nodes and compare with A^3. When A^k has ***no zeros***, that number k is the diameter of the graph – the number of edges needed to connect the most pair of nodes. What is the diameter of the second graph?

5. A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

a) AB *b*) BA *c*) ABD *d*) DBA

e) ABC *f*) ABCD *g*) A(B + C)

6. What rows or columns or matrices do you multiply to find.

a) The third column of AB?

b) The second column of AB?

c) The first row of AB?

d) The second row of AB?

e) The entry in row 3, column 4 of AB?

f) The entry in row 2, column 3 of AB?

7. Add AB to AC and compare with $A(B+C)$:

$$A=\begin{bmatrix}1 & 5\\2 & 3\end{bmatrix} \quad \textit{and} \quad B=\begin{bmatrix}0 & 2\\0 & 1\end{bmatrix} \quad \textit{and} \quad C=\begin{bmatrix}3 & 1\\0 & 0\end{bmatrix}$$

8. True or False

a) If A^2 is defined then A is necessarily square.

b) If AB and BA are defined then A and B are square.

c) If AB and BA are defined then AB and BA are square.

d) If $AB=B$, then $A=I$

9. *a*) Find a nonzero matrix A such that $A^2=0$

b) Find a matrix that has $A^2 \neq 0$ but $A^3=0$

10. Suppose you solve $Ax=b$ for three special right sides b:

$$Ax_1=\begin{bmatrix}1\\0\\0\end{bmatrix} \quad Ax_2=\begin{bmatrix}0\\1\\0\end{bmatrix} \quad Ax_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

If the three solutions x_1, x_2, x_3 are the columns of a matrix X, what is A times X?

11. Show that $(A+B)^2$ is different from $A^2+2AB+B^2$, when

$$A=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix} \quad and \quad B=\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}$$

Write down the correct rule for $(A+B)(A+B)=A^2+$ __________ $+B^2$

12. Find the product of the 2 matrices by rows or by columns:

a) $\begin{bmatrix}2 & 3\\ 5 & 1\end{bmatrix}\begin{bmatrix}4\\ 2\end{bmatrix}$

b) $\begin{bmatrix}1 & 2 & 4\\ 2 & 0 & 1\end{bmatrix}\begin{bmatrix}3\\ 1\\ 1\end{bmatrix}$

c) $\begin{bmatrix}3 & 6\\ 6 & 12\end{bmatrix}\begin{bmatrix}2\\ -1\end{bmatrix}$

d) $\begin{bmatrix}1 & 2 & 4\\ -2 & 3 & 1\\ -4 & 1 & 2\end{bmatrix}\begin{bmatrix}2\\ 2\\ 3\end{bmatrix}$

13. Given $A=\begin{bmatrix}4 & 1 & 3\\ 3 & -1 & -2\\ 0 & 0 & 4\end{bmatrix}$ $B=\begin{bmatrix}-3 & -2 & -3\\ -1 & 0 & 0\\ 8 & -2 & -4\end{bmatrix}$ Find $A+B,\ \ 2A,\ \ and\ \ -B$

14. Given $A=\begin{bmatrix}3 & 2 & -3\\ 0 & 1 & 0\end{bmatrix}$ $B=\begin{bmatrix}3 & -4\\ 0 & 1\\ 1 & 0\end{bmatrix}$ Find $AB\ \ and\ \ BA\ \ if\ possible$

15. Given $A=\begin{bmatrix}5 & 3\\ -1 & 0\end{bmatrix}$ $B=\begin{bmatrix}4 & -2\\ -2 & 0\\ 9 & 1\end{bmatrix}$ Find $AB\ \ and\ \ BA\ \ if\ possible$

16. Given $A=\begin{bmatrix}3 & 0\\ -1 & 2\\ 1 & 1\end{bmatrix}$ $B=\begin{bmatrix}4 & -1\\ 0 & 2\end{bmatrix}$ Find $AB\ \ and\ \ BA\ \ if\ possible$

17. Consider the matrices

$$A=\begin{bmatrix}3 & 0\\ -1 & 2\\ 1 & 1\end{bmatrix} \quad B=\begin{bmatrix}4 & -1\\ 0 & 2\end{bmatrix} \quad C=\begin{bmatrix}1 & 4 & 2\\ 3 & 1 & 5\end{bmatrix} \quad D=\begin{bmatrix}1 & 5 & 2\\ -1 & 0 & 1\\ 3 & 2 & 4\end{bmatrix} \quad E=\begin{bmatrix}6 & 1 & 3\\ -1 & 1 & 2\\ 4 & 1 & 3\end{bmatrix}$$

Compute the following (where possible):

a) $D+E$ ***b***) $D-E$ ***c***) $5A$ ***d***) $-7C$ ***e***) $2B-C$ ***g***) $-3(D+2E)$

Section 1.4 – Inverse Matrices - Finding A^{-1}

Definition

The matrix A is invertible if there exists a matrix A^{-1} such that:

$$A^{-1}A = AA^{-1} = I$$

where A^{-1} *read as* "A ***inverse***" and A has to be a ***square matrix***.

Not all matrices have inverses.

1. The inverse exists *iff* elimination produces n pivots (row exchanges allow).
2. The matrix A cannot have two different inverses.
3. If A is invertible, the one and only one solution to $Ax = B$ is $x = A^{-1}B$

$$\begin{aligned} AX &= B \\ A^{-1}(AX) &= A^{-1}B && \textit{Multiply both side by } A^{-1} \\ \left(A^{-1}A\right)X &= A^{-1}B && \textit{Associate property} \\ IX &= A^{-1}B && \textit{Multiplicative inverse property} \\ X &= A^{-1}B && \textit{Identity property} \end{aligned}$$

4. Suppose there is a ***nonzero*** vector x such that $Ax = 0$. Then A cannot have an inverse
5. A 2 by 2 matrix is invertible iff $ad - bc$ is not zero.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \textit{Only for 2 by 2 matrices}$$

If $ad - bc = 0$ is the determinant, then A^{-1} doesn't exist

The Inverse of a Product AB

Theorem

If an $n \times n$ matrix has an inverse, that inverse is unique.

Proof

Suppose that A has an inverse A^{-1} and B is a matrix such that $BA = I$

$$B = BI = B\left(AA^{-1}\right) = (BA)A^{-1} = IA^{-1} = A^{-1}$$

Theorem

If A and B are invertible then so is AB. The inverse of a product AB is $(AB)^{-1} = B^{-1}A^{-1}$

Proof

$$\begin{aligned}(AB)\left(B^{-1}A^{-1}\right) &= A\left(BB^{-1}\right)A^{-1} \\ &= (AI)A^{-1} \\ &= AA^{-1} \\ &= \underline{I}\end{aligned}$$

Reverse Order

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

Theorem

If A is invertible and n is a nonnegative integer, then:

a) A^{-1} is *invertible* and $\left(A^{-1}\right)^{-1} = A$

b) A^{n} is *invertible* and $\left(A^{n}\right)^{-1} = A^{-n} = \left(A^{-1}\right)^{n}$

c) kA is *invertible* for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$

Proof

$$(kA)\left(k^{-1}A^{-1}\right) = k^{-1}(kA)A^{-1} = \left(k^{-1}k\right)AA^{-1} = (1)I = I$$

$$\left(k^{-1}A^{-1}\right)(kA) = k^{-1}\left(kA^{-1}\right)A = \left(k^{-1}k\right)A^{-1}A = (1)I = I$$

Finding A^{-1} using Gauss-Jordan Elimination

$$\left[A|I\right] \to \left[I|A^{-1}\right]$$

Find A^{-1} *if* $A=\begin{bmatrix}1 & 0 & 1\\ 2 & -2 & -1\\ 3 & 0 & 0\end{bmatrix}$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 1 & 0 & 0\\ 2 & -2 & -1 & 0 & 1 & 0\\ 3 & 0 & 0 & 0 & 0 & 1\end{array}\right] \begin{matrix} \\ R_2-2R_1 \\ R_3-3R_1\end{matrix}$$

$$\begin{array}{cccccc}2 & -2 & -1 & 0 & 1 & 0\\ -2 & 0 & -2 & -2 & 0 & 0\\ \hline 0 & -2 & -3 & -2 & 1 & 0\end{array} \qquad \begin{array}{cccccc}3 & 0 & 0 & 0 & 0 & 1\\ -3 & 0 & -3 & -3 & 0 & 0\\ \hline 0 & 0 & -3 & -3 & 0 & 1\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 1 & 0 & 0\\ 0 & -2 & -3 & -2 & 1 & 0\\ 0 & 0 & -3 & -3 & 0 & 1\end{array}\right] -\tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 1 & 0 & 0\\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0\\ 0 & 0 & -3 & -3 & 0 & 1\end{array}\right] -\tfrac{1}{3}R_3$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 1 & 0 & 0\\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0\\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3}\end{array}\right] \begin{matrix} R_1-R_3 \\ R_2-\frac{3}{2}R_3 \\ \\ \end{matrix}$$

$$\begin{array}{cccccc}1 & 0 & 1 & 1 & 0 & 0\\ 0 & 0 & -1 & -1 & 0 & \frac{1}{3}\\ \hline 1 & 0 & 0 & 0 & 0 & \frac{1}{3}\end{array} \qquad \begin{array}{cccccc}0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0\\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2}\\ \hline 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 0 & 0 & \frac{1}{3}\\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3}\end{array}\right]$$

$$A^{-1}=\begin{bmatrix}0 & 0 & \frac{1}{3}\\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\\ 1 & 0 & -\frac{1}{3}\end{bmatrix}$$

- ✓ Matrix $\boldsymbol{A}$ is ***symmetric*** across its main diagonal. So is A^{-1}
- ✓ Matrix $\boldsymbol{A}$ is ***tridiagonal*** (only three nonzero diagonals). But A^{-1} is a full matrix with no zeros. (another reason we don't compute A^{-1})

Singular *versus* Invertible

A^{-1} exists when A has a full set of n pivots. (Row exchanges allowed)

- With n pivots, elimination solves all the equations $Ax_i = b_i$. The columns x_i go into A^{-1}. Then $AA^{-1} = I$ is at least a ***right-inverse***.
- Elimination is really a sequence of multiplications.

Conclusion

- If A doesn't have n pivots, elimination will lead to a ***zero row***.
- Elimination steps are taken by an invertible M. So a row of MA is zero.
- If $AB = I$ then $MAB = M$. The zero row of MA, times B, gives a zero row of M.
- The invertible matrix M can't have a zero row! A must have n pivots if $AB = I$.

Elementary Matrices

Definition

Let ***e*** be an elementary row operation. Then the $n \times n$ ***elementary matrix*** E associated with ***e*** is the matrix obtained by applying ***e*** to the $n \times n$ identity matrix. Thus

$$E = eI$$

Example

a) $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \rightarrow$ *Multiply* R_2 *of* I *by* -3

b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow$ *Multiply the third row by* -5

c) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ *Interchange the first and second rows*

d) $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ *Add* -3 *times* R_1 *to* R_2

Theorem

Let e be an elementary operation and let E be the corresponding elementary matrix $E = e(I)$. Then for every $m \times n$ matrix A

$$e(A) = EA$$

That is, an elementary row operation can be performed on A by multiplying A on the left by the corresponding elementary matrix.

Example $m \times m$

Let $A = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$ $\quad M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

$$MA = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 6 & 2 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$

This result can be obtained from A by multiplying the first row by 2.

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 & 1 \\ -6 & 2 & 1 & 5 \\ 1 & 0 & 2 & 4 \end{bmatrix}$$

This result can be obtained from A by interchanging rows 2 and 3.

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ 0 & -4 & 10 & 8 \end{bmatrix}$$

This result can be obtained from A by adding 3 times row 1 to row 3.

Uniqueness of Echelon Form

Two matrices A and B are row-equivalent if and only if they have the same reduced echelon form.

Proof

If A and B have the same reduced echelon form E, then A is row-equivalent to E and E is row-equivalent to B. It follows that A is row-equivalent to B.

Now Suppose A and B are row-equivalent. Let E_1 be a reduced echelon form of A and E_2 be a reduced echelon form of B. Then E_1 and E_2 are row equivalent.

Suppose $E_1 = IF_1 \quad and \quad E_2 = IF_2$. Since E_1 and E_2 are row equivalent, $E_2 = CE_1$ for some matrix C. This means $I = CI$ and $F_2 = CF_1$. But then $C = I \quad and \quad F_2 = F_1$.

Example

Show that the two matrices are row equivalent

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

Solution

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{array}{l} R_1 + R_2 \\ R_2 - 2R_1 \end{array}$$

$$= \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

$$= B$$

Definition

A relationship ~ (equivalent) between elements of a set is called an equivalence relation if

- ✓ A ~ A is always true,
- ✓ A ~ B always implies B ~ A,
- ✓ A ~ B and B ~ C always implies A ~ C.

Exercises Section 1.4 – Inverse Matrices - Finding A^{-1}

1. Apply Gauss-Jordan method to find the inverse of this triangular "Pascal matrix"

Triangular Pascal matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$

2. If A is invertible and $AB = AC$, prove that $B = C$

3. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find two matrices $B \ne C$ such that $AB = AC$

4. If A has ***row*** 1 + ***row*** 2 = ***row*** 3, show that A is not invertible

a) Explain why $Ax = (1,\ 0,\ 0)$ can't have a solution.

b) Which right sides $(b_1,\ b_2,\ b_3)$ might allow a solution to $Ax = b$

c) What happens to ***row*** 3 in elimination?

5. True or false (with a counterexample if false and a reason if true):

a) A 4 by 4 matrix with a row of zeros is not invertible.

b) A matrix with 1's down the main diagonal is invertible.

c) If A is invertible then A^{-1} is invertible.

d) If A is invertible then A^2 is invertible.

6. Do there exist 2 by 2 matrices A and B with real entries such that $AB - BA = I$, where I is the identity matrix?

7. If B is the inverse of A^2, show that AB is the inverse of A.

8. Find and check the inverses (assuming they exist) of these block matrices.

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}$$

9. For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

10. Find A^{-1} and B^{-1} (if they exist) by elimination.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

11. Find A^{-1} using the Gauss-Jordan method, which has a remarkable inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

12. Find the inverse.

a) $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$

c) $\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

e) $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$

f) $\begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

g) $\begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}$

13. Show that *A* is not invertible for any values of the entries

$$A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

14. Prove that if *A* is an invertible matrix and *B* is row equivalent to *A*, then *B* is also invertible.

15. Determine if the given matrix has an inverse, and find the inverse if it exists. Check your answer by multiplying $A \cdot A^{-1} = I$

a) $\begin{bmatrix} 2 & 3 \\ -3 & -5 \end{bmatrix}$ *b)* $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}$

16. Show that the inverse of $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is $\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$

Section 1.5 – Transpose, Diagonal, Triangular, and Symmetric Matrices

Transpose

Definition

The transpose of a matrix A is defined as the matrix that is obtained by interchanging the corresponding rows and columns in A. Then the transpose of **A**, denoted by A^T or A'.

The columns of A^T are the rows of A.

When A is an *m* by *n* matrix, the transpose is *n* by *m*:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \quad \textit{then} \quad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

The matrix ***flips over*** the main diagonal. The entry in row *i*, column *j* of A^T comes from row *j*, column *i* of the original A.

$$\left(A^T\right)_{ij} = A_{ji}$$

Properties of Transpose

a) $\left(A^T\right)^T = A$

b) $(A+B)^T = A^T + B^T$

c) $(A-B)^T = A^T - B^T$

d) $(kA)^T = kA^T$

e) $(AB)^T = B^T A^T$

The transpose of a product of any number of matrices is the product of the transposes in the reverse order.

Theorem

If A is an invertible matrix, then A^T is also invertible and

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

Proof

$$A^T\left(A^{-1}\right)^T = \left(A^{-1}A\right)^T = I^T = I$$

$$\left(A^{-1}\right)^T A^T = \left(AA^{-1}\right)^T = I^T = I$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad and \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\Rightarrow \left(A^T\right)^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{c}{ad-bc} \\ -\frac{b}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

Trace

Definition

If A is a square matrix, then the trace of A, denoted by **tr(*A*)**, is defined to the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \rightarrow \quad tr(A) = a_{11} + a_{22} + a_{33}$$

Diagonal

A square matrix in which all the entries off the main diagonal are zero is called a ***diagonal matrix***. A general $n \times n$ diagonal matrix can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

A diagonal matrix is invertible iff all of its diagonal entries are nonzero; the

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$$

Powers of diagonal matrices are:

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

Triangular Matrices

A square matrix in which all the entries above the main diagonal are zero is called ***lower diagonal triangular***.

A square matrix in which all the entries below the main diagonal are zero is called ***upper diagonal triangular***.

A matrix that is either upper triangular or lower triangular is called ***triangular***.

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ a_{51} & a_{51} & a_{53} & a_{54} & a_{55} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix}$$

lower diagonal triangular   ***upper diagonal triangular***

Theorem

- ✓ The transpose of a lower triangular matrix is upper triangular, and the transpose of a upper triangular matrix is lower triangular.
- ✓ The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.
- ✓ A triangular matrix is invertible iff its diagonal entries are all nonzero.
- ✓ The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

Example

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Solution</u>

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \qquad AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix} \qquad BA = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

The goal is to describe Gaussian elimination in the most useful way by looking at them closely, which are factorizations of a matrix.

The factors are triangular matrices.

The factorization that comes from elimination is $A = LU$ **.**

Symmetric Matrices

Definition

A square matrix A is said to be ***symmetric*** if $A^T = A$. That means a square matrix must satisfies $a_{ij} = a_{ji}$

Example

$$A = \begin{pmatrix} 1 & -4 \\ -4 & 1 \end{pmatrix} = A^T$$

$$A = \begin{pmatrix} 6 & 5 & 1 \\ 5 & 0 & 7 \\ 1 & 7 & -1 \end{pmatrix} = A^T$$

- The ***inverse*** of a symmetric matrix is also ***symmetric***.

Example

Given $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, show that the inverse is symmetric too?

Solution

$$A^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

Theorem

If A and B are symmetric matrices with the same size, and if k is any scalar, then:

a) A^T is symmetric

b) $A + B$ and $A - B$ are symmetric.

c) kA is symmetric

- If A is an invertible symmetric matrix, then A^{-1} is *symmetric*.

Proof

Assume that A is symmetric and invertible then $A = A^T$

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1} = A^{-1}$$

Which proves that A^{-1} is *symmetric*

- Multiplying M by M^T gives a symmetric matrix.

Proof

The entry (i, j) of $M^T M$, it is the dot product of ***row*** i of M^T (column i of M) with column j of M.

The (i, j) entry is the same dot product, column j with column i. so $M^T M$ is symmetric.

The matrix $M.M^T$ is also symmetric and $M^T M$ is a different matrix from $M.M^T$.

- If A is an invertible symmetric matrix, then AA^T and $A^T A$ are also invertible.
- Matrix $\boldsymbol{A}$ is symmetric across its main diagonal. So is A^{-1}
- Matrix $\boldsymbol{A}$ is tridiagonal (only three nonzero diagonals). But A^{-1} is a full matrix with no zeros. (another reason we don't compute A^{-1})

Example

Given $M = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $M^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find $M^T M$ and $M.M^T$

<u>Solution</u>

$$M^T M = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$MM^T = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \end{bmatrix}$$

Symmetric in LDU

When elimination is applied to a symmetric matrix, $A^T = A$ is an advantage.

$$\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$= \underset{L}{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}$$

- If $A = A^T$ can be factored into LDU with no row exchanges, then $U = L^T$. The ***symmetric factorization of a symmetric matrix is*** $A = LDL^T$

Exercises Section 1.5 – Transpose, Diagonal, Triangular, and Symmetric Matrices

1. Solve $Lc = b$ to find $\boldsymbol{c}$. Then solve $Ux = c$ to find $\boldsymbol{x}$. What was A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

2. Find L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots

3. Determine whether the given matrix is invertible

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

4. Find $A^2,\ A^{-2},\ and\ A^{-k}$ by inspection

$a)\ A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ $\qquad b)\ A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$ $\qquad c)\ A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

5. Decide whether the given matrix is symmetric

$a)\ \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ $\qquad b)\ \begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix}$ $\qquad c)\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

6. Find all values of the unknown constant(s) in order for A to be symmetric

$$A = \begin{bmatrix} 2 & a-2b+2c & 2a+b+c \\ 3 & 5 & a+c \\ 0 & -2 & 7 \end{bmatrix}$$

7. Find a diagonal matrix A that satisfies the given condition $A^{-2} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. Let A be an $n \times n$ symmetric matrix

a) Show that A^2 is symmetric

b) Show that $2A^2 - 3A + I$ is symmetric

9. Prove if $A^T A = A$, then A is symmetric and $A = A^2$

10. A square matrix A is called ***skew-symmetric*** if $A^T = -A$. Prove

a) If A is an invertible skew-symmetric matrix, then A^{-1} is skew-symmetric.

b) If A and B are skew-symmetric matrices, then so are A^T, $A+B$, $A-B$, *and* kA for any scalar k.

c) Every square matrix A can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

$$\left[\mathit{Hint}: \text{ Note the identity } A = \tfrac{1}{2}\left(A + A^T\right) + \tfrac{1}{2}\left(A - A^T\right) \right]$$

11. Suppose R is rectangular (m by n) and A is symmetric (m by m)

a) Transpose $R^T AR$ to show its symmetric

b) Show why $R^T R$ has no negative numbers on its diagonal.

12. If L is a lower-triangular matrix, then $\left(L^{-1}\right)^T$ is _______Triangular

13. True or False

a) The block matrix $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ is automatically symmetric

b) If A and B are symmetric then their product is symmetric

c) If A is not symmetric then A^{-1} is not symmetric

d) When A, B, C are symmetric, the transpose of ABC is CBA.

e) The transpose of a diagonal matrix is a diagonal.

f) The transpose of an upper triangular matrix is an upper triangular matrix.

g) The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix.

h) All entries of a symmetric matrix are determined by the entries occurring on and above the main diagonal.

i) All entries of an upper triangular matrix are determined by the entries occurring on and above the main diagonal.

j) The inverse of an invertible lower triangular matrix is an upper triangular matrix.

k) A diagonal matrix is invertible if and only if all of its diagonal entries are positive.

l) The sum of a diagonal matrix and a lower triangular matrix is a lower triangular matrix.

m) A matrix that is both symmetric and upper triangular must be a diagonal matrix.

n) If *A* and *B* are $n \times n$ matrices such that $A + B$ is symmetric, then *A* and *B* are symmetric.

o) If *A* and *B* are $n \times n$ matrices such that $A + B$ is upper triangular, then *A* and *B* are upper triangular.

p) If A^2 is a symmetric matrix, then *A* is a symmetric matrix.

q) If kA is a symmetric matrix for some $k \neq 0$, then *A* is a symmetric matrix.

14. Find 2 by 2 symmetric matrices $A = A^T$ with these properties

a) A is not invertible

b) *A* is invertible but cannot be factored into *LU* (row exchanges needed)

c) *A* can be factored into LDL^T but not into LL^T (because of negative *D*)

15. A group of matrices includes *AB* and A^{-1} if it includes *A* and *B* . "Products and inverses stay in the group." Which of these sets are groups?

Lower triangular matrices *L* with 1's on the diagonal, symmetric matrices *S*, positive matrices *M*, diagonal invertible matrices *D*, permutation matrices *P*, matrices with $Q^T = Q^{-1}$. ***Invent two more matrix groups***.

16. Write $A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$ as the product *EH* of an elementary row operation matrix *E* and a symmetric matrix *H*.

17. When is the product of two symmetric matrices symmetric? Explain your answer.

18. Express $\left((AB)^{-1}\right)^T$ in terms of $\left(A^{-1}\right)^T$ and $\left(B^{-1}\right)^T$

19. Find the transpose of the given matrix: $\begin{bmatrix} 8 & -1 \\ 3 & 5 \\ -2 & 5 \\ 1 & 2 \\ -3 & -5 \end{bmatrix}$

20. For the given matrix, compute A^T, $\left(A^T\right)^{-1}$, A^{-1}, and $\left(A^{-1}\right)^T$, then compare $\left(A^T\right)^{-1}$ and $\left(A^{-1}\right)^T$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

Section 1.6 – Determinants and Properties

The determinant is a number that contains information about matrix. It is used to find formulas for inverse matrices, pivots, and solutions $A^{-1}b$.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ has inverse } A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Determinant of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is written $\det(A)$ *or* $|A|$ and is define as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinant is zero when the matrix has no inverse.

Properties of the Determinants

There are 3 basic properties (rules 1, 2, 3), by using those rules we can compute the determinant of any square matrix.

1. ***Determinant of the n by n identity matrix is* 1.**

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad and \quad \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} = 1$$

2. ***Determinant changes sign when 2 rows are exchanged.***

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -(ad - bc) \quad \Rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

3. ***Determinant is a linear function of each row separately*.**

Multiply row 1 by any number t: $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Add row 1 of A to row 1 of A ′: $\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

- ***For 2 by 2 determinants, if you expand to a rectangle, the determinants equal areas.***
- ***For n-dimensional, the determinants equal volumes.***

4. ***If 2 rows of A are equal, then*** $\mathbf{det}\boldsymbol{A} = \mathbf{0}$.

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$$

5. ***Subtracting a multiple of one row from another row leaves*** $\mathbf{det}\boldsymbol{A}$ ***unchanged.***

$$\begin{vmatrix} a & b \\ c - ta & d - ta \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

6. ***A matrix with a row of zeros has*** $\mathbf{det}\boldsymbol{A} = \mathbf{0}$.

$$\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0 \quad and \quad \begin{vmatrix} 0 & 0 \\ b & c \end{vmatrix} = 0$$

7. ***If A is triangular then*** $\mathbf{det}\boldsymbol{A} = \boldsymbol{a}_{11}\boldsymbol{a}_{22} \ \cdots \ \boldsymbol{a}_{nn}$ ***= product of diagonal entries.***

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad \quad and \quad \begin{vmatrix} a_{11} & & 0 \\ & a_{22} & \\ & & \\ 0 & & a_{nn} \end{vmatrix} = a_{11}a_{22}\dots a_{nn}$$

8. ***If A is singular then*** $\mathbf{det}\boldsymbol{A} = \mathbf{0}$. ***If A is invertible then*** $\mathbf{det}\boldsymbol{A} \neq \mathbf{0}$.

9. ***The determinant of AB*** $\mathbf{det}\boldsymbol{A}$ ***is times*** $\mathbf{det}\boldsymbol{B}$ ***:*** $|AB| = |A||B|$

10. ***The transpose*** $\boldsymbol{A}^{\boldsymbol{T}}$ ***has the same determinant as A:*** $\det(A) = \det\left(A^T\right)$

- $\det(A+B) \neq \det(A) + \det(B)$

Big Formula for Determinants (Diagonal)

Determinant Using Diagonal Method

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix} \qquad a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \quad (1)$$

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix} \qquad -a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \quad (2)$$

Determinant: ***D* = (1) + (2)**

det = $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$

Example

Evaluate: $\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix} \begin{matrix} x & 0 \\ 2 & x \\ -3 & x \end{matrix} = x(x)(1) + 0\left(x^2\right)(2) + (-1)(2)(x) - (-1)(x)(-3) - x\left(x^2\right)(x) - 0(-3)(1)$$

$$= x^2 + 3x - 3x - x^4$$

$$= x^2 - x^4$$

Determinant by *Cofactors*

$$\boldsymbol{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor

For a square matrix $A = \left[a_{ij}\right]$, the minor M_{ij}. Of an element a_{ij} is the ***determinant*** of the matrix formed by deleting the i^{th} row and the j^{th} column of A.

Example

Let $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$ Find M_{32}

Solution

$$M_{32} = \begin{vmatrix} 3 & \cancel{1} & -4 \\ 2 & \cancel{5} & 6 \\ \cancel{1} & \cancel{4} & \cancel{8} \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix}$$

$$= 26$$

Theorem

The determinant is the dot product of any row $\boldsymbol{i}$ of $\boldsymbol{A}$ with its cofactors:

Cofactor Formula: $\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$

$$|\boldsymbol{A}| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example

Find the determinant of the matrix:

$$A = \begin{bmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{bmatrix}$$

Solution

$$|A| = \begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix}$$

$$= -8\begin{vmatrix} -6 & 7 \\ -3 & 5 \end{vmatrix} - 0\begin{vmatrix} 4 & 7 \\ -1 & 5 \end{vmatrix} + 6\begin{vmatrix} 4 & -6 \\ -1 & -3 \end{vmatrix}$$

$= -8(-30 - (-21)) - 0 + 6(-12 - 6)$

$= -8(-9) + 6(-18)$

$= -36$

✓ By the property of determinants, If A is triangular then $\mathbf{det}A = a_{11}a_{22} \ \dots \ a_{nn}$ = product of diagonal entries.

Example

$$\begin{vmatrix} 2 & 7 & -3 & 8 & 3 \\ 0 & -3 & 7 & 5 & 1 \\ 0 & 0 & 6 & 7 & 6 \\ 0 & 0 & 0 & 9 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix} = (2)(-3)(6)(9)(4) \ \underline{= -1296}$$

Theorem

Let A be any n by n matrix.

a) If A' is the matrix that results when a single row of A is multiplied by a constant k, then $\det(A') = k\det(A)$.

b) If A' is the matrix that results when two rows of A are interchanged, then $\det(A') = -\det(A)$

c) If A' is the matrix that results when a multiple of one row of A is added to another row then $\det(A') = \det(A)$

Example

Evaluate $\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$ ***Interchanged 1st and 2nd row***

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$ ***A common factor of 3 from the first row (no need)***

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \begin{matrix} \\ \\ R_3 - 2R_1 \end{matrix}$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} \begin{matrix} \\ \\ R_3 - 10R_2 \end{matrix}$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

$$= -3(1)(1)(-55)$$

$$\underline{= 165|}$$

Exercises *Section* 1.6 – Determinants and Properties

1. Verify that $\det(AB)=\det(A)\det(B)$ when: $A=\begin{pmatrix}2 & 1 & 0\\ 3 & 4 & 0\\ 0 & 0 & 2\end{pmatrix}$ *and* $B=\begin{pmatrix}1 & -1 & 3\\ 7 & 1 & 2\\ 5 & 0 & 1\end{pmatrix}$

2. For which value(s) of $\boldsymbol{k}$ does A fail to be invertible? $A=\begin{bmatrix}k-3 & -2\\ -2 & k-2\end{bmatrix}$

3. Without directly evaluating, show that $\begin{vmatrix}b+c & c+a & b+a\\ a & b & c\\ 1 & 1 & 1\end{vmatrix}=0$

4. If the entries in every row of A add to zero, solve $\boldsymbol{Ax}$ = 0 to prove det A = 0. If those entries add to one, show that det ($A - I$) = 0. Does this mean det A = I?

5. Does $\det(AB)=\det(BA)$ in general?

a) True or false if $\boldsymbol{A}$ and $\boldsymbol{B}$ are square n x n matrices?

b) True or false if $\boldsymbol{A}$ is m x n and B is n x m with $m \neq n$?

6. True or false, with a reason if true or a counterexample if false:

a) The determinant of $I+A$ is 1 + det $\boldsymbol{A}$.

b) The determinant of ABC is $|A||B||C|$.

c) The determinant of 4A is $4|A|$

d) The determinant of $AB - BA$ is zero. (try an example)

e) If A is not invertible then AB is not invertible.

f) The determinant of $A - B$ equals to det A – det B.

7. Use row operations to show the 3 by 3 "Vandermonde determinant" is

$$\det\begin{bmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{bmatrix}=(b-a)(c-a)(c-b)$$

8. The inverse of a 2 by 2 matrix seems to have determinant = 1:

$$\det A^{-1} = \det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{ad-bc}{ad-bc} = 1$$

What is wrong with this calculation? What is the correct $\det A^{-1}$

9. A *Hessenberg* matrix is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci's rule $|H_4| = |H_3| + |H_2|$. The same rule will continue for all sizes $|H_n| = |H_{n-1}| + |H_{n-2}|$. Which Fibonacci number is $|H_n|$?

$$H_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad H_3 = \begin{bmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad H_4 = \begin{bmatrix} 2 & 1 & & \\ \mathbf{1} & 2 & \mathbf{1} & \\ \mathbf{1} & 1 & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

10. Evaluate the determinant:

a) $\begin{vmatrix} -1 & 7 \\ -8 & -3 \end{vmatrix}$

b) $\begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix}$

c) $\begin{vmatrix} k-1 & 2 \\ 4 & k-3 \end{vmatrix}$

d) $\begin{vmatrix} c & -4 & 3 \\ 2 & 1 & c^2 \\ 4 & c-1 & 2 \end{vmatrix}$

e) $\begin{vmatrix} 1 & 0 & 3 \\ 4 & 0 & -1 \\ 2 & 8 & 6 \end{vmatrix}$

f) $\begin{vmatrix} x & -3 & 9 \\ 2 & 4 & x+1 \\ 1 & x^2 & 3 \end{vmatrix}$

g) $\begin{vmatrix} -3 & 1 & 2 \\ 6 & 2 & 1 \\ -9 & 1 & 2 \end{vmatrix}$

h) $\begin{vmatrix} 1 & 4 & -3 & 1 \\ 2 & 0 & 6 & 3 \\ 4 & -1 & 2 & 5 \\ 1 & 0 & -2 & 4 \end{vmatrix}$

i) $\begin{vmatrix} 1 & 3 & 2 & -1 \\ 4 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 5 & 1 & 3 & -2 \end{vmatrix}$

j) $\begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & 1 & -2 & 0 \\ 1 & 0 & 3 & -3 \end{vmatrix}$

11. Find all the values of λ for which det($\boldsymbol{A}$) = 0

$$a)\ A = \begin{bmatrix} \lambda-1 & -2 \\ 1 & \lambda-4 \end{bmatrix} \qquad b)\ A = \begin{bmatrix} \lambda-6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda-4 \end{bmatrix}$$

12. Prove that if a square matrix $\boldsymbol{A}$ has a column of zeros, then det($\boldsymbol{A}$) = 0

13. With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D| \quad \textit{but} \quad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|$$

a) Why is the first statement true? Somehow B doesn't enter.

b) Show by example that equality fails (as shown) when C enters.

c) Show by example that the answer $\det(AD - CB)$ is also wrong.

14. Show that the value of the following determinant is independent of θ.

$$\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{vmatrix}$$

15. Show that the matrices $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ *and* $B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$

commute if and only if $\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$

16. Show that $\det(A) = \dfrac{1}{2}\begin{vmatrix} tr(A) & 1 \\ tr(A^2) & tr(A) \end{vmatrix}$ for every 2×2 matrix A.

17. What is the maximum number of zeros that a 4×4 matrix can have without a zero determinant? Explain your reasoning.

18. Evaluate *det* ***A***, *det* ***E***, and *det* (***AE***). Then verify that (*det* ***A***)(*det* ***E***) = *det*(***AE***)

$$A = \begin{bmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & & \\ & 3 & \\ & & 1 \end{bmatrix}$$

19. Show that $\begin{bmatrix} \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma \\ 1 & 1 & 1 \end{bmatrix}$ is not invertible for any values of α, β, γ

Section 1.7 – Cramer's Rule

Cramer's Rule

Theorem

If $AX = B$ is a system of a linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(B_1)}{\det(A)} \quad x_2 = \frac{\det(B_2)}{\det(A)}, \quad \cdots, \quad x_n = \frac{\det(B_n)}{\det(A)}$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

$$\det(B_1) = \begin{bmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & & & \\ \vdots & & & \\ b_n & a_{n2} & & a_{nn} \end{bmatrix}$$

Example

Use Cramer's rule to solve

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ -2x_1 + x_2 \quad &= 0 \\ -4x_1 \quad + x_3 &= 0 \end{aligned}$$

Solution

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 7$$

$$|B_1| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \qquad |B_2| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 2 \qquad |B_3| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 0 \end{vmatrix} = 4$$

$$x_1 = \frac{|B_1|}{|A|} = \frac{1}{7} \qquad x_2 = \frac{|B_2|}{|A|} = \frac{2}{7} \qquad x_3 = \frac{|B_3|}{|A|} = \frac{4}{7}$$

Solution: $\left(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right)$

Example

Use Cramer's Rule to solve.

$$\begin{aligned} x_1 + \quad\quad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

Solution

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} \quad \Rightarrow \quad \det(A) = 44$$

$$A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix} \quad \Rightarrow \quad \det(A_1) = -40$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix} \quad \Rightarrow \quad \det(A_2) = 72$$

$$A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix} \quad \Rightarrow \quad \det(A_3) = 152$$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \underline{-\frac{10}{11}}$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \underline{\frac{18}{11}}$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \underline{\frac{38}{11}}$$

A Formula for A^{-1}

***Theorem**: Inverse of a matrix using its Adjoint*

The i, j entry of A^{-1} is the cofactor C_{ji} $\left(\textit{not } C_{ij}\right)$ divided by det(***A***):

$$\textbf{\textit{Formula for }} A^{-1}: \quad \left(A^{-1}\right)_{ij} = \frac{C_{ji}}{|A|} \quad \textit{and} \quad A^{-1} = \frac{C^T}{|A|}$$

$$\boxed{A^{-1} = \frac{1}{\det(A)} adj(A)}$$

Example

Find the inverse matrix of $A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ using its adjoint.

Solution

$$C_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1; \quad C_{12} = -\begin{vmatrix} -2 & 0 \\ -4 & 1 \end{vmatrix} = 2; \quad C_{13} = \begin{vmatrix} -2 & 1 \\ -4 & 0 \end{vmatrix} = 4$$

$$C_{21} = -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1; \quad C_{22} = \begin{vmatrix} 1 & 1 \\ -4 & 1 \end{vmatrix} = 5; \quad C_{23} = -\begin{vmatrix} 1 & 1 \\ -4 & 0 \end{vmatrix} = -4$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1; \quad C_{32} = -\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = -2; \quad C_{33} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3$$

$$C = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 5 & -2 \\ 4 & -4 & 3 \end{bmatrix} \text{ and } \det(A) = \tfrac{1}{7} \qquad \Rightarrow \quad A^{-1} = \tfrac{1}{7}\begin{bmatrix} 1 & -1 & -1 \\ 2 & 5 & -2 \\ 4 & -4 & 3 \end{bmatrix}$$

Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent

a) A is invertible

b) $A\mathbf{x} = 0$ has only the trivial solution

c) The reduced row echelon form of A is I_n

d) A can be expressed as a product of elementary matrices

e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$

f) $\det(A) \neq 0$

Exercises Section 1.7 – Cramer's Rule

1. Use Cramer's Rule with ratios $\dfrac{\det B_j}{\det A}$ to solve $A\boldsymbol{x} = b$. Also find the inverse matrix $A^{-1} = \dfrac{C^T}{\det A}$.

Why is the solution $\boldsymbol{x}$ is the first part the same as column 3 of A^{-1}? Which cofactors are involved in computing that column $\boldsymbol{x}$?

$$Ax = b \quad is \quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2. Verify that $\det(AB) = \det(BA)$ and determine whether the equality $\det(A+B) = \det(A) + \det(B)$ holds

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

3. Verify that $\det(kA) = k^n \det(A)$

a) $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}, \quad k = 2$

b) $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}, \quad k = -2$

c) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}, \quad k = 3$

4. Solve by using Cramer's rule

a) $\begin{cases} 7x - 2y = 3 \\ 3x + y = 5 \end{cases}$

b) $\begin{cases} 4x + 5y = 2 \\ 11x + y + 2z = 3 \\ x + 5y + 2z = 1 \end{cases}$

c) $\begin{cases} x - 4y + z = 6 \\ 4x - y + 2z = -1 \\ 2x + 2y - 3z = -20 \end{cases}$

d) $\begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 = -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 = 14 \\ -x_1 + x_2 + 3x_3 + x_4 = 11 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -4 \end{cases}$

e) $\begin{cases} 2x - y + z = -1 \\ 3x + 4y - z = -1 \\ 4x - y + 2z = -1 \end{cases}$

5. Show that the matrix A is invertible for all values of θ, then find A^{-1} using $A^{-1} = \dfrac{1}{\det(A)} adj(A)$

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$