Section 2.4 – Partial Fractions

This section shows how to express a rational; function as a sum of simpler functions, called ***partial fractions***.

Example

Evaluate $\int \frac{5x-3}{x^2-2x-3}dx$

Solution

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$5x-3 = (A+B)x - 3A + B \quad \to \begin{cases} A+B=5 \\ -3A+B=-3 \end{cases} \to \boxed{A=2, \quad B=3}$$

$$\int \frac{5x-3}{x^2-2x-3}dx = \int \left(\frac{2}{x+1} + \frac{3}{x-3}\right)dx$$

$$= 2\ln|x+1| + 3\ln|x-3| + C$$

Example

Use partial fractions to evaluate $\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}dx$

Solution

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

$$x^2+4x+1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$$

$$= Ax^2 + 4Ax + 3A + Bx^2 + 2Bx - 3B + Cx^2 - C$$

$$= (A+B+C)x^2 + (4A+2B)x + (3A-3B-C)$$

$$\to \begin{cases} A+B+C=1 \\ 4A+2B=4 \\ 3A-3B-C=1 \end{cases} \xrightarrow{rref} \boxed{A=\frac{3}{4}, \quad B=\frac{1}{2}, \quad C=-\frac{1}{4}}$$

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}dx = \int \left[\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3}\right]dx$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K$$

Method of Partial Fractions $\left(f(x)/g(x) \;\; \textbf{\textit{Proper}} \right)$

1. Let $(x-r)$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $(x-r)$ that divides $g(x)$. Then,

$$\frac{A_1}{(x-r)}+\frac{A_2}{(x-r)^2}+\cdots+\frac{A_m}{(x-r)^m}$$

2. Let x^2+px+q be an irreducible quadratic function of $g(x)$ has no real roots. Suppose that $\left(x^2+px+q\right)^n$ is the highest power. Then

$$\frac{B_1x+C_1}{\left(x^2+px+q\right)}+\frac{B_2x+C_2}{\left(x^2+px+q\right)^2}+\cdots+\frac{B_nx+C_n}{\left(x^2+px+q\right)^n}$$

3. Set the original fraction $\dfrac{f(x)}{g(x)}$ equal to the sum of these partial fractions.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Example

Use partial fractions to evaluate $\displaystyle\int \frac{6x+7}{(x+2)^2}dx$

Solution

$$\frac{6x+7}{(x+2)^2}=\frac{A}{x+2}+\frac{B}{(x+2)^2}$$

$$\begin{aligned} 6x+7 &= A(x+2)+B \\ &= Ax+2A+B \end{aligned}$$

$$\rightarrow \begin{cases} \boxed{A=6} \\ 2A+B=7 & \rightarrow \underline{B}=7-12=\underline{-5} \end{cases}$$

$$\begin{aligned} \int \frac{6x+7}{(x+2)^2}dx &= \int\left(\frac{6}{x+2}-\frac{5}{(x+2)^2}\right)dx \qquad\qquad d(x+2)=dx \\ &= \int \frac{6}{x+2}dx-5\int (x+2)^{-2}\,d(x+2) \\ &= 6\ln|x+2|+5(x+2)^{-1}+C \end{aligned}$$

Example

Use partial fractions to evaluate $\displaystyle\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}dx$

Solution

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3} = 2x + \frac{5x-3}{x^2-2x-3}$$

$$\begin{array}{r} 2x \\ x^2-2x-3 \overline{\smash{\big)}\,2x^3-4x^2-x-3} \\ \underline{2x^3-4x^2-6x} \\ 5x-3 \end{array}$$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$5x-3 = (A+B)x - 3A + B \quad \to \begin{cases} A+B=5 \\ -3A+B=-3 \end{cases} \to \boxed{A=2, \quad B=3}$$

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}dx = \int 2xdx + \int \frac{5x-3}{x^2-2x-3}dx$$

$$= \int 2xdx + \int \frac{2}{x+1}dx + \int \frac{3}{x-3}dx$$

$$= x^2 + 2\ln|x+1| + 3\ln|x-3| + C$$

Example

Use partial fractions to evaluate $\displaystyle\int \frac{-2x+4}{\left(x^2+1\right)(x-1)^2}dx$

Solution

$$\frac{-2x+4}{\left(x^2+1\right)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$-2x+4 = (Ax+B)(x-1)^2 + C(x-1)\left(x^2+1\right) + D\left(x^2+1\right)$$

$$= (Ax+B)\left(x^2-2x+1\right) + C\left(x^3-x^2+x-1\right) + Dx^2 + D$$

$$= (A+C)x^3 + (-2A+B-C+D)x^2 + (A-2B+C)x + B-C+D$$

$$\to \begin{cases} A+C=0 \\ -2A+B-C+D=0 \\ A-2B+C=-2 \\ B-C+D=4 \end{cases} \Rightarrow \boxed{A=2, \quad B=1, \quad C=-2, \quad D=1}$$

$$\int \frac{-2x+4}{\left(x^2+1\right)(x-1)^2}dx = \int \left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right)dx$$

$$= \int \left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2} \right) dx$$

$$\boxed{= \ln\left(x^2+1\right) + \tan^{-1} x - 2\ln|x-1| - \frac{1}{x-1} + K}$$

Example

Use partial fractions to evaluate $\displaystyle\int \frac{dx}{x\left(x^2+1\right)^2}$

Solution

$$\frac{1}{x\left(x^2+1\right)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{\left(x^2+1\right)^2}$$

$$1 = A\left(x^2+1\right)^2 + (Bx+C)x\left(x^2+1\right) + x(Dx+E)$$

$$1 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

$$\begin{cases} A+B=0 \\ C=0 \\ 2A+B+D=0 \\ C+E=0 \\ A=1 \end{cases} \Rightarrow \boxed{A=1,\ B=-1,\ C=0,\ D=-1,\ E=0}$$

$$\frac{1}{x\left(x^2+1\right)^2} = \frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{\left(x^2+1\right)^2}$$

$$\int \frac{dx}{x\left(x^2+1\right)^2} = \int \frac{dx}{x} - \int \frac{xdx}{x^2+1} - \int \frac{xdx}{\left(x^2+1\right)^2} \qquad u = x^2+1 \Rightarrow du = 2xdx \to \tfrac{1}{2}du = xdx$$

$$= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2}$$

$$= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2}\frac{1}{u} + K$$

$$= \ln|x| - \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\frac{1}{x^2+1} + K$$

$$= \ln|x| - \ln\sqrt{x^2+1} + \frac{1}{2}\frac{1}{x^2+1} + K$$

$$\boxed{= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2\left(x^2+1\right)} + K}$$

Exercises *Section* 2.4 – Partial Fractions

Express the integrand as a sum of partial fractions and evaluate the integrals

1. $\displaystyle\int \frac{dx}{x^2+2x}$

2. $\displaystyle\int \frac{2x+1}{x^2-7x+12}dx$

3. $\displaystyle\int \frac{x+3}{2x^3-8x}dx$

4. $\displaystyle\int \frac{x^2}{(x-1)\left(x^2+2x+1\right)}dx$

5. $\displaystyle\int \frac{8x^2+8x+2}{\left(4x^2+1\right)^2}dx$

6. $\displaystyle\int \frac{x^2+x}{x^4-3x^2-4}dx$

7. $\displaystyle\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{\left(\theta^2+1\right)^3}d\theta$

8. $\displaystyle\int \frac{x^4}{x^2-1}dx$

9. $\displaystyle\int \frac{16x^3}{4x^2-4x+1}dx$

10. $\displaystyle\int \frac{e^{4x}+2e^{2x}-e^x}{e^{2x}+1}dx$

11. $\displaystyle\int \frac{\sin\theta\, d\theta}{\cos^2\theta+\cos\theta-2}$

12. $\displaystyle\int \frac{(x-2)^2\tan^{-1}(2x)-12x^3-3x}{\left(4x^2+1\right)(x-2)^2}dx$

13. $\displaystyle\int \frac{\sqrt{x+1}}{x}dx$

14. $\displaystyle\int \frac{x^3-2x^2+3x-4}{x^2+1}dx$

15. $\displaystyle\int \frac{4x^2+2x+4}{x+1}dx$

16. $\displaystyle\int \frac{3x^2+7x-2}{x^3-x^2-2x}dx$

17. $\displaystyle\int \frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)}dx$

18. $\displaystyle\int \frac{5x^2-3x+2}{x^3-2x^2}dx$

19. $\displaystyle\int \frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}dx$

20. $\displaystyle\int \frac{1}{x^2-5x+6}dx$

21. $\displaystyle\int \frac{1}{x^2-5x+5}dx$

22. $\displaystyle\int \frac{5x^2+20x+6}{x^3+2x^2+x}dx$

23. $\displaystyle\int \frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}dx$

24. $\displaystyle\int \frac{8x^3+13x}{\left(x^2+2\right)^2}dx$

25. $\displaystyle\int \frac{\sin x}{\cos x+\cos^2 x}dx$

26. $\displaystyle\int \frac{5\cos x}{\sin^2 x+3\sin x-4}dx$

27. $\displaystyle\int \frac{e^x}{\left(e^x-1\right)\left(e^x+4\right)}dx$

28. $\displaystyle\int \frac{e^x}{\left(e^{2x}+1\right)\left(e^x-1\right)}dx$

29. $\displaystyle\int \frac{\sqrt{x}}{x-4}\,dx$

30. $\displaystyle\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$

31. $\displaystyle\int \frac{dx}{1+\sin x}$

32. $\displaystyle\int \frac{dx}{2+\cos x}$

33. $\displaystyle\int \frac{dx}{1-\cos x}$

34. $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$

35. $\displaystyle\int \frac{1}{x^2-9}\,dx$

36. $\displaystyle\int \frac{5}{x^2+3x-4}\,dx$

37. $\displaystyle\int \frac{2}{9x^2-1}\,dx$

38. $\displaystyle\int \frac{3-x}{3x^2-2x-1}\,dx$

39. $\displaystyle\int \frac{x^2+12x+12}{x^3-4x}\,dx$

40. $\displaystyle\int \frac{x^3-x+3}{x^2+x-2}\,dx$

41. $\displaystyle\int \frac{5x-2}{(x-2)^2}\,dx$

42. $\displaystyle\int \frac{2x^3-4x^2-15x+4}{x^2-2x-8}\,dx$

43. $\displaystyle\int \frac{x+2}{x^2+5x}\,dx$

44. $\displaystyle\int \frac{\sec^2 x}{\tan^2 x+5\tan x+6}\,dx$

45. $\displaystyle\int \frac{\sec^2 x}{\tan x(\tan x+1)}\,dx$

46. $\displaystyle\int_0^2 \frac{3}{4x^2+5x+1}\,dx$

47. $\displaystyle\int_1^5 \frac{x-1}{x^2(x+1)}\,dx$

48. $\displaystyle\int_0^1 \frac{x^2-x}{x^2+x+1}\,dx$

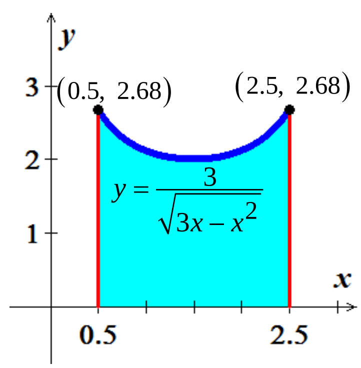
49. $\displaystyle\int_4^8 \frac{y\,dy}{y^2-2y-3}$

50. $\displaystyle\int_1^{\sqrt{3}} \frac{3x^2+x+4}{x^3+x}\,dx$

51. $\displaystyle\int_0^{\pi/2} \frac{dx}{\sin x+\cos x}$

52. $\displaystyle\int_0^{\pi/3} \frac{\sin\theta}{1-\sin\theta}\,d\theta$

53. Find the volume of the solid generated by the revolving the shaded region about x-axis



Find the area of the region bounded by the graphs of

54. $y=\dfrac{12}{x^2+5x+6}$, $y=0$, $x=0$, *and* $x=1$

55. $y=\dfrac{7}{16-x^2}$ *and* $y=1$

56. Consider the region bounded by the graphs $y=\dfrac{2x}{x^2+1}$, $y=0$, $x=0$, *and* $x=3$.

a) Find the volume of the solid generated by revolving the region about the x-*axis*

b) Find the centroid of the region.

57. Consider the region bounded by the graph $y^2 = \dfrac{(2-x)^2}{(1+x)^2}$ $\quad 0 \le x \le 1$.

Find the volume of the solid generated by revolving this region about the x-*axis* .

58. A single infected individual enters a community of n susceptible individuals. Let x be the number of newly infected individuals at time t. The common epidemic model assumes that the disease spreads at a rate proportional to the product of the total number infected and the number not yet infected. So,

$\dfrac{dx}{dt} = k(x+1)(n-x)$ and you obtain

$$\int \frac{1}{(x+1)(n-x)}\, dx = \int k\, dt$$

Solve for x as a function of t.

59. Evaluate $\displaystyle\int_0^1 \frac{x}{1+x^4}\, dx$ in ***two*** different ways.