

## ***Solution***     **Section 2.1 – Infinite Sequences and Summation Notation**

### ***Exercise***

Find the first four terms and the eight term of the sequence:  $\{12 - 3n\}$

### **Solution**

$$a_n = 12 - 3n$$

$$a_1 = 12 - 3(1) = \underline{9}$$

$$a_2 = 12 - 3(2) = 6$$

$$a_3 = 12 - 3(3) = \underline{3}$$

$$a_4 = 12 - 3(4) = \underline{0}$$

$$a_8 = 12 - 3(8) = \underline{-12}$$

### ***Exercise***

Find the first four terms and the eight term of the sequence:  $\left\{ \frac{3n-2}{n^2+1} \right\}$

### **Solution**

$$a_n = \frac{3n-2}{n^2+1}$$

$$a_1 = \frac{3-2}{1^2+1} = \underline{\frac{1}{2}}$$

$$a_2 = \frac{3(2)-2}{2^2+1} = \underline{\frac{4}{5}}$$

$$a_3 = \frac{3(3)-2}{3^2+1} = \underline{\frac{7}{10}}$$

$$a_4 = \frac{3(4)-2}{4^2+1} = \underline{\frac{10}{17}}$$

$$a_8 = \frac{3(8)-2}{8^2+1} = \underline{\frac{22}{65}}$$

### Exercise

Find the first four terms and the eight term of the sequence:  $\{9\}$

### Solution

$$\underline{a_1 = 9}$$

$$\underline{a_2 = 9}$$

$$\underline{a_3 = 9}$$

$$\underline{a_4 = 9}$$

$$\underline{a_8 = 9}$$

### Exercise

Find the first four terms and the eight term of the sequence:  $\left\{(-1)^{n-1} \frac{n+7}{2n}\right\}$

### Solution

$$a_1 = (-1)^{1-1} \frac{1+7}{2(1)} = 4$$

$$a_2 = (-1)^{2-1} \frac{2+7}{2(2)} = -\frac{9}{4}$$

$$a_3 = (-1)^{3-1} \frac{3+7}{2(3)} = \frac{5}{3}$$

$$a_4 = (-1)^{4-1} \frac{4+7}{2(4)} = -\frac{11}{8}$$

$$a_8 = (-1)^{8-1} \frac{8+7}{2(8)} = -\frac{15}{16}$$

### Exercise

Find the first four terms and the eight term of the sequence:  $\left\{\frac{2^n}{n^2 + 2}\right\}$

### Solution

$$a_1 = \frac{2^1}{1^2 + 2} = \frac{2}{3}$$

$$a_2 = \frac{2^2}{2^2 + 2} = \frac{2}{3}$$

$$a_3 = \frac{2^3}{3^2 + 2} = \frac{8}{11}$$

$$a_4 = \frac{2^4}{4^2 + 2} = \frac{8}{9}$$

$$a_8 = \frac{2^8}{8^2 + 2} = \frac{128}{33}$$

### Exercise

Find the first four terms and the eight term of the sequence:  $\left\{(-1)^{n-1} \frac{n}{2n-1}\right\}$

### Solution

$$a_1 = (-1)^0 \frac{1}{2-1} = 1$$

$$a_2 = (-1)^1 \frac{2}{4-1} = -\frac{2}{3}$$

$$a_3 = (-1)^2 \frac{3}{6-1} = \frac{3}{5}$$

$$a_4 = (-1)^3 \frac{4}{8-1} = -\frac{4}{7}$$

$$a_8 = (-1)^7 \frac{8}{16-1} = -\frac{8}{15}$$

### Exercise

Find the first four terms and the eight term of the sequence:  $\left\{\frac{2^n}{3^n + 1}\right\}$

### Solution

$$a_1 = \frac{2^1}{3^1 + 1} = \frac{2}{4} = \frac{1}{2}$$

$$a_2 = \frac{2^2}{3^2 + 1} = \frac{4}{10} = \frac{2}{5}$$

$$a_3 = \frac{2^3}{3^3 + 1} = \frac{8}{28} = \frac{2}{7}$$

$$a_4 = \frac{2^4}{3^4 + 1} = \frac{16}{82} = \frac{8}{41}$$

$$a_8 = \frac{2^8}{3^8 + 1} = \frac{256}{6562} = \frac{128}{3281}$$

### ***Exercise***

Find the first four terms and the eight term of the sequence:  $\left\{ \frac{n^2}{2^n} \right\}$

#### **Solution**

$$a_1 = \frac{1^2}{2^1} = \frac{1}{2}$$

$$a_2 = \frac{2^2}{2^2} = 1$$

$$a_3 = \frac{3^2}{2^3} = \frac{9}{8}$$

$$a_4 = \frac{4^2}{2^4} = \frac{16}{16} = 1$$

$$a_8 = \frac{8^2}{2^8} = \frac{64}{256} = \frac{1}{4}$$

### ***Exercise***

Find the first four terms and the eight term of the sequence:  $\left\{ \frac{n}{e^n} \right\}$

#### **Solution**

$$a_1 = \frac{1}{e^1} = \frac{1}{e}$$

$$a_2 = \frac{2}{e^2}$$

$$a_3 = \frac{3}{e^3}$$

$$a_4 = \frac{4}{e^4}$$

$$a_8 = \frac{8}{e^8}$$

### ***Exercise***

Find the first four terms and the eight term of the sequence:  $\{c_n\} = \{(-1)^{n+1} n^2\}$

### **Solution**

$$c_1 = (-1)^2 1^2 = \underline{1}$$

$$c_2 = (-1)^3 2^2 = \underline{-4}$$

$$c_3 = (-1)^4 3^2 = \underline{9}$$

$$c_4 = (-1)^5 4^2 = \underline{-16}$$

$$c_8 = (-1)^9 8^2 = \underline{-64}$$

### ***Exercise***

Find the first four terms and the eight term of the sequence:  $\{c_n\} = \left\{ \frac{(-1)^n}{(n+1)(n+2)} \right\}$

### **Solution**

$$c_1 = \frac{(-1)^1}{2 \cdot 3} = \underline{-\frac{1}{6}}$$

$$c_2 = \frac{(-1)^2}{3 \cdot 4} = \underline{\frac{1}{12}}$$

$$c_3 = \frac{(-1)^3}{4 \cdot 5} = \underline{-\frac{1}{20}}$$

$$c_4 = \frac{(-1)^4}{5 \cdot 6} = \underline{\frac{1}{30}}$$

$$c_8 = \frac{(-1)^8}{9 \cdot 10} = \underline{\frac{1}{90}}$$

### ***Exercise***

Find the first four terms and the eight term of the sequence:  $\{c_n\} = \left\{\left(\frac{4}{3}\right)^n\right\}$

### **Solution**

$$c_1 = \left(\frac{4}{3}\right)^1 = \underline{\frac{4}{3}}$$

$$c_2 = \left(\frac{4}{3}\right)^2 = \underline{\frac{16}{9}}$$

$$c_3 = \left(\frac{4}{3}\right)^3 = \underline{\frac{64}{27}}$$

$$c_4 = \left(\frac{4}{3}\right)^4 = \underline{\frac{256}{81}}$$

$$c_8 = \left(\frac{4}{3}\right)^8 = \underline{\frac{65,536}{6,561}}$$

### ***Exercise***

Find the first four terms and the eight term of the sequence:  $\{b_n\} = \left\{\frac{3^n}{n}\right\}$

### **Solution**

$$b_1 = \frac{3^1}{1} = \underline{3}$$

$$b_2 = \frac{3^2}{2} = \underline{\frac{9}{2}}$$

$$b_3 = \frac{3^3}{3} = \underline{9}$$

$$b_4 = \frac{3^4}{4} = \underline{\frac{81}{4}}$$

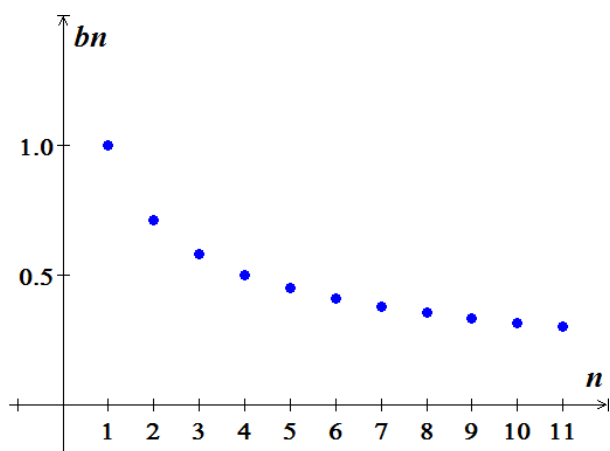
$$c_8 = \frac{3^8}{8} = \underline{\frac{6,561}{8}}$$

### Exercise

Graph the sequence  $\left\{ \frac{1}{\sqrt{n}} \right\}$

### Solution

$$\left\{ \frac{1}{\sqrt{n}} \right\} = \left\{ \frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, \dots \right\}$$
$$\approx \{1, 0.71, 0.58, 0.5, 0.45\}$$



### Exercise

Find the first four terms of the sequence of partial sums for the given sequence.  $\left\{ 3 + \frac{1}{2}n \right\}$

### Solution

$$S_1 = a_1$$
$$= 3 + \frac{1}{2}(1)$$
$$= \frac{7}{2}$$

$$S_2 = S_1 + a_2$$
$$= \frac{7}{2} + 3 + \frac{1}{2}(2)$$
$$= \frac{15}{2}$$

$$S_3 = S_2 + a_3$$
$$= \frac{15}{2} + 3 + \frac{1}{2}(3)$$
$$= 12$$

$$S_4 = S_3 + a_4$$
$$= 12 + 3 + \frac{1}{2}(4)$$
$$= 17$$

### ***Exercise***

Find the first five terms of the recursively defined infinite sequence:  $a_1 = 2$ ,  $a_{k+1} = 3a_k - 5$

#### **Solution**

$$\begin{aligned}k=1 \rightarrow a_2 &= 3a_1 - 5 \\&= 3(2) - 5 \\&= 1 \quad | \end{aligned}$$

$$\begin{aligned}k=2 \rightarrow a_3 &= 3a_2 - 5 \\&= 3(1) - 5 \\&= -2 \quad | \end{aligned}$$

$$\begin{aligned}k=3 \rightarrow a_4 &= 3a_3 - 5 \\&= 3(-2) - 5 \\&= -11 \quad | \end{aligned}$$

$$\begin{aligned}k=4 \rightarrow a_5 &= 3a_4 - 5 \\&= 3(-11) - 5 \\&= -38 \quad | \end{aligned}$$

### ***Exercise***

Find the first five terms of the recursively defined infinite sequence:  $a_1 = -3$ ,  $a_{k+1} = a_k^2$

#### **Solution**

$$\begin{aligned}k=1 \rightarrow a_2 &= a_1^2 \\&= (-3)^2 \\&= 9 \quad | \end{aligned}$$

$$\begin{aligned}k=2 \rightarrow a_3 &= a_2^2 \\&= (9)^2 \\&= 81 \quad | \end{aligned}$$

$$\begin{aligned}k=3 \rightarrow a_4 &= a_3^2 \\&= (81)^2 \\&= 6561 \quad | \end{aligned}$$



$$\begin{aligned}
 k = 4 \rightarrow a_5 &= a_4^2 \\
 &= (3^8)^2 \\
 &= \underline{3^{16}}
 \end{aligned}$$

### ***Exercise***

Find the first five terms of the recursively defined infinite sequence:  $a_1 = 5$ ,  $a_{k+1} = ka_k$

### **Solution**

$$\begin{aligned}
 k = 1 \rightarrow a_2 &= 1a_1 \\
 &= \underline{5}
 \end{aligned}$$

$$\begin{aligned}
 k = 2 \rightarrow a_3 &= 2a_2 \\
 &= 2(5) \\
 &= \underline{10}
 \end{aligned}$$

$$\begin{aligned}
 k = 3 \rightarrow a_4 &= 3a_3 \\
 &= 3(10) \\
 &= \underline{30}
 \end{aligned}$$

$$\begin{aligned}
 k = 4 \rightarrow a_5 &= 4a_4 \\
 &= 4(30) \\
 &= \underline{120}
 \end{aligned}$$

### ***Exercise***

Find the first five terms of the recursively defined infinite sequence:  $a_1 = 2$ ,  $a_n = 3 + a_{n-1}$

### **Solution**

$$a_2 = 3 + a_1 = 3 + 2 = \underline{5}$$

$$a_3 = 3 + a_2 = 3 + 5 = \underline{8}$$

$$a_4 = 3 + a_3 = 3 + 8 = \underline{11}$$

$$a_5 = 3 + a_4 = 3 + 11 = \underline{14}$$

### Exercise

Find the first five terms of the recursively defined infinite sequence:  $a_1 = 5$ ,  $a_n = 2a_{n-1}$

#### Solution

$$\begin{aligned}a_2 &= 2a_1 \\&= 2(5) \\&= 10\end{aligned}$$

$$\begin{aligned}a_3 &= 2a_2 \\&= 2(10) \\&= 20\end{aligned}$$

$$\begin{aligned}a_4 &= 2a_3 \\&= 2(20) \\&= 40\end{aligned}$$

$$\begin{aligned}a_5 &= 2a_4 \\&= 2(40) \\&= 80\end{aligned}$$

### Exercise

Find the first five terms of the recursively defined infinite sequence:  $a_1 = \sqrt{2}$ ,  $a_n = \sqrt{2 + a_{n-1}}$

#### Solution

$$\begin{aligned}a_2 &= \sqrt{2 + a_1} \\&= \sqrt{2 + \sqrt{2}}\end{aligned}$$

$$\begin{aligned}a_3 &= \sqrt{2 + a_2} \\&= \sqrt{2 + \sqrt{2 + \sqrt{2}}}\end{aligned}$$

$$\begin{aligned}a_4 &= \sqrt{2 + a_3} \\&= \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}\end{aligned}$$

$$\begin{aligned}a_5 &= \sqrt{2 + a_4} \\&= \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}\end{aligned}$$

### ***Exercise***

Find the first five terms of the recursively defined infinite sequence:  $a_1 = 2$ ,  $a_{n+1} = 7 - 2a_n$

#### **Solution**

$$\begin{aligned}a_2 &= 7 - 2a_1 \\&= 7 - 4 \\&= 3\end{aligned}$$

$$\begin{aligned}a_3 &= 7 - 2a_2 \\&= 7 - 6 \\&= 1\end{aligned}$$

$$\begin{aligned}a_4 &= 7 - 2a_3 \\&= 7 - 2 \\&= 5\end{aligned}$$

$$\begin{aligned}a_5 &= 7 - 2a_4 \\&= 7 - 10 \\&= -5\end{aligned}$$

### ***Exercise***

Find the first five terms of the recursively defined infinite sequence:  $a_1 = 128$ ,  $a_{n+1} = \frac{1}{4}a_n$

#### **Solution**

$$\begin{aligned}a_2 &= \frac{1}{4}a_1 \\&= \frac{1}{4}128 \\&= 32\end{aligned}$$

$$\begin{aligned}a_3 &= \frac{1}{4}a_2 \\&= \frac{32}{4} \\&= 8\end{aligned}$$

$$\begin{aligned}a_4 &= \frac{1}{4}a_3 \\&= 2\end{aligned}$$

$$\begin{aligned}a_5 &= \frac{1}{4}a_4 \\&= \frac{1}{2}\end{aligned}$$

### ***Exercise***

Find the first five terms of the recursively defined infinite sequence:  $a_1 = 2, \quad a_{n+1} = (a_n)^n$

#### **Solution**

$$\begin{aligned} a_2 &= (a_1)^1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} a_3 &= (a_2)^2 \\ &= 2^2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} a_4 &= (a_3)^3 \\ &= 4^3 \\ &= 64 \end{aligned}$$

$$\begin{aligned} a_5 &= (a_4)^4 \\ &= 64^4 \end{aligned}$$

### ***Exercise***

Find the first five terms of the recursively defined infinite sequence:  $a_1 = A, \quad a_n = a_{n-1} + d$

#### **Solution**

$$\begin{aligned} a_2 &= a_1 + d \\ &= A + d \end{aligned}$$

$$\begin{aligned} a_3 &= a_2 + d \\ &= A + d + d \\ &= A + 2d \end{aligned}$$

$$\begin{aligned} a_4 &= a_3 + d \\ &= A + 3d \end{aligned}$$

$$\begin{aligned} a_5 &= a_4 + d \\ &= A + 4d \end{aligned}$$

### ***Exercise***

Find the first five terms of the recursively defined infinite sequence:  $a_1 = A$ ,  $a_n = ra_{n-1}$ ,  $r \neq 0$

### **Solution**

$$\begin{aligned} a_2 &= ra_1 \\ &= rA \end{aligned}$$

$$\begin{aligned} a_3 &= ra_2 \\ &= Ar^2 \end{aligned}$$

$$\begin{aligned} a_4 &= ra_3 \\ &= Ar^3 \end{aligned}$$

$$\begin{aligned} a_5 &= ra_4 \\ &= Ar^4 \end{aligned}$$

### ***Exercise***

Find the first 5 terms of the recursively defined infinite sequence:  $a_1 = 2$ ,  $a_2 = 2$ ;  $a_n = a_{n-1} \cdot a_{n-2}$

### **Solution**

$$\begin{aligned} a_3 &= a_2 \cdot a_1 \\ &= 2 \cdot 2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} a_4 &= a_3 \cdot a_2 \\ &= 4 \cdot 2 \\ &= 8 \end{aligned}$$

$$\begin{aligned} a_5 &= a_4 \cdot a_3 \\ &= 8 \cdot 4 \\ &= 32 \end{aligned}$$

$$\begin{aligned} a_6 &= a_5 \cdot a_4 \\ &= 32 \cdot 8 \\ &= 256 \end{aligned}$$

### ***Exercise***

Express each sum using summation notation  $1 + 2 + 3 + \dots + 20$

#### **Solution**

$$1 + 2 + 3 + 4 + \dots + 20 = \sum_{k=1}^{20} k$$

### ***Exercise***

Express each sum using summation notation  $1 + 2 + 3 + \dots + 40$

#### **Solution**

$$1 + 2 + 3 + \dots + 40 = \sum_{k=1}^{40} k$$

### ***Exercise***

Express each sum using summation notation  $1^3 + 2^3 + 3^3 + \dots + 8^3$

#### **Solution**

$$1^3 + 2^3 + 3^3 + \dots + 8^3 = \sum_{k=1}^8 k^3$$

### ***Exercise***

Express each sum using summation notation  $1^2 + 2^2 + 3^2 + \dots + 15^2$

#### **Solution**

$$1^2 + 2^2 + 3^2 + \dots + 15^2 = \sum_{k=1}^{15} k^2$$

### ***Exercise***

Express each sum using summation notation  $2^2 + 2^3 + 2^4 + \dots + 2^{11}$

#### **Solution**

$$2^2 + 2^3 + 2^4 + \dots + 2^{11} = \sum_{k=2}^{11} 2^k$$

### Exercise

Express each sum using summation notation

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{13}{14}$$

### Solution

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{13}{14} = \sum_{k=1}^{13} \frac{k}{k+1}$$

### Exercise

Express each sum using summation notation

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots + (-1)^6 \frac{1}{3^6}$$

### Solution

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots + (-1)^6 \frac{1}{3^6} = \sum_{k=0}^6 (-1)^k \frac{1}{3^k}$$

### Exercise

Express each sum using summation notation

$$\frac{2}{3} - \frac{4}{9} + \frac{8}{27} - \cdots + (-1)^{12} \left(\frac{2}{3}\right)^{11}$$

### Solution

$$\frac{2}{3} - \frac{4}{9} + \frac{8}{27} - \cdots + (-1)^{12} \left(\frac{2}{3}\right)^{11} = \sum_{k=1}^{11} (-1)^{k+1} \left(\frac{2}{3}\right)^k$$

### Exercise

Express each sum using summation notation

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{14}{14+1}$$

### Solution

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{14}{14+1} = \sum_{k=1}^{14} \frac{k}{k+1}$$

### Exercise

Express each sum using summation notation

$$\frac{1}{e} + \frac{2}{e^2} + \frac{3}{e^3} + \cdots + \frac{n}{e^n}$$

### Solution

$$\frac{1}{e} + \frac{2}{e^2} + \frac{3}{e^3} + \cdots + \frac{n}{e^n} = \sum_{k=1}^n \frac{k}{e^k}$$

***Exercise***

Find the sum:  $\sum_{k=1}^5 (2k - 7)$

**Solution**

$$\sum_{k=1}^5 (2k - 7) = (-5) + (-3) + (-1) + 1 + 3$$
$$\underline{= -5}$$

***Exercise***

Find the sum:  $\sum_{k=0}^5 k(k - 2)$

**Solution**

$$\sum_{k=0}^5 k(k - 2) = 0 + (-1) + 0 + 3 + 8 + 15$$
$$\underline{= 25}$$

***Exercise***

Find the sum:  $\sum_{k=1}^5 (-3)^{k-1}$

**Solution**

$$\sum_{k=1}^5 (-3)^{k-1} = 1 + (-3) + 9 + (-27) + 81$$
$$\underline{= 61}$$

***Exercise***

Find the sum:  $\sum_{k=253}^{571} \left(\frac{1}{3}\right)$

**Solution**



$$\sum_{k=253}^{571} \left(\frac{1}{3}\right) = (571 - 253 + 1) \left(\frac{1}{3}\right)$$

$$= \underline{\underline{\frac{319}{3}}}$$

$$\sum_{k=m}^n c = (n - m + 1)c$$

### Exercise

Find the sum:  $\sum_{k=1}^{50} 8$

### Solution

$$\sum_{k=1}^{50} 8 = (50 - 1 + 1)8$$

$$= \underline{\underline{400}}$$

$$\sum_{k=m}^n c = (n - m + 1)c$$

### Exercise

Find the sum:  $\sum_{k=1}^{40} k$

### Solution

$$\sum_{k=1}^{40} k = \frac{40(41)}{2}$$

$$= \underline{\underline{820}}$$

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

### Exercise

Find the sum:  $\sum_{k=1}^5 (3k)$

### Solution

$$\sum_{k=1}^5 3k = 3(1) + 3(2) + 3(3) + 3(4) + 3(5)$$

$$= \underline{\underline{45}}$$

### Exercise

Find the sum:  $\sum_{k=1}^{10} (k^3 + 1)$

### Solution

$$\begin{aligned}\sum_{k=1}^{10} (k^3 + 1) &= \sum_{k=1}^{10} k^3 + \sum_{k=1}^{10} 1 \\ &= \frac{10^2(10+1)^2}{4} + 10(1) \\ &= \frac{12100}{4} + 10 \\ &= 3025 + 10 \\ &= \underline{3035} \quad | \end{aligned}$$

$$\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}$$

### Exercise

Find the sum:  $\sum_{k=1}^{24} (k^2 - 7k + 2)$

### Solution

$$\begin{aligned}\sum_{k=1}^{24} (k^2 - 7k + 2) &= \frac{24(24+1)(2 \cdot 24 + 1)}{6} - 7 \frac{24(24+1)}{2} + 2(24) \\ &= \underline{2848} \quad | \end{aligned}$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

### Exercise

Find the sum:  $\sum_{k=6}^{20} (4k^2)$

### Solution

$$\begin{aligned}\sum_{k=6}^{20} (4k^2) &= 4 \left( \sum_{k=1}^{20} k^2 - \sum_{k=1}^5 k^2 \right) \\ &= 4 \left( \frac{20(20+1)(2 \cdot 20 + 1)}{6} - \frac{5(5+1)(2 \cdot 5 + 1)}{6} \right) \end{aligned}$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\begin{aligned}
 &= 4 \left( \frac{20(21)(41)}{6} - \frac{5(6)(11)}{6} \right) \\
 &= 4(2870 - 55) \\
 &= \underline{11,260}
 \end{aligned}$$

### Exercise

Find the sum:  $\sum_{k=1}^{16} (k^2 - 4)$

### Solution

$$\begin{aligned}
 \sum_{k=1}^{16} (k^2 - 4) &= \sum_{k=1}^{16} k^2 - \sum_{k=1}^{16} 4 \\
 &= \frac{16(16+1)(2 \cdot 16 + 1)}{6} - 4(16) \\
 &= 1496 - 64 \\
 &= \underline{1432}
 \end{aligned}$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

### Exercise

Find the sum:  $\sum_{k=1}^6 (10 - 3k)$

### Solution

$$\begin{aligned}
 \sum_{k=1}^6 (10 - 3k) &= 7 + 4 + 1 - 2 - 5 - 8 \\
 &= \underline{-3}
 \end{aligned}$$

### Exercise

Find the sum:  $\sum_{k=1}^{10} [1 + (-1)^k]$

### Solution

$$\sum_{k=1}^{10} \left[ 1 + (-1)^k \right] = 0 + 2 + 0 + 2 + 0 + 2 + 0 + 2 + 0 + 2$$

$$\underline{= 10}$$

### Exercise

Find the sum:  $\sum_{k=1}^6 \frac{3}{k+1}$

### Solution

$$\sum_{k=1}^6 \frac{3}{k+1} = \frac{3}{2} + 1 + \frac{3}{4} + \frac{3}{5} + 2 + \frac{3}{7}$$

$$\underline{= \frac{879}{140}}$$

### Exercise

Find the sum:  $\sum_{k=137}^{428} 2.1$

### Solution

$$\sum_{k=137}^{428} 2.1 = (428 - 137 + 1)2.1 = (292)2.1$$

$$\underline{= 613.2}$$

$$\sum_{k=m}^n c = (n - m + 1)c$$

### Exercise

Write out each sum  $\sum_{k=1}^n (k+2)$

### Solution

$$\sum_{k=1}^n (k+2) = \underline{3 + 5 + 7 + 9 + \cdots + (n+2)}$$

### Exercise

Write out each sum  $\sum_{k=1}^n k^2$

### Solution

$$\begin{aligned}\sum_{k=1}^n k^2 &= 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 \\ &= \underline{1 + 4 + 9 + 16 + \cdots + n^2}\end{aligned}$$

### Exercise

Write out each sum  $\sum_{k=2}^n (-1)^k \ln k$

### Solution

$$\begin{aligned}\sum_{k=2}^n (-1)^k \ln k &= (-1)^2 \ln 2 + (-1)^3 \ln 3 + (-1)^4 \ln 4 + (-1)^5 \ln 5 + \cdots + (-1)^n \ln n \\ &= \underline{\ln 2 - \ln 3 + \ln 4 - \ln 5 + \cdots + (-1)^n \ln n}\end{aligned}$$

### Exercise

Write out each sum  $\sum_{k=3}^n (-1)^{k+1} 2^k$

### Solution

$$\begin{aligned}\sum_{k=3}^n (-1)^{k+1} 2^k &= (-1)^4 2^3 + (-1)^5 2^4 + (-1)^6 2^5 + (-1)^7 2^6 + \cdots + (-1)^{n+1} 2^n \\ &= \underline{8 - 16 + 32 - 64 + \cdots + (-1)^{n+1} 2^n}\end{aligned}$$

### Exercise

Write out each sum  $\sum_{k=0}^n \frac{1}{3^k}$

### Solution

$$\sum_{k=0}^n \frac{1}{3^k} = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^n}$$

### ***Exercise***

Fred has a balance of \$3,000 on his card which charges 1% interest per month on any unpaid balance. Fred can afford to pay \$100 toward the balance each *month*. His balance each month after making a \$100 payment is given by the recursively defined sequence

$$B_0 = \$3,000 \quad B_n = 1.01B_{n-1} - 100$$

Determine Fred's balance after making the first payment. That is, determine  $B_1$

### **Solution**

$$\begin{aligned} B_1 &= 1.01B_0 - 100 \\ &= 1.01(3,000) - 100 \\ &= \$2,930 \end{aligned}$$

Fred's balance is \$2,930 after making the first payment.

### ***Exercise***

A pond currently has 2,000 trout in it. A fish hatchery decides to add an additional 20 trout each month. Is it also known that the trout population is growing at a rate of 3% per *month*. The size of the population after  $n$  months is given by the recursively defined sequence

$$P_0 = 2,000 \quad P_n = 1.03P_{n-1} + 20$$

How many trout are in the pond after 2 months? That is, what is  $P_2$ ?

### **Solution**

$$\begin{aligned} P_1 &= 1.03P_0 + 20 \\ &= 1.03(2,000) + 20 \\ &= 2,080 \end{aligned}$$

$$\begin{aligned} P_2 &= 1.03P_1 + 20 \\ &= 1.03(2,080) + 20 \\ &= 2,162.4 \end{aligned}$$

There are approximately 2162 **trout** in the pond after 2 *months*.

### Exercise

Fred bought a car by taking out a loan for \$18,500 at 0.5% interest per month. Fred's normal monthly payment is \$434.47 per month, but he decides that he can afford to pay \$100 extra toward the balance each month. His balance each month is given by the recursively defined sequence

$$B_0 = \$18,500 \quad B_n = 1.005B_{n-1} - 534.47$$

Determine Fred's balance after making the first payment. That is, determine  $B_1$

### Solution

$$\begin{aligned} B_1 &= 1.005B_0 - 534.47 \\ &= 1.005(18,500) - 534.47 \\ &= \underline{\$18,058.03} \end{aligned}$$

Fred's balance is \$18,058.03 after making the first payment.

### Exercise

The Environmental Protection Agency (EPA) determines that Maple Lake has 250 *tons* of pollutant as a result of industrial waste and that 10% of the pollutant present is neutralized by solar oxidation every year. The EPA imposes new pollution control laws that result in 15 *tons* of new pollutant entering the lake each year. The amount of pollutant in the lake after  $n$  years is given by the recursively defined sequence

$$P_0 = 250 \quad P_n = 0.9P_{n-1} + 15$$

Determine the amount of pollutant in the lake after 2 years? That is, what is  $P_2$ ?

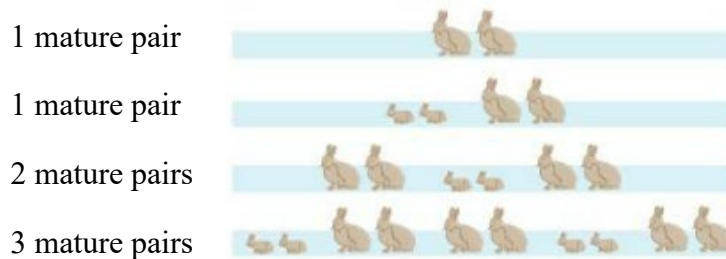
### Solution

$$\begin{aligned} P_1 &= 0.9P_0 + 15 \\ &= 0.9(250) + 15 \\ &= \underline{240} \\ P_2 &= 0.9P_1 + 15 \\ &= 0.9(240) + 15 \\ &= \underline{231} \end{aligned}$$

There are 231 *tons* of pollutants after 2 years.

### Exercise

A colony of rabbits begins with one pair of mature rabbits, which will produce a pair of offspring (one male, one female) each month. Assume that all rabbits mature in 1 month and produce a pair of offspring (one male, one female) after 2 months. If no rabbits ever die, how many pairs of mature rabbits are there after 7 months?



### Solution

$$a_1 = 1$$

$$a_2 = 1$$

$$a_3 = 2$$

$$a_4 = 3$$

$$a_5 = 5$$

$$a_6 = 8$$

$$a_7 = 13$$

$$a_8 = 21$$

$$\vdots$$

$$a_n = a_{n-1} + a_{n-2}$$

After 7 months there are 21 mature pairs of rabbits.

### Exercise

Let 
$$u_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}$$

Define the  $n$ th term of a sequence

a) Show that  $u_1 = 1$  and  $u_2 = 1$

b) Show that  $u_{n+2} = u_{n+1} + u_n$

c) Draw the conclusion that  $\{u_n\}$  is a Fibonacci sequence

d) Find the first ten terms of the sequence from part (c)

### Solution



$$\begin{aligned}
 a) \quad u_1 &= \frac{(1+\sqrt{5})^1 - (1-\sqrt{5})^1}{2^1 \sqrt{5}} \\
 &= \frac{2\sqrt{5}}{2\sqrt{5}} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 u_2 &= \frac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{2^2 \sqrt{5}} & a^2 - b^2 &= (a-b)(a+b) \\
 &= \frac{(1+\sqrt{5}-1+\sqrt{5}) - (1-\sqrt{5}+1-\sqrt{5})}{2^2 \sqrt{5}} \\
 &= \frac{4\sqrt{5}}{4\sqrt{5}} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 b) \quad u_{n+1} + u_n &= \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1} \sqrt{5}} + \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n \sqrt{5}} \\
 &= \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1} + 2(1+\sqrt{5})^n - 2(1-\sqrt{5})^n}{2^{n+1} \sqrt{5}} \\
 &= \frac{(1+\sqrt{5})^n (1+\sqrt{5}+2) - (1-\sqrt{5})^n (1-\sqrt{5}+2)}{2^{n+1} \sqrt{5}} \\
 &= \frac{(1+\sqrt{5})^n (3+\sqrt{5}) - (1-\sqrt{5})^n (3-\sqrt{5})}{2^{n+1} \sqrt{5}} \\
 &= \frac{(1+\sqrt{5})^{n+2} \frac{3+\sqrt{5}}{(1+\sqrt{5})^2} - (1-\sqrt{5})^{n+2} \frac{3-\sqrt{5}}{(1-\sqrt{5})^2}}{2^{n+1} \sqrt{5}} \\
 &= \frac{(1+\sqrt{5})^{n+2} \frac{3+\sqrt{5}}{1+2\sqrt{5}+5} - (1-\sqrt{5})^{n+2} \frac{3-\sqrt{5}}{1-2\sqrt{5}+5}}{2^{n+1} \sqrt{5}} \\
 &= \frac{\frac{1}{2}(1+\sqrt{5})^{n+2} \frac{3+\sqrt{5}}{3+\sqrt{5}} - \frac{1}{2}(1-\sqrt{5})^{n+2} \frac{3-\sqrt{5}}{3-\sqrt{5}}}{2^{n+1} \sqrt{5}}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \frac{(1+\sqrt{5})^{n+2} - (1-\sqrt{5})^{n+2}}{2^{n+1} \sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n+2} - (1-\sqrt{5})^{n+2}}{2^{n+2} \sqrt{5}} \\
&= \underline{u_{n+2}} \quad \checkmark
\end{aligned}$$

c) Since  $u_1 = 1$  and  $u_2 = 1$  and  $u_{n+2} = u_{n+1} + u_n$

$\therefore \{u_n\}$  is a Fibonacci sequence

e)  $u_1 = 1$

$u_2 = 1$

$u_3 = u_2 + u_1 = 1 + 1 = \underline{2}$

$u_4 = u_3 + u_2 = 2 + 1 = \underline{3}$

$u_5 = u_4 + u_3 = 3 + 2 = \underline{5}$

$u_6 = u_5 + u_4 = 5 + 3 = \underline{8}$

$u_7 = u_6 + u_5 = 8 + 5 = \underline{13}$

$u_8 = u_7 + u_6 = 13 + 8 = \underline{21}$

$u_9 = u_8 + u_7 = 21 + 13 = \underline{34}$

$u_{10} = u_9 + u_8 = 34 + 21 = \underline{55}$