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Section 3-1 Intro to Limits

Goal: To find limits of functions

Definition: Limit

We write

$$\lim_{x \to c} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to c$$

if the functional value $f(x)$ is close to the single real number L whenever x is close, but not equal, to c (on either side of c).

Definition: One sided limits

$\lim_{x \to c^-} f(x) = L$ is the limit of the function as x approaches the value c from the left.

$\lim_{x \to c^+} f(x) = L$ is the limit of the function as x approaches the value c from the right.

Properties of Limits:

1. $\lim_{x \to c} k = k$ for any constant k
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
4. $\lim_{x \to c} [f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$
5. $\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x)$ for any constant k
6. $\lim_{x \to c} [f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$
7. $\lim_{x \to c} \left(\dfrac{f(x)}{g(x)} \right) = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ provided $\lim_{x \to c} g(x) \neq 0$
8. $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$ (the limit value must be positive for n even.)

1 - 5 Find each limit if it exists

1. $\lim_{x\to 6}(4x+5) = (4(6)+5) = 24+5 = 29$

2. $\lim_{x\to -3} 5x = 5(-3) = -15$

3. $\lim_{x\to 3} x(2x+7) = \lim_{x\to 3} x \cdot \lim_{x\to 3}(2x+7) = 3[2(3)+7)] = 3(13) = 39$

4. $\lim_{x\to -1}\left(\dfrac{x+8}{x+2}\right) = \dfrac{\lim_{x\to -1} x+8}{\lim_{x\to -1} x+2} = \dfrac{-1+8}{-1+2} = \dfrac{7}{1} = 7$

5. $\lim_{x\to -5}\sqrt{-5x+11} = \sqrt{-5(-5)+11} = \sqrt{25+11} = \sqrt{36} = 6$

6 - 8 Find the value of the following limits given that

$\lim_{x\to 3} f(x) = 6$ and $\lim_{x\to 3} g(x) = -2.$

6. $\lim_{x\to 3} 7f(x) = 7\cdot 6 = 42$

7. $\lim_{x\to 3}[3f(x) - 2g(x)] = 3(6) - 2(-2) = 18+4 = 22$

8. $\lim_{x\to 3}\left(\dfrac{2f(x)}{3g(x)}\right) = \dfrac{2(6)}{3(-2)} = \dfrac{12}{-6} = -2$

9. Let $f(x)=\begin{cases} x^2+2 & \text{if } x<1 \\ 4x-1 & \text{if } x\ge 1 \end{cases}$. Find:

a) $\lim\limits_{x\to 1^-} f(x)$

Since we are looking for the left–hand limit, use the top function.

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x^2+2)=((1)^2+2)=3$$

b) $\lim\limits_{x\to 1^+} f(x)$

Since we are looking for the right–hand limit, use the bottom function.

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(4x-1)=(4(1)-1)=3$$

c) $\lim\limits_{x\to 1} f(x)$ Since the left– and right–hand limits are the same value, $\lim\limits_{x\to 1} f(x)=3$.

d) $f(1)$

The value of 1 is defined in the bottom function, therefore $f(1)=4(1)-1=3$.

10. Let $f(x)=\begin{cases} 5x-6 & \text{if } x\le 2 \\ 2x+4 & \text{if } x>2 \end{cases}$. Find:

a) $\lim\limits_{x\to 2^-} f(x)$

Since we are looking for the left–hand limit, use the top function.

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(5x-6)=5(2)-6=4$$

b) $\lim\limits_{x\to 2^+} f(x)$

Since we are looking for the right–hand limit, use the bottom function.

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(2x+4)=(2(2)+4)=8$$

c) $\lim_{x \to 2} f(x)$

Since the left– and right–hand limits are not the same value, $\lim_{x \to 2} f(x)$ does not exist.

d) $f(2)$

The value of 2 is defined in the top function, therefore $f(2) = 5(2) - 6 = 4.$

11. Let $f(x) = \left(\dfrac{x^2 + 4x - 5}{x - 1} \right)$. Find

a) $\lim_{x \to 1} f(x)$

Substituting in a value of 1 will result in a zero in the denominator, therefore we must try to remove the problem by first simplifying the function:

$$f(x) = \frac{x^2 + 4x - 5}{x - 1} = \frac{(x+5)(x-1)}{x-1} = x + 5, \quad x \neq 1$$

Therefore, $\lim_{x \to 1} f(x) = \lim_{x \to 1} (x + 5) = 1 + 5 = 6$

b) $$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2 + 4x - 5}{x - 1} = \frac{(-1)^2 + 4(-1) - 5}{-1 - 1} = \frac{-8}{-2} = 4$$

c) $$\lim_{x \to -5} f(x) = \lim_{x \to -5} \frac{x^2 + 4x - 5}{x - 1} = \frac{(-5)^2 + 4(-5) - 5}{-5 - 1} = \frac{0}{-6} = 0$$

12. Let $f(x) = \left(\dfrac{|x-2|}{x-2}\right)$. Find

a) $\lim\limits_{x \to 2^-} f(x)$

As x approaches 2 from the left side, the value of the numerator will be positive and the value of the denominator will be negative (because the x values are smaller than 2). The limit will therefore, be a value of –1 (because they are opposite signs).

b) $\lim\limits_{x \to 2^+} f(x)$

As x approaches 2 from the right side, the value of the numerator will be positive and the value of the denominator will be positive (because the x values are larger than 2). The limit will therefore, be a value of 1 (because they are the same sign).

c) $\lim\limits_{x \to 2} f(x)$ does not exist because the left–hand and right–hand limits are different values.

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Section 3-2 Infinite Limits and Limits at Infinity

Goal: To find limits of functions as they approach infinity

Limits of Power Functions at Infinity:

If p is a positive real number and k is any real number except 0, then

1. $\lim\limits_{x \to -\infty} \frac{k}{x^p} = 0$

2. $\lim\limits_{x \to \infty} \frac{k}{x^p} = 0$

3. $\lim\limits_{x \to -\infty} kx^p = \pm\infty$

4. $\lim\limits_{x \to \infty} kx^p = \pm\infty$

provided that x^p is a real number for negative values of x. The limits in 3 and 4 will be either positive or negative infinity, depending on k and p.

Limits of Rational Functions at Infinity:

If $f(x) = \dfrac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}, a_m \neq 0, b_n \neq 0,$

then $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \dfrac{a_m x^m}{b_n x^n}$ and $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \dfrac{a_m x^m}{b_n x^n}$

There are three cases to consider:

1. If $m < n$, then $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to -\infty} f(x) = 0.$

2. If $m = n$, $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to -\infty} f(x) = \dfrac{a_m}{b_n}.$

3. If $m > n$, then the limit will be ∞ or $-\infty$, depending on the values of m, n, a_m, and b_n.

1 - 3 Find each limit. Use ∞ or $-\infty$ when appropriate.

1. $f(x)=\dfrac{x}{x+2}$

a) $\lim\limits_{x\to -2^-} f(x)$

If the value is substituted into the function, we would have a zero in the denominator. Since there is no way to remove the problem, we will look at what happens around the value of –2 when approached from the left. These values are smaller than –2, therefore the denominator will approach 0 from the negative side (that is, the denominator will always be negative). Since the numerator will always be negative, $\lim\limits_{x\to -2^-} f(x)=\infty$.

b) $\lim\limits_{x\to -2^+} f(x)$

If the value is substituted into the function, we would have a zero in the denominator. Since there is no way to remove the problem, we will look at what happens around the value of –2 when approached from the right. These values are larger than –2, therefore the denominator will approach 0 from the positive side (that is, the denominator will always be positive). Since the numerator will always be negative, $\lim\limits_{x\to -2^+} f(x)=-\infty$.

c) $\lim\limits_{x\to -2} f(x)$ does not exist because the left–hand and right–hand limits are different infinite limits.

2. $f(x)=\dfrac{3x-4}{(x-3)^2}$

a) $\lim\limits_{x\to 3^-} f(x)$

If the value is substituted into the function, we would have a zero in the denominator. Since there is no way to remove the problem, we will look at what happens around the value of 3 when approached from the left. These values are smaller than 3, but the denominator value is being squared and will always be positive. Since the value of the numerator is 5 as the x value approaches 3, $\lim\limits_{x\to 3^-} f(x)=\infty$.

b) $\lim_{x \to 3^+} f(x)$

If the value is substituted into the function, we would have a zero in the denominator. Since there is no way to remove the problem, we will look at what happens around the value of 3 when approached from the right. These values are larger than 3, but the denominator value is being squared and will always be positive. Since the value of the numerator is 5 as the *x* value approaches 3, $\lim_{x \to 3^+} f(x) = \infty$.

c) $\lim_{x \to 3} f(x) = \infty$ because the left– and right–hand limits both approach infinity.

3. $f(x) = \dfrac{x^2 + 3x - 5}{x + 5}$

a) $\lim_{x \to -5^-} f(x)$

If the value is substituted into the function, we would have a zero in the denominator. Since there is no way to remove the problem, we will look at what happens around the value of –5 when approached from the left. These values are smaller than –5, therefore the denominator will approach 0 from the negative side (that is, the denominator will always be negative). Since the numerator is positive around –5, $\lim_{x \to -5^-} f(x) = -\infty$.

b) $\lim_{x \to -5^+} f(x)$

If the value is substituted into the function, we would have a zero in the denominator. Since there is no way to remove the problem, we will look at what happens around the value of –5 when approached from the right. These values are larger than –5, therefore the denominator will approach 0 from the positive side (that is, the denominator will always be positive). Since the numerator is positive around –5, $\lim_{x \to -5^+} f(x) = \infty$.

c) $\lim_{x \to -5} f(x)$ does not exist because the left–hand and right–hand limits are different infinite limits.

4 - 6 Find each function value and limit. Use ∞ or $-\infty$ where appropriate.

4. $f(x) = \dfrac{2x+3}{3x-8}$

a) $f(20)$

$$f(20) = \frac{2(20)+3}{3(20)-8} = \frac{43}{52} \approx 0.8269$$

b) $f(200)$

$$f(200) = \frac{2(200)+3}{3(200)-8} = \frac{403}{592} \approx 0.6807$$

c) $\lim\limits_{x\to\infty} f(x) = \dfrac{2}{3}$ because the function is a rational expression and $m = n$, the limit is the ratio of the coefficients.

5. $f(x) = \dfrac{x-5}{3x^2+2x+2}$

a) $f(10)$

$$f(10) = \frac{10-5}{3(10)^2+2(10)+2} = \frac{5}{322} \approx 0.0155$$

b) $f(100)$

$$f(100) = \frac{100-5}{3(100)^2+2(100)+2} = \frac{95}{30,222} \approx 0.0031$$

c) $\lim\limits_{x\to\infty} f(x) = 0$ because the function is a rational expression and $m < n$.

6. $f(x) = \dfrac{x^3 + 2x - 1}{3 - 8x}$

a) $f(-10)$

$$f(-10) = \frac{(-10)^3 + 2(-10) - 1}{3 - 8(-10)} = \frac{-1021}{83} \approx -12.3012$$

b) $f(-100)$

$$f(-100) = \frac{(-100)^3 + 2(-100) - 1}{3 - 8(-100)} = \frac{-1{,}000{,}201}{803} \approx -1245.5803$$

c) $\lim\limits_{x \to -\infty} f(x) = -\infty$ because the function is a rational expression, $m > n$, and when x becomes increasingly negative, the value of the function will be increasingly negative.

7 - 9 Find the vertical and horizontal asymptotes for the following functions.

7. $f(x) = \dfrac{3x}{x + 3}$

Vertical asymptotes are found by setting the denominator equal to zero. Therefore, there would be a vertical asymptote at $x + 3 = 0 \Rightarrow x = -3$.

Horizontal asymptotes are found by finding the limit of the function as it goes to infinity. The limit of this function is 3 because $m = n$ and the limit is the ratio of the coefficients. Therefore, the horizontal asymptote is $y = 3$.

8. $f(x) = \dfrac{x^2 - 3}{x^2 + 4}$

Vertical asymptotes are found by setting the denominator equal to zero. Since the denominator cannot be zero, there are no vertical asymptotes.

Horizontal asymptotes are found by finding the limit of the function as it goes to infinity. The limit of this function is 1 because $m = n$ and the limit is the ratio of the coefficients. Therefore, the horizontal asymptote is $y = 1$.

9. $f(x)=\dfrac{x^2+4x-5}{x^2+8x+15}$

The function can be reduced as follows:

$$f(x)=\frac{x^2+4x-5}{x^2+8x+15}=\frac{(x+5)(x-1)}{(x+5)(x+3)}=\frac{x-1}{x+3},\quad x\neq -5 \text{ and } x\neq -3$$

Vertical asymptotes are found by setting the denominator equal to zero. Therefore, there would be a vertical asymptote at $x+3=0 \Rightarrow x=-3$.

Horizontal asymptotes are found by finding the limit of the function as it goes to infinity. The limit of this function is 1 because $m=n$ and the limit is the ratio of the coefficients. Therefore, the horizontal asymptote is $y=1$.

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Section 3-3 Continuity

Goal: To determine if functions are continuous at specific points and intervals

Definition: Continuity

A function *f* is continuous at the point $x = c$ if

1. $\lim_{x \to c} f(x)$ exists 2. $f(c)$ exists 3. $\lim_{x \to c} f(x) = f(c)$

Continuity Properties:

1. A constant function $f(x) = k$, where *k* is a constant, is continuous for all *x*.
2. For *n* a positive integer, $f(x) = x^n$ is continuous for all *x*.
3. A polynomial function is continuous for all *x*.
4. A rational function is continuous for all *x* except those values that make a denominator 0.
5. For *n* an odd positive integer greater than 1, $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous.
6. For *n* an even positive integer, $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous and nonnegative.

Constructing Sign Charts:

1. Find all partition numbers. These are all the values that make the function discontinuous or 0.
2. Plot the numbers found in step 1 on a real–number line, dividing the number line into intervals.
3. Select a test value in each open interval and evaluate $f(x)$ at each test value to determine whether $f(x)$ is positive or negative.
4. Construct a sign chart, using the real–number line in step 2.

1 - 5 Using the continuity properties, determine where each of the functions are continuous.

1. $f(x)=3x^3-4x^2+x+7$

Since the function is a polynomial function, it is continuous for all x.

2. $f(x)=\dfrac{x^3+5x^2+x-3}{x^2-8x+15}$

Since the function is a rational function, it is continuous for all x except when the denominator is 0. Therefore, $x^2-8x+15=0$, or $(x-5)(x-3)=0$, when $x=3$ or $x=5$. So the function is continuous for all x except $x=3$ or $x=5$.

3. $f(x)=\dfrac{6x-1}{x^2+6}$

Since the function is a rational function, it is continuous for all x except when the denominator is 0. The denominator cannot have a value of 0, therefore the function is continuous for all x.

4. $f(x)=\sqrt[3]{x-10}$

Since the function is an odd positive root, the function is continuous for all values of x where the radicand is continuous. The radicand is a polynomial function, therefore the function is continuous for all x.

5. $f(x)=\sqrt{x^2-25}$

Since the function is an even root, the function is continuous for all values of x where the radicand is continuous and nonnegative. Therefore, $x^2-25\geq 0$ and the points in question are ± 5. For all values between –5 and 5, the radicand would be negative, therefore, the function is continuous on the interval $(-\infty,-5]\cup[5,\infty)$.

6 - 7 Use a sign chart to solve each inequality. Express answers in inequality and interval notation.

6. $4x^2 - 29x + 7 < 0$

Find the partition numbers:

$$4x^2 - 29x + 7 = 0$$
$$(4x - 1)(x - 7) = 0$$

Therefore, the partition numbers are $\frac{1}{4}$ and 7.

The number line would be broken into three parts involving the intervals $(-\infty, \frac{1}{4})$, $(\frac{1}{4}, 7)$, and $(7, \infty)$. Pick test values. We choose 0, 1, and 8 for the test values. Find the function values using these three test values.

$$f(0) = 4(0)^2 - 29(0) + 7$$
$$f(0) = 7$$

$$f(1) = 4(1)^2 - 29(1) + 7$$
$$f(1) = 4 - 29 + 7$$
$$f(1) = -18$$

$$f(8) = 4(8)^2 - 29(8) + 7$$
$$f(8) = 256 - 232 + 7$$
$$f(8) = 31$$

Only the test value 1 makes the inequality true, so the solution is $\frac{1}{4} < x < 7$, or $(\frac{1}{4}, 7)$.

7. $\dfrac{x^3+6x^2}{x+3} > 0$

Find the partition numbers:

$$x^3+6x^2=0 \qquad\qquad x+3=0$$
$$x^2(x+6)=0$$

Therefore, the partition numbers are 0, –6, and –3.

The number line would be broken into four parts involving the intervals $(-\infty,-6)$, $(-6,-3)$, $(-3,0)$, and $(0,\infty)$. Pick test values. We choose –7, –4, –1, and 1 for the test values. Find the function values using these four test values.

$$f(-7)=\frac{(-7)^3+6(-7)^2}{-7+3}=\frac{-49}{-4}=12.25 \qquad f(-4)=\frac{(-4)^3+6(-4)^2}{-4+3}=\frac{32}{-1}=-32$$

$$f(-1)=\frac{(-1)^3+6(-1)^2}{-1+3}=\frac{5}{2}=2.5 \qquad f(1)=\frac{(1)^3+6(1)^2}{1+3}=\frac{7}{4}=1.75$$

The test values of –7, –1, and 1 make the original inequality true, so the solution is $x<-6$ or $x>-3$, or $(-\infty,-6)\cup(-3,\infty)$.

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Section 3-4 The Derivative

Goal: To find the first derivative of a function using the four step process.

Definition: Average Rate of Change

For $y = f(x)$, the average rate of change from $x = a$ to $x = a + h$ is

$$\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h},\ h \neq 0$$

where h is the distance from the initial value of x to the final value of x.

Definition: Instantaneous Rate of Change

For $y = f(x)$, the instantaneous rate of change at $x = a$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \text{ if the limit exists.}$$

This formula is also used to find the slope of a graph at the point $(a, f(a))$ and to find the first derivative of a function, $f(x)$.

Procedure: Finding the first derivative:

1. Find $f(x+h)$.
2. Find $f(x+h) - f(x)$.
3. Find $\dfrac{f(x+h) - f(x)}{h}$.
4. Find $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

1 - 3 Use the four step procedure to find $f'(x)$ and then find $f'(1)$, $f'(2)$, and $f'(3)$.

1. $f(x) = 6x - 9$

Step 1: $f(x+h) = 6(x+h) - 9 = 6x + 6h - 9$

Step 2: $f(x+h) - f(x) = 6x + 6h - 9 - (6x - 9) = 6h$

Step 3: $\dfrac{f(x+h) - f(x)}{h} = \dfrac{6h}{h} = 6$

Step 4: $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} 6 = 6$

$f'(x) = 6$, $f'(1) = 6$, $f'(2) = 6$, and $f'(3) = 6$.

2. $f(x) = -2x^2 + 4x - 7$

Step 1:
$$f(x+h) = -2(x+h)^2 + 4(x+h) - 7$$
$$= -2(x^2 + 2xh + h^2) + 4(x+h) - 7$$
$$f(x+h) = -2x^2 - 4xh - 2h^2 + 4x + 4h - 7$$

Step 2:
$$f(x+h) - f(x) = -2x^2 - 4xh - 2h^2 + 4x + 4h - 7 - (-2x^2 + 4x - 7)$$
$$f(x+h) - f(x) = -4xh - 2h^2 + 4h$$

Step 3:
$$\frac{f(x+h) - f(x)}{h} = \frac{-4xh - 2h^2 + 4h}{h}$$
$$\frac{f(x+h) - f(x)}{h} = -4x - 2h + 4$$

Step 4:
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (-4x - 2h + 4)$$
$$= -4x - 2(0) + 4$$
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = -4x + 4$$

Therefore, $f'(x) = -4x + 4$

$f'(1) = -4(1) + 4$	$f'(2) = -4(2) + 4$	$f'(3) = -4(3) + 4$
$f'(1) = 0$	$f'(2) = -4$	$f'(3) = -8$

3. $f(x) = \dfrac{2x}{x-5}$

<u>Step 1</u>: $$f(x+h) = \frac{2(x+h)}{(x+h)-5} = \frac{2x+2h}{x+h-5}$$

<u>Step 2</u>: $$f(x+h) - f(x) = \frac{2x+2h}{x+h-5} - \frac{2x}{x-5}$$
$$= \frac{(2x+2h)(x-5)}{(x+h-5)(x-5)} - \frac{2x(x+h-5)}{(x+h-5)(x-5)}$$
$$f(x+h) - f(x) = \frac{-10h}{(x+h-5)(x-5)}$$

<u>Step 3</u>: $$\frac{f(x+h)-f(x)}{h} = \frac{-10h}{(x+h-5)(x-5)h}$$
$$\frac{f(x+h)-f(x)}{h} = \frac{-10}{(x+h-5)(x-5)}$$

<u>Step 4</u>: $$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{-10}{(x+h-5)(x-5)}$$
$$= \frac{-10}{(x+0-5)(x-5)}$$
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \frac{-10}{(x-5)^2}$$

Therefore, $f'(x) = \dfrac{-10}{(x-5)^2}$

$$f'(1) = \frac{-10}{(1-5)^2} = \frac{-10}{(-4)^2} = \frac{-10}{16} = -\frac{5}{8}$$

$$f'(2) = \frac{-10}{(2-5)^2} = \frac{-10}{(-3)^2} = \frac{-10}{9} = -\frac{10}{9}$$

$$f'(3) = \frac{-10}{(3-5)^2} = \frac{-10}{(-2)^2} = \frac{-10}{4} = -\frac{5}{2}$$

4. The profit, in hundreds of dollars, from the sale of x items is given by

$$P(x) = 2x^2 - 5x + 6$$

a. Find the average rate of change of profit from $x = 2$ to $x = 4$.
b. Find the instantaneous rate of change equation using the four–step procedure.
c. Using the equation found in part b, find the instantaneous rate of change when $x = 2$ and interpret the results.

Solution:

a. First find the function values at 2 and 4. Note that $h = 4 - 2 = 2$.

$$\begin{aligned} P(2) &= 2(2)^2 - 5(2) + 6 \\ P(2) &= 8 - 10 + 6 \\ P(2) &= 4 \end{aligned} \qquad \begin{aligned} P(4) &= 2(4)^2 - 5(4) + 6 \\ P(4) &= 32 - 20 + 6 \\ P(4) &= 18 \end{aligned}$$

Now use the average rate of change formula:

$$\begin{aligned} \frac{P(a+h) - P(a)}{h} &= \frac{P(2+2) - P(2)}{2} \\ &= \frac{P(4) - P(2)}{2} \\ &= \frac{18 - 4}{2} = 7 \end{aligned}$$

b. Step 1:

$$\begin{aligned} P(x+h) &= 2(x+h)^2 - 5(x+h) + 6 \\ &= 2(x^2 + 2xh + h^2) - 5(x+h) + 6 \\ P(x+h) &= 2x^2 + 4xh + 2h^2 - 5x - 5h + 6 \end{aligned}$$

Step 2:

$$\begin{aligned} P(x+h) - P(x) &= 2x^2 + 4xh + 2h^2 - 5x - 5h + 6 - (2x^2 - 5x + 6) \\ P(x+h) - P(x) &= 4xh + 2h^2 - 5h \end{aligned}$$

Step 3:

$$\begin{aligned} \frac{P(x+h) - P(x)}{h} &= \frac{4xh + 2h^2 - 5h}{h} \\ \frac{P(x+h) - P(x)}{h} &= 4x + 2h - 5 \end{aligned}$$

Step 4: $$\lim_{h\to 0}\frac{P(x+h)-P(x)}{h}=\lim_{h\to 0}(4x+2h-5)$$
$$=4x+2(0)-5$$
$$\lim_{h\to 0}\frac{P(x+h)-P(x)}{h}=4x-5$$
Therefore, $P'(x)=4x-5$

c. $P'(2)=4(2)-5=3$ means that when the second item was sold, your profit increased by \$300.

5. The distance of a particle from some fixed point is given by

$$s(t)=t^2+5t+2$$

where t is time measured in seconds.

a. Find the average velocity from $t = 4$ to $t = 6$.
b. Find the instantaneous rate of change equation using the four–step procedure.
c. Using the equation found in part b, find the instantaneous rate of change when $t = 4$ and interpret the results.

Solution:

a. First find the function values at 4 and 6:

$$s(4)=(4)^2+5(4)+2 \qquad s(6)=(6)^2+5(6)+2$$
$$s(4)=16+20+2 \qquad s(6)=36+30+2$$
$$s(4)=38 \qquad s(6)=68$$

Now use the average rate of change formula. Note that $h=6-4=2$.

$$\frac{s(t+h)-s(t)}{h}=\frac{s(4+2)-s(4)}{2}$$
$$=\frac{s(6)-s(4)}{2}$$
$$=\frac{68-38}{2}=15$$

b. Step 1: $s(t+h) = (t+h)^2 + 5(t+h) + 2$

$$s(t+h) = t^2 + 2th + h^2 + 5t + 5h + 2$$

Step 2: $s(t+h) - s(t) = t^2 + 2th + h^2 + 5t + 5h + 2 - (t^2 + 5t + 2)$

$$s(t+h) - s(t) = 2th + h^2 + 5h$$

Step 3: $\dfrac{s(t+h) - s(t)}{h} = \dfrac{2th + h^2 + 5h}{h}$

$$\frac{s(t+h) - s(t)}{h} = 2t + h + 5$$

Step 4: $\displaystyle\lim_{h \to 0} \frac{s(t+h) - s(t)}{h} = \lim_{h \to 0} (2t + h + 5)$

$$= 2t + 0 + 5$$

$$\lim_{h \to 0} \frac{s(t+h) - s(t)}{h} = 2t + 5$$

Therefore, $s'(t) = 2t + 5$

c. $s'(4) = 2(4) + 5 = 9$ means that after 2 seconds the particle is traveling at 9 units per second.

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Section 3-5 Basic Differentiation Properties

Goal: To find the first derivatives using the basic properties

Notation: If $y = f(x)$, then $f'(x)$, y', $\frac{dy}{dx}$ all represent the derivative of *f* at *x*.

Theorems:

1. If $y = f(x) = C$, then $f'(x) = 0$ (Constant Function Rule)
2. If $y = f(x) = x^n$, where *n* is a real number, then $f'(x) = nx^{n-1}$ (Power Rule)
3. If $y = f(x) = ku(x)$, then $f'(x) = ku'(x)$ (Constant Multiple Property)
4. If $y = f(x) = u(x) \pm v(x)$, then $f'(x) = u'(x) \pm v'(x)$ (Sum and Difference Property)

1 - 6 Find the indicated derivatives.

1. y' for $y = x^8$

   Use theorem 2 to find the derivative: $y' = 8x^7$

2. $\frac{dy}{dx}$ for $y = \frac{1}{x^9}$

   Convert the problem to the form x^n and then use theorem 2.

   $y = \frac{1}{x^9} = x^{-9}$ $\frac{dy}{dx} = -9x^{-10} = -\frac{9}{x^{10}}$

3. $\frac{d}{du}(5u^2 - 6u + 3)$

Use theorem 4 to break the original function into pieces, then use a combination of theorems 1–3.

$$\begin{aligned}\frac{d}{du}(5u^2 - 6u + 3) &= \frac{d}{du}5u^2 - \frac{d}{du}6u + \frac{d}{du}3 \\ &= 10u - 6 + 0 \\ &= 10u - 6\end{aligned}$$

4. $f'(x)$ if $f(x) = x^{2.3} + 7x^{1.2} - x + 5$

Use theorem 4 to break the original function into pieces, then use a combination of theorems 1–3.

$$\begin{aligned}f'(x) &= f'(x^{2.3}) + 7f'(x^{1.2}) - f'(x) + f'(5) \\ &= 2.3x^{1.3} + 7(1.2)x^{0.2} - 1 + 0 \\ &= 2.3x^{1.3} + 8.4x^{0.2} - 1\end{aligned}$$

5. $\frac{d}{dx}(\frac{1}{x^4} + 6\sqrt{x} - 5x + 8)$

Use theorem 4 to break the original function into pieces, then use a combination of theorems 1–3.

$$\begin{aligned}\frac{d}{dx}(\frac{1}{x^4} + 6\sqrt{x} - 5x + 8) &= \frac{d}{dx}\frac{1}{x^4} + \frac{d}{dx}6\sqrt{x} - \frac{d}{dx}5x + \frac{d}{dx}8 \\ &= \frac{d}{dx}x^{-4} + \frac{d}{dx}6x^{1/2} - \frac{d}{dx}5x + \frac{d}{dx}8 \\ &= -4x^{-5} + 6(\tfrac{1}{2})x^{-1/2} - 5 + 0 \\ &= -\frac{4}{x^5} + \frac{3}{\sqrt{x}} - 5\end{aligned}$$

6. $f'(x)$ if $f(x) = x^{4/5} + 4x^{3/5} - 2x^{-1/5} + 5$

Use theorem 4 to break the original function into pieces, then use a combination of theorems 1 - 3.

$$\begin{aligned}f'(x) &= f'(x^{4/5}) + 4f'(x^{3/5}) - 2f'(x^{-1/5}) + f'(5) \\ &= \frac{4}{5}x^{-1/5} + \frac{12}{5}x^{-2/5} + \frac{2}{5}x^{-6/5}\end{aligned}$$

7. Given the function $f(x) = 2x^2 - 8x + 3$

a. Find $f'(x)$.
b. Find the slope of the graph of *f* at $x = 3$.
c. Find the equation of the tangent line at $x = 3$.
d. Find the value of *x* where the tangent is horizontal.

<u>Solution</u>:

a.
$$f(x) = 2x^2 - 8x + 3$$
$$f'(x) = 2f'(x^2) - 8f'(x) + f'(3)$$
$$f'(x) = 4x - 8$$

b. Substitute the value into the equation in part a to find the slope.

$$f'(x) = 4x - 8$$
$$f'(3) = 4(3) - 8$$
$$m = 4$$

c. Substitute the given value into the original function to find the *y* value of the point and then use that point with the slope found in part b. The point is (3, –3).

$$y - y_1 = m(x - x_1)$$
$$y - (-3) = 4(x - 3)$$
$$y + 3 = 4x - 12$$
$$y = 4x - 15$$

d. The tangent is horizontal when the first derivative has a value of 0.

$$f'(x) = 4x - 8$$
$$0 = 4x - 8$$
$$8 = 4x$$
$$2 = x$$

8 - 9 Use the following information for both problems:

If an object moves along the y axis (marked in feet) so that its position at time x (in seconds) is given by the indicated function, find:

a. The instantaneous velocity function $v = f'(x)$
b. The velocity when $x = 0$ and $x = 4$
c. The time(s) when $v = 0$

8. $f(x) = 3x^2 - 12x - 8$

<u>Solution</u>:

a.
$$f(x) = 3x^2 - 12x - 8$$
$$f'(x) = 3f'(x^2) - 12f'(x) - f'(8)$$
$$f'(x) = 6x - 12$$
$$v = 6x - 12$$

b.
$$f'(x) = 6x - 12$$
$$v(x) = 6x - 12$$
$$v(0) = 6(0) - 12$$
$$v(0) = -12$$

$$f'(x) = 6x - 12$$
$$v(x) = 6x - 12$$
$$v(4) = 6(4) - 12$$
$$v(4) = 12$$

c.
$$v(x) = 6x - 12$$
$$0 = 6x - 12$$
$$12 = 6x$$
$$2 = x$$

9. $f(x) = x^3 - \frac{21}{2}x^2 + 30x$

<u>Solution</u>:

a.
$$f(x) = x^3 - \frac{21}{2}x^2 + 30x$$
$$f'(x) = f'(x^3) - \frac{21}{2}f'(x^2) + 30f'(x)$$
$$f'(x) = 3x^2 - 21x + 30$$
$$v = 3x^2 - 21x + 30$$

b.
$$f'(x) = 3x^2 - 21x + 30$$
$$v(x) = 3x^2 - 21x + 30$$
$$v(0) = 3(0)^2 - 21(0) + 30$$
$$v(0) = 0 - 0 + 30$$
$$v(0) = 30$$

$$f'(x) = 3x^2 - 21x + 30$$
$$v(x) = 3x^2 - 21x + 30$$
$$v(4) = 3(4)^2 - 21(4) + 30$$
$$v(4) = 48 - 84 + 30$$
$$v(4) = -6$$

c.
$$v(x) = 3x^2 - 21x + 30$$
$$0 = 3x^2 - 21x + 30$$
$$0 = 3(x^2 - 7x + 10)$$
$$0 = 3(x-5)(x-2)$$
$$x = 2, 5$$

10. Find $f'(x)$ if $f(x) = (3x-5)^2$

First simplify the function by using the FOIL to expand it.

$$f(x) = (3x-5)^2 = (3x-5)(3x-5) = 9x^2 - 30x + 25$$

Now use theorem 4 to break the original function into pieces, then use a combination of theorems 1–3.

$$f(x) = 9x^2 - 30x + 25$$
$$f'(x) = 9f'(x^2) - 30f'(x) + f'(25)$$
$$f'(x) = 18x - 30$$

11. Find $f'(x)$ if $f(x) = \frac{9x-5}{x}$.

First simplify the function.

$$f(x) = \frac{9x-5}{x} = \frac{9x}{x} - \frac{5}{x} = 9 - 5x^{-1}$$

Now use theorem 4 to break the original function into pieces, then use a combination of theorems 1 - 3.

$$f(x) = 9 - 5x^{-1}$$
$$f'(x) = f'(9) - 5f'(x^{-1})$$
$$f'(x) = 0 + 5x^{-2}$$
$$f'(x) = \frac{5}{x^2}$$

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Section 3-6 Differentials

Goal: To use differentials to solve problems

Definition: Differentials

If $y = f(x)$ defines a differentiable function, then the differential *dy* or *df* is defined as the product of $f'(x)$ and *dx,* where $dx = \Delta x$. Symbolically,

$$dy = f'(x)dx \quad \text{or} \quad df = f'(x)dx$$

where $dx = \Delta x$.

Recall that $\Delta y = f(x + \Delta x) - f(x)$

1. Given the function $y = 2x^3$, find $\Delta x, \Delta y,$ and $\frac{\Delta y}{\Delta x}$ given $x_1 = 3$ and $x_2 = 6$.

When $x_1 = 3,\ y_1 = 2(3)^3 = 54$. When $x_2 = 6,\ y_2 = 2(6)^3 = 432$.

$\Delta x = 6 - 3$ $\quad$ $\Delta y = 432 - 54$ $\quad$ $\frac{\Delta y}{\Delta x} = \frac{378}{3} = 126$

$\Delta x = 3$ $\quad$ $\Delta y = 378$

2. Given the function $y = 2x^3$, find $\Delta x, \Delta y,$ and $\frac{\Delta y}{\Delta x}$ given $x_1 = 2$ and $x_2 = 5$.

When $x_1 = 2,\ y_1 = 2(2)^3 = 16$. When $x_2 = 5,\ y_2 = 2(5)^3 = 250$.

$\Delta x = 5 - 2$ $\quad$ $\Delta y = 250 - 16$ $\quad$ $\frac{\Delta y}{\Delta x} = \frac{234}{3} = 78$

$\Delta x = 3$ $\quad$ $\Delta y = 234$

3. Given the function $y = 5x^3 - 6x^2 + 8x + 11$, find dy.

$$y = 5x^3 - 6x^2 + 8x + 11$$
$$\frac{dy}{dx} = 15x^2 - 12x + 8$$
$$dy = (15x^2 - 12x + 8)dx$$

4. Given the function $y = x^3(6 - \frac{x}{12})$, find dy.

$$y = x^3(6 - \tfrac{x}{12})$$
$$y = 6x^3 - \tfrac{1}{12}x^4$$
$$\frac{dy}{dx} = 18x^2 - \tfrac{1}{3}x^3$$
$$dy = (18x^2 - \tfrac{1}{3}x^3)dx$$

5. Given the function $y = 25 - 7x^2 - x^3$, find dy and Δy given $x = 4$ and $dx = \Delta x = 0.1$.

$$y = 25 - 7x^2 - x^3$$
$$\frac{dy}{dx} = -14x - 3x^2$$
$$dy = (-14x - 3x^2)dx$$

$$dy = (-14x - 3x^2)dx$$
$$dy = [-14(4) - 3(4)^2](0.1)$$
$$dy = (-56 - 48)(0.1)$$
$$dy = (-104)(0.1)$$
$$dy = -10.4$$

$$\Delta y = f(x + \Delta x) - f(x)$$
$$\Delta y = f(4 + 0.1) - f(4)$$
$$\Delta y = f(4.1) - f(4)$$
$$\Delta y = (25 - 7(4.1)^2 - (4.1)^3) - (25 - 7(4)^2 - (4)^3)$$
$$\Delta y = -161.591 - (-151)$$
$$\Delta y = -10.591$$

6. A company will sell N units of a product after spend x thousand dollars in advertising, as given by

$$N = 120x - x^2 \qquad 10 \le x \le 60$$

Approximately what increase in sales will result by increasing the advertising budget from \$15,000 to \$17,000? From \$25,000 to \$27,000?

$$N = 120x - x^2 \qquad dN = (120 - 2x)dx \qquad dN = (120 - 2x)dx$$
$$\tfrac{dN}{dx} = 120 - 2x \qquad dN = (120 - 2(15))(2) \qquad dN = (120 - 2(25))(2)$$
$$dN = (120 - 2x)dx \qquad dN = 180 \qquad dN = 140$$

Therefore, increasing the advertising budget from \$15,000 to \$17, 000 will result in an increase of 180 units and an increase from \$25,000 to \$27,000 will only result in a 140 unit increase.

7. The average pulse rate y (in beats per minute) of a healthy person x inches tall is given approximately by

$$y = \frac{590}{\sqrt{x}} \qquad 30 \le x \le 75$$

Approximately how will the pulse rate change for a change in height from 49 inches to 52 inches?

$$y = \frac{590}{\sqrt{x}} = 590x^{-1/2}$$
$$y = 590x^{-1/2} \qquad dy = (-295x^{-3/2})dx$$
$$\tfrac{dy}{dx} = -295x^{-3/2} \qquad dy = (-295(49)^{-3/2})(3)$$
$$dy = (-295x^{-3/2})dx \qquad dy = (-\tfrac{295}{343})(3)$$
$$dy = -\tfrac{885}{343} \approx -2.58$$

Therefore, if a person grows from 49 to 52 inches, their pulse rate would decrease by approximately 2.5 beats per minute.

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Section 3-7 Marginal Analysis in Bus. And Econ.

Goal: To solve problems involving marginal functions in business and economics

Definition: Marginal Cost, Revenue, and Profit

If x is the number of units of a product produced in some time interval, then

total cost $= C(x)$ and marginal cost $= C'(x)$
total revenue $= R(x)$ and marginal revenue $= R'(x)$
total profit $= R(x) - C(x)$ and marginal profit $= R'(x) - C'(x)$

Definition: Marginal Average Cost, Revenue, and Profit

If x is the number of units of a product produced in some time interval, then

Cost per unit: average cost $= \overline{C} = \frac{C(x)}{x}$ and marginal average cost $= \overline{C}'(x) = \frac{d}{dx}\overline{C}(x)$

Rev. per unit: average revenue $= \overline{R} = \frac{R(x)}{x}$ and marginal average revenue $= \overline{R}'(x) = \frac{d}{dx}\overline{R}(x)$

Profit per unit: average profit $= \overline{P} = \frac{P(x)}{x}$ and marginal average profit $= \overline{P}'(x) = \frac{d}{dx}\overline{P}(x)$

1 - 10 Find the indicated function if cost and revenue are given by

$C(x) = 3000 - 20x + 0.003x^2$ and $R(x) = 5000x - 100x^2$

1. Marginal cost function

$$C(x) = 3000 - 20x + 0.003x^2$$
$$C'(x) = -20 + 0.006x$$

2. Average cost function

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{3000 - 20x + 0.003x^2}{x} = \frac{3000}{x} - 20 + 0.003x$$

3. Marginal average cost function

$$\overline{C}(x) = \frac{3000}{x} - 20 + 0.003x$$
$$\overline{C}(x) = 3000x^{-1} - 20 + 0.003x$$
$$\overline{C}'(x) = -3000x^{-2} + 0.003$$
$$\overline{C}'(x) = -\frac{3000}{x^2} + 0.003$$

4. Marginal revenue function

$$R(x) = 5000x - 100x^2$$
$$R'(x) = 5000 - 200x$$

5. Average revenue function

$$\overline{R}(x) = \frac{R(x)}{x} = \frac{5000x - 100x^2}{x} = 5000 - 100x$$

6. Marginal average revenue function

$$\overline{R}(x) = 5000 - 100x$$
$$\overline{R}'(x) = -100$$

7. Profit function

$$P(x) = R(x) - C(x)$$
$$= (5000x - 100x^2) - (3000 - 20x + 0.003x^2)$$
$$P(x) = -100.003x^2 + 5020x - 3000$$

8. Marginal profit function

$$P(x) = -100.003x^2 + 5020x - 3000$$
$$P'(x) = -200.006x + 5020$$

9. Average profit function

$$\overline{P}(x) = \frac{P(x)}{x} = \frac{-100.003x^2 + 5020x - 3000}{x} = -100.003x + 5020 - \frac{3000}{x}$$

10. Marginal average profit function

$$\overline{P}'(x) = \overline{R}'(x) - \overline{C}'(x)$$

$$\overline{P}'(x) = (-100) - (-\tfrac{3000}{x^2} + 0.003)$$

$$\overline{P}'(x) = \tfrac{3000}{x^2} - 100.003$$

11. Consider the revenue (in dollars) of a stereo system given by

$$R(x) = \tfrac{1000}{x} + 1000x$$

a. Find the exact revenue from the sale of the 101st stereo.
b. Use marginal revenue to approximate the revenue from the sale of the 101st stereo.

Solution:

a. To find the exact revenue, find the revenue from the 101st and 100th and subtract their values:

$$R(x) = \tfrac{1000}{x} + 1000x \qquad R(x) = \tfrac{1000}{x} + 1000x$$

$$R(101) = \tfrac{1000}{101} + 1000(101) \qquad R(100) = \tfrac{1000}{100} + 1000(100)$$

$$R(101) = 101,009.90 \qquad R(100) = 100,010$$

$$R(101) - R(100) = 101009.90 - 100010 = 999.90$$

Therefore, the actual revenue from the 101st stereo was \$999.90.

b. Find the marginal revenue formula and then substitute in 101.

$$R(x) = \tfrac{1000}{x} + 1000x$$

$$R(x) = 1000x^{-1} + 1000x$$

$$R'(x) = -1000x^{-2} + 1000$$

$$R'(x) = -\tfrac{1000}{x^2} + 1000$$

$$R'(x) = -\tfrac{1000}{x^2} + 1000$$

$$R'(101) = -\tfrac{1000}{(101)^2} + 1000$$

$$R'(x) = -0.098 + 1000$$

$$R'(x) = 999.90$$

Therefore, the approximate revenue from the 101st stereo was \$999.90.

12. The total cost (in dollars) of manufacturing x units of a product is:

$$C(x) = 10{,}000 + 15x$$

a. Find the average cost per unit if 300 units are produced.
b. Find the marginal average cost at a production level of 300 units and interpret the results.
c. Use the results in parts a and b to estimate the average cost per unit if 301 units are produced.

Solution:

a. The cost of producing 300 units is $C(300) = 10{,}000 + 15(300) = 14{,}500$.

Therefore the average cost is $\overline{C}(300) = \frac{14500}{300} = \$48.33$.

b. The average cost function is $\overline{C}(x) = \frac{C(x)}{x} = \frac{10{,}000 + 15x}{x} = 10000x^{-1} + 15$.

Therefore, the marginal average cost function for producing the 300th unit would be:

$$\overline{C}(x) = 10000x^{-1} + 15$$
$$\overline{C}'(x) = -10000x^{-2}$$
$$\overline{C}(x) = -\frac{10000}{x^2}$$

$$\overline{C}(x) = -\frac{10000}{x^2}$$
$$\overline{C}(300) = -\frac{10000}{(300)^2}$$
$$\overline{C}(300) = -0.11$$

This means that the average cost is decreasing by \$0.11 per unit produced.

c. The average cost per unit for the 301st unit would be \$48.33 – \$0.11 = \$48.22.

13. The total profit (in dollars) from the sale of x units of a product is:

$$P(x) = 30x - 0.03x^2 + 200$$

a. Find the exact profit from the 201st unit sold.
b. Find the marginal profit from selling the 201st unit.

<u>Solution</u>:

a. To find the exact profit, find the profit from the 201st and 200th and subtract their values:

$$P(x) = 30x - 0.03x^2 + 200 \qquad P(x) = 30x - 0.03x^2 + 200$$
$$P(201) = 30(201) - 0.03(201)^2 + 200 \qquad P(200) = 30(200) - 0.03(200)^2 + 200$$
$$P(201) = 5017.97 \qquad P(200) = 5000$$

$$P(201) - P(200) = 5017.97 - 5000 = 17.97$$

Therefore, the actual profit from the 201st unit was \$17.97.

b. Find the marginal profit formula and then substitute in 201.

$$P(x) = 30x - 0.03x^2 + 200 \qquad P'(x) = 30 - 0.06x$$
$$P'(x) = 30 - 0.06x \qquad P'(201) = 30 - 0.06(201)$$
$$P'(201) = 17.94$$

Therefore, the approximate profit from the 201st stereo was \$17.94.

14. The total cost and revenue (in dollars) for the production and sale of *x* units are given, respectively, by:

$$C(x) = 32x + 36{,}000 \quad \text{and} \qquad R(x) = 300x - 0.03x^2$$

a. Find the profit function *P(x)*.
b. Determine the actual cost, revenue, and profit from making and selling 101 units.
c. Determine the marginal cost, revenue, and profit from making and selling the 101st unit.

Solution:

a.
$$P(x) = R(x) - C(x)$$
$$P(x) = (300x - 0.03x^2) - (32x + 36{,}000)$$
$$P(x) = -0.03x^2 + 268x - 36{,}000$$

b.
$$C(x) = 32x + 36{,}000$$
$$C(101) = 32(101) + 36{,}000$$
$$C(101) = 39{,}232$$

$$C(x) = 32x + 36{,}000$$
$$C(100) = 32(100) + 36{,}000$$
$$C(100) = 39{,}200$$

$$R(x) = 300x - 0.03x^2$$
$$R(101) = 300(101) - 0.03(101)^2$$
$$R(101) = 29{,}993.97$$

$$R(x) = 300x - 0.03x^2$$
$$R(100) = 300(100) - 0.03(100)^2$$
$$R(100) = 29{,}700$$

$$P(x) = R(x) - C(x)$$
$$P(101) = R(101) - C(101)$$
$$P(101) = 29{,}993.97 - 39{,}232$$
$$P(101) = -9238.03$$

$$P(x) = R(x) - C(x)$$
$$P(100) = R(100) - C(100)$$
$$P(100) = 29{,}700 - 39{,}200$$
$$P(100) = -9500$$

Actual Cost	Actual Revenue	Actual Profit
$C(101) - C(100)$	$R(101) - R(100)$	$P(101) - P(100)$
$39{,}232 - 39{,}200$	$29{,}993.97 - 29{,}700$	$-9238.03 - (-9500)$
32	293.97	261.97

c. Marginal functions would be:

$$C(x) = 32x + 36{,}000 \qquad R(x) = 300x - 0.03x^2$$
$$C'(x) = 32 \qquad R'(x) = 300 - 0.06x$$

$$P'(x) = R'(x) - C'(x)$$
$$P'(x) = (300 - 0.06x) - (32)$$
$$P'(x) = -0.06x + 268$$

Marginal values for the 101st unit are:

$$C'(x) = 32 \qquad R'(x) = 300 - 0.06x$$
$$C'(101) = 32 \qquad R'(101) = 300 - 0.06(101)$$
$$R'(101) = 293.94$$

$$P'(x) = -0.06x + 268$$
$$P'(101) = -0.06(101) + 268$$
$$P'(101) = 261.94$$