Lecture Four – Integration

Section 4.1 – Antiderivatives, Substitution and General Power Rule

Antiderivatives

$$f(x) = x^3 \qquad \Rightarrow \qquad f'(x) = 3x^2$$

Definition of Antiderivative

A Function F is an Antiderivative of a function f if for every x in the domain of f, it follows that

$$F'(x) = f(x)$$

Notation for Antiderivatives and indefinite integrals

The notation

$$\int f(x)dx = F(x) + C$$

where C is an arbitrary constant, means that F is an Antiderivative of f.

That is $F'(x) = f(x)$ for all x in the domain of f.

$\int f(x)dx$ Indefinite integral

Antiderivative

Integrand

Integral sign

$$\int f(x)dx = \overbrace{F(x)+C}^{\text{Antiderivative}}$$

Differential

Basic Integration Rules

$$\int kdx = kx + C$$

$$\int kf(x)dx = k\int f(x)dx$$

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \mathbf{n \neq -1}$$

Example

Find each indefinite integral.

a) $\int 5dx = 5x + C$

b) $\int -1dr = -r + C$

c) $\int 2dt = 2t + C$

Example

Find indefinite integral. $\int 5xdx$

Solution

$$\int 5xdx = \int 5x^{1}dx$$

$$= 5\frac{x^{1+1}}{1+1} + C$$

$$= \frac{5}{2}x^2 + C$$

Example

Find each indefinite integral.

a) $\displaystyle\int \frac{1}{x^2}dx = \int x^{-2}dx$

$$= \frac{x^{-2+1}}{-2+1} + C$$

$$= \frac{x^{-1}}{-1} + C$$

$$= -\frac{1}{x} + C$$

b) $\displaystyle\int \sqrt[3]{x}dx = \int x^{1/3}dx$

$$= \frac{x^{1/3+1}}{1/3+1} + C$$

$$= \frac{x^{4/3}}{4/3} + C$$

$$= \frac{3}{4}x^{4/3} + C \qquad or \qquad = \frac{3}{4}x\sqrt[3]{x} + C$$

Example

Find each indefinite integral.

a) $\displaystyle\int (x+4)dx = \int xdx + \int 4dx$

$$= \frac{1}{2}x^2 + 4x + C$$

b) $\displaystyle\int (4x^3 - 5x + 2)dx = \int 4x^3dx - \int 5xdx + \int 2dx$

$$= 4\frac{x^4}{4} - 5\frac{x^2}{2} + 2x + C$$

$$= x^4 - \frac{5}{2}x^2 + 2x + C$$

Example

Find the integral $\int \frac{x^2+1}{\sqrt{x}}dx$

<u>*Solution*</u>

$$\int \frac{x^2+1}{\sqrt{x}}dx = \int \left(\frac{x^2}{x^{1/2}} + \frac{1}{x^{1/2}}\right)dx$$

$$= \int \left(x^{3/2} + x^{-1/2}\right)dx$$

$$= \frac{x^{5/2}}{5/2} + \frac{x^{1/2}}{1/2} + C$$

$$\boxed{= \frac{2}{5}x^{5/2} + 2x^{1/2} + C}$$

Particular Solutions

In many applications of integrations, we are given enough information to determine a particular solution. To do this, we need to know the value of *F(x)* for one value of *x*. This information is called an initial condition.

Example

Find the general solution of $F'(x) = 4x + 2$, and find the particular solution that satisfies the initial condition $F(1) = 8$.

<u>*Solution*</u>

$$F(x) = \int (4x+2)dx$$

$$= 4\frac{x^2}{2} + 2x + C$$

$$= 2x^2 + 2x + C$$

$$F(x) = 2(1)^2 + 2(1) + C = 8$$

$$2 + 2 + C = 8$$

$$4 + C = 8$$

$$C = 4$$

$$\boxed{F(x) = 2x^2 + 2x + 4}$$

Example

The marginal cost function for producing x units of a product is modeled by

$$\frac{dC}{dx} = 28 - 0.02x$$

It costs \$40 to produce one unit. Find the cost of producing 200 units.

<u>*Solution*</u>

$$C = \int (28 - 0.02x)dx$$

$$= 28x - 0.02\frac{x^2}{2} + K$$

Cost \$40 for one unit $\Rightarrow$ C(x = 1) = 40

$$C(x=1) = 28(1) - 0.01(1)^2 + K = 40$$

$$K = 12.01$$

$$C(x) = -0.01x^2 + 28x + 12.01$$

$$C(200) = -0.01(200)^2 + 28(200) + 12.01 \ \underline{= \$5212.01}$$

General Power Rule

The Simple Power Rule is given by:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \text{n} \neq -1$$

$$\int \overbrace{\left(x^2+1\right)^3}^{\text{u}^3} \underbrace{2xdx}_{\text{du}} = \int u^3 du$$

$$= \frac{u^4}{4} + C$$

General Power Rule for Integration

If u is a differentiable function of x, then

$$\int u^n \frac{du}{dx} dx = \int u^n du = \frac{u^{n+1}}{n+1} + C \qquad \text{n} \neq -1$$

Example

Find the indefinite integral. $\int (3x^2+6)(x^3+6x)^2\,dx$

<u>*Solution*</u>

Let $u = x^3+6x \Rightarrow du = (3x^2+6)dx$

$$\int u^2 du = \frac{u^3}{3}+C$$

$$\boxed{=\frac{(x^3+6x)^3}{3}+C}$$

Example

Find the indefinite integral. $\int 2x\sqrt{x^2-2}\ dx$

<u>*Solution*</u>

$u = x^2-2 \Rightarrow du = 2xdx$

$$\int 2x\sqrt{x^2-2}\ dx = \int \sqrt{u}\ du$$

$$= \int u^{1/2}\ du$$

$$= \frac{u^{3/2}}{3/2}+C$$

$$= \frac{2}{3}u^{3/2}+C$$

$$\boxed{= \frac{2}{3}(x^2-2)^{3/2}+C}$$

Example

Evaluate $\int x^3(3x^4+1)^2 dx$

<u>*Solution*</u>

$u = 3x^4+1 \Rightarrow du = 12x^3dx \quad \Rightarrow \frac{1}{12}du = x^3dx$

$$\int x^3(3x^4+1)^2 dx = \int \frac{1}{12}u^2 du$$

$$= \frac{1}{12}\frac{(3x^4+1)^3}{3}+C$$

$$\boxed{= \frac{1}{36}(3x^4+1)^3+C}$$

Failure of the General Power Rule

Example

Find $\int 2(3x^4+1)^2\,dx$

<u>*Solution*</u>

$$\int 2(3x^4+1)^2\,dx = \int 2\left((3x^4)^2 + 2(3x^4)(1) + 1^2\right)dx \qquad (a+b)^2 = a^2 + 2ab + b^2$$

$$= \int 2\left(9x^8 + 6x^4 + 1\right)dx$$

$$= \int \left(18x^8 + 12x^4 + 2\right)dx$$

$$= 18\frac{x^9}{9} + 12\frac{x^5}{5} + 2x + C$$

$$= 2x^9 + \frac{12}{5}x^5 + 2x + C$$

Example

Find $\int 5x\sqrt{x^2-1}\;dx$

<u>*Solution*</u>

$$u = x^2 - 1 \quad \Rightarrow du = 2xdx$$

$$\Rightarrow \tfrac{1}{2}du = xdx$$

$$\int 5x\left(x^2-1\right)^{1/2}\;dx = 5\int u^{1/2}\;\tfrac{1}{2}du \qquad \textit{Substitute for x and dx}$$

$$= 5\int u^{1/2}\;\tfrac{1}{2}du$$

$$= \frac{5}{2}\int u^{1/2}du$$

$$= \frac{5}{2}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{5}{3}u^{3/2} + C$$

$$= \frac{5}{3}(x^2-1)^{3/2} + C$$

Example

The marginal propensity to consume income x can be modeled by $\frac{dQ}{dx} = \frac{0.98}{(x-19,999)^{0.02}}$; $x \ge 20,000$

Where Q represents the income consumed. Estimate the amount by a family of four whose income was \$30,000.00, with initial condition of 19,999.

<u>*Solution*</u>

$$Q = \int \frac{0.98}{(x-19,999)^{0.02}} dx$$

$$= \int 0.98(x-19,999)^{-0.02} dx$$

$$Q = (x-19,999)^{0.98} + 19,999$$

$$30,000 = (x-19,999)^{0.98} + 19,999$$

$$30,000 - 19,999 = (x-19,999)^{0.98}$$

$$10,001 = (x-19,999)^{0.98}$$

$$x - 19,999 = 10,001^{1/0.98}$$

$$x = 10,001^{1/0.98} + 19,999$$

$$= \$32,068.16$$

Exercises *Section* 4.1 – Antiderivatives, Substitution and General Power Rule

Find each indefinite integral.

1. $\int v^2 dv$

2. $\int x^{1/2} dx$

3. $\int e^{3t} dt$

4. $\int (6x^2 - 2e^x) dx$

5. $\int 4y^{-3} dy$

6. $\int (x^3 - 4x + 2) dx$

7. $\int \left(3z^2 - 4z + 5\right) dz$

8. $\int \left(x^2 - 1\right)^2 dx$

9. $\int \left(\sqrt[4]{x^3} + 1\right) dx$

10. $\int \sqrt{x}(x+1) dx$

11. $\int (1+3t)t^2 dt$

12. $\int \frac{x^2 - 5}{x^2} dx$

13. $\int (-40x + 250) dx$

14. $\int \frac{x+2}{\sqrt{x}} dx$

15. $\int\left(\frac{2}{\sqrt[3]{x}}-6\sqrt{x}\right)dx$

16. $\int\left(x^2-1\right)^3(2x)dx$

17. $\int\frac{6x}{\left(1+x^2\right)^3}dx$

18. $\int u^3\sqrt{u^4+2}\;du$

19. $\int\frac{t+2t^2}{\sqrt{t}}\;dt$

20. $\int\left(1+\frac{1}{t}\right)^3\frac{1}{t^2}\;dt$

21. $\int\left(7-3x-3x^2\right)(2x+1)\;dx$

22. $\int\sqrt{x}\left(4-x^{3/2}\right)^2\;dx$

23. $\int\frac{1}{\sqrt{x}+\sqrt{x+1}}dx$

24. $\int\sqrt{1-x}\;dx$

25. $\int x\sqrt{x^2+4}\;dx$

26. Derive the position function if a ball is thrown upward with initial velocity of 32 ft per second from an initial height of 48 ft. When does the ball hit the ground? With what velocity does the ball hit the ground?

27. Suppose a publishing company has found that the marginal cost at a level of production of x thousand books is given by

$$\frac{dC}{dx}=\frac{50}{\sqrt{x}}$$

And that the fixed cost (the cost before the first book can be produced) is a $25,000. Find the cost function $C(x)$.

28. If the marginal cost of producing x units of a commodity is given by

$$C'(x) = 0.3x^2 + 2x$$

And the fixed cost is \$2,000, find the cost function $C(x)$ and the cost of producing 20 units.

29. A satellite radio station is launching an aggressive advertising campaign in order to increase the number of daily listeners. The station currently has 27,000 daily listeners, and management expects the number of daily listeners, $S(t)$, to grow at the rate of

$$S'(t) = 60t^{1/2}$$

Listeners per day, where t is the number of days since the campaign began. How long should the campaign last if the station wants the number of daily listeners to grow to 41,000?

30. In 2007, U.S. consumption of renewable energy was 6.8 quadrillion Btu (or 6.8×10^{15} Btu). Since the 1960s, consumption has been growing at a rate (in quadrillion Btu's per year) given by

$$f'(t) = 0.004t + 0.062$$

Where t is years after 1960. Find $f(t)$ and estimate U.S. consumption of renewable energy in 2020.

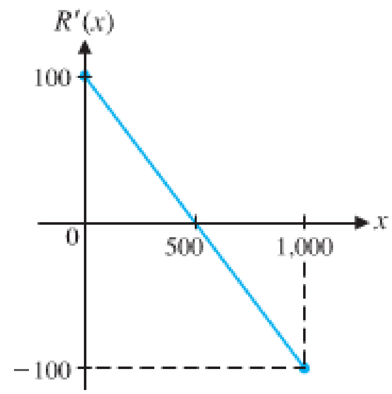
31. The graph of the marginal revenue function from the sale of x sports watches is given in the figure.

a) Using the graph shown, describe the shape of the graph of the revenue function $R(x)$ as x increases from 0 to 1.000.

b) Find the equation of the marginal revenue function. (linear function)

c) Find the equation of the revenue function that satisfies $R(0) = 0$. Graph the revenue function over the interval [0, 1,000]. Check the shape of the graph relative to the analysis in part (*a*).

d) Find the price-demand equation and determine the price when the demand is 700 units.



32. The rate of change of the monthly sales of a newly released football game is given by

$$S'(t) = 500t^{1/4} \qquad S(0) = 0$$

Where t is the number of months since the game was released and $S(t)$ is the number of games sold each month. Find $S(t)$. When will monthly sales reach 20,000 games?

33. If the rate of labor is given by: $g(x) = 2{,}000x^{-1/3}$

And if the first 8 control units require 12,000 labor-hours, how many labor-hours, $L(x)$, will be required for the first x control units? The first 27 control units?

34. The area A of a healing wound changes at a rate given approximately by

$$\frac{dA}{dt} = -4t^{-3} \qquad 1 \le t \le 10$$

Where t is time in days and $A(1) = 2\ cm^2$. What will the area of the wound be in 10 days?

35. The marginal revenue (in thousands of dollars) from the sale of x gadgets is given by the following function $R'(x) = 4x\left(x^2 + 27,000\right)^{-2/3}$

a) Find the total revenue function if the revenue from 115 gadgets is \$55,581.

b) How many gadgets must be sold for a revenue of at least \$50,000.

Section 4.2 – Exponential and Logarithmic Integrals

Using the Exponential Rule

Let u be a differentiable function of x

$$\int e^{x}dx = e^{x} + C \qquad \textit{Simple Exponential Rule}$$

$$\int e^{u}\frac{du}{dx}dx = \int e^{u}du$$

$$= e^{u} + C \qquad \textit{General Exponential Rule}$$

Example

Find each indefinite integral.

a. $\int 3e^{x}dx = 3\int e^{x}dx$

$$= 3e^{x} + C$$

b. $\int 5e^{5x}dx$

Let $u = 5x \rightarrow \text{d}u = 5\text{d}x$

$$\int e^{u}du = e^{u} + C$$

$$= e^{5x} + C$$

c. $\int \left(e^{x} - x\right)dx$

$$\int \left(e^{x} - x\right)dx = \int e^{x}dx - \int xdx$$

$$= e^{x} - \frac{x^2}{2} + C$$

Example

Find indefinite integral $\int e^{2x+3}dx$

<u>*Solution*</u>

Let $u = 2x + 3 \rightarrow du = 2dx$

$$\int e^{2x+3}dx = \int e^{u}\,\frac{1}{2}\,du$$

$$= \frac{1}{2}\int e^{u}du$$

$$= \frac{1}{2}e^{u} + C$$

$$= \frac{1}{2}e^{2x+3} + C$$

Using the Log Rule

Integrals of Logarithmic Functions

Let u be a differentiable function of x.

$$\int \frac{1}{x}\,dx = \ln|x| + C$$ ***Simple Logarithmic Rule***

$$\int \frac{du/dx}{u}\,dx = \int \frac{1}{u}du = \ln|u| + C$$

$$= \ln|u| + C$$ ***General Logarithmic Rule***

Example

Find each indefinite integral.

a) $\int \frac{2}{x}\,dx = 2\int \frac{1}{x}\,dx$

$$= 2\ln|x| + C$$

b) $\int \frac{3x^2}{x^3}\,dx = 3\int \frac{1}{x}\,dx$

$$= 3\ln|x| + C$$

c) $\int \frac{2}{2x+1}\,dx$

Let $u = 2x + 1 \rightarrow du = 2dx$

$$\int \frac{2}{2x+1}\,dx = \int \frac{2dx}{2x+1}$$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \ln|2x+1| + C$$

Example

Find the indefinite integral. $\int \frac{1}{4x+1}dx$

<u>*Solution*</u>

Let $u = 4x+1 \rightarrow du = 4dx \rightarrow \frac{1}{4}du = dx$

$$\int \frac{1}{4x+1}dx = \int \frac{1}{u}\frac{1}{4}du$$

$$= \frac{1}{4}\int \frac{1}{u}du$$

$$= \frac{1}{4}\ln|u| + C$$

$$= \frac{1}{4}\ln|4x+1| + C$$

Exercise Section 4.2 – Exponential and Logarithmic Integrals

Find each indefinite integral.

1. $\int (2x+1)e^{x^2+x}dx$

2. $\int \frac{1}{6x-5}dx$

3. $\int \frac{x^2+2x+3}{x^3+3x^2+9x+1}dx$

4. $\int \frac{1}{x(\ln x)^2}dx$

5. $\int \frac{e^x}{1+e^x}dx$

6. $\int \frac{1}{x^3}e^{1/4x^2}dx$

7. $\int \frac{e^{1/\sqrt{x}}}{x^{3/2}}dx$

8. $\int \frac{-e^{3x}}{2-e^{3x}}dx$

9. $\int (6x+e^x)\sqrt{3x^2+e^x}\,dx$

10. $\int \frac{2(e^x-e^{-x})}{(e^x+e^{-x})^2}dx$

11. $\int \frac{x-3}{x+3}dx$

12. $\int \frac{5}{e^{-5x}+7}dx$

13. $\int \frac{4x^2-3x+2}{x^2}dx$

14. $\int \frac{2}{e^{-x}+1}\,dx$

15. $\int \frac{4x^2+2x+4}{x+1}\,dx$

16. $\int 4xe^{x^2}\,dx$

17. $\int \frac{3x}{x^2+4}\,dx$

18. $\int 12t^3e^{-t^4}\,dt$

19. $\int \frac{7e^{7x}}{3+e^{7x}}dx$

20. Under certain conditions, the number of diseased cells $N(t)$ at time t increases at a rate $N'(t) = Ae^{kt}$, where A is the rate of increase at time 0 (in cells per day) and k is a constant.

a) Suppose $A = 60$, and at 4 days, the cells are growing at a rate of 180 per day. Find a formula for the number of cells after t days, given that 200 cells are present at $t = 0$.

b) Use the answer from part (*a*) to find the number of cells present after 9 days.

Section 4.3 – Integration by Parts

Integration by Parts

$\textit{Example}: \int x^2 e^x dx \quad \int x \ln x dx$

Let u and v be differentiable functions of x.

$$\int u dv = uv - \int v du$$

Guidelines for integration by Parts

1. Let dv be the most complicated portion of the integrand that fits a basic integration formula. Let u be the remaining factor.
2. Let u be the portion of the integrand whose derivative is a function simpler than u. Let dv be the remaining factor.

Example

$\textit{Find}: \int xe^{2x} dx$

<u>Solution</u>

Let: $u = x \Rightarrow du = dx$

$$dv = e^{2x} dx \Rightarrow v = \int dv = \int e^{2x} dx = \frac{1}{2} e^{2x}$$

$$\int u dv \;=\; uv - \int v du$$

$$\int xe^{2x} dx = \frac{1}{2} xe^{2x} - \int \frac{1}{2} e^{2x} dx$$

$$= \frac{1}{2} xe^{2x} - \frac{1}{2}\frac{1}{2} e^{2x} + C$$

$$= \frac{1}{2} xe^{2x} - \frac{1}{4} e^{2x} + C$$

Example

$$Find: \int x \ln x dx$$

<u>*Solution*</u>

Let: $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$$dv = xdx \Rightarrow v = \int dv = \int xdx = \frac{1}{2}x^2$$

$$\int x \ln x dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x^2 \frac{dx}{x}$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int xdx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

Example

Differentiate $y = x \ln x - x + C$ to show that it is the Antiderivative of ln*x*.

<u>*Solution*</u>

$$\frac{dy}{dx} = \ln x + x\frac{1}{x} - 1$$

$$= \ln x + 1 - 1$$

$$= \ln x$$

Integration by Parts Repeatedly

Example

$$Find : \int x^3 e^x dx$$

<u>*Solution*</u>

$$\text{Let: } u = x^3 \Rightarrow du = 3x^2 dx$$

$$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$\int x^3 e^x dx = x^3 e^x - \int e^x 3x^2 dx$$

$$= x^3 e^x - 3\int e^x x^2 dx$$

$$\text{Let: } u = x^2 \Rightarrow du = 2x dx$$

$$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$\int e^x x^2 dx = x^2 e^x - 2\int x e^x dx$$

$$\int x^3 e^x dx = x^3 e^x - 3\left[x^2 e^x - 2\int x e^x dx\right]$$

$$= x^3 e^x - 3x^2 e^x + 6\int x e^x dx$$

$$\text{Let: } u = x \Rightarrow du = dx$$

$$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x$$

$$\int x^3 e^x dx = x^3 e^x - 3x^2 e^x + 6\left[x e^x - e^x\right] + C$$

$$= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$$

$$= e^x\left(x^3 - 3x^2 + 6x - 6\right) + C$$

Exercises *Section* 4.3 – Integration by Parts

Find each integral

1. $\displaystyle\int \ln x^2 dx$

2. $\displaystyle\int \frac{2x}{e^x} dx$

3. $\displaystyle\int \ln(3x) dx$

4. $\displaystyle\int \frac{1}{x \ln x} dx$

5. $\displaystyle\int \frac{x}{\sqrt{x-1}} dx$

6. $\displaystyle\int \frac{x^3 e^{x^2}}{\left(x^2+1\right)^2} dx$

7. $\displaystyle\int x^2 e^{-3x} dx$

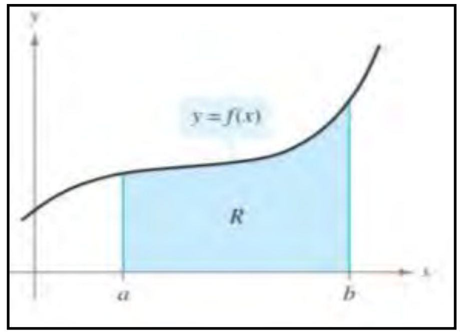
Section 4.4 – Area and the Fundamental Theorem of Calculus

Area and Definite Integrals

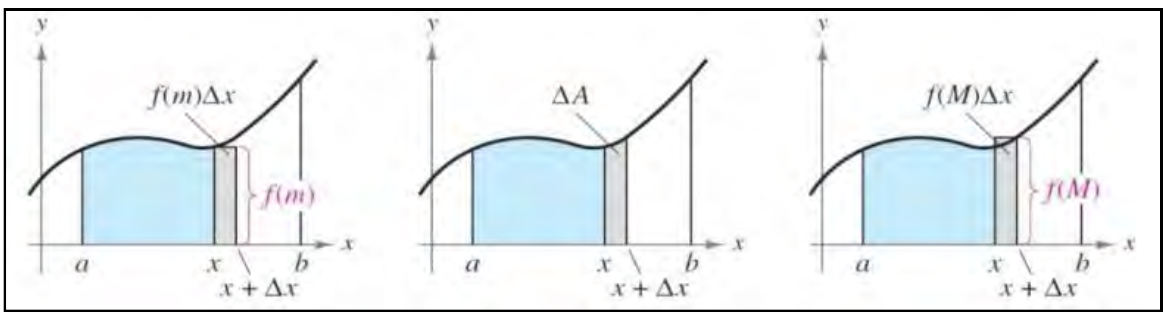
Definition of a Definite Integral

Let f be nonnegative and continuous on the closed interval $[a, b]$. The area of the region bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is denoted by

$$Area = \int_a^b f(x)dx$$



The expression $\int_a^b f(x)dx$ is called the definite integral from a to b, where a is the ***lower limit of integration*** and b is the ***upper limit of integration***.



The Fundamental Theorem of Calculus

If f is nonnegative and continuous on the closed interval $[a, b]$, then

$$\int_{\color{blue}a}^{\color{red}b} f(x)dx = F({\color{red}b}) - F({\color{blue}a})$$

Where F is any function such that $F'(x) = f(x)$ for all x in $[a, b]$.

Guidelines for Using the Fundamental Theorems of Calculus

1. The Fundamental Theorem of Calculus describes a way of evaluating a definite integral, not a procedure for finding anti-derivatives.

2. In applying the Fundamental Theorem, it is helpful to use the notation

$$\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a)$$

3. The constant of integration C can be dropped because

$$\int_a^b f(x)dx = \left[F(x)+C\right]_a^b = F(b) - F(a)$$

$$= \left[F(b)+C\right] - \left[F(a)+C\right]$$

$$= F(b) + C - F(a) - C$$

$$= F(b) - F(a)$$

Properties of Definite Integrals

Let *f* and *g* be continuous on the closed interval [*a*, *b*].

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx \,, \qquad k \text{ is a constant}$$

$$\int_a^b \left[f(x) \pm g(x)\right]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad a < c < b$$

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Example

Evaluate: $\displaystyle\int_0^3 4x\,dx$

<u>*Solution*</u>

$$\int_0^3 4x\,dx = 4\tfrac{1}{2}x^2\Big|_0^3$$

$$= 2x^2\Big|_0^3$$

$$= 2\left[3^2 - 0^2\right]$$

$$= 2(9)$$

$$\underline{= 18}$$

Example

Find the area of the region bounded by the x-axis and the graph of $f(x) = x^2 + 1, \quad 2 \le x \le 3$

<u>*Solution*</u>

$$\int_2^3 (x^2+1)\,dx = \left[\tfrac{1}{3}x^3 + x\right]_2^3$$

$$= \left(\tfrac{1}{3}3^3 + 3\right) - \left(\tfrac{1}{3}2^3 + 2\right)$$

$$= (9+3) - \left(\tfrac{8}{3} + 2\right)$$

$$= 12 - \left(\tfrac{14}{3}\right)$$

$$= \frac{22}{3}$$

$$\underline{= 7.3}$$

Example

Evaluate: $\int_0^1 (2t+3)^3 dt$

<u>*Solution*</u>

$$u = 2t+3 \Rightarrow du = 2dt \rightarrow \frac{du}{2} = dt$$

$$\int_0^1 (2t+3)^3 dt = \int_0^1 u^3 \frac{1}{2} du$$

$$= \frac{1}{2}\int_0^1 u^3 du$$

$$= \frac{1}{2}\frac{u^4}{4}\Big|_0^1$$

$$= \frac{1}{8}(2t+3)^4\Big|_0^1$$

$$= \frac{1}{8}\left[(2(1)+3)^4 - (2(0)+3)^4\right]$$

$$= \frac{1}{8}\left[5^4 - 3^4\right]$$

$$= 68$$

Example

Evaluate:

a) $\int_0^1 e^{4x} dx$

<u>*Solution*</u>

$$\int_0^1 e^{4x} dx = \frac{1}{4}e^{4x}\Big|_0^1$$

$$= \frac{1}{4}\left[e^4 - e^0\right]$$

$$= \frac{1}{4}\left[e^4 - 1\right]$$

$$\approx 13.4$$

b) $\int_2^5 -\frac{1}{x}\,dx$

<u>*Solution*</u>

$$\int_2^5 -\frac{1}{x}\,dx = -\ln x \Big|_2^5$$

$$= -(\ln 5 - \ln 2)$$

$$\boxed{\approx -0.916}$$

Example

Evaluate: $\int_0^5 |x-2|\,dx$

<u>*Solution*</u>

$$|x-2| = \begin{cases} x-2 & x>2 \\ -(x-2) & x<2 \end{cases}$$

$$\int_0^5 |x-2|\,dx = \int_0^2 -(x-2)dx + \int_2^5 (x-2)dx$$

$$= -\frac{x^2}{2} + 2x \Big|_0^2 + \left(\frac{x^2}{2} - 2x\right)\Big|_2^5$$

$$= -\frac{4}{2} + 4 - 0 + \left(\frac{25}{2} - 10 - (\frac{4}{2} - 4)\right)$$

$$= -2 + 4 + \frac{25}{2} - 10 - 2 + 4$$

$$= \frac{25}{2} - 6$$

$$\boxed{= \frac{13}{2}}$$

Marginal Analysis

Example

The marginal profit for a product is modeled by $\frac{dP}{dx} = -0.0002x + 14.2$

a. Find the change in profit when sales increase from 100 to 101 units.
b. Find the change in profit when sales increase from 100 to 110 units.

<u>*Solution*</u>

a. $\int_{100}^{101} \frac{dP}{dx}dx = \int_{100}^{101} (-0.0002x + 14.2)dx$

$$= -\frac{0.0002}{2}x^2 + 14.2x \Big|_{100}^{101}$$

$$= -\frac{0.0002}{2}101^2 + 14.2(101) - \left[-\frac{0.0002}{2}100^2 + 14.2(100)\right]$$

$$= \$14.18$$

b. $\int_{100}^{110} \frac{dP}{dx}dx = \int_{100}^{110} (-0.0002x + 14.2)dx$

$$= -\frac{0.0002}{2}x^2 + 14.2x \Big|_{100}^{110}$$

$$= -\frac{0.0002}{2}110^2 + 14.2(110) - \left[-\frac{0.0002}{2}100^2 + 14.2(100)\right]$$

$$\approx \$141.79$$

Average Value

Definition of the Average Value of a Definition

If f is continuous on $[a, b]$, then the average value of f on $[a, b]$.

$$\text{Average Value of } f \text{ on } [a, b] = \frac{1}{b-a}\int_a^b f(x)dx$$

Example

Find the average cost per unit over a two-year period if the cost per unit c of roller blades is given by $c = 0.005t^2 + 0.02t + 12.5$, for $0 \le t \le 24$, where t is the time in months.

Solution

$$\text{Average cost} = \frac{1}{24-0}\int_0^{24}(0.005t^2 + 0.02t + 12.5)dt$$

$$= \frac{1}{24}\left[\frac{0.005}{3}t^3 + \frac{0.02}{2}t^2 + 12.5t\right]_0^{24}$$

$$= \frac{1}{24}\left[\left(\frac{0.005}{3}24^3 + \frac{0.02}{2}24^2 + 12.5(24)\right) - 0\right]$$

$$\boxed{\approx 13.7}$$

Even and Odd Function

Integration of Even and Odd Functions

1. If f is an *even* function, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

2. If f is an *odd* function, then $\int_{-a}^{a} f(x)dx = 0$

Example

Evaluate each definite integral

a) $\int_{-1}^{1} x^4 dx$

$$\int_{-1}^{1} x^4 dx = 2\int_{0}^{1} x^4 dx$$

$$= 2\left.\frac{x^5}{5}\right|_0^1$$

$$= 2\left[\frac{1^5}{5} - 0\right]$$

$$= \frac{2}{5}$$

b) $\int_{-1}^{1} x^5 dx = 0$

$$or \int_{-1}^{1} x^5 dx = \left.\frac{x^6}{6}\right|_{-1}^{1} = \frac{1^6}{6} - \frac{(-1)^6}{6} = \frac{1^6}{6} - \frac{1^6}{6} = 0$$

Annuity

Amount of an Annuity

If ***c*** represents a continuous income function in dollars per year (where ***t*** is the time in years), ***r*** represents the interest rate compounded continuously and **T** represents the term of the annuity in years, then the amount of an annuity is

$$\text{Amount of an annuity} = e^{rT}\int_0^T c(t)e^{-rt}dt$$

Example

If you deposit $1000 in a savings account every year, paying 4% interest, how much will be in the account after 10 years?

Solution

$$c(t) = 1000$$

$$\begin{aligned}\text{Amount of an annuity} &= e^{rT}\int_0^T c(t)e^{-rt}dt \\ &= e^{(0.04)(10)}\int_0^{10} 1000e^{-0.04t}dt \\ &= 1000e^{0.4}\left[-\frac{e^{-0.04t}}{0.04}\right]_0^{10} \\ &= 1000e^{0.4}\left[-\frac{e^{-0.4}}{0.04}-\left(-\frac{e^{0}}{0.04}\right)\right] \\ &\approx \$\ 12{,}295.62\end{aligned}$$

Exercise *Section* 4.4 – Area and the Fundamental Theorem of Calculus

Evaluate each integral

1. $\displaystyle\int_0^3 (2x+1)dx$

2. $\displaystyle\int_{-1}^4 |x-2|dx$

3. $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$

4. $\displaystyle\int_0^1 x^2e^x dx$

5. $\displaystyle\int_0^2 x(x-3)\,dx$

6. $\displaystyle\int_0^4 \left(3x-\frac{x^3}{4}\right)dx$

7. $\displaystyle\int_{-2}^2 \left(x^3-2x+3\right)dx$

8. $\displaystyle\int_0^1 \left(x^2+\sqrt{x}\right)dx$

9. $\displaystyle\int_{-3}^{-1} \frac{y^5-2y}{y^3}\,dy$

10. $\displaystyle\int_1^8 \frac{\left(x^{1/3}+1\right)\left(2-x^{2/3}\right)}{x^{1/3}}\,dx$

11. $\displaystyle\int_0^3 \sqrt{y+1}\,dy$

12. $\displaystyle\int_{-1}^{1} r\sqrt{1-r^2}\, dr$

13. $\displaystyle\int_{0}^{1} t^3\left(1+t^4\right)^3\, dt$

14. A company manufactures x HDTVs per month. The monthly marginal profit (in dollars) is given by

$$P'(x) = 165 - 0.1x \qquad 0 \le x \le 4{,}000$$

The company is currently manufacturing 1,500 HDTVs per month, but is planning to increase production. Find the change in the monthly profit if monthly production is increased to 1,600 HDTVs.

15. An amusement company maintains records for each video game installed in an arcade. Suppose that $C(t)$ and $R(t)$ represent the total accumulated costs and revenues (in thousands of dollars), respectively, t years after a particular game has been installed. Suppose also that

$$C'(t) = 2 \quad R'(t) = 9e^{-0.5t}$$

The value of t for which $C'(t) = R'(t)$ is called the ***useful life*** of the game.

a) Find the useful life of the game, to the nearest year.
b) Find the total profit accumulated during the useful life of the game.

16. The total cost (in dollars) of printing x dictionaries is $C(x) = 20{,}000 + 10x$

a) Find the average cost per unit if 1,000 dictionaries are produced.
b) Find the average value of the cost function over the interval [0, 1,000}
c) Discuss the difference between parts (***a***) and (***b***)

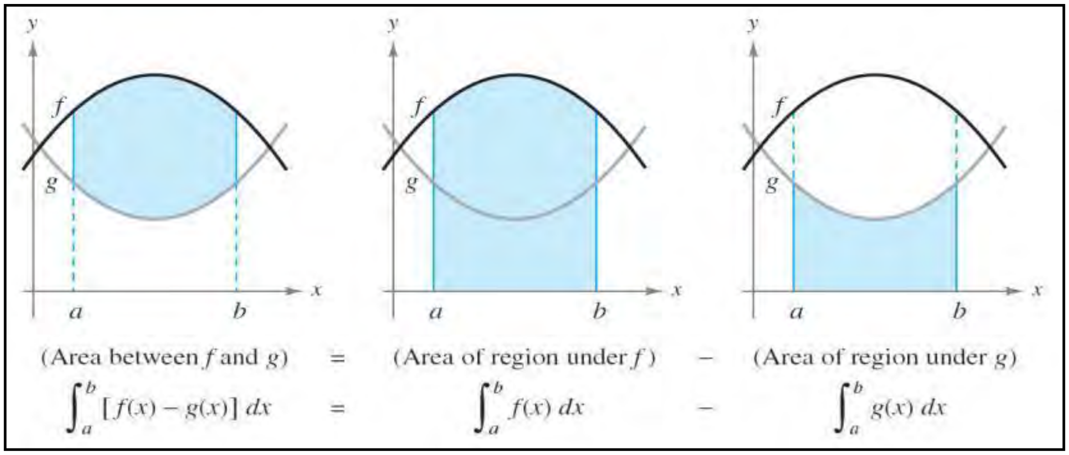
17. If the rate of labor is $g(x) = 2{,}000\, x^{-1/3}$, then approximately how many labor–hours will be required to assemble the 9th through the 27th.

Section 4.5 – Area between Two Curves

Area of a Region Bounded by Two Graphs

If *f* and *g* are continuous on [*a, b*] and $g(x) \le f(x)$ fro all *x* in the interval, then the area of the region bounded by the graphs of *f, g, x* = *a*, and *x* = *b* is given by

$$A = \int_a^b [f(x) - g(x)]dx$$



Example

Find the area of the region bounded by the graphs of $y = x^2 + 1$ and $y = x$ for $0 \le x \le 2$

<u>*Solution*</u>

$$A = \int_0^2 [(x^2 + 1) - x]dx$$

$$A = \int_0^2 [x^2 - x + 1]dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + 1x \Big|_0^2$$

$$= \frac{2^3}{3} - \frac{2^2}{2} + 1(2) - 0$$

$$= \frac{8}{3}$$

Example

Find the area of the region bounded by the graphs of $y = 3 - x^2$ and $y = 2x$

<u>*Solution*</u>

Determine the intersection between two functions: $y = 3 - x^2 = 2x$

$$3 - x^2 - 2x = 0$$

$$x^2 + 2x - 3 = 0$$

$$\rightarrow \boxed{x = 1, -3}$$

$$A = \int_{-3}^{1} [(3 - x^2) - 2x]dx$$

$$A = \int_{-3}^{1} [-x^2 - 2x + 3]dx$$

$$= -\frac{x^3}{3} - 2\frac{x^2}{2} + 3x \Big|_{-3}^{1}$$

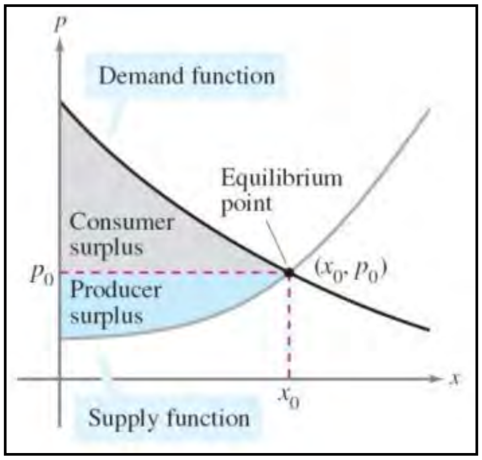
$$= -\frac{1^3}{3} - 1^2 + 3(1) - \left[-\frac{(-3)^3}{3} - (-3)^2 + 3(-3) \right]$$

$$= -\frac{1}{3} - 1 + 3 - [9 - 9 - 9]$$

$$= 11 - \frac{1}{3}$$

$$= \frac{32}{3}$$

Consumer Surplus and Producer Surplus



Demand Function: D(x)

Supply Function: S(x)

$$Consumer = \int_0^{x_0} (D - P_0)dx$$

$$Producer = \int_0^{x_0} (P_0 - S)dx$$

Example

The Demand and supply functions for a product are modeled by

Demand: $p = -0.2x + 8$ and *Supply*: $p = 0.1x + 2$

Where x is the number of units (in millions). Find the consumer and producer surpluses for this product.

<u>*Solution*</u>

$$-0.2x + 8 = 0.1x + 2$$

$$\Rightarrow -0.2x - 0.1x = 2 - 8$$

$$\Rightarrow -0.3x = -6$$

$$\Rightarrow x = 20$$

$$\text{Consumer} = \int_0^{20} \left[(-0.2x + 8) - 4\right] dx$$

$$= \int_0^{20} (-0.2x + 4) dx$$

$$= -0.2\frac{x^2}{2} + 4x \Big|_0^{20}$$

$$= -0.1(20)^2 + 4(20) - 0$$

$$= 40$$

$$\text{Producer} = \int_0^{20} \left[4 - (0.1x + 2)\right] dx$$

$$= \int_0^{20} \left[2 - 0.1x\right] dx$$

$$= 2x - 0.1\frac{x^2}{2} \Big|_0^{20}$$

$$= \left(2(20) - 0.1\frac{20^2}{2}\right) - 0$$

$$= 20$$

Example

The projected fuel cost C (in millions dollars per year) for a trucking company from 2008 through 2020 is $C_1 = 5.6 + 2.21t, \;\; 8 \le t \le 20$, where t = 8 corresponds to 2008. If the company purchases more efficient truck engines, fuel cost is expected to decrease and to follow the model $C_2 = 4.7 + 2.04t, \;\; 8 \le t \le 20$. How much can the company save with the more efficient engines?

<u>*Solution*</u>

$$\text{Petroleum saved} = \int_8^{20} (C_1 - C_2)dt$$

$$= \int_8^{20} \left[5.6 - 2.21t - (4.7 + 2.04t)\right]dt$$

$$= \int_8^{20} \left[5.6 - 2.21t - 4.7 - 2.04t\right]dt$$

$$= \int_8^{20} (0.17t + 0.9)dt$$

$$= 0.17\frac{t^2}{2} + 0.9t \Big|_8^{20}$$

$$= \left(0.17\frac{20^2}{2} + 0.9(20)\right) - \left(0.17\frac{8^2}{2} + 0.9(8)\right)$$

$= \$\ 39.36$ *millions*

Exercises Section 4.5 – Area between Two Curves

1. Find the area of the region bounded by the graphs of $y = x^2 - x - 2$ and x-axis

2. Find the area of the region bounded by the graphs of $f(x) = x^3 + 2x^2 - 3x$ and $g(x) = x^2 + 3x$

3. Find the area bounded by $f(x) = -x^2 + 1$, $g(x) = 2x + 4$, $x = -1$, and $x = 2$

4. Find the area between the curves $y = x^{1/2}$ *and* $y = x^3$

5. Find the area of the region bounded by the graphs of $y = x^2 - 2x$ and $y = x$ on [0, 4].

6. Find the area between the curves $x = 1,\quad x = 2,\quad y = x^3 + 2,\quad y = 0$

7. Find the area between the curves $y = x^2 - 18,\quad y = x - 6$

8. Find the area between the curves $x = -1,\quad x = 2,\quad y = e^{-x},\quad y = e^x$

9. Find the area between the curves $y = \sqrt{x},\quad y = x\sqrt{x}$

10. A company is considering a new manufacturing process in one of its plants. The new process provides substantial initial savings, with the savings declining with time *t* (in years) according to the rate-of-savings function

$$S'(t) = 100 - t^2$$

where $S'(t)$ is in thousands of dollars per year. At the same time, the cost of operating the new process increases with time *t* (in years), according to the rate-of-cost function (in thousands of dollars per year)

$$C'(t) = t^2 + \frac{14}{3}t$$

a) For how many years will the company realize savings?
b) What will be the net total savings during this period?

11. Find the producers' surplus if the supply function for pork bellies is given by

$$S(x) = x^{5/2} + 2x^{3/2} + 50$$

Assume supply and demand are in equilibrium at $x = 16$.

12. Suppose the supply function for concrete is given by

$$S(q) = 100 + 3q^{3/2} + q^{5/2}$$

And that supply and demand are in equilibrium at $q = 9$. Find the producers' surplus.

14. Find the consumers' surplus if the demand function for grass seed is given by

$$D(x) = \frac{200}{(3x+1)^2}$$

Assuming supply and demand are in equilibrium at $x = 3$.

15. Find the consumers' surplus if the demand function for olive oil is given by

$$D(x) = \frac{32{,}000}{(2x+8)^3}$$

And if supply and demand are in equilibrium at $x = 6$.