Solution Section 2.3 – Orthogonality

Exercise

Determine whether $\vec{u}$ and $\vec{v}$ are orthogonal

a) $\vec{u}=(-6,\ -2),\quad \vec{v}=(5,\ -7)$

b) $\vec{u}=(6,\ 1,\ 4),\quad \vec{v}=(2,\ 0,\ -3)$

c) $\vec{u}=(1,\ -5,\ 4),\quad \vec{v}=(3,\ 3,\ 3)$

d) $\vec{u}=(-2,\ 2,\ 3),\quad \vec{v}=(1,\ 7,\ -4)$

Solution

a) $\vec{u}\bullet\vec{v}=(-6)(5)+(-2)(-7)$

$$=-30+14$$

$$=-16\neq 0$$

$\therefore\ \vec{u}$ and $\vec{v}$ are not orthogonal

b) $\vec{u}\bullet\vec{v}=6(2)+1(0)+4(-3)$

$$=0$$

$\therefore\ \vec{u}$ and $\vec{v}$ are orthogonal

c) $\vec{u}\bullet\vec{v}=1(3)-5(3)+4(3)$

$$=0$$

$\therefore\ \vec{u}$ and $\vec{v}$ are orthogonal

d) $\vec{u}\bullet\vec{v}=-2(1)+2(7)+3(-4)$

$$=0$$

$\therefore\ \vec{u}$ and $\vec{v}$ are orthogonal

Exercise

Determine whether the vectors form an orthogonal set

a) $\vec{v}_1=(2,\ 3),\quad \vec{v}_2=(3,\ 2)$

b) $\vec{v}_1=(1,\ -2),\quad \vec{v}_2=(-2,\ 1)$

c) $\vec{u}=(-4,\ 6,\ -10,\ 1)\quad \vec{v}=(2,\ 1,\ -2,\ 9)$

d) $\vec{u}=(a,\ b)\quad \vec{v}=(-b,\ a)$

e) $\vec{v}_1=(-2,\ 1,\ 1),\quad \vec{v}_2=(1,\ 0,\ 2),\quad \vec{v}_3=(-2,\ -5,\ 1)$

f) $\vec{v}_1 = (1,\ 0,\ 1),\quad \vec{v}_2 = (1,\ 1,\ 1),\quad \vec{v}_3 = (-1,\ 0,\ 1)$

g) $\vec{v}_1 = (2,\ -2,\ 1),\quad \vec{v}_2 = (2,\ 1,\ -2),\quad \vec{v}_3 = (1,\ 2,\ 2)$

Solution

a) $\vec{v}_1 \bullet \vec{v}_2 = 2(3) + 3(2)$

$= 12 \neq 0$

$\therefore$ Vectors don't form an orthogonal set

b) $\vec{v}_1 \bullet \vec{v}_2 = 1(-2) - 2(1)$

$= -4 \neq 0$

$\therefore$ Vectors don't form an orthogonal set

c) $\vec{u} \bullet \vec{v} = -8 + 6 + 20 + 9$

$= 27 \neq 0$

$\therefore$ These vectors are not orthogonal

d) $\vec{u} \bullet \vec{v} = -ab + ab$

$= 0$

$\therefore$ These vectors are orthogonal

e) $\vec{v}_1 \bullet \vec{v}_2 = -2(1) + 1(0) + 1(2)$

$= 0$

$\vec{v}_1 \bullet \vec{v}_3 = -2(-2) + 1(-5) + 1(1)$

$= 0$

$\vec{v}_2 \bullet \vec{v}_3 = 1(-2) + 0(-5) + 2(1)$

$= 0$

$\therefore$ Vectors form an orthogonal set

f) $\vec{v}_1 \bullet \vec{v}_2 = 1(1) + 0(1) + 1(1)$

$= 2 \neq 0$

$\therefore$ Vectors don't form an orthogonal set

g) $\vec{v}_1 \bullet \vec{v}_2 = 2(2) - 2(1) + 1(-2)$

$= 0$

$\vec{v}_1 \bullet \vec{v}_3 = 2(1) - 2(2) + 1(2)$

$= 0$

$$\vec{v}_2 \bullet \vec{v}_3 = 2(1) + 1(2) - 2(2)$$
$$= 0$$

$\therefore$ Vectors form an orthogonal set

Exercise

Find a unit vector that is orthogonal to both $\vec{u} = (1, 0, 1)$ and $\vec{v} = (0, 1, 1)$

Solution

Let $\vec{w} = (w_1, w_2, w_3)$ be the unit vector that is orthogonal to both $\vec{u}$ and $\vec{v}$.

$$\begin{aligned} \vec{u} \bullet \vec{w} &= 1(w_1) + 0(w_2) + 1(w_3) \\ &= w_1 + w_3 = 0 \end{aligned}$$

$$w_3 = -w_1$$

$$\begin{aligned} \vec{v} \bullet \vec{w} &= 0(w_1) + 1(w_2) + 1(w_3) \\ &= w_2 + w_3 = 0 \end{aligned}$$

$$w_3 = -w_2$$

$$w_1 = w_2 = -w_3$$

The orthogonal vector to both $\vec{u}$ and $\vec{v}$ is $\vec{w} = (1, 1, -1)$, therefore the unit vector is

$$\begin{aligned} \frac{\vec{w}}{\|\vec{w}\|} &= \frac{1}{\sqrt{1^2 + 1^2 + (-1)^2}}(1, 1, -1) \\ &= \frac{1}{\sqrt{3}}(1, 1, -1) \\ &= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \end{aligned}$$

The possible vectors are: $\pm\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

Exercise

a) Show that $\vec{v} = (a, b)$ and $\vec{w} = (-b, a)$ are orthogonal vectors.

b) Use the result to find two vectors that are orthogonal to $\vec{v} = (2, -3)$.

c) Find two unit vectors that are orthogonal to $(-3, 4)$

Solution

a) $\vec{v} \bullet \vec{w} = a(-b) + b(a)$

$= -ab + ab$

$\boxed{= 0}$

$\vec{v}$ and $\vec{w}$ are orthogonal vectors.

b) (2, 3) and (–2, 3).

c) $\vec{u}_1 = \frac{1}{\sqrt{4^2 + 3^2}}(4,\ 3)$

$\boxed{= \left(\frac{4}{5},\ \frac{3}{5}\right)}$

$\vec{u}_2 = -\frac{1}{\sqrt{4^2 + 3^2}}(4,\ 3)$

$\boxed{= \left(-\frac{4}{5},\ -\frac{3}{5}\right)}$

Exercise

Find the vector component of $\vec{u}$ along $\vec{a}$ and the vector component of $\vec{u}$ orthogonal to

a) $\vec{u} = (6,\ 2),\quad \vec{a} = (3,\ -9)$

b) $\vec{u} = (3,\ 1,\ -7),\quad \vec{a} = (1,\ 0,\ 5)$

c) $\vec{u} = (1,\ 0,\ 0),\quad \vec{a} = (4,\ 3,\ 8)$

d) $\vec{u} = (1,\ 1,\ 1),\quad \vec{a} = (0,\ 2,\ -1)$

e) $\vec{u} = (2,\ 1,\ 1,\ 2),\quad \vec{a} = (4,\ -4,\ 2,\ -2)$

f) $\vec{u} = (5,\ 0,\ -3,\ 7),\quad \vec{a} = (2,\ 1,\ -1, -1)$

Solution

a) $proj_{\vec{a}}\vec{u} = \frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a}$

$= \frac{6(3) + 2(-9)}{3^2 + (-9)^2}(3,\ -9)$

$= \frac{0}{90}(3,\ -9)$

$\boxed{= (0,\ 0)}$

$\vec{u} - proj_{\vec{a}}\vec{u} = (6,\ 2) - (0,\ 0)$

$= \boxed{(6,\ 2)}$

b) $proj_{\vec{a}}\vec{u} = \frac{3(1) + 0 - 7(5)}{1^2 + 0 + 5^2}(1,\ 0,\ 5)$

$proj_{\vec{a}}\vec{u} = \frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a}$

$= \frac{-32}{26}(1,\ 0,\ 5)$

$$= \boxed{\left(-\frac{16}{13},\ 0,\ -\frac{80}{13}\right)}$$

$$\vec{u} - proj_{\vec{a}}\vec{u} = (1,0,5) - \left(-\frac{16}{13},\ 0,\ -\frac{80}{13}\right)$$

$$= \boxed{\left(\frac{55}{13},\ 1,\ -\frac{11}{13}\right)}$$

c) $$proj_{\vec{a}}\vec{u} = \frac{1(4)+0+0}{4^2+3^2+8^2}(4,\ 3,\ 8)$$ $$proj_{\vec{a}}\vec{u} = \frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a}$$

$$= \frac{4}{89}(4,\ 3,\ 8)$$

$$= \boxed{\left(\frac{16}{89},\ \frac{12}{89},\ \frac{32}{89}\right)}$$

$$\vec{u} - proj_{\vec{a}}\vec{u} = (1,\ 0,\ 0) - \left(\frac{16}{89},\ \frac{12}{89},\ \frac{32}{89}\right)$$

$$= \boxed{\left(\frac{73}{89},\ -\frac{12}{89},\ -\frac{32}{89}\right)}$$

d) $$proj_{\vec{a}}\vec{u} = \frac{1(0)+1(2)+1(-1)}{0^2+2^2+(-1)^2}(0,\ 2,\ -1)$$ $$proj_{\vec{a}}\vec{u} = \frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a}$$

$$= \frac{1}{5}(0,\ 2,\ -1)$$

$$= \boxed{\left(0,\ \frac{2}{5},\ -\frac{1}{5}\right)}$$

$$\vec{u} - proj_{\vec{a}}\vec{u} = (1,1,1) - \left(0,\ \frac{2}{5},\ -\frac{2}{5}\right)$$

$$= \boxed{\left(1,\ \frac{3}{5},\ \frac{6}{5}\right)}$$

e) $$proj_{\vec{a}}\vec{u} = \frac{2(4)+1(-4)+1(2)+2(-2)}{4^2+(-4)^2+2^2+(-2)^2}(4,\ -4,\ 2,\ -2)$$ $$proj_{\vec{a}}\vec{u} = \frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a}$$

$$= \frac{2}{40}(4,\ -4,\ 2,\ -2)$$

$$= \boxed{\left(\frac{1}{5},\ -\frac{1}{5},\ \frac{1}{10},\ -\frac{1}{10}\right)}$$

$$\vec{u} - proj_{\vec{a}}\vec{u} = (2,\ 1,\ 1,\ 2) - \left(\frac{1}{5},\ -\frac{1}{5},\ \frac{1}{10},\ -\frac{1}{10}\right)$$

$$= \boxed{\left(\frac{9}{5},\ \frac{6}{5},\ \frac{9}{10},\ \frac{21}{10}\right)}$$

f) $proj_{\vec{a}}\vec{u} = \frac{5(2)+0(1)-3(-1)+7(-1)}{2^2+1^2+(-1)^2+(-1)^2}(2,\ 1,\ -1,\ -1)$ $\qquad proj_{\vec{a}}\vec{u} = \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\vec{a}$

$= \frac{6}{7}(2,\ 1,\ -1,\ -1)$

$= \left(\frac{12}{7},\ \frac{6}{7},\ -\frac{6}{7},\ -\frac{6}{7}\right)$

$\vec{u} - proj_{\vec{a}}\vec{u} = (5,\ 0,\ -3,\ 7) - \left(\frac{12}{7},\ \frac{6}{7},\ -\frac{6}{7},\ -\frac{6}{7}\right)$

$= \left(\frac{23}{7},\ -\frac{6}{7},\ -\frac{15}{7},\ \frac{55}{7}\right)$

Exercise

Project the vector $\vec{v}$ onto the line through $\vec{a}$, check that $\vec{e} = \vec{u} - proj_{\vec{a}}\vec{u}$ is perpendicular to $\vec{a}$:

a) $\vec{v} = \begin{pmatrix}1\\2\\2\end{pmatrix}$ *and* $\vec{a} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

b) $\vec{v} = \begin{pmatrix}1\\3\\1\end{pmatrix}$ *and* $\vec{a} = \begin{pmatrix}-1\\-3\\-1\end{pmatrix}$

c) $\vec{v} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ *and* $\vec{a} = \begin{pmatrix}1\\2\\2\end{pmatrix}$

Solution

a) $proj_{\vec{a}}\vec{v} = \frac{1(1)+2(1)+2(1)}{1^2+1^2+1^2}(1,\ 1,\ 1)$ $\qquad proj_{\vec{a}}\vec{v} = \frac{\vec{v}\cdot\vec{a}}{\|\vec{a}\|^2}\vec{a}$

$= \frac{5}{3}(1,\ 1,\ 1)$

$= \left(\frac{5}{3},\ \frac{5}{3},\ \frac{5}{3}\right)$

$\vec{e} = \vec{v} - proj_{\vec{a}}\vec{v}$

$= (1,\ 2,\ 2) - \left(\frac{5}{3},\ \frac{5}{3},\ \frac{5}{3}\right)$

$= \left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$

$\vec{e}\cdot\vec{a} = \left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)\cdot(1,\ 1,\ 1)$

$$=-\frac{2}{3}+\frac{1}{3}+\frac{1}{3}$$

$$\underline{=0}$$

$\vec{e}$ is perpendicular to $\vec{a}$

b) $$proj_{\vec{a}}\vec{v}=\frac{1(-1)+3(-3)+1(-1)}{(-1)^2+(-3)^2+(-1)^2}(-1,\ -3,\ -1) \qquad proj_{\vec{a}}\vec{v}=\frac{\vec{v}\bullet\vec{a}}{\|\vec{a}\|^2}\vec{a}$$

$$=\frac{-11}{11}(-1,\ -3,\ -1)$$

$$\underline{=(1,\ 3,\ 1)}$$

$$\vec{e}=\vec{v}-proj_{\vec{a}}\vec{v}$$

$$=(1,\ 3,\ 1)-(1,\ 3,\ 1)$$

$$\underline{=(0,\ 0,\ 0)}$$

$$\vec{e}\bullet\vec{a}=(0,\ 0,\ 0)\bullet(-1,\ -3,\ -1)$$

$$\underline{=0}$$

$\vec{e}$ is perpendicular to $\vec{a}$

c) $$proj_{\vec{a}}\vec{v}=\frac{1(1)+1(2)+1(2)}{(1)^2+(2)^2+(2)^2}(1,\ 2,\ 2) \qquad proj_{\vec{a}}\vec{v}=\frac{\vec{v}\bullet\vec{a}}{\|\vec{a}\|^2}\vec{a}$$

$$=\frac{5}{9}(1,\ 2,\ 2)$$

$$\underline{=\left(\frac{5}{9},\ \frac{10}{9},\ \frac{10}{9}\right)}$$

$$\vec{e}=\vec{v}-proj_{\vec{a}}\vec{v}$$

$$=(1,\ 1,\ 1)-\left(\frac{5}{9},\ \frac{10}{9},\ \frac{10}{9}\right)$$

$$\underline{=\left(\frac{4}{9},\ -\frac{1}{9},\ -\frac{1}{9}\right)}$$

$$\vec{e}\bullet\vec{a}=\left(\frac{4}{9},\ -\frac{1}{9},\ -\frac{1}{9}\right)\bullet(1,\ 2,\ 2)$$

$$=\frac{4}{9}-\frac{2}{9}-\frac{2}{9}$$

$$\underline{=0}$$

$\vec{e}$ is perpendicular to $\vec{a}$

Exercise

Find the projection matrix $proj_{\vec{a}}\vec{u} = \dfrac{\vec{u}\bullet\vec{a}}{\|\vec{a}\|^2}\vec{a}$ onto the line through $\vec{a} = \begin{bmatrix}1\\2\\2\end{bmatrix}$

Solution

$$a^T a = \begin{bmatrix}1 & 2 & 2\end{bmatrix}\begin{bmatrix}1\\2\\2\end{bmatrix} = 9$$

$$P = \frac{1}{a^T a}a.a^T$$

$$= \frac{1}{9}\begin{pmatrix}1\\2\\2\end{pmatrix}\begin{pmatrix}1 & 2 & 2\end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix}1 & 2 & 2\\2 & 4 & 4\\2 & 4 & 4\end{pmatrix}$$

Exercise

Draw the projection of $\vec{b}$ onto $\vec{a}$ and also compute it from $proj_{\vec{a}}\vec{b} = \dfrac{\vec{b}\bullet\vec{a}}{\|\vec{a}\|^2}\vec{a}$

$$\vec{b} = \begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix} \quad \textit{and} \quad \vec{a} = \begin{bmatrix}1\\0\end{bmatrix}$$

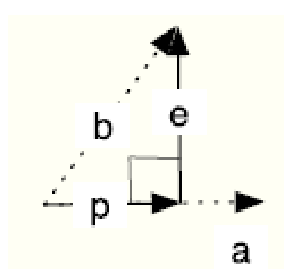
Solution

$$proj_{\vec{a}}\vec{b} = \frac{\vec{b}\bullet\vec{a}}{\|\vec{a}\|^2}\vec{a}$$

$$= \frac{\cos\theta(1) + \sin\theta(0)}{(1)^2 + 0}(1,0)$$

$$= \cos\theta(1,\ 0)$$

$$\boxed{= (\cos\theta,\ 0)}$$



$$\vec{e} = \vec{b} - proj_{\vec{a}}\vec{b}$$

$$= (\cos\theta, \sin\theta) - (\cos\theta, 0)$$

$$\boxed{= (0,\ \sin\theta)}$$

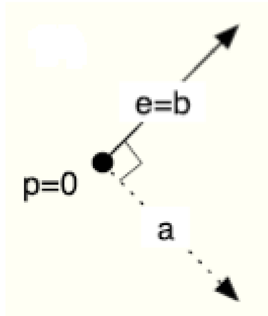
Exercise

Draw the projection of $\vec{b}$ onto $\vec{a}$ and also compute it from $proj_{\vec{a}}\vec{b} = \frac{\vec{b}\bullet\vec{a}}{\|\vec{a}\|^2}\vec{a}$

$\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $\vec{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Solution

$$\begin{aligned} proj_{\vec{a}}\vec{b} &= \frac{\vec{b}\bullet\vec{a}}{\|\vec{a}\|^2}\vec{a} \\ &= \frac{1(1)+1(-1)}{1^2+(-1)^2}(1,\ -1) \\ &= \frac{0}{2}(1,-1) \\ &= (0,\ 0) \end{aligned}$$



$$\begin{aligned} \vec{e} &= \vec{b} - proj_{\vec{a}}\vec{b} \\ &= (1,\ 1)-(0,\ 0) \\ &= (1,\ 1) \end{aligned}$$

Exercise

Show that if $\vec{v}$ is orthogonal to both $\vec{w}_1$ and $\vec{w}_2$, then $\vec{v}$ is orthogonal to $k_1\vec{w}_1 + k_2\vec{w}_2$ for all scalars k_1 and k_2.

Solution

$$\begin{aligned} \vec{v}\bullet\left(k_1\vec{w}_1 + k_2\vec{w}_2\right) &= \vec{v}\bullet\left(k_1\vec{w}_1\right) + \vec{v}\bullet\left(k_2\vec{w}_2\right) \\ &= k_1\left(\vec{v}\bullet\vec{w}_1\right) + k_2\left(\vec{v}\bullet\vec{w}_2\right) \end{aligned}$$

If $\vec{v}$ is orthogonal to $\vec{w}_1$ & $\vec{w}_2$

$\rightarrow \vec{v}\bullet\vec{w}_1 = \vec{v}\bullet\vec{w}_2 = 0$

$$\begin{aligned} &= k_1(0) + k_2(0) \\ &= 0 \end{aligned}$$

Exercise

a) Project the vector $\vec{v} = (3, 4, 4)$ onto the line through $\vec{a} = (2, 2, 1)$ and then onto the plane that also contains $\vec{a}^* = (1,\ 0,\ 0)$.

b) Check that the first error vector $\vec{v} - \vec{p}$ is perpendicular to $\vec{a}$, and the second error vector $\vec{v} - \vec{p}^*$ is also perpendicular to $\vec{a}^*$.

Solution

a) $$proj_{\vec{a}}\vec{v} = \frac{\vec{v} \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a}$$

$$= \frac{3(2)+4(2)+4(1)}{(2)^2+(2)^2+(1)^2}(2,\ 2,\ 1)$$

$$= \frac{18}{9}(2,\ 2,\ 1)$$

$$= (4,\ 4,\ 2)$$

The plane contains the vectors $\vec{a}$ and $\vec{a}^*$ is the column space of ***A***.

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\left(A^T A\right)^{-1} = \begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix}^{-1}$$

$$= \frac{1}{5}\begin{bmatrix} 1 & -2 \\ -2 & 9 \end{bmatrix}$$

$$P = A\left(A^T A\right)^{-1} A^T$$

$$= \frac{1}{5}\begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -2 & 9 \end{bmatrix}\begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & .8 & .4 \\ 0 & .4 & .2 \end{bmatrix}$$

b) The error vector:

$$\vec{e} = \vec{v} - \vec{p}$$

$$= \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$$\vec{a}\,\vec{e} = \begin{pmatrix} 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$$= 2(-1) + 2(0) + 1(2)$$

$$\underline{= 0 \,|}$$

Therefore, $\vec{e}$ is perpendicular to $\vec{a}$

$$p^* = P\vec{v}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & .8 & .4 \\ 0 & .4 & .2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 4.8 \\ 2.4 \end{pmatrix}$$

The error vector:

$$\vec{e}^* = \vec{v} - \vec{p}^*$$

$$= \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 4.8 \\ 2.4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -.8 \\ 1.6 \end{pmatrix}$$

$$\vec{a}^*\vec{e}^* = \begin{pmatrix} 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & -.8 & 1.6 \end{pmatrix}$$

$$= 2(0) + 2(-.8) + 1(1.6)$$

$$\underline{= 0 \,|}$$

Therefore, $\vec{e}^*$ is perpendicular to $\vec{a}^*$

Exercise

Compute the projection matrices $\vec{a}\vec{a}^T / \vec{a}^T\vec{a}$ onto the lines through $\vec{a}_1 = (-1,\ 2,\ 2)$ and $\vec{a}_2 = (2,\ 2,\ -1)$. Multiply those projection matrices and explain why their product P_1P_2 is what it is. Project $\vec{v} = (1,\ 0,\ 0)$ onto the lines $\vec{a}_1$, $\vec{a}_2$, and also onto $\vec{a}_3 = (2,\ -1,\ 2)$. Add up the three projections $p_1 + p_2 + p_3$.

Solution

For $\vec{a}_1 = (-1,\ 2,\ 2)$

$$\vec{a}_1\vec{a}_1^{\ T} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} -1 & 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}$$

$$\vec{a}_1^{\ T}\vec{a}_1 = \begin{pmatrix} -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

$$\underline{= 9 \,|}$$

$$P_1 = \frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\vec{a}}$$

$$= \frac{1}{9}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}$$

For $\vec{a}_2 = (2,\ 2,\ -1)$

$$\vec{a}_2\vec{a}_2^{\ T} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 2 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$\vec{a}_2^{\ T}\vec{a}_2 = \begin{pmatrix} 2 & 2 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$\underline{= 9 \,|}$$

$$P_2 = \frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\vec{a}}$$

$$= \frac{1}{9}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$P_1 P_2 = \frac{1}{9}\left(\frac{1}{9}\right)\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{81}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{= 0}$$

This because $\vec{a}_1$ *and* $\vec{a}_2$ are perpendicular.

For $\vec{a}_3 = (2,\ -1,\ 2)$

$$\vec{a}_3\vec{a}_3^{\,T} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}$$

$$a_3^{\,T} a_3 = \begin{pmatrix} 2 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

$$\underline{= 9}$$

$$P_3 = \frac{\vec{a}_3\vec{a}_3^{\,T}}{\vec{a}_3^{\,T}\vec{a}_3}$$

$$= \frac{1}{9}\begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}$$

$$p_3 = P_3\vec{v}$$

$$= \frac{1}{9}\begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ \frac{4}{9} \end{pmatrix}$$

$$p_1 = P_1\vec{v}$$

$$= \frac{1}{9}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{9} \\ -\frac{2}{9} \\ -\frac{2}{9} \end{pmatrix}$$

$$p_2 = P_2\vec{v}$$

$$= \frac{1}{9}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{2}{9} \end{pmatrix}$$

$$p_1 + p_2 + p_3 = \begin{pmatrix} \frac{1}{9} \\ -\frac{2}{9} \\ -\frac{2}{9} \end{pmatrix} + \begin{pmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{2}{9} \end{pmatrix} + \begin{pmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ \frac{4}{9} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{= \vec{v}}$$

The reason is that $\vec{a}_3$ is perpendicular to $\vec{a}_1$ *and* $\vec{a}_2$.

Hence, when you compute the three projections of a vector and add them up you get back to the vector you start with.

Exercise

If $P^2 = P$ show that $(I-P)^2 = I-P$. When P projects onto the column space of A, $I-P$ projects onto the ____.

Solution

$$\begin{aligned}
(I-P)^2\vec{v} &= (I-P)(I-P)\vec{v} \\
&= (I-P)(I\vec{v}-P\vec{v}) \\
&= I^2\vec{v} - IP\vec{v} - PI\vec{v} + P^2\vec{v} \\
&= \vec{v} - P\vec{v} - P\vec{v} + P^2\vec{v} \qquad P^2\vec{v} = P\vec{v} \quad \textit{By given definition} \\
&= \vec{v} - P\vec{v} - P\vec{v} + P\vec{v} \\
&= \vec{v} - P\vec{v}
\end{aligned}$$

$$(I-P)^2\vec{v} = (I-P)\vec{v}$$

$$\boxed{(I-P)^2 = (I-P)}$$

When P projects onto the column space of A, then $I-P$ projects onto the left nullspace.

Because $(I-P)^2\vec{v} = (I-P)\vec{v}$; if $P\vec{v}$ is in the column space of A, then $\vec{v}-P\vec{v}$ is a vector perpendicular to $C(A)$.

Exercise

What linear combination of $(1,\ 2,\ -1)$ and $(1,\ 0,\ 1)$ is closest to $\vec{v} = (2,\ 1,\ 1)$?

Solution

$$\tfrac{1}{2}(1,\ 2,\ -1) + \tfrac{3}{2}(1,\ 0,\ 1) = (2,\ 1,\ 1)$$

So, this $\boldsymbol{v}$ is actually in the span of the two given vectors.

Exercise

Show that $\vec{u}-\vec{v}$ is orthogonal to $\vec{u}+\vec{v}$ if and only if $\|\vec{u}\|=\|\vec{v}\|$

Solution

Suppose that $\vec{u}-\vec{v}$ is orthogonal to $\vec{u}+\vec{v}$. Then

$$\begin{aligned}
0 &= \langle \vec{u}-\vec{v},\ \vec{u}+\vec{v}\rangle \\
&= (\vec{u}-\vec{v})^T(\vec{u}+\vec{v}) \\
&= \left(\vec{u}^T-\vec{v}^T\right)(\vec{u}+\vec{v}) \\
&= \vec{u}^T\vec{u}+\vec{u}^T\vec{v}-\vec{v}^T\vec{u}-\vec{v}^T\vec{v} \\
&= \langle \vec{u},\ \vec{u}\rangle+\langle \vec{u},\ \vec{v}\rangle-\langle \vec{v},\ \vec{u}\rangle-\langle \vec{v},\ \vec{v}\rangle \qquad \langle \vec{u},\ \vec{v}\rangle=\langle \vec{v},\ \vec{u}\rangle \\
&= \langle \vec{u},\ \vec{u}\rangle-\langle \vec{v},\ \vec{v}\rangle
\end{aligned}$$

So $\langle \vec{u},\ \vec{u}\rangle=\langle \vec{v},\ \vec{v}\rangle$.

Therefore, $\|\vec{u}\|^2=\|\vec{v}\|^2 \quad\Rightarrow\quad \|\vec{u}\|=\|\vec{v}\|$.

Suppose $\|\vec{u}\|=\|\vec{v}\|$. Then

$$\begin{aligned}
\langle \vec{u}-\vec{v},\ \vec{u}+\vec{v}\rangle &= (\vec{u}-\vec{v})^T(\vec{u}+\vec{v}) \\
&= \left(\vec{u}^T-\vec{v}^T\right)(\vec{u}+\vec{v}) \\
&= \vec{u}^T\vec{u}+\vec{u}^T\vec{v}-\vec{v}^T\vec{u}-\vec{v}^T\vec{v} \\
&= \langle \vec{u},\ \vec{u}\rangle+\langle \vec{u},\ \vec{v}\rangle-\langle \vec{v},\ \vec{u}\rangle-\langle \vec{v},\ \vec{v}\rangle \qquad \langle \vec{u},\ \vec{v}\rangle=\langle \vec{v},\ \vec{u}\rangle \\
&= \langle \vec{u},\ \vec{u}\rangle-\langle \vec{v},\ \vec{v}\rangle \\
&= \|\vec{u}\|^2-\|\vec{v}\|^2 \\
&= 0
\end{aligned}$$

So, we can see that $\vec{u}-\vec{v}$ is orthogonal to $\vec{u}+\vec{v}$

We conclude that $\vec{u}-\vec{v}$ is orthogonal to $\vec{u}+\vec{v}$ if and only if $\|\vec{u}\|=\|\vec{v}\|$, as desired.

Exercise

Given $\vec{u}=(3,\ -1,\ 2)\quad \vec{v}=(4,\ -1,\ 5)\quad \textit{and}\quad \vec{w}=(8,\ -7,\ -6)$

a) Find $3\vec{v}-4(5\vec{u}-6\vec{w})$

b) Find $\vec{u}\bullet\vec{v}$ and then the angle θ between $\vec{u}$ and $\vec{v}$.

Solution

a) $3\vec{v}-4(5\vec{u}-6\vec{w})=3(4,\ -1,\ 5)-4\big(5(3,\ -1,\ 2)-6(8,\ -7,\ -6)\big)$

$$=(12,\ -3,\ 15)-4\left((15,\ -5,\ 10)-(48,\ -42,\ -36)\right)$$
$$=(12,\ -3,\ 15)-4(-33,\ 37,\ 46)$$
$$=(12,\ -3,\ 15)-(-132,\ 148,\ 184)$$
$$=(144,\ -151,\ -169)$$

b) $\vec{u}\bullet\vec{v}=(3,\ -1,\ 2)\bullet(1,\ 1,\ -1)$

$$=3-1-2$$
$$=0$$

$$\theta=90^\circ$$

Exercise

Given: $\vec{u}=(3,\ 1,\ 3)$ $\vec{v}=(4,\ 1,\ -2)$

a) Compute the projection $\vec{w}$ of $\vec{u}$ on $\vec{v}$

b) Find $\vec{p}=\vec{u}-\vec{v}$ and show that $\vec{p}$ is perpendicular to $\vec{v}$.

Solution

a) $\vec{w}=proj_{\vec{v}}\vec{u}$

$$=\frac{\vec{u}\bullet\vec{v}}{\|\vec{v}\|^2}\vec{v}$$
$$=\frac{(3,\ 1,\ 3)\bullet(4,\ 1,\ -2)}{4^2+1^2+(-2)^2}(4,\ 1,\ -2)$$
$$=\frac{12+1-6}{21}(4,\ 1,\ -2)$$
$$=\frac{7}{21}(4,\ 1,\ -2)$$
$$=\frac{1}{3}(4,\ 1,\ -2)$$
$$=\left(\frac{4}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)$$

b) $\vec{p}=(3,\ 1,\ 3)-\left(\frac{4}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)$

$$=\left(\frac{5}{3},\ \frac{2}{3},\ \frac{11}{3}\right)$$

$$\vec{p}\bullet\vec{u}=\left(\frac{5}{3},\ \frac{2}{3},\ \frac{11}{3}\right)\bullet(4,\ 1,\ -2)$$
$$=\frac{20}{3}+\frac{2}{3}-\frac{22}{3}$$
$$=0$$

$\vec{p}$ is perpendicular to $\vec{v}$.

Exercise

a) Show that $\vec{v} = (a, b)$ and $\vec{w} = (-b, a)$ are orthogonal vectors

b) Use the result in part (*a*) to find two vectors that are orthogonal to $\vec{v} = (2, -3)$

c) Find two unit vectors that are orthogonal to $(-3, 4)$

Solution

a) $\vec{u} \bullet \vec{v} = -ab + ba$

$$= 0$$

The 2 vectors are orthogonal vectors.

b) $\vec{v} = (2, -3)$

$\vec{w} = (-3, -2)$ and $\vec{w} = (3, 2)$

c) $(-3, 4)$

$$\vec{u} = \frac{(-3,\ 4)}{\sqrt{9+16}}$$

$$= \left(-\frac{3}{5},\ \frac{4}{5}\right)$$

$$\vec{u}_1 = \left(\frac{4}{5},\ \frac{3}{5}\right) \quad \textit{and} \quad \vec{u}_2 = \left(-\frac{4}{5},\ -\frac{3}{5}\right)$$

Exercise

Show that $A(3, 0, 2)$, .$B(4, 3, 0)$, and $C(8, 1, -1)$ are vertices of a right triangle. At which vertex is the right angle?

Solution

$$AB = (4-3,\ 3-0,\ 0-2) = (1,\ 3,\ -2)$$

$$AC = (5,\ 1,\ -3)$$

$$BC = (4,\ -2,\ -1)$$

$$AB \bullet AC = 5 + 3 + 6 = 14$$

$$AB \bullet BC = 4 - 6 + 2 = 0$$

$$AC \bullet BC = 20 - 2 + 3 = 21$$

The right triangle at point B

Exercise

Establish the identity: $\vec{u} \bullet \vec{v} = \frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2$

Solution

Let $\vec{u}\left(u_1, u_2, \ldots, u_n\right)$ *and* $\vec{v} = \left(v_1, v_2, \ldots, v_n\right)$

$\underline{\vec{u} \bullet \vec{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n}$

$\vec{u} + \vec{v} = \left(u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n\right)$

$$\|\vec{u}+\vec{v}\|^2 = \left(u_1 + v_1\right)^2 + \left(u_2 + v_2\right)^2 + \cdots + \left(u_n + v_n\right)^2$$
$$= u_1^2 + v_1^2 + 2u_1 v_1 + u_2^2 + v_2^2 + 2u_2 v_2 + \cdots + u_2^2 + v_n^2 + 2u_n v_n$$

$\vec{u} - \vec{v} = \left(u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n\right)$

$$\|\vec{u}-\vec{v}\|^2 = \left(u_1 - v_1\right)^2 + \left(u_2 - v_2\right)^2 + \cdots + \left(u_n - v_n\right)^2$$
$$= u_1^2 + v_1^2 - 2u_1 v_1 + u_2^2 + v_2^2 - 2u_2 v_2 + \cdots + u_2^2 + v_n^2 - 2u_n v_n$$

$$\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2 = u_1^2 + v_1^2 + 2u_1 v_1 + u_2^2 + v_2^2 + 2u_2 v_2 + \cdots + u_2^2 + v_n^2 + 2u_n v_n$$
$$-\left(u_1^2 + v_1^2 - 2u_1 v_1 + u_2^2 + v_2^2 - 2u_2 v_2 + \cdots + u_2^2 + v_n^2 - 2u_n v_n\right)$$
$$= u_1^2 + v_1^2 + 2u_1 v_1 + u_2^2 + v_2^2 + 2u_2 v_2 + \cdots + u_2^2 + v_n^2 + 2u_n v_n$$
$$-u_1^2 - v_1^2 + 2u_1 v_1 - u_2^2 - v_2^2 + 2u_2 v_2 - \cdots - u_2^2 - v_n^2 + 2u_n v_n$$
$$= 4u_1 v_1 + 4u_2 v_2 + \cdots + 4u_n v_n$$

$$\frac{1}{4}\left(\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2\right) = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Therefore; $\vec{u} \bullet \vec{v} = \frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2$ is true.

2nd *method*:

$$\frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2 = \frac{1}{4}\left[(\vec{u}+\vec{v})(\vec{u}+\vec{v}) - (\vec{u}-\vec{v})(\vec{u}-\vec{v})\right]$$
$$= \frac{1}{4}\left[\vec{u}\vec{u} + 2\vec{u}\vec{v} + \vec{v}\vec{v} - \left(\vec{u}\vec{u} - 2\vec{u}\vec{v} + \vec{v}\vec{v}\right)\right]$$
$$= \frac{1}{4}\left[\vec{u}\vec{u} + 2\vec{u}\vec{v} + \vec{v}\vec{v} - \vec{u}\vec{u} + 2\vec{u}\vec{v} - \vec{v}\vec{v}\right]$$

$$= \frac{1}{4}(4\vec{u}\vec{v})$$
$$= \vec{u} \bullet \vec{v}$$

Exercise

Find the Euclidean inner product $\vec{u} \bullet \vec{v}$: $\vec{u} = (-1,\ 1,\ 0,\ 4,\ -3)$ $\vec{v} = (-2,\ -2,\ 0,\ 2,\ -1)$

Solution

$$\vec{u} \bullet \vec{v} = 2 - 2 + 0 + 8 + 3$$
$$= 11$$

Exercise

Find the Euclidean distance between $\vec{u}$ and $\vec{v}$: $\vec{u} = (3,\ -3,\ -2,\ 0,\ -3)$ $\vec{v} = (-4,\ 1,\ -1,\ 5,\ 0)$

Solution

$$d(\vec{u},\ \vec{v}) = \|\vec{u} - \vec{v}\|$$
$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$$
$$= \sqrt{(3+4)^2 + (-3-1)^2 + (-2+1)^2 + (0-5)^2 + (-3-0)^2}$$
$$= \sqrt{49 + 16 + 1 + 25 + 9}$$
$$= \sqrt{100}$$
$$= 10$$

Exercise

Find for $\vec{v} = 2\hat{i} - 4\hat{j} + \sqrt{5}\hat{k}$, $\vec{u} = -2\hat{i} + 4\hat{j} - \sqrt{5}\hat{k}$

a) $\vec{v} \bullet \vec{u}$, $|\vec{v}|$, $|\vec{u}|$

b) The cosine of the angle between $\vec{v}$ and $\vec{u}$

c) The scalar component of $\vec{u}$ in the direction of $\vec{v}$

d) The vector $proj_{\vec{v}}\vec{u}$

Solution

a) $\vec{v} \bullet \vec{u} = \left(2\hat{i} - 4\hat{j} + \sqrt{5}\hat{k}\right) \bullet \left(-2\hat{i} + 4\hat{j} - \sqrt{5}\hat{k}\right)$

$$= -4 - 16 - 5$$
$$= -25$$

$$|\vec{v}| = \sqrt{2^2 + (-4)^2 + \left(\sqrt{5}\right)^2}$$

$$= \sqrt{4+16+5}$$

$$= \sqrt{25}$$

$$\underline{= 5}$$

$$|\vec{u}| = \sqrt{(-2)^2 + 4^2 + \left(-\sqrt{5}\right)^2}$$

$$= \sqrt{25}$$

$$\underline{= 5}$$

b) $\cos\theta = \dfrac{\vec{u} \bullet \vec{v}}{|\vec{u}||\vec{v}|}$

$$= \frac{-25}{(5)(5)}$$

$$\underline{= -1}$$

c) $|\vec{u}|\cos\theta = (5)(-1)$

$$\underline{= -5}$$

d) $proj_{\vec{v}}\vec{u} = \left(\dfrac{\vec{u} \bullet \vec{v}}{|\vec{v}|^2}\right)\vec{v}$

$$= \left(\frac{-25}{5^2}\right)\left(2\,\hat{i} - 4\hat{j} + \sqrt{5}\,\hat{k}\right)$$

$$= -\left(2\,\hat{i} - 4\hat{j} + \sqrt{5}\,\hat{k}\right)$$

$$\underline{= -2\,\hat{i} + 4\hat{j} - \sqrt{5}\,\hat{k}}$$

Exercise

Find for $\vec{v} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{k}, \quad \vec{u} = 5\hat{i} + 12\hat{j}$

a) $\vec{v} \bullet \vec{u}, \; |\vec{v}|, \; |\vec{u}|$

b) The cosine of the angle between $\vec{v}$ and $\vec{u}$

c) The scalar component of $\vec{u}$ in the direction of $\vec{v}$

d) The vector $proj_{\vec{v}}\vec{u}$

Solution

a) $\vec{v} \bullet \vec{u} = \left(\frac{3}{5}\,\hat{i} + \frac{4}{5}\,\hat{k}\right) \bullet \left(5\,\hat{i} + 12\hat{j}\right)$

$$\underline{= 3}$$

$$|\vec{v}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$

$$= \sqrt{\frac{9}{25} + \frac{16}{25}}$$

$$= \sqrt{\frac{25}{25}}$$

$$\underline{= 1}$$

$$|\vec{u}| = \sqrt{5^2 + 12^2}$$

$$\underline{= 13}$$

b) $\cos\theta = \dfrac{\vec{u} \bullet \vec{v}}{|\vec{u}||\vec{v}|}$

$$= \frac{3}{(1)(13)}$$

$$\underline{= \frac{3}{13}}$$

c) $|\vec{u}|\cos\theta = (13)\left(\frac{3}{13}\right)$

$$\underline{= 3}$$

d) $proj_{\vec{v}}\vec{u} = \left(\dfrac{\vec{u} \bullet \vec{v}}{|\vec{v}|^2}\right)\vec{v}$

$$= \left(\frac{3}{1^2}\right)\left(\frac{3}{5}\hat{i} + \frac{4}{5}\hat{k}\right)$$

$$\underline{= \frac{9}{5}\hat{i} + \frac{12}{5}\hat{k}}$$

Exercise

Find for $\vec{v} = 2\hat{i} + 10\hat{j} - 11\hat{k}$, $\vec{u} = 2\hat{i} + 2\hat{j} + \hat{k}$

a) $\vec{v} \bullet \vec{u}$, $|\vec{v}|$, $|\vec{u}|$

b) The cosine of the angle between $\vec{v}$ and $\vec{u}$

c) The scalar component of $\vec{u}$ in the direction of $\vec{v}$

d) The vector $proj_{\vec{v}}\vec{u}$

Solution

a) $\vec{v} \bullet \vec{u} = \left(2\hat{i} + 10\hat{j} - 11\hat{k}\right) \bullet \left(2\hat{i} + 2\hat{j} + \hat{k}\right)$

$$= 4 + 20 - 11$$

$$=13$$

$$|\vec{v}|=\sqrt{2^2+10^2+(-11)^2}$$

$$=\sqrt{4+100+121}$$

$$=\sqrt{225}$$

$$=15$$

$$|\vec{u}|=\sqrt{2^2+2^2+1^2}$$

$$=3$$

b) $$\cos\theta=\frac{\vec{u}\bullet\vec{v}}{|\vec{u}||\vec{v}|}$$

$$=\frac{13}{(3)(15)}$$

$$=\frac{13}{45}$$

c) $$|\vec{u}|\cos\theta=(3)\left(\frac{13}{45}\right)$$

$$=\frac{13}{15}$$

d) $$proj_{\vec{v}}\vec{u}=\left(\frac{\vec{u}\bullet\vec{v}}{|\vec{v}|^2}\right)\vec{v}$$

$$=\left(\frac{13}{15^2}\right)\left(2\,\hat{i}+10\hat{j}-11\hat{k}\right)\hat{j}$$

$$=\frac{13}{225}\left(2\,\hat{i}+10\,\hat{j}-11\,\hat{k}\right)$$

Exercise

Find for $\vec{v}=5\,\hat{i}+\hat{j},\quad \vec{u}=2\,\hat{i}+\sqrt{17}\,\hat{j}$

a) $\vec{v}\bullet\vec{u},\ \ |\vec{v}|,\ \ |\vec{u}|$

b) The cosine of the angle between $\vec{v}$ and $\vec{u}$

c) The scalar component of $\vec{u}$ in the direction of $\vec{v}$

d) The vector $proj_{\vec{v}}\vec{u}$

Solution

a) $$\vec{v}\bullet\vec{u}=\left(5\hat{i}+\hat{j}\right)\bullet\left(2\hat{i}+\sqrt{17}\hat{j}\right)$$

$$=10+\sqrt{17}$$

$$|\vec{v}| = \sqrt{25+1}$$
$$= \boxed{\sqrt{26}}$$
$$|\vec{u}| = \sqrt{4+17}$$
$$= \boxed{\sqrt{21}}$$

b) $\cos\theta = \dfrac{\vec{u} \bullet \vec{v}}{|\vec{u}|\ |\vec{v}|}$
$$= \frac{10+\sqrt{17}}{\sqrt{21}\sqrt{26}}$$
$$= \boxed{\frac{10+\sqrt{17}}{\sqrt{546}}}$$

c) $|\vec{u}|\cos\theta = \left(\sqrt{21}\right)\left(\dfrac{10+\sqrt{17}}{\sqrt{546}}\right)$
$$= \boxed{\frac{10+\sqrt{17}}{\sqrt{26}}}$$

d) $proj_{\vec{v}}\vec{u} = \left(\dfrac{\vec{u} \bullet \vec{v}}{|\vec{v}|^2}\right)\vec{v}$
$$= \boxed{\left(\frac{10+\sqrt{17}}{26}\right)\left(5\,\hat{i} + \hat{j}\right)}$$

Exercise

Find for $\vec{v} = \left(\dfrac{1}{\sqrt{2}},\ \dfrac{1}{\sqrt{3}}\right)$, $\vec{u} = \left(\dfrac{1}{\sqrt{2}},\ -\dfrac{1}{\sqrt{3}}\right)$

a) $\vec{v} \bullet \vec{u},\ |\vec{v}|,\ |\vec{u}|$

b) The cosine of the angle between $\vec{v}$ and $\vec{u}$

c) The scalar component of $\vec{u}$ in the direction of $\vec{v}$

d) The vector $proj_{\vec{v}}\vec{u}$

Solution

a) $\vec{v} \bullet \vec{u} = \left(\dfrac{1}{\sqrt{2}},\ \dfrac{1}{\sqrt{3}}\right) \bullet \left(\dfrac{1}{\sqrt{2}},\ -\dfrac{1}{\sqrt{3}}\right)$
$$= \frac{1}{2} - \frac{1}{3}$$
$$= \boxed{\frac{1}{6}}$$

$$|\vec{v}| = \sqrt{\frac{1}{2}+\frac{1}{3}}$$

$$= \frac{\sqrt{5}}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}}$$

$$= \frac{\sqrt{30}}{6}$$

$$|\vec{u}| = \sqrt{\frac{1}{2}+\frac{1}{3}}$$

$$= \frac{\sqrt{5}}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}}$$

$$= \frac{\sqrt{30}}{6}$$

b) $\cos\theta = \dfrac{\vec{u}\bullet\vec{v}}{|\vec{u}|\ |\vec{v}|}$

$$= \frac{\frac{1}{6}}{\frac{\sqrt{30}}{6}\frac{\sqrt{30}}{6}}$$

$$= \frac{1}{6}\left(\frac{36}{30}\right)$$

$$= \frac{1}{5}$$

c) $|\vec{u}|\cos\theta = \left(\dfrac{\sqrt{30}}{6}\right)\left(\dfrac{1}{5}\right)$

$$= \frac{\sqrt{30}}{30}$$

$$= \frac{1}{\sqrt{30}}$$

d) $proj_{\vec{v}}\vec{u} = \left(\dfrac{\vec{u}\bullet\vec{v}}{|\vec{v}|^2}\right)\vec{v}$

$$= \frac{1}{6}\left(\frac{36}{30}\right)\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{5}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}}\right)$$

Exercise

Suppose Ted weighs 180 *lb*. and he is sitting on an inclined plane that drops 3 *units* for every 4 horizontal units. The gravitational force vector is $\vec{F}_g = \begin{pmatrix} 0 \\ -180 \end{pmatrix}$.

a) Find the force pushing Ted down the slope.

b) Find the force acting to hold Ted against the slope

Solution

A vector parallel to the slope of the inclined plane is $\vec{v} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$.

a) The vector of the force acting to push Ted down the slope is

$$\begin{aligned} \vec{F}_s &= \frac{\vec{v} \cdot \vec{F}_g}{|\vec{v}|^2}\vec{v} \\ &= \frac{(4,\ -3) \cdot (0,\ -180)}{16+9}(4,\ -3) \\ &= \frac{540}{25}(4,\ -3) \\ &= \underline{\left(\frac{432}{5},\ -\frac{324}{5}\right)} \end{aligned}$$

The magnitude of the force pushing Ted down the slope is

$$\begin{aligned} \left\|\vec{F}_s\right\| &= \sqrt{\left(\frac{432}{5}\right)^2 + \left(\frac{324}{5}\right)^2} \\ &= \frac{540}{5} \\ &= \underline{108\ \ lb} \end{aligned}$$

b) The vector of the force acting to hold Ted against the slope is

$$\begin{aligned} \vec{F}_p &= \vec{F}_g - \vec{F}_s \\ &= \begin{pmatrix} 0 \\ -180 \end{pmatrix} - \begin{pmatrix} \frac{432}{5} \\ -\frac{324}{5} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{432}{5} \\ -\frac{576}{5} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \left\|\vec{F}_p\right\| &= \sqrt{\left(\frac{432}{5}\right)^2 + \left(\frac{576}{5}\right)^2} \\ &= \frac{720}{5} \\ &= \underline{144\ \ lb} \end{aligned}$$

Exercise

Prove that is two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ are orthogonal to nonzero vector $\vec{w}$ in $\mathbb{R}^2$, then $\vec{u}$ and $\vec{v}$ are scalar multiples of each other.

Solution

Since $\vec{u}$ is orthogonal to $\vec{w}$ $\rightarrow$ $\vec{u} \bullet \vec{w} = 0$

$\vec{v}$ is orthogonal to $\vec{w}$ $\rightarrow$ $\vec{v} \bullet \vec{w} = 0$

$\Rightarrow \quad \vec{u} \bullet \vec{w} = \vec{v} \bullet \vec{w} = 0$

There exist $a \in \mathbb{R}$ such that $(a\vec{v}) \bullet \vec{w} = a(\vec{v} \bullet \vec{w}) = 0$

$\vec{u} = a\vec{v}$

$\vec{u} \bullet \vec{w} = \vec{v} \bullet \vec{w} = 0 = (a\vec{v}) \bullet \vec{w}$

Therefore, $\vec{u}$ and $\vec{v}$ are scalar multiples of each other