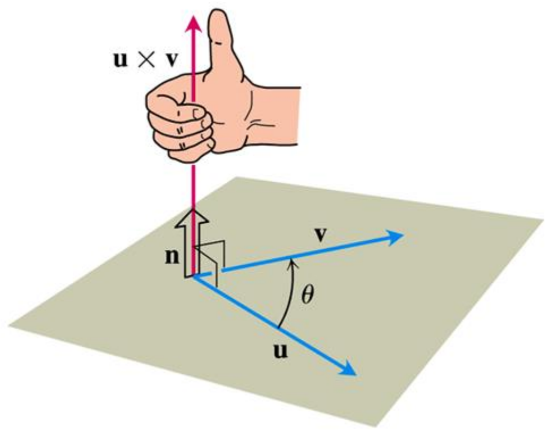
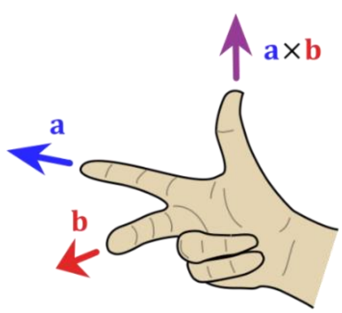
Section 1.3 – Cross Products

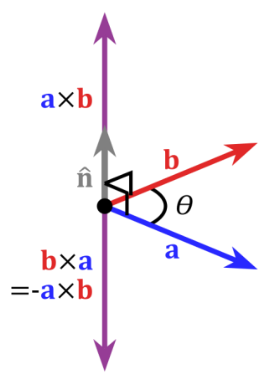
The *Cross* Product

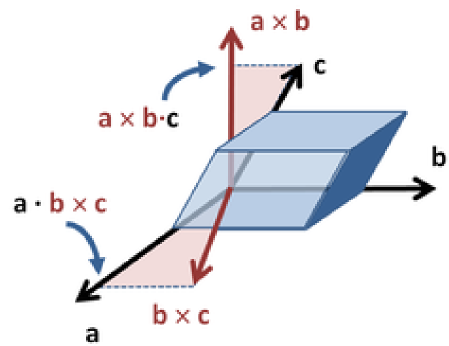
To find a vector in 3-space that is perpendicular to two vectors; the type of vector multiplication that facilities this construction is the cross product.

We start with two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in space. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are no parallel, they determine a plane. We select a unit vector $\boldsymbol{n}$ perpendicular to the plane by the ***right-hand rule***. Then the cross product $\boldsymbol{u} \times \boldsymbol{v}$ ("$\boldsymbol{u}$ cross $\boldsymbol{v}$") is the vector defined as follows









Definition

$$\boldsymbol{u} \times \boldsymbol{v} = \left(|\boldsymbol{u}||\boldsymbol{v}|\sin\theta\right)\boldsymbol{n}$$

Parallel Vectors

Nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel iff $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{0}$

Properties of the Cross Product

If ***u, v*** and ***w*** are any vectors and r, s are scalars, then

a) $\boldsymbol{u}\times\boldsymbol{v}=-(\boldsymbol{v}\times\boldsymbol{u})$

b) $\boldsymbol{u}\times(\boldsymbol{v}+\boldsymbol{w})=(\boldsymbol{u}\times\boldsymbol{v})+(\boldsymbol{u}\times\boldsymbol{w})$

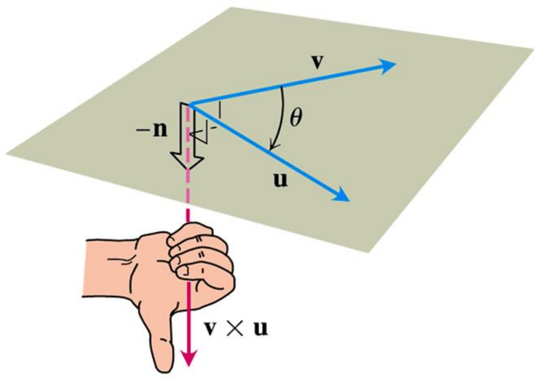
c) $(\boldsymbol{u}+\boldsymbol{v})\times\boldsymbol{w}=(\boldsymbol{u}\times\boldsymbol{w})+(\boldsymbol{v}\times\boldsymbol{w})$

d) $r(\boldsymbol{u}\times\boldsymbol{v})=(r\boldsymbol{u})\times\boldsymbol{v}=\boldsymbol{u}\times(r\boldsymbol{v})$

e) $(r\boldsymbol{u})\times(s\boldsymbol{v})=(rs)(\boldsymbol{u}\times\boldsymbol{v})$

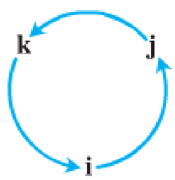
f) $\boldsymbol{u}\times 0=0\times\boldsymbol{u}=0$

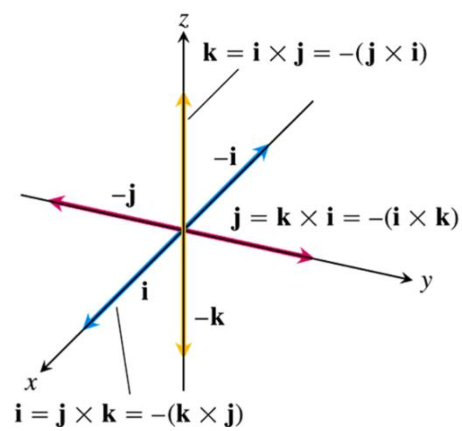
g) $\boldsymbol{u}\times\boldsymbol{u}=0$



Note:

- ✓ $\boldsymbol{i}\times\boldsymbol{i}=\boldsymbol{j}\times\boldsymbol{j}=\boldsymbol{k}\times\boldsymbol{k}=\boldsymbol{0}$
- ✓ $\boldsymbol{i}\times\boldsymbol{j}=\boldsymbol{k},\quad \boldsymbol{j}\times\boldsymbol{k}=\boldsymbol{i},\quad \boldsymbol{k}\times\boldsymbol{i}=\boldsymbol{j}$
- ✓ $\boldsymbol{j}\times\boldsymbol{i}=-\boldsymbol{k},\quad \boldsymbol{k}\times\boldsymbol{j}=-\boldsymbol{i},\quad \boldsymbol{i}\times\boldsymbol{k}=-\boldsymbol{j}$
- ✓ $\boldsymbol{i}\times\boldsymbol{j}=-(\boldsymbol{j}\times\boldsymbol{i})=\boldsymbol{k}$
- ✓ $\boldsymbol{j}\times\boldsymbol{k}=-(\boldsymbol{k}\times\boldsymbol{j})=\boldsymbol{i}$
- ✓ $\boldsymbol{k}\times\boldsymbol{i}=-(\boldsymbol{i}\times\boldsymbol{k})=\boldsymbol{j}$

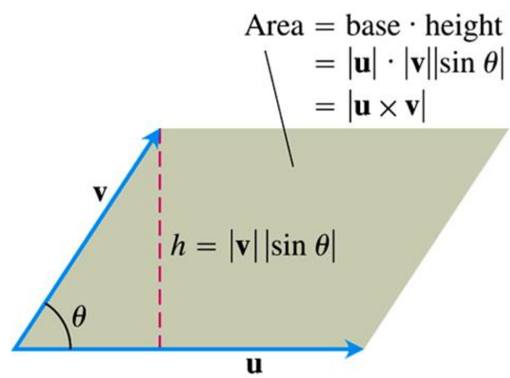




$|\boldsymbol{u}\times\boldsymbol{v}|$ Is the *Area* of the Parallelogram

Because $\boldsymbol{n}$ is a unit vector, the magnitude of $\boldsymbol{u}\times\boldsymbol{v}$ is

$$|\boldsymbol{u}\times\boldsymbol{v}|=|\boldsymbol{u}||\boldsymbol{v}||\sin\theta||n|=|\boldsymbol{u}||\boldsymbol{v}||\sin\theta|$$



Determinant Formula for $\boldsymbol{u} \times \boldsymbol{v}$

Definition

The cross product of $\boldsymbol{u} = (u_1,\ u_2,\ u_3)$ and $\boldsymbol{v} = (v_1,\ v_2,\ v_3)$ is the vector

$$\begin{aligned}
\boldsymbol{u} \times \boldsymbol{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\
&= \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \hat{\boldsymbol{i}} - \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} \hat{\boldsymbol{j}} + \begin{vmatrix} u_1 & v_2 \\ u_2 & v_1 \end{vmatrix} \hat{\boldsymbol{k}} \\
&= (u_2 v_3 - u_3 v_2)\hat{\boldsymbol{i}} - (u_1 v_3 - u_3 v_1)\hat{\boldsymbol{j}} + (u_1 v_2 - u_2 v_1)\hat{\boldsymbol{k}} \\
&= \boxed{(u_2 v_3 - u_3 v_2,\ \ u_3 v_1 - u_1 v_3,\ \ u_1 v_2 - u_2 v_1)}
\end{aligned}$$

Example

Find $\boldsymbol{u} \times \boldsymbol{v}$ and $\boldsymbol{v} \times \boldsymbol{u}$ if $\boldsymbol{u} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{v} = -4\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$

Solution

$$\begin{aligned}
\boldsymbol{u} \times \boldsymbol{v} &= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \hat{\boldsymbol{i}} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} \hat{\boldsymbol{j}} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} \hat{\boldsymbol{k}} \\
&= -2\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}} + 10\hat{\boldsymbol{k}}
\end{aligned}$$

$$\boldsymbol{v} \times \boldsymbol{u} = -\boldsymbol{u} \times \boldsymbol{v} = 2\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} - 10\hat{\boldsymbol{k}}$$

Example

a) Find a vector perpendicular to the plane of $P(1,\ -1,\ 0),\quad Q(2,\ 1,\ -1),\quad and\ \ R(-1,\ 1,\ 2)$

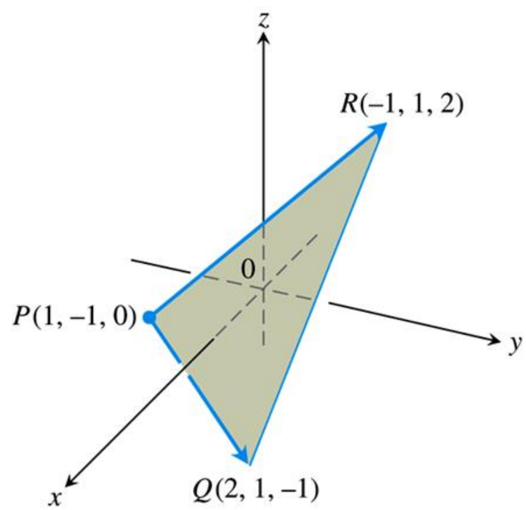
b) Find the area of the triangle with vertices *P*, *Q*, and *R*.

c) Find a unit vector perpendicular to the *P*, *Q*, and *R*

Solution

a) The vector $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to the plane.

$$\overrightarrow{PQ}=(2-1)\hat{i}+(1+1)\hat{j}+(-1-0)\hat{k}$$
$$=\hat{i}+2\hat{j}-\hat{k}$$
$$\overrightarrow{PR}=-2\hat{i}+2\hat{j}+2\hat{k}$$
$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -1\\ -2 & 2 & 2\end{vmatrix}$$
$$\boxed{=6\hat{\boldsymbol{i}}+6\hat{\boldsymbol{k}}}$$



b) The area of the triangle is equal half the parallelogram determined by *P*, *Q*, and *R*.

$$\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|=\sqrt{6^2+6^2}=\sqrt{2\bullet 36}=6\sqrt{2}$$

Area of the triangle: $\frac{1}{2}\left(6\sqrt{2}\right)=\boxed{3\sqrt{2}}$

c) Since $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to the plane, its direction *n* is a unit vector ⊥ to the plane

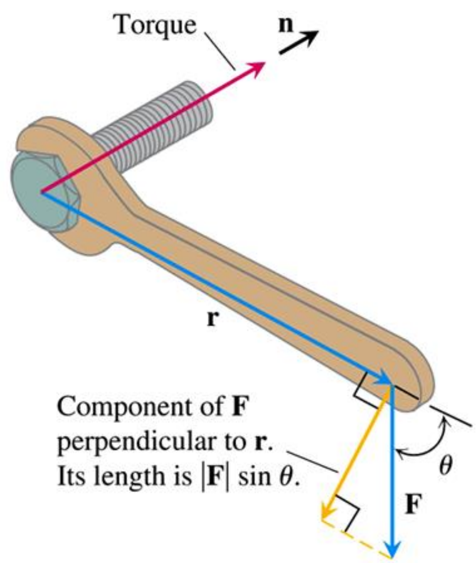
$$\boldsymbol{n}=\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|}$$
$$=\frac{6\hat{i}+6\hat{k}}{6\sqrt{2}}$$
$$\boxed{=\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{k}}$$

Torque

When we turn a bolt by applying a force ***F*** to a wrench, we produce a torque that causes the bolt to rotate. The ***torque vector*** points in the direction of the axis of the bolt according to the right-hand rule (*ccw*).

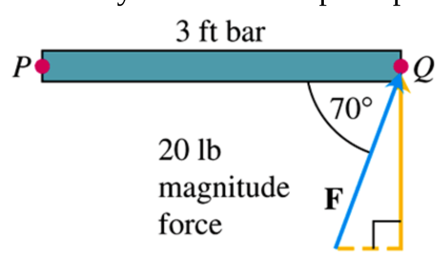
$$\textbf{\textit{Magnitude of torque vector}} = |\boldsymbol{r}||\boldsymbol{F}|\sin\theta$$

$$\textbf{\textit{Torque vector}} = \left(|\boldsymbol{r}||\boldsymbol{F}|\sin\theta\right)\boldsymbol{n}$$



Example

Find the magnitude of the torque generated by force F at the pivot point P.



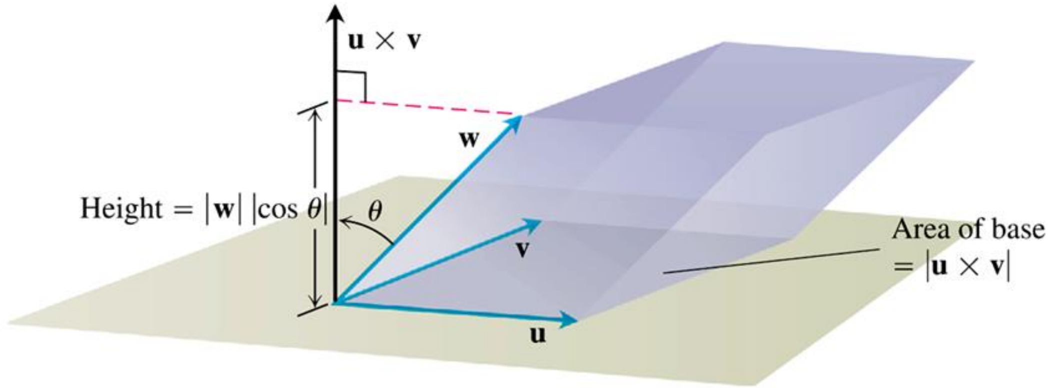
<u>***Solution***</u>

$$\begin{aligned}\left|\overrightarrow{PQ}\times\boldsymbol{F}\right| &= \left|\overrightarrow{PQ}\right|\left|\boldsymbol{F}\right|\sin 70^\circ \\ &\approx (3)(20)(0.94) \\ &\approx \boxed{56.4\ \ ft-lb}\end{aligned}$$

Triple Scalar or Box Product

The product $(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}$ is called the triple scalar product of ***u, v***, and ***w*** (in that order).

$$(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w} = |\boldsymbol{u}\times\boldsymbol{v}||\boldsymbol{w}||\cos\theta|$$



Volume

The Volume of the Parallelepiped is

$$\begin{aligned} V &= (area\ of\ \ base).(height) \\ &= |\boldsymbol{u}\times\boldsymbol{v}||\boldsymbol{w}||\cos\theta| \\ &= |\boldsymbol{u}\times\boldsymbol{v}|\frac{|(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}|}{|\boldsymbol{u}\times\boldsymbol{v}|} \\ &= |(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}| \end{aligned}$$

$$V = (\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w} = \left|\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}\right|$$

Example

Find the volume of the box (parallelepiped) determined by $\boldsymbol{u} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k},\ \ \boldsymbol{v} = -2\boldsymbol{i} + 3\boldsymbol{k},$ and $\boldsymbol{w} = 7\boldsymbol{j} - 4\boldsymbol{k}$.

Solution

$$V = (\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w} = \left|\det\begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{bmatrix}\right| = |-23| \ \underline{= 23}|$$

The volume is 23 units cubed.

Exercises *Section* 1.3 – Cross Products

Find the length and direction of $\boldsymbol{u}\times\boldsymbol{v}$ and $\boldsymbol{v}\times\boldsymbol{u}$

1. $\boldsymbol{u}=2\boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k}, \quad \boldsymbol{v}=\boldsymbol{i}-\boldsymbol{k}$

2. $\boldsymbol{u}=\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}, \quad \boldsymbol{v}=0$

3. $\boldsymbol{u}=\boldsymbol{i}\times\boldsymbol{j}, \quad \boldsymbol{v}=\boldsymbol{j}\times\boldsymbol{k}$

4. $\boldsymbol{u}=-8\boldsymbol{i}-2\boldsymbol{j}-4\boldsymbol{k}, \quad \boldsymbol{v}=2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$

5. Sketch the coordinate axes and then include the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{u}\times\boldsymbol{v}$ as vectors starting origin for $\boldsymbol{u}=\boldsymbol{i}-\boldsymbol{k}, \quad \boldsymbol{v}=\boldsymbol{j}$

6. Sketch the coordinate axes and then include the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{u}\times\boldsymbol{v}$ as vectors starting origin for $\boldsymbol{u}=\boldsymbol{i}-\boldsymbol{k}, \quad \boldsymbol{v}=\boldsymbol{j}+\boldsymbol{k}$

7. Find the area of the triangle determined by the points *P*, *Q*, and *R*, and then find a unit vector perpendicular to plane *PQ R*. $P(1,\ -1,\ 2), \quad Q(2,\ 0,\ -1), \quad and \quad R(0,\ 2,\ 1)$

8. Find the area of the triangle determined by the points *P*, *Q*, and *R*, and then find a unit vector perpendicular to plane *PQ R*. $P(1,\ 1,\ 1), \quad Q(2,\ 1,\ 3), \quad and \quad R(3,\ -1,\ 1)$

9. Find the area of the triangle determined by the points *P*, *Q*, and *R*, and then find a unit vector perpendicular to plane *PQ R*. $P(-2,\ 2,\ 0), \quad Q(0,\ 1,\ -1), \quad and \quad R(-1,\ 2,\ -2)$

10. Verify that $(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}=(\boldsymbol{v}\times\boldsymbol{w})\bullet\boldsymbol{u}=(\boldsymbol{w}\times\boldsymbol{u})\bullet\boldsymbol{v}$ and find the volume of the parallelepiped determined by $\boldsymbol{u}=2\boldsymbol{i}, \quad \boldsymbol{v}=2\boldsymbol{j}, \quad and \quad \boldsymbol{w}=2\boldsymbol{k}$

Find $|\boldsymbol{v}|$, $|\boldsymbol{u}|$, $\boldsymbol{v}\bullet\boldsymbol{u}$, $\boldsymbol{u}\bullet\boldsymbol{v}$, $\boldsymbol{v}\times\boldsymbol{u}$, $\boldsymbol{u}\times\boldsymbol{v}$, $|\boldsymbol{v}\times\boldsymbol{u}|$, the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$, the scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$, and the vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

11. $\vec{v}=\hat{i}+\hat{j}+2\hat{k}, \quad \vec{u}=-\hat{i}-\hat{k}$

12. $\vec{v}=2\hat{i}+\hat{j}-\hat{k}, \quad \vec{u}=\hat{i}+\hat{j}-5\hat{k}$

Find the area of the parallelogram determined by vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, then the volume of the parallelepiped determined by vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

13. $\vec{u}=\hat{i}+\hat{j}-\hat{k}, \quad \vec{v}=2\hat{i}+\hat{j}+\hat{k}, \quad \vec{w}=-\hat{i}-2\hat{j}+3\hat{k}$

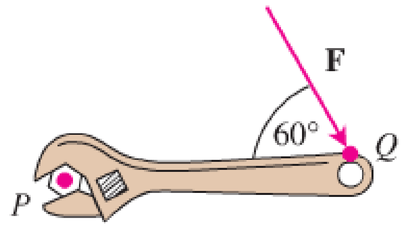
14. $\vec{u}=\hat{i}+\hat{j}, \quad \vec{v}=\hat{j}, \quad \vec{w}=\hat{i}+\hat{j}+\hat{k}$

Find the volume of the parallelepiped determined by

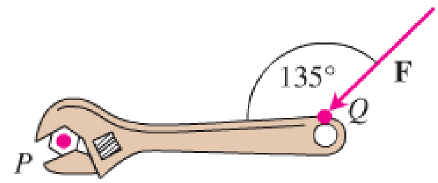
15. $\vec{u}=\hat{i}-\hat{j}+\hat{k}, \quad \vec{v}=2\hat{i}+\hat{j}-\hat{k}, \quad and \quad \vec{w}=-\hat{i}+2\hat{j}-\hat{k}$

16. $\vec{u}=\hat{i}+\hat{j}-2\hat{k}, \quad \vec{v}=-\hat{i}-\hat{k}, \quad and \quad \vec{w}=2\hat{i}+4\hat{j}-2\hat{k}$

17. Find the magnitude of the torque force exerted by $\boldsymbol{F}$ on the bolt at P if $\left|\overrightarrow{PQ}\right| = 8\ in.$ and $\left|\boldsymbol{F}\right| = 30\ lb.$



18. Find the magnitude of the torque force exerted by $\boldsymbol{F}$ on the bolt at P if $\left|\overrightarrow{PQ}\right| = 8\ in.$ and $\left|\boldsymbol{F}\right| = 30\ lb.$



Find the area of the parallelogram whose vertices are

19. $A(1,\ 0),\quad B(0,\ 1),\quad C(-1,\ 0),\quad D(0,\ -1)$

20. $A(0,\ 0),\quad B(7,\ 3),\quad C(9,\ 8),\quad D(2,\ 5)$

21. $A(-1,\ 2),\quad B(2,\ 0),\quad C(7,\ 1),\quad D(4,\ 3)$

22. $A(0,\ 0,\ 0),\quad B(3,\ 2,\ 4),\quad C(5,\ 1,\ 4),\quad D(2,\ -1,\ 0)$

23. $A(1,\ 0,\ -1),\quad B(1,\ 7,\ 2),\quad C(2,\ 4,\ -1),\quad D(0,\ 3,\ 2)$

24. $(1,\ 2,\ 3)$, $(1,\ 0,\ 6)$, and $(4,\ 2,\ 4)$

25. $(1,\ 0,\ 3)$, $(5,\ 0,\ -1)$, and $(0,\ 2,\ -2)$

Find the area of the triangle whose vertices are

26. $A(0,\ 0),\quad B(-2,\ 3),\quad C(3,\ 1)$

27. $A(-1,\ -1),\quad B(3,\ 3),\quad C(2,\ 1)$

28. $A(1,\ 0,\ 0),\quad B(0,\ 0,\ 2),\quad C(0,\ 0,\ -1)$

29. $A(0,\ 0,\ 0),\quad B(-1,\ 1,\ -1),\quad C(3,\ 0,\ 3)$

30. Find the volume of the parallelepiped if four of its eight vertices are:

$$A(0,\ 0,\ 0),\quad B(1,\ 2,\ 0),\quad C(0,\ -3,\ 2),\quad D(3,\ -4,\ 5)$$

31. Let $\vec{u} = \langle 2,\ 4,\ -5\rangle$ and $\vec{v} = \langle -6,\ 10,\ 2\rangle$

a) Compute $\boldsymbol{u} - 3\boldsymbol{v}$

b) Compute $\left|\boldsymbol{u} + \boldsymbol{v}\right|$

c) Find the unit vector with the same direction as $\boldsymbol{u}$

d) Find a vector parallel to $\boldsymbol{v}$ with length 20.

e) Compute $\vec{u} \bullet \vec{v}$ and the angle between $\vec{u}$ and $\vec{v}$.

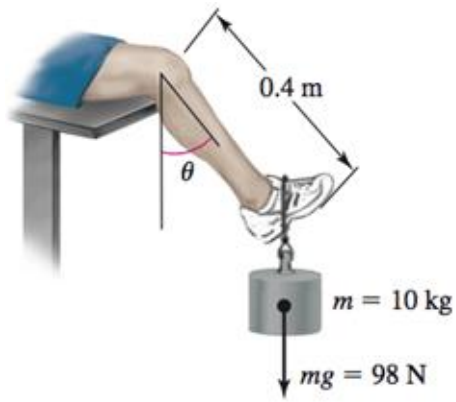
f) Compute $\vec{u} \times \vec{v}$, $\vec{v} \times \vec{u}$

g) Find the area of the triangle with vertices $(0,\ 0,\ 0)$, $(2,\ 4,\ -5)$, and $(-6,\ 10,\ 2)$

32. Find a unit vector normal to the vectors $\langle 2,\ -6,\ 9\rangle$ and $\langle -1,\ 0,\ 6\rangle$

33. Find the angle between $\langle 2,\ 0,\ -2\rangle$ and $\langle 2,\ 2,\ 0\rangle$ using the dot product then the cross product.

34. You do leg lifts with 10-*kg* weight attached to your foot, so the resulting force is $mg \approx 98N$ directed vertically downward. If the distance from your knee to the weight is 0.4*m* and her lower leg makes an angle of θ to the vertical, find the magnitude of the torque about your knee as your leg is lifted (as a function of θ).



a) What is the minimum and maximum magnitude of the torque?

b) Does the direction of the torque change as your leg is lifted?

35. An automobile wheel has center at the origin and axle along the *y*-axis. One of the retaining nuts holding the wheel is at position $P_0(0,\ 0,\ 10)$. (Distances are measured in *cm*.) A bent tire wrench with arm 25 *cm* long and inclined at an angle of 60° to the direction of its handle is fitted to the nut in an upright direction. If the horizontal force $\boldsymbol{F} = 500\boldsymbol{i}\ \ (N)$ is applied to the handle of the wrench, what is its torque on the nut? What part (component) of this torque is effective in trying to rotate the nut about its horizontal axis? What is the effective torque trying to rotate the wheel?