Solution — Section 4.2 – Calculus with Parametric Curves

Exercise

Find all the points at which the curve has the given slope. $x = 4\cos t, \quad y = 4\sin t; \quad slope = \frac{1}{2}$

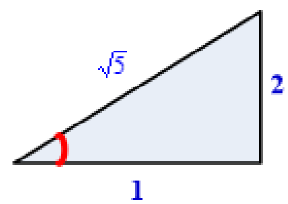
Solution

$$\frac{dy}{dx} = \frac{4\cos t}{-4\sin t} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= -\cot t = \frac{1}{2}$$

$$\cot t = -\frac{1}{2} \quad \Rightarrow \quad t = \cot^{-1}\left(-\frac{1}{2}\right) \quad t \in QII\ \&\ QIV$$

$$\begin{cases} x = -\cos\left(\cot^{-1}\frac{1}{2}\right) = -\frac{1}{\sqrt{5}} \\ y = 4\sin\left(\cot^{-1}\frac{1}{2}\right) = \frac{8}{\sqrt{5}} \end{cases} \rightarrow \left(-\frac{\sqrt{5}}{5}, \frac{8\sqrt{5}}{5}\right)$$



$$\begin{cases} x = \cos\left(\cot^{-1}\frac{1}{2}\right) = \frac{1}{\sqrt{5}} \\ y = -4\sin\left(\cot^{-1}\frac{1}{2}\right) = -\frac{8}{\sqrt{5}} \end{cases} \rightarrow \left(\frac{\sqrt{5}}{5}, -\frac{8\sqrt{5}}{5}\right)$$

Exercise

Find all the points at which the curve has the given slope. $x = 2\cos t, \quad y = 8\sin t; \quad slope = -1$

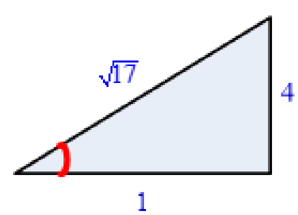
Solution

$$\frac{dy}{dx} = \frac{8\cos t}{-2\sin t} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= -4\cot t = -1$$

$$\cot t = \frac{1}{4} \quad \Rightarrow \quad t = \cot^{-1}\left(\frac{1}{4}\right) \quad t \in QI\ \&\ QIII$$

$$\begin{cases} x = 2\cos\left(\cot^{-1}\frac{1}{4}\right) = \frac{2}{\sqrt{17}} \\ y = 8\sin\left(\cot^{-1}\frac{1}{4}\right) = \frac{32}{\sqrt{5}} \end{cases} \rightarrow \left(\frac{2\sqrt{17}}{17}, \frac{32\sqrt{17}}{17}\right)$$



$$\begin{cases} x = -2\cos\left(\cot^{-1}\frac{1}{4}\right) = -\frac{2}{\sqrt{17}} \\ y = -8\sin\left(\cot^{-1}\frac{1}{4}\right) = -\frac{32}{\sqrt{5}} \end{cases} \rightarrow \left(-\frac{2\sqrt{17}}{17}, -\frac{32\sqrt{17}}{17}\right)$$

Exercise

Find all the points at which the curve has the given slope. $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}; \quad slope = 1$

Solution

$$\frac{dy}{dx} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

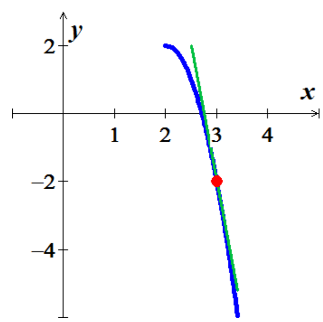
$$= \frac{t^2 + 1}{t^2 - 1} = 1$$

$t^2 + 1 \neq 1 \quad \therefore$ There are ***no*** points on this curve with slope 1.

Exercise

Find all the points at which the curve has the given slope. $x = 2 + \sqrt{t}, \quad y = 2 - 4t; \quad slope = -8$

Solution



$$\frac{dy}{dx} = \frac{-4}{\frac{1}{2\sqrt{t}}} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= -8\sqrt{t}$$

$$\underline{= -8}$$

$t = 1 \quad \rightarrow \quad (3, -2)$

Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of *t*.

$$x = \sin t, \quad y = \cos t, \quad t = \frac{\pi}{4}$$

Solution

$$\frac{dy}{dx} = \frac{-\sin t}{\cos t} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= -\tan t \Big|_{t = \frac{\pi}{4}}$$

$$\underline{= -1}$$

At $t = \frac{\pi}{4}$

$$\begin{cases} x = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \\ y = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \end{cases} \rightarrow \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

The equation of the tangent line is

$$y = -\left(x - \frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}$$

$$\underline{= -x + \sqrt{2} \,|}$$

Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of *t*.

$$x = t^2 - 1, \quad y = t^3 + t, \quad t = 2$$

Solution

$$\frac{dy}{dx} = \frac{3t^2+1}{2t}\Big|_{t=2} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\underline{= \frac{13}{4} \,|}$$

At $t = 2$

$$\begin{cases} x = 3 \\ y = 10 \end{cases} \rightarrow (3, 10)$$

The equation of the tangent line is

$$y = \frac{13}{4}(x-3) + 10$$

$$\underline{= \frac{13}{4}x + \frac{1}{4} \,|}$$

Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of *t*.

$$x = e^t, \quad y = \ln(t+1), \quad t = 0$$

Solution

$$\frac{dy}{dx} = \frac{\frac{1}{t+1}}{e^t}\Bigg|_{t=0} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\underline{= 1 \,|}$$

At $t = 0$

$$\begin{cases} x = 1 \\ y = 0 \end{cases} \quad \rightarrow \quad (1,\ 0)$$

The equation of the tangent line is

$$\boxed{y = x - 1}$$

Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of *t*.

$$x = \cos t + t\sin t, \quad y = \sin t - t\cos t, \quad t = \frac{\pi}{4}$$

Solution

$$\frac{dy}{dx} = \frac{\cos t - \cos t + t\sin t}{-\sin t + \sin t + t\cos t} \qquad\qquad \frac{dy}{dx} = \frac{dy / dt}{dx / dt}$$

$$= \tan t \Big|_{t=\frac{\pi}{4}}$$

$$\boxed{=1}$$

At $t = \frac{\pi}{4}$

$$\begin{cases} x = \frac{\sqrt{2}}{2} + \frac{\pi}{4}\frac{\sqrt{2}}{2} \\ y = \frac{\sqrt{2}}{2} - \frac{\pi}{4}\frac{\sqrt{2}}{2} \end{cases} \quad \rightarrow \quad \left(\frac{4\sqrt{2} + \pi\sqrt{2}}{8},\ \frac{4\sqrt{2} - \pi\sqrt{2}}{8}\right)$$

The equation of the tangent line is

$$y = x - \frac{4\sqrt{2}+\pi\sqrt{2}}{8} + \frac{4\sqrt{2}-\pi\sqrt{2}}{8} \qquad\qquad y = m\left(x - x_0\right) + y_0$$

$$\boxed{= x - \frac{\pi\sqrt{2}}{4}}$$

Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of *t*.

$$x = 6t, \quad y = t^2 + 4, \quad t = 1$$

Solution

$$\frac{dy}{dx} = \frac{2t}{6} \Big|_{t=1} \qquad\qquad \frac{dy}{dx} = \frac{dy / dt}{dx / dt}$$

$$\boxed{= \frac{1}{3}}$$

At $t = 1$:

$$\begin{cases} x = 6 \\ y = 5 \end{cases} \rightarrow (6,\ 5)$$

The equation of the tangent line is

$$y = \frac{1}{3}(x-6)+5 \qquad\qquad y = m\left(x - x_0\right) + y_0$$

$$\boxed{= \frac{1}{3}x + 3}$$

Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of *t*.

$$x = t-2, \quad y = \frac{1}{t}+3, \quad t = 1$$

Solution

$$\frac{dy}{dx} = -\frac{1}{t^2}\Big|_{t=1} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\boxed{= -1}$$

At $t = 1$:

$$\begin{cases} x = -1 \\ y = 4 \end{cases} \rightarrow (-1,\ 4)$$

The equation of the tangent line is

$$y = -(x+1)+4 \qquad\qquad y = m\left(x - x_0\right) + y_0$$

$$\boxed{= -x + 3}$$

Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of *t*.

$$x = t^2 - t + 2, \quad y = t^3 - 3t, \quad t = -1$$

Solution

$$\frac{dy}{dx} = \frac{3t^2-3}{2t-1}\Big|_{t=-1} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\boxed{= 0}$$

At $t = -1$

$$\begin{cases} x = 4 \\ y = 2 \end{cases} \rightarrow (4,\ 2)$$

The equation of the tangent line is

$$\boxed{y = 2} \qquad\qquad y = m\left(x - x_0\right) + y_0$$

Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of t.

$$x = -t^2 + 3t, \quad y = 2t^{3/2}, \quad t = \frac{1}{4}$$

Solution

$$\frac{dy}{dx} = \frac{3t^{1/2}}{-2t+3}\Big|_{t=1/4} \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{3}{2}\frac{1}{-\frac{1}{2}+3}$$

$$= \frac{3}{5}$$

At $t = \frac{1}{4}$:

$$\begin{cases} x = -\frac{1}{16} + \frac{3}{4} = \frac{11}{16} \\ y = \frac{1}{4} \end{cases} \rightarrow \left(\frac{11}{16}, \frac{1}{4}\right)$$

The equation of the tangent line is

$$y = \frac{3}{5}\left(x - \frac{11}{16}\right) + \frac{1}{4} \qquad y = m\left(x - x_0\right) + y_0$$

$$= \frac{3}{5}x - \frac{13}{80}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x = \sin 2\pi t, \quad y = \cos 2\pi t, \quad t = -\frac{1}{6}$

Solution

$$x = \sin 2\pi\left(-\frac{1}{6}\right)$$

$$= -\sin\left(\frac{\pi}{3}\right)$$

$$= -\frac{\sqrt{3}}{2}$$

$$y = \cos 2\pi\left(-\frac{1}{6}\right)$$

$$= \cos\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{2}$$

The point $\left(-\frac{\sqrt{3}}{2},\ \frac{1}{2}\right)$

$$\frac{dx}{dt} = 2\pi\cos 2\pi t$$

$$\frac{dy}{dt} = -2\pi\sin 2\pi t$$

$$\frac{dy}{dx} = \frac{-2\pi\sin 2\pi t}{2\pi\cos 2\pi t} \qquad \frac{dy}{dx} = \frac{dy\,/\,dt}{dx\,/\,dt}$$

$$= -\tan 2\pi t$$

$$\left.\frac{dy}{dx}\right|_{t=-\frac{1}{6}} = -\tan 2\pi\left(-\frac{1}{6}\right)$$

$$= -\tan\left(-\frac{\pi}{3}\right)$$

$$\boxed{= \sqrt{3}}$$

The tangent to the curve at the point $\left(-\frac{\sqrt{3}}{2},\ \frac{1}{2}\right)$ is:

$$y = \sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{1}{2} \qquad y = m\left(x - x_0\right) + y_0$$

$$\boxed{= \sqrt{3}x + 2}$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(-\tan 2\pi t\right)$$

$$= -2\pi\sec^2 2\pi t$$

$$\frac{d^2y}{dx^2} = \frac{-2\pi\sec^2 2\pi t}{2\pi\cos 2\pi t} \qquad \frac{d^2y}{dx^2} = \frac{dy'\,/\,dt}{dx\,/\,dt}$$

$$\boxed{= -\frac{1}{\cos^3 2\pi t}}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=-\frac{1}{6}} = -\frac{1}{\cos^3\left(-\frac{\pi}{3}\right)}$$

$$\boxed{= -8}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x = \cos t, \quad y = \sqrt{3}\cos t, \quad t = \frac{2\pi}{3}$

Solution

$x = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

$y = \sqrt{3}\cos\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

The point $\left(-\frac{1}{2},\ -\frac{\sqrt{3}}{2}\right)$

$\frac{dx}{dt} = -\sin t$

$\frac{dy}{dt} = -\sqrt{3}\sin t$

$\frac{dy}{dx} = \frac{-\sqrt{3}\sin t}{-\sin t}$ $\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$= \sqrt{3}$

$\left.\frac{dy}{dx}\right|_{t=\frac{2\pi}{3}} = \sqrt{3}$

The tangent to the curve at the point $\left(-\frac{1}{2},\ -\frac{\sqrt{3}}{2}\right)$ is:

$y = \sqrt{3}\left(x + \frac{1}{2}\right) - \frac{\sqrt{3}}{2}$ $\qquad y = m\left(x - x_0\right) + y_0$

$= \sqrt{3}x$

$\frac{dy'}{dt} = \frac{d}{dt}\left(\sqrt{3}\right) = 0$

$\frac{d^2y}{dx^2} = \frac{0}{-\sin t}$ $\qquad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$

$= 0$

$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{2\pi}{3}} = 0$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x = t, \quad y = \sqrt{t}, \quad t = \frac{1}{4}$

Solution

$x = \frac{1}{4}$

$$y = \sqrt{\frac{1}{4}}$$
$$= \boxed{\frac{1}{2}}$$

The point $\left(\frac{1}{4}, \frac{1}{2}\right)$

$$\frac{dx}{dt} = 1$$
$$\frac{dy}{dt} = \frac{1}{2\sqrt{t}}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{t}} \cdot 1 \qquad \frac{dy}{dx} = \frac{dy\,/\,dt}{dx\,/\,dt}$$
$$= \frac{1}{2\sqrt{t}}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{1}{4}} = \frac{1}{2\sqrt{\frac{1}{4}}}$$
$$\boxed{=1}$$

The tangent is:

$$y = \left(x - \frac{1}{4}\right) + \frac{1}{2} \qquad y = m\left(x - x_0\right) + y_0$$
$$\boxed{= x + \frac{1}{4}}$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1}{2\sqrt{t}}\right)$$
$$= -\frac{1}{4}t^{-3/2}$$

$$\frac{d^2y}{dx^2} = \frac{-\frac{1}{4}t^{-3/2}}{1} \qquad \frac{d^2y}{dx^2} = \frac{dy'\,/\,dt}{dx\,/\,dt}$$
$$\boxed{= -\frac{1}{4}t^{-3/2}}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{1}{4}} = -\frac{1}{4}\left(\frac{1}{4}\right)^{-3/2}$$
$$\boxed{= -2}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\dfrac{d^2y}{dx^2}$ at this point $\quad x = \sec^2 t - 1, \quad y = \tan t, \quad t = -\frac{\pi}{4}$

Solution

$$\boxed{= -\frac{1}{2}x + \frac{1}{2}}$$

$$y = \tan\left(-\tfrac{\pi}{4}\right) = -1$$

The point $(1,\ -1)$

$$\frac{dx}{dt} = 2\sec^2 t \tan t$$

$$\frac{dy}{dt} = \sec^2 t$$

$$\frac{dy}{dx} = \frac{\sec^2 t}{2\sec^2 t \tan t} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{1}{2\tan t}$$

$$\left.\frac{dy}{dx}\right|_{t=-\frac{\pi}{4}} = \frac{1}{2\tan\left(-\frac{\pi}{4}\right)}$$

$$\boxed{= -\frac{1}{2}}$$

The tangent is:

$$y = -\frac{1}{2}(x-1) - 1 \qquad\qquad y = m\left(x - x_0\right) + y_0$$

$$\boxed{= -\frac{1}{2}x + \frac{1}{2}}$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1}{2}\frac{1}{\tan t}\right)$$

$$= \frac{1}{2}\frac{-\sec^2 t}{\tan^2 t}$$

$$= -\frac{1}{2}\frac{\frac{1}{\cos^2 t}}{\frac{\sin^2 t}{\cos^2 t}}$$

$$= -\frac{1}{2}\csc^2 t$$

$$\frac{d^2y}{dx^2} = \frac{-\frac{1}{2}\csc^2 t}{2\sec^2 t\tan t} \qquad\qquad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$= -\frac{1}{4}\frac{\frac{1}{\sin^2 t}}{\frac{1}{\cos^2 t}\frac{\sin t}{\cos t}}$$

$$= -\frac{1}{4}\frac{\cos^3 t}{\sin^3 t}$$

$$\boxed{= -\frac{1}{4}\cot^3 t}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=-\frac{\pi}{4}} = -\frac{1}{4}\cot^3\left(-\frac{\pi}{4}\right)$$

$$\boxed{= \frac{1}{4}}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of d^2y/dx^2 at this point $x = \frac{1}{t+1}, \quad y = \frac{t}{t-1}, \quad t = 2$

Solution

$$x = \frac{1}{2+1} = \frac{1}{3}$$

$$y = \frac{2}{2-1} = 2$$

The point $\left(\frac{1}{3},\ 2\right)$

$$\frac{dx}{dt} = \frac{-1}{(t+1)^2}$$

$$\frac{dy}{dt} = \frac{t-1-t}{(t-1)^2}$$

$$= \frac{-1}{(t-1)^2}$$

$$\frac{dy}{dx} = \frac{\frac{-1}{(t-1)^2}}{\frac{-1}{(t+1)^2}} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{(t+1)^2}{(t-1)^2}$$

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{(2+1)^2}{(2-1)^2}$$

$$\underline{= 9 \,|}$$

The tangent is:

$$y = 9\left(x - \frac{1}{3}\right) + 2 \qquad\qquad y = m\left(x - x_0\right) + y_0$$

$$\underline{= 9x - 1 \,|}$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{t+1}{t-1}\right)^2$$

$$= 2\left(\frac{t+1}{t-1}\right)\left(\frac{t-1-t-1}{(t-1)^2}\right)$$

$$= -4\frac{t+1}{(t-1)^3}$$

$$\frac{d^2y}{dx^2} = -4\frac{t+1}{(t-1)^3}\frac{(t+1)^2}{-1} \qquad\qquad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$\boxed{= 4\frac{(t+1)^3}{(t-1)^3}}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=2} = 4\frac{(2+1)^3}{(2-1)^3}$$

$$\underline{= 108 \,|}$$

Exercise

Find the tangent to the curve at the point defined by the given value of *t*. Also find the value of d^2y/dx^2 at this point $x = t + e^t, \quad y = 1 - e^t, \quad t = 0$

Solution

$$x = 0 + e^0 = 1$$

$$y = 1 - e^0 = 0$$

The point $(1,\ 0)$

$$\frac{dx}{dt} = 1 + e^t$$

$$\frac{dy}{dt} = -e^t$$

$$\frac{dy}{dx} = \frac{-e^t}{1+e^t} \qquad\qquad \frac{dy}{dx} = \frac{dy\,/\,dt}{dx\,/\,dt}$$

$$\left.\frac{dy}{dx}\right|_{t=0} = -\frac{e^0}{1+e^0}$$

$$\boxed{= -\frac{1}{2}}$$

The tangent is:

$$y = -\frac{1}{2}(x-1)$$

$$\boxed{= -\frac{1}{2}x + \frac{1}{2}}$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{-e^t}{1+e^t}\right)$$

$$= \frac{-e^t\left(1+e^t\right) - e^t\left(-e^t\right)}{\left(1+e^t\right)^2}$$

$$= \frac{-e^t - e^{2t} + e^{2t}}{\left(1+e^t\right)^2}$$

$$= \frac{-e^t}{\left(1+e^t\right)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-e^t}{\left(1+e^t\right)^2}\frac{1}{1+e^t} \qquad\qquad \frac{d^2y}{dx^2} = \frac{dy'\,/\,dt}{dx\,/\,dt}$$

$$\boxed{= \frac{-e^t}{\left(1+e^t\right)^3}}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=0} = \frac{-e^0}{\left(1+e^0\right)^3}$$

$$\boxed{= -\frac{1}{8}}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\dfrac{d^2y}{dx^2}$ at this point $\quad x = 4t, \quad y = 3t - 2, \quad t = 3$

Solution

$$\frac{dx}{dt} = 4$$

$$\frac{dy}{dt} = 3$$

$$\frac{dy}{dx} = \frac{3}{4} \Big|_{t=3} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{3}{4}$$

$$t = 3 \Rightarrow \begin{cases} x = 12 \\ y = 7 \end{cases}$$

The tangent to the curve at the point $(12, \ 7)$

$$y = \frac{3}{4}(x - 12) + 7 \qquad\qquad y = m\left(x - x_0\right) + y_0$$

$$= \frac{3}{4}x - 2$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{3}{4}\right) = 0$$

$$\frac{d^2y}{dx^2} = 0 \qquad\qquad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\dfrac{d^2y}{dx^2}$ at this point $\quad x = \sqrt{t}, \quad y = 3t - 1, \quad t = 1$

Solution

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}}$$

$$\frac{dy}{dt} = 3$$

$$\frac{dy}{dx} = 6\sqrt{t} \Big|_{t=1} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= 6$$

$$t=1 \Rightarrow \begin{cases} x=1 \\ y=2 \end{cases}$$

The tangent to the curve at the point $(1,\ 2)$

$$y=6(x-1)+2 \qquad\qquad y=m\left(x-x_0\right)+y_0$$

$$\underline{=6x-4}\rfloor$$

$$\frac{dy'}{dt}=\frac{d}{dt}\left(6\sqrt{t}\right)$$

$$=\frac{3}{\sqrt{t}}$$

$$\frac{d^2y}{dx^2}=\frac{3}{\sqrt{t}}\cdot 2\sqrt{t} \qquad\qquad \frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}$$

$$\underline{=6}\rfloor$$

Exercise

Find the tangent to the curve at the point defined by the given value of *t*. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\qquad x=t+1, \quad y=t^2+3t, \quad t=-1$

Solution

$$\frac{dx}{dt}=1$$

$$\frac{dy}{dt}=2t+3$$

$$\frac{dy}{dx}=2t+3\Big|_{t=-1} \qquad\qquad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}$$

$$\underline{=1}\rfloor$$

$$t=-1 \Rightarrow \begin{cases} x=2 \\ y=4 \end{cases}$$

The tangent to the curve at the point $(2,\ 4)$

$$y=(x-2)+4 \qquad\qquad y=m\left(x-x_0\right)+y_0$$

$$\underline{=x+2}\rfloor$$

$$\frac{dy'}{dt}=\frac{d}{dt}(2t+3)=2$$

$$\frac{d^2y}{dx^2}\ \underline{=2}\rfloor \qquad\qquad \frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x = t^2 + 5t + 4, \quad y = 4t, \quad t = 0$

Solution

$$\frac{dx}{dt} = 2t + 5$$

$$\frac{dy}{dt} = 4$$

$$\frac{dy}{dx} = \frac{4}{2t+5}\Big|_{t=0} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{4}{5}$$

$$t = 0 \Rightarrow \begin{cases} x = 4 \\ y = 0 \end{cases}$$

The tangent to the curve at the point $(4, 0)$

$$y = \frac{4}{5}(x - 4) \qquad\qquad y = m\left(x - x_0\right) + y_0$$

$$= \frac{4}{5}x - \frac{16}{5}$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{4}{2t+5}\right)$$

$$= \frac{-8}{(2t+5)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-8}{(2t+5)^2} \cdot \frac{1}{2t+5} \qquad\qquad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$= \frac{-8}{(2t+5)^3}\Big|_{t=0}$$

$$= -\frac{8}{125}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x = 4\cos\theta, \quad y = 4\sin\theta, \quad \theta = \frac{\pi}{4}$

Solution

$$\frac{dx}{d\theta} = -4\sin\theta$$

$$\frac{dy}{d\theta} = 4\cos\theta$$

$$\frac{dy}{dx} = \frac{4\cos\theta}{-4\sin\theta} \qquad\qquad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= -\cot\theta \Big|_{\theta=\frac{\pi}{4}}$$

$$\underline{= -1}$$

$$\theta = \frac{\pi}{4} \Rightarrow \begin{cases} x = 2\sqrt{2} \\ y = 2\sqrt{2} \end{cases}$$

The tangent to the curve at the point $\left(2\sqrt{2},\ 2\sqrt{2}\right)$:

$$y = -\left(x - 2\sqrt{2}\right) + 2\sqrt{2} \qquad\qquad y = m\left(x - x_0\right) + y_0$$

$$\underline{= -x + 4\sqrt{2}}$$

$$\frac{dy'}{d\theta} = \frac{d}{d\theta}(-\cot\theta)$$

$$= \csc^2\theta$$

$$\frac{d^2y}{dx^2} = \frac{\csc^2\theta}{-4\sin\theta} \qquad\qquad \frac{d^2y}{dx^2} = \frac{dy'/d\theta}{dx/d\theta}$$

$$= -\frac{1}{4}\csc^3\theta \Big|_{\theta=\frac{\pi}{4}}$$

$$\underline{= -\frac{\sqrt{2}}{2}}$$

Exercise

Find the tangent to the curve at the point defined by the given value of *t*. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x = \cos\theta, \quad y = 3\sin\theta, \quad \theta = 0$

Solution

$$\frac{dx}{d\theta} = -\sin\theta$$

$$\frac{dy}{d\theta} = 3\cos\theta$$

$$\frac{dy}{dx} = \frac{3\cos\theta}{-\sin\theta} \qquad\qquad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= -3\cot\theta \Big|_{\theta=0}$$

$$\underline{= \infty}$$

$$\theta = 0 \Rightarrow \begin{cases} x = 1 \\ y = 0 \end{cases}$$

The tangent to the curve at the point $(1,\ 0)$: $\underline{x = 1}$

$$\frac{dy'}{d\theta} = \frac{d}{d\theta}(-3\cot\theta)$$

$$= 3\csc^2\theta$$

$$\frac{d^2y}{dx^2} = \frac{3\csc^2\theta}{-\sin\theta} \qquad\qquad \frac{d^2y}{dx^2} = \frac{dy'/d\theta}{dx/d\theta}$$

$$= -3\csc^3\theta \Big|_{\theta=0}$$

$$\underline{= \infty} \quad \textit{undefined}$$

Exercise

Find the tangent to the curve at the point defined by the given value of *t*. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\qquad x = 2 + \sec\theta, \quad y = 1 + 2\tan\theta, \quad \theta = \frac{\pi}{6}$

Solution

$$\frac{dx}{d\theta} = \sec\theta\tan\theta$$

$$\frac{dy}{d\theta} = 2\sec^2\theta$$

$$\frac{dy}{dx} = \frac{2\sec^2\theta}{\sec\theta\tan\theta} \qquad\qquad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= 2\csc\theta \Big|_{\theta=\frac{\pi}{6}}$$

$$\underline{= 4}$$

$$\theta = \frac{\pi}{6} \Rightarrow \begin{cases} x = 2 + \frac{2}{\sqrt{3}} \\ y = 1 + \frac{2\sqrt{3}}{3} \end{cases}$$

The tangent to the curve at the point $\left(2 + \frac{2\sqrt{3}}{3},\ 1 + \frac{2\sqrt{3}}{3}\right)$:

$$y = 2\left(x - 2 - \frac{2\sqrt{3}}{3}\right) + 1 + \frac{2\sqrt{3}}{3} \qquad\qquad y = m\left(x - x_0\right) + y_0$$

$$= 2x - 3 - \frac{2\sqrt{3}}{3}$$

$$\frac{dy'}{d\theta} = \frac{d}{d\theta}(2\csc\theta)$$

$$= -2\csc\theta\cot\theta$$

$$\frac{d^2y}{dx^2} = \frac{-2\csc\theta\cot\theta}{\sec\theta\tan\theta} \qquad\qquad \frac{d^2y}{dx^2} = \frac{dy'/d\theta}{dx/d\theta}$$

$$= -2\cot^3\theta \Big|_{\theta=\frac{\pi}{6}}$$

$$= -6\sqrt{3}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x = \sqrt{t}, \quad y = \sqrt{t-1}, \quad t = 2$

Solution

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}}$$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{t-1}}$$

$$\frac{dy}{dx} = \frac{2\sqrt{t}}{2\sqrt{t-1}}\Big|_{t=2} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \sqrt{2}$$

$$t = 2 \Rightarrow \begin{cases} x = \sqrt{2} \\ y = 1 \end{cases}$$

The tangent to the curve at the point $\left(\sqrt{2},\ 1\right)$

$$y = \sqrt{2}\left(x - \sqrt{2}\right) + 1 \qquad\qquad y = m\left(x - x_0\right) + y_0$$

$$= \sqrt{2}x - 1$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{\sqrt{t}}{\sqrt{t-1}}\right) \qquad\qquad \left(U^n V^m\right)' = U^{n-1}V^{m-1}\left(nU'V + mUV'\right)$$

$$=\frac{\frac{1}{2}t-\frac{1}{2}-\frac{1}{2}t}{(t-1)^{3/2}\sqrt{t}}$$

$$=-\frac{1}{2}\frac{1}{(t-1)^{3/2}\sqrt{t}}$$

$$\frac{d^2y}{dx^2}=-\frac{1}{2}\frac{1}{(t-1)^{3/2}\sqrt{t}}\cdot 2\sqrt{t} \qquad \frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}$$

$$=-\frac{1}{(t-1)^{3/2}}\Big|_{t=2}$$

$$\underline{=-1}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x=\cos^3\theta,\quad y=\sin^3\theta,\quad \theta=\frac{\pi}{4}$

Solution

$$\frac{dx}{d\theta}=-3\sin\theta\cos^2\theta$$

$$\frac{dy}{d\theta}=3\cos\theta\sin^2\theta$$

$$\frac{dy}{dx}=\frac{3\cos\theta\sin^2\theta}{-3\sin\theta\cos^2\theta} \qquad \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}$$

$$=-\tan\theta\Big|_{\theta=\frac{\pi}{4}}$$

$$\underline{=-1}$$

$$\theta=\frac{\pi}{4}\Rightarrow\begin{cases}x=\frac{\sqrt{2}}{4}\\ y=\frac{\sqrt{2}}{4}\end{cases}$$

The tangent to the curve at the point $\left(\frac{\sqrt{2}}{4},\ \frac{\sqrt{2}}{4}\right)$:

$$y=-\left(x-\frac{\sqrt{2}}{4}\right)+\frac{\sqrt{2}}{4} \qquad y=m\left(x-x_0\right)+y_0$$

$$\underline{=-x+\frac{\sqrt{2}}{2}}$$

$$\frac{dy'}{d\theta} = \frac{d}{d\theta}(-\tan\theta)$$

$$= -\sec^2\theta$$

$$\frac{d^2y}{dx^2} = \frac{-\sec^2\theta}{-3\sin\theta\cos^2\theta} \qquad \frac{d^2y}{dx^2} = \frac{dy'/d\theta}{dx/d\theta}$$

$$= \frac{1}{3\sin\theta\cos^4\theta}\Bigg|_{\theta=\frac{\pi}{4}}$$

$$\boxed{= \frac{4\sqrt{2}}{3}}$$

Exercise

Find the tangent to the curve at the point defined by the given value of *t*. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x = \theta - \sin\theta, \quad y = 1 - \cos\theta, \quad \theta = \pi$

Solution

$$\frac{dx}{d\theta} = 1 - \cos\theta$$

$$\frac{dy}{d\theta} = \sin\theta$$

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}\Bigg|_{\theta=\pi} \qquad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$\boxed{= 0}$$

$$\theta = \pi \Rightarrow \begin{cases} x = \pi \\ y = 2 \end{cases}$$

The tangent to the curve at the point $(\pi,\ 2)$:

$$\boxed{y = 2} \qquad y = m\left(x - x_0\right) + y_0$$

$$\frac{dy'}{d\theta} = \frac{d}{d\theta}\left(\frac{\sin\theta}{1-\cos\theta}\right)$$

$$= \frac{\cos\theta - \cos^2\theta - \sin^2\theta}{(1-\cos\theta)^2}$$

$$= \frac{\cos\theta - 1}{(1-\cos\theta)^2}$$

$$= \frac{-1}{1-\cos\theta}$$

$$\frac{d^2y}{dx^2} = \left(\frac{-1}{1-\cos\theta}\right)\frac{1}{1-\cos\theta}\Bigg|_{\theta=\pi} \qquad\qquad \frac{d^2y}{dx^2} = \frac{dy'/d\theta}{dx/d\theta}$$

$$= \frac{-1}{(1-\cos\theta)^2}\Bigg|_{\theta=\pi}$$

$$\boxed{= -\frac{1}{4}}$$

Exercise

Find the equations of the tangent lines at the point where the curve crosses itself

$$x = 2\sin 2t, \quad y = 3\sin t$$

Solution

$x = y$

$2\sin 2t = 3\sin t$

$\Rightarrow \boxed{t = 0, \ \pi}$

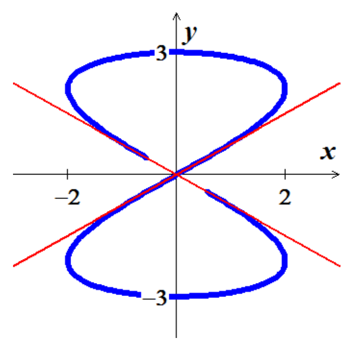
$\frac{dx}{dt} = 4\cos 2t, \quad \frac{dy}{dt} = 3\cos t$

$\frac{dy}{dx} = \frac{3\cos t}{4\cos 2t}$

At $t = 0$ $\quad \frac{dy}{dx} = \frac{3}{4}$

The point at $t = 0$ is $(0,\ 0)$

The tangent line: $\boxed{y = \frac{3}{4}x}$



At $t = \pi$ $\quad \frac{dy}{dx} = -\frac{3}{4}$

The point at $t = 0$ is $(0,\ 0)$

The tangent line: $\boxed{y = -\frac{3}{4}x}$

Exercise

Find the equations of the tangent lines at the point where the curve crosses itself

$$x = 2 - \pi\cos t, \quad y = 2t - \pi\sin t$$

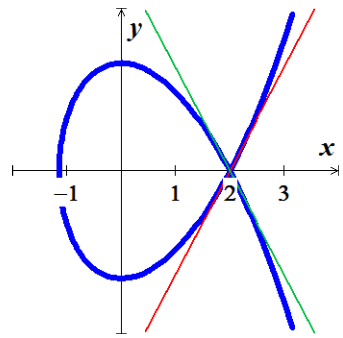
Solution

The graph crosses itself at the point $(2,\ 0)$

$x = 2 - \pi\cos t = 2$

$\cos t = 0$

$$\Rightarrow \boxed{t = \pm\frac{\pi}{2}}$$



$$\frac{dx}{dt} = \pi \sin t$$

$$\frac{dy}{dt} = 2 - \pi \cos t$$

$$\frac{dy}{dx} = \frac{2 - \pi \cos t}{\pi \sin t}$$

$At \quad t = \frac{\pi}{2} \quad \frac{dy}{dx} = \frac{2}{\pi}$

The tangent line: $y = \frac{2}{\pi}(x-2)$

$$\boxed{= \frac{2}{\pi}x - \frac{4}{\pi}}$$

$At \quad t = -\frac{\pi}{2} \quad \frac{dy}{dx} = -\frac{2}{\pi}$

The tangent line: $y = -\frac{2}{\pi}(x-2)$

$$\boxed{= -\frac{2}{\pi}x + \frac{4}{\pi}}$$

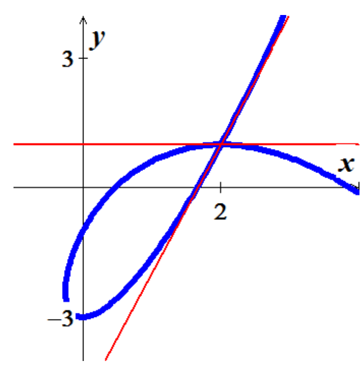
Exercise

Find the equations of the tangent lines at the point where the curve crosses itself

$$x = t^2 - t, \quad y = t^3 - 3t - 1$$

Solution

The graph crosses itself at the point $(2,\ 1)$



$x = t^2 - t = 2$

$t^2 - t - 2 = 0$

$\Rightarrow \boxed{t = -1,\ 2}$

$$\frac{dx}{dt} = 2t - 1$$

$$\frac{dy}{dt} = 3t^2 - 3$$

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t - 1}$$

$At \quad t = -1 \quad \frac{dy}{dx} = 0$

The tangent line: $\boxed{y = 1}$

$At \quad t = 2 \quad \frac{dy}{dx} = 3$

The tangent line: $y = 3(x-2)+1$

$\boxed{= 3x - 5}$

Exercise

Find the equations of the tangent lines at the point where the curve crosses itself

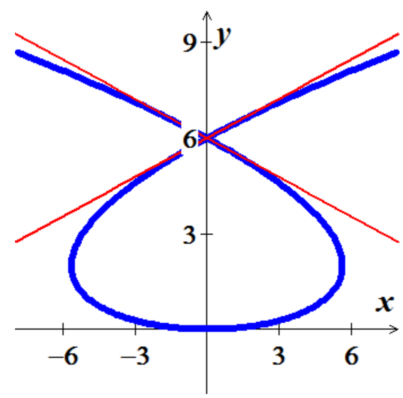
$$x = t^3 - 6t, \quad y = t^2$$

Solution

The graph crosses itself at the point $(0,\ 6)$

$y = t^2 = 6$

$\Rightarrow \boxed{t = \pm\sqrt{6}}$



$\frac{dx}{dt} = 3t^2 - 6, \quad \frac{dy}{dt} = 2t$

$\frac{dy}{dx} = \frac{2t}{3t^2 - 6}$

$At \quad t = -\sqrt{6}$

$\frac{dy}{dx} = \frac{-2\sqrt{6}}{12}$

$= -\frac{\sqrt{6}}{6}$

The tangent line: $\boxed{y = -\frac{\sqrt{6}}{6}x + 6}$

$At \quad t = \sqrt{6}$

$\frac{dy}{dx} = \frac{2\sqrt{6}}{12}$

$= \frac{\sqrt{6}}{6}$

The tangent line: $\boxed{y = \frac{\sqrt{6}}{6}x + 6}$

Exercise

Find the slope of the curve $x = f(t),\ y = g(t)$ at the given value of t. Define x and y as differentiable functions. $x^3 + 2t^2 = 9, \quad 2y^3 - 3t^2 = 4, \quad t = 2$

Solution

$x^3 + 2(2)^2 = 9$

$x^3 = 9 - 8 = 1$

$\rightarrow \underline{x = 1}$

$2y^3 - 3(2)^2 = 4$

$2y^3 = 4 + 12 = 16$

$y^3 = 8$

$\Rightarrow \underline{y = 2}$

$$x^3 + 2t^2 = 9 \quad \Rightarrow \quad 3x^2 \frac{dx}{dt} + 4t = 0$$

$$3x^2 \frac{dx}{dt} = -4t$$

$$\frac{dx}{dt} = -\frac{4t}{3x^2}$$

$$2y^3 - 3t^2 = 4 \quad \Rightarrow \quad 6y^2 \frac{dy}{dt} - 6t = 0$$

$$y^2 \frac{dy}{dt} = t$$

$$\frac{dy}{dt} = \frac{t}{y^2}$$

$$\frac{dy}{dx} = \frac{\frac{t}{y^2}}{-\frac{4t}{3x^2}} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= -\frac{3x^2}{4y^2}$$

$$\left.\frac{dy}{dx}\right|_{t=2} = -\frac{3(1)^2}{4(2)^2}$$

$$\underline{= -\frac{3}{16}}$$

Exercise

Find the slope of the curve $x = f(t),\ y = g(t)$ at the given value of t. Define x and y as differentiable functions. $\quad x + 2x^{3/2} = t^2 + t, \quad y\sqrt{t+1} + 2t\sqrt{y} = 4, \quad t = 0$

Solution

$$x + 2x^{3/2} = 0^2 + 0$$

$$x\left(1 + 2x^{1/2}\right) = 0$$

$$\rightarrow \boxed{x = 0} \qquad \cancel{x^{1/2} = -\frac{1}{2}} \quad (False)$$

$$y\sqrt{0+1} + 2(0)\sqrt{y} = 4$$

$$\Rightarrow \ y = 4$$

$$x + 2x^{3/2} = t^2 + t$$

$$\frac{dx}{dt} + 3x^{1/2}\frac{dx}{dt} = 2t + 1$$

$$\frac{dx}{dt}\left(1 + 3x^{1/2}\right) = 2t + 1$$

$$\frac{dx}{dt} = \frac{2t+1}{1 + 3x^{1/2}}$$

$$y\sqrt{t+1} + 2t\sqrt{y} = 4$$

$$\frac{dy}{dt}\sqrt{t+1} + \frac{1}{2}y(t+1)^{-1/2} + 2\sqrt{y} + 2t\left(\frac{1}{2}y^{-1/2}\right)\frac{dy}{dt} = 0$$

$$\frac{dy}{dt}\left(\sqrt{t+1} + \frac{t}{\sqrt{y}}\right) = -\frac{y}{2\sqrt{t+1}} - 2\sqrt{y}$$

$$\frac{dy}{dt}\left(\frac{\sqrt{t+1}\sqrt{y} + t}{\sqrt{y}}\right) = \frac{-y - 4\sqrt{t+1}\sqrt{y}}{2\sqrt{t+1}}$$

$$\frac{dy}{dt} = \frac{-y - 4\sqrt{t+1}\sqrt{y}}{2\sqrt{t+1}} \cdot \frac{\sqrt{y}}{\sqrt{t+1}\sqrt{y} + t}$$

$$\frac{dy}{dt} = \frac{-y\sqrt{y} - 4y\sqrt{t+1}}{2(t+1)\sqrt{y} + 2t\sqrt{t+1}}$$

$$\frac{dy}{dx} = \frac{-y\sqrt{y} - 4y\sqrt{t+1}}{2(t+1)\sqrt{y} + 2t\sqrt{t+1}} \cdot \frac{1 + 3\sqrt{x}}{2t+1} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{-4\sqrt{4} - 4(4)\sqrt{0+1}}{2(0+1)\sqrt{4} + 2(0)\sqrt{0+1}} \cdot \frac{1 + 3\sqrt{0}}{2(0)+1}$$

$$= -6$$

Exercise

Find the slope of the curve $x=f(t)$, $y=g(t)$ at the given value of t. Define x and y as differentiable functions. $\quad t=\ln(x-t), \quad y=te^t, \quad t=0$

Solution

$$0=\ln(x-0)$$

$$\ln x=0 \rightarrow \boxed{x=1}$$

$$y=(0)e^0 \Rightarrow \boxed{y=0}$$

$$t=\ln(x-t)$$

$$1=\frac{\frac{dx}{dt}-1}{x-t}$$

$$\frac{dx}{dt}-1=x-t$$

$$\frac{dx}{dt}=x-t+1$$

$$y=te^t$$

$$\frac{dy}{dt}=e^t+te^t$$

$$=e^t(1+t)$$

$$\frac{dy}{dx}=\frac{e^t(1+t)}{x-t+1} \qquad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}$$

$$\left.\frac{dy}{dx}\right|_{t=0}=\frac{e^0(1+0)}{1-0+1}$$

$$\boxed{=\frac{1}{2}}$$

Exercise

Find $\dfrac{d^2y}{dx^2}$ for $\quad x(t)=t-t^2 \quad y(t)=t-t^3$

Solution

$$\frac{dx}{dt}=1-2t$$

$$\frac{dy}{dt}=1-3t^2$$

$$\boxed{\frac{dy}{dx}=\frac{1-3t^2}{1-2t}}$$ $$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$$

$$\frac{dy'}{dt}=\frac{d}{dt}\left(\frac{-3t^2+1}{-2t+1}\right)$$ $$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)=\frac{(ae-bd)x^2+2(af-cd)x+bf-ce}{\left(dx^2+ex+f\right)^2}$$

$$=\boxed{\frac{-6t+2}{(1-2t)^2}}$$

$$\frac{d^2y}{dx^2}=\frac{-6t+2}{(1-2t)^2}\cdot\frac{1}{1-2t}$$ $$\frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}$$

$$=\boxed{\frac{-6t+2}{(1-2t)^3}}$$

Exercise

Find $\frac{d^2y}{dx^2}$ for $\quad x(t)=2\sec t \quad y(t)=4\tan t+2$

Solution

$$\frac{dx}{dt}=2\sec t\tan t$$

$$\frac{dy}{dt}=4\sec^2 t$$

$$\frac{dy}{dx}=\frac{4\sec^2 t}{2\sec t\tan t}$$ $$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$$

$$=\frac{2\sec t}{\tan t}$$

$$=\frac{2}{\sin t}$$

$$=\boxed{2\csc t}$$

$$\frac{dy'}{dt}=\frac{d}{dt}(2\csc t)$$

$$=\boxed{-2\csc t\cot t}$$

$$\frac{d^2y}{dx^2}=\frac{-2\csc t\cot t}{2\sec t\tan t}$$ $$\frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}$$

$$=-\frac{\csc t\cot t}{\frac{1}{\csc t\cot t}}$$

$$=\boxed{-\csc^2 t\cot^2 t}$$

Exercise

Find $\dfrac{d^2y}{dx^2}$ for $\quad x(t)=t^2+1 \quad y(t)=2t-1$

Solution

$$\frac{dx}{dt}=2t$$

$$\frac{dy}{dt}=2$$

$$\frac{dy}{dx}=\frac{2t}{2} \qquad\qquad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}$$

$$=t$$

$$\frac{dy'}{dt}=\frac{d}{dt}(t)$$

$$=1$$

$$\boxed{\frac{d^2y}{dx^2}=\frac{1}{2t}} \qquad\qquad \frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}$$

Exercise

Find $\dfrac{d^2y}{dx^2}$ for $\quad x(t)=2t^2-1 \quad y(t)=2t^3+t$

Solution

$$\frac{dx}{dt}=4t$$

$$\frac{dy}{dt}=6t^2$$

$$\frac{dy}{dx}=\frac{6t^2}{4t} \qquad\qquad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}$$

$$=\frac{3}{2}t$$

$$\frac{dy'}{dt}=\frac{d}{dt}\left(\frac{3}{2}t\right)$$

$$=\frac{3}{2}$$

$$\frac{d^2y}{dx^2}=\frac{3}{2}\cdot\frac{1}{4t} \qquad\qquad \frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}$$

$$=\frac{3}{8t}$$

Exercise

Find an equation of the line tangent to cycloid $x(t) = t - \sin t, \quad y(t) = 2 - \cos t$ at the points corresponding to $t = \frac{\pi}{6}$ and $t = \frac{2\pi}{3}$.

Solution

$$\frac{dx}{dt} = 1 - \cos t$$

$$\frac{dy}{dt} = \sin t$$

$$\frac{dy}{dx} = \frac{\sin t}{1 - \cos t} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

@ $t = \frac{\pi}{6}$

$$x = \frac{\pi}{6} - \sin\frac{\pi}{6}$$

$$= \frac{\pi}{6} - \frac{1}{2}$$

$$y = 2 - \cos\frac{\pi}{6}$$

$$= 2 - \frac{\sqrt{3}}{2}$$

$$\frac{dy}{dx} = \frac{\sin t}{1 - \cos t}\Bigg|_{t=\frac{\pi}{6}}$$

$$m = \frac{\sin\frac{\pi}{6}}{1 - \cos\frac{\pi}{6}}$$

$$= \frac{\frac{1}{2}}{1 - \frac{\sqrt{3}}{2}}$$

$$= \frac{1}{2 - \sqrt{3}} \cdot \frac{2 + \sqrt{3}}{2 + \sqrt{3}}$$

$$= 2 + \sqrt{3}$$

$$y = \left(2 + \sqrt{3}\right)\left(x - \frac{\pi}{6} + \frac{1}{2}\right) + 2 - \frac{\sqrt{3}}{2}$$

$$= \left(2 + \sqrt{3}\right)x - \left(2 + \sqrt{3}\right)\frac{\pi}{6} + 1 + \frac{\sqrt{3}}{2} + 2 - \frac{\sqrt{3}}{2}$$

$$= \left(2 + \sqrt{3}\right)x - \left(2 + \sqrt{3}\right)\frac{\pi}{6} + 3$$

@ $t=\frac{2\pi}{3}$

$$x = \frac{2\pi}{3} - \sin\frac{2\pi}{3}$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

$$y = 2 - \cos\frac{2\pi}{3}$$

$$= 2 + \frac{1}{2}$$

$$= \frac{5}{2}$$

$$m = \frac{\sin\frac{2\pi}{3}}{1-\cos\frac{2\pi}{3}}$$

$$= \frac{\frac{\sqrt{3}}{2}}{1+\frac{1}{2}}$$

$$= \frac{\sqrt{3}}{3}$$

$$y = \frac{\sqrt{3}}{3}\left(x - \frac{2\pi}{3} + \frac{\sqrt{3}}{2}\right) + \frac{5}{2}$$

$$= \frac{\sqrt{3}}{3}x - \frac{2\pi\sqrt{3}}{9} + \frac{1}{2} + \frac{5}{2}$$

$$= \frac{\sqrt{3}}{3}x - \frac{2\pi\sqrt{3}}{9} + 3$$

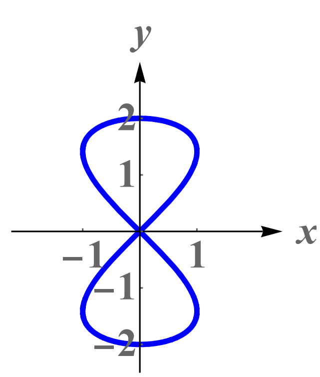
Exercise

Consider Lissajous curve, estimate the coordinates of the points on the curve at which there is

$$x = \sin 2t, \quad y = 2\sin t; \quad 0 \le t \le 2\pi$$

a) A horizontal tangent line

b) A vertical tangent line.



Solution

a) $\frac{dy}{dx} = \frac{2\cos t}{2\cos 2t} = 0$ $\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$\cos t = 0 \quad \to \quad t = \frac{\pi}{2}, \ \frac{3\pi}{2}$$

$t = \frac{\pi}{2} \rightarrow \begin{cases} x = \sin \pi = 0 \\ y = 2\sin \frac{\pi}{2} = 2 \end{cases}$

$\underline{(0,\ 2)}$

$t = \frac{3\pi}{2} \rightarrow \begin{cases} x = \sin 3\pi = 0 \\ y = 2\sin \frac{3\pi}{2} = -2 \end{cases}$

$\underline{(0,\ -2)}$

b) Vertical tangent line: $\cos 2t = 0 \quad \cos t \neq 0$

$\cos 2t = 0$

$2t = \frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2}.\ \frac{7\pi}{2}$

$t = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4}.\ \frac{7\pi}{4}$

$t = \frac{\pi}{4} \quad \begin{cases} x = 1 \\ y = \sqrt{2} \end{cases} \rightarrow \underline{(1,\ \sqrt{2})}$

$t = \frac{3\pi}{4} \quad \begin{cases} x = -1 \\ y = \sqrt{2} \end{cases} \rightarrow \underline{(-1,\ \sqrt{2})}$

$t = \frac{5\pi}{4} \quad \begin{cases} x = -1 \\ y = -\sqrt{2} \end{cases} \rightarrow \underline{(-1,\ -\sqrt{2})}$

$t = \frac{7\pi}{4} \quad \begin{cases} x = 1 \\ y = -\sqrt{2} \end{cases} \rightarrow \underline{(1,\ -\sqrt{2})}$

Exercise

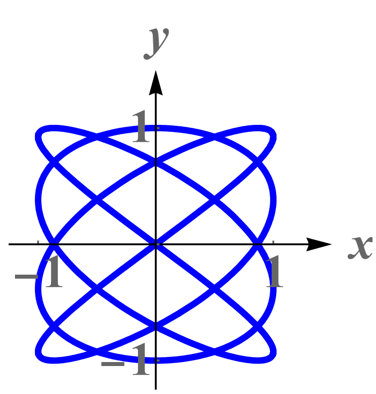
Consider Lissajous curve, estimate the coordinates of the points on the curve at which there is

$$x = \sin 4t, \quad y = \sin 3t; \quad 0 \leq t \leq 2\pi$$

a) A horizontal tangent line

b) A vertical tangent line.

Solution



a) $\frac{dy}{dx} = \frac{3\cos 3t}{4\cos 4t} = 0 \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$\cos 3t = 0 \rightarrow 3t = \frac{\pi}{2},\ \frac{3\pi}{2},\frac{5\pi}{2},\ \frac{7\pi}{2},\frac{9\pi}{2},\ \frac{11\pi}{2}$

$t = \frac{\pi}{6},\ \frac{\pi}{2},\frac{5\pi}{6},\ \frac{7\pi}{6},\frac{3\pi}{2},\ \frac{11\pi}{6}$

$$t=\frac{\pi}{6}\Rightarrow\ \begin{cases}x=-\frac{\sqrt{3}}{2}\\ y=1\end{cases}\ \rightarrow\ \boxed{\left(-\frac{\sqrt{3}}{2},\ 1\right)}$$

$$t=\frac{\pi}{2}\Rightarrow\ \begin{cases}x=0\\ y=-1\end{cases}\ \rightarrow\ \boxed{(0,\ -1)}$$

$$t=\frac{5\pi}{6}\Rightarrow\ \begin{cases}x=\frac{\sqrt{3}}{2}\\ y=-1\end{cases}\ \rightarrow\ \boxed{\left(\frac{\sqrt{3}}{2},\ -1\right)}$$

$$t=\frac{7\pi}{6}\Rightarrow\ \begin{cases}x=-\frac{\sqrt{3}}{2}\\ y=1\end{cases}\ \rightarrow\ \boxed{\left(-\frac{\sqrt{3}}{2},\ -1\right)}$$

$$t=\frac{3\pi}{2}\Rightarrow\ \begin{cases}x=0\\ y=1\end{cases}\ \rightarrow\ \boxed{(0,\ 1)}$$

$$t=\frac{11\pi}{6}\Rightarrow\ \begin{cases}x=\frac{\sqrt{3}}{2}\\ y=1\end{cases}\ \rightarrow\ \boxed{\left(\frac{\sqrt{3}}{2},\ 1\right)}$$

b) Vertical tangent line: $\cos 4t=0 \quad \cos 3t\neq 0$

$$\cos 4t=0\ \rightarrow\ 4t=\frac{(n+1)\pi}{2}$$

$$\boxed{t=\frac{(n+1)\pi}{8}}$$

$$t=\frac{(n+1)\pi}{8}\Rightarrow\ \begin{cases}x=\pm 1\\ y=\pm\sin\frac{3\pi}{8}\end{cases}$$

$$\boxed{\left(\pm 1,\ \pm\sin\frac{3\pi}{8}\right)}$$

Exercise

Find the area of the region $\quad x=2\sin^2\theta,\quad y=2\sin^2\theta\tan\theta,\quad 0\leq\theta<\frac{\pi}{2}$

Solution

$$dx=4\sin\theta\cos\theta\ d\theta$$

$$A=\int_0^{\pi/2} 2\sin^2\theta\tan\theta\ (4\sin\theta\cos\theta)\ d\theta \qquad A=\int_a^b y\,dx$$

$$= 8\int_0^{\pi/2} \sin^4\theta\, d\theta$$

$$= 2\int_0^{\pi/2} (1-\cos 2\theta)^2\, d\theta$$

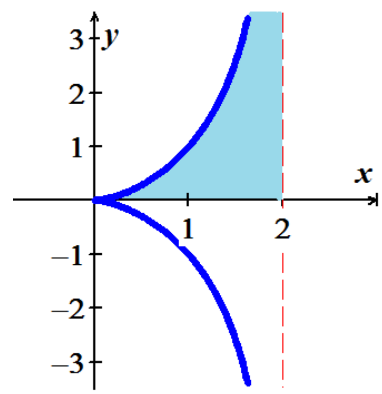
$$= 2\int_0^{\pi/2} \left(1-2\cos 2\theta+\cos^2 2\theta\right) d\theta$$

$$= 2\int_0^{\pi/2} \left(1-2\cos 2\theta+\frac{1}{2}+\frac{1}{2}\cos 4\theta\right) d\theta$$

$$= 2\left(\frac{3}{2}\theta-\sin 2\theta+\frac{1}{8}\sin 4\theta\right)\Big|_0^{\pi/2}$$

$$= 2\left(\frac{3\pi}{4}\right)$$

$$= \frac{3\pi}{2}\quad unit^2$$



Exercise

Find the area of the region $\quad x = 2\cot\theta,\quad y = 2\sin^2\theta,\quad 0 \le \theta < \pi$

Solution

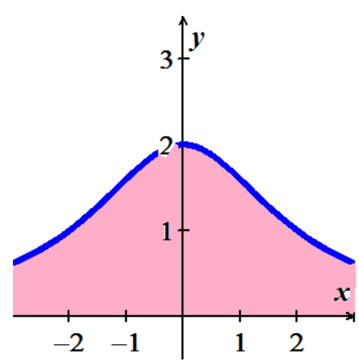
$$dx = -2\csc^2\theta\, d\theta$$

$$A = -4\int_0^{\pi} \sin^2\theta\csc^2\theta\, d\theta \qquad\qquad A = \int_a^b y\,dx$$

$$= -4\int_0^{\pi} d\theta$$

$$= |-4\theta|\Big|_0^{\pi}$$

$$= 4\pi\quad unit^2$$



Exercise

Find the area under one arch of the cycloid $x = a(t-\sin t),\quad y = a(1-\cos t)$

Solution

$$A = \int_0^{2\pi} y\, dx$$

$$= \int_0^{2\pi} a(1-\cos t)\, d\left[a(t-\sin t)\right] \qquad d\left[a(t-\sin t)\right] = a(1-\cos t)dt$$

$$= \int_0^{2\pi} a^2 (1-\cos t)^2\, dt$$

$$= a^2 \int_0^{2\pi} \left(1 - 2\cos t + \cos^2 t\right) dt$$

$$= a^2 \int_0^{2\pi} \left(1 - 2\cos t + \frac{1+\cos 2t}{2}\right) dt$$

$$= a^2 \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{1}{2}\cos 2t\right) dt$$

$$= a^2 \left(\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t \ \right|_0^{2\pi}$$

$$= a^2 \left(\frac{3}{2}(2\pi) - 2\sin(2\pi) + \frac{1}{4}\sin 2(2\pi) - 0\right)$$

$$= 3\pi a^2 \quad unit^2$$

Exercise

Find the area enclosed by the y–axis and the curve $x = t - t^2, \quad y = 1 + e^{-t}$

Solution

$$x = t - t^2 = 0 \quad \Rightarrow \quad t = 0,\ 1$$

$$A = \int_0^1 x\, dy$$

$$= \int_0^1 \left(t - t^2\right) d\left(1 + e^{-t}\right)$$

$$= \int_0^1 \left(t - t^2\right)\left(-e^{-t}\right) dt$$

		$\int e^{-t}\,dt$
+	$t-t^2$	$-e^{-t}$
−	$1-2t$	e^{-t}
+	-2	$-e^{-t}$

$$= -\int_0^1 \left(t-t^2\right)e^{-t}dt$$

$$= -\left(\left(t-t^2\right)\left(-e^{-t}\right)-(1-2t)\left(e^{-t}\right)-2\left(-e^{-t}\right)\ \right|_0^1$$

$$= -\left(e^{-t}\left(t^2-t\right)-e^{-t}(1-2t)+2e^{-t}\ \right|_0^1$$

$$= -\left[e^{-1}\left(1^2-1\right)-e^{-1}(1-2(1))+2e^{-1}-\left(e^{-0}\left(0^2-0\right)-e^{-0}(1-2(0))+2e^{-0}\right)\right]$$

$$= -\left[e^{-1}+2e^{-1}-(-1+2)\right]$$

$$= -\left(3e^{-1}-1\right)$$

$$= 1-3e^{-1}$$

$$\boxed{= 1-\frac{3}{e}\ unit^2}$$

Exercise

Find the area enclosed by the ellipse $x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi$

Solution

$$A = \int_0^{2\pi} y\,dx$$

$$= 2\left|\int_0^{\pi} y\,dx\right|$$

$$= 2\int_0^{\pi} b\sin t\ d\left(a\cos t\right)$$

$$= 2\int_0^{\pi} b\sin t\left(-a\sin t\right)\, dt$$

$$= -2ab\int_0^{\pi} \sin^2 t\, dt$$

$$= -2ab\int_0^{\pi} \left(\frac{1-\cos 2t}{2}\right) dt$$

$$= -ab\left(t - \frac{1}{2}\sin 2t\right. \Big|_0^{\pi}$$

$$= -ab\left(\pi - \frac{1}{2}\sin 2\pi - 0\right)$$

$$= \left|-\pi ab\right|$$

$$\underline{= \pi ab \ \ unit^2}$$

Exercise

Find the area of the closed curve *Ellipse* $\begin{cases} x = b\cos t \\ y = a\sin t \end{cases}$ $0 \le t \le 2\pi$

Solution

$$A = \int_0^{2\pi} ydx$$

$$= 2\left|\int_0^{\pi} ydx\right|$$

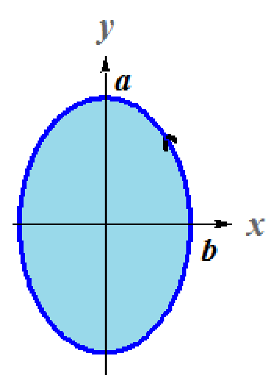
$$= 2\int_0^{\pi} a\sin t\ d\left(b\cos t\right)$$

$$= -2ab\int_0^{\pi} \sin^2 t\ dt$$

$$= -2ab\int_0^{\pi} \left(\frac{1-\cos 2t}{2}\right) dt$$

$$= -ab\left(t - \frac{1}{2}\sin 2t\right. \Big|_0^{\pi}$$

$$= -ab\left(\pi - \frac{1}{2}\sin 2\pi - 0\right)$$



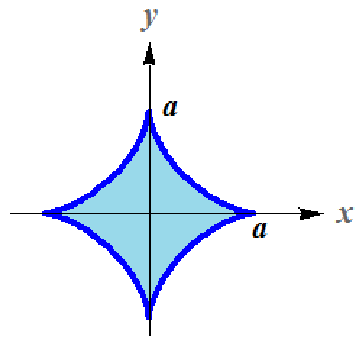
$= \left|-\pi ab\right|$

$\boxed{= \pi ab \quad unit^2}$

Exercise

Find the area of the closed curve *Astroid* $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases} \quad 0 \le t \le 2\pi$

Solution



$$A = \int_0^{2\pi} y\,dx$$

$$= 4\int_0^{\pi/2} a\sin^3 t\,\left|d\left(a\cos^3 t\right)\right|$$

$$= 12a^2\int_0^{\pi/2} \sin^4 t\cos^2 t\ dt$$

$$= 12a^2\int_0^{\pi/2} \left(\frac{1-\cos 2t}{2}\right)^2\left(\frac{1+\cos 2t}{2}\right)\ dt$$

$$= \frac{3}{2}a^2\int_0^{\pi/2} \left(1-2\cos 2t+\cos^2 2t\right)\left(1+\cos 2t\right)\ dt$$

$$= \frac{3}{2}a^2\int_0^{\pi/2} \left(1-\cos 2t-\cos^2 2t+\cos^3 2t\right)\ dt$$

$$= \frac{3}{2}a^2\int_0^{\pi/2} \left(\frac{1}{2}-\cos 2t-\cos 4t\right)\ dt+\frac{3}{2}a^2\int_0^{\pi/2} \cos^2 2t\cos 2t\ dt$$

$$= \frac{3}{2}a^2\left(\frac{1}{2}t-\frac{1}{2}\sin 2t-\frac{1}{4}\sin 4t\ \Big|_0^{\pi/2}+\frac{3}{4}a^2\int_0^{\pi/2} \left(1-\sin^2 2t\right)\ d\left(\sin 2t\right)\right.$$

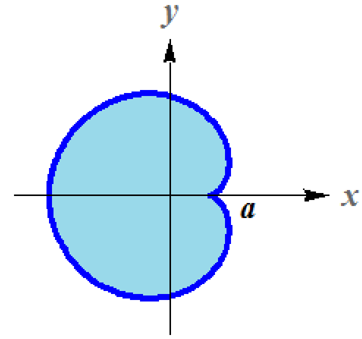
$$= \frac{3}{2}a^2\left(\frac{\pi}{4}\right)+\frac{3}{4}a^2\left(\sin 2t-\frac{1}{3}\sin^3 2t\ \Big|_0^{\pi/2}\right.$$

$$\boxed{= \frac{3}{8}\pi a^2 \quad unit^2}$$

Exercise

Find the area of the closed curve *Cardioid* $\begin{cases} x = 2a\cos t - a\cos 2t \\ y = 2a\sin t - a\sin 2t \end{cases}$ $0 \le t \le 2\pi$

Solution



$$A = \int_0^{2\pi} y\,dx$$

$$= 2\left|\int_0^{\pi} (2a\sin t - a\sin 2t)\, d(2a\cos t - a\cos 2t)\right|$$

$$= 2\left|\int_0^{\pi} (2a\sin t - a\sin 2t)(-2a\sin t + 2a\sin 2t)\, dt\right|$$

$$= 4a^2\left|\int_0^{\pi} (2\sin t - \sin 2t)(-\sin t + \sin 2t)\, dt\right|$$

$$= 4a^2\left|\int_0^{\pi} \left(-2\sin^2 t + 3\sin t\sin 2t - \sin^2 2t\right)\, dt\right| \qquad \sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$= 4a^2\left|\int_0^{\pi} \left(-1 + \cos 2t + \frac{3}{2}\cos t - \frac{3}{2}\cos 3t - \frac{1}{2} + \frac{1}{2}\cos 4t\right)\, dt\right|$$

$$= 4a^2\left|\int_0^{\pi} \left(-\frac{3}{2} + \cos 2t + \frac{3}{2}\cos t - \frac{3}{2}\cos 3t + \frac{1}{2}\cos 4t\right)\, dt\right|$$

$$= 4a^2\left|\left(-\frac{3}{2}t + \frac{1}{2}\sin 2t + \frac{3}{2}\sin t - \frac{1}{2}\sin 3t + \frac{1}{8}\sin 4t\right)\Big|_0^{\pi}\right|$$

$$= 4a^2\left|\left(-\frac{3\pi}{2}\right)\right|$$

$$\underline{= 6\pi a^2 \quad unit^2}$$

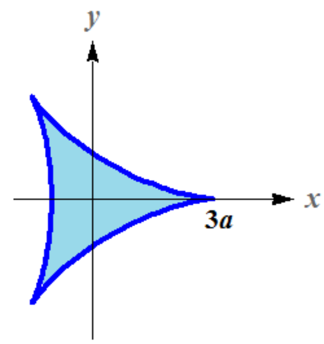
Exercise

Find the area of the closed curve *Deltoid* $\begin{cases} x = 2a\cos t + a\cos 2t \\ y = 2a\sin t - a\sin 2t \end{cases}$ $0 \le t \le 2\pi$

Solution

$$A = \int_0^{2\pi} y\,dx$$

$$= 2\left|\int_0^{\pi} (2a\sin t - a\sin 2t)\ d(2a\cos t + a\cos 2t)\right|$$



$$= 2\left|\int_0^{\pi} (2a\sin t - a\sin 2t)(-2a\sin t - 2a\sin 2t)\ dt\right|$$

$$= 4a^2\left|\int_0^{\pi} (2\sin t - \sin 2t)(\sin t + \sin 2t)\ dt\right|$$

$$= 4a^2\left|\int_0^{\pi} \left(2\sin^2 t + \sin t\sin 2t - \sin^2 2t\right)\ dt\right| \qquad \sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$= 4a^2\left|\int_0^{\pi} \left(1 - \cos 2t + \frac{1}{2}\cos t - \frac{1}{2}\cos 3t - \frac{1}{2} + \frac{1}{2}\cos 4t\right)\ dt\right|$$

$$= 4a^2\left|\int_0^{\pi} \left(\frac{1}{2} - \cos 2t + \frac{1}{2}\cos t - \frac{1}{2}\cos 3t + \frac{1}{2}\cos 4t\right)\ dt\right|$$

$$= 4a^2\left|\left(\frac{1}{2}t - \frac{1}{2}\sin 2t + \frac{1}{2}\sin t - \frac{1}{6}\sin 3t + \frac{1}{8}\sin 4t\ \right|_0^{\pi}\right|$$

$$\underline{= 2\pi a^2 \quad unit^2}$$

Exercise

Find the area of the closed curve *Hourglass* $\begin{cases} x = a\sin 2t \\ y = b\sin t \end{cases} \quad 0 \le t \le 2\pi$

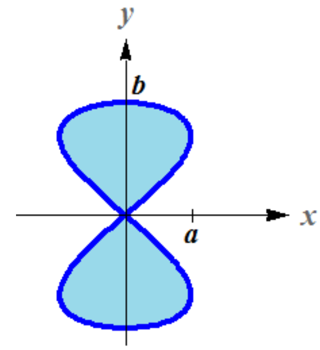
Solution

$$A = \int_0^{2\pi} y\,dx$$

$$= 2\left|\int_0^{\pi} (b\sin t)\ d(a\sin 2t)\right|$$

$$= 4ab\left|\int_0^{\pi} (\sin t\cos 2t)\ dt\right| \qquad \sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$= 2ab \left| \int_0^{\pi} (\sin 3t + \sin(-t))\, dt \right|$$



$$= 2ab \left| \int_0^{\pi} (\sin 3t - \sin t)\, dt \right|$$

$$= 2ab \left| \left(-\frac{1}{3}\cos 3t + \cos t \right|_0^{\pi} \right|$$

$$= 2ab \left| \frac{1}{3} - 1 + \frac{1}{3} - 1 \right|$$

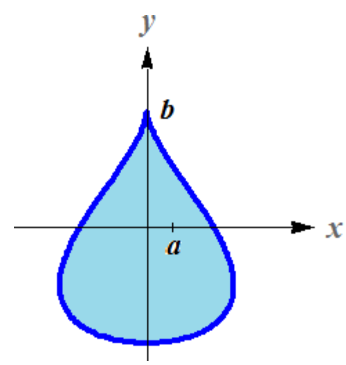
$$= \frac{8}{3} ab \quad unit^2$$

Exercise

Find the area of the closed curve *Teardrop* $\begin{cases} x = 2a\cos t - a\sin 2t \\ y = b\sin t \end{cases}$ $0 \le t \le 2\pi$

Solution

$$A = \int_0^{2\pi} y\, dx$$



$$= 2 \left| \int_{-\pi/2}^{\pi/2} (b\sin t)\, d(2a\cos t - a\sin 2t) \right|$$

$$= 2 \left| \int_{-\pi/2}^{\pi/2} (b\sin t)(-2a\sin t - 2a\cos 2t)\, dt \right|$$

$$= 4ab \left| \int_{-\pi/2}^{\pi/2} \left(\sin^2 t + \sin t\cos 2t\right) dt \right| \qquad \sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$= 2ab \left| \int_{-\pi/2}^{\pi/2} (1 - \cos 2t + \sin 3t - \sin t)\, dt \right|$$

$$= 2ab \left| \left(t - \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \cos t \right|_{-\pi/2}^{\pi/2} \right|$$

$$= 2ab\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= 2\pi ab \quad unit^2$$

Exercise

Find the lengths of the curves $x = \cos t, \quad y = t + \sin t, \quad 0 \le t \le \pi$

Solution

$$x = \cos t \quad \Rightarrow \quad \frac{dx}{dt} = -\sin t$$

$$y = t + \sin t \quad \Rightarrow \quad \frac{dy}{dt} = 1 + \cos t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\sin^2 t + (1+\cos t)^2}$$

$$= \sqrt{\sin^2 t + 1 + 2\cos t + \cos^2 t}$$

$$= \sqrt{2 + 2\cos t}$$

$$L = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{2 + 2\cos t}\, dt$$

$$= \sqrt{2}\int_0^{\pi} \sqrt{(1+\cos t)\frac{1-\cos t}{1-\cos t}}\, dt$$

$$= \sqrt{2}\int_0^{\pi} \sqrt{\frac{1-\cos^2 t}{1-\cos t}}\, dt$$

$$= \sqrt{2}\int_0^{\pi} \sqrt{\frac{\sin^2 t}{1-\cos t}}\, dt$$

$$= \sqrt{2}\int_0^{\pi} \frac{\sin t}{\sqrt{1-\cos t}}\, dt \qquad d(1-\cos t) = \sin t\, dt$$

$$= \sqrt{2}\int_0^{\pi} \frac{d(1-\cos t)}{\sqrt{1-\cos t}}$$

$$= \sqrt{2}\left(2\sqrt{1-\cos t}\right)\Big|_0^{\pi}$$

$$= 2\sqrt{2}\left(\sqrt{1-\cos\pi} - \sqrt{1-\cos 0}\right)$$

$$= 2\sqrt{2}\left(\sqrt{2} - 0\right)$$

$= 4 \quad unit$

$$L = \int_0^{\pi} \sqrt{4\sin^2 \frac{t}{2}}\, dt \qquad 2\sin^2 \frac{t}{2} = 1 + \cos t \qquad L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\int_0^{\pi} \sin \frac{t}{2}\, dt$$

$$= -4\cos \frac{t}{2} \Big|_0^{\pi}$$

$$= -4(0-1)$$

$$= 4 \quad unit$$

Exercise

Find the lengths of the curves $x = t^3, \quad y = \frac{3}{2}t^2, \quad 0 \le t \le \sqrt{3}$

Solution

$$x = t^3 \quad \Rightarrow \quad \frac{dx}{dt} = 3t^2$$

$$y = \frac{3}{2}t^2 \quad \Rightarrow \quad \frac{dy}{dt} = 3t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9t^4 + 9t^2}$$

$$= 3t\sqrt{t^2 + 1}$$

$$L = \int_0^{\sqrt{3}} 3t\, \sqrt{t^2 + 1}\, dt \qquad L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \frac{3}{2}\int_0^{\sqrt{3}} \left(t^2 + 1\right)^{1/2}\, d\left(t^2 + 1\right)$$

$$= \left(t^2 + 1\right)^{3/2} \Big|_0^{\sqrt{3}}$$

$$= 4^{3/2} - 1$$

$$= 7 \quad unit$$

Exercise

Find the lengths of the curves $x = 8\cos t + 8t\sin t, \quad y = 8\sin t - 8t\cos t, \quad 0 \le t \le \frac{\pi}{2}$

Solution

$$x = 8\cos t + 8t\sin t$$

$$\frac{dx}{dt} = -8\sin t + 8\sin t + 8t\cos t$$

$$= 8t\cos t$$

$$y = 8\sin t - 8t\cos t$$

$$\frac{dy}{dt} = 8\cos t - 8\cos t + 8t\sin t$$

$$= 8t\sin t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(8t\cos t)^2 + (8t\sin t)^2}$$

$$= \sqrt{(8t)^2\cos^2 t + (8t)^2\sin^2 t}$$

$$= 8t\sqrt{\cos^2 t + \sin^2 t} \qquad \cos^2 t + \sin^2 t = 1$$

$$= 8t$$

$$L = \int_0^{\pi/2} 8t\ dt \qquad L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= 4t^2 \Big|_0^{\pi/2}$$

$$= 4\left(\frac{\pi^2}{4} - 0\right)$$

$$= \pi^2 \quad unit$$

Exercise

Find the lengths of the curves $x = \ln(\sec t + \tan t) - \sin t, \quad y = \cos t, \quad 0 \le t \le \frac{\pi}{3}$

Solution

$$x = \ln(\sec t + \tan t) - \sin t$$

$$\frac{dx}{dt} = \frac{\sec t\tan t + \sec^2 t}{\sec t + \tan t} - \cos t$$

$$= \frac{\sec t(\tan t + \sec t)}{\sec t + \tan t} - \cos t$$

$$= \sec t - \cos t$$

$y = \cos t$

$$\frac{dy}{dt} = -\sin t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(\sec t - \cos t)^2 + (-\sin t)^2}$$

$$= \sqrt{\sec^2 t - 2\sec t \cos t + \cos^2 t + \sin^2 t}$$

$$= \sqrt{\sec^2 t - 2\frac{1}{\cos t}\cos t + 1}$$

$$= \sqrt{\sec^2 t - 2 + 1}$$

$$= \sqrt{\sec^2 t - 1}$$

$$= \sqrt{\tan^2 t}$$

$$= \tan t$$

$$L = \int_0^{\pi/3} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi/3} \tan t\ dt$$

$$= \int_0^{\pi/3} \frac{\sin t}{\cos t}\ dt$$

$$= \int_0^{\pi/3} -\frac{d(\cos t)}{\cos t}$$

$$= -\ln|\cos t|\ \Big|_0^{\pi/3}$$

$$= -\ln\cos\frac{\pi}{3} + \ln\cos 0$$

$$= -\ln\frac{1}{2} + \ln 1$$

$$= \ln 2 \quad unit$$

Exercise

Find the arc length of the Hypocycloid perimeter curve: $x = a\cos\theta, \quad y = a\sin\theta$

Solution

$$x = a\cos\theta \quad \rightarrow \quad \frac{dx}{d\theta} = -a\sin\theta$$

$$y = a\sin\theta \quad \rightarrow \quad \frac{dy}{d\theta} = a\cos\theta$$

$$L = 4\int_0^{\pi/2} \sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\; d\theta \qquad\qquad L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

$$= 4a\int_0^{\pi/2} d\theta$$

$$= 4a\ \theta \Big|_0^{\pi/2}$$

$$\underline{= 2\pi a \quad unit}$$

Exercise

Find the arc length of the circle circumference: $x = a\cos^3\theta, \quad y = a\sin^3\theta$

Solution

$$\frac{dx}{d\theta} = -3a\sin\theta\cos^2\theta$$

$$\frac{dy}{d\theta} = 3a\cos\theta\sin^2\theta$$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{9a^2\sin^2\theta\cos^4\theta + 9a^2\cos^2\theta\sin^4\theta}$$

$$= 3a\sin\theta\cos\theta\sqrt{\cos^2\theta + \sin^2\theta}$$

$$= 3a\sin\theta\cos\theta$$

$$L = 4\int_0^{\pi/2} 3a\sin\theta\cos\theta\; d\theta \qquad\qquad L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

$$= 6a\int_0^{\pi/2} \sin 2\theta\; d\theta$$

$$= -3a\cos 2\theta \Big|_0^{\pi/2}$$

$$= -3a(-1-1)$$

$$\underline{= 6a \quad unit}$$

Exercise

Find the arc length of the Cycloid arch: $x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta)$

Solution

$$x = a(\theta - \sin\theta) \quad \to \quad \frac{dx}{d\theta} = a(1 - \cos\theta)$$

$$y = a(1 - \cos\theta) \quad \to \quad \frac{dy}{d\theta} = a\sin\theta$$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2\left(1 - 2\cos\theta + \cos^2\theta\right) + a^2\sin^2\theta}$$

$$= a\sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= a\sqrt{2 - 2\cos\theta}$$

$$L = 2a\sqrt{2}\int_0^\pi \sqrt{1 - \cos\theta}\ d\theta \qquad\qquad L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= 2a\sqrt{2}\int_0^\pi \sqrt{1 - \cos\theta}\ \frac{\sqrt{1 + \cos\theta}}{\sqrt{1 + \cos\theta}}\ d\theta$$

$$= 2a\sqrt{2}\int_0^\pi \frac{\sin\theta}{\sqrt{1 + \cos\theta}}\ d\theta$$

$$= -2a\sqrt{2}\int_0^\pi (1 + \cos\theta)^{-1/2}\ d(1 + \cos\theta)$$

$$= -4a\sqrt{2}\sqrt{1 + \cos\theta}\ \Big|_0^\pi$$

$$= -4a\sqrt{2}\left(0 - \sqrt{2}\right)$$

$$= 8a \quad unit$$

Exercise

Find the arc length of the involute of a circle: $x = \cos\theta + \theta\sin\theta, \quad y = \sin\theta - \theta\cos\theta$

Solution

$$x = \cos\theta + \theta\sin\theta \quad \to \quad \frac{dx}{d\theta} = -\sin\theta + \sin\theta + \theta\cos\theta = \theta\cos\theta$$

$$y = \sin\theta - \theta\cos\theta \quad \to \quad \frac{dy}{d\theta} = \theta\sin\theta$$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}=\sqrt{\theta^2\cos^2\theta+\theta^2\sin^2\theta}$$

$$=\theta\sqrt{\cos^2\theta+\sin^2\theta}$$

$$=\theta$$

$$L=\int_0^{2\pi}\theta\,d\theta \qquad\qquad L=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$=\frac{1}{2}\theta^2\Big|_0^{2\pi}$$

$$\underline{=2\pi^2\quad unit}$$

Exercise

Find the arc length of $x=t^2,\quad y=t^3,\quad 0\le t\le 2$

Solution

$$\frac{dx}{dt}=2t$$

$$\frac{dy}{dt}=3t^2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\sqrt{4t^2+9t^4}$$

$$=t\sqrt{4+9t^2}$$

$$L=\int_0^2 t\sqrt{4+9t^2}\,dt \qquad\qquad L=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$=\frac{1}{18}\int_0^2\left(4+9t^2\right)^{1/2}\,d\left(4+9t^2\right)$$

$$=\frac{1}{27}\left(4+9t^2\right)^{3/2}\Big|_0^2$$

$$=\frac{1}{27}\left((40)^{3/2}-4^{3/2}\right)$$

$$=\frac{1}{27}\left(8(10)^{3/2}-8\right)$$

$$\underline{=\frac{8}{27}\left((10)^{3/2}-1\right)\ unit}$$

Exercise

Find the arc length of $x = 5\sin t, \quad y = 5\cos t, \quad -\frac{\pi}{3} \le t \le \frac{\pi}{2}$

Solution

$$\frac{dx}{dt} = 5\cos t$$

$$\frac{dy}{dt} = -5\sin t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{25\cos^2 t + 25\sin^2 t}$$

$$= \sqrt{25\left(\cos^2 t + \sin^2 t\right)}$$

$$= 5$$

$$L = \int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} 5\, dt \qquad\qquad L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 5t \Big|_{-\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= 5\left(\frac{\pi}{2} + \frac{\pi}{3}\right)$$

$$= \frac{25\pi}{6} \ unit$$

Exercise

Find the area of the surface generated by revolving the curve about each given axis.

$$x = \frac{1}{3}t^3, \quad y = t + 1, \quad 1 \le t \le 2, \quad y\text{-}axis$$

Solution

$$\frac{dx}{dt} = t^2, \quad \frac{dy}{dt} = 1$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{t^4 + 1}$$

$$S = 2\pi \int_1^2 \frac{1}{3}t^3 \sqrt{t^4 + 1}\, dt \qquad\qquad S = 2\pi \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$=\frac{1}{6}\pi\int_1^2 \sqrt{t^4+1}\ d\left(t^4+1\right)$$

$$=\frac{\pi}{9}\left(t^4+1\right)^{3/2}\Big|_1^2$$

$$=\frac{\pi}{9}\left(17^{3/2}-2^{3/2}\right)$$

$$=\frac{\pi}{9}\left(17\sqrt{17}-2\sqrt{2}\right)\quad unit^2$$

Exercise

Find the areas of the surfaces generated by revolving the curves

$$x=\frac{2}{3}t^{3/2},\quad y=2\sqrt{t},\quad 0\le t\le\sqrt{3};\quad x-axis$$

Solution

$$x=\frac{2}{3}t^{3/2}\quad\Rightarrow\quad \frac{dx}{dt}=t^{1/2}$$

$$y=2\sqrt{t}\quad\Rightarrow\quad \frac{dy}{dt}=t^{-1/2}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\sqrt{\left(t^{1/2}\right)^2+\left(t^{-1/2}\right)^2}$$

$$=\sqrt{t+t^{-1}}$$

$$=\sqrt{\frac{t^2+1}{t}}$$

$$A=2\pi\int_0^{\sqrt{3}} x\ ds$$

$$=2\pi\int_0^{\sqrt{3}} \frac{2}{3}t^{3/2}\sqrt{\frac{t^2+1}{t}}\ dt$$

$$=\frac{4\pi}{3}\int_0^{\sqrt{3}} t\sqrt{t^2+1}\ dt$$

$$=\frac{2\pi}{3}\int_0^{\sqrt{3}} \left(t^2+1\right)^{1/2}\ d\left(t^2+1\right)$$

$$=\frac{2\pi}{3}\left(\frac{2}{3}\left(t^2+1\right)^{3/2}\ \Big|_0^{\sqrt{3}}\right.$$

$$= \frac{4\pi}{9}\left(4^{3/2} - 1\right)$$

$$\boxed{= \frac{28\pi}{9} \quad unit^2}$$

Exercise

Find the areas of the surfaces generated by revolving the curves

$$x = t + \sqrt{2}, \quad y = \frac{t^2}{2} + \sqrt{2}t, \quad -\sqrt{2} \le t \le \sqrt{2}; \quad y - axis$$

Solution

$$x = t + \sqrt{2} \quad \Rightarrow \quad \boxed{\frac{dx}{dt} = 1}$$

$$y = \frac{t^2}{2} + \sqrt{2}t \quad \Rightarrow \quad \boxed{\frac{dy}{dt} = t + \sqrt{2}}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{1^2 + \left(t + \sqrt{2}\right)^2}$$

$$= \sqrt{1 + t^2 + 2\sqrt{2}t + 2}$$

$$= \sqrt{t^2 + 2\sqrt{2}\ t + 3}$$

$$A = 2\pi \int_{-\sqrt{2}}^{\sqrt{2}} xds$$

$$= 2\pi \int_{-\sqrt{2}}^{\sqrt{2}} \left(t + \sqrt{2}\right)\sqrt{t^2 + 2\sqrt{2}\ t + 3}\ dt$$

$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left(t^2 + 2\sqrt{2}\ t + 3\right)^{1/2}\ d\left(t^2 + 2\sqrt{2}\ t + 3\right)$$

$$= \pi\left(\frac{2}{3}\left(t^2 + 2\sqrt{2}\ t + 3\right)^{3/2}\ \Big|_{-\sqrt{2}}^{\sqrt{2}}\right.$$

$$= \frac{2\pi}{3}\left(9^{3/2} - 1\right)$$

$$\boxed{= \frac{52\pi}{9} \quad unit^2}$$

Exercise

Find the areas of the surfaces generated by revolving the curves $x = 2t, \quad y = 3t; \quad 0 \le t \le 3 \quad x\text{-}axis$

Solution

$$x = 2t \quad \rightarrow \quad \frac{dx}{dt} = 2$$

$$y = 3t \quad \rightarrow \quad \frac{dy}{dt} = 3$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{4+9}$$

$$= \sqrt{13}$$

$$S = 2\pi \int_0^3 (3t)\sqrt{13}\, dt \qquad\qquad S = 2\pi \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 3\pi\sqrt{13}\,\left(t^2 \,\right|_0^3$$

$$= 27\pi\sqrt{13}\ \ unit^2$$

Exercise

Find the areas of the surfaces generated by revolving the curves $x = 2t, \quad y = 3t; \quad 0 \le t \le 3 \quad y\text{-}axis$

Solution

$$x = 2t \quad \rightarrow \quad \frac{dx}{dt} = 2$$

$$y = 3t \quad \rightarrow \quad \frac{dy}{dt} = 3$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{13}$$

$$S = 2\pi \int_0^3 (2t)\sqrt{13}\, dt \qquad\qquad S = 2\pi \int_a^b x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\pi\sqrt{13}\,\left(t^2 \,\right|_0^3$$

$$= 18\pi\sqrt{13}\ \ unit^2$$

Exercise

Find the areas of the surfaces generated by revolving the curves $x = t, \quad y = 4 - 2t; \quad 0 \le t \le 2 \quad x\text{-}axis$

Solution

$$\frac{dx}{dt} = 1$$

$$\frac{dy}{dt} = -2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{5}$$

$$S = 2\pi \int_0^2 (4 - 2t)\sqrt{5}\; dt \qquad\qquad S = 2\pi \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

$$= 2\pi\sqrt{5}\left(4t - t^2 \right|_0^2$$

$$= 8\pi\sqrt{5}\ \ unit^2$$

Exercise

Find the areas of the surfaces generated by revolving the curves $x = t, \quad y = 4 - 2t; \quad 0 \le t \le 2 \quad y\text{-}axis$

Solution

$$\frac{dx}{dt} = 1$$

$$\frac{dy}{dt} = -2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{5}$$

$$S = 2\pi \int_0^2 (t)\sqrt{5}\; dt \qquad\qquad S = 2\pi \int_a^b x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

$$= \pi\sqrt{5}\left(t^2 \right|_0^2$$

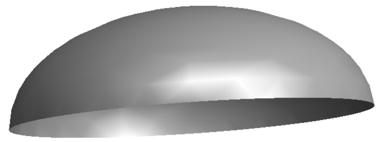
$$= 4\pi\sqrt{5}\ \ unit^2$$

Exercise

Find the area of the surface generated by revolving the curve about each given axis.

$$x = 5\cos\theta, \quad y = 5\sin\theta, \quad 0 \le \theta \le \frac{\pi}{2}, \quad y\text{-axis}$$

Solution



$$\frac{dx}{d\theta} = -5\sin\theta$$

$$\frac{dy}{d\theta} = 5\cos\theta$$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{25\sin^2\theta + 25\cos^2\theta}$$

$$= 5$$

$$S = 2\pi \int_0^{\pi/2} 5\cos\theta(5)\, d\theta \qquad\qquad S = 2\pi \int_a^b x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 50\pi \sin\theta \Big|_0^{\pi/2}$$

$$= 50\pi \ \ unit^2$$

Exercise

Find the area of the surface generated by revolving the curve about each given axis.

$$x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad 0 \le \theta \le \pi, \quad x\text{-axis}$$

Solution

$$x = a\cos^3\theta$$

$$\frac{dx}{d\theta} = -3a\sin\theta\cos^2\theta$$

$$y = a\sin^3\theta$$

$$\frac{dy}{d\theta} = 3a\cos\theta\sin^2\theta$$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{9a^2\sin^2\theta\cos^4\theta + 9a^2\cos^2\theta\sin^4\theta}$$

$$= 3a\sin\theta\cos\theta$$

$$S = 2\pi \int_0^{\pi/2} a\sin^3\theta(3a\sin\theta\cos\theta)\, d\theta \qquad\qquad S = 2\pi \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 12a^2\pi \int_0^{\pi/2} \sin^4\theta\, d(\sin\theta)$$

$$= \frac{12a^2\pi}{5}\sin^5\theta \Big|_0^{\pi/2}$$

$$\boxed{= \frac{12}{5}\pi a^2 \ \ unit^2}$$

Exercise

Find the area of the surface generated by revolving the curve about each given axis.

$$x = a\cos\theta, \quad y = b\sin\theta, \quad 0 \le \theta \le 2\pi$$

a) *x-axis* *b)* *y-axis*

Solution

$$x = a\cos\theta \quad \rightarrow \quad \frac{dx}{d\theta} = -a\sin\theta$$

$$y = b\sin\theta \quad \rightarrow \quad \frac{dy}{d\theta} = b\cos\theta$$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$$

a) $S = 4\pi \displaystyle\int_0^{\pi/2} b\sin\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$ $\qquad S = \pi\displaystyle\int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

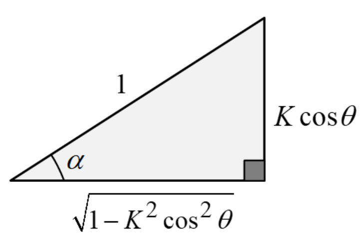
$$= 4\pi\int_0^{\pi/2} b\sin\theta\sqrt{a^2\left(1-\cos^2\theta\right) + b^2\cos^2\theta}\, d\theta$$

$$= 4\pi\int_0^{\pi/2} b\sin\theta\sqrt{a^2 + \left(b^2 - a^2\right)\cos^2\theta}\, d\theta$$

$$= 4\pi\int_0^{\pi/2} ab\sin\theta\sqrt{1 - \left(\frac{a^2-b^2}{a^2}\right)\cos^2\theta}\, d\theta$$

$$Let: \ K^2 = \frac{a^2-b^2}{a^2}$$

$$= 4\pi\int_0^{\pi/2} ab\sin\theta\sqrt{1-K^2\cos^2\theta}\, d\theta$$



$$K\cos\theta = \sin\alpha \qquad \sqrt{1-K^2\cos^2\theta} = \cos\alpha$$

$$-K\sin\theta d\theta = \cos\alpha d\alpha$$

$$= -\frac{4ab\pi}{K}\int_0^{\pi/2} \cos^2\alpha\ d\alpha$$

$$= -\frac{2ab\pi}{K}\int_0^{\pi/2} (1+\cos 2\alpha)\ d(\alpha)$$

$$= -\frac{2ab\pi}{K}\left(\alpha + \frac{1}{2}\sin 2\alpha\ \right|_0^{\pi/2}$$

$$= -\frac{2ab\pi}{K}\left(\arcsin(K\cos\theta) + K\cos\theta\sqrt{1-K^2\cos^2\theta}\ \right|_0^{\pi/2}$$

$$= -\frac{2a^2b\pi}{\sqrt{a^2-b^2}}\left(-\arcsin\left(\frac{\sqrt{a^2-b^2}}{a}\right) - \frac{\sqrt{a^2-b^2}}{a}\right)$$

$$e = \frac{\sqrt{a^2-b^2}}{a} = \frac{c}{a}: \quad \textit{eccentricity}$$

$$\boxed{= \frac{2ab\pi}{e}\left(e + \arcsin(e)\right) \quad unit^2} \qquad c = \sqrt{a^2-b^2}$$

b) $S = 4\pi\int_0^{\pi/2} a\cos\theta\ \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\ d\theta$ $\qquad S = \pi\int_a^b x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$

$$= 4a\pi\int_0^{\pi/2} \cos\theta\sqrt{\left(a^2-b^2\right)\sin^2\theta + b^2}\ d\theta$$

$$= 4a\pi\int_0^{\pi/2} \cos\theta\ \sqrt{c^2\sin^2\theta + b^2}\ d\theta$$

$$c\sin\theta = b\tan\alpha \qquad \sqrt{c^2\sin^2\theta + b^2} = b\sec\alpha$$

$$c\cos\theta d\theta = b\sec^2\alpha\ d\alpha$$

$$= 4a\pi\int_0^{\pi/2} \frac{b^2}{c}\sec^3\alpha\ d\alpha$$

$$u = \sec x \qquad dv = \sec^2 xdx$$

$$du = \sec x\tan xdx \qquad v = \tan x$$

$$\int \sec^3 xdx = \sec x\tan x - \int \tan x\left(\sec x\tan xdx\right)$$

$$= \sec x\tan x - \int \tan^2 x\sec xdx$$

$$= \sec x \tan x - \int \left(\sec^2 x - 1\right) \sec x \, dx$$

$$= \sec x \tan x - \int \left(\sec^3 x - \sec x\right) dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$2\int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

$$= \sec x \tan x + \ln\left|\sec x + \tan x\right|$$

$$\int \sec^3 x \, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right|$$

$$= \frac{2ab^2\pi}{c^2} \left(\sec\alpha \tan\alpha + \ln\left|\sec\alpha + \tan\alpha\right|\right.\left.\right|_0^{\pi/2}$$

$$= \frac{2ab^2\pi}{c}\left(\frac{c\sin\theta\sqrt{c^2\sin^2\theta + b^2}}{b^2} + \ln\left|\frac{c\sin\theta + \sqrt{c^2\sin^2\theta + b^2}}{b}\right|\right)\Bigg|_0^{\pi/2}$$

$$= \frac{2ab^2\pi}{c}\left(\frac{c\sqrt{c^2 + b^2}}{b^2} + \ln\left|\frac{c + \sqrt{c^2 + b^2}}{b}\right|\right)$$

$$= 2a\pi\sqrt{a^2 - b^2 + b^2} + \frac{2ab^2\pi}{c}\ln\left|\frac{\sqrt{a^2 - b^2} + \sqrt{a^2 - b^2 + b^2}}{b}\right|$$

$$= 2a^2\pi + \frac{2ab^2\pi}{\sqrt{a^2 - b^2}}\ln\left|\frac{\sqrt{a^2 - b^2} + a}{b}\right|$$

$$\boxed{= 2a^2\pi + \frac{2b^2\pi}{e}\ln\left|\frac{a(e+1)}{b}\right| \quad unit^2}$$

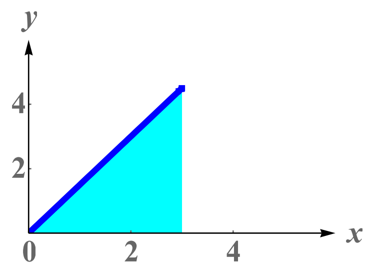
Exercise

Find the area of the surface generated by revolving the curve about each given axis.

$x = 2t, \quad y = 3t, \quad 0 \le t \le 3$

a) x-*axis*

b) *y-axis*

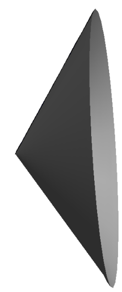


Solution

$$x = 2t \quad \rightarrow \frac{dx}{dt} = 2$$

$$y = 3t \quad \rightarrow \frac{dy}{dt} = 3$$

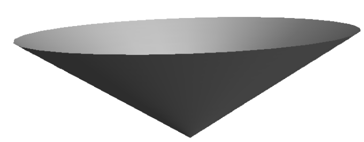
$$\textbf{a)} \quad S = 2\pi \int_0^3 3t\,\sqrt{4+9}\;dt \qquad\qquad S = 2\pi\int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$



$$= 3\pi\sqrt{13}\;\; t^2 \Big|_0^3$$

$$= 27\pi\sqrt{13}\;\; unit^2$$

$$\textbf{b)} \quad S = 2\pi \int_0^3 2t\,\sqrt{13}\;dt \qquad\qquad S = 2\pi\int_a^b x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$



$$= 2\pi\sqrt{13}\; t^2 \Big|_0^3$$

$$= 18\pi\sqrt{13}\;\; unit^2$$

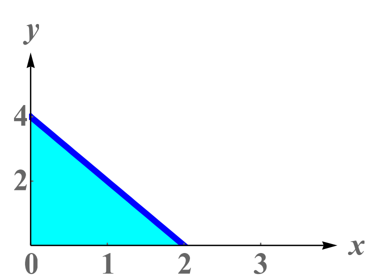
Exercise

Find the area of the surface generated by revolving the curve about each given axis.

$x = t, \quad y = 4 - 2t, \quad 0 \le t \le 2$

a) x-*axis*

b) *y-axis*



Solution

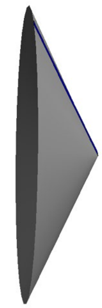
$$x = t \qquad \rightarrow \frac{dx}{dt} = 1$$

$$y = 4 - 2t \quad \rightarrow \frac{dy}{dt} = -2$$

a) $S = 2\pi \int_0^2 (4-2t)\sqrt{1+4}\ dt$ $\qquad S = 2\pi \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$

$= 2\pi\sqrt{5}\left(4t - t^2\right)\Big|_0^2$

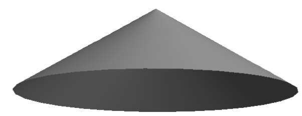
$= 8\pi\sqrt{5}\ \ unit^2$



b) $S = 2\pi \int_0^2 t\ \sqrt{5}\ dt$ $\qquad S = 2\pi \int_a^b x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$

$= \pi\sqrt{5}\ t^2\Big|_0^2$

$= 4\pi\sqrt{5}\ \ unit^2$



Exercise

Use the parametric equations $x = t^2\sqrt{3}$ *and* $y = 3t - \frac{1}{3}t^3$ to

a) Graph the curve on the interval $-3 \le t \le 3$.

b) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

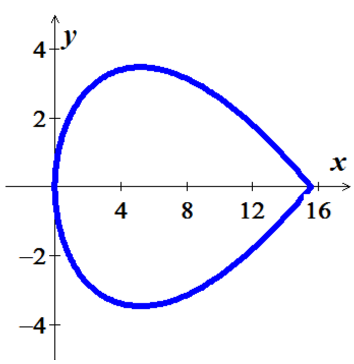
c) Find the equation of the tangent line at the point $\left(\sqrt{3},\ \frac{8}{3}\right)$

d) Find the length of the curve

e) Find the surface area generated by revolving the curve about the *x-axis*

Solution

a)



b) $\frac{dy}{dx} = \frac{3-t^2}{2t\sqrt{3}}$ $\qquad \frac{dy}{dx} = \frac{dy\ /\ dt}{dx\ /\ dt}$

$\frac{dy'}{dt} = \frac{1}{2\sqrt{3}}\frac{-2t^2 - 3 + t^2}{t^2}$

$$= -\frac{t^2+3}{2\sqrt{3}t^2}$$

$$\frac{d^2y}{dx^2} = -\frac{t^2+3}{2\sqrt{3}t^2} \cdot \frac{1}{2t\sqrt{3}}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$\boxed{= -\frac{t^2+3}{12t^3}}$$

c) $\left(\sqrt{3},\ \frac{8}{3}\right) \to x = t^2\sqrt{3} = \sqrt{3}$

$\Rightarrow \boxed{t=1}$

$$m = \left.\frac{dy}{dx}\right|_{t=1}$$

$$= \left.\frac{3-t^2}{2t\sqrt{3}}\right|_{t=1}$$

$$\boxed{= \frac{1}{\sqrt{3}}}$$

$$y = \frac{\sqrt{3}}{3}\left(x-\sqrt{3}\right) + \frac{8}{3}$$

$$\boxed{= \frac{\sqrt{3}}{3}x + \frac{5}{3}}$$

d) $\frac{dx}{dt} = 2t\sqrt{3} \qquad \frac{dy}{dt} = 3 - t^2$

$$L = \int_{-3}^{3} \sqrt{12t^2 + 9 - 6t^2 + t^4}\ dt$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= \int_{-3}^{3} \sqrt{\left(t^2+3\right)^2}\ dt$$

$$= \int_{-3}^{3} \left(t^2+3\right)\ dt$$

$$= \frac{1}{3}t^3 + 3t\ \Big|_{-3}^{3}$$

$$= 9 + 9 + 9 + 9$$

$$\boxed{= 36\ \ unit}$$

e) $S = 2\pi \int_0^3 \left(3t - \frac{1}{3}t^3\right)\left(t^2+3\right)dt$

$$S = 2\pi \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= 2\pi \int_0^3 \left(2t^3 - \frac{1}{3}t^5 + 9t\right) dt$$

$$= 2\pi \left(\frac{1}{2}t^4 - \frac{1}{18}t^6 + \frac{9}{2}t^2\right. \Big|_0^3$$

$$= 2\pi \left(\frac{81}{2} - \frac{81}{2} + \frac{81}{2}\right)$$

$$= 81\pi \ \ unit^2$$

Exercise

Use the parametric equations $x = a(\theta - \sin\theta)$ *and* $y = a(1 - \cos\theta)$ $a > 0$

a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

b) Find the equation of the tangent line at the point where $\theta = \frac{\pi}{6}$

c) Find all points (if any) of horizontal tangency.

d) Determine where the curve is concave upward or concave downward.

e) Find the length of one arc of the curve

Solution

a) $\frac{dx}{d\theta} = a(1 - \cos\theta)$

$\frac{dy}{d\theta} = a\sin\theta$

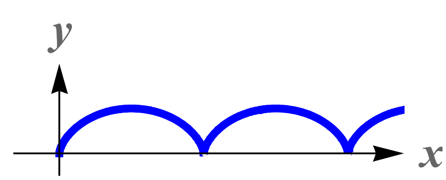
$$\frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta} \qquad\qquad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$\frac{dy'}{d\theta} = \frac{d}{d\theta}\left(\frac{\sin\theta}{1 - \cos\theta}\right)$$

$$= \frac{\cos\theta - \cos^2\theta - \sin^2\theta}{(1 - \cos\theta)^2}$$

$$= \frac{\cos\theta - 1}{(1 - \cos\theta)^2}$$

$$= \frac{-1}{1 - \cos\theta}$$



$$\frac{d^2y}{dx^2} = \left(\frac{-1}{1 - \cos\theta}\right)\frac{1}{a(1 - \cos\theta)} \qquad\qquad \frac{d^2y}{dx^2} = \frac{dy'/d\theta}{dx/d\theta}$$

$$= \frac{-1}{a(1 - \cos\theta)^2}$$

b) At $\theta = \frac{\pi}{6}$

$$x = a\left(\frac{\pi}{6} - \frac{1}{2}\right) \quad y = a\left(1 - \frac{\sqrt{3}}{2}\right)$$

$$m = \frac{dy}{dx}$$

$$= \frac{\sin\theta}{1-\cos\theta}\Bigg|_{\theta=\frac{\pi}{6}}$$

$$= \frac{\frac{1}{2}}{1-\frac{\sqrt{3}}{2}}$$

$$= \boxed{\frac{1}{2-\sqrt{3}}}$$

Tangent Line:

$$y = \frac{1}{2-\sqrt{3}}\left(x - \frac{\pi a}{6} + \frac{a}{2}\right) + a - \frac{a\sqrt{3}}{2} \qquad y = m\left(x - x_0\right) + y_0$$

$$\boxed{= \left(2+\sqrt{3}\right)\left(x - \frac{\pi a}{6} + \frac{a}{2}\right) + a - \frac{a\sqrt{3}}{2}}$$

c) $\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta} = 0$

$\sin\theta = 0 \quad \to \quad \boxed{\theta = (2n+1)\pi}$

$1 - \cos\theta \neq 0 \quad \to \theta = 2\pi n$

$x = a(2n+1)\pi, \quad y = 2a$

Points of horizontal tangency: $\boxed{(x,\ y) = \left(a(2n+1)\pi,\ 2a\right)}$

d) Concave downward on all open θ–intervals

$\ldots,\ (-2\pi,\ 0),\ (0,\ 2\pi),\ (2\pi,\ 4\pi),\ \ldots$

e) $\frac{dx}{d\theta} = a(1-\cos\theta)$

$\frac{dy}{d\theta} = a\sin\theta$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2\left(1 - 2\cos\theta + \cos^2\theta\right) + a^2\sin^2\theta}$$

$$= a\sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= a\sqrt{2 - 2\cos\theta}$$

$$L = 2a\sqrt{2}\int_0^{\pi}\sqrt{1-\cos\theta}\ d\theta \qquad\qquad L=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

$$= 2a\sqrt{2}\int_0^{\pi}\sqrt{1-\cos\theta}\ \frac{\sqrt{1+\cos\theta}}{\sqrt{1+\cos\theta}}\ d\theta$$

$$= 2a\sqrt{2}\int_0^{\pi}\frac{\sin\theta}{\sqrt{1+\cos\theta}}\ d\theta$$

$$= -2a\sqrt{2}\int_0^{\pi}(1+\cos\theta)^{-1/2}\ d(1+\cos\theta)$$

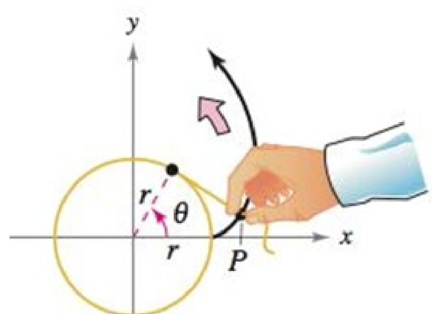
$$= -4a\sqrt{2}\ \sqrt{1+\cos\theta}\ \Big|_0^{\pi}$$

$$= -4a\sqrt{2}\left(0-\sqrt{2}\right)$$

$$= 8a\ \ unit$$

Exercise

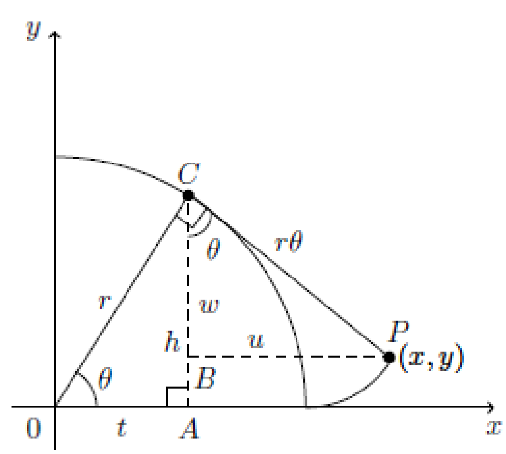
The involute of a circle is described by the endpoint P of a string that is held taut as it is unwound from a spool that does not turn.



Show that a parametric representation of the involute is

$$x = r(\cos\theta+\theta\sin\theta) \quad and \quad y = r(\sin\theta-\theta\cos\theta)$$

Solution



$\Delta OAC:\quad \cos\theta=\frac{t}{r}\quad \sin\theta=\frac{h}{r}$

$\Delta PBC:\quad \cos\theta=\frac{w}{r\theta}\quad \sin\theta=\frac{u}{r\theta}$

$x = t+u$

$\quad = r\cos\theta + r\theta\sin\theta$

$\quad = r(\cos\theta+\theta\sin\theta)$

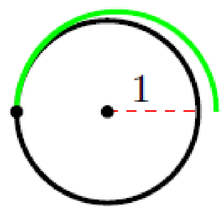
$y = h - w$

$= r\sin\theta - r\theta\cos\theta$

$\boxed{= r\left(\sin\theta - \theta\cos\theta\right)}$

Exercise

The figure shows a piece of string tied to a circle with a radius of one unit. The string is just long enough to reach the opposite side if the circle.



Find the area that is covered when the string is unwounded counterclockwise.

Solution

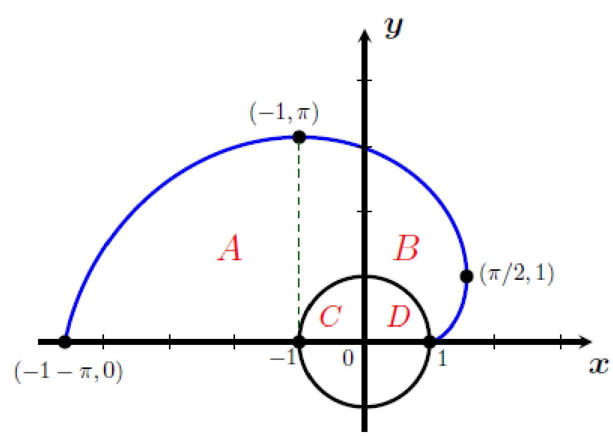
From previous exercise, we have

$$x = \cos\theta + \theta\sin\theta \quad \textit{and} \quad y = \sin\theta - \theta\cos\theta$$

At $(-1,\ \pi)$, the string is fully extended and has length x.

The area of region $\boldsymbol{A}$ is:

$$\frac{1}{4}\pi r^2 = \boxed{\frac{1}{4}\pi^3}$$



The area of region $\boldsymbol{C+D}$ is:

$$\frac{1}{2}\pi r^2 = \boxed{\frac{\pi}{2}}$$

$$\frac{dx}{d\theta} = -\sin\theta + \sin\theta + \theta\cos\theta$$

$$\boxed{= \theta\cos\theta}$$

$$\frac{dx}{d\theta} = \theta\cos\theta = 0$$

$$\boxed{= \theta\cos\theta}$$

The area of the region $B+C+D$ is given by

$$\int_{\pi}^{\pi/2} y\,dx - \int_{0}^{\pi/2} y\,dx = \int_{\pi}^{0} y\,dx$$

$$A_2 = \int_{\pi}^{0} \left(\sin\theta - \theta\cos\theta\right)\theta\cos\theta\ d\theta$$

$$= \int_{\pi}^{0} \left(\theta \cos\theta \sin\theta - \theta^2 \cos^2\theta \right) d\theta$$

$$= \int_{\pi}^{0} \left(\frac{1}{2}\theta \sin 2\theta - \frac{1}{2}\theta^2 - \frac{1}{2}\theta^2 \cos 2\theta \right) d\theta$$

		$\int \sin 2\theta$
+	θ	$-\frac{1}{2}\cos 2\theta$
−	1	$-\frac{1}{4}\sin 2\theta$

		$\int \cos 2\theta$
+	$\frac{1}{2}\theta^2$	$\frac{1}{2}\sin 2\theta$
−	θ	$-\frac{1}{4}\cos 2\theta$
+	1	$-\frac{1}{8}\sin 2\theta$

$$= -\frac{1}{4}\theta \cos 2\theta + \frac{1}{8}\sin 2\theta - \frac{1}{6}\theta^3 - \frac{1}{4}\theta^2 \sin 2\theta - \frac{1}{4}\theta \cos 2\theta + \frac{1}{8}\sin 2\theta \Big|_{\pi}^{0}$$

$$= \frac{\pi}{4} + \frac{\pi^3}{6} + \frac{\pi}{4}$$

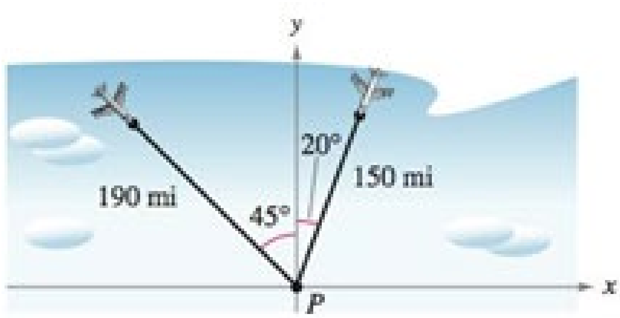
$$= \frac{\pi^3}{6} + \frac{\pi}{2}$$

Total area covered $= 2\left(\frac{\pi^3}{4} + \frac{\pi^3}{6} + \frac{\pi}{2} - \frac{\pi}{2} \right)$

$$= \frac{5\pi^3}{6} \ unit^2$$

Exercise

An Air traffic controller spots two planes at the same altitude flying toward each other.



Their flight paths are 20° and 315°. One plane is 150 *miles* from point *P* with a speed of 375 *miles per hour*. The other is 190 *miles* from point *P* with a speed of 450 *miles per hour*.

a) Find parameteric equations for the path of each plane where *t* is the time in *hours*, with $t = 0$ corresponding to the time at which the air traffic controller spots the planes.

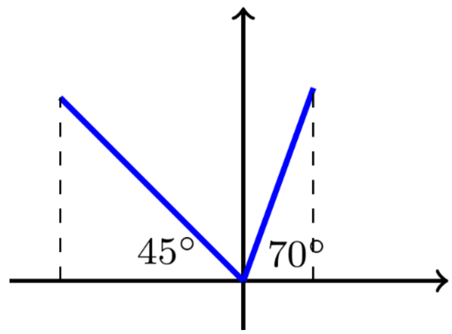
b) Use part (*a*) to write the distance between the planes as a function of *t*.

c) Graph the function in part (*b*).
d) When the distance between the planes be minimum?
e) If the planes must keep a separation of at least 3 *miles*, is the requirement met?

Solution

a) ***First*** Plane:

Given: $\theta_1 = 90° - 20° = 70° \quad d_1 = 150 \quad v_1 = 375$



$$\begin{cases} x_1 = (150 - 375t)\cos 70° \\ y_1 = (150 - 375t)\sin 70° \end{cases}$$

$$\begin{cases} x_1 = 75(2 - 5t)\cos 70° \\ y_1 = 75(2 - 5t)\sin 70° \end{cases}$$

Second Plane:

Given: $\theta_2 = 45° \quad d_2 = 190 \quad v_2 = 450$

$$\begin{cases} x_2 = -(190 - 450t)\cos 45° \\ y_2 = (190 - 450t)\sin 45° \end{cases}$$

$$\begin{cases} x_2 = -10(19 - 45t)\left(\frac{\sqrt{2}}{2}\right) \\ y_2 = 10(19 - 45t)\left(\frac{\sqrt{2}}{2}\right) \end{cases}$$

$$\begin{cases} x_2 = -5\sqrt{2}(19 - 45t) \\ y_2 = 5\sqrt{2}(19 - 45t) \end{cases}$$

b) $d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$

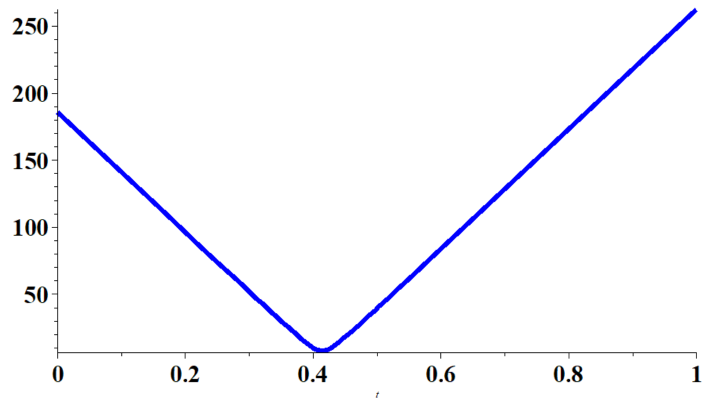
$$= \sqrt{\left(-5\sqrt{2}(19 - 45t) - 75(2 - 5t)\cos 70°\right)^2 + \left(5\sqrt{2}(19 - 45t) - 75(2 - 5t)\sin 70°\right)^2}$$

At $t = 0$

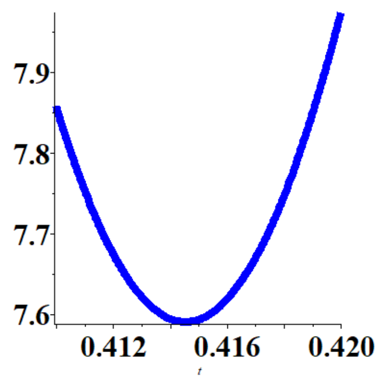
$$d = \sqrt{190^2 + 150^2 - 2(190)(150)\cos 65°}$$

$$\approx 185.77$$

c)



d) Using software:



t	***d***
0.4100000000	7.8578282443
0.4105000000	7.8029045315
0.4110000000	7.7540582513
0.4115000000	7.7114048932
0.4120000000	7.6750477077
0.4125000000	7.6450765223
0.4130000000	7.6215666744
0.4135000000	7.6045780909
0.4140000000	7.5941545371
0.4145000000	**7.5903230599**
0.4150000000	7.5930936382
0.4155000000	7.6024590542
0.4160000000	7.6183949864
0.4165000000	7.6408603242
0.4170000000	7.6697976925
0.4175000000	7.7051341728
0.4180000000	7.7467821981
0.4185000000	7.7946405981
0.4190000000	7.8485957667
0.4195000000	7.9085229209
0.4200000000	7.9742874225

The minimum distance is 7.59 ***miles*** when $t = 0.4145$

e) Yes, the planes must keep a separation of at least 3 *miles.*