Section 1.3 - Quadratic Equations

Basic Complex Number

$$i^2 = -1 \qquad \Rightarrow\ i = \sqrt{-1} \qquad \Rightarrow\ \sqrt{-1} = i$$

The number i is called the ***imaginary unit***.

Example

$\sqrt{-8} = i2\sqrt{2} \qquad 2i\sqrt{2}$

$$\begin{aligned}\sqrt{-7}\sqrt{-7} &= i\sqrt{7}\ i\sqrt{7}\\ &= i^2\left(\sqrt{7}\right)^2\\ &= -7\end{aligned}$$

Complex number is written in a form: $z = a + ib$

a is the real part

b is the imaginary part

Conjugate of a complex number $a + bi$ is $a - bi$

A ***quadratic equation*** in *x* is an equation that can be written in the general form:

$$ax^2 + bx + c = 0 \qquad \text{where } a,\ b, \text{ and c are real numbers,}$$

$$4x^2 - 3x + 2 = 0 \qquad a = 4 \quad b = -3 \quad c = 2$$

Solving Quadratic Equations by Factoring

The Zero-Product Principle

If *AB* = 0 then *A* = 0 or *B* = 0.

Example

Solve $6x^2 + 7x - 3 = 0$

Solution

$(3x-1)(2x+3) = 0$

$3x - 1 = 0 \qquad 2x + 3 = 0$

$x = \frac{1}{3} \qquad x = -\frac{3}{2}$

The Square Root Property

If *u* is an algebraic expression and *d* is a nonzero real number, then $u^2 = d$ has exactly two solutions:

$$\text{If } u^2 = d, \text{ then } u = \sqrt{d} \text{ or } u = -\sqrt{d}$$

Equivalently,

$$\text{If } u^2 = d \Rightarrow u = \pm\sqrt{d}.$$

Example

Solve $3x^2 - 21 = 0$

Solution

$3x^2 = 21$

$x^2 = 7$

$\Rightarrow \boxed{x = \pm\sqrt{7}}$

Example

Solve $5x^2 + 45 = 0$

Solution

$5x^2 = -45$

$x^2 = -9$

$x = \pm\sqrt{-9}$

$\Rightarrow \boxed{x = \pm 3i}$

Example

Solve $(x+5)^2 = 11$

Solution

$x + 5 = \pm\sqrt{11}$

$\Rightarrow \boxed{x = -5 \pm \sqrt{11}}$

Completing the Square

If $x^2 + bx$ is a binomial, then by ***adding*** $\left(\frac{b}{2}\right)^2$ which is the square of half the coefficient of *x*. a perfect square trinomial will result. That is.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2 \qquad\qquad x^2 + bx + \left(\frac{1}{2}b\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Example

Solve: $x^2 + 4x - 1 = 0$

Solution

$$x^2 + 4x \quad = 1$$

$$x^2 + 4x + \left(\frac{4}{2}\right)^2 = 1 + \left(\frac{4}{2}\right)^2$$

$$x^2 + 4x + (2)^2 = 1 + 4$$

$$(x + 2)^2 = 5$$

$$x + 2 = \pm\sqrt{5}$$

$$x = -2 \pm \sqrt{5}$$

Quadratic Formula

(*Using Completing the Square*)

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{1}{2}\frac{b}{a}\right)^2 = -\frac{c}{a} + \left(\frac{1}{2}\frac{b}{a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$= \frac{b^2}{4a^2} - \frac{c}{a}$$

$$= \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

*** $ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\begin{cases} b^2 - 4ac > 0 & \rightarrow \text{ 2 Real numbers} \\ b^2 - 4ac < 0 & \rightarrow \text{ 2 Complex numbers} \\ b^2 - 4ac = 0 & \rightarrow \text{ One solution (repeated)} \end{cases}$$

Example

Solve: $2x^2 + 2x - 1 = 0$

Solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \Rightarrow a = 2 \quad b = 2 \quad c = -1$$

$$= \frac{-2 \pm \sqrt{2^2 - 4(2)(-1)}}{2(2)}$$

$$= \frac{-2 \pm \sqrt{4+8}}{4}$$

$$= -\frac{2}{4} \pm \frac{\sqrt{12}}{4}$$

$$= -\frac{1}{2} \pm \frac{2\sqrt{3}}{4}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$$

$$= \frac{-2 \pm \sqrt{12}}{4}$$

$$= \frac{-2 \pm 2\sqrt{3}}{4}$$

$$= \frac{2\left(-1 \pm \sqrt{3}\right)}{4}$$

$$= \frac{-1 \pm \sqrt{3}}{2}$$

Example

Solve $x^2 - 4x = -2$

Solution

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} \qquad \Rightarrow a = 1 \quad b = -4 \quad c = 2 \;\; x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16-8}}{2}$$

$$= \frac{4 \pm \sqrt{8}}{2}$$

$$= \frac{4 \pm 2\sqrt{2}}{2}$$

$$= \frac{2\left(2 \pm \sqrt{2}\right)}{2}$$

$$= 2 \pm \sqrt{2}$$

Example

Solve: $x^2 - 2x + 2 = 0$

Solution

$\Rightarrow a = 1 \quad b = -2 \quad c = 2$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2}{2} \pm \frac{\sqrt{-4}}{2}$$

$$= 1 \pm \frac{2i}{2}$$

$$\boxed{= 1 \pm i}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

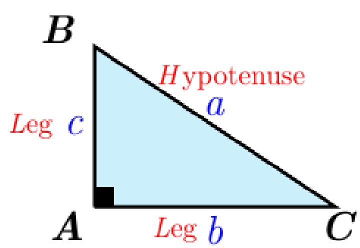
$$= \frac{2 \pm 2i}{2}$$

$$= \frac{2(1 \pm i)}{2}$$

$$\boxed{= 1 \pm i}$$

Pythagorean Theorem

The sum of the squares of the lengths of the legs of a right triangle equals the square of the length of the hypotenuse. If the legs have lengths a and b, and the hypotenuse has length c, then:

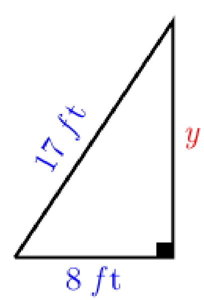


$$a^2 = b^2 + c^2$$

Example

A ladder that is 17 *feet* long is 8 *feet* from the base of a wall. How far up the wall does the ladder reach?

Solution



$8^2 + y^2 = 17^2$

$y^2 = 17^2 - 8^2$

$y = \sqrt{17^2 - 8^2} = \boxed{15\ ft}$

Height of a Projected Object *(Position Function)*

An object that is falling or vertically projected into the air has its height above the ground, $s(t)$, in feet, given by

$$s(t) = -16t^2 + v_0 t + s_0$$

v_0 is the original velocity (initial velocity) of the object, in feet per second

t is the time that the object is in motion, in second

s_0 is the original height (initial height) of the object, in feet

Example

If a projectile is shot vertically upward from the ground with an initial velocity of 100 $ft/\sec$, neglecting air resistance, its height s (in *feet*) above the ground t seconds after projection is given by

$$s = -16t^2 + 100t$$

a) After how many seconds will it be 50 *feet*. above the ground?

b) How long will it take for the projectile to return to the ground?

Solution

a) *After how many seconds will it be 50 feet above the ground?*

$$50 = -16t^2 + 100t$$

$$16t^2 - 100t + 50 = 0$$

$$8t^2 - 50t + 25 = 0$$

$$t = \frac{-(-50) \pm \sqrt{(-50)^2 - 4(8)(25)}}{2(8)}$$

$$= \frac{50 \pm \sqrt{1700}}{16}$$

$$t = \frac{50 - 10\sqrt{17}}{16} = \frac{25 - 5\sqrt{17}}{8} \approx 0.55$$

$$t = \frac{50 + 10\sqrt{17}}{16} = \frac{25 + 5\sqrt{17}}{8} \approx 5.70$$

b) *How long will it take for the projectile to return to the ground?*

$$0 = -16t^2 + 100t$$

$$0 = -4t(4t - 25)$$

$$-4t = 0 \qquad 4t - 25 = 0$$

$$t = 0 \qquad t = \frac{25}{4} = 6.25$$

Exercises *Section* 1.3 - Quadratic Equations

Solve

1. $x^2 = -25$
2. $x^2 = 49$
3. $9x^2 = 100$
4. $4x^2 + 25 = 0$
5. $5x^2 + 35 = 0$
6. $5x^2 - 45 = 0$
7. $(x-4)^2 = 12$
8. $(x+3)^2 = -16$
9. $(x-2)^2 = -20$
10. $(4x+1)^2 = 20$
11. $x^2 - 6x = -7$
12. $-6x^2 = 3x + 2$
13. $3x^2 + 2x = 7$
14. $3x^2 + 6 = 10x$
15. $5x^2 + 2 = x$
16. $5x^2 = 2x - 3$
17. $x^2 + 8x + 15 = 0$
18. $x^2 + 5x + 2 = 0$
19. $x^2 + x - 12 = 0$
20. $x^2 - 2x - 15 = 0$
21. $x^2 - 4x - 45 = 0$
22. $x^2 - 6x - 10 = 0$
23. $2x^2 + 3x - 4 = 0$
24. $x^2 - x + 8 = 0$
25. $2x^2 - 13x = 1$
26. $r^2 + 3r - 3 = 0$
27. $x^3 + 8 = 0$
28. $4x^2 - 12x + 9 = 0$
29. $9x^2 - 30x + 25 = 0$
30. $x^2 - 14x + 49 = 0$
31. $x^2 - 8x + 16 = 0$
32. $x^2 + 6x + 13 = 0$
33. $2x^2 - 2x + 13 = 0$
34. $x^2 + 2x + 29 = 0$
35. $4x^2 + 4x + 13 = 0$
36. $x^2 - 2x + 26 = 0$
37. $9x^2 - 4x + 20 = 0$
38. $x^2 + 6x + 21 = 0$
39. $9x^2 - 12x - 49 = 0$
40. $x(x-3) = 18$
41. $x(x-4) - 21 = 0$
42. $(x-1)(x+4) = 14$
43. $(x-3)(x+8) = -30$
44. $x(x+8) = 16(x-1)$
45. $x(x+9) = 4(2x+5)$
46. $(x+1)^2 = 2(x+3)$
47. $(x+1)^2 - 5(x+2) = 3x + 7$
48. $x(8x+1) = 3x^2 - 2x + 2$

Solve

49. $x^2 + 6x - 7 = 0$
50. $x^2 - 6x - 7 = 0$
51. $3x^2 + 4x - 7 = 0$
52. $3x^2 - 4x - 7 = 0$
53. $3x^2 - x - 2 = 0$
54. $3x^2 + x - 2 = 0$
55. $2x^2 + 3x - 5 = 0$
56. $2x^2 - 3x - 5 = 0$
57. $x^2 - 3x - 4 = 0$
58. $x^2 + 3x - 4 = 0$
59. $x^2 + 2x + 1 = 0$
60. $4x^2 - x - 5 = 0$

61. Solve for the specified variable $A = \frac{\pi d^2}{4}$, *for d*

62. Solve for the specified variable $rt^2 - st - k = 0 \ (r \neq 0)$, *for t*

63. A rectangular park is 6 *miles* long and 2 *miles* wide. How long is a pedestrian route that runs diagonally across the park?

64. What is the width of a 25-*inch* television set whose height is 15 *inches*?

65. The length of a rectangular sign is 3 *feet* longer than the width. If the sign's area is 54 square *feet*, find its length and width.

66. A rectangular parking lot has a length that is 3 *yards* greater than the width. The area of the parking lot is 180 square *yards*, find the length and the width.

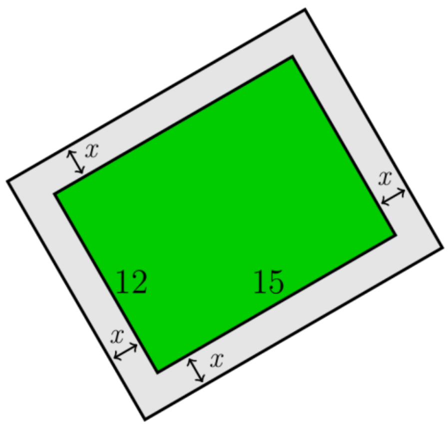
67. Each side of a square is lengthened by 3 *inches*. The area of this new, larger square is 64 square *inches*. Find the length of a side of the original square.

68. Each side of a square is lengthened by 2 *inches*. The area of this new, larger square is 36 square *inches*. Find the length of a side of the original square.

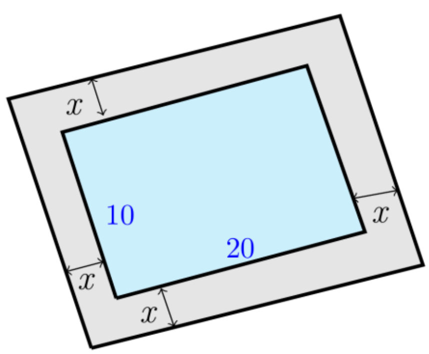
69. One number is 5 greater than another. The product of the numbers is 36. Find the numbers.

70. One number is 6 less than another. The product of the numbers is 72. Find the numbers.

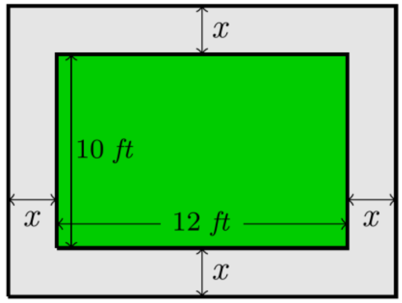
71. A vacant rectangular lot is being turned into a community vegetable garden measuring 15 *meters* by 12 *meters*. A path of uniform width is to surround the garden. If the area of the garden and path combined is 378 *square meters*, find the width of the path.



72. A pool measuring 10 *m* by 20 *m* is surrounded by a path of uniform width. If the area of the pool and the path combined is 600 m^2, what is the width of the path?



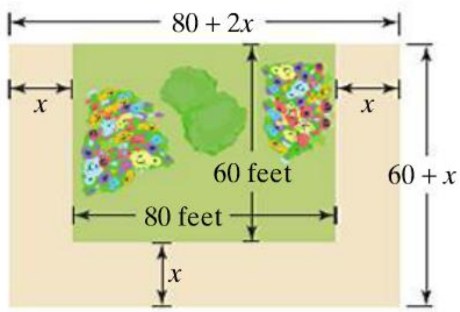
73. You put in flower bed measuring 10 *feet* by 12 *feet*. You plan to surround the bed with uniform border of low-growing plants.



a) Write a polynomial that describes the area of the uniform border that surrounds your flowers.

b) The low growing plants surrounding the flower bed require 1 square *foot* each when mature. If you have 168 of these plants, how wide a strip around the flower bed should you prepare for the border?

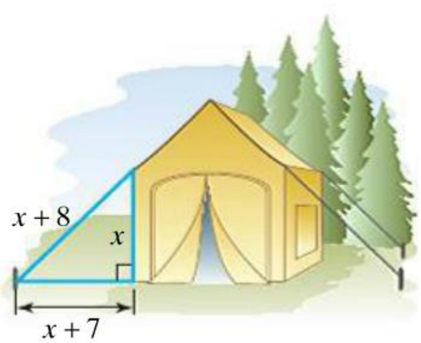
74. A rectangular garden measures 80 *feet* by 60 *feet*. A large path of uniform width is to be added along both shorter sides and one longer side of the garden. The landscape designer doing the work wants to double the garden's area with the addition of this path. How wide should the path be?



75. The length of a rectangular poster is 1 *foot* more than the width, and a diagonal of the poster is 5 *feet*. Find the length and the width.

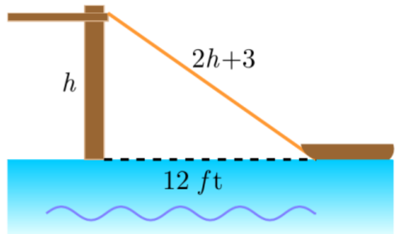
76. One leg of a right triangle is 7 *cm* less than the length of the other leg. The length of the hypotenuse is 13 *cm*. find the lengths of the legs.

77. A tent with wires attached to help stabilize it, as shown below. The length of each wire is 8 *feet* greater than the distance from the ground to where it is attached to the tent.

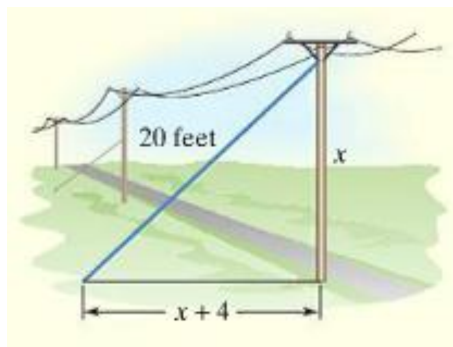


The distance from the base of the tent to where the wire is anchored exceeds this height by 7 feet, Find the length of each wire used to stabilize the tent.

78. A boat is being pulled into a dock with a rope attached to the boat at water level. Where the boat is 12 *feet*. from the dock, the length of the rope from the boat to the dock is 3 *feet*. longer than twice the height of the dock above the water. Find the height of the dock.

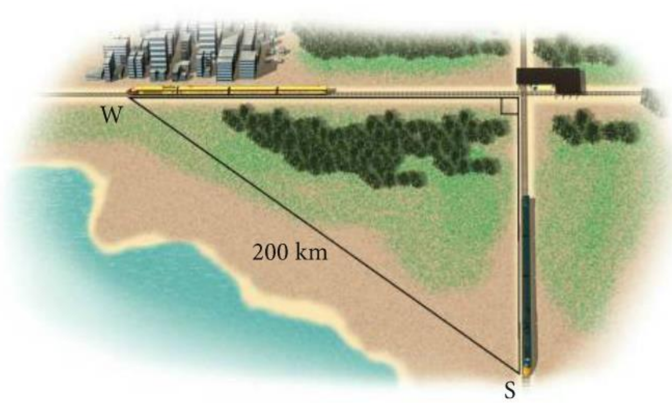


79. A piece of wire measuring 20 *feet* is attached to a telephone pole as a guy wire. The distance along the ground from the bottom of the pole to the end of the wire is 4 *feet* greater than the height where the wire is attached to the pole. How far up the pole does the guy wire reach?



80. Logan and Cassidy leave a campsite, Logan biking due north and Cassidy biking due east. Logan bikes 7 *km/h* slower than Cassidy. After 4 *hrs*, they are 68 *km* apart. Find the speed of each bicyclist.

81. Two trains leave a station at the same time. One train travels due west, and the other travels due south. The train traveling west travels 20 *km/hr* faster than the train traveling south. After 2 *hr*., the trains are 200 *km* apart. Find the speed of each train.



82. Towers are 1482 *feet.* tall. How long would it take an object dropped from the top to reach the ground? Given $s = 16t^2$

83. The formula $P = 0.01A^2 + .05A + 107$ models a woman's normal Point systolic blood pressure, *P*, an age *A*. Use this formula to find the age, to the nearest year, of a woman whose normal systolic blood pressure is 115 *mm* Hg.

84. A rectangular piece of metal is 10 *in*. longer than it is wide. Squares with sides 2 *in*. long are cut from the four corners, and the flaps folded upward to form an open box. If the volume of the box is 832 in^3, what were the original dimensions of the piece of metal?

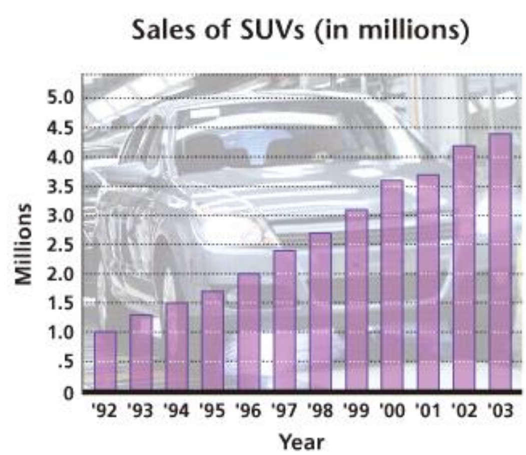
85. An astronaut on the moon throws a baseball upward. The astronaut is 6 *ft*., 6 *in*., tall, and the initial velocity of the ball is 30 *ft/sec* . The height s of the ball in feet is given by the equation

$$s = -2.7t^2 + 30t + 6.5$$

Where t is the number of seconds after the ball was thrown.

a) After how many seconds is the ball 12 *ft*. above the moon's surface?
b) How many seconds will it take for the ball to return to the surface?

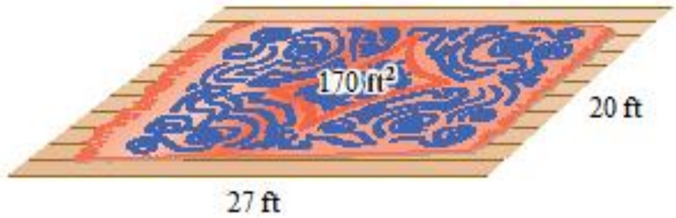
86. The bar graph shows of SUVs (sport utility vehicles in the US, in *millions*. The quadratic equation $S = .00579x^2 + .2579x + .9703$ models sales of SUVs from 1992 to 2003, where *S* represents sales in *millions*, and $x = 0$ represents 1992, $x = 1$ represents 1993 and so on.



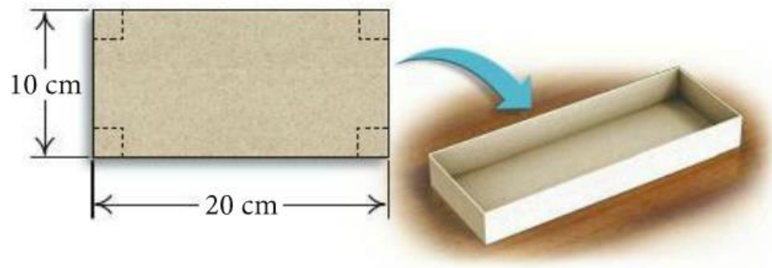
a) Use the model to determine sales in 2002 and 2003. Compare the results to the actual figures of 4.2 million and 4.4 *million* from the graph.
b) According to the model, in what year do sales reach 3.5 million? Is the result accurate?

87. Erik finds a piece of property in the shape of a right triangle. He finds that the longer leg is 20 *m* longer than twice the length of the shorter leg. The hypotenuse is 10 *m* longer than the length of the longer leg. Find the lengths of the sides of the triangular lot.

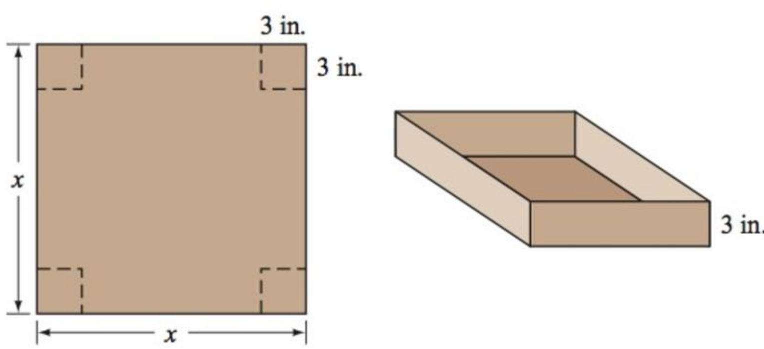
88. Cynthia wants to buy a rug for a room that is 20 *feet.* wide and 27 *feet*. long. She wants to leave a uniform strip of floor around the rug. She can afford to buy 170 square *feet* of carpeting. What dimension should the rug have?



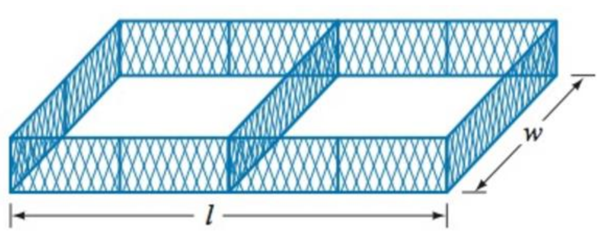
89. An open box is made from a 10-*cm* by 20-*cm* piece of tin by cutting a square from each corner and folding up the edges. The area of the resulting base is 96 cm^2. What is the length of the sides of the squares?



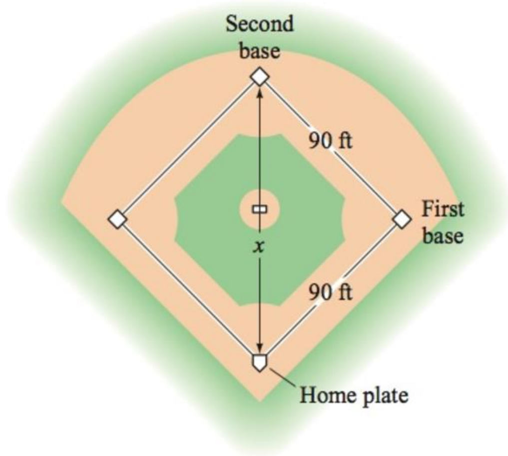
90. A square piece of cardboard is formed into a box by cutting out 3-*inch* squares from each of the corners and folding up the sides. If the volume of the box needs to be 126.75 cubic *inches*, what size square piece of cardboard is needed?



91. You want to use 132 *feet* of chain-link fencing to enclose a rectangular region and subdivide the region into two smaller rectangular regions. If the total enclosed area is 576 *square feet*, find the dimensions of the enclosed region.

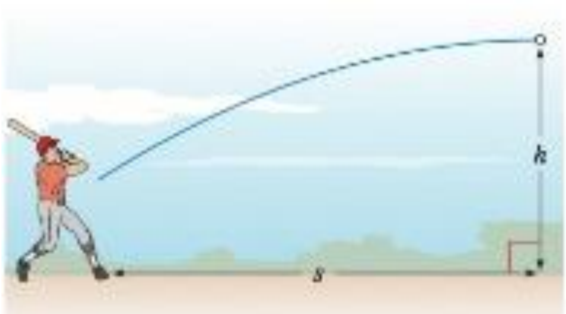


92. How far is it from home plate to second base on a baseball diamond?



93. Two equations can be used to track the position of a baseball t seconds after it is hit.

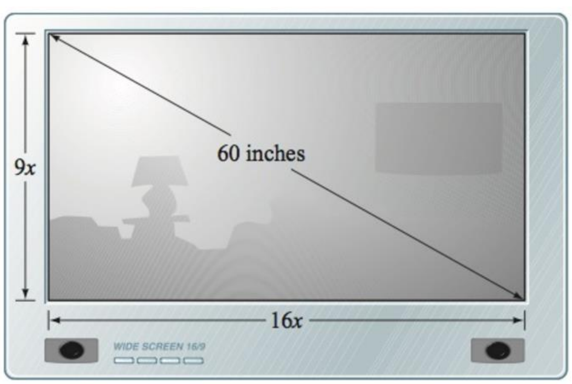
For instance, suppose $h = -16t^2 + 50t + 4.5$ gives the height, in *feet*, of a baseball t seconds after it is hit and $s = 103.9t$ gives the horizontal distance, in *feet*, of the ball from home plate t seconds after it is hit.



Use these equations to determine whether this particular baseball will clear a 10-*foot* fence positioned 360 *feet* from home plate.

94. A ball is thrown downward with an initial velocity of 5 *feet* per *second* from the Golden Gate Bridge, which is 220 *feet* above the water. How long will it take for the ball to hit the water?

95. A television screen measures 60 *inches* diagonally, and its aspect ratio is 16 to 9. This means that the ratio of the width of the screen to the height of the screen is 16 to 9.
Find the width and height of the screen.



96. A company makes rectangular solid candy bars that measures 5 *inches* by 2 *inches* by 0.5 *inch*. Due to difficult financial times, the company has decided to keep the price of the candy bar fixed and reduce the volume of the bar by 20%. What should the dimensions be for the new candy bar if the company keeps the height at 0.5 *inch* and makes length of the candy bar 3 *inches* longer than the width?

97. A company makes rectangular solid candy bars that measures 5 *inches* by 2 *inches* by 0.5 *inch*. Due to difficult financial times, the company has decided to keep the price of the candy bar fixed and reduce the volume of the bar by 20%. What should the dimensions be for the new candy bar if the company keeps the height at 0.5 *inch* and makes length of the candy bar 2.5 times as long as its width?