Lecture Two

Section 2.1 – Definition of the Derivative

Derivative

The derivative of the function f at x is defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

A function is differentiable @ x if its derivative exists at x.

The process of finding derivatives is called ***differentiation***.

$$f'(x), \quad f', \quad \frac{d}{dx}[f(x)], \quad \frac{d}{dx}f, \quad \frac{dy}{dx}, \quad y', \quad \dot{y}, \text{ and } D_x[y]$$

Differentiability ⇒ Continuity

If a function f is differentiable @ $x = c \Rightarrow f$ is continuous @ $x = c$

Example

Find the derivative of $f(x) = x^2$

<u>*Solution*</u>

$$\begin{aligned} f(x+h) &= (x+h)^2 \\ &= x^2 + 2hx + h^2 \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2hx + h^2}{h} \\ &= \lim_{h\to 0} (2x + h) \\ &= 2x \end{aligned}$$

Example

Find the derivative of $f(x) = 3x^2 - 2x$

<u>*Solution*</u>

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3(x+\Delta x)^2 - 2(x+\Delta x) - (3x^2 - 2x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3(x^2 + \Delta x^2 + 2x\Delta x) - 2x - 2\Delta x - 3x^2 + 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2 + 3\Delta x^2 + 6x\Delta x - 2x - 2\Delta x - 3x^2 + 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3\Delta x^2 + 6x\Delta x - 2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 3\Delta x + 6x - 2$$

$$= 6x - 2$$

Exercises *Section* 2.1 – Definition of the Derivative

1. Find the derivative of y with the respect to t for the function $y = \frac{4}{t}$

2. Find the equation of the tangent line to $f(x) = x^2 + 1$ that is parallel to $2x + y = 0$

3. Use the definition of limits to find the derivative: $f(x) = \frac{3}{\sqrt{x}}$

4. Use the definition of limits to find the derivative: $f(x) = \sqrt{x+2}$

Section 2.2 – Techniques for Finding Derivatives

Notations for the Derivative

The derivative of $y = f(x)$ may be written in any of the following ways:

1st *derivative*	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}\left[f(x)\right]$	$D_x[y]$

Constant Rule

$$\frac{d}{dx}[c] = f'(c) = 0 \qquad c \text{ is constant}$$

Proof:

Let $f(x) = c$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x}$$

$$= 0$$

So, $\frac{d}{dx}[c] = 0$

Example

Find the derivative

a) $f(x) = 9$

$f' = 0$

b) $h(t) = \pi$

$D_t\left[h(t)\right] = 0$

c) $y = 2^3$

$\frac{dy}{dx} = 0$

Power Rule

$$f(x) = x^n \quad \Rightarrow f'(x) = nx^{n-1}$$ $\quad$ $\boldsymbol{n}$ is any real number

Proof

Let $f(x) = x^n$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}(\Delta x)^2 + \ldots + (\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}\Delta x + \ldots + (\Delta x)^{n-1}$$

$$= nx^{n-1}$$

Example

Find the derivative

a) $y = x^6$

$y' = 6x^{6-1}$

$= 6x^5$

b) $y = t$

$y' = t^{1-1}$

$= t^0$

$= 1$

c) $y = \frac{1}{x^3}$

$y = x^{-3}$

$y' = -3x^{-3-1}$

$= -3x^{-4}$ $\quad$ *or* $\quad -\frac{3}{x^4}$

e) $D_x\left(x^{4/3}\right)$

$$D_x\left(x^{4/3}\right)=\frac{4}{3}x^{1/3}$$

f) $y=\sqrt{z}$

$$\frac{dy}{dz}=\frac{d}{dz}\left[z^{1/2}\right]$$

$$=\frac{1}{2}z^{1/2-1}$$

$$=\frac{1}{2}z^{-1/2} \qquad \frac{1}{2z^{1/2}} \qquad \frac{1}{2\sqrt{z}}$$

Constant Times a Function

If f is a differentiable function of x, and c is a real number, then

$$\text{If } f(x) = k.g(x) \qquad \Rightarrow f' = k.g'$$

Example

a) If $y = 8x^4$, find $\frac{dy}{dx}$

$$\frac{dy}{dx} = 8\left(4x^3\right)$$

$$= 32x^3$$

b) If $y = -\frac{3}{4}x^{12}$, find $\frac{dy}{dx}$

$$\frac{dy}{dx} = -\frac{3}{4}\left(12x^{11}\right)$$

$$= -9x^{11}$$

c) If $D_t(-8t)$

$$D_t(-8t) = -8$$

d) If $y = \frac{6}{x}$, find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d}{dx}\left[6x^{-1}\right]$$

$$= -6x^{-2}$$

$$= -\frac{6}{x^2}$$

e) $y = \frac{9}{4x^2}$

$$= \frac{9}{4}x^{-2}$$

$$\to y' = \frac{9}{4}(-2)x^{-3}$$

$$= -\frac{9}{2x^3}$$

Sum or Difference Rule

The derivative of the sum or difference of two differentiable functions is the sum or difference of their derivatives.

$$f(x) = u(x) \pm v(x) \qquad\qquad f'(x) = u'(x) \pm v'(x)$$

Example

Find the derivative of each function

a) $y = 6x^3 + 15x^2$

$$y' = 18x^2 + 30x$$

b) $p(t) = 12t^4 - 6\sqrt{t} + \frac{5}{t}$

$$p(t) = 12t^4 - 6t^{1/2} + 5t^{-1}$$

$$p' = 48t^3 - 3t^{-1/2} - 5t^{-2}$$

$$= 48t^3 - \frac{3}{t^{1/2}} - \frac{5}{t^2}$$

c) $f(x) = \frac{x^3 + 3\sqrt{x}}{x}$

$$f(x) = \frac{x^3}{x} + 3\frac{x^{1/2}}{x}$$

$$= x^2 + 3x^{-1/2}$$

$$f'(x) = 2x - \frac{3}{2}x^{-3/2}$$

$$= 2x - \frac{3}{2x^{3/2}}$$

$$= 2x - \frac{3}{2\sqrt{x^3}}$$

d) $f(x) = \left(4x^2 - 3x\right)^2$ $\qquad\qquad (a+b)^2 = a^2 + 2ab + b^2$

$$= 16x^4 - 24x^3 + 9x^2$$

$$f' = 64x^3 - 72x^2 + 18x$$

Example

Find the slope of the graph of $f(x) = x^2 - 5x + 1$ at the point (2, -5)

<u>*Solution*</u>

$$f'(x) = 2x - 5$$

$$\begin{aligned} \text{Slope} &= f'(2) \\ &= 2(2) - 5 \\ &= -1 \end{aligned}$$

Example

Researchers have determined that the daily energy requirements of female beagles who are at least 1 year old change with respect to age according to the function

$$E(t) = 753t^{-0.1321}$$

where $E(t)$ is the daily energy requirements $\left(in\ kJ / W^{0.67}\right)$ for a dog that is t years old.

a) Find $E'(t)$

$$\begin{aligned} E' &= 753(-0.1321)t^{-0.1321-1} \\ &= -99.4713t^{-1.1321} \end{aligned}$$

b) Determine the rate of change of the daily energy requirements of a 2-year old female beagle

$$\begin{aligned} E'(2) &= -99.4713(2)^{-1.1321} \\ &= -45.4\ \ kJ / W^{0.67} \end{aligned}$$

The daily energy requirements of a 2-year old female beagle are decreasing at the rate

Example

From 1998 through 2005, the revenue per share R (in dollars) for McDonald's Corporation can be modeled by

$$R = 0.0598t^2 - 0.379t + 8.44 \qquad\qquad 8 \le t \le 15$$

Where t represents the year, with $t = 8$ corresponding to 1998. At what rate was McDonald's revenue per share changing in 2003?

<u>*Solution*</u>

$2003 \Rightarrow t = 13$

$$R' = 0.1196t - 0.379$$

$$\begin{aligned} \rightarrow R' &= 0.1196(13) - 0.379 \\ &= 1.1758 \end{aligned}$$

Marginal Analysis

Profit = P Revenue = R Cost = C $P = R - C$

The derivatives of these quantities are called ***Marginal***

$$\frac{dP}{dx} = \text{Marginal Profit}$$

$$\frac{dR}{dx} = \text{Marginal Revenue}$$

$$\frac{dC}{dx} = \text{Marginal Cost}$$

Marginal Cost

Example

Suppose that the total cost in hundreds of dollars to produce x thousand barrels of a beverage is given by

$$C(x) = 4x^2 + 100x + 500$$

Find the marginal cost for the following values of x.

a) $x = 5$

Solution

$$C' = 8x + 100$$

$$C'(5) = 8(5) + 100$$

$$= 140$$

After 5 thousand barrels, the cost will be 140 (hundred dollars) or $14,000.00

b) $x = 30$

Solution

$$C'(30) = 8(30) + 100$$

$$= 340$$

After 30 thousand barrels, the cost will be $34,000.00

Demand Functions

The numbers of unit q that are willing to purchase at a given price per unit p

$$p = f(q)$$

Total Revenue *R*

Related to the price per unit and the quantity demanded (or sold): $R(q) = q.p$

Example

The demand function for a certain product is given by $p = \dfrac{50,000 - q}{25,000}$

Find the marginal revenue when $q = 10,000$ units and p is in dollars.

Solution

$$R = q.p$$
$$= q\left(\frac{50,000 - q}{25,000}\right)$$
$$= \frac{50,000q - q^2}{25,000}$$
$$= \frac{50,000q}{25,000} - \frac{q^2}{25,000}$$
$$= 2q - \frac{1}{25,000}q^2$$
$$R' = 2 - \frac{2}{25,000}q$$
$$R'(10000) = 2 - \frac{2}{25,000}(10000)$$
$$= 1.2$$

When $q = 10,000$ units; the marginal revenue is \$1.20 per unit.

Example

Find the revenue function and marginal revenue for a demand function of $P = 2000 - 4x$

Solution

Revenue = quantity * price

Revenue: $R = x.P$

$$= x(2000 - 4x)$$
$$= 2000x - 4x^2$$

Marginal: $R' = 2000 - 8x$

Marginal Profit

Example

Suppose that the function for the product $p = \dfrac{50,000 - x}{25,000}$ is given by

$$C(x) = 2100 + 0.25x \qquad \text{where } 0 \le x \le 30,000$$

Find the marginal profit from the production of the following numbers of units.

a) $x = 15,000$

b) $x = 25,000$

<u>*Solution*</u>

a) $R(x) = 2x - \dfrac{1}{25,000}x^2$

The profit is given by: $P = R - C$

$$P = 2x - \frac{1}{25,000}x^2 - (2100 + 0.25x)$$

$$= 2x - \frac{1}{25,000}x^2 - 2100 - 0.25x$$

$$= -\frac{1}{25,000}x^2 + 1.75x - 2100$$

$$P' = -\frac{2}{25,000}x + 1.75$$

$$P'(15000) = -\frac{2}{25,000}(15000) + 1.75$$

$$= 0.55$$

The marginal profit is \$0.55 per unit

b) $P'(25000) = -\dfrac{2}{25,000}(25000) + 1.75$

$$= -0.25$$

The marginal profit is $= -\$0.25$ per unit, which will reduce the profit.

Exercises *Section* 2.2 – Techniques for Finding Derivatives

Find the derivative of

1. $f(x) = -2$

2. $y = \pi$

3. $y = \sqrt{5}$

4. $f(x) = x^4$

5. $s(t) = \frac{1}{t}$

6. $y = 4x^2$

7. $y = \frac{9}{4x^2}$

8. $y = \frac{9}{(4x)^2}$

9. $y = \sqrt{5x}$

10. $f(x) = 16x^{1/2}$

11. $y = \sqrt[3]{x}$

12. $y = \frac{t}{4}$

13. $y = \frac{0.4}{\sqrt{x^3}}$

14. $y = -\frac{2}{\sqrt[3]{x}}$

15. $y = \frac{1}{\sqrt[3]{x}}$

16. $y = \frac{x^3 - 4x}{\sqrt{x}}$

17. $f(x) = 3x^2 + 2x$

18. $f(x) = 4 + 2x^3 - 3x^{-1}$

19. $f(x) = \frac{5}{3x^2} - \frac{2}{x^4} + \frac{x^3}{9}$

20. $f(x) = \frac{3}{x^{3/5}} - \frac{6}{x^{1/2}}$

21. $f(x) = \frac{5}{x^{1/5}} - \frac{8}{x^{3/2}}$

22. $y = \frac{1.2}{\sqrt{x}} - 3.2x^{-2} + x$

23. $f(x) = x^2 - 3x - 4\sqrt{x}$

24. $f(x) = 3\sqrt[3]{x^4} - 2x^3 + 4x$

25. $f(x) = 0.05x^4 + 0.1x^3 - 1.5x^2 - 1.6x + 3$

26. $y = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$

27. $f(t) = -3t^2 + 2t - 4$

28. $g(x) = 4\sqrt[3]{x} + 2$

29. $f(x) = x\left(x^2 + 1\right)$

30. $f(x) = \frac{2x^2 - 3x + 1}{x}$

31. $f(x) = \frac{4x^3 - 3x^2 + 2x + 5}{x^2}$

32. $f(x) = \frac{-6x^3 + 3x^2 - 2x + 1}{x}$

33. Find the slope of the graph of $f(x) = x^2 - 5x + 1$ at the point (2, –5)

34. Find an equation of the tangent line to the graph of $f(x) = -x^2 + 3x - 2$ at the point (2, 0)

35. Find the slope of the graph of $f(x) = x^3$ when $x = -1$, 0, and 1.

36. The height h (in feet) of a free-falling object at time (in seconds) is given by $h = -16t^2 + 180$. Find the average velocity of the object over each interval.

a. [0, 1]

b. [1, 2]

37. Give the position function of a diver who jumps from a board 12 feet high with initial velocity 16 feet per second. Then find the diver's velocity function.

38. An analyst has found that a company's costs and revenues in dollars for one product are given by

$$C(x) = 2x \qquad\qquad R(x) = 6x - \frac{x^2}{1000}$$

Respectively, where x is the number of items produced.

a) Find the marginal cost function

b) Find the marginal revenue function

c) Using the fact that profit is the difference between revenue and costs, find the marginal profit function.

d) What value of x makes the marginal profit is 0.

e) Find the profit when the marginal profit is 0.

39. A business sells 2000 units per month at a price \$10 each. If monthly sales increases 200 units for each \$0.10 reduction in price.

40. From 1998 through 2005, the revenue per share R (in dollars) for McDonald's Corporation can be modeled by

$$R = 0.0598t^2 - 0.379t + 8.44 \qquad 8 \le t \le 15$$

Where t represents the year, with $t = 8$ corresponding to 1998. At what rate was McDonald's revenue per share changing in 2003?

41. The cost C (in dollars) of producing x units of a product is given by $C = 3.6\sqrt{x} + 500$

a) Find the additional cost when the production increases from 9 to 10 units.

b) Find the marginal cost when $x = 9$

c) Compare the results of parts (*a*) and (*b*)

42. The revenue R (in dollars) of renting x apartments can be modeled by $R = 2x\left(900 + 32x - x^2\right)$

a) Find the additional revenue when the number of rentals is increased from 14 to 15

b) Find the marginal revenue when $x = 14$

c) Compare the results of parts (*a*) and (*b*)

43. The profit P (in dollars) of selling x units of calculus textbooks is given by

$$P = -0.05x^2 + 20x - 1000$$

a) Find the additional profit when the sales increase from 150 to 151 units.

b) Find the marginal profit when $x = 150$

c) Compare the results of parts (*a*) and (*b*)

44. From 1998 through 2005, the revenue per share R (in dollars) for McDonald's Corporation can be modeled by

$$R = 0.0598t^2 - 0.379t + 8.44 \qquad 8 \le t \le 15$$

Where t represents the year, with $t = 8$ corresponding to 1998. At what rate was McDonald's revenue per share changing in 2003?

45. The profit derived from selling x units, is given by $P = 0.0002x^3 + 10x$, find the marginal profit for a production level of 100 units. Compare this with the actual gain in profit by increasing production from 100 to 101 units.

46. The Cost of producing x hamburgers is $C = 5000 + 0.56x, \quad 0 \le x \le 50{,}000$ and the revenue function is given by

$$R = \frac{1}{20000}\left(60000x - x^2\right)$$

Compare the marginal profit when 10,000 units are produced with the actual increase in profit from 10,000 units to 10,001 units

47. An object moves along the y-axis (marked in feet) so that its position at time x (in seconds) is

$$f(x) = x^3 - 6x^2 + 9x$$

a) Find the instantaneous velocity function v.
b) Find the velocity at $x = 2$ and $x = 5$ seconds
c) Find the time(s) when the velocity is 0.

48. A company's total sales (in millions of dollars) t months from now are given by

$$S(t) = 0.03t^3 + 0.5t^2 + 2t + 3$$

a) Find $S'(t)$.
b) Find $S(5)$ and $S'(5)$ (to two decimal places). Write a brief verbal interpretation of these results.
c) Find $S(10)$ and $S'(10)$ (to two decimal places). Write a brief verbal interpretation of these results.

49. A company's total sales (in millions of dollars) t months from now are given by

$$S(t) = 0.015t^4 + 0.4t^3 + 3.4t^2 + 10t - 3$$

a) Find $S'(t)$.
b) Find $S(4)$ and $S'(4)$ (to two decimal places). Write a brief verbal interpretation of these results.
c) Find $S(8)$ and $S'(8)$ (to two decimal places). Write a brief verbal interpretation of these results.

50. A marine manufacturer will sell $N(x)$ power boats after spending \$$x$ thousand on advertising, as given by

$$N(x) = 1{,}000 - \frac{3{,}780}{x} \quad 5 \le x \le 30$$

a) Find $N'(x)$.
b) Find $N(20)$ and $N'(20)$ (to two decimal places). Write a brief verbal interpretation of these results.

51. A company manufactures and sells x transistor radios per week. If the weekly cost and revenue equations are

$$C(x) = 5{,}000 + 2x \quad R(x) = 10x - \frac{x^2}{1{,}000} \quad 0 \le x \le 8{,}000$$

Then find the approximate changes in revenue and profit if production is increased from 2,000 to 2,010 per week.

52. A company manufactures fuel tanks for cars. The total weekly cost (in dollars) of producing x tanks given by

$$C(x) = 10{,}000 + 90x - 0.05x^2$$

a) Find the marginal cost function.
b) Find the marginal cost at a production level of 500 tanks per week.
c) Interpret the result of part b.
d) Find the exact cost of producing the 501^{st} item.

53. A company's market research department recommends the manufacture and marketing of a new headphone set for MP3 players. After suitable test marketing, the research department presents the following *price-demand* equation:

$$x = 10{,}000 - 1{,}000p \quad \rightarrow \quad p = 10 - 0.001x$$

Where x is the number of headphones that retailers are likely to buy at $\$p$ per set.
The financial department provides the cost function

$$C(x) = 7{,}000 + 2x$$

Where \$7,000 is the estimate of fixed costs (tooling and overhead) and \$2 is the estimate of variable costs per headphone set (materials, labor, marketing, transportation, storage, etc.).
a) Find the domain of the function defined by the price demand function.
b) Find and interpret the marginal cost function $C'(x)$.
c) Find the revenue function as a function of x and find its domain.
d) Find the marginal revenue at x = 2,000, 5,000, and 7,000. Interpret these results.
e) Graph the cost function and the revenue function in the same coordinate system, Find the intersection points of these two graphs and interpret the results.
f) Find the profit function and its domain and sketch the graph of the function.
g) Find the marginal profit at x = 1,000, 4,000, and 6,000. Interpret these results.

54. A small machine shop manufactures drill bits used in the petroleum industry. The manager estimates that the total daily cost (in dollars) of producing x bits is

$$C(x) = 1{,}000 + 25x - 0.1x^2$$

a) Find $\overline{C}(x)$ and $\overline{C}'(x)$
b) Find $\overline{C}(10)$ and $\overline{C}'(10)$. Interpret these quantities.
c) Use the results in part (***b***) to estimate the average cost per bit at a production level of 11 bits per day.

55. The total profit (in dollars) from the sale of x calendars is

$$P(x) = 22x - 0.2x^2 - 400 \qquad 0 \le x \le 100$$

a) Find the exact profit from the sale of the 41^{st} calendar.
b) Use the marginal profit to approximate the profit from the sale of the 41^{st} calendar.

56. The total profit (in dollars) from the sale of x cameras is

$$P(x) = 12x - 0.02x^2 - 1{,}000 \qquad 0 \le x \le 600$$

Evaluate the marginal profit at the given values of x, and interpret the results.

a) $x = 200$.
b) $x = 350$.

57. The total profit (in dollars) from the sale of x gas grills is

$$P(x) = 20x - 0.02x^2 - 320 \qquad 0 \le x \le 1{,}000$$

a) Find the average profit per grill if 40 grills are produced.
b) Find the marginal average profit at a production level of 40 grills and interpret the results.
c) Use the results from parts (*a*) and (*b*) to estimate the average profit per grill if 41 grills are produced.

58. The price p (in dollars) and the demand x for a particular steam iron are related by the equation

$$x = 1{,}000 - 20p$$

a) Express the price p in terms of the demand x, and find the domain of this function.
b) Find the revenue $R(x)$ from the sale of x steam irons. What is the domain of R?
c) Find the marginal revenue at a production level of 400 steam irons and interpret the results.
d) Find the marginal revenue at a production level of 650 steam irons and interpret the results.

59. The price-demand equation and the cost function for the production of TVs are given respectively, by

$$x = 9{,}000 - 30p \quad \textit{and} \quad C(x) = 150{,}000 + 30x$$

Where x is the number of TVs that can be sold at a price of \$$p$ per TV and $C(x)$ is the total cost (in dollars) of producing x TVs.

a) Express the price p as a function of the demand x, and find the domain of this function.
b) Find the marginal cost.
c) Find the revenue function and state its domain.
d) Find the marginal revenue.
e) Find $R'(3{,}000)$ and $R'(6{,}000)$ and interpret these quantities.
f) Graph the cost function and the revenue function on the same coordinate system for $0 \le x \le 9{,}000$. Find the break–even points and indicate regions of loss and profit.
g) Find the profit function in terms of x.
h) Find the marginal profit.
i) Find $P'(1{,}500)$ and $P'(4{,}500)$ and interpret these quantities

60. The total cost and the total revenue (in dollars) for the production and sale of x hair dryers are given, respectively, by

$$C(x) = 5x + 2{,}340 \quad \textit{and} \quad R(x) = 40x - 0.1x^2 \quad 0 \le x \le 400$$

a) Find the value of x where the graph of $R(x)$ has a horizontal tangent line.

b) Find the profit function $P(x)$.

c) Find the value of x where the graph of $P(x)$ has a horizontal tangent line.

d) Graph $C(x)$, $R(x)$, and $P(x)$ on the same coordinate system for $0 \le x \le 400$. Find the break–even points. Find the x intercept of the graph of $P(x)$.

Section 2.3 – Derivatives of Products and Quotients

Product Rule

The derivative of the product of two differentiable functions is equal to the first function times the derivative of the second plus the second function times the derivative of the first,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$(f.g)' = f'.g + g'.f \qquad\qquad \boxed{(u.v)' = u'.v + v'.u}$$

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'gh + fg'h + fgh'$$

Example

Find the derivative of $f(x) = (2x+3)(3x^2)$

Solution

$$u = 2x+3 \qquad v = 3x^2$$
$$u' = 2 \qquad v' = 6x$$

$$\begin{aligned} f' &= u'v + v'u \\ &= (2)(3x^2) + (2x+3)(6x) \\ &= 12x^2 + 18x + 6x^2 \\ &= 18x^2 + 18x \\ &= 18x(x+1) \end{aligned}$$

$$f(x) = 6x^3 + 9x^2$$

Example

Find the derivative of $f(x) = (\sqrt{x}+3)(x^2-5x)$

Solution

$$u = x^{1/2} + 3 \qquad v = x^2 - 5x$$
$$u' = \frac{1}{2}x^{-1/2} \qquad v' = 2x - 5$$

$$\begin{aligned} f' &= \left(\tfrac{1}{2}x^{-1/2}\right)(x^2-5x) + (2x-5)(\sqrt{x}+3) \\ &= \tfrac{1}{2}x^{3/2} - \tfrac{5}{2}x^{1/2} + 2x^{3/2} - 5x^{1/2} + 6x - 15 \\ &= \tfrac{5}{2}x^{3/2} - \tfrac{15}{2}x^{1/2} + 6x - 15 \\ &= \tfrac{5}{2}x^{3/2} + 6x - \tfrac{15}{2}x^{1/2} - 15 \end{aligned}$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)g(x)-g'(x)f(x)}{[g(x)]^2}$$

$$=\frac{f'g-g'f}{g^2}$$

$$\boxed{\left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}}$$

Example

Find $f'(x)$ if $f(x)=\dfrac{2x-1}{4x+3}$

Solution

$$u=2x-1 \quad v=4x+3$$
$$u'=2 \quad v'=4$$

$$f'(x)=\frac{(2)(4x+3)-(2x-1)(4)}{(4x+3)^2}$$

$$\left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{8x+6-8x+4}{(4x+3)^2}$$

$$=\frac{10}{(4x+3)^2}$$

Example

Find the derivative of $y=\dfrac{x+4}{5x-2}$

Solution

$$u=x+4 \quad v=5x-2$$
$$u'=1 \quad v'=5$$

$$y'=\frac{(1)(5x-2)-(5)(x+4)}{(5x-2)^2}$$

$$\left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{5x-2-5x-20}{(5x-2)^2}$$

$$=-\frac{22}{(5x-2)^2}$$

Example

Find an equation of the tangent line to the graph of $y=\dfrac{x^2-4}{2x+5}$ when $x=0$

<u>*Solution*</u>

$$u=x^2-4 \quad v=2x+5$$
$$u'=2x \qquad v'=2$$

$$y'=\frac{(2x)(2x+5)-(2)\left(x^2-4\right)}{(2x+5)^2} \qquad \left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{4x^2+10x-2x^2+8}{(2x+5)^2}$$

$$=\frac{2x^2+10x+8}{(2x+5)^2}$$

$$\Rightarrow x=0\to y'=\frac{8}{25}=m$$

$$x=0\to y=\frac{x^2-4}{2x+5}=-\frac{4}{5}$$

$$y-y_1=m\left(x-x_1\right)$$

$$\to y+\tfrac{4}{5}=\tfrac{8}{25}(x-0) \quad \to \boxed{y=\frac{8}{25}x-\frac{4}{5}}$$

Combining the product and Quotient Rules

Example

Find the derivative of $y = \dfrac{(1+x)(2x-1)}{x-1}$

Solution

$$y' = \frac{(x-1)\frac{d}{dx}\left[(1+x)(2x-1)\right] - (1+x)(2x-1)\frac{d}{dx}[x-1]}{(x-1)^2}$$

$$= \frac{(x-1)\left[(1)(2x-1) + 2(1+x)\right] - (1+x)(2x-1)(1)}{(x-1)^2}$$

$$= \frac{(x-1)(2x-1+2+2x) - (2x-1+2x^2-x)}{(x-1)^2}$$

$$= \frac{(x-1)(4x+1) - 2x+1-2x^2+x}{(x-1)^2}$$

$$= \frac{4x^2+x-4x-1-2x+1-2x^2+x}{(x-1)^2}$$

$$= \frac{2x^2-4x}{(x-1)^2}$$

Or

$$y = \frac{(1+x)(2x-1)}{x-1}$$

$$= \frac{2x-1+2x^2-x}{x-1}$$

$$= \frac{2x^2+x-1}{x-1}$$

$$y' = \frac{(x-1)(4x+1) - (2x^2+x-1)(1)}{(x-1)^2}$$

$$= \frac{4x^2+x-4x-1-2x^2-x+1}{(x-1)^2}$$

$$= \frac{2x^2-4x}{(x-1)^2}$$

$$= \frac{2x(x-2)}{(x-1)^2}$$

Average Cost Function

To Study the effects of production levels on cost, economist use the average cost function $\overline{C}$, which is defined as

$$\overline{C} = \frac{C}{x}$$

Where $C = f(x)$ is the total cost function and x the number of units produced.

Marginal Average Cost Function $= \overline{C}\,'$

Example

Suppose the cost in dollars of manufacturing x hundred small motors is given by

$$C(x) = \frac{3x^2 + 120}{2x+1} \qquad 10 \le x \le 200$$

a) Find the average cost per hundred motors

$$\overline{C} = \frac{C}{x}$$

$$= \frac{3x^2+120}{2x+1} \cdot \frac{1}{x}$$

$$= \frac{3x^2+120}{2x^2+x}$$

b) Find the marginal average cost

$$\overline{C}\,' = \frac{(6x)\left(2x^2+x\right) - \left(3x^2+120\right)(4x+1)}{\left(2x^2+x\right)^2}$$

$$\overline{C}\,' = \frac{\left(3x^2+120\right)'\left(2x^2+x\right) - \left(3x^2+120\right)\left(2x^2+x\right)'}{\left(2x^2+x\right)^2}$$

$$= \frac{12x^3+6x^2-12x^3-3x^2-480x-120}{\left(2x^2+x\right)^2}$$

$$= \frac{3x^2-480x-120}{\left(2x^2+x\right)^2}$$

c) Average cost is generally minimized when the marginal average cost is zero. Find the level of production that minimizes average cost

$$\frac{3x^2-480x-120}{\left(2x^2+x\right)^2} = 0$$

$$3x^2 - 480x - 120 = 0 \quad \Rightarrow x = -0.25,\ 160$$

16,000 *motors* will minimize average cost.

Exercises *Section* 2.3 – Derivatives of Products and Quotients

Find the derivative

1. $y=(x+1)\left(\sqrt{x}+2\right)$

2. $y=\left(4x+3x^2\right)(6-3x)$

3. $y=\left(\frac{1}{x}+1\right)(2x+1)$

4. $y=3x\left(2x^2+5x\right)$

5. $y=3\left(2x^2+5x\right)$

6. $y=\dfrac{x^2+4x}{5}$

7. $y=\dfrac{3x^4}{5}$

8. $y=\dfrac{3-\frac{2}{x}}{x+4}$

9. $g(x)=\dfrac{x^2-4x+2}{x^2+3}$

10. $f(x)=\dfrac{(3-4x)(5x+1)}{7x-9}$

11. $f(x)=x\left(1-\dfrac{2}{x+1}\right)$

12. $g(s)=\dfrac{s^2-2s+5}{\sqrt{s}}$

13. $f(x)=\dfrac{x+1}{\sqrt{x}}$

14. $f(x)=\dfrac{x^2}{2x+1}$

15. $f(x)=\dfrac{x^2-x}{x^3+1}$

16. $f(x)=\dfrac{2x}{x^2+3}$

17. $y=\dfrac{t^3-3t}{t^2-4}$

18. $f(x) = 5x^2(x^3 + 2)$

19. $f(x) = \frac{3x - 4}{2x + 3}$

20. $f(x) = \frac{3x + 5}{x^2 - 3}$

21. $f(x) = (x^2 - 4)(x^2 + 5)$

22. Find an equation of the tangent line to the graph of $y = \frac{x^2 - 4}{2x + 5}$ when $x = 0$

23. A company that manufactures bicycles has determined that a new employee can assemble $M(d)$ bicycles per day after *d* days of on-the-job training, where

$$M(d) = \frac{100d^2}{3d^2 + 10}$$

a) Find the rate of change function for the number of bicycles assembled with respect to time.
b) Find and interpret $M'(2)$ and $M'(5)$

24. A small business invests \$25,000.00 in a new product. In addition, the product will cost \$0.75 per unit to produce. Find the cost function and the average cost function. What is the limit of the average cost function as production increase?

25. A communications company has installed a new cable TV system in a city. The total number *N* (in thousands) of subscribers *t* months after the installation of the system is given by

$$N(t) = \frac{180t}{t + 4}$$

a) Find $N'(t)$
b) Find $N(16)$ and $N'(16)$. Write a brief interpretation of these results.
c) Use the results from part (***b***) to estimate the total number of subscribers after 17 months.

26. One hour after a dose of x milligrams of a particular drug is administered to a person, the change in body temperature $T(x)$, in degrees Fahrenheit, is given approximately by

$$T(x) = x^2\left(1 - \frac{x}{9}\right) \quad 0 \le x \le 7$$

The rate $T'(x)$ at which T changes with respect to the size of the dosage *x* is called the sensitivity of the body to the dosage.

a) Find $T'(x)$
b) Find $T'(1)$, $T'(3)$, and $T'(6)$

27. According to economic theory, the supply x of a quantity in a free market increases as the price p increases. Suppose that the number x of DVD players a retail chain is willing to sell per week at a price of \$$p$ is given by

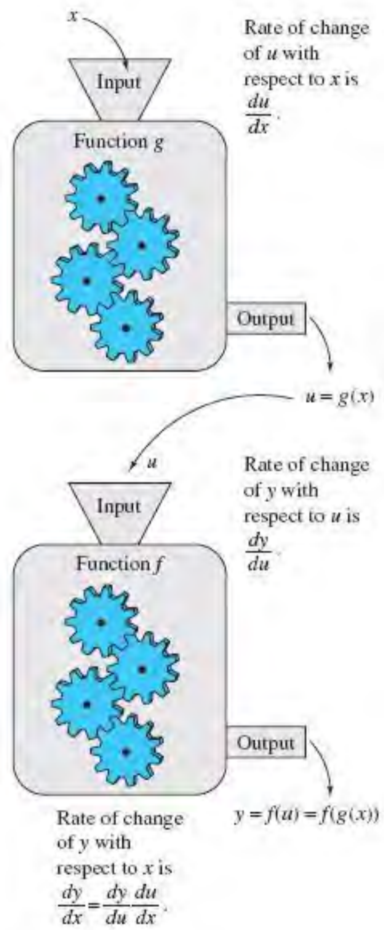
$$x = \frac{100p}{0.1p+1} \quad 10 \le p \le 70$$

a) Find $\frac{dx}{dp}$

b) Find the supply and the instantaneous rate of change of supply with respect to price is \$40. Write a brief interpretation of these results.

c) Use the results from part (***b***) to estimate the supply if the price is increased to \$41.

Section 2.4 – The Chain Rule



The Chain Rule

If $y = f(u)$ is a differentiable function of u, and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x, and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

$$y = f(g(x)) = f(u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

$$\frac{dy}{dx} = f'[g(x)] \bullet g'(x)$$

The General Power Rule

$$\frac{dy}{dx} = \frac{d}{dx}\left[u(x)^n\right]$$

$$= n\ u^{n-1} \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[u^n\right] = n\ u^{n-1}u'$$

Example

Find $\frac{dy}{dx}$ if $y=\left(3x^2-5x\right)^{1/2}$

<u>*Solution*</u>

$$u=3x^2-5x \quad \& \quad y=u^{1/2}$$

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}\cdot\frac{du}{dx}\\&=\frac{1}{2}u^{-1/2}(6x-5)\\&=\frac{1}{2}(6x-5)\left(3x^2-5x\right)^{-1/2}\\&=\frac{6x-5}{2\left(3x^2-5x\right)^{1/2}}\end{aligned}$$

Example

Find $D_x\left(x^2+5x\right)^8$

<u>*Solution*</u>

$$\begin{aligned}D_x\left(x^2+5x\right)^8&=8\left(x^2+5x\right)^7\left(x^2+5x\right)'\\&=8\left(x^2+5x\right)^7(2x+5)\\&=8(2x+5)\left(x^2+5x\right)^7\end{aligned}$$

Example

Find the derivative $y=4x(3x+5)^5$

<u>*Solution*</u>

$$y'=(4x)'(3x+5)^5+\left((3x+5)^5\right)'(4x) \qquad (uv)'=u'v+v'u$$

$$y'=4(3x+5)^5+5(3x+5)^4(3)(4x) \qquad \left((3x+5)^5\right)'=5(3x+5)^4(3x+5)'$$

$$\begin{aligned}&=4(3x+5)^5+60x(3x+5)^4\\&=4(3x+5)^4(3x+5+15x)\\&=4(3x+5)^4(18x+5)\end{aligned}$$

Example

Find the derivative $y = \dfrac{(3x+2)^7}{x-1}$

<u>*Solution*</u>

$$u = (3x+2)^7 \qquad\qquad v = x-1$$

$$u = 7(3x+2)^6(3x+2)' \qquad v = 1$$

$$\begin{aligned}
y' &= \frac{7(3)(3x+2)^6(x-1)-(1)(3x+2)^7}{(x-1)^2} \\
&= \frac{(3x+2)^6(21x-21-3x-2)}{(x-1)^2} \\
&= \frac{(3x+2)^6(18x-23)}{(x-1)^2}
\end{aligned}$$

Example

Find the tangent line to the graph of $y = \sqrt[3]{(x+4)^2}$ when $x = 4$.

<u>*Solution*</u>

$$\begin{aligned}
y &= (x+4)^{2/3} \\
y' &= \tfrac{2}{3}(x+4)^{-1/3} \\
&= \frac{2}{3}\frac{1}{(x+4)^{1/3}} \\
&= \frac{2}{3\sqrt[3]{x+4}}
\end{aligned}$$

$$x = 4 \to \underline{m = } y' = \frac{2}{3\sqrt[3]{4+4}} = \frac{2}{3\sqrt[3]{2^3}} = \frac{2}{3(2)} \ \boxed{= \frac{1}{3}}$$

$$x = 4 \to y = \sqrt[3]{(4+4)^2} = 4$$

$$\begin{aligned}
y - 4 &= \tfrac{1}{3}(x-4) \\
y &= \tfrac{1}{3}x - \tfrac{4}{3} + 4 \\
y &= \tfrac{1}{3}x + \tfrac{8}{3}
\end{aligned}$$

Exercises *Section* 2.4 – The Chain Rule

Find the derivative of

1. $y=\left(3x^4+1\right)^4\left(x^3+4\right)$

2. $p(t)=\dfrac{(2t+3)^3}{4t^2-1}$

3. $y=\left(x^3+1\right)^2$

4. $y=\left(x^2+3x\right)^4$

5. $y=\dfrac{4}{2x+1}$

6. $y=\dfrac{2}{(x-1)^3}$

7. $y=x^2\sqrt{x^2+1}$

8. $y=\left(\dfrac{x+1}{x-5}\right)^2$

9. $s(t)=\sqrt{2t^2+5t+2}$

10. $f(x)=\dfrac{1}{\left(x^2-3x\right)^2}$

11. $y=t^2\sqrt{t-2}$

12. $y=\left(\dfrac{6-5x}{x^2-1}\right)^2$

13. $y=\dfrac{1}{(9x-4)^8}$

14. $f(x)=(3x+1)^4$

15. $f(x)=\dfrac{1}{\left(x^2+x+4\right)^3}$

16. $f(x)=\sqrt{3-x}$

17. $f(x)=\left(3x^2+5\right)^5$

18. $f(x)=\left(5x^2-3\right)^6$

19. $f(x) = \left(x^4 + 1\right)^{-2}$

20. $f(x) = (4x+3)^{1/2}$

21. Suppose a demand function is given by

$$q = D(p) = 30\left(5 - \frac{p}{\sqrt{p^2+1}}\right)$$

Where q is the demand for a product and p is the price per item in dollars. Find the rate of change in the demand for the product per unit change in price (i.e. find dq/dp)

22. Find the tangent line to the graph of $y = \sqrt[3]{(x+4)^2}$ when $x = 4$.

23. The revenue realized by a small city from the collection of fines from parking tickets is given by

$$R(n) = \frac{8000n}{n+2}$$

where n is the number of work-hours each day that can be devoted to parking patrol. At the outbreak of a flu epidemic, 30 work-hours are used daily in parking patrol, but during the epidemic that number is decreasing at the rate of 6 work-hours per day. How fast is revenue from parking fines decreasing at the outbreak of the epidemic?

24. To test an individual's use of a certain mineral, a researcher injects small amount form of that material into the person's bloodstream. The mineral remaining in the bloodstream is measured each day for several days. Suppose the amount of the mineral remaining in the bloodstream (in milligrams per cubic centimeter) t days after the injection is approximated by $C(t) = \frac{1}{2}(2t+1)^{-1/2}$. Find the rate of change of the mineral level with respect to time for 4 days.

25. The total cost (in hundreds of dollars) of producing x cameras per week is

$$C(x) = 6 + \sqrt{4x+4} \quad 0 \le x \le 30$$

a) Find $C'(x)$

b) Find $C'(15)$ and $C'(24)$. Interpret the results

Section 2.5 – Higher Order Derivatives

Higher Derivatives

$\frac{d}{dx}\left[f'(x)\right] = f''(x)$ *Second derivative*

$\frac{d}{dx}\left[f''(x)\right] = f'''(x)$ *Third derivative*

Notation for Higher-Order Derivatives						
1.	***1st derivative***	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}\left[f(x)\right]$	$D_x[y]$
2.	***2nd derivative***	y''	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}\left[f(x)\right]$	$D_x^2[y]$
3.	***3rd derivative***	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}\left[f(x)\right]$	$D_x^3[y]$
4.	***4th derivative***	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}\left[f(x)\right]$	$D_x^4[y]$
5.	***nth derivative***	$y^{(n)}$	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}\left[f(x)\right]$	$D_x^n[y]$

Example

Find the first four derivatives of $f(x) = 6x^3 - 2x^2 + 1$

<u>*Solution*</u>

$$f'(x) = 18x^2 - 4x$$

$$f''(x) = 36x - 4$$

$$f'''(x) = 36$$

$$f^{(4)}(x) = 0$$

Example

Find the value of $g'''(1)$ for $g(x)=x^4-x^3+2x$

<u>*Solution*</u>

$$g'(x)=4x^3-3x^2+2$$

$$g''(x)=12x^2-6x$$

$$g'''(x)=24x-6$$

$$\Rightarrow g'''(1)=24-6=18$$

$$f(x)=a_n x^n+a_{n-1}x^{n-1}+\cdots+a_1 x+a_0 \qquad \Rightarrow \boxed{f^{(n)}(x)=n!a_n}$$

Example

Find the fourth derivative of $y=\frac{1}{x^2}$

<u>*Solution*</u>

$$\Rightarrow y=\frac{1}{x^2}=x^{-2}$$

$$y'=-2x^{-3}=-\frac{2}{x^3}$$

$$y''=6x^{-4}=\frac{6}{x^4}$$

$$y'''=-24x^{-5}=-\frac{24}{x^5}$$

$$y^{(4)}=120x^{-6}=\frac{120}{x^6}$$

Acceleration

$s = f(t)$	***P**osition function*
$\frac{ds}{dt} = f'(t)$	***V**elocity function*
$\frac{d^2s}{dt^2} = f'''(t)$	***A**cceleration function*

Example

A ball is thrown upward from the top of an 80-foot cliff with an initial velocity of 64 feet per second. Give the position function. Then find the velocity and acceleration functions.

<u>Solution</u>

$$s(t) = -16t^2 + 64t + 80$$

$$Velocity: v(t) = s'(t) = -32t + 64$$

$$Acceleration: a(t) = s''(t) = -32$$

Exercises *Section* 2.5 – Higher Order Derivatives

1. Find the second derivative: $f(x)=3\left(2-x^2\right)^3$

2. Find the third derivative: $f(x)=5x(x+4)^3$

3. Find $f'''(-5)$ the given value: $f(x)=\sqrt{4-x}$

4. Find the 4th derivative of $f(x)=x^4+2x^3+3x^2-5x+7$

5. Find the second derivative of $f(x)=\left(x^2-1\right)^2$

6. Find $f''(x)$ for $f(x)=\sqrt{x^2+36}$, then find $f''(0)$ and $f''(9)$

7. Find $f''(x)$ for $f(x)=\sqrt{x^2+81}$, then find $f''(0)$ and $f''(2)$

8. The position function on Earth, where s is measured in meters, t is measured in seconds, v_0 is the initial velocity in meters per second, and h_0 is the initial height in meters, is

$$s=-4.9t^2+v_0t+h_0$$

If the initial velocity is 2.2 and the initial height is 3.6, what is the acceleration due to gravity on Earth in meters per second per second?

Section 2.6 – Exponential & Logarithmic Functions

Exponential

Definition

The exponential function *f* with base *b* is defined by

$$f(x) = b^x \qquad \text{or} \qquad y = b^x$$

Where $b > 0$, $b \neq 1$ and x is any real number.

Example: $f(x) = 2^x \qquad f(x) = \left(\frac{1}{2}\right)^{2x+1} \qquad f(x) = 3^{-x} \qquad \cancel{f(x) = (-2)^x}$

Exponential Equations

$$b^x = b^y \quad \leftrightarrow \quad x = y \qquad \text{for any b} > 0, \neq 1$$

Example

Solve $9^x = 27$

<u>*Solution*</u>

$\left(3^2\right)^x = 3^3$

$3^{2x} = 3^3$

$2x = 3$

$x = \frac{3}{2}$

Example

Solve $32^{2x-1} = 128^{x+3}$

<u>*Solution*</u>

$\left(2^5\right)^{2x-1} = \left(2^7\right)^{x+3}$

$2^{10x-5} = 2^{7x+21}$

$10x - 5 = 7x + 21$

$3x = 26$

$x = \frac{26}{3}$

Natural Base *e*

The irrational number e is called natural base

$f(x) = e^x$ is called natural exponential function

$e^0 = 1$ $\quad\quad$ $e \approx 2.7183$ $\quad\quad$ $e^2 \approx 7.389$ $\quad\quad$ $e^{-1} \approx 0.3679$

Example

Biologists studying salmon have found that the oxygen consumption of yearling salmon (in appropriate units) increases exponentially with the speed of swimming according to the function defined by

$$f(x) = 100e^{0.6x}$$

where x is the speed in feet per second. Find the following

a) The oxygen consumption when the fish are still

$$f(x=0) = 100e^{0.6(0)}$$
$$= 100$$

b) The oxygen consumption at a speed of 2 *ft* per second

$$f(x=2) = 100e^{0.6(2)}$$
$$\approx 332$$

Logarithmic **Function** (*Definition*)

For $x > 0$ and $b > 0,\ b \neq 0$

$$y = \log_b x \quad \text{is equivalent to} \quad x = b^y$$

$$y = \log_b x \iff x = b^y$$

Base

The function $f(x) = \log_b x$ is the logarithmic function with base b.

$\log_b x$: *read* log base b of x $\qquad$ $\log x$ ***means*** $\log_{10} x$

Logarithmic Form	***Exponential Form***
$\log_2 8 = 3$	$2^3 = 8$
$\log_{1/2} 16 = -4$	$\left(\frac{1}{2}\right)^{-4} = 16$
$\log_3 \frac{1}{81} = -4$	$3^{-4} = \frac{1}{81}$
$\log_{3/4} 1 = 0$	$\left(\frac{3}{4}\right)^0 = 1$

Natural Logarithms

Definition

$$f(x) = \log_e x = \ln x$$

The logarithmic function with base ***e*** is called natural logarithmic function.

$\ln x$ *read* "el en of x"

$\log(-1) =$ *doesn't exist*	$\ln(-1) =$ *doesn't exist*
$\log 0 =$ *doesn't exist*	$\ln 0 =$ *doesn't exist*
$\log 0.5 \approx -0.3010$	$\ln 0.5 \approx -0.6931$
$\log 1 = 0$	$\ln 1 = 0$
$\log 2 \approx 0.3010$	$\ln 2 \approx 0.6931$
$\log 10 = 1$	$\ln e = 1$

Domain

The domain of a logarithmic function of the form $f(x)=\log_b x$ is the set of all positive real numbers.
(Inside the log has to be > 0)

Find the domain of $f(x)=\log_4(x-5)$

$x-5>0 \Rightarrow x>5$ *Domain:* $(5,\infty)$

Properties of Logarithmic Functions

Product Rule

$$\log_b MN=\log_b M+\log_b N$$

Power Rule

$$\log_b M^p=p\log_b M$$

Quotient Rule

$$\log_b \frac{M}{N}=\log_b M-\log_b N$$

$\log_b 1=0$ $\log_b b=1$ $\log_b b^x=x$

Example

Use the properties of logarithms to rewrite $\log_a\left(\frac{mnq}{p^2r^4}\right)$

Solution

$$\log_a\left(\frac{mnq}{p^2r^4}\right)=\log_a(mnq)-\log_a\left(p^2r^4\right) \qquad \textit{Quotient Rule}$$

$$=\log_a m+\log_a n+\log_a q-\left(\log_a p^2+\log_a r^4\right) \qquad \textit{Product Rule}$$

$$=\log_a m+\log_a n+\log_a q-\log_a p^2-\log_a r^4$$

$$=\log_a m+\log_a n+\log_a q-2\log_a p-4\log_a r \qquad \textit{Power Rule}$$

Changing Logarithmic Bases

$$\log_b M = \frac{\log_a M}{\log_a b} \qquad\qquad \log_b M = \frac{\log M}{\log b} \quad \textbf{or} \quad \log_b M = \frac{\ln M}{\ln b}$$

Example

Find: $\log_5 27$

Solution

$$\log_5 27 = \frac{\ln 27}{ln5}$$

ln(27) / ln(5)

$$\approx 2.05$$

Property of *Logarithmic*

- $M = N \quad \leftrightarrow \quad \ln M = \ln N$
  1. Isolate the exponential expression
  2. Take the natural logarithm on both sides of the equation
  3. Simplify using one of the following properties: $\ln b^x = x\ln b$ *or* $\ln e^x = x$
  4. Solve for the variable

Example

Solve $3^{2x} = 4^{x+1}$

Solution

$$\ln 3^{2x} = \ln 4^{x+1}$$

$$2x\ln 3 = (x+1)\ln 4$$

$$2x\ln 3 = x\ln 4 + \ln 4$$

$$2x\ln 3 - x\ln 4 = \ln 4$$

$$x(2\ln 3 - \ln 4) = \ln 4$$

$$x = \frac{\ln 4}{2\ln 3 - \ln 4}$$

ln(4) / (2 ln(3) − ln(4))

$$\approx 1.710$$

Solving Logarithmic Equations

1. Express the equation in the form $\log_b M = c$

2. Use the definition of a logarithm to rewrite the equation in exponential form:

$$\log_b M = c \quad \Rightarrow \quad b^c = M$$

3. Solve for the variable

4. Check proposed solution in the original equation. Include only the set for M > 0

Example

Solve $\quad \log_2 x - \log_2 (x-1) = 1$

Solution

$\log_2 \frac{x}{x-1} = 1$ *Write in exponential form*

$\frac{x}{x-1} = 2^1 = 2$

$x = 2(x-1)$

$x = 2x - 2$

$-x = -2$

$x = 2$ *Check*: $\log_2 2 - \log_2 (2-1) = 1$

For any M > 0, N > 0, b > 0, ≠ 1

$$\log_b M = \log_b N \leftrightarrow M = N$$

Check proposed solution in the original equation. Include only the set inside the log for > 0

Example

Solve: $\quad \log(x+6) - \log(x+2) = \log x$

Solution

$\log(x+6) - \log(x+2) = \log x$ *Quotient Rule*

$\log \frac{x+6}{x+2} = \log x$

$\frac{x+6}{x+2} = x$ *Multiply by x + 2*

$x + 6 = x(x+2)$

$x + 6 = x^2 + 2x$

$0 = x^2 + 2x - x - 6$

$x^2 + x - 6 = 0$ *Solve for x*

$x = -3,\ 2$

<u>*Check*</u>: $x = -3 \rightarrow \log(-3+6) - \log(-3+2) = \log(-3)$ *Or Domain*

$x = 2 \rightarrow \log(2+6) - \log(2+2) = \log(2)$

Solution: $\boxed{x = 2}$

$\log_e x = \ln x$

$log_{10}\ x = logx$

Exercises *Section* 2.6 – Exponential & Logarithmic Functions

Solve

1. $4^{2x-1} = 64$

2. $3^{1-x} = \frac{1}{27}$

3. $9^{x} = \frac{1}{\sqrt[3]{3}}$

4. $5^{3x-6} = 125$

5. $8^{x+2} = 4^{x-3}$

Solve

6. $7e^{2x} - 5 = 58$

7. $4\ln(3x) = 8$

8. $\ln(x-3) = \ln(7x-23) - \ln(x+1)$

Use the properties of logarithms to rewrite

9. $\log_b\left(\frac{x^3 y}{z^2}\right)$

10. $\log_b\left(\frac{\sqrt[3]{x}\, y^4}{z^5}\right)$

11. $\log_b \sqrt[n]{\frac{x^3 y^5}{z^m}}$

12. $\log_p \sqrt[3]{\frac{m^5 n^4}{t^2}}$

13. $log_a \sqrt[4]{\frac{m^8\, n^{12}}{a^3\, b^5}}$

Section 2.7 – Derivatives of Exponential and Logarithmic Functions

Derivative of the Natural Exponential Function

$$\frac{d}{dx}\left[e^{x}\right]=e^{x} \qquad \frac{d}{dx}\left[e^{U}\right]=e^{U}\frac{dU}{dx}$$

Derivative of a^x

$$\frac{d}{dx}\left(a^{x}\right)=\left(\ln a\right)a^{x} \qquad \frac{d}{dx}\left[a^{g(x)}\right]=\ln(a)a^{g(x)}g'(x)$$

Example

Find the derivative of each function

a) $y=e^{5x}$

$$y'=(5x)'\,e^{5x}$$

$$=5e^{5x}$$

b) $s=3^{t}$

$$\frac{ds}{dt}=\left(\ln 3\right)3^{t}$$

c) $y=10e^{3x^2}$

$$y'=10e^{3x^2}\left(3x^2\right)'$$

$$=10e^{3x^2}\left(6x\right)$$

$$=60x\,e^{3x^2}$$

d) $s=8.10^{1/t}$

$$\frac{ds}{dt}=8\left(\ln 10\right)10^{1/t}\left(t^{-1}\right)'$$

$$=-\frac{8\left(\ln 10\right)10^{1/t}}{t^{2}}$$

Example

Find the derivative of $y = e^{x^2+1}\sqrt{5x+2}$

<u>*Solution*</u>

$$f = e^{x^2+1} \qquad\qquad g = (5x+2)^{1/2}$$

$$f' = 2xe^{x^2+1} \qquad g' = \frac{1}{2}5(5x+2)^{-1/2} = \frac{5}{2\sqrt{5x+2}}$$

$$y' = (2x)e^{x^2+1}\sqrt{5x+2} + e^{x^2+1}\frac{5}{2\sqrt{5x+2}}$$

$$= 2xe^{x^2+1}\sqrt{5x+2}\,\frac{2\sqrt{5x+2}}{2\sqrt{5x+2}} + \frac{5e^{x^2+1}}{2\sqrt{5x+2}}$$

$$= \frac{4xe^{x^2+1}(5x+2)}{2\sqrt{5x+2}} + \frac{5e^{x^2+1}}{2\sqrt{5x+2}}$$

$$= \frac{20x^2e^{x^2+1} + 8xe^{x^2+1} + 5e^{x^2+1}}{2\sqrt{5x+2}}$$

$$= \frac{e^{x^2+1}\left(20x^2+8x+5\right)}{2\sqrt{5x+2}}$$

Example

The demand function for the product is modeled by $p = 50e^{-0.0000125x}$ where p is the price per unit in dollars and x is the number of units. What price will yield maximum revenue?

<u>*Solution*</u>

$$R = xp = 50xe^{-0.0000125x}$$

$$R' = 50e^{-0.0000125x} + (-0.0000125)50xe^{-0.0000125x}$$

$$R' = 50e^{-0.0000125x} - 0.000625xe^{-0.0000125x}$$

$$R' = e^{-0.0000125x}(50 - 0.000625x) = 0$$

$$50 - 0.000625x = 0$$

$$-0.000625x = -50$$

$$x = \frac{-50}{-0.000625} = 80000$$

$$p(x = 80000) = 50e^{-0.0000125(80000)}$$

$$\approx \$18.39 \ / \, unit$$

Example

A company sells 990 units of a new product in the first year and 3213 units in the fourth year. They expect that sales can be approximated by a logistic function, leveling off at around 100,000 in the long run given by the formula

$$S(t)=\frac{100,000}{1+100e^{-100,000kt}}$$

a) Find ***k*** and rewrite the function

<u>*Solution*</u>

$$S(4)=3213$$

$$3213=\frac{100,000}{1+100e^{-100,000k4}}$$

$$3213=\frac{100,000}{1+100e^{-400,000k}}$$

$$3213+321300e^{-400,000k}=100,000$$ *Subtract 3213 from both sides*

$$321300e^{-400,000k}=96787$$ *Divide both sides by 321300*

$$e^{-400,000k}=0.3012$$ *ln both sides*

$$-400,000k=\ln 0.3012$$

$$k=\frac{\ln 0.3012}{-400,000}$$

$$\approx 3\times 10^{-6}$$

$$S(t)=\frac{100,000}{1+100e^{-0.3t}}$$

b) Find the rate of change of sales after 4 years

<u>*Solution*</u>

$$S'=\frac{3,000,000e^{-0.3t}}{\left(1+100e^{-0.3t}\right)^2}$$

$$S'(4)=\frac{3,000,000e^{-0.3(4)}}{\left(1+100e^{-0.3(4)}\right)^2}$$ $3000000e((-)0.3*4)/\left(1+100e((-)0.3*4)\right)\wedge 2$

$$\approx 933$$

Derivatives of Logarithmic

***Derivative* of** $\log_a x$

$$\frac{d}{dx}\left[\log_a |x|\right] = \frac{1}{(\ln a)x} \qquad \frac{d}{dx}\left[\log_a |g(x)|\right] = \frac{1}{(\ln a)} \bullet \frac{g'(x)}{g(x)}$$

Derivative of $\ln$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx} = \frac{u'}{u}$$

Other Bases

$$\frac{d}{dx}\left[a^x\right] = a^x \ln a \qquad \frac{d}{dx}\left[a^u\right] = a^u (\ln a)\frac{du}{dx}$$

Example

Find the derivative of each function

a) $f(x) = \ln 6x$

$$f'(x) = \frac{(6x)'}{6x}$$

$$= \frac{6}{6x}$$

$$= \frac{1}{x}$$

b) $f(x) = \log x$

$$f' = \frac{1}{(\ln 10)x}$$

c) $f(x) = \ln\left(x^2 + 1\right)$ $\qquad \frac{d}{dx}(\ln u) = \frac{u'}{u}$

$$f' = \frac{2x}{x^2 + 1}$$

Example

Find the derivative of function $f(x) = \log_2\left(3x^2 - 4x\right)$

<u>*Solution*</u>

$$f' = \frac{1}{\ln 2}\frac{6x-4}{3x^2-4x}$$

$$\frac{d}{dx}\left[\log_a |g(x)|\right] = \frac{1}{(\ln a)} \cdot \frac{g'(x)}{g(x)}$$

$$\boxed{= \frac{6x-4}{\ln 2\left(3x^2-4x\right)}}$$

Example

Find the derivative of function $y = \ln|5x|$

<u>*Solution*</u>

$$y' = \frac{5}{5x}$$

$$\boxed{= \frac{1}{x}}$$

Example

Find the derivative of function $y = 3x\ln x^2$

<u>*Solution*</u>

$$y' = 3\ln x^2 + 3x\frac{2x}{x^2}$$

$$\boxed{= 3\ln x^2 + 6}$$

Example

Find the derivative of function $y = x\ 3^{x+1}$

<u>*Solution*</u>

$$f = x \qquad g = 3^{x+1}$$

$$f' = 1 \qquad g' = 3^{x+1}\ln(3)$$

$$y' = 3^{x+1} + x\ 3^{x+1}\ln 3$$

$$\boxed{= 3^{x+1}\left[1 + x\ \ln 3\right]}$$

Example

Find the derivative of function $s(t)=\dfrac{\log_8\left(t^{3/2}+1\right)}{t}$

<u>*Solution*</u>

$$f=\log_8\left(t^{3/2}+1\right) \qquad g=t$$

$$f'=\frac{1}{\ln 8}\frac{\frac{3}{2}t^{1/2}}{t^{3/2}+1} \qquad g=1$$

$$s'=\frac{\frac{1}{\ln 8}\frac{\frac{3}{2}t^{1/2}}{t^{3/2}+1}\bullet t-\log_8\left(t^{3/2}+1\right)}{t^2}$$

$$=\frac{\frac{1}{\ln 8}\frac{\frac{3}{2}t^{3/2}}{t^{3/2}+1}-\log_8\left(t^{3/2}+1\right)}{t^2}\bullet\frac{2\ln 8\left(t^{3/2}+1\right)}{2\ln 8\left(t^{3/2}+1\right)}$$

$$\boxed{=\frac{3t^{3/2}-2\left(t^{3/2}+1\right)(\ln 8)\log_8\left(t^{3/2}+1\right)}{t^2\left(t^{3/2}+1\right)\ln 8}}$$

Exercises *Section* 2.7 – Derivatives of Exponential and Logarithmic Functions

Find the derivative:

1. $f(x) = e^{3x}$

2. $f(x) = e^{-2x^3}$

3. $f(x) = 4e^{x^2}$

4. $f(x) = e^{-2x}$

5. $f(x) = x^2 e^x$

6. $f(x) = \frac{e^x + e^{-x}}{2}$

7. $f(x) = \frac{e^x}{x^2}$

8. $f(x) = x^2 e^x - e^x$

9. $f(x) = (1+2x)e^{4x}$

10. $y = x^2 e^{5x}$

11. $f(x) = \frac{100,000}{1+100e^{-0.3x}}$

12. $y = x^2 e^{-2x}$

13. $y = \frac{e^x + e^{-x}}{x}$

14. $y = \sqrt{e^{2x^2} + e^{-2x^2}}$

15. $y = \frac{x}{e^{2x}}$

16. $y = \ln\sqrt{x+5}$

17. $y = (3x+7)\ln(2x-1)$

18. $y = e^{x^2}\ln x$

19. $y = \log_7 \sqrt{4x-3}$

20. $f(x) = \ln\sqrt[3]{x+1}$

21. $f(x) = \ln\left[x^2\sqrt{x^2+1}\right]$

22. $y = \ln \frac{1+e^x}{1-e^x}$

23. $y = \ln \frac{x^2}{x^2+1}$

24. $y = \ln\left[\frac{x^2(x+1)^3}{(x+3)^{1/2}}\right]$

25. $y = \ln(x^2+1)$

26. $y = \frac{\ln x}{e^{2x}}$

27. $f(x) = \ln\left(x^2-4\right)$

28. $f(x) = x^2 \ln x$

29. $f(x) = -\frac{\ln x}{x^2}$

30. $f(x) = \frac{e^{\sqrt{x}}}{\ln\left(\sqrt{x}+1\right)}$

31. $f(x) = e^{2x}\ln\left(xe^x+1\right)$

32. $f(x) = \frac{xe^x}{\ln\left(x^2+1\right)}$

33. $f(x) = 2\ln\left(x^2-3x+4\right)$

34. $f(x) = e^{x^2+3x+1}$

35. $f(x) = 3\ln\left(1+x^2\right)$

36. $f(x) = (1+\ln x)^3$

37. $f(x) = (x-2\ln x)^4$

38. $f(x) = \frac{e^x}{x^2+1}$

39. $f(x) = \frac{1-e^x}{1+e^x}$

40. $f(x) = \frac{\ln x}{1+x}$

41. $f(x) = \frac{2x}{1+\ln x}$

42. $f(x) = x^2 e^x$

43. $f(x) = x^3 \ln x$

44. $f(x) = 6x^4 \ln x$

45. $f(x) = 2x^3 e^x$

46. $f(x) = \dfrac{3e^x}{1+e^x}$

47. $f(x) = 5e^x + 3x + 1$

48. $f(x) = \dfrac{\ln x}{2x+5}$

49. $f(x) = -2\ln x + x^2 - 4$

50. $f(x) = e^x + x - \ln x$

51. $f(x) = \ln x + 2e^x - 3x^2$

52. $f(x) = \ln x^8$

53. $f(x) = 5x - \ln x^5$

54. $f(x) = \ln x^2 + 4e^x$

55. $f(x) = \ln x^{10} + 2\ln x$

56. Find the second derivative of $y = 3\,e^{5x^3+1}$

57. Find the equation of the tangent line to $f(x) = e^x$ at the point (0, 1)

58. Find the equation of the tangent line to $f(x) = e^x$ at the point (1, e)

59. Find the equation of the tangent lines to $f(x) = 4e^{-8x}$ at the points (0, 4)

60. Find the equation of the tangent line to $y = 4xe^{-x} + 5$ at $x = 1$

61. The percentage of people of any particular age group that will die in a given year may be approximated by the formula

$$P(t) = 0.00239e^{0.0957t}$$

where t is the age of the person in years

a) Find $P(25)$

b) Find $P'(25)$

62. Assume the cost of a gallon of milk is \$2.90. With continuous compounding, find the time it would take the cost to be 5 times as much (to the nearest tenth of a year), at an annual inflation rate of 6 %.

63. The sales in thousands of a new type of product are given by $S(t) = 30 - 90e^{-0.5t}$, where t represents time in years. Find the rate of change of sales at the time when $t = 3$

64. A company's total cost, in millions of dollars, is given by $C(t) = 300 - 70e^{-t}$ where t = time in years. Find the marginal cost when $t = 3$.

65. A company's total cost, in millions of dollars, is given by $C(t) = 280 - 30e^{-t}$ where t = time in years. Find the marginal cost when $t = 2$.

66. The demand function for a certain book is given by the function $x = D(p) = 70e^{-0.006p}$. Find the marginal demand $D'(p)$

67. Suppose that the amount in grams of a radioactive substance present at time t (in years) is given by $A(t) = 840e^{-0.63t}$. Find the rate of change of the quantity present at the time when $t = 2$.

68. Researchers have found that the maximum number of successful trials that a laboratory rat can complete in a week is given by

$$P(t) = 53\left(1 - e^{-0.4t}\right)$$

where t is the number of weeks the rat has been trained. Find the rate of change $P'(t)$.

69. When a telephone wire is hung between two poles, the wire forms a U-shape curve called a Catenary. For instance, the function $y = 30\left(e^{x/60} + e^{-x/60}\right)$ $-30 \le x \le 30$ models the shape of the telephone wire strung between two poles that are 60 ft. apart (x & y are measured in ft.). Show that the lowest point on the wire is midway between two poles. How much does the wire sag between the two poles?

70. Find $f''(x)$ for $f(x) = \frac{\ln x}{7x}$, then find $f''(0)$ and $f''(2)$

71. Suppose the average test score p and was modeled by $p = 92.3 - 16.9\ln(t+1)$, where t is the time in months. How would the rate at which the average test score changed after 1 year?

72. Suppose that the population of a certain collection of rare ants is given by

$$P(t) = (t+100)\ln(t+2)$$

Where t represents the time in days. Find the rates of change of the population on the second day and on the eighth day.

73. Suppose that the demand function for x units of a certain item is $P(x) = 100 + \dfrac{180\ln(x+5)}{x}$ where P is the price per unit, in dollars. Find the marginal revenue.

74. The population of coyotes in the northwestern portion of Alabama is given by the formula $P(t)=\left(t^2+100\right)\ln\left(t+2\right)$, where t represents the time in years since 2000 (the year 2000 corresponds to $(t=0)$ Find the rate of change of the coyote population in 2013 $(t=13)$.

75. Students in a math class took a final exam. They took equivalent forms of the exam in monthly intervals thereafter. The average score $S(t)$, in percent, after t months was found to be given by

$$S(t)=73-17\ln\left(t+1\right), \qquad t\geq 0$$

Find $S'(t)$.

76. Suppose that the population of a town is given by $P(t)=8\ln\sqrt{8t+7}$ where t is the time in years after 1980 and P is the population of the town in thousands. Find $P'(t)$.

77. The following formula accurately models the relationship between the size of a certain type of tumor and the amount of time that it has been growing:

$$V(t)=450\left(1-e-0.0022t\right)^3$$

where t is in months and $V(t)$ is measured in cubic centimeters. Calculate the rate of change of tumor volume at 80 months.

78. A yeast culture at room temperature (68° F) is placed in a refrigerator set at a constant temperature of 38° F. After t hours, the temperature T of the culture is given approximately by

$$T=30e^{-0.58t}+38 \quad t\geq 0$$

What is the rate of change of temperature of the culture at the end of 1 hour? At the end of 4 hours?

79. A mathematical model for the average age of a group of people learning to type is given by

$$N\left(t\right)=10+6\ln t \quad t\geq 1$$

Where $N\left(t\right)$ is the number of words per minute typed after t hours of instruction and practice (2 hours per day, 5 days per week). What is the rate of learning after 10 hours of instruction and practice? After 100 hours?