

## ***SOLUTION***

## ***Section 3.5 – The Ratio and Root Tests***

### ***Exercise***

Use the Ratio Test to determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

### **Solution**

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} &= \lim_{n \rightarrow \infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \\ &= \lim_{n \rightarrow \infty} \frac{2}{(n+1)} \\ &= 0 < 1\end{aligned}$$

Therefore, the given series ***converges*** by the Ratio Test.

### ***Exercise***

Use the Ratio Test to determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{n3^{n-1}}$$

### **Solution**

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{2^{n+2}}{(n+1)3^n} \cdot \frac{n3^{n-1}}{2^{n+1}} &= \lim_{n \rightarrow \infty} \frac{2n}{3(n+1)} \\ &= \lim_{n \rightarrow \infty} \frac{2n}{3n} \\ &= \frac{2}{3} < 1\end{aligned}$$

Therefore, the given series ***converges*** by the Ratio Test.

### ***Exercise***

Use the Ratio Test to determine if the series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$$

### **Solution**

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{3^{n+3}}{\ln(n+1)} \cdot \frac{\ln n}{3^{n+2}} &= \lim_{n \rightarrow \infty} \frac{3 \ln n}{\ln(n+1)} \\ &= 3 \lim_{n \rightarrow \infty} \frac{\ln n}{\ln(n+1)}\end{aligned}$$

$$\begin{aligned}
&= 3 \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} \\
&= 3 \lim_{n \rightarrow \infty} \frac{n+1}{n} \\
&= 3 > 1
\end{aligned}$$

Therefore, the given series **diverges** by the Ratio Test.

### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n^2(n+2)!}{n!3^{2n}}$

### Solution

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{(n+1)^2(n+3)!}{(n+1)!3^{2(n+1)}} \cdot \frac{n!3^{2n}}{n^2(n+2)!} &= \lim_{n \rightarrow \infty} \frac{(n+1)^2(n+3)}{n^2(n+1) \cdot 3^2} \\
&= \lim_{n \rightarrow \infty} \frac{(n+1)(n+3)}{n^2 \cdot 3^2} \\
&= \lim_{n \rightarrow \infty} \frac{n^2 + 4n + 3}{9n^2} \\
&= \frac{1}{9} < 1
\end{aligned}$$

Therefore, the given series **converges** by the Ratio Test.

### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n5^n}{(2n+3)\ln(n+1)}$

### Solution

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{(n+1) \cdot 5^{n+1}}{(2n+5)\ln(n+2)} \cdot \frac{(2n+3)\ln(n+1)}{n \cdot 5^n} &= \lim_{n \rightarrow \infty} \frac{5(n+1)(2n+3)\ln(n+1)}{n(2n+5)\ln(n+2)} \\
&= \lim_{n \rightarrow \infty} \frac{10n^2 + 10n + 6}{2n^2 + 5n} \cdot \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{\ln(n+2)} \\
&= 5 \cdot \lim_{n \rightarrow \infty} \frac{\frac{1}{n+1}}{\frac{1}{n+2}} \\
&= 5 \cdot \lim_{n \rightarrow \infty} \frac{n+2}{n+1}
\end{aligned}$$

$$= 5 > 1$$

Therefore, the given series *diverges* by the Ratio Test.

### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{99^n}{n!}$

### Solution

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \frac{99^{n+1}}{(n+1)!} \bigg/ \frac{99^n}{n!} & \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{99}{n+1} \\ &= 0 < 1 \end{aligned}$$

Therefore, the given series *converges* by the Ratio Test.

### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n^5}{2^n}$

### Solution

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \frac{(n+1)^5}{2^{n+1}} \bigg/ \frac{n^5}{2^n} & \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \left( \frac{n+1}{n} \right)^5 \\ &= \frac{1}{2} < 1 \end{aligned}$$

Therefore, the given series *converges* by the Ratio Test.

### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

### Solution

$$\rho = \lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+1)^{n+1}} \bigg/ \frac{n!}{n^n} \quad \rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \\
&= \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^n \\
&= \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} \\
&= \frac{1}{e} < 1
\end{aligned}$$

Therefore, the given series **converges** by the Ratio Test.

### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

### Solution

$$\begin{aligned}
\rho &= \lim_{n \rightarrow \infty} \frac{(2(n+1))!}{((n+1)!)^2} \bigg/ \frac{(2n)!}{(n!)^2} \\
&= \lim_{n \rightarrow \infty} \frac{(2(n+1))!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!} \\
&= \lim_{n \rightarrow \infty} \frac{(2n+2)(2n+1)}{(n+1)^2} \\
&= \lim_{n \rightarrow \infty} \frac{4n^2}{n^2} \\
&= 4 > 1
\end{aligned}$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Therefore, the given series **diverges** by the Ratio Test.

### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{1}{5^n}$

### Solution

$$\rho = \lim_{n \rightarrow \infty} \frac{1}{5^{n+1}} \cdot \frac{5^n}{1} = \frac{1}{5} < 1$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Therefore, the given series **converges** by the Ratio Test.

### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{1}{n!}$

### Solution

$$\begin{aligned}\rho &= \lim_{n \rightarrow \infty} \frac{1}{(n+1)!} \cdot \frac{n!}{1} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n+1} \\ &= 0 < 1\end{aligned}$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Therefore, the given series **converges** by the Ratio Test.

### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=0}^{\infty} \frac{n!}{3^n}$

### Solution

$$\begin{aligned}\rho &= \lim_{n \rightarrow \infty} \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!} \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{3} \\ &= \infty\end{aligned}$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Therefore, the given series **diverges** by the Ratio Test.

### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

### Solution

$$\begin{aligned}\rho &= \lim_{n \rightarrow \infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \\ &= \lim_{n \rightarrow \infty} \frac{2}{n+1} \\ &= 0 < 1\end{aligned}$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Therefore, the given series **converges** by the Ratio Test.

### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} n\left(\frac{6}{5}\right)^n$

### Solution

$$\begin{aligned}\rho &= \lim_{n \rightarrow \infty} \frac{(n+1)\left(\frac{6}{5}\right)^{n+1}}{n\left(\frac{6}{5}\right)^n} \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{n} \left(\frac{6}{5}\right) \\ &= \frac{6}{5} > 1\end{aligned}$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Therefore, the given series **diverges** by the Ratio Test.

### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} n\left(\frac{7}{8}\right)^n$

### Solution

$$\begin{aligned}\rho &= \lim_{n \rightarrow \infty} \frac{(n+1)\left(\frac{7}{8}\right)^{n+1}}{n\left(\frac{7}{8}\right)^n} \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{n} \left(\frac{7}{8}\right) \\ &= \frac{7}{8} < 1\end{aligned}$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Therefore, the given series **converges** by the Ratio Test.

### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n}{4^n}$

### Solution

$$\begin{aligned}\rho &= \lim_{n \rightarrow \infty} \frac{n+1}{4^{n+1}} \cdot \frac{4^n}{n} \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{4n}\end{aligned}$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \frac{1}{4} < 1$$

Therefore, the given series *converges* by the Ratio Test.

### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{5^n}{n^4}$

### Solution

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \frac{5^{n+1}}{(n+1)^4} \cdot \frac{n^4}{5^n} & \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n \rightarrow \infty} 5 \left( \frac{n}{n+1} \right)^4 \\ &= 5 > 1 \end{aligned}$$

Therefore, the given series *diverges* by the Ratio Test.

### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

### Solution

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} & \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{1}{3} \left( \frac{n+1}{n} \right)^3 \\ &= \frac{1}{3} < 1 \end{aligned}$$

Therefore, the given series *converges* by the Ratio Test.

### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

### Solution

$$\rho = \lim_{n \rightarrow \infty} \frac{n+3}{(n+1)(n+2)} \cdot \frac{n(n+1)}{(n+2)} \quad \rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{n(n+3)}{(n+2)^2}$$

$$= 1$$

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{(n+2)}{n(n+1)} = 0$$

Therefore, the given series *converges Conditionally* by the Ratio Test.

### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

### Solution

$$\rho = \lim_{n \rightarrow \infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$$

$$= \lim_{n \rightarrow \infty} \frac{2}{n+1}$$

$$= 0 < 1$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Therefore, the given series *converges* by the Ratio Test.

### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \left(\frac{3}{2}\right)^n}{n^2}$

### Solution

$$\rho = \lim_{n \rightarrow \infty} \frac{\left(\frac{3}{2}\right)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{\left(\frac{3}{2}\right)^n}$$

$$= \lim_{n \rightarrow \infty} \frac{3}{2} \left(\frac{n}{n+1}\right)^2$$

$$= \frac{3}{2} > 1$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Therefore, the given series *diverges* by the Ratio Test.



### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

### Solution

$$\begin{aligned}\rho &= \lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+1)3^{n+1}} \cdot \frac{n3^n}{n!} \\ &= \lim_{n \rightarrow \infty} \frac{n}{3} \\ &= \infty\end{aligned}$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Therefore, the given series **diverges** by the Ratio Test.

### Exercise

Use the Ratio Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

### Solution

$$\begin{aligned}\rho &= \lim_{n \rightarrow \infty} \frac{(2n+2)!}{(n+1)^5} \cdot \frac{n^5}{(2n)!} \\ &= \lim_{n \rightarrow \infty} (2n+1)(2n+2) \left( \frac{n}{n+1} \right)^5 \\ &= \infty\end{aligned}$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Therefore, the given series **diverges** by the Ratio Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \sqrt[n]{\frac{4^n}{(3n)^n}} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{4}{3n} \right)^n} \\ &= \lim_{n \rightarrow \infty} \frac{4}{3n} = 0 < 1\end{aligned}$$

Therefore, the given series **converges** by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{4n+3}{3n-5}\right)^n} &= \lim_{n \rightarrow \infty} \left(\frac{4n+3}{3n-5}\right) \\ &= \frac{4}{3} > 1\end{aligned}$$

Therefore, the given series **diverges** by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \left(\ln\left(e^2 + \frac{1}{n}\right)\right)^{n+1}$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \sum_{n=1}^{\infty} \sqrt[n]{\left(\ln\left(e^2 + \frac{1}{n}\right)\right)^{n+1}} &= \lim_{n \rightarrow \infty} \sum_{n=1}^{\infty} \left(\ln\left(e^2 + \frac{1}{n}\right)\right)^{1+\frac{1}{n}} \\ &= \ln\left(e^2\right) \\ &= 2 > 1\end{aligned}$$

Therefore, the given series **diverges** by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \sqrt[n]{\sin^n\left(\frac{1}{\sqrt{n}}\right)} &= \lim_{n \rightarrow \infty} \sin\left(\frac{1}{\sqrt{n}}\right) \\ &= \sin(0) \\ &= 0 < 1\end{aligned}$$

Therefore, the given series **converges** by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \sqrt[n]{\left(1 - \frac{1}{n}\right)^{n^2}} &= \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n \\ &= e^{-1} < 1\end{aligned}$$

Therefore, the given series **converges** by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \sqrt[n]{\left|\frac{e^{2n}}{n^n}\right|} &= \lim_{n \rightarrow \infty} \frac{e^2}{n} \\ &= 0 < 1\end{aligned}$$

Therefore, the given series **converges absolutely** by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{1}{5^n}$

### Solution

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{5^n}} = \frac{1}{5} < 1$$

Therefore, the given series **converges** by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{1}{n^n}$

### Solution

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{1}{n^n} \right|} = \lim_{n \rightarrow \infty} \frac{1}{n} \\ = 0 < 1$$

Therefore, the given series **converges absolutely** by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \left( \frac{n}{2n+1} \right)^n$

### Solution

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \left( \frac{n}{2n+1} \right)^n \right|} = \lim_{n \rightarrow \infty} \frac{n}{2n+1} \\ = \frac{1}{2} < 1$$

Therefore, the given series **converges** by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \left( \frac{2n}{n+1} \right)^n$

### Solution

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \left( \frac{2n}{n+1} \right)^n \right|} = \lim_{n \rightarrow \infty} \frac{2n}{n+1} \\ = 2 > 1$$

Therefore, the given series **diverges** by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \left( \frac{3n+2}{n+3} \right)^n$

### Solution

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \left( \frac{3n+2}{n+3} \right)^n \right|} = \lim_{n \rightarrow \infty} \frac{3n+2}{n+3} \\ = \underline{3 > 1}$$

Therefore, the given series *diverges* by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \left( \frac{n-2}{5n+1} \right)^n$

### Solution

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \left( \frac{n-2}{5n+1} \right)^n \right|} = \lim_{n \rightarrow \infty} \left| \frac{n-2}{5n+1} \right| \\ = \underline{\frac{1}{5} < 1}$$

Therefore, the given series *converges* by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$

### Solution

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{(-1)^n}{(\ln n)^n} \right|} = \lim_{n \rightarrow \infty} \left| \frac{1}{\ln n} \right| \\ = \underline{0 < 1}$$

Therefore, the given series *converges* by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \left( \frac{-3n}{2n+1} \right)^{3n}$

### Solution

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \left( \frac{-3n}{2n+1} \right)^{3n} \right|} = \lim_{n \rightarrow \infty} \left| \left( \frac{-3n}{2n+1} \right)^3 \right| \\ = \left( \frac{3}{2} \right)^3$$

$$\boxed{= \frac{27}{8} > 1}$$

Therefore, the given series *diverges* by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} (2\sqrt[n]{n} + 1)^n$

### Solution

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| (2\sqrt[n]{n} + 1)^n \right|} = \lim_{n \rightarrow \infty} \left| (2\sqrt[n]{n} + 1) \right|$$

$$\text{Let } x = \lim_{n \rightarrow \infty} \sqrt[n]{n} \Rightarrow \ln x = \lim_{n \rightarrow \infty} \ln \sqrt[n]{n}$$

$$\ln x = \lim_{n \rightarrow \infty} \frac{\ln n}{n}$$

$$= \lim_{n \rightarrow \infty} \frac{1/n}{1}$$

$$= 0$$

$$x = e^0 = 1 = \lim_{n \rightarrow \infty} \sqrt[n]{n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| (2\sqrt[n]{n} + 1)^n \right|} = 2(1) + 1$$

$$\boxed{= 3 > 1}$$

Therefore, the given series *diverges* by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=0}^{\infty} e^{-3n}$

### Solution

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| e^{-3n} \right|} = \lim_{n \rightarrow \infty} \left| e^{-3} \right|$$

$$\boxed{= \frac{1}{e^3} < 1}$$

Therefore, the given series *converges* by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n}{3^n}$

### Solution

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{n}{3^n} \right|} = \lim_{n \rightarrow \infty} \left| \frac{\sqrt[n]{n}}{3} \right|$$

$$\text{Let } x = \sqrt[n]{n} \Rightarrow \ln x = \ln \sqrt[n]{n} = \frac{\ln n}{n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \ln x &= \lim_{n \rightarrow \infty} \frac{\ln n}{n} \\ &= \lim_{n \rightarrow \infty} \frac{1/n}{1} \\ &= 0 \end{aligned}$$

$$x = e^0 = 1 = \lim_{n \rightarrow \infty} \sqrt[n]{n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{n}{3^n} \right|} = \frac{1}{3} < 1$$

Therefore, the given series **converges** by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \left( \frac{n}{500} \right)^n$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left( \frac{n}{500} \right)^n \right|} &= \lim_{n \rightarrow \infty} \left| \frac{n}{500} \right| \\ &= \infty > 1 \end{aligned}$$

Therefore, the given series **diverges** by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n^2} \right)^n$

### Solution

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \left( \frac{1}{n} - \frac{1}{n^2} \right)^n \right|} = \lim_{n \rightarrow \infty} \left| \frac{1}{n} - \frac{1}{n^2} \right|$$

$$\underline{= 0 < 1}$$

Therefore, the given series *converges* by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \left( \frac{\ln n}{n} \right)^n$

### Solution

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \left( \frac{\ln n}{n} \right)^n \right|} = \lim_{n \rightarrow \infty} \left| \frac{\ln n}{n} \right|$$

$$\underline{= 0 < 1}$$

Therefore, the given series *converges* by the Root Test.

### Exercise

Use the Root Test to determine if the series converges or diverges.  $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

### Solution

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{n}{(\ln n)^n} \right|} = \lim_{n \rightarrow \infty} \left| \frac{n^{1/n}}{\ln n} \right|$$

$$\underline{= 0 < 1}$$

Therefore, the given series *converges* by the Root Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$

### Solution

$$\lim_{n \rightarrow \infty} \frac{(n+1)^{\sqrt{2}}}{2^{n+1}} \cdot \frac{2^n}{n^{\sqrt{2}}} = \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \frac{(n+1)^{\sqrt{2}}}{n^{\sqrt{2}}}$$

$$= \frac{1}{2} \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \right)^{\sqrt{2}}$$



$$= \frac{1}{2} < 1$$

Therefore, the given series *converges* by the Ratio Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} n^2 e^{-n}$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2} &= \lim_{n \rightarrow \infty} \frac{1}{e} \cdot \left( \frac{n+1}{n} \right)^2 \\ &= \frac{1}{e} < 1 \end{aligned}$$

Therefore, the given series *converges* by the Ratio Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} &= \lim_{n \rightarrow \infty} \frac{n+1}{10} \\ &= \infty > 1 \end{aligned} \quad \rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Therefore, the given series *converges* by the Ratio Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{\frac{(\ln n)^n}{n^n}} &= \lim_{n \rightarrow \infty} \frac{\ln n}{n} && \text{Hopital Rule} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \end{aligned}$$

$$= 0 < 1$$

Therefore, the given series **converges** by the Root Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n2^n(n+1)!}{3^n n!}$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{(n+1)2^{n+1}(n+2)!}{3^{n+1}(n+1)!} \cdot \frac{3^n n!}{n2^n(n+1)!} &= \lim_{n \rightarrow \infty} \frac{2}{3} \frac{(n+1)(n+2)}{n(n+1)} & \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{2}{3} \left( \frac{n+2}{n} \right) \\ &= \frac{2}{3} < 1 \end{aligned}$$

Therefore, the given series **converges** by the Ratio Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} &= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(2n+1)(2n+2)} & \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{2(2n+1)(n+1)} \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{4n+2} \\ &= \frac{1}{4} < 1 \end{aligned}$$

Therefore, the given series **converges** by the Ratio Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

### Solution

Using comparison method:

$$\frac{1}{n} = \frac{n^2}{n^3} < \frac{n^2+1}{n^3+1}$$

Since  $\frac{n^2+1}{n^3+1} > \frac{1}{n}$ , then the given series *diverges*

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \left| \sin \frac{1}{n^2} \right|$

### Solution

For  $x \geq 0 \Rightarrow \sin x \leq x$

$$\left| \sin \frac{1}{n^2} \right| = \sin \frac{1}{n^2} \leq \frac{1}{n^2}$$

Thus, the given series *converges* by Comparison Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=8}^{\infty} \frac{1}{\pi^n + 5}$

### Solution

The given series converges by comparison with  $\sum_{n=8}^{\infty} \left( \frac{1}{\pi} \right)^n$

$$\text{Since } 0 < \frac{1}{\pi^n + 5} < \frac{1}{\pi^n}$$

Thus, the given series *converges* by Comparison Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^3}$

### Solution

$$\text{Since } (\ln n)^3 < n \Rightarrow \frac{1}{(\ln n)^3} > \frac{1}{n}$$

Thus, the given series *diverges* by Comparison Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{1}{\pi^n - n^\pi}$

### Solution

$$a_n = \frac{1}{\pi^n - n^\pi} \Rightarrow b_n = \frac{1}{\pi^n} = \left(\frac{1}{\pi}\right)^n$$
$$\sum_{n=1}^{\infty} \frac{1}{\pi^n} \text{ converges geometric since } |r| = \frac{1}{\pi} < 1$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1}{\pi^n - n^\pi} \cdot \frac{1}{\pi^n}$$
$$= \lim_{n \rightarrow \infty} \frac{1}{1 - \frac{n^\pi}{\pi^n}}$$
$$= 1$$

Thus, the given series **converges** by Comparison Test with geometric series

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=0}^{\infty} \frac{1+n}{2+n}$

### Solution

$$\lim_{n \rightarrow \infty} \frac{1+n}{2+n} = \underline{1 > 0}$$

Thus, the given series **diverges**.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{1+n^{4/3}}{2+n^{5/3}}$

### Solution

$$\text{Let } b_n = \frac{1}{n^{1/3}}$$

$$\lim_{x \rightarrow \infty} \frac{1+x^{4/3}}{2+x^{5/3}} \bigg/ \frac{1}{x^{1/3}} = \lim_{x \rightarrow \infty} \frac{x^{1/3} + x^{5/3}}{2+x^{5/3}} = 1$$

Thus, the given series **diverges** to infinity by Comparison Test with divergent **p-series**

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n^2}{1+n\sqrt{n}}$

### Solution

$$\lim_{n \rightarrow \infty} \frac{n^2}{1+n\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{n^2}{n^{3/2}} \\ = \infty$$

Thus, the given series **diverges** to infinity.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=2}^{\infty} \frac{1}{n \ln n (\ln \ln n)^2}$

### Solution

$$\int_2^{\infty} \frac{1}{x \ln x (\ln \ln x)^2} dx = \int_2^{\infty} \frac{d(\ln \ln x)}{(\ln \ln x)^2} \\ = -\frac{1}{\ln(\ln x)} \Big|_2^{\infty} \\ = \frac{1}{\ln(\ln 2)} < \infty$$

Thus, the given series **converges** by Integral Test

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=3}^{\infty} \frac{1}{n \ln n \sqrt{\ln \ln n}}$

### Solution

$$\int_3^{\infty} \frac{1}{x \ln x \sqrt{\ln \ln x}} dx = \int_3^{\infty} \frac{d(\ln \ln x)}{(\ln \ln x)^{1/2}} \\ = 2\sqrt{\ln(\ln x)} \Big|_3^{\infty} \\ = \infty$$

Thus, the given series **diverges** by Integral Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{1+(-1)^n}{\sqrt{n}}$

### Solution

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1+(-1)^n}{\sqrt{n}} &= 0 + \frac{2}{\sqrt{2}} + 0 + \frac{2}{\sqrt{4}} + 0 + \frac{2}{\sqrt{6}} \\ &= 2 \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k}} \\ &= \sqrt{2} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}\end{aligned}$$

Thus, the given series **diverges** to infinity ( $p$ -series)

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n!}{n^2 e^n}$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+1)^2 e^{n+1}} \cdot \frac{n^2 e^n}{n!} &= \lim_{n \rightarrow \infty} \frac{1}{e} \cdot \frac{n^2}{n+1} \\ &= \infty\end{aligned}$$

Therefore, the given series **converges** by the Ratio Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{(2n)!6^n}{(3n)!}$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{(2(n+1))!6^{n+1}}{(3(n+1))!} \cdot \frac{(3n)!}{(2n)!6^n} &= 6 \lim_{n \rightarrow \infty} \frac{(2n+1)(2n+2)}{(3n+1)(3n+2)(3n+3)} \\ &= \lim_{n \rightarrow \infty} \frac{4n^2}{27n^3} \\ &= 0\end{aligned}$$

Therefore, the given series **converges** by the Ratio Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{3^n \ln n}$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{\sqrt{n+1}}{3^{n+1} \ln(n+1)} \cdot \frac{3^n \ln n}{\sqrt{n}} &= \frac{1}{3} \lim_{n \rightarrow \infty} \frac{\sqrt{n+1}}{\sqrt{n}} \frac{\ln n}{\ln(n+1)} \\ &= \frac{1}{3} \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n}} \lim_{n \rightarrow \infty} \frac{\ln n}{\ln(n+1)} \\ &= \frac{1}{3} < 1\end{aligned}$$

Thus, the given series **converges** by the Ratio Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=0}^{\infty} \frac{n^{100} 2^n}{\sqrt{n!}}$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{(n+1)^{100} 2^{n+1}}{\sqrt{(n+1)!}} \cdot \frac{\sqrt{n!}}{n^{100} 2^n} &= 2 \lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^{100} \frac{1}{\sqrt{n+1}} \\ &= 0\end{aligned}$$

Thus, the given series **converges** by the Ratio Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{1+n!}{(1+n)!}$

### Solution

$$1+n! > n!$$

$$\frac{1+n!}{(1+n)!} > \frac{n!}{(1+n)!} = \frac{1}{n+1}$$

Therefore, the given series **diverges** by comparison with the Harmonic

### Exercise

Use any method to determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n - n^3}$$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{2^{n+1}}{3^{n+1} - (n+1)^3} \cdot \frac{3^n - n^3}{2^n} &= \frac{2}{3} \lim_{n \rightarrow \infty} \frac{3^n - n^3}{3^n - \frac{1}{3}(n+1)^3} \\ &= \frac{2}{3} \lim_{n \rightarrow \infty} \frac{1 - \frac{n^3}{3^n}}{1 - \frac{(n+1)^3}{3^{n+1}}} \\ &= \frac{2}{3} < 1 \end{aligned}$$

Therefore, the given series **converges** by the Ratio Test.

### Exercise

Use any method to determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^n}{\pi^n n!}$$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{\pi^{n+1} (n+1)!} \cdot \frac{\pi^n n!}{n^n} &= \frac{1}{\pi} \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \right)^n \\ &= \frac{1}{\pi} \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^n \\ &= \frac{e}{\pi} < 1 \end{aligned}$$

Therefore, the given series **converges** by the Ratio Test.

### Exercise

Use any method to determine if the series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}$$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} &= \lim_{n \rightarrow \infty} \frac{2}{n+1} \\ &= 0 < 1 \end{aligned}$$

Therefore, the given series **converges** by the Ratio Test.



### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

### Solution

$$\lim_{n \rightarrow \infty} \frac{(n+1)^2 2^{n+2}}{3^{n+1}} \cdot \frac{3^n}{n^2 2^{n+1}} = \lim_{n \rightarrow \infty} \frac{2}{3} \left( \frac{n+1}{n} \right)^2$$
$$\boxed{= \frac{2}{3} < 1}$$

Therefore, the given series **converges** by the Ratio Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

### Solution

$$\lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n \rightarrow \infty} \frac{n+1}{n+1} \left( \frac{n+1}{n} \right)^n$$
$$= \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^n$$
$$\boxed{= e > 1}$$

Therefore, the given series **diverges** by the Ratio Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{100}{n}$

### Solution

$$\sum_{n=1}^{\infty} \frac{100}{n} = 100 \sum_{n=1}^{\infty} \frac{1}{n}$$

Therefore, the given series **diverges** by harmonic series.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$

### Solution

$$\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}} = 3 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

Therefore, the given series **converges** by *p-series*  $\left(p = \frac{3}{2} > 1\right)$

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \left(\frac{2\pi}{3}\right)^n$

### Solution

$$|r| = \frac{2\pi}{3} > 1$$

Therefore, the given series **diverges** by Geometric series.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{5n}{2n-1}$

### Solution

$$\lim_{n \rightarrow \infty} \frac{5n}{2n-1} = \frac{5}{2} \neq 0$$

Therefore, the given series **diverges** by nth-Term Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$

### Solution

$$\lim_{n \rightarrow \infty} \frac{n}{2n^2+1} = \frac{1}{2} > 0$$

Therefore, the given series **diverges** by harmonic series.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} (-1)^n \frac{3^{n-2}}{2^n}$

### Solution

$$\sum_{n=1}^{\infty} (-1)^n \frac{3^{n-2}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{9} \left(-\frac{3}{2}\right)^n$$

$$|r| = \frac{3}{2} > 1$$

Therefore, the given series **diverges** by Geometric series.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{10}{3\sqrt[n]{3}}$

### Solution

$$\sum_{n=1}^{\infty} \frac{10}{3\sqrt[n]{3}} = \frac{10}{3} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

Therefore, the given series **converges** by **p-series**  $\left(p = \frac{3}{2} > 1\right)$

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$

### Solution

$$b_n = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$$

$$\lim_{n \rightarrow \infty} \frac{10n+3}{n2^n} \cdot \frac{2^n}{1} = \lim_{n \rightarrow \infty} \frac{10n+3}{n}$$

$$= 10$$

converges

Therefore, the given series **converges** by Limit Comparison Test with Geometric series  $\left(|r| = \frac{1}{2} < 1\right)$

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{2^n}{4n^2 - 1}$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{2^n}{4n^2 - 1} &= \lim_{n \rightarrow \infty} \frac{2^n (\ln 2)}{8n} \\ &= \lim_{n \rightarrow \infty} \frac{2^n (\ln 2)^2}{8} \\ &= \infty\end{aligned}$$

Therefore, the given series **diverges** by nth-Term Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{\cos n}{3^n}$

### Solution

$$\left| \frac{\cos n}{3^n} \right| \leq \frac{1}{3^n} = \left( \frac{1}{3} \right)^n$$

Therefore, the given series **converges** by Direct Comparison Test with Geometric series  $\left( |r| = \frac{1}{3} < 1 \right)$

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n!}{n 7^n}$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+1)7^{n+1}} \cdot \frac{n7^n}{n!} &= \lim_{n \rightarrow \infty} \frac{1}{7} \\ &= \frac{1}{7}\end{aligned}$$
$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Therefore, the given series **diverges** by the Ratio Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

### Solution

$$\frac{\ln n}{n^2} \leq \frac{1}{n^{3/2}}$$

Therefore, the given series **converges** by Comparison Test with **p-series**  $\left(p = \frac{3}{2} > 1\right)$

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{(-3)^n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{(-3)^{n+2}}{3 \cdot 5 \cdot 7 \cdots (2n+1) \cdot (2n+3)} \cdot \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{(-3)^n} \\ &= \lim_{n \rightarrow \infty} \frac{9}{2n+3} \\ &= 0 \end{aligned}$$

Therefore, the given series **converges** by the Ratio Test.

### Exercise

Use any method to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{18^n (2n-1)n!}$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)}{18^{n+1} (2n-1)(2n+1)(n+1)!} \cdot \frac{18^n (2n-1)n!}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \\ &= \lim_{n \rightarrow \infty} \frac{1}{18} \frac{2n+3}{(2n+1)(n+1)} \\ &= 0 \end{aligned}$$

Therefore, the given series **converges** by the Ratio Test.

### Exercise

Use the integral test to show that  $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$  converges. Show that the sum  $s$  of the series is less than  $\frac{\pi}{2}$

### Solution

$$\int_0^{\infty} f(x) dx = \int_0^{\infty} \frac{1}{1+x^2} dx$$

$$= \tan^{-1} x \Big|_0^{\infty}$$

$$= \frac{\pi}{2}$$

Therefore, the given series **converges** by the Integral Test and its sum is less than  $\frac{\pi}{2}$

### Exercise

Use the root test to show that  $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n}$  converges

### Solution

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{2^{n+1}}{n^n}} = \lim_{n \rightarrow \infty} \frac{2^{(n+1)/n}}{n}$$

$$= \lim_{n \rightarrow \infty} \frac{2 \times 2^{1/n}}{n}$$

$$= 0$$

Therefore, the given series **converges** by the Root Test.

### Exercise

Use the root test to test that  $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$  converges

### Solution

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{n}}\right)^{n^2/n}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{n}}\right)^{\frac{1}{n}}$$

$$= \frac{1}{e} < 1$$

Therefore, the given series **converges** by the Root Test.

### Exercise

Try to use the ratio test to determine whether  $\sum_{n=1}^{\infty} \frac{2^{2n} (n!)^2}{(2n)!}$  converges. What happen?

$$\begin{aligned}\text{Now observe that } \frac{2^{2n} (n!)^2}{(2n)!} &= \frac{[2n(2n-2)(2n-4) \cdots 6 \times 4 \times 2]^2}{2n(2n-1)(2n-2) \cdots 3 \times 2 \times 1} \\ &= \frac{2n}{2n-1} \times \frac{2n-2}{2n-3} \times \frac{4}{3} \times \frac{2}{1}\end{aligned}$$

Does the given series converges? Why or why not?

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{2^{2n+2} ((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{2^{2n} (n!)^2} &= \lim_{n \rightarrow \infty} \frac{2^2 (n+1)^2}{(2n+2)(2n+1)} \\ &= \lim_{n \rightarrow \infty} \frac{4n^2}{4n^2} \\ &= 1\end{aligned}$$

Therefore, the ratio test provides no information

However from the given:

$$\begin{aligned}\frac{2^{2n} (n!)^2}{(2n)!} &= \frac{[2n(2n-2)(2n-4) \cdots 6 \times 4 \times 2]^2}{2n(2n-1)(2n-2) \cdots 3 \times 2 \times 1} \\ &= \frac{2n}{2n-1} \times \frac{2n-2}{2n-3} \times \frac{4}{3} \times \frac{2}{1} > 1\end{aligned}$$

Therefore, the given series **diverges** to infinity.

### Exercise

Suppose  $a_n > 0$  and  $\frac{a_{n+1}}{a_n} > \frac{n}{n+1}$  for all  $n$ . Show that  $\sum_{n=1}^{\infty} a_n$  diverges.

$$\left( a_n \geq \frac{K}{n} \text{ for some constant } K \right)$$

### Solution

If  $a_n > 0$  and  $\frac{a_{n+1}}{a_n} > \frac{n}{n+1}$  for all  $n$ .

Then, by using induction

$$\frac{a_2}{a_1} > \frac{1}{2} \Rightarrow a_2 > \frac{1}{2} a_1$$

$$\frac{a_3}{a_2} > \frac{2}{3} \Rightarrow a_3 > \frac{2}{3}a_2 > \frac{1}{3}a_1$$

$$\frac{a_4}{a_3} > \frac{3}{4} \Rightarrow a_4 > \frac{3}{4}a_3 > \frac{1}{4}a_1$$

$$\vdots \qquad \qquad \qquad \vdots$$

$$\frac{a_n}{a_{n-1}} > \frac{n-1}{n} \Rightarrow a_n > \frac{1}{n}a_{n-1}$$

Therefore, the given series *diverges* by comparison with the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$

### Exercise

Working in the early 1600s, the mathematicians Wallis, Pascal, and Fermat were calculating the area of the region under the curve  $y = x^p$  between  $x = 0$  and  $x = 1$ , where  $p$  is the positive integer. Using arguments that predated the Fundamental Theorem of Calculus, they were able to prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{k}{n}\right)^p = \frac{1}{p+1}$$

Use Riemann sums and integrals to verify this limit.

### Solution

The sum on the left is simply the left Riemann sum over  $n$  equal intervals between 0 and 1 for

$$f(x) = x^p.$$

The limit of the sum is:

$$\int_0^1 x^p dx = \frac{1}{p+1} x^{p+1} \Big|_0^1 = \frac{1}{p+1} \quad (p > 0)$$

### Exercise

Complete the following steps to find the values of  $p > 0$  for which the series  $\sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{p^k k!}$  converges

a) Use the Ratio Test to show that  $\sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{p^k k!}$  converges for  $p > 2$ .



- b) Use Stirling's formula,  $k! = \sqrt{2\pi k} k^k e^{-k}$  for large  $k$ , to determine whether the series converges when  $p = 2$ .

$$\left( \text{Hint: } 1 \cdot 3 \cdot 5 \cdots (2k-1) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2k-1) 2k}{2 \cdot 4 \cdot 6 \cdots 2k} \right)$$

### Solution

- a) Using the Ratio Test

$$\begin{aligned} \frac{a_{k+1}}{a_k} &= \frac{1 \cdot 3 \cdot 5 \cdots (2k-1) \cdot (2(k+1)-1)}{p^{k+1} (k+1)!} \cdot \frac{p^k k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)} \\ &= \frac{2k+1}{(k+1)p} \end{aligned}$$

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} &= \lim_{k \rightarrow \infty} \frac{2k+1}{(k+1)p} \\ &= \underline{\underline{\frac{2}{p}}} \end{aligned}$$

Therefore, the given series converges for  $p > 2$ .

- b) When  $p = 2$

$$\text{Given: } 1 \cdot 3 \cdot 5 \cdots (2k-1) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2k-1) 2k}{2 \cdot 4 \cdot 6 \cdots 2k} = \frac{(2k)!}{2 \cdot 4 \cdot 6 \cdots 2k}$$

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^k k!} &= \sum_{k=1}^{\infty} \frac{(2k)!}{2^k k! (2 \cdot 4 \cdot 6 \cdots 2k)} \\ &= \sum_{k=1}^{\infty} \frac{(2k)!}{(2^k)^2 (k!)^2} \end{aligned}$$

$$\text{Given: } k! = \sqrt{2\pi k} k^k e^{-k}$$

$$\begin{aligned} \rightarrow (2k)! &= \sqrt{2\pi(2k)} (2k)^{2k} e^{-2k} \\ &= 2\sqrt{\pi} \sqrt{k} (2^k)^2 (k^k)^2 e^{-2k} \end{aligned}$$

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(2k)!}{(2^k)^2 (k!)^2} &= \sum_{k=1}^{\infty} \frac{2\sqrt{\pi} \sqrt{k} (2^k)^2 (k^k)^2 e^{-2k}}{(2^k)^2 (\sqrt{2\pi k} k^k e^{-k})^2} \\ &= \sum_{k=1}^{\infty} \frac{2\sqrt{\pi} \sqrt{k}}{(\sqrt{2\pi k})^2} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} \frac{1}{\sqrt{\pi} \sqrt{k}} \\
&= \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{1}{k^{1/2}} \quad \left( p = \frac{1}{2} < 1 \right)
\end{aligned}$$

Therefore, the given series diverges for  $p = 2$  by the Limit Comparison with  $p$ -series  $\left( p = \frac{1}{2} < 1 \right)$