

Section 3.6 – Exponential Growth and Decay

Exponential Growth and Decay

The mathematical model for exponential growth or decay is given by

$$A(t) = A_0 e^{kt}$$

$A(t)$: Exponential Function (After time t)

A_0 : At time zero (initial value).

t : Time

k : Exponential rate.

✚ If $k > 0$, the function models of a growing entity

✚ If $k < 0$, the function models of a decay entity.

Example

In 1990, the population of Africa was 643 *million* and by 2000 it had grown to 813 *million*

- Use the exponential growth function $A(t) = A_0 e^{kt}$, in which t is the number of years after 1990, to find the exponential growth function that models data
- By which year will Africa's population reach 2000 *million*, or two *billion*?

Solution

a. $A(t) = A_0 e^{kt}$

From 1990 to 2000, is 10 years, that implies in 10 years the population grows from 643 to 813

$$813 = 643e^{k(10)}$$

$$\frac{813}{643} = e^{10k}$$

$$\ln \frac{813}{643} = \ln e^{10k}$$

$$\ln \frac{813}{643} = 10k$$

$$\frac{1}{10} \ln \frac{813}{643} = k$$

$$k \approx 0.023$$

$$\Rightarrow A(t) = 643e^{0.023t}$$

$$b. \quad 2000 = 643e^{0.023t}$$

$$\frac{2000}{643} = e^{0.023t}$$

$$\ln \frac{2000}{643} = \ln e^{0.023t}$$

$$\ln \frac{2000}{643} = 0.023t$$

$$\frac{\ln \frac{2000}{643}}{0.023} = t$$

$$t \approx 49$$

$$\text{Year : } 2039$$

Doubling Time

$$P(t) = P_0 e^{kt}$$

$$2P_0 = P_0 e^{kt} \quad P(t) = 2P_0$$

$$2 = e^{kt}$$

$$\ln 2 = \ln e^{kt}$$

$$\ln 2 = kt \ln e \quad \ln e = 1$$

$$\ln 2 = kt$$

$$t = \frac{\ln 2}{k}$$

Growth Rate and Doubling Time

$$\boxed{Tk = \ln 2} \quad \text{or} \quad T = \frac{\ln 2}{k} \quad \text{or} \quad k = \frac{\ln 2}{T}$$

Example

A country's population doubled in 45 years. What was the exponential growth rate?

Solution

$$k = \frac{\ln 2}{t}$$

$$= \frac{\ln 2}{45}$$

$$\approx 0.0154$$

Finding K or T

$$A = A_0 e^{kt} \Rightarrow \boxed{kT = \ln \frac{A}{A_0}}$$

Proof

$$A = A_0 e^{kt}$$

$$\frac{A}{A_0} = e^{kt}$$

$$\ln \frac{A}{A_0} = \ln e^{kt}$$

$$\boxed{\ln \frac{A}{A_0} = kt} \quad \checkmark$$

Example

According to the U.S. Census Bureau, the world population reached 6 *billion* people on July 18, 1999, and was growing exponentially. By the end of 2000, the population had grown to 6.079 *billion*. The projected world population (in *billion* of people) t years after 2000, is given by the function defined by

$$f(t) = 6.079e^{0.0126t}$$

- a) Based on this model, what will the world population be in 2010?
- b) In what year will the world population reach 7 *billion*?

Solution

- a) In 2010 $\rightarrow t = 10$

$$f(t=10) = 6.079e^{0.0126(10)} \\ \approx \boxed{6.895}$$

$$b) \quad t = \frac{1}{.0126} \ln \left(\frac{7}{6.079} \right) \\ \approx \boxed{11.2}$$

$$\text{Year : } \boxed{2011}$$

$$kT = \ln \frac{A}{A_0}$$

Example

Strontium-90 is a waste product from nuclear reactors. As a consequence of fallout from atmosphere nuclear tests, we all have a measurable amount of strontium-90 in our bones.

- a) The half-life of Strontium-90 is 28 *years*, meaning that all after 28 *years* a given amount of the substance will have decayed to half the original amount. Find the exponential decay model for Strontium-90.
- b) Suppose the nuclear accident occurs and releases 60 *grams* of Strontium-90 into the atmosphere. How long will it take for Strontium-90 to decay to a level of 10 *grams*?

Solution

$$\begin{aligned} a) \quad k &= \frac{1}{28} \ln \frac{1}{2} & kT &= \ln \frac{A}{A_0} \\ &\approx -0.0248 & & \\ A(t) &= A_0 e^{-0.0248t} \end{aligned}$$

$$\begin{aligned} b) \quad A &= A_0 e^{-0.0248t} \\ t &= \frac{\ln \frac{1}{6}}{-0.0248} & kT &= \ln \frac{A}{A_0} \\ &\approx 72.25 \text{ yrs} \end{aligned}$$

Logistic Model Function

Definition

The magnitude of a population at time $t \geq 0$ is given by

$$P(t) = \frac{c}{1 + ae^{-bt}}$$

where

c is the **carrying capacity** (the maximum population that can be supported by available resources as $t \rightarrow \infty$)

b is positive constant called the **growth rate constant**.

$P_0 = P(0)$ is the **initial population**.

a is related to the initial population P_0 and the carrying capacity c by the formula

$$a = \frac{c - P_0}{P_0}$$

Example

The coyote population in a wilderness area was estimated at 200 in 2007. By the beginning of 2009, the coyote population had increased to 250. A park ranger estimates that the carrying capacity of the wilderness area is 500 coyotes.

- Use the given data to determine the growth rate constant for the logistic model of this coyote population.
- Use the logistic model (part a) to predict the year in which the coyote population will first reach 400.

Solution

a) **Given:** $P(0) = 200$ $P(2) = 250$ $c = 500$

$$a = \frac{500 - 200}{200}$$

$$= \frac{3}{2}$$

$$a = \frac{c - P_0}{P_0}$$

$$P(2) = \frac{500}{1 + \frac{3}{2}e^{-2b}} = 250$$

$$P(t) = \frac{c}{1 + ae^{-bt}}$$

$$1 + \frac{3}{2}e^{-2b} = \frac{500}{250}$$

$$\frac{3}{2}e^{-2b} = 2 - 1$$

$$e^{-2b} = \frac{2}{3}$$

$$-2b = \ln \frac{2}{3}$$

$$b = -\frac{1}{2} \ln \frac{2}{3}$$

$$= \frac{1}{2} \ln \frac{3}{2}$$

$$= \ln \sqrt{\frac{3}{2}}$$

$$\approx 0.202732554$$

$$\ln \frac{1}{x} = -\ln x$$

$$P(t) = \frac{500}{1 + \frac{3}{2}e^{-\ln \sqrt{\frac{3}{2}} t}}$$
$$= \frac{1,000}{2 + 3e^{-\ln \sqrt{\frac{3}{2}} t}}$$

b) $400 = \frac{1,000}{2 + 3e^{-\ln \sqrt{\frac{3}{2}} t}}$

$$2 + 3e^{-\ln \sqrt{\frac{3}{2}} t} = \frac{1,000}{400}$$

$$3e^{-\ln \sqrt{\frac{3}{2}} t} = \frac{5}{2} - 2$$

$$e^{-\ln \sqrt{\frac{3}{2}} t} = \frac{1}{6}$$

$$-\ln \sqrt{\frac{3}{2}} t = \ln \frac{1}{6}$$

$$-\frac{1}{2} \ln \frac{3}{2} t = -\ln 6$$

$$t = 2 \frac{\ln 6}{\ln \frac{3}{2}}$$

$$= 2 \log_{\frac{3}{2}} 6$$

$$\approx 8.84 \text{ yrs.}$$

<i>Isotope</i>	<i>Half-Life</i>
Carbon $\left(^{14}\text{C}\right)$	5,730 years
Radium $\left(^{226}\text{Ra}\right)$	1,660 years
Polonium $\left(^{210}\text{Po}\right)$	138 days
Phosphorus $\left(^{32}\text{P}\right)$	14 days
Polonium $\left(^{214}\text{Po}\right)$	$\frac{1}{10,000}$ of a second

Formulas for Compound Interest

1. For n compounding per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$
2. For Continuous compounding: $A = Pe^{rt}$

P : Principal, initial value
 n : number of period per year
 t : number of years
 r : interest rate

A is also called **Future value**

P is also called **Present value**

Example

A sum of \$10,000 is invested at an annual rate of 8%. Find the balance in the account after 5 years subject to quarterly compounding

Solution

Given:

$$P = \$10,000$$

$$r = 8\% = 0.08$$

$$t = 5$$

Quarterly $n = 4$

$$A = 10000\left(1 + \frac{0.08}{4}\right)^{4(5)}$$
$$\approx \$14,859.47$$

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Example

Suppose \$5000 is deposited in an account paying 3% interest compounded continuously for 5 yrs. Find the total amount on deposit at the end of 5 yrs.

Solution

$$A = 5000e^{(.03)(5)}$$
$$\approx \$5,809.17$$

$$A = Pe^{rt}$$

Exercises **Section 3.6 – Exponential Growth and Decay**

1. Suppose that \$10,000 is invested at interest rate of 5.4% per year, compounded continuously.
 - a) Find the exponential growth function
 - b) What will the balance be after, 1 yr. 10 yrs.?
 - c) After how long will the investment be double?
2. In 1990, the population of Africa was 643 *million* and by 2000 it had grown to 813 *million*
 - a) Use the exponential growth function $A(t) = A_0 e^{kt}$, in which t is the number of years after 1990, to find the exponential growth function that models data
 - b) By which year will Africa's population reach 2000 *million*, or two *billion*?
3. The radioactive element carbon-14 has a half-life of 5750 yrs. The percentage of carbon-14 present in the remains of organic matter can be used to determine the age of that organic matter. Archaeologists discovered that the linen wrapping from one of the Dead Sea Scrolls had lost 22.3% of its carbon-14 at the time it was found. How old was the linen wrapping?
4. Suppose that \$2000 is invested at interest rate k , compounded continuously, and grows to \$2983.65 in 5 yrs.
 - a) What is the interest rate?
 - b) Find the exponential growth function
 - c) What will the balance be after 10 yrs?
 - d) After how long will the \$2000 have doubled?
5. In 2005, the population of China was about 1.306 *billion*, and the exponential growth rate was 0.6% per year.
 - a) Find the exponential growth function
 - b) Estimate the population in 2008
 - c) After how long will the population be double what it was in 2005?
6. How long will it take for the money in an account that is compounded continuously at 3% interest to double?
7. If 600 g of radioactive substance are present initially and 3 yrs. later only 300 g remain, how much of the substance will be present after 6 yrs.?
8. The population of an endangered species of bird was 4200 in 1990. Thirteen years later, in 2003, the bird population declined to 3000. The population of the birds is decreasing exponentially according to the function $A(t) = 4200e^{kt}$ where A is the bird population t years after 1990. Use this information to find the value of k .

9. A city had a population of 21,400 in 2000 and a population of 23,200 in 2005.
 - a) Find the exponential growth function for the city.
 - b) Use the growth function to predict the population of the city in 2018.
10. A city had a population of 53,700 in 2002 and a population of 58,100 in 2006.
 - a) Find the exponential growth function for the city.
 - b) Use the growth function to predict the population of the city in 2013.
11. The population of Charlotte, North Carolina, is growing exponentially. The population of Charlotte was 395,934 in 1990 and 610,949 in 2005. Find the exponential growth function that models the population of Charlotte and use it to predict the population of Charlotte in 2017.
12. The population of Las Vegas, Nevada, is growing exponentially. The population of Las Vegas was 258,295 in 1990 and 545,147 in 2005. Find the exponential growth function that models the population of Las Vegas and use it to predict the population of Las Vegas in 2017.
13. Find the decay function for the amount of Polonium (^{210}Po) that remains in a sample after t days.
14. Estimate the percentage of polonium (^{210}Po) that remains in a sample after 2 years.
15. Estimate the age of a bone if it now contains 65% of its original amount of carbon-14.
16. Geologists have determined that Crater Lake in Oregon was formed by a volcanic eruption. Chemical analysis of a wood chip assumed to be from a tree that died during the eruption has shown that it contains approximately 45% of its original carbon-14. Estimate how long ago the volcanic eruption occurred.
17. Lead shielding is used to contain radiation. The percentage of a certain radiation that can penetrate x millimeters of lead shielding is given by $I(x) = 100e^{-1.5x}$
 - a) What percentage of radiation will penetrate a lead shield that is 1 millimeter thick?
 - b) How many millimeters of lead shielding are required so that less than 0.02% of the radiations penetrates the shielding?
18. After a race, a runner's pulse rate R , in beats per minute, decreases according to the function

$$R(t) = 145e^{-0.092t}, \quad 0 \leq t \leq 15$$

Where t is measured in minutes.

- a) Find the runner's pulse rate at the end of the race and 1 minute after the end of the race.
- b) How long after the end of the race will the runner's pulse rate be 80 beats per minute?

19. A can of soda at $79^{\circ}F$ is placed in a refrigerator that maintains a constant temperature of $36^{\circ}F$. The temperature T of the soda t minutes after it is placed in the refrigerator is given by

$$T(t) = 36 + 43e^{-0.058t}$$

- a) Find the temperature of the soda 10 minutes after it is placed in the refrigerator.
b) When will the temperature of the soda be $45^{\circ}F$
20. During surgery, a patient's circulatory system requires at least 50 milligrams of an anesthetic. The amount of anesthetic present t hours after 80 milligrams of anesthetic is administered is given by

$$T(t) = 80(0.727)^t$$

- a) How much of the anesthetic is present in the patient's circulatory system 30 minutes after the anesthetic is administered?
b) How long can the operation last if the patient does not receive additional anesthetic?
21. The following function models the average typing speed S , in words per minute, for a student who has been typing for t months.

$$S(t) = 5 + 29 \ln(t + 1), \quad 0 \leq t \leq 9$$

Use S to determine how long it takes the student to achieve an average speed of 65 words per minute.

22. The exponential function

$$S(x) = 8320(0.73)^x, \quad 10 \leq x \leq 20$$

models the speed of the dragster during the 10-second period immediately following the time when the dragster crosses the finish line. This is the deceleration period.

How long after the start of the race did the dragster attain a speed of 275 miles per hour?

23. If \$8,000 is invested at an annual interest rate of 5% and compounded annually, find the balance after
- a) 4 years. b) 8 years.
24. If \$20,000 is invested at an annual interest rate of 4.5% and compounded annually, find the balance after
- a) 3 years. b) 5 years.
25. If \$10,000 is invested at an annual interest rate of 3% for 5 years, find the balance if the interest rate is compounded
- a) Annually. c) Quarterly e) Daily (365) g) Continuously
b) Semi-annually. d) Monthly f) Hourly

26. If \$20,000 is invested at an annual interest rate of 2% for 10 *years*, find the balance if the interest rate is compounded
- a) Annually. c) Quarterly e) Daily (365) g) Continuously
 b) Semi-annually. d) Monthly f) Hourly
27. Find the accumulated value of an investment of \$10,000 for 5 *years* at an interest rate of 5.5% if the money is
- a) Compounded *semiannually*
 b) Compounded *quarterly*
 c) Compounded *monthly*
 d) Compounded *Continuously*
28. Suppose \$1000 is deposited in an account paying 4% interest per year compounded *quarterly*.
- a) Find the amount in the account after 10 *years* with no withdrawals.
 b) How much interest is earned over the 10 *years* period?
29. An investment of 1,000 increased to \$13,464 in 20 *years*. If the interest was compounded continuously, find the interest rate.
30. Becky must pay a lump sum of \$6000 in 5 *yrs.*
- a) What amount deposited today at 3.1% compounded annually will grow to \$6000 in 5 *yrs.*?
 b) If only \$5000 is available to deposit now, what annual interest rate is necessary for the money to increase to \$6000 in 5 *yrs.*?
31. Find the present value of \$4,000 if the annual interest rate is 3.5% compounded *quarterly* for 6 *years*.
32. How much money will there be in an account at the end of 8 years if \$18,000 is deposited at 3% interest compounded *semi-annually*?
33. The function defined by $P(x) = 908e^{-0.0001348x}$ approximates the atmospheric pressure (in *millibars*) at an altitude of x *meters*. Use P to predict the pressure:
- a) At 0 *meters*
 b) At 12,000 *meters*
34. How long, to the nearest tenth of a year, will it take \$1000 to grow to \$3600 at 8% annual interest compounded quarterly?
35. The annual revenue R , in *dollars*, of a new company can be closely modeled by the logistic function

$$R(t) = \frac{625,000}{1 + \frac{3}{10}e^{-.045t}}$$

Where the natural number t is the time, in *years*, since the company was founded.

- a) According to the model, what will the company's annual revenue for its first year and its second year?
- b) According to the model, what will the company's annual revenue approach in the long-term future?

36. The number of cars A sold annually by an automobile dealership can be closely modeled by the logistic function

$$A(t) = \frac{1,650}{1 + \frac{12}{5}e^{-.055t}}$$

- a) According to the model, what number of cars will the dealership sell during its first year and its second year?
- b) According to the model, what will the dealership's car sales approach in the long-term future?

37. The population of wolves in a preserve satisfies a logistic model in which $P_0 = 312$ in 2008, $c = 1,600$, and $P(6) = 416$.

- a) Determine the logistic model for this population, where t is the number of years after 2008.
- b) Use the logistic model from part (a) to predict the size of the groundhog population in 2014.

38. The population of walrus on an island satisfies a logistic model in which $P_0 = 800$ in 2006, $c = 5,500$, and $P(1) = 900$.

- a) Determine the logistic model for this population, where t is the number of years after 2006.
- b) Use the logistic model from part (a) to predict the year in which the walrus population will first exceed 2000.

39. Newton's Law of Cooling states that is an object at temperature T_0 is placed into an environment at constant temperature A , then the temperature of the object, $T(t)$ (in degrees Fahrenheit), after t minutes is given by $T(t) = A + (T_0 - A)e^{-kt}$, where k is a constant that depends on the object.

- a) Determine the constant k for a canned soda drink that takes 5 minutes to cool from $75^\circ F$ to $65^\circ F$ after being placed in a refrigerator that maintains a constant temperature of $34^\circ F$
- b) What will be the temperature of the soda after 30 minutes?
- c) When will the temperature of the soda drink be $36^\circ F$?

40. According to a software company, the users of its typing tutorial can expect to type $N(t)$ words per minute after t hours of practice with the product, according to the function $N(t) = 100(1.04 + 0.99^t)$

- a) How many words per minute can a student expect to type after 2 hours of practice?
- b) How many words per minute can a student expect to type after 40 hours of practice?
- c) How many hours of practice will be required before a student can expect to type 60 words per minute?

- 41.** A lawyer has determined that the number of people $P(t)$ in a city of 1.2 *million* people who have been exposed to a news item after t *days* is given by the function

$$P(t) = 1,200,000(1 - e^{-0.03t})$$

- a) How many days after a major crime has been reported has 40% of the population heard of the crime?
- b) A defense lawyer knows it will be difficult to pick an unbiased jury after 80% of the population has heard of the crime. After how many days will 80% of the population have heard of the crime?