Lecture Three

Section 3.1 – Quadratic Functions and Models

Quadratic Function

A function f is a ***quadratic function*** if $f(x) = ax^2 + bx + c$

Vertex of a Parabola

The ***vertex*** of the graph of $f(x)$ is

V_x or $x_V = -\frac{b}{2a}$

V_y or $y_V = f\left(-\frac{b}{2a}\right)$

Vertex Point $\left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right)$

$f(x) = x^2 - 4x - 2$

$x = -\frac{b}{2a} = -\frac{-4}{2(1)} = 2$

$y = f\left(-\frac{b}{2a}\right) = f(2)$

$= (2)^2 - 4(2) - 2$

$= -6$

Vertex point: $(2, -6)$

Axis of Symmetry: $x = V_x = -\frac{b}{2a}$

Axis of Symmetry: $x = 2$

Minimum or Maximum Point

If $a > 0 \Rightarrow f(x)$ has a ***minimum*** point

If $a < 0 \Rightarrow f(x)$ has a ***maximum*** point

@ vertex point $\left(V_x, V_y\right)$

Minimum point @ $(2, -6)$

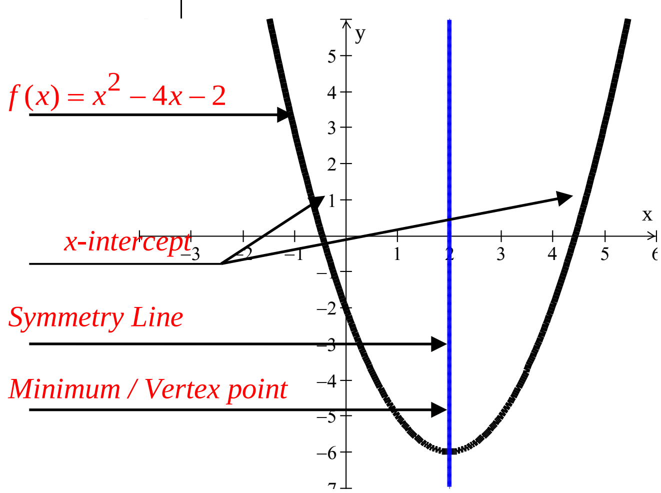
Range

If $a > 0 \Rightarrow \left[V_y,\ \infty\right)$

If $a < 0 \Rightarrow \left(-\infty,\ V_y\right]$

$[-6, \infty)$

Domain: $(-\infty, \infty)$



Example

For the graph of the function $f(x) = -x^2 - 2x + 8$

a. **Find the vertex point**

$$x = -\frac{-2}{2(-1)} = -1$$

$$y = f(-1) = -(-1)^2 - 2(-1) + 8 = 9$$

Vertex point (–1, 9)

b. **Find the line of symmetry:** $x = -1$

c. **State whether there is a maximum or minimum value *and* find that value**

Minimum point, value (–1, 9)

d. **Find the *x*-intercept**

$x = -4,\ 2$

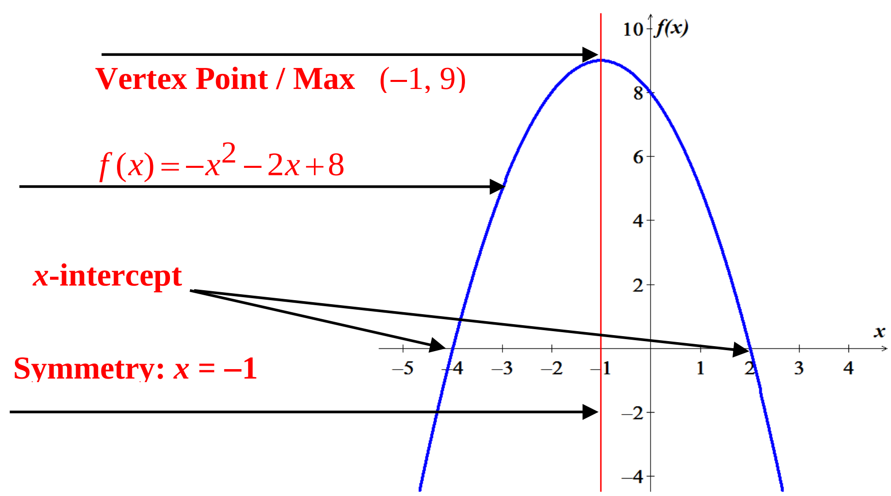
e. **Find the y-intercept**

$y = 8$

f. **Find the range and the domain of the function.**

Range: $(-\infty, 9]$ *Domain*: $(-\infty, \infty)$

g. **Graph the function and label, show part *a thru d* on the plot below**



h. **On what intervals is the function increasing? Decreasing?**

Increasing: $(-\infty, -1)$ *Decreasing*: $(-1, \infty)$

Example

Find the axis and vertex of the parabola having equation $f(x) = 2x^2 + 4x + 5$

Solution

$$x = -\frac{b}{2a}$$
$$= -\frac{4}{2(2)}$$
$$= -1$$

Axis of the parabola: $x = -1$

$$y = f(-1)$$
$$= 2(-1)^2 + 4(-1) + 5$$
$$= 3$$

Vertex point: $\boxed{(-1, 3)}$

Maximizing Area

You have 120 ft of fencing to enclose a rectangular region. Find the dimensions of the rectangle that maximize the enclosed area. What is the maximum area?

Solution

$$P = 2l + 2w$$
$$120 = 2l + 2w$$
$$60 = l + w \quad \to \boxed{l = 60 - w}$$

$$A = lw$$
$$= (60 - w)w$$
$$= 60w - w^2$$
$$= -w^2 + 60w$$

Vertex: $w = -\frac{60}{2(-1)} = 30$

$\to l = 60 - w = 30$

$A = lw = (30)(30)\ \boxed{= 900\ ft^2}$

Example

A stone mason has enough stones to enclose a rectangular patio with 60 *ft* of stone wall. If the house forms one side of the rectangle, what is the maximum area that the mason can enclose? What should the dimensions of the patio be in order to yield this area?

Solution

$$P = l + 2w = 60 \quad \Rightarrow \boxed{l = 60 - 2w}$$

$$\begin{aligned} A &= lw \\ &= (60 - 2w)w \\ &= 60w - 2w^2 \\ &= -2w^2 + 60w \end{aligned}$$

$$\begin{aligned} w &= -\frac{b}{2a} \\ &= -\frac{60}{2(-2)} \\ &= 15\,ft \end{aligned}$$

$$\Rightarrow l = 60 - 2w = 60 - 2(15) = 30\,ft$$

$$Area = (15)(30) = 450\ ft^2$$

Position Function (*Projectile Motion*)

Example

A model rocket is launched with an initial velocity of 100 ft/sec from the top of a hill that is 20 ft high. Its height t seconds after it has been launched is given by the function $s(t) = -16t^2 + 100t + 20$. Determine the time at which the rocket reaches its maximum height and find the maximum height.

Solution

$$\begin{aligned} t &= -\frac{b}{2a} \\ &= -\frac{100}{2(-16)} \\ &= 3.125 \text{ sec} \end{aligned}$$

$$s(t = 3.125) = -16(3.125)^2 + 100(3.125) + 20 = 176.25\ ft$$

Exercises *Section* 3.1 – Quadratic Functions and Models

1. Give the vertex, axis, domain, and range. Then, graph the function $f(x)=x^2+6x+5$

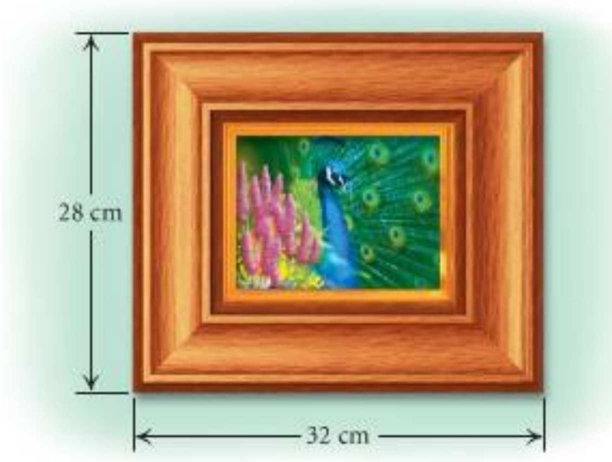
2. Give the vertex, axis, domain, and range. Then, graph the function $f(x)=-x^2-6x-5$

3. Give the vertex, axis, domain, and range. Then, graph the function $f(x)=x^2-4x+2$

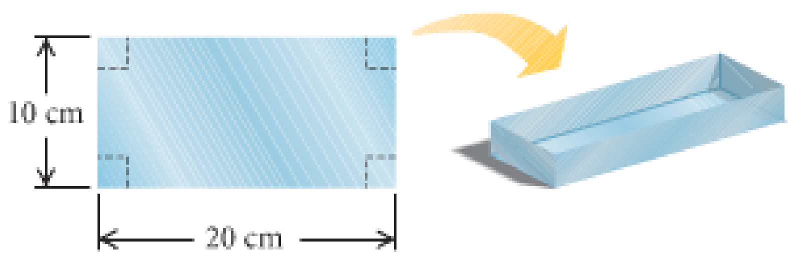
4. Give the vertex, axis, domain, and range. Then, graph the function $f(x)=-2x^2+16x-26$

5. You have 600 *ft.* of fencing to enclose a rectangular plot that borders on a river. If you do not fence the side along the river, find the length and width of the plot that will maximize the area. What is the largest area that can be enclosed?

6. A picture frame measures 28 cm by 32 cm and is of uniform width. What is the width of the frame if 192 cm^2 of the picture shows?



7. An open box is made from a 10-cm by 20-cm of tin by cutting a square from each corner and folding up the edges. The area of the resulting base is 96 cm^2. What is the length of the sides of the squares?



8. A fourth-grade class decides to enclose a rectangular garden, using the side of the school as one side of the rectangle. What is the maximum area that the class can enclose with 32 ft. of fence? What should the dimensions of the garden be in order to yield this area?

9. A rancher needs to enclose two adjacent rectangular corrals, one for cattle and one for sheep. If a river forms one side of the corrals and 240 yd of fencing is available, what is the largest total area that can be enclosed?

10. A Norman window is a rectangle with a semicircle on top. Sky Blue Windows is designing a Norman window that will require 24 ft of trim on the outer edges. What dimensions will allow the maximum amount of light to enter a house?

A Norman window

11. A frog leaps from a stump 3.5 ft. high and lands 3.5 ft. from the base of the stump.

It is determined that the height of the frog as a function of its distance, x, from the base of the stump is given by the function $h(x) = -0.5x^2 + 0.75x + 3.5$ where h is in feet.

a) How high is the frog when its horizontal distance from the base of the stump is 2 ft.?

b) At what two distances from the base of the stump after is jumped was the frog 3.6 ft. above the ground?

c) At what distance from the base did the frog reach its highest point?

d) What was the maximum height reached by the frog?

Section 3.2 – Polynomial Functions

Polynomial Function

A *Polynomial function* $P(x)$ in x is a sum of the form is given by:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$$

Where the coefficients $a_n,\ a_{n-1},\ \ldots, a_2,\ a_1, a_0$ are real numbers and the exponents are whole numbers.

$$a_n x^n$$

Degree

Leading Term

Leading Coefficient

Non-polynomial Functions: $\frac{1}{x} + 2x;\quad \sqrt{x^2 - 3} + x;\quad \frac{x-5}{x^2+2}$

Degree of f	***Form of f(x)***	***Graph of f(x)***
0	$f(x) = a_0$	A horizontal line
1	$f(x) = a_1 x + a_0$	A line with slope a_1
2	$f(x) = a_2 x^2 + a_1 x + a_0$	A parabola with a vertical axis

All polynomial functions are ***continuous functions***.

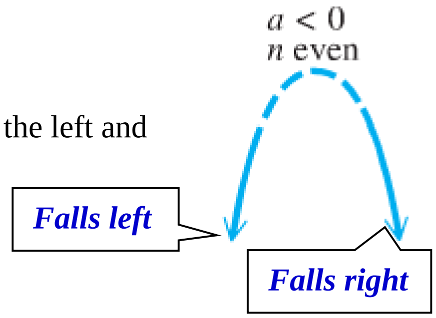
End Behavior $\left(a_n x^n\right)$

If n (degree) is even:

If $a_n < 0$ (in front x^n is negative), then the function falls from the left and right side

$$x \to -\infty \;\Rightarrow\; f(x) \to -\infty$$

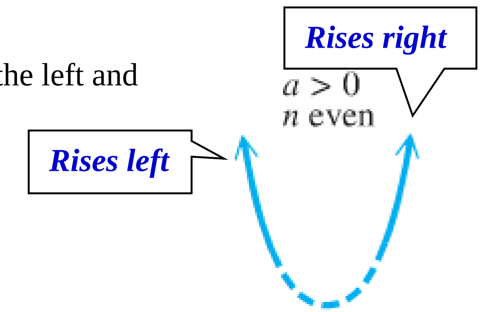
$$x \to \infty \;\Rightarrow\; f(x) \to -\infty$$



If $a_n > 0$ (in front x^n is positive), then the function rises from the left and right side

$$x \to -\infty \;\Rightarrow\; f(x) \to \infty$$

$$x \to \infty \;\Rightarrow\; f(x) \to \infty$$

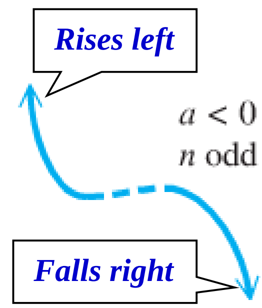


If n (degree) is odd:

If $a_n < 0$ (negative), then the function rises from the left side and falls from the right side

$$x \to -\infty \;\Rightarrow\; f(x) \to \infty$$

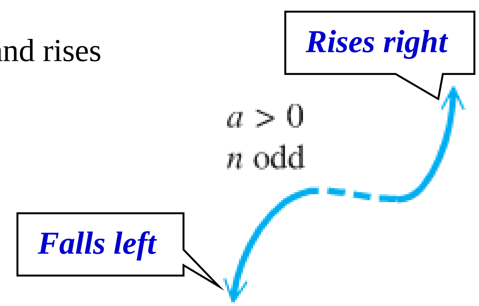
$$x \to \infty \;\Rightarrow\; f(x) \to -\infty$$



If $a_n > 0$ (positive), then the function falls from the left side and rises from the right side

$$x \to -\infty \;\Rightarrow\; f(x) \to -\infty$$

$$x \to \infty \;\Rightarrow\; f(x) \to \infty$$



Example

Determine the end behavior of the graph of the polynomial function $f(x) = -4x^5 + 7x^2 - x + 9$

Solution

Leading term: $-4x^5$ with 5th degree (n is odd)

$x \to -\infty \;\Rightarrow\; f(x) = -(-)^5 = (-)(-) = + \to \infty \qquad f(x)$ rises left

$x \to \infty \;\Rightarrow\; f(x) \to -\infty \qquad f(x)$ falls right

The Intermediate Value *Theorem*

For any polynomial function $f(x)$ with real coefficients and $f(a) \neq f(b)$ for $a < b$, then f takes on every value between $f(a)$ and $f(b)$ in the interval $[a, b]$.

$\therefore$ $f(a)$ and $f(b)$ are the opposite signs. Then the function has a real zero between a and b.

Example

Using the intermediate value theorem, determine, if possible, whether the function has a real zero between a and b.

a) $f(x) = x^3 + x^2 - 6x; \quad a = -4, \ b = -2$

b) $f(x) = x^3 + x^2 - 6x; \quad a = -1, \ b = 3$

Solution

a) $f(x) = x^3 + x^2 - 6x; \quad a = -4, \ b = -2$

$f(-4) = (-4)^3 + (-4)^2 - 6(-4) = -24$

$f(-2) = (-2)^3 + (-2)^2 - 6(-2) = 8$ $\qquad$ $f(x)$ has a zero between -4 and -2

b) $f(x) = x^3 + x^2 - 6x; \quad a = -1, \ b = 3$

$f(-1) = (-1)^3 + (-1)^2 - 6(-1) = 6$

$f(3) = (3)^3 + (3)^2 - 6(3) = 18$ $\qquad$ *Can't be determined*

Example

Show that $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3$ has a zero between 1 and 2.

Solution

$f(1) = (1)^5 + 2(1)^4 - 6(1)^3 + 2(1) - 3 = -4$

$f(2) = (2)^5 + 2(2)^4 - 6(2)^3 + 2(2) - 3 = 17$

Since $f(1)$ *and* $f(2)$ have opposite signs; therefore, $f(c) = 0$ for at least one real number c between 1 and 2.

Exercises *Section* 3.2 – Polynomial Functions

Determine the end behavior of the graph of the polynomial function

1. $f(x) = 5x^3 + 7x^2 - x + 9$

2. $f(x) = 11x^3 - 6x^2 + x + 3$

3. $f(x) = -11x^3 - 6x^2 + x + 3$

4. $f(x) = 5x^4 + 7x^2 - x + 9$

5. $f(x) = 11x^4 - 6x^2 + x + 3$

6. $f(x) = -5x^4 + 7x^2 - x + 9$

7. $f(x) = -11x^4 - 6x^2 + x + 3$

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers.

8. $f(x) = x^3 - x - 1;\quad between\ \ 1\ and\ \ 2$

9. $f(x) = x^3 - 4x^2 + 2;\quad between\ \ 0\ and\ \ 1$

10. $f(x) = 2x^4 - 4x^2 + 1;\quad between\ \ -1\ and\ \ 0$

11. $f(x) = x^4 + 6x^3 - 18x^2;\quad between\ \ 2\ and\ \ 3$

12. $f(x) = x^3 + x^2 - 2x + 1;\quad between\ \ -3\ and\ \ -2$

13. $f(x) = x^5 - x^3 - 1;\quad between\ \ 1\ and\ \ 2$

14. $f(x) = 3x^3 - 10x + 9;\quad between\ \ -3\ and\ \ -2$

15. $f(x) = 3x^3 - 8x^2 + x + 2;\quad between\ \ 2\ and\ \ 3$

16. $f(x) = 3x^3 - 8x^2 + x + 2;\quad between\ \ 1\ and\ \ 2$

17. $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3;\quad between\ \ 0\ and\ \ 1$

Section 3.3 – Properties of Division

Long Division

Divide $\left(x^3+2x^2-5x-6\right) \div (x+1)$

$$
\begin{array}{r|l}
 & \overbrace{x^2+x-6}^{\text{Quotient}} \\
\hline
\underset{\text{Divisor}}{x+1} & x^3+2x^2-5x-6 \leftarrow \text{Dividend} \\
 & \underline{x^3+x^2} \\
 & x^2-5x \\
 & \underline{x^2-x} \\
 & -6x-6 \\
 & \underline{-6x-6} \\
 & 0 \leftarrow \text{Remainder}
\end{array}
$$

$Q(x)=x^2+x-6$

$R(x)=0$

Example

Use the long division to find the quotient and the remainder: $\left(x^4-16\right)\div\left(x^2+3x+1\right)$

Solution

$$
\begin{array}{r|l}
 & x^2-3x+8 \\
\hline
x^2+3x+1 & x^4+0x^3+0x^2+0x-16 \\
 & \underline{x^4+3x^3+x^2} \\
 & -3x^3-x^2 \\
 & \underline{-3x^3-9x^2-3x} \\
 & 8x^2+\ 3x-16 \\
 & \underline{8x^2+24x+\ 8} \\
 & -21x-24
\end{array}
$$

$$\frac{x^4-16}{x^2+3x+1}=x^2-3x+8+\frac{-21x-24}{x^2+3x+1}$$

$$x^4-16=\boxed{\left(x^2+3x+1\right)\left(x^2-3x+8\right)+(-21x-24)}$$

Remainder *Theorem*

If a number c is substituted for x in the polynomial $f(x)$, then the result $f(c)$ is the remainder that would be obtained by dividing $f(x)$ by $x-c$.

That is, if $f(x)=(x-c)Q(x)+R(x)$ *then* $f(c)=R$

Example

If $f(x)=x^3-3x^2+x+5$, use the remainder theorem to find $f(2)$

Solution

$$\begin{array}{r|l} & x^2-x-1 \\ \hline x-2 & x^3-3x^2+x+5 \\ & \underline{x^3-2x^2} \\ & \quad -x^2+x \\ & \quad \underline{-x^2+2x} \\ & \qquad\quad -x+5 \\ & \qquad\quad \underline{-x+2} \\ & \qquad\qquad\quad \boxed{3} \end{array}$$

$f(2)=3$

Factor *Theorem*

A polynomial $f(x)$ has a factor $x-c$ if and only if $f(c)=0$

Example

Show that $x-2$ is a factor of $f(x)=x^3-4x^2+3x+2$.

Solution

Since $f(2)=(2)^3-4(2)^2+3(2)=0$

From the factor theorem; $x-2$ is a factor of $f(x)$.

Synthetic Division

Use synthetic division to find the quotient and the remainder of $\left(4x^3 - 3x^2 + x + 7\right) \div (x-2)$

$$
\begin{array}{c|cccc}
 & x^3 & x^2 & x^1 & x^0 \\
2 & 4 & -3 & 1 & 7 \\
 & \downarrow & 8 & 10 & 22 \\
\hline
 & 4 & 5 & 11 & \boxed{29} \\
 & x^2 & x^1 & x^0 &
\end{array}
$$

Quotient : $Q(x) = 4x^2 + 5x + 11$

Remainder : $R(x) = 29$

Example

If $f(x) = 3x^5 - 38x^3 + 5x^2 - 1$, use the synthetic division to find $f(4)$.

Solution

$$
\begin{array}{c|cccccc}
4 & 3 & 0 & -38 & 5 & 0 & -1 \\
 & & 12 & 48 & 40 & 180 & 720 \\
\hline
 & 3 & 12 & 10 & 45 & 180 & \boxed{719}
\end{array}
$$

$f(4) = 719$

Example

Show that -11 is a zero of the polynomial $f(x) = x^3 + 8x^2 - 29x + 44$

Solution

$$
\begin{array}{c|cccc}
-11 & 1 & 8 & -29 & 44 \\
 & & -11 & 33 & -44 \\
\hline
 & 1 & -3 & 4 & \boxed{0}
\end{array}
$$

Thus, $f(-11) = 0$, and -11 is a zero of f.

The Rational Zeros *Theorem*

If the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ *has integer* coefficients and if $\frac{c}{d}$ is a rational zero of $f(x)$ such that $\boldsymbol{c}$ and $\boldsymbol{d}$ have no common prime factor, then

1. The numerator c of the zero is a factor of the constant term a_0
2. The denominator d of the zero is a factor of the leading coefficient a_n

$$\text{possible rational zeros} = \frac{\text{factors of the constant term } a_0}{\text{factors of the leading coefficient } a_n} = \frac{\text{possibilities for } a_0}{\text{possibilities for } a_n}$$

Example

Find all rational solutions of the equation: $3x^4 + 14x^3 + 14x^2 - 8x - 8 = 0$

Solution

possibilities for a_0	$\pm 1, \ \pm 2, \ \pm 4, \ \pm 8$
possibilities for a_n	$\pm 1, \ \pm 3$
possibilities for c/d	$\pm 1, \ \pm 2, \ \pm 4, \ \pm 8, \ \pm\frac{1}{3}, \ \pm\frac{2}{3}, \ \pm\frac{4}{3}, \ \pm\frac{8}{3}$

Using the calculator, the result will show that –2 is a zero.

$$\begin{array}{r|rrrrr} -2 & 3 & 14 & 14 & -8 & -8 \\ & & -6 & -16 & 4 & 8 \\ \hline & 3 & 8 & -2 & -4 & \boxed{0} \end{array}$$

We have the factorization of: $(x+2)\left(3x^3 + 8x^2 - 2x - 4\right) = 0$

For $3x^3 + 8x^2 - 2x - 4 \quad \Rightarrow \quad \frac{c}{d} = \frac{\pm 1, \ \pm 2, \ \pm 4}{\pm 1, \ \pm 3}$

$x = -\frac{2}{3}$ is another solution.

$$\begin{array}{r|rrrr} -\frac{2}{3} & 3 & 8 & -2 & -4 \\ & & -2 & -4 & 4 \\ \hline & 3 & 6 & -6 & \boxed{0} \end{array}$$

We have the factorization of: $(x+2)\left(x+\frac{2}{3}\right)\left(3x^2 + 6x - 6\right) = 0$

By applying quadratic formula to solve: $3x^2 + 6x - 6 = 0 \Rightarrow x = -1 \pm \sqrt{3}$

Hence, the polynomial has two rational roots $x = -2$ *and* $-\frac{2}{3}$ and two irrational roots $x = -1 \pm \sqrt{3}$.

Exercises *Section* 3.3 – Properties of Division

1. Find the quotient and remainder if $f(x)$ is divided by $p(x)$:

$f(x)=2x^4-x^3+7x-12;\quad p(x)=x^2-3$

2. Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=3x^3+2x-4;\quad p(x)=2x^2+1$

3. Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=7x+2;\quad p(x)=2x^2-x-4$

4. Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=9x+4;\quad p(x)=2x-5$

5. Use the remainder theorem to find $f(c)$: $f(x)=x^4-6x^2+4x-8;\quad c=-3$

6. Use the remainder theorem to find $f(c)$: $f(x)=x^4+3x^2-12;\quad c=-2$

7. Use the factor theorem to show that $x-c$ is a factor of $f(x)$: $f(x)=x^3+x^2-2x+12;\quad c=-3$

8. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $2x^3-3x^2+4x-5;\quad x-2$

9. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $5x^3-6x^2+15;\quad x-4$

10. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $9x^3-6x^2+3x-4;\quad x-\frac{1}{3}$

11. Use the synthetic division to find $f(c)$: $f(x)=2x^3+3x^2-4x+4;\quad c=3$

12. Use the synthetic division to find $f(c)$: $f(x)=8x^5-3x^2+7;\quad c=\frac{1}{2}$

13. Use the synthetic division to find $f(c)$: $f(x)=x^3-3x^2-8;\quad c=1+\sqrt{2}$

14. Use the synthetic division to show that c is a zero of $f(x)$:

$f(x)=3x^4+8x^3-2x^2-10x+4;\quad c=-2$

15. Use the synthetic division to show that c is a zero of $f(x)$:

$f(x)=27x^4-9x^3+3x^2+6x+1;\quad c=-\frac{1}{3}$

16. Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

$f(x)=kx^3+x^2+k^2x+3k^2+11;\quad x+2$

17. Find all solutions of the equation: $x^3-x^2-10x-8=0$

18. Find all solutions of the equation: $x^3+x^2-14x-24=0$

19. Find all solutions of the equation: $2x^3 - 3x^2 - 17x + 30 = 0$

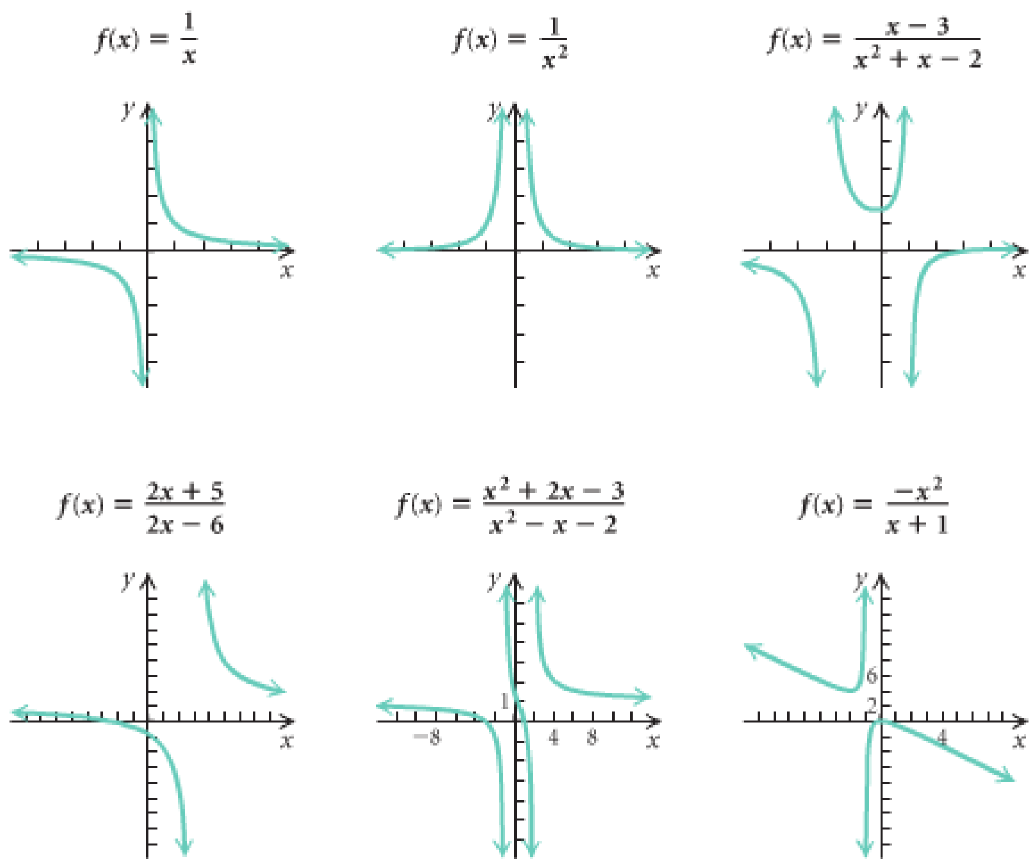
20. Find all solutions of the equation: $12x^3 + 8x^2 - 3x - 2 = 0$

21. Find all solutions of the equation: $x^4 + 3x^3 - 30x^2 - 6x + 56 = 0$

22. Find all solutions of the equation: $3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6 = 0$

23. Find all solutions of the equation: $6x^5 + 19x^4 + x^3 - 6x^2 = 0$

Section 3.4 – Rational Functions



Rational Function

A rational function is a function f that is a quotient of two polynomials, that is,

$$f(x)=\frac{g(x)}{h(x)}$$

Where $g(x)$ and $h(x)$ are polynomials. The domain of f consists of all real numbers ***except*** the zeros of the denominator $h(x)$.

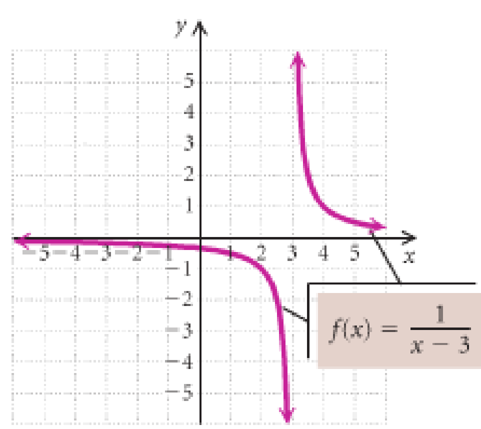
The Domain of a Rational Function

Example

Consider: $f(x)=\frac{1}{x-3}$

Find the domain and graph *f*.

Solution



$x-3=0 \quad \Rightarrow \quad \boxed{x=3}$

Thus the domain is: $\{x \mid x \neq 3\}$ *or* $(-\infty,\ 3)\cup(3,\ \infty)$

Function	*Domain*	
$f(x)=\frac{1}{x}$	$\{x \mid x\neq 0\}$	$(-\infty,\ 0)\cup(0,\ \infty)$
$f(x)=\frac{1}{x^2}$	$\{x \mid x\neq 0\}$	$(-\infty,\ 0)\cup(0,\ \infty)$
$f(x)=\frac{x-3}{x^2+x-2}$	$\{x \mid x\neq -2 \text{ and } x\neq 1\}$	$(-\infty,\ -2)\cup(-2,\ 1)\cup(1,\ \infty)$
$f(x)=\frac{2x+5}{2x-6}=\frac{2x+5}{2(x-3)}$	$\{x \mid x\neq 3\}$	$(-\infty,\ 3)\cup(3,\ \infty)$

Asymptotes

Vertical Asymptote (*VA*) - ***Think Domain***

The line $x=a$ is a ***vertical asymptote*** for the graph of a function f if

$$f(x)\to\infty \quad or \quad f(x)\to-\infty$$

As ***x*** approaches ***a*** from either the left or the right

Example

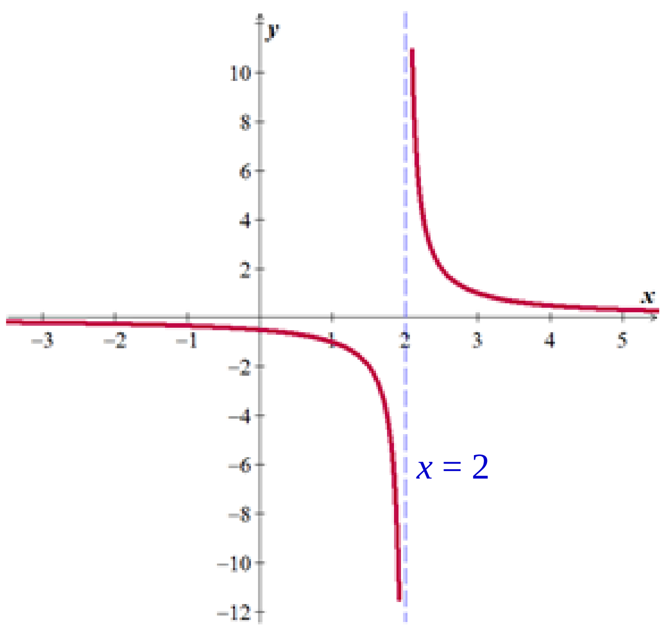
Find the vertical asymptote of $f(x)=\frac{1}{x-2}$, and sketch the graph.

Solution

VA: $x=2$

$f(x)\to\infty \quad as \quad x\to 2^+$

$f(x)\to-\infty \quad as \quad x\to 2^-$



Horizontal Asymptote (*HA*)

The line $y = c$ is a ***horizontal asymptote*** for the graph of a function f if

$$f(x) \to c \quad as \quad x \to -\infty \quad or \quad x \to -\infty$$

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0} = \dfrac{a_n x^n}{b_m x^m}$ be a rational function.

1. If the degree of numerator is less than of denominator $(n < m) \Rightarrow$ y = 0

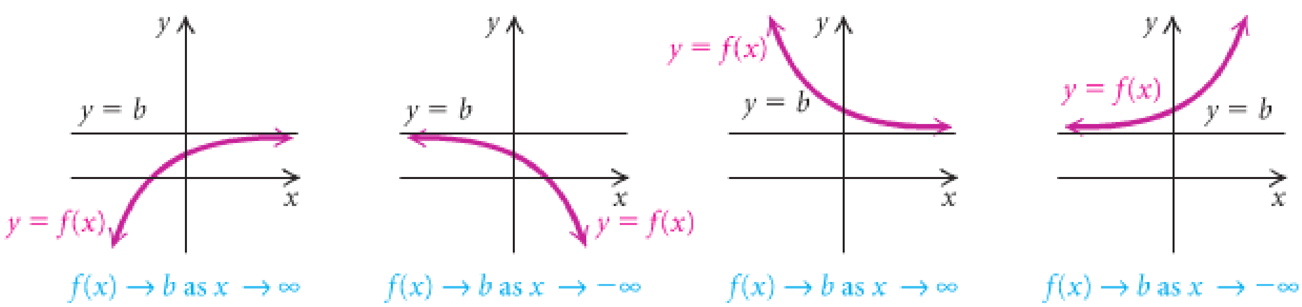
$$y = \frac{2x+1}{4x^2+5} \quad \Rightarrow \boxed{y = 0}$$

2. If the degree of numerator is equal of denominator $(n = m) \Rightarrow y = \dfrac{a_n}{b_m}$

$$y = \frac{2x^2+1}{4x^2+5} \quad \Rightarrow \boxed{y = \frac{2}{4} = \frac{1}{2}}$$

3. If the degree of numerator is greater than of denominator $(n > m) \Rightarrow$ No horizontal asymptote

$$y = \frac{2x^3+1}{4x^2+5} \quad \Rightarrow No \;\; HA$$



Example

Determine the horizontal asymptote of $f(x) = \dfrac{-7x^4 - 10x^2 + 1}{11x^4 + x - 2}$.

Solution

$$f(x) = \frac{-7x^4 - 10x^2 + 1}{11x^4 + x - 2} \to \frac{-7x^4}{11x^4} = -\frac{7}{11}$$

Therefore, the horizontal asymptote (***HA***) is: $\boxed{y = -\frac{7}{11}}$

Example

Find the vertical and the horizontal asymptote for the graph of f, if it exists

a) $f(x)=\frac{3x-1}{x^2-x-6}$

b) $f(x)=\frac{5x^2+1}{3x^2-4}$

c) $f(x)=\frac{2x^4-3x^2+5}{x^2+1}$

Solution

a) $f(x)=\frac{3x-1}{x^2-x-6}$

$$x^2-x-6=0 \ \rightarrow x=-2,\ 3$$

VA: $x=-2, \quad x=3$

HA: $y=0$

b) $f(x)=\frac{5x^2+1}{3x^2-4}$

$$3x^2-4=0 \ \rightarrow 3x^2=4 \ \rightarrow x^2=\frac{4}{3} \ \rightarrow \boxed{x=\pm\frac{2}{\sqrt{3}}}$$

VA: $x=-\frac{2}{\sqrt{3}}, \quad x=\frac{2}{\sqrt{3}}$

HA: $y=\frac{5}{3}$

c) $f(x)=\frac{2x^4-3x^2+5}{x^2+1}$

$$x^2+1=0 \ \rightarrow x^2=-1$$

VA: *n/a*

HA: *n/a*

Slant or *Oblique* Asymptotes

When the degree of the numerator is one greater than the degree of the numerator, the graph has a slant or oblique asymptote and it is a line $y = ax + b, \quad a \neq 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

$$y = \frac{3x^2 - 1}{x + 2}$$

$$\begin{array}{r} 3x - 6 \\ x + 2 \overline{\smash{\big)}\, 3x^2 + 0x - 1} \\ \underline{3x^2 + 6x} \\ -6x - 1 \\ \underline{-6x - 12} \\ R = 11 \end{array}$$

$$y = \frac{3x^2 - 1}{x + 2} = 3x - 6 + \frac{11}{x + 2}$$

The ***oblique asymptote*** is the line ***y* = 3*x* – 6**

Example

Find all the asymptotes of $f(x) = \dfrac{2x^2 - 3x - 1}{x - 2}$

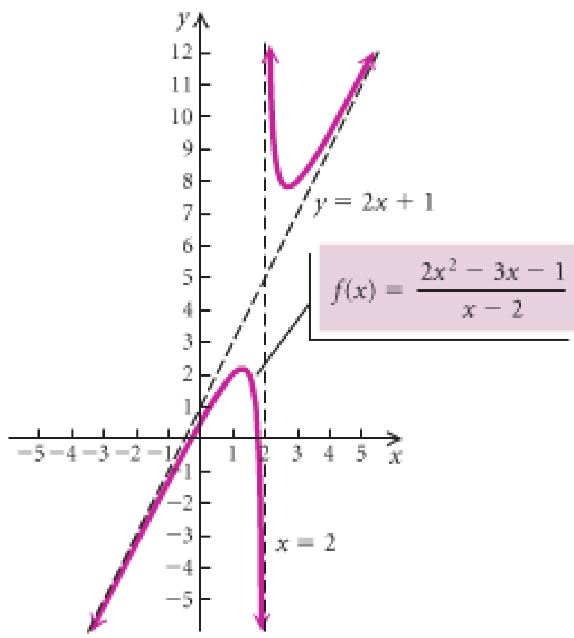
Solution

$$\begin{array}{r} 2x + 1 \\ x - 2 \overline{\smash{\big)}\, 2x^2 - 3x - 1} \\ \underline{-2x^2 + 4x} \\ x - 1 \\ \underline{-x + 2} \\ 1 \end{array}$$

$$f(x) = \frac{2x^2 - 3x - 1}{x - 2} = (2x + 1) + \frac{1}{x - 2}$$

The ***oblique asymptote*** is the line $y = 2x + 1$

VA:: $x = 2$



Graph That Has a *Hole*

Example

Sketch the graph of g if $g(x)=\dfrac{(3x+4)(x-1)}{(2x-5)(x-1)}$

Solution

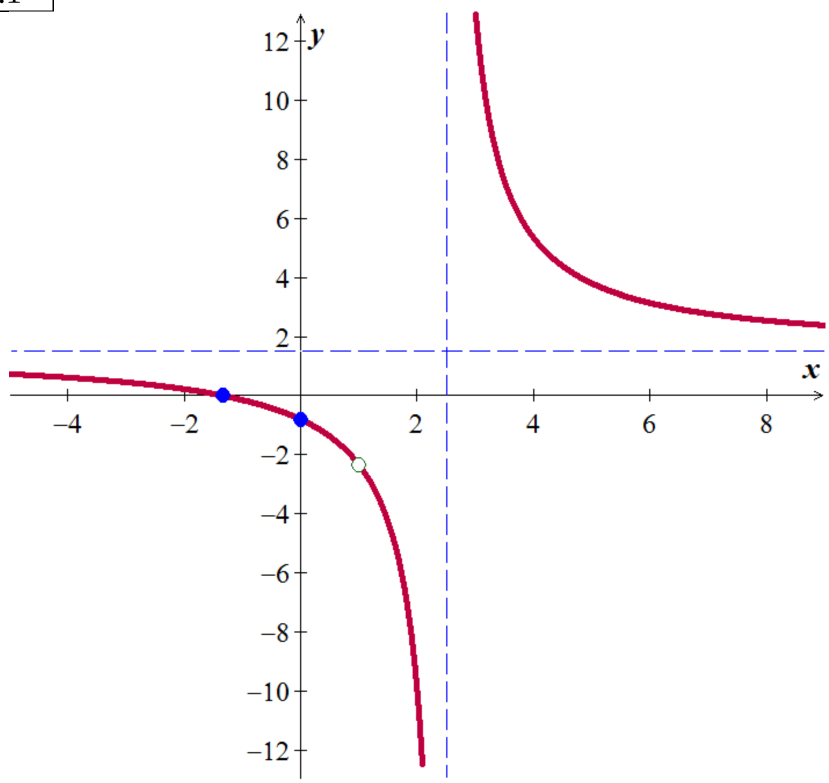
$$g(x)=\frac{(3x+4)(x-1)}{(2x-5)(x-1)}=\frac{3x+4}{2x-5}=f(x)$$

VA: $x=\frac{5}{2}$

HA: $y=\frac{3}{2}$

The only different between the graphs that g has a ***hole*** at $x=1\rightarrow f(1)=-\frac{7}{3}$

x	***y***
–4	.6
1.3	0
0	–.8
4	5.3
6	3.1



Exercises *Section* 3.4 – Rational Functions

Find the vertical and horizontal asymptotes (if any) of

1. $y=\dfrac{3x}{1-x}$

2. $y=\dfrac{x^2}{x^2+9}$

3. $y=\dfrac{x-2}{x^2-4x+3}$

4. $y=\dfrac{3}{x-5}$

5. $y=\dfrac{x^3-1}{x^2+1}$

6. $y=\dfrac{3x^2-27}{(x+3)(2x+1)}$

7. $y=\dfrac{x^3+3x^2-2}{x^2-4}$

8. $y=\dfrac{x-3}{x^2-9}$

9. $y=\dfrac{6}{\sqrt{x^2-4x}}$

10. $y=\dfrac{5x-1}{1-3x}$

11. $f(x)=\dfrac{2x-11}{x^2+2x-8}$

12. $f(x)=\dfrac{x^2-4x}{x^3-x}$

13. $f(x)=\dfrac{x-2}{x^3-5x}$

Determine all asymptotes of the function

14. $f(x)=\dfrac{4x}{x^2+10x}$

15. $f(x)=\dfrac{3-x}{(x-4)(x+6)}$

16. $f(x)=\dfrac{x^3}{2x^3-x^2-3x}$

17. $f(x)=\dfrac{3x^2+5}{4x^2-3}$

18. $f(x)=\dfrac{x+6}{x^3+2x^2}$

19. $f(x)=\dfrac{x^2+4x-1}{x+3}$

20. $f(x)=\dfrac{x^2-6x}{x-5}$

21. $f(x)=\dfrac{x^3-x^2+x-4}{x^2+2x-1}$

22. $f(x)=\dfrac{-3x}{x+2}$

23. $f(x)=\dfrac{x+1}{x^2+2x-3}$

24. $f(x)=\dfrac{2x^2-2x-4}{x^2+x-12}$

25. $f(x)=\dfrac{-2x^2+10x-12}{x^2+x}$

26. $f(x)=\dfrac{x^2-x-6}{x+1}$

27. $f(x)=\dfrac{x^3+1}{x-2}$

28. $f(x)=\dfrac{2x^2+x-6}{x^2+3x+2}$

29. $f(x)=\dfrac{x-1}{1-x^2}$

30. $f(x)=\dfrac{x^2+x-2}{x+2}$

31. $f(x)=\dfrac{x^3-2x^2-4x+8}{x-2}$

32. Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \textit{vertical asymptote}: \ x=4 \\ \textit{horizontal asymptote}: \ y=-1 \\ x-\textit{intercept}: \ 3 \end{cases}$$

33. Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \textit{vertical asymptote}: \ x=-3, x=1 \\ \textit{horizontal asymptote}: \ y=0 \\ x-\textit{intercept}: \ -1, \ f(0)=-2 \\ \textit{hole at } \ x=2 \end{cases}$$

34. Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \textit{vertical asymptote}:\ x=-4, x=5 \\ \textit{horizontal asymptote}:\ y=\dfrac{3}{2} \\ x-\textit{intercept}:\ -2 \end{cases}$$