Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a: $f(x)=e^{2x}, \quad a=0$

Solution

$$f(x)=e^{2x} \quad\to\quad f(0)=e^{2(0)}=1$$
$$f'(x)=2e^{2x} \quad\to\quad f'(0)=2$$
$$f''(x)=4e^{2x} \quad\to\quad f''(0)=4$$
$$f'''(x)=8e^{2x} \quad\to\quad f'''(0)=8$$

$$P_0(x)=f(0)=1$$
$$P_1(x)=f(0)+f'(0)(x-0)=1+2x$$
$$P_2(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2=1+2x+2x^2$$
$$P_3(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3$$
$$=1+2x+2x^2+\frac{4}{3}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a: $f(x)=\sin x, \quad a=0$

Solution

$$f(x)=\sin x \quad\to\quad f(0)=0$$
$$f'(x)=\cos x \quad\to\quad f'(0)=1$$
$$f''(x)=-\sin x \quad\to\quad f''(0)=0$$
$$f'''(x)=-\cos x \quad\to\quad f'''(0)=-1$$

$$P_0(x)=f(0)=0$$
$$P_1(x)=f(0)+f'(0)(x-0)=x$$
$$P_2(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2=x$$
$$P_3(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3$$

$$= x - \frac{1}{6}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \ln(1+x), \quad a = 0$

Solution

$$f(x) = \ln(1+x) \quad \to \quad f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \quad \to \quad f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \quad \to \quad f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \quad \to \quad f'''(0) = 2$$

$$P_0(x) = f(0) = 0$$

$$P_1(x) = f(0) + f'(0)(x-0) = x$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 = x - \frac{1}{2}x^2$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \frac{1}{x+2}, \quad a = 0$

Solution

$$f(x) = (x+2)^{-1} \quad \to \quad f(0) = \frac{1}{2}$$

$$f'(x) = -(x+2)^{-2} \quad \to \quad f'(0) = -\frac{1}{4}$$

$$f''(x) = 2(x+2)^{-3} \quad \to \quad f''(0) = \frac{1}{4}$$

$$f'''(x) = -6(x+2)^{-4} \quad \to \quad f'''(0) = -\frac{3}{8}$$

$$P_0(x) = f(0) = \frac{1}{2}$$

$$P_1(x) = f(0) + f'(0)(x-0) = \frac{1}{2} - \frac{1}{4}x$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 = \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$\boxed{= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \sqrt{1-x}, \quad a = 0$

Solution

$$f(x) = (1-x)^{1/2} \quad \to \quad f(0) = 1$$

$$f'(x) = -\tfrac{1}{2}(1-x)^{-1/2} \quad \to \quad f(0) = -\tfrac{1}{2}$$

$$f''(x) = -\tfrac{1}{4}(1-x)^{-3/2} \quad \to \quad f(0) = -\tfrac{1}{4}$$

$$f'''(x) = -\tfrac{3}{8}(1-x)^{-5/2} \quad \to \quad f(0) = -\tfrac{3}{8}$$

$$P_0(x) = f(0) = 1$$

$$P_1(x) = f(0) + f'(0)(x-0) = 1 - \tfrac{1}{2}x$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 = 1 - \frac{1}{2}x - \frac{1}{8}x^2$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$\boxed{= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = x^3, \quad a = 1$

Solution

$$f(x) = x^3 \quad \to \quad f(1) = 1$$

$$f'(x) = 3x^2 \quad \to \quad f'(1) = 3$$

$$f''(x) = 6x \quad \to \quad f''(1) = 6$$

$$f'''(x) = 6 \quad \to \quad f'''(1) = 6$$

$$P_0(x) \ \boxed{=1}$$ $$P_0(x) = f(a)$$

$$P_1(x) \boxed{=1+3(x-1)} \qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x) \boxed{=1+3(x-1)+3(x-1)^2} \qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x) \boxed{=1+3(x-1)+3(x-1)^2+(x-1)^3} \quad P_3(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=8\sqrt{x}, \quad a=1$

Solution

$$f(x)=8x^{1/2} \quad \to \quad f(1)=8$$

$$f'(x)=4x^{-1/2} \quad \to \quad f'(1)=4$$

$$f''(x)=-2x^{-3/2} \quad \to \quad f''(1)=-2$$

$$f'''(x)=3x^{-5/2} \quad \to \quad f'''(1)=3$$

$$P_0(x) \boxed{=8} \qquad P_0(x)=f(a)$$

$$P_1(x) \boxed{=8+4(x-1)} \qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x) \boxed{=8+4(x-1)-(x-1)^2} \qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x) \boxed{=8+4(x-1)-(x-1)^2+3(x-1)^3} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\sin x, \quad a=\frac{\pi}{4}$

Solution

$$f(x)=\sin x \quad \to \quad f\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

$$f'(x)=\cos x \quad \to \quad f'\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

$$f''(x)=-\sin x \quad \to \quad f''\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$

$$f'''(x)=-\cos x \quad \to \quad f'''\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$

$$P_0(x) \boxed{=\frac{\sqrt{2}}{2}} \qquad P_0(x)=f(a)$$

$$P_1(x) = \boxed{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)} \qquad P_1(x) = f(a) + f'(a)(x-a)$$

$$P_2(x) = \boxed{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2} \qquad P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x) = \boxed{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3} \qquad P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \cos x, \quad a = \frac{\pi}{6}$

Solution

$$f(x) = \cos x \quad \to \quad f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$f'(x) = -\sin x \quad \to \quad f'\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

$$f''(x) = -\cos x \quad \to \quad f''\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$f'''(x) = \sin x \quad \to \quad f'''\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$P_0(x) = \boxed{\frac{\sqrt{3}}{2}} \qquad P_0(x) = f(a)$$

$$P_1(x) = \boxed{\frac{\sqrt{2}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right)} \qquad P_1(x) = f(a) + f'(a)(x-a)$$

$$P_2(x) = \boxed{\frac{\sqrt{2}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2} \qquad P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x) = \boxed{\frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x - \frac{\pi}{6}\right)^3} \qquad P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \sqrt{x}, \quad a = 9$

Solution

$$f(x) = x^{1/2} \quad \to \quad f(9) = 3$$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \quad \to \quad f'(9) = \frac{1}{6}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} = -\frac{1}{4x\sqrt{x}} \quad \to \quad f''(9) = -\frac{1}{4 \times 3^3}$$

$$f'''(x)=\frac{3}{8}x^{-5/2}=\frac{3}{8x^2\sqrt{x}} \quad \to \quad f'''(9)=\frac{1}{2^3\times 3^4}$$

$$P_0(x)\ \underline{=3}| \qquad\qquad P_0(x)=f(a)$$

$$P_1(x)\ \underline{=3+\frac{1}{6}(x-9)}| \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)\ \underline{=3+\frac{1}{2\cdot 3}(x-9)-\frac{1}{2^3\cdot 3^3}(x-9)^2}| \qquad\qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)\ \underline{=3+\frac{1}{2\cdot 3}(x-9)-\frac{1}{2^2\cdot 3^3}(x-9)^2+\frac{1}{2^4\cdot 3^5}(x-9)^3}| \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\sqrt[3]{x}, \quad a=8$

Solution

$$f(x)=x^{1/3} \quad \to \quad f(8)=2$$

$$f'(x)=\frac{1}{3}x^{-2/3}=\frac{1}{3x^{2/3}} \quad \to \quad f'(8)=\frac{1}{2^2\times 3}$$

$$f''(x)=-\frac{2}{9}x^{-5/3}=-\frac{2}{3^2x^{5/3}} \quad \to \quad f''(8)=-\frac{1}{3^2\times 2^4}$$

$$f'''(x)=\frac{10}{3^3}x^{-8/3}=\frac{2\cdot 5}{3^3x^{8/3}} \quad \to \quad f'''(8)=\frac{5}{2^7\times 3^3}$$

$$P_0(x)\ \underline{=2}| \qquad\qquad P_0(x)=f(a)$$

$$P_1(x)\ \underline{=2+\frac{1}{2^2\cdot 3}(x-8)}| \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)\ \underline{=2+\frac{1}{2^2\cdot 3}(x-8)-\frac{1}{2^5\cdot 3^2}(x-8)^2}| \qquad\qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)\ \underline{=2+\frac{1}{2^2\cdot 3}(x-8)-\frac{1}{2^5\cdot 3^2}(x-8)^2+\frac{1}{2^8\cdot 3^4}(x-8)^3}| \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\ln x, \quad a=e$

Solution

$$f(x)=\ln x \quad \to \quad f(e)=1$$

$$f'(x)=\frac{1}{x} \quad \to \quad f'(e)=\frac{1}{e}$$

$$f''(x)=-\frac{1}{x^2} \quad \to \quad f''(e)=-\frac{1}{e^2}$$

$$f'''(x)=\frac{2}{x^3} \quad \to \quad f'''(e)=\frac{2}{e^3}$$

$$P_0(x)\ \boxed{=1} \qquad\qquad P_0(x)=f(a)$$

$$P_1(x)\ \boxed{=1+\frac{1}{e}(x-e)} \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)\ \boxed{=1+\frac{1}{e}(x-e)-\frac{1}{2e^2}(x-e)^2} \qquad\qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)\ \boxed{=1+\frac{1}{e}(x-e)-\frac{1}{2e^2}(x-e)^2+\frac{1}{3e^3}(x-e)^3} \qquad\qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\sqrt[4]{x}, \quad a=8$

Solution

$$f(x)=x^{1/4} \quad \to \quad f(8)=\sqrt[4]{8}$$

$$f'(x)=\frac{1}{4}x^{-3/4}=\frac{1}{4x^{3/4}} \quad \to \quad f'\left(8=2^3\right)=\frac{1}{2^2\times 2^{9/4}}=\frac{1}{2^4\sqrt[4]{2}}$$

$$f''(x)=-\frac{3}{16}x^{-7/4}=-\frac{3}{2^4x^{7/4}} \quad \to \quad f''(8)=-\frac{3}{2^4\ 2^{21/4}}=-\frac{3}{2^9\sqrt[4]{2}}$$

$$f'''(x)=\frac{21}{2^6}x^{-11/4}=\frac{21}{2^6x^{11/4}} \quad \to \quad f'''(8)=\frac{21}{2^6\times 2^{33/4}}=\frac{21}{2^{14}\sqrt[4]{2}}$$

$$P_0(x)\ \boxed{=\sqrt[4]{8}} \qquad\qquad P_0(x)=f(a)$$

$$P_1(x)\ \boxed{=\sqrt[4]{8}+\frac{1}{2^4\cdot\sqrt[4]{2}}(x-8)} \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)\ \boxed{=\sqrt[4]{8}+\frac{1}{2^4\cdot\sqrt[4]{2}}(x-8)-\frac{3}{2^{10}\cdot\sqrt[4]{2}}(x-8)^2} \qquad\qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)\ \boxed{=\sqrt[4]{8}+\frac{1}{2^4\cdot\sqrt[4]{2}}(x-8)-\frac{3}{2^{10}\cdot\sqrt[4]{2}}(x-8)^2+\frac{7}{2^{15}\cdot\sqrt[4]{2}}(x-8)^3} \qquad\qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\tan^{-1}x+x^2+1, \quad a=1$

Solution

$$f(x)=\tan^{-1}x+x^2+1 \quad \to \quad f(1)=\frac{\pi}{4}+2$$

$$f'(x)=\frac{1}{x^2+1}+2x \quad \to \quad f'(1)=\frac{5}{2}$$

$$f''(x)=-\frac{2x}{\left(x^2+1\right)^2}+2 \quad \to \quad f''(1)=-\frac{1}{2}+2=\frac{3}{2}$$

$$f'''(x)=-\frac{2x^2+2-8x^2}{\left(x^2+1\right)^3}=-\frac{2-2x^2}{\left(x^2+1\right)^3} \quad \to \quad f'''(1)=0 \qquad \left(U^nV^m\right)'=U^{n-1}V^{m-1}\left(nU'V+mUV'\right)$$

$$P_0(x)\ \boxed{=\frac{\pi}{4}+2} \qquad P_0(x)=f(a)$$

$$P_1(x)\ \boxed{=\frac{\pi}{4}+2+\frac{5}{2}(x-1)} \qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)\ \boxed{=\frac{\pi}{4}+2+\frac{5}{2}(x-1)-\frac{3}{4}(x-1)^2} \qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)\ \boxed{=\frac{\pi}{4}+2+\frac{5}{2}(x-1)-\frac{3}{4}(x-1)^2} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=e^x, \quad a=\ln 2$

Solution

$$f(x)=e^x \quad \to \quad f(\ln 2)=2$$

$$f'(x)=e^x \quad \to \quad f'(\ln 2)=2$$

$$f''(x)=e^x \quad \to \quad f''(\ln 2)=2$$

$$f'''(x)=e^x \quad \to \quad f'''(\ln 2)=2$$

$$P_0(x)\ \boxed{=2} \qquad P_0(x)=f(a)$$

$$P_1(x)\ \boxed{=2+2(x-\ln 2)} \qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)\ \boxed{=2+2(x-\ln 2)+(x-\ln 2)^2} \qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)\ \boxed{=2+2(x-\ln 2)+(x-\ln 2)^2+\frac{1}{3}(x-\ln 2)^3} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $\quad f(x)=e^{3x}; \quad a=0$

Solution

$f(x)=e^{3x}$	$\to f(0)=1$
$f'(x)=3e^{3x}$	$\to f'(0)=3$
$f''(x)=9e^{3x}$	$\to f''(0)=9$
$f'''(x)=27e^{3x}$	$\to f'''(0)=27$

$\boxed{P_0(x)=1}$ $\qquad P_0(x)=f(a)$

$\boxed{P_1(x)=1+3x}$ $\qquad P_1(x)=f(a)+f'(a)(x-a)$

$\boxed{P_2(x)=1+3x+\frac{9}{2}x^2}$ $\qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2$

$$P_3(x)=f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2+\frac{1}{3!}f'''(a)(x-a)^3$$

$$\boxed{=1+3x+\frac{9}{2}x^2+\frac{9}{2}x^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $\quad f(x)=\frac{1}{x}; \quad a=1$

Solution

$f(x)=\frac{1}{x}$	$\to f(1)=1$
$f'(x)=-\frac{1}{x^2}$	$\to f'(1)=-1$
$f''(x)=\frac{2}{x^3}$	$\to f''(1)=2$
$f'''(x)=-\frac{6}{x^4}$	$\to f'''(1)=-6$

$\boxed{P_0(x)=1}$ $\qquad P_0(x)=f(a)$

$P_1(x)=1-(x-1)$ $\qquad P_1(x)=f(a)+f'(a)(x-a)$

$\boxed{=2-x}$

$$\boxed{P_2(x) = 1-(x-1)+(x-1)^2} \qquad P_2(x) = f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2$$

$$P_3(x) = f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2+\frac{1}{3!}f'''(a)(x-a)^3$$

$$\boxed{= 1-(x-1)+(x-1)^2-(x-1)^3}$$

$$\boxed{f(x) = \sum_{k=0}^{\infty}(-1)^k(x-1)^k}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $\quad f(x) = \cos x; \quad a = \frac{\pi}{2}$

Solution

$f(x) = \cos x$	$\rightarrow f\left(\frac{\pi}{2}\right) = 0$
$f'(x) = -\sin x$	$\rightarrow f'\left(\frac{\pi}{2}\right) = -1$
$f''(x) = -\cos x$	$\rightarrow f''\left(\frac{\pi}{2}\right) = 0$
$f'''(x) = \sin x$	$\rightarrow f'''\left(\frac{\pi}{2}\right) = 1$

$$\boxed{P_0(x) = 0} \qquad P_0(x) = f(a)$$

$$\boxed{P_1(x) = -\left(x-\frac{\pi}{2}\right)} \qquad P_1(x) = f(a)+f'(a)(x-a)$$

$$\boxed{P_2(x) = -\left(x-\frac{\pi}{2}\right)} \qquad P_2(x) = f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2$$

$$P_3(x) = f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2+\frac{1}{3!}f'''(a)(x-a)^3$$

$$\boxed{= -\left(x-\frac{\pi}{2}\right)+\frac{1}{6}\left(x-\frac{\pi}{2}\right)^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $\quad f(x) = \frac{1}{x+1}; \quad a = 0$

Solution

$f(x)=\frac{1}{x+1} \qquad \to f(0)=1$

$f'(x)=-\frac{1}{(x+1)^2} \qquad \to f'(0)=-1$

$f''(x)=\frac{2}{(x+1)^3} \qquad \to f''(0)=2$

$f'''(x)=-\frac{6}{(x+1)^4} \qquad \to f'''(0)=-6$

$\boxed{P_0(x)=1}$ $\qquad P_0(x)=f(a)$

$\boxed{P_1(x)=1-x}$ $\qquad P_1(x)=f(a)+f'(a)(x-a)$

$\boxed{P_2(x)=1-x+x^2}$ $\qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2$

$P_3(x)=f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2+\frac{1}{3!}f'''(a)(x-a)^3$

$\boxed{=1-x+x^2-x^3}$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $\qquad f(x)=\tan^{-1}4x; \quad a=0$

Solution

$f(x)=\tan^{-1}4x \qquad \to f(0)=0$

$f'(x)=\frac{4}{1+16x^2} \qquad \to f'(0)=4$

$f''(x)=-\frac{128x}{\left(1+16x^2\right)^2} \qquad \to f''(0)=0$

$$f'''(x)=-\frac{128\left(1+16x^2\right)-2(32x)(128x)}{\left(1+16x^2\right)^3}$$

$$=\frac{6144x^2-128}{\left(1+16x^2\right)^3} \qquad \to f'''(0)=-128$$

$\boxed{P_0(x)=0}$ $\qquad P_0(x)=f(a)$

$$\boxed{P_1(x) = 4x} \qquad P_1(x) = f(a) + f'(a)(x-a)$$

$$\boxed{P_2(x) = 4x} \qquad P_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3$$

$$\boxed{= x - \frac{64}{3}x^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $\quad f(x) = \sin 2x; \quad a = -\frac{\pi}{2}$

Solution

$f(x) = \sin 2x$	$\to f\left(-\frac{\pi}{2}\right) = 0$
$f'(x) = 2\cos 2x$	$\to f'\left(-\frac{\pi}{2}\right) = -2$
$f''(x) = -4\sin 2x$	$\to f''\left(-\frac{\pi}{2}\right) = 0$
$f'''(x) = -8\cos 2x$	$\to f'''\left(-\frac{\pi}{2}\right) = 8$

$$\boxed{P_0(x) = 0} \qquad P_0(x) = f(a)$$

$$\boxed{P_1(x) = -2\left(x + \frac{\pi}{2}\right)} \qquad P_1(x) = f(a) + f'(a)(x-a)$$

$$\boxed{P_2(x) = -2\left(x + \frac{\pi}{2}\right)} \qquad P_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3$$

$$\boxed{= -2\left(x + \frac{\pi}{2}\right) + \frac{4}{3}\left(x + \frac{\pi}{2}\right)^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $\quad f(x) = \cosh 3x; \quad a = 0$

Solution

$f(x) = \cosh 3x$	$\to f(0) = 1$
$f'(x) = 3\sinh 3x$	$\to f'(0) = 0$

$f''(x) = 9\cosh 3x \quad \rightarrow f''(0) = 9$

$f'''(x) = 27\cosh 3x \quad \rightarrow f'''(0) = 0$

$\boxed{P_0(x) = 1}$ $\qquad P_0(x) = f(a)$

$\boxed{P_1(x) = 1}$ $\qquad P_1(x) = f(a) + f'(a)(x-a)$

$\boxed{P_2(x) = 1 + \frac{9}{2}x^2}$ $\qquad P_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3$$

$$\boxed{= 1 + \frac{9}{2}x^2}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a: $\qquad f(x) = \frac{1}{4+x^2}; \quad a = 0$

Solution

$f(x) = \dfrac{1}{4+x^2} \qquad \rightarrow f(0) = \frac{1}{4}$

$f'(x) = -\dfrac{2x}{\left(4+x^2\right)^2} \qquad \rightarrow f'(0) = 0$

$$f''(x) = -\frac{8 + 2x^2 - 2(2x)^2}{\left(4+x^2\right)^3}$$

$$= -\frac{8 - 6x^2}{\left(4+x^2\right)^3} \qquad \rightarrow f''(0) = -\frac{1}{8}$$

$$f'''(x) = -\frac{-12x\left(4+x^2\right) - 6x\left(8-6x^2\right)}{\left(4+x^2\right)^4}$$

$$= \frac{96x - 24x^2}{\left(4+x^2\right)^4} \qquad \rightarrow f'''(0) = 0$$

$\boxed{P_0(x) = \frac{1}{4}}$ $\qquad P_0(x) = f(a)$

$$\boxed{P_1(x)=\frac{1}{4}} \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$\boxed{P_2(x)=\frac{1}{4}-\frac{1}{16}x^2} \qquad\qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2$$

$$P_3(x)=f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2+\frac{1}{3!}f'''(a)(x-a)^3$$

$$\boxed{=\frac{1}{4}-\frac{1}{16}x^2}$$

Exercise

Find out the third term of the Maclaurin series for the following function. $f(x)=(1+x)^{1/3}$

Solution

$f(x)=(1+x)^{1/3}$	$\rightarrow f(0)=1$
$f'(x)=\frac{1}{3}(1+x)^{-2/3}$	$\rightarrow f'(0)=\frac{1}{3}$
$f''(x)=-\frac{2}{9}(1+x)^{-5/3}$	$\rightarrow f''(0)=-\frac{2}{9}$
$f'''(x)=\frac{10}{27}(1+x)^{-8/3}$	$\rightarrow f'''(0)=\frac{10}{27}$

$$P_3(x)=f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2+\frac{1}{3!}f'''(a)(x-a)^3$$

$$\boxed{=1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3}$$

Exercise

Find out the third term of the Maclaurin series for the following function. $f(x)=(1+x)^{-1/2}$

Solution

$f(x)=(1+x)^{-1/2}$	$\rightarrow f(0)=1$
$f'(x)=-\frac{1}{2}(1+x)^{-3/2}$	$\rightarrow f'(0)=-\frac{1}{2}$
$f''(x)=\frac{3}{4}(1+x)^{-5/2}$	$\rightarrow f''(0)=\frac{3}{4}$
$f'''(x)=-\frac{15}{8}(1+x)^{-7/2}$	$\rightarrow f'''(0)=-\frac{15}{8}$

$$P_3(x)=f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2+\frac{1}{3!}f'''(a)(x-a)^3$$

$$=1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3$$

Exercise

Find out the third term of the Maclaurin series for the following function. $f(x)=\left(1+\frac{x}{2}\right)^{-3}$

Solution

$f(x)=\left(1+\frac{x}{2}\right)^{-3}$	$\rightarrow f(0)=1$
$f'(x)=-\frac{3}{2}\left(1+\frac{x}{2}\right)^{-4}$	$\rightarrow f'(0)=-\frac{3}{2}$
$f''(x)=3\left(1+\frac{x}{2}\right)^{-5}$	$\rightarrow f''(0)=3$
$f'''(x)=-\frac{15}{2}\left(1+\frac{x}{2}\right)^{-6}$	$\rightarrow f'''(0)=-\frac{15}{2}$

$$P_3(x)=f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2+\frac{1}{3!}f'''(a)(x-a)^3$$

$$=1-\frac{3}{2}x+\frac{3}{2}x^2-\frac{5}{4}x^3$$

Exercise

Find out the third term of the Maclaurin series for the following function. $f(x)=(1+2x)^{-5}$

Solution

$f(x)=(1+2x)^{-5}$	$\rightarrow f(0)=1$
$f'(x)=-10(1+2x)^{-6}$	$\rightarrow f'(0)=-10$
$f''(x)=120(1+2x)^{-7}$	$\rightarrow f''(0)=120$
$f'''(x)=-1680(1+2x)^{-8}$	$\rightarrow f'''(0)=-1680$

$$P_3(x)=f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2+\frac{1}{3!}f'''(a)(x-a)^3$$

$$=1-10x+60x^2-280x^3$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x)=e^{4x}, \quad n=4$

Solution

$f(x)=e^{4x} \quad \to \quad f(0)=1$

$f'(x)=4e^{4x} \quad \to \quad f'(0)=4$

$f''(x)=16e^{4x} \quad \to \quad f''(0)=16$

$f'''(x)=64e^{4x} \quad \to \quad f'''(0)=64$

$f^{(4)}(x)=256e^{4x} \quad \to \quad f^{(4)}(0)=256$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x)=1+4x+8x^2+\frac{32}{3}x^3+\frac{32}{3}x^4}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x)=e^{-x}, \quad n=5$

Solution

$f(x)=e^{-x} \quad \to \quad f(0)=1$

$f'(x)=-e^{-x} \quad \to \quad f'(0)=-1$

$f''(x)=e^{-x} \quad \to \quad f''(0)=1$

$f'''(x)=-e^{-x} \quad \to \quad f'''(0)=-1$

$f^{(4)}(x)=e^{-x} \quad \to \quad f^{(4)}(0)=1$

$f^{(5)}(x)=-e^{-x} \quad \to \quad f^{(5)}(0)=-1$

$$P_5(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5$$

$$\boxed{P_5(x)=1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{120}x^5}$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x)=e^{-x/2}, \quad n=4$

Solution

$$f(x)=e^{-x/2} \quad \rightarrow \quad f(0)=1$$

$$f'(x)=-\frac{1}{2}e^{-x/2} \quad \rightarrow \quad f'(0)=-\frac{1}{2}$$

$$f''(x)=\frac{1}{4}e^{-x/2} \quad \rightarrow \quad f''(0)=\frac{1}{4}$$

$$f'''(x)=-\frac{1}{8}e^{-x/2} \quad \rightarrow \quad f'''(0)=-\frac{1}{8}$$

$$f^{(4)}(x)=\frac{1}{16}e^{-x/2} \quad \rightarrow \quad f^{(4)}(0)=\frac{1}{16}$$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x)=1-\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{48}x^3+\frac{1}{384}x^4$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x)=e^{x/3}, \quad n=4$

Solution

$$f(x)=e^{x/3} \quad \rightarrow \quad f(0)=1$$

$$f'(x)=\frac{1}{3}e^{x/3} \quad \rightarrow \quad f'(0)=\frac{1}{3}$$

$$f''(x)=\frac{1}{9}e^{x/3} \quad \rightarrow \quad f''(0)=\frac{1}{9}$$

$$f'''(x)=\frac{1}{27}e^{x/3} \quad \rightarrow \quad f'''(0)=\frac{1}{27}$$

$$f^{(4)}(x)=\frac{1}{81}e^{x/3} \quad \rightarrow \quad f^{(4)}(0)=\frac{1}{81}$$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x)=1+\frac{1}{3}x+\frac{1}{18}x^2+\frac{1}{162}x^3+\frac{1}{1944}x^4$$

Exercise

Find the nth Maclaurin polynomial for the function $f(x)=\sin x, \quad n=5$

Solution

$$f(x)=\sin x \quad \to \quad f(0)=0$$

$$f'(x)=\cos x \quad \to \quad f'(0)=1$$

$$f''(x)=-\sin x \quad \to \quad f''(0)=0$$

$$f'''(x)=-\cos x \quad \to \quad f'''(0)=-1$$

$$f^{(4)}(x)=\sin x \quad \to \quad f^{(4)}(0)=0$$

$$f^{(5)}(x)=\cos x \quad \to \quad f^{(5)}(0)=1$$

$$P_5(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5$$

$$\boxed{P_5(x)=x-\frac{1}{6}x^3+\frac{1}{120}x^5}$$

Exercise

Find the nth Maclaurin polynomial for the function $f(x)=\cos \pi x, \quad n=4$

Solution

$$f(x)=\cos \pi x \quad \to \quad f(0)=1$$

$$f'(x)=-\pi \sin \pi x \quad \to \quad f'(0)=0$$

$$f''(x)=-\pi^2 \cos \pi x \quad \to \quad f''(0)=-\pi^2$$

$$f'''(x)=\pi^3 \sin \pi x \quad \to \quad f'''(0)=0$$

$$f^{(4)}(x)=\pi^4 \cos \pi x \quad \to \quad f^{(4)}(0)=\pi^4$$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x)=1-\frac{\pi^2}{2}x^2+\frac{\pi^4}{24}x^4}$$

Exercise

Find the nth Maclaurin polynomial for the function $f(x)=xe^x, \quad n=4$

Solution

$$f(x)=xe^x \quad \to \quad f(0)=0$$

$$f'(x)=e^x+xe^x \quad \to \quad f'(0)=1$$

$$f''(x)=2e^x+xe^x \quad \to \quad f''(0)=2$$

$$f'''(x)=3e^x+xe^x \quad \to \quad f'''(0)=3$$

$$f^{(4)}(x)=4e^x+xe^x \quad \to \quad f^{(4)}(0)=4$$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x)=x+x^2+\frac{1}{2}x^3+\frac{1}{6}x^4}$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x)=x^2e^{-x}, \quad n=4$

Solution

$$f(x)=x^2e^{-x} \quad \to \quad f(0)=0$$

$$f'(x)=2xe^{-x}-x^2e^{-x} \quad \to \quad f'(0)=0$$

$$f''(x)=2e^{-x}-4xe^{x}+x^2e^{-x} \quad \to \quad f''(0)=2$$

$$f'''(x)=-6e^{-x}+6xe^{-x}-x^2e^{-x} \quad \to \quad f'''(0)=-6$$

$$f^{(4)}(x)=12e^{-x}-8xe^{-x}+x^2e^{-x} \quad \to \quad f^{(4)}(0)=12$$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x)=x^2-x^3+\frac{1}{2}x^4}$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x)=\frac{1}{x+1}, \quad n=5$

Solution

$$f(x)=\frac{1}{x+1} \quad \to \quad f(0)=1$$

$$f'(x)=-(x+1)^{-2} \quad \to \quad f'(0)=-1$$

$$f''(x)=2(x+1)^{-3} \quad \to \quad f''(0)=2$$

$$f'''(x)=-6(x+1)^{-4} \quad \to \quad f'''(0)=-6$$

$$f^{(4)}(x) = 24(x+1)^{-5} \quad \rightarrow \quad f^{(4)}(0) = 24$$

$$f^{(5)}(x) = -120(x+1)^{-6} \quad \rightarrow \quad f^{(5)}(0) = -120$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$P_5(x) = 1 - x + x^2 - x^3 + x^4 - x^5$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x) = \frac{x}{x+1}, \quad n = 4$

Solution

$$f(x) = \frac{x}{x+1} = 1 - \frac{1}{x+1} \quad \rightarrow \quad f(0) = 0$$

$$f'(x) = (x+1)^{-2} \quad \rightarrow \quad f'(0) = 1$$

$$f''(x) = -2(x+1)^{-3} \quad \rightarrow \quad f''(0) = -2$$

$$f'''(x) = 6(x+1)^{-4} \quad \rightarrow \quad f'''(0) = 6$$

$$f^{(4)}(x) = -24(x+1)^{-5} \quad \rightarrow \quad f^{(4)}(0) = -24$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x) = x - x^2 + x^3 - x^4$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x) = \sec x, \quad n = 2$

Solution

$$f(x) = \sec x \quad \rightarrow \quad f(0) = 1$$

$$f'(x) = \sec x \tan x \quad \rightarrow \quad f'(0) = 0$$

$$f''(x) = \sec x \tan^2 x + \sec^3 x \quad \rightarrow \quad f''(0) = 1$$

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

$$P_2(x) = 1 + \frac{1}{2}x^2$$

Exercise

Find the nth Maclaurin polynomial for the function $f(x) = \tan x, \quad n = 3$

Solution

$$f(x) = \tan x \quad \to \quad f(0) = 0$$

$$f'(x) = \sec^2 x \quad \to \quad f'(0) = 1$$

$$f''(x) = 2\sec^2 x \tan x \quad \to \quad f''(0) = 0$$

$$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x \quad \to \quad f'''(0) = 2$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3$$

$$\boxed{P_4(x) = x + \frac{1}{3}x^3}$$

Exercise

Find the Maclaurin series for: xe^x

Solution

$$f(x) = xe^x \quad \to \quad f(0) = 0$$

$$f'(x) = e^x + xe^x \quad \to \quad f'(0) = 1$$

$$f''(x) = 2e^x + xe^x \quad \to \quad f''(0) = 2$$

$$f'''(x) = 3e^x + xe^x \quad \to \quad f'''(0) = 3$$

$$\vdots \qquad\qquad \vdots \qquad \vdots$$

$$f^{(n)}(x) = ne^x + xe^x \quad \to \quad f^{(n)}(0) = n \quad \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$xe^x = x + x^2 + \frac{1}{2}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{(n-1)!}x^n$$

Exercise

Find the Maclaurin series for: $5\cos \pi x$

Solution

$$f(x) = 5\cos \pi x \quad \to \quad f(0) = 5$$

$$f'(x) = -5\pi \sin \pi x \quad \to \quad f'(0) = 0$$

$$f''(x) = -5\pi^2 \cos \pi x \quad \rightarrow \quad f''(0) = -5\pi^2$$

$$f'''(x) = 5\pi^3 \sin \pi x \quad \rightarrow \quad f'''(0) = 0$$

$$5\cos \pi x = 5 - \frac{5\pi^2 x^2}{2!} + \frac{5\pi^4 x^4}{4!} - \frac{5\pi^6 x^6}{6!} + \cdots = 5\sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!}$$

Exercise

Find the Maclaurin series for: $\dfrac{x^2}{x+1}$

Solution

$$f(x) = \frac{x^2}{x+1} \quad \rightarrow \quad f(0) = 0$$

$$f'(x) = \frac{2x(x+1) - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2} \quad \rightarrow \quad f'(0) = 0$$

$$f''(x) = \frac{(2x+2)(x+1)^2 - 2(x+1)\left(x^2 + 2x\right)}{(x+1)^4}$$

$$= \frac{2x^2 + 4x + 2 - 2x^2 - 4x}{(x+1)^3}$$

$$= \frac{2}{(x+1)^3} \quad \rightarrow \quad f''(0) = 2$$

$$f'''(x) = \frac{-6}{(x+1)^4} \quad \rightarrow \quad f'''(0) = -6$$

$$\frac{x^2}{x+1} = \frac{2}{2!}x^2 - \frac{6x^3}{3!} + \frac{24x^4}{4!} - \cdots$$

$$= x^2 - x^3 + x^4 - \cdots$$

$$= \sum_{n=2}^{\infty} (-1)^n x^n$$

Exercise

Find the Maclaurin series for: e^{3x+1}

Solution

$$e^{3x+1} = e \cdot e^{3x}$$

$$= e\left(\sum_{n=0}^{\infty} \frac{(3x)^n}{n!}\right)$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{e3^n x^n}{n!}} \quad (\,for\ all\ x)$$

Exercise

Find the Maclaurin series for: $\cos\left(2x^3\right)$

Solution

$$\cos\left(2x^3\right) = 1 - \frac{\left(2x^3\right)^2}{2!} + \frac{\left(2x^3\right)^4}{4!} - \frac{\left(2x^3\right)^6}{6!} + \cdots$$

$$= 1 - \frac{2^2 x^3}{2!} + \frac{2^4 x^{12}}{4!} - \frac{2^6 x^{18}}{6!} + \cdots$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} x^{6n}} \quad (\,for\ all\ x)$$

Exercise

Find the Maclaurin series for: $\cos(2x - \pi)$

Solution

$$\cos(2x - \pi) = \cos(2x)\cos\pi + \sin(2x)\sin\pi$$

$$= -\cos(2x)$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n}$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 4^n}{(2n)!} x^{2n}} \quad (\,for\ all\ x)$$

Exercise

Find the Maclaurin series for: $x^2 \sin\left(\frac{x}{3}\right)$

Solution

$$x^2 \sin\left(\frac{x}{3}\right) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{x}{3}\right)^{2n+1}}{(2n+1)!}$$

$$= x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{3^{2n+1}(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{3^{2n+1}(2n+1)!} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\cos^2\left(\frac{x}{2}\right)$

Solution

$$\cos^2\left(\frac{x}{2}\right) = \frac{1}{2}(1+\cos x)$$

$$= \frac{1}{2}\left(1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\right)$$

$$= \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$= 1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\sin x \cos x$

Solution

$$\sin x \cos x = \frac{1}{2}\sin(2x)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2x)^{2n+1}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 2^{2n+1} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{(2n+1)!} x^{2n+1} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\tan^{-1}\left(5x^2\right)$

Solution

$$\tan^{-1}\left(5x^2\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(5x^2\right)^{2n+1} \qquad 5x^2 \le 1 \ \rightarrow\ x^2 \le \frac{1}{5} \ \Rightarrow\ -\frac{1}{\sqrt{5}} \le x \le \frac{1}{\sqrt{5}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1}}{2n+1} x^{4n+2} \qquad \left(\text{for } -\frac{1}{\sqrt{5}} \le x \le \frac{1}{\sqrt{5}}\right)$$

Exercise

Find the Maclaurin series for: $\ln\left(2+x^2\right)$

Solution

$$\ln\left(2+x^2\right) = \ln 2\left(1+\frac{x^2}{2}\right)$$

$$= \ln 2 + \ln\left(1+\frac{x^2}{2}\right)$$

$$= \ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{x^2}{2}\right)^n \qquad \frac{x^2}{2} \le 1 \ \rightarrow\ x^2 \le 2 \ \Rightarrow\ -\sqrt{2} \le x \le \sqrt{2}$$

$$= \ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} \frac{x^{2n}}{2^n} \qquad \left(\text{for } -\sqrt{2} \le x \le \sqrt{2}\right)$$

Exercise

Find the Maclaurin series for: $\dfrac{1+x^3}{1+x^2}$

Solution

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

$$\frac{1+x^3}{1+x^2} = \left(1+x^3\right)\left(1 - x^2 + x^4 - x^6 + \cdots\right)$$

$$= 1 - x^2 + x^4 - x^6 + \cdots + x^3 - x^5 + x^7 - x^9 + \cdots$$

$$= 1 - x^2 + x^3 + x^4 - x^5 - x^6 + x^7 + x^8 - x^9 - \cdots$$

$$= 1 - x^2 + \sum_{n=2}^{\infty} (-1)^n \left(x^{2n-1} + x^{2n}\right) \quad \left(for \quad |x| < 1 \right)$$

Exercise

Find the Maclaurin series for: $\ln \dfrac{1+x}{1-x}$

Solution

$$\ln \frac{1+x}{1-x} = \ln(1+x) - \ln(1-x) \qquad = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \right) = 2x + 2\frac{x^3}{3} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty} \left((-1)^n + 1 \right) \frac{x^{n+1}}{n+1}$$

$$= 2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1} \qquad (-1 < x < 1)$$

Exercise

Find the Maclaurin series for: $\dfrac{e^{2x^2} - 1}{x^2}$

Solution

$$\frac{e^{2x^2} - 1}{x^2} = \frac{1}{x^2}\left(e^{2x^2} - 1 \right)$$

$$= \frac{1}{x^2}\left(1 + 2x^2 + \frac{\left(2x^2\right)^2}{2!} + \frac{\left(2x^2\right)^3}{3!} + \cdots - 1 \right)$$

$$= \frac{1}{x^2}\left(2x^2 + \frac{2^2 x^4}{2!} + \frac{2^3 x^6}{3!} + \cdots \right)$$

$$= 2 + \frac{2^2 x^2}{2!} + \frac{2^3 x^4}{3!} + \frac{2^4 x^6}{4!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{2^{n+1}}{(n+1)!} x^{2n} \qquad \left(for\ all \quad x \neq 0 \right)$$

Exercise

Find the Maclaurin series for: $\cosh x - \cos x$

Solution

$$\cosh x - \cos x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = 2\frac{x^2}{2!} + 2\frac{x^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} \left(1 - (-1)^n\right) \frac{x^{2n}}{(2n)!}$$

$$\boxed{= 2\sum_{n=0}^{\infty} \frac{x^{4n+2}}{(4n+2)!}} \quad (\textit{for all } x)$$

Exercise

Find the Maclaurin series for: $\sinh x - \sin x$

Solution

$$\sinh x - \sin x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \left(1 - (-1)^n\right) \frac{x^{2n+1}}{(2n+1)!}$$

$$\boxed{= 2\sum_{n=0}^{\infty} \frac{x^{4n+3}}{(4n+3)!}} \quad (\textit{for all } x)$$

Exercise

Finding Taylor and Maclaurin Series generated by f at $x = a$: $\quad f(x) = x^3 - 2x + 4, \quad a = 2$

Solution

$f(x) = x^3 - 2x + 4 \quad \to \quad f(2) = 8$

$f'(x) = 3x^2 - 2 \quad \to \quad f'(2) = 10$

$f''(x) = 6x \quad \to \quad f''(2) = 12$

$f'''(x) = 6 \quad \to \quad f'''(2) = 6$

$f^{(n)}(x) = 0 \quad \to \quad f^{(n)}(2) = 0 \quad (n > 3)$

$$P_n(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \cdots$$

$$x^3 - 2x + 4 = 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3$$

Exercise

Finding Taylor and Maclaurin Series generated by f *at* $x = a$: $\quad f(x) = 2x^3 + x^2 + 3x - 8, \quad a = 1$

Solution

$$f(x) = 2x^3 + x^2 + 3x - 8 \quad \to \quad f(1) = -2$$

$$f'(x) = 6x^2 + 2x + 3 \quad \to \quad f'(1) = 11$$

$$f''(x) = 12x + 2 \quad \to \quad f''(1) = 14$$

$$f'''(x) = 12 \quad \to \quad f'''(1) = 12$$

$$f^{(n)}(x) = 0 \quad \to \quad f^{(n)}(1) = 0 \quad (n \ge 4)$$

$$P_n(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots$$

$$2x^3 + x^2 + 3x - 8 = -2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3$$

Exercise

Finding Taylor and Maclaurin Series generated by f *at* $x = a$:

$$f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2, \quad a = -1$$

Solution

$$f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2 \quad \to \quad f(-1) = -7$$

$$f'(x) = 15x^4 - 4x^3 + 6x^2 + 2x \quad \to \quad f'(-1) = 23$$

$$f''(x) = 60x^3 - 12x^2 + 12x + 2 \quad \to \quad f''(-1) = -82$$

$$f'''(x) = 180x^2 - 24x + 12 \quad \to \quad f'''(-1) = 216$$

$$f^{(4)}(x) = 360x - 24 \quad \to \quad f^{(4)}(-1) = -384$$

$$f^{(5)}(x) = 360 \quad \to \quad f^{(5)}(-1) = 360$$

$$f^{(n)}(x) = 0 \quad \to \quad f^{(n)}(-1) = 0 \quad (n \ge 6)$$

$$P_n(x) = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3 + \frac{f^{(4)}(-1)}{4!}(x+1)^2 + \frac{f^{(4)}(-1)}{5!}(x+1)^3$$

$$3x^5 - x^4 + 2x^3 + x^2 - 2 = -7 + 23(x+1) - \frac{82}{2!}(x+1)^2 + \frac{216}{3!}(x+1)^3 - \frac{384}{4!}(x+1)^4 + \frac{360}{5!}(x+1)^3$$

$$\boxed{= -7 + 23(x+1) - 41(x+1)^2 + 36(x+1)^3 - 16(x+1)^4 + 3(x+1)^3}$$

Exercise

Finding Taylor and Maclaurin Series generated by f at $x = a$: $\quad f(x) = \cos\left(2x + \frac{\pi}{2}\right), \quad a = \frac{\pi}{4}$

Solution

$$f(x) = \cos\left(2x + \frac{\pi}{2}\right) \quad \rightarrow \quad f\left(\frac{\pi}{4}\right) = -1$$

$$f'(x) = -2\sin\left(2x + \frac{\pi}{2}\right) \quad \rightarrow \quad f'\left(\frac{\pi}{4}\right) = 0$$

$$f''(x) = -4\cos\left(2x + \frac{\pi}{2}\right) \quad \rightarrow \quad f''\left(\frac{\pi}{4}\right) = 4$$

$$f'''(x) = 8\sin\left(2x + \frac{\pi}{2}\right) \quad \rightarrow \quad f'''\left(\frac{\pi}{4}\right) = 0$$

$$f^{(4)}(x) = 16\cos\left(2x + \frac{\pi}{2}\right) \quad \rightarrow \quad f^{(4)}\left(\frac{\pi}{4}\right) = -16$$

$$f^{(5)}(x) = -32\sin\left(2x + \frac{\pi}{2}\right) \quad \rightarrow \quad f^{(5)}\left(\frac{\pi}{4}\right) = 0$$

$$\rightarrow \quad f^{(2n)}\left(\frac{\pi}{4}\right) = (-1)^n 2^{2n}$$

$$\cos\left(2x + \frac{\pi}{2}\right) = -1 + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{16}{4!}\left(x - \frac{\pi}{4}\right)^4 + \cdots$$

$$= -1 + 2\left(x - \frac{\pi}{4}\right)^2 - \frac{2}{3}\left(x - \frac{\pi}{4}\right)^4 + \cdots$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!}\left(x - \frac{\pi}{4}\right)^{2n}}$$

Exercise

Find the Taylor series of the functions, where is the series representation valid? $f(x) = e^{-2x}$ *about* -1

Solution

Let $t = x + 1 \quad \Rightarrow \quad x = t - 1$

$$f(x) = e^{-2x} = e^{-2x-2+2} = e^{-2(x+1)+2}$$

$$= e^2 \sum_{n=0}^{\infty} \frac{(-2(x+1))^n}{n!}$$

$$= e^2 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} (x+1)^n \qquad (\text{for all } x)$$

Exercise

Find the Taylor series of the functions, where is the series representation valid? $f(x) = \sin x \quad \text{about } \frac{\pi}{2}$

Solution

Let $y = x - \frac{\pi}{2} \quad \Rightarrow \quad x = y + \frac{\pi}{2}$

$$\sin x = \sin\left(y + \frac{\pi}{2}\right) = \cos y$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} y^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(x - \frac{\pi}{2}\right)^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(x - \frac{\pi}{2}\right)^{2n} \qquad (\text{for all } x)$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$f(x) = \ln x \quad \text{in powers of } x - 3$

Solution

Let $y = x - 3 \quad \Rightarrow \quad x = y + 3$

$$\ln x = \ln(y+3) = \ln 3\left(1 + \frac{y}{3}\right)$$

$$= \ln 3 + \ln\left(1 + \frac{y}{3}\right)$$

$$= \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\frac{y}{3}\right)^{n+1}$$

$$= \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \frac{(x-3)^{n+1}}{3^{n+1}}$$

$$= \ln 3 + \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)3^{n+1}}(x-3)^{n+1}} \qquad (0 < x \le 6)$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$f(x) = \ln(2+x) \quad$ *in powers of* $x-2$

Solution

$$\ln(2+x) = \ln(2+x-2+2) = \ln(4+x-2)$$

$$= \ln 4\left(1+\frac{x-2}{4}\right)$$

$$= \ln 4 + \ln\left(1+\frac{x-2}{4}\right)$$

$$= \ln 4 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\left(\frac{x-2}{4}\right)^{n+1} \qquad \frac{|x-2|}{4} < 1 \Rightarrow |x-2| < 4 \qquad -4 < x-2 < 4 \qquad -2 < x < 6$$

$$\boxed{= \ln 4 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)4^{n+1}}(x-2)^{n+1}} \qquad (-2 < x \le 6)$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$f(x) = e^{2x+3} \quad$ *in powers of* $x+1$

Solution

$$e^{2x+3} = e^{2x+2-2+3} = e^{2(x+1)+1}$$

$$= e \cdot e^{2(x+1)}$$

$$= e\sum_{n=0}^{\infty} \frac{\left(2(x+1)\right)^n}{n!}$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{2^n e(x+1)^n}{n!}} \qquad (\textit{for all } x)$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = \sin x - \cos x \quad \text{about} \ \frac{\pi}{4}$$

Solution

$$\sin x - \cos x = \sin\left(x - \frac{\pi}{4} + \frac{\pi}{4}\right) - \cos\left(x - \frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$= \cos\left(x - \frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(x - \frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)$$

$$= \cos\left(x - \frac{\pi}{4}\right)\frac{\sqrt{2}}{2} + \sin\left(x - \frac{\pi}{4}\right)\frac{\sqrt{2}}{2} - \cos\left(x - \frac{\pi}{4}\right)\frac{\sqrt{2}}{2} + \sin\left(x - \frac{\pi}{4}\right)\frac{\sqrt{2}}{2}$$

$$= \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$$

$$= \sqrt{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(x - \frac{\pi}{4}\right)^{2n+1} \qquad (\text{for all } x)$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = \cos^2 x \quad \text{about} \ \frac{\pi}{8}$$

Solution

$$\cos^2(x) = \frac{1}{2}(1 + \cos 2x) = \frac{1}{2}\left(1 + \cos\left(2x - \frac{\pi}{4} + \frac{\pi}{4}\right)\right)$$

$$= \frac{1}{2}\left(1 + \cos\left(2x - \frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(2x - \frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)\right)$$

$$= \frac{1}{2}\left(1 + \frac{\sqrt{2}}{2}\cos\left(2\left(x - \frac{\pi}{8}\right)\right) - \frac{\sqrt{2}}{2}\sin\left(2\left(x - \frac{\pi}{8}\right)\right)\right)$$

$$= \frac{1}{2} + \frac{\sqrt{2}}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(2\left(x - \frac{\pi}{8}\right)\right)^{2n} - \frac{\sqrt{2}}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(2\left(x - \frac{\pi}{8}\right)\right)^{2n+1}$$

$$= \frac{1}{2} + \frac{1}{2\sqrt{2}}\sum_{n=0}^{\infty}(-1)^n\left[\frac{2^{2n}}{(2n)!}\left(x - \frac{\pi}{8}\right)^{2n} - \frac{2^{2n+1}}{(2n+1)!}\left(x - \frac{\pi}{8}\right)^{2n+1}\right] \qquad (\text{for all } x)$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = \frac{x}{1+x} \quad \textit{in powers of } x-1$$

Solution

$$\frac{x}{1+x} = \frac{x-1+1}{1+x-1+1} = \frac{(x-1)+1}{(x-1)+2}$$

$$= 1 - \frac{1}{(x-1)+2}$$

$$= 1 - \frac{1}{2\left(1+\frac{x-1}{2}\right)}$$

$$= 1 - \frac{1}{2}\left(1 - \frac{x-1}{2} + \left(\frac{x-1}{2}\right)^2 - \left(\frac{x-1}{2}\right)^3 + \cdots\right)$$

$$= \frac{1}{2} - \frac{1}{2^2}(x-1) + \frac{1}{2^3}(x-1)^2 - \frac{1}{2^4}(x-1)^3 + \cdots \qquad |x-1| < 1 \quad -1 < x-1 < 1$$

$$\boxed{= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n+1}}(x-1)^n} \qquad (0 < x < 2)$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = xe^x \quad \textit{in powers of } x+2$$

Solution

$$xe^x = (x+2-2)e^{x+2-2}$$

$$= (x+2-2)e^{-2}e^{x+2}$$

$$= (x+2)e^{-2}e^{x+2} - 2e^{-2}e^{x+2}$$

$$= \frac{1}{e^2}(x+2)\sum_{n=0}^{\infty}\frac{(x+2)^n}{n!} - \frac{2}{e^2}\sum_{n=0}^{\infty}\frac{(x+2)^n}{n!}$$

$$= \frac{1}{e^2}\sum_{n=0}^{\infty}\frac{(x+2)^{n+1}}{n!} - \frac{2}{e^2}\sum_{n=0}^{\infty}\frac{(x+2)^n}{n!}$$

$$= \frac{1}{e^2}\sum_{n=1}^{\infty}\frac{(x+2)^n}{(n-1)!} - \frac{2}{e^2}\left(1 + \sum_{n=1}^{\infty}\frac{(x+2)^n}{n!}\right)$$

$$= -\frac{2}{e^2} + \frac{1}{e^2}\sum_{n=1}^{\infty}\frac{(x+2)^n}{(n-1)!} - \frac{2}{e^2}\sum_{n=1}^{\infty}\frac{(x+2)^n}{n!}$$

$$= -\frac{2}{e^2} + \frac{1}{e^2}\sum_{n=1}^{\infty}\left(\frac{1}{(n-1)!} - \frac{2}{n!}\right)(x+2)^n \qquad (\text{for all } x)$$

Exercise

Find the nth Taylor polynomial centered at c for the function $\quad f(x) = \frac{2}{x}, \quad n = 3, \quad c = 1$

Solution

$$f(x) = \frac{2}{x} \quad \to \quad f(1) = 2$$

$$f'(x) = -\frac{2}{x^2} \quad \to \quad f'(1) = -2$$

$$f''(x) = \frac{4}{x^3} \quad \to \quad f''(1) = 4$$

$$f'''(x) = -\frac{12}{x^4} \quad \to \quad f'''(0) = -12$$

$$P_3(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3$$

$$P_3(x) = 2 - 2(x-1) + 2(x-1)^2 - 2(x-1)^3$$

Exercise

Find the nth Taylor polynomial centered at c for the function $\quad f(x) = \frac{1}{x^2}, \quad n = 4, \quad c = 2$

Solution

$$f(x) = \frac{1}{x^2} \quad \to \quad f(2) = \frac{1}{4}$$

$$f'(x) = -\frac{2}{x^3} \quad \to \quad f'(2) = -\frac{1}{4}$$

$$f''(x) = \frac{6}{x^4} \quad \to \quad f''(2) = \frac{3}{8}$$

$$f'''(x) = -\frac{24}{x^5} \quad \to \quad f'''(2) = -\frac{3}{4}$$

$$f^{(4)}(x) = \frac{120}{x^4} \quad \to \quad f^{(4)}(2) = \frac{15}{8}$$

$$P_4(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4$$

$$\boxed{P_4(x) = \frac{1}{4} - \frac{1}{4}(x-2) + \frac{3}{16}(x-2)^2 - \frac{1}{8}(x-2)^3 + \frac{5}{64}(x-2)^4}$$

Exercise

Find the *n*th Taylor polynomial centered at *c* for the function $f(x) = \sqrt{x}, \quad n = 3, \quad c = 4$

Solution

$$f(x) = x^{1/2} \quad \to \quad f(4) = 2$$

$$f'(x) = \frac{1}{2}x^{-1/2} \quad \to \quad f'(4) = \frac{1}{4}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \quad \to \quad f''(4) = -\frac{1}{4}\frac{1}{\left(2^2\right)^{3/2}} = -\frac{1}{32}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} \quad \to \quad f'''(4) = \frac{3}{256}$$

$$P_3(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3$$

$$\boxed{P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3}$$

Exercise

Find the *n*th Taylor polynomial centered at *c* for the function $f(x) = \sqrt[3]{x}, \quad n = 3, \quad c = 8$

Solution

$$f(x) = x^{1/3} \quad \to \quad f(8) = 2$$

$$f'(x) = \frac{1}{3}x^{-2/3} \quad \to \quad f'(8) = \frac{1}{12}$$

$$f''(x) = -\frac{2}{9}x^{-5/3} \quad \to \quad f''(8) = -\frac{1}{144}$$

$$f'''(x) = \frac{10}{27}x^{-8/3} \quad \to \quad f'''(8) = \frac{5}{3456}$$

$$P_3(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3$$

$$\boxed{P_3(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20,736}(x-8)^3}$$

Exercise

Find the nth Taylor polynomial centered at c for the function $\quad f(x) = \ln x, \quad n = 4, \quad c = 2$

Solution

$$f(x) = \ln x \quad \rightarrow \quad f(2) = \ln 2$$

$$f'(x) = \frac{1}{x} \quad \rightarrow \quad f'(2) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{x^2} \quad \rightarrow \quad f''(2) = -\frac{1}{4}$$

$$f'''(x) = \frac{2}{x^3} \quad \rightarrow \quad f'''(2) = \frac{1}{4}$$

$$f^{(4)}(x) = -\frac{6}{x^4} \quad \rightarrow \quad f^{(4)}(2) = -\frac{3}{8}$$

$$P_4(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4$$

$$P_4(x) = \ln 2 + \frac{1}{2}\frac{1}{4}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4$$

Exercise

Find the nth Taylor polynomial centered at c for the function $\quad f(x) = x^2 \cos x, \quad n = 2, \quad c = \pi$

Solution

$$f(x) = x^2 \cos x \quad \rightarrow \quad f(\pi) = -\pi^2$$

$$f'(x) = 2x\cos x - x^2 \sin x \quad \rightarrow \quad f'(\pi) = -2\pi$$

$$f''(x) = 2\cos x - 4x\sin x - x^2 \cos x \quad \rightarrow \quad f''(\pi) = -2 + \pi^2$$

$$P_2(x) = -\pi^2 - 2\pi(x-\pi) + \frac{\pi^2 - 2}{2}(x-\pi)^2 \qquad P_2(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2$$

Exercise

Find the nth−order Taylor polynomial centered at c for the function $\quad f(x) = \sin 2x; \quad n = 3, \quad c = 0$

Solution

$f(x) = \sin 2x$	$\rightarrow f(0) = 0$
$f'(x) = 2\cos 2x$	$\rightarrow f'(0) = 2$
$f''(x) = -4\sin 2x$	$\rightarrow f''(0) = 0$
$f'''(x) = -8\cos 2x$	$\rightarrow f'''(0) = -8$

$$P(x) = 2x - \frac{8}{3!}x^3$$

$$\boxed{= 2x - \frac{1}{3!}(2x)^3}$$

Exercise

Find the nth−order Taylor polynomial centered at c for the function $f(x) = \cos x^2; \quad n = 2, \quad c = 0$

Solution

$f(x) = \cos x^2$	$\rightarrow f(0) = 1$
$f'(x) = -2x \sin x^2$	$\rightarrow f'(0) = 0$
$f''(x) = -2\sin x^2 - 4x^2 \cos x^2$	$\rightarrow f''(0) = 0$

$$\boxed{P(x) = 1}$$

Exercise

Find the nth−order Taylor polynomial centered at c for the function $f(x) = e^{-x}; \quad n = 2, \quad c = 0$

Solution

$f(x) = e^{-x}$	$\rightarrow f(0) = 1$
$f'(x) = -e^{-x}$	$\rightarrow f'(0) = -1$
$f''(x) = e^{-x}$	$\rightarrow f''(0) = 1$

$$\boxed{P(x) = 1 - x - \frac{1}{2}x^2}$$

Exercise

Find the nth−order Taylor polynomial centered at c for the function $f(x) = \cos x; \quad n = 2, \quad c = \frac{\pi}{4}$

Solution

$f(x) = \cos x$	$\rightarrow f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$
$f'(x) = -\sin x$	$\rightarrow f'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$$f''(x) = -\cos x \qquad \to f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\boxed{P(x) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2}$$

Exercise

Find the nth−order Taylor polynomial centered at c for the function $\quad f(x) = \ln x; \quad n = 2, \quad c = 1$

Solution

$$f(x) = \ln x \qquad \to f(1) = 0$$

$$f'(x) = \frac{1}{x} \qquad \to f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \qquad \to f''(1) = -1$$

$$\boxed{P(x) = x - 1 - \frac{1}{2}(x-1)^2}$$

Exercise

Find the nth−order Taylor polynomial centered at c for the function $\quad f(x) = \sinh 2x; \quad n = 4, \quad c = 0$

Solution

$$f(x) = \sinh 2x \qquad \to f(0) = 0$$

$$f'(x) = 2\cosh 2x \qquad \to f'(0) = 2$$

$$f''(x) = 4\sinh 2x \qquad \to f''(0) = 0$$

$$f'''(x) = 8\cosh 2x \qquad \to f'''(0) = 8$$

$$f^{(iv)}(x) = 16\sinh 2x \qquad \to f^{(iv)}(0) = 0$$

$$P(x) = 2x - \frac{8}{3!}x^3$$

$$\boxed{= 2x - \frac{1}{6}(2x)^3}$$

Exercise

Find the nth−order Taylor polynomial centered at c for the function $f(x)=\cosh x; \quad n=3, \quad c=\ln 2$

Solution

$$f(x)=\cosh x \qquad \to f(\ln 2)=\frac{e^{\ln 2}+e^{-\ln 2}}{2}$$
$$=\frac{1}{2}\left(2+\frac{1}{2}\right)$$
$$=\frac{5}{4}$$

$$f'(x)=\sinh x \qquad \to f'(\ln 2)=\frac{e^{\ln 2}-e^{-\ln 2}}{2}$$
$$=\frac{1}{2}\left(2-\frac{1}{2}\right)$$
$$=\frac{3}{4}$$

$$f''(x)=\cosh x \qquad \to f''(\ln 2)=\frac{e^{\ln 2}+e^{-\ln 2}}{2}$$
$$=\frac{1}{2}\left(2+\frac{1}{2}\right)$$
$$=\frac{5}{4}$$

$$f'''(x)=\sinh x \qquad \to f'''(\ln 2)=\frac{e^{\ln 2}-e^{-\ln 2}}{2}$$
$$=\frac{1}{2}\left(2-\frac{1}{2}\right)$$
$$=\frac{3}{4}$$

$$P(x)=\frac{5}{4}+\frac{3}{4}(x-\ln 2)+\frac{5}{4}\frac{1}{2!}(x-\ln 2)^2+\frac{3}{4}\frac{1}{3!}(x-\ln 2)^3$$
$$=\frac{5}{4}+\frac{3}{4}(x-\ln 2)+\frac{5}{8}(x-\ln 2)^2+\frac{1}{8}(x-\ln 2)^3$$

Exercise

Find the sums of the series $1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}+\cdots$

Solution

$$1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}+\cdots=1+\left(x^2\right)^1+\frac{\left(x^2\right)^2}{2!}+\frac{\left(x^2\right)^3}{3!}+\frac{\left(x^2\right)^4}{4!}+\cdots$$
$$=e^{x^2}$$

Exercise

Find the sums of the series $1+\frac{x^2}{3!}+\frac{x^4}{5!}+\frac{x^6}{7!}+\frac{x^8}{9!}+\cdots$

Solution

$$1+\frac{x^2}{3!}+\frac{x^4}{5!}+\frac{x^6}{7!}+\frac{x^8}{9!}+\cdots=\frac{1}{x}\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\frac{x^9}{9!}+\cdots\right)$$

$$=\frac{1}{x}\sinh x$$

$$=\begin{cases}\frac{e^x-e^{-x}}{2x} & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$$

Exercise

Find the sums of the series $x^3-\frac{x^9}{3!\times 4}+\frac{x^{15}}{5!\times 16}-\frac{x^{21}}{7!\times 64}+\frac{x^{27}}{9!\times 256}-\cdots$

Solution

$$x^3-\frac{x^9}{3!\times 4}+\frac{x^{15}}{5!\times 16}-\frac{x^{21}}{7!\times 64}+\frac{x^{27}}{9!\times 256}-\cdots=2\left(\frac{x^3}{2}-\frac{1}{3!}\left(\frac{x^3}{2}\right)^3+\frac{1}{5!}\left(\frac{x^3}{2}\right)^5-\frac{1}{7!}\left(\frac{x^3}{2}\right)^7+\cdots\right)$$

$$\boxed{=2\sin\left(\frac{x^3}{2}\right)} \qquad \textit{for all } x$$

Exercise

Use the geometric series $\sum_{k=0}^{\infty} x^k=\frac{1}{1-x}$, for $|x|<1$, to determine the Maclaurin series and the interval of convergence for the following function

$$f(x)=\frac{1}{1-x^2}$$

Solution

The Maclaurin series for $f(x)=\frac{1}{1-x^2}$ is $\sum_{k=0}^{\infty} x^{2k}$

By the Root test:

$$\sqrt[k]{x^{2k}}=x^2<1$$

$$-1<x<1$$

At $x=-1$, the series is $\sum(-1)^{2k}=\sum 1$ which diverges

At $x = 1$, the series is $\sum (1)^{2k} = \sum 1$ which diverges

The interval of convergence is the real line $(-1,\ 1)$

Exercise

Use the geometric series $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, for $|x| < 1$, to determine the Maclaurin series and the interval of convergence for the following function

$$f(x) = \frac{1}{1+x^3}$$

Solution

The Maclaurin series for $f(x) = \frac{1}{1+x^3}$ is $\sum_{k=0}^{\infty} (-x)^{3k} = \sum_{k=0}^{\infty} (-1)^k x^{3k}$

By the Root test:

$$\sqrt[k]{\left|(-x)^{3k}\right|} = x^3 < 1$$

$$-1 < x < 1$$

At $x = -1$, the series is $\sum (1)^{3k} = \sum 1$ which diverges

At $x = 1$, the series is $\sum (-1)^{3k} = \sum (-1)^k$ which diverges absolutely (*harmonic*)

The interval of convergence is the real line $(-1,\ 1)$

Exercise

Use the geometric series $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, for $|x| < 1$, to determine the Maclaurin series and the interval of convergence for the following function

$$f(x) = \frac{1}{1+5x}$$

Solution

The Maclaurin series for $f(x) = \frac{1}{1+5x}$ is $\sum_{k=0}^{\infty} (-5x)^k = \sum_{k=0}^{\infty} (-5)^k x^k$

By the Root test:

$$\sqrt[k]{\left|(-5x)^k\right|} = |5x| < 1$$

$$-\frac{1}{5} < x < \frac{1}{5}$$

At $x = -\frac{1}{5}$, the series is $\sum (1)^k = \sum 1$ which diverges

At $x = \frac{1}{5}$, the series is $\sum (-1)^k$ which diverges absolutely (*harmonic*)

The interval of convergence is the real line $\left(-\frac{1}{5},\ \frac{1}{5}\right)$

Exercise

Use the geometric series $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, for $|x| < 1$, to determine the Maclaurin series and the interval of convergence for the following function

$$f(x) = \frac{10}{1+x}$$

Solution

The Maclaurin series for $f(x) = \frac{10}{1+x}$ is $\sum_{k=0}^{\infty} 10(-x)^k$

By the Root test:

$$\sqrt[k]{\left|(-x)^k\right|} = |x| < 1$$

$$-1 < x < 1$$

At $x = -1$, the series is $\sum 10(1)^k = \sum 10$ which diverges

At $x = 1$, the series is $\sum 10(-1)^k$ which diverges absolutely (*harmonic*)

The interval of convergence is the real line $(-1,\ 1)$

Exercise

Use the geometric series $\sum\limits_{k=0}^{\infty} x^k = \frac{1}{1-x}$, for $|x|<1$, to determine the Maclaurin series and the interval of convergence for the following function

$$f(x) = \frac{1}{(1-10x)^2}$$

Solution

$$\frac{1}{1-10x} = \sum_{k=0}^{\infty} (10x)^k$$

$$\frac{1}{10} \cdot \frac{1}{1-10x} = \frac{1}{10} \sum_{k=0}^{\infty} (10x)^k$$

$$\left(\frac{1}{10} \cdot \frac{1}{1-10x}\right)' = \frac{1}{(1-10x)^2}$$

Thus, the Maclaurin series for $f(x)$

$$\left(\frac{1}{10} \sum_{k=0}^{\infty} (10x)^k\right)' = \frac{1}{10} \sum_{k=0}^{\infty} 10k(10x)^{k-1}$$

$$= \sum_{k=0}^{\infty} k(10x)^{k-1}$$

$$L = \lim_{k\to\infty} \left|\frac{k(10x)^{k-1}}{(k+1)(10x)^k}\right| \qquad\qquad L = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$$

$$= \left|(10x)^{-1}\right| < 1$$

$$|x| < \frac{1}{10} \quad \to \quad -\frac{1}{10} < x < \frac{1}{10}$$

At $x = -\frac{1}{10}$, the series is $\sum k(-1)^k$ which diverges absolutely

At $x = \frac{1}{10}$, the series is $\sum k$ which diverges

The interval of convergence is the real line $\left(-\frac{1}{10}, \frac{1}{10}\right)$

Exercise

Use the geometric series $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, for $|x| < 1$, to determine the Maclaurin series and the interval of convergence for the following function

$$f(x) = \ln(1-4x)$$

Solution

$$-4\int \frac{dx}{1-4x} = \int \frac{d(1-4x)}{1-4x}$$

$$= \ln(1-4x)$$

$$\int \frac{dx}{1-4x} = -\frac{1}{4}\ln(1-4x)$$

$$= -\frac{1}{4} f(x)$$

$$\int \left(-\frac{1}{4}\sum_{k=0}^{\infty} (4x)^k\right) = -\sum_{k=0}^{\infty} \frac{1}{k+1}(4x)^{k+1}$$

$$f(x) = -\sum_{k=0}^{\infty} \frac{1}{k+1}(4x)^{k+1}$$

$$L = \lim_{k\to\infty} \left|\frac{(4x)^{k+2}}{k+2} \cdot \frac{k+1}{(4x)^{k+1}}\right| \qquad L = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$$

$$= \lim_{k\to\infty} \frac{k+1}{k+2}|4x|$$

$$= |4x| < 1$$

$$|x| < \frac{1}{4} \quad \to \quad -\frac{1}{4} < x < \frac{1}{4}$$

At $x = -\frac{1}{4}$, the series is $f(x) = \ln 2$ which converges

At $x = \frac{1}{4}$, the series is $f(x) = \ln(0) = -\infty$ which diverges

The interval of convergence is the real line $\left[-\frac{1}{4}, \frac{1}{4}\right)$

Exercise

The limit $\lim\limits_{n\to\infty} \dfrac{n!}{\sqrt{2\pi}\ n^{n+1/2}e^{-n}} = 1$ that is the relative error in the approximation $n! \approx \sqrt{2\pi}\ n^{n+1/2}e^{-n}$

Approaches zero as n increases. That is $n!$ grows at a rate comparable to $\sqrt{2\pi}\ n^{n+1/2}e^{-n}$. This result, known as Stirling's Formula, is often very useful in applied mathemmatics and statistics. Prove it by carrying out the following steps.

a) Use the identity $\ln(n!) = \sum\limits_{j=1}^{n} \ln j$ and the increasing nature of ln to show that if $n \ge 1$,

$$\int_0^n \ln x\, dx < \ln(n!) < \int_1^{n+1} \ln x\, dx$$

And hence that $n\ln n - n < \ln(n!) < (n+1)\ln(n+1) - n$

b) If $c_n = \ln(n!) - \left(n + \frac{1}{2}\right)\ln n + n$, show that

$$c_n - c_{n+1} = \left(n+\frac{1}{2}\right)\ln\left(\frac{n+1}{n}\right) - 1$$

$$= \left(n+\frac{1}{2}\right)\ln\left(\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}\right) - 1$$

c) Use the Maclaurin series for $\ln\frac{1+t}{1-t}$ to show that

$$0 < c_n - c_{n+1} < \frac{1}{3}\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^4} + \frac{1}{(2n+1)^6} + \cdots\right)$$

$$= \frac{1}{12}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

and therefore that $\{c_n\}$ is decreasing and $\left\{c_n - \frac{1}{12n}\right\}$ is increasing. Hence conclude that $\lim\limits_{n\to\infty} c_n = c$ exists, and that

$$\lim_{n\to\infty} \frac{n!}{n^{n+1/2}e^{-n}} = \lim_{n\to\infty} e^{c_n} = e^c$$

Solution

a) $\ln(k-1) < \displaystyle\int_{k-1}^{k} \ln x dx < \ln k, \quad k = 1,\ 2,\ 3,\ \cdots$

$$n\ln n - n = \int_0^n \ln x\, dx < \ln(n!) < \int_1^{n+1} \ln x\, dx$$

$$= (n+1)\ln(n+1) - n - 1$$

$$< (n+1)\ln(n+1)$$

b) If $c_n = \ln(n!) - \left(n+\frac{1}{2}\right)\ln n + n$, then

$$c_n - c_{n+1} = \ln(n!) - \left(n+\frac{1}{2}\right)\ln n + n - \left[\ln((n+1)!) - \left(n+1+\frac{1}{2}\right)\ln(n+1) + n+1\right]$$

$$= \ln(n!) - \left(n+\frac{1}{2}\right)\ln n - \ln((n+1)!) + \left(n+\frac{1}{2}+1\right)\ln(n+1) - 1$$

$$= \ln(n!) - \ln((n+1)!) - \left(n+\frac{1}{2}\right)\ln n^{n+1/2} + \left(n+\frac{1}{2}\right)\ln(n+1) + \ln(n+1) - 1$$

$$= \ln\left(\frac{n!}{(n+1)!}\right) - \left(n+\frac{1}{2}\right)\ln\frac{n}{n+1} + \ln(n+1) - 1$$

$$= \ln\left(\frac{1}{n+1}\right) - \left(n+\frac{1}{2}\right)\ln\frac{n}{n+1} + \ln(n+1) - 1$$

$$= -\ln(n+1) + \left(n+\frac{1}{2}\right)\ln\left(\frac{n}{n+1}\right)^{-1} + \ln(n+1) - 1$$

$$= \left(n+\frac{1}{2}\right)\ln\left(\frac{n+1}{n}\right) - 1$$

$$= \left(n+\frac{1}{2}\right)\ln\left(\frac{2n+2}{2n}\right) - 1$$

$$= \left(n+\frac{1}{2}\right)\ln\left(\frac{\frac{2n+2}{2n+1}}{\frac{2n}{2n+1}}\right) - 1$$

$$= \left(n+\frac{1}{2}\right)\ln\left(\frac{\frac{2n+1+1}{2n+1}}{\frac{2n+1-1}{2n+1}}\right) - 1$$

$$= \left(n+\frac{1}{2}\right)\ln\left(\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}\right) - 1$$

c) $\ln\frac{1+t}{1-t} = \ln(1+t) - \ln(1-t)$

$$= t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots + t + \frac{t^2}{2} + \frac{t^3}{3} - \cdots$$

$$= 2\left(t + \frac{t^3}{3} + \frac{t^5}{5} + \cdots\right) \qquad \textit{for} \qquad -1 < t < 1$$

$$0 < c_n - c_{n+1} = \left(\frac{2n+1}{2}\right)\ln\left(\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}\right) - 1 \qquad\qquad \ln\frac{1+t}{1-t} = 2\left(t + \frac{t^3}{3} + \frac{t^5}{5} + \cdots\right)$$

$$= \frac{1}{2}(2n+1)(2)\left(\frac{1}{2n+1} + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \cdots\right) - 1$$

$$=1+\frac{1}{3(2n+1)^2}+\frac{1}{5(2n+1)^4}+\cdots-1$$

$$=\frac{1}{3(2n+1)^2}+\frac{1}{5(2n+1)^4}+\frac{1}{7(2n+1)^6}+\cdots$$

$$<\frac{1}{3}\left(\frac{1}{(2n+1)^2}+\frac{1}{(2n+1)^4}+\cdots\right)$$

Geometric series $S_n=a_1\frac{1}{1-r}$

$$=\frac{1}{3}\cdot\frac{1}{(2n+1)^2}\cdot\frac{1}{1-\frac{1}{(2n+1)^2}}$$

$$=\frac{1}{3}\cdot\frac{1}{(2n+1)^2}\cdot\frac{1}{1-\frac{1}{(2n+1)^2}}$$

$$=\frac{1}{3}\cdot\frac{1}{(2n+1)^2-1}$$

$$=\frac{1}{3}\cdot\frac{1}{4n^2+4n}$$

$$=\frac{1}{12}\cdot\frac{1}{n(n+1)}$$

$$=\frac{1}{12}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

These inequalities imply that $\{c_n\}$ is decreasing and $\left\{c_n-\frac{1}{12n}\right\}$ is increasing.

Thus $\{c_n\}$ is bounded below by $c_1-\frac{1}{12}=1-\frac{1}{12}=\frac{11}{12}$ $\quad\left(c_1=\ln(1!)-\left(1+\frac{1}{2}\right)\ln 1+1\right)$

So $\lim\limits_{n\to\infty}c_n=c$ exists.

Since $e^{c_n}=\frac{n!}{n^{(n+1/2)}e^{-n}}$, we have

$$\lim_{n\to\infty}\frac{n!}{n^{(n+1/2)}e^{-n}}=\lim_{n\to\infty}e^{c_n}$$

$$=e^{c}\quad \text{exists.}$$

Exercise

Suppose you want to approximate $\sqrt[3]{128}$ to within 10^{-4} of the exact value.

a) Use a Taylor polynomial for $f(x)=(125+x)^{1/3}$ centered at 0.

b) Use a Taylor polynomial for $f(x)=x^{1/3}$ centered at 125.

c) Compare the two approaches. Are they equivalent?

Solution

$$\boxed{\sqrt[3]{128} \approx 5.03968419958}$$

a) $a = 0$

$f(x) = (125 + x)^{1/3} \quad \rightarrow \quad f(0) = (125)^{1/3} = 5$

$\rightarrow \quad f'(0) = \frac{1}{3}(125)^{-2/3}$

$f'(x) = \frac{1}{3}(125 + x)^{-2/3} \qquad = \frac{1}{3}\left(5^3\right)^{-2/3} = \frac{1}{3}(5)^{-2}$

$= \frac{1}{75}$

$f''(x) = -\frac{2}{9}(125 + x)^{-5/3} \quad \rightarrow \quad f''(1) = -\frac{2}{9}\left(5^3\right)^{-5/3} = -\frac{2}{9}\frac{1}{5^5}$

$= -\frac{2}{28{,}125}$

$$f(x) = 5 + \frac{1}{75}x - \frac{1}{28{,}125}x^2$$

$125 + x = 128 \quad \Rightarrow \quad \boxed{x = 3}$

$$f(3) = 5 + \frac{1}{75}(3) - \frac{1}{28{,}125}(9)$$

$$= 5 + \frac{1}{25} - \frac{1}{3{,}125}$$

$$= 5 + .04 - .00032$$

$$\boxed{\approx 5.03968}$$

b) $a = 125 = 5^3$

$f(x) = x^{1/3} \quad \rightarrow \quad f(125) = 5$

$f'(x) = \frac{1}{3}x^{-2/3} \quad \rightarrow \quad f'(0) = \frac{1}{3}\left(5^3\right)^{-2/3}$

$= \frac{1}{75}$

$f''(x) = -\frac{2}{9}x^{-5/3} \quad \rightarrow \quad f''(1) = -\frac{2}{9}\left(5^3\right)^{-5/3}$

$= -\frac{2}{28{,}125}$

$$f(x) = 5 + \frac{1}{75}(x - 125) - \frac{1}{28{,}125}(x - 125)^2$$

$$f(128) = 5 + \frac{1}{75}(3) - \frac{1}{28{,}125}(3)^2$$

$$= 5 + \frac{1}{25} - \frac{1}{3{,}125}$$

$$= 5 + .04 - .00032$$

$$\approx 5.03968$$

c) Both the results from part (a) and (b) are the same since they are just shifting.

Exercise

Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

a) Use the definition of the derivative to show that $f'(0) = 0$

b) Assume the fact that $f^k(0) = 0$ for $k = 1, 2, 3, \ldots$. Write the Taylor series for f centered at 0.

c) Explain why the Taylor series for f does not converge to f for $x \neq 0$

Solution

a) $f'(0) = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$

$$= \lim_{x \to 0} \frac{e^{-1/x^2} - 0}{x}$$

$$= \lim_{x \to 0} \frac{e^{-1/x^2}}{x} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\frac{2}{x^3} e^{-1/x^2}}{1}$$

$$= \lim_{x \to 0} \frac{2e^{-1/x^2}}{x^3} = \frac{0}{0}$$

Let $y = \dfrac{1}{x^2} \quad \Rightarrow \quad x = \dfrac{1}{\sqrt{y}}$

$x \to 0 \quad \Rightarrow \quad y \to \infty$

$$f'(0) = \lim_{y \to \infty} \frac{e^{-y}}{\frac{1}{\sqrt{y}}}$$

$$= \lim_{y \to \infty} \frac{\sqrt{y}}{e^y}$$

$$= 0 \quad \checkmark$$

b) ***Given***: $f^k(0) = 0$

Since the Taylor series centered at 0 has only one term $f(x)=f(0)=0$ and $f^k(0)=0$ (derivaties are equal to 0).
Therefore; the Taylor series is zero.

c) It does not converge to $f(x)$ because when $x\neq 0$, $f(x)\neq 0$

Exercise

Teams *A* and *B* go into sudden death overtime after playing to a tie. The teams alternate possession of the ball and the first team to score wins. Each team has a $\frac{1}{6}$ chance of scoring when it has the ball, with Team *A* having the ball first.

a) The probability that Team *A* ultimately wins is $\sum\limits_{k=0}^{\infty}\frac{1}{6}\left(\frac{5}{6}\right)^{2k}$. Evaluate this series.

b) The expected number of rounds (possessions by either team) required for the overtime to end is $\frac{1}{6}\sum\limits_{k=1}^{\infty}k\left(\frac{5}{6}\right)^{k-1}$. Evaluate this series.

Solution

a) $$\sum_{k=0}^{\infty}\frac{1}{6}\left(\frac{5}{6}\right)^{2k}=\frac{1}{6}\sum_{k=0}^{\infty}\left(\frac{25}{36}\right)^{k}$$

It is a *Geometric* series with $r=\frac{25}{36}<1$, then

$$\sum_{k=0}^{\infty}\frac{1}{6}\left(\frac{5}{6}\right)^{2k}=\frac{1}{6}\cdot\frac{1}{1-\frac{25}{36}}$$

$$=\frac{1}{6}\cdot\frac{36}{11}$$

$$=\frac{6}{11}$$

b) Using the series $\sum\limits_{k=1}^{\infty}x^k=\frac{x}{1-x}$

$$\left(\sum_{k=1}^{\infty}x^k\right)'=\left(\frac{x}{1-x}\right)'$$

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

Let $x = \frac{5}{6}$

$$\frac{1}{6}\sum_{k=1}^{\infty} k\left(\frac{5}{6}\right)^{k-1} = \frac{1}{6}\frac{1}{\left(1-\frac{5}{6}\right)^2}$$

$$= \frac{1}{6}\frac{1}{\left(\frac{1}{6}\right)^2}$$

$$\boxed{= 6}$$