Chapter One

Section 1.1 – Introduction to System of Linear Equations & Matrices

1.1−1 *System Equations*

A system of equations is a set of one or more equations involving two or more variables.
The solutions to systems of equations are the variable mappings such that all these equations intersect, no intersection, or a unique infinite solution.

1.1−2 *Matrices*

A matrix is a rectangle array of real numbers, arranged in rows and columns.

$$\begin{array}{ccc} & & \text{Columns} \\ & & \begin{array}{ccc} C_1 & C_2 & C_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \text{Rows} & \begin{array}{c} Row\ 1 \rightarrow R_1 \\ Row\ 2 \rightarrow R_2 \\ Row\ 3 \rightarrow R_3 \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

This is called Matrix (*Matrices*).

Each number in the array is an ***element*** or ***entry.***

The matrix is said to be of ***order*** (or ***size***) $m\ x\ n$ if there are:

m: numbers of rows,

n: number of columns

When $m = n$, then matrix is said to be ***square***.

Given the system equations

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

The ***augmented matrix*** form can be written as:

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 31 \\ 1 & 1 & 2 & 19 \\ 1 & 3 & 2 & 25 \end{array}\right]$$

The ***matrix equation*** can be written in the ***Form*** $AX = B$

Where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}$$ is called the ***coefficient matrix*** of the system.

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$ is called the ***variables matrix*** of the system.

$$B = \begin{bmatrix} 31 \\ 29 \\ 25 \end{bmatrix}$$ is called the ***constant matrix*** of the system.

If $B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, then the system equations are called ***homogeneous equations***.

The matrix A above has 3 rows and 3 columns.

Therefore, the order of the matrix A is (3 x 3)

$$A = \begin{bmatrix} a_{ij} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

The order of the matrix A is (m x n)

1.1−3 Example

Given the linear equations

$$\begin{cases} x-2y=1 \\ 3x+2y=11 \end{cases}$$

The solution to this system is $(3,\ 1)$, which means that 2 lines meeting at a single point.

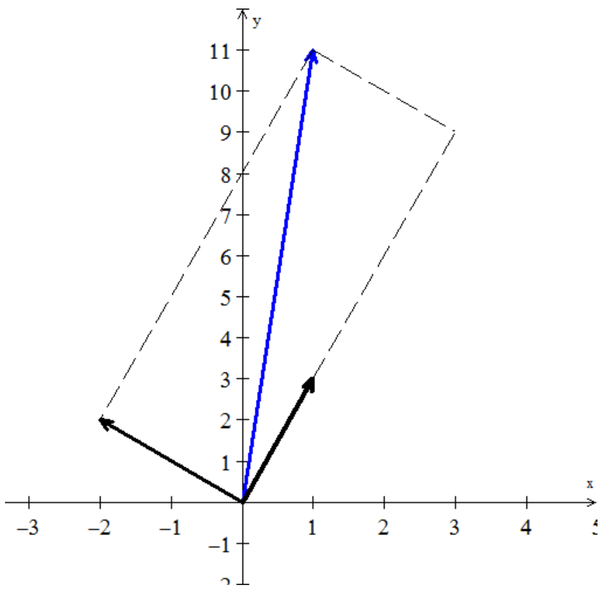
We can rewrite the system equation as linear combination:

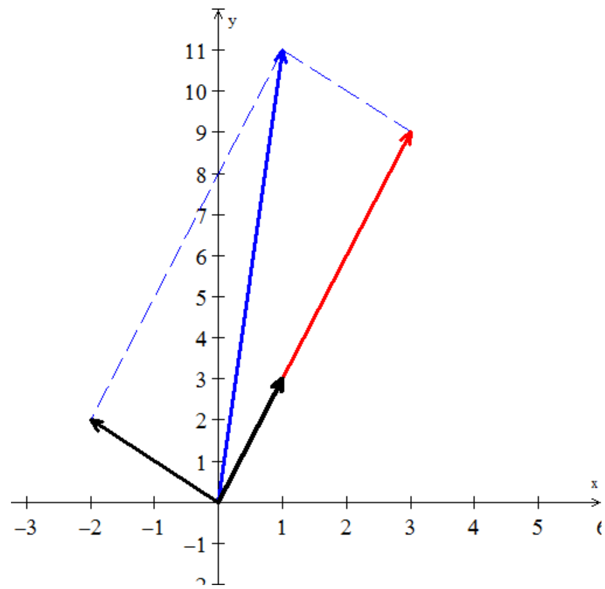
$$x\begin{bmatrix}1\\3\end{bmatrix}+y\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$$

$$x.\vec{v}_1+y.\vec{v}_2=\vec{v}$$

$$\begin{bmatrix}1+x\\3+y\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}\ \Rightarrow\ \begin{bmatrix}x=3\\y=9\end{bmatrix}$$

$$\begin{bmatrix}3\\9\end{bmatrix}=3\begin{bmatrix}1\\3\end{bmatrix}$$





Therefore, the side vectors are $\begin{bmatrix}3\\9\end{bmatrix}$ *and* $\begin{bmatrix}-2\\2\end{bmatrix}$

The diagonal sum is $\begin{bmatrix}3-2\\9+2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$

The linear combination is given by:

$$3\begin{bmatrix}1\\3\end{bmatrix}+1\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$$

Thus, the solution is $x=3 \quad y=1$

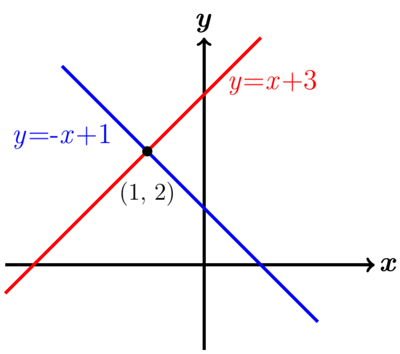
<u>Note</u>

$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$ is called the "***coefficient matrix***"

The matrix form of the system is written as $\quad Ax = b$

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

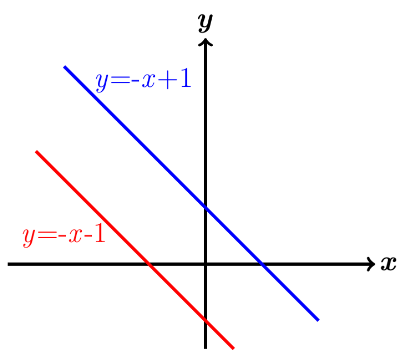
1.1–4 *Graphically*



***One solution* (*lines intersect*)**

Consistent

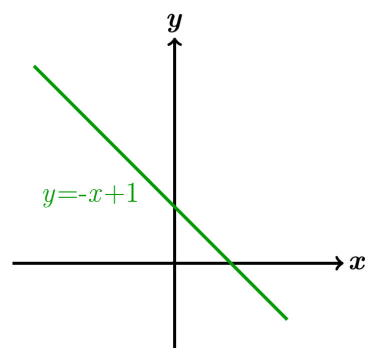
Independent



***No Solution* (*lines //*)**

Inconsistent

Independent



Unique Infinite solution

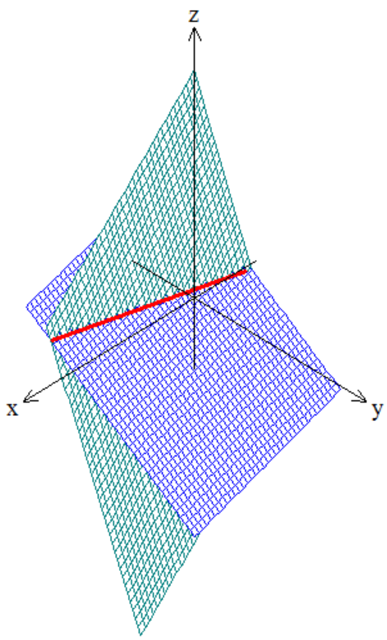
Consistent

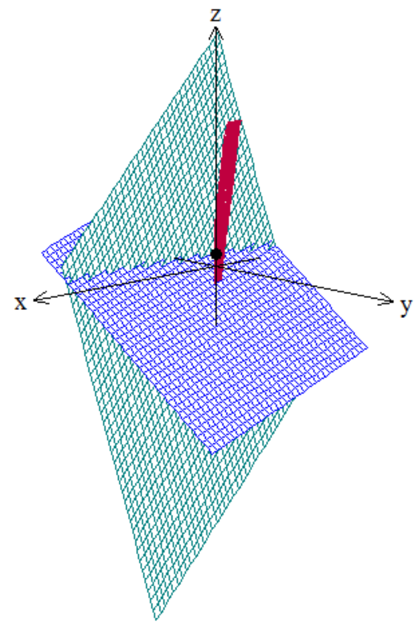
Dependent

1.1−5 *Three* Equations in 3 Unknowns

Given the system equations

$$x + 2y + 3z = 6$$
$$2x + 5y + 2z = 4$$
$$6x - 3y + z = 2$$



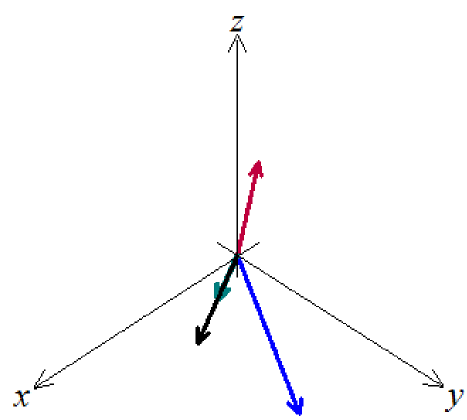


This system can be written as linear combination:

$$x\begin{bmatrix}1\\2\\6\end{bmatrix} + y\begin{bmatrix}2\\5\\-3\end{bmatrix} + z\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix}$$

Let $b = \begin{bmatrix}6\\4\\2\end{bmatrix}$

We want to multiply the three column vectors by $x,\ y,\ z$ to produce ***b***.

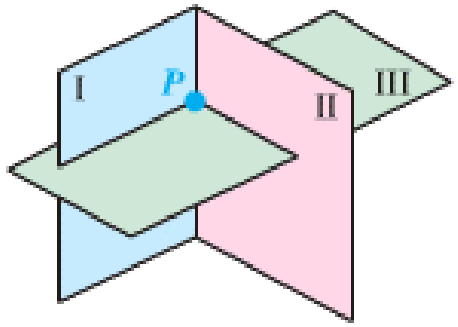


The combination of the three vectors that produces vector b is 2 times the third vector.

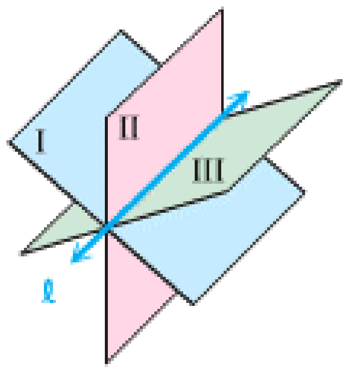
$$2(3,\ 2,\ 1) = (6,\ 4,\ 2) = b$$

Therefore, the coefficients that we need are $x = 0, \quad y = 0, \quad and\ z = 2$.

$$0\begin{bmatrix}1\\2\\6\end{bmatrix} + 0\begin{bmatrix}2\\5\\-3\end{bmatrix} + 2\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix}$$

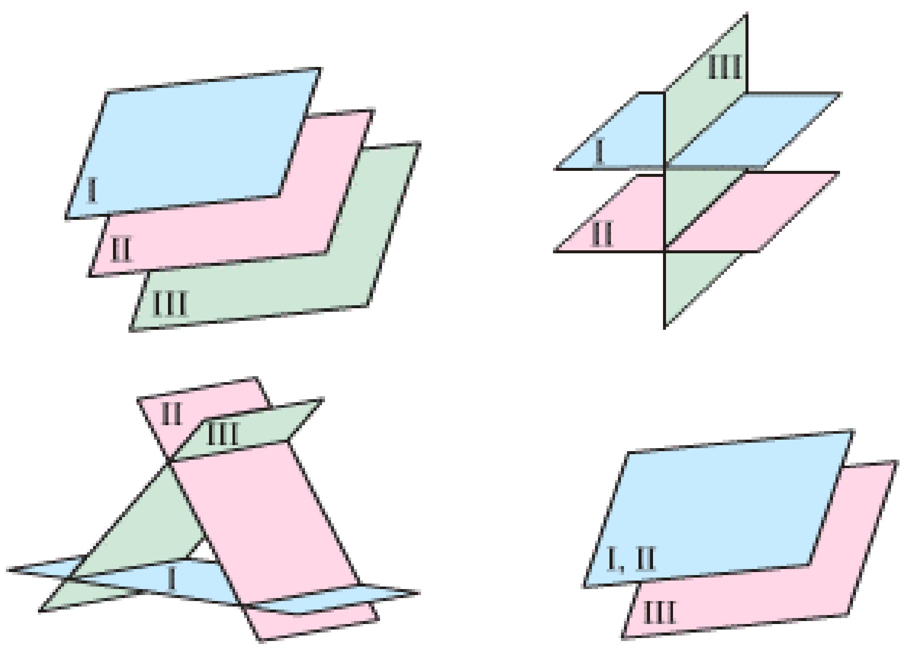


A single Solution



Points of a line in common

All points in common



No solution – No points in common

Exercises *Section* 1.1 – Introduction to System of Linear Equations

1. Find a solution for x, y, z to the system of equations

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3e+2\sqrt{2}+\pi \\ 6e+5\sqrt{2}+4\pi \\ 9e+8\sqrt{2}+7\pi \end{pmatrix}$$

2. Draw the two pictures in two planes for the equations: $x-2y=0, \quad x+y=6$

3. Normally 4 planes in 4-dimensional space meet at a ________. Normally 4 column vectors in 4-deimensional space can combine to produce b. what combinations of $(1,\ 0,\ 0,\ 0),\quad (1,\ 1,\ 0,\ 0),\quad (1,\ 1,\ 1,\ 0),\quad (1,\ 1,\ 1,\ 1)$ produces $b=(3,\ 3,\ 3,\ 2)$?
What 4 equations for $x,\ y,\ z,\ w$ are you solving?

4. What 2 by 2 matrix A rotates every vector through 45° ?
The vector (1, 0) goes to $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. The vector (0, 1) goes to $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.
Those determine the matrix. Draw these particular vectors is the xy-plane and find A.

5. What two vectors are obtained by rotating the plane vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by 30° (cw) ?
Write a matrix A such that for every vector $\vec{v}$ in the plane, $A\vec{v}$ is the vector obtained by rotating $\vec{v}$ clockwise by 30°.
Find a matrix B such that for every 3-dimensional vector $\vec{v}$, the vector $B\vec{v}$ is the reflection of $\vec{v}$ through the plane $x+y+z=0$. *Hint*: $\vec{v}=(1,\ 0,\ 0)$

6. In each part, find a system of linear equation corresponding to the given augmented matrix

a) $\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \\ 5 & -6 & 1 & 1 \\ -8 & 0 & 0 & 3 \end{bmatrix}$

7. Find the augmented matrix for the given system of linear equations.

a) $\begin{cases} -2x_1 = 6 \\ 3x_1 = 8 \\ 9x_1 = -3 \end{cases}$

b) $\begin{cases} 3x_1 - 2x_2 = -1 \\ 4x_1 + 5x_2 = 3 \\ 7x_1 + 3x_2 = 2 \end{cases}$

c) $\begin{cases} 2x_1 \qquad + 2x_3 = 1 \\ 3x_1 - x_2 + 4x_3 = 7 \\ 6x_1 + x_2 - x_3 = 0 \end{cases}$

(**8 – 10**) Determine the size of the matrix

8. $\begin{bmatrix} 1 & 2 & -4 \\ 3 & 4 & 6 \\ 0 & 1 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

10. $\begin{bmatrix} 2 & 1 & -1 & -1 \\ -6 & 2 & 0 & 1 \end{bmatrix}$

Section 1.2 – Gaussian Elimination

1.2−1 Gaussian Elimination

Gaussian elimination, also known as *R*ow *R*eduction *E*chelon *F*orm, is an algorithm in linear algebra for solving a system of linear equations.

The method is named after *Carl Friedrich Gauss* (1777–1855).

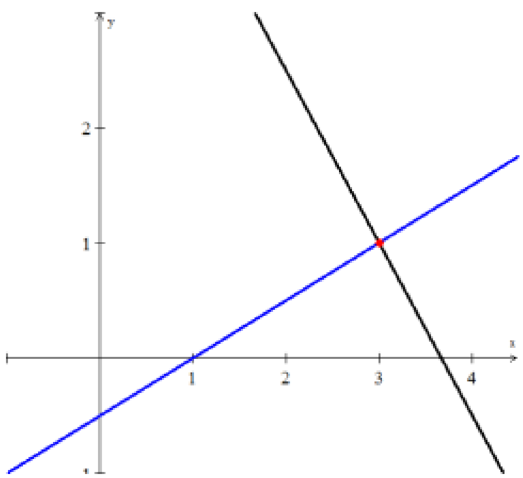
1.2−2 Elimination Procedure

Elimination produces an ***upper triangular system***.

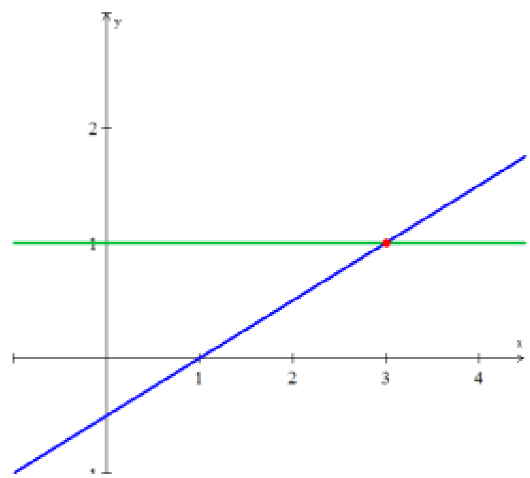
$$\begin{cases} x-2y=1 \\ 3x+2y=11 \end{cases} \Rightarrow \begin{cases} x-2y=1 & \textit{Mutliply by } 3 \\ \quad 8y=8 & \textit{and subtract} \end{cases}$$

The equation $8y=8$ *reveals* $y=1$

This process is called ***back substitution***.



Before elimination



After elimination

1.2−3 *Definitions*

Pivot: first nonzero in the row that does the elimination

Multiplier: (entry to eliminate) divide by pivot

$$\begin{array}{lll} 4x-8y=4 & \textit{Multiply equation } 1 \textit{ by } \frac{3}{4} & 4x-8y=4 \\ 3x+2y=11 & \textit{Subtract from equation } 2 & \quad 8y=8 \end{array}$$

The first pivot is 4 (the coefficient of x) and the multiplier is $l=\frac{3}{4}$

The pivots are on the diagonal of the triangle after elimination.

1.2−4 *Definition*

The operations are the elementary reduction operations, or row operations, or Gaussian operations. They are swapping, multiplying by a scalar or rescaling, and pivoting.

Reduced Row Echelon Form

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

1.2−5 *Example*

Use the Gaussian elimination method to solve the system

$$\begin{aligned} 3x + \ y + 2z &= 31 \\ x + \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 3 & 1 & 2 & 31 \\ 1 & 3 & 2 & 25 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array} \qquad \begin{array}{cccc} 3 & 1 & 2 & 31 \\ -3 & -3 & -6 & -57 \\ \hline 0 & -2 & -4 & -26 \end{array} \qquad \begin{array}{cccc} 1 & 2 & 2 & 25 \\ -1 & -1 & -2 & -19 \\ \hline 0 & 2 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & -2 & -4 & -26 \\ 0 & 2 & 0 & 6 \end{array}\right] -\tfrac{1}{2}R_2 \qquad 0 \quad 1 \quad 2 \quad 13$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 2 & 0 & 6 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_2 \end{array} \qquad \begin{array}{cccc} 0 & 2 & 0 & 6 \\ 0 & -2 & -4 & -26 \\ \hline 0 & 0 & -4 & -20 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -4 & -20 \end{array}\right] -\tfrac{1}{4}R_3 \qquad 0 \quad 0 \quad 1 \quad 5$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array}\right] \Rightarrow \begin{aligned} x + y + 2z &= 19 \quad (3) \\ y + 2z &= 13 \quad (2) \\ z &= 5 \quad (1) \end{aligned}$$

$(2) \Rightarrow \; y = 13 - 2z$
$\quad = 13 - 2(5)$
$\quad \boxed{= 3}$

$(3) \Rightarrow \; x = 19 - y - 2z$
$\quad = 19 - 3 - 10$
$\quad \boxed{= 6}$

Solution: $\boxed{(6,\ 3,\ 5)}$

1.2−6 *Example*

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\quad &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\ 5x_3 + 10x_4 \qquad + 15x_6 &= 5 \\ 2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 + 18x_6 &= 6 \end{aligned}$$

Solution

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \quad \textit{Adding } (-2) \textit{ times the 1st row to the 2nd} \\ \\ R_4 - 2R_1 \quad \textit{Adding } (-2) \textit{ times the 1st row to the 4th} \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 5R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right] \begin{array}{l} \\ \\ \\ \frac{1}{6}R_4 \quad \textit{then interchanging row3 and row4} \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \begin{cases} x_1 + 3x_2 \quad + 4x_4 + 2x_5 \quad = 0 \\ \qquad x_3 + 2x_4 \qquad = 0 \\ \qquad\qquad + x_6 = \frac{1}{3} \end{cases}$$

The general solution of the system:

$$\underline{\left(-3x_2 - 4x_4 - 2x_5,\ \ x_2,\ \ -2x_4,\ \ x_4,\ \ x_5,\ \ \tfrac{1}{3}\right)\Big|}$$

1.2−7 *Example*

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$\begin{aligned} 2x + \ 8y - \ z + \ w &= 0 \\ 4x + 16y - 3z - w &= -10 \\ -2x + 4y - z + 3w &= -6 \\ -6x + 2y + 5z + w &= 3 \end{aligned}$$

Solution

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 4 & 16 & -3 & -1 & -10 \\ -2 & 4 & -1 & 3 & -6 \\ -6 & 2 & 5 & 1 & 3 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 + 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - \frac{13}{6}R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \qquad \textit{Interchange } R_2 \textit{ and } R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & \frac{19}{3} & -\frac{14}{3} & 16 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 + \frac{19}{3}R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & 0 & -\frac{71}{3} & -\frac{142}{3} \end{array}\right] \qquad \begin{array}{c} 2x + 8y - z + w = 0 \\ 12y - 2z + 4w = -6 \\ -z - 3w = -10 \\ -\frac{71}{3}w = -\frac{142}{3} \end{array}$$

$$\begin{cases} 2x = -8y + z - w & (1) \\ 12y = 2z - 4w - 6 & (2) \\ \underline{z = 10 - 3w \ \ = 4} \\ \underline{w = 2} \end{cases}$$

$$\begin{cases} (2) \to \ 12y = 8 - 8 - 6 \ \Rightarrow \ \underline{y = -\frac{1}{2}} \\ (1) \to \ 2x = 4 + 4 - 2 \ \Rightarrow \ \underline{x = 3} \end{cases}$$

Solution: $\underline{\left(3,\ -\frac{1}{2},\ 4,\ 2\right)}$

1.2−8 *Theorem*: Free Variable Theorem for Homogeneous Systems

If a *homogeneous linear* system has $\boldsymbol{n}$ unknowns, and if the reduced row echelon form of its augmented matrix has $\boldsymbol{r}$ nonzero rows, then the system has $n - r$ free variables.

1.2−9 *Theorem*

A *homogeneous linear* system with more unknowns than equations has ***infinitely many*** *unknowns*.

1.2−10 Breakdown Elimination

Permanent failure with no solution

$$\begin{array}{lcr} x-2y=1 & \textit{Subtract 3 times} & x-2y=1 \\ 3x-6y=11 & \textit{eqn. 1 from eqn. 2} & 0y=8 \end{array}$$

The last equation $0y=8$; therefore, there is *no* solution.

This system has no second pivot, since no zero allowed as a pivot.

Permanent failure with infinitely many solutions

$$\begin{array}{lcr} x-2y=1 & \textit{Subtract 3 times} & x-2y=1 \\ 3x-6y=3 & \textit{eqn. 1 from eqn. 2} & 0y=0 \end{array}$$

Every y satisfies $0y=0$. There is only one equation $x-2y=1$.

There are ***unique infinitely*** many solutions.

Three Equations in Three Unknowns

To understand Gaussian elimination, you have to go beyond 2 by 2 systems.

Consider the system equations: $\begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases}$

$$\begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases} \quad \begin{array}{c} \textit{subtract 2 times eqn.1} \\ \textit{from eqn.2} \\ \ \end{array} \quad \begin{array}{r} 2x+4y-2z=2 \\ y+z=4 \\ -2x-3y+7z=10 \end{array}$$

$$\begin{cases} 2x+4y-2z=2 \\ y+z=4 \\ -2x-3y+7z=10 \end{cases} \quad \begin{array}{c} \textit{Add eqn.1} \\ \ \\ \textit{and eqn.3} \end{array} \quad \begin{array}{r} 2x+4y-2z=2 \\ y+z=4 \\ y+5z=12 \end{array}$$

$$\begin{cases} 2x+4y-2z=2 \\ y+z=4 \\ y+5z=12 \end{cases} \quad \begin{array}{c} \ \\ \textit{Subtract eqn.2} \\ \textit{from eqn.3} \end{array} \quad \begin{array}{r} 2x+4y-2z=2 \\ y+z=4 \\ 4z=8 \end{array}$$

$$\begin{cases} x=1-2y+z=\underline{-1} \\ y=4-z=\underline{2} \\ \underline{z=2} \end{cases}$$

The solution is $\underline{(-1,\ 2,\ 2)}$

1.2−11 *Definition*

A square matrix is nonsingular if it is the matrix of coefficient of a homogeneous system, with a unique solution. It is singular otherwise, that is, if it is the matrix of coefficients of a homogeneous system, with infinitely many solutions.

Exercises *Section* 1.2 – Gaussian Elimination

1. When elimination is applied to the matrix $A=\begin{bmatrix}3&1&0\\6&9&2\\0&1&5\end{bmatrix}$

a) What are the first and second pivots?

b) What is the multiplier l_{21} in the first step (l_{21} times row 1 is subtracted from row 2)?

c) What entry in the 2, 2 position (instead of 9) would force an exchange of rows 2 and 3?

d) What is the multiplier $l_{31}=0$, subtracting 0 times row 1 from row 3?

2. Use elimination to reach upper triangular matrices *U*. Solve by back substitution or explain why this impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the $-x$ in equation (3).

$$\begin{cases}x+y+z=7\\x+y-z=5\\x-y+z=3\end{cases}\qquad\qquad\begin{cases}x+y+z=7\\x+y-z=5\\-x-y+z=3\end{cases}$$

3. For which number(s) *a* does the elimination break down (1) permanently (2) temporarily

$$ax+3y=-3$$
$$4x+6y=6$$

Solve for *x* and *y* after fixing the second breakdown by a row change.

4. Find the pivots and the solution for these four equations:

$$\begin{aligned}2x+y\qquad\quad&=0\\x+2y+z\quad\;&=0\\y+2z+t&=0\\z+2t&=5\end{aligned}$$

5. Look for a matrix that has row sums 4 and 8, and column sums 2 and *s*.

$$A=\begin{bmatrix}a&b\\c&d\end{bmatrix}\qquad\begin{matrix}a+b=4 & a+c=2\\c+d=8 & b+d=s\end{matrix}$$

The four equations are solvable only if $s=$ _____. Then find two different matrices that have the correct row and column sums.

6. Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of *A* is a ________ of the first two rows. Find a third equation that can't be solved together with $x+y+z=0$ and $x-2y-z=1$

(7 – 14) Use the Gauss-Jordan method to solve the system

7. $\begin{cases} x-y+5z=-6 \\ 3x+3y-z=10 \\ x+3y+2z=5 \end{cases}$

8. $\begin{cases} 2x-y+4z=-3 \\ x-2y-10z=-6 \\ 3x \quad +4z=7 \end{cases}$

9. $\begin{cases} 4x+3y-5z=-29 \\ 3x-7y-z=-19 \\ 2x+5y+2z=-10 \end{cases}$

10. $\begin{cases} x+2y-3z=-15 \\ 2x-3y+4z=18 \\ -3x+y+z=1 \end{cases}$

11. $\begin{cases} x+2y+3z=10 \\ 4x+5y+6z=11 \\ 7x+8y+9z=12 \end{cases}$

12. $\begin{cases} 2x+y+2z=4 \\ 2x+2y \quad =5 \\ 2x-y+6z=2 \end{cases}$

13. $\begin{cases} x_1+x_2+2x_3=8 \\ -x_1-2x_2+3x_3=1 \\ 3x_1-7x_2+4x_3=10 \end{cases}$

14. $\begin{cases} x+2y+z=8 \\ -x+3y-2z=1 \\ 3x+4y-7z=10 \end{cases}$

(15 – 51) Use augmented elimination to solve linear system

15. $\begin{cases} 2x-5y+3z=1 \\ x-2y-2z=8 \end{cases}$

16. $\begin{cases} x+y+z=2 \\ 2x+y-z=5 \\ x-y+z=-2 \end{cases}$

17. $\begin{cases} 2x+y+z=9 \\ -x-y+z=1 \\ 3x-y+z=9 \end{cases}$

18. $\begin{cases} 3y-z=-1 \\ x+5y-z=-4 \\ -3x+6y+2z=11 \end{cases}$

19. $\begin{cases} x+3y+4z=14 \\ 2x-3y+2z=10 \\ 3x-y+z=9 \end{cases}$

20. $\begin{cases} x+4y-z=20 \\ 3x+2y+z=8 \\ 2x-3y+2z=-16 \end{cases}$

21. $\begin{cases} 2y-z=7 \\ x+2y+z=17 \\ 2x-3y+2z=-1 \end{cases}$

22. $\begin{cases} -2x+6y+7z=3 \\ -4x+5y+3z=7 \\ -6x+3y+5z=-4 \end{cases}$

23. $\begin{cases} 2x-y+z=1 \\ 3x-3y+4z=5 \\ 4x-2y+3z=4 \end{cases}$

24. $\begin{cases} 3x-4y+4z=7 \\ x-y-2z=2 \\ 2x-3y+6z=5 \end{cases}$

25. $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

26. $\begin{cases} x+y+z=3 \\ -y+2z=1 \\ -x+z=0 \end{cases}$

27. $\begin{cases} 3x+y+3z=14 \\ 7x+5y+8z=37 \\ x+3y+2z=9 \end{cases}$

28. $\begin{cases} 4x-2y+z=7 \\ x+y+z=-2 \\ 4x+2y+z=3 \end{cases}$

29. $\begin{cases} 2x-2y+z=-4 \\ 6x+4y-3z=-24 \\ x-2y+2z=1 \end{cases}$

30. $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

31. $\begin{cases} 2x-y+2z=-8 \\ x+2y-3z=9 \\ 3x-y-4z=3 \end{cases}$

32. $\begin{cases} x \quad -3z=-5 \\ 2x-y+2z=16 \\ 7x-3y-5z=19 \end{cases}$

33. $\begin{cases} x+2y-z=5 \\ 2x-y+3z=0 \\ 2y+z=1 \end{cases}$

34. $\begin{cases} x+y+z=6 \\ 3x+4y-7z=1 \\ 2x-y+3z=5 \end{cases}$

35. $\begin{cases} 3x+2y+3z=3 \\ 4x-5y+7z=1 \\ 2x+3y-2z=6 \end{cases}$

36. $\begin{cases} x-3y+z=2 \\ 4x-12y+4z=8 \\ -2x+6y-2z=-4 \end{cases}$

37. $\begin{cases} 2x-2y+z=-1 \\ x+2y-z=2 \\ 6x+4y+3z=5 \end{cases}$

38. $\begin{cases} x_1 - 3x_3 = -2 \\ 3x_1 + x_2 - 2x_3 = 5 \\ 2x_1 + 2x_2 + x_3 = 4 \end{cases}$

39. $\begin{cases} 2x_1 + 3x_3 = 3 \\ 4x_1 - 3_2 + 7x_3 = 5 \\ 8x_1 - 9x_2 + 15x_3 = 10 \end{cases}$

40. $\begin{cases} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 \\ x_2 - 3x_3 - x_4 = 0 \\ 3x_1 + 2x_3 - x_4 = 6 \\ -4x_1 + x_2 + 4x_3 + 2x_4 = -3 \end{cases}$

41. $\begin{cases} x_1 + x_2 + x_3 + x_4 = 5 \\ x_1 + 2x_2 - x_3 - 2x_4 = -1 \\ x_1 - 3x_2 - 3x_3 - x_4 = -1 \\ 2x_1 - x_2 + 2x_3 - x_4 = -2 \end{cases}$

42. $\begin{cases} 2x + 8y - z + w = 0 \\ 4x + 16y - 3z - w = -10 \\ -2x + 4y - z + 3w = -6 \\ -6x + 2y + 5z + w = 3 \end{cases}$

43. $\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 = 0 \\ x_2 + x_3 = 0 \end{cases}$

44. $\begin{cases} 2x + 2y + 4z = 0 \\ -y - 3z + w = 0 \\ 3x + y + z + 2w = 0 \\ x + 3y - 2z - 2w = 0 \end{cases}$

45. $\begin{cases} 2x + z + w = 5 \\ y - w = -1 \\ 3x - z - w = 0 \\ 4x + y + 2z + w = 9 \end{cases}$

46. $\begin{cases} 4y + z = 20 \\ 2x - 2y + z = 0 \\ x + z = 5 \\ x + y - z = 10 \end{cases}$

47. $\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x - 3w = -3 \end{cases}$

48. $\begin{cases} 2u - 3v + w - x + y = 0 \\ 4u - 6v + 2w - 3x - y = -5 \\ -2u + 3v - 2w + 2x - y = 3 \end{cases}$

49. $\begin{cases} 6x_3 + 2x_4 - 4x_5 - 8x_6 = 8 \\ 3x_3 + x_4 - 2x_5 - 4x_6 = 4 \\ 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 + x_6 = 2 \\ 6x_1 - 9x_2 + 11x_4 - 19x_5 + 3x_6 = 1 \end{cases}$

50. $\begin{cases} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{cases}$

51. $\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ 5x_3 + 10x_4 + 15x_6 = 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$

52. Add 3 times the second row to the first of $\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$

53. For what value(s) of k, if any, does the system $\begin{cases} x + y - z = 1 \\ 2x + 3y + kz = 3 \\ x + ky + 3z = 2 \end{cases}$ have

a) A unique solution?
b) Infinitely many solutions?
c) No solution?

54. Assume that the matrix is the augmented matrix of a system of linear equations, and

$$\begin{bmatrix} 1 & k & 2 \\ -3 & 4 & 1 \end{bmatrix}$$

a) Determine the number of equations and the number of variables.
b) Find the value(s) of $\boldsymbol{k}$ such that the system is consistent.
c) Determine the number of equations and the number of variables but if the matrix is the coefficient matrix of a *homogeneous* system of linear equations.
d) Find the value(s) of $\boldsymbol{k}$ from part (c).

55. Choose a coefficient b that makes the system singular.

$$\begin{cases} 3x + 4y = 16 \\ 4x + by = g \end{cases}$$

Then choose a right-hand side g that makes it solvable.
Find 2 solutions in that singular case.

56. This system us not linear, in some sense,

$$\begin{cases} 2\sin\alpha - \cos\beta + 3\tan\theta = 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\theta = 10 \\ 6\sin\alpha - 3\cos\beta + \tan\theta = 9 \end{cases}$$

Does the system have a solution?

57. Determine whether the matrix is in row-echelon form. If it is, determine whether it is also in reduced row-echelon form.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Section 1.3 – The Algebra of Matrices

$$A = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

1.3−1 Equality of Matrices

Definition of Equality of Matrices

Two matrices ***A*** and ***B*** are equal if and only if

- ✓ They have the same order (size) $m \times n$ and
- ✓ All the entries of ***A*** are the same as the corresponding entries in ***B***.

$a_{ij} = b_{ij}$

for $i = 1, 2, \ldots, m$ ***and*** $j = 1, 2, \ldots, n$

1.3−2 *Example*

Find the values of the variables for which each statement is true, if possible.

a) $\begin{bmatrix} 2 & 1 \\ p & q \end{bmatrix} = \begin{bmatrix} x & y \\ -1 & 0 \end{bmatrix}$

$x = 2, \ y = 1, \ \ p = -1, \ \ q = 0$

b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$

can't be true

c) $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

$\Rightarrow \begin{bmatrix} w = 9 & x = 17 \\ 8 = y & -12 = z \end{bmatrix}$

1.3−3 *Addition* and *Subtraction* of Matrices

Definition

If $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ are $m\times n$ matrices, their sum $A+B$, is the $m\times n$ matrix obtained by adding the corresponding entries; that is

$$\left[a_{ij}\right]+\left[b_{ij}\right]=\left[a_{ij}+b_{ij}\right]$$

Matrices can be added if their shapes are the same, meaning have the same ***order***.

$$\begin{bmatrix}1 & 2\\ 3 & 4\\ 0 & 0\end{bmatrix}+\begin{bmatrix}2 & 2\\ 4 & 4\\ 9 & 9\end{bmatrix}=\begin{bmatrix}1+2 & 2+2\\ 3+4 & 4+4\\ 0+9 & 0+9\end{bmatrix}$$
$$=\begin{bmatrix}3 & 4\\ 7 & 8\\ 9 & 9\end{bmatrix}$$

1.3−4 *Scalar* Multiplication Matrices

Definition

If k is a scalar and $A=\left[a_{ij}\right]$ is an $m\times n$ matrices, then scalar product ***kA*** is the $m\times n$ matrix obtained by multiplying each entry of A by k; that is

$$k\left[a_{ij}\right]=\left[ka_{ij}\right]$$

$$kA=k\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$$
$$=\begin{bmatrix}ka_{11} & ka_{12}\\ ka_{21} & ka_{22}\end{bmatrix}$$

1.3−5 *Example*

$$2\begin{bmatrix}1 & 2\\ 3 & 4\\ 0 & 0\end{bmatrix}=\begin{bmatrix}(2)1 & (2)2\\ (2)3 & (2)4\\ (2)0 & (2)0\end{bmatrix}$$
$$=\begin{bmatrix}2 & 4\\ 6 & 8\\ 0 & 0\end{bmatrix}$$

1.3−6 *Definition*

If $A_1, A_2, \ldots, A_n$ are matrices of the same size, and if $c_1, c_2, \ldots, c_n$ are scalars, then expression of the form

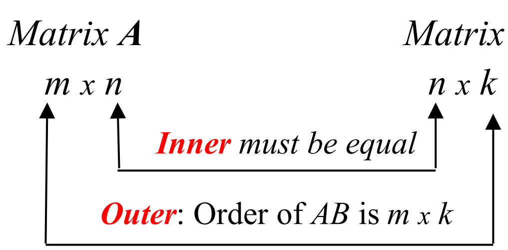
$$c_1A_1 + c_2A_2 + \ldots + c_nA_n$$

Is called a ***linear combination*** of $A_1, A_2, \ldots, A_n$ with *coefficients* $c_1, c_2, \ldots, c_n$.

1.3−7 *Matrix Multiplication*

Product of Two Matrices

Let ***A*** be an *m x n* matrix and let ***B*** be an *n x k* matrix. To find the element in the i^{th} row and j^{th} column of the product matrix ***AB***, multiply each element in the i^{th} row of ***A*** by the corresponding element in the j^{th} column of ***B***, and then add these products. The product matrix ***AB*** is an *m x k* matrix.



- ✓ To multiply ***AB*** or dot product, if ***A*** has ***n*** columns, ***B*** must have ***n*** rows.
- ✓ Squares matrices can be multiplied if and only if they have the same size.
- ✓ The entry in row *i* and column *j* of ***AB*** is $\boxed{(row\ i\ of\ A).(col\ j\ of\ B)}$

The result: $\sum a_{ik}b_{kj}$

$$\underset{4\ by\ 5}{\begin{bmatrix} * & * & & & \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{i5} \\ * & & & & \\ * & & & & \end{bmatrix}} \underset{5\ by\ 6}{\begin{bmatrix} * & * & b_{1j} & * & * & * \\ & & b_{2j} & & & \\ & & \vdots & & & \\ & & \vdots & & & \\ & & b_{5j} & & & \end{bmatrix}} = \underset{4\ by\ 6}{\begin{bmatrix} & & * & & & \\ * & * & (AB)_{ij} & * & * & * \\ & & * & & & \\ & & * & & & \end{bmatrix}}$$

$$\boldsymbol{AB} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$2x2 \qquad 2x2 \quad \rightarrow \quad 2x2$$

$$a_{11} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & \_ \\ \_ & \_ \end{bmatrix}$$

$$a_{12} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & af+bh \\ \_ & \_ \end{bmatrix}$$

$$a_{21} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ ce+dg & \_ \end{bmatrix}$$

$$a_{22} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ \_ & cf+dh \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

1.3−7 *Example*

Find: $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}$

Solution

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} 1(5)+1(1) & 1(6)+1(0) \\ 2(5)-1(1) & 2(6)-1(0) \end{bmatrix} \\ &= \begin{bmatrix} 6 & 6 \\ 9 & 12 \end{bmatrix} \end{aligned}$$

1.3−8 *Special Case*

When A is a square matrix, then

$$A \quad \text{times} \quad A = A^2 \quad \text{times} \quad A \quad \underline{= A^3}$$

$$A^p = AA \cdots A \qquad (p \text{ factors})$$

$$\left(A^p\right)\left(A^q\right) = A^{p+q}$$

$$\left(A^p\right)^q = A^{pq}$$

1.3−9 Block Multiplication

If the cuts between columns of ***A*** match the cuts between rows of ***B***, then the block multiplication of ***AB*** allowed.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \left[\begin{array}{c|c} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{21} + a_{12}b_{22} \\ \hline a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{21} + a_{22}b_{22} \end{array}\right]$$

1.3−10 Important special case

$$\begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix}$$

1.3−11 Matrix Form of the Equations

The coefficient matrix is $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}$

The equivalent matrix equation is in the form $AX = b$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Multiplication by ***rows***

$$AX = \begin{bmatrix} (row\ 1).X \\ (row\ 2).X \\ (row\ 3).X \end{bmatrix}$$

Multiplication by ***columns***

$$AX = x\ (column\ 1) + y\ (column\ 2) + z\ (column\ 3)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

1.3−12 Identity Matrix

The identity matrix is given by the form: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow \quad IM = MI = M$

1.3−13 Properties of Matrix

Addition and Scalar Multiplication

$A + B = B + A$	*Commutative Property of Addition*
$A + (B + C) = (A + B) + C$	*Associative Property of Addition*
$(kl)A = k(lA)$	*Associative Property of Scalar Multiplication*
$k(A + B) = kA + kB$	*Distributive Property*
$k(A - B) = kA - kB$	*Distributive Property*
$(k + l)A = kA + lA$	*Distributive Property*
$(k - l)A = kA - lA$	*Distributive Property*
$A + 0 = 0 + A = A$	*Additive Identity Property*
$A + (-A) = (-A) + A = 0$	*Additive Inverse Property*
$k(AB) = kA(B) = A(kB)$	

Multiplication

$AB \neq BA$	*Commutative "**law**" is usually broken*
$A(BC) = (AB)C$	*Associative Property of Multiplication* (***Parentheses not needed***)
$A(B + C) = AB + AC$	*Distributive Property*
$(B + C)A = BA + CA$	*Distributive Property*
$A(B - C) = AB - AC$	*Distributive Property*
$(B - C)A = BA - CA$	*Distributive Property*

Proof $(kl)A = k(lA)$

Let $A = \left[a_{ij}\right]$ and k and l are scalars.

$$\begin{aligned}(kl)A &= kl\left[a_{ij}\right]\\ &= k\left[l\,a_{ij}\right]\\ &= k\left(l\left[a_{ij}\right]\right)\\ &= k(lA) \quad \checkmark\end{aligned}$$

Proof $k(A+B) = kA + kB$

Let $A = \left[a_{ij}\right]$ & $B = \left[b_{ij}\right]$ and k is a scalar.

$$\begin{aligned}k(A+B) &= k\left(\left[a_{ij}\right]+\left[b_{ij}\right]\right)\\ &= k\left[a_{ij}\right]+k\left[b_{ij}\right]\\ &= \left[k\,a_{ij}\right]+\left[k\,b_{ij}\right]\\ &= kA+kB \quad \checkmark\end{aligned}$$

1.3−14 Cyclic Difference

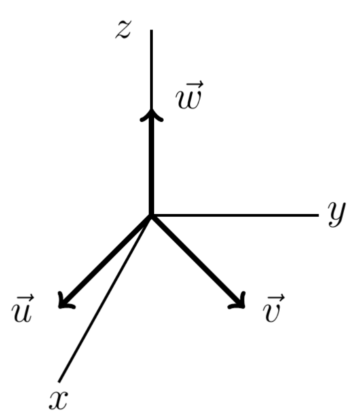
The linear combinations of three vectors $\vec{u},\ \vec{v},\ and\ \vec{w}*$ lead to a cyclic difference matrix C and is given by:

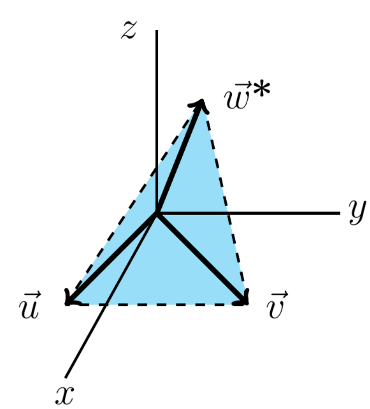
$$\vec{u}=\begin{bmatrix}1\\-1\\0\end{bmatrix} \qquad \vec{v}=\begin{bmatrix}0\\1\\-1\end{bmatrix} \qquad \vec{w}*=\begin{bmatrix}-1\\0\\1\end{bmatrix} \qquad \vec{w}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\begin{aligned} Cx &= \begin{bmatrix}1 & 0 & -1\\ -1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \\ &= \begin{bmatrix}x_1-x_3\\x_2-x_1\\x_3-x_2\end{bmatrix} \\ &= b \end{aligned}$$

The matrix C is not triangular. It is not easy to find the solution to $Cx=b$, because either we are going to have ***infinitely many solution*** or ***no solution***.

Let looks at these problems geometrically.





Exercises Section 1.3 – The Algebra of Matrices

1. For the matrices: $A=\begin{bmatrix} p & 0 \\ q & r \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, when does $AB=BA$

(2 – 9) Find values for the variables so that the matrices are equal.

2. $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix}=\begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

3. $\begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix}=\begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$

4. $\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix}=\begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$

5. $\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix}+\begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix}=\begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$

6. $\begin{bmatrix} a+11 & 12z+1 & 5m \\ 11k & 3 & 1 \end{bmatrix}+\begin{bmatrix} 9a & 9z & 4m \\ 12k & 5 & 3 \end{bmatrix}=\begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$

7. $\begin{bmatrix} x+2 & 3y+1 & 5z \\ 8w & 2 & 3 \end{bmatrix}+\begin{bmatrix} 3x & 2y & 5z \\ 2w & 5 & -5 \end{bmatrix}=\begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$

8. $\begin{bmatrix} 2x-3 & y-2 & 2z+1 \\ 5 & 2w & 7 \end{bmatrix}+\begin{bmatrix} 3x-3 & y+2 & z-1 \\ -5 & 5w+1 & 3 \end{bmatrix}=\begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$

9. $\begin{bmatrix} 16 & 4 & 5 & 4 \\ -3 & 13 & 15 & 6 \\ 0 & 2 & 4 & 0 \end{bmatrix}=\begin{bmatrix} 16 & 4 & 2x+1 & 4 \\ -3 & 13 & 16 & 3x \\ 0 & 2 & 3y-5 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 6 & 3 \\ 19 & 2 \end{bmatrix}$

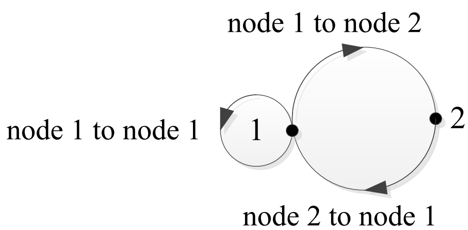
11. Find a combination $x_1w_1+x_2w_2+x_3w_3$ that gives the zero vector:

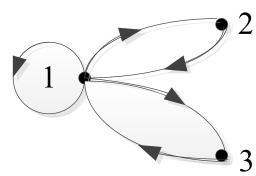
$$w_1=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix};\quad w_2=\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix};\quad w_3=\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are independent or dependent?
The vectors lie in a ______.
The matrix W with those columns is not invertible.

12. The very last words say that the 5 by 5 centered difference matrix is not invertible, Write down the 5 equations $Cx = b$. Find a combination of left sides that gives zero. What combination of b_1, b_2, b_3, b_4, b_5 must be zero?

13. A direct graph starts with n nodes. There are n^2 possible edges, each edge leaves one of the n nodes and enters one of the n nodes (possibly itself). The n by n adjacency matrix has $a_{ij} = 1$ when edge leaves node i and enter node j; if no edge then $a_{ij} = 0$. Here are directed graphs and their adjacency matrices:





$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The i, j entry of A^2 is $a_{i1}a_{1j} + \ldots + a_{in}a_{nj}$.

Why does that sum count the two-step paths from i to any node to j?

The i, j entry of A^k counts k-steps paths:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \textit{counts the paths with two edges} \quad \begin{bmatrix} 1 \textit{ to } 2 \textit{ to } 1, 1 \textit{ to } 1 \textit{ to } 1 & 1 \textit{ to } 1 \textit{ to } 2 \\ 2 \textit{ to } 1 \textit{ to } 1 & 2 \textit{ to } 1 \textit{ to } 2 \end{bmatrix}$$

List all 3-step paths between each pair of nodes and compare with A^3. When A^k has ***no zeros***, that number k is the diameter of the graph – the number of edges needed to connect the most pair of nodes. What is the diameter of the second graph?

14. A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

a) *AB* *b*) *BA* *c*) *ABD* *d*) *DBA*

e) *ABC* *f*) *ABCD* *g*) *A*(*B* + *C*)

15. What rows or columns or matrices do you multiply to find.

a) The third column of *AB*?
b) The second column of *AB*?
c) The first row of *AB*?
d) The second row of *AB*?

e) The entry in row 3, column 4 of AB?

f) The entry in row 2, column 3 of AB?

16. Add AB to AC and compare with $A(B+C)$:

$$A=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix} \quad \textit{and} \quad B=\begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix} \quad \textit{and} \quad C=\begin{bmatrix}3 & 1\\ 0 & 0\end{bmatrix}$$

17. True or False

a) If A^2 is defined then A is necessarily square.

b) If AB and BA are defined then A and B are squares.

c) If AB and BA are defined then AB and BA are squares.

d) If $AB=B$, then $A=I$

18. *a*) Find a nonzero matrix A such that $A^2=0$

b) Find a matrix that has $A^2 \neq 0$ but $A^3=0$

19. Suppose you solve $Ax=b$ for three special right sides b:

$$Ax_1=\begin{bmatrix}1\\0\\0\end{bmatrix} \quad Ax_2=\begin{bmatrix}0\\1\\0\end{bmatrix} \quad Ax_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

If the three solutions x_1, x_2, x_3 are the columns of a matrix X, what is A times X?

20. Show that $(A+B)^2$ is different from $A^2+2AB+B^2$, when

$$A=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix} \quad \textit{and} \quad B=\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}$$

Write down the correct rule for $(A+B)(A+B)=A^2+$ ________ $+B^2$

21. Let $A=\begin{pmatrix}1 & -1\\ 0 & 2\end{pmatrix} \quad B=\begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}$

Show the computations that

a) $(A+B)(A+B) \neq A^2+2AB+B^2$

b) $(A+B)(A-B) \neq A^2-B^2$

c) $(A+B)(A+B)=A^2+AB+BA+B^2$

(**22 – 25**) Find the product of the 2 matrices by rows or by columns:

22. $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

23. $\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

25. $\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

26. Given $A = \begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$ $B = \begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix}$

Find $A+B,\quad 2A,\quad and\quad -B$

(**27 – 40**) Find AB and BA, if possible

27. $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ $B = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}$

28. $A = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$ $B = \begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}$

29. $A = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$ $B = \begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}$

30. $A = \begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix}$

31. $A = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$

32. $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$

33. $A = \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$

34. $A = \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix}$

35. $A = \begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

36. $A = \begin{bmatrix} 5 & 3 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 4 & -2 \\ -2 & 0 \\ 9 & 1 \end{bmatrix}$

37. $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$

38. $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$

39. $A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix}$ $B = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

40. $A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}$ $B = \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}$

41. Consider the matrices

$$A=\begin{bmatrix}3&0\\-1&2\\1&1\end{bmatrix}\quad B=\begin{bmatrix}4&-1\\0&2\end{bmatrix}\quad C=\begin{bmatrix}1&4&2\\3&1&5\end{bmatrix}\quad D=\begin{bmatrix}1&5&2\\-1&0&1\\3&2&4\end{bmatrix}\quad E=\begin{bmatrix}6&1&3\\-1&1&2\\4&1&3\end{bmatrix}$$

Compute the following (where possible):

$a)\ D+E$ $\quad b)\ D-E$ $\quad c)\ 5A$

$d)\ -7C$ $\quad e)\ 2B-C$ $\quad f)\ -3(D+2E)$

42. Consider the matrices

$$A=\begin{bmatrix}3&0\\-1&2\\1&1\end{bmatrix}\quad B=\begin{bmatrix}1&5&2\\-1&1&0\\-4&1&3\end{bmatrix}\quad C=\begin{bmatrix}-3&-1\\2&1\\4&3\end{bmatrix}\quad D=\begin{bmatrix}4&-1\\2&0\end{bmatrix}$$

Compute the following (where possible):

$a)\ A+B$ $\quad b)\ A+C$ $\quad c)\ AB$ $\quad d)\ BA$ $\quad e)\ CD$ $\quad f)\ DC$

$g)\ BD$ $\quad h)\ DB$ $\quad i)\ A^2$ $\quad j)\ B^2$ $\quad k)\ D^2$

(**42 – 45**) Find if possible, $a)\ A+B \quad b)\ A-B \quad c)\ 2A \quad d)\ 2A-B \quad e)\ B+\frac{1}{2}A \quad f)\ AB \quad g)\ BA$

43. $A=\begin{pmatrix}1&2\\2&1\end{pmatrix}\quad B=\begin{pmatrix}-3&-2\\4&2\end{pmatrix}$

44. $A=\begin{pmatrix}1&2\\4&2\end{pmatrix}\quad B=\begin{pmatrix}2&-1\\-1&8\end{pmatrix}$

45. $A=\begin{pmatrix}6&0&3\\-1&-4&0\end{pmatrix}\quad B=\begin{pmatrix}8&-1\\4&-3\end{pmatrix}$

46. $A=\begin{pmatrix}2&1\\-3&4\\1&6\end{pmatrix}\quad B=\begin{pmatrix}0&-1&0\\4&0&2\\8&-1&7\end{pmatrix}$

47. Consider the matrices

$$A=\begin{pmatrix}x^2&x\\-3&2\end{pmatrix}\quad B=\begin{pmatrix}x&x^2\\4&x\end{pmatrix}$$

Compute the following (where possible):

$a)\ A+B$ $\quad b)\ A-B$ $\quad c)\ AB$ $\quad d)\ BA$ $\quad e)\ 2A-3B$

48. Consider the matrices

$$A=\begin{pmatrix}1&x&-2\\3&1&1\\0&-2&2\end{pmatrix}\quad B=\begin{pmatrix}2&x&1\\-3&1&0\\2&1&4\end{pmatrix}$$

Compute the following (where possible):

$a)\ A+B$ $\quad b)\ A-B$ $\quad c)\ AB$ $\quad d)\ BA$ $\quad e)\ 2A-3B$

49. Let $B=\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$, show that $B^4=\begin{pmatrix} a^4 & 0 \\ a^3+a^2b+ab^2+b^3 & b^4 \end{pmatrix}$

50. Let $B=\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$, show that $B^n=\begin{pmatrix} a^n & 0 \\ \sum_{k=0}^{n-1} a^{n-1-k}b^k & b^n \end{pmatrix}$

51. Let $A=\begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix}$. Prove that $A^n=\begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$ if $n\ge 1$

52. Let $A=\begin{bmatrix} 2a & -a^2 \\ 1 & 0 \end{bmatrix}$. Prove that $A^n=\begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix}$ if $n\ge 1$

53. The following system of recurrence relations holds for all $n\ge 0$

$$\begin{cases} x_{n+1}=7x_n+4y_n \\ y_{n+1}=-9x_n-5y_n \end{cases}$$

Solve the system for x_n and y_n in terms of x_0 and y_0

54. If $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, prove that $A^2-(a+d)A+(ad-bc)I_{2\times 2}=0$

55. If $A=\begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}$, use the fact $A^2=4A-3I$ and mathematical induction, to prove that

$$A^n=\frac{3^n-1}{2}A+\frac{3-3^n}{2}I \quad if \quad n\ge 1$$

56. A sequence of numbers $x_1, x_2, \ldots, x_n, \ldots$ satisfies the recurrence relation $x_{n+1}=ax_n+bx_{n-1}$ for $n\ge 1$, where *a* and *b* are constants. Prove that

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}=A\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

Where $A=\begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$ and hence express $\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$ in terms of $\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$.

If $a=4$ and $b=-3$, use the previous question to find a formula for x_n in terms x_1 and x_0

57. Prove $(A+B)C = AC + BC$

58. Prove $(A+B)+C = A+(B+C)$

59. Prove $A(BC)=(AB)C$

60. Prove $A(B-C)= AB - AC$

61. Prove that $(A+B)(A-B) \neq A^2 - B^2$

62. Prove that $(A+B)(A+B) \neq A^2 + 2AB + B^2$

63. Show that $AC = BC$, even though $A \neq B$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}; \quad C = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$$

64. Verify $AB = BA$ for the matrices below

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \qquad B = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

65. Solve for A:

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

66. Solve for X in the equation $3X + 2A = B$, given

$$A = \begin{pmatrix} -4 & 0 \\ 1 & -5 \\ -3 & 2 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 4 \end{pmatrix}$$

67. For

$$A = \begin{pmatrix} 4 & 1 & 3 \\ 0 & 5 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -2 & -4 \\ 3 & 0 & 1 \end{pmatrix}$$

Solve the equations for X.

a) $2X = 3A - B$

b) $3X - 2A + B = 0$

c) $-2X = -5A + 3B$

d) $4X + 5A - 3B = 0$

68. For

$$A=\begin{pmatrix}-2 & -1\\ 1 & 0\\ 3 & -4\end{pmatrix},\quad B=\begin{pmatrix}1 & 2\\ -2 & 1\\ 4 & 3\end{pmatrix}$$

Solve the equations for X.

a) $X=3A-2B$

b) $3X+2A=B$

c) $-2X=-4A+3B$

d) $2X+3A-4B=0$

69. For

$$A=\begin{pmatrix}2 & 2 & 2\\ 2 & 1 & -3\\ 1 & 0 & 4\end{pmatrix},\quad B=\begin{pmatrix}3 & 3 & 3\\ 3 & 0 & 5\\ 6 & 9 & -1\end{pmatrix},\quad C=\begin{pmatrix}4 & 4 & 4\\ 5 & -1 & 0\\ 7 & 8 & -1\end{pmatrix}$$

Solve the equations for X in few steps as you dare

e) $\frac{1}{2}(X+A)=3\left[X+(2X+B)\right]+C$

f) $2(X+B)=3\left(X+\left(\frac{1}{2}X+A\right)\right)+C$

g) $40(X+A)=47(X+B)+48(X+C)$

h) $\sqrt{2}(X+C)=31\left(X+\sqrt{2}(X+A-B)\right)$

(70 – 76) If f is the polynomial function is given by $f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$.

Then $f(A)$ is defined by $f(A)=a_0I+a_1A+a_2A^2+\cdots+a_nA^n$

Find $f(A)$

70. $f(x)=x^2+3x-2,\qquad A=\begin{pmatrix}2 & 3\\ 0 & 1\end{pmatrix}$

71. $f(x)=x^2-2x+3,\qquad A=\begin{pmatrix}1 & 3\\ 3 & 2\end{pmatrix}$

72. $f(x)=2x^2-3x+5,\qquad A=\begin{pmatrix}1 & 2\\ 3 & -4\end{pmatrix}$

73. $g(x)=x^2+3x-10,\qquad A=\begin{pmatrix}1 & 2\\ 3 & -4\end{pmatrix}$

74. $f(x)=x^2-2x-5,\qquad A=\begin{pmatrix}2 & -5 & 8\\ 3 & -6 & -7\\ 4 & 0 & -1\end{pmatrix}$

75. $g(x) = x^2 - 3x + 15,\qquad A = \begin{pmatrix} 2 & -5 & 8 \\ 3 & -6 & -7 \\ 4 & 0 & -1 \end{pmatrix}$

76. $f(x) = x^3 - 2x^2 - 3x + 5,\qquad A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{pmatrix}$

Section 1.4 – Inverse Matrices

1.4−1 *Definition*

The matrix A is invertible if there exists a matrix A^{-1} such that:

$$A^{-1}A = AA^{-1} = I$$

where A^{-1} *read as* "A ***inverse***" and A has to be a ***square matrix***.

Not all matrices have inverses.

1. The inverse exists *iff* elimination produces n pivots (row exchanges allow).
2. The matrix A cannot have two different inverses.
3. If A is invertible, the one and only one solution to $Ax = B$ is $x = A^{-1}B$

$$AX = B$$

$$A^{-1}(AX) = A^{-1}B \qquad \textit{Multiply both side by } A^{-1}$$

$$\left(A^{-1}A\right)X = A^{-1}B \qquad \textit{Associate property}$$

$$IX = A^{-1}B \qquad \textit{Multiplicative inverse property}$$

$$X = A^{-1}B \qquad \textit{Identity property}$$

4. Suppose there is a ***nonzero*** vector x such that $Ax = 0$. Then A cannot have an inverse

5. A 2 by 2 matrix is invertible iff $ad - bc$ is not zero.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \textit{Only for 2 by 2 matrices}$$

If $ad - bc = 0$ is the determinant, then A^{-1} doesn't exist

- ✓ Interchange the two elements on the main diagonal.
- ✓ Opposite sign for the other diagonal

1.4−2 The Inverse of a Product AB

1.4−3 *Theorem*

If an $n \times n$ matrix has an inverse, that inverse is unique.

Proof

Suppose that A has an inverse A^{-1} and B is a matrix such that $BA = I$

$$\begin{aligned} B &= BI \\ &= B\left(AA^{-1}\right) \\ &= (BA)A^{-1} \\ &= IA^{-1} \\ &= A^{-1} \end{aligned}$$

Therefore, the inverse is unique.

1.4−4 *Theorem*

If A and B are invertible then so is AB.

The inverse of a product AB is

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof

$$\begin{aligned} (AB)\left(B^{-1}A^{-1}\right) &= A\left(BB^{-1}\right)A^{-1} \\ &= (AI)A^{-1} \\ &= AA^{-1} \\ &= I \end{aligned}$$

1.4−5 *Theorem*

If A is invertible and n is a nonnegative integer, then:

a) A^{-1} is *invertible* and $\left(A^{-1}\right)^{-1} = A$

b) A^{n} is *invertible* and $\left(A^{n}\right)^{-1} = A^{-n} = \left(A^{-1}\right)^{n}$

c) kA is *invertible* for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$

Proof

$$\begin{aligned}(kA)\left(k^{-1}A^{-1}\right) &= k^{-1}(kA)A^{-1} \\ &= \left(k^{-1}k\right)AA^{-1} \\ &= (1)I \\ &= I\end{aligned}$$

$$\begin{aligned}\left(k^{-1}A^{-1}\right)(kA) &= k^{-1}\left(kA^{-1}\right)A \\ &= \left(k^{-1}k\right)A^{-1}A \\ &= (1)I \\ &= I\end{aligned}$$

1.4−6 Finding A^{-1} using Gauss-Jordan Elimination

$$\left[A \middle| I\right] \rightarrow \left[I \middle| A^{-1}\right]$$

Find A^{-1} *if* $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccccc} 2 & -2 & -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -2 & 0 & 0 \\ \hline 0 & -2 & -3 & -2 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 3 & 0 & 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & -3 & 0 & 0 \\ \hline 0 & 0 & -3 & -3 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] -\tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] -\tfrac{1}{3}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - \frac{3}{2}R_3 \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & \frac{1}{3} \\ \hline 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{array} \qquad \begin{array}{cccccc} 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2} \\ \hline 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

Notation

- ✓ Matrix ***A*** is ***symmetric*** across its main diagonal. So is A^{-1}
- ✓ Matrix ***A*** is ***tridiagonal*** (only three nonzero diagonals). But A^{-1} is a full matrix with no zeros. (another reason we don't compute A^{-1})

1.4−7 Singular *versus* Invertible

A^{-1} exists when A has a full set of n pivots. (Row exchanges allowed)

- With n pivots, elimination solves all the equations $Ax_i = b_i$. The columns x_i go into A^{-1}. Then $AA^{-1} = I$ is at least a ***right-inverse***.
- Elimination is really a sequence of multiplications.

1.4−8 Conclusion

- If A doesn't have n pivots, elimination will lead to a ***zero row***.
- Elimination steps are taken by an invertible M. So a row of MA is zero.
- If $AB = I$ then $MAB = M$. The zero row of MA, times B, gives a zero row of M.
- The invertible matrix M can't have a zero row! A must have n pivots if $AB = I$.

1.4−9 *Elementary Matrices*

1.4−10 *Definition*

Let ***e*** be an elementary row operation. Then the $n \times n$ ***elementary matrix*** E associated with ***e*** is the matrix obtained by applying ***e*** to the $n \times n$ identity matrix. Thus

$$E = eI$$

1.4−11 *Example*

a) $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \to$ *Multiply* R_2 *of* I *by* -3

b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to$ *Multiply the third row by* -5

c) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ *Interchange the first and second rows*

d) $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ *Add* -3 *times* R_1 *to* R_2

1.4−12 *Theorem*

Let e be an elementary operation and let E be the corresponding elementary matrix $E = e(I)$. Then for every $m \times n$ matrix A

$$e(A) = EA$$

That is, an elementary row operation can be performed on A by multiplying A on the left by the corresponding elementary matrix.

1.4−13 *Example* $m \times m$

Let $A = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$ $M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

$$\begin{aligned} MA &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -4 & 6 & 2 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} \end{aligned}$$

This result can be obtained from A by multiplying the first row by 2.

$$\begin{aligned} PA &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -2 & 3 & 1 \\ -6 & 2 & 1 & 5 \\ 1 & 0 & 2 & 4 \end{bmatrix} \end{aligned}$$

This result can be obtained from A by interchanging rows 2 and 3.

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ 0 & -4 & 10 & 8 \end{bmatrix}$$

This result can be obtained from A by adding 3 times row 1 to row 3.

1.4−14 Uniqueness of Echelon Form

Two matrices A and B are row-equivalent if and only if they have the same reduced echelon form.

Proof

If A and B have the same reduced echelon form E, then A is row-equivalent to E and E is row-equivalent to B.
It follows that A is row-equivalent to B.
Now Suppose A and B are row-equivalent.
Let E_1 be a reduced echelon form of A and E_2 be a reduced echelon form of B. Then E_1 and E_2 are row equivalent.
Suppose $E_1 = IF_1 \quad and \quad E_2 = IF_2$.
Since E_1 and E_2 are row equivalent, $E_2 = CE_1$ for some matrix C. This means $I = CI$ and $F_2 = CF_1$.
But then $C = I \quad and \quad F_2 = F_1$.

1.4−15 *Example*

Show that the two matrices are row equivalent

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

Solution

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{matrix} R_1 + R_2 \\ R_2 - 2R_1 \end{matrix}$$

$$= \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

$= B$

1.4−16 *Definition*

A relationship ~ (equivalent) between elements of a set is called an equivalence relation if

- ✓ $A \sim A$ is always true,
- ✓ $A \sim B$ always implies $B \sim A$,
- ✓ $A \sim B$ and $B \sim C$ always implies $A \sim C$.

Exercises *Section* 1.4 – Inverse Matrices - Finding A^{-1}

1. Apply Gauss-Jordan method to find the inverse of this triangular "Pascal matrix"

Triangular Pascal matrix $$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

2. If A is invertible and $AB = AC$, prove that $B = C$

3. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find two matrices $B \neq C$ such that $AB = AC$

4. If A has ***row*** 1 + ***row*** 2 = ***row*** 3, show that A is not invertible

a) Explain why $Ax = (1,\ 0,\ 0)$ can't have a solution.

b) Which right sides $(b_1,\ b_2,\ b_3)$ might allow a solution to $Ax = b$

c) What happens to ***row*** 3 in elimination?

5. True or false (with a counterexample if false and a reason if true):

a) A 4 by 4 matrix with a row of zeros is not invertible.
b) A matrix with 1's down the main diagonal is invertible.
c) If A is invertible then A^{-1} is invertible.
d) If A is invertible then A^2 is invertible.

6. Do there exist 2 by 2 matrices A and B with real entries such that $AB - BA = I$, where I is the identity matrix?

7. If B is the inverse of A^2, show that AB is the inverse of A.

8. Find and check the inverses (assuming they exist) of these block matrices.

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}$$

9. For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

10. Find A^{-1} and B^{-1} (if they exist) by elimination.

$$A=\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\qquad B=\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}$$

11. Find A^{-1} using the Gauss-Jordan method, which has a remarkable inverse.

$$A=\begin{bmatrix}1&-1&1&-1\\0&1&-1&1\\0&0&1&-1\\0&0&0&1\end{bmatrix}$$

(**12 – 55**) Find the inverse, if exists, of

12. $\begin{bmatrix}6&-4\\-3&2\end{bmatrix}$

13. $\begin{bmatrix}1&4\\2&7\end{bmatrix}$

14. $\begin{bmatrix}-3&6\\4&5\end{bmatrix}$

15. $A=\begin{bmatrix}2&-6\\1&-2\end{bmatrix}$

16. $A=\begin{bmatrix}10&-2\\-5&1\end{bmatrix}$

17. $A=\begin{bmatrix}-2&3\\-3&4\end{bmatrix}$

18. $A=\begin{bmatrix}a&b\\3&3\end{bmatrix}$

19. $A=\begin{bmatrix}-2&a\\4&a\end{bmatrix}$

20. $A=\begin{bmatrix}4&4\\b&a\end{bmatrix}$

21. $A=\begin{pmatrix}-1&2\\-2&1\end{pmatrix}$

22. $A=\begin{pmatrix}1&-2\\2&-1\end{pmatrix}$

23. $A=\begin{pmatrix}2&4\\3&-1\end{pmatrix}$

24. $A=\begin{pmatrix}-1&3\\2&-1\end{pmatrix}$

25. $A=\begin{pmatrix}1&3\\-2&5\end{pmatrix}$

26. $A=\begin{pmatrix}4&6\\2&3\end{pmatrix}$

27. $A=\begin{pmatrix}-6&9\\2&-3\end{pmatrix}$

28. $A=\begin{pmatrix}-2&7\\0&2\end{pmatrix}$

29. $A=\begin{pmatrix}4&-16\\1&-4\end{pmatrix}$

30. $A=\begin{pmatrix}2&1\\1&1\end{pmatrix}$

31. $A=\begin{pmatrix}2&1\\a&a\end{pmatrix}$

32. $A=\begin{pmatrix}b&3\\b&2\end{pmatrix}$

33. $A=\begin{pmatrix}1&a\\3&a\end{pmatrix}$

34. $A=\begin{pmatrix}a&2\\2&a\end{pmatrix}$

35. $A=\begin{pmatrix}4&-2\\2&-1\end{pmatrix}$

36. $A=\begin{pmatrix}-3&\frac{1}{2}\\6&-1\end{pmatrix}$

37. $A=\begin{pmatrix}-1&1\\3&-3\end{pmatrix}$

38. $A=\begin{pmatrix}\sin\theta&\cos\theta\\-\cos\theta&\sin\theta\end{pmatrix}$

39. $A=\begin{bmatrix}1&0&1\\2&-2&-1\\3&0&0\end{bmatrix}$

40. $A=\begin{bmatrix}1&2&-1\\3&5&3\\2&4&3\end{bmatrix}$

41. $A=\begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

42. $A=\begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$

43. $A=\begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{pmatrix}$

44. $A=\begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$

45. $A=\begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

46. $A=\begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 1 & 2 \end{pmatrix}$

47. $A=\begin{pmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$

48. $A=\begin{pmatrix} -3 & 1 & -1 \\ 1 & -4 & -7 \\ 1 & 2 & 5 \end{pmatrix}$

49. $A=\begin{pmatrix} 1 & 1 & -3 \\ 2 & -4 & 1 \\ -5 & 7 & 1 \end{pmatrix}$

50. $A=\begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{pmatrix}$

51. $A=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

52. $A=\begin{pmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{pmatrix}$

53. $A=\begin{pmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

54. $A=\begin{pmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{pmatrix}$

55. $A=\begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -6 & 1 \\ -2 & -2 & 5 & 1 \end{bmatrix}$

56. $A=\begin{bmatrix} 1 & -14 & 7 & 38 \\ -1 & 2 & 1 & -2 \\ 1 & 2 & -1 & -6 \\ 1 & -2 & 3 & 6 \end{bmatrix}$

57. $A=\begin{bmatrix} 10 & 20 & -30 & 15 \\ 3 & -7 & 14 & -8 \\ -7 & -2 & -1 & 2 \\ 4 & 4 & -3 & 1 \end{bmatrix}$

58. Show that A is not invertible for any values of the entries

$$A=\begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

59. Prove that if A is an invertible matrix and B is row equivalent to A, then B is also invertible.

60. Determine if the given matrix has an inverse and find the inverse if it exists. Check your answer by multiplying $A \cdot A^{-1} = I$

a) $\begin{bmatrix} 2 & 3 \\ -3 & -5 \end{bmatrix}$ *b)* $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}$

61. Show that the inverse of $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is $\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$

62. If the product $C = AB$ is invertible (and A & B are square matrices), find a formula for A^{-1} that involves C^{-1} and B.

Hence, it is not possible to multiply a non-invertible matrix by another matric and obtain an invertible matrix as a result.

63. Prove that if A is an $m \times n$ matrix, there is an invertible matrix C such that CA is in reduced row-echelon form.

64. Prove that 2 $m \times n$ matrices A and B are row equivalent if and only if there exists a nonsingular matrix P such that $B = PA$

65. Let A and B be 2 $m \times n$ matrices. Suppose A is row equivalent to B. Prove that A is nonsingular if and only if B is nonsingular.

66. Show that if A and B are two $n \times n$ invertible matrices then A is row equivalent to B.

67. Prove that a square matrix A is nonsingular if and only if A is a product of elementary matrices.

68. Show that if $A \sim B$ (that is, if they are row equivalent), then $EA = B$ for some matrix E which is a product of elementary matrices.

69. Show that if $EA = B$ for some matrix E which is a product of elementary matrices, then $AC \sim BC$ for every $n \times n$ matrix C.

70. Let $A\vec{x}=0$ be a homogeneous system of *n* linear equations in *n* unknowns that has only the trivial solution. Show that of *k* is any positive integer, then the system $A^k\vec{x}=0$ also has only trivial solution.

71. Let $A\vec{x}=0$ be a homogeneous system of *n* linear equations in *n* unknowns, and let *Q* be an invertible $n\times n$ matrix. Show that $A\vec{x}=0$ has just trivial solution if and only if $(QA)\vec{x}=0$ has just trivial solution.

72. Let $A\vec{x}=b$ be any consistent system of linear equations, and let $\vec{x}_1$ be a fixed solution. Show that every solution to the system can be written in the form $\vec{x}=\vec{x}_1+\vec{x}_0$ where $\vec{x}_0$ is a solution to $A\vec{x}=0$. Show also that every matrix of this form is a solution.

73. If *A* and *B* are $n\times n$ matrices satisfying $A^2=B^2=(AB)^2=I_n$. Prove that $AB=BA$.

74. Let $A=\begin{pmatrix}0&1&0\\0&0&1\\5&0&0\end{pmatrix}$. Verify that $A^3=5I$, then find A^{-1} in term of *A*.

75. Consider $B(A,\ I)=(BA,\ B)$, thus if *B* is the inverse of *A*, then $(BA,\ B)$ becomes $\left(I,\ A^{-1}\right)$. On the other hand *B* is a product of elementary matrices since it is invertible. This indicates that the inverse of *A* can be obtained by applying elementary row operations to $(A,\ I)$ to get $\left(I,\ A^{-1}\right)$.

Find the inverses of

a) $A=\begin{pmatrix}1&0&0\\0&1&0\\a&b&1\end{pmatrix}$

b) $B=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\a&b&c&d\end{pmatrix}$

76. Let $A,\ B,\ C,\ X,\ Y,\ Z\in M_n(\mathbb{C})$, *A* and *C* are invertible. Find

a) $\begin{pmatrix}A&B\\0&C\end{pmatrix}^{-1}$

b) $\begin{pmatrix}I&X&Y\\0&I&Z\\0&0&I\end{pmatrix}^{-1}$

77. Suppose that *A, B*, and $A-B$ are invertible $n\times n$ matrices. Show that

$$(A-B)^{-1}=A^{-1}+A^{-1}\left(B^{-1}-A^{-1}\right)^{-1}A^{-1}$$

78. Suppose P is invertible and $A = PBP^{-1}$. Solve for B in terms of A.

79. Suppose $(A-B)C = 0$, where A and B are $m \times n$ matrices and C is invertible. Show that $A = B$.

80. Prove $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ where A, B, and C are invertible.

81. Prove $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$ where A, B, C and D are invertible.

82. Prove that if $A^2 = A$, then $I - 2A = (I - 2A)^{-1}$

83. Let $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

a) Show that $A^2 - 2A + 5I = 0$

b) Show that $A^{-1} = \frac{1}{5}(2I - A)$

c) Show that for any square matrix satisfying $A^2 - 2A + 5I = 0$, the inverse of A is $A^{-1} = \frac{1}{5}(2I - A)$

84. find the inverse of the 3×3 Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{pmatrix}$$

When $a_1,\ a_2,\ and\ a_3$ are distinct from each other

Section 1.5 – Transpose, Diagonal, Triangular, and Symmetric Matrices

Transpose

1.5−1 *Definition*

The transpose of a matrix A is defined as the matrix that is reflected in its main diagonal, so that all columns become rows and all rows become columns with changing their relative order or the order of the entries in the columns and rows.

Then the transpose of A, denoted by A^T or A'.

The columns of A^T are the rows of A.

When A is an m by n matrix, the transpose is n by m:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \quad \text{then} \quad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

The matrix ***flips over*** the main diagonal. The entry in row i, column j of A^T comes from row j, column i of the original A.

$$\left(A^T\right)_{ij} = A_{ji}$$

1.5−2 Properties of Transpose

a) $\left(A^T\right)^T = A$

b) $(A+B)^T = A^T + B^T$

c) $(A-B)^T = A^T - B^T$

d) $(kA)^T = kA^T$

e) $(AB)^T = B^T A^T$

The transpose of a product of any number of matrices is the product of the transposes in the reverse order.

1.5−3 *Theorem*

If A is an invertible matrix, then A^T is also invertible and

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

Proof

$$A^T\left(A^{-1}\right)^T = \left(A^{-1}A\right)^T$$
$$= I^T$$
$$= I$$

$$\left(A^{-1}\right)^T A^T = \left(AA^{-1}\right)^T$$
$$= I^T$$
$$= I$$

****

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad and \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\left(A^T\right)^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$= \begin{bmatrix} \frac{d}{ad-bc} & -\frac{c}{ad-bc} \\ -\frac{b}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

Trace

1.5−4 *Definition*

If A is a square matrix, then the trace of A, denoted by **tr(A)**, is defined to the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$tr(A) = a_{11} + a_{22} + a_{33}$$

Diagonal

1.5−5 *Definition*

A square matrix in which all the entries off the main diagonal are ***zero*** is called a ***diagonal matrix***. A general $n \times n$ diagonal matrix can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

A diagonal matrix is invertible iff all of its diagonal entries are nonzero; the

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$$

Powers of diagonal matrices are:

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

Triangular Matrices

1.5−6 *Definition*

A square matrix in which all the entries above the main diagonal are zero is called ***lower diagonal triangular***.

A square matrix in which all the entries below the main diagonal are zero is called ***upper diagonal triangular***.

A matrix that is either upper triangular or lower triangular is called ***triangular***.

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ a_{51} & a_{51} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

lower diagonal triangular

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix}$$

upper diagonal triangular

1.5−7 *Theorem*

- ✓ The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.
- ✓ The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.
- ✓ A triangular matrix is invertible iff its diagonal entries are all nonzero.
- ✓ The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

Example

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

The goal is to describe Gaussian elimination in the most useful way by looking at them closely, which are factorizations of a matrix.

The factors are triangular matrices.

The factorization that comes from elimination is $A = LU$ **.**

Symmetric Matrices

1.5−8 *Definition*

A square matrix A is said to be ***symmetric*** if $A^T = A$. That means a square matrix must satisfies $a_{ij} = a_{ji}$

A square matrix A is said to be ***skew symmetric*** when $A^T = -A$

Example

$$A = \begin{pmatrix} 1 & -4 \\ -4 & 1 \end{pmatrix} = A^T$$

$$A = \begin{pmatrix} 6 & 5 & 1 \\ 5 & 0 & 7 \\ 1 & 7 & -1 \end{pmatrix} = A^T$$

- The ***inverse*** of a symmetric matrix is also ***symmetric***.

Example

Given $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, show that the inverse is symmetric too?

Solution

$$A^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

1.5–9 *Theorem*

If A and B are symmetric matrices with the same size, and if k is any scalar, then:

a) A^T is symmetric

b) $A+B$ and $A-B$ are symmetric.

c) kA is symmetric

- If A is an invertible symmetric matrix, then A^{-1} is *symmetric*.

Proof

Assume that A is symmetric and invertible then $A=A^T$

$$\left(A^{-1}\right)^T=\left(A^T\right)^{-1}=A^{-1}$$

Which proves that A^{-1} is *symmetric*

- Multiplying M by M^T gives a symmetric matrix.

Proof

The entry (i,j) of M^TM, it is the dot product of ***row*** *i* of M^T (column *i* of *M*) with column *j* of *M*.

The (i,j) entry is the same dot product, column *j* with column *i*. so M^TM is symmetric.

The matrix $M.M^T$ is also symmetric and M^TM is a different matrix from $M.M^T$.

- If A is an invertible symmetric matrix, then AA^T and A^TA are also invertible.
- Matrix A is symmetric across its main diagonal. So is A^{-1}
- Matrix A is tridiagonal (only three nonzero diagonals). But A^{-1} is a full matrix with no zeros. (another reason we don't compute A^{-1})

Example

Given $M=\begin{bmatrix}1 & 2\end{bmatrix}$ and $M^T=\begin{bmatrix}1\\2\end{bmatrix}$. Find M^TM and $M.M^T$

Solution

$$M^TM=\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1 & 2\end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$MM^T = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \end{bmatrix}$$

1.5−10 *Symmetric in* LDU

When elimination is applied to a symmetric matrix, $A^T = A$ is an advantage.

$$\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{U}$$

- If $A = A^T$ can be factored into LDU with no row exchanges, then $U = L^T$. The ***symmetric factorization of a symmetric matrix is*** $A = LDL^T$

Exercises *Section* 1.5 – Transpose, Diagonal, Triangular, and Symmetric Matrices

1. Consider the matrices

$$A=\begin{pmatrix}1 & -2\\ 3 & 5\end{pmatrix} \qquad B=\begin{pmatrix}-7 & -3\\ 6 & 4\end{pmatrix} \qquad \vec{v}=\begin{pmatrix}4\\ -1\end{pmatrix}$$

Find:

a) A^T

b) B^T

c) A^T+B^T

d) $(A+B)^T$

e) $(A\vec{v})^T$

f) $A^T\vec{v}^T$

g) $(AB)^T$

h) A^TB^T

i) B^TA^T

j) $(B\vec{v})^T$

k) $B^T\vec{v}^T$

l) $(2A)^T$

m) $2A^T$

2. Solve $L\vec{c}=b$ to find $\vec{c}$. Then solve $U\vec{x}=c$ to find $\vec{x}$. What was A?

$$L=\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{bmatrix} \quad U=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix} \quad b=\begin{bmatrix}4\\ 5\\ 6\end{bmatrix}$$

3. Find L and U for the symmetric matrix

$$A=\begin{bmatrix}a & a & a & a\\ a & b & b & b\\ a & b & c & c\\ a & b & c & d\end{bmatrix}$$

Find four conditions on a, b, c, d to get $A=LU$ with four pivots

4. Determine whether the given matrix is invertible

$$\begin{bmatrix}-1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & \frac{1}{3}\end{bmatrix}$$

(5 – 7) Find $A^2,\ A^{-2},\ and\ A^{-k}$

5. $A=\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}$

6. $A=\begin{bmatrix}\frac{1}{2} & 0 & 0\\ 0 & \frac{1}{3} & 0\\ 0 & 0 & \frac{1}{4}\end{bmatrix}$

7. $A=\begin{bmatrix}-2 & 0 & 0 & 0\\ 0 & -4 & 0 & 0\\ 0 & 0 & -3 & 0\\ 0 & 0 & 0 & 2\end{bmatrix}$

(8 – 9) Find A^3, *and* A^{-1}

8. $A = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$

9. $B = \begin{pmatrix} 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{5} \end{pmatrix}$

10. Find the power of A^{16} for the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

(11 – 13) Decide whether the given matrix is symmetric

11. $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

12. $\begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix}$

13. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

14. Find all values of the unknown constant(s) in order for A to be symmetric

$$A = \begin{bmatrix} 2 & a-2b+2c & 2a+b+c \\ 3 & 5 & a+c \\ 0 & -2 & 7 \end{bmatrix}$$

15. Find x and B, if the given matrix B is symmetric

$$B = \begin{pmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{pmatrix}$$

16. Find a diagonal matrix A that satisfies the given condition $A^{-2} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

17. Let A be an $n \times n$ symmetric matrix

a) Show that A^2 is symmetric.

b) Show that $2A^2 - 3A + I$ is symmetric.

c) Show that $\frac{1}{2}\left(A + A^T\right)$ is symmetric.

18. Prove if $A^T A = A$, then A is symmetric and $A = A^2$

19. A square matrix A is called ***skew-symmetric*** if $A^T = -A$. Prove

a) If A is an invertible skew-symmetric matrix, then A^{-1} is skew-symmetric.

b) If A and B are skew-symmetric matrices, then so are A^T, $A+B$, $A-B$, *and* kA for any scalar k.

c) Every square matrix A can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

$$\left[\textit{Hint}: \text{ Note the identity } A = \tfrac{1}{2}\left(A + A^T\right) + \tfrac{1}{2}\left(A - A^T\right)\right]$$

20. Suppose R is rectangular (m by n) and A is symmetric (m by m)

a) Transpose $R^T AR$ to show its symmetric

b) Show why $R^T R$ has no negative numbers on its diagonal.

21. If L is a lower-triangular matrix, then $\left(L^{-1}\right)^T$ is _______Triangular

22. True or False

a) The block matrix $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ is automatically symmetric

b) If A and B are symmetric then their product is symmetric

c) If A is not symmetric then A^{-1} is not symmetric

d) When A, B, C are symmetric, the transpose of ABC is CBA.

e) The transpose of a diagonal matrix is a diagonal.

f) The transpose of an upper triangular matrix is an upper triangular matrix.

g) The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix.

h) All entries of a symmetric matrix are determined by the entries occurring on and above the main diagonal.

i) All entries of an upper triangular matrix are determined by the entries occurring on and above the main diagonal.

j) The inverse of an invertible lower triangular matrix is an upper triangular matrix.

k) A diagonal matrix is invertible if and only if all of its diagonal entries are positive.

l) The sum of a diagonal matrix and a lower triangular matrix is a lower triangular matrix.

m) A matrix that is both symmetric and upper triangular must be a diagonal matrix.

n) If A and B are $n \times n$ matrices such that $A+B$ is symmetric, then A and B are symmetric.

o) If A and B are $n \times n$ matrices such that $A+B$ is upper triangular, then A and B are upper triangular.

p) If A^2 is a symmetric matrix, then A is a symmetric matrix.

q) If kA is a symmetric matrix for some $k \neq 0$, then A is a symmetric matrix.

23. Find 2 by 2 symmetric matrices $A = A^T$ with these properties

a) A is not invertible

b) A is invertible but cannot be factored into LU (row exchanges needed)

c) A can be factored into LDL^T but not into LL^T (because of negative D)

24. A group of matrices includes AB and A^{-1} if it includes A and B. "Products and inverses stay in the group." Which of these sets are groups?

Lower triangular matrices L with 1's on the diagonal, symmetric matrices S, positive matrices M, diagonal invertible matrices D, permutation matrices P, matrices with $Q^T = Q^{-1}$. ***Invent two more matrix groups***.

25. Write $A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$ as the product EH of an elementary row operation matrix E and a symmetric matrix H.

26. When is the product of two symmetric matrices symmetric? Explain your answer.

27. Express $\left((AB)^{-1}\right)^T$ in terms of $\left(A^{-1}\right)^T$ and $\left(B^{-1}\right)^T$

(28 – 29) Find the transpose of the given matrix:

28. $\begin{bmatrix} 8 & -1 \\ 3 & 5 \\ -2 & 5 \\ 1 & 2 \\ -3 & -5 \end{bmatrix}$

29. $D = \begin{pmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -1 \end{pmatrix}$

30. Show that if A is symmetric and invertible, then A^{-1} is also symmetric.

31. Prove that $(AB)^T = B^T A^T$

32. Prove that $(A+B)^T = A^T + B^T$

33. Prove that if A, B, and C are square symmetric matrices and $ABC = I$, then B is an invertible and $B^{-1} = CA$.

34. Prove that each statement is true when A and B are square matrices of order n and c is a scalar.

a) $Tr(A+B) = Tr(A) + Tr(B)$

b) $Tr(cA) = c\,Tr(A)$

35. Verify that $(AB)^T = B^T A^T$ given $A = \begin{bmatrix} -1 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix}$; $B = \begin{bmatrix} -3 & 0 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}$

36. For the given matrix, compute A^T, $\left(A^T\right)^{-1}$, A^{-1}, and $\left(A^{-1}\right)^T$, then compare $\left(A^T\right)^{-1}$ and $\left(A^{-1}\right)^T$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

37. Show that a (2×2) lower triangular matrix is invertible if and only if $a_{11}a_{22} \neq 0$ and in this case the inverse is also lower triangular.

38. Let A be any (2×2) diagonal matrix. Give a necessary and sufficient condition on the diagonal entries so that A has an inverse. Compute the inverse of any such matrix.

39. Find (2×2) matrices A and B such that A and B are symmetric, but AB is not symmetric.

40. Let A and B be $(n \times n)$ symmetric matrices. Give a necessary and sufficient condition for AB to be symmetric.

41. Let A be any (2×2) matrix, and let B and C be given by

$$B = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

Find the matrix A if:

a) $A^T + B = C$

b) $A^T B = C$

42. Given the matrix $A = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

a) Find $A^T A$, show it is symmetric

b) Find AA^T, show it is symmetric

43. Prove each of the following:

a) If A is any (2×2) matrix, then $A^T A$ and AA^T are symmetric

b) If A, B, and C are matrices such that the product ABC is defined then $(ABC)^T = C^T B^T A^T$

44. Let A be any $(n \times n)$ real matrix

a) Show that $A + A^T$ is symmetric

b) Show that $A - A^T$ is skew symmetric

45. Suppose that L is unit lower triangular and U is the upper triangular. Show that if $L = U$, then $L = U = I$

Section 1.6 – Determinants and Properties

1.6−1 *Definition*

A determinant is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the linear transformation described by the matrix.

The determinant of a matrix A is denoted by

$$\det(A) \quad or \quad |A|$$

Geometrically, it can be viewed as the volume scaling factor of the linear transformation described by the matrix.

This is also the signed volume of the $n-$dimensional parallelepiped spanned by the column or row vectors of the matrix. The determinant is positive or negative according to whether the linear transformation preserves or reverses the orientation of a real vector space.

Determinant of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined by *Leibniz* formula as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinant is a number that contains information about matrix. It is used to find formulas for inverse matrices, pivots, and solutions $A^{-1}b$.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has inverse

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The determinant is zero when the matrix has no inverse.

- ***For 2 by 2 determinants, if you expand to a rectangle, the determinants equal areas.***
- ***For n-dimensional, the determinants equal volumes.***

1.6−2 Properties of the Determinants

By using these property rules, we can compute the determinant of any square matrix.

1. ***Determinant of the n by n identity matrix is* 1.**

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad \text{and} \quad \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} = 1$$

2. ***Determinant changes sign when* 2 *rows are exchanged.***

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{aligned} \begin{vmatrix} c & d \\ a & b \end{vmatrix} &= bc - ad \\ &= -(ad - bc) \end{aligned}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

3. ***Determinant is a linear function of each row separately*.**

Multiply row 1 by any number t:

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Add row 1 of A to row 1 of A':

$$\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

4. ***If* 2 *rows of A are equal, then*** $\det A = 0$.

$$\begin{aligned} \begin{vmatrix} a & b \\ a & b \end{vmatrix} &= ab - ab \\ &\underline{= 0\,|} \end{aligned}$$

5. ***Subtracting a multiple of one row from another row leaves*** $\det A$ ***unchanged.***

$$\begin{vmatrix} a & b \\ c - ta & d - ta \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

6. ***A matrix with a row of zeros has*** $\det A = 0$.

$$\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0 \quad and \quad \begin{vmatrix} 0 & 0 \\ b & c \end{vmatrix} = 0$$

7. ***If A is triangular then*** $\det A = a_{11}a_{22} \; \cdots \; a_{nn}$
$= product\ of\ diagonal\ entries.$

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad$$

$$\begin{vmatrix} a_{11} & & 0 \\ & a_{22} & \\ & & \\ 0 & & a_{nn} \end{vmatrix} = a_{11}a_{22}\ldots a_{nn}$$

8. ***If A is singular, then*** $\det A = 0$.

9. ***If A is invertible, then*** $\det A \neq 0$.

10. ***The determinant of AB*** **det*A*** ***is times*** $\det B$:

$$|AB| = |A||B|$$

11. ***The transpose*** A^T ***has the same determinant as A:*** $\det(A) = \det\left(A^T\right)$

- $\det(A+B) \neq \det(A) + \det(B)$

1.6−3 *Formulas* for Determinants

There are several methods to determine the determinant of any square matrix $(n \times n)$.

The total terms (summation) of a determinant is equal to $n!$ with $\boldsymbol{n}$ entries (elements) to each product term.

Half are ***positive*** (product sign stay the same) and the other half sign has to be multiplied by ***negative*** sign (or opposite sign).

1.6–4 *Block Method*

$$\left|\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right| = |A| \cdot |B|$$

Proof

$$\left|\begin{array}{cc:cc} a & b & 0 & 0 \\ c & d & 0 & 0 \\ \hdashline 0 & 0 & e & f \\ 0 & 0 & g & h \end{array}\right| = a\begin{vmatrix} d & 0 & 0 \\ 0 & e & f \\ 0 & g & h \end{vmatrix} - b\begin{vmatrix} c & 0 & 0 \\ 0 & e & f \\ 0 & g & h \end{vmatrix}$$

$$= ad\begin{vmatrix} e & f \\ g & h \end{vmatrix} - bc\begin{vmatrix} e & f \\ g & h \end{vmatrix}$$

$$= (ad - bc)\begin{vmatrix} e & f \\ g & h \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix}\begin{vmatrix} e & f \\ g & h \end{vmatrix}$$

$$= |A| \cdot |B|$$

Example

Evaluate: $\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & -2 & 7 \end{vmatrix}$

Solution

$$\left|\begin{array}{cc:cc} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ \hdashline 0 & 0 & -2 & 5 \\ 0 & 0 & -2 & 7 \end{array}\right|$$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \underline{= -2}$$

$$\begin{vmatrix} -2 & 5 \\ -2 & 7 \end{vmatrix} \underline{= -4}$$

$$\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & -2 & 7 \end{vmatrix} = (-2)(-4)$$

$$\underline{= 8}$$

Example

Evaluate: $\begin{vmatrix} 1 & 2 & 2 & 0 & 0 \\ 3 & 4 & 5 & 0 & 0 \\ 6 & 7 & 8 & 0 & 0 \\ 0 & 0 & 0 & -5 & 7 \\ 0 & 0 & 0 & 3 & 4 \end{vmatrix}$

Solution

$$\left|\begin{array}{ccc:cc} 1 & 2 & 2 & 0 & 0 \\ 3 & 4 & 5 & 0 & 0 \\ 6 & 7 & 8 & 0 & 0 \\ \hdashline 0 & 0 & 0 & -5 & 7 \\ 0 & 0 & 0 & 3 & 4 \end{array}\right|$$

$$\begin{vmatrix} 1 & 2 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix} \underline{=3}$$

$$\begin{vmatrix} -5 & 7 \\ 3 & 4 \end{vmatrix} \underline{=-41}$$

$$\begin{vmatrix} 1 & 2 & 2 & 0 & 0 \\ 3 & 4 & 5 & 0 & 0 \\ 6 & 7 & 8 & 0 & 0 \\ 0 & 0 & 0 & -5 & 7 \\ 0 & 0 & 0 & 3 & 4 \end{vmatrix} = (3)(-41)$$

$$\underline{=-123}$$

1.6−5 ***Diagonal Method*** (Sarrus' Scheme)

The diagonal method only works on 3×3 matrices

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix} \qquad a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \quad (1)$$

$$\begin{array}{ccccc} & & - & - & - \\ a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \qquad -a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \quad (2)$$

Determinant: $\boldsymbol{D} = \mathbf{(1) + (2)}$

$$\det = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Example

Evaluate: $\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix} \begin{matrix} x & 0 \\ 2 & x \\ -3 & x \end{matrix}$$

$$= x(x)(1) + 0\left(x^2\right)(2) + (-1)(2)(x) - (-1)(x)(-3) - x\left(x^2\right)(x) - 0(-3)(1)$$

$$= x^2 - 2x - 3x - x^4$$

$$\underline{= x^2 - 5x - x^4}$$

1.6−6 *Co-factor Method*

The knowledge of Minors and Cofactors is compulsory in the computation of adjoint of a matrix and hence in its inverse as well as in the computation of determinant of a square matrix. This technique of computing determinant is known as Cofactor Expansion. This method can be used for any $(n \times n)$ and most important this method in programing.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

1.6−6*a* *Minor*

For a square matrix $A=\left[a_{ij}\right]$, the minor M_{ij}. Of an element a_{ij} is the ***determinant*** of the matrix formed by deleting the i^{th} row and the j^{th} column of A.

$$A=[a_{ij}]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Example

Let $A=\begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$

Find M_{32}

Solution

$$M_{32}=\begin{vmatrix} 3 & \cancel{1} & -4 \\ 2 & \cancel{5} & 6 \\ \cancel{1} & \cancel{4} & \cancel{8} \end{vmatrix}$$

$$=\begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix}$$

$$=26$$

1.6−6*b* *Theorem*

The determinant is the dot product of any row ***i*** of ***A*** with its cofactors:

Cofactor Formula:

$$C_{ij}=(-1)^{i+j}M_{ij}$$

$$\det(A)=\sum_{i=1}^{n}(-1)^{i+j}a_{i,j}M_{i,j}$$

$$a_{1*}a_{2*}a_{3*}\ \cdots\ a_{n*}$$

The ***Laplace*** formula for the determinant of a (3×3) is

$$\begin{array}{c} \quad\ \ + \qquad - \qquad + \\ |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{array}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Example

Find the determinant of the matrix:

$$A = \begin{bmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{bmatrix}$$

Solution

$$|A| = \begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix}$$

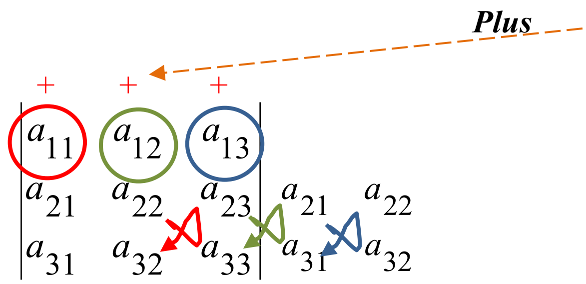
$$= -8\begin{vmatrix} -6 & 7 \\ -3 & 5 \end{vmatrix} - 0\begin{vmatrix} 4 & 7 \\ -1 & 5 \end{vmatrix} + 6\begin{vmatrix} 4 & -6 \\ -1 & -3 \end{vmatrix}$$

$$= -8(-30 - (-21)) - 0 + 6(-12 - 6)$$

$$= -8(-9) + 6(-18)$$

$$\underline{= -36}$$

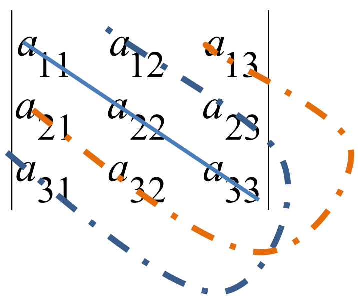
1.6−7 *Another Method*

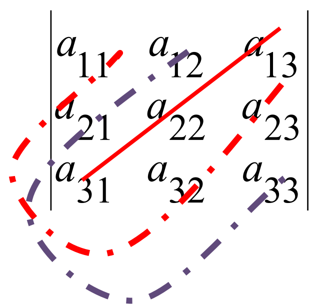


$$= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) + a_{12}\left(a_{23}a_{31} - a_{21}a_{33}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)$$

1.6−8 *Visualization Method*

This method required a good solid calculation that your multiply and add & subtract at the same time.





(*Opposite sign*)

1.6−9 *Theorem*

Let $\boldsymbol{A}$ be any n by n matrix.

a) If $\boldsymbol{A'}$ is the matrix that results when a single row of $\boldsymbol{A}$ is multiplied by a constant $\boldsymbol{k}$, then $\det(A') = k\det(A)$.

b) If $\boldsymbol{A'}$ is the matrix that results when two rows of $\boldsymbol{A}$ are interchanged, then $\det(A') = -\det(A)$

c) If $\boldsymbol{A'}$ is the matrix that results when a multiple of one row of $\boldsymbol{A}$ is added to another row then $\det(A') = \det(A)$

Example

Evaluate $\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$ *Interchanged* 1^{st} *and* 2^{nd} *row*

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$ *A common factor of 3 from the first row (no need)*

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \; R_3 - 2R_1$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} \; R_3 - 10R_2$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

$$= -3(1)(1)(-55)$$

$$\underline{= 165}$$

1.6−10 A Formula for A^{-1}

1.6−11 *Theorem*: *Inverse of a matrix using its Adjoint*

The i, j entry of A^{-1} is the cofactor C_{ji} $\left(not\ C_{ij}\right)$ divided by det(***A***):

Formula for A^{-1}:

$$\left(A^{-1}\right)_{ij} = \frac{C_{ji}}{|A|} \quad and \quad A^{-1} = \frac{C^T}{|A|}$$

$$\boxed{A^{-1} = \frac{1}{\det(A)} adj(A)}$$

Example

Find the inverse matrix of $A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix}$ using its adjoint.

Solution

$$|A| = \begin{vmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{vmatrix}$$

$$= 1$$

$$a_{11}: \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix} \quad a_{11} = \begin{vmatrix} 1 & 3 \\ -1 & -4 \end{vmatrix} = -1 \qquad A^{-1} = \frac{1}{|A|}\begin{pmatrix} -1 & & \\ & & \\ & & \end{pmatrix}$$

$$a_{12}: \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix} \quad a_{12} = -\begin{vmatrix} -2 & 2 \\ -1 & -4 \end{vmatrix} = 10 \qquad A^{-1} = \begin{pmatrix} -1 & -10 & \\ & & \\ & & \end{pmatrix}$$

$$a_{13}: \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix} \quad a_{13} = \begin{vmatrix} -2 & 2 \\ 1 & 3 \end{vmatrix} = -8 \qquad A^{-1} = \begin{pmatrix} -1 & -10 & -8 \\ & & \\ & & \end{pmatrix}$$

$$a_{21}: \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix} \quad a_{21} = -\begin{vmatrix} -1 & 3 \\ 1 & -4 \end{vmatrix} = -1 \qquad A^{-1} = \begin{pmatrix} -1 & -10 & -8 \\ -1 & & \\ & & \end{pmatrix}$$

$$a_{22}: \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix} \quad a_{22} = \begin{vmatrix} 1 & 2 \\ 1 & -4 \end{vmatrix} = -6 \qquad A^{-1} = \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & \\ & & \end{pmatrix}$$

$$a_{23}: \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix} \quad a_{23} = -\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = -5 \qquad A^{-1} = \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ & & \end{pmatrix}$$

$$a_{31}: \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix} \quad a_{31} = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0 \qquad A^{-1} = \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & & \end{pmatrix}$$

$$a_{32}: \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix} \quad a_{32} = -\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} = -1 \qquad A^{-1} = \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & \end{pmatrix}$$

$$a_{33}: \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix} \quad a_{33} = \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} = -1 \qquad A^{-1} = \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix}$$

Exercises Section 1.6 – Determinants and Properties

1. Verify that $\det(AB)=\det(A)\det(B)$ when: $A=\begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ *and* $B=\begin{pmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{pmatrix}$

2. For which value(s) of $\boldsymbol{k}$ does A fail to be invertible? $A=\begin{bmatrix} k-3 & -2 \\ -2 & k-2 \end{bmatrix}$

3. Without directly evaluating, show that $\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}=0$

4. If the entries in every row of A add to zero, solve $\boldsymbol{Ax}=0$ to prove $\det(A)=0$. If those entries add to one, show that $\det(A-I)=0$. Does this mean $\det(A)=I$?

5. Does $\det(AB)=\det(BA)$ in general?

a) True or false if $\boldsymbol{A}$ and $\boldsymbol{B}$ are square n x n matrices?

b) True or false if $\boldsymbol{A}$ is m x n and B is n x m with $m\neq n$?

6. True or false, with a reason if true or a counterexample if false:

a) The determinant of $I+A$ is $1+\det(A)$.

b) The determinant of ABC is $|A||B||C|$.

c) The determinant of $4A$ is $4|A|$

d) The determinant of $AB-BA$ is zero. (try an example)

e) If A is not invertible then AB is not invertible.

f) The determinant of $A-B$ equals to $\det(A)-\det(B)$.

7. Use row operations to show the 3 by 3 "Vandermonde determinant" is

$$\det\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}=(b-a)(c-a)(c-b)$$

8. The inverse of a 2 by 2 matrix seems to have determinant = 1:

$$\begin{aligned}\det A^{-1} &= \det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{ad-bc}{ad-bc} \\ &= 1\end{aligned}$$

What is wrong with this calculation? What is the correct $\det A^{-1}$

9. A *Hessenberg* matrix is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci's rule $\left|H_4\right| = \left|H_3\right| + \left|H_2\right|$. The same rule will continue for all sizes $\left|H_n\right| = \left|H_{n-1}\right| + \left|H_{n-2}\right|$. Which Fibonacci number is $\left|H_n\right|$?

$$H_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad H_3 = \begin{bmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad H_4 = \begin{bmatrix} 2 & 1 & & \\ \mathbf{1} & 2 & \mathbf{1} & \\ \mathbf{1} & 1 & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

(10 – 60) Evaluate

10. $\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix}$

11. $\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix}$

12. $\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix}$

13. $\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix}$

14. $\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix}$

15. $\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix}$

16. $\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}$

17. $\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix}$

18. $\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix}$

19. $\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix}$

20. $\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix}$

21. $\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix}$

22. $\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix}$

23. $\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix}$

24. $\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix}$

25. $\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix}$

26. $\begin{vmatrix} \lambda-3 & 2 \\ 4 & \lambda-1 \end{vmatrix}$

27. $\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix}$

28. $\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix}$

29. $\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix}$

30. $\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix}$

31. $\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix}$

32. $\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}$

33. $\begin{vmatrix} 5 & 3 \\ -6 & 3 \end{vmatrix}$

34. $\begin{vmatrix} \sin\theta & 1 \\ 1 & \sin\theta \end{vmatrix}$

35. $\begin{vmatrix} 0 & 8 \\ 0 & 4 \end{vmatrix}$

36. $\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}$

37. $\begin{vmatrix} x & \ln x \\ 1 & \frac{1}{x} \end{vmatrix}$

38. $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$

39. $\begin{vmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{vmatrix}$

40. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{vmatrix}$

41. $\begin{vmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

42. $\begin{vmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{vmatrix}$

43. $\begin{vmatrix} x & y & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix}$

44. $\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$

45. $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

46. $\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix}$

47. $\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix}$

48. $\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix}$

49. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix}$

50. $\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix}$

51. $\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix}$

52. $\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix}$

53. $\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix}$

54. $\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix}$

55. $\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix}$

56. $\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix}$

57. $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix}$

58. $\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix}$

59. $\begin{vmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{vmatrix}$

60. $\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix}$

(61 – 63) Find $a)\ |A|\quad b)\ |B|\quad c)\ AB\quad d)\ |AB|\quad d)\ |A+B|$. Then verify that $|A|\ |B| = |AB|$ & $|A|+|B| \neq |A+B|$

61. $A=\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix},\quad B=\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

62. $A=\begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix},\quad B=\begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$

63. $A=\begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\quad B=\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(64 – 67) Find all the values of λ for which $\det(A)=0$

64. $A=\begin{bmatrix} \lambda-1 & -2 \\ 1 & \lambda-4 \end{bmatrix}$

65. $A=\begin{pmatrix} \lambda+2 & 2 \\ 1 & \lambda \end{pmatrix}$

66. $A=\begin{pmatrix} \lambda & 2 & 0 \\ 0 & \lambda+1 & 2 \\ 0 & 1 & \lambda \end{pmatrix}$

67. $A=\begin{bmatrix} \lambda-6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda-4 \end{bmatrix}$

(65 – 66) Use the fact that $|cA| = c^n |A|$ to evaluate the determinant of the $n\times n$ matrix

65. $A=\begin{pmatrix} 5 & 15 \\ 10 & -20 \end{pmatrix}$

66. $A=\begin{pmatrix} -3 & 6 & 9 \\ 6 & 9 & 12 \\ 9 & 12 & 15 \end{pmatrix}$

(67 – 68) Verify that $\left|A^{-1}\right| = \dfrac{1}{|A|}$

67. $A=\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

68. $A=\begin{pmatrix} 2 & -2 & 3 \\ 1 & -1 & 2 \\ 3 & 0 & 3 \end{pmatrix}$

69. Prove that if a square matrix $\boldsymbol{A}$ has a column of zeros, then $\det(A)=0$

70. With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D| \quad \textit{but} \quad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|$$

a) Why is the first statement true? Somehow B doesn't enter.

b) Show by example that equality fails (as shown) when C enters.

c) Show by example that the answer $\det(AD-CB)$ is also wrong.

71. Show that the value of the following determinant is independent of θ.

$$\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta-\cos\theta & \sin\theta+\cos\theta & 1 \end{vmatrix}$$

72. Show that the matrices $A=\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ *and* $B=\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$

commute if and only if $\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix}=0$

73. Show that $\det(A)=\frac{1}{2}\begin{vmatrix} tr(A) & 1 \\ tr\left(A^2\right) & tr(A) \end{vmatrix}$ for every 2×2 matrix *A*.

74. What is the maximum number of zeros that a 4×4 matrix can have without a zero determinant? Explain your reasoning.

75. Evaluate $\det(A)$, $\det(E)$, and $\det(AE)$. Then verify that $\det(A)\cdot\det(E)=\det(AE)$

$$A=\begin{bmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix}, \qquad E=\begin{bmatrix} 1 & & \\ & 3 & \\ & & 1 \end{bmatrix}$$

76. Show that $\begin{bmatrix} \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma \\ 1 & 1 & 1 \end{bmatrix}$ is not invertible for any values of $\alpha,\ \beta,\ \gamma$

77. The determinant of a 2×2 matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det(A)=ad-bc$.

Assuming no rows swaps are required, perform elimination on *A* and show explicitly that $ad-bc$ is the product of the pivots.

78. If *A* is a 7×7 matrix and let $\det(A)=17$. What is $\det\left(3A^2\right)$?

79. Let *A* be $n\times n$ real matrix.

a) Show that if $A^t=-A$ and *n* is odd, then $|A|=0$.

b) Show that if $A^2+I=0$, then *n* must be even.

c) Does part (*b*) remain true for complex matrices?

80. Explain without computations why the following determinant is equal to zero

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & 0 & 0 & 0 \\ d_1 & d_2 & 0 & 0 & 0 \\ e_1 & e_2 & 0 & 0 & 0 \end{vmatrix}$$

81. Let A and C be $m\times m$ and $n\times n$ matrices, respectively.

a) Show that $\begin{vmatrix} A & B \\ 0 & C \end{vmatrix} = \begin{vmatrix} A & 0 \\ B & C \end{vmatrix} = |A||C|$

b) Evaluate

i. $\begin{vmatrix} I_m & 0 \\ 0 & I_n \end{vmatrix}$

ii. $\begin{vmatrix} 0 & I_m \\ I_n & 0 \end{vmatrix}$

iii. $\begin{vmatrix} I_m & B \\ 0 & I_n \end{vmatrix}$

iv. $\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix}_{n\times n}$

c) Find a formula for $\begin{vmatrix} 0 & A \\ C & B \end{vmatrix}_{n\times n}$

82. Let $f(x)=(p_1-x)(p_2-x)\ldots(p_n-x)$ and let

$$\Delta_n = \begin{vmatrix} p_1 & a & a & \cdots & a & a \\ b & p_2 & a & \cdots & a & a \\ b & b & p_3 & \cdots & a & a \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b & b & b & \cdots & p_{n-1} & a \\ b & b & b & \cdots & b & p_n \end{vmatrix}$$

a) Show that, if $a \neq b$,

$$\Delta_n = \frac{bf(a) - af(b)}{b-a}$$

b) Show that, if $a = b$,

$$\Delta_n = a\sum_{i=1}^{n} f_i(a) + p_n f_n(a)$$

Where $f_i(a)$ means $f(a)$ with factor $(p_i - a)$ missing.

c) Use part (b) to evaluate

$$\begin{vmatrix} a & b & b & \dots & b & b \\ b & a & b & \dots & b & b \\ b & b & a & \dots & a & a \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b & b & b & \cdots & a & a \\ b & b & b & \dots & b & a \end{vmatrix}_{n\times n}$$

83. Let $A, B, C, D \in M_n(\mathbb{C})$

a) Show that when A is invertible: $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A||D - CA^{-1}B|$

b) Show that when $AC = CA$: $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|$

c) Can B and C on the right-hand side of the identity be switched?

d) Does part (b) remain true if the condition $AC = CA$ is dropped?

84. Show that the matrix A is invertible for all values of θ, then find A^{-1} using $A^{-1} = \frac{1}{\det(A)} adj(A)$

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(85 – 92) Find the inverse matrix of using its ***adjoint***.

85. $A = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}$

86. $A = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$

87. $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

88. $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$

89. $A=\begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{pmatrix}$

90. $A=\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

91. $A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & -4 & -12 \end{pmatrix}$

92. $A=\begin{pmatrix} -3 & -5 & -7 \\ 2 & 4 & 3 \\ 0 & 1 & -1 \end{pmatrix}$

93. Let A and B be $n\times n$ matrices such that $AB=I$. Prove that $|A|\neq 0$ and $|B|\neq 0$

94. Let A and B be $n\times n$ matrices such that AB is singular. Prove that A or B is singular.

95. Find two 2×2 matrices such that $|A|+|B|=|A+B|$

96. Verify the equation $\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix}=b^2(3a+b)$

97. Let A be an $n\times n$ matrix in which the entries of each row sum to zero. Find $|A|$

98. Show that if A is an $n\times n$ matrix with entries 1 and -1, then $|A|$ is divisible by 2^{n-1}.

99. Show that (the Vandermonde determinant)

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{vmatrix} = \prod_{1\le i<j\le n}\left(a_j-a_i\right)$$

In Particular, If V is the $n\times n$ matrix, with $(i,j)-entry\ j^{i-1}$, then

$$|V|=(n-1)\,(n-2)^2\,\cdots\,2^{n-2}$$

100. Show that if $a\neq b$, then the determinant for $n\times n$ matrix is:

$$\begin{vmatrix} a+b & ab & 0 & \cdots & 0 & 0 \\ 1 & a+b & ab & \cdots & 0 & 0 \\ 0 & 1 & a+b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a+b & ab \\ 0 & 0 & 0 & \cdots & 1 & a+b \end{vmatrix} = \frac{a^{n+1}-b^{n+1}}{a-b}$$

What if $a=b$?

101. Let $a = \left(-a_0,\ -a_1,\ \cdots,\ -a_{n-2}\right)$ and let

$$A = \begin{pmatrix} 0 & I_{n-1} \\ a & -a_{n-1} \end{pmatrix}$$

Show that $\left|\lambda I - A\right| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$

Section 1.7 – Cramer's Rule

Cramer's Rule

1.7−1 *Theorem*

If $AX = B$ is a system of a linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(B_1)}{\det(A)}$$

$$x_2 = \frac{\det(B_2)}{\det(A)}$$

$$\vdots \qquad \vdots$$

$$x_n = \frac{\det(B_n)}{\det(A)}$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

$$\det(B_1) = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & & & \\ \vdots & & & \\ b_n & a_{n2} & & a_{nn} \end{vmatrix}$$

1.7–2 *Example*

Use Cramer's rule to solve

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ -2x_1 + x_2 \qquad &= 0 \\ -4x_1 \qquad + x_3 &= 0 \end{aligned}$$

Solution

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 7$$

$$|B_1| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$|B_2| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 2$$

$$|B_3| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 0 \end{vmatrix} = 4$$

$$x_1 = \frac{|B_1|}{|A|} = \frac{1}{7} \qquad\qquad x_1 = \frac{\det(A_1)}{\det(A)}$$

$$x_2 = \frac{|B_2|}{|A|} = \frac{2}{7}$$

$$x_3 = \frac{|B_3|}{|A|} = \frac{4}{7}$$

Solution: $\left(\frac{1}{7},\ \frac{2}{7},\ \frac{4}{7}\right)$

1.7−3 *Example*

Use Cramer's Rule to solve.

$$\begin{aligned} x_1 + \quad\; + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

Solution

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} \quad \Rightarrow \quad \det(A) = 44$$

$$\det(A_1) = \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix} = -40$$

$$\det(A_2) = \begin{vmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{vmatrix} = 72$$

$$\det(A_3) = \begin{vmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{vmatrix} = 152$$

$$x_1 = \frac{-40}{44} \qquad x_1 = \frac{\det(A_1)}{\det(A)}$$

$$\boxed{= -\frac{10}{11}}$$

$$x_2 = \frac{72}{44} \qquad x_2 = \frac{\det(A_2)}{\det(A)}$$

$$\boxed{= \frac{18}{11}}$$

$$x_3 = \frac{152}{44} \qquad x_3 = \frac{\det(A_3)}{\det(A)}$$

$$\boxed{= \frac{38}{11}}$$

Solution: $\boxed{\left(-\frac{10}{11}, \frac{18}{11}, \frac{38}{11}\right)}$

1.7−6 *Theorem*

If A is an $n \times n$ matrix, then the following statements are equivalent

- ***a)*** A is invertible
- ***b)*** $A\boldsymbol{x} = 0$ has only the trivial solution
- ***c)*** The reduced row echelon form of A is I_n
- ***d)*** A can be expressed as a product of elementary matrices
- ***e)*** $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $n \times 1$ matrix $\boldsymbol{b}$
- ***f)*** $\det(A) \neq 0$

Exercises *Section* 1.7 – Cramer's Rule

1. Use Cramer's Rule with ratios $\dfrac{\det B_j}{\det A}$ to solve $A\boldsymbol{x} = b$. Also find the inverse matrix $A^{-1} = \dfrac{C^T}{\det A}$. Why is the solution $\boldsymbol{x}$ is the first part the same as column 3 of A^{-1}? Which cofactors are involved in computing that column $\boldsymbol{x}$?

$$Ax = b \quad is \quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2. Verify that $\det(AB) = \det(BA)$ and determine whether the equality $\det(A+B) = \det(A) + \det(B)$ holds

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

3. Verify that $\det(kA) = k^n \det(A)$

$a)\quad A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}, \quad k = 2$

$b)\quad A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}, \quad k = -2$

$c)\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}, \quad k = 3$

(4 – 61) Use Cramer's rule to solve the system

4. $\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$

5. $\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$

6. $\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$

7. $\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$

8. $\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$

9. $\begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$

10. $\begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$

11. $\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$

12. $\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$

13. $\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$

14. $\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$

15. $\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$

16. $\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$

17. $\begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases}$

18. $\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$

19. $\begin{cases} x+y=7 \\ x-y=3 \end{cases}$

20. $\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

21. $\begin{cases} 12x+3y=15 \\ 2x-3y=13 \end{cases}$

22. $\begin{cases} x-2y=5 \\ 5x-y=-2 \end{cases}$

23. $\begin{cases} 4x-5y=17 \\ 2x+3y=3 \end{cases}$

24. $\begin{cases} 3x+2y=2 \\ 2x+2y=3 \end{cases}$

25. $\begin{cases} x-3y=4 \\ 3x-4y=12 \end{cases}$

26. $\begin{cases} 2x-9y=5 \\ 3x-3y=11 \end{cases}$

27. $\begin{cases} 3x-4y=4 \\ x+y=6 \end{cases}$

28. $\begin{cases} 3x=7y+1 \\ 2x=3y-1 \end{cases}$

29. $\begin{cases} 2x=3y+2 \\ 5x=51-4y \end{cases}$

30. $\begin{cases} y=-4x+2 \\ 2x=3y-1 \end{cases}$

31. $\begin{cases} 3x=2-3y \\ 2y=3-2x \end{cases}$

32. $\begin{cases} x+2y-3=0 \\ 12=8y+4x \end{cases}$

33. $\begin{cases} 7x-2y=3 \\ 3x+y=5 \end{cases}$

34. $\begin{cases} x_1+2x_2=5 \\ -x_1+x_2=1 \end{cases}$

35. $\begin{cases} 3x+2y-z=4 \\ 3x-2y+z=5 \\ 4x-5y-z=-1 \end{cases}$

36. $\begin{cases} x+y+z=2 \\ 2x+y-z=5 \\ x-y+z=-2 \end{cases}$

37. $\begin{cases} 2x+y+z=9 \\ -x-y+z=1 \\ 3x-y+z=9 \end{cases}$

38. $\begin{cases} 3y-z=-1 \\ x+5y-z=-4 \\ -3x+6y+2z=11 \end{cases}$

39. $\begin{cases} x+3y+4z=14 \\ 2x-3y+2z=10 \\ 3x-y+z=9 \end{cases}$

40. $\begin{cases} x+4y-z=20 \\ 3x+2y+z=8 \\ 2x-3y+2z=-16 \end{cases}$

41. $\begin{cases} -2x+6y+7z=3 \\ -4x+5y+3z=7 \\ -6x+3y+5z=-4 \end{cases}$

42. $\begin{cases} 2x-y+z=1 \\ 3x-3y+4z=5 \\ 4x-2y+3z=4 \end{cases}$

43. $\begin{cases} 3x-4y+4z=7 \\ x-y-2z=2 \\ 2x-3y+6z=5 \end{cases}$

44. $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

45. $\begin{cases} x+y+z=3 \\ -y+2z=1 \\ -x+z=0 \end{cases}$

46. $\begin{cases} 3x+y+3z=14 \\ 7x+5y+8z=37 \\ x+3y+2z=9 \end{cases}$

47. $\begin{cases} 4x-2y+z=7 \\ x+y+z=-2 \\ 4x+2y+z=3 \end{cases}$

48. $\begin{cases} 2y-z=7 \\ x+2y+z=17 \\ 2x-3y+2z=-1 \end{cases}$

49. $\begin{cases} 2x-2y+z=-4 \\ 6x+4y-3z=-24 \\ x-2y+2z=1 \end{cases}$

50. $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

51. $\begin{cases} 2x-y+2z=-8 \\ x+2y-3z=9 \\ 3x-y-4z=3 \end{cases}$

52. $\begin{cases} x \qquad -3z=-5 \\ 2x-y+2z=16 \\ 7x-3y-5z=19 \end{cases}$

53. $\begin{cases} x+2y-z=5 \\ 2x-y+3z=0 \\ 2y+z=1 \end{cases}$

54. $\begin{cases} x+y+z=6 \\ 3x+4y-7z=1 \\ 2x-y+3z=5 \end{cases}$

55. $\begin{cases} 3x+2y+3z=3 \\ 4x-5y+7z=1 \\ 2x+3y-2z=6 \end{cases}$

56. $\begin{cases} 4x+5y \qquad =2 \\ 11x+y+2z=3 \\ x+5y+2z=1 \end{cases}$

57. $\begin{cases} x-4y+z=6 \\ 4x-y+2z=-1 \\ 2x+2y-3z=-20 \end{cases}$

58. $\begin{cases} 2x-\ \ y+\ \ z=-1 \\ 3x+4y-\ z=-1 \\ 4x-\ \ y+2z=-1 \end{cases}$

59. $\begin{cases} 4x-y-z=1 \\ 2x+2y+3z=10 \\ 5x-2y-2z=-1 \end{cases}$

60. $\begin{cases} 3x+4y+4z=11 \\ 4x-4y+6z=11 \\ 6x-6y\qquad =3 \end{cases}$

61. $\begin{cases} -x_1-4x_2+2x_3+x_4=-32 \\ 2x_1-x_2+7x_3+9x_4=14 \\ -x_1+x_2+3x_3+x_4=11 \\ -x_1-2x_2+x_3-4x_4=-4 \end{cases}$

(62 – 64) Determine the polynomial function whose graph through the points and sketch the graph of the polynomial function, showing the points.

62. $(2,\ 5),\quad (3,\ 2),\quad (4,\ 5)$

63. $(2,\ 4),\quad (3,\ 6),\quad (5,\ 10)$

64. $(-1,\ 3),\quad (0,\ 0),\quad (1,\ 1),\quad (4,\ 58)$

65. The U.S. census lists the population of the United States as 249 million in 1990, 282 million in 2000, and 309 million in 2010. Fit a second-degree polynomial passing through these three points and use it to predict the populations in 2020 and 2030.

Section 1.8 – Applications

Network Analysis

1.8−1 *Definition*

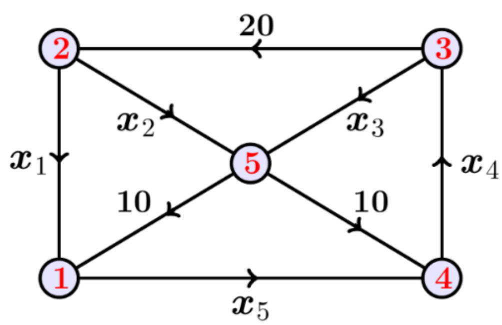
Networks composed of branches and junctions or nodes are used as models in such fields as economics, traffic analysis, and electrical engineering.

The direction of flow in each branch is indicated, and the flow amount is either shown or is denoted by a variable.

In a network model, you assume that the total flow into a junction is equal to the total flow out of the junction.

1.8−2 *Example*

Set up a system of linear equations to represent the network shown below. Then solve the system for $x_i, \quad i = 1,\ 2,\ 3,\ 4,\ 5.$



Solution

$1 \rightarrow \quad x_1 + 10 = x_5 \quad \Rightarrow \quad x_1 - x_5 = -10$

$2 \rightarrow \quad x_1 + x_2 = 20$

$3 \rightarrow \quad x_4 = x_3 + 20 \quad \Rightarrow \quad -x_3 + x_4 = 20$

$4 \rightarrow \quad x_4 = x_5 + 10 \quad \Rightarrow \quad x_4 - x_5 = 10$

$5 \rightarrow \quad x_2 + x_3 = 10 + 10 = 20$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 1 & 1 & 0 & 0 & 20 \end{array}\right) \quad R_2 - R_1$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 1 & 1 & 0 & 0 & 20 \end{array}\right) \quad R_5 - R_2$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 1 & 0 & -1 & -10 \end{array}\right) \quad R_5 + R_3$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 10 \end{array}\right) \quad R_5 - R_4$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{ll} \rightarrow\ x_1 - x_5 = -10 & (1) \\ \rightarrow\ x_2 + x_5 = 30 & (2) \\ \rightarrow\ -x_3 + x_4 = 20 & (3) \\ \rightarrow\ x_4 - x_5 = 10 & (4) \end{array}$$

$$\Rightarrow \begin{cases} (1) \rightarrow & \underline{x_1 = x_5 - 10} \\ (2) \rightarrow & \underline{x_2 = 30 - x_5} \\ (3) \rightarrow & \underline{x_3 = x_5 - 10} \\ (4) \rightarrow & \underline{x_4 = x_5 + 10} \end{cases}$$

Solution: $\underline{\left(x_5 - 10,\ \ 30 - x_5,\ \ x_5 - 10,\ \ 10 + x_5,\ \ x_5\right)}$

2nd Method

$$\begin{vmatrix} 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{vmatrix} - 1 \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= -1 + 1$$

$$\boxed{= 0}$$

Infinite solution:

$1 \rightarrow \quad \boxed{x_1 = x_5 - 10}$

$2 \rightarrow \quad x_2 = 20 - x_1 \ \boxed{= 30 - x_5}$

$4 \rightarrow \quad \boxed{x_4 = x_5 + 10}$

$3 \rightarrow \quad x_3 = x_4 - 20 \ \boxed{= x_5 - 10}$

Electrical network

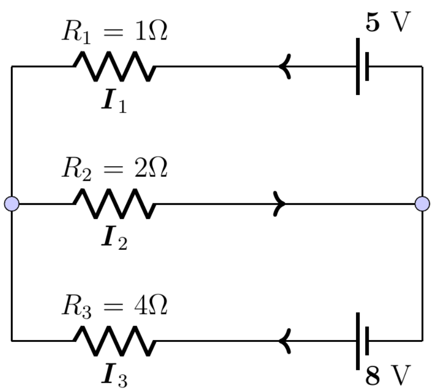
1.8−3 *Definition*

An electrical network is another type of network where analysis is commonly applied. An analysis of such a system uses two properties of electrical networks known as **Kirchhoff's Laws.**

- All the current flowing into a junction must flow out of it.
- The sum of the products *IR* (*I* is current and *R* is resistance) around a closed path is equal to the total voltage in the path.

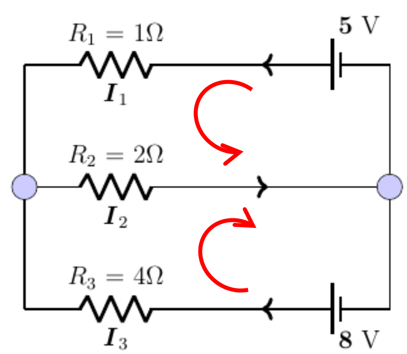
1.8−4 *Example*

Determine the currents I_1, I_2, and I_3 for the electrical network



Solution

$$I_2 = I_1 + I_3$$
$$I_1 + 2I_2 = 5$$
$$2I_2 + 4I_3 = 8$$



$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ I_1 + 2I_2 \qquad = 5 \\ \qquad I_2 + 2I_3 = 4 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 7 \qquad D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 5 & 2 & 0 \\ 4 & 1 & 2 \end{vmatrix} = 7 \qquad D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 5 & 0 \\ 0 & 4 & 2 \end{vmatrix} = 14 \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 5 \\ 0 & 1 & 4 \end{vmatrix} = 7$$

$\underline{I_1 = 1 \ A}$ $\underline{I_2 = 2 \ A}$ $\underline{I_3 = 1 \ A}$

Cryptography

1.8−5 Definition

A ***cryptogram*** is a message written according to a secret code (the Greek word ***kryptos*** means "*hidden*"). One method of using matrix multiplication to ***encode*** and ***decode*** messages.

Let assign a number to each letter in the alphabet (with 0 assigned to a blank space), as shown

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

1.8−6 Example

Consider the invertible matrix: $A=\begin{pmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{pmatrix}$

The message: ***MEET ME MONDAY***

a) Write the uncoded row matrices 1×3 for the message.
b) Use the matrix A to encode the message.
c) Decode a message from part *b*) given the matrix A.

Solution

a)

$$\begin{array}{ccccc} M\ E\ E & T\ \_\ M & E\ \_\ M & O\ N\ D & A\ Y\ \_ \\ \begin{bmatrix}13 & 5 & 5\end{bmatrix} & \begin{bmatrix}20 & 0 & 13\end{bmatrix} & \begin{bmatrix}5 & 0 & 13\end{bmatrix} & \begin{bmatrix}15 & 14 & 4\end{bmatrix} & \begin{bmatrix}1 & 25 & 0\end{bmatrix}\end{array}$$

b) Let encode the message ***MEET ME MONDAY***

$$\begin{bmatrix}13 & 5 & 5\end{bmatrix}\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}=\begin{bmatrix}13 & -26 & 21\end{bmatrix}$$

$$\begin{bmatrix}20 & 0 & 13\end{bmatrix}\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}=\begin{bmatrix}33 & -53 & -12\end{bmatrix}$$

$$\begin{bmatrix}5 & 0 & 13\end{bmatrix}\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}=\begin{bmatrix}18 & -23 & -42\end{bmatrix}$$

$$\begin{bmatrix}15 & 14 & 4\end{bmatrix}\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}=\begin{bmatrix}5 & -20 & 56\end{bmatrix}$$

$$\begin{bmatrix}1 & 25 & 0\end{bmatrix}\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}=\begin{bmatrix}-24 & 23 & 77\end{bmatrix}$$

The sequence of coded row matrices is

$$\begin{bmatrix}13 & -26 & -21\end{bmatrix}\ \begin{bmatrix}33 & -53 & -12\end{bmatrix}\ \begin{bmatrix}18 & -23 & -42\end{bmatrix}\ \begin{bmatrix}5 & -20 & 56\end{bmatrix}\ \begin{bmatrix}-24 & 23 & 77\end{bmatrix}$$

The cryptogram:

$$13\ \ -26\ \ -21\ \ 33\ \ -53\ \ -12\ \ 18\ \ -23\ \ -42\ \ 5\ \ -20\ \ 56\ \ -24\ \ 23\ \ 77$$

c) To decode a message given the matrix A.

$$|A|=\begin{vmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{vmatrix}=1$$

$$A^{-1}=\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}$$

With the cryptogram:

$$\begin{bmatrix}13 & -26 & -21\end{bmatrix}\ \begin{bmatrix}33 & -53 & -12\end{bmatrix}\ \begin{bmatrix}18 & -23 & -42\end{bmatrix}\ \begin{bmatrix}5 & -20 & 56\end{bmatrix}\ \begin{bmatrix}-24 & 23 & 77\end{bmatrix}$$

$$\begin{bmatrix}13 & -26 & 21\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}13 & 5 & 5\end{bmatrix}$$

$$\begin{bmatrix}33 & -53 & -1\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}20 & 0 & 13\end{bmatrix}$$

$$\begin{bmatrix}18 & -23 & -42\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}5 & 0 & 13\end{bmatrix}$$

$$\begin{bmatrix}5 & -20 & 56\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}15 & 14 & 4\end{bmatrix}$$

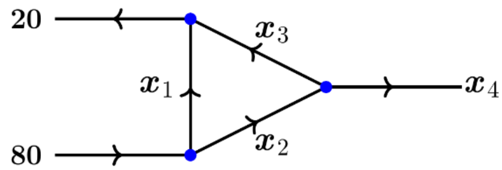
$$\begin{bmatrix} -24 & 23 & 77 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 25 & 0 \end{bmatrix}$$

The message is:

13	5	5	20	0	13	5	0	13	15	14	4	1	25	0
M	*E*	*E*	*T*	_	*M*	*E*	_	*M*	*O*	*N*	*D*	*A*	*Y*	_

Exercises Section 1.8 – Applications

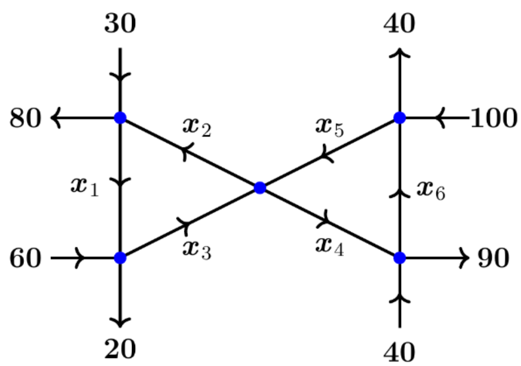
1. Through a network



a) Solve this system for x_i, $i = 1, 2, 3, 4, 5, 6$

b) What is the largest value of x_3 ?

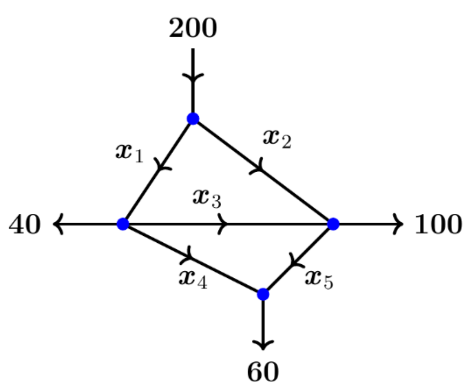
2. The flow of traffic, through a network of streets as is shown below



a) Solve this system for x_i, $i = 1, 2, 3, 4, 5, 6$

b) Find the minimum flows in the branches denoted by $x_2, x_3, x_4,$ *and* x_5

3. The flow of traffic, through a network of streets as is shown below



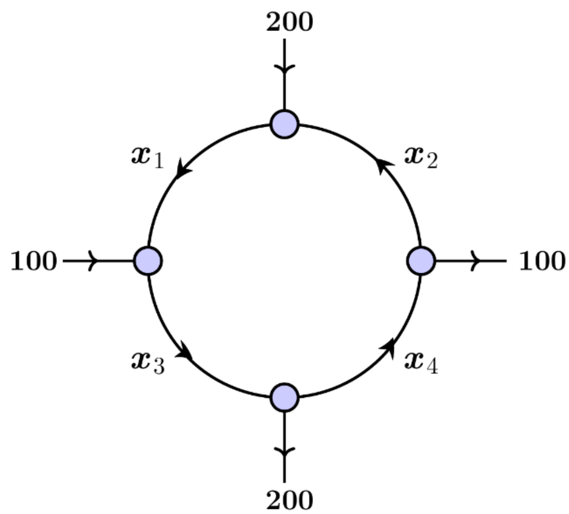
a) Solve this system for x_i, $i = 1, 2, 3, 4, 5$

b) Find the traffic flow when $x_4 = 0$.

c) Find the traffic flow when $x_4 = 100$.

d) Find the traffic flow when $x_1 = 2x_2$.

4. The flow of traffic, in vehicles per hour, through a network of streets as is shown below



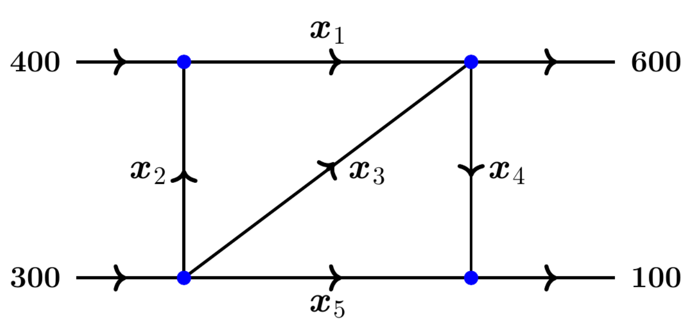
a) Solve this system for x_i, $i = 1, 2, 3, 4$.

b) Find the traffic flow when $x_4 = 0$.

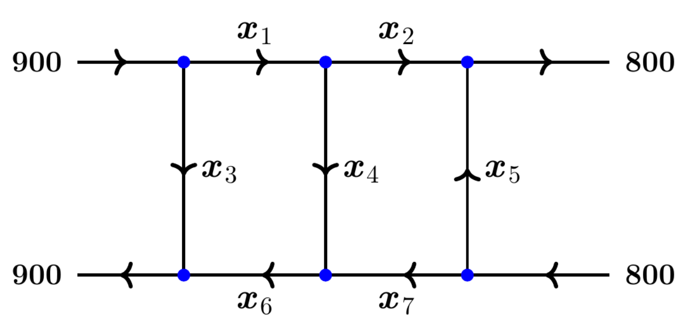
c) Find the traffic flow when $x_4 = 100$.

d) Find the traffic flow when $x_1 = 2x_2$.

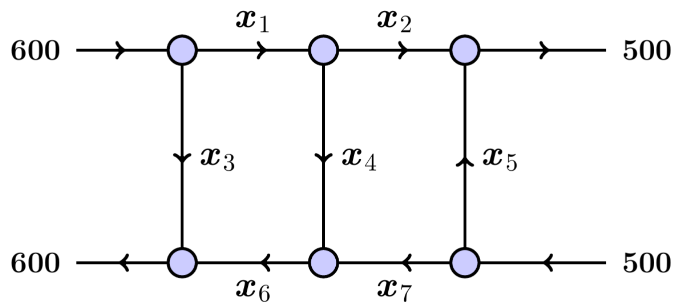
5. Through a network, Express x_n's in terms of the parameters ***s*** and ***t***.



6. Water is flowing through a network of pipes. Express x_n's in terms of the parameters ***s*** and ***t***.



7. Water is flowing through a network of pipes (in thousands of cubic meters per hour)



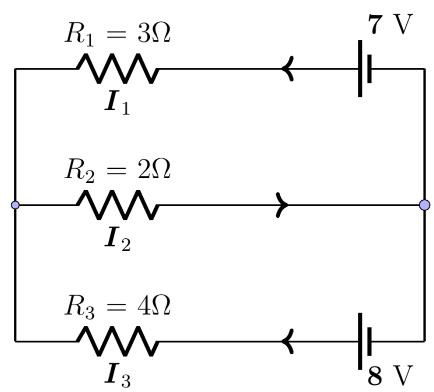
a) Solve this system for the water flow represented by x_i, $i = 1, 2, \ldots, 7$.

b) Find the water flow when $x_1 = x_2 = 100$

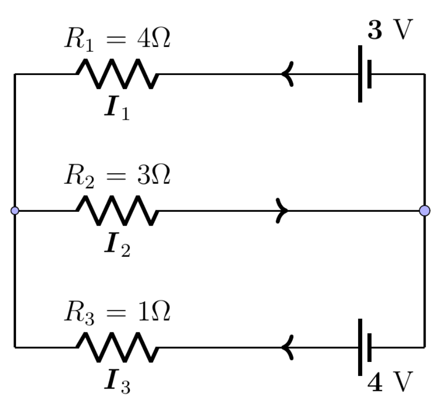
c) Find the water flow when $x_6 = x_7 = 0$

d) Find the water flow when $x_5 = 1000$ *and* $x_6 = 0$

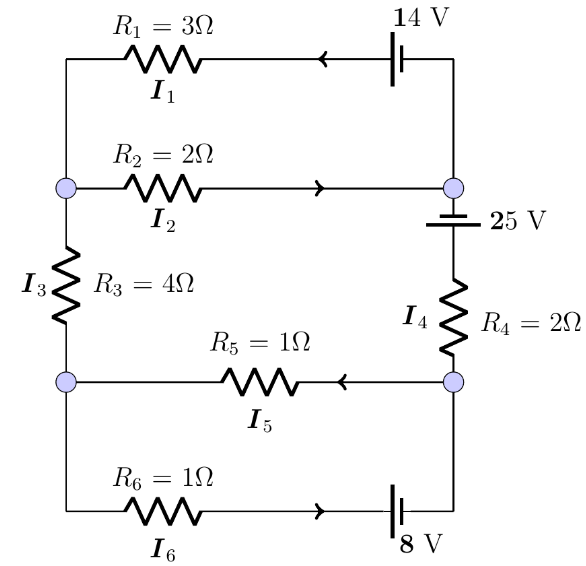
8. Determine the currents $I_1, I_2,$ *and* I_3 for the electrical network shown below



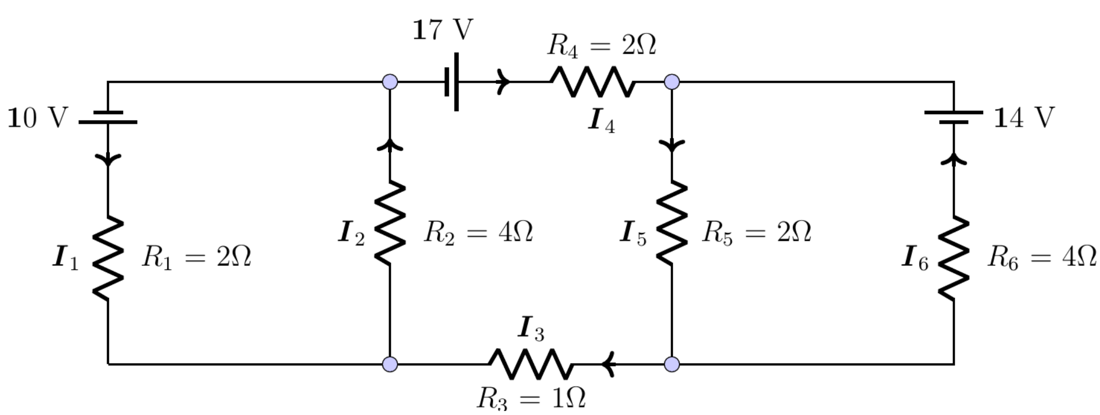
9. Determine the currents $I_1, I_2,$ *and* I_3 for the electrical network shown below



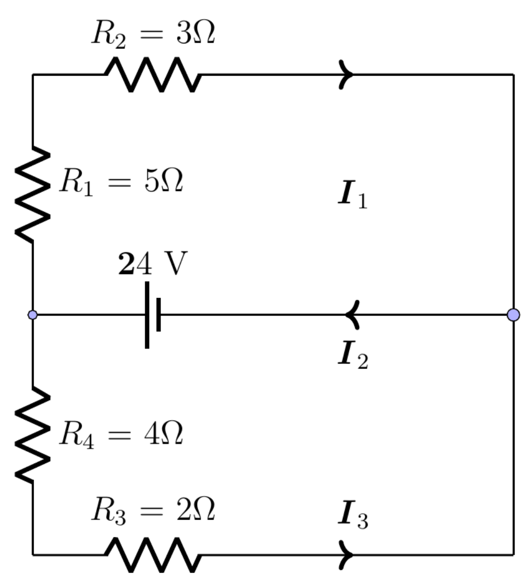
10. Determine the currents $I_1, I_2, I_3, I_4, I_5,$ *and* I_6 for the electrical network shown below



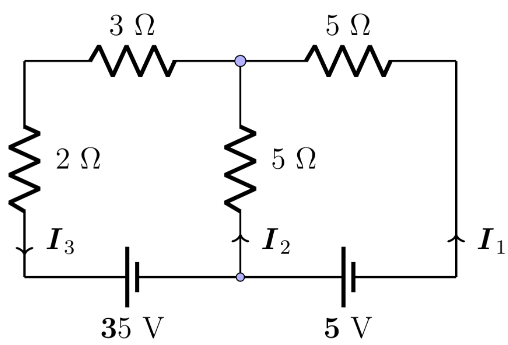
11. Determine the currents $I_1, I_2, I_3, I_4, I_5,$ *and* I_6 for the electrical network shown below



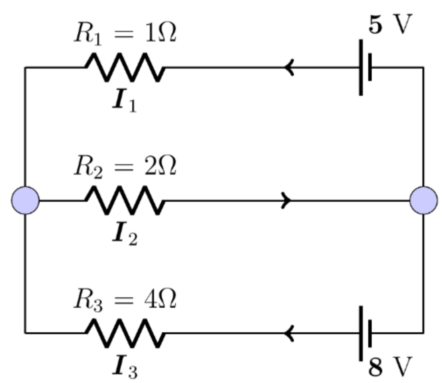
12. Determine the currents $I_1, I_2,$ *and* I_3 for the electrical network shown below



13. Determine the currents I_1, I_2, *and* I_3 for the electrical network shown below

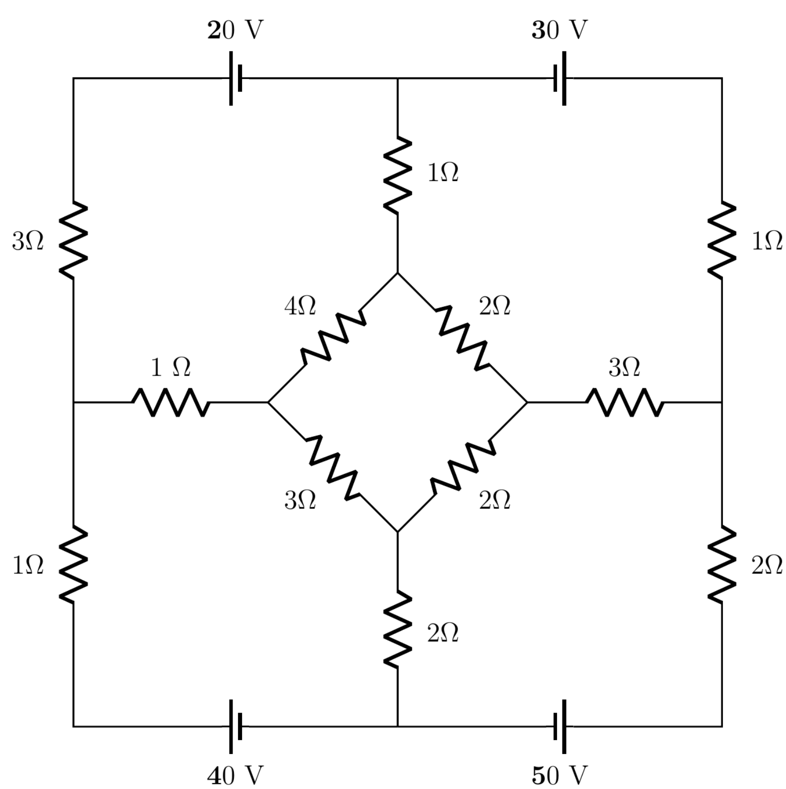


14. Determine the currents I_1, I_2, *and* I_3 for the electrical network shown below



How is the result affected when the 5 V is changed to 2 V and 8V to 6 V?

15. Determine the currents for the electrical network



16. Use A^{-1} to decode the cryptogram

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$$

11 21 64 112 25 50 29 53 23 46 40 75 55 92

17. Use A^{-1} to decode the cryptogram

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{pmatrix}$$

13 19 10 −1 −33 −77 3 −2 −14 4 1 −9 −5 −25 −47 4 1 −9

18. Consider the invertible matrix: $A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -2 & -2 & 7 \end{pmatrix}$

The message: ***ICEBERG DEAD AHEAD***

a) Write the uncoded row matrices 1×3 for the message.
b) Use the matrix A to encode the message.
c) Decode a message from part *b*) given the matrix A.

19. You want to send the message: ***LINEAR ALGEBRA*** with a key word ***MATH***

a) Write the matrix A.
b) Write the uncoded row matrices 1×2 for the message.
c) Use the matrix A to encode the message.
d) Decode a message from part *b*) given the matrix A.

20. You want to send the message: ***CRYPTOGRAPHY IS A METHOD OF PROTECTING INFORMATIONS*** with a key word ***CODE***

a) Write the matrix A.
b) Write the uncoded row matrices 1×2 for the message.
c) Use the matrix A to encode the message.
d) Decode a message from part *b*) given the matrix A.

21. Write the matrix A with a key word ***MATH,*** then decode the cryptogram

117 9 456 132 386 62 260 104 413 161 104 8

22. Write the matrix A with a key word ***MATH,*** then decode the cryptogram

438 150 145 37 240 96 635 191 445 157 260 104 413 161 104 8

23. Consider the invertible matrix: $A=\begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix}$

Decode the cryptogram

$1 \quad -5 \quad 11 \quad 19 \quad -25 \quad -45 \quad 11 \quad -16 \quad -28 \quad 20 \quad -29 \quad -27$

$12 \quad -12 \quad -53 \quad 40 \quad -61 \quad -35 \quad 8 \quad -17 \quad 7$

24. Determine the key word, then decode the given cryptogram

6	18	5	4	15	13	1	20	8
102	649	238	57	324	112	128	622	207
180	613	290	102	360	259	151	580	297

Hint: First row is the key

25. Determine the key word, then decode the given cryptogram

5	17	21	1	20	9	15	14	19
259	863	783	77	378	357	301	448	565
106	266	318	325	365	485	301	522	653
326	653	738	103	566	495	115	640	555
290	791	762	115	474	507	119	332	279
305	454	513	339	645	611	226	341	426
260	338	368	406	657	830	270	649	590
110	337	418	74	318	330	261	561	469
114	426	390	160	543	372	89	535	441
323	842	783	97	344	245	84	601	444
424	851	944	175	262	339	379	698	755
226	341	426	37	454	217	156	694	536