Solution Section 1.4 – Slack Variables and the Pivot

Exercise

Write the initial simplex tableau for each linear programing problem

a) *Maximized* : $z = 7x_1 + x_2$

subject to : $4x_1 + 2x_2 \le 5$

$x_1 + 2x_2 \le 4$

$x_1, x_2 \ge 0$

b) *Maximized* : $z = x_1 + 3x_2$

subject to : $2x_1 + 3x_2 \le 100$

$5x_1 + 4x_2 \le 200$

$x_1, x_2 \ge 0$

c) *Maximized* : $z = x_1 + 3x_2$

subject to : $x_1 + x_2 \le 10$

$5x_1 + 2x_2 \le 4$

$x_1 + 2x_2 \le 36$

$x_1, x_2 \ge 0$

d) *Maximized* : $z = 5x_1 + 3x_2$

subject to : $x_1 + x_2 \le 25$

$4x_1 + 3x_2 \le 48$

$x_1, x_2 \ge 0$

Solution

a)

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 4 & 2 & 1 & 0 & 0 & 5 \\ 1 & 2 & 0 & 1 & 0 & 4 \\ \hline -7 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

b)

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 2 & 1 & 1 & 0 & 0 & 100 \\ 5 & 4 & 0 & 1 & 0 & 200 \\ \hline -1 & -3 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

c)

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ 5 & 2 & 0 & 1 & 0 & 0 & 4 \\ 1 & 2 & 0 & 0 & 1 & 0 & 36 \\ \hline -1 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

d)

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 25 \\ 4 & 3 & 0 & 1 & 0 & 48 \\ \hline -5 & -3 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Exercise

Pivot once as indicated in each simplex tableau. Read the solution from the result

a)

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 2 & 4 & 1 & 0 & 0 & 56 \\ 2 & \{2\} & 1 & 0 & 1 & 0 & 40 \\ \hline -1 & -3 & -2 & 0 & 0 & 1 & 0 \end{array}\right]$$

b)

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array}$$

$$\left[\begin{array}{cccccc|c} 2 & 3 & 4 & 1 & 0 & 0 & 18 \\ 6 & \{3\} & 2 & 0 & 1 & 0 & 15 \\ \hline -1 & -6 & -2 & 0 & 0 & 1 & 0 \end{array}\right]$$

c)

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array}$$

$$\left[\begin{array}{ccccccc|c} 2 & 2 & \{1\} & 1 & 0 & 0 & 0 & 12 \\ 1 & 2 & 3 & 0 & 1 & 0 & 0 & 45 \\ 3 & 1 & 1 & 0 & 0 & 1 & 0 & 20 \\ \hline -2 & -1 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

d)

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array}$$

$$\left[\begin{array}{ccccccc|c} 2 & \{2\} & 3 & 1 & 0 & 0 & 0 & 500 \\ 4 & 1 & 1 & 0 & 1 & 0 & 0 & 300 \\ 7 & 2 & 4 & 0 & 0 & 1 & 0 & 700 \\ \hline -3 & -4 & -2 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Solution

a)

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 2 & 4 & 1 & 0 & 0 & 56 \\ 2 & \{2\} & 1 & 0 & 1 & 0 & 40 \\ \hline -1 & -3 & -2 & 0 & 0 & 1 & 0 \end{array}\right] \quad \frac{1}{2}R_2$$

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 2 & 4 & 1 & 0 & 0 & 56 \\ 1 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 20 \\ \hline -1 & -3 & -2 & 0 & 0 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 3R_2 \end{array}$$

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array}$$

$$\left[\begin{array}{cccccc|c} -2 & 0 & 3 & 1 & -1 & 0 & 16 \\ 1 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 20 \\ \hline 2 & 0 & -\frac{1}{2} & 0 & \frac{3}{2} & 1 & 60 \end{array}\right]$$

$$\boxed{x_1 = 0, \quad x_2 = 20, \quad x_3 = 0, \quad s_1 = 16, \quad s_2 = 0, \quad z = 60}$$

b)

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array}$$

$$\left[\begin{array}{cccccc|c} 2 & 3 & 4 & 1 & 0 & 0 & 18 \\ 6 & \{3\} & 2 & 0 & 1 & 0 & 15 \\ \hline -1 & -6 & -2 & 0 & 0 & 1 & 0 \end{array}\right] \quad \frac{1}{3}R_2$$

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 2 & 3 & 4 & 1 & 0 & 0 & 18 \\ 2 & 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 5 \\ \hline -1 & -6 & -2 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \begin{array}{l} R_1 - 3R_2 \\ \\ R_3 + 6R_2 \end{array}$$

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} -4 & 0 & 2 & 1 & -1 & 0 & 3 \\ 2 & 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 5 \\ \hline 11 & 0 & 2 & 0 & 2 & 1 & 30 \end{array}\right] \end{array}$$

$$\boxed{x_1 = 0, \quad x_2 = 5, \quad x_3 = 0, \quad s_1 = 3, \quad s_2 = 0, \quad z = 30}$$

c)

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 2 & 2 & \{1\} & 1 & 0 & 0 & 0 & 12 \\ 1 & 2 & 3 & 0 & 1 & 0 & 0 & 45 \\ 3 & 1 & 1 & 0 & 0 & 1 & 0 & 20 \\ \hline -2 & -1 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \\ R_4 + 3R_1 \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 2 & 2 & 1 & 1 & 0 & 0 & 0 & 12 \\ -5 & -4 & 0 & -3 & 1 & 0 & 0 & 9 \\ 1 & -1 & 0 & -1 & 0 & 1 & 0 & 8 \\ \hline 4 & 5 & 0 & 3 & 0 & 0 & 1 & 36 \end{array}\right] \end{array}$$

$$\boxed{x_1, x_2 = 0, \quad x_3 = 12, \quad s_1 = 0, \quad s_2 = 9, \quad s_3 = 8, \quad z = 36}$$

d)

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 2 & \{2\} & 3 & 1 & 0 & 0 & 0 & 500 \\ 4 & 1 & 1 & 0 & 1 & 0 & 0 & 300 \\ 7 & 2 & 4 & 0 & 0 & 1 & 0 & 700 \\ \hline -3 & -4 & -2 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \begin{array}{l} \frac{1}{2}R_1 \\ \\ \\ \\ \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 1 & \frac{3}{2} & \frac{1}{2} & 0 & 0 & 0 & 250 \\ 4 & 1 & 1 & 0 & 1 & 0 & 0 & 300 \\ 7 & 2 & 4 & 0 & 0 & 1 & 0 & 700 \\ \hline -3 & -4 & -2 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 + 4R_1 \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 1 & \frac{3}{2} & \frac{1}{2} & 0 & 0 & 0 & 250 \\ 3 & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 & 50 \\ 5 & 0 & 1 & -1 & 0 & 1 & 0 & 200 \\ \hline 1 & 0 & 4 & 2 & 0 & 0 & 1 & 100 \end{array}\right] \end{array}$$

$$\boxed{x_1 = x_3 = 0, \quad x_2 = 250, \quad s_1 = 0, \quad s_2 = 50, \quad s_3 = 200, \quad z = 100}$$

Exercise

The authors of a best-selling textbook in finite mathematics are told that, for the next edition of their book, each simple figure would cost the project \$20, each figure with additions would cost \$35, and each computer-drawn sketch would cost \$60. They are limited to 400 figures, for which they are allowed to spend up to \$2200. The number of computer-drawn sketches must be no more than the number of the other two types combined, and there must be at least twice as many simple figures as there are figures with additions. If each simple figure increases the royalties by \$95, each figure with additions increases royalties by \$200, and each computer-drawn figure increases royalties by \$325, how many of each type of figure should be included to maximize royalties, assuming that all art costs are borne by the publisher?

Solution

$x_1 = \textit{simple figure}$

$x_2 = \textit{additions figure}$

$x_3 = \textit{computer} - \textit{drawn sketch}$

	x_1	x_2	x_3	
Cost	20	35	60	2200
Royalties	95	200	325	

$$20x_1 + 35x_2 + 60x_3 \le 2200$$

$$x_1 + x_2 + x_3 \le 400$$

$$x_3 \le x_1 + x_2$$

$$x_1 \ge 2x_2$$

$$\textit{Maximized}: \ z = 95x_1 + 200x_2 + 325x_3$$

$$\textit{Subject to: } \begin{cases} 20x_1 + 35x_2 + 60x_3 \le 2200 \\ \ \ x_1 + x_2 + x_3 \le 400 \\ -x_1 - x_2 + x_3 \le 0 \\ -x_1 + 2x_2 \le 0 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & z \end{array} \\ \left[\begin{array}{cccccccc|c} 20 & 35 & 60 & 1 & 0 & 0 & 0 & 0 & 2200 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 400 \\ -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline -95 & -200 & -325 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & z \end{array} \\ \left[\begin{array}{cccccccc|c} .16 & 0 & 1 & .01 & 0 & .37 & 0 & 0 & 23.6 \\ .32 & 0 & 0 & -.02 & 1 & .26 & 0 & 0 & 353.68 \\ .84 & 1 & 0 & .01 & 0 & -.63 & 0 & 0 & 23.16 \\ -2.68 & 0 & 0 & -.02 & 0 & 1.26 & 1 & 0 & -46.32 \\ \hline 22.11 & 0 & 0 & 5.5 & 0 & -6.58 & 0 & 1 & 12{,}157.89 \end{array}\right] \end{array}$$

To maximize royalties of \$12,157.89, 23 of additional figure and 23 computer

Exercise

A manufacturer of bicycles builds racing, touring, and mountain models. The bicycles are made of both aluminum and steel. The company has available 91,800 units of steel and 42,000 units of aluminum. The racing, touring, and mountain models need 17, 27, and 34 units steel, and 12, 21, and 15 units of aluminum respectively. How many of each type of bicycle should be made in order to maximize profit if the company makes \$8 per racing bike, \$12 per touring bike, and \$22 per mountain bike? What is the maximum possible profit?

Solution

	Racing x_1	***Touring*** x_2	***Mountain*** x_3	
Steel	17	27	34	91,800
Aluminum	12	21	15	42,000
Profit	\$8	\$12	\$22	

Maximized: $z = 8x_1 + 12x_2 + 22x_3$

Subject to: $\begin{cases} 17x_1 + 27x_2 + 34x_3 \le 91{,}800 \\ 12x_1 + 21x_2 + 15x_3 \le 42{,}000 \\ x_1, x_2, x_3 \ge 0 \end{cases}$

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 17 & 27 & 34 & 1 & 0 & 0 & 91{,}800 \\ 12 & 21 & 15 & 0 & 1 & 0 & 42{,}000 \\ \hline -8 & -12 & -22 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} .5 & .79 & 1 & .03 & 0 & 0 & 2,700 \\ 4.5 & 9.1 & 0 & .44 & 1 & 0 & 1,500 \\ \hline 3 & 5.47 & 0 & .65 & 0 & 1 & 59,400 \end{array}\right] \end{array}$$

To maximize the profit of \$59,400. The should make 2,700 mountain bike only.