Solution Section 3.3 – Double Integrals in Polar Coordinates

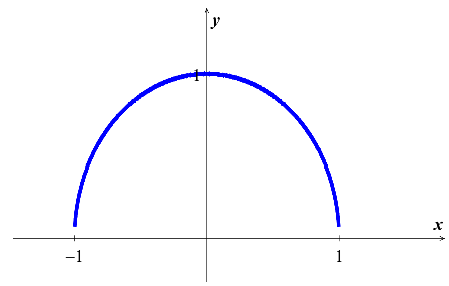
Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dydx$$

Solution

$$y=\sqrt{1-x^2} \quad\Rightarrow\quad y^2=1-x^2 \to x^2+y^2=1=r^2$$



$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dydx=\int_{0}^{\pi}\int_{0}^{1} r\,drd\theta$$

$$=\int_{0}^{\pi}\frac{1}{2}\left(r^2\right|_{0}^{1}d\theta$$

$$=\frac{1}{2}\int_{0}^{\pi}d\theta$$

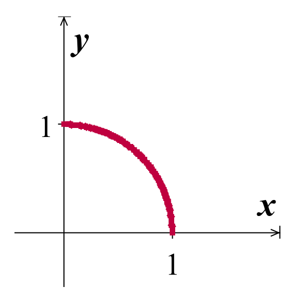
$$=\frac{\pi}{2}$$

Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}}\left(x^2+y^2\right)dxdy$$

Solution



$$\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}}\left(x^2+y^2\right)dxdy=\int_{0}^{\pi/2}d\theta\int_{0}^{1}r^2\,rdr$$

$$=\frac{\pi}{2}\,\frac{1}{4}\left(r^4\right|_{0}^{1}$$

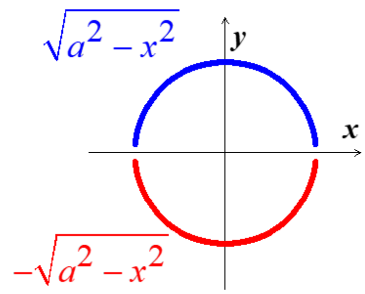
$$=\frac{\pi}{8}$$

Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dydx$$

Solution



$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dydx = \int_0^{2\pi} d\theta \int_0^a rdr$$

$$= (2\pi)\ \frac{1}{2}\left(r^2\ \Big|_0^a\right.$$

$$\underline{= \pi a^2}$$

Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_0^6\int_0^y xdxdy$$

Solution

$$x = r\cos\theta, \quad \sin\theta = \frac{6}{r} \to r = \frac{6}{\sin\theta} = 6\csc\theta$$

$$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$$

$$\int_0^6\int_0^y x\,dxdy = \int_{\pi/4}^{\pi/2}\int_0^{6\csc\theta} r^2\cos\theta\ drd\theta$$

$$= \frac{1}{3}\int_{\pi/4}^{\pi/2} \cos\theta\left(r^3\ \Big|_0^{6\csc\theta}\right.\ d\theta$$

$$= \frac{216}{3}\int_{\pi/4}^{\pi/2} \cos\theta\csc^3\theta\ d\theta$$

$$= 72\int_{\pi/4}^{\pi/2} \cot\theta\ \csc^2\theta\ d\theta \qquad d(\cot\theta) = -\csc^2\theta\ d\theta$$

$$= -72\int_{\pi/4}^{\pi/2} \cot\theta\ d(\cot\theta)$$

$$= -36\left(\cot^2\theta \,\Big|_{\pi/4}^{\pi/2}\right.$$

$$= -36(0-1)$$

$$\boxed{= 36}$$

Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\,dydx$$

Solution

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\,dydx = \int_{\pi}^{3\pi/2}\int_{0}^{1} \frac{2}{1+r}r\;drd\theta$$

$$= 2\int_{\pi}^{3\pi/2} d\theta \int_{0}^{1}\left(1-\frac{1}{1+r}\right)dr$$

$$= 2\left(\frac{3\pi}{2}-\pi\right)\left(r-\ln(1+r)\right)\Big|_{0}^{1}$$

$$\boxed{= (1-\ln 2)\pi}$$

Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{0}^{\ln 2}\int_{0}^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\,dxdy$$

Solution

$$\int_{0}^{\ln 2}\int_{0}^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\,dxdy = \int_{0}^{\pi/2} d\theta \int_{0}^{\ln 2} e^{r}r\;dr$$

		$\int e^r$
+	r	e^r
−	1	e^r

$$= \frac{\pi}{2}\left(re^r - e^r\right)\Big|_{0}^{\ln 2}$$

$$= \frac{\pi}{2}\left(\ln 2e^{\ln 2} - e^{\ln 2}+1\right)$$

$$\boxed{= \frac{\pi}{2}(2\ln 2-1)}$$

Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)dxdy$$

Solution

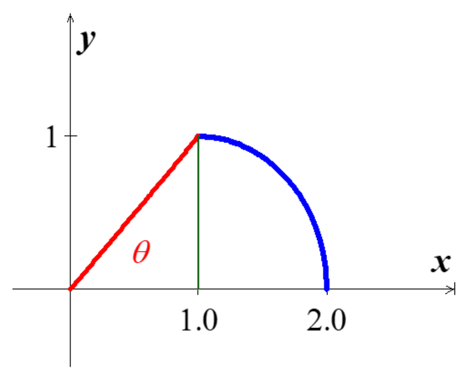
$$\begin{aligned}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)dxdy &= \int_0^{2\pi} d\theta \int_0^1 \ln\left(r^2+1\right)r\, dr\\ &= (2\pi)\int_0^1 \ln\left(r^2+1\right)\frac{1}{2}\, d\left(r^2+1\right) \qquad \int \ln au\ du = u\ln au\ -u\\ &= \pi\left(\left(r^2+1\right)\ln\left(r^2+1\right)-\left(r^2+1\right)\right)\Big|_0^1\\ &= \pi\left(2\ln 2-2+1\right)\\ &= \boxed{\pi\left(\ln 4-1\right)}\end{aligned}$$

Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_1^2\int_0^{\sqrt{2x-x^2}} \frac{1}{\left(x^2+y^2\right)^2}dydx$$

Solution



$$y^2=2x-x^2 \Rightarrow x^2-2x+1-1+y^2=0 \quad (x-1)^2+y^2=1$$

$$r=\frac{x}{\cos\theta}=\frac{1}{\cos\theta}=\sec\theta$$

$$y=\sqrt{2x-x^2} \quad \rightarrow \quad y^2=2x-x^2 \Rightarrow x^2+y^2=2x$$

$$r^2=2r\cos\theta \rightarrow r=2\cos\theta$$

$$\int_1^2\int_0^{\sqrt{2x-x^2}} \frac{1}{\left(x^2+y^2\right)^2}dydx=\int_0^{\pi/4}\int_{\sec\theta}^{2\cos\theta}\frac{1}{r^4}r\, drd\theta$$

$$= \int_0^{\pi/4} \int_{\sec\theta}^{2\cos\theta} r^{-3}\, dr d\theta$$

$$= \int_0^{\pi/4} \left(-\frac{1}{2r^2} \Big|_{\sec\theta}^{2\cos\theta} \right) d\theta$$

$$= \int_0^{\pi/4} \left(-\frac{1}{8\cos^2\theta} + \frac{1}{2\sec^2\theta} \right) d\theta$$

$$= \int_0^{\pi/4} \left(-\frac{1}{8}\sec^2\theta + \frac{1}{2}\cos^2\theta \right) d\theta \qquad \int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$= -\frac{1}{8}\tan\theta + \frac{1}{2}\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \right) \Big|_0^{\pi/4}$$

$$= \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta - \frac{1}{8}\tan\theta \Big|_0^{\pi/4}$$

$$= \frac{1}{4}\frac{\pi}{4} + \frac{1}{8} - \frac{1}{8} - (0)$$

$$= \frac{\pi}{16}$$

Exercise

Evaluate the integral by changing to polar coordinates $\displaystyle \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2dydx}{\left(1+x^2+y^2\right)^2}$

Solution

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2dydx}{\left(1+x^2+y^2\right)^2} = \int_0^{2\pi} d\theta \int_0^1 \frac{2r}{\left(1+r^2\right)^2} dr$$

$$= (2\pi) \int_0^1 \left(1+r^2\right)^{-2} d\left(1+r^2\right)$$

$$= 2\pi \ \frac{-1}{1+r^2} \Big|_0^1$$

$$= 2\pi \left(-\frac{1}{2} + 1 \right)$$

$$= \pi$$

Exercise

Evaluate the integral by changing to polar coordinates $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)dxdy$

Solution

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)dxdy = \int_0^{2\pi} d\theta \int_0^1 \ln\left(r^2+1\right)r\ dr$$

$$= (2\pi)\ \frac{1}{2}\int_0^1 \ln\left(r^2+1\right)\ d\left(r^2+1\right)$$

$$r^2+1=w$$

$$u=\ln w \quad\rightarrow\quad du=\frac{dw}{w} \qquad v=\int dw=w$$

$$\int \ln w\ dw = w\ln w - \int dw$$

$$= w\ln w - w$$

$$= \pi\left(\left(r^2+1\right)\left(\ln\left(r^2+1\right)-1\right)\right)\Bigg|_0^1$$

$$= \pi\left(2\ln 2 - 2 + 1\right)$$

$$\underline{= \pi\left(2\ln 2 - 1\right)}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\infty}\int_0^{\infty} \frac{1}{\left(1+x^2+y^2\right)^2}\ dxdy$

Solution

$$\int_0^{\infty}\int_0^{\infty} \frac{1}{\left(1+x^2+y^2\right)^2}\ dxdy = \int_0^{\pi/2} d\theta \int_0^{\infty} \frac{1}{\left(1+r^2\right)^2} rdr \qquad d\left(1+r^2\right)=2rdr$$

$$= \frac{\pi}{2}\int_0^{\infty}\left(1+r^2\right)^{-2}\frac{1}{2}d\left(1+r^2\right)$$

$$= \frac{\pi}{4}\left(-\frac{1}{1+r^2}\right)\Bigg|_0^{\infty} \qquad \frac{1}{\infty}=0$$

$$\underline{= \frac{\pi}{4}}$$

Exercise

Evaluate the integral $\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\, dydx$

Solution

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\, dydx = \int_0^{\frac{\pi}{2}} \int_0^3 r\ rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^3 r^2\, dr$$

$$= \frac{\pi}{2}\left(\frac{1}{3}r^3\right|_0^3$$

$$= \frac{9\pi}{2}$$

Exercise

Evaluate the integral $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(x^2+y^2\right)^{3/2}\, dydx$

Solution

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(x^2+y^2\right)^{3/2}\, dydx = \int_0^{2\pi} \int_0^1 \left(r^2\right)^{3/2}\ rdrd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 r^4\, dr$$

$$= 2\pi\left(\frac{1}{5}r^5\right|_0^1$$

$$= \frac{2\pi}{5}$$

Exercise

Evaluate the integral $\int_{-4}^4 \int_0^{\sqrt{16-y^2}} \left(16-x^2-y^2\right)\, dxdy$

Solution

$$\int_{-4}^{4}\int_{0}^{\sqrt{16-y^2}}\left(16-x^2-y^2\right)\,dxdy=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{4}\left(16-r^2\right)\,rdrd\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_{0}^{4}\left(16r-r^3\right)\,dr$$

$$=\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\left(8r^2-\frac{1}{4}r^4\right.\Big|_{0}^{4}$$

$$=\pi\left(128-64\right)$$

$$=64\pi$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{\frac{\pi}{4}}\int_{0}^{\sec\theta}r^3\,drd\theta$

Solution

$$\int_{0}^{\frac{\pi}{4}}\int_{0}^{\sec\theta}r^3\,drd\theta=\frac{1}{4}\int_{0}^{\frac{\pi}{4}}r^4\Big|_{0}^{\sec\theta}\,d\theta$$

$$=\frac{1}{4}\int_{0}^{\frac{\pi}{4}}\sec^4\theta\,d\theta$$

$$=\frac{1}{4}\int_{0}^{\frac{\pi}{4}}\sec^2\theta\sec^2\theta\,d\theta$$

$$=\frac{1}{4}\int_{0}^{\frac{\pi}{4}}\left(1+\tan^2\theta\right)\,d\left(\tan\theta\right)$$

$$=\frac{1}{4}\left(\tan\theta+\frac{1}{3}\tan^3\theta\right.\Big|_{0}^{\frac{\pi}{4}}$$

$$=\frac{1}{4}\left(1+\frac{1}{3}\right)$$

$$=\frac{1}{3}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\frac{\pi}{2}}\int_1^{\infty}\frac{\cos\theta}{r^3}r\ drd\theta$

Solution

$$\int_0^{\frac{\pi}{2}}\int_1^{\infty}\frac{\cos\theta}{r^3}r\ drd\theta=\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta\int_1^{\infty}\frac{1}{r^2}dr$$

$$=\sin\theta\Big|_0^{\frac{\pi}{2}}\left(-\frac{1}{r}\Big|_1^{\infty}\right.$$

$$=-(1)(0-1) \qquad \frac{1}{\infty}=0$$

$$\underline{=1}$$

Exercise

Find the area of the region cut from the first quadrant by the curve $r=2(2-\sin 2\theta)^{1/2}$

Solution

$$\int_0^{\pi/2}\int_0^{2\sqrt{2-\sin 2\theta}}rdrd\theta=\frac{1}{2}\int_0^{\pi/2}\left(r^2\Big|_0^{2\sqrt{2-\sin 2\theta}}\right.d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}4(2-\sin 2\theta)d\theta$$

$$=2\left(2\theta+\frac{1}{2}\cos 2\theta\right.\Big|_0^{\pi/2}$$

$$=2\left[\pi-\frac{1}{2}-\left(\frac{1}{2}\right)\right]$$

$$\underline{=2(\pi-1)}$$

Exercise

Find the area of the region lies inside the cardioid $r=1+\cos\theta$ and outside the circle $r=1$

Solution

$$A=2\int_0^{\pi/2}\int_1^{1+\cos\theta}rdrd\theta$$

$$= \int_0^{\pi/2} \left(r^2 \Big|_1^{1+\cos\theta} \right. d\theta$$

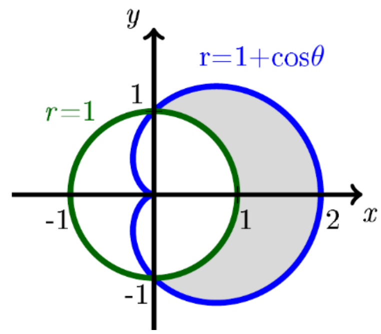
$$= \int_0^{\pi/2} \left((1+\cos\theta)^2 - 1 \right) d\theta$$

$$= \int_0^{\pi/2} \left(1 + 2\cos\theta + \cos^2\theta - 1 \right) d\theta$$

$$= \int_0^{\pi/2} \left(2\cos\theta + \cos^2\theta \right) d\theta$$

$$= 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \Big|_0^{\pi/2}$$

$$\boxed{= 2 + \frac{\pi}{4}}$$



$$\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

Exercise

Find the area enclosed by one leaf of the rose $r = 12\cos 3\theta$

Solution

$$A = 2\int_0^{\pi/6} \int_0^{12\cos 3\theta} r dr d\theta$$

$$= \int_0^{\pi/6} \left(r^2 \Big|_0^{12\cos 3\theta} \right. d\theta$$

$$= 144 \int_0^{\pi/6} \cos^2 3\theta d\theta$$

$$\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$= 144 \left(\frac{\theta}{2} + \frac{\sin 6\theta}{12} \Big|_0^{\pi/6} \right.$$

$$= 144 \left(\frac{\pi}{12} \right)$$

$$\boxed{= 12\pi}$$

Exercise

Find the area of the region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$

Solution

$$A = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2} \left(r^2 \Big|_0^{1-\cos\theta}\right) d\theta$$

$$= 2\int_0^{\pi/2} (1-\cos\theta)^2\, d\theta$$

$$= 2\int_0^{\pi/2} \left(1 - 2\cos\theta + \cos^2\theta\right) d\theta \qquad \int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$= 2\left(\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right.\Big|_0^{\pi/2}$$

$$= 2\left(\frac{\pi}{2} - 2 + \frac{\pi}{4}\right)$$

$$\boxed{= \frac{3\pi}{2} - 4}$$

Exercise

Find the area of the region bounded by all leaves of the rose $r = 3\cos 2\theta$

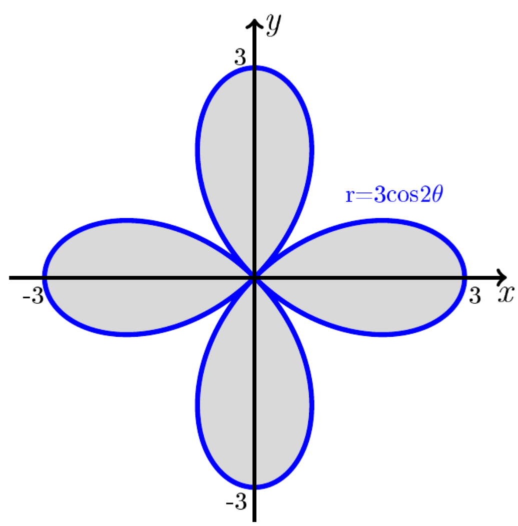
Solution

$$A = 4\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{3\cos 2\theta} r\;dr\,d\theta$$

$$= 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2 \Big|_0^{3\cos 2\theta}\; d\theta$$

$$= 18\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta\; d\theta$$

$$= 9\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 + \cos 4\theta)\; d\theta$$



$$= 9\left(\theta + \frac{1}{4}\sin 4\theta\right)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= 9\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$\boxed{= \frac{9\pi}{2} \quad unit^2}$$

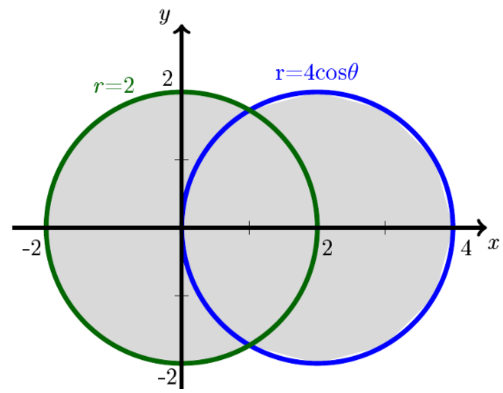
Exercise

Find the area of the region inside both the circles $r = 2$ and $r = 4\cos\theta$

Solution

$$r = 4\cos\theta = 2 \quad \rightarrow \quad \cos\theta = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}, \ \frac{5\pi}{3}$$



$$A = 2\int_0^{\frac{\pi}{3}}\int_0^2 r\ drd\theta \ + \ 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{4\cos\theta} r\ drd\theta$$

$$= \int_0^{\frac{\pi}{3}} d\theta \quad r^2 \Big|_0^2 \ + \ \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} r^2 \Big|_0^{4\cos\theta} d\theta$$

$$= \frac{4\pi}{3} \ + \ 16\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2\theta\ d\theta$$

$$= \frac{4\pi}{3} + 8\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1 + \cos 2\theta)\ d\theta$$

$$= \frac{4\pi}{3} + 8\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \frac{4\pi}{3} + 8\left(\frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$

$$= \frac{4\pi}{3} + \frac{4\pi}{3} - 2\sqrt{3}$$

$$\boxed{= \frac{8\pi}{3} - 2\sqrt{3} \ \ unit^2}$$

Exercise

Find the area of the region that lies inside both the cardioids $r = 2 - 2\cos\theta$ and $r = 2 + 2\cos\theta$

Solution

$$A = 4\int_0^{\frac{\pi}{2}}\int_0^{2-2\cos\theta} r\ drd\theta$$

$$= 2\int_0^{\frac{\pi}{2}} r^2 \Big|_0^{2-2\cos\theta} d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (2-2\cos\theta)^2\, d\theta$$

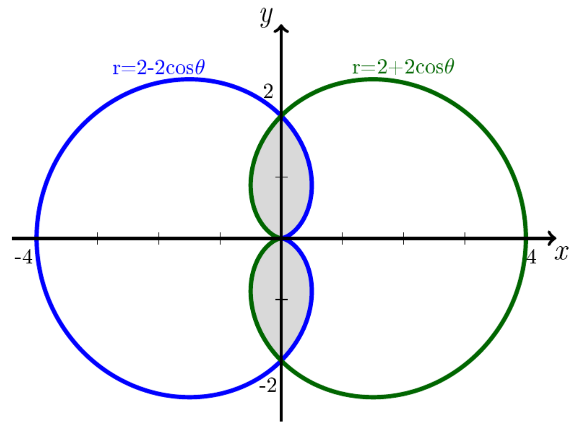
$$= 2\int_0^{\frac{\pi}{2}} \left(4-8\cos\theta+4\cos^2\theta\right)\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (6-8\cos\theta+2\cos 2\theta)\ d\theta$$

$$= 2\left(6\theta-8\sin\theta+\sin 2\theta\right. \Big|_0^{\frac{\pi}{2}}$$

$$= 2(3\pi-8)$$

$$\underline{= 6\pi-16\ \ unit^2}$$



Exercise

Find the area of the annular region $\{(r,\ \theta):\ \ 1\le r\le 2,\ \ \ 0\le\theta\le 2\pi\}$

Solution

$$\int_0^{2\pi}\int_1^2 r\ drd\theta = \int_0^{2\pi} d\theta\ \ \left(\frac{1}{2}r^2\ \Big|_1^2\right.$$

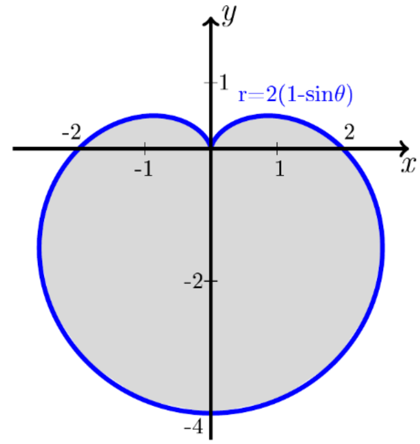
$$= 2\pi\frac{1}{2}(4-1)$$

$$\underline{= 3\pi\ \ unit^2}$$

Exercise

Find the area of the region bounded by the cardioid $r = 2(1-\sin\theta)$

Solution



$$A = \int_0^{2\pi}\int_0^{2(1-\sin\theta)} r\, dr d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2}r^2 \Big|_0^{2(1-\sin\theta)}\right) d\theta$$

$$= 2\int_0^{2\pi} \left(1-2\sin\theta+\sin^2\theta\right) d\theta$$

$$= 2\int_0^{2\pi} \left(\frac{3}{2}-2\sin\theta-\frac{1}{2}\cos 2\theta\right) d\theta$$

$$= 2\left(\frac{3}{2}\theta+2\cos\theta-\frac{1}{4}\sin 2\theta\right. \Big|_0^{2\pi}$$

$$= 2(3\pi+2-2)$$

$$\underline{= 6\pi \ \ unit^2}$$

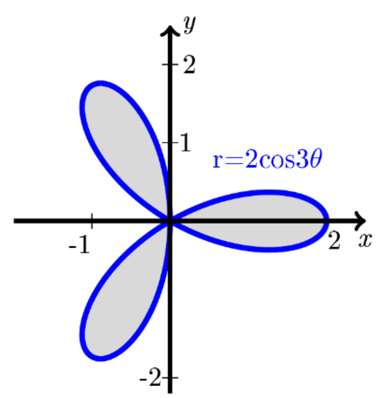
Exercise

Find the area of the region bounded by all leaves of the rose $r = 2\cos 3\theta$

Solution

$r = 2\cos 3\theta = 2 \quad \to 3\theta = 0+2n\pi \quad \Rightarrow \theta = 0, \ \ldots$

$r = 2\cos 3\theta = 0 \quad \to 3\theta = \frac{\pi}{2}+2n\pi \quad \Rightarrow \theta = \frac{\pi}{6}, \ \ldots$



$$A = 6\int_0^{\frac{\pi}{6}}\int_0^{2\cos 3\theta} r\, dr d\theta$$

$$= 3\int_0^{\frac{\pi}{6}} \left(r^2 \Big|_0^{2\cos 3\theta}\right. d\theta$$

$$= 12\int_0^{\frac{\pi}{6}} \cos^2 3\theta\, d\theta$$

$$= 6\int_0^{\frac{\pi}{6}} (1+\cos 6\theta)\, d\theta$$

$$= 6\left(\theta + \frac{1}{6}\sin 6\theta \,\right|_0^{\pi/6}$$

$$= 6\left(\frac{\pi}{6}\right)$$

$$\boxed{= \pi \ \ unit^2}$$

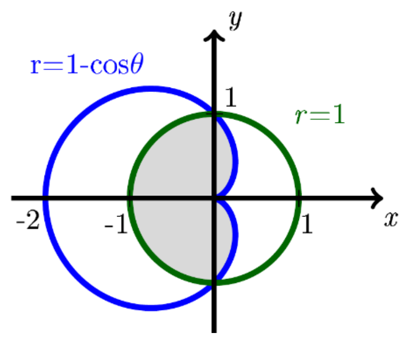
Exercise

Find the area of the region inside both the cardioid $r = 1 - \cos\theta$ and the circle $r = 1$

Solution

$r = 1 - \cos\theta = 1 \quad \to \quad \cos\theta = 0$

$\theta = \frac{\pi}{2}, \ \frac{3\pi}{2}$



$$A = \left(area\ of\ \frac{1}{2}\ circle\right) + 2\int_0^{\pi/2}\int_0^{1-\cos\theta} r\ drd\theta$$

$$= \frac{\pi}{2} + \int_0^{\pi/2} r^2 \Big|_0^{1-\cos\theta} d\theta$$

$$= \frac{\pi}{2} + \int_0^{\pi/2} (1-\cos\theta)^2\, d\theta$$

$$= \frac{\pi}{2} + \int_0^{\pi/2} \left(1 - 2\cos\theta + \cos^2\theta\right)d\theta$$

$$= \frac{\pi}{2} + \int_0^{\pi/2} \left(\frac{3}{2} - 2\cos\theta + \cos 2\theta\right)d\theta$$

$$= \frac{\pi}{2} + \left(\frac{3}{2}\theta - 2\sin\theta + \frac{1}{2}\sin 2\theta \,\right|_0^{\pi/2}$$

$$= \frac{\pi}{2} + \frac{3\pi}{4} - 2$$

$$\boxed{= \frac{5\pi}{4} - 2 \ \ unit^2}$$

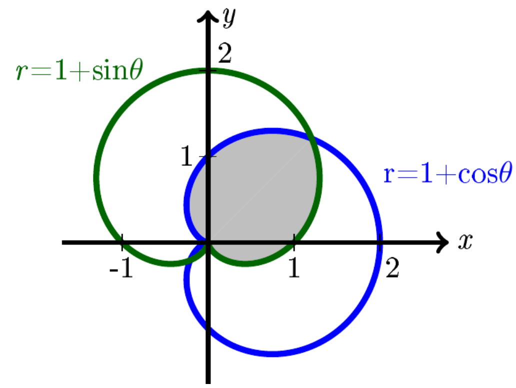
Exercise

Find the area of the region inside both the cardioid $r = 1 + \sin\theta$ and the cardioid $r = 1 + \cos\theta$

Solution

$$r = 1 + \sin\theta = 1 + \cos\theta \quad \rightarrow \quad \sin\theta = \cos\theta$$

$\theta = \frac{\pi}{4}, \ \frac{5\pi}{4}$, and due to the symmetry;



$$A = 2\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \int_0^{1+\cos\theta} r \, dr d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} r^2 \Big|_0^{1+\cos\theta} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1+\cos\theta)^2 \ d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left(\frac{3}{2} + 2\cos\theta + \cos 2\theta\right) d\theta$$

$$= \frac{3}{2}\theta + 2\sin\theta + \frac{1}{2}\sin 2\theta \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \frac{15\pi}{8} - \sqrt{2} + \frac{1}{2} - \frac{3\pi}{8} - \sqrt{2} - \frac{1}{2}$$

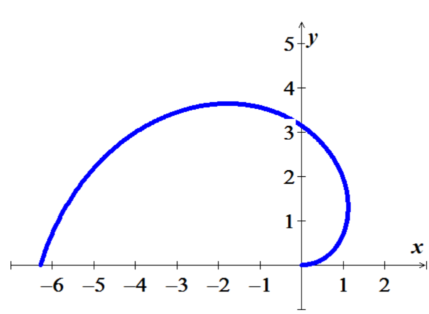
$$= \frac{3\pi}{2} - 2\sqrt{2} \ \ unit^2$$

Exercise

Find the area of the region bounded by the spiral $r = 2\theta$, for $0 \le \theta \le \pi$, and the x-axis.

Solution

$$A = \int_0^{\pi} \int_0^{2\theta} r \, dr d\theta$$



$$= \frac{1}{2}\int_0^{\pi} r^2 \Big|_0^{2\theta} \, d\theta$$

$$= 2\int_0^{\pi} \theta^2 \, d\theta$$

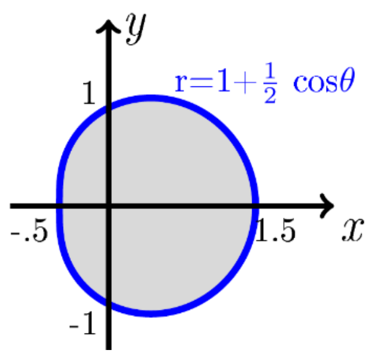
$$= \frac{2}{3}\theta^3 \Big|_0^{\pi}$$

$$= \frac{2\pi^3}{3} \ unit^2$$

Exercise

Find the area of the region inside the limaçon $r = 1 + \frac{1}{2}\cos\theta$

Solution



$$A = \int_0^{2\pi}\int_0^{1+\frac{1}{2}\cos\theta} r \, dr d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} r^2 \Big|_0^{1+\frac{1}{2}\cos\theta} \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(1 + \frac{1}{2}\cos\theta\right)^2 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(1 + \cos\theta + \frac{1}{4}\cos^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(\frac{9}{8} + \cos\theta + \frac{1}{8}\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\left(\frac{9}{8}\theta + \sin\theta + \frac{1}{16}\sin 2\theta \Big|_0^{2\pi}\right.$$

$$= \frac{9\pi}{8} \ unit^2$$

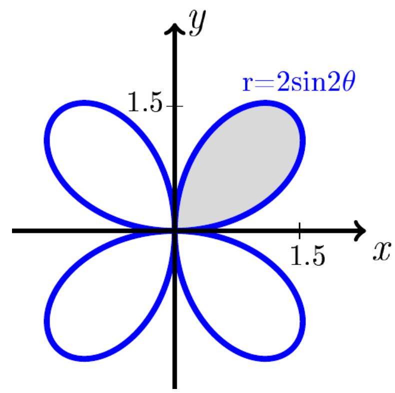
Exercise

Find the area of the region bounded by $r = 2\sin 2\theta$ in QI.

Solution

$$r = 2\sin 2\theta = 0$$

$$2\theta = n\pi \quad \Rightarrow \quad \boxed{\theta = 0,\ \frac{\pi}{2}}$$



$$A = \int_0^{\frac{\pi}{2}} \int_0^{2\sin 2\theta} r\, dr d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \Big|_0^{2\sin 2\theta} d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^2 2\theta\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} (1 - \cos 4\theta)\ d\theta$$

$$= \theta - \frac{1}{4}\sin 4\theta \Big|_0^{\frac{\pi}{2}}$$

$$\boxed{= \frac{\pi}{2}\ \ unit^2}$$

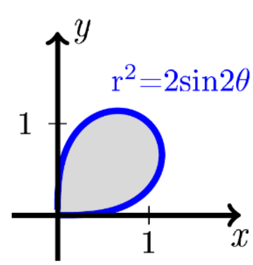
Exercise

Find the area of the region bounded by $r^2 = 2\sin 2\theta$ in QI.

Solution

$$r^2 = 2\sin 2\theta = 0$$

$$2\theta = n\pi \quad \Rightarrow \quad \boxed{\theta = 0,\ \frac{\pi}{2}}$$



$$A = \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2\sin 2\theta}} r\, dr d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \Big|_0^{\sqrt{2\sin 2\theta}} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin 2\theta \, d\theta$$

$$= -\frac{1}{2}\cos 2\theta \Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{2}(-1-1)$$

$$= 1 \ \ unit^2$$

Exercise

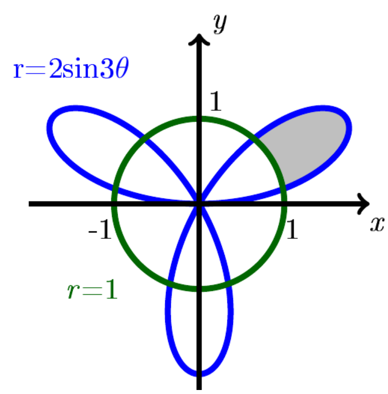
Find the area of the region outside the circle $r = 1$ and inside the rose $r = 2\sin 3\theta$ in QI.

Solution

$$r = 2\sin 3\theta = 1$$

$$3\theta = \frac{\pi}{6}, \ \frac{5\pi}{6} \quad \Rightarrow \quad \theta = \frac{\pi}{18}, \ \frac{5\pi}{18}$$

$$A = \int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} \int_1^{2\sin 3\theta} r \, dr d\theta$$



$$= \frac{1}{2}\int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} r^2 \Big|_1^{2\sin 3\theta} d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} \left(4\sin^2 3\theta - 1\right) d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} \left(1 - 2\cos 6\theta\right) d\theta$$

$$= \frac{1}{2}\left(\theta - \frac{1}{3}\cos 6\theta \right. \Big|_{\frac{\pi}{18}}^{\frac{5\pi}{18}}$$

$$= \frac{1}{2}\left(\frac{5\pi}{18} - \frac{1}{3}\cos\frac{5\pi}{3} - \frac{\pi}{18} + \frac{1}{3}\cos\frac{\pi}{3}\right)$$

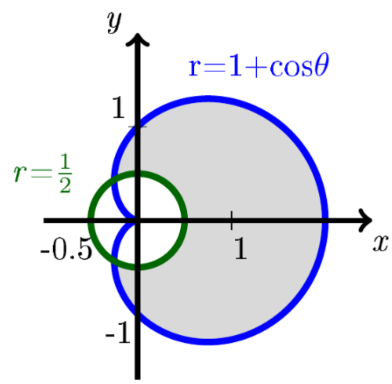
$$= \frac{1}{2}\left(\frac{2\pi}{9} - \frac{1}{6} + \frac{1}{6}\right)$$

$$= \frac{\pi}{9} \ \ unit^2$$

Exercise

Find the area of the region outside the circle $r=\frac{1}{2}$ and inside the circle $r=1+\cos\theta$

Solution



$$A=2\int_0^{\pi/2}\int_{\frac{1}{2}}^{1+\cos\theta} r\,dr\,d\theta$$

$$=\int_0^{\pi/2}\left(r^2 \Big|_{\frac{1}{2}}^{1+\cos\theta}\right.d\theta$$

$$=\int_0^{\pi/2}\left[(1+\cos\theta)^2-\frac{1}{4}\right]d\theta$$

$$=\int_0^{\pi/2}\left(\frac{3}{4}+2\cos\theta+\cos^2\theta\right)d\theta$$

$$=\int_0^{\pi/2}\left(\frac{5}{4}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$=\frac{5}{4}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\Big|_0^{\pi/2}$$

$$=\frac{5\pi}{8}+2\ \ unit^2$$

Exercise

Integrate $f(x,y)=\dfrac{\ln\left(x^2+y^2\right)}{\sqrt{x^2+y^2}}$ over the region $1\le x^2+y^2\le e$

Solution

$$\int_0^{2\pi}\int_1^{\sqrt{e}}\left(\frac{\ln r^2}{r}\right)r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_1^{\sqrt{e}}2\ln r\,dr$$

$$=2\pi\ \left(r\ln r-r\Big|_1^{\sqrt{e}}\right.$$

$$=2\pi\left(\sqrt{e}\ln e^{1/2}-\sqrt{e}-(0-1)\right)$$

$$=2\pi\left(2-\sqrt{e}\right)$$

Exercise

The region enclosed by the lemniscates $r^2 = 2\cos 2\theta$ is the base of a solid right cylinder whose top is bounded by the sphere $z = \sqrt{2-r^2}$. Find the cylinder's volume.

Solution

$$V = 4\int_0^{\pi/4}\int_0^{\sqrt{2\cos 2\theta}} r\sqrt{2-r^2}\,dr\,d\theta \qquad\qquad d\left(2-r^2\right) = -2r\,dr$$

$$= -2\int_0^{\pi/4}\int_0^{\sqrt{2\cos 2\theta}} \left(2-r^2\right)^{1/2} d\left(2-r^2\right)\,d\theta$$

$$= -2\int_0^{\pi/4} \tfrac{2}{3}\left(2-r^2\right)^{3/2}\Big|_0^{\sqrt{2\cos 2\theta}}\,d\theta$$

$$= -\tfrac{4}{3}\int_0^{\pi/4}\left[(2-2\cos 2\theta)^{3/2} - 2^{3/2}\right]d\theta$$

$$= -\tfrac{4}{3}\int_0^{\pi/4}\left[2^{3/2}(1-\cos 2\theta)^{3/2}\right]d\theta + \tfrac{4}{3}\int_0^{\pi/4} 2^{3/2}\,d\theta$$

$$= -\tfrac{4}{3}2\sqrt{2}\int_0^{\pi/4}\left(2\sin^2\theta\right)^{3/2} d\theta + \tfrac{4}{3}2\sqrt{2}\left(\tfrac{\pi}{4}\right)$$

$$= -\frac{8\sqrt{2}}{3}\int_0^{\pi/4} 2\sqrt{2}\,\sin^3\theta\,d\theta + \frac{2\pi}{3}\sqrt{2}$$

$$= -\frac{32}{3}\int_0^{\pi/4}\sin^2\theta\,\sin\theta\,d\theta + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\int_0^{\pi/4}\left(1-\cos^2\theta\right)d(\cos\theta) + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\left(\cos\theta - \tfrac{1}{3}\cos^3\theta\right|_0^{\pi/4} + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\left[\frac{\sqrt{2}}{2} - \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3 - \left(1-\frac{1}{3}\right)\right] + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12} - \frac{2}{3}\right) + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\left(\frac{5\sqrt{2}-8}{12}\right) + \frac{2\pi\sqrt{2}}{3}$$

$$= 8\left(\frac{5\sqrt{2}-8}{9}\right)+\frac{2\pi\sqrt{2}}{3}$$

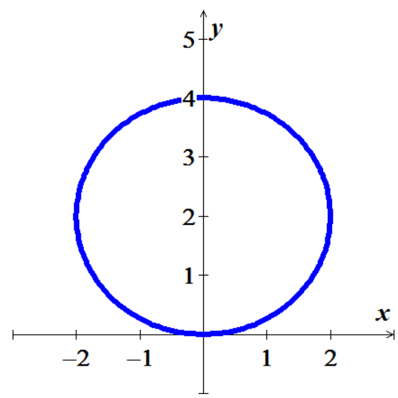
$$= \frac{40\sqrt{2}-64+6\pi\sqrt{2}}{9}\ \ unit^3$$

Exercise

Evaluate $\displaystyle\iint_R (x+y)\,dA$; R is the disk bounded by circle $r = 4\sin\theta$

Solution

$$\iint_R (x+y)\,dA = \int_0^{\pi}\int_0^{4\sin\theta} (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta$$



$$= \int_0^{\pi}\int_0^{4\sin\theta} (\cos\theta+\sin\theta)\,r^2\ dr\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi} (\cos\theta+\sin\theta)\ r^3 \Big|_0^{4\sin\theta}\ d\theta$$

$$= \frac{64}{3}\int_0^{\pi} (\cos\theta+\sin\theta)\sin^3\theta\ d\theta$$

$$= \frac{64}{3}\int_0^{\pi} \cos\theta\sin^3\theta\ d\theta + \frac{64}{3}\int_0^{\pi} \sin^4\theta\ d\theta$$

$$= \frac{64}{3}\int_0^{\pi} \sin^3\theta\ d(\sin\theta) + \frac{64}{3}\int_0^{\pi} \frac{1}{4}(1-\cos 2\theta)^2\ d\theta$$

$$= \frac{16}{3}\sin^4\theta\Big|_0^{\pi} + \frac{64}{3}\int_0^{\pi} \frac{1}{4}\left(1-2\cos 2\theta+\cos^2 2\theta\right)\ d\theta$$

$$= \frac{16}{3}\int_0^{\pi} \left(\frac{3}{2}-2\cos 2\theta+\cos 4\theta\right)\ d\theta$$

$$= \frac{16}{3}\left(\frac{3}{2}\theta - \sin 2\theta + \frac{1}{4}\sin 4\theta\right)\Big|_0^{\pi}$$

$$= 8\pi$$

Exercise

Find the volume of the solid bounded above by the paraboloid $z = 2 - x^2 - y^2$ and below by the plane $z = 1$

Solution

$$z = 2 - x^2 - y^2 - 1 \quad \rightarrow \quad x^2 + y^2 = 1$$

$$0 \le r \le 1 \quad \& \quad 0 \le \theta \le 2\pi$$

$$V = \iint_R \left(2 - x^2 - y^2 - 1\right) dA$$

$$= \int_0^{2\pi} d\theta \int_0^1 \left(1 - r^2\right) r\, dr$$

$$= 2\pi \int_0^1 \left(r - r^3\right) dr$$

$$= 2\pi \left(\frac{1}{2} r^2 - \frac{1}{4} r^4 \right|_0^1$$

$$= 2\pi \left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{\pi}{2} \ unit^3$$

Exercise

Find the volume of the solid bounded above by the paraboloid $z = 8 - x^2 - 3y^2$ and below by the hyperbolic paraboloid $z = x^2 - y^2$

Solution

$$z = 8 - x^2 - 3y^2 = x^2 - y^2 \quad \rightarrow \quad x^2 + y^2 = 4$$

$$0 \le r \le 2 \quad \& \quad 0 \le \theta \le 2\pi$$

$$V = \iint_R \left(8 - x^2 - 3y^2 - x^2 + y^2\right) dA$$

$$= \iint_R \left(8 - 2\left(x^2 + y^2\right)\right) dA$$

$$= 2\int_0^{2\pi} d\theta \int_0^2 \left(4 - r^2\right) r\, dr$$

$$= 4\pi \int_0^2 \left(4r - r^3\right) dr$$

$$= 4\pi\left(2r^2 - \frac{1}{4}r^4 \Big|_0^2\right.$$

$$= 4\pi(8-4)$$

$$\underline{= 16\pi \ \ unit^3}$$

Exercise

Evaluate the integral over R using polar coordinates

$$\iint_R \left(x^2 + y^2\right) dA; \quad R = \{(r, \theta): \ 0 \le r \le 4, \quad 0 \le \theta \le 2\pi\}$$

Solution

$$\iint_R \left(x^2 + y^2\right) dA = \int_0^{2\pi} \int_0^4 \left(r^2\right) r \ dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^4 r^3 \ dr$$

$$= 2\pi\left(\frac{1}{4}r^4 \Big|_0^4\right.$$

$$\underline{= 128\pi}$$

Exercise

Evaluate the integral over R using polar coordinates

$$\iint_R 2xy dA; \quad R = \left\{(r, \theta): \ 1 \le r \le 3, \quad 0 \le \theta \le \frac{\pi}{2}\right\}$$

Solution

$$\iint_R (2xy) \, dA = \int_0^{\frac{\pi}{2}} \int_1^3 2(r\cos\theta)(r\sin\theta) r \ dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin 2\theta \ d\theta \int_1^3 r^3 \ dr$$

$$= -\frac{1}{2}\cos 2\theta \Big|_0^{\frac{\pi}{2}} \left(\frac{1}{4}r^4\right|_1^3$$

$$= -\frac{1}{8}(-1-1)\ (81-1)$$

$$\underline{= 20}$$

Exercise

Evaluate the integral over R using polar coordinates

$$\iint_R 2xy\ dA; \quad R = \left\{(x,\ y):\ \ x^2 + y^2 \le 9, \quad y \ge 0\right\}$$

Solution

$$x^2 + y^2 = 9 \quad \to \quad 0 \le r \le 3$$

$$y \ge 0 \quad \to \quad 0 \le \theta \le \pi$$

$$\iint_R (2xy)\,dA = \int_0^\pi \int_0^3 2(r\cos\theta)(r\sin\theta) r\ drd\theta$$

$$= \int_0^\pi \sin 2\theta\ d\theta \ \int_0^3 r^3\ dr$$

$$= -\frac{1}{2}\cos 2\theta \Big|_0^{\pi} \left(\frac{1}{4}r^4\right|_0^3$$

$$= -\frac{1}{8}(-1+1)\ (81-1)$$

$$\underline{= 0}$$

Exercise

Evaluate the integral over R using polar coordinates

$$\iint_R \frac{dA}{1+x^2+y^2}; \quad R = \left\{(r,\ \theta):\ \ 1 \le r \le 2, \quad 0 \le \theta \le \pi\right\}$$

Solution

$$\iint_R \frac{dA}{1+x^2+y^2} = \int_0^\pi \int_1^2 \frac{1}{1+r^2} r\ drd\theta$$

$$= \frac{1}{2}\int_0^{\pi} d\theta \int_1^2 \frac{1}{1+r^2}\, d\left(1+r^2\right)$$

$$= \frac{\pi}{2}\ln\left(1+r^2\right)\Big|_1^2$$

$$= \frac{\pi}{2}(\ln 5 - \ln 2)$$

$$\underline{= \frac{\pi}{2}\ln\frac{5}{2}\,\Big|}$$

Exercise

Evaluate the integral over R using polar coordinates

$$\iint\limits_R \frac{dA}{\sqrt{16-x^2-y^2}}; \quad R = \left\{(x,\ y):\ \ x^2+y^2 \le 4, \quad y \ge 0\right\}$$

Solution

$$x^2+y^2=4 \quad \to \quad 0 \le r \le 2$$

$$y \ge 0 \quad \to \quad 0 \le \theta \le \pi$$

$$\iint\limits_R \frac{dA}{\sqrt{16-x^2-y^2}} = \int_0^{\pi}\int_0^2 \frac{1}{\sqrt{16-r^2}}\, r\, dr d\theta$$

$$= -\frac{1}{2}\int_0^{\pi} d\theta \int_0^2 \left(16-r^2\right)^{-1/2} d\left(16-r^2\right)$$

$$= -\pi\left(16-r^2\right)^{1/2}\Big|_0^2$$

$$= -\pi\left(2\sqrt{3}-4\right)$$

$$\underline{= 2\pi\left(2-\sqrt{3}\right)\Big|}$$

Exercise

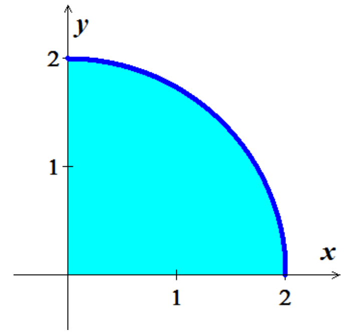
Evaluate the integral over R using polar coordinates

$$\iint\limits_R \frac{dA}{\sqrt{16-x^2-y^2}}; \quad R = \left\{(x,\ y):\ \ x^2+y^2 \le 4, \quad x,\ y \ge 0\right\}$$

Solution

$$x^2+y^2=4 \quad \to \quad 0 \le r \le 2$$

$$x,\ y \ge 0 \quad \to \quad 0 \le \theta \le \frac{\pi}{2}$$



$$\iint\limits_R \frac{dA}{\sqrt{16-x^2-y^2}} = \int_0^{\frac{\pi}{2}} \int_0^2 \frac{1}{\sqrt{16-r^2}} r\ dr d\theta$$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta \int_0^2 \left(16-r^2\right)^{-1/2} d\left(16-r^2\right)$$

$$= -\frac{\pi}{2}\left(16-r^2\right)^{1/2} \Big|_0^2$$

$$= -\frac{\pi}{2}\left(2\sqrt{3}-4\right)$$

$$= \pi\left(2-\sqrt{3}\right)$$

Exercise

Evaluate the integral over R using polar coordinates

$$\iint\limits_R e^{-x^2-y^2} dA; \quad R = \left\{(x,\ y):\ x^2+y^2 \le 9\right\}$$

Solution

$$\iint\limits_R e^{-x^2-y^2} dA = \int_0^{2\pi} \int_0^3 e^{-r^2} r\ dr d\theta$$

$$= -\frac{1}{2} \int_0^{2\pi} d\theta \int_0^3 e^{-r^2} d\left(-r^2\right)$$

$$= -\pi e^{-r^2} \Big|_0^3$$

$$= -\pi\left(e^{-9}-1\right)$$

$$= \pi\left(1-e^{-9}\right)$$

Exercise

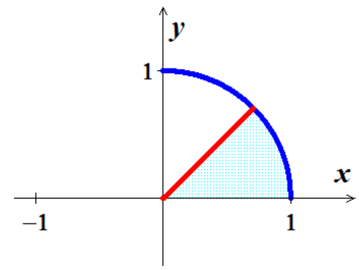
Evaluate the integral over R using polar coordinates

$$\iint_R \sqrt{x^2+y^2}\, dA; \quad R=\{(x,\ y):\ \ y \le x \le 1,\quad 0 \le y \le 1\}$$

Solution

$$y = x \quad \to \quad \cos\theta = \sin\theta \ \Rightarrow \ \theta = \frac{\pi}{4}$$

$$y = r\sin\theta \le 1 \quad \to \quad r \le \sec\theta$$



$$\iint_R \sqrt{x^2+y^2}\, dA = \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} r^2\, dr d\theta$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{4}} r^3 \Big|_0^{\sec\theta}\, d\theta$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta$$

$$= \frac{1}{6}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_0^{\frac{\pi}{4}}$$

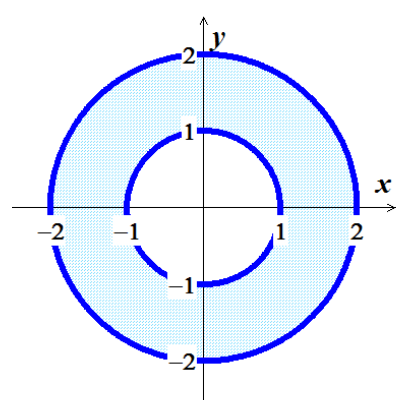
$$= \frac{1}{6}\left(\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right)$$

Exercise

Evaluate the integral over R using polar coordinates

$$\iint_R \sqrt{x^2+y^2}\, dA; \quad R=\{(x,\ y):\ \ 1 \le x^2+y^2 \le 2\}$$

Solution



$$\iint_R \sqrt{x^2+y^2}\, dA = \int_0^{2\pi} \int_1^{2} r^2\, dr d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \quad r^3 \Big|_1^{2}$$

$$= \frac{2\pi}{3}(8-1)$$

$$= \frac{14\pi}{3}$$

Exercise

Evaluate the integral over R using polar coordinates

$$\iint_R \frac{dA}{\left(x^2+y^2\right)^{5/2}}; \quad R=\left\{(r,\ \theta):\ \ 1\le r\le \infty,\quad 0\le \theta \le 2\pi\right\}$$

Solution

$$\iint_R \frac{dA}{\left(x^2+y^2\right)^{5/2}} = \int_0^{2\pi}\int_1^{\infty} \frac{1}{r^5}\ r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_1^{\infty} r^{-4}\ dr$$

$$= 2\pi\left(-\frac{1}{3}\frac{1}{r^3}\ \Big|_1^{\infty}\right.$$

$$= -\frac{2\pi}{3}(0-1)$$

$$= \frac{2\pi}{3}$$

Exercise

Evaluate the integral over R using polar coordinates

$$\iint_R e^{-x^2-y^2}\,dA; \quad R=\left\{(r,\ \theta):\ \ 0\le r\le \infty,\quad 0\le \theta \le \frac{\pi}{2}\right\}$$

Solution

$$\iint_R e^{-x^2-y^2}\,dA = \int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}\ r\,dr\,d\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} e^{-r^2}\,d\left(-r^2\right)$$

$$= -\frac{\pi}{4}e^{-r^2}\ \Big|_0^{\infty}$$

$$= -\frac{\pi}{4}(0-1)$$

$$= \frac{\pi}{4}$$

Exercise

Evaluate the integral over R using polar coordinates

$$\iint_R \frac{dA}{\left(1+x^2+y^2\right)^2}; \quad R \in QI$$

Solution

$$\iint_R \frac{dA}{\left(1+x^2+y^2\right)^2} = \int_0^{\frac{\pi}{2}} \int_0^{\infty} \frac{1}{\left(1+r^2\right)^2} r\, dr d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \quad \int_0^{\infty} \frac{1}{\left(1+r^2\right)^2}\, d\left(1+r^2\right)$$

$$= \frac{1}{2}\frac{\pi}{2}\left(-\frac{1}{1+r^2} \,\Big|_0^{\infty}\right.$$

$$= \frac{\pi}{4}(-0+1)$$

$$= \frac{\pi}{4}$$

Exercise

Find the volume of a bowl holds water if it is filled to a depth of four units?

a) The paraboloid $z = x^2 + y^2$, for $0 \le z \le 4$

b) The cone $z = \sqrt{x^2+y^2}$, for $0 \le z \le 4$

c) The hyperboloid $z = \sqrt{1+x^2+y^2}$, for $1 \le z \le 5$

d) Which bowl holds more water?

e) To what weight (above the bottom of the bowl) must the cone and paraboloid bowls be filled to hold the same volume of water as the hyperboloid bowl filled to a depth of 4 units $(1 \le z \le 5)$

Solution

a) $V = \iint_R \left(4-\left(x^2+y^2\right)\right) dA$

$$= \int_0^{2\pi} \int_0^2 \left(4-r^2\right) r\, dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^2 \left(4r-r^3\right) dr$$

$$= 2\pi\left(2r^2 - \frac{1}{4}r^4 \right|_0^2$$

$$= 2\pi(8-4)$$

$$\underline{= 8\pi \ \ unit^3}$$

b) $0 \le z = \sqrt{x^2+y^2} \le 4$

$0 \le x^2 + y^2 \le 16$

$$V = \iint\limits_R \left(4 - \sqrt{x^2+y^2}\right) dA$$

$$= \int_0^{2\pi} \int_0^4 (4-r)r \ drd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^4 \left(4r - r^2\right) dr$$

$$= 2\pi\left(2r^2 - \frac{1}{3}r^3 \right|_0^4$$

$$= 2\pi\left(32 - \frac{64}{3}\right)$$

$$\underline{= \frac{64\pi}{3} \ unit^3}$$

c) $1 \le z = \sqrt{1+x^2+y^2} \le 5$

$1 \le 1 + x^2 + y^2 \le 25 \quad \to \quad 0 \le x^2 + y^2 \le 24$

$$V = \iint\limits_R \left(5 - \sqrt{1+x^2+y^2}\right) dA$$

$$= \int_0^{2\pi} \int_0^{\sqrt{24}} \left(5 - \sqrt{1+r^2}\right) r \ drd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{24}} \left(5r - r\sqrt{1+r^2}\right) dr$$

$$= 2\pi \int_0^{\sqrt{24}} 5r \ dr \ - \ \pi \int_0^{\sqrt{24}} \left(1+r^2\right)^{1/2} \ d\left(1+r^2\right)$$

$$= 5\pi r^2 \Big|_0^{\sqrt{24}} \ - \ \frac{2\pi}{3}\left(1+r^2\right)^{3/2} \Big|_0^{\sqrt{24}}$$

$$= 5\pi(24) - \frac{2}{3}\pi(125-1)$$

$$\boxed{= \frac{112\pi}{3}\ unit^3}$$

d) The hyperboloid bowl holds most water of $\frac{112\pi}{3}\ unit^3$.

e) Let the height = h

$Paraboloid$: $z = x^2 + y^2 = h$

$$V = \int_0^{2\pi} \int_0^{\sqrt{h}} \left(r^2\right) r\ dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{h}} r^3\ dr$$

$$= \frac{\pi}{2} r^4 \Big|_0^{\sqrt{h}}$$

$$= \frac{\pi}{2} h^2 = \frac{112\pi}{3}$$

$$h^2 = \frac{224}{3}$$

$$\Rightarrow \boxed{h = \sqrt{\frac{224}{3}}\ \ units}$$

$Cone$: $z = \sqrt{x^2 + y^2} = h$

$$V = \int_0^{2\pi} \int_0^{h} r^2\ dr d\theta$$

$$= \int_0^{2\pi} d\theta\ \ \frac{1}{3} r^3 \Big|_0^{h}$$

$$= \frac{2\pi}{3} h^3 = \frac{112\pi}{3}$$

$$h^3 = 56$$

$$\Rightarrow \boxed{h = \sqrt[3]{56}\ \ units}$$

Exercise

Consider the surface $z = x^2 - y^2$

a) Find the region in the *xy*-plane in polar coordinates for which $z \ge 0$.

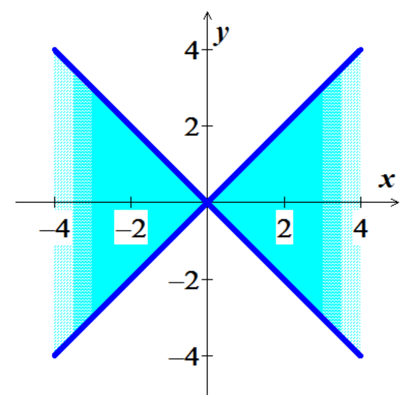
b) Let $R = \left\{ (r,\ \theta):\ \ 0 \le r \le a,\ \ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \right\}$, which is a sector of a circle of radius a. Find the volume of the region below the hyperbolic paraboloid and above the region *R*.

Solution

a) $z = x^2 - y^2 \ge 0 \quad \to \quad x^2 \ge y^2$

$-|y| \le x \le |y|$

$R = \left\{ (r,\ \theta):\ \ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4},\ \ \frac{3\pi}{4} \le \theta \le \frac{5\pi}{4} \right\}$



b) $V = \iint\limits_R \left(x^2 - y^2\right) dA$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^a \left(r^2 \cos^2\theta - r^2 \sin^2\theta\right) r\ dr d\theta$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\cos^2\theta - \sin^2\theta\right) d\theta \ \int_0^a r^3\ dr$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos 2\theta) d\theta \ \left(\frac{1}{4} r^4 \ \Big|_0^a\right.$

$= \frac{1}{2} \sin 2\theta \ \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \ \left(\frac{1}{4} a^4\right)$

$= \frac{1}{2}(1+1) \ \left(\frac{1}{4} a^4\right)$

$= \frac{1}{4} a^4$

Exercise

A cake is shaped like a hemisphere of radius 4 with its base on the *xy*-plane. A wedge of the cake is removed by making two slices from the center of the cake outward, perpendicular to the *xy*-plane and separated by an angle of φ.

a) Use a double integral to find the volume of the slice for $\varphi = \frac{\pi}{4}$.

b) Suppose the cake is sliced by a plane perpendicular to the *xy*-plane at $x = a > 0$. Let D be the smaller of the two pieces produced. For what value of a is the volume of D equal to the volume in part (*a*)?

Solution

a)
$$V = \iint_R \left(4^2 - \left(x^2 + y^2\right)\right) dA$$
$$= \int_0^{\frac{\pi}{4}} \int_0^4 \sqrt{16 - r^2}\ r\ dr d\theta$$
$$= -\frac{1}{2}\int_0^{\frac{\pi}{4}} d\theta \int_0^4 \left(16 - r^2\right)^{1/2}\ d\left(16 - r^2\right)$$
$$= -\frac{1}{2}\left(\frac{\pi}{4}\right)\ \frac{2}{3}\left(16 - r^2\right)^{3/2}\ \Big|_0^4$$
$$= -\frac{\pi}{12}(-64)$$
$$= \frac{16\pi}{3}\ unit^3$$

Geometrically, this slice is $\frac{1}{8}$ of the hemispherical cake.

The formula for the volume of a sphere is $\frac{4\pi}{3}$, them the volume of the slice is

$$V = \frac{1}{8}\frac{1}{2}\frac{4\pi}{3}$$
$$= \frac{16\pi}{3}\ unit^3 \qquad \checkmark$$

b)
$$V = \iint_R \left(16 - \left(x^2 + y^2\right)\right) dA$$
$$= \int_0^{\varphi} d\theta \int_0^4 \sqrt{16 - r^2}\ r\ dr$$
$$= -\frac{\varphi}{2}\int_0^4 \left(16 - r^2\right)^{1/2}\ d\left(16 - r^2\right)$$

$$= -\frac{\varphi}{3}\left(16 - r^2\right)^{3/2} \Big|_0^4$$

$$\boxed{= \frac{64\pi}{3}\ unit^3}$$

Exercise

Suppose the density of a thin plate represented by the region R is $\rho(r,\ \theta)$ (in units of mass per area). The mass of the plate is $\iint\limits_R \rho(r,\ \theta)\, dA$. Find the mass of the thin half annulus

$R = \{(r,\ \theta):\ \ 1 \le r \le 4,\ \ \ 0 \le \theta \le \pi\}$ with a density $\rho(r,\ \theta) = 4 + r\sin\theta$

Solution

$$\iint\limits_R \rho(r,\ \theta)\, dA = \int_0^{\pi}\int_1^4 (4 + r\sin\theta)r\ drd\theta$$

$$= \int_0^{\pi}\int_1^4 \left(4r + r^2\sin\theta\right)\ drd\theta$$

$$= \int_0^{\pi} \left(2r^2 + \frac{1}{3}r^3\sin\theta\right) \Big|_1^4\ d\theta$$

$$= \int_0^{\pi} \left(32 + \frac{64}{3}\sin\theta - 2 - \frac{1}{3}\sin\theta\right)\ d\theta$$

$$= \int_0^{\pi} (30 + 21\sin\theta)\ d\theta$$

$$= 30\theta - 21\cos\theta \Big|_0^{\pi}$$

$$= 30\pi + 21 + 21$$

$$\boxed{= 30\pi + 42}$$

Exercise

An important integral in statistics associated with the normal distribution is $I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$. It is evaluated in the following steps.

a) Assume that $I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dxdy$$

Where we have chosen the variables of integration to be x and y and then written the product as an iterated integral. Evaluate this integral in polar coordinates and show that $I = \sqrt{\pi}$. Why is the solution $I = -\sqrt{\pi}$ rejected?

b) Evaluate $\int_0^{\infty} e^{-x^2}\, dx$, $\int_0^{\infty} xe^{-x^2}\, dx$, and $\int_0^{\infty} x^2e^{-x^2}\, dx$.

Solution

a) $I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)} dxdy$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\, drd\theta$$

$$= -\frac{1}{2}\int_0^{2\pi} d\theta \quad \int_0^{\infty} e^{-r^2}\, d\left(-r^2\right)$$

$$= -\frac{1}{2}(2\pi)(0-1)$$

$$\boxed{= \pi}$$

The integrand is positive everywhere, so the integral of a positive function is positive.

b) $\int_0^{\infty} e^{-x^2}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\, dx$

$$\boxed{= \frac{\sqrt{\pi}}{2}}$$

$$\int_0^{\infty} xe^{-x^2}\, dx = -\frac{1}{2}\int_0^{\infty} e^{-x^2}\, d\left(-x^2\right)$$

$$= -\frac{1}{2}e^{-x^2}\Big|_0^{\infty}$$

$$= -\frac{1}{2}(0-1)$$

$$\boxed{= \frac{1}{2}}$$

$$u = x \qquad dv = xe^{-x^2}\,dx$$

$$du = dx \qquad v = -\frac{1}{2}e^{-x^2}$$

$$\int_0^\infty x^2 e^{-x^2}\,dx = -\frac{1}{2}xe^{-x^2}\Big|_0^\infty + \frac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

$$= 0 + \frac{1}{2}\frac{\sqrt{\pi}}{2}$$

$$\boxed{= \frac{\sqrt{\pi}}{4}}$$

Exercise

For what values of p does the integral $\iint_R \frac{k}{\left(x^2+y^2\right)^p}\,dA$ exist in the following cases?

a) $R = \{(r,\ \theta):\ 1 \le r \le \infty,\quad 0 \le \theta \le 2\pi\}$

b) $R = \{(r,\ \theta):\ 0 \le r \le 1,\quad 0 \le \theta \le 2\pi\}$

Solution

a) $$\iint_R \frac{k}{\left(x^2+y^2\right)^p}\,dA = \int_0^{2\pi}\int_1^\infty \frac{k}{r^{2p}}\ r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_1^\infty kr^{1-2p}\,dr$$

$$= \frac{k\pi}{1-p}\left(r^{2-2p}\ \Big|_1^\infty\right.$$

$$= \frac{k\pi}{1-p}\left(r^{-2(p-1)}\ \Big|_1^\infty\right.$$

If $p-1<0 \quad \rightarrow \quad p<1$ the integral diverges.

If $p-1>0 \quad \rightarrow \quad p>1$ the integral converges.

$$\iint_R \frac{k}{\left(x^2+y^2\right)^p}\,dA = \frac{k\pi}{1-p}\left(r^{-2(p-1)}\ \Big|_1^\infty\right.$$

$$= \frac{k\pi}{1-p}(0-1)$$

$$\boxed{= \frac{k\pi}{p-1}}$$

b) $$\iint_R \frac{k}{\left(x^2+y^2\right)^p} dA = \int_0^{2\pi}\int_0^1 \frac{k}{r^{2p}}\ r\ drd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 kr^{1-2p}\ dr$$

$$= \frac{k\pi}{1-p}\left(\frac{1}{r^{2(p-1)}}\right|_0^1$$

$$= \frac{k\pi}{1-p}\left(1-\frac{1}{0}\right)$$

If $p-1>0 \quad \rightarrow \quad p>1$ the integral diverges.

If $p-1<0 \quad \rightarrow \quad p<1$ the integral converges.

$$\iint_R \frac{k}{\left(x^2+y^2\right)^p} dA = \frac{k\pi}{1-p}(1-0)$$

$$\boxed{= \frac{k\pi}{1-p}}$$

Exercise

Consider the integral $\displaystyle\iint_R \frac{1}{\left(1+x^2+y^2\right)^2} dA$ where $R = \{(x,\ y):\ \ 0 \le x \le 1,\ \ \ 0 \le y \le a\}$

a) Evaluate I for $a=1$.

b) Evaluate I for arbitrary $a>0$.

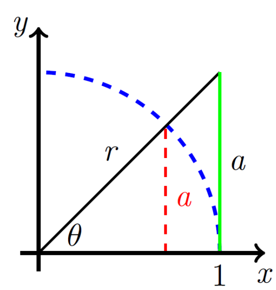
c) Let $a \to \infty$ in part (*b*) to find I over the infinite strip $R = \{(x,\ y):\ \ 0 \le x \le 1,\ \ \ 0 \le y \le \infty\}$

Solution

$0 \le x = r\cos\theta \le 1 \quad \rightarrow \quad 0 \le r \le \sec\theta$

$0 \le y = r\sin\theta \le a \quad \rightarrow \quad 0 \le r \le a\csc\theta$

$\tan\theta = \frac{a}{1} \quad \rightarrow \quad \theta = \tan^{-1} a$



a) $\theta = \tan^{-1} 1 = \frac{\pi}{4}$

$$\iint_R \frac{1}{\left(1+x^2+y^2\right)^2} dA = \int_0^1 \int_0^1 \frac{1}{\left(1+x^2+y^2\right)^2} dydx$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} \frac{1}{\left(1+r^2\right)^2} rdrd\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\csc\theta} \frac{1}{\left(1+r^2\right)^2} rdrd\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} \frac{d\left(1+r^2\right)}{\left(1+r^2\right)^2} d\theta + \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\csc\theta} \frac{d\left(1+r^2\right)}{\left(1+r^2\right)^2} d\theta$$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{1}{1+r^2} \Big|_0^{\sec\theta} d\theta - \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{1+r^2} \Big|_0^{\csc\theta} d\theta$$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{4}} \left(\frac{1}{1+\sec^2\theta} - 1\right) d\theta - \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1}{1+\csc^2\theta} - 1\right) d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{2+\tan^2\theta} d\theta + \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\csc^2\theta}{2+\cot^2\theta} d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{d(\tan\theta)}{2+\tan^2\theta} - \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{d(\cot\theta)}{2+\cot^2\theta} \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{\tan\theta}{\sqrt{2}}\right) \Big|_0^{\frac{\pi}{4}} - \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{\cot\theta}{\sqrt{2}}\right) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2\sqrt{2}} \left(\tan^{-1}\frac{1}{\sqrt{2}} + \tan^{-1}\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

b) $$\iint_R \frac{1}{\left(1+x^2+y^2\right)^2} dA = \int_0^1 \int_0^a \frac{1}{\left(1+x^2+y^2\right)^2} dydx$$

$$= \int_0^{\tan^{-1} a} \int_0^{\sec\theta} \frac{1}{\left(1+r^2\right)^2} rdrd\theta + \int_{\tan^{-1} a}^{\frac{\pi}{2}} \int_0^{a\csc\theta} \frac{1}{\left(1+r^2\right)^2} rdrd\theta$$

$$=\frac{1}{2}\int_0^{\tan^{-1}a}\int_0^{\sec\theta}\frac{d\left(1+r^2\right)}{\left(1+r^2\right)^2}d\theta+\frac{1}{2}\int_{\tan^{-1}a}^{\frac{\pi}{2}}\int_0^{a\csc\theta}\frac{d\left(1+r^2\right)}{\left(1+r^2\right)^2}d\theta$$

$$=-\frac{1}{2}\int_0^{\tan^{-1}a}\frac{1}{1+r^2}\Bigg|_0^{\sec\theta}d\theta-\frac{1}{2}\int_{\tan^{-1}a}^{\frac{\pi}{2}}\frac{1}{1+r^2}\Bigg|_0^{a\csc\theta}d\theta$$

$$=-\frac{1}{2}\int_0^{\tan^{-1}a}\left(\frac{1}{1+\sec^2\theta}-1\right)d\theta-\frac{1}{2}\int_{\tan^{-1}a}^{\frac{\pi}{2}}\left(\frac{1}{1+a^2\csc^2\theta}-1\right)d\theta$$

$$=\frac{1}{2}\int_0^{\tan^{-1}a}\frac{\sec^2\theta}{2+\tan^2\theta}d\theta+\frac{1}{2}\int_{\tan^{-1}a}^{\frac{\pi}{2}}\frac{a^2\csc^2\theta}{1+a^2\csc^2\theta}d\theta$$

$$=\frac{1}{2}\int_0^{\tan^{-1}a}\frac{\sec^2\theta}{2+\tan^2\theta}d\theta+\frac{1}{2}\int_{\tan^{-1}a}^{\frac{\pi}{2}}\frac{\csc^2\theta}{\frac{1}{a^2}+\csc^2\theta}d\theta$$

$$=\frac{1}{2}\int_0^{\tan^{-1}a}\frac{\sec^2\theta}{2+\tan^2\theta}d\theta+\frac{1}{2}\int_{\tan^{-1}a}^{\frac{\pi}{2}}\frac{\csc^2\theta}{\frac{1}{a^2}+1+\cot^2\theta}d\theta$$

$$=\frac{1}{2}\int_0^{\tan^{-1}a}\frac{d(\tan\theta)}{2+\tan^2\theta}-\frac{1}{2}\int_{\tan^{-1}a}^{\frac{\pi}{2}}\frac{d(\cot\theta)}{\frac{1+a^2}{a^2}+\cot^2\theta}$$

$$=\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\tan\theta}{\sqrt{2}}\right)\Bigg|_0^{\tan^{-1}a}-\frac{a}{2\sqrt{1+a^2}}\tan^{-1}\left(\frac{a}{\sqrt{1+a^2}}\cot\theta\right)\Bigg|_{\tan^{-1}a}^{\frac{\pi}{2}}$$

$$=\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{a}{\sqrt{2}}\right)+\frac{a}{2\sqrt{1+a^2}}\tan^{-1}\left(\frac{a}{\sqrt{1+a^2}}\cot\left(\tan^{-1}a\right)\right)$$

$$=\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{a}{\sqrt{2}}\right)+\frac{a}{2\sqrt{1+a^2}}\tan^{-1}\left(\frac{a}{\sqrt{1+a^2}}\frac{1}{a}\right)$$

$$=\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{a}{\sqrt{2}}\right)+\frac{a}{2\sqrt{1+a^2}}\tan^{-1}\left(\frac{1}{\sqrt{1+a^2}}\right)$$

c) $$\lim_{a\to\infty}\left(\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{a}{\sqrt{2}}\right)+\frac{a}{2\sqrt{1+a^2}}\tan^{-1}\left(\frac{1}{\sqrt{1+a^2}}\right)\right)=\frac{1}{2\sqrt{2}}\tan^{-1}(\infty)+\frac{a}{2\sqrt{1+a^2}}\tan^{-1}(0)$$

$$= \frac{1}{2\sqrt{2}} \frac{\pi}{2} - 0$$

$$\boxed{= \frac{\pi\sqrt{2}}{8}}$$

Exercise

In polar coordinates an equation of an ellipse with eccentricity $0 < e < 1$ and semimajor axis a is

$$r = \frac{a\left(1-e^2\right)}{1+e\cos\theta}$$

a) Write the integral that gives the area of the ellipse.

b) Show that the area of an ellipse is πab, where $b^2 = a^2\left(1-e^2\right)$

Solution

a) $A = \iint\limits_R 1dA$

$$= \int_0^{2\pi} \int_0^{\frac{a\left(1-e^2\right)}{1+e\cos\theta}} r\ drd\theta$$

b) $A = \int_0^{2\pi} \int_0^{\frac{a\left(1-e^2\right)}{1+e\cos\theta}} r\ drd\theta$

$$= \frac{1}{2}\int_0^{2\pi} r^2 \Big|_0^{a\left(1-e^2\right)/(1+e\cos\theta)} d\theta$$

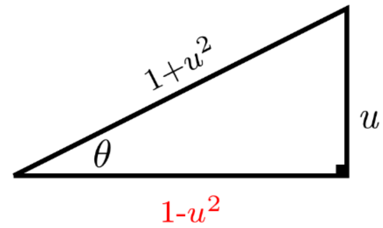
$$= \frac{1}{2}\int_0^{2\pi} \frac{a^2\left(1-e^2\right)^2}{(1+e\cos\theta)^2} d\theta$$

$$= a^2\left(1-e^2\right)^2 \int_0^{\pi} \frac{1}{(1+e\cos\theta)^2} d\theta$$

$$\tan^2\alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha}$$

$$\tan^2\alpha + \tan^2\alpha\cos 2\alpha = 1 - \cos 2\alpha$$

$$\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$



$$\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

$$(1+e\cos\theta)^2 = \left(1+e\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)^2$$

$$= \frac{1}{\left(1+\tan^2\frac{\theta}{2}\right)^2}\left(1+e+(1-e)\tan^2\frac{\theta}{2}\right)^2 \qquad \tan\frac{\theta}{2} = u$$

$$= \frac{1}{\left(1+u^2\right)^2}\left(1+e+(1-e)u^2\right)^2$$

$$\tan\frac{\theta}{2} = u \quad \rightarrow \quad \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta = du$$

$$d\theta = 2\cos^2\frac{\theta}{2}\;\; du$$

$$= \frac{2}{1+u^2}du$$

$$= a^2\left(1-e^2\right)^2\int_0^{\pi}\frac{\left(1+u^2\right)^2}{\left(1+e+(1-e)u^2\right)^2}\frac{2}{1+u^2}du$$

$$= 2a^2\left(1-e^2\right)^2\int_0^{\pi}\frac{1+u^2}{\left(1+e+(1-e)u^2\right)^2}du$$

$$\frac{1+u^2}{\left(1+e+(1-e)u^2\right)^2} = \frac{Au+B}{1+e+(1-e)u^2} + \frac{Cu+D}{\left(1+e+(1-e)u^2\right)^2}$$

$$1+u^2 = (1+e)Au + (1-e)Au^3 + (1+e)B + (1-e)Bu^2 + Cu + D$$

$$u^3 \qquad (1-e)A = 0 \qquad \rightarrow A = 0$$

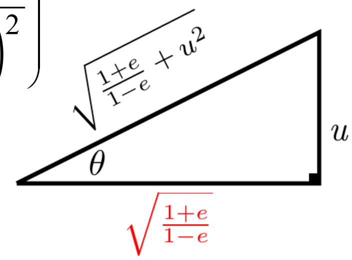
$$u^2 \qquad (1-e)B = 1 \qquad \rightarrow B = \frac{1}{1-e}$$

$$u \qquad (1+e)A + C = 0 \qquad \rightarrow C = 0$$

$$1 \qquad (1+e)B + D = 1 \quad \rightarrow D = 1 - \frac{1+e}{1-e} = -\frac{2e}{1-e}$$

$$= \frac{2a^2}{1-e}\left(1-e^2\right)^2 \int_0^{\pi} \frac{du}{1+e+(1-e)u^2} - \frac{4ea^2}{1-e}\left(1-e^2\right)^2 \int_0^{\pi} \frac{du}{\left(1+e+(1-e)u^2\right)^2}$$

$$= \frac{2a^2}{1-e}\left(1-e^2\right)^2 \left(\frac{1}{1-e}\int_0^{\pi} \frac{du}{\frac{1+e}{1-e}+u^2} - \frac{2e}{(1-e)^2}\int_0^{\pi} \frac{du}{\left(\frac{1+e}{1-e}+u^2\right)^2} \right)$$



$$u = \sqrt{\frac{1+e}{1-e}}\tan\alpha \quad \rightarrow du = \sqrt{\frac{1+e}{1-e}}\sec^2\alpha\, d\alpha$$

$$\frac{1+e}{1-e}+u^2 = \frac{1+e}{1-e}\sec^2\alpha$$

$$= \frac{2a^2}{1-e}\left(1-e^2\right)^2 \left(\frac{1}{1-e}\sqrt{\frac{1-e}{1+e}}\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\ \tan\frac{\theta}{2}\right)\Big|_0^{\pi} - \frac{2e}{(1-e)^2}\int_0^{\pi}\sqrt{\frac{1+e}{1-e}}\frac{\sec^2\alpha\, d\alpha}{\left(\frac{1+e}{1-e}\right)^2\sec^4\alpha} \right)$$

$$= \frac{2a^2}{1-e}\left(1-e^2\right)^2 \left(\frac{1}{1-e}\sqrt{\frac{1-e}{1+e}\cdot\frac{1-e}{1-e}}\tan^{-1}(\infty) - \frac{2e}{(1-e)^{1/2}(1+e)^{3/2}}\int_0^{\pi}\frac{1}{\sec^2\alpha}d\alpha \right)$$

$$= 2a^2\frac{\left(1-e^2\right)^2}{1-e}\left(\frac{\pi}{2\sqrt{1-e^2}} - \frac{2e}{(1+e)\sqrt{1-e^2}}\int_0^{\pi}\cos^2\alpha\ d\alpha \right)$$

$$= 2a^2\frac{\left(1-e^2\right)^2}{(1-e)\sqrt{1-e^2}}\left(\frac{\pi}{2} - \frac{e}{(1+e)}\int_0^{\pi}(1+\cos 2\alpha)\ d\alpha \right)$$

$$= 2a^2\frac{\left(1-e^2\right)^2}{(1-e)\sqrt{1-e^2}}\left(\frac{\pi}{2} - \frac{e}{(1+e)}\left(\alpha+\frac{1}{2}\sin 2\alpha\ \Big|_0^{\pi}\right)\right)$$

$$\frac{1}{2}\sin 2\alpha = \sin\alpha\cos\alpha$$

$$= \frac{u}{\sqrt{\frac{1+e}{1-e}+u^2}}\frac{\sqrt{\frac{1+e}{1-e}}}{\sqrt{\frac{1+e}{1-e}+u^2}} \qquad u = \tan\frac{\theta}{2}$$

$$= \sqrt{\frac{1+e}{1-e}}\frac{\tan\frac{\theta}{2}}{\frac{1+e}{1-e}+\tan^2\frac{\theta}{2}}$$

$$= 2a^2\frac{\left(1-e^2\right)^2}{(1-e)\sqrt{1-e^2}}\left(\frac{\pi}{2} - \frac{e}{(1+e)}\left(\arctan\left(\sqrt{\frac{1-e}{1-e}}\tan\frac{\theta}{2}\right) + \sqrt{\frac{1+e}{1-e}}\frac{\tan\frac{\theta}{2}}{\frac{1+e}{1-e}+\tan^2\frac{\theta}{2}}\right)\Bigg|_0^{\pi}\right)$$

$$\lim_{\theta\to\pi} \frac{\tan\frac{\theta}{2}}{\frac{1+e}{1-e}+\tan^2\frac{\theta}{2}} = \frac{\infty}{\infty}$$

$$= \lim_{\theta\to\pi} \frac{\tan\frac{\theta}{2}}{\tan^2\frac{\theta}{2}}$$

$$= \lim_{\theta\to\pi} \frac{1}{\tan\frac{\theta}{2}}$$

$$= 0$$

$$= 2a^2\frac{\left(1-e^2\right)^{3/2}}{1-e}\left(\frac{\pi}{2}-\frac{e\pi}{2(1+e)}\right)$$

$$= \pi a^2\frac{\left(1-e^2\right)^{3/2}}{1-e}\left(\frac{1+e-e}{1+e}\right)$$

$$= \pi a^2\frac{\left(1-e^2\right)^{3/2}}{1-e^2}$$

$$= \pi a^2\left(1-e^2\right)^{1/2} \qquad b^2 = a^2\left(1-e^2\right)$$

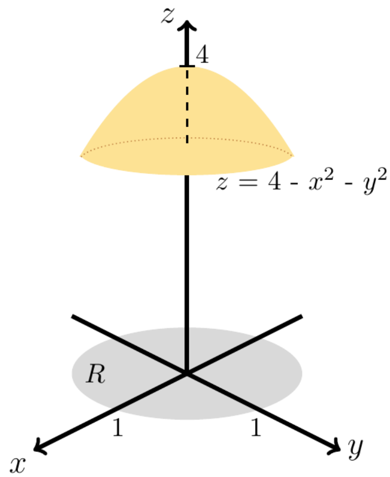
$$= \pi a\sqrt{a\left(1-e^2\right)}$$

$$\boxed{= \pi ab}$$

Exercise

Find the volume of the solid below the paraboloid $z = 4 - x^2 - y^2$ and above

$$R = \{(r,\ \theta):\quad 0 \le r \le 1,\quad 0 \le \theta \le 2\pi\}$$



Solution

$$V = \iint_R \left(4 - x^2 - y^2\right) dA$$

$$= \int_0^{2\pi}\int_0^1 \left(4 - r^2\right) r\ dr d\theta$$

$$= \int_0^{2\pi} d\theta \quad \int_0^1 \left(4r - r^3\right)\ dr$$

$$= 2\pi\left(2r^2 - \frac{1}{4}r^4 \,\Big|_0^1\right.$$

$$= 2\pi\left(2 - \frac{1}{4}\right)$$

$$= \frac{7\pi}{2}\ unit^3$$

Exercise

Find the volume of the solid below the paraboloid $z = 4 - x^2 - y^2$ and above

$$R = \{(r,\ \theta):\ \ 0 \le r \le 2,\quad 0 \le \theta \le 2\pi\}$$

Solution

$$V = \iint_R \left(4 - x^2 - y^2\right) dA$$

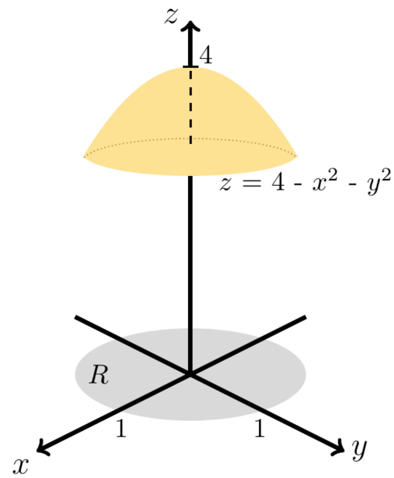
$$= \int_0^{2\pi}\int_0^2 \left(4 - r^2\right) r\ dr d\theta$$

$$= \int_0^{2\pi} d\theta\ \int_0^2 \left(4r - r^3\right)\ dr$$

$$= 2\pi\left(2r^2 - \frac{1}{4}r^4 \,\Big|_0^2\right.$$

$$= 2\pi(8 - 4)$$

$$= 8\pi\ \ unit^3$$

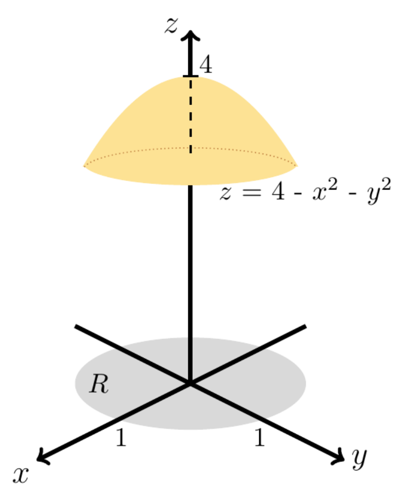


Exercise

Find the volume of the solid below the paraboloid $z = 4 - x^2 - y^2$ and above

$$R = \{(r,\ \theta):\ \ 1 \le r \le 2,\quad 0 \le \theta \le 2\pi\}$$

Solution



$$V = \iint_R \left(4 - x^2 - y^2\right) dA$$

$$= \int_0^{2\pi}\int_1^2 \left(4 - r^2\right) r\ dr d\theta$$

$$= \int_0^{2\pi} d\theta\ \int_1^2 \left(4r - r^3\right)\ dr$$

$$= 2\pi\left(2r^2 - \frac{1}{4}r^4 \right|_1^2$$

$$= 2\pi\left(8 - 4 - 2 + \frac{1}{4}\right)$$

$$\boxed{= \frac{9\pi}{2}\ \ unit^3}$$

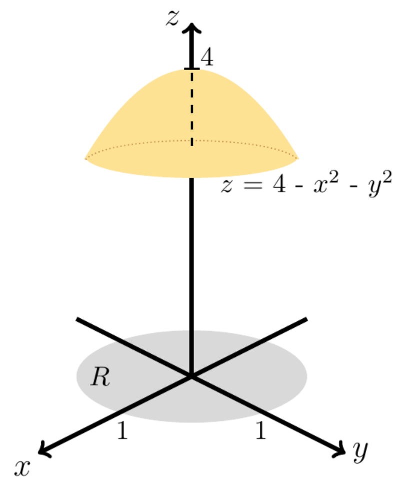
Exercise

Find the volume of the solid below the paraboloid $z = 4 - x^2 - y^2$ and above

$$R = \left\{(r,\ \theta):\ \ 1 \le r \le 2,\ \ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$$

Solution

$$V = \iint_R \left(4 - x^2 - y^2\right) dA$$



$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^2 \left(4 - r^2\right) r\ dr d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\ \ \int_1^2 \left(4r - r^3\right)\ dr$$

$$= \pi\left(2r^2 - \frac{1}{4}r^4 \right|_1^2$$

$$= \pi\left(8 - 4 - 2 + \frac{1}{4}\right)$$

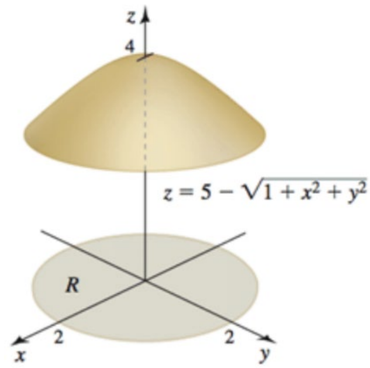
$$\boxed{= \frac{9\pi}{4}\ unit^3}$$

Exercise

Find the volume of the solid below the hyperboloid $z = 5 - \sqrt{1+x^2+y^2}$ and above

$$R = \{(r,\ \theta):\ \ 0 \le r \le 2,\ \ \ 0 \le \theta \le 2\pi\}$$

Solution



$$V = \iint\limits_R \left(5 - \sqrt{1+x^2+y^2}\right) dA$$

$$= \int_0^{2\pi} \int_0^2 \left(5 - \sqrt{1+r^2}\right) r\ drd\theta$$

$$= \int_0^{2\pi} d\theta \ \int_0^2 \left(5r - r\sqrt{1+r^2}\right)\ dr$$

$$= 2\pi \int_0^2 5r\ dr\ -\ 2\pi \int_0^2 r\ \sqrt{1+r^2}\ dr$$

$$= 5\pi \left(r^2\ \Big|_0^2\ -\ \pi \int_0^2 \left(1+r^2\right)^{1/2}\ d\left(1+r^2\right)\right)$$

$$= 20\pi - \frac{2\pi}{3}\left(1+r^2\right)^{3/2}\ \Big|_0^2$$

$$= 20\pi - \frac{2\pi}{3}\left(5\sqrt{5} - 1\right)$$

$$= 20\pi - \frac{10\pi\sqrt{5}}{3} + \frac{2\pi}{3}$$

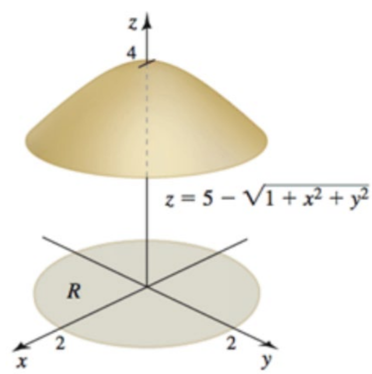
$$= \frac{\pi}{3}\left(62 - 10\sqrt{5}\right)\ \ unit^3$$

Exercise

Find the volume of the solid below the hyperboloid $z = 5 - \sqrt{1+x^2+y^2}$ and above

$$R = \{(r,\ \theta):\ \ 0 \le r \le 1,\ \ \ 0 \le \theta \le \pi\}$$

Solution



$$V = \iint\limits_R \left(5 - \sqrt{1+x^2+y^2}\right) dA$$

$$= \int_0^{\pi} \int_0^1 \left(5 - \sqrt{1+r^2}\right) r\ drd\theta$$

$$= \int_0^{\pi} d\theta \int_0^1 \left(5r - r\sqrt{1+r^2}\right) dr$$

$$= \pi \int_0^1 5r\, dr - \pi \int_0^1 r\sqrt{1+r^2}\, dr$$

$$= \frac{5}{2}\pi \left(r^2 \Big|_0^1 - \frac{\pi}{2} \int_0^1 \left(1+r^2\right)^{1/2} d\left(1+r^2\right)\right.$$

$$= \frac{5\pi}{2} - \frac{\pi}{3}\left(1+r^2\right)^{3/2} \Big|_0^1$$

$$= \frac{5\pi}{2} - \frac{\pi}{3}\left(2\sqrt{2} - 1\right)$$

$$= \frac{5\pi}{2} - \frac{2\pi\sqrt{2}}{3} + \frac{\pi}{3}$$

$$= \frac{\pi}{6}\left(17 - 4\sqrt{2}\right) \ \ unit^3$$

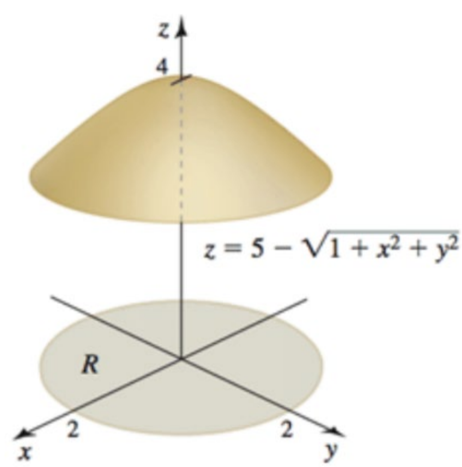
Exercise

Find the volume of the solid below the hyperboloid $z = 5 - \sqrt{1 + x^2 + y^2}$ and above

$$R = \left\{(r,\ \theta):\ \ \sqrt{3} \le r \le 2\sqrt{2},\ \ 0 \le \theta \le 2\pi\right\}$$

Solution

$$V = \iint\limits_R \left(5 - \sqrt{1+x^2+y^2}\right) dA$$



$$= \int_0^{2\pi} \int_{\sqrt{3}}^{2\sqrt{2}} \left(5 - \sqrt{1+r^2}\right) r\, dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_{\sqrt{3}}^{2\sqrt{2}} \left(5r - r\sqrt{1+r^2}\right) dr$$

$$= 2\pi \int_{\sqrt{3}}^{2\sqrt{2}} 5r\, dr - 2\pi \int_{\sqrt{3}}^{2\sqrt{2}} r\sqrt{1+r^2}\, dr$$

$$= 5\pi \left(r^2 \Big|_{\sqrt{3}}^{2\sqrt{2}} - \pi \int_{\sqrt{3}}^{2\sqrt{2}} \left(1+r^2\right)^{1/2} d\left(1+r^2\right)\right.$$

$$= 5\pi(8-3) - \frac{2\pi}{3}\left(1+r^2\right)^{3/2} \Big|_{\sqrt{3}}^{2\sqrt{2}}$$

$$= 25\pi - \frac{2\pi}{3}(27-8)$$

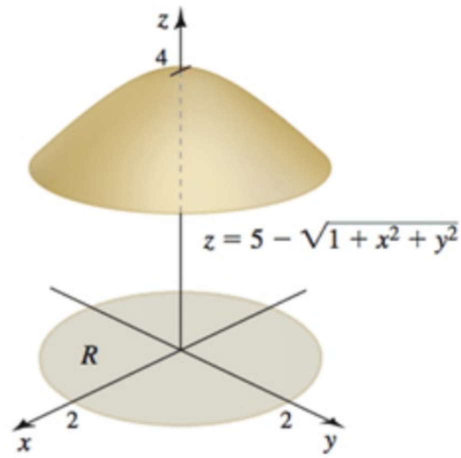
$$= \boxed{\frac{37\pi}{3}\ unit^3}$$

Exercise

Find the volume of the solid below the hyperboloid $z = 5 - \sqrt{1+x^2+y^2}$ and above

$$R = \left\{(r,\ \theta):\ \ \sqrt{3} \le r \le \sqrt{15},\ \ -\frac{\pi}{2} \le \theta \le \pi\right\}$$

Solution



$$V = \iint\limits_R \left(5 - \sqrt{1+x^2+y^2}\right) dA$$

$$= \int_{-\frac{\pi}{2}}^{\pi} \int_{\sqrt{3}}^{\sqrt{15}} \left(5 - \sqrt{1+r^2}\right) r\ dr d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\pi} d\theta \ \int_{\sqrt{3}}^{\sqrt{15}} \left(5r - r\sqrt{1+r^2}\right)\ dr$$

$$= \left(\pi + \frac{\pi}{2}\right) \int_{\sqrt{3}}^{\sqrt{15}} 5r\ dr\ -\ \frac{3\pi}{2} \int_{\sqrt{3}}^{\sqrt{15}} r\sqrt{1+r^2}\ dr$$

$$= \frac{15\pi}{4}\left(r^2\ \Big|_{\sqrt{3}}^{\sqrt{15}}\ -\ \frac{3\pi}{4} \int_{\sqrt{3}}^{\sqrt{15}} \left(1+r^2\right)^{1/2}\ d\left(1+r^2\right)\right.$$

$$= \frac{15\pi}{4}(12)\ -\ \frac{\pi}{2}\left(1+r^2\right)^{3/2}\ \Big|_{\sqrt{3}}^{\sqrt{15}}$$

$$= 45\pi - \frac{\pi}{2}(64-8)$$

$$= \boxed{17\pi\ \ unit^3}$$

Exercise

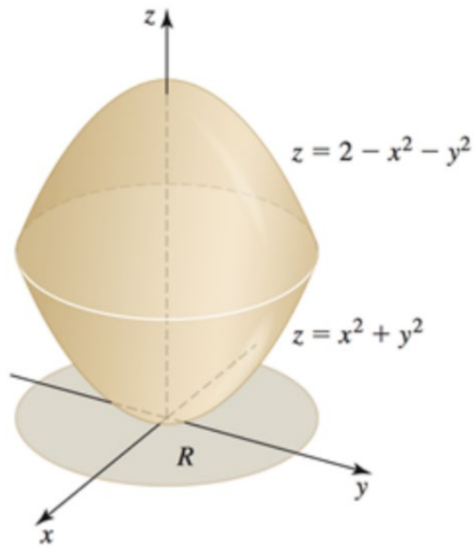
Find the volume of the solid between the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$

Solution

$$z = x^2 + y^2 = 2 - x^2 - y^2$$

$$2x^2 + 2y^2 = 2 \quad \rightarrow \quad x^2 + y^2 = 1$$

$$0 \le r \le 1 \quad \& \quad 0 \le \theta \le 2\pi$$



$$V = \iint_R \left(\left(2 - x^2 - y^2\right) - \left(x^2 + y^2\right) \right) dA$$

$$= \int_0^{2\pi} \int_0^1 \left(2 - r^2 - r^2\right) r \, dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 \left(2r - 2r^3\right) dr$$

$$= 2\pi \left(r^2 - \frac{1}{2} r^4 \right|_0^1$$

$$= 2\pi \left(1 - \frac{1}{2}\right)$$

$$= \pi \ \ unit^3$$

Exercise

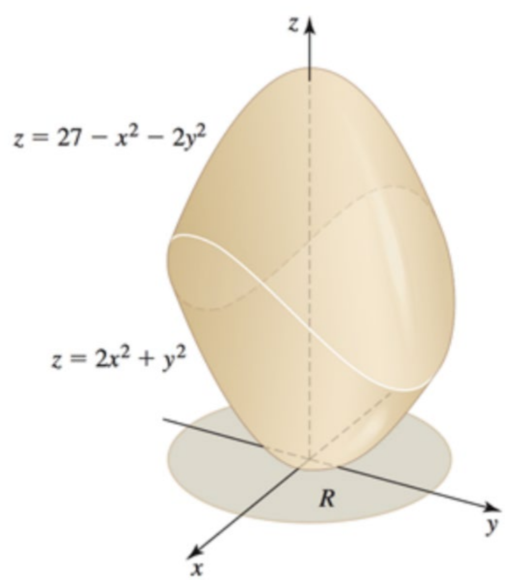
Find the volume of the solid between the paraboloids $z = 2x^2 + y^2$ and $z = 27 - x^2 - 2y^2$

Solution

$$z = 2x^2 + y^2 = 27 - x^2 - 2y^2$$

$$3x^2 + 3y^2 = 27 \quad \rightarrow \quad x^2 + y^2 = 9$$

$$0 \le r \le 3 \quad \& \quad 0 \le \theta \le 2\pi$$



$$V = \iint_R \left(\left(27 - x^2 - 2y^2\right) - \left(2x^2 + y^2\right) \right) dA$$

$$= \iint_R \left(27 - 3\left(x^2 + y^2\right)\right) dA$$

$$= 3 \int_0^{2\pi} \int_0^3 \left(9 - r^2\right) r \, dr d\theta$$

$$= 3\int_0^{2\pi} d\theta \int_0^3 \left(9r - r^3\right) dr$$

$$= 6\pi\left(\frac{9}{2}r^2 - \frac{1}{4}r^4\right.\Big|_0^3$$

$$= 6\pi\left(\frac{81}{2} - \frac{81}{4}\right)$$

$$\boxed{= \frac{243\pi}{2} \quad unit^3}$$

Exercise

Find the volume of island $z = e^{-\left(x^2+y^2\right)/8} - e^{-2}$

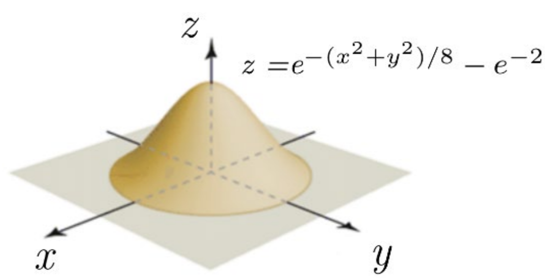
Solution

$$z = e^{-\left(x^2+y^2\right)/8} - e^{-2} = 0$$

$$e^{-\left(x^2+y^2\right)/8} = e^{-2}$$

$$-\frac{x^2+y^2}{8} = -2$$

$$x^2 + y^2 = 16$$



$$V = \int_0^{2\pi}\int_0^4 \left(e^{-r^2/8} - e^{-2}\right) r\, dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^4 \left(re^{-r^2/8} - re^{-2}\right) dr$$

$$= -8\pi \int_0^4 e^{-r^2/8}\, d\left(-\frac{1}{8}r^2\right) - 2\pi e^{-2}\int_0^4 r\, dr$$

$$= -8\pi e^{-r^2/8}\Big|_0^4 - \pi e^{-2}r^2\Big|_0^4$$

$$= -8\pi\left(e^{-2} - 1\right) - 16\pi e^{-2}$$

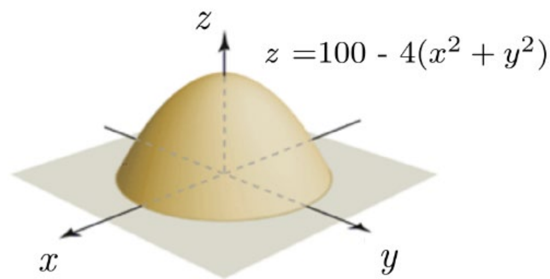
$$\boxed{= 8\pi - 24\pi e^{-2} \quad unit^3}$$

Exercise

Find the volume of island $z = 100 - 4\left(x^2 + y^2\right)$

Solution

$$z = 100 - 4\left(x^2 + y^2\right) = 0 \quad \to \quad x^2 + y^2 = 25$$



$$V = \int_0^{2\pi} \int_0^{5} \left(100 - 4r^2\right) r dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{5} \left(100r - 4r^3\right) dr$$

$$= 2\pi\left(50r^2 - r^4 \Big|_0^5\right.$$

$$= 2\pi\,(1250 - 625)$$

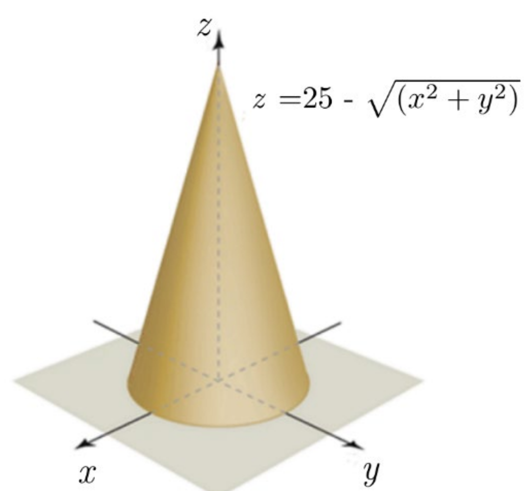
$$= 1,250\pi \ \ unit^3$$

Exercise

Find the volume of island $z = 25 - \sqrt{x^2 + y^2}$

Solution

$$z = 25 - \sqrt{x^2 + y^2} \quad \to \quad x^2 + y^2 \le 25^2$$



$$V = \int_0^{2\pi} \int_0^{25} (25 - r)r \ dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{25} \left(25r - r^2\right) dr$$

$$= 2\pi\left(\frac{25}{2}r^2 - \frac{1}{3}r^3 \Big|_0^{25}\right.$$

$$= 2\pi\,(15,625)\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{15,625\pi}{3} \ \ unit^3$$