

Solution **Section 4.1 – System of linear Equations**

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$$

Solution

$$\begin{cases} 3x + 2y = -4 \\ 2 \times 2x - y = -5 \end{cases}$$

$$3x + 2y = -4$$

$$\frac{4x - 2y = -10}{7x = -14}$$

$$x = -2$$

$$y = 2x + 5$$

$$= -4 + 5$$

$$= 1$$

$$\text{Solution: } \underline{(-2, 1)}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$$

Solution

$$\begin{cases} -5 \times 2x + 5y = 7 \\ 2 \times 5x - 2y = -3 \end{cases}$$

$$-10x - 25y = -35$$

$$\frac{10x - 4y = -6}{-29y = -41}$$

$$y = \frac{41}{29}$$

$$x = \frac{1}{2} \left(7 - 5 \left(\frac{41}{29} \right) \right)$$

$$x = \frac{1}{2} \left(-\frac{2}{29} \right)$$

$$= -\frac{1}{29}$$

$$\therefore \text{Solution: } \left(-\frac{1}{29}, \frac{41}{29} \right)$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$$

Solution

$$\begin{cases} 4x - 7y = -16 \\ -2 \times 2x + 5y = 9 \end{cases}$$

$$4x - 7y = -16$$

$$\begin{array}{r} -4x - 10y = -18 \\ \hline -17y = -34 \end{array}$$

$$y = 2$$

$$x = \frac{9 - 5y}{2}$$

$$= \frac{9 - 10}{2}$$

$$= -\frac{1}{2}$$

$$\therefore \text{Solution: } \left(-\frac{1}{2}, 2 \right)$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$$

Solution

$$\begin{cases} 3x + 2y = 4 & (1) \\ 2x + y = 1 & (2) \end{cases}$$

$$(2) \rightarrow y = 1 - 2x \quad (3)$$

$$(1) \rightarrow 3x + 2 - 4x = 4$$

$$\underline{x = -2}$$

$$(3) \rightarrow y = 1 + 4$$

$$\underline{= 5}$$

$$\therefore \text{Solution: } \underline{(-2, 5)}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$$

Solution

$$\begin{cases} -2 \times & 3x + 4y = 2 \\ 3 \times & 2x + 5y = -1 \end{cases}$$

$$-6x - 8y = -4$$

$$\frac{6x + 15y = -3}{7y = -7}$$

$$\underline{y = -1}$$

$$2x = -1 + 5$$

$$x = \frac{4}{2}$$

$$\underline{= 2}$$

$$\therefore \text{Solution: } \underline{(2, -1)}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method) $\begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$

Solution

$$\begin{cases} 2 \times & 5x - 2y = 4 \\ & -10x + 4y = 7 \end{cases}$$

$$10x - 4y = 8$$

$$\frac{-10x + 4y = 7}{0 = 15} \quad (\text{impossible})$$

\therefore **No Solution**

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$$

Solution

$$\begin{cases} x - 4y = -8 & (1) \\ 5x - 20y = -40 & (2) \end{cases}$$

$$(1) \rightarrow x = 4y - 8$$

$$(2) \rightarrow 5(4y - 8) - 20y = -40$$

$$20y - 40 - 20y = -40$$

$$-40 = -40 \quad (\text{True})$$

$$\therefore \text{Solution: } \underline{x - 4y = -8}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$$

Solution

$$\begin{cases} 2x + y = 3 & (1) \\ x - y = 3 & (2) \end{cases}$$

$$(2) \rightarrow x = 3 + y \quad (3)$$

$$(1) \rightarrow 6 + 2y + y = 3$$

$$3y = -3$$

$$\underline{y = -1}$$

$$(3) \rightarrow \underline{x = 2}$$

$$\therefore \text{Solution: } \underline{(2, -1)}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$$

Solution

$$\begin{cases} 2x + 10y = -14 \\ 5 \times 7x - 2y = -16 \end{cases}$$

$$2x + 10y = -14$$

$$\begin{array}{r} 35x - 10y = -80 \\ \hline 37x = -94 \end{array}$$

$$x = -\frac{94}{37}$$

$$2y = 7\left(-\frac{94}{37}\right) + 16$$

$$y = -\frac{329}{37} + 8$$

$$= -\frac{33}{37}$$

$$\therefore \text{Solution: } \left(-\frac{94}{37}, -\frac{33}{37}\right)$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$$

Solution

$$\begin{cases} 3 \times 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$$

$$12x - 9y = 72$$

$$\begin{array}{r} -3x + 9y = -1 \\ \hline -9x = -71 \end{array}$$

$$x = \frac{71}{9}$$

$$3y = 4\left(\frac{71}{9}\right) - 24$$

$$y = \frac{284}{27} - 8$$

$$= \frac{68}{27} \Big|$$

$$\therefore \textbf{Solution:} \quad \underline{\left(\frac{71}{9}, \frac{68}{27} \right) \Big|}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$$

Solution

$$4x + 2y = 12$$

$$3x - 2y = 16$$

$$\underline{7x = 28}$$

$$\underline{x = 4 \Big|}$$

$$2y = 12 - 4(4)$$

$$y = -\frac{4}{2}$$

$$\underline{\underline{= -2 \Big|}}$$

$$\therefore \textbf{Solution:} \quad \underline{(4, -2) \Big|}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$$

Solution

$$x + 2y = -1$$

$$4x - 2y = 6$$

$$\underline{5x = 5}$$

$$\underline{x = 1 \Big|}$$

$$2y = -x - 1$$

$$y = -\frac{2}{2}$$

$$\underline{\underline{= -1 \Big|}}$$

$$\therefore \textbf{Solution:} \quad \underline{(1, -1) \Big|}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$$

Solution

$$x - 2y = 5$$

$$\begin{array}{r} -10x + 2y = 4 \\ \hline -9x = 9 \end{array}$$

$$\underline{x = -1}$$

$$2y = x - 5$$

$$y = -\frac{6}{2}$$

$$\underline{= -3}$$

$$\therefore \text{Solution: } \underline{(-1, -3)}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases}$$

Solution

$$12x + 15y = -27$$

$$\begin{array}{r} 30x - 15y = -15 \\ \hline 42x = -42 \end{array}$$

$$\underline{x = -1}$$

$$15y = -27 - 12(-1)$$

$$y = -\frac{15}{15}$$

$$\underline{= -1}$$

$$\therefore \text{Solution: } \underline{(-1, -1)}$$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$$

Solution

$$4x - 4y = -12$$

$$4x + 4y = -20$$

$$\hline 8x = -32$$

$$\underline{x = -4}$$

$$4y = 4(-4) + 12$$

$$y = -\frac{4}{4}$$

$$\underline{= -1}$$

$$\therefore \text{Solution: } \underline{(-4, -1)}$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 1 & 4 & 7 \\ 3 & 5 & 0 \end{array} \right] \quad R_2 - 3R_1$$

Solution

$$3 \quad 5 \quad 0$$

$$-3 \quad -12 \quad -21$$

$$\hline 0 \quad -7 \quad -21$$

$$\left[\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & -7 & -21 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array} \right] \quad R_2 - 2R_1$$

Solution

$$\begin{array}{rrr} 2 & 1 & -5 \\ -2 & 6 & -2 \\ \hline 0 & 7 & -7 \end{array}$$

$$\left[\begin{array}{rr|r} 1 & -3 & 1 \\ 0 & 7 & -7 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{rr|r} 1 & -3 & 3 \\ 5 & 2 & 19 \end{array} \right] \quad R_2 - 5R_1$$

Solution

$$\begin{array}{rrr} 5 & 2 & 19 \\ -5 & 15 & -15 \\ \hline 0 & 17 & -4 \end{array}$$

$$\left[\begin{array}{rr|r} 1 & -3 & 3 \\ 0 & 17 & -4 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{rr|r} 2 & -3 & 8 \\ -6 & 9 & 4 \end{array} \right] \quad R_2 + 3R_1$$

Solution

$$\begin{array}{rrr} -6 & 9 & 4 \\ 6 & -9 & 24 \\ \hline 0 & 0 & 28 \end{array}$$

$$\left[\begin{array}{rr|r} 2 & -3 & 8 \\ 0 & 0 & 28 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 2 & 3 & 11 \\ 1 & 2 & 8 \end{array} \right] \quad 2R_2 - R_1$$

Solution

$$\begin{array}{ccc} 2 & 4 & 16 \\ -2 & -3 & -11 \\ \hline 0 & 1 & 5 \end{array}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 11 \\ 0 & 1 & 5 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 3 & 5 & -13 \\ 2 & 3 & -9 \end{array} \right] \quad 3R_2 - 2R_1$$

Solution

$$\begin{array}{ccc} 6 & 9 & -27 \\ -6 & -10 & 26 \\ \hline 0 & -1 & -1 \end{array}$$

$$\left[\begin{array}{cc|c} 3 & 5 & -13 \\ 0 & -1 & -1 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 5 & 4 & 1 \end{array} \right] \quad R_3 - 5R_2$$

Solution

$$\begin{array}{cccc} 0 & 5 & 4 & 1 \\ 0 & -5 & 5 & -10 \\ \hline 0 & 0 & 9 & -9 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 9 & -9 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array} \right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

Solution

$$\begin{array}{cccc} 3 & 3 & -1 & 10 \\ -3 & 3 & -15 & 18 \\ \hline 0 & 6 & -16 & 28 \end{array} \quad \begin{array}{cccc} 1 & 3 & 2 & 5 \\ -1 & 1 & -5 & 6 \\ \hline 0 & 4 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 4 & -3 & 11 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 2 & 4 & 4 & 22 \\ -1 & -2 & 3 & 15 \end{array} \right] \quad \begin{array}{l} 3R_2 - 2R_1 \\ 3R_3 + R_1 \end{array}$$

Solution

$$\begin{array}{cccc} 6 & 12 & 12 & 66 \\ -6 & -4 & -2 & -2 \\ \hline 0 & 8 & 10 & 64 \end{array} \quad \begin{array}{cccc} -3 & -6 & 9 & 45 \\ 3 & 2 & 1 & 1 \\ \hline 0 & -4 & 10 & 46 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & 8 & 10 & 64 \\ 0 & -4 & 10 & 46 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & -4 & 2 & -7 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

Solution

$$\begin{array}{cccc} 2 & 1 & 1 & 3 \\ -2 & -2 & -2 & -4 \\ \hline 0 & -1 & -1 & -1 \end{array} \quad \begin{array}{cccc} 3 & -4 & 2 & -7 \\ -3 & -3 & -3 & -6 \\ \hline 0 & -7 & -1 & -13 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & -1 \\ 0 & -7 & -1 & -13 \end{array} \right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ 2 & -3 & 5 & -1 & 0 \\ 1 & 0 & 3 & 1 & -4 \\ -4 & 3 & 2 & -1 & 3 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 + 4R_1 \end{array}$$

Solution

$$\begin{array}{ccccc} 2 & -3 & 5 & -1 & 0 \\ -2 & 4 & -2 & -6 & 4 \\ \hline 0 & 1 & 3 & -7 & 4 \end{array} \quad \begin{array}{ccccc} 1 & 0 & 3 & 1 & -4 \\ -1 & 2 & -1 & -3 & 2 \\ \hline 0 & 2 & 2 & -2 & -2 \end{array} \quad \begin{array}{ccccc} -4 & 3 & 2 & -1 & 3 \\ 4 & -8 & 4 & 12 & -8 \\ \hline 0 & -5 & 6 & 11 & -5 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ 0 & 1 & 3 & -7 & 4 \\ 0 & 2 & 2 & -2 & -2 \\ 0 & -5 & 6 & 11 & -5 \end{array} \right]$$

Exercise

Use the Gauss-Jordan method to solve the system

$$x - y + 5z = -6$$

$$3x + 3y - z = 10$$

$$x + 3y + 2z = 5$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array} \right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{ccc|c} 3 & 3 & -1 & 10 \\ -3 & 3 & -15 & 18 \\ \hline 0 & 6 & -16 & 28 \end{array} \quad \begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ -1 & 1 & -5 & 6 \\ \hline 0 & 4 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 4 & -3 & 11 \end{array} \right] \frac{1}{6}R_2$$

$$0 \quad 1 \quad -\frac{8}{3} \quad \frac{14}{3}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 4 & -3 & 11 \end{array} \right] \begin{array}{l} R_1 + R_2 \\ \\ R_3 - 4R_2 \end{array}$$

$$\begin{array}{ccc|c} 0 & 4 & -3 & 11 \\ 0 & -4 & \frac{32}{3} & -\frac{56}{3} \\ \hline 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array} \quad \begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ \hline 1 & 0 & \frac{7}{3} & -\frac{4}{3} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3} \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array} \right] \frac{3}{23}R_3$$

$$0 \quad 0 \quad 1 \quad -1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3} \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & 1 & -1 \end{array} \right] \begin{array}{l} R_1 - \frac{7}{3}R_3 \\ R_2 + \frac{8}{3}R_3 \\ \end{array}$$

$$\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3} \\ 0 & 0 & -\frac{7}{3} & \frac{7}{3} \\ \hline 1 & 0 & 0 & 1 \end{array} \quad \begin{array}{ccc|c} 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & \frac{8}{3} & -\frac{8}{3} \\ \hline 0 & 1 & 0 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Solution: $\underline{(1, 2, -1)}$

Exercise

Use the Gauss-Jordan method to solve the system

$$\begin{cases} 2x - y + 4z = -3 \\ x - 2y - 10z = -6 \\ 3x \quad \quad + 4z = 7 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & -3 \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array} \right] \frac{1}{2}R_1$$

$$\begin{array}{cccc} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array} \right] \begin{array}{l} R_2 - R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc|cccc} 1 & -2 & -10 & -6 & 3 & 0 & 4 & 7 \\ -1 & \frac{1}{2} & -2 & \frac{3}{2} & -3 & \frac{3}{2} & -6 & \frac{9}{2} \\ \hline 0 & -\frac{3}{2} & -12 & -\frac{9}{2} & 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array} \right] -\frac{2}{3}R_2$$

$$\begin{array}{cccc} 0 & 1 & 8 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & 1 & 8 & 3 \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array} \right] \begin{array}{l} R_1 + \frac{1}{2}R_2 \\ R_3 - \frac{3}{2}R_2 \end{array}$$

$$\begin{array}{cccc|cccc} 0 & \frac{3}{2} & -2 & \frac{23}{2} & 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} & 0 & \frac{1}{2} & 4 & \frac{3}{2} \\ \hline 0 & 0 & -14 & 7 & 1 & 0 & 6 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & -14 & 7 \end{array} \right] -\frac{1}{14}R_3$$

$$\begin{array}{cccc} 0 & 0 & 1 & -\frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array} \right] \begin{array}{l} R_1 - 6R_3 \\ R_2 - 8R_3 \end{array}$$

$$\begin{array}{cccc|cccc} 1 & 0 & 6 & 0 & 0 & 1 & 8 & 3 \\ 0 & 0 & -6 & 3 & 0 & 0 & -8 & 4 \\ \hline 1 & 0 & 0 & 3 & 0 & 1 & 0 & 7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array} \right]$$

Solution: $\left(3, 7, -\frac{1}{2} \right)$

Exercise

Use the Gauss-Jordan method to solve the system
$$\begin{cases} 4x + 3y - 5z = -29 \\ 3x - 7y - z = -19 \\ 2x + 5y + 2z = -10 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 4 & 3 & -5 & -29 \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array} \right] \frac{1}{4}R_1$$

$$1 \quad \frac{3}{4} \quad -\frac{5}{4} \quad -\frac{29}{4}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array} \right] \begin{array}{l} R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\begin{array}{ccc|c} 3 & -7 & -1 & -19 \\ -3 & -\frac{9}{4} & \frac{15}{4} & \frac{87}{4} \\ \hline 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \end{array}$$

$$\begin{array}{ccc|c} 2 & 5 & 2 & -10 \\ -2 & -\frac{3}{2} & \frac{5}{2} & \frac{29}{2} \\ \hline 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array} \right] -\frac{4}{37}R_2$$

$$0 \quad 1 \quad -\frac{11}{37} \quad -\frac{11}{37}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array} \right] \begin{array}{l} R_1 - \frac{3}{4}R_2 \\ R_3 - \frac{7}{2}R_2 \end{array}$$

$$\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{3}{4} & \frac{33}{148} & \frac{33}{148} \\ \hline 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \end{array}$$

$$\begin{array}{ccc|c} 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \\ 0 & -\frac{7}{2} & \frac{77}{72} & \frac{77}{72} \\ \hline 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array} \right] \frac{72}{401}R_3$$

$$0 \quad 0 \quad 1 \quad 1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & 1 & 1 \end{array} \right] \begin{array}{l} R_1 + \frac{38}{37}R_3 \\ R_2 + \frac{11}{37}R_3 \end{array}$$

$$\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 0 & \frac{38}{37} & \frac{38}{37} \\ \hline 1 & 0 & 0 & -6 \end{array}$$

$$\begin{array}{ccc|c} 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{11}{37} & \frac{11}{37} \\ \hline 0 & 1 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

Solution: $\underline{(-6, 0, 1)}$

Exercise

Use the Gauss-Jordan method to solve the system
$$\begin{cases} x + 2y - 3z = -15 \\ 2x - 3y + 4z = 18 \\ -3x + y + z = 1 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array} \right] \begin{array}{l} \mathbf{R_2 - 2R_1} \\ \mathbf{R_3 + 3R_1} \end{array}$$

$$\begin{array}{cccc} -2 & -4 & 6 & 30 \\ 2 & -3 & 4 & 18 \\ \hline 0 & -7 & 10 & 48 \end{array} \quad \begin{array}{cccc} 3 & 6 & -9 & -45 \\ -3 & 1 & 1 & 1 \\ \hline 0 & 7 & -8 & -44 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \end{array} \right] -\frac{1}{7}\mathbf{R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 7 & -8 & -44 \end{array} \right] \begin{array}{l} \mathbf{R_1 - 2R_2} \\ \mathbf{R_3 - 7R_2} \end{array}$$

$$\begin{array}{cccc} 1 & 2 & -3 & -15 \\ 0 & -2 & \frac{20}{7} & \frac{96}{7} \\ \hline 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \end{array} \quad \begin{array}{cccc} 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \\ \hline 0 & 0 & 2 & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 2 & 4 \end{array} \right] \frac{1}{2}\mathbf{R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 1 & 2 \end{array} \right] \begin{array}{l} \mathbf{R_1 + \frac{1}{7}R_3} \\ \mathbf{R_2 + \frac{10}{7}R_3} \end{array}$$

$$\begin{array}{cccc} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 0 & \frac{1}{7} & \frac{2}{7} \\ \hline 1 & 0 & 0 & -1 \end{array} \quad \begin{array}{cccc} 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & \frac{10}{7} & \frac{20}{7} \\ \hline 0 & 1 & 0 & -4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

Solution: $(-1, -4, 2)$

Exercise

Use the Gauss-Jordan method to solve the system
$$\begin{cases} x + 2y + 3z = 10 \\ 4x + 5y + 6z = 11 \\ 7x + 8y + 9z = 12 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 11 \\ 7 & 8 & 9 & 12 \end{array} \right] \begin{array}{l} \\ R_2 - 4R_1 \\ R_3 - 7R_1 \end{array} \quad \begin{array}{cccc} -4 & -8 & -12 & -40 \\ 4 & 5 & 6 & 11 \\ \hline 0 & -3 & -6 & -29 \end{array} \quad \begin{array}{cccc} -7 & -14 & -21 & -70 \\ 7 & 8 & 9 & 12 \\ \hline 0 & -6 & -12 & -58 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -3 & -6 & -29 \\ 0 & -6 & -12 & -58 \end{array} \right] \frac{1}{3}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & \frac{29}{3} \\ 0 & -6 & -12 & -58 \end{array} \right] R_3 + 6R_2 \quad \begin{array}{cccc} 0 & -6 & -12 & -58 \\ 0 & 6 & 12 & 58 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & \frac{29}{3} \\ 0 & 0 & 0 & 0 \end{array} \right]$$

let z be the variable

$$\text{From Row 1} \Rightarrow y + 2z = \frac{29}{3}$$

$$\underline{y = \frac{29}{3} - 2z}$$

$$\text{From Row 1} \Rightarrow x + 2y + 3z = 10$$

$$x = 10 - 2y - 3z$$

$$x = 10 - 2\left(\frac{29}{3} - 2z\right) - 3z$$

$$x = 10 - \frac{58}{3} + 4z - 3z$$

$$\underline{x = z - \frac{28}{3}}$$

$$\text{Solution: } \underline{\left(z - \frac{28}{3}, \frac{29}{3} - 2z, z \right)}$$

Exercise

Use the Gauss-Jordan method to solve the system
$$\begin{cases} 2x + y + 2z = 4 \\ 2x + 2y = 5 \\ 2x - y + 6z = 2 \end{cases}$$

Solution

$$\begin{aligned} & \left[\begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array} \right] \begin{array}{l} \frac{1}{2}R_1 \\ \\ \end{array} \quad \begin{array}{cccc} & & 1 & \frac{1}{2} \\ & & & 1 \\ & & & 2 \end{array} \\ & \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array} \quad \begin{array}{cccc} 2 & 2 & 0 & 5 \\ -2 & -1 & -2 & -4 \\ 0 & 1 & -2 & 1 \end{array} \quad \begin{array}{cccc} 2 & -1 & 6 & 2 \\ -2 & -1 & -2 & -4 \\ 0 & -2 & 4 & -2 \end{array} \\ & \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & -2 \end{array} \right] \begin{array}{l} \\ R_1 - \frac{1}{2}R_2 \\ R_3 + 2R_2 \end{array} \quad \begin{array}{cccc} 1 & \frac{1}{2} & 1 & 2 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 1 & 0 & 2 & \frac{3}{2} \end{array} \quad \begin{array}{cccc} 0 & -2 & 4 & -2 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array} \\ & \left[\begin{array}{ccc|c} 1 & 0 & 2 & \frac{3}{2} \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

From Row 3: $0 = 0$ is a true statement. Let z be the variable.

From Row 2: $y - 2z = 1$

$$\underline{y = 1 + 2z}$$

From Row 1: $x + 2z = \frac{3}{2}$

$$\underline{x = -2z + \frac{3}{2}}$$

\therefore **Solution:** $\underline{\left(-2z + \frac{3}{2}, 2z + 1, z \right)}$

Exercise

Use the Gauss-Jordan method to solve the system

$$\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array} \right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} -1 & -2 & 3 & 1 \\ 1 & 1 & 2 & 8 \\ \hline 0 & -1 & 5 & 9 \end{array}$$

$$\begin{array}{cccc} 3 & -7 & 4 & 10 \\ -3 & -3 & -6 & -24 \\ \hline 0 & -10 & -2 & -14 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array} \right] -R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array} \right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 10R_2 \end{array}$$

$$\begin{array}{cccc} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ \hline 1 & 0 & 7 & 17 \end{array}$$

$$\begin{array}{cccc} 0 & -10 & -2 & -14 \\ 0 & 10 & -50 & -90 \\ \hline 0 & 0 & -52 & -104 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array} \right] -\frac{1}{52}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array} \right] \begin{array}{l} R_1 - 7R_3 \\ R_2 + 5R_3 \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & 7 & 17 \\ 0 & 0 & -7 & -14 \\ \hline 1 & 0 & 0 & 3 \end{array}$$

$$\begin{array}{cccc} 0 & 1 & -5 & -9 \\ 0 & 0 & 5 & 10 \\ \hline 0 & 1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

\therefore **Solution:** (3, 1, 2)

Exercise

Use augmented elimination to solve linear system

$$2x - 5y + 3z = 1$$

$$x - 2y - 2z = 8$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 2 & -5 & 3 & 1 \end{array} \right] \quad R_2 - 2R_1$$

$$\begin{array}{cccc} 2 & -5 & 3 & 1 \\ -2 & 4 & 4 & -16 \\ \hline 0 & -1 & 7 & -15 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & -1 & 7 & -15 \end{array} \right] \quad -R_2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & 1 & -7 & 15 \end{array} \right] \quad R_1 + 2R_2$$

$$\begin{array}{cccc} 1 & -2 & -2 & 8 \\ 0 & 2 & -14 & 30 \\ \hline 1 & 0 & -16 & 38 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -16 & 38 \\ 0 & 1 & -7 & 15 \end{array} \right] \quad \begin{array}{l} \rightarrow x - 16z = 38 \\ \rightarrow y - 7z = 15 \end{array}$$

$$\begin{cases} x = 16z + 38 \\ y = 7z + 15 \end{cases}$$

$$\therefore \text{Solution: } \underline{(16z + 38, 7z + 15, z)}$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 5 \\ 1 & -1 & 1 & -2 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccc} 2 & 1 & -1 & 5 \\ -2 & -2 & -2 & -4 \\ \hline 0 & -1 & -3 & 1 \end{array}$$

$$\begin{array}{cccc} 1 & -1 & 1 & -2 \\ -1 & -1 & -1 & -2 \\ \hline 0 & -2 & 0 & -4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & 1 \\ 0 & -2 & 0 & -4 \end{array} \right] \quad \begin{array}{l} (2) \\ (1) \\ -2y = -4 \end{array}$$

$$\underline{y = 2}$$

$$(1) \rightarrow -y - 3z = 1$$

$$3z = -1 - 2$$

$$\underline{z = -1}$$

$$(2) \rightarrow x + y + z = 2$$

$$x = 2 - 2 + 1$$

$$\underline{\underline{=1}}$$

$$\therefore \text{Solution: } \underline{(1, 2, -1)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ -1 & -1 & 1 & 1 \\ 3 & -1 & 1 & 9 \end{array} \right] \begin{array}{l} \\ 2R_2 + R_1 \\ 2R_3 - 3R_1 \end{array} \quad \begin{array}{cccc} -2 & -2 & 2 & 2 \\ 2 & 1 & 1 & 9 \\ \hline 0 & -1 & 3 & 11 \end{array} \quad \begin{array}{cccc} 6 & -2 & 2 & 18 \\ -6 & -3 & -3 & -27 \\ \hline 0 & -5 & -1 & -9 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 0 & -1 & 3 & 11 \\ 0 & -5 & -1 & -9 \end{array} \right] R_3 - 5R_2 \quad \begin{array}{cccc} 0 & -5 & -1 & -9 \\ 0 & 5 & -15 & -55 \\ \hline 0 & 0 & -16 & -64 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 0 & -1 & 3 & 11 \\ 0 & 0 & -16 & -64 \end{array} \right] \begin{array}{l} (2) \\ (1) \\ -16z = -64 \end{array}$$

$$\underline{z = 4}$$

$$(1) \rightarrow -y + 3z = 11$$

$$y = 12 - 11$$

$$\underline{\underline{=1}}$$

$$(2) \rightarrow 2x + y + z = 9$$

$$2x = 9 - 1 - 4$$

$$\underline{\underline{x = 2}}$$

$$\therefore \text{Solution: } \underline{(2, 1, 4)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ -3 & 6 & 2 & 11 \end{array} \right] \quad R_3 + 3R_1 \quad \begin{array}{cccc} -3 & 6 & 2 & 11 \\ 3 & 15 & -3 & -12 \\ \hline 0 & 21 & -1 & -1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ 0 & 21 & -1 & -1 \end{array} \right] \quad R_3 - 7R_2 \quad \begin{array}{cccc} 0 & 21 & -1 & -1 \\ 0 & -21 & 7 & 7 \\ \hline 0 & 0 & 6 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ 0 & 0 & 6 & 6 \end{array} \right] \rightarrow \begin{array}{l} x + 5y - z = -4 \quad (2) \\ 3y - z = -1 \quad (1) \\ 6z = 6 \end{array}$$

$$\underline{z = 1}$$

$$(1) \rightarrow 3y = -1 + 1$$

$$\underline{y = 0}$$

$$(2) \rightarrow x = -4 + 1$$

$$\underline{x = -3}$$

$$\therefore \text{Solution: } \underline{(-3, 0, 1)}$$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 2 & -3 & 2 & 10 \\ 3 & -1 & 1 & 9 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \quad \begin{array}{cccc} 2 & -3 & 2 & 10 \\ -2 & -6 & -8 & -28 \\ \hline 0 & -9 & -6 & -18 \end{array} \quad \begin{array}{cccc} 3 & -1 & 1 & 9 \\ -3 & -9 & -12 & -42 \\ \hline 0 & -10 & -11 & -33 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -9 & -6 & -18 \\ 0 & -10 & -11 & -33 \end{array} \right] \quad 9R_3 - 10R_2 \quad \begin{array}{cccc} 0 & -90 & -99 & -297 \\ 0 & 90 & 60 & 180 \\ \hline 0 & 0 & -39 & -117 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -9 & -6 & -18 \\ 0 & 0 & -39 & -117 \end{array} \right] \rightarrow \begin{array}{l} x + 3y + 4z = 14 \quad (3) \\ -9y - 6z = -18 \quad (2) \\ -39z = -117 \quad (1) \end{array}$$

$$(1) \rightarrow z = \frac{117}{39} \\ \underline{= 3}$$

$$(2) \rightarrow 9y = 18 - 6(3)$$

$$9y = 0$$

$$\underline{y = 0}$$

$$(3) \rightarrow x = 14 - 12$$

$$\underline{x = 2}$$

$$\therefore \text{Solution: } \underline{(2, 0, 3)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 3 & 2 & 1 & 8 \\ 2 & -3 & 2 & -16 \end{array} \right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \quad \begin{array}{cccc} 3 & 2 & 1 & 8 \\ -3 & -12 & 3 & -60 \\ 0 & -10 & 4 & -52 \end{array} \quad \begin{array}{cccc} 2 & -3 & 2 & -16 \\ -2 & -8 & 2 & -40 \\ 0 & -11 & 4 & -56 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 0 & -10 & 4 & -52 \\ 0 & -11 & 4 & -56 \end{array} \right] \begin{array}{l} \\ \\ 10R_3 - 11R_2 \end{array} \quad \begin{array}{cccc} 0 & -110 & 40 & -560 \\ 0 & 110 & -44 & 572 \\ 0 & 0 & -4 & 12 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 0 & -10 & 4 & -52 \\ 0 & 0 & -4 & 12 \end{array} \right] \begin{array}{l} x + 4y - z = 20 \quad (3) \\ -10y + 4z = -52 \quad (2) \\ -4z = 12 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -3}$$

$$(2) \rightarrow -10y = -52 + 12$$

$$-10y = -40$$

$$\underline{y = 4}$$

$$(3) \rightarrow x = 20 - 16 - 3$$

$$\underline{x = 1}$$

$$\therefore \text{Solution: } \underline{(1, 4, -3)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 2 & -1 & 7 \\ 2 & -3 & 2 & -1 \end{array} \right] R_3 - 2R_1 \quad \begin{array}{cccc} 2 & -3 & 2 & -1 \\ -2 & -4 & -2 & -34 \\ \hline 0 & -7 & 0 & -35 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 2 & -1 & 7 \\ 0 & -7 & 0 & -35 \end{array} \right] \begin{array}{l} x + 2y + z = 17 \quad (3) \\ 2y - z = 7 \quad (2) \\ -7y = -35 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{y = 5}$$

$$(2) \rightarrow \underline{z = 10 - 7} \\ \underline{= 3}$$

$$(3) \rightarrow \underline{x = 17 - 10 - 3} \\ \underline{= 4}$$

$$\therefore \text{Solution: } \underline{(4, 5, 3)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ -4 & 5 & 3 & 7 \\ -6 & 3 & 5 & -4 \end{array} \right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \quad \begin{array}{cccc} -4 & 5 & 3 & 7 \\ 4 & -12 & -14 & -6 \\ \hline 0 & -7 & -11 & 1 \end{array} \quad \begin{array}{cccc} -6 & 3 & 5 & -4 \\ 6 & -18 & -21 & -9 \\ \hline 0 & -15 & -16 & -13 \end{array}$$

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ 0 & -7 & -11 & 1 \\ 0 & -15 & -16 & -13 \end{array} \right] 7R_3 - 15R_1 \quad \begin{array}{cccc} 0 & -105 & -112 & -91 \\ 0 & 105 & 165 & -15 \\ \hline 0 & 0 & 53 & -106 \end{array}$$

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ 0 & -7 & -11 & 1 \\ 0 & 0 & 53 & -106 \end{array} \right] \begin{array}{l} -2x + 6y + 7z = 3 \quad (3) \\ -7y - 11z = 1 \quad (2) \\ 53z = -106 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -2}$$

$$(2) \rightarrow -7y = 1 - 22$$

$$-7y = -21$$

$$\underline{y = 3}$$

$$(3) \rightarrow -2x = 3 - 18 + 14$$

$$-2x = -1$$

$$\underline{x = \frac{1}{2}}$$

$$\therefore \text{Solution: } \underline{\left(\frac{1}{2}, 3, -2\right)}$$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 3 & -3 & 4 & 5 \\ 4 & -2 & 3 & 4 \end{array} \right] \quad \begin{array}{l} 2R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \quad \begin{array}{cccc} 6 & -6 & 8 & 10 \\ -6 & 3 & -3 & -3 \\ \hline 0 & -3 & 5 & 7 \end{array} \quad \begin{array}{cccc} 4 & -2 & 3 & 4 \\ -4 & 2 & -2 & -2 \\ \hline 0 & 0 & 1 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 0 & -3 & 5 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right] \quad \begin{array}{l} 2x - y + z = 1 \quad (2) \\ -3y + 5z = 7 \quad (1) \\ \underline{z = 2} \end{array}$$

$$(1) \rightarrow -3y = 7 - 10$$

$$-3y = -3$$

$$\underline{y = 1}$$

$$(3) \rightarrow 2x = 1 + 1 - 2$$

$$\underline{x = 0}$$

$$\therefore \text{Solution: } \underline{(0, 1, 2)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 3x - 4y + 4z = 7 \\ x - y - 2z = 2 \\ 2x - 3y + 6z = 5 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 3 & -4 & 4 & 7 \\ 2 & -3 & 6 & 5 \end{array} \right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \quad \begin{array}{cccc} 3 & -4 & 4 & 7 \\ -3 & 3 & 6 & -6 \\ \hline 0 & -1 & 10 & 1 \end{array} \quad \begin{array}{cccc} 2 & -3 & 6 & 5 \\ -2 & 2 & 4 & -4 \\ \hline 0 & -1 & 10 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & -1 & 10 & 1 \\ 0 & -1 & 10 & 1 \end{array} \right] \begin{array}{l} \\ \\ R_3 = R_2 \end{array} \quad \begin{array}{l} \rightarrow x - y - 2z = 2 \quad (2) \\ \rightarrow -y + 10z = 1 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{y = 10z - 1}$$

$$(2) \rightarrow \underline{x = 2 + 10z - 1 + 2z} \\ \underline{= 12z + 1}$$

$$\therefore \text{Solution: } \underline{(12z + 1, 10z - 1, z)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 2 & -1 & 1 & 4 \\ -1 & 1 & 1 & 4 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array} \quad \begin{array}{cccc} 2 & -1 & 1 & 4 \\ -2 & 4 & 2 & -4 \\ \hline 0 & 3 & 3 & 0 \end{array} \quad \begin{array}{cccc} -1 & 1 & 1 & 4 \\ 1 & -2 & -1 & 2 \\ \hline 0 & -1 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & 0 & 6 \end{array} \right] \begin{array}{l} x - 2y - z = 2 \quad (3) \\ 3y + 3z = 0 \quad (2) \\ -y = 6 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{y = -6}$$

$$(2) \rightarrow \underline{z = -y} \\ \underline{= 6}$$

$$(3) \rightarrow \underline{x = 2 - 12 + 6} \\ \underline{= -4}$$

$$\therefore \text{Solution: } \underline{(-4, -6, 6)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x + y + z = 3 \\ -y + 2z = 1 \\ -x + z = 0 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ -1 & 0 & 1 & 0 \end{array} \right] R_3 + R_1 \quad \begin{array}{cccc} -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 3 \\ \hline 0 & 1 & 2 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{array} \right] R_3 + R_2 \quad \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 0 & -1 & 2 & 1 \\ \hline 0 & 0 & 4 & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 4 & 4 \end{array} \right] \begin{array}{ll} x + y + z = 3 & (3) \\ -y + 2z = 1 & (2) \\ 4z = 4 & (1) \end{array}$$

$$(1) \rightarrow \underline{z=1}$$

$$(2) \rightarrow -y = 1 - 2$$

$$\underline{y=1}$$

$$(3) \rightarrow x = 3 - 1 - 1$$

$$\underline{\underline{x=1}}$$

$$\therefore \text{Solution: } \underline{(1, 1, 1)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 3x + y + 3z = 14 \\ 7x + 5y + 8z = 37 \\ x + 3y + 2z = 9 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 3 & 1 & 3 & 14 \\ 7 & 5 & 8 & 37 \end{array} \right] \begin{array}{l} R_2 - 3R_1 \\ R_3 - 7R_1 \end{array} \quad \begin{array}{cccc} 3 & 1 & 3 & 14 \\ -3 & -9 & -6 & -27 \\ \hline 0 & -8 & -3 & -13 \end{array} \quad \begin{array}{cccc} 7 & 5 & 8 & 37 \\ -7 & -21 & -14 & -63 \\ \hline 0 & -16 & -6 & -26 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & -8 & -3 & -13 \\ 0 & -16 & -6 & -26 \end{array} \right] R_3 - 2R_2 \quad \begin{array}{cccc} 0 & -16 & -6 & -26 \\ 0 & 16 & 6 & 26 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & -8 & -3 & -13 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} x+3y+2z=9 \quad (2) \\ -8y-3z=-13 \quad (1) \end{array}$$

$$(1) \rightarrow -8y = 3z - 13$$

$$\underline{y = -\frac{3}{8}z + \frac{13}{8}}$$

$$(3) \rightarrow x = 9 - 3\left(\frac{13}{8} - \frac{3}{8}z\right) - 2z$$

$$= 9 - \frac{39}{8} + \frac{9}{8}z - 2z$$

$$\underline{= \frac{33}{8} - \frac{7}{8}z}$$

$$\therefore \text{Solution: } \underline{\left(\frac{33}{8} - \frac{7}{8}z, \frac{13}{8} - \frac{3}{8}z, z \right)}$$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 4x - 2y + z = 7 \\ x + y + z = -2 \\ 4x + 2y + z = 3 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 4 & -2 & 1 & 7 \\ 4 & 2 & 1 & 3 \end{array} \right] \quad \begin{array}{l} R_2 - 4R_1 \\ R_3 - 4R_1 \end{array} \quad \begin{array}{cccc} 4 & -2 & 1 & 7 \\ -4 & -4 & -4 & 8 \\ 0 & -6 & -3 & 15 \end{array} \quad \begin{array}{cccc} 4 & 2 & 1 & 3 \\ -4 & -4 & -4 & 8 \\ 0 & -2 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 15 \\ 0 & -2 & -3 & 11 \end{array} \right] \quad -3R_3 + R_2 \quad \begin{array}{cccc} 0 & 6 & 9 & -33 \\ 0 & -6 & -3 & 15 \\ 0 & 0 & 6 & -18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 15 \\ 0 & 0 & 6 & -18 \end{array} \right] \quad \begin{array}{l} x+y+z=-2 \quad (3) \\ -6y-3z=15 \quad (2) \\ 6z=-18 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -3}$$

$$(2) \rightarrow -6y = 15 - 9$$

$$\underline{y = -1}$$

$$(3) \rightarrow x = -2 + 1 + 3$$

$$\underline{= 2}$$

$$\therefore \text{Solution: } \underline{(2, -1, -3)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 2x - 2y + z = -4 \\ 6x + 4y - 3z = -24 \\ x - 2y + 2z = 1 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 2 & -2 & 1 & -4 \\ 6 & 4 & -3 & -24 \end{array} \right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 6R_1 \end{array} \quad \begin{array}{ccc|c} 2 & -2 & 1 & -4 \\ -2 & 4 & -4 & -2 \\ \hline 0 & 2 & -3 & -6 \end{array} \quad \begin{array}{ccc|c} 6 & 4 & -3 & -24 \\ -6 & 12 & -12 & -6 \\ \hline 0 & 16 & -15 & -30 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 16 & -15 & -30 \end{array} \right] \begin{array}{l} \\ R_3 - 8R_2 \end{array} \quad \begin{array}{ccc|c} 0 & 16 & -15 & -30 \\ 0 & -16 & 24 & 48 \\ \hline 0 & 0 & 9 & 18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 0 & 9 & 18 \end{array} \right] \begin{array}{l} x - 2y + 2z = 1 \quad (3) \\ 2y - 3z = -6 \quad (2) \\ 9z = 18 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = 2}$$

$$(2) \rightarrow 2y = -6 + 6$$

$$\underline{y = 0}$$

$$(3) \rightarrow x = 1 - 4$$

$$\underline{\underline{= -3}}$$

$$\therefore \text{Solution: } \underline{(-3, 0, 2)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 9x + 3y + z = 4 \\ 16x + 4y + z = 2 \\ 25x + 5y + z = 2 \end{cases}$$

Solution

$$\begin{cases} z + 9x + 3y = 4 \\ z + 16x + 4y = 2 \\ z + 25x + 5y = 2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 1 & 16 & 4 & 2 \\ 1 & 25 & 5 & 2 \end{array} \right] \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array} \quad \begin{array}{ccc|c} 1 & 16 & 4 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 7 & 1 & -2 \end{array} \quad \begin{array}{ccc|c} 1 & 25 & 5 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 16 & 2 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 16 & 2 & -2 \end{array} \right] \quad 7R_3 - 16R_2$$

$$\begin{array}{cccc} 0 & 112 & 14 & -14 \\ 0 & -112 & -16 & 32 \\ \hline 0 & 0 & -2 & 18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 0 & -2 & 18 \end{array} \right] \quad \begin{array}{l} z + 9x + 3y = 4 \quad (3) \\ 7x + y = -2 \quad (2) \\ -2y = 18 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{y = -9}$$

$$(2) \rightarrow 7x = -2 + 9 \\ \underline{= 1}$$

$$(3) \rightarrow z = 4 - 9 + 27 \\ \underline{= 22}$$

$$\therefore \text{Solution: } \underline{(1, -9, 22)}$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 2 & -8 \\ 3 & -1 & -4 & 3 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 2 & -8 \\ -2 & -4 & 6 & -18 \\ \hline 0 & -5 & 8 & -26 \end{array} \quad \begin{array}{cccc} 3 & -1 & -4 & 3 \\ -3 & -6 & 9 & -27 \\ \hline 0 & -7 & 5 & -24 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & -7 & 5 & -24 \end{array} \right] \quad 5R_3 - 7R_2$$

$$\begin{array}{cccc} 0 & -35 & 25 & -120 \\ 0 & 35 & -56 & 182 \\ \hline 0 & 0 & -31 & 62 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & 0 & -31 & 62 \end{array} \right] \quad \begin{array}{l} x + 2y - 3z = 9 \quad (3) \\ -5y + 8z = -26 \quad (2) \\ -31z = 62 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -2}$$

$$(2) \rightarrow -5y = -26 + 16 \\ -5y = 10 \\ \underline{y = 2}$$

$$(3) \rightarrow x = 9 - 4 - 6$$

$$\underline{\underline{= -1}}$$

$$\therefore \text{Solution: } \underline{\underline{(-1, 2, -2)}}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x - 3z = -5 \\ 2x - y + 2z = 16 \\ 7x - 3y - 5z = 19 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 2 & -1 & 2 & 16 \\ 7 & -3 & -5 & 19 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 7R_1 \end{array} \quad \begin{array}{ccc|c} 2 & -1 & 2 & 16 \\ -2 & 0 & 6 & 10 \\ \hline 0 & -1 & 8 & 26 \end{array} \quad \begin{array}{ccc|c} 7 & -3 & -5 & 19 \\ -7 & 0 & 21 & 35 \\ \hline 0 & -3 & 16 & 54 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & -3 & 16 & 54 \end{array} \right] \begin{array}{l} \\ \\ R_3 - 3R_2 \end{array} \quad \begin{array}{ccc|c} 0 & -3 & 16 & 54 \\ 0 & 3 & -24 & -78 \\ \hline 0 & 0 & -8 & -24 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & 0 & -8 & -24 \end{array} \right] \begin{array}{l} x - 3z = -5 \quad (3) \\ -y + 8z = 26 \quad (2) \\ -8z = -24 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{\underline{z = 3}}$$

$$(2) \rightarrow -y = 26 - 24$$

$$\underline{\underline{y = -2}}$$

$$(3) \rightarrow x = -5 + 9$$

$$\underline{\underline{= 4}}$$

$$\therefore \text{Solution: } \underline{\underline{(4, -2, 3)}}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x + 2y - z = 5 \\ 2x - y + 3z = 0 \\ 2y + z = 1 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 2 & -1 & 3 & 0 \\ 0 & 2 & 1 & 1 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ \end{array} \quad \begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ -2 & -4 & 2 & -10 \\ \hline 0 & -5 & 5 & -10 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 2 & 1 & 1 \end{array} \right] \quad 5R_3 + 2R_2 \quad \begin{array}{cccc} 0 & 10 & 5 & 5 \\ 0 & -10 & 10 & -20 \\ \hline 0 & 0 & 15 & -15 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 0 & 15 & -15 \end{array} \right] \quad \begin{array}{l} x + 2y - z = 5 \quad (3) \\ -5y + 5z = -10 \quad (2) \\ 15z = -15 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -1}$$

$$(2) \rightarrow -5y = -10 + 5 \\ \underline{y = 1}$$

$$(3) \rightarrow x = 5 - 2 - 1 \\ \underline{= 2}$$

$$\therefore \text{Solution: } \underline{(2, 1, -1)}$$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + y + z = 6 \\ 3x + 4y - 7z = 1 \\ 2x - y + 3z = 5 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 4 & -7 & 1 \\ 2 & -1 & 3 & 5 \end{array} \right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \quad \begin{array}{cccc} 3 & 4 & -7 & 1 \\ -3 & -3 & -3 & -18 \\ \hline 0 & 1 & -10 & -17 \end{array} \quad \begin{array}{cccc} 2 & -1 & 3 & 5 \\ -2 & -2 & -2 & -12 \\ \hline 0 & -3 & 1 & -7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -10 & -17 \\ 0 & -3 & 1 & -7 \end{array} \right] \quad R_3 + 3R_2 \quad \begin{array}{cccc} 0 & -3 & 1 & -7 \\ 0 & 3 & -30 & -51 \\ \hline 0 & 0 & -29 & -58 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -10 & -17 \\ 0 & 0 & -29 & -58 \end{array} \right] \quad \begin{array}{l} x + y + z = 6 \quad (3) \\ y - 10z = -17 \quad (2) \\ -29z = -58 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = 2}$$

$$(2) \rightarrow y = -17 + 20 \\ \underline{= 3}$$

$$(3) \rightarrow x = 6 - 3 - 2 \\ \underline{= 1}$$

$$\therefore \text{Solution: } \underline{(1, 3, 2)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 3x + 2y + 3z = 3 \\ 4x - 5y + 7z = 1 \\ 2x + 3y - 2z = 6 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 4 & -5 & 7 & 1 \\ 2 & 3 & -2 & 6 \end{array} \right] \begin{array}{l} \\ 3R_2 - 4R_1 \\ 3R_3 - 2R_1 \end{array} \quad \begin{array}{cccc} 12 & -15 & 21 & 3 \\ -12 & -8 & -12 & -12 \\ \hline 0 & -23 & 9 & -9 \end{array} \quad \begin{array}{cccc} 6 & 9 & -6 & 18 \\ -6 & -4 & -6 & -6 \\ \hline 0 & 5 & -12 & 12 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 0 & -23 & 9 & -9 \\ 0 & 5 & -12 & 12 \end{array} \right] 23R_3 + 5R_2 \quad \begin{array}{cccc} 0 & 115 & -276 & 276 \\ 0 & -115 & 45 & -45 \\ \hline 0 & 0 & -231 & 231 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 0 & -23 & 9 & -9 \\ 0 & 0 & -231 & 231 \end{array} \right] \begin{array}{l} 3x + 2y + 3z = 3 \quad (3) \\ -23y + 9z = -9 \quad (2) \\ -231z = 231 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -1}$$

$$(2) \rightarrow -23y = -9 + 9$$

$$\underline{y = 0}$$

$$(3) \rightarrow 3x = 3 + 3$$

$$\underline{x = 2}$$

$$\therefore \text{Solution: } \underline{(2, 0, -1)}$$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x - 3y + z = 2 \\ 4x - 12y + 4z = 8 \\ -2x + 6y - 2z = -4 \end{cases}$$

Solution

$$\begin{cases} x - 3y + z = 2 \\ \frac{1}{4} \times 4x - 12y + 4z = 8 \\ -\frac{1}{2} \times -2x + 6y - 2z = -4 \end{cases}$$

$$\begin{cases} x - 3y + z = 2 \\ x - 3y + z = 2 \\ x - 3y + z = 2 \end{cases}$$

Since all three equations are the same.

∴ **Solution:** is the plane $\underline{x - 3y + z = 2}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x - 2y + z = -1 \\ x + 2y - z = 2 \\ 6x + 4y + 3z = 5 \end{cases}$

Solution

$$\begin{bmatrix} 1 & 2 & -1 & | & 2 \\ 2 & -2 & 1 & | & -1 \\ 6 & 4 & 3 & | & 5 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 6R_1 \end{array} \quad \begin{array}{cccc} 2 & -2 & 1 & -1 \\ -2 & -4 & 2 & -4 \\ \hline 0 & -6 & 3 & -5 \end{array} \quad \begin{array}{cccc} 6 & 4 & 3 & 5 \\ -6 & -12 & 6 & -12 \\ \hline 0 & -8 & 9 & -7 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 & | & 2 \\ 0 & -6 & 3 & | & -5 \\ 0 & -8 & 9 & | & -7 \end{bmatrix} \begin{array}{l} \\ \\ 3R_3 - 4R_2 \end{array} \quad \begin{array}{cccc} 0 & -24 & 27 & -21 \\ 0 & 24 & -12 & 20 \\ \hline 0 & 0 & 15 & -1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 & | & 2 \\ 0 & -6 & 3 & | & -5 \\ 0 & 0 & 15 & | & -1 \end{bmatrix} \begin{array}{l} x + 2y - z = 2 \quad (3) \\ -6y + 3z = -5 \quad (2) \\ 15z = -1 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -\frac{1}{15}}$$

$$(2) \rightarrow -6y = -5 + \frac{1}{5}$$

$$-6y = -\frac{24}{5}$$

$$\underline{y = \frac{4}{5}}$$

$$(3) \rightarrow x = 2 - \frac{8}{5} - \frac{1}{15}$$

$$\underline{= \frac{1}{3}}$$

∴ **Solution:** $\underline{\left(\frac{1}{3}, \frac{4}{5}, -\frac{1}{15}\right)}$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 \\ x_2 - 3x_3 - x_4 = 0 \\ 3x_1 + 2x_3 - x_4 = 6 \\ -4x_1 + x_2 + 4x_3 + 2x_4 = -3 \end{cases}$$

Solution

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 3 & 0 & 2 & -1 & 6 \\ -4 & 1 & 4 & 2 & -3 \end{array} \right] \begin{array}{l} R_3 - 3R_1 \\ R_4 + 4R_1 \end{array}$$

$$\begin{array}{ccccc|ccccc} 3 & 0 & 2 & -1 & 6 & -4 & 1 & 4 & 2 & -3 \\ -3 & 15 & -6 & 6 & -12 & 4 & -20 & 8 & -8 & 16 \\ \hline 0 & 15 & -4 & 5 & -6 & 0 & -19 & 12 & -6 & 13 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 15 & -4 & 5 & -6 \\ 0 & -19 & 12 & -6 & 13 \end{array} \right] \begin{array}{l} R_3 - 15R_2 \\ R_4 + 19R_2 \end{array}$$

$$\begin{array}{ccccc|ccccc} 0 & 15 & -4 & 5 & -6 & 0 & -19 & 12 & -6 & 13 \\ 0 & -15 & 45 & 15 & 0 & 0 & 19 & -57 & -19 & 0 \\ \hline 0 & 0 & 41 & 20 & -6 & 0 & 0 & -45 & -25 & 13 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & -45 & -25 & 13 \end{array} \right] 41R_4 + 45R_2$$

$$\begin{array}{ccccc|ccccc} 0 & 0 & -1845 & -1025 & 533 & & & & & \\ 0 & 0 & 1845 & 900 & -270 & & & & & \\ \hline 0 & 0 & 0 & -125 & 263 & & & & & \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & 0 & -125 & 263 \end{array} \right] \begin{array}{l} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 \quad (4) \\ x_2 - 3x_3 - x_4 = 0 \quad (3) \\ 41x_3 + 20x_4 = -6 \quad (2) \\ -125x_4 = 263 \quad (1) \end{array}$$

$$(1) \rightarrow x_4 = -\frac{263}{125}$$

$$(2) \rightarrow 41x_3 = -6 + \frac{1,052}{25}$$

$$= \frac{902}{25}$$

$$x_3 = \frac{22}{25}$$

$$(3) \rightarrow x_2 = \frac{66}{25} - \frac{263}{125}$$

$$= \frac{67}{125}$$

$$(4) \rightarrow x_1 = 4 + \frac{67}{25} - \frac{44}{25} - \frac{526}{125}$$

$$\begin{aligned}
&= 4 + \frac{23}{25} - \frac{526}{125} \\
&= \frac{500 + 115 - 526}{125} \\
&= \frac{89}{125}
\end{aligned}$$

$$\therefore \text{Solution: } \left(\frac{89}{125}, \frac{67}{125}, \frac{22}{25}, -\frac{263}{125} \right)$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases}
x_1 + x_2 + x_3 + x_4 = 5 \\
x_1 + 2x_2 - x_3 - 2x_4 = -1 \\
x_1 - 3x_2 - 3x_3 - x_4 = -1 \\
2x_1 - x_2 + 2x_3 - x_4 = -2
\end{cases}$$

Solution

$$\left[\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 5 \\
1 & 2 & -1 & -2 & -1 \\
1 & -3 & -3 & -1 & -1 \\
2 & -1 & 2 & -1 & -2
\end{array} \right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - 2R_1 \end{array}$$

$$\begin{array}{ccccc}
1 & 2 & -1 & -2 & -1 \\
-1 & -1 & -1 & -1 & -5 \\
\hline
0 & 1 & -2 & -3 & -6
\end{array}$$

$$\begin{array}{ccccc}
1 & -3 & -3 & -1 & -1 \\
-1 & -1 & -1 & -1 & -5 \\
\hline
0 & -4 & -4 & -2 & -6
\end{array}$$

$$\begin{array}{ccccc}
2 & -1 & 2 & -1 & -2 \\
-2 & -2 & -2 & -2 & -10 \\
\hline
0 & -3 & 0 & -3 & -12
\end{array}$$

$$\left[\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 5 \\
0 & 1 & -2 & -3 & -6 \\
0 & -4 & -4 & -2 & -6 \\
0 & -3 & 0 & -3 & -12
\end{array} \right] \begin{array}{l} \\ R_3 + 4R_2 \\ R_4 + 3R_2 \end{array}$$

$$\begin{array}{ccccc}
0 & -4 & -4 & -2 & -6 \\
0 & 4 & -8 & -12 & -24 \\
\hline
0 & 0 & -12 & -14 & -30
\end{array}$$

$$\begin{array}{ccccc}
0 & -3 & 0 & -3 & -12 \\
0 & 3 & -6 & -9 & -18 \\
\hline
0 & 0 & -6 & -12 & -30
\end{array}$$

$$\left[\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 5 \\
0 & 1 & -2 & -3 & -6 \\
0 & 0 & -12 & -14 & -30 \\
0 & 0 & -6 & -12 & -30
\end{array} \right] -2R_4 + R_3$$

$$\begin{array}{ccccc}
0 & 0 & 12 & 24 & 60 \\
0 & 0 & -12 & -14 & -30 \\
\hline
0 & 0 & 0 & 10 & 30
\end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & -12 & -14 & -30 \\ 0 & 0 & 0 & 10 & 30 \end{array} \right] \quad \begin{array}{l} x_1 + x_2 + x_3 + x_4 = 5 \quad (4) \\ x_2 - 2x_3 - 3x_4 = -6 \quad (3) \\ -12x_3 - 14x_4 = -30 \quad (2) \\ 10x_4 = 30 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{x_4 = 3}$$

$$(2) \rightarrow -12x_3 = -30 + 42 \\ = 12$$

$$\underline{x_3 = -1}$$

$$(3) \rightarrow x_2 = -6 - 2 + 9 \\ = 1$$

$$(4) \rightarrow x_1 = 5 - 1 + 1 - 3 \\ = 2$$

$$\therefore \text{Solution: } \underline{(2, 1, -1, 3)}$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 2x + 8y - z + w = 0 \\ 4x + 16y - 3z - w = -10 \\ -2x + 4y - z + 3w = -6 \\ -6x + 2y + 5z + w = 3 \end{cases}$$

Solution

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 4 & 16 & -3 & -1 & -10 \\ -2 & 4 & -1 & 3 & -6 \\ -6 & 2 & 5 & 1 & 3 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 + 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 26 & 2 & 4 & 3 \end{array} \right] \quad R_4 - \frac{13}{6}R_2$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 26 & 2 & 4 & 3 \end{array} \right] \quad \text{Interchange } R_2 \text{ and } R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & \frac{19}{3} & -\frac{14}{3} & 16 \end{array} \right] R_4 + \frac{19}{3}R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & 0 & -\frac{71}{3} & -\frac{142}{3} \end{array} \right] \begin{array}{l} 2x + 8y - z + w = 0 \quad (3) \\ 12y - 2z + 4w = -6 \quad (2) \\ -z - 3w = -10 \quad (1) \\ -\frac{71}{3}w = -\frac{142}{3} \rightarrow \underline{w = 2} \end{array}$$

$$(1) \rightarrow z = 10 - 3w = \underline{4}$$

$$(2) \rightarrow 12y = 2z - 4w - 6$$

$$\underline{y = -\frac{1}{2}}$$

$$(3) \rightarrow 2x = -8y + z - w$$

$$2x = 4 + 4 - 2$$

$$2x = 6$$

$$\underline{x = 3}$$

$$\therefore \text{Solution: } \underline{\left(3, -\frac{1}{2}, 4, 2 \right)}$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

Solution

$$\begin{cases} x_1 = -2x_2 \\ x_3 = -x_2 \end{cases}$$

$$2x_1 + x_2 + 3x_3 = 0$$

$$-4x_2 + x_2 - 3x_2 = 0 \rightarrow \underline{x_2 = 0}$$

$$\therefore \text{Solution: } \underline{(0, 0, 0)}$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 2x + 2y + 4z = 0 \\ -y - 3z + w = 0 \\ 3x + y + z + 2w = 0 \\ x + 3y - 2z - 2w = 0 \end{cases}$$

Solution

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & -1 & -3 & 1 & 0 \\ 3 & 1 & 1 & 2 & 0 \\ 1 & 3 & -2 & -2 & 0 \end{array} \right] \begin{array}{l} \\ -R_2 \\ 2R_3 - 3R_1 \\ 2R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & -4 & -10 & 4 & 0 \\ 0 & 4 & -8 & -4 & 0 \end{array} \right] \begin{array}{l} \\ R_3 + 4R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -20 & 0 & 0 \end{array} \right] \begin{array}{l} 2x + 2y - 4z = 0 \quad (1) \\ y + 3z - w = 0 \quad (2) \\ \rightarrow \underline{z = 0} \\ \end{array}$$

$$(2) \rightarrow \underline{y = w}$$

$$(1) \rightarrow 2x = -2y \quad \underline{x = -w}$$

$$\therefore \text{Solution: } \underline{(-w, w, 0, w)}$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 2x + z + w = 5 \\ y - w = -1 \\ 3x - z - w = 0 \\ 4x + y + 2z + w = 9 \end{cases}$$

Solution

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 3 & 0 & -1 & -1 & 0 \\ 4 & 1 & 2 & 1 & 9 \end{array} \right] \begin{array}{l} \\ \\ 2R_3 - 3R_1 \\ 2R_4 - 4R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -5 & -5 & -15 \\ 0 & 2 & 0 & -2 & -2 \end{array} \right] \quad R_4 - 2R_2$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -5 & -5 & -15 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} 2x + z + w = 5 \quad (1) \\ y - w = -1 \quad (2) \\ -5z - 5w = -15 \quad (3) \end{array}$$

$$(2) \rightarrow \underline{y = 1 + w}$$

$$(3) \rightarrow \underline{z = 3 - w}$$

$$(1) \rightarrow 2x = 5 - (3 - w) - w \Rightarrow \underline{x = 1}$$

$$\therefore \text{Solution: } \underline{(1, 1 + w, 3 - w, w)}$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 4y + z = 20 \\ 2x - 2y + z = 0 \\ x + z = 5 \\ x + y - z = 10 \end{cases}$$

Solution

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 10 & 10 \\ 2 & -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 5 & 5 \\ 0 & 4 & 1 & 20 & 20 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 10 & 10 \\ 0 & -4 & 3 & -20 & -20 \\ 0 & -1 & 2 & -5 & -5 \\ 0 & 4 & 1 & 20 & 20 \end{array} \right] \quad \begin{array}{l} 4R_3 - R_2 \\ R_4 + R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 10 & 10 \\ 0 & -4 & 3 & -20 & -20 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{array} \right] \quad \begin{array}{l} x + y = 10 \\ \rightarrow -4y = -20 \\ \rightarrow z = 0 \end{array}$$

$$\therefore \text{Solution: } \underline{(5, 5, 0)}$$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \quad \quad - 3w = -3 \end{cases}$$

Solution

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & -1 \\ -1 & 3 & -2 & -2 & -2 \\ 3 & 4 & -7 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array} \right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & -1 \\ 0 & 5 & -1 & -1 & -3 \\ 0 & -2 & -10 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} \\ \\ 5R_3 + 2R_2 \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & -1 \\ 0 & 5 & -1 & -1 & -3 \\ 0 & 0 & -52 & 6 & 6 \end{array} \right] \begin{array}{l} x + 2y + z = -1 \quad (3) \\ 5y - z = -3 \quad (2) \\ -52z = 6 \quad (1) \end{array}$$

$$(1) \Rightarrow z = -\frac{6}{52} = -\frac{3}{26}$$

$$(2) \Rightarrow 5y - (-\frac{3}{26}) = -3 \Rightarrow 5y = -3 - \frac{3}{26} = -\frac{78}{26} - \frac{3}{26} = -\frac{81}{26} \Rightarrow y = -\frac{81}{130}$$

$$(3) \Rightarrow x + 2(-\frac{81}{130}) + (-\frac{3}{26}) = -1 \Rightarrow x - \frac{162}{65} - \frac{3}{26} = -1 \Rightarrow x = -1 + \frac{162}{65} + \frac{3}{26} = -\frac{65}{65} + \frac{162}{65} + \frac{7.5}{65} = \frac{104.5}{65} = \frac{209}{130}$$

$$\therefore \text{Solution: } \left(\frac{209}{130}, -\frac{81}{130}, -\frac{3}{26} \right)$$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} 2u - 3v + w - x + y = 0 \\ 4u - 6v + 2w - 3x - y = -5 \\ -2u + 3v - 2w + 2x - y = 3 \end{cases}$$

Solution

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 4 & -6 & 2 & -3 & -1 & -5 \\ -2 & 3 & -2 & 2 & -1 & 3 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -3 & -5 \\ 0 & 0 & -1 & 1 & 0 & 3 \end{array} \right] \quad \begin{array}{l} 2u - 3v + w - x + y = 0 \quad (3) \\ -x - 3y = -5 \quad (2) \\ -w + x = 3 \quad (1) \end{array}$$

$$(2) \Rightarrow x = 5 - 3y$$

$$(1) \Rightarrow w = x - 3 = 2 - 3y$$

$$(3) \Rightarrow 2u = 3v - 2 + 3y + 5 - 3y - y = 3v - y + 3$$

$$u = \frac{3}{2}v - \frac{1}{2}y + \frac{3}{2}$$

$$\therefore \text{Solution: } \left(\frac{3}{2}v - \frac{1}{2}y + \frac{3}{2}, v, 2 - 3y, 5 - 3y, y \right)$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 6x_3 + 2x_4 - 4x_5 - 8x_6 = 8 \\ 3x_3 + x_4 - 2x_5 - 4x_6 = 4 \\ 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 + x_6 = 2 \\ 6x_1 - 9x_2 + 11x_4 - 19x_5 + 3x_6 = 1 \end{cases}$$

Solution

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 6 & -9 & 0 & 11 & -19 & 3 & 1 \end{array} \right] \quad R_4 - 3R_1$$

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & -3 & -1 & 2 & 0 & -5 \end{array} \right] \quad \begin{array}{l} R_3 - 2R_2 \\ R_4 + R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & -1 \end{array} \right] \quad \begin{array}{l} 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 = 2 - x_6 \\ 3x_3 + x_4 - 2x_5 = 4 + 4x_6 \end{array}$$

$$\rightarrow \underline{x_6 = \frac{1}{4}}$$

$$\begin{cases} 3x_3 = 5 - x_4 + 2x_5 \\ 2x_1 = \frac{7}{4} + 3x_2 - \frac{1}{3}(5 - x_4 + 2x_5) - 4x_4 + 7x_5 \end{cases}$$

$$\left\{ \begin{array}{l} x_3 = \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5 \\ 2x_1 = \frac{1}{12} + 3x_2 - \frac{11}{3}x_4 + \frac{19}{3}x_5 \end{array} \right|$$

$$\therefore \text{Solution: } \left(\frac{1}{24} + \frac{3}{2}x_2 - \frac{11}{6}x_4 + \frac{19}{6}x_5, x_2, \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5, x_4, x_5, \frac{1}{4} \right) |$$

Exercise

Use augmented elimination to solve linear system

$$\left\{ \begin{array}{l} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{array} \right.$$

Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{array} \right] \begin{array}{l} \\ 3R_2 - 5R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & -1 & 4 & 26 \\ 0 & 0 & 0 & 0 \end{array} \right] R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} 3x_1 + 2x_2 - x_3 = -15 \quad (3) \\ -x_2 + 11x_3 = 75 \quad (2) \\ -7x_3 = -49 \quad (1) \end{array}$$

$$(1) \rightarrow x_3 = 7$$

$$(2) \rightarrow x_2 = 77 - 75 = 2$$

$$(1) \rightarrow 3x_1 = -15 - 4 + 7 = -12$$

$$\underline{x_1 = -4}$$

$$\therefore \text{Solution: } \underline{(-4, 2, 7)} |$$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ 5x_3 + 10x_4 + 15x_6 = 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$$

Solution

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array} \right] -R_2$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array} \right] \begin{array}{l} \\ R_3 - 5R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array} \right] \frac{1}{6}R_4 \text{ then interchanging row3 and row4}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] R_2 - 3R_3$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \begin{cases} x_1 + 3x_2 + 4x_4 + 2x_5 = 0 \\ x_3 + 2x_4 = 0 \\ + x_6 = \frac{1}{3} \end{cases}$$

The general solution of the system: $x_6 = \frac{1}{3}, \quad x_3 = -2x_4, \quad x_1 = -3x_2 - 4x_4 - 2x_5$

\therefore **Solution:** $\left(-3x_2 - 4x_4 - 2x_5, x_2, -2x_4, x_4, x_5, \frac{1}{3} \right)$

Exercise

At SnackMix, caramel corn worth \$2.50 per *pound* is mixed with honey roasted missed nuts worth \$7.50 per *pound* in order to get 20 *lbs.* of a mixture worth \$4.50 per *pound*. How much of each snack is used?

Solution

$$x + y = 20 \quad (1)$$

$$2.50x + 7.50y = 90 \quad (2)$$

$$(1) \quad y = 20 - x$$

$$(2) \quad 2.5x + 7.5(20 - x) = 90$$

$$2.5x + 150 - 7.5x = 90$$

$$-5x = 90 - 150$$

$$-5x = -60$$

$$x = \frac{-60}{-5} = 12$$

$$y = 20 - x$$

$$= 20 - 12$$

$$= 8$$

The mixture consists of 12 *lbs.* of caramel and 8 *lbs.* of nuts

Solution **Section 4.2 – Matrix operations and Their Applications**

Exercise

Find values for the variables so that the matrices are equal. $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

Solution

$$\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$$

$$\Rightarrow \begin{cases} w = 9 & x = 17 \\ y = 8 & z = -12 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal. $\begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$

Solution

$$\begin{cases} x = 12 \\ y + 3 = 5 \rightarrow y = 2 \\ 2z = 6 \rightarrow z = 3 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal. $\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix} = \begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$

Solution

$$\begin{bmatrix} 5 = y+3 & x-4 = 2 & 9 = 9 \\ 2 = z+4 & -3 = -3 & 8 = 8 \\ 6 = 6 & 0 = 0 & 5 = w \end{bmatrix}$$

$$\rightarrow \begin{cases} y = 2 & z = -2 \\ x = 6 & w = 5 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

Solution

$$\begin{bmatrix} a-5 & 5b & 4c+6 \\ -2d & 7f-6 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

$$\begin{cases} a-5=15 & \rightarrow & a=20 \\ 5b=25 & \rightarrow & b=5 \\ 4c+6=6 & \rightarrow & 4c=0 \rightarrow c=0 \\ -2d=-8 & \rightarrow & d=4 \\ 7f-6=1 & \rightarrow & 7f=7 \rightarrow f=1 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+11 & 12z+1 & 5m \\ 11k & 3 & 1 \end{bmatrix} + \begin{bmatrix} 9a & 9z & 4m \\ 12k & 5 & 3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

Solution

$$\begin{bmatrix} a+11+9a & 12z+1+9z & 5m+4m \\ 11k+12k & 3+5 & 1+3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 10a+11 & 21z+1 & 9m \\ 23k & 8 & 4 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

$$10a+11=41 \rightarrow 10a=30$$

$$\underline{a=3}$$

$$21z+1=-62 \rightarrow 21z=-63$$

$$\underline{z=-3}$$

$$9m=72 \rightarrow \underline{m=8}$$

$$23k=92 \rightarrow \underline{k=\frac{92}{23}=4}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} x+2 & 3y+1 & 5z \\ 8w & 2 & 3 \end{bmatrix} + \begin{bmatrix} 3x & 2y & 5z \\ 2w & 5 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 4x+2 & 5y+1 & 10z \\ 10w & 7 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$$

$$\begin{cases} 4x+2=10 & \rightarrow \underline{x=2} \\ 5y+1=-14 & \rightarrow \underline{y=-3} \\ 10z=80 & \rightarrow \underline{z=8} \\ 10w=10 & \rightarrow \underline{w=1} \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} 2x-3 & y-2 & 2z+1 \\ 5 & 2w & 7 \end{bmatrix} + \begin{bmatrix} 3x-3 & y+2 & z-1 \\ -5 & 5w+1 & 3 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 5x-6 & 2y & 3z \\ 0 & 7w+1 & 10 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$$

$$\begin{cases} 5x-6=20 & \rightarrow \underline{x=\frac{26}{5}} \\ 2y=8 & \rightarrow \underline{y=4} \\ 3z=9 & \rightarrow \underline{z=3} \\ 7w+1=8 & \rightarrow \underline{w=1} \end{cases}$$

Exercise

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix}$$

Solution

$$\begin{aligned} A - B &= \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix} - \begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 & -5 \\ 2 & 1 & 9 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
 3A + 2B &= 3 \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix} + 2 \begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix} \\
 &= \begin{bmatrix} 9 & 3 & 3 \\ -3 & 6 & 15 \end{bmatrix} + \begin{bmatrix} 4 & -6 & 12 \\ -6 & 2 & -8 \end{bmatrix} \\
 &= \begin{bmatrix} 13 & -3 & 15 \\ -9 & 8 & 7 \end{bmatrix}
 \end{aligned}$$

Exercise

Given $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ $F = \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}$ Find $3F + 2A$

Solution

$$\begin{aligned}
 3F + 2A &= 3 \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} 3(3) & 3(3) \\ 3(-1) & 3(-1) \end{bmatrix} + \begin{bmatrix} 2(1) & 2(2) \\ 2(4) & 2(3) \end{bmatrix} \\
 &= \begin{bmatrix} 9 & 9 \\ -3 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 8 & 6 \end{bmatrix} \\
 &= \begin{bmatrix} 9+2 & 9+4 \\ -3+8 & -3+6 \end{bmatrix} \\
 &= \begin{bmatrix} 11 & 13 \\ 5 & 3 \end{bmatrix}
 \end{aligned}$$

Exercise

Evaluate $\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \\ 12 \end{bmatrix}$

Solution

$$\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \\ 12 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ 20 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 5 & 8 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 9 & 1 \\ 4 & 2 & 5 \end{bmatrix}$

Solution

$$\begin{bmatrix} 5 & 8 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 9 & 1 \\ 4 & 2 & 5 \end{bmatrix}$$

It is ***impossible***; 2×2 and 2×3 are not the same size.

Exercise

Evaluate $\begin{bmatrix} -5 & 0 \\ 4 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix}$

Solution

$$\begin{bmatrix} -5 & 0 \\ 4 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -5+6 & 0+(-3) \\ 4+2 & \frac{1}{2}+3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -3 \\ 6 & \frac{7}{2} \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix}$

Solution

$$\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix} = \begin{bmatrix} 5-4 & -6+6 \\ 8+8 & 9-3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 16 & 6 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} -5 & 6 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ 5 & -8 \end{bmatrix}$

Solution

$$\begin{bmatrix} -5 & 6 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ 5 & -8 \end{bmatrix} = \begin{bmatrix} -5 - (-3) & 6 - 2 \\ 2 - 5 & 4 - (-8) \end{bmatrix} \\ = \begin{bmatrix} -2 & 4 \\ -3 & 12 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 8 & 6 & -4 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -8 \end{bmatrix}$

Solution

$$\begin{bmatrix} 8 & 6 & -4 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -8 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 4 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1(4) + 3(1) & 1(6) + 3(0) \\ 2(4) + 5(1) & 2(6) + 5(0) \end{bmatrix} \\ = \begin{bmatrix} 7 & 6 \\ 13 & 12 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix}$

Solution

$$\begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -3(-6) + 4(2) + 2(3) & -3(4) + 4(3) + 2(-2) \\ 5(-6) + 0(2) + 4(3) & 5(4) + 0(3) + 4(-2) \end{bmatrix} \\ = \begin{bmatrix} 32 & -4 \\ -18 & 12 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1(1) - 1(1) + 4(1) & 1(1) - 1(2) + 4(-1) & 1(0) - 1(4) + 4(3) \\ 4(1) - 1(1) + 3(1) & 4(1) - 1(2) + 3(-1) & 4(0) - 1(4) + 3(3) \\ 2(1) + 0(1) - 2(1) & 2(1) + 0(2) - 2(-1) & 2(0) + 0(4) - 2(3) \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -5 & 8 \\ 6 & -1 & 5 \\ 0 & 4 & -6 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1(1) + 1(4) + 0(2) & 1(-1) + 1(-1) + 0(0) & 1(4) + 1(3) + 0(-2) \\ 1(1) + 2(4) + 4(2) & 1(-1) + 2(-1) + 4(0) & 1(4) + 2(3) + 4(-2) \\ 1(1) - 1(4) + 3(2) & 1(-1) - 1(-1) + 3(0) & 1(4) - 1(3) + 3(-2) \end{bmatrix}$$
$$= \begin{bmatrix} 5 & -2 & 7 \\ 17 & -3 & 2 \\ 3 & 0 & -5 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} -2 & -3 & -4 \\ 2 & -1 & 0 \\ 4 & -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$

Solution

$$\begin{bmatrix} -2 & -3 & -4 \\ 2 & -1 & 0 \\ 4 & -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix} = \begin{bmatrix} -3 - 12 & -2 - 6 - 8 & -8 + 3 + 8 \\ -1 & 2 - 2 & 8 + 1 \\ -2 + 9 & 4 - 4 + 6 & 16 + 2 - 6 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & -16 & 3 \\ -1 & 0 & 9 \\ 7 & 6 & 12 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{18} \\ \sqrt{3} & \sqrt{27} & 0 \end{bmatrix} \begin{bmatrix} 8 & -10 \\ 9 & 12 \\ 0 & 2 \end{bmatrix}$

Solution

$$\begin{bmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{18} \\ \sqrt{3} & \sqrt{27} & 0 \end{bmatrix} \begin{bmatrix} 8 & -10 \\ 9 & 12 \\ 0 & 2 \end{bmatrix} = \begin{pmatrix} 17\sqrt{2} & -4\sqrt{2} \\ 35\sqrt{3} & 26\sqrt{3} \end{pmatrix}$$

Exercise

Evaluate $\begin{bmatrix} x & 2x+1 & 4 \\ 5 & x-1 & 8 \\ -2 & 3x & 2x+1 \end{bmatrix} + \begin{bmatrix} 2x-1 & -2x-1 & 4x \\ -5 & 6 & x+1 \\ -5 & 2 & -2x \end{bmatrix}$

Solution

$$\begin{bmatrix} x & 2x+1 & 4 \\ 5 & x-1 & 8 \\ -2 & 3x & 2x+1 \end{bmatrix} + \begin{bmatrix} 2x-1 & -2x-1 & 4x \\ -5 & 6 & x+1 \\ -5 & 2 & -2x \end{bmatrix} = \begin{bmatrix} 3x-1 & 0 & 4x+4 \\ 0 & x+5 & x+9 \\ -7 & 3x+2 & 1 \end{bmatrix}$$

Exercise

Given $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}$. Find AB and BA .

Solution

$$AB = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 13 \\ 4 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} -16 & 29 \\ -4 & 10 \end{bmatrix}$$

Note: $AB \neq BA$

Exercise

Given $A = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$ $B = \begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}$. Find AB and BA .

Solution

$$\begin{aligned} AB &= \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix} \\ &= \begin{pmatrix} -6 & 17 \\ 6 & -8 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 14 \\ 1 & -20 \end{pmatrix} \end{aligned}$$

Exercise

Given $A = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$ $B = \begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}$. Find AB and BA .

Solution

$$\begin{aligned} AB &= \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -3 & -11 \\ 4 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -7 & 1 \\ 16 & 4 \end{pmatrix} \end{aligned}$$

Exercise

Given $A = \begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix}$. Find AB and BA .

Solution

$$\begin{aligned} AB &= \begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 10 & 6 \\ 14 & -7 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
 BA &= \begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix} \\
 &= \begin{pmatrix} 14 & -1 \\ 0 & -11 \end{pmatrix}
 \end{aligned}$$

Exercise

Given $A = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$. Find AB and BA .

Solution

$$\begin{aligned}
 AB &= \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix} \\
 &= \begin{pmatrix} -4 & 2 \\ 4 & -4 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 BA &= \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix} \\
 &= \begin{pmatrix} 4 & -4 \\ 14 & -12 \end{pmatrix}
 \end{aligned}$$

Exercise

Given $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$. Find AB and BA .

Solution

$$\begin{aligned}
 AB &= \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & -4 & 5 \\ 6 & 0 & 3 \\ -3 & -2 & 1 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 BA &= \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix} \\
 &= \begin{pmatrix} 5 & -13 \\ 3 & -4 \end{pmatrix}
 \end{aligned}$$

Exercise

Given $A = \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$. Find AB and BA .

Solution

$$\begin{aligned} AB &= \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 5 & 4 \\ 2 & -3 & 8 \\ -3 & 8 & -5 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -14 & 7 \\ -4 & 5 \end{pmatrix} \end{aligned}$$

Exercise

Given $A = \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix}$. Find AB and BA .

Solution

$$\begin{aligned} AB &= \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 24 & 4 \\ 2 & -6 & -2 \\ -13 & 12 & 10 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 12 & 8 \\ -10 & 10 \end{pmatrix} \end{aligned}$$

Exercise

Given $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$. Find AB and BA .

Solution

$$AB = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$$
$$= \begin{pmatrix} 11 & -2 & -6 \\ 0 & -1 & 2 \\ 5 & -3 & -4 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 6 & -3 & 9 \\ 2 & -3 & 4 \\ 4 & -6 & 3 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix}$ $B = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$. Find AB and BA .

Solution

$$AB = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 8 & -2 \\ 3 & -8 & 4 \\ -2 & 13 & -5 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} -4 & 5 & -1 \\ 11 & -5 & 6 \\ -8 & 7 & -4 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}$ $B = \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}$. Find AB and BA .

Solution

$$AB = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} -5 & -7 & 2 \\ -6 & 2 & 2 \\ -8 & -6 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 6 & 1 \\ 7 & 0 & 5 \\ 4 & -4 & -2 \end{pmatrix}$$

Exercise

Given $A = \begin{bmatrix} -3 & 4 \\ 2 & -3 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 1 \\ 1 & -2 \\ 3 & -4 \end{bmatrix}$, Find

a) $A + B$

c) $3A$

e) $2A + 3B$

g) AB

b) $A - B$

d) $-2B$

f) A^2

h) BA

Solution

a) $A + B = \begin{bmatrix} -3 & 4 \\ 2 & -3 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 1 & -2 \\ 3 & -4 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 5 \\ 3 & -5 \\ 2 & -4 \end{bmatrix}$$

b) $A - B = \begin{bmatrix} -3 & 4 \\ 2 & -3 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ 1 & -2 \\ 3 & -4 \end{bmatrix}$

$$= \begin{bmatrix} -7 & 3 \\ 1 & -1 \\ -4 & 4 \end{bmatrix}$$

$$c) \quad 3A = 3 \begin{bmatrix} -3 & 4 \\ 2 & -3 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 12 \\ 6 & -9 \\ -3 & 0 \end{bmatrix}$$

$$d) \quad -2B = -2 \begin{bmatrix} 4 & 1 \\ 1 & -2 \\ 3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & -2 \\ -2 & 4 \\ -6 & 8 \end{bmatrix}$$

$$e) \quad 2A + 3B = 2 \begin{bmatrix} -3 & 4 \\ 2 & -3 \\ -1 & 0 \end{bmatrix} + 3 \begin{bmatrix} 4 & 1 \\ 1 & -2 \\ 3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 8 \\ 4 & -6 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 12 & 3 \\ 3 & -6 \\ 9 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 11 \\ 7 & -12 \\ 7 & -12 \end{bmatrix}$$

$$f) \quad A^2 = \text{doesn't exist} \quad (\text{not a square matrix})$$

$$g) \quad AB = \cancel{\text{exists}} \quad (2 \times 3 \quad 2 \times 3) \text{ the inner not equal}$$

$$h) \quad BA = \cancel{\text{exists}} \quad (2 \times 3 \quad 2 \times 3) \text{ the inner not equal}$$

Exercise

Given $A = \begin{bmatrix} 2 & -2 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 8 \\ 2 & -2 \\ -4 & 3 \end{bmatrix}$, Find

a) $A + B$

c) $3A$

e) $2A + 3B$

g) AB

b) $A - B$

d) $-2B$

f) A^2

h) BA

Solution

$$\begin{aligned} \text{a) } A + B &= \begin{bmatrix} 2 & -2 \\ 3 & 4 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 8 \\ 2 & -2 \\ -4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 6 \\ 5 & 2 \\ -3 & 3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{b) } A - B &= \begin{bmatrix} 2 & -2 \\ 3 & 4 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 8 \\ 2 & -2 \\ -4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -10 \\ 1 & 6 \\ 5 & -3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{c) } 3A &= 3 \begin{bmatrix} 2 & -2 \\ 3 & 4 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -6 \\ 9 & 12 \\ 3 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{d) } -2B &= -2 \begin{bmatrix} -1 & 8 \\ 2 & -2 \\ -4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -16 \\ -4 & 4 \\ 8 & -6 \end{bmatrix} \end{aligned}$$

$$\text{e) } 2A + 3B = 2 \begin{bmatrix} 2 & -2 \\ 3 & 4 \\ 1 & 0 \end{bmatrix} + 3 \begin{bmatrix} -1 & 8 \\ 2 & -2 \\ -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 \\ 6 & 8 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 24 \\ 6 & -6 \\ -12 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 20 \\ 12 & 2 \\ -10 & 9 \end{bmatrix}$$

f) $A^2 = \text{doesn't exist}$ (not a square matrix)

g) $AB = \nexists$ (2×3 2×3) the inner not equal

h) $BA = \nexists$ (2×3 2×3) the inner not equal

Exercise

Given $A = \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$, Find

a) $A + B$

c) $3A$

e) $2A + 3B$

g) AB

b) $A - B$

d) $-2B$

f) A^2

h) BA

Solution

a) $A + B = \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 1 & -1 \\ 2 & 2 & 1 \\ -1 & 2 & 5 \end{bmatrix}$$

b) $A - B = \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} -3 & 5 & -1 \\ -2 & -4 & 3 \\ -7 & 4 & 1 \end{bmatrix}$$

c) $3A = 3 \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -6 & 9 & -3 \\ 0 & -3 & 6 \\ -12 & 9 & 9 \end{bmatrix}$$

$$\begin{aligned} d) \quad -2B &= -2 \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 4 & 0 \\ -4 & -6 & 2 \\ -6 & 2 & -4 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} e) \quad 2A + 3B &= 2 \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix} + 3 \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 6 & -2 \\ 0 & -2 & 4 \\ -8 & 6 & 6 \end{bmatrix} + \begin{bmatrix} 3 & -6 & 0 \\ 6 & 9 & -3 \\ 9 & -3 & 6 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & -2 \\ 6 & 7 & 1 \\ 1 & 3 & 12 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} f) \quad A^2 &= \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix} \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 4+4 & -6-3-3 & 2+6-3 \\ -8 & 1+6 & -2+6 \\ 8-12 & -12-3+9 & 4+6+9 \end{bmatrix} \\ &= \begin{bmatrix} 8 & -12 & 5 \\ -8 & 7 & 4 \\ -4 & -6 & 19 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} g) \quad AB &= \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -2+6-3 & 4+9+1 & -3-2 \\ -2+6 & -3-2 & 1+4 \\ -4+6+9 & 8+9-3 & -3+6 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 1 & 14 & -5 \\ 4 & -5 & 5 \\ 11 & 14 & 3 \end{bmatrix}$$

$$\begin{aligned} h) \quad BA &= \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 3+2 & -1-4 \\ -4+4 & 6-3-3 & -2+6-3 \\ -6-8 & 9+1+6 & -3-2+6 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 5 & -5 \\ 0 & 0 & 1 \\ -14 & 16 & 1 \end{bmatrix} \end{aligned}$$

Exercise

Given $A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$, Find

a) $A + B$

c) $3A$

e) $2A + 3B$

g) AB

b) $A - B$

d) $-2B$

f) A^2

h) BA

Solution

$$\begin{aligned} a) \quad A + B &= \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 4 & 4 \\ 4 & 0 & 1 \\ 1 & 8 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} b) \quad A - B &= \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -4 \\ -2 & -6 & 5 \\ 9 & 0 & -5 \end{bmatrix} \end{aligned}$$

$$c) \quad 3A = 3 \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 6 & 0 \\ 3 & -9 & 9 \\ 15 & 12 & -6 \end{bmatrix}$$

$$\begin{aligned} d) \quad -2B &= -2 \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -4 & -8 \\ -6 & -6 & 4 \\ 8 & -8 & -6 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} e) \quad 2A + 3B &= 2 \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix} + 3 \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 4 & 0 \\ 2 & -6 & 6 \\ 10 & 8 & -4 \end{bmatrix} + \begin{bmatrix} -3 & 6 & 12 \\ 9 & 9 & -6 \\ -12 & 12 & 9 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 10 & 12 \\ 11 & 3 & 0 \\ -2 & 20 & 5 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} f) \quad A^2 &= \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -6 & 6 \\ -3+15 & 2+9+12 & -9-6 \\ 4-10 & 10-12-8 & 12+4 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -6 & 6 \\ 12 & 23 & -15 \\ -6 & -10 & 16 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} g) \quad AB &= \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 6 & -4 \\ -1-9-12 & 2-9+12 & 4+6+9 \\ -5+12+8 & 10+12-8 & 20-8-6 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 6 & 6 & -4 \\ -22 & 5 & 19 \\ 15 & 14 & 6 \end{bmatrix}$$

$$\begin{aligned} h) \quad BA &= \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2+10 & -2-6+16 & 6-8 \\ 3-10 & 6-9-8 & 9+4 \\ 4+15 & -8-12+12 & 12-6 \end{bmatrix} \\ &= \begin{bmatrix} 12 & 8 & -2 \\ -7 & -11 & 13 \\ 19 & -8 & 6 \end{bmatrix} \end{aligned}$$

Exercise

Given $A = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$ $C = \begin{pmatrix} 4 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix}$ $D = \begin{pmatrix} -2 & 3 \\ 2 & -1 \\ 3 & 2 \end{pmatrix}$, Find

a) $4A - 2B$

d) $2A - 3B$

g) A^2

j) CA

b) $3A + C$

e) AB

h) B^3

k) CD

c) $3A + B$

f) BA

i) AC

l) DC

Solution

$$\begin{aligned} a) \quad 4A - 2B &= 4 \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 8 \\ -8 & 4 \end{pmatrix} - \begin{pmatrix} 2 & -4 \\ 4 & -2 \end{pmatrix} \\ &= \begin{pmatrix} -6 & 12 \\ -12 & 6 \end{pmatrix} \end{aligned}$$

b) $3A + C = \cancel{A}$

They are not the same order.

$$\begin{aligned} c) \quad 3A + B &= 3 \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -3 & 6 \\ -6 & 3 \end{pmatrix} + \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 4 \\ -4 & 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
 d) \quad 2A - 3B &= 2 \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} - 3 \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \\
 &= \begin{pmatrix} -2 & 4 \\ -4 & 2 \end{pmatrix} - \begin{pmatrix} 3 & -6 \\ 6 & -3 \end{pmatrix} \\
 &= \begin{pmatrix} -5 & 10 \\ -10 & 5 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 e) \quad AB &= \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \\
 &= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 f) \quad BA &= \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 g) \quad A^2 &= \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 h) \quad B^3 &= \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \\
 &= \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \\
 &= \begin{pmatrix} -3 & 6 \\ -6 & 3 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 i) \quad AC &= \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix} & 2 \times 2 \quad 2 \times 3 \quad \rightarrow 2 \times 3 \\
 &= \begin{pmatrix} -6 & 1 & 0 \\ -9 & -4 & -3 \end{pmatrix}
 \end{aligned}$$

$$j) \quad CB = \text{not possible} \quad 2 \times 3 \quad 2 \times 2$$

C and B are not the same order.

$$\begin{aligned}
 k) \quad CD &= \begin{pmatrix} 4 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 2 & -1 \\ 3 & 2 \end{pmatrix} & 2 \times 3 \quad 3 \times 2 \quad \rightarrow 2 \times 2
 \end{aligned}$$

$$= \begin{pmatrix} -8+6+6 & 12-3+4 \\ 2+4+3 & -3+2+2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 13 \\ 9 & 1 \end{pmatrix}$$

$$l) \quad DC = \begin{pmatrix} -2 & 3 \\ 2 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 4 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix} \quad 3 \times 2 \quad 2 \times 3 \rightarrow 3 \times 3$$

$$= \begin{pmatrix} -8-3 & -6+6 & -4+3 \\ 8+1 & 6-2 & 4-1 \\ 12-2 & 9+4 & 6+2 \end{pmatrix}$$

$$= \begin{pmatrix} -11 & 0 & -1 \\ 9 & 4 & 3 \\ 10 & 13 & 8 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$ $B = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$ $C = \begin{pmatrix} 1 & 4 & 5 \\ -2 & 3 & 4 \\ -1 & 0 & -2 \end{pmatrix}$ $D = \begin{pmatrix} 2 & 4 & -2 \\ 0 & 3 & 5 \\ -3 & 1 & 1 \end{pmatrix}$, Find

a) $4A - 2B$

d) $2A - 3B$

g) A^2

j) CB

b) $3A + C$

e) AB

h) B^3

k) CD

c) $3A + B$

f) BA

i) AC

l) DC

Solution

$$a) \quad 4A - 2B = 4 \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} - 2 \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 16 \\ 12 & -4 \end{pmatrix} - \begin{pmatrix} -2 & 6 \\ 4 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 10 \\ 8 & -2 \end{pmatrix}$$

b) $3A + C = \cancel{A}$

They are not the same order.

$$c) \quad 3A + B = 3 \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} + \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 12 \\ 9 & -3 \end{pmatrix} + \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 15 \\ 11 & -4 \end{pmatrix}$$

$$\begin{aligned} d) \quad 2A - 3B &= 2 \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} - 3 \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 8 \\ 6 & -2 \end{pmatrix} - \begin{pmatrix} -3 & 9 \\ 6 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 7 & -1 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} e) \quad AB &= \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 2 \\ -5 & 10 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} f) \quad BA &= \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 7 & -7 \\ 1 & 9 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} g) \quad A^2 &= \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 14 & 4 \\ 3 & 13 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} h) \quad B^3 &= \left(\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix} \right) \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 7 & -6 \\ -4 & 7 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -19 & 27 \\ 18 & -19 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} i) \quad AC &= \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix} & 2 \times 2 \quad 2 \times 3 \quad \rightarrow 2 \times 3 \\ &= \begin{pmatrix} -6 & 1 & 0 \\ -9 & -4 & -3 \end{pmatrix} \end{aligned}$$

$$j) \quad CB = \text{not possible} \quad 2 \times 3 \quad 2 \times 2$$

C and B are not the same order.

$$k) \quad CD = \begin{pmatrix} 1 & 4 & 5 \\ -2 & 3 & 4 \\ -1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 0 & 3 & 5 \\ -3 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -12 & 21 & 13 \\ -16 & 5 & 23 \\ 4 & -6 & 0 \end{pmatrix}$$

$$l) \quad DC = \begin{pmatrix} 2 & 4 & -2 \\ 0 & 3 & 5 \\ -3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 5 \\ -2 & 3 & 4 \\ -1 & 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2-8+2 & 8+12 & 10+16+4 \\ -6-5 & 9 & 12-10 \\ -3-2-1 & -12+3 & -15+4-2 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 20 & 30 \\ -11 & 9 & 2 \\ -6 & -9 & -12 \end{pmatrix}$$

Exercise

A contractor builds three kinds of houses, models *A*, *B*, and *C*, with a choice of two styles, Spanish and contemporary. Matrix *P* shows the number of each kind of house planned for a new 100-home subdivision. The amounts for each of the exterior materials depend primarily on the style of the house. These amounts are shown in matrix *Q*. (concrete is in cubic yards, lumber in units of 1000 board feet, brick in 1000s, and shingles in units of 100 ft^2 .) Matrix *R* gives the cost in dollars for each kind of material.

- What is the total cost of these materials for each model?
- How much of each of four kinds of material must be ordered
- What is the total cost for exterior materials?

Solution

$$\begin{array}{cc} \text{Spanish} & \text{Contemporary} \\ \text{Model A} & \begin{bmatrix} 0 & 30 \end{bmatrix} \\ \text{Model B} & \begin{bmatrix} 10 & 20 \end{bmatrix} \\ \text{Model C} & \begin{bmatrix} 20 & 20 \end{bmatrix} \end{array} = P$$

$$\begin{array}{ccccc} & \text{Concrete} & \text{Lumber} & \text{Brick} & \text{Shingles} \\ \text{Spanish} & \begin{bmatrix} 10 & 2 & 0 & 2 \end{bmatrix} \\ \text{Contemporary} & \begin{bmatrix} 50 & 1 & 20 & 2 \end{bmatrix} \end{array} = Q$$

	<i>Cost per unit</i>
Concrete	20
Lumber	180
Brick	60
Shingles	25

$$= R$$

a) What is the total cost of these materials for each model?

$$PQ = \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix}$$

	<i>Concrete</i>	<i>Lumber</i>	<i>Brick</i>	<i>Shingles</i>	
=	1500	30	600	60	<i>Model A</i>
	100	40	400	60	<i>Model B</i>
	1200	60	400	80	<i>Model C</i>

$$(PQ)R = \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix}$$

$$= \begin{bmatrix} 72,900 \\ 54,700 \\ 60,800 \end{bmatrix} \begin{matrix} \textit{Model A} \\ \textit{Model B} \\ \textit{Model C} \end{matrix}$$

The total cost of materials is \$72,900 for model A, \$54,700 for model B, \$60,800 for model C.

b) How much of each of four kinds of material must be ordered

1500	30	600	60
100	40	400	60
1200	60	400	80
3800	130	1400	200

$$T = [3800 \quad 130 \quad 1400 \quad 200]$$

3800 yd^3 of concrete, 130,000 board feet of lumber, 1,400,000 bricks, and 20,000 ft^2 of shingles are needed.

c) What is the total cost for exterior materials?

$$TR = \begin{bmatrix} 3800 & 130 & 1400 & 200 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix}$$

$$= [188,400]$$

The total cost for exterior materials is \$188,400.

Exercise

Mitchell Fabricators manufactures three styles of bicycle frames in its two plants. The following table shows the number of each style produced at each plant

	<i>Mountain Bike</i>	<i>Racing Bike</i>	<i>Touring Bike</i>
<i>North Plant</i>	150	120	100
<i>South Plant</i>	180	90	130

- Write a 2×3 matrix A that represents the information in the table
- The manufacturer increased production of each style by 20%. Find a Matrix M that represents the increased production figures.
- Find the matrix $A + M$ and tell what it represents

Solution

$$a) \quad A = \begin{bmatrix} 150 & 120 & 100 \\ 180 & 90 & 130 \end{bmatrix}$$

- The 20% production will represent

$$A + 20\%(A)$$

$$\rightarrow A + .2A = 1.2A$$

$$M = (1.2) \begin{bmatrix} 150 & 120 & 100 \\ 180 & 90 & 130 \end{bmatrix}$$

$$= \begin{bmatrix} 180 & 144 & 120 \\ 216 & 108 & 156 \end{bmatrix}$$

$$c) \quad A + M = \begin{bmatrix} 150 & 120 & 100 \\ 180 & 90 & 130 \end{bmatrix} + \begin{bmatrix} 180 & 144 & 120 \\ 216 & 108 & 156 \end{bmatrix}$$

$$= \begin{bmatrix} 330 & 264 & 220 \\ 396 & 198 & 286 \end{bmatrix}$$

The matrix $A + M$ represents the total production of each style at each plant for the time period (2 months)

Exercise

Sal's Shoes and Fred's Footwear both have outlets in California and Arizona. Sal's sells shoes for \$80, sandals for \$40, and boots for \$120. Fred's prices are \$60, \$30, and \$150 for shoes, sandals and boots, respectively. Half of all sales in California stores are shoes, 1/4 are *sandals*, and 1/4 are *boots*. In Arizona the fractions are 1/5 *shoes*, 1/5 are *sandals*, and 3/5 are *boots*.

- Write a 2×3 matrix called P representing prices for the two stores and three types of footwear.
- Write a 2×3 matrix called F representing fraction of each type of footwear sold in each state.
- Only one of the two products PF and FP is meaningful. Determine which one it is, calculate the product, and describe what the entries represent.

Solution

- Write a 2×3 matrix called P representing prices for the two stores and three types of footwear.

$$P = \begin{bmatrix} 80 & 40 & 120 \\ 60 & 30 & 150 \end{bmatrix} \quad \begin{matrix} \text{Sal's} \\ \text{Fred's} \end{matrix}$$

- Write a 2×3 matrix called F representing fraction of each type of footwear sold in each state.

$$F = \begin{matrix} \begin{matrix} CA & AR \end{matrix} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{3}{5} \end{bmatrix} \end{matrix}$$

$$\begin{aligned} c) \quad PF &= \begin{bmatrix} 80 & 40 & 120 \\ 60 & 30 & 150 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{3}{5} \end{bmatrix} \\ &= \begin{bmatrix} 80\frac{1}{2} + 40\frac{1}{4} + 120\frac{1}{4} & 80\frac{1}{5} + 40\frac{1}{5} + 120\frac{3}{5} \\ 60\frac{1}{2} + 30\frac{1}{4} + 150\frac{1}{4} & 60\frac{1}{5} + 30\frac{1}{5} + 150\frac{3}{5} \end{bmatrix} \\ &= \begin{bmatrix} 80 & 96 \\ 75 & 108 \end{bmatrix} \end{aligned}$$

Solution

Section 4.3 – Multiplicative Inverses of Matrices

Exercise

Show that B is Multiplicative inverse of A

$$A = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

Solution

$$\begin{aligned} AB &= \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \\ &= \begin{pmatrix} -6 & \\ & \end{pmatrix} \\ &\neq I \end{aligned}$$

B is not multiplicative inverse of A

Exercise

Show that B is Multiplicative inverse of A

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

Solution

$$\begin{aligned} AB &= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I \end{aligned}$$

$\therefore B$ is Multiplicative inverse of A

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{-4+6} \begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix} \end{aligned}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{10-10} \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix} \\ &= \frac{1}{0} \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix} \end{aligned}$$

\therefore Inverse *doesn't exist*

Exercise

Find the inverse of $A = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$

Solution

$$\begin{array}{l} \left[\begin{array}{cc|cc} -2 & 3 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{array} \right] \quad -\frac{1}{2}R_1 \qquad \begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ -3 & 4 & 0 & 1 \end{array} \\ \\ \left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ -3 & 4 & 0 & 1 \end{array} \right] \quad R_2 + 3R_1 \qquad \begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array} \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array} \right] \quad -2R_2$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 3 & -2 \end{array} \right] \quad R_1 + \frac{3}{2}R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 4 & -3 \\ 0 & 1 & 3 & -2 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} a & b \\ 3 & 3 \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{3a-3b} \begin{bmatrix} 3 & -b \\ -3 & a \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{3(a-b)} & \frac{-b}{3(a-b)} \\ \frac{-3}{3(a-b)} & \frac{a}{3(a-b)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{a-b} & \frac{-b}{3(a-b)} \\ \frac{-1}{a-b} & \frac{a}{3(a-b)} \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} -2 & a \\ 4 & a \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{-2a-4a} \begin{bmatrix} a & -a \\ -4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{-6a} & \frac{-a}{-6a} \\ \frac{-4}{-6a} & \frac{-2}{-6a} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{6} & \frac{1}{6} \\ \frac{2}{3a} & \frac{1}{3a} \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} 4 & 4 \\ b & a \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{4a-4b} \begin{bmatrix} a & -4 \\ -b & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{4(a-b)} & \frac{-4}{4(a-b)} \\ \frac{-b}{4(a-b)} & \frac{4}{4(a-b)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{4(a-b)} & \frac{-1}{a-b} \\ \frac{-b}{4(a-b)} & \frac{1}{a-b} \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{-1+4} \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{3} \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= -\frac{1}{14} \begin{pmatrix} -1 & -4 \\ -3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{14} & \frac{2}{7} \\ \frac{3}{14} & -\frac{1}{7} \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= -\frac{1}{5} \begin{pmatrix} -1 & -3 \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{11} \begin{pmatrix} 5 & -3 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{11} & -\frac{3}{11} \\ \frac{2}{11} & \frac{1}{11} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{\textcolor{red}{0}} \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$$

\therefore Inverse *doesn't exist*

Exercise

Find the inverse of $A = \begin{pmatrix} -6 & 9 \\ 2 & -3 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{\textcolor{red}{18-18}} \begin{pmatrix} & \\ & \end{pmatrix}$$

\therefore Inverse *doesn't exist*

Exercise

Find the inverse of $A = \begin{pmatrix} -2 & 7 \\ 0 & 2 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{4} \begin{pmatrix} 2 & -7 \\ 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{7}{4} \\ 0 & -\frac{1}{2} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$

Solution

$$A = \frac{1}{-16+16} \begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$$

\therefore Inverse *doesn't exist*

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Solution

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 1 \\ a & a \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{a} \begin{pmatrix} a & -1 \\ -a & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\frac{1}{a} \\ -1 & \frac{2}{a} \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} b & 3 \\ b & 2 \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= -\frac{1}{b} \begin{pmatrix} 2 & -3 \\ -b & b \end{pmatrix} \\ &= \begin{pmatrix} -\frac{2}{b} & \frac{3}{b} \\ 1 & -1 \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & a \\ 3 & a \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= -\frac{1}{2a} \begin{pmatrix} a & -a \\ -3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2a} & -\frac{1}{2a} \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{a^2 - 4} \begin{pmatrix} a & -2 \\ -2 & a \end{pmatrix} \\ &= \begin{pmatrix} \frac{a}{a^2 - 4} & \frac{-2}{a^2 - 4} \\ \frac{-2}{a^2 - 4} & \frac{a}{a^2 - 4} \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{\textcolor{red}{0}} \begin{pmatrix} & \\ & \end{pmatrix}$$

\therefore Inverse *doesn't exist*

Exercise

Find the inverse of $A = \begin{pmatrix} -3 & \frac{1}{2} \\ 6 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{\textcolor{red}{0}} \begin{pmatrix} & & \end{pmatrix}$$

\therefore Inverse *doesn't exist*

Exercise

Find A^{-1} if $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \quad \begin{array}{cccccc} 2 & -2 & -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -2 & 0 & 0 \\ \hline 0 & -2 & -3 & -2 & 1 & 0 \end{array} \quad \begin{array}{cccccc} 3 & 0 & 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & -3 & 0 & 0 \\ \hline 0 & 0 & -3 & -3 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array} \right] -\frac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array} \right] -\frac{1}{3}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array} \right] \begin{array}{l} R_1 - R_3 \\ R_2 - \frac{3}{2}R_3 \\ \end{array} \quad \begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & \frac{1}{3} \\ \hline 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{array} \quad \begin{array}{cccccc} 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2} \\ \hline 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} \textcolor{blue}{0} & \textcolor{blue}{0} & \textcolor{blue}{\frac{1}{3}} \\ \textcolor{blue}{-\frac{1}{2}} & \textcolor{blue}{-\frac{1}{2}} & \textcolor{blue}{\frac{1}{2}} \\ \textcolor{blue}{1} & \textcolor{blue}{0} & \textcolor{blue}{-\frac{1}{3}} \end{bmatrix}$$

Exercise

Find A^{-1} , where $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 + 2R_1 \end{array}$$

$$\begin{array}{cccccc} 3 & 5 & 3 & 0 & 1 & 0 \\ -3 & -6 & 3 & -3 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \end{array} \quad \begin{array}{cccccc} 2 & 4 & 3 & 0 & 0 & 1 \\ -2 & -4 & 2 & -2 & 0 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right] -R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right] R_1 - 2R_2$$

$$\begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 12 & -6 & 2 & 0 \\ 1 & 0 & 11 & -5 & 2 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right] \frac{1}{5}R_3$$

$$0 \quad 0 \quad 1 \quad -\frac{2}{5} \quad 0 \quad \frac{1}{5}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array} \right] \begin{array}{l} R_1 - 11R_3 \\ R_2 + 6R_3 \end{array}$$

$$\begin{array}{cccccc} 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 6 & -\frac{12}{5} & 0 & \frac{6}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \end{array} \quad \begin{array}{cccccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 0 & -11 & \frac{22}{5} & 0 & -\frac{11}{5} \\ 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{bmatrix}$$

Exercise

Find A^{-1} , where $A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccccc} -2 & 0 & 1 & 0 & 1 & 0 \\ 2 & 4 & -2 & 2 & 0 & 0 \\ 0 & 4 & -1 & 2 & 1 & 0 \end{array} \quad \begin{array}{cccccc} 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & -2 & 1 & -1 & 0 & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 4 & -1 & 2 & 1 & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array} \right] \frac{1}{4}R_2$$

$$0 \quad 1 \quad -\frac{1}{4} \quad \frac{1}{2} \quad \frac{1}{4} \quad 0$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 3R_2 \end{array}$$

$$\begin{array}{cccccc} 0 & -3 & 1 & -1 & 0 & 1 \\ 0 & 3 & -\frac{3}{4} & \frac{3}{2} & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{array} \quad \begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & \frac{1}{2} & -1 & -\frac{1}{2} & 0 \\ 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{array} \right] 4R_3$$

$$0 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array} \right] \begin{array}{l} R_1 + \frac{1}{2}R_3 \\ R_2 + \frac{1}{4}R_3 \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2 \\ 1 & 0 & 0 & 1 & 1 & 2 \end{array} \quad \begin{array}{cccccc} 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

Exercise

Find A^{-1} , where $A = \begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} -2 & 5 & 3 & 1 & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array} \right] \quad \frac{1}{-2}R_1 \quad \begin{array}{cccccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array} \right] \quad R_2 - 4R_1 \quad \begin{array}{cccccc} 4 & -1 & 3 & 0 & 1 & 0 \\ -4 & 10 & 6 & 2 & 0 & 0 \\ \hline 0 & 9 & 9 & 2 & 1 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array} \right] \quad R_3 - 7R_1 \quad \begin{array}{cccccc} 7 & -2 & 5 & 0 & 0 & 1 \\ -7 & \frac{35}{2} & \frac{21}{2} & \frac{7}{2} & 0 & 0 \\ \hline 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array} \right] \quad \frac{1}{9}R_2 \quad \begin{array}{cccccc} 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array} \right] \quad R_3 - \frac{31}{2}R_2 \quad \begin{array}{cccccc} 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \\ 0 & -\frac{31}{2} & -\frac{31}{2} & -\frac{31}{9} & -\frac{31}{18} & 0 \\ \hline 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array} \right]$$

\therefore The inverse matrix **doesn't exist**

OR

$$\left[\begin{array}{ccc|ccc} -2 & 5 & 3 & 1 & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array} \right] \quad \begin{array}{l} R_2 + 2R_1 \\ -2R_3 + 7R_1 \end{array} \quad \begin{array}{cccccc} 4 & -1 & 3 & 0 & 1 & 0 \\ -4 & 10 & 6 & 2 & 0 & 0 \\ \hline 0 & 9 & 9 & 2 & 1 & 0 \end{array} \quad \begin{array}{cccccc} -14 & 4 & -10 & 0 & 0 & -2 \\ 14 & 35 & 21 & 7 & 0 & 0 \\ \hline 0 & 39 & 11 & 7 & 0 & -2 \end{array}$$

$$\begin{array}{l}
 \left[\begin{array}{ccc|ccc} -2 & 5 & 3 & 1 & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 0 & 39 & 11 & 7 & 0 & -2 \end{array} \right] \begin{array}{l} 9R_1 - 5R_2 \\ \\ 9R_3 - 39R_2 \end{array} \\
 \begin{array}{cccccc} -18 & 45 & 27 & 9 & 0 & 0 \\ 0 & -45 & -45 & -10 & -5 & 0 \\ \hline -18 & 0 & -18 & -1 & -5 & 0 \\ \\ 0 & 351 & -99 & 63 & 0 & -18 \\ 0 & -351 & 99 & -78 & -39 & 0 \\ \hline 0 & 0 & 0 & -15 & -39 & -18 \end{array}
 \end{array}$$

$$\left[\begin{array}{ccc|ccc} -18 & 0 & -18 & -1 & -5 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 0 & 0 & 0 & -15 & -39 & -18 \end{array} \right]$$

\therefore The inverse matrix *doesn't exist*

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array} \right) R_2 + R_1 \quad \begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ \hline 0 & 4 & 4 & 1 & 1 & 0 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 4 & 1 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array} \right) \frac{1}{4}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - R_2 \\ R_3 - 4R_2 \end{array} \quad \begin{array}{cccccc} 0 & 4 & 3 & 0 & 0 & 1 \\ 0 & -4 & -4 & -1 & -1 & 0 \\ \hline 0 & 0 & -1 & -1 & -1 & 1 \end{array} \quad \begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ \hline 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array} \right) -R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array} \right) \begin{array}{l} R_1 + R_3 \\ R_2 - R_3 \end{array} \quad \begin{array}{cccccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ \hline 1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & -1 \end{array} \quad \begin{array}{cccccc} 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \\ \hline 0 & 1 & 0 & -\frac{3}{4} & -\frac{3}{4} & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & -1 \\ 0 & 1 & 0 & -\frac{3}{4} & -\frac{3}{4} & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{7}{4} & \frac{3}{4} & -1 \\ -\frac{3}{4} & -\frac{3}{4} & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ -2 & -3 & 0 & 0 & 0 & 1 \end{array} \right) \quad R_3 + 2R_1 \quad \begin{array}{cccccc} -2 & -3 & 0 & 0 & 0 & 1 \\ 2 & -2 & 2 & 2 & 0 & 0 \\ \hline 0 & -5 & 2 & 2 & 0 & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array} \right) \quad -\frac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array} \right) \quad \begin{array}{l} R_1 + R_2 \\ R_3 + 5R_2 \end{array} \quad \begin{array}{cccccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ \hline 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \end{array} \quad \begin{array}{cccccc} 0 & -5 & 2 & 2 & 0 & 1 \\ 0 & 5 & -\frac{5}{2} & 0 & -\frac{5}{2} & 0 \\ \hline 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1 \end{array} \right) \quad -2R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -4 & 5 & -2 \end{array} \right) \quad \begin{array}{l} R_1 - \frac{1}{2}R_3 \\ R_2 + \frac{1}{2}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & -4 & 5 & -2 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array} \right) \frac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array} \right) R_3 + R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array} \right) 2R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array} \right) \begin{array}{l} R_1 - 2R_3 \\ R_2 - \frac{5}{2}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 1 & 2 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & -1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -4 & -3 & 1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array} \right) -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 2R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & 0 & 7 & 3 & -2 & 1 \end{array} \right) \frac{1}{7}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{array} \right) \begin{array}{l} R_1 + 3R_3 \\ R_2 - 4R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ 0 & 1 & 0 & \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 3 & 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array} \right) \frac{1}{3}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & -3 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - R_2 \\ R_3 + 3R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{7}{3} & -\frac{5}{3} & 3 & 1 \end{array} \right) \frac{3}{7}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array} \right) \begin{array}{l} R_1 + \frac{1}{3}R_3 \\ R_2 - \frac{2}{3}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ 0 & 1 & 0 & \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} -3 & 1 & -1 \\ 1 & -4 & -7 \\ 1 & 2 & 5 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} -3 & 1 & -1 & 1 & 0 & 0 \\ 1 & -4 & -7 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array} \right) \quad -\frac{1}{3}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 1 & -4 & -7 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array} \right) \quad \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{11}{3} & -\frac{22}{3} & \frac{1}{3} & 1 & 0 \\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1 \end{array} \right) \quad -\frac{3}{11}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0 \\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1 \end{array} \right) \quad R_3 - \frac{7}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0 \\ 0 & 0 & 0 & -\frac{1}{11} & -\frac{3}{11} & 0 \end{array} \right)$$

\therefore Inverse **does not exist**

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 1 & -3 \\ 2 & -4 & 1 \\ -5 & 7 & 1 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 2 & -4 & 1 & 0 & 1 & 0 \\ -5 & 7 & 1 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + 5R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & -6 & 7 & -2 & 1 & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array} \right) -\frac{1}{6}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 12R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

\therefore Inverse **does not exist**

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_3 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right) -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right) R_1 - 2R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right) \frac{1}{5}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array} \right) \begin{array}{l} R_1 - 11R_3 \\ R_2 + 6R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -6 & 1 \\ -2 & -2 & 5 & 1 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} -2 & -3 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array} \right] -\frac{1}{2}R_1$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array} \right] R_4 + 2R_1$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array} \right] R_4 - R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

\therefore Inverse *does not exist*

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 1 & -14 & 7 & 38 \\ -1 & 2 & 1 & -2 \\ 1 & 2 & -1 & -6 \\ 1 & -2 & 3 & 6 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & -6 & 0 & 0 & 1 & 0 \\ 1 & -2 & 3 & 6 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_2 + R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & -12 & 8 & 36 & 1 & 1 & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array} \right] -\frac{1}{12}R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 + 14R_2 \\ R_3 - 16R_2 \\ R_4 - 12R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & \frac{8}{3} & 4 & \frac{1}{3} & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array} \right] \frac{3}{8}R_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 + \frac{7}{3}R_3 \\ R_2 + \frac{2}{3}R_3 \\ \\ R_4 - 4R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & -2 & -\frac{1}{2} & -1 & -\frac{3}{2} & 1 \end{array} \right] -\frac{1}{2}R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array} \right] \begin{array}{l} R_1 + \frac{1}{2}R_4 \\ R_2 + 2R_4 \\ R_3 - \frac{3}{2}R_4 \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ 0 & 0 & 1 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{bmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 10 & 20 & -30 & 15 \\ 3 & -7 & 14 & -8 \\ -7 & -2 & -1 & 2 \\ 4 & 4 & -3 & 1 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} 10 & 20 & -30 & 15 & 1 & 0 & 0 & 0 \\ 3 & -7 & 14 & -8 & 0 & 1 & 0 & 0 \\ -7 & -2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & -3 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \quad \frac{1}{10}R_1$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 3 & -7 & 14 & -8 & 0 & 1 & 0 & 0 \\ -7 & -2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & -3 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 + 7R_1 \\ R_4 - 4R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 0 & -13 & 23 & -\frac{25}{2} & -\frac{3}{10} & 1 & 0 & 0 \\ 0 & 12 & -22 & \frac{25}{2} & \frac{7}{10} & 0 & 1 & 0 \\ 0 & -4 & 9 & -5 & -\frac{2}{5} & 0 & 0 & 1 \end{array} \right] \quad -\frac{1}{13}R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 12 & -22 & \frac{25}{2} & \frac{7}{10} & 0 & 1 & 0 \\ 0 & -4 & 9 & -5 & -\frac{2}{5} & 0 & 0 & 1 \end{array} \right] \quad \begin{array}{l} R_1 - 2R_2 \\ R_3 - 12R_2 \\ R_4 + 4R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \frac{7}{13} & -\frac{11}{26} & \frac{7}{130} & \frac{2}{13} & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 0 & -\frac{10}{13} & \frac{25}{26} & \frac{11}{26} & \frac{12}{13} & 1 & 0 \\ 0 & 0 & \frac{25}{13} & -\frac{15}{13} & -\frac{4}{13} & -\frac{4}{13} & 0 & 1 \end{array} \right] \quad -\frac{13}{10}R_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \frac{7}{13} & -\frac{11}{26} & \frac{7}{130} & \frac{2}{13} & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & \frac{25}{13} & -\frac{15}{13} & -\frac{4}{13} & -\frac{4}{13} & 0 & 1 \end{array} \right] \begin{array}{l} R_1 - \frac{7}{13}R_3 \\ R_2 + \frac{23}{13}R_3 \\ R_4 - \frac{25}{13}R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{19}{20} & -\frac{11}{5} & -\frac{23}{10} & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array} \right] \begin{array}{l} R_2 + R_4 \\ R_3 + R_4 \end{array}.$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array} \right] \frac{4}{5}R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array} \right] R_1 - \frac{1}{4}R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{bmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $A = \begin{bmatrix} x \end{bmatrix}$

Solution

For A^{-1} exists, $x \neq 0$

$$AA^{-1} = I$$

$$\begin{bmatrix} x \end{bmatrix} \begin{bmatrix} a \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

$$xa = 1$$

$$\underline{a = \frac{1}{x}}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{x} \end{bmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $A = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$

Solution

For A^{-1} exists, $\underline{x, y \neq 0}$

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad AA^{-1} = I$$

$$\begin{cases} ax = 1 & bx = 0 \\ cy = 0 & dy = 1 \end{cases}$$

$$\begin{cases} a = \frac{1}{x} & b = 0 \\ c = 0 & d = \frac{1}{y} \end{cases}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{x} & 0 \\ 0 & \frac{1}{y} \end{bmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1}

$$A = \begin{bmatrix} 0 & 0 & x \\ 0 & y & 0 \\ z & 0 & 0 \end{bmatrix}$$

Solution

For A^{-1} exists, $\underline{x, y, z \neq 0}$

$$\begin{pmatrix} 0 & 0 & x \\ 0 & y & 0 \\ z & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad AA^{-1} = I$$

$$\begin{pmatrix} xg & xh & xi \\ yd & ye & yf \\ za & zb & zc \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{cases} xg = 1 & xh = 0 & xi = 0 \\ yd = 0 & ye = 1 & yf = 0 \\ za = 0 & zb = 0 & zc = 1 \end{cases}$$

$$\begin{cases} g = \frac{1}{x} & h = 0 & i = 0 \\ d = 0 & e = \frac{1}{y} & f = 0 \\ a = 0 & b = 0 & c = \frac{1}{z} \end{cases}$$

$$A^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{z} \\ 0 & \frac{1}{y} & 0 \\ \frac{1}{z} & 0 & 0 \end{pmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1}

$$A = \begin{bmatrix} x & 1 & 1 & 1 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{bmatrix}$$

Solution

For A^{-1} exists, $\underline{x, y, z, w \neq 0}$

$$\begin{pmatrix} x & 1 & 1 & 1 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad AA^{-1} = I$$

$$\begin{cases} xa_{11} + xa_{21} + xa_{31} + xa_{41} = 1 \\ xa_{12} + xa_{22} + xa_{32} + xa_{42} = 0 \\ xa_{13} + xa_{23} + xa_{33} + xa_{43} = 0 \\ xa_{14} + xa_{24} + xa_{34} + xa_{44} = 0 \end{cases}$$

$$\begin{cases} ya_{21} = 0 & \underline{a_{21} = 0} \\ ya_{22} = 1 & \underline{a_{22} = \frac{1}{y}} \\ ya_{23} = 0 & \underline{a_{23} = 0} \\ ya_{24} = 0 & \underline{a_{24} = 0} \end{cases}$$

$$\begin{cases} za_{31} = 0 & \underline{a_{31} = 0} \\ za_{32} = 0 & \underline{a_{32} = 0} \\ za_{33} = 1 & \underline{a_{33} = \frac{1}{z}} \\ za_{34} = 0 & \underline{a_{34} = 0} \end{cases}$$

$$\begin{cases} wa_{41} = 0 & \underline{a_{41} = 0} \\ wa_{42} = 0 & \underline{a_{42} = 0} \\ wa_{43} = 0 & \underline{a_{43} = 0} \\ wa_{44} = 1 & \underline{a_{44} = \frac{1}{w}} \end{cases}$$

$$\rightarrow \begin{cases} xa_{11} = 1 & \underline{a_{11} = \frac{1}{x}} \\ xa_{12} + \frac{x}{y} = 0 & \underline{a_{12} = -\frac{1}{y}} \\ xa_{13} + \frac{x}{z} = 0 & \underline{a_{13} = -\frac{1}{z}} \\ xa_{14} + \frac{x}{w} = 0 & \underline{a_{14} = -\frac{1}{w}} \end{cases}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{x} & -\frac{1}{y} & -\frac{1}{z} & -\frac{1}{w} \\ 0 & \frac{1}{y} & 0 & 0 \\ 0 & 0 & \frac{1}{z} & 0 \\ 0 & 0 & 0 & \frac{1}{w} \end{pmatrix}$$

Exercise

Solve the system using A^{-1} $\begin{cases} x + 2z = 6 \\ -x + 2y + 3z = -5 \\ x - y = 6 \end{cases}$ Given $A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

Solution

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3(6) - 2(-5) - 4(6) \\ 3(6) - 2(-5) - 5(6) \\ -1(6) + 1(-5) + 2(6) \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

Solution: $(4, -2, 1)$

Exercise

Solve the system using A^{-1} $\begin{cases} x + 2y + 5z = 2 \\ 2x + 3y + 8z = 3 \\ -x + y + 2z = 3 \end{cases}$

- Write the linear system as a matrix equation in the form $AX = B$
- Solve the system using the inverse that is given for the coefficient matrix

the inverse of $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix}$ is $\begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}$

Solution

$$a) \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

$$b) \begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$$

Exercise

Solve the system using A^{-1}
$$\begin{cases} x - y + z = 8 \\ 2y - z = -7 \\ 2x + 3y = 1 \end{cases}$$

- a) Write the linear system as a matrix equation in the form $AX = B$
- b) Solve the system using the inverse that is given for the coefficient matrix

the inverse is
$$\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

Solution

$$a) \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 1 \end{bmatrix}$$

$$b) \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ -7 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system
$$\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$$

Solution

$$A = \begin{pmatrix} 3 & 2 \\ 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{7} \begin{pmatrix} -1 & -2 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{pmatrix} \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{6}{7} \\ -\frac{23}{7} \end{pmatrix}$$

$$\therefore \text{Solution: } \left(\frac{6}{7}, -\frac{23}{7} \right)$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$$

Solution

$$A = \begin{pmatrix} 2 & 5 \\ 5 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{29} \begin{pmatrix} -2 & -5 \\ -5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{29} & \frac{5}{29} \\ \frac{5}{29} & -\frac{2}{29} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{2}{29} & \frac{5}{29} \\ \frac{5}{29} & -\frac{2}{29} \end{pmatrix} \begin{pmatrix} 7 \\ -3 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{29} \\ -\frac{41}{7} \end{pmatrix}$$

$$\therefore \text{Solution: } \left(-\frac{1}{29}, -\frac{41}{29} \right)$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system
$$\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$$

Solution

$$A = \begin{pmatrix} 4 & -7 \\ 2 & 5 \end{pmatrix} \quad B = \begin{pmatrix} -16 \\ 9 \end{pmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{34} \begin{pmatrix} 5 & 7 \\ -2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} \frac{5}{34} & \frac{7}{34} \\ -\frac{1}{17} & \frac{2}{17} \end{pmatrix} \end{aligned}$$

$$X = \begin{pmatrix} \frac{5}{34} & \frac{7}{34} \\ -\frac{1}{17} & \frac{2}{17} \end{pmatrix} \begin{pmatrix} -16 \\ 9 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix}$$

$$\therefore \text{Solution: } \left(-\frac{1}{2}, 2\right)$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system
$$\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$$

Solution

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\begin{aligned} A^{-1} &= -1 \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} \end{aligned}$$

$$X = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

$$\therefore \text{Solution: } (-2, 5)$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$$

Solution

$$A = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{7} \begin{pmatrix} 5 & -4 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{3}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{5}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{3}{7} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(2, -1)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$$

Solution

$$A = \begin{pmatrix} 5 & -2 \\ -10 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\textcolor{red}{0}} \begin{pmatrix} 4 & 2 \\ 10 & 5 \end{pmatrix}$$

Inverse matrix *doesn't exist*.

$$\textcolor{red}{-\frac{1}{2}} \begin{cases} 5x - 2y = 4 \\ 5x - 2y = -\frac{7}{2} \end{cases}$$

$$\textcolor{red}{4} \neq \textcolor{red}{-\frac{7}{2}}$$

\therefore *No Solution*

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system
$$\begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$$

Solution

$$A = \begin{pmatrix} 1 & -4 \\ 5 & -20 \end{pmatrix} \quad B = \begin{pmatrix} -8 \\ -40 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\textcolor{red}{0}} \begin{pmatrix} & \\ & \end{pmatrix}$$

Inverse matrix *doesn't exist*.

$$\textcolor{red}{\frac{1}{5}} \begin{cases} x - 4y = -8 \\ x - 4y = -8 \end{cases}$$

$$\therefore \text{Solution: } \underline{(4y - 8, y)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system
$$\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$$

Solution

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{3} \begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$\textcolor{green}{X = A^{-1}B}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \textcolor{blue}{2} \\ \textcolor{blue}{-1} \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(\textcolor{blue}{2}, \textcolor{blue}{-1})}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system
$$\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$$

Solution

$$A = \begin{pmatrix} 2 & 10 \\ 7 & -2 \end{pmatrix} \quad B = \begin{pmatrix} -14 \\ -16 \end{pmatrix}$$

$$\begin{aligned} A^{-1} &= -\frac{1}{74} \begin{pmatrix} -2 & -10 \\ -7 & 2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{37} & \frac{5}{37} \\ \frac{7}{74} & -\frac{1}{37} \end{pmatrix} \end{aligned}$$

$$X = \begin{pmatrix} \frac{1}{37} & \frac{5}{37} \\ \frac{7}{74} & -\frac{1}{37} \end{pmatrix} \begin{pmatrix} -14 \\ -16 \end{pmatrix} \quad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{94}{37} \\ -\frac{33}{37} \end{pmatrix}$$

$$\therefore \text{Solution: } \left(-\frac{94}{37}, -\frac{33}{37} \right)$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system
$$\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$$

Solution

$$A = \begin{pmatrix} 4 & -3 \\ -3 & 9 \end{pmatrix} \quad B = \begin{pmatrix} 24 \\ -1 \end{pmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{27} \begin{pmatrix} 9 & 3 \\ 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{27} \end{pmatrix} \end{aligned}$$

$$X = \begin{pmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{27} \end{pmatrix} \begin{pmatrix} 24 \\ -1 \end{pmatrix} \quad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{71}{9} \\ \frac{68}{27} \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{\left(\frac{71}{9}, \frac{68}{27} \right)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system $\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$

Solution

$$A = \begin{pmatrix} 4 & 2 \\ 3 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 12 \\ 16 \end{pmatrix}$$

$$\begin{aligned} A^{-1} &= -\frac{1}{14} \begin{pmatrix} -2 & -2 \\ -3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{3}{14} & -\frac{2}{7} \end{pmatrix} \end{aligned}$$

$$X = \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{3}{14} & -\frac{2}{7} \end{pmatrix} \begin{pmatrix} 12 \\ 16 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(4, -2)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system $\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 2 \\ 4 & -2 \end{pmatrix} \quad B = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{10} \begin{pmatrix} -2 & -2 \\ -4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{10} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(1, -1)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$$

Solution

$$\frac{1}{2} \begin{cases} x - 2y = 5 \\ -5x + y = 2 \end{cases}$$

$$A = \begin{pmatrix} 1 & -2 \\ -5 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{9} \begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{9} & -\frac{2}{9} \\ -\frac{5}{9} & -\frac{1}{9} \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{1}{9} & -\frac{2}{9} \\ -\frac{5}{9} & -\frac{1}{9} \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(-1, -3)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases}$$

Solution

$$\frac{1}{3} \rightarrow \begin{cases} 4x + 5y = -9 \\ 2x - y = -1 \end{cases}$$

$$\frac{1}{15}$$

$$A = \begin{pmatrix} 4 & 5 \\ 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} -9 \\ -1 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{14} \begin{pmatrix} -1 & -5 \\ -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{14} & \frac{5}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{14} & \frac{5}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{pmatrix} \begin{pmatrix} -9 \\ -1 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(-1, -1)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$$

Solution

$$\frac{1}{4} \rightarrow \begin{cases} x - y = -3 \\ x + y = -5 \end{cases}$$

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$$

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(-4, -1)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} -2x + 3y = 4 \\ -3x + 4y = 5 \end{cases}$$

Solution

$$A = \begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}$$

$$X = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(1, 2)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} x - 2y = 6 \\ 4x + 3y = 2 \end{cases}$$

Solution

$$A = \begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{11} \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{11} & \frac{2}{11} \\ -\frac{4}{11} & \frac{1}{11} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{3}{11} & \frac{2}{11} \\ -\frac{4}{11} & \frac{1}{11} \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(2, -2)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} 2x - 3y = 7 \\ 4x + y = -7 \end{cases}$$

Solution

$$A = \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 7 \\ -7 \end{pmatrix}$$

$$A^{-1} = \frac{1}{14} \begin{pmatrix} 1 & 3 \\ -4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{14} & \frac{3}{14} \\ -\frac{2}{7} & \frac{1}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{14} & \frac{3}{14} \\ -\frac{2}{7} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} 7 \\ -7 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(-1, -3)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$$

Solution

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array} \right) -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 2R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & 6 & 3 & -2 & 1 \end{array} \right) \frac{1}{6}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \end{array} \right) \begin{array}{l} R_1 + 2R_3 \\ R_2 - 3R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

∴ Solution: (1, 2, -1)

Exercise

Use the **inverse** of the coefficient matrix to solve the linear system

$$\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$$

Solution

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -1 \\ 9 \\ 9 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{array} \right) \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_2 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & -4 & 4 & -3 & 0 & 1 \end{array} \right) \quad -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & -4 & 4 & -3 & 0 & 1 \end{array} \right) \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 4R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & -8 & 5 & -4 & 1 \end{array} \right) \quad -\frac{1}{8}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{array} \right) \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 + 3R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{8} & \frac{1}{2} & -\frac{3}{8} \\ 0 & 0 & 1 & -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & -\frac{3}{8} \\ -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & -\frac{3}{8} \\ -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{pmatrix} \begin{pmatrix} -1 \\ 9 \\ 9 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

∴ Solution: (2, 1, 4)

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$$

Solution

$$A = \begin{pmatrix} 1 & 5 & -1 \\ 0 & 3 & -1 \\ -3 & 6 & 2 \end{pmatrix} \quad B = \begin{pmatrix} -4 \\ -1 \\ 11 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ -3 & 6 & 2 & 0 & 0 & 1 \end{array} \right) \quad R_3 + 3R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & 21 & -1 & 3 & 0 & 1 \end{array} \right) \quad \frac{1}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 21 & -1 & 3 & 0 & 1 \end{array} \right) \quad \begin{array}{l} R_1 - 5R_2 \\ R_3 - 21R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & 1 & -\frac{5}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 6 & 3 & -7 & 1 \end{array} \right) \frac{1}{6}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & 1 & -\frac{5}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{array} \right) \begin{array}{l} R_1 - \frac{2}{3}R_3 \\ R_2 + \frac{1}{3}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & \frac{8}{9} & -\frac{1}{9} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{18} & \frac{1}{18} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{8}{9} & -\frac{1}{9} \\ \frac{1}{6} & -\frac{1}{18} & \frac{1}{18} \\ \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{2}{3} & \frac{8}{9} & -\frac{1}{9} \\ \frac{1}{6} & -\frac{1}{18} & \frac{1}{18} \\ \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} -4 \\ -1 \\ 11 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

\therefore Solution: $(-3, 0, 1)$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$$

Solution

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -3 & 2 \\ 3 & -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 14 \\ 10 \\ 9 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 2 & -3 & 2 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -9 & -6 & -2 & 1 & 0 \\ 0 & -10 & -11 & -3 & 0 & 1 \end{array} \right) -\frac{1}{9}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & -\frac{1}{9} & 0 \\ 0 & -10 & -11 & -3 & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - 3R_2 \\ \\ R_3 + 10R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & -\frac{1}{9} & 0 \\ 0 & 0 & -\frac{13}{3} & -\frac{7}{9} & -\frac{10}{9} & 1 \end{array} \right) -\frac{3}{13}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & -\frac{1}{9} & 0 \\ 0 & 0 & 1 & \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{array} \right) \begin{array}{l} R_2 - 2R_3 \\ R_2 - \frac{2}{3}R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{39} & -\frac{7}{39} & \frac{6}{13} \\ 0 & 1 & 0 & \frac{4}{39} & -\frac{11}{39} & \frac{2}{13} \\ 0 & 0 & 1 & \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} -\frac{1}{39} & -\frac{7}{39} & \frac{6}{13} \\ \frac{4}{39} & -\frac{11}{39} & \frac{2}{13} \\ \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{1}{39} & -\frac{7}{39} & \frac{6}{13} \\ \frac{4}{39} & -\frac{11}{39} & \frac{2}{13} \\ \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{pmatrix} \begin{pmatrix} 14 \\ 10 \\ 9 \end{pmatrix} \quad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

∴ **Solution:** (2, 0, 3)

Exercise

Use the **inverse** of the coefficient matrix to solve the linear system

$$\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -10 & 4 & -3 & 1 & 0 \\ 0 & -11 & 4 & -2 & 0 & 1 \end{array} \right) \begin{array}{l} 5R_1 + 2R_2 \\ \\ 10R_3 - 11R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 5 & 0 & 3 & -1 & 2 & 0 \\ 0 & -10 & 4 & -3 & 1 & 0 \\ 0 & 0 & -4 & 13 & -11 & 10 \end{array} \right) \begin{array}{l} 4R_1 + 3R_3 \\ R_2 + R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 20 & 0 & 0 & 35 & -25 & 30 \\ 0 & -10 & 0 & 10 & -10 & 10 \\ 0 & 0 & -4 & 13 & -11 & 10 \end{array} \right) \begin{array}{l} \frac{1}{20}R_1 \\ -\frac{1}{10}R_2 \\ -\frac{1}{4}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & -\frac{5}{4} & \frac{3}{2} \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{13}{4} & \frac{11}{4} & -\frac{5}{2} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{7}{4} & -\frac{5}{4} & \frac{3}{2} \\ -1 & 1 & -1 \\ -\frac{13}{4} & \frac{11}{4} & -\frac{5}{2} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{7}{4} & -\frac{5}{4} & \frac{3}{2} \\ -1 & 1 & -1 \\ -\frac{13}{4} & \frac{11}{4} & -\frac{5}{2} \end{pmatrix} \begin{pmatrix} 20 \\ 8 \\ -16 \end{pmatrix}$$

$$X = A^{-1}B$$

$$= \begin{pmatrix} 35 - 10 - 24 \\ -20 + 8 + 16 \\ -65 + 22 + 40 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

∴ Solution: (1, 4, -3)

Exercise

Use the **inverse** of the coefficient matrix to solve the linear system

$$\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$$

Solution

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 2 & -3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 17 \\ 7 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array} \right) \quad R_3 - 2R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & -7 & 0 & -2 & 0 & 1 \end{array} \right) \quad \frac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -7 & 0 & -2 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 7R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{7}{2} & -2 & \frac{7}{2} & 1 \end{array}\right) -\frac{2}{7}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{4}{7} & -1 & -\frac{2}{7} \end{array}\right) \begin{array}{l} R_1 - 2R_3 \\ R_2 + \frac{1}{2}R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{7} & 1 & \frac{4}{7} \\ 0 & 1 & 0 & \frac{2}{7} & 0 & -\frac{1}{7} \\ 0 & 0 & 1 & \frac{4}{7} & -1 & -\frac{2}{7} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{1}{7} & 1 & \frac{4}{7} \\ \frac{2}{7} & 0 & -\frac{1}{7} \\ \frac{4}{7} & -1 & -\frac{2}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{1}{7} & 1 & \frac{4}{7} \\ \frac{2}{7} & 0 & -\frac{1}{7} \\ \frac{4}{7} & -1 & -\frac{2}{7} \end{pmatrix} \begin{pmatrix} 17 \\ 7 \\ -1 \end{pmatrix} \quad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$$

∴ Solution: (4, 5, 3)

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$$

Solution

$$A = \begin{pmatrix} -2 & 6 & 7 \\ -4 & 5 & 3 \\ -6 & 3 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} -2 & 6 & 7 & 1 & 0 & 0 \\ -4 & 5 & 3 & 0 & 1 & 0 \\ -6 & 3 & 5 & 0 & 0 & 1 \end{array} \right) \quad -\frac{1}{2}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & -\frac{7}{2} & -\frac{1}{2} & 0 & 0 \\ -4 & 5 & 3 & 0 & 1 & 0 \\ -6 & 3 & 5 & 0 & 0 & 1 \end{array} \right) \quad \begin{array}{l} R_2 + 4R_1 \\ R_3 + 6R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & -\frac{7}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & -7 & -11 & -2 & 1 & 0 \\ 0 & -15 & -16 & -3 & 0 & 1 \end{array} \right) \quad -\frac{1}{7}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & -\frac{7}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & -15 & -16 & -3 & 0 & 1 \end{array} \right) \quad \begin{array}{l} R_1 + 3R_2 \\ R_3 + 15R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{17}{14} & \frac{5}{14} & -\frac{3}{7} & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & \frac{53}{7} & \frac{9}{7} & -\frac{15}{7} & 1 \end{array} \right) \quad \frac{7}{53}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{17}{14} & \frac{5}{14} & -\frac{3}{7} & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & 1 & \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{array} \right) \quad \begin{array}{l} R_1 - \frac{17}{14}R_3 \\ R_2 - \frac{11}{7}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{8}{53} & -\frac{9}{106} & -\frac{17}{106} \\ 0 & 1 & 0 & \frac{1}{53} & \frac{16}{53} & -\frac{11}{53} \\ 0 & 0 & 1 & \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{8}{53} & -\frac{9}{106} & -\frac{7}{106} \\ \frac{1}{53} & \frac{16}{53} & -\frac{11}{53} \\ \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{8}{53} & -\frac{9}{106} & -\frac{7}{106} \\ \frac{1}{53} & \frac{16}{53} & -\frac{11}{53} \\ \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{pmatrix} \begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix} \quad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 3 \\ -2 \end{pmatrix}$$

$$\therefore \text{Solution: } \left(\frac{1}{2}, 3, -2 \right)$$

Exercise

Use the **inverse** of the coefficient matrix to solve the linear system

$$\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$$

Solution

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -3 & 4 \\ 4 & -2 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 3 & -3 & 4 & 0 & 1 & 0 \\ 4 & -2 & 3 & 0 & 0 & 1 \end{array} \right) \quad \begin{array}{l} \\ 2R_2 - 3R_1 \\ 2R_3 - 4R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 5 & -3 & 2 & 0 \\ 0 & 0 & 2 & -4 & 0 & 2 \end{array} \right) \quad \begin{array}{l} 3R_1 - R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 6 & 0 & -2 & 6 & -2 & 0 \\ 0 & -3 & 5 & -3 & 2 & 0 \\ 0 & 0 & 2 & -4 & 0 & 2 \end{array} \right) \quad \begin{array}{l} R_1 + R_3 \\ 2R_2 - 5R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 6 & 0 & 0 & 2 & -2 & 2 \\ 0 & -6 & 0 & 14 & 4 & -10 \\ 0 & 0 & 2 & -4 & 0 & 2 \end{array} \right) \quad \begin{array}{l} \frac{1}{6}R_1 \\ -\frac{1}{6}R_2 \\ \frac{1}{2}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{7}{6} & -\frac{2}{3} & \frac{5}{3} \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{7}{6} & -\frac{2}{3} & \frac{5}{3} \\ -2 & 0 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{7}{6} & -\frac{2}{3} & \frac{5}{3} \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

∴ Solution: (0, 1, 2)

Exercise

Use the **inverse** of the coefficient matrix to solve the linear system

$$\begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$$

Solution

$$A = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 3 & -2 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{array}\right) \frac{1}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 + 2R_2 \\ R_3 + R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & 1 \end{array}\right) \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & -1 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & -1 \\ -1 & 0 & -1 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & -1 \\ -1 & 0 & -1 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \quad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \\ 6 \end{pmatrix}$$

∴ Solution: $(-4, -6, 6)$

Solution **Section 4.4 – Determinants**

Exercise

Evaluate $\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix}$

Solution

$$\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix} = -9 - (-6) \\ \underline{\underline{= -3}}$$

Exercise

Evaluate $\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix} = -6 - (0) \\ \underline{\underline{= -6}}$$

Exercise

Evaluate $\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix} = x(8x) - 4x(2x) \\ = 8x^2 - 8x^2 \\ \underline{\underline{= 0}}$$

Exercise

Evaluate $\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix}$

Solution

$$\begin{aligned}\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix} &= 3x - 2x(4) \\ &= 3x - 8x \\ &= \underline{-5x}\end{aligned}$$

Exercise

Evaluate $\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix}$

Solution

$$\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix} = \underline{-3x^4 - 2x}$$

Exercise

Evaluate $\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix}$

Solution

$$\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix} = \underline{-8a + 5b}$$

Exercise

Evaluate $\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}$

Solution

$$\begin{aligned}\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix} &= 15 - 14 \\ &= \underline{1}\end{aligned}$$

Exercise

Evaluate $\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix} = 5 - 20$$

$$\underline{\underline{= -16}}$$

Exercise

Evaluate $\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix}$

Solution

$$\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix} = 15 + 6$$

$$\underline{\underline{= 21}}$$

Exercise

Evaluate $\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix}$

Solution

$$\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix} = -24 + 5$$

$$\underline{\underline{= -19}}$$

Exercise

Evaluate $\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix}$

Solution

$$\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix} = 3 - 6$$

$$\underline{\underline{= -3}}$$

Exercise

Evaluate $\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix}$

Solution

$$\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix} = 7 + 18$$

$$\underline{\underline{= 25}}$$

Exercise

Evaluate $\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix} = \underline{\underline{2\sqrt{5} + 6}}$$

Exercise

Evaluate $\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix}$

Solution

$$\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix} = -\frac{3}{8} - \frac{1}{16}$$

$$\underline{\underline{= -\frac{7}{16}}}$$

Exercise

Evaluate $\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix}$

Solution

$$\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix} = -1 + 1$$

$$\underline{\underline{= 0}}$$

Exercise

Evaluate $\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix}$

Solution

$$\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix} = \frac{1}{2} + \frac{1}{6}$$
$$= \frac{2}{3}$$

Exercise

Evaluate $\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix}$

Solution

$$\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix} = x^2 - 4x^2$$
$$= -3x^2$$

Exercise

Evaluate $\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix} = 9x - x^3$$

Exercise

Evaluate $\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix} = 2x^2 + 3x$$

Exercise

Evaluate $\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix}$

Solution

$$\begin{aligned} \begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix} &= 4(x+2) - 6(x-2) \\ &= 4x + 8 - 6x + 12 \\ &= \underline{-2x + 20} \end{aligned}$$

Exercise

Evaluate $\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix}$

Solution

$$\begin{aligned} \begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix} &= -3x - 3 + 6x + 18 \\ &= \underline{-2x + 20} \end{aligned}$$

Exercise

Evaluate $\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix}$

Solution

$$\begin{aligned} \begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix} &= \begin{matrix} 3 & 0 \\ 2 & 1 \\ 2 & 5 \end{matrix} \\ &= -3 + 0 + 0 - 0 + 75 - 0 \\ &= \underline{72} \end{aligned}$$

Exercise

Evaluate $\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix}$

Solution

$$\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix} \begin{matrix} 4 & 0 \\ 3 & -1 \\ 2 & -3 \end{matrix}$$

$$= -24 + 48$$

$$= \underline{24}$$

$$\text{or} = 4 \begin{vmatrix} -1 & 4 \\ -3 & 6 \end{vmatrix}$$

Exercise

Evaluate $\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix} \begin{matrix} 3 & 1 \\ -3 & -4 \\ -1 & 3 \end{matrix}$$

$$= -60 + 15$$

$$= \underline{-45}$$

Exercise

Evaluate $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix} \begin{matrix} 1 & 1 \\ 2 & 2 \\ 3 & -4 \end{matrix}$$

$$= 10 + 6 - 8 - 6 + 8 - 10$$

$$= \underline{0}$$

Exercise

Evaluate $\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix} \begin{array}{cc} x & 0 \\ 2 & 1 \\ -3 & x \end{array} \\
 = x - 2x - 3 - x^4 \\
 = \underline{\underline{-x^4 - x - 3}}$$

Exercise

Evaluate $\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix} \begin{array}{cc} x & 1 \\ x^2 & x \\ 0 & x \end{array} \\
 = x^2 - x^3 - x^3 - x^2 \\
 = \underline{\underline{-2x^3}}$$

Exercise

Evaluate $\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix}$

Solution

$$\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix} = 0 + 126 + 48 - (-48 + 36 + 0) \\
 = \underline{\underline{90}}$$

Exercise

Evaluate $\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix} = 0 - 4 - 16 - (-14 - 16 + 0) \\ = 10$$

Exercise

Evaluate $\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix}$

Solution

$$\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix} \begin{matrix} 3 & 1 \\ -2 & 3 \\ 3 & 4 \end{matrix} \\ = -54 + 3 - 16 - 18 - 12 - 12 \\ = -109$$

Exercise

Evaluate $\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix} \begin{matrix} 2x & 1 \\ 0 & 4 \\ 3 & 0 \end{matrix} \\ = 16x + 3x + 12 \\ = 19x + 12$$

Exercise

Evaluate $\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix} \begin{matrix} 0 & x \\ x & x^2 \\ x & 7 \end{matrix} \\
 = 5x^2 + 7x^2 - x^4 + 5x^2 \\
 = \underline{17x^2 - x^4}$$

Exercise

Evaluate $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix} \begin{matrix} 2 & x \\ -3 & 1 \\ 2 & 1 \end{matrix} \\
 = 8 - 3 - 2 + 12x \\
 = \underline{12x + 3}$$

Exercise

Evaluate $\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} \begin{matrix} 1 & x \\ 3 & 1 \\ 0 & -2 \end{matrix} \\
 = 2 + 12 + 2 - 6x \\
 = \underline{-6x + 16}$$

Exercise

Solve for x. $\begin{vmatrix} x & 3 \\ 2 & 1 \end{vmatrix} = 12$

Solution

$$\begin{vmatrix} x & 3 \\ 2 & 1 \end{vmatrix} = x - 6 = 12$$

$$\therefore \text{Solution: } \underline{x = 18}$$

Exercise

Solve for x . $\begin{vmatrix} x & 1 \\ 2 & x \end{vmatrix} = -1$

Solution

$$\begin{vmatrix} x & 1 \\ 2 & x \end{vmatrix} = x^2 - 2 = -1$$

$$x^2 = 1$$

$$\therefore \text{Solution: } \underline{x = \pm 1}$$

Exercise

Solve for x . $\begin{vmatrix} 3 & x \\ x & 4 \end{vmatrix} = -13$

Solution

$$\begin{vmatrix} 3 & x \\ x & 4 \end{vmatrix} = 12 - x^2 = -13$$

$$x^2 = 25$$

$$\therefore \text{Solution: } \underline{x = \pm 5}$$

Exercise

Solve for x . $\begin{vmatrix} x & 2 \\ 3 & x \end{vmatrix} = x$

Solution

$$\begin{vmatrix} x & 2 \\ 3 & x \end{vmatrix} = x^2 - 6 = x$$

$$x^2 - x - 6 = 0$$

$$\therefore \text{Solution: } \underline{x = -2, 3}$$

Exercise

Solve for x . $\begin{vmatrix} 4 & 6 \\ -2 & x \end{vmatrix} = 32$

Solution

$$\begin{vmatrix} 4 & 6 \\ -2 & x \end{vmatrix} = 4x + 12 = 32$$

$$4x = 20$$

$$\therefore \text{Solution: } \underline{x = 5}$$

Exercise

Solve for x . $\begin{vmatrix} x+2 & -3 \\ x+5 & -4 \end{vmatrix} = 3x - 5$

Solution

$$\begin{vmatrix} x+2 & -3 \\ x+5 & -4 \end{vmatrix} = -4x - 8 + 3x + 15 = 3x - 5$$

$$-4x = -12$$

$$\therefore \text{Solution: } \underline{x = 3}$$

Exercise

Solve for x . $\begin{vmatrix} x+3 & -6 \\ x-2 & -4 \end{vmatrix} = 28$

Solution

$$\begin{vmatrix} x+3 & -6 \\ x-2 & -4 \end{vmatrix} = -4x - 12 + 6x - 12 = 28$$

$$2x = 52$$

$$\therefore \text{Solution: } \underline{x = 26}$$

Exercise

Solve for x . $\begin{vmatrix} x & -3 \\ -1 & x \end{vmatrix} \geq 0$

Solution

$$\begin{vmatrix} x & -3 \\ -1 & x \end{vmatrix} = x^2 - 3 \geq 0$$

$$x^2 \geq 3$$

$$\therefore \text{Solution: } \underline{x \leq -\sqrt{3} \quad x \geq \sqrt{3}}$$

Exercise

Solve for x . $\begin{vmatrix} 2 & x & 1 \\ 1 & 2 & -1 \\ 3 & 4 & -2 \end{vmatrix} = -6$

Solution

$$\begin{vmatrix} 2 & x & 1 \\ 1 & 2 & -1 \\ 3 & 4 & -2 \end{vmatrix} = -8 - 3x + 4 - 6 + 8 + 2x = -6$$

$$-x = -4$$

$$\therefore \text{Solution: } \underline{x = 4}$$

Exercise

Solve for x . $\begin{vmatrix} 1 & x & -3 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} = 8$

Solution

$$\begin{vmatrix} 1 & x & -3 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} = 2 + 18 + 2 - 6x = 8$$

$$-6x = -14$$

$$\therefore \text{Solution: } \underline{x = \frac{7}{3}}$$

Exercise

Solve for x . $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix} = 39$

Solution

$$\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix} = 8 - 3 - 2 + 12x = 39$$

$$12x = 36$$

$$\therefore \text{Solution: } \underline{x = 3}$$

Exercise

Solve for x . $\begin{vmatrix} x & 0 & 0 \\ 7 & x & 1 \\ 7 & 2 & 1 \end{vmatrix} = -1$

Solution

$$\begin{vmatrix} x & 0 & 0 \\ 7 & x & 1 \\ 7 & 2 & 1 \end{vmatrix} = x^2 - 2x = -1$$

$$x^2 - 2x + 1 = 0$$

$$\therefore \text{Solution: } \underline{x = 1}$$