Solution *Section* 4.6 – Substitution Rule

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int 2(2x+4)^5\,dx, \quad u = 2x+4$$

Solution

Let $u = 2x+4 \quad \Rightarrow \quad du = 2xdx$

$$\int 2(2x+4)^5\,dx = \int u^5 du$$

$$= \frac{1}{6}u^6 + C$$

$$\boxed{= \frac{1}{6}(2x+4)^6 + C}$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int \frac{4x^3}{(x4+1)^2}\,dx, \quad u = x^4+1$$

Solution

Let $u = x^4+1 \quad \Rightarrow \quad du = 4x^3dx$

$$\int \frac{4x^3}{(x4+1)^2}\,dx = \int \frac{du}{u^2}$$

$$= \int u^{-2}du$$

$$= \frac{u^{-1}}{-1} + C$$

$$\boxed{= \frac{-1}{x^2+1} + C}$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int x\sin\left(2x^2\right)dx, \quad u = 2x^2$$

Solution

Let $u = 2x^2 \quad \Rightarrow \quad du = 4xdx \to \ \frac{1}{4}du = xdx$

$$\int x\sin\left(2x^2\right)dx = \int \frac{1}{4}\sin u du$$

$$= -\frac{1}{4}\cos u + C$$

$$\boxed{= -\frac{1}{4}\cos\left(2x^2\right) + C}$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int 12\left(y^4 + 4y^2 + 1\right)^2\left(y^3 + 2y\right)dx, \quad u = y^4 + 4y^2 + 1$$

Solution

Let $u = y^4 + 4y^2 + 1 \quad \Rightarrow \quad du = \left(4y^3 + 8y\right)dx \to \ \ du = 4\left(y^3 + 2y\right)dx$

$$\int 12\left(y^4 + 4y^2 + 1\right)^2\left(y^3 + 2y\right)dx = \int 12u^2\left(\frac{1}{4}du\right)$$

$$= 3\int u^2\ du$$

$$= 3\frac{u^3}{3} + C$$

$$\boxed{= \left(y^4 + 4y^2 + 1\right)^3 + C}$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int \csc^2 2\theta \cot 2\theta\ d\theta \quad \to \begin{cases} a)\ \text{Using} \ \ u = \cot 2\theta \\ b)\ \text{Using} \ \ u = \csc 2\theta \end{cases}$$

Solution

Let $u = \cot 2\theta \quad \Rightarrow \quad du = -2\csc^2 2\theta d\theta \to \quad -\frac{1}{2}du = \csc^2 2\theta dx$

$$\int \csc^2 2\theta \cot 2\theta \, d\theta = -\int \frac{1}{2} u du$$

$$= -\frac{1}{2}\frac{u^2}{2} + C$$

$$\boxed{= -\frac{1}{4}\cot^2 2\theta + C}$$

Let $u = \csc 2\theta \quad \Rightarrow \quad du = -2\csc 2\theta \cot 2\theta d\theta \to \quad -\frac{1}{2}du = \csc 2\theta \cot 2\theta dx$

$$\int \csc^2 2\theta \cot 2\theta \, d\theta = \int \csc 2\theta \left(\csc 2\theta \cot 2\theta \, d\theta\right)$$

$$= -\int \frac{1}{2} u du$$

$$= -\frac{1}{2}\frac{u^2}{2} + C$$

$$\boxed{= -\frac{1}{4}\csc^2 2\theta + C}$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{1}{\sqrt{5s+4}} ds$

Solution

Let $u = 5s + 4 \quad \Rightarrow \quad du = 5ds \quad \to \quad \frac{1}{5}du = ds$

$$\int \frac{1}{\sqrt{5s+4}} ds = \frac{1}{5}\int u^{-1/2} du$$

$$= \frac{1}{5}\frac{u^{1/2}}{1/2} + C$$

$$\boxed{= \frac{2}{5}\sqrt{5s+4} + C}$$

Exercise

Evaluate the integrals $\displaystyle\int \theta \sqrt[4]{1-\theta^2} \, d\theta$

Solution

Let $u = 1 - \theta^2 \quad \Rightarrow \quad du = -2\theta d\theta \quad \to \quad -\frac{1}{2}du = \theta d\theta$

$$\int \theta \sqrt[4]{1-\theta^2}\, d\theta = -\frac{1}{2}\int u^{1/4} du$$

$$= -\frac{1}{2}\frac{u^{5/4}}{5/4} + C$$

$$\boxed{= -\frac{2}{5}\left(1-\theta^2\right)^{5/4} + C}$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2} dx$

Solution

Let $u = 1+\sqrt{x} \quad \Rightarrow \quad du = \frac{1}{2\sqrt{x}} dx \quad \to \quad 2du = \frac{1}{\sqrt{x}} dx$

$$\int \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2} dx = \int \frac{2}{u^2} du$$

$$= 2\int u^{-2} du$$

$$= 2\frac{u^{-1}}{-1} + C$$

$$\boxed{= \frac{-2}{1+\sqrt{x}} + C}$$

Exercise

Evaluate the integrals $\displaystyle\int \tan^2 x \sec^2 x\, dx$

Solution

Let $u = \tan x \quad \Rightarrow \quad du = \sec^2 x dx$

$$\int \tan^2 x \sec^2 x\, dx = \int u^2 du$$

$$= \frac{1}{3}u^3 + C$$

$$\boxed{= \frac{1}{3}\tan^3 x + C}$$

Exercise

Evaluate the integrals $\int \sin^5 \frac{x}{3} \cos \frac{x}{3}\, dx$

Solution

Let $u = \sin\left(\frac{x}{3}\right) \Rightarrow du = \frac{1}{3}\cos\left(\frac{x}{3}\right)dx \to 3du = \cos\left(\frac{x}{3}\right)dx$

$$\int \sin^5 \frac{x}{3} \cos \frac{x}{3}\, dx = \int u^5 (3du)$$

$$= 3\frac{u^6}{6} + C$$

$$= \frac{1}{2}\sin^6\left(\frac{x}{3}\right) + C$$

Exercise

Evaluate the integrals $\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2}\, dx$

Solution

Let $u = \tan\left(\frac{x}{2}\right)$

$du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx \to 2du = \sec^2\left(\frac{x}{2}\right)dx$

$$\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2}\, dx = 2\int u^7 du$$

$$= 2\frac{1}{8}u^8 + C$$

$$= \frac{1}{4}\tan^8 \frac{x}{2} + C$$

$$\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2}\, dx = 2\int \tan^7 \frac{x}{2}\, d\left(\tan \frac{x}{2}\right)$$

$$= \frac{1}{4}\tan^8 \frac{x}{2} + C$$

Exercise

Evaluate the integrals $\int r^4 \left(7 - \frac{r^5}{10}\right)^3 dr$

Solution

Let $u = 7 - \frac{r^5}{10}$

$du = -\frac{1}{10}5r^4 dr \to -2du = r^4 dr$

$$\int r^4 \left(7 - \frac{r^5}{10}\right)^3 dr = \int u^3 (-2du)$$

$$= -2\int u^3 du$$

$$= -2\frac{u^4}{4} + C$$

$$\boxed{= -\frac{1}{2}\left(7 - \frac{r^5}{10}\right)^4 + C}$$

Exercise

Evaluate the integrals $\int x^{1/2} \sin\left(x^{3/2} + 1\right) dx$

Solution

Let $u = x^{3/2} + 1 \quad \Rightarrow \quad du = \frac{3}{2} x^{1/2} dx \rightarrow \; \frac{2}{3} du = x^{1/2} dx$

$$\int x^{1/2} \sin\left(x^{3/2} + 1\right) dx = \int \sin u \; \left(\frac{2}{3} du\right)$$

$$= \frac{2}{3} \int \sin u \;\; du$$

$$= \frac{2}{3}(-\cos u) + C$$

$$\boxed{= -\frac{2}{3} \cos\left(x^{3/2} + 1\right) + C}$$

Exercise

Evaluate the integrals $\int \csc\left(\frac{v-\pi}{2}\right) \cot\left(\frac{v-\pi}{2}\right) dv$

Solution

Let $u = \csc\left(\frac{v-\pi}{2}\right) \quad \Rightarrow \quad du = -\frac{1}{2} \csc\left(\frac{v-\pi}{2}\right) \cot\left(\frac{v-\pi}{2}\right) dv$

$$-2du = \csc\left(\frac{v-\pi}{2}\right) \cot\left(\frac{v-\pi}{2}\right) dv$$

$$\int \csc\left(\frac{v-\pi}{2}\right) \cot\left(\frac{v-\pi}{2}\right) dv = \int -2du$$

$$= -2u + C$$

$$\boxed{= -2\csc\left(\frac{v-\pi}{2}\right) + C}$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{\sin(2t+1)}{\cos^2(2t+1)}dt$

Solution

$$\int \frac{\sin(2t+1)}{\cos^2(2t+1)}dt = -\frac{1}{2}\int \frac{d(\cos(2t+1))}{\cos^2(2t+1)}$$

$$\boxed{= \frac{1}{2\cos(2t+1)} + C}$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{\sec z \ \tan z}{\sqrt{\sec z}}dz$

Solution

Let $u = \sec z \quad \Rightarrow \quad du = \sec z \tan z dz$

$$\int \frac{\sec z \ \tan z}{\sqrt{\sec z}}dz = \int \frac{du}{u^{1/2}}$$

$$= \int u^{-1/2} du$$

$$= 2u^{1/2} + C$$

$$\boxed{= 2\sqrt{\sec z} + C}$$

$$\int \frac{\sec z \ \tan z}{\sqrt{\sec z}}dz = \int (\sec z)^{-1/2} d(\sec z)$$

$$\boxed{= 2\sqrt{\sec z} + C}$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{1}{\sqrt{t}}\cos\left(\sqrt{t}+3\right)dt$

Solution

$u = \sqrt{t} + 3 \quad \Rightarrow \quad du = \frac{1}{2\sqrt{t}}dt \to 2du = \frac{1}{\sqrt{t}}dt$

$$\int \frac{1}{\sqrt{t}}\cos\left(\sqrt{t}+3\right)dt = \int (\cos u)(2du)$$

$$= 2\int \cos u \ du$$

$$= 2\sin u + C$$

$$\boxed{= 2\sin\left(\sqrt{t}+3\right) + C}$$

$$\int \frac{1}{\sqrt{t}}\cos\left(\sqrt{t}+3\right)dt = 2\int \cos\left(\sqrt{t}+3\right)d\left(\sqrt{t}+3\right)$$

$$\boxed{= 2\sin\left(\sqrt{t}+3\right) + C}$$

Exercise

Evaluate the integrals $\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\ d\theta$

Solution

Let $u=\sin\frac{1}{\theta} \quad \Rightarrow \quad du=\left(\cos\frac{1}{\theta}\right)\left(\frac{1}{\theta}\right)'=\left(\cos\frac{1}{\theta}\right)\left(-\frac{1}{\theta^2}\right)d\theta$

$-du=\frac{1}{\theta^2}\cos\frac{1}{\theta}d\theta$

$$\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\ d\theta=-\int udu$$

$$=-\frac{1}{2}u^2+C$$

$$\boxed{=-\frac{1}{2}\sin^2\frac{1}{\theta}+C}$$

Exercise

Evaluate the integrals $\int t^3\left(1+t^4\right)^3 dt$

Solution

$u=1+t^4 \quad \Rightarrow \quad du=4t^3dt \to \frac{1}{4}du=t^3dt$

$$\int t^3\left(1+t^4\right)^3 dt=\frac{1}{4}\int u^3du$$

$$=\frac{1}{4}\left(\frac{u^4}{4}\right)+C$$

$$\boxed{=\frac{1}{16}\left(1+t^4\right)^4+C}$$

$d\left(1+t^4\right)=4t^3dt$

$$\int t^3\left(1+t^4\right)^3 dt=\frac{1}{4}\int \left(1+t^4\right)^3 d\left(1+t^4\right)$$

$$\boxed{=\frac{1}{16}\left(1+t^4\right)^4+C}$$

Exercise

Evaluate the integrals $\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}dx$

Solution

Let $u=\frac{x^2-1}{x^2}=1-\frac{1}{x^2}=1-x^{-2}$

$du=2x^{-3}dx \quad \to \quad \frac{1}{2}du=\frac{1}{x^3}dx$

$$\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}dx = \int u^{1/2}\left(\tfrac{1}{2}du\right)$$

$$= \tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2}\right) + C$$

$$\boxed{= \tfrac{1}{3}\left(1 - \frac{1}{x^2}\right)^{3/2} + C}$$

Exercise

Evaluate the integrals $\displaystyle\int x^3\sqrt{x^2+1}\ dx$

Solution

Let $u = x^2 + 1 \quad \Rightarrow \quad du = 2xdx \quad \rightarrow \quad \tfrac{1}{2}du = xdx$

$x^2 = u - 1$

$$\int x^3\sqrt{x^2+1}\ dx = \int x^2\sqrt{x^2+1}\ xdx$$

$$= \int (u-1)u^{1/2}\left(\tfrac{1}{2}du\right)$$

$$= \tfrac{1}{2}\int \left(u^{3/2} - u^{1/2}\right)du$$

$$= \tfrac{1}{2}\left(\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right) + C$$

$$\boxed{= \tfrac{1}{5}\left(x^2+1\right)^{5/2} - \tfrac{1}{3}\left(x^2+1\right)^{3/2} + C}$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{x}{\left(x^2-4\right)^3}dx$

Solution

$u = x^2 - 4 \quad \Rightarrow \quad du = 2xdx \rightarrow \tfrac{1}{2}du = xdx$

$$\int \frac{x}{\left(x^2-4\right)^3}dx = \tfrac{1}{2}\int u^{-3}du$$

$$= -\tfrac{1}{4}u^{-2} + C$$

$d\left(x^2-4\right) = 2xdx$

$$\int \frac{x}{\left(x^2-4\right)^3}dx = \tfrac{1}{2}\int \left(x^2-4\right)^{-3} d\left(x^2-4\right)$$

$$= -\frac{1}{4\left(x^2-4\right)^2} + C$$

$$= -\frac{1}{4\left(x^2-4\right)^2} + C$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}dr$

Solution

Let $u = \sqrt{3(2r-1)^2+6} \quad \Rightarrow \quad du = \frac{1}{2}\left(3(2r-1)^2+6\right)^{-1/2}\left(6(2r-1)(2)\right)dr$

$$= \frac{6(2r-1)}{\left(3(2r-1)^2+6\right)^{1/2}}dr$$

$$\to \quad \frac{1}{6}du = \frac{2r-1}{\sqrt{3(2r-1)^2+6}}dr$$

$$\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}dr = \int \cos u\left(\frac{1}{6}du\right)$$

$$= \frac{1}{6}\sin u + C$$

$$= \frac{1}{6}\sin\sqrt{3(2r-1)^2+6} + C$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}d\theta$

Solution

Let $u = \cos\sqrt{\theta} \quad \Rightarrow \quad du = \left(-\sin\sqrt{\theta}\right)\left(\frac{1}{2\sqrt{\theta}}\right)d\theta$

$-2du = \frac{1}{\sqrt{\theta}}\sin\sqrt{\theta}\,d\theta$

$$\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}d\theta = \int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\sqrt{\cos^3\sqrt{\theta}}}d\theta$$

$$= \int \frac{1}{u^{3/2}}(-2du)$$

$$= -2\int u^{-3/2}du$$

$$= -2\frac{u^{-1/2}}{-1/2} + C$$

$$\boxed{= \frac{4}{\sqrt{\cos\sqrt{\theta}}} + C}$$

Exercise

Evaluate the integrals. $\int 2x\sqrt{x^2-2}\ dx$

Solution

$$\int 2x\sqrt{x^2-2}\ dx = \int \left(x^2-2\right)^{1/2}\ d\left(x^2-2\right)$$

$$\boxed{= \frac{2}{3}(x^2-2)^{3/2} + C}$$

Exercise

Evaluate the integrals $\int x^3\left(3x^4+1\right)^2 dx$

Solution

$$\int x^3\left(3x^4+1\right)^2 dx = \int \left(3x^4+1\right)^2 d\left(3x^4+1\right)$$

$$\boxed{= \frac{1}{36}(3x^4+1)^3 + C}$$

Exercise

Evaluate the integrals $\int 2\left(3x^4+1\right)^2 dx$

Solution

$$\int 2\left(3x^4+1\right)^2 dx = \int 2\left(9x^8+6x^4+1\right)dx \qquad (a+b)^2 = a^2+2ab+b^2$$

$$= \int \left(18x^8+12x^4+2\right)dx$$

$$\boxed{= 2x^9 + \frac{12}{5}x^5 + 2x + C}$$

Exercise

Evaluate the integrals $\int 5x\sqrt{x^2-1}\,dx$

Solution

$$u = x^2 - 1 \quad \Rightarrow du = 2xdx \quad \Rightarrow \frac{1}{2}du = xdx$$

$$\int 5x\left(x^2-1\right)^{1/2}\,dx = 5\int u^{1/2}\ \frac{1}{2}du \qquad \textit{Substitute for x and dx}$$

$$= 5\int u^{1/2}\ \frac{1}{2}du$$

$$= \frac{5}{2}\int u^{1/2}du$$

$$= \frac{5}{2}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{5}{3}u^{3/2} + C$$

$$\boxed{= \frac{5}{3}\left(x^2-1\right)^{3/2} + C}$$

Exercise

Find the integral $\int \left(x^2-1\right)^3(2x)dx$

Solution

$$\int \left(x^2-1\right)^3(2x)dx = \int \left(x^2-1\right)^3 d\left(x^2-1\right)$$

$$\boxed{= \frac{1}{4}\left(x^2-1\right)^4 + C}$$

Exercise

Find the integral $\int \frac{6x}{\left(1+x^2\right)^3}dx$

Solution

$$\int \frac{6x}{\left(1+x^2\right)^3}dx = 3\int \left(1+x^2\right)^3 d\left(1+x^2\right)$$

$$= -\frac{3}{2}\left(1+x^2\right)^{-2} + C$$

$$\boxed{= -\frac{3}{2}\frac{1}{\left(1+x^2\right)^2} + C}$$

Exercise

Find the integral $\int u^3\sqrt{u^4+2}\ du$

Solution

$$\int u^3\sqrt{u^4+2}\ du = \frac{1}{4}\int \left(u^4+2\right)^{1/2} d\left(u^4+2\right)$$

$$\boxed{= \frac{1}{6}\left(u^4+2\right)^{3/2} + C}$$

Exercise

Find the integral $\int \frac{t+2t^2}{\sqrt{t}}\ dt$

Solution

$$\int \frac{t+2t^2}{\sqrt{t}}\ dt = \int \left(\frac{t}{t^{1/2}} + 2\frac{t^2}{t^{1/2}}\right) dt$$

$$= \int \left(t^{1/2} + 2t^{3/2}\right) dt$$

$$= \frac{2}{3}t^{3/2} + 2\frac{2}{5}t^{5/2} + C$$

$$\boxed{= \frac{2}{3}t^{3/2} + \frac{4}{5}t^{5/2} + C}$$

Exercise

Find the integral $\int \left(1+\frac{1}{t}\right)^3 \frac{1}{t^2}\ dt$

Solution

$$\int \left(1+\frac{1}{t}\right)^3 \frac{1}{t^2}\ dt = -\int \left(1+\frac{1}{t}\right)^3 d\left(1+\frac{1}{t}\right)$$

$$\boxed{= -\frac{1}{4}\left(1+\frac{1}{t}\right)^4 + C}$$

Exercise

Find the integral $\int \left(7-3x-3x^2\right)(2x+1)\ dx$

Solution

$$d\left(7-3x-3x^2\right)=\left(-3-6x^2\right)dx=-3\left(1+2x^2\right)dx$$

$$\int \left(7-3x-3x^2\right)(2x+1)\ dx=-\frac{1}{3}\int \left(7-3x-3x^2\right)\left(7-3x-3x^2\right)\ dx$$

$$=-\frac{1}{6}\left(7-3x-3x^2\right)^2+C$$

Exercise

Find the integral $\int \sqrt{x}\left(4-x^{3/2}\right)^2\ dx$

Solution

$$u=4-x^{3/2}\Rightarrow du=-\frac{3}{2}x^{1/2}dx$$

$$\to -\frac{2}{3}du=\sqrt{x}dx$$

$$\int \sqrt{x}\left(4-x^{3/2}\right)^2\ dx=\int u^2\left(-\frac{2}{3}\right)du$$

$$=-\frac{2}{3}\int u^2 du$$

$$=-\frac{2}{9}u^3+C$$

$$=-\frac{2}{9}\left(4-x^{3/2}\right)^3+C$$

$$d\left(4-x^{3/2}\right)=-\frac{3}{2}x^{1/2}dx$$

$$\int \sqrt{x}\left(4-x^{3/2}\right)^2\ dx=-\frac{2}{3}\int \left(4-x^{3/2}\right)^2 d\left(4-x^{3/2}\right)$$

$$=-\frac{2}{9}\left(4-x^{3/2}\right)^3+C$$

Exercise

Find the integral $\int \frac{1}{\sqrt{x}+\sqrt{x+1}}dx$

Solution

$$\int \frac{1}{\sqrt{x}+\sqrt{x+1}}dx=\int \frac{1}{\sqrt{x}+\sqrt{x+1}}\frac{\sqrt{x}-\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}dx$$

$$=\int \frac{\sqrt{x}-\sqrt{x+1}}{x-(x+1)}dx$$

$$= \int \frac{\sqrt{x} - \sqrt{x+1}}{-1}\, dx$$

$$= -\int \left(x^{1/2} - (x+1)^{1/2} \right) dx$$

$$= -\left(\frac{2}{3} x^{3/2} - \frac{2}{3} (x+1)^{3/2} \right) + C$$

$$\underline{= \frac{2}{3}(x+1)^{3/2} - \frac{2}{3} x^{3/2} + C|}$$

Exercise

Find the integral $\int \sqrt{1-x}\ dx$

Solution

$$\int \sqrt{1-x}\ dx = -\int (1-x)^{1/2}\ d(1-x) \qquad d(1-x) = -dx$$

$$\underline{= -\frac{2}{3}(1-x)^{3/2} + C|}$$

Exercise

Find the integral $\int x\sqrt{x^2+4}\ dx$

Solution

$$\int \sqrt{x^2+4}\ xdx = \frac{1}{2}\int \left(x^2+4\right)^{1/2} d\left(x^2+4\right) \qquad d\left(x^2+4\right) = 2xdx$$

$$\underline{= \frac{1}{3}\left(x^2+4\right)^{3/2} + C|}$$

Exercise

Find the integral $\int \sin^2\left(\theta + \frac{\pi}{6}\right) d\theta$

Solution

$$\int \sin^2\left(\theta + \frac{\pi}{6}\right) d\theta = \frac{1}{2}\int \left(1 - \cos\left(2\theta + \frac{\pi}{3}\right)\right) d\theta$$

$$= \frac{1}{2}\left(\theta - \frac{1}{2}\sin\left(2\theta + \frac{\pi}{3}\right)\right) + C$$

$$\underline{= \frac{\theta}{2} - \frac{1}{4}\sin\left(2\theta + \frac{\pi}{3}\right) + C|}$$

Exercise

Find the integral $\displaystyle\int \cos^2(8\theta)\,d\theta$

Solution

$$\int \cos^2(8\theta)\,d\theta = \frac{1}{2}\int \left(1+\cos(16\theta)\right)d\theta$$

$$= \frac{1}{2}\left(1+\frac{1}{16}\sin(16\theta)\right)+C$$

$$\boxed{= \frac{1}{2}+\frac{1}{32}\sin(16\theta)+C}$$

Exercise

Find the integral $\displaystyle\int \sin^2(2\theta)\,d\theta$

Solution

$$\int \sin^2(2\theta)\,d\theta = \frac{1}{2}\int \left(1-\cos(4\theta)\right)d\theta$$

$$= \frac{1}{2}\left(1-\frac{1}{4}\sin(4\theta)\right)+C$$

$$\boxed{= \frac{1}{2}-\frac{1}{8}\sin(4\theta)+C}$$

Exercise

Evaluate the integral $\displaystyle\int 8\cos^4 2\pi x\,dx$

Solution

$$\int 8\cos^4 2\pi x\,dx = 8\int (\cos 2\pi x)^4\,dx \qquad \cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$= 8\int \left(\frac{1+\cos 4\pi x}{2}\right)^2 dx$$

$$= 2\int (1+\cos 4\pi x)^2\,dx$$

$$= 2\int \left(1+2\cos 4\pi x+\cos^2 4\pi x\right)dx$$

$$= 2\int dx + 4\int \cos 4\pi x dx + 2\int \cos^2 4\pi x dx$$

$$= 2x + 4\frac{1}{4\pi}\cos 4\pi x + 2\int \frac{1+\cos 8\pi x}{2}dx$$

$$= 2x + \frac{1}{\pi}\cos 4\pi x + \int (1+\cos 8\pi x)\,dx$$

$$= 2x + \frac{1}{\pi}\sin 4\pi x + x + \frac{1}{8\pi}\sin 8\pi x + C$$

$$\boxed{= 3x + \frac{1}{\pi}\sin 4\pi x + \frac{1}{8\pi}\sin 8\pi x + C}$$

Exercise

Evaluate the integral $\int \sec x dx$

Solution

$$\int \sec x dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}dx$$

$$= \int \frac{\sec^2 x + \sec\tan x}{\sec x + \tan x}dx \qquad d(\sec x + \tan x) = (\sec x \tan x + \sec^2 x)dx$$

$$= \int \frac{d(\sec x + \tan x)}{\sec x + \tan x}$$

$$\boxed{= \ln\left|\sec x + \tan x\right| + C}$$

Exercise

Evaluate $\int \frac{dx}{\sqrt{1-4x^2}}$

Solution

Let $u = 2x \quad \Rightarrow du = 2dx \to \boxed{\frac{1}{2}du = dx}$

$$\int \frac{dx}{\sqrt{1-4x^2}} = \int \frac{dx}{\sqrt{1-(2x)^2}}$$

$$= \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{1}{2}\sin^{-1} u + C$$

$$\boxed{= \frac{1}{2}\sin^{-1}(2x) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{3-4x^2}}$

Solution

$a^2 = 3 \to a = \sqrt{3} \qquad u^2 = 4x^2 = (2x)^2 \to u = 2x \quad du = 2dx$

$$\int \frac{dx}{\sqrt{3-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{a^2-u^2}}$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-6}}$

Solution

$a^2 = 6 \to a = \sqrt{6} \qquad u^2 = e^{2x} \to u = e^x \quad du = e^x dx \to dx = \frac{du}{e^x} = \frac{du}{u}$

$$\int \frac{dx}{\sqrt{e^{2x}-6}} = \int \frac{du}{u\sqrt{u^2-a^2}}$$

$$= \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

$$= \frac{1}{\sqrt{6}}\sec^{-1}\left|\frac{e^x}{\sqrt{6}}\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{4x-x^2}}$

Solution

$$4x - x^2 = -\left(x^2-4x\right) - 4 + 4 \qquad \textit{Using Completing the Square}$$

$$= -\left(x^2-4x+4\right) + 4$$

$$= 4 - (x-2)^2$$

$a = 2 \qquad u = x-2 \quad \to du = dx$

$$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}}$$

$$= \sin^{-1}\left(\frac{x-2}{2}\right) + C$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{4x^2+4x+2}$

Solution

$$4x^2+4x+2 = 4\left(x^2+x\right)+2$$

$$= 4\left(x^2+x+\frac{1}{4}\right)+2-4\left(\frac{1}{4}\right)$$

$$= 4\left(x+\frac{1}{2}\right)^2+1$$

$$= (2x+1)^2+1$$

$a = 1 \qquad u = 2x+1 \quad \to du = 2dx$

$$\int \frac{dx}{4x^2+4x+2} = \int \frac{dx}{(2x+1)^2+1}$$

$$= \frac{1}{2}\int \frac{du}{u^2+1} \qquad\qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{1}{2}\cdot\frac{1}{1}\tan^{-1}\left(\frac{2x+1}{1}\right)+C$$

$$= \frac{1}{2}\tan^{-1}(2x+1)+C$$

Exercise

Find the integral $\displaystyle\int \frac{1}{6x-5}dx$

Solution

$$\int \frac{1}{6x-5}dx = \frac{1}{6}\int \frac{d(6x-5)}{6x-5}$$

$$= \frac{1}{6}\ln|6x-5|+C$$

Exercise

Find the integral $\displaystyle\int \frac{x^2+2x+3}{x^3+3x^2+9x+1}dx$

Solution

$$d\left(x^3+3x^2+9x+1\right)=\left(3x^2+6x+9\right)dx$$

$$\int \frac{x^2+2x+3}{x^3+3x^2+9x+1}dx=\frac{1}{3}\int \frac{d\left(x^3+3x^2+9x+1\right)}{x^3+3x^2+9x+1}$$

$$\boxed{=\tfrac{1}{3}\ln\left|x^3+3x^2+9x+1\right|+C}$$

Exercise

Find the integral $\displaystyle\int \frac{1}{x(\ln x)^2}dx$

Solution

$$\int \frac{1}{x(\ln x)^2}dx=\int \frac{1}{(\ln x)^2}d(\ln x)$$

$$\boxed{=-\frac{1}{\ln x}+C}$$

Exercise

Find the integral $\displaystyle\int \frac{x-3}{x+3}dx$

Solution

$$\int \frac{x-3}{x+3}dx=\int\left(1-\frac{6}{x+3}\right)dx$$

$$\boxed{=x-6\ln|x+3|+C}$$

Exercise

Find the indefinite integral. $\displaystyle\int \frac{3x}{x^2+4}dx$

Solution

$$u=x^2+4\to du=2xdx\to \tfrac{1}{2}du=xdx$$

$$\int \frac{3x}{x^2+4}dx = \int \frac{3}{u}\frac{1}{2}du$$

$$= \frac{3}{2}\int \frac{1}{u}du$$

$$= \frac{3}{2}\ln|u| + C$$

$$\boxed{= \frac{3}{2}\ln\left(x^2+4\right) + C}$$

$$\int \frac{3x}{x^2+4}dx = \frac{3}{2}\int \frac{d\left(x^2+4\right)}{x^2+4}$$

$$\boxed{= \frac{3}{2}\ln\left(x^2+4\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{2\sqrt{x}+2x}$

Solution

$$\int \frac{dx}{2\sqrt{x}+2x} = \int \frac{dx}{2\sqrt{x}\left(1+\sqrt{x}\right)}$$

$$u = 1+\sqrt{x} \quad \Rightarrow \quad du = \frac{1}{2\sqrt{x}}dx$$

$$= \int \frac{du}{u}$$

$$= \ln u + C$$

$$\boxed{= \ln\left(1+\sqrt{x}\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sec x dx}{\sqrt{\ln(\sec x + \tan x)}}$

Solution

Let $u = \sec x + \tan x \quad \Rightarrow \quad du = \left(\sec x \tan x + \sec^2 x\right)dx = \sec x\left(\tan x + \sec x\right)dx$

$$\sec x dx = \frac{du}{\tan x + \sec x} = \frac{du}{u}$$

$$\int \frac{\sec x dx}{\sqrt{\ln(\sec x + \tan x)}} = \int \frac{du}{u\sqrt{\ln u}}$$

$$= \int (\ln u)^{-1/2}\ d(\ln u) \qquad\qquad d(\ln u) = \frac{1}{u}du$$

$$= 2(\ln u)^{1/2} + C$$

$$\boxed{= 2\sqrt{\ln(\sec x + \tan x)} + C}$$

Exercise

Evaluate the integral $\int 8e^{(x+1)}dx$

Solution

$$\int 8e^{(x+1)}dx = \underline{8e^{(x+1)} + C}$$

Exercise

Find the indefinite integral. $\int 4xe^{x^2}dx$

Solution

$$\int 4xe^{x^2}dx = 2\int e^{x^2}d\left(x^2\right)$$

$$= 2e^{x^2} + C$$

Exercise

Evaluate the integral $\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}dr$

Solution

$$u = -r^{1/2} \quad \to du = -\tfrac{1}{2}r^{-1/2}dr$$

$$\Rightarrow -2du = \frac{1}{r^{1/2}}dr$$

$$\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}dr = \int e^{u}(-2du)$$

$$= -2e^{u} + C$$

$$= -2e^{-\sqrt{r}} + C$$

Exercise

Evaluate the integral $\int t^3e^{t^4}dt$

Solution

$$\int t^3 e^{t^4}\,dt = \frac{1}{4}\int e^{t^4}\,d\left(t^4\right)$$

$$= \boxed{\frac{1}{4}e^{t^4} + C}$$

Exercise

Evaluate the integral $\displaystyle\int e^{\sec \pi t}\sec\pi\ \tan\pi t\ dt$

Solution

$u = \sec \pi t \quad \to du = \pi \sec \pi t \tan \pi t\ dt$

$\frac{1}{\pi}du = \sec \pi t \tan \pi t\ dt$

$$\int e^{\sec \pi t}\sec\pi\ \tan\pi t\ dt = \frac{1}{\pi}\int e^u du$$

$$= \frac{1}{\pi}e^u + C$$

$$= \boxed{\frac{1}{\pi}e^{\sec \pi t} + C}$$

$d(\sec \pi t) = \pi \sec \pi t \tan \pi t\ dt$

$$\int e^{\sec \pi t}\sec\pi\ \tan\pi t\ dt = \frac{1}{\pi}\int e^{\sec \pi t} d(\sec \pi)$$

$$= \boxed{\frac{1}{\pi}e^{\sec \pi t} + C}$$

Exercise

Find the integral $\displaystyle\int (2x+1)e^{x^2+x}dx$

Solution

$$\int (2x+1)e^{x^2+x}\ dx = \int e^{x^2+x}\ d\left(x^2+x\right)$$

$$\boxed{= e^{x^2+x} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{1+e^x}$

Solution

$$\int \frac{dx}{1+e^x} = \int \frac{e^{-x}}{e^{-x}}\frac{dx}{1+e^x}$$

$$= \int \frac{e^{-x}dx}{e^{-x}+1}$$

$$= -\int \frac{1}{e^{-x}+1} d\left(e^{-x}+1\right)$$

$$= \underline{-\ln\left(e^{-x}+1\right)+C}$$

Exercise

Find the integral $\int \frac{e^x}{1+e^x}dx$

Solution

$u = 1+e^x \Rightarrow du = e^x dx$

$$\int \frac{1}{u}du = \ln|u| + C$$

$$= \underline{\ln(1+e^x)+C}$$

$$\int \frac{e^x}{1+e^x}dx = \int \frac{1}{1+e^x} d\left(1+e^x\right)$$

$$= \underline{\ln(1+e^x)+C}$$

Exercise

Find the integral $\int \frac{2}{e^{-x}+1}dx$

Solution

$$\int \frac{2}{e^{-x}+1}dx = \int \frac{2}{e^{-x}+1}\frac{e^x}{e^x}dx$$

$$= 2\int \frac{e^x}{1+e^x}dx$$

$$= 2\int \frac{d(e^x+1)}{1+e^x}$$

$$= \underline{2\ln(e^x+1)+C}$$

Exercise

Find the integral $\int \frac{1}{x^3} e^{1/4x^2} dx$

Solution

$$u = \frac{1}{4x^2} = \frac{1}{4}x^{-2} \Rightarrow du = -\frac{1}{2}x^{-3}dx$$

$$\Rightarrow -2du = \frac{1}{x^3}dx$$

$$\int e^u (-2)du = -2\int e^u du$$

$$= -2e^u + C$$

$$= -2e^{1/4x^2} + C$$

$$d\left(\frac{1}{4}x^{-2}\right) = -\frac{1}{2}x^{-3}dx$$

$$\int \frac{1}{x^3} e^{1/4x^2} dx = -2\int e^{1/4x^2} d\left(\frac{1}{4x^2}\right)$$

$$= -2e^{1/4x^2} + C$$

Exercise

Find the integral $\int \frac{e^{1/\sqrt{x}}}{x^{3/2}} dx$

Solution

$$u = \frac{1}{\sqrt{x}} = x^{-1/2} \Rightarrow du = -\frac{1}{2}x^{-3/2}dx \Rightarrow -2du = \frac{1}{x^{3/2}}dx$$

$$\int \frac{e^{1/\sqrt{x}}}{x^{3/2}} dx = \int e^u (-2du)$$

$$= -2\int e^u du$$

$$= -2e^u + C$$

$$= -2e^{1/\sqrt{x}} + C$$

Exercise

Find the integral $\int \frac{-e^{3x}}{2 - e^{3x}} dx$

Solution

$$\int \frac{-e^{3x}}{2-e^{3x}} dx = \frac{1}{3}\int \frac{1}{2 - e^{3x}}\, d\left(2 - e^{3x}\right)$$

$$= \frac{1}{3}\ln\left|2 - e^{3x}\right| + C$$

Exercise

Evaluate the integral $\int \frac{7e^{7x}}{3+e^{7x}}dx$

Solution

$$u = 3 + e^{7x} \quad \rightarrow \quad du = 7e^{7x}dx$$

$$\int \frac{7e^{7x}}{3+e^{7x}}dx = \int \frac{du}{u}$$

$$= \ln|u|$$

$$= \ln\left(3 + e^{7x}\right) + C$$

$$\int \frac{7e^{7x}}{3+e^{7x}}dx = \int \frac{1}{3+e^{7x}}d\left(3 + e^{7x}\right)$$

$$= \ln\left(3 + e^{7x}\right) + C$$

Exercise

Find the integral $\int \frac{2(e^x - e^{-x})}{(e^x + e^{-x})^2}dx$

Solution

$$u = e^x + e^{-x} \Rightarrow du = (e^x - e^{-x})dx$$

$$\int \frac{2(e^x - e^{-x})}{(e^x + e^{-x})^2}dx = 2\int \frac{1}{u^2}du$$

$$= 2\int u^{-2}du$$

$$= 2\frac{u^{-1}}{-1} + C$$

$$= -2\frac{1}{u} + C$$

$$= -\frac{2}{e^x + e^{-x}} + C$$

Exercise

Evaluate the integral $\int \frac{3^x}{3-3^x}dx$

Solution

Let $u = 3 - 3^x \quad \rightarrow du = -3^x \ln 3 dx$

$$\Rightarrow -\frac{1}{\ln 3}du = 3^x dx$$

$$\int \frac{3^x}{3-3^x}dx = -\frac{1}{\ln 3}\int \frac{du}{u}$$

$$= -\frac{1}{\ln 3}\ln|u| + C$$

$$\boxed{= -\frac{1}{\ln 3}\ln\left|3-3^x\right| + C}$$

Exercise

Find the integral $\int \left(6x+e^x\right)\sqrt{3x^2+e^x}\,dx$

Solution

$$u = 3x^2 + e^x \Rightarrow du = (6x+e^x)dx$$

$$\frac{du}{6x+e^x} = dx$$

$$\int (6x+e^x)\sqrt{u}\,\frac{du}{6x+e^x} = \int u^{1/2}du$$

$$= \frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{3}u^{3/2} + C$$

$$\boxed{= \frac{2}{3}\left(3x^2+e^x\right)^{3/2} + C}$$

$$\int \left(6x+e^x\right)\sqrt{3x^2+e^x}\,dx = \int \left(3x^2+e^x\right)^{1/2} d\left(3x^2+e^x\right)$$

$$\boxed{= \frac{2}{3}\left(3x^2+e^x\right)^{3/2} + C}$$

Exercise

Evaluate the integral $\int \frac{x\,2^{x^2}}{1+2^{x^2}}dx$

Solution

Let $u = 1+2^{x^2} \quad \Rightarrow \quad du = 2x2^{x^2}\ln(2)\,dx$

$$\Rightarrow \frac{du}{2\ln 2} = x2^{x^2}\,dx$$

$$\int \frac{x2^{x^2}}{1+2^{x^2}}dx = \frac{1}{2\ln 2}\int \frac{du}{u}$$

$$= \frac{1}{2\ln 2}\ln u + C$$

$$\boxed{= \frac{1}{2\ln 2}\ln\left(1+2^{x^2}\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{x\left(\log_8 x\right)^2}$

Solution

$$\int \frac{dx}{x\left(\log_8 x\right)^2} = \int \frac{dx}{x\left(\frac{\ln x}{\ln 8}\right)^2}$$

$$= (\ln 8)^2 \int \frac{dx}{x(\ln x)^2} \qquad d(\ln x) = \frac{1}{x}dx$$

$$= (\ln 8)^2 \int \frac{d(\ln x)}{(\ln x)^2}$$

$$= -(\ln 8)^2 \frac{1}{\ln x} + C$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{x\sqrt{25x^2 - 2}}$

Solution

Let $u = 5x \quad \Rightarrow du = 5dx \rightarrow \frac{1}{5}du = dx$

$$\int \frac{dx}{x\sqrt{25x^2 - 2}} = \int \frac{du/5}{\frac{u}{5}\sqrt{u^2 - 2}}$$

$$= \int \frac{du}{u\sqrt{u^2 - \left(\sqrt{2}\right)^2}} \qquad \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}$$

$$= \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{u}{\sqrt{2}}\right| + C$$

$$= \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{5x}{\sqrt{2}}\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{6dr}{\sqrt{4 - (r+1)^2}}$

Solution

$u = r + 1 \quad \Rightarrow du = dr$

$a^2 = 4 \qquad \to a = 2$

$$\int \frac{6dr}{\sqrt{4-(r+1)^2}} = 6\int \frac{du}{\sqrt{4-u^2}} \qquad\qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$= 6\sin^{-1}\frac{u}{2} + C$$

$$\boxed{= 6\sin^{-1}\left(\frac{r+1}{2}\right) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{2+(x-1)^2}$

Solution

$u = x - 1 \quad \Rightarrow du = dx$

$a^2 = 2 \qquad \to a = \sqrt{2}$

$$\int \frac{dx}{2+(x-1)^2} = \int \frac{du}{2+u^2} \qquad\qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$\boxed{= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{\sec^2 y\, dy}{\sqrt{1-\tan^2 y}}$

Solution

$$\int \frac{\sec^2 y\, dy}{\sqrt{1-\tan^2 y}} = \int \frac{du}{\sqrt{1-u^2}}$$

$u = \tan y \quad \Rightarrow du = \sec^2 y dy$

$a^2 = 1 \qquad \to a = 1$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$= \sin^{-1}(u) + C$$

$$\boxed{= \sin^{-1}(\tan y) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{-x^2+4x-3}}$

Solution

$$\int \frac{dx}{\sqrt{-x^2+4x-3}} = \int \frac{dx}{\sqrt{1-x^2+4x-3-1}}$$

$$= \int \frac{dx}{\sqrt{1-\left(x^2-4x+4\right)}}$$

$$= \int \frac{dx}{\sqrt{1-(x+2)^2}} \qquad u = x+2 \quad \Rightarrow du = dx$$

$$= \int \frac{du}{\sqrt{1-u^2}} \int \frac{dx}{\sqrt{2x-x^2}} \qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$= \sin^{-1} u + C$$

$$\boxed{= \sin^{-1}(x-2) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{2x-x^2}}$

Solution

$$\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1+2x-x^2-1}}$$

$$= \int \frac{dx}{\sqrt{1-\left(x^2-2x+1\right)}}$$

$$= \int \frac{dx}{\sqrt{1-(x-1)^2}} \qquad u = x-1 \quad \Rightarrow du = dx$$

$$= \int \frac{du}{\sqrt{1-u^2}} \qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$\boxed{= \sin^{-1}(x-1) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x-2}{x^2-6x+10}dx$

Solution

$$\int \frac{x-2}{x^2-6x+10}dx = \int \frac{x-2}{x^2-6x+9+1}dx$$

$$= \int \frac{x-2-1+1}{(x-3)^2+1}dx$$

$$= \int \frac{x-3+1}{(x-3)^2+1}dx \qquad u = x-3 \quad \Rightarrow \quad du = dx$$

$$= \int \frac{u+1}{u^2+1}du$$

$$= \int \frac{u}{u^2+1}du + \int \frac{1}{u^2+1}du \qquad w = u^2+1 \quad \Rightarrow dw = 2udu \quad \to \tfrac{1}{2}dw = udu$$

$$= \frac{1}{2}\int \frac{dw}{w} + \int \frac{1}{u^2+1}du$$

$$= \frac{1}{2}\ln w + \tan^{-1}u + C$$

$$= \frac{1}{2}\ln\left((x-3)^2+1\right) + \tan^{-1}(x-3) + C$$

$$= \frac{1}{2}\ln\left(x^2-6x+10\right) + \tan^{-1}(x-3) + C$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

Solution

$$\int \frac{dx}{(x+1)\sqrt{x^2+2x}} = \int \frac{dx}{(x+1)\sqrt{x^2+2x+1-1}}$$

$$= \int \frac{dx}{(x+1)\sqrt{(x+1)^2-1}} \qquad \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u|$$

$$= \sec^{-1}|x+1| + C$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

Solution

$$\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}} = \int \frac{dx}{(x-2)\sqrt{x^2-4x+4-1}}$$

$$= \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}}$$

$$= \int \frac{du}{u\sqrt{u^2-1}} \qquad \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u|$$

$$= \sec^{-1} u + C$$

$$\boxed{= \sec^{-1}|x-2| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{e^{\cos^{-1}x}dx}{\sqrt{1-x^2}}$

Solution

$$\int \frac{e^{\cos^{-1}x}dx}{\sqrt{1-x^2}} = -\int e^{\cos^{-1}x}\, d\left(\cos^{-1}x\right) \qquad d\left(\cos^{-1}x\right) = \frac{-1}{\sqrt{1-x^2}}dx$$

$$\boxed{= -e^{\cos^{-1}x} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{\left(\sin^{-1}x\right)^2 dx}{\sqrt{1-x^2}}$

Solution

$$\int \frac{\left(\sin^{-1}x\right)^2 dx}{\sqrt{1-x^2}} = \int \left(\sin^{-1}x\right)^2 d\left(\sin^{-1}x\right) \qquad d\left(\sin^{-1}x\right) = \frac{dx}{\sqrt{1-x^2}}$$

$$\boxed{= \frac{1}{3}\left(\sin^{-1}x\right)^3 + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dy}{\left(\sin^{-1} y\right)\sqrt{1+y^2}}$

Solution

$$\int \frac{dy}{\left(\sin^{-1} y\right)\sqrt{1+y^2}} = \int \frac{1}{\sin^{-1} y}\, d\left(\sin^{-1} y\right) \qquad d\left(\sin^{-1} y\right) = \frac{dy}{\sqrt{1-y^2}}$$

$$= \ln\left|\sin^{-1} y\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{1}{\sqrt{x}\,(x+1)\left(\left(\tan^{-1}\sqrt{x}\right)^2+9\right)}\,dx$

Solution

$$d\left(\tan^{-1}\sqrt{x}\right) = \frac{1}{2\sqrt{x}}\frac{1}{1+\left(\sqrt{x}\right)^2}\,dx$$

$$= \frac{1}{2\sqrt{x}\,(1+x)}\,dx$$

$$\int \frac{1}{\sqrt{x}\,(x+1)\left(\left(\tan^{-1}\sqrt{x}\right)^2+9\right)}\,dx = 2\int \frac{1}{\left(\tan^{-1}\sqrt{x}\right)^2+9}\, d\left(\tan^{-1}\sqrt{x}\right)$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{\tan^{-1}\sqrt{x}}{3}\right) + C$$

Exercise

Evaluate the integral $\displaystyle\int 2x\left(x^2+1\right)^4 dx$

Solution

$$\int 2x\left(x^2+1\right)^4 dx = \int \left(x^2+1\right)^4 d\left(x^2+1\right) \qquad d\left(x^2+1\right) = 2x\,dx$$

$$= \frac{1}{5}\left(x^2+1\right)^5 + C$$

Exercise

Evaluate the integral $\displaystyle\int 8x\cos\left(4x^2+3\right)dx$

Solution

$$\int 8x\cos\left(4x^2+3\right)dx=\int\cos\left(4x^2+3\right)d\left(4x^2+3\right) \qquad d\left(4x^2+3\right)=8xdx$$

$$=\sin\left(4x^2+3\right)+C$$

Exercise

Evaluate the integral $\displaystyle\int \sin^3 x\cos x\,dx$

Solution

$$\int \sin^3 x\cos x\,dx=\int \sin^3 x\,d(\sin x) \qquad d(\sin x)=\cos xdx$$

$$=\frac{1}{4}\sin^4 x+C$$

Exercise

Evaluate the integral $\displaystyle\int (6x+1)\sqrt{3x^2+x}\,dx$

Solution

$$\int (6x+1)\sqrt{3x^2+x}\,dx=\int\left(3x^2+x\right)^{1/2}\,d\left(3x^2+x\right) \qquad d\left(3x^2+x\right)=(6x+1)dx$$

$$=\frac{2}{3}\left(3x^2+x\right)^{3/2}+C$$

Exercise

Evaluate the integral $\displaystyle\int 2x\left(x^2-1\right)^{99}dx$

Solution

$$\int 2x\left(x^2-1\right)^{99}dx=\int\left(x^2-1\right)^{99}d\left(x^2-1\right) \qquad d\left(x^2-1\right)=2xdx$$

$$=\frac{1}{100}\left(x^2-1\right)^{100}+C$$

Exercise

Evaluate the integral $\int xe^{x^2}dx$

Solution

$$\int xe^{x^2}dx = \frac{1}{2}\int e^{x^2}d\left(x^2\right) \qquad d\left(x^2\right) = 2xdx$$

$$\boxed{= \frac{1}{2}e^{x^2} + C}$$

Exercise

Evaluate the integral $\int \frac{2x^2}{\sqrt{1-4x^3}}dx$

Solution

$$\int \frac{2x^2}{\sqrt{1-4x^3}}dx = -\frac{1}{6}\int \left(1-4x^3\right)^{-1/2}d\left(1-4x^3\right) \qquad d\left(1-4x^3\right) = -12x^2dx$$

$$\boxed{= -\frac{1}{3}\sqrt{1-4x^3} + C}$$

Exercise

Evaluate the integral $\int \frac{\left(\sqrt{x}+1\right)^4}{2\sqrt{x}}dx$

Solution

$$\int \frac{\left(\sqrt{x}+1\right)^4}{2\sqrt{x}}dx = \int \left(\sqrt{x}+1\right)^4 d\left(\sqrt{x}+1\right) \qquad d\left(\sqrt{x}+1\right) = \frac{1}{2\sqrt{x}}dx$$

$$\boxed{= \frac{1}{5}\left(\sqrt{x}+1\right)^5 + C}$$

Exercise

Evaluate the integral $\int \left(x^2+x\right)^{10}(2x+1)dx$

Solution

$$\int \left(x^2+x\right)^{10}(2x+1)dx = \int \left(x^2+x\right)^{10}d\left(x^2+x\right) \qquad d\left(x^2+x\right) = (2x+1)dx$$

$$\boxed{= \frac{1}{11}\left(x^2+x\right)^{11} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{10x-3}$

Solution

$$\int \frac{dx}{10x-3} = \frac{1}{10}\int \frac{d(10x-3)}{10x-3} \qquad d(10x-3) = 10dx$$

$$\boxed{= \ln|10x-3| + C}$$

Exercise

Evaluate the integral $\displaystyle\int x^3\left(x^4+16\right)^6 dx$

Solution

$$\int x^3\left(x^4+16\right)^6 dx = \frac{1}{4}\int \left(x^4+16\right)^6 d\left(x^4+16\right) \qquad d\left(x^4+16\right) = 4x^3dx$$

$$\boxed{= \frac{1}{28}\left(x^4+16\right)^7 + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sin^{10}\theta\ \cos\theta\ d\theta$

Solution

$$\int \sin^{10}\theta\ \cos\theta\ d\theta = \int \sin^{10}\theta\ \ d(\sin\theta) \qquad d(\sin\theta) = \cos\theta d\theta$$

$$\boxed{= \frac{1}{11}\sin^{11}\theta + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{1-9x^2}}$

Solution

$$\int \frac{dx}{\sqrt{1-9x^2}} = \frac{1}{3}\int \frac{d(3x)}{\sqrt{1-(3x)^2}} \qquad d(3x) = 3dx \qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$\boxed{= \frac{1}{3}\arcsin 3x + C}$$

Exercise

Evaluate the integral $\displaystyle\int x^9 \sin x^{10}\, dx$

Solution

$$\int x^9 \sin x^{10}\, dx = \frac{1}{10}\int \sin x^{10}\, d\left(x^{10}\right) \qquad d\left(x^{10}\right) = 10x^9 dx$$

$$\boxed{= -\frac{1}{10}\cos x^{10} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \left(x^6 - 3x^2\right)^4 \left(x^5 - x\right) dx$

Solution

$$\int \left(x^6 - 3x^2\right)^4 \left(x^5 - x\right) dx = \frac{1}{6}\int \left(x^6 - 3x^2\right)^4\, d\left(x^6 - 3x^2\right) \qquad d\left(x^6 - 3x^2\right) = 6\left(x^5 - x\right) dx$$

$$\boxed{= \frac{1}{30}\left(x^6 - 3x^2\right)^5 + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x}{x-2}\, dx$

Solution

$$\begin{array}{r|l} & 1 \\ \hline x-2 & x \\ & -x+2 \\ \hline & \quad 2 \end{array}$$

$$\int \frac{x}{x-2}\, dx = \int \left(1 + \frac{2}{x-2}\right) dx$$

$$\boxed{= x + 2\ln|x-2| + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{1+4x^2}$

Solution

$$\int \frac{dx}{1+4x^2} = \frac{1}{2}\int \frac{d(2x)}{1+(2x)^2} \qquad d(2x) = 2dx \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\boxed{= \frac{1}{2}\arctan 2x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{3}{1+25y^2}dy$

Solution

$$\int \frac{3}{1+25y^2}dy = \frac{3}{5}\int \frac{d(5y)}{1+(5y)^2} \qquad d(5y) = 5dy \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{3}{5}\arctan 5y + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{2}{x\sqrt{4x^2-1}}dx \quad \left(x > \frac{1}{2}\right)$

Solution

$$\int \frac{2}{x\sqrt{4x^2-1}}dx = \int \frac{d(2x)}{x\sqrt{(2x)^2-1}} \qquad \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|$$

$$= \operatorname{arcsec}(2x) + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{8x+6}{2x^2+3x}dx$

Solution

$$\int \frac{8x+6}{2x^2+3x}dx = 2\int \frac{1}{2x^2+3x}d\left(2x^2+3x\right) \qquad 2d\left(2x^2+3x\right) = 2(4x+3)dx$$

$$= 2\ln\left|2x^2+3x\right| + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x}{\sqrt{x-4}}dx$

Solution

$u = x-4 \quad \to \quad x = u+4$

$dx = du$

$$\int \frac{x}{\sqrt{x-4}}dx = \int \frac{u+4}{u^{1/2}}\,du$$

$$= \int \left(u^{1/2} + 4u^{-1/2} \right) du$$

$$= \frac{2}{3} u^{3/2} + 8u^{1/2} + C$$

$$\boxed{= \frac{2}{3} (x-4)^{3/2} + 8(x-4)^{1/2} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^2}{(x+1)^4} dx$

Solution

$u = x+1 \quad \rightarrow \quad x = u - 1$

$dx = du$

$$\int \frac{x^2}{(x+1)^4} dx = \int \frac{(u-1)^2}{u^4} du$$

$$= \int \frac{u^2 - 2u + 1}{u^4} du$$

$$= \int \left(\frac{1}{u^2} - 2u^{-3} + u^{-4} \right) du$$

$$= -\frac{1}{u} + u^{-2} - \frac{1}{3} u^{-3} + C$$

$$\boxed{= -\frac{1}{x+1} + \frac{1}{(x+1)^2} - \frac{1}{3(x+1)^3} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x}{\sqrt[3]{x+4}} dx$

Solution

$u = x+4 \quad \rightarrow \quad x = u - 4$

$dx = du$

$$\int \frac{x}{\sqrt[3]{x+4}} dx = \int \frac{u-4}{u^{1/3}} du$$

$$= \int \left(u^{2/3} - 4u^{-1/3} \right) du$$

$$= \frac{3}{5} u^{5/3} - 6u^{2/3} + C$$

$$= \boxed{\frac{3}{5}(x+4)^{5/3} - 6(x+4)^{2/3} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

Solution

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx = \int \frac{1}{e^x + e^{-x}}\, d\left(e^x + e^{-x}\right) \qquad d\left(e^x + e^{-x}\right) = \left(e^x - e^{-x}\right)dx$$

$$= \boxed{\ln\left(e^x + e^{-x}\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int x\,\sqrt[3]{2x+1}\, dx$

Solution

$$u = 2x+1 \quad \rightarrow \quad x = \frac{1}{2}(u-1)$$

$$dx = \frac{1}{2}du$$

$$\int x\,\sqrt[3]{2x+1}\, dx = \int \frac{1}{2}(u-1)u^{1/3}\left(\frac{1}{2}du\right)$$

$$= \frac{1}{4}\int \left(u^{4/3} - u^{1/3}\right)du$$

$$= \frac{1}{4}\left(\frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3}\right) + C$$

$$= \boxed{\frac{3}{28}(2x+1)^{7/3} - \frac{3}{16}(2x+1)^{4/3} + C}$$

Exercise

Evaluate the integral $\displaystyle\int (x+1)\sqrt{3x+2}\, dx$

Solution

$$u = 3x+2 \quad \rightarrow \quad x = \frac{1}{3}(u-2)$$

$$dx = \frac{1}{3}du$$

$$\begin{aligned}\int (x+1)\sqrt{3x+2}\ dx &= \int \left(\tfrac{1}{3}u-2+1\right)u^{1/2}\ \tfrac{1}{3}du\\ &= \tfrac{1}{3}\int \left(\tfrac{1}{3}u-1\right)u^{1/2}\ du\\ &= \tfrac{1}{3}\int \left(\tfrac{1}{3}u^{3/2}-u^{1/2}\right)\ du\\ &= \tfrac{1}{3}\left(\tfrac{2}{15}u^{5/2}-\tfrac{2}{3}u^{3/2}\right)+C\\ &= \tfrac{2}{45}(3x+2)^{5/2}-\tfrac{2}{9}(3x+2)^{3/2}+C\end{aligned}$$

Exercise

Evaluate the integral $\int \sin^2 x\ dx$

Solution

$$\int \sin^2 x\ dx = \tfrac{1}{2}\int (1-\cos 2x)dx \qquad \sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta)$$

$$= \tfrac{1}{2}\left(x-\tfrac{1}{2}\sin 2x\right)+C$$

Exercise

Evaluate the integral $\int \sin^2\left(\theta+\frac{\pi}{6}\right)d\theta$

Solution

$$\int \sin^2\left(\theta+\tfrac{\pi}{6}\right)d\theta = \int \tfrac{1}{2}\left(1-\cos 2\left(\theta+\tfrac{\pi}{6}\right)\right)d\theta \qquad \sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta)$$

$$= \tfrac{1}{2}\int d\theta - \tfrac{1}{4}\int \cos\left(2\theta+\tfrac{\pi}{3}\right)\ d\left(2\theta+\tfrac{\pi}{3}\right) \qquad d\left(2\theta+\tfrac{\pi}{3}\right)=2d\theta$$

$$= \tfrac{1}{2}\theta-\tfrac{1}{4}\sin\left(2\theta+\tfrac{\pi}{3}\right)+C$$

Exercise

Evaluate the integral $\int x\cos^2\left(x^2\right)dx$

Solution

$$\int x\cos^2\left(x^2\right)dx = \frac{1}{2}\int \cos^2\left(x^2\right)\,d\left(x^2\right) \qquad d\left(x^2\right) = 2xdx$$

$$= \frac{1}{4}\int \left(1+\cos\left(2x^2\right)\right)\,d\left(x^2\right)$$

$$= \frac{1}{4}\int d\left(x^2\right) + \frac{1}{8}\int \cos\left(2x^2\right)\,d\left(2x^2\right) \qquad d\left(x^2\right) = 2d\left(x^2\right)$$

$$\boxed{= \frac{1}{4}x^2 + \frac{1}{8}\sin\left(2x^2\right) + C}$$

Or

$$\int x\cos^2\left(x^2\right)dx = \frac{1}{2}\int x\left(1+\cos\left(2x^2\right)\right)\,dx \qquad \cos^2\theta = \frac{1}{2}\left(1+\cos 2\theta\right)$$

$$= \frac{1}{2}\int x\,dx + \frac{1}{2}\int x\cos\left(2x^2\right)\,dx$$

$$= \frac{1}{4}x^2 + \frac{1}{8}\int \cos\left(2x^2\right)\,d\left(2x^2\right)$$

$$\boxed{= \frac{1}{4}x^2 + \frac{1}{8}\sin\left(2x^2\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sec 4x\ \tan 4x\ dx$

Solution

$$\int \sec 4x\ \tan 4x\ dx = \frac{1}{4}\int d\left(\sec 4x\right) \qquad d\left(\sec 4x\right) = 4\sec 4x\ \tan 4x$$

$$\boxed{= \frac{1}{4}\sec 4x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sec^2 10x\ dx$

Solution

$$\int \sec^2 10x\ dx = \frac{1}{10}\int \sec^2 10x\ d\left(10x\right)$$

$$\boxed{= \frac{1}{10}\tan 10x + C}$$

Exercise

Evaluate the integral $\int \left(\sin^5 x + 3\sin^3 x - \sin x\right)\cos x \, dx$

Solution

$$\int \left(\sin^5 x + 3\sin^3 x - \sin x\right)\cos x \, dx = \int \left(\sin^5 x + 3\sin^3 x - \sin x\right) d\left(\sin x\right) \qquad d\left(\sin x\right) = \cos x dx$$

$$= \frac{1}{6}\sin^6 x + \frac{3}{4}\sin^4 x - \frac{1}{2}\sin^2 x + C$$

Exercise

Evaluate the integral $\int \frac{\csc^2 x}{\cot^3 x} dx$

Solution

$$\int \frac{\csc^2 x}{\cot^3 x} dx = -\int \cot^{-3} x \, d\left(\cot x\right) \qquad d\left(\cot x\right) = -\csc^2 x dx$$

$$= \frac{1}{2}\cot^{-2} x + C$$

$$= \frac{1}{2\cot^2 x} + C$$

$$= \frac{1}{2}\tan^2 x + C$$

Exercise

Evaluate the integral $\int \left(x^{3/2} + 8\right)^5 \sqrt{x} \, dx$

Solution

$$\int \left(x^{3/2} + 8\right)^5 \sqrt{x} \, dx = \frac{2}{3}\int \left(x^{3/2} + 8\right)^5 d\left(x^{3/2} + 8\right) \qquad d\left(x^{3/2} + 8\right) = \frac{3}{2}x^{1/2} dx$$

$$= \frac{1}{9}\left(x^{3/2} + 8\right)^6 + C$$

Exercise

Evaluate the integral $\int \sin x \, \sec^8 x \, dx$

Solution

$$\int \sin x \, \sec^8 x \, dx = -\int \cos^{-8} x \, d\left(\cos x\right) \qquad d\left(\cos x\right) = -\sin x dx; \quad \sec x = \frac{1}{\cos x}$$

$$= \frac{1}{7}\cos^{-7} x + C$$

$$\boxed{= \frac{1}{7}\sec^{7} x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{e^{2x}}{e^{2x}+1}dx$

Solution

$$\int \frac{e^{2x}}{e^{2x}+1}dx = \frac{1}{2}\int \frac{1}{e^{2x}+1}d\left(e^{2x}+1\right) \qquad d\left(e^{2x}+1\right) = 2e^{2x}dx$$

$$\boxed{= \frac{1}{2}\ln\left(e^{2x}+1\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sec^3\theta \ \tan\theta \ d\theta$

Solution

$$\int \sec^3\theta \ \tan\theta \ d\theta = \int \sec^2\theta \ \sec\theta \tan\theta \ d\theta \qquad d(\sec\theta) = \tan\theta\sec\theta d\theta$$

$$= \int \sec^2\theta \ d(\sec\theta)$$

$$\boxed{= \frac{1}{3}\sec^3\theta + C}$$

Exercise

Evaluate the integral $\displaystyle\int x \ \sin^4 x^2 \ \cos x^2 \ dx$

Solution

$$\int x \ \sin^4 x^2 \ \cos x^2 \ dx = \frac{1}{2}\int \sin^4 x^2 \ d\left(\sin x^2\right) \qquad d\left(\sin x^2\right) = 2x\cos x^2 \ dx$$

$$\boxed{= \frac{1}{10}\sin^5\left(x^2\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{1+\sqrt{1+x}}}$

Solution

$$u = 1+\sqrt{1+x} \quad \rightarrow \quad \sqrt{1+x} = u-1$$

$$du = \frac{1}{2\sqrt{1+x}}dx \quad \rightarrow dx = 2(u-1)du$$

$$\begin{aligned}\int \frac{dx}{\sqrt{1+\sqrt{1+x}}} &= \int \frac{2(u-1)}{u^{1/2}}du \\ &= 2\int \left(u^{1/2} - u^{-1/2}\right)du \\ &= 2\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right) + C \\ &= \frac{4}{3}\left(1+\sqrt{1+x}\right)^{3/2} - 4\left(1+\sqrt{1+x}\right)^{1/2} + C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \tan^{10} 4x \ \sec^2 4x \ dx$

Solution

$$\begin{aligned}\int \tan^{10} 4x \ \sec^2 4x \ dx &= \frac{1}{4}\int \tan^{10} 4x \ \ d(\tan 4x) \qquad d(\tan 4x) = 4\sec^2(4x)dx \\ &= \frac{1}{44}\tan^{11} 4x + C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^2}{x^3+27}dx$

Solution

$$\begin{aligned}\int \frac{x^2}{x^3+27}dx &= \frac{1}{3}\int \frac{1}{x^3+27}d\left(x^3+27\right) \qquad d\left(x^3+27\right) = 3x^2dx \\ &= \frac{1}{3}\ln\left|x^3+27\right| + C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int y^2\left(3y^3+1\right)^4 dy$

Solution

$$\int y^2\left(3y^3+1\right)^4 dy=\frac{1}{9}\int\left(3y^3+1\right)^4 d\left(3y^3+1\right) \qquad d\left(3y^3+1\right)=9y^2dy$$

$$=\frac{1}{45}\left(3y^3+1\right)^5+C$$

Exercise

Evaluate the integral $\displaystyle\int x\sin x^2\cos^8 x^2\ dx$

Solution

$$\int x\sin x^2\cos^8 x^2\ dx=-\frac{1}{2}\int\cos^8 x^2\ d\left(\cos x^2\right) \qquad d\left(\cos x^2\right)=-2x\sin x^2\ dx$$

$$=-\frac{1}{18}\cos^9\left(x^2\right)+C$$

Exercise

Evaluate the integral $\displaystyle\int\frac{\sin 2x}{1+\cos^2 x}dx$

Solution

$$\int\frac{\sin 2x}{1+\cos^2 x}dx=-\int\frac{1}{1+\cos^2 x}d\left(1+\cos^2 x\right) \qquad d\left(1+\cos^2 x\right)=-2\cos x\sin x\ dx=-\sin 2x\ dx$$

$$=-\ln\left|1+\cos^2 x\right|+C$$

Exercise

Evaluate the integral $\displaystyle\int\frac{\sin^{-1}x}{\sqrt{1-x^2}}dx$

Solution

$$\int\frac{\sin^{-1}x}{\sqrt{1-x^2}}dx=\int\sin^{-1}x\ d\left(\sin^{-1}x\right) \qquad d\left(\sin^{-1}x\right)=\frac{dx}{\sqrt{1-x^2}}$$

$$=\frac{1}{2}\left(\sin^{-1}x\right)^2+C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\left(\tan^{-1} x\right)\left(1+x^2\right)}$

Solution

$$\int \frac{dx}{\left(\tan^{-1} x\right)\left(1+x^2\right)} = \int \frac{1}{\tan^{-1} x}\, d\left(\tan^{-1} x\right) \qquad d\left(\tan^{-1} x\right) = \frac{dx}{1+x^2}$$

$$\boxed{= \ln\left|\tan^{-1} x\right| + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\left(\tan^{-1} x\right)^5}{1+x^2}\, dx$

Solution

$$\int \frac{\left(\tan^{-1} x\right)^5}{1+x^2}\, dx = \int \left(\tan^{-1} x\right)^5\, d\left(\tan^{-1} x\right) \qquad d\left(\tan^{-1} x\right) = \frac{dx}{1+x^2}$$

$$\boxed{= \frac{1}{6}\left(\tan^{-1} x\right)^6 + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{1}{x^2} \sin\frac{1}{x}\, dx$

Solution

$$\int \frac{1}{x^2} \sin\frac{1}{x}\, dx = -\int \sin\frac{1}{x}\, d\left(\frac{1}{x}\right) \qquad d\left(\frac{1}{x}\right) = -\frac{1}{x^2}dx$$

$$\boxed{= \cos\frac{1}{x} + C}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{2} x^2 e^{x^3+1} dx$

Solution

$$\int_{-1}^{2} x^2 e^{x^3+1}\, dx = \frac{1}{3}\int_{-1}^{2} e^{x^3+1}\, d\left(x^3+1\right) \qquad d\left(x^3+1\right) = 3x^2 dx$$

$$= \frac{1}{3} e^{x^3+1} \Big|_{-1}^{2}$$

$$= \frac{1}{3}\left(e^9 - 1\right)$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 x^2 e^{x^3} dx$

Solution

$$\int_0^2 x^2 e^{x^3}\, dx = \frac{1}{3}\int_0^2 e^{x^3}\, d\left(x^3\right) \qquad d\left(x^3\right) = 3x^2 dx$$

$$= \frac{1}{3} e^{x^3} \Big|_0^2$$

$$= \frac{1}{3}\left(e^8 - e\right)$$

Exercise

Evaluate the integral $\displaystyle\int_0^4 \frac{x}{x^2+1} dx$

Solution

$$\int_0^4 \frac{x}{x^2+1}\, dx = \frac{1}{2}\int_0^4 \frac{1}{x^2+1}\, d\left(x^2+1\right)$$

$$= \frac{1}{2}\ln\left(x^2+1\right) \Big|_0^4$$

$$= \frac{1}{2}(\ln 17 - \ln 1)$$

$$= \frac{1}{2}\ln 17$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{18\tan^2 x\sec^2 x}{\left(2+\tan^3 x\right)^2}dx$

a) $u=\tan x$, followed by $v=u^3$ *then by* $w=2+v$

b) $u=\tan^3 x$, followed by $v=2+u$

c) $u=2+\tan^3 x$

Solution

a) Let $u=\tan x \quad\Rightarrow\quad du=\sec^2 x dx$

$v=u^3 \quad\Rightarrow\quad dv=3u^2 du$

$w=2+v \quad\Rightarrow\quad dw=dv$

$$\int \frac{18\tan^2 x\sec^2 x}{\left(2+\tan^3 x\right)^2}dx=\int \frac{18u^2 du}{\left(2+u^3\right)^2}$$

$$=\int \frac{6dv}{(2+v)^2}$$

$$=\int \frac{6dw}{w^2}$$

$$=6\int w^{-2}dw$$

$$=6\frac{w^{-1}}{-1}+C$$

$$=-\frac{6}{w}+C$$

$$=-\frac{6}{2+v}+C$$

$$=-\frac{6}{2+u^3}+C$$

$$\boxed{=-\frac{6}{2+\tan^3 x}+C}$$

b) Let $u=\tan^3 x \quad\Rightarrow\quad du=3\tan^2 x\sec^2 x dx$

$v=2+u \quad\Rightarrow\quad dv=du$

$$\int \frac{18\tan^2 x\sec^2 x}{\left(2+\tan^3 x\right)^2}dx=\int \frac{6du}{(2+u)^2}$$

$$= \int \frac{6dv}{v^2}$$

$$= \int 6v^{-2}dv$$

$$= -6v^{-1} + C$$

$$= -\frac{6}{v} + C$$

$$= -\frac{6}{2+u} + C$$

$$\boxed{= -\frac{6}{2+\tan^3 x} + C}$$

c) Let $u = 2 + \tan^3 x \quad \Rightarrow \quad du = 3\tan^2 x \sec^2 x dx \to \frac{1}{3}du = \tan^2 x \sec^2 x dx$

$du = 3\tan^2 x \sec^2 x dx \ \to \ \frac{1}{3}du = \tan^2 x \sec^2 x dx$

$$\int \frac{18\tan^2 x \sec^2 x}{\left(2+\tan^3 x\right)^2} dx = \int \frac{18}{u^2}\left(\frac{1}{3}du\right)$$

$$= 6\int u^{-2}du$$

$$= -6u^{-1} + C$$

$$= -\frac{6}{u} + C$$

$$\boxed{= -\frac{6}{2+\tan^3 x} + C}$$

Exercise

Evaluate: $\displaystyle\int_0^1 (2t+3)^3\, dt$

Solution

$$\int_0^1 (2t+3)^3\, dt = \frac{1}{2}\int_0^1 (2t+3)^3\, d(2t+3) \qquad d(2t+3) = 2dt \ \to \ \frac{1}{2}d(2t+3) = dt$$

$$= \frac{1}{8}(2t+3)^4 \Big|_0^1$$

$$= \frac{1}{8}\left[(2(1)+3)^4 - (2(0)+3)^4\right]$$

$$= \frac{1}{8}\left[5^4 - 3^4\right]$$

$$\boxed{= 68}$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$

Solution

$$\int_0^2 \sqrt{4-x^2}\,dx = \left[\frac{1}{2}x\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$\sqrt{4-x^2}$ is a semi-circle with center (0, 0) and radius = 2. Since x from 0 to 2

$\Rightarrow$ Area $= \frac{1}{4}$(Area of this circle)

$$= \frac{1}{4}2\pi 2^2$$

$$\boxed{= 2\pi}$$

Exercise

Evaluate the integral $\displaystyle\int_0^3 \sqrt{y+1}\;dy$

Solution

$$\int_0^3 \sqrt{y+1}\;dy = \int_0^3 (y+1)^{1/2}\;d(y+1) \qquad\qquad d(y+1) = dy$$

$$= \frac{2}{3}(y+1)^{3/2}\Big|_0^3$$

$$= \frac{2}{3}\left[(3+1)^{3/2} - (0+1)^{3/2}\right]$$

$$= \frac{2}{3}[8-1]$$

$$\boxed{= \frac{14}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^1 r\sqrt{1-r^2}\;dr$

Solution

$$\int_{-1}^1 r\sqrt{1-r^2}\;dr = -\frac{1}{2}\int_{-1}^1 \left(1-r^2\right)^{1/2}\;d\left(1-r^2\right) \qquad\qquad d\left(1-r^2\right) = -2rdr$$

$$= -\frac{1}{3}\left[\left(1-r^2\right)^{3/2}\right]_{-1}^{1}$$

$$= -\frac{1}{3}\left[\left(1-(1)^2\right)^{3/2} - \left(1-(-1)^2\right)^{3/2}\right]$$

$$= -\frac{1}{3}[0-0]$$

$$\underline{= 0}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/4} \tan x \sec^2 x\, dx$

Solution

$$\int_0^{\pi/4} \tan x \sec^2 x\, dx = \int_0^{\pi/4} \tan x\, d(\tan x) \qquad d(tanx) = \sec^2 x\, dx$$

$$= \frac{1}{2}\tan^2 x \Big|_0^{\pi/4}$$

$$= \frac{1}{2}\left[1^2 - 0^2\right]$$

$$\underline{= \frac{1}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\, dx$

Solution

$$\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\, dx = -\int_{2\pi}^{3\pi} 3\cos^2 x\, d(\cos x) \qquad d(\cos x) = -\sin x\, dx$$

$$= -\cos^3 x \Big|_{2\pi}^{3\pi}$$

$$= -\left[(-1)^3 - 1^3\right]$$

$$\underline{= 2}$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 t^3\left(1+t^4\right)^3\,dt$

Solution

$$\int_0^1 t^3\left(1+t^4\right)^3\,dt = \frac{1}{4}\int_0^1 \left(1+t^4\right)^3 d\left(1+t^4\right) \qquad d\left(1+t^4\right)=4t^3\,dt$$

$$= \frac{1}{16}\left(1+t^4\right)^4 \Big|_0^1$$

$$= \frac{1}{16}\left(2^4-1^4\right)$$

$$= \frac{15}{16}$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 \frac{r}{\left(4+r^2\right)^2}\,dr$

Solution

$$\int_0^1 \frac{r}{\left(4+r^2\right)^2}\,dr = \frac{1}{2}\int_0^1 \frac{d\left(4+r^2\right)}{\left(4+r^2\right)^2} \qquad d\left(4+r^2\right)=2rdr$$

$$= -\frac{1}{2}\left[\frac{1}{4+r^2}\right]_0^1$$

$$= -\frac{1}{2}\left(\frac{1}{5}-\frac{1}{4}\right)$$

$$= -\frac{1}{40}$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 \frac{10\sqrt{v}}{\left(1+v^{3/2}\right)^2}\,dv$

Solution

$$\int_0^1 \frac{10\sqrt{v}}{\left(1+v^{3/2}\right)^2}\,dv = \frac{20}{3}\int_0^1 \frac{d\left(1+v^{3/2}\right)}{\left(1+v^{3/2}\right)^2} \qquad d\left(1+v^{3/2}\right)=\frac{3}{2}\sqrt{v}\,dv$$

$$= -\frac{20}{3}\left[\frac{1}{1+v^{3/2}}\right]_0^1$$

$$= -\frac{20}{3}\left(\frac{1}{2}-1\right)$$

$$\underline{= \frac{10}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$

Solution

Let $u = x^2+1$

$du = 2xdx \quad \rightarrow \quad \frac{1}{2}du = xdx$

$$\rightarrow \begin{cases} x = \sqrt{3} & \rightarrow u = 4 \\ x = -\sqrt{3} & \rightarrow u = 4 \end{cases}$$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx = \int_4^4 4u^{-1}\left(\frac{1}{2}du\right)$$

$$\underline{= 0}$$

OR

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx = 2\int_{-\sqrt{3}}^{\sqrt{3}} \left(x^2+1\right)^{-1/2} d\left(x^2+1\right)$$

$$= 4\sqrt{x^2+1}\,\Big|_{-\sqrt{3}}^{\sqrt{3}}$$

$$= 4(2-2)$$

$$\underline{= 0}$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$

Solution

$u = x^4+9$

$$du = 4x^3dx \rightarrow \frac{1}{4}du = x^3dx \rightarrow \begin{cases} x = 1 \rightarrow u = 10 \\ x = 0 \rightarrow u = 9 \end{cases}$$

$$\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx = \frac{1}{4}\int_3^{10} u^{-1/2}du$$

$$= \frac{1}{4}\left[2u^{1/2}\right]_9^{10}$$

$$= \frac{1}{2}\left[10^{1/2} - 9^{1/2}\right]$$

$$=\boxed{\frac{\sqrt{10}-3}{2}}$$

Or

$$\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx = \frac{1}{4}\int_0^1 \left(x^4+9\right)^{-1/2} d\left(x^4+9\right)$$

$$=\frac{1}{2}\left(x^4+9\right)^{1/2}\Big|_0^1$$

$$=\frac{1}{2}\left[10^{1/2}-9^{1/2}\right]$$

$$=\boxed{\frac{\sqrt{10}-3}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/6} (1-\cos 3t)\sin 3t\ dt$

Solution

$$\int_0^{\pi/6} (1-\cos 3t)\sin 3t\ dt = \frac{1}{3}\int_0^{\pi/6} (1-\cos 3t)\ d(1-\cos 3t)$$

$$=\frac{1}{6}\left[(1-\cos 3t)^2\right]_0^{\pi/6}$$

$$=\frac{1}{6}\left(1^2-0^2\right)$$

$$=\boxed{\frac{1}{6}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\ dt$

Solution

$$d\left(2+\tan\frac{t}{2}\right)=\frac{1}{2}\sec^2\frac{t}{2}\ dt$$

$$\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\ dt = 2\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right) d\left(2+\tan\frac{t}{2}\right)$$

$$= \left(2+\tan\frac{t}{2}\right)^2 \Bigg|_{-\pi/2}^{\pi/2}$$

$$= 3^2 - 1$$

$$\underline{= 8}$$

Or

$$u = 2 + \tan\frac{t}{2}$$

$$du = \frac{1}{2}\sec^2\frac{t}{2}\ dt \quad \to \quad 2du = \sec^2\frac{t}{2}\ dt$$

$$\begin{cases} t = \frac{\pi}{2} & \to u = 3 \\ t = -\frac{\pi}{2} & \to u = 1 \end{cases}$$

$$\int_{-\pi/2}^{\pi/2}\left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\ dt = \int_1^3 u(2du)$$

$$= 2\left[\frac{u^2}{2}\right]_1^3$$

$$= \left[3^2 - 1^2\right]$$

$$\underline{= 8}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi}^{\pi}\frac{\cos z}{\sqrt{4+3\sin z}}dz$

Solution

Let $u = 4 + 3\sin z$

$$du = 3\cos z\ dz \quad \to \quad \frac{1}{3}du = \cos z\ dz \qquad \begin{cases} z = \pi & \to u = 4 \\ z = -\pi & \to u = 4 \end{cases}$$

$$\int_{-\pi}^{\pi}\frac{\cos z}{\sqrt{4+3\sin z}}dz = \int_4^4 \frac{1}{\sqrt{u}}\frac{1}{3}du$$

$$\underline{= 0}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi/2}^{0}\frac{\sin w}{(3+2\cos w)^2}dw$

Solution

$$\int_{-\pi/2}^{0} \frac{\sin w}{(3+2\cos w)^2}dw = -\frac{1}{2}\int_{-\pi/2}^{0} \frac{d(3+2\cos w)}{(3+2\cos w)^2}$$

$$d\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$= \frac{1}{2}\left[\frac{1}{3+2\cos w}\right]_{-\pi/2}^{0}$$

$$= \frac{1}{2}\left(\frac{1}{5}-\frac{1}{3}\right)$$

$$\boxed{= -\frac{1}{15}}$$

Exercise

Evaluate the integral $\int_0^1 \sqrt{t^5+2t}\,\left(5t^4+2\right)dt$

Solution

$$\int_0^1 \sqrt{t^5+2t}\,\left(5t^4+2\right)dt = \int_0^1 \left(t^5+2t\right)^{1/2}\, d\left(t^5+2t\right)$$

$$= \frac{2}{3}\left(t^5+2t\right)^{3/2}\Big|_0^1$$

$$= \frac{2}{3}\left(3^{3/2}\right)$$

$$\boxed{= 2\sqrt{3}}$$

Exercise

Evaluate the integral $\int_1^4 \frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2}$

Solution

$$\int_1^4 \frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2} = \int_1^4 \frac{1}{\left(1+\sqrt{y}\right)^2}\, d\left(1+\sqrt{y}\right)$$

$$d\left(1+\sqrt{y}\right) = \frac{1}{2\sqrt{y}}dy$$

$$= -\frac{1}{1+\sqrt{y}}\Big|_1^4$$

$$= -\left(\frac{1}{3}-\frac{1}{2}\right)$$

$$\boxed{= \frac{1}{6}}$$

Exercise

Evaluate the integral $\int_0^1 \left(4y - y^2 + 4y^3 + 1\right)^{-2/3}\left(12y^2 - 2y + 4\right)dy$

Solution

Let $\quad u = 4y - y^2 + 4y^3 + 1 \quad \Rightarrow \quad du = \left(4 - 2y + 12y^2\right)dy$

$$\rightarrow \begin{cases} y = 1 & \rightarrow u = 8 \\ y = 0 & \rightarrow u = 1 \end{cases}$$

$$\begin{aligned}\int_0^1 \left(4y - y^2 + 4y^3 + 1\right)^{-2/3}\left(12y^2 - 2y + 4\right)dy &= \int_1^8 u^{-2/3}du \\ &= 3u^{1/3}\Big|_1^8 \\ &= 3\left(8^{1/3} - 1^{1/3}\right) \\ &= 3\end{aligned}$$

Exercise

Evaluate the integral $\int_0^5 |x-2|\,dx$

Solution

$$|x-2| = \begin{cases} x-2 & x > 2 \\ -(x-2) & x < 2 \end{cases}$$

$$\begin{aligned}\int_0^5 |x-2|\,dx &= \int_0^2 -(x-2)dx + \int_2^5 (x-2)dx \\ &= -\frac{x^2}{2} + 2x\Big|_0^2 + \left(\frac{x^2}{2} - 2x\right)\Big|_2^5 \\ &= -\frac{4}{2} + 4 - 0 + \left(\frac{25}{2} - 10 - \left(\frac{4}{2} - 4\right)\right) \\ &= -2 + 4 + \frac{25}{2} - 10 - 2 + 4 \\ &= \frac{25}{2} - 6 \\ &= \frac{13}{2}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/2} e^{\sin x} \cos x \, dx$

Solution

$$\int_0^{\pi/2} e^{\sin x} \cos x \, dx = \int_0^{\pi/2} e^{\sin x} d(\sin x) \qquad d(\sin x) = \cos x \, dx$$

$$= e^{\sin x} \Big]_0^{\pi/2}$$

$$= e^{\sin\frac{\pi}{2}} - e^{\sin 0}$$

$$= e^1 - e^0$$

$$\underline{= e - 1}$$

Exercise

Evaluate $\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$

Solution

$$\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big|_{\sqrt{2}/2}^{\sqrt{3}/2} \qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$\underline{= \frac{\pi}{12}}$$

Exercise

Evaluate the integral $\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta} d\theta$

Solution

$$\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta} d\theta = \int_0^{\pi/3} \frac{d(1-4\cos\theta)}{1-4\cos\theta}$$

$$= \ln\left|1-4\cos\theta\right| \Big|_0^{\pi/3}$$

$$= \left(\ln\left|1-4\cos\frac{\pi}{3}\right| - \ln\left|1-4\cos 0\right|\right)$$

$$= \ln|-1| - \ln|-3|$$

$$= \ln 1 - \ln 3$$

$$= -\ln 3$$

$$\boxed{= \frac{1}{\ln 3}}$$

Exercise

Evaluate the integral $\displaystyle\int_1^2 \frac{2\ln x}{x}dx$

Solution

$$u = \ln x \quad \to du = \frac{1}{x}dx \quad \to \begin{cases} x = 1 & u = \ln 1 = 0 \\ x = 2 & u = \ln 2 \end{cases}$$

$$\int_1^2 \frac{2\ln x}{x}dx = \int_0^{\ln 2} 2udu$$

$$= u^2 \Big|_0^{\ln 2}$$

$$\boxed{= (\ln 2)^2}$$

Exercise

Evaluate the integral $\displaystyle\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$

Solution

$$u = \ln x \quad \to du = \frac{1}{x}dx \quad \to \begin{cases} x = 2 & u = \ln 2 \\ x = 16 & u = \ln 16 = \ln 2^4 \end{cases}$$

$$\int_2^{16} \frac{dx}{2x\sqrt{\ln x}} = \int_{\ln 2}^{4\ln 2} \frac{1}{2}u^{-1/2}du$$

$$= u^{1/2} \Big|_{\ln 2}^{4\ln 2}$$

$$=(4\ln 2)^{1/2}-(\ln 2)^{1/2}$$

$$=2\sqrt{\ln 2}-\sqrt{\ln 2}$$

$$=\sqrt{\ln 2}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/2} \tan\frac{x}{2}\,dx$

Solution

$$d\cos\frac{x}{2}=-\frac{1}{2}\sin\frac{x}{2}\,dx$$

$$\int_0^{\pi/2} \tan\frac{x}{2}\,dx=\int_0^{\pi/2} \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\,dx$$

$$=\int_0^{\pi/2} \frac{-2}{\cos\frac{x}{2}}\,d\cos\frac{x}{2}$$

$$=-2\ln\left|\cos\frac{x}{2}\right|\Big|_0^{\pi/2}$$

$$=-2\left[\ln\left|\cos\frac{\pi}{4}\right|-\ln\left|\cos 0\right|\right]$$

$$=-2\left[\ln\left|\frac{1}{\sqrt{2}}\right|-\ln|1|\right]$$

$$=-2\ln 2^{-1/2}$$

$$=\ln 2$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/4}^{\pi/2} \cot x\,dx$

Solution

$$\int_{\pi/4}^{\pi/2} \cot x\,dx=\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x}\,dx$$

$$=\int_{\pi/4}^{\pi/2} \frac{d(\sin x)}{\sin x}$$

$$= \ln(\sin x)\Big|_{\pi/4}^{\pi/2}$$

$$= \ln 1 - \ln\frac{1}{\sqrt{2}}$$

$$= -\ln\frac{1}{\sqrt{2}}$$

$$\boxed{= \ln\sqrt{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\ln 2}^{0} e^{-x}\, dx$

Solution

$$\int_{-\ln 2}^{0} e^{-x} dx = -e^{-x}\Big|_{-\ln 2}^{0}$$

$$= -\left(e^{0} - e^{\ln 2}\right)$$

$$= -(1-2)$$

$$\boxed{=1}$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/4}^{\pi/2} \left(1+e^{\cot\theta}\right)\csc^2\theta\, d\theta$

Solution

Let $u = \cot\theta \quad \Rightarrow du = -\csc^2\theta d\theta \quad \rightarrow \begin{cases} \theta = \frac{\pi}{2} & \Rightarrow u = 0 \\ \theta = \frac{\pi}{4} & \Rightarrow u = 1 \end{cases}$

$$\int_{\pi/4}^{\pi/2} \left(1+e^{\cot\theta}\right)\csc^2\theta\, d\theta = \int_{\pi/4}^{\pi/2} \csc^2\theta\, d\theta + \int_{\pi/4}^{\pi/2} e^{\cot\theta}\csc^2\theta\, d\theta$$

$$= -\cot\theta\Big|_{\pi/4}^{\pi/2} + \int_0^1 e^u du$$

$$= -\left(\cot\frac{\pi}{2} - \cot\frac{\pi}{4}\right) + e^u\Big|_0^1$$

$$= -(0-1) + e^1 - 1$$

$$= \boxed{e}$$

Exercise

Evaluate the integral $\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx$

Solution

Let $u=\sqrt{x} \quad \to du=\frac{1}{2\sqrt{x}}dx \;\Rightarrow 2du=\frac{1}{\sqrt{x}}dx \quad \to \begin{cases} x=1 & u=1 \\ x=4 & u=\sqrt{4}=2 \end{cases}$

$$\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx=\int_1^2 2^u(2du)$$

$$=2\int_1^2 2^u\,du$$

$$=2\left[\frac{2^u}{\ln 2}\right]_1^2$$

$$=\frac{2}{\ln 2}\left[2^2-2^1\right]$$

$$=\frac{2}{\ln 2}(2)$$

$$=\boxed{\frac{4}{\ln 2}}$$

Exercise

Evaluate the integral $\int_0^{\pi/4}\left(\frac{1}{3}\right)^{\tan t}\sec^2 t\,dt$

Solution

$u=\tan t \quad \Rightarrow du=\sec^2 tdt \quad \to \begin{cases} t=\frac{\pi}{4} & \to u=1 \\ t=0 & \to u=0 \end{cases}$

$$\int_0^{\pi/4}\left(\frac{1}{3}\right)^{\tan t}\sec^2 t\,dt=\int_0^1\left(\frac{1}{3}\right)^u du$$

$$=\left[\frac{1}{\ln\frac{1}{3}}\left(\frac{1}{3}\right)^u\right]_0^1$$

$$=\frac{1}{-\ln 3}\left[\frac{1}{3}-1\right]$$

$$= \frac{1}{-\ln 3}\left(\frac{-2}{3}\right)$$

$$\underline{= \frac{2}{3\ln 3}}$$

Exercise

Evaluate the integral $\int_1^e x^{(\ln 2)-1}dx$

Solution

$$\int_1^e x^{(\ln 2)-1}dx = \frac{x^{\ln 2}}{\ln 2}\Big|_1^e$$

$$= \frac{1}{\ln 2}\left(e^{\ln 2} - 1\right)$$

$$= \frac{1}{\ln 2}(2-1)$$

$$\underline{= \frac{1}{\ln 2}}$$

Exercise

Evaluate the integral $\int_1^e \frac{2\ln 10 \log_{10} x}{x}dx$

Solution

$$\int_1^e \frac{2\ln 10 \log_{10} x}{x}dx = 2\ln 10\int_1^e \frac{1}{x}\frac{\ln x}{\ln 10}dx$$

$$= 2\int_1^e \frac{\ln x}{x}dx \qquad d(\ln x) = \frac{1}{x}dx$$

$$= 2\int_1^e \ln x \ \ d(\ln x)$$

$$= 2\left[\frac{1}{2}(\ln x)^2\right]_1^e$$

$$= (\ln e)^2 - (\ln 1)^2$$

$$\underline{= 1}$$

Exercise

Evaluate the integral $\displaystyle\int_0^9 \frac{2\log_{10}(x+1)}{x+1}dx$

Solution

$$\int_0^9 \frac{2\log_{10}(x+1)}{x+1}dx = 2\int_0^9 \frac{1}{x+1}\frac{\ln(x+1)}{\ln 10}dx$$

$$= \frac{2}{\ln 10}\int_0^9 \frac{\ln(x+1)}{x+1}dx \qquad d(\ln(x+1)) = \frac{1}{x+1}dx$$

$$= \frac{2}{\ln 10}\int_0^9 \ln(x+1)\ d(x+1)$$

$$= \frac{2}{\ln 10}\left[\frac{1}{2}(\ln(x+1))^2\right]_0^9$$

$$= \frac{1}{\ln 10}\left[(\ln 10)^2 - (\ln 1)^2\right]$$

$$= \frac{1}{\ln 10}\left[(\ln 10)^2\right]$$

$$= \ln 10$$

Exercise

Evaluate the integral $\displaystyle\int_1^{e^x} \frac{1}{t}\ dt$

Solution

$$\int_1^{e^x} \frac{1}{t}dt = \ln|t|\Big|_1^{e^x}$$

$$= \ln\left|e^x\right| - \ln 1$$

$$= x$$

Exercise

Evaluate the integral $\displaystyle\frac{1}{\ln a}\int_1^x \frac{1}{t}\ dt \quad x > 0$

Solution

$$\frac{1}{\ln a}\int_1^x \frac{1}{t}\,dt = \frac{1}{\ln a}\Big[\ln|t|\ \Big]_1^x$$

$$= \frac{1}{\ln a}(\ln x - \ln 1)$$

$$= \frac{\ln x}{\ln a}$$

$$\boxed{= \log_a x}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\sqrt{\ln \pi}} 2x\, e^{x^2} \cos\left(e^{x^2}\right) dx$

Solution

$$\int_0^{\sqrt{\ln \pi}} 2x\, e^{x^2} \cos\left(e^{x^2}\right) dx = \int_0^{\sqrt{\ln \pi}} \cos\left(e^{x^2}\right) d\left(e^{x^2}\right)$$

$$= \sin\left(e^{x^2}\right)\Big|_0^{\sqrt{\ln \pi}}$$

$$= \sin \pi - \sin 1$$

$$\boxed{= -\sin 1} \qquad \approx \boxed{-0.84147}$$

Exercise

Evaluate $\displaystyle\int_0^{3\sqrt{2}/4} \frac{dx}{\sqrt{9-4x^2}}$

Solution

Let: $\quad u = 2x \quad \Rightarrow \quad du = 2dx \quad \to \quad \dfrac{du}{2} = dx$

$$\begin{cases} x = \dfrac{3\sqrt{2}}{4} & \to u = \dfrac{3\sqrt{2}}{2} \\ x = 0 & \to u = 0 \end{cases}$$

$$\int_0^{3\sqrt{2}/4} \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int_0^{3\sqrt{2}/2} \frac{du}{\sqrt{9-u^2}} \qquad\qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$= \frac{1}{2}\sin^{-1}\frac{u}{3}\Big|_0^{3\sqrt{2}/2}$$

$$= \frac{1}{2}\left(\sin^{-1}\frac{\sqrt{2}}{2} - \sin^{-1} 0\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi}{8}$$

Exercise

Evaluate $\displaystyle\int_{\pi/6}^{\pi 4} \frac{\csc^2 x dx}{1 + (\cot x)^2}$

Solution

$u = \cot x \quad \Rightarrow du = -\csc^2 x dx$

$a^2 = 1 \qquad \rightarrow a = 1$

$$\begin{cases} x = \frac{\pi}{4} & \rightarrow u = \cot\frac{\pi}{4} = 1 \\ x = \frac{\pi}{6} & \rightarrow u = \cot\frac{\pi}{6} = \sqrt{3} \end{cases}$$

$$\int_{\pi/6}^{\pi 4} \frac{\csc^2 x dx}{1 + (\cot x)^2} = -\int_{\sqrt{3}}^{1} \frac{du}{1 + u^2}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= -\tan^{-1} u \Big|_{\sqrt{3}}^{1}$$

$$= -\left(\tan^{-1} 1 - \tan^{-1}\sqrt{3}\right)$$

$$= -\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$$

$$= \frac{\pi}{12}$$

Exercise

Evaluate $\displaystyle\int_{1}^{e^{\pi/4}} \frac{4dt}{t\left(1 + \ln^2 t\right)}$

Solution

$u = \ln t \quad \Rightarrow du = \frac{dt}{t}$

$a^2 = 1 \quad \rightarrow a = 1$

$$\begin{cases} u = e^{\pi/4} & \rightarrow u = \ln e^{\pi/4} = \frac{\pi}{4} \\ u = 1 & \rightarrow u = \ln 1 = 0 \end{cases}$$

$$\int_1^{e^{\pi/4}} \frac{4dt}{t\left(1+\ln^2 t\right)} = 4\int_0^{\pi/4} \frac{du}{1+u^2}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \tan^{-1} u \Big|_0^{\pi/4}$$

$$= 4\left(\tan^{-1}\frac{\pi}{4} - \tan^{-1} 0\right)$$

$$\boxed{= 4\tan^{-1}\frac{\pi}{4}}$$

Exercise

Evaluate $\displaystyle\int_{1/2}^{1} \frac{6dx}{\sqrt{-4x^2+4x+3}}$

Solution

$$-4x^2+4x+3 = -4x^2+4x+3+1-1$$

$$4-4x^2+4x-1$$

$$4-\left(4x^2-4x+1\right)$$

$$2^2-(2x-1)^2$$

$$\int_{1/2}^{1} \frac{6dx}{\sqrt{-4x^2+4x+3}} = \int_{1/2}^{1} \frac{6dx}{\sqrt{2^2-(2x-1)^2}}$$

$$u = 2x-1 \quad \Rightarrow du = 2dx \to \frac{du}{2} = dx$$

$$= \int_{1/2}^{1} \frac{3du}{\sqrt{2^2-u^2}}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$= 3\sin^{-1}\left(\frac{2x-1}{2}\right)\Big|_{1/2}^{1}$$

$$= 3\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right]$$

$$= 3\left(\frac{\pi}{6} - 0\right)$$

$$\boxed{= \frac{\pi}{2}}$$

Exercise

Evaluate $\displaystyle\int_{2/\sqrt{3}}^{2} \frac{\cos\left(\sec^{-1} x\right) dx}{x\sqrt{x^2-1}}$

Solution

$$u = \sec^{-1} x \quad \Rightarrow du = \frac{dx}{x\sqrt{x^2-1}}$$

$$\begin{cases} x = 2 & \to u = \sec^{-1} 2 = \frac{\pi}{3} \\ x = \frac{2}{\sqrt{3}} & \to u = \sec^{-1} \frac{2}{\sqrt{3}} = \frac{\pi}{6} \end{cases}$$

$$\begin{aligned}
\int_{2/\sqrt{3}}^{2} \frac{\cos\left(\sec^{-1} x\right) dx}{x\sqrt{x^2-1}} &= \int_{\pi/6}^{\pi/3} \cos u \; du \\
&= \sin u \Big|_{\pi/6}^{\pi/3} \\
&= \sin\frac{\pi}{3} - \sin\frac{\pi}{6} \\
&= \frac{\sqrt{3}}{2} - \frac{1}{2} \\
&= \frac{\sqrt{3}-1}{2}
\end{aligned}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^3 \frac{x}{\sqrt{25-x^2}} dx$

Solution

$$\begin{aligned}
\int_0^3 \frac{x}{\sqrt{25-x^2}} dx &= -\frac{1}{2}\int_0^3 \left(25-x^2\right)^{-1/2} d\left(25-x^2\right) \qquad d\left(25-x^2\right) = -2x dx \\
&= -\left(25-x^2\right)^{1/2} \Big|_0^3 \\
&= -(4-5) \\
&= 1
\end{aligned}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\pi} \sin^2 5\theta \, d\theta$

Solution

$$\int_0^{\pi} \sin^2 5\theta \, d\theta = \frac{1}{2}\int_0^{\pi} (1-\cos 10\theta) \, d\theta \qquad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$= \frac{1}{2}\left(\theta - \frac{1}{10}\sin 10\theta\right)\Big|_0^{\pi}$$

$$= \frac{\pi}{2}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\pi} \left(1-\cos^2 3\theta\right) d\theta$

Solution

$$\int_0^{\pi} \left(1-\cos^2 3\theta\right) d\theta = \int_0^{\pi} \left(1-\frac{1}{2}-\cos 6\theta\right) d\theta \qquad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$= \int_0^{\pi} \left(\frac{1}{2}-\cos 6\theta\right) d\theta$$

$$= \frac{1}{2}\theta - \frac{1}{6}\sin 6\theta\Big|_0^{\pi}$$

$$= \frac{\pi}{2}$$

Exercise

Evaluate the definite integral $\displaystyle\int_2^{3} \frac{x^2+2x-2}{x^3+3x^2-6x} dx$

Solution

$$d\left(x^3+3x^2-6x\right) = \left(3x^2+6x-6\right)dx$$

$$= 3\left(x^2+2x-2\right)dx$$

$$\int_2^3 \frac{x^2+2x-2}{x^3+3x^2-6x}dx = \frac{1}{3}\int_2^3 \frac{1}{x^3+3x^2-6x}\, d\left(x^3+3x^2-6x\right)$$

$$= \frac{1}{3}\ln\left|x^3+3x^2-6x\right|\Big|_2^3$$

$$= \frac{1}{3}(\ln 36 - \ln 8)$$

$$= \frac{1}{3}\left(\ln 6^2 - \ln 2^3\right)$$

$$= \frac{1}{3}(2\ln 6 - 3\ln 2)$$

$$\boxed{= \frac{2}{3}\ln 6 - \ln 2}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\ln 2} \frac{e^x}{1+e^{2x}}dx$

Solution

$$\int_0^{\ln 2} \frac{e^x}{1+e^{2x}}dx = \int_0^{\ln 2} \frac{1}{1+\left(e^x\right)^2} d\left(e^x\right) \qquad d\left(e^x\right) = e^x dx$$

$$= \arctan e^x \Big|_0^{\ln 2} \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \arctan e^{\ln 2} - \arctan 1$$

$$\boxed{= \arctan 2 - \frac{\pi}{4}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_1^3 x\sqrt[3]{x^2-1}\, dx$

Solution

$$\int_1^3 x\sqrt[3]{x^2-1}\, dx = \frac{1}{2}\int_1^3 \left(x^2-1\right)^{1/3} d\left(x^2-1\right) \qquad d\left(x^2-1\right) = 2xdx$$

$$= \frac{3}{8}\left(x^2-1\right)^{4/3}\Big|_1^3$$

$$= \frac{3}{8}\left(8^{4/3} - 0\right)$$

$$= \frac{3}{8} 2^4$$

$$\underline{= 6}$$

Exercise

Evaluate the definite integral $\int_0^2 (x+3)^3 \, dx$

Solution

$$\int_0^2 (x+3)^3 \, dx = \int_0^2 (x+3)^3 \, d(x+3) \qquad d(x+3) = dx$$

$$= \frac{1}{4}(x+3)^4 \Big|_0^2$$

$$= \frac{1}{4}\left(5^4 - 3^4\right)$$

$$= \frac{1}{4}(625 - 81)$$

$$= \frac{544}{4}$$

$$\underline{= 136}$$

Exercise

Evaluate the definite integral $\int_{-2}^2 e^{4x+8} dx$

Solution

$$\int_{-2}^2 e^{4x+8} \, dx = \frac{1}{4}\int_{-2}^2 e^{4x+8} \, d(4x+8) \qquad d(4x+8) = 4dx$$

$$= \frac{1}{4} e^{4x+8} \Big|_{-2}^2$$

$$\underline{= \frac{1}{4}\left(e^{16} - 1\right)}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^1 \sqrt{x}\left(\sqrt{x}+1\right)dx$

Solution

$$\int_0^1 \sqrt{x}\left(\sqrt{x}+1\right)dx = \int_0^1 \left(x + x^{1/2}\right)\, dx$$

$$= \frac{1}{2}x + \frac{2}{3}x^{3/2}\Big|_0^1$$

$$= \frac{1}{2} + \frac{2}{3}$$

$$\boxed{= \frac{7}{6}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

Solution

$$\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\frac{x}{2}\Big|_0^1 \qquad\qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$= \sin^{-1}\frac{1}{2} - \sin^{-1}0$$

$$\boxed{= \frac{\pi}{3}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^2 \frac{2x}{\left(x^2+1\right)^2}dx$

Solution

$$\int_0^2 \frac{2x}{\left(x^2+1\right)^2}dx = \int_0^2 \frac{1}{\left(x^2+1\right)^2}d\left(x^2+1\right)$$

$$= -\frac{1}{x^2+1}\Big|_0^2$$

$$= -\left(\frac{1}{5} - 1\right)$$

$$\underline{= \frac{4}{5}}\Big|$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\pi/2} \sin^2\theta \, \cos\theta \, d\theta$

Solution

$$\int_0^{\pi/2} \sin^2\theta \, \cos\theta \, d\theta = \int_0^{\pi/2} \sin^2\theta \, d(\sin\theta) \qquad d(\sin\theta) = \cos\theta \, d\theta$$

$$= \frac{1}{3}\sin^3\theta \Big|_0^{\pi/2}$$

$$\underline{= \frac{1}{3}}\Big|$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\pi/4} \frac{\sin x}{\cos^2 x} dx$

Solution

$$\int_0^{\pi/4} \frac{\sin x}{\cos^2 x} dx = -\int_0^{\pi/4} \frac{1}{\cos^2 x} d(\cos x)$$

$$= \frac{1}{\cos x}\Big|_0^{\pi/4}$$

$$\underline{= \sqrt{2} - 1}\Big|$$

Exercise

Evaluate the definite integral $\displaystyle\int_{1/3}^{1/\sqrt{3}} \frac{4}{9x^2 + 1} dx$

Solution

$$\int_{1/3}^{1/\sqrt{3}} \frac{4}{9x^2 + 1} dx = \frac{4}{3}\int_{1/3}^{1/\sqrt{3}} \frac{1}{(3x)^2 + 1} d(3x) \qquad d(3x) = 3dx$$

$$= \frac{4}{3}\arctan(3x)\Big|_{1/3}^{1/\sqrt{3}}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{4}{3}\left(\arctan\left(\sqrt{3}\right) - \arctan 1\right)$$

$$= \frac{4}{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{9}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\ln 4} \frac{e^x}{3+2e^x}\,dx$

Solution

$$\int_0^{\ln 4} \frac{e^x}{3+2e^x}\,dx = \frac{1}{2}\int_0^{\ln 4} \frac{1}{3+2e^x}\,d\left(3+2e^x\right)$$

$$= \frac{1}{2}\ln\left(3+2e^x\right)\Big|_0^{\ln 4}$$

$$= \frac{1}{2}\left(\ln\left(3+2e^{\ln 4}\right) - \ln 5\right)$$

$$= \frac{1}{2}(\ln 11 - \ln 5)$$

$$= \frac{1}{2}\ln\frac{11}{5}$$

Exercise

Evaluate the definite integral $\displaystyle\int_{-\pi}^{\pi} \cos^2 x\,dx$

Solution

$$\int_{-\pi}^{\pi} \cos^2 x\,dx = \frac{1}{2}\int_{-\pi}^{\pi} (1+\cos 2x)\,dx$$

$$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right)\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2}(\pi + \pi)$$

$$= \pi$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\pi/4} \cos^2 8\theta \, d\theta$

Solution

$$\int_0^{\pi/4} \cos^2 8\theta \, d\theta = \frac{1}{2}\int_0^{\pi/4} (1+\cos 16\theta)\, d\theta \qquad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$= \frac{1}{2}\left(x + \frac{1}{16}\sin 16\theta\right)\Big|_0^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right)$$

$$= \boxed{\frac{\pi}{8}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_{-\pi/4}^{\pi/4} \sin^2 2\theta \, d\theta$

Solution

$$\int_{-\pi/4}^{\pi/4} \sin^2 2\theta \, d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} (1-\cos 4\theta)\, d\theta \qquad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$= \frac{1}{2}\left(\theta - \frac{1}{4\theta}\sin 4\theta\right)\Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$= \boxed{\frac{\pi}{4}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\pi/6} \frac{\sin 2x}{\sin^2 x + 2} dx$

Solution

$$d\left(\sin^2 x + 2\right) = 2\sin x \cos x \, dx$$

$$= \sin 2x \, dx$$

$$\int_0^{\pi/6} \frac{\sin 2x}{\sin^2 x + 2} dx = \int_0^{\pi/6} \frac{1}{\sin^2 x + 2} d\left(\sin^2 x + 2\right)$$

$$= \ln\left|\sin^2 x + 2\right|\Big|_0^{\pi/6}$$

$$= \ln\frac{9}{4} - \ln 2$$

$$\boxed{= \ln\frac{9}{8}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\pi/2} \sin^4\theta \, d\theta$

Solution

$$\int_0^{\pi/2} \sin^4\theta \, d\theta = \int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta \qquad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(1 - 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(1 - 2\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(\frac{3}{2} - 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{1}{4}\left(\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta\right)\Big|_0^{\pi/2}$$

$$= \frac{1}{4}\left(\frac{3}{2}\frac{\pi}{2}\right)$$

$$\boxed{= \frac{3\pi}{16}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^1 x\sqrt{1-x^2}\, dx$

Solution

$$\int_0^1 x\sqrt{1-x^2}\, dx = -\frac{1}{2}\int_0^1 \left(1-x^2\right)^{1/2} d\left(1-x^2\right) \qquad d\left(1-x^2\right) = -2x\,dx$$

$$= -\frac{1}{3}\left(1-x^2\right)^{3/2}\Big|_0^1$$

$$= -\frac{1}{3}(0-1)$$

$$\boxed{= \frac{1}{3}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{1/4} \frac{x}{\sqrt{1-16x^2}}dx$

Solution

$$\int_0^{1/4} \frac{x}{\sqrt{1-16x^2}}dx = -\frac{1}{32}\int_0^{1/4}\left(1-16x^2\right)^{-1/2} d\left(1-16x^2\right)$$

$$= -\frac{1}{16}\left(1-16x^2\right)^{1/2}\Big|_0^{1/4}$$

$$= -\frac{1}{16}(0-1)$$

$$\boxed{= \frac{1}{16}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_2^{3} \frac{x}{\sqrt[3]{x^2-1}}dx$

Solution

$$\int_2^{3} \frac{x}{\sqrt[3]{x^2-1}}dx = \frac{1}{2}\int_2^{3}\left(x^2-1\right)^{-1/3} d\left(x^2-1\right) \qquad d\left(x^2-1\right) = 2xdx$$

$$= \frac{3}{4}\left(x^2-1\right)^{2/3}\Big|_2^3$$

$$= \frac{3}{4}\left(8^{2/3}-1\right)$$

$$= \frac{3}{4}(4-1)$$

$$\boxed{= \frac{9}{4}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{6/5} \frac{dx}{25x^2+36}$

Solution

$$\int_0^{6/5} \frac{dx}{25x^2+36} = \int_0^{6/5} \frac{dx}{25\left(x^2+\frac{36}{25}\right)}$$

$$= \int_0^{6/5} \frac{dx}{25\left(x^2+\left(\frac{6}{5}\right)^2\right)} \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{1}{25}\left(\frac{5}{6}\right)\tan^{-1}\frac{5x}{6}\Bigg|_0^{6/5}$$

$$= \frac{1}{30}\left(\tan^{-1}1 - \tan^{-1}0\right)$$

$$= \frac{1}{30}\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{120}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{2} x^3\sqrt{16-x^4}\,dx$

Solution

$$\int_0^{2} x^3\sqrt{16-x^4}\,dx = -\frac{1}{4}\int_0^2 \left(16-x^4\right)^{1/2}\,d\left(16-x^4\right) \qquad d\left(16-x^4\right) = -4x^3dx$$

$$= -\frac{1}{6}\left(16-x^4\right)^{3/2}\Bigg|_0^2$$

$$= -\frac{1}{6}\left(0-4^3\right)$$

$$= \frac{32}{3}$$

Exercise

Evaluate the definite integral $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^2 x}\,dx$

Solution

$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^2 x}\,dx = \int_{\pi/4}^{\pi/2} \frac{1}{\sin^2 x}\,d(\sin x) \qquad d(\sin x) = \cos x\,dx$$

$$= -\frac{1}{\sin x}\Big|_{\pi/4}^{\pi/2}$$

$$= -\left(1-\sqrt{2}\right)$$

$$= \sqrt{2}-1$$

Exercise

Evaluate the definite integral $\displaystyle\int_{-1}^{1} (x-1)\left(x^2-2x\right)^7 dx$

Solution

$$d\left(x^2-2x\right) = (2x-2)dx$$

$$= 2(x-1)dx$$

$$\int_{-1}^{1} (x-1)\left(x^2-2x\right)^7 dx = \frac{1}{2}\int_{-1}^{1} \left(x^2-2x\right)^7 d\left(x^2-2x\right)$$

$$= \frac{1}{16}\left(x^2-2x\right)^8\Big|_{-1}^{1}$$

$$= \frac{1}{16}\left(1-3^8\right)$$

$$= \frac{6560}{16}$$

$$= 410$$

Exercise

Evaluate the definite integral $\displaystyle\int_{-\pi}^{0} \frac{\sin x}{2+\cos x}\,dx$

Solution

$$\int_{-\pi}^{0} \frac{\sin x}{2+\cos x} dx = -\int_{-\pi}^{0} \frac{1}{2+\cos x}\, d(2+\cos x) \qquad d(2+\cos x) = -\sin x\; dx$$

$$= -\ln|2+\cos x| \Big|_{-\pi}^{0}$$

$$= -(\ln 3 - \ln 1)$$

$$\boxed{= -\ln 3}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^1 \frac{(x+1)(x+2)}{2x^3+9x^2+12x+36} dx$

Solution

$$d\left(2x^3+9x^2+12x+36\right) = \left(6x^2+18x+12\right)dx$$

$$= 6\left(x^2+3x+2\right)dx$$

$$\int_0^1 \frac{(x+1)(x+2)}{2x^3+9x^2+12x+36}\, dx = \int_0^1 \frac{x^2+3x+2}{2x^3+9x^2+12x+36}\, dx$$

$$= \frac{1}{6}\int_0^1 \frac{1}{2x^3+9x^2+12x+36}\, d\left(2x^3+9x^2+12x+36\right)$$

$$= \frac{1}{6}\ln\left|2x^3+9x^2+12x+36\right| \Big|_0^1$$

$$= \frac{1}{6}(\ln 59 - \ln 36)$$

$$\boxed{= \frac{1}{6}\ln\frac{59}{36}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_1^2 \frac{4}{9x^2+6x+1}\, dx$

Solution

$$\int_1^2 \frac{4}{9x^2+6x+1}\, dx = \int_1^2 \frac{4}{(3x+1)^2}\, dx$$

$$= \frac{4}{3}\int_1^2 \frac{1}{(3x+1)^2}\, d(3x+1) \qquad\qquad d(3x+1) = 3xdx$$

$$= -\frac{4}{3}\left.\frac{1}{3x+1}\right|_1^2$$

$$= -\frac{4}{3}\left(\frac{1}{7} - \frac{1}{4}\right)$$

$$= -\frac{4}{3}\left(-\frac{3}{28}\right)$$

$$= \frac{1}{7}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\pi/4} e^{\sin^2 x} \sin 2x\, dx$

Solution

$$d\left(\sin^2 x\right) = 2\sin x \cos x\, dx$$

$$= \sin 2x\, dx$$

$$\int_0^{\pi/4} e^{\sin^2 x} \sin 2x\, dx = \int_0^{\pi/4} e^{\sin^2 x}\, d\left(\sin^2 x\right)$$

$$= e^{\sin^2 x}\Big|_0^{\pi/4}$$

$$= e^{\sin^2 \frac{\pi}{4}} - e^{\sin^2 0}$$

$$= e^{\frac{1}{2}} - 1$$

$$= \sqrt{e} - 1$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^1 x\sqrt{x+a}\, dx \quad (a > 0)$

Solution

Let $u = x + a \quad \to \quad x = u - a \quad \Rightarrow \quad du = dx$

$$\int_0^1 x\sqrt{x+a}\, dx = \int_0^1 (u-a)u^{1/2}\, du$$

$$= \int_0^1 \left(u^{3/2} - au^{1/2}\right) du$$

$$= \frac{2}{5}u^{5/2} - \frac{2}{3}au^{3/2} \Big|_0^1$$

$$= \frac{2}{5}(x+a)^{5/2} - \frac{2}{3}a(x+a)^{3/2} \Big|_0^1$$

$$= \frac{2}{5}(1+a)^{5/2} - \frac{2}{3}a(1+a)^{3/2} - \frac{2}{5}a^{5/2} + \frac{2}{3}a\left(a^{3/2}\right)$$

$$= \frac{2}{5}(1+a)^{5/2} - \frac{2}{3}a(1+a)^{3/2} - \frac{2}{5}a^{5/2} + \frac{2}{3}a^{5/2}$$

$$= \frac{2}{5}(1+a)^{5/2} - \frac{2}{3}a(1+a)^{3/2} + \frac{4}{15}a^{5/2}$$

$$= \frac{2}{5}(1+a)^2\sqrt{1+a} - \frac{2}{3}a(1+a)\sqrt{1+a} + \frac{4}{15}a^2\sqrt{a}$$

$$= \left(\frac{2}{5}(1+a)^2 - \frac{2}{3}\left(a+a^2\right)\right)\sqrt{1+a} + \frac{4}{15}a^2\sqrt{a}$$

Exercise

Evaluate the definite integral $\int_0^1 x\sqrt[p]{x+a}\, dx \quad (a>0)$

Solution

Let $u = x + a \quad \rightarrow \quad x = u - a$

$du = dx$

$$\int_0^1 x\sqrt[p]{x+a}\, dx = \int_0^1 (u-a)u^{1/p}\, du$$

$$= \int_0^1 \left(u^{1+1/p} - au^{1/p}\right) du$$

$$= \frac{p}{2p+1}u^{2+1/p} - \frac{p}{p+1}au^{1+1/p} \Big|_0^1$$

$$= \frac{p}{2p+1}(x+a)^{2+1/p} - \frac{p}{p+1}a(x+a)^{1+1/p} \Big|_0^1$$

$$= \frac{p}{2p+1}(1+a)^{2+1/p} - \frac{p}{p+1}a(1+a)^{1+1/p} - \frac{p}{2p+1}a^{2+1/p} + \frac{p}{p+1}a(a)^{1+1/p}$$

$$= \frac{p}{2p+1}(1+a)^2 \sqrt[p]{1+a} - \frac{p}{p+1}a(1+a)\sqrt[p]{1+a} - \frac{p}{2p+1}a^{2+1/p} + \frac{p}{p+1}a^{2+1/p}$$

$$= \left(\frac{p}{2p+1}(1+a)^2 - \frac{p}{p+1}a(1+a)\right)\sqrt[p]{1+a} + \left(\frac{p}{p+1} - \frac{p}{2p+1}\right)a^{2+1/p}$$

$$= \left(\frac{p}{2p+1}(1+a)^2 - \frac{p}{p+1}\left(a+a^2\right)\right)\sqrt[p]{1+a} + \left(\frac{2p^2+p-p^2-p}{(p+1)(2p+1)}\right)a^{2+1/p}$$

$$= \left(\frac{p}{2p+1}(1+a)^2 - \frac{p}{p+1}\left(a+a^2\right)\right)\sqrt[p]{1+a} + \frac{p^2}{(p+1)(2p+1)}a^{2+1/p}$$

Or

Let $u = \sqrt[p]{x+a} \quad \to \quad u^p = x+a$

$x = u^p - a \quad \to \quad dx = pu^{p-1}du$

$$\int_0^1 x \sqrt[p]{x+a}\, dx = \int_0^1 \left(u^p - a\right)\cdot u \cdot \left(pu^{p-1}\right) du$$

$$= p\int_0^1 \left(u^p - a\right)\cdot u^p \, du$$

$$= p\int_0^1 \left(u^{2p} - au^p\right) du$$

$$= p\left(\frac{1}{2p+1}\left(\sqrt[p]{x+a}\right)^{2p+1} - \frac{1}{p+1}a\left(\sqrt[p]{x+a}\right)^{p+1}\right)\Bigg|_0^1$$

$$= p\left(\frac{1}{2p+1}\left(\sqrt[p]{1+a}\right)^{2p+1} - \frac{1}{p+1}a\left(\sqrt[p]{1+a}\right)^{\frac{p+1}{p}} - \frac{1}{2p+1}\left(\sqrt[p]{a}\right)^{2p+1} + \frac{1}{p+1}a\left(\sqrt[p]{a}\right)^{p+1}\right)$$

$$= p\left(\begin{array}{l} \frac{1}{2p+1}(1+a)^{(2p+1)/p} - \frac{1}{p+1}a(1+a)^{(p+1)/p} \\ -\frac{1}{2p+1}(a)^{(2p+1)/p} + \frac{1}{p+1}a(a)^{(p+1)/p} \end{array}\right)$$

$$= p\left(\begin{array}{l} \frac{1}{2p+1}(1+a)^{(2p+1)/p} - \frac{1}{p+1}a(1+a)^{(p+1)/p} \\ -\frac{1}{2p+1}(a)^{(2p+1)/p} + \frac{1}{p+1}(a)^{(2p+1)/p} \end{array}\right)$$

Exercise

Evaluate the definite integral $\int_0^1 x\sqrt{1-\sqrt{x}}\,dx$

Solution

$u = 1-\sqrt{x} \quad \to \quad x = (1-u)^2$

$dx = -2(1-u)\,du$

$$\int_0^1 x\sqrt{1-\sqrt{x}}\,dx = -2\int_0^1 (1-u)^2\ u^{1/2}\ (1-u)\,du$$

$$= -2\int_0^1 (1-u)^3\ u^{1/2}\ du$$

$$= -2\int_0^1 \left(1-3u+3u^2-u^3\right)\ u^{1/2}\ du$$

$$= -2\int_0^1 \left(u^{1/2}-3u^{3/2}+3u^{5/2}-u^{7/2}\right)\ du$$

$$= -2\left(\frac{2}{3}\left(1-\sqrt{x}\right)^{3/2}-\frac{6}{5}\left(1-\sqrt{x}\right)^{5/2}+\frac{6}{7}\left(1-\sqrt{x}\right)^{7/2}-\frac{2}{9}\left(1-\sqrt{x}\right)^{9/2}\right)\Bigg|_0^1$$

$$= -2\left(0-\frac{2}{3}+\frac{6}{5}-\frac{6}{7}+\frac{2}{9}\right)$$

$$= -2\left(-\frac{32}{315}\right)$$

$$\boxed{= \frac{34}{315}}$$

Exercise

Evaluate the definite integral $\int_0^1 \sqrt{x-x\sqrt{x}}\,dx$

Solution

$u = 1-\sqrt{x} \quad \to \quad x = (1-u)^2 \qquad \Rightarrow\ dx = -2(1-u)\,du$

$$\int_0^1 \sqrt{x-x\sqrt{x}}\,dx = \int_0^1 \sqrt{x\left(1-\sqrt{x}\right)}\,dx$$

$$= -2\int_0^1 \sqrt{(1-u)^2 u}\ (1-u)\,du$$

$$= -2\int_0^1 \sqrt{(1-u)^2 u}\ (1-u)\,du$$

$$= -2\int_0^1 (1-u)^2 \sqrt{u}\ du$$

$$= -2\int_0^1 \left(1-2u+u^2\right)u^{1/2}\ du$$

$$= -2\int_0^1 \left(u^{1/2}-2u^{3/2}+u^{5/2}\right)\ du$$

$$= -2\left(\frac{2}{3}\left(1-\sqrt{x}\right)^{3/2}-\frac{4}{5}\left(1-\sqrt{x}\right)^{5/2}+\frac{2}{7}\left(1-\sqrt{x}\right)^{7/2}\right)\Bigg|_0^1$$

$$= -2\left(0-\frac{2}{3}+\frac{4}{5}-\frac{2}{7}\right)$$

$$= -2\left(\frac{-16}{105}\right)$$

$$\boxed{= \frac{32}{105}}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\pi/2} \frac{\cos\theta\sin\theta}{\sqrt{\cos^2\theta+16}}\,d\theta$

Solution

$$d\left(\cos^2\theta+16\right) = -2\cos\theta\sin\theta\ d\theta$$

$$\int_0^{\pi/2} \frac{\cos\theta\sin\theta}{\sqrt{\cos^2\theta+16}}\,d\theta = -\frac{1}{2}\int_0^{\pi/2}\left(\cos^2\theta+16\right)^{-1/2} d\left(\cos^2\theta+16\right)$$

$$= -\sqrt{\cos^2\theta+16}\Bigg|_0^{\pi/2}$$

$$= -\left(4-\sqrt{17}\right)$$

$$\boxed{= \sqrt{17}-4}$$

Exercise

Evaluate the definite integral $\displaystyle\int_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \frac{dx}{x\sqrt{25x^2-1}}$

Solution

$$\int_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \frac{dx}{x\sqrt{25x^2-1}} = \int_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \frac{d(5x)}{(5x)\sqrt{(5x)^2-1}}$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|$$

$$= \sec^{-1}(5x)\Big|_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}}$$

$$= \sec^{-1}(2) - \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$$

$$= \frac{\pi}{3} - \frac{\pi}{6}$$

$$= \frac{\pi}{6}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^4 \frac{x}{\sqrt{9+x^2}}dx$

Solution

$$\int_0^4 \frac{x}{\sqrt{9+x^2}}dx = \frac{1}{2}\int_0^4 \left(9+x^2\right)^{-1/2} d\left(9+x^2\right)$$

$$d\left(9+x^2\right) = 2xdx$$

$$= \sqrt{9+x^2}\Big|_0^4$$

$$= 5 - 3$$

$$= 2$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^{\pi/4} \frac{\sin\theta}{\cos^3\theta}d\theta$

Solution

$$\int_0^{\pi/4} \frac{\sin\theta}{\cos^3\theta}\, d\theta = -\int_0^{\pi/4} \cos^{-3}\theta \; d(\cos\theta) \qquad d(\cos\theta) = -\sin\theta$$

$$= \frac{1}{2}\frac{1}{\cos^2\theta}\Big|_0^{\pi/4}$$

$$= \frac{1}{2}(2-1)$$

$$= \frac{1}{2}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^1 2x\left(4-x^2\right)\, dx$

Solution

$$\int_0^1 2x\left(4-x^2\right)\, dx = -\int_0^1 \left(4-x^2\right)\, d\left(4-x^2\right) \qquad d\left(4-x^2\right) = -2xdx$$

$$= -\frac{1}{2}\left(4-x^2\right)^2\Big|_0^1$$

$$= -\frac{1}{2}(9-16)$$

$$= \frac{7}{2}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^3 \frac{x^2+1}{\sqrt{x^3+3x+4}}\, dx$

Solution

$$\int_0^3 \frac{x^2+1}{\sqrt{x^3+3x+4}}\, dx = \frac{1}{3}\int_0^3 \left(x^3+3x+4\right)^{-1/2} d\left(x^3+3x+4\right) \qquad d\left(x^3+3x+4\right) = \left(3x^2+3\right)dx$$

$$= \frac{2}{3}\sqrt{x^3+3x+4}\Big|_0^3$$

$$= \frac{2}{3}\left(\sqrt{40}-2\right)$$

$$= \frac{4}{3}\left(\sqrt{10}-1\right)$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^4 \frac{x}{x^2+1}dx$

Solution

$$\int_0^4 \frac{x}{x^2+1}dx = \frac{1}{2}\int_0^4 \frac{1}{x^2+1}d\left(x^2+1\right) \qquad d\left(x^2+1\right) = 2xdx$$

$$= \frac{1}{2}\ln\left(x^2+1\right)\Big|_0^4$$

$$= \frac{1}{2}(\ln 17 - \ln 1)$$

$$= \frac{1}{2}\ln 17$$

Exercise

Evaluate the definite integral $\displaystyle\int_1^{e^2} \frac{\ln x}{x}dx$

Solution

$$\int_1^{e^2} \frac{\ln x}{x}dx = \int_1^{e^2} \ln x \, d(\ln x) \qquad d(\ln x) = \frac{1}{x}dx$$

$$= \frac{1}{2}(\ln x)^2\Big|_1^{e^2}$$

$$= \frac{1}{2}\left(\left(\ln e^2\right)^2 - (\ln 1)^2\right)$$

$$= \frac{1}{2}(2)^2$$

$$= 2$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^3 \frac{x^2+1}{\sqrt{x^3+3x+4}}dx$

Solution

$$d\left(x^3+3x+4\right) = \left(3x^2+3\right)dx$$

$$= 3\left(x^2+1\right)dx$$

$$\int_0^3 \frac{x^2+1}{\sqrt{x^3+3x+4}}\,dx = \frac{1}{3}\int_0^3 \left(x^3+3x+4\right)^{-1/2} d\left(x^3+3x+4\right)$$

$$= \frac{2}{3}\sqrt{x^3+3x+4}\,\Big|_0^3$$

$$= \frac{2}{3}\left(\sqrt{40}-2\right)$$

$$= \frac{2}{3}\left(\sqrt{10}-1\right)$$

Exercise

Evaluate the definite integral $\displaystyle\int_{-\pi/4}^{\pi/4} \sin^2 2\theta\, d\theta$

Solution

$$\int_{-\pi/4}^{\pi/4} \sin^2 2\theta\, d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} (1-\cos 4\theta)\, d\theta \qquad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$= \frac{1}{2}\left(\theta - \frac{1}{4}\sin 4\theta\right)\Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi}{4}+\frac{\pi}{4}\right)$$

$$= \frac{\pi}{4}$$

Exercise

Evaluate the definite integral $\displaystyle\int_0^1 \left(y^3+6y^2-12y+9\right)^{-1/2}\left(y^2+4y-4\right)dy$

Solution

$$d\left(y^3+6y^2-12y+9\right) = \left(3y^2+12y-12\right)dy$$

$$= 3\left(y^2+4y-4\right)dy$$

$$\int_0^1 \left(y^3+6y^2-12y+9\right)^{-1/2}\left(y^2+4y-4\right)dy$$

$$= \frac{1}{3}\int_0^1 \left(y^3 + 6y^2 - 12y + 9\right)^{-1/2} \left(y^3 + 6y^2 - 12y + 9\right) dy$$

$$= \frac{2}{3}\sqrt{y^3 + 6y^2 - 12y + 9}\,\Big|_0^1$$

$$= \frac{2}{3}(2 - 3)$$

$$= -\frac{2}{3}$$

Exercise

Solve the initial value problem $\dfrac{dy}{dt} = e^t \sin\left(e^t - 2\right), \quad y(\ln 2) = 0$

Solution

$$\frac{dy}{dt} = e^t \sin\left(e^t - 2\right) \quad \Rightarrow y = \int e^t \sin\left(e^t - 2\right) dt$$

Let $u = e^t - 2 \quad \to du = e^t dt$

$$y = \int \sin u \; du$$

$$= -\cos u + C$$

$$= -\cos\left(e^t - 2\right) + C$$

$$y(\ln 2) = -\cos\left(e^{\ln 2} - 2\right) + C = 0$$

$$C = \cos(2 - 2)$$

$$C = \cos(0) = 1$$

$$y(t) = -\cos\left(e^t - 2\right) + 1$$

Exercise

Solve the initial value problem $\dfrac{dy}{dt} = e^{-t} \sec^2\left(\pi e^{-t}\right), \quad y(\ln 4) = \dfrac{2}{\pi}$

Solution

$$\frac{dy}{dt} = e^{-t} \sec^2\left(\pi e^{-t}\right) \quad \Rightarrow y = \int e^{-t} \sec^2\left(\pi e^{-t}\right) dt$$

Let $u = \pi e^{-t} \quad \rightarrow du = -\pi e^{-t}dt \;\Rightarrow\; -\frac{1}{\pi}du = e^{-t}dt$

$$y = \int e^{-t}\sec^2\left(\pi e^{-t}\right)dt = -\frac{1}{\pi}\int \sec^2 u\ du$$

$$= -\frac{1}{\pi}\tan u + C$$

$$= -\frac{1}{\pi}\tan\left(\pi e^{-t}\right) + C$$

$$y(\ln 4) = -\frac{1}{\pi}\tan\left(\pi e^{-\ln 4}\right) + C = \frac{2}{\pi}$$

$$C = \frac{2}{\pi} + \frac{1}{\pi}\tan\left(\pi \cdot \frac{1}{4}\right)$$

$$= \frac{2}{\pi} + \frac{1}{\pi}$$

$$= \frac{3}{\pi}$$

$$y(t) = -\frac{1}{\pi}\tan\left(\pi e^{-t}\right) + \frac{3}{\pi}$$

Exercise

Verify the integration formula: $\displaystyle\int \frac{\tan^{-1}x}{x^2}dx = \ln x - \frac{1}{2}\ln\left(1+x^2\right) - \frac{\tan^{-1}x}{x} + C$

Solution

If $y = \ln x - \frac{1}{2}\ln\left(1+x^2\right) - \frac{\tan^{-1}x}{x} + C$

$$dy = \left[\frac{1}{x} - \frac{1}{2}\frac{2x}{1+x^2} - \frac{\frac{x}{1+x^2} - \tan^{-1}x}{x^2}\right]dx$$

$$dy = \left[\frac{1}{x} - \frac{x}{1+x^2} - \frac{x - \left(1+x^2\right)\tan^{-1}x}{x^2\left(1+x^2\right)}\right]dx$$

$$dy = \left(\frac{x\left(1+x^2\right) - x^3 - x + \left(1+x^2\right)\tan^{-1}x}{x^2\left(1+x^2\right)}\right)dx$$

$$dy = \left(\frac{x + x^3 - x^3 - x + \left(1+x^2\right)\tan^{-1}x}{x^2\left(1+x^2\right)}\right)dx$$

$$dy = \frac{\left(1+x^2\right)\tan^{-1}x}{x^2\left(1+x^2\right)}dx$$

$$dy = \frac{\tan^{-1}x}{x^2}dx \qquad \surd \qquad \text{Which verifies the formula}$$

Exercise

Verify the integration formula: $\int \ln\left(a^2+x^2\right)dx = x\ln\left(a^2+x^2\right) - 2x + 2a\tan^{-1}\frac{x}{a} + C$

Solution

If $y = x\ln\left(a^2+x^2\right) - 2x + 2a\tan^{-1}\frac{x}{a} + C$

$$dy = \left[\ln\left(a^2+x^2\right) + x\frac{2x}{a^2+x^2} - 2 + 2a\frac{\frac{1}{a}}{1+\frac{x^2}{a^2}}\right]dx$$

$$dy = \left[\ln\left(a^2+x^2\right) + \frac{2x^2}{a^2+x^2} - 2 + \frac{2}{\frac{a^2+x^2}{a^2}}\right]dx$$

$$dy = \left[\ln\left(a^2+x^2\right) + \frac{2x^2}{a^2+x^2} - 2 + \frac{2a^2}{a^2+x^2}\right]dx$$

$$dy = \left[\ln\left(a^2+x^2\right) + \frac{2x^2+2a^2}{a^2+x^2} - 2\right]dx$$

$$dy = \left[\ln\left(a^2+x^2\right) + \frac{2\left(x^2+a^2\right)}{a^2+x^2} - 2\right]dx$$

$$dy = \left[\ln\left(a^2+x^2\right) + 2 - 2\right]dx$$

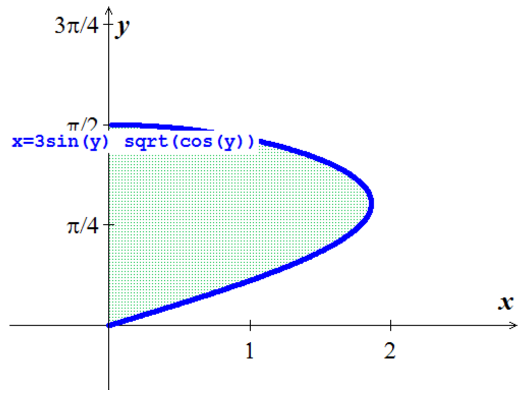
$$dy = \ln\left(a^2+x^2\right)dx \qquad \surd \qquad \text{Which verifies the formula}$$

Exercise

Find the area of the region bounded by the graphs of $x = 3\sin y\sqrt{\cos y}$, *and* $x = 0$, $0 \le y \le \frac{\pi}{2}$

Solution

$$A = \int_0^{\pi/2} \left(3\sin y\sqrt{\cos y} - 0\right)dx \qquad d(\cos y) = -\sin y\ dy$$



$$= -3\int_0^{\pi/2} \cos^{1/2} y\ \ d(\cos y)$$

$$= -3\left[\frac{2}{3}\cos^{3/2} y\right]_0^{\pi/2}$$

$$= -2(0-1)$$

$$= 2\ \ unit^2$$

Exercise

Find the area of the region bounded by the graph of $f(x) = \dfrac{x}{\sqrt{x^2-9}}$ *on* $3 \le x \le 4$

Solution

$$A = \int_3^4 \frac{x}{\sqrt{x^2-9}}dx$$

$$= \frac{1}{2}\int_3^4 \left(x^2-9\right)^{-1/2} d\left(x^2-9\right) \qquad d\left(x^2-9\right) = 2xdx$$

$$= \sqrt{x^2-9}\,\Big|_3^4$$

$$= \sqrt{7} - 0$$

$$= \sqrt{7}\ \ unit^2$$

Exercise

Find the area of the region bounded by the graph of $f(x) = \dfrac{x}{\sqrt{x^2-9}}$ and the *x-axis* between $x = 4$ and $x = 5$.

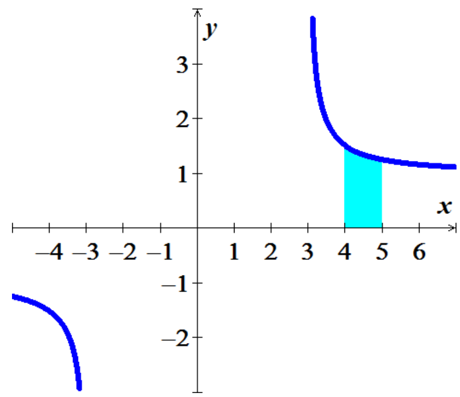
Solution

$$A = \int_4^5 \frac{x}{\sqrt{x^2-9}}\,dx$$

$$= \frac{1}{2}\int_4^5 \left(x^2-9\right)^{-1/2} d\left(x^2-9\right) \qquad d\left(x^2-9\right) = 2xdx$$

$$= \sqrt{x^2-9}\,\Big|_4^5$$

$$= 4-\sqrt{7}\ \ unit^2$$



Exercise

Find the area of the region bounded by the graph of $f(x) = x\ \sin x^2$ and the *x-axis* between $x = 0$ and $x = \sqrt{\pi}$.

Solution

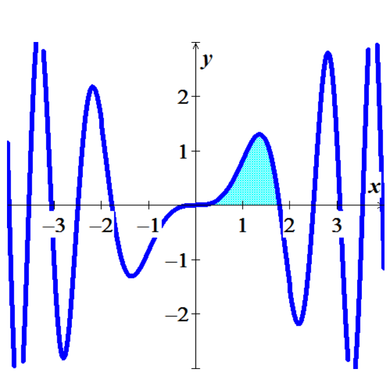
$$A = \int_0^{\sqrt{\pi}} x\ \sin x^2\ dx$$

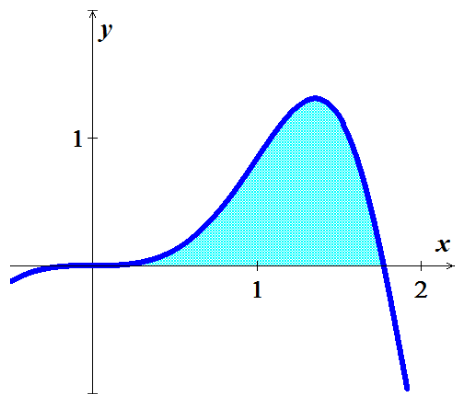
$$= \frac{1}{2}\int_0^{\sqrt{\pi}} \sin x^2\ d\left(x^2\right)$$

$$= -\frac{1}{2}\cos\left(x^2\right)\Big|_0^{\sqrt{\pi}}$$

$$= -\frac{1}{2}(-1-1)$$

$$= 1\ \ unit^2$$





Exercise

Find the area of the region bounded by the graph of $f(\theta) = \cos\theta\ \sin\theta$ and the θ*-axis* between $\theta = 0$ and $\theta = \frac{\pi}{2}$.

Solution

$$A = \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta\ d(\sin\theta)$$

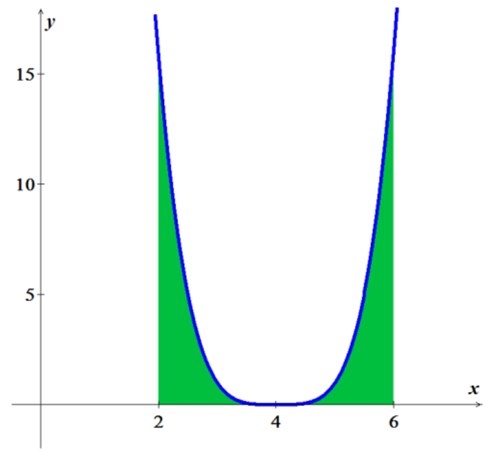
$$= \frac{1}{2}\sin^2\theta \Big|_0^{\pi/2}$$

$$\boxed{= 1 \ \ unit^2}$$

Exercise

Find the area of the region bounded by the graph of $f(x) = (x-4)^4$ and the x-axis between $x = 2$ and $x = 6$.

Solution



$$A = \int_2^6 (x-4)^4 dx$$

$$= \int_2^6 (x-4)^4 d(x-4)$$

$$= 2 \times \frac{1}{5}(x-4)^5 \Big|_2^4$$

$$\boxed{= \frac{64}{5} \ \ unit^2}$$

Exercise

Perhaps the simplest change of variables is the shift or translation given by $u = x + c$, where c is a real number.

a) Prove that shifting a function does not change the net area under the curve, in the sense that

$$\int_a^b f(x+c)dx = \int_{a+c}^{b+c} f(u)du$$

b) Draw a picture to illustrate this change of variables in the case that $f(x) = \sin x$, $a = 0$, $b = \pi$, and $c = \frac{\pi}{2}$

Solution

a) Let $u = x + c \quad \to \quad du = dx$

$$\begin{cases} x = b & \to \ u = b + c \\ x = a & \to \ u = a + c \end{cases}$$

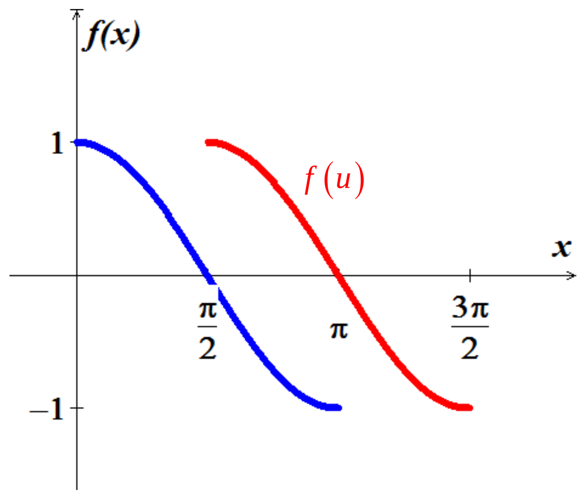
$$\int_a^b f(x+c)dx = \int_{a+c}^{b+c} f(u)du$$

b) Given: $\quad f(x) = \sin x, \quad a = 0, \quad b = \pi, \quad \& \quad c = \frac{\pi}{2}$

$$f(x+c)=\sin\left(x+\frac{\pi}{2}\right)$$

$$\begin{cases} b=\pi & \rightarrow f\left(\pi+\frac{\pi}{2}\right)=\sin\frac{3\pi}{2}=-1 \\ a=0 & \rightarrow f\left(0+\frac{\pi}{2}\right)=\sin\frac{\pi}{2}=1 \end{cases}$$

$$f(u) \rightarrow \begin{cases} b+c=\frac{3\pi}{2} \\ a+c=\frac{\pi}{2} \end{cases}$$



Exercise

Another change of variables that can be interpreted geometrically is the scaling $u=cx$, where c is a real number. Prove and interpret the fact that

$$\int_a^b f(cx)dx=\frac{1}{c}\int_{ac}^{bc} f(u)du$$

Draw a picture to illustrate this change of variables in the case that $f(x)=\sin x$, $a=0$, $b=\pi$, and $c=\frac{1}{2}$

Solution

Let $u=cx \quad \rightarrow \quad du=cdx$

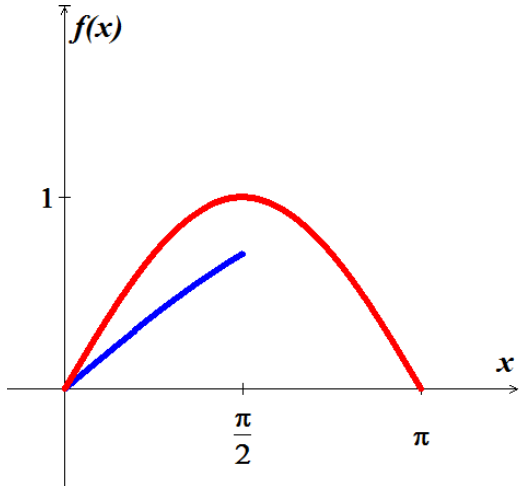
$$\begin{cases} x=b & \rightarrow u=bc \\ x=a & \rightarrow u=ac \end{cases}$$

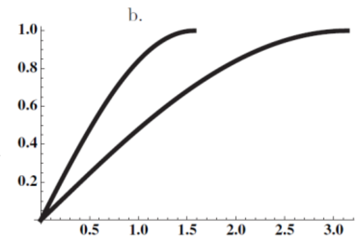
$$\int_a^b f(cx)dx=\frac{1}{c}\int_{ac}^{bc} f(u)du$$

Given: $f(x)=\sin x, \quad a=0, \quad b=\pi, \quad \& \quad c=\frac{1}{2}$

$$f(cx) = f\left(\frac{x}{2}\right) = \sin\frac{x}{2}$$

$$\begin{cases} a = 0 & \to ac = 0 \\ b = \pi & \to bc = \frac{\pi}{2} \end{cases}$$





Exercise

The function f satisfies the equation $3x^4 - 48 = \int_2^x f(t)\,dt$. Find f and check your answer by substitution.

Solution

$$\frac{d}{dx}\left(3x^4 - 48\right) = \frac{d}{dx}\int_2^x f(t)\,dt$$

$$12x^3 = f(x)$$

$$\int_2^x 12t^3\,dt = 3t^4 \Big|_2^x$$

$$= 3x^4 - 3(2)^4$$

$$\underline{= 3x^4 - 48} \quad \checkmark$$

Exercise

Assume f' is continuous on $[2,\ 4]$, $\int_1^2 f'(2x)\,dx = 10$, and $f(2) = 4$. Evaluate $f(4)$.

Solution

$$\int_1^2 f'(2x)\,dx = f(2x)\Big|_1^2$$

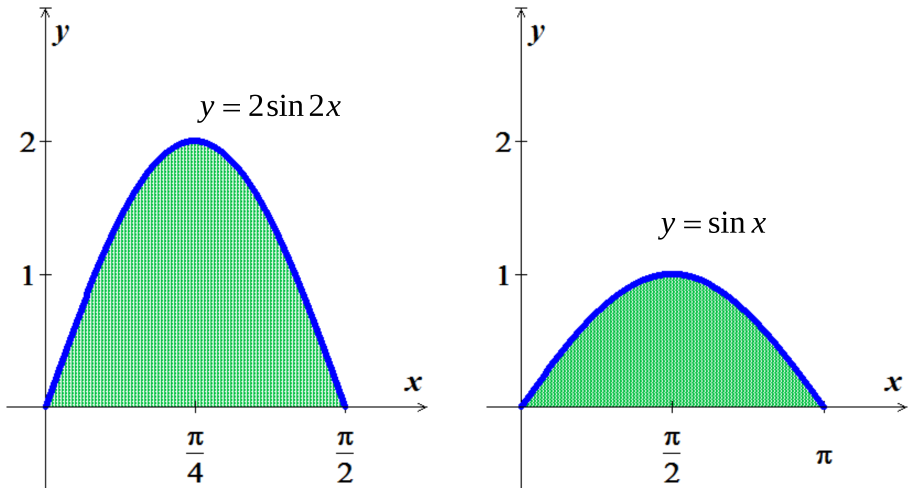
$$= f(4) - f(2) = 10$$

$$f(4) - 4 = 10$$

$$\boxed{f(4) = 14}$$

Exercise

The area of the shaded region under the curve $y = 2\sin 2x$ in



a) Equals the area on the shaded region under the curve $y = \sin x$

b) Explain why this is true without computation areas.

Solution

a) $A = \int_0^{\pi/2} 2\sin 2x\, dx \qquad\qquad u = 2x \;\rightarrow\; du = 2dx$

$$\begin{cases} x = \frac{\pi}{2} & \rightarrow\; u = \pi \\ x = 0 & \rightarrow\; u = 0 \end{cases}$$

$$= \int_0^{\pi} \sin u\, du$$

$$= \int_0^{\pi} \sin x\, dx$$

b) Let A_1 = area of $\sin x \quad 0 \le x \le \pi$

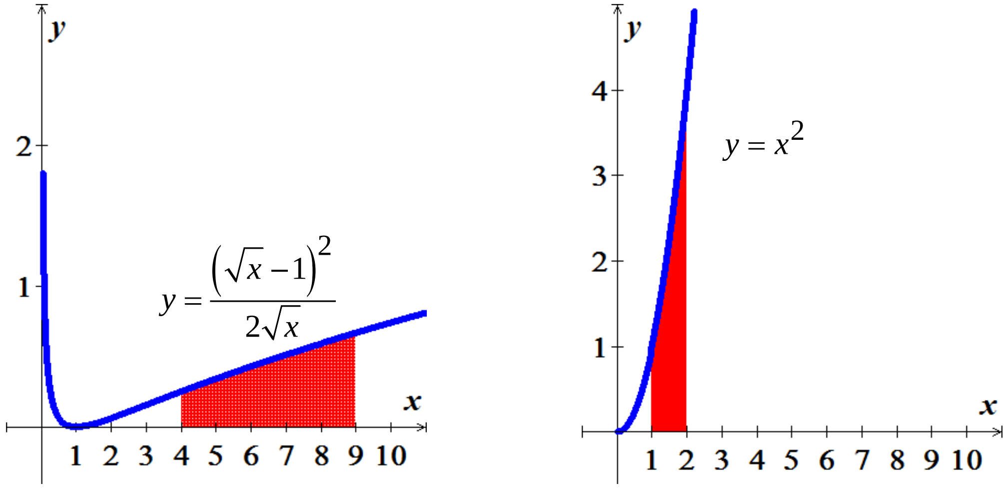
A_2 = area of $\sin 2x \quad 0 \le 2x \le \pi \;\rightarrow\; 0 \le x \le \frac{\pi}{2}$

Area of $0 \le x \le \frac{\pi}{2}$ is $\frac{1}{2} A_1$

$A_2 = 2\frac{1}{2}A_1 = A_1$ ✓

Exercise

The area of the shaded region under the curve $y=\dfrac{\left(\sqrt{x}-1\right)^2}{2\sqrt{x}}$ on the interval $[4,\ 9]$



a) Equals the area on the shaded region under the curve $y=x^2$ on the interval $[1,\ 2]$

b) Explain why this is true without computation areas.

Solution

a) Let $u=\sqrt{x}-1 \quad \rightarrow \quad x=(u+1)^2$

$$dx=2(u+1)\,du$$

$$\begin{cases} x=9 & \rightarrow \ u=2 \\ x=4 & \rightarrow \ u=1 \end{cases}$$

$$A_1=\int_4^9 \frac{\left(\sqrt{x}-1\right)^2}{2\sqrt{x}}\,dx$$

$$=\int_1^2 \frac{u^2}{2(u+1)}\ 2(u+1)\,du$$

$$=\int_1^2 u^2\,du \qquad \checkmark$$

$$=\frac{1}{3}\left(\sqrt{x}-1\right)^3\Big|_4^9$$

$$=\frac{1}{3}\left(2^3-1\right)$$

$$=\frac{7}{3}$$

$$A_2 = \int_1^2 x^2\, dx$$

$$= \frac{1}{3}x^3 \Big|_1^2$$

$$= \frac{1}{3}\left(2^3 - 1\right)$$

$$\boxed{= \frac{7}{3}}$$

b) $\displaystyle\int_4^9 \frac{\left(\sqrt{x}-1\right)^2}{2\sqrt{x}}\, dx = \int_1^2 u^2\, du = \int_1^2 x^2\, dx$ ✓

Exercise

The family of parabolas $y = \frac{1}{a} - \frac{x^2}{a^3}$, where $a > 0$, has the property that for $x \ge 0$, the x-intercept is $(a,\ 0)$ and the y-intercept is $\left(0,\ \frac{1}{a}\right)$. Let $A(a)$ be the area of the region in the first quadrant bounded by the parabola and the x-*axis*. Find $A(a)$ and determine whether it is increasing, decreasing, or constant function of *a*.

Solution

Given: $y = \frac{1}{a} - \frac{x^2}{a^3}$ $\quad (a,\ 0)$ & $\left(0,\ \frac{1}{a}\right)$

$$A = \int_0^a \left(\frac{1}{a} - \frac{x^2}{a^3}\right) dx$$

$$= \left(\frac{1}{a}x - \frac{1}{3}\frac{x^3}{a^3}\right)\Big|_0^a$$

$$= 1 - \frac{1}{3}$$

$$\boxed{= \frac{2}{3}}$$

$A(a) = \frac{2}{3}$ is a constant function.

Exercise

Consider the right triangle with vertices $(0,\ 0)$, $(0,\ b)$, and $(a,\ 0)$, where $a>0$ and $b>0$. Show that the average vertical distance from points on the *x-axis* to the hypotenuse is $\frac{b}{2}$, for all $a>0$.

Solution

$$y=\frac{b-0}{0-a}(x-0)+b \qquad\qquad y=m\left(x-x_0\right)+y_0$$

$$=-\frac{b}{a}x+b$$

Average vertical distance is:

$$\frac{1}{a-0}\int_0^a\left(-\frac{b}{a}x+b\right)dx=\frac{1}{a}\int_0^a\left(b-\frac{b}{a}x\right)dx$$

$$=\frac{1}{a}\left(bx-\frac{b}{2a}x^2\right)\Big|_0^a$$

$$=\frac{1}{a}\left(ba-\frac{b}{2a}a^2\right)$$

$$=b-\frac{b}{2}$$

$$=\frac{b}{2}$$

Exercise

Consider the integral $I=\int\sin^2 x\ \cos^2 x\ dx$

a) Find I using the identity $\sin 2x=2\sin x\cos x$

b) Find I using the identity $\cos^2 x=1-\sin^2 x$

c) Confirm that the results in part (*a*) and (*b*) are consistent and compare the work involved in each method.

Solution

a) $\sin 2x=2\sin x\cos x$

$$\sin^2 2x=4\sin^2 x\cos^2 x$$

$$\sin^2 x\cos^2 x=\frac{1}{4}\sin^2 2x$$

$$I=\int\sin^2 x\ \cos^2 x\ dx$$

$$=\frac{1}{4}\int\sin^2 2x\ dx \qquad\qquad \sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$$

$$=\frac{1}{8}\int(1-\cos 4x)\ dx$$

$$=\frac{1}{8}\left(x-\frac{1}{4}\sin 4x\right)+C$$

$$\boxed{=\frac{1}{8}x-\frac{1}{32}\sin 4x+C}$$

b) $\cos^2 x=1-\sin^2 x$ $\qquad\qquad\qquad$ $\sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$

$$=1-\frac{1}{2}+\frac{1}{2}\cos 2x$$

$$=\frac{1}{2}+\frac{1}{2}\cos 2x$$

$$I=\int\frac{1}{2}(1-\cos 2x)\ \frac{1}{2}(1+\cos 2x)\ dx \qquad \sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$$

$$=\frac{1}{4}\int\left(1-\cos^2 2x\right)\ dx \qquad \cos^2 x=1-\sin^2 x$$

$$=\frac{1}{4}\int\sin^2 2x\ dx \qquad \text{From part } (\boldsymbol{a})$$

$$\boxed{=\frac{1}{8}x-\frac{1}{32}\sin 4x+C}$$

c) The results from part *a* & *b* are consistent.

Exercise

Let $H(x)=\int_0^x\sqrt{4-t^2}\ dt$, for $-2\le x\le 2$.

a) Evaluate $H(0)$

b) Evaluate $H'(1)$

c) Evaluate $H'(2)$

d) Use geometry to evaluate $H(2)$

e) Find the value of *s* such that $H(x)=sH(-x)$

Solution

a) $H(0)=\int_0^0\sqrt{4-t^2}\ dt\ \boxed{=0}$

b) $H'(x)=\sqrt{4-x^2}\ \frac{d}{dx}(x)$

$$=\sqrt{4-x^2}$$

$H'(1)\ \boxed{=\sqrt{3}}$

c) $H'(2)=\sqrt{4-4}\ \boxed{=0}$

d) $H(2)=\int_0^2 \sqrt{4-t^2}\ dt$ is the area inside a circle in the first quadrant of radius 2

$$=\frac{1}{4}\pi(2)^2$$

$$\boxed{=\pi}$$

e) $H(x)=\int_0^{-x} \sqrt{4-t^2}\ dt$ $\sqrt{4-t^2}$ is an even function

$$=-\int_{-x}^{0} \sqrt{4-t^2}\ dt$$

$$=-H(x)$$

$\therefore\ \boxed{s=-1}$

$t=2\sin u \quad \rightarrow \quad dt=2\cos u\ du$

$\sqrt{4-t^2}=2\cos u$

$$H(x)=\int_0^{x} \sqrt{4-t^2}\ dt$$

$$=\int_0^{x} 2\cos u\ 2\cos u\ du$$

$$=\int_0^{x} 4\cos^2 u\ du$$

$$=\int_0^{x} 2(1+\cos 2u)\ du$$

$$=2\left(u+\frac{1}{2}\sin 2u\right)\Big|_0^x \qquad t=2\sin u \ \rightarrow\ u=\sin^{-1}\frac{t}{2}$$

$$=2\left(\sin^{-1}\frac{t}{2}+\sin u\cos u\right)\Big|_0^x \qquad \sqrt{4-t^2}=2\cos u \ \rightarrow\ \cos u=\frac{1}{2}\sqrt{4-t^2}$$

$$= 2\left(\sin^{-1}\frac{t}{2} + \frac{t}{4}\sqrt{4-t^2}\right)\Bigg|_0^x$$

$$= 2\left(\sin^{-1}\frac{x}{2} + \frac{x}{4}\sqrt{4-x^2}\right)$$

$$\boxed{= 2\sin^{-1}\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2}}$$

Exercise

Evaluate the limits $\displaystyle\lim_{x\to 2}\frac{\int_2^x e^{t^2}\,dt}{x-2}$

Solution

$$\lim_{x\to 2}\frac{\int_2^x e^{t^2}\,dt}{x-2} = \frac{\int_2^2 e^{t^2}\,dt}{2-2} = \frac{0}{0}$$

$$= \lim_{x\to 2}\frac{e^{x^2}\frac{d}{dx}(x)}{1}$$

$$= \lim_{x\to 2} e^{x^2}$$

$$\boxed{= e^4}$$

Exercise

Evaluate the limits $\displaystyle\lim_{x\to 1}\frac{\int_1^{x^2} e^{t^3}\,dt}{x-1}$

Solution

$$\lim_{x\to 1}\frac{\int_1^{x^2} e^{t^3}\,dt}{x-1} = \lim_{x\to 1}\frac{\int_1^{1} e^{t^3}\,dt}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to 1}\frac{2xe^{x^3}}{1}$$

$$\boxed{= 2e}$$

Exercise

Prove that for nonzero constants a and b, $\displaystyle\int \frac{dx}{a^2x^2+b^2} = \frac{1}{ab}\tan^{-1}\left(\frac{ax}{b}\right)+C$

Solution

$$\int \frac{dx}{a^2x^2+b^2} = \int \frac{dx}{a^2\left(x^2+\left(\frac{b}{a}\right)^2\right)}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{1}{a^2}\frac{a}{b}\tan^{-1}\frac{x}{\frac{b}{a}}+C$$

$$\boxed{= \frac{1}{ab}\tan^{-1}\left(\frac{ax}{b}\right)+C}\quad \checkmark$$

Exercise

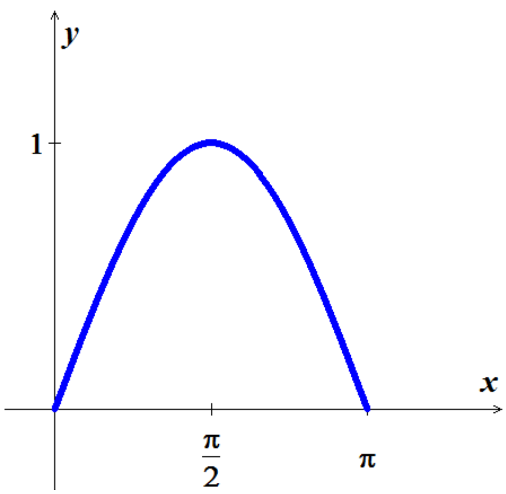
Let $a>0$ be a real number and consider the family of functions $f(x)=\sin ax$ on the interval $\left[0,\ \frac{\pi}{a}\right]$.

a) Graph f, for a = 1, 2, 3.

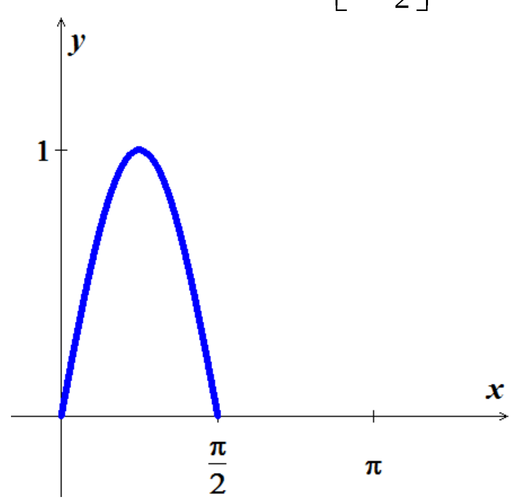
b) Let $g(a)$ be the area of the region bounded by the graph of f and the *x-axis* on the interval $\left[0,\ \frac{\pi}{a}\right]$. Graph g for $0<a<\infty$. Is g an increasing function, a decreasing function, or neither?

Solution

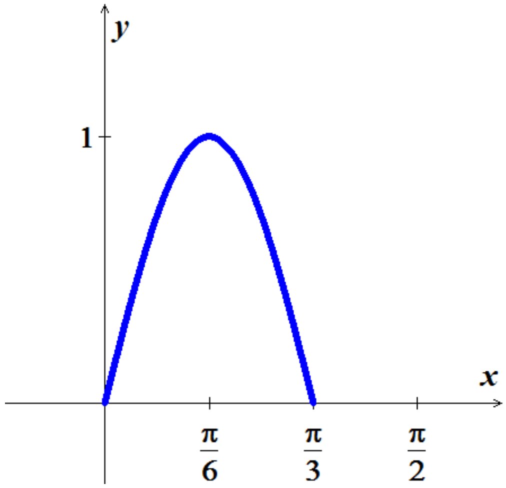
a) $a=1\rightarrow \quad f(x)=\sin x \quad x\in[0,\ \pi]$

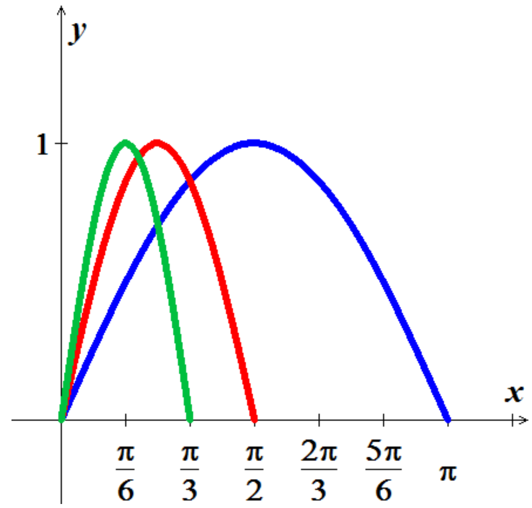


$a=2\rightarrow \quad f(x)=\sin 2x \quad x\in\left[0,\ \frac{\pi}{2}\right]$



$a = 3 \to \quad f(x) = \sin 3x \quad x \in \left[0, \ \frac{\pi}{3}\right]$





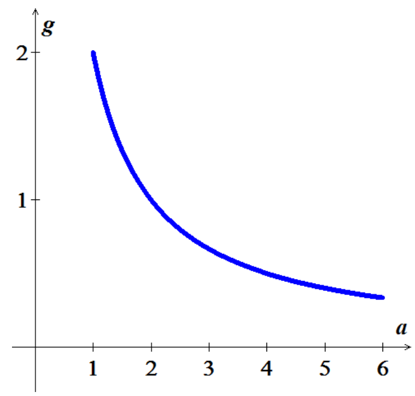
b) $g(x) = \displaystyle\int_0^{\pi/a} \sin ax \, dx$

$$= -\frac{1}{a} \cos ax \Big|_0^{\pi/a}$$

$$= -\frac{1}{a}(\cos \pi - \cos 0)$$

$$= -\frac{1}{a}(-1-1)$$

$$\boxed{= \frac{2}{a}}$$



The function is decreasing as $a \geq 1$ is increasing.

Exercise

Explain why if a function u satisfies the equation $u(x) + 2\int_0^x u(t)dt = 10$, then it also satisfies the equation $u'(x) + 2u(x) = 0$. Is it true that is u satisfies the second equation, then it satisfies the first equation?

Solution

$$\frac{d}{dx}u(x) + 2\frac{d}{dx}\int_0^x u(t)dt = \frac{d}{dx}(10)$$

$$u'(x) + 2\frac{d}{dx}u(x)\frac{d}{dx}x = 0$$

$$u'(x) + 2u(x) = 0 \qquad \checkmark$$

Exercise

Let $f(x)=\int_0^x (t-1)^{15}(t-2)^9\,dt$

a) Find the interval on which f is increasing and the intervals on which f is decreasing.

b) Find the intervals on which f is concave up and the intervals on which f is concave down.

c) For what values of x does f have local minima? Local maxima?

d) Where are the inflection points of f ?

Solution

a) $f'(x)=(x-1)^{15}(x-2)^9=0$

$CN:\ \ x=1,\ 2$

Where $x=1$ is multiplicity of 15

$x=2$ is multiplicity of 9

0	1	2
+	–	+

Therefore, the sign will change.

f is increasing on $(-\infty,\ 1)\cup(2,\ \infty)$

f is decreasing on $(1,\ 2)$

b) $f''(x)=(x-1)^{14}(x-2)^8\big(15(x-2)+9(x-1)\big)$

$=(x-1)^{14}(x-2)^8(24x-39)=0$

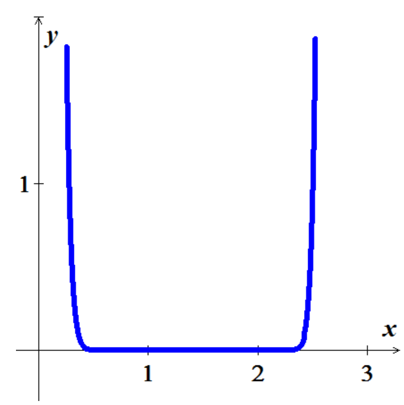
$x=1,\ 2,\ \frac{13}{8}$

0	1	$\frac{13}{8}$	2
–	–	+	+

$(x-1)^{14}(x-2)^8\geq 0$ $\quad(always)$

Concave up: $\left(\frac{13}{8},\ \infty\right)$

Concave down: $\left(-\infty,\ \frac{13}{8}\right)$



c) LMIN: $(1,\ 0)$

LMAX: $(2,\ 0)$

d) *point of inflection* : $x=\frac{13}{8}$

Exercise

A company is considering a new manufacturing process in one of its plants. The new process provides substantial initial savings, with the savings declining with time t (in years) according to the rate-of-savings function

$$S'(t) = 100 - t^2$$

where $S'(t)$ is in thousands of dollars per year. At the same time, the cost of operating the new process increases with time t (in years), according to the rate-of-cost function (in thousands of dollars per year)

$$C'(t) = t^2 + \frac{14}{3}t$$

a) For how many years will the company realize savings?

b) What will be the net total savings during this period?

Solution

a) For how many years will the company realize savings?

$$C'(t) = S'(t)$$

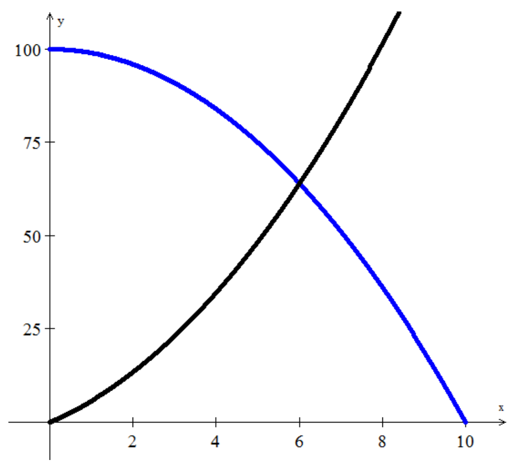
$$t^2 + \frac{14}{3}t = 100 - t^2$$

$$2t^2 + \frac{14}{3}t - 100 = 0$$

$$\to t = -\frac{25}{3} \ or \ \ 6$$

$$\boxed{t = 6}$$

The company should use this type for 6 years.



b) What will be the net total savings during this period?

$$\text{Total savings } = \int_0^6 \left[\left(100 - t^2\right) - \left(t^2 + \frac{14}{3}t\right)\right] dt$$

$$= \int_0^6 \left[100 - 2t^2 - \frac{14}{3}t\right] dt$$

$$= 100t - \frac{2}{3}t^3 - \frac{7}{3}t^2 \Big|_0^6$$

$$= 100(6) - \frac{2}{3}(6)^3 - \frac{7}{3}(6)^2 - \left(100(0) - \frac{2}{3}(0)^3 - \frac{7}{3}(0)^2\right)$$

$$= 372$$

The company will save a total of \$372,000. Over the 6-*year* period

Exercise

Find the producers' surplus if the supply function for pork bellies is given by

$$S(x) = x^{5/2} + 2x^{3/2} + 50$$

Assume supply and demand are in equilibrium at $x = 16$.

Solution

The equilibrium price:

$$\boxed{p_0 = S(x=16) = 16^{5/2} + 2(16)^{3/2} + 50 = 1202}$$

$$\text{Producer's surplus } = \int_0^{x_0} \left[p_0 - S(x)\right]dx$$

$$= \int_0^{16} \left[1202 - \left(x^{5/2} + 2x^{3/2} + 50\right)\right]dx$$

$$= \int_0^{16} \left[1152 - x^{5/2} - 2x^{3/2}\right]dx$$

$$= 1152x - \tfrac{2}{7}x^{7/2} - \tfrac{4}{5}x^{5/2}\Big|_0^{16}$$

$$= \left(1152(16) - \tfrac{2}{7}(16)^{7/2} - \tfrac{4}{5}(16)^{5/2}\right) - \left(1152(0) - \tfrac{2}{7}(0)^{7/2} - \tfrac{4}{5}(0)^{5/2}\right)$$

$$\boxed{= \$12{,}931.66}$$

The producers' surplus is \$12,931.66

Exercise

An object moves along a line with a velocity in m/s given by $v(t) = 8\cos\left(\frac{\pi t}{6}\right)$. Its initial position is $s(0) = 0$.

a) Graph the velocity function.

b) The position of the object is given by $s(t) = \int_0^t v(y)\,dy$, for $t \ge 0$. Find the position function, for $t \ge 0$.

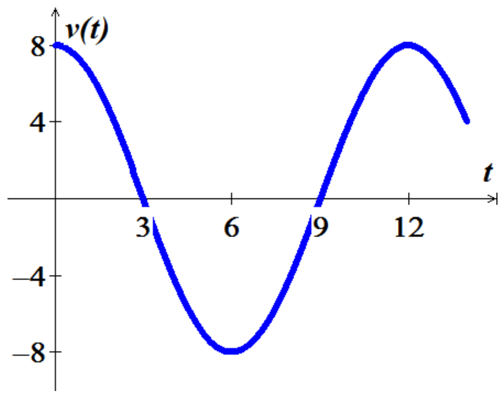
c) What is the period of the motion – that is, starting at any point, how long does it take the object to return to that position?

Solution

a) $v(t) = 8\cos\left(\frac{\pi t}{6}\right)$

$|A| = 8 \quad P = 12$

t	$v(t)$
0	8
3	0
6	–8
9	0
12	8



b) $s(t) = \int_0^t v(y)\,dy$

$$= \int_0^t 8\cos\left(\frac{\pi}{6}y\right) dy$$

$$= \frac{48}{\pi}\sin\left(\frac{\pi}{6}y\right)\Big|_0^t$$

$$\boxed{= \frac{48}{\pi}\sin\frac{\pi}{6}t}$$

c) Period: $P = \dfrac{2\pi}{\frac{\pi}{6}}$ $\boxed{=12}$

Exercise

The population of a culture of bacteria has a growth rate given by $p'(t) = \dfrac{200}{(t+1)^r}$ bacteria per hour, for $t \ge 0$, where $r > 1$ is a real number. It is shown that the increase in the population over time interval $[0,\ t]$ is given by $\int_0^t p'(s)\,ds$. (note that the growth rate decreases in time, reflecting competition for space and food.)

a) Using the population model with $r = 2$, what is the increase in the population over the time interval $0 \le t \le 4$?

b) Using the population model with $r = 3$, what is the increase in the population over the time interval $0 \le t \le 6$?

c) Let ΔP be the increase in the population over a fixed time interval $[0,\ T]$. For fixed *T*, does ΔP increase or decrease with the parameter *r*? Explain.

d) A lab technician measures an increase in the population of 350 bacteria over the 10-*hr* period $[0,\ 10]$. Estimate the value of *r* that best fits this data point.

e) Use the population model in part (*b*) to find the increase in population over time interval $[0,\ T]$, for any $T > 0$. If the culture is allowed to grow indefinitely $(T \to \infty)$, does the bacteria population increase without bound? Or does it approach a finite limit?

Solution

a) $r = 2\ \&\ 0 \le t \le 4$

$$\int_0^4 \frac{200}{(t+1)^2}\, dt = \int_0^4 \frac{200}{(t+1)^2}\, d(t+1)$$

$$= -\frac{200}{t+1}\Big|_0^4$$

$$= -(40 - 200)$$

$$\boxed{= 160}$$

b) $r = 3\ \&\ 0 \le t \le 6$

$$\int_0^6 \frac{200}{(t+1)^3}\, dt = 200\int_0^6 (t+1)^{-3}\, d(t+1)$$

$$= -100\frac{1}{(t+1)^2}\Big|_0^6$$

$$= -100\left(\frac{1}{49} - 1\right)$$

$$\boxed{= \frac{4800}{49}}$$

c) $\Delta P = \int_0^T \frac{200}{(t+1)^r}\, dt$ decreases as r increases.

Because $\frac{200}{(t+1)^r} > \frac{200}{(t+1)^{r+1}}$

d) $$\int_0^{10} \frac{200}{(t+1)^r}\, dt = 350$$

$$200\int_0^{10} (t+1)^{-r}\, d(t+1) = 350$$

$$\frac{1}{1-r}(t+1)^{1-r}\Big|_0^{10} = \frac{7}{4}$$

$$\frac{1}{1-r}\left(11^{1-r} - 1\right) = \frac{7}{4}$$

$$4(11)^{1-r} - 4 = 7 - r$$

$$4(11)^{1-r} + r - 44 = 0 \xrightarrow{\text{using software}} \boxed{r \approx 1.278}$$

e) $$\int_0^T \frac{200}{(t+1)^3}\,dt = -100\frac{1}{(t+1)^2}\Big|_0^T$$

$$= -100\left(\frac{1}{(T+1)^2} - 1\right)$$

$$= 100 - \frac{100}{(T+1)^2}$$

$$\lim_{T\to\infty}\left(100 - \frac{100}{(T+1)^2}\right) \boxed{= 100}$$

∴ The bacteria approach a finite limit of 100.

Exercise

Consider the function $f(x) = x^2 - 5x + 4$ and the area function $A(x) = \int_0^x f(t)\,dt$.

a) Graph f on the interval [0, 6].

b) Compute and graph A on the interval [0, 6].

c) Show that the local extrema of A occur at the zeros of f.

d) Give a geometric and analytical explanation for the observation in part (c).

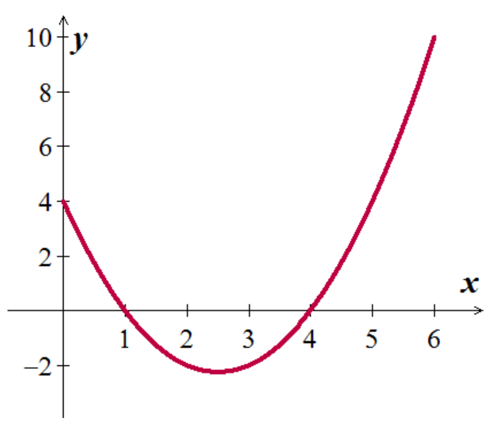
e) Find the approximate zeros of A, other than 0, and call them x_1 and x_2.

f) Find b such that the area bounded by the graph of f and the x-axis on the interval $\left[0,\ x_1\right]$ equals the area bounded by the graph of f and the x-axis on the interval $\left[x_1,\ b\right]$.

g) If f is an integrable function and $A(x) = \int_0^x f(t)\,dt$, is it always true that the local extrema of A occur at the zeros of f? Explain

Solution

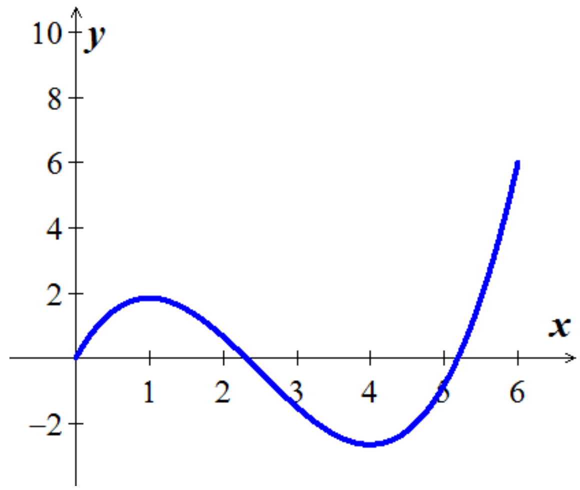
a)



b) $A(x) = \int_0^x f(t)\,dt = \int_0^x \left(t^2 - 5t + 4\right)dt$

$= \left(\frac{1}{3}t^3 - \frac{5}{2}t^2 + 4t\right)_0^x$

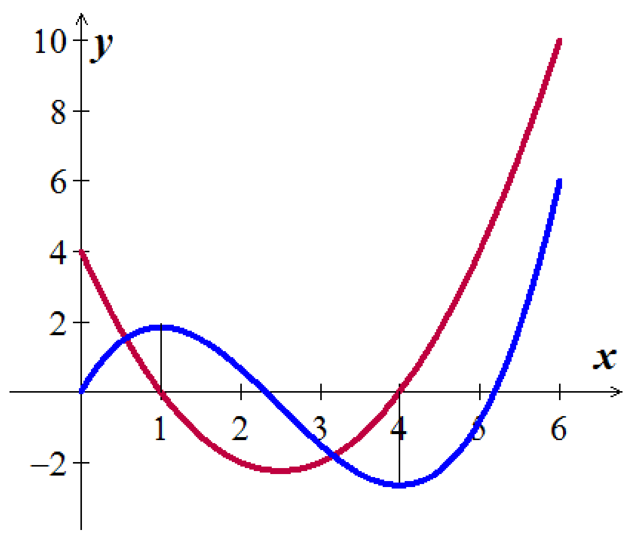
$= \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$



c) $f(x) = x^2 - 5x + 4 = 0 \quad \Rightarrow \quad \boxed{x = 0,\ 4}$

$A(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x \quad \to \quad A'(x) = f(x)$

The zeros of *f* are at 1 and 4, and *A* has a local maximum at *x* = 1 and local minimum at *x* = 4.



d) Since *f* is above the axis from 0 to 1, the net area *A* is increasing and switches to decreasing to the right of 1. When *x* is between 1 and 4, the function *f* is below *x*-axis (negative sign), the Area *A* is decreasing.

By the fundamental Theorem: $A'(x) = f(x)$, the zeros of *f* are critical points of *A*.

e) $A(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x = \frac{1}{6}x\left(2x^2 - 15x + 24\right)$

$$x = \frac{15 \pm \sqrt{33}}{4} \to \begin{cases} x_1 = \frac{15-\sqrt{33}}{4} \approx 2.31386 \\ x_2 = \frac{15+\sqrt{33}}{4} \approx 5.18614 \end{cases}$$

f) $A_1 = \int_0^{x_1} f(x)\,dx = \int_0^1 f(x)\,dx + \left| \int_1^{x_1} f(x)\,dx \right|$

$$= \left[\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x \right]_0^1 + \left| \left[\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x \right]_1^{x_1} \right|$$

$$= \left[\left(\frac{1}{3} - \frac{5}{2} + 4 \right) - 0 \right] + \left| 0 - \left(\frac{1}{3} - \frac{5}{2} + 4 \right) \right|$$

$$= 2\left(\frac{1}{3} - \frac{5}{2} + 4 \right)$$

$$= 2\left(\frac{11}{6} \right)$$

$$= \frac{11}{3}$$

$A_2 = \left| \int_{x_1}^{x_2} f(x)\,dx \right| + \int_{x_2}^{b} f(x)\,dx$

$$= \left| \left[\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x \right]_{x_1}^{x_2} \right| + \left[\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x \right]_{x_2}^{b}$$

$$= 0 + \left[\left(\frac{1}{3}b^3 - \frac{5}{2}b^2 + 4b \right) - 0 \right]$$

$$= \frac{1}{3}b^3 - \frac{5}{2}b^2 + 4b$$

Since $A_1 = A_2$

$\frac{1}{3}b^3 - \frac{5}{2}b^2 + 4b = \frac{11}{3}$

$\frac{1}{3}b^3 - \frac{5}{2}b^2 + 4b - \frac{11}{3} = 0 \quad \to \quad \boxed{b = 5.744348}$ (and 2 complex numbers)

g) No, if the function is a piecewise function.

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ -1 & \text{if } 1 \le x \le 2 \end{cases}$$

Then $A(x)$ has a maximum at $x = 1$ even though f is never zero.

This is a case where an extreme point occurs at a singular point rather than a stationary point.