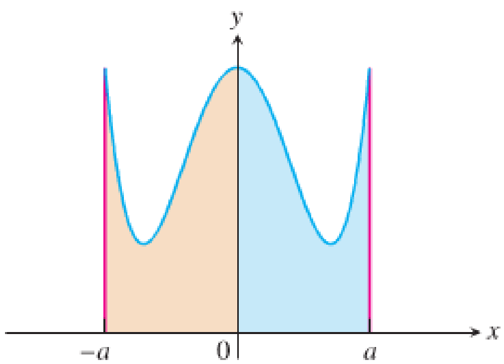
Section 4.5 – Working with Integrals

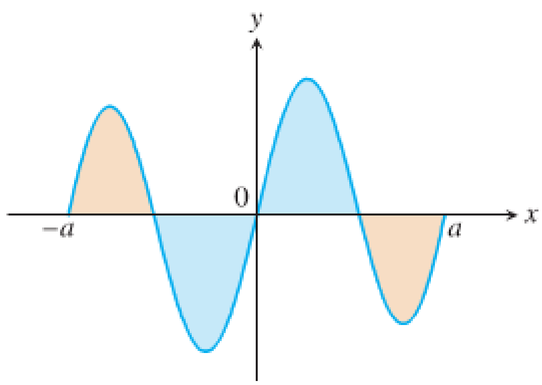
Definite Integrals of Symmetric Functions

Theorem

Let f be continuous on the symmetric interval $[-a, a]$

- ✓ If f is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
- ✓ If f is odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$





Example

Evaluate $\displaystyle\int_{-2}^{2} \left(x^4 - 4x^2 + 6\right) dx$

Solution

Since $f(-x) = f(x) \quad \Rightarrow \quad f(x)$ *is even*

$$\int_{-2}^{2} \left(x^4 - 4x^2 + 6\right) dx = 2\int_{0}^{2} \left(x^4 - 4x^2 + 6\right) dx$$

$$= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2$$

$$= 2\left[\left(\frac{2^5}{5} - \frac{4}{3}2^3 + 6(2)\right) - 0\right]$$

$$= \frac{232}{15}$$

Average Value of a Continuous Function Revisited

The average value of a nonnegative continuous function *f* over an interval [*a*, *b*], leading us to define this average as the area under the graph of $y = f(x)$ divided by $b - a$.

$$Average = \frac{1}{b-a}\int_a^b f(x)dx$$

Definition

If *f* is integrable on [*a*, *b*], then its *average value* on [*a*, *b*], also called its ***mean***, is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

Example

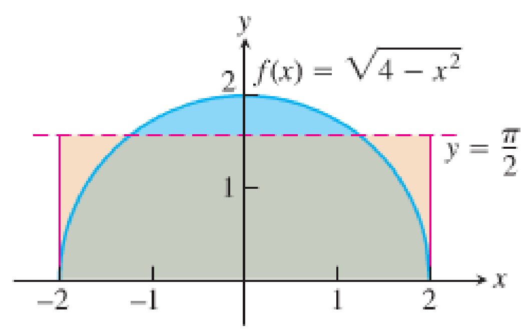
Find the average value of $f(x) = \sqrt{4 - x^2}$ *on* $[-2,\ 2]$

Solution

$f(x) = \sqrt{4 - x^2}$ is a function of an upper semicircle with a radius 2 and centered at the origin.

The area between the semicircle and the x-axis from –2 to 2 can be computed using the geometry formula:

$$Area = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(2)^2 = 2\pi$$ -



$$Area = \int_{-2}^{2}\sqrt{4 - x^2}\ dx = 2\pi$$

$$av(f) = \frac{1}{2-(-2)}\int_{-2}^{2}\sqrt{4 - x^2}dx$$

$$= \frac{1}{4}(2\pi)$$

$$= \boxed{\frac{\pi}{2}}$$

Example

We can model the voltage in the electrical wiring of a typical home with the sine function

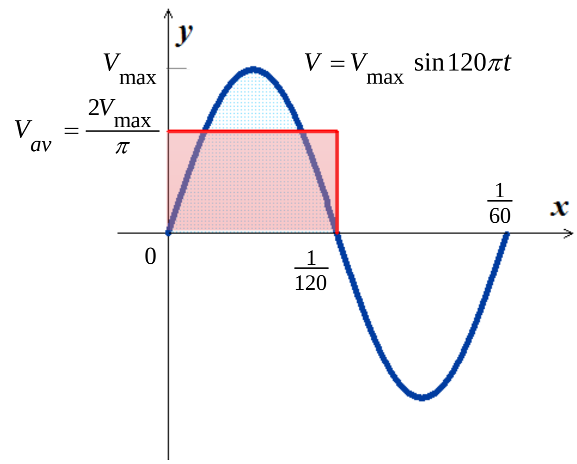
$$V = V_{\max} \sin 120\pi t$$

Which express the voltage V in volts as a function of time t in seconds. The function runs through 60 cycles each second (its frequency is 60 Hz (*hertz*)). The positive constant $V_{\max}$ is the ***peak voltage***.

Solution

The average value of V over the half-cycle from 0 to $\frac{1}{120}$ sec is

$$\begin{aligned}
V_{av} &= \frac{1}{\frac{1}{120}-0}\int_0^{1/120} V_{\max} \sin 120\pi t \; dt \\
&= 120V_{\max}\int_0^{1/120} \sin 120\pi t \; dt \\
&= 120V_{\max}\left[-\frac{1}{120\pi}\cos 120\pi t\right]_0^{1/120} \\
&= -\frac{1}{\pi}V_{\max}\left(\cos\left(120\pi\frac{1}{120}\right)-\cos\left(120\pi\cdot 0\right)\right) \\
&= -\frac{1}{\pi}V_{\max}\left(\cos\pi-\cos 0\right) \\
&= -\frac{1}{\pi}V_{\max}\left(-1-1\right) \\
&= \boxed{\frac{2}{\pi}V_{\max}}
\end{aligned}$$



The average value of the voltage over a full cycle is zero.

To measure the voltage effectively, we can use an instrument the square root of the average value of the square of the voltage, namely:

$$V_{rms} = \sqrt{\left(V^2\right)_{av}}$$

"rms" : root mean square.

$$V^2_{av} = \frac{1}{\frac{1}{60}-0}\int_0^{1/60} \left(V_{\max}\right)^2 \sin^2 120\pi t \; dt = \frac{\left(V_{\max}\right)^2}{2}$$

The rms voltage is: $V_{rms} = \sqrt{\left(V^2\right)_{av}} = \frac{V_{\max}}{\sqrt{2}}$

The "115 volts ac" means that the rms voltage is 115. The peak voltage is:

$$V_{\max} = \sqrt{2}\;V_{rms} = \sqrt{2}\;(115)\;\boxed{\approx 163 \;\; volts}$$

Exercises Section 4.5 – Working with Integrals

1. If f is an odd function, why is $\int_{-a}^{a} f(x)\,dx = 0$?

2. If f is an even function, why is $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

3. Is x^{12} an even or odd function? Is $\sin\left(x^2\right)$ an even or odd function?

Use symmetry to evaluate the following integrals

4. $\int_{-2}^{2} x^9 dx$

5. $\int_{-200}^{200} 2x^5 dx$

6. $\int_{-\pi/4}^{\pi/4} \cos x\; dx$

7. $\int_{-2}^{2} \left(x^9 - 3x^5 + 2x^2 - 10\right) dx$

8. $\int_{-\pi/2}^{\pi/2} \left(\cos 2x + \cos x \sin x - 3\sin x^5\right)\, dx$

Find the average value of the following functions on the given interval.

9. $f(x) = x^3 \quad on\ [-1,\ 1]$

10. $f(x) = \dfrac{1}{x^2+1} \quad on\ [-1,\ 1]$

11. $f(x) = \dfrac{1}{x} \quad on\ [1,\ e]$

12. $f(x) = e^{2x} \quad on\ [0,\ \ln 2]$

Suppose that $\int_{0}^{4} f(x)\,dx = 10$ and $\int_{0}^{4} g(x)\,dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integrals

13. $\int_{-4}^{4} f(x)\,dx$

14. $\int_{-4}^{4} 3g(x)\,dx$

15. $\int_{0}^{1} 8xf\left(4x^2\right)dx$

16. $\int_{-2}^{2} 3xf(x)\,dx$

17. $\int_{-4}^{4} \left(4f(x) - 3g(x)\right)dx$

Zuppose that f is an even function with $\int_0^8 f(x)\,dx = 9$. Evaluate the integral

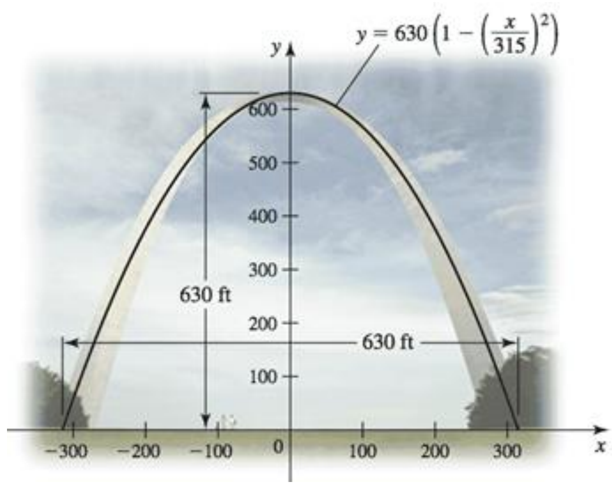
18. $\int_{-1}^{1} x\, f\left(x^2\right) dx$ **19.** $\int_{-2}^{2} x^2\, f\left(x^3\right) dx$

Suppose that p is a nonzero real number and f is an odd integrable function with $\int_0^1 f(x)\,dx = \pi$.

Evaluate the integral

20. $\int_0^{\frac{\pi}{2p}} (\cos px)\, f(\sin px)\, dx$ **21.** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)\, f(\sin x)\, dx$

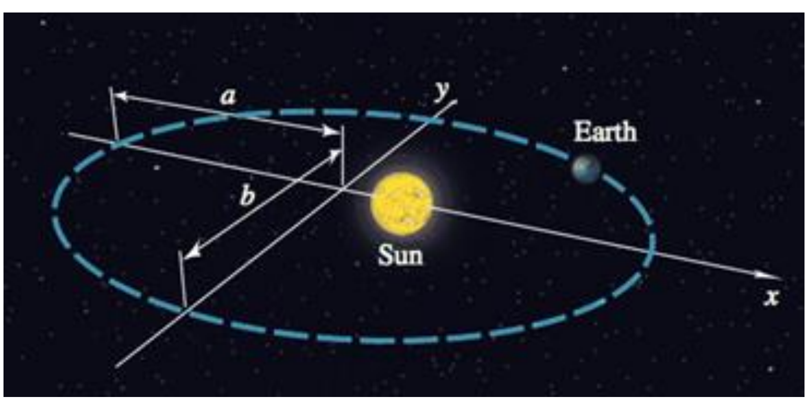
22. The Gateway Arch in St. Louis is 630 *ft* high and has a 630-*ft* base. Its shape can be modeled by the parabola



$$y = 630\left(1 - \left(\frac{x}{315}\right)^2\right)$$

Find the average height of the arch above the ground.

23. The planets orbit the Sun in elliptical orbits with the Sun at one focus. The equation of an ellipse whose dimensions are 2 a in the x-direction and 2 b in the y-direction is



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ W}$$

a) Let d^2 denote the square of the distance from a planet to the center of the ellipse at (0, 0). Integrate over the interval $[-a,\ a]$ to show that the average value of d^2 is $\frac{a^2+2b^2}{3}$

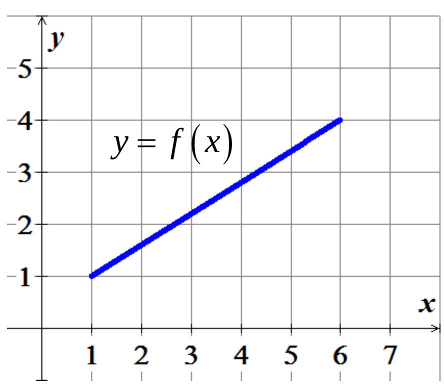
b) Show that in the case of a circle (a = b = R), the average value in part (a) is R^2.

c) Assuming 0 < b < a, the coordinates of the Sun are $\left(\sqrt{a^2-b^2},\ 0\right)$. Let D^2 denote the square of the distance from the planet to the Sun. Integrate over the interval $[-a,\ a]$ to show that the average value of D^2 is $\frac{4a^2-b^2}{3}$.

24. A particle moves along a line with a velocity given by $v(t)=5\sin \pi t$ starting with an initial position $s(0)=0$. Find the displacement of the particle between $t=0$ and $t=2$, which is given by $s(t)=\int_0^2 v(t)\,dt$. Find the distance traveled by the particle during this interval, which is $\int_0^2 |v(t)|\,dt$.

25. A baseball is launched into the outfield on a parabolic trajectory given by $y=0.01x(200-x)$. Find the average height of the baseball over the horizontal extent of its flight.

26. Find the average value of f shown in the figure on the interval $[1,\ 6]$ and then find the point(s) c in $(1,\ 6)$ guaranteed to exist by the Mean Value Theorem for Integrals



27. Find the average value of f shown in the figure on the interval $[2,\ 6]$ and then find the point(s) c in $(2,\ 6)$ guaranteed to exist by the Mean Value Theorem for Integrals