Section 1.6 – Determinants and Properties

The determinant is a number that contains information about matrix. It is used to find formulas for inverse matrices, pivots, and solutions $A^{-1}b$.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ has inverse } A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Determinant of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is written $\det(A)$ *or* $|A|$ and is define as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinant is zero when the matrix has no inverse.

Properties of the Determinants

By using these property rules, we can compute the determinant of any square matrix.

1. *Determinant of the n by n identity matrix is* 1.

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad \text{and} \quad \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} = 1$$

2. *Determinant changes sign when 2 rows are exchanged.*

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -(ad - bc) \quad \Rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

3. *Determinant is a linear function of each row separately*.

Multiply row 1 by any number t: $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Add row 1 of A to row 1 of A': $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

- ***For 2 by 2 determinants, if you expand to a rectangle, the determinants equal areas.***
- ***For n-dimensional, the determinants equal volumes.***

4. ***If 2 rows of A are equal, then*** $\det A = 0$.

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$$

5. ***Subtracting a multiple of one row from another row leaves*** $\det A$ ***unchanged.***

$$\begin{vmatrix} a & b \\ c - ta & d - ta \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

6. ***A matrix with a row of zeros has*** $\det A = 0$.

$$\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0 \quad and \quad \begin{vmatrix} 0 & 0 \\ b & c \end{vmatrix} = 0$$

7. ***If A is triangular then*** $\det A = a_{11} a_{22} \ \ldots \ a_{nn}$ ***= product of diagonal entries.***

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad \quad and \quad \begin{vmatrix} a_{11} & & 0 \\ & a_{22} & \\ & & \\ 0 & & a_{nn} \end{vmatrix} = a_{11} a_{22} \ldots a_{nn}$$

8. ***If A is singular then*** $\det A = 0$.

9. ***If A is invertible then*** $\det A \neq 0$.

10. ***The determinant of AB*** $\det A$ ***is times*** $\det B$ ***:*** $|AB| = |A||B|$

11. ***The transpose*** A^T ***has the same determinant as A:*** $\det(A) = \det\left(A^T\right)$

- $\det(A + B) \neq \det(A) + \det(B)$

Big Formula for Determinants (Diagonal)

Determinant Using Diagonal Method

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix} \qquad a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \quad (1)$$

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix} \qquad -a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \quad (2)$$

Determinant: $\boldsymbol{D} = (1) + (2)$

$$\textbf{det} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Example

Evaluate: $\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix} \begin{matrix} x & 0 \\ 2 & x \\ -3 & x \end{matrix} = x(x)(1) + 0\left(x^2\right)(2) + (-1)(2)(x) - (-1)(x)(-3) - x\left(x^2\right)(x) - 0(-3)(1)$$

$$= x^2 - 2x - 3x - x^4$$

$$\underline{= x^2 - 5x - x^4}$$

Determinant by *Cofactors*

$$\boldsymbol{A}=[a_{ij}]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor

For a square matrix $A=\left[a_{ij}\right]$, the minor M_{ij}. Of an element a_{ij} is the ***determinant*** of the matrix formed by deleting the i^{th} row and the j^{th} column of A.

Example

Let $A=\begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$ Find M_{32}

Solution

$$M_{32}=\begin{vmatrix} 3 & \cancel{1} & -4 \\ 2 & \cancel{5} & 6 \\ \cancel{1} & \cancel{4} & \cancel{8} \end{vmatrix}$$

$$=\begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix}$$

$$=26$$

Theorem

The determinant is the dot product of any row $\boldsymbol{i}$ of $\boldsymbol{A}$ with its cofactors:

Cofactor Formula: $\boxed{C_{ij}=(-1)^{i+j}M_{ij}}$

$$|\boldsymbol{A}|=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}$$

$$=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example

Find the determinant of the matrix:

$$A=\begin{bmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{bmatrix}$$

Solution

$$|A|=\begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix}$$

$$=-8\begin{vmatrix} -6 & 7 \\ -3 & 5 \end{vmatrix}-0\begin{vmatrix} 4 & 7 \\ -1 & 5 \end{vmatrix}+6\begin{vmatrix} 4 & -6 \\ -1 & -3 \end{vmatrix}$$

$$=-8(-30-(-21))-0+6(-12-6)$$

$$=-8(-9)+6(-18)$$

$$\underline{=-36}$$

✓ By the property of determinants, If A is triangular then $\det A = a_{11}a_{22}\ \ldots\ a_{nn}$ = product of diagonal entries.

Example

$$\begin{vmatrix} 2 & 7 & -3 & 8 & 3 \\ 0 & -3 & 7 & 5 & 1 \\ 0 & 0 & 6 & 7 & 6 \\ 0 & 0 & 0 & 9 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix}=(2)(-3)(6)(9)(4)$$

$$\underline{=-1296}$$

Theorem

Let A be any n by n matrix.

a) If A' is the matrix that results when a single row of A is multiplied by a constant k, then $\det(A') = k\det(A)$.

b) If A' is the matrix that results when two rows of A are interchanged, then $\det(A') = -\det(A)$

c) If A' is the matrix that results when a multiple of one row of A is added to another row then $\det(A') = \det(A)$

Example

Evaluate $\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$ ***Interchanged 1st and 2nd row***

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$ ***A common factor of 3 from the first row (no need)***

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \; R_3 - 2R_1$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} \; R_3 - 10R_2$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

$$= -3(1)(1)(-55)$$

$$\underline{= 165}$$

Exercises *Section* 1.6 – Determinants and Properties

1. Verify that $\det(AB)=\det(A)\det(B)$ when: $A=\begin{pmatrix}2 & 1 & 0\\ 3 & 4 & 0\\ 0 & 0 & 2\end{pmatrix}$ *and* $B=\begin{pmatrix}1 & -1 & 3\\ 7 & 1 & 2\\ 5 & 0 & 1\end{pmatrix}$

2. For which value(s) of ***k*** does A fail to be invertible? $A=\begin{bmatrix}k-3 & -2\\ -2 & k-2\end{bmatrix}$

3. Without directly evaluating, show that $\begin{vmatrix}b+c & c+a & b+a\\ a & b & c\\ 1 & 1 & 1\end{vmatrix}=0$

4. If the entries in every row of A add to zero, solve $\boldsymbol{Ax}=0$ to prove $\det(A)=0$. If those entries add to one, show that $\det(A-I)=0$. Does this mean $\det(A)=I$?

5. Does $\det(AB)=\det(BA)$ in general?

a) True or false if ***A*** and ***B*** are square *n* x *n* matrices?

b) True or false if ***A*** is *m* x *n* and B is *n* x *m* with $m \neq n$?

6. True or false, with a reason if true or a counterexample if false:

a) The determinant of $I+A$ is $1+\det(A)$.

b) The determinant of ABC is $|A||B||C|$.

c) The determinant of $4A$ is $4|A|$

d) The determinant of $AB-BA$ is zero. (try an example)

e) If A is not invertible then AB is not invertible.

f) The determinant of $A-B$ equals to $\det(A)-\det(B)$.

7. Use row operations to show the 3 by 3 "Vandermonde determinant" is

$$\det\begin{bmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{bmatrix}=(b-a)(c-a)(c-b)$$

8. The inverse of a 2 by 2 matrix seems to have determinant = 1:

$$\det A^{-1} = \det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{ad-bc}{ad-bc} = 1$$

What is wrong with this calculation? What is the correct $\det A^{-1}$

9. A *Hessenberg* matrix is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci's rule $|H_4| = |H_3| + |H_2|$. The same rule will continue for all sizes $|H_n| = |H_{n-1}| + |H_{n-2}|$. Which Fibonacci number is $|H_n|$?

$$H_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad H_3 = \begin{bmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad H_4 = \begin{bmatrix} 2 & 1 & & \\ \mathbf{1} & 2 & \mathbf{1} & \\ \mathbf{1} & 1 & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

(10 – 44) Evaluate

10. $\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix}$

11. $\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix}$

12. $\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix}$

13. $\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix}$

14. $\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix}$

15. $\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix}$

16. $\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}$

17. $\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix}$

18. $\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix}$

19. $\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix}$

20. $\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix}$

21. $\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix}$

22. $\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix}$

23. $\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix}$

24. $\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix}$

25. $\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix}$

26. $\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix}$

27. $\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix}$

28. $\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix}$

29. $\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix}$

30. $\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix}$

31. $\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix}$

32. $\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix}$

33. $\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix}$

34. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix}$

35. $\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix}$

36. $\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix}$

37. $\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix}$

38. $\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix}$

39. $\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix}$

40. $\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix}$

41. $\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix}$

42. $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix}$

43. $\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix}$

44. $\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix}$

45. Find all the values of λ for which $\det(A)=0$

a) $A=\begin{bmatrix} \lambda-1 & -2 \\ 1 & \lambda-4 \end{bmatrix}$ *b)* $A=\begin{bmatrix} \lambda-6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda-4 \end{bmatrix}$

46. Prove that if a square matrix $\boldsymbol{A}$ has a column of zeros, then $\det(A)=0$

47. With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D| \quad \textit{but} \quad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D|-|C||B|$$

a) Why is the first statement true? Somehow B doesn't enter.

b) Show by example that equality fails (as shown) when C enters.

c) Show by example that the answer $\det(AD-CB)$ is also wrong.

48. Show that the value of the following determinant is independent of θ.

$$\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta-\cos\theta & \sin\theta+\cos\theta & 1 \end{vmatrix}$$

49. Show that the matrices $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ *and* $B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$

commute if and only if $\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$

50. Show that $\det(A) = \frac{1}{2}\begin{vmatrix} tr(A) & 1 \\ tr\left(A^2\right) & tr(A) \end{vmatrix}$ for every 2×2 matrix A.

51. What is the maximum number of zeros that a 4×4 matrix can have without a zero determinant? Explain your reasoning.

52. Evaluate $\det(A)$, $\det(E)$, and $\det(AE)$. Then verify that $\det(A) \cdot \det(E) = \det(AE)$

$$A = \begin{bmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & & \\ & 3 & \\ & & 1 \end{bmatrix}$$

53. Show that $\begin{bmatrix} \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma \\ 1 & 1 & 1 \end{bmatrix}$ is not invertible for any values of α, β, γ

54. The determinant of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det(A) = ad - bc$.

Assuming no rows swaps are required, perform elimination on A and show explicitly that $ad - bc$ is the product of the pivots.

55. If A is a 7×7 matrix and let $\det(A) = 17$. What is $\det\left(3A^2\right)$?

56. Explain without computations why the following determinant is equal to zero

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & 0 & 0 & 0 \\ d_1 & d_2 & 0 & 0 & 0 \\ e_1 & e_2 & 0 & 0 & 0 \end{vmatrix}$$

57. Let A be $n \times n$ real matrix.

a) Show that if $A^t = -A$ and n is odd, then $|A| = 0$.

b) Show that if $A^2 + I = 0$, then n must be even.

c) Does part (b) remain true for complex matrices?

58. Let A and C be $m \times m$ and $n \times n$ matrices, respectively.

a) Show that $\begin{vmatrix} A & B \\ 0 & C \end{vmatrix} = \begin{vmatrix} A & 0 \\ B & C \end{vmatrix} = |A||C|$

b) Evaluate

i. $\begin{vmatrix} I_m & 0 \\ 0 & I_n \end{vmatrix}$

ii. $\begin{vmatrix} 0 & I_m \\ I_n & 0 \end{vmatrix}$

iii. $\begin{vmatrix} I_m & B \\ 0 & I_n \end{vmatrix}$

iv. $\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix}_{n \times n}$

c) Find a formula for $\begin{vmatrix} 0 & A \\ C & B \end{vmatrix}_{n \times n}$

59. Let $f(x) = (p_1 - x)(p_2 - x)\ldots(p_n - x)$ and let

$$\Delta_n = \begin{vmatrix} p_1 & a & a & \ldots & a & a \\ b & p_2 & a & \ldots & a & a \\ b & b & p_3 & \ldots & a & a \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b & b & b & \cdots & p_{n-1} & a \\ b & b & b & \ldots & b & p_n \end{vmatrix}$$

a) Show that, if $a \neq b$,

$$\Delta_n = \frac{bf(a) - af(b)}{b - a}$$

b) Show that, if $a = b$,

$$\Delta_n = a\sum_{i=1}^{n} f_i(a) + p_n f_n(a)$$

Where $f_i(a)$ means $f(a)$ with factor $(p_i - a)$ missing.

c) Use part (b) to evaluate

$$\begin{vmatrix} a & b & b & \dots & b & b \\ b & a & b & \dots & b & b \\ b & b & a & \dots & a & a \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b & b & b & \cdots & a & a \\ b & b & b & \dots & b & a \end{vmatrix}_{n\times n}$$

60. Let $A, B, C, D \in M_n(\mathbb{C})$

a) Show that when A is invertible: $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|\left|D - CA^{-1}B\right|$

b) Show that when $AC = CA$: $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|$

c) Can B and C on the right-hand side of the identity be switched?

d) Does part (b) remain true if the condition $AC = CA$ is dropped?