Lecture R – *Review*

Section R.1 – Basic Algebra Review

R.1–1 Adding and Subtracting Polynomials

Properties of Real numbers

For all real numbers a, b, and c:

$a+b=b+a$ $ab=ba$	***Commutative properties***
$(a+b)+c=a+(b+c)$ $(ab)c=a(bc)$	***Associative properties***
$a(b+c)=ab+ac$	***Distributive properties***

Example 1

Perform the indicated operation:

$$\left(8x^3-4x^2+6x\right)+\left(3x^3+5x^2-9x+8\right)=8x^3-4x^2+6x+3x^3+5x^2-9x+8$$

$$=\left(8x^3+3x^3\right)+\left(-4x^2+5x^2\right)+(6x-9x)+8$$

$$\underline{=11x^3+x^2-3x+8}$$

R.1–2 Multiplication Polynomials

Example 2

Perform the indicated operation:

$$8x(6x-4)=8x(6x)-8x(4)$$

$$\underline{=48x^2-32x}$$

Example 3

Perform the indicated operation:

$$(3p-2)\left(p^2+5p-1\right)=3p^3+15p^2-3p-2p^2-10p+2$$

$$=3p^3+13p^2-13p+2$$

Example 4

Perform the indicated operations:

$$2\left(3x^2+4x+2\right)-3\left(-x^2+4x-5\right)=6x^2+8x+4+3x^2-12x+15$$

$$=9x^2-4x+19$$

R.1–3 Factoring

Prime Factorization

A process that allows us to write a composite number as a product of two or more prime numbers.

Factoring Trinomial

Example 5

Factor $y^2+8y+15$

Solution

Product 15	***Sum*** 8
15 *x* 1	15 + 1
3 *x* 5	3 + 5

$$y^2+8y+15=(y+3)(y+5)$$

Special Factorization

$a^2 - b^2 = (a-b)(a+b)$

$a^2 - 2ab + b^2 = (a-b)^2$

$a^2 + 2ab + b^2 = (a+b)^2$

$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$

$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$

R.1–4 Foiling

Pascal's Triangle

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}
$$

$$(a+b)^n = a^n + {}^*a^{n-1}b + {}^*a^{n-2}b^2 + \ldots + {}^*a^2b^{n-2} + {}^*ab^{n-1} + b^n$$

Example 6

Use Pascal's triangle to expand $(a+b)^7$

Solution

$$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

R.1–5 Exponents

Integer Exponents

Definition of exponent

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n-times}$$

a appears as a factor *n* times

$$a^0 = 1 \qquad a^{-n} = \frac{1}{a^n} \qquad a^{m/n} = \left(a^{1/n}\right)^m$$

$$a^m \cdot a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (ab)^m = a^m b^m$$

$$\left(a^m\right)^n = a^{mn} \qquad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \qquad \left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n}$$

Example 7

Simplify $27^{2/3}$

Solution

$$\begin{aligned} 27^{2/3} &= \left(27^{1/3}\right)^2 \\ &= \left(\left(3^3\right)^{1/3}\right)^2 \\ &= \left(3^{3 \cdot \frac{1}{3}}\right)^2 \\ &= (3)^2 \\ &= 9 \end{aligned}$$

R.1–6 Radical

$$a^{1/n} = \sqrt[n]{a}$$

Properties

$$\left(\sqrt[n]{a}\right)^n = a \qquad \sqrt[n]{a}.\sqrt[n]{b} = \sqrt[n]{ab} \qquad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} \qquad \sqrt[n]{a^n} = \begin{cases} |a| & \text{if } n \text{ is even} \\ a & \text{if } n \text{ is odd} \end{cases}$$

Example 7

Simplify $\dfrac{1}{1-\sqrt{2}}$

Solution

$$\frac{1}{1-\sqrt{2}} = \frac{1}{1-\sqrt{2}}\,\frac{1+\sqrt{2}}{1+\sqrt{2}}$$

$$= \frac{1+\sqrt{2}}{1-2}$$

$$= \frac{1+\sqrt{2}}{-1}$$

$$= -1-\sqrt{2}$$

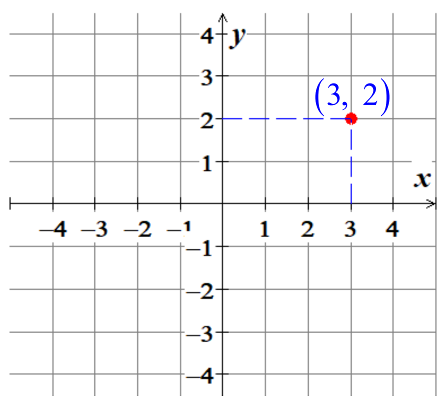
R.1–7 Cartesian / Rectangular Coordinates

In geometry, a point is a location that defines a point. A point has no length or no size. A point is notated by a dot.

Example 8

Draw the point $(3,\ 2)$

Solution



R.1–8 Lines

A ***linear*** equation, in mathematics, is an equation between two points (x_1, y_1) and (x_2, y_2) that give a straight ***line*** when plotted on a graph.

A line can be written as $y = mx + b$

m: is the slope can be found by: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

b: is y-intercept.

The general line formula can be written: $y = m\left(x - x_1\right) + y_1$

A *line* ***segment*** is a piece or part of a line having two endpoints.

Unlike a line, a line segment has a definite length.

Distance

A ***distance***, d, between the two points in the rectangular coordinate system is

$$d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

To complete the distance between two points. Find the square of the difference between the x-coordinate plus the square of the difference between the y-coordinates. The principal square root of this sum is the distance.

Midpoint

Consider a line segment whose endpoints are $\left(x_1, y_1\right)$ and $\left(x_2, y_2\right)$. The coordinates of the segment's midpoint are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

To find the midpoint, take the average of the two x-coordinates and the average of the y-coordinates

Example 9

For the given 2 points: $A\left(2,\ 3\right)$ & $B\left(-3, -1\right)$

1. Find an equation of a line that passes through the 2 points A & B. Then, plot the line
2. Find the distance between the 2 points.
3. Find the middle point.
4. Plot the segment AB.

Solution

a) $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{-1-3}{-3-2}$$

$$= \frac{-4}{-5}$$

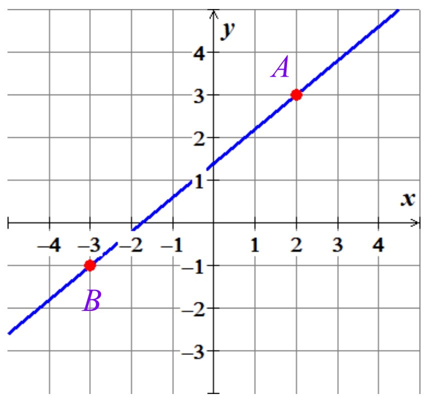
$$= \frac{4}{5}$$

$$y = \frac{4}{5}(x-2)+3 \qquad\qquad y = m\left(x-x_1\right)+y_1$$

$$= \frac{4}{5}x - \frac{8}{5} + 3$$

$$= \frac{4}{5}x + \frac{7}{5}$$

Or $\quad 5y - 4x = 7$



b) The distance between $A(2,\ 3)$ & $B(-3,-1)$ is:

$$d = \sqrt{(-2-3)^2 + (-1-3)^2} \qquad\qquad d = \sqrt{\left(x_2-x_1\right)^2 + \left(y_2-y_1\right)^2}$$

$$= \sqrt{(-5)^2 + (-4)^2}$$

$$= \sqrt{25+9}$$

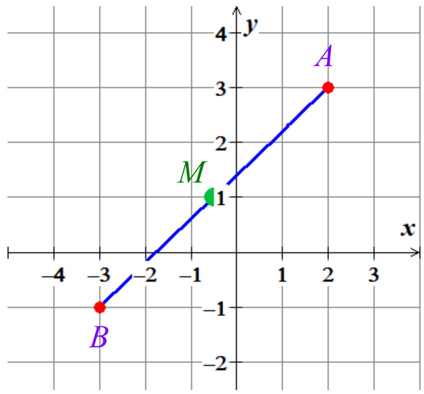
$$= \sqrt{34} \quad \textit{unit}$$

c) Let M be the middle point between $A(2,\ 3)$ & $B(-3,-1)$

$$M = \left(\frac{2-3}{2},\ \frac{3-1}{2}\right) \qquad\qquad M = \left(\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2}\right)$$

$$= \left(-\frac{1}{2},\ 1\right)$$

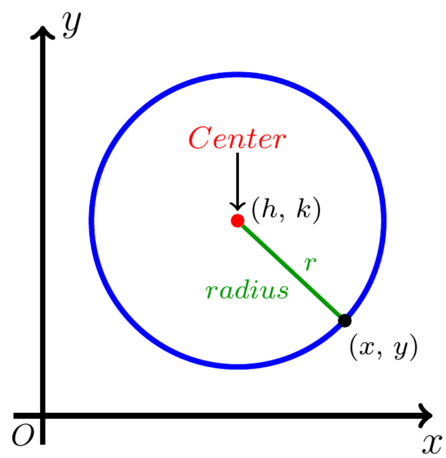
d)



R.1–9 The Standard Form of the Equation of a *Circle*

The *standard form of the equation of a circle* with center $(h,\ k)$ and radius r is

$$(x-h)^2+(y-k)^2=r^2$$



A circle with center $(0,\ 0)$ and radius r has equation: $x^2+y^2=r^2$

Example 10

Write in a standard form of a circle center at $(-3,4)$, radius 6.

Solution

$$(x-h)^2+(y-k)^2=r^2$$

$$(x-(-3))^2+(y-4)^2=6^2$$

$$(x+3)^2+(y-4)^2=36$$

Exercises *Section R.1 – Basic Algebra Review*

(1 – 7) Expand and simplify:

1. $(4x-y)^3$

2. $\left(\sqrt{x}-\sqrt{3}\right)^4$

3. $(ax+by)^5$

4. $(x+y)^6$

5. $(2x+5y)^7$

6. $(a+b)^8$

7. $\left(x-\frac{1}{x^2}\right)^9$

(8 – 11) Find a line equation that passes through the given points

8. $P(-4,\ 3)$ & $Q(2,\ -5)$

9. $P(8,\ 2)$ & $Q(3,\ 5)$

10. $\left(2\sqrt{3},\ \sqrt{6}\right)$ & $\left(-\sqrt{3},\ 5\sqrt{6}\right)$

11. $(-4,\ 9)$ & $(1,\ -3)$

(12 – 15) Find the distance between the two given points

12. $P(-4,\ 3)$ & $Q(2,\ -5)$

13. $P(8,\ 2)$ & $Q(3,\ 5)$

14. $\left(2\sqrt{3},\ \sqrt{6}\right)$ & $\left(-\sqrt{3},\ 5\sqrt{6}\right)$

15. $(-4,\ 9)$ & $(1,\ -3)$

(16 – 19) Find the midpoint of the line segment with endpoints

16. $P(-2,\ \text{-}1)$ & $Q(-8,\ 6)$

17. $P(8,\ 2)$ & $Q(3,\ 5)$

18. $(1,\ 2)$ & $(7,\ -3)$

19. $(7,\ -2)$ & $(9,\ 5)$

(20 – 23) Write the standard form of the equation of the circle

20. center $\left(-\sqrt{3},-\sqrt{3}\right)$, radius $\sqrt{3}$.

21. center $(-5,-3)$ and $r=\sqrt{5}$.

22. center $(6,\ -5)$ that passes through $(1,\ 7)$

23. diameter whose endpoints are $(4,\ 4)$ and $(-2,\ 3)$.

Section R.2 – Solving Equations

The numbers of solutions to a polynomial with ***n*** degree, where ***n*** is Natural Number, are ***n*** solutions.

R.2–1 Definition of a Linear Equation

A linear equation in one variable x is an equation that can be written in the form

$$ax + b = 0$$

where a and b are real number, and $a \neq 0$

Addition and Multiplication Properties of Equalities

$$If\ \ a = b,\ then\ \ a + c = b + c$$

$$If\ \ a = b,\ then\ \ ac = bc$$

Example 1

Solve: $3(2x-4) = 7-(x+5)$

Solution

$$6x - 12 = 7 - x - 5$$
$$6x - 12 + x = 2 - x + x$$
$$7x - 12 = 2$$
$$7x - 12 + 12 = 2 + 12$$
$$7x = 14$$
$$\frac{7}{7}x = \frac{14}{7}$$
$$\underline{x = 2}$$

Example 2

Solve: $\dfrac{x}{x-2} = \dfrac{2}{x-2} - \dfrac{2}{3}$

Solution

$$3(x-2)\frac{x}{x-2} = 3(x-2)\frac{2}{x-2} - (x-2)\frac{2}{3}$$

Restriction: $\boldsymbol{x \neq 2}$

$3x = 6 - 2(x-2)$

$3x = 6 - 2x + 4$

$3x + 2x = 10$

$5x = 10$

$x = \frac{10}{5}$

$\quad = 2$

No Solution ***or*** $\{\varnothing\}$, since the restriction $x \neq 2$

R.2–2 The ***Square Root*** Property

If u is an algebraic expression and d is a nonzero real number, then $u^2 = d$ has exactly two solutions:

$$\text{If } u^2 = d, \text{ then } u = \sqrt{d} \text{ or } u = -\sqrt{d}$$

Equivalently,

$$\text{If } u^2 = d \Rightarrow u = \pm\sqrt{d}.$$

Example 3

Solve $3x^2 - 21 = 0$

Solution

$3x^2 = 21$

$x^2 = 7$

$x = \pm\sqrt{7}$

R.2–3 Completing the Square

If $x^2 + bx$ is a binomial, then by ***adding*** $\left(\frac{b}{2}\right)^2$ which is the square of half the coefficient of *x*. a perfect square trinomial will result. That is.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2 \qquad x^2 + bx + \left(\frac{1}{2}b\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Example 4

Solve: $x^2+4x-1=0$

Solution

$$x^2+4x \qquad =1$$

$$x^2+4x+\left(\frac{4}{2}\right)^2=1+\left(\frac{4}{2}\right)^2$$

$$x^2+4x+(2)^2=1+4$$

$$(x+2)^2=5$$

$$x+2=\pm\sqrt{5}$$

$$\underline{x=-2\pm\sqrt{5}}$$

R.2–4 Quadratic Formula

$$ax^2+bx+c=0 \quad\Rightarrow\quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Example 5

Solve $x^2-4x=-2$

Solution

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(2)}}{2(1)} \qquad \Rightarrow a=1 \quad b=-4 \quad c=2 \qquad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{4\pm\sqrt{16-8}}{2}$$

$$=\frac{4\pm\sqrt{8}}{2}$$

$$=\frac{4\pm 2\sqrt{2}}{2}$$

$$=\frac{2\left(2\pm\sqrt{2}\right)}{2}$$

$$\underline{=2\pm\sqrt{2}}$$

$$ax^2 + bx + c = 0$$

If $\boldsymbol{a + b + c = 0}$ $\quad \Rightarrow \quad x = 1, \ \frac{c}{a}$

Example **6**

$$2x^2 + x - 3 = 0$$
$$2 + 1 - 3 = 0$$
$$\Rightarrow \ \underline{x = 1, \quad -\frac{3}{2}}|$$

If $\boldsymbol{a - b + c = 0}$ $\quad \Rightarrow \quad x = -1, \ -\frac{c}{a}$

Example **7**

$$2x^2 - x - 3 = 0$$
$$2 - (-1) - 3 = 0$$
$$\Rightarrow \ \underline{x = -1, \quad \frac{3}{2}}|$$

R.2–5 Solving a Polynomial Equation by factoring

Example **8**

Solve: $\quad 4x^4 = 12x^2$

Solution

$$4x^4 - 12x^2 = 0$$
$$4x^2\left(x^2 - 3\right) = 0$$

$4x^2 = 0 \qquad\qquad x^2 - 3 = 0$

$x^2 = 0 \qquad\qquad x^2 = 3$

$\rightarrow \ \underline{x = 0, 0}| \qquad\qquad \underline{x = \pm\sqrt{3}}|$

Example 9

Solve: $2x^3 + 3x^2 = 8x + 12$

Solution

$$2x^3 + 3x^2 - 8x - 12 = 0 \qquad \textit{Factor by grouping}$$

$$x^2(2x+3) - 4(2x+3) = 0$$

$$(2x+3)\left(x^2 - 4\right) = 0$$

$$2x + 3 = 0 \qquad x^2 - 4 = 0$$

$$2x = -3 \qquad x^2 = 4$$

$$x = -\frac{3}{2} \qquad x = \pm\sqrt{4}$$

$$= \pm 2$$

R.2−6 Equations that Are Quadratic in Form

$$ax^2 + bx + c = 0$$

$$a(x)^2 + b(x)^1 + c = 0$$

$$a(u)^2 + b(u)^1 + c = 0$$

$$a\left(x^n\right)^2 + b\left(x^n\right)^1 + c = 0$$

$$au^2 + bu + c = 0$$

Example 10

Solve: $x^4 - 5x^2 + 6 = 0$

Solution

$$\left(x^2\right)^2 - 5\left(x^2\right) + 6 = 0 \qquad \textit{or} \quad \left(x^2 - 2\right)\left(x^2 - 3\right) = 0$$

$$(U)^2 - 5(U) + 6 = 0 \qquad x^2 - 2 = 0 \quad x^2 - 3 = 0$$

$$U^2 - 5U + 6 = 0 \qquad x^2 = 2 \quad x^2 = 3$$

Solve for U $\qquad x = \pm\sqrt{2} \quad x = \pm\sqrt{3}$

$$U = \frac{5 \pm \sqrt{(-5)^2 - 4(6)(6)}}{2(1)}$$
$$= \frac{5 \pm \sqrt{25-24}}{2}$$
$$= \frac{5 \pm \sqrt{1}}{2}$$
$$\to \begin{cases} U = \frac{5-1}{2} = 2 \\ U = \frac{5+1}{2} = 3 \end{cases}$$

$$x^2 = U$$

$$\to \begin{cases} x^2 = 2 \to \underline{x = \pm\sqrt{2}} \\ x^2 = 3 \to \underline{x = \pm\sqrt{3}} \end{cases}$$

R.2−7 Solving a *Radical* Equation

Power Property

If P and Q are algebraic expressions, then every solution of the equation $P = Q$ is also a solution of the equation $P^n = Q^n$; for any positive integer n.

Example 11

Solve $x - \sqrt{15-2x} = 0$

Solution

$$x = \sqrt{15-2x}$$
$$x^2 = \left(\sqrt{15-2x}\right)^2$$
$$x^2 = 15 - 2x$$
$$x^2 + 2x - 15 = 0$$
$$(x-3)(x+5) = 0$$

$x - 3 = 0$ $\quad$ $x + 5 = 0$

$x = 3$ $\quad$ $x = -5$

Check

$x = 3$ $\quad$ $x = -5$

$3-\sqrt{15-2(3)}=0$ $\qquad$ $-5-\sqrt{15-2(-5)}=0$

$3-\sqrt{9}=0$ $\qquad$ $-5-\sqrt{25}=0$

$3-3=0 \quad (true)$ $\qquad$ $-5-5\neq 0 \quad (false)$

$\therefore\ x=3$ is the only solution

R.2–8 Interval/Set Notations

Type of Interval	*Set*	*Interval Notation*
Open Interval	$\{x \mid x<a\}$ or $x<a$	$(-\infty,\ a)$
	$\{x \mid a<x<b\}$ or $a<x<b$	$(a,\ b)$
	$\{x \mid x>b\}$ or $x>b$	$(b,\ \infty)$
Closed Interval	$\{x \mid a\le x\le b\}$ or $a\le x\le b$	$[a,\ b]$
Other Intervals	$\{x \mid x\le a\}$ or $x\le a$	$(-\infty,\ a]$
	$\{x \mid a\le x<b\}$ or $a\le x<b$	$[a,\ b)$
	$\{x \mid a<x\le b\}$ or $a<x\le b$	$(a,\ b]$
	$\{x \mid x\ge b\}$ or $x\ge b$	$[b,\ \infty)$
Disjoint Interval	$\{x \mid x<a \text{ or } x>b\}$	$(-\infty,\ a)\cup(b,\ \infty)$
All Real Numbers $\mathbb{R}$	$\{x \mid x\in\mathbb{R}\}$	$(-\infty,\ \infty)$

R.2–9 Solving an *Absolute Value* Equation

If c is a positive real number and X represents any algebraic expression, then $|X|=c$ is equivalent to $X=c$ or $X=-c$

$$|X|=c \quad\rightarrow\quad X=c \text{ or } X=-c$$

Properties of Absolute Value

1. For b > 0, $|a|=b$ if and only if (*iff*) $a=b$ or $a=-b$
2. $|a|=|b|$ *iff* $a=b$ or $a=-b$

For any positive number b:

3. $|a| < b \; iff \;\; -b < a < b$

4. $|a| < b \; iff \;\; a < -b \;\; or \;\; a > b$

Example 12

Solve: $|2x - 1| = 5$

Solution

$2x - 1 = 5$ $\qquad$ $2x - 1 = -5$

$2\boldsymbol{x} = 6$ $\qquad$ $2\boldsymbol{x} = -4$

$\boldsymbol{x} = 3$ $\qquad$ $\boldsymbol{x} = -2$

Solutions: $\underline{x = -2, \; 3 \,|}$

Properties of inequality

1. *If a < b, then a + c < b + c*
2. *If a < b and if c > 0, then ac < bc*
3. *If a < b and if c < 0, then ac > bc*

Example 13

Solve $\quad 3x + 1 > 7x - 15$

Solution

$3x - 7x > -15 - 1$

$-4x > -16$ $\qquad$ *Divide by – 4 both sides*

$\underline{x < 4 \,|}$ $\qquad$ *or* $(-\infty, \; 4)$ $\qquad$ *or* $\{x | \; x < 4\}$

R.2−10 Solving an *Absolute* Value Inequality:

If X is an algebraic expression and c is a positive number,

1. The solutions of $|X| < c$ are the numbers that satisfy $-c < X < c$.
2. The solutions of $|X| > c$ are the numbers that satisfy $X < -c$ or $X > c$.

Example 14

Solve: $\quad -3\left|5x-2\right|+20 \ \ge \ -19$

Solution

$$-3\left|5x-2\right| \ \ge \ -39$$

$$|5x-2| \le 13$$

$$-13 \le 5x-2 \le 13$$

$$-11 \le 5x \le 15$$

$$\boxed{-\frac{11}{5} \le x \le 3} \qquad \text{or} \qquad \boxed{\left[-\frac{11}{5},\ 3\right]}$$

R.2−11 *Definition* of a Polynomial Inequality

A polynomial inequality is any inequality that can be put into one of the forms

$$f(x)<0 \qquad f(x)>0 \qquad f(x)\le 0 \qquad f(x)\ge 0$$

✓ $ax^2+bx+c\ge 0 \ \rightarrow \ \text{if } a>0 \ \Rightarrow \ x\le x_1, \ \ x\ge x_2$

✓ $ax^2+bx+c\le 0 \ \rightarrow \ \text{if } a>0 \ \Rightarrow \ x_1\le x\le x_2$

Example 15

Solve $\ 2x^2+5x-12\ge 0$

Solution

$$2x^2+5x-12=0$$
$$(2x-3)(x+4)=0$$

$$x=-4, \ \frac{3}{2}$$

Solution: $\boxed{x\le -4 \quad x\ge \frac{3}{2}}$

-4	0	$\frac{3}{2}$
+	−	+

$$(-\infty,\ -4] \cup \left[\frac{3}{2},\ \infty\right)$$

Example 16

Solve $\qquad \dfrac{2x-1}{3x+4} < 5$

Solution

$$\frac{2x-1}{3x+4} - 5 = 0 \qquad \textit{Restriction: } 3x+4 \neq 0 \;\Rightarrow\; \boxed{x \neq -\tfrac{4}{3}}$$

$$(3x+4)\frac{2x-1}{3x+4} - 5(3x+4) = 0$$

$$2x - 1 - 15x - 20 = 0$$

$$-13x - 21 = 0$$

$$\boxed{x = -\frac{21}{13}}$$

	$-\frac{21}{13}$		$-\frac{4}{3}$		0
$-$		$+$		$-$	

Solution: $\boxed{x < -\frac{21}{13} \quad x > -\frac{4}{3}}$ $\qquad \boxed{\left(-\infty,\ -\frac{21}{13}\right) \cup \left(-\frac{4}{3},\ \infty\right)}$

R.2−12 The Rational Zeros *Theorem*

If the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ *has integer* coefficients and if $\frac{c}{d}$ is a rational zero of $f(x)$ such that ***c*** and ***d*** have no common prime factor, then

1. The numerator c of the zero is a factor of the constant term a_0
2. The denominator d of the zero is a factor of the leading coefficient a_n

$$\textit{possible rational zeros} = \frac{\textit{factors of the constant term } a_0}{\textit{factors of the leading coefficient } a_n} = \frac{\textit{possibilities for } a_0}{\textit{possibilities for } a_n}$$

Example 17

Find all rational solutions of the equation: $\quad 3x^4 + 14x^3 + 14x^2 - 8x - 8 = 0$

Solution

possibilities for a_0	$\pm 1,\ \pm 2,\ \pm 4,\ \pm 8$
possibilities for a_n	$\pm 1,\ \pm 3$
possibilities for c/d	$\pm 1,\ \pm 2,\ \pm 4,\ \pm 8,\ \pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm\frac{4}{3},\ \pm\frac{8}{3}$

The result will show that –2 is a ***zero*** (plug in).

$$\begin{array}{r|rrrrr} -2 & 3 & 14 & 14 & -8 & -8 \\ & & -6 & -16 & 4 & 8 \\ \hline & 3 & 8 & -2 & -4 & \boxed{0} \end{array}$$

We have the factorization of: $(x+2)\left(3x^3+8x^2-2x-4\right)=0$

For $3x^3+8x^2-2x-4 \quad \Rightarrow \quad \frac{c}{d}=\frac{\pm1,\ \pm2,\ \pm4}{\pm1,\ \pm3}$

$x=-\frac{2}{3}$ is another solution.

$$\begin{array}{r|rrrr} -\frac{2}{3} & 3 & 8 & -2 & -4 \\ & & -2 & -4 & 4 \\ \hline & 3 & 6 & -6 & \boxed{0} \end{array}$$

We have the factorization of: $(x+2)\left(x+\frac{2}{3}\right)\left(3x^2+6x-6\right)=0$

By applying quadratic formula to solve: $3x^2+6x-6=0 \Rightarrow x=-1\pm\sqrt{3}$

Hence, the polynomial has two rational roots $x=-2$ *and* $-\frac{2}{3}$ and two irrational roots $x=-1\pm\sqrt{3}$.

Exercises Section R.2 – Solving Equations

(1 – 78) Solving Equations

1. $3\left[2x-(4-x)+5\right]=7x-2$

2. $-4(2x-6)+8x=5x+24+x$

3. $-8(3x+4)+6x=4(x-8)+4x$

4. $\frac{1}{2}(4x+8)-16=-\frac{2}{3}(9x-12)$

5. $\frac{3}{4}(24-8x)-16=-\frac{2}{3}(6x-9)$

6. $\frac{x-3}{4}=\frac{5}{14}-\frac{x+5}{7}$

7. $\frac{x+1}{4}=\frac{1}{6}+\frac{2-x}{3}$

8. $\frac{3x+2}{x-2}+\frac{1}{x}=\frac{-2}{x^2-2x}$

9. $\frac{6}{x+1}-\frac{5}{x+2}=\frac{10}{x^2+3x+2}$

10. $x^2=-25$

11. $x^2=49$

12. $9x^2=100$

13. $4x^2+25=0$

14. $5x^2-45=0$

15. $(x-4)^2=12$

16. $(x+3)^2=-16$

17. $x^2+8x+15=0$

18. $x^2+5x+2=0$

19. $x^2+x-12=0$

20. $x^2-2x-15=0$

21. $x(8x+1)=3x^2-2x+2$

22. $3x^2-x-2=0$

23. $3x^2+x-2=0$

24. $2x^2+3x-5=0$

25. $2x^2-3x-5=0$

26. $3x^3+2x^2=12x+8$

27. $x^3+x^2-4x-4=0$

28. $x^3-x^2=16x-16$

29. $x^4+3x^2=10$

30. $x^4-4x^3+3x^2=0$

31. $x^4+6x^2-7=0$

32. $3x^4-x^2-2=0$

33. $x-3\sqrt{x}-4=0$

34. $\left(5x^2-6\right)^{1/4}=x$

35. $\sqrt[3]{6x-3}=3$

36. $\sqrt{2x+3}=5$

37. $\sqrt{x-3}+6=5$

38. $\sqrt{3x-2}=4$

39. $\sqrt{2x+5}+11=6$

40. $\sqrt{x+2}+\sqrt{x-1}=3$

41. $\sqrt{x+2}+\sqrt{3x+7}=1$

42. $|x|=-9$

43. $|x|=9$

44. $|x-2|=7$

45. $|x-2|=0$

46. $2|x-6|=8$

47. $3|2x-1|=21$

48. $2|3x-2|=14$

49. $|4x+1|+4=10$

50. $|3x-1|+2=16$

51. $|x+1|=|1-3x|$

52. $|3x-1|=|x+5|$

53. $|2x-4|=|x-1|$

54. $-3x+5>-7$

55. $4-3x\le 7+2x$

56. $4(x+1)+2\ge 3x+6$

57. $8x+3>3(2x+1)+x+5$

58. $5(3-x)\le 3x-1$

59. $\frac{2x-5}{-8}\le 1-x$

60. $8(x+1)\le 7(x+5)+x$

61. $|x-2|<1$

62. $|x+2|\ge 1$

63. $|2(x-1)+4|\le 8$

64. $\left|\frac{2x+6}{3}\right|>2$

65. $|12-9x|\ge -12$

66. $|6-3x|<-11$

67. $2x^2-9x\le 18$

68. $x^2-5x+4>0$

69. $x^2+7x+10<0$

70. $x^3-3x^2-9x+27<0$

71. $x^3+3x^2\le x+3$

72. $x^4-20x^2+64\le 0$

73. $x^4-10x^2+9\ge 0$

74. $\frac{x+4}{x-1}<0$

75. $\frac{x-2}{x+3}>0$

76. $\frac{x-4}{x+6}\le 1$

77. $\frac{x}{2x+7}\ge 4$

78. $\frac{x-3}{x+4}\ge\frac{x+2}{x-5}$

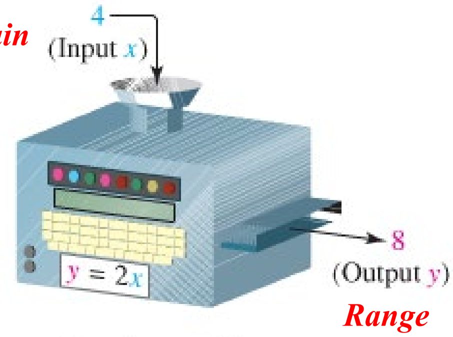
Section R.3 – Functions

R.3–1 Relations

A ***relation*** is any set of ordered pairs. The set of all first components of ordered pairs is called the domain of the relation and the set of second components is called the range of the relation.

R.3–2 *Definition* of a Function

A ***function*** is a relation between two variables such that to matches each element of a first set (called ***domain***) to an element of a second set (called ***range***) in such way that no element in the first set is assigned to two different elements in the second set.



Function machine

The ***domain*** of the function is the set of all values of the independent variable for which the function is defined.

The ***range*** of the function is the set of all values taken on by the dependent variable.

Notation for Function

$f(x)$ read "f of x" or "f at x" represents the value of the function at the number x.

Example 1

Let $f(x) = -x^2 + 5x - 3$ Find: $f(2)$

Solution

$$f(x) = -x^2 + 5x - 3$$

$$f(\text{——}) = -(\text{——})^2 + 5(\text{——}) - 3$$

$$f(2) = -(2)^2 + 5(2) - 3$$
$$\underline{= 3}$$

R.3−3 *Piecewise*-Defined Functions

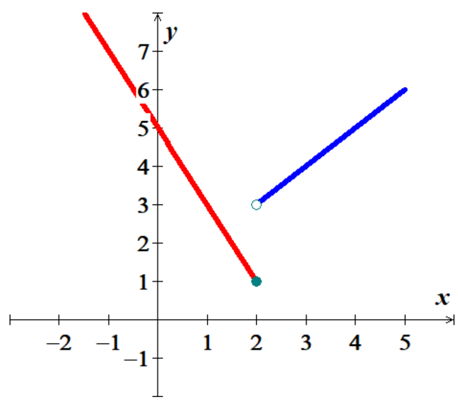
Function is sometimes described by more than one expression; we call such functions ***piecewise-defined functions***.

Example 2

Graph the piecewise function

$$f(x) = \begin{cases} -2x+5 & \text{if } x \le 2 \\ x+1 & \text{if } x > 2 \end{cases}$$

Find: $f(2),\ f(0),\ f(4)$



Solution

$$f(2) = -2(2) + 5 = 1$$
$$f(0) = -2(0) + 5 = 5$$
$$f(4) = 4 + 1 = 5$$

R.3−4 The *Domain* of a Function

1. ***Rational*** function: $\frac{f(x)}{h(x)}$ $\Rightarrow$ ***Domain***: $h(x) \ne 0$

Example: $f(x) = \frac{1}{x-3}$

Domain: $\underline{x \ne 3}$ $\{x \mid x \ne 3\}$

Or $(-\infty, 3) \cup (3, \infty)$ *Interval Notation*

Or $\mathbb{R} - \{3\}$

2. ***Irrational*** function: $\sqrt{g(x)}$ $\Rightarrow$ ***Domain***: $g(x) \ge 0$

Example: $g(x) = \sqrt{3-x} + 5$

$$3-x\ge 0$$
$$-x\ge -3$$

Domain: $\underline{x<3}\rfloor$ $\qquad (-\infty,\ 3]$

3. ***Otherwise***: Domain all real numbers $(-\infty,\ \infty)$

Example: $f(x)=x^3+|x|$

Domain: All real numbers $\qquad \underline{\mathbb{R}}\rfloor$

$(-\infty,\ \infty)$

(1) & (2) → Find the domain: $f(x)=\frac{x+1}{\sqrt{x-3}}$

$x>3$

Domain: $(3,\ \infty)$

Example 3

Find the domain

a) $f(x)=x^2+3x-17$

Domain: $\underline{\mathbb{R}}\rfloor$

b) $g(x)=\frac{5x}{x^2-49}$

$x^2\ne 49$

$\underline{x\ne \pm 7}\rfloor$

Domain: $\begin{cases}\{x\,|\,x\ne \pm 7\} & \text{or}\\ (-\infty,-7)\cup(-7,7)\cup(7,\infty)\end{cases}$

c) $h(x)=\sqrt{9x-27}$

$9x-27\ge 0$

$9x\ge 27$

Domain: $\underline{x\ge 3}\rfloor$ $[3,\ \infty)$

R.3–5 The *Algebra* of Functions

$$(f+g)(x)=f(x)+g(x)$$

$$(f-g)(x)=f(x)-g(x)$$

$$(f\cdot g)(x)=f(x)\cdot g(x)$$

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$$

Example 4

Let $f(x)=x^2+1$ and $g(x)=3x+5$. Find each of the following $(f+g)(1)$, $(f-g)(-3)$, $(fg)(5)$, and $\left(\frac{f}{g}\right)(0)$

Solution

$$\begin{aligned}(f+g)(1)&=f(1)+g(1)\\&=1^2+1+3(1)+5\\&=1+1+3+5\\&=10\end{aligned}$$

$$\begin{aligned}(f-g)(-3)&=f(-3)-g(-3)\\&=(-3)^2+1-(3(-3)+5)\\&=14\end{aligned}$$

$$\begin{aligned}(fg)(5)&=f(5)\cdot g(5)\\&=\left(5^2+1\right)\cdot(3(5)+5)\\&=(26)\cdot(20)\\&=520\end{aligned}$$

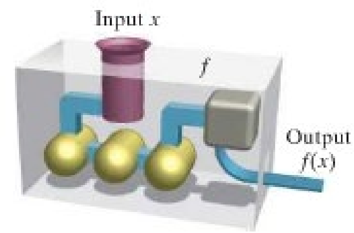
$$\left(\frac{f}{g}\right)(0)=\frac{f(0)}{g(0)}$$

$$= \frac{0^2+1}{3(0)+5}$$

$$\boxed{= \frac{1}{5}}$$

R.3–6 *Composition* of Functions

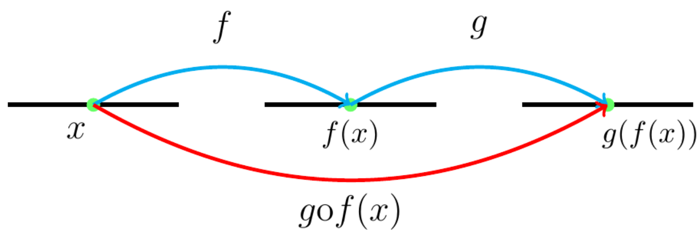
The composite function $g \circ f$, the composite of *f* and *g*, is defined as



$$(g \circ f)(x) = g\big(f(x)\big)$$

Where x is in the domain of *f*

and $g(x)$ is in the domain of *f*



Example 5

Given that $f(x) = 5x + 6$ and $g(x) = 2x^2 - x - 1$, find $(f \circ g)(x)$ and $(g \circ f)(x)$

Solution

$(f \circ g)(x) = f\big(g(x)\big)$ $\qquad = f(2x^2 - x - 1)$ ***Domain:*** All real numbers

$$= 5(\text{---------}) + 6$$

$$= 5(2x^2 - x - 1) + 6$$

$$= 10x^2 - 5x - 5 + 6$$

$$\boxed{= 10x^2 - 5x + 1}$$

Domain: All real numbers

$$(g \circ f)(x) = g\big(f(x)\big)$$

$$= g(5x + 6)$$

Domain: All real numbers

$$= 2(\qquad)^2 - (\qquad) - 1$$

$$= 2(5x + 6)^2 - (5x + 6) - 1$$

$$= 2(25x^2 + 60x + 36) - 5x - 6 - 1$$

$$= 50x^2 + 120x + 72 - 5x - 7$$

$$\boxed{= 50x^2 + 115x + 65}$$

Domain: All real numbers

R.3–8 Polynomial Function

A *Polynomial function* $P(x)$ in x is a sum of the form is given by:

$$P(x)=a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$$

Where the coefficients $a_n,\ a_{n-1},\ \ldots, a_2,\ a_1, a_0$ are real numbers and the exponents are whole numbers.

$$a_n x^{n}$$

Degree

Leading Term

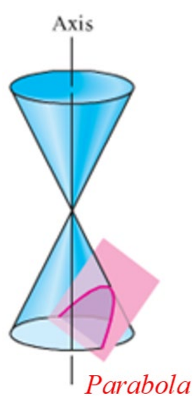
Leading Coefficient

Non-polynomial Functions: $\frac{1}{x}+2x;\quad \sqrt{x^2-3}+x;\quad \frac{x-5}{x^2+2}$

R.3–9 Graphing a Quadratic Function

Definition of a Parabola

A ***parabola*** is the set of all points in a plane equidistant from a fixed–point F (the ***focus***) and a fixed line l (the ***directrix***) that lie in the plane



A function f is a ***quadratic function*** if $f(x)=ax^2+bx+c$

Formula

Vertex of a Parabola

The ***vertex*** of the graph of $f(x)$ is

$$V_x \text{ or } x_v = -\frac{b}{2a}$$

Example 6

$$f(x)=x^2-4x-2$$

$$x=-\frac{b}{2a}=-\frac{-4}{2(1)}=2$$

$$y=f\left(-\frac{b}{2a}\right)=f(2)$$

V_y or $y_V = f\left(-\frac{b}{2a}\right)$

Vertex Point $\left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right)$

$= (2)^2 - 4(2) - 2$

$= -6$

Vertex point: $(2,\ -6)$

Axis of Symmetry:

$$x = V_x = -\frac{b}{2a}$$

Axis of Symmetry: $x = 2$

Minimum or Maximum Point

If $a > 0 \Rightarrow$ *f(x)* has a ***minimum*** point

If $a < 0 \Rightarrow$ *f(x)* has a ***maximum*** point

@ vertex point $\left(V_x,\ V_y\right)$

Minimum point @ $(2,\ -6)$

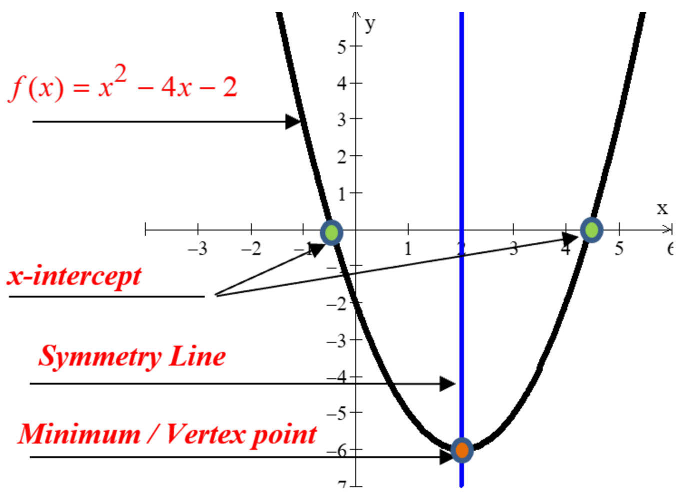
Range

If $a > 0 \Rightarrow \left[V_y,\ \infty\right)$

If $a < 0 \Rightarrow \left(-\infty,\ V_y\right]$

$[-6,\ \infty)$

Domain: $(-\infty,\ \infty)$



R.3–10 Graphing a Polynomial Function

Example 7

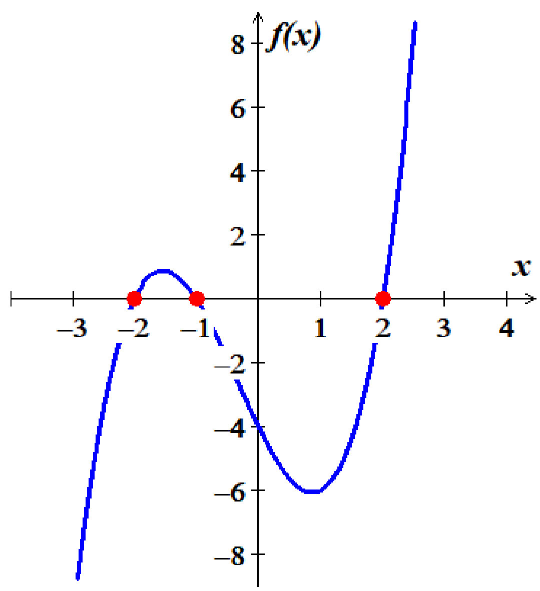
Let $f(x)=x^3+x^2-4x-4$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

Solution

$$\begin{aligned} f(x) &= x^3+x^2-4x-4 \\ &= x^2(x+1)-4(x+1) \\ &= (x+1)\left(x^2-4\right) \\ &= (x+1)(x+2)(x-2)=0 \end{aligned}$$

The zeros of $f(x)$ (x-intercepts) are: $-2,\ -1,\ and\ 2$

Interval	$-\infty$ -2	-1	**0**	2 ∞
Sign of $f(x)$	$-$	$+$	$-$	$+$
Position	Below x-axis	Above x-axis	Below x-axis	Above x-axis



We can conclude from the chart and the graph that:

$$f(x)>0 \quad \text{if } x \text{ is in } (-2,\ -1)\cup(2,\ \infty)$$

$$f(x)<0 \quad \text{if } x \text{ is in } (-\infty,\ -2)\cup(-1,\ 2)$$

R.3−11 Graphing Rational Functions

R.3−12 Vertical Asymptote (*VA*) - *Think Domain*

The line $x=a$ is a ***vertical asymptote*** for the graph of a function f if

$$f(x)\to\infty \quad or \quad f(x)\to-\infty$$

As ***x*** approaches ***a*** from either the left or the right

R.3−13 Horizontal Asymptote (*HA*)

The line $y=c$ is a ***horizontal asymptote*** for the graph of a function f if

$$f(x)\to c \quad as \quad x\to-\infty \quad or \quad x\to-\infty$$

Let $f(x)=\dfrac{p(x)}{q(x)}=\dfrac{a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+\ldots+b_1x+b_0}=\dfrac{a_nx^n}{b_mx^m}$ be a rational function.

1. If the degree of numerator is less than of denominator $(n<m)\Rightarrow$ y = 0

$$y=\frac{2x+1}{4x^2+5} \quad \Rightarrow \boxed{y=0}$$

2. If the degree of numerator is equal of denominator $(n=m)\Rightarrow y=\dfrac{a_n}{b_m}$

$$y=\frac{2x^2+1}{4x^2+5} \quad \Rightarrow \boxed{y=\frac{2}{4}=\frac{1}{2}}$$

3. If the degree of numerator is greater than of denominator $(n>m)\Rightarrow$ No horizontal asymptote

$$y=\frac{2x^3+1}{4x^2+5} \quad \Rightarrow No \;\; HA$$

R.3−14 *Slant* or *Oblique* Asymptotes

When the degree of the numerator is one greater than the degree of the numerator, the graph has a slant or oblique asymptote, and it is a line $y=ax+b$, $\quad a\neq 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

$$y=\frac{3x^2-1}{x+2}$$

$$\begin{array}{r|l} & 3x-6 \\ \hline x+2 & 3x^2+0x-1 \\ & \underline{3x^2+6x} \\ & \quad -6x-1 \\ & \quad \underline{-6x-12} \\ & \qquad R=11 \end{array}$$

$$y=\frac{3x^2-1}{x+2}$$

$$=3x-6+\frac{11}{x+2}$$

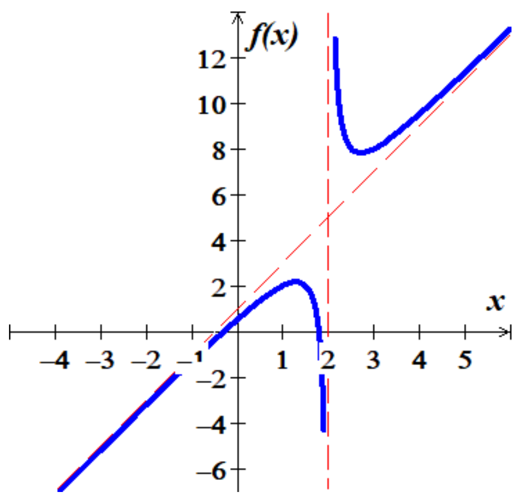
∴ The ***oblique asymptote*** is the line ***y* = 3x – 6**

Example 8

Find all the asymptotes of $f(x)=\frac{2x^2-3x-1}{x-2}$

Solution

$$\begin{array}{r|l} & 2x+1 \\ \hline x-2 & 2x^2-3x-1 \\ & \underline{-2x^2+4x} \\ & \quad x-1 \\ & \quad \underline{-x+2} \\ & \qquad 1 \end{array}$$



$$f(x)=\frac{2x^2-3x-1}{x-2}$$

$$=(2x+1)+\frac{1}{x-2}$$

The ***oblique asymptote*** is the line $y=2x+1$

VA: $x=2$

R.3−15 Graph That Has a *Hole*

Example 9

Sketch the graph of g if $g(x)=\frac{3x^2+x-4}{2x^2-7x+5}$

Solution

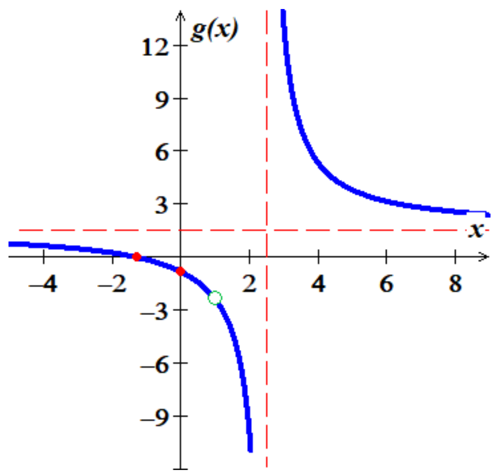
$$g(x)=\frac{(3x+4)(x-1)}{(2x-5)(x-1)}$$
$$=\frac{3x+4}{2x-5}=f(x)$$

VA: $x=\frac{5}{2}$

HA: $y=\frac{3}{2}$

The only different between the graphs that g has a ***hole*** at $x=1 \to f(1)=-\frac{7}{3}$

x	y
$-\frac{4}{3}$	0
0	$-\frac{4}{5}$



Exercises Section R.3– Functions

1. $f(x)=\begin{cases} 2+x & if \;\; x<-4 \\ -x & if \;\; -4\le x\le 2 \\ 3x & if \;\; x>2 \end{cases}$ **Find**: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

2. $f(x)=\begin{cases} -2x & if \;\; x<-3 \\ 3x-1 & if \;\; -3\le x\le 2 \\ -4x & if \;\; x>2 \end{cases}$ **Find**: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

3. $f(x)=\begin{cases} 3x+5 & if \;\; x<0 \\ 4x+7 & if \;\; x\ge 0 \end{cases}$ **Find**

a) $f(0)$ *c)* $f(1)$ *e)* Graph $f(x)$

b) $f(-2)$ *d)* $f(3)+f(-3)$

$(\mathbf{4-37})$ Find the Domain

4. $f(x)=7x+4$

5. $f(x)=|3x-2|$

6. $f(x)=3x+\pi$

7. $f(x)=-2x^2+3x-5$

8. $f(x)=x^3-2x^2+x-3$

9. $f(x)=4-\frac{2}{x}$

10. $f(x)=\frac{1}{x^4}$

11. $y=\frac{2}{x-3}$

12. $f(x)=\frac{3x}{x+2}$

13. $f(x)=x-\frac{2}{x-3}$

14. $f(x)=\frac{1}{2}x-\frac{8}{x+7}$

15. $f(x)=\frac{1}{x-3}-\frac{8}{x+7}$

16. $f(x)=\frac{1}{x+4}-\frac{2x}{x-4}$

17. $f(x)=\frac{x}{x^2+3x+2}$

18. $f(x)=\frac{x^2}{x^2-5x+4}$

19. $g(x)=\frac{2}{x^2+x-12}$

20. $h(x)=\frac{5}{\frac{4}{x}-1}$

21. $y=\sqrt{x}$

22. $f(x)=\sqrt{3-2x}$

23. $f(x)=\sqrt{3+2x}$

24. $f(x)=\sqrt{6-3x}$

25. $f(x)=\sqrt{2x+7}$

26. $f(x)=\sqrt{x^2-16}$

27. $f(x)=\sqrt{16-x^2}$

28. $f(x)=\sqrt{x^2-5x+4}$

29. $f(x)=\sqrt{x^2+5x+4}$

30. $f(x)=\frac{\sqrt{x+1}}{x}$

31. $g(x)=\frac{\sqrt{x-3}}{x-6}$

32. $f(x)=\frac{\sqrt{x+4}}{\sqrt{x-1}}$

33. $f(x)=\frac{\sqrt{5-x}}{x}$

34. $f(x)=\frac{x}{\sqrt{5-x}}$

35. $f(x)=\frac{1}{(x-3)\sqrt{x+3}}$

36. $f(x)=\frac{\sqrt{x+2}}{\sqrt{x^2+3x+2}}$

37. $f(x)=\frac{\sqrt{2x+3}}{x^2-6x+5}$

38. Let $f(x) = 2x^2 + 3$ and $g(x) = 3x - 4$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

39. Let $f(x) = x^2 - 2x - 3$ and $g(x) = x^2 + 3x - 2$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

40. Given that $f(x) = x + 1$ and $g(x) = \sqrt{x+3}$

a) Find $(f+g)(x)$

b) Find the domain of $(f+g)(x)$

c) Find: $(f+g)(6)$

(41 – 55) For the given function, find:

a) Find $(f \circ g)(x)$ and the ***domain*** of $f \circ g$

b) Find $(g \circ f)(x)$ and the ***domain*** of $g \circ f$

41. $f(x) = x - 3$ *and* $g(x) = x + 3$

42. $f(x) = \frac{2}{3}x$ *and* $g(x) = \frac{3}{2}x$

43. $f(x) = x - 1$ *and* $g(x) = 3x^2 - 2x - 1$

44. $f(x) = x^2 - 2$ *and* $g(x) = 4x - 3$

45. $f(x) = \sqrt{x}$ *and* $g(x) = x + 3$

46. $f(x) = \sqrt{x}$ *and* $g(x) = 2 - 3x$

47. $f(x) = x^4$ *and* $g(x) = \sqrt[4]{x}$

48. $f(x) = x^n$ *and* $g(x) = \sqrt[n]{x}$

49. $f(x) = x^2 - 3x$ *and* $g(x) = \sqrt{x+2}$

50. $f(x) = \sqrt{x-2}$ *and* $g(x) = \sqrt{x+5}$

51. $f(x) = x^5 - 2$ *and* $g(x) = \sqrt[5]{x+2}$

52. $f(x) = 1 - x^2$ *and* $g(x) = \sqrt{x^2 - 25}$

53. $f(x) = \frac{1}{x-2}$ *and* $g(x) = \frac{x+2}{x}$

54. $f(x) = \frac{3x+5}{2}$ *and* $g(x) = \frac{2x-5}{3}$

55. $f(x) = \frac{2x+3}{x+4}$ *and* $g(x) = \frac{-4x+3}{x-2}$

(56 – 80) Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

56. $f(x) = x^2 + 6x + 3$	**59.** $f(x) = x^2 - 4x + 2$
57. $f(x) = x^2 + 6x + 5$	**60.** $f(x) = -2x^2 + 16x - 26$
58. $f(x) = -x^2 - 6x - 5$	**61.** $f(x) = x^2 + 4x + 1$

62. $f(x)=x^2+6x-1$

63. $f(x)=x^2+6x+3$

64. $f(x)=x^2-10x+3$

65. $f(x)=x^2-3x+4$

66. $f(x)=x^2-4x-5$

67. $f(x)=2x^2-3x+1$

68. $f(x)=-x^2-4x+5$

69. $f(x)=x^3+2x^2-4x-8$

70. $f(x)=x^3+2x^2-5x-6$

71. $f(x)=x^3-3x^2-9x+27$

72. $f(x)=2x^3+11x^2-7x-6$

73. $f(x)=x^3+8x^2+11x-20$

74. $f(x)=-x^4+12x^2-27$

75. $f(x)=x^4+x^2-2$

76. $f(x)=x^4-x^3-6x^2+4x+8$

77. $f(x)=2x^4-x^3-5x^2+2x+2$

78. $f(x)=6x^5+19x^4+x^3-6x^2$

79. $f(x)=x^5-x^4-7x^3+7x^2+12x-12$

80. $f(x)=x^5-7x^4+19x^3-37x^2+60x-36$

(81 – 98) Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

81. $f(x)=\dfrac{-3x}{x+2}$

82. $f(x)=\dfrac{x+1}{x^2+2x-3}$

83. $f(x)=\dfrac{2x^2-2x-4}{x^2+x-12}$

84. $f(x)=\dfrac{-2x^2+10x-12}{x^2+x}$

85. $f(x)=\dfrac{x^2-x-6}{x+1}$

86. $f(x)=\dfrac{2x^2+x-6}{x^2+3x+2}$

87. $f(x)=\dfrac{x-1}{1-x^2}$

88. $f(x)=\dfrac{x^2+x-2}{x+2}$

89. $f(x)=\dfrac{x^3-2x^2-4x+8}{x-2}$

90. $f(x)=\dfrac{2x^2-3x-1}{x-2}$

91. $f(x)=\dfrac{2x+3}{3x^2+7x-6}$

92. $f(x)=\dfrac{x^2-1}{x^2+x-6}$

93. $f(x)=\dfrac{1}{x-3}$

94. $f(x)=\dfrac{-2}{x+3}$

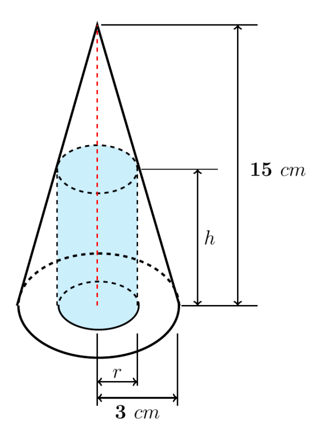
95. $f(x)=\dfrac{x}{x+2}$

96. $f(x)=\dfrac{x-5}{x+4}$

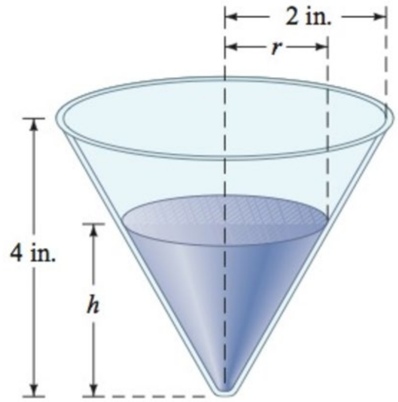
97. $f(x)=\dfrac{2x^2-2}{x^2-9}$

98. $f(x)=\dfrac{x^2-3}{x^2+4}$

99. A cone has an altitude of 15 *cm* and a radius of 3 *cm*.
A right circular cylinder of radius *r* and height *h* is inscribed in the cone.
Use similar triangles to write *h* as a function of *r*.

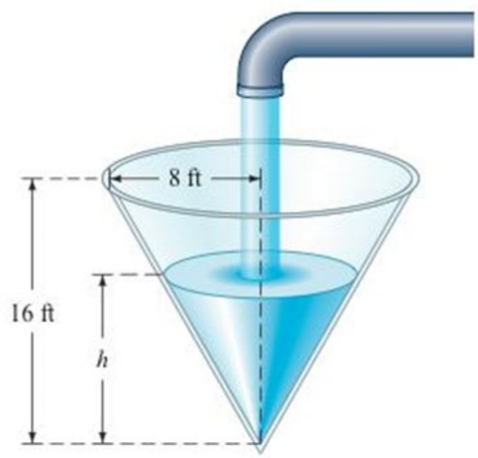


100. Water is flowing into a conical drinking cup with an altitude of 4 *inches* am a radius of 2 *inches*.



a) Write the radius r of the surface of the water as a function of its depth h.
b) Write the volume V of the water as a function of its depth h.

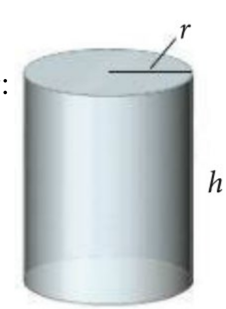
101. A water tank has the shape of a right circular cone with height 16 *feet* and radius 8 *feet*. Water is running into the tank so that the radius r (in *feet*) of the surface of the water is given by $r = 1.5t$, where t is the time (in *minutes*) that the water has been running.



a) The area A of the surface of the water is $A = \pi r^2$. Find $A(t)$ and use it to determine the area of the surface of the water when $t = 2$ *minutes*.

b) The volume V of the water is given by $V = \frac{1}{3}\pi r^2 h$. Find $V(t)$ and use it to determine the volume of the water when $t = 3$ *minutes*

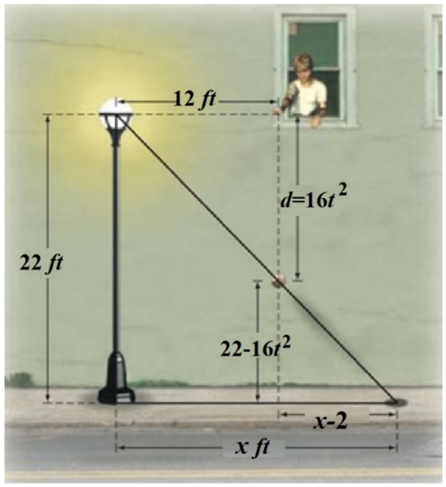
102. The surface area S of a right circular cylinder is given by the formula $S = 2\pi rh + 2\pi r^2$. if the height is twice the radius, find each of the following:



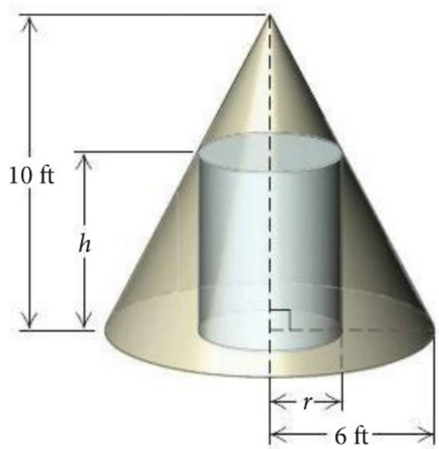
a) A function $S(r)$ for the surface area as a function of r.

b) A function $S(h)$ for the surface area as a function of h.

103. The light from a lamppost casts a shadow from a ball that was dropped from a height of 22 *feet* above the ground. The distance d, in *feet*, the ball has dropped t *seconds* after it is released is given by $d(t)=16t^2$. Find the distance x, in *feet*, of the shadow from the base of the lamppost as a function of time t.

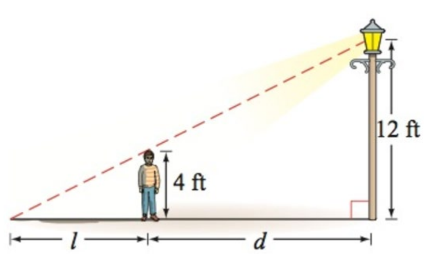


104. A right circular cylinder of height h and a radius r is inscribed in a right circular cone with a height of 10 *feet* and a base with radius 6 *feet*.



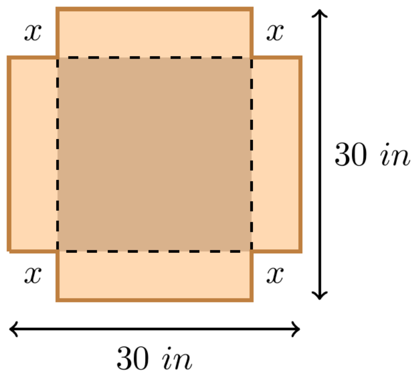
a) Express the height h of the cylinder as a function of r.
b) Express the volume V of the cylinder as a function of r.
c) Express the volume V of the cylinder as a function of h.

105. A child 4 *feet* tall is standing near a streetlamp that is 12 *feet* high. The light from the lamp casts a shadow.



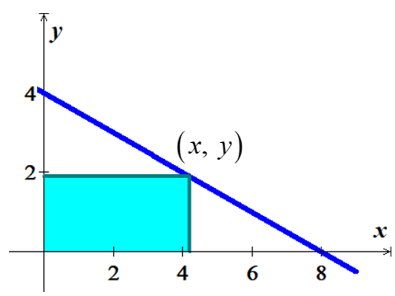
a) Find the length l of the shadow as a function of the distance d of the child from the lamppost.
b) What is the domain of the function?
c) What is the length of the shadow when the child is 8 *feet* from the base of the lamppost?

106. An open box is to be made from a square piece of cardboard with the dimensions 30 *inches* by 30 *inches* by cutting out squares of area x^2 from each corner.



a) Express the volume V of the box as a function of x.
b) Determine the domain of V.

107. A rectangle is bounded by the x- and y-*axis* of $y=-\frac{1}{2}x+4$



a) Find the area of the rectangle as a function of x.
b) What is the domain of this function?

Section R.4 – Exponential & Logarithm

R.4–1 *Definition*: Exponential

The exponential function f with base $\boldsymbol{b}$ is defined by

$$f(x) = \underbrace{b}_{\text{Base}}{}^{x} \qquad \text{or} \qquad y = b^x$$

where $b > 0$, $b \neq 1$ and $\boldsymbol{x}$ is any real number.

$$f(x) = 2^x \qquad f(x) = \left(\frac{1}{2}\right)^{2x+1} \qquad f(x) = 3^{-x} \qquad \cancel{f(x) = (-2)^x}$$

Example 1

If $f(x) = 2^x$, find each of the following. $f(-1)$, $f(3)$, $f\left(\frac{5}{2}\right)$

Solution

a) $f(-1) = 2^{-1}$

$\quad = \frac{1}{2}$

b) $f(3) = 2^3$

$\quad = 8$

c) $f\left(\frac{5}{2}\right) = 2^{\frac{5}{2}}$

$\quad = 4\sqrt{2}$

R.4–2 Graphing Exponential

1. Define the Horizontal Asymptote $f(x) = b^x \pm d$

$\quad y = 0 \pm \text{d}$

The exponential function always equals to 0

$\quad x \to \infty \ \ or \ \ x \to -\infty \Rightarrow f(x) \to 0$

Example 2

$f(x) = 3^x$

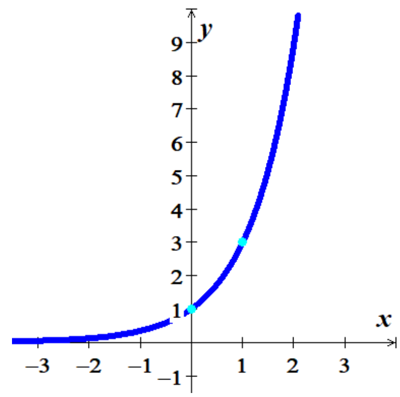
Asymptote: $y = 0$

2. Define/Make a table

(Force your exponential to = 0, then solve for *x*)

x	*f(x)*
$x-1$	
x	
$x+1$	
$x+2$	

x	*f(x)*
–1	1/3
0	1
1	3
2	9



Domain: $(-\infty, \infty)$

Range: (d, ∞)

Example 3

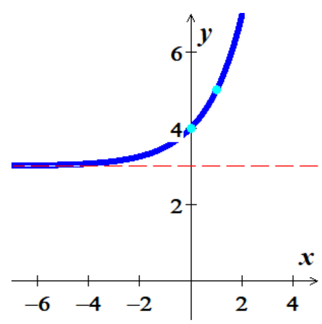
Graph $f(x) = 2^x + 3$

Solution

Asymptote: $y = 3$

Domain: $(-\infty, \infty)$

Range: $(3, \infty)$



R.4–3 Natural Base *e*

The irrational number e is called natural base.

$f(x) = e^x$ is called natural exponential function.

$e^0 = 1$ $\qquad$ $e \approx 2.7183$ $\qquad$ $e^2 \approx 7.3891$ $\qquad$ $e^{-1} \approx 0.3679$

Example 4

The exponential function $f(x) = 1066e^{0.042x}$ models the gray wolf population of the Western Great Lakes, $f(x)$, in billions, *x years* after 1978. Project the gray population in the recovery area in 2012.

Solution

$x = 2012 - 1978 = 34$

$f(x = 34) = 1066e^{0.042(34)}$

$= 4445.6$

$\approx 4,446$

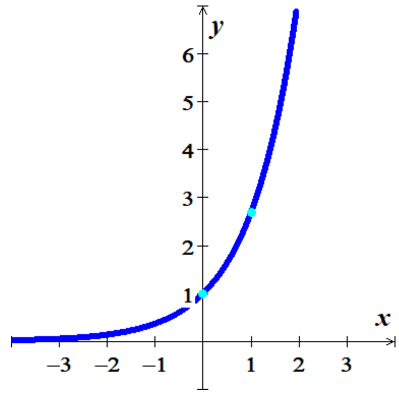
Example 5

Graph $\quad f(x) = e^{x}$

Solution

Asymptote: $y = 0$

x	$f(x)$
-1	.4
0	1
1	2.7



Domain: $(-\infty,\ \infty)$

Range: $(0, \infty)$

R.4−4 Logarithmic Function (*Definition*)

For $\boldsymbol{x > 0}$ and $b > 0,\ b \neq 1$

$$y = \log_b x \quad \text{is equivalent to} \quad x = b^y$$

$$y = \log_b x \iff x = b^y$$

Base

The function $f(x) = \log_b x$ is the lo unction with base b.

$\log_b x$: ***read*** log base b of x

$\log x$ ***means*** $\log_{10} x$

$\ln x$ ***means*** $\log_e x$

The logarithmic function with base $\boldsymbol{e}$ is called natural logarithmic function.

$\ln x$ *read* "el en of x"

Change-of-Base Logarithmic

$$\log_b M = \frac{\log_a M}{\log_a b} \qquad \log_b M = \frac{\log M}{\log b} \quad \textit{or} \quad \log_b M = \frac{\ln M}{\ln b}$$

R.4–5 Logarithm *Domain*

The domain of a logarithmic function of the form $f(x) = \log_b x$ is the set of all positive real numbers.
(*Inside* the log has to be > 0)

Range: $(-\infty,\ \infty)$

Example 6

Find the ***domain*** of

a) $f(x) = \log_4(x-5)$

$$x - 5 > 0 \quad \Rightarrow \ x > 5$$

Domain: $\underline{(5,\ \infty)\,|}$

b) $f(x) = \ln(4-x)$

$$4 - x > 0$$
$$-x > -4$$
$$x < 4$$

Domain: $\underline{(-\infty,\ 4)\,|}$

c) $h(x) = \ln\left(x^2\right)$

$x^2 > 0 \ \Rightarrow$ all real numbers except 0.

Domain: $\{x|\ x \neq 0\}$

or $\underline{(-\infty,\ 0)\cup(0,\ \infty)\,|}$ ***or*** $\underline{\mathbb{R}-\{0\}\,|}$

R.4–6 Graphs of *Logarithmic* Functions

Example 7

Graph $g(x) = \log x$

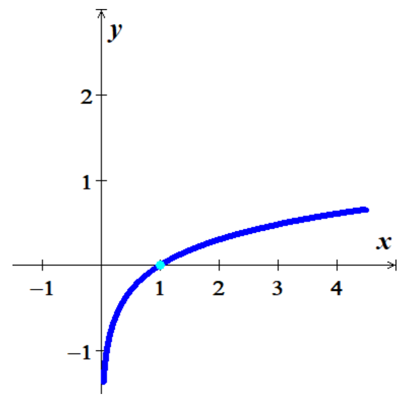
Solution

Asymptote: $x = 0$
(*Force inside log to be equal to zero, then solve for* x)

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

x	$g(x)$
0	
0.5	–.3
1	0
2	.3
3	.5



R.4–7 Properties of Logarithms

Product **Rule**

$$\log_b MN = \log_b M + \log_b N$$

For M > 0 and N > 0

Example

Use the product rule to expand the logarithmic expression

$\log(100x) = \log 100 + \log x$

Power **Rule**

$$\log_b M^p = p \log_b M$$

Example

Use the power rule to expand each logarithmic expression

$\ln \sqrt[3]{x} = \ln(x)^{1/3} = \frac{1}{3}\ln x$

Quotient **Rule**

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

Example

Use the quotient rule to expand the logarithmic expression

$$\ln\left(\frac{e^5}{11}\right) = \ln e^5 - \ln 11$$

$$\boxed{= 5 - \ln 11}$$

Example 8

Express each of the following in terms of sums and differences of logarithm: $\log_a\left(\frac{mnq}{p^2r^4}\right)$

Solution

$$\log_a\left(\frac{mnq}{p^2r^4}\right) = \log_a(mnq) - \log_a\left(p^2r^4\right) \qquad \textit{Quotient Rule}$$

$$= \log_a m + \log_a n + \log_a q - \left(\log_a p^2 + \log_a r^4\right) \qquad \textit{Product Rule}$$

$$= \log_a m + \log_a n + \log_a q - \log_a p^2 - \log_a r^4$$

$$\boxed{= \log_a m + \log_a n + \log_a q - 2\log_a p - 4\log_a r} \qquad \textit{Power Rule}$$

Example 9

Express each of the following in terms of sums and differences of logarithm: $\log_5\left(\frac{\sqrt{x}}{25y^3}\right)$

Solution

$$\log_5\left(\frac{\sqrt{x}}{25y^3}\right) = \log_5\left(x^{1/2}\right) - \log_5\left(25y^3\right) \qquad \textit{Quotient Rule}$$

$$= \log_5\left(x^{1/2}\right) - \left[\log_5\left(5^2\right) + \log_5\left(y^3\right)\right] \qquad \textit{Product Rule}$$

$$= \log_5\left(x^{1/2}\right) - \log_5\left(5^2\right) - \log_5\left(y^3\right) \qquad \log_5\left(5^2\right) = 2$$

$$\boxed{= \frac{1}{2}\log_5 x - 2 - 3\log_5 y}$$

Example 10

Write as a single logarithmic $\quad 2\ln x + \frac{1}{3}\ln(x+5)$

Solution

$$2\ln x + \frac{1}{3}\ln(x+5) = \ln x^2 + \ln(x+5)^{1/3} \qquad \textit{Power Rule}$$

$= \ln x^2 (x+5)^{1/3}$ *Product Rule*

$= \ln\left(x^2 \sqrt[3]{x+5}\right)$

***Example* 11**

Write as a single logarithmic $2\log(x-3) - \log x$

Solution

$2\log(x-3) - \log x = \log(x-3)^2 - \log x$ *Power Rule*

$= \log\frac{(x-3)^2}{x}$ *Quotient Rule*

R.4–8 Solving Exponential Equations

1. Exponential Equations

$$b^M = b^N \quad \leftrightarrow \quad M = N \quad \text{for any b} > 0, \neq 1$$

***Example* 12**

Solve $5^{3x-6} = 125$

Solution

$5^{3x-6} = 5^3$

$3x - 6 = 3$

$3x = 9$

$x = 3$

2. Using *Natural Logarithms*

a. Isolate the exponential expression

b. Take the natural logarithm on both sides of the equation

c. Simplify using one of the following properties: $\ln b^x = x\ln b$ *or* $\ln e^x = x$

d. Solve for the variable.

Note: This method will work on any exponential equation.

Example 13

Solve: $7e^{2x} - 5 = 58$

Solution

$7e^{2x} - 5 = 58$	*Isolate the exponential expression*
$7e^{2x} = 63$	*Divide by 7 both sides*
$e^{2x} = 9$	*Natural logarithm on both sides*
$\ln e^{2x} = \ln 9$	*Use inverse Property*
$2x = \ln 9$	

$$\boxed{x = \frac{\ln 9}{2}} \qquad \boxed{\approx 1.0986}$$

R.4–9 Solving Logarithm Equations

1. Logarithmic Equations

a. Express the equation in the form $\log_b M = c$

b. Use the definition of a logarithm to rewrite the equation in exponential form:

$$\log_b M = c \quad \Rightarrow \quad b^c = M$$

c. Solve for the variable

d. Check proposed solution in the original equation. Include only the set for $M > 0$

Example 14

Solve: $\log(x) + \log(x-3) = 1$

Solution

$\log\big(x(x-3)\big) = 1$	*Product Rule*
$x(x-3) = 10^1$	*Convert to exponential form*
$x^2 - 3x = 10$	
$x^2 - 3x - 10 = 0$	*Solve for x*
$x = -2, 5$	

Check: $x = -2 \Rightarrow \log(-2) + \cancel{\log(x-3) = 1}$

$x = 5 \Rightarrow \log(5) + \log(5-3) = 1$

$\therefore$ ***Solution***: $\boxed{x = 5}$

2. Property of Logarithmic Equality

For any $M > 0,\ \ N > 0,\ \ \&\ \ b > 0,\ \ \neq 1$

$$\log_b M = \log_b N \quad \Rightarrow \quad M = N$$

Example 15

Solve: $\ln(x-3) = \ln(7x-23) - \ln(x+1)$

Solution

$\ln(x-3) = \ln\left(\frac{7x-23}{x+1}\right)$ ***Quotient Rule***

$x - 3 = \frac{7x-23}{x+1}$

$(x-3)(x+1) = 7x - 23$

$x^2 - 2x - 3 = 7x - 23$

$x^2 - 9x + 20 = 0$

$\underline{x = 4,\ 5|}$

Check: $x = 4 \Rightarrow \ln(4-3) = \ln(7(4)-23) - \ln(4+1)$

$x = 5 \Rightarrow \ln(5-3) = \ln(7(5)-23) - \ln(5+1)$

$\therefore$ ***Solution***: $\underline{x = 4,\ 5\ |}$

R.4–10 Exponential Growth and Decay

The mathematical model for exponential growth or decay is given by

$$A(t) = A_0 e^{kt}$$

$A(t)$: Exponential Function (After time t)

A_0 : At time zero (initial value).

t : Time

k : Exponential rate.

- If $\boldsymbol{k > 0}$, the function models of a growing entity
- If $\boldsymbol{k < 0}$, the function models of a decay entity.

Example 16

In 1990, the population of Africa was 643 *million* and by 2000 it had grown to 813 *million*

a. Use the exponential growth function $A(t) = A_0 e^{kt}$, in which t is the number of years after 1990, to find the exponential growth function that models data

b. By which year will Africa's population reach 2000 *million*, or two *billion*?

Solution

a. $A(t) = A_0 e^{kt}$

From 1990 to 2000, is 10 years, that implies in 10 years the population grows from 643 to 813

$$813 = 643e^{k(10)}$$

$$\frac{813}{643} = e^{10k}$$

$$\ln\frac{813}{643} = \ln e^{10k}$$

$$\ln\frac{813}{643} = 10k$$

$$\frac{1}{10}\ln\frac{813}{643} = k$$

$$k \approx 0.023$$

$$\Rightarrow \boxed{A(t) = 643e^{0.023t}}$$

b. $2000 = 643e^{0.023t}$

$$\frac{2000}{643} = e^{0.023t}$$

$$\ln\frac{2000}{643} = \ln e^{0.023t}$$

$$\ln\frac{2000}{643} = 0.023t$$

$$\frac{\ln\frac{2000}{643}}{0.023} = t$$

$$t \approx 49$$

$$\boxed{Year:\ 2039}$$

Example 17

Strontium-90 is a waste product from nuclear reactors. As a consequence of fallout from atmosphere nuclear tests, we all have a measurable amount of strontium-90 in our bones.

a) The half-life of Strontium-90 is 28 *years*, meaning that all after 28 *years* a given amount of the substance will have decayed to half the original amount. Find the exponential decay model for Strontium-90.

b) Suppose the nuclear accident occurs and releases 60 *grams* of Strontium-90 into the atmosphere. How long will it take for Strontium-90 to decay to a level of 10 *grams*?

Solution

a) $k = \frac{1}{28}\ln\frac{1}{2}$ $\qquad kT = \ln\frac{A}{A_0}$

≈ -0.0248

$\boxed{A(t) = A_0 e^{-0.0248t}}$

b) $A = A_0 e^{-0.0248t}$

$t = \dfrac{\ln\frac{1}{6}}{-0.0248}$ $\qquad kT = \ln\frac{A}{A_0}$

$\approx 72.25\ \ yrs$

Exercises *Section R.4* – Exponential & Logarithm

$(1-20)$ Find the ***asymptote***, ***domain***, and ***range*** of the given functions. Then, sketch the graph

1. $f(x)=2^x+3$
2. $f(x)=2^{3-x}$
3. $f(x)=-\left(\frac{1}{2}\right)^x+4$
4. $f(x)=4^x$
5. $f(x)=2-4^x$
6. $f(x)=-3+4^{x-1}$
7. $f(x)=1+\left(\frac{1}{4}\right)^{x+1}$
8. $f(x)=e^{x-2}$
9. $f(x)=3-e^{x-2}$
10. $f(x)=e^{x+4}$
11. $f(x)=2+e^{x-1}$
12. $f(x)=\log_4(x-2)$
13. $f(x)=\log_4|x|$
14. $f(x)=\left(\log_4 x\right)-2$
15. $f(x)=\log(3-x)$
16. $f(x)=2-\log(x+2)$
17. $f(x)=\ln(x-2)$
18. $f(x)=\ln(3-x)$
19. $f(x)=2+\ln(x+1)$
20. $f(x)=1-\ln(x-2)$

$(21-32)$ Write the equation in its equivalent logarithmic form

21. $2^6=64$
22. $5^4=625$
23. $5^{-3}=\frac{1}{125}$
24. $\sqrt[3]{64}=4$
25. $b^3=343$
26. $8^y=300$
27. $\sqrt[n]{x}=y$
28. $\left(\frac{2}{3}\right)^{-3}=\frac{27}{8}$
29. $\left(\frac{1}{2}\right)^{-5}=32$
30. $e^{x-2}=2y$
31. $e=3x$
32. $\sqrt[3]{e^{2x}}=y$

$(33-41)$ Write the equation in its equivalent exponential form

33. $\log_5 125=y$
34. $\log_4 16=x$
35. $\log_5 \frac{1}{5}=x$
36. $\log_6 \sqrt{6}=x$
37. $\log_3 \frac{1}{\sqrt{3}}=x$
38. $6=\log_2 64$
39. $\log_{\sqrt{3}} 81=8$
40. $\log_4 \frac{1}{64}=-3$
41. $\ln M=c$

$(42-50)$ Simplify

42. $\log_5 1$
43. $\log_7 7^2$
44. $3^{\log_3 8}$
45. $10^{\log 3}$
46. $e^{2+\ln 3}$
47. $\ln e^{-3}$
48. $\ln e^{x-5}$
49. $\log_b b^n$
50. $\ln e^{x^2+3x}$

$(\mathbf{51-67})$ Express the following in terms of sums and differences of logarithms

51. $\log_5\left(\frac{125}{y}\right)$

52. $\log_b x^7$

53. $\ln\sqrt[7]{x}$

54. $\log_b\left(\frac{x^3 y}{z^2}\right)$

55. $\log_b\left(\frac{\sqrt[3]{x}y^4}{z^5}\right)$

56. $\log_a \sqrt[4]{\frac{m^8 n^{12}}{a^3 b^5}}$

57. $\log_a \sqrt[3]{\frac{a^2 b}{c^5}}$

58. $\log_b\left(x^4\sqrt[3]{y}\right)$

59. $\log_5\left(\frac{\sqrt{x}}{25y^3}\right)$

60. $\ln\left(x^2\sqrt{x^2+1}\right)$

61. $\ln\frac{x^2}{x^2+1}$

62. $\ln\left(\frac{x^2(x+1)^3}{(x+3)^{1/2}}\right)$

63. $\ln\sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$

64. $\ln\frac{\left(x^2+1\right)^5}{\sqrt{1-x}}$

65. $\ln\left(\sqrt[3]{\frac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}}\right)$

66. $\ln\left(\sqrt{\frac{1}{x(x+1)}}\right)$

67. $\ln\left(\sqrt{\left(x^2+1\right)(x-1)^2}\right)$

$(\mathbf{68-81})$ Write the expression as a single logarithm and simplify if necessary

68. $\log\left(x^3y^2\right)-2\log\left(x\sqrt[3]{y}\right)-3\log\left(\frac{x}{y}\right)$

69. $\ln y^3+\frac{1}{3}\ln\left(x^3y^6\right)-5\ln y$

70. $4\ln x+7\ln y-3\ln z$

71. $\frac{1}{3}\left[5\ln(x+6)-\ln x-\ln(x^2-25)\right]$

72. $\frac{2}{3}\left[\ln\left(x^2-4\right)-\ln(x+2)\right]+\ln(x+y)$

73. $\frac{2}{3}\left[\ln\left(x^2-9\right)-\ln(x+3)\right]+\ln(x+y)$

74. $\frac{1}{4}\log_b x-2\log_b 5-10\log_b y$

75. $2\ln(x+4)-\ln x-\ln\left(x^2-3\right)$

76. $\ln\left(x^2-25\right)-2\ln(x+5)+\ln(x-5)$

77. $5\log_a x-\frac{1}{2}\log_a(3x-4)-3\log_a(5x+1)$

$(\mathbf{78-120})$ Solve the equations

78. $2^x=128$

79. $3^x=243$

80. $5^x=70$

81. $2^{5x+3}=\frac{1}{16}$

82. $\left(\frac{2}{5}\right)^x=\frac{8}{125}$

83. $2^{3x-7}=32$

84. $4^{2x-1}=64$

85. $2^{x+4}=8^{x-6}$

86. $5^{x+4}=4^{x+5}$

87. $3^{x-1}=7^{2x+5}$

88. $3^{x+4}=2^{1-3x}$

89. $e^x=15$

90. $e^{x+1}=20$

91. $e^{x\ln 3}=27$

92. $e^{x^2}=e^{7x-12}$

93. $f(x)=xe^x+e^x$

94. $f(x)=x^3\left(4e^{4x}\right)+3x^2e^{4x}$

95. $e^{2x}-2e^x-3=0$

96. $e^{2x+1}\cdot e^{-4x}=3e$

97. $e^{2x}-8e^x+7=0$

98. $e^{2x}+2e^x-15=0$

99. $e^x+e^{-x}-6=0$

100. $6\ln(2x)=30$

101. $\log_4(5+x)=3$

102. $\log(4x-18)=1$

103. $\log_5 x+\log_5(4x-1)=1$

104. $\log x-\log(x+3)=1$

105. $\log x+\log(x-9)=1$

106. $\ln(4x+6)-\ln(x+5)=\ln x$

107. $\ln(5+4x)-\ln(x+3)=\ln 3$

108. $\ln\sqrt[4]{x}=\sqrt{\ln x}$

109. $\sqrt{\ln x}=\ln\sqrt{x}$

110. $\log x^2=(\log x)^2$

111. $\log x^3=(\log x)^2$

112. $\ln(\ln x)=2$

113. $\ln\left(e^{x^2}\right)=64$

114. $e^{\ln(x-1)}=4$

115. $\ln x^2=\ln(12-x)$

116. $\ln x=1-\ln(x+2)$

117. $\ln x=1+\ln(x+1)$

118. $\log_3(x+3)+\log_3(x+5)=1$

119. $\ln x=\frac{1}{2}\ln\left(2x+\frac{5}{2}\right)+\frac{1}{2}\ln 2$

120. $\ln(-4-x)+\ln 3=\ln(2-x)$

121. Lead shielding is used to contain radiation. The percentage of a certain radiation that can penetrate x *millimeters* of lead shielding is given by $I(x)=100e^{-1.5x}$

a) What percentage of radiation will penetrate a lead shield that is 1 *millimeter* thick?
b) How many *millimeters* of lead shielding are required so that less than 0.02% of the radiations penetrates the shielding?

122. After a race, a runner's pulse rate R, in *beats* per *minute*, decreases according to the function

$$R(t)=145e^{-0.092t},\quad 0\le t\le 15$$

Where t is measured in minutes.

a) Find the runner's pulse rate at the end of the race and 1 *minute* after the end of the race.
b) How long after the end of the race will the runner's pulse rate be 80 *beats* per *minute*?

123. A can of soda at $79°F$ is placed in a refrigerator that maintains a constant temperature of $36°F$. The temperature T of the soda t *minutes* after it is placed in the refrigerator is given by

$$T(t)=36+43e^{-0.058t}$$

a) Find the temperature of the soda 10 *minutes* after it is placed in the refrigerator.

b) When will the temperature of the soda be $45°F$

124. During surgery, a patient's circulatory system requires at least 50 *milligrams* of an anesthetic. The amount of anesthetic present *t hours* after 80 *milligrams* of anesthetic is administered is given by

$$T(t) = 80(0.727)^t$$

a) How much of the anesthetic is present in the patient's circulatory system 30 *minutes* after the anesthetic is administered?
b) How long can the operation last if the patient does not receive additional anesthetic?

125. The following function models the average typing speed *S*, in *words* per *minute*, for a student who has been typing for *t months*.

$$S(t) = 5 + 29\ln(t+1), \quad 0 \le t \le 9$$

Use *S* to determine how long it takes the student to achieve an average speed of 65 *words* per *minute*.

126. A lawyer has determined that the number of people $P(t)$ in a city of 1.2 *million* people who have been exposed to a news item after *t days* is given by the function

$$P(t) = 1,200,000\left(1 - e^{-0.03t}\right)$$

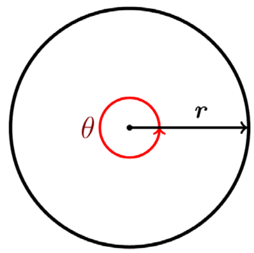
a) How many days after a major crime has been reported has 40% of the population heard of the crime?
b) A defense lawyer knows it will be difficult to pick an unbiased jury after 80% of the population has heard of the crime. After how many days will 80% of the population have heard of the crime?

127. Newton's Law of Cooling states that is an object at temperature T_0 is placed into an environment at constant temperature *A*, then the temperature of the object, $T(t)$ (in degrees Fahrenheit), after *t minutes* is given by $T(t) = A + \left(T_0 - A\right)e^{-kt}$, where *k* is a constant that depends on the object.

a) Determine the constant *k* for a canned soda drink that takes 5 *minutes* to cool from $75°F$ to $65°F$ after being placed in a refrigerator that maintains a constant temperature of $34°F$
b) What will be the temperature of the soda after 30 *minutes*?
c) When will the temperature of the soda drink be $36°F$?

Section R.5 – Trigonometry

R.5–1 Degrees – Radians



θ measures one full rotation $\theta = 2\pi$ *The measure of θ in radians is* 2π

$$\boxed{1 = 1\ rad}$$

$$1° = 1\ degree$$

If no unit of angle measure is specified, then the angle is to be measured in radians.

$$Full\ Rotation:\ 360° = 2\pi\ rad$$

$$180° = \pi\ rad$$

Converting from Degrees to Radians

$$\frac{180°}{180} = \frac{\pi}{180}\ rad \qquad \Rightarrow 1° = \frac{\pi}{180}\ rad$$

Multiply a degree measure by $\frac{\pi}{180}\ rad$ and simplify to convert to radians.

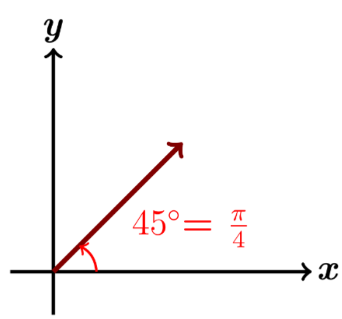
Example 1

Convert 45° to radians

Solution

$$45° = 45\left(\frac{\pi}{180}\right)rad$$

$$= \frac{\pi}{4}\ rad$$



Converting from Radians to Degrees

Multiply a radian measure by $\frac{180°}{\pi}$ radian and simplify to convert to degrees.

$$\frac{180°}{\pi} = \frac{\pi}{\pi}\ rad$$

$$\boxed{\frac{180^\circ}{\pi} = 1\ \ rad}$$

Example 2

Convert 1 to *degrees*

Solution

$$1\ rad = 1\left(\frac{180^\circ}{\pi}\right)$$

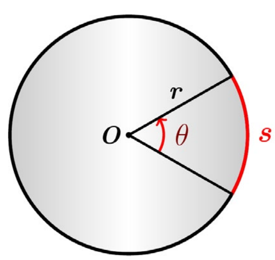
$$= 1\left(\frac{180^\circ}{3.14}\right)$$

$$= 57.3^\circ$$

R.5–2 Arc Length

Definition

If a central angle θ, in a circle of a radius r, cuts off an arc of length s, then the measure of θ, in radians is:

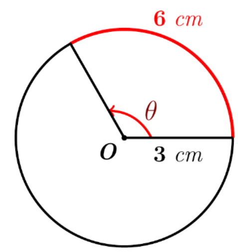


$\theta r = \frac{s}{r} r$ $\qquad$ $s = r\theta \qquad (\theta \text{ in radians})$

$\boxed{\textit{Note}: \textit{When applying the formula, the value of } \theta \textbf{ must} \text{ be in } \textbf{radian}}$

Example 3

A central angle θ in a circle of radius 3 *cm* cuts off an arc of length 6 *cm*.
What is the radian measure of θ.



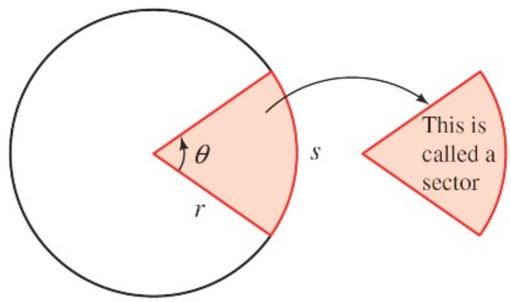
Solution

$$\theta = \frac{6\ cm}{3\ cm} \qquad\qquad \theta = \frac{s}{r}$$

$$\boxed{= 2\ \ rad}$$

R.5–3 Area of a Sector

A sector of a circle is a portion of the interior of a circle intercepted by a central angle.



$$\begin{array}{rcccl} \textit{Area of sector} & \rightarrow & \dfrac{A}{\pi r^2} = \dfrac{\theta}{2\pi} & \leftarrow & \textit{Central angle } \theta \\ \textit{Area of circle} & \rightarrow & & \leftarrow & \textit{One full rotation} \end{array}$$

Definition

If θ (in *radians*) is a central angle in a circle with radius *r*, then the area of the sector formed by an angle θ is given by

$$A = \frac{1}{2} r^2 \theta \qquad\qquad (\boldsymbol{\theta} \ \textbf{in radian})$$

Example 4

Find the area of the sector formed by a central angle of 1.4 *radians* in a circle of radius 2.1 *meters*.

Solution

Given: $r = 2.1\ m, \ \ \theta = 1.4$

$$A = \frac{1}{2} r^2 \theta$$

$$= \frac{1}{2}\left(\frac{21}{10}\right)^2\left(\frac{14}{10}\right)$$

$$= \frac{3,087}{1,000}$$

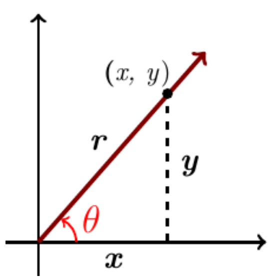
$$\underline{= 3.087\ m^2}$$

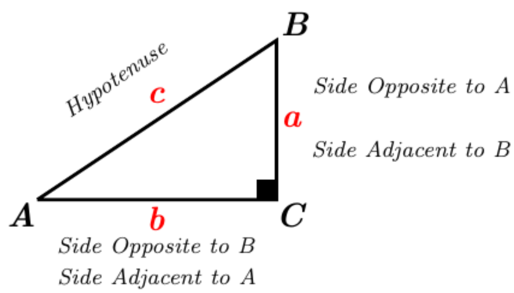
R.5−4 *Six* Trigonometry Functions

The relationships between side lengths and angles of triangles are defined as ***Trigonometry*** (from Greek trigōnon, "triangle" and metron, "measure").

Let (x, y) be a point on the terminal side of an angle θ in standard position

The distance from the point to the origin is given by: $r = \sqrt{x^2 + y^2}$





$$\sin\theta = \frac{Opposite}{Hypotenuse} = \frac{opp}{hyp} = \frac{y}{r}$$

$$\sin A = \frac{a}{c} = \cos B$$

$$\csc\theta = \frac{hyp}{opp} = \frac{1}{\sin\theta} = \frac{r}{y}$$

$$\csc A = \frac{c}{a} = \sec B$$

$$\cos\theta = \frac{Adjacent}{Hypotenuse} = \frac{adj}{hyp} = \frac{x}{r}$$

$$\cos A = \frac{b}{c} = \sin B$$

$$\sec\theta = \frac{hyp}{adj} = \frac{1}{\cos\theta} = \frac{r}{x}$$

$$\sec A = \frac{c}{b} = \csc B$$

$$\tan\theta = \frac{opp}{adj} = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$$

$$\tan A = \frac{a}{b} = \cot B$$

$$\cot\theta = \frac{adj}{opp} = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} = \frac{x}{y}$$

$$\cot A = \frac{b}{a} = \tan B$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

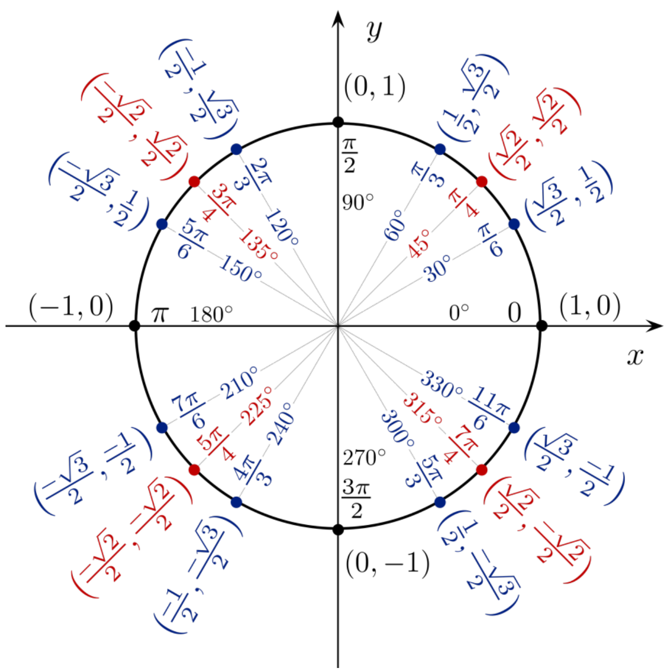
$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Ratio Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$\cos^2\theta + \sin^2\theta = 1$	$\cos\theta = \pm\sqrt{1-\sin^2\theta}$	$1+\tan^2\theta = \sec^2\theta$
	$\sin\theta = \pm\sqrt{1-\cos^2\theta}$	$1+\cot^2\theta = \csc^2\theta$



Example 5

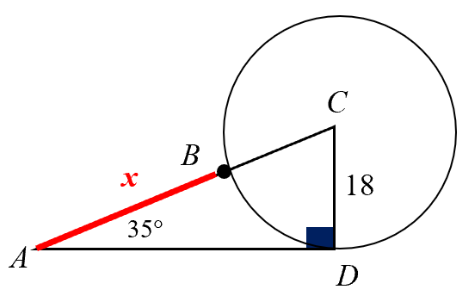
A circle has its center at *C* and a radius of 18 *inches*. If triangle *ADC* is a right triangle and *A* = 35°. Find *x*, the distance from *A* to *B*.

Solution

$$\sin 35^\circ = \frac{18}{x+18}$$

$$(x+18)\sin 35^\circ = 18$$

$$x+18 = \frac{18}{\sin 35^\circ}$$



$$x = \frac{18}{\sin 35°} - 18$$
$$\approx 13 \ in$$

Example 6

Prove: $\tan x + \cos x = \sin x(\sec x + \cot x)$

Solution

$$\tan x + \cos x = \frac{\sin x}{\cos x} + \cos x$$
$$= \sin x \frac{1}{\cos x} + \cos x \frac{\sin x}{\sin x}$$
$$= \sin x \sec x + \sin x \frac{\cos x}{\sin x}$$
$$= \sin x(\sec x + \cot x) \quad \checkmark$$

$\boxed{or}$

$$\sin x(\sec x + \cot x) = \sin x\left(\frac{1}{\cos x} + \frac{\cos x}{\sin x}\right)$$
$$= \frac{\sin x}{\cos x} + \sin x \frac{\cos x}{\sin x}$$
$$= \tan x + \cos x$$

Example 7

Prove $\quad \dfrac{\cos^4 t - \sin^4 t}{\cos^2 t} = 1 - \tan^2 t$

Solution

$$\frac{\cos^4 t - \sin^4 t}{\cos^2 t} = \frac{\left(\cos^2 t - \sin^2 t\right)\left(\cos^2 t + \sin^2 t\right)}{\cos^2 t} \qquad a^2 - b^2 = (a-b)(a+b)$$
$$= \frac{\left(\cos^2 t - \sin^2 t\right)(1)}{\cos^2 t} \qquad \cos^2 t + \sin^2 t = 1$$
$$= \frac{\cos^2 t - \sin^2 t}{\cos^2 t}$$
$$= \frac{\cos^2 t}{\cos^2 t} - \frac{\sin^2 t}{\cos^2 t}$$
$$= 1 - \tan^2 t \quad \checkmark$$

R.5–5 Sum and Difference Formulas

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \qquad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Example 8

Show that $\cos(x + 2\pi) = \cos x$

Solution

$$\begin{aligned} \cos(x+2\pi) &= \cos x \cos 2\pi - \sin x \sin 2\pi \\ &= \cos x \cdot (1) - \sin x \cdot (0) \\ &= \cos x \quad \checkmark \end{aligned}$$

Example 9

Establish the identity: $\dfrac{\cos(x-y)}{\sin x \sin y} = \cot x \cot y + 1$

Solution

$$\begin{aligned} \frac{\cos(x-y)}{\sin x \sin y} &= \frac{\cos x \cos y + \sin x \sin y}{\sin x \sin y} \\ &= \frac{\cos x \cos y}{\sin x \sin y} + \frac{\sin x \sin y}{\sin x \sin y} \\ &= \cot x \cot y + 1 \quad \checkmark \end{aligned}$$

Example 10

Establish the identity: $\cot(x+y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$

Solution

$$\begin{aligned} \cot(x+y) &= \frac{\cos(x+y)}{\sin(x+y)} \\ &= \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y} \end{aligned}$$

$$=\frac{\frac{\cos x\cos y}{\sin x\sin y}-\frac{\sin x\sin y}{\sin x\sin y}}{\frac{\sin x\cos y}{\sin x\sin y}+\frac{\cos x\sin y}{\sin x\sin y}}$$

$$=\frac{\cot x\cot y-1}{\cot x+\cot y}\quad \checkmark$$

R.5−6 Double & Half −Angle

$$\cos 2A=\cos^2 A-\sin^2 A$$
$$=2\cos^2 A-1$$
$$=1-2\sin^2 A$$

$$\sin 2A=2\sin A\cos A$$

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}$$

$$\cos\frac{A}{2}=\pm\sqrt{\frac{1+\cos A}{2}}$$

$$\sin\frac{A}{2}=\pm\sqrt{\frac{1-\cos A}{2}}$$

$$\tan\frac{A}{2}=\frac{\sin A}{1+\cos A}=\frac{1-\cos A}{\sin A}$$

Example 11

Prove $\quad (\sin\theta+\cos\theta)^2=1+\sin 2\theta$

Solution

$$(\sin\theta+\cos\theta)^2=\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta$$
$$=\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta$$
$$=1+2\sin\theta\cos\theta$$
$$=1+\sin 2\theta\quad \checkmark$$

Example 12

Prove $\quad \dfrac{2\cot x}{1+\cot^2 x}=\sin 2x$

Solution

$$\frac{2\cot x}{1+\cot^2 x}=\frac{2\frac{\cos x}{\sin x}}{1+\frac{\cos^2 x}{\sin^2 x}}$$

$$=2\frac{\cos x}{\sin x}\frac{\sin^2 x}{\sin^2 x+\cos^2 x}$$

$$= 2\frac{\cos x}{1}\frac{\sin x}{1}$$
$$= 2\cos x \sin x$$
$$\underline{= \sin 2x} \quad \checkmark$$

Example 13

Prove $\quad \cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

Solution

$$\begin{aligned}
\cos 4x &= \cos(2.2x) \\
&= 2\cos^2 2x - 1 \\
&= 2(\cos 2x)^2 - 1 \\
&= 2\left(2\cos^2 x - 1\right)^2 - 1 \\
&= 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1 \\
&= 8\cos^4 x - 8\cos^2 x + 2 - 1 \\
&\underline{= 8\cos^4 x - 8\cos^2 x + 1} \quad \checkmark
\end{aligned}$$

Example 14

Prove $\quad \dfrac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$

Solution

$$\begin{aligned}
\frac{1-\cos 2\theta}{\sin 2\theta} &= \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta} \\
&= \frac{1-1+2\sin^2\theta}{2\sin\theta\cos\theta} \\
&= \frac{2\sin^2\theta}{2\sin\theta\cos\theta} \\
&= \frac{\sin\theta}{\cos\theta} \\
&\underline{= \tan\theta} \quad \checkmark
\end{aligned}$$

Example 15

Prove $\sin^2\frac{x}{2}=\frac{\tan x-\sin x}{2\tan x}$

Solution

$$\sin^2\frac{x}{2}=\frac{1-\cos x}{2}$$

$$=\frac{\tan x}{\tan x}\frac{1-\cos x}{2}$$

$$=\frac{\tan x-\tan x\cos x}{2\tan x}$$

$$=\frac{\tan x-\frac{\sin x}{\cos x}\cos x}{2\tan x}$$

$$=\frac{\tan x-\sin x}{2\tan x}\quad\checkmark$$

R.5–7 Solving Trigonometry Equations

Example 16

Find the solutions of the equation $\sin\theta=\frac{1}{2}$ if

a) θ is in the interval $[0,\ 2\pi)$

b) θ is any real number

Solution

a) $\theta=\sin^{-1}\frac{1}{2}=\frac{\pi}{6}$

$\theta=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$

b) Since the sine function has period 2π.

$\theta=\frac{\pi}{6}+2\pi n$ *and* $\theta=\frac{5\pi}{6}+2\pi n$

Example 17

Solve the equation $4\sin^2x\tan x-\tan x=0$ in the interval $[0,\ 2\pi)$.

Solution

$4\sin^2x\tan x-\tan x=0$

$\tan x\left(4\sin^2 x - 1\right) = 0$ *Factor out* $\tan x$

$\tan x = 0$ $\quad 4\sin^2 x - 1 = 0$

$\sin^2 x = \frac{1}{4}$

$\tan x = 0$ $\quad \sin x = \frac{1}{2}$ $\quad \sin x = -\frac{1}{2}$

$\underline{x = 0,\ \pi}$ $\quad \underline{x = \frac{\pi}{6},\ \frac{5\pi}{6}}$ $\quad \underline{x = \frac{7\pi}{6},\ \frac{11\pi}{6}}$

Example 18

Solve the equation $2\sin^2 t - \cos t - 1 = 0$, and express the solutions both in radians and degrees.

Solution

$2\sin^2 t - \cos t - 1 = 0$

$2\left(1 - \cos^2 t\right) - \cos t - 1 = 0$ $\quad \sin^2 t + \cos^2 t = 1$

$2 - 2\cos^2 t - \cos t - 1 = 0$

$-2\cos^2 t - \cos t + 1 = 0$ *Multiply by* -1

$2\cos^2 t + \cos t - 1 = 0$ *Factor or use quadratic formula*

$(2\cos t - 1)(\cos t + 1) = 0$

$2\cos t - 1 = 0$ $\quad \cos t + 1 = 0$

$2\cos t = 1$ $\quad \cos t = -1$

$\cos t = \frac{1}{2}$

$t = \frac{\pi}{3}$ *or* $t = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$ $\quad t = \pi$

$t = \frac{\pi}{3} + 2\pi n, \quad t = \frac{5\pi}{3} + 2\pi n,$ $\quad t = \pi + 2\pi n$

$\underline{t = 60° + 360° n, \quad 300° + 360° n, \quad \textit{and} \quad 180° + 360° n}$

R.5–8 Graphing *Sine* & Cosine

We consider graphs of the equation: $y = A\sin(Bx+C)+D \quad y = A\cos(Bx+C)+D$

Amplitude $= |A|$ Period $= \dfrac{2\pi}{|B|}$ Phase Shift $= \phi = -\dfrac{C}{B}$

Vertical Shift: $y = D$

Example 19

Graph $y = 4\cos\left(2x - \frac{3\pi}{2}\right)$ *for* $0 \le x \le 2\pi$

Solution

Amplitude: $|A| = 4$

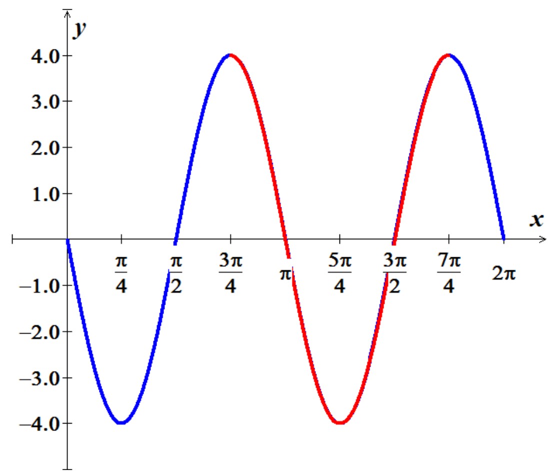
Period: $P = \dfrac{2\pi}{2} = \pi$

Phase Shift:

$$\phi = \frac{\frac{3\pi}{2}}{2} = \frac{3\pi}{4}$$

VT: $y = 0$

	x	y
$0 + \frac{3\pi}{4}$	$\frac{3\pi}{4}$	4
$\frac{\pi}{4} + \frac{3\pi}{4}$	π	0
$\frac{\pi}{2} + \frac{3\pi}{4}$	$\frac{5\pi}{4}$	–4
$\frac{3\pi}{4} + \frac{3\pi}{4}$	$\frac{3\pi}{2}$	0
$\pi + \frac{3\pi}{4}$	$\frac{7\pi}{4}$	4



Example 20

Graph one complete cycle $y = 3 - 5\sin\left(\pi\, x + \frac{\pi}{4}\right)$

Solution

Amplitude: $|A| = 5$

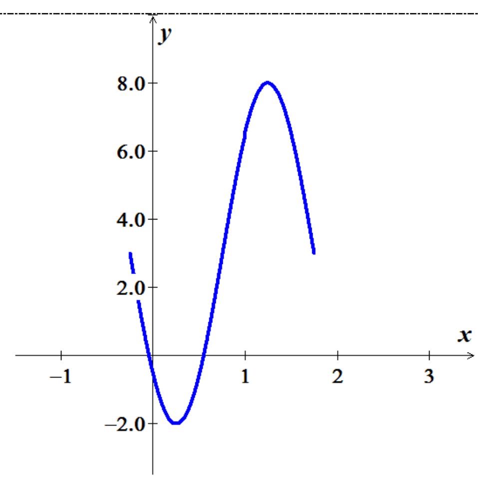
Period: $P = \dfrac{2\pi}{\pi} = 2$

Phase Shift:

$$\phi = -\frac{\frac{\pi}{4}}{\pi} = -\frac{1}{4}$$

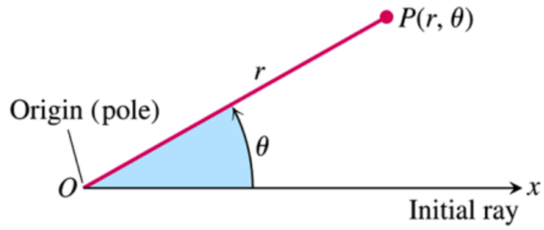
VT: $y = 3$

	x	y
$0 - \frac{1}{4}$	$-\frac{1}{4}$	3
$\frac{1}{2} - \frac{1}{4}$	$\frac{1}{4}$	–2
$1 - \frac{1}{4}$	$\frac{3}{4}$	3
$\frac{3}{2} - \frac{1}{4}$	$\frac{5}{4}$	8
$2 - \frac{1}{4}$	$\frac{7\pi}{4}$	3



R.5−9 *Definition* of Polar Coordinates

To define polar coordinates, let an ***origin*** O (called the ***pole***) and an ***initial ray*** from O. Then each point P can be located by assigning to it a ***polar coordinate pair*** $(r,\ \theta)$ in which r gives the directed from O to P and θ gives the directed angle from the initial ray to yay OP.



$$P(r,\ \theta)$$

Directed distance from O to P

Directed angle from intial ray to OP

R.5−10 *Definition* – Relationships between Rectangular and Polar Coordinates

The rectangular coordinates (x, y) and polar coordinates $(r,\ \theta)$ of a point P are related as follows:

1. $x = r\cos\theta, \quad y = r\sin\theta$

2. $r^2 = x^2 + y^2 \quad \tan\theta = \dfrac{y}{x} \quad \text{if } x \neq 0$

Example 21

If $(r,\ \theta) = \left(4,\ \frac{7\pi}{6}\right)$ are polar coordinates of a point P, find the rectangular coordinates of P.

Solution

$$\begin{aligned} x &= r\cos\theta \\ &= 4\cos\frac{7\pi}{6} \\ &= 4\left(-\frac{\sqrt{3}}{2}\right) \\ &= -2\sqrt{3} \end{aligned}$$

$$y = r\sin\theta$$

$$= 4\sin\frac{7\pi}{6}$$

$$= 4\left(-\frac{1}{2}\right)$$

$$= -2$$

The rectangular coordinates of P are $(x, y) = \left(-2\sqrt{3},\ -2\right)$

Example 22

If $(x, y) = \left(-1,\ \sqrt{3}\right)$ are rectangular coordinates of a point P, find three different pairs the polar coordinates of P.

Solution

$$r = \pm\sqrt{x^2 + y^2}$$

$$= \pm\sqrt{(-1)^2 + \left(\sqrt{3}\right)^2}$$

$$= \pm\sqrt{1+3}$$

$$= \pm\sqrt{4}$$

$$= \pm 2$$

$$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{-1}$$

$$= -\sqrt{3}$$

$$\hat{\theta} = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

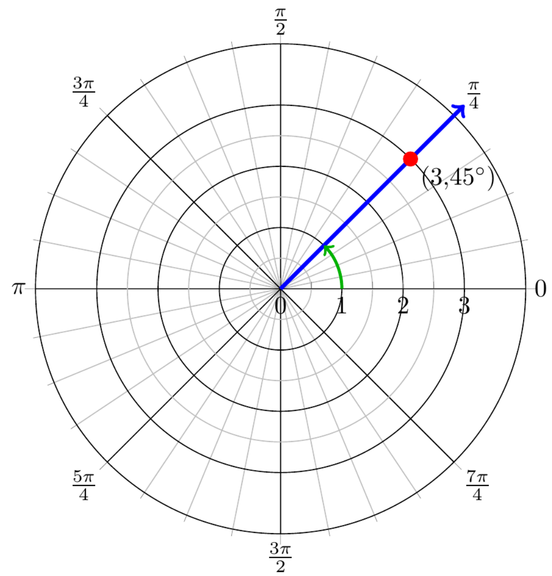
$$\theta_1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\theta_2 = \frac{2\pi}{3} + 2\pi = \frac{3\pi}{3}$$

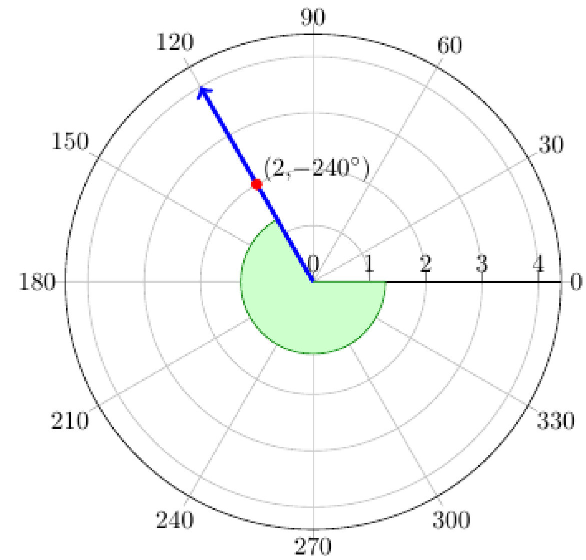
$$\theta_3 = -\frac{\pi}{3}$$

The polar coordinates of P are: $\left(2,\ \frac{2\pi}{3}\right)$, $\left(-2,\ \frac{5\pi}{3}\right)$, $\left(2,\ -\frac{4\pi}{3}\right)$, and $\left(-2,\ -\frac{\pi}{3}\right)$

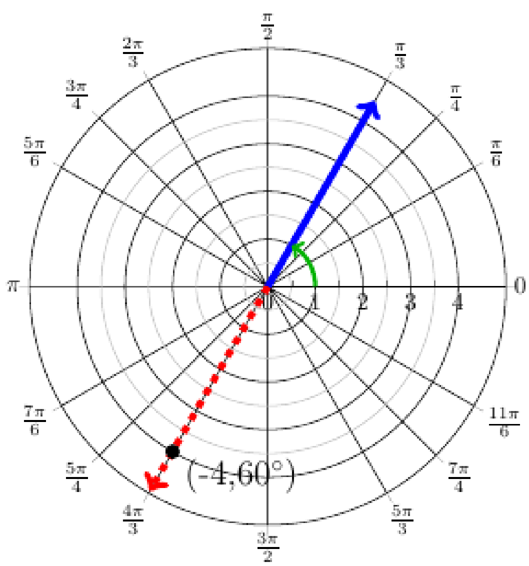
R.5–11 Graphing Polar Coordinates



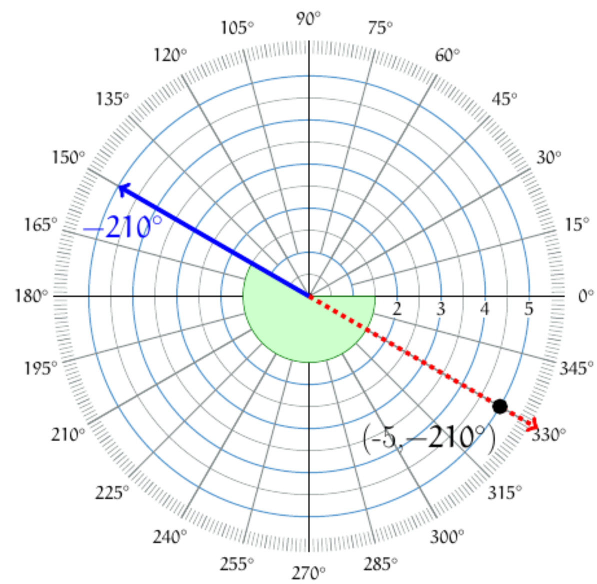
$\left(3,\ 45^\circ\right)$



$\left(2,\ -\frac{4\pi}{3}\right)$



$\left(-4,\ \frac{\pi}{3}\right)$



$\left(-5,\ -210^\circ\right)$

Example 23

Find a polar equation of the hyperbola $x^2 - y^2 = 16$.

Solution

$$(r\cos\theta)^2 - (r\sin\theta)^2 = 16$$

$$r^2\cos^2\theta - r^2\sin^2\theta = 16$$

$$r^2\left(\cos^2\theta - \sin^2\theta\right) = 16$$

$$r^2\left(\cos 2\theta\right) = 16$$

$$\boxed{r^2 = \frac{16}{\cos 2\theta}} \qquad \cos 2\theta \neq 0$$

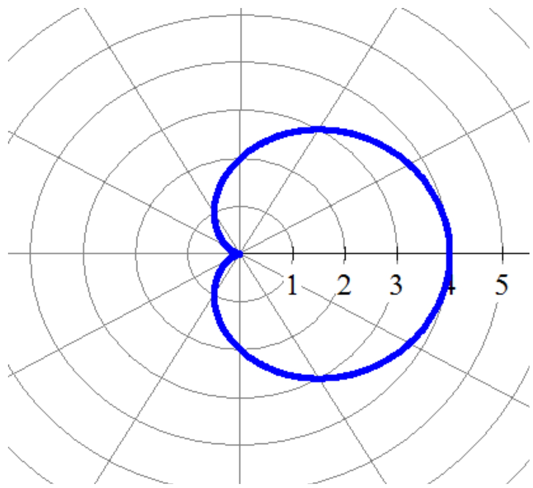
or $\quad r^2 = 16\sec 2\theta$

Example 24

Sketch the graph of the polar equation $r = 2 + 2\cos\theta$.

Solution

θ	r
0	4
$\frac{\pi}{4}$	$2+\sqrt{2}$
$\frac{\pi}{2}$	2
$\frac{3\pi}{4}$	$2-\sqrt{2}$
π	0
$\frac{3\pi}{2}$	2
2π	4



Exercises Section R.5 – Trigonometry

1. Convert to radians

a) $256°\ 20'$ *b)* $-78.4°$ *c)* $330°$ *d)* $-60°$ *e)* $-225°$

2. Convert to degrees

a) $\frac{11\pi}{6}$ *b)* $-\frac{5\pi}{3}$ *c)* $\frac{\pi}{6}$ *d)* 2.4 *e)* $\frac{\pi}{3}$ *f)* $-\frac{5\pi}{12}$ *g)* -4π *h)* $\frac{7\pi}{13}$

$(\mathbf{2-27})$ Verify the identity of each equation

3. $\dfrac{\tan\theta\cot\theta}{\csc\theta}=\sin\theta$

4. $\dfrac{\sec^2\theta}{\tan\theta}=\sec\theta\csc\theta$

5. $\dfrac{\sin\theta}{\csc\theta}+\dfrac{\cos\theta}{\sec\theta}=1$

6. $\cot\theta+\tan\theta=\csc\theta\sec\theta$

7. $\tan x(\cos x+\cot x)=\sin x+1$

8. $\dfrac{\cos x}{1+\sin x}-\dfrac{1-\sin x}{\cos x}=0$

9. $\tan^2 x=\sec^2 x-\sin^2 x-\cos^2 x$

10. $\dfrac{\sin\theta}{1+\sin\theta}-\dfrac{\sin\theta}{1-\sin\theta}=-2\tan^2\theta$

11. $\dfrac{\cos x}{1+\sin x}+\dfrac{1+\sin x}{\cos x}=2\sec x$

12. $\dfrac{\tan x+\cot x}{\tan x-\cot x}=\dfrac{1}{\sin^2 x-\cos^2 x}$

13. $\sec x+\tan x=\dfrac{\cos x}{1-\sin x}$

14. $\dfrac{\cos x}{\cos x-\sin x}=\dfrac{1}{1-\tan x}$

15. $\dfrac{\cot^2 x}{\csc x-1}=\dfrac{1+\sin x}{\sin x}$

16. $\sec^4 x-\tan^4 x=\sec^2 x+\tan^2 x$

17. $\left(1+\tan^2 x\right)\left(1-\sin^2 x\right)=1$

18. $1-\dfrac{\cos^2 x}{1+\sin x}=\sin x$

19. $\dfrac{\sin(x-y)}{\sin x\cos y}=1-\cot x\tan y$

20. $\dfrac{\sin(x-y)}{\sin x\sin y}=\cot y-\cot x$

21. $\cos 3x=\cos^3 x-3\cos x\sin^2 x$

22. $\cos^4 x-\sin^4 x=\cos 2x$

23. $\dfrac{\cos 2x}{\cos^2 x}=\sec^2 x-2\tan^2 x$

24. $\tan^2 x(1+\cos 2x)=1-\cos 2x$

25. $\dfrac{\cos 2x}{\sin^2 x}=2\cot^2 x-\csc^2 x$

26. $2\sin^2\left(\dfrac{x}{2}\right)=\dfrac{\sin^2 x}{1+\cos x}$

27. $\sec^2\left(\dfrac{x}{2}\right)=\dfrac{2\sec x+2}{\sec x+2+\cos x}$

$(\mathbf{28-37})$ Find the solutions of the equation that are in the interval $[0,\ 2\pi)$

28. $2\sin^2 x=1-\sin x$

29. $\tan^2 x\sin x=\sin x$

31. $2\sin^2 x-\cos x-1=0$

32. $\sin x+\cos x\cot x=\csc x$

30. $1-\sin x=\sqrt{3}\cos x$

33. $2\sin^3 x+\sin^2 x-2\sin x-1=0$

34. $2\cos^2 t-9\cos t=5$

36. $\tan x+\sqrt{3}=\sec x$

35. $\tan^2 x+\tan x-2=0$

37. $4\cos^2 x+4\sin x-5=0$

(38 – 60) Find the amplitude, the period, the phase shift, and the vertical translation and sketch the graph of the equation

38. $y=2\sin(x-\pi)$

39. $y=\frac{2}{3}\sin\left(x+\frac{\pi}{2}\right)$

40. $y=2-\sin\left(3x-\frac{\pi}{5}\right)$

41. $y=2\sin\left(x-\frac{\pi}{3}\right)$

42. $y=-\frac{2}{3}\sin\left(3x-\frac{\pi}{2}\right)$

43. $y=-2\sin(2\pi x+\pi)$

44. $y=3\cos\left(\frac{\pi}{2}\left(x-\frac{1}{2}\right)\right)$

45. $y=-\cos\pi\left(x-\frac{1}{3}\right)$

46. $y=\frac{5}{2}-3\cos\left(\pi x-\frac{\pi}{4}\right)$

47. $y=-3+\sin\left(\pi x+\frac{\pi}{2}\right)$

48. $y=4\cos\left(\frac{1}{2}x+\frac{\pi}{2}\right)$

49. $y=-2\sin(2x-\pi)+3$

50. $y=3\cos(x+3\pi)-2$

51. $y=2\tan\left(2x+\frac{\pi}{2}\right)$

52. $y=-\frac{1}{4}\tan\left(\frac{1}{2}x+\frac{\pi}{3}\right)$

53. $y=2\cot\left(2x+\frac{\pi}{2}\right)$

54. $y=-\frac{1}{2}\cot\left(\frac{1}{2}x+\frac{\pi}{4}\right)$

55. $y=1-2\cot 2\left(x+\frac{\pi}{2}\right)$

56. $y=3+2\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)$

57. $y=1-\frac{1}{2}\csc\left(x-\frac{3\pi}{4}\right)$

58. $y=2+\frac{1}{4}\sec\left(\frac{1}{2}x-\pi\right)$

59. $y=-3\sec\left(\frac{1}{3}x+\frac{\pi}{3}\right)$

60. $y=2\sec\left(2x-\frac{\pi}{2}\right)$

(61 – 66) Convert to rectangular coordinates

61. $(4,\ 30^\circ)$

62. $\left(-\sqrt{2},\ \frac{3\pi}{4}\right)$

63. $(3,\ 270^\circ)$

64. $(2,\ 60^\circ)$

65. $\left(\sqrt{2},\ -225^\circ\right)$

66. $\left(4\sqrt{3},\ -\frac{\pi}{6}\right)$

(67 – 72) Convert to polar coordinates

67. $(3,\ 3)$

68. $(-2,\ 0)$

69. $(-1,\ \sqrt{3})$

70. $(-3,\ -3)\quad r\ge 0\quad 0^\circ\le\theta<360^\circ$

71. $\left(2,\ -2\sqrt{3}\right)\quad r\ge 0\quad 0^\circ\le\theta<360^\circ$

72. $(-2,\ 0)\quad r\ge 0\quad 0\le\theta<2\pi$

(73 – 85) Write the equation in rectangular coordinates

73. $r^2=4$

74. $r=6\cos\theta$

75. $r^2=4\cos 2\theta$

76. $r(\cos\theta-\sin\theta)=2$

78. $r\sin\theta=-2$

79. $\theta=\frac{\pi}{4}$

80. $r^2\left(4\sin^2\theta-9\cos^2\theta\right)=36$

82. $r(\sin\theta-2\cos\theta)=6$

83. $r=8\sin\theta-2\cos\theta$

84. $r=\tan\theta$

77. $r^2 = 4\sin 2\theta$

81. $r^2\left(\cos^2\theta + 4\sin^2\theta\right) = 16$

85. $r\left(\sin\theta + r\cos^2\theta\right) = 1$

(86 – 94) Write the equation in polar coordinates

86. $y^2 = 6x$

87. $xy = 8$

88. $x + y = 5$

89. $x^2 + y^2 = 9$

90. $(x+2)^2 + (y-3)^2 = 13$

91. $y^2 - x^2 = 4$

92. $x^2 + y^2 = 4x$

93. $y = -x$

94. $x + y = 4$

(95 – 105) Sketch the graph of the polar equation

95. $r = 5$

96. $\theta = \frac{\pi}{4}$

97. $r = 4\cos\theta + 2\sin\theta$

98. $r = 2 + 4\sin\theta$

99. $r = 2 - \cos\theta$

100. $r = 4\csc\theta$

101. $r^2 = 4\cos 2\theta$

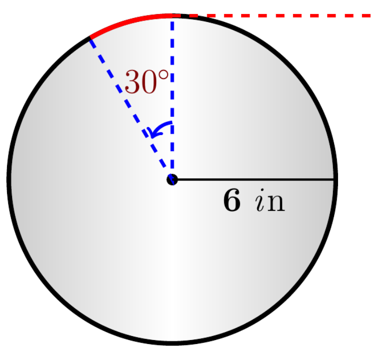
102. $r = 2^\theta \quad \theta \ge 0$

103. $r = e^{2\theta} \quad \theta \ge 0$

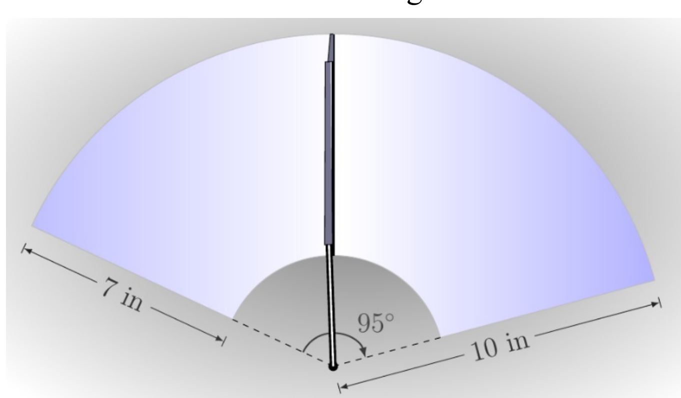
104. $r\theta = 1 \quad \theta > 0$

105. $r = 2 + 2\sec\theta$

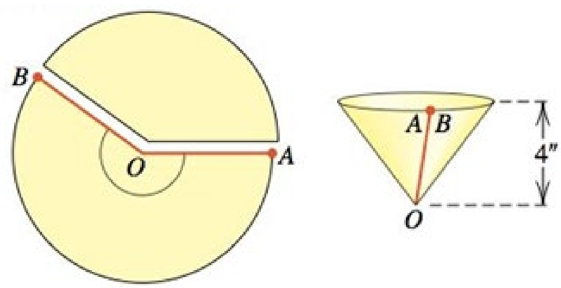
106. A rope is being wound around a drum with radius 6 *inches*. How much rope will be wound around the drum if the drum is rotated through an angle of 30°?



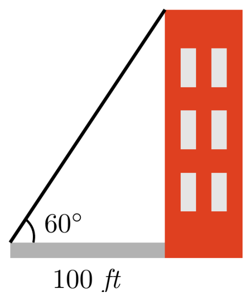
107. The total arm and blade of a single windshield wiper was 10 *in*. long and rotated back and forth through an angle of 95°. The shaded region in the figure is the portion of the windshield cleaned by the 7-*in*. wiper blade. What is the area of the region cleaned?



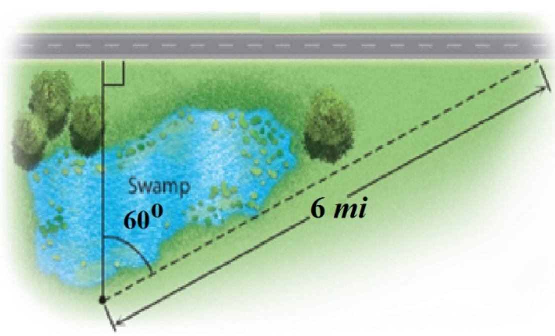
108. A conical paper cup is constructed by removing a sector from a circle of radius 5 *inches* and attaching edge *OA* to *OB*. Find angle *AOB* so that the cap has a depth of 4 *inches*.



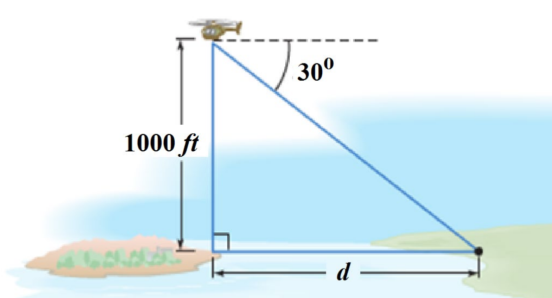
109. The shadow of a vertical tower is 100 *feet* long when the angle of elevation of the sun is 60°. Find the height of the tower.



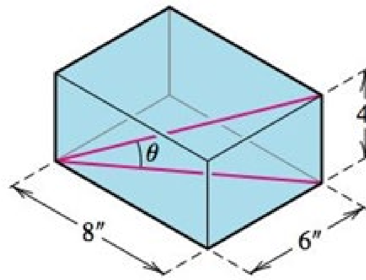
110. You were hiking directly toward a long straight road when you encountered a swamp. you turned 60° to the right and hiked 6 *mi* in that direction to reach the road. How far were you from the road when you encountered the swamp?



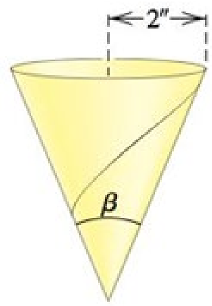
111. A helicopter hovers 1,000 *feet* above a small island. The angle of depression from the helicopter to point *P* on the coast is 30°. How far off the coast is the island?



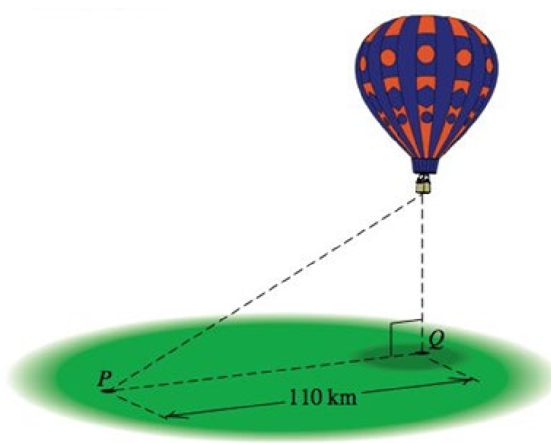
112. A rectangular box has dimensions $8'' \times 6'' \times 4''$. Approximate, to the nearest tenth of a degree, the angle θ formed by a diagonal of the base and the diagonal of the box.



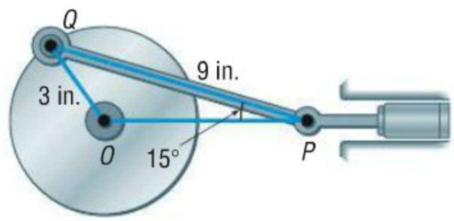
113. A conical paper cup has a radius of 2 *inches*, approximate, to the nearest degree, the angle β so that the cone will have a volume of $20\ in^3$.



114. As a hot-air balloon rises vertically, its angle of elevation from a point *P* on level ground 100 *km* from the point *Q* directly underneath the balloon changes from $19°\ 20'$ to $31°\ 50'$. Approximately how far does the balloon rise during this period?

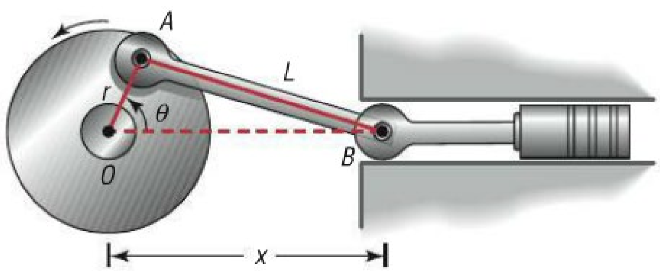


115. On a certain automobile, the crankshaft is 3 *inches* long and the connecting rod is 9 *inches* long. At the time when $\angle OPQ$ is 15°, how far is the piston *P* from the center *O* of the crankshaft?



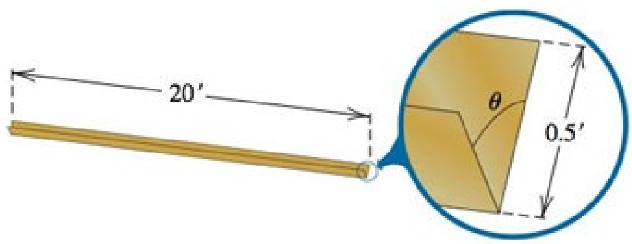
116. Rod OA rotates about the fixed point O so that point A travels on a circle of radius r. Connected to point A is another rod AB of length $L > 2r$, and point B is connected to a piston. Show that the distance x between point O and point B is given by

$$x = r\cos\theta + \sqrt{r^2\cos^2\theta + L^2 - r^2}$$



Where θ is the angle of rotation of rod OA.

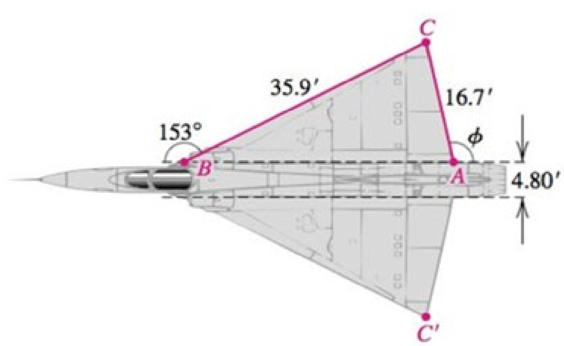
117. Shown in the figure is a design for a rain gutter.



a) Express the volume V as a function of θ.

b) Approximate the acute angle θ that results in a volume of $2\ ft^3$

118. Shown in the figure is a plan for the top of a wing of a jet fighter.

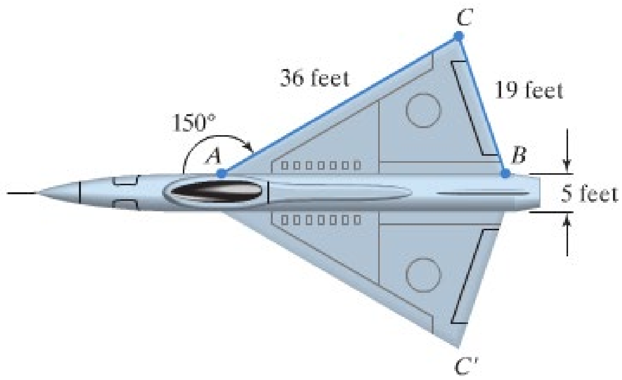


a) Approximate angle ϕ.

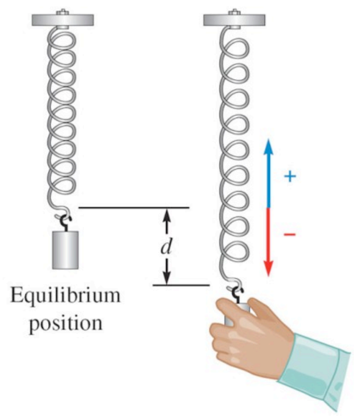
b) If the fuselage is 4.80 *feet* wide, approximate the wing span CC'.

c) Approximate the area of the triangle ABC.

119. Shown in the figure is a plan for the top of a wing of a jet fighter. The fuselage is 5 *feet* wide. Find the wing span CC'



120. A mass attached to a spring oscillates upward and downward. The length L of the spring after t seconds is given by the function $L = 15 - 3.5\cos(2\pi t)$, where L is measured in cm.



a) Sketch the graph of this function for $0 \le t \le 5$
b) What is the length the spring when it is at equilibrium?
c) What is the length the spring when it is shortest?
d) What is the length the spring when it is longest?

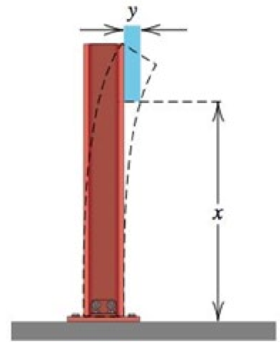
121. Based on years of weather data, the expected low temperature T (in °F) in Fairbanks, Alaska, can be approximated by

$$T = 36\sin\left(\frac{2\pi}{365}(t-101)\right) + 14$$

a) Sketch the graph T for $0 \le t \le 365$
b) Predict when the coldest day of the year will occur.

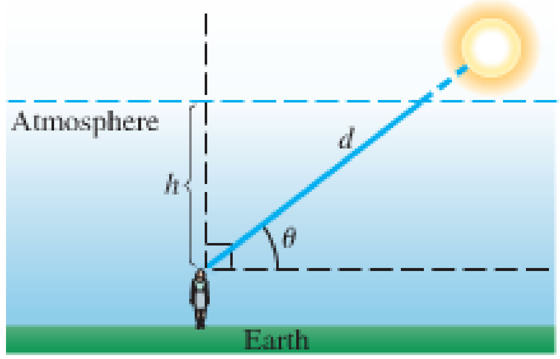
122. To simulate the response of a structure to an earthquake, an engineer must choose a shape for the initial displacement of the beams in the building. When the beam has length L *feet* and the maximum displacement is a *feet*, the equation

$$y = a - a\cos\frac{\pi}{2L}x$$



Has been used by engineers to estimate the displacement y. if $a = 1$ and $L = 10$, sketch the graph of the equation for $0 \le x \le 10$.

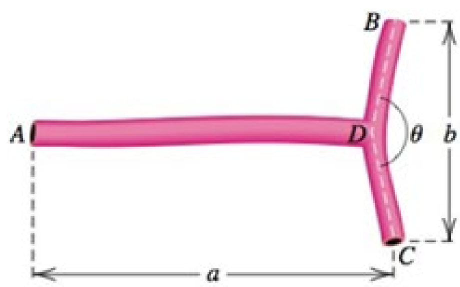
123. The shortest path for the sun's rays through Earth's atmosphere occurs when the sun is directly overhead. Disregarding the curvature of Earth, as the sun moves lower on the horizon, the distance that sunlight passes through the atmosphere increases by a factor of $\csc\theta$, where θ is the angle of elevation of the sun. This increased distance reduces both the intensity of the sun and the amount of ultraviolet light that reached Earth's surface.



a) Verify that $d = h\csc\theta$

b) Determine θ when $d = 2h$

124. A common form of cardiovascular branching is bifurcation, in which an artery splits into two smaller blood vessels. The bifurcation angle θ is the angle formed by the two smaller arteries. The line through A and D bisects θ and is perpendicular to the line through B and C.



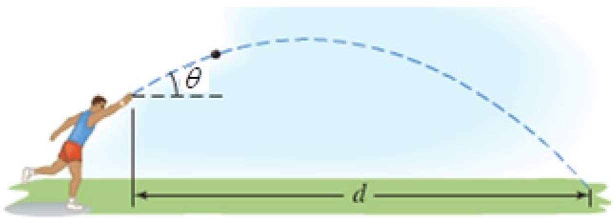
a) Show that the length l of the artery from A to B is given by $l = a + \frac{b}{2}\tan\frac{\theta}{4}$.

b) Estimate the length l from the three measurements $a = 10\ mm$, $b = 6\ mm$, and $\theta = 156°$.

125. Throwing events in track and field include the shot put, the discus throw, the hammer throw, and the javelin throw. The distance that the athlete can achieve depends on the initial speed of the object thrown and the angle above the horizontal at which the object leaves the hand. This angle is represented by θ. The distance, d, in *feet*, that the athlete throws is modeled by the formula

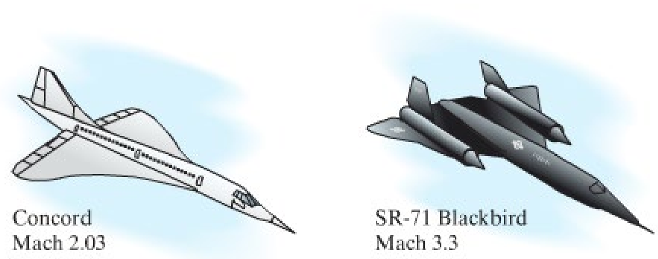
$$d = \frac{v_0^2}{16} \sin\theta \cos\theta$$

In which v_0 is the initial speed of the object thrown, in *feet* per *second*, and θ is the angle, in *degrees*, at which the object leaves the hand.

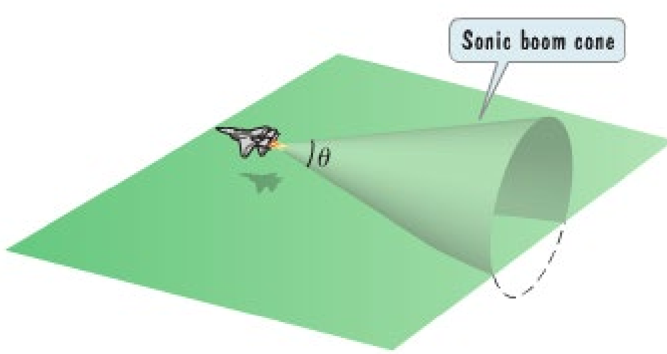


a) Use the identity to express the formula so that it contains the since function only.
b) Use the formula from part (*a*) to find the angle, θ, that produces the maximum distance, d, for a given initial speed, v_0.

126. The speed of a supersonic aircraft is usually represented by a Mach number. A Mach number is the speed of the aircraft, in *miles* per *hour*, divided by the speed of sound, approximately 740 *mph*. Thus, a plane flying at twice the speed of sound has a speed, M, of Mach 2.



If an aircraft has a speed greater than Mach 1, a sonic boom is heard, created by sound waves that form a cone with a vertex angle θ.



The relationship between the cone's vertex angle θ, and the Mach speed, M, of an aircraft that is flying faster than the speed of sound is given by

$$\sin\frac{\theta}{2}=\frac{1}{M}$$

a) If $\theta=\frac{\pi}{6}$, determine the Mach speed, M, of the aircraft. Express the speed as an exact value and as decimal to the nearest tenth.

b) If $\theta=\frac{\pi}{4}$, determine the Mach speed, M, of the aircraft. Express the speed as an exact value and as decimal to the nearest tenth.

Section R.6 – Inverse of a Function

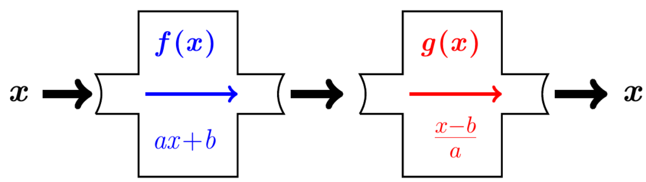
R.6–1 Definition of the Inverse of a Function

Let *f* and *g* be two functions such that

$$f(g(x)) = x \qquad \text{and} \qquad g(f(x)) = x$$

$$x \underset{g=f^{-1}}{\overset{f}{\rightleftarrows}} f(x)$$

$$g\left(f(x)\right) = f^{-1}\left(f(x)\right) = \left(f^{-1} \circ f\right)(x) = x$$



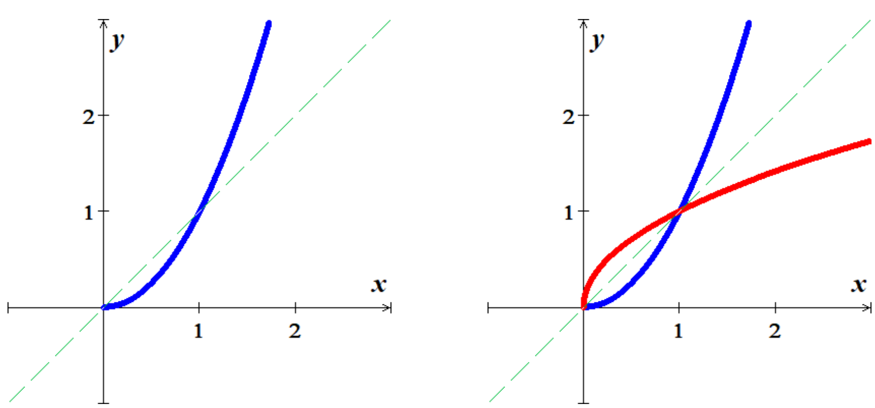
If the inverse of a function *f* is also a function, it is named f^{-1} read "*f* – inverse"

The -1 in f^{-1} is not an exponent! And is not equal to $\cancel{\dfrac{1}{f(x)}}$

***Domain* and *Range* of f and f^{-1}**

$$\text{domain of } f^{-1} = \text{range of } f$$
$$\text{range of } f^{-1} = \text{domain of } f$$

Graphing



R.6–2 *One-to-One* Functions

A function *f* is one-to-one (1 – 1) if different inputs have different outputs that is,

$$\text{if } a \neq b, \quad \text{then } f(a) \neq f(b)$$

A function *f* is one-to-one (1 – 1) if different outputs the same, the inputs are the same – that is,

$$\text{if } f(a) = f(b), \quad \text{then } a = b$$

Example

Given the function *f* described by $f(x) = 2x - 3$, prove that *f* is one-to-one.

Solution

$f(a) = f(b)$

$2a - 3 = 2b - 3$ *Add 3 on both sides*

$2a = 2b$ *Divide by 2*

$a = b$

$\therefore f$ is one-to-one

R.6–3 Finding the *Inverse* Function

	Example
Finding an Inverse Function	$f(x) = 2x + 7$
1. Replace $f(x)$ with *y*	$y = 2x + 7$
2. Interchange *x* and *y*	$x = 2y + 7$
3. Solve for *y*	$x - 7 = 2y$ $\frac{x-7}{2} = y$
4. Replace *y* with $f^{-1}(x)$	$f^{-1}(x) = \frac{x-7}{2}$

Example 1

Find the inverse of $f(x) = 4x^3 - 1$

Solution

$$y = 4x^3 - 1$$

$$x = 4y^3 - 1$$

$$x + 1 = 4y^3$$

$$\frac{x+1}{4} = y^3$$

$$y = \left(\frac{x+1}{4}\right)^{1/3}$$

$$f^{-1}(x) = \sqrt[3]{\frac{x+1}{4}}$$

R.6–4 The Inverse *Sine* Function

Definition

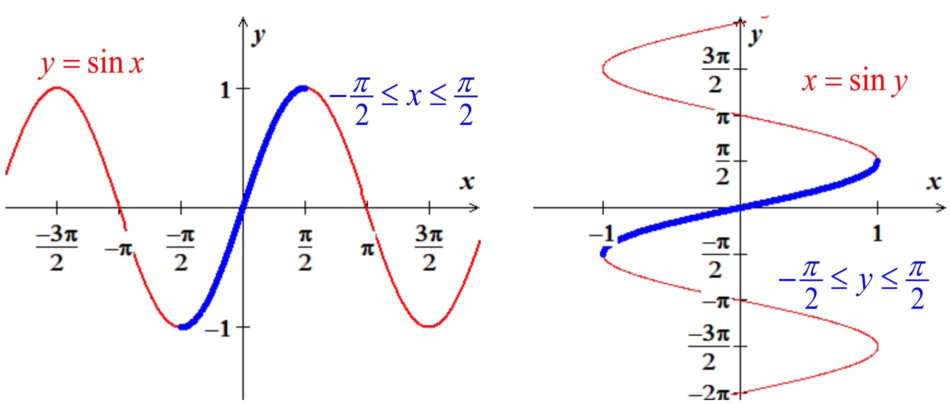
The inverse sine function, denoted by $\sin^{-1}$ *or* $\arcsin$, is defined by

$$y = \sin^{-1} x \quad or \quad y = \arcsin x \quad \textit{iff} \quad x = \sin y \quad for\ -\frac{\pi}{2} \le y \le \frac{\pi}{2} \quad and \quad -1 \le x \le 1$$

Properties of $\sin^{-1}$

$$\sin\left(\sin^{-1} x\right) = \sin(\arcsin x) = x \quad if \quad -1 \le x \le 1$$

$$\sin^{-1}(\sin y) = \arcsin(\sin y) = y \quad if \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$



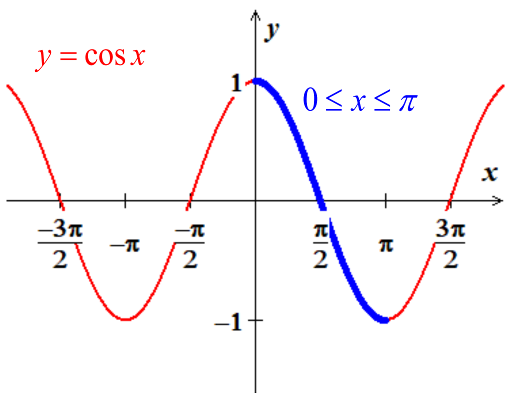
R.6–5 The Inverse *Cosine* Function

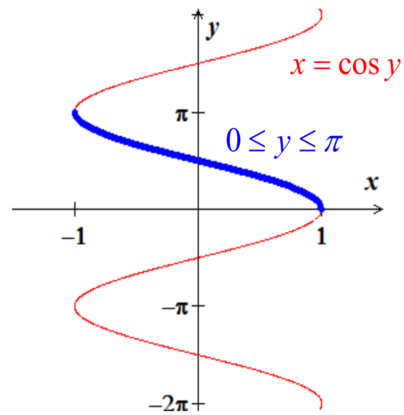
Definition

The inverse cosine function, denoted by $\cos^{-1}$ *or* arccos, is defined by

$$y = \cos^{-1} x \quad \textit{iff} \quad x = \cos y \quad \textit{for } 0 \le y \le \pi \quad \textit{and} \quad -1 \le x \le 1$$

Notation	***Meaning***
$y = \cos^{-1} x \quad \textit{or} \quad y = \arccos x$	$x = \cos y \quad \textit{and} \quad 0 \le y \le \pi$





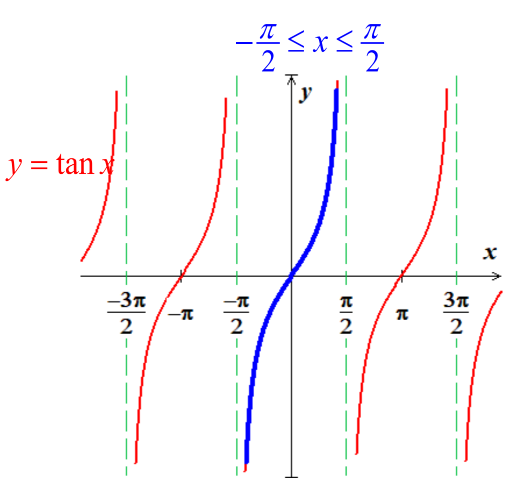
R.6–6 The Inverse *Tangent* Function

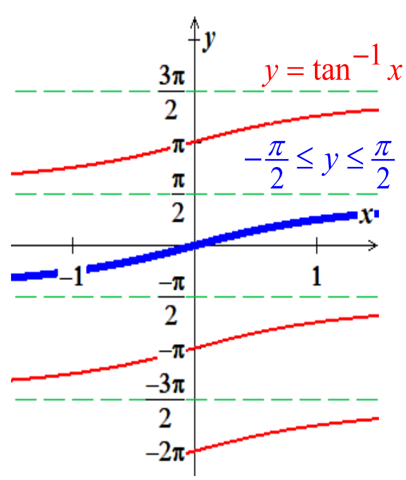
Definition

The inverse cosine function, denoted by $\tan^{-1}$, is defined by

$y = \tan^{-1} x \quad \textit{iff} \quad x = \tan y$ for any real number x and for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$y = \tan^{-1} x \quad \textit{or} \quad y = \arctan x$





Example 3

Find the exact value: $\cos\left(\cos^{-1}(-0.5)\right)$, $\cos^{-1}\left(\cos(3.14)\right)$, $\cos^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)$ $\arctan(\tan\pi)$

Solution

$\cos\left(\cos^{-1}(-0.5)\right)=-0.5$ Since $-1\le -0.5\le 1$

$\cos^{-1}\left(\cos(3.14)\right)=3.14$ Since $0\le 3.14\le \pi$

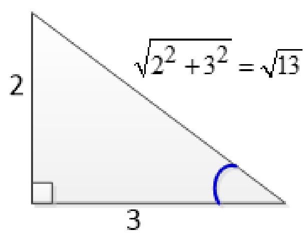
$$\cos^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)=\cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$$

$\arctan(\tan\pi)=\arctan(0)=0$ $\therefore \pi>\frac{\pi}{2}$

Example 4

Find the exact value: $\sec\left(\arctan\frac{2}{3}\right)$

Solution



$$\alpha=\arctan\frac{2}{3}\to\tan\alpha=\frac{2}{3}$$

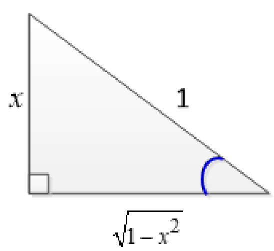
$$\sec\left(\arctan\frac{2}{3}\right)=\sec\alpha$$

$$=\frac{\sqrt{13}}{3}$$

Example 5

If $-1\le x\le 1$, rewrite $\cos\left(\sin^{-1}x\right)$ as an algebraic expression in x.

Solution



$$\alpha=\sin^{-1}x \quad\to\quad \sin\alpha=x=\frac{x}{1}$$

$$\cos\left(\sin^{-1}x\right)=\cos\alpha$$

$$=\frac{\sqrt{1-x^2}}{1}$$

$$=\sqrt{1-x^2}$$

Exercises Section *R*.6 – Inverse of a Function

$(1-12)$ For the given functions

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

d) Graph both functions (if $f^{-1}(x)$ exists)

1. $f(x)=\dfrac{x}{x-2}$

2. $f(x)=\dfrac{x+1}{x-1}$

3. $f(x)=\dfrac{2x+1}{x+3}$

4. $f(x)=\dfrac{3x-1}{x-2}$

5. $f(x)=\sqrt{x-1} \quad x\ge 1$

6. $f(x)=\sqrt{2-x} \quad x\le 2$

7. $f(x)=x^2+4x \quad x\ge -2$

8. $f(x)=3x+5$

9. $f(x)=2x^3-5$

10. $f(x)=\sqrt{3-x}$

11. $f(x)=\sqrt[3]{x}+1$

12. $f(x)=\left(x^3+1\right)^5$

$(12-24)$ Find the exact value of the expression whenever it is defined

13. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

14. $\arccos\left(\frac{\sqrt{2}}{2}\right)$

15. $\arctan\left(-\frac{\sqrt{3}}{3}\right)$

16. $\cos\left(\sin^{-1}\frac{1}{2}\right)$

17. $\arcsin\left[\sin\left(-\frac{\pi}{2}\right)\right]$

18. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

19. $\arctan\left[\tan\left(-\frac{\pi}{4}\right)\right]$

20. $\sin\left[2\arccos\left(-\frac{3}{5}\right)\right]$

21. $\cot\left(\tan^{-1}\frac{1}{2}\right)$

22. $\tan\left(\cos^{-1}\frac{3}{5}\right)$

23. $\cos\left[\arctan\left(-\frac{3}{4}\right)-\arcsin\frac{4}{5}\right]$

24. $\tan\left[\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)\right]$

$(25-35)$ Write an equivalent expression that involves x only for

25. $\cos\left(\cos^{-1}x\right)$

26. $\tan\left(\cos^{-1}x\right)$

27. $\csc\left(\sin^{-1}\frac{1}{x}\right)$

28. $\sin\left(\tan^{-1}x\right); \quad x>0$

29. $\sec\left(\sin^{-1}\dfrac{x}{\sqrt{x^2+4}}\right) \quad x>0$

30. $\cot\left(\sin^{-1}\dfrac{\sqrt{x^2-9}}{x}\right) \quad x>0$

31. $\sin\left(2\sin^{-1}x\right) \quad x>0$

32. $\tan\left(\frac{1}{2}\cos^{-1}\frac{1}{x}\right), \quad x>0$

33. $\sec\left(\tan^{-1}\dfrac{2}{\sqrt{x^2-4}}\right) \quad x>0$

34. $\sec\left(\sin^{-1}\dfrac{\sqrt{x^2-25}}{x}\right) \quad x>0$

(36 – 38) Sketch the graph of the equation:

35. $y=\sin^{-1}2x$

36. $y=\sin^{-1}(x-2)+\frac{\pi}{2}$

37. $y=\cos^{-1}\frac{1}{2}x$