Lecture Two – Techniques of Integration

Section 2.1 – Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx$$

Example: $\int x\cos x\,dx, \quad \int x^2e^x dx, \quad \text{and} \quad \int x\ln x\,dx$

Integration by Parts Formula

$$\int f(x)g'(x) = f(x)g(x) - \int f'(x)g(x)\,dx$$

Let *u* and *v* be differentiable functions of *x*. $\int udv = uv - \int vdu$

Guidelines for integration by Parts

1. Let *dv* be the most complicated portion of the integrand that fits a basic integration formula. Let *u* be the remaining factor.
2. Let *u* be the portion of the integrand whose derivative is a function simpler than *u*. Let *dv* be the remaining factor.

Example

Evaluate: $\int x\cos x\,dx$

Solution

Let: $u = x \qquad dv = \cos x\,dx$

$du = dx \qquad v = \int dv = \int \cos x\,dx = \sin x$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx \qquad \int udv = uv - \int vdu$$

$$\underline{= x\sin x + \cos x + C}$$

Example

Evaluate: $\displaystyle\int \ln x\, dx$

Solution

Let:

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \quad v = \int dx = x$

$$\int \ln x dx = x \ln x - \int x\frac{1}{x}dx \qquad\qquad \int udv = uv - \int vdu$$

$$= x\ln x - \int dx$$

$$= x\ln x - x + C$$

Tabular Integration

Example

Evaluate $\displaystyle\int x^2e^x dx$

Solution

$f(x) = x^2 \quad and \quad g(x) = e^x$

$f(x)$ **& derivatives**		$\int g(x) = \int e^x$
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x

It is called ***tabular integration***

$$\int x^2e^x dx = x^2e^x - 2xe^x + 2e^x + C$$

$u = x^2 \qquad dv = e^x dx$

$du = 2xdx \quad v = \int e^x dx = e^x$

$$\int x^2e^x dx = x^2e^x - 2\int xe^x dx$$

Let: $u = x \qquad dv = e^x dx$

$du = dx \quad v = \int e^x dx = e^x$

$$\int x^2e^x dx = x^2e^x - 2\int xe^x dx$$

$$= x^2e^x - 2\left[xe^x - \int e^x dx\right]$$

$$= x^2e^x - 2\left(xe^x - e^x\right) + C$$

$$= x^2e^x - 2xe^x + 2e^x + C$$

Example

Evaluate $\int x^3 \sin x\, dx$

Solution

$$\int x^3 \sin x dx = \boxed{-x^3 \cos x + 3x^2 \sin x + 6x\cos x - 6\sin x + C}$$

		$\int \sin x$
+	x^3	$-\cos x$
−	$3x^2$	$-\sin x$
+	$6x$	$\cos x$
−	6	$\sin x$

Example

Evaluate $\int e^x \cos x\, dx$

Solution

$$\int e^x \cos x\, dx = e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$

$$2\int e^x \cos x\, dx = e^x (\sin x + \cos x)$$

$$\int e^x \cos x\, dx = \boxed{\frac{1}{2} e^x (\sin x + \cos x) + C}$$

		$\int \cos x\, dx$
+	e^x	$\sin x$
-	e^x	$-\cos x$
+	e^x	$-\int \cos x\, dx$

Let: $u = e^x \qquad dv = \cos x dx$

$du = e^x dx \qquad v = \int \cos x dx = \sin x$

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

Let: $u = e^x \qquad dv = \sin x dx$

$du = e^x dx \qquad v = \int \sin x dx = -\cos x$

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

$$= e^x \sin x - \left[-e^x \cos x - \int (-\cos x) e^x dx \right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x dx$$

$$\int e^x \cos x dx + \int e^x \cos x dx = e^x \sin x + e^x \cos x - \int e^x \cos x dx + \int e^x \cos x dx$$

$$2\int e^x \cos x dx = e^x \sin x + e^x \cos x + C_1$$

$$\int e^x \cos x\, dx = \boxed{\frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x + C}$$

Example

Obtain a formula that expresses the integral $\int \cos^n x dx$

Solution

Let:

$$u = \cos^{n-1} x \qquad dv = \cos x dx$$

$$du = (n-1)\cos^{n-2} x(-\sin x dx) = -(n-1)\cos^{n-2} x \sin x dx \qquad v = \int \cos x dx = \sin x$$

$$\int u dv = uv - \int v du$$

$$\int \cos^n x dx = \cos^{n-1} x \sin x - \int \sin x\left(-(n-1)\cos^{n-2} x \sin x dx\right)$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\left(1-\cos^2 x\right) dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \left(\cos^{n-2} x - \cos^n x\right) dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^n x dx$$

$$\int \cos^n x dx + (n-1)\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx$$

$$(1+n-1)\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx$$

$$n\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx$$

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

Example:

$$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\int \cos x dx$$

$$= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$$

Evaluating Definite Integrals by Parts

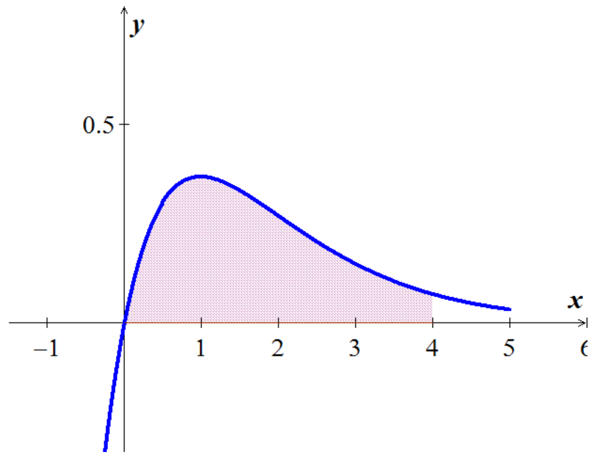
Example

Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

Solution

$$A = \int_0^4 xe^{-x}dx$$

		$\int e^{-x}$
$+$	x	$-e^{-x}$
$-$	1	e^{-x}



$$A = (-x-1)e^{-x}\Big|_0^4$$

$$= -5e^{-4} + 1$$

$$\boxed{\approx 0.91\ unit^2}$$

2nd *Method*

Let: $u = x \qquad dv = e^{-x}dx$

$du = dx \qquad v = \int e^{-x}dx = -e^{-x}$

$$\int udv = uv - \int vdu$$

$$\int_0^4 xe^{-x}dx = -xe^{-x}\Big]_0^4 - \int_0^4 \left(-e^{-x}\right)dx$$

$$= -\left[4e^{-4} - 0\right] + \int_0^4 e^{-x}dx$$

$$= -4e^{-4} + \left[-e^{-x}\right]_0^4$$

$$= -4e^{-4} - \left[e^{-4} - 1\right]$$

$$= -4e^{-4} - e^{-4} + 1$$

$$= 1 - 5e^{-4}$$

$$\boxed{\approx 0.91\ unit^2}$$

Formula

Evaluate $\int x^n e^{ax}\, dx$

		$\int e^{ax}$
+	x^n	$\frac{1}{a}e^{ax}$
−	nx^{n-1}	$\frac{1}{a^2}e^{ax}$
+	$n(n-1)x^{n-2}$	$\frac{1}{a^3}e^{ax}$
−	$n(n-1)(n-2)x^{n-3}$	$\frac{1}{a^4}e^{ax}$
	$\vdots\ \vdots$	$\vdots\ \vdots$

$$\int x^n e^{ax}\, dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a^2}x^{n-1}e^{ax} + \frac{n(n-1)}{a^3}x^{n-2}e^{ax} - \frac{n(n-1)(n-2)}{a^4}x^{n-3}e^{ax} + \dots$$

$$= e^{ax}\sum_{k=0}^{n}(-1)^k \cdot \frac{n!}{(n-k)!}\cdot\frac{1}{a^{k+1}}\cdot x^{n-k}$$

Exercises *Section* 2.1 – Integration by Parts

Evaluate the integrals

1. $\int xe^{2x}dx$
2. $\int x\ln x\,dx$
3. $\int x^3e^x dx$
4. $\int \ln x^2 dx$
5. $\int \frac{2x}{e^x}dx$
6. $\int \ln(3x)dx$
7. $\int \frac{1}{x\ln x}dx$
8. $\int \frac{x}{\sqrt{x-1}}dx$
9. $\int \frac{x^3e^{x^2}}{\left(x^2+1\right)^2}dx$
10. $\int x^2e^{-3x}dx$
11. $\int \theta\cos\pi\theta\,d\theta$
12. $\int x^2\sin x\,dx$
13. $\int x(\ln x)^2\,dx$
14. $\int \left(x^2-2x+1\right)e^{2x}dx$
15. $\int \tan^{-1}y\,dy$
16. $\int \sin^{-1}y\,dy$
17. $\int 4x\sec^2 2x\,dx$
18. $\int e^{2x}\cos 3x\,dx$
19. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$
20. $\int \frac{(\ln x)^3}{x}dx$
21. $\int x^5e^{x^3}dx$
22. $\int x^2\ln x^3\,dx$
23. $\int \ln\left(x+x^2\right)dx$
24. $\int e^{-x}\sin 4x\,dx$
25. $\int e^{-2\theta}\sin 6\theta\,d\theta$
26. $\int xe^{-4x}dx$
27. $\int x\ln(x+1)dx$
28. $\int \frac{(\ln x)^2}{x}dx$
29. $\int \frac{xe^{2x}}{(2x+1)^2}dx$
30. $\int \frac{5x}{e^{2x}}dx$
31. $\int \frac{e^{1/x}}{x^2}dx$
32. $\int x^5\ln 3x\,dx$
33. $\int x\sqrt{x-5}\,dx$
34. $\int \frac{x}{\sqrt{6x+1}}dx$
35. $\int x\cos x\,dx$
36. $\int x\csc x\cot x\,dx$
37. $\int x^3\sin x\,dx$
38. $\int x^2\cos x\,dx$
39. $\int e^{-3x}\sin 5x\,dx$
40. $\int e^{-3x}\sin 4x\,dx$
41. $\int e^{4x}\cos 2x\,dx$
42. $\int e^{3x}\cos 3x\,dx$
43. $\int x^2e^{4x}\,dx$
44. $\int x^3e^{-3x}\,dx$
45. $\int x^3\cos 2x\,dx$
46. $\int x^3\sin x\,dx$
47. $\int x^5\ln x\,dx$

48. $\displaystyle\int_0^{1/\sqrt{2}} 2x\sin^{-1}\left(x^2\right)dx$

49. $\displaystyle\int_1^{e} x^3 \ln x\ dx$

50. $\displaystyle\int_0^{1} x\sqrt{1-x}\ dx$

51. $\displaystyle\int_0^{\pi/3} x\tan^2 x\ dx$

52. $\displaystyle\int_0^{\pi} x\sin x\ dx$

53. $\displaystyle\int_1^{e} \ln 2x\ dx$

54. $\displaystyle\int_0^{\pi/2} x\cos 2x\ dx$

55. $\displaystyle\int_0^{\ln 2} xe^x\ dx$

56. $\displaystyle\int_1^{e^2} x^2 \ln x\ dx$

57. $\displaystyle\int_0^{3} xe^{x/2}dx$

58. $\displaystyle\int_0^{2} x^2e^{-2x}dx$

59. $\displaystyle\int_0^{\pi/4} x\cos 2x\ dx$

60. $\displaystyle\int_0^{\pi} x\sin 2x\ dx$

61. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate aces, the cure $y=e^x$, and the line $x=\ln 2$ about the line $x=\ln 2$

62. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the cure $y=e^{-x}$, and the line $x=1$, about

a) the line $y-axis$

b) the line $x=1$

63. Find the volume of the solid that is generated by the region bounded by $f(x)=e^{-x}$, $x=\ln 2$, and the coordinate axes is revolved about the *y-axis*.

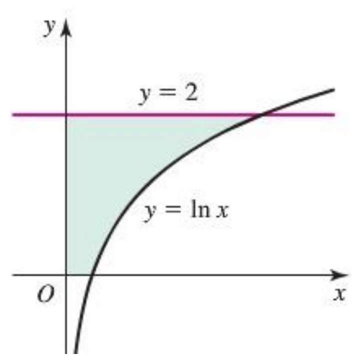
64. Find the volume of the solid that is generated by the region bounded by $f(x)=\sin x$, and the *x-axis* on $[0,\ \pi]$ is revolved about the *y-axis*.

65. Find the area of the region generated when the region bounded by $y=\sin x$ and $y=\sin^{-1} x$ on the interval $\left[0,\ \frac{1}{2}\right]$.

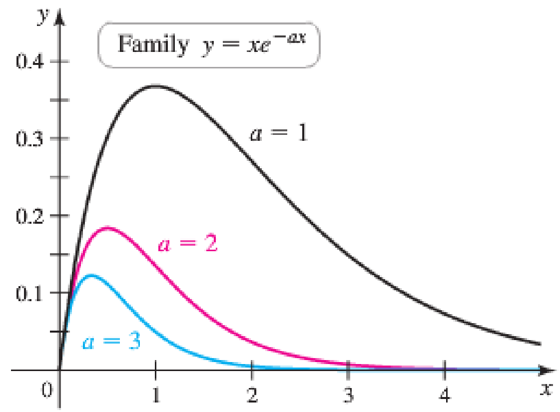
66. Find the area between the curves $y=\ln x^2,\quad y=\ln x,\quad and\ \ x=e^2$

67. Determine the area of the shaded region bounded by

$y=\ln x,\quad y=2,\quad y=0,\quad and\ \ x=0$



68. The curves $y = xe^{-ax}$ are shown in the figure for $a = 1,\ 2,\ and\ 3$.



a) Find the area of the region bounded by $y = xe^{-x}$ and the x-axis on the interval [0, 4].

b) Find the area of the region bounded by $y = xe^{-ax}$ and the x-axis on the interval [0, 4] where $a > 0$

c) Find the area of the region bounded by $y = xe^{-ax}$ and the x-axis on the interval [0, b]. Because this area depends on a and b, we call it $A(a,\ b)$ where $a > 0$ and $b > 0$.

d) Use part (c) to show that $A(1,\ \ln b) = 4A\left(2,\ \frac{1}{2}\ln b\right)$

e) Does this pattern continue? Is it true that $A(1,\ \ln b) = a^2 A\left(a,\ \frac{1}{a}\ln b\right)$

69. Suppose a mass on a spring that is slowed by friction has the position function $s(t) = e^{-t}\sin t$

a) Graph the position function. At what times does the oscillator pass through the position $s = 0$?

b) Find the average value of the position on the interval $[0,\ \pi]$.

c) Generalize part (b) and find the average value of the position on the interval $[n\pi,\ (n+1)\pi]$, for $n = 0,\ 1,\ 2,\ \ldots$

70. Given the region bounded by the graphs of $y = x\sin x, \quad y = 0, \quad x = 0, \quad x = \pi$, find

a) The area of the region.

b) The volume of the solid generated by revolving the region about the x-*axis*

c) The volume of the solid generated by revolving the region about the y-*axis*

d) The centroid of the region