Lecture Three – Laplace and Linear Systems

Section 3.1 – Definition of the Laplace Transform

Definition

Suppose $f(t)$ is a function of t defined for $0 < t < \infty$. The ***Laplace transform*** of f is the function

$$\mathcal{L}(f)(s) = F(s) = \int_0^\infty f(t)e^{-st}dt$$

The integral of the Laplace transform is an improper integral because the upper limit is ∞.

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \lim_{T\to\infty}\int_0^T f(t)e^{-st}dt$$

Example

Assume $f(t) = e^{at}$

Solution

$$F(s) = \int_0^\infty e^{at}e^{-st}dt$$

$$= \int_0^\infty e^{-(s-a)t}dt$$

$$F(s) = \lim_{T\to\infty}\int_0^T e^{-(s-a)t}dt$$

$$= \lim_{T\to\infty} \left.\frac{-e^{-(s-a)t}}{s-a}\right|_0^T \qquad e^{-(s-a)0} = 1$$

$$= \lim_{T\to\infty}\left(\frac{-e^{-(s-a)T}}{s-a} + \frac{1}{s-a}\right) \qquad e^{-(s-a)\infty} = \frac{1}{e^\infty} = 0$$

$$dv = \int e^{-st}dt \qquad = \frac{1}{s-a}$$

$$\mathcal{L}\left(e^{at}\right)(s) = F(s) = \frac{1}{s-a} \qquad for \;\; s > a$$

Example

Assume $f(t) = t$

Solution

$$F(s) = \int_0^\infty te^{-st}dt$$

$$u = t$$

$$du = dt \qquad v = -\frac{1}{s}e^{-st}$$

$$\int te^{-st}dt = -\frac{1}{s}te^{-st} - \int\left(-\frac{1}{s}\right)e^{-st}dt$$

$$= -\frac{1}{s}te^{-st} + \frac{1}{s}\int e^{-st}dt$$

$$= -\frac{1}{s}te^{-st} + \frac{1}{s}\left(-\frac{1}{s}\right)e^{-st}$$

$$= -\frac{1}{s}te^{-st} - \frac{1}{s^2}e^{-st}$$

$$F(s) = \lim_{T\to\infty}\left(-\frac{1}{s}te^{-st} - \frac{1}{s^2}e^{-st}\right)_{t=0}^{T}$$

$$= \lim_{T\to\infty}\left(-\frac{1}{s}Te^{-sT} - \frac{1}{s^2}e^{-sT} + \frac{1}{s^2}\right) \qquad \lim_{T\to\infty}\left(e^{-sT}\right) = 0$$

$$= \frac{1}{s^2}$$

Laplace transform to any power t^n

$$\mathcal{L}\left(t^n\right)(s) = \frac{n!}{s^{n+1}}$$

Example

Assume $f(t) = \sin at$

Solution

$$F(s) = \int_0^{\infty} e^{-st} \sin at \; dt$$

$$u = e^{-st} \qquad dv = \int \sin at \; dt$$

$$du = -se^{-st} dt \qquad v = -\frac{1}{a}\cos at$$

$$\int e^{-st} \sin at dt = -\frac{1}{a} e^{-st} \cos at - \int \left(-\frac{1}{a}\cos at\right)\left(-se^{-st}\right) dt$$

$$= -\frac{1}{a} e^{-st} \cos at - \frac{s}{a} \int \left(e^{-st} \cos at\right) dt$$

$$\int \left(e^{-st} \cos at\right) dt \qquad u = e^{-st} \qquad dv = \int \cos at \; dt$$

$$du = -se^{-st} dt \qquad v = \frac{1}{a}\sin at$$

$$\int e^{-st} \sin at \; dt = -\frac{1}{a} e^{-st} \cos at - \frac{s}{a}\left[\frac{1}{a} e^{-st} \sin at - \frac{1}{a}\int \left(-se^{-st}\right)(\sin at) dt\right]$$

$$\int e^{-st} \sin at \; dt = -\frac{1}{a} e^{-st} \cos at - \frac{s}{a^2} e^{-st} \sin at - \frac{s^2}{a^2} \int e^{-st} \sin at \; dt$$

$$\int e^{-st} \sin at \; dt + \frac{s^2}{a^2} \int e^{-st} \sin at \; dt = -\frac{1}{a} e^{-st} \cos at - \frac{s}{a^2} e^{-st} \sin at$$

$$\frac{a^2+s^2}{a^2} \int e^{-st} \sin at \; dt = -\frac{1}{a} e^{-st} \cos at - \frac{s}{a^2} e^{-st} \sin at$$

$$\int e^{-st} \sin at \; dt = -\frac{ae^{-st}}{a^2+s^2} \cos at - \frac{se^{-st}}{a^2+s^2} \sin at$$

$$F(s) = \lim_{T\to\infty} \int_0^T e^{-st} \sin at \; dt$$

$$= \lim_{T\to\infty} \left(\left(-\frac{ae^{-sT}}{a^2+s^2} \cos aT - \frac{se^{-sT}}{a^2+s^2} \sin aT \right) - \left(-\frac{ae^{-s(0)}}{a^2+s^2} \cos a(0) - \frac{se^{-s(0)}}{a^2+s^2} \sin a(0) \right) \right)$$

$$= \lim_{T\to\infty} \left(\left(-\frac{ae^{-sT}}{a^2+s^2} \cos aT - \frac{se^{-sT}}{a^2+s^2} \sin aT \right) - \left(-\frac{a}{a^2+s^2} \right) \right)$$

$$= \lim_{T\to\infty} \left(-\frac{ae^{-sT}}{a^2+s^2} \cos aT - \frac{se^{-sT}}{a^2+s^2} \sin aT \right) + \frac{a}{a^2+s^2} \qquad \lim_{T\to\infty} \left(e^{-sT}\right) = 0$$

$$= \boxed{\frac{a}{a^2+s^2}}$$

Exercises ***Section*** **3.1 - The Definition of the Laplace Transform**

Use Definition of Laplace transform to find the Laplace transform of:

1. $f(t) = 3$

2. $f(t) = e^{-2t}$

3. $f(t) = te^{-3t}$

4. $f(t) = e^{2t}\cos 3t$

5. $f(t) = \sin 3t$

Use Definition of Laplace transform to show the Laplace transform of

6. $f(t) = \cos \omega t$ is $F(s) = \dfrac{s}{s^2 + \omega^2}$

Section 3.2 – Basic Properties of the Laplace Transform

The Laplace Transform of Derivatives

Proposition

Suppose y is a piecewise differentiable function of exponential order. Suppose also that y' is of the exponential order.

$$\begin{aligned}\mathcal{L}(y')(s) &= s.\,\mathcal{L}(y)(s) - y(0)\\ &= sY(s) - y(0)\end{aligned}$$

Proof

$$\begin{aligned}\mathcal{L}(y')(s) &= \int_0^\infty y'(t)e^{-st}dt\\ &= \lim_{T\to\infty}\int_0^T y'(t)e^{-st}dt\\ &= \lim_{T\to\infty}\left[e^{-st}y(t)\Big|_{t=0}^{T} + s\int_0^T y(t)e^{-st}dt\right]\\ &= \lim_{T\to\infty} e^{-st}y(t) - y(0) + s.\,\mathcal{L}(y)(s)\end{aligned}$$

Let: $|y(t)| \le Ce^{at}$

$e^{-sT}|y(T)| \le Ce^{aT}e^{-sT}$

$e^{-sT}|y(T)| \le Ce^{-(s-a)T}$; which converges to 0 for $s > a$ as $T \to \infty$. Therefore,

$$\mathcal{L}(y')(s) = s.\,\mathcal{L}(y)(s) - y(0)$$

Proposition

$$\begin{aligned}\mathcal{L}(y'')(s) &= s^2.\,\mathcal{L}(y)(s) - sy(0) - y'(0)\\ &= s^2.Y(s) - sy(0) - y'(0)\end{aligned}$$

Proposition

$$\begin{aligned}\mathcal{L}\left(y^{(k)}\right)(s) &= s^k.\,\mathcal{L}(y)(s) - s^{k-1}y(0) - \cdots - sy^{(k-2)}(0) - y^{(k-1)}(0)\\ &= s^k.Y(s) - s^{k-1}y(0) - \cdots - sy^{(k-2)}(0) - y^{(k-1)}(0)\end{aligned}$$

Laplace Transform Linear

$$\mathcal{L}[\alpha f(t)+\beta.g(t)](s)=\alpha.\mathcal{L}[f(t)](s)+\beta.\mathcal{L}[g(t)](s)$$

Example

Find the Laplace transform of $f(t)=3\sin 2t-4t+5e^{3t}$

Solution

$$\mathcal{L}\left[3\sin 2t-4t+5e^{3t}\right](s)=3\mathcal{L}[\sin 2t](s)-4\mathcal{L}[-4t](s)+5\mathcal{L}\left[e^{3t}\right](s)$$

$$=3\left(\frac{2}{4+s^2}\right)-4\left(\frac{1}{s^2}\right)+5\left(\frac{1}{s-3}\right)$$

Example

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

$$y''-y=e^{2t} \quad with \quad y(0)=0 \;\; and \;\; y'(0)=1$$

Solution

For the right-hand side

$$\mathcal{L}\left(e^{2t}\right)(s)=\frac{1}{s-2}$$

$$\begin{aligned}\mathcal{L}\{y''-y\}(s)&=\mathcal{L}\{y''\}(s)-\mathcal{L}\{y\}(s)\\&=s^2.\,\mathcal{L}(y)(s)-sy(0)-y'(0)-\mathcal{L}(y)(s)\\&=s^2.Y(s)-sy(0)-y'(0)-Y(s) && y(0)=0 \;\; and \;\; y'(0)=1\\&=s^2.Y(s)-1-Y(s)\end{aligned}$$

$$s^2.Y(s)-Y(s)-1=\frac{1}{s-2}$$

$$Y(s)\left(s^2-1\right)=\frac{1}{s-2}+1$$

$$\begin{aligned}Y(s)&=\frac{1}{s^2-1}\left[\frac{1}{s-2}+1\right]\\&=\frac{1}{(s-1)(s+1)}\left[\frac{s-1}{s-2}\right]\\&=\frac{1}{(s-2)(s+1)}\end{aligned}$$

Laplace Transform of the Product of an Exponential with a Function

The result is a translation in the Laplace transform

$$\mathcal{L}\left(e^{ct}f(t)\right)(s) = F(s-c)$$

Example

Compute the Laplace transform of the function $g(t) = e^{2t}\sin 3t$

Solution

Let $f(t) = \sin 3t \rightarrow F(s) = \dfrac{3}{s^2+9}$

With $c = 2$

$$\mathcal{L}\left(e^{2t}f(t)\right)(s) = F(s-c)$$

$$= \frac{3}{(s-2)^2+9}$$

$$= \frac{3}{s^2-4s+13}$$

Proposition: Derivative of a Laplace Transform

$$\mathcal{L}(s) = -F'(s)$$

$$\mathcal{L}\left[t^n.f(t)\right](s) = (-1)^n F^{(n)}(s)$$

Example

Compute the Laplace transform of t^2e^{3t}

Solution

$f(t) = e^{3t} \Rightarrow F(s) = \dfrac{1}{s-3}$

$$F'(s) = \frac{-1}{(s-3)^2}$$

$$F''(s) = \frac{2}{(s-3)^3}$$

$$\mathcal{L}\left[t^2e^{3t}\right](s) = (-1)^2 F''(s)$$

$$= \frac{2}{(s-3)^3}$$

Exercises *Section* 3.2 - Basic Properties of the Laplace Transform

Find the Laplace transform and defined the time domain of

1. $y(t) = t^2 + 4t + 5$

2. $y(t) = -2\cos t + 4\sin 3t$

3. $y(t) = 2\sin 3t + 3\cos 5t$

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

4. $y' - 5y = e^{-2t}, \quad with \quad y(0) = 1$

5. $y' - 4y = \cos 2t, \quad with \quad y(0) = -2$

6. $y'' + 2y' + 2y = \cos 2t; \quad with \quad y(0) = 1 \;\; and \quad y'(0) = 0$

7. $y'' + 3y' + 5y = t + e^{-t}; \quad with \quad y(0) = -1 \;\; and \quad y'(0) = 0$

Find the Laplace transform of

8. $y(t) = e^{2t}\cos 2t$

9. $y(t) = t\sin 3t$

10. $y(t) = t^2\cos 2t$

11. $y(t) = e^{-2t}(2t+3)$

12. $y(t) = t^2 e^{2t}$

13. $y(t) = e^{-t}\left(t^2 + 3t + 4\right)$

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

14. $y' + 2y = t\sin t, \quad with \quad y(0) = 1$

15. $y' + 2y = t^2 e^{-2t}, \quad with \quad y(0) = 0$

16. $y'' + y' + 2y = e^{-t}\cos 2t, \quad with \quad y(0) = 1 \;\; and \quad y'(0) = -1$

Section 3.3 – Inverse Laplace Transform

Definition

If f is a continuous function of exponential order and $\mathcal{L}(f)(s) = F(s)$, then we call f the inverse Laplace transform of F,

$$f(t) = \mathcal{L}^{-1}(F(s))$$

$$F(s) = \mathcal{L}(f(t)) \quad \Leftrightarrow \quad f(t) = \mathcal{L}^{-1}(F(s))$$

$$f(t) \quad \underset{\textit{Inverse Laplace tansform - }\mathcal{L}^{-1}}{\overset{\textit{Laplace tansform - }\mathcal{L}}{\rightleftarrows}} \quad F(S)$$

Note: Inverse transforms are not unique. If f_1 and f_2 are identical except at a discrete set of points, then $\mathcal{L}(f_1(t)) = \mathcal{L}(f_2(t))$. However, there is at most one continuous function f satisfying $\mathcal{L}\{f(t)\} = F(s)$

Laplace Transform Linear

Proposition

$$\mathcal{L}^{-1}[aF(s) + bG(s)] = a.\mathcal{L}^{-1}(F(s)) + b.\mathcal{L}^{-1}(G(s))$$
$$= af(t) + bg(t)$$

Example

Compute the inverse Laplace transform of $F(s) = \dfrac{1}{s-2} - \dfrac{16}{s^2+4}$

Solution

$$\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \sin 2t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-2} - 8\frac{2}{s^2+4}\right\} \boxed{= e^{2t} - 8\sin 2t}$$

Example

Compute the inverse Laplace transform of $F(s) = \dfrac{1}{s^2 - 2s - 3}$; $\quad s > 3$

Solution

$$\frac{1}{s^2 - 2s - 3} = \frac{A}{s-3} + \frac{B}{s+1}$$

$$= \frac{As + A + Bs - 3B}{(s-3)(s+1)}$$

$$= \frac{(A+B)s + A - 3B}{(s-3)(s+1)}$$

$$\begin{cases} A + B = 0 \\ A - 3B = 1 \end{cases} \rightarrow A = \tfrac{1}{4} \quad B = -\tfrac{1}{4}$$

$$= \frac{A(s+1) + B(s-3)}{(s-3)(s+1)}$$

$$\begin{cases} s = 3 & \Rightarrow 1 = 4A & A = \tfrac{1}{4} \\ s = 1 & \Rightarrow 1 = -4A & B = -\tfrac{1}{4} \end{cases}$$

$$\frac{1}{s^2 - 2s - 3} = \frac{1}{4}\left(\frac{1}{s-3} - \frac{1}{s+1}\right)$$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-3} - \frac{1}{s+1}\right\}$$

$$= \frac{1}{4}\left(e^{3t} - e^{-t}\right)$$

Example

Compute the inverse Laplace transform of $F(s) = \dfrac{1}{s^2 + 4s + 13}$

Solution

$$s^2 + 4s + 13 = s^2 + 4s + 4 + 9$$

$$= (s+2)^2 + 3^2$$

$$\mathcal{L}^{-1}\left\{\frac{1}{3}\frac{3}{(s+2)^2 + 3^2}\right\} \;\underline{= \frac{1}{3}e^{-2t}\sin 3t|}$$

Example

Find the inverse Laplace transform of $F(s)=\dfrac{2s^2+s+13}{(s-1)\left((s+1)^2+4\right)}$

Solution

$$\frac{2s^2+s+13}{(s-1)\left((s+1)^2+4\right)}=\frac{A}{(s-1)}+\frac{Bs+C}{(s+1)^2+4}$$

$$=\frac{As^2+2As+5A+Bs^2+(C-B)s-C}{(s-1)\left(s^2+2s+5\right)}$$

$$=\frac{(A+B)s^2+(2A+C-B)s+5A-C}{(s-1)\left(s^2+2s+5\right)}$$

$$\begin{cases} A+B=2 & \rightarrow B=2-A \\ 2A-B+C=1 & \qquad 2A-2+A+5A-13=1 \Rightarrow A=2 \\ 5A-C=13 & \rightarrow C=5A-13 \end{cases}$$

$$\begin{cases} B=2-2=0 \\ C=5(2)-13=-3 \end{cases}$$

$$F(s)=\frac{2}{(s-1)}-\frac{3}{(s+1)^2+4}$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{2}{(s-1)}-\frac{3}{(s+1)^2+4}\right\}$$

$$=2\mathcal{L}^{-1}\left\{\frac{1}{(s-1)}\right\}-3\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\}$$

$$\boxed{=2e^{-t}-\frac{3}{2}e^{-t}\sin 2t}$$

Exercises *Section* 3.3 - Inverse Laplace Transform

Find the inverse Laplace transform of

1. $Y(s)=\dfrac{1}{3s+2}$

2. $Y(s)=\dfrac{2}{3-5s}$

3. $Y(s)=\dfrac{1}{s^2+4}$

4. $Y(s)=\dfrac{3}{s^2}$

5. $Y(s)=\dfrac{3s+2}{s^2+25}$

6. $Y(s)=\dfrac{2-5s}{s^2+9}$

7. $Y(s)=\dfrac{5}{(s+2)^3}$

8. $Y(s)=\dfrac{1}{(s-1)^6}$

9. $Y(s)=\dfrac{4(s-1)}{(s-1)^2+4}$

10. $Y(s)=\dfrac{2s-3}{(s-1)^2+5}$

11. $Y(s)=\dfrac{2s-1}{(s+1)(s-2)}$

12. $Y(s)=\dfrac{2s-2}{(s-4)(s+2)}$

13. $Y(s)=\dfrac{7s^2+3s+16}{(s+1)\left(s^2+4\right)}$

14. $Y(s)=\dfrac{1}{(s+2)^2\left(s^2+9\right)}$

15. $Y(s)=\dfrac{s}{(s+2)^2\left(s^2+9\right)}$

16. $Y(s)=\dfrac{1}{(s+1)^2\left(s^2-4\right)}$

17. $Y(s)=\dfrac{7s^2+20s+53}{(s-1)\left(s^2+2s+5\right)}$

18. $F(s)=\dfrac{s^2+1}{s^3-2s^2-8s}$

Section 3.4 – Using Laplace Transform to Solve Differential Equations

Example

Use Laplace transform to find the solution to the initial value problem

$$y'' - y = e^{2t} \qquad y(0) = 0 \quad \textit{and} \quad y'(0) = 1$$

Solution

$$\mathcal{L}\{y'' - y\} = \mathcal{L}\left\{e^{2t}\right\}$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = \mathcal{L}\left\{e^{2t}\right\}$$

$$\left(s^2Y(s) - sy(0) - y'(0)\right) - Y(s) = \frac{1}{s-2}$$

$$\left(s^2 - 1\right)Y(s) - 1 = \frac{1}{s-2}$$

$$\left(s^2 - 1\right)Y(s) = \frac{1}{s-2} + 1$$

$$(s-1)(s+1)Y(s) = \frac{s-1}{s-2}$$

$$\begin{aligned} Y(s) &= \frac{1}{(s+1)(s-2)} \\ &= \frac{A}{(s+1)} + \frac{B}{(s-2)} \\ &= \frac{(A+B)s + B - 2A}{(s+1)(s-2)} \end{aligned} \qquad \begin{cases} A + B = 0 \\ -2A + B = 1 \end{cases} \Rightarrow A = -\frac{1}{3};\ B = \frac{1}{3}$$

$$Y(s) = \frac{1}{3}\left[-\frac{1}{(s+1)} + \frac{1}{(s-2)}\right]$$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] \\ &= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s-2)} - \frac{1}{(s+1)}\right\} \\ &= \boxed{\frac{1}{3}\left(e^{2t} - e^{-t}\right)} \end{aligned}$$

Homogeneous Equations

Example

Use Laplace transform to find the solution to the initial value problem

$$y'' - 2y' - 3y = 0 \qquad y(0) = 1 \quad \text{and} \quad y'(0) = 0$$

Solution

$$\mathcal{L}\left(y'' - 2y' - 3y\right) = s^2Y(s) - sy(0) - y'(0) - 2\left(sY(s) - y(0)\right) - 3Y(s)$$

$$= \left(s^2 - 2s - 3\right)Y(s) - s + 2$$

$$= 0$$

$$Y(s) = \frac{s-2}{s^2 - 2s - 3}$$

$$= \frac{A}{s-3} + \frac{B}{s+1}$$

$$= \frac{(A+B)s + A - 3B}{(s-3)(s+1)} \qquad \begin{cases} A + B = 1 \\ A - 3B = -2 \end{cases} \Rightarrow A = \frac{1}{4}, \;\; B = \frac{3}{4}$$

$$= \frac{1}{4}\frac{1}{s-3} + \frac{3}{4}\frac{1}{s+1}$$

$$y(t) = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \frac{1}{4}e^{3t} + \frac{3}{4}e^{-t}$$

Inhomogeneous Equations

Example

Use Laplace transform to find the solution to the initial value problem

$$y'' + 2y' + 2y = \cos 2t \qquad y(0) = 0 \quad \textit{and} \quad y'(0) = 1$$

Solution

$$\mathcal{L}\left(y'' + 2y' + 2y\right) = s^2 Y(s) - sy(0) - y'(0) + 2\left(sY(s) - y(0)\right) + 2Y(s)$$

$$= \left(s^2 + 2s + 2\right)Y(s) - 1$$

$$\mathcal{L}(\cos 2t) = \frac{s}{s^2 + 4}$$

$$\left(s^2 + 2s + 2\right)Y(s) - 1 = \frac{s}{s^2 + 4}$$

$$\left(s^2 + 2s + 2\right)Y(s) = \frac{s}{s^2 + 4} + 1$$

$$= \frac{s^2 + s + 4}{s^2 + 4}$$

$$Y(s) = \frac{s^2 + s + 4}{\left(s^2 + 4\right)\left(s^2 + 2s + 2\right)}$$

$$= \frac{As + B}{\left(s^2 + 2s + 2\right)} + \frac{Cs + D}{\left(s^2 + 4\right)}$$

$$= \frac{(A + C)s^3 + (2C + B + D)s^2 + (4A + 2C + 2D)s + 4B + 2D}{\left(s^2 + 2s + 2\right)\left(s^2 + 4\right)}$$

$$\begin{cases} A + C = 0 \\ B + 2C + D = 1 \\ 4A + 2C + 2D = 1 \\ 4B + 2D = 4 \end{cases} \rightarrow \begin{matrix} A = \frac{1}{10} & B = \frac{4}{5} = \frac{8}{10} \\ C = -\frac{1}{10} & D = \frac{2}{5} = \frac{4}{10} \end{matrix}$$

$$= \frac{1}{10}\frac{s + 8}{\left((s+1)^2 + 1\right)} - \frac{1}{10}\frac{s - 4}{\left(s^2 + 4\right)}$$

$$= \frac{1}{10}\frac{s + 1 + 7}{\left((s+1)^2 + 1\right)} - \frac{1}{10}\frac{s - 4}{\left(s^2 + 4\right)}$$

$$= \frac{1}{10}\frac{s + 1}{\left((s+1)^2 + 1\right)} + \frac{7}{10}\frac{1}{\left((s+1)^2 + 1\right)} - \frac{1}{10}\frac{s}{\left(s^2 + 4\right)} + \frac{1}{10}\frac{4}{\left(s^2 + 4\right)}$$

$$y(t) = \frac{1}{10}e^{-t}\cos t + \frac{7}{10}e^{-t}\sin t - \frac{1}{10}\cos 2t + \frac{1}{10}2\sin 2t$$

$$\boxed{y(t) = \frac{1}{10}\left(e^{-t}\left(\cos t + 7\sin t\right) + 2\sin 2t - \cos 2t\right)}$$

Higher-Order Equations

Example

Find the solution to the initial value problem

$$y^{(4)} - y = 0 \qquad y(0) = 0,\ y'(0) = 1,\ y''(0) = 0,\ and\ y'''(0) = 0$$

Solution

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - Y(s) = 0$$

$$\left(s^4 - 1\right)Y(s) - s^2 = 0$$

$$\left(s^4 - 1\right)Y(s) = s^2$$

$$Y(s) = \frac{s^2}{s^4 - 1}$$

$$= \frac{s^2}{(s-1)(s+1)\left(s^2+1\right)}$$

$$= \frac{A}{(s-1)} + \frac{B}{(s+1)} + \frac{Cs+D}{\left(s^2+1\right)}$$

$$= \frac{As^3+As^2+As+A+Bs^3-Bs^2+Bs-B+Cs^3-Cs+Ds^2-D}{(s-1)(s+1)\left(s^2+1\right)}$$

$$= \frac{(A+B+C)s^3+(A-B+D)s^2+(A+B-C)s+A-B-D}{(s-1)(s+1)\left(s^2+1\right)}$$

$$\begin{cases} A+B+C=0 \\ A-B+D=1 \\ A+B-C=0 \\ A-B-D=0 \end{cases} \Rightarrow \begin{cases} 2A+2B=1 \\ 2A-2B=0 \end{cases} \Rightarrow \begin{cases} \boxed{A=\frac{1}{4}} \\ \boxed{B=-\frac{1}{4}} \end{cases} \to \begin{matrix} \boxed{C=0} \\ \boxed{D=\frac{1}{2}} \end{matrix}$$

$$Y(s) = \frac{1}{4}\frac{1}{s-1} - \frac{1}{4}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s^2+1}$$

$$\boxed{y(t) = \tfrac{1}{4}e^{t} - \tfrac{1}{4}e^{-t} + \tfrac{1}{2}\sin t}$$

Exercises *Section* 3.4 - Using Laplace Transform to Solve Differential Equations

Solve using the Laplace transform:

1. $y' + 3y = e^{2t}, \quad y(0) = -1$

2. $y' + 9y = e^{-t}, \quad y(0) = 0$

3. $y' + 4y = \cos t, \quad y(0) = 0$

4. $y' + 16y = \sin 3t, \quad y(0) = 1$

5. $y' + y = te^{t}, \quad y(0) = -2$

6. $y' - 4y = t^2 e^{-2t}, \quad y(0) = 1$

7. $y'' - 4y = e^{-t}, \quad y(0) = -1 \quad y'(0) = 0$

8. $y'' + 9y = 2\sin 2t, \quad y(0) = 0 \quad y'(0) = -1$

9. $y'' - y = 2t, \quad y(0) = 0 \quad y'(0) = -1$

10. $y'' + 3y' = -3t, \quad y(0) = -1 \quad y'(0) = 1$

11. $y'' - y' - 2y = e^{2t}, \quad y(0) = -1 \quad y'(0) = 0$

12. $y'' + y = -2\cos 2t, \quad y(0) = 1 \quad y'(0) = -1$

13. $y'' + 9y = 3\sin 2t, \quad y(0) = 0 \quad y'(0) = -1$

14. $x'' - x' - 6x = 0; \quad x(0) = 2, \quad x'(0) = -1$

15. $x'' + 4x' + 4x = t^2; \quad x(0) = x'(0) = 0$

16. Solve the initial value problem $y^{(4)} + 2y'' + y = 4te^{t}; \quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0$

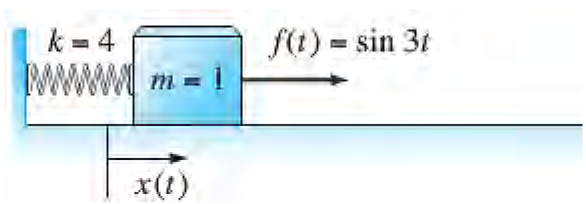
17. Given: $y'' - 4y' + 3y = 0, \quad y(0) = 1 \quad y'(0) = -1$

a) Show that the general solution is: $y(t) = C_1 e^{3t} + C_2 e^{t}$ and find C_1 *and* C_2

b) Use Laplace transform to solve the system

18. Solve the initial value problem $x'' + 4x = \sin 3t; \quad x(0) = x'(0) = 0.$

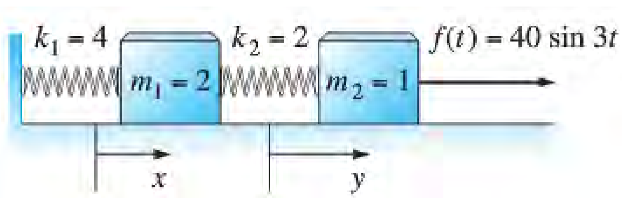
Such problem arises in the motion of a mass-and-spring system with external force as shown below.



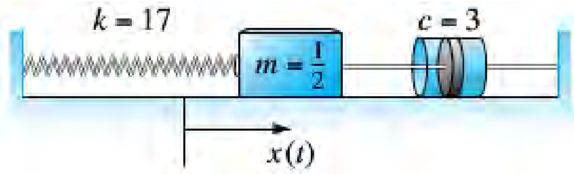
19. Solve the system $\begin{cases} 2x'' = -6x + 2y \\ y'' = 2x - 2y + 40\sin 3t \end{cases}$

Subject to the initial conditions $x(0) = x'(0) = y(0) = y'(0) = 0$

Thus the force $f(t) = 40\sin 3t$ is applied to the second mass as shown below, beginning at time $t = 0$ when the system is at rest in its equilibrium position.



20. Consider a mass-spring system with $m = \frac{1}{2}, \quad k = 17, \quad and\ c = 3$.



Let $x(t)$ be the displacement of the mass *m* from its equilibrium position. If the mass is set in motion with $x(0) = 3\ and\ x'(0) = 1$, find $x(t)$ for the resulting damped free oscillations.

21. Consider a mass-spring-dashpot system with $m = \frac{1}{2}, \quad k = 17, \quad c = 3, \quad and\ f(t) = 15\sin 2t$ with initial conditions $x(0) = x'(0) = 0$. Let $x(t)$ be the displacement of the mass *m* from its equilibrium position. Find the resulting transient motion and steady periodic motion of the mass..

Section 3.5 – Electrical Circuit

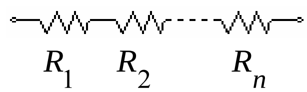
Resistor: (**O**hm's **L**aw)

A ***resistor*** is a component of a circuit that resists the flow of electrical current. It has two terminals across which electricity must pass, and it is designed to drop the voltage of the current as it flows from one terminal to the other. Resistors are primarily used to create and maintain known safe currents within electrical components.

A voltage $V(t)$ across the terminals of a resistor is proportional to the current $I(t)$ in it. The constant proportional $\boldsymbol{R}$ is called the resistance of the resistor in **V**olt/**A**mpere or **O**hms (Ω), and is given by the equation:

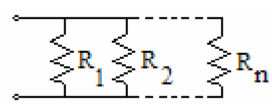
$$V_R = RI$$

For series resistors, the equivalent resistor is:



Then: $R_{eq} = R_1 + R_2 + \cdots + R_n$

For resistors in parallels:

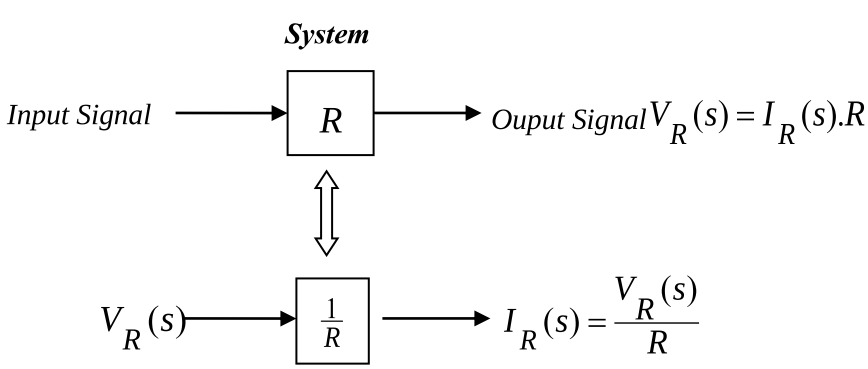


Then: $\dfrac{1}{R_{eq}} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \cdots + \dfrac{1}{R_n}$

Laplace Transform

$$\rightarrow I_R(s).R = V_R(s)$$

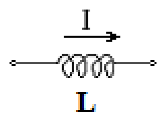
The block diagram is shown below



Inductor: (**F**araday's **L**aw)

When a current in a circuit is changing, then the magnetic flux is linking the same circuit changes. This change in flux causes an *emf* ***v*** to be induced in the circuit.

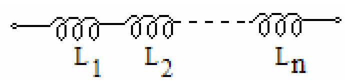
Inductance is symbolized by letter $\boldsymbol{L}$, is measured in ***h*****enrys** (***H***), and is represented graphically as a coiled wire – a reminder that inductance is a consequence of a conductor linking a magnetic field.



The voltage $V(t)$ is proportional to the time rate of change of the current, and is given by:

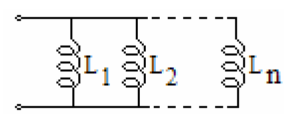
$$V_L = L\frac{dI}{dt} \qquad \text{and} \qquad I(t) = \frac{1}{L}\int Vdt$$

For series inductors, the equivalent inductor is:



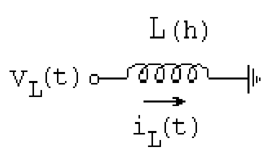
$$L_{eq} = L_1 + L_2 + \cdots + L_n$$

For inductors in parallels:



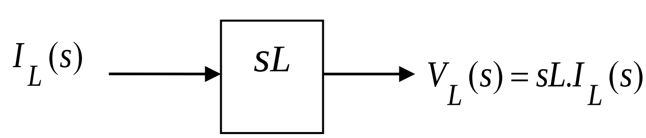
$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n}$$

Laplace Transform



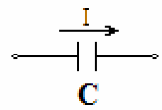
$$v_L(t) = L\frac{di_L(t)}{dt} \overset{£}{\leftrightarrow} V_L(s) = L\left(sI_L(s) - I_L(0)\right)$$

The block diagram:



Capacitance: (Coulomb's Law)

The circuit parameter of ***capacitance*** is represented by letter ***C***, is measured in *farads* (***F***), and is symbolized graphically by two short parallel conductive plates.



The farad is an extremely large quantity of capacitance, practical capacitor values usually lie in the picofarad (pF) to microfarad (μF) range.

The graphic symbol for a capacitor is a reminder that capacitance occurs whenever electrical conductors are separated by a dielectric, or insulating, material. This condition implies that electric charge is not transported through the capacitor. Although applying a voltage to the terminals of the capacitor cannot move a charge through the dielectric, it can displace a charge within the dielectric. As the voltage varies with time, the displacement of charge also varies with time, causing what is known as the displacement current.

The potential ***v*** between the terminals of a capacitor is proportional to the charge q on it.

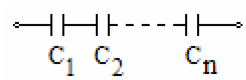
$$Q(t) = Cv(t)$$

$$I = \frac{dq}{dt} = C\frac{dv}{dt}$$

$$\Rightarrow \quad v(t) = \frac{1}{C}\int I\ dt$$

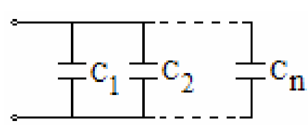
C is **C**oulombs/**V**olts or farads.

For capacitances in series, the equivalent capacitance is given by:



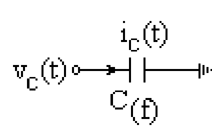
$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}$$

For capacitances in parallels:



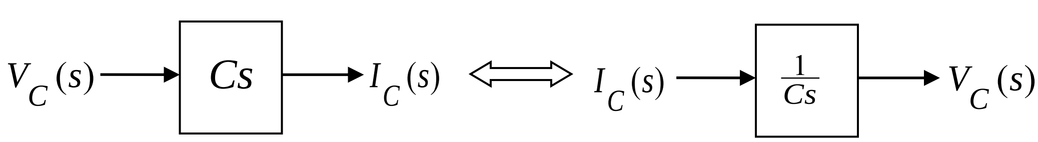
$$C_{eq} = C_1 + C_2 + \cdots + C_n$$

Laplace Transform



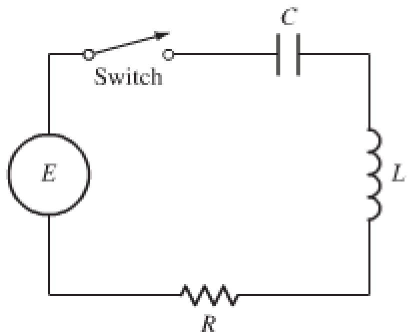
$v_C(0) = E_0(0)$

$$i_c(t) = C\frac{dV_c(t)}{dt} \overset{\pounds}{\leftrightarrow} I_c(s) = Cs.V_c(s)$$



RLC circuit

RLC circuit is a basic building block in electrical circuits and networks. A second order linear differential equations with constant coefficients is their use as a model of the flow of electric current in the simple series circuit



The current *I*, measured in amperes (***A***), is a function of time *t*.

A ***resistor*** with a resistance of *R ohms* (**Ω**)

An ***inductor*** with an inductance of *L henries* (***H***)

A ***capacitor*** with a capacitance of *C farads* (***F***)

The impressed ***voltage*** *E* in volts (***V***) is a given function of time.

Circuit Element	***Voltage Drop***
Inductor	$L\frac{dI}{dt}$
Resistor	RI
Capacitor	$\frac{1}{C}Q$

In series with a source of electromotive force (such as a battery or a generator) that supplies a voltage of $E(t)$ volts at time *t*. If the switch shown in the circuit is closed, this results in a current of $I(t)$ *amperes* in the circuit and a charge of $Q(t)$ *coulombs* on the capacitor at time *t*. The relation between the functions *Q* and the current *I* is

$$\frac{dQ}{dt} = I(t)$$

We use ***mks*** electric units, in which time is measured in seconds.

According to elementary principles of electricity, the voltage drops across the three circuit elements.

Kirchhoff's Current Law (***KCL***) (also known as Kirchhoff's ***First Law***)

> ***The algebraic sum of all the currents at any node in a circuit equals to zero.***
>
> ***Current is distributed when it reaches a junction: the amount of current entering a junction must equal the amount of current leaving that junction.***

Kirchhoff's Voltage Law (***KVL***) (also known as Kirchhoff's ***Second Law***)

> ***The algebraic sum of all the voltages around any closed path in a circuit equals to zero.***
>
> ***In a closed circuit the impressed voltage is equal to the sum of the voltage drops in the rest of the circuit.***

According to the elementary laws of electricity, we know that

The voltage drop across the resistor is *IR*.

The voltage drop across the capacitor is $\frac{Q}{C}$.

The voltage drop across the inductor is $L\frac{dI}{dt}$.

- When the switch is open, no current flows in the circuit; when the switch is closed, there is a current $I(t)$ and a charge $Q(t)$ on the capacitor.

The current and charge in the simple RLC circuit satisfy the basic electrical equation

$$L\frac{dI}{dt}+RI+\frac{1}{C}Q=E(t)$$

The units for voltage, resistance, current, charge, capacitance, inductance, and time are all related:

$$1\ volt\ =\ 1\ ohm\times 1\ ampere=\frac{1\ coulomb}{1\ farad}=\frac{1\ henry\times 1\ ampere}{1\ second}$$

$$\boxed{1\ V\ =\ 1\ \Omega\times 1\ A=\frac{1\ C}{1\ F}=\frac{1\ H\times 1\ A}{1\ sec}}$$

Since $\frac{dQ}{dt}=I(t)\ \Rightarrow\ \frac{d^2Q}{dt^2}=\frac{dI}{dt}$, we can get the second-order linear differential equation

$$LQ''+RQ'+\frac{1}{C}Q=E(t)$$

For the charge $Q(t)$, under the assumption that the voltage $E(t)$ is known.
It is the current, in most problems, rather than the charge Q that us of primary interest, so we differentiate both sides and substitute I for Q' to obtain

$$LI''+RI'+\frac{1}{C}I=E'(t)$$

With initial conditions are

$$Q(t_0)=Q_0,\quad Q'(t_0)=I(t_0)=I_0$$

$$Q''(t_0)=I'_0,\quad I'(t_0)=I'_0$$

$$LQ''+RQ'+\frac{1}{C}Q=E(t)$$

$$LI'_0+RI_0+\frac{1}{C}Q=E(t_0)$$

$$I'_0=\frac{E(t_0)-RI_0-\frac{1}{C}Q}{L}$$

Hence I'_0 is also determined by the initial charge and current, which are physically measurable quantities.

- The most important conclusion is that the flow of current in the circuit is precisely the same form as the one that describes the motion of a spring-mass system.

Summary

In RLC circuit:

$$\boxed{L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=E(t)}$$

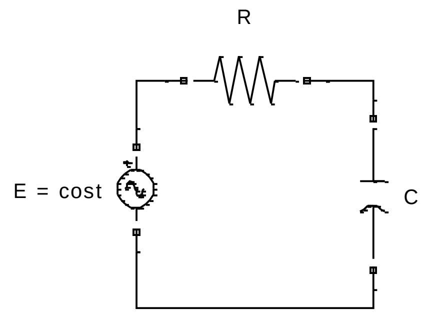
In terms of current: $I(t)=\dfrac{dQ(t)}{dt}$

$$\boxed{L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}I=\frac{d}{dt}E(t)}$$

Without capacitor $$\boxed{L\frac{dI}{dt}+RI=E(t)}$$

Where $Q(t)$ is the change on the capacitor and $E(t)$ is the applied voltage.

Example



Suppose the electrical circuit has a resistor of $R = 2\Omega$ and a capacitor of $C = \frac{1}{5}F$. Assume the voltage source is $E = \cos t\ (V)$. If the initial current is 0 *A*, find the resulting current.

Solution

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E$$

$$2Q' + 5Q = \cos t$$

$$\mathcal{L}(2Q' + 5Q) = \mathcal{L}(\cos t)$$

$$2sQ(s) - 2Q(0) + 5Q(s) = \frac{s}{s^2+1}$$

$$(2s+5)Q(s) = \frac{s}{s^2+1}$$

$$2\left(s+\frac{5}{2}\right)Q(s) = \frac{s}{s^2+1}$$

$$Q(s) = \frac{1}{2}\frac{s}{\left(s+\frac{5}{2}\right)\left(s^2+1\right)}$$

$$\frac{s}{\left(s+\frac{5}{2}\right)\left(s^2+1\right)} = \frac{A}{s+\frac{5}{2}} + \frac{Bs+C}{s^2+1}$$

$$s = As^2 + A + Bs^2 + \frac{5}{2}Bs + Cs + \frac{5}{2}C$$

$$s = (A+B)s^2 + \left(\frac{5}{2}B + C\right)s + A + \frac{5}{2}C$$

$$\begin{cases} A+B=0 \\ \frac{5}{2}B + C = 1 \\ A + \frac{5}{2}C = 0 \end{cases} \Rightarrow A = -\frac{10}{29} \quad B = \frac{10}{29} \quad C = \frac{4}{29}$$

$$Q(s) = \frac{1}{2}\left(-\frac{10}{29}\frac{1}{s+\frac{5}{2}} + \frac{10}{29}\frac{s}{s^2+1} + \frac{4}{29}\frac{1}{s^2+1}\right)$$

$$= \frac{1}{29}\left(-5\frac{1}{s+\frac{5}{2}} + 5\frac{s}{s^2+1} + 2\frac{1}{s^2+1}\right)$$

$$Q(t) = \frac{1}{29}\left(-5\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{5}{2}}\right\} + 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right)$$

$$\boxed{= \frac{1}{29}\left(-5e^{-5t/2} + 5\cos t + 2\sin t\right)}$$

OR

$$2Q' + 5Q = \cos t \;\rightarrow\; Q' + \frac{5}{2}Q = \frac{1}{2}\cos t$$

$$e^{\int \frac{5}{2}dt} = e^{\frac{5}{2}t}$$

$$\int e^{5t/2}\left(\frac{1}{2}\cos t\right)dt = \frac{1}{2}e^{5t/2}\frac{1}{\left(\frac{5}{2}\right)^2 + 1^2}\left(\frac{5}{2}\cos t - \sin t\right) + C$$

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2}$$

$$= \frac{1}{4}e^{5t/2}\frac{1}{\frac{29}{4}}\left(5\cos t + 2\sin t\right) + C$$

$$= \frac{1}{29}e^{5t/2}\left(5\cos t + 2\sin t\right) + C$$

$$Q(t) = \frac{1}{e^{5t/2}}\left[\frac{1}{29}e^{5t/2}\left(5\cos t + 2\sin t\right) + C\right]$$

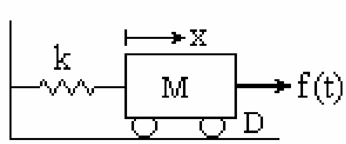
$$= \frac{1}{29}\left(5\cos t + 2\sin t\right) + Ce^{-5t/2} \qquad I(0) = 0$$

$$0 = \frac{1}{29}(5) + C \;\Rightarrow\; \boxed{C = -\frac{5}{29}}$$

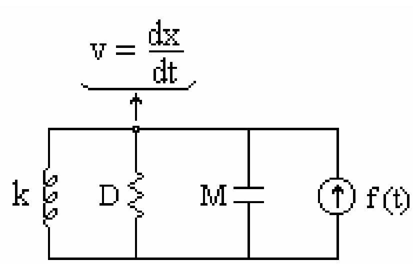
$$\boxed{Q(t) = \frac{5}{29}\cos t + \frac{2}{29}\sin t - \frac{5}{29}e^{-5t/2}}$$

Example

The electrical analog of a carriage on wheels, coupled to the wall through a spring.



***a*)** Mechanical system.



***b*)** Electrical analog.

A mechanical system with a one coordinates movement.

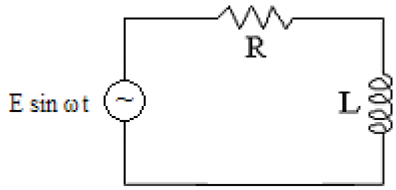
In the case of the electrical network, the equation was obtained by applying Kirchhoff's current law at the node ***v***, and is seen to be identical to the equation that would have been obtained by applying D'Alembert's principle to the mechanical system.

The differential equation for both systems is:

$$M\frac{d^2x}{dt^2} + D\frac{dx}{dt} + kx = f(t)$$

In particular, if one uses the force current analogy (or force-torque for a rational system). The topology of the electrical analog is very similar to that of the mechanical system.

***Example*:** Alternating Circuit



Alternating Circuit

Translating the circuit into differential equation:

$$L\frac{di}{dt}+Ri=E\sin\omega t$$

$$\Rightarrow\ \frac{di}{dt}+\frac{R}{L}i=\frac{E}{L}\sin\omega t$$

$$e^{\int\frac{R}{L}dt}=e^{\frac{R}{L}t}$$

$$\int e^{\frac{R}{L}t}\,\frac{E}{L}\sin\omega t\ dt=\frac{E}{L}\int e^{\frac{R}{L}t}\sin\omega t\ dt \qquad\qquad \int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}\left(a\sin bx-b\cos bx\right)$$

$$=\frac{E}{L}e^{\frac{R}{L}t}\frac{1}{\frac{R^2}{L^2}+\omega^2}\left(\frac{R}{L}\sin\omega t-\omega\cos\omega t\right)$$

$$=\frac{EL}{R^2+L^2\omega^2}e^{\frac{R}{L}t}\left(\frac{R}{L}\sin\omega t-\omega\cos\omega t\right)$$

$$i(t)=\frac{1}{e^{\frac{R}{L}t}}\left[\frac{EL}{R^2+L^2\omega^2}e^{\frac{R}{L}t}\left(\frac{R}{L}\sin\omega t-\omega\cos\omega t\right)+C\right]$$

$$=\frac{EL}{R^2+L^2\omega^2}\left(\frac{R}{L}\sin\omega t-\omega\cos\omega t\right)+Ce^{-\frac{R}{L}t}$$

At $t=0;\quad i=0$

$$0=\frac{EL}{R^2+L^2\omega^2}\left(\frac{R}{L}\sin\omega 0-\omega\cos\omega 0\right)+Ce^{-\frac{R}{L}(0)}$$

$$0=\frac{EL}{R^2+L^2\omega^2}(-\omega)+C$$

$$C=\frac{EL\omega}{R^2+L^2\omega^2}$$

$$i(t)=\frac{EL}{R^2+L^2\omega^2}\left(\frac{R}{L}\sin\omega t-\omega\cos\omega t\right)+\frac{EL\omega}{R^2+L^2\omega^2}e^{-\frac{R}{L}t}$$

Important facts that the differential equations for electrical and mechanical (Translation and Rotational) are identical in some forms.

TABLE *A*: Relationships between the variables of the analog system components.

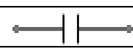
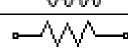
Electrical	***Mechanical Translation***	***Mechanical Rotational***
$i = C\frac{dv}{dt}$ $= Gv$ $= N\phi$ $= r^2\left(1 - r^2\right)$ $= \frac{1}{L}\int vdt$	$f = M\frac{dv}{dt}$ $= Dv$ $= kx$ $= k\int vdt$	$T = J$ $= D\omega$ $= k\theta$ $= k\int \omega dt$

Engineers sometimes utilize the similarity by determining the properties of a proposed mechanical system with a simple electrical analog.

TABLE *B*: Analogous between electrical and mechanical systems.

Electrical	***Mechanical Translation***	***Mechanical Rotational***
Current, ***i***	Force , ***f*** N, lb	Torque, ***T*** N-m, lb-ft
Voltage, ***V***	Velocity, ***v***	Angular velocity, ω
Flux linkages	Displacement	Angular displacement, $N\phi$,*xh* or θ*rad*
Capacitance. ***C***	Mass, **M** kg, slug	Moment of inertia, **J** kg-m^2, lb-ft/sec^2**.**
Conductance **G = 1/R**	Damping coefficient (of dash pot) **D** or **B** **N/m/sec**, lb/ft/sec	Rotational damping Coefficient friction: **D** or **B**
Inductance, ***L***	Compliance **τ = 1/k** of spring	Torsional compliance **τ = 1/k** of spring **k** →N.m/rad

Summary

	Abv.		***Unit***
Capacitor	*C*		Farad (**F**)
Current	*I*		Ampere (**A**)
Electric Charge	*q*		Coulomb (**C**)
Electromotive Force	*emf*		**Emf**
Inductor	*L*		Henry (**H**)
Resistor	*R*		Ohm (**Ω**)
Time	*t*		Second (**s**)
Voltage	*V*		Volt (**V**)

Exercises Section 3.5 - Basic Electrical Circuit

A resistor $R = 20\ \Omega$ and a capacitor of $C = 0.1\ F$ are joined in series with an electronic force (*emf*) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge on the capacitor at time *t* for the given:

1. $E(t) = 100\sin 2t$

2. $E(t) = 100e^{-0.1t}$

3. $E(t) = 100\left(1 - e^{-0.1t}\right)$

4. $E(t) = 100\cos 3t$

An inductor $(L = 1\ H)$ and a resistor $(R = 0.1\ \Omega)$ are joined in series with an electronic force (*emf*) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing current in the current at time *t* for the given:

5. $E(t) = 10 - 2t$

6. $E(t) = 4\cos 3t$

7. $E(t) = 4\sin 2\pi t$

8. Solve the general initial value problem modeling the *RC* circuit

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E, \quad Q(0) = 0$$

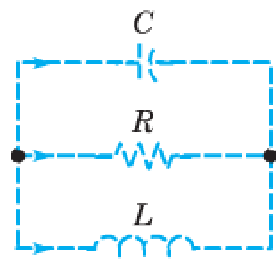
Where *E* is a constant source of *emf*

9. Solve the general initial value problem modeling the *LR* circuit

$$L\frac{dI}{dt} + RI = E, \quad I(0) = I_0$$

Where *E* is a constant source of *emf*

10. Consider the circuit shown and let I_1, I_2, and I_3 be the currents through the capacitor, resistor, and inductor, respectively. Let V_1, V_2, and V_3 be the corresponding voltage drops. The arrows denote the arbitrary chosen directions in which currents and voltage drops will be taken to be positive.



a) Applying Kirchhoff's second law to the upper loop in the circuit, show that $V_1 - V_2 = 0$ *and* $V_2 - V_3 = 0$

b) Applying Kirchhoff's first law to either node in the circuit, show that

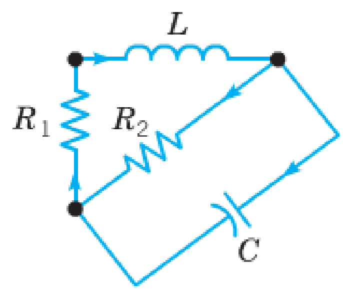
$$I_1 + I_2 + I_3 = 0$$

c) Use the current-voltage relation through each element in the circuit to obtain the equations

$$CV_1' = I_1, \quad V_2 = RI_2, \quad LI_3' = V_3$$

d) Eliminate V_2, V_3, I_1 and I_2 to obtain $CV_1' = -I_3 - \dfrac{V_1}{R}, \quad LI_3' = V_1$

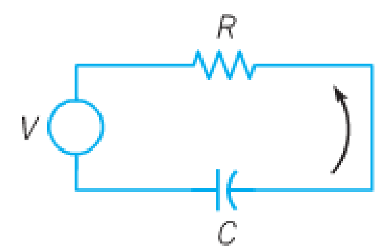
11. Consider the circuit. Use the method outlined to show that the current *I* through the inductor and the voltage *V* across the capacitor satisfy the system of differential equations.



$$L\frac{dI}{dt} = -R_1 I - V, \quad C\frac{dV}{dt} = I - \frac{V}{R_2}$$

12. Consider an electric circuit containing a capacitor, resistor, and battery. The charge $Q(t)$ on the capacitor satisfies the equation

$$R\frac{dQ}{dt} + \frac{Q}{C} = V$$



Where *R* is the resistance, *C* is the capacitance, and *V* is the constant voltage supplied by the battery.

a) If $Q(0) = 0$, find $Q(t)$ at time *t*.

b) Find the limiting value Q_L that $Q(t)$ approaches after a long time.

c) Suppose that $Q(t_1) = Q(t)$ and that at time $t = t_1$ the battery is removed and the circuit is closed again. Find $Q(t)$ for $t > t_1$.

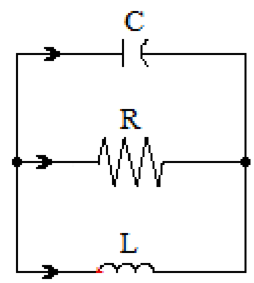
Section 3.6 – Introduction to Systems

A system of differential equations is a set of one or more equations, involving one or more differential equations.

There are several physical problems that involve number of separate elements such as an example: electrical networks, mechanical, and more other fields.

Example of a *parallel* LRC circuit

Consider the parallel LRC circuit as shown below



Let V be the voltage drop across the capacitor and I current through the inductor.
The current is described by the ***equation***

$$\frac{dI}{dt} = \frac{V}{L}$$

The voltage is described by the ***equation***

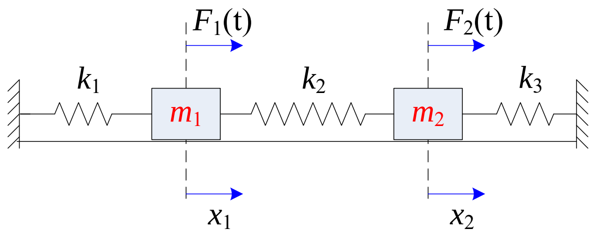
$$\frac{dV}{dt} = -\frac{1}{C} - \frac{V}{RC}$$

Therefore, we can rewrite the equation as ***system equations***

$$\begin{cases} I' = \frac{1}{L}V \\ V' = -\frac{1}{C} - \frac{1}{RC}V \end{cases}$$

Example of a *Spring-Mass* (*mechanical*)

Two masses move on a frictionless surface under the influence of external forces $F_1(t)$ and $F_2(t)$, and they are also constraint by the three springs whose constant are $k_1,\ k_2,\ and\ k_3$



Let examine the forces acting on m_1

The ***first*** spring exerts a force of: $f_1 = -k_1 x_1$ k_1 ***is the spring constant***.

The ***second*** spring exerts a force of: $f_2 = k_2\left(x_2 - x_1\right)$

By Newton's second law:

$$m_1 \frac{d^2 x_1}{dt^2} = \sum forces = f_1 + f_2 + F_1(t)$$

$$m_1 \frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2\left(x_2 - x_1\right) + F_1(t)$$

$$= -\left(k_1 + k_2\right)x_1 + k_2 x_2 + F_1(t)$$

The forces acting on m_2

The ***second*** spring exerts a force of: $-f_2 = -k_2\left(x_2 - x_1\right)$

The ***third*** spring exerts a force of: $f_3 = -k_3 x_2$

$$m_2 \frac{d^2 x_2}{dt^2} = -k_2\left(x_2 - x_1\right) - k_3 x_2 + F_2(t)$$

$$= k_2 x_1 - \left(k_2 + k_3\right)x_2 + F_2(t)$$

Example of *Predator-Prey* Systems (*Ecology*)

The dynamical of biological growth of populations is a branch of ecology.

The growth rate of species is depending on their population. The rate is increased by birth and food supply causes the species to live, and decreased by death, overcrowded, etc….

Consider two species that exist together and interact as an example such as wolves and deer, shark and food fish, etc... Vito Volterra, an Italian mathematician, formulated a predator-prey system model.

Let the *prey* population denoted by $F(t)$.

Let the *predator* population denoted by $G(t)$.

For each $F(t)$ and $G(t)$, we have a reproductive rate which denoted by r_F and r_G for prey and predator respectively.

Therefore, that will imply to:

$$F' = r_F F(t)$$
$$G' = r_G G(t)$$

Let assume there is absence of predator, by using *Malthusian*model, the prey population will be given by

$$G = 0 \quad \Rightarrow R_F = a > 0$$

When there are predator activities, then , and the decrease in the reproductive rate would also be proportional to $G(t)$.

$$R_F = a - bG \quad a,\ b > 0$$

In the absence of prey, by using *Malthusian* model, the predator population will be given by

$$F = 0 \quad \Rightarrow R_G = -c < 0$$

The presence of the prey would decrease in the reproductive rate would be proportional to the size of the prey population..

$$R_G = -c + d\ F \quad c,\ d > 0$$

That will give us a system of:

$$F' = (a - bG)F$$
$$G' = (-c + d\ F)G$$

This model is ***nonlinear*** because the right-side contains the product FG.

It is ***autonomous*** because the right-side doesn't depend explicitly on the independent variable.

Summary of Predator-Prey

The Predator-Prey or ***Lotka–Volterra*** system is given by:

$$\begin{cases} \dot{x} = -ax + bxy \\ \dot{y} = cy - dxy \end{cases}$$

Where x is the predator, their prey is 'y', and the coefficient a, b, c, and d are positive real numbers and they are defined as follow:

> ***a***: is the natural decay.
>
> ***ax***: is a rate term, which shows that without prey to eat, the predator population diminishes.
>
> ***c***: is the natural growth coefficient.
>
> ***cy***: is a rate term, where the prey population increases.
>
> ***b** and **c***: predator efficiency in converting food into fertility and the probability that are predator-prey encounter removes of the prey.
>
> ***bxy** and **cxy***: Food promotes the predator population's growth rate, while serving as food diminishes the prey populations' growth rate.

The predator-prey system or model is based on the population Law of Mass Action.
The Law of Mass Action is defined as:

> "*The rate of change of one population due to interaction of another is proportional to the product of the two populations*."

Section 3.7 – Basic Theory of Linear Systems

Definition

A linear system of differential equations is any set of differential equations having the following ***standard form***:

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + f_1(t) \\ x_2' &= a_{21}(t)x_1 + \cdots + a_{2n}(t)x_n + f_2(t) \\ &\vdots \qquad\qquad \vdots \\ x_n' &= a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + f_n(t) \end{aligned}$$

Where $x_1, \cdots, \text{ and } x_n$ are the unknown functions. The ***coefficients*** $a_{ij}(t)$ and $f_i(t)$ are known functions of the independent variable $t \in (a,\ b)$ is an interval in **R**.

If all of the $f_i(t) = 0$, the system said to be ***homogeneous***. Otherwise it is ***inhomogeneous***. The inhomogeneous part is sometimes called the ***forcing term***.

Matrix Notation for Linear Systems

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

In simple form, we can rewrite:

$$x'(t) = A(t)x(t) + f(t)$$

$$x' = Ax + f$$

Example

Given the linear system

$$\begin{cases} x_1' = x_1 + 2x_2 \\ x_2' = 2x_1 + x_2 \end{cases}$$

Write in the form $x' = Ax + f$

Solution

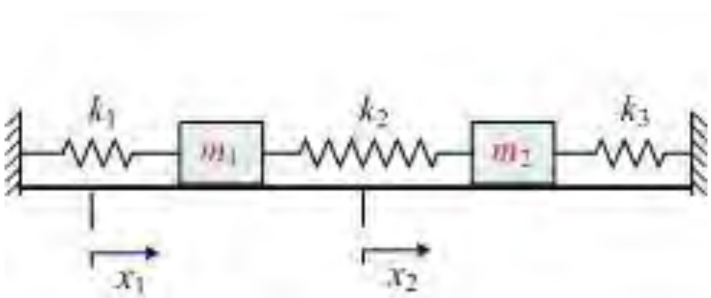
$$x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Example

Consider the spring-mass system consisting of two masses that are constraint by the three springs whose constant are $k_1, k_2, \text{ and } k_3$. Assume there is no damping and there are no external forces.

$u_4(t) = x_2'(t)$



Write an equivalent linear system of the first-order differential equations.

Solution

$$m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2$$

$$m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2$$

$$\begin{cases} x_1'' = -\dfrac{k_1 + k_2}{m_1} x_1 + \dfrac{k_2}{m_1} x_2 \\ x_2'' = \dfrac{k_2}{m_2} x_1 - \dfrac{k_2 + k_3}{m_2} x_2 \end{cases}$$

To write an equivalent first-order system, let $U = (u_1, u_2, u_3, u_4)^T$

Where $u_1(t) = x_1(t) \qquad u_2(t) = x_1'(t) \qquad u_3(t) = x_2(t)$

$$\begin{cases} u_2' = x_1'' = -\dfrac{k_1+k_2}{m_1}u_1 + \dfrac{k_2}{m_1}u_3 \\ u_4' = x_2'' = \dfrac{k_2}{m_2}u_1 - \dfrac{k_2+k_3}{m_2}u_3 \end{cases}$$

$$\begin{cases} u_1' = u_2 \\ u_2' = -\dfrac{k_1+k_2}{m_1}u_1 + \dfrac{k_2}{m_1}u_3 \\ u_3' = u_4 \\ u_4' = \dfrac{k_2}{m_2}u_1 - \dfrac{k_2+k_3}{m_2}u_3 \end{cases}$$

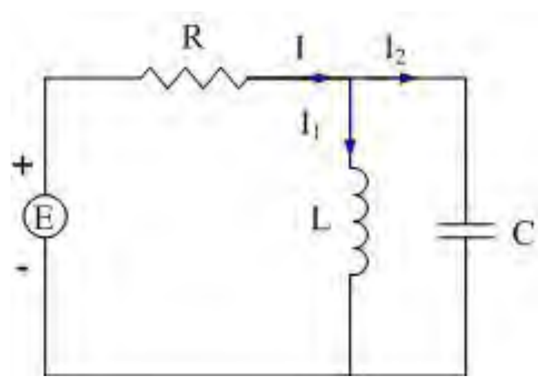
$$\begin{pmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{k_1+k_2}{m_1} & 0 & \dfrac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_2}{m_2} & 0 & -\dfrac{k_2+k_3}{m_2} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

The initial conditions for this system involve the initial position and velocity of both masses.

$$x_1(0) = a_1 \qquad x_1'(0) = b_1 \qquad x_2(0) = a_2 \qquad x_2'(0) = b_2$$

Example

Find a first-order system the models the circuit below



<u>Solution</u>

Using Kirchhoff's current law:

$$I = I_1 + I_2$$

Kirchhoff's voltage law applied to the loop containing the source and the inductor:

$$E = RI + LI_1'$$

$$E = R\left(I_1 + I_2\right) + LI_1'$$

$$LI_1' = E - R\left(I_1 + I_2\right)$$

$$I_1' = \frac{1}{L}\left[E - R\left(I_1 + I_2\right)\right]$$

Kirchhoff's voltage law applied to the loop containing the source, resistor and the capacitor:

$$E = RI + \frac{1}{C}Q$$

Differentiate the equation:

$$E' = RI' + \frac{1}{C}Q' \qquad\qquad Q' = I_2$$

$$E' = R\left(I_1' + I_2'\right) + \frac{1}{C}I_2$$

$$E' = RI_1' + RI_2' + \frac{1}{C}I_2$$

$$RI_2' = E' - RI_1' - \frac{1}{C}I_2$$

$$RI_2' = E' - R\frac{1}{L}\left[E - R\left(I_1 + I_2\right)\right] - \frac{1}{C}I_2$$

$$\begin{aligned} I_2' &= \frac{1}{R}E' - \frac{1}{L}\left[E - R\left(I_1 + I_2\right)\right] - \frac{1}{RC}I_2 \\ &= \frac{1}{R}E' - \frac{1}{L}E + \frac{R}{L}I_1 + \frac{R}{L}I_2 - \frac{1}{RC}I_2 \\ &= \frac{R}{L}I_1 + \left(\frac{R}{L} - \frac{1}{RC}\right)I_2 + \frac{E'}{R} - \frac{E}{L} \end{aligned}$$

$$\begin{cases} I_1' = -\frac{R}{L}I_1 - \frac{R}{L}I_2 + \frac{E}{L} & (1) \\ I_2' = \frac{R}{L}I_1 + \left(\frac{R}{L} - \frac{1}{RC}\right)I_2 + \frac{E'}{R} - \frac{E}{L} & (2) \end{cases}$$

Properties of Linear Systems

Properties of Homogeneous Systems

Theorem

Suppose x_1 and x_2 are solution to the homogeneous linear system

$$x' = Ax$$

If C_1 and C_2 are any constants, then $x = C_1x_1 + C_2x_2$ is also a solution

Theorem

Suppose $x_1, x_2, \ldots, \text{ and } x_n$ are solution to the homogeneous linear system

If $C_1, C_2, \ldots, \text{ and } C_n$ are any constants, then

$$x(t) = C_1x_1(t) + C_2x_2(t) + \cdots + C_nx_n(t)$$

is also a solution to $x' = Ax$

Linearly Independence and Dependence

Proposition

Suppose $y_1, y_2, \ldots, \text{ and } y_n$ are solution to the n-dimensional system $y' = Ay$ defined on the interval $I = (\alpha, \beta)$.

1. If the vectors $y_1(0), y_2(0), \ldots, \text{ and } y_n(0)$ are linearly dependent for some $t_0 \in I$, then there are constants $C_1, C_2, \ldots, \text{ and } C_n$ not all zero, such that $C_1y_1(t) + C_2y_2(t) + \cdots + C_ny_n(t) = 0$ for all $t \in I$. In particular, $y_1(t), y_2(t), \ldots, \text{ and } y_n(t)$ are linearly dependent for all $t \in I$.

2. If for some $t_0 \in I$ the vectors $y_1(0), y_2(0), \ldots, \text{ and } y_n(0)$ are linearly independent, then $y_1(t), y_2(t), \ldots, \text{ and } y_n(t)$ are linearly independent for all $t \in I$.

Definition

A set of ***n*** solutions to the linear system $x' = Ax$ is linearly independent if it is linearly independent for any one value of *t*.

Example

Given $x_1(t)=\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$ and $x_2(t)=\begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$ are solutions to the homogeneous system

$x'(t)=\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}x(t)$

Show that all solutions to this system can be expressed as linear combination of x_1 and x_2

Solution

$$x_1(t)=\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}=e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$x_2(t)=\begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}=e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x(t)=C_1x_1(t)+C_2x_2(t)$$

$$\begin{aligned} x(t=0) &= C_1x_1(0)+C_2x_2(0) \\ &= C_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}+C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} C_1+C_2 \\ -C_1+C_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}=\left[x_1(0),\ x_2(0)\right]$$

$\det=\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}=2\neq 0 \Rightarrow$ The matrix is nonsingular and $x_1(0)$, $x_2(0)$ are linearly independent

$$x(0)=\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Example

Consider the system of homogeneous equations

$$x'(t) = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix} x(t)$$

We can show that

$$x_1(t) = \begin{pmatrix} e^t \cos t \\ e^t(\cos t - \sin t) \end{pmatrix} \qquad x_2(t) = \begin{pmatrix} e^t \sin t \\ e^t(\cos t + \sin t) \end{pmatrix}$$

are solutions the given system

$$x_1(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned} x(t) &= C_1 x_1(t) + C_2 x_2(t) \\ &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \end{aligned}$$

$$\det = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow x_1(0) \text{ and } x_2(0) \text{ are linearly independent}$$

Exercises *Section* 3.7 – Basic Theory of Linear Systems

For the linear systems which are homogeneous? Which are inhomogeneous?

1. $\begin{cases} x_1' = -2x_1 + x_1 x_2 \\ x_2' = -3x_1 - x_2 \end{cases}$

2. $\begin{cases} x_1' = -x_2 \\ x_2' = \sin x_1 \end{cases}$

3. $\begin{cases} x_1' = x_1 + (\sin t) x_2 \\ x_2' = 2t x_1 - x_2 \end{cases}$

Write the given system of equations in matrix-form then show that the given vector is a solution to the system

4. $\begin{cases} x_1' = -3x_1 + x_2 \\ x_2' = -2x_1 \end{cases} \qquad v = \left(-e^{-2t} + e^{-t}, -e^{-2t} + 2e^{-t}\right)^T$

5. $\begin{cases} x_1' = -x_1 + 4x_2 \\ x_2' = \quad\quad 3x_2 \end{cases} \qquad v = \left(e^{3t} - e^{-t}, e^{3t}\right)^T$

Verify by substitution that $x_1(t)$ and $x_2(t)$ are solutions of the given homogenous equation. Show also that the solutions $x_1(t)$ and $x_2(t)$ are linearly independent. Find the solution of the given homogeneous equation with the initial condition $x(0) = x_0$

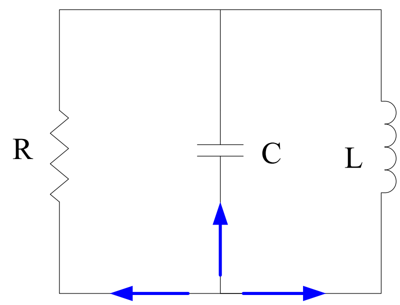
6. $\begin{cases} x_1(t) = \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix}, & x_2(t) = \begin{pmatrix} -e^{-2t} \\ e^{-2t} \end{pmatrix} \\ x' = \begin{pmatrix} -6 & -4 \\ 8 & 6 \end{pmatrix} x & x(0) = \begin{pmatrix} -5 \\ 8 \end{pmatrix} \end{cases}$

7. $\begin{cases} x_1(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}, & x_2(t) = \begin{pmatrix} e^{2t}(t+2) \\ e^{2t}(t+1) \end{pmatrix} \\ x' = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} x & x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$

8. $\begin{cases} x_1(t) = \begin{pmatrix} \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \cos t \end{pmatrix}, & x_2(t) = \begin{pmatrix} \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ \sin t \end{pmatrix} \\ x' = \begin{pmatrix} 3 & -1 \\ 2 & -1 \end{pmatrix} x & x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$

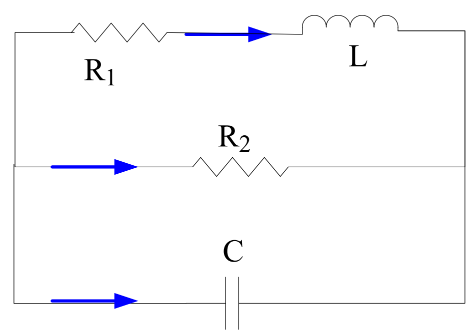
9. Consider the *RLC* parallel circuit below. Let *V* represent the voltage drop across the capacitor and

I represent the current across the inductor.



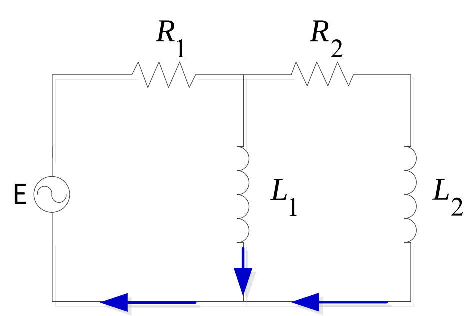
Show that: $$\begin{cases} V' = -\dfrac{V}{RC} - \dfrac{1}{C} \\ I' = \dfrac{V}{L} \end{cases}$$

10. Consider the *RLC* parallel circuit below. Let *V* represent the voltage drop across the capacitor and *I* represent the current across the inductor.



Show that: $$\begin{cases} CV' = -I - \dfrac{V}{R_2} \\ LI' = -R_1 I + V \end{cases}$$

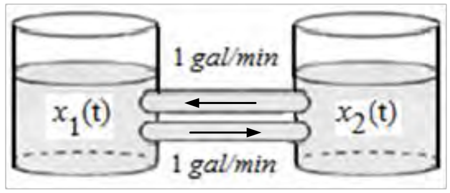
11. Consider the circuit below.



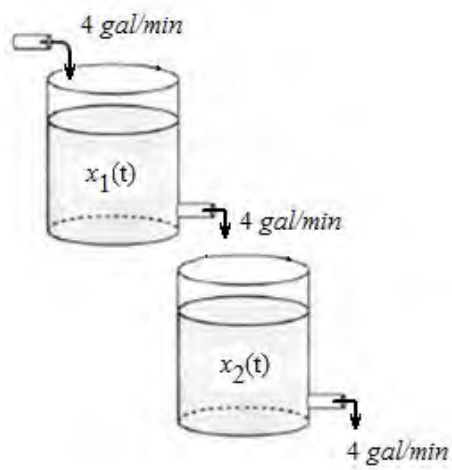
Let I_1 and I_2 represent the current flow across the indicators L_1 and L_2 respectively. Show that the circuit is modeled by the system

$$\begin{cases} L_1 I_1' = -R_1 I_1 - R_1 I_2 + E \\ L_2 I_2' = -R_1 I_1 - \left(R_1 + R_2\right) I_2 + E \end{cases}$$

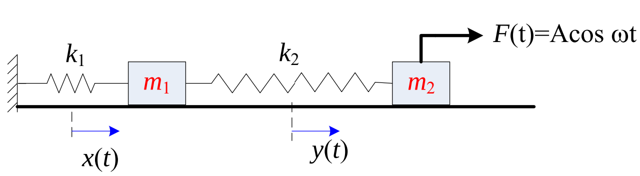
12. Two tanks are connected by two pipes. Each tank contains 500 gallons of a salt solution. Through on pipe solution is pumped from the first tank to the second at 1 *gal/min*. Through the other pipe, solution is pumped at the same rate from the second to the first tank. Show the salt content in each tank varies with time.



13. Each tank contains 100 gallons of a salt solution.Pure water flows into the upper tank at a rate of 4 *gal/min*. Salt solution drains from the upper tank into the lower tank at a rate of 4 *gal/min*. Finally, salt solution drains from the lower tank at a rate of 4 *gal/min*, effectively keeping the volume of solution in each tank at a constant 100 *gal*. If the initial salt content of the upper and lower tanks is 10 and 20 pounds, respectively. Set up an initial value problem that models the amount of salt in each tank over time (do not solve).Write the model in matrix-vector form. Is the system homogeneous or inhomogeneous?



14. Two masses on a frictionless tabletop are connected with a spring having spring constant k_2. The first mass is connected to a vertical support with a spring having spring constant k_1. The second mass is shaken harmonically via a force equaling $F = A\cos\omega t$. Let $x(t)$ and $y(t)$ measure the displacements of the masses m_1 and m_2 ., respectively, from their equilibrium positions as a function of time. If both masses start from rest at their equilibrium positions at time $t = 0$.



Set up an initial value problem that models the position of the masses over time (do not solve). Write the model in matrix-vector form. Is the system homogeneous or inhomogeneous?

Section 3.8 – Linear Systems with Constant Coefficients

Consider the system equation:

$$x' = Ax$$

Where A is a matrix with constant entries $\left\lfloor a_{ij} \right\rfloor$

The 1st-order homogeneous equation can be written as

$$x' = ax$$

The solution to this system is given by:

$$x = Ce^{at}$$

We can rewrite the solution in form of vector:

$$x = ve^{\lambda t}$$

The first derivative of the solution: $x' = \lambda ve^{\lambda t}$

$$x' = Ax$$

$$\lambda ve^{\lambda t} = Ave^{\lambda t}$$

$$\lambda v = Av$$

Definition

Suppose A is an nxn matrix and

$$Av = \lambda v$$

The values of λ are called eigenvalues of the matrix $\boldsymbol{A}$ and the nonzero vectors $\boldsymbol{v}$ are called the eigenvectors corresponding to that eigenvalue.

Eigenvalues

Let's change the form of the system to a general matrix form and is defined by the form:

$$X' = AX(t)$$

Where A is a square matrix (n x n)

The behavior of a system can be determined from equilibrium point(s) by finding the eigenvalues and the eigenvectors of the system.

Therefore; the equation can be rewritten into the form:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

Let's rewrite the equation $Av = \lambda v$.

$Av - \lambda v = 0$ $\quad\quad$ λ : are the eigenvalues and not a vector

$Av - \lambda I v = 0$

$(A - \lambda I)v = 0$

$$\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} a_{11} - \lambda_1 & a_{12} \\ a_{21} & a_{22} - \lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

Since v is a nonzero vector that implies that the matrix $A - \lambda I$ has a nontrivial null space.

This exists if and only if (*iff*):

$$\det(A - \lambda I) = 0$$

Therefore, the eigenvalues (λ's) are the roots which can be determined by solving the determinant:

$$\begin{vmatrix} a_{11} - \lambda_1 & a_{12} \\ a_{21} & a_{22} - \lambda_2 \end{vmatrix} = 0$$

$$(a_{11} - \lambda_1)(a_{22} - \lambda_2) - a_{12}a_{21} = 0$$

where the eigenvalues $= \lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2\sigma b(r - 1)$ are the solutions

Example

Find the eigenvalues of the matrix

$$A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$$

Solution

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{vmatrix} = (-4-\lambda)(5-\lambda) + 18$$

$$= -20 + 4\lambda - 5\lambda + \lambda^2 + 18$$

$$= \lambda^2 - \lambda - 2$$

The characteristic polynomial is: $\lambda^2 - \lambda - 2 = 0$

Thus, the eigenvalues of ***A*** are 2 and -1.

Eigenvectors

From the eigenvalues, the eigenvectors, in the form $V_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$, of the system can be determined by letting:

$$\left(A-\lambda_1 I\right)V_1 = 0 \qquad \text{and} \left(A-\lambda_2 I\right)V_2 = 0$$

The general solution can be written as:

$$x(t) = V_i e^{\lambda t}$$

Example

Find the eigenvectors of the matrix

$$A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$$

Solution

The eigenvalues of $\boldsymbol{A}$ are 2 and -1.

For $\lambda = 2$, we have

$$\left(A-\lambda_1 I\right)V_1 = 0$$

$$\begin{pmatrix} -4-2 & 6 \\ -3 & 5-2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -6x+6y=0 \\ -3x+3y=0 \end{cases} \Rightarrow x = y$$

If $x = c \quad \Rightarrow y = c$

$$V_1 = c\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda = -1$, we have

$$\left(A-\lambda_2 I\right)V_2 = 0$$

$$\begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -3x+6y=0 \\ -3x+6y=0 \end{cases} \Rightarrow x = 2y$$

If $y = c \quad \Rightarrow x = 2c$

$$V_2 = c\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \Rightarrow V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x(t) = V_1 e^{\lambda_1 t} + V_2 e^{\lambda_2 t}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}$$

Summary

In general, a polynomial of degree n has n roots. Each root λ is an eigenvalue, and for each we can find an eigenvector v. From these, we can form the solution $y(t) = ve^{\lambda t}$.
However, the numbers of the eigenvalue solutions are as follow:

1. Two Distinct real roots $\left(T^2 - 4D > 0\right)$
2. Two complex conjugate roots $\left(T^2 - 4D < 0\right)$
3. One real Repeated roots $\left(T^2 - 4D = 0\right)$

Exercises *Section* 3.8 – Linear Systems with Constant Coefficients

Find the eigenvalues and the eigenvectors for each of the matrices.

1. $A=\begin{pmatrix} 12 & 14 \\ -7 & -9 \end{pmatrix}$

2. $A=\begin{pmatrix} -4 & 1 \\ -2 & 1 \end{pmatrix}$

3. $A=\begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$

4. $A=\begin{pmatrix} -2 & 3 \\ 0 & -5 \end{pmatrix}$

5. $A=\begin{pmatrix} 6 & 10 \\ -5 & -9 \end{pmatrix}$

6. $A=\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

7. $A=\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

8. $A=\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

9. $A=\begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -8 & -4 & -3 \end{pmatrix}$

10. $A=\begin{pmatrix} -1 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & -12 & 2 \end{pmatrix}$

11. $A=\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

12. $A=\begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}$

13. $A=\begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Find a fundamental set of solutions for the system $x' = Ax$, where A is the given matrices.

14. $A=\begin{pmatrix} 2 & 0 \\ -4 & -1 \end{pmatrix}$

15. $A=\begin{pmatrix} -1 & 0 & 0 \\ 2 & -5 & -6 \\ -2 & 3 & 4 \end{pmatrix}$

Section 3.9 – Planar Systems – *Distinct, Complex, and Repeated Eigenvalues*

Planar Systems

2-dimension linear systems are also called planar systems, we will enable to solve the system

$$y' = Ay$$

$$\text{where} \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix}$$

$$= \lambda^2 - \left(a_{11} + a_{22}\right)\lambda + a_{11}a_{22} - a_{12}a_{21}$$

$$D = \det(A) = a_{11}a_{22} - a_{12}a_{21}$$

$$T = tr(A) = a_{11} + a_{22} \qquad tr(A): trace$$

$$\lambda^2 - T\lambda + D = 0$$

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

Proposition

Suppose λ_1 and λ_2 are eigenvalues of an $n \times n$ matrix A. Suppose $V_1 \neq 0$ is an eigenvector for λ_1 and $V_2 \neq 0$ is an eigenvector for λ_2. If $\lambda_1 \neq \lambda_2$ then V_1 and V_2 are linearly independent.

Distinct Real Eigenvalues

If $T^2 - 4D > 0$, then the solutions of the characteristic equation are:

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} \quad \textit{and} \quad \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

Then $\lambda_1 < \lambda_2$, and both are real eigenvalues of A.

Let V_1 and V_2 be the associated eigenvectors. Then we have two exponential solutions:

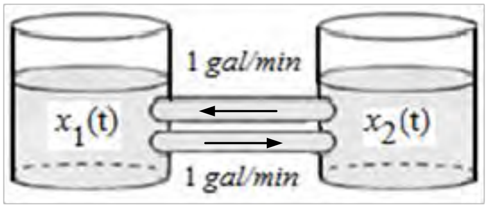
$$y_1(t) = V_1 e^{\lambda_1 t} \quad \textit{and} \quad y_2(t) = V_2 e^{\lambda_2 t}$$

The general solution is:

$$\begin{aligned} y(t) &= C_1 y_1(t) + C_2 y_2(t) \\ &= C_1 V_1 e^{\lambda_1 t} + C_2 V_2 e^{\lambda_2 t} \end{aligned}$$

Example

Two tanks are connected by two pipes. Each tank contains 500 gallons of a salt solution. Through on pipe solution is pumped from the first tank to the second at 1 *gal/min*. Through the other pipe, solution is pumped at the same rate from the second to the first tank. Suppose that at time $t = 0$ there is no salt in the tank on the right and 100 *lb* in the tank on the left. Show the salt content in each tank varies with time.



Solution

$$\textit{Rate in} = 1 gal/min \frac{x_2}{500} lb/gal = \frac{x_2}{500} lb/min$$

$$\textit{Rate out} = 1 gal/min \frac{x_1}{500} lb/gal = \frac{x_1}{500} lb/min$$

$$\begin{aligned} \frac{dx_1}{dt} &= \textit{Rate in} - \textit{Rate out} = \frac{x_2}{500} - \frac{x_1}{500} \\ &= -\frac{x_1}{500} + \frac{x_2}{500} \end{aligned}$$

$$\frac{dx_2}{dt} = \frac{x_1}{500} - \frac{x_2}{500}$$

$$\begin{cases} x_1' = -\dfrac{x_1}{500} + \dfrac{x_2}{500} \\ x_2' = \dfrac{x_1}{500} - \dfrac{x_2}{500} \end{cases}$$

The system is: $x' = Ax(t)$

Where $A = \begin{pmatrix} -\frac{1}{500} & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500} \end{pmatrix}$

$$\det\left(A - \lambda I\right) = \begin{vmatrix} -\frac{1}{500} - \lambda & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500} - \lambda \end{vmatrix}$$

$$= \left(-\frac{1}{500} - \lambda\right)\left(-\frac{1}{500} - \lambda\right) - \frac{1}{500}\frac{1}{500}$$

$$= \frac{1}{500^2} + \frac{2}{500}\lambda + \lambda^2 - \frac{1}{500^2}$$

$$= \lambda^2 + \frac{1}{250}\lambda$$

$$= \lambda\left(\lambda + \frac{1}{250}\right)$$

The eigenvalues are: $\lambda_1 = -\frac{1}{250} \qquad \lambda_2 = 0$

For $\lambda_1 = -\frac{1}{250}$, we have

$$\left(A - \lambda_1 I\right)V_1 = 0$$

$$\begin{pmatrix} -\frac{1}{500} + \frac{1}{250} & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500} + \frac{1}{250} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{500} & \frac{1}{500} \\ \frac{1}{500} & \frac{1}{500} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \frac{1}{500}x + \frac{1}{500}y = 0 \\ \frac{1}{500}x + \frac{1}{500}y = 0 \end{cases} \Rightarrow \begin{matrix} x + y = 0 \\ x + y = 0 \end{matrix} \Rightarrow x = -y$$

$$V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad \boxed{x_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t/250}}$$

For $\lambda_2 = 0$, we have

$$(A - \lambda_2 I)V_2 = 0$$

$$\begin{pmatrix} -\frac{1}{500} & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -\frac{1}{500}x + \frac{1}{500}y = 0 \\ \frac{1}{500}x - \frac{1}{500}y = 0 \end{cases} \Rightarrow \begin{matrix} -x + y = 0 \\ x - y = 0 \end{matrix} \Rightarrow x = y$$

$$V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \boxed{x_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{0t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

The general solution:

$$x(t) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t/250} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} C_1 e^{-t/250} + C_2 \\ -C_1 e^{-t/250} + C_2 \end{pmatrix}$$

$$x(0) = \begin{pmatrix} C_1 e^{-0/250} + C_2 \\ -C_1 e^{-0/250} + C_2 \end{pmatrix}$$

$$\begin{pmatrix} 100 \\ 0 \end{pmatrix} = \begin{pmatrix} C_1 + C_2 \\ -C_1 + C_2 \end{pmatrix}$$

$$\rightarrow \begin{cases} C_1 + C_2 = 100 \\ -C_1 + C_2 = 0 \end{cases} \Rightarrow \quad C_1 = 50 \quad C_2 = 50$$

$$x(t) = \begin{pmatrix} 50 + 50e^{-t/250} \\ 50 - 50e^{-t/250} \end{pmatrix}$$

Complex Eigenvalues

If $T^2 - 4D < 0$, then the solutions of the characteristic equation are the complex conjugate:

$$\lambda_1 = \frac{T + i\sqrt{4D - T^2}}{2} \quad \textit{and} \quad \lambda_2 = \frac{T - i\sqrt{4D - T^2}}{2}$$

Example

Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$$

Solution

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -\lambda & 1 \\ -2 & 2-\lambda \end{vmatrix} \\ &= -\lambda(2-\lambda) + 2 \\ &= -\lambda^2 - 2\lambda + 2 \\ &= 0 \end{aligned}$$

$$-\lambda^2 - 2\lambda + 2 = 0 \quad \Rightarrow \lambda = 1 + i \quad \bar{\lambda} = 1 - i$$

For $\lambda = 1 + i$;

$$(A - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} -1-i & 1 \\ -2 & 2-i-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} -1-i & 1 \\ -2 & 1-i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\rightarrow \begin{cases} -(1+i)x + y = 0 \\ -2x + (1-i)y = 0 \end{cases}$$

$$\rightarrow \begin{cases} -(1+i)(1-i)x + (1-i)y = 0 \\ \qquad -2x + (1-i)y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -2x + (1-i)y = 0 \\ -2x + (1-i)y = 0 \end{cases} \rightarrow x = 1 \Rightarrow y = \frac{2}{1-i}\frac{1+i}{1+i} = 1 + i$$

$$V_1 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} \quad \Rightarrow \quad x_1(t) = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^{(1+i)t}$$

For $\bar{\lambda} = 1 - i$;

$$V_2 = \bar{V}_1 \quad \Rightarrow \quad x_2(t) = V_2 e^{\bar{\lambda} t}$$

$$V_2 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} \quad \Rightarrow \quad x_2(t) = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{(1-i)t}$$

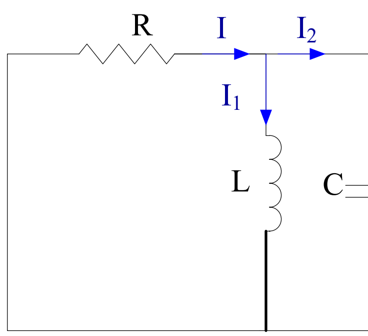
Theorem

Suppose that A is a 2×2 matrix with complex conjugate eigenvalues λ and $\bar{\lambda}$. Suppose that V is an eigenvector associated with λ. Then the general solution to the system $x' = Ax$ is

$$x(t) = C_1 V_1 e^{\lambda t} + C_2 V_2 e^{\bar{\lambda} t}$$

Example

Find the current I_1 and I_2 for the circuit below, where $R = 1\,\Omega$, $L = 1\ henry$, and $C = \frac{5}{4}\ farad$.
Assume that $I_1(0) = 5A$ and $I_2(0) = 1A$.



Solution

Kirchoff's current law: $I = I_1 + I_2$

Since there is no voltage: $0 = RI + LI_1' \quad \Rightarrow LI_1' = -RI = -R\left(I_1 + I_2\right)$

$$I_1' = -\frac{R}{L} I_1 - \frac{R}{L} I_2 \qquad (1)$$

Kirchoff's law applied to the large loop: $0 = RI + \frac{1}{C}Q \quad \Rightarrow \quad \frac{1}{C}Q + R\left(I_1 + I_2\right) = 0$

$$\frac{1}{C}Q' + R\left(I_1' + I_2'\right) = 0$$

$$\frac{1}{C}I_2 + R\left(I_1' + I_2'\right) = 0$$

$$RI_2' = -\frac{1}{C}I_2 - RI_1'$$

$$I_2' = -\frac{1}{RC}I_2 - \left(-\frac{R}{L}I_1 - \frac{R}{L}I_2\right)$$

$$I'_2 = \frac{R}{L} I_1 + \left(\frac{R}{L} - \frac{1}{RC}\right) I_2 \quad (2)$$

$$\begin{cases} I'_1 = -\frac{R}{L} I_1 - \frac{R}{L} I_2 \\ I'_2 = \frac{R}{L} I_1 + \left(\frac{R}{L} - \frac{1}{RC}\right) I_2 \end{cases}$$

$$\Rightarrow \begin{cases} I'_1 = -I_1 - I_2 \\ I'_2 = I_1 + \left(1 - \frac{1}{5/4}\right) I_2 \end{cases}$$

$$\begin{cases} I'_1 = -I_1 - I_2 \\ I'_2 = I_1 + \frac{1}{5} I_2 \end{cases}$$

The system can be written as: $I' = AI$, where $A = \begin{pmatrix} -1 & -1 \\ 1 & \frac{1}{5} \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & -1 \\ 1 & \frac{1}{5} - \lambda \end{vmatrix} = (-1-\lambda)\left(\frac{1}{5} - \lambda\right) + 1 = 0$$

$$\lambda^2 + \frac{4}{5}\lambda + \frac{4}{5} = 0$$

$$5\lambda^2 + 4\lambda + 4 = 0 \qquad \lambda = \frac{-2 \pm 4i}{5}$$

For $\lambda = \frac{-2+4i}{5}$;

$$\begin{pmatrix} -1 + \frac{2}{5} - \frac{4i}{5} & -1 \\ 1 & \frac{1}{5} + \frac{2}{5} - \frac{4i}{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} -\frac{3}{5} - \frac{4i}{5} & -1 \\ 1 & \frac{3}{5} - \frac{4i}{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} -\frac{1}{5}(3+4i)x - y = 0 \\ x + \frac{1}{5}(3-4i)y = 0 \end{cases} \to x = -5 \qquad y = 3 + 4i$$

$$V_1 = \begin{pmatrix} -5 \\ 3+4i \end{pmatrix}$$

$$x_1(t) = V_1 e^{\lambda t}$$

$$= \begin{pmatrix} -5 \\ 3+4i \end{pmatrix} e^{\left(\frac{-2+4i}{5}\right)t}$$

$$=\begin{pmatrix}-5\\3+4i\end{pmatrix}e^{-\frac{2}{5}t}e^{\frac{4}{5}it}$$

$$=e^{-\frac{2}{5}t}\begin{pmatrix}-5\\3+4i\end{pmatrix}\left(\cos\left(\tfrac{4}{5}t\right)+\boldsymbol{i}\sin\left(\tfrac{4}{5}t\right)\right)$$

$$=e^{-\frac{2}{5}t}\begin{pmatrix}-5\cos\left(\tfrac{4}{5}t\right)-i5\sin\left(\tfrac{4}{5}t\right)\\3\cos\left(\tfrac{4}{5}t\right)-4\sin\left(\tfrac{4}{5}t\right)+i\left(4\cos\left(\tfrac{4}{5}t\right)+3\sin\left(\tfrac{4}{5}t\right)\right)\end{pmatrix}$$

$$x_1(t)=e^{-\frac{2}{5}t}\begin{pmatrix}-5\cos\left(\tfrac{4}{5}t\right)\\3\cos\left(\tfrac{4}{5}t\right)-4\sin\left(\tfrac{4}{5}t\right)\end{pmatrix}$$

$$x_2(t)=e^{-\frac{2}{5}t}\begin{pmatrix}-5\sin\left(\tfrac{4}{5}t\right)\\4\cos\left(\tfrac{4}{5}t\right)+3\sin\left(\tfrac{4}{5}t\right)\end{pmatrix}$$

$$I(t)=C_1x_1+C_2x_2$$

$$I(0)=C_1e^{-\frac{2}{5}(0)}\begin{pmatrix}-5\cos\left(\tfrac{4}{5}(0)\right)\\3\cos\left(\tfrac{4}{5}(0)\right)-4\sin\left(\tfrac{4}{5}(0)\right)\end{pmatrix}+C_2e^{-\frac{2}{5}(0)}\begin{pmatrix}-5\sin\left(\tfrac{4}{5}(0)\right)\\4\cos\left(\tfrac{4}{5}(0)\right)+3\sin\left(\tfrac{4}{5}(0)\right)\end{pmatrix}$$

$$\begin{pmatrix}5\\1\end{pmatrix}=C_1\begin{pmatrix}-5\\3\end{pmatrix}+C_2\begin{pmatrix}0\\4\end{pmatrix}$$

$$\begin{pmatrix}5\\1\end{pmatrix}=\begin{pmatrix}-5C_1\\3C_1+4C_2\end{pmatrix}$$

$$\begin{cases}-5C_1=5\\3C_1+4C_2=1\end{cases}\Rightarrow C_1=-1\quad C_2=1$$

The general solution is:

$$I(t)=-e^{-\frac{2}{5}t}\begin{pmatrix}-5\cos\left(\tfrac{4}{5}t\right)\\3\cos\left(\tfrac{4}{5}t\right)-4\sin\left(\tfrac{4}{5}t\right)\end{pmatrix}+e^{-\frac{2}{5}t}\begin{pmatrix}-5\sin\left(\tfrac{4}{5}t\right)\\4\cos\left(\tfrac{4}{5}t\right)+3\sin\left(\tfrac{4}{5}t\right)\end{pmatrix}$$

$$I(t)=e^{-\frac{2}{5}t}\begin{pmatrix}5\cos\left(\tfrac{4}{5}t\right)-5\sin\left(\tfrac{4}{5}t\right)\\\cos\left(\tfrac{4}{5}t\right)+7\sin\left(\tfrac{4}{5}t\right)\end{pmatrix}$$

One Real Eigenvalue of Multiplicity 2

If $T^2 - 4D = 0 \Rightarrow T^2 = 4D$, then the solutions of the characteristic equation:

$$\lambda_1 = \lambda_2 = \frac{T}{2}$$

$$\left.\begin{aligned} x_1(t) &= V_1 e^{\lambda t} \\ x_2(t) &= \left(V_1 t + V_2\right) e^{\lambda t} \end{aligned}\right\} \Rightarrow \; x(t) = C_1 x_1(t) + C_2 x_2(t)$$

Example

Find all exponential solutions for $A = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$

Solution

$$\left|A - \lambda I\right| = \begin{vmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix}$$

$$= \left(-1-\lambda\right)\left(-3-\lambda\right) + 1$$

$$= \lambda^2 + 4\lambda + 4$$

$$= \left(\lambda + 2\right)^2 = 0$$

$$\lambda_{1,2} = -2$$

$$\left(A - \lambda I\right)^2 V_2 = 0$$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} V_2 = 0$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} V_2 = 0$$

$$V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left(A + 2I\right) V_2 = V_1$$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = V_1$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = V_1$$

$$x_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$$

$$x_2(t) = \left(V_1 t + V_2\right)e^{-2t}$$

$$= \left(\begin{pmatrix}1\\1\end{pmatrix}t + \begin{pmatrix}1\\0\end{pmatrix}\right)e^{-2t}$$

$$= \begin{pmatrix}t+1\\t\end{pmatrix}e^{-2t}$$

$$x(t) = C_1 x_1(t) + C_2 x_2(t)$$

$$= C_1\begin{pmatrix}1\\1\end{pmatrix}e^{-2t} + C_2\begin{pmatrix}t+1\\t\end{pmatrix}e^{-2t}$$

Exercises   Section 3.9 – Planar Systems – *Distinct, Complex, and Repeated Eigenvalues*

Find the general solution of the system $y' = Ay$

1. $A=\begin{pmatrix} -1 & 6 \\ -3 & 8 \end{pmatrix} \qquad y(0)=\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

2. $A=\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \qquad y(0)=\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

3. $A=\begin{pmatrix} -4 & -8 \\ 4 & 4 \end{pmatrix} \qquad y(0)=\begin{pmatrix} 0 & 2 \end{pmatrix}^T$

4. $A=\begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix} \qquad y(0)=\begin{pmatrix} 0 & 1 \end{pmatrix}^T$

5. $A=\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \qquad y(0)=\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

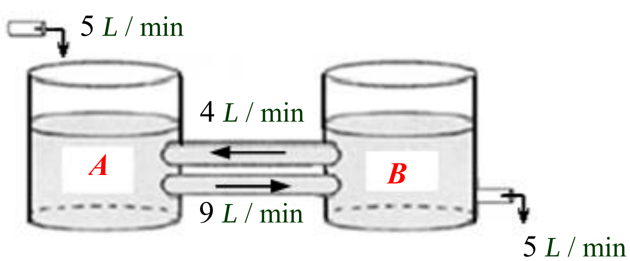
6. $A=\begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} \qquad y(0)=\begin{pmatrix} 0 \\ -3 \end{pmatrix}$

7. $A=\begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix} \qquad y(0)=\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

8. $A=\begin{pmatrix} -8 & -10 \\ 5 & 7 \end{pmatrix} \qquad y(0)=\begin{pmatrix} 3 & 1 \end{pmatrix}^T$

9. Find the real and imaginary part of $z(t)=e^{2it}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$

10. Two tanks, each containing 360 liters of a salt solution. Pure water pours into tank *A* at a rate of 5 *L/min*. There are two pipes connecting tank *A* to tank *B*. The first pumps salt solution from tank *B* into tank *A* at a rate of 4 *L/min*. The second pumps salt solution from tank *A* into tank *B* at a rate of 9 *L/min*. Finally, there is a drain on tank *B* from which salt solution drains at a rate of 5 *L/min*. Thus, each tank maintains a constant volume of 360 *liters* of salt solution. Initially, there are 60 *kg* of salt present in tank *A*, but tank *B* contains pure water.



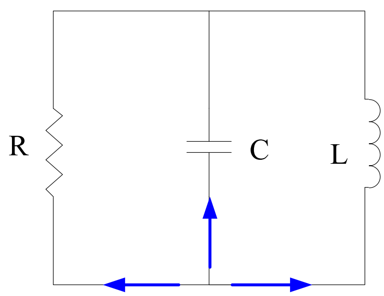
a) Set up, in matrix-vector form, an initial value problem that models the salt content in each tank over time.

b) Find the eigenvalues and eigenvectors of the coefficient matrix in part (*a*), then find the general solution in vector form. Find the solution that satisfies the initial conditions posed in part (*a*).

c) Plot each component of your solution in part (*b*) over a period of four time constants $\left[0,\ 4T_c\right]$. What is the eventual salt content in each tank? Give both a physical and a mathematical reason for your answer.

15. Consider the *RLC* parallel circuit below. Let *V* represent the voltage drop across the capacitor and *I* represent the current across the inductor that satisfied the system.

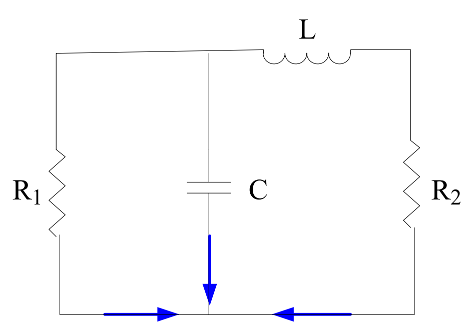
$$\begin{cases} V' = -\dfrac{V}{RC} - \dfrac{1}{C} \\ I' = \dfrac{V}{L} \end{cases}$$



Suppose that the resistance is $R = \frac{1}{2}\Omega$, the capacitor is *C* = 1 *farad*, and the inductance is $L = \frac{1}{2}$ *henry*. If the initial voltage across the capacitor is $V(0) = 10$ *volts* and there is no initial current across the inductor, solve the system to determine the voltage and current as a function of time. Plot the voltage and current as a function of time. Assume current flows in the directions indicated.

16. Show that the voltage *V* across the capacitor and the current *I* through the inductor satisfy the system

$$\begin{cases} I' = -\dfrac{R_1}{L} I + \dfrac{1}{L} V \\ V' = -\dfrac{1}{C} I - \dfrac{1}{R_2 C} V \end{cases}$$



Suppose that the capacitance is *C* = 1 *farad*, the inductance is $L = 1$ *henry*, the leftmost resistor has resistance $R_2 = 1\ \Omega$, and the rightmost resistor has resistance $R_1 = 5\ \Omega$. If the initial voltage across the capacitor is 12 volts and the initial current through the inductor is zero, determine the voltage *V* across the capacitor and the current *I* through the inductor as functions of time. Plot the voltage and current as functions of time. Assume current flows in the directions indicated.

Section 3.10 – Phase Plane Portraits

Equilibrium Points (*Review*)

The dynamical behavior of a linear system is easier than non-linear system. We need to determine a set of points to satisfy the autonomous system $y' = 0\ \left(y' = f(y(t), t) \equiv 0\right)$. These set of points are called ***equilibrium points***.

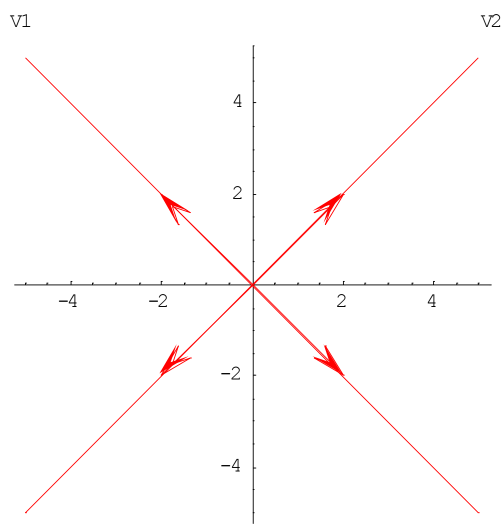
From these equilibrium points, we can determine the stability of the system.

The equilibrium point O_1 is the intersection of the eigenvectors, and we can plot those two lines by joining these points V_1O_1 and V_2O_1 together.

The general solution for the system is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 V_1 e^{\lambda_1 t} + C_2 V_2 e^{\lambda_2 t}$$

The behavior of the system or the solutions is depending on the value of λ_1 and λ_2, and if they are real or complex values.



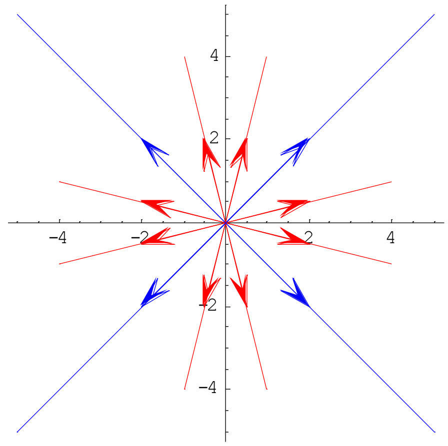
Eigenvectors V_1 and V_2 plot.

The family of all solution curves without the presence of the independent variable is **called** ***phase portrait***.

Stability of the equilibrium point condition

- An equilibrium point is *stable* if all nearby solutions stay nearby
- An equilibrium point is *asymptotically stable* if all nearby solutions not only stay nearby, but also tend the equilibrium point.

Case 1: If $\lambda_1 > 0$ and $\lambda_2 > 0$ are real values.



$\lambda_1 > 0$ and $\lambda_2 > 0$ <u>*source*</u> or <u>*repel*</u> (unstable at point (0, 0))

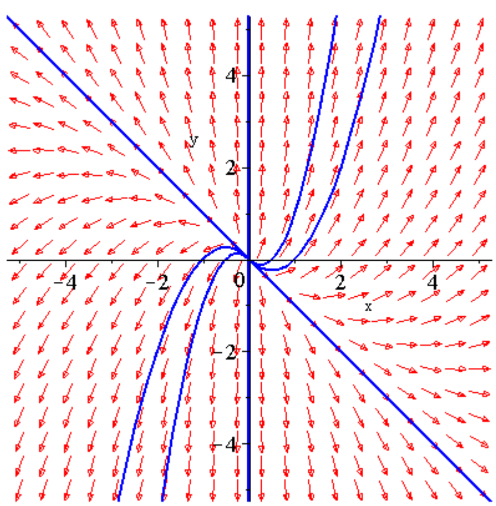
The system is unstable and the solution as the time go by, will diverge away from the equilibrium point

Example

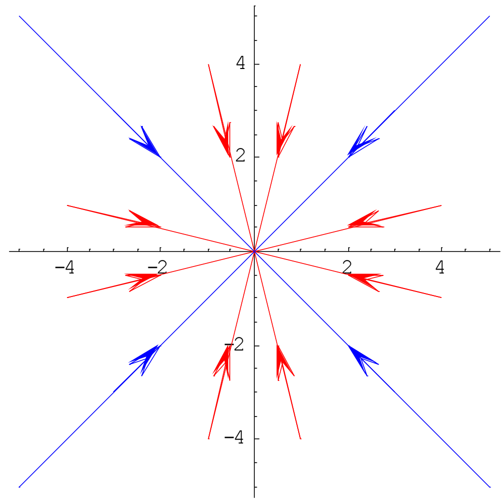
$$y' = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} y$$

<u>***Solution***</u>

$$\begin{cases} \lambda = 1 & \rightarrow & V = \begin{pmatrix} -1 & 1 \end{pmatrix}^T \\ \lambda = 2 & \rightarrow & V = \begin{pmatrix} 0 & 1 \end{pmatrix}^T \end{cases} \qquad y(t) = C_1 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



Case 2: If $\lambda_1 \& \lambda_2 < 0$



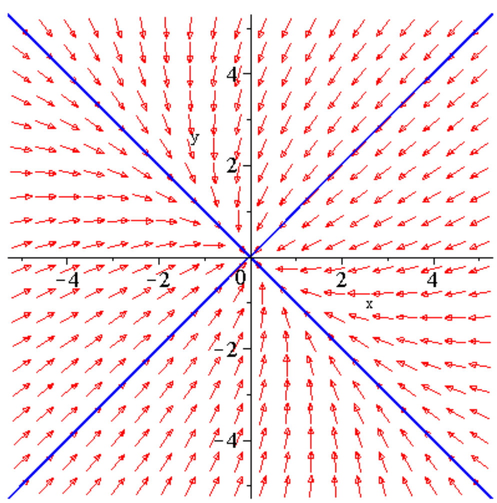
$\lambda_1 \& \lambda_2 < 0$ <u>*sink*</u> or <u>*attractor*</u> ((0, 0) is asymptotically stable) $\lambda_1 = \lambda_2 < 0$ <u>*proper node*</u>.

Example

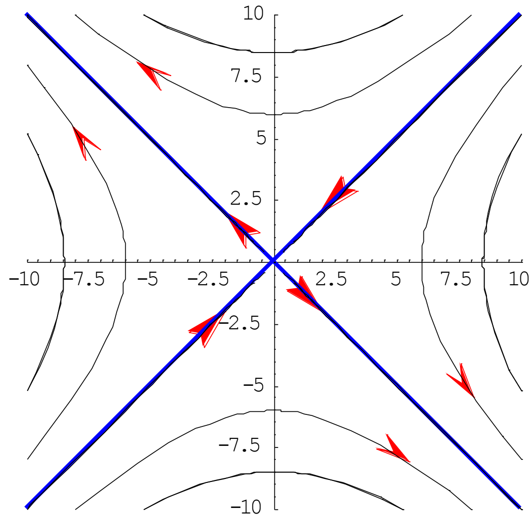
$$y' = \begin{pmatrix} -3 & -1 \\ -1 & -3 \end{pmatrix} y$$

<u>Solution</u>

$$\begin{cases} \lambda = -4 & \rightarrow \quad V = \begin{pmatrix} 1 & 1 \end{pmatrix}^T \\ \lambda = -2 & \rightarrow \quad V = \begin{pmatrix} -1 & 1 \end{pmatrix}^T \end{cases} \qquad y(t) = C_1 e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



Case 3: If $\lambda_1 > 0$ & $\lambda_2 < 0$



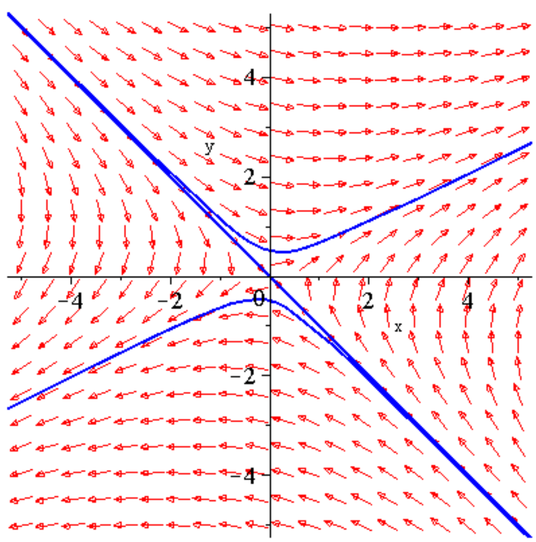
$\lambda_1 > 0$ & $\lambda_2 < 0$ A <u>*saddle point*</u>. ((0,0) is semi-stable)

Example

$$y' = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} y$$

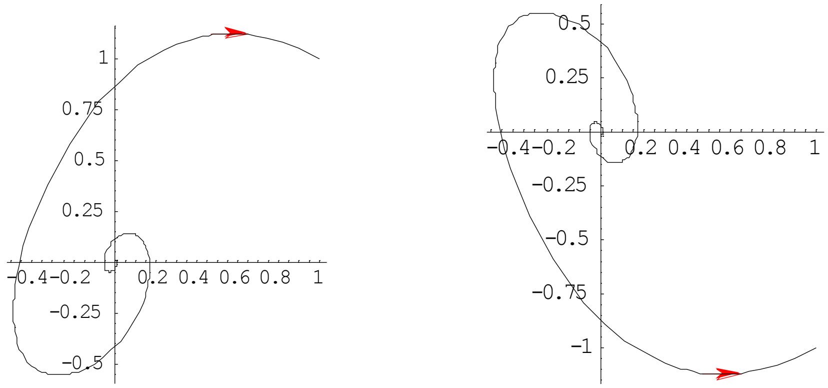
Solution

$$\begin{cases} \lambda = 3 & \rightarrow \quad V = \begin{pmatrix} 2 & 1 \end{pmatrix}^T \\ \lambda = -3 & \rightarrow \quad V = \begin{pmatrix} -1 & 1 \end{pmatrix}^T \end{cases}$$



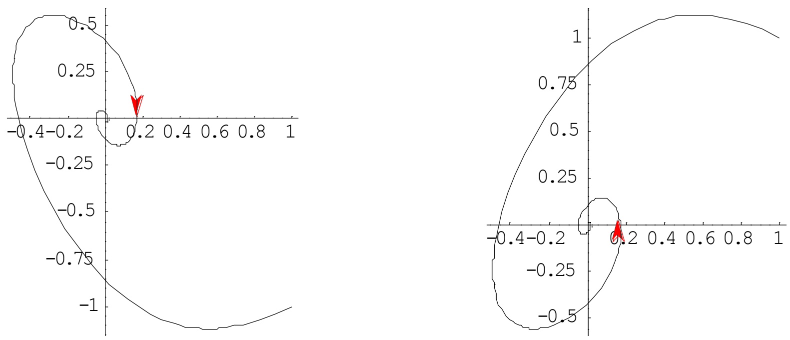
Case 4: If λ_1 & λ_2 are complex values: $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$

If $b > 0$, the behavior of the system is spiral clockwise (*cw*), then otherwise is *ccw*.

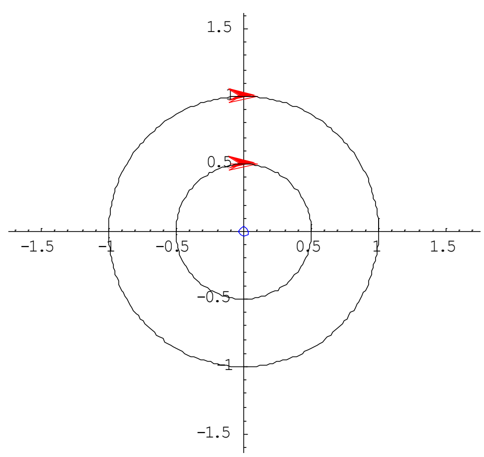


<u>*spiral out*</u>. (unstable at (0,0) point)

or



$a < 0$ <u>*spiral in*</u>. (asymptotically stable at (0,0) point)



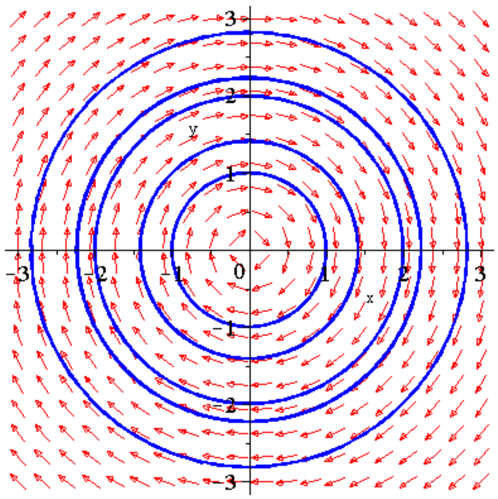
$a = 0 \quad \lambda_{1,2} = \pm\, ib$ ‘circle’ periodic solution- (0, 0) is a <u>*center*</u> stable.

Example

$$y' = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} y$$

Solution

$$\begin{cases} \lambda = 2i & \rightarrow \quad V = \begin{pmatrix} -i & 1 \end{pmatrix}^T \\ \lambda = -2i & \rightarrow \quad V = \begin{pmatrix} i & 1 \end{pmatrix}^T \end{cases} \qquad y(t) = C_1 \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + C_2 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}$$



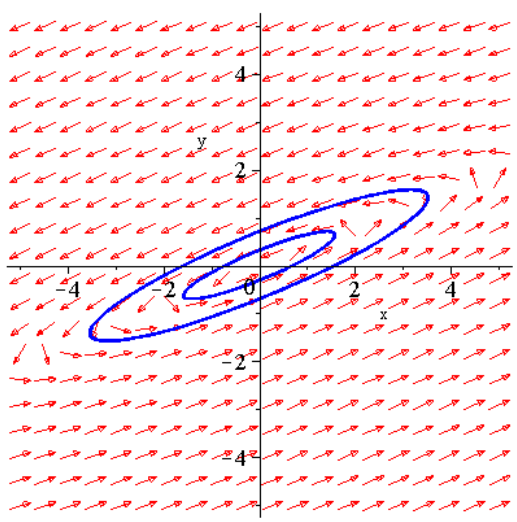
The equilibrium point is the center, but the solution curves are circles.

Example

$$y' = \begin{pmatrix} 4 & -10 \\ 2 & -4 \end{pmatrix} y$$

Solution

$$\begin{cases} \lambda = 2i & \rightarrow \quad V = \begin{pmatrix} 2+i & 1 \end{pmatrix}^T \\ \lambda = -2i & \rightarrow \quad V = \begin{pmatrix} 2-i & 1 \end{pmatrix}^T \end{cases}$$



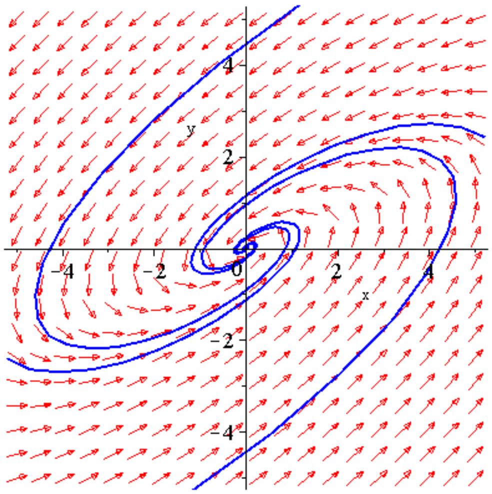
The equilibrium point is the center, but the solution curves are ellipses.

Example

$$y' = \begin{pmatrix} 1 & -4 \\ 2 & -3 \end{pmatrix} y$$

Solution

$$\begin{cases} \lambda = -1+2i & \rightarrow & V = \begin{pmatrix} 1+i & 1 \end{pmatrix}^T \\ \lambda = -1-2i & \rightarrow & V = \begin{pmatrix} 1-i & 1 \end{pmatrix}^T \end{cases}$$



The behavior of the system at the equilibrium point center is an asymptotically stable and spiral in.

Stability properties of linear systems (in 2-dimensions)

Eigenvalues	***Type of critical point***	***Stability***
$\lambda_1 > \lambda_2 > 0$	*Improper node*	*Unstable.*
$\lambda_1 < \lambda_2 < 0$	*Improper node*	*Asymptotically stable*
$\lambda_2 < 0 < \lambda_1$	*Saddle point*	*Unstable.*
$\lambda_1 = \lambda_2 > 0$	*Proper/improper node*	*Unstable*
$\lambda_1 = \lambda_2 < 0$	*Proper/improper node*	*Asymptotically stable*
$\lambda_{1,2} = a \pm ib$	*Spiral point*	
a > 0	*spiral out*	*Unstable*
a < 0	*spiral in*	*Asymptotically stable*
$\lambda_{1,2} = \pm ib$	*Center*	*Stable*

Exercises *Section* 3.10 – Phase Plane Portraits

Sketch a rough approximation of a solution in each region determined by the half-line solutions. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

1. $y(t) = C_1 e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

2. $y(t) = C_1 e^{t}\begin{pmatrix} -1 \\ -2 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

3. $y(t) = C_1 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

4. $y(t) = C_1 e^{-t}\begin{pmatrix} -5 \\ 2 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

Sketch a rough approximation of the given system. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

5. $y' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} y$

6. $y' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} y$

Calculate the eigenvalues to determine the behavior of the system whether the equilibrium point at the origin is the center, a spiral sink or a source. Calculate and sketch the vector generated by the right-hand side of the system at the point (1, 0). Use this to help sketch the solution trajectory for the system passing through the point (1, 0).

7. $y' = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} y$

8. $y' = \begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix} y$

9. $y' = \begin{pmatrix} 7 & -10 \\ 4 & -5 \end{pmatrix} y$

10. $y' = \begin{pmatrix} -4 & 8 \\ -4 & 4 \end{pmatrix} y$

11. $y' = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} y$

12. For the given system $y' = \begin{pmatrix} -1 & 6 \\ -3 & 8 \end{pmatrix} y$

a) Sketch a rough approximation of the given system. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

b) Find the solution of the initial-value problem $y(0) = (0,\ 1)^T$