

Lecture Two – Functions

Section 2.1 – Functions and Graphs

Increasing and Decreasing Functions

- ✚ A function *ris*es from left to right (*x*-coordinate), the function *f* is said to be **increasing** on an open interval *I* (*a, b*) (*x*-coordinate)

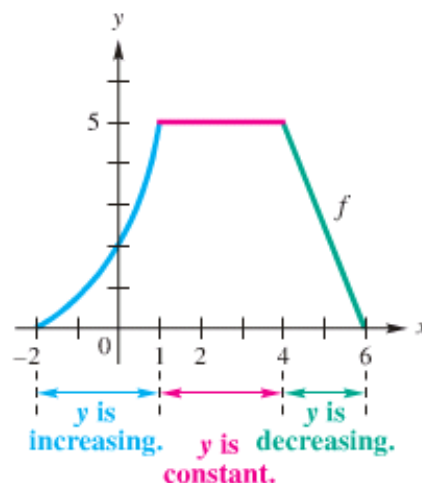
$$a < b \Rightarrow f(a) < f(b)$$

- ✚ A function *f* is said to be **decreasing** on an open interval *I*

$$a < b \Rightarrow f(a) > f(b)$$

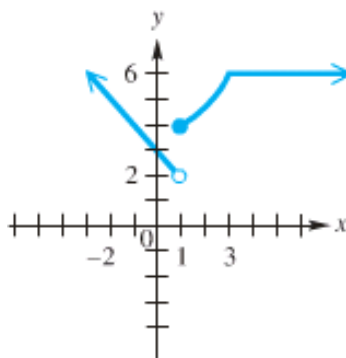
- ✚ A function *f* is said to be **constant** on an open interval *I*

$$a < b \Rightarrow f(a) = f(b)$$



Example

Determine the intervals over which the function is increasing, decreasing, or constant



Increasing: $[1, 3]$

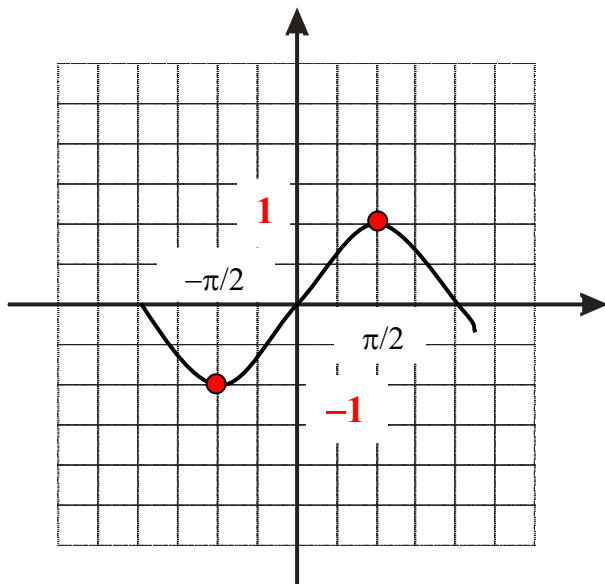
Decreasing: $(-\infty, 1)$

Constant: $[3, \infty)$

Relative *Maxima* (um) and *Minima* (um)

$f(a)$ is a relative maximum if there exists an open interval I about a such that $f(a) > f(x)$, for all x in I .

$f(a)$ is a relative minimum if there exists an open interval I about a such that $f(a) < f(x)$, for all x in I .

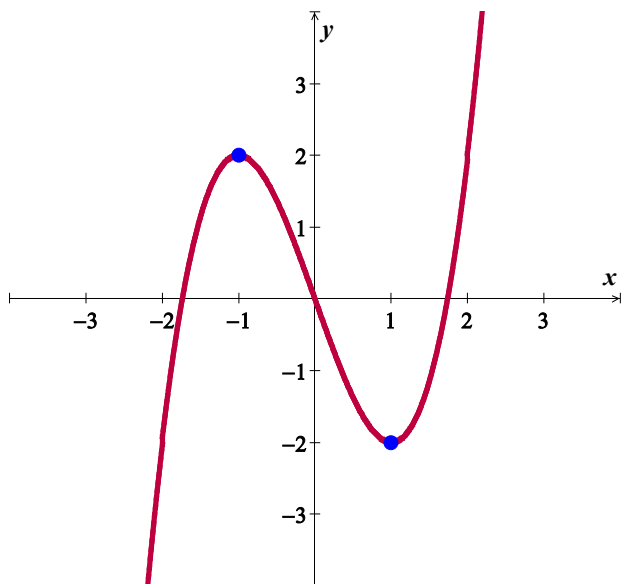


The relative minimum value of the function is -1 @ $x = -\pi/2$

The relative maximum value of the function is 1 @ $x = \pi/2$

Example

State the intervals on which the given function $f(x) = x^3 - 3x$ is increasing, decreasing, or constant, and determine the extreme values



Increasing $(-\infty, -1) \cup (1, \infty)$

Decreasing $(-1, 1)$

RMIN $(1, -2)$

RMAX $(-1, 2)$

Piecewise-Defined Functions

Function are sometimes described by more than one expression, we call such functions *piecewise-defined functions*.

Example

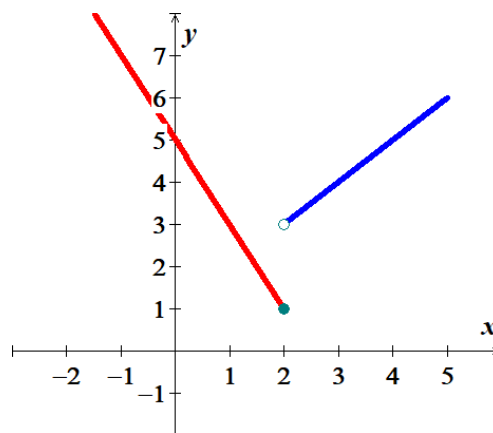
Graph function

$$f(x) = \begin{cases} -2x + 5 & \text{if } x \leq 2 \\ x + 1 & \text{if } x > 2 \end{cases}$$

Find: $f(2) = -2(\textcolor{red}{2}) + 5 = \textcolor{blue}{1}$

$$f(0) = -2(\textcolor{red}{0}) + 5 = \textcolor{blue}{5}$$

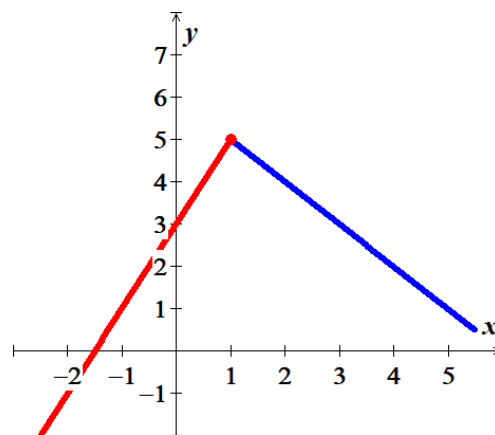
$$f(4) = \textcolor{red}{4} + 1 = \textcolor{blue}{5}$$



Example

Graph function

$$f(x) = \begin{cases} 2x + 3 & \text{if } x \leq 1 \\ -x + 6 & \text{if } x > 1 \end{cases}$$



Example

$$C(t) = \begin{cases} 20 & \text{if } 0 \leq t \leq 60 \\ 20 + 0.40(t - 60) & \text{if } t > 60 \end{cases}$$

Find $C(40)$, $C(80)$, and $C(60)$

Solution

a) $C(40) = 20$

b) $C(80) = 20 + 0.40(80 - 60) = 28$

c) $C(60) = 20$

Exercise Section 2.1 – Functions and Graphs

1. $f(x) = \begin{cases} 2+x & \text{if } x < -4 \\ -x & \text{if } -4 \leq x \leq 2 \\ 3x & \text{if } x > 2 \end{cases}$ **Find:** $f(-5)$, $f(-1)$, $f(0)$, and $f(3)$

2. $f(x) = \begin{cases} -2x & \text{if } x < -3 \\ 3x-1 & \text{if } -3 \leq x \leq 2 \\ -4x & \text{if } x > 2 \end{cases}$ **Find:** $f(-5)$, $f(-1)$, $f(0)$, and $f(3)$

3. $f(x) = \begin{cases} x^3+3 & \text{if } -2 \leq x \leq 0 \\ x+3 & \text{if } 0 < x < 1 \\ 4+x-x^2 & \text{if } 1 \leq x \leq 3 \end{cases}$ **Find:** $f(-5)$, $f(-1)$, $f(0)$, and $f(3)$

4. $h(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$ **Find:** $h(5)$, $h(0)$, and $h(3)$

5. $f(x) = \begin{cases} 3x+5 & \text{if } x < 0 \\ 4x+7 & \text{if } x \geq 0 \end{cases}$ **Find**

a) $f(0)$ b) $f(-2)$ c) $f(1)$ d) $f(3)+f(-3)$ e) Graph $f(x)$

6. $f(x) = \begin{cases} 6x-1 & \text{if } x < 0 \\ 7x+3 & \text{if } x \geq 0 \end{cases}$ **Find**

a) $f(0)$ b) $f(-1)$ c) $f(4)$ d) $f(2)+f(-2)$ e) Graph $f(x)$

7. $f(x) = \begin{cases} 2x+1 & \text{if } x \leq 1 \\ 3x-2 & \text{if } x > 1 \end{cases}$ **Find**

a) $f(0)$ b) $f(2)$ c) $f(-2)$ d) $f(1)+f(-1)$ e) Graph $f(x)$

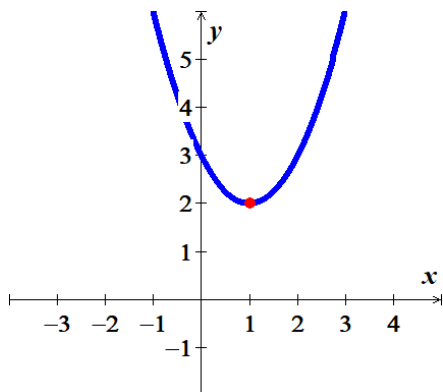
8. Graph the piecewise function defined by $f(x) = \begin{cases} 3 & \text{if } x \leq -1 \\ x-2 & \text{if } x > -1 \end{cases}$

9. Sketch the graph $f(x) = \begin{cases} x+2 & \text{if } x \leq -1 \\ x^3 & \text{if } -1 < x < 1 \\ -x+3 & \text{if } x \geq 1 \end{cases}$

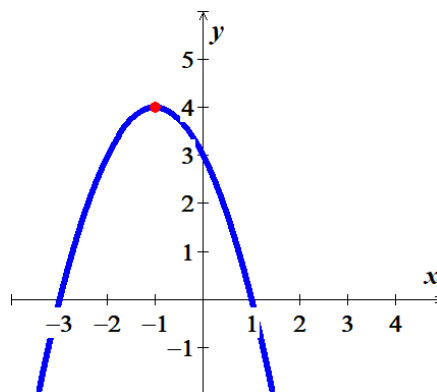
10. Sketch the graph $f(x) = \begin{cases} x-3 & \text{if } x \leq -2 \\ -x^2 & \text{if } -2 < x < 1 \\ -x+4 & \text{if } x \geq 1 \end{cases}$

(37 – 42) Determine any **relative maximum** or **minimum** of the function, determine the intervals on which the function **increasing** or **decreasing**, and then find the **domain** and the **range**.

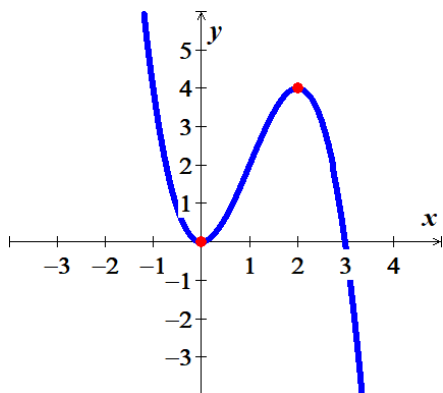
11. $f(x) = x^2 - 2x + 3$



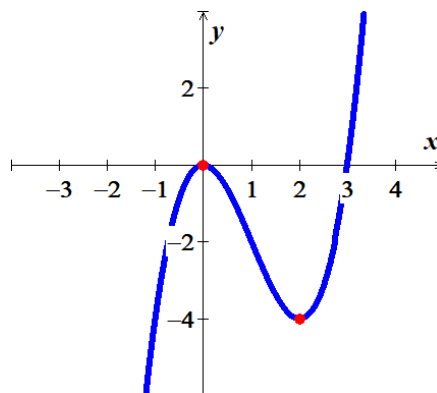
12. $f(x) = -x^2 - 2x + 3$



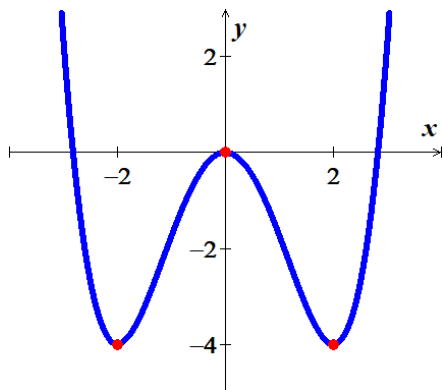
13. $f(x) = -x^3 + 3x^2$



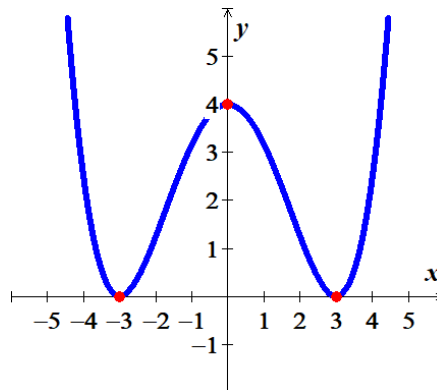
14. $f(x) = x^3 - 3x^2$



15. $f(x) = \frac{1}{4}x^4 - 2x^2$



16. $f(x) = \frac{4}{81}x^4 - \frac{8}{9}x^2 + 4$

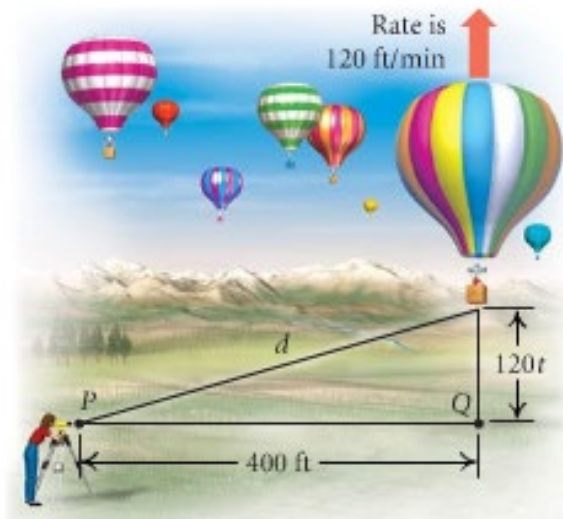


17. The elevation H , in *meters*, above sea level at which the boiling point of water is in t *degrees Celsius* is given by the function

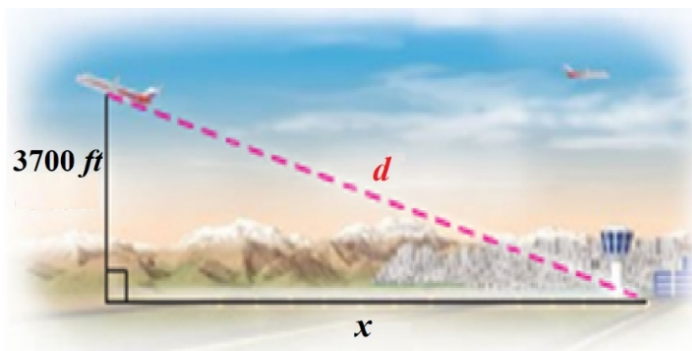
$$H(t) = 1000(100 - t) + 580(100 - t)^2$$

At what elevation is the boiling point 99.5° .

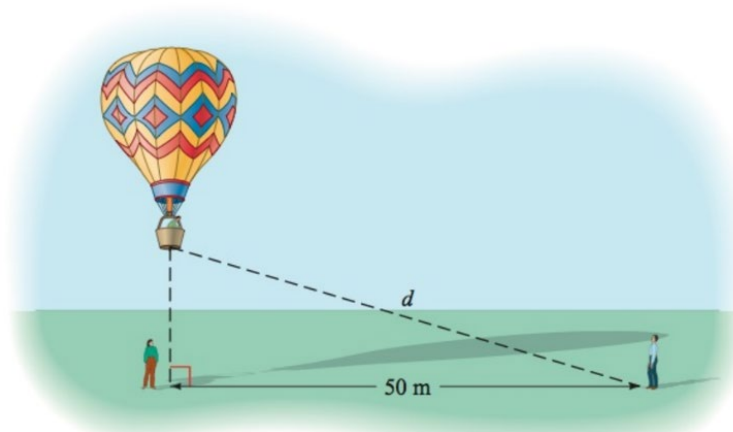
18. A hot-air balloon rises straight up from the ground at a rate of 120 ft./min. The balloon is tracked from a rangefinder on the ground at point P , which is 400 feet. from the release point Q of the balloon. Let d be the distance from the balloon to the rangefinder and t – the time, in *minutes*, since the balloon was released. Express d as a function of t .



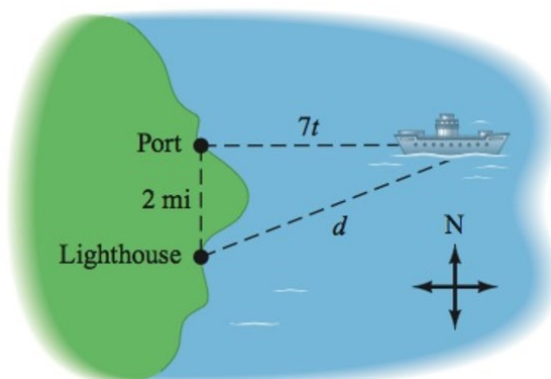
19. An airplane is flying at an altitude of 3700 feet. The slanted distance directly to the airport is $d \text{ feet.}$ Express the horizontal distance x as a function of d .



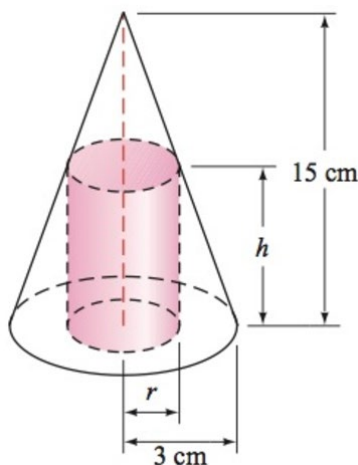
20. For the first minute of flight, a hot air balloon rises vertically at a rate of 3 m/sec . If t is the time in *seconds* that the balloon has been airborne, write the distance d between the balloon and a point on the ground 50 meters from the point to lift off as a function of t .



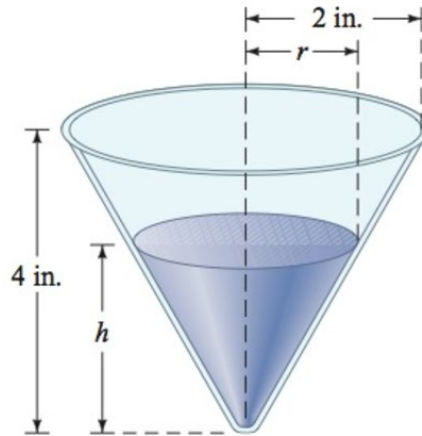
21. A light house is 2 miles south of a port. A ship leaves port and sails east at a rate of 7 miles per hour . Express the distance d between the ship and the lighthouse as a function of time, given that the ship has been sailing for $t \text{ hours}$.



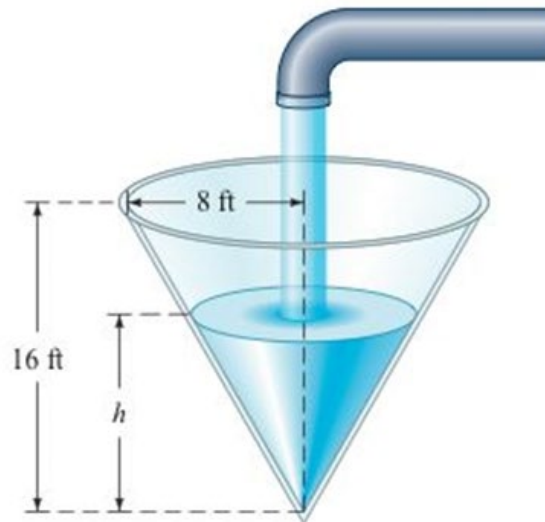
22. A cone has an altitude of 15 cm and a radius of 3 cm . A right circular cylinder of radius r and height h is inscribed in the cone. Use similar triangles to write h as a function of r .



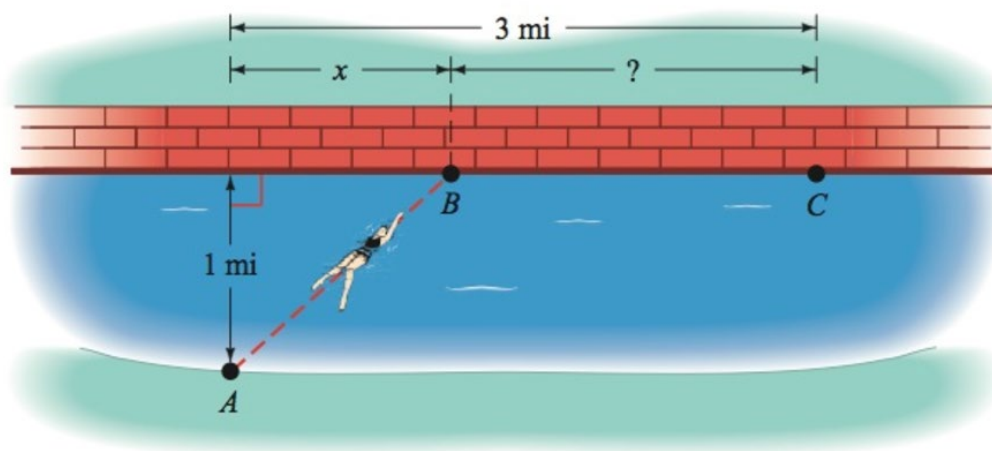
23. Water is flowing into a conical drinking cup with an altitude of 4 inches and a radius of 2 inches.



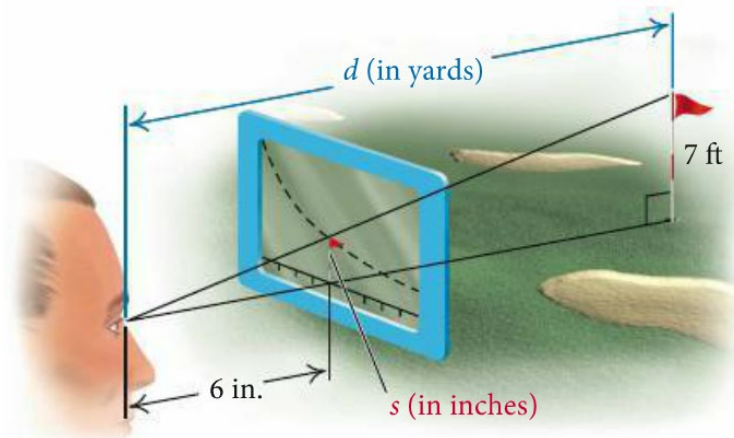
- Write the radius r of the surface of the water as a function of its depth h .
 - Write the volume V of the water as a function of its depth h .
24. A water tank has the shape of a right circular cone with height 16 feet and radius 8 feet. Water is running into the tank so that the radius r (in feet) of the surface of the water is given by $r = 1.5t$, where t is the time (in minutes) that the water has been running.



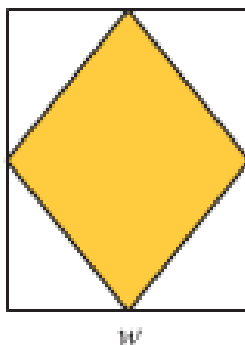
- The area A of the surface of the water is $A = \pi r^2$. Find $A(t)$ and use it to determine the area of the surface of the water when $t = 2$ minutes.
 - The volume V of the water is given by $V = \frac{1}{3}\pi r^2 h$. Find $V(t)$ and use it to determine the volume of the water when $t = 3$ minutes.
25. An athlete swims from point A to point B at a rate of 2 miles per hour and runs from point B to point C at a rate of 8 miles per hour. Use the dimensions in the figure to write the time t required to reach point C as a function of x .



26. A device used in golf to estimate the distance d , in yards, to a hole measures the size s , in inches, that the 7-foot pin appears to be in a viewfinder. Express the distance d as a function of s .



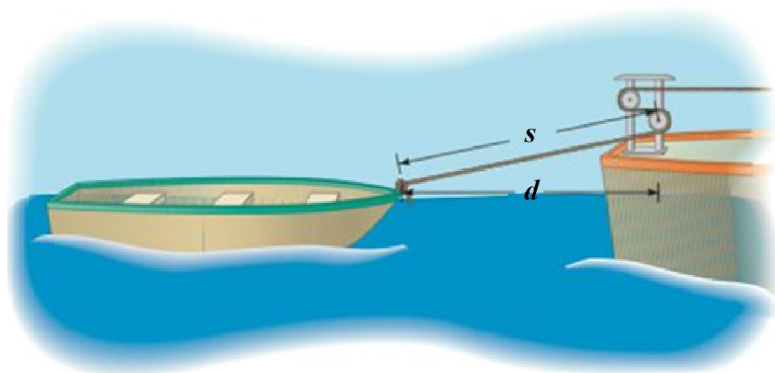
27. A rhombus is inscribed in a rectangle that is w meters wide with a perimeter of 40 m. Each vertex of the rhombus is a midpoint of a side of the rectangle. Express the area of the rhombus as a function of the rectangle's width.



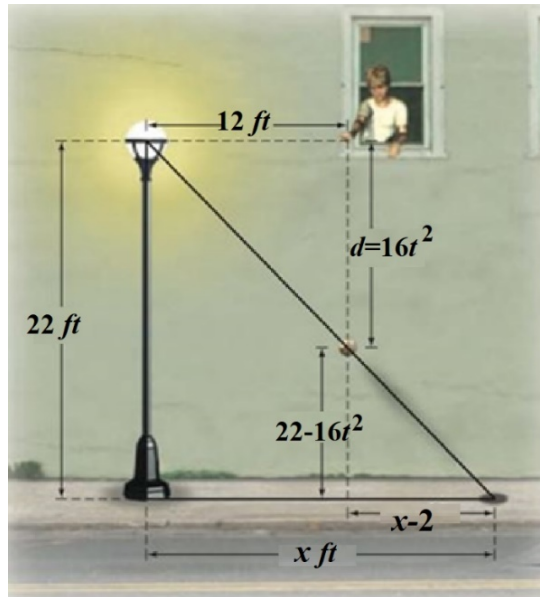
28. The surface area S of a right circular cylinder is given by the formula $S = 2\pi rh + 2\pi r^2$. If the height is twice the radius, find each of the following.



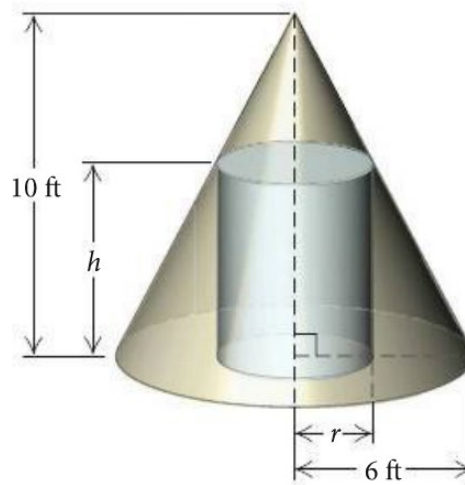
- a) A function $S(r)$ for the surface area as a function of r .
b) A function $S(h)$ for the surface area as a function of h .
29. A boat is towed by a rope that runs through a pulley that is 4 feet above the point where the rope is tied to the boat. The length (in feet) of the rope from the boat to the pulley is given by $s = 48 - t$, where t is the time in seconds that the boat has been in tow. The horizontal distance from the pulley to the boat is d .



- a) Find $d(t)$
b) Evaluate $s(35)$ and $d(35)$
30. The light from a lamppost casts a shadow from a ball that was dropped from a height of 22 feet above the ground. The distance d , in feet, the ball has dropped t seconds after it is released is given by $d(t) = 16t^2$. Find the distance x , in feet, of the shadow from the base of the lamppost as a function of time t .



31. *A right circular cylinder of height h and a radius r is inscribed in a right circular cone with a height of 10 feet and a base with radius 6 feet.



- Express the height h of the cylinder as a function of r .
- Express the volume V of the cylinder as a function of r .
- Express the volume V of the cylinder as a function of h .

Section 2.2 – Function Operations

The *Domain* of a Function

1. **Rational** function: $\frac{f(x)}{h(x)} \Rightarrow \text{Domain: } \boxed{h(x) \neq 0}$

Example: $f(x) = \frac{1}{x-3}$

Domain: $x \neq 3 \mid \{x \mid x \neq 3\}$

Or $(-\infty, 3) \cup (3, \infty)$ *Interval Notation*

Or $\mathbb{R} - \{3\}$

2. **Irrational** function: $\sqrt{g(x)} \Rightarrow \text{Domain: } \boxed{g(x) \geq 0}$

Example: $g(x) = \sqrt{3-x} + 5$

$$3 - x \geq 0$$

$$-x \geq -3$$

Domain: $x < 3 \mid (-\infty, 3]$

3. **Otherwise:** Domain all real numbers $(-\infty, \infty)$

Example: $f(x) = x^3 + |x|$

Domain: All real numbers $\mathbb{R} \mid (-\infty, \infty)$

(1) & (2) \rightarrow Find the domain: $f(x) = \frac{x+1}{\sqrt{x-3}}$

$$x > 3$$

Domain: $(3, \infty)$

Example

Find the domain

a) $f(x) = x^2 + 3x - 17$

Domain: \mathbb{R}

b) $g(x) = \frac{5x}{x^2 - 49}$

$$x^2 \neq 49$$

$$\underline{x \neq \pm 7}$$

Domain: $\begin{cases} \{x \mid x \neq \pm 7\} \\ (-\infty, -7) \cup (-7, 7) \cup (7, \infty) \end{cases}$ **or**

c) $h(x) = \sqrt{9x - 27}$

$$9x - 27 \geq 0$$

$$9x \geq 27$$

Domain: $\underline{x \geq 3}$ $[3, \infty)$

The *Algebra* of Functions

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Example

Let $f(x) = x^2 + 1$ and $g(x) = 3x + 5$. Find each of the following $(f + g)(1)$, $(f - g)(-3)$, $(fg)(5)$, and $\left(\frac{f}{g}\right)(0)$

Solution

$$\begin{aligned}(f + g)(1) &= f(1) + g(1) \\ &= 1^2 + 1 + 3(1) + 5 \\ &= 1 + 1 + 3 + 5 \\ &= 10\end{aligned}$$

$$\begin{aligned}(f - g)(-3) &= f(-3) - g(-3) \\ &= (-3)^2 + 1 - (3(-3) + 5) \\ &= 14\end{aligned}$$

$$\begin{aligned}(fg)(5) &= f(5) \cdot g(5) \\ &= (5^2 + 1) \cdot (3(5) + 5) \\ &= (26) \cdot (20) \\ &= 520\end{aligned}$$

$$\begin{aligned}\left(\frac{f}{g}\right)(0) &= \frac{f(0)}{g(0)} \\ &= \frac{0^2 + 1}{3(0) + 5} \\ &= \frac{1}{5}\end{aligned}$$

Example

Let $f(x) = 8x - 9$ and $g(x) = \sqrt{2x - 1}$. Find each of the following and give the domain

$$(f + g)(x), \quad (f - g)(x), \quad (fg)(x), \quad \left(\frac{f}{g}\right)(x)$$

Solution

Domain of f : $(-\infty, \infty)$

Domain of g : $\left[\frac{1}{2}, \infty\right)$ $\sqrt{2x-1} \geq 0 \rightarrow 2x \geq 1 \Rightarrow x \geq \frac{1}{2}$

a) $(f + g)(x) = 8x - 9 + \sqrt{2x - 1}$

Domain: $\underline{x \geq \frac{1}{2}} \quad \left[\frac{1}{2}, \infty\right)$

b) $(f - g)(x) = 8x - 9 - \sqrt{2x - 1}$

Domain: $\underline{x \geq \frac{1}{2}} \quad \left[\frac{1}{2}, \infty\right)$

c) $(fg)(x) = (8x - 9)\sqrt{2x - 1}$

Domain: $\underline{x \geq \frac{1}{2}} \quad \left[\frac{1}{2}, \infty\right)$

d) $\left(\frac{f}{g}\right)(x) = \frac{8x - 9}{\sqrt{2x - 1}}$

Domain: $\underline{x > \frac{1}{2}} \quad \left(\frac{1}{2}, \infty\right)$

Example

Let $f(x) = \sqrt{x - 3}$ and $g(x) = \sqrt{x + 1}$

Find $(f + g)(x)$ and its domain, $\left(\frac{f}{g}\right)(x)$ and its domain

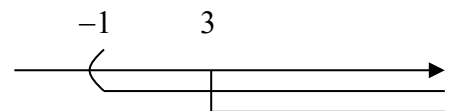
Solution

Domain $f(x)$: $x \geq 3$ and **Domain** $g(x)$: $x \geq -1$

a) $(f + g)(x) = \sqrt{x - 3} + \sqrt{x + 1}$

b) $x \geq 3$ and $x \geq -1 \Rightarrow$ **Domain:** $x \geq 3$

c) $\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x - 3}}{\sqrt{x + 1}}$



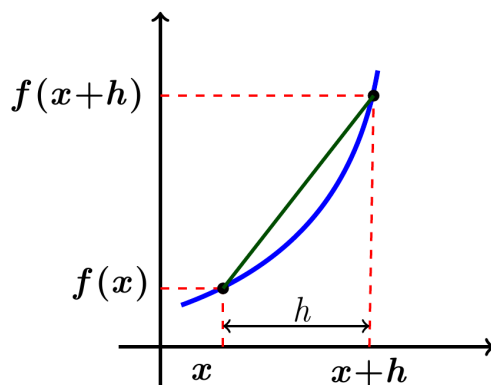
$$\rightarrow \begin{cases} x-3 \geq 0 \Rightarrow \underline{x \geq 3} \\ x+1 > 0 \Rightarrow \underline{x > -1} \end{cases}$$

Domain: $x \geq 3$ $[3, \infty)$

Difference Quotients

$$\frac{f(x+h) - f(x)}{(x+h) - x}$$

The difference quotient is given by: $\frac{f(x+h) - f(x)}{h}$



Example

For the function f given by $f(x) = 2x - 3$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned} f(x+h) &= 2(\text{---}) - 3 \\ &= 2(x+h) - 3 \\ &= 2x + 2h - 3 \end{aligned}$$

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\underline{f(x+h)} - \underline{f(x)}}{h} \\ &= \frac{2x + 2h - 3 - (2x - 3)}{h} \\ &= \frac{2x + 2h - 3 - 2x + 3}{h} \\ &= \frac{2h}{h} \\ &= \underline{2} \end{aligned}$$

Example

For the function f given by $f(x) = -2x^2 + x + 5$, find the difference quotient $\frac{f(x+h)-f(x)}{h}$

Solution

$$f(\mathbf{x+h}) = -2(\mathbf{x+h})^2 + (\mathbf{x+h}) + 5$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

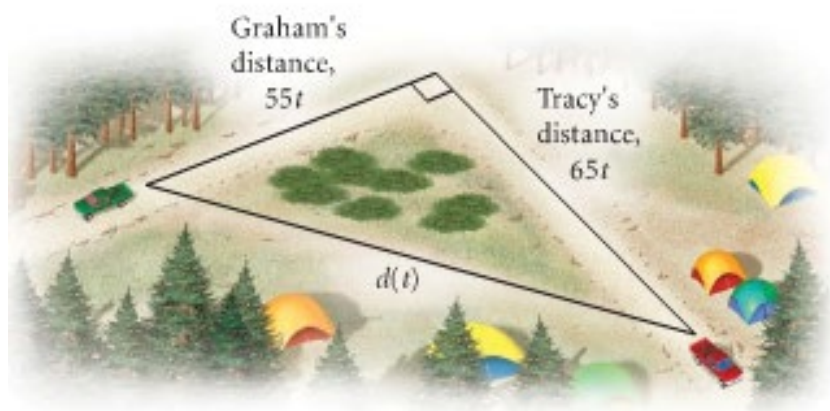
$$f(x+h) = -2(x^2 + 2hx + h^2) + x + h + 5$$

$$f(x+h) = -2x^2 - 4hx - 2h^2 + x + h + 5$$

$$\begin{aligned}\frac{f(\mathbf{x+h}) - \mathbf{f(x)}}{h} &= \frac{-2\mathbf{x^2} - 4\mathbf{hx} - 2\mathbf{h^2} + \mathbf{x+h+5} - (-2\mathbf{x^2} + \mathbf{x+5})}{h} \\&= \frac{-2x^2 - 4hx - 2h^2 + x + h + 5 + 2x^2 - x - 5}{h} \\&= \frac{-4hx - 2h^2 + h}{h} \\&= \frac{-4hx}{h} - \frac{2h^2}{h} + \frac{h}{h} \\&= \underline{-4x - 2h + 1}\end{aligned}$$

Example

Tracy and Graham drive away from a camp-ground at right angles to each other. Tracy's speed is 65 mph and Graham's is 55 mph.



- Express the distance between the cars as a function of time.
- Find the domain of the function.

Solution

a) $Distance = velocity * time$

Use Pythagorean Theorem:

$$d^2(t) = (65t)^2 + (55t)^2$$

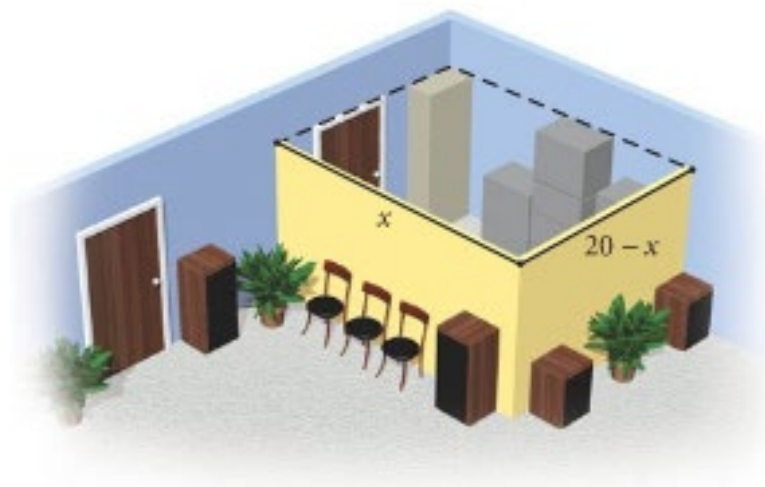
$$\begin{aligned} d^2 &= 4225t^2 + 3025t^2 \\ &= 7250t^2 \end{aligned}$$

$$\begin{aligned} d(t) &= \sqrt{7250t^2} \\ &= \sqrt{7250} \sqrt{t^2} \\ &\approx 85.15|t| \\ &= \underline{85.15 t} \end{aligned}$$

b) Domain: $t \geq 0$

Example: (storage area)

The sound Shop has 20 *feet*. of dividers with which to set off a rectangular area for the storage of overstock. If a corner of the store is used for the storage area, the partition need only form two sides of a rectangle.



- Express the floor area of the storage space as a function of the length of the partition.
- Find the domain of the function.

Solution

Let x = the length

$$\text{width} + \text{length} = 20$$

$$\text{width} = 20 - \text{length}$$

a) Area = length * width

$$= x(20 - x)$$

$$= \underline{20x - x^2}$$

b) Domain: x value varies from 0 to 20 $\Rightarrow (0, 20)$

Exercises Section 2.2 – Function Operations

(1 – 80) Find the Domain

1. $f(x) = 7x + 4$
2. $f(x) = |3x - 2|$
3. $f(x) = 3x + \pi$
4. $f(x) = \sqrt{7}x + \frac{1}{2}$
5. $f(x) = -2x^2 + 3x - 5$
6. $f(x) = x^3 - 2x^2 + x - 3$
7. $f(x) = x^2 - 2x - 15$
8. $f(x) = 4 - \frac{2}{x}$
9. $f(x) = \frac{1}{x^4}$
10. $g(x) = \frac{3}{x-4}$
11. $y = \frac{2}{x-3}$
12. $y = \frac{-7}{x-5}$
13. $f(x) = \frac{x+5}{2-x}$
14. $f(x) = \frac{8}{x+4}$
15. $f(x) = \frac{1}{x+4}$
16. $f(x) = \frac{1}{x-4}$
17. $f(x) = \frac{3x}{x+2}$
18. $f(x) = x - \frac{2}{x-3}$
19. $f(x) = x + \frac{3}{x-5}$
20. $f(x) = \frac{1}{2}x - \frac{8}{x+7}$
21. $f(x) = \frac{1}{x-3} - \frac{8}{x+7}$
22. $f(x) = \frac{1}{x+4} - \frac{2x}{x-4}$
23. $f(x) = \frac{3x^2}{x+3} - \frac{4x}{x-2}$
24. $f(x) = \frac{1}{x^2 - 2x + 1}$
25. $f(x) = \frac{x}{x^2 + 3x + 2}$
26. $f(x) = \frac{x^2}{x^2 - 5x + 4}$
27. $f(x) = \frac{1}{x^2 - 4x - 5}$
28. $g(x) = \frac{2}{x^2 + x - 12}$
29. $h(x) = \frac{5}{\frac{4}{x} - 1}$
30. $y = \sqrt{x}$
31. $f(x) = \sqrt{8 - 3x}$
32. $y = \sqrt{4x + 1}$
33. $y = \sqrt{7 - 2x}$
34. $f(x) = \sqrt{8 - x}$
35. $f(x) = \sqrt{3 - 2x}$
36. $f(x) = \sqrt{3 + 2x}$
37. $f(x) = \sqrt{5 - x}$
38. $f(x) = \sqrt{x - 5}$
39. $f(x) = \sqrt{6 - 3x}$
40. $f(x) = \sqrt{3x - 6}$
41. $f(x) = \sqrt{2x + 7}$
42. $f(x) = \sqrt{x^2 - 16}$
43. $f(x) = \sqrt{16 - x^2}$
44. $f(x) = \sqrt{9 - x^2}$
45. $f(x) = \sqrt{x^2 - 25}$
46. $f(x) = \sqrt{x^2 - 5x + 4}$
47. $f(x) = \sqrt{x^2 + 5x + 4}$
48. $f(x) = \sqrt{x^2 + 3x + 2}$
49. $f(x) = \sqrt{x^2 - 3x + 2}$
50. $f(x) = \sqrt{x-4} + \sqrt{x+1}$
51. $f(x) = \sqrt{3-x} + \sqrt{x-2}$
52. $f(x) = \sqrt{1-x} + \sqrt{4-x}$
53. $f(x) = \sqrt{1-x} - \sqrt{x-3}$
54. $f(x) = \sqrt{x+4} - \sqrt{x-1}$
55. $f(x) = \frac{\sqrt{x+1}}{x}$
56. $g(x) = \frac{\sqrt{x-3}}{x-6}$
57. $f(x) = \frac{\sqrt{x+4}}{\sqrt{x-1}}$
58. $f(x) = \frac{\sqrt{5-x}}{x}$
59. $f(x) = \frac{x}{\sqrt{5-x}}$

$$60. f(x) = \frac{1}{x\sqrt{5-x}}$$

$$67. f(x) = \frac{\sqrt{x-2}}{\sqrt{x+2}}$$

$$75. f(x) = \frac{4x}{6x^2 + 13x - 5}$$

$$61. f(x) = \frac{x+1}{x^3 - 4x}$$

$$68. f(x) = \frac{\sqrt{2-x}}{\sqrt{x+2}}$$

$$76. f(x) = \frac{\sqrt{2x-3}}{x^2 - 5x + 4}$$

$$62. f(x) = \frac{\sqrt{x+5}}{x}$$

$$69. f(x) = \frac{x-4}{\sqrt{x-2}}$$

$$77. f(x) = \frac{x^2}{\sqrt{x^2 - 5x + 4}}$$

$$63. f(x) = \frac{x}{\sqrt{x+5}}$$

$$70. f(x) = \frac{1}{(x-3)\sqrt{x+3}}$$

$$78. f(x) = \frac{x+2}{\sqrt{x^2 + 5x + 4}}$$

$$64. f(x) = \frac{1}{x\sqrt{x+5}}$$

$$71. f(x) = \sqrt{x+2} + \sqrt{2-x}$$

$$79. f(x) = \frac{\sqrt{x+2}}{\sqrt{x^2 + 3x + 2}}$$

$$65. f(x) = \frac{x+3}{\sqrt{x-3}}$$

$$72. f(x) = \sqrt{(x-2)(x-6)}$$

$$80. f(x) = \frac{\sqrt{2x+3}}{x^2 - 6x + 5}$$

$$66. f(x) = \frac{\sqrt{x+3}}{\sqrt{x-3}}$$

$$74. f(x) = \frac{\sqrt{4x-3}}{x^2 - 4}$$

81. Let $f(x) = 4x - 3$ and $g(x) = 5x + 7$. Find each of the following and give the domain

$$a) (f+g)(x) \quad b) (f-g)(x) \quad c) (fg)(x) \quad d) \left(\frac{f}{g}\right)(x)$$

82. Let $f(x) = 2x^2 + 3$ and $g(x) = 3x - 4$. Find each of the following and give the domain

$$a) (f+g)(x) \quad b) (f-g)(x) \quad c) (fg)(x) \quad d) \left(\frac{f}{g}\right)(x)$$

83. Let $f(x) = x^2 - 2x - 3$ and $g(x) = x^2 + 3x - 2$. Find each of the following and give the domain

$$a) (f+g)(x) \quad b) (f-g)(x) \quad c) (fg)(x) \quad d) \left(\frac{f}{g}\right)(x)$$

84. Let $f(x) = \sqrt{4x-1}$ and $g(x) = \frac{1}{x}$. Find each of the following and give the domain

$$a) (f+g)(x) \quad b) (f-g)(x) \quad c) (fg)(x) \quad d) \left(\frac{f}{g}\right)(x)$$

85. Given that $f(x) = x+1$ and $g(x) = \sqrt{x+3}$

$$a) \text{ Find } (f+g)(x)$$

$$b) \text{ Find the domain of } (f+g)(x)$$

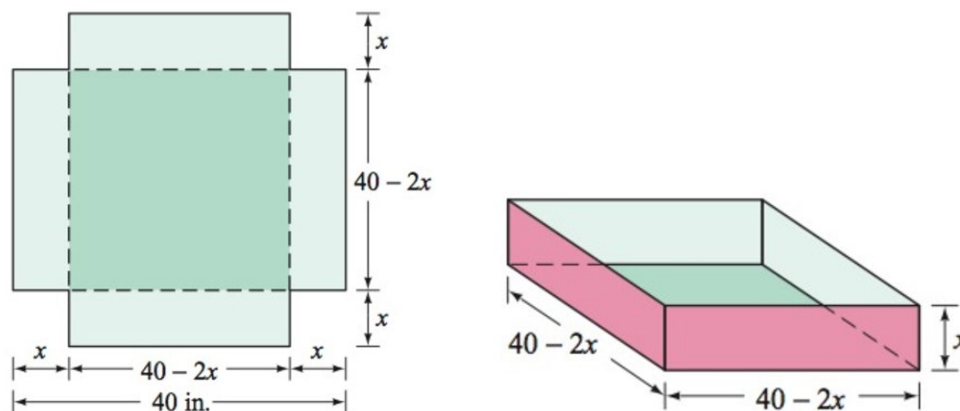
$$c) \text{ Find: } (f+g)(6)$$

86. Given that $f(x) = x^2 - 4$ and $g(x) = x + 2$
- Find $(f + g)(x)$ and its domain
 - Find $(f / g)(x)$ and its domain
87. Let $f(x) = x^2 + 1$ and $g(x) = 3x + 5$. Find $(f + g)(1)$, $(f - g)(-3)$, $(fg)(5)$, and $\left(\frac{f}{g}\right)(0)$
88. Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $(f / g)(x)$ and the domain of
 $f(x) = \sqrt{3 - 2x}$, $g(x) = \sqrt{x + 4}$
89. Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $(f / g)(x)$ and the domain of
 $f(x) = \frac{2x}{x - 4}$, $g(x) = \frac{x}{x + 5}$
90. Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $(f / g)(x)$ of $f(x) = x - 5$ and $g(x) = x^2 - 1$

(88 – 103) Find and simplify the difference quotient $\frac{f(x+h) - f(x)}{h}$ for the given function

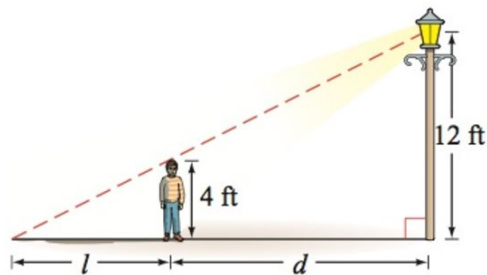
- | | | |
|----------------------|-------------------------|------------------------------|
| 91. $f(x) = 9x + 5$ | 97. $f(x) = 3x - 6$ | 102. $f(x) = 2x^2 - 3x$ |
| 92. $f(x) = 6x + 2$ | 98. $f(x) = -5x - 7$ | 103. $f(x) = 2x^2 - x - 3$ |
| 93. $f(x) = 4x + 11$ | 99. $f(x) = 2x^2$ | 104. $f(x) = x^2 - 2x + 5$ |
| 94. $f(x) = 3x - 5$ | 100. $f(x) = 5x^2$ | 105. $f(x) = 3x^2 - 2x + 5$ |
| 95. $f(x) = -2x - 3$ | 101. $f(x) = 3x^2 - 4x$ | 106. $f(x) = -2x^2 - 3x + 7$ |
| 96. $f(x) = -4x + 3$ | | |

107. An open box is to be made from a square piece of cardboard that measures 40 inches on each side, to construct the box, squares that measure x inches on each side are cut from each corner of the cardboard.

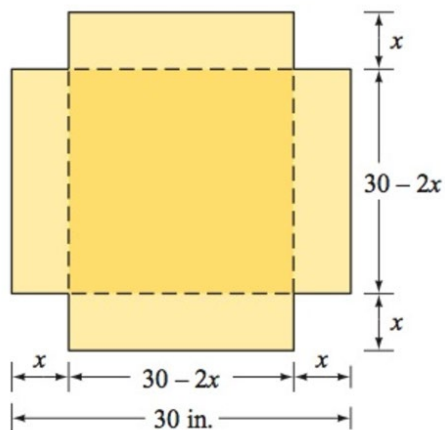


- Express the volume V of the box as a function of x .
- Determine the domain of V .

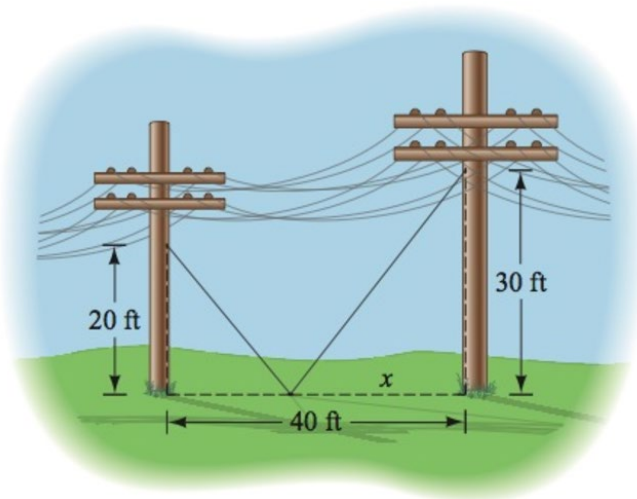
108. A child 4 feet tall is standing near a street lamp that is 12 feet high. The light from the lamp casts a shadow.



- Find the length l of the shadow as a function of the distance d of the child from the lamppost.
 - What is the domain of the function?
 - What is the length of the shadow when the child is 8 feet from the base of the lamppost?
109. An open box is to be made from a square piece of cardboard with the dimensions 30 inches by 30 inches by cutting out squares of area x^2 from each corner.



- Express the volume V of the box as a function of x .
 - Determine the domain of V .
110. Two guy wires are attached to utility poles that are 40 feet apart.



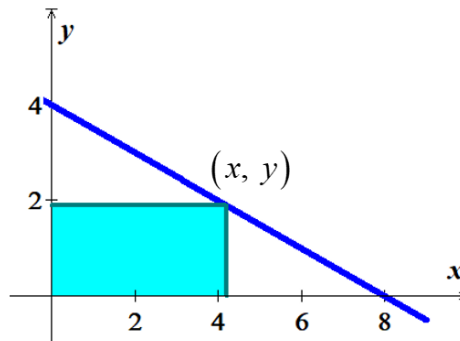
- a) Find the total length of the two guy wires as a function of x .
- b) What is the domain of this function?

- 111.** A rancher has 360 yards. of fencing with which to enclose two adjacent rectangular corrals, one for sheep and one for cattle. A river forms one side of the corrals. Suppose the width of each corral is x yards.



- a) Express the total area of the two corrals as a function of x .
- b) Find the domain of the function.

- 112.** A rectangle is bounded by the x - and y -axis of $y = -\frac{1}{2}x + 4$



- a) Find the area of the rectangle as a function of x .
- b) What is the domain of this function.

Section 2.3 – Composition Functions

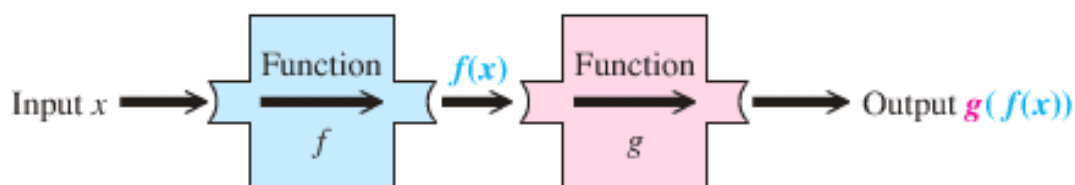
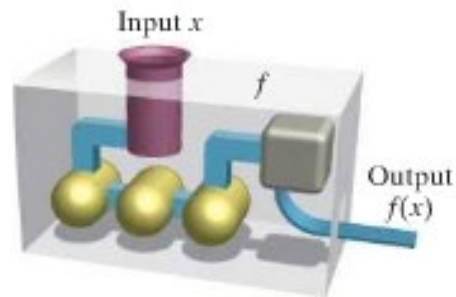
Composition of Functions

The composite function $g \circ f$, the composite of f and g , is defined as

$$(g \circ f)(x) = g(f(x))$$

Where x is in the domain of f

and $g(x)$ is in the domain of f



Example

Given that $f(x) = 5x + 6$ and $g(x) = 2x^2 - x - 1$, find $(f \circ g)(x)$ and $(g \circ f)(x)$

Solution

$$(f \circ g)(x) = f(g(x)) = f(2x^2 - x - 1) \quad \text{Domain: All real numbers}$$

$$= 5(\text{-----}) + 6$$

$$= 5(2x^2 - x - 1) + 6$$

$$= 10x^2 - 5x - 5 + 6$$

$$= \underline{10x^2 - 5x + 1}$$

Domain: All real numbers

$$(g \circ f)(x) = g(f(x))$$

$$= g(5x + 6)$$

Domain: All real numbers

$$= 2(\quad)^2 - (\quad) - 1$$

$$= 2(5x + 6)^2 - (5x + 6) - 1$$

$$= 2(25x^2 + 60x + 36) - 5x - 6 - 1$$

$$= 50x^2 + 120x + 72 - 5x - 7$$

$$= \underline{50x^2 + 115x + 65}$$

Domain: All real numbers

Example

Let $f(x) = \sqrt{x}$ and $g(x) = 4x + 2$, find each of the following and its domain.

a) $(f \circ g)(x)$

b) $(g \circ f)(x)$

Solution

$$\begin{aligned} \text{a) } (f \circ g)(x) &= f(g(x)) \\ &= f(4x + 2) & (-\infty, \infty) \\ &= \sqrt{4x + 2} \\ & \quad 4x + 2 \geq 0 \\ & \quad 4x \geq -2 \\ & \quad x \geq -\frac{2}{4} \end{aligned}$$

$$\text{Domain: } \underline{x \geq -\frac{1}{2}} \mid \left[-\frac{1}{2}, \infty \right)$$

$$\begin{aligned} \text{b) } (g \circ f)(x) &= g(f(x)) \\ &= g(\sqrt{x}) & x \geq 0 \\ &= 4\sqrt{x} + 2 & x \geq 0 \end{aligned}$$

$$\text{Domain: } \underline{x \geq 0} \mid [0, \infty)$$

Example

Let $f(x) = 2x - 1$ and $g(x) = \frac{4}{x-1}$ Find:

a) $(f \circ g)(2)$

b) $(g \circ f)(-3)$

Solution

$$\begin{aligned} \text{a) } (f \circ g)(2) &= f(g(2)) \\ &= f\left(\frac{4}{2-1}\right) \\ &= f(4) \\ &= 2(4) - 1 \\ &= 7 \end{aligned}$$

$$\begin{aligned} \text{b) } (g \circ f)(-3) &= g(f(-3)) \\ &= g(2(-3) - 1) \end{aligned}$$

$$\begin{aligned}
 &= g(-7) \\
 &= \frac{4}{-7-1} \\
 &= \frac{4}{-8} \\
 &= -\frac{1}{2}
 \end{aligned}$$

Example

Given that $f(x) = \frac{4}{x+2}$ and $g(x) = \frac{1}{x}$, find

a) $(f \circ g)(x)$

b) Domain of $(f \circ g)(x)$

Solution

a) $(f \circ g)(x) = f(g(x))$

$$= f\left(\frac{1}{x}\right)$$

Domain:: $x \neq 0$

$$= \frac{4}{\frac{1}{x} + 2}$$

$$= \frac{4}{\frac{1+2x}{x}}$$

$$= 4 \div \frac{1+2x}{x}$$

$$= 4 \frac{x}{1+2x}$$

$$= \frac{4x}{1+2x}$$

Domain:: $x \neq -\frac{1}{2}$

b) Domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right) \cup (0, \infty)$

Exercises Section 2.3 – Composition Functions

1. Given that $f(x) = 2x - 5$ and $g(x) = x^2 - 3x + 8$, find $(f \circ g)(x)$, $(g \circ f)(x)$ and their domain then find $(f \circ g)(7)$
2. Given that $f(x) = \sqrt{x}$ and $g(x) = x - 1$, find
 - a) $(f \circ g)(x) = f(g(x))$
 - b) $(g \circ f)(x) = g(f(x))$
 - c) $(f \circ g)(2) = f(g(2))$
3. Given that $f(x) = \frac{x}{x+5}$ and $g(x) = \frac{6}{x}$, find
 - a) $(f \circ g)(x) = f(g(x))$
 - b) $(g \circ f)(x) = g(f(x))$
 - c) $(f \circ g)(2) = f(g(2))$
4. Find $(f \circ g)(x)$, $(g \circ f)(x)$, $f(g(-2))$ and $g(f(3))$: $f(x) = 2x^2 + 3x - 4$, $g(x) = 2x - 1$
5. Find $(f \circ g)(x)$, $(g \circ f)(x)$, $f(g(-2))$ and $g(f(3))$: $f(x) = x^3 + 2x^2$, $g(x) = 3x$
6. Find $(f \circ g)(x)$, $(g \circ f)(x)$, $f(g(-2))$ and $g(f(3))$: $f(x) = |x|$, $g(x) = -7$

(7 – 36) For the given function; find:

- a) Find $(f \circ g)(x)$ and the **domain** of $f \circ g$
 - b) Find $(g \circ f)(x)$ and the **domain** of $g \circ f$
- | | |
|--|---|
| 7. $f(x) = x - 3$ and $g(x) = x + 3$ | 15. $f(x) = 3x + 2$ and $g(x) = \sqrt{x}$ |
| 8. $f(x) = \frac{2}{3}x$ and $g(x) = \frac{3}{2}x$ | 16. $f(x) = x^4$ and $g(x) = \sqrt[4]{x}$ |
| 9. $f(x) = x - 1$ and $g(x) = 3x^2 - 2x - 1$ | 17. $f(x) = x^n$ and $g(x) = \sqrt[n]{x}$ |
| 10. $f(x) = 3x - 2$ and $g(x) = x^2 - 5$ | 18. $f(x) = x^2 - 3x$ and $g(x) = \sqrt{x+2}$ |
| 11. $f(x) = x^2 - 2$ and $g(x) = 4x - 3$ | 19. $f(x) = \sqrt{x-2}$ and $g(x) = \sqrt{x+5}$ |
| 12. $f(x) = 4x^2 - x + 10$ and $g(x) = 2x - 7$ | 20. $f(x) = x^2 + 2$ and $g(x) = \sqrt{3-x}$ |
| 13. $f(x) = \sqrt{x}$ and $g(x) = x + 3$ | 21. $f(x) = x^5 - 2$ and $g(x) = \sqrt[5]{x+2}$ |
| 14. $f(x) = \sqrt{x}$ and $g(x) = 2 - 3x$ | 22. $f(x) = 1 - x^2$ and $g(x) = \sqrt{x^2 - 25}$ |

23. $f(x) = 2x + 3$ and $g(x) = \frac{x-3}{2}$
24. $f(x) = 4x - 5$ and $g(x) = \frac{x+5}{4}$
25. $f(x) = \frac{4}{1-5x}$ and $g(x) = \frac{1}{x}$
26. $f(x) = \frac{1}{x-2}$ and $g(x) = \frac{x+2}{x}$
27. $f(x) = \frac{1}{1+x}$ and $g(x) = \frac{1-x}{x}$
28. $f(x) = \frac{3x+5}{2}$ and $g(x) = \frac{2x-5}{3}$
29. $f(x) = \frac{x-1}{x-2}$ and $g(x) = \frac{x-3}{x-4}$
30. $f(x) = \frac{6}{x-3}$ and $g(x) = \frac{1}{x}$
31. $f(x) = \frac{6}{x}$ and $g(x) = \frac{1}{2x+1}$
32. $f(x) = 3x - 7$ and $g(x) = \frac{x+7}{3}$
33. $f(x) = \frac{2x+3}{x-4}$ and $g(x) = \frac{4x+3}{x-2}$
34. $f(x) = \frac{2x+3}{x+4}$ and $g(x) = \frac{-4x+3}{x-2}$
35. $f(x) = x + 1$ and $g(x) = x^3 - 5x^2 + 3x + 7$
36. $f(x) = x - 1$ and $g(x) = x^3 + 2x^2 - 3x - 9$

(37 – 48) Evaluate each composite function, where $f(x) = 2x - 3$ and $g(x) = x^2 - 5x$

- | | | | |
|-----------------------|-----------------------|-----------------------------|------------------------|
| 37. $(f \circ g)(4)$ | 40. $(g \circ f)(-2)$ | 43. $(f \circ g)(\sqrt{2})$ | 46. $(g \circ f)(3b)$ |
| 38. $(g \circ f)(4)$ | 41. $(f \circ f)(-3)$ | 44. $(g \circ f)(\sqrt{3})$ | 47. $(f \circ g)(k+1)$ |
| 39. $(f \circ g)(-2)$ | 42. $(g \circ g)(7)$ | 45. $(f \circ g)(2a)$ | 48. $(g \circ f)(k-1)$ |

Section 2.4 – Properties of Division

Long Division

Divide $(x^3 + 2x^2 - 5x - 6) \div (x + 1)$

$$\begin{array}{r}
 \text{Quotient} \\
 \overline{x^2 + x - 6} \\
 x + 1 \overline{) x^3 + 2x^2 - 5x - 6} \quad \leftarrow \text{Dividend} \\
 \underline{x^3 + x^2} \\
 x^2 - 5x \\
 \underline{x^2 - x} \\
 -6x - 6 \\
 \underline{-6x - 6} \\
 0 \quad \leftarrow \text{Remainder}
 \end{array}$$

Divisor

$$\underline{Q(x) = x^2 + x - 6}$$

$$\underline{R(x) = 0}$$

Example

Use the long division to find the quotient and the remainder: $(x^4 - 16) \div (x^2 + 3x + 1)$

Solution

$$\begin{array}{r}
 \overline{x^2 - 3x + 8} \\
 x^2 + 3x + 1 \overline{) x^4 + 0x^3 + 0x^2 + 0x - 16} \\
 \underline{x^4 + 3x^3 + x^2} \\
 -3x^3 - x^2 \\
 \underline{-3x^3 - 9x^2 - 3x} \\
 8x^2 + 3x - 16 \\
 \underline{8x^2 + 24x + 8} \\
 -21x - 24
 \end{array}$$

$$\frac{x^4 - 16}{x^2 + 3x + 1} = x^2 - 3x + 8 + \frac{-21x - 24}{x^2 + 3x + 1}$$

$$\underline{x^4 - 16 = (x^2 + 3x + 1)(x^2 - 3x + 8) + (-21x - 24)}$$

Remainder Theorem

If a number c is substituted for x in the polynomial $f(x)$, then the result $f(c)$ is the remainder that would be obtained by dividing $f(x)$ by $x - c$.

That is, if $f(x) = (x - c)Q(x) + R(x)$ then $f(c) = R$

Example

If $f(x) = x^3 - 3x^2 + x + 5$, use the remainder theorem to find $f(2)$

Solution

$$\begin{array}{r} x^2 - x - 1 \\ x - 2 \overline{) x^3 - 3x^2 + x + 5} \\ \underline{x^3 - 2x^2} \\ -x^2 + x \\ \underline{-x^2 + 2x} \\ -x + 5 \\ \underline{-x + 2} \\ 3 \end{array}$$

$$\underline{f(2) = 3}$$

Factor Theorem

A polynomial $f(x)$ has a factor $x - c$ if and only if $f(c) = 0$

Example

Show that $x - 2$ is a factor of $f(x) = x^3 - 4x^2 + 3x + 2$.

Solution

$$\begin{aligned} \text{Since } f(2) &= (2)^3 - 4(2)^2 + 3(2) \\ &= 0 \end{aligned}$$

From the factor theorem; $x - 2$ is a factor of $f(x)$.

Synthetic Division

Use synthetic division to find the quotient and the remainder of $(4x^3 - 3x^2 + x + 7) \div (x - 2)$

$$\begin{array}{r|rrrr}
 & x^3 & x^2 & x^1 & x^0 \\
 2 & 4 & -3 & 1 & 7 \\
 & & 8 & 10 & 22 \\
 \hline
 & 4 & 5 & 11 & 29
 \end{array}$$

$x^2 \quad x^1 \quad x^0$

Quotient : $Q(x) = 4x^2 + 5x + 11$

Remainder : $R(x) = 29$

Example

If $f(x) = 3x^5 - 38x^3 + 5x^2 - 1$, use the synthetic division to find $f(4)$.

Solution

$$\begin{array}{r|rrrrrr}
 4 & 3 & 0 & -38 & 5 & 0 & -1 \\
 & & 12 & 48 & 40 & 180 & 720 \\
 \hline
 & 3 & 12 & 10 & 45 & 180 & 719
 \end{array}$$

$f(4) = 719$

Example

Show that -11 is a zero of the polynomial $f(x) = x^3 + 8x^2 - 29x + 44$

Solution

$$\begin{array}{r|rrrr}
 -11 & 1 & 8 & -29 & 44 \\
 & & -11 & 33 & -44 \\
 \hline
 & 1 & -3 & 4 & 0
 \end{array}$$

Thus, $f(-11) = 0$, and -11 is a zero of f .

The Rational Zeros *Theorem*

If the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ has integer coefficients and if $\frac{c}{d}$ is a rational zero of $f(x)$ such that c and d have no common prime factor, then

1. The numerator c of the zero is a factor of the constant term a_0
2. The denominator d of the zero is a factor of the leading coefficient a_n

$$\text{possible rational zeros} = \frac{\text{factors of the constant term } a_0}{\text{factors of the leading coefficient } a_n} = \frac{\text{possibilities for } a_0}{\text{possibilities for } a_n}$$

Example

Find all rational solutions of the equation: $3x^4 + 14x^3 + 14x^2 - 8x - 8 = 0$

Solution

possibilities for a_0	$\pm 1, \pm 2, \pm 4, \pm 8$
possibilities for a_n	$\pm 1, \pm 3$
possibilities for c/d	$\pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{8}{3}$

Using the calculator, the result will show that -2 is a zero.

$$\begin{array}{r|rrrrr} -2 & 3 & 14 & 14 & -8 & -8 \\ & & -6 & -16 & 4 & 8 \\ \hline & 3 & 8 & -2 & -4 & \boxed{0} \end{array}$$

We have the factorization of: $(x+2)(3x^3 + 8x^2 - 2x - 4) = 0$

$$\text{For } 3x^3 + 8x^2 - 2x - 4 \Rightarrow \frac{c}{d} = \frac{\pm 1, \pm 2, \pm 4}{\pm 1, \pm 3}$$

$x = -\frac{2}{3}$ is another solution.

$$\begin{array}{r|rrrr} -\frac{2}{3} & 3 & 8 & -2 & -4 \\ & & -2 & -4 & 4 \\ \hline & 3 & 6 & -6 & \boxed{0} \end{array}$$

We have the factorization of: $(x+2)\left(x + \frac{2}{3}\right)(3x^2 + 6x - 6) = 0$

By applying quadratic formula to solve: $3x^2 + 6x - 6 = 0 \Rightarrow x = -1 \pm \sqrt{3}$

Hence, the polynomial has two rational roots $x = -2$ and $-\frac{2}{3}$ and two irrational roots $x = -1 \pm \sqrt{3}$.

Exercises **Section 2.4 – Properties of Division**

1. Find the quotient and remainder if $f(x)$ is divided by $p(x)$:

$$f(x) = 2x^4 - x^3 + 7x - 12; \quad p(x) = x^2 - 3$$

- (2 – 4) Find the quotient and remainder if $f(x)$ is divided by $p(x)$

2. $f(x) = 3x^3 + 2x - 4; \quad p(x) = 2x^2 + 1$

3. $f(x) = 7x + 2; \quad p(x) = 2x^2 - x - 4$

4. $f(x) = 9x + 4; \quad p(x) = 2x - 5$

5. Use the remainder theorem to find $f(c)$: $f(x) = x^4 - 6x^2 + 4x - 8; \quad c = -3$

6. Use the remainder theorem to find $f(c)$: $f(x) = x^4 + 3x^2 - 12; \quad c = -2$

7. Use the factor theorem to show that $x - c$ is a factor of $f(x)$: $f(x) = x^3 + x^2 - 2x + 12; \quad c = -3$

8. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $2x^3 - 3x^2 + 4x - 5; \quad x - 2$

9. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $5x^3 - 6x^2 + 15; \quad x - 4$

10. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $9x^3 - 6x^2 + 3x - 4; \quad x - \frac{1}{3}$

- (11 – 13) Use the synthetic division to find $f(c)$:

11. $f(x) = 2x^3 + 3x^2 - 4x + 4; \quad c = 3$

12. $f(x) = 8x^5 - 3x^2 + 7; \quad c = \frac{1}{2}$

13. $f(x) = x^3 - 3x^2 - 8; \quad c = 1 + \sqrt{2}$

14. Use the synthetic division to show that c is a zero of $f(x)$:

$$f(x) = 3x^4 + 8x^3 - 2x^2 - 10x + 4; \quad c = -2$$

15. Use the synthetic division to show that c is a zero of $f(x)$:

$$f(x) = 27x^4 - 9x^3 + 3x^2 + 6x + 1; \quad c = -\frac{1}{3}$$

16. Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

$$f(x) = kx^3 + x^2 + k^2x + 3k^2 + 11; \quad x + 2$$

(17 – 62) Find all solutions of the equation

17. $x^3 - x^2 - 10x - 8 = 0$

18. $x^3 + x^2 - 14x - 24 = 0$

19. $2x^3 - 3x^2 - 17x + 30 = 0$

20. $12x^3 + 8x^2 - 3x - 2 = 0$

21. $x^3 + x^2 - 6x - 8 = 0$

22. $x^3 - 19x - 30 = 0$

23. $2x^3 + x^2 - 25x + 12 = 0$

24. $3x^3 + 11x^2 - 6x - 8 = 0$

25. $2x^3 + 9x^2 - 2x - 9 = 0$

26. $x^3 + 3x^2 - 6x - 8 = 0$

27. $3x^3 - x^2 - 6x + 2 = 0$

28. $x^3 - 8x^2 + 8x + 24 = 0$

29. $x^3 - 7x^2 - 7x + 69 = 0$

30. $x^3 - 3x - 2 = 0$

31. $x^3 - 2x + 1 = 0$

32. $x^3 - 2x^2 - 11x + 12 = 0$

33. $x^3 - 2x^2 - 7x - 4 = 0$

34. $x^3 - 10x - 12 = 0$

35. $x^3 - 5x^2 + 17x - 13 = 0$

36. $6x^3 + 25x^2 - 24x + 5 = 0$

37. $8x^3 + 18x^2 + 45x + 27 = 0$

38. $3x^3 - x^2 + 11x - 20 = 0$

39. $x^4 - x^3 - 9x^2 + 3x + 18 = 0$

40. $2x^4 - 9x^3 + 9x^2 + x - 3 = 0$

41. $6x^4 + 5x^3 - 17x^2 - 6x = 0$

42. $x^4 - 2x^2 - 16x - 15 = 0$

43. $x^4 - 2x^3 - 5x^2 + 8x + 4 = 0$

44. $2x^4 - 17x^3 + 4x^2 + 35x - 24 = 0$

45. $x^4 + x^3 - 3x^2 - 5x - 2 = 0$

46. $6x^4 - 17x^3 - 11x^2 + 42x = 0$

47. $x^4 - 5x^2 - 2x = 0$

48. $3x^4 - 4x^3 - 11x^2 + 16x - 4 = 0$

49. $6x^4 + 23x^3 + 19x^2 - 8x - 4 = 0$

50. $4x^4 - 12x^3 + 3x^2 + 12x - 7 = 0$

51. $2x^4 - 9x^3 - 2x^2 + 27x - 12 = 0$

52. $2x^4 - 19x^3 + 51x^2 - 31x + 5 = 0$

53. $4x^4 - 35x^3 + 71x^2 - 4x - 6 = 0$

54. $2x^4 + 3x^3 - 4x^2 - 3x + 2 = 0$

55. $x^4 + 3x^3 - 30x^2 - 6x + 56 = 0$

56. $6x^5 + 19x^4 + x^3 - 6x^2 = 0$

57. $3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6 = 0$

58. $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = 0$

59. $x^5 - x^4 - 7x^3 + 7x^2 + 12x - 12 = 0$

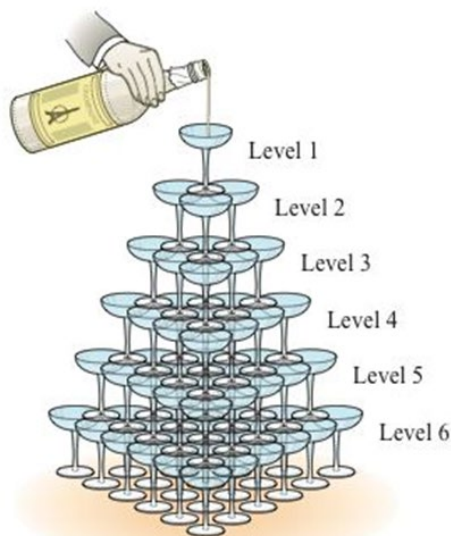
60. $x^5 - 2x^3 - 8x = 0$

61. $x^5 - 32 = 0$

62. $3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24 = 0$

63. Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

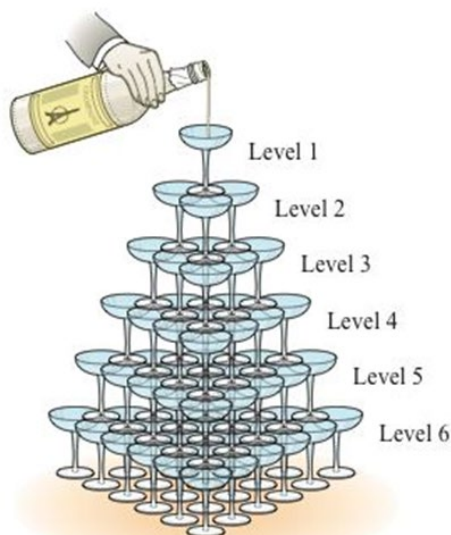
$$T(k) = \frac{1}{6}(k^3 + 3k^2 + 2k)$$



Where k is the number of levels in the pyramid. If 220 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

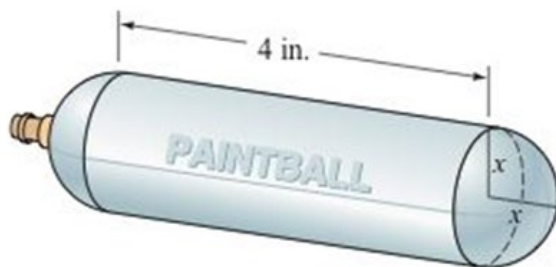
64. Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k) = \frac{1}{6}(2k^3 + 3k^2 + k)$$



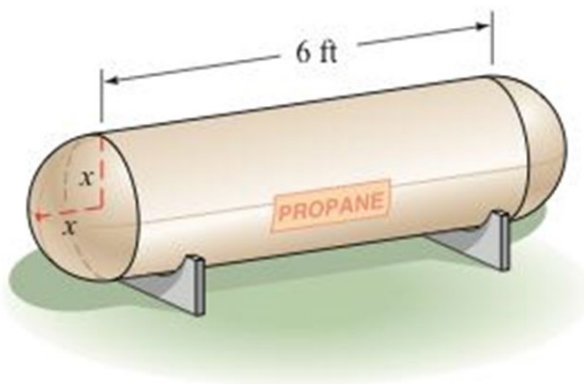
Where k is the number of levels in the pyramid. If 140 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

65. A carbon dioxide cartridge for a paintball rifle has the shape of a right circular cylinder with a hemisphere at each end. The cylinder is 4 inches long, and the volume of the cartridge is $2\pi \text{ in}^3$.

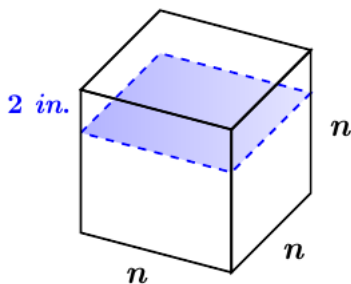


The common interior radius of the cylinder and the hemispheres is denoted by x . Estimate the length of the radius x .

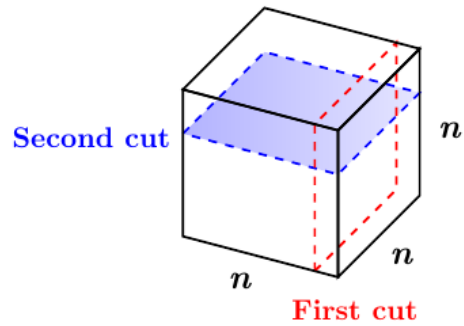
66. A propane tank has the shape of a circular cylinder with a hemisphere at each end. The cylinder is 6 feet long and the volume of the tank is $9\pi \text{ ft}^3$. Find the length of the radius x .



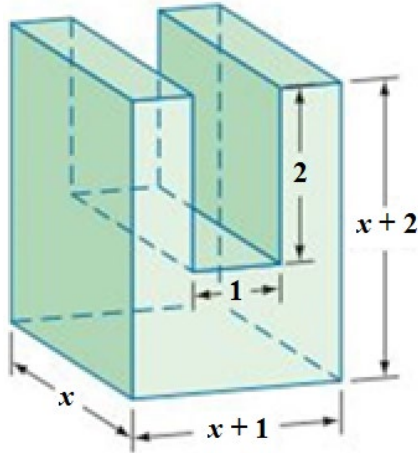
67. A cube measures n inches on each edge. If a slice 2 inches thick is cut from one face of the cube, the resulting solid has a volume of 567 in^3 . Find n .



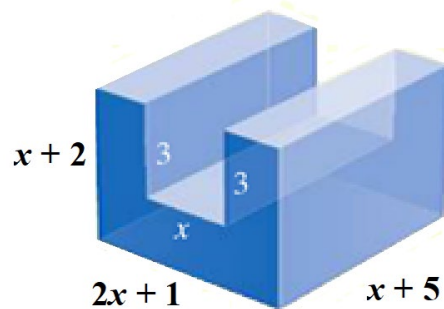
68. A cube measures n inches on each edge. If a slice 1 inch thick is cut from one face of the cube and then a slice 3 inches thick is cut from another face of the cube, the resulting solid has a volume of 1560 in^3 . Find the dimensions of the original cube.



69. For what value of x will the volume of the following solid be 112 in^3



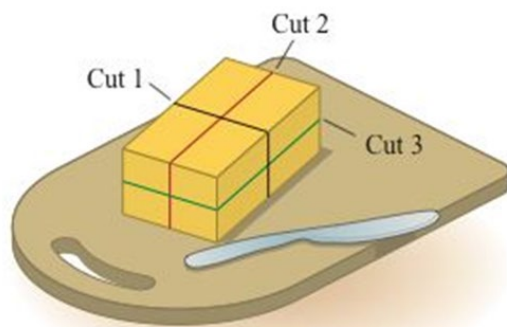
70. For what value of x will the volume of the following solid be 208 in^3



71. The length of rectangular box is 1 inch more than twice the height of the box, and the width is 3 inches more than the height. If the volume of the box is 126 in^3 , find the dimensions of the box.



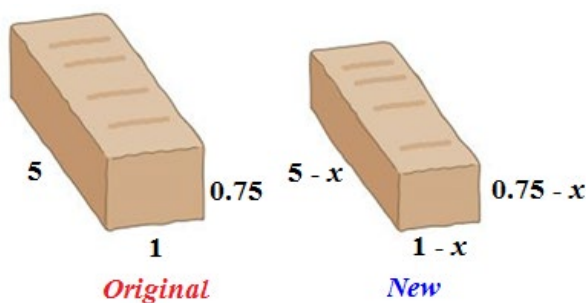
72. One straight cut through a thick piece of cheese produces two pieces. Two straight cuts can produce a maximum of four pieces. Two straight cuts can produce a maximum of four pieces. Three straight cuts can produce a maximum of eight pieces.



You might be inclined to think that every additional cut double numbers of pieces. However, for four straight cuts, you get a maximum of 15 pieces. The maximum number of pieces P that can be produced by n straight cuts is given by

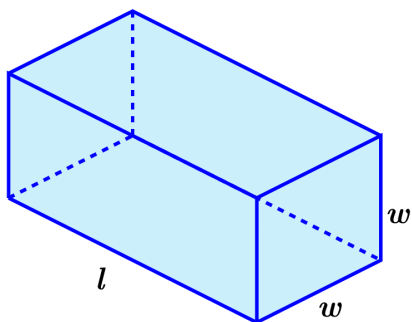
$$P(n) = \frac{n^3 + 5n + 6}{6}$$

- Determine number of pieces that can be produced by five straight cuts.
 - What is the fewest number of straight cuts that are needed to produce 64 pieces?
73. The number of ways one can select three cards from a group of n cards (the order of the selection matters), where $n \geq 3$, is given by $P(n) = n^3 - 3n^2 + 2n$. For a certain card trick, a magician has determined that there are exactly 504 ways to choose three cards from a given group. How many cards are in the group?
74. A nutrition bar in the shape of a rectangular solid measure 0.75 in. by 1 in. by 5 inches.



To reduce costs, the manufacturer has decided to decrease each of the dimensions of the nutrition bar by x inches, what value of x will produce a new bar with a volume that is 0.75 in^3 less than the present bar's volume.

75. A rectangular box is square on two ends and has length plus girth of 81 *inches*. (Girth: distance *around* the box). Determine the possible lengths l ($l > w$) of the box if its volume is 4900 in^3 .



Section 2.5 – Graphing Polynomial Functions

Polynomial Function

A *Polynomial function* $P(x)$ in x is a sum of the form is given by:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x + a_0$$

Where the coefficients $a_n, a_{n-1}, \dots, a_2, a_1, a_0$ are real numbers and the exponents are whole numbers.

$a_n x^n$
 ↑ ↙
 Leading Coefficient Leading Term

Non-polynomial Functions: $\frac{1}{x} + 2x$; $\sqrt{x^2 - 3} + x$; $\frac{x-5}{x^2+2}$

<i>Degree of f</i>	<i>Form of $f(x)$</i>	<i>Graph of $f(x)$</i>
0	$f(x) = a_0$	A horizontal line
1	$f(x) = a_1 x + a_0$	A line with slope a_1
2	$f(x) = a_2 x^2 + a_1 x + a_0$	A parabola with a vertical axis

All polynomial functions are *continuous functions*.

End Behavior ($a_n x^n$)

If n (degree) is **even**:

If $a_n < 0$ (in front x^n is negative).

Then the function falls from the left and right side

$$x \rightarrow -\infty \Rightarrow f(x) \rightarrow -\infty$$

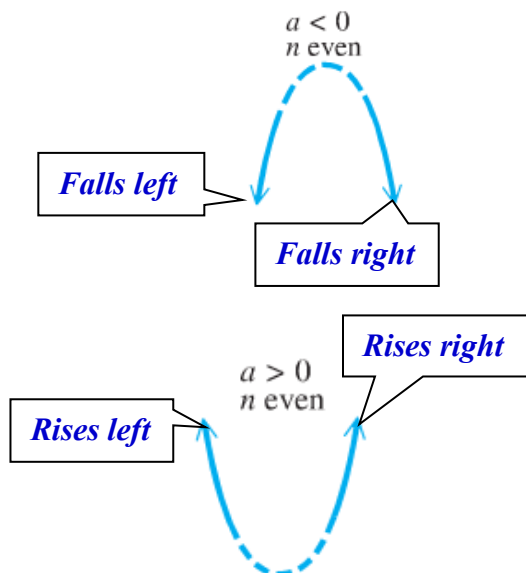
$$x \rightarrow \infty \Rightarrow f(x) \rightarrow -\infty$$

If $a_n > 0$ (in front x^n is positive).

Then the function rises from the left and right side

$$x \rightarrow -\infty \Rightarrow f(x) \rightarrow \infty$$

$$x \rightarrow \infty \Rightarrow f(x) \rightarrow \infty$$



If n (degree) is **odd**:

If $a_n < 0$ (negative).

Then the function rises from the left side and falls from the right side

$$x \rightarrow -\infty \Rightarrow f(x) \rightarrow \infty$$

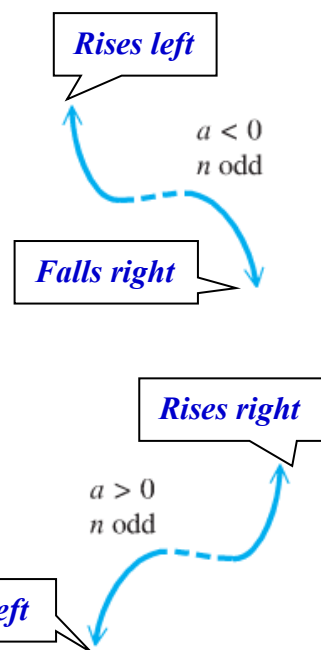
$$x \rightarrow \infty \Rightarrow f(x) \rightarrow -\infty$$

If $a_n > 0$ (positive).

Then the function falls from the left side and rises from the right side

$$x \rightarrow -\infty \Rightarrow f(x) \rightarrow -\infty$$

$$x \rightarrow \infty \Rightarrow f(x) \rightarrow \infty$$



Example

Determine the end behavior of the graph of the polynomial function $f(x) = -4x^5 + 7x^2 - x + 9$

Solution

Leading term: $-4x^5$ with 5th degree (n is odd)

$$x \rightarrow -\infty \Rightarrow f(x) = -(-)^5 = (-)(-) = + \rightarrow \infty \quad f(x) \text{ rises left}$$

$$x \rightarrow \infty \Rightarrow f(x) \rightarrow -\infty \quad f(x) \text{ falls right}$$

The Intermediate Value *Theorem*

For any polynomial function $f(x)$ with real coefficients and $f(a) \neq f(b)$ for $a < b$, then f takes on every value between $f(a)$ and $f(b)$ in the interval $[a, b]$.

$\therefore f(a)$ and $f(b)$ are the **opposite signs**. Then the function has a real zero between a and b .

Example

Using the intermediate value theorem, determine, if possible, whether the function has a real zero between a and b .

a) $f(x) = x^3 + x^2 - 6x$; $a = -4$, $b = -2$

b) $f(x) = x^3 + x^2 - 6x$; $a = -1$, $b = 3$

Solution

a) $f(x) = x^3 + x^2 - 6x$; $a = -4$, $b = -2$

$$f(-4) = (-4)^3 + (-4)^2 - 6(-4) \\ = -24$$

$$f(-2) = (-2)^3 + (-2)^2 - 6(-2) \\ = 8$$

$\therefore f(x)$ has a zero between -4 and -2

b) $f(x) = x^3 + x^2 - 6x$; $a = -1$, $b = 3$

$$f(-1) = (-1)^3 + (-1)^2 - 6(-1) \\ = 6$$

$$f(3) = (3)^3 + (3)^2 - 6(3) = 18 \\ = 18$$

$\therefore f(x)$ zeros *can't be determined*

Example

Show that $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3$ has a zero between 1 and 2.

Solution

$$f(1) = 1 + 2 - 6 + 2 - 3 \\ = -4$$

$$f(2) = (2)^5 + 2(2)^4 - 6(2)^3 + 2(2) - 3 \\ = 17$$

Since $f(1)$ and $f(2)$ have opposite signs.

Therefore, $f(c) = 0$ for at least one real number c between 1 and 2.

Sketching

Example

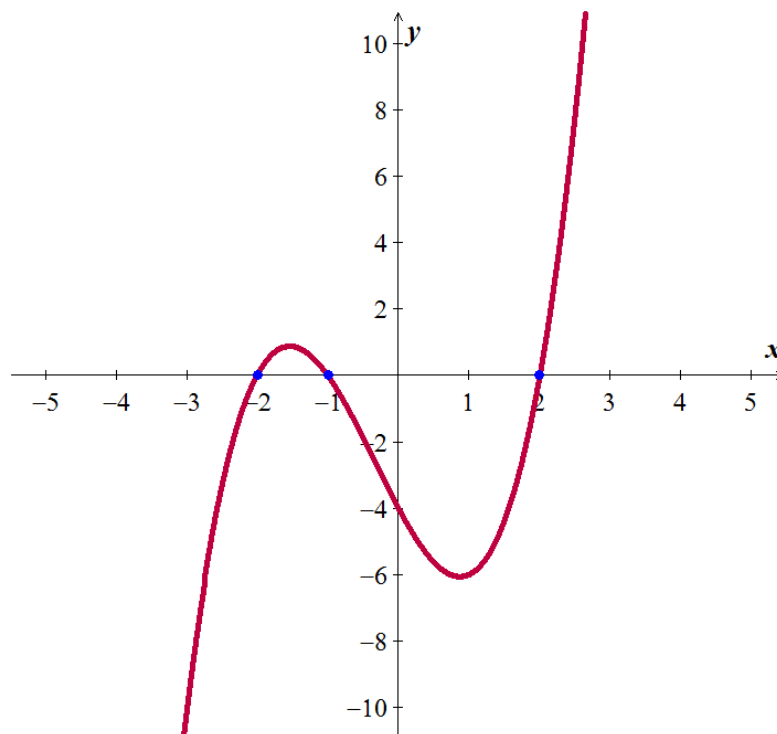
Let $f(x) = x^3 + x^2 - 4x - 4$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f .

Solution

$$\begin{aligned}f(x) &= x^3 + x^2 - 4x - 4 \\&= x^2(x+1) - 4(x+1) \\&= (x+1)(x^2 - 4) \\&= (x+1)(x+2)(x-2)\end{aligned}$$

The zeros of $f(x)$ (x -intercepts) are: -2 , -1 , and 2

<i>Interval</i>	$-\infty$	-2	-1	0	2	∞
Sign of $f(x)$		−	+	−	+	
Position		Below x-axis	Above x-axis	Below x-axis	Above x-axis	



We can conclude from the chart and the graph that:

$$f(x) > 0 \quad \text{if } x \text{ is in } (-2, -1) \cup (2, \infty)$$

$$f(x) < 0 \quad \text{if } x \text{ is in } (-\infty, -2) \cup (-1, 2)$$

Example

Let $f(x) = x^4 - 4x^3 + 3x^2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f .

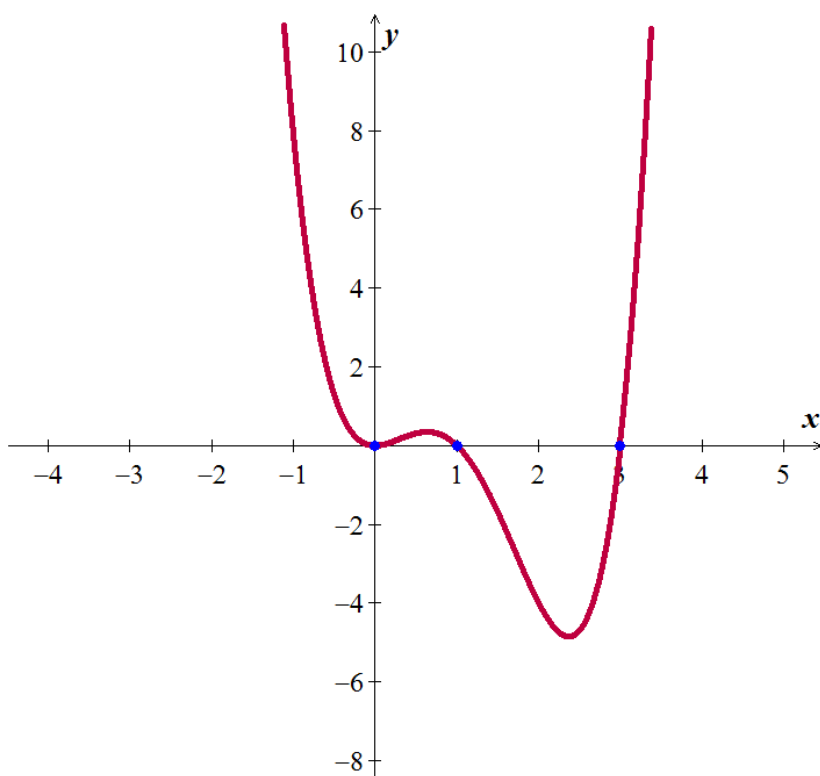
Solution

$$\begin{aligned} f(x) &= x^2(x^2 - 4x + 3) \\ &= x^2(x-1)(x-3) \end{aligned}$$

The zeros are: 0, 1, 3.

Since the factor x^2 is always positive, it has no factor

$-\infty$	1	2	3	∞
+		-		+



$$f(x) > 0 \Rightarrow x \text{ is in } (-\infty, 0) \cup (0, 1) \cup (3, \infty)$$

$$f(x) < 0 \Rightarrow x \text{ is in } (1, 3)$$

Exercises Section 2.5 – Polynomial Functions

(1 – 12) Determine the end behavior of the graph of the polynomial function

1. $f(x) = 5x^3 + 7x^2 - x + 9$

7. $f(x) = -5x^4 + 7x^2 - x + 9$

2. $f(x) = 11x^3 - 6x^2 + x + 3$

8. $f(x) = -11x^4 - 6x^2 + x + 3$

3. $f(x) = -11x^3 - 6x^2 + x + 3$

9. $f(x) = 5x^5 - 16x^2 - 20x + 64$

4. $f(x) = 2x^3 + 3x^2 - 23x - 42$

10. $f(x) = -5x^5 - 16x^2 - 20x + 64$

5. $f(x) = 5x^4 + 7x^2 - x + 9$

11. $f(x) = -3x^6 - 16x^3 + 64$

6. $f(x) = 11x^4 - 6x^2 + x + 3$

12. $f(x) = 3x^6 - 16x^3 + 4$

(13 – 32) Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers.

13. $f(x) = x^3 - x - 1$; between 1 and 2

14. $f(x) = x^3 - 4x^2 + 2$; between 0 and 1

15. $f(x) = 2x^4 - 4x^2 + 1$; between -1 and 0

16. $f(x) = x^4 + 6x^3 - 18x^2$; between 2 and 3

17. $f(x) = x^3 + x^2 - 2x + 1$; between -3 and -2

18. $f(x) = x^5 - x^3 - 1$; between 1 and 2

19. $f(x) = 3x^3 - 10x + 9$; between -3 and -2

20. $f(x) = 3x^3 - 8x^2 + x + 2$; between 2 and 3

21. $f(x) = 3x^3 - 8x^2 + x + 2$; between 1 and 2

22. $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3$; between 0 and 1

23. $P(x) = 2x^3 + 3x^2 - 23x - 42$, $a = 3$, $b = 4$

24. $P(x) = 4x^3 - x^2 - 6x + 1$, $a = 0$, $b = 1$

25. $P(x) = 3x^3 + 7x^2 + 3x + 7$, $a = -3$, $b = -2$

26. $P(x) = 2x^3 - 21x^2 - 2x + 25$, $a = 1$, $b = 2$

27. $P(x) = 4x^4 + 7x^3 - 11x^2 + 7x - 15$, $a = 1$, $b = \frac{3}{2}$

28. $P(x) = 5x^3 - 16x^2 - 20x + 64$, $a = 3$, $b = \frac{7}{2}$

$$29. \quad P(x) = x^4 - x^2 - x - 4, \quad a = 1, \quad b = 2$$

$$30. \quad P(x) = x^3 - x - 8, \quad a = 2, \quad b = 3$$

$$31. \quad P(x) = x^3 - x - 8, \quad a = 0, \quad b = 1$$

$$32. \quad P(x) = x^3 - x - 8, \quad a = 2.1, \quad b = 2.2$$

(33 – 91) Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

$$33. \quad f(x) = x^4 - 4x^2$$

$$34. \quad f(x) = x^4 + 3x^3 - 4x^2$$

$$35. \quad f(x) = x^3 + 2x^2 - 4x - 8$$

$$36. \quad f(x) = x^3 - 3x^2 - 9x + 27$$

$$37. \quad f(x) = -x^4 + 12x^2 - 27$$

$$38. \quad f(x) = x^2(x+2)(x-1)^2(x-2)$$

$$39. \quad f(x) = 2x^3 + 11x^2 - 7x - 6$$

$$40. \quad f(x) = x^3 + 2x^2 - 5x - 6$$

$$41. \quad f(x) = x^3 + 8x^2 + 11x - 20$$

$$42. \quad f(x) = x^4 + x^2 - 2$$

$$43. \quad f(x) = x^4 - x^3 - 6x^2 + 4x + 8$$

$$44. \quad f(x) = 4x^5 - 8x^4 - x + 2$$

$$45. \quad f(x) = 2x^4 - x^3 - 5x^2 + 2x + 2$$

$$46. \quad f(x) = x^3 - x^2 - 10x - 8$$

$$47. \quad f(x) = x^3 + x^2 - 14x - 24$$

$$48. \quad f(x) = 2x^3 - 3x^2 - 17x + 30$$

$$49. \quad f(x) = 12x^3 + 8x^2 - 3x - 2$$

$$50. \quad f(x) = x^3 + x^2 - 6x - 8$$

$$51. \quad f(x) = x^3 - 19x - 30$$

$$53. \quad f(x) = 3x^3 + 11x^2 - 6x - 8$$

$$54. \quad f(x) = 2x^3 + 9x^2 - 2x - 9$$

$$55. \quad f(x) = x^3 + 3x^2 - 6x - 8$$

$$56. \quad f(x) = 3x^3 - x^2 - 6x + 2$$

$$57. \quad f(x) = x^3 - 8x^2 + 8x + 24$$

$$58. \quad f(x) = x^3 - 7x^2 - 7x + 69$$

$$59. \quad f(x) = x^3 - 3x - 2$$

$$60. \quad f(x) = x^3 - 2x + 1$$

$$61. \quad f(x) = x^3 - 2x^2 - 11x + 12$$

$$62. \quad f(x) = x^3 - 2x^2 - 7x - 4$$

$$63. \quad f(x) = x^3 - 10x - 12$$

$$64. \quad f(x) = x^3 - 5x^2 + 17x - 13$$

$$65. \quad f(x) = 6x^3 + 25x^2 - 24x + 5$$

$$66. \quad f(x) = 8x^3 + 18x^2 + 45x + 27$$

$$67. \quad f(x) = 3x^3 - x^2 + 11x - 20$$

$$68. \quad f(x) = x^4 - x^3 - 9x^2 + 3x + 18$$

$$69. \quad f(x) = 2x^4 - 9x^3 + 9x^2 + x - 3$$

$$70. \quad f(x) = 6x^4 + 5x^3 - 17x^2 - 6x$$

$$71. \quad f(x) = x^4 - 2x^2 - 16x - 15$$

$$72. \quad f(x) = x^4 - 2x^3 - 5x^2 + 8x + 4$$

$$73. \quad f(x) = 2x^4 - 17x^3 + 4x^2 + 35x - 24$$

$$74. \quad f(x) = x^4 + x^3 - 3x^2 - 5x - 2$$

$$52. \quad f(x) = 2x^3 + x^2 - 25x + 12$$

$$75. \quad f(x) = 6x^4 - 17x^3 - 11x^2 + 42x$$

$$76. \quad f(x) = x^4 - 5x^2 - 2x$$

$$77. \quad f(x) = 3x^4 - 4x^3 - 11x^2 + 16x - 4$$

$$78. \quad f(x) = 6x^4 + 23x^3 + 19x^2 - 8x - 4$$

$$79. \quad f(x) = 4x^4 - 12x^3 + 3x^2 + 12x - 7$$

$$80. \quad f(x) = 2x^4 - 9x^3 - 2x^2 + 27x - 12$$

$$81. \quad f(x) = 2x^4 - 19x^3 + 51x^2 - 31x + 5$$

$$82. \quad f(x) = 4x^4 - 35x^3 + 71x^2 - 4x - 6$$

$$83. \quad f(x) = 2x^4 + 3x^3 - 4x^2 - 3x + 2$$

$$84. \quad f(x) = x^4 + 3x^3 - 30x^2 - 6x + 56$$

$$85. \quad f(x) = 3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6$$

$$86. \quad f(x) = 6x^5 + 19x^4 + x^3 - 6x^2$$

$$87. \quad f(x) = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$$

$$88. \quad f(x) = x^5 - x^4 - 7x^3 + 7x^2 + 12x - 12$$

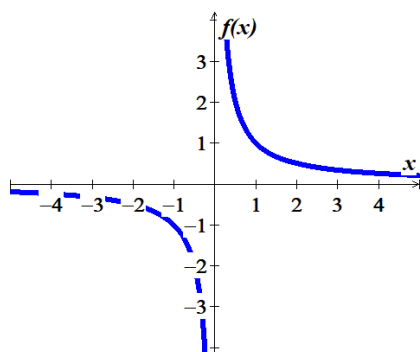
$$89. \quad f(x) = x^5 - 2x^3 - 8x$$

$$90. \quad f(x) = x^5 - 7x^4 + 19x^3 - 37x^2 + 60x - 36$$

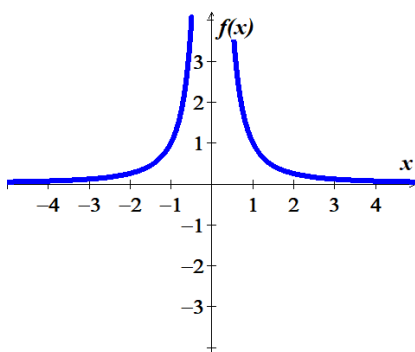
$$91. \quad f(x) = 3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24$$

Section 2.6 – Graphing Rational Functions

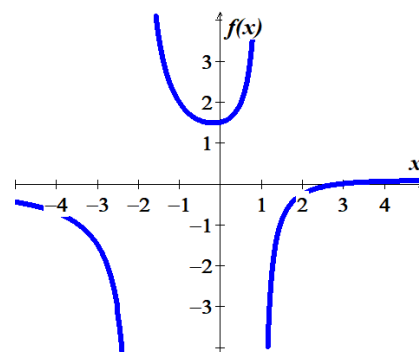
$$f(x) = \frac{1}{x}$$



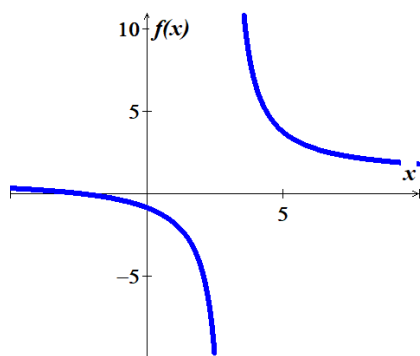
$$f(x) = \frac{1}{x^2}$$



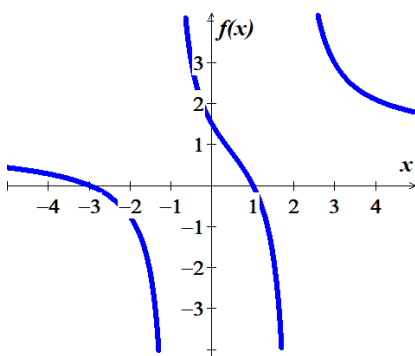
$$f(x) = \frac{x-3}{x^2+x-2}$$



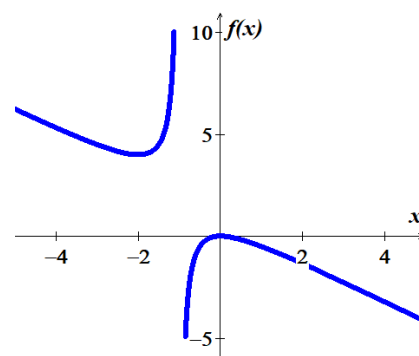
$$f(x) = \frac{2x+5}{2x-6}$$



$$f(x) = \frac{x^2+2x-3}{x^2-x-2}$$



$$f(x) = -\frac{x^2}{x+1}$$



Rational Function

A rational function is a function f that is a quotient of two polynomials, that is,

$$f(x) = \frac{g(x)}{h(x)}$$

Where $g(x)$ and $h(x)$ are polynomials. The domain of f consists of all real numbers **except** the zeros of the denominator $h(x)$.

The Domain of a Rational Function

Example

Consider: $f(x) = \frac{1}{x-3}$

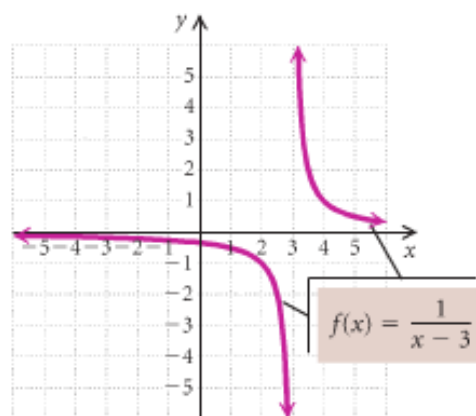
Find the domain and graph f .

Solution

$$x - 3 = 0$$

$$\underline{x = 3}$$

Thus, the domain is: $\{x | x \neq 3\}$ *or* $(-\infty, 3) \cup (3, \infty)$



<i>Function</i>	<i>Domain</i>	
$f(x) = \frac{1}{x}$	$\{x x \neq 0\}$	$(-\infty, 0) \cup (0, \infty)$
$f(x) = \frac{1}{x^2}$	$\{x x \neq 0\}$	$(-\infty, 0) \cup (0, \infty)$
$f(x) = \frac{x-3}{x^2 + x - 2}$	$\{x x \neq -2 \text{ and } x \neq 1\}$	$(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$
$f(x) = \frac{2x+5}{2x-6} = \frac{2x+5}{2(x-3)}$	$\{x x \neq 3\}$	$(-\infty, 3) \cup (3, \infty)$

Asymptotes

Vertical Asymptote (VA) - Think Domain

The line $x = a$ is a **vertical asymptote** for the graph of a function f if

$$f(x) \rightarrow \infty \quad \text{or} \quad f(x) \rightarrow -\infty$$

As x approaches a from either the left or the right

Example

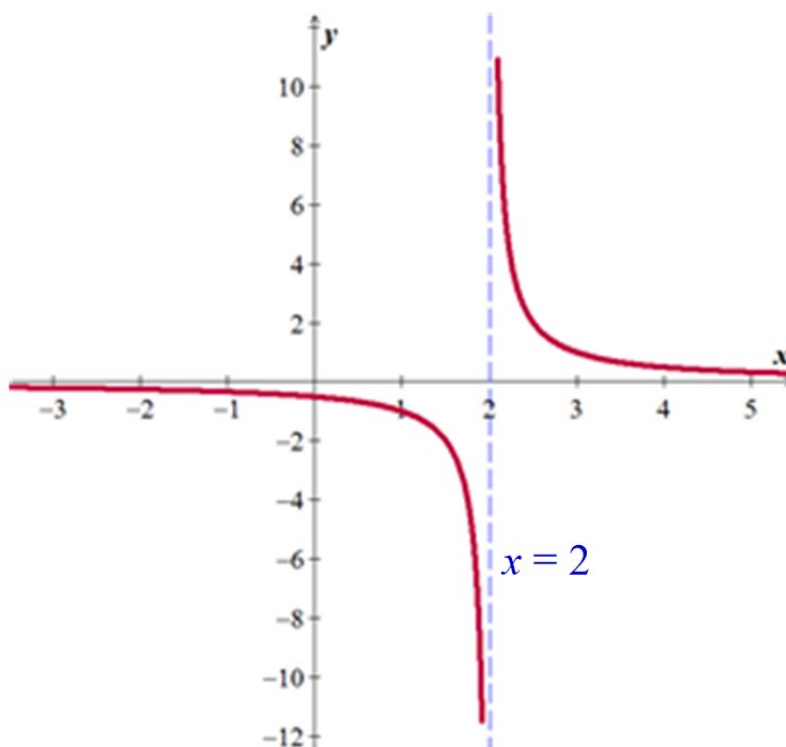
Find the vertical asymptote of $f(x) = \frac{1}{x-2}$, and sketch the graph.

Solution

$$VA: x = 2$$

$$f(x) \rightarrow \infty \quad \text{as} \quad x \rightarrow 2^+$$

$$f(x) \rightarrow -\infty \quad \text{as} \quad x \rightarrow 2^-$$



Horizontal Asymptote (**HA**)

The line $y = c$ is a **horizontal asymptote** for the graph of a function f if

$$f(x) \rightarrow c \text{ as } x \rightarrow -\infty \text{ or } x \rightarrow \infty$$

Let $f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0} = \frac{a_n x^n}{b_m x^m}$ be a rational function.

1. If the degree of numerator is less than of denominator ($n < m$) $\Rightarrow y = 0$

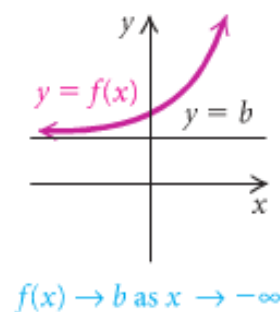
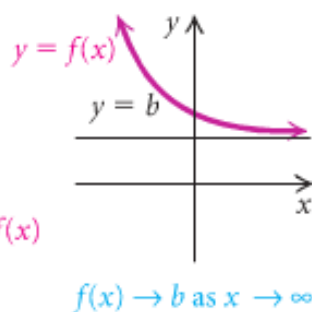
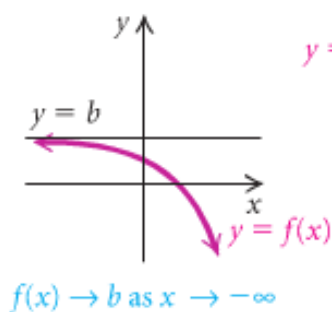
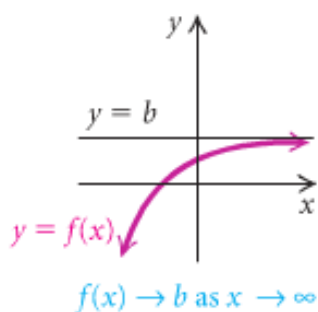
$$y = \frac{2x+1}{4x^2+5} \Rightarrow \boxed{y=0}$$

2. If the degree of numerator is equal of denominator ($n = m$) $\Rightarrow y = \frac{a_n}{b_m}$

$$y = \frac{2x^2+1}{4x^2+5} \Rightarrow \boxed{y = \frac{2}{4} = \frac{1}{2}}$$

3. If the degree of numerator is greater than of denominator ($n > m$) \Rightarrow No horizontal asymptote

$$y = \frac{2x^3+1}{4x^2+5} \Rightarrow \text{No HA}$$



Example

Determine the horizontal asymptote of $f(x) = \frac{-7x^4 - 10x^2 + 1}{11x^4 + x - 2}$

Solution

$$f(x) = \frac{-7x^4}{11x^4} = -\frac{7}{11}$$

Therefore, the horizontal asymptote (**HA**) is: $\boxed{y = -\frac{7}{11}}$

Example

Find the vertical and the horizontal asymptote for the graph of f , if it exists

$$a) \quad f(x) = \frac{3x-1}{x^2-x-6}$$

$$b) \quad f(x) = \frac{5x^2+1}{3x^2-4}$$

$$c) \quad f(x) = \frac{2x^4-3x^2+5}{x^2+1}$$

Solution

$$a) \quad f(x) = \frac{3x-1}{x^2-x-6}$$

$$x^2 - x - 6 = 0 \rightarrow x = -2, 3$$

$$VA: x = -2, \quad x = 3$$

$$HA: y = 0$$

$$b) \quad f(x) = \frac{5x^2+1}{3x^2-4}$$

$$3x^2 - 4 = 0$$

$$3x^2 = 4$$

$$x^2 = \frac{4}{3}$$

$$x = \pm \frac{2}{\sqrt{3}}$$

$$VA: x = -\frac{2}{\sqrt{3}}, \quad x = \frac{2}{\sqrt{3}}$$

$$HA: y = \frac{5}{3}$$

$$c) \quad f(x) = \frac{2x^4-3x^2+5}{x^2+1}$$

$$x^2 + 1 = 0 \rightarrow x^2 = -1$$

$$VA: n/a$$

$$HA: n/a$$

Slant or Oblique Asymptotes

When the degree of the numerator is one greater than the degree of the denominator, the graph has a slant or oblique asymptote and it is a line $y = ax + b$, $a \neq 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

$$y = \frac{3x^2 - 1}{x + 2}$$

$$\begin{array}{r} 3x - 6 \\ x + 2 \overline{) 3x^2 + 0x - 1} \\ \underline{3x^2 + 6x} \\ -6x - 12 \\ \underline{-6x - 12} \\ R = 11 \end{array}$$

$$y = \frac{3x^2 - 1}{x + 2} = 3x - 6 + \frac{11}{x + 2}$$

The **oblique asymptote** is the line $y = 3x - 6$

Example

Find all the asymptotes of $f(x) = \frac{2x^2 - 3x - 1}{x - 2}$

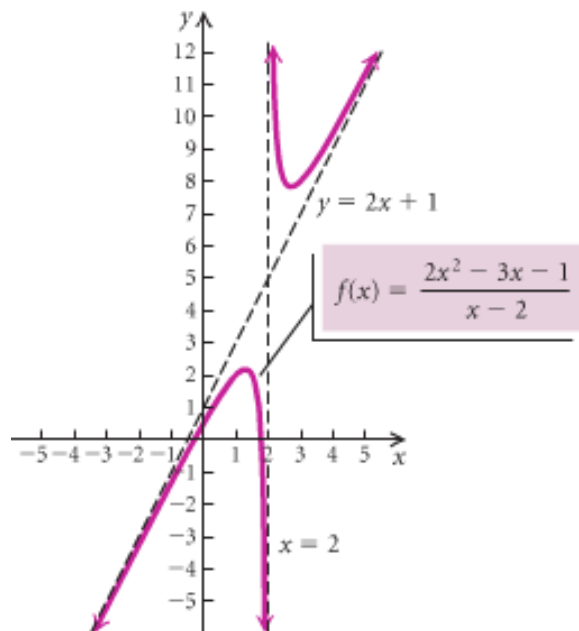
Solution

$$\begin{array}{r} 2x + 1 \\ x - 2 \overline{) 2x^2 - 3x - 1} \\ \underline{-2x^2 + 4x} \\ x - 1 \\ \underline{-x + 2} \\ 1 \end{array}$$

$$f(x) = \frac{2x^2 - 3x - 1}{x - 2} = (2x + 1) + \frac{1}{x - 2}$$

The **oblique asymptote** is the line $y = 2x + 1$

VA:: $x = 2$



Graph That Has a *Hole*

Example

Sketch the graph of g if $g(x) = \frac{3x^2 + x - 4}{2x^2 - 7x + 5}$

Solution

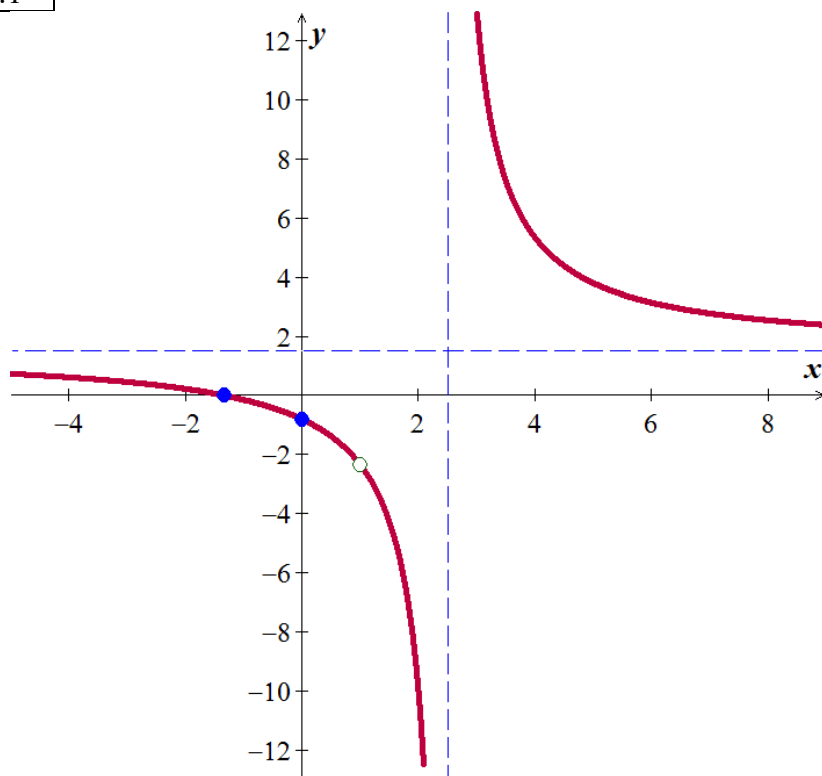
$$\begin{aligned} g(x) &= \frac{(3x+4)(x-1)}{(2x-5)(x-1)} \\ &= \frac{3x+4}{2x-5} = f(x) \end{aligned}$$

$$VA: x = \frac{5}{2}$$

$$HA: y = \frac{3}{2}$$

The only different between the graphs that g has a *hole* at $x = 1 \rightarrow f(1) = -\frac{7}{3}$

x	y
-4	.6
1.3	0
0	-.8
4	5.3
6	3.1



Exercises Section 2.6 – Rational Functions

(1 – 21) Determine all asymptotes of the function

1. $y = \frac{3x}{1-x}$

8. $y = \frac{x-3}{x^2-9}$

15. $f(x) = \frac{3-x}{(x-4)(x+6)}$

2. $y = \frac{x^2}{x^2+9}$

9. $y = \frac{6}{\sqrt{x^2-4x}}$

16. $f(x) = \frac{x^3}{2x^3-x^2-3x}$

3. $y = \frac{x-2}{x^2-4x+3}$

10. $y = \frac{5x-1}{1-3x}$

17. $f(x) = \frac{3x^2+5}{4x^2-3}$

4. $y = \frac{3}{x-5}$

11. $f(x) = \frac{2x-11}{x^2+2x-8}$

18. $f(x) = \frac{x+6}{x^3+2x^2}$

5. $y = \frac{x^3-1}{x^2+1}$

12. $f(x) = \frac{x^2-4x}{x^3-x}$

19. $f(x) = \frac{x^2+4x-1}{x+3}$

6. $y = \frac{3x^2-27}{(x+3)(2x+1)}$

13. $f(x) = \frac{x-2}{x^3-5x}$

20. $f(x) = \frac{x^2-6x}{x-5}$

7. $y = \frac{x^3+3x^2-2}{x^2-4}$

14. $f(x) = \frac{4x}{x^2+10x}$

21. $f(x) = \frac{x^3-x^2+x-4}{x^2+2x-1}$

(22 – 53) Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

22. $f(x) = \frac{-3x}{x+2}$

29. $f(x) = \frac{x-1}{1-x^2}$

36. $f(x) = \frac{1}{x-3}$

23. $f(x) = \frac{x+1}{x^2+2x-3}$

30. $f(x) = \frac{x^2+x-2}{x+2}$

37. $f(x) = \frac{-2}{x+3}$

24. $f(x) = \frac{2x^2-2x-4}{x^2+x-12}$

31. $f(x) = \frac{x^3-2x^2-4x+8}{x-2}$

38. $f(x) = \frac{x}{x+2}$

25. $f(x) = \frac{-2x^2+10x-12}{x^2+x}$

32. $f(x) = \frac{2x^2-3x-1}{x-2}$

39. $f(x) = \frac{x-5}{x+4}$

26. $f(x) = \frac{x^2-x-6}{x+1}$

33. $f(x) = \frac{2x+3}{3x^2+7x-6}$

40. $f(x) = \frac{2x^2-2}{x^2-9}$

27. $f(x) = \frac{x^3+1}{x-2}$

34. $f(x) = \frac{x^2-1}{x^2+x-6}$

41. $f(x) = \frac{x^2-3}{x^2+4}$

28. $f(x) = \frac{2x^2+x-6}{x^2+3x+2}$

35. $f(x) = \frac{-2x^2-x+15}{x^2-x-12}$

42. $f(x) = \frac{x^2+4}{x^2-3}$

$$43. \quad f(x) = \frac{x^2}{x^2 - 6x + 9}$$

$$44. \quad f(x) = \frac{x^2 + x + 4}{x^2 + 2x - 1}$$

$$45. \quad f(x) = \frac{2x^2 + 14}{x^2 - 6x + 5}$$

$$46. \quad f(x) = \frac{x^2 - 4x - 5}{2x + 5}$$

$$47. \quad f(x) = \frac{x - 3}{x^2 - 3x + 2}$$

$$48. \quad f(x) = \frac{x^2 + 2}{x^2 + 3x + 2}$$

$$49. \quad f(x) = \frac{x - 2}{x^2 - 3x + 2}$$

$$50. \quad f(x) = \frac{x^2 + x}{x + 1}$$

$$51. \quad f(x) = \frac{x^2 - 2x}{x - 2}$$

$$52. \quad f(x) = \frac{x^2 - 3x}{x + 3}$$

$$53. \quad f(x) = \frac{x^3 + 3x^2 - 4x + 6}{x + 2}$$

(54 – 59) Find an equation of a rational function f that satisfies the given conditions

$$54. \quad \begin{cases} \text{vertical asymptote: } x = 4 \\ \text{horizontal asymptote: } y = -1 \\ x\text{-intercept: } 3 \end{cases}$$

$$57. \quad \begin{cases} \text{vertical asymptote: } x = -2, x = 0 \\ \text{horizontal asymptote: } y = 0 \\ x\text{-intercept: } 2, \quad f(3) = 1 \end{cases}$$

$$55. \quad \begin{cases} \text{vertical asymptote: } x = -4, x = 5 \\ \text{horizontal asymptote: } y = \frac{3}{2} \\ x\text{-intercept: } -2 \end{cases}$$

$$58. \quad \begin{cases} \text{vertical asymptote: } x = -3, x = 1 \\ \text{horizontal asymptote: } y = 0 \\ x\text{-intercept: } -1, \quad f(0) = -2 \\ \text{hole at } x = 2 \end{cases}$$

$$56. \quad \begin{cases} \text{vertical asymptote: } x = 5 \\ \text{horizontal asymptote: } y = -1 \\ x\text{-intercept: } 2 \end{cases}$$

$$59. \quad \begin{cases} \text{vertical asymptote: } x = -1, x = 3 \\ \text{horizontal asymptote: } y = 2 \\ x\text{-intercept: } -2, 1 \\ \text{hole: } x = 0 \end{cases}$$

