Solution   *Section* 2.1 – Integration by Parts

Exercise

Evaluate the integral $\int x\ln x\, dx$

Solution

Let: $u = \ln x \Rightarrow du = \frac{1}{x}dx$

$$dv = xdx \Rightarrow v = \int dv = \int xdx = \frac{1}{2}x^2$$

$$\int x\ln xdx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x^2 \frac{dx}{x}$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\, dx$$

$$\boxed{= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}$$

Exercise

Evaluate the integral $\int \ln x^2\, dx$

Solution

$$\int \ln x^2\, dx = 2\int \ln x\, dx \qquad u = \ln x \Rightarrow du = \frac{1}{x}dx \qquad v = \int dx = x$$

$$= 2\left[x\ln x - \int x\frac{1}{x}dx\right]$$

$$= 2\left[x\ln x - \int dx\right]$$

$$= 2(x\ln x - x) + C$$

$$\boxed{= 2x(\ln x - 1) + C}$$

Exercise

Evaluate the integral $\int \ln(3x)\, dx$

Solution

$$u = \ln 3x \Rightarrow du = \frac{3}{3x}dx = \frac{1}{x}dx$$

$$dv = dx \Rightarrow v = x$$

$$\int \ln(3x)\ dx = x\ln(3x) - \int x\frac{1}{x}dx$$

$$= x\ln(3x) - \int dx$$

$$= x\ln(3x) - x + C$$

$$= x\left(\ln(3x) - 1\right) + C$$

Exercise

Evaluate the integral $\int \frac{1}{x\ln x}dx$

Solution

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{\ln x}\,d\left(\ln x\right)$$

$$= \ln\left|\ln x\right| + C$$

Exercise

Evaluate the integrals $\int x(\ln x)^2\ dx$

Solution

$$u = \ln x \quad \rightarrow \quad x = e^u$$

$$du = \frac{1}{x}dx \quad \Rightarrow \quad xdu = dx \rightarrow dx = e^u du$$

$$\int x(\ln x)^2\ dx = \int e^u u^2 e^u du$$

$$= \int u^2 e^{2u}\ du$$

$$= \frac{1}{2}u^2 e^{2u} - \frac{1}{2}ue^{2u} + \frac{1}{4}e^{2u} + C$$

$$= \frac{1}{4}e^{2u}\left(2u^2 - 2u + 1\right) + C$$

$$= \frac{1}{4}x^2\left(2(\ln x)^2 - 2\ln x + 1\right) + C$$

		$\int e^{2u}du$
+	u^2	$\frac{1}{2}e^{2u}$
−	$2u$	$\frac{1}{4}e^{2u}$
+	2	$\frac{1}{8}e^{2u}$
−	0	

2^{nd} Method

$$u = \ln x \qquad dv = \int (x\ln x)\,dx$$

$$du = \frac{1}{x}dx \quad v = \tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2$$

$$\int x(\ln x)^2\,dx = (\ln x)\left(\tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2\right) - \int \left(\tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2\right)\frac{1}{x}dx$$

$$= \tfrac{1}{2}x^2(\ln x)^2 - \tfrac{1}{4}x^2 \ln x - \int \left(\tfrac{1}{2}x \ln x - \tfrac{1}{4}x\right)dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$

$$dv = xdx \Rightarrow v = \frac{1}{2}x^2$$

$$\int x \ln xdx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x^2 \frac{dx}{x}$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int xdx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2$$

$$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{4}x^2 \ln x - \left(\frac{1}{2}\left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right) - \frac{1}{8}x^2\right) + C$$

$$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{4}x^2 \ln x - \frac{1}{4}x^2 \ln x + \frac{1}{8}x^2 + \frac{1}{8}x^2 + C$$

$$\boxed{= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + C}$$

3^{nd} Method

$$u = (\ln x)^2 \qquad dv = \int x\,dx$$

$$du = 2(\ln x)\frac{1}{x}dx \qquad v = \tfrac{1}{2}x^2$$

$$\int x(\ln x)^2\,dx = \tfrac{1}{2}x^2(\ln x)^2 - \int \tfrac{1}{2}x^2(2\ln x)\frac{1}{x}dx$$

$$= \tfrac{1}{2}x^2(\ln x)^2 - \int x \ln x\,dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$

$$dv = xdx \Rightarrow v = \frac{1}{2}x^2$$

$$\int x \ln x\,dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x^2 \frac{dx}{x}$$

$$=\frac{1}{2}x^2\ln x-\frac{1}{2}\int x\,dx$$

$$=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2$$

$$=\frac{1}{2}x^2(\ln x)^2-\frac{1}{4}x^2\ln x-\left(\frac{1}{2}x^2\ln x-\frac{1}{4}x^2\right)+C$$

$$=\frac{1}{2}x^2(\ln x)^2-\frac{1}{4}x^2\ln x-\frac{1}{2}x^2\ln x+\frac{1}{4}x^2+C$$

$$=\frac{1}{2}x^2(\ln x)^2-\frac{1}{2}x^2\ln x+\frac{1}{4}x^2+C$$

Exercise

Evaluate the integral $\int x^2(\ln x)^2\,dx$

Solution

$u=(\ln x)^2 \qquad v=\int x^2dx$

$du=2\frac{\ln x}{x}dx \qquad v=\frac{1}{3}x^3$

$$\int x^2(\ln x)^2\,dx=\frac{1}{3}x^3(\ln x)^2-\frac{2}{3}\int x^2\ln x dx$$

$u=\ln x \qquad v=\int x^2dx$

$du=\frac{1}{x}dx \qquad v=\frac{1}{3}x^3$

$$\int x^2(\ln x)^2\,dx=\frac{1}{3}x^3(\ln x)^2-\frac{2}{3}\left(\frac{1}{3}x^3\ln x-\frac{1}{3}\int x^2dx\right)$$

$$=\frac{1}{3}x^3(\ln x)^2-\frac{2}{9}x^3\ln x+\frac{2}{27}x^3+C$$

$$=\frac{1}{27}x^3\left(9\ln^2 x-6\ln x+2\right)+C$$

Or

Let $y=\ln x \quad\Rightarrow\quad x=e^y$

$dx=e^ydy$

$$\int x^2(\ln x)^2\,dx=\int\left(e^y\right)^2y^2e^ydy$$

$$= \int y^2 e^{3y}\,dy$$

		$\int e^{3y}\,dy$
$+$	y^2	$\frac{1}{3}e^{3y}$
$-$	$2y$	$\frac{1}{9}e^{3y}$
$+$	2	$\frac{1}{27}e^{3y}$

$$\int x^2(\ln x)^2\,dx = e^{3y}\left(\frac{1}{3}y^2 - \frac{2}{9}y + \frac{2}{27}\right) + C$$

$$= x^3\left(\frac{1}{3}\ln^2 x - \frac{2}{9}\ln x + \frac{2}{27}\right) + C$$

$$\boxed{= \frac{1}{27}x^3\left(9\ln^2 x - 6\ln x + 2\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{(\ln x)^3}{x}\,dx$

Solution

$$\int \frac{(\ln x)^3}{x}\,dx = \int (\ln x)^3\,d(\ln x) \qquad d(\ln x) = \frac{dx}{x}$$

$$\boxed{= \frac{1}{4}(\ln x)^4 + C}$$

Exercise

Evaluate the integral $\displaystyle\int x^2 \ln x^3\,dx$

Solution

$$u = \ln x \qquad v = \int 3x^2 dx = x^3 \qquad \int u\,dv = uv - \int v\,du$$

$$du = \frac{1}{x}dx$$

$$\int x^2 \ln x^3\,dx = \int 3x^2 \ln x\,dx$$

$$= x^3 \ln x - \int x^2\, dx$$

$$\boxed{= x^3 \ln x - \frac{1}{3}x^3 + C}$$

Exercise

Evaluate the integral $\int \ln\left(x + x^2\right) dx$

Solution

Let: $\quad u = \ln\left(x + x^2\right) \qquad dv = dx$

$\qquad du = \dfrac{2x+1}{x+x^2}dx \qquad v = x$

$$\int \ln\left(x+x^2\right)dx = x\ln\left(x+x^2\right) - \int x\frac{2x+1}{x+x^2}dx$$

$$= x\ln\left(x+x^2\right) - \int \frac{2x+1}{x(1+x)}x\,dx$$

$$= x\ln\left(x+x^2\right) - \int \frac{2x+2-1}{1+x}\,dx$$

$$= x\ln\left(x+x^2\right) - \int \frac{2(x+1)-1}{x+1}dx$$

$$= x\ln\left(x+x^2\right) - \int \left(2 - \frac{1}{x+1}\right)dx$$

$$= x\ln\left(x+x^2\right) - \left(2x - \ln|x+1|\right) + C$$

$$\boxed{= x\ln\left(x+x^2\right) - 2x + \ln|x+1| + C}$$

Exercise

Evaluate the integral $\qquad \int x\ln(x+1)\,dx$

Solution

$u = \ln(x+1) \Rightarrow du = \frac{1}{x+1}dx$

$dv = xdx \Rightarrow v = \frac{1}{2}x^2$

$$\int x\ln(x+1)\,dx = \frac{1}{2}x^2\ln(x+1) - \frac{1}{2}\int \frac{x^2}{x+1}dx$$

$$= \frac{1}{2}x^2\ln(x+1) - \frac{1}{2}\int\left(x-1+\frac{1}{x+1}\right)dx$$

$$= \frac{1}{2}x^2\ln(x+1) - \frac{1}{2}\left(\frac{1}{2}x^2 - x + \ln(x+1)\right) + C$$

$$= \frac{1}{2}x^2\ln(x+1) - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{2}\ln(x+1) + C$$

$$= -\frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{2}\left(x^2-1\right)\ln(x+1) + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{(\ln x)^2}{x}dx$

Solution

$$\int \frac{(\ln x)^2}{x}dx = \int (\ln x)^2\, d(\ln x)$$

$$= \frac{1}{3}(\ln x)^3 + C$$

Exercise

Evaluate the integral $\displaystyle\int x^5 \ln 3x\; dx$

Solution

		$\int x^5\ dx$
+	$\ln 3x$	$\frac{1}{6}x^6$
−	$\frac{1}{x}$	$\int \frac{1}{6}x^6$

$$\int x^5 \ln 3x\; dx = \frac{1}{6}x^6\ln 3x - \frac{1}{6}\int x^6\frac{1}{x}dx$$

$$= \frac{1}{6}x^6\ln 3x - \frac{1}{6}\int x^5 dx$$

$$= \frac{1}{6}x^6\ln 3x - \frac{1}{36}x^6 + C$$

Exercise

Evaluate the integral $\int x^5 \ln x \, dx$

Solution

		$\int x^5 \, dx$
$+$	$\ln x$	$\frac{1}{6}x^6$
$-$	$\frac{1}{x}$	$\int \frac{1}{6}x^6$

$$\int x^5 \ln x \, dx = \frac{1}{6}x^6 \ln x - \frac{1}{6}\int x^6 \frac{1}{x} dx$$

$$= \frac{1}{6}x^6 \ln x - \frac{1}{6}\int x^5 \, dx$$

$$= \frac{1}{6}x^6 \ln x - \frac{1}{36}x^6 + C$$

Exercise

Evaluate the integral $\int \ln(x+1)\,dx$

Solution

		$\int dx$
$+$	$\ln(x+1)$	$\frac{1}{2}x$
$-$	$\frac{1}{x+1}$	$\frac{1}{2}\int x$

$$\int \ln(x+1)\,dx = \frac{1}{2}x\ln(x+1) - \frac{1}{2}\int \frac{x}{x+1} \, dx$$

$$= \frac{1}{2}x\ln(x+1) - \frac{1}{2}\int \left(1 - \frac{1}{x+1}\right)dx$$

$$= \frac{1}{2}x\ln(x+1) - \frac{1}{2}\left(x - \ln(x+1)\right) + C$$

$$= \frac{1}{2}x\ln(x+1) - \frac{1}{2}x + \frac{1}{2}\ln(x+1) + C$$

$$= \frac{1}{2}(x+1)\ln(x+1) - \frac{1}{2}x + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\ln x}{x^{10}}\,dx$

Solution

		$\int x^{-10}\ dx$
+	$\ln x$	$-\frac{1}{9}x^{-9}$
−	$\frac{1}{x}$	$-\frac{1}{9}\int x^{-9}$

$$\int \frac{\ln x}{x^{10}}\,dx = -\frac{1}{9x^9}\ln x + \frac{1}{9}\int \frac{1}{x}x^{-9}dx$$

$$= -\frac{1}{9x^9}\ln x + \frac{1}{9}\int x^{-10}dx$$

$$\boxed{= -\frac{\ln x}{9x^9} - \frac{1}{81x^9} + C}$$

Exercise

Evaluate the integral $\displaystyle\int xe^{2x}dx$

Solution

Let: $u = x \Rightarrow du = dx$

$$dv = e^{2x}dx \Rightarrow v = \int dv = \int e^{2x}dx = \frac{1}{2}e^{2x}$$

$$\int udv \ = \ uv - \int vdu$$

$$\int x\,e^{2x}dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx$$

$$\boxed{= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C}$$

		$\int e^{2x}\ dx$
+	x	$\frac{1}{2}e^{2x}$
−	1	$\frac{1}{4}e^{2x}$

$$\int xe^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

Exercise

Evaluate the integral $\int x^3 e^x dx$

Solution

		$\int e^x \, dx$
$+$	x^3	e^x
$-$	$3x^2$	e^x
$+$	$6x$	e^x
$-$	6	e^x

$$\int x^3 e^x dx = e^x\left(x^3 - 3x^2 + 6x - 6\right) + C$$

Or

Let: $u = x^3 \Rightarrow du = 3x^2 dx$

$$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$\int x^3 e^x dx = x^3 e^x - \int e^x 3x^2 dx$$

$$= x^3 e^x - 3\int e^x x^2 dx$$

Let: $u = x^2 \Rightarrow du = 2xdx$

$$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$\int e^x x^2 dx = x^2 e^x - 2\int xe^x dx$$

$$\int x^3 e^x dx = x^3 e^x - 3\left[x^2 e^x - 2\int xe^x dx\right]$$

$$= x^3 e^x - 3x^2 e^x + 6\int xe^x dx$$

Let: $u = x \Rightarrow du = dx$

$$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$\int xe^{x}dx = xe^{x} - \int e^{x}dx = xe^{x} - e^{x}$$

$$\int x^{3}e^{x}dx = x^{3}e^{x} - 3x^{2}e^{x} + 6\left[xe^{x} - e^{x}\right] + C$$

$$= x^{3}e^{x} - 3x^{2}e^{x} + 6xe^{x} - 6e^{x} + C$$

$$\boxed{= e^{x}\left(x^{3} - 3x^{2} + 6x - 6\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{2x}{e^{x}}dx$

Solution

		$\int e^{-x}\,dx$
+	$2x$	$-e^{-x}$
−	2	e^{-x}

$$\int \frac{2x}{e^{x}}dx \boxed{= -e^{-x}\left(2x+2\right) + C}$$

Or

$$u = 2x \Rightarrow du = 2dx$$

$$dv = e^{-x}dx \Rightarrow v = -e^{-x}$$

$$\int \frac{2x}{e^{x}}dx = 2x(-e^{-x}) - \int -e^{-x}2dx$$

$$= -2xe^{-x} + 2\int e^{-x}dx$$

$$= -2xe^{-x} - 2e^{-x} + C$$

$$= -2e^{-x}(x+1) + C$$

$$\boxed{= -\frac{2(x+1)}{e^{x}} + C}$$

Exercise

Evaluate the integral $\int \frac{x^3 e^{x^2}}{\left(x^2+1\right)^2}dx$

Solution

Let: $u = x^2 \mathrm{e}^{x^2} \quad \Rightarrow du = \left(2x\mathrm{e}^{x^2} + 2xx^2\mathrm{e}^{x^2}\right)dx$

$$du = 2x\mathrm{e}^{x^2}\left(1+x^2\right)dx$$

$dv = x\left(x^2+1\right)^{-2}dx \qquad \Rightarrow v = \int x(x^2+1)^{-2}dx$

$$= \frac{1}{2}\int (x^2+1)^{-2}d(x^2+1)$$

$$= \frac{(x^2+1)^{-1}}{-1}$$

$$= -\frac{1}{2(x^2+1)}$$

$$\int \frac{x^3 e^{x^2}}{(x^2+1)^2}dx = x^2 e^{x^2}\left(-\frac{1}{2(x^2+1)}\right) - \int -\frac{1}{2(x^2+1)}2xe^{x^2}(x^2+1)dx$$

$$= -\frac{x^2 e^{x^2}}{2(x^2+1)} + \int xe^{x^2}dx$$

$$\int \frac{x^3 e^{x^2}}{\left(x^2+1\right)^2}dx = -\frac{x^2 e^{x^2}}{2(x^2+1)} + \frac{1}{2}\int e^{x^2}d\left(x^2\right)$$

$$= -\frac{x^2 e^{x^2}}{2(x^2+1)} + \frac{1}{2}e^{x^2} + C$$

$$= \frac{1}{2}e^{x^2}\left(-\frac{x^2}{(x^2+1)} + 1\right) + C$$

$$= \frac{1}{2}e^{x^2}\left(\frac{-x^2+x^2+1}{(x^2+1)}\right) + C$$

$$= \frac{e^{x^2}}{2(x^2+1)} + C$$

Exercise

Evaluate the integral $\int x^2 e^{-3x}dx$

Solution

		$\int e^{-3x}$
+	x^2	$-\frac{1}{3}e^{-3x}$
−	$2x$	$\frac{1}{9}e^{-3x}$
+	2	$-\frac{1}{27}e^{-3x}$

$$\boxed{\int x^2 e^{-3x}dx = -\frac{1}{3}x^2 e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C}$$

$$u = x^2 \Rightarrow du = 2xdx$$

$$dv = e^{-3x}dx \Rightarrow v = -\frac{1}{3}e^{-3x}$$

$$\int x^2 e^{-3x}dx = -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\int xe^{-3x}dx$$

$$u = x \Rightarrow du = dx$$

$$dv = e^{-3x}dx \Rightarrow v = -\frac{1}{3}e^{-3x}$$

$$\int x^2 e^{-3x}dx = -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\left[-\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}dx\right]$$

$$= -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\left(-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\right) + C$$

$$= -\frac{1}{3}x^2 e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C$$

$$\boxed{= -\frac{9x^2 + 6x + 2}{27}e^{-3x} + C}$$

Exercise

Evaluate the integral $\int \left(x^2 - 2x + 1\right) e^{2x} dx$

Solution

		$\int e^{2x}$
+	$x^2 - 2x + 1$	$\frac{1}{2}e^{2x}$
−	$2x - 2$	$\frac{1}{4}e^{2x}$
+	2	$\frac{1}{8}e^{2x}$

$$\int \left(x^2 - 2x + 1\right) e^{2x}\, dx = \frac{1}{2}\left(x^2 - 2x + 1\right)e^{2x} - \frac{1}{4}(2x - 2)e^{2x} + \frac{1}{8}(2)e^{2x} + C$$

$$= \left(\frac{1}{2}x^2 - x + \frac{1}{2} - \frac{1}{2}x + \frac{1}{2} + \frac{1}{4}\right)e^{2x} + C$$

$$\boxed{= \left(\frac{1}{2}x^2 - \frac{3}{2}x + \frac{5}{4}\right)e^{2x} + C}$$

Exercise

Evaluate the integral $\int x^5 e^{x^3} dx$

Solution

Let:

$$u = x^3 \quad dv = x^2 e^{x^3} dx = \frac{1}{3} d\left(e^{x^3}\right)$$

$$d\left(e^{x^3}\right) = 3x^2 e^{x^3} dx$$

$$du = 3x^2 dx \quad v = \frac{1}{3}e^{x^3}$$

$$\int x^5 e^{x^3} dx = x^3 \frac{1}{3}e^{x^3} - \int \frac{1}{3}e^{x^3} 3x^2 dx \qquad d\left(e^{x^3}\right) = 3x^2 e^{x^3} dx \qquad \int u dv = uv - \int v du$$

$$= \frac{1}{3}x^3 e^{x^3} - \frac{1}{3}\int d\left(e^{x^3}\right)$$

$$\boxed{= \frac{1}{3}x^3 e^{x^3} - \frac{1}{3}e^{x^3} + C}$$

Exercise

Evaluate the integral $\int xe^{-4x}\,dx$

Solution

		$\int e^{-4x}\,dx$
+	x	$-\frac{1}{4}e^{-4x}$
−	1	$\frac{1}{16}e^{-4x}$

$$\int xe^{-4x}\,dx \boxed{=\left(-\frac{x}{4}-\frac{1}{16}\right)e^{-4x}+C}$$

Exercise

Evaluate the integral $\int \frac{xe^{2x}}{(2x+1)^2}dx$

Solution

$$u = xe^{2x} \quad \to \quad du = (2x+1)e^{2x}dx$$

$$v = \int \frac{dx}{(2x+1)^2}$$

$$= \frac{1}{2}\int \frac{1}{(2x+1)^2}d(2x+1)$$

$$\boxed{= -\frac{1}{2}\frac{1}{2x+1}}$$

$$\int \frac{xe^{2x}}{(2x+1)^2}dx = -\frac{xe^{2x}}{4x+2}+\frac{1}{2}\int e^{2x}dx$$

$$\boxed{= -\frac{x}{4x+2}e^{2x}+\frac{1}{4}e^{2x}+C}$$

Exercise

Evaluate the integral $\int \frac{5x}{e^{2x}}dx$

Solution

		$\int e^{-2x}\, dx$
+	$5x$	$-\frac{1}{2}e^{-2x}$
−	5	$\frac{1}{4}e^{-2x}$

$$\int \frac{5x}{e^{2x}}\, dx = \int 5xe^{-2x}\, dx$$

$$\boxed{=\left(-\frac{5}{2}x-\frac{5}{4}\right)e^{-2x}+C}$$

Exercise

Evaluate the integral $\int \frac{e^{1/x}}{x^2}\, dx$

Solution

$$\int \frac{e^{1/x}}{x^2}\, dx = -\int e^{1/x} d\left(\frac{1}{x}\right)$$

$$\boxed{=-e^{1/x}+C}$$

Exercise

Evaluate the integral $\int x^2 e^{4x}\, dx$

Solution

$$\int x^2 e^{4x}\, dx = \boxed{\left(\frac{1}{4}x^2-\frac{1}{8}x+\frac{1}{32}\right)e^{4x}+C}$$

$$\int x^n e^{ax}\, dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!a^{k+1}}x^{n-k}$$

Exercise

Evaluate the integral $\int x^3 e^{-3x}\, dx$

Solution

$$\int x^3 e^{-3x}\, dx = \boxed{\left(-\frac{1}{3}x^3+\frac{1}{3}x^2-\frac{2}{9}x+\frac{2}{27}\right)e^{-3x}+C}$$

$$\int x^n e^{ax}\, dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!a^{k+1}}x^{n-k}$$

Exercise

Evaluate the integral $\int x^4 e^x dx$

Solution

		$\int e^x\ dy$
$+$	x^4	e^x
$-$	$4x^3$	e^x
$+$	$12x^2$	e^x
$-$	$24x$	e^x
$+$	24	e^x

$$\int x^4 e^x dx = \left(x^4 + 4x^3 + 12x^2 + 24x + 24\right)e^x + C$$

Exercise

Evaluate the integral $\int x^3 e^{4x} dx$

Solution

$$\int x^3 e^{4x} dx = e^{4x}\left(\frac{1}{4}x^3 - \frac{3}{16}x^2 + \frac{3}{32}x - \frac{3}{128}\right) + C$$

$$\int x^n e^{ax}\ dx = e^{ax}\sum_{k=0}^{n} \frac{(-1)^k\, n!}{(n-k)!a^{k+1}}x^{n-k}$$

Exercise

Evaluate the integral $\int (x+1)^2\, e^x dx$

Solution

		$\int e^x\ dx$
$+$	$(x+1)^2$	e^x
$-$	$2(x+1)$	e^x
$+$	2	e^x

$$\int (x+1)^2\, e^x dx = e^x\left[(x+1)^2 - 2(x+1) + 2\right] + C$$

$$= e^{x}\left(x^{2}+2x+1-2x-2+2\right)+C$$

$$= e^{x}\left(x^{2}+1\right)+C$$

Exercise

Evaluate the integral $\int 2xe^{3x}\ dx$

Solution

		$\int e^{3x}\ dx$
$+$	$2x$	$\frac{1}{3}e^{3x}$
$-$	2	$\frac{1}{9}e^{3x}$

$$\int 2xe^{3x}\ dx = e^{3x}\left(\frac{2}{3}x-\frac{2}{9}\right)+C$$

$$= \frac{2}{9}e^{3x}\left(3x-1\right)+C$$

Exercise

Evaluate the integral $\int x^{2}\sin x\ dx$

Solution

		$\int \sin x$
x^{2}	$(+)$	$-\cos x$
$2x$	$(-)$	$-\sin x$
2	$(+)$	$\cos x$
0		

$$\int x^{2}\sin x\ dx = -x^{2}\cos x+2x\sin x+2\cos x+C$$

Exercise

Evaluate the integral $\int \theta \cos \pi\theta \, d\theta$

Solution

Let:

$u = \theta \quad \to \quad du = d\theta$

$v = \int \cos \pi\theta d\theta$

$= \frac{1}{\pi}\sin \pi\theta$

$$\int \theta \cos \pi\theta \, d\theta = \frac{\theta}{\pi}\sin \pi\theta - \int \frac{1}{\pi}\sin \pi\theta \, d\theta$$

$$= \frac{\theta}{\pi}\sin \pi\theta + \frac{1}{\pi}\frac{1}{\pi}\cos \pi\theta + C$$

$$\boxed{= \frac{\theta}{\pi}\sin \pi\theta + \frac{1}{\pi^2}\cos \pi\theta + C}$$

Exercise

Evaluate the integral $\int 4x \sec^2 2x \, dx$

Solution

Let: $u = 4x \quad \to \quad du = 4$

$v = \int \sec^2 2x \, dx$

$= \frac{1}{2}\tan 2x$

$$\int 4x \sec^2 2x \, dx = 2x \tan 2x - \int 4\left(\frac{1}{2}\tan 2x\right)dx$$

$$= 2x \tan 2x - 2\frac{1}{2}\ln\left|\sec 2x\right| + C$$

$$\boxed{= 2x \tan 2x - \ln\left|\sec 2x\right| + C}$$

Exercise

Evaluate the integral $\int x^3 \sin x \, dx$

Solution

		$\int \sin x$
+	x^3	$-\cos x$
–	$3x^2$	$-\sin x$
+	$6x$	$\cos x$
–	6	$\sin x$

$$\int x^3 \sin x\, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$$

Exercise

Evaluate the integral $\int \left(x^3 - 2x\right) \sin 2x\, dx$

Solution

		$\int \sin 2x\, dx$
+	$x^3 - 2x$	$-\frac{1}{2} \cos 2x$
–	$3x^2 - 2$	$-\frac{1}{4} \sin 2x$
+	$6x$	$\frac{1}{8} \cos 2x$
–	6	$\frac{1}{16} \sin 2x$

$$\int \left(x^3 - 2x\right) \sin 2x\, dx = -\frac{1}{2}\left(x^3 - 2x\right) \cos 2x + \frac{1}{4}\left(3x^2 - 2\right) \sin 2x + \frac{3}{4} x \cos 2x - \frac{3}{8} \sin 2x + C$$

$$= \left(-\frac{1}{2} x^3 + x + \frac{3}{4} x\right) \cos 2x + \left(\frac{3}{4} x^2 - \frac{1}{2} - \frac{3}{8}\right) \sin 2x + C$$

$$= \left(-\frac{1}{2} x^3 + \frac{7}{4} x\right) \cos 2x + \left(\frac{3}{4} x^2 - \frac{7}{8}\right) \sin 2x + C$$

Exercise

Evaluate the integral $\displaystyle\int x^2 \sin 2x \, dx$

Solution

		$\int \sin 2x dx$
$+$	x^2	$-\frac{1}{2}\cos 2x$
$-$	$2x$	$-\frac{1}{4}\sin 2x$
$+$	2	$\frac{1}{8}\cos 2x$

$$\int x^2 \sin 2x \, dx = -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x + \frac{1}{4}\cos 2x + C$$

$$= -\frac{1}{4}\left(2x^2 - 1\right)\cos 2x + \frac{1}{2}x \sin 2x + C$$

Exercise

Evaluate the integral $\displaystyle\int x^2 \sin(1-x) \, dx$

Solution

		$\int \sin(1-x) dx$
$+$	x^2	$\cos(1-x)$
$-$	$2x$	$-\sin(1-x)$
$+$	2	$\cos(1-x)$

$$\int x^2 \sin(1-x) \, dx = x^2 \cos(1-x) + 2x \sin(1-x) + 2\cos(1-x) + C$$

$$= \left(x^2 + 2\right)\cos(1-x) + 2x \sin(1-x) + C$$

Exercise

Evaluate the integral $\displaystyle\int x \sin x \cos x \, dx$

Solution

		$\int \sin 2x\ dx$
+	x	$-\frac{1}{2}\cos 2x$
−	1	$-\frac{1}{4}\sin 2x$

$$\int x\sin x\cos x\ dx = \frac{1}{2}\int x\sin 2x\ dx$$

$$= \frac{1}{2}\left(-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right)$$

$$\boxed{= -\frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C}$$

Exercise

Evaluate the integral $\int x\cos x\ dx$

Solution

		$\int \cos x$
+	x	$\sin x$
−	1	$-\cos x$

$$\boxed{\int x\cos x\ dx = x\sin x + \cos x + C}$$

Exercise

Evaluate the integral $\int x\csc x\cot x\ dx$

Solution

$u = x \quad \rightarrow \quad du = dx$

$dv = \csc x\cot x\ dx \quad \rightarrow \quad v = -\csc x$

$$\int x\csc x\cot x\ dx = -x\csc x + \int \csc x\, dx$$

$$\boxed{= -x\csc x - \ln\left|\csc x + \cot x\right| + C}$$

Exercise

Evaluate the integral $\int x^2 \cos x \, dx$

Solution

		$\int \cos x$
$+$	x^2	$\sin x$
$-$	$2x$	$-\cos x$
$+$	2	$-\sin x$

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C$$

Exercise

Evaluate the integral $\int x^3 \cos 2x \, dx$

Solution

		$\int \cos 2x$
$+$	x^3	$\frac{1}{2} \sin 2x$
$-$	$3x^2$	$-\frac{1}{4} \cos 2x$
$+$	$6x$	$-\frac{1}{8} \sin 2x$
$-$	6	$\frac{1}{16} \cos 2x$

$$\int x^3 \cos 2x \, dx = \frac{1}{2} x^3 \sin 2x + \frac{3}{4} x^2 \cos 2x - \frac{3}{4} x \sin 2x - \frac{3}{8} \cos 2x + C$$

$$= \left(\frac{1}{2} x^3 - \frac{3}{4} x \right) \sin 2x + \left(\frac{3}{4} x^2 - \frac{3}{8} \right) \cos 2x + C$$

Exercise

Evaluate the integral $\int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx$

Solution

$$d\left(\sqrt{x}\right)=\frac{1}{2\sqrt{x}}\,dx$$

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx=2\int \cos\sqrt{x}\;d\left(\sqrt{x}\right)$$

$$\boxed{=2\sin\sqrt{x}+C}$$

Exercise

Evaluate the integral $\displaystyle\int x\sinh x\,dx$

Solution

		$\int \sinh x\,dx$
+	x	$\cosh x$
–	1	$\sinh x$

$$\int x\sinh x\,dx\;\boxed{=x\cosh x-\sinh x+C}$$

Exercise

Evaluate the integral $\displaystyle\int x^2\cosh x\,dx$

Solution

		$\int \cosh x$
+	x^2	$\sinh x$
–	$2x$	$\cosh x$
+	2	$\sinh x$

$$\int x^2\cosh x\,dx=x^2\sinh x-2x\cosh x+2\sinh x+C$$

$$\boxed{=\left(x^2+2\right)\sinh x-2x\cosh x+C}$$

Exercise

Evaluate the integral $\int e^{2x}\cos 3x\,dx$

Solution

		$\int \cos 3x\ dx$
$+$	e^{2x}	$\frac{1}{3}\sin 3x$
$-$	$2e^{2x}$	$-\frac{1}{9}\cos 3x$
$+$	$4e^{2x}$	$-\frac{1}{9}\int \cos 3x\ dx$

$$\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x - \frac{4}{9}\int e^{2x}\cos 3x\,dx$$

$$\left(1+\frac{4}{9}\right)\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x$$

$$\frac{13}{9}\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x$$

$$\int e^{2x}\cos 3x\,dx = \boxed{\frac{e^{2x}}{13}(3\sin 3x + 2\cos 3x) + C}$$

Exercise

Evaluate the integral $\int e^{-3x}\sin 5x\ dx$

Solution

		$\int \sin 5x$
$+$	e^{-3x}	$-\frac{1}{5}\cos 5x$
$-$	$-3e^{-3x}$	$-\frac{1}{25}\sin 5x$
$+$	$9e^{-3x}$	$-\int \frac{1}{25}\sin 5x$

$$\int e^{-3x}\sin 5x\ dx = -\frac{1}{5}e^{-3x}\cos 5x - \frac{3}{25}e^{-3x}\sin 5x - \frac{9}{25}\int e^{-3x}\sin 5x\ dx$$

$$\left(1+\frac{9}{25}\right)\int e^{-3x}\sin 5x\,dx=-\frac{1}{25}(5\cos 5x+3\sin 5x)e^{-3x}$$

$$\frac{34}{25}\int e^{-3x}\sin 5x\,dx=-\frac{1}{25}(5\cos 5x+3\sin 5x)e^{-3x}$$

$$\int e^{-3x}\sin 5x\,dx=-\frac{1}{34}(5\cos 5x+3\sin 5x)e^{-3x}+C$$

Exercise

Evaluate the integral $\int e^{-x}\sin 4x\,dx$

Solution

		$\int \sin 4x\,dx$
+	e^{-x}	$-\frac{1}{4}\cos 4x$
−	$-e^{-x}$	$-\frac{1}{16}\sin 4x$
+	e^{-x}	$-\frac{1}{16}\int \sin 4x\,dx$

$$\int e^{-x}\sin 4x\,dx=-\frac{1}{4}e^{-x}\cos 4x-\frac{1}{16}e^{-x}\sin 4x-\frac{1}{16}\int e^{-x}\sin 4x\,dx$$

$$\left(1+\frac{1}{16}\right)\int e^{-x}\sin 4x\,dx=-\frac{1}{4}e^{-x}\cos 4x-\frac{1}{16}e^{-x}\sin 4x$$

$$\frac{17}{16}\int e^{-x}\sin 4x\,dx=-\frac{1}{16}e^{-x}(4\cos 4x+\sin 4x)$$

$$\int e^{-x}\sin 4x\,dx=-\frac{e^{-x}}{17}(4\cos 4x+\sin 4x)+C$$

Exercise

Evaluate the integral $\int e^{-2\theta}\sin 6\theta\,d\theta$

Solution

		$\int \sin 6\theta \, d\theta$
+	$e^{-2\theta}$	$-\frac{1}{6}\cos 6\theta$
−	$-2e^{-2\theta}$	$-\frac{1}{36}\sin 6\theta$
+	$4e^{-2\theta}$	$-\frac{1}{36}\int \sin 6\theta \, d\theta$

$$\int e^{-2\theta} \sin 6\theta \, d\theta = -\frac{1}{6} e^{-2\theta} \cos 6\theta - \frac{1}{18} e^{-2\theta} \sin 6\theta - \frac{1}{9} \int e^{-2\theta} \sin 6\theta \, d\theta$$

$$\left(1 + \frac{1}{9}\right) \int e^{-2\theta} \sin 6\theta \, d\theta = -\frac{1}{18} e^{-2\theta} \left(3\cos 6\theta + \sin 6\theta\right)$$

$$\frac{10}{9} \int e^{-2\theta} \sin 6\theta \, d\theta = -\frac{1}{18} e^{-2\theta} \left(3\cos 6\theta + \sin 6\theta\right)$$

$$\int e^{-2\theta} \sin 6\theta \, d\theta = -\frac{e^{-2\theta}}{20} \left(3\cos 6\theta + \sin 6\theta\right) + C$$

Exercise

Evaluate the integral $\int e^{-3x} \sin 4x \, dx$

Solution

		$\int \sin 4x$
+	e^{-3x}	$-\frac{1}{4}\cos 4x$
−	$-3e^{-3x}$	$-\frac{1}{16}\sin 4x$
+	$9e^{-3x}$	$-\frac{1}{16}\int \sin 4x$

$$\int e^{-3x} \sin 4x \, dx = -\frac{1}{4} e^{-3x} \cos 4x - \frac{3}{16} e^{-3x} \sin 4x - \frac{9}{16} \int e^{-3x} \sin 4x \, dx$$

$$\left(1 + \frac{9}{16}\right) \int e^{-3x} \sin 4x \, dx = -\frac{1}{16} \left(4\cos 4x + 3\sin 4x\right) e^{-3x}$$

$$\frac{25}{16} \int e^{-3x} \sin 4x \, dx = -\frac{1}{16} \left(4\cos 4x + 3\sin 4x\right) e^{-3x}$$

$$\int e^{-3x} \sin 4x \, dx = -\frac{1}{25} \left(4\cos 4x + 3\sin 4x\right) e^{-3x} + C$$

Exercise

Evaluate the integral $\int e^{4x}\cos 2x\,dx$

Solution

		$\int \cos 2x\,dx$
+	e^{4x}	$\frac{1}{2}\sin 2x$
−	$4e^{4x}$	$-\frac{1}{4}\cos 2x$
+	$16e^{4x}$	$-\frac{1}{4}\int \cos 2x$

$$\int e^{4x}\cos 2x\,dx = \frac{1}{2}e^{4x}\sin 2x + e^{4x}\cos 2x - 4\int e^{4x}\cos 2x\,dx$$

$$5\int e^{4x}\cos 2x\,dx = \frac{1}{2}(\sin 2x + 2\cos 2x)e^{4x}$$

$$\int e^{4x}\cos 2x\,dx = \boxed{\frac{1}{10}(\sin 2x + 2\cos 2x)e^{4x} + C}$$

Exercise

Evaluate the integral $\int e^{3x}\cos 3x\,dx$

Solution

		$\int \cos 3x$
+	e^{3x}	$\frac{1}{3}\sin 3x$
−	$3e^{3x}$	$-\frac{1}{9}\cos 3x$
+	$9e^{3x}$	$-\frac{1}{9}\int \cos 3x$

$$\int e^{3x}\cos 3x\,dx = \frac{1}{3}e^{3x}\sin 3x + \frac{1}{3}e^{3x}\cos 3x - \int e^{3x}\cos 3x\,dx$$

$$2\int e^{3x}\cos 3x\,dx = \frac{1}{3}(\sin 3x + \cos 3x)e^{3x}$$

$$\int e^{3x}\cos 3x\,dx = \boxed{\frac{1}{6}(\sin 3x + \cos 3x)e^{3x} + C}$$

Exercise

Evaluate the integral $\displaystyle\int e^{3x}\cos 2x\,dx$

Solution

		$\int \cos 2x$
$+$	e^{3x}	$\frac{1}{2}\sin 2x$
$-$	$3e^{3x}$	$-\frac{1}{4}\cos 2x$
$+$	$9e^{3x}$	$-\frac{1}{4}\int \cos 2x$

$$\int e^{3x}\cos 2x\,dx = e^{3x}\left(\frac{1}{2}\sin 2x+\frac{3}{4}\cos 2x\right)-\frac{9}{4}\int e^{3x}\cos 2x\,dx$$

$$\left(1+\frac{9}{4}\right)\int e^{3x}\cos 2x\,dx = \frac{1}{4}e^{3x}\left(2\sin 2x+3\cos 2x\right)$$

$$\frac{13}{4}\int e^{3x}\cos 2x\,dx = \frac{1}{4}e^{3x}\left(2\sin 2x+3\cos 2x\right)$$

$$\boxed{\int e^{3x}\cos 2x\,dx = \frac{1}{13}e^{3x}\left(2\sin 2x+3\cos 2x\right)+C}$$

Exercise

Evaluate the integral $\displaystyle\int e^{x}\sin x\,dx$

Solution

		$\int \sin x$
$+$	e^{x}	$-\cos x$
$-$	e^{x}	$-\sin x$
$+$	e^{x}	$-\int \sin x$

$$\int e^{x}\sin x\,dx = e^{x}\left(-\cos x+\sin x\right)-\int e^{x}\sin x\,dx$$

$$2\int e^{x}\sin x\,dx = e^{x}\left(\sin x-\cos x\right)$$

$$\int e^{x}\sin x\,dx = \boxed{\frac{1}{2}e^{x}\left(\sin x-\cos x\right)+C}$$

Exercise

Evaluate the integral $\displaystyle\int e^{-2x}\sin 3x dx$

Solution

		$\int \sin 3x$
+	e^{-2x}	$-\frac{1}{3}\cos 3x$
−	$-2e^{-2x}$	$-\frac{1}{9}\sin 3x$
+	$4e^{-2x}$	$-\frac{1}{9}\int \sin 3x$

$$\int e^{-2x}\sin 3x\,dx = e^{-2x}\left(-\frac{1}{3}\cos 3x-\frac{2}{9}\sin 3x\right)-\frac{4}{9}\int e^{-2x}\sin 3x\,dx$$

$$\left(1+\frac{4}{9}\right)\int e^{-2x}\sin 3x\,dx = -\frac{1}{9}e^{-2x}\left(3\cos 3x+2\sin 3x\right)$$

$$\frac{13}{9}\int e^{-2x}\sin 3x\,dx = -\frac{1}{9}e^{-2x}\left(3\cos 3x+2\sin 3x\right)$$

$$\int e^{-2x}\sin 3x\,dx = \boxed{-\frac{1}{13}e^{-2x}\left(3\cos 3x+2\sin 3x\right)+C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x}{\sqrt{x-1}}dx$

Solution

Let: $u = x \ \Rightarrow du = dx$

$$dv = \frac{dx}{\sqrt{x-1}} \Rightarrow v = \int (x-1)^{-1/2}d(x-1)$$

$$= \frac{(x-1)^{1/2}}{1/2}$$

$$\boxed{=2(x-1)^{1/2}}$$

$$\int \frac{x}{\sqrt{x-1}}dx = 2x\sqrt{x-1} - 2\int (x-1)^{1/2}dx$$

$$= 2x\sqrt{x-1} - 2\frac{(x-1)^{3/2}}{3/2} + C$$

$$= 2x\sqrt{x-1} - \frac{4}{3}(x-1)\sqrt{x-1} + C$$

$$= \sqrt{x-1}\left[2x - \frac{4}{3}x + \frac{4}{3}\right] + C$$

$$= \sqrt{x-1}\left[\frac{6x-4x+4}{3}\right] + C$$

$$= \sqrt{x-1}\left[\frac{2x+4}{3}\right] + C$$

$$\boxed{= \frac{2}{3}\sqrt{x-1}(x+2) + C}$$

Or

Let: $u = x-1 \quad \Rightarrow \quad x = u+1$

$du = dx$

$$\int \frac{x}{\sqrt{x-1}}dx = \int (u+1)u^{-1/2}du$$

$$= \int \left(u^{1/2} + u^{-1/2}\right)du$$

$$= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C$$

$$= (x-1)^{1/2}\left(\frac{2}{3}x - \frac{2}{3} + 2\right) + C$$

$$= \sqrt{x-1}\left[\frac{2x+4}{3}\right] + C$$

$$\boxed{= \frac{2}{3}\sqrt{x-1}(x+2) + C}$$

Exercise

Evaluate the integral $\displaystyle\int x\sqrt{x-5}\,dx$

Solution

Let $u = \sqrt{x-5}$

$u^2 = x-5 \Rightarrow x = u^2 + 5$

$$2udu = dx$$

$$\int x\sqrt{x-5}\ dx = \int \left(u^2+5\right)u\left(2u\,du\right)$$

$$= \int \left(2u^4+10u^2\right)du$$

$$\boxed{= \frac{2}{5}u^5 + \frac{10}{3}u^3 + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x}{\sqrt{6x+1}}\,dx$

Solution

$$u = x \quad \to \quad du = dx$$

$$dv = (6x+1)^{-1/2}\ dx$$

$$= \frac{1}{6}(6x+1)^{-1/2}\,d\left(6x+1\right)$$

$$v = \frac{1}{3}(6x+1)^{1/2}$$

$$\int \frac{x}{\sqrt{6x+1}}\,dx = \frac{1}{3}x\sqrt{6x+1} - \frac{1}{3}\int (6x+1)^{1/2}\,dx$$

$$= \frac{1}{3}x\sqrt{6x+1} - \frac{1}{18}\int (6x+1)^{1/2}\,d\left(6x+1\right)$$

$$\boxed{= \frac{1}{3}x\sqrt{6x+1} - \frac{1}{27}(6x+1)^{3/2} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x}{2\sqrt{x+2}}\,dx$

Solution

		$\int (x+2)^{-1/2}\ d\left(x+2\right)$
+	x	$2\left(x+2\right)^{1/2}$
−	1	$\frac{4}{3}(x+2)^{3/2}$

$$\int \frac{x}{2\sqrt{x+2}}\,dx = \frac{1}{2}\int x(x+2)^{-1/2}\,dx$$

$$= \frac{1}{2}\left[2x\sqrt{x+2} - \frac{4}{3}(x+2)^{3/2}\right] + C$$

$$= \frac{1}{3}\sqrt{x+2}\,(3x - 2(x+2)) + C$$

$$= \frac{1}{3}\sqrt{x+2}\,(3x - 2x - 4) + C$$

$$= \frac{1}{3}\sqrt{x+2}\,(x-4) + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{2x^2-3x}{(x-1)^3}\,dx$

Solution

		$\int (x-1)^{-3}\,d(x-1)$
+	$2x^2-3x$	$-\frac{1}{2}(x-1)^{-2}$
–	$4x-3$	$\frac{1}{2}(x-1)^{-1}$
+	4	$\frac{1}{2}\ln\lvert x-1\rvert$

$$\int \frac{2x^2-3x}{(x-1)^3}\,dx = -\frac{1}{2}\frac{2x^2-3x}{(x-1)^2} - \frac{1}{2}\frac{4x-3}{x-1} + 2\ln\lvert x-1\rvert + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^2+3x+4}{\sqrt[3]{2x+1}}\,dx$

Solution

		$\frac{1}{2}\int (2x+1)^{-1/3}\,d(2x+1)$
+	x^2+3x+x	$\frac{3}{4}(2x+1)^{2/3}$
–	$2x+3$	$\frac{1}{2}\frac{9}{20}(2x+1)^{5/3}$
+	2	$\frac{1}{2}\frac{27}{320}(2x+1)^{8/3}$

$$\int \frac{x^2+3x+4}{\sqrt[3]{2x+1}}\,dx = \frac{3}{4}\left(x^2+3x+4\right)(2x+1)^{2/3} - \frac{9}{40}(2x+3)(2x+1)^{5/3} + \frac{27}{320}(2x+1)^{8/3} + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$

Solution

		$\int (x+1)^{-1/2}\,dx$
+	x	$2(x+1)^{1/2}$
−	1	$\frac{4}{3}(x+1)^{3/2}$

$$\int \frac{x}{\sqrt{x+1}}\,dx = 2x(x+1)^{1/2} - \frac{4}{3}(x+1)^{3/2} + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^5}{\sqrt{1-2x^3}}\,dx$

Solution

		$\int x^2\left(1-2x^3\right)^{-1/2}\,dx = -\frac{1}{6}\int \left(1-2x^3\right)^{-1/2}\,d\left(1-2x^3\right)$
+	x^3	$-\frac{1}{3}\left(1-2x^3\right)^{1/2}$
−	$3x^2$	$\int -\frac{1}{3}\left(1-2x^3\right)^{1/2}$

$$\int \frac{x^5}{\sqrt{1-2x^3}}\,dx = -\frac{1}{3}x^3\sqrt{1-2x^3} + \int x^2\left(1-2x^3\right)^{1/2}\,dx$$

$$= -\frac{1}{3}x^3\sqrt{1-2x^3} - \frac{1}{6}\int \left(1-2x^3\right)^{1/2}\,d\left(1-2x^3\right)$$

$$= -\frac{1}{3}x^3\sqrt{1-2x^3} - \frac{1}{9}\left(1-2x^3\right)^{3/2} + C$$

Exercise

Evaluate the integral $\int x\sqrt{1-3x}\ dx$

Solution

		$\int (1-3x)^{1/2}\,dx = -\frac{1}{3}\int (1-3x)^{1/2}\,d(1-3x)$
+	x	$-\frac{2}{9}(1-3x)^{3/2}$
−	1	$-\left(-\frac{1}{3}\right)\frac{2}{9}\frac{2}{5}(1-3x)^{5/2}$

$$\int x\sqrt{1-3x}\ dx = -\frac{2x}{9}(1-3x)^{3/2} - \frac{4}{135}(1-3x)^{5/2} + C$$

Exercise

Evaluate the integral $\int \sin(\ln x)\ dx$

Solution

		$\int dx$
+	$\sin(\ln x)$	x
−	$\frac{\cos(\ln x)}{x}$	$\int x\ dx$

$$\int \sin(\ln x)\ dx = x\sin(\ln x) - \int x\frac{\cos(\ln x)}{x}dx$$

$$= x\sin(\ln x) - \int \cos(\ln x)\ dx$$

		$\int dx$
+	$\cos(\ln x)$	x
−	$-\frac{\sin(\ln x)}{x}$	$\int x\ dx$

$$= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\ dx$$

$$2\int \sin(\ln x)\ dx = x\sin(\ln x) - x\cos(\ln x)$$

$$\int \sin(\ln x)\,dx = \boxed{\tfrac{1}{2}x\sin(\ln x)-\tfrac{1}{2}x\cos(\ln x)+C}$$

Exercise

Evaluate the integral $\int \tan^{-1} y\,dy$

Solution

Let: $u=\tan^{-1} y \quad dv=dy$

$du=\dfrac{dy}{1+y^2} \qquad v=y$

$$\int \tan^{-1} y\,dy = y\tan^{-1} y-\int \frac{y}{1+y^2}dy \qquad d\left(1+y^2\right)=2ydy \ \to\ \tfrac{1}{2}d\left(1+y^2\right)=ydy$$

$$= y\tan^{-1} y-\frac{1}{2}\int \frac{1}{1+y^2}\,d\left(1+y^2\right)$$

$$= y\tan^{-1} y-\frac{1}{2}\ln\left(1+y^2\right)+C$$

$$\boxed{= y\tan^{-1} y-\ln\sqrt{1+y^2}+C}$$

Exercise

Evaluate the integral $\int \sin^{-1} y\,dy$

Solution

Let: $u=\sin^{-1} y \quad dv=dy$

$du=\dfrac{dy}{\sqrt{1-y^2}} \qquad v=y$

$$\int \sin^{-1} y\,dy = y\sin^{-1} y-\int \frac{y}{\sqrt{1-y^2}}dy \qquad d\left(1-y^2\right)=-2ydy \ \to\ -\tfrac{1}{2}d\left(1-y^2\right)=ydy$$

$$= y\sin^{-1} y+\frac{1}{2}\int \left(1-y^2\right)^{-1/2}d\left(1-y^2\right)$$

$$= y\sin^{-1} y+\frac{1}{2}(2)\left(1-y^2\right)^{1/2}+C$$

$$\boxed{= y\sin^{-1} y+\sqrt{1-y^2}+C}$$

Or

		$\int dy$
$+$	$\sin^{-1} y$	y
$-$	$\frac{1}{\sqrt{1-y^2}}$	$\int y$

$$\int \sin^{-1} y\ dy = y\sin^{-1} y - \int \frac{y}{\sqrt{1-y^2}} dy$$

$$= y\sin^{-1} y + \frac{1}{2}\int \left(1-y^2\right)^{-1/2} d\left(1-y^2\right)$$

$$\boxed{= y\sin^{-1} y + \sqrt{1-y^2} + C}$$

Exercise

Evaluate the integral $\int x\tan^{-1} x\ dx$

Solution

$u = \tan^{-1} x \qquad v = \int xdx$

$du = \dfrac{dx}{x^2+1} \qquad v = \frac{1}{2}x^2$

$$\int x\tan^{-1} x\ dx = \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2}{x^2+1} dx$$

$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{x^2+1}\right) dx$$

$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\left(x - \tan^{-1} x\right) + C$$

$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + C$$

$$\boxed{= \frac{1}{2}\left(x^2+1\right)\tan^{-1} x - \frac{1}{2}x + C}$$

Exercise

Evaluate the integral $\int \sinh^{-1} x \, dx$

Solution

		$\int dx$
+	$\sinh^{-1} x$	x
−	$\frac{1}{\sqrt{x^2+1}}$	$\int x \, dx$

$$\int \sinh^{-1} x \, dx = x \sinh^{-1} x - \int \frac{x}{\sqrt{x^2+1}} dx$$

$$= x \sinh^{-1} x - \frac{1}{2} \int \left(x^2+1\right)^{-1/2} d\left(x^2+1\right)$$

$$= x \sinh^{-1} x - \sqrt{x^2+1} + C$$

Exercise

Evaluate the integral $\int \tan^{-1} 3x \, dx$

Solution

		$\int dx$
+	$\tan^{-1} 3x$	x
−	$\frac{3}{9x^2+1}$	$\int x \, dx$

$$\int \tan^{-1} 3x \, dx = x \tan^{-1} 3x - \int \frac{3x}{9x^2+1} dx$$

$$= x \tan^{-1} 3x - \frac{1}{6} \int \frac{1}{9x^2+1} \, d\left(9x^2+1\right)$$

$$= x \tan^{-1} 3x - \frac{1}{6} \ln\left(9x^2+1\right) + C$$

Exercise

Evaluate the integral $\int \cos^{-1}\left(\frac{x}{2}\right)\, dx$

Solution

		$\int dx$
+	$\cos^{-1}\left(\frac{x}{2}\right)$	x
−	$\dfrac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}x^2}}$	$\int x\, dx$

$$\frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}x^2}} = \frac{\frac{1}{2}}{\sqrt{\frac{4-x^2}{4}}}$$

$$= \frac{1}{\sqrt{4-x^2}}$$

$$\int \cos^{-1}\left(\frac{x}{2}\right) dx = x\cos^{-1}\left(\frac{x}{2}\right) - \int \frac{x}{\sqrt{4-x^2}} dx$$

$$= x\cos^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}\int \left(4-x^2\right)^{-1/2} d\left(4-x^2\right)$$

$$= x\cos^{-1}\left(\frac{x}{2}\right) + \sqrt{4-x^2} + C$$

Exercise

Evaluate the integral $\int x\sec^{-1} x\, dx \quad x \ge 1$

Solution

		$\int x\, dx$
+	$\sec^{-1} x$	$\frac{1}{2}x^2$
−	$\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{1}{2}\int x^2$

$$\int x\sec^{-1}x\,dx = \frac{1}{2}x^2\sin^{-1}x - \frac{1}{2}\int \frac{1}{x\sqrt{x^2-1}}x^2dx$$

$$= \frac{1}{2}x^2\sin^{-1}x - \frac{1}{2}\int x\left(x^2-1\right)^{-1/2}dx$$

$$= \frac{1}{2}x^2\sin^{-1}x - \frac{1}{4}\int \left(x^2-1\right)^{-1/2}d\left(x^2-1\right)$$

$$\boxed{= \frac{1}{2}x^2\sin^{-1}x - \frac{1}{2}\sqrt{x^2-1}+C}$$

Exercise

Evaluate the integral $\int_{-1}^{0} 2x^2\sqrt{x+1}\,dx$

Solution

		$\int (x+1)^{1/2}\,d(x+1)$
+	$2x^2$	$\frac{2}{3}(x+1)^{3/2}$
−	$4x$	$\frac{4}{15}(x+1)^{5/2}$
+	4	$\frac{8}{105}(x+1)^{7/2}$

$$\int_{-1}^{0} 2x^2\sqrt{x+1}\,dx = \frac{4}{3}x^2(x+1)^{3/2} - \frac{16}{15}x(x+1)^{5/2} + \frac{32}{105}(x+1)^{7/2}\Big|_{-1}^{0}$$

$$\boxed{=\frac{32}{105}}$$

Exercise

Evaluate the integral $\int_{0}^{1/\sqrt{2}} x\tan^{-1}x^2\,dx$

Solution

$$\int_{0}^{1/\sqrt{2}} x\tan^{-1}x^2\,dx = \frac{1}{2}\int_{0}^{1/\sqrt{2}} \tan^{-1}x^2\,d\left(x^2\right)$$

$$=\frac{1}{2}\int_0^{1/\sqrt{2}} \tan^{-1} y\, dy \qquad \left(Let\ \ y=x^2\right)$$

		$\int dy$
$+$	$\tan^{-1} y$	y
$-$	$\frac{1}{1+y^2}$	$\int y\, dy$

$$\int_0^{1/\sqrt{2}} x\tan^{-1} x^2\ dx=\frac{1}{2}\left[\ y\tan^{-1} y\ \Big|_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} \frac{y}{1+y^2}\, dy\right]$$

$$=\frac{1}{2}x^2\tan^{-1} x^2\ \Big|_0^{1/\sqrt{2}}\ -\frac{1}{4}\int_0^{1/\sqrt{2}} \frac{1}{1+y^2}\, d\left(1+y^2\right)$$

$$=\frac{1}{4}\tan^{-1}\frac{1}{2}\ -\frac{1}{4}\ln\left(1+x^4\right)\ \Big|_0^{1/\sqrt{2}}$$

$$=\frac{1}{4}\tan^{-1}\frac{1}{2}\ -\frac{1}{4}\ln\left(1+\frac{1}{4}\right)$$

$$=\frac{1}{4}\tan^{-1}\frac{1}{2}\ -\frac{1}{4}\ln\frac{5}{4}$$

Exercise

Evaluate the integral $\displaystyle\int_1^e x^2\ln x\, dx$

Solution

		$\int x^2\ dx$
$+$	$\ln x$	$\frac{1}{3}x^3$
$-$	$\frac{1}{x}$	$\frac{1}{3}\int x^3 dx$

$$\int_1^e x^2\ln x\, dx=\frac{1}{3}x^3\ln x\ \Big|_1^e -\frac{1}{3}\int_1^e \frac{1}{x}x^3\ dx$$

$$=\frac{1}{3}\left(e^3-0\right)-\frac{1}{3}\int_1^e x^2\ dx$$

$$=\frac{1}{3}e^3-\frac{1}{9}\left(x^3\ \right|_1^e$$

$$=\frac{1}{3}e^3-\frac{1}{9}\left(e^3-1\right)$$

$$=\frac{1}{3}e^3-\frac{1}{9}e^3+\frac{1}{9}$$

$$\boxed{=\frac{1}{9}\left(2e^3+1\right)}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{\ln 2}\frac{3t}{e^t}\,dt$

Solution

		$\int e^{-t}$
+	t	$-e^{-t}$
−	1	e^{-t}

$$\int_{-1}^{\ln 2}\frac{3t}{e^t}\,dt=3e^{-t}(-t-1)\ \Big|_{-1}^{\ln 2}$$

$$=-3\left(e^{-\ln 2}(\ln 2+1)-e(0)\right)$$

$$\boxed{=-\frac{3}{2}(\ln 2+1)}$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi}^{2\pi}\cot\frac{x}{3}\,dx$

Solution

$$\int_{\pi}^{2\pi}\cot\frac{x}{3}\,dx=\int_{\pi}^{2\pi}\frac{\cos\frac{x}{3}}{\sin\frac{x}{3}}\,dx$$

$$=3\int_{\pi}^{2\pi}\frac{1}{\sin\frac{x}{3}}\,d\left(\sin\frac{x}{3}\right)$$

$$=3\ln\left|\sin\frac{x}{3}\right|\ \Big|_{\pi}^{2\pi}$$

$$= 3\left(\ln\left|\sin\frac{2\pi}{3}\right| - \ln\left|\sin\frac{\pi}{3}\right|\right)$$

$$= 3\left(\ln\frac{\sqrt{3}}{2} - \ln\frac{\sqrt{3}}{2}\right)$$

$$\underline{= 0\,}|$$

Exercise

Evaluate the integral $\int_0^{1/\sqrt{2}} 2x\sin^{-1}\left(x^2\right)dx$

Solution

$u = \sin^{-1}\left(x^2\right) \qquad dv = 2xdx$

$du = \frac{2x}{\sqrt{1-x^4}}dx \qquad v = x^2$

$\int udv = uv - \int vdu$

$$\int_0^{1/\sqrt{2}} 2x\sin^{-1}\left(x^2\right)dx = \left[x^2\sin^{-1}\left(x^2\right)\right]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} x^2\frac{2x}{\sqrt{1-x^4}}dx \qquad d\left(1-x^4\right) = -4x^3dx$$

$$= \left(\left(\frac{1}{\sqrt{2}}\right)^2\sin^{-1}\left(\left(\frac{1}{\sqrt{2}}\right)^2\right) - 0\right) + \int_0^{1/\sqrt{2}} \frac{d\left(1-x^4\right)}{2\sqrt{1-x^4}}$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) + \left[\sqrt{1-x^4}\right]_0^{1/\sqrt{2}}$$

$$= \frac{1}{2}\frac{\pi}{6} + \left(\sqrt{1-\frac{1}{4}} - 1\right)$$

$$= \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

$$\underline{= \frac{1}{12}\left(\pi + 6\sqrt{3} - 12\right)\,}|$$

Exercise

Evaluate the integral $\int_1^e x^3\ln xdx$

Solution

$u = \ln x$

$du = \frac{1}{x}dx \qquad v = \int x^3dx = \frac{1}{4}x^4$

$$\int_1^e x^3 \ln x dx = \frac{1}{4}\left(x^4 \ln x \,\Big|_1^e - \frac{1}{4}\int_1^e x^4 \frac{dx}{x}\right.$$

$$= \frac{1}{4}\left(e^4 \ln e - 1^4 \ln 1\right) - \frac{1}{4}\int_1^e x^3 dx$$

$$= \frac{e^4}{4} - \frac{1}{16}\left(x^4 \,\Big|_1^e\right.$$

$$= \frac{e^4}{4} - \frac{1}{16}\left(e^4 - 1\right)$$

$$= \frac{4}{4}\frac{e^4}{4} - \frac{1}{16}e^4 + \frac{1}{16}$$

$$\boxed{= \frac{3e^4 + 1}{16}}$$

Exercise

Evaluate the integral $\int_0^1 x\sqrt{1-x}dx$

Solution

$$u = x \quad \rightarrow \quad du = dx$$

$$v = -\int (1-x)^{1/2} d(1-x)$$

$$= -\frac{2}{3}(1-x)^{2/3}$$

$$\int_0^1 x\sqrt{1-x}dx = -\frac{2}{3}\left(x(1-x)^{2/3} \,\Big|_0^1 - \int_0^1 -\frac{2}{3}(1-x)^{2/3} dx\right. \qquad \int udv = uv - \int vdu$$

$$= 0 - \frac{2}{3}\int_0^1 (1-x)^{2/3} d(1-x)$$

$$= -\frac{4}{15}\,(1-x)^{5/3} \,\Big|_0^1$$

$$\boxed{= \frac{4}{15}}$$

		$-\int (1-x)^{1/2}\ d(1-x)$
+	x	$-\frac{2}{3}(1-x)^{3/2}$

−	1	$\frac{4}{15}(1-x)^{5/2}$

$$\int_0^1 x\sqrt{1-x}dx = -\frac{2}{3}x(1-x)^{3/2} - \frac{4}{15}(1-x)^{5/2} \Big|_0^1$$

$$= \frac{4}{15}$$

Exercise

Evaluate the integral $\int_0^{\pi/3} x\tan^2 x\,dx$

Solution

$$u = x \quad \to \quad du = dx$$

$$v = \int \tan^2 x\,dx$$

$$= \int \frac{\sin^2 x}{\cos^2 x}dx$$

$$= \int \frac{1-\cos^2 x}{\cos^2 x}dx$$

$$= \int \left(\frac{1}{\cos^2 x} - 1\right)dx$$

$$= \int \left(\sec^2 x - 1\right)dx$$

$$= \tan x - x$$

$$\int_0^{\pi/3} x\tan^2 x\ dx = \left(x(\tan x - x)\right)\Big|_0^{\pi/3} - \int_0^{\pi/3} (\tan x - x)\,dx \qquad \int udv = uv - \int vdu$$

$$= \frac{\pi}{3}\left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) - 0\ +\ \left(\ln|\cos x| + \frac{x^2}{2}\right)\Big|_0^{\pi/3}$$

$$= \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \ln\left|\cos\frac{\pi}{3}\right| + \frac{1}{2}\left(\frac{\pi}{3}\right)^2 - \ln|1|$$

$$= \frac{\pi}{3}\sqrt{3} - \frac{\pi^2}{9} + \ln\left|\frac{1}{2}\right| + \frac{\pi^2}{18}$$

$$= \frac{\pi}{3}\sqrt{3} - \ln 2 - \frac{\pi^2}{18}$$

Exercise

Evaluate the integral $\int_0^{\pi} x\sin x\,dx$

Solution

		$\int \sin x\,dx$
$+$	x	$-\cos x$
$-$	1	$-\sin x$

$$\int_0^{\pi} x\sin x\,dx = -x\cos x + \sin x\,\Big|_0^{\pi}$$

$$\boxed{=\pi}$$

Exercise

Evaluate the integral $\int_1^{e} \ln 2x\,dx$

Solution

$$\int_1^{e} \ln 2x\,dx = \frac{1}{2}\int_1^{e} \ln 2x\,d(2x)$$

$$\int \ln x\,dx = x\ln x - x$$

$$= x\ln 2x - x\,\Big|_1^{e}$$

$$= e\ln 2e - e - \ln 2 + 1$$

$$= e(\ln 2 + \ln e) - e - \ln 2 + 1$$

$$= e\ln 2 - \ln 2 + 1$$

$$\boxed{=(e-1)\ln 2 + 1}$$

Exercise

Evaluate the integral $\int_0^{\pi/2} x\cos 2x\,dx$

Solution

		$\int \cos 2x\,dx$
$+$	x	$\frac{1}{2}\sin 2x$
$-$	1	$-\frac{1}{4}\cos 2x$

$$\int_0^{\pi/2} x\cos 2x\ dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\ \Big|_0^{\pi/2}$$

$$= -\frac{1}{4} - \frac{1}{4}$$

$$\boxed{= -\frac{1}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\ln 2} xe^x\ dx$

Solution

		$\int e^x\ dx$
+	x	e^x
−	1	e^x

$$\int_0^{\ln 2} xe^x\ dx = e^x(x-1)\ \Big|_0^{\ln 2}$$

$$= 2(\ln 2 - 1) + 1$$

$$\boxed{= 2\ln 2 - 1}$$

Exercise

Evaluate the integral $\displaystyle\int_1^{e^2} x^2 \ln x\ dx$

Solution

$$u = \ln x$$

$$du = \frac{1}{x}dx \qquad v = \int x^2 dx = \frac{1}{3}x^3$$

$$\int x^2 \ln x\ dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx$$

$$\int_1^{e^2} x^2 \ln x\ dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3\ \Big|_1^{e^2}$$

$$= \frac{2}{3}e^6 - \frac{1}{9}e^6 + \frac{1}{9}$$

$$\boxed{= \frac{5}{9}e^6 + \frac{1}{9}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^3 xe^{x/2}dx$

Solution

$$\int x^n e^{ax}\, dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k\, n!}{(n-k)!a^{k+1}}x^{n-k}$$

$$\int_0^3 xe^{x/2}dx = (2x-4)e^{x/2}\Big|_0^3$$

$$\boxed{= 2e^{3/2} + 4}$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 x^2e^{-2x}dx$

Solution

$$\int x^n e^{ax}\, dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k\, n!}{(n-k)!a^{k+1}}x^{n-k}$$

$$\int_0^2 x^2e^{-2x}\,dx = \left(-\frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{4}\right)e^{-2x}\Big|_0^2$$

$$= \left(-2 + 1 - \frac{1}{4}\right)e^{-4} + \frac{1}{4}$$

$$\boxed{= \frac{1}{4} - \frac{5}{4}e^{-4}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/4} x\cos 2x\, dx$

Solution

		$\int \cos 2x\, dx$
+	x	$\frac{1}{2}\sin 2x$
−	1	$-\frac{1}{4}\cos 2x$

$$\int_0^{\pi/4} x\cos 2x\, dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\Big|_0^{\pi/4}$$

$$\boxed{= \frac{\pi}{8} - \frac{1}{4}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi} x \sin 2x \, dx$

Solution

		$\int \sin 2x \, dx$
+	x	$-\frac{1}{2}\cos 2x$
−	1	$-\frac{1}{4}\sin 2x$

$$\int_0^{\pi} x \sin 2x \, dx = -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x \Big|_0^{\pi}$$

$$\boxed{= -\frac{\pi}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_1^{4} e^{\sqrt{x}} \, dx$

Solution

$$u = \sqrt{x} \quad \to \quad u^2 = x$$

$$2u \, du = dx$$

$$\int_1^{4} e^{\sqrt{x}} \, dx = 2\int_1^{4} u e^{u} \, du$$

		$\int e^{u} \, du$
+	u	e^{u}
−	1	e^{u}

$$\int_1^{4} e^{\sqrt{x}} \, dx = 2e^{u}(u-1) \Big|_1^{4}$$

$$= 2e^{\sqrt{x}}\left(\sqrt{x}-1\right) \Big|_1^{4}$$

$$= 2\left[e^{2}(2-1) - e(1-1)\right]$$

$$\boxed{= 2e^{2}}$$

Exercise

Use integration by parts to establish the reduction formula

$$\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1}\cos x\,dx$$

Solution

$$u = x^n \qquad\qquad dv = \sin x dx$$

$$du = nx^{n-1}dx \qquad v = -\cos x$$

$$\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1}\cos x\,dx \quad \checkmark$$

Exercise

Use integration by parts to establish the reduction formula

$$\int x^n e^{ax}dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx, \quad a \neq 0$$

Solution

$$u = x^n \qquad\qquad dv = e^{ax}dx$$

$$du = nx^{n-1}dx \qquad v = \frac{1}{a}e^{ax}$$

$$\int x^n e^{ax}dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx, \quad a \neq 0 \quad \checkmark$$

Exercise

Use integration by parts to establish the reduction formula

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$$

Solution

$$u = (\ln x)^n \qquad\qquad dv = \int dx$$

$$du = n(\ln x)^{n-1}\frac{1}{x}dx \qquad v = x$$

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx \quad \checkmark$$

Exercise

Use integration by parts to establish the reduction formula

$$\int_a^b \left(\int_x^b f(t)\,dt \right) dx = \int_a^b (x-a)\, f(x)\,dx$$

Solution

$$u = x - a \qquad dv = f(x)\,dx$$

$$du = dx \qquad v = \int_b^x f(t)\,dt$$

$$\int_a^b (x-a)\,f(x)\,dx = \left[(x-a)\int_b^x f(t)\,dt \right]_a^b - \int_a^b \left(\int_b^x f(t)\,dt \right) dx$$

$$= (b-a)\int_b^b f(t)\,dt - (a-a)\int_b^a f(t)\,dt - \int_a^b \left(-\int_x^b f(t)\,dt \right) dx$$

$$= \int_a^b \left(\int_x^b f(t)\,dt \right) dx \quad \checkmark \qquad\qquad \int_b^b f(t)\,dt = 0, \quad a - a = 0$$

Exercise

Use integration by parts to establish the reduction formula

$$\int \sqrt{1-x^2}\; dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx$$

Solution

$$u = \sqrt{1-x^2} \qquad dv = dx$$

$$du = \frac{-x}{\sqrt{1-x^2}}\,dx \qquad v = x$$

$$\int \sqrt{1-x^2}\; dx = x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}}\,dx$$

$$= x\sqrt{1-x^2} - \int \left(\frac{1-x^2-1}{\sqrt{1-x^2}} \right) dx$$

$$= x\sqrt{1-x^2} - \int \left(\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \right) dx$$

$$= x\sqrt{1-x^2} - \int \sqrt{1-x^2}\ dx + \int \frac{1}{\sqrt{1-x^2}}\ dx$$

$$\int \sqrt{1-x^2}\ dx + \int \sqrt{1-x^2}\ dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\ dx$$

$$2\int \sqrt{1-x^2}\ dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\ dx$$

$$\int \sqrt{1-x^2}\ dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\ dx \quad \checkmark$$

Exercise

Find the indefinite integral: $\int 5x^n \ln ax\ dx \qquad a \neq 0,\ n \neq -1$

Solution

$$u = \ln ax \qquad\qquad dv = x^n dx$$

$$du = \frac{a}{ax}dx = \frac{dx}{x} \qquad\qquad v = \frac{x^{n+1}}{n+1}$$

$$\int 5x^n \ln ax\ \ dx \ = 5\left[\frac{x^{n+1}}{n+1}\ln ax - \frac{1}{n+1}\int \frac{x^{n+1}}{x}dx\right]$$

$$= 5\left[\frac{x^{n+1}}{n+1}\ln ax - \frac{1}{n+1}\int x^n dx\right]$$

$$= 5\left[\frac{x^{n+1}}{n+1}\ln ax - \frac{1}{n+1}\frac{x^{n+1}}{n+1}\right] + C$$

$$\boxed{= \frac{5x^{n+1}}{n+1}\left(\ln ax - \frac{1}{n+1}\right) + C}$$

Exercise

Find the volume of the solid generated by the region bounded by $f(x) = x\ln x$, and the $x-axis$ on $\left[1,\ e^2\right]$ is revolved about the $y-axis$.

Solution

Using ***Disk*** Method:

$$V = \pi \int_{1}^{e^2} (x \ln x)^2 \, dx$$

$$= \pi \int_{1}^{e^2} x^2 \ln^2 x \, dx$$

Let $y = \ln x \quad \Rightarrow \quad x = e^y$

$$dx = e^y dy$$

$$\int x^2 (\ln x)^2 \, dx = \int \left(e^y\right)^2 y^2 e^y dy$$

$$= \int y^2 e^{3y} \, dy$$

		$\int e^{3y} \, dy$
+	y^2	$\frac{1}{3} e^{3y}$
−	$2y$	$\frac{1}{9} e^{3y}$
+	2	$\frac{1}{27} e^{3y}$

$$V = \pi e^{3y} \left(\frac{1}{3} y^2 - \frac{2}{9} y + \frac{2}{27} \right|_{1}^{e^2}$$

$$= \pi \left(x^3 \left(\frac{1}{3} \ln^2 x - \frac{2}{9} \ln x + \frac{2}{27} \right) \right|_{1}^{e^2}$$

$$= \pi \left(e^2\right)^3 \left(\frac{1}{3} \left(\ln e^2\right)^2 - \frac{2}{9} \ln e^2 + \frac{2}{27} \right) - \pi \left(\frac{2}{27} \right)$$

$$= \pi e^6 \left(\frac{4}{3} - \frac{4}{9} + \frac{2}{27} \right) - \frac{2}{27} \pi$$

$$= \pi e^6 \left(\frac{36 - 12 + 2}{27} \right) - \frac{2\pi}{27}$$

$$= \pi e^6 \left(\frac{26}{27} \right) - \frac{2\pi}{27}$$

$$= \frac{2\pi}{27} \left(13 e^2 - 1 \right) \quad unit^3$$

Exercise

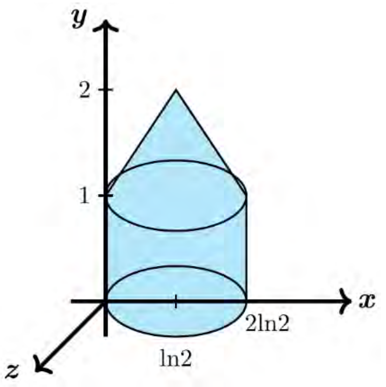
Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the cure $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$

Solution

$$V = 2\pi \int_0^{\ln 2} (\ln 2 - x)e^x dx$$

$$= 2\pi \int_0^{\ln 2} \left(\ln 2e^x - xe^x\right) dx$$

$$= 2\pi \ln 2\left(e^x \Big|_0^{\ln 2} - 2\pi \int_0^{\ln 2} xe^x dx\right.$$



		$\int e^x\, dx$
+	x	e^x
−	1	e^x

$$= 2\pi \ln 2\left(e^{\ln 2} - e^0\right) - 2\pi\left(xe^x - e^x \Big|_0^{\ln 2}\right.$$

$$= 2\pi \ln 2(2-1) - 2\pi\left[\ln 2e^{\ln 2} - e^{\ln 2} - (0-1)\right]$$

$$= 2\pi \ln 2 - 2\pi[2\ln 2 - 2 + 1]$$

$$= 2\pi \ln 2 - 4\pi \ln 2 + 2\pi$$

$$= -2\pi \ln 2 + 2\pi$$

$$= 2\pi(1 - \ln 2) \quad unit^3$$

Exercise

Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate aces, the cure $y = e^{-x}$, and the line $x = 1$, about

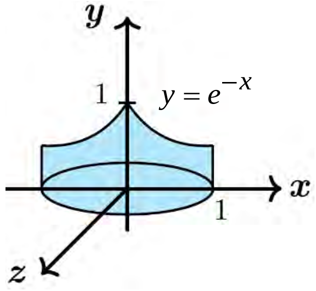
a) the line $y - axis$

b) the line $x = 1$

Solution

a) $V = 2\pi \int_0^1 xe^{-x}\, dx$

		$\int e^{-x}\,dx$
$(+)$	x	$-e^{-x}$
$(-)$	1	e^{-x}



$$=2\pi\left(-xe^{-x}-e^{-x}\right|_0^1$$

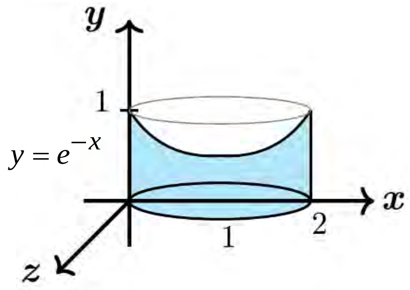
$$=2\pi\left(-e^{-1}-e^{-1}+0+1\right)$$

$$=2\pi\left(-\frac{1}{e}-\frac{1}{e}+1\right)$$

$$=2\pi\left(-\frac{2}{e}+1\right)$$

$$=2\pi-\frac{4\pi}{e}\quad unit^3$$

b) $V=2\pi\int_0^1(1-x)e^{-x}dx$



$$=2\pi\left(\int_0^1 e^{-x}dx-\int_0^1 xe^{-x}dx\right)$$

$$=2\pi\left(-e^{-x}-\left(-xe^{-x}-e^{-x}\right)\right|_0^1$$

$$=2\pi\left(e^{-x}+xe^{-x}-e^{-x}\right|_0^1$$

$$=2\pi\left(xe^{-x}\right|_0^1$$

$$=2\pi\left(e^{-1}\right)$$

$$=\frac{2\pi}{e}\quad unit^3$$

Exercise

Find the volume of the solid that is generated by the region bounded by $f(x)=e^{-x}$, $x=\ln 2$, and the coordinate axes is revolved about the *y-axis*.

Solution

Using ***Shells*** *Method*:

$$V = 2\pi \int_0^{\ln 2} x\, e^{-x}\, dx \qquad\qquad V = \int_a^b 2\pi\left(radius\right)\left(height\right)dx$$

		$\int e^{-x}\, dx$
$+$	x	$-e^{-x}$
$-$	1	e^{-x}

$$= 2\pi\left(e^{-x}\left(-x-1\right)\right) \Big|_0^{\ln 2}$$

$$= 2\pi\left(e^{-\ln 2}\left(-\ln 2-1\right)+1\right)$$

$$= 2\pi\left(\frac{1}{2}\left(-\ln 2-1\right)+1\right)$$

$$= 2\pi\left(-\frac{1}{2}\ln 2+\frac{1}{2}\right)$$

$$\boxed{= \pi\left(1-\ln 2\right) \quad unit^3}$$

Exercise

Find the volume of the solid that is generated by the region bounded by $f\left(x\right)=e^{-x}$, and the $x-axis$ on $\left[1,\ \ln 2\right]$ is revolved about the line $x=\ln 2$.

Solution

Using ***Shells*** *Method*:

$$V = 2\pi \int_0^{\ln 2} \left(\ln 2 - x\right)e^{-x}\, dx \qquad\qquad V = \int_a^b 2\pi\left(radius\right)\left(height\right)dx$$

$$= 2\pi \ln 2 \int_0^{\ln 2} e^{-x}\, dx - 2\pi \int_0^{\ln 2} xe^{-x}\, dx$$

		$\int e^{-x}\, dx$
$+$	x	$-e^{-x}$
$-$	1	e^{-x}

$$= 2\pi\left(-\left(\ln 2\right)e^{-x} - \left(-x-1\right)e^{-x}\right. \Big|_0^{\ln 2}$$

$$= 2\pi\left(-\left(\ln 2\right)e^{-\ln 2} + \left(\ln 2+1\right)e^{-\ln 2} - \ln 2 - 1\right)$$

$$= 2\pi\left(-\frac{1}{2}\ln 2 + \frac{1}{2}(\ln 2 + 1) + \ln 2 - 1\right)$$

$$= 2\pi\left(-\frac{1}{2}\ln 2 + \frac{1}{2}\ln 2 + \frac{1}{2} + \ln 2 - 1\right)$$

$$= 2\pi\left(\ln 2 - \frac{1}{2}\right)$$

$$= \pi(2\ln 2 - 1)$$

$$= \pi(\ln 4 - 1) \quad unit^3$$

Exercise

Find the volume of the solid that is generated by the region bounded by $f(x) = \sin x$, and the *x-axis* on $[0,\ \pi]$ is revolved about the *y-axis*.

Solution

Using ***Shells*** *Method*:

$$V = 2\pi\int_0^\pi x\sin x\ dx \qquad\qquad V = \int_a^b 2\pi(radius)(height)\,dx$$

		$\int \sin x\ dx$
$+$	x	$-\cos x$
$-$	1	$-\sin x$

$$= 2\pi\left(-x\cos x + \sin x\right.\Big|_0^\pi$$

$$= 2\pi^2 \quad unit^3$$

Exercise

Find the area of the region generated when the region bounded by $y = \sin x$ and $y = \sin^{-1} x$ on the interval $\left[0,\ \frac{1}{2}\right]$.

Solution

$$A = \int_0^{1/2}\left(\sin^{-1} x - \sin x\right)dx \qquad u = \sin^{-1} x \qquad du = \frac{dx}{\sqrt{1-x^2}} \qquad v = \int dx = x$$

$$= x\sin^{-1} x\,\Big|_0^{1/2} - \int_0^{1/2}\frac{x\ dx}{\sqrt{1-x^2}} + \cos x\,\Big|_0^{1/2}$$

$$= x\sin^{-1}x + \cos x\Big|_0^{1/2} + \frac{1}{2}\int_0^{1/2}\left(1-x^2\right)^{-1/2} d\left(1-x^2\right)$$

$$= x\sin^{-1}x + \cos x + \left(1-x^2\right)^{1/2}\Big|_0^{1/2}$$

$$= \frac{1}{2}\sin^{-1}\frac{1}{2} + \cos\frac{1}{2} + \left(1-\frac{1}{4}\right)^{1/2} - 1 - 1$$

$$= \frac{\pi}{12} + \cos\frac{1}{2} + \frac{\sqrt{3}}{2} - 2 \quad unit^2$$

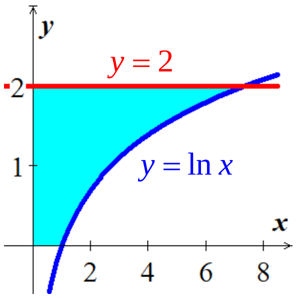
Exercise

Determine the area of the shaded region bounded by $y = \ln x, \quad y = 2, \quad y = 0, \quad and \ x = 0$

Solution

$$y = \ln x = 0 \quad \to \quad x = 1$$

$$y = \ln x = 2 \quad \to \quad x = e^2$$



$$A = 1\times 2 + \int_1^2 (2-\ln x)\,dx$$

$$= 2 + \left(2x - x\ln x + x\right.\Big|_1^2$$

$$= 2 + 4 - 2\ln 2 + 2 - 2 - 1$$

$$= 5 - 2\ln 2 \quad unit^2$$

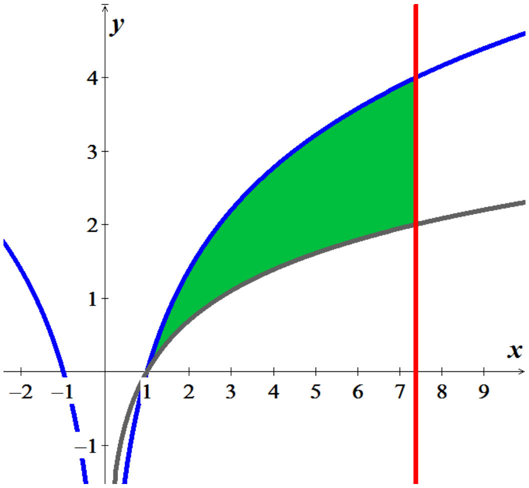
Exercise

Find the area between the curves $y = \ln x^2, \quad y = \ln x, \quad and \ \ x = e^2$

Solution

$$y = \ln x^2 = \ln x \quad with \quad x > 0$$

$$x^2 = x \quad \Rightarrow \quad x = 1$$



$$A = \int_1^{e^2}\left(\ln x^2 - \ln x\right)dx$$

$$= \int_1^{e^2}(2\ln x - \ln x)\,dx$$

$$= \int_1^{e^2} \ln x \, dx$$

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x}dx \quad v = \int dx = x$$

$$\int \ln x dx = x \ln x - \int x \frac{1}{x} dx$$

$$= x \ln x - \int dx$$

$$\underline{= x \ln x - x \,|}$$

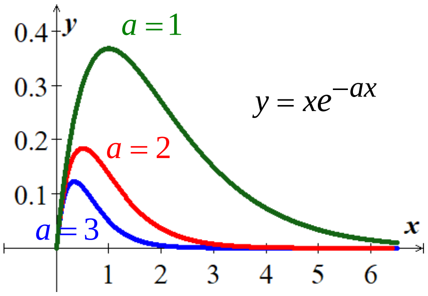
$$= x \ln x - x \Big|_1^{e^2}$$

$$= e^2 \ln e^2 - e^2 + 1$$

$$\underline{= e^2 + 1 \quad unit^2 \,|}$$

Exercise

The curves $y = xe^{-ax}$ are shown in the figure for $a = 1,\ 2,\ and\ 3$.



a) Find the area of the region bounded by $y = xe^{-x}$ and the *x*-axis on the interval [0, 4].

b) Find the area of the region bounded by $y = xe^{-ax}$ and the *x*-axis on the interval [0, 4] where $a > 0$

c) Find the area of the region bounded by $y = xe^{-ax}$ and the *x*-axis on the interval [0, *b*]. Because this area depends on *a* and *b*, we call it $A(a,\ b)$ where $a > 0$ and $b > 0$.

d) Use part (*c*) to show that $A(1,\ \ln b) = 4A\left(2,\ \frac{1}{2}\ln b\right)$

e) Does this pattern continue? Is it true that $A(1,\ \ln b) = a^2 A\left(a,\ \frac{1}{a}\ln b\right)$

Solution

a) $\int_0^4 xe^{-x}dx = e^{-x}(-x-1)\Big|_0^4$

		$\int e^{-x}\ dx$
$+$	x	$-e^{-x}$
$-$	1	e^{-x}

$$= e^{-4}(-5)-(-1)$$

$$\boxed{= 1-\frac{5}{e^4} \quad unit^2}$$

b) $\int_0^4 xe^{-ax}\ dx = e^{-ax}\left(-\frac{1}{a}x-\frac{1}{a^2}\right)\Big|_0^4$

		$\int e^{-ax}\ dx$
$+$	x	$-\frac{1}{a}e^{-ax}$
$-$	1	$\frac{1}{a^2}e^{-ax}$

$$= e^{-4a}\left(-\frac{4}{a}-\frac{1}{a^2}\right)-\left(-\frac{1}{a^2}\right)$$

$$= \frac{1}{a^2}-e^{-4a}\left(\frac{4a+1}{a^2}\right)$$

$$\boxed{= \frac{1}{a^2}\left(1-\frac{4a+1}{e^{-4a}}\right) \quad unit^2}$$

c) $\int_0^b xe^{-ax}\ dx = e^{-ax}\left(-\frac{1}{a}x-\frac{1}{a^2}\right)\Big|_0^b$

$$= e^{-ab}\left(-\frac{b}{a}-\frac{1}{a^2}\right)-\left(-\frac{1}{a^2}\right)$$

$$= \frac{1}{a^2}-e^{-ab}\left(\frac{ab+1}{a^2}\right)$$

$$\boxed{= \frac{1}{a^2}\left(1-\frac{ab+1}{e^{ab}}\right) \quad unit^2}$$

d) $A(a,b) = \frac{1}{a^2}\left(1-\frac{ab+1}{e^{ab}}\right)$

$$A(1,\ \ln b) = 1-\frac{\ln b+1}{e^{\ln b}}$$

$$\boxed{= 1-\frac{\ln b+1}{b}}$$

$$A\left(2,\ \frac{1}{2}\ln b\right) = \frac{1}{4}\left(1-\frac{\ln b+1}{e^{\ln b}}\right)$$

$$=\frac{1}{4}\left(1-\frac{\ln b+1}{b}\right)$$

$$=\frac{1}{4}A(1,\ \ln b)$$

$$\therefore\quad \underline{A(1,\ \ln b)=4A\left(2,\ \tfrac{1}{2}\ln b\right)}\,\Big|$$

e) $A\left(a,\ \frac{1}{a}\ln b\right)=\frac{1}{a^2}\left(1-\frac{\ln b+1}{e^{\ln b}}\right)$

$$=\frac{1}{a^2}\left(1-\frac{\ln b+1}{b}\right)$$

$$=\frac{1}{a^2}A(1,\ \ln b)$$

Yes, there is a pattern: $\underline{A(1,\ \ln b)=a^2A\left(a,\ \frac{1}{a}\ln b\right)}\,\Big|$

Exercise

Suppose a mass on a spring that is slowed by friction has the position function $s(t)=e^{-t}\sin t$

a) Graph the position function. At what times does the oscillator pass through the position $s=0$?

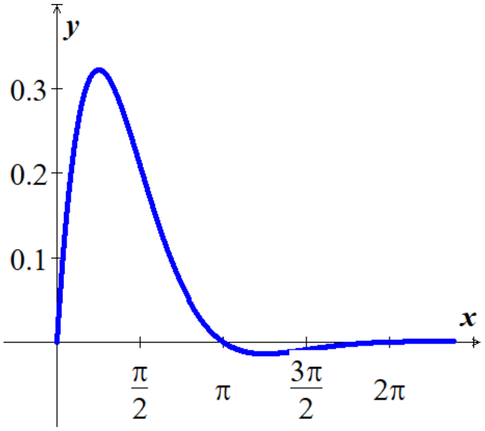
b) Find the average value of the position on the interval $[0,\ \pi]$.

c) Generalize part (*b*) and find the average value of the position on the interval $[n\pi,\ (n+1)\pi]$, for $n=0,\ 1,\ 2,\ \dots$

Solution

a) $s(t)=e^{-t}\sin t=0$

$$\sin t=0\quad\rightarrow\quad \underline{t=n\pi}\,\Big|$$



b) $\int e^{-t}\sin t\ dt=-e^{-t}(\cos t+\sin t)-\int e^{-t}\sin t\ dt$

$$2\int e^{-t}\sin t\ dt=-e^{-t}(\cos t+\sin t)$$

		$\int \sin t\ dt$
+	e^{-t}	$-\cos t$
−	$-e^{-t}$	$-\sin t$
+	e^{-t}	$-\int \sin t\ dt$

$$Average = \frac{1}{\pi}\int_0^{\pi} e^{-t}\sin t\ dt$$

$$= -\frac{1}{2\pi}e^{-t}\left(\cos t - \sin t\right)\Big|_0^{\pi}$$

$$= -\frac{1}{2\pi}\left(-e^{-\pi} - 1\right)$$

$$= \frac{1}{2\pi}\left(e^{-\pi} + 1\right)$$

c) $$Average = \frac{1}{\pi}\int_{n\pi}^{(n+1)\pi} e^{-t}\sin t\ dt$$

$$= -\frac{1}{2\pi}e^{-t}\left(\cos t - \sin t\right)\Big|_{n\pi}^{(n+1)\pi}$$

$$= -\frac{1}{2\pi}\left(e^{-(n+1)\pi}\left(\cos\left((n+1)\pi\right) - \sin\left((n+1)\pi\right)\right) - e^{-n\pi}\left(\cos n\pi - \sin n\pi\right)\right)$$

$$= -\frac{1}{2\pi}\left(e^{-(n+1)\pi}\cos\left((n+1)\pi\right) - e^{-n\pi}\cos n\pi\right)$$

$$= \frac{e^{-n\pi}}{2\pi}\left(\cos n\pi - e^{-\pi}\cos(n+1)\pi\right)$$

$$= \frac{e^{-n\pi}}{2\pi}\left((-1)^n - e^{-\pi}(-1)^{n+1}\right)$$

$$= (-1)^n\frac{e^{-n\pi}}{2\pi}\left(1 + e^{-\pi}\right)$$

Exercise

Given the region bounded by the graphs of $y = x\sin x,\quad y = 0,\quad x = 0,\quad x = \pi$, find

a) The area of the region.

b) The volume of the solid generated by revolving the region about the *x-axis*

c) The volume of the solid generated by revolving the region about the *y-axis*

d) The centroid of the region

Solution

a) $$A = \int_0^{\pi} x\ \sin x\ dx$$

		$\int \sin x\, dx$
+	x	$-\cos x$
–	1	$-\sin x$

$$= -x\cos x + \sin x\ \Big|_0^{\pi}$$

$$= \pi \quad unit^2$$

b) $V = \pi \int_0^{\pi} (x \sin x)^2 \, dx$

$$= \pi \int_0^{\pi} x^2 \sin^2 x \, dx$$

$$= \frac{\pi}{2} \int_0^{\pi} x^2 (1 - \cos 2x) \, dx$$

$$= \frac{\pi}{2} \int_0^{\pi} \left(x^2 - x^2 \cos 2x \right) dx$$

		$\int \cos 2x \, dx$
+	x^2	$\frac{1}{2} \sin 2x$
–	$2x$	$-\frac{1}{4} \cos 2x$
+	2	$-\frac{1}{8} \sin 2x$

$$= \frac{\pi}{2} \left(\frac{1}{3} x^3 - \frac{1}{2} x^2 \sin 2x - \frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x \right|_0^{\pi}$$

$$= \frac{\pi}{2} \left(\frac{1}{3} \pi^3 - \frac{\pi}{2} \right)$$

$$= \frac{\pi^4}{6} - \frac{\pi^2}{4} \quad unit^3$$

c) $V = 2\pi \int_0^{\pi} x (x \sin x) \, dx$

$$= 2\pi \int_0^{\pi} \left(x^2 \sin x \right) dx$$

		$\int \sin x \, dx$
+	x^2	$-\cos x$
–	$2x$	$-\sin x$
+	2	$\cos x$

$$= 2\pi \left(-x^2 \cos x + 2x \sin x + 2 \cos x \right|_0^{\pi}$$

$$= 2\pi \left(\pi^2 - 2 - 2 \right)$$

$$\boxed{= 2\pi^3 - 8\pi \quad unit^3}$$

d) $m = \int_0^{\pi} x\sin x\, dx$ From (***a***)

$$= -x\cos x + \sin x \Big|_0^{\pi}$$

$$\boxed{= \pi}$$

$M_x = \frac{1}{2}\int_0^{\pi} (x\sin x)^2\, dx$ From (***b***)

$$\boxed{= \frac{1}{2}\left(\frac{\pi^3}{6} - \frac{\pi}{4}\right)}$$

$M_y = \int_0^{\pi} x(x\sin x)\, dx$ From (***c***)

$$= \frac{2\pi^3 - 8\pi}{2\pi}$$

$$\boxed{= \pi^2 - 4}$$

$$\overline{x} = \frac{M_y}{m} = \boxed{\frac{\pi^2 - 4}{\pi}} \quad \boxed{\approx 1.8684}$$

$$\overline{y} = \frac{M_x}{m} = \frac{1}{\pi}\left(\frac{\pi^3}{12} - \frac{\pi}{8}\right)$$

$$\boxed{= \frac{\pi^2}{12} - \frac{1}{8}} \quad \boxed{\approx 0.6975}$$

$$\therefore \text{ The centre of mass: } \boxed{\left(\frac{\pi^2 - 4}{\pi},\ \frac{\pi^2}{12} - \frac{1}{8}\right)}$$

Exercise

The region R is bounded by the curve $y = \ln x$ and the x-*axis* on the interval $[1, e]$. Find the volume of the solid that is generated when R is revolved in the following ways

a) About the x-*axis*

b) About the y-*axis*

c) About the line $x = 1$

d) About the line $y = 1$

Solution

a) About the x-*axis*

$$V = \pi \int_1^e (\ln x)^2\, dx$$

Let $y = \ln x \quad \Rightarrow \quad x = e^y$

$$dx = e^y dy$$

$$V = \pi \int_1^e y^2\, e^y\ dy$$

		$\int e^y\ dy$
+	y^2	e^y
−	$2y$	e^y
+	2	e^y

$$= \pi \left(y^2 - 2y + 2\right) e^y \ \Big|_1^e$$

$$= \pi x \left((\ln x)^2 - 2\ln x + 2\right) \ \Big|_1^e$$

$$= \pi \left(e(1-2+2) - 2\right)$$

$$= \pi (e-2) \ \ unit^3$$

b) About the *y-axis*

$$V = 2\pi \int_1^e x \ln x\ dx$$

		$\int x\ dx$
+	$\ln x$	$\frac{1}{2}x^2$
−	$\frac{1}{x}$	$\int \frac{1}{2}x^2$

$$= 2\pi \left[\frac{1}{2}x^2 \ln x - \int_1^e \frac{1}{2}x^2 \frac{1}{x}\ dx \right]$$

$$= 2\pi \left[\frac{1}{2}x^2 \ln x - \frac{1}{2}\int_1^e x\ dx \right]$$

$$= \pi\left(x^2 \ln x - \frac{1}{2}x^2 \right|_1^e$$

$$= \pi\left(e^2 \ln e - \frac{1}{2}e^2 + \frac{1}{2}\right)$$

$$\boxed{= \frac{\pi}{2}\left(e^2 + 1\right) \ \ unit^3}$$

c) About the line $x = 1$

$$V = 2\pi \int_1^e (x-1)\ln x \, dx$$

		$\int (x-1)\,dx$
+	$\ln x$	$\frac{1}{2}x^2 - x$
−	$\frac{1}{x}$	$\int\left(\frac{1}{2}x^2 - x\right)$

$$= 2\pi\left[\left(\frac{1}{2}x^2 - x\right)\ln x - \int_1^e \left(\frac{1}{2}x^2 - x\right)\frac{1}{x}\, dx\right]$$

$$= 2\pi\left(\frac{1}{2}x^2 - x\right)\ln x \ - \ 2\pi \int_1^e \left(\frac{1}{2}x - 1\right) dx$$

$$= 2\pi\left(\left(\frac{1}{2}x^2 - x\right)\ln x - \left(\frac{1}{4}x^2 - x\right) \right|_1^e$$

$$= 2\pi\left(\frac{1}{2}e^2 - e - \frac{1}{4}e^2 + e + \frac{1}{4} - 1\right)$$

$$= 2\pi\left(\frac{1}{4}e^2 - \frac{3}{4}\right)$$

$$\boxed{= \frac{\pi}{2}\left(e^2 - 3\right) \ \ unit^3}$$

d) About the line $y = 1$

$$V = \pi \int_1^e \left(1 - (1 - \ln x)^2\right) dx$$

$$= \pi \int_1^e \left(1 - 1 + 2\ln x - (\ln x)^2\right) dx$$

$$= \pi \int_1^e \left(2\ln x - (\ln x)^2\right) dx$$

Let $y = \ln x \quad \Rightarrow \quad x = e^y$

$$dx = e^y dy$$

		$\int e^y \ dy$
+	y^2	e^y
−	$2y$	e^y
+	2	e^y

$$\int (\ln x)^2 \, dx = \left(y^2 - 2y + 2\right) e^y$$

$$= x\left((\ln x)^2 - 2\ln x + 2\right)$$

		$\int 2dx$
+	$\ln x$	$2x$
−	$\frac{1}{x}$	$\int 2x$

$$\int 2\ln x \ dx = 2x\ln x - \int 2x\frac{1}{x}dx$$

$$= 2x\ln x - 2\int dx$$

$$= 2x\ln x - 2x$$

$$V = \pi\int_1^e \left(2\ln x - (\ln x)^2\right) \, dx$$

$$= \pi\left(2x\ln x - 2x - x\left((\ln x)^2 - 2\ln x + 2\right)\right) \Big|_1^e$$

$$= \pi\left(2x\ln x - 2x - x(\ln x)^2 + 2x\ln x - 2x\right) \Big|_1^e$$

$$= \pi\left(4x\ln x - 4x - x(\ln x)^2\right) \Big|_1^e$$

$$= \pi(4e - 4e - e + 4)$$

$$= \pi(4 - e) \ \ unit^3$$

Exercise

A string stretched between the two points $(0,\ 0)$ and $(2,\ 0)$ is plucked by displacing the string h units at its midpoint. The motion of the string is modeled by a ***Fourier Sine series*** whose coefficients are given by

$$b_n = h\int_0^1 x\sin\frac{n\pi x}{2}dx + h\int_1^2 (-x+2)\sin\frac{n\pi x}{2}dx$$

Find b_n

Solution

$$b_n = h\int_0^1 x\sin\frac{n\pi x}{2}dx + h\int_1^2 (-x+2)\sin\frac{n\pi x}{2}dx$$

		$\int \sin\frac{n\pi x}{2}dx$	
+	x	$-\frac{2}{n\pi}\cos\frac{n\pi x}{2}$	$-x+2$
−	1	$-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}$	-1

$$= h\left(-\frac{2}{n\pi}x\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\ \Big|_0^1 + h\left(-\frac{2}{n\pi}(2-x)\cos\frac{n\pi x}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\ \Big|_1^2\right.\right.$$

$$= h\left(-\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right) + h\left(-\frac{4}{n^2\pi^2}\sin n\pi + \frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right)$$

$$= h\left(\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\sin n\pi + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right) \qquad \left(\cos\frac{n\pi}{2} = 0 \quad \sin n\pi = 0\right)$$

$$= \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\boxed{= (-1)^n\frac{8h}{n^2\pi^2}}$$

Solution *Section* 2.2 – Trigonometric Integrals

Exercise

Evaluate the integral $\displaystyle\int \sin^5 \frac{x}{2}\,dx$

Solution

$$
\begin{aligned}
\sin^5 \frac{x}{2} &= \left(\sin^2 \frac{x}{2}\right)^2 \sin \frac{x}{2} \\
&= \left(1-\cos^2 \frac{x}{2}\right)^2 \sin \frac{x}{2} \\
&= \left(1-2\cos^2 \frac{x}{2}+\cos^4 \frac{x}{2}\right)\sin \frac{x}{2}
\end{aligned}
$$

$$d\left(\cos \frac{x}{2}\right) = -\frac{1}{2}\sin \frac{x}{2}\,dx \quad\rightarrow\quad -2d\left(\cos \frac{x}{2}\right) = \sin \frac{x}{2}\,dx$$

$$
\begin{aligned}
\int \sin^5 \frac{x}{2}\,dx &= -2\int \left(1-2\cos^2 \frac{x}{2}+\cos^4 \frac{x}{2}\right) d\left(\cos \frac{x}{2}\right) \\
&= -2\left(\cos \frac{x}{2}-\frac{2}{3}\cos^3 \frac{x}{2}+\frac{1}{5}\cos^5 \frac{x}{2}\right)+C \\
&\boxed{= -2\cos \frac{x}{2}+\frac{4}{3}\cos^3 \frac{x}{2}-\frac{2}{5}\cos^5 \frac{x}{2}+C}
\end{aligned}
$$

Exercise

Evaluate $\displaystyle\int \sin^4 6\theta\,d\theta$

Solution

$$\int \sin^4 6\theta\,d\theta = \int \left(\frac{1-\cos 12\theta}{2}\right)^2 d\theta \qquad\qquad \sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$$

$$= \frac{1}{4}\int \left(1-2\cos 12\theta+\cos^2 12\theta\right)d\theta \qquad\qquad \cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$$

$$= \frac{1}{4}\int \left(1-2\cos 12\theta+\frac{1}{2}+\frac{1}{2}\cos 24\theta\right)d\theta$$

$$\boxed{= \frac{1}{4}\left(\frac{3}{2}\theta-\frac{1}{6}\sin 12\theta+\frac{1}{48}\sin 24\theta\right)+C}$$

Exercise

Evaluate $\displaystyle\int x^2 \sin^2 x\,dx$

Solution

		$\int \cos 2x\, dx$
+	x^2	$\frac{1}{2}\sin 2x$
−	$2x$	$-\frac{1}{4}\cos 2x$
+	2	$-\frac{1}{8}\sin 2x$

$$\begin{aligned}\int x^2 \sin^2 x\, dx &= \frac{1}{2}\int x^2 (1-\cos 2x)\, dx \\ &= \frac{1}{2}\int \left(x^2 - x^2\cos 2x\right) dx \\ &\boxed{= \frac{1}{2}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 \sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x\right) + C}\end{aligned}$$

Exercise

Evaluate $\int \sin^3 3x\, dx$

Solution

$$\begin{aligned}\int \sin^3 3x\, dx &= \int \sin^2 3x(\sin 3x)\, dx \qquad\qquad d(\cos 3x) = -3\sin 3x\, dx \qquad \cos^2\alpha + \sin^2\alpha = 1 \\ &= -\frac{1}{3}\int \left(1-\cos^2 3x\right)\, d(\cos 3x) \\ &= -\frac{1}{3}\left(\cos 3x - \frac{1}{3}\cos^3 3x\right) + C \\ &\boxed{= \frac{1}{9}\cos^3 3x - \frac{1}{3}\cos 3x + C}\end{aligned}$$

Exercise

Evaluate the integral $\int \sin^5 x\, dx$

Solution

$$\begin{aligned}\int \sin^5 x\, dx &= \int \sin^4 x\, \sin x\, dx \\ &= -\int \left(1-\cos^2 x\right)^2 d(\cos x)\end{aligned}$$

$$= -\int \left(1 - 2\cos^2 x + \cos^4 x\right) d\left(\cos x\right)$$

$$\boxed{= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C}$$

Exercise

Evaluate the integral $\int 8\cos^4 2\pi x \; dx$

Solution

$$\int 8\cos^4 2\pi x \; dx = 8\int \left(\cos 2\pi x\right)^4 dx \qquad \cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$= 8\int \left(\frac{1+\cos 4\pi x}{2}\right)^2 dx$$

$$= 2\int \left(1+\cos 4\pi x\right)^2 dx$$

$$= 2\int \left(1 + 2\cos 4\pi x + \cos^2 4\pi x\right) dx$$

$$= 2\int dx + 4\int \cos 4\pi x dx + 2\int \cos^2 4\pi x dx$$

$$= 2x + 4\frac{1}{4\pi}\cos 4\pi x + 2\int \frac{1+\cos 8\pi x}{2} dx$$

$$= 2x + \frac{1}{\pi}\cos 4\pi x + \int \left(1 + \cos 8\pi x\right) dx$$

$$= 2x + \frac{1}{\pi}\sin 4\pi x + x + \frac{1}{8\pi}\sin 8\pi x + C$$

$$\boxed{= 3x + \frac{1}{\pi}\sin 4\pi x + \frac{1}{8\pi}\sin 8\pi x + C}$$

Exercise

Evaluate the integral $\int x\cos^3 x \; dx$

Solution

$$\int x\cos^3 x dx = \int x\cos^2 x \cos x \; dx$$

$$= \int x\left(1 - \sin^2 x\right)\cos x \; dx \qquad \cos^2\alpha + \sin^2\alpha = 1$$

$$= \int x\cos x\,dx - \int x\sin^2 x\cos x\,dx$$

$$u = x \qquad dv = \cos x dx \qquad\qquad u = x \qquad dv = \sin^2 x\cos x dx$$

$$du = dx \qquad v = \sin x \qquad\qquad du = dx \qquad v = \frac{1}{3}\sin^3 x$$

$$= x\sin x - \int \sin x\,dx - \left(\frac{1}{3}x\sin^3 x - \frac{1}{3}\int \sin^3 x\,dx\right)$$

$$= x\sin x + \cos x - \frac{1}{3}x\sin^3 x + \frac{1}{3}\int \sin^2 x\sin x\,dx$$

$$= x\sin x + \cos x - \frac{1}{3}x\sin^3 x - \frac{1}{3}\int \left(1-\cos^2 x\right)d\left(\cos x\right)$$

$$= x\sin x + \cos x - \frac{1}{3}x\sin^3 x - \frac{1}{3}\left(\cos x - \frac{1}{3}\cos^3 x\right) + C$$

$$= x\sin x + \cos x - \frac{1}{3}x\sin^3 x - \frac{1}{3}\cos x + \frac{1}{9}\cos^3 x + C$$

$$= x\sin x + \frac{2}{3}\cos x - \frac{1}{3}x\sin^3 x + \frac{1}{9}\cos^3 x + C$$

Exercise

Evaluate the integral $\int \cos^4 x\,dx$

Solution

$$\int \cos^4 x\,dx = \frac{1}{4}\int (1+\cos 2x)^2\,dx \qquad\qquad \cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \cos^2 2x\right)dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right)dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right)dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 4x\right) + C$$

Exercise

Evaluate the integral $\displaystyle\int \cos^4 5x\, dx$

Solution

$$\int \cos^4 5x\, dx = \frac{1}{4}\int (1+\cos 10x)^2\, dx \qquad\qquad \cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$= \frac{1}{4}\int \left(1+2\cos 10x+\cos^2 10x\right) dx$$

$$= \frac{1}{4}\int \left(1+2\cos 10x+\frac{1}{2}+\frac{1}{2}\cos 20x\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2}+2\cos 10x+\frac{1}{2}\cos 20x\right) dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x+\frac{1}{5}\sin 10x+\frac{1}{40}\sin 20x\right)+C$$

$$\boxed{= \frac{3}{8}x+\frac{1}{20}\sin 10x+\frac{1}{160}\sin 20x+C}$$

Exercise

Evaluate $\displaystyle\int \cos^2 3x\, dx$

Solution

$$\int \cos^2 3x\, dx = \frac{1}{2}\int (1+\cos 6x)\, dx \qquad\qquad \cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$$

$$\boxed{= \frac{1}{2}\left(x+\frac{1}{6}\sin 6x\right)+C}$$

Exercise

Evaluate $\displaystyle\int \cos^3 \frac{x}{3}\, dx$

Solution

$$\int \cos^3 \frac{x}{3}\, dx = \int \cos^2 \frac{x}{3}\cos\frac{x}{3}\, dx$$

$$= 3\int \left(1-\sin^2 \frac{x}{3}\right)\, d\left(\sin\frac{x}{3}\right)$$

$$\boxed{= 3\sin\frac{x}{3}-\sin^3\frac{x}{3}+C}$$

Exercise

Evaluate the integral $\displaystyle\int \cos^2 4x\, dx$

Solution

$$\int \cos^2 4x\, dx = \frac{1}{2}\int (1+\cos 8x)\, dx \qquad\qquad \cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$$

$$\boxed{= \frac{1}{2}\left(x + \frac{1}{8}\sin 8x\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sqrt{1+\cos\frac{x}{2}}\, dx$

Solution

$$2\cos^2\alpha = 1+\cos 2\alpha$$

$$2\alpha = \frac{x}{2} \quad \to \quad \alpha = \frac{x}{4}$$

$$1+\cos\frac{x}{2} = 2\cos^2\frac{x}{4}$$

$$\int \sqrt{1+\cos\frac{x}{2}}\, dx = \int \sqrt{2\cos^2\frac{x}{4}}\, dx$$

$$= \sqrt{2}\int \cos\frac{x}{4}\, dx$$

$$\boxed{= 4\sqrt{2}\sin\frac{x}{4} + C}$$

Exercise

Evaluate $\displaystyle\int \sec^4 2x\, dx$

Solution

$$\int \sec^4 2x\, dx = \int \left(1+\tan^2 2x\right)\sec^2 2x\, dx$$

$$= \frac{1}{2}\int \left(1+\tan^2 2x\right)\, d(\tan 2x)$$

$$\boxed{= \frac{1}{2}\tan 2x + \frac{1}{6}\tan^3 2x + C}$$

Exercise

Evaluate the integral $\displaystyle\int 6\sec^4 x\, dx$

Solution

$$\begin{aligned}\int 6\sec^4 x\, dx &= 6\int \sec^2 x\ \sec^2 x\, dx\\ &= 6\int \sec^2 x\left(\tan^2 x+1\right) dx\\ &= 6\int \left(\tan^2 x+1\right) d\left(\tan x\right)\\ &= 6\left(\tfrac{1}{3}\tan^3 x+\tan x\right)+C\\ &\boxed{= 2\tan^3 x+6\tan x+C}\end{aligned}$$

Exercise

Evaluate $\displaystyle\int \sec^3 \pi x\, dx$

Solution

$$u=\sec \pi x \qquad dv=\sec^2 \pi x\, dx$$

$$du=\pi \sec \pi x \tan \pi x \qquad v=\frac{1}{\pi}\tan \pi x$$

$$\begin{aligned}\int \sec^3 \pi x\, dx &= \frac{1}{\pi}\sec \pi x \tan \pi x - \int \sec \pi x \tan^2 \pi x\, dx\\ &= \frac{1}{\pi}\sec \pi x \tan \pi x - \int \sec \pi x\left(\sec^2 \pi x-1\right) dx\end{aligned}$$

$$\int \sec^3 \pi x\, dx = \frac{1}{\pi}\sec \pi x \tan \pi x - \int \sec^3 \pi x\, dx + \int \sec \pi x\, dx$$

$$2\int \sec^3 \pi x\, dx = \frac{1}{\pi}\sec \pi x \tan \pi x + \ln\left|\sec \pi x+\tan \pi x\right| + C_1$$

$$\boxed{\int \sec^3 \pi x\, dx = \frac{1}{2\pi}\sec \pi x \tan \pi x + \frac{1}{2}\ln\left|\sec \pi x+\tan \pi x\right| + C}$$

Exercise

Evaluate the integral $\int \sec 4x\, dx$

Solution

$$\begin{aligned}\int \sec 4x\, dx &= \frac{1}{4}\int \sec 4x\, d(4x) \\ &= \frac{1}{4}\int \sec 4x \frac{\sec 4x + \tan 4x}{\sec 4x + \tan 4x} d(4x) \\ &= \frac{1}{4}\int \frac{\sec^2 4x + \sec 4x \tan 4x}{\sec 4x + \tan 4x}\, d(4x) \\ &= \frac{1}{4}\int \frac{1}{\sec 4x + \tan 4x}\, d(\sec 4x + \tan 4x) \\ &\underline{= \frac{1}{4}\ln\left|\sec 4x + \tan 4x\right| + C}\,\Big|\end{aligned}$$

Exercise

Evaluate the integral $\int \csc^6 x\, dx$

Solution

$$\begin{aligned}\int \csc^6 x\, dx &= \int \csc^4 x\ \csc^2 x\, dx \\ &= -\int \left(\cot^2 x + 1\right)^2\, d(\cot x) \\ &= -\int \left(\cot^4 x + 2\cot^2 x + 1\right)\, d(\cot x) \\ &\underline{= -\frac{1}{5}\cot^5 x - \frac{2}{3}\cot^3 x - \cot x + C}\,\Big|\end{aligned}$$

Exercise

Evaluate the integral $\int \tan^5 \frac{x}{2}\, dx$

Solution

$$\int \tan^5 \frac{x}{2}\, dx = 2\int \tan^2 \frac{x}{2}\ \tan^3 \frac{x}{2}\, d\left(\frac{x}{2}\right)$$

$$= 2\int \left(\sec^2\frac{x}{2} - 1\right)\ \tan^3\frac{x}{2}\ d\left(\frac{x}{2}\right)$$

$$= 2\int \sec^2\frac{x}{2}\tan^3\frac{x}{2}\ d\left(\frac{x}{2}\right) - 2\int \tan^2\frac{x}{2}\ \ \tan\frac{x}{2}\ d\left(\frac{x}{2}\right)$$

$$= 2\int \tan^3\frac{x}{2}\ d\left(\tan\frac{x}{2}\right) - 2\int \left(\sec^2\frac{x}{2} - 1\right)\ \tan\frac{x}{2}\ d\left(\frac{x}{2}\right)$$

$$= \frac{1}{2}\tan^4\frac{x}{2} - 2\int \sec^2\frac{x}{2}\ \ \tan\frac{x}{2}\ d\left(\frac{x}{2}\right) + 2\int \tan\frac{x}{2}\ d\left(\frac{x}{2}\right)$$

$$= \frac{1}{2}\tan^4\frac{x}{2} - 2\int \tan\frac{x}{2}\ d\left(\tan\frac{x}{2}\right) + 2\int \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\ d\left(\frac{x}{2}\right)$$

$$= \frac{1}{2}\tan^4\frac{x}{2} - \tan^2\frac{x}{2} - 2\int \frac{1}{\cos\frac{x}{2}}\ d\left(\cos\frac{x}{2}\right)$$

$$= \frac{1}{2}\tan^4\frac{x}{2} - \tan^2\frac{x}{2} - 2\ln\left|\cos\frac{x}{2}\right| + C$$

$$= \frac{1}{2}\tan^4\frac{x}{2} - \tan^2\frac{x}{2} + 2\ln\left|\sec\frac{x}{2}\right| + C$$

Exercise

Evaluate the integral $\int \tan^5 5\ dx$

Solution

$$\int \tan^5 x\ dx = \int \tan^2 x\ \tan^3 x\ dx$$

$$= \int \left(\sec^2 x - 1\right)\ \tan^3 x\ dx$$

$$= \int \sec^2 x\ \tan^3 x\ dx - \int \tan^2 x\ \tan x\ dx$$

$$= \int \tan^3 x\ d\left(\tan x\right) - \int \left(\sec^2 x - 1\right)\ \tan x\ dx$$

$$= \frac{1}{4}\tan^4 x - \int \sec^2 x\ \tan x\ dx + \int \tan x\ dx$$

$$= \frac{1}{4}\tan^4 x - \int \tan x\ d\left(\tan x\right) + \int \frac{\sin x}{\cos x}\ dx$$

$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \int \frac{1}{\cos x}\, d(\cos x)$$

$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C$$

$$\boxed{= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + C}$$

Exercise

Evaluate $\int \tan^6 3x\, dx$

Solution

$$\int \tan^6 3x\, dx = \int \left(\sec^2 3x - 1\right)\tan^4 3x\, dx$$

$$= \frac{1}{3}\int \tan^4 3x\, d(\tan 3x) - \int \left(\sec^2 3x - 1\right)\tan^2 3x\, dx$$

$$= \frac{1}{15}\tan^5 3x - \int \sec^2 3x \tan^2 3x\, dx + \int \tan^2 3x\, dx$$

$$= \frac{1}{15}\tan^5 3x - \frac{1}{3}\int \tan^2 3x\, d(\tan 3x) + \int \left(\sec^2 3x - 1\right)\, dx$$

$$= \frac{1}{15}\tan^5 3x - \frac{1}{9}\tan^3 3x + \frac{1}{3}\int d(\tan 3x) - \int dx$$

$$\boxed{= \frac{1}{15}\tan^5 3x - \frac{1}{9}\tan^3 3x + \frac{1}{3}\tan 3x - x + C}$$

Exercise

Evaluate the integral $\int 20\tan^6 x\, dx$

Solution

$$\int 20\tan^6 x\, dx = 20\int \left(\sec^2 x - 1\right)\tan^4 x\, dx$$

$$= 20\int \tan^4 x\, d(\tan x) - 20\int \left(\sec^2 x - 1\right)\tan^2 x\, dx$$

$$= 4\tan^5 x - 20\int \sec^2 x \tan^2 x\, dx + 20\int \tan^2 x\, dx$$

$$= 4\tan^5 x - 20\int \tan^2 x\, d(\tan x) + 20\int \left(\sec^2 x - 1\right)\, dx$$

$$\boxed{= 4\tan^5 x - \frac{20}{3}\tan^3 x + 20\tan x - 20x + C}$$

Exercise

Evaluate the integral $\int \tan^4 x \, dx$

Solution

$$\begin{aligned}
\int \tan^4 x \, dx &= \int \left(\tan^2 x\right)\left(\tan^2 x\right) \, dx \\
&= \int \tan^2 x\left(\sec^2 x - 1\right) \, dx \\
&= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\
&= \int \tan^2 x \, d(\tan x) - \int \left(\sec^2 x - 1\right) \, dx \\
&\boxed{= \frac{1}{3}\tan^3 x - \tan x + x + C}
\end{aligned}$$

Exercise

Evaluate the integral $\int \tan^3 \theta \, d\theta$

Solution

$$\begin{aligned}
\int \tan^3 \theta \, d\theta &= \int \tan^2 \theta \;\; \tan\theta \, d\theta \\
&= \int \left(\sec^2 \theta - 1\right) \;\; \tan\theta \, d\theta \\
&= \int \sec^2 \theta \;\; \tan\theta \, d\theta - \int \tan\theta \, d\theta \\
&= \int \tan\theta \, d(\tan\theta) - \ln\left|\sec\theta\right| \\
&\boxed{= \frac{1}{2}\tan^2 \theta - \ln\left|\sec\theta\right| + C}
\end{aligned}$$

Exercise

Evaluate the integral $\int \tan^3 4x\, dx$

Solution

$$\begin{aligned}\int \tan^3 4x\, dx &= \int \tan^2 4x\ \tan 4x\, dx\\ &= \int \left(\sec^2 4x - 1\right)\ \tan 4x\, dx\\ &= \int \sec^2 4x\ \tan 4x\, dx - \int \tan 4x\, dx\\ &= \frac{1}{4}\int \tan 4x\, d\left(\tan 4x\right) - \int \frac{\sin 4x}{\cos 4x} dx\\ &= \frac{1}{8}\tan^2 4x + \frac{1}{4}\int \frac{1}{\cos 4x} d\left(\cos 4x\right)\\ &= \frac{1}{8}\tan^2 4x + \frac{1}{4}\ln\left|\cos 4x\right| + C\end{aligned}$$

Exercise

Evaluate the integral $\int \cot^4 x\, dx$

Solution

$$\begin{aligned}\int \cot^4 x\, dx &= \int \cot^2 x\left(\csc^2 x - 1\right)\ dx\\ &= \int \cot^2 x\ \csc^2 x\, dx - \int \cot^2 x\, dx\\ &= -\int \cot^2 x\, d\left(\cot x\right) - \int \left(\csc^2 x - 1\right)\ dx\\ &= -\frac{1}{3}\cot^3 x - \cot x + x + C\end{aligned}$$

Exercise

Evaluate the integral $\int \cot^5 3x\, dx$

Solution

$$\int \cot^5 3x\ dx = \int \cot^3 3x\left(\csc^2 3x - 1\right)\ dx$$

$$= \int \cot^3 3x\ \csc^2 3x\ dx - \int \cot^3 3x\ dx$$

$$= -\frac{1}{3}\int \cot^3 3x\ d\left(\cot 3x\right) - \int \cot 3x\left(\csc^2 3x - 1\right)\ dx$$

$$= -\frac{1}{12}\cot^4 3x - \int \cot 3x \csc^2 3x\ dx + \int \cot 3x\ dx$$

$$= -\frac{1}{12}\cot^4 3x + \frac{1}{3}\int \cot 3x\ d\left(\cot 3x\right) + \int \frac{\cos 3x}{\sin 3x}\ dx$$

$$= -\frac{1}{12}\cot^4 3x + \frac{1}{6}\cot^2 3x + \frac{1}{3}\int \frac{1}{\sin 3x}\ d\left(\sin 3x\right)$$

$$\boxed{= -\frac{1}{12}\cot^4 x + \frac{1}{6}\cot^2 3x + \frac{1}{3}\ln\left|\sin 3x\right| + C}$$

Exercise

Evaluate the integral $\int \sin^2 x\ \cos^2 x\ dx$

Solution

$$\int \sin^2 x\ \cos^2 x\ dx = \frac{1}{4}\int (1 - \cos 2x)(1 + \cos 2x)\,dx$$

$$= \frac{1}{4}\int \left(1 - \cos^2 2x\right)dx$$

$$= \frac{1}{4}\int \left(1 - \frac{1}{2} - \frac{1}{2}\cos 4x\right)dx$$

$$= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right)dx$$

$$= \frac{1}{4}\left(\frac{1}{2}x - \frac{1}{8}\sin 4x\right) + C$$

$$\boxed{= \frac{1}{8}x - \frac{1}{32}\sin 4x + C}$$

Exercise

Evaluate the integral $\int \sin^2 x \ \cos^3 x \, dx$

Solution

$$\int \sin^2 x \ \cos^3 x \, dx = \int \sin^2 x \ \cos^2 x \ \cos x \, dx$$

$$= \int \sin^2 x \left(1 - \sin^2 x\right) \, d\left(\sin x\right)$$

$$= \int \left(\sin^2 x - \sin^4 x\right) \, d\left(\sin x\right)$$

$$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

Exercise

Evaluate the integral $\int \sin^2 x \ \cos^5 x \, dx$

Solution

$$\int \sin^2 x \ \cos^5 x \, dx = \int \sin^2 x \ \cos^4 x \ \cos x \, dx$$

$$= \int \sin^2 x \left(1 - \sin^2 x\right)^2 \, d\left(\sin x\right)$$

$$= \int \sin^2 x \left(1 - 2\sin^2 x + \sin^4 x\right) \, d\left(\sin x\right)$$

$$= \int \left(\sin^2 x - 2\sin^4 x + \sin^6 x\right) \, d\left(\sin x\right)$$

$$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C$$

Exercise

Evaluate the integral $\int \sin^3 x \ \cos^5 x \, dx$

Solution

$$\int \sin^3 x \ \cos^5 x \, dx = \int \sin^2 x \ \cos^5 x \ \sin x \, dx$$

$$= -\int \left(1-\cos^2 x\right)\ \cos^5 x\ d\left(\cos x\right)$$

$$= -\int \left(\cos^5 x - \cos^7 x\right)\ d\left(\cos x\right)$$

$$\boxed{= -\frac{1}{6}\cos^6 x + \frac{1}{8}\cos^8 x + C}$$

Exercise

Evaluate the integral $\int \sin^3 x\ \cos^4 x\ dx$

Solution

$$\int \sin^3 x\ \cos^4 x\ dx = \int \sin^2 x\ \sin x\ \cos^4 x\ dx$$

$$= -\int \left(1-\cos^2 x\right)\cos^4 x\ d\left(\cos x\right)$$

$$= -\int \left(\cos^4 x - \cos^6 x\right)\ d\left(\cos x\right)$$

$$= -\left(\frac{1}{5}\cos^5 x - \frac{1}{7}\cos^7 x\right) + C$$

$$\boxed{= \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C}$$

Exercise

Evaluate the integral $\int \sin^3 2x\ \cos^4 x\ dx$

Solution

$$\int \sin^3 2x\ \cos^4 x\ dx = \int \left(2\sin x\cos x\right)^3\ \cos^4 x\ dx$$

$$= 8\int \sin^3 x\ \cos^7 x\ dx$$

$$= -8\int \sin^2 x\ \cos^7 x\ d\left(\cos x\right)$$

$$= -8\int \left(1-\cos^2 x\right)\cos^7 x\ d\left(\cos x\right)$$

$$= -8\int\left(\cos^7 x - \cos^9 x\right)\, d\left(\cos x\right)$$

$$= -8\left(\frac{1}{8}\cos^8 x - \frac{1}{10}\cos^{10} x\right) + C$$

$$\boxed{= -\cos^8 x + \frac{4}{5}\cos^{10} x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sin^3 2x\ \cos^3 2x\, dx$

Solution

$$\int \sin^3 2x\ \cos^3 2x\, dx = \frac{1}{2}\int \sin^3 2x\ \cos^2 2x\ d\left(\sin 2x\right)$$

$$= \frac{1}{2}\int \sin^3 2x\ \left(1 - \sin^2 2x\right)\ d\left(\sin 2x\right)$$

$$= \frac{1}{2}\int \left(\sin^3 2x - \sin^5 2x\right)\ d\left(\sin 2x\right)$$

$$= \frac{1}{2}\left(\frac{1}{4}\sin^4 2x - \frac{1}{6}\sin^6 2x\right) + C$$

$$\boxed{= \frac{1}{8}\sin^4 2x - \frac{1}{12}\sin^6 2x + C}$$

Exercise

Evaluate $\displaystyle\int \sin^4 x \cos^2 x\, dx$

Solution

$$\int \sin^4 x \cos^2 x\, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{8}\int \left(1 - 2\cos 2x + \cos^2 2x\right)\left(1 + \cos 2x\right) dx$$

$$= \frac{1}{8}\int \left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right) dx$$

$$= \frac{1}{8}\int \left(1 - \cos 2x - \frac{1}{2} - \frac{1}{2}\cos 4x\right) dx + \frac{1}{8}\int \cos^2 2x \cos 2x\, dx$$

$$= \frac{1}{8}\int \left(\frac{1}{2} - \cos 2x - \frac{1}{2}\cos 4x\right) dx + \frac{1}{16}\int \left(1 - \sin^2 2x\right)\ d\left(\sin 2x\right)$$

$$=\frac{1}{8}\left(\frac{1}{2}x-\frac{1}{2}\sin 2x-\frac{1}{4}\sin 4x\right)+\frac{1}{16}\sin 2x-\frac{1}{48}\sin^3 2x+C$$

$$\boxed{=\frac{1}{16}x-\frac{1}{64}\sin 4x-\frac{1}{48}\sin^3 2x+C}$$

Exercise

Evaluate the integral $\int \sin^4 x\cos^3 x\ dx$

Solution

$$\int \sin^4 x\cos^3 x\ dx=\int \sin^4 x\left(1-\sin^2 x\right)\ d\left(\sin x\right)$$

$$=\int \left(\sin^4 x-\sin^6 x\right)\ d\left(\sin x\right)$$

$$\boxed{=\frac{1}{5}\sin^5 x-\frac{1}{7}\sin^7 x+C}$$

Exercise

Evaluate the integral $\int \sin^4 x\ \cos^4 x\ dx$

Solution

$$\int \sin^4 x\ \cos^4 x\ dx=\int \left(\sin^2 x\right)^2\left(\cos^2 x\right)^2\ dx$$

$$=\int \left(\frac{1-\cos 2x}{2}\right)^2\left(\frac{1+\cos 2x}{2}\right)^2\ dx$$

$$=\frac{1}{16}\int \left(1-\cos^2 2x\right)^2\ dx$$

$$=\frac{1}{16}\int \left(1-2\cos^2 2x+\cos^4 2x\right)\ dx$$

$$=\frac{1}{16}\int \left(1-1-\cos 4x+\left(\frac{1+\cos 4x}{2}\right)^2\right)\ dx$$

$$=\frac{1}{16}\int \left(-\cos 4x+\frac{1}{4}\left(1+2\cos 4x+\cos^2 4x\right)\right)\ dx$$

$$=\frac{1}{64}\int \left(-4\cos 4x+1+2\cos 4x+\cos^2 4x\right)\ dx$$

$$= \frac{1}{64}\int \left(1 - 2\cos 4x + \frac{1}{2} + \frac{1}{2}\cos 8x\right)\, dx$$

$$= \frac{1}{128}\int (3 - 4\cos 4x + \cos 8x)\, dx$$

$$= \frac{1}{128}\left(3x - \sin 4x + \frac{1}{8}\sin 8x\right) + C$$

Exercise

Evaluate the integral $\int \sin^4 x \, \cos^5 x \, dx$

Solution

$$\int \sin^4 x \, \cos^5 x \, dx = \int \sin^4 x \, \cos^4 x \, \cos x \, dx$$

$$= \int \sin^4 x \left(1 - \sin^2 x\right)^2 \, d(\sin x)$$

$$= \int \sin^4 x \left(1 - 2\sin^2 x + \sin^4 x\right) \, d(\sin x)$$

$$= \int \left(\sin^4 x - 2\sin^6 x + \sin^8 x\right) \, d(\sin x)$$

$$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$$

Exercise

Evaluate the integral $\int \sin^5 x \, \cos^5 x \, dx$

Solution

$$\int \sin^5 x \, \cos^5 x \, dx = \int \sin^5 x \, \cos^4 x (\cos x) \, dx$$

$$= \int \sin^5 x \left(1 - \sin^2 x\right)^2 \, d(\sin x)$$

$$= \int \sin^5 x \left(1 - 2\sin^2 x + \sin^4 x\right) \, d(\sin x)$$

$$= \int \left(\sin^5 x - 2\sin^7 x + \sin^9 x\right) \, d(\sin x)$$

$$= \frac{1}{6}\sin^6 x - \frac{1}{4}\sin^8 x + \frac{1}{10}\sin^{10} x + C$$

Exercise

Evaluate the integral $\int \sin^5 x \ \cos^{-2} x \ dx$

Solution

$$\int \sin^5 x \ \cos^{-2} x \ dx = \int \sin^4 x \ \cos^{-2} x \ \sin x \ dx$$

$$= -\int \left(1-\cos^2 x\right)^2 \cos^{-2} x \ d\left(\cos x\right)$$

$$= -\int \left(1-2\cos^2 x+\cos^4 x\right)\cos^{-2} x \ d\left(\cos x\right)$$

$$= -\int \left(\cos^{-2} x-2+\cos^2 x\right) \ d\left(\cos x\right)$$

$$= \cos^{-1} x+2\cos x-\frac{1}{3}\cos^3 x+C$$

$$\boxed{= \sec x+2\cos x-\frac{1}{3}\cos^3 x+C}$$

Exercise

Evaluate the integral $\int \sin 3x \cos^6 3x \ dx$

Solution

$$\int \sin 3x \cos^6 3x \ dx = -\frac{1}{3}\int \cos^6 3x \ d\left(\cos 3x\right)$$

$$\boxed{= -\frac{1}{21}\cos^7 3x+C}$$

Exercise

Evaluate the integral $\int \sin^4 2x \ \cos 2x \, dx$

Solution

$$d\left(\sin 2x\right) = 2\cos 2x dx$$

$$\int \sin^4 2x \ \cos 2x \, dx = \frac{1}{2}\int \sin^4 2x \ d\left(\sin 2x\right)$$

$$\boxed{= \frac{1}{10}\sin^5 2x+C}$$

Exercise

Evaluate the integral $\int \cos^3 2x \sin^5 2x\ dx$

Solution

$$\int \cos^3 2x \sin^5 2x\ dx = \int \left(\cos^2 2x\right)\cos 2x \sin^5 2x\ dx \qquad d\left(\sin 2x\right) = 2\cos 2x\ dx$$

$$= \int \left(1-\sin^2 2x\right)\sin^5 2x\ \left(\tfrac{1}{2} d\sin 2x\right)$$

$$= \frac{1}{2}\int \left(\sin^5 2x - \sin^7 2x\right)\ \left(d\sin 2x\right)$$

$$= \frac{1}{2}\left(\frac{1}{6}\sin^6 2x - \frac{1}{8}\sin^8 2x\right) + C$$

$$\boxed{= \frac{1}{12}\sin^6 2x - \frac{1}{16}\sin^8 2x + C}$$

Exercise

Evaluate the integral $\int 16\sin^2 x\cos^2 x dx$

Solution

$$\int 16\sin^2 x\cos^2 xdx = 16\int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx \qquad \cos^2\alpha = \frac{1+\cos 2\alpha}{2} \quad \sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$= 4\int \left(1-\cos^2 2x\right)dx$$

$$= 4\int \left(1-\frac{1+\cos 4x}{2}\right)dx$$

$$= 4\int \frac{1-\cos 4x}{2}dx$$

$$= 2\left(x-\frac{1}{4}\sin 4x\right)+C$$

$$= 2x - \frac{1}{2}\left(2\sin 2x\cos 2x\right)+C$$

$$= 2x - \left(2\sin x\cos x\right)\left(2\cos^2 x - 1\right)+C$$

$$\boxed{= 2x - 4\sin x\cos^3 x + 2\sin x\cos x + C}$$

Exercise

Evaluate the integral $\int \sin 2x \cos 3x \; dx$

Solution

$$\int \sin 2x \cos 3x \; dx = \frac{1}{2}\int \left(\sin 5x + \sin(-x)\right) \; dx \qquad \sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$= \frac{1}{2}\int \left(\sin 5x - \sin x\right) \; dx$$

$$= \frac{1}{2}\left(-\frac{1}{5}\cos 5x + \cos x\right) + C$$

$$\boxed{= \frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C}$$

Exercise

Evaluate the integral $\int \sin^2 \theta \cos 3\theta \; d\theta$

Solution

$$\int \sin^2\theta \cos 3\theta \; d\theta = \int \frac{1-\cos 2\theta}{2}\cos 3\theta \; d\theta \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \frac{1}{2}\int \left(\cos 3\theta - \cos 2\theta \cos 3\theta\right) \; d\theta$$

$$= \frac{1}{2}\int \cos 3\theta \; d\theta - \frac{1}{2}\int \cos 2\theta \cos 3\theta \; d\theta$$

$$= \frac{1}{6}\sin 3\theta - \frac{1}{2}\int \frac{1}{2}\left(\cos(5\theta) + \cos(-\theta)\right) \; d\theta \qquad \cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$

$$= \frac{1}{6}\sin 3\theta - \frac{1}{4}\left(\frac{1}{5}\sin 5\theta + \sin\theta\right) + C$$

$$\boxed{= \frac{1}{6}\sin 3\theta - \frac{1}{20}\sin 5\theta - \frac{1}{4}\sin\theta + C}$$

Exercise

Evaluate the integral $\int \cos^3\theta \sin 2\theta \; d\theta$

Solution

$$\int \cos^3\theta \sin 2\theta \; d\theta = \int \cos^3\theta \left(2\sin\theta\cos\theta\right) \; d\theta \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= -2\int \cos^4\theta \; d(\cos\theta) \qquad\qquad d(\cos\theta) = -\sin\theta d\theta$$

$$\boxed{= -\frac{2}{5}\cos^5\theta + C}$$

Exercise

Evaluate the integral $\int \sin^{-3/2} x \;\cos^3 x \; dx$

Solution

$$\int \sin^{-3/2} x \;\cos^3 x \; dx = \int \sin^{-3/2} x \;\cos^2 x \;\cos x \; dx$$

$$= \int \sin^{-3/2} x \;\left(1-\sin^2 x\right)\; d(\sin x)$$

$$= \int \left(\sin^{-3/2} x - \sin^{1/2} x\right)\; d(\sin x)$$

$$\boxed{= -2\sin^{-1/2} x - \frac{2}{3}\sin^{3/2} x + C}$$

Exercise

Evaluate the integral $\int \sin^3 x \;\cos^{3/2} x \; dx$

Solution

$$\int \sin^3 x \;\cos^{3/2} x \; dx = \int \sin^2 x \;\cos^{3/2} x \;\sin x \; dx$$

$$= -\int \left(1-\cos^2 x\right)\;\cos^{3/2} x \; d(\cos x)$$

$$= \int \left(-\cos^{3/2} x + \cos^{7/2} x\right)\; d(\cos x)$$

$$\boxed{= \frac{2}{9}\cos^{9/2} x - \frac{2}{5}\cos^{5/2} + C}$$

Exercise

Evaluate the integral $\int \sin\theta \sin 2\theta \sin 3\theta \; d\theta$

Solution

$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$

$$\int \sin\theta \sin 2\theta \sin 3\theta \, d\theta = \int \tfrac{1}{2}\left(\cos(1-2)\theta - \cos(1+2)\theta\right) sin3\theta \, d\theta$$

$$= \frac{1}{2}\int \left(\cos(-\theta) - \cos(3\theta)\right)\sin 3\theta \, d\theta$$

$$= \frac{1}{2}\int \cos\theta \ \sin 3\theta \, d\theta - \frac{1}{2}\int \cos 3\theta \ \sin 3\theta \, d\theta$$

$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$

$$= \frac{1}{4}\int (\sin 4\theta + \sin 2\theta) \, d\theta - \frac{1}{4}\int \left(\sin 6\theta + \sin(0)\right) \, d\theta$$

$$= \frac{1}{4}\left(-\frac{1}{4}\cos 4\theta - \frac{1}{2}\cos 2\theta\right) + \frac{1}{24}\cos 6\theta + C$$

$$\boxed{= -\frac{1}{16}\cos 4\theta - \frac{1}{8}\cos 2\theta + \frac{1}{24}\cos 6\theta + C}$$

Exercise

Evaluate the integral $\int \sin 3x \ \cos 6x \, dx$

Solution

$$\int \sin 3x \ \cos 6x \, dx = \frac{1}{2}\int \left(\sin 9x + \sin(-3x)\right) dx$$

$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$

$$= \frac{1}{2}\int (\sin 10x - \sin 3x) \, dx$$

$$= \frac{1}{2}\left(-\frac{1}{10}\cos 10x + \frac{1}{3}\cos 3x\right) + C$$

$$\boxed{= \frac{1}{6}\cos 3x - \frac{1}{20}\cos 10x + C}$$

Exercise

Evaluate the integral $\int \sin 3x \cos 7x \, dx$

Solution

$$\int \sin 3x \cos 7x \, dx = \frac{1}{2}\int \left(\sin 10x + \sin(-4x)\right) dx$$

$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$

$$= \frac{1}{2}\int \left(\sin 10x - \sin 4x\right) dx$$

$$= \frac{1}{2}\left(-\frac{1}{10}\cos 10x + \frac{1}{4}\cos 4x\right) + C$$

$$\boxed{= \frac{1}{8}\cos 4x - \frac{1}{20}\cos 10x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sin 5x \cos 4x \; dx$

Solution

$$\int \sin 5x \cos 4x \; dx = \frac{1}{2}\int \left(\sin x + \sin 9x\right) dx$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$= \frac{1}{2}\left(-\cos x - \frac{1}{9}\cos x9x\right) + C$$

$$\boxed{= \frac{1}{2} - \cos x - \frac{1}{18}\cos x9x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \cos 2\theta \;\cos 6\theta \; d\theta$

Solution

$$\int \cos 2\theta \;\cos 6\theta \; d\theta = \frac{1}{2}\int \left(\cos 8\theta + \cos(-4\theta)\right) \; d\theta$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$= \frac{1}{2}\int \left(\cos 8\theta + \cos 4\theta\right) \; d\theta$$

$$= \frac{1}{2}\left(\frac{1}{8}\sin 8\theta + \frac{1}{4}\sin 4\theta\right) + C$$

$$\boxed{= \frac{1}{16}\sin 8\theta + \frac{1}{8}\sin 4\theta + C}$$

Exercise

Evaluate the integral $\displaystyle\int \cos 5\theta \;\cos 3\theta \; d\theta$

Solution

$$\int \cos 5\theta \ \cos 3\theta \ d\theta = \frac{1}{2}\int (\cos 8\theta + \cos 2\theta) \ d\theta \qquad \cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

$$= \frac{1}{2}\left(\frac{1}{8}\sin 8\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$\boxed{= \frac{1}{16}\sin 8\theta + \frac{1}{4}\sin 2\theta + C}$$

Exercise

Evaluate the integral $\int \sin 2\theta \ \cos 4\theta \ d\theta$

Solution

$$\int \sin 2\theta \ \cos 4\theta \ d\theta = \frac{1}{2}\int (\sin 6\theta + \sin(-2\theta)) \ d\theta \qquad \sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$= \frac{1}{2}\int (\sin 6\theta - \sin 2\theta) \ d\theta$$

$$= \frac{1}{2}\left(-\frac{1}{6}\cos 6\theta + \frac{1}{2}\cos 2\theta\right) + C$$

$$\boxed{= \frac{1}{4}\cos 4\theta - \frac{1}{12}\cos 6\theta + C}$$

Exercise

Evaluate the integral $\int \sin(-7\theta) \ \cos 6\theta \ d\theta$

Solution

$$\int \sin(-7\theta) \ \cos 6\theta \ d\theta = -\int \sin 7\theta \ \cos 6\theta \ d\theta$$

$$= -\frac{1}{2}\int (\sin 13\theta + \sin\theta) \ d\theta \qquad \sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$= -\frac{1}{2}\left(-\frac{1}{13}\cos 13\theta - \cos\theta\right) + C$$

$$\boxed{= \frac{1}{26}\cos 13\theta + \frac{1}{2}\cos\theta + C}$$

Exercise

Evaluate the integral $\int \sin\theta \;\sin 3\theta \; d\theta$

Solution

$$\int \sin\theta \;\sin 3\theta \; d\theta = \frac{1}{2}\int \left(\cos(-2\theta) - \cos 4\theta\right)\; d\theta \qquad \sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$= \frac{1}{2}\int \left(\cos 2\theta - \cos 4\theta\right)\; d\theta$$

$$= \frac{1}{2}\left(\frac{1}{2}\sin 2\theta - \frac{1}{4}\sin 4\theta\right) + C$$

$$= \frac{1}{4}\sin 2\theta - \frac{1}{8}\sin 4\theta + C$$

Exercise

Evaluate the integral $\int \sin 5\theta \;\sin 4\theta \; d\theta$

Solution

$$\int \sin 5\theta \;\sin 4\theta \; d\theta = \frac{1}{2}\int \left(\cos\theta - \cos 9\theta\right)\; d\theta \qquad \sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$= \frac{1}{2}\left(\sin\theta - \frac{1}{9}\sin 9\theta\right) + C$$

$$= \frac{1}{2}\sin\theta - \frac{1}{18}\sin 9\theta + C$$

Exercise

Evaluate the integral $\int \sin x \cos^5 x \; dx$

Solution

$$\int \sin x \cos^5 x \; dx = -\int \cos^5 x \; d(\cos x)$$

$$= -\frac{1}{6}\cos^6 x + C$$

Exercise

Evaluate the integral $\int \sin^7 2x \ \cos 2x \ dx$

Solution

$$\int \sin^7 2x \ \cos 2x \ dx = \frac{1}{2}\int \sin^7 2x \ d\left(\sin 2x\right)$$

$$= \frac{1}{16}\sin^8 2x + C$$

Exercise

Evaluate the integral $\int \sin^3 2x \ \sqrt{\cos 2x} \ dx$

Solution

$$\int \sin^3 2x \ \sqrt{\cos 2x} \ dx = -\frac{1}{2}\int \left(1-\cos^2 2x\right)\left(\cos 2x\right)^{1/2} \ d\left(\cos 2x\right)$$

$$= -\frac{1}{2}\int \left(\left(\cos 2x\right)^{1/2} - \left(\cos 2x\right)^{5/2}\right) \ d\left(\cos 2x\right)$$

$$= -\frac{1}{2}\left(\frac{2}{3}\left(\cos 2x\right)^{3/2} - \frac{2}{7}\left(\cos 2x\right)^{7/2}\right) + C$$

$$= \frac{1}{7}\left(\cos 2x\right)^{7/2} - \frac{1}{3}\left(\cos 2x\right)^{3/2} + C$$

Exercise

Evaluate $\int \sin^3 x \cos^2 x \ dx$

Solution

$$\int \sin^3 x \cos^2 x \ dx = \int \sin^2 x \cos^2 x \ \sin x dx \qquad d\left(\cos x\right) = -\sin x \ dx$$

$$= -\int \left(1-\cos^2 x\right)\cos^2 x \ d\left(\cos x\right)$$

$$= \int \left(\cos^4 x - \cos^2 x\right) \ d\left(\cos x\right)$$

$$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\cos^3\theta}{\sqrt{\sin\theta}}\,d\theta$

Solution

$$\begin{aligned}\int \frac{\cos^3\theta}{\sqrt{\sin\theta}}\,d\theta &= \int (\sin\theta)^{-1/2}\cos^2\theta\cos\theta\,d\theta \\ &= \int (\sin\theta)^{-1/2}\left(1-\sin^2\theta\right)\,d(\sin\theta) \\ &= \int \left(\sin^{-1/2}\theta - \sin^{3/2}\theta\right)\,d(\sin\theta) \\ &\boxed{= 2\sqrt{\sin\theta} - \frac{2}{5}\sin^{5/2}\theta + C}\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\cos^5\theta}{\sqrt{\sin\theta}}\,d\theta$

Solution

$$\begin{aligned}\int \frac{\cos^5\theta}{\sqrt{\sin\theta}}\,d\theta &= \int (\sin\theta)^{-1/2}\left(1-\sin^2\theta\right)^2\,d(\sin\theta) \\ &= \int (\sin\theta)^{-1/2}\left(1-2\sin^2\theta+\sin^4\theta\right)\,d(\sin\theta) \\ &= \int \left((\sin\theta)^{-1/2} - 2(\sin\theta)^{3/2} + (\sin\theta)^{7/2}\right)\,d(\sin\theta) \\ &\boxed{= 2(\sin\theta)^{1/2} - \frac{1}{5}(\sin\theta)^{5/2} + \frac{2}{9}(\sin\theta)^{9/2} + C}\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\cos^2 x}{\sin^5 x}\,dx$

Solution

$$\begin{aligned}\int \frac{\cos^2 x}{\sin^5 x}\,dx &= \int \frac{\cos^2 x}{\sin^2 x}\frac{1}{\sin^3 x}\,dx \\ &= \int \cot^2 x\,\csc^3 x\,dx\end{aligned}$$

$$= \int \left(\csc^2 x - 1\right)\ \csc^3 x\ dx$$

$$= \int \csc^5 x\ dx - \int \csc^3 x\ dx$$

$$u = \csc^3 x \qquad dv = \csc^2 x dx$$

$$du = -3\csc^3 x \cot x dx \qquad v = -\cot x$$

$$\int \csc^5 x dx = -\csc^3 x \cot x - 3\int \csc^3 x\ \cot^2 x\ dx$$

$$= -\csc^3 x \cot x - 3\int \csc^3 x\ \left(\csc^2 x - 1\right)\ dx$$

$$= -\csc^3 x \cot x - 3\int \csc^5 x\ dx + 3\int \csc^3 x\ \ dx$$

$$5\int \csc^5 x\ dx = -\csc^3 x \cot x + 3\int \csc^3 x\ \ dx$$

$$\boxed{\int \csc^5 x\ dx = -\frac{1}{5}\csc^3 x \cot x + \frac{3}{5}\int \csc^3 x\ \ dx}$$

$$u = \csc x \qquad dv = \csc^2 x dx$$

$$du = -\csc x \cot x dx \qquad v = -\cot x$$

$$\int \csc^3 x dx = -\csc x \cot x - \int \csc x \cot^2 x\ dx$$

$$= -\csc x \cot x - \int \csc x\left(\csc^2 x - 1\right)\ dx$$

$$= -\csc x \cot x - \int \csc^3 x\ dx + \int \csc x \frac{\csc x + \cot x}{\csc x + \cot x}\ dx$$

$$2\int \csc^3 x\ dx = -\csc x \cot x + \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\ dx$$

$$2\int \csc^3 x\ dx = -\csc x \cot x - \int \frac{1}{\csc x + \cot x}\ d\left(\csc x + \cot x\right)$$

$$\boxed{\int \csc^3 x\ dx = -\frac{1}{2}\csc x \cot x - \ln\left|\csc x + \cot x\right|}$$

$$\int \frac{\cos^2 x}{\sin^5 x} dx = -\frac{1}{5}\csc^3 x \cot x + \frac{3}{5}\left(-\frac{1}{2}\csc x \cot x - \ln\left|\csc x + \cot x\right|\right) + \frac{1}{2}\csc x \cot x + \ln\left|\csc x + \cot x\right|$$

$$= -\frac{1}{5}\csc^3 x\cot x - \frac{3}{10}\csc x\cot x - \frac{3}{5}\ln\left|\csc x + \cot x\right| + \frac{1}{2}\csc x\cot x + \ln\left|\csc x + \cot x\right|$$

$$= \frac{1}{5}\csc x\cot x\left(1 - \csc^2 x\right) + + \frac{2}{5}\ln\left|\csc x + \cot x\right| + C$$

$$\boxed{= \frac{1}{5}\csc x\cot^3 x + \frac{2}{5}\ln\left|\csc x + \cot x\right| + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sin^3 x}{\cos^4 x}dx$

Solution

$$\int \frac{\sin^3 x}{\cos^4 x}dx = \int \frac{\sin^2 x\sin x}{\cos^4 x}dx \qquad \cos^2\alpha + \sin^2\alpha = 1$$

$$= \int \frac{\left(1 - \cos^2 x\right)\sin x}{\cos^4 x}dx$$

$$= -\int\left(\frac{1}{\cos^4 x} - \frac{\cos^2 x}{\cos^4 x}\right)d\left(\cos x\right)$$

$$= -\int\left(\cos^{-4} x - \cos^{-2} x\right)d\left(\cos x\right)$$

$$= -\left(-\frac{1}{3}\cos^{-3} x + \cos^{-1} x\right) + C$$

$$= \frac{1}{3}\frac{1}{\cos^3 x} - \frac{1}{\cos x} + C$$

$$\boxed{= \frac{1}{3}\csc^3 x - \csc x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sin^4 x}{\cos^6 x}\,dx$

Solution

$$\int \frac{\sin^4 x}{\cos^6 x}\,dx = \int \frac{\sin^4 x}{\cos^4 x}\cdot\frac{1}{\cos^2 x}\,dx$$

$$= \int \tan^4 x\ \sec^2 x\,dx$$

$$= \int \tan^4 x \ d\left(\tan x\right)$$

$$\boxed{= \frac{1}{5}\tan^5 x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{2\cos x + 3\sin x}{\sin^3 x}\,dx$

Solution

$$\int \frac{2\cos x + 3\sin x}{\sin^3 x}\,dx = 2\int \frac{\cos x}{\sin^3 x}\,dx + 3\int \frac{\sin x}{\sin^3 x}\,dx$$

$$= 2\int \sin^{-3} x \ d\left(\sin x\right) + 3\int \frac{1}{\sin^2 x}\,dx$$

$$= -\sin^{-2} x + 3\int \csc^2 x\,dx$$

$$\boxed{= -\csc^2 x - 3\cot x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{2 + \sin x + 2\cos x}{1 + \cos x}\,dx$

Solution

$$\int \frac{2 + \sin x + 2\cos x}{1 + \cos x}\,dx = \int \frac{2\left(1 + \cos x\right)}{1 + \cos x}\,dx + \int \frac{\sin x}{1 + \cos x}\,dx$$

$$= 2\int dx - \int \frac{1}{1 + \cos x}\,d\left(1 + \cos x\right)$$

$$\boxed{= 2x - \ln\left|1 + \cos x\right| + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{1 - \cos x}$

Solution

$$\begin{aligned}
\int \frac{dx}{1-\cos x} &= \int \frac{1}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x}\,dx \\
&= \int \frac{1+\cos x}{1-\cos^2 x}\,dx \\
&= \int \frac{1+\cos x}{\sin^2 x}\,dx \\
&= \int \frac{1}{\sin^2 x}\,dx + \int \frac{\cos x}{\sin^2 x}\,dx \\
&= \int \csc^2 x\,dx + \int \frac{1}{\sin^2 x}\,d(\sin x) \\
&= -\cot x - \frac{1}{\sin x} + C \\
&\boxed{= -\cot x - \csc x + C}
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{1-\sin x}$

Solution

$$\begin{aligned}
\int \frac{dx}{1-\sin x} &= \int \frac{1}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}\,dx \\
&= \int \frac{1+\sin x}{1-\sin^2 x}\,dx \\
&= \int \frac{1+\sin x}{\cos^2 x}\,dx \\
&= \int \frac{1}{\cos^2 x}\,dx + \int \frac{\sin x}{\cos^2 x}\,dx \\
&= \int \sec^2 x\,dx - \int \frac{1}{\cos^2 x}\,d(\cos x) \\
&= \tan x + \frac{1}{\cos x} + C \\
&\boxed{= \tan x + \sec x + C}
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\sin\theta\cos\theta}{2-\cos\theta}\,d\theta$

Solution

$$\int \frac{\sin\theta\cos\theta}{2-\cos\theta}\,d\theta = \int \frac{\sin\theta\left(\cos\theta+2-2\right)}{2-\cos\theta}\,d\theta$$

$$= \int \frac{\sin\theta\left(\cos\theta-2\right)+2\sin\theta}{2-\cos\theta}\,d\theta$$

$$= -\int \frac{\sin\theta\left(2-\cos\theta\right)}{2-\cos\theta}\,d\theta + \int \frac{2\sin\theta}{2-\cos\theta}\,d\theta$$

$$= -\int \sin\theta\,d\theta + \int \frac{2}{2-\cos\theta}\,d\left(2-\cos\theta\right)$$

$$= \cos\theta + \ln\left|2-\cos\theta\right| + C$$

Exercise

Evaluate the integral $\int \tan^3 x\,\sec^3 x\,dx$

Solution

$$\int \tan^3 x\,\sec^3 x\,dx = \int \tan^2 x\,\sec^2 x\,\left(\tan x\,\sec x\right)\,dx$$

$$= \int \left(\sec^2 x-1\right)\,\sec^2 x\,d\left(\sec x\right)$$

$$= \int \left(\sec^4 x-\sec^2 x\right)\,d\left(\sec x\right)$$

$$= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$$

Exercise

Evaluate the integral $\int \sec x\tan^2 x\,dx$

Solution

$$\int \sec x\tan^2 x\,dx = \int \sec x\tan x\tan x\,dx$$

$u = \tan x \qquad dv = \sec x \tan x dx$

$du = \sec^2 xdx \qquad v = \sec x$

$$\begin{aligned}\int \sec x \tan^2 x\, dx &= \tan x \sec x - \int \sec x \sec^2 x\, dx\\ &= \tan x \sec x - \int \sec x\left(1+\tan^2 x\right)\, dx\\ &= \tan x \sec x - \left[\int \sec x\, dx + \int \sec x \tan^2 x\, dx\right]\\ &= \tan x \sec x - \ln\left|\sec x + \tan x\right| - \int \sec x \tan^2 x\, dx\end{aligned}$$

$$\int \sec x \tan^2 x\, dx + \int \sec x \tan^2 x\, dx = \tan x \sec x - \ln\left|\sec x + \tan x\right|$$

$$2\int \sec x \tan^2 x\, dx = \tan x \sec x - \ln\left|\sec x + \tan x\right|$$

$$\int \sec x \tan^2 x\, dx = \underline{\frac{1}{2}\tan x \sec x - \frac{1}{2}\ln\left|\sec x + \tan x\right| + C}\Big|$$

Exercise

Evaluate the integral $\int \sec^2 x \tan^2 x\, dx$

Solution

$$\int \sec^2 x \tan^2 xdx = \int \tan^2 x\, d(\tan x) \qquad d(\tan x) = \sec^2 xdx$$

$$= \underline{\frac{1}{3}\tan^3 x + C}\Big|$$

Exercise

Evaluate the integral $\int \sec^4 x \tan^2 x\, dx$

Solution

$$\int \sec^4 x \tan^2 xdx = \int \sec^2 x \sec^2 x \tan^2 x\, dx \qquad d(\tan x) = \sec^2 xdx \quad \sec^2 x = 1+\tan^2 x$$

$$= \int \left(1+\tan^2 x\right)\tan^2 x\, d(\tan x)$$

$$= \int \left(\tan^2 x + \tan^4 x\right) d\left(\tan x\right)$$

$$\boxed{= \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C}$$

Exercise

Evaluate the integral $\int \sec^6 4x \ \tan 4x \ dx$

Solution

$$\int \sec^6 4x \ \tan 4x \ dx = \frac{1}{4}\int \sec^5 4x \ \left(\sec 4x \ \tan 4x\right) \ d\left(4x\right)$$

$$= \frac{1}{4}\int \sec^5 4x \ d\left(\sec 4x\right)$$

$$\boxed{= \frac{1}{24}\sec^6 4x + C}$$

Exercise

Evaluate the integral $\int \sec^2 \frac{x}{2} \ \tan \frac{x}{2} \ dx$

Solution

$$\int \sec^2 \frac{x}{2} \ \tan \frac{x}{2} \ dx = 2\int \sec \frac{x}{2}\left(\sec \frac{x}{2} \tan \frac{x}{2}\right) \ d\left(\frac{x}{2}\right)$$

$$= 2\int \sec \frac{x}{2} \ d\left(\sec \frac{x}{2}\right)$$

$$\boxed{= \sec^2 \frac{x}{2} + C}$$

Exercise

Evaluate the integral $\int \tan^3 2x \ \sec^3 2x \ dx$

Solution

$$\int \tan^3 2x \ \sec^3 2x \ dx = \frac{1}{2}\int \left(\tan^2 2x\right) \ \sec^2 2x \ \left(\sec 2x \tan 2x\right) \ d\left(2x\right)$$

$$= \frac{1}{2}\int \left(\sec^2 2x - 1\right) \ \sec^2 2x \ d\left(\sec 2x\right)$$

$$=\frac{1}{2}\int\left(\sec^4 2x-\sec^2 2x\right)\,d\left(\sec 2x\right)$$

$$=\frac{1}{2}\left(\frac{1}{5}\sec^5 2x-\frac{1}{3}\sec^3 2x\right)+C$$

$$\boxed{=\frac{1}{10}\sec^5 2x-\frac{1}{6}\sec^3 2x+C}$$

Exercise

Evaluate the integral $\int \tan^5 2x\ \sec^4 2x\ dx$

Solution

$$\int \tan^5 2x\ \sec^4 2x\ dx=\frac{1}{2}\int \tan^5 2x\ \sec^2 2x\ \sec^2 2x\ d\left(2x\right)$$

$$=\frac{1}{2}\int \tan^5 2x\ \left(\tan^2 2x+1\right)\ d\left(\tan 2x\right)$$

$$=\frac{1}{2}\int \left(\tan^7 2x+\tan^5 2x\right)\ d\left(\tan 2x\right)$$

$$=\frac{1}{2}\left(\frac{1}{8}\tan^8 2x+\frac{1}{6}\tan^6 2x\right)+C$$

$$\boxed{=\frac{1}{16}\tan^8 2x+\frac{1}{12}\tan^6 2x+C}$$

Exercise

Evaluate the integral $\int \tan^3 x\ \sec^5 x\ dx$

Solution

$$\int \tan^3 x\ \sec^5 x\ dx=\int \tan^4 x\ \sec^4 x\ \left(\sec x\tan x\right)dx$$

$$=\int\left(\sec^2 x-1\right)^2\ \sec^4 x\ d\left(\sec x\right)$$

$$=\int\left(\sec^4 x-2\sec^2 x+1\right)\sec^4 x\ d\left(\sec x\right)$$

$$=\int\left(\sec^8 x-2\sec^6 x+\sec^4 x\right)\ d\left(\sec x\right)$$

$$\boxed{=\frac{1}{9}\sec^9 x-\frac{2}{7}\sec^7 x+\frac{1}{5}\sec^5 x+C}$$

Exercise

Evaluate $\displaystyle\int \tan^3 x\ \sec^4 x\ dx$

Solution

$$\int \tan^3 x \sec^4 x\ dx = \int \tan^3 x\left(1+\tan^2 x\right)\sec^2 x\ dx \qquad \sec^2 x = 1+\tan^2 x$$

$$= \int \left(\tan^3 x + \tan^5 x\right)\ d\left(\tan x\right) \qquad d\left(\tan x\right) = \sec^2 xdx$$

$$\boxed{= \frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \tan^5\theta\ \sec^4\theta\ d\theta$

Solution

$$\int \tan^5\theta\ \sec^4\theta\ d\theta = \int \tan^5\theta\ \sec^2\theta\ \sec^2\theta\ d\theta$$

$$= \int \tan^5\theta\ \left(1+\tan^2\theta\right)\ d\left(\tan\theta\right)$$

$$= \int \left(\tan^5\theta + \tan^7\theta\right)\ d\left(\tan\theta\right)$$

$$\boxed{= \frac{1}{6}\tan^6\theta + \frac{1}{8}\tan^8\theta + C}$$

Exercise

Evaluate the integral $\displaystyle\int \tan^5\theta\ \sec^7\theta\ d\theta$

Solution

$$\int \tan^5\theta\ \sec^7\theta\ d\theta = \int \tan^4\theta\ \sec^6\theta\left(\tan\theta\ \sec\theta\right)\ d\theta$$

$$= \int \left(\sec^2\theta - 1\right)^2\ \sec^6\theta\ d\left(\sec\theta\right)$$

$$= \int \left(\sec^4\theta - 2\sec^2\theta + 1\right)\sec^6\theta\ d\left(\sec\theta\right)$$

$$= \int \left(\sec^{10}\theta - 2\sec^{8}\theta + \sec^{6}\theta\right)\, d\left(\sec\theta\right)$$

$$\boxed{= \frac{1}{11}\sec^{11}\theta - \frac{2}{9}\sec^{9}\theta + \frac{1}{7}\sec^{7}\theta + C}$$

Exercise

Evaluate the integral $\int \tan^{7}\theta\ \sec^{5}\theta\ d\theta$

Solution

$$\int \tan^{7}\theta\ \sec^{5}\theta\ d\theta = \int \tan^{6}\theta\ \sec^{4}\theta\left(\tan\theta\ \sec\theta\right)\ d\theta$$

$$= \int \left(\sec^{2}\theta - 1\right)^{3} \sec^{4}\theta\ d\left(\sec\theta\right)$$

$$= \int \left(\sec^{6}\theta - 3\sec^{4}\theta + 3\sec^{2}\theta - 1\right)\sec^{4}\theta\ d\left(\sec\theta\right)$$

$$= \int \left(\sec^{10}\theta - 3\sec^{8}\theta + 3\sec^{6}\theta - \sec^{4}\theta\right)\ d\left(\sec\theta\right)$$

$$\boxed{= \frac{1}{11}\sec^{11}\theta - \frac{1}{3}\sec^{9}\theta + \frac{3}{7}\sec^{7}\theta - \frac{1}{5}\sec^{5}\theta + C}$$

Exercise

Evaluate the integral $\int \sec^{4}3x\tan^{3}3x\ dx$

Solution

$$\int \sec^{4}3x\tan^{3}3x\ dx = \int \sec^{2}3x\tan^{3}3x\ \sec^{2}3x\ dx$$

$$= \frac{1}{3}\int \left(1 + \tan^{2}3x\right)\tan^{3}3x\ d\left(\tan 3x\right)$$

$$= \frac{1}{3}\int \left(\tan^{3}3x + \tan^{5}3x\right)\ d\left(\tan 3x\right)$$

$$= \frac{1}{3}\left(\frac{1}{4}\tan^{4}3x + \frac{1}{6}\tan^{6}3x\right) + C$$

$$\boxed{= \frac{1}{12}\tan^{4}3x + \frac{1}{18}\tan^{6}3x + C}$$

Exercise

Evaluate $\displaystyle\int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2}\, dx$

Solution

$$\int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2}\, dx = \frac{2}{\pi}\int \tan^3 \frac{\pi x}{2}\, d\left(\tan \frac{\pi x}{2}\right)$$

$$= \frac{1}{2\pi}\tan^4 \frac{\pi x}{2} + C$$

Exercise

Evaluate the integral $\displaystyle\int \sec^{-2} x \tan^3 x\, dx$

Solution

$$\begin{aligned}\int \sec^{-2} x \tan^3 x\, dx &= \int \sec^{-2} x \tan^2 x \tan x\, dx \\ &= \int \sec^{-2} x\left(\sec^2 x - 1\right)\tan x\, dx \\ &= \int \left(1 - \sec^{-2} x\right)\tan x\, dx \\ &= \int \tan x\, dx - \int \sec^{-2} x \tan x\, dx \\ &= \int \frac{\sin x}{\cos x}\, dx - \int \cos^2 x \cdot \frac{\sin x}{\cos x}\, dx \\ &= -\int \frac{1}{\cos x}\, d(\cos x) + \int \cos x d(\cos x) \\ &= -\ln\left|\cos x\right| + \frac{1}{2}\cos^2 x + C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \sqrt{\tan x}\, \sec^4 x\, dx$

Solution

$$\int \sqrt{\tan x}\, \sec^4 x\, dx = \int (\tan x)^{1/2}\left(\tan^2 x + 1\right)\, \sec^2 x\, dx$$

$$= \int \left(\tan^{5/2} x + \tan^{1/2} x \right) d\left(\tan x \right)$$

$$= \frac{2}{7} \tan^{7/2} x + \frac{2}{3} \tan^{3/2} x + C$$

Exercise

Evaluate the integral $\int \tan^5 \theta \ \csc^2 \theta \ d\theta$

Solution

$$\int \tan^5 \theta \ \csc^2 \theta \ d\theta = \int \frac{\sin^5 \theta}{\cos^5 \theta} \cdot \frac{1}{\sin^2 \theta} \ d\theta$$

$$= \int \frac{\sin^3 \theta}{\cos^5 \theta} \ d\theta$$

$$= \int \frac{\sin^3 \theta}{\cos^3 \theta} \cdot \frac{1}{\cos^2 \theta} \ d\theta$$

$$= \int \tan^3 \theta \ \sec^2 \theta \ d\theta$$

$$= \int \tan^3 \theta \ d\left(\tan \theta \right)$$

$$= \frac{1}{4} \tan^4 \theta + C$$

Exercise

Evaluate the integral $\int \csc^2 x \ \cot x \ dx$

Solution

$$\int \csc^2 x \ \cot x \ dx = -\int \cot x \ d\left(\cot x \right)$$

$$= -\frac{1}{2} \cot^2 x + C$$

Exercise

Evaluate the integral $\int \csc^{10} x \ \cot x \ dx$

Solution

$$\int \csc^{10} x \cot x\, dx = -\int \csc^9 x\, d(\csc x)$$

$$= -\frac{1}{10}\csc^{10} x + C$$

Exercise

Evaluate the integral $\int (\cot 2x - \csc 2x)^2\, dx$

Solution

$$\int (\cot 2x - \csc 2x)^2\, dx = \int \left(\cot^2 2x - 2\cot 2x \csc 2x + \csc^2 2x\right) dx$$

$$= \int \cot^2 2x\, dx - \int (2\cot 2x \csc 2x)\, dx + \int \csc^2 2x\, dx$$

$$= \int \left(\csc^2 2x - 1\right) dx - \int (\cot 2x \csc 2x)\, d(2x) - \frac{1}{2}\cot 2x$$

$$= -\frac{1}{2}\cot 2x - x + \csc 2x - \frac{1}{2}\cot 2x + C$$

$$= \csc 2x - \cot 2x - x + C$$

Exercise

Evaluate the integral $\int \operatorname{sech}^4 x\, dx$

Solution

$$\int \operatorname{sech}^4 x\, dx = \int \operatorname{sech}^2 x \operatorname{sech}^2 x\, dx$$

$$= \int \operatorname{sech}^2 x \left(1 - \tanh^2 x\right) dx$$

$$= \int \operatorname{sech}^2 x\, dx - \int \operatorname{sech}^2 x \tanh^2 x\, dx$$

$$= \tanh x - \int \tanh^2 x\, d(\tanh x)$$

$$= \tanh x - \frac{1}{3}\tanh^3 x + C$$

Exercise

Evaluate the integral $\displaystyle\int \sinh^3 x\ \cosh^2 x\, dx$

Solution

$$\begin{aligned}
\int \sinh^3 x\ \cosh^2 x\, dx &= \int \sinh^2 x\ \cosh^2 x\ (\sinh x\, dx) \\
&= \int \left(\cosh^2 x - 1\right)\ \cosh^2 x\ \ d\left(\cosh x\right) \\
&= \int \left(\cosh^4 x - \cosh^2 x\right)\ d\left(\cosh x\right) \\
&\boxed{= \frac{1}{5}\cosh^5 x - \frac{1}{3}\cosh^3 x + C}
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \operatorname{sech}^2 x\ \sinh x\, dx$

Solution

$$\begin{aligned}
\int \operatorname{sech}^2 x\ \sinh x\, dx &= \int \frac{d\left(\cosh x\right)}{\cosh^2 x} \\
&= -\frac{1}{\cosh x} + C \\
&\boxed{= -\operatorname{sech} x + C}
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\tan^3 x}{\sqrt{\sec x}}\, dx$

Solution

$$\begin{aligned}
\int \frac{\tan^3 x}{\sqrt{\sec x}}\, dx &= \int \frac{\tan^2 x \tan x}{(\sec x)^{1/2}} \frac{\sec x}{\sec x}\, dx \\
&= \int (\sec x)^{-3/2}\left(\sec^2 x - 1\right) d\left(\sec x\right) \\
&= \int \left((\sec x)^{1/2} - (\sec x)^{-3/2}\right) d\left(\sec x\right) \\
&\boxed{= \frac{2}{3}(\sec x)^{3/2} + 2(\sec x)^{-1/2} + C}
\end{aligned}$$

$1 + \tan^2 \alpha = \sec^2 \alpha$

Exercise

Evaluate the integral $\displaystyle\int \frac{\tan^2 x}{\sec x}\,dx$

Solution

$$\begin{aligned}
\int \frac{\tan^2 x}{\sec x}\,dx &= \int \frac{\sec^2 x - 1}{\sec x}\,dx \\
&= \int \left(\sec x - \frac{1}{\sec x}\right) dx \\
&= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\,dx - \int \cos x\,dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx - \sin x \\
&= \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} - \sin x \\
&\boxed{= \ln\left|\sec x + \tan x\right| - \sin x + C}
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sec x}{\tan^2 x}\,dx$

Solution

$$\begin{aligned}
\int \frac{\sec x}{\tan^2 x}\,dx &= \int \frac{1}{\cos x}\frac{\cos^2 x}{\sin^2 x}\,dx \\
&= \int \frac{\cos x}{\sin^2 x}\,dx \\
&= \int \frac{1}{\sin^2 x}\,d(\sin x) \\
&= -\frac{1}{\sin x} + C \\
&\boxed{= -\csc x + C}
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sec^2 x}{\tan^5 x}\,dx$

Solution

$$\int \frac{\sec^2 x}{\tan^5 x}\,dx = \int \tan^{-5} x\; d(\tan x)$$

$$= -\frac{1}{4}\tan^{-4} x + C$$

Exercise

Evaluate the integral $\int \frac{\csc^4 x}{\cot^2 x}\,dx$

Solution

$$\begin{aligned}
\int \frac{\csc^4 x}{\cot^2 x}\,dx &= \int \frac{\csc^2 x\left(\cot^2 x + 1\right)}{\cot^2 x}\,dx \\
&= -\int \frac{\cot^2 x + 1}{\cot^2 x}\,d(\cot x) \\
&= -\int \left(1 + \frac{1}{\cot^2 x}\right) d(\cot x) \\
&= -\cot x + \frac{1}{\cot x} + C \\
&= -\cot x + \tan x + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\sec^4(\ln x)}{x}\,dx$

Solution

$$\begin{aligned}
\int \frac{\sec^4(\ln x)}{x}\,dx &= \int \sec^4(\ln x)\; d(\ln x) \\
&= \int \sec^2(\ln x)\left(\tan^2(\ln x) + 1\right) d(\ln x) \\
&= \int \sec^2(\ln x)\tan^2(\ln x)\; d(\ln x) + \int \sec^2(\ln x)\; d(\ln x) \\
&= \int \tan^2(\ln x)\; d(\tan(\ln x)) + \tan(\ln x) \\
&= \frac{1}{3}\tan^3(\ln x) + \tan(\ln x) + C
\end{aligned}$$

Exercise

Evaluate the integral $\int e^x \sec\left(e^x+1\right)dx$

Solution

$$\int e^x \sec\left(e^x+1\right)dx = \int \sec\left(e^x+1\right)d\left(e^x+1\right)$$

$$= \int \sec\left(e^x+1\right)\frac{\sec\left(e^x+1\right)+\tan\left(e^x+1\right)}{\sec\left(e^x+1\right)+\tan\left(e^x+1\right)}\,d\left(e^x+1\right)$$

$$= \int \frac{\sec^2\left(e^x+1\right)+\sec\left(e^x+1\right)\tan\left(e^x+1\right)}{\sec\left(e^x+1\right)+\tan\left(e^x+1\right)}\,d\left(e^x+1\right)$$

$$= \int \frac{1}{\sec\left(e^x+1\right)+\tan\left(e^x+1\right)}\,d\left(\sec\left(e^x+1\right)+\tan\left(e^x+1\right)\right)$$

$$\boxed{= \ln\left|\sec\left(e^x+1\right)+\tan\left(e^x+1\right)\right|+C}$$

Exercise

Evaluate the integral $\int e^x \sec^3\left(e^x\right)\,dx$

Solution

$u = \sec\left(e^x\right)$ $\qquad dv = \sec\left(e^x\right)e^x dx$

$du = \sec\left(e^x\right)\tan\left(e^x\right)e^x dx$ $\qquad v = \int \sec\left(e^x\right)d\left(e^x\right) = \tan\left(e^x\right)$

$$\int e^x \sec^3\left(e^x\right)\,dx = \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec\left(e^x\right)\tan^2\left(e^x\right)e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec\left(e^x\right)\left(\sec^2\left(e^x\right)-1\right)e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \left(\sec^3\left(e^x\right)-\sec\left(e^x\right)\right)e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec^3\left(e^x\right)e^x dx + \int \sec\left(e^x\right)e^x dx \qquad d\left(e^x\right) = e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec^3\left(e^x\right)e^x dx + \int \sec\left(e^x\right)d\left(e^x\right)$$

$$\int \sec^3\left(e^x\right)e^x dx = \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec^3\left(e^x\right)e^x dx + \ln\left|\sec\left(e^x\right) + \tan\left(e^x\right)\right|$$

$$2\int \sec^3\left(e^x\right)e^x dx = \sec\left(e^x\right)\tan\left(e^x\right) + \ln\left|\sec\left(e^x\right) + \tan\left(e^x\right)\right| + C$$

$$\int \sec^3\left(e^x\right)e^x dx = \boxed{\frac{1}{2}\sec\left(e^x\right)\tan\left(e^x\right) + \frac{1}{2}\ln\left|\sec\left(e^x\right) + \tan\left(e^x\right)\right| + C}$$

Exercise

Evaluate the integral $\int e^x\sqrt{\tan^2 e^x + 1}\, dx$

Solution

$$\begin{aligned}\int e^x\sqrt{\tan^2\left(e^x\right)+1}\, dx &= \int \sqrt{\tan^2\left(e^x\right)+1}\, d\left(e^x\right)\\ &= \int \sqrt{\sec^2\left(e^x\right)}\, d\left(e^x\right)\\ &= \int \sec e^x \cdot \frac{\sec e^x + \tan e^x}{\sec e^x + \tan e^x}\, d\left(e^x\right)\\ &= \int \frac{\sec^2 e^x + \sec e^x \tan e^x}{\sec e^x + \tan e^x}\, d\left(e^x\right)\\ &= \int \frac{1}{\sec e^x + \tan e^x}\, d\left(\sec e^x + \tan e^x\right)\\ &\boxed{= \ln\left|\sec e^x + \tan e^x\right| + C}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\sqrt{\frac{\pi}{2}}} x\sin^3\left(x^2\right)dx$

Solution

$$\begin{aligned}\int_0^{\sqrt{\frac{\pi}{2}}} x\sin^3\left(x^2\right)dx &= \frac{1}{2}\int_0^{\sqrt{\frac{\pi}{2}}} \sin^3\left(x^2\right)d\left(x^2\right)\\ &= \frac{1}{2}\int_0^{\sqrt{\frac{\pi}{2}}} \sin^2\left(x^2\right)\ \sin\left(x^2\right)\, d\left(x^2\right)\end{aligned}$$

$$= -\frac{1}{2}\int_{0}^{\sqrt{\frac{\pi}{2}}}\left(1-\cos^{2}\left(x^{2}\right)\right)\, d\left(\cos x^{2}\right)$$

$$= -\frac{1}{2}\left(\cos x^{2} - \frac{1}{3}\cos^{3}\left(x^{2}\right)\right)\Bigg|_{0}^{\sqrt{\frac{\pi}{2}}}$$

$$= -\frac{1}{2}\left(\cos\frac{\pi}{2} - \frac{1}{3}\cos^{3}\frac{\pi}{2} - 1 + \frac{1}{3}\right)$$

$$= -\frac{1}{2}\left(-\frac{2}{3}\right)$$

$$\boxed{= \frac{1}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/6}^{\pi/3}\frac{\cos^{3}x}{\sqrt{\sin x}}\,dx$

Solution

$$\int_{\pi/6}^{\pi/3}\frac{\cos^{3}x}{\sqrt{\sin x}}\,dx = \int_{\pi/6}^{\pi/3}(\sin x)^{-1/2}\left(1-\sin^{2}x\right)\, d(\sin x)$$

$$= \int_{\pi/6}^{\pi/3}\left((\sin x)^{-1/2} - (\sin x)^{3/2}\right)\, d(\sin x)$$

$$= 2(\sin x)^{1/2} - \frac{2}{5}(\sin x)^{5/2}\ \Bigg|_{\pi/6}^{\pi/3}$$

$$= 2\left(\frac{\sqrt{3}}{2}\right)^{1/2} - \frac{2}{5}\left(\frac{\sqrt{3}}{2}\right)^{5/2} - 2\left(\frac{1}{2}\right)^{1/2} + \frac{2}{5}\left(\frac{1}{2}\right)^{5/2}$$

$$= \sqrt[4]{3}\sqrt{2} - \frac{3}{10}\frac{\sqrt[4]{3}}{\sqrt{2}} - \sqrt{2} + \frac{\sqrt{2}}{20}$$

$$\boxed{= \frac{\sqrt{2}}{20}\left(17\sqrt[4]{3} - 19\right)}$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/6}^{\pi/2}\frac{dx}{\sin x}$

Solution

$$\int_{\pi/6}^{\pi/2} \frac{dx}{\sin x} = \int_{\pi/6}^{\pi/2} \csc x \ \frac{\csc x + \cot x}{\csc x + \cot x} \ dx$$

$$= \int_{\pi/6}^{\pi/2} \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \ dx$$

$$= -\int_{\pi/6}^{\pi/2} \frac{1}{\csc x + \cot x} \ d\left(\csc x + \cot x\right)$$

$$= -\ln\left|\csc x + \cot x\right| \Big|_{\pi/6}^{\pi/2}$$

$$= -\ln\left|1+0\right| + \ln\left|2+\sqrt{3}\right|$$

$$\boxed{= \ln\left(2+\sqrt{3}\right)}$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/6}^{\pi/3} \cot^3\theta \ d\theta$

Solution

$$\int_{\pi/6}^{\pi/3} \cot^3\theta \ d\theta = \int_{\pi/6}^{\pi/3} \cot\theta\left(\csc^2\theta - 1\right) \ d\theta$$

$$= \int_{\pi/6}^{\pi/3} \cot\theta \csc^2\theta \ d\theta - \int_{\pi/6}^{\pi/3} \cot\theta \ d\theta$$

$$= -\int_{\pi/6}^{\pi/3} \cot\theta \ d\left(\cot\theta\right) - \int_{\pi/6}^{\pi/3} \frac{\cos\theta}{\sin\theta} \ d\theta$$

$$= -\frac{1}{2}\cot^2\theta - \int_{\pi/6}^{\pi/3} \frac{1}{\sin\theta} \ d\left(\sin\theta\right)$$

$$= -\frac{1}{2}\cot^2\theta - \ln\left|\sin\theta\right| \Big|_{\pi/6}^{\pi/3}$$

$$= \frac{3}{2} - \frac{1}{6} + \ln\frac{1}{2} - \ln\frac{\sqrt{3}}{2}$$

$$= \frac{4}{3} + \ln\frac{1}{\sqrt{3}}$$

$$\boxed{= \frac{4}{3} - \ln\sqrt{3}}$$

Exercise

Evaluate the integral $\int_0^{\pi/3} \tan^2 x \, dx$

Solution

$$\int_0^{\pi/3} \tan^2 x \, dx = \int_0^{\pi/3} \left(\sec^2 x - 1\right) dx$$

$$= \tan x - x \Big|_0^{\pi/3}$$

$$\boxed{= \sqrt{3} - \frac{\pi}{3}}$$

Exercise

Evaluate the integral $\int_0^{\pi/4} 6\tan^3 x \, dx$

Solution

$$\int_0^{\pi/4} 6\tan^3 x \, dx = 6\int_0^{\pi/4} \tan x \tan^2 x \, dx$$

$$= 6\int_0^{\pi/4} \tan x\left(\sec^2 x - 1\right) dx$$

$$= 6\int_0^{\pi/4} \tan x \sec^2 x \, dx - 6\int_0^{\pi/4} \tan x \, dx$$

$$= 6\int_0^{\pi/4} \tan x \, d(\tan x) - 6\int_0^{\pi/4} \frac{\sin x}{\cos x} \, dx$$

$$= 3\tan^2 x \Big|_0^{\pi/4} + 6\int_0^{\pi/4} \frac{1}{\cos x} \, d(\cos x)$$

$$= 3 + 6\ln(\cos x) \Big|_0^{\pi/4}$$

$$= 3 + 6\ln\frac{\sqrt{2}}{2} - 6\ln 1$$

$$\boxed{= 3 + 6\ln\frac{\sqrt{2}}{2}}$$

Exercise

Evaluate the integral $\int_0^{\pi/4} \tan^4 x\, dx$

Solution

$$\int_0^{\pi/4} \tan^4 x\, dx = \int_0^{\pi/4} \tan^2 x\left(\sec^2 x - 1\right) dx$$

$$= \int_0^{\pi/4} \tan^2 x\left(\sec^2 x - 1\right) dx$$

$$= \int_0^{\pi/4} \tan^2 x \sec^2 x\, dx - \int_0^{\pi/4} \tan^2 x\, dx$$

$$= \int_0^{\pi/4} \tan^2 x\, d\left(\tan x\right) - \int_0^{\pi/4} \left(\sec^2 x - 1\right)\, dx$$

$$= \frac{1}{3}\tan^3 x - \tan x + x \Big|_0^{\pi/4}$$

$$= \frac{1}{3} - 1 + \frac{\pi}{4}$$

$$= \frac{\pi}{4} - \frac{2}{3}$$

Exercise

Evaluate the integral $\int_0^{\pi} 8\sin^4 y \cos^2 y\, dy$

Solution

$$\int_0^{\pi} 8\sin^4 y \cos^2 y\, dy = 8\int_0^{\pi} \left(\frac{1-\cos 2y}{2}\right)^2 \left(\frac{1+\cos 2y}{2}\right) dy$$

$$= \int_0^{\pi} \left(1 - 2\cos 2y + \cos^2 2y\right)\left(1 + \cos 2y\right) dy$$

$$= \int_0^{\pi} \left(1 - 2\cos 2y + \cos^2 2y + \cos 2y - 2\cos^2 2y + \cos^3 2y\right) dy$$

$$= \int_0^{\pi} \left(1 - \cos 2y - \cos^2 2y + \cos^3 2y\right) dy$$

$$= \int_0^{\pi} \left(1 - \cos 2y - \frac{1}{2} - \frac{1}{2}\cos 4y\right) dy + \int_0^{\pi} \cos^2 2y \cos 2y \, dy$$

$$= \int_0^{\pi} \left(\frac{1}{2} - \cos 2y - \frac{1}{2}\cos 4y\right) dy + \frac{1}{2}\int_0^{\pi} \left(1 - \sin^2 2y\right) d\left(\sin 2y\right)$$

$$= \frac{1}{2}y - \frac{1}{2}\sin 2y - \frac{1}{8}\sin 4y + \frac{1}{2}\left(\sin 2y - \frac{1}{3}\sin^3 2y\right) \Bigg|_0^{\pi}$$

$$\boxed{= \frac{\pi}{2}}$$

Exercise

Evaluate the integral $\int_0^{\pi/6} 3\cos^5 3x \, dx$

Solution

$$\int_0^{\pi/6} 3\cos^5 3x \, dx = \int_0^{\pi/6} 3\left(\cos^2 3x\right)^2 \cos 3x \, dx$$

$$= \int_0^{\pi/6} \left(1 - \sin^2 3x\right)^2 d\left(\sin 3x\right)$$

$$= \int_0^{\pi/6} \left(1 - 2\sin^2 3x + \sin^4 3x\right) d\left(\sin 3x\right)$$

$$= \sin 3x - \frac{2}{3}\sin^2 3x + \frac{1}{5}\sin^4 3x \Bigg|_0^{\pi/9}$$

$$= \sin\frac{\pi}{2} - \frac{2}{3}\sin^2\frac{\pi}{2} + \frac{1}{5}\sin^4\frac{\pi}{2} - 0$$

$$= 1 - \frac{2}{3} + \frac{1}{5}$$

$$\boxed{= \frac{8}{15}}$$

Exercise

Evaluate the integral $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta$

Solution

$$\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta = \int_0^{\pi/2} \sin^2 2\theta \left(\cos^2 2\theta\right) \cos 2\theta \, d\theta \qquad d\left(\sin 2\theta\right) = 2\cos 2\theta d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^2 2\theta\left(1-\sin^2 2\theta\right) d\left(\sin 2\theta\right)$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(\sin^2 2\theta - \sin^4 2\theta\right) d\left(\sin 2\theta\right)$$

$$= \frac{1}{2}\left(\frac{1}{3}\sin^3 2\theta - \frac{1}{5}\sin^5 2\theta \right|_0^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{1}{3}\sin^3 \pi - \frac{1}{5}\sin^5 \pi - 0\right)$$

$$\underline{= 0\,|}$$

Exercise

Evaluate the integral $\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\, dx$

Solution

$$\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\, dx = \int_0^{2\pi} \sin\frac{x}{2}\, dx$$

$$\left|\sin\left(\frac{\alpha}{2}\right)\right| = \sqrt{\frac{1-\cos\alpha}{2}}$$

$$= -2\cos\frac{x}{2}\Big|_0^{2\pi}$$

$$= -2\left(\cos\pi - \cos 0\right)$$

$$\underline{= 2\,|}$$

Exercise

Evaluate the integral $\int_0^{\pi} \sqrt{1-\cos^2\theta}\, d\theta$

Solution

$$\int_0^{\pi} \sqrt{1-\cos^2\theta}\, d\theta = \int_0^{\pi} \left|\sin\theta\right|\, d\theta$$

$$= -\cos\theta\Big|_0^{\pi}$$

$$= -\cos\pi + \cos 0$$

$$\underline{= 2\,|}$$

Exercise

Evaluate the integral $\int_0^{\pi/6} \sqrt{1+\sin x}\, dx$

Solution

$$\int_0^{\pi/6} \sqrt{1+\sin x}\, dx = \int_0^{\pi/6} \sqrt{1+\sin x}\, \frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}\, dx$$

$$= \int_0^{\pi/6} \frac{\sqrt{1-\sin^2 x}}{\sqrt{1-\sin x}}\, dx \qquad \cos x = \sqrt{1-\sin^2 x}$$

$$= \int_0^{\pi/6} \frac{\cos x}{\sqrt{1-\sin x}}\, dx \qquad d\left(1-\sin x\right) = -\cos x dx$$

$$= -\int_0^{\pi/6} \left(1-\sin x\right)^{-1/2}\, d\left(1-\sin x\right)$$

$$= -2\,\left(1-\sin x\right)^{1/2} \Big|_0^{\pi/6}$$

$$= -2\left(\sqrt{1-\sin\frac{\pi}{6}} - 1\right)$$

$$= -2\left(\sqrt{1-\frac{1}{2}} - 1\right)$$

$$= -2\left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= -2\left(\frac{\sqrt{2}}{2} - 1\right)$$

$$\boxed{= 2-\sqrt{2}}$$

Exercise

Evaluate the integral $\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2\theta - 1}\, d\theta$

Solution

$$\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2\theta - 1}\, d\theta = \int_{-\pi/3}^{\pi/3} \tan\theta\, d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \frac{\sin\theta}{\cos\theta}\, d\theta$$

$$= -\int_{-\pi/3}^{\pi/3} \frac{1}{\cos\theta}\, d(\cos\theta)$$

$$= -\ln|\cos\theta| \Big|_{-\pi/3}^{\pi/3}$$

$$= -\left(\ln\left(\cos\frac{\pi}{3}\right) - \ln\left(\cos\frac{\pi}{3}\right)\right)$$

$$\underline{= 0}$$

Exercise

Evaluate the integral $\int_0^{\pi} (1-\cos 2x)^{3/2}\, dx$

Solution

$$\int_0^{\pi} (1-\cos 2x)^{3/2}\, dx = \int_0^{\pi} \left(2\sin^2 x\right)^{3/2}\, dx$$

$$= 2\sqrt{2}\int_0^{\pi} \sin^3 x\, dx$$

$$= 2\sqrt{2}\int_0^{\pi} \sin^2 x\ \sin x\, dx$$

$$= -2\sqrt{2}\int_0^{\pi} \left(1-\cos^2 x\right)\, d(\cos x)$$

$$= -2\sqrt{2}\ \left(\cos x - \frac{1}{3}\cos^3 x\ \Big|_0^{\pi}\right.$$

$$= -2\sqrt{2}\left(-1+\frac{1}{3}-1+\frac{1}{3}\right)$$

$$= -2\sqrt{2}\left(-\frac{4}{3}\right)$$

$$\underline{= \frac{8\sqrt{2}}{3}}$$

Exercise

Evaluate the integral $\int_0^{\pi} \left(1-\cos^2 x\right)^{3/2}\, dx$

Solution

$$\int_0^{\pi}\left(1-\cos^2 x\right)^{3/2} dx = \int_0^{\pi}\left(\sin^2 x\right)^{3/2} dx$$

$$= \int_0^{\pi} \sin^3 x\ dx$$

$$= \int_0^{\pi} \sin^3 x\ dx \qquad\qquad \sin^2 x = 1-\cos^2 x$$

$$= \int_0^{\pi}\left(1-\cos^2 x\right)\sin x\ dx \qquad\qquad d\left(\cos x\right) = -\sin x dx$$

$$= -\int_0^{\pi}\left(1-\cos^2 x\right) d\left(\cos x\right)$$

$$= -\left(\cos x - \tfrac{1}{3}\cos^3 x\ \right|_0^{\pi}$$

$$= -\left(-1+\tfrac{1}{3}-\left(1-\tfrac{1}{3}\right)\right)$$

$$= 1-\tfrac{1}{3}+1-\tfrac{1}{3}$$

$$= 2-\tfrac{2}{3}$$

$$\boxed{= \frac{4}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi}^{\pi}\left(1-\cos^2 x\right)^{3/2} dx$

Solution

$$\int_{-\pi}^{\pi}\left(1-\cos^2 x\right)^{3/2} dx = \int_{-\pi}^{\pi}\left(\sin^2 x\right)^{3/2} dx$$

$$= \int_{-\pi}^{\pi}\left|\sin^3 x\right| dx$$

$$= -\int_{-\pi}^{0} \sin^3 x\ dx + \int_0^{\pi} \sin^3 x\ dx \qquad\qquad \sin^2 x = 1-\cos^2 x$$

$$= -\int_{-\pi}^{0}\left(1-\cos^2 x\right)\sin x\ dx + \int_0^{\pi}\left(1-\cos^2 x\right)\sin x\ dx \qquad\qquad d\left(\cos x\right) = -\sin x dx$$

$$= \int_{-\pi}^{0} \left(1 - \cos^2 x\right) d\left(\cos x\right) - \int_{0}^{\pi} \left(1 - \cos^2 x\right) d\left(\cos x\right)$$

$$= \left(\cos x - \tfrac{1}{3}\cos^3 x \right|_{-\pi}^{0} - \left(\cos x - \tfrac{1}{3}\cos^3 x \right|_{0}^{\pi}$$

$$= \left(1 - \tfrac{1}{3} - \left(-1 + \tfrac{1}{3}\right)\right) - \left(-1 + \tfrac{1}{3} - \left(1 - \tfrac{1}{3}\right)\right)$$

$$= 1 - \tfrac{1}{3} + 1 - \tfrac{1}{3} + 1 - \tfrac{1}{3} + 1 - \tfrac{1}{3}$$

$$= 4 - \tfrac{4}{3}$$

$$\boxed{= \frac{8}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/4}^{\pi/2} \csc^4 \theta d\theta$

Solution

$$\int_{\pi/4}^{\pi/2} \csc^4 \theta d\theta = \int_{\pi/4}^{\pi/2} \left(1 + \cot^2 \theta\right) \csc^2 \theta d\theta \qquad \csc^2 \theta = 1 + \cot^2 \theta$$

$$= \int_{\pi/4}^{\pi/2} \csc^2 \theta d\theta + \int_{\pi/4}^{\pi/2} \cot^2 \theta \csc^2 \theta d\theta \qquad d\left(\cot \theta\right) = -\csc^2 \theta d\theta$$

$$= \int_{\pi/4}^{\pi/2} \csc^2 \theta d\theta - \int_{\pi/4}^{\pi/2} \cot^2 \theta \, d\left(\cot \theta\right)$$

$$= \left(-\cot \theta - \tfrac{1}{3}\cot^3 \theta \right|_{\pi/4}^{\pi/2}$$

$$= -\left(\cot \tfrac{\pi}{2} + \tfrac{1}{3}\cot^3 \tfrac{\pi}{2} - \cot \tfrac{\pi}{4} - \tfrac{1}{3}\cot^3 \tfrac{\pi}{4}\right)$$

$$= -\left(0 + \tfrac{1}{3}(0) - 1 - \tfrac{1}{3}\right)$$

$$\boxed{= \frac{4}{3}}$$

Exercise

Evaluate the integral $\int_{-\pi}^{\pi} \sin 3x \sin 3x \; dx$

Solution

$$\int_{-\pi}^{\pi} \sin 3x \sin 3x \; dx = \frac{1}{2}\int_{-\pi}^{\pi} \left(\cos 0 - \cos 6x\right) \; dx$$

$\sin\alpha \sin\beta = \frac{1}{2}\left[\cos\left(\alpha-\beta\right) - \cos\left(\alpha+\beta\right)\right]$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \left(1 - \cos 6x\right) \; dx$$

$$= \frac{1}{2} \left(x - \frac{1}{6}\sin 6x \; \right|_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left(\pi - \frac{1}{6}\sin 6\pi - \left(-\pi - \frac{1}{6}\sin\left(-6\pi\right)\right)\right)$$

$$= \frac{1}{2}\left(\pi + \pi\right)$$

$$\underline{= \pi \;|}$$

Exercise

Evaluate the integral $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x \; dx$

Solution

$$\int_{-\pi/2}^{\pi/2} \cos x \cos 7x dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(\cos 8x + \cos(-6x)\right) dx$$

$\cos\alpha \cos\beta = \frac{1}{2}\left[\cos\left(\alpha+\beta\right) + \cos\left(\alpha-\beta\right)\right]$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(\cos 8x + \cos 6x\right) dx$$

$$= \frac{1}{2}\left(\frac{1}{6}\sin 6x + \frac{1}{8}\sin 8x \; \right|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{1}{6}\sin\left(3\pi\right) + \frac{1}{8}\sin\left(4\pi\right) - \frac{1}{6}\sin\left(-3\pi\right) - \frac{1}{8}\sin\left(-4\pi\right)\right)$$

$$\underline{= 0 \;|}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/4} \cos^5 2x\, \sin^2 2x\, dx$

Solution

$$\int_0^{\pi/4} \cos^5 2x\, \sin^2 2x\, dx = \int_0^{\pi/4} \cos^4 2x\, \cos 2x\, \sin^2 2x\, dx$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(1-\sin^2 2x\right)^2 \sin^2 2x\, d\left(\sin 2x\right)$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(1-2\sin^2 2x+\sin^4 2x\right) \sin^2 2x\, d\left(\sin 2x\right)$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(\sin^2 2x-2\sin^4 2x+\sin^6 2x\right) d\left(\sin 2x\right)$$

$$= \frac{1}{2}\left(\frac{1}{3}\sin^3 2x-\frac{2}{5}\sin^5 2x+\frac{1}{7}\sin^7 2x\right.\Bigg|_0^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{1}{3}\sin^3\frac{\pi}{2}-\frac{2}{5}\sin^5\frac{\pi}{2}+\frac{1}{7}\sin^7\frac{\pi}{2}-0\right)$$

$$= \frac{1}{2}\left(\frac{1}{3}-\frac{2}{5}+\frac{1}{7}\right)$$

$$= \frac{1}{2}\left(\frac{35-42+25}{105}\right)$$

$$\boxed{= \frac{4}{105}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/6} \sin^5 x\, dx$

Solution

$$\int_0^{\pi/6} \sin^5 x\, dx = \int_0^{\pi/6} \sin^4 x \sin x\, dx$$

$$= -\int_0^{\pi/6} \left(1-\cos^2 x\right)^2 d\left(\cos x\right)$$

$$= -\int_0^{\pi/6} \left(1-2\cos^2 x+\cos^4 x\right) d\left(\cos x\right)$$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x \,\Big|_0^{\pi/6}$$

$$= -\frac{\sqrt{3}}{2} + \frac{2}{3}\frac{3\sqrt{3}}{8} - \frac{1}{5}\frac{9\sqrt{3}}{32} + 1 - \frac{2}{3} + \frac{1}{5}$$

$$= -\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4} - \frac{9\sqrt{3}}{160} + \frac{15-10+3}{15}$$

$$= \left(\frac{-80+40-9}{160}\right)\sqrt{3} + \frac{8}{15}$$

$$= -\frac{49\sqrt{3}}{160} + \frac{8}{15}$$

$$= \frac{256 - 147\sqrt{3}}{480}$$

Exercise

Evaluate the integral $\int_{-\pi}^{\pi} \sin^2 x\, dx$

Solution

$$\int_{-\pi}^{\pi} \sin^2 x\, dx = 2\int_0^{\pi} \sin^2 x\, dx$$

$$= 2\int_0^{\pi} \frac{1}{2}(1 - \cos 2x)\, dx$$

$$= x - \frac{1}{2}\sin 2x \,\Big|_0^{\pi}$$

$$= \pi$$

Exercise

Evaluate the integral $\int_{-\pi/2}^{\pi/2} \left(\sin^2 x + 1\right) dx$

Solution

$$\int_{-\pi/2}^{\pi/2} \left(\sin^2 x + 1\right) dx = \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2} - \frac{1}{2}\cos 2x + 1\right) dx$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} - \frac{1}{2}\cos 2x\right) dx$$

$$= \frac{3}{2}x - \frac{1}{4}\sin 2x \Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{3\pi}{4} + \frac{3\pi}{4}$$

$$\boxed{= \frac{3\pi}{2}}$$

Exercise

Evaluate the integral $\int_0^{\pi/3} \sec^{3/2} x \, \tan x \, dx$

Solution

$$\int_0^{\pi/3} \sec^{3/2} x \, \tan x \, dx = \int_0^{\pi/3} \sec^{1/2} x \, (\sec x \tan x) \, dx$$

$$= \int_0^{\pi/3} \sec^{1/2} x \, d(\sec x)$$

$$= \frac{2}{3}\sec^{3/2} x \Big|_0^{\pi/3}$$

$$= \frac{2}{3}\left((2)^{3/2} - 1\right)$$

$$\boxed{= \frac{2}{3}\left(2\sqrt{2} - 1\right)}$$

Exercise

Evaluate the integral $\int_0^{\pi/2} \frac{\cos x}{1+\sin x} dx$

Solution

$$\int_0^{\pi/2} \frac{\cos x}{1+\sin x} dx = \int_0^{\pi/2} \frac{1}{1+\sin x} d(1+\sin x)$$

$$= \ln|1+\sin x| \Big|_0^{\pi/2}$$

$$= \ln 2 - \ln 1$$

$$\boxed{= \ln 2}$$

Exercise

Evaluate the integral $\int_{\pi/6}^{\pi/3} \sin 6x \ \cos 4x \ dx$

Solution

$$\int_{\pi/6}^{\pi/3} \sin 6x \ \cos 4x \ dx = \frac{1}{2}\int_{\pi/6}^{\pi/3} (\sin 10x + \sin 2x) \ dx \qquad \sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$= \frac{1}{2}\left(-\frac{1}{10}\cos 10x - \frac{1}{2}\cos 2x \ \right|_{\pi/6}^{\pi/3}$$

$$= -\frac{1}{2}\left(\frac{1}{10}\cos\frac{10\pi}{3} + \frac{1}{2}\cos\frac{2\pi}{3} - \frac{1}{10}\cos\frac{5\pi}{3} - \frac{1}{2}\cos\frac{\pi}{3}\right)$$

$$= -\frac{1}{2}\left(-\frac{1}{20} - \frac{1}{4} - \frac{1}{20} - \frac{1}{4}\right)$$

$$= \frac{1}{2}\left(\frac{1}{10} + \frac{1}{2}\right)$$

$$= \frac{6}{20}$$

$$\underline{= \frac{3}{10}}$$

Exercise

Evaluate the integral $\int_{-\pi/2}^{\pi/2} 3\cos^3 x \ dx$

Solution

$$\int_{-\pi/2}^{\pi/2} 3\cos^3 x \ dx = 3\int_{-\pi/2}^{\pi/2} \cos^2 x \ \cos x \ dx$$

$$= 3\int_{-\pi/2}^{\pi/2} \left(1 - \sin^2 x\right) \ d(\sin x)$$

$$= 3\left(\sin x - \frac{1}{3}\sin^3 x \ \right|_{-\pi/2}^{\pi/2}$$

$$= 3\left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right)$$

$$= 3\left(2 - \frac{2}{3}\right)$$

$$\underline{= 4}$$

Exercise

Evaluate the integral $\int_0^{\pi} \sec^2 x\, dx$

Solution

$$\int_0^{\pi} \sec^2 x\, dx = \tan x \Big|_0^{\pi}$$

$$= \tan \pi - \tan 0$$

$$\boxed{= 0}$$

Exercise

Evaluate the integral $\int_0^{\ln\left(\sqrt{3}+2\right)} \frac{\cosh x}{\sqrt{4-\sinh^2 x}}\, dx$

Solution

$$\int_0^{\ln\left(\sqrt{3}+2\right)} \frac{\cosh x}{\sqrt{4-\sinh^2 x}}\, dx = \int_0^{\ln\left(\sqrt{3}+2\right)} \frac{1}{\sqrt{4-\sinh^2 x}}\, d\left(\sinh x\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\sinh x\right) \Big|_0^{\ln\left(\sqrt{3}+2\right)}$$

$$= \sin^{-1}\left(\frac{1}{2}\sinh\left(\ln\left(\sqrt{3}+2\right)\right)\right) - \sin^{-1}\left(\frac{1}{2}\sinh 0\right)$$

$$= \sin^{-1}\left(\frac{1}{4}\left(e^{\ln\left(\sqrt{3}+2\right)} - e^{-\ln\left(\sqrt{3}+2\right)}\right)\right)$$

$$= \sin^{-1}\left(\frac{1}{4}\left(\sqrt{3}+2-\frac{1}{\sqrt{3}+2}\right)\right)$$

$$= \sin^{-1}\left(\frac{1}{4}\left(\frac{3+4\sqrt{3}+4-1}{\sqrt{3}+2}\right)\right)$$

$$= \sin^{-1}\left(\frac{1}{4}\left(\frac{6+4\sqrt{3}}{\sqrt{3}+2}\,\frac{\sqrt{3}-2}{\sqrt{3}-2}\right)\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\left(\frac{-\sqrt{3}}{3-4}\right)\right)$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\boxed{= \frac{\pi}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/2} \cos^4 x\, dx$

Solution

$$\int_0^{\pi/2} \cos^4 x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{\pi}{2}\right)$$

$$\int_0^{\pi/2} \cos^n x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

$$\boxed{=\frac{3\pi}{16}}$$

Exercise

Evaluate $\displaystyle\int_0^{\pi/2} \cos^{10} x\, dx$

Solution

$$\int_0^{\pi/2} \cos^{10}\theta\, d\theta = \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^5 d\theta$$

$$= \frac{1}{32}\int_0^{\pi/2} \left(1+5\cos 2\theta+10\cos^2 2\theta+10\cos^3 2\theta+5\cos^4 2\theta+\cos^5 2\theta\right) d\theta$$

$$= \frac{1}{32}\int_0^{\pi/2} \left(\begin{array}{l} 1+5\cos 2\theta+5+5\cos 4\theta+\frac{5}{4}(1+\cos 4\theta)^2 \\ \quad +\left(10+\cos^2 2\theta\right)\cos^3 2\theta \end{array}\right) d\theta$$

$$= \frac{1}{32}\int_0^{\pi/2} \left(6+5\cos 2\theta+5\cos 4\theta+\frac{5}{4}\left(1+2\cos 4\theta+\cos^2 4\theta\right)\right) d\theta$$

$$+\frac{5}{16}\int_0^{\pi/2} \cos^3 2\theta\, d\theta+\frac{1}{32}\int_0^{\pi/2} \cos^5 2\theta\, d\theta$$

$$= \frac{1}{32}\int_0^{\pi/2} \left(\frac{63}{8}+5\cos 2\theta+\frac{11}{2}\cos 4\theta+\frac{5}{8}\cos 8\theta\right) d\theta$$

$$+\frac{5}{32}\int_0^{\pi/2} \left(1-\sin^2 2\theta\right) d(\sin 2\theta)+\frac{1}{64}\int_0^{\pi/2} \left(1-\sin^2 2\theta\right)^2 d(\sin 2\theta)$$

$$= \frac{1}{32}\left(\frac{63}{8}\theta+\frac{5}{2}\sin 2\theta+\frac{11}{8}\sin 4\theta+\frac{5}{64}\sin 8\theta\right)+\frac{5}{32}\left(\sin 2\theta-\frac{1}{3}\sin^3 2\theta\right)\Bigg|_0^{\pi/2}$$

$$+\frac{1}{64}\int_0^{\pi/2}\left(1-2\sin^2 2\theta+\sin^4 2\theta\right)\,d\left(\sin 2\theta\right)$$

$$=\frac{1}{32}\left(\frac{63\pi}{16}\right)+\frac{1}{64}\left(\sin 2\theta-\frac{2}{3}\sin^3 2\theta+\frac{1}{5}\sin^5 2\theta\right)\Big|_0^{\pi/2}$$

$$\boxed{=\frac{63\,\pi}{512}}$$

Or

$$\int_0^{\pi/2}\cos^{10}x\,dx=\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\left(\frac{9}{10}\right)\left(\frac{\pi}{2}\right)$$

$$\int_0^{\pi/2}\cos^n x\,dx=\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

$$\boxed{=\frac{63\,\pi}{512}}$$

Exercise

Evaluate $\displaystyle\int_0^{\pi/2}\cos^7 x\,dx$

Solution

$$\int_0^{\pi/2}\cos^7 x\,dx=\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)$$

$$\int_0^{\pi/2}\cos^n\theta\,d\theta=\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$$

$$\boxed{=\frac{16}{35}}$$

Or

$$\int_0^{\pi/2}\cos^7 x\,dx=\int_0^{\pi/2}\left(\cos^2 x\right)^3\,d\left(\sin x\right)$$

$$=\int_0^{\pi/2}\left(1-\sin^2 x\right)^3\,d\left(\sin x\right)$$

$$=\int_0^{\pi/2}\left(1-3\sin^2 x+3\sin^4 x-\sin^6 x\right)\,d\left(\sin x\right)$$

$$=\left(\sin x-\sin^3 x+\frac{3}{5}\sin^5 x-\frac{1}{7}\sin^7 x\right)_0^{\pi/2}$$

$$=\frac{3}{5}-\frac{1}{7}$$

$$\boxed{=\frac{16}{37}}$$

Exercise

Evaluate $\int_0^{\pi/2} \cos^9 x\, dx$

Solution

$$\int_0^{\pi/2} \cos^9 x\, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\left(\frac{8}{9}\right)$$

$$\int_0^{\pi/2} \cos^n \theta\, d\theta = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\ldots\left(\frac{n-1}{n}\right)$$

$$= \frac{128}{315}$$

Or

$$\int_0^{\pi/2} \cos^9 \theta\, d\theta = \int_0^{\pi/2} \left(1-\sin^2 x\right)^4\, d(\sin x)$$

$$= \int_0^{\pi/2} \left(1-4\sin^2 x+6\sin^4 x-4\sin^6 x+\sin^8 x\right)\, d(\sin x)$$

$$= \sin x-\frac{4}{3}\sin^3 x+\frac{6}{5}\sin^5 x-\frac{4}{7}\sin^7 x+\frac{1}{9}\sin^9 x \Big|_0^{\pi/2}$$

$$= 1-\frac{4}{3}+\frac{6}{5}-\frac{4}{7}+\frac{1}{9}$$

$$= \frac{128}{315}$$

Exercise

Evaluate $\int_0^{\pi/2} \sin^5 x\, dx$

Solution

$$\int_0^{\pi/2} \sin^5 x\, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)$$

$$\int_0^{\pi/2} \cos^n \theta\, d\theta = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\ldots\left(\frac{n-1}{n}\right)$$

$$= \frac{8}{15}$$

Or

$$\int_0^{\pi/2} \sin^5 x\, dx = \int_0^{\pi/2} \left(1-\cos^2 x\right)^2\, d(\cos x)$$

$$= \int_0^{\pi/2} \left(1-2\cos^2 x+\cos^4 x\right)\, d(\cos x)$$

$$= \cos x - \frac{2}{3}\cos^3 x + \frac{1}{5}\cos^5 x \Big|_0^{\pi/2}$$

$$= -1 + \frac{2}{3} - \frac{1}{5}$$

$$\underline{= -\frac{8}{15}}\Big|$$

Exercise

Evaluate the integral $\int_0^{\pi/2} \sin^6 x\, dx$

Solution

$$\int_0^{\pi/2} \sin^6 x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{\pi}{2}\right)$$

$$\int_0^{\pi/2} \sin^n x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

$$\underline{= \frac{5\pi}{32}}\Big|$$

Exercise

Evaluate the integral $\int_0^{\pi/2} \sin^8 x\, dx$

Solution

$$\int_0^{\pi/2} \sin^8 x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\left(\frac{\pi}{2}\right)$$

$$\int_0^{\pi/2} \sin^n x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

$$\underline{= \frac{35\pi}{256}}\Big|$$

Exercise

Evaluate the integral $\int_0^{\pi/2} \tan^2 \frac{x}{2}\, dx$

Solution

$$\int_0^{\pi/2} \tan^2 \frac{x}{2}\, dx = \int_0^{\pi/2} \left(\sec^2 \frac{x}{2} - 1\right) dx$$

$$= 2\tan\frac{x}{2} - x \Big|_0^{\frac{\pi}{2}}$$

$$= 2\tan\frac{\pi}{4} - \frac{\pi}{2}$$

$$\boxed{= 2 - \frac{\pi}{2}}$$

Exercise

Find the area of the region bounded by the graphs of $y = \tan x$ and $y = \sec x$ on the interval $\left[0,\ \frac{\pi}{4}\right]$

Solution

$$A = \int_0^{\pi/4} (\sec x - \tan x)\,dx$$

$$= \ln|\sec x + \tan x| + \ln|\cos x| \Big|_0^{\pi/4}$$

$$= \ln\left(\sqrt{2}+1\right) + \ln\frac{\sqrt{2}}{2} - 0$$

$$= \ln\left(\frac{\sqrt{2}}{2}\left(\sqrt{2}+1\right)\right)$$

$$\boxed{= \ln\left(1+\frac{\sqrt{2}}{2}\right) \quad unit^2}$$

Exercise

Find the area of the region bounded by the graphs of the equations $y = \sin x, \quad y = \sin^3 x, \quad x = 0, \quad x = \frac{\pi}{2}$

Solution

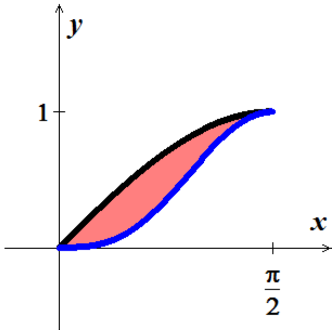
$$A = \int_0^{\pi/2} \left(\sin x - \sin^3 x\right) dx$$

$$= \int_0^{\pi/2} \sin x\,dx - \int_0^{\pi/2} \sin^3 x\,dx$$

$$= -\cos x \Big|_0^{\pi/2} - \frac{2}{3}$$

$$= 1 - \frac{2}{3}$$

$$\boxed{= \frac{1}{3} \quad unit^2}$$



Exercise

Find the area of the region bounded by the graphs of the equations $y = \sin^2 \pi x, \quad y = 0, \quad x = 0, \quad x = 1$

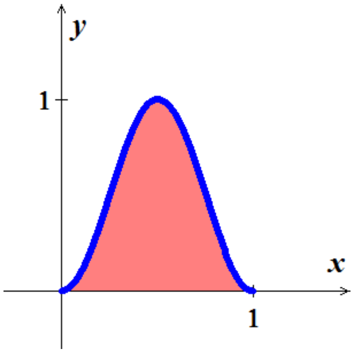
Solution

$$A = \int_0^1 \sin^2 \pi x \, dx$$

$$= \frac{1}{2}\int_0^1 (1+\cos 2\pi x)\, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{2\pi}\sin 2\pi x \right|_0^1$$

$$\boxed{= \frac{1}{2} \quad unit^2}$$



Exercise

Find the area of the region bounded by the graphs of the equations

$$y = \cos^2 x, \quad y = \sin^2 x, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$$

Solution

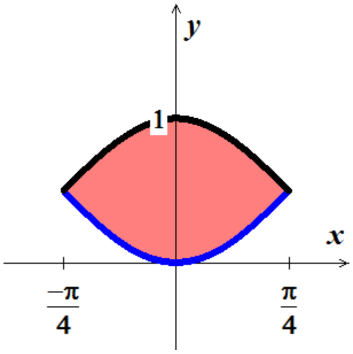
$$A = \int_{-\pi/4}^{\pi/4} \left(\cos^2 x - \sin^2 x\right) dx$$

$$= \int_{-\pi/4}^{\pi/4} \cos 2x \, dx$$

$$= \frac{1}{2}\sin 2x \Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{2}(1+1)$$

$$\boxed{= 1 \quad unit^2}$$



Exercise

Find the area of the region bounded by the graphs of the equations

$$y = \cos^2 x, \quad y = \sin x \cos x, \quad x = -\frac{\pi}{2}, \quad x = \frac{\pi}{4}$$

Solution

$$A = \int_{-\pi/2}^{\pi/4} \left(\cos^2 x - \sin x \cos x\right) dx$$

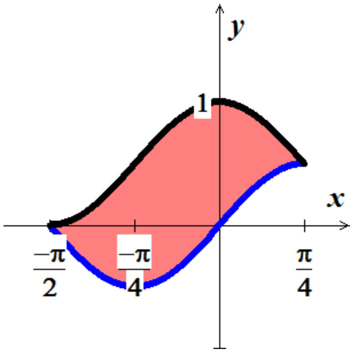
$$= \int_{-\pi/2}^{\pi/4} \left(\frac{1}{2}+\frac{1}{2}\cos 2x - \frac{1}{2}\sin 2x\right) dx$$

$$= \frac{1}{2}\left(x+\frac{1}{2}\sin 2x + \frac{1}{2}\cos 2x\right|_{-\pi/2}^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{2}+\frac{\pi}{2}+\frac{1}{2}\right)$$

$$= \frac{1}{2}\left(\frac{3\pi}{4}+1\right)$$

$$= \frac{3\pi}{8}+\frac{1}{2} \quad unit^2$$



Exercise

Find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the *x-axis* $y=\tan x, \quad y=0, \quad x=-\frac{\pi}{4}, \quad x=\frac{\pi}{4}$

Solution

Disks Method:

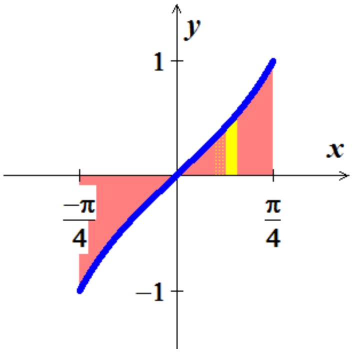
$$V = 2\pi \int_0^{\pi/4} \tan^2 x\, dx$$

$$= 2\pi \int_0^{\pi/4} \left(\sec^2 x - 1\right) dx$$

$$= 2\pi \left(\tan x - x\right|_0^{\pi/4}$$

$$= 2\pi\left(1-\frac{\pi}{4}\right)$$

$$= 2\pi - \frac{1}{2} \quad unit^3$$



Exercise

Find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the *x-axis* $y=\cos\frac{x}{2}, \quad y=\sin\frac{x}{2}, \quad x=0, \quad x=\frac{\pi}{2}$

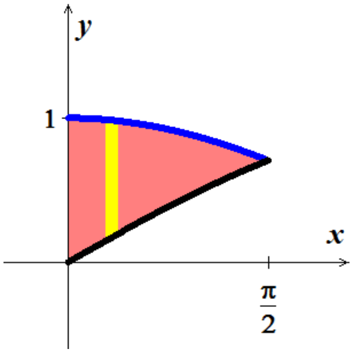
Solution

$$V = \pi \int_0^{\pi/2} \left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) dx \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= \pi \int_0^{\pi/2} \cos x \, dx$$

$$= \pi \sin x \Big|_0^{\pi/2}$$

$$\boxed{= \pi \quad unit^3}$$



Exercise

Find the ***volume*** of the solid generated by revolving the region bounded by the graphs of the equations about the x-*axis*, then find the ***centroid*** of the region

$$y = \sin x, \quad y = 0, \quad x = 0, \quad x = \pi$$

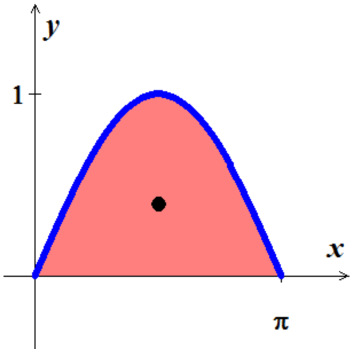
Solution

$$V = \pi \int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x) \, dx$$

$$= \frac{\pi}{2} \left(x - \frac{1}{2} \sin 2x \right|_0^{\pi}$$

$$\boxed{= \frac{\pi^2}{2} \quad unit^3}$$



$$A = \int_0^{\pi} \sin x \, dx$$

$$= -\cos x \Big|_0^{\pi}$$

$$= -(-1 - 1)$$

$$\boxed{= 2 \quad unit^2}$$

$$\overline{x} = \frac{1}{A} \int_0^{\pi} x \sin x \, dx$$

		$\int \sin x$
+	x	$-\cos x$
–	1	$-\sin x$

$$=\frac{1}{2}\left(-x\cos x+\sin x\right|_0^{\pi}$$

$$\boxed{=\frac{\pi}{2}}$$

$$\bar{y}=\frac{1}{2A}\int_0^{\pi}\sin^2 x\,dx$$

$$=\frac{1}{8}\int_0^{\pi}(1-\cos 2x)\,dx$$

$$=\frac{1}{8}\left(x-\frac{1}{2}\sin 2x\right|_0^{\pi}$$

$$\boxed{=\frac{\pi}{8}}$$

$$\boxed{(\bar{x},\bar{y})=\left(\frac{1}{2},\ \frac{\pi}{8}\right)}$$

Exercise

Find the ***volume*** of the solid generated by revolving the region bounded by the graphs of the equations about the x-*axis*, then find the ***centroid*** of the region

$$y=\cos x,\quad y=0,\quad x=0,\quad x=\frac{\pi}{2}$$

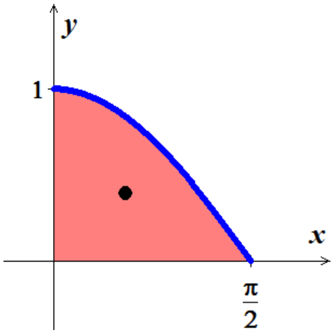
Solution

$$V=\pi\int_0^{\pi/2}\cos^2 x\,dx$$

$$=\frac{\pi}{2}\int_0^{\pi/2}(1+\cos 2x)\,dx$$

$$=\frac{\pi}{2}\left(x+\frac{1}{2}\sin 2x\right|_0^{\pi/2}$$

$$\boxed{=\frac{\pi^2}{4}\quad unit^3}$$



$$A=\int_0^{\pi/2}\cos x\,dx$$

$$=\sin x\Big|_0^{\pi/2}$$

$$\boxed{=1\quad unit^2}$$

$$\overline{x} = \frac{1}{A}\int_0^{\pi/2} x\ \cos x\ dx$$

		$\int \cos x\ dx$
$+$	x	$\sin x$
$-$	1	$-\cos x$

$$= x\sin x + \cos x \Big|_0^{\pi/2}$$

$$\boxed{= \frac{\pi}{2} - 1}$$

$$\overline{y} = \frac{1}{2A}\int_0^{\pi/2} \cos^2 x\ dx$$

$$= \frac{1}{4}\int_0^{\pi/2} (1 + \cos 2x)\ dx$$

$$= \frac{1}{4}\ \left(x + \frac{1}{2}\sin 2x\ \right|_0^{\pi/2}$$

$$\boxed{= \frac{\pi}{8}}$$

$$\boxed{(\overline{x},\ \overline{y}) = \left(\frac{\pi-2}{2},\ \frac{\pi}{8}\right)}$$

Solution *Section* 2.3 – Trigonometric Substitutions

Exercise

Evaluate the integral $\displaystyle\int \frac{3\,dx}{\sqrt{1+9x^2}}$

Solution

$$3x = \tan t \quad \Rightarrow \quad dx = \frac{1}{3}\sec^2 t\ dt$$

$$\sqrt{1+9x^2} = 3\sec^2 t$$

$$\int \frac{3\,dx}{\sqrt{1+9x^2}} = \frac{1}{3}\int \frac{\sec^2 t}{3\sec t}\,dt$$

$$= \int \sec t\ dt$$

$$= \int \sec t\,\frac{\sec t + \tan t}{\sec t + \tan t}\,dt$$

$$= \int \frac{\sec^2 t + \sec t\tan t}{\sec t + \tan t}\ dt$$

$$= \int \frac{1}{\sec t + \tan t}\ d\left(\sec t + \tan t\right)$$

$$= \ln\left|\sec t + \tan t\right| + C$$

$$= \ln\left|\sqrt{1+u^2} + u\right| + C$$

$$\boxed{= \ln\left|\sqrt{1+9x^2} + 3x\right| + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^2}{4+x^2}\,dx$

Solution

$$x = 2\tan\theta \quad dx = 2\sec^2\theta d\theta$$

$$4 + x^2 = 4\sec^2\theta$$

$$\int \frac{x^2}{4+x^2}\,dx = \int \frac{4\tan^2\theta}{4\sec^2\theta}\ 2\sec^2\theta\,d\theta$$

$$= 2\int \tan^2\theta \; d\theta$$

$$= 2\int \left(\sec^2\theta - 1\right) d\theta \qquad\qquad \int \sec^2\theta d\theta = \tan\theta$$

$$= 2\left(\tan\theta - \theta\right) + C$$

$$= 2\left(\frac{x}{2} - \tan^{-1}\left(\frac{x}{2}\right)\right) + C$$

$$= x - 2\tan^{-1}\left(\frac{x}{2}\right) + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}}$

Solution

$x = \tan\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = \sec^2\theta d\theta$

$\sqrt{x^2+1} = \sqrt{\tan^2\theta + 1} = \sec\theta$

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta \; d\theta}{\tan^2\theta\sec\theta}$$

$$= \int \frac{\sec\theta}{\tan^2\theta} d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta\cos\theta} d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta} d\theta$$

$$= \int \sin^{-2}\theta d\left(\sin\theta\right)$$

$$= -\frac{1}{\sin\theta} + C$$

$$= -\frac{\sec\theta}{\tan\theta} + C$$

$$= -\frac{\sqrt{x^2+1}}{x} + C$$

Exercise

Evaluate: $\displaystyle\int \frac{dx}{\sqrt{x^2+4}}$

Solution

Let: $x = 2\tan\theta \quad \to dx = 2\sec^2\theta d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\sqrt{x^2+4} = 2|\sec\theta|$

$$\int \frac{dx}{\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\ d\theta}{\sqrt{4\sec^2\theta}}$$

$$= \int \frac{2\sec^2\theta\ d\theta}{2|\sec\theta|}$$

$$= \int \sec\theta\ d\theta$$

$$= \int \sec\theta\ \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\ d\theta$$

$$= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\ d\theta$$

$$= \int \frac{1}{\sec\theta + \tan\theta}\ d\left(\sec\theta + \tan\theta\right)$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$\underline{= \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\left(1+x^2\right)^2}$

Solution

$x = \tan\theta \qquad 1 + x^2 = \left(\sec^2\theta\right)^2$

$dx = \sec^2\theta\ d\theta$

$$\int \frac{dx}{\left(1+x^2\right)^2} = \int \frac{\sec^2\theta}{\sec^4\theta}\ d\theta$$

$$= \int \frac{1}{\sec^2 \theta}\, d\theta$$

$$= \int \cos^2 \theta \, d\theta$$

$$= \frac{1}{2} \int (1 + \cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}\tan^{-1} x + \frac{1}{2}\sin\theta\cos\theta + C$$

$$= \frac{1}{2}\tan^{-1} x + \frac{1}{2}\frac{x}{\sqrt{1+x^2}}\frac{1}{\sqrt{1+x^2}} + C$$

$$\boxed{= \frac{1}{2}\tan^{-1} x + \frac{x}{2\left(1+x^2\right)} + C}$$

Exercise

Evaluate $\displaystyle \int \frac{dx}{\sqrt{4x^2+1}}$

Solution

$2x = \tan\theta \qquad \sqrt{4x^2+1} = \sec\theta$

$dx = \frac{1}{2}\sec^2\theta \, d\theta$

$$\int \frac{dx}{\sqrt{4x^2+1}} = \frac{1}{2}\int \frac{\sec^2\theta}{\sec\theta}\, d\theta$$

$$= \frac{1}{2}\int \sec\theta \, d\theta$$

$$= \int \sec\theta \, \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\, d\theta$$

$$= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\, d\theta$$

$$= \int \frac{1}{\sec\theta + \tan\theta}\, d\left(\sec\theta + \tan\theta\right)$$

$$= \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$\boxed{= \frac{1}{2}\ln\left|\sqrt{4x^2+1} + 2x\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\left(x^2+1\right)^{3/2}}$

Solution

$x = \tan\theta \qquad \sqrt{x^2+1} = \sec\theta$

$dx = \sec^2\theta\, d\theta$

$$\int \frac{dx}{\left(x^2+1\right)^{3/2}} = \int \frac{\sec^2\theta}{(\sec\theta)^3}\, d\theta$$

$$= \int \frac{d\theta}{\sec\theta}$$

$$= \int \cos\theta\, d\theta$$

$$= \sin\theta + C \qquad\qquad \sin\theta = \frac{\tan\theta}{\sec\theta} = \frac{x}{\sqrt{x^2+1}}$$

$$= \frac{x}{\sqrt{x^2+1}} + C$$

Exercise

Evaluate $\displaystyle\int \frac{9x^3}{\sqrt{x^2+1}}\, dx$

Solution

$x = \tan\theta \qquad \sqrt{x^2+1} = \sec\theta$

$dx = \sec^2\theta\, d\theta$

$$\int \frac{9x^3}{\sqrt{x^2+1}}\, dx = \int \frac{9\tan^3\theta}{\sec\theta}\left(\sec^2\theta\right) d\theta$$

$$= 9\int \tan^2\theta \tan\theta \sec\theta\, d\theta$$

$$= 9\int \left(\sec^2\theta - 1\right)\, d(\sec\theta)$$

$$= 9\left(\tfrac{1}{3}\sec^3\theta - \sec\theta\right) + C$$

$$= 3\left(x^2+1\right)\sqrt{x^2+1} - 9\sqrt{x^2+1} + C$$

$$=3\sqrt{x^2+1}\left(x^2+1-3\right)+C$$

$$\boxed{=3\sqrt{x^2+1}\left(x^2-2\right)+C}$$

Exercise

Evaluate $\displaystyle\int \sqrt{16x^2+9}\ dx$

Solution

$$4x=3\tan\theta \qquad \sqrt{4x^2+9}=3\sec\theta$$

$$4dx=3\sec^2\theta\ d\theta$$

$$\int \sqrt{16x^2+9}\ dx=\int 3\sec\theta\left(\frac{3}{4}\sec^2\theta\right)d\theta$$

$$=\frac{9}{4}\int \sec^3\theta\ d\theta$$

$$u=\sec x \qquad dv=\sec^2 xdx$$

$$du=\sec x\tan xdx \qquad v=\tan x$$

$$\int \sec^3 xdx=\sec x\tan x-\int \tan x\left(\sec x\tan xdx\right)$$

$$=\sec x\tan x-\int \tan^2 x\sec xdx$$

$$=\sec x\tan x-\int \left(\sec^2 x-1\right)\sec x\ dx$$

$$=\sec x\tan x-\int \sec^3 x\ dx+\int \sec x\ dx$$

$$2\int \sec^3 x\ dx=\sec x\tan x+\ln\left|\sec x+\tan x\right|$$

$$\int \sec^3 x\ dx=\frac{1}{2}\sec x\tan x+\frac{1}{2}\ln\left|\sec x+\tan x\right|$$

$$=\frac{9}{8}\sec\theta\tan\theta+\frac{9}{8}\ln\left|\sec\theta+\tan\theta\right|+C$$

$$=\frac{9}{8}\frac{\sqrt{4x^2+9}}{3}\frac{4x}{3}+\frac{9}{8}\ln\left|\frac{\sqrt{4x^2+9}}{3}+\frac{4x}{3}\right|+C$$

$$\boxed{=\frac{1}{2}x\sqrt{4x^2+9}+\frac{9}{8}\ln\left|\frac{2x+\sqrt{4x^2+9}}{3}\right|+C}$$

Exercise

Evaluate $\displaystyle\int x\sqrt{x^2+1}\,dx$

Solution

$$\int x\sqrt{x^2+1}\,dx = \frac{1}{2}\int \left(x^2+1\right)^{1/2}\,d\left(x^2+1\right)$$

$$\boxed{=\frac{1}{3}\left(x^2+1\right)^{3/2}+C}$$

OR

$x=\tan\theta \qquad \sqrt{x^2+1}=\sec\theta$

$dx=\sec^2\theta\,d\theta$

$$\int x\sqrt{x^2+1}\,dx = \int \tan\theta\sec^3\theta\,d\theta$$

$$= \int \sec^2\theta\,d(\sec\theta)$$

$$= \frac{1}{3}\sec^3\theta + C$$

$$\boxed{=\frac{1}{3}\left(x^2+1\right)^{3/2}+C}$$

Exercise

Evaluate $\displaystyle\int \frac{\sqrt{25x^2+4}}{x^4}\,dx$

Solution

$5x=2\tan\theta \qquad \sqrt{25x^2+4}=2\sec\theta$

$5dx=2\sec^2\theta\,d\theta$

$$\int \frac{\sqrt{25x^2+4}}{x^4}\,dx = \int \frac{2\sec\theta}{\left(\frac{2}{5}\right)^4\tan^4\theta}\,\frac{2}{5}\sec^2\theta\,d\theta$$

$$= \frac{125}{4}\int \frac{\sec^3\theta}{\tan^4\theta}\,d\theta$$

$$= \frac{125}{4}\int \frac{\cos\theta}{\sin^4\theta}\,d\theta$$

$$= \frac{125}{4}\int \sin^{-4}\theta \, d(\sin\theta)$$

$$= -\frac{125}{12}\frac{1}{\sin^3\theta} + C$$

$$= -\frac{125}{12}\left(\frac{\tan\theta}{\sec\theta}\right)^3 + C$$

$$= -\frac{125}{12}\left(\frac{\sqrt{25x^2+4}}{5x}\right)^3 + C$$

$$\boxed{= -\frac{\left(25x^2+4\right)^{3/2}}{12x^3} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x\sqrt{4x^2+9}}\,dx$

Solution

$2x = 3\tan\theta \qquad \sqrt{4x^2+9} = 3\sec\theta$

$dx = \frac{3}{2}\sec^2\theta\, d\theta$

$$\int \frac{1}{x\sqrt{4x^2+9}}\,dx = \int \frac{1}{\frac{9}{2}\tan\theta\sec\theta}\left(\frac{3}{2}\sec^2\theta\right)\,d\theta$$

$$= \int \frac{1}{\frac{9}{2}\tan\theta\sec\theta}\left(\frac{3}{2}\sec^2\theta\right)\,d\theta$$

$$= \frac{1}{3}\int \frac{\sec\theta}{\tan\theta}\,d\theta$$

$$= \frac{1}{3}\int \frac{1}{\sin\theta}\,d\theta$$

$$= \frac{1}{3}\int \csc\theta\,\frac{\csc\theta+\cot\theta}{\csc\theta+\cot\theta}\,d\theta$$

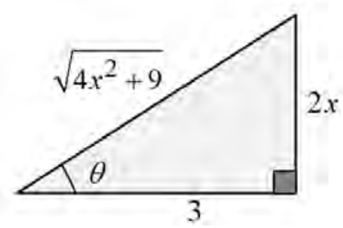
$$= \frac{1}{3}\int \frac{\csc^2\theta+\csc\theta\cot\theta}{\csc\theta+\cot\theta}\,d\theta$$

$$= -\frac{1}{3}\int \frac{1}{\csc\theta+\cot\theta}\,d\left(\csc\theta+\cot\theta\right)$$

$$= -\frac{1}{3}\ln\left|\csc\theta + \cot\theta\right| + C$$

$$= -\frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9}}{2x} + \frac{3}{2x}\right| + C$$

$$\boxed{= -\frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9}+3}{2x}\right| + C}$$



Exercise

Evaluate $\displaystyle\int \frac{1}{\left(x^2+5\right)^{3/2}}\,dx$

Solution

$$x = \sqrt{5}\tan\theta \qquad \sqrt{x^2+5} = \sqrt{5}\sec\theta$$

$$dx = \sqrt{5}\sec^2\theta\; d\theta$$

$$\int \frac{1}{\left(x^2+5\right)^{3/2}}\,dx = \int \frac{1}{5\sqrt{5}\sec^3\theta}\left(\sqrt{5}\sec^2\theta\right)\,d\theta$$

$$= \frac{1}{5}\int \frac{1}{\sec\theta}\,d\theta$$

$$= \frac{1}{5}\int \cos\theta\; d\theta$$

$$= \frac{1}{5}\sin\theta$$

$$= \frac{1}{5}\frac{\tan\theta}{\sec\theta}$$

$$= \frac{1}{5}\frac{x}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{x^2+5}}$$

$$\boxed{= \frac{1}{5}\frac{x}{\sqrt{x^2+5}} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+4}}$

Solution

$$x = 2\tan\theta \qquad \sqrt{x^2+4} = 2\sec\theta$$

$$dx = 2\sec^2\theta\, d\theta$$

$$\int \frac{x\,dx}{\sqrt{x^2+4}} = \int \frac{2\tan\theta}{2\sec\theta}\left(2\sec^2\theta\, d\theta\right)$$

$$= 2\int \tan\theta\sec\theta\, d\theta$$

$$= 2\sec\theta$$

$$\underline{= \sqrt{x^2+4} + C}\,\big|$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^3}{\sqrt{x^2+4}}\,dx$

Solution

$$x = 2\tan\theta \qquad \sqrt{x^2+4} = 2\sec\theta$$

$$dx = 2\sec^2\theta\, d\theta$$

$$\int \frac{x^3\,dx}{\sqrt{x^2+4}} = \int \frac{8\tan^3\theta}{2\sec\theta}\left(2\sec^2\theta\, d\theta\right)$$

$$= 8\int \tan^2\theta\tan\theta\sec\theta\, d\theta$$

$$= 8\int \left(\sec^2\theta - 1\right) d\left(\sec\theta\right)$$

$$= 8\left(\frac{1}{3}\sec^3\theta - \sec\theta\right) + C$$

$$= \frac{8}{3}\frac{\left(x^2+4\right)^{3/2}}{8} - 8\frac{\left(x^2+4\right)^{1/2}}{2} + C$$

$$= \sqrt{x^2+4}\left(\frac{1}{3}\left(x^2+4\right) - 4\right) + C$$

$$= \frac{1}{3}\sqrt{x^2+4}\,\left(x^2+4-12\right) + C$$

$$\underline{= \frac{1}{3}\sqrt{x^2+4}\,\left(x^2-8\right) + C}\,\big|$$

Exercise

Evaluate the integral $\int \frac{dx}{\left(1+4x^2\right)^{3/2}}$

Solution

$$x = \frac{1}{2}\tan\theta \qquad \sqrt{4x^2+1} = \sec\theta$$

$$dx = \frac{1}{2}\sec^2\theta\, d\theta$$

$$\int \frac{dx}{\left(1+4x^2\right)^{3/2}} = \frac{1}{2}\int \frac{\sec^2\theta}{\left(\sec^2\theta\right)^{3/2}}\, d\theta$$

$$= \frac{1}{2}\int \frac{\sec^2\theta}{\sec^3\theta}\, d\theta$$

$$= \frac{1}{2}\int \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{1}{2}\int \cos\theta\, d\theta$$

$$= \frac{1}{2}\sin\theta + C$$

$$= \frac{1}{2}\frac{\tan\theta}{\sec\theta} + C$$

$$= \frac{1}{2}\frac{2x}{\sqrt{4x^2+1}} + C$$

$$= \frac{x}{\sqrt{4x^2+1}} + C$$

Exercise

Evaluate the integral $\int \frac{dx}{x^2\sqrt{x^2+9}}$

Solution

$$x = 3\tan\theta \qquad \sqrt{x^2+9} = 3\sec\theta$$

$$dx = 3\sec^2\theta\, d\theta$$

$$\int \frac{dx}{x^2\sqrt{x^2+9}} = \int \frac{3\sec^2\theta}{9\tan^2\theta\,(3\sec\theta)}\, d\theta$$

$$= \frac{1}{9}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

$$= \frac{1}{9}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{9}\int \frac{1}{\sin^2\theta}\,d(\sin\theta)$$

$$= -\frac{1}{9}\frac{1}{\sin\theta}+C$$

$$= -\frac{1}{9}\frac{\sec\theta}{\tan\theta}+C$$

$$\boxed{= -\frac{1}{9}\frac{\sqrt{x^2+9}}{x}+C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+4}}$

Solution

$$x = 2\tan\theta \qquad \sqrt{x^2+4} = 2\sec\theta$$

$$dx = 2\sec^2\theta\,d\theta$$

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta}{4\tan^2\theta\,(2\sec\theta)}\,d\theta$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{1}{\sin^2\theta}\,d(\sin\theta)$$

$$= -\frac{1}{4}\frac{1}{\sin\theta}+C$$

$$= -\frac{1}{4}\frac{\sec\theta}{\tan\theta}+C$$

$$\boxed{= -\frac{1}{4}\frac{\sqrt{x^2+4}}{x}+C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^2+1}}$

Solution

$$x = \tan\theta \qquad \sqrt{x^2+1} = \sec\theta$$

$$dx = \sec^2\theta\,d\theta$$

$$\int \frac{x^2\,dx}{\sqrt{x^2+1}} = \int \frac{\tan^2\theta}{\sec\theta}\sec^2\theta\,d\theta$$

$$= \int \tan^2\theta\sec\theta\,d\theta$$

$$= \int \left(\sec^2\theta - 1\right)\sec\theta\,d\theta$$

$$= \int \sec^3\theta\,d\theta - \int \sec\theta\,d\theta$$

$$u = \sec\theta \qquad dv = \sec^2\theta d\theta$$

$$du = \sec\theta\tan\theta d\theta \qquad v = \tan\theta$$

$$\int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \tan\theta\left(\sec\theta\tan\theta d\theta\right)$$

$$= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta\,d\theta$$

$$= \sec\theta\tan\theta - \int \left(\sec^2\theta - 1\right)\sec\theta\,d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta\,d\theta + \int \sec\theta\,d\theta$$

$$2\int \sec^3\theta d\theta = \sec\theta\tan\theta + \int \sec\theta\,d\theta$$

$$\int \sec^3\theta d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\,d\theta$$

$$\int \sec\theta\,d\theta = \int \sec\theta\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\,d\theta$$

$$= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\,d\theta$$

$$= \int \frac{1}{\sec\theta + \tan\theta}\,d\left(\sec\theta + \tan\theta\right)$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$\int \frac{x^2\,dx}{\sqrt{x^2+1}} = \int \sec^3\theta\,d\theta - \int \sec\theta\,d\theta$$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\,d\theta - \int \sec\theta\,d\theta$$

$$= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\int \sec\theta\,d\theta$$

$$= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \frac{1}{2}x\sqrt{x^2+1} - \frac{1}{2}\ln\left|\sqrt{x^2+1} + x\right| + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^3\,dx}{\left(x^2+a^2\right)^{3/2}}$

Solution

$x = a\tan\theta \qquad \sqrt{x^2+a^2} = a\sec\theta$

$dx = a\sec^2\theta\,d\theta$

$$\int \frac{x^3\,dx}{\left(x^2+a^2\right)^{3/2}} = \int \frac{a^3\tan^3\theta}{a^3\sec^3\theta}\left(a\sec^2\theta\right)\,d\theta$$

$$= a\int \frac{\tan^3\theta}{\sec\theta}\,d\theta$$

$$= a\int \frac{\sin^3\theta}{\cos^3\theta}\cos\theta\,d\theta$$

$$= a\int \frac{\sin^3\theta}{\cos^2\theta}\,d\theta$$

$$= a\int \sin^2\theta\cos^{-2}\theta\,\left(\sin\theta\,d\theta\right)$$

$$= -a\int \left(1-\cos^2\theta\right)\cos^{-2}\theta\,d\left(\cos\theta\right)$$

$$= -a\int \left(\cos^{-2}\theta - 1\right)\,d\left(\cos\theta\right)$$

$$= -a\left(-\frac{1}{\cos\theta} - \cos\theta\right) + C$$

$$= a\left(\sec\theta + \frac{1}{\sec\theta}\right) + C$$

$$= a\left(\frac{\sqrt{x^2 + a^2}}{a} + \frac{a}{\sqrt{x^2 + a^2}}\right) + C$$

$$\boxed{= \sqrt{x^2 + a^2} + \frac{a^2}{\sqrt{x^2 + a^2}} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\left(x^2 + 4\right)^2}$

Solution

$$x = 2\tan\theta \qquad x^2 + 4 = 4\sec^2\theta$$

$$dx = 2\sec^2\theta\, d\theta$$

$$\int \frac{dx}{\left(x^2 + 4\right)^2} = \int \frac{2\sec^2\theta}{\left(4\sec^2\theta\right)^2}\, d\theta$$

$$= \frac{1}{8}\int \frac{\sec^2\theta}{\sec^4\theta}\, d\theta$$

$$= \frac{1}{8}\int \frac{1}{\sec^2\theta}\, d\theta$$

$$= \frac{1}{8}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{16}\int (1 + \cos 2\theta)\, d\theta$$

$$= \frac{1}{16}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{16}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{1}{16}\left(\theta + \frac{\tan\theta}{\sec\theta}\frac{1}{\sec\theta}\right) + C$$

$$= \frac{1}{16}\left(\arctan\frac{x}{2} + \frac{x}{\sqrt{x^2 + 4}}\frac{2}{\sqrt{x^2 + 4}}\right) + C$$

$$\boxed{= \frac{1}{16}\arctan\frac{x}{2} + \frac{1}{8}\frac{x}{x^2 + 4} + C}$$

Exercise

Evaluate the integral $\int \frac{dx}{\sqrt{4x^2+16}}$

Solution

$$\int \frac{dx}{\sqrt{4x^2+16}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2+4}}$$

$$x = 2\tan\theta \qquad \sqrt{x^2+4} = 2\sec\theta$$

$$dx = 2\sec^2\theta\, d\theta$$

$$\begin{aligned}\int \frac{dx}{\sqrt{4x^2+16}} &= \frac{1}{2}\int \frac{2\sec^2\theta}{2\sec\theta}\, d\theta \\ &= \frac{1}{2}\int \sec\theta\, d\theta \\ &= \frac{1}{2}\int \sec\theta \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta} d\theta \\ &= \frac{1}{2}\int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\, d\theta \\ &= \frac{1}{2}\int \frac{1}{\sec\theta+\tan\theta}\, d\left(\sec\theta+\tan\theta\right) \\ &= \frac{1}{2}\ln\left|\sec\theta+\tan\theta\right| + C \\ &\boxed{= \frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right| + C}\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x^4}{x^2+1}\, dx$

Solution

$$x = \tan\theta \qquad x^2+1 = \sec^2\theta$$

$$dx = \sec^2\theta\, d\theta$$

$$\int \frac{x^4}{x^2+1}\, dx = \int \frac{\tan^4\theta}{\sec^2\theta}\sec^2\theta\, d\theta$$

$$= \int \tan^4 \theta \, d\theta$$

$$= \int \tan^2 \theta \left(\sec^2 \theta - 1\right) \, d\theta$$

$$= \int \tan^2 \theta \sec^2 \theta \, d\theta - \int \tan^2 \theta \, d\theta$$

$$= \int \tan^2 \theta \, d(\tan \theta) - \int \left(\sec^2 \theta - 1\right) d\theta$$

$$= \frac{1}{3} \tan^3 \theta - \tan \theta + \theta + C$$

$$\boxed{= \frac{1}{3} x^3 - x + \tan^{-1} x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{e^{2x}}{\left(1+e^{4x}\right)^{3/2}} dx$

Solution

$$\int \frac{e^{2x}}{\left(1+e^{4x}\right)^{3/2}} dx = \frac{1}{2} \int \frac{1}{\left(1+\left(e^{2x}\right)^2\right)^{3/2}} d\left(e^{2x}\right) \qquad \textbf{\textit{Let}}\ y = e^{2x}$$

$$= \frac{1}{2} \int \frac{1}{\left(1+y^2\right)^{3/2}} dy$$

$y = \tan \theta \qquad y^2 + 1 = \sec^2 \theta$

$dy = \sec^2 \theta \, d\theta$

$$\int \frac{e^{2x}}{\left(1+e^{4x}\right)^{3/2}} dx = \frac{1}{2} \int \frac{1}{\left(\sec^2 \theta\right)^{3/2}} \sec^2 \theta \, d\theta$$

$$= \frac{1}{2} \int \frac{1}{\sec^3 \theta} \sec^2 \theta \, d\theta$$

$$= \frac{1}{2} \int \frac{1}{\sec \theta} \, d\theta$$

$$=\frac{1}{2}\int \cos\theta\, d\theta$$

$$=\frac{1}{2}\sin\theta + C$$

$$=\frac{1}{2}\frac{\tan\theta}{\sec\theta} + C$$

$$=\frac{1}{2}\frac{y}{\sqrt{1+y^2}} + C$$

$$\boxed{=\frac{1}{2}\frac{e^{2x}}{\sqrt{1+e^{4x}}} + C}$$

Exercise

Evaluate the integral $\int \frac{dx}{1+\cos x}$

Solution

$$\int \frac{dx}{1+\cos x} = \int \frac{1}{1+\cos x}\frac{1-\cos x}{1-\cos x}\, dx$$

$$= \int \frac{1-\cos x}{1-\cos^2 x}\, dx$$

$$= \int \frac{1-\cos x}{\sin^2 x}\, dx$$

$$= \int \left(\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}\right) dx$$

$$= \int \csc^2 x\, dx - \int \frac{1}{\sin^2 x}\, d(\sin x)$$

$$= -\cot x + \frac{1}{\sin x} + C$$

$$\boxed{= -\cot x + \csc x + C}$$

Exercise

Evaluate the integral $\int \frac{dx}{\sqrt{x^2-a^2}}$

Solution

$$x = a\sec\theta \qquad \sqrt{x^2-a^2} = a\tan\theta$$

$$dx = a\sec\theta\tan\theta\, d\theta$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta}{a\tan\theta}\, d\theta$$

$$= \int \sec\theta \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\, d\theta$$

$$= \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\, d\theta$$

$$= \int \frac{1}{\sec\theta+\tan\theta}\, d\left(\sec\theta+\tan\theta\right)$$

$$= \ln\left|\sec\theta+\tan\theta\right| + C$$

$$\boxed{= \ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right| + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\, dx$

Solution

$x = \sec\theta \qquad \sqrt{x^2-1} = \tan\theta$

$dx = \sec\theta\tan\theta\ d\theta$

$$\int \frac{\sqrt{x^2-1}}{x}\, dx = \int \frac{\tan\theta}{\sec\theta}\left(\sec\theta\tan\theta\right)\, d\theta$$

$$= \int \tan^2\theta\ d\theta$$

$$= \int \left(\sec^2\theta - 1\right)\, d\theta$$

$$= \tan\theta - \theta + C$$

$$\boxed{= \sqrt{x^2-1} - \operatorname{arcsec}\theta + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sqrt{y^2-49}}{y} dy, \quad y > 7$

Solution

$y = 7\sec\theta \quad \to \quad dy = 7\sec\theta\tan\theta d\theta$

$$\sqrt{y^2-49} = 7\tan\theta$$

$$\int \frac{\sqrt{y^2-49}}{y}\,dy = \int \frac{(7\tan\theta)}{7\sec\theta}(7\sec\theta\tan\theta)\,d\theta$$

$$= 7\int \tan^2\theta d\theta$$

$$= 7\int \left(\sec^2\theta - 1\right)d\theta$$

$$= 7(\tan\theta - \theta) + C$$

$$\boxed{= 7\left[\frac{\sqrt{y^2-49}}{7} - \sec^{-1}\left(\frac{y}{7}\right)\right] + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{5\,dx}{\sqrt{25x^2-9}}, \quad x > \frac{3}{5} = \sin^{-1}\frac{1}{2} - \sin^{-1}0$

Solution

$$5x = 3\sec\theta \quad \rightarrow \quad dx = \frac{3}{5}\sec\theta\tan\theta d\theta$$

$$\sqrt{25x^2-9} = 3\tan\theta$$

$$\int \frac{5dx}{\sqrt{25x^2-9}} = \int \frac{5\left(\frac{3}{5}\sec\theta\tan\theta d\theta\right)}{3\tan\theta}$$

$$= \int \sec\theta d\theta$$

$$= \int \sec\theta\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}d\theta$$

$$= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\,d\theta$$

$$= \int \frac{1}{\sec\theta + \tan\theta}\,d(\sec\theta + \tan\theta)$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$\boxed{= \ln\left|\frac{5}{3}x + \frac{1}{3}\frac{\sqrt{25x^2-9}}{3}\right| + C}$$

Exercise

Evaluate the integral $\int \frac{2dx}{x^3\sqrt{x^2-1}}, \quad x>1$

Solution

$x=\sec\theta \quad dx=\sec\theta\tan\theta\ d\theta$

$\sqrt{x^2-1}=\sqrt{\sec^2\theta-1}=\tan\theta$

$$\int \frac{2\ dx}{x^3\sqrt{x^2-1}}=\int \frac{2\sec\theta\tan\theta\ d\theta}{\sec^3\theta\tan\theta}$$

$$=2\int \cos^2\theta d\theta \qquad \cos^2\theta=\frac{1+\cos 2\theta}{2}$$

$$=2\int \frac{1+\cos 2\theta}{2}\ d\theta$$

$$=\int (1+\cos 2\theta)d\theta$$

$$=\theta+\frac{1}{2}\sin 2\theta+C$$

$$=\theta+\sin\theta\cos\theta+C \qquad x=\sec\theta=\frac{1}{\cos\theta}\Rightarrow\cos\theta=\frac{1}{x}$$

$$\sin\theta=\tan\theta\cos\theta=\sqrt{x^2-1}\left(\frac{1}{x}\right)$$

$$\boxed{=\sec^{-1}x+\frac{\sqrt{x^2-1}}{x^2}+C}$$

Exercise

Evaluate the integral $\int \frac{dx}{x^3\ \sqrt{x^2-100}}$

Solution

$x=10\sec\theta \qquad \sqrt{x^2-100}=10\tan\theta$

$dx=10\sec\theta\tan\theta d\theta$

$$\int \frac{dx}{x^3\ \sqrt{x^2-100}}=\int \frac{10\sec\theta\tan\theta\ d\theta}{10^3\sec^3\theta(10\tan\theta)}$$

$$=\frac{1}{10^3}\int \frac{1}{\sec^2\theta}\ d\theta$$

$$=\frac{1}{10^3}\int \cos^2\theta\ d\theta$$

$$=\frac{1}{2\times 10^3}\int (1+\cos 2\theta)\ d\theta$$

$$=\frac{1}{2\times 10^3}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C$$

$$=\frac{1}{2\times 10^3}(\theta+\sin\theta\cos\theta)+C$$

$$=\frac{1}{2\times 10^3}\left(\tan^{-1}\frac{\sqrt{x^2-100}}{10}+\frac{\tan\theta}{\sec\theta}\frac{1}{\sec\theta}\right)+C$$

$$=\frac{1}{2\times 10^3}\left(\tan^{-1}\frac{\sqrt{x^2-100}}{10}+\frac{\sqrt{x^2-100}}{x}\frac{10}{x}\right)+C$$

$$=\frac{1}{2\times 10^3}\left(\tan^{-1}\frac{\sqrt{x^2-100}}{10}+\frac{10\ \sqrt{x^2-100}}{x^2}\right)+C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^3\,dx}{x^2-1}$

Solution

$$\begin{array}{r|l} & \quad x \\ \hline x^2-1 & \ x^3 \\ & \ \underline{x^3-x} \\ & \qquad x \end{array}$$

$$d\left(x^2-1\right)=2xdx\ \Rightarrow\ \frac{1}{2}d\left(x^2-1\right)=xdx$$

$$\int\frac{x^3\,dx}{x^2-1}=\int\left(x+\frac{x}{x^2-1}\right)dx$$

$$=\int xdx+\int\frac{x}{x^2-1}dx$$

$$=\int xdx+\frac{1}{2}\int\frac{d\left(x^2-1\right)}{x^2-1}$$

$$=\frac{1}{2}x^2+\frac{1}{2}\ln\left|x^2-1\right|+C$$

Exercise

Evaluate the integral $\int \frac{\left(1-x^2\right)^{1/2}}{x^4}\,dx$

Solution

$$x = \sin\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$dx = \cos\theta\, d\theta$$

$$\left(1-x^2\right)^{1/2} = \left(1-\sin^2 x\right)^{1/2} = \cos\theta$$

$$\begin{aligned}\int \frac{\left(1-x^2\right)^{1/2}}{x^4}\,dx &= \int \frac{\cos\theta}{\sin^4\theta}\cos\theta\, d\theta \\ &= \int \frac{\cos^2\theta}{\sin^2\theta}\frac{1}{\sin^2\theta}\, d\theta \\ &= \int \cot^2\theta\csc^2\theta\, d\theta \\ &= -\frac{1}{3}\cot^3\theta + C \\ &= -\frac{1}{3}\left(\frac{\cos\theta}{\sin\theta}\right)^3 + C \\ &\boxed{= -\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C}\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}\,dx$

Solution

$$\ln x = \sin\theta \qquad 0 < \theta \le \frac{\pi}{2}$$

$$\frac{1}{x}dx = \cos\theta\, d\theta$$

$$\sqrt{1-(\ln x)^2} = \sqrt{1-\sin^2\theta} = \cos\theta$$

$$\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}\,dx = \int \frac{\cos\theta}{\sin\theta}\cos\theta\, d\theta$$

$$= \int \frac{\cos^2\theta}{\sin\theta}\, d\theta$$

$$= \int \frac{1-\sin^2\theta}{\sin\theta}\, d\theta$$

$$= \int \frac{1}{\sin\theta}\, d\theta - \int \sin\theta\, d\theta$$

$$= \int \csc\theta\, d\theta - \int \sin\theta\, d\theta$$

$$= -\ln\left|\csc\theta + \cot\theta\right| + \cos\theta + C$$

$$= -\ln\left|\frac{1}{\ln x} + \frac{\sqrt{1-(\ln x)^2}}{\ln x}\right| + \sqrt{1-(\ln x)^2} + C$$

Exercise

Evaluate the integral $\int \sqrt{x}\,\sqrt{1-x}\; dx$

Solution

$u = \sqrt{x} \;\; \to u^2 = x \quad \Rightarrow dx = 2udu$

$u = \sin\theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$du = \cos\theta\, d\theta$

$$\int \sqrt{x}\,\sqrt{1-x}\; dx = \int u\,\sqrt{1-u^2}\,(2udu)$$

$$= 2\int u^2\,\sqrt{1-u^2}\; du \qquad\qquad \sqrt{1-u^2} = \sqrt{1-\sin^2\theta} = \cos\theta$$

$$= 2\int \sin^2\theta\cos\theta\cos\theta\; d\theta$$

$$= 2\int \sin^2\theta\cos^2\theta\; d\theta \qquad\qquad \sin 2\theta = 2\sin\theta\cos\theta \quad \to \sin^2 2\theta = 4\sin^2\theta\cos^2\theta$$

$$= \frac{1}{2}\int \sin^2 2\theta\, d\theta \qquad\qquad \sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$= \frac{1}{2}\int \frac{1-\cos 4\theta}{2}\; d\theta$$

$$=\frac{1}{4}\int d\theta-\frac{1}{4}\int \cos 4\theta d\theta$$

$$=\frac{1}{4}\theta-\frac{1}{16}\sin 4\theta+C$$

$$=\frac{1}{4}\theta-\frac{1}{16}2\sin 2\theta\cos 2\theta+C$$

$$=\frac{1}{4}\theta-\frac{1}{8}2\sin\theta\cos\theta\left(2\cos^2\theta-1\right)+C$$

$$=\frac{1}{4}\theta-\frac{1}{2}\sin\theta\cos^3\theta+\frac{1}{4}\sin\theta\cos\theta+C$$

$$=\frac{1}{4}\sin^{-1}u-\frac{1}{2}u\left(1-u^2\right)^{3/2}+\frac{1}{4}u\sqrt{1-u^2}+C$$

$$\boxed{=\frac{1}{4}\sin^{-1}\sqrt{x}-\frac{1}{2}\sqrt{x}\,(1-x)^{3/2}+\frac{1}{4}\sqrt{x}\sqrt{1-x}+C}$$

Exercise

Evaluate the integral $\displaystyle\int\frac{\sqrt{x-2}}{\sqrt{x-1}}dx$

Solution

$$u=\sqrt{x-1}\quad\to u^2=x-1\quad\Rightarrow 2udu=dx$$

$$u=\sec\theta\qquad 0<\theta<\frac{\pi}{2}$$

$$du=\sec\theta\tan\theta\,d\theta$$

$$\int\frac{\sqrt{x-2}}{\sqrt{x-1}}dx=\int\frac{\sqrt{u^2-1}}{u}2udu$$

$$=2\int\sqrt{u^2-1}\;du$$

$$\sqrt{u^2-1}=\sqrt{\sec^2\theta-1}=\tan\theta$$

$$=2\int\tan\theta\sec\theta\tan\theta\;d\theta$$

$$w=\tan\theta\qquad dv=\sec\theta\tan\theta d\theta$$

$$dw=\sec^2\theta d\theta\qquad v=\sec\theta$$

$$2\int\tan\theta\sec\theta\tan\theta\,d\theta=2\sec\theta\tan\theta-2\int\sec^3\theta\,d\theta$$

$$=2\sec\theta\tan\theta-2\int\sec^2\theta\sec\theta\,d\theta$$

$$= 2\sec\theta\tan\theta - 2\int\left(\tan^2\theta + 1\right)\sec\theta\ d\theta$$

$$= 2\sec\theta\tan\theta - 2\int\tan^2\theta\sec\theta\ d\theta - 2\int\sec\theta\ d\theta$$

$$\int\sec\theta\ d\theta = \int\sec\theta\ \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\ d\theta$$

$$= \int\frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\ d\theta$$

$$= \int\frac{1}{\sec\theta+\tan\theta}\ d\left(\sec\theta+\tan\theta\right)$$

$$= \ln\left|\sec\theta+\tan\theta\right|$$

$$2\int\tan^2\theta\sec\theta d\theta = 2\sec\theta\tan\theta - 2\int\tan^2\theta\sec\theta\ d\theta - 2\ln\left|\sec\theta+\tan\theta\right|$$

$$4\int\tan^2\theta\sec\theta\ d\theta = 2\sec\theta\tan\theta - 2\ln\left|\sec\theta+\tan\theta\right|$$

$$2\int\tan^2\theta\sec\theta\ d\theta = \sec\theta\tan\theta - \ln\left|\sec\theta+\tan\theta\right|$$

$$\int\frac{\sqrt{x-2}}{\sqrt{x-1}}\ dx = 2\int\tan\theta\sec\theta\tan\theta\ d\theta$$

$$= \sec\theta\tan\theta - \ln\left|\sec\theta+\tan\theta\right| + C$$

$$= u\sqrt{u^2-1} - \ln\left|u+\sqrt{u^2-1}\right| + C$$

$$\boxed{= \sqrt{x-1}\sqrt{x-2} - \ln\left|\sqrt{x-1}+\sqrt{x-2}\right| + C}$$

Exercise

Evaluate: $\displaystyle\int\frac{dx}{\sqrt{4x^2-49}}$

Solution

$$2x = 7\sec\theta \quad \to dx = \frac{7}{2}\sec\theta\tan\theta\ d\theta$$

$$\sqrt{4x^2-49} = \frac{7}{2}\tan\theta$$

$$\int\frac{dx}{\sqrt{4x^2-49}} = \frac{1}{2}\int\frac{dx}{\sqrt{x^2-\left(\frac{7}{2}\right)^2}}$$

$$=\frac{1}{2}\int \frac{\frac{7}{2}\sec\theta\tan\theta\ d\theta}{\frac{7}{2}\tan\theta}$$

$$=\frac{1}{2}\int \sec\theta\ d\theta$$

$$=\frac{1}{2}\int \sec\theta\ \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\ d\theta$$

$$=\frac{1}{2}\int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\ d\theta$$

$$=\frac{1}{2}\int \frac{1}{\sec\theta+\tan\theta}\ d\left(\sec\theta+\tan\theta\right)$$

$$=\frac{1}{2}\ln\left|\sec\theta+\tan\theta\right|+C$$

$$\boxed{=\frac{1}{2}\ln\left|\frac{2x}{7}+\frac{\sqrt{4x^2-49}}{7}\right|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2-25}}$

Solution

$$\int \frac{dx}{\sqrt{x^2-25}}=\int \frac{5\sec\theta\tan\theta}{5\tan\theta}\ d\theta$$

$x=5\sec\theta$

$dx=5\sec\theta\tan\theta\ d\theta$

$\sqrt{x^2-25}=5\tan\theta$

$$=\int \sec\theta\ d\theta$$

$$=\int \sec\theta\ \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\ d\theta$$

$$=\int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\ d\theta$$

$$=\int \frac{1}{\sec\theta+\tan\theta}\ d\left(\sec\theta+\tan\theta\right)$$

$$=\ln\left|\sec\theta+\tan\theta\right|+C$$

$$\boxed{=\ln\left|\frac{x}{5}+\frac{1}{5}\sqrt{x^2-25}\right|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\,dx$

Solution

$x = 5\sec\theta \qquad \sqrt{x^2-25} = 5\tan\theta$

$dx = 5\sec\theta\tan\theta\, d\theta$

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \int \frac{5\tan\theta}{5\sec\theta}\left(5\sec\theta\tan\theta\right)d\theta$$

$$= 5\int \tan^2\theta\, d\theta$$

$$= 5\int \left(\sec^2\theta - 1\right)d\theta$$

$$= 5\left(\tan\theta - \theta\right) + C$$

$$= \sqrt{x^2-25} - 5\operatorname{arcsec}\frac{x}{5} + C$$

Exercise

Evaluate $\displaystyle\int \frac{x^3}{\sqrt{x^2-a^2}}\,dx$

Solution

$x = a\sec\theta \qquad \sqrt{x^2-a^2} = a\tan\theta$

$dx = a\sec\theta\tan\theta\, d\theta$

$$\int \frac{x^3}{\sqrt{x^2-a^2}}\,dx = \int \frac{a^3\sec^3\theta}{a\tan\theta}\left(a\sec\theta\tan\theta\right)d\theta$$

$$= a^3\int \sec^4\theta\, d\theta$$

$$= a^3\int \left(1+\tan^2\theta\right)\sec^2\theta\, d\theta$$

$$= a^3\int \left(1+\tan^2\theta\right)d\left(\tan\theta\right)$$

$$= a^3\left(\tan\theta + \frac{1}{3}\tan^3\theta\right) + C$$

$$= a^3\left(\frac{\sqrt{x^2-a^2}}{a}+\frac{1}{3}\frac{\left(x^2-a^2\right)^{3/2}}{a^3}\right)+C$$

$$= \sqrt{x^2-a^2}\left(a^2+\frac{1}{3}\left(x^2-a^2\right)\right)+C$$

$$\underline{= \frac{1}{3}\sqrt{x^2-a^2}\left(x^2+2a^2\right)+C}$$

Exercise

Evaluate $\displaystyle\int \frac{x^3}{\sqrt{x^2-25}}\,dx$

Solution

$$x = 5\sec\theta \qquad \sqrt{x^2-25} = 5\tan\theta$$

$$dx = 5\sec\theta\tan\theta\;d\theta$$

$$\int \frac{x^3}{\sqrt{x^2-25}}\,dx = \int \frac{5^3\sec^3\theta}{5\tan\theta}\left(5\sec\theta\tan\theta\right)\,d\theta$$

$$= 125\int \sec^4\theta\;d\theta$$

$$= 125\int \left(1+\tan^2\theta\right)\sec^2\theta\;d\theta$$

$$= 125\int \left(1+\tan^2\theta\right)\,d\left(\tan\theta\right)$$

$$= 125\left(\tan\theta+\frac{1}{3}\tan^3\theta\right)+C$$

$$= 125\left(\frac{\sqrt{x^2-25}}{5}+\frac{1}{3}\frac{\left(x^2-25\right)^{3/2}}{125}\right)+C$$

$$= \sqrt{x^2-25}\left(25+\frac{x^2-25}{3}\right)+C$$

$$\underline{= \frac{1}{3}\sqrt{x^2-25}\left(x^2+50\right)+C}$$

Exercise

Evaluate $\displaystyle\int x^3\sqrt{x^2-25}\,dx$

Solution

$$x = 5\sec\theta \qquad \sqrt{x^2-25} = 5\tan\theta$$

$$dx = 5\sec\theta\tan\theta\, d\theta$$

$$\begin{aligned}
\int x^3\sqrt{x^2-25}\,dx &= \int 5^3\sec^3\theta\,(5\tan\theta)(5\sec\theta\tan\theta)\,d\theta \\
&= 5^5\int \sec^4\theta\tan^2\theta\,d\theta \\
&= 5^5\int \sec^2\theta\left(1+\tan^2\theta\right)\tan^2\theta\,d\theta \\
&= 5^5\int \left(\tan^2\theta+\tan^4\theta\right)\,d\left(\tan\theta\right) \\
&= 5^5\left(\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta\right)+C \\
&= 5^5\left(\frac{1}{3}\frac{1}{5^3}\left(x^2-25\right)^{3/2}+\frac{1}{5^6}\left(x^2-25\right)^{5/2}\right)+C \\
&= \left(x^2-25\right)^{3/2}\left(\frac{25}{3}+\frac{1}{5}\left(x^2-25\right)\right)+C \\
&= \frac{1}{15}\left(x^2-25\right)^{3/2}\left(125+3x^2-75\right)+C \\
&= \boxed{\frac{1}{15}\left(x^2-25\right)^{3/2}\left(3x^2+50\right)+C}
\end{aligned}$$

Exercise

Evaluate $\displaystyle\int \sqrt{5x^2-1}\,dx$

Solution

$$\sqrt{5}x = \sec\theta \qquad \sqrt{5x^2-1} = \tan\theta$$

$$dx = \frac{1}{\sqrt{5}}\sec\theta\tan\theta\,d\theta$$

$$\int \sqrt{5x^2-1}\,dx = \frac{1}{\sqrt{5}}\int \sec\theta\,d\theta$$

$$= \frac{1}{\sqrt{5}} \int \sec\theta \, \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} \, d\theta$$

$$= \frac{1}{\sqrt{5}} \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta} \, d\theta$$

$$= \frac{1}{\sqrt{5}} \int \frac{1}{\sec\theta + \tan\theta} \, d\left(\sec\theta + \tan\theta\right)$$

$$= \frac{1}{\sqrt{5}} \ln\left|\sec\theta + \tan\theta\right| + C$$

$$\boxed{= \frac{1}{\sqrt{5}} \ln\left|\frac{x}{\sqrt{5}} + \sqrt{5x^2 - 1}\right| + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{9x^2 - 25}}, \quad x > \frac{5}{3}$

Solution

$$x = \frac{5}{3}\sec\theta \qquad \sqrt{9x^2 - 25} = 5\tan\theta$$

$$dx = \frac{5}{3}\sec\theta\tan\theta \, d\theta$$

$$\int \frac{dx}{\sqrt{9x^2 - 25}} = \int \frac{1}{5\tan\theta}\left(\frac{5}{3}\sec\theta\tan\theta\right) d\theta$$

$$= \frac{1}{3}\int \sec\theta \, d\theta$$

$$= \frac{1}{3}\int \sec\theta \, \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} \, d\theta$$

$$= \frac{1}{3}\int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta} \, d\theta$$

$$= \frac{1}{3}\int \frac{1}{\sec\theta + \tan\theta} \, d\left(\sec\theta + \tan\theta\right)$$

$$= \frac{1}{3}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$\boxed{= \frac{1}{3}\ln\left|\frac{3x}{5} + \frac{\sqrt{9x^2 - 25}}{5}\right| + C}$$

Exercise

Evaluate the integral $\int \frac{x\,dx}{\sqrt{4x^2-1}}$

Solution

$$x = \frac{1}{2}\sec\theta \qquad \sqrt{4x^2-1} = \tan\theta$$

$$dx = \frac{1}{2}\sec\theta\tan\theta\ d\theta$$

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{4x^2-1}} &= \frac{1}{2}\int \frac{\sec\theta}{\tan\theta}\left(\frac{1}{2}\sec\theta\tan\theta\right)d\theta \\ &= \frac{1}{4}\int \sec^2\theta\ d\theta \\ &= \frac{1}{4}\tan\theta + C \\ &= \boxed{\frac{1}{4}\sqrt{4x^2-1} + C}\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{\sqrt{x^2-81}}$

Solution

$$x = 9\sec\theta \qquad \sqrt{x^2-81} = 9\tan\theta$$

$$dx = 9\sec\theta\tan\theta\ d\theta$$

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2-81}} &= \int \frac{9\sec\theta\tan\theta}{9\tan\theta}\ d\theta \\ &= \int \sec\theta\ d\theta \\ &= \int \sec\theta\ \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\ d\theta \\ &= \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\ d\theta \\ &= \int \frac{1}{\sec\theta+\tan\theta}\ d\left(\sec\theta+\tan\theta\right) \\ &= \ln\left|\sec\theta+\tan\theta\right| + C \\ &= \boxed{\ln\left|\frac{x}{9}+\frac{\sqrt{x^2-81}}{9}\right| + C}\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\sqrt{x^2-9}}{x}\,dx$

Solution

$$x = 3\sec\theta \qquad \sqrt{x^2-9} = 3\tan\theta$$

$$dx = 3\sec\theta\tan\theta\, d\theta$$

$$\begin{aligned}
\int \frac{\sqrt{x^2-9}}{x}\,dx &= \int \frac{3\tan\theta}{3\sec\theta}(3\sec\theta\tan\theta)\,d\theta \\
&= 3\int \tan^2\theta\, d\theta \\
&= 3\int \left(\sec^2\theta - 1\right)\, d\theta \\
&= 3\left(\tan\theta - \theta\right) + C \\
&\boxed{= \sqrt{x^2-9} - 3\sec^{-1}\frac{x}{3} + C}
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{1}{x^2\sqrt{9x^2-1}}\,dx$

Solution

$$3x = \sec\theta \qquad \sqrt{9x^2-1} = \tan\theta$$

$$dx = \frac{1}{3}\sec\theta\tan\theta\, d\theta$$

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{9x^2-1}}\,dx &= \int \frac{1}{\frac{1}{9}\sec^2\theta(\tan\theta)}\cdot\frac{1}{3}\sec\theta\tan\theta\, d\theta \\
&= 3\int \frac{1}{\sec\theta}\, d\theta \\
&= 3\int \cos\theta\, d\theta \\
&= 3\sin\theta + C \qquad\qquad \sin\theta = \frac{\tan\theta}{\sec\theta} \\
&= 3\frac{\sqrt{9x^2-1}}{3x} + C \\
&\boxed{= \frac{\sqrt{9x^2-1}}{x} + C}
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\left(x^2-36\right)^{3/2}}$

Solution

$$x = 6\sec\theta \qquad x^2 - 36 = 36\tan^2\theta$$

$$dx = 6\sec\theta\tan\theta\, d\theta$$

$$\begin{aligned}
\int \frac{dx}{\left(x^2-36\right)^{3/2}} &= \int \frac{6\sec\theta\tan\theta\, d\theta}{\left(36\tan^2\theta\right)^{3/2}} \\
&= \int \frac{6\sec\theta\tan\theta\, d\theta}{6^3\tan^3\theta} \\
&= \frac{1}{36}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta \\
&= \frac{1}{36}\int \frac{1}{\cos\theta}\frac{\cos^2\theta}{\sin^2\theta}\, d\theta \\
&= \frac{1}{36}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta \\
&= \frac{1}{36}\int \frac{1}{\sin^2\theta}\, d\left(\sin\theta\right) \\
&= -\frac{1}{36}\frac{1}{\sin\theta} + C \qquad\qquad \sin\theta = \frac{\tan\theta}{\sec\theta} \\
&= -\frac{1}{36}\cdot\frac{x}{6}\cdot\frac{6}{\sqrt{x^2-36}} + C \\
&= -\frac{1}{36}\frac{x}{\sqrt{x^2-36}} + C
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{36-x^2}}$

Solution

$$x = 6\sin\theta \qquad \sqrt{36-x^2} = 6\cos\theta$$

$$dx = 6\cos\theta\, d\theta$$

$$\int \frac{dx}{\sqrt{36-x^2}} = \int \frac{6\cos\theta\, d\theta}{6\cos\theta}$$

$$= \int d\theta$$

$$= \theta + C$$

$$\boxed{= \sin^{-1}\frac{x}{6} + C}$$

Exercise

Evaluate the integral $\int \sqrt{a^2 - x^2}\ dx$

Solution

$$x = a\sin\theta \qquad \sqrt{a^2 - x^2} = a\cos\theta$$

$$dx = a\cos\theta\ d\theta$$

$$\int \sqrt{a^2 - x^2}\ dx = \int a\cos\theta\left(a\cos\theta\ d\theta\right)$$

$$= a^2\int \cos^2\theta\ d\theta$$

$$= \frac{a^2}{2}\int (1 + \cos 2\theta)\ d\theta$$

$$= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{a^2}{2}\left(\arcsin\frac{x}{a} + \sin\theta\cos\theta\right) + C$$

$$= \frac{a^2}{2}\left(\arcsin\frac{x}{a} + \frac{x}{a}\frac{\sqrt{a^2 - x^2}}{a}\right) + C$$

$$\boxed{= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{1}{2}x\ \sqrt{a^2 - x^2} + C}$$

Exercise

Evaluate the integral $\int \frac{dx}{2 - \sqrt{3x}}$

Solution

Let $u^2 = \sqrt{3x} \quad \rightarrow \quad u^4 = 3x$

$$4u^3 du = 3dx$$

$$\int \frac{dx}{2-\sqrt{3x}} = \frac{4}{3}\int \frac{u^3}{2-u^2}\,du$$

$$u = \sqrt{2}\sin\theta \qquad 2-u^2 = 2\cos^2\theta$$

$$du = \sqrt{2}\cos\theta\; d\theta$$

$$\int \frac{dx}{2-\sqrt{3x}} = \frac{4}{3}\int \frac{2\sqrt{2}\sin^3\theta}{2\cos^2\theta}\left(\sqrt{2}\cos\theta\; d\theta\right)$$

$$= \frac{8}{3}\int \frac{\sin^3\theta}{\cos\theta}\; d\theta$$

$$= \frac{8}{3}\int \frac{\sin^2\theta}{\cos\theta}\;\left(\sin\theta\; d\theta\right)$$

$$= -\frac{8}{3}\int \frac{\left(1-\cos^2\theta\right)}{\cos\theta}\; d\left(\cos\theta\right)$$

$$= -\frac{8}{3}\int \left(\frac{1}{\cos\theta}-\cos\theta\right)\; d\left(\cos\theta\right)$$

$$= -\frac{8}{3}\left(\ln\left|\cos\theta\right| - \frac{1}{2}\cos^2\theta\right) + C$$

$$= -\frac{8}{3}\left(\ln\left|\frac{\sqrt{2-u^2}}{\sqrt{2}}\right| - \frac{1}{2}\,\frac{2-u^2}{2}\right) + C$$

$$= -\frac{8}{3}\left(\ln\left|\frac{\sqrt{2-\sqrt{3x}}}{\sqrt{2}}\right| - \frac{2-\sqrt{3x}}{4}\right) + C$$

$$= -\frac{8}{3}\ln\left|\frac{\sqrt{2-\sqrt{3x}}}{\sqrt{2}}\right| + \frac{2}{3}\left(2-\sqrt{3x}\right) + C$$

$$= -\frac{8}{3}\ln\left(\frac{2-\sqrt{3x}}{2}\right)^{1/2} + \frac{4}{3} - \frac{2}{3}\sqrt{3x} + C$$

$$\boxed{= -\frac{4}{3}\ln\left(\frac{2-\sqrt{3x}}{2}\right) - \frac{2}{3}\sqrt{3x} + C_1}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x\,dx}{1-\sqrt{x}}$

Solution

Let $u^2 = \sqrt{x} \quad \to \quad u^4 = x$

$$4u^3du = dx$$

$$\int \frac{x\,dx}{1-\sqrt{x}} = 4\int \frac{u^4}{1-u^2}u^3du$$

$$= 4\int \frac{u^7}{1-u^2}\ du$$

$u=\sin\theta \qquad 1-u^2=\cos^2\theta$

$du=\cos\theta\ d\theta$

$$\int \frac{x\,dx}{1-\sqrt{x}} = 4\int \frac{\sin^7\theta}{\cos^2\theta}\cos\theta\ d\theta$$

$$= 4\int \frac{\sin^6\theta}{\cos\theta}(\sin\theta\ d\theta)$$

$$= -4\int \frac{\left(1-\cos^2\theta\right)^3}{\cos\theta}d\left(\cos\theta\right)$$

$$= -4\int \frac{1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta}{\cos\theta}\ d\left(\cos\theta\right)$$

$$= -4\int \left(\frac{1}{\cos\theta}-3\cos\theta+3\cos^3\theta-\cos^5\theta\right)\ d\left(\cos\theta\right)$$

$$= -4\left(\ln\left|\cos\theta\right|-\frac{3}{2}\cos^2\theta+\frac{3}{4}\cos^4\theta-\frac{1}{6}\cos^6\theta\right)+C$$

$$= -4\ln\left|\cos\theta\right|+6\cos^2\theta-3\cos^4\theta+\frac{2}{3}\cos^6\theta+C$$

$$= -4\ln\left|\sqrt{1-u^2}\right|+6\left(1-u^2\right)-3\left(1-u^2\right)^2+\frac{2}{3}\left(1-u^2\right)^3+C$$

$$= -4\ln\left(1-\sqrt{x}\right)^{1/2}+6-6\sqrt{x}-3\left(1-\sqrt{x}\right)^2+\frac{2}{3}\left(1-\sqrt{x}\right)^3+C$$

$$= -2\ln\left(1-\sqrt{x}\right)-6\sqrt{x}-3\left(1-2\sqrt{x}+x\right)+\frac{2}{3}\left(1-3\sqrt{x}+3x-x\sqrt{x}\right)$$

$$= -2\ln\left(1-\sqrt{x}\right)-6\sqrt{x}+6\sqrt{x}-3x-2\sqrt{x}+2x-\frac{2}{3}x\sqrt{x}$$

$$\underline{= -2\ln\left(1-\sqrt{x}\right)-x-2\sqrt{x}-\frac{2}{3}x\sqrt{x}+C_1\ \Big|}$$

Exercise

Evaluate the integral $\int \frac{dx}{\sqrt{1-2x^2}}$

Solution

$$\sqrt{2}\,x = \sin\theta \qquad \sqrt{1-2x^2} = \cos\theta$$

$$dx = \frac{1}{\sqrt{2}}\cos\theta\ d\theta$$

$$\int \frac{dx}{\sqrt{1-2x^2}} = \frac{1}{\sqrt{2}}\int \frac{\cos\theta\ d\theta}{\cos\theta}$$

$$= \frac{1}{\sqrt{2}}\theta + C$$

$$\boxed{= \frac{1}{\sqrt{2}}\sin^{-1}\left(\sqrt{2}\,x\right) + C}$$

Exercise

Evaluate the integral $\int \frac{x^2}{\left(100-x^2\right)^{3/2}}\ dx$

Solution

$$x = 10\sin\theta \qquad \sqrt{100-x^2} = 10\cos\theta$$

$$dx = 10\cos\theta\ d\theta$$

$$\int \frac{x^2}{\left(100-x^2\right)^{3/2}}\ dx = \int \frac{100\sin^2\theta}{(10\cos\theta)^3}(10\cos\theta)\ d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta}\ d\theta$$

$$= \int \tan^2\theta\ d\theta$$

$$= \int \left(\sec^2\theta - 1\right)\ d\theta$$

$$= \tan\theta - \theta + C$$

$$\boxed{= \frac{x}{\sqrt{100-x^2}} - \sin^{-1}\frac{x}{10} + C}$$

Exercise

Evaluate: $\displaystyle\int \frac{2dx}{\sqrt{1-4x^2}}$

Solution

$$\int \frac{2dx}{\sqrt{1-4x^2}} = \int \frac{d(2x)}{\sqrt{1-(2x)^2}}$$

$$\boxed{= \sin^{-1} 2x + C}$$

Exercise

Evaluate $\displaystyle\int \sqrt{\frac{x}{1-x}}\, dx$

Solution

$$x = \sin^2\theta \qquad \sqrt{1-x} = \cos\theta$$

$$dx = 2\sin x\cos\theta\, d\theta$$

$$\begin{aligned}
\int \sqrt{\frac{x}{1-x}}\, dx &= \int \frac{\sin\theta}{\cos\theta} 2\sin\theta\cos\theta\, d\theta \\
&= 2\int \sin^2\theta\, d\theta \\
&= \int (1-\cos 2\theta)\, d\theta \\
&= \theta - \frac{1}{2}\sin 2\theta + C \\
&= \theta - \sin\theta\cos\theta + C \\
&\boxed{= \arcsin\sqrt{x} - \sqrt{x}\sqrt{1-x} + C}
\end{aligned}$$

Exercise

Evaluate $\displaystyle\int x\sqrt{2x-x^2}\, dx$

Solution

$$\begin{aligned}
2x-x^2 &= 1-1+2x-x^2 \\
&= 1-\left(1-2x+x^2\right) \\
&= 1-(1-x)^2
\end{aligned}$$

$$\int x\sqrt{2x-x^2}\,dx=\int x\sqrt{1-(1-x)^2}\,dx$$

$$1-x=\sin\theta \qquad \sqrt{1-(1-x)^2}=\cos\theta$$

$$dx=-\cos\theta\,d\theta$$

$$\begin{aligned}\int x\sqrt{2x-x^2}\,dx&=\int(1-\sin\theta)(\cos\theta)(-\cos\theta\,d\theta)\\&=-\int(1-\sin\theta)\left(\cos^2\theta\right)\,d\theta\\&=-\int\cos^2\theta\,d\theta+\int\left(\sin\theta\cos^2\theta\right)\,d\theta\\&=-\frac{1}{2}\int(1+\cos 2\theta)\,d\theta-\int\cos^2\theta\,d(\cos\theta)\\&=-\frac{1}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)-\frac{1}{3}\cos^3\theta+C\\&=-\frac{1}{2}\theta-\frac{1}{2}\sin\theta\cos\theta-\frac{1}{3}\cos^3\theta+C\\&=-\frac{1}{2}\arcsin(1-x)-\frac{1}{2}(1-x)\sqrt{2x-x^2}-\frac{1}{3}\left(\sqrt{2x-x^2}\right)^3+C\end{aligned}$$

Exercise

Evaluate $\int x^2\sqrt{a^2-x^2}\,dx$

Solution

$$x=a\sin\theta \qquad \sqrt{a^2-x^2}=a\cos\theta$$

$$dx=a\cos\theta\,d\theta$$

$$\begin{aligned}\int x^2\sqrt{a^2-x^2}\,dx&=\int a^2\sin^2\theta(a\cos\theta)(a\cos\theta\,d\theta)\\&=a^4\int\sin^2\theta\cos^2\theta\,d\theta\\&=\frac{a^4}{4}\int(1-\cos 2\theta)(1+\cos 2\theta)\,d\theta\\&=\frac{1}{4}a^4\int\left(1-\cos^2 2\theta\right)\,d\theta\end{aligned}$$

$$= \frac{1}{4}a^4 \int \left(1 - \frac{1}{2} - \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{1}{4}a^4 \int \left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{1}{8}a^4 \left(\theta - \frac{1}{4}\sin 4\theta\right) + C$$

$$= \frac{1}{8}a^4 \left(\theta - \frac{1}{2}\sin 2\theta \cos 2\theta\right) + C$$

$$= \frac{1}{8}a^4 \left(\theta - \sin\theta \cos\theta \left(1 - 2\sin^2\theta\right)\right) + C$$

$$= \frac{1}{8}a^4 \left(\arcsin\frac{x}{a} - \frac{x}{a}\frac{\sqrt{a^2 - x^2}}{a}\left(1 - 2\frac{x^2}{a^2}\right)\right) + C$$

$$= \frac{1}{8}a^4 \left(\arcsin\frac{x}{a} - \frac{x\sqrt{a^2 - x^2}}{a^2} + 2\frac{x^3\sqrt{a^2 - x^2}}{a^4}\right) + C$$

$$\boxed{= \frac{1}{8}a^4 \arcsin\frac{x}{a} - \frac{1}{8}a^2 x\sqrt{a^2 - x^2} + \frac{1}{4}x^3\sqrt{a^2 - x^2} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x}{\sqrt{4x - x^2}}\,dx$

Solution

$$4x - x^2 = 4 - 4 + 4x - x^2$$

$$= 4 - \left(4 - 4x + x^2\right)$$

$$\boxed{= 4 - (2 - x)^2}$$

$$\int \frac{x}{\sqrt{4x - x^2}}\,dx = \int \frac{x}{\sqrt{4 - (2 - x)^2}}\,dx$$

$$2 - x = 2\sin\theta \qquad \sqrt{4 - (2 - x)^2} = 2\cos\theta$$

$$dx = -2\cos\theta\, d\theta$$

$$\int \frac{x}{\sqrt{4x - x^2}}\,dx = \int \frac{2 - 2\sin\theta}{2\cos\theta}(-2\cos\theta\, d\theta)$$

$$= -2\int (1 - \sin\theta)\, d\theta$$

$$= -2(\theta + \cos\theta) + C$$

$$= -2\left(\arcsin\left(1 - \frac{1}{2}x\right) + \frac{\sqrt{4 - (2-x)^2}}{2} \right) + C$$

$$\boxed{= -2\arcsin(2-x) - \sqrt{4x - x^2} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x}{\sqrt{ax - x^2}}\, dx, \quad a > 0$

Solution

$$ax - x^2 = \frac{a^2}{4} - \frac{a^2}{4} + ax - x^2$$

$$= \frac{a^2}{4} - \left(\frac{a^2}{4} - ax + x^2 \right)$$

$$\boxed{= \frac{a^2}{4} - \left(\frac{a}{2} - x \right)^2}$$

$$\int \frac{x}{\sqrt{ax - x^2}}\, dx = \int \frac{x}{\sqrt{\frac{a^2}{4} - \left(\frac{a}{2} - x\right)^2}}\, dx$$

$$\frac{a}{2} - x = \frac{a}{2}\sin\theta \qquad \sqrt{\frac{a^2}{4} - \left(\frac{a}{2} - x\right)^2} = \frac{a}{2}\cos\theta$$

$$dx = -\frac{a}{2}\cos\theta\; d\theta$$

$$\int \frac{x}{\sqrt{ax - x^2}}\, dx = \int \frac{\frac{a}{2} - \frac{a}{2}\sin\theta}{\frac{a}{2}\cos\theta} \left(-\frac{a}{2}\cos\theta\; d\theta \right)$$

$$= -\frac{a}{2} \int (1 - \sin\theta)\; d\theta$$

$$= -\frac{a}{2}(\theta + \cos\theta) + C$$

$$= -\frac{a}{2}\arcsin\left(1 - \frac{2}{a}x\right) - \sqrt{ax - x^2} + C$$

$$\boxed{= \frac{a}{2}\left(\arcsin\left(\frac{2}{a}x - 1\right) + \frac{2}{a}\sqrt{ax - x^2} \right) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx$

Solution

$$x = a\sin\theta \qquad \sqrt{a^2-x^2} = a\cos\theta$$

$$dx = a\cos\theta\, d\theta$$

$$\begin{aligned}
\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx &= \int \frac{a^2\sin^2\theta}{a\cos\theta}\left(a\cos\theta\, d\theta\right)\\
&= a^2\int \sin^2\theta\, d\theta\\
&= \frac{1}{2}a^2\int \left(1-\cos 2\theta\ \right)d\theta\\
&= \frac{1}{2}a^2\left(\theta-\frac{1}{2}\sin 2\theta\right)+C\\
&= \frac{1}{2}a^2\left(\theta-\sin\theta\cos\theta\right)+C\\
&= \frac{1}{2}a^2\left(\arcsin\frac{x}{a}-\frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}\right)+C\\
&\boxed{= \frac{1}{2}a^2\arcsin\left(\frac{x}{a}\right)-\frac{1}{2}x\sqrt{a^2-x^2}+C}
\end{aligned}$$

Exercise

Evaluate $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

Solution

$$x = 4\sin\theta \qquad \sqrt{16-x^2} = 4\cos\theta$$

$$dx = 4\cos\theta\, d\theta$$

$$\begin{aligned}
\int \frac{x^2}{\sqrt{16-x^2}}\,dx &= \int \frac{16\sin^2\theta}{4\cos\theta}\left(4\cos\theta\, d\theta\right)\\
&= 16\int \sin^2\theta\, d\theta\\
&= 8\int \left(1-\cos 2\theta\right)\, d\theta
\end{aligned}$$

$$= 8\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

$$= 8\left(\sin^{-1}\frac{x}{4} - \sin\theta\cos\theta\right) + C$$

$$= 8\left(\sin^{-1}\frac{x}{4} - \frac{x}{4}\frac{\sqrt{16-x^2}}{4}\right) + C$$

$$\boxed{= 8\sin^{-1}\frac{x}{4} - \frac{1}{2}x\sqrt{16-x^2} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\left(16-x^2\right)^{3/2}}$

Solution

$x = 4\sin\theta \qquad \sqrt{16-x^2} = 4\cos\theta$

$dx = 4\cos\theta d\theta$

$$\int \frac{dx}{\left(16-x^2\right)^{3/2}} = \int \frac{4\cos\theta}{(4\cos\theta)^3} d\theta$$

$$= \frac{1}{16}\int \frac{1}{\cos^2\theta} d\theta$$

$$= \frac{1}{16}\int \sec^2\theta\ d\theta$$

$$\boxed{= \frac{1}{16}\tan\theta + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$

Solution

$x = 3\sin\theta \qquad \sqrt{9-x^2} = 3\cos\theta$

$dx = 3\cos\theta d\theta$

$$\int \frac{dx}{x^2\sqrt{9-x^2}} = \int \frac{3\cos\theta}{9\sin^2\theta(3\cos\theta)} d\theta$$

$$= \frac{1}{9}\int \csc^2\theta\ d\theta$$

$$= -\frac{1}{9}\cot\theta + C$$

$$= -\frac{1}{9}\frac{\cos\theta}{\sin\theta} + C$$

$$= -\frac{1}{9}\frac{\sqrt{9-x^2}}{3}\cdot\frac{3}{x} + C$$

$$\boxed{= -\frac{1}{9}\frac{\sqrt{9-x^2}}{x} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$

Solution

$x = 2\sin\theta \qquad \sqrt{4-x^2} = 2\cos\theta$

$dx = 2\cos\theta\ d\theta$

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta}{4\sin^2\theta(2\cos\theta)}d\theta$$

$$= \frac{1}{4}\int \csc^2\theta\ d\theta$$

$$= -\frac{1}{4}\cot\theta + C$$

$$= -\frac{1}{4}\frac{\cos\theta}{\sin\theta} + C$$

$$= -\frac{1}{4}\frac{\sqrt{4-x^2}}{2}\cdot\frac{2}{x} + C$$

$$\boxed{= -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{4}{x^2\ \sqrt{16-x^2}}\ dx$

Solution

$x = 4\sin\theta \qquad \sqrt{16-x^2} = 4\cos\theta$

$dx = 4\cos\theta\, d\theta$

$$\int \frac{4}{x^2\ \sqrt{16-x^2}}\ dx = \int \frac{16\cos\theta}{16\sin^2\theta(4\cos\theta)}\ d\theta$$

$$=\frac{1}{4}\int \csc^2\theta\, d\theta$$

$$=-\frac{1}{4}\cot\theta + C$$

$$=-\frac{1}{4}\frac{\cos\theta}{\sin\theta} + C$$

$$\boxed{=-\frac{1}{4}\frac{\sqrt{16-x^2}}{x} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\, dx$

Solution

$$x = 3\sin\theta \qquad \sqrt{9-x^2} = 3\cos\theta$$

$$dx = 3\cos\theta\, d\theta$$

$$\int \frac{x^3}{\sqrt{9-x^2}}\, dx = \int \frac{27\sin^3\theta}{3\cos\theta}\,(3\cos\theta)\, d\theta$$

$$= 27\int \sin^3\theta\, d\theta$$

$$= 27\int \left(1-\cos^2\theta\right)\, d\left(\cos\theta\right)$$

$$= 27\left(\cos\theta - \frac{1}{3}\cos^3\theta\right) + C$$

$$= 27\cos\theta - 9\cos^3\theta + C$$

$$= 27\frac{\sqrt{9-x^2}}{3} - 9\left(\frac{\sqrt{9-x^2}}{3}\right)^3 + C$$

$$= 9\sqrt{9-x^2} - \frac{1}{3}\left(9-x^2\right)\sqrt{9-x^2} + C$$

$$= \frac{1}{3}\sqrt{9-x^2}\left(27-9+x^2\right) + C$$

$$\boxed{= \frac{1}{3}\sqrt{9-x^2}\left(18+x^2\right) + C}$$

Exercise

Evaluate $\int \sqrt{25-4x^2}\ dx$

Solution

$2x = 5\sin\theta \qquad \sqrt{25-4x^2} = 5\cos\theta$

$dx = \frac{5}{2}\cos\theta\, d\theta$

$$\begin{aligned}\int \sqrt{25-4x^2}\ dx &= \frac{25}{2}\int \cos^2\theta\ d\theta \\ &= \frac{25}{4}\int (1+\cos 2\theta)\ d\theta \\ &= \frac{25}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C \\ &= \frac{25}{4}(\theta + \sin\theta\cos\theta) + C \\ &= \frac{25}{4}\left(\sin^{-1}\frac{2x}{5} + \frac{2x}{5}\frac{\sqrt{25-4x^2}}{5}\right) + C \\ &= \boxed{\frac{25}{4}\sin^{-1}\frac{2x}{5} + \frac{1}{2}x\sqrt{25-4x^2} + C}\end{aligned}$$

Exercise

Evaluate $\int e^x\sqrt{1-e^{2x}}\ dx$

Solution

$e^x = \sin\theta \qquad \sqrt{1-e^{2x}} = \cos\theta$

$e^x dx = \cos\theta\ d\theta$

$$\begin{aligned}\int e^x\sqrt{1-e^{2x}}\ dx &= \int \cos^2\theta\ d\theta \\ &= \frac{1}{2}\int (1+\cos 2\theta)\ d\theta \\ &= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C \\ &= \frac{1}{2}(\theta + \sin\theta\cos\theta) + C \\ &= \boxed{\frac{1}{2}\left(\arcsin e^x + e^x\sqrt{1-e^{2x}}\right) + C}\end{aligned}$$

Exercise

Evaluate $\displaystyle\int \frac{\sqrt{1-x}}{\sqrt{x}}\ dx$

Solution

$$\sqrt{x} = \sin\theta \rightarrow x = \sin^2\theta \quad \sqrt{1-x} = \cos\theta$$

$$dx = 2\sin\theta\cos\theta\ d\theta$$

$$\begin{aligned}
\int \frac{\sqrt{1-x}}{\sqrt{x}}\ dx &= \int \frac{\cos\theta}{\sin\theta}(2\sin\theta\cos\theta)\ d\theta \\
&= 2\int \cos^2\theta\ d\theta \\
&= \int (1+\cos 2\theta)\ d\theta \\
&= \theta + \frac{1}{2}\sin 2\theta + C \\
&= \theta + 2\sin\theta\cos\theta + C \\
&\boxed{= \arcsin\sqrt{x} + 2\sqrt{x}\sqrt{1-x} + C}
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{y\ dy}{\sqrt{16-y^2}}$

Solution

$$y = 4\sin\theta \qquad \sqrt{16-y^2} = 4\cos\theta$$

$$dy = 4\cos\theta\ d\theta$$

$$\begin{aligned}
\int \frac{y\ dy}{\sqrt{16-y^2}} &= \int \frac{4\sin\theta}{4\cos\theta}(4\cos\theta\ d\theta) \\
&= 4\int \sin\theta\ d\theta \\
&= -4\cos\theta + C \\
&\boxed{= -\sqrt{16-y^2} + C}
\end{aligned}$$

OR

$$\int \frac{y\ dy}{\sqrt{16-y^2}} = -\frac{1}{2}\int \left(16-y^2\right)^{-1/2} d\left(16-y^2\right)$$

$$= -\left(16 - y^2\right)^{1/2} + C$$

$$\boxed{= -\sqrt{16 - y^2} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^3}{\sqrt{4 - x^2}}\,dx$

Solution

$x = 2\sin\theta \qquad \sqrt{4 - x^2} = 2\cos\theta$

$dx = 2\cos\theta\,d\theta$

$$\int \frac{x^3}{\sqrt{4 - x^2}}\,dx = \int \frac{8\sin^3\theta}{2\cos\theta}(2\cos\theta\,d\theta)$$

$$= 8\int \sin^2\theta \sin\theta\,d\theta$$

$$= -8\int \left(1 - \cos^2\theta\right)\,d(\cos\theta)$$

$$= -8\left(\cos\theta - \frac{1}{3}\cos^3\theta\right) + C$$

$$= -8\frac{\sqrt{4 - x^2}}{2} + \frac{8}{3}\frac{\left(4 - x^2\right)\sqrt{4 - x^2}}{8} + C$$

$$= \frac{1}{3}\sqrt{4 - x^2}\left(-12 + 4 - x^2\right) + C$$

$$\boxed{= -\frac{1}{3}\sqrt{4 - x^2}\left(x^2 + 8\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{4 - x^2}}$

Solution

$x = 2\sin\theta \qquad \sqrt{4 - x^2} = 2\cos\theta$

$dx = 2\cos\theta\,d\theta$

$$\int \frac{1}{\sqrt{4 - x^2}}\,dx = \int \frac{1}{2\cos\theta}(2\cos\theta\,d\theta)$$

$$= \int d\theta$$

$$= \theta + C$$

$$\boxed{= \sin^{-1}\frac{x}{2} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\left(1-x^2\right)^{3/2}}$

Solution

$$x = \sin\theta \qquad \sqrt{1-x^2} = \cos\theta$$

$$dx = \cos\theta\ d\theta$$

$$\int \frac{dx}{\left(1-x^2\right)^{3/2}} = \int \frac{\cos\theta\ d\theta}{\cos^3\theta}$$

$$= \int \frac{d\theta}{\cos^2\theta}$$

$$= \int \sec^2\theta\ d\theta$$

$$= \tan\theta + C$$

$$= \frac{\sin\theta}{\cos\theta} + C$$

$$\boxed{= \frac{x}{\sqrt{1-x^2}} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\left(1-x^2\right)^{5/2}}{x^8}dx$

Solution

$$x = \sin\theta \qquad 1-x^2 = \cos^2\theta$$

$$dx = \cos\theta\ d\theta$$

$$\int \frac{\left(1-x^2\right)^{5/2}}{x^8}dx = \int \frac{\left(\cos^2\theta\right)^{5/2}}{\sin^8\theta}\cos\theta\ d\theta$$

$$= \int \frac{\cos^5\theta}{\sin^8\theta}\cos\theta\, d\theta$$

$$= \int \frac{\cos^6\theta}{\sin^6\theta}\cdot\frac{1}{\sin^2\theta}\, d\theta$$

$$= \int \cot^6\theta \csc^2\theta\, d\theta$$

$$= -\int \cot^6\theta\, d(\cot\theta)$$

$$= -\frac{1}{7}\cot^7\theta + C$$

$$= -\frac{1}{7}\left(\frac{\cos\theta}{\sin\theta}\right)^7 + C$$

$$\boxed{= -\frac{1}{7}\left(\frac{\sqrt{1-x^2}}{x}\right)^7 + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{3-2x-x^2}}$

Solution

$$3-2x-x^2 = 3+1-1-2x-x^2$$

$$= 4-\left(1+2x+x^2\right)$$

$$\boxed{= 4-(1+x)^2}$$

$$\int \frac{dx}{\sqrt{3-2x-x^2}} = \int \frac{dx}{\sqrt{4-(1+x)^2}}$$

$x+1 = 2\sin\theta \qquad \sqrt{4-(1+x)^2} = 2\cos\theta$

$dx = 2\cos\theta\, d\theta$

$$\int \frac{dx}{\sqrt{3-2x-x^2}} = \int \frac{2\cos\theta}{2\cos\theta}\, d\theta$$

$$= \int d\theta$$

$$= \theta + C$$

$$\boxed{= \sin^{-1}\left(\frac{x+1}{2}\right) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x^4+4x^2+4}\,dx$

Solution

$$x=\sqrt{2}\tan\theta \qquad x^2+2=2\sec^2\theta$$

$$dx=\sqrt{2}\sec^2\theta\,d\theta$$

$$\int \frac{1}{x^4+4x^2+4}\,dx=\int \frac{dx}{\left(x^2+2\right)^2}$$

$$=\int \frac{\sqrt{2}\sec^2\theta}{4\sec^4\theta}\,d\theta$$

$$=\frac{\sqrt{2}}{4}\int \cos^2\theta\,d\theta$$

$$=\frac{\sqrt{2}}{8}\int (1+\cos 2\theta)\,d\theta$$

$$=\frac{\sqrt{2}}{8}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C$$

$$=\frac{\sqrt{2}}{8}(\theta+\sin\theta\cos\theta)+C$$

$$=\frac{\sqrt{2}}{8}\left(\theta+\frac{\tan\theta}{\sec\theta}\frac{1}{\sec\theta}\right)+C$$

$$=\frac{\sqrt{2}}{8}\left(\arctan\frac{x}{\sqrt{2}}+\frac{x\sqrt{2}}{x^2+2}\right)+C$$

Exercise

Evaluate $\displaystyle\int \frac{x^3+x+1}{x^4+2x^2+1}\,dx$

Solution

$$x=\tan\theta \qquad x^2+1=\sec^2\theta$$

$$dx=\sec^2\theta\,d\theta$$

$$\int \frac{x^3+x+1}{x^4+2x^2+1}\,dx=\int \frac{x^3+x}{x^4+2x^2+1}\,dx+\int \frac{1}{\left(x^2+1\right)^2}dx$$

$$= \frac{1}{4}\int \frac{1}{x^4+2x^2+1}\, d\left(x^4+2x^2+1\right) + \int \frac{1}{\sec^2\theta}\, d\theta$$

$$= \frac{1}{4}\ln\left(x^2+1\right)^2 + \int \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}(\theta+\sin\theta\cos\theta) + C$$

$$= \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\left(\theta+\frac{\tan\theta}{\sec\theta}\frac{1}{\sec\theta}\right) + C$$

$$= \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\left(\arctan x+\frac{x}{x^2+1}\right) + C$$

Exercise

Evaluate $\int \operatorname{arcsec} 2x\, dx \quad x > \frac{1}{2}$

Solution

$$u = \operatorname{arcsec} 2x \qquad dv = dx$$

$$du = \frac{dx}{x\sqrt{4x^2-1}} \qquad v = x$$

$$\int \operatorname{arcsec} 2x\, dx = x\operatorname{arcsec} 2x - \int \frac{dx}{\sqrt{4x^2-1}}$$

$$2x = \sec\theta \qquad \sqrt{4x^2-1} = \tan\theta$$

$$dx = \frac{1}{2}\sec\theta\tan\theta\, d\theta$$

$$= x\operatorname{arcsec} 2x - \frac{1}{2}\int \frac{\sec\theta\tan\theta}{\tan\theta}\, d\theta$$

$$= x\operatorname{arcsec} 2x - \frac{1}{2}\int \sec\theta\, d\theta$$

$$= x\operatorname{arcsec} 2x - \frac{1}{2}\ln\left|\sec\theta+\tan\theta\right| + C$$

$$= x\operatorname{arcsec} 2x - \frac{1}{2}\ln\left|2x+\sqrt{4x^2-1}\right| + C$$

Exercise

Evaluate $\displaystyle\int x \ \arcsin x \ dx$

Solution

$$u = \arcsin x \qquad dv = xdx$$

$$du = \frac{dx}{\sqrt{1-x^2}} \qquad v = \frac{1}{2}x^2$$

$$\int x \ \arcsin x \ dx = \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}} dx$$

$$x = \sin\theta \qquad \sqrt{1-x^2} = \cos\theta$$

$$dx = \cos\theta \ d\theta$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta}\cos\theta \ d\theta$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\int \sin^2\theta \ d\theta$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{4}\int (1-\cos 2\theta) \ d\theta$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{4}(\theta - \sin\theta\cos\theta) + C$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{4}\left(\arcsin x - \frac{1}{x\sqrt{1-x^2}}\right) + C$$

$$= \frac{1}{4}\left(2x^2 - 1\right)\arcsin x + \frac{1}{4}\frac{1}{x\sqrt{1-x^2}} + C$$

Exercise

Evaluate $\displaystyle\int_0^2 \sqrt{1+4x^2} \ dx$

Solution

$$2x = \tan\theta \qquad \sqrt{1+4x^2} = \sec\theta$$

$$dx = \frac{1}{2}\sec^2\theta \, d\theta$$

$$\int_0^2 \sqrt{1+4x^2} \ dx = \frac{1}{2}\int_0^2 \sec^3\theta \ d\theta$$

$$u = \sec x \qquad dv = \sec^2 x dx$$

$$du = \sec x \tan x \; dx \qquad v = \tan x$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \tan x \left(\sec x \tan x \; dx\right)$$

$$= \sec x \tan x - \int \tan^2 x \sec x \; dx$$

$$= \sec x \tan x - \int \left(\sec^2 x - 1\right) \sec x \; dx$$

$$= \sec x \tan x - \int \sec^3 x \; dx + \int \sec x \; dx$$

$$2\int \sec^3 x \; dx = \sec x \tan x + \ln\left|\sec x + \tan x\right|$$

$$\int \sec^3 x \; dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right|$$

$$\int_0^2 \sqrt{1+4x^2} \; dx = \frac{1}{4} \left(\sec\theta \tan\theta + \ln\left|\sec\theta + \tan\theta\right| \right|_0^2$$

$$= \frac{1}{4}\left(2x\sqrt{1+4x^2} + \ln\left|2x + \sqrt{1+4x^2}\right| \right|_0^2$$

$$= \frac{1}{4}\left(4\sqrt{17} + \ln\left|4+\sqrt{17}\right|\right)$$

$$\boxed{= \sqrt{17} + \frac{1}{4}\ln\left(4+\sqrt{17}\right)}$$

Exercise

Evaluate $\displaystyle\int_0^3 \frac{x^3}{\sqrt{x^2+9}} dx$

Solution

$$x = 3\tan\theta \qquad \sqrt{x^2+9} = 3\sec\theta$$

$$dx = 3\sec^2\theta \; d\theta$$

$$\int_0^3 \frac{x^3}{\sqrt{x^2+9}} dx = \int_0^3 \frac{27\tan^3\theta}{3\sec\theta} 3\sec^2\theta \; d\theta$$

$$= 27\int_0^3 \tan^2\theta \tan\theta \sec\theta \; d\theta$$

$$= 27\int_0^3 \left(\sec^2\theta - 1\right) \; d\left(\sec\theta\right)$$

$$= 27\left(\frac{1}{3}\sec^3\theta - \sec\theta \right|_0^3$$

$$= 9\sqrt{x^2+9}\left(\frac{x^2+9}{27} - 1\right) \Big|_0^3$$

$$= \frac{1}{3}\sqrt{x^2+9}\left(x^2 - 18\right) \Big|_0^3$$

$$\underline{= -9\sqrt{2} + 18}\,\Big|$$

Exercise

Evaluate the integral $\displaystyle\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^x \, dx}{\left(1+e^{2x}\right)^{3/2}}$

Solution

$e^x = \tan\theta$ $\qquad\qquad$ $\tan^{-1}\left(\frac{3}{4}\right) < \theta < \tan^{-1}\left(\frac{4}{3}\right)$

$x = \ln\left(\tan\theta\right)$

$dx = \dfrac{\sec^2\theta}{\tan\theta} d\theta$

$$\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^x \; dx}{\left(1+e^{2x}\right)^{3/2}} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{\tan\theta}{\left(\sec^2\theta\right)^{3/2}} \; \frac{\sec^2\theta}{\tan\theta} \; d\theta$$

$$= \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{\sec^2\theta}{\sec^3\theta} d\theta$$

$$= \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{1}{\sec\theta} d\theta$$

$$= \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \cos\theta \; d\theta$$

$$= \sin\theta \Big|_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)}$$

$$= \sin\left(\tan^{-1}(3/4)\right) - \sin\left(\tan^{-1}(4/3)\right)$$

$$= \frac{4}{5} - \frac{3}{5}$$

$$\boxed{= \frac{1}{5}}$$

Exercise

Evaluate the integral $\int_1^e \frac{e^x dx}{\left(1+e^{2x}\right)^{3/2}}$

Solution

$$\int_1^e \frac{e^x dx}{\left(1+e^{2x}\right)^{3/2}} = \int_1^e \frac{1}{\left(1+\left(e^x\right)^2\right)^{3/2}} d\left(e^x\right) \qquad \text{Let } y = e^x$$

$$= \int_1^e \frac{1}{\left(1+y^2\right)^{3/2}} dy$$

$$y = \tan\theta \qquad \sqrt{y^2+1} = \sec\theta$$

$$dy = \sec^2\theta \, d\theta$$

$$\int_1^e \frac{e^x dx}{\left(1+e^{2x}\right)^{3/2}} = \int_1^e \frac{1}{\left(\sec^2\theta\right)^{3/2}} \sec^2\theta \, d\theta$$

$$= \int_1^e \frac{1}{\sec^3\theta} \sec^2\theta \, d\theta$$

$$= \int_1^e \frac{1}{\sec\theta} \, d\theta$$

$$= \int_1^e \cos\theta \, d\theta$$

$$= \sin\theta \Big|_1^e$$

$$= \left. \frac{\tan\theta}{\sec\theta} \right|_1^e$$

$$= \left. \frac{y}{\sqrt{1+y^2}} \right|_1^e$$

$$= \left. \frac{e^x}{\sqrt{1+e^{2x}}} \right|_1^e$$

$$\boxed{= \frac{e^e}{\sqrt{1+e^{2e}}} - \frac{e}{\sqrt{1+e^2}}}$$

Exercise

Evaluate the integral $\displaystyle\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}}$

Solution

$$y = e^{\tan\theta} \quad 1 \le y \le e \to \quad 0 \le \theta = \tan^{-1}(\ln y) \le \frac{\pi}{4}$$

$$dy = e^{\tan\theta}\sec^2\theta \, d\theta$$

$$\sqrt{1+(\ln y)^2} = \sqrt{1+\tan^2\theta}$$

$$\boxed{= \sec\theta}$$

$$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_0^{\pi/4} \frac{e^{\tan\theta}\sec^2\theta}{e^{\tan\theta}\sec\theta}\, d\theta$$

$$= \int_0^{\pi/4} \sec\theta \, d\theta$$

$$= \left(\ln\left|\sec\theta + \tan\theta\right| \right|_0^{\pi/4}$$

$$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln\left|\sec 0 + \tan 0\right|$$

$$\boxed{= \ln\left(1+\sqrt{2}\right)}$$

Exercise

Evaluate the integral $\displaystyle\int_{1/2}^{1/4} \frac{dy}{y\sqrt{1+(\ln y)^2}}$

Solution

Let $x = \ln y \quad \rightarrow \quad y = e^x$

$dy = e^x dx$

$$\int_{1/2}^{1/4} \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_{1/2}^{1/4} \frac{e^x\, dx}{e^x\sqrt{1+x^2}}$$

$$= \int_{1/2}^{1/4} \frac{dx}{\sqrt{1+x^2}}$$

$x = \tan\theta \qquad \sqrt{x^2+1} = \sec\theta$

$dx = \sec^2\theta\, d\theta$

$$\int_{1/2}^{1/4} \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_{1/2}^{1/4} \frac{\sec^2\theta\, d\theta}{\sec\theta}$$

$$= \int_{1/2}^{1/4} \sec\theta\, d\theta$$

$$= \int_{1/2}^{1/4} \sec\theta\, \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\, d\theta$$

$$= \int_{1/2}^{1/4} \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\, d\theta$$

$$= \int_{1/2}^{1/4} \frac{1}{\sec\theta+\tan\theta}\, d(\sec\theta+\tan\theta)$$

$$= \ln(\sec\theta+\tan\theta)\Big|_{1/2}^{1/4}$$

$$= \ln\left(\sqrt{x^2+1}+x\right)\Big|_{1/2}^{1/4}$$

$$= \ln\left(\sqrt{\ln^2 y+1}+\ln y\right)\Big|_{1/2}^{1/4}$$

$$= \ln\left|\sqrt{\left(\ln\frac{1}{4}\right)^2+1}+\ln\frac{1}{4}\right| - \ln\left|\sqrt{\left(\ln\frac{1}{2}\right)^2+1}+\ln\frac{1}{2}\right|$$

$$= \ln\left|\sqrt{(-\ln 4)^2+1}-\ln 4\right| - \ln\left|\sqrt{(-\ln 2)^2+1}-\ln 2\right|$$

$$\boxed{= \ln\left|\frac{\sqrt{(\ln 4)^2+1}-\ln 4}{\sqrt{(\ln 2)^2+1}-\ln 2}\right|}$$

Exercise

Evaluate $\displaystyle\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}}$

Solution

Let: $u = 3y \quad \Rightarrow \quad du = 3dy \quad \to \quad \dfrac{du}{3} = dy$

$$\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}} = \frac{1}{3}\int_{-2/3}^{-\sqrt{2}/3} \frac{1}{\frac{u}{3}\sqrt{u^2-1}}\,du$$

$$= \int_{-2/3}^{-\sqrt{2}/3} \frac{du}{u\sqrt{u^2-1}} \qquad\qquad \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|$$

$$= \sec^{-1}|3y| \Big|_{-2/3}^{-\sqrt{2}/3}$$

$$= \sec^{-1}\left|-\sqrt{2}\right| - \sec^{-1}|-2|$$

$$= \frac{\pi}{4} - \frac{\pi}{3}$$

$$\boxed{= -\frac{\pi}{12}}$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{\sqrt{3}/2} \frac{4}{9+4x^2}\,dx$

Solution

$$x = \frac{3}{2}\tan\theta \qquad 9+4x^2 = 9\sec^2\theta$$

$$dx = \frac{3}{2}\sec^2\theta\; d\theta$$

$$\int_{0}^{\sqrt{3}/2} \frac{4}{9+4x^2}\,dx = \int_{0}^{\sqrt{3}/2} \frac{4}{9\sec^2\theta}\left(\frac{3}{2}\sec^2\theta\; d\theta\right)$$

$$= \frac{2}{3}\int_0^{\sqrt{3}/2} d\theta$$

$$= \frac{2}{3}\theta \Big|_0^{\sqrt{3}/2}$$

$$= \frac{2}{3}\arctan\left(\frac{2}{3}x\right) \Big|_0^{\sqrt{3}/2}$$

$$= \frac{2}{3}\left(\arctan\left(\frac{\sqrt{3}}{3}\right) - \arctan(0)\right)$$

$$= \frac{2}{3}\arctan\frac{1}{\sqrt{3}}$$

$$= \frac{2}{3}\frac{\pi}{6}$$

$$\boxed{= \frac{\pi}{9}}$$

Exercise

Evaluate the integral $\displaystyle\int_{1/12}^{1/4} \frac{dx}{\sqrt{x}(1+4x)}$

Solution

Let $u = \sqrt{x} \quad \rightarrow \quad x = u^2$

$dx = 2u\ du$

$$\int_{1/12}^{1/4} \frac{dx}{\sqrt{x}(1+4x)} = \int_{1/12}^{1/4} \frac{2u\ du}{u\left(1+4u^2\right)}$$

$$= \int_{1/12}^{1/4} \frac{d(2u)}{1+(2u)^2}$$

$$= \arctan 2u \Big|_{1/12}^{1/4}$$

$$= \arctan 2\sqrt{x} \Big|_{1/12}^{1/4}$$

$$= \arctan 1 - \arctan\frac{1}{\sqrt{3}}$$

$$= \frac{\pi}{4} - \frac{\pi}{6}$$

$$\boxed{= \frac{\pi}{12}}$$

Exercise

Evaluate the integral $\int_{8\sqrt{2}}^{16} \frac{dx}{\sqrt{x^2-64}}$

Solution

$$x = 8\sec\theta \qquad \sqrt{x^2-64} = 8\tan\theta$$

$$dx = 8\sec\theta\tan\theta\, d\theta$$

$$\int_{8\sqrt{2}}^{16} \frac{dx}{\sqrt{x^2-64}} = \int_{8\sqrt{2}}^{16} \frac{8\sec\theta\tan\theta\, d\theta}{8\tan\theta}$$

$$= \int_{8\sqrt{2}}^{16} \sec\theta\, d\theta$$

$$= \int_{8\sqrt{2}}^{16} \sec\theta \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\, d\theta$$

$$= \int_{8\sqrt{2}}^{16} \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\, d\theta$$

$$= \int_{8\sqrt{2}}^{16} \frac{1}{\sec\theta+\tan\theta}\, d\left(\sec\theta+\tan\theta\right)$$

$$= \ln\left|\sec\theta+\tan\theta\right| \Big|_{8\sqrt{2}}^{16}$$

$$= \ln\left|\frac{x}{8}+\frac{\sqrt{x^2-64}}{8}\right| \Big|_{8\sqrt{2}}^{16}$$

$$= \ln\left|2+\frac{\sqrt{16^2-64}}{8}\right| - \ln\left|\sqrt{2}+\frac{\sqrt{128-64}}{8}\right|$$

$$= \ln\left|2+\frac{8\sqrt{4-1}}{8}\right| - \ln\left(\sqrt{2}+1\right)$$

$$= \ln\left(2+\sqrt{3}\right) - \ln\left(\sqrt{2}+1\right)$$

$$\underline{= \ln\left(\frac{2+\sqrt{3}}{1+\sqrt{2}}\right)\Big|}$$

Exercise

Evaluate the integral $\int_{\sqrt{2}}^{2} \frac{\sqrt{x^2-1}}{x}dx$

Solution

$$x = \sec\theta \qquad \sqrt{x^2-1} = \tan\theta$$

$$dx = \sec\theta\tan\theta\, d\theta$$

$$\int_{\sqrt{2}}^{2} \frac{\sqrt{x^2-1}}{x}dx = \int_{\sqrt{2}}^{2} \frac{\tan\theta}{\sec\theta}\ \sec\theta\tan\theta\, d\theta$$

$$= \int_{\sqrt{2}}^{2} \tan^2\theta\, d\theta$$

$$= \int_{\sqrt{2}}^{2} \left(\sec^2\theta - 1\right)\, d\theta$$

$$= \tan\theta - \theta \Big|_{\sqrt{2}}^{2}$$

$$x = 2 = \sec\theta \quad \rightarrow\ \theta = \arccos\frac{1}{2} = \frac{\pi}{3}$$

$$x = \sqrt{2} = \sec\theta \quad \rightarrow\ \theta = \arccos\frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

$$= \tan\theta - \theta \Big|_{\pi/4}^{\pi/3}$$

$$= \tan\frac{\pi}{3} - \frac{\pi}{3} - \tan\frac{\pi}{4} + \frac{\pi}{4}$$

$$= \sqrt{3} - 1 - \frac{\pi}{12}$$

Exercise

Evaluate $\int_{4}^{6} \frac{x^2}{\sqrt{x^2-9}}\, dx$

Solution

$$x = 3\sec\theta \qquad \sqrt{x^2-9} = 3\tan\theta$$

$$dx = 3\sec\theta\tan\theta\, d\theta$$

$$\int_{4}^{6} \frac{x^2}{\sqrt{x^2-9}}\, dx = \int_{4}^{6} \frac{9\sec^2\theta}{3\tan\theta}\ \left(3\sec\theta\tan\theta\right)d\theta$$

$$=9\int_4^6 \sec^3\theta\, d\theta$$

$$u=\sec x \qquad dv=\sec^2 x\, dx$$

$$du=\sec x\tan x\, dx \qquad v=\tan x$$

$$\int \sec^3 x\, dx = \sec x\tan x - \int \tan x\left(\sec x\tan x\, dx\right)$$

$$= \sec x\tan x - \int \tan^2 x\sec x\, dx$$

$$= \sec x\tan x - \int \left(\sec^2 x - 1\right)\sec x\, dx$$

$$= \sec x\tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$2\int \sec^3 x\, dx = \sec x\tan x + \ln\left|\sec x + \tan x\right|$$

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right|$$

$$= \frac{9}{2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right.\Bigg|_4^6$$

$$= \frac{1}{2}x\sqrt{x^2-9} + \frac{9}{2}\ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| \Bigg|_4^6$$

$$= \frac{9}{2}\left(2\sqrt{3} + \ln\left(2+\sqrt{3}\right) - \frac{4\sqrt{7}}{9} - \ln\left(\frac{4+\sqrt{7}}{3}\right)\right)$$

$$\boxed{= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{6+3\sqrt{3}}{4+\sqrt{7}}\right)}$$

Exercise

Evaluate $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\, dx$

Solution

$$x = \sqrt{3}\sec\theta \qquad \sqrt{x^2-3} = \sqrt{3}\tan\theta$$

$$dx = \sqrt{3}\sec\theta\tan\theta\, d\theta$$

$$\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\, dx = \int_{\sqrt{3}}^{2} \frac{\sqrt{3}\tan\theta}{\sqrt{3}\sec\theta}\left(\sqrt{3}\sec\theta\tan\theta\right) d\theta$$

$$=\sqrt{3}\int_{\sqrt{3}}^{2}\tan^2\theta\,d\theta$$

$$=\sqrt{3}\int_{\sqrt{3}}^{2}\left(\sec^2\theta-1\right)\,d\theta$$

$$=\sqrt{3}\left(\tan\theta-\theta\right.\Big|_{\sqrt{3}}^{2}$$

$$=\sqrt{3}\left(\frac{\sqrt{x^2-3}}{\sqrt{3}}-\operatorname{arcsec}\frac{x}{\sqrt{3}}\right.\Bigg|_{\sqrt{3}}^{2}$$

$$=\sqrt{3}\left(\frac{1}{\sqrt{3}}-\frac{\pi}{6}\right)$$

$$\underline{=1-\frac{\pi\sqrt{3}}{6}}\,\Big|$$

Exercise

Evaluate $\displaystyle\int_{0}^{\sqrt{3}/2}\frac{x^2}{\left(1-x^2\right)^{3/2}}dx$

Solution

$x=\sin\theta \qquad \sqrt{1-x^2}=\cos\theta$

$dx=\cos\theta\,d\theta$

$$\int_{0}^{\sqrt{3}/2}\frac{x^2}{\left(1-x^2\right)^{3/2}}\,dx=\int_{0}^{\sqrt{3}/2}\frac{\sin^2\theta}{\cos^3\theta}(\cos\theta)d\theta$$

$$=\int_{0}^{\sqrt{3}/2}\tan^2\theta\,d\theta$$

$$=\int_{0}^{\sqrt{3}/2}\left(\sec^2\theta-1\right)\,d\theta$$

$$=\tan\theta-\theta\Big|_{0}^{\sqrt{3}/2}$$

$$=\frac{x}{\sqrt{1-x^2}}-\arcsin x\Big|_{0}^{\sqrt{3}/2}$$

$$=\frac{\sqrt{3}}{2}\frac{1}{\sqrt{1-\frac{3}{4}}}-\frac{\pi}{3}$$

$$\underline{=\sqrt{3}-\frac{\pi}{3}\,\big|}$$

Exercise

Evaluate $\displaystyle\int_0^{\sqrt{3}/2}\frac{1}{\left(1-x^2\right)^{5/2}}dx$

Solution

$$x=\sin\theta \qquad \sqrt{1-x^2}=\cos\theta$$

$$dx=\cos\theta\ d\theta$$

$$\int_0^{\sqrt{3}/2}\frac{1}{\left(1-x^2\right)^{5/2}}dx=\int_0^{\sqrt{3}/2}\frac{1}{\cos^5\theta}\cos\theta\ d\theta$$

$$=\int_0^{\sqrt{3}/2}\sec^4\theta\ d\theta$$

$$=\int_0^{\sqrt{3}/2}\left(1+\tan^2\theta\right)\sec^2\theta\ d\theta$$

$$=\int_0^{\sqrt{3}/2}\left(1+\tan^2\theta\right)\ d\left(\tan\theta\right)$$

$$=\tan\theta+\frac{1}{3}\tan^3\theta\ \Big|_0^{\sqrt{3}/2}$$

$$=\frac{x}{\sqrt{1-x^2}}+\frac{1}{3}\frac{x^3}{\left(1-x^2\right)^{3/2}}\ \Big|_0^{\sqrt{3}/2}$$

$$=\frac{\sqrt{3}}{2}\frac{1}{\sqrt{1-\frac{3}{4}}}+\frac{\sqrt{3}}{8}\frac{1}{\left(\frac{1}{4}\right)^{3/2}}$$

$$=\sqrt{3}+\sqrt{3}$$

$$\underline{=2\sqrt{3}\,\big|}$$

Exercise

Evaluate $\displaystyle\int_0^{3/5} \sqrt{9-25x^2}\ dx$

Solution

$$5x = 3\sin\theta \qquad \sqrt{9-25x^2} = 3\cos\theta$$

$$dx = \frac{3}{5}\cos\theta\, d\theta$$

$$\begin{aligned}
\int_0^{3/5} \sqrt{9-25x^2}\ dx &= \frac{9}{5}\int_0^{3/5} \cos^2\theta\ d\theta \\
&= \frac{9}{10}\int_0^{3/5} (1+\cos 2\theta)\ d\theta \\
&= \frac{9}{10}\left(\theta + \frac{1}{2}\sin 2\theta\right.\Bigg|_0^{3/5} \\
&= \frac{9}{10}\left(\theta + \sin\theta\cos\theta\right.\Bigg|_0^{3/5} \\
&= \frac{9}{10}\left(\arcsin\frac{5x}{3} + 2\cdot\frac{5x}{3}\cdot\frac{5\sqrt{9-25x^2}}{3}\right.\Bigg|_0^{3/5} \\
&= \frac{9}{10}\left(\arcsin\frac{5x}{3} + \frac{25}{9}x\sqrt{9-25x^2}\right.\Bigg|_0^{3/5} \\
&= \frac{9\pi}{20}
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}}$

Solution

$$\begin{aligned}
\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}} &= \sin^{-1}\frac{x}{3}\Bigg|_0^{3/2} \\
&= \frac{\pi}{6}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

Solution

$$\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\frac{x}{3}\Big|_0^3$$

$$= \sin^{-1}1 - \sin^{-1}0$$

$$\boxed{= \frac{\pi}{2}}$$

Exercise

Evaluate $\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx$

Solution

$x+2 = 3\sec\theta \qquad \sqrt{(x+2)^2-9} = 3\tan\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$$\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx = \int_1^4 \frac{\sqrt{(x+2)^2-9}}{x+2}\,dx$$

$$= \int_1^4 \frac{3\tan\theta}{3\sec\theta}(3\sec\theta\tan\theta)\,d\theta$$

$$= 3\int_1^4 \tan^2\theta\,d\theta$$

$$= 3\int_1^4 \left(\sec^2\theta - 1\right)\,d\theta$$

$$= 3\,(\tan\theta - \theta\,\Big|_1^4$$

$$= \sqrt{(x+2)^2-9} - 3\sec^{-1}\left(\frac{x+2}{3}\right)\Big|_1^4$$

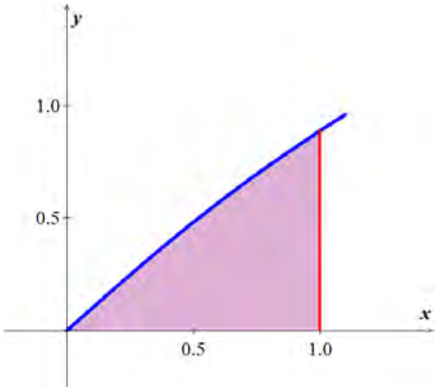
$$= \sqrt{27} - 3\sec^{-1}(2) + 3\sec^{-1}(1)$$

$$\boxed{= 3\sqrt{3} - \pi}$$

Exercise

Consider the region bounded by the graph $y=\sqrt{x\tan^{-1}x}$ and $y=0$ for $0\le x\le 1$. Find the volume of the solid formed by revolving this region about the *x*-axis.

Solution



$$V=\pi\int_0^1\left(\sqrt{x\tan^{-1}x}\right)^2 dx$$

$$=\pi\int_0^1 x\tan^{-1}x\ dx$$

$$u=\tan^{-1}x \qquad v=\int x\,dx$$

$$du=\frac{1}{x^2+1}dx \qquad v=\frac{1}{2}x^2$$

$$V=\pi\left(\frac{1}{2}\left(x^2\tan^{-1}x\ \Big|_0^1 - \frac{1}{2}\int_0^1\frac{x^2}{1+x^2}dx\right)\right.$$

$$=\frac{\pi}{2}\left(\left(\tan^{-1}1-0\right) - \int_0^1\left(1-\frac{1}{1+x^2}\right)dx\right)$$

$$=\frac{\pi}{2}\left(\frac{\pi}{4} - \int_0^1 dx + \int_0^1\frac{1}{1+x^2}dx\right)$$

$$=\frac{\pi}{2}\left(\frac{\pi}{4} - \left(x\ \Big|_0^1 + \left(\tan^{-1}x\ \Big|_0^1\right)\right.\right.$$

$$=\frac{\pi}{2}\left(\frac{\pi}{4}-1+\tan^{-1}1\right)$$

$$=\frac{\pi}{2}\left(\frac{\pi}{4}-1+\frac{\pi}{4}\right)$$

$$=\frac{\pi}{2}\left(\frac{\pi}{2}-1\right)$$

$$\boxed{=\frac{\pi^2}{4}-\frac{\pi}{2}\quad unit^3}$$

Exercise

Use two approach to show that the area of a cap (or segment) of a circle of radius r subtended by an angle θ is given by

$$A_{seg}=\frac{1}{2}r^2(\theta-\sin\theta)$$

a) Find the area using geometry (no calculus).

b) Find the area using calculus

Solution

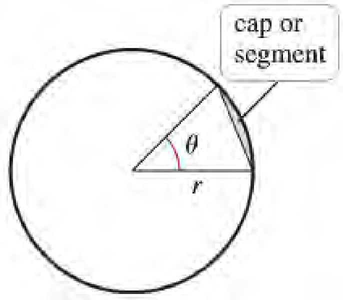
a) Area of a segment (*cap*) = Area of a sector ***minus*** Area of the isosceles triangle

The area of a sector: $A=\frac{1}{2}\theta r^2$

Area of the isosceles triangle: $A=\frac{1}{2}r^2\sin\theta$

$$A_{seg}=\frac{1}{2}r^2\theta-\frac{1}{2}r^2\sin\theta$$

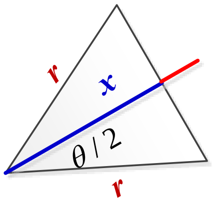
$$=\frac{1}{2}r^2(\theta-\sin\theta)$$



b) $0\le\theta\le\pi \quad\to\quad 0\le\frac{\theta}{2}\le\frac{\pi}{2}$

$x=r\cos\frac{\alpha}{2} \quad\to\quad dx=-\frac{1}{2}r\sin\frac{\alpha}{2}\ d\alpha$

$\sqrt{r^2-x^2}=r\sin\frac{\alpha}{2}$



$$A_{cap}=2\int_{r\cos\theta/2}^{r}\sqrt{r^2-x^2}\ dx$$

$$=2\int_{\theta}^{0}\left(r\sin\frac{\alpha}{2}\right)\left(-\frac{1}{2}r\sin\frac{\alpha}{2}\right)d\alpha$$

$$=r^2\int_{0}^{\theta}\left(\sin^2\frac{\alpha}{2}\right)\ d\alpha$$

$$=\frac{1}{2}r^2\int_{0}^{\theta}(1-\cos\alpha)\ d\alpha$$

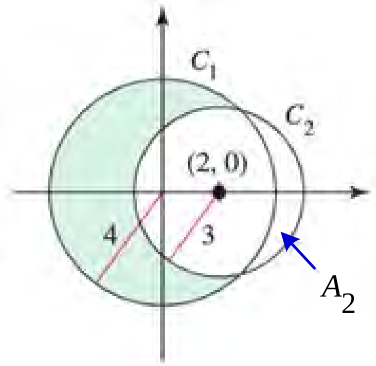
$$=\frac{1}{2}r^2(\alpha-\sin\alpha)\Big|_{0}^{\theta}$$

$$\boxed{=\frac{1}{2}r^2(\theta-\sin\theta)\quad unit^2}$$

Exercise

A lune is a crescent-shaped region bounded by the arcs of two circles. Let C_1 be a circle of radius 4 centered at the origin. Let C_2 be a circle of radius 3 centered at the point $(2,\ 0)$. Find the area of the lune that lies inside C_1 and outside C_2.

Solution



$$C_1 \quad \rightarrow \quad x^2+y^2=16$$

$$y^2=16-x^2$$

$$C_2 \quad \rightarrow \quad (x-2)^2+y^2=9$$

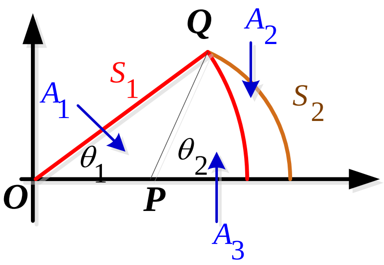
$$y^2=9-(x-2)^2$$

$$16-x^2=9-x^2+4x-4$$

$$11=4x \quad \rightarrow \quad \boxed{x=\frac{11}{4}}$$

$$y=\pm\frac{\sqrt{135}}{4}$$

$$\boxed{=\pm\frac{3\sqrt{15}}{4}}$$



For **sector** C_1: $\theta_1=\tan^{-1}\frac{y}{x}$

$$\boxed{=\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right)}$$

Area: $S_1=\frac{1}{2}r^2\theta_1$

$$\boxed{=8\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right)}$$

For **sector** C_2: $x_2=\frac{11}{4}-2=\frac{3}{4}$

$$\theta_2=\tan^{-1}\frac{y}{x_2}$$

$$=\tan^{-1}\sqrt{15}$$

Area: $S_2=\frac{1}{2}r_2^2\theta_2$

$$\boxed{=\frac{9}{2}\tan^{-1}\left(\sqrt{15}\right)}$$

$$OQ=4, \quad PQ=3, \quad OP=2$$

$$Area(\Delta APQ)=A_1=\frac{1}{2}(4)(2)\sin\theta_1$$

$$= 4\frac{y}{4}$$

$$\boxed{= \frac{3\sqrt{15}}{4}}$$

$$A_2 = S_2 - S_1 + A_1$$

$$\boxed{= \frac{9}{2}\tan^{-1}\left(\sqrt{15}\right) - 8\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right) + \frac{3\sqrt{15}}{4}}$$

$$A_{lune} = A_{C_1} - A_{C_2} + 2A_2$$

$$= 16\pi - 9\pi + 9\tan^{-1}\left(\sqrt{15}\right) - 16\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right) + \frac{3\sqrt{15}}{2}$$

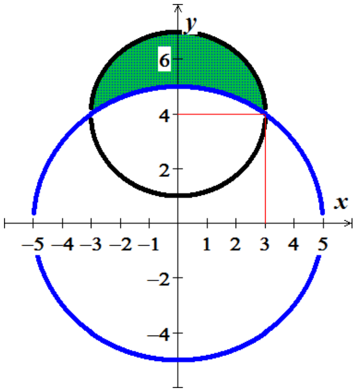
$$\boxed{= 7\pi + 9\tan^{-1}\left(\sqrt{15}\right) - 16\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right) + \frac{3\sqrt{15}}{2} \quad unit^2}$$

$$\boxed{\approx 26.66 \quad unit^2}$$

Exercise

The crescent-shaped region bounded by two circles forms a lune. Find the area of the lune given that the radius of the smaller circle is 3 and the radius of the larger circle is 5.

Solution



Large Circle: $x^2 + y^2 = 25$

$$y = \sqrt{25 - x^2}$$

Small Circle: $r = 3 \rightarrow y = \sqrt{25 - 9} \ \boxed{= 4}$

$$x^2 + (y-4)^2 = 9$$

$$y = 4 + \sqrt{9 - x^2}$$

$$A = 2\int_0^3 \left(4 + \sqrt{9 - x^2} - \sqrt{25 - x^2}\right) dx$$

$$= 2\left(4x + \frac{1}{2}\left(9\arcsin\left(\frac{x}{3}\right) + x\sqrt{9 - x^2}\right) - \frac{1}{2}\left(25\arcsin\left(\frac{x}{5}\right) + x\sqrt{25 - x^2}\right)\right)\Bigg|_0^3$$

$$= 2\left[12 + \frac{1}{2}\left(9\frac{\pi}{2}\right) - \frac{1}{2}\left(25\arcsin\left(\frac{3}{5}\right) + 12\right)\right]$$

$$\boxed{= 12 + \frac{9\pi}{2} - 25\ \arcsin\left(\frac{3}{5}\right) \quad unit^2}$$

Exercise

The surface of a machine part is the region between the graphs of $y=|x|$ *and* $x^2+(y-k)^2=25$

a) Find k when the circle is tangent to the graph of $y=|x|$

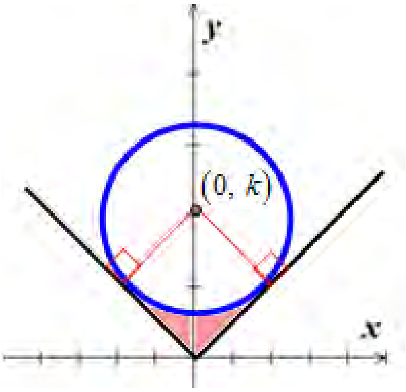
b) Find the area of the surface of the machine part.

c) Find the area of the surface of the machine part as a function of the radius r of the circle.

Solution

a) $x^2+(y-k)^2=25 \quad \rightarrow \quad \boxed{r=5}$

$k^2=5^2+5^2=50 \quad \rightarrow \quad \boxed{k=5\sqrt{2}}$



b) $Area = area\ square - \frac{1}{4}(area\ circle)$

$$=5^2-\frac{1}{4}\pi 5^2$$

$$\boxed{=25\left(1-\frac{\pi}{4}\right)\quad unit^2}$$

c) $Area = area\ square - \frac{1}{4}(area\ circle)$

$$=r^2-\frac{1}{4}\pi r^2$$

$$\boxed{=r^2\left(1-\frac{\pi}{4}\right)\quad unit^2}$$

Exercise

Consider the function $f(x)=\left(9+x^2\right)^{-1/2}$ and the region ***R*** on the interval [0, 4].

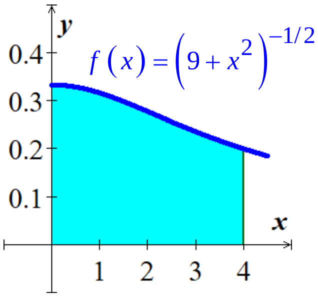
a) Find the area of *R*.

b) Find the volume of the solid generated when *R* is revolved about the *x-axis*.

c) Find the volume of the solid generated when *R* is revolved about the *y-axis*.

Solution

a) $A=\int_0^4 \frac{dx}{\sqrt{9+x^2}}$



$$x=3\tan\theta \ \rightarrow \ dx=3\sec^2\theta\ d\theta$$

$$\sqrt{9+x^2}=3\sec\theta$$

$$=\int_0^4 \frac{3\sec^2\theta}{3\sec\theta}\ d\theta$$

$$=\int_0^4 \sec\theta\ d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| \Big|_0^4$$

$$= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| \Big|_0^4$$

$$= \ln\left(\frac{5}{3} + \frac{4}{3}\right) - 0$$

$$\boxed{= \ln 3 \quad unit^2}$$

b) $x = 3\tan\theta \ \rightarrow \ dx = 3\sec^2\theta \, d\theta$

$9 + x^2 = 9\sec^2\theta$

$$V = \pi\int_0^4 \frac{dx}{9+x^2}$$

$$= \pi\int_0^4 \frac{3\sec^2\theta \, d\theta}{9\sec^2\theta}$$

$$= \frac{\pi}{3}\int_0^4 d\theta$$

$$= \frac{\pi}{3}\theta \Big|_0^4$$

$$= \frac{\pi}{3}\tan^{-1}\frac{x}{3} \Big|_0^4$$

$$\boxed{= \frac{\pi}{3}\tan^{-1}\frac{4}{3} \quad unit^3}$$

c) $V = 2\pi\int_0^4 \frac{x}{\sqrt{9+x^2}} dx$ $\qquad d\left(9+x^2\right) = 2xdx$

$$= \pi\int_0^4 \left(9+x^2\right)^{-1/2} d\left(9+x^2\right)$$

$$= 2\pi\left(9+x^2\right)^{1/2} \Big|_0^4$$

$$= 2\pi(5-3)$$

$$\boxed{= 4\pi \quad unit^3}$$

Exercise

A total of $\boldsymbol{Q}$ is distributed uniformly on a line segment of length $2L$ along the $y\text{-}axis$. The x-component of the electric field at a point $(a,\ 0)$ is given by

$$E_x = \frac{kQa}{2L}\int_{-L}^{L} \frac{dy}{\left(a^2+y^2\right)^{3/2}}$$

Where k is a physical constant and $a > 0$

a) Confirm that $E_x(a) = \dfrac{kQ}{a\sqrt{a^2+L^2}}$

b) Letting $\rho = \dfrac{Q}{2L}$ be the charge density on the line segment, show that if $L \to \infty$, *then* $E_x = \dfrac{2k\rho}{a}$

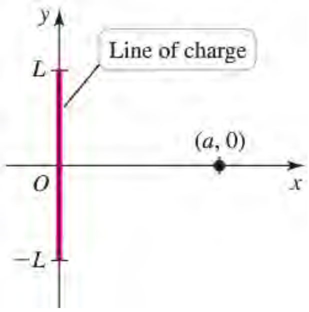
Solution

a) $E_x = \dfrac{kQa}{2L}\displaystyle\int_{-L}^{L} \frac{dy}{\left(a^2+y^2\right)^{3/2}}$

$$y = a\tan\theta \ \rightarrow dy = a\sec^2\theta\, d\theta$$

$$\sqrt{a^2+y^2} = a\sec\theta$$

$$= \frac{kQa}{2L}\int_{-L}^{L} \frac{a\sec^2\theta\, d\theta}{a^3\sec^3\theta}$$



$$= \frac{kQ}{2aL}\int_{-L}^{L} \frac{d\theta}{\sec\theta}$$

$$= \frac{kQ}{2aL}\int_{-L}^{L} \cos\theta d\theta$$

$$= \frac{kQ}{2aL}\sin\theta\ \Big|_{-L}^{L}$$

$$= \frac{kQ}{2aL}\left(\frac{y}{\sqrt{a^2+y^2}}\ \Big|_{-L}^{L}\right.$$

$$= \frac{kQ}{2aL}\left(\frac{2L}{\sqrt{a^2+L^2}}\right)$$

$$= \frac{kQ}{a\sqrt{a^2+L^2}}$$

b) Let $\rho = \dfrac{Q}{2L} \quad \rightarrow \quad Q = 2\rho L$

$$E_x(a) = \frac{kQa}{2L} \lim_{L\to\infty} \int_{-L}^{L} \frac{dy}{\left(a^2+y^2\right)^{3/2}}$$

$$= \frac{kQa}{2L} \lim_{L\to\infty} \left(\frac{2L}{a^2\sqrt{a^2+L^2}} \right)$$

$$= k\rho a \, \frac{2}{a^2}$$

$$\boxed{= \frac{2k\rho}{a}}$$

Exercise

A long, straight wire of length 2*L* on the *y-axis* carries a current *I*. according to the Biot-Savart Law, the magnitude of the field due to the current at a point $(a,\ 0)$ is given by

$$B(a) = \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{\sin\theta}{r^2} dy$$

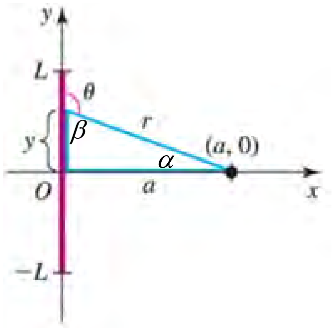
Where μ_0 is a physical constant, $a > 0$, and θ, r, and y are related to the figure

a) Show that the magnitude of the magnetic field at $(a,\ 0)$ is $B(a) = \dfrac{\mu_0 IL}{2\pi a\sqrt{a^2+L^2}}$

b) What is the magnitude of the magnetic field at $(a,\ 0)$ due to an infinitely long wire $(L\to\infty)$?

Solution

a) $\beta = \pi - \theta \quad \& \quad \alpha + \beta = \frac{\pi}{2}$



$$\sin\theta = \sin(\pi - \beta) = \sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha = \frac{a}{r}$$

$$r^2 = y^2 + a^2$$

$$\frac{\sin\theta}{r^2} = \frac{a}{r^3} = \frac{a}{\left(a^2+y^2\right)^{3/2}}$$

$$B(a) = \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{\sin\theta}{r^2} dy$$

$$= \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{a}{\left(a^2+y^2\right)^{3/2}} dy$$

$$y = a\tan u \qquad \sqrt{a^2+y^2} = a\sec u$$

$$dy = a\sec^2 u\ du$$

$$= \frac{\mu_0 I}{2\pi} \int_0^L \frac{a^2 \sec^2 u \; du}{a^3 \sec^3 u}$$

$$= \frac{\mu_0 I}{2a\pi} \int_0^L \frac{1}{\sec u} du$$

$$= \frac{\mu_0 I}{2a\pi} \int_0^L \cos u \; du$$

$$= \frac{\mu_0 I}{2a\pi} \sin u \Big|_0^L$$

$$= \frac{\mu_0 I}{2a\pi} \frac{y}{\sqrt{a^2 + y^2}} \Big|_0^L$$

$$\boxed{= \frac{\mu_0 IL}{2a\pi\sqrt{a^2 + L^2}}}$$

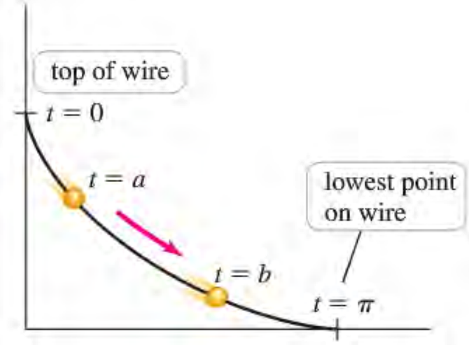
b) $\lim\limits_{L\to\infty} B(a) = \lim\limits_{L\to\infty} \frac{\mu_0 IL}{2a\pi\sqrt{a^2 + L^2}}$

$$= \frac{\mu_0 I}{2a\pi} \lim_{L\to\infty} \frac{L}{\sqrt{a^2 + L^2}} \qquad \lim_{L\to\infty} \frac{L}{\sqrt{a^2 + L^2}} = \lim_{L\to\infty} \frac{L}{\sqrt{L^2}} = 1$$

$$\boxed{= \frac{\mu_0 I}{2a\pi}}$$

Exercise

The cycloid is the curve traced by a point on the rim of a rolling wheel. Imagine a wire shaped like an inverted cycloid.



A bead sliding down this wire without friction has some remarkable properties. Among all wire shapes, the cycloid is the shape that produces the fastest descent time. It can be shown that the descent time between any two points $0 \le a < b \le \pi$ on the curve is

$$descent\ time = \int_a^b \sqrt{\frac{1-\cos t}{g\left(\cos a - \cos t\right)}}\, dt$$

Where g is the acceleration due to gravity, $t = 0$ corresponds to the top of the wire, and $t = \pi$ corresponds to the lowest point on the wire.

a) Find the descent time on the interval $[a,\ b]$.

b) Show that when $b = \pi$, the descent time is the same for all values of a; that is, the descent time to the bottom of the wire is the same for all starting points.

Solution

a) $$\int_a^b \sqrt{\frac{1-\cos t}{g\left(\cos a - \cos t\right)}}\, dt = \int_a^b \sqrt{\frac{(1-\cos t)(1+\cos t)}{g\left(\cos a - \cos t\right)(1+\cos t)}}\, dt$$

$$= \frac{1}{\sqrt{g}} \int_a^b \sqrt{\frac{\left(1-\cos^2 t\right)}{\cos a + (\cos a - 1)\cos t - \cos^2 t}}\, dt$$

$$= \frac{1}{\sqrt{g}} \int_a^b \frac{\sin t}{\sqrt{\cos a + \left(\frac{\cos a - 1}{2}\right)^2 - \left(\frac{\cos a - 1}{2}\right)^2 + (\cos a - 1)\cos t - \cos^2 t}}\, dt$$

$$= \frac{1}{\sqrt{g}} \int_a^b \frac{\sin t}{\sqrt{\cos a + \left(\frac{\cos a - 1}{2}\right)^2 - \left(\left(\frac{\cos a - 1}{2}\right) - \cos t\right)^2}}\, dt$$

Let: $v = \sqrt{\cos a + \left(\frac{\cos a - 1}{2}\right)^2}$

$$= \frac{1}{2}\sqrt{4\cos a + \cos^2 a - 2\cos a + 1}$$

$$= \frac{1}{2}\left(\cos a + 1\right)$$

$$\frac{\cos a - 1}{2} - \cos t = v\sin\theta \quad \to \quad \sin t\ dt = v\cos\theta\ d\theta$$

$$\sqrt{v - \left(\left(\frac{\cos a - 1}{2}\right) - \cos t\right)^2} = v\cos\theta$$

$$= \frac{1}{\sqrt{g}} \int_a^b \frac{v\cos\theta}{v\cos\theta}\, d\theta$$

$$= \frac{1}{\sqrt{g}}\,\theta\ \Big|_a^b$$

$$\theta = \sin^{-1}\left(\frac{\cos a - 1 - 2\cos t}{2}\,\frac{2}{1+\cos a}\right)$$

$$= \frac{1}{\sqrt{g}} \sin^{-1}\left(\frac{\cos a - 1 - 2\cos t}{1+\cos a}\right) \Bigg|_a^b$$

$$= \frac{1}{\sqrt{g}} \left(\sin^{-1}\left(\frac{\cos a - 1 - 2\cos b}{1+\cos a}\right) - \sin^{-1}(-1)\right)$$

$$= \frac{1}{\sqrt{g}} \left(\sin^{-1}\left(\frac{\cos a - 1 - 2\cos b}{1+\cos a}\right) + \frac{\pi}{2}\right)$$

b) $\frac{1}{\sqrt{g}} \left(\sin^{-1}\left(\frac{\cos a - 1 - 2\cos b}{1+\cos a}\right) + \frac{\pi}{2}\right) \Bigg|_{b=\pi} = \frac{1}{\sqrt{g}} \left(\sin^{-1}\left(\frac{\cos a - 1 + 2}{1+\cos a}\right) + \frac{\pi}{2}\right)$

$$= \frac{1}{\sqrt{g}} \left(\sin^{-1}(1) + \frac{\pi}{2}\right)$$

$$= \frac{\pi}{\sqrt{g}}$$

Exercise

Find the area of the region bounded by the curve $f(x) = \left(16 + x^2\right)^{-3/2}$ and the x-*axis* on the interval $[0,\ 3]$

Solution

$$A = \int_0^3 \frac{dx}{\left(16+x^2\right)^{3/2}}$$

$$x = 4\tan\theta \quad \rightarrow dx = 4\sec^2\theta\ d\theta$$

$$16 + x^2 = 16\sec^2\theta$$

$$= \int_0^3 \frac{4\sec^2\theta d\theta}{\left(16\sec^2\theta\right)^{3/2}}$$

$$= \int_0^3 \frac{4\sec^2\theta}{4^3\sec^3\theta} d\theta$$

$$= \frac{1}{16}\int_0^3 \cos\theta\ d\theta$$

$$= \frac{1}{16}\sin\theta \Bigg|_0^3$$

$$= \frac{1}{16}\frac{\tan\theta}{\sec\theta} \Bigg|_0^3$$

$$= \frac{1}{16}\frac{x}{\sqrt{16+x^2}} \Bigg|_0^3$$

$$=\frac{1}{16}\left(\frac{3}{5}-0\right)$$

$$=\frac{3}{80}\quad unit^2$$

Exercise

Find the length of the curve $y = ax^2$ from $x = 0$ to $x = 10$, where $a > 0$ is a real number.

Solution

$$1+(y')^2 = 1+(2ax)^2$$

$$L=\int_0^{10}\sqrt{1+4a^2x^2}\ dx$$

$$=\int_0^{10} 2a\sqrt{\frac{1}{4a^2}+x^2}\ dx$$

$$x=\frac{1}{2a}\tan\theta \qquad \frac{1}{4a^2}+x^2=\frac{1}{4a^2}\sec^2\theta$$

$$dx=\frac{1}{4a^2}\sec^2\theta\ d\theta$$

$$=\int_0^{10} 2a\frac{1}{2a}\sec\theta\frac{1}{4a^2}\sec^2\theta\ d\theta$$

$$=\frac{1}{2a}\int_0^{10}\sec^3\theta\ d\theta$$

$$u=\sec x \qquad dv=\sec^2 xdx$$

$$du=\sec x\tan xdx \qquad v=\tan x$$

$$\int\sec^3 xdx=\sec x\tan x-\int\tan x(\sec x\tan xdx)$$

$$=\sec x\tan x-\int\tan^2 x\sec xdx$$

$$=\sec x\tan x-\int\left(\sec^2 x-1\right)\sec x\ dx$$

$$=\sec x\tan x-\int\sec^3 x\ dx+\int\sec x\ dx$$

$$\int\sec^3 x\ dx+\int\sec^3 x\ dx=\sec x\tan x-\int\sec^3 x\ dx+\int\sec^3 x\ dx+\int\sec x\ dx$$

$$2\int\sec^3 x\ dx=\sec x\tan x+\int\sec x\ dx$$

$$= \sec x \tan x + \ln\left|\sec x + \tan x\right|$$

$$= \frac{1}{4a}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right.\Big|_{0}^{10}$$

$$= \frac{1}{4a}\left(2a\sqrt{\frac{1}{4a^2} + x^2}\,(2ax) + \ln\left|\sqrt{1+4a^2x^2} + 2ax\right|\right.\Big|_{0}^{10}$$

$$= \frac{1}{4a}\left((2ax)\sqrt{1+4a^2x^2} + \ln\left|\sqrt{1+4a^2x^2} + 2ax\right|\right.\Big|_{0}^{10}$$

$$\boxed{= \frac{1}{4a}\left((20a)\sqrt{1+400a^2} + \ln\left|\sqrt{1+400a^2} + 20a\right|\right) \quad unit}$$

Exercise

Find the arc length of the graph of $f(x) = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$

Solution

$$1 + (f')^2 = 1 + x^2$$

$$L = \int_0^1 \sqrt{1+x^2}\, dx \qquad x = \tan\theta \qquad \sqrt{x^2+1} = \sec\theta$$

$$dx = \sec^2\theta\, d\theta$$

$$= \int_0^1 \sec^3\theta\, d\theta$$

$$= \frac{1}{2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right.\Big|_{0}^{1}$$

$$= \frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left|x + \sqrt{x^2+1}\right|\right.\Big|_{0}^{1}$$

$$\boxed{= \frac{1}{2}\left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right) \quad unit}$$

Exercise

A projectile is launched from the ground with an initial speed V at an angle θ from the horizontal. Assume that the *x-axis* is the horizontal ground and y is the height above the ground. Neglecting air resistance and letting g be the acceleration due to gravity, it can be shown that the trajectory of the projectile is given by

$$y = -\frac{1}{2}kx^2 + y_{max} \qquad \textit{where} \;\; k = \frac{g}{(V\cos\theta)^2}$$

$$\textit{and} \qquad\qquad y_{max} = \frac{(V\sin\theta)^2}{2g}$$

a) Note that the high point of the trajectory occurs at $\left(0,\ y_{max}\right)$. If the projectile is on the ground at $(-a,\ 0)$ and $(a,\ 0)$, what is a?

b) Show that the length of the trajectory (arc length) is $2\displaystyle\int_0^a \sqrt{1+k^2x^2}\; dx$

c) Evaluate the arc length integral and express your result in the terms of V, g, and θ.

d) For fixed value of V and g, show that the launch angle θ that maximizes the length of the trajectory satisfies $(\sin\theta)\ln(\sec\theta+\tan\theta)=1$

Solution

a) At $(\pm a,\ 0) \rightarrow\ y = 0 = -\frac{1}{2}ka^2 + y_{max}$

$$a^2 = \frac{2}{k}y_{max}$$

$$\boxed{a = \sqrt{\frac{2y_{max}}{k}}}$$

b) $y' = -kx \;\; \Rightarrow \;\; 1+(y')^2 = 1+k^2x^2$

$$L = \int_{-a}^{a} \sqrt{1+k^2x^2}\; dx \qquad\qquad \text{since } y(x) \text{ is an even function}$$

$$= 2\int_0^a \sqrt{1+k^2x^2}\; dx$$

c) $L = 2\displaystyle\int_0^a \sqrt{1+k^2x^2}\; dx$

$$x = \frac{1}{k}\tan\theta \;\; \Rightarrow \;\; dx = \frac{1}{k}\sec^2\theta\; d\theta$$

$$1+k^2x^2 = \sec^2\theta$$

$$= 2\int_0^a \frac{1}{k}\sec\theta\sec^2\theta\; d\theta$$

$$= \frac{2}{k}\int_0^a \sec^3\theta\; d\theta$$

$$= \frac{1}{k}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right| \;\right|_0^a$$

$$= \frac{1}{k}\left(\sqrt{1+k^2x^2}\,(kx) + \ln\left|\sqrt{1+k^2x^2} + kx\right| \;\right|_0^a$$

$$\boxed{= \frac{1}{k}\left(ak\sqrt{1+k^2a^2} + \ln\left|\sqrt{1+k^2a^2} + ka\right|\right)}$$

$$a = \sqrt{\frac{2}{k}\frac{(V\sin\theta)^2}{2g}}$$

$$= \frac{V\sin\theta}{\sqrt{g\dfrac{g}{(V\cos\theta)^2}}}$$

$$= \frac{V^2}{g}\sin\theta\cos\theta$$

$$k = \frac{g}{(V\cos\theta)^2}$$

$$ak = \tan\theta$$

$$L(\theta) = \frac{(V\cos\theta)^2}{g}\left(\tan\theta\sqrt{1+\tan^2\theta} + \ln\left|\sqrt{1+\tan^2\theta} + \tan\theta\right|\right)$$

$$= \frac{V^2\cos^2\theta}{g}\left(\tan\theta\sec\theta + \ln\left|\sec\theta + \tan\theta\right|\right)$$

$$= \frac{V^2}{g}\sin\theta + \frac{V^2}{g}\cos^2\theta\ln\left|\sec\theta + \tan\theta\right|$$

$$\boxed{= \frac{V^2}{g}\left(\sin\theta + \cos^2\theta\sinh^{-1}(\tan\theta)\right) \quad \textit{unit}}$$

d) $$L'(\theta) = \frac{V^2}{g}\left(\cos\theta - 2\cos\theta\sin\theta\sinh^{-1}(\tan\theta) + \cos^2\theta\frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\right)$$

$$= \frac{V^2}{g}\left(\cos\theta - 2\cos\theta\sin\theta\sinh^{-1}(\tan\theta) + \cos^2\theta\sec\theta\right)$$

$$= \frac{2V^2\cos\theta}{g}\left(1 - \sin\theta\sinh^{-1}(\tan\theta)\right) = 0$$

$$\sin\theta\sinh^{-1}(\tan\theta) = 1$$

$$\sin\theta\ln(\sec\theta + \tan\theta) = 1$$ ✓

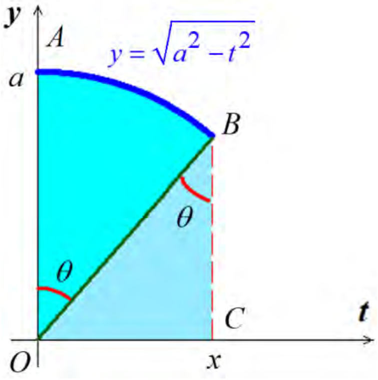
Exercise

Let $F(x)=\int_0^x \sqrt{a^2-t^2}\,dt$. The figure shows that $F(x)=\textit{area of sector OAB + area of triangle OBC}$

a) Use the figure to prove that $F(x)=\dfrac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\dfrac{x\sqrt{a^2-x^2}}{2}$

b) Conclude that $\displaystyle\int \sqrt{a^2-x^2}\,dx=\dfrac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\dfrac{x\sqrt{a^2-x^2}}{2}+C$

Solution



a) Area of sector *OAB* is $\frac{1}{2}\theta a^2$

From the triangle *OBC*: $\sin\theta=\dfrac{x}{a}\ \to\ \theta=\sin^{-1}\dfrac{x}{a}$

$$|BC|=\sqrt{a^2-x^2}$$

Area of sector *OAB* is $\frac{1}{2}a^2\sin^{-1}\dfrac{x}{a}$

Area of triangle *OBC*: $\frac{1}{2}x\sqrt{a^2-x^2}$

$$F(x)=\textit{area of sector OAB + area of triangle OBC}$$

$$=\dfrac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\dfrac{x\sqrt{a^2-x^2}}{2}$$

b) $$\frac{d}{dx}\left(\dfrac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\dfrac{x\sqrt{a^2-x^2}}{2}+C\right)=\frac{a^2}{2}\dfrac{\frac{1}{a}}{\sqrt{1-\left(\frac{x}{a}\right)^2}}+\frac{1}{2}\sqrt{a^2-x^2}-\frac{1}{2}\frac{x^2}{\sqrt{a^2-x^2}}$$

$$=\frac{1}{2}\frac{a^2}{\sqrt{a^2-x^2}}+\frac{1}{2}\frac{a^2-2x^2}{\sqrt{a^2-x^2}}$$

$$=\frac{1}{2}\frac{2a^2-2x^2}{\sqrt{a^2-x^2}}$$

$$=\frac{a^2-x^2}{\sqrt{a^2-x^2}}$$

$$=\sqrt{a^2-x^2}$$

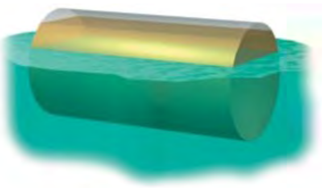
By the antiderivative:

$$\int \sqrt{a^2-x^2}\,dx=\dfrac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\dfrac{x\sqrt{a^2-x^2}}{2}+C\quad\checkmark$$

Exercise

A sealed barrel of oil (weighing 48 pounds per cubic foot) is floating in seawater (weighing 64 pounds per cubic foot). The barrel is not completely full of oil. With the barrel lying on its side, the top 0.2 *foot* of the barrel is empty.

Compare the fluid forces against one end of the barrel from the inside and from the outside.



Solution

$$x^2+y^2=1 \quad \rightarrow \quad 2x=2\sqrt{1-y^2}$$

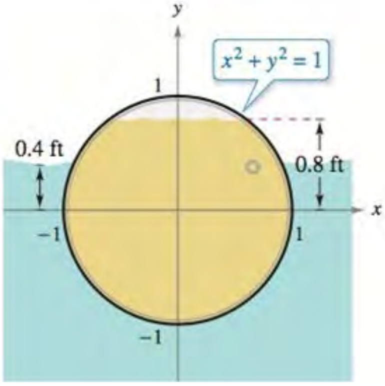
$$F_{inside}=48\int_{-1}^{0.8}(0.8-y)(2)\sqrt{1-y^2}\;dy \qquad\qquad F=w\int_c^d h(y)L(y)dy$$

$$=76.8\int_{-1}^{0.8}\sqrt{1-y^2}\;dy\;-\;96\int_{-1}^{0.8}y\sqrt{1-y^2}\;dy$$

$$=76.8\int_{-1}^{0.8}\sqrt{1-y^2}\;dy\;+\;48\int_{-1}^{0.8}\left(1-y^2\right)^{1/2}\;d\left(1-y^2\right)$$

$$y=\sin\theta \qquad \sqrt{1-y^2}=\cos\theta$$

$$dy=\cos\theta\;d\theta$$



$$=76.8\int_{-1}^{0.8}\cos^2\theta\;d\theta\;+\;32\left(1-y^2\right)^{3/2}\Big|_{-1}^{0.8}$$

$$=38.4\int_{-1}^{0.8}(1+\cos 2\theta)\;d\theta+32(0.16)^{3/2}$$

$$=38.4\left(\theta+\frac{1}{2}\sin 2\theta\right.\Big|_{-1}^{0.8}\;+\;32(0.4)^3$$

$$=38.4\left(\arcsin y+y\sqrt{1-y^2}\right.\Big|_{-1}^{0.8}\;+\;2.048$$

$$=38.4\left(\arcsin 0.8+0.32+\frac{\pi}{2}\right)+2.048$$

$$\approx 121.3\;\; lbs$$

$$F_{outside}=64\int_{-1}^{0.4}(0.4-y)(2)\sqrt{1-y^2}\;dy \qquad\qquad F=w\int_c^d h(y)L(y)dy$$

$$=51.2\int_{-1}^{0.4}\sqrt{1-y^2}\;dy-128\int_{-1}^{0.4}y\sqrt{1-y^2}\;dy$$

$$= 25.6\left(\arcsin y + y\sqrt{1-y^2}\right)\Big|_{-1}^{0.4} + \frac{128}{3}\left(1-y^2\right)^{3/2}\Big|_{-1}^{0.4}$$

$$\underline{\approx 93.0 \ \ lbs\ |}$$

Exercise

The axis of a storage tank in the form of a right circular cylinder is horizontal. The radius and length of the tank are 1 *meter* and 3 *meters*, respectively.

a) Determine the volume of fluid in the tank as a function of its depth *d*.

b) Graph the function in part (a).

c) Design a dip stick for the tank with markings of $\frac{1}{4}, \frac{1}{2}, \textit{ and } \frac{3}{4}$

d) Fluid is entering the tank at a rate of $\frac{1}{4}\ m^3/min$. Determine the rate of change of the depth of the fluid as a function of its depth *d*.

e) Graph the function in part (*d*).\When will the rate of change of the depth be minimum?

Solution

a) Consider the center at $(0,\ 1)$: $x^2 + (y-1)^2 = 1$

$$x = \sqrt{1-(y-1)^2}$$

The depth: $0 \le d \le 2$

$$V = \int_0^d (3)\left(2\sqrt{1-(y-1)^2}\right)dy$$

$$= 6\int_0^d \sqrt{1-(y-1)^2}\ d(y-1)$$

$$y-1 = \sin\theta \qquad \sqrt{1-(y-1)^2} = \cos\theta$$

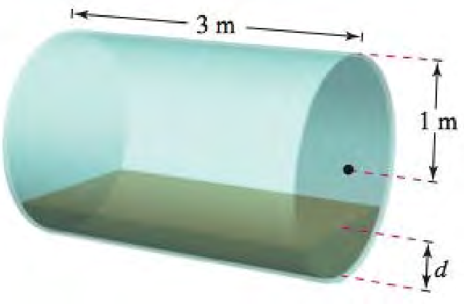
$$d(y-1) = \cos\theta\ d\theta$$

$$= 6\int_0^d \cos^2\theta\ d\theta$$

$$= 3\int_0^d (1+\cos 2\theta)\ d\theta$$

$$= 3\left(\theta + \frac{1}{2}\sin 2\theta\right.\Big|_0^d$$

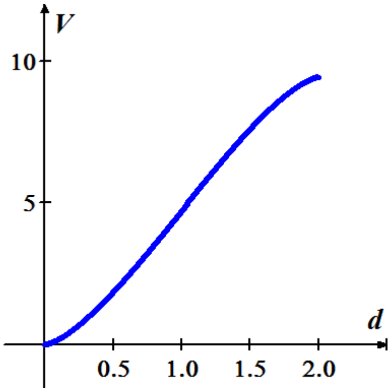
$$= 3\ (\theta + \sin\theta\cos\theta\ \Big|_0^d$$



$$=3\left(\arcsin(y-1)+(y-1)\sqrt{1-(y-1)^2}\right|_0^d$$

$$\boxed{=3\arcsin(d-1)+3(d-1)\sqrt{2d-d^2}+\frac{3\pi}{2}\quad unit^3}$$

b)



c) The full tank holds $3\pi\ m^3$

A dip stick for the tank with markings of $\frac{1}{4}, \frac{1}{2}, and\ \frac{3}{4}$

The horizontal lines are: $y=\frac{3\pi}{4},\quad y=\frac{3\pi}{2},\quad y=\frac{9\pi}{4}$

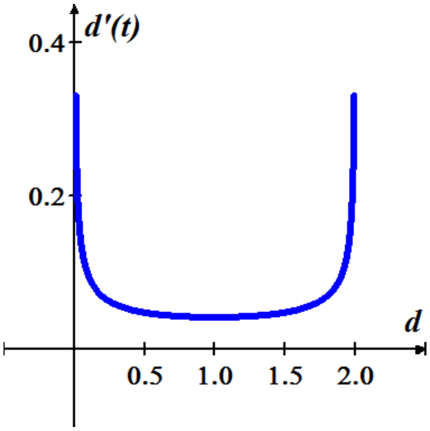
Intersect the curve at $\boxed{d=0.596,\quad d=1.0,\quad d=1.404}$

d) $V=6\int_0^d\sqrt{1-(y-1)^2}\,dy \quad\rightarrow\quad \frac{dV}{dt}=\frac{dV}{dd}\frac{dd}{dt}$

$$\frac{dV}{dt}=6\sqrt{1-(d-1)^2}\cdot d'(t)=\frac{1}{4}$$

$$\boxed{d'(t)=\frac{1}{24\sqrt{1-(d-1)^2}}}$$

e)



From the graph, the minimum occurs at $d=1$, which is the widest part of the tank.

Exercise

The field strength H of a magnet of length $2L$ on a particle r units from the center of the magnet is

$$H = \frac{2mL}{\left(r^2 + L^2\right)^{3/2}}$$

Where $\pm m$ are the poles of the magnet.

Find the average field strength as the particle moves from 0 to R units from the center by evaluating the integral

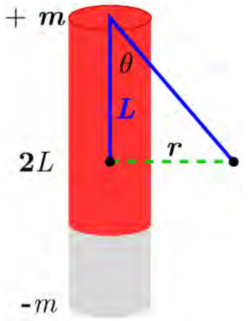
$$\frac{1}{R}\int_0^R \frac{2mL}{\left(r^2+L^2\right)^{3/2}}\,dr$$

Solution

$$r = L\tan\theta \quad \to \quad dr = L\sec^2\theta d\theta$$

$$r^2 + L^2 = L^2\tan^2\theta + L^2$$

$$= L^2\sec^2\theta$$



$$\frac{1}{R}\int_0^R \frac{2mL}{\left(r^2+L^2\right)^{3/2}}\,dr = \frac{1}{R}\int_0^R \frac{2mL}{(L\sec\theta)^3}L\sec^2\theta\ d\theta$$

$$= \frac{2m}{RL}\int_0^R \frac{1}{\sec\theta}\ d\theta$$

$$= \frac{2m}{RL}\int_0^R \cos\theta\ d\theta$$

$$= \frac{2m}{RL}\sin\theta\ \Big|_0^R$$

$$= \frac{2m}{RL}\frac{r}{\sqrt{r^2+L^2}}\ \Big|_0^R$$

$$= \frac{2m}{L\ \sqrt{R^2+L^2}}$$

Solution *Section* 2.4 – Integration of Rational Functions by Partial Fractions

Exercise

Evaluate $\displaystyle\int \frac{dx}{x^2+2x}$

Solution

$$\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2}$$

$$1 = Ax + 2A + Bx$$

$$x \qquad 2A = 1 \qquad \rightarrow A = \frac{1}{2}$$

$$x^0 \qquad A + B = 0 \quad \rightarrow B = -\frac{1}{2}$$

$$\int \frac{1}{x^2+2x}dx = \frac{1}{2}\int \frac{1}{x}dx \; - \; \frac{1}{2}\int \frac{1}{x+2}dx$$

$$\boxed{= \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{2x+1}{x^2-7x+12}dx$

Solution

$$\frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3}$$

$$2x+1 = Ax - 3A + Bx - 4B$$

$$x \qquad A + B = 2$$

$$x^0 \qquad -3A - 4B = 1$$

$$A = \frac{\begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -3 & -4 \end{vmatrix}} = \frac{-9}{-1} \boxed{= 9} \qquad\qquad B = \frac{\begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix}}{-1} = \frac{7}{-1} \; \boxed{= -7}$$

$$\int \frac{2x+1}{x^2-7x+12}dx = 9\int \frac{dx}{x-4} - 7\int \frac{dx}{x-3}$$

$$= 9\ln|x-4| - 7\ln|x-3| + C$$

$$\boxed{= \ln\left|\frac{(x-4)^9}{(x-3)^7}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x+3}{2x^3-8x}\,dx$

Solution

$$\frac{x+3}{2x^3-8x} = \frac{1}{2}\frac{x+3}{x\left(x^2-4\right)}$$

$$= \frac{1}{2}\left(\frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}\right)$$

$$= \frac{1}{2}\frac{A(x+2)(x-2) + Bx(x-2) + Cx(x+2)}{x(x+2)(x-2)}$$

$$Ax^2 - 4A + Bx^2 - 2Bx + Cx^2 + 2Cx = x+3$$

$$x^2 \quad A+B+C=0$$

$$x \quad -2B+2C=1$$

$$x^0 \quad -4A=3 \quad \rightarrow \boxed{A=-\frac{3}{4}}$$

$$\begin{cases} B+C=\frac{3}{4} \\ -2B+2C=1 \end{cases} \Rightarrow \quad \begin{matrix} 2B+2C=\frac{3}{2} \\ -2B+2C=1 \end{matrix}$$

$$4C=\frac{5}{2} \quad \rightarrow \quad \boxed{C=\frac{5}{8}}$$

$$B=\frac{3}{4}-\frac{5}{8} \quad \rightarrow \quad \boxed{B=\frac{1}{8}}$$

$$\int \frac{x+3}{2x^3-8x}dx = \frac{1}{2}\int -\frac{3}{4}\frac{dx}{x} + \frac{1}{2}\int \frac{1}{8}\frac{dx}{x+2} + \frac{1}{2}\int \frac{5}{8}\frac{dx}{x-2}$$

$$= -\frac{3}{8}\ln|x| + \frac{1}{16}\ln|x+2| + \frac{5}{16}\ln|x-2| + K$$

$$= \frac{1}{16}\left(\ln|x+2| + 5\ln|x-2| - 6\ln|x|\right) + K$$

$$\boxed{= \frac{1}{16}\ln\left|\frac{(x+2)(x-2)^5}{x^6}\right| + K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^2}{(x-1)\left(x^2+2x+1\right)}dx$

Solution

$$\frac{x^2}{(x-1)\left(x^2+2x+1\right)} = \frac{x^2}{(x-1)(x+1)^2}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$$= Ax^2 + 2Ax + A + Bx^2 - B + Cx - C$$

x^2 $\quad A + B = 1 \quad \rightarrow B = 1 - A$

x $\quad 2A + C = 0 \quad \rightarrow C = -2A$

x^0 $\quad A - B - C = 0 \quad \rightarrow A - 1 + A + 2A = 0$

$$\boxed{A = \tfrac{1}{4} \quad B = \tfrac{3}{4} \quad C = -\tfrac{1}{2}}$$

$$\int \frac{x^2}{(x-1)\left(x^2+2x+1\right)}dx = \frac{1}{4}\int \frac{dx}{x-1} + \frac{3}{4}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{(x+1)^2}$$

$$= \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2}\frac{1}{(x+1)} + K$$

$$= \frac{1}{4}\left(\ln|x-1| + \ln|x+1|^3\right) + \frac{1}{2(x+1)} + K$$

$$\boxed{= \frac{1}{4}\ln\left|(x-1)(x+1)^3\right| + \frac{1}{2(x+1)} + K}$$

Exercise

Evaluate $\displaystyle\int \frac{8x^2+8x+2}{\left(4x^2+1\right)^2}dx$

Solution

$$\frac{8x^2+8x+2}{\left(4x^2+1\right)^2} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{\left(4x^2+1\right)^2}$$

$$= \frac{(Ax+B)\left(4x^2+1\right) + Cx + D}{\left(4x^2+1\right)^2}$$

$$8x^2+8x+2=4Ax^3+4Bx^2+(A+C)x+B+D$$

$$\begin{cases} x^3 & A=0 \\ x^2 & 4B=8 \\ x & A+C=8 \\ x^0 & B+D=2 \end{cases} \rightarrow \boxed{A=0} \quad \boxed{B=2} \quad \boxed{C=8} \quad \boxed{D=0}$$

$$\int \frac{8x^2+8x+2}{\left(4x^2+1\right)^2}dx = \int \frac{2}{4x^2+1}dx + \int \frac{8x}{\left(4x^2+1\right)^2}dx$$

$$d\left(4x^2+1\right)=8xdx$$

$$= \int \frac{2}{4x^2+1}\,dx + \int \frac{d\left(4x^2+1\right)}{\left(4x^2+1\right)^2}$$

$$\int \frac{du}{u^2}=-\frac{1}{u} \qquad \int \frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \tan^{-1}2x-\frac{1}{4x^2+1}+K$$

Exercise

Evaluate $\displaystyle\int \frac{x^2+x}{x^4-3x^2-4}dx$

Solution

$$\frac{x^2+x}{x^4-3x^2-4}=\frac{x^2+x}{\left(x^2-4\right)\left(x^2+1\right)}$$

$$=\frac{A}{x-2}+\frac{B}{x+2}+\frac{Cx+D}{x^2+1}$$

$$x^2+x=A(x+2)\left(x^2+1\right)+B(x-2)\left(x^2+1\right)+(Cx+D)\left(x^2-4\right)$$

$$=Ax^3+Ax+2Ax^2+2A+Bx^3+Bx-2Bx^2-2B+Cx^3-4Cx+Dx^2-4D$$

$$=(A+B+C)x^3+(2A-2B+D)x^2+(A+B-4C)x+2A-2B-4D$$

$$\begin{cases} x^3 & A+B+C=0 & (1) \\ x^2 & 2A-2B+D=1 & (2) \\ x & A+B-4C=1 & (3) \\ x^0 & 2A-2B-4D=0 & (4) \end{cases}$$

$$(1)-(3)\rightarrow \quad 5C=-1 \quad C=-\frac{1}{5}$$

$$(2)-(4)\rightarrow \quad 5D=1 \quad D=\frac{1}{5}$$

$$\begin{cases} A+B=\frac{1}{5} \\ 2A-2B=\frac{4}{5} \end{cases} \to \begin{cases} 2A+2B=\frac{2}{5} \\ 2A-2B=\frac{4}{5} \end{cases}$$

$$4A=\frac{6}{5} \quad \to \quad \boxed{A=\frac{3}{10}}$$

$$B=\frac{1}{5}-\frac{3}{10} \quad \to \quad \boxed{B=-\frac{1}{10}}$$

$$\int \frac{x^2+x}{x^4-3x^2-4}dx=\frac{3}{10}\int \frac{1}{x-2}dx-\frac{1}{10}\int \frac{1}{x+2}dx+\frac{1}{5}\int \frac{-x+1}{x^2+1}dx$$

$$=\frac{3}{10}\ln|x-2|-\frac{1}{10}\ln|x+2|-\frac{1}{5}\int \frac{x}{x^2+1}dx+\frac{1}{5}\int \frac{1}{x^2+1}dx \qquad d\left(x^2+1\right)=2xdx$$

$$=\frac{3}{10}\ln|x-2|-\frac{1}{10}\ln|x+2|-\frac{1}{10}\int \frac{d\left(x^2+1\right)}{x^2+1}+\frac{1}{5}\int \frac{1}{x^2+1}dx \qquad \int \frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\boxed{=\frac{3}{10}\ln|x-2|-\frac{1}{10}\ln|x+2|-\frac{1}{10}\ln\left(x^2+1\right)+\frac{1}{5}\tan^{-1}x+K}$$

Exercise

Evaluate $\displaystyle\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{\left(\theta^2+1\right)^3}d\theta$

Solution

$$\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{\left(\theta^2+1\right)^3}=\frac{A\theta+B}{\theta^2+1}+\frac{C\theta+D}{\left(\theta^2+1\right)^2}+\frac{E\theta+F}{\left(\theta^2+1\right)^3}$$

$$\theta^4-4\theta^3+2\theta^2-3\theta+1=(A\theta+B)\left(\theta^2+1\right)^2+(C\theta+D)\left(\theta^2+1\right)+E\theta+F$$

$$=(A\theta+B)\left(\theta^4+2\theta^2+1\right)+C\theta^3+C\theta+D\theta^2+D+E\theta+F$$

$$=A\theta^5+B\theta^4+(2A+C)\theta^3+(2B+D)\theta^2+(A+C+E)\theta+B+D+F$$

$$\begin{cases} \boxed{A=0} \\ \boxed{B=1} \\ 2A+C=-4 \\ 2B+D=2 \\ A+C+E=-3 \\ B+D+F=1 \end{cases} \to \boxed{C=-4} \quad \boxed{D=0} \quad \boxed{E=1} \quad \boxed{F=0}$$

$$\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{\left(\theta^2+1\right)^3} d\theta = \int \frac{1}{\theta^2+1} d\theta - 4\int \frac{\theta}{\left(\theta^2+1\right)^2} d\theta + \int \frac{\theta}{\left(\theta^2+1\right)^3} d\theta$$

$$= \int \frac{1}{\theta^2+1} d\theta - 2\int \frac{d\left(\theta^2+1\right)}{\left(\theta^2+1\right)^2} + \frac{1}{2} \int \frac{d\left(\theta^2+1\right)}{\left(\theta^2+1\right)^3} \qquad d\left(\theta^2+1\right) = 2\theta d\theta$$

$$= \tan^{-1}\theta + 2\frac{1}{\theta^2+1} - \frac{1}{4}\frac{1}{\left(\theta^2+1\right)^2} + K$$

Exercise

Evaluate $\displaystyle\int \frac{x^4}{x^2-1} dx$

Solution

$$\frac{x^4}{x^2-1} = x^2 + 1 + \frac{1}{(x-1)(x+1)}$$

$$\begin{array}{r} x^2+1 \\ x^2-1\overline{)\,x^4\qquad} \\ \underline{x^4-x^2} \\ x^2 \\ \underline{x^2-1} \\ 1 \end{array}$$

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$1 = Ax + A + Bx - B$$

$$\begin{cases} A + B = 0 \\ A - B = 1 \end{cases} \rightarrow \quad A = \frac{1}{2}, \quad B = -\frac{1}{2}$$

$$\int \frac{x^4}{x^2-1} dx = \int \left(x^2+1\right) dx + \frac{1}{2}\int \frac{1}{x-1} dx - \frac{1}{2}\int \frac{1}{x+1} dx$$

$$= \frac{1}{3}x^3 + x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

$$= \frac{1}{3}x^3 + x + \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right) + C$$

$$= \frac{1}{3}x^3 + x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{16x^3}{4x^2-4x+1} dx$

Solution

$$\frac{16x^3}{4x^2-4x+1}=4x+4+\frac{12x-4}{(2x-1)^2}$$

$$=4x+4+\frac{A}{2x-1}+\frac{B}{(2x-1)^2}$$

$$\begin{array}{r}
4x+4 \\
4x^2-4x+1\overline{\big)16x^3} \\
\underline{16x^3-16x^2+4x} \\
16x^2-4x \\
\underline{16x^2-16x+4} \\
12x-4
\end{array}$$

$$12x-4=2Ax-A+B$$

$$\begin{cases} 2A=12 \\ -A+B=-4 \end{cases} \quad \rightarrow \quad \boxed{A=6, \quad B=2}$$

$$\int \frac{16x^3}{4x^2-4x+1}dx=\int(4x+4)\,dx+6\int\frac{dx}{2x-1}+2\int\frac{dx}{(2x-1)^2}$$

$$=2x^2+4x+6\left(\frac{1}{2}\right)\ln|2x-1|+2\left(-\frac{1}{2}\right)\frac{1}{2x-1}+C$$

$$\boxed{=2x^2+4x+3\ln|2x-1|-\frac{1}{2x-1}+C}$$

Exercise

Evaluate $\displaystyle\int \frac{e^{4x}+2e^{2x}-e^x}{e^{2x}+1}dx$

Solution

$$\int \frac{e^{4x}+2e^{2x}-e^x}{e^{2x}+1}dx=\int\frac{e^x\left(e^{3x}+2e^x-1\right)}{e^{2x}+1}dx \qquad y=e^x \quad \Rightarrow \quad dy=e^x dx$$

$$=\int\frac{y^3+2y-1}{y^2+1}dy$$

$$=\int\left(y+\frac{y-1}{y^2+1}\right)dy$$

$$=\int y\,dy+\int\frac{y}{y^2+1}dy-\int\frac{1}{y^2+1}dy$$

$$=\int y\,dy+\frac{1}{2}\int\frac{1}{y^2+1}d\left(y^2+1\right)-\int\frac{1}{y^2+1}dy \qquad d\left(y^2+1\right)=2y\,dy$$

$$=\frac{1}{2}y^2+\frac{1}{2}\ln\left(y^2+1\right)-\tan^{-1}y+C$$

$$\boxed{=\frac{1}{2}e^{2x}+\frac{1}{2}\ln\left(e^{2x}+1\right)-\tan^{-1}e^x+C}$$

Exercise

Evaluate $$\int \frac{\sin\theta d\theta}{\cos^2\theta+\cos\theta-2}$$

Solution

Let $y=\cos\theta \quad \Rightarrow \quad dy=-\sin\theta\ d\theta$

$$\int \frac{\sin\theta d\theta}{\cos^2\theta+\cos\theta-2}=-\int \frac{dy}{y^2+y-2}$$

$$\frac{1}{y^2+y-2}=\frac{1}{(y+2)(y-1)}=\frac{A}{y+2}+\frac{B}{y-1}$$

$$1=(A+B)y-A+2B$$

$$\begin{cases} A+B=0 \\ -A+2B=1 \end{cases} \quad \to \boxed{A=-\frac{1}{3},\ B=\frac{1}{3}}$$

$$\begin{aligned}
\int \frac{\sin\theta d\theta}{\cos^2\theta+\cos\theta-2} &= -\left(-\frac{1}{3}\int\frac{dy}{y+2}+\frac{1}{3}\int\frac{dy}{y-1}\right) \\
&= \frac{1}{3}\ln|y+2|-\frac{1}{3}\ln|y-1|+C \\
&= \frac{1}{3}\left(\ln|y+2|-\ln|y-1|\right)+C \\
&= \frac{1}{3}\ln\left|\frac{y+2}{y-1}\right|+C \\
&\boxed{= \frac{1}{3}\ln\left|\frac{\cos\theta+2}{\cos\theta-1}\right|+C}
\end{aligned}$$

Exercise

Evaluate $$\int \frac{(x-2)^2\tan^{-1}(2x)-12x^3-3x}{\left(4x^2+1\right)(x-2)^2}\ dx$$

Solution

$$\begin{aligned}
\int \frac{(x-2)^2\tan^{-1}(2x)-12x^3-3x}{\left(4x^2+1\right)(x-2)^2}\ dx &= \int \frac{(x-2)^2\tan^{-1}(2x)}{\left(4x^2+1\right)(x-2)^2}\ dx-\int \frac{12x^3+3x}{\left(4x^2+1\right)(x-2)^2}\ dx \\
&= \int \frac{\tan^{-1}(2x)}{4x^2+1}\ dx-\int \frac{3x\left(4x^2+1\right)}{\left(4x^2+1\right)(x-2)^2}\ dx \\
&= \int \frac{\tan^{-1}(2x)}{4x^2+1}\ dx-\int \frac{3x}{(x-2)^2}\ dx
\end{aligned}$$

$$d\left(\tan^{-1}2x\right)=\frac{dx}{(2x)^2+1}=\frac{dx}{4x^2+1}$$

$$\frac{3x}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}$$

$$=\frac{Ax-2A+B}{(x-2)^2}$$

$$\begin{cases} \underline{A=3} \\ -2A+B=0 \end{cases} \rightarrow \underline{B=6}$$

$$\int \frac{(x-2)^2\tan^{-1}(2x)-12x^3-3x}{\left(4x^2+1\right)(x-2)^2}\,dx=\frac{1}{2}\int \tan^{-1}(2x)\,d\left(\tan^{-1}(2x)\right)\ -3\int\frac{dx}{x-2}\ -6\int\frac{dx}{(x-2)^2}$$

$$=\frac{1}{4}\left(\tan^{-1}(2x)\right)^2-3\int\frac{d(x-2)}{x-2}\ -6\int\frac{d(x-2)}{(x-2)^2}$$

$$\underline{=\frac{1}{4}\left(\tan^{-1}(2x)\right)^2-3\ln|x-2|-\frac{6}{x-2}+C}$$

Exercise

Evaluate $\displaystyle\int\frac{\sqrt{x+1}}{x}dx$

Solution

$$\begin{array}{r|l} & 1 \\ \hline u^2-1 & u^2 \\ & \underline{u^2-1} \\ & 1 \end{array}$$

Let $x+1=u^2 \quad \Rightarrow \quad dx=2udu$

$$\int\frac{\sqrt{x+1}}{x}dx=\int\frac{u}{u^2-1}2udu$$

$$=2\int\frac{u^2}{u^2-1}du$$

$$=2\int\left(1+\frac{1}{u^2-1}\right)du$$

$$=2\int du+2\int\frac{1}{u^2-1}du$$

$$\frac{1}{u^2-1}=\frac{A}{u-1}+\frac{B}{u+1}$$

$$=\frac{(A+B)u+A-B}{(u-1)(u+1)}$$

$$\begin{cases} A+B=0 \\ A-B=1 \end{cases} \quad \Rightarrow \quad \boxed{A=\tfrac{1}{2},\ \ B=-\tfrac{1}{2}}$$

$$= 2\int du + 2\int \left(\frac{1}{2}\frac{1}{u-1} - \frac{1}{2}\frac{1}{u+1}\right) du$$

$$= 2u + \int \frac{1}{u-1}\,du - \int \frac{1}{u+1}\,du$$

$$= 2u + \ln|u-1| - \ln|u+1| + C$$

$$= 2\sqrt{x+1} + \ln\left|\sqrt{x+1}-1\right| - \ln\left|\sqrt{x+1}+1\right| + C$$

$$\boxed{= 2\sqrt{x+1} + \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x^3-2x^2+3x-4}{x^2+1}\,dx$

Solution

$$\int \frac{x^3-2x^2+3x-4}{x^2+1}\,dx = \int \left(x-2+\frac{2x-2}{x^2+1}\right) dx$$

$$\begin{array}{rl} & x-2 \\ x^2+1 \big) & x^3-2x^2+3x-4 \\ & \underline{x^3 \qquad\quad +x} \\ & \quad -2x^2+2x-4 \\ & \quad \underline{-2x^2 \qquad\quad -2} \\ & \qquad\qquad 2x-2 \end{array}$$

$$= \int (x-2)\,dx + \int \frac{2x}{x^2+1}\,dx - 2\int \frac{1}{x^2+1}\,dx$$

$$= \int (x-2)\,dx + \int \frac{d\left(x^2+1\right)}{x^2+1} - 2\int \frac{1}{x^2+1}\,dx$$

$$\boxed{= \frac{1}{2}x^2 - 2x + \ln\left(x^2+1\right) - 2\tan^{-1}(x) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{4x^2+2x+4}{x+1}\,dx$

Solution

$$\int \frac{4x^2+2x+4}{x+1}\,dx = \int \left(4x+2+\frac{6}{x+1}\right) dx$$

$$= \int (4x-2)dx + \int \frac{6}{x+1}dx$$

$$= \int (4x-2)\ dx\ +\ 6\int \frac{d(x+1)}{x+1} \qquad\qquad \int \frac{d(U)}{U} = \ln|U|$$

$$\boxed{= 2x^2 - 2x + 6\ln|x+1| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{3x^2+7x-2}{x^3-x^2-2x}dx$

Solution

$$\frac{3x^2+7x-2}{x^3-x^2-2x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$$

$$3x^2+7x-2 = A(x+1)(x-2) + Bx(x-2) + Cx(x+1)$$

$$= Ax^2 - Ax - 2A$$

$$Bx^2 - 2Bx$$

$$Cx^2 + Cx$$

$$\begin{cases} A+B+C=3 \\ -A-2B+C=7 \\ -2A=-2 & \rightarrow \boxed{A=1} \end{cases}$$

$$\begin{cases} B+C=2 \\ -2B+C=8 \end{cases} \rightarrow \boxed{B=-2} \quad \boxed{C=4}$$

$$\int \frac{3x^2+7x-2}{x^3-x^2-2x}\ dx = \int \left(\frac{1}{x} - \frac{2}{x+1} + \frac{4}{x-2}\right)dx$$

$$= \ln|x| - 2\ln|x+1| + 4\ln|x-2| + K$$

$$\boxed{= \ln\frac{|x|(x-2)^4}{(x+1)^2} + K}$$

Exercise

Evaluate $\displaystyle\int \frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)}dx$

Solution

$$\frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)}=\frac{A}{x-1}+\frac{B}{x-5}+\frac{C}{x+4}$$

$$3x^2+2x+5=(A+B+C)x^2+(-A+3B-6C)x-20A-4B+5C$$

$$\begin{cases} x^2 & A+B+C=3 \\ x & -A+3B-6C=2 \\ x^0 & -20A-4B+5C=5 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 & 1 \\ -1 & 3 & -6 \\ -20 & -4 & 5 \end{vmatrix}=180 \qquad D_A=\begin{vmatrix} 3 & 1 & 1 \\ 2 & 3 & -6 \\ 5 & -4 & 5 \end{vmatrix}=-90$$

$$D_B=\begin{vmatrix} 1 & 3 & 1 \\ -1 & 2 & -6 \\ -20 & 5 & 5 \end{vmatrix}=450 \qquad D_C=\begin{vmatrix} 1 & 1 & 3 \\ -1 & 3 & 2 \\ -20 & -4 & 5 \end{vmatrix}=180$$

$$\boxed{A=\frac{1}{2}, \quad B=\frac{5}{2}, \quad C=1}$$

$$\int \frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)}\,dx=\int\left(\frac{1}{2}\frac{1}{x-1}+\frac{5}{2}\frac{1}{x-5}+\frac{1}{x+4}\right)dx$$

$$\boxed{=\frac{1}{2}\ln|x-1|+\frac{5}{2}\ln|x-5|+\ln|x+4|+K}$$

Exercise

Evaluate $\displaystyle\int \frac{5x^2-3x+2}{x^3-2x^2}\,dx$

Solution

$$\frac{5x^2-3x+2}{x^3-2x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}$$

$$5x^2-3x+2=Ax^2-2Ax+Bx-2B+Cx^2$$

$$\begin{cases} x^2 & A+C=5 & \boxed{C=4} \\ x & -2A+B=-3 & \boxed{A=1} \\ x^0 & -2B=2 & \rightarrow \boxed{B=-1} \end{cases}$$

$$\int \frac{5x^2-3x+2}{x^3-2x^2}\,dx=\int\frac{dx}{x}-\int\frac{dx}{x^2}+4\int\frac{dx}{x-2}$$

$$\boxed{=\ln|x|+\frac{1}{x}+4\ln|x-2|+K}$$

Exercise

Evaluate $\displaystyle\int \frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}dx$

Solution

$$\frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}=\frac{A}{x-2}+\frac{Bx+C}{x^2-2x+3}$$

$$7x^2-13x+13=Ax^2-2Ax+3A+Bx^2-2Bx+Cx-2C$$

$$\begin{cases} x^2 & A+B=7 \\ x^1 & -2A-2B+C=-13 \\ x^0 & 3A-2C=13 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 & 0 \\ -2 & -2 & 1 \\ 3 & 0 & -2 \end{vmatrix}=3 \qquad D_A=\begin{vmatrix} 7 & 1 & 0 \\ -13 & -2 & 1 \\ 13 & 0 & -2 \end{vmatrix}=15$$

$$D_B=\begin{vmatrix} 1 & 7 & 0 \\ -2 & -13 & 1 \\ 3 & 13 & -2 \end{vmatrix}=6 \qquad D_C=\begin{vmatrix} 1 & 1 & 7 \\ -2 & -2 & -13 \\ 3 & 0 & 13 \end{vmatrix}=3$$

$$A=5;\quad B=2;\quad C=1$$

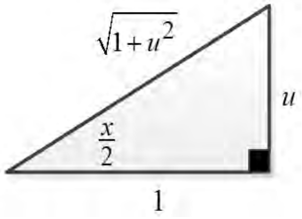
$$\begin{aligned}\int \frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}dx &= 5\int \frac{dx}{x-2}+\int \frac{2x+1}{x^2-2x+3}dx \\ &= 5\ln|x-2|+\int \frac{2x-2+3}{x^2-2x+3}dx \\ &= 5\ln|x-2|+\int \frac{2x-2}{x^2-2x+3}dx+\int \frac{3}{(x-1)^2+3}dx \\ &= 5\ln|x-2|+\ln\left(x^2-2x+3\right)+\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+K\end{aligned}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{1+\sin x}$

Solution

Let $u=\tan\left(\frac{x}{2}\right) \quad x=2\tan^{-1}u$



$$dx = \frac{2du}{1+u^2}$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}}$$

$$= \frac{2u}{1+u^2}$$

$$\int \frac{dx}{1+\sin x} = \int \frac{1}{1+\frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2}\, du$$

$$= \int \frac{2}{u^2+2u+1}\, du$$

$$= \int \frac{2}{(u+1)^2}\, d(u+1)$$

$$= -\frac{2}{u+1} + C$$

$$\boxed{= -\frac{2}{\tan\left(\frac{x}{2}\right)+1} + C}$$

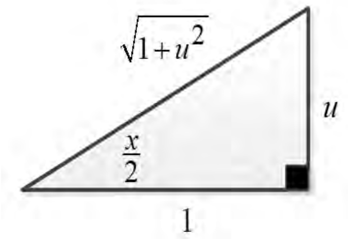
Exercise

Evaluate $\displaystyle\int \frac{dx}{2+\cos x}$

Solution

Let $u = \tan\left(\frac{x}{2}\right) \quad x = 2\tan^{-1} u$

$$dx = \frac{2du}{1+u^2}$$



$$\cos x = 2\cos^2\frac{x}{2} - 1$$

$$= 2\frac{1}{1+u^2} - 1$$

$$= \frac{1-u^2}{1+u^2}$$

$$\int \frac{dx}{2+\cos x} = \int \frac{1}{2+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}\, du$$

$$= 2\int \frac{1}{u^2+3}\,du \qquad \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C$$

$$\boxed{= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) + C}$$

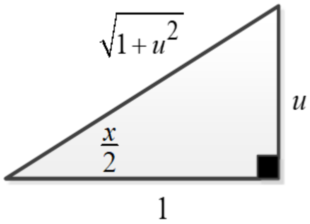
Exercise

Evaluate $\displaystyle\int \frac{dx}{1-\cos x}$

Solution

Let $u = \tan\left(\frac{x}{2}\right) \quad x = 2\tan^{-1} u$

$$dx = \frac{2du}{1+u^2}$$



$$\cos x = 2\cos^2\frac{x}{2} - 1$$

$$= 2\frac{1}{1+u^2} - 1$$

$$= \frac{1-u^2}{1+u^2}$$

$$\int \frac{dx}{1-\cos x} = \int \frac{1}{1-\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}\,du$$

$$= \int \frac{1}{u^2}\,du$$

$$= -\frac{1}{u} + C$$

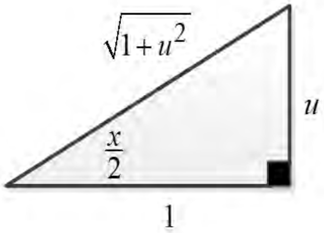
$$= -\frac{1}{\tan\frac{x}{2}} + C$$

$$\boxed{= -\cot\frac{x}{2} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$

Solution



Let $u=\tan\left(\frac{x}{2}\right) \rightarrow x=2\tan^{-1}u$

$$dx=\frac{2du}{1+u^2}$$

$$\cos x=2\cos^2\frac{x}{2}-1$$

$$=2\frac{1}{1+u^2}-1$$

$$=\frac{1-u^2}{1+u^2}$$

$$\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$=2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}}$$

$$=\frac{2u}{1+u^2}$$

$$\int\frac{dx}{1+\sin x+\cos x}=\int\frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\cdot\frac{2}{1+u^2}\,du$$

$$=2\int\frac{1}{2+2u}\,du$$

$$=\int\frac{1}{1+u}\,d(1+u)$$

$$=\ln|1+u|+C$$

$$\boxed{=\ln\left|1+\tan\frac{x}{2}\right|+C}$$

Exercise

Evaluate $\displaystyle\int\frac{1}{x^2-5x+6}\,dx$

Solution

$$\frac{1}{x^2-5x+6}=\frac{A}{x-2}+\frac{B}{x-3}$$

$$Ax - 3A + Bx - 2B = 1$$

$$\rightarrow \begin{cases} A + B = 0 \\ -3A - 2B = 1 \end{cases} \rightarrow \quad A = -1 \quad B = 1$$

$$\int \frac{1}{x^2 - 5x + 6}\, dx = \int \left(\frac{-1}{x-2} + \frac{1}{x-3} \right) dx$$

$$= \ln|x-3| - \ln|x-2| + C$$

$$\boxed{= \ln\left|\frac{x-3}{x-2}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x^2 - 5x + 5}\, dx$

Solution

$$\frac{1}{x^2 - 5x + 5} = \frac{A}{x - \frac{5+\sqrt{5}}{2}} + \frac{B}{x - \frac{5-\sqrt{5}}{2}}$$

$$Ax - \left(\frac{5-\sqrt{5}}{2}\right)A + Bx - \left(\frac{5+\sqrt{5}}{2}\right)B = 1$$

$$\begin{cases} A + B = 0 \\ -\frac{5-\sqrt{5}}{2}A - \frac{5+\sqrt{5}}{2}B = 1 \end{cases}$$

$$\begin{cases} \frac{5-\sqrt{5}}{2}A + \frac{5-\sqrt{5}}{2}B = 0 \\ -\frac{5-\sqrt{5}}{2}A - \frac{5+\sqrt{5}}{2}B = 1 \end{cases}$$

$$-\sqrt{5}B = 1 \rightarrow \boxed{B = -\frac{1}{\sqrt{5}}} \quad \Rightarrow \quad \boxed{A = \frac{1}{\sqrt{5}}}$$

$$\int \frac{1}{x^2 - 5x + 5}\, dx = \int \left(\frac{\sqrt{5}}{5} \frac{2}{2x - 5 - \sqrt{5}} - \frac{\sqrt{5}}{5} \frac{2}{2x - 5 + \sqrt{5}} \right) dx$$

$$\boxed{= \frac{\sqrt{5}}{5} \ln\left|2x - 5 - \sqrt{5}\right| - \frac{\sqrt{5}}{5} \ln\left|2x - 5 + \sqrt{5}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$

Solution

$$\frac{5x^2+20x+6}{x^3+2x^2+x}=\frac{5x^2+20x+6}{x(x+1)^2}$$

$$=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$Ax^2+2Ax+A+Bx^2+Bx+Cx=5x^2+20x+6$$

$$\begin{cases} A+B=5 \\ 2A+B+C=20 \quad \rightarrow \quad \underline{B=-1} \quad \underline{C=9} \\ \underline{A=6} \end{cases}$$

$$\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx=\int\left(\frac{6}{x}-\frac{1}{x+1}+\frac{9}{(x+1)^2}\right)dx$$

$$=6\ln|x|-\ln|x+1|-\frac{9}{x+1}+C$$

$$\boxed{=\ln\frac{x^6}{|x+1|}-\frac{9}{x+1}+C}$$

Exercise

Evaluate $\displaystyle\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}\,dx$

Solution

$$\frac{2x^3-4x-8}{(x^2-x)(x^2+4)}=\frac{2x^3-4x-8}{x(x-1)(x^2+4)}=\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+4}$$

$$Ax^3-Ax^2+4Ax-4A+Bx^3+4Bx+Cx^3-Cx^2+Dx^2-Dx=2x^3-4x-8$$

$$\begin{cases} x^3 & A+B+C=2 \\ x^2 & -A-C+D=0 \\ x^1 & 4A+4B-D=-4 \\ x^0 & -4A=-8 \end{cases} \rightarrow \begin{cases} B+C=0 \\ -C+D=2 \\ 4B-D=-12 \\ \underline{A=2} \end{cases}$$

$$\Rightarrow \begin{cases} B+D=2 \\ 4B-D=-12 \end{cases}$$

$$\boxed{A=2\quad B=-2\quad C=2\quad D=4}$$

$$\int \frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}\,dx=\int\left(\frac{2}{x}-\frac{2}{x-1}+\frac{2x}{x^2+4}+\frac{4}{x^2+4}\right)dx \qquad \int\frac{dx}{x^2+a^2}=\frac{1}{a}\arctan\frac{x}{a}$$

$$\boxed{=2\ln|x|-2\ln|x-1|+\ln\left(x^2+4\right)+2\tan^{-1}\frac{x}{2}+C}$$

Exercise

Evaluate $\displaystyle\int\frac{8x^3+13x}{\left(x^2+2\right)^2}\,dx$

Solution

$$\frac{8x^3+13x}{\left(x^2+2\right)^2}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{\left(x^2+2\right)^2}$$

$$Ax^3+2Ax+Bx^2+2B+Cx+D=8x^3+13x$$

$$\begin{cases} x^3 & \boxed{A=8} \\ x^2 & \boxed{B=0} \\ x^1 & 2A+C=13 \\ x^0 & \boxed{D=0} \end{cases} \to \boxed{C=-3}$$

$$\int\frac{8x^3+13x}{\left(x^2+2\right)^2}\,dx=\int\frac{8x}{x^2+2}\,dx-\int\frac{3x}{\left(x^2+2\right)^2}\,dx$$

$$=2\int\frac{1}{x^2+2}\,d\left(x^2+2\right)\ -\ \frac{3}{2}\int\frac{1}{\left(x^2+2\right)^2}\,d\left(x^2+2\right)$$

$$\boxed{=2\ln\left(x^2+2\right)+\frac{3}{2}\frac{1}{x^2+2}+C}$$

Exercise

Evaluate $\displaystyle\int\frac{\sin x}{\cos x+\cos^2 x}\,dx$

Solution

$$\frac{\sin x}{\cos x+\cos^2 x}=\frac{A}{\cos x}+\frac{B}{1+\cos x}$$

$$A+A\cos x+B\cos x=\sin x$$

$$\begin{cases} \boxed{A = \sin x} \\ A + B = 0 \end{cases} \quad \to \quad \boxed{B = -\sin x}$$

$$\int \frac{\sin x}{\cos x + \cos^2 x}\, dx = \int \frac{\sin x}{\cos x} dx - \int \frac{\sin x}{1 + \cos x} dx$$

$$= -\int \frac{1}{\cos x}\, d(\cos x) + \int \frac{1}{1 + \cos x}\, d(1 + \cos x)$$

$$= -\ln\left|\cos x\right| + \ln\left|1 + \cos x\right| + C$$

$$\boxed{= \ln\left|\frac{1 + \cos x}{\cos x}\right| + C}$$

$$\boxed{= \ln\left|\sec x + 1\right| + C}$$

Exercise

Evaluate $\displaystyle \int \frac{5\cos x}{\sin^2 x + 3\sin x - 4}\, dx$

Solution

$$\frac{5\cos x}{\sin^2 x + 3\sin x - 4} = \frac{A}{\sin x - 1} + \frac{B}{\sin x + 4}$$

$$A\sin x + 4A + B\sin x - B = 5\cos x$$

$$\begin{cases} 4A - B = 5\cos x \\ \quad A + B = 0 \end{cases}$$

$$\boxed{A = \cos x} \quad \boxed{B = -\cos x}$$

$$\int \frac{5\cos x}{\sin^2 x + 3\sin x - 4}\, dx = \int \frac{\cos x}{\sin x - 1} dx - \int \frac{\cos x}{\sin x + 4} dx$$

$$= \int \frac{1}{\sin x - 1} d(\sin x - 1) - \int \frac{1}{\sin x + 4} d(\sin x + 4)$$

$$= \ln\left|\sin x - 1\right| - \ln\left|\sin x + 4\right| + C$$

$$\boxed{= \ln\left|\frac{\sin x - 1}{\sin x + 4}\right| + C}$$

Exercise

Evaluate $\displaystyle \int \frac{e^x}{\left(e^x - 1\right)\left(e^x + 4\right)}\, dx$

Solution

Let $\quad u = e^x \quad \rightarrow \quad du = e^x dx$

$$\int \frac{e^x}{\left(e^x - 1\right)\left(e^x + 4\right)}\, dx = \int \frac{du}{(u-1)(u+4)}$$

$$\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4}$$

$$Au + 4A + Bu - B = 1$$

$$\Rightarrow \begin{cases} A + B = 0 \\ 4A - B = 1 \end{cases} \rightarrow \quad \boxed{A = \frac{1}{5}, \quad B = -\frac{1}{5}}$$

$$\int \frac{du}{(u-1)(u+4)} = \frac{1}{5}\int \frac{1}{u-1}\, du \; + \; \frac{4}{5}\int \frac{1}{u+4}\, du$$

$$= \frac{1}{5}\int \frac{1}{u-1}\, d(u-1) \; + \; \frac{4}{5}\int \frac{1}{u+4}\, d(u+4)$$

$$= \frac{1}{5}\ln\left|e^x - 1\right| - \frac{1}{5}\ln\left(e^x + 4\right) + C$$

$$\boxed{= \frac{1}{5}\ln\left|\frac{e^x - 1}{e^x + 4}\right| + C}$$

Exercise

Evaluate $\qquad \displaystyle\int \frac{e^x}{\left(e^{2x} + 1\right)\left(e^x - 1\right)}\, dx$

Solution

Let $\quad u = e^x \quad \rightarrow \quad du = e^x dx$

$$\int \frac{e^x}{\left(e^{2x} + 1\right)\left(e^x - 1\right)}\, dx = \int \frac{du}{\left(u^2 + 1\right)(u-1)}$$

$$\frac{1}{\left(u^2 + 1\right)(u-1)} = \frac{Au + B}{u^2 + 1} + \frac{C}{u-1}$$

$$Au^2 - Au + Bu - B + Cu^2 + C = 1$$

$$\begin{cases} u^2 & A+C=0 \\ u^1 & -A+B=0 \rightarrow \begin{cases} B + C = 0 \\ -B + C = 1 \end{cases} \\ u^0 & -B+C=1 \end{cases}$$

$$\boxed{C=\frac{1}{2}\quad B=-\frac{1}{2}\quad A=-\frac{1}{2}}$$

$$\int\frac{du}{\left(u^2+1\right)(u-1)}=-\frac{1}{2}\int\frac{u}{u^2+1}du-\frac{1}{2}\int\frac{du}{u^2+1}+\frac{1}{2}\int\frac{du}{u-1}$$

$$=-\frac{1}{4}\int\frac{1}{u^2+1}d\left(u^2+1\right)-\frac{1}{2}\arctan u+\frac{1}{2}\ln|u-1|$$

$$\boxed{=-\frac{1}{4}\ln\left(e^{2x}+1\right)-\frac{1}{2}\arctan e^x+\frac{1}{2}\ln\left|e^x-1\right|+C}$$

Exercise

Evaluate $\displaystyle\int\frac{\sqrt{x}}{x-4}\,dx$

Solution

Let $u=\sqrt{x}$

$$du=\frac{1}{2\sqrt{x}}dx\ \Rightarrow\ 2udu=dx$$

$$\int\frac{\sqrt{x}}{x-4}\,dx=\int\frac{u}{u^2-4}\ 2u\ du$$

$$=\int\frac{2u^2}{u^2-4}\,du$$

$$=\int\left(2+\frac{8}{u^2-4}\right)du$$

$$\frac{8}{u^2-4}=\frac{A}{u-2}+\frac{B}{u+2}$$

$$Au+2A+Bu-2B=8$$

$$\begin{cases}A+B=0\\2A-2B=8\end{cases}\Rightarrow\ \boxed{A=2\quad B=-2}$$

$$=\int\left(2+\frac{2}{u-2}-\frac{2}{u+2}\right)du$$

$$=2\sqrt{x}+2\ln\left|\sqrt{x}-2\right|-2\ln\left|\sqrt{x}+2\right|+C$$

$$\boxed{=2\sqrt{x}+2\ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$

Solution

Let $u = x^{1/6} \rightarrow u^6 = x \quad \rightarrow \quad 6u^5du = dx$

$u^2 = x^{1/3} \quad u^3 = x^{1/2}$

$$\begin{array}{r|l} & 6u^2+6u+6 \\ \hline u-1 & 6u^3 \\ & \underline{-6u^3+6u^2} \\ & \quad 6u^2 \\ & \quad \underline{-6u^2+6u} \\ & \qquad 6u \\ & \qquad \underline{-6u+6} \\ & \qquad\quad 6 \end{array}$$

$$\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx = \int \frac{6u^5}{u^3-u^2}\,du$$

$$= \int \frac{6u^3}{u-1}\,du$$

$$= \int \left(6u^2+6u+6+\frac{6}{u-1}\right)du$$

$$= 2u^3+3u^2+6u+6\ln|u-1|+C$$

$$\boxed{= 2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+6\ln\left|\sqrt[6]{x}-1\right|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x^2-9}\,dx$

Solution

$$\frac{1}{x^2-9} = \frac{A}{x-3}+\frac{B}{x+3}$$

$$Ax+3A+Bx-3B=1$$

$$\begin{cases} A+B=0 \\ 3A-3B=1 \end{cases} \rightarrow \boxed{A=\frac{1}{6} \quad B=-\frac{1}{6}}$$

$$\int \frac{1}{x^2-9}\,dx = \frac{1}{6}\int \frac{1}{x-3}\,dx - \frac{1}{6}\int \frac{1}{x+3}\,dx$$

$$= \frac{1}{6}\ln|x-3| - \frac{1}{6}\ln|x+3| + C$$

$$\boxed{= \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{2}{9x^2-1}\,dx$

Solution

$$\frac{2}{9x^2-1}=\frac{A}{3x-1}+\frac{B}{3x+1}$$

$$3Ax+A+3Bx-B=2$$

$$\begin{cases}3A+3B=0\\ A-B=2\end{cases} \rightarrow \boxed{A=1 \quad B=-1}$$

$$\int \frac{2}{9x^2-1}\,dx=\int \frac{1}{3x-1}\,dx-\int \frac{1}{3x+1}\,dx$$

$$=\frac{1}{3}\ln|3x-1|-\frac{1}{3}\ln|3x+1|+C$$

$$\boxed{=\frac{1}{3}\ln\left|\frac{3x-1}{3x+1}\right|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{5}{x^2+3x-4}\,dx$

Solution

$$\frac{5}{x^2+3x-4}=\frac{A}{x-1}+\frac{B}{x+4}$$

$$Ax+4A+Bx-B=5$$

$$\begin{cases}A+B=0\\ 4A-B=5\end{cases} \rightarrow \boxed{A=1 \quad B=-1}$$

$$\int \frac{5}{x^2+3x-4}\,dx=\int \frac{1}{x-1}\,dx-\int \frac{1}{x+4}\,dx$$

$$=\ln|x-1|-\ln|x+4|+C$$

$$\boxed{=\ln\left|\frac{x-1}{x+4}\right|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{3-x}{3x^2-2x-1}\,dx$

Solution

$$\frac{3-x}{3x^2-2x-1}=\frac{A}{x-1}+\frac{B}{3x+1}$$

$$3Ax+A+Bx-B=3-x$$

$$\begin{cases}3A+B=-1\\ A-B=3\end{cases}\to \boxed{A=\frac{1}{2}\quad B=-\frac{5}{2}}$$

$$\int\frac{3-x}{3x^2-2x-1}\,dx=\frac{1}{2}\int\frac{1}{x-1}\,dx-\frac{5}{2}\int\frac{1}{3x+1}\,dx$$

$$\boxed{=\frac{1}{2}\ln|x-1|-\frac{5}{6}\ln|3x+1|+C}$$

Exercise

Evaluate $\displaystyle\int\frac{x^2+12x+12}{x^3-4x}\,dx$

Solution

$$\frac{x^2+12x+12}{x^3-4x}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2}$$

$$Ax^2-4A+Bx^2+2Bx+Cx^2-2Cx=x^2+12x+12$$

$$\begin{cases}x^2 & A+B+C=1\\ x^1 & 2B-2C=12\\ x^0 & -4A=12\quad\to\boxed{A=-3}\end{cases}$$

$$\begin{cases}B+C=4\\ B-C=6\end{cases}\quad\boxed{B=5\quad C=-1}$$

$$\int\frac{x^2+12x+12}{x^3-4x}\,dx=-\frac{3}{x}+\frac{5}{x-2}-\frac{1}{x+2}$$

$$\boxed{=-3\ln|x|+5\ln|x-2|-\ln|x+2|+C}$$

Exercise

Evaluate $\displaystyle\int\frac{x^3-x+3}{x^2+x-2}\,dx$

Solution

$$\frac{x^3-x+3}{x^2+x-2}=x-1+\frac{2x+1}{x^2+x-2}$$

$$\begin{array}{r} x-1\\ x^2+x-2\,\overline{)\,x^3-x+3}\\ \underline{-x^3-x^2+2x}\\ -x^2+x+3\\ \underline{x^2+x-2}\\ 2x-1\end{array}$$

$$\frac{2x+1}{x^2+x-2}=\frac{A}{x-1}+\frac{B}{x+2}$$

$$Ax+2A+Bx-B=2x+1$$

$$\begin{cases} A+B=2 \\ 2A-B=1 \end{cases} \rightarrow \boxed{A=1 \quad B=1}$$

$$\int \frac{x^3-x+3}{x^2+x-2}\,dx=\int\left(x-1+\frac{1}{x-1}+\frac{1}{x+2}\right)dx$$

$$\boxed{=\frac{1}{2}x^2-x+\ln|x-1|+\ln|x+2|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{5x-2}{(x-2)^2}\,dx$

Solution

$$\frac{5x-2}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}$$

$$Ax-2A+B=5x-2$$

$$\Rightarrow\begin{cases} \boxed{A=5} \\ -2A+B=-2 \end{cases} \rightarrow \boxed{B=8}$$

$$\int \frac{5x-2}{(x-2)^2}\,dx=\frac{5}{x-2}+\frac{8}{(x-2)^2}$$

$$\boxed{=5\ln|x-2|-\frac{8}{x-2}+C}$$

Exercise

Evaluate $\displaystyle\int \frac{2x^3-4x^2-15x+4}{x^2-2x-8}\,dx$

Solution

$$\begin{array}{r|l} & \quad\quad 2x \\ \hline x^2-2x-8 & 2x^3-4x^2-15x+4 \\ & \underline{2x^3-4x^2-16x} \\ & \quad\quad\quad\quad x+4 \end{array}$$

$$\int \frac{2x^3-4x^2-15x+4}{x^2-2x-8}\,dx=\int 2x\,dx+\int \frac{x+4}{x^2-2x-8}\,dx$$

$$\frac{x+4}{x^2-2x-8}=\frac{A}{x-4}+\frac{B}{x+2}$$

$$Ax+2A+Bx-4B=x+4$$

$$\begin{cases} A+B=1 \\ 2A-4B=4 \end{cases} \rightarrow \boxed{A=\frac{4}{3} \quad B=-\frac{1}{3}}$$

$$\int \frac{2x^3-4x^2-15x+4}{x^2-2x-8}\,dx = x^2+\frac{4}{3}\int \frac{1}{x-4}\,dx-\frac{1}{3}\int \frac{1}{x+2}\,dx$$

$$\boxed{=x^2+\frac{4}{3}\ln|x-4|-\frac{1}{3}\ln|x+2|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{x+2}{x^2+5x}\,dx$

Solution

$$\frac{x+2}{x^2+5x}=\frac{A}{x}+\frac{B}{x+5}$$

$$Ax+5A+Bx=x+2$$

$$\begin{cases} A+B=1 \\ 5A=2 \end{cases} \rightarrow \boxed{A=\frac{2}{5} \quad B=\frac{3}{5}}$$

$$\int \frac{x+2}{x^2+5x}\,dx=\frac{2}{5}\int \frac{1}{x}\,dx+\frac{3}{5}\int \frac{1}{x+5}\,dx$$

$$\boxed{=\frac{2}{5}\ln|x|+\frac{3}{5}\ln|x+5|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{\sec^2 x}{\tan^2 x+5\tan x+6}\,dx$

Solution

Let $u=\tan x \quad du=\sec^2 x\,dx$

$$\int \frac{\sec^2 x}{\tan^2 x+5\tan x+6}\,dx=\int \frac{1}{u^2+5u+6}\,du$$

$$\frac{1}{u^2+5u+6}=\frac{1}{(u+2)(u+3)}=\frac{A}{u+2}+\frac{B}{u+3}$$

$$1=Au+3A+Bu+2B$$

$$\begin{cases} A+B=0 \\ 3A+2B=1 \end{cases} \rightarrow \boxed{A=1, \quad B=-1}$$

$$\int \frac{\sec^2 x}{\tan^2 x+5\tan x+6}\,dx = \int \frac{1}{u+2}du - \int \frac{1}{u+3}du$$

$$= \ln|\tan x+2| - \ln|\tan x+3| + C$$

$$\boxed{= \ln\left|\frac{\tan x+2}{\tan x+3}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{\sec^2 x}{\tan x(\tan x+1)}\,dx$

Solution

Let $u=\tan x \quad du=\sec^2 x\,dx$

$$\int \frac{\sec^2 x}{\tan x(\tan x+1)}\,dx = \int \frac{du}{u(u+1)}$$

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

$$1 = Au + A + Bu$$

$$\begin{cases} A+B=0 \\ \boxed{A=1} \end{cases} \rightarrow \boxed{B=-1}$$

$$\int \frac{\sec^2 x}{\tan x(\tan x+1)}\,dx = \int \frac{du}{u} - \int \frac{du}{u+1}$$

$$= \ln|\tan x| - \ln|\tan x+1| + C$$

$$\boxed{= \ln\left|\frac{\tan x}{\tan x+1}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x\;dx}{x^2+4x+3}$

Solution

$$\frac{x}{x^2+4x+3} = \frac{A}{x+1} + \frac{B}{x+3}$$

$$x = Ax+3A+Bx+B$$

$$\begin{cases} x & A+B=1 \\ x^0 & 3A+B=0 \end{cases} \rightarrow \boxed{A=-\frac{1}{2} \quad B=\frac{3}{2}}$$

$$\int \frac{x\ dx}{x^2+4x+3} = -\frac{1}{2}\int \frac{dx}{x+1} + \frac{3}{2}\int \frac{dx}{x+3}$$

$$= -\frac{1}{2}\ln|x+1| + \frac{3}{2}\ln|x+3| + C$$

$$= \frac{1}{2}\left(3\ln|x+3| - \ln|x+1|\right) + C$$

$$= \frac{1}{2}\ln\left|\frac{(x+3)^3}{x+1}\right| + C$$

$$\boxed{= \ln\sqrt{\left|\frac{(x+3)^3}{x+1}\right|} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x+1}{x^2(x-1)}\,dx$

Solution

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$x+1 = Ax^2 - Ax + Bx - B + Cx^2$$

$$\begin{array}{lll} x^2 & A+C=0 & \boxed{C=2} \\ x & -A+B=1 & \boxed{A=-2} \\ x^0 & -B=1 & \rightarrow \boxed{B=-1} \end{array}$$

$$\int \frac{x+1}{x^2(x-1)}\,dx = -2\int \frac{dx}{x} - \int \frac{dx}{x^2} + 2\int \frac{dx}{x-1}$$

$$\boxed{= -2\ln|x| + \frac{1}{x} + 2\ln|x-1| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\,dx$

Solution

$$\begin{array}{r} 2x-3 \\ x^2-2x-8\overline{\smash{)}\,2x^3+x^2-21x+24} \\ \underline{2x^3-4x^2-16x} \\ -3x^2-5x+24 \\ \underline{-3x^2-6x+24} \\ x \end{array}$$

$$\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\,dx = \int\left(2x-3+\frac{x}{x^2+2x-8}\right)dx$$

$$\frac{x}{x^2+2x-8} = \frac{A}{x+4}+\frac{B}{x-2}$$

$$x = Ax-2A+Bx+4B$$

$$\begin{cases} x & A+B=1 \\ x^0 & -2A+4B=0 \end{cases} \rightarrow \ B=\frac{1}{3} \quad A=\frac{2}{3}$$

$$\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\,dx = \int(2x-3)\,dx+\frac{2}{3}\int\frac{dx}{x+4}+\frac{1}{3}\int\frac{dx}{x-2}$$

$$\boxed{=x^2-3x+\frac{2}{3}\ln|x+4|+\frac{1}{3}\ln|x-2|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{8x+5}{2x^2+3x+1}\,dx$

Solution

$$\frac{8x+5}{2x^2+3x+1} = \frac{A}{2x+1}+\frac{B}{x+1}$$

$$8x+5 = Ax+A+2Bx+B$$

$$\begin{cases} x & A+2B=8 \\ x^0 & A+B=5 \end{cases} \rightarrow \ B=3 \quad A=2$$

$$\int \frac{8x+5}{2x^2+3x+1}\,dx = \int\frac{2}{2x+1}\,dx+\int\frac{3}{x+1}\,dx$$

$$=\int\frac{1}{2x+1}\,d(2x+1)+3\int\frac{1}{x+1}\,d(x+1)$$

$$\boxed{=\ln|2x+1|+3\ln|x+1|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{3x^3+4x^2+6x}{(x+1)^2\left(x^2+4\right)}\,dx$

Solution

$$\frac{3x^3+4x^2+6x}{(x+1)^2\left(x^2+4\right)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+4}$$

$$3x^3+4x^2+6x = A(x+1)\left(x^2+4\right)+Bx^2+4B+(Cx+D)\left(x^2+2x+1\right)$$

$$= Ax^3+4Ax+Ax^2+4A+Bx^2+4B+Cx^3+2Cx^2+Cx+Dx^2+2Dx+D$$

$$\begin{cases} x^3 & A+C=3 \qquad \rightarrow A=3-C \\ x^2 & A+B+2C+D=4 \\ x & 4A+C+2D=6 \\ x^0 & 4A+4B+D=0 \end{cases}$$

$$\begin{cases} B+C+D=1 \\ -3C+2D=-6 \\ 4B-4C+D=-12 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & 2 \\ 4 & -4 & 1 \end{vmatrix} = 25 \qquad \Delta_B = \begin{vmatrix} 1 & 1 & 1 \\ -6 & -3 & 2 \\ -12 & -4 & 1 \end{vmatrix} = -25 \qquad \Delta_C = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -6 & 2 \\ 4 & -12 & 1 \end{vmatrix} = 50$$

$$B = \frac{-25}{25}\ \underline{=-1}\big| \qquad C = \frac{50}{25}\ \underline{=2}\big| \quad \rightarrow \quad A = 3-2\ \underline{=1}\big|$$

$$2D = -6+6 \quad \rightarrow \quad \underline{D=0}\big|$$

$$\int \frac{3x^3+4x^2+6x}{(x+1)^2\left(x^2+4\right)}\,dx = \int \frac{1}{x+1}\,dx - \int \frac{1}{(x+1)^2}\,dx + \int \frac{2x}{x^2+4}\,dx$$

$$= \int \frac{1}{x+1}\,d(x+1) - \int \frac{1}{(x+1)^2}\,d(x+1) + \int \frac{1}{x^2+4}\,d\left(x^2+4\right)$$

$$\underline{= \ln|x+1| + \frac{1}{x+1} + \ln\left(x^2+4\right) + K}\Big|$$

Exercise

Evaluate $\displaystyle\int \frac{x^2-4}{x^2+4}\,dx$

Solution

$$\begin{array}{r|l} & \quad 1 \\ \hline x^2+4 & x^2-4 \\ & \underline{x^2+4} \\ & \quad 8 \end{array}$$

$$\int \frac{x^2-4}{x^2+4}\,dx = \int\left(1+\frac{8}{x^2+4}\right)dx$$

$$\boxed{= x+8\arctan\frac{x}{2}+C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{x^2-2x-15}$

Solution

$$\frac{1}{x^2-2x-15}=\frac{A}{x-5}+\frac{B}{x+3}$$

$$1 = Ax+3A+Bx-5B$$

$$\begin{cases} x & A+B=0 \qquad \rightarrow B=-A \\ x^0 & 3A-5B=1 \quad \rightarrow 8A=1 \;\Rightarrow\; \boxed{A=\frac{1}{8}} \end{cases}$$

$$\boxed{B=-\frac{1}{8}}$$

$$\int \frac{dx}{x^2-2x-15}=\frac{1}{8}\int\frac{dx}{x-5}-\frac{1}{8}\int\frac{dx}{x+3}$$

$$=\frac{1}{8}\ln|x-5|-\frac{1}{8}\ln|x+3|+C$$

$$\boxed{=\frac{1}{8}\ln\left|\frac{x-5}{x+3}\right|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{3x^2+x-3}{x^2-1}\,dx$

Solution

$$\begin{array}{r|l} & \quad 3 \\ \hline x^2-1 & 3x^2+x-3 \\ & \underline{3x^2-3} \\ & \quad x \end{array}$$

$$\int \frac{3x^2+x-3}{x^2-1}\,dx = \int \left(3+\frac{x}{x^2-1}\right) dx$$

$$= 3x + \frac{1}{2}\int \frac{1}{x^2-1}\, d\left(x^2-1\right)$$

$$\boxed{= 3x + \frac{1}{2}\ln\left|x^2-1\right| + C}$$

Exercise

Evaluate $\displaystyle \int \frac{2x^2-4x}{x^2-4}\,dx$

Solution

$$\begin{array}{r} 2 \\ x^2-4\,\overline{\big)\,2x^2-4x} \\ \underline{2x^2 \quad -8} \\ -4x+8 \end{array}$$

$$\int \frac{2x^2-4x}{x^2-4}\,dx = \int \left(2-4\frac{x-2}{x^2-4}\right) dx$$

$$= \int \left(2-4\frac{x-2}{(x-2)(x+2)}\right) dx$$

$$= \int \left(2-\frac{4}{x+2}\right) dx$$

$$\boxed{= 2x + 4\ln|x+2| + C}$$

Exercise

Evaluate $\displaystyle \int \frac{dx}{x^3-2x^2}$

Solution

$$\frac{1}{x^3-2x^2} = \frac{1}{x^2(x-2)}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$1 = Ax(x-2) + Bx - 2B + Cx^2$$

$$x^2 \qquad A+C=0 \qquad \boxed{C=\frac{1}{4}}$$

$$x \qquad -2A+B=0 \qquad \boxed{A=-\frac{1}{4}}$$

$$x^0 \qquad -2B=1 \qquad \to \boxed{B=-\frac{1}{2}}$$

$$\int \frac{dx}{x^3-2x^2} = -\frac{1}{4}\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{1}{x^2}\,dx + \frac{1}{4}\int \frac{1}{x-2}\,dx$$

$$= -\frac{1}{4}\ln|x| + \frac{1}{2}\frac{1}{x} + \frac{1}{4}\ln|x-2| + K$$

$$\boxed{= \frac{1}{2x} + \frac{1}{4}\ln\left|\frac{x-2}{x}\right| + K}$$

Exercise

Evaluate $\displaystyle \int \frac{dx}{x^2-x-2}$

Solution

$$\frac{1}{x^2-x-2} = \frac{A}{x+1} + \frac{B}{x-2}$$

$$1 = Ax - 2A + Bx + B$$

$$\begin{cases} x & A+B=0 \\ x^0 & -2A+B=1 \end{cases}$$

$$\boxed{A=-\frac{1}{3} \quad B=\frac{1}{3}}$$

$$\int \frac{dx}{x^2-x-2} = -\frac{1}{3}\int \frac{dx}{x+1} + \frac{1}{3}\int \frac{dx}{x-2}$$

$$= -\frac{1}{3}\ln|x+1| + \frac{1}{3}\ln|x-2| + K$$

$$\boxed{= \frac{1}{3}\ln\left|\frac{x-2}{x+1}\right| + C}$$

Exercise

Evaluate $\displaystyle \int \frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx$

Solution

$$\frac{4x^2+13x-9}{x^3+2x^2-3x}=\frac{4x^2+13x-9}{x\left(x^2+2x-3\right)}$$

$$=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+3}$$

$$4x^2+13x-9=A\left(x^2+2x-3\right)+Bx(x+3)+Cx(x-1)$$

$$x^2 \qquad A+B+C=4$$

$$x \qquad 2A+3B-C=13$$

$$x^0 \qquad -3A=-9 \qquad \to \underline{A=3}$$

$$\begin{cases} B+C=1 \\ 3B-C=7 \end{cases} \qquad \to \underline{B=2 \quad C=-1}$$

$$\int \frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx=\int\left(\frac{3}{x}+\frac{2}{x-1}-\frac{1}{x+3}\right)dx$$

$$\underline{=3\ln|x|+2\ln|x-1|-\ln|x+3|+K}$$

Exercise

Evaluate $\displaystyle\int \frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3}\,dx$

Solution

$$\frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3}=\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}+\frac{D}{(x-2)^3}$$

$$3x^3-18x^2+29x-4=A(x-2)^3+B(x+1)(x-2)^2+C(x+1)(x-2)+Dx+D$$

$$=A\left(x^3-6x^2+12x-8\right)+B(x+1)\left(x^2-4x+4\right)+C\left(x^2-x-2\right)+Dx+D$$

$$x^3 \qquad A+B=3 \qquad \to A=3-B$$

$$x^2 \qquad -6A-3B+C=-18$$

$$x \qquad 12A-C+D=29$$

$$x^0 \qquad -8A+4B-2C+D=-4$$

$$\begin{cases} -18+6B-3B+C=-18 \\ 36-12B-C+D=29 \\ -24+8B+4B-2C+D=-4 \end{cases}$$

$$\begin{cases} 3B + C = 0 \\ -12B - C + D = -7 \\ 12B - 2C + D = 20 \end{cases}$$

$$\Delta = \begin{vmatrix} 3 & 1 & 0 \\ -12 & -1 & 1 \\ 12 & -2 & 1 \end{vmatrix} = 27 \qquad \Delta_B = \begin{vmatrix} 0 & 1 & 0 \\ -7 & -1 & 1 \\ 20 & -2 & 1 \end{vmatrix} = 27$$

$B = \frac{27}{27} \underline{= 1}$

$C = -3B \underline{= -3}$

$D = -7 + 12 - 3 \underline{= 2}$

$A = 3 - 1 \underline{= 2}$

$$\int \frac{3x^3 - 18x^2 + 29x - 4}{(x+1)(x-2)^3} dx = \int \frac{2}{x+1} dx + \int \frac{1}{x-2} dx - 3\int \frac{1}{(x-2)^2} dx + 2\int \frac{1}{(x-2)^3} dx$$

$$= 2\int \frac{d(x+1)}{x+1} + \int \frac{d(x-2)}{x-2} - 3\int \frac{d(x-2)}{(x-2)^2} + 2\int (x-2)^{-3}\, d(x-2)$$

$$= 2\ln|x+1| + \ln|x-2| + \frac{3}{x-2} - \frac{1}{(x-2)^2} + K$$

Exercise

Evaluate $\displaystyle\int \frac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4} dx$

Solution

$$2x^3 - x^2 + 8x - 4 = x^2(2x-1) + 4(2x-1)$$

$$= (2x-1)\left(x^2 + 4\right)$$

$$\frac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4} = \frac{A}{2x-1} + \frac{Bx + C}{x^2 + 4}$$

$$x^2 - x - 21 = Ax^2 + 4A + 2Bx^2 - Bx + 2Cx - C$$

$x^2 \quad A + 2B = 1 \quad A = 1 - 2B$

$x \quad -B + 2C = -1 \quad C = \frac{1}{2}(B-1)$

$x^0 \quad 4A - C = -21 \quad (1)$

$(1) \quad 4 - 8B - \frac{1}{2}B + \frac{1}{2} = -21$

$$-\frac{17}{2}B=-21-\frac{9}{2}$$

$$\frac{17}{2}B=\frac{51}{2}\quad\to\quad \underline{B=3}$$

$$C=\frac{1}{2}(3-1)\ \underline{=1}$$

$$A=1-6\ \underline{=5}$$

$$\int \frac{x^2-x-21}{2x^3-x^2+8x-4}\,dx=\int \frac{5}{2x-1}\,dx+\int \frac{3x}{x^2+4}\,dx+\int \frac{1}{x^2+4}\,dx$$

$$=\frac{5}{2}\int \frac{1}{2x-1}\,d(2x-1)+\frac{3}{2}\int \frac{1}{x^2+4}\,d\left(x^2+4\right)+\int \frac{1}{x^2+4}\,dx$$

$$\underline{=\frac{5}{2}\ln|2x-1|+\frac{3}{2}\ln\left(x^2+4\right)+\frac{1}{2}\arctan\left(\frac{x}{2}\right)+K}$$

Exercise

Evaluate $\displaystyle\int \frac{5x^3-3x^2+7x-3}{\left(x^2+1\right)^2}\,dx$

Solution

$$\frac{5x^3-3x^2+7x-3}{\left(x^2+1\right)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{\left(x^2+1\right)^2}$$

$$5x^3-3x^2+7x-3=Ax^3+Ax+Bx^2+B+Cx+D$$

x^3 $\quad \underline{A=5}$

x^2 $\quad \underline{B=-3}$

x $\quad A+C=7 \quad\to\quad \underline{C=2}$

x^0 $\quad B+D=-3 \quad\to\quad \underline{D=0}$

$$\int \frac{5x^3-3x^2+7x-3}{\left(x^2+1\right)^2}\,dx=\int \frac{5x-3}{x^2+1}\,dx+\int \frac{2x}{\left(x^2+1\right)^2}\,dx$$

$$=\frac{5}{2}\int \frac{1}{x^2+1}\,d\left(x^2+1\right)-3\int \frac{1}{x^2+1}\,dx+\int \frac{1}{\left(x^2+1\right)^2}\,d\left(x^2+1\right)$$

$$\underline{=\frac{5}{2}\ln\left(x^2+1\right)-3\arctan(x)-\frac{1}{x^2+1}+K}$$

Exercise

Evaluate $\displaystyle\int \frac{2x^4-2x^3+6x^2-5x+1}{x^3-x^2+x-1}\,dx$

Solution

$$\begin{array}{r l}
 & \qquad\qquad 2x \\
x^3-x^2+x-1 & \big)\overline{\,2x^4-2x^3+6x^2-5x+1} \\
 & \;\;\underline{2x^4-2x^3-2x^2+2x} \\
 & \qquad\quad 8x^2-7x+1
\end{array}$$

$$\frac{2x^4-2x^3+6x^2-5x+1}{x^3-x^2+x-1}=2x+\frac{8x^2-7x+1}{x^3-x^2+x-1}$$

$$\begin{aligned}
x^3-x^2+x-1 &= x^2(x-1)+(x-1) \\
&= (x-1)\left(x^2+1\right)
\end{aligned}$$

$$\frac{8x^2-7x+1}{x^3-x^2+x-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}$$

$$8x^2-7x+1=Ax^2+A+Bx^2-Bx+Cx-C$$

$$\begin{array}{lll}
x^2 & A+B=8 & A=8-B \\
x & -B+C=-7 & C=B-7 \\
x^0 & A-C=1 & (1)
\end{array}$$

$$(1)\to\quad 8-B-B+7=1\quad\Rightarrow \underline{B=7}$$

$$A=8-7\ \underline{=1}$$

$$C=7-7\ \underline{=0}$$

$$\begin{aligned}
\int \frac{2x^4-2x^3+6x^2-5x+1}{x^3-x^2+x-1}\,dx &= \int 2x\,dx+\int\frac{1}{x-1}\,dx+\int\frac{7x}{x^2+1}\,dx \\
&= x^2+\ln|x-1|+\frac{7}{2}\int\frac{1}{x^2+1}\,d\left(x^2+1\right) \\
&= \underline{x^2+\ln|x-1|+\tfrac{7}{2}\ln\left(x^2+1\right)+K}
\end{aligned}$$

Exercise

Evaluate $\displaystyle\int \frac{81}{x^3-9x^2}\,dx$

Solution

$$\frac{81}{x^3-9x^2}=\frac{81}{x^2(x-9)}$$

$$=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-9}$$

$$81=Ax^2-9Ax+Bx-9B+Cx^2$$

$$x^2 \qquad A+C=0 \qquad \to \boxed{C=1}$$

$$x \qquad -9A+B=0 \qquad \to \boxed{A=-1}$$

$$x^0 \qquad -9B=81 \quad \to \boxed{B=-9}$$

$$\int \frac{81}{x^3-9x^2}\,dx=\int\left(-\frac{1}{x}-\frac{9}{x^2}+\frac{1}{x-9}\right)dx$$

$$=-\ln|x|+\frac{9}{x}+\ln|x-9|+K$$

$$\boxed{=\frac{9}{x}+\ln\left|\frac{x-9}{x}\right|+K}$$

Exercise

Evaluate $\displaystyle\int \frac{10x}{x^2-2x-24}\,dx$

Solution

$$\frac{10x}{x^2-2x-24}=\frac{A}{x-6}+\frac{B}{x+4}$$

$$10x=Ax+4A+Bx-6B$$

$$x \qquad A+B=10$$

$$x^0 \qquad 4A-6B=0$$

$$\Delta=\begin{vmatrix}1 & 1\\ 4 & -6\end{vmatrix}=-10 \qquad \Delta_A=\begin{vmatrix}10 & 1\\ 0 & -6\end{vmatrix}=-60$$

$$\boxed{A=6 \quad B=4}$$

$$\int \frac{10x}{x^2-2x-24}\,dx=\int\left(\frac{6}{x-6}+\frac{4}{x+4}\right)dx$$

$$\boxed{=6\ln|x-6|+4\ln|x+4|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{x+1}{x^2\left(x^2+4\right)}\,dx$

Solution

$$\frac{x+1}{x^2\left(x^2+4\right)} = \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+4}$$

$$x+1 = Ax^3+4Ax+Bx^2+4B+Cx^3+Dx^2$$

$$x^3 \quad A+C=0 \qquad \to \boxed{C=-\frac{1}{4}}$$

$$x^2 \quad B+D=0 \qquad \to \boxed{D=-\frac{1}{4}}$$

$$x \quad 4A=1 \qquad \to \boxed{A=\frac{1}{4}}$$

$$x^0 \quad 4B=1 \qquad \to \boxed{B=\frac{1}{4}}$$

$$\int \frac{x+1}{x^2\left(x^2+4\right)}\,dx = \frac{1}{4}\int \frac{1}{x}\,dx + \frac{1}{4}\int \frac{1}{x^2}\,dx - \frac{1}{4}\int \frac{x}{x^2+4}\,dx - \frac{1}{4}\int \frac{1}{x^2+4}\,dx$$

$$= \frac{1}{4}\ln|x| - \frac{1}{4}\frac{1}{x} - \frac{1}{8}\int \frac{x}{x^2+4}\,d\left(x^2+4\right) - \frac{1}{8}\arctan\left(\frac{x}{2}\right)+K$$

$$\boxed{= \frac{1}{4}\ln|x| - \frac{1}{4x} - \frac{1}{8}\ln\left(x^2+4\right) - \frac{1}{8}\arctan\left(\frac{x}{2}\right)+K}$$

Exercise

Evaluate $\displaystyle\int \frac{1+x^2}{(x+1)^3}\,dx$

Solution

$$\frac{1+x^2}{(x+1)^3} = \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$

$$x^2+1 = A(x+1)^2+Bx+B+C$$

$$x^2 \quad \boxed{A=1}$$

$$x \quad 2A+B=0 \quad \to \boxed{B=-2}$$

$$x^0 \quad A+B+C=1 \quad \to \boxed{C=2}$$

$$\int \frac{1+x^2}{(x+1)^3}\,dx = \int \frac{1}{x+1}\,dx - \int \frac{2}{(x+1)^2}\,dx + \int \frac{2}{(x+1)^3}\,dx$$

$$= \int \frac{1}{x+1}\, d(x+1) - 2\int \frac{1}{(x+1)^2}\, d(x+1) + 2\int (x+1)^{-3}\, d(x+1)$$

$$= \ln|x+1| + \frac{2}{x+1} - \frac{1}{(x+1)^2} + K$$

Exercise

Evaluate $\int \frac{6}{x^2-1}\, dx$

Solution

$$\frac{6}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$6 = Ax + A + Bx - B$$

$$x \quad A + B = 0$$

$$x^0 \quad A - B = 6$$

$$2A = 6 \quad \rightarrow \quad A = 3 \quad B = -3$$

$$\int \frac{6}{x^2-1}\, dx = \int \frac{3}{x-1}\, dx - \int \frac{3}{x+1}\, dx$$

$$= 3\ln|x-1| - 3\ln|x+1| + C$$

$$= 3\ln\left|\frac{x-1}{x+1}\right| + C$$

Exercise

Evaluate $\int \frac{21x^2}{x^3-x^2-12x}\, dx$

Solution

$$\int \frac{21x^2}{x^3-x^2-12x}\, dx = \int \frac{21x}{x^2-x-12}\, dx$$

$$\frac{21x^2}{x^3-x^2-12x} = \frac{21x^2}{x\left(x^2-x-12\right)}$$

$$= \frac{21x^2}{x(x+3)(x-4)}$$

$$\frac{21x}{x^2-x-12} = \frac{A}{x+3} + \frac{B}{x-4}$$

$$21x = Ax - 4A + Bx + 3B$$

$x \quad A+B=21$

$x^0 \quad -4A+3B=0$

$$\Delta = \begin{vmatrix} 1 & 1 \\ -4 & 3 \end{vmatrix} = 7 \qquad \Delta_A = \begin{vmatrix} 21 & 1 \\ 0 & 3 \end{vmatrix} = 63$$

$\boxed{A=9 \quad B=12}$

$$\int \frac{21x^2}{x^3-x^2-12x}\,dx = \int \frac{9}{x+3}\,dx + \int \frac{12}{x-4}\,dx$$

$$\boxed{= 9\ln|x+3| + 12\ln|x-4| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x+1}{x^3+3x^2-18x}\,dx$

Solution

$$\frac{x+1}{x^3+3x^2-18x} = \frac{x+1}{x\left(x^2+3x-18\right)}$$

$$= \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+6}$$

$x+1 = Ax^2+3Ax-18A+Bx^2+6Bx+Cx^2-3Cx$

$x^2 \quad A+B+C=0$

$x \quad 3A+6B-3C=1$

$x^0 \quad -18A=1 \quad \rightarrow \boxed{A=-\frac{1}{18}}$

$$\begin{cases} B+C=\frac{1}{18} \\ 6B-3C=\frac{7}{6} \end{cases} \rightarrow \begin{cases} 3B+3C=\frac{1}{6} \\ 6B-3C=\frac{7}{6} \end{cases}$$

$9B=\frac{4}{3} \quad \rightarrow \boxed{B=\frac{4}{27}}$

$C=\frac{1}{18}-\frac{4}{27} \boxed{=-\frac{5}{54}}$

$$\int \frac{x+1}{x^3+3x^2-18x}\,dx = \int \left(-\frac{1}{18}\frac{1}{x} + \frac{4}{27}\frac{1}{x-3} - \frac{5}{54}\frac{1}{x+6}\right)dx$$

$$\boxed{= -\frac{1}{18}\ln|x| + \frac{4}{27}\ln|x-3| - \frac{5}{54}\ln|x+6| + K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^2+12x-4}{x^3-4x}\,dx$

Solution

$$\frac{x^2+12x-4}{x^3-4x}=\frac{x^2+12x-4}{x\left(x^2-4\right)}$$

$$=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2}$$

$$x^2+12x-4=Ax^2-4A+Bx^2+2Bx+Cx^2-2Cx$$

$$\begin{aligned}
&x^2 && A+B+C=1\\
&x && 2B-2C=12\\
&x^0 && -4A=-4 \quad \rightarrow \boxed{A=1}
\end{aligned}$$

$$\rightarrow\begin{cases} B+C=0\\ 2B-2C=12\end{cases}$$

$$3B=12 \quad\rightarrow\quad \boxed{B=4} \quad \boxed{C=-4}$$

$$\int \frac{x^2+12x-4}{x^3-4x}\,dx=\int\left(\frac{1}{x}+\frac{4}{x-2}-\frac{4}{x+2}\right)dx$$

$$\boxed{=\ln|x|+4\ln|x-2|-4\ln|x+2|+K}$$

Exercise

Evaluate $\displaystyle\int \frac{6x^2}{x^4-5x^2+4}\,dx$

Solution

$$\frac{6x^2}{x^4-5x^2+4}=\frac{6x^2}{\left(x^2-1\right)\left(x^2-4\right)}$$

$$=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x-2}+\frac{D}{x+2}$$

$$6x^2=A(x+1)\left(x^2-4\right)+B(x-1)\left(x^2-4\right)+C(x+2)\left(x^2-1\right)+D(x-2)\left(x^2-1\right)$$

$$\begin{aligned}
&x^3 && A+B+C+D=0\\
&x^2 && A-B+2C-2D=6\\
&x^1 && -4A-4B-C-D=0\\
&x^0 && -4A+4B-2C+2D=0
\end{aligned}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -2 \\ -4 & -4 & -1 & -1 \\ -4 & 4 & -2 & 2 \end{vmatrix} = 72 \qquad \Delta_A = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 6 & -1 & 2 & -2 \\ 0 & -4 & -1 & -1 \\ 0 & 4 & -2 & 2 \end{vmatrix} = -72$$

$$\Delta_B = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 6 & 2 & -2 \\ -4 & 0 & -1 & -1 \\ -4 & 0 & -2 & 2 \end{vmatrix} = 72 \qquad \Delta_C = \begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 6 & -2 \\ -4 & -4 & 0 & -1 \\ -4 & 4 & 0 & 2 \end{vmatrix} = 144$$

$$\underline{A=-1 \quad B=1 \quad C=2 \quad D=-2}\,\rvert$$

$$\int \frac{6x^2}{x^4-5x^2+4}\,dx = \int \left(\frac{-1}{x-1} + \frac{1}{x+1} + \frac{2}{x-2} - \frac{2}{x+2} \right) dx$$

$$\underline{= -\ln|x-1| + \ln|x+1| + 2\ln|x-2| - 2\ln|x+2| + K}\,\rvert$$

Exercise

Evaluate $\displaystyle\int \frac{4x-2}{x^3-x}\,dx$

Solution

$$\frac{4x-2}{x^3-x} = \frac{4x-2}{x\left(x^2-1\right)}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$4x-2 = Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx$$

$$x^2 \quad A+B+C=0$$

$$x^1 \quad B-C=4$$

$$x^0 \quad -A=-2 \quad \rightarrow \underline{A=2}\,\rvert$$

$$\begin{cases} B+C=-2 \\ B-C=4 \end{cases} \rightarrow \underline{B=1}\,\rvert \quad \underline{C=-3}\,\rvert$$

$$\int \frac{4x-2}{x^3-x}\,dx = \int \left(\frac{2}{x} + \frac{1}{x-1} - \frac{3}{x+1} \right) dx$$

$$\underline{= 2\ln|x| + \ln|x-1| - 3\ln|x+1| + K}\,\rvert$$

Exercise

Evaluate $\displaystyle\int \frac{16x^2}{(x-6)(x+2)^2}\,dx$

Solution

$$\frac{16x^2}{(x-6)(x+2)^2}=\frac{A}{x-6}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$$

$$16x^2=A(x+2)^2+Bx^2-4Bx-12B+Cx-6C$$

$x^2 \quad A+B=16$

$x^1 \quad 4A-4B+C=0$

$x^0 \quad 4A-12B-6C=0$

$$\Delta=\begin{vmatrix}1&1&0\\4&-4&1\\4&-12&-6\end{vmatrix}=64 \qquad \Delta_A=\begin{vmatrix}16&1&0\\0&-4&1\\0&-12&-6\end{vmatrix}=576$$

$$A=\frac{576}{64}\ \underline{=9\,}\big|$$

$$B=16-9\ \underline{=7\,}\big|$$

$$C=-36+28\ \underline{=-8\,}\big|$$

$$\int \frac{16x^2}{(x-6)(x+2)^2}\,dx=\int\frac{9}{x-6}\,dx+\int\frac{7}{x+2}\,dx-\int\frac{8}{(x+2)^2}\,dx$$

$$=\int\frac{9}{x-6}\,d(x-6)+\int\frac{7}{x+2}\,d(x+2)-\int\frac{8}{(x+2)^2}\,d(x+2)$$

$$\underline{=9\ln|x-6|+7\ln|x+2|+\frac{8}{x+2}+K\,}\Big|$$

Exercise

Evaluate $\displaystyle\int \frac{8\left(x^2+4\right)}{x\left(x^2+8\right)}\,dx$

Solution

$$\int \frac{8\left(x^2+4\right)}{x\left(x^2+8\right)}\,dx=8\int\frac{x^2+4}{x\left(x^2+8\right)}\,dx$$

$$\frac{x^2+4}{x\left(x^2+8\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+8}$$

$$x^2+4=Ax^2+8A+Bx^2+Cx$$

$$x^2 \quad A+B=1 \qquad \boxed{B=\frac{1}{2}}$$

$$x^1 \quad \boxed{C=0}$$

$$x^0 \quad 8A=4 \quad \boxed{A=\frac{1}{2}}$$

$$8\int \frac{x^2+4}{x\left(x^2+8\right)}\,dx=8\int\left(\frac{1}{2}\frac{1}{x}+\frac{1}{2}\frac{x}{x^2+8}\right)dx$$

$$=4\int \frac{1}{x}\,dx+2\int \frac{x}{x^2+8}\,d\left(x^2+8\right)$$

$$\boxed{=4\ln|x|+2\ln\left(x^2+8\right)+K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^2+x+2}{(x+1)\left(x^2+1\right)}\,dx$

Solution

$$\frac{x^2+x+2}{(x+1)\left(x^2+1\right)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$$

$$x^2+x+2=Ax^2+A+Bx^2+Bx+Cx+C$$

$$x^2 \quad A+B=1 \quad \rightarrow A=1-B$$

$$x^1 \quad B+C=1 \quad \rightarrow C=1-B$$

$$x^0 \quad A+C=2 \qquad \rightarrow \qquad 1-B+1-B=2$$

$$\boxed{B=0 \quad A=1 \quad C=1}$$

$$\int \frac{x^2+x+2}{(x+1)\left(x^2+1\right)}\,dx=\int \frac{1}{x+1}\,dx+\int \frac{1}{x^2+1}\,dx$$

$$\boxed{=\ln|x+1|+\arctan(x)+K}$$

Exercise

Evaluate $\displaystyle\int \frac{2}{x\left(x^2+1\right)^2}\,dx$

Solution

$$\frac{2}{x\left(x^2+1\right)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{\left(x^2+1\right)^2}$$

$$2=A\left(x^2+1\right)^2+(Bx+C)\left(x^3+x\right)+Dx^2+Ex$$

$$\begin{array}{lll}
x^4 & A+B=0 & \to \boxed{B=-2} \\
x^3 & \boxed{C=0} & \\
x^2 & 2A+B+D=0 & \to \boxed{D=-2} \\
x^1 & C+E=0 & \to \boxed{E=0} \\
x^0 & \boxed{A=2} &
\end{array}$$

$$\int \frac{2}{x\left(x^2+1\right)^2}\,dx=\int \frac{2}{x}\,dx-\int \frac{2x}{x^2+1}\,dx-\int \frac{2x}{\left(x^2+1\right)^2}\,dx$$

$$=2\ln|x|-\int \frac{1}{x^2+1}\,d\left(x^2+1\right)-\int \frac{1}{\left(x^2+1\right)^2}\,d\left(x^2+1\right)$$

$$\boxed{=2\ln|x|-\ln\left(x^2+1\right)+\frac{1}{x^2+1}+K}$$

Exercise

Evaluate $\displaystyle\int \frac{1}{(x+1)\left(x^2+2x+2\right)^2}\,dx$

Solution

$$\frac{1}{(x+1)\left(x^2+2x+2\right)^2}=\frac{A}{x+1}+\frac{Bx+C}{x^2+2x+2}+\frac{Dx+E}{\left(x^2+2x+2\right)^2}$$

$$1=A\left(x^2+2x+2\right)^2+(Bx+C)(x+1)\left(x^2+2x+2\right)+(Dx+E)(x+1)$$

$$=Ax^4+4Ax^3+8Ax^2+8Ax+4A+\left(Bx^2+Bx+Cx+C\right)\left(x^2+2x+2\right)+Dx^2+Dx+Ex+E$$

$x^4 \qquad A + B = 0$

$x^3 \qquad 4A + 3B + C = 0$

$x^2 \qquad 8A + 4B + 3C + D = 0$

$x^1 \qquad 8A + 2B + 4C + D + E = 0$

$x^0 \qquad 4A + 2C + E = 1$

$$\boxed{A = 1, \quad B = -1, \quad C = -1, \quad D = -1, \quad E = -1}$$

$$\int \frac{1}{(x+1)\left(x^2+2x+2\right)^2}\,dx = \int \frac{dx}{x+1} - \int \frac{x+1}{x^2+2x+2}\,dx - \int \frac{x+1}{\left(x^2+2x+2\right)^2}\,dx$$

$$= \ln|x+1| - \frac{1}{2}\int \frac{d\left(x^2+2x+2\right)}{x^2+2x+2} - \frac{1}{2}\int \frac{d\left(x^2+2x+2\right)}{\left(x^2+2x+2\right)^2}$$

$$\boxed{= \ln|x+1| - \frac{1}{2}\ln\left|x^2+2x+2\right| + \frac{1}{2}\frac{1}{x^2+2x+2} + K}$$

Exercise

Evaluate $\displaystyle \int \frac{2-x}{x^2+x}\,dx$

Solution

$$\frac{2-x}{x^2+x} = \frac{A}{x} + \frac{B}{x+1}$$

$$2 - x = Ax + A + Bx$$

$x^1 \qquad A + B = -1 \quad \rightarrow \boxed{B = -3}$

$x^0 \qquad \boxed{A = 2}$

$$\int \frac{2-x}{x^2+x}\,dx = \int \left(\frac{2}{x} - \frac{3}{x+1}\right)dx$$

$$\boxed{= 2\ln|x| - 3\ln|x+1| + C}$$

Exercise

Evaluate $\displaystyle \int \frac{3x+11}{(x+2)(x+3)}\,dx$

Solution

$$\frac{3x+11}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$3x+11 = Ax+3A+Bx+2B$$

$$x^1 \qquad A+B=3$$

$$x^0 \qquad 3A+2B=11$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -1 \qquad \Delta_A = \begin{vmatrix} 3 & 1 \\ 11 & 2 \end{vmatrix} = -5 \qquad \Delta_B = \begin{vmatrix} 1 & 3 \\ 3 & 11 \end{vmatrix} = 2$$

$$\boxed{A=5 \quad B=-2}$$

$$\int \frac{3x+11}{(x+2)(x+3)}\,dx = \int \left(\frac{5}{x+2} - \frac{1}{x+3}\right)dx$$

$$\boxed{= 5\ln|x+2| - 2\ln|x+3| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x^2-a^2}\,dx$

Solution

$$\frac{1}{x^2-a^2} = \frac{A}{x-a} + \frac{B}{x+a}$$

$$1 = Ax+aA+Bx-aB$$

$$x^1 \qquad A+B=0$$

$$x^0 \qquad aA-aB=1$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ a & -a \end{vmatrix} = -2a \qquad \Delta_A = \begin{vmatrix} 0 & 1 \\ 1 & -a \end{vmatrix} = -1 \qquad \Delta_B = \begin{vmatrix} 1 & 0 \\ a & 1 \end{vmatrix} = 1$$

$$\boxed{A = \frac{1}{2a} \quad B = -\frac{1}{2a}}$$

$$\int \frac{1}{x^2-a^2}\,dx = \int \left(\frac{1}{2a}\frac{1}{x-a} - \frac{1}{2a}\frac{1}{x+a}\right)dx$$

$$\boxed{= \frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x^2+5x+6}\,dx$

Solution

$$\frac{1}{x^2+5x+6} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$1 = Ax + 3A + Bx + 2B$$

$$x^1 \qquad A + B = 0$$

$$x^0 \qquad 3A + 2B = 1$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -1 \qquad \Delta_A = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1 \qquad \Delta_B = \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 1$$

$$\boxed{A = 1 \quad B = -1}$$

$$\int \frac{1}{x^2 + 5x + 6}\,dx = \int \left(\frac{1}{x+2} - \frac{1}{x+3} \right) dx$$

$$= \ln|x+2| - \ln|x-3| + C$$

$$\boxed{= \ln\left|\frac{x+2}{x-3}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x^3 + 6x^2 + 3x + 6}{x^3 + 2x^2}\,dx$

Solution

$$\begin{array}{r|l} & \quad 1 \\ \hline x^3 + 2x^2 & x^3 + 6x^2 + 3x + 6 \\ & \underline{x^3 + 2x^2} \\ & \quad 4x^2 + 3x + 6 \end{array}$$

$$\frac{x^3 + 6x^2 + 3x + 6}{x^3 + 2x^2} = 1 + \frac{4x^2 + 3x + 6}{x^3 + 2x^2}$$

$$\frac{4x^2 + 3x + 6}{x^3 + 2x^2} = \frac{4x^2 + 3x + 6}{x^2(x+2)}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

$$4x^2 + 3x + 6 = Ax(x+2) + Bx + 2B + Cx^2$$

$$x^2 \qquad A + C = 4 \qquad \rightarrow \boxed{C = 4}$$

$$x^1 \qquad 2A + B = 3 \qquad \rightarrow \boxed{A = 0}$$

$$x^0 \qquad 2B = 6 \quad \rightarrow \boxed{B = 3}$$

$$\int \frac{x^3 + 6x^2 + 3x + 6}{x^3 + 2x^2}\,dx = \int \left(1 + \frac{0}{x} + \frac{3}{x^2} + \frac{4}{x+2} \right) dx$$

$$\boxed{= x - \frac{3}{x} + 4\ln|x+2| + K}$$

Exercise

Evaluate $\displaystyle\int \frac{2x^2+5x-1}{x^3+x^2-2x}\,dx$

Solution

$$\frac{2x^2+5x-1}{x^3+x^2-2x}=\frac{2x^2+5x-1}{x\left(x^2+x-2\right)}$$

$$=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+2}$$

$$2x^2+5x-1=Ax^2+Ax-2A+Bx^2+2Bx+Cx^2-Cx$$

$$x^2 \quad A+B+C=2 \qquad\qquad B+C=\frac{3}{2}$$

$$x^1 \quad A+2B-C=5 \qquad\qquad 2B-C=\frac{9}{2}$$

$$x^0 \quad -2A=-1 \quad \rightarrow \boxed{A=\frac{1}{2}}$$

$$3B=6 \quad \rightarrow \boxed{B=2}$$

$$C=\frac{3}{2}-2 \quad \rightarrow \boxed{C=-\frac{1}{2}}$$

$$\int \frac{2x^2+5x-1}{x^3+x^2-2x}\,dx=\int\left(\frac{1}{2}\frac{1}{x}+\frac{2}{x-1}-\frac{1}{2}\frac{1}{x+2}\right)dx$$

$$\boxed{=\frac{1}{2}\ln|x|+2\ln|x-1|-\frac{1}{2}\ln|x+2|+K}$$

Exercise

Evaluate $\displaystyle\int \frac{3x+6}{x^3+2x^2-3x}\,dx$

Solution

$$\frac{3x+6}{x^3+2x^2-3x}=\frac{3x+6}{x\left(x^2+2x-3\right)}$$

$$=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+3}$$

$$3x+6=Ax^2+2Ax-3A+Bx^2+3Bx+Cx^2-Cx$$

$$x^2 \quad A+B+C=0 \qquad\qquad B+C=2$$

$$x^1 \quad 2A+3B-C=3 \qquad\qquad 3B-C=7$$

$$x^0 \quad -3A=6 \quad \rightarrow \boxed{A=-2}$$

$$4B=9 \quad \rightarrow \boxed{B=\frac{9}{4}}$$

$$C = 2 - \frac{9}{4} \quad \rightarrow \quad \boxed{C = -\frac{1}{4}}$$

$$\int \frac{3x+6}{x^3+2x^2-3x}\,dx = \int \left(-\frac{2}{x} + \frac{9}{4}\frac{1}{x-1} - \frac{1}{4}\frac{1}{x+3}\right)dx$$

$$\boxed{= -2\ln|x| + \frac{9}{4}\ln|x-1| - \frac{1}{4}\ln|x+3| + K}$$

Exercise

Evaluate $\displaystyle\int \frac{3x^2+2x-2}{x^3-1}\,dx$

Solution

$$\frac{3x^2+2x-2}{x^3-1} = \frac{3x^2+2x-2}{(x-1)\left(x^2+x+1\right)}$$

$$= \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$3x^2+2x-2 = Ax^2+Ax+A+Bx^2-Bx+Cx-C$$

$$\begin{array}{llll} x^2 & A+B=3 & \rightarrow B=3-A & \\ x^1 & A-B+C=2 & & \rightarrow A-3+A+A+2=2 \\ x^0 & A-C=-2 & \rightarrow C=A+2 & \end{array}$$

$$3A = 3 \quad \rightarrow \quad \boxed{A=1}$$

$$B = 3-1 \boxed{= 2}$$

$$C = 1+2 \boxed{= 3}$$

$$\int \frac{3x^2+2x-2}{x^3-1}\,dx = \int \left(\frac{1}{x-1} + \frac{2x+3}{x^2+x+1}\right)dx$$

$$= \ln|x-1| + \int \frac{2x+1+2}{x^2+x+1}\,dx$$

$$= \ln|x-1| + \int \frac{2x+1}{x^2+x+1}\,dx + \int \frac{2}{x^2+x+1}\,dx$$

$$= \ln|x-1| + \int \frac{d\left(x^2+x+1\right)}{x^2+x+1} + \int \frac{2}{\left(x+\frac{1}{2}\right)^2+1-\frac{1}{4}}\,dx$$

$$= \ln|x-1| + \ln\left(x^2+x+1\right) + \int \frac{2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,d\left(x+\frac{1}{2}\right)$$

$$= \ln|x-1| + \ln\left(x^2+x+1\right) + \int \frac{2}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} d\left(x+\frac{1}{2}\right)$$

$$= \ln|x-1| + \ln\left(x^2+x+1\right) + 2\frac{2}{\sqrt{3}}\arctan\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + K$$

$$= \ln|x-1| + \ln\left(x^2+x+1\right) + \frac{4\sqrt{3}}{3}\arctan\frac{2x+1}{\sqrt{3}} + K$$

Exercise

Evaluate $\displaystyle\int \frac{x^3+5x^2+2x-4}{x^4-1}\,dx$

Solution

$$\frac{x^3+5x^2+2x-4}{x^4-1} = \frac{x^3+5x^2+2x-4}{(x-1)(x+1)\left(x^2+1\right)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$x^3+5x^2+2x-4 = A(x+1)\left(x^2+1\right) + B(x-1)\left(x^2+1\right) + (Cx+D)\left(x^2-1\right)$$

$$\begin{array}{lll} x^3 & A+B+C=1 & (1) \\ x^2 & A-B+D=5 & (2) \\ x^1 & A+B-C=2 & (3) \\ x^0 & A-B-D=-4 & (4) \end{array}$$

$$(1)+(3) \rightarrow \quad 2A+2B=3$$
$$(2)+(4) \rightarrow \quad 2A-2B=1$$

$$\Delta = \begin{vmatrix} 2 & 2 \\ 2 & -2 \end{vmatrix} = -8 \qquad \Delta_A = \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix} = -8 \qquad \Delta_B = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = -4$$

$$A=1 \quad B=\frac{1}{2}$$

$$(1) \rightarrow \quad C = 1-1-\frac{1}{2} = -\frac{1}{2}$$

$$(2) \rightarrow \quad D = 5-1+\frac{1}{2} = \frac{9}{2}$$

$$\int \frac{x^3+5x^2+2x-4}{x^4-1}\,dx = \int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{x}{x^2+1}\,dx + \frac{9}{2}\int \frac{1}{x^2+1}\,dx$$

$$= \ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\int \frac{1}{x^2+1}\, d\left(x^2+1\right) + \frac{9}{2}\arctan x + K$$

$$\boxed{= \ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln\left(x^2+1\right) + \frac{9}{2}\arctan x + K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^2+4x}{\left(x^2+4\right)(x-2)^2}\,dx$

Solution

$$\frac{x^2+4x}{\left(x^2+4\right)(x-2)^2} = \frac{Ax+B}{x^2+4} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$$

$$x^2+4x = (Ax+B)\left(x^2-4x+4\right) + C\left(x^2+4\right)(x-2) + Dx^2 + 4D$$

$$\begin{array}{lll} x^3 & A+C=0 & \to A=-C \\ x^2 & -4A+B-2C+D=1 & \\ x^1 & 4A-4B+4C=4 & \\ x^0 & 4B-8C+4D=0 & \end{array}$$

$$\begin{cases} B+2C+D=1 \\ -4B=4 \\ 4B-8C+4D=0 \end{cases} \quad \boxed{B=-1} \quad \begin{array}{l} 2C+D=2 \\ \\ -8C+4D=4 \end{array}$$

$$\begin{cases} 2C+D=2 \\ -2C+D=1 \end{cases}$$

$$2D=3 \quad \to \quad \boxed{D=\frac{3}{2}}$$

$$2C = 2-\frac{3}{2} = \frac{1}{2} \quad \to \quad \boxed{C=\frac{1}{4}}$$

$$\boxed{A=-\frac{1}{4}}$$

$$\int \frac{x^2+4x}{\left(x^2+4\right)(x-2)^2}\,dx = -\frac{1}{4}\int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx + \frac{1}{4}\int \frac{1}{x-2}\,dx + \frac{3}{2}\int \frac{1}{(x-2)^2}\,dx$$

$$= -\frac{1}{8}\int \frac{1}{x^2+4}\, d\left(x^2+4\right) - \frac{1}{2}\arctan\frac{x}{2} + \frac{1}{4}\ln|x-2| + \frac{3}{2}\int \frac{1}{(x-2)^2}\, d(x-2)$$

$$\boxed{= -\frac{1}{8}\ln\left(x^2+4\right) - \frac{1}{2}\arctan\frac{x}{2} + \frac{1}{4}\ln|x-2| - \frac{3}{2}\frac{1}{x-2} + K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^2+2x+3}{(x-1)(x+1)^2}\,dx$

Solution

$$\frac{x^2+2x+3}{(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$x^2+2x+3=Ax^2+2Ax+A+B(x+1)(x-1)+Cx-C$$

$$\begin{array}{lll} x^2 & A+B=1 & B=1-A \\ x^1 & 2A+C=2 & C=2-2A \\ x^0 & A-B-C=3 & (1) \end{array}$$

$$(1)\to \quad A-1+A-2+2A=3 \quad \Rightarrow \boxed{A=\frac{3}{2}}$$

$$B=1-\frac{3}{2}\ \boxed{=-\frac{1}{2}}$$

$$C=2-3\ \boxed{=-1}$$

$$\int \frac{x^2+2x+3}{(x-1)(x+1)^2}\,dx=\frac{3}{2}\int\frac{1}{x-1}\,dx-\frac{1}{2}\int\frac{1}{x+1}\,dx-\int\frac{1}{(x+1)^2}\,d(x+1)$$

$$\boxed{=\frac{3}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+\frac{1}{x+1}+K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^4-x^3+3x^2-x+2}{(x-1)\left(x^2+2\right)^2}\,dx$

Solution

$$\frac{x^4-x^3+3x^2-x+2}{(x-1)\left(x^2+2\right)^2}=\frac{A}{x-1}+\frac{Bx+C}{x^2+2}+\frac{Dx+E}{\left(x^2+2\right)^2}$$

$$x^4-x^3+3x^2-x+2=A\left(x^4+4x^2+4\right)+(Bx+C)(x-1)\left(x^2+2\right)+(Dx+E)(x-1)$$

$$=Ax^4+4Ax^2+4A+(Bx+C)\left(x^3+2x-x^2-2\right)+Dx^2-Dx+Ex-E$$

$$\begin{array}{lll} x^4 & A+B=1 & \rightarrow A=1-B \\ x^3 & -B+C=-1 & \rightarrow C=B-1 \\ x^2 & 4A+2B-C+D=3 & \\ x^1 & -2B+2C-D+E=1 & \\ x^0 & 4A-2C-E=2 & \end{array}$$

$$\begin{cases} 4-4B+2B-B+1+D=3 \\ -2B+2B-2-D+E=1 \\ 4-4B-2B+2-E=2 \end{cases}$$

$$\begin{cases} \left.\begin{array}{l} -3B+D=-2 \\ -D+E=3 \end{array}\right\} \\ -6B-E=-4 \end{cases} \rightarrow \begin{array}{l} -3B+E=1 \\ -6B-E=-4 \end{array}$$

$$-9B=-3 \quad \rightarrow \quad \left.\underline{B=\frac{1}{3}}\right|$$

$$E=1+1\left.\underline{=2}\right|$$

$$A=1-\frac{1}{3}\left.\underline{=\frac{2}{3}}\right|$$

$$C=\frac{1}{3}-1\left.\underline{=-\frac{2}{3}}\right|$$

$$D=-2+1\left.\underline{=-1}\right|$$

$$\int \frac{x^4-x^3+3x^2-x+2}{(x-1)\left(x^2+2\right)^2}\,dx=\frac{2}{3}\int\frac{1}{x-1}dx+\frac{1}{3}\int\frac{x}{x^2+2}dx-\frac{2}{3}\int\frac{1}{x^2+2}dx-\int\frac{x}{\left(x^2+2\right)^2}dx$$

$$+\int\frac{2}{\left(x^2+2\right)^2}dx$$

$$=\frac{2}{3}\ln|x-1|+\frac{1}{6}\int\frac{d\left(x^2+2\right)}{x^2+2}-\frac{2}{3\sqrt{2}}\arctan\frac{x}{\sqrt{2}}-\frac{1}{2}\int\frac{d\left(x^2+2\right)}{\left(x^2+2\right)^2}$$

$$+\int\frac{2}{\left(x^2+2\right)^2}dx$$

$$=\frac{2}{3}\ln|x-1|+\frac{1}{6}\ln\left(x^2+2\right)-\frac{2}{3\sqrt{2}}\arctan\frac{x}{\sqrt{2}}+\frac{1}{2\left(x^2+2\right)}+\int\frac{2}{\left(x^2+2\right)^2}dx$$

$$x=\sqrt{2}\tan\theta \quad \rightarrow dx=\sqrt{2}\sec^2\theta\, d\theta$$

$$x^2+2=2\sec^2\theta$$

$$\begin{aligned}
\int \frac{2}{\left(x^2+2\right)^2}\,dx &= \int \frac{2}{4\sec^4\theta}\sqrt{2}\sec^2\theta\,d\theta \\
&= \frac{\sqrt{2}}{2}\int \frac{d\theta}{\sec^2\theta} \\
&= \frac{\sqrt{2}}{2}\int \cos^2\theta\,d\theta \\
&= \frac{\sqrt{2}}{4}\int (1+\cos 2\theta)\,d\theta \\
&= \frac{\sqrt{2}}{4}\left(\theta+\frac{1}{2}\sin 2\theta\right) \\
&= \frac{\sqrt{2}}{4}(\theta+\sin\theta\cos\theta) \\
&= \frac{\sqrt{2}}{4}\left(\theta+\frac{\tan\theta}{\sec^2\theta}\right) \\
&= \frac{\sqrt{2}}{4}\left(\arctan\frac{x}{\sqrt{2}}+\frac{x}{\sqrt{2}}\frac{2}{x^2+2}\right)
\end{aligned}$$

$$\begin{aligned}
\int \frac{x^4-x^3+3x^2-x+2}{(x-1)\left(x^2+2\right)^2}\,dx &= \frac{2}{3}\ln|x-1|+\frac{1}{6}\ln\left(x^2+2\right)-\frac{2}{3\sqrt{2}}\arctan\frac{x}{\sqrt{2}}+\frac{1}{2\left(x^2+2\right)} \\
&\quad +\frac{\sqrt{2}}{4}\arctan\frac{x}{\sqrt{2}}+\frac{1}{2}\frac{x}{x^2+2}+K
\end{aligned}$$

Exercise

Evaluate $\displaystyle\int \frac{-x^2+11x+18}{(x-1)(x+1)\left(x^2+3x+3\right)}\,dx$

Solution

$$\frac{-x^2+11x+18}{(x-1)(x+1)\left(x^2+3x+3\right)} = \frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+3x+3}$$

$$-x^2+11x+18 = A(x+1)\left(x^2+3x+3\right)+B(x-1)\left(x^2+3x+3\right)+(Cx+D)\left(x^2-1\right)$$

$$\begin{array}{lll} x^3 & A+B+C=0 & (1) \\ x^2 & 4A+2B+D=-1 & (2) \\ x^1 & 6A-C=11 & \rightarrow\ C=6A-11 \\ x^0 & 3A-3B-D=18 & (3) \end{array}$$

$$\begin{cases} (1) \rightarrow & 7A+B=11 \\ (2)+(3) \rightarrow & 7A-B=17 \end{cases}$$

$$14A=28 \quad \rightarrow \quad \underline{A=2}$$

$$B=11-14 \ \underline{=-3}$$

$$C=12-11 \ \underline{=1}$$

$$(2) \rightarrow \quad D=-1-8+6 \ \underline{=-3}$$

$$\int \frac{-x^2+11x+18}{(x-1)(x+1)\left(x^2+3x+3\right)}\,dx = \int \frac{2}{x-1}\,dx - \int \frac{3}{x+1}\,dx + \int \frac{x-3}{x^2+3x+3}\,dx$$

$$= 2\ln|x-1| - 3\ln|x+1| + \int \frac{x-3}{x^2+3x+3}\,dx$$

$$\int \frac{x-3}{x^2+3x+3}\,dx = \frac{1}{2}\int \frac{2x-6+3-3}{x^2+3x+3}\,dx$$

$$= \frac{1}{2}\int \frac{2x+3}{x^2+3x+3}\,dx - \frac{9}{2}\int \frac{1}{\left(x+\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x^2+3x+3}\,d\left(x^2+3x+3\right) - \frac{9}{2}\frac{2}{\sqrt{3}}\arctan\frac{x+\frac{3}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{2}\ln\left|x^2+3x+3\right| - 3\sqrt{3}\arctan\frac{2x+3}{\sqrt{3}}$$

$$\int \frac{-x^2+11x+18}{(x-1)(x+1)\left(x^2+3x+3\right)}\,dx \ \underline{= 2\ln|x-1| - 3\ln|x+1| + \frac{1}{2}\ln\left|x^2+3x+3\right| - 3\sqrt{3}\arctan\frac{2x+3}{\sqrt{3}} + K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^3+5x^2+2x-4}{x\left(x^2+4\right)^2}\,dx$

Solution

$$\frac{x^3+5x^2+2x-4}{x\left(x^2+4\right)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+4}+\frac{Dx+E}{\left(x^2+4\right)^2}$$

$$x^3+5x^2+2x-4=A\left(x^4+8x^2+16\right)+\left(Bx^2+Cx\right)\left(x^2+4\right)+Dx^2+Ex$$

x^4 $\quad A+B=0 \quad \rightarrow \boxed{B=\frac{1}{4}}$

x^3 $\quad \boxed{C=1}$

x^2 $\quad 8A+4B+D=5 \quad \rightarrow \boxed{D=6}$

x^1 $\quad 4C+E=2 \quad \rightarrow \boxed{E=-2}$

x^0 $\quad 16A=-4 \quad \rightarrow \boxed{A=-\frac{1}{4}}$

$$\int \frac{x^3+5x^2+2x-4}{x\left(x^2+4\right)^2}\,dx=-\frac{1}{4}\int\frac{dx}{x}+\frac{1}{4}\int\frac{x}{x^2+4}\,dx+\int\frac{dx}{x^2+4}+\int\frac{6x}{\left(x^2+4\right)^2}\,dx-\int\frac{2}{\left(x^2+4\right)^2}\,dx$$

$$=-\frac{1}{4}\ln|x|+\frac{1}{8}\int\frac{d\left(x^2+4\right)}{x^2+4}+\frac{1}{2}\arctan\frac{x}{2}+3\int\frac{d\left(x^2+4\right)}{\left(x^2+4\right)^2}-\int\frac{2}{\left(x^2+4\right)^2}\,dx$$

$$=-\frac{1}{4}\ln|x|+\frac{1}{8}\ln\left(x^2+4\right)+\frac{1}{2}\arctan\frac{x}{2}-\frac{3}{x^2+4}-\int\frac{2}{\left(x^2+4\right)^2}\,dx$$

$x=2\tan\theta \quad \rightarrow dx=2\sec^2\theta\, d\theta$

$x^2+4=4\sec^2\theta$

$$\int\frac{2}{\left(x^2+4\right)^2}\,dx=\int\frac{2}{16\sec^4\theta}2\sec^2\theta\, d\theta$$

$$=\frac{1}{4}\int\frac{d\theta}{\sec^2\theta}$$

$$=\frac{1}{4}\int\cos^2\theta\, d\theta$$

$$=\frac{1}{8}\int(1+\cos 2\theta)\, d\theta$$

$$=\frac{1}{8}\left(\theta+\frac{1}{2}\sin 2\theta\right)$$

$$=\frac{1}{8}(\theta+\sin\theta\cos\theta)$$

$$= \frac{1}{8}\left(\theta + \frac{\tan\theta}{\sec^2\theta}\right)$$

$$= \frac{1}{8}\left(\arctan\frac{x}{2} + \frac{x}{2}\frac{4}{x^2+4}\right)$$

$$\underline{= \frac{1}{8}\arctan\frac{x}{2} + \frac{1}{4}\frac{x}{x^2+4}}\Big|$$

$$\int \frac{x^3+5x^2+2x-4}{x\left(x^2+4\right)^2}\,dx = -\frac{1}{4}\ln|x| + \frac{1}{8}\ln\left(x^2+4\right) + \frac{1}{2}\arctan\frac{x}{2} - \frac{3}{x^2+4} - \frac{1}{8}\arctan\frac{x}{2} - \frac{1}{4}\frac{x}{x^2+4} + K$$

$$\underline{= -\frac{1}{4}\ln|x| + \frac{1}{8}\ln\left(x^2+4\right) + \frac{3}{8}\arctan\frac{x}{2} - \frac{1}{4}\frac{x+12}{x^2+4} + K}\Big|$$

Exercise

Evaluate $\displaystyle\int_{-1}^{2} \frac{5x}{x^2-x-6}\,dx$

Solution

$$\frac{5x}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$5x = Ax + 2A + Bx - 3B$$

$x^1 \quad A + B = 5$

$x^0 \quad 2A - 3B = 0$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5 \qquad \Delta_A = \begin{vmatrix} 5 & 1 \\ 0 & -3 \end{vmatrix} = -15 \qquad \Delta_B = \begin{vmatrix} 1 & 5 \\ 2 & 0 \end{vmatrix} = -10$$

$$A = \frac{-15}{-5}\ \underline{= 3}\,\Big| \qquad B = \frac{-10}{-5}\ \underline{= 2}\,\Big|$$

$$\int_{-1}^{2} \frac{5x}{x^2-x-6}\,dx = \int_{-1}^{2}\left(\frac{3}{x-3} + \frac{2}{x+2}\right)dx$$

$$= 3\ln|x-3| + 2\ln|x+2|\ \Big|_{-1}^{2}$$

$$= 3\ln|-1| + 2\ln 4 - 3\ln|-4| - 2\ln 1$$

$$= 2\ln 4 - 3\ln 4$$

$$\underline{= -\ln 4}\,\Big|$$

Exercise

Evaluate $\displaystyle\int_0^5 \frac{2}{x^2-4x-32}\,dx$

Solution

$$\frac{2}{x^2-4x-32}=\frac{A}{x-8}+\frac{B}{x+4}$$

$$2=Ax+4A+Bx-8B$$

$$x^1 \quad A+B=0$$

$$x^0 \quad 4A-8B=2$$

$$\Delta=\begin{vmatrix}1 & 1\\4 & -8\end{vmatrix}=-12 \qquad \Delta_A=\begin{vmatrix}0 & 1\\2 & -8\end{vmatrix}=-2 \qquad \Delta_B=\begin{vmatrix}1 & 0\\4 & 2\end{vmatrix}=2$$

$$A=\frac{-2}{-12}=\underline{\frac{1}{6}} \qquad B=\frac{2}{-12}=\underline{-\frac{1}{6}}$$

$$\begin{aligned}\int_0^5 \frac{2}{x^2-4x-32}\,dx&=\frac{1}{6}\int_0^5\left(\frac{1}{x-8}-\frac{1}{x+4}\right)dx\\&=\frac{1}{6}\left(\ln|x-8|-\ln|x+4|\right.\Big|_0^5\\&=\frac{1}{6}\left(\ln|-3|-\ln 9-\ln|-8|+\ln 4\right)\\&=\frac{1}{6}\left(\ln 3-\ln 3^2-\ln 8+\ln 4\right)\\&=\frac{1}{6}\left(\ln 3-2\ln 3-3\ln 2+2\ln 2\right)\\&=\frac{1}{6}\left(-\ln 3-\ln 2\right)\\&=-\frac{1}{6}\left(\ln 3+\ln 2\right)\\&=\underline{-\frac{\ln 6}{6}}\end{aligned}$$

Exercise

Evaluate $\displaystyle\int_0^1 \frac{dx}{(x+1)\left(x^2+1\right)}$

Solution

$$\frac{1}{(x+1)\left(x^2+1\right)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$$

$$1 = Ax^2 + A + Bx^2 + Bx + Cx + C$$

$$\begin{array}{lll} x^2 & A + B = 0 & A = -B \\ x & B + C = 0 & C = -B \\ x^0 & A + C = 1 & (1) \end{array}$$

$$(1) \rightarrow \quad -B - B = 1 \quad \boxed{B = -\frac{1}{2}}$$

$$\boxed{A = C = \frac{1}{2}}$$

$$\int_0^1 \frac{dx}{(x+1)\left(x^2+1\right)} = \frac{1}{2}\int_0^1 \frac{1}{x+1}\,dx - \frac{1}{2}\int_0^1 \frac{x}{x^2+1}\,dx + \frac{1}{2}\int_0^1 \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln(x+1) - \frac{1}{4}\int_0^1 \frac{1}{x^2+1}\,d\left(x^2+1\right) + \frac{1}{2}\arctan x$$

$$= \left(\frac{1}{2}\ln(x+1) - \frac{1}{4}\ln\left(x^2+1\right) + \frac{1}{2}\arctan x\right.\Bigg|_0^1$$

$$= \frac{1}{2}\ln 2 - \frac{1}{4}\ln 2 + \frac{1}{2}\arctan 1 - \frac{1}{2}\ln 1 - \frac{1}{4}\ln 1 + \frac{1}{2}\arctan 0$$

$$\boxed{= \frac{1}{4}\ln 2 + \frac{\pi}{8}}$$

Exercise

Evaluate $\displaystyle\int_{-1/2}^{1/2} \frac{x^2+1}{x^2-1}\,dx$

Solution

$$\int_{-1/2}^{1/2} \frac{x^2+1}{x^2-1}\,dx = \int_{-1/2}^{1/2} \left(1 + \frac{2}{x^2-1}\right) dx$$

$$\frac{2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$2 = Ax + A + Bx - B$$

$$\begin{array}{ll} x & A + B = 0 \\ x^0 & A - B = 2 \end{array} \quad \rightarrow \quad A = 1 \quad B = -1$$

$$\int_{-1/2}^{1/2} \frac{x^2+1}{x^2-1}\,dx = \int_{-1/2}^{1/2} \left(1 + \frac{1}{x-1} - \frac{1}{x+1}\right) dx$$

$$= x + \ln|x-1| - \ln|x+1| \ \Big|_{-1/2}^{1/2}$$

$$= x + \ln\left|\frac{x-1}{x+1}\right| \ \Big|_{-1/2}^{1/2}$$

$$= \frac{1}{2} + \ln\left|\frac{-\frac{1}{2}}{\frac{3}{2}}\right| + \frac{1}{2} - \ln\left|\frac{-\frac{3}{2}}{\frac{1}{2}}\right|$$

$$= 1 + \ln\left|\frac{1}{3}\right| - \ln|-3|$$

$$= 1 - \ln 3 - \ln 3$$

$$\underline{= 1 - 2\ln 3\ |}$$

Exercise

Evaluate $\displaystyle\int_0^2 \frac{3}{4x^2+5x+1}\, dx$

Solution

$$\frac{3}{4x^2+5x+1} = \frac{A}{x+1} + \frac{B}{4x+1}$$

$$4Ax + A + Bx + B = 3$$

$$\Rightarrow \begin{cases} 4A + B = 0 \\ A + B = 3 \end{cases} \rightarrow \underline{A = -1 \quad B = 4|}$$

$$\int_0^2 \frac{3}{4x^2+5x+1}\, dx = -\int_0^2 \frac{1}{x+1}\, dx + \int_0^2 \frac{4}{4x+1}\, dx$$

$$= -\ln(x+1) + \ln(4x+1) \ \Big|_0^2$$

$$= \ln\frac{4x+1}{x+1} \ \Big|_0^2$$

$$\underline{= \ln 3\ |}$$

Exercise

Evaluate $\displaystyle\int_1^5 \frac{x-1}{x^2(x+1)}\, dx$

Solution

$$\frac{x-1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$Ax^2 + Ax + Bx + B + Cx^2 = x - 1$$

$$\begin{cases} x^2 & A + C = 0 \\ x^1 & A + B = 1 \quad \to \underline{A = 2 \quad C = -2 \,\big|} \\ x^0 & \underline{B = -1\,\big|} \end{cases}$$

$$\int_1^5 \frac{x-1}{x^2(x+1)}\,dx = \int_1^5 \left(\frac{2}{x} - \frac{1}{x^2} - \frac{2}{x+1} \right) dx$$

$$= 2\ln x + \frac{1}{x} - 2\ln(x+1)\ \Big|_1^5$$

$$= 2\ln 5 + \frac{1}{5} - 2\ln 6 - 1 + 2\ln 2$$

$$\underline{= 2\ln\frac{5}{3} - \frac{4}{5}\,\Big|}$$

Exercise

Evaluate $\displaystyle\int_1^2 \frac{x+1}{x\left(x^2+1\right)}\,dx$

Solution

$$\frac{x+1}{x\left(x^2+1\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$Ax^2 + A + Bx^2 + Cx = x + 1$$

$$\begin{cases} x^2 & A+B=0 \\ x^1 & \underline{C=1\,\big|} \quad \to \underline{B=-1\,\big|} \\ x^0 & \underline{A=1\,\big|} \end{cases}$$

$$\int_1^2 \frac{x+1}{x\left(x^2+1\right)}\,dx = \int_1^2 \frac{1}{x}\,dx - \int_1^2 \frac{x}{x^2+1}\,dx + \int_1^2 \frac{1}{x^2+1}\,dx$$

$$= \int_1^2 \frac{1}{x}\,dx - \frac{1}{2}\int_1^2 \frac{1}{x^2+1}\,d\left(x^2+1\right) + \int_1^2 \frac{1}{x^2+1}\,dx$$

$$= \ln x - \frac{1}{2}\ln\left(x^2+1\right) + \arctan x\ \Big|_1^2$$

$$= \ln 2 - \frac{1}{2}\ln 5 + \arctan 2 + \frac{1}{2}\ln 2 - \frac{\pi}{4}$$

$$= \frac{1}{2}(3\ln 2 - \ln 5) - \frac{\pi}{4} + \arctan 2$$

$$\boxed{=\frac{1}{2}\ln\frac{8}{5}-\frac{\pi}{4}+\arctan 2}$$

Exercise

Evaluate $\displaystyle\int_0^1 \frac{x^2-x}{x^2+x+1}\,dx$

Solution

$$\int_0^1 \frac{x^2-x}{x^2+x+1}\,dx=\int_0^1 \left(1-\frac{2x+1}{x^2+x+1}\right)dx$$

$$=\int_0^1 dx-\int_0^1 \frac{1}{x^2+x+1}\,d\left(x^2+x+1\right)$$

$$=x-\ln\left(x^2+x+1\right)\Big|_0^1$$

$$\boxed{=1-\ln 3}$$

Exercise

Evaluate $\displaystyle\int_4^8 \frac{y\,dy}{y^2-2y-3}$

Solution

$$\frac{y}{y^2-2y-3}=\frac{A}{y-3}+\frac{B}{y+1}$$

$$y=Ay+A+By-3B$$

$$\rightarrow\begin{cases} A+B=1\\ A-3B=0\end{cases}$$

$$\Delta=\begin{vmatrix}1&1\\1&-3\end{vmatrix}=-4 \qquad \Delta_A=\begin{vmatrix}1&1\\0&-3\end{vmatrix}=-3 \qquad \Delta_B=\begin{vmatrix}1&1\\1&0\end{vmatrix}=-1$$

$$\Rightarrow \boxed{A=\frac{3}{4}}\quad \boxed{B=\frac{1}{4}}$$

$$\int_4^8 \frac{y\,dy}{y^2-2y-3}=\frac{3}{4}\int_4^8 \frac{dy}{y-3}+\frac{1}{4}\int_4^8 \frac{dy}{y+1}$$

$$=\frac{3}{4}\ln|y-3|+\frac{1}{4}\ln|y+1|\ \Big|_4^8$$

$$=\frac{3}{4}\ln|5|+\frac{1}{4}\ln|9|-\left(\frac{3}{4}\ln|1|+\frac{1}{4}\ln|5|\right)$$

$$=\frac{3}{4}\ln 5+\frac{1}{4}\ln 9-\frac{1}{4}\ln 5$$

$$=\frac{1}{2}\ln 5+\frac{1}{4}\ln 3^2 \qquad \textbf{\textit{Power Rule}}$$

$$=\frac{1}{2}\ln 5+\frac{1}{2}\ln 3$$

$$=\frac{1}{2}\left(\ln 5+\ln 3\right) \qquad \textbf{\textit{Product Rule}}$$

$$\boxed{=\frac{1}{2}\ln 15}$$

Exercise

Evaluate $\displaystyle\int_1^{\sqrt{3}}\frac{3x^2+x+4}{x^3+x}dx$

Solution

$$\frac{3x^2+x+4}{x^3+x}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$$

$$3x^2+x+4=Ax^2+A+Bx^2+Cx$$

$$\begin{cases} A+B=3 & \rightarrow\ B=3-4\ \boxed{=-1} \\ \boxed{C=1} \\ \boxed{A=4} \end{cases}$$

$$\int_1^{\sqrt{3}}\frac{3x^2+x+4}{x^3+x}dx=\int_1^{\sqrt{3}}\frac{4}{x}dx+\int_1^{\sqrt{3}}\frac{-x+1}{x^2+1}dx$$

$$=4\int_1^{\sqrt{3}}\frac{1}{x}dx-\int_1^{\sqrt{3}}\frac{x}{x^2+1}dx+\int_1^{\sqrt{3}}\frac{1}{x^2+1}dx \qquad d\left(x^2+1\right)=2xdx$$

$$=4\int_1^{\sqrt{3}}\frac{1}{x}dx-\frac{1}{2}\int_1^{\sqrt{3}}\frac{d\left(x^2+1\right)}{x^2+1}+\int_1^{\sqrt{3}}\frac{1}{x^2+1}dx \qquad \int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$=\left[4\ln|x|-\frac{1}{2}\ln\left(x^2+1\right)+\tan^{-1}x\right]_1^{\sqrt{3}}$$

$$=4\ln\sqrt{3}-\frac{1}{2}\ln 4+\tan^{-1}\sqrt{3}-\left(4\ln 1-\frac{1}{2}\ln 2+\tan^{-1}1\right)$$

$$=4\ln 3^{1/2}-\frac{1}{2}\ln 2^2+\frac{\pi}{3}+\frac{1}{2}\ln 2-\frac{\pi}{4}$$

$$=2\ln 3-\ln 2+\frac{\pi}{12}+\frac{1}{2}\ln 2$$

$$= 2\ln 3 - \frac{1}{2}\ln 2 + \frac{\pi}{12}$$

$$\boxed{= \ln\left(\frac{9}{\sqrt{2}}\right) + \frac{\pi}{12}}$$

Exercise

Evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$

Solution

$$u = \tan\left(\frac{x}{2}\right) \quad \rightarrow \quad x = 2\tan^{-1} u$$

$$dx = \frac{2du}{1+u^2}$$

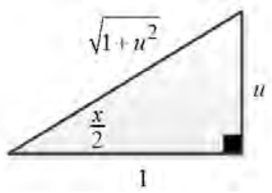
$$\cos x = 2\cos^2 \frac{x}{2} - 1$$

$$= 2\frac{1}{1+u^2} - 1$$

$$= \frac{1-u^2}{1+u^2}$$

$$\sin x = 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}}$$

$$= \frac{2u}{1+u^2}$$



$$\int_0^{\pi/2} \frac{dx}{\sin x + \cos x} = \int_0^{\pi/2} \frac{1}{\frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2} du$$

$$= 2\int_0^{\pi/2} \frac{du}{2u + 1 - u^2}$$

$$= -2\int_0^{\pi/2} \frac{du}{u^2 - 2u - 1}$$

$$= -\frac{1}{\sqrt{2}}\int_0^{\pi/2} \left(\frac{1}{u-1-\sqrt{2}} - \frac{1}{u-1+\sqrt{2}}\right) du$$

$$\frac{2}{u^2 - 2u - 1} = \frac{A}{u-1-\sqrt{2}} + \frac{B}{u-1+\sqrt{2}}$$

$$2 = Au + \left(-1+\sqrt{2}\right)A + Bu + \left(-1-\sqrt{2}\right)B$$

$$\begin{cases} x & A+B=0 \\ x^0 & \left(-1+\sqrt{2}\right)A-\left(1+\sqrt{2}\right)B=2 \end{cases}$$

$$\to \begin{cases} B=-A=-\dfrac{1}{\sqrt{2}} \\ 2\sqrt{2}A=2 \end{cases}$$

$$=-\frac{1}{\sqrt{2}}\left(\ln\left|\frac{1}{u-1-\sqrt{2}}\right|-\ln\left|\frac{1}{u-1+\sqrt{2}}\right|\right|_0^{\pi/2}$$

$$=\frac{1}{\sqrt{2}}\left(\ln\left|\frac{u-1+\sqrt{2}}{u-1-\sqrt{2}}\right|\right|_0^{\pi/2}$$

$$=\frac{1}{\sqrt{2}}\left(\ln\left|\frac{\tan\frac{x}{2}-1+\sqrt{2}}{\tan\frac{x}{2}-1-\sqrt{2}}\right|\right|_0^{\pi/2}$$

$$=\frac{1}{\sqrt{2}}\left(\ln|-1|-\ln\left|\frac{-1+\sqrt{2}}{-1-\sqrt{2}}\right|\right)$$

$$=\frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

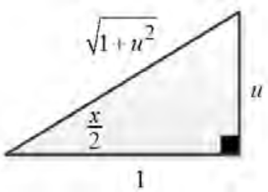
Exercise

Evaluate $\displaystyle\int_0^{\pi/3}\frac{\sin\theta}{1-\sin\theta}\,d\theta$

Solution

$$u=\tan\left(\frac{x}{2}\right)\quad\to\quad x=2\tan^{-1}u$$

$$dx=\frac{2du}{1+u^2}$$



$$\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$=2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}}$$

$$=\frac{2u}{1+u^2}$$

$$\int_0^{\pi/3}\frac{\sin\theta}{1-\sin\theta}\,d\theta=\int_0^{\pi/3}\frac{1}{\csc\theta-1}\,d\theta$$

$$= \int_0^{\pi/3} \frac{1}{\frac{1+u^2}{2u} - 1} \cdot \frac{2}{1+u^2}\, du$$

$$= \int_0^{\pi/3} \frac{4u}{\left(1+u^2-2u\right)\left(1+u^2\right)}\, du$$

$$= \int_0^{\pi/3} \frac{4u}{(u-1)^2\left(1+u^2\right)}\, du$$

$$\frac{4u}{(u-1)^2\left(1+u^2\right)} = \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{Cu+D}{1+u^2}$$

$$4u = Au + Au^3 - A - Au^2 + B + Bu^2 + Cu^3 - 2Cu^2 + Cu + Du^2 - 2Du + D$$

$$\begin{cases} A + C = 0 & \rightarrow A = -C \\ -A + B - 2C + D = 0 & \\ C - 2D = 4 & \rightarrow D = \frac{1}{2}C - 2 \\ -A + B + D = 0 & \end{cases}$$

$$\begin{cases} C + B - 2C + \frac{1}{2}C - 2 = 0 \\ C + B + \frac{1}{2}C - 2 = 0 \end{cases}$$

$$\begin{cases} B - \frac{1}{2}C = 2 \\ B + \frac{3}{2}C = 2 \end{cases} \quad \rightarrow \quad \underline{C = 0\,|} \quad \underline{B = 2\,|}$$

$$\underline{A = 0; \quad D = -2\,|}$$

$$= \int_0^{\pi/3} \left(\frac{2}{(u-1)^2} - \frac{2}{1+u^2} \right) du$$

$$= \frac{-2}{u-1} - 2\tan^{-1} u \;\Big|_0^{\pi/3}$$

$$= \frac{-2}{\tan\frac{x}{2} - 1} - 2\tan^{-1}\left(\tan\frac{x}{2}\right) \;\Big|_0^{\pi/3}$$

$$= \frac{-2}{\tan\frac{x}{2} - 1} - x \;\Big|_0^{\pi/3}$$

$$= \frac{-2}{\frac{1}{\sqrt{3}} - 1} - \frac{\pi}{3} - 2$$

$$= \frac{-2\sqrt{3}}{1-\sqrt{3}} - \frac{\pi}{3} - 2$$

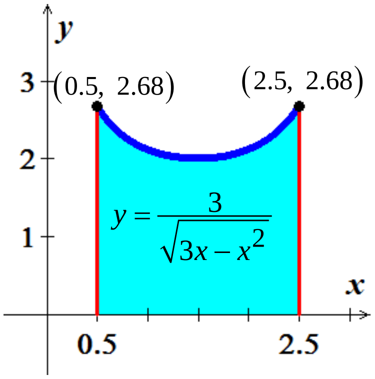
$$= \frac{-2}{1-\sqrt{3}} - \frac{\pi}{3} \qquad = \frac{-2}{1-\sqrt{3}}\,\frac{1+\sqrt{3}}{1+\sqrt{3}} - \frac{\pi}{3}$$

$$\boxed{= 1 + \sqrt{3} - \frac{\pi}{3}}$$

Exercise

Find the volume of the solid generated by the revolving the shaded region about *x*-axis

Solution



$$V = \pi \int_{0.5}^{2.5} y^2\, dx$$

$$= \pi \int_{0.5}^{2.5} \frac{9}{3x-x^2}\, dx$$

$$= 9\pi \int_{0.5}^{2.5} \frac{1}{3x-x^2}\, dx$$

$$\frac{1}{3x-x^2} = \frac{1}{x(3-x)}$$

$$= \frac{A}{x} + \frac{B}{3-x}$$

$$1 = 3A - Ax + Bx$$

$$\begin{cases} B - A = 0 \\ 3A = 1 \end{cases} \Rightarrow \quad \boxed{A = \frac{1}{3}} \quad \boxed{B = \frac{1}{3}}$$

$$= 9\pi \int_{0.5}^{2.5} \frac{1}{3}\left(\frac{1}{x} + \frac{1}{3-x}\right) dx$$

$$= 3\pi \int_{0.5}^{2.5} \left(\frac{1}{x} - \frac{1}{x-3}\right) dx$$

$$= 3\pi \left[\int_{0.5}^{2.5} \frac{1}{x}\, dx - \int_{0.5}^{2.5} \frac{1}{x-3}\, dx\right]$$

$$= 3\pi \left(\ln|x| - \ln|x-3|\right)\Big|_{0.5}^{2.5}$$

$$= 3\pi \left(\ln\left|\frac{x}{x-3}\right| \ \right|_{0.5}^{2.5}$$

$$= 3\pi \left[\ln\left|\frac{2.5}{-.5}\right| - \ln\left|\frac{0.5}{-2.5}\right| \right]$$

$$= 3\pi \left(\ln 5 - \ln \frac{1}{5} \right)$$

$$= 3\pi \ (\ln 5 + \ln 5)$$

$$\underline{= 6\pi \ln 5 \ |}$$

Exercise

Find the area of the region bounded by the graphs of $y = \frac{12}{x^2+5x+6}, \quad y = 0, \quad x = 0, \quad and \quad x = 1$

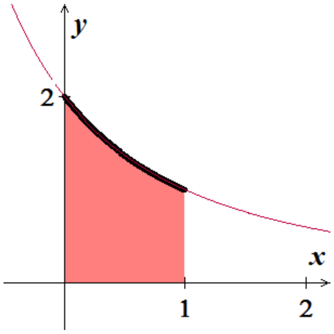
Solution

$$A = \int_0^1 \frac{12}{x^2+5x+6} dx$$

$$\frac{12}{x^2+5x+6} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$12 = Ax + 3A + Bx + 2B$$

$$\begin{cases} A + B = 0 \\ 3A + 2B = 12 \end{cases} \rightarrow A = 12 \quad B = -12$$



$$A = \int_0^1 \frac{12}{x+2} dx - \int_0^1 \frac{12}{x+3} dx$$

$$= 12 \ \left(\ln|x+2| - \ln|x+3| \ \right|_0^1$$

$$= 12(\ln 3 - \ln 4 - \ln 2 + \ln 3)$$

$$= 12(2\ln 3 - 3\ln 2)$$

$$= 12(\ln 9 - \ln 8)$$

$$\underline{= 12 \ln \frac{9}{8} \ |}$$

Exercise

Find the area of the region bounded by the graphs of $y=\dfrac{7}{16-x^2}$ *and* $y=1$

Solution

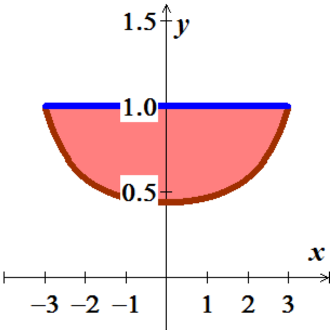
$$A=2\int_0^3\left(1-\frac{7}{16-x^2}\right)dx$$

$$=2\int_0^3 dx\ -\ 2\int_0^3\frac{7}{16-x^2}\,dx$$

$x=4\sin\theta \qquad 16-x^2=16\cos^2\theta$

$dx=4\cos\theta\, d\theta$

$$=2x\Big|_0^3-14\int_0^3\frac{1}{4\cos\theta}\,d\theta$$



$$=6-\frac{7}{2}\int_0^3\sec\theta\ d\theta$$

$$=6-\frac{7}{2}\ln\left|\sec\theta+\tan\theta\right|\ \Big|_0^3$$

$$=6-\frac{7}{2}\ln\left|\frac{4+x}{\sqrt{16-x^2}}\right|\ \Big|_0^3$$

$$=6-\frac{7}{2}\ln\left|\frac{7}{\sqrt{7}}\right|$$

$$=6-\frac{7}{2}\ln\sqrt{7}$$

$$\boxed{=6-\frac{7}{4}\ln 7}\quad\approx 2.595$$

Exercise

The region in the first quadrant that is enclosed by the *x*-axis, the curve $y=\dfrac{5}{x\sqrt{5-x}}$, and the lines $x=1$ and $x=4$ is revolved about the *x*-axis to generate a solid. Find the volume of the solid.

Solution

$$V=\pi\int_1^4 y^2\ dx$$

$$=\pi\int_1^4\left(\frac{5}{x\sqrt{5-x}}\right)^2\ dx$$

$$= \pi \int_1^4 \frac{25}{x^2(5-x)}\, dx$$

$$\frac{25}{x^2(5-x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{5-x}$$

$$25 = 5Ax - Ax^2 + 5B - Bx + Cx^2$$

$$x^2 \quad -A + C = 0 \qquad \rightarrow \ \underline{C = 1}\,\big|$$

$$x^1 \quad 5A - B = 0 \qquad \rightarrow \ \underline{A = 1}\,\big|$$

$$x^0 \quad 5B = 25 \quad \rightarrow \ \underline{B = 5}\,\big|$$

$$= \pi \int_1^4 \left(\frac{1}{x} + \frac{5}{x^2} + \frac{1}{5-x} \right) dx$$

$$= \pi \left(\ln x - \frac{5}{x} - \ln|5-x| \ \right|_1^4$$

$$= \pi \left(\ln\left|\frac{x}{5-x}\right| - \frac{5}{x} \ \right|_1^4$$

$$= \pi \left(\ln 4 - \frac{5}{4} - \ln\frac{1}{4} + 5 \right)$$

$$= \pi \left(\ln 4 + \ln 4 + \frac{15}{4} \right)$$

$$\underline{= \pi \left(2\ln 4 + \frac{15}{4} \right)}\,\Big|$$

Exercise

Find the length of the graph of the function $y = \ln\left(1 - x^2\right), \quad 0 \le x \le \frac{1}{2}$

Solution

$$\frac{dy}{dx} = \frac{-2x}{1-x^2}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{-2x}{1-x^2}\right)^2$$

$$= 1 + \frac{4x^2}{\left(1-x^2\right)^2}$$

$$= \frac{1 - 2x^2 + x^4 + 4x^2}{\left(1-x^2\right)^2}$$

$$=\frac{1+2x^2+x^4}{\left(1-x^2\right)^2}$$

$$=\left(\frac{1+x^2}{1-x^2}\right)^2$$

$$L=\int_0^{1/2}\frac{1+x^2}{1-x^2}\,dx$$

$$\begin{array}{r} -1 \\ -x^2+1\overline{\smash{)}\,x^2+1} \\ \underline{x^2-1} \\ 2 \end{array}$$

$$=\int_0^{1/2}\left(-1+\frac{2}{1-x^2}\right)dx$$

$$\frac{2}{1-x^2}=\frac{A}{1-x}+\frac{B}{1+x}$$

$$2=A+Ax+B-Bx$$

$$\begin{cases} x & A-B=0 \\ x^0 & A+B=2 \end{cases}$$

$$\boxed{A=1 \quad B=1}$$

$$=\int_0^{1/2}\left(-1+\frac{1}{1-x}+\frac{1}{1+x}\right)dx$$

$$=-x-\ln\left|1-x\right|+\ln\left|1+x\right|\,\Bigg|_0^{1/2}$$

$$=-x+\ln\left|\frac{1+x}{1-x}\right|\,\Bigg|_0^{1/2}$$

$$=-\frac{1}{2}+\ln\left|\frac{\frac{3}{2}}{\frac{1}{2}}\right|+0-\ln 1$$

$$\boxed{=-\frac{1}{2}+\ln 3}$$

Exercise

Consider the region bounded by the graphs $y=\frac{2x}{x^2+1}, \quad y=0, \quad x=0, \quad \text{and } x=3$.

a) Find the volume of the solid generated by revolving the region about the *x-axis*

b) Find the centroid of the region.

Solution

a) $$V=\pi\int_0^3\left(\frac{2x}{x^2+1}\right)^2 dx$$

$$=4\pi\int_0^3\frac{x^2}{\left(x^2+1\right)^2}dx$$

$$\frac{x^2}{\left(x^2+1\right)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{\left(x^2+1\right)^2}$$

$$x^2=Ax^3+Ax+Bx^2+B+Cx+D$$

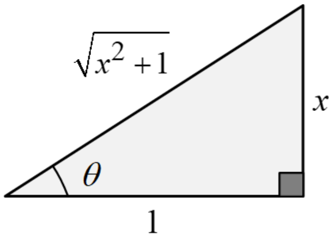
$$\begin{cases} x^3 & A=0 \\ x^2 & B=1 \\ x & A+C=0\to C=0 \\ x^0 & B+D=0\to D=-1 \end{cases}$$

$$=4\pi\int_0^3\frac{1}{x^2+1]}dx-4\pi\int_0^3\frac{1}{\left(x^2+1\right)^2}dx$$

$x=\tan\theta \qquad x^2+1=\sec^2\theta$

$dx=\sec^2\theta\, d\theta$

$$=4\pi\arctan x\Big|_0^3-4\pi\int_0^3\frac{1}{\sec^2\theta}d\theta$$



$$=4\pi\arctan 3-4\pi\int_0^3\cos^2\theta\, d\theta$$

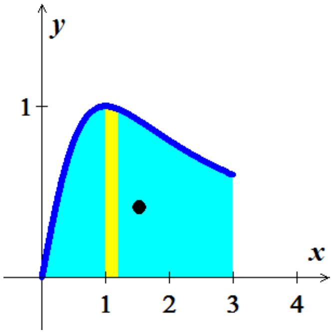
$$=4\pi\arctan 3-2\pi\int_0^3(1+\cos 2\theta)\, d\theta$$

$$=4\pi\arctan 3-2\pi\left(\theta+\sin\theta\cos\theta\right.\Big|_0^3$$

$$=4\pi\arctan 3-2\pi\left(\arctan x+\frac{x}{x^2+1}\right.\Big|_0^3$$

$$=4\pi\arctan 3-2\pi\left(\arctan 3+\frac{3}{10}\right)$$

$$= 2\pi \arctan 3 - \frac{3\pi}{5} \quad unit^3 \qquad \approx 5.963 \quad unit^3$$



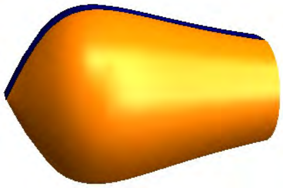
b) $A = \int_0^3 \frac{2x}{x^2+1}\,dx$

$$= \int_0^3 \frac{1}{x^2+1}\,d\left(x^2+1\right)$$

$$= \ln\left(x^2+1\right)\Big|_0^3$$

$$= \ln 10 \quad unit^2$$

$$\bar{x} = \frac{1}{\ln 10}\int_0^3 \frac{2x^2}{x^2+1}dx \qquad\qquad \bar{x} = \frac{1}{A}\int_a^b x\cdot f(x)\,dx$$



$$= \frac{1}{\ln 10}\int_0^3 \left(2 - \frac{2}{x^2+1}\right)dx$$

$$= \frac{1}{\ln 10}\ \left(2x - 2\arctan x\ \right|_0^3$$

$$= \frac{2}{\ln 10}(3 - \arctan 3) \qquad \approx 1.521$$

$$\bar{y} = \frac{1}{2}\frac{1}{\ln 10}\int_0^3 \left(\frac{2x}{x^2+1}\right)^2 dx \qquad\qquad \bar{x} = \frac{1}{A}\int_a^b x\cdot f(x)\,dx$$

$$= \frac{2}{\ln 10}\int_0^3 \frac{x^2}{\left(x^2+1\right)^2}dx$$

$$= \frac{2}{\ln 10}\int_0^3 \frac{1}{x^2+1}dx \ - \ \frac{2}{\ln 10}\int_0^3 \frac{1}{\left(x^2+1\right)^2}dx$$

$$= \frac{2}{\ln 10}\left(\arctan x - \frac{1}{2}\arctan x - \frac{1}{2}\frac{x}{x^2+1}\right|_0^3$$

$$= \frac{2}{\ln 10}\left(\frac{1}{2}\arctan 3 - \frac{3}{20}\right)$$

$$= \frac{1}{\ln 10}\left(\arctan 3 - \frac{3}{10}\right) \qquad \approx 0.412$$

$$(\bar{x},\ \bar{y}) = \left(\frac{2}{\ln 10}(3 - \arctan 3),\quad \frac{1}{\ln 10}\left(\arctan 3 - \frac{3}{10}\right)\right) \qquad \approx (1.521,\quad 0.412)$$

Exercise

Consider the region bounded by the graph $y^2 = \dfrac{(2-x)^2}{(1+x)^2} \quad 0 \le x \le 1$.

Find the volume of the solid generated by revolving this region about the x-*axis*.

Solution

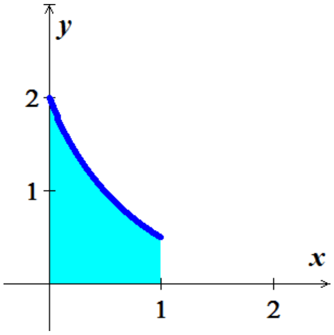
$$V = \pi \int_0^1 \frac{(2-x)^2}{(1+x)^2}\, dx$$

$$= 4\pi \int_0^1 \frac{1}{(1+x)^2}\, dx \; - \; 4\pi \int_0^1 \frac{x}{(1+x)^2}\, dx \; + \; \pi \int_0^1 \frac{x^2}{(1+x)^2}\, dx$$

$$\frac{x}{(1+x)^2} = \frac{A}{x+1} + \frac{B}{(1+x)^2}$$

$$Ax + A + B = x$$

$$A = 1, \quad B = -1$$



$$\frac{x^2}{x^2+2x+1} = 1 - \frac{2x+1}{(1+x)^2}$$

$$= 1 - \left(\frac{C}{x+1} + \frac{D}{(1+x)^2} \right)$$

$$Cx + C + D = 2x + 1$$

$$C = 2, \quad D = -1$$

$$= -4\pi \frac{1}{1+x}\Big|_0^1 - 4\pi \int_0^1 \frac{1}{x+1}\, dx + 4\pi \int_0^1 \frac{1}{(1+x)^2}\, dx + \pi \int_0^1 dx - 2\pi \int_0^1 \frac{1}{x+1}\, dx + \pi \int_0^1 \frac{1}{(1+x)^2}\, dx$$

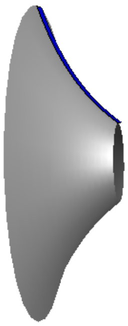
$$= 2\pi \; + \; \left(-4\pi \ln(x+1) - 4\pi \frac{1}{x+1} + \pi x - 2\pi \ln(x+1) - \pi \frac{1}{x+1} \right|_0^1$$

$$= 2\pi - \left(6\pi \ln(x+1) + 5\pi \frac{1}{x+1} - \pi x \right|_0^1$$

$$= 2\pi - \left(6\pi \ln(2) + \frac{5}{2}\pi - \pi - 5\pi \right)$$

$$= 2\pi - 6\pi \ln 2 + \frac{7}{2}\pi$$

$$= \frac{\pi}{2}(11 - 12\ln 2) \quad unit^3$$



Exercise

A single infected individual enters a community of n susceptible individuals. Let x be the number of newly infected individuals at time t. The common epidemic model assumes that the disease spreads at a rate proportional to the product of the total number infected and the number not yet infected. So,

$\frac{dx}{dt} = k(x+1)(n-x)$ and you obtain

$\int \frac{1}{(x+1)(n-x)}\,dx = \int k\,dt$ Solve for x as a function of t.

Solution

$$\frac{1}{(x+1)(n-x)} = \frac{A}{x+1} + \frac{B}{n-x}$$

$$1 = An - Ax + Bx + B$$

$$\begin{cases} -A+B=0 \\ nA+B=1 \end{cases}$$

$$(n+1)A = 1 \Rightarrow \boxed{A = \frac{1}{n+1} = B}$$

$$\int \frac{1}{(x+1)(n-x)}\,dx = \frac{1}{n+1}\int \frac{1}{x+1}\,dx + \frac{1}{n+1}\int \frac{1}{n-x}\,dx$$

$$= \frac{1}{n+1}\left(\ln|x+1| - \ln|n-x|\right)$$

$$= \frac{1}{n+1}\ln\left|\frac{x+1}{n-x}\right|$$

$$\int k\,dt = kt + C$$

$$\frac{1}{n+1}\ln\left|\frac{x+1}{n-x}\right| = kt + C$$

$$x(t=0) = 0$$

$$\boxed{C = \frac{1}{n+1}\ln\left|\frac{1}{n}\right|}$$

$$\frac{1}{n+1}\ln\left|\frac{x+1}{n-x}\right| = kt + \frac{1}{n+1}\ln\left|\frac{1}{n}\right|$$

$$\ln\left|\frac{x+1}{n-x}\right| - \ln\left|\frac{1}{n}\right| = (n+1)kt$$

$$\ln\left|\frac{nx+n}{n-x}\right| = (n+1)kt$$

$$\frac{nx+n}{n-x} = e^{(n+1)kt}$$

$$nx + n = ne^{(n+1)kt} - xe^{(n+1)kt}$$

$$\left(n+e^{(n+1)kt}\right)x = ne^{(n+1)kt} - n$$

$$\boxed{x=\frac{ne^{(n+1)kt}-n}{n+e^{(n+1)kt}}} \qquad\qquad \lim_{t\to\infty} x = n$$

Exercise

Evaluate $\displaystyle\int_0^1 \frac{x}{1+x^4}\,dx$ in ***two*** different ways.

Solution

1- Partial method

$$\frac{x}{1+x^4}=\frac{Ax+B}{x^2+\sqrt{2}x+1}+\frac{Cx+D}{x^2-\sqrt{2}x+1}$$

$$x = Ax^3-\sqrt{2}Ax^2+Ax+Bx^2-\sqrt{2}Bx+B+Cx^3+\sqrt{2}Cx^2+Cx+Dx^2+\sqrt{2}Dx+D$$

$$\begin{array}{ll}
x^3 & A+C=0\to C=-A\\
x^2 & -\sqrt{2}A+B+\sqrt{2}C+D=0\\
x & A-\sqrt{2}B+C+\sqrt{2}D=1\\
x^0 & B+D=0\to D=-B
\end{array}$$

$$\begin{cases} -2\sqrt{2}A=0\to \boxed{A=0=C}\\ -2\sqrt{2}B=1\Rightarrow \boxed{B=-\frac{1}{2\sqrt{2}}=-\frac{\sqrt{2}}{4}}\end{cases}$$

$$\to\ \boxed{D=\frac{\sqrt{2}}{4}}$$

$$\int_0^1\frac{x}{1+x^4}\,dx=-\frac{\sqrt{2}}{4}\int_0^1\frac{1}{x^2+\sqrt{2}x+1}dx\ +\ \frac{\sqrt{2}}{4}\int_0^1\frac{1}{x^2-\sqrt{2}x+1}dx$$

$$=-\frac{\sqrt{2}}{4}\int_0^1\frac{1}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}dx+\frac{\sqrt{2}}{4}\int_0^1\frac{1}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}dx$$

$$=\frac{\sqrt{2}}{4}\left(-\sqrt{2}\arctan\sqrt{2}\left(x+\frac{\sqrt{2}}{2}\right)+\sqrt{2}\arctan\sqrt{2}\left(x-\frac{\sqrt{2}}{2}\right)\right)\Big|_0^1$$

$$=\frac{1}{2}\left(-\arctan\left(\sqrt{2}x+1\right)+\arctan\left(\sqrt{2}x-1\right)\right)\Big|_0^1$$

$$=\frac{1}{2}\left(-\arctan\left(\sqrt{2}+1\right)+\arctan\left(\sqrt{2}-1\right)+\arctan 1-\arctan(-1)\right)$$

$$\boxed{=\frac{1}{2}\left(-\arctan\left(\sqrt{2}+1\right)+\arctan\left(\sqrt{2}-1\right)+\frac{\pi}{2}\right)}$$

2- Let $u=x^2 \rightarrow du=2xdx$

$$\int_0^1 \frac{x}{1+x^4}\,dx=\frac{1}{2}\int_0^1 \frac{1}{1+u^2}\,du$$

$$=\frac{1}{2}\arctan x^2\Big|_0^1$$

$$\boxed{=\frac{\pi}{8}}$$

$$\frac{1}{2}\left(\arctan\left(\sqrt{2}-1\right)-\arctan\left(\sqrt{2}+1\right)+\frac{\pi}{2}\right)=\frac{1}{2}\left(\arctan\left(\frac{\sqrt{2}-1-\sqrt{2}-1}{1+\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}\right)+\frac{\pi}{2}\right)$$

$$=\frac{1}{2}\left(\arctan(-1)+\frac{\pi}{2}\right)$$

$$=\frac{1}{2}\left(-\frac{\pi}{4}+\frac{\pi}{2}\right)$$

$$\boxed{=\frac{\pi}{8}}$$

Solution *Section* 2.5 – Numerical Integration

Exercise

Find the Midpoint Rule approximations to: $\int_0^1 \sin \pi x \, dx$ *using* $n = 6$ *subintervals*

Solution

$$\Delta x = \frac{1-0}{6} \qquad\qquad \Delta x = \frac{b-a}{n}$$

$$= \boxed{\frac{1}{6}}$$

$$x_k = a + k\Delta x$$

$$\boxed{x_0 = 0}$$

$$x_1 = 0 + \frac{1}{6}$$

$$= \boxed{\frac{1}{6}}$$

$$x_2 = 0 + 2\left(\frac{1}{6}\right)$$

$$= \boxed{\frac{1}{3}}$$

$$x_3 = 0 + 3\left(\frac{1}{6}\right)$$

$$= \boxed{\frac{1}{2}}$$

$$x_4 = 0 + 4\left(\frac{1}{6}\right)$$

$$= \boxed{\frac{2}{3}}$$

$$x_5 = 0 + 5\left(\frac{1}{6}\right)$$

$$= \boxed{\frac{5}{6}}$$

$$x_6 = 0 + 6\left(\frac{1}{6}\right)$$

$$\boxed{=1}$$

$$m_1 = \frac{1}{2}\left(0 + \frac{1}{6}\right) \qquad\qquad m_k = \frac{x_{k-1} + x_k}{2}$$

$$= \boxed{\frac{1}{12}}$$

$$m_2 = \frac{1}{2}\left(\frac{1}{6} + \frac{1}{3}\right)$$

$$=\boxed{\frac{1}{4}}$$

$$m_3 = \frac{1}{2}\left(\frac{1}{3}+\frac{1}{2}\right)$$

$$=\boxed{\frac{5}{12}}$$

$$m_4 = \frac{1}{2}\left(\frac{1}{2}+\frac{2}{3}\right)$$

$$=\boxed{\frac{7}{12}}$$

$$m_5 = \frac{1}{2}\left(\frac{2}{3}+\frac{5}{6}\right)$$

$$=\boxed{\frac{3}{4}}$$

$$m_6 = \frac{1}{2}\left(\frac{5}{6}+1\right)$$

$$=\boxed{\frac{11}{12}}$$

$$M(n) = f\left(m_1\right)\Delta x + f\left(m_2\right)\Delta x + \cdots + f\left(m_n\right)\Delta x$$

$$M(6) = \left(\sin\left(\frac{\pi}{12}\right)+\sin\left(\frac{\pi}{4}\right)+\sin\left(\frac{5\pi}{12}\right)+\sin\left(\frac{7\pi}{12}\right)+\sin\left(\frac{9\pi}{12}\right)+\sin\left(\frac{11\pi}{12}\right)\right)\left(\frac{1}{6}\right)$$

$$\boxed{\approx 0.6439505509}$$

Exercise

Find the Midpoint Rule approximations to: $\int_0^{\pi} x^2 \sin x\, dx \quad n = 8 \ \textit{subintervals}$

Solution

$$\Delta x = \frac{\pi - 0}{8} = \boxed{\frac{\pi}{8}} \qquad \Delta x = \frac{b-a}{n}$$

$$\boxed{x_0 = 0}$$

$$x_1 = 0 + \frac{\pi}{8} = \boxed{\frac{\pi}{8}} \qquad x_k = x_0 + k\Delta x$$

$$x_2 = 0 + 2\left(\frac{\pi}{8}\right) = \boxed{\frac{\pi}{4}}$$

$$x_3 = 0 + 3\left(\frac{\pi}{8}\right) = \boxed{\frac{3\pi}{8}}$$

$$x_4 = 0 + 4\left(\frac{\pi}{8}\right) = \boxed{\frac{\pi}{2}}$$

$$x_5 = 0+5\left(\frac{\pi}{8}\right) \boxed{=\frac{5\pi}{8}}$$

$$x_6 = 0+6\left(\frac{\pi}{8}\right) \boxed{=\frac{3\pi}{4}}$$

$$x_7 = 0+7\left(\frac{\pi}{8}\right) \boxed{=\frac{7\pi}{8}}$$

$$x_8 = 0+8\left(\frac{\pi}{8}\right) \boxed{=\pi}$$

$$m_1 = \frac{1}{2}\left(0+\frac{\pi}{8}\right) \boxed{=\frac{\pi}{16}} \qquad m_k = \frac{x_{k-1}+x_k}{2}$$

$$m_2 = \frac{1}{2}\left(\frac{\pi}{8}+\frac{\pi}{4}\right) \boxed{=\frac{3\pi}{16}}$$

$$m_3 = \frac{1}{2}\left(\frac{\pi}{4}+\frac{3\pi}{8}\right) \boxed{=\frac{5\pi}{16}}$$

$$m_4 = \frac{1}{2}\left(\frac{3\pi}{8}+\frac{\pi}{2}\right) \boxed{=\frac{7\pi}{16}}$$

$$m_5 = \frac{1}{2}\left(\frac{\pi}{2}+\frac{5\pi}{8}\right) \boxed{=\frac{9\pi}{16}}$$

$$m_6 = \frac{1}{2}\left(\frac{5\pi}{8}+\frac{3\pi}{4}\right) \boxed{=\frac{11\pi}{16}}$$

$$m_7 = \frac{1}{2}\left(\frac{3\pi}{4}+\frac{7\pi}{8}\right) \boxed{=\frac{13\pi}{16}}$$

$$m_8 = \frac{1}{2}\left(\frac{7\pi}{8}+\pi\right) \boxed{=\frac{15\pi}{16}}$$

$$M(n) = f\left(m_1\right)\Delta x + f\left(m_2\right)\Delta x + \cdots + f\left(m_n\right)\Delta x$$

$$M(8) = \left(\begin{array}{l} \left(\frac{\pi}{16}\right)^2 \sin\frac{\pi}{16} + \left(\frac{3\pi}{16}\right)^2 \sin\frac{3\pi}{16} + \left(\frac{5\pi}{16}\right)^2 \sin\frac{5\pi}{16} + \left(\frac{7\pi}{16}\right)^2 \sin\frac{7\pi}{16} + \left(\frac{9\pi}{16}\right)^2 \sin\frac{9\pi}{16} \\ +\left(\frac{11\pi}{16}\right)^2 \sin\frac{11\pi}{16} + \left(\frac{13\pi}{16}\right)^2 \sin\frac{13\pi}{16} + \left(\frac{15\pi}{16}\right)^2 \sin\frac{15\pi}{16} \end{array}\right)\left(\frac{\pi}{8}\right)$$

$$= \left(\begin{array}{l} \sin\frac{\pi}{16} + 9\sin\frac{3\pi}{16} + 25\sin\frac{5\pi}{16} + 49\sin\frac{7\pi}{16} + 81\sin\frac{9\pi}{16} + 121\sin\frac{11\pi}{16} \\ +169\sin\frac{13\pi}{16} + 225\sin\frac{15\pi}{16} \end{array}\right)\frac{\pi^3}{2,048}$$

$$\approx \left(\begin{array}{l} 0.19509 + 5.000132 + 20.7867403 + 67.3478925 + 79.4436077 \\ 100.607823 + 93.89136938 + 43.895322453 \end{array}\right)\frac{\pi^3}{2,048}$$

$$\boxed{\approx 6.22414635}$$

Exercise

Find the Midpoint Rule approximations to: $\int_0^1 e^{-\sqrt{x}}\,dx \quad n = 6 \ \ subintervals$

Solution

$$\Delta x = \frac{1-0}{6} \boxed{= \frac{1}{6}}$$

$$\Delta x = \frac{b-a}{n}$$

$$\boxed{x_0 = 0}$$

$$x_1 = 0 + \frac{1}{6} \boxed{= \frac{1}{6}}$$

$$x_k = x_0 + k\Delta x$$

$$x_2 = 0 + 2\left(\frac{1}{6}\right) \boxed{= \frac{1}{3}}$$

$$x_3 = 0 + 3\left(\frac{1}{6}\right) \boxed{= \frac{1}{2}}$$

$$x_4 = 0 + 4\left(\frac{1}{6}\right) \boxed{= \frac{2}{3}}$$

$$x_5 = 0 + 5\left(\frac{1}{6}\right) \boxed{= \frac{5}{6}}$$

$$x_6 = 0 + 6\left(\frac{1}{6}\right) \boxed{= 1}$$

$$m_1 = \frac{1}{2}\left(0 + \frac{1}{6}\right) \boxed{= \frac{1}{12}}$$

$$m_k = \frac{x_{k-1} + x_k}{2}$$

$$m_2 = \frac{1}{2}\left(\frac{1}{6} + \frac{1}{3}\right) \boxed{= \frac{1}{4}}$$

$$m_3 = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{2}\right) \boxed{= \frac{5}{12}}$$

$$m_4 = \frac{1}{2}\left(\frac{1}{2} + \frac{2}{3}\right) \boxed{= \frac{7}{12}}$$

$$m_5 = \frac{1}{2}\left(\frac{2}{3} + \frac{5}{6}\right) \boxed{= \frac{3}{4}}$$

$$m_6 = \frac{1}{2}\left(\frac{5}{6} + 1\right) \boxed{= \frac{11}{12}}$$

$$M(n) = f\left(m_1\right)\Delta x + f\left(m_2\right)\Delta x + \ \cdots + f\left(m_n\right)\Delta x$$

$$M(6) = \left(e^{-\sqrt{1/12}} + e^{-\sqrt{1/4}} + e^{-\sqrt{5/12}} + e^{-\sqrt{7/12}} + e^{-\sqrt{3/4}} + e^{-\sqrt{11/12}}\right)\left(\frac{1}{6}\right)$$

$$\approx (.74925557 + .6065306597 + .52440173 + .46591 + .42062 + .383879)\left(\frac{1}{6}\right)$$

$$\boxed{\approx 0.67787732}$$

Exercise

Find the Midpoint Rule approximations to: $\int_0^1 e^{-x}dx \quad using \quad n=8 \;\; subintervals$

Solution

$$\Delta x = \frac{1-0}{8} \;\; \boxed{=\frac{1}{8}}$$

$$\boxed{x_0 = 0}$$

$$x_1 = 0 + \frac{1}{8} \;\; \boxed{=\frac{1}{8}} \qquad\qquad x_k = x_0 + k\Delta x$$

$$x_2 = 0 + 2\left(\frac{1}{8}\right) \;\; \boxed{=\frac{1}{4}}$$

$$x_3 = 0 + 3\left(\frac{1}{8}\right) \;\; \boxed{=\frac{3}{8}}$$

$$x_4 = 0 + 4\left(\frac{1}{8}\right) \;\; \boxed{=\frac{1}{2}}$$

$$x_5 = 0 + 5\left(\frac{1}{8}\right) \;\; \boxed{=\frac{5}{8}}$$

$$x_6 = 0 + 6\left(\frac{1}{8}\right) \;\; \boxed{=\frac{3}{4}}$$

$$x_7 = 0 + 7\left(\frac{1}{8}\right) \;\; \boxed{=\frac{7}{8}}$$

$$x_8 = 0 + 8\left(\frac{1}{8}\right) \;\; \boxed{=1}$$

$$m_1 = \frac{1}{2}\left(0 + \frac{1}{8}\right) \;\; \boxed{=\frac{1}{16}} \qquad\qquad m_k = \frac{x_{k-1} + x_k}{2}$$

$$m_2 = \frac{1}{2}\left(\frac{1}{8} + \frac{1}{4}\right) \;\; \boxed{=\frac{3}{16}}$$

$$m_3 = \frac{1}{2}\left(\frac{1}{4} + \frac{3}{8}\right) \;\; \boxed{=\frac{5}{16}}$$

$$m_4 = \frac{1}{2}\left(\frac{3}{8} + \frac{1}{2}\right) \;\; \boxed{=\frac{7}{16}}$$

$$m_5 = \frac{1}{2}\left(\frac{1}{2} + \frac{5}{8}\right) \;\; \boxed{=\frac{9}{16}}$$

$$m_6 = \frac{1}{2}\left(\frac{5}{8} + \frac{3}{4}\right) \;\; \boxed{=\frac{11}{16}}$$

$$m_7 = \frac{1}{2}\left(\frac{3}{4} + \frac{7}{8}\right) \;\; \boxed{=\frac{13}{16}}$$

$$m_8 = \frac{1}{2}\left(\frac{7}{8} + 1\right) \;\; \boxed{=\frac{15}{16}}$$

$$M(8) = \frac{1}{8}\left(e^{-1/16} + e^{-3/16} + e^{-5/16} + e^{-7/16} + e^{-9/16} + e^{-11/16} + e^{-13/16} + e^{-15/16}\right)$$

$$\boxed{\approx 0.6317092095}$$

Exercise

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of 10^{-4} by (***a***) the Trapezoid Rule and (***b***) Simpson's Rule. $\int_1^3 (2x-1)\,dx$

Solution

a) ***i***) $\Delta x = \frac{3-1}{4}$ $\qquad\qquad$ $\Delta x = \frac{b-a}{n}$

$$\underline{= \frac{1}{2}}\,\big|$$

$$T = \frac{1}{2}\Delta x\left(m\, f\left(x_i\right)\right)$$

$$= \frac{1}{2}\frac{1}{2}(24)$$

$$\underline{= 6}\,\big|$$

	x_i	$f\left(x_i\right) = 2x_i - 1$	m	$mf\left(x_i\right)$
x_0	1	1	1	1
x_1	$1+\frac{1}{2}=\frac{3}{2}$	2	2	4
x_2	2	3	2	6
x_3	$\frac{5}{2}$	4	2	8
x_4	3	5	1	5
				24

$$f(x) = 2x-1 \quad \Rightarrow \quad f'(x) = 2$$

$$\Rightarrow \quad f''(x) = 0 = M$$

$$\Rightarrow \; Error = 0$$

ii) $\int_1^3 (2x-1)\,dx = \left(x^2 - x \right|_1^3$

$$= \left(3^2 - 3\right) - \left(1^2 - 1\right)$$

$$\underline{= 6}\,\big|$$

iii) $Error = \frac{\left|E_T\right|}{True\;\; Value} \times 100$

$$\underline{= 0\%}\,\big|$$

b) ***i***) $\Delta x = \frac{3-1}{4}$ $\qquad\qquad$ $\Delta x = \frac{b-a}{n}$

$$\underline{= \frac{1}{2}}\,\big|$$

$$S = \frac{1}{3}\Delta x\left(\sum m\, f\left(x_i\right)\right)$$

$$= \frac{1}{3}\frac{1}{2}(36)$$

$$\underline{= 6}\,\big|$$

	x_i	$f\left(x_i\right) = 2x_i - 1$	m	$mf\left(x_i\right)$
x_0	1	1	1	1
x_1	$\frac{3}{2}$	2	4	8
x_2	2	3	2	6
x_3	$\frac{5}{2}$	4	4	16
x_4	3	5	1	5
				36

$$f(x) = 2x-1$$

$$\Rightarrow \quad f^{(4)}(x) = 0 = M$$

$$\Rightarrow \left|E_s\right| = 0$$

ii) $\displaystyle\int_1^3 (2x-1)\,dx = 6$

$$\left|E_s\right| = \int_1^3 (2x-1)\,dx - S$$

$$= 6 - 6$$

$$= 0$$

iii) $\displaystyle Error = \frac{\left|E_T\right|}{True\ \ Value} \times 100$

$$= 0\%$$

Exercise

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of 10^{-4} by (***a***) the Trapezoid Rule and (***b***) Simpson's Rule. $\displaystyle\int_{-1}^{1} \left(x^2+1\right)dx$

Solution

a) ***i***) $\displaystyle\Delta x = \frac{b-a}{n}$

$$= \frac{1+1}{4}$$

$$= \frac{1}{2}$$

	x_i	$f\left(x_i\right)$	m	$m\,f\left(x_i\right)$
x_0	-1	2	1	2
x_1	$-\frac{1}{2}$	$\frac{5}{4}$	2	$\frac{5}{2}$
x_2	0	1	2	2
x_3	$\frac{1}{2}$	$\frac{5}{4}$	2	$\frac{5}{2}$
x_4	1	2	1	2
				11

$$T = \frac{1}{2}\Delta x\left(m\,f\left(x_i\right)\right)$$

$$= \frac{1}{2}\frac{1}{2}(11)$$

$$= \frac{11}{4}$$

$$f(x) = x^2 + 1$$

$$f'(x) = 2x$$

$$f''(x) = 2 = M$$

$$\left|E_T\right| = \frac{1-(-1)}{12}\left(\frac{1}{2}\right)^2(2)$$

$$= \frac{1}{12}$$

$$= 0.0833\ldots$$

ii) $$\int_{-1}^{1}\left(x^2+1\right)dx=\left(\tfrac{1}{3}x^3+x\ \right|_{-1}^{1}$$

$$=\left(\tfrac{1}{3}+1\right)-\left(-\tfrac{1}{3}-1\right)$$

$$\boxed{=\tfrac{8}{3}}$$

$$E_T=\int_{-1}^{1}\left(x^2+1\right)dx-T$$

$$=\frac{8}{3}-\frac{11}{4}$$

$$\boxed{=-\frac{1}{12}}$$

iii) $$Error=\frac{\left|E_T\right|}{True\ \ Value}\times 100$$

$$=\frac{\frac{1}{12}}{\frac{8}{3}}$$

$$\boxed{=\frac{100}{32}}\qquad \boxed{\approx 3\%}$$

b) ***i***) $$\Delta x=\frac{b-a}{n}$$

$$=\frac{-1-(-1)}{4}$$

$$\boxed{=\frac{1}{2}}$$

$$S=\tfrac{1}{3}\Delta x\left(\sum m\,f\left(x_i\right)\right)$$

$$=\tfrac{1}{3}\tfrac{1}{2}(16)$$

$$\boxed{=\frac{8}{3}}$$

	x_i	$f\left(x_i\right)$	m	$m\,f\left(x_i\right)$
x_0	-1	2	1	2
x_1	$-\frac{1}{2}$	$\frac{5}{4}$	4	5
x_2	0	1	2	2
x_3	$\frac{1}{2}$	$\frac{5}{4}$	4	5
x_4	1	2	1	2
				16

$$f\left(x\right)=x^2+1$$

$$f^{(4)}\left(x\right)=0=M$$

$$\Rightarrow\left|E_s\right|=0$$

ii) $$\int_{-1}^{1}\left(x^2+1\right)dx\boxed{=\frac{8}{3}}$$

$$E_S = \int_{-1}^{1}\left(x^2+1\right)dx - S$$

$$= \frac{8}{3} - \frac{8}{3}$$

$$\underline{= 0\,}|$$

iii) $Error = \dfrac{\left|E_T\right|}{True\ \ Value}\times 100$

$$\underline{= 0\%\,}|$$

Exercise

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of 10^{-4} by (***a***) the Trapezoid Rule and (***b***) Simpson's Rule. $\displaystyle\int_2^4 \frac{1}{(s-1)^2}\,ds$

Solution

a) $\Delta x = \dfrac{4-2}{4}$ $\qquad\qquad\qquad \Delta x = \dfrac{b-a}{n}$

$$= \frac{1}{2}\,\Big|$$

$x_0 = 2$

$x_1 = 2 + \frac{1}{2} = \frac{5}{2}$

$x_2 = 2 + 2\left(\frac{1}{2}\right) = 3$

$x_3 = 2 + 3\left(\frac{1}{2}\right) = \frac{7}{2}$

$x_4 = 4$

$$T = \frac{1}{2}\Delta x\left(m\, f\left(x_i\right)\right)$$

$$= \frac{1}{2}\frac{1}{2}\left(\frac{1}{(2-1)^2} + 2\frac{1}{\left(\frac{5}{2}-1\right)^2} + 2\frac{1}{(3-1)^2} + 2\frac{1}{\left(\frac{7}{2}-1\right)^2} + \frac{1}{(4-1)^2}\right)$$

$$= \frac{1}{4}\left(\frac{1}{1} + \frac{8}{9} + \frac{1}{2} + \frac{8}{25} + \frac{1}{9}\right)$$

$$= \frac{1269}{1800}\,\Big|$$

$$\underline{\approx 0.705\,}|$$

$$f(s)=\frac{1}{(s-1)^2}$$

$$f'(s)=-\frac{2}{(s-1)^3}$$

$$f''(s)=\frac{6}{(s-1)^4} \quad \Rightarrow \quad \boxed{M=6}$$

$$\int_2^4 \frac{1}{(s-1)^2}\,ds=-\frac{1}{s-1}\Big|_2^4$$

$$=-\left(\frac{1}{3}-1\right)$$

$$\boxed{=\frac{2}{3}}$$

The percentage error: $\approx \dfrac{|0.705-.6667|}{.6667}$

$$\boxed{\approx 0.0575} \quad \boxed{5.75\%}$$

b) $\Delta x=\dfrac{4-2}{4}$ $\qquad\qquad$ $\Delta x=\dfrac{b-a}{n}$

$$\boxed{=\frac{1}{2}}$$

$$x_0=2$$

$$x_1=2+\frac{1}{2}=\frac{5}{2}$$

$$x_2=2+2\left(\frac{1}{2}\right)=3$$

$$x_3=2+3\left(\frac{1}{2}\right)=\frac{7}{2}$$

$$x_4=4$$

$$S=\frac{1}{3}\Delta x\left(m\,f\left(x_i\right)\right)$$

$$=\frac{1}{3}\frac{1}{2}\left(\frac{1}{(2-1)^2}+4\frac{1}{\left(\frac{5}{2}-1\right)^2}+2\frac{1}{(3-1)^2}+4\frac{1}{\left(\frac{7}{2}-1\right)^2}+\frac{1}{(4-1)^2}\right)$$

$$=\frac{1}{6}\left(\frac{1}{1}+\frac{16}{9}+\frac{1}{2}+\frac{16}{25}+\frac{1}{9}\right)$$

$$\boxed{=\frac{1813}{450}}$$

$$\boxed{\approx 0.67148}$$

$$\boxed{\int_2^4 \frac{1}{(s-1)^2}\,ds = \frac{2}{3}}$$

The percentage error: $= \dfrac{|0.67148 - .6667|}{.6667}$

≈ 0.0072 | 0.72%

Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^1 \sin \pi x\, dx \quad n=6 \ \ subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1666666667	0.5000000000	1.0000000000
2	0.3333333333	0.8660254000	1.7320508000
3	0.5000000000	1.0000000000	2.0000000000
4	0.6666666667	0.8660254000	1.7320508000
5	0.8333333333	0.5000000000	1.0000000000
6	1.0000000000	0.0000000000	0.0000000000

Trapezoid Rule approximation ≈ 0.62200847

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1666666667	0.5000000000	2.0000000000
2	0.3333333333	0.8660254000	1.7320508000
3	0.5000000000	1.0000000000	2.0000000000
4	0.6666666667	0.8660254000	1.7320508000
5	0.8333333333	0.5000000000	1.0000000000
6	1.0000000000	0.0000000000	0.0000000000

Simpson's Rule approximation ≈ 0.63689453

	Exact	***Trapezoid***	***Simpson***
Value:	0.63661977	0.62200847	0.63689453
Error:		2.2951 %	0.0432 %

Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations to and error to $\int_0^1 e^{-x}dx \quad n=8 \;\; subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.1250000000	0.8824969000	1.7649938000
2	0.2500000000	0.7788007800	1.5576015600
3	0.3750000000	0.6872892800	1.3745785600
4	0.5000000000	0.6065306600	1.2130613200
5	0.6250000000	0.5352614300	1.0705228600
6	0.7500000000	0.4723665500	0.9447331000
7	0.8750000000	0.4168620200	0.8337240400
8	1.0000000000	0.3678794400	0.3678794400

Trapezoid Rule approximation ≈ 0.63294342

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.1250000000	0.8824969000	3.5299876000
2	0.2500000000	0.7788007800	1.5576015600
3	0.3750000000	0.6872892800	2.7491571200
4	0.5000000000	0.6065306600	1.2130613200
5	0.6250000000	0.5352614300	2.1410457200
6	0.7500000000	0.4723665500	0.9447331000
7	0.8750000000	0.4168620200	1.6674480800
8	1.0000000000	0.3678794400	0.3678794400

Simpson's Rule approximation ≈ 0.63212141

	Exact	***Trapezoid***	***Simpson***
Value:	0.63212056	0.63294342	0.63212141
Error:		0.1302 %	0.0001 %

Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error to:

$$\int_1^5 \left(3x^2 - 2x\right) dx \quad n = 8 \ \ subintervals$$

Solution

Trapezoid Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.5000000000	3.7500000000	7.5000000000
2	2.0000000000	8.0000000000	16.0000000000
3	2.5000000000	13.7500000000	27.5000000000
4	3.0000000000	21.0000000000	42.0000000000
5	3.5000000000	29.7500000000	59.5000000000
6	4.0000000000	40.0000000000	80.0000000000
7	4.5000000000	51.7500000000	103.500000000
8	5.0000000000	65.0000000000	65.0000000000

Trapezoid Rule approximation ≈ 100.50000000

Simpson's Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.5000000000	3.7500000000	15.0000000000
2	2.0000000000	8.0000000000	16.0000000000
3	2.5000000000	13.7500000000	55.0000000000
4	3.0000000000	21.0000000000	42.0000000000
5	3.5000000000	29.7500000000	119.000000000
6	4.0000000000	40.0000000000	80.0000000000
7	4.5000000000	51.7500000000	207.000000000
8	5.0000000000	65.0000000000	65.0000000000

Simpson's Rule approximation ≈ 100.00000000

	Exact	***Trapezoid***	***Simpson***
Value:	100.000000	100.500000	100.00000000
Error:		0.5000%	0.0000 %

Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^{\pi/4} 3\sin 2x\, dx \quad n=8 \;\; subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.0981747704	0.5852709700	1.1705419400
2	0.1963495408	1.1480503000	2.2961006000
3	0.2945243113	1.6667107000	3.3334214000
4	0.3926990817	2.1213203400	4.2426406800
5	0.4908738521	2.4944088400	4.9888176800
6	0.5890486225	2.7716386000	5.5432772000
7	0.6872233930	2.9423558400	5.8847116800
8	0.7853981634	3.0000000000	3.0000000000

Trapezoid Rule approximation ≈ 1.49517776

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.0981747704	0.5852709700	2.3410838800
2	0.1963495408	1.1480503000	2.2961006000
3	0.2945243113	1.6667107000	6.6668428000
4	0.3926990817	2.1213203400	4.2426406800
5	0.4908738521	2.4944088400	9.9776353600
6	0.5890486225	2.7716386000	5.5432772000
7	0.6872233930	2.9423558400	11.7694233600
8	0.7853981634	3.0000000000	3.0000000000

Simpson's Rule approximation ≈ 1.50001244

	Exact	***Trapezoid***	***Simpson***
Value:	1.500000	1.49517776	1.50001244
Error:		0.3215 %	0.0008 %

Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^8 e^{-2x}\,dx \quad n=8 \ \ subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f\left(x_n\right)$	$m\cdot f\left(x_n\right)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.0000000000	0.1353352800	0.2706705600
2	2.0000000000	0.0183156400	0.0366312800
3	3.0000000000	0.0024787500	0.0049575000
4	4.0000000000	0.0003354600	0.0006709200
5	5.0000000000	0.0000454000	0.0000908000
6	6.0000000000	0.0000061400	0.0000122800
7	7.0000000000	0.0000008300	0.0000016600
8	8.0000000000	0.0000001100	0.0000001100

Trapezoid Rule approximation ≈ 0.65651755

Simpson's Rule Method

n	x_n	$f\left(x_n\right)$	$m\cdot f\left(x_n\right)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.0000000000	0.1353352800	0.5413411200
2	2.0000000000	0.0183156400	0.0366312800
3	3.0000000000	0.0024787500	0.0099150000
4	4.0000000000	0.0003354600	0.0006709200
5	5.0000000000	0.0000454000	0.0001816000
6	6.0000000000	0.0000061400	0.0000122800
7	7.0000000000	0.0000008300	0.0000033200
8	8.0000000000	0.0000001100	0.0000001100

Simpson's Rule approximation ≈ 0.52958521

	Exact	***Trapezoid***	***Simpson***
Value:	0.49999994	0.65651755	0.52958521
Error:		31.3035 %	5.9171 %

Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_{-1}^{1} \sqrt{x^2+1}\, dx \quad n=8 \;\; subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	–1.0000000000	1.4142135600	1.4142135600
1	–0.7500000000	1.2500000000	2.5000000000
2	–0.5000000000	1.1180339900	2.2360679800
3	–0.2500000000	1.0307764100	2.0615528200
4	0.0000000000	1.0000000000	2.0000000000
5	0.2500000000	1.0307764100	2.0615528200
6	0.5000000000	1.1180339900	2.2360679800
7	0.7500000000	1.2500000000	2.5000000000
8	1.0000000000	1.4142135600	1.4142135600

Trapezoid Rule approximation ≈ 2.30295859

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	–1.0000000000	1.4142135600	1.4142135600
1	–0.7500000000	1.2500000000	5.0000000000
2	–0.5000000000	1.1180339900	2.2360679800
3	–0.2500000000	1.0307764100	4.1231056400
4	0.0000000000	1.0000000000	2.0000000000
5	0.2500000000	1.0307764100	4.1231056400
6	0.5000000000	1.1180339900	2.2360679800
7	0.7500000000	1.2500000000	5.0000000000
8	1.0000000000	1.4142135600	1.4142135600

Simpson's Rule approximation ≈ 2.29556453

	Exact	***Trapezoid***	***Simpson***
Value:	2.29558715	2.30295859	2.29556453
Error:		0.3211 %	0.0010 %

Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^{1/2} \sin\left(x^2\right)\,dx \quad n=4$ *subintervals*

Solution

Trapezoid Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1250000000	0.0156243642	0.0312487284
2	0.2500000000	0.0624593178	0.1249186357
3	0.3750000000	0.1401619723	0.2803239447
4	0.5000000000	0.2474039593	0.2474039593

Trapezoid Rule approximation ≈ 0.0427434543

Simpson's Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1250000000	0.0156243642	0.0624974569
2	0.2500000000	0.0624593178	0.1249186357
3	0.3750000000	0.1401619723	0.5606478894
4	0.5000000000	0.2474039593	0.2474039593

Simpson Rule approximation ≈ 0.0414778309

	Exact	***Trapezoid***	***Simpson***
Value:	0.0414810243	0.0427434543	0.0414778309
Error:		3.04339%	0.00770 %

Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_{\pi/2}^{\pi} \frac{\sin x}{x}\,dx \quad n=6 \;\; subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	1.5708000000	0.6366182800	0.6366182800
1	1.8323333333	0.5271932200	1.0543864400
2	2.0938666667	0.4137271600	0.8274543200
3	2.3554000000	0.3004450800	0.6008901600
4	2.6169333333	0.1914141900	0.3828283800
5	2.8784666667	0.0903606800	0.1807213600
6	3.1400000000	0.0005072100	0.0005072100

Trapezoid Rule approximation ≈ 0.48166674

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	1.5708000000	0.6366182800	0.6366182800
1	1.8323333333	0.5271932200	2.1087728800
2	2.0938666667	0.4137271600	0.8274543200
3	2.3554000000	0.3004450800	1.2017803200
4	2.6169333333	0.1914141900	0.3828283800
5	2.8784666667	0.0903606800	0.3614427200
6	3.1400000000	0.0005072100	0.0005072100

Simpson's Rule approximation ≈ 0.48116938

	Exact	***Trapezoid***	***Simpson***
Value:	0.48117214	0.48166674	0.48116938
Error:		0.1028 %	0.0006 %

Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^{\pi/4} x\tan x\, dx \quad n = 6 \;\; subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1308996939	0.0172332716	0.0344665433
2	0.2617993878	0.0701489345	0.1402978691
3	0.3926990817	0.1626612856	0.3253225711
4	0.5235987756	0.3022998940	0.6045997881
5	0.6544984695	0.5022143392	1.0044286785
6	0.7853981634	0.7853981634	0.7853981634

Trapezoid Rule approximation ≈ 0.1894454730

Simpson's Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1308996939	0.0172332716	0.0689330865
2	0.2617993878	0.0701489345	0.1402978691
3	0.3926990817	0.1626612856	0.6506451423
4	0.5235987756	0.3022998940	0.6045997881
5	0.6544984695	0.5022143392	2.0088573569
6	0.7853981634	0.7853981634	0.7853981634

Simpson Rule approximation ≈ 0.1858222125

	Exact	***Trapezoid***	***Simpson***
Value:	0.1857845357	0.1894454730	0.1858222125
Error:		1.97053%	0.02028 %

Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^1 e^{-x^2}\,dx \quad n=10 \;\; subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f\left(x_n\right)$	$m\cdot f\left(x_n\right)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.1000000000	0.9900498337	1.9800996675
2	0.2000000000	0.9607894392	1.9215788783
3	0.3000000000	0.9139311853	1.8278623705
4	0.4000000000	0.8521437890	1.7042875779
5	0.5000000000	0.7788007831	1.5576015661
6	0.6000000000	0.6976763261	1.3953526521
7	0.7000000000	0.6126263942	1.2252527884
8	0.8000000000	0.5272924240	1.0545848481
9	0.9000000000	0.4448580662	0.8897161324
10	1.0000000000	0.3678794412	0.3678794412

Trapezoid Rule approximation ≈ 0.7462107961

Simpson's Rule Method

n	x_n	$f\left(x_n\right)$	$m\cdot f\left(x_n\right)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.1000000000	0.9900498337	3.9601993350
2	0.2000000000	0.9607894392	1.9215788783
3	0.3000000000	0.9139311853	3.6557247411
4	0.4000000000	0.8521437890	1.7042875779
5	0.5000000000	0.7788007831	3.1152031323
6	0.6000000000	0.6976763261	1.3953526521
7	0.7000000000	0.6126263942	2.4505055767
8	0.8000000000	0.5272924240	1.0545848481
9	0.9000000000	0.4448580662	1.7794322649
10	1.0000000000	0.3678794412	0.3678794412

Simpson Rule approximation ≈ 0.7468249483

	Exact	***Trapezoid***	***Simpson***
Value:	0.7468241328	0.7462107961	0.7468249483
Error:		0.08213%	0.00011 %

Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^2 \frac{1}{\sqrt{1+x^2}}\,dx \quad n=10 \ \ subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f\left(x_n\right)$	$m\cdot f\left(x_n\right)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.2000000000	0.9805806800	1.9611613600
2	0.4000000000	0.9284766900	1.8569533800
3	0.6000000000	0.8574929300	1.7149858600
4	0.8000000000	0.7808688100	1.5617376200
5	1.0000000000	0.7071067800	1.4142135600
6	1.2000000000	0.6401844000	1.2803688000
7	1.4000000000	0.5812381900	1.1624763800
8	1.6000000000	0.5299989400	1.0599978800
9	1.8000000000	0.4856429300	0.9712858600
10	2.0000000000	0.4472136000	0.4472136000

Trapezoid Rule approximation ≈ 1.44303943

Simpson's Rule Method

n	x_n	$f\left(x_n\right)$	$m\cdot f\left(x_n\right)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.2000000000	0.9805806800	3.9223227200
2	0.4000000000	0.9284766900	1.8569533800
3	0.6000000000	0.8574929300	3.4299717200
4	0.8000000000	0.7808688100	1.5617376200
5	1.0000000000	0.7071067800	2.8284271200
6	1.2000000000	0.6401844000	1.2803688000
7	1.4000000000	0.5812381900	2.3249527600
8	1.6000000000	0.5299989400	1.0599978800
9	1.8000000000	0.4856429300	1.9425717200
10	2.0000000000	0.4472136000	0.4472136000

Simpson's Rule approximation ≈ 1.44363449

	Exact	***Trapezoid***	***Simpson***
Value:	1.44363548	1.44303943	1.44363449
Error:		0.0413 %	0.0001 %

Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^{1/2} \sin\left(e^{x/2}\right)\, dx \quad n=8\ \ subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.8414709848	0.8414709848
1	0.0625000000	0.8581952249	1.7163904498
2	0.1250000000	0.8745438796	1.7490877592
3	0.1875000000	0.8904281963	1.7808563927
4	0.2500000000	0.9057510229	1.8115020459
5	0.3125000000	0.9204063003	1.8408126006
6	0.3750000000	0.9342785616	1.8685571232
7	0.4375000000	0.9472424468	1.8944848937
8	0.5000000000	0.9591622435	0.9591622435

Trapezoid Rule approximation ≈ 0.4519476404

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.8414709848	0.8414709848
1	0.0625000000	0.8581952249	3.4327808996
2	0.1250000000	0.8745438796	1.7490877592
3	0.1875000000	0.8904281963	3.5617127853
4	0.2500000000	0.9057510229	1.8115020459
5	0.3125000000	0.9204063003	3.6816252012
6	0.3750000000	0.9342785616	1.8685571232
7	0.4375000000	0.9472424468	3.7889697874
8	0.5000000000	0.9591622435	0.9591622435

Simpson Rule approximation ≈ 0.4519764340

	Exact	***Trapezoid***	***Simpson***
Value:	0.4519764600	0.4519476404	0.4519764340
Error:		0.00638%	0.00001%

Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_2^3 \frac{1}{\ln x}\,dx \quad n = 10 \;\; subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	2.0000000000	1.4426950400	1.4426950400
1	2.1000000000	1.3478227100	2.6956454200
2	2.2000000000	1.2682994000	2.5365988000
3	2.3000000000	1.2006111700	2.4012223400
4	2.4000000000	1.1422452400	2.2844904800
5	2.5000000000	1.0913566700	2.1827133400
6	2.6000000000	1.0465599400	2.0931198800
7	2.7000000000	1.0067940700	2.0135881400
8	2.8000000000	0.9712326500	1.9424653000
9	2.9000000000	0.9392222400	1.8784444800
10	3.0000000000	0.9102392300	0.9102392300

Trapezoid Rule approximation $\approx$ 1.11906112

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	2.0000000000	1.4426950400	1.4426950400
1	2.1000000000	1.3478227100	5.3912908400
2	2.2000000000	1.2682994000	2.5365988000
3	2.3000000000	1.2006111700	4.8024446800
4	2.4000000000	1.1422452400	2.2844904800
5	2.5000000000	1.0913566700	4.3654266800
6	2.6000000000	1.0465599400	2.0931198800
7	2.7000000000	1.0067940700	4.0271762800
8	2.8000000000	0.9712326500	1.9424653000
9	2.9000000000	0.9392222400	3.7568889600
10	3.0000000000	0.9102392300	0.9102392300

Simpson's Rule approximation $\approx$ 1.11842787

	Exact	***Trapezoid***	***Simpson***
Value:	1.11842481	1.11906112	1.11842787
Error:		0.0569 %	0.0003 %

Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_1^2 e^{1/x}\,dx \quad n=4\ \ subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f\left(x_n\right)$	$m\cdot f\left(x_n\right)$
0	1.0000000000	2.7182818300	2.7182818300
1	1.2500000000	2.2255409300	4.4510818600
2	1.5000000000	1.9477340400	3.8954680800
3	1.7500000000	1.7707949500	3.5415899000
4	2.0000000000	1.6487212700	1.6487212700

Trapezoid Rule approximation ≈ 2.03189287

Simpson's Rule Method

n	x_n	$f\left(x_n\right)$	$m\cdot f\left(x_n\right)$
0	1.0000000000	2.7182818300	2.7182818300
1	1.2500000000	2.2255409300	8.9021637200
2	1.5000000000	1.9477340400	3.8954680800
3	1.7500000000	1.7707949500	7.0831798000
4	2.0000000000	1.6487212700	1.6487212700

Simpson's Rule approximation ≈ 2.02065122

	Exact	***Trapezoid***	***Simpson***
Value:	2.02005862	2.03189287	2.02065122
Error:		0.5858 %	0.0293 %

Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^1 \ln\left(1+e^x\right)\,dx \quad n=8 \;\; subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	0.6931471800	0.6931471800
1	0.1250000000	0.7575990400	1.5151980800
2	0.2500000000	0.8259394200	1.6518788400
3	0.3750000000	0.8981232600	1.7962465200
4	0.5000000000	0.9740769800	1.9481539600
5	0.6250000000	1.0537006800	2.1074013600
6	0.7500000000	1.1368710100	2.2737420200
7	0.8750000000	1.2234445800	2.4468891600
8	1.0000000000	1.3132616900	1.3132616900

Trapezoid Rule approximation ≈ 0.98411993

Simpson's Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	0.6931471800	0.6931471800
1	0.1250000000	0.7575990400	3.0303961600
2	0.2500000000	0.8259394200	1.6518788400
3	0.3750000000	0.8981232600	3.5924930400
4	0.5000000000	0.9740769800	1.9481539600
5	0.6250000000	1.0537006800	4.2148027200
6	0.7500000000	1.1368710100	2.2737420200
7	0.8750000000	1.2234445800	4.8937783200
8	1.0000000000	1.3132616900	1.3132616900

Simpson's Rule approximation ≈ 0.98381891

	Exact	***Trapezoid***	***Simpson***
Value:	0.98381904	0.98411993	0.98381891
Error:		0.0306 %	0.0000 %

Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^1 x^5 e^x \, dx \quad n = 10 \ \ subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1000000000	0.0000110500	0.0000221000
2	0.2000000000	0.0003908500	0.0007817000
3	0.3000000000	0.0032801600	0.0065603200
4	0.4000000000	0.0152762800	0.0305525600
5	0.5000000000	0.0515225400	0.1030450800
6	0.6000000000	0.1416879600	0.2833759200
7	0.7000000000	0.3384514200	0.6769028400
8	0.8000000000	0.7292652500	1.4585305000
9	0.9000000000	1.4523710400	2.9047420800
10	1.0000000000	2.7182818300	2.7182818300

Trapezoid Rule approximation ≈ 0.40913975

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1000000000	0.0000110500	0.0000442000
2	0.2000000000	0.0003908500	0.0007817000
3	0.3000000000	0.0032801600	0.0131206400
4	0.4000000000	0.0152762800	0.0305525600
5	0.5000000000	0.0515225400	0.2060901600
6	0.6000000000	0.1416879600	0.2833759200
7	0.7000000000	0.3384514200	1.3538056800
8	0.8000000000	0.7292652500	1.4585305000
9	0.9000000000	1.4523710400	5.8094841600
10	1.0000000000	2.7182818300	2.7182818300

Simpson's Rule approximation ≈ 0.39580225

	Exact	***Trapezoid***	***Simpson***
Value:	0.39559955	0.40913975	0.39580225
Error:		3.4227 %	0.0512 %

Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^4 \sqrt{x}\ \sin x\, dx \quad n = 8\ \ subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.5000000000	0.3390050500	0.6780101000
2	1.0000000000	0.8414709800	1.6829419600
3	1.5000000000	1.2216768700	2.4433537400
4	2.0000000000	1.2859407500	2.5718815000
5	2.5000000000	0.9462675500	1.8925351000
6	3.0000000000	0.2444270200	0.4888540400
7	3.5000000000	–0.6562553300	–1.3125106600
8	4.0000000000	–1.5136049900	–1.5136049900

Trapezoid Rule approximation ≈ 1.73286520

Simpson's Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.5000000000	0.3390050500	1.3560202000
2	1.0000000000	0.8414709800	1.6829419600
3	1.5000000000	1.2216768700	4.8867074800
4	2.0000000000	1.2859407500	2.5718815000
5	2.5000000000	0.9462675500	3.7850702000
6	3.0000000000	0.2444270200	0.4888540400
7	3.5000000000	–0.6562553300	–2.6250213200
8	4.0000000000	–1.5136049900	–1.5136049900

Simpson's Rule approximation ≈ 1.77214151

	Exact	***Trapezoid***	***Simpson***
Value:	1.76874870	1.73286520	1.77214151
Error:		2.0288 %	0.1918 %

Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_0^3 \frac{1}{1+x^5}\,dx \quad n = 6\ \ subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.5000000000	0.9696969700	1.9393939400
2	1.0000000000	0.5000000000	1.0000000000
3	1.5000000000	0.1163636400	0.2327272800
4	2.0000000000	0.0303030300	0.0606060600
5	2.5000000000	0.0101362100	0.0202724200
6	3.0000000000	0.0040983600	0.0040983600

Trapezoid Rule approximation ≈ 1.06427452

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.5000000000	0.9696969700	3.8787878800
2	1.0000000000	0.5000000000	1.0000000000
3	1.5000000000	0.1163636400	0.4654545600
4	2.0000000000	0.0303030300	0.0606060600
5	2.5000000000	0.0101362100	0.0405448400
6	3.0000000000	0.0040983600	0.0040983600

Simpson's Rule approximation ≈ 1.07491528

	Exact	***Trapezoid***	***Simpson***
Value:	1.06587854	1.06427452	1.07491528
Error:		0.150488%	.84782%

Exercise

Find the ***Trapezoid*** & ***Simpson's*** Rule approximations and error: $\int_1^4 \frac{e^x}{x}\,dx \quad n = 10 \;\; subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	1.0000000000	2.7182818300	2.7182818300
1	1.3000000000	2.8225359000	5.6450718000
2	1.6000000000	3.0956452700	6.1912905400
3	1.9000000000	3.5188918100	7.0377836200
4	2.2000000000	4.1022788600	8.2045577200
5	2.5000000000	4.8729975800	9.7459951600
6	2.8000000000	5.8730881300	11.7461762600
7	3.1000000000	7.1606294500	14.3212589000
8	3.4000000000	8.8129706000	17.6259412000
9	3.7000000000	10.9317038800	21.8634077600
10	4.0000000000	13.6495375100	13.6495375100

Trapezoid Rule approximation $\approx$ 17.81239534

Simpson's Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	1.0000000000	2.7182818300	2.7182818300
1	1.3000000000	2.8225359000	11.2901436000
2	1.6000000000	3.0956452700	6.1912905400
3	1.9000000000	3.5188918100	14.0755672400
4	2.2000000000	4.1022788600	8.2045577200
5	2.5000000000	4.8729975800	19.4919903200
6	2.8000000000	5.8730881300	11.7461762600
7	3.1000000000	7.1606294500	28.6425178000
8	3.4000000000	8.8129706000	17.6259412000
9	3.7000000000	10.9317038800	43.7268155200
10	4.0000000000	13.6495375100	13.6495375100

Simpson's Rule approximation $\approx$ 17.73628195

	Exact	***Trapezoid***	***Simpson***
Value:	17.73575665	17.81239534	17.73628195
Error:		0.4321 %	0.0030 %

Exercise

Find the ***Trapezoid*** & ***Simpson’s*** Rule approximations and error: $\int_1^2 \frac{dx}{x} \quad n=10 \;\; subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	1.0000000000	1.0000000000	1.0000000000
1	1.1000000000	0.9090909100	1.8181818200
2	1.2000000000	0.8333333300	1.6666666600
3	1.3000000000	0.7692307700	1.5384615400
4	1.4000000000	0.7142857100	1.4285714200
5	1.5000000000	0.6666666700	1.3333333400
6	1.6000000000	0.6250000000	1.2500000000
7	1.7000000000	0.5882352900	1.1764705800
8	1.8000000000	0.5555555600	1.1111111200
9	1.9000000000	0.5263157900	1.0526315800
10	2.0000000000	0.5000000000	0.5000000000

Trapezoid Rule approximation ≈ 0.69377140

Simpson's Rule Method

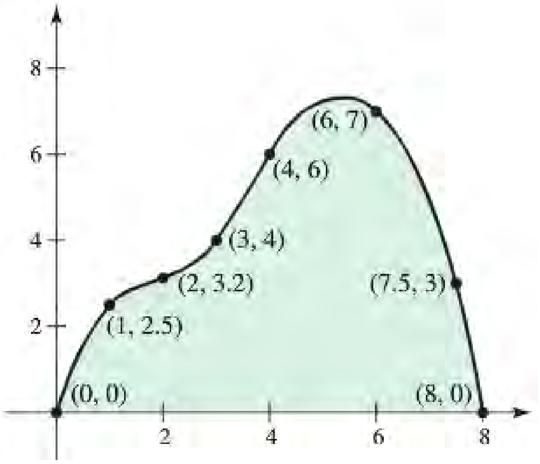
n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	1.0000000000	1.0000000000	1.0000000000
1	1.1000000000	0.9090909100	3.6363636400
2	1.2000000000	0.8333333300	1.6666666600
3	1.3000000000	0.7692307700	3.0769230800
4	1.4000000000	0.7142857100	1.4285714200
5	1.5000000000	0.6666666700	2.6666666800
6	1.6000000000	0.6250000000	1.2500000000
7	1.7000000000	0.5882352900	2.3529411600
8	1.8000000000	0.5555555600	1.1111111200
9	1.9000000000	0.5263157900	2.1052631600
10	2.0000000000	0.5000000000	0.5000000000

Simpson's Rule approximation ≈ 0.69315023

	Exact	***Trapezoid***	***Simpson***
Value:	0.69314718	0.69377140	0.69315023
Error:		0.0901%	0.0004%

Exercise

A piece of wood paneling must be cut in the shape shown below. The coordinates of several point on its curved surface are also shown (with units of inches).



a) Estimate the surface area of the paneling using the Trapezoid Rule

b) Estimate the surface area of the paneling using a left Riemann sum.

c) Could two identical pieces be cut from a 9-in by 9-in piece of wood?

Solution

a) The *trapezoid* Rule gives

$$\frac{(0+.25)\cdot 1}{2}+\frac{(2.5+3.2)\cdot 1}{2}+\frac{(3.2+4)\cdot 1}{2}+\frac{(4+6)\cdot 1}{2}+\frac{(6+7)\cdot 2}{2}+\frac{(7+5.3)\cdot 1.5}{2}+\frac{(3+0)\cdot 0.5}{2}$$

$$\underline{=35.675\,|}$$

b) The left *Riemann* sum gives

$$0\cdot 1+2.5\cdot 1+3.2\cdot 1+4\cdot 1+6\cdot 2+7\cdot 1.5+5.3\cdot 0.5\ \underline{=34.85\,|}$$

c) Although the surface area of the piece appears to be less than half of $81=9^2$ (area of 9×9 piece of wood), the shape prohibits the creation of two identical pieces.

Exercise

The region bounded by the curves $y=\frac{1}{1+e^{-x}}$, $x=0$ and $x=10$ is rotated about $x-axis$. Use Simpson's Rule with $n=10$ to estimate the volume of the resulting solid.

Solution

Using *Disk* method:

$$V=\pi\int_0^{10}\frac{1}{\left(1+e^{-x}\right)^2}\,dx$$

Simpson's Rule Method

n	x_n	$f\left(x_n\right)$	$m\cdot f\left(x_n\right)$
0	0.0000000000	0.2500000000	0.2500000000
1	1.0000000000	0.5344466454	2.1377865816
2	2.0000000000	0.7758034926	1.5516069851
3	3.0000000000	0.9073974671	3.6295898684
4	4.0000000000	0.9643510838	1.9287021676
5	5.0000000000	0.9866590924	3.9466363696
6	6.0000000000	0.9950608676	1.9901217351
7	7.0000000000	0.9981787276	3.9927149105
8	8.0000000000	0.9993294122	1.9986588244
9	9.0000000000	0.9997532261	3.9990129043
10	10.000000000	0.9999092063	0.9999092063

Simpson Rule approximation ≈ 8.8082465177

$$V=\pi\int_0^{10}\frac{1}{\left(1+e^{-x}\right)^2}\,dx$$

$$\approx \pi\left(8.8082465177\right)$$

$$\boxed{\approx 27.6719226 \quad unit^3}$$

Exercise

A pendulum with length L that makes a maximum angle θ_0 with the vertical. Using Newton's Second Law it can be shown that the period T (the time for one complete swing) is given by

$$T=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2 x}}$$

Where $k=\sin\left(\frac{1}{2}\theta_0\right)$ and g is the acceleration due to gravity. If $L=1\,m$ and $\theta_0=42^\circ$, use Simpson's Rule with $n=10$ to find the period.

Solution

$$T=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2 x}}$$

$$=4\sqrt{\frac{1}{9.8}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-\sin^2\left(\frac{1}{2}42^\circ\right)\sin^2 x}}$$

$$=\frac{4}{\sqrt{9.8}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-\sin^2(21^\circ)\sin^2 x}}$$

Simpson's Rule Method $\displaystyle\int_0^{\pi/2}\frac{dx}{\sqrt{1-\sin^2(21^\circ)\sin^2 x}}$

n	x_n	$f(x_n)$	$m\cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.1570796327	1.0031527554	4.0126110216
2	0.3141592654	1.0124160101	2.0248320201
3	0.4712388980	1.0271895774	4.1087583096
4	0.6283185307	1.0464308046	2.0928616093
5	0.7853981634	1.0686201540	4.2744806162
6	0.9424777961	1.0917709315	2.1835418629
7	1.0995574288	1.1135333115	4.4541332459
8	1.2566370614	1.1314314233	2.2628628466
9	1.4137166941	1.1432291699	4.5729166795
10	1.5707963268	1.1473515974	1.1473515974

Simpson Rule approximation ≈ 1.6825506215

$$T=\frac{4}{\sqrt{9.8}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-\sin^2(21^\circ)\sin^2 x}}$$

$$\approx\frac{4}{\sqrt{9.8}}(1.6825506215)$$

$$\underline{\approx 2.149884\,|}$$

Exercise

Evaluate the integral $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$

Solution

$$\int_0^\infty \frac{dx}{x^2+1} = \lim_{b\to\infty}\int_0^b \frac{dx}{x^2+1}$$

$$= \lim_{b\to\infty}\left(\tan^{-1} x \,\Big|_0^b\right.$$

$$= \lim_{b\to\infty}\left(\tan^{-1} b - \tan^{-1} 0\right)$$

$$= \frac{\pi}{2} - 0$$

$$\boxed{= \frac{\pi}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$

Solution

$$\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{b\to 4^-}\int_0^b (4-x)^{-1/2}\,dx$$

$$= \lim_{b\to 4^-}\int_0^b -(4-x)^{-1/2}\,d(4-x)$$

$$= -2\lim_{b\to 4^-}\left((4-x)^{1/2}\,\Big|_0^b\right.$$

$$= -2\lim_{b\to 4^-}\left((4-b)^{1/2} - (4)^{1/2}\right)$$

$$= -2(0-2)$$

$$\boxed{= 4}$$

Exercise

Evaluate the integral $\int_{-\infty}^{2} \frac{2dx}{x^2+4}$

Solution

$$\int_{-\infty}^{2} \frac{2dx}{x^2+4} = 2\lim_{b\to-\infty}\int_{b}^{2} \frac{dx}{x^2+2^2}$$

$$= 2\lim_{b\to-\infty}\frac{1}{2}\left(\tan^{-1}\frac{x}{2}\right|_{b}^{2}$$

$$= \lim_{b\to-\infty}\left(\tan^{-1}1-\tan^{-1}\frac{b}{2}\right)$$

$$= \frac{\pi}{4}-\left(-\frac{\pi}{2}\right)$$

$$\boxed{= \frac{3\pi}{4}}$$

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} \frac{xdx}{\left(x^2+4\right)^{3/2}}$

Solution

$$\int_{-\infty}^{\infty} \frac{xdx}{\left(x^2+4\right)^{3/2}} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{d\left(x^2+4\right)}{\left(x^2+4\right)^{3/2}} \qquad u = x^2+4 \quad \to du = 2xdx$$

$$= \frac{1}{2}\left(-2\left(x^2+4\right)^{-1/2}\right|_{-\infty}^{\infty}$$

$$= -\frac{1}{\sqrt{x^2+4}}\Bigg|_{-\infty}^{\infty}$$

$$= -(0-0)$$

$$\boxed{= 0}$$

Exercise

Evaluate the integral $\int_{1}^{\infty} \frac{dx}{x\sqrt{x^2-1}}$

Solution

$$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^\infty \frac{dx}{x\sqrt{x^2-1}}$$

$$= \lim_{b\to 1^+} \int_b^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{c\to\infty} \int_2^c \frac{dx}{x\sqrt{x^2-1}}$$

$$= \lim_{b\to 1^+} \left(\sec^{-1}|x|\right|_b^2 + \lim_{c\to\infty} \left(\sec^{-1}|x|\right|_2^c$$

$$= \lim_{b\to 1^+} \left(\sec^{-1}2 - \sec^{-1}b\right) + \lim_{c\to\infty} \left(\sec^{-1}c - \sec^{-1}2\right)$$

$$= \left(\frac{\pi}{3} - 0\right) + \left(\frac{\pi}{2} - \frac{\pi}{3}\right)$$

$$= \frac{\pi}{2}$$

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx$

Solution

$$\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx = \int_{-\infty}^{0} 2xe^{-x^2}\,dx + \int_{0}^{\infty} 2xe^{-x^2}\,dx$$

$$d\left(-x^2\right) = -2xdx$$

$$= -\lim_{b\to-\infty} \int_b^0 e^{-x^2}\,d\left(-x^2\right) - \lim_{c\to\infty} \int_0^c e^{-x^2}\,d\left(-x^2\right)$$

$$= -\lim_{b\to-\infty} \left(e^{-x^2}\right|_b^0 - \lim_{c\to\infty} \left(e^{-x^2}\right|_0^c$$

$$= -\lim_{b\to-\infty} \left(1 - e^{-b^2}\right) - \lim_{c\to\infty} \left(e^{-c^2} - 1\right)$$

$$= -(1-0) - (0-1)$$

$$= 0$$

Exercise

Evaluate the integral $\int_0^1 (-\ln x)\,dx$

Solution

$$u = \ln x \qquad v = \int dx = x$$

$$du = \frac{1}{x}dx$$

$$\int_0^1 (-\ln x)\,dx = -x\ln x + \int_0^1 x\frac{1}{x}dx$$

$$= -x\ln x + \int_0^1 dx$$

$$= -x\ln x + x\,\Big|_0^1$$

$$= 0 + 1 + 0 - 0$$

$$\underline{= 1\,|}$$

Exercise

Evaluate the integral $\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$

Solution

$$\int_{-1}^4 \frac{dx}{\sqrt{|x|}} = \int_{-1}^{0^-} \frac{dx}{\sqrt{-x}} + \int_{0^+}^4 \frac{dx}{\sqrt{x}}$$

$$= \left(-2\sqrt{-x}\,\right|_{-1}^{0^-} + \left(2\sqrt{x}\,\right|_{0^+}^{4}$$

$$= 2 + 4$$

$$\underline{= 6\,|}$$

Exercise

Evaluate the integral $\int_0^\infty e^{-3x}\,dx$

Solution

$$\int_0^{\infty} e^{-3x}\ dx = -\frac{1}{3}e^{-3x}\Big|_0^{\infty}$$

$$= -\frac{1}{3}\left(e^{-\infty} - 1\right)$$

$$\boxed{= \frac{1}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Solution

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \tan^{-1} x\Big|_{-\infty}^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1}(-\infty)$$

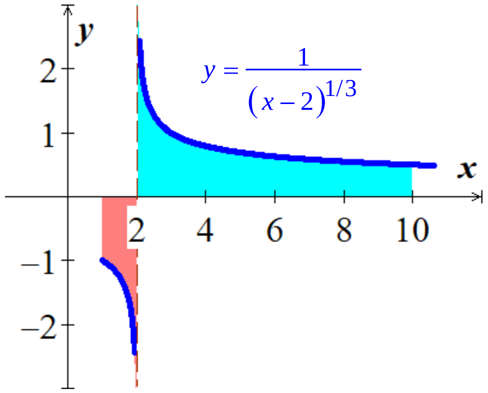
$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$\boxed{= \pi}$$

Exercise

Evaluate the integral $\displaystyle\int_1^{10} \frac{dx}{(x-2)^{1/3}}$

Solution



$$\int_1^{10} (x-2)^{-1/3}\ dx = \frac{3}{2}\ (x-2)^{2/3}\ \Big|_1^{10}$$

$$= \frac{3}{2}\left(8^{2/3} - (-1)^{2/3}\right)$$

$$= \frac{3}{2}(4-1)$$

$$\boxed{= \frac{9}{2}}$$

$$\int_1^{10} (x-2)^{-1/3}\ dx = \int_1^{2} (x-2)^{-1/3}\ dx + \int_2^{10} (x-2)^{-1/3}\ dx$$

$$= \frac{3}{2}\ (x-2)^{2/3}\ \Big|_1^{2}\ +\ (x-2)^{2/3}\ \Big|_2^{10}$$

$$= \frac{3}{2}\left(0-(-1)^{2/3}\right)+\frac{3}{2}\left(8^{2/3}-0\right)$$

$$= \frac{3}{2}(-1+4)$$

$$\underline{= \frac{9}{2}}\,\Big|$$

Exercise

Evaluate the integral $\int_{1}^{\infty} \frac{dx}{x^{2}}$

Solution

$$\int_{1}^{\infty} \frac{dx}{x^{2}} = -\frac{1}{x}\Big|_{1}^{\infty}$$

$$= -\left(\frac{1}{\infty}-1\right)$$

$$= -(0-1)$$

$$\underline{=1}\,\Big|$$

Exercise

Evaluate the integral $\int_{0}^{\infty} \frac{dx}{(x+1)^{3}}$

Solution

$$\int_{0}^{\infty} (x+1)^{-3}dx = -\frac{2}{(x+1)^{2}}\Big|_{0}^{\infty}$$

$$= -2\left(\frac{1}{\infty}-1\right)$$

$$= -2(0-1)$$

$$\underline{=2}\,\Big|$$

Exercise

Evaluate the integral $\int_{-\infty}^{0} e^{x}\, dx$

Solution

$$\int_{-\infty}^{0} e^{x}\, dx = e^{x}\Big|_{-\infty}^{0}$$

$$=\left(1-e^{-\infty}\right) \qquad e^{-\infty}\to 0$$

$$\underline{=1}\,\rfloor$$

Exercise

Evaluate the integral $\displaystyle\int_1^{\infty} 2^{-x}\,dx$

Solution

$$\int_1^{\infty} 2^{-x}\,dx=-\int_1^{\infty} 2^{-x}\,d(-x) \qquad \int a^x dx=\frac{a^x}{\ln a}$$

$$=-\frac{2^{-x}}{\ln 2}\Big|_1^{\infty}$$

$$=-\frac{1}{\ln 2}\left(0-\frac{1}{2}\right)$$

$$\boxed{=\frac{1}{2\ln 2}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{0} \frac{dx}{\sqrt[3]{2-x}}$

Solution

$$\int_{-\infty}^{0} \frac{dx}{\sqrt[3]{2-x}}=-\int_{-\infty}^{0} (2-x)^{-1/3}\,d(2-x)$$

$$=-\frac{3}{2}(2-x)^{2/3}\Big|_{-\infty}^{0}$$

$$=-\frac{3}{2}\left(2^{2/3}-\infty\right)$$

$$\underline{=\infty}\,\rfloor \quad \textit{diverges}$$

Exercise

Evaluate the integral $\displaystyle\int_{4/\pi}^{\infty} \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right)dx$

Solution

$$\int_{4/\pi}^{\infty} \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right) dx = -\int_{4/\pi}^{\infty} \sec^2\left(\frac{1}{x}\right) d\left(\frac{1}{x}\right) \qquad d\left(\frac{1}{x}\right) = -\frac{1}{x^2}dx$$

$$= -\tan\left(\frac{1}{x}\right) \Big|_{4/\pi}^{\infty}$$

$$= -\left(\tan 0 - \tan\frac{\pi}{4}\right)$$

$$\boxed{=1}$$

Exercise

Evaluate the integral $\displaystyle\int_{e^2}^{\infty} \frac{dx}{x \ln^p x} \quad p > 1$

Solution

$$\int_{e^2}^{\infty} \frac{dx}{x \ln^p x} = \int_{e^2}^{\infty} (\ln x)^{-p}\ d(\ln x)$$

$$= \frac{1}{1-p}\ (\ln x)^{1-p}\ \Big|_{e^2}^{\infty}$$

$$= \frac{1}{1-p}\left((\ln x)^{-\infty} - \left(\ln e^2\right)^{1-p}\right)$$

$$= \frac{-1}{1-p}2^{1-p}$$

$$\boxed{= \frac{1}{(p-1)2^{p-1}}}$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{\infty} \frac{p}{\sqrt[5]{p^2+1}}\, dp$

Solution

$$\int_{0}^{\infty} \frac{p}{\sqrt[5]{p^2+1}}\, dp = \frac{1}{2}\int_{0}^{\infty} \left(p^2+1\right)^{-1/5}\ d\left(p^2+1\right) \qquad d\left(p^2+1\right) = 2pdp$$

$$= \frac{5}{8}\left(p^2+1\right)^{4/5}\ \Big|_{0}^{\infty}$$

$$\boxed{=\infty} \quad \textit{diverges}$$

Exercise

Evaluate the integral $\int_{-1}^{1} \ln y^2 \; dy$

Solution

$$\int_{-1}^{1} \ln y^2 \; dy = 2\int_{0}^{1} \ln y^2 \; dy$$

$$\int \ln x^2 \; dx = 2\int \ln x \; dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$

$$v = \int dx = x$$

$$= 2\left[x\ln x - \int dx \right]$$

$$\underline{= 2\left(x\ln x - x\right)}\,\Big|$$

$$\int_{-1}^{1} \ln y^2 \; dy = 4\left(y\ln y - y\;\right|_{0}^{1}$$

$$= 4(-1-0)$$

$$\underline{= -4}\,\Big|$$

Exercise

Evaluate the integral $\int_{-2}^{6} \frac{dx}{\sqrt{|x-2|}}$

Solution

$$\int_{-2}^{6} \frac{dx}{\sqrt{|x-2|}} = \int_{-2}^{2} \frac{dx}{\sqrt{2-x}} + \int_{2}^{6} \frac{dx}{\sqrt{x-2}}$$

$$= -\int_{-2}^{2} (2-x)^{-1/2} \, d(2-x) + \int_{2}^{6} (x-2)^{-1/2} \, d(x-2)$$

$$= -2\left(\sqrt{2-x}\;\right|_{-2}^{2} + 2\left(\sqrt{x-2}\;\right|_{2}^{6}$$

$$= -2(0-2) + 2(2-0)$$

$$\underline{= 8}\,\Big|$$

Exercise

Evaluate $\displaystyle\int_0^{\infty} xe^{-x}\, dx$

Solution

		$\int e^{-x}dx$
+	x	$-e^{-x}$
−	1	e^{-x}

$$\int_0^{\infty} xe^{-x}\, dx = -xe^{-x} - e^{-x} \Big|_0^{\infty}$$

$$= 0 - (-1)$$

$$\boxed{= 1}$$

Exercise

Evaluate $\displaystyle\int_0^{1} x \ln x \, dx$

Solution

$u = \ln x \qquad dv = x\, dx$

$du = \dfrac{dx}{x} \qquad v = \dfrac{1}{2}x^2$

$$\int x \ln x \, dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\, dx$$

$$\boxed{= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2}$$

$$\int_0^{1} x \ln x \, dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 \Big|_0^{1}$$

$$\boxed{= -\frac{1}{4}}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\, dx$

Solution

$$u = \ln x \qquad dv = \frac{1}{x^2}\,dx$$

$$du = \frac{dx}{x} \qquad v = -\frac{1}{x}$$

$$\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x + \int \frac{1}{x^2}\,dx$$

$$= -\frac{1}{x}\ln x - \frac{1}{x}$$

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = -\frac{1}{x}(\ln x + 1)\,\Big|_1^\infty$$

$$\boxed{= 1}$$

Exercise

Evaluate $\displaystyle\int_1^\infty (1-x)e^{-x}\,dx$

Solution

$$\int_1^\infty (1-x)e^{-x}\,dx = \left(-e^{-x} - (-x-1)e^{-x}\right.\Big|_1^\infty$$

$$= xe^{-x}\Big|_1^\infty \qquad\qquad e^{-\infty} \to 0$$

$$= 0 - e^{-1}$$

$$\boxed{= -\frac{1}{e}}$$

Exercise

Evaluate $\displaystyle\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx$

Solution

$$\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx = \int_{-\infty}^\infty \frac{d\left(e^x\right)}{1+\left(e^x\right)^2}$$

$$= \arctan e^x\,\Big|_{-\infty}^\infty$$

$$= \arctan\infty - \arctan 0$$

$$\boxed{= \frac{\pi}{2}}$$

Exercise

Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}$

Solution

$$\int_0^1 x^{-1/3}\, dx = \frac{3}{2}x^{2/3}\ \Big|_0^1$$

$$\boxed{= \frac{3}{2}}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} 4x^{-1/4}\, dx$

Solution

$$\int_1^{\infty} 4x^{-1/4}\, dx = \frac{16}{3}x^{3/4}\ \Big|_1^{\infty}$$

$$\boxed{= \infty}$$ ***Diverges***

Exercise

Evaluate $\displaystyle\int_0^2 \frac{dx}{x^3}$

Solution

$$\int_0^2 \frac{dx}{x^3} = -\frac{1}{2x^2}\ \Big|_0^2$$

$$= -\frac{1}{8} + \infty$$

$$\boxed{= \infty}$$ ***Diverges***

Exercise

Evaluate $\displaystyle\int_1^{\infty} \frac{dx}{x^3}$

Solution

$$\int_1^{\infty} \frac{dx}{x^3} = -\frac{1}{2x^2}\ \Big|_1^{\infty}$$

$$= -\frac{1}{\infty} + \frac{1}{2}$$

$$\underline{= \frac{1}{2}}\Big|$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} 6x^{-4}\, dx$

Solution

$$\int_1^{\infty} 6x^{-4}\, dx = -2\frac{1}{x^3}\Big|_1^{\infty}$$

$$\underline{= 2}\Big|$$

Exercise

Evaluate $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x}\,(x+1)}$

Solution

$$u = \sqrt{x} \;\rightarrow\; u^2 = x$$

$$dx = 2udu$$

$$\int_0^{\infty} \frac{dx}{\sqrt{x}\,(x+1)} = \int_0^{\infty} \frac{2u}{u\left(u^2+1\right)}du$$

$$= 2\int_0^{\infty} \frac{1}{u^2+1}\, du$$

$$= 2\arctan\sqrt{x}\,\Big|_0^{\infty}$$

$$= 2\left(\frac{\pi}{2} - 0\right)$$

$$\underline{= \pi}\Big|$$

Exercise

Evaluate $\displaystyle\int_{-\infty}^{0} xe^{-4x}\, dx$

Solution

$$\int_{-\infty}^{0} xe^{-4x}\,dx = \left(-\frac{x}{4}-\frac{1}{16}\right)e^{-4x}\Big|_{-\infty}^{0} \qquad\qquad \int x^n e^{ax}\,dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k\,n!}{a^{k+1}(n-k)!}x^{n-k}$$

$$= -\frac{1}{16}-\infty$$

$$\underline{= -\infty\,|}$$ ***Diverges***

Exercise

Evaluate $\displaystyle\int_{0}^{\infty} xe^{-x/3}\,dx$

Solution

$$\int_{0}^{\infty} xe^{-x/3}\,dx = (-3x-9)e^{-x/3}\Big|_{0}^{\infty} \qquad\qquad \int x^n e^{ax}\,dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k\,n!}{a^{k+1}(n-k)!}x^{n-k}$$

$$\underline{= 9\,|}$$

Exercise

Evaluate $\displaystyle\int_{0}^{\infty} x^2e^{-x}dx$

Solution

$$\int_{0}^{\infty} x^2e^{-x}dx = \left(-x^2-2x-2\right)e^{-x}\Big|_{0}^{\infty} \qquad\qquad \int x^n e^{ax}\,dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k\,n!}{a^{k+1}(n-k)!}x^{n-k}$$

$$\underline{= 2\,|}$$

Exercise

Evaluate $\displaystyle\int_{0}^{\infty} e^{-x}\cos x\,dx$

Solution

		$\int \cos x\,dx$
+	e^{-x}	$\sin x$
−	$-e^{-x}$	$-\cos x$
+	e^{-x}	$-\int \cos x\,dx$

$$\int e^{-x}\cos x\,dx = e^{-x}(\sin x - \cos x) - \int e^{-x}\cos x\,dx$$

$$2\int e^{-x}\cos x\,dx = e^{-x}(\sin x - \cos x)$$

$$\int_0^{\infty} e^{-x}\cos x\,dx = \frac{1}{2}e^{-x}(\sin x - \cos x)\ \Big|_0^{\infty}$$

$$\boxed{=\frac{1}{2}}$$

Exercise

Evaluate $\displaystyle\int_4^{\infty}\frac{1}{x(\ln x)^3}\,dx$

Solution

$$\int_4^{\infty}\frac{1}{x(\ln x)^3}\,dx = \int_4^{\infty}(\ln x)^{-3}\,d(\ln x)$$

$$= -\frac{1}{2}\frac{1}{(\ln x)^2}\ \Big|_4^{\infty}$$

$$= \frac{1}{2}\left(0 - \frac{1}{(\ln 4)^2}\right)$$

$$\boxed{= \frac{1}{2(\ln 4)^2}}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty}\frac{\ln x}{x}\,dx$

Solution

$$\int_1^{\infty}\frac{\ln x}{x}\,dx = \int_1^{\infty}\ln x\,d(\ln x)$$

$$= \frac{1}{2}(\ln x)^2\ \Big|_1^{\infty}$$

$$\boxed{= \infty}\quad \textit{diverges}$$

Exercise

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{4}{16+x^2}\, dx$

Solution

$$\int_{-\infty}^{\infty} \frac{4}{16+x^2}\, dx = \arctan\left(\frac{x}{4}\right) \Big|_{-\infty}^{\infty}$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$

$$\underline{= \pi \,|}$$

Exercise

Evaluate $\displaystyle\int_{0}^{\infty} \frac{x^3}{\left(x^2+1\right)^2} dx$

Solution

$$\frac{x^3}{\left(x^2+1\right)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{\left(x^2+1\right)^2}$$

$$x^3 = Ax^3 + Ax + Bx^2 + B + Cx + D$$

$$\begin{cases} x^3 & & \underline{A=1\,|} \\ x^2 & & \underline{B=0\,|} \\ x & A+C=0 \rightarrow & \underline{C=-1\,|} \\ x^0 & B+D=0 \rightarrow & \underline{D=0\,|} \end{cases}$$

$$\int_0^{\infty} \frac{x^3}{\left(x^2+1\right)^2} dx = \int_0^{\infty} \frac{x}{x^2+1} dx - \int_0^{\infty} \frac{x}{\left(x^2+1\right)^2} dx$$

$$= \frac{1}{2}\int_0^{\infty} \frac{1}{x^2+1} d\left(x^2+1\right) - \frac{1}{2}\int_0^{\infty} \frac{1}{\left(x^2+1\right)^2} d\left(x^2+1\right)$$

$$= \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\frac{1}{x^2+1} \Big|_0^{\infty}$$

$$\underline{= \infty \,|} \quad \textit{diverges}$$

Exercise

Evaluate $\displaystyle\int_0^\infty \frac{1}{e^x+e^{-x}}\,dx$

Solution

$$\int_0^\infty \frac{1}{e^x+e^{-x}}\,dx = \int_0^\infty \frac{1}{e^x+e^{-x}}\frac{e^x}{e^x}\,dx$$

$$= \int_0^\infty \frac{e^x}{e^{2x}+1}\,dx$$

$$= \int_0^\infty \frac{1}{\left(e^x\right)^2+1}\,d\left(e^x\right)$$

$$= \arctan e^x \Big|_0^\infty$$

$$= \arctan(\infty) - \arctan(1)$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$\boxed{= \frac{\pi}{4}}$$

Exercise

Evaluate $\displaystyle\int_0^\infty \frac{e^x}{1+e^x}\,dx$

Solution

$$\int_0^\infty \frac{e^x}{1+e^x}\,dx = \int_0^\infty \frac{1}{1+e^x}\,d\left(e^x\right)$$

$$= \ln\left(1+e^x\right)\Big|_0^\infty$$

$\boxed{= \infty}$ *diverges*

Exercise

Evaluate $\displaystyle\int_0^\infty \cos \pi x\,dx$

Solution

$$\int_0^\infty \cos \pi x \, dx = \frac{1}{\pi} \sin \pi x \Big|_0^\infty$$

$$\underline{= \infty \,\big|} \quad \textit{diverges}$$

Exercise

Evaluate $\displaystyle\int_0^\infty \sin \frac{x}{2}\, dx$

Solution

$$\int_0^\infty \sin \frac{x}{2}\, dx = -2 \cos \frac{x}{2} \Big|_0^\infty$$

$$\underline{= \infty \,\big|} \quad \textit{diverges}$$

Exercise

Evaluate the integral $\displaystyle\int_1^\infty \frac{dx}{(x+1)^9}$

Solution

$$\int_1^\infty \frac{dx}{(x+1)^9} = \int_1^\infty (x+1)^{-9}\, d(x+1)$$

$$= -\frac{1}{8}(x+1)^{-8} \Big|_1^\infty$$

$$= -\frac{1}{8}\left(0 - \frac{1}{2^8}\right)$$

$$\underline{= \frac{1}{2,048} \,\Big|}$$

Exercise

Evaluate the integral $\displaystyle\int_1^\infty \frac{3x-1}{4x^3 - x^2}\, dx$

Solution

$$\frac{3x-1}{4x^3 - x^2} = \frac{3x-1}{x^2(4x-1)}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{4x-1}$$

$$3x - 1 = 4Ax^2 - Ax + 4Bx - B + Cx^2$$

$x^2 \quad 4A + C = 0 \qquad \rightarrow \boxed{C = -4}$

$x^1 \quad -A + 4B = 3 \qquad \rightarrow \boxed{A = 1}$

$x^0 \quad -B = -1 \qquad \rightarrow \boxed{B = 1}$

$$\int_1^\infty \frac{3x-1}{4x^3 - x^2}\, dx = \int_1^\infty \left(\frac{1}{x} + \frac{1}{x^2} - \frac{4}{4x-1} \right) dx$$

$$= \ln x - \frac{1}{x} - \int_1^\infty \frac{1}{4x-1}\, d(4x-1)$$

$$= \ln x - \frac{1}{x} - \ln(4x-1) \Big|_1^\infty$$

$$= \ln \frac{x}{4x-1} - \frac{1}{x} \Big|_1^\infty$$

$$= \ln \frac{1}{4} - 0 - \ln \frac{1}{3} - 1$$

$$= -\ln 4 + \ln 3 - 1$$

$$\boxed{= \ln \frac{3}{4} - 1}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{4}{x^2 + 16}\, dx$

Solution

$$\int_{-\infty}^{\infty} \frac{4}{x^2 + 16}\, dx = 2\int_0^\infty \frac{4}{x^2 + 4^2}\, dx$$

$$= 2 \arctan \frac{x}{4} \Big|_0^\infty$$

$$= 2 \arctan \infty - 2 \arctan 0$$

$$= 2\left(\frac{\pi}{2} - 0 \right)$$

$$\boxed{= \pi}$$

Exercise

Evaluate the integral $\int_{-\infty}^{-1} \frac{dx}{(x-1)^4}$

Solution

$$\int_{-\infty}^{-1} \frac{dx}{(x-1)^4} = \int_{-\infty}^{-1} (x-1)^{-4}\, d(x-1)$$

$$= -\frac{1}{3}\frac{1}{(x-1)^3} \Big|_{-\infty}^{-1}$$

$$= -\frac{1}{3}\left(-\frac{1}{8} - 0\right)$$

$$\underline{= \frac{1}{24}}\,\Big|$$

Exercise

Evaluate the integral $\int_{0}^{\infty} xe^{-x}\, dx$

Solution

$$\int_{0}^{\infty} xe^{-x}\, dx = -e^{-x}(x+1)\Big|_{0}^{\infty}$$

$$= -0 + 1$$

$$\underline{= 1}\,\Big|$$

Exercise

Evaluate the integral $\int_{0}^{\infty} \frac{6x}{1+x^6}\, dx$

Solution

Let $u = x^2$

$du = 2xdx$

$$\int_{0}^{\infty} \frac{6x}{1+x^6}\, dx = \int_{0}^{\infty} \frac{3\, du}{1+u^3}$$

$$\frac{3}{1+u^3}=\frac{A}{1+u}+\frac{Bu+C}{1-u+u^2}$$

$$3=A-Au+Au^2+Bu+Bu^2+C+Cu$$

$$\begin{cases} u^2 & A+B=0 \\ u^1 & -A+B+C=0 \\ u^0 & A+C=3 \end{cases}$$

$$\Delta=\begin{vmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}=3 \qquad \Delta_A=\begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix}=3$$

$$A=\frac{3}{3}\ \underline{=1}|$$

$$B=-A\ \underline{=-1}|$$

$$C=3-A\ \underline{=2}|$$

$$\int_0^\infty \frac{6x}{1+x^6}\,dx=\int_0^\infty \frac{1}{1+u}\,du+\int_0^\infty \frac{-u+2}{1-u+u^2}\,du$$

$$=\int_0^\infty \frac{1}{1+u}\,d(u+1)-\frac{1}{2}\int_0^\infty \frac{2u-1+3}{1-u+u^2}\,du$$

$$=\ln(u+1)\ \Big|_0^\infty-\frac{1}{2}\int_0^\infty \frac{2u-1}{1-u+u^2}\,du-\frac{3}{2}\int_0^\infty \frac{1}{\left(u-\frac{1}{2}\right)^2+\frac{3}{4}}\,du$$

$$=\ln\left(x^2+1\right)\ \Big|_0^\infty-\frac{1}{2}\int_0^\infty \frac{1}{1-u+u^2}\,d\left(1-u+u^2\right)-\frac{3}{2}\int_0^\infty \frac{1}{\left(u-\frac{1}{2}\right)^2+\frac{3}{4}}\,d\left(u-\frac{1}{2}\right)$$

$$=\ln\left(x^2+1\right)-\frac{1}{2}\ln\left|1-u+u^2\right|-\frac{3}{2}\frac{2}{\sqrt{3}}\tan\left(\left(u-\frac{1}{2}\right)\frac{2}{\sqrt{3}}\right)\ \Big|_0^\infty$$

$$=\ln\left(x^2+1\right)-\frac{1}{2}\ln\left|1-x^2+x^4\right|-\sqrt{3}\ \tan\left(\frac{2x^2-1}{\sqrt{3}}\right)\ \Big|_0^\infty$$

$$=\ln\left(x^2+1\right)-\ln\sqrt{1-x^2+x^4}-\sqrt{3}\ \tan\left(\frac{2x^2-1}{\sqrt{3}}\right)\ \Big|_0^\infty$$

$$=\ln\frac{x^2+1}{\sqrt{1-x^2+x^4}}-\sqrt{3}\ \tan\left(\frac{2x^2-1}{\sqrt{3}}\right)\ \Big|_0^\infty$$

$$= \ln 1 - \sqrt{3}\ \tan(\infty) - \ln 1 + \sqrt{3}\tan\left(-\frac{1}{\sqrt{3}}\right)$$

$$= -\sqrt{3}\ \frac{\pi}{2} - \sqrt{3}\frac{\pi}{6}$$

$$\underline{= -\frac{2\pi}{3}\sqrt{3}\ \Big|}$$

Exercise

Evaluate the integral $\int_0^2 \frac{dx}{\sqrt[3]{|x-1|}}$

Solution

$$\int_0^2 \frac{dx}{\sqrt[3]{|x-1|}} = \int_0^1 \frac{dx}{\sqrt[3]{|x-1|}} + \int_1^2 \frac{dx}{\sqrt[3]{|x-1|}}$$

$$= -\int_0^1 (1-x)^{-1/3}\, d(1-x) + \int_1^2 (x-1)^{-1/3}\, d(x-1)$$

$$= -\frac{3}{2}(1-x)^{2/3}\ \Big|_0^1 + \frac{3}{2}(x-1)^{2/3}\ \Big|_1^2$$

$$= -\frac{3}{2}(0-1) + \frac{3}{2}(1-0)$$

$$= \frac{3}{2} + \frac{3}{2}$$

$$\underline{= 3\ \Big|}$$

$$\int_0^2 \frac{dx}{\sqrt[3]{|x-1|}} = \int_0^2 (x-1)^{-1/3}\, d(x-1)$$

$$= \frac{3}{2}(x-1)^{2/3}\ \Big|_0^2$$

$$= \frac{3}{2}(1+1)$$

$$\underline{= 3\ \Big|}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{1} \frac{dx}{x^2+2x+5}$

Solution

$$\int_{-1}^{1} \frac{dx}{x^2+2x+5} = \int_{-1}^{1} \frac{dx}{(x+1)^2+4}$$

$$= \frac{1}{2}\arctan\frac{x+1}{2}\Big|_{-1}^{1}$$

$$= \frac{1}{2}(\arctan 1 - \arctan 0)$$

$$\boxed{= \frac{\pi}{8}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5}$

Solution

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5} = \int_{-\infty}^{\infty} \frac{dx}{(x+1)^2+4}$$

$$= \frac{1}{2}\arctan\frac{x+1}{2}\Big|_{-\infty}^{\infty}$$

$$= \frac{1}{2}(\arctan\infty - \arctan(-\infty))$$

$$= \frac{1}{2}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$\boxed{= \frac{\pi}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{\infty} \cos x\, dx$

Solution

$$\int_0^{\infty} \cos x \, dx = \sin x \Big|_0^{\infty}$$

$$= \sin \infty - \sin 0$$

$$= \nexists$$

$$= \text{doesn't exist} \qquad \rightarrow \textbf{\textit{diverges}}$$

Exercise

Evaluate the integral $\displaystyle\int_2^{\infty} \frac{\cos\left(\frac{\pi}{x}\right)}{x^2} \, dx$

Solution

$$\int_2^{\infty} \frac{\cos\left(\frac{\pi}{x}\right)}{x^2} \, dx = -\frac{1}{\pi} \int_2^{\infty} \cos\left(\frac{\pi}{x}\right) \, d\left(\frac{\pi}{x}\right) \qquad\qquad d\left(\frac{\pi}{x}\right) = -\frac{\pi}{x^2} dx$$

$$= -\frac{1}{\pi} \sin\left(\frac{\pi}{x}\right) \Big|_2^{\infty}$$

$$= -\frac{1}{\pi}\left(\sin 0 - \sin \frac{\pi}{2}\right)$$

$$= -\frac{1}{\pi}(0 - 1)$$

$$\boxed{= \frac{1}{\pi}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{a} \sqrt{e^x} \, dx$

Solution

$$\int_{-\infty}^{a} \sqrt{e^x} \, dx = \int_{-\infty}^{a} e^{x/2} \, dx$$

$$= 2e^{x/2} \Big|_{-\infty}^{a}$$

$$= 2\left(e^{a/2} - e^{-\infty}\right)$$

$$\boxed{= 2e^{a/2}}$$

Exercise

Evaluate the integral $\int_0^{\infty} \frac{e^x}{e^{2x}+1}\,dx$

Solution

$$\int_0^{\infty} \frac{e^x}{e^{2x}+1}\,dx = \int_0^{\infty} \frac{d\left(e^x\right)}{\left(e^x\right)^2+1}$$

$$= \tan^{-1} e^x \Big|_0^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1}1$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$\underline{= \frac{\pi}{4}}$$

Exercise

Evaluate the integral $\int_1^{\infty} \frac{dx}{x(x+1)}$

Solution

$$\int_1^{\infty} \frac{dx}{x(x+1)} = \int_1^{\infty} \left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= \ln|x| - \ln|x+1| \Big|_1^{\infty}$$

$$= \ln\left(\frac{x}{x+1}\right) \Big|_1^{\infty}$$

$$= \ln 1 - \ln\frac{1}{2}$$

$$\underline{= \ln 2}$$

Exercise

Evaluate the integral $\int_1^{\infty} \frac{dx}{x^2(x+1)}$

Solution

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$1 = Ax^2 + Ax + Bx + B + Cx^2$$

$$\begin{cases} x^2 & A + C = 0 \quad \rightarrow \underline{C = 1|} \\ x^1 & A + B = 0 \quad \rightarrow \underline{A = -1|} \\ x^0 & \underline{B = 1|} \end{cases}$$

$$\int_1^\infty \frac{dx}{x^2(x+1)} = \int_1^\infty \left(-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1} \right) dx$$

$$= -\ln x - \frac{1}{x} + \ln(x+1) \Big|_1^\infty$$

$$= \ln \frac{x+1}{x} - \frac{1}{x} \Big|_1^\infty$$

$$= \ln 1 - 0 - (\ln 2 - 1)$$

$$\underline{= 1 - \ln 2 \,|}$$

Exercise

Evaluate the integral $\int_1^\infty \frac{3x^2+1}{x^3+x} dx$

Solution

$$\int_1^\infty \frac{3x^2+1}{x^3+x} dx = \int_1^\infty \frac{1}{x^3+x} d\left(x^3+x\right)$$

$$= \ln\left(x^3+x\right) \Big|_1^\infty$$

$$= \ln \infty - \ln 2$$

$$\underline{= \infty \,|} \quad \textit{diverges}$$

Exercise

Evaluate the integral $\int_1^\infty \frac{1}{x^2} \sin \frac{\pi}{x} dx$

Solution

$$\int_1^\infty \frac{1}{x^2}\sin\frac{\pi}{x}\ dx = -\frac{1}{\pi}\int_1^\infty \sin\frac{\pi}{x}\ d\left(\frac{\pi}{x}\right) \qquad\qquad d\left(\frac{\pi}{x}\right) = -\frac{\pi}{x^2}dx$$

$$= \frac{1}{\pi}\cos\frac{\pi}{x}\ \Big|_1^\infty$$

$$= \frac{1}{\pi}\left(\cos 0 - \cos\pi\right)$$

$$= \frac{1}{\pi}(1+1)$$

$$\boxed{= \frac{2}{\pi}}$$

Exercise

Evaluate the integral $\displaystyle\int_2^\infty \frac{dx}{(x+2)^2}$

Solution

$$\int_2^\infty \frac{dx}{(x+2)^2} = \int_2^\infty \frac{d(x+2)}{(x+2)^2}$$

$$= -\frac{1}{x+2}\ \Big|_2^\infty$$

$$= -\left(0-\frac{1}{4}\right)$$

$$\boxed{= \frac{1}{4}}$$

Exercise

Evaluate the integral $\displaystyle\int_1^\infty \frac{\tan^{-1}x}{x^2+1}\ dx$

Solution

$$\int_1^\infty \frac{\tan^{-1}x}{x^2+1}\ dx = \int_1^\infty \tan^{-1}x\ d\left(\tan^{-1}x\right)$$

$$= \frac{1}{2}\left(\tan^{-1}x\right)^2\ \Big|_1^\infty$$

$$= \frac{1}{2}\left(\left(\tan^{-1}\infty\right)^2 - \left(\tan^{-1}1\right)^2\right)$$

$$= \frac{1}{2}\left(\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right)$$

$$= \frac{\pi^2}{2}\left(\frac{1}{4} - \frac{1}{16}\right)$$

$$= \frac{\pi^2}{2}\left(\frac{3}{16}\right)$$

$$\underline{= \frac{3\pi^2}{32}}\Big|$$

Exercise

Evaluate the integral $\int_{-3}^{1} \frac{dx}{(2x+6)^{2/3}}$

Solution

$$\int_{-3}^{1} \frac{dx}{(2x+6)^{2/3}} = \frac{1}{2}\int_{-3}^{1} (2x+6)^{-2/3}\, d(2x+6)$$

$$= \frac{3}{2}(2x+6)^{1/3}\Big|_{-3}^{1}$$

$$= \frac{3}{2}\left(\sqrt[3]{8} - 0\right)$$

$$\underline{= 3}\,\Big|$$

Exercise

Evaluate the integral $\int_{0}^{1} \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

Solution

$$\int_{0}^{1} \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx = 2\int_{0}^{1} e^{\sqrt{x}}\, d\left(\sqrt{x}\right)$$

$$= 2e^{\sqrt{x}}\Big|_{0}^{1}$$

$$\underline{= 2(e-1)}\,\Big|$$

Exercise

Evaluate the integral $\int_0^{\ln 3} \frac{e^x}{\left(e^x - 1\right)^{2/3}}\, dx$

Solution

$$\int_0^{\ln 3} \frac{e^x}{\left(e^x - 1\right)^{2/3}}\, dx = \int_0^{\ln 3} \left(e^x - 1\right)^{-2/3} d\left(e^x - 1\right)$$

$$= 3\left(e^x - 1\right)^{1/3} \Big|_0^{\ln 3}$$

$$= 3\left(\left(e^{\ln 3} - 1\right)^{1/3} - 0\right)$$

$$= 3(3-1)^{1/3}$$

$$\underline{= 3\sqrt[3]{2}}$$

Exercise

Evaluate the integral $\int_1^2 \frac{dx}{\sqrt{x-1}}$

Solution

$$\int_1^2 \frac{dx}{\sqrt{x-1}} = \int_1^2 (x-1)^{-1/2}\, d(x-1)$$

$$= 2(x-1)^{1/2} \Big|_1^2$$

$$= 2(1-0)$$

$$\underline{= 2}$$

Exercise

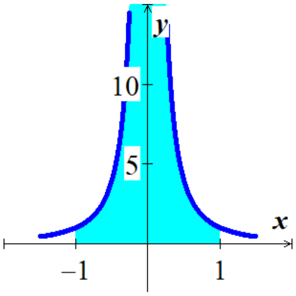
Evaluate the integral $\int_{-1}^1 \frac{dx}{x^2}$

Solution

$$\int_{-1}^1 \frac{dx}{x^2} = \int_{-1}^0 \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2}$$

$$= -\frac{1}{x}\Bigg|_{-1}^{0} - \frac{1}{x}\Bigg|_{0}^{1}$$

$$\underline{= -\infty}\,\big| \qquad \textit{diverges}$$



$$\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\Bigg|_{-1}^{1}$$

$$= -1 - 1$$

$$\underline{= -2}\,\big|$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 \frac{dx}{(x-1)^2}$

Solution

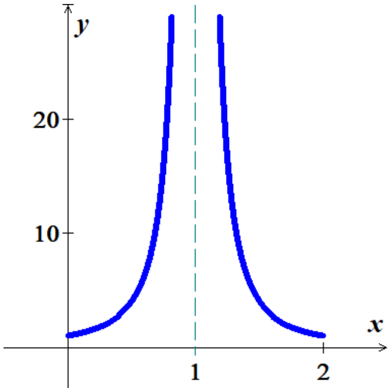
$$\int_0^2 \frac{dx}{(x-1)^2} = \int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2}$$

$$= \int_0^1 \frac{d(x-1)}{(x-1)^2} + \int_1^2 \frac{d(x-1)}{(x-1)^2}$$

$$= -\frac{1}{x-1}\Bigg|_0^1 - \frac{1}{x-1}\Bigg|_1^2$$

$$= -\infty - 1 - 1 + \infty$$

$$\underline{= \infty}\,\big| \qquad \textit{diverges}$$



$$\int_0^2 \frac{dx}{(x-1)^2} = -\frac{1}{x-1}\Bigg|_0^2$$

$$= -(1+1)$$

$$\underline{= -2}\,\big|$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^2 \frac{dx}{(x-1)^2}$

Solution

$$\int_{-1}^{2} \frac{dx}{(x-1)^2} = \int_{-1}^{1} \frac{dx}{(x-1)^2} + \int_{1}^{2} \frac{dx}{(x-1)^2}$$

$$= \int_{-1}^{1} \frac{d(x-1)}{(x-1)^2} + \int_{1}^{2} \frac{d(x-1)}{(x-1)^2}$$

$$= -\frac{1}{x-1}\Bigg|_{-1}^{1} - \frac{1}{x-1}\Bigg|_{1}^{2}$$

$$= -\infty - \frac{1}{2} - 1 + \infty$$

$\boxed{= \infty}$ ***diverges***

Exercise

Evaluate the integral $\displaystyle\int_{1}^{\infty} \frac{dx}{x\sqrt{x^2-1}}$

Solution

$$\int_{1}^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x \Big|_{1}^{\infty}$$

$$= \sec^{-1} \infty - \sec^{-1} 1$$

$$= \frac{\pi}{2} - 0$$

$$\boxed{= \frac{\pi}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{\infty} xe^{-x^2}\, dx$

Solution

$$\int_{0}^{\infty} xe^{-x^2}\, dx = -\frac{1}{2}\int_{0}^{\infty} e^{-x^2}\, d\left(-x^2\right)$$

$$= -\frac{1}{2} e^{-x^2} \Big|_{0}^{\infty}$$

$$= -\frac{1}{2}(0-1)$$

$$\boxed{= \frac{1}{2}}$$

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$

Solution

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = -\frac{1}{2}\int_{-\infty}^{0} e^{-x^2}\,d\left(-x^2\right) - \frac{1}{2}\int_{0}^{\infty} e^{-x^2}\,d\left(-x^2\right)$$

$$= -\frac{1}{2}e^{-x^2}\ \Big|_{-\infty}^{0}\ -\frac{1}{2}e^{-x^2}\ \Big|_{0}^{\infty}$$

$$= -\frac{1}{2}(1-0) - \frac{1}{2}(0-1)$$

$$= -\frac{1}{2} + \frac{1}{2}$$

$$\boxed{= 0}$$

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\,d\left(-x^2\right)$$

$$= -\frac{1}{2}e^{-x^2}\ \Big|_{-\infty}^{\infty}$$

$$= -\frac{1}{2}(0-0)$$

$$\boxed{= 0}$$

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx$

Solution

$$\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,d\left(x^2+1\right)$$

$$= \frac{1}{2}\ln\left(x^2+1\right)\ \Big|_{-\infty}^{\infty}$$

$$= \frac{1}{2}(\infty - \infty)$$

$\boxed{= \infty}$ ***diverges***

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$

Solution

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \int_{-\infty}^{\infty} \frac{dx}{(x+1)^2+1}$$

$$= \int_{-\infty}^{\infty} \frac{d(x+1)}{(x+1)^2+1}$$

$$= \tan^{-1}(x+1) \Big|_{-\infty}^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1}(-\infty)$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$\underline{= \pi \ \big|}$$

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} \frac{dx}{x^2+6x+12}$

Solution

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+6x+12} = \int_{-\infty}^{\infty} \frac{dx}{(x+3)^2+3}$$

$$= \int_{-\infty}^{\infty} \frac{d(x+3)}{(x+3)^2+3}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{x+3}{\sqrt{3}} \Big|_{-\infty}^{\infty}$$

$$= \frac{1}{\sqrt{3}} \left(\tan^{-1}\infty - \tan^{-1}(-\infty) \right)$$

$$= \frac{1}{\sqrt{3}} \left(\frac{\pi}{2} + \frac{\pi}{2} \right)$$

$$\underline{= \frac{\pi}{\sqrt{3}} \ \Big|}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{2-\sqrt{3x}}$

Solution

Let $u = 2-\sqrt{3x} \quad \rightarrow \quad \sqrt{3x} = 2-u$

$$du = -\frac{3}{2}(3x)^{-1/2}\ dx$$

$$dx = -\frac{2}{3}\sqrt{3x}\ du$$

$$= -\frac{2}{3}(2-u)\ du$$

$$\int \frac{dx}{2-\sqrt{3x}} = -\frac{2}{3}\int \frac{1}{u}(2-u)\ du$$

$$= -\frac{2}{3}\int \left(\frac{2}{u}-1\right)\ du$$

$$= -\frac{2}{3}\left(2\ln|u| - u\right) + C$$

$$= -\frac{2}{3}\left(2\ln\left|2-\sqrt{3x}\right| - 2 - \sqrt{3x}\right) + C$$

$$\underline{= -\frac{2}{3}\left(\ln\left(2-\sqrt{3x}\right)^2 - 2 - \sqrt{3x}\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \theta\ \cos(2\theta+1)\ d\theta$

Solution

		$\int \cos(2\theta+1)\ d\theta$
+	θ	$\frac{1}{2}\sin(2\theta+1)$
–	1	$-\frac{1}{4}\cos(2\theta+1)$

$$\int \theta\ \cos(2\theta+1)\ d\theta \underline{= \frac{1}{2}\theta\sin(2\theta+1) + \frac{1}{4}\cos(2\theta+1) + C}$$

Exercise

Evaluate the integral $\int \sqrt{x}\,\sqrt{1+\sqrt{x}}\;dx$

Solution

Let $u=\sqrt{x} \;\Rightarrow\; u^2=x$

$2udu=dx$

$$\int \sqrt{x}\,\sqrt{1+\sqrt{x}}\;dx=\int u\,\sqrt{1+u}\;(2udu)$$

$$=\int 2u^2\,(1+u)^{1/2}\;du$$

		$\int(1+u)^{1/2}$
+	$2u^2$	$\frac{2}{3}(1+u)^{3/2}$
−	$4u$	$\frac{4}{15}(1+u)^{5/2}$
+	4	$\frac{8}{105}(1+u)^{7/2}$

$$\int \sqrt{x}\,\sqrt{1+\sqrt{x}}\;dx=\frac{4}{3}u^2(1+u)^{3/2}-\frac{16}{15}u(1+u)^{5/2}+\frac{32}{105}(1+u)^{7/2}+C$$

$$=\frac{4}{3}x\left(1+\sqrt{x}\right)^{3/2}-\frac{16}{15}\sqrt{x}\left(1+\sqrt{x}\right)^{5/2}+\frac{32}{105}\left(1+\sqrt{x}\right)^{7/2}+C$$

Exercise

Find the area of the unbounded shaded region $y=e^x, \quad -\infty<x\le 1$

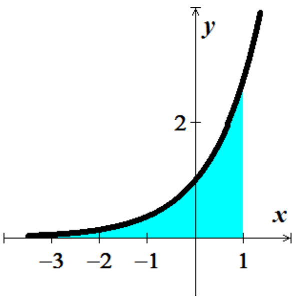
Solution

$$A=\int_{-\infty}^{1} e^x\;dx$$

$$=e^x\Big|_{-\infty}^{1}$$

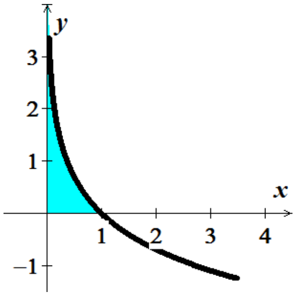
$$=e-0$$

$$=e \quad unit^2$$



Exercise

Find the area of the unbounded shaded region $y = -\ln x$



Solution

$$A = -\int_0^1 \ln x \, dx$$

$$= -(x\ln x - x) \Big|_0^1$$

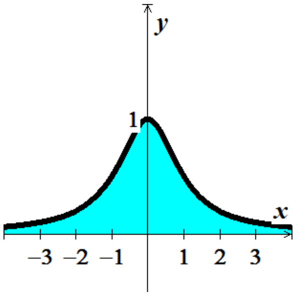
$$\underline{= 1 \quad unit^2}$$

Exercise

Find the area of the unbounded shaded region $y = \dfrac{1}{x^2+1}$

Solution

$$A = \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx$$



$$= \arctan x \Big|_{-\infty}^{\infty}$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$

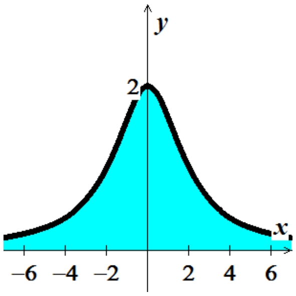
$$\underline{= \pi \quad unit^2}$$

Exercise

Find the area of the unbounded shaded region $y = \dfrac{8}{x^2+4}$

Solution

$$A = \int_{-\infty}^{\infty} \frac{8}{x^2+4} dx$$



$$= 4\arctan\frac{x}{2} \Big|_{-\infty}^{\infty}$$

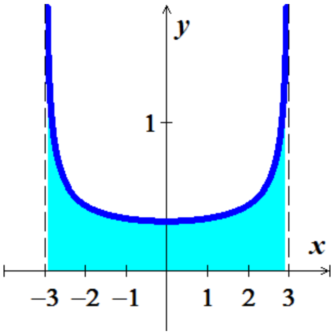
$$= 4\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$\underline{= 4\pi \quad unit^2}$$

Exercise

Find the area of the region R between the graph of $f(x)=\dfrac{1}{\sqrt{9-x^2}}$ and the x-*axis* on the interval $(-3,\ 3)$ (if it exists)

Solution



$$A=\int_{-3}^{3}\frac{dx}{\sqrt{9-x^2}}$$

$$=2\int_{0}^{3}\frac{dx}{\sqrt{9-x^2}}$$

$$=2\sin^{-1}\frac{x}{3}\Big|_0^3$$

$$=2\left(\sin^{-1}1-\sin^{-1}0\right)$$

$$=\pi\quad unit^2$$

Exercise

Find the volume of the region bounded by $f(x)=\left(x^2+1\right)^{-1/2}$ and the x-*axis* on the interval $[2,\ \infty)$ is revolved about the x-*axis*.

Solution

$$V=\pi\int_{2}^{\infty}\frac{1}{x^2+1}dx \qquad V=\pi\int_a^b\left(f(x)\right)^2dx$$

$$=\pi\ \tan^{-1}x\Big|_2^\infty$$

$$=\pi\left(\tan^{-1}\infty-\tan^{-1}2\right)$$

$$=\pi\left(\frac{\pi}{2}-\tan^{-1}2\right)\quad unit^3$$

Exercise

Find the volume of the region bounded by $f(x)=\sqrt{\dfrac{x+1}{x^3}}$ and the x-*axis* on the interval $[1,\ \infty)$ is revolved about the x-*axis*.

Solution

$$V=\pi\int_1^{\infty}\frac{x+1}{x^3}\,dx \qquad V=\pi\int_a^b (f(x))^2\,dx$$

$$=\pi\int_1^{\infty}\left(\frac{1}{x^2}+x^{-3}\right)dx$$

$$=\pi\left(-\frac{1}{x}-\frac{1}{2}\frac{1}{x^2}\right|_1^{\infty}$$

$$=\pi\left(1+\frac{1}{2}\right)$$

$$\boxed{=\frac{3\pi}{2}\quad unit^3}$$

Exercise

Find the volume of the region bounded by $f(x)=(x+1)^{-3}$ and the x-axis on the interval $[0,\ \infty)$ is revolved about the y-axis.

Solution

$$V=2\pi\int_0^{\infty}x\frac{1}{(x+1)^3}dx \qquad V=2\pi\int_a^b x\cdot f(x)\,dx \quad (Shell\ method)$$

$$=2\pi\int_0^{\infty}\left(\frac{1}{(x+1)^2}-\frac{1}{(x+1)^3}\right)d(x+1)$$

$$\frac{x}{(x+1)^3}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$

$$x=Ax^2+2Ax+A+Bx+B+C$$

$$\begin{cases}\boxed{A=0}\\ 2A+B=1 \ \rightarrow \boxed{B=1} \quad \boxed{C=-1}\\ B+C=0\end{cases}$$

$$=2\pi\left(\frac{-1}{x+1}+\frac{1}{2}\frac{1}{(x+1)^2}\right|_0^{\infty}$$

$$=2\pi\left(1-\frac{1}{2}\right)$$

$$\boxed{=\pi\quad unit^3}$$

Exercise

Find the volume of the region bounded by $f(x)=\frac{1}{\sqrt{x}\ln x}$ and the *x-axis* on the interval $[2,\ \infty)$ is revolved about the *x-axis*.

Solution

$$V=\pi\int_2^\infty \frac{1}{x\ln^2 x}\,dx \qquad\qquad V=\pi\int_a^b (f(x))^2\,dx$$

$$=\pi\int_2^\infty \frac{1}{\ln^2 x}\,d(\ln x)$$

$$=\pi\left(-\frac{1}{\ln x}\,\right|_2^\infty$$

$$=\pi\left(-0+\frac{1}{\ln 2}\right)$$

$$=\frac{\pi}{\ln 2}\quad unit^3$$

Exercise

Find the volume of the region bounded by $f(x)=\frac{\sqrt{x}}{\sqrt[3]{x^2+1}}$ and the *x-axis* on the interval $[0,\ \infty)$ is revolved about the *x-axis*.

Solution

$$V=\pi\int_0^\infty \frac{x}{\left(x^2+1\right)^{2/3}}\,dx \qquad\qquad V=\pi\int_a^b (f(x))^2\,dx$$

$$=\frac{\pi}{2}\int_0^\infty \left(x^2+1\right)^{-2/3}\,d\left(x^2+1\right)$$

$$=\frac{3\pi}{2}\left(x^2+1\right)^{1/3}\Big|_0^\infty$$

$$=\frac{3\pi}{2}(\infty-1)$$

$$=\infty\quad diverges$$

Therfore, the volume doesn't exist

Exercise

Find the volume of the region bounded by $f(x)=\left(x^2-1\right)^{-1/4}$ and the *x-axis* on the interval $(1,\ 2]$ is revolved about the *y-axis*.

Solution

$$V=2\pi\int_1^2 x\left(x^2-1\right)^{-1/4}dx \qquad V=2\pi\int_a^b x\cdot f(x)\,dx \quad (\textit{Shell method})$$

$$=\pi\int_1^2\left(x^2-1\right)^{-1/4}d\left(x^2-1\right)$$

$$=\frac{4\pi}{3}\left(x^2-1\right)^{3/4}\Big|_1^2$$

$$=\frac{4\pi}{3}(3)^{3/4}$$

$$=\frac{4\pi}{3^{1/4}}\quad unit^3$$

Exercise

Find the volume of the region bounded by $f(x)=\tan x$ and the *x-axis* on the interval $\left[0,\ \frac{\pi}{2}\right)$ is revolved about the *x-axis*.

Solution

$$V=\pi\int_0^{\pi/2}\tan^2 x\,dx \qquad V=\pi\int_a^b\left(f(x)\right)^2dx$$

$$=\pi\int_0^{\pi/2}\left(\sec^2 x-1\right)dx$$

$$=\pi(\tan x-x)\Big|_0^{\pi/2} \qquad \left(\tan\frac{\pi}{2}=\infty\right)$$

$$=\infty\quad diverges$$

Therfore, the volume doesn't exist

Exercise

Find the volume of the region bounded by $f(x)=-\ln x$ and the *x-axis* on the interval $(0,\ 1]$ is revolved about the *x-axis*.

Solution

$$V = \pi \int_0^1 \ln^2 x\, dx \qquad V = \pi \int_a^b \left(f(x)\right)^2 dx$$

$$u = \ln x \qquad dv = \ln x\, dx$$

$$du = \frac{dx}{x} \qquad v = x \ln x - x$$

$$u = \ln x \qquad dv = dx$$

$$du = \frac{dx}{x} \qquad v = x$$

$$\int \ln x dx = x \ln x - \int dx$$

$$= x \ln x - x$$

$$\int \ln^2 x\, dx = \ln x\left(x \ln x - x\right) - \int (\ln x - 1)\, dx$$

$$= x \ln^2 x - x \ln x - \left(x \ln x - x - x\right)$$

$$\underline{= x \ln^2 x - 2x \ln x + 2x}$$

$$V = \pi\left(x \ln^2 x - 2x \ln x + 2x\right|_0^1$$

$$\underline{= 2\pi \quad unit^3}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by the graphs of $y = xe^{-x}$, $y = 0$, and $x = 0$ about the x-*axis*.

Solution

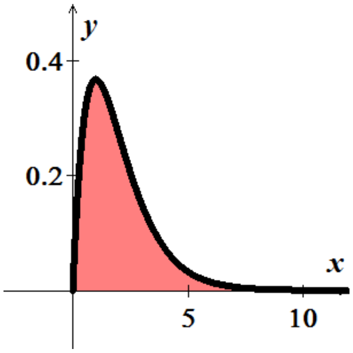
$$V = \pi \int_0^\infty \left(xe^{-x}\right)^2 dx$$

$$= \pi \int_0^\infty x^2 e^{-2x}\, dx$$

$$= \pi e^{-2x}\left(-\frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{4}\right)\Bigg|_0^\infty$$

$$= \pi\left(0 + \frac{1}{4}\right)$$

$$\underline{= \frac{\pi}{4} \quad unit^3}$$

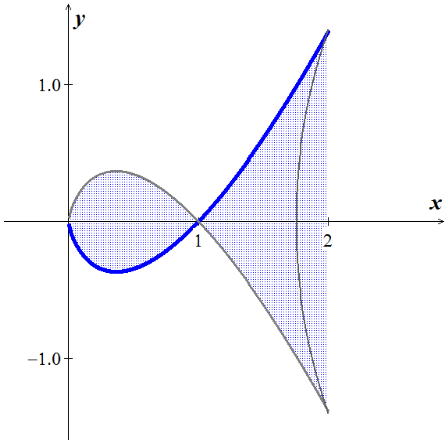


Exercise

The region between the x-axis and the curve

$$f(x)=\begin{cases} 0, & x=0 \\ x\ln x, & 0<x\le 2 \end{cases}$$

is revolved about the x-axis to generate the solid.
Find the volume of the solid.



Solution

$$V=\pi\int_0^2 y^2\,dx$$

$$=\pi\int_0^2 (x\ln x)^2\,dx$$

$$u=(\ln x)^2 \qquad dv=x^2dx$$

$$du=2\frac{\ln x}{x}dx \qquad v=\frac{1}{3}x^3$$

$$=\frac{\pi}{3}x^3(\ln x)^2\Big|_0^2-\frac{2\pi}{3}\int_0^2 x^2\ln x\,dx$$

$$u=\ln x \qquad dv=x^2dx$$

$$du=\frac{1}{x}dx \qquad v=\frac{1}{3}x^3$$

$$=\frac{\pi}{3}\left(8(\ln 2)^2-0\right)-\frac{2\pi}{9}x^3\ln x\Big|_0^2+\frac{2\pi}{9}\int_0^2 x^2\,dx$$

$$=\frac{8\pi}{3}(\ln 2)^2-\frac{2\pi}{9}(8\ln 2-0)+\frac{2\pi}{27}x^3\Big|_0^2$$

$$=\frac{8\pi}{3}(\ln 2)^2-\frac{16\pi}{9}\ln 2+\frac{16\pi}{27}$$

$$=\frac{8\pi}{3}\left((\ln 2)^2-\frac{2}{3}\ln 2+\frac{2}{9}\right)\quad unit^3$$

Exercise

Consider the region satisfying the inequalities $y\le e^{-x},\quad y\ge 0,\quad x\ge 0$

a) Find the area of the region

b) Find the volume of the solid generated by revolving the region about the *x-axis*.

c) Find the volume of the solid generated by revolving the region about the *y-axis*.

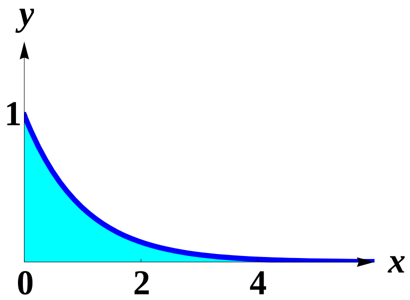
Solution

a) $A = \int_0^\infty e^{-x}\,dx$

$= -e^{-x}\Big|_0^\infty$

$= -(0-1)$

$\underline{= 1 \quad unit^2}\Big|$



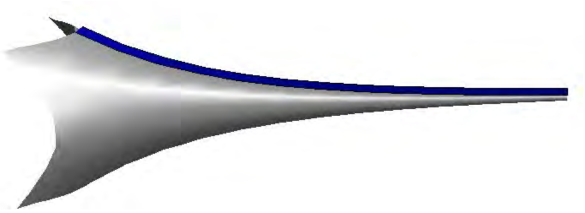
b) $V = \pi \int_0^\infty \left(e^{-x}\right)^2 dx$ $\qquad V = \pi \int_a^b \left(R(x)^2 - r(x)^2\right) dx$ (*Disk Method*)

$= \pi \int_0^\infty e^{-2x}\,dx$

$= -\frac{\pi}{2} e^{-2x}\Big|_0^\infty$

$= -\frac{\pi}{2}(0-1)$

$\underline{= \frac{\pi}{2} \quad unit^3}\Big|$

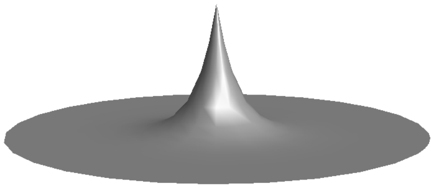


c) $V = 2\pi \int_0^\infty x\,e^{-x}\,dx$ $\qquad V = 2\pi \int_a^b xf(x)\,dx$ (*Shell Method*)

$= -2\pi\left(e^{-x}(x+1)\right)\Big|_0^\infty$

$= -2\pi(0-1)$

$\underline{= 2\pi \quad unit^3}\Big|$



Exercise

Consider the region satisfying the inequalities $\qquad y \le \frac{1}{x^2}, \quad y \ge 0, \quad x \ge 1$

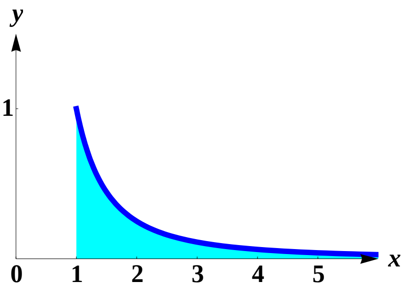
a) Find the area of the region

b) Find the volume of the solid generated by revolving the region about the *x-axis*.

c) Find the volume of the solid generated by revolving the region about the *y-axis*.

Solution

a) $A = \int_1^\infty \frac{1}{x^2}\,dx$



$$= -\frac{1}{x}\Big|_{1}^{\infty}$$

$$= -(0-1)$$

$$\boxed{= 1 \quad unit^2}$$

b) $V = \pi \int_0^\infty \left(\frac{1}{x^2}\right)^2 dx$ $\qquad V = \pi \int_a^b \left(R(x)^2 - r(x)^2\right) dx \quad$ (*Disk Method*)

$$= \pi \int_0^\infty x^{-4}\, dx$$

$$= -\frac{\pi}{3x^3}\Big|_{1}^{\infty}$$

$$= -\frac{\pi}{3}(0-1)$$

$$\boxed{= \frac{\pi}{3} \quad unit^3}$$

c) $V = 2\pi \int_0^\infty x\left(\frac{1}{x^2}\right) dx$ $\qquad V = 2\pi \int_a^b x f(x)\, dx \quad$ (*Shell Method*)

$$= 2\pi \int_0^\infty \frac{1}{x}\, dx$$

$$= 2\pi \ln x \Big|_{1}^{\infty}$$

$$\boxed{= \infty} \quad \textit{Diverges}$$

Exercise

Find the perimeter of the hypocycloid of four cusps $x^{2/3} + y^{2/3} = 4$

Solution

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{y^{2/3}}{x^{2/3}}}$$

$$= \frac{\sqrt{x^{2/3}+y^{2/3}}}{x^{1/3}}$$

$$= \frac{\sqrt{4}}{x^{1/3}}$$

$$= 2x^{-1/3}$$

$$S = 4\int_0^8 2x^{-1/3}\, dx$$

$$= 12x^{2/3} \Big|_0^8$$

$$= 12(4 - 0)$$

$$\underline{= 48 \quad unit^2}$$

Exercise

Find the arc length of the graph $y = \sqrt{16 - x^2}$ over the interval $[0,\ 4]$

Solution

$$y' = -\frac{x}{\sqrt{16 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{16 - x^2}}$$

$$= \frac{4}{\sqrt{16 - x^2}}$$

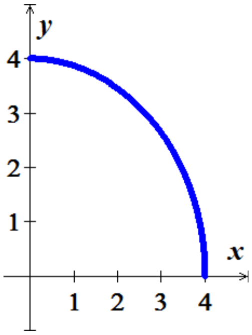
$$L = \int_0^4 \frac{4}{\sqrt{16 - x^2}}\, dx$$

$$= 4\arcsin\frac{x}{4} \Big|_0^4$$

$$= 4(\arcsin 1 - \arcsin 0)$$

$$= 4\left(\frac{\pi}{2}\right)$$

$$\underline{= 2\pi \quad unit}$$



Exercise

The region bounded by $(x - 2)^2 + y^2 = 1$ is revolved about the *y-axis* to form a torus. Find the surface area of the torus.

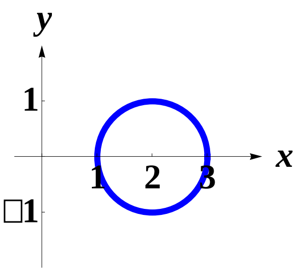
Solution

$$2(x-2)+2yy'=0$$

$$y'=-\frac{x-2}{y}$$

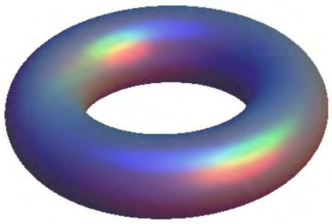
$$\sqrt{1+(y')^2}=\sqrt{1+\frac{(x-2)^2}{y^2}}$$

$$=\sqrt{\frac{y^2+(x-2)^2}{y^2}} \qquad (x-2)^2+y^2=1$$



$$=\frac{1}{y}$$

$$=\frac{1}{\sqrt{1-(x-2)^2}}$$



$$S=4\pi\int_1^3 \frac{x}{\sqrt{1-(x-2)^2}}\,dx$$

$$=4\pi\int_1^3 \frac{x-2+2}{\sqrt{1-(x-2)^2}}\,dx$$

$$=4\pi\int_1^3 \frac{x-2}{\sqrt{1-(x-2)^2}}\,dx \;+\; 4\pi\int_1^3 \frac{2}{\sqrt{1-(x-2)^2}}\,dx$$

$$=-2\pi\int_1^3 \left(1-(x-2)^2\right)^{-1/2}\,d\left(1-(x-2)^2\right) \;+\; 8\pi\arctan(x-2)\Big|_1^3$$

$$=-4\pi\sqrt{1-(x-2)^2}\Big|_1^3+8\pi\left(\arctan(1)-\arctan(-1)\right)$$

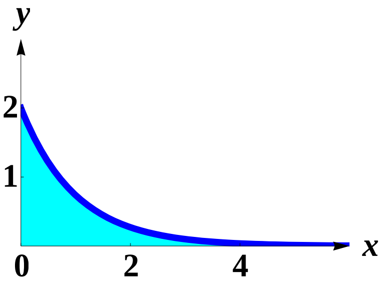
$$=-4\pi(0-0)+8\pi\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$=8\pi^2 \quad unit^2$$

Exercise

Find the surface area formed by revolving the graph $y=2e^{-x}$ on the interval $[0,\ \infty)$ about the *x-axis*

Solution



$$y'=-2e^{-x}$$

$$\sqrt{1+(y')^2}=\sqrt{1+4e^{-2x}}$$

$$S = 2\pi \int_0^{\infty} 2e^{-x}\sqrt{1+4e^{-2x}}\; dx$$

$$= -4\pi \int_0^{\infty} \sqrt{1+4\left(e^{-x}\right)^2}\; d\left(e^{-x}\right)$$

$$\int \sqrt{1+4u^2}\; du = \frac{1}{2}\int \sec^3\theta\; d\theta$$

$2u = \tan\theta \qquad \sqrt{4u^2+1} = \sec\theta$

$du = \frac{1}{2}\sec^2\theta\; d\theta$

$$u = \sec\theta \qquad dv = \sec^2\theta d\theta$$

$$du = \sec\theta\tan\theta d\theta \qquad v = \tan\theta$$

$$\int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \tan\theta\left(\sec\theta\tan\theta d\theta\right)$$

$$= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta\; d\theta$$

$$= \sec\theta\tan\theta - \int \left(\sec^2\theta - 1\right)\sec\theta\; d\theta$$

$$2\int \sec^3\theta d\theta == \sec\theta\tan\theta + \int \sec\theta\; d\theta$$

$$\int \sqrt{1+4u^2}\; du = \frac{1}{4}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)$$

$$S = -\pi\left(2e^{-x}\sqrt{1+4e^{-2x}} + \ln\left|2e^{-x} + \sqrt{1+4e^{-2x}}\right|\right)\Bigg|_0^{\infty}$$

$$= -\pi\left(-2\sqrt{5} + \ln\left(2+\sqrt{5}\right)\right)$$

$$= \pi\left(2\sqrt{5} - \ln\left(2+\sqrt{5}\right)\right) \quad unit^2$$

Exercise

The magnetic potential *P* at a point on the axis of a circular coil is given by

$$P = \frac{2\pi NIr}{k}\int_c^{\infty} \frac{1}{\left(r^2+x^2\right)^{3/2}}\; dx$$

Where *N*, *I*, *r*, *k*, and *c* are constants. Find *P*.

Solution

Let $K = \dfrac{2\pi NIr}{k}$

$$P = K\int_c^\infty \frac{1}{\left(r^2+x^2\right)^{3/2}}\,dx$$

$x = r\tan\theta \qquad \sqrt{x^2+r^2} = r\sec\theta$

$dx = r\sec^2\theta\; d\theta$

$$= K\int_c^\infty \frac{r\sec^2\theta}{r^3\sec^3\theta}\,d\theta$$

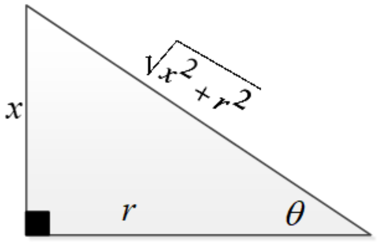
$$= \frac{K}{r^2}\int_c^\infty \cos\theta\; d\theta$$

$$= \frac{K}{r^2}\sin\theta\Big|_c^\infty$$

$$= \frac{K}{r^2}\frac{x}{\sqrt{r^2+x^2}}\Big|_c^\infty$$

$$= \frac{K}{r^2}\left(1-\frac{c}{\sqrt{r^2+c^2}}\right)$$

$$\boxed{= \frac{2\pi NI\left(\sqrt{r^2+c^2}-c\right)}{kr\sqrt{r^2+c^2}}}$$



Exercise

A “semi-infinite” uniform rod occupies the nonnegative *x-axis*. The rod has a linear density *δ*, which means that a segment of length dx has a mass of δdx. A particle of mass *M* is located at the point $(-a,\ 0)$. The gravitational force *F* that the rod exerts on the mass is given by

$$F = \int_0^\infty \frac{GM\delta}{(a+x)^2}\,dx$$

Where *G* is the gravitational constant. Find *F*.

Solution

$$F = \int_0^\infty \frac{GM\delta}{(a+x)^2}\,dx$$

$$= -\frac{GM\delta}{a+x}\Big|_0^\infty$$

$$\boxed{= \frac{GM\delta}{a}}$$

Exercise

Let *R* be the region bounded by the graph of $f(x) = x^{-p}$ and the *x-axis*

a) Let *S* be the solid generated when *R* is revolved about the *x-axis*. For what values of *p* is the volume of *S* finite for $0 < x \le 1$?

b) Let *S* be the solid generated when *R* is revolved about the *y-axis*. For what values of *p* is the volume of *S* finite for $0 < x \le 1$?

c) Let *S* be the solid generated when *R* is revolved about the *x-axis*. For what values of *p* is the volume of *S* finite for $x \ge 1$?

d) Let *S* be the solid generated when *R* is revolved about the *y-axis*. For what values of *p* is the volume of *S* finite for $x \ge 1$?

Solution

a) $V = \pi \int_0^1 \left(x^{-p}\right)^2 dx$ $\qquad V = \pi \int_a^b f(x)^2\, dx$

$$= \pi \int_0^1 x^{-2p} dx$$

$$= \pi \frac{x^{-2p+1}}{1-2p}\Big|_0^1$$

$$= \frac{\pi}{1-2p}\left(1 - 0^{-2p+1}\right)$$

The volume of *S* finite when $1 - 2p > 0 \;\Rightarrow\; \underline{p < \frac{1}{2}}\,|$

b) $V = 2\pi \int_0^1 x \cdot x^{-p} dx$ $\qquad V = 2\pi \int_a^b xf(x)\, dx$

$$= 2\pi \int_0^1 x^{1-p} dx$$

$$= \frac{2\pi}{2-p} x^{2-p}\Big|_0^1$$

$$= \frac{2\pi}{2-p}\left(1 - 0^{2-p}\right)$$

The volume of *S* finite when $2 - p > 0 \;\Rightarrow\; \underline{p < 2}\,|$

c) $V = \pi \int_1^\infty \left(x^{-p}\right)^2 dx$ $\qquad V = \pi \int_a^b f(x)^2\, dx$

$$= \pi \int_1^\infty x^{-2p}\, dx$$

$$= \pi \frac{x^{-2p+1}}{1-2p}\Big|_{1}^{\infty}$$

$$= \frac{\pi}{1-2p}\left(\infty^{1-2p} - 1\right)$$

The volume of S finite when $1-2p<0 \;\Rightarrow\; \boxed{p>\frac{1}{2}}$ $\left(\frac{1}{\infty}=0\right)$

d) $V = 2\pi \int_0^1 x \cdot x^{-p} dx$ $\qquad V = 2\pi \int_a^b xf(x)dx$

$$= 2\pi \int_0^1 x^{1-p} dx$$

$$= \frac{2\pi}{2-p} x^{2-p}\Big|_0^1$$

$$= \frac{2\pi}{2-p}\left(1 - 0^{2-p}\right)$$

The volume of S finite when $2-p>0 \;\Rightarrow\; \boxed{p<2}$

Exercise

The solid formed by revolving (about the *x-axis*) the unbounded region lying between the graph of $f(x) = \frac{1}{x}$ and the *x-axis* $(x \ge 1)$ is called ***Gabriel's Horn***.

Show that this solid has a finite volume and an infinite surface area

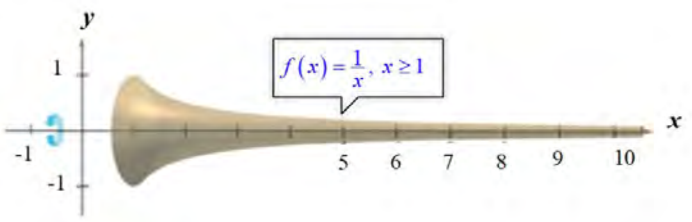
Solution

$V = \pi \int_1^{\infty} \frac{1}{x^2} dx$ $\qquad V = \pi \int_x^b (f(x))^2\, dx \ \ (disk\ method)$

$$= -\pi \frac{1}{x}\Big|_1^{\infty}$$

$$= -\pi(0-1)$$

$$\boxed{= \pi \quad unit^3}$$



$$f'(x) = -\frac{1}{x^2}$$

$S = 2\pi \int_1^{\infty} \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\, dx$ $\qquad S = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\, dx$

Since $1+\dfrac{1}{x^4}>1$ and $\displaystyle\int_1^{\infty}\frac{1}{x}\,dx$ diverges

Therefore the surface area in ***infinite***.

Exercise

Water is drained from a 3000-*gal* tank at a rate that starts at 100 *gal/hr*. and decreases continuously by 5% /*hr*. If the drain left open indefinitely, how much water drains from the tank? Can a full tank be emptied at this rate?

Solution

Rate of the drain water: $r(t)=100(1-.05)^t$

$$=100(0.95)^t$$

$$=100\,e^{(\ln 0.95)t}$$

Total water amount drained:

$$D=\int_0^{\infty}100e^{(\ln 0.95)t}dt$$

$$=\frac{100}{\ln 0.95}e^{(\ln 0.95)t}\Big|_0^{\infty}$$

$$=\frac{100}{\ln 0.95}(0-1)$$

$$\ln 0.95<0\xrightarrow[t\to\infty]{}e^{(\ln 0.95)t}=e^{-\infty}=0$$

$$=-\frac{100}{\ln 0.95}\quad\approx 1950\ \ gal$$

Since $1950\ gal<3000\ gal$ which it takes infinite time.

Therefore; the full 3,000–gallon tank cannot be emptied at this rate.

Exercise

Let $I(a)=\int_0^{\infty}\frac{dx}{\left(1+x^a\right)\left(1+x^2\right)}$, where a is a real number.

a) Evaluate $I(a)$ and show that its value is independent of a.

$\big($***Hint***: split the integral into two integrals over $[0,\ 1]$ and $[1,\ \infty)$; then use a change of variables to convert the second integral into an integral over $[0,\ 1]$.$\big)$

b) Let f be any positive continuous function on $\left[0,\ \frac{\pi}{2}\right]$

Evaluate $\int_0^{\pi/2}\frac{f(\cos x)}{f(\cos x)+f(\sin x)}dx$

(***Hint***: Use the identity $\cos\left(\frac{\pi}{2}-x\right)=\sin x$)

Solution

a) $I(a)=\int_0^{\infty}\frac{dx}{\left(1+x^a\right)\left(1+x^2\right)}$

$$=\int_0^{1}\frac{dx}{\left(1+x^a\right)\left(1+x^2\right)}+\int_1^{\infty}\frac{dx}{\left(1+x^a\right)\left(1+x^2\right)}$$

Let $u=\frac{1}{x}\ \Rightarrow\ x=\frac{1}{u}$

$$dx=-\frac{1}{u^2}du$$

$x=1\ \rightarrow\ u=1$

$x=\infty\ \rightarrow\ u=0$

$$I(a)=\int_0^{1}\frac{dx}{\left(1+x^a\right)\left(1+x^2\right)}-\int_1^{0}\frac{du}{u^2\left(1+u^{-a}\right)\left(1+u^{-2}\right)}$$

$$=\int_0^{1}\frac{dx}{\left(1+x^a\right)\left(1+x^2\right)}+\int_0^{1}\frac{du}{u^2\left(1+\frac{1}{u^a}\right)\left(1+\frac{1}{u^2}\right)}$$

$$=\int_0^{1}\frac{dx}{\left(1+x^a\right)\left(1+x^2\right)}+\int_0^{1}\frac{u^a du}{\left(1+u^a\right)\left(1+u^2\right)}\qquad (x=u)$$

$$= \int_0^1 \frac{dx}{\left(1+x^a\right)\left(1+x^2\right)} + \int_0^1 \frac{x^a du}{\left(1+x^a\right)\left(1+x^2\right)}$$

$$= \int_0^1 \frac{1+x^a}{\left(1+x^a\right)\left(1+x^2\right)} dx$$

$$= \int_0^1 \frac{dx}{1+x^2}$$

$$= \tan^{-1} x \Big|_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4}$$

b) $I = \int_0^{\pi/2} \frac{f(\cos x)}{f(\cos x) + f(\sin x)} dx$

Let $u = \frac{\pi}{2} - x \quad \Rightarrow \quad x = \frac{\pi}{2} - u$

$dx = -du$

$\cos\left(\frac{\pi}{2} - x\right) = \sin x$

$x = 1 \quad \rightarrow \quad u = 1$

$x = \infty \quad \rightarrow \quad u = 0$

$$I = -\int_{\pi/2}^0 \frac{f(\sin u)}{f(\sin u) + f(\cos u)} du$$

$$= \int_0^{\pi/2} \frac{f(\sin u)}{f(\sin u) + f(\cos u)} du$$

$$2I = \int_0^{\pi/2} \frac{f(\cos x)}{f(\cos x) + f(\sin x)} dx + \int_0^{\pi/2} \frac{f(\sin x)}{f(\sin x) + f(\cos x)} dx$$

$$= \int_0^{\pi/2} dx$$

$$= x \Big|_0^{\pi/2}$$

$$\boxed{= \frac{\pi}{2}}$$

$$\boxed{I = \frac{\pi}{4}}$$

Exercise

Let R be the region bounded by $y = \ln x$, the x-*axis*, and the line $x = a$, where $a > 1$.

a) Find the volume $V_1(a)$ of the solid generated when R is revolved about the x-*axis* (as a function of a).

b) Find the volume $V_2(a)$ of the solid generated when R is revolved about the y-*axis* (as a function of a).

c) Graph V_1 and V_2. For what values of $a > 1$ is $V_1(a) > V_2(a)$?

Solution

a) $V_1(a) = \pi \int_1^a (\ln x)^2\, dx$

Let $\quad z = \ln x \quad \Rightarrow \quad x = e^z$

$$dx = e^z dz$$

$$V_1(a) = \pi \int_1^a z^2 e^z\, dz$$

		$\int e^z dz$
+	z^2	e^z
−	$2z$	e^z
+	2	e^z

$$V_1(a) = \pi e^z \left(z^2 - 2z + 2\right) \Big|_1^a$$

$$= \pi x \left((\ln x)^2 - 2\ln x + 2\right) \Big|_1^a$$

$$= \pi \left[a\left((\ln a)^2 - 2\ln a + 2\right) - (\ln 1 - 2\ln 1 + 2)\right]$$

$$\boxed{= \pi\left(a \ln^2 a - 2a\ln a + 2a + 2\right) \quad unit^3}$$

b) About y-*axis*

$$V_2(a) = 2\pi \int_1^a x \ln x \, dx$$

		$\int x\,dx$
$+$	$\ln x$	$\frac{1}{2}x^2$
$-$	$\frac{1}{x}$	$\int \frac{1}{2}x^2\,dx$

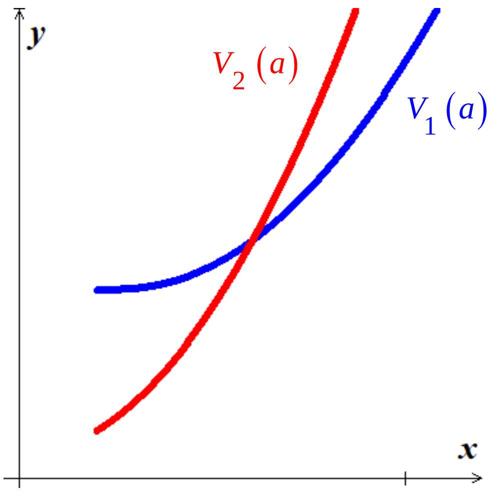
$$V_2(a) = 2\pi\left(\frac{1}{2}x^2 \ln x \Big|_1^a - \frac{1}{2}\int_1^a x\,dx\right)$$

$$= 2\pi\left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right.\Big|_1^a$$

$$= \pi\left(a^2 \ln a - \frac{1}{2}a^2 - \ln 1 + \frac{1}{2}\right)$$

$$\boxed{= \frac{\pi}{2}\left(2a^2 \ln a - a^2 + 1\right) \quad unit^3}$$

c)



$V_2(a) > V_1(a)$ when $a > 1$

Exercise

Let R be the region bounded by the graph of $f(x)=x^{-p}$ and the x-*axis*, for $x\geq 1$. Let V_1 and V_2 be the volumes of the solids generated when R is revolved about the x-*axis* and the y-*axis*, respectively, if they exist.

a) For what values of p (if any) is $V_1=V_2$?

b) Repeat part (a) on the interval $(0,\ 1]$.

Solution

a) $p=?\ \ if\ \ V_1=V_2$

$$V_1=\pi\int_1^{\infty}\left(x^{-p}\right)^2\,dx$$

$$=\pi\int_1^{\infty}x^{-2p}\,dx$$

$$=\frac{\pi}{1-2p}x^{-2p+1}\Big|_1^{\infty}$$

If $-2p+1\geq 0\ \ \rightarrow\ \ p\leq\frac{1}{2}$

$$\boxed{V_1=\infty}$$

If $p>\frac{1}{2}$

$$V_1=\frac{\pi}{1-2p}(0-1)$$

$$\boxed{=\frac{\pi}{2p-1}\quad unit^3}$$

$$V_2=2\pi\int_1^{\infty}x^{1-p}\,dx$$

$$=\frac{2\pi}{2-p}x^{2-p}\Big|_1^{\infty}$$

If $2-p\geq 0\ \ \rightarrow\ \ p\leq 2$

$$\boxed{V_2=\infty}$$

If $p>2$

$$V_2=\frac{2\pi}{2-p}(0-1)$$

$$\boxed{=\frac{2\pi}{p-2}\quad unit^3}$$

$$V_1 = V_2$$

$$\frac{\pi}{2p-1} = \frac{2\pi}{2-p}$$

$$\frac{1}{2p-1} = \frac{2}{2-p}$$

$$2 - p = 4p - 2$$

$$5p = 4$$

$$p = \frac{4}{5} < 2$$

$\therefore V_1 = V_2$ only when both volumes are infinite.

b) $V_1 = \pi \int_0^1 x^{-2p}\, dx$

$$= \left. \frac{\pi}{1-2p} x^{-2p+1} \right|_0^1$$

$$= \frac{\pi}{1-2p}(1-0)$$

$$\boxed{= \frac{\pi}{1-2p} \quad unit^3}$$

$$V_2 = 2\pi \int_0^1 x^{1-p}\, dx$$

$$= \left. \frac{2\pi}{2-p} x^{2-p} \right|_0^1$$

$$= \frac{2\pi}{2-p}(1-0)$$

$$\boxed{= \frac{2\pi}{2-p} \quad unit^3}$$

$$\frac{\pi}{1-2p} = \frac{2\pi}{2-p}$$

$$\frac{1}{1-2p} = \frac{2}{2-p}$$

$$2 - p = 2 - 4p$$

$$\boxed{p = 0}$$

$\therefore V_1 \neq V_2$ for any p.

Exercise

Let R_1 be the region bounded by the graph of $y = e^{-ax}$ and the *x-axis* on the interval $[0,\ b]$ where $a > 0$ and $b > 0$. Let R_2 be the region bounded by the graph of $y = e^{-ax}$ and the *x-axis* on the interval $[b,\ \infty)$. Let V_1 and V_2 be the volumes of the solids generated when R_1 and R_2 are revolved about the *x-axis*. Find and graph the relationship between a and b for which $V_1 = V_2$.

Solution

Given: $R_1 :\ \ y = e^{-ax} \quad [0,\ b]$

$R_2 :\ \ y = e^{-ax} \quad [b,\ \infty]$

$$\begin{aligned}V_1 &= \pi \int_0^b \left(e^{-ax}\right)^2 dx \\ &= \pi \int_0^b e^{-2ax}\, dx \\ &= -\frac{\pi}{2a} e^{-2ax} \Big|_0^b \\ &= -\frac{\pi}{2a}\left(e^{-2ab} - 1\right) \quad unit^3\end{aligned}$$

$$\begin{aligned}V_2 &= \pi \int_b^\infty \left(e^{-ax}\right)^2 dx \\ &= \pi \int_b^\infty e^{-2ax}\, dx \\ &= -\frac{\pi}{2a} e^{-2ax} \Big|_b^\infty \\ &= \frac{\pi}{2a} e^{-2ab} \quad unit^3\end{aligned}$$

$$V_1 = V_2$$

$$-\frac{\pi}{2a}\left(e^{-2ab} - 1\right) = \frac{\pi}{2a} e^{-2ab}$$

$$1 - e^{-2ab} = e^{-2ab}$$

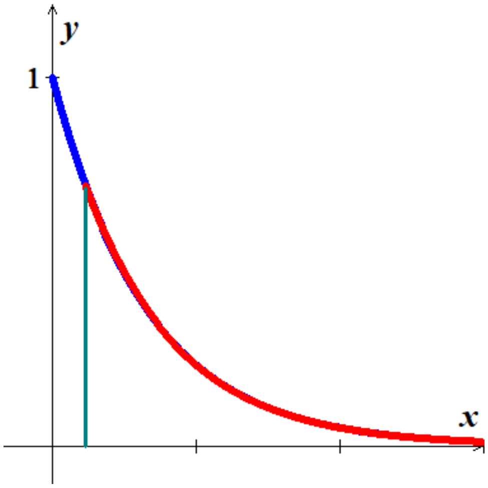
$$2e^{-2ab} = 1$$

$$\frac{1}{e^{2ab}} = \frac{1}{2}$$

$e^{2ab} = 2$

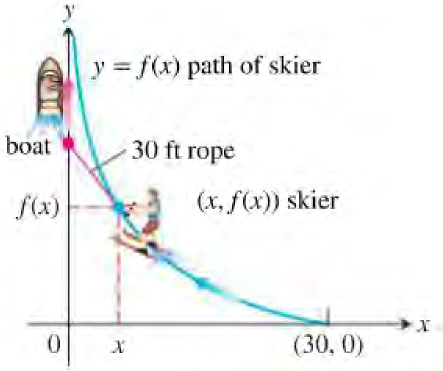
$2ab = \ln 2$

$\boxed{ab = \frac{1}{2}\ln 2}$



Exercise

Suppose that a boat is positioned at the origin with a water skier tethered to the boat at the point (30, 0) on a rope 30 *ft*. long. As the boat travels along the positive y-axis, the skier is pulled behind the boat along an unknown path $y = f(x)$, as shown



a) Show that $f'(x) = \frac{-\sqrt{900 - x^2}}{x}$

(*Hint*: Assume that the skier is always pointed directly at the boat and the rope is on line is on a line tangent to the path $y = f(x)$.)

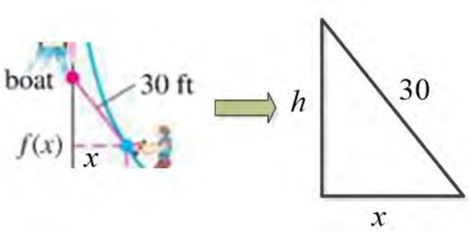
b) Solve the equation in part (a) for $f(x)$, using $f(30) = 0$

Solution

a) From the triangle: $h^2 + x^2 = 30^2 \quad \Rightarrow \quad h = \sqrt{900 - x^2}$

The slope of the tangent line (line is going down) is: $m = \frac{-\sqrt{900 - x^2}}{x}$

Thus, $f'(x) = \dfrac{-\sqrt{900 - x^2}}{x}$



b) $f(x) = \displaystyle\int \dfrac{-\sqrt{900 - x^2}}{x}\, dx$

$$x = 30\sin\theta \quad \to dx = 30\cos\theta d\theta, \quad 0 < \theta < \tfrac{\pi}{2}$$

$$\sqrt{900 - x^2} = \sqrt{900 - 900\sin^2\theta} = 30\cos\theta$$

$$f(x) = -\int \frac{30\cos\theta}{30\sin\theta}(30\cos\theta)\, d\theta$$

$$= -30\int \frac{\cos^2\theta}{\sin\theta} d\theta$$

$$= -30\int \frac{1 - \sin^2\theta}{\sin\theta} d\theta$$

$$= -30\int (\csc\theta - \sin\theta)\, d\theta$$

$$= -\ln\left|\csc\theta + \cot\theta\right| - 30\cos\theta + C$$

$$= -\ln\left|\frac{30}{x} + \frac{\sqrt{900 - x^2}}{x}\right| - \sqrt{900 - x^2} + C$$

Given (30, 0), then

$$0 = -\ln\left|\frac{30}{30} + \frac{\sqrt{900 - 30^2}}{30}\right| - \sqrt{900 - 30^2} + C$$

$$0 = -\ln|1| + C \quad \Rightarrow \quad \boxed{C = 0}$$

$$\boxed{f(x) = -\ln\left|\frac{30}{x} + \frac{\sqrt{900 - x^2}}{x}\right| - \sqrt{900 - x^2}}$$

Exercise

Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If a is the amount of substance A and b is the substance B at time $t = 0$, and if x is the amount of product at time t, then the rate of formation of x may be given by the differential equation

$$\frac{dx}{dt} = k(a-x)(b-x) \quad or \quad \frac{1}{(a-x)(b-x)}\frac{dx}{dt} = k$$

Where k is a constant for the reaction. Integrate both sides of this equation to obtain a relation between x and t.

a) If $a = b$

b) If $a \ne b$

Assume in each case that $x = 0$ when $t = 0$

Solution

$$\frac{1}{(a-x)(b-x)}dx = kdt$$

a) $a = b \quad \Rightarrow \quad \frac{1}{(a-x)^2}dx = kdt$

$$\int \frac{1}{(a-x)^2}dx = \int kdt$$

$$\frac{1}{a-x} = kt + C$$

$$x(t=0) = 0 \quad \Rightarrow \quad \frac{1}{a} = C$$

$$\frac{1}{a-x} = kt + \frac{1}{a} = \frac{kat+1}{a}$$

$$a - x = \frac{a}{kat+1}$$

$$x = a - \frac{a}{kat+1}$$

$$= \frac{a^2kt}{kat+1}$$

b) $a \ne b \quad \Rightarrow \quad \frac{1}{(a-x)(b-x)}dx = kdt$

$$\int \frac{1}{(a-x)(b-x)}dx = \int k\,dt$$

$$\frac{1}{(a-x)(b-x)} = \frac{A}{a-x} + \frac{B}{b-x}$$

$$\begin{cases} -A-B=0 \\ bA+aB=1 \end{cases} \rightarrow \begin{cases} B=\dfrac{1}{a-b} \\ A=-\dfrac{1}{a-b} \end{cases}$$

$$\frac{-1}{a-b}\int \frac{1}{a-x}dx+\frac{1}{a-b}\int \frac{1}{b-x}dx=\int kdt$$

$$\frac{1}{a-b}\ln|a-x|-\frac{1}{a-b}\ln|b-x|=kt+C$$

$$\frac{1}{a-b}\ln\left|\frac{a-x}{b-x}\right|=kt+C$$

$$x(0)=0 \quad \Rightarrow \quad \boxed{\frac{1}{a-b}\ln\left(\frac{a}{b}\right)=C}$$

$$\frac{1}{a-b}\ln\left|\frac{a-x}{b-x}\right|=kt+\frac{1}{a-b}\ln\left(\frac{a}{b}\right)$$

$$\ln\left|\frac{a-x}{b-x}\right|=(a-b)kt+\ln\left(\frac{a}{b}\right)$$

$$\frac{a-x}{b-x}=e^{(a-b)kt+\ln\left(\frac{a}{b}\right)}$$

$$\frac{a-x}{b-x}=\frac{a}{b}e^{(a-b)kt}$$

$$a-x=b\frac{a}{b}e^{(a-b)kt}-x\frac{a}{b}e^{(a-b)kt}$$

$$x\left(\frac{a}{b}e^{(a-b)kt}-1\right)=ae^{(a-b)kt}-a$$

$$\boxed{x=\frac{abe^{(a-b)kt}-ab}{ae^{(a-b)kt}-b}}$$

Solution Section 2.7 – First-Order Linear Equations

Exercise

Write an equivalent first-order differential equation and initial condition for y. $y=\int_1^x \frac{1}{t}\,dt$

Solution

$$\int_1^x \frac{1}{t}\,dt \quad \Rightarrow \quad \frac{dy}{dx}=\frac{1}{x}$$

$$y(1)=\int_1^1 \frac{1}{t}\,dt \qquad\qquad \int_a^a f(x)dx=0$$

$$=\ln t\Big|_1^1$$

$$=\ln 1-\ln 1$$

$$\underline{=0}\,\big|$$

$$\boxed{\frac{dy}{dx}=\frac{1}{x};\quad y(1)=0}$$

Exercise

Write an equivalent first-order differential equation and initial condition for $y=2-\int_0^x \left(1+y(t)\right)\sin t\;dt$

Solution

$$y=2-\int_0^x \left(1+y(t)\right)\sin t\;dt$$

$$\frac{dy}{dx}=-\left(1+y(x)\right)\sin x$$

$$y(0)=2-\int_0^0 \left(1+y(t)\right)\sin t\;dt \qquad\qquad \int_a^a f(x)dx=0$$

$$\underline{=2}\,\big|$$

$$\boxed{\frac{dy}{dx}=-\left(1+y(x)\right)\sin x;\quad y(0)=2}$$

Exercise

Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals

$$y' = 1 - \frac{y}{x}, \quad y(2) = -1, \quad dx = 0.5$$

Solution

$$y_1 = y_0 + \left(1 - \frac{y_0}{x_0}\right)dx$$

$$= -1 + \left(1 - \frac{-1}{2}\right)(0.5)$$

$$\underline{= -0.25}$$

$$y_2 = y_1 + \left(1 - \frac{y_1}{x_1}\right)dx$$

$$= -0.25 + \left(1 - \frac{-0.25}{2.5}\right)(0.5)$$

$$\underline{= 0.3}$$

$$y_3 = y_2 + \left(1 - \frac{y_2}{x_2}\right)dx$$

$$= 0.3 + \left(1 - \frac{0.3}{3}\right)(0.5)$$

$$\underline{= 0.75}$$

$$y' + \frac{1}{x}y = 1 \qquad P(x) = \frac{1}{x}, \quad Q(x) = 1$$

$$y_h = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int (1)e^{\int \frac{1}{x}dx}dx = \int x\, dx = \frac{1}{2}x^2$$

$$y(x) = \frac{1}{x}\left(\frac{1}{2}x^2 + C\right)$$

$$= \frac{1}{2}x + \frac{C}{x}$$

$$y(2) = \frac{1}{2}(2) + \frac{C}{2} = -1$$

$$1 + \frac{C}{2} = -1$$

$$\frac{C}{2} = -2$$

$$\underline{C = -4}$$

$$\underline{y(x) = \frac{x}{2} - \frac{4}{x}}$$

$$y(3.5)=\frac{3.5}{2}-\frac{4}{3.5}$$
$$\boxed{\approx 0.6071}$$

Exercise

Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals

$y'=x(1-y), \quad y(1)=0, \quad dx=0.2$

Solution

$$y_1=y_0+x_0\left(1-y_0\right)dx$$
$$=0+1(1-0)(0.2)$$
$$\boxed{=0.2}$$

$$y_2=y_1+x_1\left(1-y_1\right)dx$$
$$=0.2+1.2(1-0.2)(0.2)$$
$$\boxed{=0.392}$$

$$y_3=y_2+x_2\left(1-y_2\right)dx$$
$$=0.392+1.4(1-0.392)(0.2)$$
$$\boxed{=0.5622}$$

$$\frac{y'}{1-y}=x\,dx$$
$$\int\frac{dy}{1-y}=\int x\,dx$$
$$\ln|1-y|=\frac{1}{2}x^2+C$$
$$1-y=e^{\frac{1}{2}x^2+C}$$
$$\boxed{y=1-e^{\frac{1}{2}x^2+C}}$$
$$y(1)=1-e^{\frac{1}{2}1^2+C}=0$$
$$e^{\frac{1}{2}+C}=1$$
$$\frac{1}{2}+C=0$$
$$\boxed{C=-\frac{1}{2}}$$

$$\boxed{y(x) = 1 - e^{\frac{1}{2}\left(x^2-1\right)}}$$

$$y(1.6) = 1 - e^{\frac{1}{2}\left(1.6^2-1\right)}$$

$$\approx \boxed{0.5416}$$

Exercise

Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals

$$y' = y^2(1+2x), \quad y(-1) = 1, \quad dx = 0.5$$

Solution

$$y_1 = y_0 + y_0^2\left(1+2x_0\right)dx$$

$$= 1 + 1^2(1+2(-1))(0.5)$$

$$\boxed{= .5}$$

$$y_2 = y_1 + y_1^2\left(1+2x_1\right)dx$$

$$= 0.5 + 0.5^2(1+2(-0.5))(0.5)$$

$$\boxed{= .5}$$

$$y_3 = y_2 + y_2^2\left(1+2x_2\right)dx$$

$$= .5 + .5^2(1+2(0))(0.5)$$

$$\boxed{= .625}$$

$$\frac{dy}{y^2} = (1+2x)dx$$

$$\int \frac{dy}{y^2} = \int (1+2x)dx$$

$$-\frac{1}{y} = x + x^2 + C$$

$$y = -\frac{1}{x + x^2 + C}$$

$$y(-1) = -\frac{1}{-1+(-1)^2+C}$$

$$1 = -\frac{1}{C}$$

$$\boxed{C = -1}$$

$$y(x) = -\frac{1}{x + x^2 - 1}$$

$$= \frac{1}{1 - x - x^2}$$

$$y(.5) = \frac{1}{1 - .5 - .5^2}$$

$$= 4$$

Exercise

Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals

$y' = ye^x, \quad y(0) = 2, \quad dx = 0.5$

Solution

$$y_1 = y_0 + \left(y_0 e^{x_0} \right) dx$$

$$= 2 + \left(2e^0 \right)(0.5)$$

$$= 3$$

$$y_2 = y_1 + \left(y_1 e^{x_1} \right) dx$$

$$= 3 + \left(3e^{0.5} \right)(0.5)$$

$$= 5.47308$$

$$y_3 = y_2 + \left(y_2 e^{x_2} \right) dx$$

$$= 5.47308 + \left(5.47308 e^1 \right)(0.5)$$

$$= 12.9118$$

$$\frac{dy}{dx} = ye^x$$

$$\int \frac{dy}{y} = \int e^x dx$$

$$\ln y = e^x + C$$

$$\ln 2 = e^0 + C$$

$$C = \ln 2 - 1$$

$$\ln |y| = e^x + \ln 2 - 1$$

$$y(x)=e^{e^x+\ln 2-1}$$

$$=e^{\ln 2}e^{e^x-1}$$

$$\underline{=2e^{e^x-1}}|$$

$$y(1.5)=2e^{e^{1.5}-1}$$

$$\underline{\approx 65.0292}|$$

Exercise

Use the Euler method with $dx=0.2$ to estimate $y(2)$ if $y'=\frac{y}{x}$ and $y(1)=2$. What is the exact value of $y(2)$?

Solution

$$y_1(1)=y_0+\left(\frac{y_0}{x_0}\right)dx$$

$$=2+\left(\tfrac{2}{1}\right)(0.2)$$

$$\underline{=2.4}|$$

$$y_2(1.2)=y_1+\left(\frac{y_1}{x_1}\right)dx$$

$$=2.4+\left(\tfrac{2.4}{1.2}\right)(0.2)$$

$$\underline{=2.8}|$$

$$y_3=y_2+\left(\frac{y_2}{x_2}\right)dx$$

$$=2.8+\left(\tfrac{2.8}{1.4}\right)(0.2)$$

$$\underline{=3.2}|$$

$$y_4=y_3+\left(\frac{y_3}{x_3}\right)dx$$

$$=3.2+\left(\tfrac{3.2}{1.6}\right)(0.2)$$

$$\underline{=3.6}|$$

$$y_5=y_4+\left(\frac{y_4}{x_4}\right)dx$$

$$=3.6+\left(\tfrac{3.6}{1.8}\right)(0.2)$$

$$\underline{=4}|$$

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + C$$

$$\ln 2 = \ln 1 + C$$

$$\boxed{C = \ln 2}$$

$$\ln y = \ln x + \ln 2$$

$$= \ln 2x$$

$$\boxed{y(x) = 2x}$$

$$y(2) = 2(2)$$

$$\boxed{= 4}$$

Exercise

Use Euler's Method to solve $y' = 1 + y, \quad y(0) = 1$ on the interval $0 \le x \le 1$ and taking $dx = 0.05$. Compare the approximation to the values of the exact solution.

Solution

y(t) = 2*exp(t) – 1

```
 Euler Method
 t          Approx.            Exact         Difference
-------------------------------------------------------
0.00  |  1.00000000  |  1.00000000  |  0.00000000
0.05  |  1.10000000  |  1.10254219  |  0.00254219
0.10  |  1.20500000  |  1.21034184  |  0.00534184
0.15  |  1.31525000  |  1.32366849  |  0.00841849
0.20  |  1.43101250  |  1.44280552  |  0.01179302
0.25  |  1.55256313  |  1.56805083  |  0.01548771
0.30  |  1.68019128  |  1.69971762  |  0.01952633
0.35  |  1.81420085  |  1.83813510  |  0.02393425
0.40  |  1.95491089  |  1.98364940  |  0.02873851
0.45  |  2.10265643  |  2.13662437  |  0.03396794
0.50  |  2.25778925  |  2.29744254  |  0.03965329
0.55  |  2.42067872  |  2.46650604  |  0.04582732
0.60  |  2.59171265  |  2.64423760  |  0.05252495
0.65  |  2.77129828  |  2.83108166  |  0.05978337
0.70  |  2.95986320  |  3.02750541  |  0.06764222
0.75  |  3.15785636  |  3.23400003  |  0.07614367
0.80  |  3.36574918  |  3.45108186  |  0.08533268
```

0.85	3.58403664	3.67929370	0.09525707
0.90	3.81323847	3.91920622	0.10596775
0.95	4.05390039	4.17141932	0.11751893
1.00	4.30659541	4.43656366	0.12996825

Exercise

Use Euler's Method to solve $y' = 2xy + 2y, \quad y(0) = 3$ on the interval $0 \le x \le 1$ and taking $dx = 0.1$. Compare the approximation to the values of the exact solution.

Solution

$$\frac{dy}{y} = (2x+2)\,dx$$

$$\ln y = x^2 + 2x + C_1$$

$$y = Ce^{x^2+2x} \quad \rightarrow 3 = C$$

$$\boxed{y(x) = 3e^{x^2+2x}}$$

t	***Euler Method***	***Exact***	***Difference***
0.000	3.0	3.0	0.0
0.100	3.6	3.70103418	0.10103418
0.200	4.392	4.65812166	0.26612166
0.300	5.44608	5.9811466	0.5350666
0.400	6.8620608	7.83508942	0.97302862
0.500	8.78343782	10.47102887	1.68759105
0.600	11.41846917	14.27646374	2.85799457
0.700	15.07237931	19.85810604	4.78572673
0.800	20.19698827	28.17999386	7.98300559
0.900	27.46790405	40.79715256	13.32924851
1.000	37.90570759	60.25661077	22.35090318

Exercise

Verify that the given function *y* is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y = Ce^{-5t}; \quad y'(t) + 5y = 0$

Solution

$$y = Ce^{-5t} \quad \Rightarrow \quad y' = -5Ce^{-5t} = -5y$$

$$y'(t) + 5y = -5y + 5y$$

$$= 0 \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y = Ct^{-3};\quad ty'(t) + 3y = 0$

Solution

$$y = Ct^{-3}$$

$$y' = -3Ct^{-4}$$

$$t\left(-3Ct^{-4}\right) + 3Ct^{-3} = -3Ct^{-3} + 3Ct^{-3}$$

$$\underline{=0\,|}\quad \checkmark$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y = C_1 \sin 4t + C_2 \cos 4t;\quad y''(t) + 16y = 0$

Solution

$$y' = 4C_1 \cos 4t - 4C_2 \sin 4t$$

$$y'' = -16C_1 \sin 4t - 16C_2 \cos 4t$$

$$y''(t) + 16y = -16C_1 \sin 4t - 16C_2 \cos 4t + 16C_1 \sin 4t + 16C_2 \cos 4t$$

$$\underline{=0\,|}\quad \checkmark$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y = C_1 e^{-x} + C_2 e^{x};\quad y''(x) - y = 0$

Solution

$$y' = -C_1 e^{-x} + C_2 e^{x}$$

$$y'' = C_1 e^{-x} + C_2 e^{x}$$

$$y''(x) - y = C_1 e^{-x} + C_2 e^{x} - C_1 e^{-x} - C_2 e^{x}$$

$$\underline{=0\,|}\quad \checkmark$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $y' + 4y = \cos t, \quad y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t + Ce^{-4t}, \quad y(0) = -1$

Solution

$$y(0) = \frac{4}{17}\cos(0) + \frac{1}{17}\sin(0) + Ce^{-4(0)}$$

$$-1 = \frac{4}{17} + C$$

$$C = -1 - \frac{4}{17} = -\frac{21}{17}$$

$$\boxed{y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t - \frac{21}{17}e^{-4t}}$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $ty' + (t+1)y = 2te^{-t}, \quad y(t) = e^{-t}\left(t + \frac{C}{t}\right), \quad y(1) = \frac{1}{e}$

Solution

$$y(1) = \frac{1}{e} = e^{-1}$$

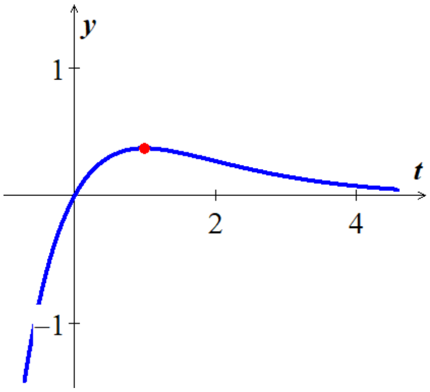
$$y(1) = e^{-1}\left(1 + \frac{C}{1}\right)$$

$$e^{-1} = e^{-1}(1 + C)$$

$$\Rightarrow\ 1 = 1 + C$$

Hence, $C = 0$

The solution is: $y(t) = te^{-t}$



This function is defined and differentiable on the whole real line. Hence, the interval of existence is the whole real line.

Exercise

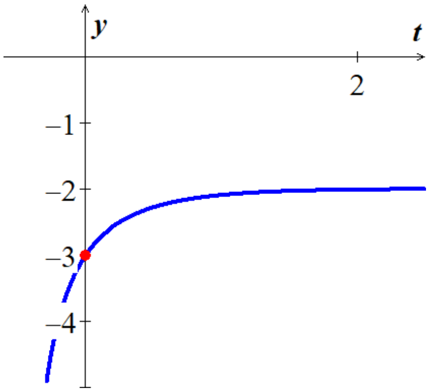
Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $y' = y(2+y), \quad y(t) = \frac{2}{-1 + Ce^{-2t}}, \quad y(0) = -3$

Solution

$$y(0) = \frac{2}{-1 + Ce^{-2(0)}}$$

$$-3 = \frac{2}{-1 + C}$$

$$3 - 3C = 2$$
$$-3C = -1$$
$$\boxed{C = \frac{1}{3}}$$



The solution is:

$$y(t) = \frac{2}{-1 + \frac{1}{3}e^{-2t}}$$
$$\boxed{= \frac{6}{-3 + e^{-2t}}}$$

Exercise

Verify that the given function y is a solution of the initial value problem that follows it.

$$y = 16e^{2t} - 10; \quad y' - 2y = 20, \quad y(0) = 6$$

Solution

$$y(0) = 6$$
$$y(0) = 16 - 10 = 6 \quad \checkmark$$

$$y = 16e^{2t} - 10$$
$$y' = 32e^{2t}$$

$$y' - 2y = 32e^{2t} - 32e^{2t} + 20$$
$$\boxed{= 20} \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the initial value problem that follows it.

$$y = 8t^6 - 3; \quad ty' - 6y = 18, \quad y(1) = 5$$

Solution

$$y = 8t^6 - 3$$
$$y(1) = 8 - 3$$
$$\boxed{= 5} \quad \checkmark$$

$$y' = 48t^5$$

$$ty' - 6y = 48t^6 - 48t^6 + 18$$
$$\boxed{= 18} \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the initial value problem that follows it.

$$y=-3\cos 3t; \quad y''+9y=0, \quad y(0)=-3, \;\; y'(0)=0$$

Solution

$$y=-3\cos 3t$$

$$y(0)=-3\cos 0$$

$$\boxed{=-3} \;\; \checkmark$$

$$y'=9\sin 3t$$

$$\boxed{y(0)=0} \;\; \checkmark$$

$$y''=27\cos 3t$$

$$y''+9y=27\cos 3t-27\cos 3t$$

$$\boxed{=0}$$

Exercise

Verify that the given function y is a solution of the initial value problem that follows it.

$$y=\frac{1}{4}\left(e^{2x}-e^{-2x}\right); \quad y''-4y=0, \quad y(0)=0, \;\; y'(0)=1$$

Solution

$$y=\frac{1}{4}\left(e^{2x}-e^{-2x}\right)$$

$$y(0)=\frac{1}{4}(1-1)$$

$$\boxed{=0} \;\; \checkmark$$

$$y'=\frac{1}{2}\left(e^{2x}+e^{-2x}\right)$$

$$y'(0)=\frac{1}{2}(1+1)$$

$$\boxed{=1} \;\; \checkmark$$

$$y''=e^{2x}-e^{-2x}$$

$$y''-4y=e^{2x}-e^{-2x}-e^{2x}+e^{-2x}$$

$$\boxed{=0} \;\; \checkmark$$

Exercise

Find the general solution of the differential equation $y'=xy$

Solution

$$\frac{dy}{dx} = xy$$

$$\frac{dy}{y} = xdx$$

$$\int \frac{dy}{y} = \int x\,dx$$

$$\ln|y| = \frac{1}{2}x^2 + C$$

$$|y| = e^{x^2/2+C}$$

$$y(x) = \pm e^{x^2/2}e^C$$

$$\underline{= Ae^{x^2/2}}\Big| \qquad \text{Where } A = \pm e^C$$

Exercise

Find the general solution of the differential equation $xy' = 2y$

Solution

$$x\frac{dy}{dx} = 2y$$

$$\frac{dy}{y} = 2\frac{dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{2}{x}dx$$

$$\ln|y| = 2\ln|x| + C$$

$$= \ln x^2 + C$$

$$y(x) = \pm e^{\ln x^2 + C}$$

$$= \pm e^C x^2$$

$$\underline{= Ax^2}\Big|$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = e^{x-y}$

Solution

$$\frac{dy}{dx} = e^x e^{-y}$$

$$\frac{dy}{e^{-y}} = e^x dx$$

$$\int e^y dy = \int e^x dx$$

$$e^y = e^x + C$$

$$\boxed{y(x) = \ln\left(e^x + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \left(1 + y^2\right)e^x$

Solution

$$\frac{dy}{dx} = \left(1 + y^2\right)e^x$$

$$\frac{dy}{1 + y^2} = e^x dx$$

$$\int \frac{dy}{1 + y^2} = \int e^x dx$$

$$\tan^{-1} y = e^x + C$$

$$\boxed{y(x) = \tan\left(e^x + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = xy + y$

Solution

$$\frac{dy}{dx} = (x + 1)y$$

$$\frac{dy}{y} = (x + 1)dx$$

$$\int \frac{dy}{y} = \int (x + 1)dx$$

$$\ln y = \frac{1}{2}x^2 + x + C$$

$$\boxed{y = e^{x^2/2 + x + C}}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$y' = ye^x - 2e^x + y - 2$$

Solution

$$\frac{dy}{dx} = (y-2)e^x + y - 2$$

$$\frac{dy}{dx} = (y-2)\left(e^x + 1\right)$$

$$\frac{dy}{y-2} = \left(e^x + 1\right)dx$$

$$\int \frac{dy}{y-2} = \int \left(e^x + 1\right)dx$$

$$\ln|y-2| = e^x + x + C$$

$$y - 2 = \pm e^{e^x + x + C}$$

$$y - 2 = \pm e^C e^{e^x + x}$$

$$\underline{y(x) = De^{e^x + x} + 2} \quad D = \pm e^C$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \dfrac{x}{y+2}$

Solution

$$\frac{dy}{dx} = \frac{x}{y+2}$$

$$(y+2)\,dy = xdx$$

$$\int (y+2)\,dy = \int x\,dx$$

$$\frac{1}{2}y^2 + 2y = \frac{1}{2}x^2 + C$$

$$\underline{y^2 + 4y = x^2 + 2C}$$

$$y^2 + 4y - x^2 - D = 0, \quad (D = 2C)$$

$$y = \frac{-4 \pm \sqrt{16 - 4\left(-x^2 - D\right)}}{2} = \frac{-4 \pm \sqrt{16 + 4x^2 + 4D}}{2}$$

$$= \frac{-4 \pm 2\sqrt{x^2 + (4+D)}}{2} \qquad E = 4 + D$$

$$= -2 \pm \sqrt{x^2 + E}$$

$$\boxed{y(x) = -2 \pm \sqrt{x^2 + E}}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \frac{xy}{x-1}$

Solution

$$\frac{dy}{dx} = y\left(\frac{x}{x-1}\right)$$

$$\frac{dy}{y} = \left(\frac{x}{x-1}\right)dx$$

$$\int \frac{dy}{y} = \int \left(1 + \frac{1}{x-1}\right) dx$$

$$\ln|y| = x + \ln|x-1| + C$$

$$y(x) = \pm e^{x + \ln|x-1| + C}$$

$$= \pm e^C e^x e^{\ln|x-1|}$$

$$\boxed{= De^x |x-1|}$$

Exercise

Solve the differential equations: $x\frac{dy}{dx} + y = e^x, \quad x > 0$

Solution

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{e^x}{x}$$

$$y_h = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int \frac{e^x}{x} e^{\int \frac{1}{x}dx} dx = \int x\frac{e^x}{x} dx$$

$$= \int e^x dx$$

$$= e^x$$

$$\boxed{y(x) = \frac{1}{x}\left(e^x + C\right)}, \quad x > 0$$

Exercise

Solve the differential equations: $y' + (\tan x)y = \cos^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

Solution

$$y' + (\tan x)y = \cos^2 x$$

$$y_h = e^{\int \tan x\, dx} = e^{\ln(\cos x)^{-1}} \qquad \int \tan x dx = -\ln|\cos x| = \ln(\cos x)^{-1}$$

$$= (\cos x)^{-1}$$

$$\int \cos^2 x (\cos x)^{-1} dx = \int \cos x\, dx$$

$$= \sin x$$

$$y(x) = \frac{1}{(\cos x)^{-1}}(\sin x + C)$$

$$y(x) = \cos x(\sin x + C)$$

$$y(x) = \cos x \sin x + C\cos x$$

Exercise

Solve the differential equations: $x\frac{dy}{dx} + 2y = 1 - \frac{1}{x}, \quad x > 0$

Solution

$$y' + \frac{2}{x}y = \frac{1}{x} - \frac{1}{x^2}$$

$$y_h = e^{\int \frac{2}{x}dx}$$

$$= e^{2\ln x}$$

$$= e^{\ln x^2}$$

$$= x^2$$

$$\int \left(\frac{1}{x} - \frac{1}{x^2}\right)x^2\, dx = \int (x-1)\, dx$$

$$= \frac{1}{2}x^2 - x$$

$$y(x) = \frac{1}{x^2}\left(\frac{1}{2}x^2 - x + C\right)$$

$$y(x) = \frac{1}{2} - \frac{1}{x} + \frac{C}{x^2}, \quad x > 0$$

Exercise

Solve the differential equations: $(1+x)\,y' + y = \sqrt{x}$

Solution

$$y' + \frac{1}{1+x}y = \frac{\sqrt{x}}{1+x}$$

$$e^{\int \frac{1}{1+x}dx} = e^{\ln(1+x)}$$

$$\boxed{=1+x}$$

$$\int \frac{\sqrt{x}}{1+x}(1+x)\,dx = \int x^{1/2}\,dx$$

$$= \frac{2}{3}x^{3/2}$$

$$y(x) = \frac{1}{1+x}\left(\frac{2}{3}x^{3/2} + C\right)$$

$$\boxed{= \frac{2x^{3/2}}{3(1+x)} + \frac{C}{1+x}}$$

Exercise

Solve the differential equations: $e^{2x}y' + 2e^{2x}y = 2x$

Solution

$$y' + 2y = 2xe^{-2x}$$

$$e^{\int 2dx} = e^{2x}$$

$$\int 2xe^{-2x}\left(e^{2x}\right)dx = 2\int x\,dx$$

$$= x^2$$

$$y(x) = \frac{1}{e^{2x}}\left(x^2 + C\right)$$

$$\boxed{= x^2e^{-2x} + Ce^{-2x}}$$

Exercise

Solve the differential equations: $(t+1)\frac{ds}{dt} + 2s = 3(t+1) + \frac{1}{(t+1)^2}, \quad t > -1$

Solution

$$s' + \frac{2}{t+1}s = 3 + \frac{1}{(t+1)^3}$$

$$e^{\int \frac{2}{t+1}dt} = e^{2\ln(t+1)}$$
$$= e^{\ln(t+1)^2}$$
$$= (t+1)^2$$

$$\int \left(3 + \frac{1}{(t+1)^3}\right)(t+1)^2\, dt = \int \left(3(t+1)^2 + \frac{1}{t+1}\right)dt \qquad d(t+1) = dt$$
$$= 3\int (t+1)^2\, d(t+1) + \int \frac{1}{t+1}d(t+1)$$
$$= (t+1)^3 + \ln(t+1)$$

$$s(t) = \frac{1}{(t+1)^2}\left((t+1)^3 + \ln(t+1) + C\right)$$
$$= t + 1 + \frac{\ln(t+1)}{(t+1)^2} + \frac{C}{(t+1)^2}, \quad t > -1$$

Exercise

Solve the differential equations: $\tan\theta \frac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \frac{\pi}{2}$

Solution

$$\frac{dr}{d\theta} + \frac{1}{\tan\theta}r = \frac{\sin^2\theta}{\tan\theta}$$
$$\frac{dr}{d\theta} + \frac{1}{\tan\theta}r = \sin^2\theta\frac{\cos\theta}{\sin\theta}$$
$$\frac{dr}{d\theta} + \frac{1}{\tan\theta}r = \sin\theta\cos\theta$$

$$e^{\int \cot\theta d\theta} = e^{\ln|\sin\theta|}$$
$$= \sin\theta, \quad 0 < \theta < \frac{\pi}{2}$$

$$\int (\sin\theta\cos\theta)(\sin\theta)\,d\theta = \int \left(\sin^2\theta\cos\theta\right)d\theta \qquad d(\sin\theta) = \cos\theta d\theta$$
$$= \int \sin^2\theta d(\sin\theta)$$

$$=\frac{1}{3}\sin^3\theta$$

$$r(\theta)=\frac{1}{\sin\theta}\left(\frac{1}{3}\sin^3\theta+C\right)$$

$$\boxed{=\frac{1}{3}\sin^2\theta+\frac{C}{\sin\theta}}$$

Exercise

Find the general solution of $y'=\cos x-y\sec x$

Solution

$$y'+(\sec x)y=\cos x$$

$$e^{\int \sec x dx}=e^{\ln|\sec x+\tan x|}$$

$$=\sec x+\tan x$$

$$\int \cos x(\sec x+\tan x)dx=\int \cos x\left(\frac{1}{\cos x}+\frac{\sin x}{\cos x}\right)dx$$

$$=\int (1+\sin x)dx$$

$$=x-\cos x$$

$$\boxed{y(x)=\frac{1}{\sec x+\tan x}(x-\cos x+C)}$$

Exercise

Find the general solution of $\left(1+x^3\right)y'=3x^2y+x^2+x^5$

Solution

$$y'-\frac{3x^2}{1+x^3}y=\frac{x^2\left(1+x^3\right)}{1+x^3}$$

$$y'-\frac{3x^2}{1+x^3}y=x^2$$

$$e^{\int -\frac{3x^2}{1+x^3}dx}=e^{-\int \frac{d\left(1+x^3\right)}{1+x^3}}$$

$$=e^{-\ln\left(1+x^3\right)}$$

$$=e^{\ln\left(1+x^3\right)^{-1}}$$

$$=\frac{1}{1+x^3}$$

$$\int \frac{1}{1+x^3}\left(x^2\right)dx=\frac{1}{3}\int \frac{d\left(1+x^3\right)}{1+x^3}$$

$$=\frac{1}{3}\ln\left(1+x^3\right)$$

$$y(x)=\left(1+x^3\right)\left(\frac{1}{3}\ln\left(1+x^3\right)+C\right)$$

$$=\frac{1}{3}\left(1+x^3\right)\ln\left(1+x^3\right)+C\left(1+x^3\right)$$

Exercise

Find the general solution of $\frac{dy}{dt}-2y=4-t$

Solution

$$e^{\int -2dt}=e^{-2t}$$

$$\int (4-t)e^{-2t}\,dt=\int\left(4e^{-2t}-te^{-2t}\right)dt$$

$$=-2e^{-2t}+\frac{1}{2}te^{-2t}+\frac{1}{4}e^{-2t}$$

$$=-\frac{7}{4}e^{-2t}+\frac{1}{2}te^{-2t}$$

		$\int e^{-2t}$
+	t	$-\frac{1}{2}e^{-2t}$
−	1	$\frac{1}{4}e^{-2t}$

$$y(t)=\frac{1}{e^{-2t}}\left(-\frac{7}{4}e^{-2t}+\frac{1}{2}te^{-2t}+C\right)$$

$$y(t)=\frac{1}{2}t-\frac{7}{4}+Ce^{2t}$$

Exercise

Find the general solution of $y'+y=\frac{1}{1+e^t}$

Solution

$$e^{\int dt}=e^t$$

$$\int \frac{1}{1+e^t}e^t\,dt=\int \frac{1}{1+e^t}d\left(1+e^t\right)$$

$$=\ln\left(1+e^t\right)$$

$$y(t)=\frac{1}{e^t}\left(\ln\left(1+e^t\right)+C\right)$$

$$\boxed{y(t)=e^{-t}\ln\left(1+e^t\right)+Ce^{-t}}$$

Exercise

Solve the differential equation $\quad y'=3y-4$

Solution

$$y'-3y=-4$$

$$e^{\int -3dx}=e^{-3x}$$

$$\int -4e^{-3x}\,dx=\frac{4}{3}e^{-3x}$$

$$y(x)=\frac{1}{e^{-3x}}\left(\frac{4}{3}e^{-3x}+C\right)$$

$$\boxed{=\frac{4}{3}+Ce^{3x}}$$

Exercise

Solve the differential equation $\quad y'=-2y-4$

Solution

$$y'+2y=-4$$

$$e^{\int 2dx}=e^{2x}$$

$$\int -4e^{2x}dx=2e^{2x}$$

$$y(x)=\frac{1}{e^{2x}}\left(2e^{2x}+C\right)$$

$$\boxed{=2+Ce^{-2x}}$$

Exercise

Solve the differential equation $\quad y'=-y+2$

Solution

$$y'+y=2$$

$$e^{\int dx} = e^x$$

$$\int 2e^x dx = 2e^x$$

$$y(x) = \frac{1}{e^x}\left(2e^x + C\right)$$

$$\underline{= 2 + Ce^{-x} \,|}$$

Exercise

Solve the differential equation $y' = 2y + 6$

Solution

$$y' - 2y = 6$$

$$e^{\int -2dx} = e^{-2x}$$

$$\int 6e^{-2x}\, dx = -3e^{-2x}$$

$$y(x) = e^{2x}\left(-3e^{-2x} + C\right)$$

$$\underline{= -3 + Ce^{2x} \,|}$$

Exercise

Solve the differential equation $x(x-1)dy - ydx = 0$

Solution

$$x(x-1)dy = ydx$$

$$\frac{dy}{y} = \frac{dx}{x(x-1)}$$

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$Ax - A + Bx = 1$$

$$\begin{cases} x & A + B = 0 & \rightarrow \underline{B = 1\,|} \\ x^0 & -A = 1 & \rightarrow \underline{A = -1\,|} \end{cases}$$

$$\int \frac{dy}{y} = \int \left(-\frac{1}{x} + \frac{1}{x-1}\right) dx$$

$$\ln y = -\ln|x| + \ln|x-1| + \ln C$$

$$\ln y = \ln\left|\frac{C(x-1)}{x}\right|$$

$$\boxed{y(x) = C\frac{(x-1)}{x}}$$

Exercise

Solve the differential equation $xy' + 2y = 1 - x^{-1}$

Solution

$$xy' + 2y = 1 - \frac{1}{x}$$

$$y' + \frac{2}{x}y = \frac{1}{x} - \frac{1}{x^2}$$

$$e^{\int \frac{2}{x}dx} = e^{2\ln|x|}$$

$$= e^{\ln x^2}$$

$$= x^2$$

$$\int \left(\frac{1}{x} - \frac{1}{x^2}\right)x^2\, dx = \int (x-1)\, dx$$

$$= \frac{1}{2}x^2 - x$$

$$y(x) = \frac{1}{x^2}\left(\frac{1}{2}x^2 - x + C\right)$$

$$\boxed{= \frac{1}{2} - \frac{1}{x} + \frac{C}{x^2}}$$

Exercise

Solve the differential equation $xy' - y = 2x\ln x$

Solution

$$y' - \frac{1}{x}y = 2\ln x$$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln|x|}$$

$$= e^{\ln x^{-1}}$$

$$= \frac{1}{x}$$

$$\int (2\ln x)\frac{1}{x}dx = 2\int \ln x \, d(\ln x)$$

$$= (\ln x)^2$$

$$y(x) = x\left(\ln^2 x + C\right)$$

$$\boxed{= x(\ln x)^2 + Cx}$$

Exercise

Solve the differential equation $\left(1+e^x\right)dy + \left(ye^x + e^{-x}\right)dx = 0$

Solution

$$\left(1+e^x\right)\frac{dy}{dx} + ye^x + e^{-x} = 0$$

$$\left(1+e^x\right)y' + e^x y = -\frac{1}{e^x}$$

$$y' + \frac{e^x}{1+e^x}y = -\frac{1}{e^x\left(1+e^x\right)}$$

$$e^{\int \frac{e^x}{1+e^x}dx} = e^{\int \frac{1}{1+e^x}d\left(1+e^x\right)}$$

$$= e^{\ln\left(1+e^x\right)}$$

$$= 1+e^x$$

$$\int \frac{-1}{e^x\left(1+e^x\right)}\left(1+e^x\right)dx = -\int e^{-x}\, dx$$

$$= e^{-x}$$

$$y(x) = \frac{1}{1+e^x}\left(e^{-x} + C\right)$$

$$\boxed{= \frac{e^{-x}+C}{1+e^x}}$$

Exercise

Solve the differential equation $\left(x+3y^2\right)dy + ydx = 0$

Solution

$$xdy + 3y^2dy + ydx = 0$$

$$xdy + ydx = -3y^2dy$$

$$d(xy) = -3y^2dy$$

$$\int d(xy) = -3\int y^2dy$$

$$\boxed{xy = -y^3 + C}$$

Exercise

Solve the differential equation $y' = \dfrac{y^2 + ty + t^2}{t^2}$

Solution

$$y' = \frac{y^2}{t^2} + \frac{y}{t} + 1$$

Let $x = \dfrac{y}{t} \Rightarrow y = xt \rightarrow y' = x + tx'$

$$y' = \frac{y^2}{t^2} + \frac{y}{t} + 1 = x^2 + x + 1$$

$$x + tx' = x^2 + x + 1$$

$$t\frac{dx}{dt} = x^2 + 1$$

$$\int \frac{dx}{x^2+1} = \int \frac{dt}{t}$$

$$\tan^{-1} x = \ln|t| + C$$

$$\tan^{-1}\frac{y}{t} = \ln|t| + C$$

$$\frac{y}{t} = \tan(\ln|t| + C)$$

$$\boxed{y(t) = t\tan(\ln|t| + C)}$$

Exercise

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{4x - x^3}{4 + y^3}$

Solution

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}$$

$$\left(4+y^3\right)dy=\left(4x-x^3\right)dx$$

$$\int\left(4+y^3\right)dy=\int\left(4x-x^3\right)dx$$

$$4y+\frac{1}{4}y^4=2x^2-\frac{1}{4}x^4+C_1$$

$$16y+y^4=8x^2-x^4+C$$

$$\boxed{y^4+16y+x^4-8x^2=C}$$

Exercise

Solve the differential equation $y'=\dfrac{2xy+2x}{x^2-1}$

Solution

$$\frac{dy}{dx}=\frac{2x(y+1)}{x^2-1}$$

$$\frac{dy}{y+1}=\frac{2x}{x^2-1}dx \qquad d\left(x^2-1\right)=2xdx$$

$$\int\frac{d(y+1)}{y+1}=\int\frac{d\left(x^2-1\right)}{x^2-1}$$

$$\ln|y+1|=\ln\left|x^2-1\right|+C$$

$$y+1=e^{\ln\left|x^2-1\right|+C}$$

$$y=e^Ce^{\ln\left|x^2-1\right|}-1$$

$$\boxed{y(x)=Ae^{\ln\left|x^2-1\right|}-1}$$

Exercise

Find the general solution of the differential equation $\dfrac{dy}{dx}=\sin 5x$

Solution

$$\int dy=\int\sin 5x\,dx$$

$$\boxed{y(x)=-\frac{1}{5}\cos 5x+C}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx}=(x+1)^2$

Solution

$$\int dy=\int\left(x^2+2x+1\right)dx$$

$$\boxed{y(x)=\frac{1}{3}x^3+x^2+x+C}$$

Exercise

Find the general solution of the differential equation $dx+e^{3x}dy=0$

Solution

$$\int dy=-\int e^{-3x}\,dx$$

$$\boxed{y(x)=\frac{1}{3}e^{-3x}+C}$$

Exercise

Find the general solution of the differential equation $dy-(y-1)^2\,dx=0$

Solution

$$\int\frac{dy}{(y-1)^2}=\int dx$$

$$\int\frac{d(y-1)}{(y-1)^2}=\int dx$$

$$-\frac{1}{y-1}=x+C$$

$$\boxed{y(x)=1-\frac{1}{x+C}}$$

Exercise

Find the general solution of the differential equation $x\frac{dy}{dx}=4y$

Solution

$$\int\frac{dy}{y}=4\int\frac{dx}{x}$$

$$\ln y = 4\ln x + \ln C$$

$$\ln y = \ln Cx^4$$

$$\boxed{y(x) = Cx^4}$$

Exercise

Find the general solution of the differential equation $\frac{dx}{dy} = y^2 - 1$

Solution

$$\int dx = \int \left(y^2 - 1\right) dy$$

$$\boxed{x(y) = \frac{1}{3}y^3 - y + C}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} = e^{2y}$

Solution

$$\int e^{-2y} dy = \int dx$$

$$-\frac{1}{2}e^{-2y} = x + C$$

$$e^{-2y} = -2x + C_1$$

$$-2y = \ln\left(C_1 - 2x\right)$$

$$\boxed{y(x) = -\frac{1}{2}\ln\left(C_1 - 2x\right)}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} + 2xy^2 = 0$

Solution

$$\frac{dy}{dx} = -2xy^2$$

$$-\int \frac{dy}{y^2} = \int 2x\, dx$$

$$\frac{1}{y} = x^2 + C$$

$$\boxed{y(x)=\frac{1}{x^2+C}}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx}=e^{3x+2y}$

Solution

$$\frac{dy}{dx}=e^{3x}e^{2y}$$

$$\int e^{-2y}\,dy=\int e^{3x}\,dx$$

$$-\frac{1}{2}e^{-2y}=\frac{1}{3}e^{3x}+C$$

$$e^{-2y}=C_1-\frac{2}{3}e^{3x}$$

$$-2y=\ln\left(C_1-\frac{2}{3}e^{3x}\right)$$

$$\boxed{y(x)=-\frac{1}{2}\ln\left(C_1-\frac{2}{3}e^{3x}\right)}$$

Exercise

Find the general solution of the differential equation $e^{x}y\frac{dy}{dx}=e^{-y}+e^{-2x-y}$

Solution

$$e^{x}y\frac{dy}{dx}=e^{-y}\left(1+e^{-2x}\right)$$

$$ye^{y}dy=e^{-x}\left(1+e^{-2x}\right)dx$$

$$\int ye^{y}dy=\int\left(e^{-x}+e^{-3x}\right)dx$$

		$\int e^{y}\,dy$
+	y	e^{y}
−	1	e^{y}

$$\boxed{(y-1)e^{y}=-e^{-x}-\frac{1}{3}e^{-3x}+C}$$

Exercise

Find the general solution of the differential equation $y\ln x\frac{dx}{dy}=\left(\frac{y+1}{x}\right)^2$

Solution

$$x^2\ln x\,dx=\frac{1}{y}\left(y^2+2y+1\right)dy$$

$$\int x^2\ln x\,dx=\int\left(y+2+\frac{1}{y}\right)dy$$

$$u=\ln x \qquad dv=x^2dx$$

$$du=\frac{dx}{x} \qquad v=\frac{1}{3}x^3$$

$$\frac{1}{3}x^3\ln x-\frac{1}{3}\int x^2\,dx=\frac{1}{2}y^2+2y+\ln|y|+C$$

$$\boxed{\frac{1}{3}x^3\ln x-\frac{1}{9}x^3=\frac{1}{2}y^2+2y+\ln|y|+C}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx}=\left(\frac{2y+3}{4x+5}\right)^2$

Solution

$$\int\frac{dy}{(2y+3)^2}=\int\frac{dx}{(4x+5)^2}$$

$$\frac{1}{2}\int\frac{d(2y+3)}{(2y+3)^2}=\frac{1}{4}\int\frac{d(4x+5)}{(4x+5)^2}$$

$$\frac{1}{2}\frac{-1}{2y+3}=\frac{1}{4}\frac{-1}{4x+5}+C$$

$$\boxed{\frac{2}{2y+3}=\frac{1}{4x+5}+C}$$

Exercise

Find the general solution of the differential equation $\csc y\,dx+\sec^2 x\,dy=0$

Solution

$$\csc y\,dx=-\sec^2 x\,dy$$

$$\frac{dy}{\csc y}=-\frac{dx}{\sec^2 x}$$

$$\sin y dy = -\cos^2 x dx$$

$$\int \sin y\ dy = -\frac{1}{2}\int (1+\cos 2x)\,dx$$

$$-\cos y = -\frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$

$$\boxed{\cos y = \frac{1}{2}x + \frac{1}{4}\sin 2x + C}$$

Exercise

Find the general solution of the differential equation $\sin 3x\,dx + 2y\cos^3 3x\,dy = 0$

Solution

$$\sin 3x\,dx = -2y\cos^3 3x\,dy$$

$$\int \frac{\sin 3x}{\cos^3 3x}dx = -\int 2ydy$$

$$-\frac{1}{3}\int \cos^{-3} 3x\ d\left(\cos 3x\right) = -\int 2y\,dy$$

$$-\frac{1}{6}\cos^{-2} 3x + C = y^2$$

$$\boxed{y^2 = -\frac{1}{6}\sec^2 3x + C}$$

Exercise

Find the general solution of the differential equation. $\dfrac{dy}{dx} = (64xy)^{1/3}$

Solution

$$\int y^{-1/3}dy = \int 4x^{1/3}dx$$

$$\frac{3}{2}y^{2/3} = 3x^{4/3} + C_1$$

$$\boxed{y^{2/3} = 2x^{4/3} + C}$$

Exercise

Find the general solution of the differential equation. $\dfrac{dy}{dx} = 2x\sec y$

Solution

$$\int \cos y\ dy = \int 2x\ dx$$

$$\boxed{\sin y = x^2 + C}$$

Exercise

Find the general solution of the differential equation. $\dfrac{dy}{dx} = \dfrac{x}{ye^{x+2y}}$

Solution

$$\frac{dy}{dx} = \frac{x}{ye^{2y}e^{x}}$$

$$\int ye^{2y}dy = \int x\,e^{-x}dx$$

		$\int e^{2y}$
+	y	$\frac{1}{2}e^{2y}$
−	1	$\frac{1}{4}e^{2y}$

		$\int e^{-x}$
+	x	$-e^{-x}$
−	1	e^{-x}

$$\frac{1}{2}ye^{2y} - \frac{1}{4}e^{2y} = -xe^{-x} - e^{-x} + C_1$$

$$\boxed{(2y-1)e^{2y} = -4(x+1)e^{-x} + C}$$

Exercise

Find the general solution of $y' - y = 3e^t$

Solution

$$y_h = e^{\int -dt} = e^{-t}$$

$$\int 3e^t e^{-t} dt = \int 3\,dt = 3t$$

$$y(t) = \frac{1}{e^{-t}}(3t + C)$$

$$\boxed{y(t) = 3te^t + Ce^t}$$

Exercise

Find the general solution of $y' - y = e^{2t} - 1$

Solution

$$e^{-\int dt} = e^{-t}$$

$$\int \left(e^{2t} - 1\right)e^{-t}\ dt = \int \left(e^{t} - e^{-t}\right)dt$$

$$= e^{t} + e^{-t}$$

$$y(t) = \frac{1}{e^{-t}}\left(e^{t} + e^{-t} + C\right)$$

$$\underline{= e^{2t} + 1 + Ce^{t}\ \big|}$$

Exercise

Find the general solution of $y' + y = te^{-t} + 1$

Solution

$$e^{\int dt} = e^{t}$$

$$\int \left(te^{-t} + 1\right)e^{t}\ dt = \int \left(t + e^{t}\right)dt$$

$$= t + e^{t}$$

$$y(t) = \frac{1}{e^{t}}\left(t + e^{t} + C\right)$$

$$\underline{= te^{-t} + 1 + Ce^{-t}\ \big|}$$

Exercise

Find the general solution of $y' + y = 1 + e^{-x}\cos 2x$

Solution

$$e^{\int dx} = e^{x}$$

$$\int \left(1 + e^{-x}\cos 2x\right)e^{x}dx = \int \left(e^{x} + \cos 2x\right)dx$$

$$= e^{x} + \frac{1}{2}\sin 2x$$

$$y(x) = e^{-x}\left(e^{x} + \frac{1}{2}\sin 2x + C\right)$$

$$\boxed{= e^{x} + \frac{1}{2}e^{-x}\sin 2x + Ce^{-x}}$$

Exercise

Solve the differential equation: $y' + y\cot x = \cos x$

Solution

$$e^{\int \cot x dx} = e^{\ln|\sin x|}$$

$$= \sin x$$

$$\int \cos x \sin x \, dx = \int \sin x \, d(\sin x)$$

$$= \frac{1}{2}\sin^2 x$$

$$y(x) = \frac{1}{\sin x}\left(\frac{1}{2}\sin^2 x + C\right)$$

$$\boxed{= \frac{1}{2}\sin x + \frac{C}{\sin x}}$$

Exercise

Solve the differential equation: $y' + y\sin t = \sin t$

Solution

$$e^{\int \sin t dt} = e^{-\cos t}$$

$$\int (\sin t) e^{-\cos t} \, dt = \int e^{-\cos t} \, d(-\cos t)$$

$$= e^{-\cos t}$$

$$y(t) = \frac{1}{e^{-\cos t}}\left(e^{-\cos t} + C\right)$$

$$\boxed{= 1 + Ce^{\cos t}}$$

Exercise

Solve the differential equation: $y' + (\cot t)\, y = 2t \csc t$

Solution

$$e^{\int \cot t\, dt} = e^{\ln|\sin t|}$$
$$= \sin t$$

$$\int 2t \csc t \sin t\, dt = \int 2t\, dt$$
$$= t^2$$

$$y(t) = \frac{1}{\sin t}\left(t^2 + C\right)$$
$$\boxed{= \left(t^2 + C\right)\csc t}$$

Exercise

Solve the differential equation: $y' + (1 + \sin t)y = 0$

Solution

$$e^{\int (1+\sin t)dt} = e^{t-\cos t}$$

$$y(t) = \frac{C}{e^{t-\cos t}}$$
$$\boxed{= C\ e^{\cos t - t}}$$

Exercise

Find the general solution of $y' + \left(\frac{1}{2}\cos x\right)y = -\frac{3}{2}\cos x$

Solution

$$e^{\int \frac{1}{2}\cos x dx} = e^{\frac{1}{2}\sin x}$$

$$\int \left(-\frac{3}{2}\cos x\right)e^{\frac{1}{2}\sin x} dx = -3\int e^{\frac{1}{2}\sin x} d\left(\frac{1}{2}\sin x\right)$$
$$= -3e^{\frac{1}{2}\sin x}$$

$$y(x) = e^{-\frac{1}{2}\sin x}\left(-3e^{\frac{1}{2}\sin x} + C\right)$$
$$\boxed{= -3 + Ce^{-\frac{1}{2}\sin x}}$$

Exercise

Solve the differential equation: $\frac{dy}{dx} + y = e^{3x}$

Solution

$$e^{\int dx} = e^{x}$$

$$\int e^{x}e^{3x}dx = \int e^{4x}dx$$

$$= \frac{1}{4}e^{4x}$$

$$y(x) = \frac{1}{e^{x}}\left(\frac{1}{4}e^{4x} + C\right)$$

$$\boxed{= \frac{1}{4}e^{3x} + Ce^{-x}}$$

Exercise

Solve the differential equation: $y' - ty = t$

Solution

$$e^{\int -tdt} = e^{-\frac{1}{2}t^2}$$

$$\int te^{-\frac{1}{2}t^2}dt = -\int e^{-\frac{1}{2}t^2}d\left(-\frac{1}{2}t^2\right)$$

$$= -e^{-\frac{1}{2}t^2}$$

$$y(t) = e^{\frac{1}{2}t^2}\left(e^{-\frac{1}{2}t^2} + C\right)$$

$$\boxed{= 1 + Ce^{\frac{1}{2}t^2}}$$

Exercise

Solve the differential equation: $y' = 2y + x^2 + 5$

Solution

$$y' - 2y = x^2 + 5$$

$$e^{\int -2dx} = e^{-2x}$$

$$\int \left(x^2+5\right)e^{-2x}\,dx = \left(-\frac{1}{2}x^2-\frac{5}{2}-\frac{1}{2}x-\frac{1}{4}\right)e^{-2x}$$

$$= \left(-\frac{1}{2}x^2-\frac{1}{2}x-\frac{11}{4}\right)e^{-2x}$$

$$= -\frac{1}{4}\left(2x^2+2x+11\right)e^{-2x}$$

		$\int e^{-2x}$
+	x^2+5	$-\frac{1}{2}e^{-2x}$
−	$2x$	$\frac{1}{4}e^{-2x}$
+	2	$-\frac{1}{8}e^{-2x}$

$$y(x) = e^{2x}\left(-\frac{1}{4}\left(2x^2+2x+11\right)e^{-2x}+C\right)$$

$$\boxed{= -\frac{1}{4}\left(2x^2+2x+11\right)+Ce^{2x}}$$

Exercise

Solve the differential equation: $xy'+2y=3$

Solution

$$y'+\frac{2}{x}y=\frac{3}{x}$$

$$e^{\int \frac{2}{x}dx} = e^{2\ln x}$$

$$= x^2$$

$$\int x^2\frac{3}{x}\,dx = \int 3x\,dx$$

$$= \frac{3}{2}x^2$$

$$y(x) = \frac{1}{x^2}\left(\frac{3}{2}x^2+C\right)$$

$$\boxed{= \frac{3}{2}+\frac{C}{x^2}}$$

Exercise

Solve the differential equation: $y'+2y=1$

Solution

$$e^{\int 2dx} = e^{2x}$$

$$\int e^{2x}\,dx = \frac{1}{2}e^{2x}$$

$$y(x) = \frac{1}{e^{2x}}\left(\frac{1}{2}e^{2x}+C\right)$$

$$\boxed{=\frac{1}{2}+Ce^{-2x}}$$

Exercise

Solve the differential equation: $y'+2y=e^{-t}$

Solution

$$e^{\int 2dt}=e^{2t}$$

$$\int e^{-t}e^{2t}\ dt=\int e^{t}\ dt$$

$$=e^{t}$$

$$y(t)=\frac{1}{e^{2t}}\left(e^{t}+C\right)$$

$$\boxed{=e^{-t}+Ce^{-2t}}$$

Exercise

Solve the differential equation: $y'+2y=e^{-2t}$

Solution

$$e^{\int 2dt}=e^{2t}$$

$$\int e^{-2t}e^{2t}\ dt=t$$

$$\boxed{y(t)=(t+C)e^{-2t}}$$

Exercise

Find the general solution of $y'-2y=e^{3t}$

Solution

$$e^{\int -2dt}=e^{-2t}$$

$$\int e^{3t}e^{-2t}dt=e^{t}$$

$$y(t)=e^{2t}\left(e^{t}+C\right)$$

$$\boxed{= e^{3t} + Ce^{2t}}$$

Exercise

Find the general solution of $y' + 2y = e^{-x} + x + 1$

Solution

$$e^{\int 2dx} = e^{2x}$$

$$\int \left(e^{-x} + x + 1\right) e^{2x} dx = \int \left(e^{x} + (x+1)e^{2x}\right) dx$$

$$= e^{x} + \left(\frac{1}{2}x + \frac{1}{2} - \frac{1}{4}\right) e^{2x}$$

$$= e^{x} + \left(\frac{1}{2}x + \frac{1}{4}\right) e^{2x}$$

		$\int e^{2x}\, dx$
+	$x+1$	$\frac{1}{2}e^{2x}$
−	1	$\frac{1}{4}e^{2x}$

$$y(x) = e^{-2x}\left(e^{x} + \left(\frac{1}{2}x + \frac{1}{4}\right)e^{2x} + C\right)$$

$$\boxed{= e^{-x} + \frac{1}{2}x + \frac{1}{4} + Ce^{-2x}}$$

Exercise

Solve the differential equation: $y' + 2xy = x$

Solution

$$e^{\int 2xdx} = e^{x^2}$$

$$\int xe^{x^2}\, dx = \frac{1}{2}\int e^{x^2}\, d\left(x^2\right)$$

$$= \frac{1}{2}e^{x^2}$$

$$y(x) = \frac{1}{e^{x^2}}\left(\frac{1}{2}e^{x^2} + C\right)$$

$$\boxed{= \frac{1}{2} + Ce^{-x^2}}$$

Exercise

Solve the differential equation: $y' - 2ty = t$

Solution

$$e^{\int -2tdt} = e^{-t^2}$$

$$\int te^{-t^2}\,dt = -\frac{1}{2}\int e^{-t^2}\,d\left(-t^2\right)$$

$$= -\frac{1}{2}e^{-t^2}$$

$$y(t) = \frac{1}{e^{-t^2}}\left(-\frac{1}{2}e^{-t^2} + C\right)$$

$$\boxed{= Ce^{t^2} - \frac{1}{2}}$$

Exercise

Find the general solution of $y' + 2ty = 5t$

Solution

$$u(t) = e^{\int 2tdt} = e^{t^2}$$

$$e^{t^2}y' + 2te^{t^2}y = 5te^{t^2}$$

$$\left(e^{t^2}y\right)' = 5te^{t^2}$$

$$e^{t^2}y = \int 5t\,e^{t^2}\,dt \qquad\qquad de^{t^2} = 2te^{t^2}\,dt$$

$$= 5\int \frac{1}{2}\,d\left(e^{t^2}\right)$$

$$= \frac{5}{2}e^{t^2} + C$$

$$\boxed{y(t) = \frac{5}{2} + Ce^{-t^2}}$$

Exercise

Solve the differential equation: $y' - 2xy = e^{x^2}$

Solution

$$e^{\int -2xdx} = e^{-x^2}$$

$$\int e^{x^2}e^{-x^2}\,dx = \int dx = x$$

$$\boxed{y(x) = e^{x^2}(x + C)}$$

Exercise

Solve the differential equation: $y' + 2xy = x^3$

Solution

$$e^{\int 2x\,dx} = e^{x^2}$$

$$\int x^3 e^{x^2}\,dx = \frac{1}{2}\int x^2 e^{x^2}\,d\left(x^2\right)$$

$$= \frac{1}{2}\int u e^u\,d(u)$$

$$= \frac{1}{2}\left(x^2 - 1\right)e^{x^2}$$

		$\int e^u\,du$
$+$	u	e^u
$-$	1	e^u

$$y(x) = e^{-x^2}\left(\frac{1}{2}\left(x^2 - 1\right)e^{x^2} + C\right)$$

$$\boxed{= \frac{1}{2}\left(x^2 - 1\right) + Ce^{-x^2}}$$

Exercise

Solve the differential equation: $y' - 2y = t^2 e^{2t}$

Solution

$$e^{\int -2dt} = e^{-2t}$$

$$\int e^{-2t} t^2 e^{2t}\,dt = \int t^2\,dt$$

$$= \frac{1}{3}t^3$$

$$y(t) = \frac{1}{e^{-2t}}\left(\frac{1}{3}t^3 + C\right)$$

$$\boxed{= e^{2t}\left(\frac{1}{3}t^3 + C\right)}$$

Exercise

Find the general solution of $x' - 2\frac{x}{t+1} = (t+1)^2$

Solution

$$e^{\int -\frac{2}{t+1}dt} = e^{-2\ln(t+1)}$$

$$= e^{\ln(t+1)^{-2}}$$

$$= (t+1)^{-2}$$

$$\int (t+1)^2 (t+1)^{-2}\, dt = \int dt$$

$$= t$$

$$x(t) = \frac{1}{(t+1)^{-2}}(t+C)$$

$$= (t+1)^2 (t+C)$$

$$\boxed{= t(t+1)^2 + C(t+1)^2}$$

Exercise

Find the general solution of $y' + \frac{2}{t} y = \frac{\cos t}{t^2}$

Solution

$$e^{\int \frac{2}{t} dt} = e^{2\ln t}$$

$$= e^{\ln t^2}$$

$$= t^2$$

$$\int \frac{\cos t}{t^2} t^2\, dt = \int \cos t\, dt$$

$$= \sin t$$

$$\boxed{y(t) = \frac{1}{t^2}(\sin t + C)}$$

Exercise

Solve the differential equation: $y' - 2(\cos 2t) y = 0$

Solution

$$e^{\int -2\cos 2t\, dt} = e^{-\sin 2t}$$

$$\boxed{y(t) = C\, e^{\sin 2t}}$$

Exercise

Find the general solution of $y' + 2y = \cos 3t$

Solution

$$e^{\int 2dt} = e^{2t}$$

$$\int (\cos 3t) e^{2t} dt = \left(\frac{1}{3}\sin 3t + \frac{1}{18}\cos 3t\right)e^{2t} - \frac{1}{36}\int (\cos 3t) e^{2t} dt$$

$$\frac{37}{36}\int (\cos 3t) e^{2t} dt = \frac{1}{18}(6\sin 3t + \cos 3t)e^{2t}$$

$$\int (\cos 3t) e^{2t} dt = \frac{2}{37}(6\sin 3t + \cos 3t)e^{2t}$$

		$\int \cos 3t\,dt$
+	e^{2t}	$\frac{1}{3}\sin 3t$
−	$\frac{1}{2}e^{2t}$	$-\frac{1}{9}\cos 3t$
+	$\frac{1}{4}e^{2t}$	

$$y(t) = e^{-2t}\left(\frac{2}{37}(6\sin 3t + \cos 3t)e^{2t} + C\right)$$

$$\boxed{= \frac{2}{37}(6\sin 3t + \cos 3t) + Ce^{-2t}}$$

Exercise

Find the general solution of $y' - 3y = 5$

Solution

$$u(t) = e^{-\int 3dt}$$

$$= e^{-3t}$$

$$e^{-3t}y' - 3e^{-3t}y = 5e^{-3t}$$

$$\left(e^{-3t}y\right)' = 5e^{-3t}$$

$$e^{-3t}y = \int 5e^{-3t}\,dt$$

$$e^{-3t}y = -\frac{5}{3}e^{-3t} + C$$

$$\boxed{y(t) = -\frac{5}{3} + Ce^{3t}}$$

Exercise

Solve the differential equation: $y' + 3y = 2xe^{-3x}$

Solution

$$e^{\int 3dx} = e^{3x}$$

$$\int 2xe^{-3x}e^{3x}\ dx = \int 2x\ dx$$

$$= x^2$$

$$\boxed{y(x) = \frac{1}{e^{3x}}\left(x^2 + C\right)}$$

Exercise

Solve the differential equation: $y' + 3x^2 y = x^2$

Solution

$$e^{\int 3x^2 dx} = e^{x^3}$$

$$\int x^2 e^{x^3} dx = \frac{1}{3}\int e^{x^3} d\left(e^{x^3}\right)$$

$$= \frac{1}{3}e^{x^3}$$

$$y(x) = \frac{1}{e^{x^3}}\left(\frac{1}{3}e^{x^3} + C\right)$$

$$\boxed{= \frac{1}{3} + Ce^{-x^3}}$$

Exercise

Find the general solution of $y' + \frac{3}{t}y = \frac{\sin t}{t^3}, \quad (t \neq 0)$

Solution

$$e^{\int \frac{3}{t}dt} = e^{3\ln t} = e^{\ln t^3} = t^3$$

$$\int \frac{\sin t}{t^3} t^3\ dt = \int \sin t\ dt$$

$$= -\cos t$$

$$y(t) = \frac{1}{t^3}(-\cos t + C)$$

$$\boxed{= \frac{C}{t^3} - \frac{\cos t}{t^3}}$$

Exercise

Find the general solution of $y' + \frac{3}{x} y = 1 + \frac{1}{x}$

Solution

$$e^{\int \frac{3}{x} dx} = e^{3\ln x}$$
$$= x^3$$

$$\int \left(1 + \frac{1}{x}\right) x^3 \, dx = \int \left(x^3 + x^2\right) dx$$
$$= \frac{1}{4} x^4 + \frac{1}{3} x^3$$

$$y(x) = \frac{1}{x^3}\left(\frac{1}{4} x^4 + \frac{1}{3} x^3 + C\right)$$
$$\boxed{= \frac{1}{4} x + \frac{1}{3} x + \frac{C}{x^3}}$$

Exercise

Find the general solution of $y' + \frac{3}{2} y = \frac{1}{2} e^x$

Solution

$$e^{\int \frac{3}{2} dx} = e^{3x/2}$$

$$\int \left(\frac{1}{2} e^x\right) e^{3x/2} dx = \frac{1}{2} \int e^{5x/2} dx$$
$$= \frac{1}{5} e^{5x/2}$$

$$y(x) = e^{-3x/2}\left(\frac{1}{5} e^{5x/2} + C\right)$$
$$\boxed{= \frac{1}{5} e^x + Ce^{-3x/2}}$$

Exercise

Find the general solution of $y' + 5y = t + 1$

Solution

$$e^{\int 5dt} = e^{5t}$$

$$\int (t+1)e^{5t}dt = \left(\frac{1}{5}t+\frac{1}{5}+\frac{1}{25}\right)e^{5t}$$

$$= \frac{1}{5}\left(t+\frac{6}{5}\right)e^{5t}$$

$$y(t) = \frac{1}{e^{5t}}\left(\frac{1}{5}\left(t+\frac{6}{5}\right)e^{5t}+C\right)$$

$$\boxed{=\frac{1}{5}\left(t+\frac{6}{5}\right)+Ce^{-5t}}$$

		$\int e^{5t}\,dt$
+	$t+1$	$\frac{1}{5}e^{5t}$
−	1	$\frac{1}{25}e^{5t}$

Exercise

Solve the differential equation: $xy' - y = x^2 \sin x$

Solution

$$y' - \frac{1}{x}y = x\sin x$$

$$e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int x^2 \sin x\, dx = -x^2\cos x + 2x\sin x + 2\cos x$$

$$y(x) = \frac{1}{x}\left(-x^2\cos x + 2x\sin x + 2\cos x + C\right)$$

$$\boxed{= -x\cos x + 2\sin x + \frac{2}{x}\cos x + \frac{C}{x}}$$

		$\int \sin x\,dx$
+	x^2	$-\cos x$
−	$2x$	$-\sin x$
+	2	$\cos x$

Exercise

Find the exact solution of the initial value problem. $y' = \frac{1-2t}{y}, \quad y(1) = -2$

Solution

$$y\frac{dy}{dt} = 1-2t$$

$$\int y\,dy = \int (1-2t)\,dt$$

$$\frac{1}{2}y^2 = t - t^2 + C_1$$

$$y^2 = 2t - 2t^2 + C$$

$$(-2)^2 = 2(1) - 2(1)^2 + C \quad \Rightarrow \quad \boxed{C=4}$$

$$\boxed{y(t) = -\sqrt{2t-2t^2+4}}$$

The negative value is taken to satisfy the initial condition.

Exercise

Find the exact solution of the initial value problem. $y' = y^2 - 4, \quad y(0) = 0$

Solution

$$\frac{dy}{dt} = y^2 - 4$$

$$\frac{dy}{y^2 - 4} = dt$$

$$\frac{1}{y^2-4} = \frac{A}{y-2} + \frac{B}{y+2}$$

$$\frac{1}{y^2 - 4} = \frac{(A+B)y + 2A - 2B}{y - 2}$$

$$\rightarrow \begin{cases} A + B = 0 \\ 2A - 2B = 1 \end{cases} \Rightarrow A = \frac{1}{4} \quad B = -\frac{1}{4}$$

$$\left(\frac{1}{4(y-2)} - \frac{1}{4(y+2)}\right)dy = dt$$

$$\int \left(\frac{1}{4(y-2)} - \frac{1}{4(y+2)}\right)dy = \int dt$$

$$\frac{1}{4}\left(\ln|y-2| - \ln|y+2|\right) = t + C$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4t + C$$

$$\frac{y-2}{y+2} = \pm e^{4t+C}$$

$$\frac{y-2}{y+2} = \pm e^{C} e^{4t} = ke^{4t}$$

$$y - 2 = ke^{4t} y + 2ke^{4t}$$

$$y - ke^{4t} y = 2 + 2ke^{4t}$$

$$y\left(1 - ke^{4t}\right) = 2 + 2ke^{4t}$$

$$y = \frac{2 + 2ke^{4t}}{1 - ke^{4t}}$$

$$0 = \frac{2 + 2ke^{4(0)}}{1 - ke^{4(0)}}$$

$$0 = 2 + 2k \quad \Rightarrow \quad \boxed{k = -1}$$

$$y(t) = \frac{2 - 2e^{4t}}{1 + e^{4t}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dx}=\frac{3x^2+4x+2}{2(y-1)}, \quad y(0)=-1$

Solution

$$\frac{dy}{dx}=\frac{3x^2+4x+2}{2y-2}$$

$$(2y-2)\,dy=\left(3x^2+4x+2\right)dx$$

$$\int(2y-2)\,dy=\int\left(3x^2+4x+2\right)dx$$

$$y^2-2y=x^3+2x^2+2x+C$$

$$y(0)=-1$$

$$(-1)^2-2(-1)=(0)^3+2(0)^2+2(0)+C$$

$$\Rightarrow \quad \boxed{C=3}$$

$$\boxed{y^2-2y=x^3+2x^2+2x+3}$$

Exercise

Find the exact solution of the initial value problem. $y'=\frac{x}{1+2y}, \quad y(-1)=0$

Solution

$$\frac{dy}{dx}=\frac{x}{1+2y}$$

$$\int(1+2y)\,dy=\int x\,dx$$

$$y+y^2=\frac{1}{2}x^2+C \qquad y(-1)=0$$

$$0=\tfrac{1}{2}(-1)^2+C$$

$$\boxed{C=-\frac{1}{2}}$$

$$\boxed{y+y^2=\frac{1}{2}x^2-\frac{1}{2}}$$

Exercise

Find the exact solution of the initial value problem $\left(e^{2y}-y\right)\cos x\frac{dy}{dx}=e^y\sin 2x, \quad y(0)=0$

Solution

$$\frac{e^{2y}-y}{e^{y}}dy=\frac{2\sin x\cos x}{\cos x}dx$$

$$\int\left(e^{y}-ye^{-y}\right)dy=\int 2\sin x\,dx$$

$$e^{y}+ye^{-y}+e^{-y}=-2\cos x+C$$

$$y(0)=0 \quad 1+1=-2+C$$

$$\rightarrow \quad \boxed{C=4}$$

$$\boxed{e^{y}+ye^{-y}+e^{-y}=4-2\cos x}$$

		$\int e^{-y}\,dy$
+	y	$-e^{-y}$
−	1	e^{-y}

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx}=e^{-x^{2}}, \quad y(3)=5$

Solution

$$\int_{3}^{x}\frac{dy}{dt}dt=\int_{3}^{x}e^{-t^{2}}dt$$

$$y(x)-y(3)=\int_{3}^{x}e^{-t^{2}}dt$$

$$\boxed{y(x)=5+\int_{3}^{x}e^{-t^{2}}dt}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dx}+2y=1, \quad y(0)=\frac{5}{2}$

Solution

$$\frac{dy}{dx}=1-2y$$

$$\frac{dy}{1-2y}=dx$$

$$-\frac{1}{2}\int\frac{d(1-2y)}{1-2y}=\int dx$$

$$-\frac{1}{2}\ln|1-2y|=x+C$$

$$\ln|1-2y|=-2x+C \qquad y(0)=\frac{5}{2}$$

$$\ln|1-5| = C$$

$$\boxed{C = \ln 4}$$

$$1-2y = e^{-2x+\ln 4}$$

$$1-2y = e^{-2x}e^{\ln 4}$$

$$\boxed{y(x) = \frac{1}{2} - 2e^{-2x}}$$

Exercise

Find the exact solution of the initial value problem. $\sqrt{1-y^2}\,dx - \sqrt{1-x^2}\,dy = 0, \quad y(0) = \frac{\sqrt{3}}{2}$

Solution

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{dy}{\sqrt{1-y^2}} \qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$\sin^{-1}x + C = \sin^{-1}y \qquad y(0) = \frac{\sqrt{3}}{2}$$

$$\sin^{-1}0 + C = \sin^{-1}\frac{\sqrt{3}}{2} \quad \Rightarrow \quad \boxed{C = \frac{\pi}{3}}$$

$$\sin^{-1}y = \sin^{-1}x + \frac{\pi}{3}$$

$$y = \sin\left(\sin^{-1}x + \frac{\pi}{3}\right)$$

$$= \sin\left(\sin^{-1}x\right)\cos\frac{\pi}{3} + \cos\left(\sin^{-1}x\right)\sin\frac{\pi}{3} \qquad \alpha = \sin^{-1}x \to \sin\alpha = x \quad \cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-x^2}$$

$$\boxed{y(x) = \frac{x}{2} + \frac{\sqrt{3}}{2}\sqrt{1-x^2}}$$

Exercise

Find the exact solution of the initial value problem. $\left(1+x^4\right)dy + x\left(1+4y^2\right)dx = 0, \quad y(1) = 0$

Solution

$$\int \frac{1}{1+(2y)^2}\,dy = -\int \frac{x}{1+\left(x^2\right)^2}\,dx$$

$$\int \frac{1}{1+(2y)^2}\,dy = -\frac{1}{2}\int \frac{1}{1+\left(x^2\right)^2}\,d\left(x^2\right) \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\frac{1}{2}\tan^{-1}2y = -\frac{1}{2}\tan^{-1}x^2 + C$$

$$\tan^{-1}2y+\tan^{-1}x^2=C_1$$

$$y(1)=0$$

$$\tan^{-1}0+\tan^{-1}1=C_1$$

$$\Rightarrow \boxed{C_1=\frac{\pi}{4}}$$

$$\boxed{\tan^{-1}2y+\tan^{-1}x^2=\frac{\pi}{4}}$$

$$2y=\tan\left(\frac{\pi}{4}-\tan^{-1}x^2\right)$$

$$=\frac{\tan\frac{\pi}{4}-\tan\left(\tan^{-1}x^2\right)}{1+\tan\left(\frac{\pi}{4}\right)\tan\left(\tan^{-1}x^2\right)}$$

$$\boxed{y(x)=\frac{1}{2}\frac{1-x^2}{1+x^2}}$$

Exercise

Find the exact solution of the initial value problem. $e^{-2t}\frac{dy}{dt}=\frac{1+e^{-2t}}{y}, \quad y(0)=0$

Solution

$$ydy=\left(1+e^{-2t}\right)e^{2t}dt$$

$$\int ydy=\int\left(e^{2t}+1\right)dt$$

$$\frac{1}{2}y^2=\frac{1}{2}e^{2t}+t+C_1$$

$$y^2=e^{2t}+2t+C \qquad y(0)=0$$

$$0=1+C \quad \rightarrow \boxed{C=-1}$$

$$\boxed{y^2=e^{2t}+2t-1}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt}=\frac{t+2}{y}, \quad y(0)=2$

Solution

$$\int y\,dy=\int(t+2)\,dt$$

$$\frac{1}{2}y^2 = \frac{1}{2}t^2 + 2t + C_1$$

$$y^2 = t^2 + 4t + C$$

$$y(0) = 2$$

$$\underline{C = 4}$$

$$\boxed{y(t) = \sqrt{t^2 + 4t + 4}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{1}{t^2}\frac{dy}{dt} = y, \quad y(0) = 1$

Solution

$$\int \frac{1}{y}dy = \int t^2\, dt$$

$$\ln|y| = \frac{1}{3}t^3 + C$$

$$y(0) = 1$$

$$\ln|1| = C \quad \to \quad \underline{C = 0}$$

$$\ln|y| = \frac{1}{3}t^3$$

$$\boxed{y(t) = e^{t^3/3}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt} = -y^2 e^{2t}; \quad y(0) = 1$

Solution

$$-\int \frac{1}{y^2}dy = \int e^{2t}dt$$

$$\frac{1}{y} = \frac{1}{2}e^{2t} + C$$

$$y(0) = 1$$

$$1 = \frac{1}{2} + C \quad \to \quad \underline{C = \frac{1}{2}}$$

$$\frac{1}{y} = \frac{1}{2}\left(e^{2t} + 1\right)$$

$$\boxed{y(t) = \frac{2}{e^{2t} + 1}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt}-(2t+1)y=0; \quad y(0)=2$

Solution

$$\frac{dy}{dt}=(2t+1)y$$

$$\int\frac{dy}{y}=\int(2t+1)\,dt$$

$$\ln|y|=t^2+t+C$$

$$y(0)=2$$

$$\ln 2=C$$

$$\ln|y|=t^2+t+\ln 2$$

$$y(t)=e^{\ln 2}e^{t^2+t}$$

$$\boxed{=2e^{t^2+t}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt}+4ty^2=0; \quad y(0)=1$

Solution

$$-\int\frac{dy}{y^2}=\int 4t\,dt$$

$$\frac{1}{y}=2t^2+C$$

$$y(0)=1$$

$$1=C$$

$$\frac{1}{y}=2t^2+1$$

$$\boxed{y(t)=\frac{1}{2t^2+1}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx}=ye^x; \quad y(0)=2e$

Solution

$$\int\frac{dy}{y}=\int e^x dx$$

$$\ln|y| = e^x + \ln C$$

$$y(0) = 2e \quad \rightarrow \quad \ln|2e| = 1 + \ln C$$

$$\ln 2 + 1 = 1 + \ln C \quad \Rightarrow \quad \underline{C = 2}$$

$$\ln|y| = e^x + \ln 2$$

$$y(x) = e^{e^x + \ln 2}$$

$$= e^{e^x} e^{\ln 2}$$

$$\underline{= 2e^{e^x}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 3x^2\left(y^2+1\right)$; $y(0) = 1$

Solution

$$\int \frac{1}{y^2+1}\,dy = \int 3x^2\,dx$$

$$\arctan y = x^3 + C$$

$$y(0) = 1$$

$$\arctan 1 = C$$

$$\underline{C = \frac{\pi}{4}}$$

$$\underline{y(x) = \tan\left(x^3 + \frac{\pi}{4}\right)}$$

Exercise

Find the exact solution of the initial value problem $2y\frac{dy}{dx} = \frac{x}{\sqrt{x^2-16}}$; $y(5) = 2$

Solution

$$\int 2y\,dy = \int \frac{x}{\sqrt{x^2-16}}\,dx$$

$$y^2 = \frac{1}{2}\int \left(x^2-16\right)^{-1/2} d\left(x^2-16\right)$$

$$y^2 = \left(x^2-16\right)^{1/2} + C$$

$$y(5) = 2$$

$$4 = (9)^{1/2} + C$$

$$C = 4 - 3 \boxed{= 1}$$

$$\boxed{y^2 = 1 + \sqrt{x^2 - 16}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 4x^3 y - y; \quad y(1) = -3$

Solution

$$\frac{dy}{dx} = \left(4x^3 - 1\right) y$$

$$\int \frac{dy}{y} = \int \left(4x^3 - 1\right) dx$$

$$\ln|y| = x^4 - x + C$$

$$y = Ce^{x^4 - x}$$

$$y(1) = -3 \quad \rightarrow \quad \boxed{-3 = C}$$

$$\boxed{y(x) = -3e^{x^4 - x}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} + 1 = 2y; \quad y(1) = 1$

Solution

$$\int \frac{dy}{2y - 1} = \int dx$$

$$\frac{1}{2}\ln(2y - 1) = x + C$$

$$\ln(2y - 1) = 2x + C$$

$$2y - 1 = e^{2x + C}$$

$$y(x) = Ae^{2x} + \frac{1}{2}$$

$$y(1) = 1$$

$$1 = Ae^2 + 1 \quad \rightarrow \quad \boxed{A = e^{-2}}$$

$$\boxed{y(x) = e^{2x - 2} + \frac{1}{2}}$$

Exercise

Find the exact solution of the initial value problem $(\tan x)\frac{dy}{dx}=y; \quad y\left(\frac{\pi}{2}\right)=\frac{\pi}{2}$

Solution

$$\int\frac{dy}{y}=\int\frac{dx}{\tan x}$$

$$=\int\frac{\cos x\,dx}{\sin x}$$

$$\ln y=\ln(\sin x)+\ln C$$

$$y(x)=C\sin x$$

$$y\left(\frac{\pi}{2}\right)=\frac{\pi}{2} \quad \rightarrow \underline{\frac{\pi}{2}=C}|$$

$$\underline{y(x)=\frac{\pi}{2}\sin x}|$$

Exercise

Find the exact solution of the initial value problem $x\frac{dy}{dx}-y=2x^2y; \quad y(1)=1$

Solution

$$x\frac{dy}{dx}=2x^2y+y$$

$$x\frac{dy}{dx}=\left(2x^2+1\right)y$$

$$\int\frac{dy}{y}=\int\left(2x+\frac{1}{x}\right)dx$$

$$\ln y=x^2+\ln x+\ln C$$

$$y(x)=e^{x^2+\ln x+\ln C}$$

$$=Cxe^{x^2}$$

$$y(1)=1$$

$$1=Ce \;\Rightarrow\; \underline{C=e^{-1}}|$$

$$\underline{y(x)=xe^{x^2-1}}|$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 2xy^2 + 3x^2y^2; \quad y(1) = -1$

Solution

$$\frac{dy}{dx} = \left(2x + 3x^2\right)y^2$$

$$\int \frac{dy}{y^2} = \int \left(2x + 3x^2\right)dx$$

$$-\frac{1}{y} = x^2 + x^3 + C$$

$$y(x) = \frac{-1}{x^2 + x^3 + C}$$

$$y(1) = -1$$

$$-1 = \frac{-1}{C} \quad \Rightarrow \quad \underline{C = 1}$$

$$\boxed{y(x) = \frac{-1}{x^2 + x^3 + 1}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 6e^{2x-y}; \quad y(0) = 0$

Solution

$$\int e^y dy = \int 6e^{2x}dx$$

$$e^y = 3e^{2x} + C$$

$$y(x) = \ln\left(3e^{2x} + C\right)$$

$$y(0) = 0 \quad \to \quad 0 = \ln(3 + C)$$

$$3 + C = 1 \to \underline{C = -2}$$

$$\boxed{y(x) = \ln\left(3e^{2x} - 2\right)}$$

Exercise

Find the exact solution of the initial value problem $2\sqrt{x}\frac{dy}{dx} = \cos^2 y; \quad y(4) = \frac{\pi}{4}$

Solution

$$\frac{dy}{\cos^2 y} = \frac{1}{2}x^{-1/2}dx$$

$$\int \sec^2 y\ dy = \int \frac{1}{2} x^{-1/2}\ dx$$

$$\tan y = \sqrt{x} + C$$

$$y(x) = \tan^{-1}\left(\sqrt{x} + C\right)$$

$$y(4) = \frac{\pi}{4}$$

$$\frac{\pi}{4} = \arctan(2 + C)$$

$$2 + C = 1 \quad \Rightarrow \quad \underline{C = -1|}$$

$$\underline{y(x) = \tan^{-1}\left(\sqrt{x} - 1\right) |}$$

Exercise

Find the exact solution of the initial value problem $y' + 3y = 0; \quad y(0) = -3$

Solution

$$\frac{dy}{dx} = -3y$$

$$\int \frac{dy}{y} = -3\int dx$$

$$\ln|y| = -3x + C$$

$$y(x) = e^{-3x+C}$$

$$= Ae^{-3x}$$

$$y(0) = -3 \quad \to \quad \underline{A = -3|}$$

$$\underline{y(x) = -3e^{-3x} |}$$

Exercise

Find the exact solution of the initial value problem $2y' - y = 0; \quad y(-1) = 2$

Solution

$$2\frac{dy}{dx} = y$$

$$\int \frac{dy}{y} = \frac{1}{2}\int dx$$

$$\ln|y| = \frac{1}{2}x + C$$

$$y(x) = e^{x/2+C}$$

$$= Ae^{x/2}$$

$$y(-1) = 2$$

$$2 = Ae^{-1/2}$$

$$\boxed{A = 2e^{1/2}}$$

$$y(x) = 2e^{1/2}e^{x/2}$$

$$\boxed{= 2e^{(x+1)/2}}$$

Exercise

Find the exact solution of the initial value problem $\sqrt{y}dx + (1+x)dy = 0; \quad y(0) = 1$

Solution

$$\int y^{-1/2}dy = -\int \frac{1}{x+1}dx$$

$$2\sqrt{y} = -\ln|x+1| + C$$

$$y(0) = 1 \quad \rightarrow \boxed{2 = C}$$

$$\boxed{2\sqrt{y} = -\ln|x+1| + 2}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 6y^2x, \quad y(1) = \frac{1}{25}$

Solution

$$\int \frac{dy}{y^2} = \int 6xdx$$

$$-\frac{1}{y} = 3x^2 + C$$

$$y(1) = \frac{1}{25}$$

$$-25 = 3 + C$$

$$\boxed{C = -28}$$

$$-\frac{1}{y} = 3x^2 - 28$$

$$\boxed{y(x) = \frac{1}{28 - 3x^2}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx}=\frac{3x^2+4x-4}{2y-4}, \quad y(1)=3$

Solution

$$\int(2y-4)\,dy=\int\left(3x^2+4x-4\right)dx$$

$$y^2-4y=x^3+2x^2-4x+C$$

$$y(1)=3$$

$$9-12=1+2-4+C$$

$$\underline{C=-2}$$

$$\underline{y^2-4y=x^3+2x^2-4x-2}$$

Exercise

Find the general solution of $y'-3y=4$; $y(0)=2$

Solution

$$e^{-\int 3dt}=e^{-3t}$$

$$\int 4e^{-3t}dt=-\frac{4}{3}e^{-3t}$$

$$y(t)=e^{3t}\left(-\frac{4}{3}e^{-3t}+C\right)$$

$$\underline{=-\frac{4}{3}+Ce^{3t}}$$

$$y(0)=-\frac{4}{3}+Ce^{3(0)}$$

$$2=-\frac{4}{3}+C$$

$$C=\frac{4}{3}+2=\underline{\frac{10}{3}}$$

$$\underline{y(t)=-\frac{4}{3}+\frac{10}{3}e^{3t}}$$

Exercise

Find the general solution of $y'=y+2xe^{2x}$; $y(0)=3$

Solution

$$y'-y=2xe^{2x}$$

$$e^{\int -1\,dx} = e^{-x}$$

$$\int 2xe^{2x}\left(e^{-x}\right)dx = 2\int xe^{x}dx$$

$$= 2\left(xe^{x} - e^{x}\right)$$

$$y(x) = \frac{1}{e^{-x}}\left(2xe^{x} - 2e^{x} + C\right)$$

$$= e^{x}\left(2xe^{x} - 2e^{x} + C\right)$$

$$\boxed{= 2xe^{2x} - 2e^{2x} + Ce^{x}}$$

$$y(x=0) = 2(0)e^{2(0)} - 2e^{2(0)} + Ce^{(0)}$$

$$3 = -2 + C$$

$$\to \boxed{C = 5}$$

$$\boxed{y(x) = 2xe^{2x} - 2e^{2x} + 5e^{x}}$$

Exercise

Find the solution of the initial value problem $\left(1+t^2\right)y' + 4ty = \left(1+t^2\right)^{-2}, \quad y(1) = 0$

Solution

$$y' + \frac{4t}{1+t^2}y = \frac{\left(1+t^2\right)^{-2}}{1+t^2}$$

$$y' + \frac{4t}{1+t^2}y = \left(1+t^2\right)^{-3}$$

$$e^{\int \frac{4t}{1+t^2}dt} = e^{2\int \frac{d\left(1+t^2\right)}{1+t^2}}$$

$$= e^{2\ln\left(1+t^2\right)}$$

$$= e^{\ln\left(1+t^2\right)^2}$$

$$= \left(1+t^2\right)^2$$

$$\int \left(1+t^2\right)^2\left(1+t^2\right)^{-3}dt = \int \frac{dt}{1+t^2}$$

$$\boxed{= \tan^{-1}|t|}$$

$$y(t)=\frac{1}{\left(1+t^2\right)^2}\left(\tan^{-1}|t|+C\right)$$

Given $y(1)=0$, then

$$0=\frac{1}{\left(1+1^2\right)^2}\left(\tan^{-1}|1|+C\right)$$

$$0=\frac{\pi}{4}+C$$

$$\boxed{C=-\frac{\pi}{4}}$$

$$\boxed{y(t)=\frac{1}{\left(1+t^2\right)^2}\left(\tan^{-1}|t|-\frac{\pi}{4}\right)}$$

Exercise

Solve the initial value problem: $y'+y=e^t, \quad y(0)=1$

Solution

$$e^{\int dt}=e^t$$

$$\int e^t e^t\ dt=\int e^{2t}\ dt$$

$$=\frac{1}{2}e^{2t}$$

$$y(t)=\frac{1}{e^t}\left(\frac{1}{2}e^{2t}+C\right)$$

$$\boxed{=\frac{1}{2}e^t+\frac{C}{e^t}}$$

$$y(0)=1$$

$$\frac{1}{2}+C=1$$

$$\boxed{C=\frac{1}{2}}$$

$$\boxed{y(t)=\frac{1}{2}\left(e^t+e^{-t}\right)}$$

Exercise

Solve the initial value problem: $y' + \frac{1}{2}y = t, \quad y(0) = 1$

Solution

$$e^{\int \frac{1}{2}dt} = e^{t/2}$$

$$\int te^{t/2}\, dt = (2t-4)e^{t/2}$$

$$y(t) = \frac{1}{e^{t/2}}\left((2t-4)e^{t/2} + C\right)$$

$$\boxed{= 2t - 4 + Ce^{-t/2}}$$

		$\int e^{t/2}\, dt$
+	t	$2e^{t/2}$
−	1	$4e^{t/2}$

$y(0) = 1$

$-4 + C = 1$

$\boxed{C = 5}$

$$\boxed{y(t) = 2t - 4 + 5e^{-t/2}}$$

Exercise

Solve the initial value problem: $y' = x + 5y, \quad y(0) = 3$

Solution

$$y' - 5y = x$$

$$\boxed{e^{\int -5dx} = e^{-5x}}$$

$$\int xe^{-5x}dx = \left(-\frac{1}{5}x - \frac{1}{25}\right)e^{-5x}$$

$$y(x) = e^{5x}\left(\left(-\frac{1}{5}x - \frac{1}{25}\right)e^{-5x} + C\right)$$

$$\boxed{= -\frac{1}{5}x - \frac{1}{25} + Ce^{5x}}$$

		$\int e^{-5x}dx$
+	x	$-\frac{1}{5}e^{-5x}$
−	1	$\frac{1}{25}e^{-5x}$

$y(0) = 3$

$3 = -\frac{1}{25} + C$

$\boxed{C = \frac{76}{25}}$

$$\boxed{y(x) = -\frac{1}{5}x - \frac{1}{25} + \frac{76}{25}e^{5x}}$$

Exercise

Solve the initial value problem: $y' = 2x - 3y, \quad y(0) = \frac{1}{3}$

Solution

$$y' + 3y = 2x$$

$$e^{\int 3dx} = \boxed{e^{3x}}$$

$$\int 2xe^{3x}dx = \left(\frac{2}{3}x - \frac{2}{9}\right)e^{3x}$$

		$\int e^{3x}\,dx$
$+$	$2x$	$\frac{1}{3}e^{3x}$
$-$	2	$\frac{1}{9}e^{3x}$

$$y(x) = e^{-3x}\left(\left(\frac{2}{3}x - \frac{2}{9}\right)e^{3x} + C\right)$$

$$\boxed{= \frac{2}{3}x - \frac{2}{9} + Ce^{-3x}}$$

$$y(0) = \frac{1}{3}$$

$$\frac{1}{3} = -\frac{2}{9} + C$$

$$\boxed{C = \frac{5}{9}}$$

$$\boxed{y(x) = \frac{2}{3}x - \frac{2}{9} + \frac{5}{9}e^{-3x}}$$

Exercise

Solve the initial value problem: $xy' + y = e^x, \quad y(1) = 2$

Solution

$$y' + \frac{1}{x}y = \frac{e^x}{x}$$

$$e^{\int \frac{1}{x}\,dx} = e^{\ln x}$$

$$\boxed{= x}$$

$$\int x\frac{e^x}{x}dx = \int e^x dx$$

$$\boxed{= e^x}$$

$$y(x) = \frac{1}{x}\left(e^x + C\right)$$

$$y(1) = 2 \quad 2 = e + C \quad \rightarrow \quad \boxed{C = 2 - e}$$

$$\boxed{y(x) = \frac{1}{x}\left(e^x + 2 - e\right)}$$

Exercise

Solve the initial value problem: $\qquad y\dfrac{dx}{dy} - x = 2y^2, \quad y(1) = 5$

Solution

$$\frac{dx}{dy} - \frac{1}{y}x = 2y$$

$$e^{\int -\frac{1}{y}dy} = e^{-\ln y}$$

$$= e^{\ln y^{-1}}$$

$$\boxed{= y^{-1}}$$

$$\int 2yy^{-1}\,dx = 2\int dy$$

$$\boxed{= 2y}$$

$$\boxed{x(y) = y(2y + C)}$$

$$y(1) = 5$$

$$1 = 5(10 + C)$$

$$\boxed{C = -\frac{49}{5}}$$

$$\boxed{x(y) = 2y^2 - \frac{49}{5}y}$$

Exercise

Solve the initial value problem: $\qquad xy' + y = 4x + 1, \quad y(1) = 8$

Solution

$$y' + \frac{1}{x}y = \frac{4x+1}{x}$$

$$e^{\int \frac{1}{x}dx} = e^{\ln x}$$

$$\boxed{= x}$$

$$\int x\frac{4x+1}{x}dx = \int (4x+1)\,dx$$

$$\boxed{= 2x^2 + x}$$

$$\boxed{y(x) = \frac{1}{x}\left(2x^2 + x + C\right)}$$

$y(1)=8$

$8=3+C$

$\boxed{C=5}$

$$\boxed{y(x)=2x+1+\frac{5}{x}}$$

Exercise

Solve the initial value problem: $y'+4xy=x^3e^{x^2}, \quad y(0)=-1$

Solution

$$e^{\int 4xdx}\ \boxed{=e^{2x^2}}$$

$$\int x^3e^{x^2}e^{2x^2}dx=\int x^3e^{3x^2}dx$$

$$=\frac{1}{6}\int x^2e^{3x^2}d\left(3x^2\right)$$

$$=\frac{1}{18}\int u\,e^u\,du$$

$$=\frac{1}{18}\left(3x^2-1\right)e^{3x^2}$$

	$u=3x^2$	$\int e^u\,du$
+	u	e^u
−	1	e^u

$$\boxed{y(x)=\frac{1}{e^{2x^2}}\left(\frac{1}{18}\left(3x^2-1\right)e^{3x^2}+C\right)}$$

$y(0)=-1$

$-1=-\frac{1}{18}+C$

$\boxed{C=-\frac{17}{18}}$

$$\boxed{y(x)=\frac{1}{18}\left(3x^2-1\right)e^{x^2}-\frac{17}{18}e^{2x^2}}$$

Exercise

Solve the initial value problem: $(x+1)y'+y=\ln x, \quad y(1)=10$

Solution

$$y'+\frac{1}{x+1}y=\frac{\ln x}{x+1}$$

$$e^{\int \frac{dx}{x+1}} = e^{\ln(x+1)}$$

$$\boxed{= x+1}$$

$$\int \frac{\ln x}{x+1}(x+1)\,dx = \int \ln x \, dx$$

$$\boxed{= x\ln x - x}$$

$$\boxed{y(x) = \frac{1}{x+1}(x\ln x - x + C)}$$

$$y(1) = 10$$

$$10 = \frac{1}{2}(-1 + C)$$

$$\boxed{C = 21}$$

$$\boxed{y(x) = \frac{1}{x+1}(x\ln x - x + 21)}$$

Exercise

Solve the initial value problem: $y' - (\sin x)y = 2\sin x, \quad y\left(\frac{\pi}{2}\right) = 1$

Solution

$$e^{\int -\sin x \, dx} \boxed{= e^{\cos x}}$$

$$\int 2\sin x e^{\cos x} dx = -2\int e^{\cos x} \, d(\cos x)$$

$$\boxed{= -2e^{\cos x}}$$

$$\boxed{y(x) = \frac{1}{e^{\cos x}}\left(-2e^{\cos x} + C\right)}$$

$$y\left(\frac{\pi}{2}\right) = 1$$

$$1 = -2 + C$$

$$\boxed{C = 3}$$

$$\boxed{y(x) = -2 + \frac{3}{e^{\cos x}}}$$

Exercise

Solve the initial value problem: $y' + y = 2, \quad y(0) = 0$

Solution

$$e^{\int dx} = e^{x}$$

$$\int 2e^{x}dx = 2e^{x}$$

$$y(x) = \frac{1}{e^{x}}\left(2e^{x} + C\right)$$

$$\underline{= 2 + Ce^{-x}}\,\big|$$

$$y(0) = 0$$

$$0 = 2 + C$$

$$\Rightarrow \ \underline{C = -2}\,\big|$$

$$\underline{y(x) = 2 - 2e^{-x}}\,\big|$$

Exercise

Solve the initial value problem: $y' - 2y = 3e^{2x}, \quad y(0) = 0$

Solution

$$e^{\int -2dx} = e^{-2x}$$

$$\int 3e^{2x}e^{-2x}dx = 3x$$

$$\underline{y(x) = e^{2x}(3x + C)}\,\big|$$

$$y(0) = 0 \quad \rightarrow \underline{0 = C}\,\big|$$

$$\underline{y(x) = 3xe^{2x}}\,\big|$$

Exercise

Solve the initial value problem: $xy' + 2y = 3x, \quad y(1) = 5$

Solution

$$y' + \frac{2}{x}y = 3$$

$$e^{\int \frac{2}{x}\,dx} = e^{2\ln x}$$

$$= x^2$$

$$\int 3x^2 \; dx = x^3$$

$$y(x) = \frac{1}{x^2}\left(x^3 + C\right)$$

$$\boxed{= x + \frac{C}{x^2}}$$

$$y(1) = 5$$

$$5 = 1 + C$$

$$\Rightarrow \boxed{C = 4}$$

$$\boxed{y(x) = x + \frac{4}{x^2}}$$

Exercise

Solve the initial value problem: $xy' + 5y = 7x^2, \quad y(2) = 5$

Solution

$$y' + \frac{5}{x}y = 7x$$

$$e^{\int \frac{5}{x}dx} = e^{5\ln x}$$

$$= x^5$$

$$\int 7x^2x^5 \; dx = \frac{7}{8}x^8$$

$$y(x) = \frac{1}{x^5}\left(\frac{7}{8}x^8 + C\right)$$

$$\boxed{= \frac{7}{8}x^3 + \frac{C}{x^5}}$$

$$y(2) = 5$$

$$5 = 7 + \frac{1}{32}C$$

$$\Rightarrow \boxed{C = -64}$$

$$\boxed{y(x) = \frac{7}{8}x^3 - \frac{64}{x^5}}$$

Exercise

Solve the initial value problem: $xy' - y = x, \quad y(1) = 7$

Solution

$$y' - \frac{1}{x}y = 1$$

$$e^{\int \frac{-1}{x}\,dx} = e^{-\ln x}$$

$$= \frac{1}{x}$$

$$\int \frac{1}{x}\,dx = \ln x$$

$$y(x) = x(\ln x + C)$$

$$\boxed{= x\ln x + Cx}$$

$$y(1) = 7 \quad \rightarrow \boxed{7 = C}$$

$$\boxed{y(x) = x\ln x + 7x}$$

Exercise

Solve the initial value problem: $xy' + y = 3xy, \quad y(1) = 0$

Solution

$$xy' + (1 - 3x)y = 0$$

$$y' + \left(\frac{1}{x} - 3\right)y = 0$$

$$e^{\int \left(\frac{1}{x} - 3\right)dx} = e^{\ln x - 3x}$$

$$= e^{\ln x}e^{-3x}$$

$$= xe^{-3x}$$

$$y(x) = \frac{1}{xe^{-3x}}C$$

$$\boxed{= \frac{Ce^{3x}}{x}}$$

$$y(1) = 0 \quad \rightarrow \boxed{0 = C}$$

$$\boxed{y(x) = 0}$$

Exercise

Solve the initial value problem: $xy' + 3y = 2x^5, \quad y(2) = 1$

Solution

$$y' + \frac{3}{x}y = 2x^4$$

$$e^{\int \frac{3}{x}dx} = e^{3\ln x}$$

$$= x^3$$

$$\int 2x^4 x^3 \, dx = \frac{1}{4}x^8$$

$$y(x) = \frac{1}{x^3}\left(\frac{1}{4}x^8 + C\right)$$

$$= \frac{1}{4}x^5 + Cx^{-3}$$

$$y(2) = 1$$

$$1 = 8 + \frac{C}{8}$$

$$\Rightarrow \; C = -56$$

$$y(x) = \frac{1}{4}x^5 - 56x^{-3}$$

Exercise

Solve the initial value problem: $y' + y = e^x, \quad y(0) = 1$

Solution

$$e^{\int dx} = e^x$$

$$\int e^x e^x \, dx = \int e^{2x} \, dx$$

$$= \frac{1}{2}e^{2x}$$

$$y(x) = \frac{1}{e^x}\left(\frac{1}{2}e^{2x} + C\right)$$

$$= \frac{1}{2}e^x + Ce^{-x}$$

$$y(0) = 1$$

$$1 = \frac{1}{2} + C$$

$$\Rightarrow \quad \underline{C = \frac{1}{2}}$$

$$\underline{y(x) = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}}$$

Exercise

Solve the initial value problem: $xy' - 3y = x^3, \quad y(1) = 10$

Solution

$$y' - \frac{3}{x}y = x^2$$

$$e^{\int -\frac{3}{x}dx} = e^{-3\ln x}$$

$$= x^{-3}$$

$$\int x^{-3}x^3 \, dx = \int dx$$

$$= x$$

$$y(x) = x^3(x + C)$$

$$\underline{= x^4 + Cx^3}$$

$$y(1) = 10$$

$$10 = 1 + C$$

$$\Rightarrow \underline{C = 9}$$

$$\underline{y(x) = x^4 + 9x^3}$$

Exercise

Solve the initial value problem: $y' + 2xy = x, \quad y(0) = -2$

Solution

$$e^{\int 2x \, dx} = e^{x^2}$$

$$\int xe^{x^2} \, dx = \frac{1}{2}e^{x^2}$$

$$y(x) = e^{-x^2}\left(\frac{1}{2}e^{x^2} + C\right)$$

$$\boxed{=\frac{1}{2}+Ce^{-x^2}}$$

$$y(0)=-2$$

$$-2=\frac{1}{2}+C$$

$$\Rightarrow \ \boxed{C=-\frac{5}{2}}$$

$$\boxed{y(x)=\frac{1}{2}-\frac{5}{2}e^{-x^2}}$$

Exercise

Solve the initial value problem: $y'=(1-y)\cos x, \quad y(\pi)=2$

Solution

$$y'+(\cos x)y=\cos x$$

$$e^{\int \cos x\, dx}=e^{\sin x}$$

$$\int \cos x\, e^{\sin x}\, dx=e^{\sin x}$$

$$y(x)=\frac{1}{e^{\sin x}}\left(e^{\sin x}+C\right)$$

$$\boxed{=1+Ce^{-\sin x}}$$

$$y(\pi)=2$$

$$2=1+C$$

$$\Rightarrow \ \boxed{C=1}$$

$$\boxed{y(x)=1+e^{-\sin x}}$$

Exercise

Solve the initial value problem: $(1+x)y'+y=\cos x, \quad y(0)=1$

Solution

$$y'+\frac{1}{x+1}y=\frac{\cos x}{1+x}$$

$$e^{\int \frac{1}{1+x}\, dx}=e^{\ln(1+x)}$$

$$=1+x$$

$$\int \frac{\cos x}{1+x}(1+x)\ dx = \sin x$$

$$\boxed{y(x) = \frac{1}{1+x}(\sin x + C)}$$

$$y(0) = 1 \quad \Rightarrow \ \boxed{C = 1}$$

$$\boxed{y(x) = \frac{1}{1+x}(\sin x + 1)}$$

Exercise

Solve the initial value problem: $y' = 1 + x + y + xy, \quad y(0) = 0$

Solution

$$y' - (1+x)y = 1 + x$$

$$e^{-\int (1+x)dx} = e^{-x-\frac{1}{2}x^2}$$

$$\int (1+x)e^{-\left(x+x^2/2\right)}\ dx = -e^{-\left(x+x^2/2\right)}$$

$$y(x) = e^{x+\frac{1}{2}x^2}\left(-e^{-\left(x+\frac{1}{2}x^2\right)} + C\right)$$

$$\boxed{= -1 + Ce^{x+\frac{1}{2}x^2}}$$

$$y(0) = 0$$

$$0 = -1 + C$$

$$\Rightarrow \ \boxed{C = 1}$$

$$\boxed{y(x) = -1 + e^{x+\frac{1}{2}x^2}}$$

Exercise

Solve the initial value problem: $xy' = 3y + x^4 \cos x, \quad y(2\pi) = 0$

Solution

$$y' - \frac{3}{x}y = x^3 \cos x$$

$$e^{-\int \frac{3}{x}dx} = e^{-3\ln x}$$

$$= x^{-3}$$

$$\int x^{-3}x^3 \cos x\, dx = \int \cos x\, dx$$

$$\underline{= \sin x\,|}$$

$$y(x) = x^3(\sin x + C)$$

$$y(2\pi) = 0 \quad \rightarrow \quad \underline{0 = C\,|}$$

$$\underline{y(x) = x^3 \sin x\,|}$$

Exercise

Solve the initial value problem: $y' = 2xy + 3x^2 e^{x^2}, \quad y(0) = 5$

Solution

$$y' - 2xy = 3x^2 e^{x^2}$$

$$e^{-\int 2x\, dx} = e^{-x^2}$$

$$\int 3x^2 e^{x^2} e^{-x^2}\, dx = \int 3x^2\, dx$$

$$= x^3$$

$$y(x) = e^{x^2}\left(x^3 + C\right)$$

$$y(0) = 5 \quad \rightarrow \quad \underline{5 = C\,|}$$

$$\underline{y(x) = e^{x^2}\left(x^3 + 5\right)\,|}$$

Exercise

Solve the initial value problem: $\left(x^2 + 4\right)y' + 3xy = x, \quad y(0) = 1$

Solution

$$y' + \frac{3x}{x^2 + 4}y = \frac{x}{x^2 + 4}$$

$$e^{\int \frac{3x}{x^2+4}dx} = e^{\frac{3}{2}\int \frac{1}{x^2+4}d\left(x^2+4\right)}$$

$$= e^{\frac{3}{2}\ln\left(x^2+4\right)}$$

$$= \left(x^2 + 4\right)^{3/2}$$

$$\int \frac{x}{x^2+4}\left(x^2+4\right)^{3/2}\ dx = \frac{1}{2}\int \left(x^2+4\right)^{1/2}\ d\left(x^2+4\right)$$

$$= \frac{1}{3}\left(x^2+4\right)^{3/2}$$

$$y(x) = \left(x^2+4\right)^{-3/2}\left(\frac{1}{3}\left(x^2+4\right)^{3/2} + C\right)$$

$$= \frac{1}{3} + C\left(x^2+4\right)^{-3/2}$$

$$y(0) = 1$$

$$1 = \frac{1}{3} + \frac{1}{8}C$$

$$\Rightarrow \boxed{C = \frac{16}{3}}$$

$$\boxed{y(x) = \frac{1}{3} + \frac{16}{3}\left(x^2+4\right)^{-3/2}}$$

Exercise

Solve the initial value problem: $\left(x^2+1\right)y' + 3x^3y = 6xe^{-3x^2/2}, \quad y(0) = 1$

Solution

$$y' + \frac{3x^3}{x^2+1}y = \frac{6xe^{-3x^2/2}}{x^2+1}$$

$$y' + \left(3x - \frac{3x}{x^2+1}\right)y = \frac{6xe^{-3x^2/2}}{x^2+1}$$

$$e^{\int\left(3x - \frac{3x}{x^2+1}\right)dx} = e^{\frac{3}{2}x^2 - \frac{3}{2}\ln\left(x^2+1\right)}$$

$$= e^{\frac{3}{2}x^2}\ e^{\ln\left(x^2+1\right)^{-3/2}}$$

$$= e^{\frac{3}{2}x^2}\left(x^2+1\right)^{-3/2}$$

$$\int \frac{6xe^{-3x^2/2}}{x^2+1}e^{\frac{3}{2}x^2}\left(x^2+1\right)^{-3/2}\ dx = 3\int\left(x^2+1\right)^{-5/2}d\left(x^2+1\right)$$

$$= -2\left(x^2+1\right)^{-3/2}$$

$$y(x) = e^{-3x^2/2}\left(x^2+1\right)^{3/2}\left(-2\left(x^2+1\right)^{-3/2} + C\right)$$

$$= e^{-3x^2/2}\left(-2 + C\left(x^2+1\right)^{3/2}\right)$$

$$y(0) = 1$$

$$1 = -2 + C$$

$$\Rightarrow \quad \underline{C = 3\,}|$$

$$\underline{y(x) = e^{-3x^2/2}\left(-2 + 3\left(x^2+1\right)^{3/2}\right)}\,\Big|$$

Exercise

Solve the initial value problem: $y' - 2y = e^{3x}; \quad y(0) = 3$

Solution

$$e^{\int -2dx} = e^{-2x}$$

$$\int e^{3x}e^{-2x}dx = \int e^{x}dx$$

$$= e^{x}$$

$$y(x) = e^{2x}\left(e^{x} + C\right)$$

$$\underline{= e^{3x} + Ce^{2x}\,}|$$

$$y(0) = 1$$

$$1 = 1 + C$$

$$\Rightarrow \underline{C = 0\,}|$$

$$\underline{y(x) = e^{3x}\,}|$$

Exercise

Solve the initial value problem: $y' - 3y = 6; \quad y(0) = 1$

Solution

$$e^{\int -3dx} = e^{-3x}$$

$$\int 6e^{-3x}dx = -2e^{-3x}$$

$$y(x) = e^{3x}\left(-2e^{-3x} + C\right)$$

$$= -2 + Ce^{3x}$$

$$y(0) = 1$$

$$1 = -2 + C$$

$$\Rightarrow \ C = 3$$

$$y(x) = -2 + 3e^{3x}$$

Exercise

Solve the initial value problem: $2y' + 3y = e^{x}; \quad y(0) = 0$

Solution

$$y' + \frac{3}{2}y = \frac{1}{2}e^{x}$$

$$e^{\int \frac{3}{2}dx} = e^{3x/2}$$

$$\int e^{x}e^{3x/2}\,dx = \int e^{5x/2}\,dx$$

$$= \frac{2}{5}e^{5x/2}$$

$$y(x) = e^{-3x/2}\left(\frac{2}{5}e^{5x/2} + C\right)$$

$$= \frac{2}{5}e^{x} + Ce^{-3x/2}$$

$$y(0) = 0$$

$$0 = \frac{2}{5} + C$$

$$\Rightarrow \ C = -\frac{2}{5}$$

$$y(x) = \frac{2}{5}e^{x} - \frac{2}{5}e^{-3x/2}$$

Exercise

Solve the initial value problem: $y' + y = 1 + e^{-x}\cos 2x; \quad y\left(\frac{\pi}{2}\right) = 0$

Solution

$$e^{\int dx} = e^{x}$$

$$\int e^{x}\left(1 + e^{-x}\cos 2x\right)dx = \int \left(e^{x} + \cos 2x\right)dx$$

$$= e^x + \frac{1}{2}\sin 2x$$

$$y(x) = e^{-x}\left(e^x + \frac{1}{2}\sin 2x + C\right)$$

$$\boxed{= 1 + \frac{1}{2}e^{-x}\sin 2x + Ce^{-x}}$$

$$y\left(\frac{\pi}{2}\right) = 0$$

$$0 = 1 + Ce^{-\pi/2}$$

$$\Rightarrow \boxed{C = -e^{\pi/2}}$$

$$\boxed{y(x) = 1 + \frac{1}{2}e^{-x}\sin 2x - e^{-x+\pi/2}}$$

Exercise

Solve the initial value problem: $2y' + (\cos x)y = -3\cos x; \quad y(0) = -4$

Solution

$$y' + \left(\frac{1}{2}\cos x\right)y = -\frac{3}{2}\cos x$$

$$e^{\frac{1}{2}\int \cos x\, dx} = e^{\frac{1}{2}\sin x}$$

$$\int e^{\frac{1}{2}\sin x}(-3\cos x)dx = -6\int e^{\frac{1}{2}\sin x} d\left(\frac{1}{2}\sin x\right)$$

$$= -6e^{\frac{1}{2}\sin x}$$

$$y(x) = e^{-\frac{1}{2}\sin x}\left(-6e^{\frac{1}{2}\sin x} + C\right)$$

$$\boxed{= -6 + Ce^{-\frac{1}{2}\sin x}}$$

$$y(0) = -4$$

$$-4 = -6 + C$$

$$\Rightarrow \boxed{C = 2}$$

$$\boxed{y(x) = -6 + 2e^{-\frac{1}{2}\sin x}}$$

Exercise

Solve the initial value problem: $y' + 2y = e^{-x} + x + 1; \quad y(-1) = e$

Solution

$$e^{\int 2dx} = e^{2x}$$

$$\int \left(e^{-x} + x + 1\right)e^{2x}dx = \int \left(e^{x} + (x+1)e^{2x}\right)dx$$

$$= e^{x} + \left(\frac{1}{2}x + \frac{1}{4}\right)e^{2x}$$

		$\int e^{2x}\,dx$
+	$x+1$	$\frac{1}{2}e^{2x}$
–	1	$\frac{1}{4}e^{2x}$

$$y(x) = e^{-2x}\left(e^{x} + \left(\frac{1}{2}x + \frac{1}{4}\right)e^{2x} + C\right)$$

$$\boxed{= e^{-x} + \frac{1}{2}x + \frac{1}{4} + Ce^{-2x}}$$

$$y(-1) = e$$

$$e = e - \frac{1}{2} + \frac{1}{4} + Ce^{2}$$

$$\Rightarrow \boxed{C = \frac{1}{4}e^{-2}}$$

$$\boxed{y(x) = e^{-x} + \frac{1}{2}x + \frac{1}{4} + \frac{1}{4}e^{-2x-2}}$$

Exercise

Solve the initial value problem: $y' + \frac{y}{x} = xe^{-x}; \quad y(1) = e - 1$

Solution

$$e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int x^{2}e^{-x}\,dx = \left(-x^{2} - 2x - 2\right)e^{-x}$$

		$\int e^{-x}dx$
+	x^{2}	$-e^{-x}$
–	$2x$	e^{-x}
+	2	$-e^{-x}$

$$y(x) = \frac{1}{x}\left(-\left(x^{2} + 2x + 2\right)e^{-x} + C\right)$$

$$y(1) = e - 1$$

$$e - 1 = -5e^{-1} + C$$

$$\Rightarrow \boxed{C = 5e^{-1} + e - 1}$$

$$\boxed{y(x) = \frac{1}{x}\left(-\left(x^{2} + 2x + 2\right)e^{-x} + 5e^{-1} + e - 1\right)}$$

Exercise

Solve the initial value problem: $y' + 4y = e^{-x}; \quad y(1) = \frac{4}{3}$

Solution

$$e^{\int 4dx} = e^{4x}$$

$$\int e^{-x}e^{4x}dx = \int e^{3x}dx$$

$$= \frac{1}{3}e^{3x}$$

$$y(x) = e^{-4x}\left(\frac{1}{3}e^{3x} + C\right)$$

$$= \frac{1}{3}e^{-x} + Ce^{-4x}$$

$$y(1) = \frac{4}{3}$$

$$\frac{4}{3} = \frac{1}{3}e^{-1} + Ce^{-4}$$

$$\Rightarrow \underline{C = \frac{1}{3}\left(4e^4 - e^3\right)}$$

$$\underline{y(x) = \frac{1}{3}e^{-x} + \frac{1}{3}\left(4e^4 - e^3\right)e^{-4x}}$$

Exercise

Solve the initial value problem: $x^2y' + 3xy = x^4 \ln x + 1; \quad y(1) = 0$

Solution

$$y' + \frac{3}{x}y = x^2 \ln x + \frac{1}{x^2}$$

$$e^{\int \frac{3}{x}dx} = e^{3\ln x}$$

$$= x^3$$

$$\int \left(x^2 \ln x + \frac{1}{x^2}\right)x^3 \, dx = \int \left(x^5 \ln x + x\right)dx$$

$$u = \ln x \qquad dv = x^5$$

$$du = \frac{1}{x} \qquad v = \frac{1}{6}x^6$$

$$= \frac{1}{6}x^6 \ln x - \frac{1}{6}\int x^5dx + \frac{1}{2}x^2$$

$$=\frac{1}{6}x^6\ln x-\frac{1}{36}x^6+\frac{1}{2}x^2$$

$$y(x)=\frac{1}{x^3}\left(\frac{1}{6}x^6\ln x-\frac{1}{36}x^6+\frac{1}{2}x^2+C\right)$$

$$=\frac{1}{6}x^3\ln x-\frac{1}{36}x^3+\frac{1}{2x}+\frac{C}{x^3}$$

$y(1)=0$

$$0=-\frac{1}{36}+\frac{1}{2}+C$$

$$\Rightarrow\ C=-\frac{17}{36}$$

$$y(x)=\frac{1}{6}x^3\ln x-\frac{1}{36}x^3+\frac{1}{2x}-\frac{17}{36x^3}$$

Exercise

Find the solution of the initial value problem $y'+\frac{3}{x}y=3x-2\quad y(1)=1$

Solution

$$e^{\int\frac{3}{x}dx}=e^{3\ln x}$$

$$=x^3$$

$$\int(3x-2)x^3\,dx=\int\left(3x^4-2x^3\right)dx$$

$$=\frac{3}{5}x^5-\frac{1}{2}x^4$$

$$y(x)=\frac{1}{x^3}\left(\frac{3}{5}x^5-\frac{1}{2}x^4+C\right)$$

$$=\frac{3}{5}x^2-\frac{1}{2}x+\frac{C}{x^3}$$

$y(1)=1$

$$1=\frac{3}{5}-\frac{1}{2}+C$$

$$\Rightarrow\ C=\frac{9}{10}$$

$$y(x)=\frac{3}{5}x^2-\frac{1}{2}x+\frac{9}{10}x^{-3}$$

Exercise

Find the solution of the initial value problem $y' - (\sin x)y = 2\sin x, \quad y\left(\frac{\pi}{2}\right) = 1$

Solution

$$e^{\int -\sin x dx} = e^{\cos x}$$

$$\int (2\sin x)e^{\cos x}\, dx = -2\int e^{\cos x}\, d(\cos x)$$

$$= -2e^{\cos x}$$

$$y(x) = \frac{1}{e^{\cos x}}\left(-2e^{\cos x} + C\right)$$

$$\underline{= -2 + Ce^{-\cos x}}|$$

$$y\left(\frac{\pi}{2}\right) = 1$$

$$1 = -2 + C$$

$$\Rightarrow \underline{C = 3}|$$

$$\underline{y(x) = Ce^{-\cos x} - 2}|$$

Exercise

Find the solution of the initial value problem $y' + (\tan x)y = \cos^2 x, \quad y(0) = -1$

Solution

$$e^{\int \tan x\, dx} = e^{\ln(\sec x)}$$

$$= \sec x$$

$$\int \cos^2 x(\sec x)dx = \int \cos x\, dx$$

$$= \sin x$$

$$y(x) = \frac{1}{\sec x}(\sin x + C)$$

$$\underline{= \sin x \cos x + C\cos x}|$$

$$y(0) = -1 \quad \rightarrow \quad \underline{-1 = C}|$$

$$\underline{y(x) = \sin x \cos x - \cos x}|$$

Exercise

Find the solution of the initial value problem $t\,y' + 2y = t^2 - t + 1 \quad y(1) = \frac{1}{2}$

Solution

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

$$e^{\int \frac{2}{t}\,dt} = e^{2\ln|t|}$$

$$= t^2$$

$$\int \left(t - 1 + \frac{1}{t}\right)t^2\,dt = \int \left(t^3 - t^2 + t\right)dt$$

$$= \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2$$

$$y(t) = \frac{1}{t^2}\left(\frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + C\right)$$

$$\boxed{= \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{C}{t^2}}$$

$$y(1) = \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C$$

$$\Rightarrow \boxed{C = \frac{1}{12}}$$

$$\boxed{y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{2t^2}}$$

Exercise

Find the solution of the initial value problem $y' + (\cos t)y = \cos t \;; \quad y(\pi) = 2$

Solution

$$e^{\int \cos t\,dt} = e^{\sin t}$$

$$\int (\cos t)e^{\sin t}dt = \int e^{\sin t}d(\sin t)$$

$$= e^{\sin t}$$

$$y(t) = \frac{1}{e^{\sin t}}\left(e^{\sin t} + C\right)$$

$$\boxed{= 1 + Ce^{-\sin t}}$$

$$y(\pi)=2$$

$$1+C=2$$

$$\Rightarrow \underline{C=1}$$

$$\underline{y(t)=1+e^{-\sin t}}$$

Exercise

Find the solution of the initial value problem $y'+2ty=2t$; $y(0)=1$

Solution

$$e^{\int 2t\,dt}=e^{t^2}$$

$$\int (2t)e^{t^2}dt=\int e^{t^2}d\left(t^2\right)$$

$$=e^{t^2}$$

$$y(t)=\frac{1}{e^{t^2}}\left(e^{t^2}+C\right)$$

$$\underline{=1+Ce^{-t^2}}$$

$$y(0)=1$$

$$1+C=1$$

$$\Rightarrow \underline{C=0}$$

$$\underline{y(t)=1}$$

Exercise

Find the solution of the initial value problem $y'+y=\dfrac{e^{-t}}{t^2}$; $y(1)=0$

Solution

$$e^{\int dt}=e^{t}$$

$$\int \left(e^{t}\right)\frac{e^{-t}}{t^2}dt=\int t^{-2}\,dt$$

$$=-\frac{1}{t}$$

$$\underline{y(t)=\frac{1}{e^{t}}\left(-\frac{1}{t}+C\right)}$$

$$y(1)=0$$

$$\frac{1}{e}(-1+C)=0$$

$$\Rightarrow \boxed{C=1}$$

$$\boxed{y(t)=\frac{1}{e^t}\left(-\frac{1}{t}+1\right)}$$

Exercise

Find the solution of the initial value problem $ty'+2y=\sin t\ ;\quad y(\pi)=\frac{1}{\pi}$

Solution

$$y'+\frac{2}{t}y=\frac{\sin t}{t}$$

$$e^{\int\frac{2}{t}dt}=e^{2\ln t}$$

$$=e^{\ln t^2}$$

$$=t^2$$

$$\int\left(t^2\right)\frac{\sin t}{t}\,dt=\int(t\sin t)\,dt$$

$$=-t\cos t+\sin t$$

		$\int \sin t\ dt$
+	t	$-\cos t$
–	1	$-\sin t$

$$\boxed{y(t)=\frac{1}{t^2}(\sin t-t\cos t+C)}$$

$$y(\pi)=\frac{1}{\pi}$$

$$\frac{1}{\pi^2}(\pi+C)=\frac{1}{\pi}$$

$$\Rightarrow \boxed{C=0}$$

$$\boxed{y(t)=\frac{1}{t^2}(\sin t-t\cos t)}$$

Exercise

Solve the initial value problem: $t\frac{dy}{dt}+2y=t^3,\quad t>0,\quad y(2)=1$

Solution

$$y'+\frac{2}{t}y=t^2$$

$$e^{\int \frac{2}{t}dt} = e^{2\ln t}$$

$$= e^{\ln t^2}$$

$$= t^2$$

$$\int t^2 t^2 \; dt = \int t^4 \, dt$$

$$= \frac{1}{5}t^5$$

$$y(t) = \frac{1}{t^2}\left(\frac{1}{5}t^5 + C\right) = \frac{1}{5}t^3 + \frac{C}{t^2}$$

$$y(2) = \frac{1}{5}2^3 + \frac{C}{2^2}$$

$$1 = \frac{8}{5} + \frac{C}{4}$$

$$\frac{C}{4} = 1 - \frac{8}{5} = -\frac{3}{5}$$

$$\boxed{C = -\frac{12}{5}}$$

$$\boxed{y(t) = \frac{1}{5}t^3 - \frac{12}{5t^2}}$$

Exercise

Solve the initial value problem: $\theta\frac{dy}{d\theta} + y = \sin\theta, \quad \theta > 0, \quad y\left(\frac{\pi}{2}\right) = 1$

Solution

$$y' + \frac{1}{\theta}y = \frac{\sin\theta}{\theta}$$

$$e^{\int \frac{1}{\theta}d\theta} = e^{\ln|\theta|}$$

$$= \theta \quad (> 0)$$

$$\int \frac{\sin\theta}{\theta}\theta \, d\theta = \int \sin\theta \, d\theta$$

$$= -\cos\theta$$

$$y(\theta) = \frac{1}{\theta}(-\cos\theta + C)$$

$$y\left(\frac{\pi}{2}\right) = \frac{2}{\pi}\left(-\cos\frac{\pi}{2} + C\right)$$

$$1 = \frac{2}{\pi}(0 + C)$$

$$1 = \frac{2}{\pi} C$$

$$\boxed{C = \frac{\pi}{2}}$$

$$\boxed{y(\theta) = -\frac{\cos\theta}{\theta} + \frac{\pi}{2\theta}}$$

Exercise

Solve the initial value problem: $\frac{dy}{dx} + xy = x, \quad y(0) = -6$

Solution

$$y' + xy = x$$

$$e^{\int x\,dx} = e^{x^2/2}$$

$$\int xe^{x^2/2}\,dx = \int e^{x^2/2}\,d\left(\frac{x^2}{2}\right)$$

$$= e^{x^2/2}$$

$$y(x) = \frac{1}{e^{x^2/2}}\left(e^{x^2/2} + C\right)$$

$$y(0) = \frac{1}{e^{0^2/2}}\left(e^{0^2/2} + C\right)$$

$$-6 = 1(1 + C)$$

$$-6 = 1 + C$$

$$\rightarrow \boxed{C = -7}$$

$$y(x) = \frac{1}{e^{x^2/2}}\left(e^{x^2/2} - 7\right)$$

$$\boxed{= 1 - \frac{7}{e^{x^2/2}}}$$

Exercise

Solve the initial value problem $y' = \frac{y}{x}, \quad y(1) = -2$

Solution

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln|y| = \ln|x| + C$$

$$y = \pm e^{\ln|x|+C}$$

$$= \pm e^{C} e^{\ln|x|}$$

$$= Dx$$

$$y = Dx$$

$$D = \frac{y}{x} = \frac{-2}{1} = -2$$

$$\boxed{y = -2x}$$

Exercise

Solve the initial value problem $y' = \frac{\sin x}{y}, \quad y\left(\frac{\pi}{2}\right) = 1$

Solution

$$\frac{dy}{dx} = \frac{\sin x}{y}$$

$$y\,dy = \sin x\,dx$$

$$\int y\,dy = \int \sin x\,dx$$

$$\frac{1}{2}y^2 = -\cos x + C_1$$

$$y^2 = -2\cos x + C \quad \left(C = 2C_1\right)$$

$$y(x) = \pm\sqrt{-2\cos x + C}$$

$$y\left(\frac{\pi}{2}\right) = \sqrt{-2\cos\frac{\pi}{2} + C}$$

$$1 = \sqrt{C}$$

$$\boxed{C = 1}$$

$$\boxed{y(x) = \sqrt{1 - 2\cos x}}$$

The interval of existence will be the interval containing $\frac{\pi}{2}$ and $1 - 2\cos x > 0$

$$\cos x < \frac{1}{2} \quad \Rightarrow \quad \boxed{\frac{\pi}{3} < x < \frac{5\pi}{3}}$$

Exercise

Find the general solution of $y' = y + 2xe^{2x}$; $y(0) = 3$

Solution

$$y' - y = 2xe^{2x}$$

$$e^{\int -1dx} = e^{-x}$$

$$\int 2xe^{2x}\left(e^{-x}\right)dx = 2\int xe^{x}dx$$

$$= 2\left(xe^{x} - e^{x}\right)$$

$$y(x) = \frac{1}{e^{-x}}\left(2xe^{x} - 2e^{x} + C\right)$$

$$= e^{x}\left(2xe^{x} - 2e^{x} + C\right)$$

$$\boxed{= 2xe^{2x} - 2e^{2x} + Ce^{x}}$$

$$y(x=0) = 2(0)e^{2(0)} - 2e^{2(0)} + Ce^{(0)}$$

$$3 = -2 + C$$

$$\rightarrow \boxed{C = 5}$$

$$\boxed{y(x) = 2xe^{2x} - 2e^{2x} + 5e^{x}}$$

Exercise

Find the general solution of $\left(x^2+1\right)y' + 3xy = 6x$; $y(0) = -1$

Solution

$$y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$$

$$e^{\int \frac{3x}{x^2+1}dx} = e^{\frac{3}{2}\ln\left(x^2+1\right)}$$

$$= e^{\ln\left(x^2+1\right)^{\frac{3}{2}}}$$

$$\boxed{=\left(x^2+1\right)^{\frac{3}{2}}}$$

$$\int \left(x^2+1\right)^{\frac{3}{2}}\frac{6x}{x^2+1}dx = 3\int \left(x^2+1\right)^{\frac{1}{2}}d\left(x^2+1\right)$$

$$= 2\left(x^2+1\right)^{\frac{3}{2}}$$

$$y(x) = 2 + C\left(x^2+1\right)^{-\frac{3}{2}}$$

$$y(0) = 2 + C\left((0)^2+1\right)^{-\frac{3}{2}}$$

$$-1 = 2 + C(1)^{-\frac{3}{2}}$$

$$\rightarrow \boxed{C = -3}$$

$$\boxed{y(x) = 2 - 3\left(x^2+1\right)^{-\frac{3}{2}}}$$

Exercise

Solve the initial value problem $y' = \left(4t^3+1\right)y, \quad y(0) = 4$

Solution

$$\frac{dy}{dt} = \left(4t^3+1\right)y$$

$$\int \frac{dy}{y} = \int \left(4t^3+1\right)dt$$

$$\ln y = t^4 + t + C$$

$$y(t) = e^{t^4+t+C}$$

$$\boxed{= Ae^{t^4+t}}$$

$$y(0) = 4 \quad \boxed{\rightarrow\ 4 = A}$$

$$\boxed{y(t) = 4e^{t^4+t}}$$

Exercise

Solve the initial value problem $y' = \dfrac{e^t}{2y}, \quad y(\ln 2) = 1$

Solution

$$\int 2y\,dy = \int e^t\,dt$$

$$y^2 = e^t + C$$

$$y(\ln 2) = 1$$

$$1 = 2 + C$$

$$\Rightarrow \ \boxed{C = -1}$$

$$\boxed{y^2 = e^t - 1}$$

Exercise

Solve the initial value problem $(\sec x)\,y' = y^3, \quad y(0) = 3$

Solution

$$\int y^{-3}\,dy = \int \frac{dx}{\sec x}$$

$$= \int \cos x\,dx$$

$$-\frac{1}{2}\frac{1}{y^2} = \sin x + C_1$$

$$y^2 = \frac{1}{-2\sin x + C}$$

$$y = \pm\sqrt{\frac{1}{-2\sin x + C}}$$ *Since the initial value is positive*

$$y = \frac{1}{\sqrt{-2\sin x + C}}$$

$$3 = \sqrt{\frac{1}{C}} \ \Rightarrow \ \boxed{C = \frac{1}{9}}$$

$$y(x) = \frac{1}{\sqrt{-2\sin x + \frac{1}{9}}}$$

$$\boxed{= \frac{3}{\sqrt{-2\sin x + 1}}}$$

Exercise

Solve the initial value problem $\frac{dy}{dx} = e^{x-y}, \quad y(0) = \ln 3$

Solution

$$dy = \left(e^x e^{-y}\right)dx$$

$$\int e^y\,dy = \int e^x\,dx$$

$$e^{y}=e^{x}+C$$
$$y=\ln\left(e^{x}+C\right)$$

$$y(0)=\ln 3$$
$$\ln 3=\ln(1+C)$$
$$1+C=3$$
$$\Rightarrow\ \underline{C=2}\,|$$

$$\underline{y(x)=\ln\left(e^{x}+2\right)}\,\Big|$$

Exercise

Solve the initial value problem $y'=2e^{3y-t}, \quad y(0)=0$

Solution

$$\frac{dy}{dt}=2e^{3y}e^{-t}$$
$$\int e^{-3y}dy=\int 2e^{-t}\,dt$$
$$-\frac{1}{3}e^{-3y}=-2e^{-t}+C_1$$
$$e^{-3y}=6e^{-t}+C$$

$$y(0)=0$$
$$1=6+C$$
$$\Rightarrow\ \underline{C=-5}\,|$$

$$e^{-3y}=6e^{-t}-5$$
$$-3y=\ln\left(6e^{-t}-5\right)$$
$$\underline{y(t)=-\frac{1}{3}\ln\left(6e^{-t}-5\right)}\,\Big|$$

Exercise

Solve the initial value problem $y'=3y-6, \quad y(0)=9$

Solution

$$y'-3y=-6$$

$$e^{\int -3dx}=e^{-3x}$$

$$\int -6e^{-3x}\,dx = 2e^{-3x}$$

$$y = \frac{1}{e^{-3x}}\left(2e^{-3x} + C\right)$$

$$\boxed{= 2 + Ce^{3x}}$$

$$y(0) = 9$$

$$9 = 2 + C$$

$$\rightarrow \boxed{C = 7}$$

$$\boxed{y(x) = 7e^{3x} + 2}$$

Exercise

Solve the initial value problem $y' = -y + 2, \quad y(0) = -2$

Solution

$$y' + y = 2$$

$$e^{\int dx} = e^{x}$$

$$\int 2e^{x}\,dx = 2e^{x}$$

$$y = \frac{1}{e^{x}}\left(2e^{x} + C\right)$$

$$\boxed{= 2 + Ce^{-x}}$$

$$y(0) = -2$$

$$-2 = 2 + C$$

$$\rightarrow \boxed{C = -4}$$

$$\boxed{y(x) = 2 - 4e^{-x}}$$

Exercise

Solve the initial value problem $y' = -2y - 4, \quad y(0) = 0$

Solution

$$y' + 2y = -4$$

$$e^{\int 2dx} = e^{2x}$$

$$\int -4e^{2x}dx = -2e^{2x}$$

$$y = \frac{1}{e^{2x}}\left(-2e^{2x} + C\right)$$

$$\underline{= -2 + Ce^{-2x}}\Big|$$

$$y(0) = 0$$

$$0 = -2 + C$$

$$\to \ \underline{C = 2}\Big|$$

$$\underline{y(x) = 2e^{-2x} - 2}\Big|$$

Exercise

Solve the initial value problem $\frac{dy}{dx} + 3x^2y = x^2, \quad y(0) = -1$

Solution

$$y' + 3x^2y = x^2$$

$$e^{\int 3x^2dx} = e^{x^3}$$

$$\int e^{x^3}\left(x^2\right)dx = \frac{1}{3}\int e^{x^3}d\left(x^3\right)$$

$$= \frac{1}{3}e^{x^3}$$

$$y(x) = \frac{1}{e^{x^3}}\left(\frac{1}{3}e^{x^3} + C\right)$$

$$= \frac{1}{3} + \frac{C}{e^{x^3}}$$

$$y(0) = \frac{1}{3} + \frac{C}{e^0} = -1$$

$$C = -1 - \frac{1}{3}$$

$$\underline{= -\frac{4}{3}}\Big|$$

$$\underline{y(x) = \frac{1}{3} - \frac{4}{3e^{x^3}}}\Big|$$

Exercise

Solve the initial value problem $xdy+(y-\cos x)dx=0,\quad y\left(\frac{\pi}{2}\right)=0$

Solution

$$xdy+(y-\cos x)dx=0$$

$$x\frac{dy}{dx}+y-\cos x=0$$

$$xy'+y=\cos x$$

$$y'+\frac{1}{x}y=\frac{\cos x}{x}$$

$$e^{\int \frac{1}{x}dx}=e^{\ln x}$$

$$=x$$

$$\int \frac{\cos x}{x}(x)dx=\int \cos x\ dx$$

$$=\sin x$$

$$y(x)=\frac{1}{x}(\sin x+C)$$

$$y\left(\frac{\pi}{2}\right)=\frac{2}{\pi}\left(\sin\frac{\pi}{2}+C\right)=0$$

$$1+C=0$$

$$\underline{C=-1\,|}$$

$$\underline{y(x)=\frac{\sin x-1}{x}\,|}$$

Exercise

Solve the initial value problem $\frac{dy}{dt}=\frac{t+1}{2ty},\quad y(1)=4$

Solution

$$\frac{dy}{dt}=\frac{t+1}{2ty}$$

$$\int 2ydy=\int \frac{t+1}{t}dt$$

$$\int 2ydy=\int \left(1+\frac{1}{t}\right)dt$$

$$y^2=t+\ln t+C$$

$$y(1)=4$$

$$16 = 1 + \ln 1 + C$$

$$\underline{C = 15}\,\big|$$

$$y^2 = t + \ln t + 15$$

$$\boxed{y(t) = \sqrt{t + \ln t + 15}} \qquad \text{Since } y(t \ge 1)$$

Exercise

Solve the initial value problem $\frac{dy}{dt} = \sqrt{y}\ \sin t, \quad y(0) = 4$

Solution

$$\frac{dy}{dt} = \sqrt{y}\ \sin t$$

$$\int y^{-1/2}\ dy = \int \sin t\ dt$$

$$2y^{1/2} = -\cos t + C$$

$$y(0) = 4$$

$$2\sqrt{4} = -\cos 0 + C$$

$$4 = -1 + C$$

$$\underline{C = 5}\,\big|$$

$$2y^{1/2} = -\cos t + 5$$

$$y^{1/2} = -\frac{1}{2}\cos t + \frac{5}{2}$$

$$\boxed{y(t) = \frac{1}{4}(5 - \cos t)^2}$$

Exercise

Solve the initial value problem $y'(t) + 3y = 0, \quad y(0) = 6$

Solution

$$\frac{dy}{dt} = -3y$$

$$\int \frac{dy}{y} = -3\int dt$$

$$\ln y = -3t + C_1$$

$$y(t) = e^{-3t + C_1}$$

$$= e^{-3t}e^{C_1}$$

$$= Ce^{-3t} \qquad\qquad C = e^{C_1}$$

$$y(0) = 6$$

$$\underline{C = 6|}$$

$$\boxed{y(t) = 6e^{-3t}}$$

Exercise

Solve the initial value problem $y'(t) = 2y + 4, \quad y(0) = 8$

Solution

$$y' - 2y = 4$$

$$e^{\int -2dt} = e^{-2t}$$

$$\int 4e^{-2t}dt = -2e^{-2t}$$

$$y(t) = \frac{1}{e^{-2t}}\left(-2e^{-2t} + C\right)$$

$$= -2 + Ce^{2t}$$

$$y(0) = 8$$

$$8 = -2 + C$$

$$\underline{C = 10|}$$

$$\boxed{y(t) = 10e^{2t} - 2}$$

Solution *Section* 2.8 – Applications

Exercise

A 66-*kg* cyclist on a 7-*kg* bicycle starts coasting on level ground at 9 *m/sec*. The $k \approx 3.9\ kg\,/\,\sec$

a) About how far will the cyclist coast before reaching a complete stop?

b) How long will it take the cyclist's speed to drop to 1 *m/sec*?

Solution

Mass: $m = 66 + 7 = 73\ kg$

$$v = v_0 e^{-(k/m)t}$$

$$\underline{= 9e^{-(3.9/73)t}}\,\Big|$$

a)
$$s(t) = \int v(t)\,dt$$
$$= \int 9e^{-(3.9/73)t}\,dt$$
$$= 9\left(-\frac{73}{3.9}\right)e^{-(3.9/73)t} + C$$
$$= -\frac{219}{1.3}e^{-(3.9/73)t} + C$$
$$= -\frac{2190}{13}e^{-(3.9/73)t} + C$$

$$s(0) = -\frac{2190}{13}e^{-(3.9/73)(0)} + C$$
$$0 = -\frac{2190}{13} + C$$
$$\underline{C = \frac{2190}{13}}\,\Big|$$

$$s(t) = -\frac{2190}{13}e^{-(3.9/73)t} + \frac{2190}{13}$$
$$\underline{= \frac{2190}{13}\left(1 - e^{-(3.9/73)t}\right)}\,\Big|$$

$$\lim_{t\to\infty} s(t) = \frac{2190}{13}\lim_{t\to\infty}\left(1 - e^{-(3.9/73)t}\right)$$
$$= \frac{2190}{13}(1 - 0)$$
$$\underline{\approx 168.5}\,\Big|$$

The cyclist coast about 168.5 *meters*.

b) $1 = 9e^{-(3.9/76)t}$

$$\frac{1}{9} = e^{-(3.9/73)t}$$

$$-\frac{3.9}{73}t = \ln\frac{1}{9}$$

$$t = -\frac{73}{3.9}\ln\frac{1}{9}$$

$$\boxed{\approx 41.13 \ \text{sec}}$$

It will take about 41.13 *seconds*.

Exercise

Suppose that an Iowa class battleship has mass 51,000 metric tons (51,000,000 *kg*) and $k \approx 59{,}000 \ kg / \sec$. Assume that the ship loses power when it is moving at a speed of 9 *m/sec*.

a) About how far will the ship coast before it is dead in the water?

b) About how long will it take the ship's speed to drop to 1 *m/sec*?

Solution

$$\begin{aligned} v &= v_0 e^{-(k/m)t} \\ &= 9e^{-(59{,}000/51{,}000{,}000)t} \\ &= 9e^{-(59/51{,}000)t} \end{aligned}$$

a) $s(t) = \int v(t)\,dt$

$$\begin{aligned} &= \int 9e^{-(59/51{,}000)t}\,dt \\ &= 9\left(-\frac{51000}{59}\right)e^{-(59/51{,}000)t} + C \\ &= -\frac{459{,}000}{59}e^{-(59/51{,}000)t} + C \end{aligned}$$

$$s(0) = -\frac{51{,}000}{59}e^{-(59/51{,}000)(0)} + C$$

$$0 = -\frac{51{,}000}{59} + C$$

$$\boxed{C = \frac{51{,}000}{59}}$$

$$\begin{aligned} s(t) &= -\frac{459{,}000}{59}e^{-(59/51{,}000)t} + \frac{459{,}000}{59} \\ &= \boxed{\frac{459{,}000}{59}\left(1 - e^{-(59/51{,}000)t}\right)} \end{aligned}$$

$$\lim_{t\to\infty} s(t) = \frac{459{,}000}{59}\lim_{t\to\infty}\left(1 - e^{-(59/51{,}000)t}\right)$$

$$= \frac{51,000}{59}(1-0)$$

$$\approx 7780\ \ m$$

The ship will coast about 7780 ***meters*** or 7.78 ***km***.

b) $1 = 9e^{-(59/51,000)t}$

$$e^{-(59/51,000)t} = \frac{1}{9}$$

$$-\frac{59}{51000}t = \ln\frac{1}{9}$$

$$t = -\frac{51000}{59}\ln\frac{1}{9}$$

$$\approx 1899.3\ \ sec$$

It will take about $\frac{1899.3}{60}$ $\approx 61.65\ \ \ minutes$

Exercise

A 200-*gal* tank is half full of distilled water. At time *t* = 0, a solution containing 0.5 *lb./gal* of concentrate enters the tank at the rate of 5 *gal/min*, and the well-stirred mixture is withdrawn at the rate of 3 *gal/min*.

a) At what time will the tank be full?

b) At the time the tank is full, how many pounds of concentrate will it contain?

Solution

a) $V(t) = 100 + \left(5\frac{gal}{\min} - 3\frac{gal}{\min}\right)(t\ \ \min)$

$$= 100 + 2t\ \ \ gal$$

$$200 = 100 + 2t$$

$$100 = 2t$$

$$t = 50\ \ \ min$$

b) Let *y*(*t*) be the amount of concentrate in the tank at time *t*.

$$\frac{dy}{dt} = Rate\ in - \ Rate\ out$$

$$\frac{dy}{dt} = \left(0.5\ \frac{lb}{gal}\right)\left(5\ \frac{gal}{\min}\right) - \left(\frac{y}{100+2t}\ \frac{lb}{gal}\right)\left(3\ \frac{gal}{\min}\right)$$

$$= \frac{5}{2} - \frac{3y}{100+2t}$$

$$\frac{dy}{dt} + \frac{3}{100+2t}y = \frac{5}{2} \quad \rightarrow \quad P(t) = \frac{3}{100+2t} \quad\quad Q(t) = \frac{5}{2}$$

$$e^{\int \frac{3dt}{100+2t}} = e^{\frac{3}{2}\int \frac{dt}{50+t}}$$

$$= e^{\frac{3}{2}\ln(50+t)}$$

$$= e^{\ln(50+t)^{3/2}}$$

$$= (50+t)^{3/2}$$

$$\int \frac{5}{2}(50+t)^{3/2}\,dt = (t+50)^{5/2}$$

$$y(t) = \frac{1}{(t+50)^{3/2}}\left((t+50)^{5/2} + C\right)$$

$$= t + 50 + \frac{C}{(t+50)^{3/2}}$$

$$y(0) = 0 + 50 + \frac{C}{(0+50)^{3/2}}$$

$$0 = 50 + \frac{C}{50^{3/2}}$$

$$\frac{C}{50^{3/2}} = -50$$

$$\Rightarrow \boxed{C = -50^{5/2}}$$

$$\boxed{y(t) = t + 50 - \frac{50^{5/2}}{(t+50)^{3/2}}}$$

$$y(t=50) = 50 + 50 - \frac{50^{5/2}}{(50+50)^{3/2}}$$

$$\boxed{\approx 83.22 \quad lb\ of\ \ concentrate}$$

Exercise

A tank contains 100 *gal* of fresh water. A solution containing 1 *lb./gal* of soluble lawn fertilizer runs into the tank at the rate of 1 *gal/min*, and the mixture is pumped out of the tank at a rate of 3 *gal/min*. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.

Solution

Volume of the tank at time t is:

$$V(t) = 100\ gal + \left(1\frac{gal}{\text{min}} - 3\frac{gal}{\text{min}}\right)(t\ \ \text{min})$$

$$\boxed{= 100 - 2t \quad gal}$$

$$\frac{dy}{dt} = Rate\ in - \ Rate\ out$$

$$\frac{dy}{dt}=\left(1\ \frac{lb}{gal}\right)\left(1\ \frac{gal}{\min}\right)-\left(\frac{y}{100-2t}\ \frac{lb}{gal}\right)\left(3\ \frac{gal}{\min}\right)$$

$$\frac{dy}{dt}=1-\frac{3y}{100-2t}$$

$$\frac{dy}{dt}+\frac{3}{100-2t}y=1 \quad\rightarrow\quad P(t)=\frac{3}{100-2t} \qquad Q(t)=1$$

$$e^{\int \frac{3dt}{100-2t}}=e^{\frac{3}{2}\int \frac{-dt}{100-2t}}$$

$$=e^{-\frac{3}{2}\ln(100-2t)}$$

$$=e^{\ln(100-2t)^{-3/2}}$$

$$=(100-2t)^{-3/2}$$

$$\int 1(100-2t)^{-3/2}dt=-\frac{1}{2}\int (100-2t)^{-3/2}d(100-2t)$$

$$=(100-2t)^{-1/2}$$

$$y(t)=\frac{1}{(100-2t)^{-3/2}}\left[(100-2t)^{-1/2}+C\right]$$

$$y(t)=100-2t+C(100-2t)^{3/2}$$

$$y(0)=100-2(0)+C\left(100-2(0)\right)^{3/2}$$

$$0=100+C(100)^{3/2}$$

$$C=-100^{-1/2}$$

$$\boxed{=-\frac{1}{10}}$$

$$y(t)=100-2t-0.1(100-2t)^{3/2}$$

$$\frac{dy}{dx}=-2-0.1\frac{3}{2}(100-2t)^{1/2}(-2)$$

$$\frac{dy}{dx}=-2+0.3(100-2t)^{1/2}=0$$

$$(100-2t)^{1/2}=\frac{2}{0.3}$$

$$100-2t=\left(\frac{2}{0.3}\right)^2$$

$$=\frac{4}{0.09}$$

$$=\frac{400}{9}$$

$$2t = 100 - \frac{400}{9}$$

$$= \frac{500}{9}$$

$$t = \frac{500}{18}$$

$$\underline{\approx 12.78 \quad min}\,|$$

The maximum amount is:

$$y(t = 12.78) = 100 - 2(12.78) - 0.1(100 - 2(12.78))^{3/2}$$

$$\underline{y \approx 14.8 \quad lb}\,|$$

Exercise

An Executive conference room of a corporation contains 4500 ft^3 of air initially free of carbon monoxide. Starting at time *t* = 0, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 ft^3 / min. A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3 ft^3 / min. Find the time when the concentration of carbon monoxide in the room reaches 0.01%.

Solution

Let $y(t)$ be the amount of carbon monoxide (CO) in the room at time *t*.

$$\frac{dy}{dt} = Rate\ in - \ Rate\ out$$

$$\frac{dy}{dt} = (0.04)(0.3) - \left(\frac{y}{4500}\right)(0.3)$$

$$\frac{dy}{dt} = \frac{12}{1000} - \frac{y}{15,000}$$

$$\frac{dy}{dt} + \frac{1}{15,000}y = \frac{12}{1000} \quad \rightarrow \quad P(t) = \frac{1}{15,000} \quad Q(t) = \frac{12}{1000}$$

$$e^{\int \frac{dt}{15000}} = e^{\frac{1}{15000}t}$$

$$\int \frac{12}{1000} e^{\frac{1}{15000}t} dt = \frac{12}{1000} 15000 e^{\frac{1}{15000}t}$$

$$= 180 e^{\frac{1}{15000}t}$$

$$y(t) = \frac{1}{e^{\frac{1}{15000}t}} \left(180 e^{\frac{1}{15000}t} + C\right)$$

$$y(t) = 180 + Ce^{\frac{-1}{15000}t}$$

$$y(0) = 180 + Ce^{\frac{-1}{15000}0}$$

$$0 = 180 + C$$

$$\boxed{C = -180}$$

$$\boxed{y(t) = 180 - 180e^{\frac{-1}{15000}t}}$$

When the concentration of CO is 0.01% in the room, the amount of CO satisfies

$$\frac{y}{4500} = \frac{.01}{100}$$

$$\boxed{y = 0.45 \quad ft^3}$$

When the room contains the amount $y = 0.45\, ft^3$

$$0.45 = 180 - 180e^{\frac{-1}{15000}t}$$

$$180e^{\frac{-1}{15000}t} = 179.55$$

$$e^{\frac{-1}{15000}t} = \frac{179.55}{180}$$

$$\frac{-1}{15000}t = \ln\left(\frac{179.55}{180}\right)$$

$$t = -15000\ln\left(\frac{179.55}{180}\right)$$

$$\boxed{t \approx 37.55 \quad min}$$

Exercise

Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If *a* is the amount of substance *A* and *b* is the substance *B* at time *t* = 0, and if *x* is the amount of product at time *t*, then the rate of formation of *x* may be given by the differential equation

$$\frac{dx}{dt} = k(a-x)(b-x) \quad or \quad \frac{1}{(a-x)(b-x)}\frac{dx}{dt} = k$$

Where *k* is a constant for the reaction. Integrate both sides of this equation to obtain a relation between *x* and *t*.

a) If $a = b$

b) If $a \neq b$

Assume in each case that $x = 0$ when $t = 0$

Solution

$$\frac{1}{(a-x)(b-x)}dx = kdt$$

a) $a = b$

$$\frac{1}{(a-x)^2}\,dx = k\,dt$$

$$\int \frac{1}{(a-x)^2}\,dx = \int k\,dt$$

$$\frac{1}{a-x} = kt + C$$

$$x(t=0) = 0$$

$$\boxed{C = \frac{1}{a}}$$

$$\frac{1}{a-x} = kt + \frac{1}{a}$$

$$= \frac{kat+1}{a}$$

$$a - x = \frac{a}{kat+1}$$

$$x = a - \frac{a}{kat+1}$$

$$\boxed{= \frac{a^2kt}{kat+1}}$$

b) $a \neq b$

$$\frac{1}{(a-x)(b-x)}\,dx = kdt$$

$$\int \frac{1}{(a-x)(b-x)}\,dx = \int k\,dt$$

$$\frac{1}{(a-x)(b-x)} = \frac{A}{a-x} + \frac{B}{b-x}$$

$$\begin{cases} -A - B = 0 \\ bA + aB = 1 \end{cases} \rightarrow \begin{cases} B = \dfrac{1}{a-b} \\ A = -\dfrac{1}{a-b} \end{cases}$$

$$\frac{-1}{a-b}\int \frac{1}{a-x}\,dx + \frac{1}{a-b}\int \frac{1}{b-x}\,dx = \int kdt$$

$$\frac{1}{a-b}\ln|a-x| - \frac{1}{a-b}\ln|b-x| = kt + C$$

$$\frac{1}{a-b}\ln\left|\frac{a-x}{b-x}\right| = kt + C$$

$$x(0) = 0 \quad \Rightarrow \quad \boxed{\frac{1}{a-b}\ln\left(\frac{a}{b}\right) = C}$$

$$\frac{1}{a-b}\ln\left|\frac{a-x}{b-x}\right| = kt + \frac{1}{a-b}\ln\left(\frac{a}{b}\right)$$

$$\ln\left|\frac{a-x}{b-x}\right| = (a-b)kt + \ln\left(\frac{a}{b}\right)$$

$$\frac{a-x}{b-x} = e^{(a-b)kt+\ln\left(\frac{a}{b}\right)}$$

$$\frac{a-x}{b-x} = \frac{a}{b}e^{(a-b)kt}$$

$$a - x = b\frac{a}{b}e^{(a-b)kt} - x\frac{a}{b}e^{(a-b)kt}$$

$$x\left(\frac{a}{b}e^{(a-b)kt} - 1\right) = ae^{(a-b)kt} - a$$

$$\boxed{x = \frac{abe^{(a-b)kt} - ab}{ae^{(a-b)kt} - b}}$$

Exercise

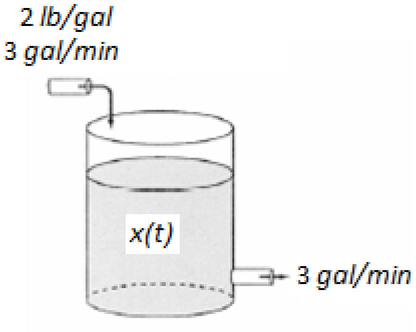
The tank initially holds 100 *gal* of pure water. At time $t = 0$, a solution containing 2 *lb* of salt per gallon begins to enter the tank at the rate of 3 gallons per minute. At the same time a drain is opened at the bottom of the tank so that the volume of solution in the tank remains constant.
How much salt is in the tank after 60 *min*?
What will be the eventual salt content in the tank?

Solution

$x(t)$: number of pounds of salt in the tank after t min.

Volume: $V(t) = 100 + (3-3)t = 100$

Concentration at time t: $c(t) = \frac{x(t)}{V(t)} = \frac{x(t)}{100} \quad lb / gal$



Rate in = Volume Rate x Concentration

$$= 3\frac{gal}{min} \times 2\frac{lb}{gal}$$

$$= 6 \quad lb / min$$

Rate out = Volume Rate x Concentration

$$= 3\frac{gal}{\min} \times \frac{x(t)}{100}\frac{lb}{gal}$$

$$= \frac{3x(t)}{100} \quad lb / min$$

$\frac{dx}{dt}$ = rate of change

= rate in − rate out

$$= 6 - \frac{3x}{100}$$

$$\frac{dx}{dt}+\frac{3}{100}x=6$$

$$u(t)=e^{\int\left(\frac{3}{100}\right)dt}=e^{0.03t}$$

$$\int 6e^{0.03t}\,dt=\frac{6}{0.03}e^{0.03t}$$

$$=200e^{0.03t}$$

$$x(t)=e^{-0.03t}\left(200e^{0.03t}+C\right)$$

$$\boxed{x(t)=200+Ce^{-0.03t}}$$

Since there was no salt present in the tank initially, the initial condition is $x(0)=0$

$$x(t=0)=200+Ce^{-0.03(0)}=0$$

$$200+C=0$$

$$\rightarrow\ \boxed{C=-200}$$

$$\boxed{x(t)=200-200e^{-0.03t}}$$

After 60 *min*:

$$x(60)=200-200e^{-0.03(60)}$$

$$\boxed{\approx 167\ \ lb}$$

As $t\to\infty$ then $x(t)=\lim\limits_{t\to\infty}\left(200-200e^{-0.03t}\right)$

$$=200-200\lim_{t\to\infty}\left(e^{-0.03t}\right) \qquad \lim_{t\to\infty}\left(e^{-0.03t}\right)=e^{-\infty}=0$$

$$\boxed{=200\ \ lb}$$

Exercise

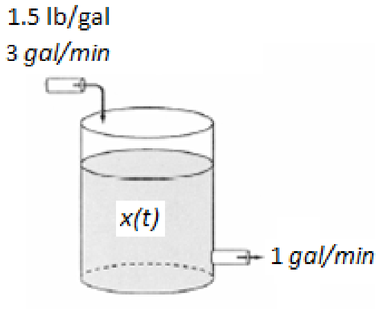
The 600-*gal* tank is filled with 300 *gal* of pure water. A spigot is opened above the tank and a salt solution containing 1.5 *lb*. of salt per gallon of solution begins flowing into the tank at the rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave tank at a rate of 1 *gal/min*. What will be the salt content in the tank at the precise moment that the volume of solution in the tank is equal to the tank's capacity (600 *gal*)?

Solution

$$V(t)=300+(3-1)t$$

$$=300+2t$$

$$c(t)=\frac{x(t)}{300+2t}$$



$$Rate\ in\ = 3\frac{gal}{min}\ \times 1.5\frac{lb}{gal}$$

$$= 4.5\ \ lb/min$$

$$Rate\ out\ = 1\ \times\ \frac{x}{300+2t}$$

$$= \frac{x}{300+2t}\ \ lb/min$$

$$\frac{dx}{dt} = 4.5 - \frac{x}{300+2t}$$

$$\frac{dx}{dt} + \frac{1}{300+2t}x = 4.5$$

$$u(t) = e^{\int \frac{1}{300+2t}dt} \qquad d(300+2t) = 2dt$$

$$= e^{\frac{1}{2}\int \frac{1}{300+2t}d(300+2t)}$$

$$= e^{\frac{1}{2}\ln(300+2t)}$$

$$= e^{\ln(300+2t)^{1/2}}$$

$$= \sqrt{300+2t}$$

$$\int 4.5\sqrt{300+2t}\ dt = 4.5\frac{1}{2}\frac{2}{3}(300+2t)^{2/3}$$

$$= \frac{3}{2}(300+2t)^{2/3}$$

$$x(t) = \frac{1}{\sqrt{300+2t}}\left(\frac{3}{2}(300+2t)^{3/2} + C\right)$$

$$= \frac{3}{2}(300+2t) + \frac{C}{\sqrt{300+2t}}$$

$$\boxed{= 450 + 3t + \frac{C}{\sqrt{300+2t}}}$$

$$x(0) = 450 + 3(0) + \frac{C}{\sqrt{300+2(0)}} = 0$$

$$450 + \frac{C}{\sqrt{300}} = 0$$

$$C = -450\sqrt{300}$$

$$\boxed{= -4500\sqrt{3}}$$

$$x(t) = 450 + 3t - \frac{4500\sqrt{3}}{\sqrt{300+2t}}$$

$$V\ = 300 + 2t = 600$$

$$t = 150\ min$$

$$x(t=150)=450+3(150)-\frac{4500\sqrt{3}}{\sqrt{300+2(150)}}$$

$$\underline{\approx 582\ lb}\Big|$$

Exercise

A tank with a 2,000 *gal* capacity initially contains 500 *gal* of brine containing 100 *lbs*. of salt starting at time *t* = 0, brine containing 0.1 *lb/gal* of salt is added at a rate of 60 *gal/min* and the mixed solution is drained off at a rate of 40 *gal/min*. How much salt is in the tank when it reaches the point of over flowing?

Solution

$$V(t)=500+(60-40)t$$

$$=500+20t$$

$$c(t)=\frac{x(t)}{500+20t}$$

$$Rate\ in\ =60\frac{gal}{min}\ \times 0.1\frac{lb}{gal}$$

$$=6\ \ lb\,/\,min$$

$$Rate\ out\ =40\ \times\ \frac{x}{500+20t}$$

$$=\frac{2x}{25+t}\ \ lb\,/\,\text{min}$$

$$\frac{dx}{dt}=6-\frac{2x}{25+t}$$

$$\frac{dx}{dt}+\frac{2}{25+t}x=6$$

$$u(t)=e^{\int\frac{2}{25+t}dt}$$

$$=e^{\int\frac{2}{25+t}d(25+t)}$$

$$=e^{2\ln(25+t)}$$

$$=e^{\ln(25+t)^2}$$

$$=(25+t)^2$$

$$\int 6(25+t)^2\ dt=6\int(25+t)^2\ d(25+t)$$

$$=2(25+t)^3$$

$$x(t)=\frac{1}{(25+t)^2}\left(2(25+t)^3+C\right)$$

$$= 50 + 2t + \frac{C}{(25+t)^2}$$

$x(0) = 100$

$$100 = 50 + \frac{C}{25^2}$$

$$\frac{C}{625} = 50$$

$$\underline{C = 31,250}$$

$$x(t) = 50 + 2t + \frac{31,250}{(25+t)^2}$$

Exercise

The amount of drug in the blood of a patient (in *mg*) due to an intravenous line is governed by the initial value problem

$$y'(t) = -0.02y + 3, \quad y(0) = 0 \ \text{ for } t \geq 0$$

Where t is measured in hours

a) Find and graph the solution of the initial value problem.
b) What is the steady-state level of the drug?
c) When does the drug level reach 90% of the steady-state value?

Solution

a) $y' + 0.02y = 3$

$$e^{\int 0.02dt} = e^{0.02t}$$

$$\int 3e^{0.02t}dt = 150e^{0.02t}$$

$$y = \frac{1}{e^{0.02t}}\left(150e^{0.02t} + C\right)$$

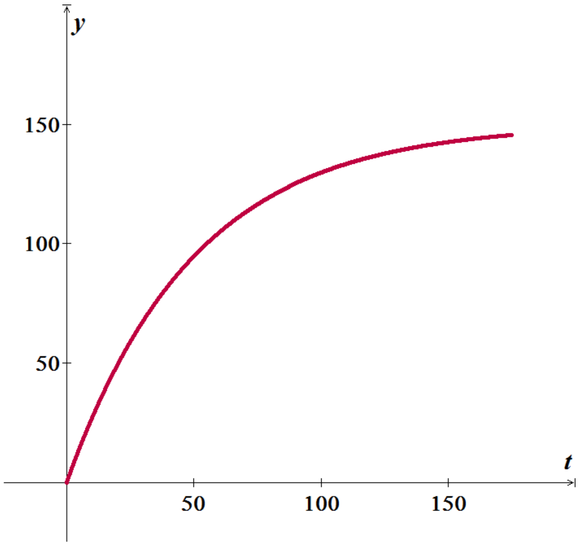
$$\underline{= 150 + Ce^{-0.02t}}$$

$y(0) = 0$

$0 = 150 + C$

$\rightarrow \underline{C = -150}$

$$y(t) = 150\left(1 - e^{-0.02t}\right)$$



b) The steady-state level is

$$\lim_{t\to\infty} 150\left(1-e^{-0.02t}\right) \underline{=150 \quad mg}$$

c) $150\left(1-e^{-0.02t}\right)=0.9(150)$

$$1-e^{-0.02t}=0.9$$

$$e^{-0.02t}=0.1$$

$$-0.02t=\ln 0.1$$

$$t=\frac{\ln 0.1}{-0.02}$$

$$\underline{\approx 115 \quad hrs}$$

Exercise

A fish hatchery has 500 *fish* at time $t=0$, when harvesting begins at a rate of *b fish/yr*. where $b>0$. The fish population is modeled by the initial value problem.

$$y'(t)=0.1y-b, \quad y(0)=500 \qquad for \ \ t\geq 0$$

Where *t* is measured in years.

a) Find the fish population for $t\geq 0$ in terms of the harvesting rate *b*.

b) Graph the solution in the case that $b=40\ fish\,/\,yr$. Describe the solution.

c) Graph the solution in the case that $b=60\ fish\,/\,yr$. Describe the solution.

Solution

a) $y'-0.1y=-b$

$$e^{\int -0.1dt}=e^{-0.1t}$$

$$\int -be^{-0.1t}dt=10be^{-0.1t}$$

$$y(t)=e^{0.1t}\left(10be^{-0.1t}+C\right)$$

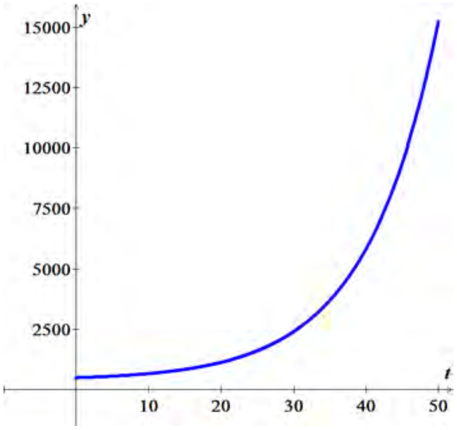
$$\underline{=10b+Ce^{0.1t}}$$

$$y(0)=500$$

$$500=10b+C$$

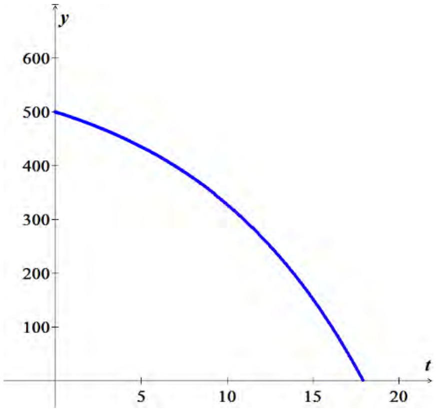
$$\Rightarrow \underline{C=500-10b}$$

$$y(t)\ \underline{=10b+(500-10b)e^{0.1t}}$$



b) For $b=40$

$$\underline{y(t)=400+100e^{0.1t}}$$



c) For $b=60$

$$\underline{y(t)=600-100e^{0.1t}}$$

Exercise

A community of hares on an island has a population of 50 when observations begin at $t = 0$. The population for $t \geq 0$ is modeled by the initial value problem.

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{200}\right), \quad P(0) = 50$$

d) Find the solution of the initial value problem.

e) What is the steady-state population?

Solution

a) $$\int \frac{200}{P(200-P)} dP = \int 0.08 dt$$

$$\int \left(\frac{1}{P} + \frac{1}{200-P}\right) dP = \int 0.08\, dt$$

$$\ln P + \ln|200 - P| = 0.08t + C$$

$$\ln\left|\frac{P}{200-P}\right| = 0.08t + C$$

$$P(0) = 50$$

$$\ln \frac{50}{150} = C$$

$$\Rightarrow \underline{C = -\ln 3}$$

$$\ln\left|\frac{P}{200-P}\right| = 0.08t - \ln 3$$

$$\frac{P}{200-P} = e^{0.08t - \ln 3}$$

$$\frac{P}{200-P} = e^{0.08t} e^{\ln 3^{-1}}$$

$$\frac{P}{200-P} = \frac{1}{3} e^{0.08t}$$

$$3P = 200e^{0.08t} - Pe^{0.08t}$$

$$P(t) = \frac{200e^{0.08t}}{3 + e^{0.08t}}$$

$$\underline{= \frac{200}{3e^{-0.08t} + 1}}$$

b) $$\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \frac{200}{3e^{-0.08t} + 1}$$

$$\underline{= 200}$$

Exercise

When an infected person is introduced into a closed and otherwise healthy community, the number of people who become infected with the disease (in the absence of any intervention) may be modeled by the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right), \quad P(0) = P_0$$

Where *k* is a positive infection rate, *A* is the number of people in the community, and P_0 is the number of infected people at $t = 0$. The model assumes no recovery or intervention.

a) Find the solution of the initial value problem in terms of *k*, *A*, and P_0.

b) Graph the solution in the case that $k = 0.025, \quad A = 300, \quad and \;\; P_0 = 1$.

c) For fixed values of *k* and *A*, describe the long-term behavior of the solutions for any P_0 with $0 < P_0 < A$

Solution

a) $\frac{dP}{dt} = kP\left(\frac{A-P}{A}\right)$

$$\int \frac{A}{P(A-P)} dP = \int k\,dt$$

$$\int \left(\frac{1}{P} + \frac{1}{A-P}\right) dP = \int k\,dt$$

$$\ln P - \ln|A-P| = kt + C_1$$

$$\ln\left|\frac{P}{A-P}\right| = kt + C_1$$

$$\frac{P}{A-P} = Ce^{kt}$$

$$P(0) = P_0$$

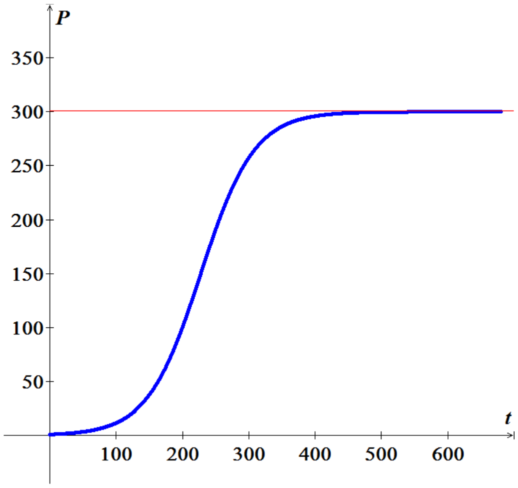
$$\underline{C = \frac{P_0}{A-P_0}\Big|}$$

$$\frac{P}{A-P} = \frac{P_0}{A-P_0} e^{kt}$$

$$P = (A-P)\frac{P_0}{A-P_0} e^{kt}$$

$$\left(A - P_0 + P_0 e^{kt}\right)P = AP_0 e^{kt}$$

$$\underline{P(t) = \frac{AP_0 e^{kt}}{A - P_0 + P_0 e^{kt}}\Big|} \quad = \frac{AP_0}{P_0 + (A-P_0)e^{-kt}}$$



b) $k = 0.025, \quad A = 300, \quad \text{and} \quad P_0 = 1$

$$P(t) = \frac{300}{1 + 299e^{-0.025t}}$$

c) $\lim\limits_{t\to\infty} P(t) = \lim\limits_{t\to\infty} \dfrac{AP_0}{P_0 + \left(A - P_0\right)e^{-kt}}$

$$= \frac{AP_0}{P_0}$$

$\boxed{= A}$ Which is the *steady-state* solution

Exercise

An object of mass ***m*** is released from a balloon. Find the distance it falls in ***t*** seconds, if the force of resistance due to the air is directly proportional to the speed of the object.

Solution

Force resistance due to the air is kv.

Downward force $F = ma = m\dfrac{dv}{dt}$

$$F = mg - kv$$

$$m\frac{dv}{dt} = mg - kv$$

$$\frac{dv}{dt} = g - \frac{k}{m}v$$

$$\frac{dv}{dt} + \frac{k}{m}v = g$$

$$e^{\int \frac{k}{m}dt} = e^{\frac{k}{m}t}$$

$$\int g\, e^{\frac{k}{m}t}dt = \frac{mg}{k}e^{\frac{k}{m}t}$$

$$v(t) = \frac{1}{e^{\frac{k}{m}t}}\left(\frac{mg}{k}e^{\frac{k}{m}t} + C\right)$$

$$\boxed{= \frac{mg}{k} + C\, e^{-\frac{k}{m}t}}$$

$$v(0) = 0$$

$$0 = \frac{mg}{k} + C$$

$$C = -\frac{mg}{k}$$

$$v(t) = \frac{mg}{k} - \frac{mg}{k}\ e^{-\frac{k}{m}t}$$

$$= \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

$$= \frac{ds}{dt}$$

$$\int ds = \frac{mg}{k}\int\left(1 - e^{-\frac{k}{m}t}\right)dt$$

$$s(t) = \frac{mg}{k}\left(t + \frac{m}{k}e^{-\frac{k}{m}t}\right) + C_1$$

$$s(0) = 0$$

$$0 = \frac{mg}{k}\left(\frac{m}{k}\right) + C_1$$

$$\boxed{C_1 = -\frac{m^2 g}{k^2}}$$

$$\boxed{s(t) = \frac{mg}{k}\left(t + \frac{m}{k}e^{-\frac{k}{m}t}\right) - \frac{m^2 g}{k^2}}$$

Exercise

A body falls from a height of 300 *ft*. What distance has it traveled after 4 *sec*. if subject to g, the earth's acceleration?

Solution

$$a(t) = -g$$

$$v(t) = -\int g\ dt$$

$$= -gt + C_1$$

$$v(0) = 0 \quad \to \quad \boxed{C_1 = 0}$$

$$\boxed{v(t) = -32.2t}$$

$$v(t) = \frac{dh}{dt} = -32.2t$$

$$\int dh = -32.2\int t\ dt$$

$$h(t) = -16.1t^2 + C_2$$

$$h(0) = 300$$

$$\boxed{C_2 = 300}$$

$$\boxed{h(t) = -16.1t^2 + 300}$$

$$h(t = 4) = -16.1(16) + 300$$

$$\boxed{= 42.4\ ft}$$

Exercise

A body falls from an initial velocity of $1,000\ ft/s$. What distance has it traveled after 3 *sec*. if subject to $g = 32\ ft/s^2$, the earth's acceleration?

Solution

$$a(t) = g$$

$$v(t) = \int g\ dt$$

$$= gt + C_1$$

$$v(0) = 1,000 \quad \rightarrow \quad \boxed{C_1 = 1,000}$$

$$v(t) = 32t + 1,000$$

$$v(t) = \frac{dh}{dt} = 32t + 1,000$$

$$\int dh = \int (32t + 1,000)\ dt$$

$$h(t) = 16t^2 + 1,000t + C_2$$

$$h(0) = 0 \quad \rightarrow \quad \boxed{C_2 = 0}$$

$$\boxed{h(t) = 16t^2 + 1,000t}$$

$$h(t = 3) = 16(9) + 3,000$$

$$\boxed{= 3,144\ ft}$$

Exercise

A projectile is fired straight upwards with an initial velocity of $1,600\ ft/s$. What is its velocity at $40,000\ ft$.

$\left(g = 32\ ft/s^2\right)$

Solution

$$a(t) = -g$$

$$v(t) = -\int 32\,dt$$

$$= -32t + C_1$$

$$v(0) = 1,600 \quad \rightarrow \quad \underline{C_1 = 1,600}\rfloor$$

$$\underline{v(t) = -32t + 1,600}\rfloor$$

$$v(t) = \frac{dh}{dt} = -32t + 1,600$$

$$\int dh = \int (-32t + 1,600)\ dt$$

$$h(t) = -16t^2 + 1,600t + C_2$$

$$h(0) = 0 \quad \rightarrow \quad \underline{C_2 = 0}\rfloor$$

$$h(t) = -16t^2 + 1,600t$$

$$-16t^2 + 1,600t = 40,000$$

$$t^2 - 100t + 2,500 = 0$$

$$(t-50)^2 = 0 \quad \rightarrow \quad \underline{t = 50}\rfloor$$

$$v(50) = -32(50) + 1,600$$

$$\underline{= 0\ \ ft/sec}\rfloor$$

Exercise

A projectile is fired straight upwards with an initial velocity of $1,000\ ft/s$. What is its velocity at $8,000\ ft$.

$\left(g = 32\ ft/s^2\right)$

Solution

$$v(t) = -\int 32\,dt$$

$$= -32t + C_1$$

$$v(0)=1,000 \quad \to \quad \boxed{C_1=1,000}$$

$$\boxed{v(t)=-32t+1,000}$$

$$v(t)=\frac{dh}{dt}=-32t+1,000$$

$$\int dh=\int(-32t+1,000)\ dt$$

$$h(t)=-16t^2+1,000t+C_2$$

$$h(0)=0 \quad \to \quad \boxed{C_2=0}$$

$$h(t)=-16t^2+1,000t$$

$$-16t^2+1,000t=8,000$$

$$2t^2-125t+1,000=0$$

$$\begin{aligned} t &= \frac{125\pm\sqrt{15625-8000}}{4} \\ &= \frac{125\pm\sqrt{7,625}}{4} \\ &\approx \begin{cases} \boxed{9.42} \\ \boxed{53.08} \end{cases} \end{aligned}$$

$$\begin{aligned} v(9.42) &= -32(9.42)+1,000 \\ &\approx \boxed{698.56\ ft/sec} \end{aligned}$$

$$\begin{aligned} v(53.08) &= -32(53.08)+1,000 \\ &\approx \boxed{-698.56\ ft/sec} \end{aligned}$$

Exercise

A stone is released from rest and dropped into a deep well. Eight seconds later, the sound of the stone splashing into the water at the bottom of the well returns to the ear of the person who released the stone. How long does it take the stone to drop to the bottom of the well? How deep is the well? Ignore air resistance. The speed of sound is 340 *m/s*.

Solution

$$\begin{aligned} d &= \frac{1}{2}gt^2 \\ &= \frac{1}{2}9.8t^2 \\ &= 4.9t^2 \end{aligned}$$

$$\begin{aligned} d &= 340s \\ &= 340(8-t) \end{aligned}$$

$$4.9t^2 = 2720 - 340t$$

$$4.9t^2 + 340t - 2720 = 0$$

$$\underline{t = 7.2438\ sec}$$

$$d = 340(8 - 7.2438)$$

$$\underline{= 257.1\ m}$$

Exercise

A rocket is fired vertically and ascends with constant acceleration $a = 100\ m/s^2$ for 1.0 *min*. At that point, the rocket motor shuts off and the rocket continues upward under the influence of gravity. Find the maximum altitude acquired by the rocket and the total time elapsed from the take-off until the rocket returns to the earth. *Ignore air resistance*.

Solution

$$d = \frac{1}{2}(a - g)t^2$$

$$= \frac{1}{2}(100 - 9.8)t^2$$

$$d(1hr = 60min) = \frac{1}{2}(100 - 9.8)(60)^2$$

$$= 162,360\ m$$

$$v = d'$$

$$= \left(\frac{1}{2}(100 - 9.8)t^2\right)'$$

$$= (100 - 9.8)t$$

$$v(60) = (100 - 9.8)(60)$$

$$= 5412\ m/s$$

The velocity will be reduced: $5412\ - 9.8t = 0$

$$\underline{t = 552.2\ \ sec}$$

The altitude: $d(t) = -\frac{9.8}{2}t^2 + 5412\ t + 162,360$

$$d(552.2) = -\frac{9.8}{2}(552.2)^2 + 5412(552.2) + 162,360$$

$$\underline{= 1.657 \times 10^6\ \ m}$$

Back to the ground: $4.9t^2 = 1.657 \times 10^6$

$$\underline{t_b = 581.5\ \ sec}$$

Total time: $t = 552.2 + 581.5 = \underline{1193.7\ \ sec}$

Exercise

A ball is projected vertically upward with initial velocity v_0 from ground level. Ignore air resistance.

a) What is the maximum height acquired by the ball?

b) How long does it take the ball to reach its maximum height? How long does it take the ball to return to the ground? Are these times identical?

c) What is the speed of the ball when it impacts with the ground on its return?

Solution

The position: $x(t) = -\frac{1}{2}gt^2 + v_0 t$

a) The maximum height when the velocity is zero

$$v = x' = -gt + v_0 = 0$$

$$t = \frac{v_0}{g}$$

$$\text{Maximum height } = -\frac{1}{2}g\left(\frac{v_0}{g}\right)^2 + v_0\frac{v_0}{g}$$

$$= -\frac{1}{2}\frac{v_0^2}{g} + \frac{v_0^2}{g}$$

$$\boxed{= \frac{v_0^2}{2g}}$$

b) The ball will take to reach the maximum height $t = \frac{v_0}{g}$ and the same to return to the ground, both are equal to $t = \frac{v_0}{g}$

c) When the ball hits the ground the time is equal to zero.

$$v = -g(0) + v_0$$

$$\boxed{= v_0}$$

Exercise

An object having mass 70 *kg* falls from rest under the influence of gravity. The terminal velocity of the object is $-20\ m/s$. Assume that the air resistance is proportional to the velocity.

a) Find the velocity and distance traveled at the end of 2 *seconds*.

b) How long does it take the object to reach 80% of its terminal velocity?

Solution

a) The terminal velocity: $v = -\frac{mg}{r}$

$$-20 = -\frac{70(9.8)}{r}$$

$$r = \frac{70(9.8)}{20}$$

$$\boxed{= 34.3}$$

$$v(t) = Ce^{-rt/m} - \frac{mg}{r}$$

$$v(t=0) = Ce^{-r(0)/m} - \frac{mg}{r}$$

$$0 = C - \frac{mg}{r}$$

$$\boxed{C = \frac{mg}{r}}$$

$$v(t) = \frac{mg}{r}\left(e^{-rt/m} - 1\right)$$

$$v(t=2) = \frac{70(9.8)}{34.3}\left(e^{-34.3(2)/70} - 1\right)$$

$$\boxed{\approx -12.4938 \quad m/s}$$

$$x = \int_0^t v(t)\,dt$$

$$= \frac{mg}{r}\int_0^t \left(e^{-rs/m} - 1\right)ds$$

$$= \frac{mg}{r}\left(-\frac{m}{r}e^{-rs/m} - s \right|_0^t$$

$$= \frac{mg}{r}\left[-\frac{m}{r}e^{-rt/m} - t - \left(-\frac{m}{r} - 0\right)\right]$$

$$= \frac{mg}{r}\left[\frac{m}{r}\left(1 - e^{-rt/m}\right) - t\right]$$

$$x(2) = \frac{70(9.8)}{34.3}\left[\frac{70}{34.3}\left(1 - e^{-34.3(2)/70}\right) - 2\right]$$

$$\boxed{\approx -14.5025}$$

b) The velocity is 80% of its terminal velocity when $.8 = 1 - e^{-rt/m}$

$$e^{-rt/m} = .2$$

$$-\frac{rt}{m} = \ln(.2)$$

$$t = \frac{m}{r}\ln(.2)$$

$$\boxed{\approx 3.285 \quad sec}$$

Exercise

An object in free fall may be modeled by assuming that the only forces at work are the gravitational force and resistance (friction due to the medium in which the objects falls). By Newton's second law (mass × acceleration = the sum of the external forces), the velocity of the object satisfies the differential equation

$$\underbrace{m}_{mass} \cdot \underbrace{v'(t)}_{acceleration} = \underbrace{mg + f(v)}_{external\ force}$$

Where f is a function that models the resistance and the positive direction is downward. One common assumption (often used for motion in air) is that $f(v) = -kv^2$, where $k > 0$ is a drag coefficient.

a) Show that the equation can be written in the form $v'(t) = g - av^2$ where $a = \frac{k}{m}$

b) For what (positive) value of v is $v'(t) = 0$? (This equilibrium solution is called the ***terminal velocity***.)

c) Find the solution of this separable equation assuming $v(0) = 0$ *and* $0 < v(t)^2 < \frac{g}{a}$ for $t \geq 0$

d) Graph the solution found in part (***c***) with $g = 9.8\ m/s^2$, $m = 1\ kg$, *and* $k = 0.1\ kg/m$, and verify the terminal velocity agrees with the value found in part (***b***).

Solution

a) Given: $f(v) = -kv^2$

$$mv'(t) = mg + f(v)$$

$$mv'(t) = mg - kv^2$$

$$v'(t) = g - \frac{k}{m}v^2$$

$$\boxed{v'(t) = g - av^2} \qquad \text{where } a = \frac{k}{m}$$

b) $v'(t) = g - av^2 = 0$

$$v^2 = \frac{g}{a} \rightarrow \boxed{v = \sqrt{\frac{g}{a}}}$$

c) $\frac{dv}{dt} = g - av^2$

$$\int \frac{dv}{g - av^2} = \int dt$$

$$-\frac{1}{a}\int \frac{dv}{v^2 - \frac{g}{a}} = \int dt$$

$$\frac{1}{v^2 - \frac{g}{a}} = \frac{A}{v - \sqrt{\frac{g}{a}}} + \frac{B}{v + \sqrt{\frac{g}{a}}}$$

$$1 = A\sqrt{\frac{g}{a}} + Av + Bv - B\sqrt{\frac{g}{a}}$$

$$\begin{cases} A + B = 0 \ \rightarrow \underline{A = -B|} \\ A\sqrt{\frac{g}{a}} - B\sqrt{\frac{g}{a}} = 1 \end{cases}$$

$$\underline{A = -B = \frac{1}{2}\sqrt{\frac{a}{g}}\,\Big|}$$

$$-\frac{1}{2a}\sqrt{\frac{a}{g}}\int\frac{dv}{v-\sqrt{\frac{g}{a}}} + \frac{1}{2a}\sqrt{\frac{a}{g}}\int\frac{dv}{v+\sqrt{\frac{g}{a}}} = \int dt$$

$$\frac{1}{2}\sqrt{\frac{1}{ag}}\left(-\ln\left|\sqrt{\frac{g}{a}} - v\right| + \ln\left|\sqrt{\frac{g}{a}} + v\right|\right) = t + C_1$$

$$\ln\frac{\sqrt{\frac{g}{a}} + v}{\sqrt{\frac{g}{a}} - v} = 2\sqrt{ag}\,t + C_2$$

$$\frac{\sqrt{\frac{g}{a}} + v}{\sqrt{\frac{g}{a}} - v} = e^{2\sqrt{ag}\,t + C_2}$$

$$\sqrt{\frac{g}{a}} + v = Ce^{2\sqrt{ag}\,t}\left(\sqrt{\frac{g}{a}} - v\right)$$

$$v(0) = 0 \quad \Rightarrow \quad \sqrt{\frac{g}{a}} = \sqrt{\frac{g}{a}}C \ \rightarrow \ \underline{C = 1|}$$

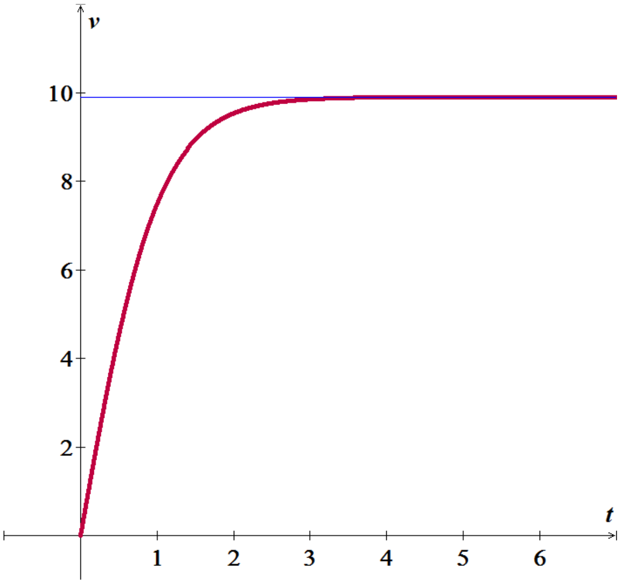
$$v\left(1 + e^{2\sqrt{ag}\,t}\right) = \sqrt{\frac{g}{a}}e^{2\sqrt{ag}\,t} - \sqrt{\frac{g}{a}}$$

$$\underline{v(t) = \frac{e^{2\sqrt{ag}\,t} - 1}{1 + e^{2\sqrt{ag}\,t}}\sqrt{\frac{g}{a}}\,\Big|}$$

d) $g = 9.8\ m/s^2, \quad m = 1\ kg, \quad and \quad k = 0.1\ kg/m$

$$\rightarrow a = \frac{k}{m} \ \underline{= 0.1|}$$

$$v(t) = \sqrt{98}\,\frac{e^{2\sqrt{.98}t} - 1}{1 + e^{2\sqrt{.98}t}}$$



Exercise

An open cylindrical tank initially filled with water drains through a hole in the bottom of the tank according to Torricelli's Law. If $h(t)$ is the depth of water in the tank for $t \geq 0$, then Torricelli's Law implies $h'(t) = -2k\sqrt{h}$, where k is a constant that includes the acceleration due to gravity, the radius of the tank, and the radius of the drain. Assume that the initial depth of the water is $h(0) = H$.

a) Find the solution of the initial value problem.
b) Find the solution in the case that $k = 0.1$ and $H = 0.5\ m$.
c) In general, how long does it take the tank to drain in terms of k and H?

Solution

a) $\frac{dh}{dt} = -2k\sqrt{h}$

$$\int \frac{dh}{\sqrt{h}} = -2\int k\,dt$$

$$2\sqrt{h} = 2kt + C_1$$

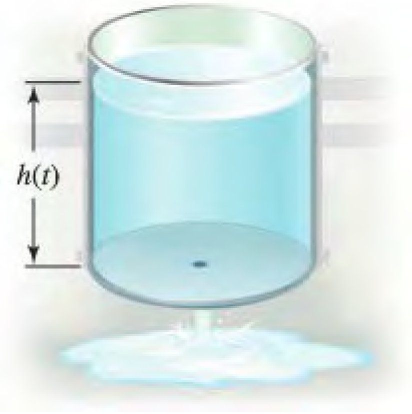
$$h(t) = (kt + C)^2$$

$$h(0) = H$$

$$H = C^2$$

$$\Rightarrow \underline{C = \sqrt{H}\,|}$$

$$\boxed{h(t) = \left(kt + \sqrt{H}\right)^2}$$



b) Given: $k = 0.1 \quad H = 0.5\ m$

$$h(t) = \left(0.1t + \sqrt{0.5}\right)^2$$

$$\boxed{=(0.1t+0.707)^2}$$

c) The tank is drained when $h(t)=0$

$$\left(kt+\sqrt{H}\right)^2=0$$

$$kt+\sqrt{H}=0$$

$$\boxed{t=-\frac{\sqrt{H}}{k}}$$

Exercise

The reaction of chemical compounds can often be modeled by differential equations. Let $y(t)$ be the concentration of a substance in reaction for $t\geq 0$ (typical units of *y* are *moles/L*). The change in the concentration of a substance, under appropriate conditions, is $\frac{dy}{dt}=-ky^n$, where $k>0$ is a rate constant and the positive integer *n* is the order of the reaction.

a) Show that for a first-order reaction $(n=1)$, the concentration obeys an exponential decay law.

b) Solve the initial value problem for a second-order reaction $(n=2)$ assuming $y(0)=y_0$

c) Graph and compare the concentration for a first-order and second-order reaction with $k=0.1$ and $y_0=1$

Solution

a) $$\int\frac{dy}{y}=-\int k\,dt$$

$$\ln|y|=-kt+C_1$$

$$\boxed{y(t)=Ce^{-kt}}$$

b) $n=2 \;\rightarrow\; \frac{dy}{dt}=-ky^2$

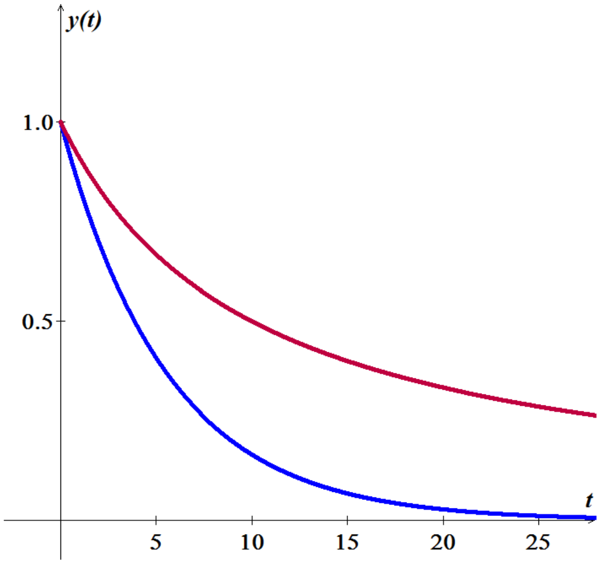
$$-\int\frac{dy}{y^2}=\int k\,dt$$

$$\frac{1}{y}=kt+C$$

$$y(0)=y_0 \quad\rightarrow\quad \boxed{\frac{1}{y_0}=C}$$

$$\frac{1}{y}=kt+\frac{1}{y_0}$$

$$\boxed{y(t)=\frac{y_0}{1+ky_0t}}$$



c) $y(t)=\dfrac{1}{1+0.1t}$

$y_0=1 \quad \rightarrow \quad \boxed{C=1}$

$\boxed{y(t)=e^{-0.1t}}$

Exercise

The consumption of a substrate in a reaction involving an enzyme is often modeled using Michaelis-Menton Kinetics, which involves the initial value problem $\dfrac{ds}{dt}=\dfrac{Qs}{K+s}$, $s(0)=s_0$, where $s(t)$ is the amount of substrate present at time $t\geq 0$, and Q and K are positive constants. Solve the initial value problem with $Q=10$, $K=5$, and $s_0=50$. Notice that the solution can be expressed explicitly only with *t* as a function of ***s***. Describe how ***s*** behaves as $t\rightarrow\infty$.

Solution

Given: $\quad Q=10 \quad K=5 \quad s_0=50$

$$\frac{ds}{dt}=\frac{Qs}{K+s}$$

$$=\frac{-50s}{5+s}$$

$$\int\frac{5+s}{s}\,ds=-\int 50\,dt$$

$$\int\left(\frac{5}{s}+1\right)ds=-\int 50\,dt$$

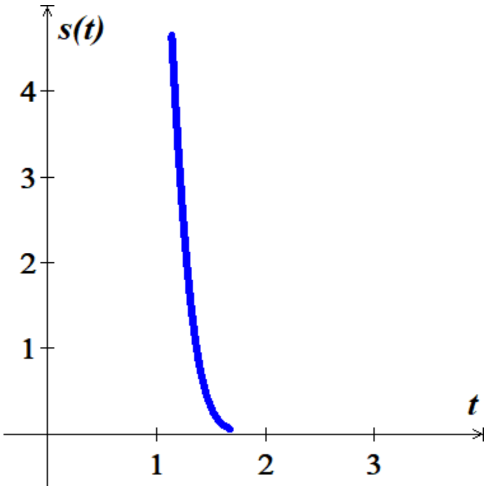
$$5\ln s+s=-50t+C$$

$s_0=50$

$\boxed{5\ln 50+50=C}$

$$50t=C-5\ln s-s$$

$$\boxed{t=\frac{1}{50}\left(5\ln 50+50-5\ln s-s\right)}$$



As $t\rightarrow\infty$

$$\lim\left(\frac{1}{50}\left(5\ln 50+50-5\ln s-s\right)\right)=\infty$$

$$\lim\left(5\ln 50+50-5\ln s-s\right)=\infty$$

$$\lim\left(-5\ln s-s\right)=\infty$$

$$\lim\left(5\ln s+s\right)=-\infty$$

$$\lim\left(\ln s\right)=-\infty \quad \textit{when} \quad s=0$$

$$\boxed{\lim_{t\to\infty} s(t)=0}$$

Exercise

An investment account, which earns interest and has regular deposits, can be modeled by the initial value problem $B'(t) = aB + m$ for $t \geq 0$, with $B(0) = B_0$. The constant a reflects the monthly interest rate, m is the rate of monthly deposits, and B_0 is the initial balance in the account. Solve the initial value problem with $a = 0.005$, $m = \$100 / month$, and $B_0 = \$100$. After how many months does the account have a balance of \$7,500?

Solution

Given: $a = 0.005 \quad m = 100 \quad B_0 = 100$

$$B'(t) = aB + m$$

$$B' - .005B = 100$$

$$e^{\int -.005\, dt} = e^{-.005t}$$

$$\int 100e^{-.005t}\, dt = -\frac{100}{.005}e^{-.005t}$$

$$= -2 \times 10^4 e^{-.005t}$$

$$B(t) = \frac{1}{e^{-.005t}}\left(-2 \times 10^4 e^{-.005t} + C\right)$$

$$= -2 \times 10^4 + Ce^{.005t}$$

$$B(0) = 100$$

$$-2 \times 10^4 + C = 100$$

$$C = 20,000 + 100$$

$$\boxed{= 20,100}$$

$$\boxed{B(t) = 20,100e^{.005t} - 2 \times 10^4}$$

$$7,500 = 20,100e^{.005t} - 20,000$$

$$20,100e^{.005t} = 27,500$$

$$e^{.005t} = \frac{27,500}{20,100}$$

$$.005t = \ln\frac{275}{201}$$

$$\boxed{t = 200\ln\frac{275}{201}}$$

$$\boxed{t \approx 63 \quad years}$$

Exercise

The growth of cancer turmors may be modeled by the Gomperts growth equation. Let $M(t)$ be the mass of the tumor for $t \geq 0$. The relevant intial value problem is

$$\frac{dM}{dt} = -aM \ln\frac{M}{K}, \quad M(0) = M_0$$

Where a and K are positive constants and $0 < M_0 < K$

a) Graph the growth rate function $R(M) = -aM \ln\frac{M}{K}$ assuming $a = 1$ and $K = 4$. For what values of *M* is the growth rate positive? For what values of *M* is maximum?

b) Solve the initial cvalue problem and graph the solution for $a = 1$, $K = 4$, and $M_0 = 1$. Describe the groath pattern of the tumor. Is the growth unbounded? If not, what is the limiting size of the tumor?

c) In the general equation, what is the meaning of *K*?

Solution

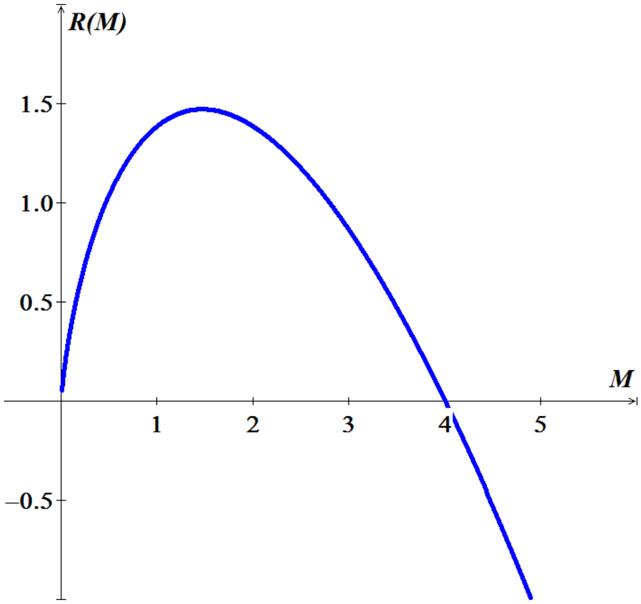
a) $R'(M) = -a\left(\ln\frac{M}{K} + M\frac{1}{K}\frac{K}{M}\right)$

$$= -a\left(\ln\frac{M}{K} + 1\right) = 0$$

$$\ln\frac{M}{K} = -1$$

$$M = Ke^{-1}$$

$$\boxed{= \frac{K}{e}}$$



For $a = 1$ and $K = 4$

$$\rightarrow \boxed{R(M) = -M \ln\frac{M}{4}}$$

b) $\displaystyle\int \frac{dM}{M(\ln M - \ln K)} = -\int a\,dt$

$$d(\ln M - \ln K) = \frac{1}{M}dM$$

$$\int \frac{d(\ln M - \ln K)}{\ln M - \ln K} = -\int a\,dt$$

$$\ln|\ln M - \ln K| = -at + C_1$$

$$\ln\frac{M}{K} = Ce^{-at}$$

$$\boxed{M(t) = Ke^{Ce^{-at}}}$$

For $a = 1$, $K = 4$, and $M_0 = 1$

$$M(0) = 4e^{C} = 1 \quad \Rightarrow \quad C = \ln\frac{1}{4} = -\ln 4$$

$$\boxed{M(t) = 4e^{-(\ln 4)e^{-t}}}$$

$$\lim_{t\to\infty} M(t) = \lim_{t\to\infty} 4e^{-(\ln 4)e^{-t}}$$

$$\underline{= 4\,|}$$

So the limiting size of the tumor is 4.

c) $\lim_{t\to\infty} M(t) = \lim_{t\to\infty} Ke^{Ce^{-at}}$ since $a > 0$

$$\underline{= K\,|}$$

Exercise

An endowment is an investment account in which the balance ideally remains constant and withdrawals are made on the interest earned by the account. Such an account may be modeled by the initial value problem $B'(t) = aB - m$ for $t \ge 0$, with $B(0) = B_0$. The constant a reflects the annual interest rate, m is the annual rate of withdrawal, and B_0 is the initial balance in the account.

a) Solve the initial value problem with $a = 0.05$, $m = \$1000\,/\,yr.$ and $B_0 = \$15{,}000$. Does the balance in the account increase or decrease?

b) If $a = 0.05$ and $B_0 = \$50{,}000$, what is the annual withdrawal rate m that ensures a constant balance in the account? What is the constant balance?

Solution

a) $B'(t) - aB = -m$

$$e^{\int -a\,dt} = e^{-at}$$

$$\int -me^{-at}\,dt = \frac{m}{a}e^{-at}$$

$$B(t) = \frac{1}{e^{-at}}\left(\frac{m}{a}e^{-at} + C\right)$$

$$\boxed{= \frac{m}{a} + Ce^{at}}$$

Given: $a = 0.05$, $m = \$1000\,/\,yr.$ $B_0 = \$15{,}000$

$$B(0) = \frac{1000}{.05} + C = 15{,}000$$

$$C = 15{,}000 - 20{,}000$$

$$\boxed{= -5{,}000}$$

$$\boxed{B(t) = 20,000 - 5,000\, e^{0.05t}}$$

The balance decreases since the exponential increases with time and subtract from 20,000.

b) ***Given***: $a = 0.05 \quad B_0 = \$50,000$

$$B = \frac{m}{a} = 50,000$$

$$m = 0.05 \times 50,000$$

$$\boxed{= \$2,500}$$

Exercise

The halibut fishery has been modeled by the differential equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$

Where $y(t)$ is the biomass (the total mass of the members of the population) in kilograms at time t

(measured in years), the carrying capacity is estimated to be $M = 8 \times 10^7\ kg$ and $k = 0.71\ per\ year$.

a) If $y(0) = 2 \times 10^7\ kg$, find the biomass a year later.

b) How long will it take for the biomass to reach $4 \times 10^7\ kg$.

Solution

a)
$$\frac{M}{ky(M-y)}dy = dt$$

$$\frac{M}{k}\frac{1}{y(M-y)}dy = dt$$

$$\frac{1}{y(M-y)} = \frac{A}{y} + \frac{B}{M-y}$$

$$AM - Ay + By = 1$$

$$\begin{cases} AM = 1 & \Rightarrow \boxed{A = \frac{1}{M}} \\ -A + B = 0 & \Rightarrow B = A \boxed{= \frac{1}{M}} \end{cases}$$

$$\frac{M}{k}\frac{1}{M}\int\left(\frac{1}{y} + \frac{1}{M-y}\right)dy = \int dt$$

$$\frac{1}{k}(\ln y - \ln(M-y)) = t + C_1$$

$$\ln\frac{y}{M-y} = kt + C_2$$

$$\frac{y}{M-y} = e^{kt + C_2}$$

$$y = Me^{kt}e^{C_2} - ye^{kt}e^{C_2} \qquad C = e^{C_2}$$

$$y\left(1+Ce^{kt}\right)=MCe^{kt}$$

$$y=\frac{MCe^{kt}}{1+Ce^{kt}}$$

$$=\frac{M}{1+Ce^{-kt}}$$

$$\boxed{=\frac{8\times10^{7}}{1+Ce^{-0.71t}}}$$

$$y(0)=\frac{8\times10^{7}}{1+C}=2\times10^{7}$$

$$C=\frac{8\times10^{7}}{2\times10^{7}}-1$$

$$\boxed{=3}$$

$$\boxed{y(t)=\frac{8\times10^{7}}{1+3e^{-0.71t}}}$$

$$y(1)=\frac{8\times10^{7}}{1+3e^{-0.71}}$$

$$\boxed{\approx 3.23\times10^{7}\ kg}$$

b) $y(t)=\dfrac{8\times10^{7}}{1+3e^{-0.71t}}=4\times10^{7}$

$$1+3e^{-0.71t}=\frac{8\times10^{7}}{4\times10^{7}}=2$$

$$3e^{-0.71t}=1$$

$$e^{-0.71t}=\frac{1}{3}$$

$$-.071t=\ln\frac{1}{3}$$

$$t=\frac{\ln 3}{0.71}$$

$$\boxed{\approx 1.55\ \ years}$$

Exercise

Suppose a population $P(t)$ satisfies $\dfrac{dP}{dt}=0.4P-0.001P^{2},\quad P(0)=50$

Where t is measured in years.

a) What is the carrying capacity?

b) What is $P'(0)$?

c) When will the population reach 50% of the carrying capacity?

Solution

a) $\dfrac{1}{0.4P(1-0.0025P)}dP = dt$

$$\frac{1}{P(1-0.0025P)} = \frac{A}{P} + \frac{B}{1-0.0025P}$$

$$A - .0025PA + PB = 1$$

$$\rightarrow \begin{cases} \underline{A=1} \\ -.0025A + B = 0 \quad \underline{B = .0025} \end{cases}$$

$$\int \left(\frac{1}{P} + \frac{.0025}{1-.0025P}\right) dP = 0.4\int dt$$

$$\ln P - \ln(1-.0025P) = 0.4t + C_1$$

$$\ln\frac{P}{1-.0025P} = 0.4t + C_1$$

$$\frac{P}{1-.0025P} = e^{0.4t+C_1} = Ce^{0.4t} \qquad C = e^{C_1}$$

$$Ce^{-0.4t}P = 1 - .0025P$$

$$Ce^{-0.4t}P + .0025P = 1$$

$$\left(Ce^{-0.4t} + .0025\right)P = 1$$

$$P(t) = \frac{1}{Ce^{-0.4t} + .0025}$$

$$P(0) = \frac{1}{C + .0025} = 50$$

$$C = \frac{1}{50} - .0025$$

$$\underline{= .0175}$$

$$P(t) = \frac{1}{.0175e^{-0.4t} + .0025}$$

$$\boxed{P(t) = \frac{400}{7e^{-0.4t} + 1}}$$

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{400}{1 + 7e^{-0.4t}}$$

$$\underline{= 400}$$

The carrying capacity is 400.

b) $P'(0) = \dfrac{dP}{dt}\Big|_{t=0}$

$$= 0.4(50) - 0.001(50)^2$$

$$\underline{= 17.5}$$

c) $P(t)=\dfrac{400}{7e^{-0.4t}+1}=200$

$$7e^{-0.4t}+1=2$$

$$e^{-0.4t}=\frac{1}{7}$$

$$-0.4t=\ln\left(\frac{1}{7}\right)$$

$$t=\frac{\ln\left(\frac{1}{7}\right)}{-0.4}$$

$$\approx 4.86\ \ years$$

Exercise

Let $P(t)$ be the performance level of someone learning a skill as a function of the training time *t*. The graph of *P* is called a ***learning curve***. We proposed the differential equation

$$\frac{dP}{dt}=k\left(M-P(t)\right)$$

As a reasonable model for learning, where *k* is a positive constant. Solve it as a linear differential equation and use your solution to graph the learning curve.

Solution

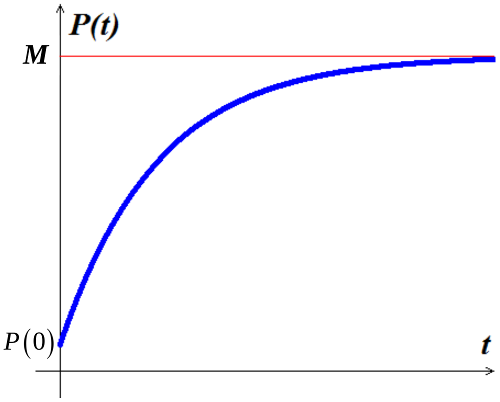
$$\frac{dP}{dt}+kP=kM$$

$$e^{\int kdt}=e^{kt}$$

$$\int kMe^{kt}dt=Me^{kt}$$

$$P(t)=\frac{1}{e^{kt}}\left(Me^{kt}+C\right)$$

$$=M+Ce^{-kt}\qquad k>o$$



Exercise

An inductor $(L=1\ H)$ and a resistor $(R=0.1\ \Omega)$ are joined in series with an electronic force (emf) $E=E(t)$ and no charge on the capacitor at $t=0$. Find the ensuing charge current in the current at time *t* for the given $E(t)=10-2t$

Solution

$$\frac{dI}{dt}+0.1I=10-2t \qquad L\frac{dI}{dt}+RI=E(t)$$

$$e^{\int .1dt} = e^{t/10}$$

$$\int (10-2t)e^{t/10}dt = (100-20t+200)e^{t/10}$$

$$= (300-20t)e^{t/10}$$

$$I(t) = e^{-10t}\left((300-20t)e^{t/10}+K\right)$$

$$= 300-20t+Ke^{-10t}$$

$$I(0)=0 \quad \rightarrow \boxed{K=-300}$$

$$\boxed{I(t) = 300-20t-300e^{-t/10}}$$

Exercise

An inductor $(L=1\ H)$ and a resistor $(R=0.1\ \Omega)$ are joined in series with an electronic force (*emf*) $E=E(t)$ and no charge on the capacitor at $t=0$. Find the ensuing current in the current at time t for the given $E(t)=4\cos 3t$

Solution

$$\frac{dI}{dt}+0.1I = 4\cos 3t \qquad\qquad L\frac{dI}{dt}+RI=E(t)$$

$$e^{\int .1dt} = e^{t/10}$$

$$\int (4\cos 3t)e^{t/10} = \left(\frac{4}{3}\sin 3t+\frac{2}{45}\cos 3t\right)e^{t/10} - \frac{1}{900}\int (4\cos 3t)e^{t/10}$$

$$\frac{901}{900}\int (4\cos 3t)e^{t/10} = \frac{2}{45}(30\sin 3t+\cos 3t)e^{t/10}$$

$$\int (4\cos 3t)e^{t/10} = \frac{40}{901}(30\sin 3t+\cos 3t)e^{t/10}$$

$$I(t) = e^{-t/10}\left(\frac{40}{901}(30\sin 3t+\cos 3t)e^{t/10}+K\right)$$

$$= \frac{40}{901}(30\sin 3t+\cos 3t)+Ke^{-t/10}$$

$$I(0)=0 \quad \rightarrow \boxed{K=-\frac{40}{901}}$$

$$\boxed{I(t) = \frac{40}{901}\left(30\sin 3t+\cos 3t-e^{-t/10}\right) \quad A}$$

		$\int 4\cos 3t\ dt$
+	$e^{t/10}$	$\frac{4}{3}\sin 3t$
−	$\frac{1}{10}e^{t/10}$	$-\frac{4}{9}\cos 3t$
+	$\frac{1}{100}e^{t/10}$	

Exercise

An inductor $\left(L = 1\,H\right)$ and a resistor $\left(R = 0.1\,\Omega\right)$ are joined in series with an electronic force (*emf*) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing current in the current at time t for the given $E(t) = 4\sin 2\pi t$

Solution

$$\frac{dI}{dt} + 0.1I = 4\sin 2\pi t \qquad\qquad L\frac{dI}{dt} + RI = E(t)$$

$$e^{\int .1dt} = e^{t/10}$$

$$\int (4\sin 2\pi t)e^{t/10} = \left(-\frac{2}{\pi}\cos 2\pi t + \frac{1}{10\pi^2}\sin 2\pi t\right)e^{t/10} - \frac{1}{400\pi^2}\int (4\sin 2\pi t)e^{t/10}$$

		$\int 4\sin 2\pi t\,dt$
+	$e^{t/10}$	$-\frac{2}{\pi}\cos 2\pi t$
−	$\frac{1}{10}e^{t/10}$	$-\frac{1}{\pi^2}\sin 2\pi t$
+	$\frac{1}{100}e^{t/10}$	

$$\frac{1+400\pi^2}{400\pi^2}\int (4\sin 2\pi t)e^{t/10} = \frac{1}{10\pi^2}\left(-20\pi\cos 2\pi t + \sin 2\pi t\right)e^{t/10}$$

$$\int (4\sin 2\pi t)e^{t/10} = \frac{40}{1+400\pi^2}\left(-20\pi\cos 2\pi t + \sin 2\pi t\right)e^{t/10}$$

$$I(t) = e^{-t/10}\left(\frac{40}{1+400\pi^2}\left(-20\pi\cos 2\pi t + \sin 2\pi t\right)e^{t/10} + K\right)$$

$$= \frac{40}{1+400\pi^2}\left(-20\pi\cos 2\pi t + \sin 2\pi t\right) + Ke^{-t/10}$$

$$I(0) = 0 \quad \to 0 = \frac{40}{1+400\pi^2}(-20\pi) + K$$

$$\Rightarrow \boxed{K = \frac{800\pi}{1+25\pi^2}}$$

$$I(t) = \frac{40}{1+400\pi^2}\left(-20\pi\cos 2\pi t + \sin 2\pi t\right) + \frac{800}{1+400\pi^2}e^{-t/10}$$

$$\boxed{= \frac{40}{1+400\pi^2}\left(\sin 2\pi t - 20\pi\cos 2\pi t + 20\pi e^{-t/10}\right) \quad A}$$

Exercise

An *RL* circuit with a $1-\Omega$ resistor and a $0.01\text{-}H$ inductor is driven by a voltage $E(t)=\sin 100t\ V$. If the initial inductor current is zero, determine the subsequence resistor and inductor current and the voltages.

Solution

$$0.01\frac{dI}{dt}+I=\sin 100t \qquad\qquad L\frac{dI}{dt}+RI=E(t)$$

$$\frac{dI}{dt}+100I=100\sin 100t$$

$$e^{\int 100dt}=e^{100t}$$

$$\int(100\sin 100t)e^{100t}dt=(-\cos 100t+\sin 100t)e^{100t}-\int(100\sin 100t)e^{100t}dt$$

		$\int 100\sin 100t\,dt$
+	e^{100t}	$-\cos 100t$
−	$100e^{100t}$	$-\frac{1}{100}\sin 100t$
+	$10000e^{100t}$	

$$2\int(100\sin 100t)e^{100t}dt=(-\cos 100t+\sin 100t)e^{100t}$$

$$\int(100\sin 100t)e^{100t}dt=\frac{1}{2}(-\cos 100t+\sin 100t)e^{100t}$$

$$I(t)=e^{-100t}\left(\frac{1}{2}(-\cos 100t+\sin 100t)e^{100t}+K\right)$$

$$\boxed{=\frac{1}{2}(-\cos 100t+\sin 100t)+Ke^{-100t}\quad A}$$

$$I(0)=0$$

$$0=\frac{1}{2}(-1)+K$$

$$\Rightarrow\ \boxed{K=\frac{1}{2}}$$

$$\boxed{I(t)=\frac{1}{2}(-\cos 100t+\sin 100t)+\frac{1}{2}e^{-100t}}$$

The voltage at the resistor:

$$E_R(t)=RI$$

$$\boxed{=\frac{1}{2}\left(\sin 100t-\cos 100t+e^{-100t}\right)}$$

The voltage at the inductor:

$$E_L\left(t\right)=L\frac{dI}{dt}$$

$$=\left(0.01\right)\frac{1}{2}\left(100\cos 100t+100\sin 100t-100e^{-100t}\right)$$

$$\boxed{=\frac{1}{2}\left(\cos 100t+\sin 100t+e^{-10t}\right)}$$

Exercise

An *RL* circuit with a $5-\Omega$ resistor and a $0.05\text{-}H$ inductor is driven by a voltage $E\left(t\right)=5\cos 120t\ V$. If the initial inductor current is 1 *A*, determine the subsequence resistor and inductor current and the voltages.

Solution

$$0.05\frac{dI}{dt}+5I=5\cos 120t \qquad L\frac{dI}{dt}+RI=E(t)$$

$$\frac{dI}{dt}+100I=100\cos 120t$$

$$e^{\int 100dt}=e^{100t}$$

$$\int\left(100\cos 120t\right)e^{100t}dt=\left(\frac{5}{6}\sin 120t+\frac{25}{36}\cos 120t\right)e^{100t}-\frac{25}{36}\int\left(100\cos 120t\right)e^{100t}dt$$

		$\int 100\cos 120t\ dt$
+	e^{100t}	$\frac{5}{6}\sin 120t$
−	$100e^{100t}$	$-\frac{1}{144}\cos 120t$
+	10^4e^{100t}	

$$\left(1+\frac{25}{36}\right)\int\left(100\cos 120t\right)e^{100t}dt=\left(\frac{5}{6}\sin 120t+\frac{25}{36}\cos 120t\right)e^{100t}$$

$$\frac{61}{36}\int\left(100\cos 120t\right)e^{100t}dt=\frac{5}{36}\left(6\sin 120t+5\cos 120t\right)e^{100t}$$

$$\int\left(100\cos 120t\right)e^{100t}dt=\frac{5}{61}\left(6\sin 120t+5\cos 120t\right)e^{100t}$$

$$I\left(t\right)=e^{-100t}\left(\frac{5}{61}\left(6\sin 120t+5\cos 120t\right)e^{100t}+K\right)$$

$$\boxed{=\frac{5}{61}\left(6\sin 120t+5\cos 120t\right)+Ke^{-100t}}$$

$$I\left(0\right)=1$$

$$1 = \frac{5}{61}(5) + K$$

$$\Rightarrow \ \boxed{K = \frac{36}{61}}$$

$$\boxed{I(t) = \frac{5}{61}(6\sin 120t + 5\cos 120t) + \frac{36}{61}e^{-100t}}$$

The voltage at the resistor:

$$E_R(t) = RI$$

$$\boxed{= \frac{25}{61}(6\sin 120t + 5\cos 120t) + \frac{180}{61}e^{-100t}}$$

The voltage at the inductor:

$$E_L(t) = L\frac{dI}{dt}$$

$$= (0.05)\left(\frac{25}{61}(720\cos 120t - 600\sin 120t) - \frac{18000}{61}e^{-100t}\right)$$

$$\boxed{= 14.754\cos 120t - 12.295\sin 120t - 14.754e^{-100t}}$$

Exercise

For the given $RL-$circuit

Which has a constant impressed voltage E, a resistor of resistance R, and a coil of impedance L.

Find the current $I(t)$ flowing in the circuit.

Solution

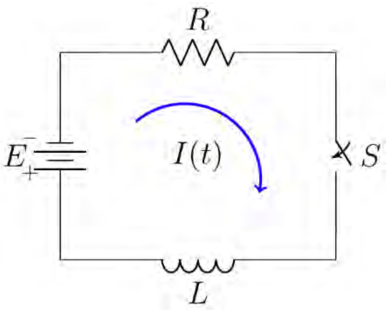
$$L\frac{dI}{dt} + RI = E$$

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}$$

$$e^{\int \frac{R}{L}dt} = e^{(R/L)t}$$

$$\int \frac{E}{L}e^{(R/L)t}dt = \frac{E}{L}\frac{L}{R}e^{(R/L)t}$$

$$= \frac{E}{R}e^{(R/L)t}$$



$$I(t) = \frac{1}{e^{(R/L)t}}\left(\frac{E}{R}e^{(R/L)t} - K\right)$$

$$= \frac{E}{R} - Ke^{-(R/L)t}$$

$$I(t = 0) = \frac{E}{R} - K = 0$$

$$\boxed{K = \frac{E}{R}}$$

$$\boxed{I(t)=\frac{E}{R}-\frac{E}{R}e^{-(R/L)t}}$$

$$\lim_{t\to\infty} I(t)=\lim_{t\to\infty}\left(\frac{E}{R}-\frac{E}{R}e^{-(R/L)t}\right)$$

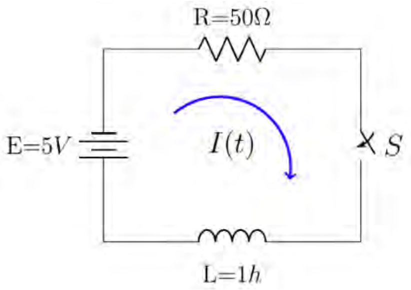
$$\boxed{=\frac{E}{R}}$$

Exercise

For the given $RL-$circuit

Which has a constant impressed voltage E, a resistor of resistance R, and a coil of impedance L. Find the current $I(t)$ flowing in the circuit.

Solution



$$L\frac{dI}{dt}+RI=E$$

$$\frac{dI}{dt}+50I=5$$

$$e^{\int 50dt}=e^{50t}$$

$$\int 5e^{50t}dt=\frac{1}{10}e^{50t}$$

$$I(t)=\frac{1}{e^{50t}}\left(\frac{1}{10}e^{50t}+K\right)$$

$$=\frac{1}{10}+Ke^{-50t}$$

$$I(t=0)=\frac{1}{10}+K=0$$

$$\boxed{K=-\frac{1}{10}}$$

$$\boxed{I(t)=\frac{1}{10}\left(1-e^{-50t}\right)}$$

Exercise

A circuit containing an electromotive force, a capacitor with a capacitance of *C* farads (*F*), and a resistor with a resistance of *R* ohms (Ω). The voltage drop across the capacitor is $\frac{Q}{C}$, where *Q* is the charge (in coulombs), so in this case ***Kirchhoff's Law*** gives

$$RI+\frac{Q}{C}=E(t)$$

But $I=\frac{dQ}{dt}$, so we have $\qquad R\frac{dQ}{dt}+\frac{1}{C}Q=E(t)$

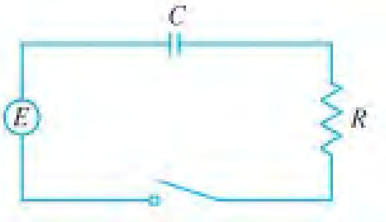
Find the charge and the current at time *t*

a) Suppose the resistance is $5\ \Omega$, the capacitance is 0.05 *F*, a battery gives voltage of 60 *V* and initial charge is $Q(0)=0\ C$

b) Suppose the resistance is $2\ \Omega$, the capacitance is 0.01 *F*, $E(t)=10\sin 60t$ and initial charge is $Q(0)=0\ C$

Solution

a) $5\frac{dQ}{dt}+\frac{1}{.05}Q=60 \ \rightarrow\ \frac{dQ}{dt}+4Q=12$



$$e^{\int 4dt}=e^{4t}$$

$$\int 12e^{4t}dt=3e^{4t}$$

$$Q(t)=\frac{1}{e^{4t}}\left(3e^{4t}+C\right)$$

$$=3+Ce^{-4t}$$

$$Q(0)=3+C=0$$

$$\Rightarrow\ C=-3$$

$$Q(t)=3\left(1-e^{-4t}\right)$$

$$I=\frac{dQ}{dt}$$

$$=12e^{-4t}$$

b) $2\frac{dQ}{dt}+\frac{1}{.01}Q=10\sin 60t$

$$\frac{dQ}{dt}+50Q=5\sin 60t$$

$$e^{\int 50dt}=e^{50t}$$

$$5\int e^{50t}\left(\sin 60t\right)dt=$$

		$\int \sin 60t\, dt$
+	e^{50t}	$-\frac{1}{60}\cos 60t$
−	$50e^{50t}$	$-\frac{1}{3600}\sin 60t$
+	$2500e^{50t}$	$-\frac{1}{3600}\int \sin 60t$

$$\int e^{50t}\left(\sin 60t\right)dt=\left(-\frac{1}{60}\cos 60t+\frac{1}{72}\sin 60t\right)e^{50t}-\frac{25}{36}\int e^{50t}\left(\sin 60t\right)dt$$

$$\frac{61}{36}\int e^{50t}\left(\sin 60t\right)dt=\left(-\frac{1}{60}\cos 60t+\frac{1}{72}\sin 60t\right)e^{50t}$$

$$\int e^{50t}\left(\sin 60t\right)dt=\frac{36}{21,960}\left(-6\cos 60t+5\sin 60t\right)e^{50t}$$

$$5\int e^{50t}\left(\sin 60t\right)dt=\frac{1}{122}\left(-6\cos 60t+5\sin 60t\right)e^{50t}$$

$$Q(t)=\frac{1}{e^{50t}}\left(\frac{1}{122}\left(-6\cos 60t+5\sin 60t\right)e^{50t}+C\right)$$

$$=\frac{1}{122}\left(-6\cos 60t+5\sin 60t\right)+Ce^{-50t}$$

$$Q(0)=-\frac{6}{122}+C=0$$

$$\Rightarrow\ \boxed{C=\frac{3}{61}}$$

$$\boxed{Q(t)=\frac{1}{122}\left(-5\cos 60t+6\sin 60t+6e^{-50t}\right)}$$

$$I=\frac{dQ}{dt}=\frac{1}{122}\left(300\sin 60t+360\cos 60t-300e^{-50t}\right)$$

$$\boxed{=\frac{30}{61}\left(5\sin 60t+6\cos 60t-5e^{-50t}\right)}$$

Exercise

A 30–*volt* electromotive force is applied to an *LR*-series circuit in which the inductance is 0.1 *henry* and the resistance is 50 *ohms*.

a) Find the current $i(t)$ if $i(0)=0$

b) Determine the current as $t \to \infty$

c) Solve the equation when $E(t)=E_0 \sin \omega t$ and $i(0)=i_0$

Solution

a) $0.1\frac{di}{dt}+50i=30 \qquad\qquad L\frac{di}{dt}+Ri=E(t)$

$$\frac{di}{dt}+500i=300$$

$$e^{\int 500dt}=e^{500t}$$

$$\int 300e^{500t}\,dt=\frac{3}{5}e^{500t}$$

$$i(t)=e^{-500t}\left(\frac{3}{5}e^{500t}+C\right)$$

$$i(0)=0$$

$$0=\frac{3}{5}+C$$

$$\to \boxed{C=-\frac{3}{5}}$$

$$\boxed{i(t)=\frac{3}{5}-\frac{3}{5}e^{-500t}}$$

b) $\lim\limits_{t\to\infty} i(t)=\lim\limits_{t\to\infty}\left(\frac{3}{5}-\frac{3}{5}e^{-500t}\right)$

$$\boxed{=\frac{3}{5}}$$

c) $\frac{di}{dt}+500i=10E_0 \sin \omega t$

$$\int 10E_0(\sin \omega t)e^{500t}dt=$$

		$\int \sin \omega t\, dt$
+	e^{500t}	$-\frac{1}{\omega}\cos \omega t$
−	$500e^{500t}$	$-\frac{1}{\omega^2}\sin \omega t$
+	$25\times 10^4 e^{500t}$	$-\int \frac{1}{\omega^2}\sin \omega t$

$$\int (\sin \omega t) e^{500t} dt = \left(-\frac{1}{\omega}\cos \omega t + \frac{500}{\omega^2}\sin \omega t\right)e^{500t} - \frac{25\times 10^4}{\omega^2}\int (\sin \omega t) e^{500t} dt$$

$$\left(\frac{\omega^2 + 25\times 10^4}{\omega^2}\right)\int (\sin \omega t) e^{500t} dt = \frac{1}{\omega^2}(-\omega \cos \omega t + 500 \sin \omega t) e^{500t}$$

$$\int 10E_0 \, (\sin \omega t) e^{500t} dt = \frac{10E_0}{\omega^2 + 25\times 10^4}(-\omega \cos \omega t + 500 \sin \omega t) e^{500t}$$

$$i(t) = e^{-500t}\left(\frac{10E_0}{\omega^2 + 25\times 10^4}(-\omega \cos \omega t + 500 \sin \omega t) e^{500t} + C\right) \qquad i(0) = i_0$$

$$0 = -\frac{10\omega t E_0}{\omega^2 + 25\times 10^4} + C \quad \rightarrow \quad \boxed{C = \frac{10\omega t E_0}{\omega^2 + 25\times 10^4}}$$

$$\boxed{i(t) = \frac{10E_0}{\omega^2 + 25\times 10^4}\left(-\omega \cos \omega t + 500 \sin \omega t - \omega e^{-500t}\right)}$$

Exercise

A tank contains 50 *gallons* of a solution composed of 90% water and 10% alcohol. A second solution containing 50% water and 50% alcohol is added to the tank at the rate of $4 \ gal / \min$. As the second solution is being added, the tank is being drained at a rate of $5 \ gal / \min$. The solution in the tank is stirred constantly. How much alcohol is in the tank after 10 *minutes*?

Solution

Let *y* be the amount (in *lb*.) of additive in the tank at time *t* and $y(0) = 100$

$$V(t) = 50 + \left(4\frac{gal}{\min} - 5\frac{gal}{\min}\right)(t \ \min)$$

$$\boxed{= 50 - t}$$

$$Rate\ out = \frac{y}{50-t}(5)$$

$$= \frac{5y}{50-t} \quad \frac{lb}{min}$$

$$Rate\ in = \left(\frac{1}{2}\ \frac{lb}{gal}\right)\left(4\ \frac{gal}{\min}\right)$$

$$\boxed{= 2 \ \frac{lb}{min}}$$

$$\frac{dy}{dt} = 2 - \frac{5}{50-t}y$$

$$\frac{dy}{dt} + \frac{5}{50-t}y = 2$$

$$e^{\int \frac{5}{50-t}dt} = e^{\int \frac{-5}{50-t}d(50-t)}$$

$$= e^{-5\ln|50-t|}$$

$$= (50-t)^{-5}$$

$$\int 2(50-t)^{-5}\, dt = -2\int (50-t)^{-5}\, d(50-t)$$

$$= \frac{1}{2}(50-t)^{-4}$$

$$y(t) = \frac{1}{(50-t)^{-5}}\left(\frac{1}{2}(50-t)^{-4} + C\right)$$

$$\boxed{= \frac{1}{2}(50-t) + C(50-t)^5}$$

$$y(0) = \frac{1}{2}(50) + C(50)^5 = 5$$

$$\boxed{C = -\frac{20}{50^5}}$$

$$\boxed{y(t) = \frac{1}{2}(50-t) - \frac{20}{50^5}(50-t)^5}$$

$$y(t=20) = \frac{1}{2}(30) - \frac{20}{50^5}(30)^5$$

$$= 15 - \frac{20}{5^5}3^5$$

$$\boxed{\approx 13.45 \ \ gal}$$

Exercise

A 200-*gallon* tank is half full of distilled water. At time $t = 0$, a concentrate solution containing 0.5 *lb/gal* enters the tank at the rate of $5\ \ gal / \min$, and well-stirred mixture is withdrawn at the rate of $3\ \ gal / \min$.

a) At what time will the tank be full?

b) At the time the tank is full, how many pounds of concentrate will it contain?

Solution

a) $V(t) = 100 + (5-3)t = 200$

$2t = 100$

$\boxed{t = 50 \ \ min}$

b) $Rate\ out = \frac{y}{100+2t}(3)$

$$= \frac{3y}{100+2t} \ \frac{lb}{min}$$

$$Rate\ in = \left(0.5\ \frac{lb}{gal}\right)\left(5\ \frac{gal}{\text{min}}\right)$$

$$\underline{= 2.5\ \frac{lb}{min}\ \Big|}$$

$$\frac{dy}{dt} = 2.5 - \frac{3y}{100+2t}$$

$$\frac{dy}{dt} + \frac{3}{100+2t}y = 2.5$$

$$e^{\int \frac{3}{100+2t}dt} = e^{\frac{3}{2}\int \frac{1}{50+t}d(50+t)}$$

$$= e^{\frac{3}{2}\ln|50+t|}$$

$$= (50+t)^{3/2}$$

$$2.5\int (50+t)^{3/2}\, dt = \frac{5}{2}\int (50+t)^{3/2}\, d(50+t)$$

$$= (50+t)^{5/2}$$

$$y(t) = \frac{1}{(50+t)^{3/2}}\left((50+t)^{5/2} + C\right)$$

$$\underline{= 50 + t + C(50+t)^{-3/2}\ \Big|}$$

$$y(0) = 50 + C(50)^{-3/2} = 0$$

$$\underline{C = -(50)^{5/2}\ \Big|}$$

$$\underline{y(t) = 50 + t - (50)^{5/2}(50+t)^{-3/2}\ \Big|}$$

$$y(50) = 50 + 50 - (50)^{5/2}(100)^{-3/2}$$

$$\underline{\approx 82.32\ \ lb\ \Big|}$$

Exercise

A 200-*gallon* tank is half full of distilled water. At time $t=0$, a concentrate solution containing 1 *lb/gal* enters the tank at the rate of $5\ gal / \min$, and well-stirred mixture is withdrawn at the rate of $3\ gal / \min$.

a) At what time will the tank be full?

b) At the time the tank is full, how many pounds of concentrate will it contain?

Solution

c) $V(t)=100+(5-3)t=200$

$2t=100$

$\boxed{t=50\ \ min}$

d) $Rate\ out=\dfrac{y}{100+2t}(3)$

$$=\frac{3y}{100+2t}\ \ \frac{lb}{min}$$

$$Rate\ in=\left(1\ \frac{lb}{gal}\right)\left(5\ \frac{gal}{\min}\right)$$

$$\boxed{=5\ \ \frac{lb}{min}}$$

$$\frac{dy}{dt}=5-\frac{3y}{100+2t}$$

$$\frac{dy}{dt}+\frac{3}{100+2t}y=5$$

$$e^{\int\frac{3}{100+2t}dt}=e^{\frac{3}{2}\int\frac{1}{50+t}d(50+t)}$$

$$=e^{\frac{3}{2}\ln|50+t|}$$

$$=(50+t)^{3/2}$$

$$5\int(50+t)^{3/2}\,dt=5\int(50+t)^{3/2}\,d(50+t)$$

$$=2(50+t)^{5/2}$$

$$y(t)=\frac{1}{(50+t)^{3/2}}\left(2(50+t)^{5/2}+C\right)$$

$$\boxed{=100+2t+C(50+t)^{-3/2}}$$

$$y(0)=100+C(50)^{-3/2}=0$$

$$\to C=-(100)(25\times 2)^{3/2}$$

$$\boxed{=25000\sqrt{2}}$$

$$\underline{y(t) = 100 + 2t - 25{,}000\sqrt{2}\,(50+t)^{-3/2}}\,\Big|$$

$$\begin{aligned} y(50) &= 100 + 100 - 25{,}000\sqrt{2}\,(100)^{-3/2} \\ &= 200 - 25\sqrt{2} \\ &\underline{\approx 164.64 \ \ lb}\,\Big| \end{aligned}$$

Exercise

A 200-*gallon* tank is full of a concentrate solution containing $25\ lb$. Starting at time $t = 0$, distilled water is admitted to the tank at the rate of $10\ gal/\min$, and well-stirred mixture is withdrawn at the same rate.

a) Find the amount of concentrate in the solution as a function of *t*.

b) Find the time at which the amount of concentrate in the tank reaches 15 *pounds*.

c) Find the quantity of the concentrate in the solution as $t \to \infty$.

Solution

a) $$\begin{aligned} V(t) &= 200 + (10-10)t \\ &\underline{= 200 \ \ gal}\,\Big| \end{aligned}$$

$$\begin{aligned} Rate\ out &= \frac{10y}{200} \\ &= \frac{y}{20} \ \ \frac{lb}{\min} \end{aligned}$$

$$\underline{Rate\ in = 0}\,\Big|$$

$$\frac{dy}{dt} = -\frac{y}{20}$$

$$\int \frac{dy}{y} = -\frac{1}{20}\int dt$$

$$\ln y = -\frac{1}{20}t + C_1$$

$$\underline{y(t) = Ce^{-t/20}}\,\Big|$$

$$y(0) = \underline{C = 25}\,\Big|$$

$$\underline{y(t) = 25e^{-t/20}}\,\Big|$$

b) $$y(t) = 25e^{-t/20} = 15$$

$$e^{-t/20} = \frac{3}{5}$$

$$-\frac{t}{20} = \ln\left(\frac{3}{5}\right)$$

$$t = -20\ln\left(\frac{3}{5}\right)$$

$$\underline{\approx 10.2 \ \ min \,|}$$

c) $\lim\limits_{t\to\infty} y(t) = \lim\limits_{t\to\infty} 25e^{-t/20}$

$$\underline{= 0 \,|}$$

Exercise

A tank contains 300 *litres* of fluid in which 20 *grams* of salt is dissolved. Brine containing 1 *g* of salt per *litre* is then pumped into the tank at a rate of 4 *L/min*; the well-mixed solution is pumped out at the same rate. Find the number $x(t)$ of grams of salt in the tank at time *t*.

Solution

$y(0) = 20$

$V(t) = 300 + \left(4\frac{L}{min} - 4\frac{L}{min}\right)(t \ min)$

$\underline{= 300 \quad gal \,|}$

Let *y* be the amount (in *g*.) of additive in the tank at time *t* and

$Rate\ out = \frac{y}{300}(4)$

$\underline{= \frac{y}{75} \,|}$

$Rate\ in = (1)(4)$

$\underline{= 4 \,|}$

$\frac{dy}{dt} = 4 - \frac{1}{75}y$

$\frac{dy}{dt} + \frac{1}{75}y = 4$

$e^{\int \frac{1}{75}dt} = e^{t/75}$

$\int 4e^{t/75}\,dt = 300e^{t/75}$

$y(t) = \frac{1}{e^{t/75}}\left(300e^{t/75} + C\right)$

$\underline{= 300 + Ce^{-t/75} \,|}$

$y(0) = 20 \quad \rightarrow \quad 20 = 300 + C$

$\Rightarrow \ \underline{C = -280 \,|}$

$\underline{y(t) = 300 - 280e^{-t/75} \,|}$

Exercise

A 1500 *gallon* tank initially contains 600 *gallon* of water with 5 *lbs*. of salt dissolved in it. Water enters the tank at a rate of 9 *gal/hr.* and the water entering the tank at a rate has a salt concentration of $\frac{1}{5}(1+\cos t)$ *lbs./gal*. If a well mixed solution leaves the tank at a rate of 6 *gal/hr.*, how much salt is in the tank when it overflows?

Solution

Given: $y(0)=5$

$$V(t)=600+\left(9\frac{gal}{hr}-6\frac{gal}{hr}\right)(t\ hr)$$

$$\underline{=600+3t\,|}$$

$$\frac{dy}{dt}=Rate\ in-\ Rate\ out$$

$$\frac{dy}{dt}=\left(\frac{1}{5}(1+\cos t)\ \frac{lb}{gal}\right)\left(9\ \frac{gal}{hr}\right)-\left(\frac{y}{600+3t}\ \frac{lb}{gal}\right)\left(6\ \frac{gal}{hr}\right)$$

$$=\frac{9}{5}(1+\cos t)-\frac{2y}{200+t}$$

$$\frac{dy}{dt}+\frac{2}{200+t}y=\frac{9}{5}(1+\cos t)$$

$$e^{\int\frac{2}{200+t}dt}=e^{2\ln(200+t)}$$

$$=(200+t)^2$$

$$\frac{9}{5}\int(1+\cos t)(200+t)^2\,dt=\frac{9}{5}\int(200+t)^2\,d(200+t)+\frac{9}{5}\int\cos t\left(4\times10^4+400t+t^2\right)dt$$

		$\int\cos t\,dt$
+	$4\times10^4+400t+t^2$	$\sin t$
−	$400+2t$	$-\cos t$
+	400	$-\sin t$

$$=\frac{3}{5}(200+t)^3+\frac{9}{5}\left[(200+t)^2\sin t+(400+2t)\cos t-400\sin t\right]$$

$$y(t)=\frac{1}{(200+t)^2}\left[\frac{3}{5}(200+t)^3+\frac{9}{5}\left((200+t)^2\sin t+(400+2t)\cos t-400\sin t\right)+C\right]$$

$$=\frac{3}{5}(200+t)+\frac{9}{5}\sin t+\frac{18}{5}\frac{\cos t}{200+t}-\frac{720\sin t}{(200+t)^2}+\frac{C}{(200+t)^2}$$

$$y(0)=5$$

$$5=120+\frac{18}{5}\frac{1}{200}+\frac{C}{200^2}$$

$$\Rightarrow \boxed{C=-4,600,720}$$

$$\boxed{y(t)=120+\frac{3}{5}t+\frac{9}{5}\sin t+\frac{18}{5}\frac{\cos t}{200+t}-\frac{720\sin t}{(200+t)^2}-\frac{4,600,720}{(200+t)^2}}$$

$$V(t)=600+3t=1500$$

$$\boxed{t=300\ \ hrs}$$

$$y(300)=120+180+\frac{9}{5}\sin(300)+\frac{18}{5}\frac{\cos 300}{500}-\frac{720\sin 300}{500^2}-\frac{4,600,720}{500^2}$$

$$\boxed{=279.797\ \ lbs}$$

The amount of salt in the full tank is $279.797\ \ lbs$

Exercise

A tank contains 100 *gal* of pure water. At time zero, a sugar-water solution containing 0.2 *lb* of sugar per gal enters the tank at a rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the sugar solution to leave the tank at 3 *gal/min*. Assume that the solution in the tank is kept perfectly mixed at all times.

a) What will be the sugar content in the tank after 20 *minutes*?

b) How long will it take the sugar content in the tank to reach 15 *lb*?

c) What will be the eventual sugar content in the tank?

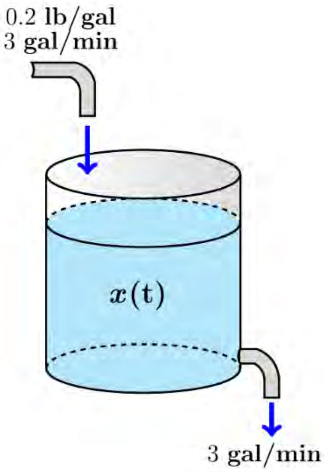
Solution

a) $Rate\ in=3\frac{gal}{min}\times 0.2\frac{lb}{gal}$

$$=0.6\quad\frac{lb}{min}$$

$$Rate\ out=3\frac{gal}{min}\times\frac{x(t)}{100}\frac{lb}{gal}$$

$$=\frac{3x(t)}{100}\quad\frac{lb}{min}$$



$$\frac{dx}{dt}=0.6-\frac{3x}{100}$$

$$x'+\frac{3}{100}x=0.6$$

$$e^{\int\frac{3}{100}dt}=e^{0.03t}$$

$$\int 0.6e^{0.03t}\,dt=\frac{0.6}{0.03}e^{.03t}$$

$$= 20e^{.03t}$$

$$x(t) = \frac{1}{e^{.03t}}\left(20e^{.03t} + C\right)$$

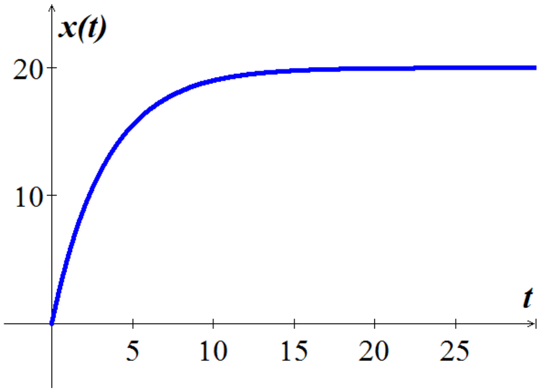
$$x(t) = 20 + Ce^{-.03t}$$

$$x(t=0) = 20 + Ce^{-.03(0)}$$

$$0 = 20 + C$$

$$\rightarrow \boxed{C = -20}$$

$$\boxed{x(t) = 20 - 20e^{-.03t}}$$



$$x(20) = 20 - 20e^{-.03(20)}$$

$$\boxed{\approx 9.038 \ \ lb}$$

b) $15 = 20 - 20e^{-.03t}$

$$-5 = -20e^{-.03t}$$

$$e^{-.03t} = \frac{5}{20}$$

$$-.03t = \ln\frac{1}{4}$$

$$t = \frac{\ln\frac{1}{4}}{-.03}$$

$$\boxed{\approx 46 \ \ min}$$

c) $t \to \infty$

$$\rightarrow \boxed{C = -20}$$

$$\boxed{x(t) \ \rightarrow \ 20}$$

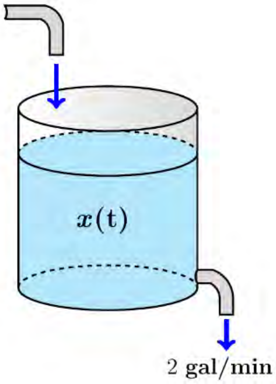
Exercise

A tank initially contains 50 *gal* of sugar water having a concentration of 2 *lb*. of sugar for each gal of water. At time zero, pure water begins pouring into the tank at a rate of 2 *gal* per *minute*. Simultaneously, a drain is opened at the bottom of the tank so that the volume of sugar-water solution in the tank remains constant.

a) How much sugar is in the tank after 10 *minutes*?

b) How long will it take the sugar content in the tank to dip below 20 *lb*.?

c) What will be the eventual sugar content in the tank?



Solution

$x(t)$ represents the number of pounds of sugar.

a) *Rate in* $= 0$

$$Rate\ out = 2\frac{gal}{min} \times \frac{x(t)}{50}\frac{lb}{gal}$$

$$= \frac{x(t)}{25} \;\; \frac{lb}{min}$$

$$\frac{dx}{dt} = 0 - \frac{x}{25}$$

$$x(t) = Ae^{-t/25}$$

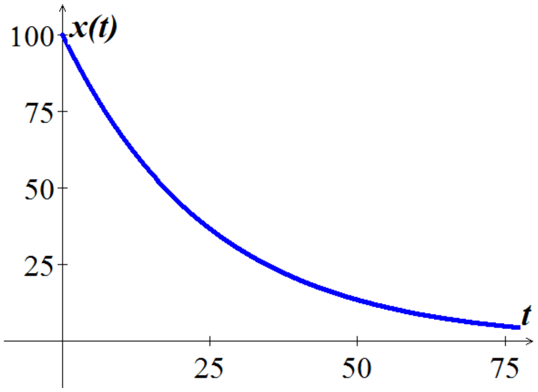
The initial condition: $x(0) = 50 gal \times 2\frac{lb}{gal} = 100\ lb$

$$A = 100$$

$$x(t) = 100e^{-.04t}$$

$$x(t = 10) = 100e^{-.04(10)}$$

$$\underline{= 67.032\ \ lb}\,|$$



b) $x(t) = 100e^{-.04t} = 20$

$$e^{-.04t} = .2$$

$$-.04t = \ln(.2)$$

$$t = \frac{\ln(.2)}{-.04}$$

$$\underline{\approx 40.236\ \ min}\,|$$

c) $x(t) = \lim\limits_{t\to\infty} 100e^{-.04t}$

$$\underline{= 0}\,|$$

Exercise

Suppose that in the cascade tank 1 initially 100 *gal* of pure ethanol and tank 2 initially contains 100 *gal* of pure water. Pure water flows into tank 1 at 10 *gal/min*, and the other two flow rates are also 10 *gal/min*.

a) Find the amounts $x(t)$ and $y(t)$ of ethanol in the two tanks at time $t \ge 0$.

b) Find the maximum amount of ethanol ever in tank 2.

Solution

a) The initial value problem $\frac{dx}{dt} = -\frac{x}{10}, \quad x(0) = 100$

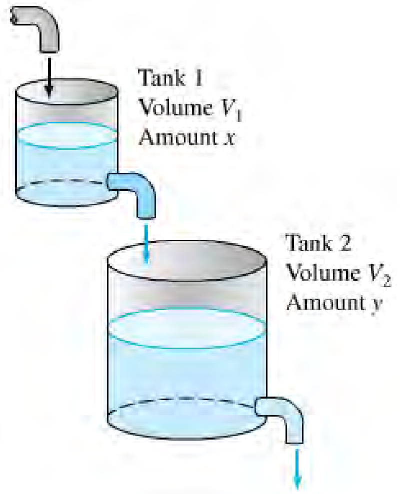
For ***Tank*** 1:

$$\frac{1}{x}dx = -\frac{1}{10}dt$$

$$\int \frac{1}{x}dx = -\frac{1}{10}\int dt$$

$$\ln|x| = -\frac{1}{10}t + C$$

$$x = e^{-t/10+C}$$



$$x = e^C e^{-t/10}$$

$$x = Ae^{-t/10}$$

$$100 = Ae^{-0/10}$$

$$\underline{100 = A}$$

$$\boxed{x(t) = 100e^{-t/10}}$$

The initial value problem $\frac{dy}{dt} = \frac{x}{10} - \frac{y}{10}, \quad y(0) = 0$

For ***Tank*** 2:

$$\frac{dy}{dt} = \frac{100e^{-t/10}}{10} - \frac{y}{10}$$

$$= 10e^{-t/10} - \frac{y}{10}$$

$$\frac{dy}{dt} + \frac{y}{10} = 10e^{-t/10}$$

$$e^{\int \frac{1}{10} dt} = e^{t/10}$$

$$\int 10e^{-t/10} e^{t/10}\, dt = 10\int dt$$

$$= 10t$$

$$y(t) = \frac{1}{e^{t/10}}(10t + C)$$

$$y(t = 0) = \frac{1}{e^{0/10}}(10(0) + C)$$

$$\to \underline{C = 0}$$

$$y(t) = \frac{1}{e^{t/10}}(10t)$$

$$\boxed{y(t) = 10te^{-t/10}}$$

b) The maximum value of y occurs when

$$y'(t) = 10e^{-t/10} - te^{-t/10} = 0$$

$$(10 - t)e^{-t/10} = 0$$

$$10 - t = 0 \quad \to \quad \underline{t = 10}$$

Thus when t = 10,

$$y_{\text{max}} = 10(10)e^{-10/10}$$

$$= 100e^{-1}$$

$$\boxed{\approx 36.79 \ gal}$$