Section 2.9 – Rank and the Fundamental Matrix Spaces

The **R**educed **R**ow **E**chelon **F**orm (***rref***) is a matrix (R) with each pivot column has only one nonzero entry (the pivots which is always 1).

$$R = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{R = rref(A)}$$

Rank of a Matrix

The rank of a matrix A (***m*** by ***n***) is the number of ***nonzero rows*** in the row-reduced echelon form of A (it is the number of pivot). The common dimension of the row space and column space of a matrix A is called the ***rank*** of A and is denoted by

$$\boxed{rank(A) = r}$$

Note:

The rank of a matrix is well defined due to the uniqueness of the row-reduced echelon form. No matter what sequence of elementary row operations is performed to put the given matrix in row-reduced echelon form; there will always be the same number of nonzero rows.

Theorem

The row space and column space of a matrix A have the same dimension

The objective is to connect ***rank*** and ***dimension***.

- The ***rank*** of a matrix is the number of pivots.
- The ***dimension*** of a subspace is the number of vectors in a basis.

✓ ***A has full row rank if every row has a pivot:*** $r = m$***. No zero in R.***

✓ ***A has full column rank if every column has a pivot:*** $r = n$***. No free variables.***

Example

Find the rank of $A=\begin{bmatrix}1 & -1 & 2 & 1\\ 0 & 1 & 1 & -2\\ 1 & -3 & 0 & 5\end{bmatrix}$

Solution

Use the calculator $R=rref\,(A)$

$$R=\begin{bmatrix}1 & 0 & 3 & -1\\ 0 & 1 & 1 & -2\\ 0 & 0 & 0 & 0\end{bmatrix}$$

The matrix R has 2 nonzero rows, therefore the $rank(A)=2$

Example

The columns of A are dependent. $Ax=0$ has a nonzero solution.

$$Ax=\begin{bmatrix}1 & 0 & 3\\ 2 & 1 & 5\\ 1 & 0 & 3\end{bmatrix}\begin{bmatrix}-3\\ 1\\ 1\end{bmatrix} \quad \Rightarrow \quad -3\begin{pmatrix}1\\ 2\\ 1\end{pmatrix}+1\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}+1\begin{pmatrix}3\\ 5\\ 3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

The rank of A is only $r = 2$.
Independent columns would give full column rank $r = n = 3$.

- The columns of A are independent exactly when the rank is $r = n$. There are $\boldsymbol{n}$ pivots and no free variables. Only $\boldsymbol{x} = \mathbf{0}$ is the nullspace.

Example

When all rows are multiply of one pivot row, the rank is $r=1$:

$$\begin{bmatrix}1 & 3 & 4\\ 2 & 6 & 8\end{bmatrix}, \quad \begin{bmatrix}0 & 3\\ 0 & 5\end{bmatrix}, \quad \begin{bmatrix}5\\ 2\end{bmatrix}, \quad [6]$$

Solution

The row-reduced echelon form $R=rref\,(A)$:

$$\begin{bmatrix}1 & 3 & 4\\ 0 & 0 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \quad \begin{bmatrix}1\\ 0\end{bmatrix}, \quad [1]$$

These matrices have only one pivot.

Dimension *Theorem* for Matrices

If A is a matrix with n columns, then $\boxed{rank(A) + nullity(A) = n}$

Theorem

If A is an $m \times n$ matrix, then

- $rank(A)$ = the number of leading variables in the general solution of $A\mathbf{x} = \mathbf{0}$
- $nullity(A)$ = the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$

Theorem

If A is any matrix, then $rank(A) = rank\left(A^T\right)$

- $Ax = 0$ ***has*** $n - r$ ***free variables and special solutions: n columns minus r pivot columns. The null matrix N has*** $n - r$ ***columns (the special solutions).***
- ***The particular solution solves:*** $Ax_p = b$
- ***Full column rank*** $R = \begin{bmatrix} n \text{ by } n \text{ identity matrix} \\ m - n \text{ rows of zeros} \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$

The reduced row echelon form looks like:

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{array}{l} \boldsymbol{r} \text{ pivot rows} \\ \boldsymbol{m - r} \text{ zero rows} \end{array}$$

The pivot variables in the $n - r$ special columns come by changing F to $-F$:

Nullspace matrix: $$N = \begin{pmatrix} -F \\ I \end{pmatrix} \begin{array}{l} \boldsymbol{r} \text{ pivot variables} \\ \boldsymbol{n - r} \text{ free variables} \end{array}$$

- Every matrix A with ***full column rank*** ($r = n$) has all these properties:
  1. All columns of A are pivot columns
  2. There are no free variables or special solutions.
  3. The nullspace $NS(A)$ contains only the zero vector $\boldsymbol{x} = \mathbf{0}$.
  4. If $A\vec{x} = b$ has a solution (might not) then it has only one solution.

Example

Suppose A is a square invertible matrix, $m = n = r$. What are $\vec{x}_p$ and $\vec{x}_n$ **?**

Solution

The particular solution is the one and only solution $A^{-1}b$.
There are no special solutions or free variables. $R = I$ has no zero rows.
The only vector in the null space is $x_n = 0$.

The complete solution is

$$\begin{aligned}\vec{x} &= \vec{x}_p + \vec{x}_n \\ &= A^{-1}b + 0 \\ &\boxed{= A^{-1}b}\end{aligned}$$

Example

Compute $N(A)$ for $A : \mathbf{R}^2 \to \mathbf{R}^3$ given by $A = (x + y,\ x,\ 2x - y)$

Solution

To find $N(A)$, we must solve the equation $A(x,\ y) = (0,\ 0,\ 0)$

$$\begin{pmatrix} x+y \\ x \\ 2x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} x + y = 0 \Rightarrow \boxed{y = 0} \\ \qquad\quad \boxed{x = 0} \end{array}$$

Thus $NS(A) = \{0\}$, the set that consists solely of the zero vector.

- If $Ax = 0$ has more unknowns than equations (more columns than rows) then it has nonzero solutions. There must be free columns, without pivots.

For an m x n matrix of rank r:

Fundamental Space	***Subspace of***	***Dimension***
Nullspace	$\mathbb{R}^n$	$n - r$
Column Space	$\mathbb{R}^m$	r
Row space	$\mathbb{R}^n$	r
Left nullspace	$\mathbb{R}^m$	$m - r$

Definition

If W is a subspace of $\boldsymbol{R}^n$ that are orthogonal to every vector in W is called orthogonal complement of W and is denoted nu the symbol $W^{\perp}$. $N(A)^{\perp}$ is exactly the row space $C\left(A^T\right)$

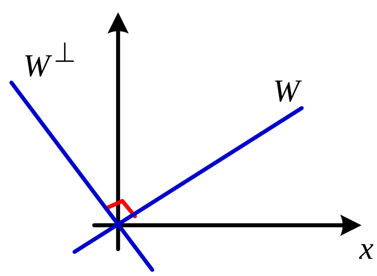
Fundamental Theorem of Linear Algebra

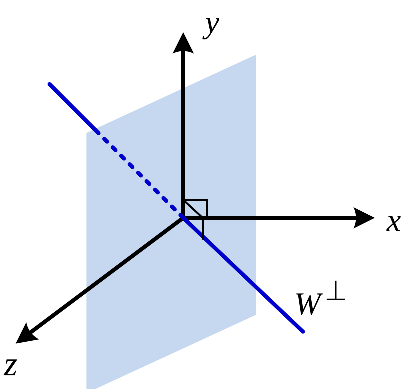
The nullspace is the orthogonal complement of the row space $\left(in\ \boldsymbol{R}^n\right)$.

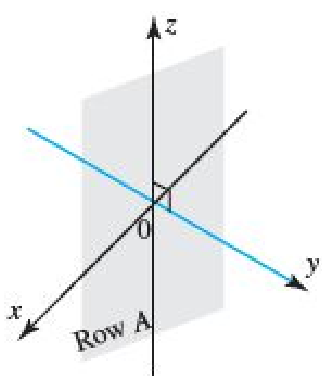
The left nullspace is the orthogonal complement of the column space $\left(in\ \boldsymbol{R}^m\right)$.

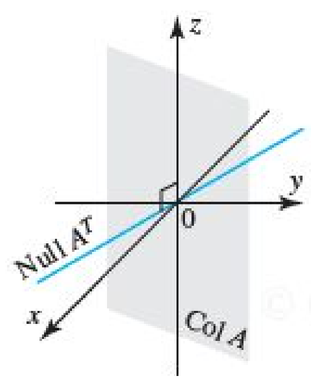
If W is a subspace of $\boldsymbol{R}^n$

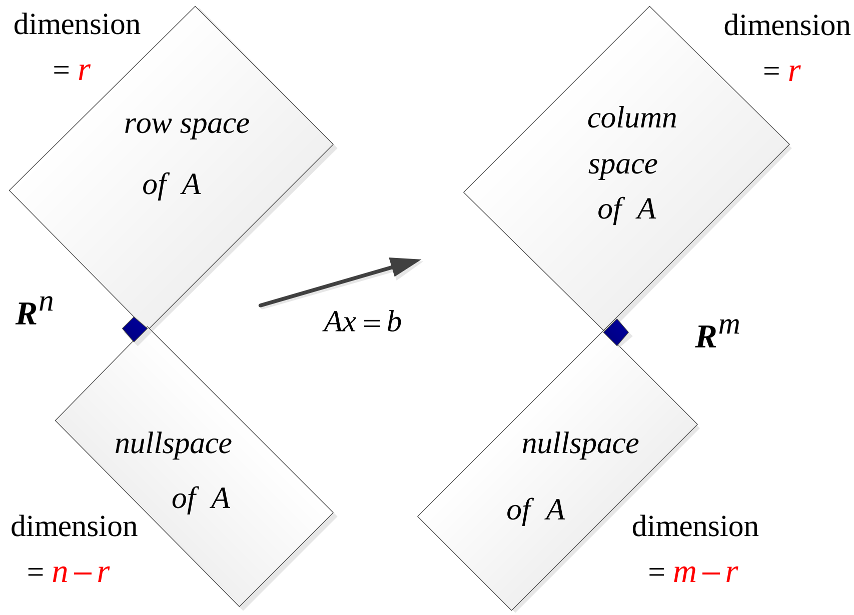
- $W^{\perp}$ is a subspace of $\boldsymbol{R}^n$
- The only vector common to W and $W^{\perp}$ is 0.
- The orthogonal complement of $W^{\perp}$ is W.







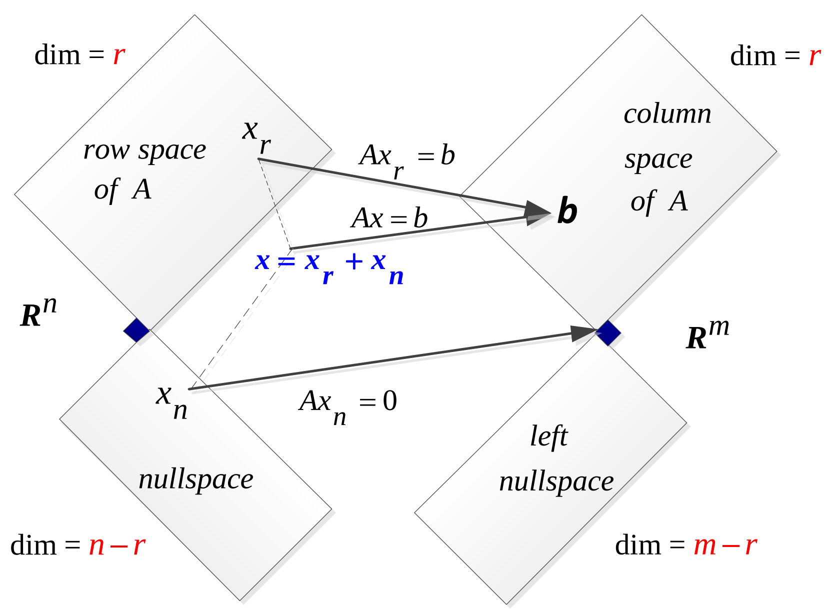




Two pairs of orthogonal subspaces.

Combining Bases from Subspaces

- Any n linearly independent vectors in $\boldsymbol{R}^n$ must span $\boldsymbol{R}^n$. They are basis. Any n vectors that span $\boldsymbol{R}^n$ must be independent. They are a basis.
- If the n columns of $\boldsymbol{A}$ are independent, they span $\boldsymbol{R}^n$, So $A\vec{x} = \vec{b}$ is solvable,
- If the n columns span $\boldsymbol{R}^n$, they are independent. So $A\vec{x} = \vec{b}$ has only one solution.



- When the orthogonal complement of a subspace S is defined to be the subspace whose vectors pairs to zero with the vectors in S. The larger the S is, the more restriction $S^{\perp}$ has, and hence the smaller $S^{\perp}$ is.

Theorem – Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

a) A is invertible
b) $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution
c) The reduced row echelon form of A is I_n
d) A is expressible as a product of elementary matrices
e) $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $n \times 1$ matrix $\boldsymbol{b}$
f) $A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution for every $n \times 1$ matrix $\boldsymbol{b}$
g) $\det(A) \neq 0$
h) The column vectors of A are linearly independent
i) The row vectors of A are linearly independent
j) The column vectors of A span $\boldsymbol{R}^n$
k) The row vectors of A span $\boldsymbol{R}^n$
l) The column vectors of A form a basis for $\boldsymbol{R}^n$
m) The row vectors of A form a basis for $\boldsymbol{R}^n$
n) A has a rank n.
o) A has nullity 0.
p) The orthogonal complement of the null space of A is $\boldsymbol{R}^n$
q) The orthogonal complement of the row space of A is $\{0\}$

Exercises *Section* 2.9 – Rank and the Fundamental Matrix Spaces

1. Verify that $rank(A) = rank\left(A^T\right)$

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

2. Find the rank and nullity of the matrix; then verify that the values obtained satisfy $rank(A) + N(A) = n$

a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

3. If *A* is an $m \times n$ matrix, what is the largest possible value for its rank and the smallest possible value of the nullity of *A*.

4. Discuss how the rank of *A* varies with *t*.

a) $A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{bmatrix}$

5. Are there values of *r* and *s* for which

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

Has rank 1? Has rank 2? If so, find those values.

6. Find the row reduced form R and the rank r of A (those depend on c).
Which are the pivot columns of A? Which variables are free? What are the special solutions and the nullspace matrix N (always depending on c)?

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & c \end{bmatrix} \quad and \quad A = \begin{bmatrix} c & c \\ c & c \end{bmatrix}$$

7. Find the row reduced form R and the rank r of A (those depend on c).
Which are the pivot columns of A? Which variables are free? What are the special solutions and the nullspace matrix N (always depending on c)?

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad and \quad A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$$

8. If A has a rank r, then it has an r by r sub-matrix S that is invertible. Remove $m-r$ rows and $n-r$ columns to find an invertible sub-matrix S inside each A (you could keep the pivot rows and pivot columns of A).

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

9. Suppose that column 3 of 4 x 6 matrix is all zero. Then x_3 must be a ______ variable. Give one special solution for this matrix.

10. Fill in the missing numbers to make A rank 1, rank 2, rank 3. (your solution should be 3 matrices)

$$A = \begin{pmatrix} & -3 & \\ 1 & 3 & -1 \\ & 9 & -3 \end{pmatrix}$$

11. Fill out these matrices so that they have rank 1:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{pmatrix} \quad B = \begin{pmatrix} 2 & & \\ 1 & & \\ 2 & 6 & -3 \end{pmatrix} \quad M = \begin{pmatrix} a & b \\ c & \end{pmatrix}$$

12. Suppose A and B are n by n matrices, and $AB = I$. Prove from $rank(AB) \le rank(A)$ that the $rank(A) = n$. So A is invertible and B must be its two-sided inverse. Therefore $BA = I$ (which is not so obvious!).

13. Every m by n matrix of rank r reduces to (m by r) times (r by n):

$$A = \text{(pivot columns of } A\text{) (first } r \text{ rows of } R\text{)} = (COL)(ROW)^T$$

Write the 3 by 4 matrix $A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{pmatrix}$ as the product of the 3 by 2 from the pivot columns and the 2 by 4 matrix from R.

14. Suppose R is m by n matrix of rank r, with pivot columns first: $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

a) What are the shapes of those 4 blocks?

b) Find the right-inverse B with $RB = I$ if $r = m$.

c) Find the right-inverse C with $CR = I$ if $r = n$.

d) What is the reduced row echelon form of R^T (with shapes)?

e) What is the reduced row echelon form of $R^T R$ (with shapes)?
Prove that $R^T R$ has the same nullspace as R. Then show that $A^T A$ always has the same nullspace as A (a value fact).

f) Suppose you allow elementary column operations on $\boldsymbol{A}$ as well as elementary row operations (which get to $\boldsymbol{R}$). What is the "row-and-column reduced form" for an m by n matrix of rank $\boldsymbol{r}$?

15. True or False (check addition or give a counterexample)

a) The symmetric matrices in M $\left(with\ A^T = A\right)$ from a subspace.

b) The skew-symmetric matrices in M $\left(with\ A^T = -A\right)$ from a subspace.

c) The un-symmetric matrices in M $\left(with\ A^T \neq A\right)$ from a subspace.

d) Invertible matrices

e) Singular matrices

16. Let $A = \begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 2 & 4 & -3 & 7 & 0 \\ 3 & 6 & -5 & 10 & -2 \\ 5 & 10 & -9 & 16 & 0 \end{pmatrix}$

a) Reduce A to row-reduced echelon form.

b) What is the rank of A?

c) What are the pivots?

d) What are the free variables?

e) Find the special solutions. What is the nullspace $N(A)$?

f) Exhibit an r x r submatrix of A which is invertible, where $r = rank(A)$. (An r x r submatrix of A is obtained by keeping r rows and r columns of A)

17. Let $A = \begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 2 & 1 & 0 & 0 & -15 \\ 6 & -1 & -8 & -1 & -47 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix}$

a) Reduce A to row-reduced echelon form.

b) What is the rank of A?

c) What the pivots?

d) What are the free variables?

e) Find the special solutions. What is the nullspace $N(A)$?

f) Give the complete solution to $Ax = b,$ *where* $b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

18. Let $A = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

a) Reduce A to row-reduced echelon form.

b) What is the rank of A?

c) What the pivots?

d) What are the free variables?

e) Find the special solutions.

f) What is the nullspace $N(A)$?

19. Let $A = \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix}$

a) Reduce A to row-reduced echelon form.

b) What is the rank of A?

c) What the pivots?

d) What are the free variables?

e) Find the special solutions.

f) What is the nullspace $N(A)$?

20. The 3 by 3 matrix A has rank 2.

$$Ax=b \quad is \quad \begin{aligned} x_1+2x_2+3x_3+5x_4&=b_1\\ 2x_1+4x_2+8x_3+12x_4&=b_2\\ 3x_1+6x_2+7x_3+13x_4&=b_3 \end{aligned}$$

a) Reduce $\begin{bmatrix} A & b \end{bmatrix}$ to $\begin{bmatrix} U & c \end{bmatrix}$, so that $A\vec{x}=b$ becomes triangular system $U\vec{x}=c$.

b) Find the condition on $\left(b_1,\ b_2,\ b_3\right)$ for $A\vec{x}=b$ to have a solution

c) Describe the column space of A. Which plane in $\boldsymbol{R}^3$?

d) Describe the nullspace of A. Which special solutions in $\boldsymbol{R}^4$?

e) Find a particular solution to $Ax=(0,\ 6,\ -6)$ and then complete solution.

21. Find the special solutions and describe the complete solution to $Ax=0$ for

A_1 = 3 by 4 zero matrix $\qquad A_2=\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \qquad A_3=\begin{bmatrix} A_1 & A_2 \end{bmatrix}$

Which are the pivot columns? Which are the free variables? What is the R (Reduced Row Echelon matrix) in each case?

22. Create a 3 by 4 matrix whose special solutions to $A\vec{x}=0$ are s_1 *and* s_2:

$$s_1=\begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad and \quad s_2=\begin{pmatrix} -2 \\ 0 \\ -6 \\ 1 \end{pmatrix}$$

You could create the matrix A in row reduced form R. Then describe all possible matrices A with the required Nullspace $N(A)=$ all combinations of s_1 *and* s_2.

23. The plane $x-3y-z=12$ is parallel to the plane $x-3y-z=0$. One particular point on this plane is $(12,\ 0,\ 0)$. All points on the plane have the form (fill the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} \\ 0 \\ 0 \end{bmatrix}+y\begin{bmatrix} \\ 1 \\ 0 \end{bmatrix}+z\begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}$$

24. Construct a matrix whose column space contains (1, 1, 5) and (0, 3, 1) and whose Nullspace contains $(1,\ 1,\ 2)$.

25. Construct a matrix whose column space contains (1, 1, 0) and (0, 1, 1) and whose Nullspace contains (1, 0, 1) and (0, 0, 1).

26. Construct a matrix whose column space contains (1, 1, 1) and whose Nullspace contains $(1, 1, 1, 1)$.

27. How is the Nullspace N(C) related to the spaces N(A) and N(B), if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

28. Why does no 3 by 3 matrix have a nullspace that equals its column space?

29. If $AB = 0$ then the column space B is contained in the _______ of A. Give an example of A and B.

30. True or false (with reason if true or example to show it is false)

a) A square matrix has no free variables.

b) An invertible matrix has no free variables.

c) An m by n matrix has no more than $\boldsymbol{n}$ pivot variables.

d) An m by n matrix has no more than $\boldsymbol{m}$ pivot variables.

31. Suppose an m by n matrix has r pivots. The number of special solutions is ______.
The Nullspace contains only $x = 0$ when $r =$ _______.
The column space is all of $\boldsymbol{R}^m$ when $r =$ ______.

32. Find the complete solution in the form $\vec{x}_p + \vec{x}_n$ to these full rank system:

$a)\ x + y + z = 4$ $\qquad\qquad$ $b)\ \begin{aligned} x + y + z &= 4 \\ x - y + z &= 4 \end{aligned}$

33. Find the complete solution in the form $\vec{x}_p + \vec{x}_n$ to the system:

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

34. If $\boldsymbol{A}$ is 3 x 7 matrix, its largest possible rank is ________. In this case, there is a pivot in every ________ of $\boldsymbol{U}$ and $\boldsymbol{R}$, the solution to $A\vec{x} = \vec{b}$ _________ (always exists or is unique), and the column space of $\boldsymbol{A}$ is _________. Construct an example of such a matrix $\boldsymbol{A}$.

35. If $\boldsymbol{A}$ is 6 x 3 matrix, its largest possible rank is ________. In this case, there is a pivot in every ________ of $\boldsymbol{U}$ and R, the solution to $A\vec{x} = \vec{b}$ _________ (always exists or is unique), and the nullspace of $\boldsymbol{A}$ is _________. Construct an example of such a matrix $\boldsymbol{A}$.

36. Find the rank of $A, A^T A$ and AA^T for $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix}$

37. Explain why these are all false:

a) The complete solution is any linear combination of $\vec{x}_p$ *and* $\vec{x}_n$.

b) A system $A\vec{x} = \vec{b}$ has at most one particular solution.

c) The solution $\vec{x}_p$ with all free variables zero is the shortest solution (minimum length $\|\vec{x}\|$). Find a 2 by 2 counterexample.

d) If A is invertible there is no solution $\vec{x}_n$ in the null space.

38. Write down all known relation between r and m and n if $A\vec{x} = \vec{b}$ has

a) No solution for some ***b***.

b) Infinitely many solutions for every ***b***.

c) Exactly one solution for some ***b***, no solution for other ***b***.

d) Exactly one solution for every ***b***.

39. Find a basis for its row space, find a basis for its column space, and determine its rank

$$a)\ \begin{bmatrix} 0 & 2 & -3 & 4 & 1 & 2 & 1 & 7 \\ 0 & 0 & 3 & -2 & 0 & 4 & -5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 6 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad b)\ \begin{bmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{bmatrix}$$

40. Find a basis for the row space, find a basis for the null space, find $\dim RS$, find $\dim NS$, and verify $\dim RS + \dim NS = n$

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{bmatrix}$$

41. Determine if ***b*** lies in the column space of the given matrix. If it does, express ***b*** as linear combination of the column.

$$\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

42. Find the transition matrix from B to C and find $[\boldsymbol{x}]_c$

a) $B = \{(3, 1), (-1, -2)\}, \quad C = \{(1, -3), (5, 0)\}, \quad [x]_B = [-1 \quad -2]^T$

b) $B = \{(1, 1, 1),\ (-2, -1, 0),\ (2, 1, 2)\}, \quad C = \{(-6, -2, 1),\ (-1, 1, 5),\ (-1, -1, 1)\},$

$[\boldsymbol{x}]_B = [-3 \quad 2 \quad 4]^T$

43. Does A and A^T have the same number of pivots.

For the given matrix A, which is given in row reduction echelon form

- ***a)*** What is the rank of A?
- ***b)*** What is the dimension of A?
- ***c)*** What are the pivots?
- ***d)*** What are the free variables?
- ***e)*** Find the special (homogeneous) solutions.
- ***f)*** What is the nullspace $N(A)$?
- ***g)*** Find the particular solution $Ax = b$
- ***h)*** Give the complete solution.

44. $A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \textit{where} \;\; b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

45. $A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \textit{where} \;\; b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

46. $A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \textit{where} \;\; b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

47. $A = \begin{pmatrix} 1 & 0 & 0 & \frac{13}{11} \\ 0 & 1 & 0 & -\frac{17}{11} \\ 0 & 0 & 1 & \frac{6}{11} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \textit{where} \;\; b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

48. $A = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \textit{where} \;\; b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

49. $A = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ *where* $b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$