

Section 2.2 – Rules of Differentiation

Notations for the Derivative

The derivative of $y = f(x)$ may be written in any of the following ways:

1st derivative	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$	$D_x[y]$
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Derivative of a *constant Function*

If f has the constant value $f(x) = c$

$$\frac{d}{dx}[c] = f'(c) = 0 \quad c \text{ is constant}$$

Proof

Let $f(x) = c$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{c - c}{h} \\ &= 0 \end{aligned}$$

$$\text{So, } \frac{d}{dx}[c] = 0$$

Example

Find the derivative

$$a) \quad f(x) = 9$$

$$f' = 0$$

$$b) \quad h(t) = \pi$$

$$D_t[h(t)] = 0$$

Power Rule

$$f(x) = x^n \Rightarrow f'(x) = nx^{n-1} \quad n \text{ is any real number}$$

Proof

Let $f(x) = x^n$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \dots + h^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h + \dots + h^{n-1} \\ &= nx^{n-1} \end{aligned}$$

Example

Find the derivative of: a) x^3 b) $x^{2/3}$ c) $\frac{1}{x^4}$ d) $x^{\sqrt{2}}$ e) $\sqrt{x^{2+\pi}}$

Solution

a) $y = x^3$

$$\begin{aligned} \frac{dy}{dx} &= 3x^{3-1} \\ &= 3x^2 \end{aligned}$$

b) $y = x^{2/3}$

$$\begin{aligned} y' &= \frac{2}{3}x^{2/3-1} \\ &= \frac{2}{3}x^{-1/3} \end{aligned}$$

c) $y = \frac{1}{x^4} = x^{-4}$

$$\begin{aligned} y' &= -4x^{-4-1} \\ &= -4x^{-5} \\ &= -\frac{4}{x^5} \end{aligned}$$

$$d) \quad D_x \left(x^{\sqrt{2}} \right) = \underline{\sqrt{2} x^{\sqrt{2}-1}}$$

$$e) \quad y = \left(x^{2+\pi} \right)^{1/2} = x^{(2+\pi)/2}$$

$$y' = \left(\frac{2+\pi}{2} \right) x^{1+\pi/2-1}$$

$$= \underline{\frac{1}{2}(2+\pi)\sqrt{x^\pi}}$$

Derivative Constant Multiple Rule

If f is a differentiable function of x , and c is a real number (constant), then $\frac{d}{dx}(cf) = c \frac{df}{dx}$

In particular, if n is any real number, then $\frac{d}{dx}(cx^n) = cnx^{n-1}$

Proof

$$\frac{d}{dx}(cf) = \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h}$$

$$= c \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Factor c

$$= c \frac{df}{dx}$$

Example

If $y = 8x^4$, find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = 8(4x^3) = \underline{32x^3}$$

Example

If $y = -\frac{3}{4}x^{12}$, find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = -\frac{3}{4}(12x^{11})$$

$$= \underline{-9x^{11}}$$

Sum or Difference Rule

The derivative of the sum or difference of two differentiable functions is the sum or difference of their derivatives.

$$\begin{aligned}\frac{d}{dx}(u+v) &= \frac{du}{dx} + \frac{dv}{dx} & \frac{d}{dx}(u-v) &= \frac{du}{dx} - \frac{dv}{dx} \\ &= u' + v' & &= u' - v'\end{aligned}$$

Proof

$$f(x) = u(x) + v(x)$$

$$\begin{aligned}\frac{d}{dx}[u(x) + v(x)] &= \lim_{h \rightarrow 0} \frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h} \\ &= \lim_{h \rightarrow 0} \frac{u(x+h) + v(x+h) - u(x) - v(x)}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h} \right] \\ &= \lim_{h \rightarrow 0} \left[\frac{u(x+h) - u(x)}{h} \right] + \lim_{h \rightarrow 0} \left[\frac{v(x+h) - v(x)}{h} \right] \\ &= \frac{du}{dx} + \frac{dv}{dx}\end{aligned}$$

Example

Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1) \\ &= 3x^2 + \frac{8}{3}x - 5 + 0 \\ &= \underline{3x^2 + \frac{8}{3}x - 5}\end{aligned}$$

Example

Find the derivative of $y = x^{5/2} + x^3 + \frac{1}{2}x^2 + 4$

Solution

$$y' = \underline{\frac{5}{2}x^{3/2} + 3x^2 + x}$$

Example

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

Solution

$$y' = 4x^3 - 4x$$

$$y' = 0 \Rightarrow 4x^3 - 4x = 0$$

$$4x(x^2 - 1) = 0$$

$$x = 0, \pm 1$$

The curve has horizontal tangents at $x = 0, 1$, and -1 .

The corresponding points on the curve are; $(0, 2)$, $(1, 1)$ and $(-1, 1)$

Second– and Higher–Order Derivatives

Notation for Higher-Order Derivatives							
1.	1st derivative	y'	y prime	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$	$D_x[y]$
2.	2nd derivative	y''	y double prime	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}[f(x)]$	$D_x^2[y]$
3.	3rd derivative	y'''	y triple prime	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}[f(x)]$	$D_x^3[y]$
4.	4th derivative	$y^{(4)}$		$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}[f(x)]$	$D_x^4[y]$
5.	nth derivative	$y^{(n)}$		$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}[f(x)]$	$D_x^n[y]$

Example

Find the first four derivatives of $y = x^3 - 3x^2 + 2$

Solution

$$y' = 3x^2 - 6x$$

$$y'' = 6x - 6$$

$$y''' = 6$$

$$y^{(4)} = 0$$

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \Rightarrow \boxed{f^{(n)}(x) = n! a_n}$$

Exercises Section 2.2 – Rules of Differentiation

Find the derivative of each function

1. $y = \frac{1}{x^3}$
2. $D_x \left(x^{4/3} \right)$
3. $y = \sqrt{z}$
4. $D_t (-8t)$
5. $y = \frac{9}{4x^2}$
6. $y = 6x^3 + 15x^2$
7. $y = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$
8. $p(t) = 12t^4 - 6\sqrt{t} + \frac{5}{t}$
9. $f(x) = \frac{x^3 + 3\sqrt{x}}{x}$
10. $y = \frac{x^3 - 4x}{\sqrt{x}}$
11. $f(x) = (4x^2 - 3x)^2$
12. $y = 3x(2x^2 + 5x)$
13. $y = 3(2x^2 + 5x)$
14. $y = (3x - 2)(2x + 3)$
15. $y = \frac{x^2 + 4x}{5}$
16. $y = \frac{3x^4}{5}$
17. $g(s) = \frac{s^2 - 2s + 5}{\sqrt{s}}$
18. $f(x) = \frac{x+1}{\sqrt{x}}$
19. $f(x) = 4x^{5/3} + 6x^{-3/2} - 11x$
20. $f(x) = \frac{2}{3}x^3 + \pi x^2 + 7x + 1$
21. $f(x) = \frac{x^5 - x^3}{15}$
22. $f(x) = x^{1/3} + 2x^{1/4} - 3x^{1/5}$
23. $f(t) = 3\sqrt[3]{t^2} - \frac{2}{\sqrt{t^3}}$
24. $f(t) = \sqrt{t} \left(5 - t - \frac{1}{3}t^2 \right)$
25. $f(x) = \frac{3}{5}x^{5/3} + \frac{5}{3}x^{-3/5}$
26. $f(x) = x^{23} - x^{-23}$

Find the *first* and *second* derivatives

27. $y = -x^3 + 3$
28. $y = 3x^7 - 7x^3 + 21x^2$
29. $y = 6x^2 - 10x - \frac{1}{x}$
30. $f(x) = \frac{1}{2}x^4 + \pi x^3 - 7x + 1$
31. $y = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$
32. $y = (2x - 3)(1 - 5x)$

Find the derivatives

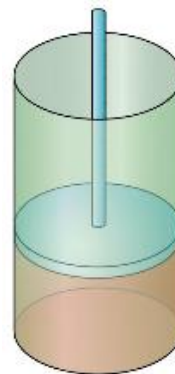
33. $f(x) = 3x^4 - 6x^3 + \frac{x^2}{8} + 5, \quad f^{(4)}(x)$
34. $f(x) = 3x^4 - 6x^3 + \frac{x^2}{8} + 5, \quad f^{(5)}(x)$
35. $f(x) = 2x^6 + 4x^4 - x + 2, \quad f^{(6)}(x)$
36. $f(x) = 4x^5 + 4x^4 + x^2 - 2, \quad f^{(5)}(x)$
37. $f(x) = 4x^5 + 4x^4 + x^2 - 2, \quad f^{(6)}(x)$
38. $f(x) = 4x^4 - 2x^3 + x + 2, \quad f^{(4)}(x)$

39. Find an equation for the line perpendicular to the tangent to the curve $y = x^3 - 4x + 1$ at the point $(2, 1)$.

40. If gas in a cylinder is maintained at a constant temperature T , the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2}$$

In which a , b , n , and R are constants. Find $\frac{dP}{dV}$



41. Show that if $(a, f(a))$ is any point on the graph of $f(x) = x^2$, then the slope of the tangent line at that point is $m = 2a$
42. Show that if $(a, f(a))$ is any point on the graph of $f(x) = bx^2 + cx + d$, then the slope of the tangent line at that point is $m = 2ab + c$
43. Let $f(x) = x^2$
- Show that $\frac{f(x) - f(y)}{x - y} = f'\left(\frac{x + y}{2}\right)$, for all $x \neq y$
 - Is this property true for $f(x) = ax^2$, where a is a nonzero real number?
 - Give a geometrical interpretation of this property.
 - Is this property true for $f(x) = ax^3$?