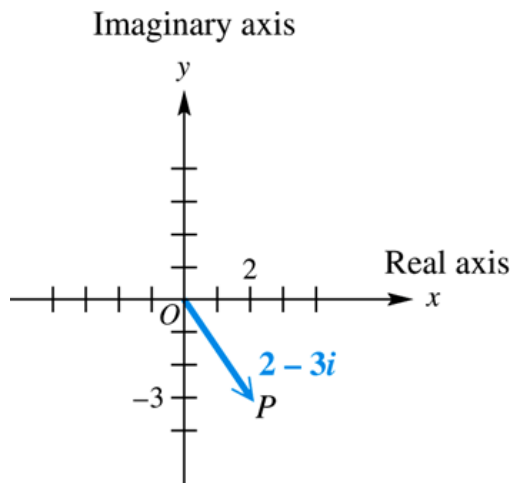


Section 3.7 – Trigonometric Form

$$\sqrt{-1} = i$$

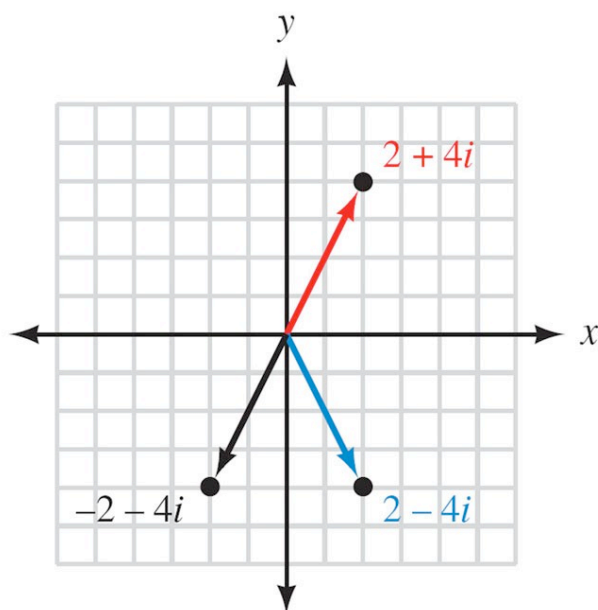
The graph of the complex number $x = yi$ is a vector (arrow) that extends from the origin out to the point (x, y)

- Horizontal axis: *real axis*
- Vertical axis: *imaginary axis*



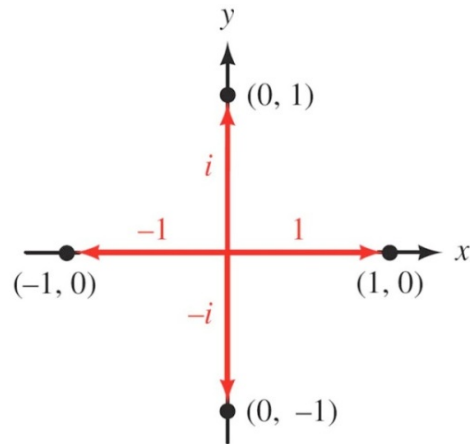
Example

Graph each complex number: $2 + 4i$, $-2 - 4i$, and $2 - 4i$



Example

Graph each complex number: 1 , i , -1 , and $-i$

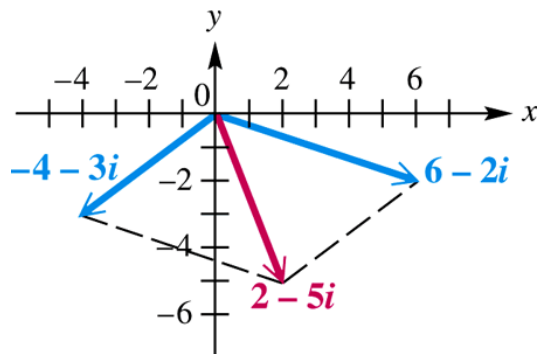


Example

Find the sum of $6 - 2i$ and $-4 - 3i$. Graph both complex numbers and their resultant.

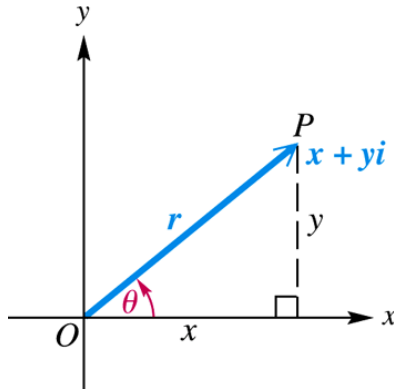
Solution

$$(6 - 2i) + (-4 - 3i) = 6 - 4 - 2i - 3i \\ = 2 - 5i$$



Definition

The *absolute value* or **modulus** of the complex number $z = x + yi$ is the distance from the origin to the point (x, y) . If this distance is denoted by r , then



$$\begin{aligned} r &= |z| = |x + yi| \\ &= \sqrt{x^2 + y^2} \end{aligned}$$

The **argument** of the complex number $z = x + yi$ denoted $\arg(z)$ is the smallest possible angle θ from the positive real axis to the graph of z .

$$\cos \theta = \frac{x}{r} \quad \Rightarrow \quad x = r \cos \theta$$

$$\sin \theta = \frac{y}{r} \quad \Rightarrow \quad y = r \sin \theta$$

$$\begin{aligned} z &= x + yi \\ &= r \cos \theta + (r \sin \theta) i \\ &= r(\cos \theta + i \sin \theta) \quad \rightarrow \text{is called the } \textit{trigonometric form} \text{ of } z. \end{aligned}$$

Definition

If $z = x + yi$ is a complex number in standard form then the **trigonometric form** for z is given by

$$z = r(\cos \theta + i \sin \theta) = r \operatorname{cis} \theta$$

Where r is the modulus or absolute value of z and

θ is the argument of z .

We can convert back and forth between standard form and trigonometric form by using the relationships that follow

$$\text{For } z = x + yi = r(\cos \theta + i \sin \theta) = r \operatorname{cis} \theta$$

$$r = \sqrt{x^2 + y^2}$$

$$\cos \theta = \frac{x}{r}, \quad \sin \theta = \frac{y}{r}, \quad \text{and} \quad \tan \theta = \frac{y}{x}$$

Example

Write $z = -1 + i$ in trigonometric form

Solution

The modulus r :

$$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$\cos \theta = \frac{x}{r} = \frac{-1}{\sqrt{2}}$$

$$\sin \theta = \frac{y}{r} = \frac{1}{\sqrt{2}}$$

$$\rightarrow \theta = 135^\circ$$

$$z = x + y i$$

$$= \sqrt{2}(\cos 135^\circ + i \sin 135^\circ)$$

$$= \sqrt{2} \operatorname{cis} 135^\circ$$

$$\text{In radians: } z = \sqrt{2} \operatorname{cis} \left(\frac{3\pi}{4} \right)$$

Example

Write $z = 2 \operatorname{cis} 60^\circ$ in rectangular form.

Solution

$$z = 2 \operatorname{cis} 60^\circ$$

$$= 2(\cos 60^\circ + i \sin 60^\circ)$$

$$= 2\left(\frac{1}{2} + i \frac{\sqrt{3}}{2}\right)$$

$$= 1 + i\sqrt{3}$$

Example

Express $2(\cos 300^\circ + i \sin 300^\circ)$ in rectangular form.

Solution

$$2(\cos 300^\circ + i \sin 300^\circ) = 2\left(\frac{1}{2} - i \frac{\sqrt{3}}{2}\right)$$

$$= 1 - i\sqrt{3}$$

Example

Find the modulus of each of the complex numbers $5i$, 7 , and $3 + 4i$

Solution

For $z = 5i$

$$= 0 + 5i$$

$$\begin{aligned} r &= |z| \\ &= \sqrt{0^2 + 5^2} \\ &= \underline{5} \end{aligned}$$

For $z = 7$

$$= 7 + 0i$$

$$\begin{aligned} r &= |z| \\ &= \sqrt{7^2 + 0^2} \\ &= \underline{7} \end{aligned}$$

For $3 + 4i$

$$\begin{aligned} \Rightarrow r &= \sqrt{3^2 + 4^2} \\ &= \underline{5} \end{aligned}$$

Product Theorem

If $r_1(\cos \theta_1 + i \sin \theta_1)$ and $r_2(\cos \theta_2 + i \sin \theta_2)$ are any two complex numbers, then

$$\left[r_1(\cos \theta_1 + i \sin \theta_1) \right] \left[r_2(\cos \theta_2 + i \sin \theta_2) \right] = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right]$$

$$(r_1 \operatorname{cis} \theta_1)(r_2 \operatorname{cis} \theta_2) = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

$$\boxed{(a + bi)(a - bi) = a^2 + b^2}$$

$$\boxed{(\sqrt{a} + \sqrt{b}i)(\sqrt{a} - \sqrt{b}i) = a + b}$$

Example

Find the product of $3(\cos 45^\circ + i \sin 45^\circ)$ and $2(\cos 135^\circ + i \sin 135^\circ)$. Write the result in rectangular form.

Solution

$$\begin{aligned} & \left[3(\cos 45^\circ + i \sin 45^\circ) \right] \left[2(\cos 135^\circ + i \sin 135^\circ) \right] \\ &= (3)(2) \left[\cos(45^\circ + 135^\circ) + i \sin(45^\circ + 135^\circ) \right] \\ &= 6(\cos 180^\circ + i \sin 180^\circ) \\ &= 6(-1 + i \cdot 0) \\ &= \underline{\underline{-6}} \end{aligned}$$

Quotient Theorem

If $r_1(\cos \theta_1 + i \sin \theta_1)$ and $r_2(\cos \theta_2 + i \sin \theta_2)$ are any two complex numbers, then

$$\frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$$

$$\frac{r_1 \operatorname{cis} \theta_1}{r_2 \operatorname{cis} \theta_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

Example

Find the quotient $\frac{10 \operatorname{cis}(-60^\circ)}{5 \operatorname{cis}(150^\circ)}$. Write the result in rectangular form.

Solution

$$\begin{aligned} \frac{10 \operatorname{cis}(-60^\circ)}{5 \operatorname{cis}(150^\circ)} &= \frac{10}{5} \operatorname{cis}(-60^\circ - 150^\circ) \\ &= 2 \operatorname{cis}(-210^\circ) \\ &= 2 [\cos(-210^\circ) + i \sin(-210^\circ)] \\ &= 2 \left[-\frac{\sqrt{3}}{2} + i \left(\frac{1}{2} \right) \right] \\ &= \underline{-\sqrt{3} + i} \end{aligned}$$

De Moivre's Theorem

If $r(\cos \theta + i \sin \theta)$ is a complex number, then

$$\left[r(\cos \theta + i \sin \theta) \right]^n = r^n (\cos n\theta + i \sin n\theta)$$

$$\boxed{(rcis\theta)^n = r^n (cisn\theta)}$$

Example

Find $(1 + i\sqrt{3})^8$ and express the result in rectangular form.

Solution

$$1 + i\sqrt{3} \Rightarrow \begin{cases} x = 1 \\ y = \sqrt{3} \end{cases}$$

$$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

$$\tan \theta = \frac{\sqrt{3}}{1} = \sqrt{3}$$

θ is in QI, that implies: $\theta = 60^\circ$

$$1 + i\sqrt{3} = 2cis60^\circ$$

Apply De Moivre's theorem:

$$\begin{aligned} (1 + i\sqrt{3})^8 &= (2cis60^\circ)^8 \\ &= 2^8 [cis(8 \cdot 60^\circ)] \\ &= 256 [cis(480^\circ)] && 480^\circ - 360^\circ = 120^\circ \\ &= 256 [cis(120^\circ)] \\ &= 256 \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2} \right) \\ &= \underline{-128 + 128i\sqrt{3}} \end{aligned}$$

n^{th} Root Theorem

For a positive integer n , the complex number $a + bi$ is an n^{th} root of the complex number $x + iy$ if

$$(a + bi)^n = x + yi$$

If n is any positive integer, r is a positive real number, and θ is in degrees, then the nonzero complex number $r(\cos \theta + i \sin \theta)$ has exactly n distinct n th roots, given by

$$\sqrt[n]{r}(\cos \alpha + i \sin \alpha) \text{ or } \sqrt[n]{r} \operatorname{cis} \alpha$$

Where $\alpha = \frac{\theta + 360^\circ k}{n}$, $k = 0, 1, 2, \dots, n-1$

$$\alpha = \frac{\theta}{n} + \frac{360^\circ k}{n}$$

$$\alpha = \frac{\theta + 2\pi k}{n}, \quad k = 0, 1, 2, \dots, n-1$$

$$\alpha = \frac{\theta}{n} + \frac{2\pi k}{n}$$

Example

Find the two square root of $4i$. Write the roots in rectangular form.

Solution

$$4i \rightarrow \begin{cases} x = 0 \\ y = 4 \end{cases}$$

$$r = \sqrt{0^2 + 4^2} \\ = 4$$

$$\tan \theta = \frac{4}{0} = \infty$$

$$\theta = \frac{\pi}{2}$$

$$4i = 4 \operatorname{cis} \frac{\pi}{2}$$

The absolute value: $\sqrt{4} = 2$

$$\begin{aligned} \text{Argument: } \alpha &= \frac{\frac{\pi}{2} + 2\pi k}{2} \\ &= \frac{\frac{\pi}{2}}{2} + \frac{2\pi k}{2} \\ &= \frac{\pi}{4} + \pi k \end{aligned}$$

Since there are **two** square root, then $k = 0$ and 1 .

If $k = 0$

$$\Rightarrow \alpha = \frac{\pi}{4} + \pi(0) = \underline{\frac{\pi}{4}}$$

If $k = 1$

$$\Rightarrow \alpha = \frac{\pi}{4} + \pi(1) = \underline{\frac{5\pi}{4}}$$

The square roots are: $2cis \frac{\pi}{4}$ and $2cis \frac{5\pi}{4}$

$$\begin{aligned} 2cis \frac{\pi}{4} &= 2\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) \\ &= 2\left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}\right) \\ &= \underline{\sqrt{2} + i\sqrt{2}} \end{aligned}$$

$$\begin{aligned} 2cis \frac{5\pi}{4} &= 2\left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right) \\ &= 2\left(-\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2}\right) \\ &= \underline{-\sqrt{2} - i\sqrt{2}} \end{aligned}$$

Example

Find all fourth roots of $-8 + 8i\sqrt{3}$. Write the roots in rectangular form.

Solution

$$-8 + 8i\sqrt{3} \Rightarrow \begin{cases} x = -8 \\ y = 8\sqrt{3} \end{cases}$$

$$r = \sqrt{(-8)^2 + (8\sqrt{3})^2} \\ = 16$$

$$\tan \theta = \frac{8\sqrt{3}}{-8} \\ = -\sqrt{3}$$

$$\theta = 120^\circ$$

$$-8 + 8i\sqrt{3} = 16\text{cis}120^\circ$$

The fourth roots have absolute value: $\sqrt[4]{16} = 2$

$$\alpha = \frac{120^\circ}{4} + \frac{360^\circ k}{4} \\ = 30^\circ + 90^\circ k$$

Since there are **four** roots, then $k = 0, 1, 2$, and 3 .

$$\text{If } k = 0 \Rightarrow \alpha = 30^\circ + 90^\circ(0) = 30^\circ$$

$$\text{If } k = 1 \Rightarrow \alpha = 30^\circ + 90^\circ(1) = 120^\circ$$

$$\text{If } k = 2 \Rightarrow \alpha = 30^\circ + 90^\circ(2) = 210^\circ$$

$$\text{If } k = 3 \Rightarrow \alpha = 30^\circ + 90^\circ(3) = 300^\circ$$

The fourth roots are: $2\text{cis}30^\circ$, $2\text{cis}120^\circ$, $2\text{cis}210^\circ$, and $2\text{cis}300^\circ$

$$2\text{cis}30^\circ = 2(\cos 30^\circ + i \sin 30^\circ)$$

$$= 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) \\ = \sqrt{3} + i$$

$$2\text{cis}120^\circ = 2(\cos 120^\circ + i \sin 120^\circ)$$

$$= 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ = -1 + i\sqrt{3}$$

$$2\text{cis}210^\circ = 2(\cos 210^\circ + i \sin 210^\circ)$$

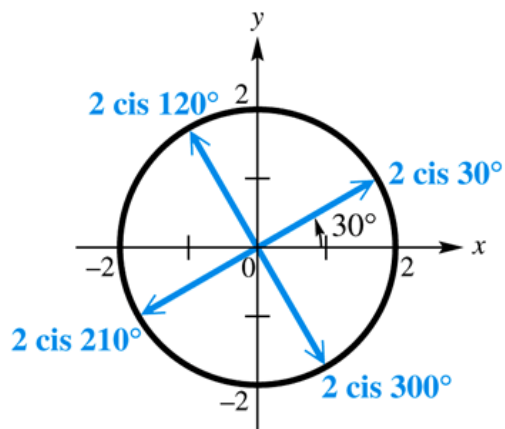
$$= 2\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$$

$$= -\sqrt{3} - i$$

$$2\text{cis}300^\circ = 2(\cos 300^\circ + i \sin 300^\circ)$$

$$= 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= 1 - i\sqrt{3}$$



Example

Find all complex number solutions of $x^5 - 1 = 0$. Graph them as vectors in the complex plane.

Solution

$$x^5 - 1 = 0 \Rightarrow x^5 = 1$$

There is one real solution, 1, while there are five complex solutions.

$$1 = 1 + 0i$$

$$r = \sqrt{1^2 + 0^2} \\ = 1$$

$$\tan \theta = \frac{0}{1} = 0$$

$$\theta = 0^\circ$$

$$1 = 1 \text{cis} 0^\circ$$

The fifth roots have absolute value: $\sqrt[5]{1} = 1$

$$\alpha = \frac{0^\circ}{5} + \frac{360^\circ k}{5} \\ = 0^\circ + 72^\circ k \\ = 72^\circ k$$

Since there are **fifth** roots, then $k = 0, 1, 2, 3$, and 4 .

$$\text{If } k = 0 \Rightarrow \alpha = 72^\circ(0) = 0^\circ$$

$$\text{If } k = 1 \Rightarrow \alpha = 72^\circ(1) = 72^\circ$$

$$\text{If } k = 2 \Rightarrow \alpha = 72^\circ(2) = 144^\circ$$

$$\text{If } k = 3 \Rightarrow \alpha = 72^\circ(3) = 216^\circ$$

$$\text{If } k = 4 \Rightarrow \alpha = 72^\circ(4) = 288^\circ$$

Solution: $\text{cis} 0^\circ$, $\text{cis} 72^\circ$, $\text{cis} 144^\circ$, $\text{cis} 216^\circ$, and $\text{cis} 288^\circ$

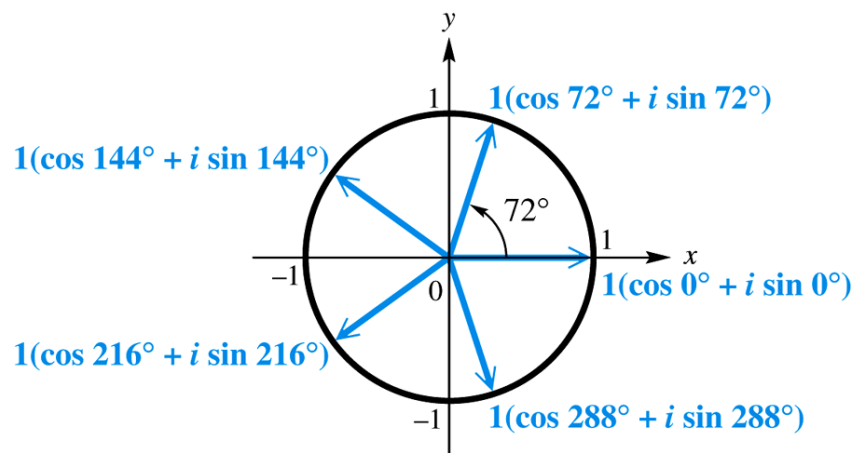
$$\text{cis} 0^\circ = 1$$

$$\text{cis} 72^\circ = \cos 72^\circ + i \sin 72^\circ$$

$$\text{cis} 144^\circ = \cos 144^\circ + i \sin 144^\circ$$

$$\text{cis} 216^\circ = \cos 216^\circ + i \sin 216^\circ$$

$$\text{cis} 288^\circ = \cos 288^\circ + i \sin 288^\circ$$



The graphs of the roots lie on a unit circle. The roots are equally spaced about the circle, 72° apart.

Exercises Section 3.7 – Trigonometric Form

(1 – 8) Write complex form in trigonometric form

- | | | | |
|--------------------|----------------|--------------------|---------------------|
| 1. $-\sqrt{3} + i$ | 3. $-21 - 20i$ | 5. $\sqrt{3} - i$ | 7. $9\sqrt{3} + 9i$ |
| 2. $3 - 4i$ | 4. $11 + 2i$ | 6. $1 - \sqrt{3}i$ | 8. $-2 + 3i$ |

(9 – 13) Write in standard form

- | | | |
|---|---|-------------------------|
| 9. $4(\cos 30^\circ + i \sin 30^\circ)$ | 11. $3cis210^\circ$ | 13. $4cis\frac{\pi}{2}$ |
| 10. $\sqrt{2} cis \frac{7\pi}{4}$ | 12. $4\left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}\right)$ | |

14. Find the quotient $\frac{20cis(75^\circ)}{4cis(40^\circ)}$. Write the result in rectangular form.

15. Divide $z_1 = 1 + i\sqrt{3}$ by $z_2 = \sqrt{3} + i$. Write the result in rectangular form.

(16 – 25) Find and express the result in rectangular form

- | | | | |
|--------------------|-------------------------------|------------------------|---|
| 16. $(1 + i)^8$ | 19. $(1 - \sqrt{5}i)^8$ | 22. $(\sqrt{2} - i)^6$ | 24. $(2cis30^\circ)^5$ |
| 17. $(1 + i)^{10}$ | 20. $(3cis80^\circ)^3$ | 23. $(4cis40^\circ)^6$ | 25. $\left(\frac{1}{2}cis72^\circ\right)^5$ |
| 18. $(1 - i)^5$ | 21. $(\sqrt{3}cis10^\circ)^6$ | | |

26. Find fifth complex roots of $z = 1 + i\sqrt{3}$ and express the result in rectangular form.

(27 – 30) Find the fourth roots of

- | | | | |
|-------------------------|--------------------|----------------------|------------|
| 27. $z = 16cis60^\circ$ | 28. $\sqrt{3} - i$ | 29. $4 - 4\sqrt{3}i$ | 30. $-16i$ |
|-------------------------|--------------------|----------------------|------------|

(31 – 33) Find the cube roots of

- | | | |
|--------|--------------|--------|
| 31. 27 | 32. $8 - 8i$ | 33. -8 |
|--------|--------------|--------|

34. Find all complex number solutions of $x^3 + 1 = 0$.