

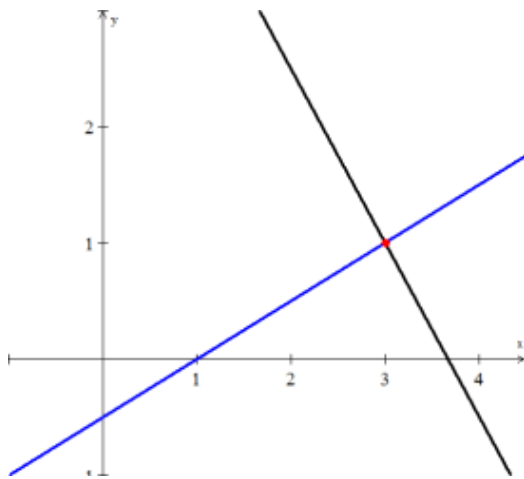
Section 3.2 – Gaussian Elimination

Elimination produces an *upper triangular system*.

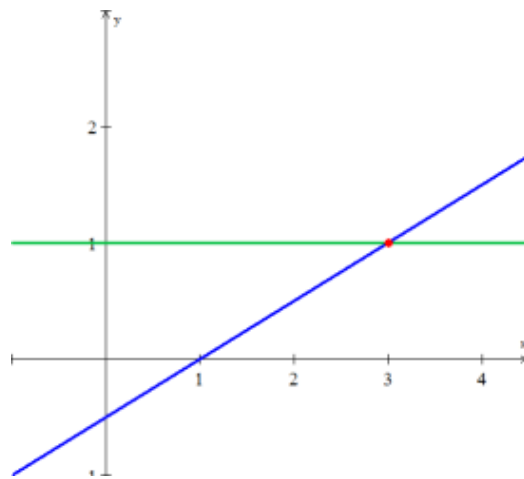
$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases} \Rightarrow \begin{cases} x - 2y = 1 & \text{Multiply by 3} \\ 8y = 8 & \text{and subtract} \end{cases}$$

The equation $8y = 8$ *reveals* $y = 1$

This process is called *back substitution*.



Before elimination



After elimination

Definitions

Pivot: first nonzero in the row that does the elimination

Multiplier: (entry to eliminate) divide by pivot

$$\begin{array}{lll} 4x - 8y = 4 & \text{Multiply equation 1 by } \frac{3}{4} & 4x - 8y = 4 \\ 3x + 2y = 11 & \text{Subtract from equation 2} & 8y = 8 \end{array}$$

The first pivot is 4 (the coefficient of x) and the multiplier is $l = \frac{3}{4}$

The pivots are on the diagonal of the triangle after elimination.

Reduced Row Echelon Form

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

Use the Gaussian elimination method to solve the system

$$3x + y + 2z = 31$$

$$x + y + 2z = 19$$

$$x + 3y + 2z = 25$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 3 & 1 & 2 & 31 \\ 1 & 3 & 2 & 25 \end{array} \right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array} \quad \begin{array}{cccc} 3 & 1 & 2 & 31 \\ -3 & -3 & -6 & -57 \\ 0 & -2 & -4 & -26 \end{array} \quad \begin{array}{cccc} 1 & 2 & 2 & 25 \\ -1 & -1 & -2 & -19 \\ 0 & 2 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & -2 & -4 & -26 \\ 0 & 2 & 0 & 6 \end{array} \right] -\frac{1}{2}R_2 \quad \begin{array}{cccc} 0 & 1 & 2 & 13 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 2 & 0 & 6 \end{array} \right] R_3 - 2R_2 \quad \begin{array}{cccc} 0 & 2 & 0 & 6 \\ 0 & -2 & -4 & -26 \\ 0 & 0 & -4 & -20 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -4 & -20 \end{array} \right] -\frac{1}{4}R_3 \quad \begin{array}{cccc} 0 & 0 & 1 & 5 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array} \right] \Rightarrow \begin{array}{l} x + y + 2z = 19 \quad (3) \\ y + 2z = 13 \quad (2) \\ z = 5 \quad (1) \end{array}$$

$$(2) \Rightarrow y = 13 - 2z = 13 - 2(5) = 3$$

$$(3) \Rightarrow x = 19 - y - 2z = 19 - 3 - 10 = 6$$

$$\Rightarrow (6, 3, 5)$$

Example

Use Gauss-Jordan elimination to solve the linear system

$$\begin{aligned}x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\5x_3 + 10x_4 + 15x_6 &= 5 \\2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 6\end{aligned}$$

Solution

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \text{ Adding } (-2) \text{ times the 1st row to the 2nd} \\ R_4 - 2R_1 \text{ Adding } (-2) \text{ times the 1st row to the 4th} \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array} \right] \quad -R_2$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array} \right] \quad \begin{array}{l} R_3 - 5R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array} \right] \quad \frac{1}{6}R_4 \text{ then interchanging row3 and row4}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad R_2 - 3R_3$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \begin{cases} x_1 + 3x_2 + 4x_4 + 2x_5 = 0 \\ x_3 + 2x_4 = 0 \\ + x_6 = \frac{1}{3} \end{cases}$$

The general solution of the system: $x_6 = \frac{1}{3}$, $x_3 = -2x_4$, $x_1 = -3x_2 - 4x_4 - 2x_5$

Example

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$2x + 8y - z + w = 0$$

$$4x + 16y - 3z - w = -10$$

$$-2x + 4y - z + 3w = -6$$

$$-6x + 2y + 5z + w = 3$$

Solution

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 4 & 16 & -3 & -1 & -10 \\ -2 & 4 & -1 & 3 & -6 \\ -6 & 2 & 5 & 1 & 3 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 + 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 26 & 2 & 4 & 3 \end{array} \right] R_4 - \frac{13}{6}R_2$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 26 & 2 & 4 & 3 \end{array} \right] \text{Interchange } R_2 \text{ and } R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & \frac{19}{3} & -\frac{14}{3} & 16 \end{array} \right] R_4 + \frac{19}{3}R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & 0 & -\frac{71}{3} & -\frac{142}{3} \end{array} \right] \begin{array}{l} 2x + 8y - z + w = 0 \rightarrow 2x = -8y + z - w = 6 \Rightarrow \boxed{x=3} \\ 12y - 2z + 4w = -6 \rightarrow 12y = 2z - 4w - 6 = -6 \Rightarrow \boxed{y = -\frac{1}{2}} \\ -z - 3w = -10 \\ -\frac{71}{3}w = -\frac{142}{3} \rightarrow \boxed{z = 10 - 3w = 4} \\ \rightarrow \boxed{w = 2} \end{array}$$

$$\text{Solution: } \left(3, -\frac{1}{2}, 4, 2 \right)$$

Exercises Section 3.2 – Gaussian Elimination

- When elimination is applied to the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix}$
 - What are the first and second pivots?
 - What is the multiplier l_{21} in the first step (l_{21} times row 1 is subtracted from row 2)?
 - What entry in the 2, 2 position (instead of 9) would force an exchange of rows 2 and 3?
 - What is the multiplier $l_{31} = 0$, subtracting 0 times row 1 from row 3?
- Use elimination to reach upper triangular matrices U. Solve by back substitution or explain why this is impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the $-x$ in equation (3).

$$\begin{cases} x + y + z = 7 \\ x + y - z = 5 \\ x - y + z = 3 \end{cases} \quad \begin{cases} x + y + z = 7 \\ x + y - z = 5 \\ -x - y + z = 3 \end{cases}$$

- For which numbers a does the elimination break down (1) permanently (2) temporarily

$$ax + 3y = -3$$

$$4x + 6y = 6$$

Solve for x and y after fixing the second breakdown by a row change.

- Find the pivots and the solution for these four equations:

$$2x + y = 0$$

$$x + 2y + z = 0$$

$$y + 2z + t = 0$$

$$z + 2t = 5$$

- Look for a matrix that has row sums 4 and 8, and column sums 2 and s .

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{array}{ll} a + b = 4 & a + c = 2 \\ c + d = 8 & b + d = s \end{array}$$

The four equations are solvable only if $s = \underline{\hspace{1cm}}$. Then find two different matrices that have the correct row and column sums.

- Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of A is a _____ of the first two rows. Find a third equation that can't be solved together with $x + y + z = 0$ and $x - 2y - z = 1$

7. Solve the linear system by Gauss-Jordan elimination.

$$a) \begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$$

$$b) \begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x - 3w = -3 \end{cases}$$

$$c) \begin{cases} x + 2y + z = 8 \\ -x + 3y - 2z = 1 \\ 3x + 4y - 7z = 10 \end{cases}$$

$$d) \begin{cases} 2u - 3v + w - x + y = 0 \\ 4u - 6v + 2w - 3x - y = -5 \\ -2u + 3v - 2w + 2x - y = 3 \end{cases}$$

8. Solve the given linear system by any method

$$a) \begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

$$b) \begin{cases} 2x + 2y + 4z = 0 \\ -y - 3z + w = 0 \\ 3x + y + z + 2w = 0 \\ x + 3y - 2z - 2w = 0 \end{cases}$$

9. Add 3 times the second row to the first of

$$\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

10. Solve the system using Gaussian elimination $\begin{cases} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{cases}$

11. For what value(s) of k , if any, does the system $\begin{cases} x + y - z = 1 \\ 2x + 3y + kz = 3 \\ x + ky + 3z = 2 \end{cases}$ have

- a) A unique solution?
- b) Infinitely many solutions?
- c) No solution?