Solution *Section* 1.1 – Propositional Logic

Exercise

Which of these sentences are propositions? What are truth values of those that are propositions?

a) Boston is the capital of Massachusetts.
b) Miami is the capital of Florida
c) $2+3=5$
d) $5+7=10$
e) $x+2=11$
f) Answer this question
g) Do not pass go
h) What time is it?
i) The moon is made of green cheese
j) $2^n \geq 100$

Solution

a) This is a true proposition

b) This is a false proposition, Tallahassee is the capital

c) This is a true proposition

d) This is a false proposition since $5+7=12 \neq 10$

e) This is not a proposition. (The truth value depends on the value assigned to x.)

f) This is not a proposition, it is a command

g) This is not a proposition; it is a command

h) This is not a proposition; it's a question

i) This is a proposition that is false

j) This is not a proposition; its truth value depends on the value of n.

Exercise

What is the negation if each of these propositions?

a) Mei has an MP3 player
b) There is no pollution in Texas
c) $2+1=3$
d) There are 13 items in a baker's dozen,
e) 121 is a perfect square

Solution

a) Mei does not have an MP3 player

b) There is pollution in Texas

c) $2+1\neq 3$

d) There are not 13 items in a baker's dozen.

e) 121 is not a perfect square

Exercise

Suppose the Smartphone *A* has 256 MB RAM and 32 GB ROM, and the resolution of its camera is 8 MP; Smartphone *B* has 288 MB RAM and 64 GB ROM, and the resolution of its camera is 4 MP; Smartphone *C* has 128 MB RAM and 32 GB ROM, and the resolution of its camera is 5 MP. Determine the truth value of each of these propositions.

a) Smartphone *B* has the most RAM of these three smartphones

b) Smartphone *C* has more ROM or higher resolution camera than Smartphone *B*.

c) Smartphone *B* has more RAM, more ROM, and a higher resolution camera than Smartphone *A*.

d) If Smartphone *B* has more RAM and more ROM than Smartphone *C*, then it also has a higher resolution camera.

e) Smartphone *A* has more RAM than Smartphone *B* if and only if Smartphone *B* has more RAM than Smartphone *A*.

Solution

a) This is a true proposition because 288 > 256 and 288 > 128

b) This is a true proposition, because the resolution is higher, C has 5 MP resolution compared to B's 4 MP.

c) This is not a true proposition. The resolution is not higher that *A*.

d) This is not a true proposition. Not necessary, the resolution in *B* is not higher that *C*.

e) This is not a true proposition

Exercise

Let *p* and *q* be the proposition

p: I bought a lottery ticket this week

q: I won the million dollars jackpot

$a)\ \neg p$ $\quad b)\ p\vee q$ $\quad c)\ p\to q$ $\quad d)\ p\wedge q$

$e)\ p\leftrightarrow q$ $\quad f)\ \neg p\to\neg q$ $\quad g)\ \neg p\wedge\neg q$ $\quad h)\ \neg p\vee\left(p\wedge q\right)$

Solution

a) I did not buy a lottery ticket this week.

b) I bought a lottery ticket this week or I won the million dollars jackpot.

c) I bought a lottery ticket this week then I won the million dollars jackpot.

d) I bought a lottery ticket this week and I won the million dollars jackpot.

e) I bought a lottery ticket this week if and only if I won the million dollars jackpot.

f) I did not buy a lottery ticket this week then I did not win the million dollars jackpot.

g) I did not buy a lottery ticket this week and I did not win the million dollars jackpot.

h) I did not buy a lottery ticket this week or either I bought a lottery ticket this week and I won the million dollars jackpot.

Exercise

Let *p* and *q* be the proposition

p: Swimming at the New Jersey shore is allowed

q: Sharks have been spotted new the shore

$a)\ \ \neg q$ $\qquad b)\ \ p \wedge q$ $\qquad c)\ \ \neg p \vee q$ $\qquad d)\ \ p \to \neg q$

$e)\ \ \neg q \to p$ $\qquad f)\ \ \neg\, p \to \neg\, q$ $\qquad g)\ \ p \leftrightarrow \neg\, q$ $\qquad h)\ \ \neg p \wedge (p \vee \neg q)$

Solution

a) Sharks have not been spotted new the shore.

b) Swimming at the New Jersey shore is allowed, and Sharks have been spotted new the shore.

c) Swimming at the New Jersey shore is not allowed, or Sharks have been spotted new the shore.

d) Swimming at the New Jersey shore is allowed then Sharks have not been spotted new the shore.

e) Sharks have not been spotted new the shore then Swimming at the New Jersey shore is allowed.

f) Swimming at the New Jersey shore is not allowed then Sharks have not been spotted new the shore.

g) Swimming at the New Jersey shore is allowed if and only if Sharks have not been spotted new the shore.

h) Swimming at the New Jersey shore is not allowed and either Swimming at the New Jersey shore is allowed, or Sharks have not been spotted new the shore

Exercise

Let *p*, *q* and *r* be the proposition

p: You have the flu

q: You miss the final examination

r: You pass the course

Express each of these propositions as an English sentence

$a)\ \ p \to q$ $\qquad b)\ \ \neg q \leftrightarrow r$ $\qquad c)\ \ q \to \neg r$ $\qquad d)\ \ p \vee q \vee r$

$e)\ \ (p \to \neg r) \vee (q \to \neg r)$ $\qquad f)\ \ (p \wedge q) \vee (\neg q \wedge r)$

Solution

a) You have the flu then you miss the final examination.

b) You don't miss the final examination if and only if you pass the course.

c) You miss the final examination then you don't pass the course.

d) You have the flu or you miss the final examination or you pass the course.

e) You have the flu then you don't pass the course or you miss the final examination then you don't pass the course.

f) You have the flu and you miss the final examination or both you don't miss the final examination and you pass the course.

Exercise

Determine whether each of these conditional statements is true or false.

a) If $1+1=2$, then 2 + 2 = 5
b) If 1 + 1 = 3, then 2 + 2 = 4
c) If 1 + 1 = 3, then 2 + 2 = 5
d) If monkeys can fly, then 1 + 1 = 3
e) If 1 + 1 = 3, then unicorns exist
f) If 1 + 1 = 3, then dogs can fly
g) If 1 + 1 = 2, the dogs can fly
h) If 2 + 2 = 4, then 1 + 2 = 3

Solution

a) Since the hypothesis is true and the conclusion is false, this implication is false.

b) Since the hypothesis is false and the conclusion is true, this implication is true.

c) Since the hypothesis is false and the conclusion is false, this implication is true.

d) Since the hypothesis is false and the conclusion is false, this implication is true.

e) Since the hypothesis is false and the conclusion is false, this implication is true.

f) Since the hypothesis is false and the conclusion is false, this implication is true.

g) Since the hypothesis is true and the conclusion is false, this implication is false.

h) Since the hypothesis is true and the conclusion is true, this implication is true.

Exercise

Write each of these propositions in the form "*p* if and only if *q*" in English

a) If it is hot outside you buy an ice cream cone, and if you buy an ice cream cone it is hot outside.
b) For you to win the contest it is necessary and sufficient that you have only winning ticket.
c) You get promoted only if you have connections, and you have connections only if you get promoted.
d) If you watch television your mind will decay, and conversely.
e) The trains run late on exactly those days when I take it.
f) For you to get an *A* in this course, it is necessary and sufficient that you learn how to solve discrete mathematics problems.

g) If you read the newspaper every day, you will be informed, and conversely.
h) It rains if it is a weekend day, and it is a weekend day if it rains.
i) You can see the wizard only if the wizard is not in, and the wizard is not in only if you can see him

Solution

a) You buy an ice cream cone if and only if it is hot outside.

b) You win the contest if and only if you hold the only winning ticket.

c) You get promoted if and only if you have connection.

d) Your mind will decay if and only if you watch television.

e) The trains run late if and only if you watch television.

f) For you to get an *A* in this course if and only if you learn how to solve discrete mathematics problems.

Exercise

Construct a truth table for each of these compound propositions.

a) $p \wedge \neg p$
b) $p \vee \neg p$
c) $p \to \neg p$
d) $p \leftrightarrow \neg p$
e) $p \to \neg q$
f) $\neg p \leftrightarrow q$
g) $(p \vee \neg q) \to q$
h) $(p \vee q) \to (p \wedge q)$
i) $(p \to q) \leftrightarrow (\neg q \to \neg p)$
j) $(p \to q) \to (q \to p)$
k) $p \oplus (p \vee q)$
l) $(p \wedge q) \to (p \vee q)$
m) $(q \to \neg p) \leftrightarrow (p \leftrightarrow q)$
n) $(p \to q) \vee (\neg p \to q)$
o) $(p \to q) \wedge (\neg p \to q)$
p) $(p \vee q) \vee r$
q) $(p \vee q) \wedge r$
r) $(p \wedge q) \vee r$
s) $(p \wedge q) \wedge r$
t) $(p \vee q) \wedge \neg r$

Solution

a)

p	$\neg p$	$p \wedge \neg p$
T	F	***F***
F	T	***F***

b)

p	$\neg p$	$p \vee \neg p$
T	F	***T***
F	T	***T***

c)

p	$\neg p$	$p \to \neg p$
T	F	***F***
F	T	***T***

d)

p	$\neg p$	$p \leftrightarrow \neg p$
T	F	**F**
F	T	**F**

e)

p	q	$\neg q$	$p \to \neg q$
T	T	F	**F**
T	F	T	**T**
F	T	F	**T**
F	F	T	**T**

f)

p	q	$\neg q$	$\neg p \leftrightarrow q$
T	T	F	***F***
T	F	T	***T***
F	T	F	***T***
F	F	T	***F***

g)

p	q	$\neg q$	$p \vee \neg q$	$(p \vee \neg q) \to q$
T	T	F	T	***T***
T	F	T	T	***F***
F	T	F	F	***T***
F	F	T	T	***F***

h)

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \to (p \wedge q)$
T	T	T	T	***T***
T	F	T	F	***F***
F	T	T	F	***F***
F	F	F	F	***T***

i)

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$	$(p \to q) \leftrightarrow (\neg q \wedge \neg p)$
T	T	**T**	F	F	**T**	***T***
T	F	**F**	T	F	**F**	***T***
F	T	**T**	F	T	**F**	***T***
F	F	**T**	T	T	**F**	***T***

j)

p	q	$p \to q$	$q \to p$	$(p \to q) \to (q \to p)$
T	T	**T**	**T**	***T***
T	F	**F**	**T**	***T***
F	T	**T**	**F**	***F***
F	F	**T**	**T**	***T***

k)

p	q	$p \vee q$	$p \oplus (p \vee q)$
T	T	**T**	***F***
T	F	**T**	***F***
F	T	**T**	***T***
F	F	**F**	***F***

l)

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \to (p \vee q)$
T	T	**T**	**T**	***T***
T	F	**F**	**T**	***T***
F	T	**F**	**T**	***T***
F	F	**F**	**F**	***T***

m)

p	q	$\neg p$	$q \to \neg p$	$p \leftrightarrow q$	$(q \to \neg p) \leftrightarrow (p \leftrightarrow q)$
T	T	F	**F**	**T**	***F***
T	F	F	**T**	**F**	***F***
F	T	T	**T**	**F**	***F***
F	F	T	**T**	**T**	***T***

n)

p	q	$\neg p$	$p \to q$	$\neg p \to q$	$(p \to q) \vee (\neg p \to q)$
T	T	F	**T**	**T**	***T***
T	F	F	**F**	**T**	***T***
F	T	T	**T**	**T**	***T***
F	F	T	**T**	**F**	***T***

o)

p	q	$\neg p$	$p \to q$	$\neg p \to q$	$(p \to q) \wedge (\neg p \to q)$
T	T	F	**T**	**T**	***T***
T	F	F	**F**	**T**	***F***
F	T	T	**T**	**T**	***T***
F	F	T	**T**	**F**	***F***

p)

p	q	r	$p \vee q$	$(p \vee q) \vee r$
T	T	T	**T**	***T***
T	T	F	**T**	***T***
T	F	T	**T**	***T***
T	F	F	**T**	***T***
F	T	T	**T**	***T***
F	T	F	**T**	***T***
F	F	T	**F**	***T***
F	F	F	**F**	***F***

q)

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
T	T	T	**T**	***T***
T	T	F	**T**	***F***
T	F	T	**T**	***T***
T	F	F	**T**	***F***
F	T	T	**T**	***T***
F	T	F	**T**	***F***
F	F	T	**F**	***F***
F	F	F	**F**	***F***

r)

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$
T	T	T	**T**	***T***
T	T	F	**T**	***T***
T	F	T	**F**	***T***
T	F	F	**F**	***F***
F	T	T	**F**	***T***
F	T	F	**F**	***F***
F	F	T	**F**	***T***
F	F	F	**F**	***F***

s)

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$
T	T	T	**T**	***T***
T	T	F	**T**	***F***
T	F	T	**F**	***F***
T	F	F	**F**	***F***
F	T	T	**F**	***F***
F	T	F	**F**	***F***
F	F	T	**F**	***F***
F	F	F	**F**	***F***

t)

p	q	r	$\neg r$	$p \vee q$	$(p \vee q) \wedge \neg r$
T	T	T	**F**	**T**	***F***
T	T	F	**T**	**T**	***T***
T	F	T	**F**	**T**	***F***
T	F	F	**T**	**T**	***T***
F	T	T	**F**	**T**	***F***
F	T	F	**T**	**T**	***T***
F	F	T	**F**	**F**	***F***
F	F	F	**T**	**F**	***F***

Solution Section 1.2 – Propositional Equivalences

Exercise

Use the truth table to verify these equivalences

a) $p \wedge \mathbf{T} \equiv p$	*b)* $p \vee \mathbf{F} \equiv p$	*c)* $p \wedge \mathbf{F} \equiv \mathbf{F}$
d) $p \vee \mathbf{T} \equiv \mathbf{T}$	*e)* $p \vee p \equiv p$	*f)* $p \wedge p \equiv p$

Solution

p	$p \wedge \mathbf{T}$	$p \vee \mathbf{F}$	$p \wedge \mathbf{F}$	$p \vee \mathbf{T}$	$p \vee p$	$p \wedge p$
T	T	T	F	T	T	T
F	F	F	F	T	F	F
	hold	***hold***	***hold***	***hold***	***hold***	***hold***

Exercise

Show that $\neg(\neg p)$ and p are logically equivalent

Solution

p	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

Therefore, $\neg(\neg p)$ and p are logically equivalent

Exercise

Use the truth table to verify the commutative laws

a) $p \vee q \equiv q \vee p$

b) $p \wedge q \equiv q \wedge p$

Solution

p	q	$p \vee q$	$q \vee p$
T	T	***T***	***T***
T	F	***T***	***T***
F	T	***T***	***T***
F	F	***F***	***F***

Identical

p	q	$p \wedge q$	$q \wedge p$
T	T	***T***	***T***
T	F	***F***	***F***
F	T	***F***	***F***
F	F	***F***	***F***

Identical

Exercise

Use the truth table to verify the associative laws

a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Solution

a)

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$
T	T	T	**T**	***T***	**T**	***T***
T	T	F	**T**	***T***	**T**	***T***
T	F	T	**T**	***T***	**T**	***T***
T	F	F	**T**	***T***	**F**	***T***
F	T	T	**T**	***T***	**T**	***T***
F	T	F	**T**	***T***	**T**	***T***
F	F	T	**F**	***T***	**T**	***T***
F	F	F	**F**	***F***	**F**	***F***

$(p \vee q) \vee r \equiv p \vee (q \vee r)$ is true

b)

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$	$q \wedge r$	$p \wedge (q \wedge r)$
T	T	T	**T**	***T***	**T**	***T***
T	T	F	**T**	***F***	**F**	***F***
T	F	T	**F**	***F***	**F**	***F***
T	F	F	**F**	***F***	**F**	***F***
F	T	T	**F**	***F***	**T**	***F***
F	T	F	**F**	***F***	**F**	***F***
F	F	T	**F**	***F***	**F**	***F***
F	F	F	**F**	***F***	**F**	***F***

$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ is true

Exercise

Show that each of these conditional statements is a tautology by using truth result tables.

a) $(p \wedge q) \rightarrow p$

b) $p \rightarrow (p \vee q)$

c) $\neg p \rightarrow (p \rightarrow q)$

d) $(p \wedge q) \rightarrow (p \rightarrow q)$

e) $\neg(p \to q) \to p$

f) $\left[\neg p \wedge (p \vee q)\right] \to q$

g) $\left[(p \to q) \wedge (q \to r)\right] \to (p \to r)$

h) $\left[p \wedge (p \to q)\right] \to q$

Solution

a)

p	q	$p \vee q$	$p \to (p \vee q)$
T	T	***T***	***T***
T	F	***T***	***T***
F	T	***T***	***T***
F	F	***F***	***T***

b)

p	q	$p \wedge q$	$(p \wedge q) \to p$
T	T	***T***	***T***
T	F	***F***	***T***
F	T	***F***	***T***
F	F	***F***	***T***

c)

p	q	$\neg p$	$p \to q$	$\neg p \to (p \to q)$
T	T	***F***	***T***	***T***
T	F	***F***	***F***	***T***
F	T	***T***	***T***	***T***
F	F	***T***	***T***	***T***

d)

p	q	$p \wedge q$	$p \to q$	$(p \wedge q) \to (p \to q)$
T	T	***T***	***T***	***T***
T	F	***F***	***F***	***T***
F	T	***F***	***T***	***T***
F	F	***F***	***T***	***T***

e)

p	q	$p \to q$	$\neg(p \to q)$	$\neg(p \to q) \to p$
T	T	***T***	***F***	***T***
T	F	***F***	***T***	***T***
F	T	***T***	***F***	***T***
F	F	***T***	***F***	***T***

f)

p	q	$p \to q$	$\neg(p \to q)$	$\left[\neg p \wedge (p \vee q)\right] \to q$
T	T	***T***	***F***	***T***
T	F	***F***	***T***	***T***
F	T	***T***	***F***	***T***
F	F	***T***	***F***	***T***

g)

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	**T**	**T**	**T**	**T**	***T***
T	T	F	**T**	**F**	**F**	**F**	***T***
T	F	T	**F**	**T**	**F**	**T**	***T***
T	F	F	**F**	**T**	**F**	**F**	***T***
F	T	T	**T**	**T**	**T**	**T**	***T***
F	T	F	**T**	**F**	**F**	**T**	***T***
F	F	T	**T**	**T**	**T**	**T**	***T***
F	F	F	**T**	**T**	**T**	**T**	***T***

h)

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	***T***	***T***	***T***
T	F	***F***	***F***	***T***
F	T	***T***	***F***	***T***
F	F	***T***	***F***	***T***

Exercise

Show that $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent

Solution

The proposition $p \leftrightarrow q$ is true when p and q have the same true or false value. Since p and q are truth, then $p \wedge q$ only true. When p and q are false, then the negation $\neg p$ and $\neg q$ are true, then $\neg p \wedge \neg q$ is true. Therefore $(p \wedge q) \vee (\neg p \wedge \neg q)$ is true only when both are true. Therefore these two expressions are logically equivalent.

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$	$p \leftrightarrow q$
T	T	***T***	F	F	***F***	***T***	***T***
T	F	***F***	F	T	***F***	***F***	***F***
F	T	***F***	T	F	***F***	***F***	***F***
F	F	***F***	T	T	***T***	***T***	***T***

Exercise

Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent

Solution

The proposition $\neg(p \leftrightarrow q)$ is true when $p \leftrightarrow q$ is false. Since $p \leftrightarrow q$ is true when p and q have the same truth value, it is false when p and q have different truth values (either p is true and q is false, or vice versa). These are precisely the cases in which $p \leftrightarrow \neg q$ is true. Therefore these two expressions are logically equivalent.

p	q	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$\neg q$	$p \leftrightarrow \neg q$
T	T	***T***	***F***	F	***F***
T	F	***F***	***T***	T	***T***
F	T	***F***	***T***	F	***T***
F	F	***T***	***F***	T	***F***

Exercise

Show that $p \to q$ and $\neg q \to \neg p$ are logically equivalent

Solution

It is easy to see from the definitions of conditional statement and negation of these propositions is false in the case which *p* is true and *q* is false the proposition is false, and true in the other three cases. Therefore these two expressions are logically equivalent.

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$
T	T	***T***	F	F	***T***
T	F	***F***	T	F	***F***
F	T	***T***	F	T	***T***
F	F	***T***	T	T	***T***

Exercise

Show that $\neg p \leftrightarrow q$ and $p \leftrightarrow \neg q$ are logically equivalent

Solution

The proposition $\neg p \leftrightarrow q$ is true when $\neg p$ and q have the same truth values, which means that p and q have different truth values (either p is true and q is false, or vice versa). By the same reasoning, these are exactly the cases in which $p \leftrightarrow \neg q$ is true. Therefore these two expressions are logically equivalent.

Exercise

Show that $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$ are logically equivalent

Solution

$(p \to q) \vee (p \to r)$ will be true when either of the conditional statements is true. The conditional statement will be true if *p* is false, or if *q* in one case or r in the other case is true, when $q \vee r$ is true, which is precisely $p \to (q \vee r)$ is true. Since the two propositions are true in exactly the same situation, they are logically equivalent.

Exercise

Show that $(p \to r) \vee (q \to r)$ and $(p \wedge q) \to r$ are logically equivalent

Solution

In order for $(p \to r) \vee (q \to r)$ to be false, we must have both of the two implications false, which happens exactly when *r* is false and both *p* and *q* are true. But this precisely the case in which $p \wedge q$ is true and *r* is false, which is $(p \wedge q) \to r$ is false. Therefore these two expressions are logically equivalent.

Exercise

Show that $(p \to q) \wedge (q \to r) \to (p \to r)$ is a tautology

Solution

Given that *p* and $p \to q$ are both true, we conclude that *q* is true; from that and $q \to r$ we conclude that *r* is true.

Exercise

Show that $(p \vee q) \vee (\neg p \vee r) \to (q \vee r)$ is a tautology

Solution

The conclusion $q \vee r$ will be true in every case except when *q* and *r* are both false. But if *q* and *r* are both false, then one of $p \vee q$ or $\neg p \vee r$ is false, because one of *p* or $\neg p$ is false. Thus in this case $(p \vee q) \wedge (\neg p \vee r)$ is false. An conditional statement in which the conclusion is true or the hypothesis is false.

Exercise

Show that | (NAND) is functionally complete

Solution

Equivalence of NOT:

$p|p \equiv \neg p$

$\neg(p \wedge p) \equiv \neg p$ *Equivalence of* NAND

$\neg(p) \equiv \neg p$ *Idempotent law*

Equivalence of AND:

$p \wedge q \equiv \neg(p|q)$ *Definition of* NAND

$p|p$

$(p|q)|(p|p)q$ *Negation of* $(p|q)$

Equivalence of OR:

$p \vee q \equiv \neg(\neg p \wedge \neg q)$ **DeMorgan's *equivalence of*** OR

We can do AND and OR with NANDs, also do ORs with NANDs

Thus, NAND is functionally complete.

Solution   *Section* 1.3 – Predicates and Quantifiers

Exercise

Let $P(x)$ denote the statement "$x \le 4$". What are these truth values?

$a)\ P(0) \qquad b)\ P(4) \qquad c)\ P(6)$

Solution

a) True, since $0 \le 4$

b) True, since $4 \le 4$

c) False, since $7 \not\le 4$

Exercise

Let $P(x)$ be the statement "*the word* **x** *contains the letter* ***a***". What are these truth values?

$a)\ P(orange) \qquad b)\ P(lemon) \qquad c)\ P(true) \qquad d)\ P(false)$

Solution

a) True, since there is an *a* in *orange*.

b) False, since there is no *a* in *lemon*.

c) False, since there is no *a* in *true*.

d) True, since there is an *a* in *false*.

Exercise

State the value of *x* after the statement **if** $P(x)$ **then** $x := 1$ is executed, where $P(x)$ is the statement "$x > 1$", if the value of *x* when the statement is reached is

$a)\ x = 0 \qquad b)\ x = 1 \qquad c)\ x = 2$

Solution

a) x is still equal to 0, since the condition is false.

b) x is still equal to 0, since the condition is false.

c) x is still equal to 1 at the end, since the condition is true, so the statement $x := 1$ is executed.

Exercise

Let $P(x)$ be the statement "$\boldsymbol{x}$ *spends more than five hours every weekday in class,*" where the domain for x consists of all students. Express each of these quantifications in English.

$a)\ \exists x\ P(x)$ $\quad b)\ \forall x\ P(x)$ $\quad c)\ \exists x\ \neg P(x)$ $\quad d)\ \forall x\ \neg P(x)$

Solution

a) There is a student who spends more than five hours every weekday in class.

b) Every student spends more than five hours every weekday in class.

c) There is a student who does not spend more than five hours every weekday in class.

d) No student spends more than five hours every weekday in class.

Exercise

Let $N(x)$ be the statement "$\boldsymbol{x}$ *has visited North Dakota,*" where the domain consists of the students in your class. Express each of these quantifications in English.

$a)\ \exists x\ N(x)$ $\quad b)\ \forall x\ N(x)$ $\quad c)\ \neg\exists x\ N(x)$ $\quad d)\ \exists x\ \neg N(x)$

$e)\ \neg\forall x\ N(x)$ $\quad f)\ \forall x\ \neg N(x)$

Solution

a) Some student in the school has visited North Dakota.
Or, there exists a student in the school who has visited North Dakota

b) Every student in the school has visited North Dakota
Or, all students in the school have visited North Dakota

c) No student in the school has visited North Dakota.
Or, there does not exist a student in the school who has visited North Dakota

d) Some student in the school has not visited North Dakota.
Or, there exists a student in the school who has not visited North Dakota

e) It is not true that every student in the school has visited North Dakota
Or, not all students in the school have visited North Dakota

f) All students in the school has not visited North Dakota.
Or, no student has visited North Dakota

Exercise

Let $C(x)$ be the statement "***x** has a cat,*" let $D(x)$ be the statement "***x** has a dog,*", and let $F(x)$ be the statement "***x** has a ferret.*" Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

a) A student in your class has a cat, a dog, and a ferret.
b) All students in your class have a cat, a dog, or a ferret.
c) Some student in your class has a cat and a ferret, but not a dog.
d) No student in your class has a cat, a dog, and a ferret.
e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

Solution

a) $\exists x\big(C(x)\wedge D(x)\wedge F(x)\big)$

b) $\forall x\big(C(x)\vee D(x)\vee F(x)\big)$

c) $\exists x\big(C(x)\wedge F(x)\wedge \neg D(x)\big)$

d) $\neg\exists x\big(C(x)\wedge D(x)\wedge F(x)\big)$

e) $\big(\exists x C(x)\big)\wedge\big(\exists x D(x)\big)\wedge\big(\exists x F(x)\big)$

Exercise

Let $Q(x)$ be the statement "$x+1>2x$". If the domain consists of all integers, what are these truth values?

a) $Q(0)$ *b)* $Q(-1)$ *c)* $Q(1)$ *d)* $\exists x\ Q(x)$

e) $\forall x\ Q(x)$ *f)* $\exists x\ \neg Q(x)$ *g)* $\forall x\ \neg Q(x)$

Solution

a) Since $0+1>2\cdot 0$, that implies to $Q(0)$ is ***true***.

b) Since $(-1)+1>2\cdot(-1)$, that implies to $Q(-1)$ is ***true***.

c) Since $1+1\not> 2\cdot 1$, that implies to $Q(1)$ is ***false***.

d) We showed that $Q(0)$ is *true*, therefore there is at least one x that makes $Q(x)$ *true*, so $\exists x\ Q(x)$ is ***true***.

e) We showed that $Q(1)$ is *false*, therefore there is at least one x that makes $Q(x)$ *false*, so $\forall x\ Q(x)$ is ***false***.

f) We showed that $Q(1)$ is *false*, therefore there is at least one x that makes $Q(x)$ *false*, so $\exists x\ \neg Q(x)$ is ***true***.

g) We showed that $Q(0)$ is *true*, therefore there is at least one x that makes $Q(x)$ *true*, so $\forall x\ \neg Q(x)$ is ***false***.

Exercise

Determine the truth value of each of these statements if the domain consists of all integers

$a)\ \forall n(n+1>n)$ $\quad b)\ \exists n(2n=3n)$ $\quad c)\ \exists n(n=-n)$ $\quad d)\ \forall n(3n\le 4n)$

Solution

a) True, since adding 1 to a number makes it larger.

b) True, since $2\cdot 0=3\cdot 0$

c) True, since $0=-0$

d) True for all integers, $3n\le 4n \Rightarrow 3\le 4$

Exercise

Determine the truth value of each of these statements if the domain consists of all real numbers

$a)\ \exists x\left(x^3=-1\right)$ $\quad b)\ \exists x\left(x^4<x^2\right)$ $\quad c)\ \forall x\left((-x)^2=x^2\right)$ $\quad d)\ \forall x(2x>x)$

Solution

a) Since $(-1)^3=-1$, the statement $\exists x\left(x^3=-1\right)$ is ***true***.

b) Since $\left(\frac{1}{2}\right)^4<\left(\frac{1}{2}\right)^2$, the statement $\exists x\left(x^4<x^2\right)$ is ***true***.

c) Since $(-x)^2=(-1)^2x^2=x^2$, the statement $\forall x\left((-x)^2=x^2\right)$ is ***true***.

d) Since $2(-1)\not> -1$, the statement $\forall x(2x>x)$ is ***false***.

Exercise

Suppose that the domain of the propositional function $P(x)$ consists of the integers 1, 2, 3, 4, and 5.

Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

$a)\ \exists x\ P(x)$ $\quad b)\ \forall x\ P(x)$ $\quad c)\ \neg\exists x\ P(x)$ $\quad d)\ \neg\forall x\ P(x)$

$e)\ \forall x\ \left((x\ne 3)\to P(x)\right)\ \vee\ \exists x\ \neg P(x)$

Solution

a) The statement to be true, so either $P(1)$ is true or $P(2)$ is true or $P(3)$ is true or $P(4)$ is true or $P(5)$ is true. Thus, $P(1)\vee P(2)\vee P(3)\vee P(4)\vee P(5)$

b) $P(1)\wedge P(2)\wedge P(3)\wedge P(4)\wedge P(5)$

c) $\neg\left(P(1)\vee P(2)\vee P(3)\vee P(4)\vee P(5)\right)$

d) $\neg\left(P(1)\wedge P(2)\wedge P(3)\wedge P(4)\wedge P(5)\right)$

e) $\left(\left((1\neq 3)\to P(1)\right)\wedge\left((2\neq 3)\to P(2)\right)\wedge\left((3\neq 3)\to P(3)\right)\wedge\left((4\neq 3)\to P(4)\right)\wedge\left((5\neq 3)\to P(5)\right)\right)$

$\vee\left(\neg\left(P(1)\vee\neg P(2)\vee\neg P(3)\vee\neg P(4)\vee\neg P(5)\right)\right)$

Since the hypothesis $x\neq 3$ is false when $x=3$ and true when *x* is anything other than 3, we have

$\left(P(1)\wedge P(2)\wedge P(3)\wedge P(4)\wedge P(5)\right)\vee\left(\neg P(1)\vee\neg P(2)\vee\neg P(3)\vee\neg P(4)\vee\neg P(5)\right)$

This statement is always true, since the first part is not true, then the second part must be true.

Exercise

For each of these statements find a domain for which the statement is true and a domain for which the statement is false.

a) Everyone is studying discrete mathematics.
b) Everyone is older than 21 years.
c) Every two people have the same mother.
d) No Two different people have the same grandmother.

Solution

Let $A(x)$ *be "x everyone at the school"*

a) Let $B(x)$ *be "x is studying discrete mathematics"*. Then we have $\forall x\ B(x)$, or $\forall x\ \left(A(x)\to B(x)\right)$

b) Let $C(x)$ *be "x is older than 21 years"*. Then we have $\forall x\ C(x)$, or $\forall x\ \left(A(x)\to C(x)\right)$

c) Let $D(x)$ *be "x has the same mother,"* $E(x)$ *two people*. $\forall x\ \left(E(x)\to D(x)\right)$

d) Let $D(x)$ *be "x has the same grandmother,"* $E(x)$ *two people*. $\neg\forall x\ \left(E(x)\to D(x)\right)$

Exercise

Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) No one is perfect.
b) Not everyone is perfect.
c) All your friends are perfect.
d) At least one of your friends is perfect.
e) Everyone is your friend and is perfect.
f) Not everybody is your friend or someone is not perfect.

Solution

Let $X(x)$ *be "x is perfect"*; let $Y(x)$ *be "x is your friend"*

a) $\forall x\ \neg X(x)$. Alternatively, we can rewrite $\neg\exists x\ X(x)$

b) $\forall x\ \neg X(x)$

c) $\forall x\ (Y(x) \rightarrow X(x))$

d) $\exists x\ (Y(x) \wedge X(x))$

e) $\forall x\ (Y(x) \wedge X(x))$

f) This is a disjunction. The expression is $(\forall x\ \neg Y(x)) \vee (\exists x\ \neg X(x))$

Exercise

Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) Something is not in the correct place.
b) All tools are in the correct place and are in excellent condition.
c) Everything is in the correct place and in excellent condition.
d) Nothing is in the correct place and is in excellent condition.
e) One of your tools is not in the correct place, but it is in excellent condition.

Solution

Let $A(x)$ *be "x in the correct place"*; let $B(x)$ *be "x is in excellent condition"*; let $C(x)$ *be "x is a tool"*

a) $\exists x\ \neg A(x)$ *There exists something is not in the correct place.*

b) $\forall x\big(C(x) \rightarrow (A(x) \wedge B(x))\big)$

c) $\forall x\big(A(x) \wedge B(x)\big)$

d) $\forall x \neg\big(A(x) \wedge B(x)\big)$

e) $\exists x\big(C(x) \wedge \neg A(x) \wedge B(x)\big)$

Solution Section 1.4 – Nested Quantifiers

Exercise

Translate these statements into English, where the domain for each variable consists of all real numbers

a) $\forall x \exists y (x < y)$

b) $\exists x \forall y (xy = y)$

c) $\forall x \forall y \big(\big((x \ge 0) \wedge (y < 0) \big) \to (x - y > 0) \big)$

d) $\forall x \forall y \big(\big((x \ge 0) \wedge (y \ge 0) \big) \to (xy \ge 0) \big)$

e) $\forall x \forall y \exists z (xy = z)$

f) $\forall x \forall y \exists z (x = y + z)$

Solution

a) For every real number *x* there exists a real number *y* such that *x* is less than *y*. Basically, there is a larger number.

b) There exists real number *x* such that for every a real number *y*, $xy = y$. This is asserting the existence of a multiplication identity for the real numbers, and the statement is true, since we can take $x = 1$.

c) For every real number *x* and real number *y*, if *x* is nonnegative and *y* is negative, then the difference $x - y$ is positive. More simply, a nonnegative number minus a negative number is positive which is true.

d) For every real number *x* and real number *y*, if *x* is positive and *y* is positive, then the product xy is positive. More simply, a product of 2 positive numbers is positive.

e) For every real number *x* and real number *y*, there exists a real number *z* such that the product $xy = z$. More simply, the real numbers are closed under multiplication.

f) For every real number *x* and real number *y*, there exists a real number *z* such that the product $x = y + z$. This is a true statement, since we can take $z = x - y$ in each case.

Exercise

Let $Q(x, y)$ be the statement "*x* has sent an e-mail message to *y*," where the domain for both *x* and *y* consists of all students in your class. Express each of these quantifications in English

a) $\exists x \exists y Q(x, y)$

b) $\exists x \forall y Q(x, y)$

c) $\forall x \exists y Q(x, y)$

d) $\exists y \forall x Q(x, y)$

e) $\forall y \exists x Q(x, y)$

f) $\forall x \forall y Q(x, y)$

Solution

a) There exist students *x* and *y* such that *x* has sent a message to *y*.
In other words, there is some student in your class who has sent a message to some student in your class.

b) There exists a student *x* for every student *y* such that *x* has sent a message to every *y*.
In other words, there is a student in your class who has sent a message to every student in your class.

c) For every student *x* in your class there exists a student *y* such that *x* has sent a message to *y*.
In other words, every student in your class has sent a message to at least one student in your class.

d) There exists a student *y* for every student *x* such that *y* has sent a message to every *x*.
In other words, there is a student in your class who has sent a message to every student in your class.

e) For every student *y* in your class there exists a student *x* such that *y* has sent a message to *x*.
In other words, every student in your class has sent a message to at least one student in your class.

f) Every student in your class has sent a message to every student in your class.

Exercise

Express each of these statements using predicates, quantifiers, logical connectives, and mathematical operators where the domain consists of all integers.

a) The product of two negative integers is positive.

b) The average of two positive integers is positive.

c) The difference of two negative integers is not necessarily negative.

d) The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers.

Solution

a) $\forall x \forall y \left((x<0) \land (y<0) \rightarrow (xy>0) \right)$

b) $\forall x \forall y \left((x>0) \land (y>0) \ \rightarrow \ \dfrac{x+y}{2} > 0 \right)$

c) $\exists x \exists y\left((x<0)\wedge(y<0)\wedge(x-y\geq 0)\right)$

d) $\forall x \forall y\left(|x+y|\leq|x|+|y|\right)$

Exercise

Rewrite these statements so that the negations only appear within the predicates

a) $\neg\exists y \forall x P(x,y)$

b) $\neg\forall x \exists y P(x,y)$

c) $\neg\exists y\left(Q(y)\wedge\forall x\neg R(x,y)\right)$

Solution

a) $\neg\exists y \forall x P(x,y) \equiv \forall y \neg\forall x P(x,y)$

$\equiv \forall y \exists x \neg P(x,y)$

b) $\neg\forall x \exists y P(x,y) \equiv \exists x \neg\exists y P(x,y)$

$\equiv \exists x \forall y \neg P(x,y)$

c) $\neg\exists y\left(Q(y)\wedge\forall x\neg R(x,y)\right) \equiv \forall y \neg\left(Q(y)\wedge\forall x\neg R(x,y)\right)$

$\equiv \forall y\left(\neg Q(y)\vee\neg\left(\forall x\neg R(x,y)\right)\right)$

$\equiv \forall y\left(\neg Q(y)\vee\exists x R(x,y)\right)$

Exercise

Express the negations of each of these statements so that all negation symbols immediately precede predicates.

a) $\forall x \exists y \forall z T(x,y,z)$

b) $\forall x \exists y P(x,y)\vee\forall x \exists y Q(x,y)$

Solution

a) $\neg\left(\forall x \exists y \forall z T(x,y,z)\right) \equiv \neg\forall x \exists y \forall z T(x,y,z)$

$\equiv \exists x \neg\exists y \forall z T(x,y,z)$

$\equiv \exists x \forall y \neg\forall z T(x,y,z)$

$\equiv \exists x \forall y \exists z \neg T(x,y,z)$

b) $\neg\left(\forall x \exists y P(x,y)\vee\forall x \exists y Q(x,y)\right) \equiv \neg\left(\forall x \exists y P(x,y)\right)\wedge\neg\left(\forall x \exists y Q(x,y)\right)$

$\equiv \exists x \neg\left(\exists y P(x,y)\right)\wedge\exists x \neg\left(\exists y Q(x,y)\right)$

$\equiv \exists x \forall y \neg P(x,y)\wedge\exists x \forall y \neg Q(x,y)$

Exercise

Let $T(x,\ y)$ mean that student *x* likes cuisine *y*, where the domain for *x* consists of all students at your school and the domain for *y* consists of all cuisines. Express each of these statements by a simple English sentence.

a) $\neg T(A,\ J)$

b) $\exists x T(x,\ Korean) \wedge \forall x Tx(x,\ Mexican)$

c) $\exists y\left(T(Monique,\ y) \vee T(Jay,\ y)\right)$

d) $\forall x \forall z \exists y\left((x \neq z) \rightarrow \neg\left(T(x,y) \wedge T(z,y)\right)\right)$

e) $\exists x \exists z \forall y\left(T(x,y) \leftrightarrow T(z,y)\right)$

f) $\forall x \forall z \exists y\left(T(x,y) \leftrightarrow T(z,y)\right)$

Solution

a) *A* does not like *J* cuisine

b) Some student at your school likes Korean cuisine, and everyone at your school likes Mexican cuisine.

c) There is some cuisine that Monique and Jay likes.

d) For every pair of distinct students at your school, there is some cuisine that at least one them does not like.

e) There are two students at your school who have exactly the same cuisines (tastes).

f) For every pair of students at your school, there is some cuisine about which they have the same opinion (either they both like it or they both do not like it).

Exercise

Let $L(x,\ y)$ be the statement " *x* loves *y*", where the domain for both *x* and *y* consists of all people in the world. Use quantifiers to express each of these statements.

a) Everybody loves Jerry.
b) Everybody loves somebody.
c) There is somebody whom everybody loves.
d) Nobody loves everybody.
e) There is somebody whom Lois does not love.
f) There is somebody whom no one loves.
g) There is exactly one person whom everybody loves.
h) There are exactly two people whom ***L*** loves.
i) Everyone loves himself or herself.
j) There is someone who loves no one besides himself or herself.

Solution

a) $\forall x\ L(x,\ Jerry)$

b) $\forall x \exists y\, L(x, y)$

c) $\exists y \forall x\, L(x, y)$

d) $\neg \exists x \forall y\, L(x, y)$

e) $\exists x\, \neg L(Lois, x)$

f) $\exists x \forall y\, \neg L(x, y)$

g) $\exists x\left(\forall y\, L(y, x) \wedge \forall z\left(\left(\forall w\, L(w, z)\right) \rightarrow z = x\right)\right)$

h) $\exists x \exists y\left(x \neq y \wedge\, L(\boldsymbol{L}, x) \wedge\, L(\boldsymbol{L}, y) \wedge \forall z\left(L(\boldsymbol{L}, z) \rightarrow (z = x \vee z = y)\right)\right)$

i) $\forall x\, L(x, x)$

j) $\exists x \forall y\, \left(L(x, y) \leftrightarrow x = y\right)$

Exercise

Let $S(x)$ be the predicate "*x* is a student," $F(x)$ the predicate "*x* is a faculty member," $A(x, y)$ the predicate "*x* has asked *y* a question," where the domain consists of all people associated with your school. Use quantifiers to express each of these statements.

a) Lois asked Professor Fred a question.
b) Every student has asked Professor Fred a question.
c) Every faculty member has either asked Professor Fred a question or been asked a question by Professor Miller.
d) Some student has not asked any faculty member a question.
e) There is a faculty member who has never been asked a question by a student.
f) Some student has asked every faculty member a question.
g) There is a faculty member who has asked every other faculty member a question.
h) Some student has never been asked a question by a faculty member.

Solution

a) $A(Lois,\ \mathrm{Prof}.\, Fred)$

b) $\forall x\left(S(x) \rightarrow A(x,\ \mathrm{Prof}.\, Fred)\right)$

c) $\forall x\left(F(x) \rightarrow \left(A(x,\ \mathrm{Prof}.\, Fred) \vee A(\mathrm{Prof}.\, Miller,\ x)\right)\right)$

d) $\exists x\left(S(x) \wedge \forall y\left(F(y) \rightarrow \neg A(x, y)\right)\right)$ ***or*** $\exists x\left(S(x) \wedge \neg \exists y\left(F(y) \rightarrow A(x, y)\right)\right)$

e) $\exists x\left(F(x) \wedge \forall y\left(S(y) \rightarrow \neg A(y, x)\right)\right)$

f) $\exists x\left(S(x) \wedge \forall y\left(F(y) \rightarrow A(x, y)\right)\right)$

g) $\exists x\Big(F(x)\wedge \forall y\big(\big(F(y)\wedge y\neq x\big)\rightarrow A(x,\ y)\big)\Big)$

h) $\exists x\Big(S(x)\wedge \forall y\big(F(y)\rightarrow \neg A(y,\ x)\big)\Big)$

Exercise

Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.

a) Every user has access to exactly one mailbox.

b) There is a process that continues to run during all error conditions only if the kernel is working correctly.

c) All users on the campus network can access all websites whose url has a .edu extension.

Solution

a) $\forall y\exists m\Big(A(u,\ m)\wedge \forall n\big(n\neq m\rightarrow \neg A(u,\ n)\big)\Big)$, where $A(u,\ m)$ means that user *u* has access to mailbox *m*.

b) $\exists p\forall e\big(H(e)\rightarrow S(p,\ running)\big)\rightarrow S(kernel,\ working\ correctly)$, where $H(e)$ means that error condition *e* is in effect and $S(x,\ y)$ means that the status of *x* is *y*.

c) $\forall u\forall s\big(E(s,\ edu)\rightarrow A(u,\ s)\big)$, where $E(s,x)$ means that website *s* has extension *x*, and $A(u,\ s)$ means that user *u* can access website *s*.

Exercise

Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers

a) $\exists x\forall y(x+y=y)$

b) $\forall x\forall y\Big(\big((x\geq 0)\wedge(y<0)\big)\rightarrow(x-y>0)\Big)$

c) $\exists x\exists y\Big(\big((x\leq 0)\wedge(y\leq 0)\big)\wedge(x-y>0)\Big)$

d) $\forall x\forall y\Big(\big((x\neq 0)\wedge(y\neq 0)\big)\leftrightarrow(xy\neq 0)\Big)$

Solution

a) There exists an additive identity for all real numbers

b) A non-negative number minus a negative number is greater than zero.

c) The difference between two non-positive numbers is not necessarily non-positive (i.e. can be positive)

d) The product of two non-zero numbers is non-zero if and only if both factors are non-zero

Exercise

Determine the truth value of each of these statements if the domain for all variables consists of all integers

a) $\forall n \exists m \left(n^2 < m\right)$

b) $\exists n \forall m \left(n < m^2\right)$

c) $\forall n \exists m \ (n + m = 0)$

d) $\exists n \forall m \ (nm = m)$

e) $\exists n \exists m \left(n^2 + m^2 = 5\right)$

f) $\exists n \exists m \left(n^2 + m^2 = 6\right)$

g) $\exists n \exists m \ (n + m = 4 \ \wedge \ n - m = 1)$

h) $\exists n \exists m \ (n + m = 4 \ \wedge \ n - m = 2)$

i) $\forall n \forall m \exists p \left(p = \frac{m+n}{2}\right)$

Solution

a) No matter how large *n* might be, we can always find an integer *m* bigger than n^2. This is certainly true, i.e. $m = n^2 + 1$.

b) There is an *n* that is smaller than the square of every integer. This statement is true since we could take $n = -1$, and then *n* would be less than every square, since squares are always greater than or equal to 0.

c) The order of quantifiers: *m* is allowed to depend on *n*. since we can take $m = -n$, this statement is true.

d) Clearly $n = 1$, so the statement is true.

e) $n^2 + m^2 = 5$ has a solution over the integers. This is true statement, since $n = \pm 1, \ m = \pm 2$ and vice versa (8 solutions).

f) $n^2 + m^2 = 6$ there is no integer solution. Therefore; this statement is false.

g) There is a unique solution for the statement $\{n + m = 4, \ n - m = 1\}$, namely $n = \frac{5}{2} \ and \ m = \frac{3}{2}$.

Since there do not exist integers that make the equations true, the statement is false.

h) There is a unique solution for the statement $\{n + m = 4, \ n - m = 2\}$, namely $n = 3 \ and \ m = 1$.

Therefore; the statement is true.

i) If we take $n = 1 \ and \ m = 2$ then $p = \frac{3}{2}$ which is not an integer. Therefore; the statement is false.

Solution *Section* 1.5 – Introduction to Proofs

Exercise

Show that the square of an even number is an even number

Solution

We can rewrite the statement as: if n is even, then n^2 is even

Assume n is even, thus $n = 2k$ for some k.

$n^2 = (2k)^2 = 4k^2 = 2\left(2k^2\right)$

As n^2 is 2 times an integer, n^2 is thus even

Exercise

Prove that if n is an integer and $n^3 + 5$ is odd, then n is even

Solution

By indirect proof:

Using the contrapositive: If n is odd, then $n^3 + 5$ is even

Assume n is odd, let show that $n^3 + 5$ is even

$n = 2k + 1$ for some integer k (definition of odd numbers)

$n^3 + 5 = (2k+1)^3 + 5 = 8k^3 + 12k^2 + 6k + 6 = 2\left(4k^3 + 6k^2 + 3k + 3\right)$

As $n^3 + 5$ is 2 times an integer, it is even

Assume that $n^3 + 5$ is odd, let show that n is odd, and Assume p is true and q is false

$n = 2k + 1$ for some integer k (definition of odd numbers)

$n^3 + 5 = (2k+1)^3 + 5 = 8k^3 + 12k^2 + 6k + 6 = 2\left(4k^3 + 6k^2 + 3k + 3\right)$

As $n^3 + 5$ is 2 times an integer, it must be even. ***Contradiction***!

The indirect proof proved that the contrapositive: $\neg q \to \neg p$

If n is odd, then $n^3 + 5$ is even

The proof by contradiction assumed that the implication was false, and showed a contradiction

- If we assume p and $\neg q$, we can show that implies q
- The contradiction is q and $\neg q$
- Note that both used similar steps, but are different means of proving the implication

Exercise

Show that $m^2 = n^2$ if and only if $m = n \quad or \quad m = -n$

Solution

Rephrased: $m^2 = n^2 \leftrightarrow \left[(m=n) \vee (m=-n)\right]$. Proof by cases!

Case 1: $(m=n) \to \left(m^2 = n^2\right)$

$$(m)^2 = m^2 \quad and \quad (n)^2 = n^2$$, this case is proven.

Case 1: $(m=-n) \to \left(m^2 = n^2\right)$

$$(m)^2 = m^2 \quad and \quad (-n)^2 = n^2$$, this case is proven.

$m^2 = n^2 \leftrightarrow \left[(m=n) \vee (m=-n)\right]$

$m^2 - n^2 = n^2 - n^2$

$$m^2 - n^2 = 0 \quad \Rightarrow \quad (m-n)(m+n) = 0$$

$$m - n = 0 \quad or \quad m + n = 0$$

$$m = n \quad or \quad m = -n$$

Exercise

Use a direct proof to show that the sum of two odd integers is even.

Solution

Let m and n be two odd integers. Then there exists a and b such that $n = 2a+1$ and $m = 2b+1$.

$$\begin{aligned} n + m &= 2a + 1 + 2b + 1 \\ &= 2a + 2b + 2 \\ &= 2(a + b + 1) \end{aligned}$$

Since this represents $n+m$ as 2 times $a+b+1$, we conclude that $n+m$ is even, as desired.

Exercise

Use a direct proof to show that the sum of two even integers is even.

Solution

Let m and n be two even integers. Then there exists a and b such that $n = 2a$ and $m = 2b$.

$$\begin{aligned} n + m &= 2a + 2b \\ &= 2(a + b) \end{aligned}$$

Since this represents $n+m$ as 2 times $a+b$, we conclude that $n+m$ is even, as desired.

Exercise

Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.

Solution

Let *r* is a rational number an *s* is irrational number then $t = r + s$ is an irrational.
Suppose that *t* is rational, then if $t = \frac{a}{b}$ and $r = \frac{c}{d}$ where *a*, *b*, *c*, and *d* are integers with $b \neq 0$ and $d \neq 0$. Then, $t + (-r) = \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$ which is rational.
$t + (-r) = r + s - r = s$, forcing that *s* is rational. This contradicts the hypothesis that *s* is irrational.
Therefore the assumption that *t* was rational was incorrect, and we conclude that *t* is irrational.

Exercise

Prove or disprove that the product of two irrational numbers is irrational.

Solution

Let $\sqrt{2}$ be the irrational number,. If we take the product of the irrational number $\sqrt{2}$ and the irrational number $\sqrt{2}$, then we obtain the rational number 2. This counterexample refutes the proposition.

Exercise

Prove that if *x* is irrational, then $\frac{1}{x}$ is irrational.

Solution

The contrapositive is: if $\frac{1}{x}$ is rational, then *x* is rational.

Since $\frac{1}{x}$ exists, then $x \neq 0$. If $\frac{1}{x}$ is rational then by definition $\frac{1}{x} = \frac{q}{p}$ and $p \neq 0$. Since $\frac{1}{x}$ can't be zero, then we would have the contradiction $1 = x \cdot 0$.

Exercise

Prove that if *x* is rational and $x \neq 0$, then $\frac{1}{x}$ is rational.

Solution

if *x* is rational and $x \neq 0$, then by definition we can write $x = \frac{p}{q}$, where *p* and *q* are nonzero integers.

Since $\frac{1}{x} = \frac{q}{p}$ and $p \neq 0$, we can conclude that $\frac{1}{x}$ is rational.

Exercise

Prove the proposition $P(0)$, where $P(n)$ is the proposition "If *n* is a positive integer greater than 1, then $n^2 > n$." What kind of proof did you use?

Solution

The proposition that we are trying to prove is If 0 is a positive integer gr2ater than 1, then $0^2 = 0$. Our proof is a vacuous one.
Since the hypothesis is false, the implication is automatically true.

Exercise

Let $P(n)$ be the proposition "If *a* and *b* are positive real numbers, then $(a+b)^n \ge a^n + b^n$." Prove that $P(1)$ is true. What kind of proof did you use?

Solution

Our proof is a direct one. By the definition of exponential, any real number to the power 1 is itself. Hence $(a+b)^1 = a+b = a^1 + b^1$. Finally, by the addition rule, we can conclude from $(a+b)^1 = a^1 + b^1$ that $(a+b)^1 \ge a^1 + b^1$.

Exercise

Show that these statements about the integer *x* are equivalent:

$\boldsymbol{i})\ 3x+2$ *is even* $\quad \boldsymbol{ii})\ x+5$ *is odd* $\quad \boldsymbol{iii})\ x^2$ *is even*

Solution

If *x* is even, then $x = 2k$ for some integer *k*.

$3x+2 = 3\cdot 2k + 2 = 6k+2 = 2(3k+1)$ which is even.

$x+5 = 2k+4+1 = 2(k+2)+1$, so $x+5$ is odd

$x^2 = (2k)^2 = 4k^2 = 2\left(2k^2\right)$, so x^2 is odd

If *x* is odd, then $x = 2k+1$ for some integer *k*.

$3x+2 = 3\cdot(2k+1)+2 = 6k+3+2 = 6k+4+1 = 2(3k+2)+1$ which is odd *not* even.

$x+5 = 2k+1+5 = 2k+6 = 2(k+3)$, so $x+5$ is even not odd.

$x^2 = (2k+1)^2 = 4k^2+4k+1 = 2\left(2k^2+2k\right)+1$, so x^2 is odd

Exercise

Show that these statements about the real number x are equivalent:

$\boldsymbol{i})\ x$ *is irrational* $\quad \boldsymbol{ii})\ 3x+2$ *is irrational* $\quad \boldsymbol{iii})\ \frac{x}{2}$ *is irrational*

Solution

The simplest way is to approach in indirect proof.

$\boldsymbol{i}) \to \boldsymbol{ii})$

Suppose that $3x+2$ is rational, that $3x+2=\frac{p}{q}$ for some integers p and q with $q \neq 0$. Then

$3x=\frac{p}{q}-2=\frac{p-2q}{q} \quad \Rightarrow \quad x=\frac{p-2q}{3q}$ where $3q \neq 0$. This shows that x is rational.

Suppose that x is rational, that $x=\frac{p}{q}$ for some integers p and q with $q \neq 0$. Then

$3x+2=3\frac{p}{q}-2=\frac{3p-2q}{q}$ where $q \neq 0$. This shows that $3x+2$ is rational.

$\boldsymbol{i}) \to \boldsymbol{iii})$

Suppose that $\frac{x}{2}$ is rational, that $\frac{x}{2}=\frac{p}{q}$ for some integers p and q with $q \neq 0$. Then $x=\frac{2p}{q}$ where $q \neq 0$. This shows that x is rational.

Suppose that x is rational, that $x=\frac{p}{q}$ for some integers p and q with $q \neq 0$. Then $\frac{x}{2}=\frac{p}{2q}$ where $2q \neq 0$. This shows that $\frac{x}{2}$ is rational.

Exercise

Prove that at least one of the real numbers $a_1,\ a_2,\ \ldots,\ a_n$ is greater than or equal to the average of these numbers. What kind of proof did you use?

Solution

Using proof of contradiction, then suppose all the number $a_1,\ a_2,\ \ldots,\ a_n$ are less than their average.

$a_1+a_2+\ldots+a_n<nA$

By definition: $A=\frac{a_1+a_2+\ldots+a_n}{n}$

The two displayed formulas clearly contradict each other, however: they imply that $nA<nA$. Thus our assumption must have been incorrect, and we conclude that at least one of the numbers a_1 is greater than or equal to their average.

Exercise

Prove that $n^2 + 1 \ge 2^n$ when n is a positive integer with $1 \le n \le 4$

Solution

$n = 1 \quad \rightarrow \quad 1^2 + 1 \ge 2^1 \Rightarrow 2 \ge 2 \quad \surd$

$n = 2 \quad \rightarrow \quad 2^2 + 1 \ge 2^2 \Rightarrow 5 \ge 4 \quad \surd$

$n = 3 \quad \rightarrow \quad 3^2 + 1 \ge 2^3 \Rightarrow 10 \ge 8 \quad \surd$

$n = 4 \quad \rightarrow \quad 4^2 + 1 \ge 2^4 \Rightarrow 17 \ge 16 \quad \surd$

Exercise

Prove that there are no positive perfect cubes less than 1000 that are the sum of the cubes of two positive integers.

Solution

The cubes are: 1, 8, 27, 64, 125, 216, 343, 512, and 729.

1 + 8 = 9, 1 + 27 = 28, 1 + 64, 1 + 125, …

8 + 8, 8 + 27, 8 + 64, 8 + 125, …

27 + 27, 27 + 64, 27 + 125, …

64 + 64, 64 + 125, 64 + 216, …

125 + 125, 125 + 216, …

216 + 216, 216 + 343, …

343 + 343, 343 + 512, 343 + 729

512 + 512, 512 + 729

729 + 729

None of them works.

We can conclude the no cube less than 1000 is the sum of two cubes.

Exercise

Prove that if x and y are real numbers, then $\max(x, y) + \min(x, y) = x + y$. (*Hint*: Use a proof by cases, with the two cases corresponding to $x \ge y$ and $x < y$, respectively.)

Solution

Suppose that $x \ge y$, then by definition $max(x,\ y) = x$ and $min(x,\ y) = y$. Therefore; in this case $\max(x, y) + \min(x, y) = x + y$.

In the second case $x < y$, then by definition $max(x,\ y) = y$ and $min(x,\ y) = x$. Therefore; in this case, $\max(x, y) + \min(x, y) = y + x = x + y$.

Hence in all cases, the equality holds.

Exercise

Prove the triangle inequality, which states that if x and y are real numbers, then $|x| + |y| \ge |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \ge 0$ and equals $-x$ if $x < 0$

Solution

If x and y are both nonnegative, then $|x| + |y| = x + y = |x + y|$.

If x and y are both negative, then $|x| + |y| = (-x) + (-y) = -(x + y) = |x + y|$.

If $x \ge 0$ and $y < 0$, then there are two subcases to consider for x and $-y$:

Case 1: Suppose that $x \ge -y$, then $x + y \ge 0$. Therefore $x + y = |x + y|$, as desired.

$|x| + |y| = x + |y|$ is a positive number greater than x. Therefore $|x + y| < x < |x| + |y|$

Case 2: Suppose that $x < -y$, then $x + y < 0$. Therefore $|x + y| = -(x + y) = (-x) + (-y)$.

is a positive number less than or equal to $-y$. Therefore $|x + y| \le -y \le |x| + |y|$, as desired.

Exercise

Prove that either $2 \cdot 10^{500} + 15$ or $2 \cdot 10^{500} + 16$ is not a perfect square

Solution

A perfect square is a square of an integer

Rephrased: Show that a non-perfect square exists in the set $\left\{2 \cdot 10^{500} + 15,\ \ 2 \cdot 10^{500} + 16\right\}$

Proof: The only two perfect squares that differ by 1 are 0 and 1

Thus, any other numbers that differ by 1 cannot both be perfect squares

Thus, a non-perfect square must exist in any set that contains two numbers that differ by 1

Note that we didn't specify which one it was!

Exercise

Prove that there exists a pair of consecutive integers such that one of these integers is a perfect square and the other is a perfect cube.

Solution

$8 = 2^3 \qquad 9 = 3^2$

Exercise

Suppose that a and b are odd integers with $a \neq b$. Show there is a unique integer c such that $|a-c| = |b-c|$

Solution

The equation $|a-c| = |b-c|$ is equivalent to the disjunction of two equations:

$a-c = b-c$ *or* $a-c = -b+c$

Case: $a-c = b-c$ is equivalent to $a = b$, which contradicts the assumption $a \neq b$, so the original equation is equivalent to $a-c = -b+c$. By adding $b+c$ to both sides and dividing by 2, we see that this equation is equivalent to $c = \frac{a+b}{2}$. Thus there is a unique solution. Furthermore, this c is an integer, because the sum of the odd integers a and b is even.

Solution   Section 1.7 – Sets

Exercise

List the members of these sets

a) $\left\{x \middle| x \text{ is a real number such that } x^2 = 1\right\}$

b) $\left\{x \middle| x \text{ is a positive integer less than } 12\right\}$

c) $\left\{x \middle| x \text{ is the square of an integer and } x < 100\right\}$

d) $\left\{x \middle| x \text{ is an integer such that } x^2 = 2\right\}$

Solution

a) $\{-1,\ 1\}$

b) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

c) $\{0, 1, 4, 9, 16, 25, 36, 49, 64, 81\}$

d) $\varnothing$ $\left\{\sqrt{2} \text{ is not an integer}\right\}$

Exercise

Determine whether each these pairs of sets are equal.

a) $\{1, 3, 3, 3, 5, 5, 5, 5, 5\}$, $\{5, 3, 1\}$

b) $\{\{1\}\}$, $\{1,\ \{1\}\}$

c) $\varnothing$, $\{\varnothing\}$

Solution

a) Yes; order and repetition do not matter.

b) No; the first set has one element, and the second has two elements.

c) No; the first set has no elements, and the second has one element (namely the empty set).

Exercise

For each of the following sets, determine whether 2 is an element of that set.

a) $\left\{x \in \mathbb{R} \middle| x \text{ is an integer greater than } 1\right\}$

b) $\left\{x \in \mathbb{R} \middle| x \text{ is the square of an integer}\right\}$

c) $\{2,\ \{2\}\}$

d) $\{\{2\}, \{\{2\}\}\}$

e) $\{\{2\}, \{2, \{2\}\}\}$

f) $\{\{\{2\}\}\}$

Solution

a) Since 2 is an integer greater than 1, 2 is an element of this set.

b) Since 2 is not a perfect square, 2 is not an element of this set.

c) This set has two elements, and clearly one of those elements is 2.

d) This set has two elements, and clearly neither of those elements is 2. Both of the elements of this set are seats; 2 is a number, not a set.

e) This set has two elements, and clearly neither of those elements is 2. Both of the elements of this set are seats; 2 is a number, not a set.

f) This set has just one element, namely the set $\{\{2\}\}$. So 2 is not an element of this set. Note $\{2\} \neq \{\{2\}\}$

Exercise

Determine whether each of these statements is true or false

a) $0 \in \varnothing$

b) $\varnothing \in \{0\}$

c) $\{0\} \subset \varnothing$

d) $\varnothing \subset \{0\}$

e) $\{0\} \in \{0\}$

f) $\{0\} \subset \{0\}$

g) $\{\varnothing\} \subseteq \{\varnothing\}$

h) $x \in \{x\}$

i) $\{x\} \subseteq \{x\}$

j) $\{x\} \in \{x\}$

k) $\{x\} \in \{\{x\}\}$

l) $\varnothing \subseteq \{x\}$

m) $\varnothing \in \{x\}$

Solution

a) *False*, since the empty set has no elements.

b) *False*, the set on the right has only one element, namely 0, not the empty set.

c) *False*, the empty set has no proper subsets.

d) *True*, every element of the set on the left, is vacuously, an element of the set on the right; and the set on the right contains an element, namely 0, that is not the set on the left.

e) *False*, the set on the right has only one element, namely 0, not the set containing the number 0.

f) *False*, for one set to be a proper subset of another, the two sets cannot be equal.

g) *True*, every set is a subset of itself.

h) *True*, x is the only element.

i) *True*, every set is a subset of itself.

j) *False*, the only element of $\{x\}$ is a letter, not a set.

k) *True*, $\{x\}$ is the only element

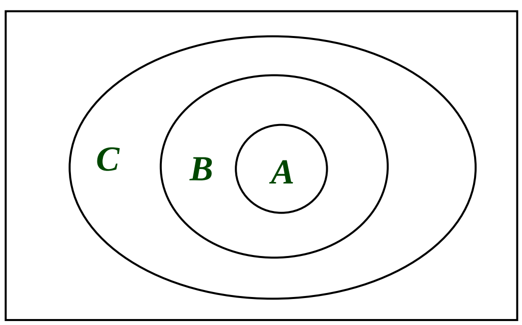
l) *True*, the empty set is a subset of every set.

m) *False*, the only element of $\{x\}$ is a letter, not a set.

Exercise

Use a Venn Diagram to illustrate the relationships $A \subset B$ and $B \subset C$.

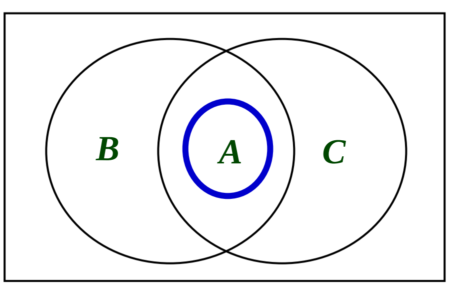
Solution



Exercise

Use a Venn Diagram to illustrate the relationships $A \subset B$ and $A \subset C$.

Solution



Since no information about the relationship between B and C, then B and C can be overlap. The set A must be a subset of each of them.

Exercise

Suppose that *A*, *B*, and *C* are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$

Solution

Let $x \in A$, then since $A \subseteq B$, we can conclude that $x \in B$. Furthermore, since $B \subseteq C$, the fact that $x \in B$ implies that $x \in C$. Therefore, $A \subseteq C$

Exercise

What is the cardinality of each of these sets?

a) $\{a\}$

b) $\{\{a\}\}$

c) $\{a, \{a\}\}$

d) $\{a, \{a\}, \{a, \{a\}\}\}$

Solution

a) 1

b) 1

c) 2

d) 3

Exercise

How many elements does each of these sets have where *a* and *b* are distinct elements?

a) $\mathcal{P}(\{a, b, \{a, b\}\})$

b) $\mathcal{P}(\{\varnothing, a, \{a\}, \{\{a\}\}\})$

c) $\mathcal{P}(\mathcal{P}(\varnothing))$

Solution

a) Since the set has 3 elements, the power of the set has $2^3 = 8$ elements

b) Since the set has 4 elements, the power of the set has $2^4 = 16$ elements

c) Since the set has 0 elements, the power of the set has $2^0 = 1$ elements. The power of this set therefore has $2^1 = 2$ elements.

Exercise

What is the Cartesian product $A \times B \times C$, where *A* is the set of all airlines and *B* and *C* are both the set of all cities in the United States? Give an example of how this Cartesian product can be used.

Solution

This is the set of triples $(a,\ b,\ c)$, where *a* is an airline and *b* and *c* are cities. A useful subset of this set of triples $(a,\ b,\ c)$ for which *a* flies between *b* and *c*. For example, Continental, Houston, Chicago) is in this subset.

Exercise

What is the Cartesian product $A \times B$, where *A* is the set of all courses offered by the mathematics department and *B* is the set of mathematics professors at this university? Give an example of how this Cartesian product can be used.

Solution

By definition it is the set of all ordered pairs $(c,\ p)$ such that *c* is a course and *p* is a professor. The elements of this set are the possible teaching assignments for the mathematics department.

Exercise

Let *A* be a set. Show that $\varnothing \times A = A \times \varnothing = \varnothing$

Solution

By definition, $\varnothing \times A$ consists of al pairs $(x,\ a)$ such that $x \in \varnothing$ and $a \in A$. Since there are no elements $x \in \varnothing$. There are no such pairs, so $\varnothing \times A = \varnothing$. Similar reasoning $A \times \varnothing = \varnothing$.

Solution Section 1.8 – Set Operations

Exercise

Let A be the set of students who live within one mile of school and let B be the set of students who walk to classes. Describe the students in each of these sets

a) $A \cap B$
b) $A \cup B$
c) $A - B$
d) $B - A$

Solution

a) The set of students who live one mile of school and walk to classes.

b) The set of students who live one mile of school or walk to classes.

c) The set of students who live one mile of school but not walk to class.

d) The set of students who live more than one mile from school but nevertheless walk to class.

Exercise

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$

a) $A \cup B$
b) $A \cap B$
c) $A - B$
d) $B - A$

Solution

a) $\{0, 1, 2, 3, 4, 5, 6\}$

b) $\{3\}$

c) $(1, 2, 4, 5\}$

d) $\{0, 6\}$

Exercise

Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$

a) $A \cup B$
b) $A \cap B$
c) $A - B$
d) $B - A$

Solution

a) $\{a, b, c, d, e, f, g, h\} = B$

b) $\{a, b, c, d, e\} = A$

c) $\varnothing$, since there are no elements in A that are not in B.

d) $\{f, g, h\}$

Exercise

Prove the domination laws by showing that

a) $A \bigcup U = U$

b) $A \bigcap U = A$

c) $A \bigcup \varnothing = A$

d) $A \bigcap \varnothing = \varnothing$

Solution

a)
$$\begin{aligned} A \bigcup U &= \left\{ x \middle| x \in A \vee x \in U \right\} \\ &= \left\{ x \middle| x \in A \vee \boldsymbol{T} \right\} \\ &= \left\{ x \middle| \boldsymbol{T} \right\} \\ &= U \end{aligned}$$

b)
$$\begin{aligned} A \bigcap U &= \left\{ x \middle| x \in A \wedge x \in U \right\} \\ &= \left\{ x \middle| x \in A \wedge \boldsymbol{T} \right\} \\ &= \left\{ x \middle| x \in A \right\} \\ &= A \end{aligned}$$

c)
$$\begin{aligned} A \bigcup \varnothing &= \left\{ x \middle| x \in A \vee x \in \varnothing \right\} \\ &= \left\{ x \middle| x \in A \vee \boldsymbol{F} \right\} \\ &= \left\{ x \middle| x \in A \right\} \\ &= A \end{aligned}$$

d)
$$\begin{aligned} A \bigcap \varnothing &= \left\{ x \middle| x \in A \wedge x \in \varnothing \right\} \\ &= \left\{ x \middle| x \in A \wedge \boldsymbol{F} \right\} \\ &= \left\{ x \middle| \boldsymbol{F} \right\} \\ &= \varnothing \end{aligned}$$

Exercise

Prove the complement laws by showing that

a) $A \bigcup \overline{A} = U$

b) $A \bigcap \overline{A} = \varnothing$

Solution

a) $A \cup \overline{A} = \{x \mid x \in A \vee x \in \overline{A}\}$

$= \{x \mid x \in A \vee x \notin A\}$

$= \{x \mid \boldsymbol{T}\}$

$= U$

b) $A \cap \overline{A} = \{x \mid x \in A \wedge x \in \overline{A}\}$

$= \{x \mid x \in A \wedge x \notin A\}$

$= \{x \mid \boldsymbol{F}\}$

$= \varnothing$

Exercise

Show that

a) $A - \varnothing = A$

b) $\varnothing - A = \varnothing$

Solution

a) $A - \varnothing = \{x \mid x \in A \wedge x \notin \varnothing\}$

$= \{x \mid x \in A \wedge \boldsymbol{T}\}$

$= \{x \mid x \in A\}$

$= A$

b) $\varnothing - A = \{x \mid x \in \varnothing \wedge x \notin A\}$

$= \{x \mid \boldsymbol{F} \wedge x \notin A\}$

$= \{x \mid \boldsymbol{F}\}$

$= \varnothing$

Exercise

Prove the absorption law by showing that if *A* and *B* are sets, then

a) $A \cap (A \cup B) = A$

b) $A \cup (A \cap B) = A$

Solution

a) Suppose $x \in A \cap (A \cup B)$, then $x \in A$ and $x \in A \cup B$ by the definition of intersection. We have $x \in A$ and in the latter case $x \in A$ *or* $x \in B$ by the definition of union. Since both of these are true, $x \in A \cup B$ by the definition of intersection, and we have shown that the right-hand side is a subset of the left-hand side.

b) Suppose $x \in A \bigcup (A \bigcap B) \quad \Rightarrow \quad x \in A$ *or* $x \in (A \bigcap B)$ by definition of union.

$x \in A$ *or* $(x \in A$ *and* $x \in B)$

By the definition of the intersection, in any event, $x \in A$. Therefore, $x \in A \bigcup (A \bigcap B)$ as well.

That proves that the right-hand side is a subset of the left-hand side.

Exercise

Show that if *A*, *B*, and *C* are sets, then $\overline{A \bigcap B \bigcap C} = \overline{A} \bigcup \overline{B} \bigcup \overline{C}$

Solution

Suppose $x \in \overline{A \bigcap B \bigcap C}$, then $x \notin A \bigcap B \bigcap C$, which means that *x* fails to be in at least one of these three sets. In other words, $x \notin A$ *or* $x \notin B$ *or* $x \notin C$. This is equivalent to saying that $x \in \overline{A}$ *or* $x \in \overline{B}$ *or* $x \in \overline{C}$. Therefore $x \in \overline{A} \bigcup \overline{B} \bigcup \overline{C}$.

Conversely, if $x \in \overline{A} \bigcup \overline{B} \bigcup \overline{C}$, then $x \in \overline{A}$ *or* $x \in \overline{B}$ *or* $x \in \overline{C}$. This means $x \notin A$ *or* $x \notin B$ *or* $x \notin C$, so *x* cannot be in the intersection of *A*, *B*, and *C*. Since $x \notin A \bigcap B \bigcap C$, we conclude that $x \in \overline{A \bigcap B \bigcap C}$, as desired.

Or

A	B	C	$A \bigcap B \bigcap C$	$\overline{A \bigcap B \bigcap C}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A} \bigcup \overline{B} \bigcup \overline{C}$
1	1	1	**1**	**0**	**0**	**0**	**0**	**0**
1	1	0	**0**	**1**	**0**	**0**	**1**	**1**
1	0	1	**0**	**1**	**0**	**1**	**0**	**1**
1	0	0	**0**	**1**	**0**	**1**	**1**	**1**
0	1	1	**0**	**1**	**1**	**0**	**0**	**1**
0	1	0	**0**	**1**	**1**	**0**	**1**	**1**
0	0	1	**0**	**1**	**1**	**1**	**0**	**1**
0	0	0	**0**	**1**	**1**	**1**	**1**	**1**

Exercise

Let A and B be sets. Show that

a) $(A \cap B) \subseteq A$

b) $A \subseteq (A \cup B)$

c) $(A - B) \subseteq A$

d) $A \cap (B - A) = \varnothing$

e) $A \cup (B - A) = A \cup B$

Solution

a) If x is in $A \cap B$, then, by definition of intersection, it is in A.

b) If x is in A, then perforce, by definition of union, it is in $A \cup B$.

c) If x is in $A - B$, then perforce, by definition of difference, it is in A.

d) Is $x \in A$ then $x \notin B - A$. Therefore there can be no elements in $A \cap (B - A)$, so $A \cap (B - A) = \varnothing$.

e) The left-hand side consists of elements of either A or B or both. This is precisely the definition of the right-hand side.

Exercise

Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

a) $A \cap (B - C)$

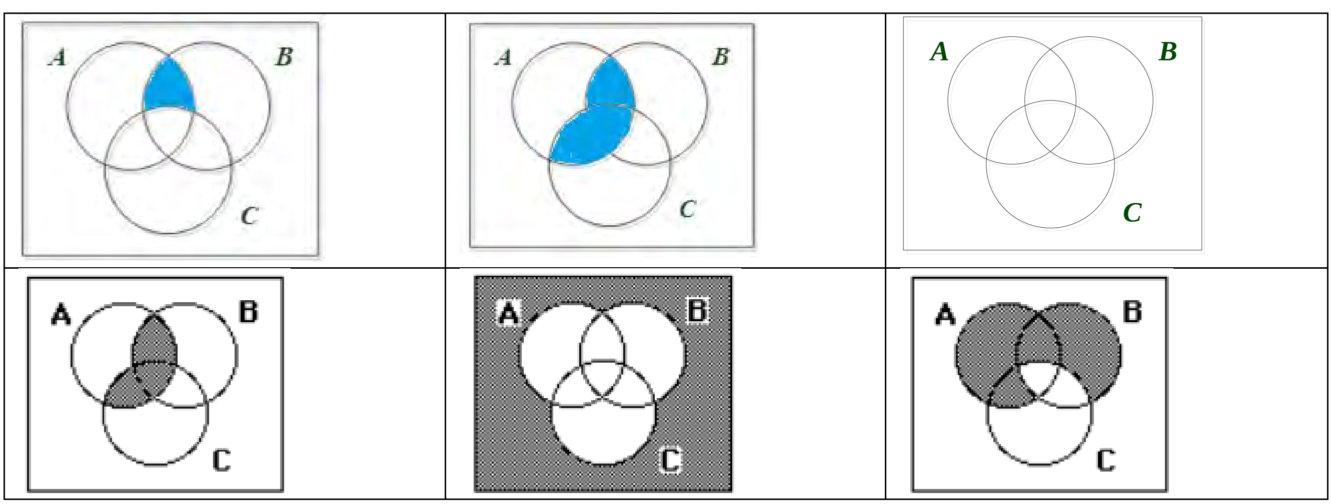
b) $(A \cap B) \cup (A \cap C)$

c) $(A \cap \overline{B}) \cup (A \cap \overline{C})$

d) $\overline{A} \cap \overline{B} \cap \overline{C}$

e) $(A - B) \cup (A - C) \cup (B - C)$

Solution



Exercise

Show that $A \oplus B = (A \cup B) - (A \cap B)$

Solution

This is just a restatement of the definition. An element is in $(A \cup B) - (A \cap B)$ if it in the union that in either *A* or *B*, but not in the intersection (i.e., not in both *A* and *B*)

Exercise

Show that $A \oplus B = (A - B) \cup (B - A)$

Solution

There are two ways that an item can be in either *A* or *B* but not both. It can be in *A* but not *B* (which is equivalent to saying that it is in $A - B$), or it can be in *B* but not *A* (which is equivalent to saying that it is in $B - A$).

Thus an element is in $A \oplus B$ if and only if it is in $(A - B) \cup (B - A)$.

Solution Section 1.9 – Functions

Exercise

Why is *f* not a function from $\mathbb{R}$ to $\mathbb{R}$ if

a) $f(x)=\frac{1}{x}$?

b) $f(x)=\sqrt{x}$?

c) $f(x)=\pm\sqrt{x^2+1}$?

Solution

a) Because for $x=0$ the value of $f(x)$ is not defined by the given rule.

b) Because for $x<0$ the value of $f(x)$ is not defined in $\mathbb{R}$

c) Because for $f(1)=\sqrt{2}$ *or* $f(1)=-\sqrt{2}$

Exercise

Determine whether *f* is a function from $\mathbb{Z}$ to $\mathbb{R}$ if

a) $f(x)=\pm x$?

b) $f(x)=\sqrt{x^2+1}$?

c) $f(x)=\frac{1}{x^2-4}$?

Solution

a) This is not a function because $f(1)=1$ *or* $f(1)=-1$,

b) This is a function for all integers *x*.

c) This is not a function since for $x=\pm 2$ the value of $f(x)$ is not defined by the given rule.

Exercise

Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.

a) The function that assigns to each bit string the number of ones in the string minus the number of zeros in the string.

b) The function that assigns to each bit string twice the number of zeros in that string.

c) The function that assigns the number of bits over when a bit string is split into bytes (which are blocks of 8 bits).

Solution

a) The domain is the set of all bit strings. Depending on how we read the word "difference" the output values can be either integers or natural numbers. For example. If the input is 1010000, then one might read the rule as stating that the function value is 3, since there are three more 0s than 1s; but most people would probably consider the function value to be –3, obtained by subtracting in the order stated: $2-5=-3$. Then the range is $\mathbb{Z}$.

b) The domain is the set of all bit strings. Since there can be any natural number of 0s in a bit string, the value of the function can be 0, 2, 4, …. Therefore the range is the set of even natural numbers.

c) The domain is the set of all bit strings. Since the number of leftover bits can be any whole number from 0 to 7 (if it were more, then we could form another byte), the range is (0, 1, 2, 3, 4, 5, 6, 7).

Exercise

Determine whether each of these functions from {*a*, *b*, *c*, *d*} to itself is one-to-one and onto.

a) $f(a)=b,\ f(b)=a,\ f(c)=c,\ f(d)=d$

b) $f(a)=b,\ f(b)=b,\ f(c)=d,\ f(d)=c$

c) $f(a)=d,\ f(b)=b,\ f(c)=c,\ f(d)=d$

Solution

a) This is one–to–one.

b) This is not one–to–one, since *b* is the image of both *a* and *b*.

c) This is not one–to–one, since *d* is the image of both *a* and *d*.

Exercise

Determine whether the function $f:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ is onto if

a) $f(m,\ n)=m+n$

b) $f(m,\ n)=m^2+n^2$

c) $f(m,\ n)=m$

d) $f(m,\ n)=|n|$

e) $f(m,\ n)=m-n$

f) $f(m,\ n)=2m-n$

g) $f(m,\ n)=m^2-n^2$

h) $f(m,\ n)=m+n+1$

i) $f(m,\ n)=|m|-|n|$

j) $f(m,\ n)=m^2-4$

Solution

a) Given any integer *n*, we have $f(0,\ n)=n$, so the function is onto.

b) The range contains no negative integers, so the function is not onto.

c) Given any integer *m*, we have $f(m,\ 4)=m$, so the function is onto.

d) The range contains no negative integers, so the function is not onto.

e) Given any integer *m*, we have $f(m,\ 0)=m$, so the function is onto.

f) For any integer *n*, we have $f(0,\ -n)=n$, so the function is onto.

g) $m^2-n^2=(m-n)(m+n)\neq 2$, since 2 is not in the range, so the function is not onto.

h) For any integer *n*, we have $f(0,\ n-1)=n$, so the function is onto.

i) This is onto. To achieve negative values we set $m=0$, and to achieve nonnegative value we set $n=0$.

j) $m^2-4=(m-2)(m+2)\neq 2$ since 2 is not in the range, so the function is not onto.

Exercise

Determine whether each of these functions is a bijection from $\mathbb{R}\to\mathbb{R}$

a) $f(x)=2x+1$

b) $f(x)=x^2+1$

c) $f(x)=x^3$

d) $f(x)=\dfrac{x^2+1}{x^2+2}$

e) $f(x)=x^5+1$

Solution

a) $f(x)=2x+1 \quad\Rightarrow\quad f^{-1}(x)=\frac{x-1}{2}$. Therefore the function is a bijection.

b) It is not one-to-one since $f(1)=f(-1)=2$ and it is also not onto since the range is the interval $[1,\ \infty)$.

c) $f(x)=x^3 \quad\Rightarrow\quad f^{-1}(x)=\sqrt[3]{x}$. Therefore the function is a bijection.

d) It is not one-to-one since $f(1)=f(-1)=\frac{2}{3}$ and it is also not onto since the range is the interval $\left[\frac{1}{2},\ \infty\right)$.

e) $f(x)=x^5+1 \quad\Rightarrow\quad f^{-1}(x)=\sqrt[5]{x-1}$. Therefore the function is a bijection.

Exercise

Suppose that g is a function from A to B and f is a function from B to C.

a) Show that if both f and g are one–to–one functions, then $f \circ g$ is also one–to–one.

b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.

Solution

a) Assume that both f and g are one–to–one. We need to show that $f \circ g$ is also one–to–one. If x and y are two distinct elements of A, then $f\left(g(x)\right) \neq f\left(g(y)\right)$. First, since g is one–to–one, by definition $g(x) \neq g(y)$. Second, since now $g(x)$ and $g(y)$ are distinct elements of B, and since f is one–to–one, we conclude that $f\left(g(x)\right) \neq f\left(g(y)\right)$ as desired.

b) Assume that both f and g are onto. We need to show that $f \circ g$ is onto. If z is any element of C, then there is some element $x \in A$ such that $f\left(g(x)\right) = z$. First, since f is onto, we can conclude that there is an element $y \in B$ such that $f(y) = z$. Second, since g is onto and $y \in B$, we can conclude that there is an element $x \in A$ such that $g(x) = y$. therefore $z = f(y) = f\left(g(x)\right)$ as desired.