Solution *Section* 4.1 – Infinite Sequences and Summation Notation

Exercise

Find the first four terms and the eight term of the sequence: $\{12-3n\}$

Solution

$$a_n = 12 - 3n$$

$$a_1 = 12 - 3(1) = 9, \quad a_2 = 12 - 3(2) = 6, \quad a_3 = 12 - 3(3) = 3, \quad a_4 = 12 - 3(4) = 0$$

$$a_8 = 12 - 3(8) = -12$$

$$\boxed{9,\ 6,\ 3,\ 0;\ -12}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\dfrac{3n-2}{n^2+1}\right\}$

Solution

$$a_n = \frac{3n-2}{n^2+1}$$

$$a_1 = \frac{3-2}{1^2+1} = \frac{1}{2}, \quad a_2 = \frac{3(2)-2}{2^2+1} = \frac{4}{5}, \quad a_3 = \frac{3(3)-2}{3^2+1} = \frac{7}{10}, \quad a_4 = \frac{3(4)-2}{4^2+1} = \frac{10}{17}$$

$$a_8 = \frac{3(8)-2}{8^2+1} = \frac{22}{65}$$

$$\boxed{\frac{1}{2},\ \frac{4}{5},\ \frac{7}{10},\ \frac{10}{17};\ \frac{22}{65}}$$

Exercise

Find the first four terms and the eight term of the sequence: $\{9\}$

Solution

$$\boxed{9,\ 9,\ 9,\ 9;\ 9}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{(-1)^{n-1}\dfrac{n+7}{2n}\right\}$

Solution

$$a_1 = (-1)^{1-1}\frac{1+7}{2(1)} = 4, \quad a_2 = (-1)^{2-1}\frac{2+7}{2(2)} = -\frac{9}{4},$$

$$a_3 = (-1)^{3-1}\frac{3+7}{2(3)} = \frac{5}{3}, \quad a_4 = (-1)^{4-1}\frac{4+7}{2(4)} = -\frac{11}{8}$$

$$a_8 = (-1)^{8-1}\frac{8+7}{2(8)} = -\frac{15}{16}$$

$$\boxed{4,\ -\frac{9}{4},\ \frac{5}{3},\ -\frac{11}{8};\ -\frac{15}{16}}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\dfrac{2^n}{n^2+2}\right\}$

Solution

$$a_1 = \frac{2^1}{1^2+2} = \frac{2}{3}, \quad a_2 = \frac{2^2}{2^2+2} = \frac{2}{3},$$

$$a_3 = \frac{2^3}{3^2+2} = \frac{8}{11}, \quad a_4 = \frac{2^4}{4^2+2} = \frac{8}{9}$$

$$a_8 = \frac{2^8}{8^2+2} = \frac{128}{33}$$

$$\boxed{\frac{2}{3},\ \frac{2}{3},\ \frac{8}{11},\ \frac{8}{9};\ \frac{128}{33}}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{(-1)^{n-1}\dfrac{n}{2n-1}\right\}$

Solution

$$a_1 = (-1)^0\frac{1}{2-1} = 1; \quad a_2 = (-1)^1\frac{2}{4-1} = -\frac{2}{3}; \quad a_3 = (-1)^2\frac{3}{6-1} = \frac{3}{5}; \quad a_4 = (-1)^3\frac{4}{8-1} = -\frac{4}{7}$$

$$a_8 = (-1)^7\frac{8}{16-1} = -\frac{8}{15}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\dfrac{2^n}{3^n+1}\right\}$

Solution

$$a_1 = \frac{2^1}{3^1+1} = \frac{2}{4} = \frac{1}{2}; \quad a_2 = \frac{2^2}{3^2+1} = \frac{4}{10} = \frac{2}{5}; \quad a_3 = \frac{2^3}{3^3+1} = \frac{8}{28} = \frac{2}{7}; \quad a_4 = \frac{2^4}{3^4+1} = \frac{16}{82} = \frac{8}{41}$$

$$a_8 = \frac{2^8}{3^8+1} = \frac{256}{6562} = \frac{128}{3281}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\dfrac{n^2}{2^n}\right\}$

Solution

$$a_1 = \frac{1^2}{2^1} = \frac{1}{2}; \quad a_2 = \frac{2^2}{2^2} = 1 \quad a_3 = \frac{3^2}{2^3} = \frac{9}{8}; \quad a_4 = \frac{4^2}{2^4} = \frac{16}{16} = 1$$

$$a_8 = \frac{8^2}{2^8} = \frac{64}{256} = \frac{1}{4}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\dfrac{n}{e^n}\right\}$

Solution

$$a_1 = \frac{1}{e^1} = \frac{1}{e}; \quad a_2 = \frac{2}{e^2}; \quad a_3 = \frac{3}{e^3}; \quad a_4 = \frac{4}{e^4}$$

$$a_8 = \frac{8}{e^8}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{c_n\right\} = \left\{(-1)^{n+1} n^2\right\}$

Solution

$$c_1 = (-1)^2 1^2 = 1; \quad c_2 = (-1)^3 2^2 = -4; \quad c_3 = (-1)^4 3^2 = 9; \quad c_4 = (-1)^5 4^2 = -16$$

$$c_8 = (-1)^9 8^2 = -64$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{c_n\right\} = \left\{\dfrac{(-1)^n}{(n+1)(n+2)}\right\}$

Solution

$$c_1 = \frac{(-1)^1}{2\cdot 3} = -\frac{1}{6}; \quad c_2 = \frac{(-1)^2}{3\cdot 4} = \frac{1}{12}; \quad c_3 = \frac{(-1)^3}{4\cdot 5} = -\frac{1}{20}; \quad c_4 = \frac{(-1)^4}{5\cdot 6} = \frac{1}{30}$$

$$c_8 = \frac{(-1)^8}{9\cdot 10} = \frac{1}{90}$$

Exercise

Find the first four terms and the eight term of the sequence: $\{c_n\} = \left\{\left(\frac{4}{3}\right)^n\right\}$

Solution

$$c_1 = \left(\frac{4}{3}\right)^1 \boxed{= \frac{4}{3}}; \quad c_2 = \left(\frac{4}{3}\right)^2 \boxed{= \frac{16}{9}}; \quad c_3 = \left(\frac{4}{3}\right)^3 \boxed{= \frac{64}{27}}; \quad c_4 = \left(\frac{4}{3}\right)^4 \boxed{= \frac{256}{81}}$$

$$c_8 = \left(\frac{4}{3}\right)^8 \boxed{= \frac{65,536}{6,561}}$$

Exercise

Find the first four terms and the eight term of the sequence: $\{b_n\} = \left\{\frac{3^n}{n}\right\}$

Solution

$$b_1 = \frac{3^1}{1} \boxed{= 3}; \quad b_2 = \frac{3^2}{2} \boxed{= \frac{9}{2}}; \quad b_3 = \frac{3^3}{3} \boxed{= 9}; \quad b_4 = \frac{3^4}{4} \boxed{= \frac{81}{4}}$$

$$c_8 = \frac{3^8}{8} \boxed{= \frac{6,561}{8}}$$

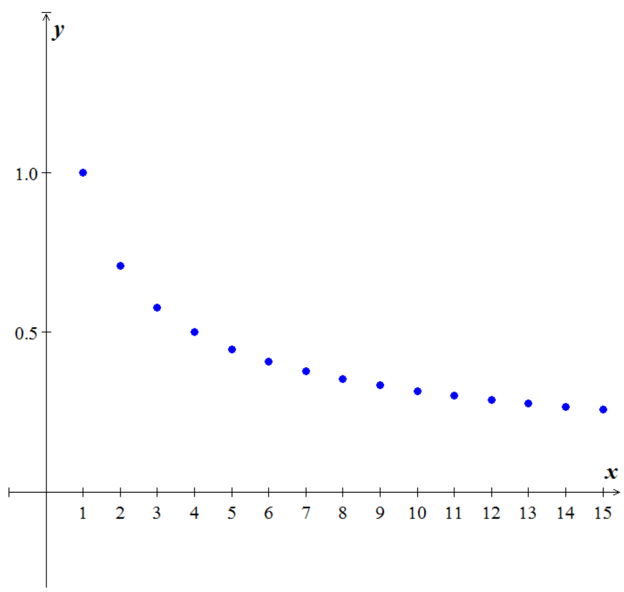
Exercise

Graph the sequence $\left\{\frac{1}{\sqrt{n}}\right\}$

Solution

$$\left\{\frac{1}{\sqrt{n}}\right\} = \frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, \dots$$

$$\approx 1,\ 0.71,\ 0.58,\ 0.5,\ 0.45$$



Exercise

Find the first four terms of the sequence of partial sums for the given sequence. $\left\{3 + \frac{1}{2}n\right\}$

Solution

$$\boxed{S_1} = a_1 = 3 + \frac{1}{2}(1) \boxed{= \frac{7}{2}} \qquad \boxed{S_2} = S_1 + a_2 = \frac{7}{2} + 3 + \frac{1}{2}(2) \boxed{= \frac{15}{2}}$$

$$\boxed{S_3} = S_2 + a_3 = \frac{15}{2} + 3 + \frac{1}{2}(3) \boxed{= 12} \qquad \boxed{S_4} = S_3 + a_4 = 12 + 3 + \frac{1}{2}(4) \boxed{= 17}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_{k+1} = 3a_k - 5$

Solution

$k = 1 \rightarrow \quad a_2 = 3a_1 - 5 = 3(2) - 5 = 1$ $\qquad k = 2 \rightarrow \quad a_3 = 3a_2 - 5 = 3(1) - 5 = -2$

$k = 3 \rightarrow \quad a_4 = 3a_3 - 5 = 3(-2) - 5 = -11$ $\qquad k = 4 \rightarrow \quad a_5 = 3a_4 - 5 = 3(-11) - 5 = -38$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = -3, \quad a_{k+1} = a_k^2$

Solution

$k = 1 \rightarrow \quad a_2 = a_1^2 = (-3)^2 = 9$ $\qquad k = 2 \rightarrow \quad a_3 = a_2^2 = (9)^2 = 81$

$k = 3 \rightarrow \quad a_4 = a_3^2 = \left(3^4\right)^2 = 3^8$ $\qquad k = 4 \rightarrow \quad a_5 = a_4^2 = \left(3^8\right)^2 = 3^{16}$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 5, \quad a_{k+1} = ka_k$

Solution

$k = 1 \rightarrow \quad a_2 = 1a_1 = 5$ $\qquad k = 2 \rightarrow \quad a_3 = 2a_2 = 2(5) = 10$

$k = 3 \rightarrow \quad a_4 = 3a_3 = 3(10) = 30$ $\qquad k = 4 \rightarrow \quad a_5 = 4a_4 = 4(30) = 120$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_n = 3 + a_{n-1}$

Solution

$a_2 = 3 + a_1 = 3 + 2 = 5$ $\qquad a_3 = 3 + a_2 = 3 + 5 = 8$

$a_4 = 3 + a_3 = 3 + 8 = 11$ $\qquad a_5 = 3 + a_4 = 3 + 11 = 14$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 5, \quad a_n = 2a_{n-1}$

Solution

$a_2 = 2a_1 = 2(5) = 10$ $\qquad a_3 = 2a_2 = 2(10) = 20$

$a_4 = 2a_3 = 2(20) = 40$ $\qquad a_5 = 2a_4 = 2(40) = 80$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = \sqrt{2}, \quad a_n = \sqrt{2 + a_{n-1}}$

Solution

$a_2 = \sqrt{2 + a_1} = \boxed{\sqrt{2+\sqrt{2}}}$ $\qquad a_3 = \sqrt{2 + a_2} = \boxed{\sqrt{2+\sqrt{2+\sqrt{2}}}}$

$a_4 = \sqrt{2 + a_3} = \boxed{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}$ $\qquad a_5 = \sqrt{2 + a_4} = \boxed{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_{n+1} = 7 - 2a_n$

Solution

$a_2 = 7 - 2a_1 = 7 - 4 \ \underline{= 3}|$ $\qquad a_3 = 7 - 2a_2 = 7 - 6 \ \underline{= 1}|$

$a_4 = 7 - 2a_3 = 7 - 2 \ \underline{= 5}|$ $\qquad a_5 = 7 - 2a_4 = 7 - 10 \ \underline{= -5}|$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 128, \quad a_{n+1} = \frac{1}{4} a_n$

Solution

$a_2 = \frac{1}{4} a_1 = \frac{1}{4} 128 \ \underline{= 32}|$ $\qquad a_3 = \frac{1}{4} a_2 = \frac{32}{4} \ \underline{= 8}|$

$a_4 = \frac{1}{4} a_3 = \frac{8}{4} \ \underline{= 2}|$ $\qquad a_5 = \frac{1}{4} a_4 = \frac{2}{4} \ \underline{= \frac{1}{2}}|$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_{n+1} = \left(a_n\right)^n$

Solution

$a_2 = \left(a_1\right)^1 \ \underline{= 2}|$ $\qquad a_3 = \left(a_2\right)^2 = 2^2 \ \underline{= 4}|$

$a_4 = \left(a_3\right)^3 = 4^3 \ \underline{= 64}|$ $\qquad a_5 = \left(a_4\right)^4 \ \underline{= 64^4}|$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = A, \quad a_n = a_{n-1} + d$

Solution

$a_2 = a_1 + d \ \underline{= A + d}|$ $\qquad a_3 = a_2 + d = A + d + d \ \underline{= A + 2d}|$

$a_4 = a_3 + d \ \underline{= A + 3d}|$ $\qquad a_5 = a_4 + d \ \underline{= A + 4d}|$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = A, \quad a_n = ra_{n-1}, \quad r \neq 0$

Solution

$a_2 = ra_1 \ \underline{= rA}$ $\qquad a_3 = ra_2 \ \underline{= Ar^2}$ $\qquad a_4 = ra_3 \ \underline{= Ar^3}$ $\qquad a_5 = ra_4 \ \underline{= Ar^4}$

Exercise

Find the first 5 terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_2 = 2; \quad a_n = a_{n-1} \bullet a_{n-2}$

Solution

$a_3 = a_2 \cdot a_1 = 2 \cdot 2 \ \underline{= 4}$ $\qquad a_4 = a_3 \cdot a_2 = 4 \cdot 2 \ \underline{= 8}$

$a_5 = a_4 \cdot a_3 = 8 \cdot 4 \ \underline{= 32}$ $\qquad a_6 = a_5 \cdot a_4 = 32 \cdot 8 \ \underline{= 256}$

Exercise

Express each sum using summation notation $\qquad 1 + 2 + 3 + \ldots + 20$

Solution

$$1 + 2 + 3 + 4 + \cdots + 20 = \sum_{k=1}^{20} k$$

Exercise

Express each sum using summation notation $\qquad 1 + 2 + 3 + \ldots + 40$

Solution

$$1 + 2 + 3 + \ldots + 40 = \sum_{k=1}^{40} k$$

Exercise

Express each sum using summation notation $\qquad 1^3 + 2^3 + 3^3 + \ldots + 8^3$

Solution

$$1^3 + 2^3 + 3^3 + \ldots + 8^3 = \sum_{k=1}^{8} k^3$$

Exercise

Express each sum using summation notation $1^2 + 2^2 + 3^2 + \ldots + 15^2$

Solution

$$1^2 + 2^2 + 3^2 + \ldots + 15^2 = \sum_{k=1}^{15} k^2$$

Exercise

Express each sum using summation notation $2^2 + 2^3 + 2^4 + \ldots + 2^{11}$

Solution

$$2^2 + 2^3 + 2^4 + \ldots + 2^{11} = \sum_{k=2}^{11} 2^k$$

Exercise

Express each sum using summation notation $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{13}{14}$

Solution

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{13}{14} = \sum_{k=1}^{13} \frac{k}{k+1}$$

Exercise

Express each sum using summation notation $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots + (-1)^6 \frac{1}{3^6}$

Solution

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots + (-1)^6 \frac{1}{3^6} = \sum_{k=0}^{6} (-1)^k \frac{1}{3^k}$$

Exercise

Express each sum using summation notation $\frac{2}{3} - \frac{4}{9} + \frac{8}{27} - \cdots + (-1)^{12} \left(\frac{2}{3}\right)^{11}$

Solution

$$\frac{2}{3} - \frac{4}{9} + \frac{8}{27} - \cdots + (-1)^{12} \left(\frac{2}{3}\right)^{11} = \sum_{k=1}^{11} (-1)^{k+1} \left(\frac{2}{3}\right)^k$$

Exercise

Express each sum using summation notation $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{14}{14+1}$

Solution

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{14}{14+1}=\sum_{k=1}^{14}\frac{k}{k+1}$$

Exercise

Express each sum using summation notation $\frac{1}{e}+\frac{2}{e^2}+\frac{3}{e^3}+\cdots+\frac{n}{e^n}$

Solution

$$\frac{1}{e}+\frac{2}{e^2}+\frac{3}{e^3}+\cdots+\frac{n}{e^n}=\sum_{k=1}^{n}\frac{k}{e^k}$$

Exercise

Find the sum: $\sum_{k=1}^{5}(2k-7)$

Solution

$$\sum_{k=1}^{5}(2k-7)=(-5)+(-3)+(-1)+1+3\ \underline{=-5}$$

Exercise

Find the sum: $\sum_{k=0}^{5}k(k-2)$

Solution

$$\sum_{k=0}^{5}k(k-2)=0+(-1)+0+3+8+15\ \underline{=25}$$

Exercise

Find the sum: $\sum_{k=1}^{5}(-3)^{k-1}$

Solution

$$\sum_{k=1}^{5}(-3)^{k-1}=1+(-3)+9+(-27)+81 \underline{=61}$$

Exercise

Find the sum: $\displaystyle\sum_{k=253}^{571}\left(\frac{1}{3}\right)$

Solution

$$\sum_{k=253}^{571}\left(\frac{1}{3}\right)=(571-253+1)\left(\frac{1}{3}\right) \underline{=\frac{319}{3}}$$

$$\sum_{k=m}^{n} c=(n-m+1)c$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{50} 8$

Solution

$$\sum_{k=1}^{50} 8=(50-1+1)8 \underline{=400}$$

$$\sum_{k=m}^{n} c=(n-m+1)c$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{40} k$

Solution

$$\sum_{k=1}^{40} k=\frac{40(41)}{2} \underline{=820}$$

$$\sum_{k=1}^{n} k=\frac{n(n+1)}{2}$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{5}(3k)$

Solution

$$\sum_{k=1}^{5} 3k=3(1)+3(2)+3(3)+3(4)+3(5) \underline{=45}$$

Exercise

Find the sum: $\sum_{k=1}^{10}\left(k^3+1\right)$

Solution

$$\sum_{k=1}^{10}\left(k^3+1\right)=\sum_{k=1}^{10}k^3+\sum_{k=1}^{10}1 \qquad \sum_{k=1}^{n}k^3=\frac{n^2(n+1)^2}{4}$$

$$=\frac{10^2(10+1)^2}{4}+10(1)$$

$$=\frac{12100}{4}+10$$

$$=3025+10$$

$$\underline{=3035}$$

Exercise

Find the sum: $\sum_{k=1}^{24}\left(k^2-7k+2\right)$

Solution

$$\sum_{k=1}^{24}\left(k^2-7k+2\right)=\frac{24(24+1)(2\cdot 24+1)}{6}-7\frac{24(24+1)}{2}+2(24) \qquad \sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

$$\underline{=2848}$$

Exercise

Find the sum: $\sum_{k=6}^{20}\left(4k^2\right)$

Solution

$$\sum_{k=6}^{20}\left(4k^2\right)=4\left(\sum_{k=1}^{20}k^2-\sum_{k=1}^{5}k^2\right) \qquad \sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

$$=4\left(\frac{20(20+1)(2\cdot 20+1)}{6}-\frac{5(5+1)(2\cdot 5+1)}{6}\right)$$

$$=4\left(\frac{20(21)(41)}{6}-\frac{5(6)(11)}{6}\right)$$

$$=4(2870-55)$$

$$\underline{=11,260}$$

Exercise

Find the sum: $\sum_{k=1}^{16}\left(k^2-4\right)$

Solution

$$\sum_{k=1}^{16}\left(k^2-4\right)=\sum_{k=1}^{16}k^2-\sum_{k=1}^{16}4$$

$$\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

$$=\frac{16(16+1)(2\cdot 16+1)}{6}-4(16)$$

$$=1496-64$$

$$\underline{=1432}$$

Exercise

Find the sum: $\sum_{k=1}^{6}(10-3k)$

Solution

$$\sum_{k=1}^{6}(10-3k)=7+4+1-2-5-8\ \underline{=-3}$$

Exercise

Find the sum: $\sum_{k=1}^{10}\left[1+(-1)^k\right]$

Solution

$$\sum_{k=1}^{10}\left[1+(-1)^k\right]=0+2+0+2+0+2+0+2+0+2\ \underline{=10}$$

Exercise

Find the sum: $\sum_{k=1}^{6}\frac{3}{k+1}$

Solution

$$\sum_{k=1}^{6}\frac{3}{k+1}=\frac{3}{2}+1+\frac{3}{4}+\frac{3}{5}+2+\frac{3}{7}\ \underline{=\frac{879}{140}}$$

Exercise

Find the sum: $\displaystyle\sum_{k=137}^{428} 2.1$

Solution

$$\sum_{k=137}^{428} 2.1 = (428-137+1)2.1 = (292)2.1 \underline{= 613.2}$$

$$\sum_{k=m}^{n} c = (n-m+1)c$$

Exercise

Write out each sum $\displaystyle\sum_{k=1}^{n}(k+2)$

Solution

$$\sum_{k=1}^{n}(k+2) \underline{= 3+5+7+9+\cdots+(n+2)}$$

Exercise

Write out each sum $\displaystyle\sum_{k=1}^{n}k^2$

Solution

$$\sum_{k=1}^{n}k^2 = 1^2+2^2+3^2+4^2+\cdots+n^2 \underline{= 1+4+9+16+\cdots+n^2}$$

Exercise

Write out each sum $\displaystyle\sum_{k=2}^{n}(-1)^k \ln k$

Solution

$$\sum_{k=2}^{n}(-1)^k \ln k = (-1)^2\ln 2+(-1)^3\ln 3+(-1)^4\ln 4+(-1)^5\ln 5+\cdots+(-1)^n\ln n$$

$$\underline{= \ln 2-\ln 3+\ln 4-\ln 5+\cdots+(-1)^n\ln n}$$

Exercise

Write out each sum $\sum_{k=3}^{n}(-1)^{k+1}2^k$

Solution

$$\sum_{k=3}^{n}(-1)^{k+1}\ 2^k = (-1)^4 2^3 + (-1)^5 2^4 + (-1)^6 2^5 + (-1)^7 2^6 + \cdots + (-1)^{n+1} 2^n$$

$$\boxed{= 8 - 16 + 32 - 64 + \cdots + (-1)^{n+1} 2^n}$$

Exercise

Write out each sum $\sum_{k=0}^{n}\frac{1}{3^k}$

Solution

$$\sum_{k=0}^{n}\frac{1}{3^k} \boxed{= 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^n}}$$

Exercise

Fred has a balance of \$3,000 on his card which charges 1% interest per month on any unpaid balance. Fred can afford to pay \$100 toward the balance each month. His balance each month after making a \$100 payment is given by the recursively defined sequence

$$B_0 = \$3,000 \quad B_n = 1.01B_{n-1} - 100$$

Determine Fred's balance after making the first payment. That is, determine B_1

Solution

$$\begin{aligned} B_1 &= 1.01B_0 - 100 \\ &= 1.01(3,000) - 100 \\ &\boxed{= \$2,930} \end{aligned}$$

Fred's balance is \$2,930 after making the first payment.

Exercise

A pond currently has 2,000 trout in it. A fish hatchery decides to add an additional 20 trout each month. Is it also known that the trout population is grwoing at a rate of 3% per month. The size of the population after *n* months is given bu the recursively defined sequence

$$P_0 = 2,000 \quad P_n = 1.03P_{n-1} + 20$$

How many trout are in the pond after 2 months? That is, what is P_2?

Solution

$$P_1 = 1.03P_0 + 20 = 1.03(2,000) + 20 = 2,080$$

$$P_2 = 1.03P_1 + 20 = 1.03(2,080) + 20 = 2,162.4$$

There are approximately 2162 trout in the pond after 2 months.

Exercise

Fred bought a car by taking out a loan for $18,500 at 0.5% interest per month. Fred's normal monthly payment is $434.47 per month, but he decides that he can afford to pay $100 extra toward the balance each month. His balance each month is given by the recursively defined sequence

$$B_0 = \$18,500 \quad B_n = 1.005B_{n-1} - 534.47$$

Determine Fred's balance after making the first payment. That is, determine B_1

Solution

$$\begin{aligned} B_1 &= 1.005B_0 - 534.47 \\ &= 1.005(18,500) - 534.47 \\ &= \$18,058.03 \end{aligned}$$

Fred's balance is $18.058.03 after making the first payment.

Exercise

The Environmental Protection Agency (EPA) determines that Maple Lake has 250 *tons* of pollutant as a result of industrial waste and that 10% of the pollutant present is neuttralized by solar oxidation every year. The EPA imposes new pollution control laws that result in 15 *tons* of new pollutant entering the lake each year. The amount of pollutant in the lake after *n* years is given by the recursively defined sequence

$$P_0 = 250 \quad P_n = 0.9P_{n-1} + 15$$

Determine the amount of pollutant in the lake after 2 years? That is, what is P_2?

Solution

$$P_1 = 0.9P_0 + 15 = 0.9(250) + 15 = 240$$

$$P_2 = 0.9P_1 + 15 = 0.9(240) + 15 = 231$$

There are 231 tons of pollutants after 2 years.

Exercise

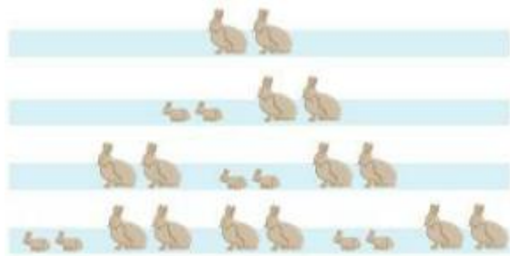
A colony of rabbits begins with one pair of mature rabbits, which will produce a pair of offspring (one male, one female) each month. Assume that all rabbits mature in 1 month and produce a pair of offspring (one male, one female) after 2 months. If no rabbits ever die, how many pairs of mature rabbits are there after 7 months?

1 mature pair

1 mature pair

2 mature pairs

3 mature pairs



Solution

$$a_1 = 1, \quad a_2 = 1, \quad a_3 = 2, \quad a_4 = 3, \quad a_5 = 5$$

$$a_6 = 8, \quad a_7 = 13, \quad a_8 = 21, \quad \cdots \quad a_n = a_{n-1} + a_{n-2}$$

After 7 months there are 21 mature pairs of rabbits.

Exercise

Let $\quad u_n = \dfrac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}$

Define the nth term of a sequence

a) Show that $u_1 = 1$ and $u_2 = 1$

b) Show that $u_{n+2} = u_{n+1} + u_n$

c) Draw the conclusion that $\{u_n\}$ is a Fibonacci sequence

d) Find the first ten terms of the sequence from part (c)

Solution

a) $u_1 = \dfrac{\left(1+\sqrt{5}\right)^1 - \left(1-\sqrt{5}\right)^1}{2^1\sqrt{5}} = \dfrac{2\sqrt{5}}{2\sqrt{5}} \; \underline{= 1}$

$$u_2 = \frac{\left(1+\sqrt{5}\right)^2 - \left(1-\sqrt{5}\right)^2}{2^2\sqrt{5}} \qquad\qquad a^2 - b^2 = (a-b)(a+b)$$

$$= \frac{\left(1+\sqrt{5}-1+\sqrt{5}\right) - \left(1-\sqrt{5}+1-\sqrt{5}\right)}{2^2\sqrt{5}}$$

$$= \frac{4\sqrt{5}}{4\sqrt{5}}$$

$\underline{= 1}$

b) $u_{n+1} + u_n = \dfrac{\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}}{2^{n+1}\sqrt{5}} + \dfrac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}$

$$=\frac{\left(1+\sqrt{5}\right)^{n+1}-\left(1-\sqrt{5}\right)^{n+1}+2\left(1+\sqrt{5}\right)^{n}-2\left(1-\sqrt{5}\right)^{n}}{2^{n+1}\sqrt{5}}$$

$$=\frac{\left(1+\sqrt{5}\right)^{n}\left(1+\sqrt{5}+2\right)-\left(1-\sqrt{5}\right)^{n}\left(1-\sqrt{5}+2\right)}{2^{n+1}\sqrt{5}}$$

$$=\frac{\left(1+\sqrt{5}\right)^{n}\left(3+\sqrt{5}\right)-\left(1-\sqrt{5}\right)^{n}\left(3-\sqrt{5}\right)}{2^{n+1}\sqrt{5}}$$

$$=\frac{\left(1+\sqrt{5}\right)^{n+2}\dfrac{3+\sqrt{5}}{\left(1+\sqrt{5}\right)^{2}}-\left(1-\sqrt{5}\right)^{n+2}\dfrac{3-\sqrt{5}}{\left(1-\sqrt{5}\right)^{2}}}{2^{n+1}\sqrt{5}}$$

$$=\frac{\left(1+\sqrt{5}\right)^{n+2}\dfrac{3+\sqrt{5}}{1+2\sqrt{5}+5}-\left(1-\sqrt{5}\right)^{n+2}\dfrac{3-\sqrt{5}}{1-2\sqrt{5}+5}}{2^{n+1}\sqrt{5}}$$

$$=\frac{\frac{1}{2}\left(1+\sqrt{5}\right)^{n+2}\dfrac{3+\sqrt{5}}{3+\sqrt{5}}-\frac{1}{2}\left(1-\sqrt{5}\right)^{n+2}\dfrac{3-\sqrt{5}}{3-\sqrt{5}}}{2^{n+1}\sqrt{5}}$$

$$=\frac{1}{2}\frac{\left(1+\sqrt{5}\right)^{n+2}-\left(1-\sqrt{5}\right)^{n+2}}{2^{n+1}\sqrt{5}}$$

$$=\frac{\left(1+\sqrt{5}\right)^{n+2}-\left(1-\sqrt{5}\right)^{n+2}}{2^{n+2}\sqrt{5}}$$

$$=u_{n+2} \quad \checkmark$$

c) Since $u_1=1$ and $u_2=1$ and $u_{n+2}=u_{n+1}+u_n$

$\therefore \{u_n\}$ is a Fibonacci sequence

e) $u_1=1,\quad u_2=1$

$u_3=u_2+u_1=1+1=2$ $\quad u_4=u_3+u_2=2+1=3$ $\quad u_5=u_4+u_3=3+2=5$

$u_6=u_5+u_4=5+3=8$ $\quad u_7=u_6+u_5=8+5=13$ $\quad u_8=u_7+u_5=13+8=21$

$u_9=u_8+u_7=21+13=34$ $\quad u_{10}=u_9+u_8=34+21=55$

Solution Section 4.2 – Arithmetic and Geometric Sequences

Exercise

Show that the sequence $-6,\ -2,\ 2,\ \ldots,\ 4n-10,\ \ldots$ is arithmetic, and find the common difference.

Solution

We to show that $a_{k+1}-a_k$ equals to a constant.

$$\begin{aligned} a_{k+1}-a_k &= 4(k+1)-10-(4k-10) \\ &= 4k+4-10-4k+10 \\ &= 4 \end{aligned}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $2,\ 6,\ 10,\ 14,\ \ldots$

Solution

$d = 6-2 = 4$

$$\begin{aligned} a_n &= 2+(n-1)4 \qquad\qquad a_n = a_1+(n-1)d \\ &= 2+4n-4 \\ &= 4n-2 \end{aligned}$$

$a_{10} = 4(10)-2 = 38$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $3,\ 2.7,\ 2.4,\ 2.1,\ \ldots$

Solution

$d = 2.7-3 = -0.3$

$$\begin{aligned} a_n &= 3+(n-1)(-0.3) \qquad\qquad a_n = a_1+(n-1)d \\ &= 3-0.3n+0.3 \\ &= 3.3-0.3n \end{aligned}$$

$a_{10} = 3.3-0.3(10) = 0.3$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $-6,\ -4.5,\ -3,\ -1.5,\ \ldots$

Solution

$d = -4.5 - (-6) = 1.5$

$a_n = -6 + (n-1)(1.5)$ $\qquad a_n = a_1 + (n-1)d$

$= -6 + 1.5n - 1.5$

$\underline{= 1.5n - 7.5|}$

$a_{10} = 1.5(10) - 7.5 \ \underline{= 7.5|}$

Exercise

Find the *n*th term, and the tenth term of the arithmetic sequence: $\ln 3,\ \ln 9,\ \ln 27,\ \ln 81,\ \ldots$

Solution

$\ln 3,\ \ln 3^2,\ \ln 3^3,\ \ln 3^4,\ \ldots$

$\ln 3,\ 2\ln 3,\ 3\ln 3,\ 4\ln 3,\ \ldots$

$d = 2\ln 3 - \ln 3 = \ln 3$

$a_n = \ln 3 + (n-1)\ln 3$ $\qquad a_n = a_1 + (n-1)d$

$= \ln 3 + n\ln 3 - \ln 3$

$\underline{= n\ln 3|}$

$a_{10} = 10\ln 3 \ \underline{= \ln 3^{10}|}$

Exercise

Find the *n*th term, and the tenth term of the arithmetic sequence: $a_1 = 2, \quad d = 3$

Solution

$a_n = 2 + 3(n-1)$ $\qquad a_n = a_1 + (n-1)d$

$= 2 + 3n - 3$

$\underline{= 3n - 1|}$

Exercise

Find the *n*th term, and the tenth term of the arithmetic sequence: $a_1 = 5, \quad d = -3$

Solution

$a_n = 5 + (n-1)(-3)$ $\qquad a_n = a_1 + (n-1)d$

$= 5 - 3n + 3$

$\underline{= 8 - 3n|}$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = 1, \quad d = -\frac{1}{2}$

Solution

$$a_n = 1 + (n-1)\left(-\tfrac{1}{2}\right) \qquad\qquad a_n = a_1 + (n-1)d$$

$$= 1 - \frac{1}{2}n + \frac{1}{2}$$

$$\underline{= \frac{3}{2} - \frac{1}{2}n}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = -2, \quad d = 4$

Solution

$$a_n = -2 + (n-1)(4) \qquad\qquad a_n = a_1 + (n-1)d$$

$$= -2 + 4n - 4$$

$$\underline{= 4n - 6}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = \sqrt{2}, \quad d = \sqrt{2}$

Solution

$$a_n = \sqrt{2} + (n-1)\sqrt{2} \qquad\qquad a_n = a_1 + (n-1)d$$

$$= \sqrt{2} + \sqrt{2}n - \sqrt{2}$$

$$\underline{= \sqrt{2}\, n}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = 0, \quad d = \pi$

Solution

$$a_n = 0 + (n-1)(\pi) \qquad\qquad a_n = a_1 + (n-1)d$$

$$\underline{= \pi n - \pi}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = 13, \quad d = 4$

Solution

$$a_n = 13 + (n-1)(4) \qquad\qquad a_n = a_1 + (n-1)d$$
$$\underline{= 4n + 9}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = -40, \quad d = 5$

Solution

$$a_n = -40 + (n-1)(5) \qquad\qquad a_n = a_1 + (n-1)d$$
$$\underline{= 5n - 45}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = -32, \quad d = 4$

Solution

$$a_n = -32 + (n-1)(4) \qquad\qquad a_n = a_1 + (n-1)d$$
$$\underline{= 4n - 36}$$

Exercise

Find the common difference for the arithmetic sequence with the specified terms: $a_4 = 14, \quad a_{11} = 35$

Solution

$$a_n = a_1 + (n-1)d$$
$$a_{11} = a_1 + 10d \rightarrow 35 = a_1 + 10d$$
$$a_4 = a_1 + 3d \rightarrow \underline{14 = a_1 + 3d}$$
$$21 = 7d \rightarrow \boxed{d = 3}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: $a_{12}; \quad a_1 = 9.1, \quad a_2 = 7.5$

Solution

$$d = a_2 - a_1 = 7.5 - 9.1 = -1.6$$
$$a_n = a_1 + (n-1)d$$
$$a_{12} = 9.1 + (11)(-1.6) \ \underline{= -8.5}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_1; $a_8 = 47$, $a_9 = 53$

Solution

$$d = a_9 - a_8 = 53 - 47 = 6$$

$$a_8 = a_1 + (7)(6) \qquad a_n = a_1 + (n-1)d$$

$$\boxed{a_1 = 47 - 42 = 5}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_{10}; $a_2 = 1$, $a_{18} = 49$

Solution

$$a_2 = a_1 + d \Rightarrow a_1 = a_2 - d$$

$$a_{18} = a_1 + (17)d = a_2 - d + 17d = a_2 + 16d$$

$$49 = 1 + 16d \Rightarrow 16d = 48 \Rightarrow \boxed{d = \frac{48}{16} = 3}$$

$$a_1 = a_2 - d = 1 - 3 = -2$$

$$\boxed{a_{10} = -2 + 9(3) = 25}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_{10}; $a_8 = 8$, $a_{20} = 44$

Solution

$$\boxed{d = \frac{44-8}{20-8} = \frac{36}{12} = 3} \qquad d = \frac{a_y - a_x}{y - x}$$

$$a_8 = a_1 + (8-1)(3) \qquad a_n = a_1 + (n-1)d$$

$$8 = a_1 + 21$$

$$a_1 = -13$$

$$\boxed{a_{10} = -13 + 9(3) = 14}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_{12}; $a_8 = 4$, $a_{18} = -96$

Solution

$$\boxed{d = \frac{-96-4}{18-8} = \frac{-100}{10} = -10} \qquad d = \frac{a_y - a_x}{y - x}$$

$$a_8 = a_1 + (8-1)(-10) \qquad a_n = a_1 + (n-1)d$$

$$4 = a_1 - 70$$

$$a_1 = 74$$

$$\boxed{a_{12} = 74 + (11)(-10) = -36}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: $a_8;\quad a_{15} = 0,\quad a_{40} = -50$

Solution

$$\boxed{d = \frac{-50-0}{40-15} = \frac{-50}{25} = -2} \qquad d = \frac{a_y - a_x}{y - x}$$

$$a_{15} = a_1 + (15-1)(-2) = 0 \qquad a_n = a_1 + (n-1)d$$

$$a_1 = 28$$

$$\boxed{a_8 = 28 + (7)(-2) = 14}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: $a_{20};\quad a_9 = -5,\quad a_{15} = 31$

Solution

$$\boxed{d = \frac{31+5}{15-9} = \frac{36}{6} = 6} \qquad d = \frac{a_y - a_x}{y - x}$$

$$a_9 = a_1 + (9-1)(6) \qquad a_n = a_1 + (n-1)d$$

$$-5 = a_1 + 42$$

$$a_1 = -47$$

$$\boxed{a_{20} = -47 + (19)(6) = 67}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: $a_n;\quad a_8 = 8,\quad a_{20} = 44$

Solution

$$\boxed{d = \frac{44-8}{20-8} = \frac{36}{12} = 3} \qquad d = \frac{a_y - a_x}{y - x}$$

$$a_8 = a_1 + 3(8-1) = 8 \qquad a_n = a_1 + (n-1)d$$

$$\boxed{a_1 = -13}$$

$$\boxed{a_n = -13 + 3(n-1) = 3n - 16}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_n; $a_8 = 4$, $a_{18} = -96$

Solution

$$d = \frac{-96-4}{18-8} = -10 \qquad d = \frac{a_y - a_x}{y-x}$$

$$a_8 = a_1 - 10(8-1) = 4 \qquad a_n = a_1 + (n-1)d$$

$$a_1 = 74$$

$$a_n = 74 - 10(n-1) = -10n + 84$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_n; $a_{14} = -1$, $a_{15} = 31$

Solution

$$d = \frac{31+1}{15-14} = 32 \qquad d = \frac{a_y - a_x}{y-x}$$

$$a_{14} = a_1 + 32(14-1) = -1 \qquad a_n = a_1 + (n-1)d$$

$$a_1 = -417$$

$$a_n = -417 + 32(n-1) = 32n + 449$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_n; $a_9 = -5$, $a_{15} = 31$

Solution

$$d = \frac{31+5}{15-9} = 6 \qquad d = \frac{a_y - a_x}{y-x}$$

$$a_9 = a_1 + 6(9-1) = -5 \qquad a_n = a_1 + (n-1)d$$

$$a_1 = -53$$

$$a_n = -53 + 6(n-1) = 6n - 59$$

Exercise

Find the sum S_n of the arithmetic sequence that satisfies the conditions: $a_1 = 40$, $d = -3$, $n = 30$

Solution

$$S_n = \frac{30}{2}\left[2(40) + (30-1)(-3)\right] \qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$= -105$$

Exercise

Find the sum S_n of the arithmetic sequence that satisfies the conditions: $a_7 = \frac{7}{3}, \quad d = -\frac{2}{3}, \quad n = 15$

Solution

$$a_7 = a_1 + (6)\left(-\frac{2}{3}\right) = \frac{7}{3}$$

$$a_1 = \frac{7}{3} + 4 = \frac{19}{3}$$

$$S_n = \frac{15}{2}\left[2\left(\frac{19}{3}\right) + (15-1)\left(-\frac{2}{3}\right)\right] \qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$\underline{= 25}$$

Exercise

Find the number of integers between 32 and 390 that are divisible by 6, find their sum

Solution

Number of terms: $n = \frac{390 - 36}{6} + 1 = 60$

$$S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$= \frac{60}{2}(36 + 390)$$

$$\underline{= 12780}$$

Exercise

Find the number of terms in the arithmetic sequence with the given conditions:

$a_1 = -2, \quad d = \frac{1}{4}, \quad S = 21$

Solution

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$21 = \frac{n}{2}\left[2(-2) + (n-1)\frac{1}{4}\right]$$

$$21 = -2n + \frac{1}{8}n(n-1)$$

$$(8)21 = -2n(8) + \frac{1}{8}n(n-1)(8)$$

$$168 = -16n + \left(n^2 - n\right)$$

$$0 = n^2 - 17n - 168$$

$$\boxed{n = 24} \qquad n = -7$$

Exercise

Express the sum in terms of summation notation and find the sum $2+11+20+\ldots+16,058$.

Solution

Difference in terms: $d = 11 - 2 = 9$

Number of terms: $n=\frac{16058-2}{9}+1=1785$

$\underline{|a_n}=2+(n-1)(9)=2+9n-9\ \underline{=9n-7|}$ $\qquad a_n=a_1+(n-1)d$

Hence the nth term is: $\displaystyle\sum_{n=1}^{1785}(9n-7)$

$S_{1785}=\frac{1789}{2}(2+16058)$ $\qquad S_n=\frac{n}{2}(a_1+a_n)$

$\underline{=14,333,550|}$

Exercise

Express the sum in terms of summation notation and find the sum $60+64+68+72+\cdots+120$.

Solution

Difference in terms: $d = 64 - 60 = 4$

Number of terms: $n=\frac{120-60}{4}+1\ \underline{=16|}$ $\qquad n=\frac{a_n-a_1}{d}+1$

$\underline{|a_n}=60+(n-1)(4)\ \underline{=4n-54|}$

Hence the nth term is: $\displaystyle\sum_{n=1}^{16}(4n-54)$

$S=\frac{16}{2}(60+120)$ $\qquad S_n=\frac{n}{2}(a_1+a_n)$

$\underline{=1440|}$

Exercise

Find each arithmetic sum $\qquad 1+3+5+\cdots+(2n-1)$

Solution

Difference in terms: $d = 3 - 1 = 2$

Number of terms: $n=\frac{(2n-1)-1}{2}+1$ $\qquad n=\frac{a_n-a_1}{d}+1$

$=\frac{2n-2}{2}+1$

$=n-1+1$

$\underline{=n|}$

$$S = \frac{n}{2}\left(1 + (2n-1)\right) \qquad\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$= \frac{n}{2}(2n)$$

$$\underline{= n^2}$$

Exercise

Find each arithmetic sum $2 + 4 + 6 + \cdots + 2n$

Solution

Difference in terms: $d = 4 - 2 = 2$

Number of terms: $n = \frac{2n-2}{2} + 1 \qquad\qquad n = \frac{a_n - a_1}{d} + 1$

$$= n - 1 + 1$$

$$\underline{= n}$$

$$S = \frac{n}{2}(2 + 2n) \qquad\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$= n(n+1)$$

$$\underline{= n^2 + n}$$

Exercise

Find each arithmetic sum $2 + 5 + 8 + \cdots + 41$

Solution

Difference in terms: $d = 5 - 2 = 3$

Number of terms: $n = \frac{41-2}{3} + 1 = 14 \qquad\qquad n = \frac{a_n - a_1}{d} + 1$

$$S = \frac{14}{2}(2 + 41) \qquad\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$\underline{= 301}$$

Exercise

Find each arithmetic sum $7 + 12 + 17 + \cdots + (2 + 5n)$

Solution

Difference in terms: $d = 12 - 7 = 5$

Number of terms: $n = \frac{2 + 5n - 7}{5} + 1 \qquad\qquad n = \frac{a_n - a_1}{d} + 1$

$$= \frac{5n-5}{5} + 1$$

$$= \frac{5n}{5} - \frac{5}{5} + 1$$

$= n$

$S = \frac{n}{2}(7 + 2 + 5n)$ $\qquad S_n = \frac{n}{2}(a_1 + a_n)$

$= \frac{n}{2}(9 + 5n)$

Exercise

Find each arithmetic sum $\quad 73 + 78 + 83 + 88 + \cdots + 558$

Solution

Difference in terms: $d = 78 - 73 = 5$

Number of terms: $n = \frac{558 - 73}{5} + 1 = 98$ $\qquad n = \frac{a_n - a_1}{d} + 1$

$S = \frac{98}{2}(73 + 558)$ $\qquad S_n = \frac{n}{2}(a_1 + a_n)$

$= 30{,}919$

Exercise

Find each arithmetic sum $\quad 7 + 1 - 5 - 11 - \cdots - 299$

Solution

Difference in terms: $d = 1 - 7 = -6$

Number of terms: $n = \frac{-299 - 7}{-6} + 1 = 52$ $\qquad n = \frac{a_n - a_1}{d} + 1$

$S = \frac{52}{2}(7 - 299)$ $\qquad S_n = \frac{n}{2}(a_1 + a_n)$

$= -7592$

Exercise

Find each arithmetic sum $\quad -1 + 2 + 7 + \cdots + (4n - 5)$

Solution

$S = \frac{n}{2}(-1 + 4n - 5)$ $\qquad S_n = \frac{n}{2}(a_1 + a_n)$

$= n(2n - 3)$

Exercise

Find each arithmetic sum $\quad 5 + 9 + 13 + \cdots + 49$

Solution

Difference in terms: $d = 9 - 5 = 4$

Number of terms: $n = \frac{49-5+4}{4} \underline{=12|}$ $\qquad n = \frac{a_n - a_1 + d}{d}$

$S = \frac{12}{2}(5+49)$ $\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$

$\underline{=324|}$

Exercise

Find each arithmetic sum $\qquad 2+4+6+\cdots+70$

Solution

Difference in terms: $d = 4-2 = 2$

Number of terms: $n = \frac{70-2+2}{2} \underline{=35|}$ $\qquad n = \frac{a_n - a_1 + d}{d}$

$S = \frac{35}{2}(70+2)$ $\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$

$\underline{=1,260|}$

Exercise

Find each arithmetic sum $\qquad 1+3+5+\cdots+59$

Solution

Difference in terms: $d = 3-1 = 2$

Number of terms: $n = \frac{59-1+2}{2} \underline{=30|}$ $\qquad n = \frac{a_n - a_1 + d}{d}$

$S = \frac{30}{2}(59+1)$ $\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$

$\underline{=900|}$

Exercise

Find each arithmetic sum $\qquad 4+4.5+5+5.5+\cdots+100$

Solution

Difference in terms: $d = 4.5-4 = 0.5$

Number of terms: $n = \frac{100-4+0.5}{0.5} \underline{=193|}$ $\qquad n = \frac{a_n - a_1 + d}{d}$

$S = \frac{193}{2}(4+100)$ $\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$

$\underline{=10,036|}$

Exercise

Find each arithmetic sum $8+8\frac{1}{4}+8\frac{1}{2}+8\frac{3}{4}+9+\cdots+50$

Solution

Difference in terms: $d=8\frac{1}{4}-8=\frac{1}{4}$

Number of terms: $n=\dfrac{50-8+0.25}{0.25}\ \underline{=169}$ $\qquad n=\dfrac{a_n-a_1+d}{d}$

$S=\frac{169}{2}(8+50)$ $\qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$

$\underline{=4{,}901}$

Exercise

Show that the given sequence is geometric, and find the common ratio

$$5,\ -\frac{5}{4},\ \frac{5}{16},\ \ldots,\ 5\left(-\frac{1}{4}\right)^{n-1},\ \ldots$$

Solution

To be geometric, we must show that $\dfrac{a_{k+1}}{a_k}=r$ is equal to some constant, which is the common ratio.

The common ratio: $r=\dfrac{a_{k+1}}{a_k}=\dfrac{a_2}{a_1}=\dfrac{-\frac{5}{4}}{5}\ \underline{=-\frac{1}{4}}$

Exercise

Find the ***n***th term, the fifth term, and the eighth term of the geometric sequence $8,\ 4,\ 2,\ 1,\ \ldots$

Solution

Given: $a_1=8,\quad r=\frac{4}{8}=\frac{1}{2}$

$$a_n=a_1r^{n-1}=8\left(\frac{1}{2}\right)^{n-1}$$

$$=2^3\left(2^{-1}\right)^{n-1}$$

$$=2^32^{-n+1}$$

$$\underline{=2^{4-n}}$$

$$\underline{a_5}=2^{4-5}=2^{-1}\ \underline{=\frac{1}{2}}$$

$$\underline{a_8}=2^{4-8}=2^{-4}\ \underline{=\frac{1}{16}}$$

Exercise

Find the $\boldsymbol{n}$th term, the fifth term, and the eighth term of the geometric sequence

$$300,\ -30,\ 3,\ -0.3,\ \ldots$$

Solution

Given: $a_1 = 300, \quad r = \frac{-30}{300} = -0.1$

$a_n = a_1 r^{n-1} = 300(-0.1)^{n-1}$

$$3\left(10^2\right)\left(-10^{-1}\right)^{n-1} = 3(10)^2(-10)^{-n+1} = 3(-10)^{-n+3}$$

$a_5 = 300(-0.1)^{5-1} = 300\left(-10^{-1}\right)^4 \ \underline{= 0.03}$

$a_8 = 3(-10)^{-8+3} = 3(-10)^{-5} \ \underline{= -.00003}$

Exercise

Find the $\boldsymbol{n}$th term, the fifth term, and the eighth term of the geometric sequence $1,\ -\sqrt{3},\ 3,\ -3\sqrt{3},\ \ldots$

Solution

Given: $a_1 = 1, \quad r = \frac{a_2}{a_1} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$

$a_n = a_1 r^{n-1} = 1\left(-\sqrt{3}\right)^{n-1} = \left(-\sqrt{3}\right)^{n-1}$

$a_5 = 1\left(-\sqrt{3}\right)^{5-1} \ \underline{= 9}$

$a_8 = 1\left(-\sqrt{3}\right)^{8-1} = \left(-\sqrt{3}\right)^7 \ \underline{= -27\sqrt{3}}$

Exercise

Find the $\boldsymbol{n}$th term, the fifth term, and the eighth term of the geometric sequence $4,\ -6,\ 9,\ -13.5,\ \ldots$

Solution

Given: $a_1 = 4, \quad r = \frac{a_2}{a_1} = \frac{-6}{4} = -\frac{3}{2}$

$a_n = a_1 r^{n-1} = 4\left(-\frac{3}{2}\right)^{n-1}$

$a_5 = 4\left(-\frac{3}{2}\right)^{5-1} = 4\left(-\frac{3}{2}\right)^4 = 4\left(\frac{3^4}{2^4}\right) \ \underline{= \frac{81}{4} \approx 20.25}$

$a_8 = 4\left(-\frac{3}{2}\right)^7 = -4\left(\frac{3^7}{2^7}\right) \ \underline{= -\frac{2187}{32} \approx -68.34375}$

Exercise

Find the **n**th term, the fifth term, and the eighth term of the geometric sequence $1,\ -x^2,\ x^4,\ -x^6,\ \ldots$

Solution

Given: $a_1 = 1,\quad r = \frac{a_2}{a_1} = \frac{-x^2}{1} = -x^2$

$$a_n = a_1 r^{n-1} = \left(-x^2\right)^{n-1}$$

$$a_5 = \left(-x^2\right)^4 = x^8$$

$$a_8 = \left(-x^2\right)^7 = -x^{14}$$

Exercise

Find the **n**th term, the fifth term, and the eighth term of the geometric sequence

$$10,\ 10^{2x-1},\ 10^{4x-3},\ 10^{6x-5},\ \ldots$$

Solution

Given: $a_1 = 10,\quad r = \frac{a_2}{a_1} = \frac{10^{2x-1}}{10} = 10^{2x-1-1} = 10^{2x-2}$

$$a_n = a_1 r^{n-1} = 10\left(10^{2x-2}\right)^{n-1} = 10\left(10^{(2x-2)(n-1)}\right) = 10\left(10^{(2x-2)n-2x+2}\right)$$

$$= 10^{2nx-2n-2x+2+1}$$

$$= 10^{2(n-1)x-2n+3}$$

$$a_5 = 10^{2(5-1)x-2(5)+3} = 10^{8x-7}$$

$$a_8 = 10^{2(8-1)x-2(8)+3} = 10^{14x-13}$$

Exercise

Find the **n**th term, the fifth term, and the eighth term of the geometric sequence $a_1 = 2,\quad r = 3$

Solution

Given: $a_1 = 10,\quad r = 3$

$$a_n = 2 \bullet 3^{n-1} \qquad\qquad a_n = a_1 r^{n-1}$$

$a_5 = 2 \bullet 3^4 = 162$ $\qquad a_8 = 2 \bullet 3^7 = 4374$

Exercise

Find the **n**th term, the fifth term, and the eighth term of the geometric sequence $a_1 = 1, \quad r = -\frac{1}{2}$

Solution

Given: $\quad a_1 = 1, \quad r = -\frac{1}{2}$

$a_n = \left(-\frac{1}{2}\right)^{n-1}$ $\qquad a_n = a_1 r^{n-1}$

$a_5 = \left(-\frac{1}{2}\right)^4 = \frac{1}{16} \qquad a_8 = \left(-\frac{1}{2}\right)^7 = -\frac{1}{128}$

Exercise

Find the **n**th term, the fifth term, and the eighth term of the geometric sequence $a_1 = -2, \quad r = 4$

Solution

Given: $\quad a_1 = -2, \quad r = 4$

$a_n = -2 \bullet (4)^{n-1}$ $\qquad a_n = a_1 r^{n-1}$

$a_5 = \left(-\frac{1}{2}\right)^4 = \frac{1}{16} \qquad a_8 = \left(-\frac{1}{2}\right)^7 = -\frac{1}{128}$

Exercise

Find the **n**th term, the fifth term, and the eighth term of the geometric sequence $a_1 = \sqrt{2}, \quad r = \sqrt{2}$

Solution

Given: $\quad a_1 = \sqrt{2}, \quad r = \sqrt{2}$

$a_n = \sqrt{2}\left(\sqrt{2}\right)^{n-1} = \left(\sqrt{2}\right)^n$ $\qquad a_n = a_1 r^{n-1}$

$a_5 = \left(\sqrt{2}\right)^5 = 4\sqrt{2} \qquad a_8 = \left(\sqrt{2}\right)^8 = 16$

Exercise

Find the **n**th term, the fifth term, and the eighth term of the geometric sequence $a_1 = 0, \quad r = \pi$

Solution

Given: $\quad a_1 = 0, \quad r = \pi$

$a_n = 0(\pi)^{n-1} = 0$ $\qquad a_n = a_1 r^{n-1}$

$a_5 = 0^5 = 0 \qquad a_8 = 0^8 = 0$

Exercise

Find the ***n***th term, the fifth term, and the eighth term of the geometric sequence $\left\{s_n\right\}=\left\{3^n\right\}$

Solution

$$a_5 \ \boxed{=3^5} \quad a_8 \ \boxed{=3^8}$$

Exercise

Find the ***n***th term, the fifth term, and the eighth term of the geometric sequence $\left\{s_n\right\}=\left\{(-5)^n\right\}$

Solution

$$a_5=(-5)^5 \ \boxed{=-5^5} \quad a_8=(-5)^8 \ \boxed{=5^8}$$

Exercise

Find the ***n***th term, the fifth term, and the eighth term of the geometric sequence $\left\{s_n\right\}=\left\{-3\left(\frac{1}{2}\right)^n\right\}$

Solution

$$a_5=-3\left(\frac{1}{2}\right)^5 \ \boxed{=-\frac{3}{32}} \quad a_8=-3\left(\frac{1}{2}\right)^8 \ \boxed{=-\frac{3}{256}}$$

Exercise

Find the ***n***th term, the fifth term, and the eighth term of the geometric sequence $\left\{u_n\right\}=\left\{\frac{3^{n-1}}{2^n}\right\}$

Solution

$$a_5=\frac{3^4}{2^5} \ \boxed{=\frac{81}{32}} \quad a_8=\frac{3^7}{2^8} \ \boxed{=\frac{3^7}{256}}$$

Exercise

Find the ***n***th term, the fifth term, and the eighth term of the geometric sequence $\left\{u_n\right\}=\left\{\frac{2^n}{3^{n-1}}\right\}$

Solution

$$a_5=\frac{2^5}{3^4} \ \boxed{=\frac{32}{81}} \quad a_8=\frac{2^8}{3^7} \ \boxed{=\frac{256}{3^7}}$$

Exercise

Find all possible values of ***r*** for a geometric sequence with the two given terms $a_4=3, \quad a_6=9$

Solution

$$\frac{a_6}{a_4} = \frac{9}{3} = 3$$

$$\frac{a_6}{a_4} = \frac{a_1 r^5}{a_1 r^3} = r^2 = 3 \Rightarrow \boxed{r = \pm\sqrt{3}}$$

Exercise

Find the sixth term of the geometric sequence whose first two terms are 4 and 6

Solution

Given: $a_1 = 4, \quad a_2 = 6$

$$r = \frac{a_2}{a_1} = \frac{6}{4} = \frac{3}{2}$$

$$a_6 = a_1 r^{n-1} = 4\left(\frac{3}{2}\right)^5 \underline{= \frac{243}{8}}$$

Exercise

Given a geometric sequence with $a_4 = 4, \quad a_7 = 12$, find r and a_{10}

Solution

$$r = \left(\frac{12}{4}\right)^{1/(7-4)} = 3^{1/3} \quad \Rightarrow \boxed{r = \sqrt[3]{3}}$$

$$r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$a_4 = a_1 r^{n-1} \Rightarrow a_1 = \frac{a_4}{r^3} = \frac{4}{3}$$

$$a_{10} = a_1 r^{n-1} = \frac{4}{3}\left(\sqrt[3]{3}\right)^9 \quad \underline{= 36}$$

Exercise

Find the specified term of the geometric sequence $\quad a_6; \quad a_1 = 4, \quad a_2 = 6$

Solution

$$r = \left(\frac{6}{4}\right)^{1/(2-1)} = \frac{3}{2}$$

$$r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$a_6 = 4\left(\frac{3}{2}\right)^5 \underline{= \frac{3^5}{8}}$$

$$a_n = a_1 r^{n-1}$$

Exercise

Find the specified term of the geometric sequence $\quad a_7; \quad a_2 = 3, \quad a_3 = -\sqrt{3}$

Solution

$$r = \left(\frac{-\sqrt{3}}{3}\right)^{1/(3-2)} = -\frac{\sqrt{3}}{3} \qquad\qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$a_2 = a_1\left(-\frac{\sqrt{3}}{3}\right)^1 = 3 \qquad\qquad a_n = a_1 r^{n-1}$$

$$a_1 = -\frac{9}{\sqrt{3}} = -3\sqrt{3}$$

$$a_7 = -3\sqrt{3}\left(-\frac{\sqrt{3}}{3}\right)^6 = -3\sqrt{3}\,\frac{3^3}{3^6} \;\; \underline{= -\frac{\sqrt{3}}{9}}$$

Exercise

Find the specified term of the geometric sequence $\quad a_6; \quad a_2 = 3, \quad a_3 = -\sqrt{2}$

Solution

$$r = \left(\frac{-\sqrt{2}}{3}\right)^{1/(3-2)} = -\frac{\sqrt{2}}{3} \qquad\qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$a_2 = a_1\left(-\frac{\sqrt{2}}{3}\right)^1 = 3 \qquad\qquad a_n = a_1 r^{n-1}$$

$$a_1 = -\frac{9}{\sqrt{2}}$$

$$a_6 = -\frac{9}{\sqrt{2}}\left(-\frac{\sqrt{2}}{3}\right)^5 = 9\,\frac{\sqrt{2}^4}{3^5} \;\; \underline{= \frac{4}{27}}$$

Exercise

Find the specified term of the geometric sequence $\quad a_5; \quad a_1 = 4, \quad a_2 = 7$

Solution

$$r = \frac{7}{4} \qquad\qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$a_5 = 4\left(\frac{7}{4}\right)^4 \;\; \underline{= \frac{7^4}{64}} \qquad\qquad a_n = a_1 r^{n-1}$$

Exercise

Find the specified term of the geometric sequence $a_9;\quad a_2 = 3,\quad a_5 = -81$

Solution

$$r = \left(\frac{-81}{3}\right)^{1/(5-2)} = (-27)^{1/3} = -3 \qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$a_2 = a_1(-3)^3 = 3 \qquad a_n = a_1 r^{n-1}$$

$$a_1 = -\tfrac{1}{9}$$

$$a_9 = -\tfrac{1}{9}(-3)^8 \;\underline{= -3^6}$$

Exercise

Find the specified term of the geometric sequence $a_7;\quad a_1 = -4,\quad a_3 = -1$

Solution

$$r = \left(\frac{-1}{-4}\right)^{1/(3-1)} = \left(\frac{1}{4}\right)^{1/2} = \frac{1}{2} \qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$a_7 = -4\left(\frac{1}{2}\right)^6 \;\underline{= -\frac{1}{16}} \qquad a_n = a_1 r^{n-1}$$

Exercise

Find the specified term of the geometric sequence $a_8;\quad a_2 = 3,\quad a_4 = 6$

Solution

$$r = \left(\frac{-81}{3}\right)^{1/(5-2)} = (-27)^{1/3} = -3 \qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$a_2 = a_1(-3)^3 = 3 \qquad a_n = a_1 r^{n-1}$$

$$a_1 = -\tfrac{1}{9}$$

$$a_8 = -\tfrac{1}{9}(-3)^7 \;\underline{= 3^5}$$

Exercise

Express the sum in terms of summation notation: $4 + 11 + 18 + 25 + 32$. (Answers are not unique)

Solution

$$n = 5 \qquad d = 11 - 4 = 7$$

$$a_n = 4 + (n-1)7 \qquad a_n = a_1 + (n-1)d$$

$$= 4 + 7n - 7$$

$$\underline{= 7n - 3}$$

$$4 + 11 + 18 + 25 + 32 = \sum_{n=1}^{5} (7n - 3)$$

Exercise

Express the sum in terms of summation notation: $4 + 11 + 18 + \ldots + 466$. (Answers are not unique)

Solution

Difference in terms: $d = 11 - 4 = 7$

Number of terms: $n = \frac{466 - 4}{7} + 1 = 67$

$$a_n = 4 + (n - 1)7 \qquad\qquad a_n = a_1 + (n-1)d$$

$$= 4 + 7n - 7$$

$$\underline{= 7n - 3}$$

$$4 + 11 + 18 + \ldots + 466 = \sum_{n=1}^{67} (7n - 3)$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*) $2 + 4 + 8 + 16 + 32 + 64 + 128$

Solution

$$2 + 4 + 8 + 16 + 32 + 64 + 128 = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7$$

$$= \sum_{n=1}^{7} 2^n$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*) $2 - 4 + 8 - 16 + 32 - 64$

Solution

$$r = \frac{-4}{2} = -2 \qquad\qquad r = \frac{a_2}{a_1}$$

$$a_n = 2(-2)^{n-1} = (-1)^{n-1} 2^n \qquad\qquad a_n = a_1 r^{n-1}$$

$$2 - 4 + 8 - 16 + 32 - 64 = \sum_{n=1}^{6} (-1)^{n-1} 2^n$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*) $3+8+13+18+23$

Solution

$d = 8-3 = 5$ $\qquad d = a_2 - a_1$

$a_n = 3+5(n-1) = 5n-2$ $\qquad a_n = a_1 + (n-1)d$

$$3+8+13+18+23 = \sum_{n=1}^{5}(5n-2)$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*) $256+192+144+108+\cdots$

Solution

$r = \frac{192}{256} = \frac{3}{4}$ $\qquad r = \frac{a_2}{a_1}$

$a_n = 256\left(\frac{3}{4}\right)^{n-1}$ $\qquad a_n = a_1 r^{n-1}$

$$256+192+144+108+\cdots = \sum_{n=1}^{\infty} 256\left(\frac{3}{4}\right)^{n-1}$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*): $\frac{5}{13}+\frac{10}{11}+\frac{15}{9}+\frac{20}{7}$

Solution

Number of terms: $n = 4$

Numerators : $5, 10, 15, 20$ *common difference* 5

Denominators : $13, 11, 9, 7$ *common difference* -2

Using the formula for nth term $a_n = a_1 + (n-1)d$:

Numerator: $a_n = 5+(n-1)5 = 5+5n-5 = 5n$

Denominator: $a_n = 13+(n-1)(-2) = 13-2n+2 = 15-2n$

Hence the nth term is: $\frac{5}{13}+\frac{10}{11}+\frac{15}{9}+\frac{20}{7} = \sum_{n=1}^{4}\frac{5n}{15-2n}$

Exercise

Express the sum in terms of summation notation (Answers are not unique.) $\frac{1}{4}-\frac{1}{12}+\frac{1}{36}-\frac{1}{108}$

Solution

$$\frac{1}{4}-\frac{1}{12}+\frac{1}{36}-\frac{1}{108}=\frac{1}{4}-\frac{1}{4}\frac{1}{3^1}+\frac{1}{4}\frac{1}{3^2}-\frac{1}{4}\frac{1}{3^3}$$

$$=\sum_{n=1}^{4}(-1)^{n+1}\frac{1}{4}\left(\frac{1}{3}\right)^{n-1}$$

Exercise

Express the sum in terms of summation notation (Answers are not unique.) $3+\frac{3}{5}+\frac{3}{25}+\frac{3}{125}+\frac{3}{625}$

Solution

$$3+\frac{3}{5}+\frac{3}{25}+\frac{3}{125}+\frac{3}{625}=\frac{3}{5^0}+\frac{3}{5^1}+\frac{3}{5^2}+\frac{3}{5^3}+\frac{3}{5^4}$$

$$=\sum_{n=0}^{4}\frac{3}{5^n}$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*): $\frac{3}{7}+\frac{6}{11}+\frac{9}{15}+\frac{12}{19}+\frac{15}{23}+\frac{18}{27}$

Solution

Numerators : 3, 6, 9, 12, 15,18 *common difference* 3

Denominators : 7, 11,15, 19, 13, 27 *common difference* 4

Numerator: $a_n=3+3(n-1)\ \underline{=3n|}$

Denominator: $a_n=7+4(n-1)\ \underline{=4n+3|}$

$a_n=a_1+(n-1)d$

$$\frac{3}{7}+\frac{6}{11}+\frac{9}{15}+\frac{12}{19}+\frac{15}{23}+\frac{18}{27}=\sum_{n=1}^{6}\frac{3n}{4n+3}$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique.*) $\frac{x}{3}+\frac{x^2}{9}+\frac{x^3}{27}+\cdots,\quad |x|<3$

Solution

$$\frac{x}{3}+\frac{x^2}{9}+\frac{x^3}{27}+\cdots=\frac{x}{3}+\frac{x^2}{3^2}+\frac{x^3}{3^3}+\cdots=\sum_{n=1}^{\infty}\left(\frac{x}{3}\right)^n$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*.) $2x+4x^2+8x^3+\cdots, \quad |x|<\frac{1}{2}$

Solution

$$2x+4x^2+8x^3+\cdots=2x+(2x)^2+(2x)^3+\cdots=\sum_{n=1}^{\infty}(2x)^n$$

Exercise

Find the sum of the infinite geometric series if it exists: $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$

Solution

$a_1=1, \quad r=-\frac{1}{2}$

$S=\frac{a_1}{1-r}$

$$S=\frac{1}{1+\frac{1}{2}}$$

$$=\frac{1}{\frac{3}{2}}$$

$$=\frac{2}{3}$$

Exercise

Find the sum of the infinite geometric series if it exists: $1.5+0.015+0.00015+\ldots$

Solution

$a_1=0.015, \; a_2=.00015, \quad r=\frac{.00015}{.015}=.01$

$$S=1.5+\frac{a_1}{1-r}$$

$$=1.5+\frac{.015}{1-.01}$$

$$=\frac{15}{10}+\frac{.015}{.99}$$

$$=\frac{15}{10}+\frac{15}{990}$$

$$=\frac{15}{10}+\frac{15}{990}$$

$$=\frac{1500}{990}$$

$$=\frac{50}{33}$$

Exercise

Find the sum of the infinite geometric series if it exists: $\sqrt{2}-2+\sqrt{8}-4+\ldots$

Solution

$$a_1=\sqrt{2},\ a_2=-2,\quad r=\frac{-2}{\sqrt{2}}=-\sqrt{2}$$

$|r|=\sqrt{2}>1 \Rightarrow$ The sum doesn't exist.

Exercise

Find the sum of the infinite geometric series if it exists: $256+192+144+108+\ldots$

Solution

$$a_1=256, a_2=192,\quad r=\frac{192}{256}=\frac{3}{4}$$

$$S=\frac{256}{1-.75}\ \underline{=1024|}$$

$$S=\frac{a_1}{1-r}$$

Exercise

Find the sum of the infinite geometric series if it exists: $\frac{1}{4}+\frac{2}{4}+\frac{2^2}{4}+\frac{2^3}{4}+\ldots+\frac{2^{n-1}}{4}$

Solution

$$r=\frac{\frac{2}{4}}{\frac{1}{4}}=2$$

$$S_n=a_1\frac{1-r^n}{1-r}$$

$$S_n=\frac{1}{4}\left(\frac{1-2^n}{1-2}\right)$$

$$\underline{=-\frac{1}{4}\left(1-2^n\right)|}$$

Exercise

Find the sum of the infinite geometric series if it exists: $\frac{3}{9}+\frac{3^2}{9}+\frac{3^3}{9}+\ldots+\frac{3^n}{9}$

Solution

$$r=\frac{\frac{3^2}{9}}{\frac{3}{9}}=3$$

$$S_n=\frac{3}{9}\left(\frac{1-3^n}{1-3}\right)$$

$$S_n=a_1\frac{1-r^n}{1-r}$$

$$\underline{=-\frac{1}{6}\left(1-3^n\right)|}$$

Exercise

Find the sum of the infinite geometric series if it exists: $-1-2-4-8-\cdots-2^{n-1}$

Solution

$$r=\frac{-2}{-1}=2$$

$$S_n=-1\left(\frac{1-2^n}{1-2}\right) \qquad S_n=a_1\frac{1-r^n}{1-r}$$

$$\boxed{=1-2^n}$$

Exercise

Find the sum of the infinite geometric series if it exists: $2+\frac{6}{5}+\frac{18}{25}+\ldots+2\left(\frac{3}{5}\right)^{n-1}$

Solution

$$r=\frac{\frac{6}{5}}{2}=\frac{3}{5}<1$$

$$S_n=2\cdot\frac{1-\left(\frac{3}{5}\right)^n}{1-\frac{3}{5}} \qquad S_n=a_1\frac{1-r^n}{1-r}$$

$$=2\cdot\frac{1-\left(\frac{3}{5}\right)^n}{\frac{2}{5}}$$

$$\boxed{=5\left(1-\left(\frac{3}{5}\right)^n\right)}$$

Exercise

Find the sum of the infinite geometric series if it exists: $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$

Solution

$$r=\frac{1}{3}<1$$

$$S=\frac{1}{1-\frac{1}{3}} \qquad S=\frac{a_1}{1-r}$$

$\boxed{=\frac{3}{2}}$ The series ***converges***

Exercise

Find the sum of the infinite geometric series if it exists: $2+\frac{4}{3}+\frac{8}{9}+\frac{16}{27}+\cdots$

Solution

$$a_1=2 \quad r=\frac{\frac{4}{3}}{2}=\frac{2}{3}<1$$

$$S=\frac{2}{1-\frac{2}{3}}$$

$$S=\frac{a_1}{1-r}$$

$$\underline{=6|}$$ The series ***converges***

Exercise

Find the sum of the infinite geometric series if it exists: $2-\frac{1}{2}+\frac{1}{8}-\frac{1}{32}+\cdots$

Solution

$$a_1=2 \quad r=-\frac{1}{4}, \ |r|<1$$

$$S=\frac{2}{1+\frac{1}{4}}$$

$$S=\frac{a_1}{1-r}$$

$$\underline{=\frac{8}{5}|}$$ The series ***converges***

Exercise

Find the sum of the infinite geometric series if it exists: $1-\frac{3}{4}+\frac{9}{16}-\frac{27}{64}+\cdots$

Solution

$$a_1=1 \quad r=-\frac{3}{4}, \ |r|<1$$

$$S=\frac{1}{1+\frac{3}{4}}$$

$$S=\frac{a_1}{1-r}$$

$$\underline{=\frac{4}{7}|}$$ The series ***converges***

Exercise

Find the sum of the infinite geometric series if it exists: $9+12+16+\frac{64}{3}+\cdots$

Solution

$a_1=9 \quad r=\frac{4}{3}>1$ The series ***diverges***

Exercise

Find the sum of the infinite geometric series if it exists: $8+12+18+27+\cdots$

Solution

$a_1 = 8 \quad r = \frac{3}{2} > 1$ The series ***diverges***

Exercise

Find the sum of the infinite geometric series if it exists: $6+2+\frac{2}{3}+\frac{2}{9}+\cdots$

Solution

$a_1 = 6 \quad r = \frac{1}{3}, \ |r| < 1$

$$S = \frac{6}{1-\frac{1}{3}}$$

$$S = \frac{a_1}{1-r}$$

$$= \frac{6}{\frac{2}{3}}$$

$$\underline{= 9}$$ The series ***converges***

Exercise

Find the sum: $\sum_{k=1}^{20}(3k-5)$

Solution

$$a_1 = 3(1)-5 = -2 \quad \textit{and} \quad a_{20} = 3(20)-5 = 55$$

$$\sum_{k=1}^{20}(3k-5) = \frac{20}{2}(-2+55)$$

$$S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$\underline{= 530}$$

Exercise

Find the sum: $\sum_{k=1}^{18}\left(\frac{1}{2}k+7\right)$

Solution

$$a_1 = \frac{1}{2}(1)+7 = \frac{15}{2} \quad \textit{and} \quad a_{18} = \frac{1}{2}(18)+7 = 16$$

$$\sum_{k=1}^{18}\left(\frac{1}{2}k+7\right) = \frac{18}{2}\left(\frac{15}{2}+16\right)$$

$$S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$\underline{= \frac{423}{2}}$$

Exercise

Find the sum: $\sum_{k=1}^{80}(2k-5)$

Solution

$a_1 = 2(1)-5 = -3 \quad and \quad a_{80} = 2(80)-5 = 155$

$\sum_{k=1}^{80}(2k-5) = \frac{80}{2}(-3+155)$ $\qquad S_n = \frac{n}{2}(a_1 + a_n)$

$= 40(152)$

$\underline{= 6080}$

Exercise

Find the sum: $\sum_{n=1}^{90}(3-2n)$

Solution

$a_1 = 3-2(1) = 1 \quad and \quad a_{90} = 3-2(90) = -177$

$\sum_{n=1}^{80}(3-2n) = \frac{90}{2}(1-177)$ $\qquad S_n = \frac{n}{2}(a_1 + a_n)$

$= 45(-176)$

$\underline{= -7920}$

Exercise

Find the sum: $\sum_{n=1}^{100}\left(6-\frac{1}{2}n\right)$

Solution

$a_1 = 6-\frac{1}{2}(1) = \frac{11}{2} \quad and \quad a_{100} = 6-\frac{1}{2}(100) = -44$

$\sum_{n=1}^{100}\left(6-\frac{1}{2}n\right) = \frac{100}{2}\left(\frac{11}{2}-44\right)$ $\qquad S_n = \frac{n}{2}(a_1 + a_n)$

$= 50\left(-\frac{77}{2}\right)$

$\underline{= -1925}$

Exercise

Find the sum: $\sum_{n=1}^{80}\left(\frac{1}{3}n+\frac{1}{2}\right)$

Solution

$$a_1=\frac{1}{3}(1)+\frac{1}{2}=\frac{5}{6} \quad \textit{and} \quad a_{80}=\frac{1}{3}(80)+\frac{1}{2}=\frac{163}{6}$$

$$\sum_{n=1}^{80}\left(\frac{1}{3}n+\frac{1}{2}\right)=\frac{80}{2}\left(\frac{5}{6}+\frac{163}{6}\right) \qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$$

$$=40\left(\frac{168}{6}\right)$$

$$=1,120$$

Exercise

Find the sum: $\sum_{k=1}^{10}3^k$

Solution

$$\sum_{k=1}^{10}3^k=3\frac{1-3^{10}}{1-3} \qquad S_n=a_1\frac{1-r^n}{1-r}$$

$$=3\frac{-59048}{-2}$$

$$=88,572$$

Exercise

Find the sum: $\sum_{k=1}^{9}\left(-\sqrt{5}\right)^k$

Solution

$$\begin{cases} a_1=-\sqrt{5} \\ a_2=\left(-\sqrt{5}\right)^2=5 \end{cases} \Rightarrow r=\frac{a_2}{a_1}=\frac{5}{-\sqrt{5}}=-\sqrt{5} \qquad S_n=a_1\frac{1-r^n}{1-r}$$

$$\sum_{k=1}^{9}\left(-\sqrt{5}\right)^k=\left(-\sqrt{5}\right)\frac{1-\left(-\sqrt{5}\right)^9}{1-\left(-\sqrt{5}\right)}$$

$$=\frac{\left(-\sqrt{5}\right)\left(1+625\sqrt{5}\right)}{1+\sqrt{5}}\frac{1-\sqrt{5}}{1-\sqrt{5}}$$

$$=\frac{3124\sqrt{5}-3120}{-4}$$

$$=780-781\sqrt{5}$$

Exercise

Find the sum: $\sum_{k=0}^{9}\left(-\frac{1}{2}\right)^{k+1}$

Solution

$$\sum_{k=0}^{9}\left(-\frac{1}{2}\right)^{k+1}=\left(-\frac{1}{2}\right)\frac{1-\left(-\frac{1}{2}\right)^{10}}{1+\frac{1}{2}} \qquad S_n=a_1\frac{1-r^n}{1-r}$$

$$=-\frac{1}{2}\frac{1-\frac{1}{2^{10}}}{\frac{3}{2}}$$

$$=-\frac{\frac{1024-1}{1024}}{3}$$

$$=-\frac{1023}{3072}$$

$$\boxed{=-\frac{341}{1024}}$$

Exercise

Find the sum : $\sum_{n=1}^{\infty}2\left(\frac{2}{3}\right)^{n-1}$

Solution

$$\sum_{n=1}^{\infty}2\left(\frac{2}{3}\right)^{n-1}=\frac{2}{1-\frac{2}{3}} \qquad S=\frac{a_1}{1-r}$$

$$=\frac{2}{\frac{1}{3}}$$

$\boxed{=6}$, the series ***converges***

Exercise

Find the sum: $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n}$

Solution

$$\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n}=\frac{2}{3}\frac{1}{1-\frac{2}{3}} \qquad |r|=\frac{2}{3}<1 \qquad S=\frac{a_1}{1-r}$$

$$=\frac{2}{3}(3)$$

$\boxed{=2}$, the series ***converges***

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{\infty} 3\left(\frac{3}{2}\right)^{n}$

Solution

Since $|r| = \frac{3}{2} > 1$, the series ***diverges***

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{\infty} 5\left(\frac{1}{4}\right)^{n-1}$

Solution

$$\sum_{n=1}^{\infty} 5\left(\frac{1}{4}\right)^{n-1} = \frac{5}{1-\frac{1}{4}} \qquad a_1 = 5 \quad |r| = \frac{1}{4} < 1 \qquad S = \frac{a_1}{1-r}$$

$$= \frac{20}{3}$$

The series ***converges***

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{\infty} 8\left(\frac{1}{3}\right)^{n-1}$

Solution

$$\sum_{n=1}^{\infty} 8\left(\frac{1}{3}\right)^{n-1} = \frac{8}{1-\frac{1}{3}} \qquad a_1 = 8 \quad |r| = \frac{1}{3} < 1 \qquad S = \frac{a_1}{1-r}$$

$$= 12$$

The series ***converges***

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2}\cdot 3^{k-1}$

Solution

Since $|r| = 3 > 1$, the series ***diverges***

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{\infty} 6\left(-\frac{2}{3}\right)^{k-1}$

Solution

$$\sum_{k=1}^{\infty} 6\left(-\frac{2}{3}\right)^{k-1} = \frac{6}{1+\frac{2}{3}} \qquad a_1 = 6 \quad |r| = \frac{2}{3} < 1 \qquad S = \frac{a_1}{1-r}$$

$$= \frac{18}{5}$$ The series ***converges***

Exercise

Find the sum: $\sum_{k=1}^{\infty} 4\left(-\frac{1}{2}\right)^{k-1}$

Solution

$$\sum_{k=1}^{\infty} 4\left(-\frac{1}{2}\right)^{k-1} = \frac{4}{1+\frac{1}{2}} \qquad a_1 = 4 \quad |r| = \frac{1}{2} < 1 \qquad S = \frac{a_1}{1-r}$$

$$= \frac{8}{3}$$ The series ***converges***

Exercise

Find the sum: $\sum_{k=8}^{14} \left(3^{k-7} + 2j^2\right)$

Solution

$$a_n = 3^{n-7} \to a_1 = 3^{-6}; \ \ r = 3 \quad \& \quad n = 14 - 8 + 1 = 7$$

$$\sum_{k=8}^{14} \left(3^{k-7} + 2j^2\right) = \sum_{k=8}^{14} 3^{k-7} + 2\sum_{k=8}^{14} j^2$$

$$= 3^{-6} \cdot \frac{1-3^7}{1-3} + 2(7)\,j^2$$

$$= -\frac{1}{2}\left(\frac{1-3^7}{3^6}\right) + 14j^2$$

$$= -\frac{1}{2}\left(\frac{-2{,}186}{729}\right) + 14j^2$$

$$= \frac{1{,}093}{729} + 14j^2$$

Exercise

Find the sum: 14, 16, 18, 20, $\cdots$

Solution

$$n = 120; \quad a_1 = 14, \quad d = 16 - 14 = 2$$

$$S_{120} = \frac{120}{2}\left[2(14)+2(120-1)\right] \qquad S_n = \frac{n}{2}\left[2a_1+(n-1)d\right]$$

$$= 60(48+238)$$

$$\boxed{=17,160}$$

Exercise

Find the sum of the first 46 terms of $2,\ -1,\ -4,\ -7,\ \cdots$

Solution

$n = 46; \quad a_1 = 2, \quad d = -1-2 = -3$

$$S_{46} = \frac{46}{2}\left[2(2)-3(46-1)\right] \qquad S_n = \frac{n}{2}\left[2a_1+(n-1)d\right]$$

$$= 23(4-135)$$

$$\boxed{=-3,013}$$

Exercise

Find the rational number represented by the repeating decimal $0.\overline{23}$

Solution

$0.\overline{23} = 0.23 + 0.0023 + .000023 + \dots \qquad a_1 = 0.23, \quad r = \frac{.0023}{.23} = 0.01$

$$S = \frac{0.23}{1-0.01} \qquad S = \frac{a_1}{1-r}$$

$$= \frac{0.23}{0.99}$$

$$\boxed{= \frac{23}{99}}$$

Exercise

Find the rational number represented by the repeating decimal $0.0\overline{71}$

Solution

$0.0\overline{71} = 0.071 + 0.00071 + .0000071 + \dots \qquad a_1 = 0.071, \quad r = \frac{.00071}{.071} = 0.01$

$$S = \frac{0.071}{1-0.01} \qquad S = \frac{a_1}{1-r}$$

$$= \frac{0.071}{0.990}$$

$$\boxed{= \frac{71}{990}}$$

Exercise

Find the rational number represented by the repeating decimal $2.4\overline{17}$

Solution

$$2.4\overline{17} = 2.4 + 0.017 + 0.00017 + .0000017 + \dots$$

$$a_1 = 0.017, \quad r = \frac{.00017}{.017} = 0.01$$

$$S = 2.4 + \frac{0.017}{1 - 0.01} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= \frac{24}{10} + \frac{0.017}{0.990}$$

$$= \frac{24}{10} + \frac{17}{990}$$

$$= \frac{240 + 17}{990}$$

$$= \frac{2,393}{990}$$

Exercise

Find the rational number represented by the repeating decimal $10.\overline{5}$

Solution

$$10.\overline{5} = 10 + 0.5 + 0.05 + .005 + \dots$$

$$a_1 = 0.5, \quad r = \frac{0.05}{0.5} = 0.1$$

$$S = 10 + \frac{0.5}{1 - 0.1} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= 10 + \frac{0.5}{0.9}$$

$$= 10 + \frac{5}{9}$$

$$= \frac{95}{9}$$

Exercise

Find the rational number represented by the repeating decimal $5.\overline{146}$

Solution

$$5.\overline{146} = 5 + 0.146 + 0.000146 + .000000146 + \dots$$

$$a_1 = 0.146, \quad r = \frac{0.000146}{0.146} = 0.001$$

$$S = 5 + \frac{0.146}{1 - 0.001} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= 5 + \frac{0.146}{0.999}$$

$$= 5 + \frac{146}{999}$$

$$\boxed{= \frac{5,141}{999}}$$

Exercise

Find the rational number represented by the repeating decimal $3.2\overline{394}$

Solution

$$3.2\overline{394} = 3.2 + 0.0394 + 0.0000394 + \dots$$

$$a_1 = 0.0394, \quad r = \frac{0.0000394}{0.0394} = 0.001$$

$$S = 3.2 + \frac{0.0394}{1 - 0.001} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= \frac{32}{10} + \frac{0.0394}{0.9990}$$

$$= \frac{32}{10} + \frac{394}{9990}$$

$$= \frac{31968 + 394}{9990}$$

$$= \frac{32,362}{9,990}$$

$$\boxed{= \frac{16,181}{4,995}}$$

Exercise

Find the rational number represented by the repeating decimal $1.\overline{6124}$

Solution

$$1.\overline{6124} = 1 + 0.6124 + 0.00006124 + \dots$$

$$a_1 = 0.6124, \quad r = \frac{0.00006124}{0.6124} = 0.0001$$

$$S = 1 + \frac{0.6124}{1 - 0.0001} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= 1 + \frac{0.6124}{0.9999}$$

$$= 1 + \frac{6124}{9999}$$

$$\boxed{= \frac{16,123}{9,999}}$$

Exercise

Find x so that $x+3$, $2x+1$, and $5x+2$ are consecutive terms of an arithmetic sequence.

Solution

$$d = 2x+1-(x+3) = x-2$$

$$d = 5x+2-(2x+1) = 3x+1$$

$$d = 3x+1 = x-2$$

$$2x = -3 \quad \rightarrow \quad \boxed{x = -\frac{3}{2}}$$

Exercise

Find x so that $2x$, $3x+2$, and $5x+3$ are consecutive terms of an arithmetic sequence.

Solution

$$d = 3x+2-2x = x+2$$

$$d = 5x+3-(3x+2) = 2x+1$$

$$d = 2x+1 = x+2 \quad \rightarrow \quad \boxed{x = 1}$$

Exercise

Find x so that x, $x+2$, and $x+3$ are consecutive terms of a geometric sequence.

Solution

$$r = \frac{x+2}{x}, \quad r = \frac{x+3}{x+2}$$

$$r = \frac{x+2}{x} = \frac{x+3}{x+2}$$

$$(x+2)^2 = x^2+3x$$

$$x^2+4x+4-x^2-3x = 0$$

$$x+4 = 0 \quad \rightarrow \quad \boxed{x = -4}$$

Exercise

Find x so that $x-1$, x and $x+2$ are consecutive terms of a geometric sequence.

Solution

$$r = \frac{x}{x-1} = \frac{x+2}{x}$$

$$x^2 = x^2+x-2$$

$$x-2 = 0 \quad \rightarrow \quad \boxed{x = 2}$$

Exercise

How many terms must be added in an arithmetic sequence whose first term is 11 and whose common difference is 3 to obtain a sum of 1092?

Solution

Given: $a_1 = 11; \quad d = 3; \quad S = 1092$

$1092 = \frac{n}{2}\left(22 + 3(n-1)\right)$ $\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

$n(3n+19) = 2184$

$3n^2 + 19n - 2184 = 0$ $\qquad n = \frac{-19 \pm \sqrt{361 + 26208}}{6} = \frac{-19 \pm 163}{6}$

$\underline{n = 24|}$ & $\cancel{n = -\frac{91}{3}}$

Exercise

How many terms must be added in an arithmetic sequence whose first term is 78 and whose common difference is –4 to obtain a sum of 702?

Solution

Given: $a_1 = 78; \quad d = -4; \quad S = 702$

$702 = \frac{n}{2}\left(2(78) - 4(n-1)\right)$ $\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

$n(160 - 4n) = 1404$

$-4n^2 + 160n - 1404 = 0$ $\qquad n = \frac{-160 \pm \sqrt{25,600 - 22464}}{-8} = \frac{160 \pm 56}{8}$

$\underline{n = 13|}$ & $\underline{n = 27|}$

Exercise

The first ten rows of seating in a certain section of a stadium have 30 seats, 32 seats, 34 seats, and so on. The eleventh through the twentieth rows each contain 50 seats. Find the total number of seats in the section.

Solution

Given: $a_1 = 30; \quad d = 2$

$S = S_{10} + 50(20 - 11 + 1)$ $\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

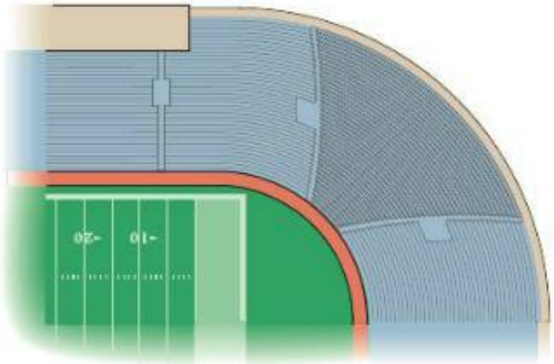
$= \frac{10}{2}\left(2(30) + 2(9)\right) + 50(10)$

$= 5(78) + 500$

$\underline{= 890 \ \ seats|}$

Exercise

The corner section of a football stadium has 15 seats in the first row and 40 rows in all. Each successive row contains two additional seats. How many seats are in this section?



Solution

Given: $a_1 = 15; \quad d = 2; \quad n = 40$

$$S_{40} = \frac{40}{2}\big(30 + 2(40-1)\big) \qquad\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$= 20(30 + 78)$$

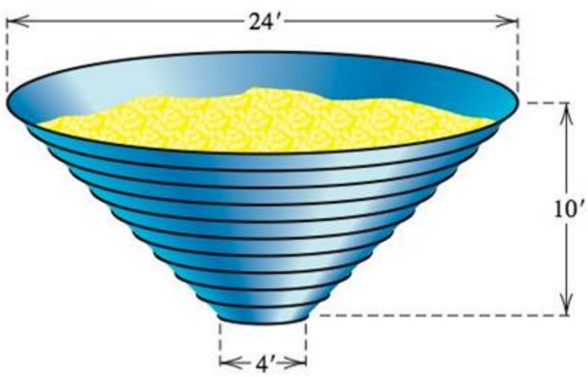
$$= 20(108)$$

$$= 2,160$$

The corner section has 2,160 seats.

Exercise

A gain bin is to be constructed in the shape of a frustum of a cone.



The bin is to be 10 *feet* tall with 11 metal rings positioned uniformly around it, from the 4-foot opening at the bottom to the 24-foot opening at the top. Find the total length of metal needed to make the rings.

Solution

The circumference of each ring is πD.

$a_1 = 4\pi; \quad a_{11} = 24\pi$

$$24 = 4 + (11-1)d \quad \rightarrow \quad 10d = 20 \ \Rightarrow \underline{d = 2} \qquad\qquad a_n = a_1 + (n-1)d$$

$$S_{11} = \frac{11}{2}(4\pi + 24\pi) \ \underline{= 154\pi \ \ ft} \qquad\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

Exercise

A bicycle rider coasts downhill, traveling 4 *feet* the first second. In each succeeding second, the rider travels 5 *feet* farther than in the preceding second. If the rider reaches the bottom of the hill in 11 *seconds*, find the total distance traveled.

Solution

Given: $a_1 = 4\ ft \quad \& \quad d = 5\ ft$

$$S_{11} = \frac{11}{2}\left(8 + 5(10)\right) \underline{= 319\ ft|} \qquad\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

∴ the total distance traveled 319 *feet*.

Exercise

A contest will have five each prizes totaling $5,000, and there will be a $100 difference between successive prices. Find the first prize.

Solution

Given: $n = 5 \quad S_5 = 5000 \quad d = -100$

$$5,000 = \frac{5}{2}\left[2a_1 + 4(-100)\right] \qquad\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$2,000 = 2a_1 - 400$$

$$\underline{a_1 = \$1,200|}$$

Exercise

A Company is to distribute $46,000 in bonuses to its top ten salespeople. The tenth salesperson on the list will receive $1,000, and the difference in bonus money between successively ranked salesperson is to be constant. Find the bonus for each salesperson.

Solution

Given: $n = 10 \quad S_{10} = 46,000 \quad a_{10} = 1,000$

$$46,000 = \frac{10}{2}\left(a_1 + 1000\right) \qquad\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$9,200 = a_1 + 1000$$

$$a_1 = 8,200$$

$$\underline{|d} = \frac{1,000 - 8,200}{9} \underline{= -800|} \qquad\qquad a_n = a_1 + (n-1)d$$

\$8,200 \$7,400 \$6,600 \$5,800 \$5,000 \$4,200 \$3,400 \$2,600 \$1,800 \$1,000

Exercise

Assuming air resistance is negligible, a small object that is dropped from a hot air balloon falls 16 *feet* during the first second, 48 *feet* during the second second, 80 *feet* during the third second, 112 *feet* during the fourth second, and so on. Find an expression for the distance the object falls in n seconds.

Solution

Given the sequence: 16, 48, 80, 112, $\cdots$

This is an arithmetic sequence with: $a_1 = 16 \quad \& \quad d = 48 - 16 = 32$

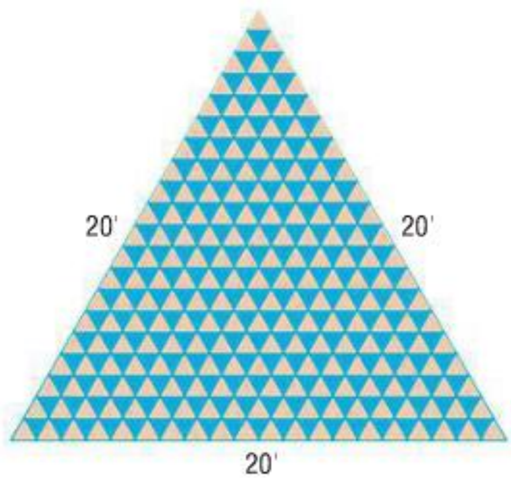
$$S_n = \frac{n}{2}\left(32 + 32(n-1)\right) \qquad\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$= \frac{n}{2}(32n)$$

$$= 16n^2$$

Exercise

A mosaic is designed in the shape of an equilateral triangle, 20 *feet* on each side. Each tile in the mosaic is in the shape of an equilateral triangle, 12 *inches* to a side. The tiles are to alternate in color as shown below.



How many tiles of each color will be required?

Solution

Bottom row has 20 lighter colored tiles.

Top row has 1 lighter colored tile.

The number decreases by 1 as we move up the triangle.

$\therefore$ This is an arithmetic sequence with: $a_1 = 20; \quad d = -1; \quad n = 20$

$$S_{20} = \frac{20}{2}\left(40 + (-1)(20-1)\right) \qquad\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$= 10(40 - 19)$$

$$= 10(21)$$

$$= 210$$

$\therefore$ There are 210 lighter colored tiles.

Bottom row has 19 darker colored tiles.
Top row has 1 darker colored tile.
∴ This is an arithmetic sequence with: $a_1 = 1; \quad d = -1; \quad n = 19$

$$S_{19} = \frac{19}{2}\left(2(19) + (-1)(19-1)\right) \qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$= \frac{19}{2}(38 - 18)$$

$$\underline{= 190|}$$

∴ There are 190 darker colored tiles.

Exercise

A brick staircase has a total of 30 steps. The bottom step requires 100 bricks. Each successive step requires two fewer bricks than the prior step.

a) How many bricks are required for the top step?
b) How many bricks are required to build the staircase?

Solution

a) ***Given***: $n = 30 \quad a_1 = 100 \quad d = -2$

$$a_n = 100 - 2(n-1) \ \underline{= -2n + 102|} \qquad a_n = a_1 + (n-1)d$$

$$a_{30} = 102 - 60 \ \underline{= 42|}$$

b) $S_{30} = 15(100 + 42) \ \underline{= 2{,}130|} \qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$

It required 2130 bricks to build the staircase.

Solution Section 4.3 – Mathematical Induction

Exercise

Find all positive integers n for which the given statement is not true

$a)\quad 3^n > 6n \qquad b)\quad 3^n > 2n+1 \qquad c)\quad 2^n > n^2 \qquad d)\quad n! > 2n$

Solution

a) $n=1 \quad 3<6$

$n=2 \quad 3^2 < 18$

$n=3, \quad 27 > 18$

The statement is true for all $n \ge 3 \quad 3^n > 6n$

The statement is not true for $\boxed{n=1,\ 2}$

b) $n=1; \quad 3=3$

$n=2; \quad 9>5$

The statement is true for all $n \ge 2 \quad 3^n > 2n+1$

The statement is not true for $\boxed{n=1}$

c) $n=1; \quad 2<4$

$n=2; \quad 4=4$

$n=3; \quad 8<9$

$n=4; \quad 16=16$

$n=5; \quad 32>25$

The statement is true for all $n \ge 5; \quad 2^n > n^2$

The statement is not true for $\boxed{n=1,\ 2,\ 3,\ 4}$

d) $n=1; \quad 1<2$

$n=2; \quad 2<4$

$n=3; \quad 6=6$

$n=4; \quad 12>8$

The statement is true for all $n \ge 4; \quad n! > 2n$

The statement is not true for $\boxed{n=1,\ 2,\ 3}$

Exercise

Prove that the statement is true for every positive integer n. $2+4+6+\ldots+2n=n(n+1)$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 2=1(1+1)=2$; hence P_1 is true.

(2) Assume $2+4+6+\ldots+2k=k(k+1)$ is true

$$\Rightarrow 2+4+6+\ldots+2k+2(k+1)=(k+1)(k+1+1)?$$

$$\begin{aligned} 2+4+6+\ldots+2k+2(k+1) &= 2+4+6+\ldots+2k+2(k+1) \\ &= k(k+1)+2(k+1) \qquad \textbf{\textit{Factor} (k + 1)} \\ &= (k+1)(k+2) \\ &= (k+1)(k+1+1) \quad \checkmark \end{aligned}$$

Hence P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $1+3+5+\ldots+(2n-1)=n^2$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 1\overset{?}{=}1^2=1$; hence P_1 is true.

(2) Assume $1+3+5+\ldots+(2k-1)=k^2$ is true

$$\Rightarrow 1+3+5+\ldots+(2(k+1)-1)=(k+1)^2?$$

$$\begin{aligned} 1+3+5+\ldots+(2k-1)+(2(k+1)-1) &= 1+3+5+\ldots+(2k-1)+(2k+2-1) \\ &= k^2+(2k+1) \\ &= k^2+2k+1 \\ &= (k+1)^2 \quad \checkmark \end{aligned}$$

Hence P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $2+7+12+\ldots+(5n-3)=\frac{1}{2}n(5n-1)$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 2\overset{?}{=}\frac{1}{2}(1)(5(1)-1)=\frac{1}{2}(4)=2$; hence P_1 is true.

(2) Assume $2+7+12+\ldots+(5k-3)=\frac{1}{2}k(5k-1)$ is true

$$2+7+12+\ldots+(5(k+1)-3)=\frac{1}{2}(k+1)(5(k+1)-1)?$$

$$\begin{aligned}
2+7+12+\ldots+(5k-3)+(5(k+1)-3) &= 2+7+12+\ldots+(5k-3)+(5k+5-3) \\
&= \frac{1}{2}k(5k-1)+(5k+2)\frac{2}{2} \\
&= \frac{1}{2}\left[5k^2-k+10k+4\right] \\
&= \frac{1}{2}\left[5k^2-k+5k+5k+5-1\right] \\
&= \frac{1}{2}\left[k(5k-1+5)+5k+5-1\right] \\
&= \frac{1}{2}\left[(k+1)(5k+5-1)\right] \\
&= \frac{1}{2}\left[(k+1)(5(k+1)-1)\right] \quad \surd \quad P_{k+1} \text{ is also true.}
\end{aligned}$$

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $1+2.2+3.2^2+\ldots+n.2^{n-1}=1+(n-1).2^n$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 1\overset{?}{=}1+(1-1)2^1=1-0=1$; hence P_1 is true.

(2) $1+2.2+3.2^2+\ldots+k.2^{k-1}=1+(k-1).2^k$ is true

$$1+2.2+3.2^2+\ldots+k.2^{k-1}+(k+1).2^{(k+1)-1}=1+((k+1)-1).2^{k+1}?$$

$$\begin{aligned}
1+2.2+3.2^2+\ldots+k.2^{k-1}+(k+1).2^{(k+1)-1} &= 1+(k-1).2^k+(k+1).2^{k+1-1} \\
&= 1+k.2^k-1.2^k+(k+1).2^k \\
&= 1+k.2^k-1.2^k+k.2^k+1.2^k \\
&= 1+2^1k.2^k \\
&= 1+(k+0).2^{k+1} \\
&= 1+((k+1)-1).2^{k+1} \quad \surd \quad P_{k+1} \text{ is also true.}
\end{aligned}$$

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow 1^2 \overset{?}{=} \frac{1(1+1)(2(1)+1)}{6}=\frac{1(2)(3)}{6}=\frac{6}{6}=1$ √ ; hence P_1 is true.

(2) $1^2+2^2+3^2+\ldots+k^2=\frac{k(k+1)(2k+1)}{6}$ is true

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}?$$

$$\begin{aligned}1^2+2^2+3^2+\ldots+k^2+(k+1)^2&=\frac{k(k+1)(2k+1)}{6}+(k+1)^2\\&=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}\\&=\frac{(k+1)\left[k(2k+1)+6(k+1)\right]}{6}\\&=\frac{(k+1)\left[2k^2+k+6k+6\right]}{6}\\&=\frac{(k+1)\left[2k^2+7k+6\right]}{6}\\&=\frac{(k+1)((k+2)(2k+3))}{6}\\&=\frac{(k+1)((k+1+1)(2k+2+1))}{6}\\&=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}\ \surd \qquad P_{k+1} \text{ is also true.}\end{aligned}$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $\frac{1}{1\bullet 2}+\frac{1}{2\bullet 3}+\frac{1}{3\bullet 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow \frac{1}{1.2}\overset{?}{=}\frac{1}{1+1}=\frac{1}{2}=\frac{1}{1.2}$ √ ; hence P_1 is true.

(2) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$ is true

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{(k+1)+1}?$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)(k+1)}{(k+1)(k+2)}$$

$$=\frac{k+1}{(k+1+1)}$$

$$=\frac{k+1}{(k+1)+1} \quad \surd \quad P_{k+1} \text{ is also true.}$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow \frac{1}{2}\overset{?}{=}1-\frac{1}{2}=\frac{1}{2}$ $\surd$; P_1 is true.

(2) $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}=1-\frac{1}{2^k}$ is true

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^{k+1}}\,?$$

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^k}+\frac{1}{2^{k+1}}$$

$$=1-\frac{1}{2^k}+\frac{1}{2^k\cdot 2}$$

$$=\frac{2^{k+1}-2+1}{2^{k+1}}$$

$$=\frac{2^{k+1}-1}{2^{k+1}}$$

$$=\frac{2^{k+1}}{2^{k+1}}-\frac{1}{2^{k+1}}$$

$$=1-\frac{1}{2^{k+1}} \quad \surd \quad P_{k+1} \text{ is also true.}$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{(3n-2)\cdot(3n+1)}=\frac{n}{3n+1}$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow \frac{1}{1\cdot 4}\overset{?}{=}\frac{1}{3(1)+1}=\frac{1}{4}$ √ ; P_1 is true.

(2) $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}$ is true

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3(k+1)-2)(3(k+1)+1)}\overset{?}{=}\frac{k+1}{3(k+1)+1}$$

$$\begin{aligned}\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3(k+1)-2)(3(k+1)+1)}&=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}\\&=\frac{3k^2+4k+1}{(3k+1)(3k+4)}\\&=\frac{(3k+1)(k+1)}{(3k+1)(3k+3+1)}\\&=\frac{k+1}{3(k+1)+1}\end{aligned}$$

√ P_{k+1} is also true

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\frac{4}{5}+\frac{4}{5^2}+\frac{4}{5^3}+\cdots+\frac{4}{5^n}=1-\frac{1}{5^n}$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow \frac{4}{5}\overset{?}{=}1-\frac{1}{5}=\frac{4}{5}$ √ ; P_1 is true.

(2) $\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}=1-\frac{1}{5^k}$ is true

$$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}+\frac{4}{5^{k+1}}\overset{?}{=}1-\frac{1}{5^{k+1}}$$

$$\begin{aligned}\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}+\frac{4}{5^{k+1}}&=1-\frac{1}{5^k}+\frac{4}{5^{k+1}}\\&=1-\left(\frac{1}{5^k}-\frac{4}{5^{k+1}}\right)\\&=1-\frac{5-4}{5^{k+1}}\\&=1-\frac{1}{5^{k+1}}\end{aligned}$$

√ P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $1^3+2^3+3^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow 1^3\overset{?}{=}\dfrac{1^2(1+1)^2}{4}=\dfrac{4}{4}=1$ $\checkmark$; P_1 is true.

(2) $\dfrac{4}{5}+\dfrac{4}{5^2}+\cdots+\dfrac{4}{5^k}=1-\dfrac{1}{5^k}$ is true

$$1^3+2^3+\cdots+k^3+(k+1)^3\overset{?}{=}\frac{(k+1)^2\left((k+1)+1\right)^2}{4}$$

$$\begin{aligned}1^3+2^3+\cdots+k^3+(k+1)^3&=\frac{k^2(k+1)^2}{4}+(k+1)^3\\&=\frac{k^2(k+1)^2+4(k+1)^3}{4}\\&=\frac{(k+1)^2\left[k^2+4(k+1)\right]}{4}\\&=\frac{(k+1)^2\left(k^2+4k+4\right)}{4}\\&=\frac{(k+1)^2(k+2)^2}{4}\\&=\frac{(k+1)^2\left((k+1)+1\right)^2}{4}\quad\checkmark\end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $3+3^2+3^3+\ldots+3^n=\frac{3}{2}\left(3^n-1\right)$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow 3\overset{?}{=}\frac{3}{2}\left(3^1-1\right)=\frac{3}{2}2=3$ $\checkmark$; P_1 is true.

(2) $3+3^2+\cdots+3^k=\frac{3}{2}\left(3^k-1\right)$ is true **→** Is $\quad 3+3^2+\cdots+3^k+3^{k+1}\overset{?}{=}\frac{3}{2}\left(3^{k+1}-1\right)$

$$\begin{aligned}3+3^2+\cdots+3^k+3^{k+1}&=\frac{3}{2}\left(3^k-1\right)+3^{k+1}\\&=\frac{1}{2}3^{k+1}-\frac{3}{2}+3^{k+1}\\&=\frac{3}{2}3^{k+1}-\frac{3}{2}\\&=\frac{3}{2}\left(3^{k+1}-1\right)\quad\checkmark\end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $x^{2n}+x^{2n-1}y+\cdots+xy^{2n-1}+y^{2n}=\dfrac{x^{2n+1}-y^{2n+1}}{x-y}$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow$ $x^2+xy+y^2\overset{?}{=}\dfrac{x^3-y^3}{x-y}$

$$=\frac{(x-y)\left(x^2+xy+y^2\right)}{x-y}$$

$$=x^2+xy+y^2 \quad \checkmark \; ; \; P_1 \text{ is true.}$$

(2) $x^{2k}+x^{2k-1}y+\cdots+xy^{2k-1}+y^{2k}=\dfrac{x^{2k+1}-y^{2k+1}}{x-y}$ is true

$$x^{2(k+1)}+x^{2(k+1)-1}y+\cdots+xy^{2(k+1)-1}+y^{2(k+1)}\overset{?}{=}\frac{x^{2(k+1)+1}-y^{2(k+1)+1}}{x-y}$$

$$x^{2k+2}+x^{2k+1}y+\cdots+xy^{2k+1}+y^{2k+2}=x^2\left(x^{2k}+x^{2k-1}y+\cdots+y^{2k}\right)+xy^{2k+1}+y^{2k+2}$$

$$=x^2\left(\frac{x^{2k+1}-y^{2k+1}}{x-y}\right)+xy^{2k+1}+y^{2k+2}$$

$$=\frac{x^{2k+3}-x^2y^{2k+1}+x^2y^{2k+1}+xy^{2k+2}-xy^{2k+2}-y^{2(k+1)+1}}{x-y}$$

$$=\frac{x^{2(k+1)+1}-y^{2(k+1)+1}}{x-y} \quad \checkmark \qquad P_{k+1} \text{ is also true.}$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $5{\bullet}6+5{\bullet}6^2+5{\bullet}6^3+\cdots+5{\bullet}6^n=6\left(6^n-1\right)$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow$ $5\cdot 6\overset{?}{=}6\left(6^1-1\right)=6(5)$ $\checkmark$; P_1 is true.

(2) $5\cdot 6+5\cdot 6^2+\cdots+5\cdot 6^k=6\left(6^k-1\right)$ is true

$$5\cdot 6+5\cdot 6^2+\cdots+5\cdot 6^k+5\cdot 6^{k+1}\overset{?}{=}6\left(6^{k+1}-1\right)$$

$$5\cdot 6+5\cdot 6^2+\cdots+5\cdot 6^k+5\cdot 6^{k+1}=6\left(6^k-1\right)+5\cdot 6^{k+1}$$

$$=6^{k+1}-6+5\cdot 6^{k+1}$$

$$=6^{k+1}(1+5)-6$$

$$=6\cdot 6^{k+1}-6$$

$$= 6\left(6^{k+1}-1\right) \quad \checkmark \qquad P_{k+1} \text{ is also true.}$$

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\quad 7{\bullet}8+7{\bullet}8^2+7{\bullet}8^3+\cdots+7{\bullet}8^n=8\left(8^n-1\right)$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 7\cdot 8\overset{?}{=}8\left(8^1-1\right)=8(7)$ $\checkmark$; P_1 is true.

(2) $7\cdot 8+7\cdot 8^2+\cdots+7\cdot 8^k=8\left(8^k-1\right)$ is true

$$7\cdot 8+7\cdot 8^2+\cdots+7\cdot 8^k+7\cdot 8^{k+1}\overset{?}{=}8\left(8^{k+1}-1\right)$$

$$\begin{aligned}7\cdot 8+7\cdot 8^2+\cdots+7\cdot 8^k+7\cdot 8^{k+1}&=8\left(8^k-1\right)+7\cdot 8^{k+1}\\&=8^{k+1}-8+7\cdot 8^{k+1}\\&=8^{k+1}(1+7)-8\\&=8\cdot 8^{k+1}-8\\&=8\left(8^{k+1}-1\right) \quad \checkmark \qquad P_{k+1} \text{ is also true.}\end{aligned}$$

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\quad 3+6+9+\cdots+3n=\dfrac{3n(n+1)}{2}$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 3\overset{?}{=}\dfrac{3(1)(1+1)}{2}=3$ $\checkmark$; P_1 is true.

(2) $3+6+9+\cdots+3k=\dfrac{3k(k+1)}{2}$ is true

$$3+6+9+\cdots+3k+3(k+1)\overset{?}{=}\frac{3(k+1)(k+2)}{2}$$

$$\begin{aligned}3+6+9+\cdots+3k+3(k+1)&=\frac{3k(k+1)}{2}+3(k+1)\\&=\frac{3k(k+1)+6(k+1)}{2}\\&=\frac{(k+1)(3k+6)}{2}\\&=\frac{3(k+1)(k+2)}{2} \quad \checkmark \qquad P_{k+1} \text{ is also true.}\end{aligned}$$

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $5+10+15+\cdots+5n=\dfrac{5n(n+1)}{2}$

Solution

(1) For ***n* = 1** $\Rightarrow 5 \overset{?}{=} \dfrac{5(1)(1+1)}{2} = 5$ ✓; P_1 is true.

(2) $5+10+15+\cdots+5k=\dfrac{5k(k+1)}{2}$ is true

$$5+10+15+\cdots+5k+5(k+1) \overset{?}{=} \frac{5(k+1)(k+2)}{2}$$

$$\begin{aligned}5+10+15+\cdots+5k+5(k+1) &= \frac{5k(k+1)}{2}+5(k+1)\\ &= \frac{5k(k+1)+10(k+1)}{2}\\ &= \frac{(k+1)(5k+10)}{2}\\ &= \frac{5(k+1)(k+2)}{2} \quad ✓\end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $1+3+5+\cdots+(2n-1)=n^2$

Solution

(1) For ***n* = 1** $\Rightarrow 1 \overset{?}{=} 1^2 = 1$ ✓ ; P_1 is true.

(2) $1+3+5+\cdots+(2k-1)=k^2$ is true

$$1+3+5+\cdots+(2k-1)+(2(k+1)-1) \overset{?}{=} (k+1)^2$$

$$\begin{aligned}1+3+5+\cdots+(2k-1)+(2(k+1)-1) &= k^2+2k+2-1\\ &= k^2+2k+1\\ &= (k+1)^2 \quad ✓\end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $4+7+10+\cdots+(3n+1)=\dfrac{n(3n+5)}{2}$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow$ $4\overset{?}{=}\dfrac{1(3+5)}{2}=4$ $\checkmark$; P_1 is true.

(2) $4+7+10+\cdots+(3k+1)=\dfrac{k(3k+5)}{2}$ is true

$$4+7+10+\cdots+(3k+1)+(3(k+1)+1)\overset{?}{=}\frac{(k+1)(3(k+1)+5)}{2}=\frac{(k+1)(3k+8)}{2}$$

$$\begin{aligned}4+7+10+\cdots+(3k+1)+(3k+4)&=\frac{k(3k+5)}{2}+3k+4\\&=\frac{3k^2+5k+6k+8}{2}\\&=\frac{3k^2+5k+3k+3k+8}{2}\\&=\frac{k(3k+8)+(3k+8)}{2}\\&=\frac{(3k+8)(k+1)}{2}\quad\checkmark\end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $\left(a^m\right)^n=a^{mn}$ (a and m are constant)

Solution

- For $\boldsymbol{n}$ **= 1** $\Rightarrow$ $\left(a^m\right)^1\overset{?}{=}a^{m(1)}$ $\rightarrow$ $a^m=a^m$ $\checkmark$; P_1 is true.
- $\left(a^m\right)^k=a^{mk}$ is true

$$\left(a^m\right)^{(k+1)}\overset{?}{=}a^{m(k+1)}$$

$$\begin{aligned}\left(a^m\right)^{(k+1)}&=\left(a^m\right)^k a^m\\&=a^{km}a^m\\&=a^{km+m}\\&=a^{m(k+1)}\quad\checkmark\end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $n < 2^n$

Solution

Step 1. For ***n* = 1** $\Rightarrow 1 \overset{?}{<} 2^1$ √ $\Rightarrow P_1$ is true.

Step 2. Assume that P_k is true $k < 2^k$

We need to prove that P_{k+1} is true, that is $k+1 < 2^{k+1}$

$$\begin{aligned} k+1 &< k+k = 2k \\ &< 2 \cdot 2^k \\ &= 2^{k+1} \end{aligned}$$ √ P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. 3 is a factor of $n^3 - n + 3$

Solution

- For ***n* = 1** $\Rightarrow 1^3 - 1 + 3 = 3 = 3(1)$ √ $\Rightarrow P_1$ is true.
- Assume that P_k is true 3 is a factor of $k^3 - k + 3$

We need to prove that P_{k+1} is true, that is $(k+1)^3 - (k+1) + 3$

$$\begin{aligned} (k+1)^3 - (k+1) + 3 &= k^3 + 3k^2 + 3k + 1 - k - 1 + 3 \\ &= \left(k^3 - k + 3\right) + 3k^2 + 3k \\ &= 3K + 3k^2 + 3k \\ &= 3\left(K + k^2 + k\right) \end{aligned}$$ √ P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. 4 is a factor of $5^n - 1$

Solution

- For ***n* = 1** $\Rightarrow 5^1 - 1 = 4 = 4(1)$ √ $\Rightarrow P_1$ is true.
- Assume that P_k is true 4 is a factor of $5^k - 1$

We need to prove that P_{k+1} is true, that is $5^{k+1} - 1$

$$5^{k+1} - 1 = 5^k 5^1 - 5 + 4$$

$$= 5\left(5^k - 1\right) + 4$$

$$= 5\left(5^k - 1\right) + 4$$

By the induction hypothesis, 4 is a factor of $5^k - 1$ and 4 is a factor of 4, so 4 is a factor of the $(k+1)$ term. $\surd$

Thus, P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $2^n > 2n \quad if \quad n \ge 3$

Solution

- For ***n* = 3** $\Rightarrow 2^3 \ge 2(3) \quad \Rightarrow \quad 8 \ge 6$ $\surd$ $\Rightarrow P_1$ is true.
- Assume that P_k is true: $2^k > 2k$; we need to prove that P_{k+1} : $2^{k+1} > 2(k+1)$ is true

$$2^k > 2k$$

$$2^k \cdot 2 > 2k \cdot 2$$

$$\begin{aligned} 2^{k+1} &> 4k = 2k + 2k \qquad k \ge 3 \\ &> 2k + 2 \\ &= 2(k+1) \ \surd \end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: If $0 < a < 1$, then $a^n < a^{n-1}$

Solution

- For ***n* = 1** $\Rightarrow a^1 < a^{1-1} \quad \Rightarrow \quad a < 1$ $\surd$ since $0 < a < 1 \Rightarrow P_1$ is true.
  - Assume that P_k is true: $a^k < a^{k-1}$; we need to prove that P_{k+1} : $a^{k+1} < a^k$ is true

$$a^k < a^{k-1} \quad \to \quad a^k \cdot a < a^{k-1} \cdot a$$

$$a^{k+1} < a^k \ \surd$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: If $n \geq 4$, then $n! > 2^n$

Solution

- For $\boldsymbol{n} = \mathbf{4} \Rightarrow 4! > 2^4 \quad \Rightarrow \quad 24 > 16$ ✓ $\Rightarrow P_1$ is true.
- Assume that P_k is true: $k! > 2^k$; we need to prove that $P_{k+1}: (k+1)! > 2^{k+1}$ is true

$$
\begin{aligned}
(k+1)! &= k! \cdot (k+1) \\
&> 2^k (k+1) \qquad\qquad k \geq 4 \Rightarrow k+1 > 2 \\
&> 2^k \cdot 2 \\
&= 2^{k+1} \quad ✓ \qquad \text{Thus, } P_{k+1} \text{ is also true.}
\end{aligned}
$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $3^n > 2n+1 \quad if \quad n \geq 2$

Solution

- For $\boldsymbol{n} = \mathbf{2} \Rightarrow 3^2 > 2(2)+1 \quad \Rightarrow \quad 9 > 5$ ✓ $\Rightarrow P_1$ is true.
- Assume that P_k is true: $3^k > 2k+1$; we need to prove that $P_{k+1}: 3^{k+1} > 2(k+1)+1$ is true

$$
\begin{aligned}
3^k > 2k+1 \quad \Rightarrow \quad 3^k \cdot 3 &> (2k+1) \cdot 3 \\
3^{k+1} &> 6k+3 \\
&> 2k+2+1 \\
&= 2(k+1)+1 \quad ✓ \qquad \text{Thus, } P_{k+1} \text{ is also true.}
\end{aligned}
$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $2^n > n^2 \quad for \quad n > 4$

Solution

- For $\boldsymbol{n} = \mathbf{5} \Rightarrow 2^5 > 5^2 \quad \Rightarrow \quad 32 > 25$ ✓ $\Rightarrow P_1$ is true.
- Assume that P_k is true: $2^k > k^2$; we need to prove that $P_{k+1}: 2^{k+1} > (k+1)^2$ is true

$$
\begin{aligned}
2^k > k^2 \quad \Rightarrow \quad 2^k \cdot 2 &> k^2 \cdot 2 \\
2^{k+1} &> 2k^2 \qquad\qquad k < k+1 \quad \Rightarrow \quad k^2 > 2k+1 \\
&> (k+1)^2 \quad ✓ \qquad \text{Thus, } P_{k+1} \text{ is also true}
\end{aligned}
$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $4^n > n^4 \quad \textit{for} \quad n \ge 5$

Solution

- For $\boldsymbol{n}$ **= 5** $\Rightarrow 4^5 > 5^4 \quad \Rightarrow \quad 1024 > 625$ √ $\quad \Rightarrow P_1$ is true.
- Assume that P_k is true: $4^k > k^4$; we need to prove that P_{k+1}: $4^{k+1} > (k+1)^4$ is true

$$4^k > k^4 \quad \Rightarrow \quad 4^k \cdot 4 > k^4 \cdot 4$$

$$4^{k+1} > 4k^4 \qquad\qquad k < k+1 \quad \Rightarrow \quad k^2 > 2k+1$$

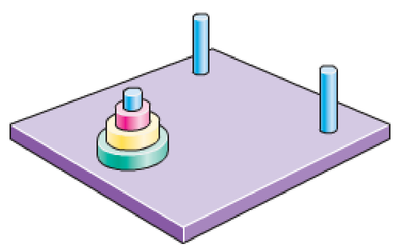
$$> (k+1)^4 \ \surd \qquad \text{Thus, } P_{k+1} \text{ is also true}$$

∴ By the mathematical induction, the proof is completed.

Exercise

A pile of *n* rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must moved, one at a time, to a different peg with no ring ever placed on top of a smaller ring. Find the least number of moves that would be required. Prove your result by mathematical induction.

Solution



With 1 ring, 1 move is required.

With 2 rings, 3 moves are required $\Rightarrow \quad 3 = 2 + 1$

With 3 rings, 7 moves are required $\Rightarrow \quad 7 = 2^2 + 2 + 1$

With *n* rings, $2^{n-1} + \cdots + 2^2 + 2^1 + 2^0 = 2^n - 1$ moves are required

- For $\boldsymbol{n}$ **= 1** $\Rightarrow 2^0 = 2^1 - 1 = 1$ √ $\quad \Rightarrow P_1$ is true.
- Assume that P_k is true: $2^{k-1} + \cdots + 2^2 + 2^1 + 2^0 = 2^k - 1$

$$2^k + 2^{k-1} + \cdots + 2^2 + 2^1 + 1 \overset{?}{=} 2^{k+1} - 1$$

$$\begin{aligned} 2^k + 2^{k-1} + \cdots + 2^2 + 2^1 + 1 &= 2^k + 2^k - 1 \\ &= 2 \cdot 2^k - 1 \\ &= 2^{k+1} - 1 \quad \surd \end{aligned}$$

Exercise

Find the fifth term in the expansion $\left(x^3+\sqrt{y}\right)^{13}$

Solution

$$\binom{13}{4}\left(x^3\right)^9\left(\sqrt{y}\right)^4=\frac{13!}{4!(13-4)!}x^{27}y^2=715x^{27}y^2$$

Exercise

Find the term involving q^{10} in the binomial expansion $\left(\frac{1}{3}p+q^2\right)^{12}$

Solution

Given: $a=\frac{1}{3}p,\quad b=q^2,\quad n=12$

$$q^{10}=\left(q^2\right)^5=b^5$$

$$\binom{n}{k}a^{n-k}b^k=\binom{12}{5}\left(\frac{1}{3}p\right)^{12-5}\left(q^2\right)^5=\frac{12!}{5!(12-5)!}\left(\frac{1}{3}p\right)^7q^{10}=\frac{88}{243}p^7q^{10}$$

Exercise

Use the binomial theorem to expand and simplify: $(4x-y)^3$

Solution

$$\begin{aligned}(4x-y)^3&=\binom{3}{0}(4x)^3(-y)^0+\binom{3}{1}(4x)^2(-y)^1+\binom{3}{2}(4x)^1(-y)^2+\binom{3}{3}(4x)^0(-y)^3\\&=64x^3+3\left(16x^2\right)(-y)+3(4x)y^2-y^3\\&=64x^3-48x^2y+12xy^2-y^3\end{aligned}$$

Exercise

Use the binomial theorem to expand and simplify: $(x+y)^6$

Solution

$$(x+y)^6=x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$$

Exercise

Use the binomial theorem to expand and simplify: $(x-y)^7$

Solution

$$(x-y)^7 = x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$$

Exercise

Use the binomial theorem to expand and simplify: $(3t-5x)^4$

Solution

$$\begin{aligned}(3t-5x)^4 &= (3t)^4 + 4(3t)^3(-5x)^1 + 6(3t)^2(-5x)^2 + 4(3t)^1(-5x)^3 + (-5x)^4 \\ &= 81t^4 - 540t^3x + 1350t^2x^2 - 1500tx^3 + 625x^4\end{aligned}$$

Exercise

Use the binomial theorem to expand and simplify: $\left(\frac{1}{3}x+y^2\right)^5$

Solution

$$\begin{aligned}\left(\tfrac{1}{3}x+y^2\right)^5 &= \left(\tfrac{1}{3}x\right)^5 + 5\left(\tfrac{1}{3}x\right)^4y^2 + 10\left(\tfrac{1}{3}x\right)^3\left(y^2\right)^2 + 10\left(\tfrac{1}{3}x\right)^2\left(y^2\right)^3 + 5\tfrac{1}{3}x\left(y^2\right)^4 + \left(y^2\right)^5 \\ &= \tfrac{1}{243}x^5 + \tfrac{5}{81}x^4y^2 + \tfrac{10}{27}x^3y^4 + \tfrac{10}{9}x^2y^6 + \tfrac{5}{3}xy^8 + y^{10}\end{aligned}$$

Exercise

Use the binomial theorem to expand and simplify: $\left(\frac{1}{x^2}+3x\right)^6$

Solution

$$\begin{aligned}\left(\frac{1}{x^2}+3x\right)^6 &= \left(x^{-2}+3x\right)^6 \\ &= \left(x^{-2}\right)^6 + 6\left(x^{-2}\right)^5 3x + 15\left(x^{-2}\right)^4(3x)^2 + 20\left(x^{-2}\right)^3(3x)^3 + 15\left(x^{-2}\right)^2(3x)^4 + 15x^{-2}(3x)^5 + (3x)^6 \\ &= x^{-12} + 18x^{-9} + 135x^{-6} + 540x^{-3} + 1215 + 1458x^3 + 729x^6\end{aligned}$$

Exercise

Use the binomial theorem to expand and simplify: $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^5$

Solution

$$\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^5=\left(x^{1/2}+x^{-1/2}\right)^5$$

$$=\left(x^{1/2}\right)^5+5\left(x^{1/2}\right)^4x^{-1/2}+10\left(x^{1/2}\right)^3\left(x^{-1/2}\right)^2+10\left(x^{1/2}\right)^2\left(x^{-1/2}\right)^3+5x^{1/2}\left(x^{-1/2}\right)^4+\left(x^{-1/2}\right)^5$$

$$=x^{5/2}+5x^2x^{-1/2}+10x^{3/2}x^{-1}+10xx^{-3/2}+5x^{1/2}x^{-2}+x^{-5/2}$$

$$\underline{=x^{5/2}+5x^{3/2}+10x^{1/2}+10x^{-1/2}+5x^{-3/2}+x^{-5/2}}$$

Exercise

Expand and simplify: $(2y-3)^4$

Solution

$$(2y-3)^4=(2y)^4+4(2y)^3(-3)+6(2y)^2(-3)^2+4(2y)(-3)^3+(-3)^4$$

$$\underline{=16y^4-96y^3+216y^2-216y+81}$$

Exercise

Expand and simplify: $(x+2)^5$

Solution

$$(x+2)^5=x^5+5x^4(2)+10x^3(2)^2+10x^2(2)^3+5x(2)^4+(2)^5$$

$$\underline{=x^5+10x^4+40x^3+80x^2+80x+32}$$

Exercise

Expand and simplify: $\left(x^2-y^2\right)^6$

Solution

$$\left(x^2-y^2\right)^6$$

$$=\left(x^2\right)^6+6\left(x^2\right)^5\left(-y^2\right)+15\left(x^2\right)^4\left(-y^2\right)^2+20\left(x^2\right)^3\left(-y^2\right)^3+15\left(x^2\right)^2\left(-y^2\right)^4+15\left(x^2\right)\left(-y^2\right)^5+\left(-y^2\right)^6$$

$$\underline{=x^{12}-6x^{10}y^2+15x^8y^4-20x^6y^6+15x^4y^8-15x^2y^{10}+y^{12}}$$

Exercise

Expand and simplify: $(ax - by)^4$

Solution

$$(ax-by)^4 = (ax)^4 + 4(ax)^3(-by) + 6(ax)^2(-by)^2 + 4(ax)(-by)^3 + (-by)^4$$

$$= a^4x^4 - 4a^3x^3by + 6a^2x^2b^2y^2 - 4axb^3y^3 + b^4y^4$$

Exercise

Expand and simplify: $(ax + by)^5$

Solution

$$(ax+by)^5 = (ax)^5 + 5(ax)^4(by) + 10(ax)^3(by)^2 + 10(ax)^2(by)^3 + 5(ax)(by)^4 + (by)^5$$

$$= a^5x^5 + 5a^4x^4by + 10a^3x^3b^2y^2 + 10a^2x^2b^3y^3 + 5axb^4y^4 + b^5y^5$$

Exercise

Expand and simplify: $\left(\sqrt{x} - \sqrt{3}\right)^4$

Solution

$$\left(\sqrt{x}-\sqrt{3}\right)^4 = \left(\sqrt{x}\right)^4 + 4\left(\sqrt{x}\right)^3\left(-\sqrt{3}\right) + 6\left(\sqrt{x}\right)^2\left(-\sqrt{3}\right)^2 + 4\left(\sqrt{x}\right)\left(-\sqrt{3}\right)^3 + \left(-\sqrt{3}\right)^4$$

$$= x^2 - 4x\sqrt{3x} + 18x^2 - 13\sqrt{3x} + 9$$

Exercise

Expand and simplify: $\left(\sqrt{x} - \sqrt{2}\right)^6$

Solution

$$\left(\sqrt{x}-\sqrt{2}\right)^6 = \left(\sqrt{x}\right)^6 + 6\left(\sqrt{x}\right)^5\left(-\sqrt{2}\right) + 15\left(\sqrt{x}\right)^4\left(-\sqrt{2}\right)^2 + 20\left(\sqrt{x}\right)^3\left(-\sqrt{2}\right)^3$$

$$+ 15\left(\sqrt{x}\right)^2\left(-\sqrt{2}\right)^4 + 15\left(\sqrt{x}\right)\left(-\sqrt{2}\right)^5 + \left(-\sqrt{2}\right)^6$$

$$= x^3 - 6x^2\sqrt{2x} + 30x^2 - 40x\sqrt{2x} + 60x - 60\sqrt{2x} + 8$$

Exercise

Expand and simplify: $(2x-1)^{12}$

Solution

$$\begin{aligned}(2x-1)^{12} &= (2x)^{12} + 12(2x)^{11}(-1) + 66(2x)^{10}(-1)^2 + 240(2x)^9(-1)^3 + 535(2x)^8(-1)^4 \\ &\quad + 812(2x)^7(-1)^5 + 924(2x)^6(-1)^6 + 812(2x)^5(-1)^7 + 535(2x)^4(-1)^8 \\ &\quad + 240(2x)^3(-1)^9 + 66(2x)^2(-1)^{10} + 12(2x)(-1)^{11} + (-1)^{12} \\ &= 4096x^{12} - 24576x^{11} + 67584x^{10} - 122880x^9 + 136960x^8 - 103936x^7 \\ &\quad + 59136x^6 - 25984x^5 + 8560x^4 - 1920x^3 + 264x^2 - 24x + 1\end{aligned}$$

Exercise

Expand and simplify: $\left(x-\frac{1}{x^2}\right)^9$

Solution

$$\begin{aligned}\left(x-\frac{1}{x^2}\right)^9 &= x^9 + 9x^8\left(-\frac{1}{x^2}\right) + 36x^7\left(-\frac{1}{x^2}\right)^2 + 84x^6\left(-\frac{1}{x^2}\right)^3 + 126x^5\left(-\frac{1}{x^2}\right)^4 + 126x^4\left(-\frac{1}{x^2}\right)^5 \\ &\quad + 84x^3\left(-\frac{1}{x^2}\right)^6 + 36x^2\left(-\frac{1}{x^2}\right)^7 + 9x\left(-\frac{1}{x^2}\right)^8 + \left(-\frac{1}{x^2}\right)^9 \\ &= x^9 - 9x^6 + 36x^3 - 84 + 126x^{-3} - 126x^{-6} + 84x^{-9} - 36x^{-12} + 9x^{-15} - x^{-18}\end{aligned}$$

Exercise

Expand and simplify: $\left(\frac{2}{x}-3y\right)^5$

Solution

$$\begin{aligned}\left(\frac{2}{x}-3y\right)^5 &= \left(\frac{2}{x}\right)^5 + 5\left(\frac{2}{x}\right)^4(-3y) + 10\left(\frac{2}{x}\right)^3(-3y)^2 + 10\left(\frac{2}{x}\right)^2(-3y)^3 + 5\left(\frac{2}{x}\right)(-3y)^4 + (-3y)^5 \\ &= \frac{32}{x^5} - 240\frac{y}{x^4} + 720\frac{y^2}{x^3} - 1{,}080\frac{y^3}{x^2} + 810\frac{y^4}{x} - 243y^5\end{aligned}$$

Exercise

Expand and simplify: $\left(3\sqrt{x}+\sqrt[4]{x}\right)^4$

Solution

$$\left(3\sqrt{x}+\sqrt[4]{x}\right)^4=\left(3\sqrt{x}\right)^4+4\left(3\sqrt{x}\right)^3\left(\sqrt[4]{x}\right)+6\left(3\sqrt{x}\right)^2\left(\sqrt[4]{x}\right)^2+4\left(3\sqrt{x}\right)\left(\sqrt[4]{x}\right)^3+\left(\sqrt[4]{x}\right)^4$$

$$=81x^2+108x^{3/2}x^{1/4}+54x\sqrt{x}+12x^{1/2}x^{3/4}+x$$

$$=81x^2+108x^{7/4}+54x\sqrt{x}+12x^{5/4}+x$$

$$\underline{=81x^2+108x\sqrt[4]{x^3}+54x\sqrt{x}+12x\sqrt[4]{x}+x}$$

Exercise

Expand and simplify: $(x+1)^5$

Solution

$$(x+1)^5\ \underline{=x^5+5x^4+10x^3+10x^2+5x+1}$$

Exercise

Expand and simplify: $(x-1)^5$

Solution

$$(x-1)^5\ \underline{=x^5-5x^4+10x^3-10x^2+5x-1}$$

Exercise

Expand and simplify: $(x-2)^6$

Solution

$$(x-2)^6\ \underline{=x^6-12x^5+60x^4-160x^3+240x^2-192x+64}$$

Exercise

Expand and simplify: $\left(\dfrac{1}{x^3}-2x\right)^5$

Solution

$$\left(\frac{1}{x^3}-2x\right)^5=\frac{1}{x^{15}}-10\frac{x}{x^{12}}+10\frac{4x^2}{x^9}-10\frac{8x^3}{x^6}+5\frac{16x^4}{x^3}-32x^5$$

$$\underline{=\frac{1}{x^{15}}-\frac{10}{x^{11}}+\frac{40}{x^7}-\frac{80}{x^3}+80x-32x^5}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{4}{x(x-1)}$

Solution

$$\frac{4}{x(x-1)}=\frac{A}{x}+\frac{B}{x-1}$$

$$4=A(x-1)+Bx$$

$$4=Ax-A+Bx$$

$$4=(A+B)x-A$$

$$\begin{cases}A+B=0\\ -A=4\end{cases}\to\begin{cases}B=-A=4\\ A=-4\end{cases}$$

$$\boxed{\frac{4}{x(x-1)}=-\frac{4}{x}+\frac{4}{x-1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{3x}{(x+2)(x-1)}$

Solution

$$\frac{3x}{(x+2)(x-1)}=\frac{A}{x+2}+\frac{B}{x-1}$$

$$3x=A(x-1)+B(x+2)$$

$$3x=Ax-A+Bx+2B$$

$$3x=(A+B)x-A+2B$$

$$\begin{cases}A+B=3\\ -A+2B=0\end{cases}\qquad \begin{array}{r}A+B=3\\ -A+2B=0\\ \hline 3B=3\end{array}\ \Rightarrow B=1$$

$$\begin{pmatrix}1&1&3\\-1&2&0\end{pmatrix}\xrightarrow{rref}\begin{pmatrix}1&0&2\\0&1&1\end{pmatrix}\qquad A=2B\ \to\ \begin{array}{l}2B+B=3\ \Rightarrow 3B=3\ \Rightarrow B=1\\ A=2\end{array}$$

$$\boxed{\frac{3x}{(x+2)(x-1)}=\frac{2}{x+2}+\frac{1}{x-1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{1}{x\left(x^2+1\right)}$

Solution

$$\frac{1}{x\left(x^2+1\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$$

$$1=A\left(x^2+1\right)+x\left(Bx+C\right)$$

$$1=Ax^2+A+Bx^2+Cx$$

$$1=\left(A+B\right)x^2+Cx+A$$

$$\begin{cases} A+B=0 \\ C=0 \\ A=1 \end{cases} \quad \to B=-A=-1$$

$$\boxed{\frac{1}{x\left(x^2+1\right)}=\frac{1}{x}-\frac{x}{x^2+1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{1}{(x+1)\left(x^2+4\right)}$

Solution

$$\frac{1}{(x+1)\left(x^2+4\right)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+4}$$

$$1=A\left(x^2+4\right)+(x+1)\left(Bx+C\right)$$

$$1=Ax^2+4A+Bx^2+Cx+Bx+C$$

$$1=\left(A+B\right)x^2+\left(B+C\right)x+4A+C$$

$$\begin{cases} A+B=0 \\ B+C=0 \\ 4A+C=1 \end{cases} \to \begin{cases} A=-B & \to A=C=\frac{1}{5} \\ C=-B \\ -4B-B=1 & \to B=-\frac{1}{5} \end{cases}$$

$$\frac{1}{(x+1)\left(x^2+4\right)}=\frac{\frac{1}{5}}{x+1}+\frac{\frac{1}{5}x+\frac{1}{5}}{x^2+4}$$

$$\boxed{=\frac{1}{5}\frac{1}{x+1}+\frac{1}{5}\frac{x+1}{x^2+4}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2}{(x-1)^2(x+1)^2}$

Solution

$$\frac{x^2}{(x-1)^2(x+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

$$x^2 = A(x-1)(x+1)^2 + B(x+1)^2 + C(x-1)^2(x+1) + D(x-1)^2$$

$$= A(x-1)\left(x^2+2x+1\right) + B\left(x^2+2x+1\right) + C\left(x^2-2x+1\right)(x+1) + D\left(x^2-2x+1\right)$$

$$= Ax^3 + 2Ax^2 + Ax - Ax^2 - 2Ax - A + Bx^2 + 2Bx + B$$
$$+Cx^3 - 2Cx^2 + Cx + Cx^2 - 2Cx + C + Dx^2 - 2Dx + D$$

$$x^2 = (A+C)x^3 + (A+B-C+D)x^2 + (-A+2B-C-2D)x - A + B + C + D$$

$$\begin{cases} A+C=0 \\ A+B-C+D=1 \\ -A+2B-C-2D=0 \\ -A+B+C+D=0 \end{cases} \quad \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 & 1 \\ -1 & 2 & -1 & -2 & 0 \\ -1 & 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{4} \end{array}\right)$$

$$\frac{x^2}{(x-1)^2(x+1)^2} = \frac{\frac{1}{4}}{x-1} + \frac{\frac{1}{4}}{(x-1)^2} + \frac{-\frac{1}{4}}{x+1} + \frac{\frac{1}{4}}{(x+1)^2}$$

$$= \frac{1}{4}\frac{1}{x-1} + \frac{1}{4}\frac{1}{(x-1)^2} - \frac{1}{4}\frac{1}{x+1} + \frac{1}{4}\frac{1}{(x+1)^2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x+1}{x^2(x-2)^2}$

Solution

$$\frac{x+1}{x^2(x-2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$$

$$x+1 = Ax(x-2)^2 + B(x-2)^2 + Cx^2(x-2) + Dx^2$$

$$= Ax\left(x^2-4x+4\right) + B\left(x^2-4x+4\right) + Cx^3 - 2Cx^2 + Dx^2$$

$$= Ax^3 - 4Ax^2 + 4Ax + Bx^2 - 4Bx + 4B + Cx^3 - 2Cx^2 + Dx^2$$

$$= (A+C)x^3 + (-4A - B - 2C + D)x^2 + (4A-4B)x + 4B$$

$$\begin{cases} A+C=0 \\ -4A-B-2C+D=0 \\ 4A-4B=1 \\ 4B=1 \end{cases} \qquad \begin{cases} C=-\frac{1}{2} \\ D=2+\frac{1}{4}-1=\frac{5}{4} \\ A=\frac{1}{2} \\ B=\frac{1}{4} \end{cases}$$

$$\boxed{\frac{x+1}{x^2(x-2)^2}=\frac{\frac{1}{2}}{x}+\frac{\frac{1}{4}}{x^2}+\frac{-\frac{1}{2}}{x-2}+\frac{\frac{5}{4}}{(x-2)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x-3}{(x+2)(x+1)^2}$

Solution

$$\frac{x-3}{(x+2)(x+1)^2}=\frac{A}{x+2}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$x-3=A(x+1)^2+B(x+1)(x+2)+C(x+2)$$

$$=Ax^2+2Ax+A+B\left(x^2+3x+2\right)+Cx+2C$$

$$=(A+B)x^2+(2A+3B+C)x+A+2B+2C$$

$$\begin{cases} A+B=0 \\ 2A+3B+C=1 \\ A+2B+2C=-3 \end{cases} \rightarrow \begin{cases} A=-B \\ -2B+3B+C=1 \\ -B+2B+2C=-3 \end{cases} \rightarrow \begin{cases} B+C=1 \\ B+2C=-3 \end{cases}$$

$$C=-4, \quad B=5, \quad A=-5$$

$$\boxed{\frac{x-3}{(x+2)(x+1)^2}=-\frac{5}{x+2}+\frac{5}{x+1}-\frac{4}{(x+1)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2+x}{(x+2)(x-1)^2}$

Solution

$$\frac{x^2+x}{(x+2)(x-1)^2}=\frac{A}{x+2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$$x^2+x=A(x-1)^2+B(x-1)(x+2)+C(x+2)$$

$$=Ax^2-2Ax+A+Bx^2+Bx-2B+Cx+2C$$

$$\begin{matrix} x^2 \\ x \\ x^0 \end{matrix}\begin{cases} A+B=1 \\ -2A+B+C=1 \\ A-2B+2C=0 \end{cases} \longrightarrow \quad A=\frac{2}{9} \quad B=\frac{7}{9} \quad C=\frac{2}{3}$$

$$\frac{x^2+x}{(x+2)(x-1)^2} = \boxed{\frac{\frac{2}{9}}{x+2} + \frac{\frac{7}{9}}{x-1} + \frac{\frac{2}{3}}{(x-1)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{10x^2+2x}{(x-1)^2\left(x^2+2\right)}$

Solution

$$\frac{10x^2+2x}{(x-1)^2\left(x^2+2\right)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+2}$$

$$\begin{aligned} 10x^2+2x &= A(x-1)\left(x^2+2\right) + B\left(x^2+2\right) + (Cx+D)(x-1)^2 \\ &= Ax^3+2Ax-Ax^2-2A+Bx^2+2B+(Cx+D)\left(x^2-2x+1\right) \\ &= Ax^3+2Ax-Ax^2-2A+Bx^2+2B+Cx^3-2Cx^2+Cx+Dx^2-2Dx+D \\ &= (A+C)x^3+(B-2A-2C+D)x^2+(2A+C-2D)x-2A+2B+D \end{aligned}$$

$$\begin{cases} A+C=0 \\ B-2A-2C+D=10 \\ 2A+C-2D=2 \\ -2A+2B+D=0 \end{cases} \rightarrow \quad \boxed{A=\frac{42}{5}} \quad \boxed{B=\frac{34}{5}} \quad \boxed{C=-\frac{42}{5}} \quad \boxed{D=\frac{16}{5}}$$

$$\boxed{\frac{10x^2+2x}{(x-1)^2\left(x^2+2\right)} = \frac{\frac{42}{5}}{x-1} + \frac{\frac{34}{5}}{(x-1)^2} + \frac{-\frac{42}{5}x+\frac{16}{5}}{x^2+2}}$$

$$\boxed{= \frac{42}{5(x-1)} + \frac{34}{5(x-1)^2} + \frac{-42x+16}{5\left(x^2+2\right)}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2+2x+3}{(x+1)\left(x^2+2x+4\right)}$

Solution

$$\frac{x^2+2x+3}{(x+1)\left(x^2+2x+4\right)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+2x+4}$$

$$x^2+2x+3=A\left(x^2+2x+4\right)+(Bx+C)(x+1)$$

$$=Ax^2+2Ax+4A+Bx^2+Bx+Cx+C$$

$$=(A+B)x^2+(2A+B+C)x+4A+C$$

$$\begin{cases} A+B=1 \\ 2A+B+C=2 \\ 4A+C=3 \end{cases} \rightarrow \boxed{A=\frac{2}{3}} \quad \boxed{B=\frac{1}{3}} \quad \boxed{C=\frac{1}{3}}$$

$$\frac{x^2+2x+3}{(x+1)\left(x^2+2x+4\right)}=\frac{\frac{2}{3}}{x+1}+\frac{\frac{1}{3}x+\frac{1}{3}}{x^2+2x+4}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2-11x-18}{x\left(x^2+3x+3\right)}$

Solution

$$\frac{x^2-11x-18}{x\left(x^2+3x+3\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+3x+3}$$

$$x^2-11x-18=Ax^2+3Ax+3A+Bx^2+Cx$$

$$=(A+B)x^2+(3A+C)x+3A$$

$$\begin{cases} A+B=1 \\ 3A+C=-11 \\ 3A=-18 \end{cases} \rightarrow \boxed{A=-6} \quad \boxed{B=7} \quad \boxed{C=7}$$

$$\frac{x^2-11x-18}{x\left(x^2+3x+3\right)}=-\frac{6}{x}+\frac{7x+7}{x^2+3x+3}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{1}{(2x+3)(4x-1)}$

Solution

$$\frac{1}{(2x+3)(4x-1)} = \frac{A}{2x+3} + \frac{B}{4x-1}$$

$$1 = 4Ax - A + 2Bx + 3B$$

$$1 = (4A+2B)x - A + 3B$$

$$\begin{cases} 4A+2B=0 \\ -A+3B=1 \end{cases} \rightarrow \begin{cases} 4A+2B=0 \\ -4A+12B=4 \end{cases} \quad 14B = 4 \Rightarrow \boxed{B = -\frac{2}{7}} \quad A = 3\left(-\frac{2}{7}\right) - 1 = \boxed{\frac{1}{7}}$$

$$\boxed{\frac{1}{(2x+3)(4x-1)} = \frac{\frac{1}{7}}{2x+3} - \frac{\frac{2}{7}}{4x-1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2+2x+3}{\left(x^2+4\right)^2}$

Solution

$$\frac{x^2+2x+3}{\left(x^2+4\right)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{\left(x^2+4\right)^2}$$

$$x^2+2x+3 = (Ax+B)\left(x^2+4\right) + Cx + D$$

$$= Ax^3 + 4Ax + Bx^2 + 4B + Cx + D$$

$$= Ax^3 + Bx^2 + (4A+C)x + 4B + D$$

$$\begin{cases} A=0 \\ B=1 \\ 4A+C=2 \\ 4B+D=3 \end{cases} \rightarrow \begin{matrix} \\ \\ C=2 \\ D=3-4B=-1 \end{matrix}$$

$$\boxed{\frac{x^2+2x+3}{\left(x^2+4\right)^2} = \frac{1}{x^2+4} + \frac{2x-1}{\left(x^2+4\right)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^3+1}{\left(x^2+16\right)^2}$

Solution

$$\frac{x^3+1}{\left(x^2+16\right)^2}=\frac{Ax+B}{x^2+16}+\frac{Cx+D}{\left(x^2+16\right)^2}$$

$$x^3+1=(Ax+B)\left(x^2+16\right)+Cx+D$$

$$=Ax^3+16Ax+Bx^2+16B+Cx+D$$

$$\begin{cases} x^3 & \underline{A=1} \\ x^2 & \underline{B=0} \\ x & 16A+C=0 \\ x^0 & 16B+D=1 \end{cases} \rightarrow \begin{array}{l} \underline{C=-16} \\ \underline{D=1} \end{array}$$

$$\boxed{\frac{x^3+1}{\left(x^2+16\right)^2}=\frac{x}{x^2+16}+\frac{-16x+1}{\left(x^2+16\right)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{7x+3}{x^3-2x^2-3x}$

Solution

$$\frac{7x+3}{x^3-2x^2-3x}=\frac{7x+3}{x(x+1)(x-3)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-3}$$

$$7x+3=A(x+1)(x-3)+Bx(x-3)+Cx(x+1)$$

$$=Ax^2-2Ax-3A+Bx^2-3B+Cx^2+Cx$$

$$=(A+B+C)x^2+(C-2A)x-3A-3B$$

$$\begin{cases} A+B+C=0 \\ C-2A=7 \\ -3A-3B=3 \end{cases} \rightarrow \boxed{A=-3} \quad \boxed{B=2} \quad \boxed{C=1}$$

$$\boxed{\frac{7x+3}{x^3-2x^2-3x}=\frac{-3}{x}+\frac{2}{x+1}+\frac{1}{x-3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2}{x^3-4x^2+5x-2}$

Solution

$$\frac{x^2}{x^3-4x^2+5x-2}=\frac{x^2}{(x-2)(x-1)^2}=\frac{A}{x-2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$$\begin{aligned}x^2&=A(x-1)^2+B(x-2)(x-1)+C(x-2)\\&=Ax^2-2Ax+A+Bx^2-3Bx+2B+Cx-2C\\&=(A+B)x^2+(-2A-3B+C)x+A+2B-2C\end{aligned}$$

$$\begin{cases}A+B=1\\-2A-3B+C=0\\A+2B-2C=0\end{cases}\rightarrow\ \boxed{A=4}\quad\boxed{B=-3}\quad\boxed{C=-1}$$

$$\boxed{\frac{x^2}{x^3-4x^2+5x-2}=\frac{4}{x-2}-\frac{3}{x-1}-\frac{1}{(x-1)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^3}{\left(x^2+16\right)^3}$

Solution

$$\frac{x^3}{\left(x^2+16\right)^3}=\frac{Ax+B}{x^2+16}+\frac{Cx+D}{\left(x^2+16\right)^2}+\frac{Ex+F}{\left(x^2+16\right)^3}$$

$$\begin{aligned}x^3&=(Ax+B)\left(x^2+16\right)^2+(Cx+D)\left(x^2+16\right)+Ex+F\\&=(Ax+B)\left(x^4+32x^2+256\right)+Cx^3+16Cx+Dx^2+16D+Ex+F\\&=Ax^5+32Ax^3+256Ax+Bx^4+32Bx^2+256B+Cx^3+Dx^2+(16C+E)x+16D+F\\&=Ax^5+Bx^4+(32A+C)x^3+(32B+D)x^2+(256A+16C+E)x+256B+16D+F\end{aligned}$$

$$\begin{cases}A=B=0\\32A+C=1\\32B+D=0\\256A+16C+E=0\\256B+16D+F=0\end{cases}\rightarrow\ \boxed{A=B=D=F=0}\quad\boxed{C=1}\quad\boxed{E=-16}$$

$$\boxed{\frac{x^3}{\left(x^2+16\right)^3}=\frac{x}{\left(x^2+16\right)^2}+\frac{-16x}{\left(x^2+16\right)^3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{4}{2x^2-5x-3}$

Solution

$$\frac{4}{2x^2-5x-3}=\frac{4}{(2x+1)(x-3)}=\frac{A}{2x+1}+\frac{B}{x-3}$$

$$\begin{aligned}4&=Ax-3A+2Bx+B\\&=(A+2B)x-3A+B\end{aligned}$$

$$\begin{cases}A+2B=0\\-3A+B=4\end{cases}\rightarrow\ \boxed{A=-\frac{8}{7}}\quad\boxed{B=\frac{4}{7}}$$

$$\frac{4}{2x^2-5x-3}=\frac{-\frac{8}{7}}{2x+1}+\frac{\frac{4}{7}}{x-3}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{2x+3}{x^4-9x^2}$

Solution

$$\frac{2x+3}{x^4-9x^2}=\frac{2x+3}{x^2(x-3)(x+3)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-3}+\frac{D}{x+3}$$

$$\begin{aligned}2x+3&=Ax\left(x^2-9\right)+B\left(x^2-9\right)+Cx^2(x+3)+Dx^2(x-3)\\&=Ax^3-9Ax+Bx^2-9B+Cx^3+3Cx^2+Dx^3-3Dx^2\\&=(A+C+D)x^3+(B+3C-3D)x^2-9Ax-9B\end{aligned}$$

$$\begin{cases}A+C+D=0\\B+3C-3D=0\\-9A=2\\-9B=3\end{cases}\rightarrow\quad\begin{aligned}C&=\frac{1}{6}\\D&=\frac{1}{18}\\A&=-\frac{2}{9}\\B&=-\frac{1}{3}\end{aligned}$$

$$\frac{2x+3}{x^4-9x^2}=-\frac{\frac{2}{9}}{x}-\frac{\frac{1}{3}}{x^2}+\frac{\frac{1}{6}}{x-3}+\frac{\frac{1}{18}}{x+3}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2+9}{x^4-2x^2-8}$

Solution

$$\frac{x^2+9}{x^4-2x^2-8}=\frac{A}{x-2}+\frac{B}{x+2}+\frac{Cx+D}{x^2+2}$$

$$x^2+9=A(x+2)\left(x^2+2\right)+B(x-2)\left(x^2+2\right)+(Cx+D)\left(x^2-4\right)$$

$$=Ax^3+2Ax+2Ax^2+4A+Bx^3+2Bx-2Bx^2-4B+Cx^3-4Cx+Dx^2-4D$$

$$=(A+B+C)x^3+(2A-2B+D)x^2+(2A+2B-4C)x+4A-4B-4D$$

$$\begin{cases} A+B+C=0 \\ 2A-2B+D=1 \\ 2A+2B-4C=0 \\ 4A-4B-4D=9 \end{cases} \to \boxed{A=\frac{13}{24}} \quad \boxed{B=-\frac{13}{24}} \quad \boxed{C=0} \quad \boxed{D=-\frac{7}{6}}$$

$$\frac{x^2+9}{x^4-2x^2-8}=\frac{\frac{13}{24}}{x-2}-\frac{\frac{13}{24}}{x+2}-\frac{\frac{7}{6}}{x^2+2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{y}{y^2-2y-3}$

Solution

$$\frac{y}{y^2-2y-3}=\frac{A}{y-3}+\frac{B}{y+1}$$

$$y=(A+B)y+A-3B$$

$$\to \begin{cases} A+B=1 \\ A-3B=0 \end{cases} \Rightarrow \boxed{A=\frac{3}{4}} \quad \boxed{B=\frac{1}{4}}$$

$$\frac{y}{y^2-2y-3}=\frac{\frac{3}{4}}{y-3}+\frac{\frac{1}{4}}{y+1}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x+3}{2x^3-8x}$

Solution

$$\frac{x+3}{2x^3-8x}=\frac{1}{2}\frac{x+3}{x\left(x^2-4\right)}=\frac{1}{2}\left(\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-2}\right)$$

$$=\frac{1}{2}\frac{A(x+2)(x-2)+Bx(x-2)+Cx(x+2)}{x(x+2)(x-2)}$$

$$(A+B+C)x^2+(2C-2B)x-4A=x+3$$

$$\begin{cases} A+B+C=0 \\ 2C-2B=1 \\ -4A=3 \end{cases} \rightarrow \boxed{A=-\frac{3}{4}} \quad \boxed{B=\frac{1}{8}} \quad \boxed{C=\frac{5}{8}}$$

$$\boxed{\frac{x+3}{2x^3-8x}=\frac{1}{2}\left(-\frac{\frac{3}{4}}{x}+\frac{\frac{1}{8}}{x+2}+\frac{\frac{5}{8}}{x-2}\right)}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2}{(x-1)\left(x^2+2x+1\right)}$

Solution

$$\frac{x^2}{(x-1)\left(x^2+2x+1\right)}=\frac{x^2}{(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$x^2=A(x+1)^2+B(x-1)(x+1)+C(x-1)$$

$$=(A+B)x^2+(2A+C)x+A-B-C$$

$$\begin{cases} A+B=1 \\ 2A+C=0 \\ A-B-C=0 \end{cases} \rightarrow \boxed{A=\frac{1}{4}} \quad \boxed{B=\frac{3}{4}} \quad \boxed{C=-\frac{1}{2}}$$

$$\boxed{\frac{x^2}{(x-1)\left(x^2+2x+1\right)}=\frac{\frac{1}{4}}{x-1}+\frac{\frac{3}{4}}{x+1}-\frac{\frac{1}{2}}{(x+1)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{3x^2+x+4}{x^3+x}$

Solution

$$\frac{3x^2+x+4}{x^3+x}=\frac{A}{x}+\frac{Bx+C}{x^2+1}=\frac{(A+B)x^2+Cx+A}{x(x^2+1)}$$

$$3x^2+x+4=(A+B)x^2+Cx+A$$

$$\begin{cases} A+B=3 \\ C=1 \\ A=4 \end{cases} \to \boxed{A=4} \quad \boxed{B=-1} \quad \boxed{C=1}$$

$$\boxed{\frac{3x^2+x+4}{x^3+x}=\frac{4}{x}+\frac{-x+1}{x^2+1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{8x^2+8x+2}{\left(4x^2+1\right)^2}$

Solution

$$\frac{8x^2+8x+2}{\left(4x^2+1\right)^2}=\frac{Ax+B}{4x^2+1}+\frac{Cx+D}{\left(4x^2+1\right)^2}=\frac{(Ax+B)\left(4x^2+1\right)+Cx+D}{\left(4x^2+1\right)^2}$$

$$\begin{aligned} 8x^2+8x+2&=(Ax+B)\left(4x^2+1\right)+Cx+D \\ &=4Ax^3+4Bx^2+(A+C)x+B+D \end{aligned}$$

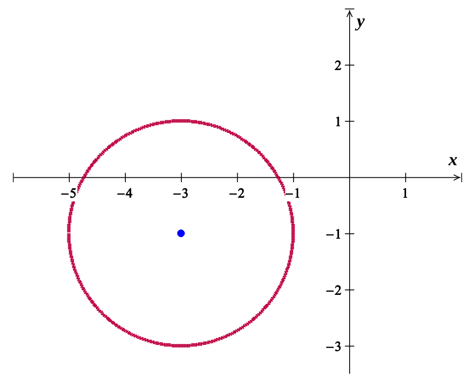
$$\begin{cases} A=0 \\ 4B=8 \\ A+C=8 \\ B+D=2 \end{cases} \to \boxed{A=0} \quad \boxed{B=2} \quad \boxed{C=8} \quad \boxed{D=0}$$

$$\boxed{\frac{8x^2+8x+2}{\left(4x^2+1\right)^2}=\frac{2}{4x^2+1}+\frac{8x}{\left(4x^2+1\right)^2}}$$

Solution Section 4.6 – Circles and Parabolas

Exercise

Find the center and the radius of $x^2 + y^2 + 6x + 2y + 6 = 0$



Solution

$$x^2 + 6x + \left(\frac{6}{2}\right)^2 + y^2 + 2y + \left(\frac{2}{2}\right)^2 = -6 + 9 + 1$$

$$(x+3)^2 + (y+1)^2 = 4$$

Center $(-3, -1)$ and $r = 2$

Exercise

Find the center and the radius of $x^2 + y^2 + 8x + 4y + 16 = 0$

Solution

$$x^2 + 8x + \left(\frac{8}{2}\right)^2 + y^2 + 4y + \left(\frac{4}{2}\right)^2 = -16 + 16 + 4$$

$$(x+4)^2 + (y+2)^2 = 4$$

Center $(-4, -2)$ and $r = 2$

Exercise

Find the center and the radius of $x^2 + y^2 - 10x - 6y - 30 = 0$

Solution

$$x^2 - 10x + \left(\frac{-10}{2}\right)^2 + y^2 - 6y + \left(\frac{-6}{2}\right)^2 = 30 + 25 + 9$$

$$(x-5)^2 + (y-3)^2 = 64$$

Center $(5, 3)$ and $r = 8$

Exercise

Find the center and the radius of $x^2 - 6x + y^2 + 10y + 25 = 0$

Solution

$$x^2 - 6x \qquad + y^2 + 10y \qquad = -25$$

$$x^2 - 6x + \left(\frac{1}{2}(-6)\right)^2 + y^2 + 10y + \left(\frac{1}{2}(10)\right)^2 = -25 + \left(\frac{1}{2}(-6)\right)^2 + \left(\frac{1}{2}(10)\right)^2$$

$$(x-3)^2 + (y+5)^2 = -25 + 9 + 25$$

$$(x-3)^2 + (y+5)^2 = 9$$

The equation represents a circle with ***center*** at (3, –5) and ***radius*** 3

Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $20x = y^2$

Solution

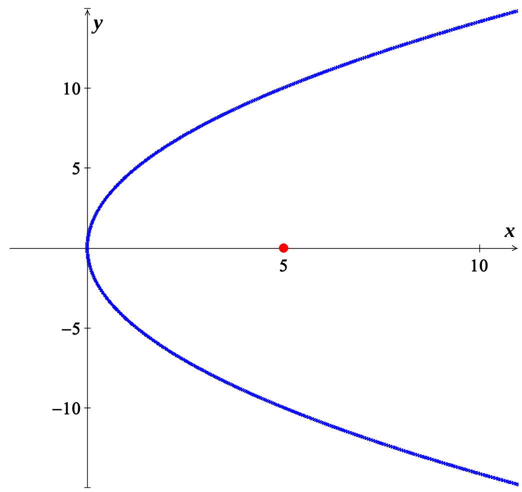
$20x = y^2 \qquad 4px = y^2$

$4p = 20 \quad \Rightarrow \quad \boxed{p = 5}$

Vertex: (0, 0)

Focus $(5,\ 0)$

Directrix: $x = -5$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. ..

Solution

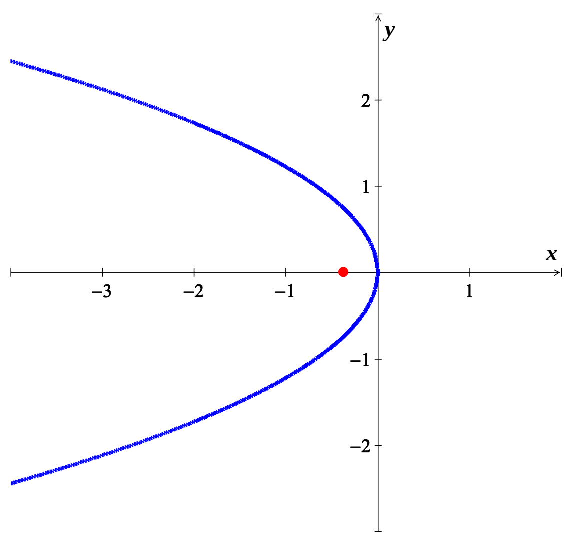
$y^2 = -\frac{3}{2}x = 4px$

$4p = -\frac{3}{2} \quad \Rightarrow \quad \boxed{p = -\frac{3}{8}}$

Vertex: (0, 0)

Focus: $\left(-\frac{3}{8},\ 0\right)$

Directrix: $x = \frac{3}{8}$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $(x+2)^2 = -8(y-1)$

Solution

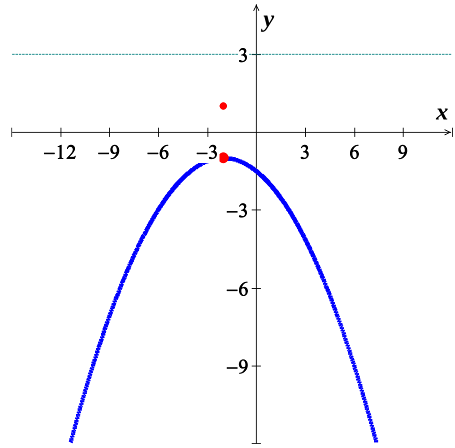
$$(x+2)^2 = 4p(y-1)$$

$4p = -8 \quad \Rightarrow \quad \boxed{p = -2}$

Vertex: $(-2,\ 1)$

Focus: $(-2,\ 1-2) = (-2,\ -1)$

Directrix: $y = 1 + 2 = \boxed{3}$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $(x-3)^2 = \frac{1}{2}(y+1)$

Solution

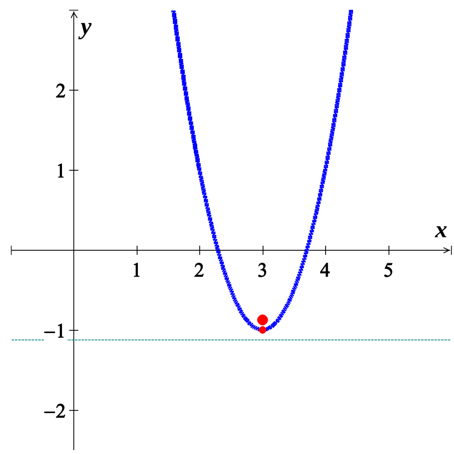
$$(x-3)^2 = 4p(y+1)$$

$4p = \frac{1}{2} \quad \Rightarrow \quad \boxed{p = \frac{1}{8}}$

Vertex: $(3,\ -1)$

Focus: $\left(3,\ -1+\frac{1}{8}\right) = \left(3,\ -\frac{7}{8}\right)$

Directrix: $\boxed{y = -1 - \frac{1}{8} = -\frac{9}{8}}$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $(y+1)^2 = -12(x+2)$

Solution

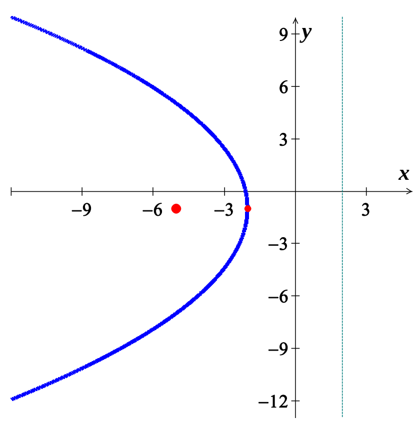
$$(y+1)^2 = 4p(x+2)$$

$4p = -12 \quad \Rightarrow \quad \boxed{p = -3}$

Vertex: $(-2,\ -1)$

Focus: $(-2-3,\ -1) = \boxed{(-5,\ -1)}$

Directrix: $\boxed{x = -1 + 3 = 2}$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $y = x^2 - 4x + 2$

Solution

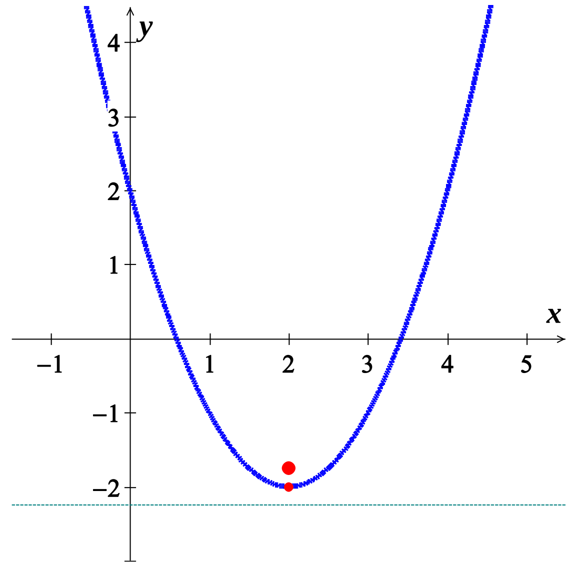
$$y = ax^2 + bx + c \quad \Rightarrow \quad a = 1$$

$$p = \frac{1}{4a} = \frac{1}{4} \quad \Rightarrow \quad \boxed{p = \frac{1}{4}}$$

$$\textbf{\textit{Vertex}}: \begin{cases} h = -\frac{b}{2a} = -\frac{-4}{2(1)} = 2 \\ k = 2^2 - 4(2) + 2 = -2 \end{cases} \rightarrow (2,\ -2)$$

$$\textbf{\textit{Focus}}: \left(2,\ -2 + \frac{1}{4}\right) = \left(2,\ -\frac{7}{4}\right)$$

$$\textbf{\textit{Directrix}}: \boxed{y = -2 - \frac{1}{4} = -\frac{9}{4}}$$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $y^2 + 14y + 4x + 45 = 0$

Solution

$$y^2 + 14y = -4x - 45$$

$$y^2 + 14y + (7)^2 = -4x - 45 + (7)^2$$

$$(y + 7)^2 = -4x + 4$$

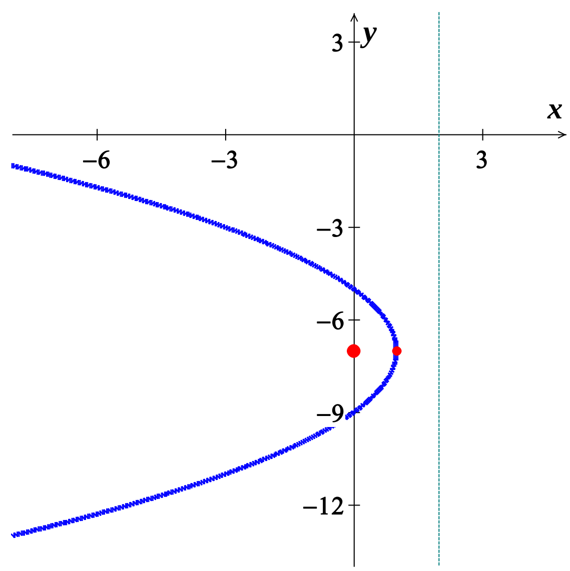
$$(y + 7)^2 = -4(x - 1)$$

$$4p = -4 \quad \Rightarrow \quad \boxed{p = -1}$$

$$\textbf{\textit{Vertex}}: (1,\ -7)$$

$$\textbf{\textit{Focus}}: (1 - 1,\ -7) = \boxed{(0,\ -7)}$$

$$\textbf{\textit{Directrix}}: \boxed{x = 1 + 1 = 2}$$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $x^2 + 20y = 10$

Solution

$$x^2 = -20y + 10$$

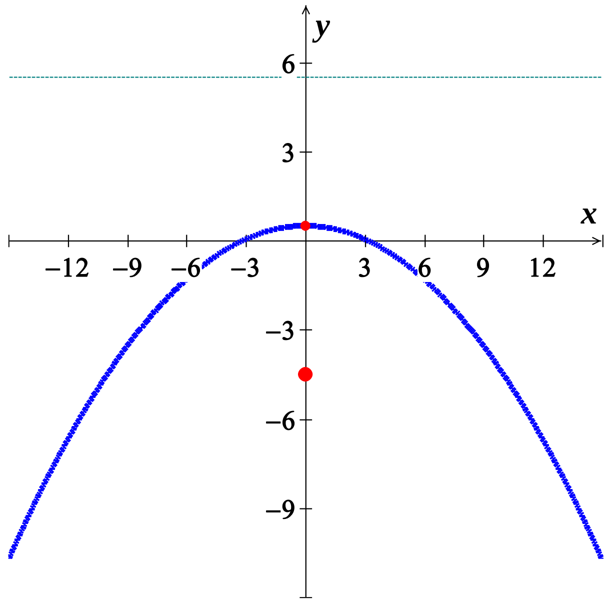
$$x^2 = -20\left(y - \frac{1}{2}\right)$$

$$4p = -20 \quad \Rightarrow \quad \boxed{p = -5}$$

Vertex: $\boxed{\left(0,\ \frac{1}{2}\right)}$

Focus: $\left(0,\ \frac{1}{2} - 5\right) = \boxed{\left(0,\ -\frac{9}{2}\right)}$

Directrix: $\boxed{y = \frac{1}{2} + 5 = \frac{11}{2}}$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $x^2 = 16y$

Solution

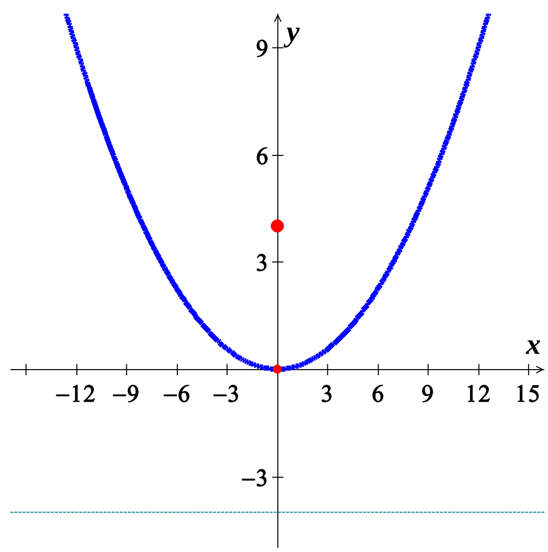
$$x^2 = 16y = 4py$$

$$4p = 16 \quad \Rightarrow \quad \boxed{p = 4}$$

Vertex: (0, 0)

Focus: $(0,\ 4)$

Directrix: $y = -4$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $x^2 = -\frac{1}{2}y$

Solution

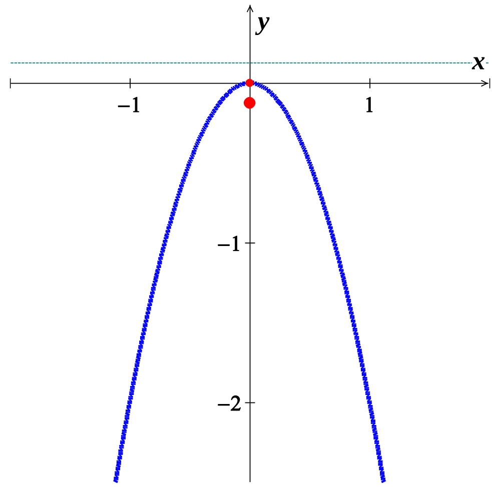
$$x^2 = -\frac{1}{2}y = 4py$$

$$4p = -\frac{1}{2} \quad \Rightarrow \quad \boxed{p = -\frac{1}{8}}$$

Vertex: (0, 0)

Focus: $\left(0,\ -\frac{1}{8}\right)$

Directrix: $y = \frac{1}{8}$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $(y+1)^2 = -4(x-2)$

Solution

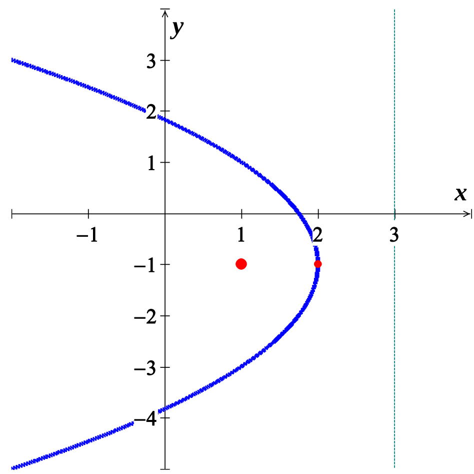
$$(y+1)^2 = 4p(x-2)$$

$$4p = -4 \quad \Rightarrow \quad \boxed{p = -1}$$

Vertex: $(2,\ -1)$

Focus: $(2-1,\ -1) = \boxed{(1,\ -1)}$

Directrix: $\lfloor x = 2+1 = 3 \rfloor$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $x^2 + 6x - 4y + 1 = 0$

Solution

$$x^2 + 6x + \left(\frac{6}{2}\right)^2 = 4y - 1 + (3)^2$$

$$(x+3)^2 = 4y + 8$$

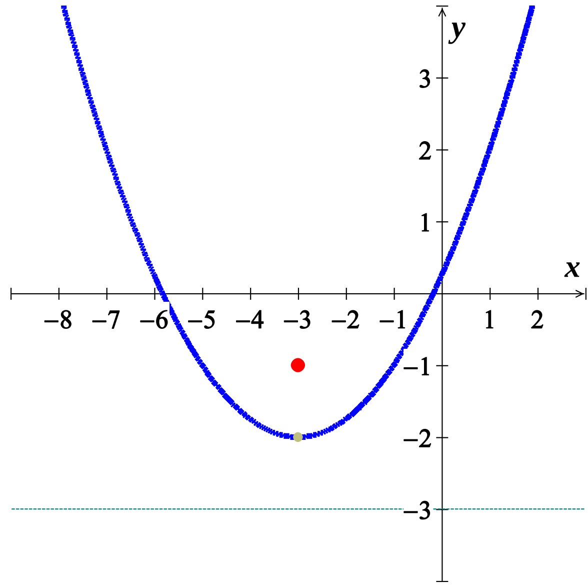
$$(x+3)^2 = 4(y+2)$$

$$4p = 4 \quad \Rightarrow \quad \boxed{p = 1}$$

Vertex: $\boxed{(-3,\ -2)}$

Focus: $(-3,\ -2+1)=\boxed{(-3,\ -1)}$

Directrix: $\boxed{y=-2-1=-3}$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $y^2+2y-x=0$

Solution

$$y^2+2y=x$$

$$y^2+2y+\left(\frac{2}{2}\right)^2=x+(1)^2$$

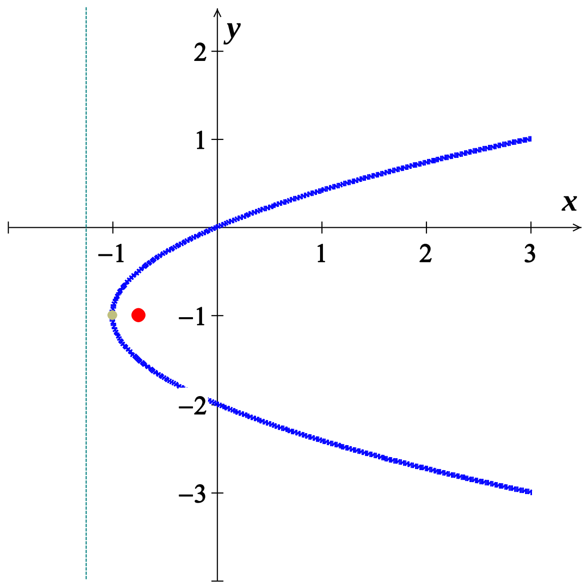
$$(y+1)^2=(x+1)$$

$$4p=1 \quad \Rightarrow \quad \boxed{p=\frac{1}{4}}$$

Vertex: $(-1,\ -1)$

Focus: $\left(-1+\frac{1}{4},\ -1\right)=\boxed{\left(-\frac{3}{4},\ -1\right)}$

Directrix: $\boxed{x=-1-\frac{1}{4}=-\frac{5}{4}}$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $y^2 - 4y + 4x + 4 = 0$

Solution

$$y^2 - 4y = -4x - 4$$

$$y^2 - 4y + \left(\frac{-4}{2}\right)^2 = -4x - 4 + (-2)^2$$

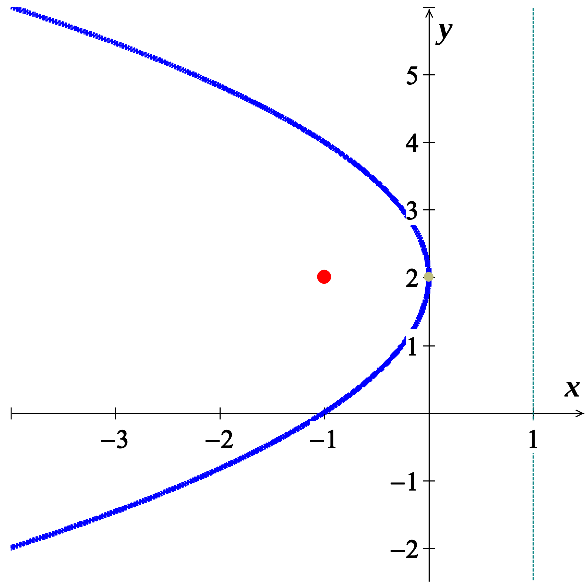
$$(y-2)^2 = -4x$$

$$4p = -4 \quad \Rightarrow \quad \boxed{p = -1}$$

Vertex: $(0,\ 2)$

Focus: $= \boxed{(-1,\ 2)}$

Directrix: $\lfloor x = 1 \rfloor$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $x^2 - 4x - 4y = 4$

Solution

$$x^2 - 4x = 4y + 4$$

$$x^2 - 4x + \left(\frac{-4}{2}\right)^2 = 4y + 4 + (-2)^2$$

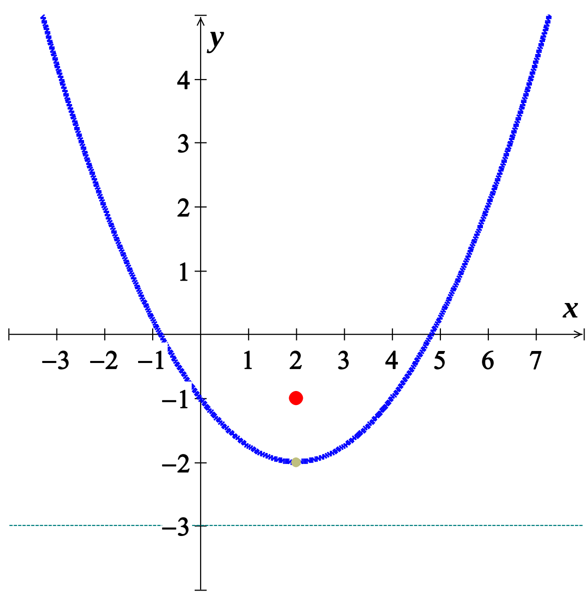
$$(x-2)^2 = 4(y+2)$$

$$4p = 4 \quad \Rightarrow \quad \boxed{p = 1}$$

Vertex: $\boxed{(2,\ -2)}$

Focus: $(2,\ -2+1) = \boxed{(2,\ -1)}$

Directrix: $\lfloor y = -2 - 1 = \underline{-3} \rfloor$



Exercise

Find an equation of the parabola that satisfies the given conditions $Focus: F(2,0)$ $directrix: x=-2$

Solution

$$x=-2=-p \quad \rightarrow \quad p=2$$

$$y^2=4px$$

$$\boxed{y^2=8x}$$

Exercise

Find an equation of the parabola that satisfies the given conditions $Focus: F(0,-40)$ $directrix: y=4$

Solution

$$y=4=-p \quad \rightarrow \quad p=-4$$

$$x^2=4py$$

$$\boxed{x^2=-16y}$$

Exercise

Find an equation of the parabola that satisfies the given conditions $Focus: F(-3,-2)$ $directrix: y=1$

Solution

$$y=1=k-p \quad \rightarrow \quad k-p=1$$

$$\begin{cases} \boxed{h=-3} \\ k+p=-2 \end{cases} \rightarrow \begin{cases} k+p=-2 \\ k-p=1 \end{cases} \Rightarrow 2k=-1 \rightarrow \boxed{k=-\frac{1}{2}}$$

$$k-p=1 \quad \rightarrow \quad \boxed{p=k-1=-\frac{1}{2}-1=-\frac{3}{2}}$$

Vertex: $\boxed{\left(-3,\ -\frac{1}{2}\right)}$

$$(x+3)^2=4\left(-\frac{3}{2}\right)\left(y+\frac{1}{2}\right)$$

$$\boxed{(x+3)^2=-6\left(y+\frac{1}{2}\right)}$$

Exercise

Find an equation of the parabola that satisfies the given conditions $Vertex: V(3,-5)$ $\quad directrix: x=2$

Solution

$$Vertex: V(3,-5) \quad \begin{cases} h=3 \\ k=-5 \end{cases}$$

$$directrix: x = 2 = h - p \quad \Rightarrow \boxed{p = h - 2 = 3 - 2 = 1}$$

$$(y-k)^2 = 4p(x-h)$$

$$\boxed{(y+5)^2 = 4(x-3)}$$

Exercise

Find an equation of the parabola that satisfies the given conditions $Vertex: V(-2,3)$ $\quad directrix: y=5$

Solution

$$Vertex: V(-2,\ 3) \quad \begin{cases} h=-2 \\ k=3 \end{cases}$$

$$directrix: y = 5 = k - p \quad \Rightarrow \boxed{p = k - 5 = 3 - 5 = -2}$$

$$(x-h)^2 = 4p(y-k)$$

$$\boxed{(x+2)^2 = -8(y-3)}$$

Exercise

Find an equation of the parabola that satisfies the given conditions $Vertex: V(-1,0)$ $\quad focus: F(-4,0)$

Solution

$$Vertex: V(-1,\ 0) \quad \begin{cases} h=-1 \\ k=0 \end{cases}$$

$$focus: F(-4,0) \quad \begin{cases} h+p=-4 \\ k=0 \end{cases} \Rightarrow \boxed{p = -4 - h = -4 + 1 = -3}$$

$$(y-k)^2 = 4p(x-h)$$

$$\boxed{y^2 = -12(x+1)}$$

Exercise

Find an equation of the parabola that satisfies the given conditions $Vertex: V(1,-2)$ $focus: F(1,0)$

Solution

$$Vertex: V(1,\ -2) \quad \begin{cases} h=1 \\ k=-2 \end{cases}$$

$$focus: F(1,\ 0) \quad \begin{cases} h=1 \\ k+p=0 \end{cases} \Rightarrow \underline{p=-k=2}$$

$$(x-h)^2 = 4p(y-k)$$

$$\boxed{(x-1)^2 = 8(y+2)}$$

Exercise

Find an equation of the parabola that satisfies the given conditions $Vertex: V(0,\ 1)$ $focus: F(0,\ 2)$

Solution

$$Vertex: V(0,\ 1) \quad \begin{cases} h=0 \\ k=1 \end{cases}$$

$$focus: F(0,\ 2) \quad \begin{cases} h=0 \\ k+p=2 \end{cases} \Rightarrow \underline{p=2-1=1}$$

$$(x-h)^2 = 4p(y-k)$$

$$\boxed{x^2 = 4(y-1)}$$

Exercise

Find an equation of the parabola that satisfies the given conditions $Vertex: V(3,\ 2)$ $focus: F(-1,\ 2)$

Solution

$$Vertex: V(3,\ 2) \quad \begin{cases} h=3 \\ k=2 \end{cases}$$

$$focus: F(-1,2) \quad \begin{cases} h+p=-1 \\ k=2 \end{cases} \Rightarrow \underline{p=-1-3=-4}$$

$$(y-k)^2 = 4p(x-h)$$

$$\boxed{(y-2)^2 = -16(x-3)}$$

Exercise

An arch in the shape of a parabola has the dimensions shown in the figure. How wide is the arch 9 *feet* up?

Solution

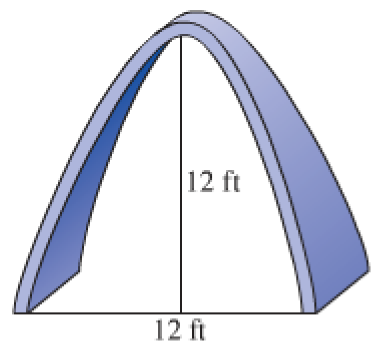
$Vertex: \ V(0,\ 12)$

$(x-h)^2 = 4p(y-k)$

$(x-0)^2 = 4p(y-12) \quad \Rightarrow \quad x^2 = 4p(y-12)$

The parabola passes through the point $(6,\ 0) \quad \Rightarrow \quad 6^2 = 4p(0-12)$

$-48p = 36 \rightarrow \boxed{p = -\frac{36}{48} = -\frac{3}{4}}$



The equation is: $x^2 = -3(y-12)$

The arch is 9 feet up that is the y-coordinate,

$x^2 = -3(9-12) = 9 \quad \Rightarrow \quad x = 3$

The width is $2(3) \ \boxed{= 6 \ \ feet}$

Exercise

The cable in the center portion of a bridge is supported as shown in the figure to form a parabola. The center support is 10 *feet* high, the tallest supports are 210 *feet* high, and the distance between the two tallest supports is 400 *feet*. Find the height of the remaining supports if the supports are evenly spaced.

Solution

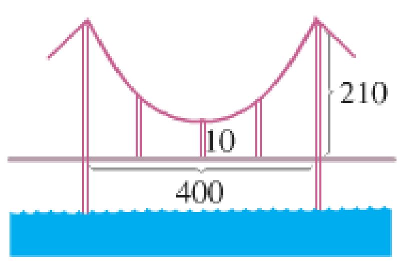
$Vertex: \ V(0,\ 10)$

$(x-h)^2 = 4p(y-k)$

$(x-0)^2 = 4p(y-10) \quad \Rightarrow \quad x^2 = 4p(y-10)$

The parabola passes through the point $(200,\ 210) \quad \Rightarrow \quad 200^2 = 4p(210-10)$

$800p = 200^2 \rightarrow \boxed{p = \frac{40000}{800} = 50}$



The equation is: $x^2 = 200(y-10)$

The x-coordinate of one of the supports is 100.

$100^2 = 200(y-10)$

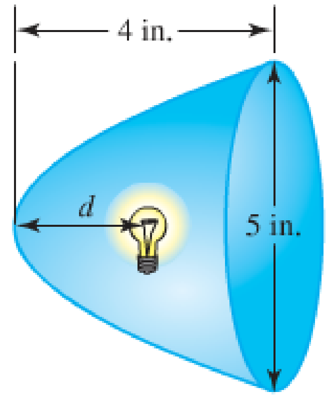
$y - 10 = \frac{10000}{200} = 50$

$\boxed{y = 50 + 10 = 60 \ \ feet}$ The height is $60 \ \ feet$

Exercise

A headlight is being constructed in the shape of a paraboloid with depth 4 *inches* and diameter 5 *inches*. Determine the distance *d* that the bulb should be form the vertex in order to have the beam of light shine straight ahead.

Solution



Let the vertex be at the origin $V(0,\ 0)$

The equation is: $y^2 = 4px$

Which it passes through the point $V(4,\ 2.5)$

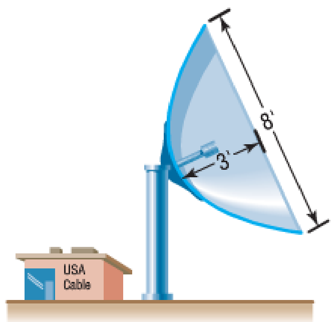
$(2.5)^2 = 4p(4)$

$p = \frac{(2.5)^2}{16} = \frac{25}{64}$

The bulb should be $\frac{25}{64} \approx 0.39$ inch from the vertex

Exercise

A satellite dish is shaped like a paraboloid of revolution. The signals that emanate from a satellite strike the surface of the dish and are reflected to a single point, where the receiver is located. If the dish is 8 *feet* across at its opening and 3 *feet* deep at its center, at what position should the receiver be placed? That is, where is the focus?



Solution

From the figure, we can draw the parabola used to form the dish on a rectangular coordinate system so that the vertex of the parabola is at the origin and its focus on the positive *y*-axis.

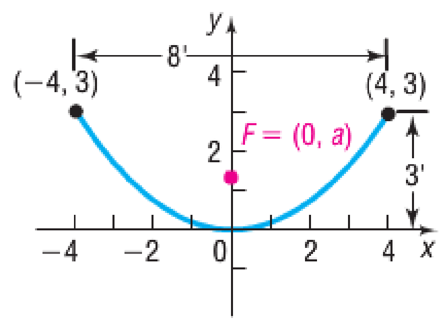
The equation from of the parabola is: $x^2 = 4py$

Since $(4,\ 3)$ is a point on the graph

$4^2 = 4p(3)$

$p = \frac{16}{12} = \frac{4}{3}$

Therefore, the receiver should be located $\frac{4}{3}\ ft$ from the base of the dish, along its axis of symmetry.



Exercise

A cable TV receiving dish is in the shape of a paraboloid of revolution. Find the location of the receiver, which is placed at the focus, if the dish is 6 *feet* across at its opening and 2 feet deep.

Solution

Given: Parabola is 6 *feet* across and 2 *feet* deep.

Let the vertex of the parabola is at (0, 0) and it opens up, then the equation of the parabola has the form $x^2 = 4ay$

Therefore, the point (3, 2) and (–3, 2) are on the parabola.

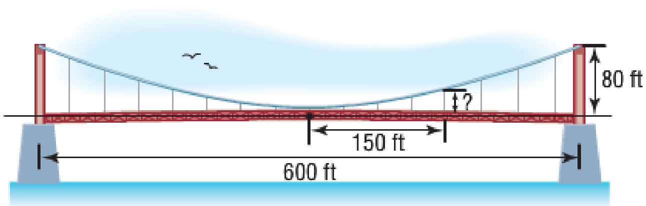
$$3^2 = 4a(2) \quad \to \quad a = \frac{9}{8} = 1.125$$

Where a is the distance from the vertex to the focus.

Thus, the receiver (located at the focus) is 1.125 *feet* or 13.5 *inches* from the base of the dish, along the axis of the parabola.

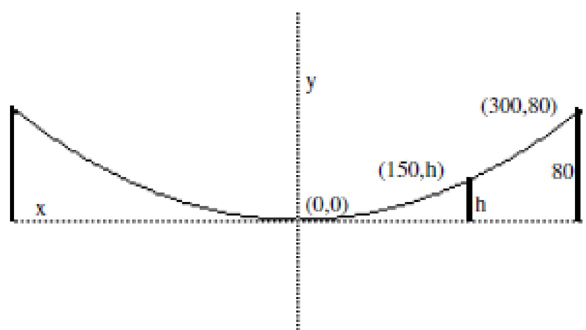
Exercise

The cables of a suspension bridge are in the shape of a parabola, as shown below. The towers supporting the cable are 600 *feet* apart and 80 *feet* high. If the cables touch the road surface midway between the towers, what is the height of the cable from the road at a point 150 *feet* from the center of the bridge?



Solution

Let the vertex of the parabola is at (0, 0) and it opens up, then the equation of the parabola has the form $x^2 = cy$



The point (300, 80) is a point on the parabola.

$$300^2 = c(80) \quad \to \quad c = \frac{300^2}{80} = 1125$$

$$x^2 = 1125y$$

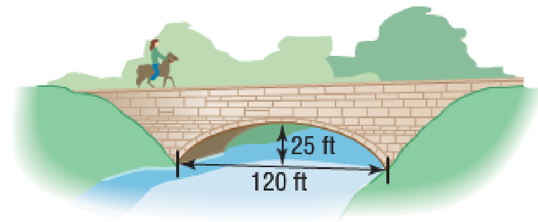
The height of the cable 150 feet from the center is:

$$150^2 = 1125h \quad \to \quad h = \frac{150^2}{1125} \underline{= 20|}$$

The height of the cable 150 feet from the center is 20 feet.

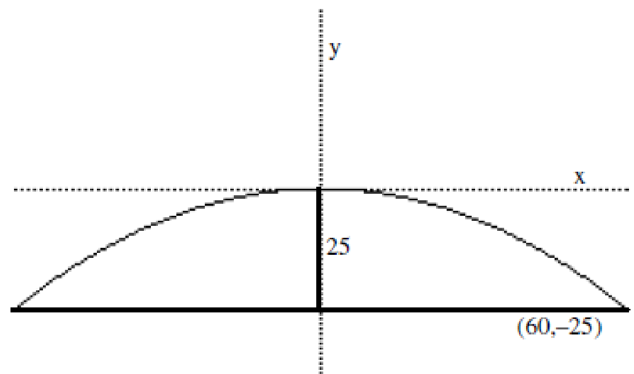
Exercise

A bridge is built in the shape of a parabolic arch. The bridge has a span of 120 *feet* and a maximum height of 25 *feet*. Choose a suitable rectangular coordinate system and find the height of the arch at distances of 10, 30, and 50 *feet* from the center.



Solution

Let the vertex of the parabola is at (0, 0) and it opens down, then the equation of the parabola has the form $x^2 = cy$



The point (60, –25) is a point on the parabola.

$$60^2 = c(-25) \quad \to \quad c = \frac{60^2}{-25} = -144$$

$$x^2 = -144y$$

The height of the arch at

Distance 10:

$$10^2 = -144y \quad \to \quad y = \frac{100}{-144} \approx -0.69$$

The height of the bridge 10 feet from the center is about $25 - 0.69 = 24.31 \ ft$

Distance 30:

$$30^2 = -144y \quad \to \quad y = \frac{900}{-144} \approx -6.25$$

The height of the bridge 30 feet from the center is about $25 - 6.25 = 18.75 \ ft$

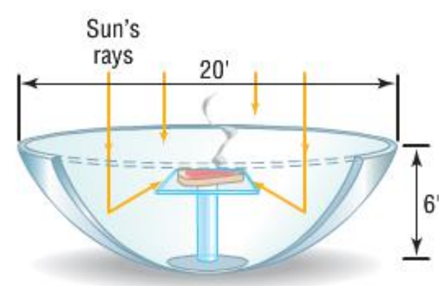
Distance 50:

$$50^2 = -144y \quad \to \quad y = \frac{2500}{-144} \approx -17.36$$

The height of the bridge 10 feet from the center is about $25 - 17.36 = 7.64 \ ft$

Exercise

A mirror is shaped like a paraboloid of revolution and will be used to concentrate the rays of the sun at its focus, creating a heat source. If the mirror is 20 *feet* across at its opening and is 6 *feet* deep, where will the heat source be concentrated?



Solution

Let the vertex of the parabola is at (0, 0) and it opens up, then the equation of the parabola has the form $x^2 = 4ay$

Since the parabola is 20 feet across and 6 feet deep.

The points (10, 6) and (–10, 6) are on the parabola.

$$10^2 = 4a(6) \quad \to \quad a = \frac{100}{24} \approx 4.17 \ ft$$

The heat will be concentrated about 4.17 *feet* from the base, along the axis of symmetry.

Exercise

A reflecting telescope contains a mirror shaped a paraboloid of revolution. If the mirror is 4 *inches* across at its opening and is 3 *inches* deep, where will the collected light be concentrated?

Solution

Let the vertex of the parabola is at (0, 0) and it opens up.

Then the equation of the parabola has the form $x^2 = 4ay$

Since the parabola is 4 *inches* across and 3 *inches* deep.

The points (2, 3) and (–2, 3) are on the parabola.

$$2^2 = 4a(3) \quad \to \quad a = \frac{4}{12} \approx \boxed{\frac{1}{2}\ in}$$

The collected light will be concentrated 1/3 inch from the base of the mirror along the axis of symmetry.

Exercise

Show that the graph of an equation of the form $Ax^2 + Dx + Ey + F = 0 \qquad A \neq 0$

a) Is a parabola if $E \neq 0$

b) Is a vertical line if $E = 0$ *and* $D^2 - 4AF = 0$

c) Is two vertical lines if $E = 0$ *and* $D^2 - 4AF > 0$

d) Contains no points if $E = 0$ *and* $D^2 - 4AF < 0$

Solution

a) If $E \neq 0 \quad \to \quad Ax^2 + Dx + Ey + F = 0$

The x-vertex: $x = -\frac{b}{2a} = -\frac{D}{2A}$

$$A\left(-\frac{D}{2A}\right)^2 + D\left(-\frac{D}{2A}\right) + Ey + F = 0$$

$$\frac{D^2}{4A} - \frac{D^2}{2A} + Ey + F = 0$$

$$Ey = \frac{D^2}{4A} - F$$

$$y = \frac{D^2 - 4AF}{4AE}$$

This is the equation of a parabola whose vertex is: $\left(-\frac{D}{2A}, \frac{D^2 - 4AF}{4AE}\right)$ and whose axis of symmetry is parallel to the y-axis.

b) If $E = 0 \quad \to \quad Ax^2 + Dx + F = 0$

$$x = \frac{-D \pm \sqrt{D^2 - 4AF}}{2A}$$

$$= -\frac{D}{2A} \qquad \textit{Since } D^2 - 4AF = 0$$

This is a single vertical line.

c) If $E = 0 \quad \to \quad Ax^2 + Dx + F = 0$

$$x = \frac{-D \pm \sqrt{D^2 - 4AF}}{2A}$$

If $D^2 - 4AF > 0$, then

$x = \dfrac{-D - \sqrt{D^2 - 4AF}}{2A}$ *and* $x = \dfrac{-D + \sqrt{D^2 - 4AF}}{2A}$ are two vertical lines.

d) If $E = 0 \quad \to \quad Ax^2 + Dx + F = 0$

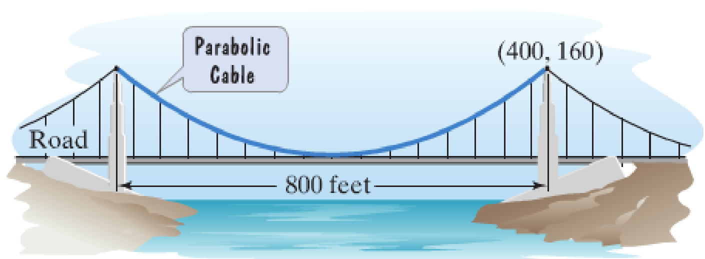
$$x = \frac{-D \pm \sqrt{D^2 - 4AF}}{2A}$$

If $D^2 - 4AF < 0$, then there is no real solution. The graph contains no points.

Exercise

The towers of a suspension bridge are 800 *feet* apart and rise 160 *feet* above the road. The cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 100 *feet* from a tower?

Solution



Given the point: $(400,\ 160)$

$(400)^2 = 4p(160) \qquad x^2 = 4py$

$p = \dfrac{400^2}{640} = \underline{250}$

$x^2 = 1{,}000y$

$x = 400 - 100 = 300$

$(300)^2 = 1{,}000y \qquad x^2 = 4py$

$y = \dfrac{300^2}{1{,}000} = \underline{90}$

The height is 90 *feet*.

Exercise

The cables of a suspension bridge are in the shape of a parabola. The towers supporting the cable are 400 *feet* apart and 100 *feet* high. If the cables are at a height of 10 *feet* midway between the towers, what is the height of the cable at a point 50 *feet* from the center of the bridge?

Solution

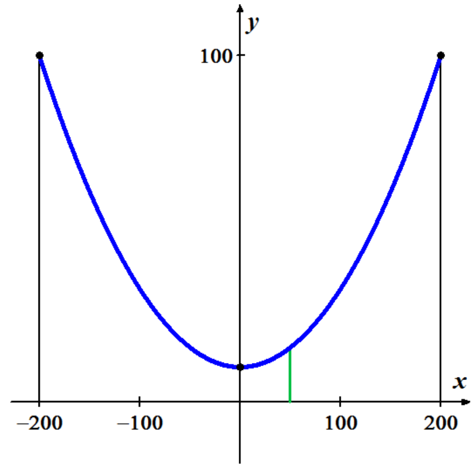
Vertex point: $(0,\ 10)$ and the parabola is open up

A point on parabola: $(200,\ 100)$

$200^2 = c(100-10)$ $\qquad (x-h)^2 = c(y-k)$

$c = \dfrac{40,000}{90} = \dfrac{4000}{9}$

$\boxed{x^2 = \dfrac{4000}{9}(y-10)}$



The height of the cable 50 feet from the center – $(50,\ h)$

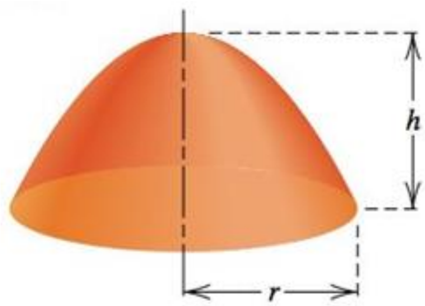
$y = \dfrac{9}{4000}x^2 + 10$

$h = \dfrac{9}{4000}(50)^2 + 10 \ \boxed{\approx 15.625\ ft}$

The height of the cable 50 *feet* from the center is about 15.625 *feet*.

Exercise

The focal length of the (finite) paraboloid is the distance p between its vertex and focus



a) Express p in terms of r and h.

b) A reflector is to be constructed with a focal length of 10 *feet* and a depth of 5 *feet*. Find the radius of the reflector.

Solution

a) The point $(r,\ h)$ is on the parabola.

$r^2 = 4p(h)$ $\qquad x^2 = 4py$

$\boxed{p = \dfrac{r^2}{4h}}$

b) Given: $p = 10;\quad h = 5$

$r = \sqrt{4(10)(5)}\ \boxed{= 10\sqrt{2}}$

Solution *Section* 4.7 – Ellipses

Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $\frac{x^2}{9}+\frac{y^2}{4}=1$

Solution

$$\begin{cases} a^2 = 9 \to a = 3 \\ b^2 = 4 \to b = 2 \end{cases}$$

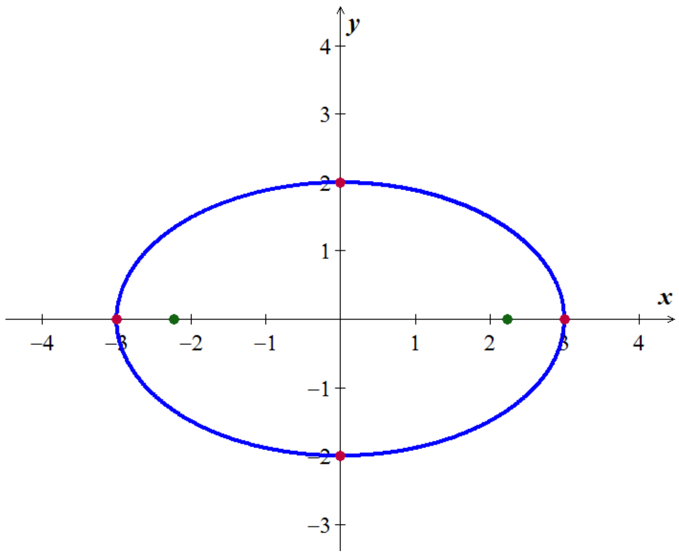
$$c=\sqrt{a^2-b^2}=\sqrt{9-4}=\sqrt{5}$$

Center: $C(0,\ 0)$

Vertices: $V(\pm 3,\ 0)$

Minor $M(0,\ \pm 2)$

Foci $F\left(\pm\sqrt{5},\ 0\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $\frac{x^2}{16}+\frac{y^2}{36}=1$

Solution

$$\begin{cases} a^2 = 36 \to a = 6 \\ b^2 = 16 \to b = 4 \end{cases}$$

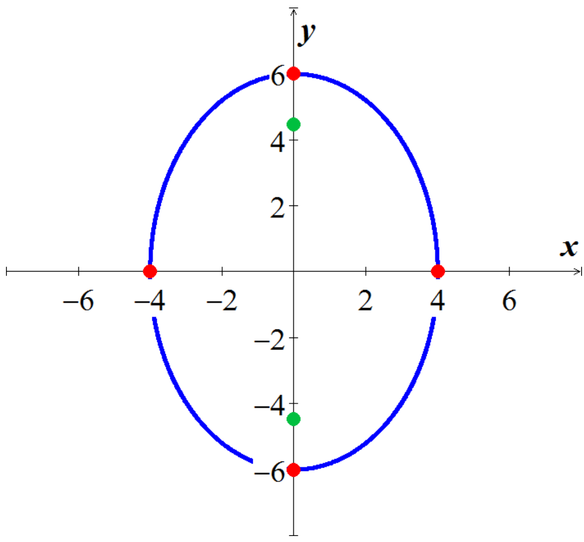
$$c=\sqrt{a^2-b^2}=\sqrt{36-16}=2\sqrt{5}$$

Center: $C(0,\ 0)$

Vertices: $V(0,\ \pm 6)$

Minors $M(\pm 4,\ 0)$

Foci $F\left(0,\ \pm 2\sqrt{5}\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $\frac{x^2}{15}+\frac{y^2}{16}=1$

Solution

$$\begin{cases} a^2=16 \rightarrow a=4 \\ b^2=15 \rightarrow b=\sqrt{15} \end{cases}$$

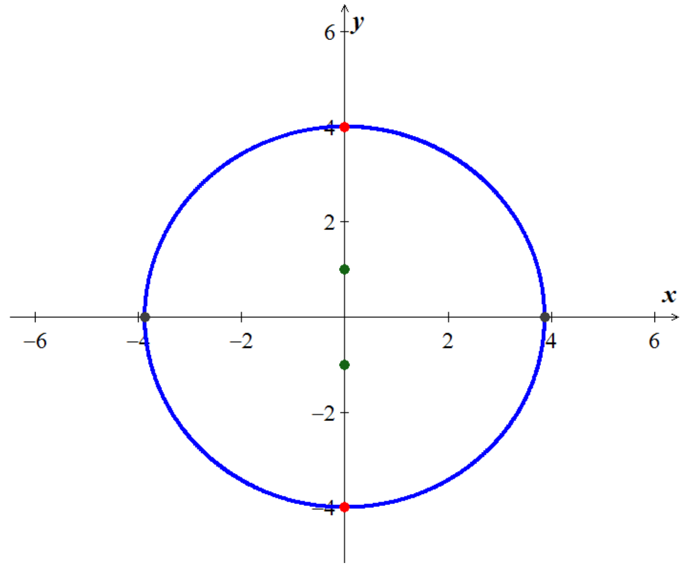
$$c=\sqrt{a^2-b^2}=\sqrt{16-15}=1$$

Center: $C(0,\ 0)$

Vertices: $V(0,\ \pm 4)$

Minors $M(\pm\sqrt{15},0)$

Foci $F(0,\ \pm 1)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $\frac{25x^2}{36}+\frac{64y^2}{9}=1$

Solution

$$\frac{x^2}{\frac{36}{25}}+\frac{y^2}{\frac{9}{64}}=1$$

$$\begin{cases} a^2=\frac{36}{25} \rightarrow a=\frac{6}{5} \\ b^2=\frac{9}{64} \rightarrow b=\frac{3}{8} \end{cases}$$

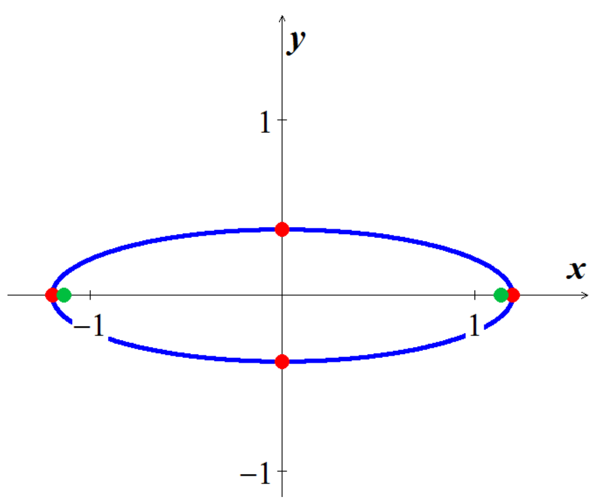
$$c=\sqrt{a^2-b^2}=\sqrt{\frac{36}{25}-\frac{9}{64}}=\sqrt{\frac{2079}{1600}}=\frac{3\sqrt{231}}{40}$$

Center: $C(0,\ 0)$

Vertices: $V\left(\pm\frac{6}{5},\ 0\right)$

Minor $M\left(0,\ \pm\frac{3}{8}\right)$

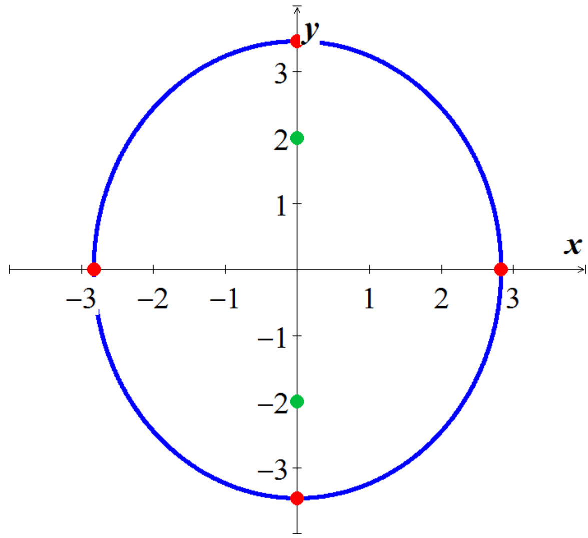
Foci $F\left(\pm\frac{3\sqrt{231}}{40},\ 0\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $12x^2 + 8y^2 = 96$

Solution



$$\frac{12}{96}x^2 + \frac{8}{96}y^2 = \frac{96}{96}$$

$$\frac{x^2}{8} + \frac{y^2}{12} = 1$$

$$\to \begin{cases} a^2 = 12 \to a = 2\sqrt{3} \\ b^2 = 8 \to b = 2\sqrt{2} \end{cases}$$

$$c = \sqrt{a^2 - b^2} = \sqrt{12 - 8} = 2$$

Center: $C(0,\ 0)$

Vertices: $V\left(0,\ \pm 2\sqrt{3}\right)$

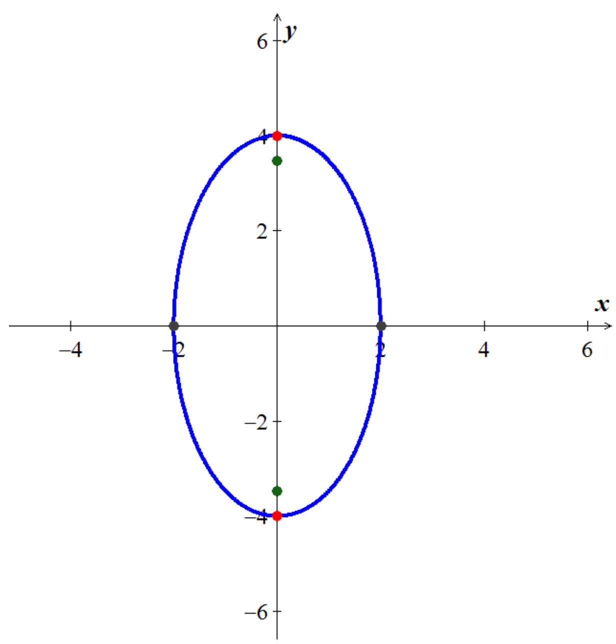
Minors $M\left(\pm 2\sqrt{2}, 0\ \right)$

Foci $F(0,\ \pm 2)$

Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $4x^2 + y^2 = 16$

Solution



$$\frac{1}{16}4x^2 + \frac{1}{16}y^2 = \frac{1}{16}16$$

$$\frac{x^2}{4} + \frac{y^2}{16} = 1$$

$$\to \begin{cases} a^2 = 16 \to a = 4 \\ b^2 = 4 \to b = 2 \end{cases}$$

$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 4} = \sqrt{12} = 2\sqrt{3}$$

Center: $C(0,\ 0)$

Vertices: $V(0,\ \pm 4)$

Minors $M(\pm 2, 0\ )$

Foci $F\left(0,\ \pm 2\sqrt{3}\right)$

Exercise

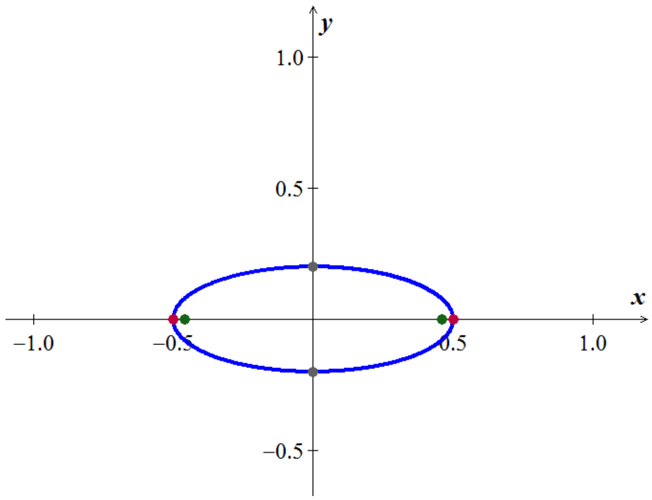
Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $4x^2 + 25y^2 = 1$

Solution

$$\frac{x^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{25}} = 1$$

$$\begin{cases} a^2 = \frac{1}{4} \rightarrow a = \frac{1}{2} \\ b^2 = \frac{1}{25} \rightarrow b = \frac{1}{5} \end{cases}$$

$$c = \sqrt{a^2 - b^2} = \sqrt{\frac{1}{4} - \frac{1}{25}} = \sqrt{\frac{21}{100}} = \frac{\sqrt{21}}{10}$$



Center: $C(0,\ 0)$

Vertices: $V\left(\pm\frac{1}{2},\ 0\right)$

Minor $M\left(0,\ \pm\frac{1}{5}\right)$

Foci $F\left(\pm\frac{\sqrt{21}}{10},\ 0\right)$

Exercise

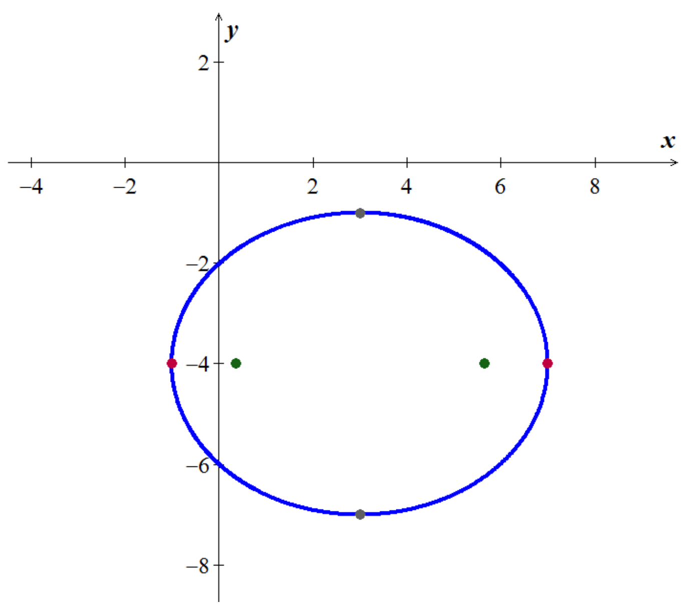
Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$$\frac{(x-3)^2}{16} + \frac{(y+4)^2}{9} = 1$$

Solution

$$\begin{cases} a^2 = 16 \rightarrow a = 4 \\ b^2 = 9 \rightarrow b = 3 \end{cases}$$

$$c = \sqrt{a^2 - b^2} = \sqrt{16-9} = \sqrt{7}$$



Center: $C(3,\ -4)$

Vertices: $V(3 \pm 4,\ -4)$

Minor $M(3,\ -4 \pm 3)$

Foci $F\left(3 \pm \sqrt{7},\ -4\right)$

Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$$\frac{(x+3)^2}{16}+\frac{(y-2)^2}{36}=1$$

Solution

$$\begin{cases} a^2=36 \to a=6 \\ b^2=16 \to b=4 \end{cases}$$

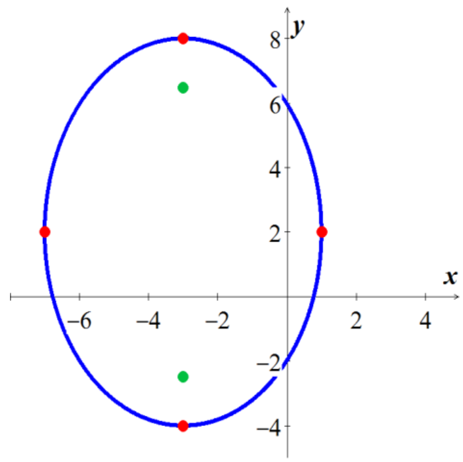
$c=\sqrt{36-16}=2\sqrt{5}$

Center: $C(-3,\ 2)$

Vertices: $V(-3,\ 2\pm 6)$

Minor $M(-3\pm 4,\ 2)$

Foci $F\left(-3,\ 2\pm 2\sqrt{5}\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$$\frac{(x+1)^2}{64}+\frac{(y-2)^2}{49}=1$$

Solution

$$\begin{cases} a^2=64 \to a=8 \\ b^2=49 \to b=7 \end{cases}$$

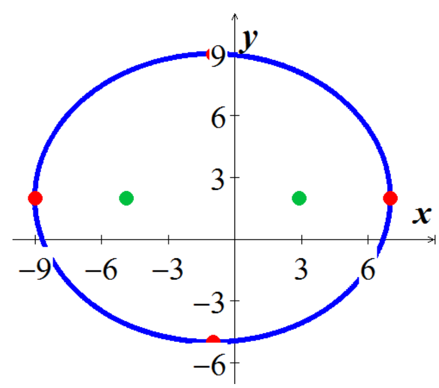
$c=\sqrt{a^2-b^2}=\sqrt{64-49}=\sqrt{15}$

Center: $C(-1,\ 2)$

Vertices: $V(-1\pm 8,\ 2)$

Minor $M(-1,\ 2\pm 7)$

Foci $F\left(-1\pm\sqrt{15},\ 2\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$4x^2 + 9y^2 - 32x - 36y + 64 = 0$

Solution

$$4\left(x^2 - 8x + \left(\frac{8}{2}\right)^2\right) + 9\left(y^2 - 4y + \left(\frac{4}{2}\right)^2\right) = -64 + 4(16) + 9(4)$$

$$4(x-4)^2 + 9(y-2)^2 = 36$$

$$\frac{(x-4)^2}{9} + \frac{(y-2)^2}{4} = 1$$

$$\begin{cases} a^2 = 9 \to a = 3 \\ b^2 = 4 \to b = 2 \end{cases}$$

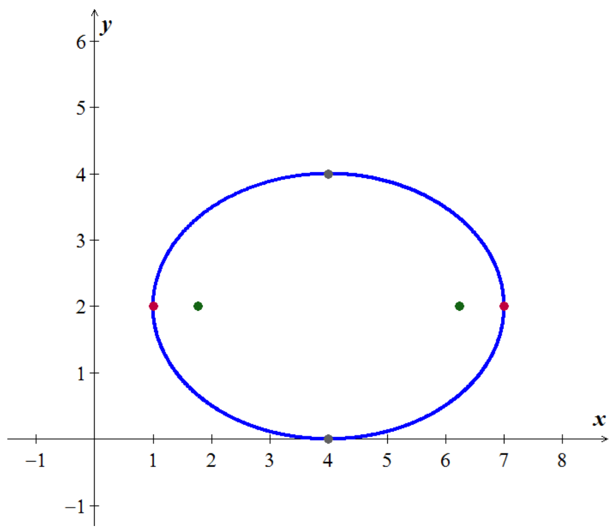
$$c = \sqrt{a^2 - b^2} = \sqrt{9-4} = \sqrt{5}$$

Center: $C(4,\ 2)$

Vertices: $(4 \pm 3,\ 2)$ $V'(1,\ 2)$ $V(7,\ 2)$

Minor $(4,\ 2 \pm 2)$ $M'(4,\ 0)$ $M(4,\ 4)$

Foci $F\left(4 \pm \sqrt{5},\ 2\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$x^2 + 2y^2 + 2x - 20y + 43 = 0$

Solution

$$\left(x^2 + 2x + \left(\frac{2}{2}\right)^2\right) + 2\left(y^2 - 10y + \left(\frac{10}{2}\right)^2\right) = -43 + 1 + 2(100)$$

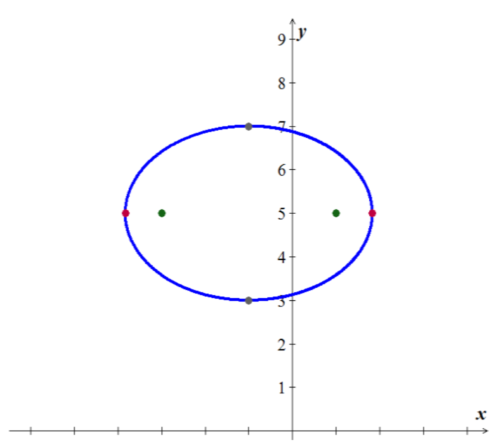
$$(x+1)^2 + 2(y-5)^2 = 8$$

$$\frac{(x+1)^2}{8} + \frac{(y-5)^2}{4} = 1$$

$$\begin{cases} a^2 = 8 \to a = 2\sqrt{2} \\ b^2 = 4 \to b = 2 \end{cases}$$

$$c = \sqrt{a^2 - b^2} = \sqrt{8-4} = 2$$

Center: $C(-1,\ 5)$



Vertices: $V\left(-1\pm 2\sqrt{2},\ 5\right)$

Minor $(-1,\ 5\pm 2) \ \rightarrow\ M'(-1,\ 3)\quad M(-1,\ 7)$

Foci $(-1\pm 2,\ 5) \ \rightarrow\ F'(-3,\ 5)\quad F(1,\ 5)$

Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$25x^2 + 4y^2 - 250x - 16y + 541 = 0$

Solution

$$25\left(x^2 - 10x + \left(\frac{10}{2}\right)^2\right) + 4\left(y^2 - 4y + \left(\frac{4}{2}\right)^2\right) = -541 + 25(25) + 4(4)$$

$$25(x-5)^2 + 4(y-2)^2 = 100$$

$$\frac{(x-5)^2}{4} + \frac{(y-2)^2}{25} = 1$$

$$\begin{cases} a^2 = 25 \rightarrow a = 5 \\ b^2 = 4 \rightarrow b = 2 \end{cases}$$

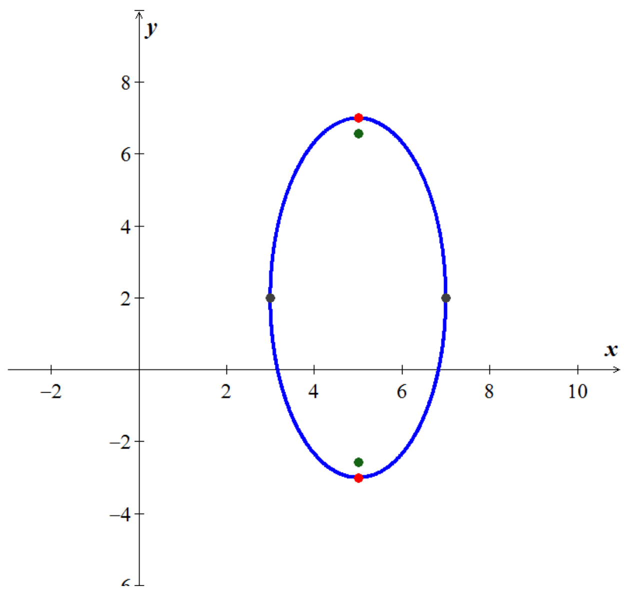
$$c = \sqrt{a^2 - b^2} = \sqrt{25-4} = \sqrt{21}$$

Center: $C(5,\ 2)$

Vertices: $(5,\ 2\pm 5) \ \rightarrow\ V'(5,\ -3)\quad V(5,\ 7)$

Minor $(5\pm 2,\ 2) \rightarrow\ M(3,\ 2)\quad M(7,\ 2)$

Foci $F\left(5,\ 2\pm\sqrt{21}\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $4x^2 + y^2 = 2y$

Solution

$$4x^2 + y^2 - 2y = 0$$

$$4x^2 + \left(y^2 - 2y + \left(\frac{2}{2}\right)^2\right) = (1)^2$$

$$4x^2 + (y-1)^2 = 1$$

$$\frac{x^2}{\frac{1}{4}} + \frac{(y-1)^2}{1} = 1$$

$$\begin{cases} a^2 = 1 \to a = 1 \\ b^2 = \frac{1}{4} \to b = \frac{1}{2} \end{cases}$$

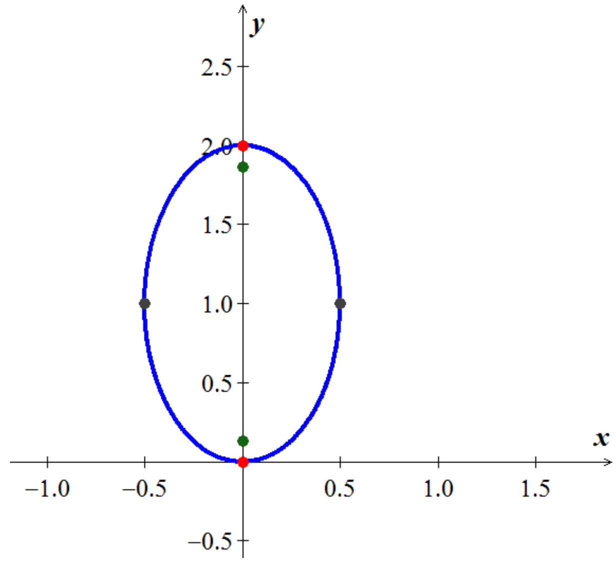
$$c = \sqrt{a^2 - b^2} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

Center: $C(0,\ 1)$

Vertices: $(0,\ 1 \pm 1) \to \quad V'(0,\ 0) \quad V(0,\ 2)$

Minor $\left(0 \pm \frac{1}{2},\ 1\right) \to \quad M'\left(-\frac{1}{2},\ 1\right) \quad M\left(\frac{1}{2},\ 1\right)$

Foci $F\left(0,\ 1 \pm \frac{\sqrt{3}}{2}\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse Sketch the graph: $2x^2 + 3y^2 - 8x + 6y + 5 = 0$

Solution

$$2x^2 - 8x + 3y^2 + 6y = -5$$

$$2\left(x^2 - 4x + \left(\frac{-4}{2}\right)^2\right) + 3\left(y^2 + 2y + \left(\frac{2}{2}\right)^2\right) = -5 + 2\left(\frac{-4}{2}\right)^2 + 3\left(\frac{2}{2}\right)^2$$

$$2(x-2)^2 + 3(y+1)^2 = -5 + 8 + 3$$

$$2(x-2)^2 + 3(y+1)^2 = 6$$

$$\frac{2(x-2)^2}{6} + \frac{3(y+1)^2}{6} = 1$$

$$\frac{(x-2)^2}{3} + \frac{(y+1)^2}{2} = 1$$

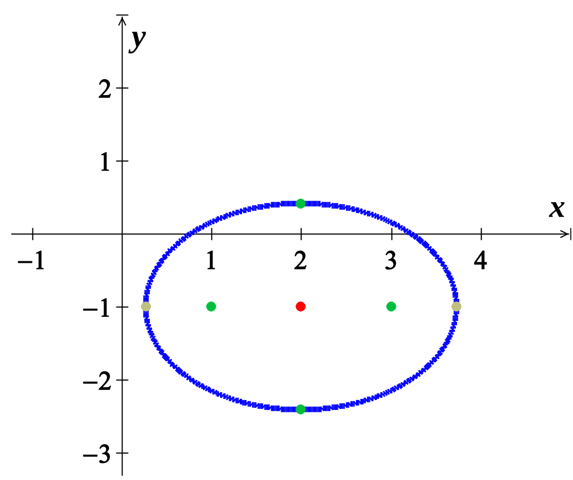
$$\begin{cases} a^2 = 3 \to a = \pm\sqrt{3} \\ b^2 = 2 \to b = \pm\sqrt{2} \\ \hline c = \sqrt{a^2 - b^2} = \sqrt{1} = 1 \end{cases}$$

Center: $(2,\ -1)$

Vertices: $V\left(2 \pm \sqrt{3},\ -1\right)$

Minor $M\left(2,\ -1 \pm \sqrt{2}\right)$

Foci $(2 \pm 1,\ -1) \to \quad F' = (1,\ -1) \quad F = (3,\ -1)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$4x^2+3y^2+8x-6y-5=0$

Solution

$$4x^2+8x+3y^2-6y=5$$

$$4\left(x^2+2x+\left(\frac{2}{2}\right)^2\right)+3\left(y^2-2y+\left(\frac{-2}{2}\right)^2\right)=5+4\left(\frac{2}{2}\right)^2+3\left(\frac{-2}{2}\right)^2$$

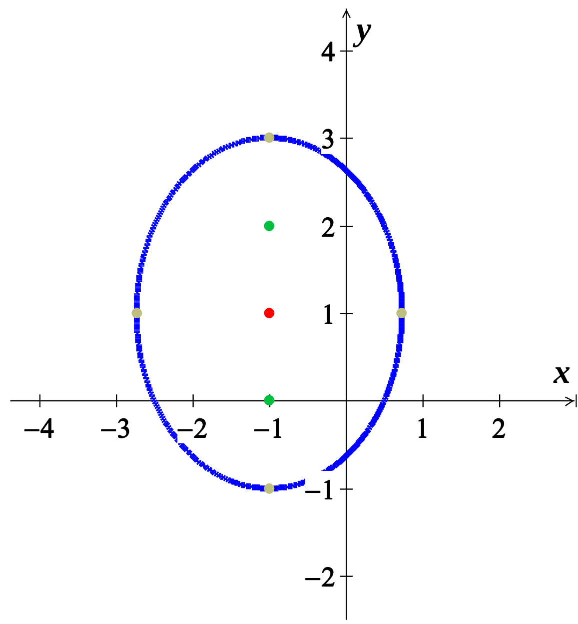
$$4(x+1)^2+3(y-1)^2=5+4+3$$

$$4(x+1)^2+3(y-1)^2=12$$

$$\frac{4(x+1)^2}{12}+\frac{3(y-1)^2}{12}=1$$

$$\frac{(x+1)^2}{3}+\frac{(y-1)^2}{4}=1$$

$$\begin{cases} a^2=4\to a=\pm 2 \\ b^2=3\to b=\pm\sqrt{3} \\ \hline c=\pm\sqrt{a^2-b^2}=\pm\sqrt{4-3}=\pm 1 \end{cases}$$



Center: $(-1,\ 1)$

Vertices: $(-1,\ 1\pm 2)\to\quad V'(-1,\ -1)\quad V(-1,\ 3)$

Minor $M\left(-1\pm\sqrt{3},\ 1\right)$

Foci $(-1,\ 1\pm 1)\to\quad F'=(-1,\ 0)\quad F=(-1,\ 2)$

Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$9x^2 + 4y^2 - 18x + 16y - 11 = 0$

Solution

$$9x^2 - 18x + 4y^2 + 16y = 11$$

$$9\left(x^2 - 2x + \left(\frac{-2}{2}\right)^2\right) + 4\left(y^2 + 4y + \left(\frac{4}{2}\right)^2\right) = 11 + 9\left(\frac{-2}{2}\right)^2 + 4\left(\frac{4}{2}\right)^2$$

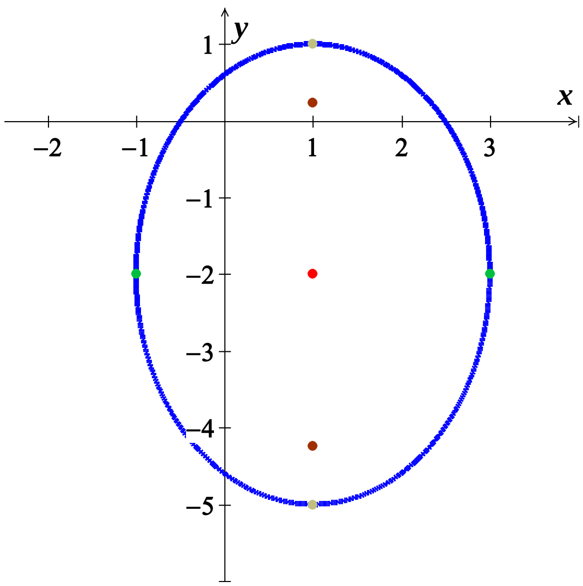
$$9(x-1)^2 + 4(y+2)^2 = 11 + 9 + 16$$

$$9(x-1)^2 + 4(y+2)^2 = 36$$

$$\frac{9(x-1)^2}{36} + \frac{4(y+2)^2}{36} = 1$$

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$$

$$\begin{cases} a^2 = 9 \to a = \pm 3 \\ b^2 = 4 \to b = \pm 2 \\ \hline c = \pm\sqrt{a^2 - b^2} = \pm\sqrt{9-4} = \pm\sqrt{5} \end{cases}$$



Center: $(1,\ -2)$

Vertices: $(1,\ -2 \pm 3) \to \quad V'(1,\ -5) \quad V(1,\ 1)$

Minor: $(1 \pm 2,\ -2) \to \quad M'(-1,\ -2) \quad M(3,\ -2)$

Foci $\left(1,\ -2 \pm \sqrt{5}\right)$

Exercise

Find an equation for an ellipse with: $x-intercepts:\ \pm 4;\quad foci\ (-2,\ 0)\ and\ (2,\ 0)$

Solution

The ellipse is centered at $(0,\ 0)$

Major axis: $a = 4$

Foci: $(\pm 2,\ 0) \quad \Rightarrow \quad c = 2$

$b^2 = a^2 - c^2 = 16 - 4 = 12$

The equation is: $\quad \dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$

Exercise

Find an equation for an ellipse with: *Endpoints of major axis at* $(6,\ 0)$ *and* $(-6,\ 0)$; $c = 4$

Solution

The ellipse is centered at $(0,\ 0)$ between the endpoint of the major axis

Major axis: $a = 6$

$b^2 = a^2 - c^2 = 36 - 16 = 20$

The equation is: $\dfrac{x^2}{36} + \dfrac{y^2}{20} = 1$

Exercise

Find an equation for an ellipse with: Center $(3, -2)$; $a = 5$; $c = 3$; major axis vertical

Solution

The ellipse is centered at $(3, -2)$

$b^2 = a^2 - c^2 = 25 - 9 = 16$

The equation is: $\boxed{\dfrac{(x-3)^2}{25} + \dfrac{(y+2)^2}{16} = 1}$

Exercise

Find an equation for an ellipse with: *major axis of length* 6; *foci* $(0,\ 2)$ *and* $(0,\ -2)$

Solution

The ellipse is centered between the foci at $(0,\ 0)$

Major axis is the vertical with $a = 3$

Foci: $(0,\ \pm 2) \quad \Rightarrow \quad c = 2$

$b^2 = a^2 - c^2 = 9 - 4 = 5$

The equation is: $\boxed{\dfrac{y^2}{9} + \dfrac{x^2}{5} = 1}$

Exercise

A patient's kidney stone is placed 12 *units* away from the source of the shock waves of a lithotripter. The lithotripter is based on an ellipse with a minor axis that measures 16 *units*. Find an equation of an ellipse that would satisfy this situation.

Solution

The patient and the emitter are 12 units apart $\Rightarrow$ these represent the foci of an ellipse, so $c = 6$.

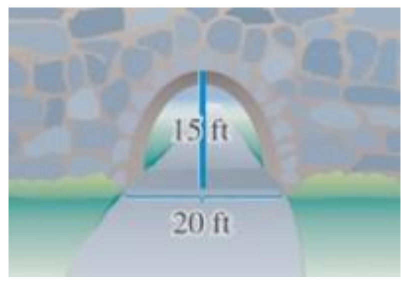
The minor axis: 16 units $\Rightarrow b = 8$.

$\therefore\ a^2 = b^2 + c^2 = 64 + 36 = 100$.

The equation is: $\dfrac{x^2}{100} + \dfrac{y^2}{64} = 1$

Exercise

A one-way road passes under an overpass in the form of half of an ellipse 15 *feet* high at the center and 20 *feet* wide. Assuming that a truck is 12 *feet* wide, what is the height of the tallest truck that can pass under the overpass?



Solution

Using a vertical major axis $\Rightarrow a = 15$.

The minor axis: 20 *ft*. $\Rightarrow b = 10$.

The equation is: $\dfrac{y^2}{225} + \dfrac{x^2}{100} = 1$

Assuming the truck drives through the middle, we want to find y when $x = 6$

$$\frac{y^2}{225} = 1 - \frac{6^2}{100} = \frac{64}{100}$$

$$\Rightarrow \quad y^2 = 225\frac{64}{100}$$

$$y = \sqrt{\frac{225(64)}{100}} \underline{= 12|}$$

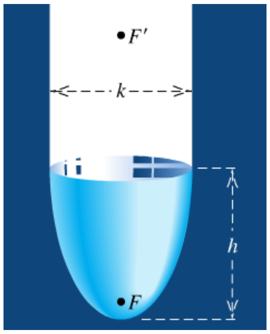
The truck must be just under 12 *feet* high to pass through.

Exercise

The basic shape of an elliptical reflector is a hemi-ellipsoid of height h and diameter k. Waves emitted from focus F will reflect off the surface into focus F'

a) Express the distance $d(V, F)$ and $d(V, F')$ in terms of h and k.

b) An elliptical reflector of height 17 *cm* is to be constructed so that waves emitted from F are reflected to a point F' that is 32 *cm* from V. Find the diameter of the reflector and the location of F.



Solution

Given: $b = \frac{k}{2}, \quad a = h$

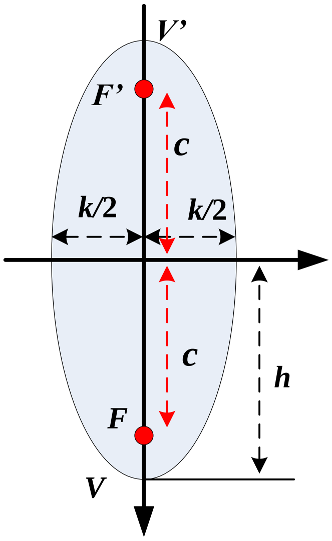
$$c^2 = a^2 - b^2 = h^2 - \left(\frac{k}{2}\right)^2$$

a) $d(V, F) = h - c$

$$= h - \sqrt{h^2 - \frac{1}{4}k^2}$$

$d(V, F') = h + c$

$$= h + \sqrt{h^2 - \frac{1}{4}k^2}$$



b) **Given**: $h = 17\ cm, \quad h + c = 32\ cm$

$c = 32 - h = 32 - 17 = 15\ cm$

$d(V, F) = h - c = 17 - 15 = 2$

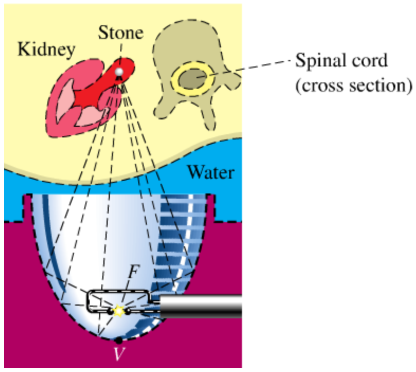
The location of F is $16\ cm;\ \ 2\ cm\ \ from\ V'$

Exercise

A lithotripter of height 15 *cm* and diameter 18 *cm* is to be constructed. High-energy underwater shock waves will be emitted from the focus F that is closest to the vertex V.

a) Find the distance from V to F.

b) How far from V (in the vertical direction) should a kidney stone located?



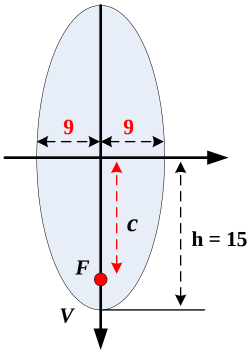
Solution

Given: $b = \frac{18}{2} = 9, \quad a = h = 15$

$$c = \sqrt{a^2 - b^2} = \sqrt{15^2 - 9^2} = 12\ cm$$

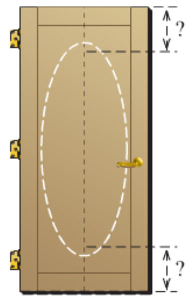
a) $d(V, F) = h - c = 15 - 12 = 3\ cm$

b) $h + c = 15 + 12 = 27\ cm$



Exercise

An Artist plans to create an elliptical design with major axis 60″ and minor axis 24″, centered on a door that measures 80″ by 36″. On a vertical line that dissects the door, approximately how far from each end of the door should the push-pins be inserted? How long should the string be?



Solution

Given: $b = \frac{24}{2} = 12''$, $a = \frac{60}{2} = 30''$

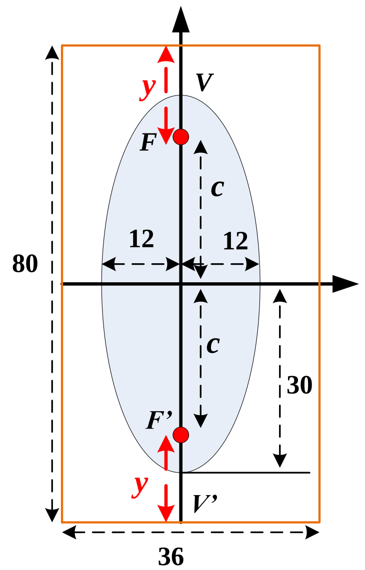
$c = \sqrt{a^2 - b^2} = \sqrt{30^2 - 12^2} = 27.5$

$2y + 2c = 80$

$y = \frac{80 - 2c}{2}$

$= \frac{80 - 2\sqrt{756}}{2}$

$\approx 12.5''$



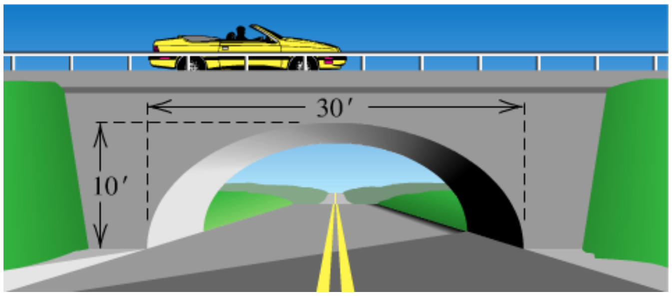
Therefore, the distance from each end of the door should the push-pins be inserted, is 12.5 *in.*

The string should be $= 30 + 30 = 60$ *in.*

Exercise

An arch of a bridge is semi-elliptical, with major axis horizontal. The base of the arch is 30 *feet*. across, and the highest part of the arch is 10 *feet*. above the horizontal roadway. Find the height of the arch 6 *feet*. from the center of the base.

Solution



Given: $b = 10'$, $a = \frac{30}{2} = 15'$

$c = \sqrt{a^2 - b^2} = \sqrt{15^2 - 10^2} = \sqrt{125} = 5\sqrt{5}$

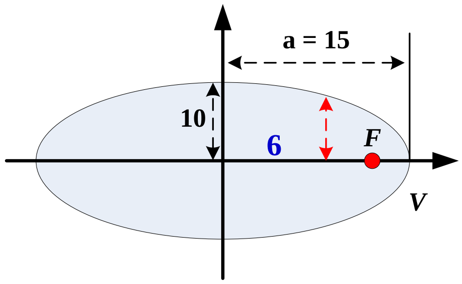
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow \frac{x^2}{225} + \frac{y^2}{100} = 1$

$\frac{y^2}{100} = 1 - \frac{6^2}{225}$

$y^2 = 100\left(1 - \frac{36}{225}\right)$

$y = \sqrt{100\left(1 - \frac{36}{225}\right)}$

$\sqrt{84} \approx 9.2$ *ft*

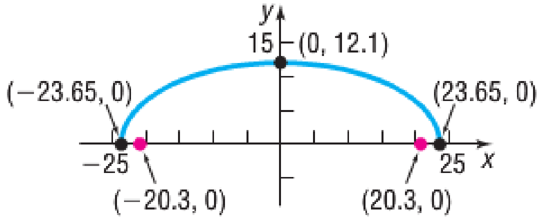


Exercise

The whispering gallery in the Museum of Science and Industry in Chicago is 47.3 *feet* long. The distance from the center of the room to the foci is 20.3 *feet*. Find an equation that describes the shape of the room. How high is the room at its center?

Solution

Set up a rectangular coordinate so that the center of the ellipse is at the origin and the major axis along the x-axis. The equation of the ellipse is



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Length of the room: 47.3 *ft*.

Distance from the center of the room to each vertex:

$$a=\frac{47.3}{2}=23.65$$

Distance from the center of the room to each focus is $c=20.3\ ft$

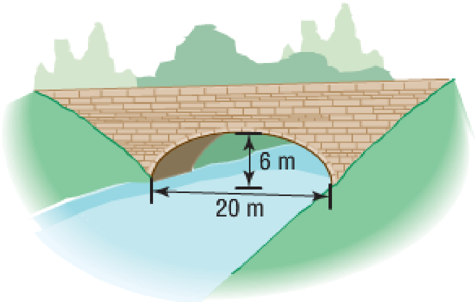
$$b^2=a^2-c^2=23.65^2-20.3^2=147.2325$$

Therefore, the equation is given: $\dfrac{x^2}{559.3225}+\dfrac{y^2}{147.2325}=1$

The Height of the room: $b=\sqrt{147.2325}\approx 12.1\ ft$

Exercise

An arch in the shape of the upper half of an ellipse is used to support a bridge that is to span a river 20 *meters* wide. The center of the arch is 6 *meters* above the center of the river. Write an equation for the ellipse in which the *x*-axis coincides with the water level and the *y*-axis passes through the center of the arch.



Solution

The center of the ellipse is $(0,\ 0)$. The length of the major axis is 20, so $a=10$.

The length of the half minor axis is 6, so $b=6$.

The ellipse is situated with its major axis on the *x*-axis.

The equation: $\dfrac{x^2}{10^2}+\dfrac{y^2}{6^2}=1 \quad\rightarrow\quad \dfrac{x^2}{100}+\dfrac{y^2}{36}=1$

Exercise

A bridge is built in the shape of a semielliptical arch. The bridge has a span of 120 *feet* and a maximum height of 25 *feet*. Choose a rectangular coordinate system and find the height of the arch at distances of 10, 30, and 50 *feet* from the center.

Solution

Since the bridge has a span of 120 feet, the length of the major axis is $120 = 2a \rightarrow a = 60$

The maximum height of the bridge is 25 feet, so $b = 25$.

The equation: $\frac{x^2}{60^2} + \frac{y^2}{25^2} = 1 \rightarrow \boxed{\frac{x^2}{3600} + \frac{y^2}{625} = 1}$

At distance 10 feet:

$$\frac{10^2}{3600} + \frac{y^2}{625} = 1 \rightarrow \frac{y^2}{625} = 1 - \frac{100}{3600}$$

$$y^2 = 625\left(1 - \frac{1}{36}\right)$$

$$y = \sqrt{625\left(\frac{35}{36}\right)}$$

The height from the center is $\boxed{y \approx 24.65\ ft}$

At distance **30** *feet*:

$$\frac{30^2}{3600} + \frac{y^2}{625} = 1 \rightarrow \frac{y^2}{625} = 1 - \frac{900}{3600}$$

$$y^2 = 625\left(1 - \frac{9}{36}\right)$$

$$y = \sqrt{625\left(\frac{27}{36}\right)}$$

The height from the center is $\boxed{y \approx 21.65\ ft}$

At distance **50** *feet*:

$$\frac{50^2}{3600} + \frac{y^2}{625} = 1 \rightarrow \frac{y^2}{625} = 1 - \frac{2500}{3600}$$

$$y^2 = 625\left(1 - \frac{25}{36}\right)$$

$$y = \sqrt{625\left(\frac{11}{36}\right)}$$

The height from the center is $\boxed{y \approx 13.82\ ft}$

Exercise

A bridge is built in the shape of a semielliptical arch. The bridge has a span of 100 *feet*. The height of the arch is 10 *feet*. Find the height of the arch at its center.

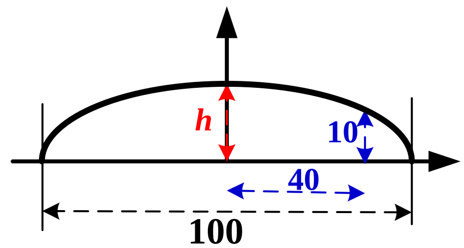
Solution

Since the bridge has a span of 100 *feet*.

Length of the major axis is $100 = 2a \rightarrow a = 50$

The maximum height of the bridge is 25 *feet*, so $b = 25$.

The equation: $\frac{x^2}{2500} + \frac{y^2}{h^2} = 1$



The height of the arch 40 *feet* from the center is 10 *feet*.

So $(40, 10)$ is a point on the ellipse.

$$\frac{40^2}{2500} + \frac{10^2}{h^2} = 1$$

$$\frac{10^2}{h^2} = 1 - \frac{1600}{2500}$$

$$\frac{100}{h^2} = 1 - \frac{16}{25}$$

$$\frac{100}{h^2} = \frac{9}{25}$$

$$h^2 = \frac{100 \cdot 25}{9}$$

$$h = \sqrt{\frac{100 \cdot 25}{9}}$$

$$h \approx 16.67$$

The height of the arch at its center is 16.67 *feet*.

Exercise

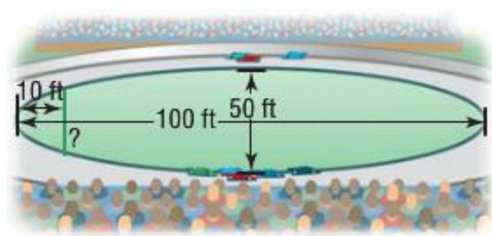
A racetrack is in the shape of an ellipse, 100 *feet* long and 50 *feet* wide. What is the width 10 *feet* from a vertex?

Solution

Length of the major axis is $100 = 2a \rightarrow a = 50$

The maximum height of the bridge is $50 = 2b \rightarrow b = 25$.

The equation: $\frac{x^2}{2500} + \frac{y^2}{625} = 1$



We need to find y at $x = 50 - 10 = 40$

$$\frac{40^2}{2500} + \frac{y^2}{625} = 1$$

$$\frac{y^2}{625} = 1 - \frac{1600}{2500}$$

$$y^2 = 625\frac{9}{25}$$

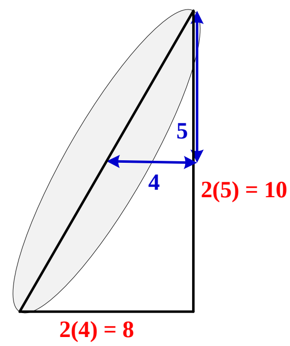
$$\boxed{y = 15\ ft}$$

The width of the ellipse at 10 *feet* from a vertex $x = 40$ is $\boxed{2 \times 15 = 30\ ft}$

Exercise

A homeowner is putting in a fireplace that has a 4-*inch* radius vent pipe. He needs to cut an elliptical hole in his roof to accommodate the pipe. If the pitch of his roof is $\frac{5}{4}$ (a rise of 5, run of 4) what are the dimensions of the hole?

Solution



The length of the major axis can be determined from the pitch by using Pythagorean Theorem:

$$a = \sqrt{4^2 + 5^2} = \sqrt{41}$$

The length of the major axis $2a \ \boxed{= 2\sqrt{41}\ in}$

The length of the minor axis:

$$2b = 2(4) = \boxed{= 8\ in}$$

Exercise

A football is in the shape of a ***prolate spheroid***, which is simply a solid obtained by rotating an ellipse about its major axis. An inflated NFL football averages 11.125 *inches* in length and 28.25 *inches* in center circumference. If the volume of a prolate spheroid is $\frac{4}{3}\pi ab^2$, how much air does the football contain? (Neglect material thickness)

Solution

The length of the football is $2a = 11.125 \quad \Rightarrow \quad a = 5.5625$

The center circumference is $28.25 = 2\pi b \quad \Rightarrow \quad b = \frac{28.25}{2\pi}$

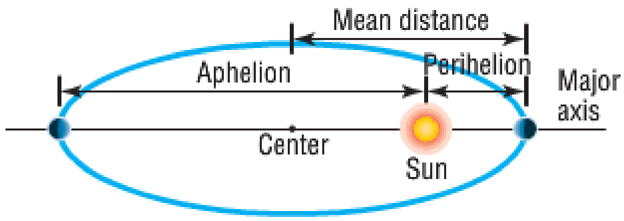
The volume is:

$$V = \frac{4}{3}\pi ab^2 = \frac{4}{3}\pi(5.5625)\left(\frac{28.25}{2\pi}\right)^2 \boxed{\approx 472\ in^3}$$

The football contains approximately 471 cubic inches of air.

Exercise

The fact that the orbit of a planet about the Sun is an ellipse with the Sun at one focus. The ***aphelion*** of a planet is its greatest distance from the Sun, and the ***perihelion*** is its shortest distance. The ***mean distance*** of a planet from the Sun is the length of the semi-major axis of the elliptical orbit.



a) The mean distance of Earth from the Sun is 93 million miles. If the aphelion of Earth is 94.5 million *miles*, what is the perihelion? Write an equation for the orbit of Earth around the Sun.

b) The mean distance of Mars from the Sun is 142 million *miles*. If the perihelion of Mars is 128.5 million *miles*, what is the aphelion? Write an equation for the orbit of Mars about the Sun.

c) The aphelion of Jupiter is 507 million *miles*. If the distance from the center of it elliptical orbit to the Sun is 23.2 million *miles*, what is the perihelion? What is the mean distance? Write an equation for the orbit of Jupiter around the Sun.

d) The perihelion of Pluto is 4551 million *miles*, and the distance from the center of its elliptical orbit to the Sun is 897.5 million *miles*. Find the aphelion of Pluto. What is the mean distance of Pluto from the Sun? Write an equation for the orbit of Pluto about the Sun.

Solution

a) The mean distance is 93 million *miles* $\Rightarrow \underline{a=93}$

The length of the major axis is 186 million

The perihelion is 186 – 94.5 = 91.5 million *miles*

Distance from the ellipse center to the sun is the focus: c = 93 – 91.5 = 1.5 million *miles*.

$b^2 = a^2 - c^2 = 93^2 - 1.5^2$

$b = \sqrt{93^2 - 1.5^2} \ \underline{= 92.99 \ \ million}$

Therefore: $a = 93 \times 10^6 \quad and \quad b = 92.99 \times 10^6$

The equation is given by: $\dfrac{x^2}{\left(93 \times 10^6\right)^2} + \dfrac{y^2}{\left(92.99 \times 10^6\right)^2} = 1$

Let x and y in millions miles: $\dfrac{x^2}{93^2} + \dfrac{y^2}{92.99^2} = 1 \qquad (in\ millions\ miles)$

The equation of the orbit is: $\boxed{\dfrac{x^2}{8649} + \dfrac{y^2}{8647.14} = 1}$

b) The mean distance is 142 million *miles* $\Rightarrow \underline{a=142}$

The length of the major axis is 284 million

The perihelion is 284 – 128.5 = 155.5 million *miles*

Distance from the ellipse center to the sun is the focus: c = 142 – 128.5 = 13.5 million *miles*.

$b^2 = a^2 - c^2 = 142^2 - 13.5^2 = 19{,}981.75$

$b=\sqrt{142^2-13.5^2}\ \underline{=141.36\ \ million|}$

Let x and y in millions miles: $\frac{x^2}{142^2}+\frac{y^2}{141.36^2}=1$ $\quad$ $(in\ millions\ miles)$

The equation of the orbit is: $\boxed{\frac{x^2}{20,164}+\frac{y^2}{19,981.75}=1}$

c) The mean distance is 507 – 23.2 = 483.8 million *miles* $\Rightarrow \underline{a=483.8|}$

The perihelion is 483.8 – 23.2 = 460.6 million *miles*

Distance from the ellipse center to the sun is the focus: *c* = 23.2 million *miles*.

$b^2=a^2-c^2=438.8^2-23.2^2=233,524.2$

$b=\sqrt{438.8^2-23.2^2}\ \underline{=483.2\ \ million|}$

Let x and y in millions miles: $\frac{x^2}{483.8^2}+\frac{y^2}{483.2^2}=1$ $\quad$ $(in\ millions\ miles)$

The equation of the orbit is: $\boxed{\frac{x^2}{234,062.44}+\frac{y^2}{233,524.2}=1}$

d) The mean distance is 4551 + 897.5 = 5448.5 million *miles* $\Rightarrow \underline{a=5448.5|}$

The aphelion is 5448.5 + 897.5 = 6346 million *miles*

Distance from the ellipse center to the sun is the focus: *c* = 897.5 million *miles*.

$b^2=a^2-c^2=5448.5^2-897.5^2=28,880,646$

$b=\sqrt{5448.5^2-897.5^2}\ \underline{=5374.07\ \ million|}$

Let x and y in millions miles: $\frac{x^2}{5448.5^2}+\frac{y^2}{5374.07^2}=1$ $\quad$ $(in\ millions\ miles)$

The equation of the orbit is: $\boxed{\frac{x^2}{29,686,152.25}+\frac{y^2}{28,880,646}=1}$

Solution   Section 4.8 – Hyperbolas

Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $\frac{x^2}{9}-\frac{y^2}{4}=1$

Solution

$$\rightarrow \begin{cases} a^2=9 \rightarrow a=3 \\ b^2=4 \rightarrow b=2 \end{cases}$$

$$\Rightarrow c=\sqrt{a^2+b^2}=\sqrt{9+4}=\sqrt{13}$$

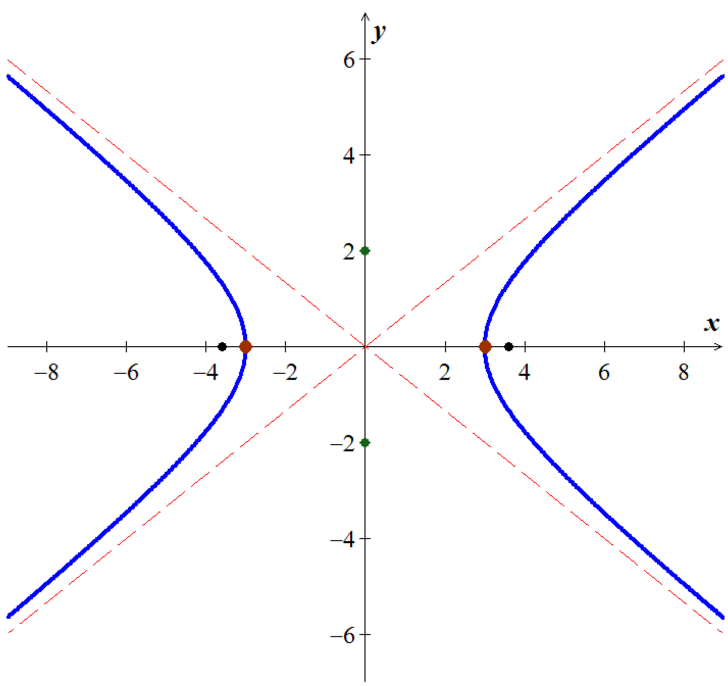
Center: $C=(0,\ 0)$

Vertices: $V=(\pm 3,\ 0)$

Endpoints: $W=(0,\ \pm 2)$

Foci: $F=\left(\pm\sqrt{13},\ 0\right)$

Equations of the ***asymptotes***: $\lfloor y=\pm\frac{b}{a}x \ \underline{=\pm\frac{2}{3}x}\,\rfloor$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $\frac{y^2}{9} - \frac{x^2}{4} = 1$

Solution

$$\rightarrow \begin{cases} a^2 = 9 \rightarrow a = 3 \\ b^2 = 4 \rightarrow b = 2 \end{cases}$$

$$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{9+4} = \sqrt{13}$$

Center: $C = (0,\ 0)$

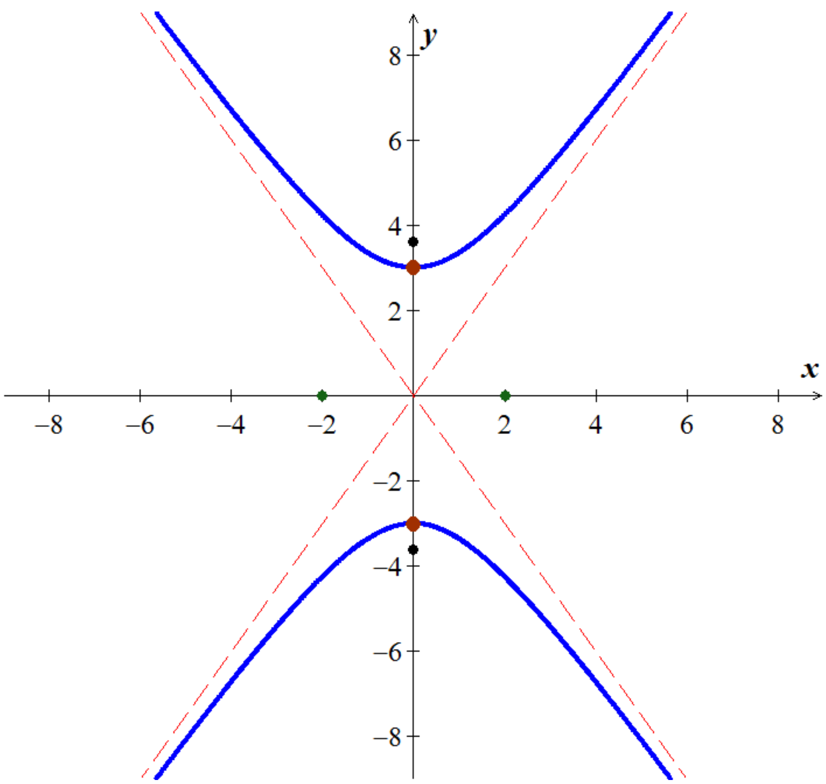
Vertices: $V = (0,\ \pm 3)$

Endpoints: $W = (\pm 2,\ 0)$

Foci: $F = \left(0,\ \pm\sqrt{13}\right)$

Equations of the ***asymptotes***:

$$y = \pm \frac{a}{b} x = \pm \frac{3}{2} x$$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $x^2 - \frac{y^2}{24} = 1$

Solution

$$\rightarrow \begin{cases} a^2 = 1 \rightarrow a = 1 \\ b^2 = 24 \rightarrow b = 2\sqrt{6} \end{cases}$$

$$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{1+24} = 5$$

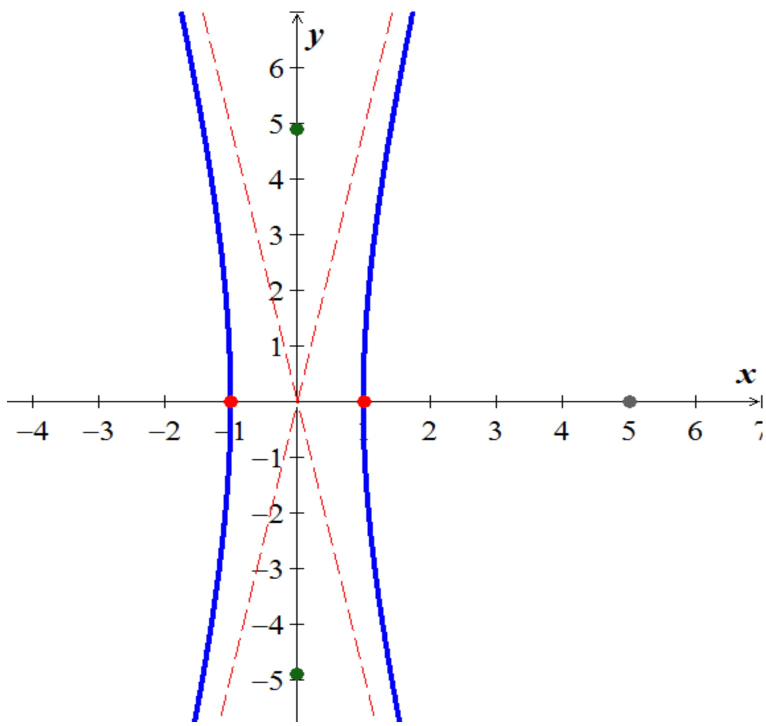
Center: $C = (0,\ 0)$

Vertices: $V = (\pm 1,\ 0)$

Endpoints: $W = \left(0,\ \pm 2\sqrt{6}\right)$

Foci: $F = (\pm 5,\ 0)$

Equations of the ***asymptotes***: $\underline{\lfloor y = \pm \frac{b}{a} x = \pm 4\sqrt{3}x \rfloor}$



Exercise

Find the ***center***, ***vertices***, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $y^2 - 4x^2 = 16$

Solution

$$\frac{y^2}{16} - \frac{x^2}{4} = 1$$

$$\to \begin{cases} a^2 = 16 \to a = 4 \\ b^2 = 4 \to b = 2 \end{cases}$$

$$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{16+4} = 2\sqrt{5}$$

Center: $C = (0,\ 0)$

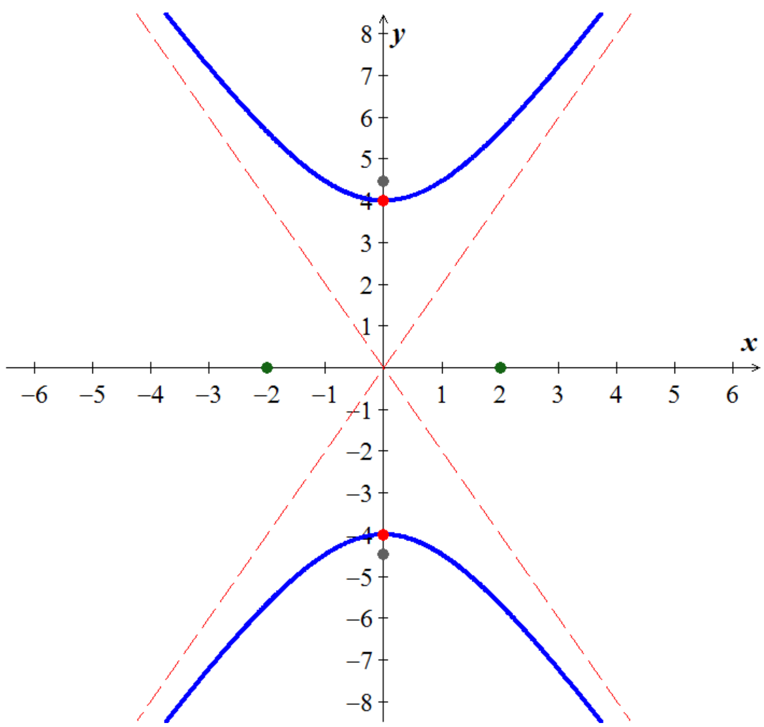
Vertices: $V = (0,\ \pm 4)$

Endpoints: $W = (\pm 2,\ 0)$

Foci: $F = \left(0,\ \pm 2\sqrt{5}\right)$

Equations of the ***asymptotes***:

$$\underline{\lfloor y = \pm \frac{a}{b} x = \pm \frac{4}{2} x = \pm 2x \rfloor}$$



Exercise

Find the ***center***, ***vertices***, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $16x^2 - 36y^2 = 1$

Solution

$$\frac{x^2}{\frac{1}{16}} - \frac{y^2}{\frac{1}{36}} = 1$$

$$\rightarrow \begin{cases} a^2 = \frac{1}{16} \rightarrow a = \frac{1}{4} \\ b^2 = \frac{1}{36} \rightarrow b = \frac{1}{6} \end{cases}$$

$$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{\frac{1}{16} + \frac{1}{36}} = \sqrt{\frac{9+4}{144}} = \pm\frac{\sqrt{13}}{12}$$

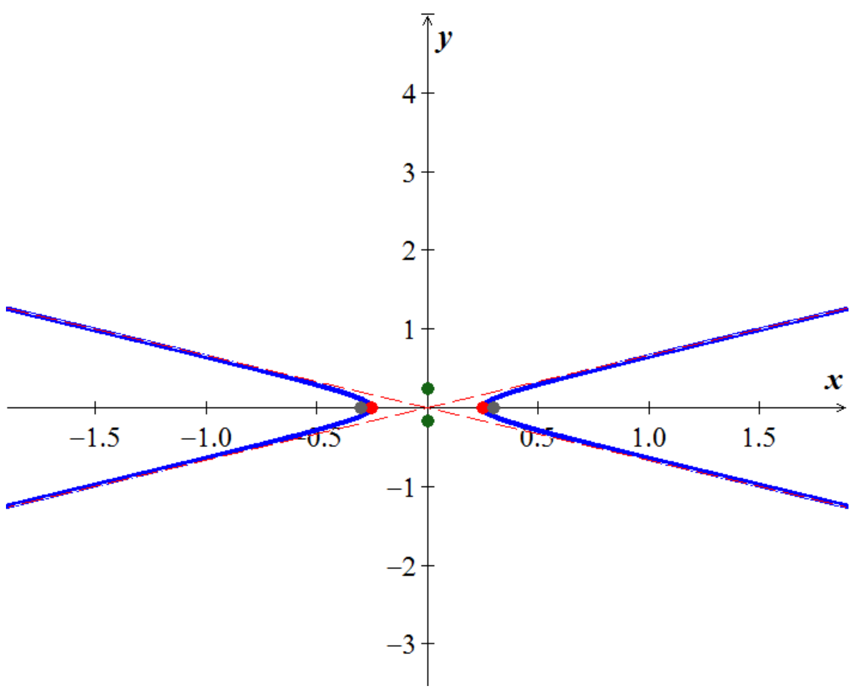
Center: $C = (0,\ 0)$

Vertices: $V = \left(\pm\frac{1}{4},\ 0\right)$

Endpoints: $W = \left(0,\ \pm\frac{1}{6}\right)$

Foci: $F = \left(\pm\frac{\sqrt{13}}{12},\ 0\right)$

*Equations of the **asymptotes***: $y = \pm\frac{b}{a}x = \pm\frac{\frac{1}{6}}{\frac{1}{4}}x = \pm\frac{4}{6}x \boxed{= \pm\frac{2}{3}x}$



Exercise

Find the ***center***, ***vertices***, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $\frac{(y+2)^2}{9}-\frac{(x+2)^2}{4}=1$

Solution

$$\to \begin{cases} a^2=9 \to a=3 \\ b^2=4 \to b=2 \end{cases}$$

$$\Rightarrow c=\sqrt{a^2+b^2}=\sqrt{9+4}=\pm\sqrt{13}$$

Center: $C=(-2,\ -2)$

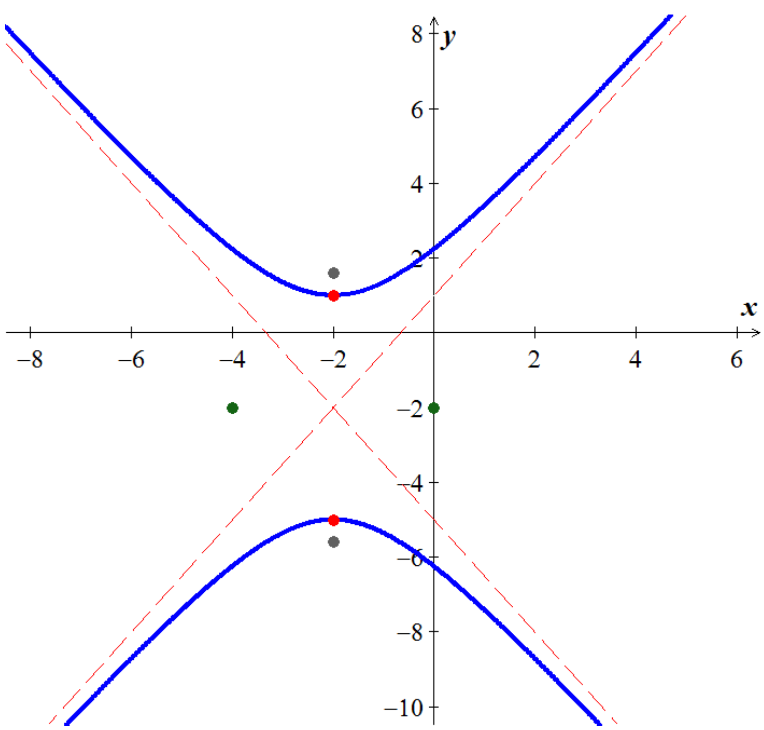
Vertices: $V=(-2,\ -2\pm 3)$

Endpoints: $W=(-2\pm 2,\ -2)$

Foci: $F=\left(-2,\ -2\pm\sqrt{13}\right)$

Equations of the ***asymptotes***: $y+2=\pm\frac{a}{b}(x+2)=\pm\frac{3}{2}(x+2)$

$y+2=-\frac{3}{2}(x+2)$	$y+2=\frac{3}{2}(x+2)$
$y+2=-\frac{3}{2}x-3$	$y+2=\frac{3}{2}x+3$
$y=-\frac{3}{2}x-5$	$y=\frac{3}{2}x+1$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$

Solution

$$\begin{cases} a^2 = 4 \to a = \pm 2 \\ b^2 = 9 \to b = \pm 3 \\ \hline c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{9+4} = \pm\sqrt{13} \end{cases}$$

Center: $C = (2,\ -3)$

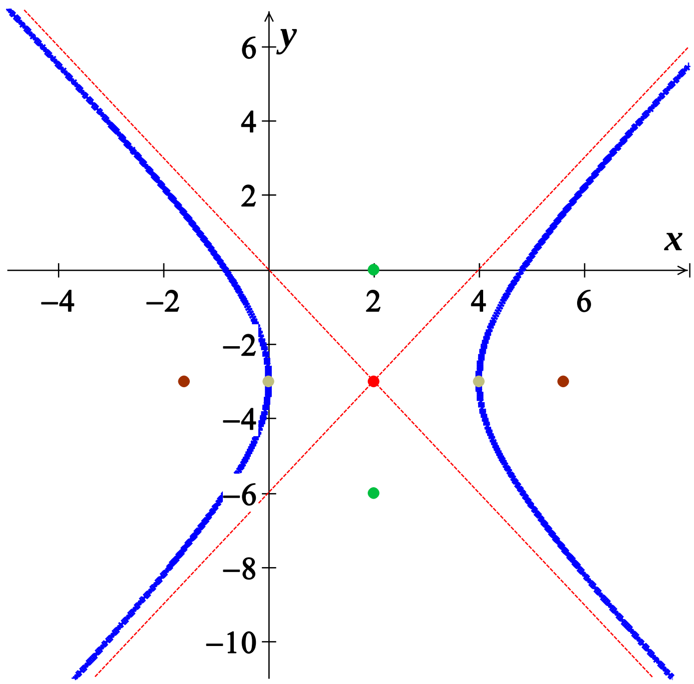
Vertices: $(2 \pm 2,\ -3) \rightarrow\ V' = (0,\ -3)\ \ V = (4,\ -3)$

Endpoints: $(2,\ -3 \pm 3) \rightarrow\ W'(2,\ -6)\ \ W = (2,\ 0)$

Foci: $F = \left(2 \pm \sqrt{13},\ -3\right)$

*Equations of the **asymptotes***: $y + 3 = \pm\frac{b}{a}(x-2) = \pm\frac{3}{2}(x-2)$

$y + 3 = -\frac{3}{2}(x-2)$	$y + 3 = \frac{3}{2}(x-2)$
$y + 3 = -\frac{3}{2}x + 3$	$y + 3 = \frac{3}{2}x - 3$
$y = -\frac{3}{2}x$	$y = \frac{3}{2}x - 6$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $(y-2)^2 - 4(x+2)^2 = 4$

Solution

$$\frac{(y-2)^2}{4} - \frac{4(x+2)^2}{4} = 1$$

$$\frac{(y-2)^2}{4} - \frac{(x+2)^2}{1} = 1$$

$$\begin{cases} a^2 = 4 \to a = \pm 2 \\ b^2 = 1 \to b = \pm 1 \\ c = \pm\sqrt{a^2+b^2} = \pm\sqrt{4+1} = \pm\sqrt{5} \end{cases}$$

Center: $C = (-2,\ 2)$

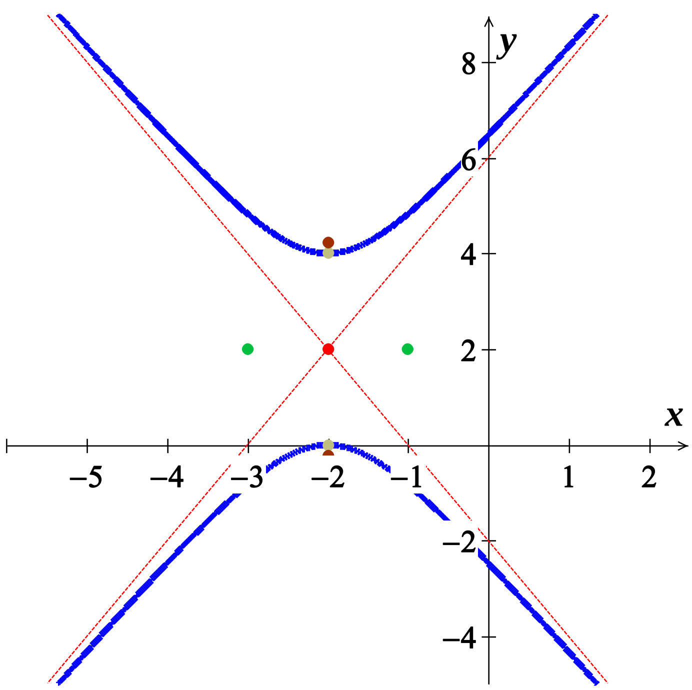
Vertices: $(-2,\ 2\pm 2) \to \quad V' = (-2,\ 0) \quad V = (-2,\ 4)$

Endpoints: $(-2\pm 1,\ 2) \to \quad W' = (-3,\ 2) \quad W = (-1,\ 2)$

Foci: $F = \left(-2,\ 2\pm\sqrt{5}\right)$

Equations of the ***asymptotes***: $y - 2 = \pm\frac{a}{b}(x+2) = \pm\frac{2}{1}(x+2)$

$$\begin{array}{l|l} y-2 = -2(x+2) & y-2 = 2(x+2) \\ y-2 = -2x-4 & y-2 = 2x+4 \\ y = -2x-2 & y = 2x+6 \end{array}$$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $(x+4)^2-9(y-3)^2=9$

Solution

$$\frac{(x+4)^2}{9}-\frac{9(y-3)^2}{9}=1$$

$$\frac{(x+4)^2}{9}-\frac{(y-3)^2}{1}=1$$

$$\begin{cases} a^2=9 \to a=\pm 3 \\ b^2=1 \to b=\pm 1 \\ \hline c=\pm\sqrt{a^2+b^2}=\pm\sqrt{9+1}=\pm\sqrt{10} \end{cases}$$

Center: $C=(-4,\ 3)$

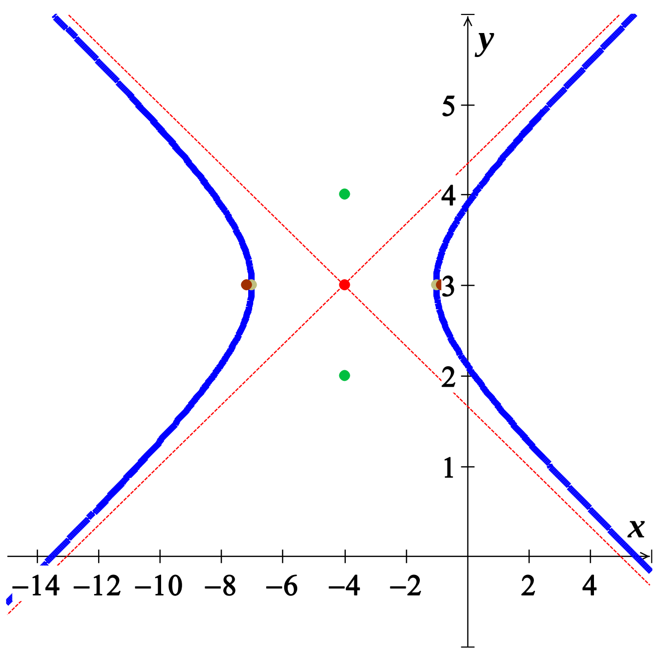
Vertices: $(-4\pm 3,\ 3)\to\ V'=(-7,\ 3)\quad V=(-1,\ 3)$

Endpoints: $(-4,\ 3\pm 1)\to\ W'(-4,\ 2)\quad W=(-4,\ 4)$

Foci: $F=\left(-4\pm\sqrt{10},\ 3\right)$

Equations of the ***asymptotes***: $y-3=\pm\frac{b}{a}(x+4)=\pm\frac{1}{3}(x+4)$

$y-3=-\frac{1}{3}(x+4)$	$y-3=\frac{1}{3}(x+4)$
$y-3=-\frac{1}{3}x-\frac{4}{3}$	$y-3=\frac{1}{3}x+\frac{4}{3}$
$y=-\frac{1}{3}x+\frac{5}{3}$	$y=\frac{1}{3}x+\frac{13}{3}$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $144x^2 - 25y^2 + 864x - 100y - 2404 = 0$

Solution

$$144\left(x^2 + 6x + \left(\frac{6}{2}\right)^2\right) - 25\left(y^2 + 4y + \left(\frac{4}{2}\right)^2\right) = 2404 + 144(4) - 25(4)$$

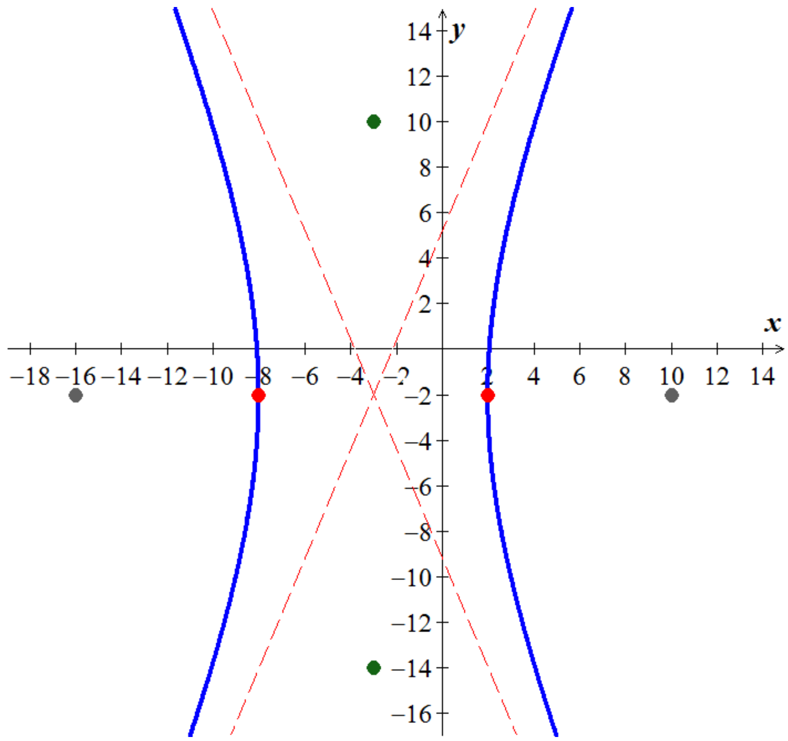
$$144(x+3)^2 - 25(y+2)^2 = 3600$$

$$\frac{(x+3)^2}{25} - \frac{(y+2)^2}{144} = 1$$

$$\rightarrow \begin{cases} a^2 = 25 \rightarrow a = 5 \\ b^2 = 144 \rightarrow b = 12 \end{cases}$$

$$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{25+144}$$

$$\underline{= \pm 13|}$$



Center: $C = (-3, \ -2)$

Vertices: $V = (-3 \pm 5, \ -2)$

Endpoints: $W = (-3, \ -2 \pm 12)$

Foci: $F = (-3 \pm 13, \ -2)$

Equations of the ***asymptotes***:

$$y + 2 = \pm \frac{b}{a}(x+3) = \pm \frac{12}{5}(x+3)$$

$y + 2 = -\frac{12}{5}(x+3)$	$y + 2 = \frac{12}{5}(x+3)$
$y + 2 = -\frac{12}{5}x - \frac{36}{5}$	$y + 2 = \frac{12}{5}x + \frac{36}{5}$
$y = -\frac{12}{5}x - \frac{46}{5}$	$y = \frac{12}{5}x + \frac{26}{5}$

Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $4y^2 - x^2 + 40y - 4x + 60 = 0$

Solution

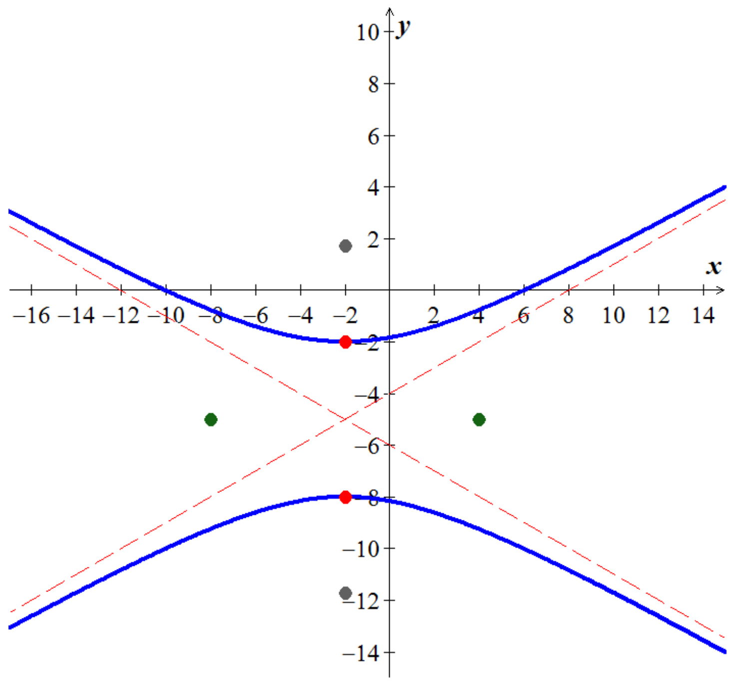
$$4\left(y^2 + 10y + \left(\frac{10}{2}\right)^2\right) - \left(x^2 + 4x + \left(\frac{4}{2}\right)^2\right) = -60 + 4(25) - (4)$$

$$4(y+5)^2 - (x+2)^2 = 36$$

$$\frac{(y+5)^2}{9} - \frac{(x+2)^2}{36} = 1$$

$$\to \begin{cases} a^2 = 9 \to a = 3 \\ b^2 = 36 \to b = 6 \end{cases}$$

$$\Rightarrow c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{9+36}$$
$$= \pm\sqrt{45}$$
$$= \pm 3\sqrt{5}$$



Center: $C = (-5,\ -2)$

Vertices: $V = (-2,\ -5 \pm 3)$

Endpoints: $W = (-2 \pm 6,\ -5)$

Foci: $F = \left(-2,\ -5 \pm 3\sqrt{5}\right)$

*Equations of the **asymptotes***:

$$y + 5 = \pm\frac{a}{b}(x+2) = \pm\frac{3}{6}(x+2) = \pm\frac{1}{2}(x+2)$$

$y + 5 = -\frac{1}{2}(x+2)$	$y + 5 = \frac{1}{2}(x+2)$
$y + 5 = -\frac{1}{2}x - 1$	$y + 5 = \frac{1}{2}x + 1$
$y = -\frac{1}{2}x - 6$	$y = \frac{1}{2}x - 4$

Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $4x^2 - 16x - 9y^2 + 36y = -16$

Solution

$$4\left(x^2 - 4x\right) - 9\left(y^2 - 4y\right) = -16$$

$$4\left(x^2 - 4x + 2^2\right) - 9\left(y^2 - 4y + 2^2\right) = -16 + 4\left(2^2\right) - 9\left(2^2\right)$$

$$4(x-2)^2 - 9(y-2)^2 = -16 + 16 - 36$$

$$4(x-2)^2 - 9(y-2)^2 = -36$$

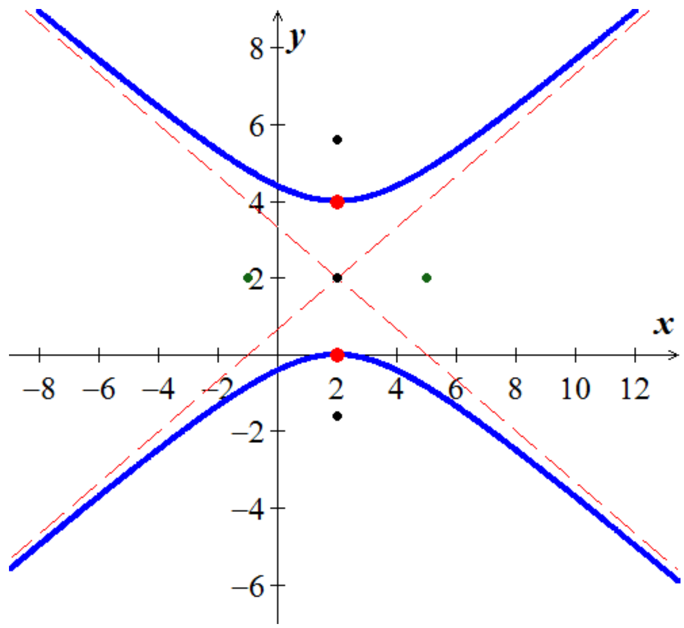
$$\frac{4(x-2)^2}{-36} - \frac{9(y-2)^2}{-36} = 1$$

$$-\frac{4(x-2)^2}{36} + \frac{9(y-2)^2}{36} = 1$$

$$\frac{9(y-2)^2}{36} - \frac{4(x-2)^2}{36} = 1$$

$$\frac{(y-2)^2}{\frac{36}{9}} - \frac{(x-2)^2}{\frac{36}{4}} = 1$$

$$\frac{(y-2)^2}{4} - \frac{(x-2)^2}{9} = 1$$



$$\rightarrow \begin{cases} a^2 = 4 \rightarrow a = \pm 2 \\ b^2 = 9 \rightarrow b = \pm 3 \end{cases}$$

$$\Rightarrow c = \mp\sqrt{a^2 + b^2} = \pm\sqrt{9+4} = \pm\sqrt{13}$$

Center: $(2,\ 2)$

The ***endpoints***: $(2 \pm 3,\ -2) \Rightarrow (-1,\ 2) \quad (5,\ 2)$

The ***vertices***: $(2,\ 2 \pm 2) \Rightarrow (2,\ 0) \quad (2,\ 4)$

The ***foci*** are $\left(2,\ 2 \pm \sqrt{13}\right)$

The equations of the ***asymptotes*** are: $y - 2 = \pm\frac{a}{b}(x-2) \ \Rightarrow \underline{y = \pm\frac{2}{3}(x-2) + 2}$

Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $2x^2 - y^2 + 4x + 4y = 4$

Solution

$$2\left(x^2 + 2x + \left(\frac{2}{2}\right)^2\right) - \left(y^2 - 4y + \left(\frac{-4}{2}\right)^2\right) = 4 + 2\left(\frac{2}{2}\right)^2 + (-1)\left(\frac{-4}{2}\right)^2$$

$$2(x+1)^2 - (y-2)^2 = 4 + 2 - 4$$

$$2(x+1)^2 - (y-2)^2 = 2$$

$$\frac{(x+1)^2}{1} - \frac{(y-2)^2}{2} = 1$$

$$\begin{cases} a^2 = 1 \to a = \pm 1 \\ b^2 = 2 \to b = \pm\sqrt{2} \\ c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{1+2} = \pm\sqrt{3} \end{cases}$$

Center: $C = (-1,\ 2)$

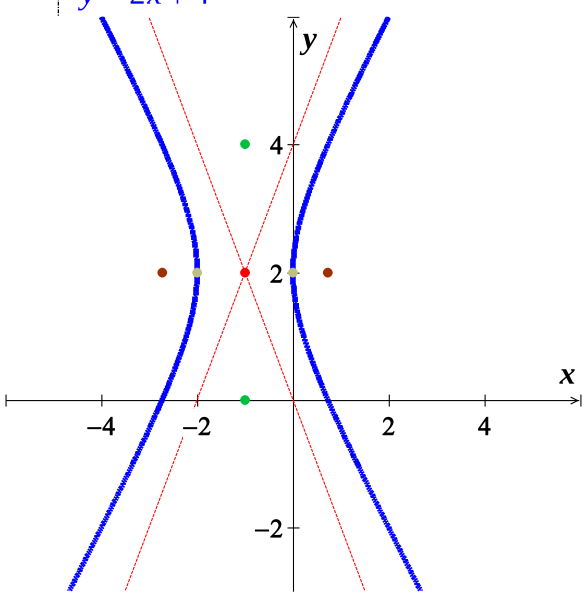
Vertices: $(-1 \pm 1,\ 2) \to \quad V' = (-2,\ 2) \quad V = (0,\ 2)$

Endpoints: $(-1,\ 2 \pm 2) \to \quad W' = (-1,\ 0) \quad W = (-1,\ 4)$

Foci: $F = \left(-1 \pm \sqrt{3},\ 2\right)$

Equations of the ***asymptotes***: $y - 2 = \pm\frac{b}{a}(x+1) = \pm\frac{2}{1}(x+1)$

$y - 2 = -2(x+1)$	$y - 2 = 2(x+1)$
$y - 2 = -2x - 2$	$y - 2 = 2x + 2$
$y = -2x$	$y = 2x + 4$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $2y^2 - x^2 + 2x + 8y + 3 = 0$

Solution

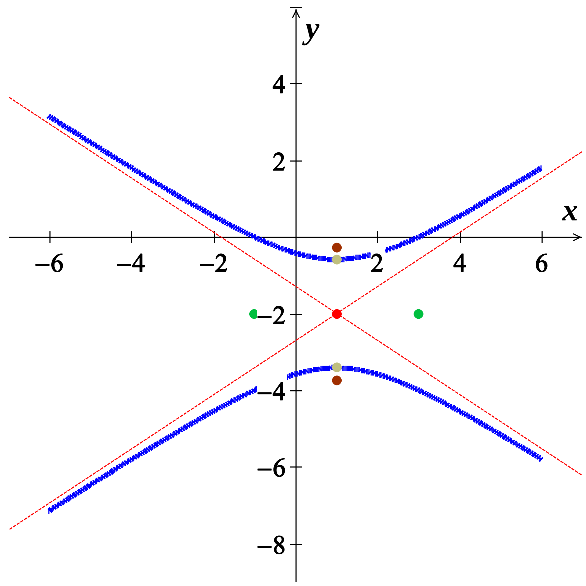
$$2y^2 + 8y - x^2 + 2x = -3$$

$$2\left(y^2 + 4y + \left(\frac{4}{2}\right)^2\right) - \left(x^2 - 2x + \left(\frac{-2}{2}\right)^2\right) = -3 + 2\left(\frac{4}{2}\right)^2 + (-1)\left(\frac{-2}{2}\right)^2$$

$$2(y+2)^2 - (x-1)^2 = -3 + 8 - 1$$

$$2(y+2)^2 - (x-1)^2 = 4$$

$$\frac{(y+2)^2}{2} - \frac{(x-1)^2}{4} = 1$$



$$\begin{cases} a^2 = 2 \to a = \pm\sqrt{2} \\ b^2 = 4 \to b = \pm 2 \\ c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{2+4} = \pm\sqrt{6} \end{cases}$$

Center: $C = (1,\ -2)$

Vertices: $V = \left(1,\ -2 \pm \sqrt{2}\right)$

Endpoints: $(1 \pm 2,\ -2) \to \quad W' = (-1,\ -2) \quad W = (3,\ -2)$

Foci: $F = \left(1,\ -2 \pm \sqrt{3}\right)$

Equations of the ***asymptotes***: $y + 2 = \pm\frac{a}{b}(x-1) = \pm\frac{\sqrt{2}}{2}(x-1)$

$y + 2 = -\frac{\sqrt{2}}{2}(x-1)$	$y + 2 = \frac{\sqrt{2}}{2}(x-1)$
$y = -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2} - 2$	$y = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2} - 2$

Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $2y^2-4x^2-16x-2y-19=0$

Solution

$$2y^2-2y-4x^2-16x=19$$

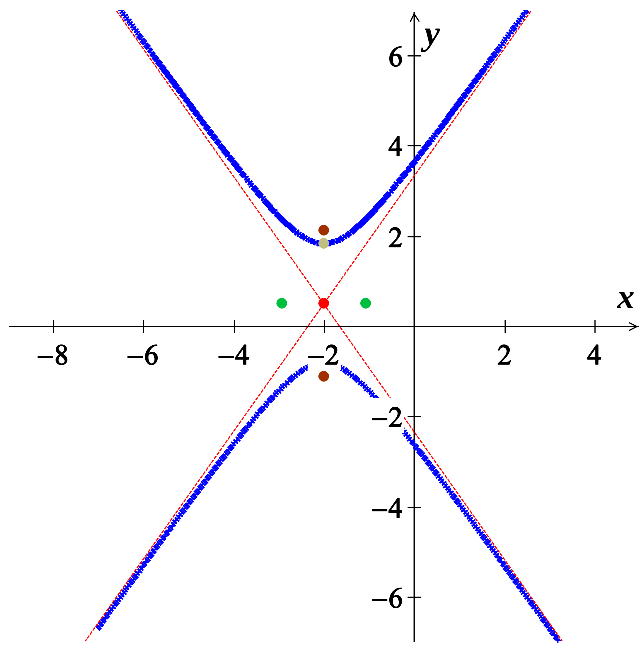
$$2\left(y^2-y+\left(\frac{-1}{2}\right)^2\right)-4\left(x^2+4x+\left(\frac{4}{2}\right)^2\right)=19+2\left(\frac{-1}{2}\right)^2-4\left(\frac{4}{2}\right)^2$$

$$2\left(y-\frac{1}{2}\right)^2-4(x+2)^2=19+\frac{1}{2}-16$$

$$2\left(y-\frac{1}{2}\right)^2-4(x+2)^2=\frac{7}{2}$$

$$\frac{2\left(y-\frac{1}{2}\right)^2}{\frac{7}{2}}-\frac{4(x+2)^2}{\frac{7}{2}}=1$$

$$\frac{\left(y-\frac{1}{2}\right)^2}{\frac{7}{4}}-\frac{4(x+2)^2}{\frac{7}{8}}=1$$



$$\begin{cases} a^2=\frac{7}{4}\to a=\pm\frac{\sqrt{7}}{2} \\ b^2=\frac{7}{8}\to b=\pm\frac{\sqrt{7}}{2\sqrt{2}}=\pm\frac{\sqrt{14}}{4} \\ c=\pm\sqrt{a^2+b^2}=\pm\sqrt{\frac{7}{4}+\frac{7}{8}}=\pm\sqrt{\frac{21}{8}} \end{cases}$$

Center: $C=\left(-2,\ \frac{1}{2}\right)$

Vertices: $V=\left(1,\ -2\pm\frac{\sqrt{7}}{2}\right)$

Endpoints: $W\left(-2\pm\frac{\sqrt{14}}{4},\ \frac{1}{2}\right)$

Foci: $F=\left(-2,\ \frac{1}{2}\pm\sqrt{\frac{21}{8}}\right)$

Equations of the ***asymptotes***: $y-\frac{1}{2}=\pm\frac{a}{b}(x+2)=\pm\frac{\frac{\sqrt{7}}{2}}{\frac{\sqrt{7}}{2\sqrt{2}}}(x+2)=\pm\sqrt{2}(x+2)$

$y-\frac{1}{2}=-\sqrt{2}(x+2)$	$y-\frac{1}{2}=\sqrt{2}(x+2)$
$y=-\sqrt{2}x-2\sqrt{2}+\frac{1}{2}$	$y=\sqrt{2}x+2\sqrt{2}+\frac{1}{2}$

Exercise

Suppose a hyperbola has center at the origin, foci at $F'(-c,\ 0)$ and $F(c,\ 0)$, and equation $d(P,\ F') - d(P,\ F) = 2a$. Let $b^2 = c^2 - a^2$, and show that an equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Solution

$$d(P,\ F') - d(P,\ F) = 2a \qquad\qquad d = \sqrt{\left(x - x_0\right)^2 + \left(y - y_0\right)^2}$$

$$\sqrt{(x+c)^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2} = 2a$$

$$\sqrt{x^2 + 2cx + c^2 + y^2} - \sqrt{x^2 - 2cx + c^2 + y^2} = 2a$$

$$\sqrt{x^2 + 2cx + c^2 + y^2} = 2a + \sqrt{x^2 - 2cx + c^2 + y^2}$$

$$\left(\sqrt{x^2 + 2cx + c^2 + y^2}\right)^2 = \left(2a + \sqrt{x^2 - 2cx + c^2 + y^2}\right)^2$$

Square both sides

$$x^2 + 2cx + c^2 + y^2 = 4a^2 + x^2 - 2cx + c^2 + y^2 + 4a\sqrt{x^2 - 2cx + c^2 + y^2}$$

$$4cx - 4a^2 = 4a\sqrt{x^2 - 2cx + c^2 + y^2}$$

Divide by 4

$$cx - a^2 = a\sqrt{x^2 - 2cx + c^2 + y^2}$$

$$\left(cx - a^2\right)^2 = \left(a\sqrt{x^2 - 2cx + c^2 + y^2}\right)^2$$

Square both sides

$$c^2x^2 - 2a^2cx + a^4 = a^2\left(x^2 - 2cx + c^2 + y^2\right)$$

$$c^2x^2 - 2a^2cx + a^4 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$$

$$c^2x^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4$$

$$\left(c^2 - a^2\right)x^2 - a^2y^2 = a^2\left(c^2 - a^2\right)$$

$$\frac{\left(c^2 - a^2\right)x^2}{a^2\left(c^2 - a^2\right)} - \frac{a^2y^2}{a^2\left(c^2 - a^2\right)} = 1$$

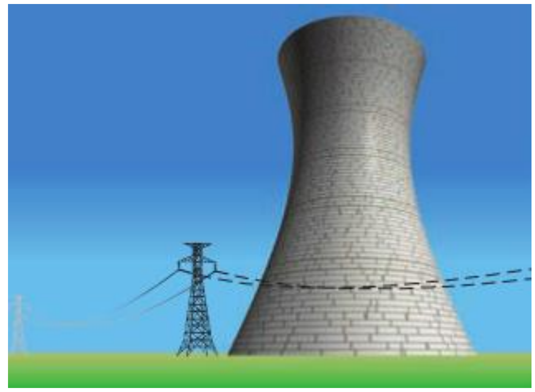
$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1 \qquad\qquad b^2 = c^2 - a^2$$

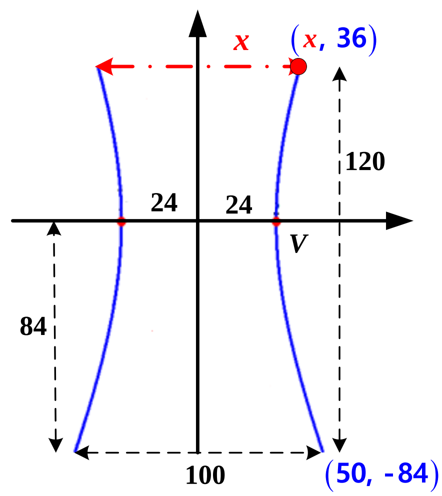
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Exercise

A cooling tower is a hydraulic structure. Suppose its base diameter is 100 *meters* and its smallest diameter of 48 *meters* occurs 84 *meters* from the base. If the tower is 120 *meters* high approximate its diameter at the top.

Solution





Given: $a = \frac{48}{2} = 24$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \rightarrow \quad \frac{x^2}{24^2} - \frac{y^2}{b^2} = 1$$

At the point $(50, \ -84)$:

$$\frac{50^2}{24^2} - \frac{(-84)^2}{b^2} = 1$$

$$\frac{50^2}{24^2} - 1 = \frac{84^2}{b^2}$$

$$\frac{50^2 - 24^2}{24^2} = \frac{84^2}{b^2}$$

$$b^2 = \frac{84^2 \cdot 24^2}{50^2 - 24^2} = 2112.4$$

$$\Rightarrow \quad \frac{x^2}{576} - \frac{y^2}{2112.4} = 1$$

At the point $(x, \ 36)$:

$$\frac{x^2}{576} - \frac{36^2}{2112.4} = 1$$

$$\frac{x^2}{576} = 1 + \frac{1296}{2112.4}$$

$$\frac{x^2}{576} = 1.61$$

$$x^2 = 929.45$$

$$x = \sqrt{929.45} \approx 30.49$$

The diameter at the top: $= 2x = \underline{60.97 \ m.}$

Exercise

An airplane is flyting along the hyperbolic path. If an equation of the path is $2y^2 - x^2 = 8$, determine how close the airplane comes to town located at (3, 0). (Hunt: Let *S* denote the square of the distance from a point (*x*, *y*) on the path to (3, 0), and find the minimum value of *S*.)

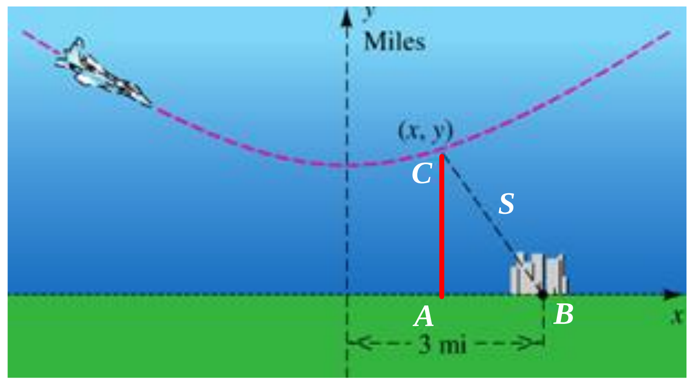
Solution

$$2y^2 - x^2 = 8 \quad \rightarrow \quad y^2 = \frac{1}{2}x^2 + 4$$

$$S^2 = (3-x)^2 + y^2$$

$$= 9 - 6x + x^2 + \frac{1}{2}x^2 + 4$$

$$= \frac{3}{2}x^2 - 6x + 13$$



The vertex point of S^2

$$x = -\frac{b}{2a} = -\frac{-6}{2\left(\frac{3}{2}\right)} = 2$$

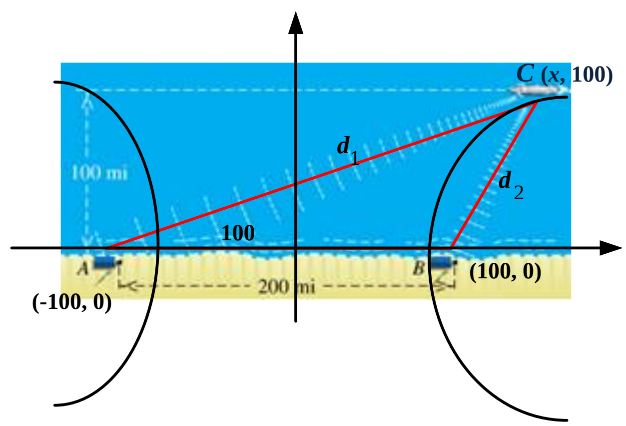
$$S^2 = \frac{3}{2}(2)^2 - 6(2) + 13 = 7$$

Therefore the close the town to the airplane is $\underline{S = \sqrt{7}\ miles}$

Exercise

A ship is traveling a course that is 100 *miles* from, and parallel tom a straight shoreline. The ship sends out a distress signal that is received by two Coast Guard stations *A* and *B*, located 200 *miles* apart. By measuring the difference in signal reception times, it is determined that the ship is 160 *miles* closer to *B* than to *A*. Where is the ship?

Solution



Given: $c = 100 \quad and \quad BC = AC - 160$

$$d_1 - d_2 = 160 = 2a \quad \rightarrow \quad a = 80$$

$$b^2 = c^2 - a^2 = 100^2 - 80^2 = 3600$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \rightarrow \quad \frac{x^2}{6400} - \frac{y^2}{3600} = 1$$

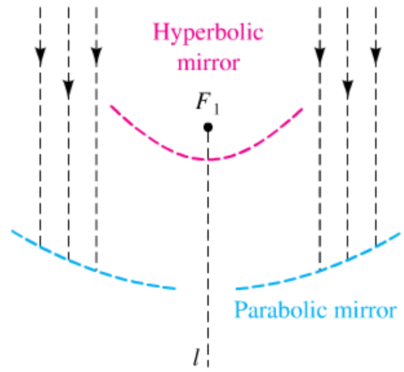
At point C(*x*, 100): $\frac{x^2}{6400} - \frac{100^2}{3600} = 1$

$$\frac{x^2}{6400} = 1 + \frac{100^2}{3600} \quad \Rightarrow \quad x^2 = 6400\left(1 + \frac{100^2}{3600}\right) \rightarrow x = 80\sqrt{\frac{13600}{3600}} = \frac{80}{3}\sqrt{34}$$

The ship position is $\left(\frac{80}{3}\sqrt{34},\ 100\right) = \underline{(155.5,\ 100)}$

Exercise

The Cassegrain telescope design (dating back to 1672) makes use of the reflective properties of both the parabola and the hyperbola. The figure shows a (***split***) parabolic mirror, with one focus at F_1 and axis along the line l, and a hyperbolic mirror, with one focus also at F_1 and transverse axis along l. Where do incoming light waves parallel to the common axis finally collect?



Solution

$\textit{Exterior focus of hyperbolic mirror (below parabolic mirror)}$

Exercise

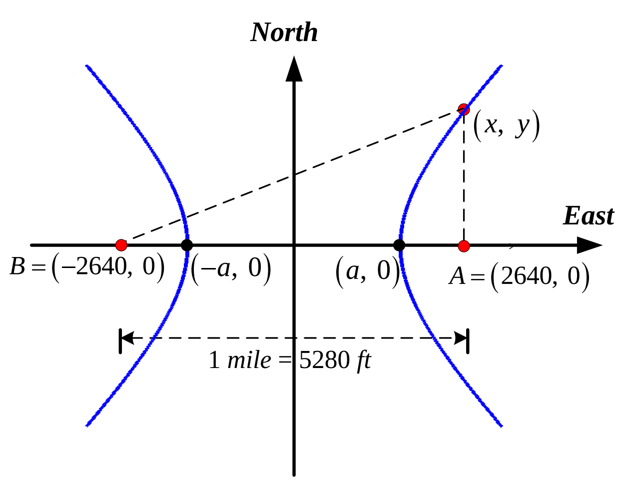
Suppose that two people standing 1 *mile* apart both see a flash of lightning. After a period of time, the person standing at point *A* hears the thunder. One second later, the person standing at point *B* hears the thunder. If the person at *B* is due west of the person at *A* and the lightning strike is known to occur due north of the person standing at point *A*, where did the lightning strike occur?
(Sound travels at 1100 $ft\,/\,sec$ and $1\ mile = 5280\ ft$)

Solution

Person ***A*** is 1100 feet closer to the lightning strike than the person at point ***B***.

Distance from (x, y) to ***B*** ***minus*** distance from (x, y) to ***A*** is 1100.

The point (x, y) lies on a hyperbola whose foci are at ***A*** and ***B***.



$2a = 1100 \quad \Rightarrow \quad \boxed{a = 550}$

$2c = 5280 \quad \Rightarrow \quad \boxed{c = 2640}$

$b^2 = c^2 - a^2 = 2640^2 - 550^2 = 6,667,100$

An equation of the hyperbola: $\quad \dfrac{x^2}{550^2} - \dfrac{y^2}{6,667,100} = 1 \qquad\qquad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

At point $A = (2640,\ 0)$, let $x = 2640$, and solve for y at that x value:

$$\frac{2640^2}{550^2} - \frac{y^2}{6,667,100} = 1$$

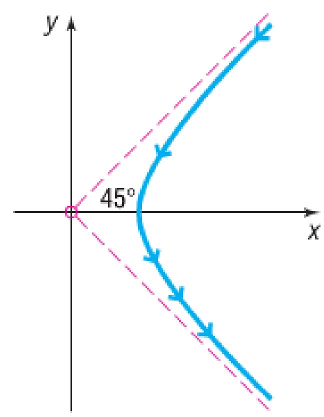
$$y^2 = 6,667,100\left(\frac{2640^2}{550^2} - 1\right)$$

$$y = \sqrt{6,667,100\left(\frac{2640^2}{550^2} - 1\right)} \ \boxed{= 12,122}$$

he lightning strike occurred 12,122 *ft*. north of the person standing at point *A*.

Exercise

Ernest Rutherford published a paper that he described the motion of alpha particles as they are shot at a piece of gold foil 0.00004 *cm* thick. Before conducting this experiment, Rutherford expected that the alpha particles would shoot through the foil just as a bullet would shoot through the foil just as a bullet would shoot through snow. Instead, a small fraction of the alpha particles bounced off the foil. This led to the conclusion that the nucleus of an atom is dense, while the remainder of the atom is sparse. Only the density of the nucleus could cause the alpha particles to deviate from their path. The figure shows a diagram from Rutherford's paper that indicates that the deflected alpha particles follow the path of one branch of a hyperbola.



a) Find an equation of the asymptotes under this scenario.

b) If the vertex of the path of the alpha particles is 10 *cm* from the center of the hyperbola, find a model that describes the path of the particle.

Solution

a) Since the particles are deflected at a 45° angle, the asymptotes will be $\boxed{y = \pm x}$

b) Since the vertex is 10 *cm* from the center of the hyperbola, so $a = 10$

The slope of the asymptotes is given by $\pm \dfrac{b}{a}$

Therfore: $\dfrac{b}{a} = 1 \quad \rightarrow \quad b = a = 10$

The equation of the particle path is: $\quad \dfrac{x^2}{100} - \dfrac{y^2}{100} = 1 \quad (x \ge 0)$

Exercise

Hyperbolas have interesting reflective properties that make them useful for lenses and mirrors. For example, if a ray of light strikes a convex hyperbolic mirror on a line that would (theoretically) pass through its rear focus, it is reflected through its rear focus, it is reflected through the front focus. This property and that of the parabola were used to develop the *Cassegrain* telescope in 1672. The focus of the parabolic mirror and the rear focus of the hyperbolic mirror are the same point. The rays are collected by the parabolic mirror, reflected toward the common focus, and thus are reflected by the hyperbolic mirror through the opening to its front focus, where the eyepiece is located. If the equation of the hyperbola is $\frac{y^2}{9}-\frac{x^2}{16}=1$ and the focal length (distance from the vertex to the focus) of the parabola is 6, find the equation of the parabola.

Solution

Assume the origin lies at the center of the hyperbola. The foci of the hyperbola are located on *y*-axis at $(0,\ \pm c)$, since the hyperbola has a transverse axis that is parallel to the *y*-axis.

Given: $a^2=9 \quad \textit{and} \quad b^2=16$

$c^2=a^2+b^2=9+6=25$

$c=\sqrt{25}\ \underline{=5}$

Therefore, the foci of the foci of the hyperbola are at $(0,\ -5)$ & $(0,\ 5)$

Assume that he parabola opens up, the common focus is at $(0,\ 5)$.

The equation of the parabola: $x^2=4a(y-k)$

The focal length of the parabola is given as $a=6$

The distance focus of the parabola is located at $(0,\ k+a)=(0,\ 5)$

$$k+6=5 \quad \Rightarrow \quad k=1$$

The equation of the parabola becomes $x^2=4(8)(y-(-1))$

$\underline{x^2=24(y+1)}$ *or* $y=\frac{1}{24}x^2-1$

Exercise

The ***eccentricity e*** of a hyperbola is defined as the number $\frac{c}{a}$, where *a* is the distance of a vertex from the center and *c* is the distance of a focus from the center. Because *c* > *a*, it follows that $e>1$. Describe the general shape of a hyperbola whose eccentricity is close to 1. What is the shape if *e* is very large?

Solution

Assume $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

If the eccentricity is close to 1, then $c\approx a$ and $b\approx 0$.

When *b* is close to 0, the hyperbola is very narrow, because the slopes of asymptotes are close to 0.

If the eccentricity is very large, then c is much larger than a and b. The result is a hyperbola is very wide, because the slopes of the asymptotes are very large.

For $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the opposite is true.

When the eccentricity is close to 1, the hyperbola is very wide because the slopes of the asymptotes are close to 0.

When the eccentricity is very large, the hyperbola is very narrow because the slopes of asymptotes are very large.