

Lecture Three

Section 3.1 – Inner Products

Definition

An **inner product** on a real vector space V is a function that associates a real number $\langle \vec{u}, \vec{v} \rangle$ with each pair of vectors in V in such a way that the following axioms are satisfied for all vectors \vec{u}, \vec{v} , and \vec{w} in V and all scalars k .

1. $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ *Symmetry axiom*
2. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ *Additivity axiom*
3. $\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle$ *Homogeneity axiom*
4. $\langle \vec{v}, \vec{v} \rangle \geq 0$ and $\langle \vec{v}, \vec{v} \rangle = 0$ iff $\vec{v} = 0$ *Positivity axiom*

A real vector space with an inner product is called a **real inner product space**.

$$\langle \vec{u}, \vec{u} \rangle = \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

This is called the **Euclidean inner product** (or the **standard inner product**)

Definition

If V is a real inner product space, then the norm (or length) of a vector \vec{v} in V is denoted by $\|\vec{v}\|$ and is defined by

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

And the **distance** between two vectors is denoted by $d(\vec{u}, \vec{v})$ and is defined by

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{\langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle}$$

A vector of norm 1 is called a **unit vector**.

Theorem

If \vec{u} and \vec{v} are vectors in a real inner product space V , and if k is a scalar, then:

- a) $\|\vec{v}\| \geq 0$ with equality iff $\vec{v} = 0$
- b) $\|k\vec{v}\| = |k| \|\vec{v}\|$
- c) $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$
- d) $d(\vec{u}, \vec{v}) \geq 0$ with equality iff $\vec{u} = \vec{v}$

Although the Euclidean inner product is the most important inner product on \mathbb{R}^n , there are various applications in which is desirable to modify it by weighing each term differently. More precisely, if w_1, w_2, \dots, w_n are positive real numbers, which we will call weighs, and if $\vec{u} = (u_1, u_2, \dots, u_n)$ and $\vec{v} = (v_1, v_2, \dots, v_n)$ are vectors in \mathbb{R}^n , then it can be shown that the formula

$$\langle \vec{u}, \vec{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \dots + w_n u_n v_n$$

Defines an inner product on \mathbb{R}^n that we call the **weighted Euclidean inner product** with weights w_1, w_2, \dots, w_n

Example

Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ be vectors in \mathbb{R}^2 , verify that the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 3u_1 v_1 + 2u_2 v_2$ satisfies the four inner product axioms.

Solution

$$\begin{aligned} \text{Axiom 1: } \langle \vec{u}, \vec{v} \rangle &= 3u_1 v_1 + 2u_2 v_2 \\ &= 3v_1 u_1 + 2v_2 u_2 \\ &= \langle \vec{v}, \vec{u} \rangle \end{aligned}$$

$$\begin{aligned} \text{Axiom 2: } \langle \vec{u} + \vec{v}, \vec{w} \rangle &= 3(u_1 + v_1)w_1 + 2(u_2 + v_2)w_2 \\ &= 3(u_1 w_1 + v_1 w_1) + 2(u_2 w_2 + v_2 w_2) \\ &= 3u_1 w_1 + 3v_1 w_1 + 2u_2 w_2 + 2v_2 w_2 \\ &= (3u_1 w_1 + 2u_2 w_2) + (3v_1 w_1 + 2v_2 w_2) \\ &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \end{aligned}$$

$$\begin{aligned} \text{Axiom 3: } \langle k\vec{u}, \vec{v} \rangle &= 3(ku_1)v_1 + 2(ku_2)v_2 \\ &= k(3u_1 v_1 + 2u_2 v_2) \\ &= k \langle \vec{u}, \vec{v} \rangle \end{aligned}$$

$$\begin{aligned} \text{Axiom 4: } \langle \vec{v}, \vec{v} \rangle &= 3v_1 v_1 + 2v_2 v_2 \\ &= 3v_1^2 + 2v_2^2 \geq 0 \\ v_1 = v_2 = 0 &\text{ iff } \vec{v} = \vec{0} \end{aligned}$$

Exercises Section 3.1 – Inner Products

1. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on \mathbb{R}^2 , and let $\vec{u} = (1, 1)$, $\vec{v} = (3, 2)$, $\vec{w} = (0, -1)$, and $k = 3$. Compute the following.

$$\begin{array}{lll} a) \langle \vec{u}, \vec{v} \rangle & c) \langle \vec{u} + \vec{v}, \vec{w} \rangle & e) d(\vec{u}, \vec{v}) \\ b) \langle k\vec{v}, \vec{w} \rangle & d) \|\vec{v}\| & f) \|\vec{u} - k\vec{v}\| \end{array}$$

2. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on \mathbb{R}^2 , and let $\vec{u} = (1, 1)$, $\vec{v} = (3, 2)$, $\vec{w} = (0, -1)$ and $k = 3$. Compute the following for the weighted Euclidean inner product

$$\langle \vec{u}, \vec{v} \rangle = 2u_1v_1 + 3u_2v_2.$$

$$\begin{array}{lll} a) \langle \vec{u}, \vec{v} \rangle & c) \langle \vec{u} + \vec{v}, \vec{w} \rangle & e) d(\vec{u}, \vec{v}) \\ b) \langle k\vec{v}, \vec{w} \rangle & d) \|\vec{v}\| & f) \|\vec{u} - k\vec{v}\| \end{array}$$

3. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on \mathbb{R}^2 , and let $\vec{u} = (3, -2)$, $\vec{v} = (4, 5)$, $\vec{w} = (-1, 6)$, and $k = -4$. Verify the following.

$$\begin{array}{ll} a) \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle & d) \langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle \\ b) \langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle & e) \langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0 \\ c) \langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle & \end{array}$$

4. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on \mathbb{R}^2 , and let $\vec{u} = (3, -2)$, $\vec{v} = (4, 5)$, $\vec{w} = (-1, 6)$, and $k = -4$. Verify the following for the weighted Euclidean inner product

$$\langle \vec{u}, \vec{v} \rangle = 4u_1v_1 + 5u_2v_2$$

$$\begin{array}{ll} a) \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle & d) \langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle \\ b) \langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle & e) \langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0 \\ c) \langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle & \end{array}$$

5. Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$. Show that the following are inner product on \mathbb{R}^2 by verifying that the inner product axioms hold. $\langle \vec{u}, \vec{v} \rangle = 3u_1v_1 + 5u_2v_2$

6. Show that the following identity holds for the vectors in any inner product space

$$\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

7. Show that the following identity holds for the vectors in any inner product space

$$\langle \vec{u}, \vec{v} \rangle = \frac{1}{4} \left(\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 \right)$$

8. Prove that $\|k\vec{v}\| = |k| \|\vec{v}\|$

Section 3.2 – Angle and Orthogonality in Inner Product Spaces

Cosine Formula

If \vec{u} and \vec{v} are nonzero vectors that implies

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$\theta = \cos^{-1} \left(\frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|} \right)$$

$$-1 \leq \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \leq 1$$

Example

Let \mathbb{R}^4 have the Euclidean inner product. Find the cosine angle θ between the vectors $\vec{u} = (4, 3, 1, -2)$ and $\vec{v} = (-2, 1, 2, 3)$.

Solution

$$\begin{aligned} \|\vec{u}\| &= \sqrt{4^2 + 3^2 + 1^2 + (-2)^2} \\ &= \sqrt{30} \end{aligned}$$

$$\begin{aligned} \|\vec{v}\| &= \sqrt{4 + 1 + 4 + 9} \\ &= \sqrt{18} \\ &= 3\sqrt{2} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= 4(-2) + 3(1) + 1(2) - 2(3) \\ &= -9 \end{aligned}$$

$$\begin{aligned} \cos \theta &= -\frac{9}{3\sqrt{30}\sqrt{2}} \\ &= -\frac{3}{\sqrt{60}} \\ &= -\frac{3}{2\sqrt{15}} \end{aligned}$$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

Theorem – Cauchy-Schwarz Inequality

If \vec{u} and \vec{v} are vectors in a real inner product space V , then

$$\|\langle \vec{u}, \vec{v} \rangle\| \leq \|\vec{u}\| \|\vec{v}\|$$

Proof

If either \vec{u} or \vec{v} is equal to zero, then both sides equal to zero
Inequality holds.

Suppose that $\vec{u}, \vec{v} \neq \mathbf{0}$ and if \vec{w} any vector

$$\|\vec{w}\|^2 = \vec{w} \cdot \vec{w} \geq 0$$

Let $\vec{w} = \vec{u} - t\vec{v}$, then:

$$\begin{aligned} 0 &\leq \vec{w} \cdot \vec{w} \\ &= (\vec{u} - t\vec{v}) \cdot (\vec{u} - t\vec{v}) \\ &= \vec{u} \cdot \vec{u} - t(\vec{u} \cdot \vec{v}) - t(\vec{v} \cdot \vec{u}) + t^2(\vec{v} \cdot \vec{v}) \\ &= \vec{u} \cdot \vec{u} - 2t(\vec{u} \cdot \vec{v}) + t^2(\vec{v} \cdot \vec{v}) \quad \text{Let } t = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \\ &= \vec{u} \cdot \vec{u} - 2\left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right)(\vec{u} \cdot \vec{v}) + \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right)^2(\vec{v} \cdot \vec{v}) \\ &= \vec{u} \cdot \vec{u} - 2\frac{(\vec{u} \cdot \vec{v})^2}{\vec{v} \cdot \vec{v}} + \frac{(\vec{u} \cdot \vec{v})^2}{\vec{v} \cdot \vec{v}} \\ &= \vec{u} \cdot \vec{u} - \frac{(\vec{u} \cdot \vec{v})^2}{\vec{v} \cdot \vec{v}} \\ &= \frac{(\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v}) - (\vec{u} \cdot \vec{v})^2}{\vec{v} \cdot \vec{v}} \quad \text{Since } \vec{v} \cdot \vec{v} > 0 \\ &\leq (\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v}) - (\vec{u} \cdot \vec{v})^2 \\ (\vec{u} \cdot \vec{v})^2 &\leq (\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v}) \\ \|\langle \vec{u}, \vec{v} \rangle\| &\leq \|\vec{u}\| \|\vec{v}\| \end{aligned}$$

The following two alternative forms of the Cauchy-Schwarz inequality are useful to know:

$$\langle \vec{u}, \vec{v} \rangle^2 \leq \langle \vec{u}, \vec{u} \rangle \langle \vec{v}, \vec{v} \rangle$$

$$\langle \vec{u}, \vec{v} \rangle^2 \leq \|\vec{u}\|^2 \|\vec{v}\|^2$$

Theorem

If \vec{u} , \vec{v} and \vec{w} are vectors in a real inner product space V , and if k is any scalar, then

$$a) \quad \|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\| \quad (\text{Triangle inequality for vectors})$$

$$b) \quad d(\vec{u}, \vec{v}) \leq d(\vec{u}, \vec{w}) + d(\vec{w}, \vec{v}) \quad (\text{Triangle inequality for distances})$$

Proof (a)

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= \langle \vec{u} + \vec{v}, \vec{u} + \vec{v} \rangle \\ &= \langle \vec{u}, \vec{u} \rangle + 2\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{v} \rangle \\ &\leq \langle \vec{u}, \vec{u} \rangle + 2|\langle \vec{u}, \vec{v} \rangle| + \langle \vec{v}, \vec{v} \rangle \\ &\leq \langle \vec{u}, \vec{u} \rangle + 2\|\vec{u}\| \|\vec{v}\| + \langle \vec{v}, \vec{v} \rangle \\ &= \|\vec{u}\|^2 + 2\|\vec{u}\| \|\vec{v}\| + \|\vec{v}\|^2 \\ &= (\|\vec{u}\| + \|\vec{v}\|)^2 \end{aligned}$$

$$\|\vec{u} + \vec{v}\|^2 \leq (\|\vec{u}\| + \|\vec{v}\|)^2$$

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

Definition

Two vectors \vec{u} and \vec{v} in an inner product space are called orthogonal if $\langle \vec{u}, \vec{v} \rangle = 0$

Example

The vectors $\vec{u} = (1, 1)$ and $\vec{v} = (1, -1)$ are orthogonal with respect to the Euclidean inner product on \mathbb{R}^2 , since

$$\begin{aligned} \vec{u} \cdot \vec{v} &= 1(1) + 1(-1) \\ &= 0 \end{aligned}$$

They are not orthogonal with the respect to the weighted Euclidean inner product

$\langle \vec{u}, \vec{v} \rangle = 3u_1v_1 + 2u_2v_2$, since

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= 3(1)(1) + 2(1)(-1) \\ &= 1 \neq 0 \end{aligned}$$

Example

$U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ are orthogonal, since

$$\begin{aligned} U \cdot V &= 1(0) + 0(2) + 1(0) + 1(0) \\ &= 0 \end{aligned}$$

Definition

If W is a subspace of an inner product space V , then the set of all vectors are orthogonal to every vector in W is called the **orthogonal complement** of W and is denoted by the symbol W^\perp

Theorem

If W is a subspace of an inner product space V , then:

a) W^\perp is a subspace of V .

b) $W \cap W^\perp = \{0\}$

Proof

a) Let set W^\perp contains at least the zero vector, since $\langle \vec{0}, \vec{w} \rangle = 0$ for every vector \vec{w} in W . We need to show that W^\perp is closed under addition and scalar multiplication.

Suppose that \vec{u} and \vec{v} are vectors in W^\perp , so every vector \vec{w} in W we have $\langle \vec{u}, \vec{w} \rangle = 0$ and $\langle \vec{v}, \vec{w} \rangle = 0$

$$\begin{aligned} \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Closed under addition

$$\begin{aligned} \langle k\vec{u}, \vec{w} \rangle &= k \langle \vec{u}, \vec{w} \rangle \\ &= k(0) \\ &= 0 \end{aligned}$$

Closed under scalar multiplication

Which proves that $\vec{u} + \vec{w}$ and $k\vec{u}$ are in W^\perp

b) If \vec{v} is any vector in both W and W^\perp , then \vec{v} is orthogonal to itself; that is, $\langle \vec{v}, \vec{v} \rangle = 0$. It follows from the positivity axiom for inner products that $\vec{v} = \vec{0}$

Theorem

If W is a subspace of a finite-dimensional inner product space V , then the orthogonal complement of W^\perp is W ; that is

$$(W^\perp)^\perp = W$$

Example

Let W be the subspace of \mathbb{R}^6 spanned by the vectors

$$\begin{aligned}\bar{w}_1 &= (1, 3, -2, 0, 2, 0), & \bar{w}_2 &= (2, 6, -5, -2, 4, -3) \\ \bar{w}_3 &= (0, 0, 5, 10, 0, 15), & \bar{w}_4 &= (2, 6, 0, 8, 4, 18)\end{aligned}$$

Find a basis for the orthogonal complement of W .

Solution

The Space W is the same as the row space of the matrix

$$A = \begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix} \quad \begin{array}{l} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{array}$$

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 0 & 0 & 4 & 8 & 0 & 18 \end{pmatrix} \quad \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 5R_2 \\ R_4 + 4R_2 \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 \\ 0 & 0 & -1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix} \quad \begin{array}{l} -R_2 \\ \\ \frac{1}{6}R_3 \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} R_1 - 6R_3 \\ R_2 - 3R_3 \\ \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution

$$\begin{aligned}\begin{pmatrix} x_1, x_2, x_3, x_4, x_5, x_6 \end{pmatrix} &= \begin{pmatrix} -3x_2 - 4x_4 - 2x_5, x_2, -2x_4, x_4, x_5, 0 \end{pmatrix} \\ &= x_2 \begin{pmatrix} -3, 1, 0, 0, 0, 0 \end{pmatrix} + x_4 \begin{pmatrix} -4, 0, -2, 1, 0, 0 \end{pmatrix} + x_5 \begin{pmatrix} -2, 0, 0, 0, 1, 0 \end{pmatrix} \\ \vec{v}_1 &= \begin{pmatrix} -3, 1, 0, 0, 0, 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -4, 0, -2, 1, 0, 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -2, 0, 0, 0, 1, 0 \end{pmatrix}\end{aligned}$$

Definition

A collection of vectors in \mathbb{R}^n (or inner space) is called orthogonal if any 2 are perpendicular.

$$\vec{v}_i \cdot \vec{v}_j = \vec{v}_i^T \vec{v}_j = \begin{cases} 0 & \text{for } i \neq j \quad (\text{orthogonal vectors}) \\ 1 & \text{for } i = j \quad (\text{unit vectors}) \end{cases}$$

Theorem

If $\vec{v}_1, \dots, \vec{v}_m$ are nonzero orthogonal vectors, then they are linearly independent.

Definition

A vector \vec{v} is called normal if $\|\vec{v}\| = 1$

A collection of vectors $\vec{v}_1, \dots, \vec{v}_m$ is called orthonormal if they are orthogonal and each $\|\vec{v}_i\| = 1$.

An orthonormal basis is a basis made up of orthonormal vectors.

Example

Q rotates every vector in the plane through the angle θ .

$$Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$Q^{-1} = \frac{1}{\cos^2 \theta + \sin^2 \theta} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \quad \cos^2 \theta + \sin^2 \theta = 1$$

$$= \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$= \underline{Q^T}$$

The dot product $(\cos \theta \sin \theta - \sin \theta \cos \theta = 0)$, the columns are orthogonal.

They are unit vectors because $\cos^2 \theta + \sin^2 \theta = 1$. Those columns give an orthonormal basis for the plane \mathbb{R}^2 .

We have: $QQ^T = I = Q^T Q$ (This type is called ***rotation***)

Exercises **Section 3.2 – Angle and Orthogonality in Inner Product Spaces**

1. Which of the following form orthonormal sets?

- a) $(1, 0), (0, 2)$ in \mathbb{R}^2
- b) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in \mathbb{R}^2
- c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in \mathbb{R}^2
- d) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ in \mathbb{R}^3
- e) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ in \mathbb{R}^3
- f) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ in \mathbb{R}^3

2. Find the cosine of the angle between \vec{u} and \vec{v} .

- a) $\vec{u} = (1, -3), \vec{v} = (2, 4)$
- b) $\vec{u} = (-1, 0), \vec{v} = (3, 8)$
- c) $\vec{u} = (-1, 5, 2), \vec{v} = (2, 4, -9)$
- d) $\vec{u} = (4, 1, 8), \vec{v} = (1, 0, -3)$
- e) $\vec{u} = (1, 0, 1, 0), \vec{v} = (-3, -3, -3, -3)$
- f) $\vec{u} = (2, 1, 7, -1), \vec{v} = (4, 0, 0, 0)$
- g) $\vec{u} = (1, 3, -5, 4), \vec{v} = (2, -4, 4, 1)$
- h) $\vec{u} = (1, 2, 3, 4), \vec{v} = (-1, -2, -3, -4)$

3. Find the cosine of the angle between A and B .

- a) $A = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix}, B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$
- b) $A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}, B = \begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$
- c) $A = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$
- d) $A = \begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix}, B = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$

4. Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

- a) $\vec{u} = (-1, 3, 2), \vec{v} = (4, 2, -1)$
- b) $\vec{u} = (a, b), \vec{v} = (-b, a)$
- c) $\vec{u} = (-2, -2, -2), \vec{v} = (1, 1, 1)$
- d) $\vec{u} = (-4, 6, -10, 1), \vec{v} = (2, 1, -2, 9)$
- e) $\vec{u} = (-4, 6, -10, 1), \vec{v} = (2, 1, -2, 9)$

5. Do there exist scalars k and l such that the vectors

$\vec{u} = (2, k, 6), \vec{v} = (l, 5, 3),$ and $\vec{w} = (1, 2, 3)$ are mutually orthogonal with respect to the Euclidean inner product?

6. Let \mathbb{R}^3 have the Euclidean inner product. For which values of k are \vec{u} and \vec{v} orthogonal?
- a) $\vec{u} = (2, 1, 3)$, $\vec{v} = (1, 7, k)$ b) $\vec{u} = (k, k, 1)$, $\vec{v} = (k, 5, 6)$
7. Let V be an inner product space. Show that if \vec{u} and \vec{v} are orthogonal unit vectors in V , then $\|\vec{u} - \vec{v}\| = \sqrt{2}$
8. Let \mathcal{S} be a subspace of \mathbb{R}^n . Explain what $(\mathcal{S}^\perp)^\perp = \mathcal{S}$ means and why it is true.
9. The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ – (**note** that all six edges have length $\sqrt{2}$, so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ to the vertices?
10. Determine if the given vectors are orthogonal.
- $\vec{x}_1 = (1, 0, 1, 0)$, $\vec{x}_2 = (0, 1, 0, 1)$, $\vec{x}_3 = (1, 0, -1, 0)$, $\vec{x}_4 = (1, 1, -1, -1)$
11. Which of the following sets of vectors are orthogonal with respect to the Euclidean inner
- a) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ $\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$
- b) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$ $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$
12. Consider vectors $\vec{u} = (2, 3, 5)$ $\vec{v} = (1, -4, 3)$ in \mathbb{R}^3
- a) $\langle \vec{u}, \vec{v} \rangle$ b) $\|\vec{u}\|$ c) $\|\vec{v}\|$ d) Cosine between \vec{u} and \vec{v}
13. Consider vectors $\vec{u} = (1, 1, 1)$ $\vec{v} = (1, 2, -3)$ in \mathbb{R}^3
- a) $\langle \vec{u}, \vec{v} \rangle$ b) $\|\vec{u}\|$ c) $\|\vec{v}\|$ d) Cosine θ between \vec{u} and \vec{v}
14. Consider vectors $\vec{u} = (1, 2, 5)$ $\vec{v} = (2, -3, 5)$ $\vec{w} = (4, 2, -3)$ in \mathbb{R}^3
- a) $\langle \vec{u}, \vec{v} \rangle$ d) $\|\vec{u}\|$ g) Cosine α between \vec{u} and \vec{v}
- b) $\langle \vec{u}, \vec{w} \rangle$ e) $\|\vec{v}\|$ h) Cosine β between \vec{u} and \vec{w}
- c) $\langle \vec{v}, \vec{w} \rangle$ f) $\|\vec{w}\|$ i) Cosine θ between \vec{v} and \vec{w}
- j) $(\vec{u} + \vec{v}) \cdot \vec{w}$

15. Consider polynomial $f(t) = 3t - 5$; $g(t) = t^2$ in $\mathbb{P}(t)$
- a) $\langle f, g \rangle$ b) $\|f\|$ c) $\|g\|$ d) Cosine between f and g
16. Consider polynomial $f(t) = t + 2$; $g(t) = 3t - 2$; $h(t) = t^2 - 2t - 3$ in $\mathbb{P}(t)$
- a) $\langle f, g \rangle$ d) $\|f\|$ g) Cosine α between f and g
b) $\langle f, h \rangle$ e) $\|g\|$ h) Cosine β between f and h
c) $\langle g, h \rangle$ f) $\|h\|$ i) Cosine θ between g and h
17. Suppose $\langle \vec{u}, \vec{v} \rangle = 3 + 2i$ in a complex inner product space V . Find:
- a) $\langle (2 - 4i)\vec{u}, \vec{v} \rangle$ b) $\langle \vec{u}, (4 + 3i)\vec{v} \rangle$ c) $\langle (3 - 6i)\vec{u}, (5 - 2i)\vec{v} \rangle$ d) $\|\vec{u}, \vec{v}\|$
18. Find the Fourier coefficient c and the projection $c\vec{v}$ of $\vec{u} = (3 + 4i, 2 - 3i)$ along $\vec{v} = (5 + i, 2i)$ in \mathbb{C}^2
19. Suppose $\vec{v} = (1, 3, 5, 7)$. Find the projection of \vec{v} onto W or find $\vec{w} \in W$ that minimizes $\|\vec{v} - \vec{w}\|$, where W is the subspace of \mathbb{R}^4 spanned by:
- a) $\vec{u}_1 = (1, 1, 1, 1)$ and $\vec{u}_2 = (1, -3, 4, -2)$
b) $\vec{v}_1 = (1, 1, 1, 1)$ and $\vec{v}_2 = (1, 2, 3, 2)$
20. Suppose $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ is an orthogonal set of vectors. Prove that (Pythagoras)
- $$\|\vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_n\|^2 = \|\vec{u}_1\|^2 + \|\vec{u}_2\|^2 + \dots + \|\vec{u}_n\|^2$$
21. Suppose A is an orthogonal matrix. Show that $\langle \vec{u}A, \vec{v}A \rangle = \langle \vec{u}, \vec{v} \rangle$ for any $\vec{u}, \vec{v} \in V$
22. Suppose A is an orthogonal matrix. Show that $\|\vec{u}A\| = \|\vec{u}\|$ for every $\vec{u} \in V$
23. Let V be an inner product space over \mathbb{R} or \mathbb{C} . Show that
- $$\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$$
- If and only if
- $$\|s\vec{u} + t\vec{v}\| = s\|\vec{u}\| + t\|\vec{v}\| \quad \text{for all } s, t \geq 0$$
24. Let V be an inner product vector space over \mathbb{R} .
- a) If e_1, e_2, e_3 are three vectors in V with pairwise product negative, that is,
- $$\langle e_i, e_j \rangle < 0, \quad i, j = 1, 2, 3, \quad i \neq j$$

Show that e_1, e_2, e_3 are linearly independent.

- b) Is it possible for three vectors on the xy -plane to have pairwise negative products?
- c) Does part (a) remain valid when the word “negative: is replaced with positive?
- d) Suppose \vec{u}, \vec{v} , and \vec{w} are three unit vectors in the xy -plane. What are the maximum and minimum values that

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle$$

Can attain? And when?

Section 3.3 – Gram-Schmidt Process

Definition

A set of two or more vectors in a real inner product space is said to be **orthogonal** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be **orthonormal**.

Theorem

1. If $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is an orthogonal basis for an inner product space V , and if \vec{u} is any vector in V , then

$$\vec{u} = \frac{\langle \vec{u}, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 + \frac{\langle \vec{u}, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 + \dots + \frac{\langle \vec{u}, \vec{v}_n \rangle}{\|\vec{v}_n\|^2} \vec{v}_n$$

2. If $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is an orthonormal basis for an inner product space V , and if \vec{u} is any vector in V , then

$$\vec{u} = \langle \vec{u}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{u}, \vec{v}_2 \rangle \vec{v}_2 + \dots + \langle \vec{u}, \vec{v}_n \rangle \vec{v}_n$$

Proof

1. Since $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis for V , every vector \vec{u} in V can be expressed in the form

$$\vec{u} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n$$

Let show that $c_i = \frac{\langle \vec{u}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2}$ for $i = 1, 2, \dots, n$

$$\begin{aligned} \langle \vec{u}, \vec{v}_i \rangle &= \langle c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n, \vec{v}_i \rangle \\ &= c_1 \langle \vec{v}_1, \vec{v}_i \rangle + c_2 \langle \vec{v}_2, \vec{v}_i \rangle + \dots + c_n \langle \vec{v}_n, \vec{v}_i \rangle \end{aligned}$$

Since S is an orthogonal set, all of the inner products in the last equality are zero except the i^{th} , so we have

$$\begin{aligned} \langle \vec{u}, \vec{v}_i \rangle &= c_i \langle \vec{v}_i, \vec{v}_i \rangle \\ &= c_i \|\vec{v}_i\|^2 \end{aligned}$$

The Gram-Schmidt Process

To convert a basis $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_r\}$ into an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$, perform the following computations:

$$\text{Step 1: } \vec{v}_1 = \vec{u}_1$$

$$\text{Step 2: } \vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$\text{Step 3: } \vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$\text{Step 4: } \vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3$$

To convert the orthogonal basis into an orthonormal basis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$, normalize the orthogonal basis

vectors.
$$\vec{q}_i = \frac{\vec{v}_i}{\|\vec{v}_i\|}$$

Example

Assume that the vector space \mathbb{R}^3 has the Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vectors

$$\vec{u}_1 = (1, 1, 1) \quad \vec{u}_2 = (0, 1, 1) \quad \vec{u}_3 = (0, 0, 1)$$

Into the orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, and then normalize the **orthogonal** basis vectors to obtain an orthonormal basis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$

Solution

$$\begin{aligned} \vec{v}_1 &= \vec{u}_1 \\ &= (1, 1, 1) \end{aligned}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$\begin{aligned}
&= (0, 1, 1) - \frac{0+1+1}{1^2+1^2+1^2} (1, 1, 1) \\
&= (0, 1, 1) - \frac{2}{3} (1, 1, 1) \\
&= \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right)
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \text{proj}_{\vec{v}_2} \vec{u}_3 \\
&= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
&= (0, 0, 1) - \frac{0+0+1}{1^2+1^2+1^2} (1, 1, 1) - \frac{0+0+\frac{1}{3}}{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right) \\
&= (0, 0, 1) - \frac{1}{3} (1, 1, 1) - \frac{\frac{1}{3}}{\frac{2}{3}} \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right) \\
&= (0, 0, 1) - \frac{1}{3} (1, 1, 1) - \frac{1}{2} \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right) \\
&= \left(0, -\frac{1}{2}, \frac{1}{2} \right)
\end{aligned}$$

$$\begin{aligned}
\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \frac{(1, 1, 1)}{\sqrt{3}} \\
&= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right)}{\sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2}} \\
&= \frac{\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right)}{\frac{\sqrt{6}}{3}} \\
&= \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right)
\end{aligned}$$

$$\begin{aligned}
 \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
 &= \frac{\left(0, -\frac{1}{2}, \frac{1}{2}\right)}{\sqrt{0^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}} \\
 &= \frac{\left(0, -\frac{1}{2}, \frac{1}{2}\right)}{\frac{\sqrt{2}}{2}} \\
 &= \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)
 \end{aligned}$$

Gram-Schmidt Process (Orthonormal)

Suppose $\vec{v}_1, \dots, \vec{v}_n$ linearly independent in \mathbb{R}^n , construct n **orthonormal** $\vec{u}_1, \dots, \vec{u}_n$ that span the same space: $\text{span} \{ \vec{u}_1, \dots, \vec{u}_k \} = \text{span} \{ \vec{v}_1, \dots, \vec{v}_k \}$

Step 1: Since \vec{v}_i are linearly independent ($\neq 0$), so $\|\vec{v}_1\| \neq 0$ (to create a normal vector)

Let $\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \vec{q}_1$, then $\|\vec{u}_1\| = 1$ since \vec{u}_1 is orthonormal and $\text{span} \{ \vec{u}_1 \} = \text{span} \{ \vec{v}_1 \}$

$$\vec{w}_1 = \vec{v}_1 \Rightarrow \vec{v}_1 = \|\vec{w}_1\| \vec{u}_1$$

Step 2: $\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{q}_1) \vec{q}_1$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{u}_1}{\|\vec{v}_1\|} \vec{v}_1 \quad (\vec{w}_2 \perp \vec{u}_1)$$

$$\vec{v}_2 = \|\vec{w}_2\| \vec{u}_2 + (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1 \quad \vec{w}_2 = \|\vec{w}_2\| \vec{u}_2$$

$$\vec{q}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

Step 3: $\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{q}_1) \vec{q}_1 - (\vec{v}_3 \cdot \vec{q}_2) \vec{q}_2$

$$\vec{q}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

	$\vec{q}_1 = \frac{\vec{v}_1}{\ \vec{v}_1\ }$
$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{q}_1) \vec{q}_1$	$\vec{q}_2 = \frac{\vec{w}_2}{\ \vec{w}_2\ }$
$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{q}_1) \vec{q}_1 - (\vec{v}_3 \cdot \vec{q}_2) \vec{q}_2$	$\vec{q}_3 = \frac{\vec{w}_3}{\ \vec{w}_3\ }$
$\vec{w}_n = \vec{v}_n - (\vec{v}_n \cdot \vec{q}_1) \vec{q}_1 - (\vec{v}_n \cdot \vec{q}_2) \vec{q}_2 - \dots - (\vec{v}_n \cdot \vec{q}_{n-1}) \vec{q}_{n-1}$	$\vec{q}_n = \frac{\vec{w}_n}{\ \vec{w}_n\ }$

Example

Use the Gram-Schmidt process to find an **orthonormal** basis for the subspaces of

$$\vec{v}_1 = (1, 1, 0, 0) \quad \vec{v}_2 = (0, 1, 1, 0) \quad \vec{v}_3 = (1, 0, 1, 1)$$

Solution

$$\begin{aligned} \text{Step 1: } \vec{q}_1 &= \frac{(1, 1, 0, 0)}{\sqrt{1^2 + 1^2 + 0 + 0}} & \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(1, 1, 0, 0)}{\sqrt{2}} \\ &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right) \end{aligned}$$

$$\begin{aligned} \text{Step 2: } \vec{w}_2 &= \vec{v}_2 - (\vec{v}_2 \cdot \vec{q}_1) \vec{q}_1 \\ &= (0, 1, 1, 0) - \left[(0, 1, 1, 0) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right) \right] \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right) \\ &= (0, 1, 1, 0) - \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right) \\ &= (0, 1, 1, 0) - \left(\frac{1}{2}, \frac{1}{2}, 0, 0 \right) \\ &= \left(-\frac{1}{2}, \frac{1}{2}, 1, 0 \right) \end{aligned}$$

$$\begin{aligned} \|\vec{w}_2\| &= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1} \\ &= \frac{\sqrt{3}}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{6}}{2} \end{aligned}$$

$$\begin{aligned} \vec{q}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\ &= \frac{\left(-\frac{1}{2}, \frac{1}{2}, 1, 0\right)}{\frac{\sqrt{6}}{2}} \\ &= \frac{2}{\sqrt{6}} \left(-\frac{1}{2}, \frac{1}{2}, 1, 0\right) \\ &= \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0\right) \end{aligned}$$

$$\text{Step 3: } \vec{v}_3 \cdot \vec{q}_1 = (1, 0, 1, 1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right) \\ = \frac{1}{\sqrt{2}} \quad \Big|$$

$$\vec{v}_3 \cdot \vec{q}_2 = (1, 0, 1, 1) \cdot \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0 \right) \\ = -\frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} \\ = \frac{1}{\sqrt{6}} \quad \Big|$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{q}_1) \vec{q}_1 - (\vec{v}_3 \cdot \vec{q}_2) \vec{q}_2 \\ = (1, 0, 1, 1) - \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right) - \frac{1}{\sqrt{6}} \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0 \right) \\ = (1, 0, 1, 1) - \left(\frac{1}{2}, \frac{1}{2}, 0, 0 \right) - \left(-\frac{1}{6}, \frac{1}{6}, \frac{2}{6}, 0 \right) \\ = \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, 1 \right) \quad \Big|$$

$$\vec{q}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|} \\ = \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + 1^2}} \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, 1 \right) \\ = \frac{1}{\sqrt{\frac{21}{9}}} \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, 1 \right) \\ = \frac{3}{\sqrt{21}} \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, 1 \right) \\ = \left(\frac{2}{\sqrt{21}}, -\frac{2}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{3}{\sqrt{21}} \right) \quad \Big|$$

The **orthonormal** basis:

$$\left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right), \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0 \right), \left(\frac{2}{\sqrt{21}}, -\frac{2}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{3}{\sqrt{21}} \right) \right\}$$

QR-Decomposition

Problem

If A is an $m \times n$ matrix with linearly independent column vectors, and if Q is the matrix that results by applying the Gram-Schmidt process to the column vectors of A , what relationship, if any, exists between A and Q ?

To solve this problem, suppose that the column vectors of A are $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ and the orthonormal column vectors of Q are $\vec{q}_1, \vec{q}_2, \dots, \vec{q}_n$.

$$\begin{aligned}\vec{u}_1 &= \langle \vec{u}_1, \vec{q}_1 \rangle \vec{q}_1 + \langle \vec{u}_1, \vec{q}_2 \rangle \vec{q}_2 + \cdots + \langle \vec{u}_1, \vec{q}_n \rangle \vec{q}_n \\ \vec{u}_2 &= \langle \vec{u}_2, \vec{q}_1 \rangle \vec{q}_1 + \langle \vec{u}_2, \vec{q}_2 \rangle \vec{q}_2 + \cdots + \langle \vec{u}_2, \vec{q}_n \rangle \vec{q}_n \\ &\vdots \\ \vec{u}_n &= \langle \vec{u}_n, \vec{q}_1 \rangle \vec{q}_1 + \langle \vec{u}_n, \vec{q}_2 \rangle \vec{q}_2 + \cdots + \langle \vec{u}_n, \vec{q}_n \rangle \vec{q}_n\end{aligned}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \cdots & \langle \vec{u}_n, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \cdots & \langle \vec{u}_n, \vec{q}_2 \rangle \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \langle \vec{u}_n, \vec{q}_n \rangle \end{bmatrix}$$

The equation $A = QR$ is a factorization of A into the product of a matrix Q with orthonormal column vectors and an invertible upper triangular matrix R . We call it the ***QR-decomposition of A*** .

Theorem

If A is an $m \times n$ matrix with linearly independent column vectors, then A can be factored as

$$A = QR$$

Where Q is an $m \times n$ matrix with orthonormal column vectors, and R is an $n \times n$ invertible upper triangular matrix.

Example

Find the QR -decomposition of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Solution

The column vectors of are

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

From the previous example

$$\vec{q}_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \quad \vec{q}_2 = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) \quad \vec{q}_3 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \langle \vec{u}_3, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \langle \vec{u}_3, \vec{q}_2 \rangle \\ 0 & 0 & \langle \vec{u}_3, \vec{q}_3 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \textcolor{red}{(1)}\frac{1}{\sqrt{3}} + \textcolor{red}{(1)}\frac{1}{\sqrt{3}} + \textcolor{red}{(1)}\frac{1}{\sqrt{3}} & \textcolor{red}{0} + \textcolor{red}{(1)}\frac{1}{\sqrt{3}} + \textcolor{red}{(1)}\frac{1}{\sqrt{3}} & \textcolor{red}{0} + \textcolor{red}{0} + \textcolor{red}{(1)}\frac{1}{\sqrt{3}} \\ 0 & \textcolor{red}{0}\left(\frac{-2}{\sqrt{6}}\right) + \textcolor{red}{(1)}\frac{1}{\sqrt{6}} + \textcolor{red}{(1)}\frac{1}{\sqrt{6}} & \textcolor{red}{0}\left(\frac{-2}{\sqrt{6}}\right) + \textcolor{red}{0}\frac{1}{\sqrt{6}} + \textcolor{red}{(1)}\frac{1}{\sqrt{6}} \\ 0 & 0 & \textcolor{red}{0} + \textcolor{red}{(0)}\frac{-1}{\sqrt{2}} + \textcolor{red}{(1)}\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\textcolor{blue}{A} = \textcolor{blue}{Q} \textcolor{blue}{R}$$

Calculus: Applying the Gram-Schmidt Process

We can apply the Gram-Schmidt orthogonalization procedure to generate some polynomials that are orthonormal on the interval $x \in [-1, 1]$ with inner product

$$\langle p, q \rangle = \int_{-1}^1 p(x)q(x)dx$$

Example

Apply the Gram-Schmidt orthonormalization process to the basis $B = \{1, x, x^2\}$ in \mathbb{P}_2 using the inner product

Solution

$$B = \{1, x, x^2\}$$

$$\text{Let } \vec{u}_1 = 1, \quad \vec{u}_2 = x, \quad \vec{u}_3 = x^2$$

$$\underline{\vec{v}_1 = \vec{u}_1 = 1}$$

$$\begin{aligned} \langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 dx \\ &= x \Big|_{-1}^1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 x dx \\ &= \frac{1}{2}x^2 \Big|_{-1}^1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= x - \frac{0}{2}(1) \\ &= x \end{aligned}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^1 x^2 dx$$

$$= \frac{1}{3} x^3 \Big|_{-1}^1$$

$$= \frac{2}{3} \Big|$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^1 x^2 \, dx$$

$$= \frac{1}{3} x^3 \Big|_{-1}^1$$

$$= \frac{2}{3} \Big|$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^1 x^3 \, dx$$

$$= \frac{1}{4} x^4 \Big|_{-1}^1$$

$$= 0 \Big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= x^2 - \frac{3}{2}(x)(0) - \frac{1}{2} \frac{2}{3}$$

$$= x^2 - \frac{1}{3} \Big|$$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^1 \left(x^2 - \frac{1}{3} \right)^2 \, dx$$

$$= \int_{-1}^1 \left(x^4 - \frac{2}{3} x^2 + \frac{1}{9} \right) \, dx$$

$$= \left(\frac{1}{5} x^5 - \frac{2}{9} x^3 + \frac{1}{9} x \right) \Big|_{-1}^1$$

$$= \frac{1}{5} - \frac{2}{9} + \frac{1}{9} + \frac{1}{5} - \frac{2}{9} + \frac{1}{9}$$

$$= \frac{8}{45} \Big|$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{2}} \Big|$$

$$\begin{aligned}\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{x}{\sqrt{2/3}} \\ &= \frac{\sqrt{3}}{\sqrt{2}}x\end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3} \right) \\ &= \frac{3\sqrt{5}}{2\sqrt{2}} \left(x^2 - \frac{1}{3} \right)\end{aligned}$$

The **orthonormal** basis is $\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \frac{1}{2}\sqrt{\frac{5}{2}}(3x^2-1) \right\}$

Exercises Section 3.3 – Gram-Schmidt Process

(1 – 14) Use the Gram-Schmidt process to find an *orthonormal* basis for the subspaces of \mathbb{R}^m .

1. $\vec{u}_1 = (1, -3), \vec{u}_2 = (2, 2)$
2. $\vec{u}_1 = (1, 0), \vec{u}_2 = (3, -5)$
3. $\{(1, 1, 1), (-1, 1, 0), (1, 2, 1)\}$
4. $\{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$
5. $\{(1, 1, 1), (0, 2, 1), (1, 0, 3)\}$
6. $\{(2, 2, 2), (1, 0, -1), (0, 3, 1)\}$
7. $\{(1, -1, 0), (0, 1, 0), (2, 3, 1)\}$
8. $\{(3, 0, 4), (-1, 0, 7), (2, 9, 11)\}$
9. $\{(1, 1, 1, 1), (1, 2, 1, 0), (1, 3, 0, 0)\}$
10. $\{(0, 2, -1, 1), (0, 0, 1, 1), (-2, 1, 1, -1)\}$
11. $\vec{u}_1 = (1, 0, 0), \vec{u}_2 = (3, 7, -2), \vec{u}_3 = (0, 4, 1)$
12. $\vec{u}_1 = (1, 1, 1, 1), \vec{u}_2 = (1, 2, 4, 5), \vec{u}_3 = (1, -3, -4, -2)$
13. $\vec{u}_1 = (1, 1, 1, 1), \vec{u}_2 = (1, 1, 2, 4), \vec{u}_3 = (1, 2, -4, -3)$
14. $\vec{u}_1 = (0, 2, 1, 0), \vec{u}_2 = (1, -1, 0, 0), \vec{u}_3 = (1, 2, 0, -1), \vec{u}_4 = (1, 0, 0, 1)$

(15 – 26) Use the Gram-Schmidt process to find an *orthogonal* basis for the subspaces of \mathbb{R}^m .

15. $\{(1, 1, 0), (1, 2, 0), (0, 1, 2)\}$
16. $\{(1, -2, 2), (2, 2, 1), (2, -1, -2)\}$
17. $\{(1, 0, 0), (1, 1, 1), (1, 1, -1)\}$
18. $\{(4, -3, 0), (1, 2, 0), (0, 0, 4)\}$
19. $\{(0, 1, 2), (2, 0, 0), (1, 1, 1)\}$
20. $\{(0, 1, 1), (1, 1, 0), (1, 0, 1)\}$
21. $\{(1, 2, -2), (1, 0, -4), (5, 2, 0), (1, 1, -1)\}$
22. $\{(-3, 1, 2), (1, 1, 1), (2, 0, -1), (1, -3, 2)\}$
23. $\{(2, 1, 1), (0, 3, -1), (3, -4, -2), (-1, -1, 3)\}$
24. $\vec{u}_1 = (1, 1, 0, -1), \vec{u}_2 = (1, 3, 0, 1), \vec{u}_3 = (4, 2, 2, 0)$
25. $\vec{u}_1 = (1, 1, 1, 1), \vec{u}_2 = (1, 1, 2, 4), \vec{u}_3 = (1, 2, -4, -3)$

26. $\{(3, 4, 0, 0), (-1, 1, 0, 0), (2, 1, 0, -1), (0, 1, 1, 0)\}$

27. Find the **QR**-decomposition of

a) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

28. Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product

$$\vec{u} = (0, -2, 2, 1), \quad \vec{v} = (-1, -1, 1, 1)$$

(29 – 33) Apply the Gram-Schmidt **orthonormalization** process in $\mathbb{C}^0[-1, 1]$ spanned by the functions, using the inner product

29. $f_1(x) = x + 2, \quad f_2(x) = x^2 - 3x + 4$

30. $f_1(x) = x, \quad f_2(x) = x^3, \quad f_3(x) = x^5$

31. $f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = \frac{1}{2}(3x^2 - 1)$

32. $f_1(x) = 1, \quad f_2(x) = \sin \pi x, \quad f_3(x) = \cos \pi x$

33. $f_1(x) = \sin \pi x, \quad f_2(x) = \sin 2\pi x, \quad f_3(x) = \sin 3\pi x$

34. For $\mathbb{P}_3[x]$, define the inner product over \mathbb{R} as

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx$$

a) If $f(x) = 1$ is a unit vector in $\mathbb{P}_3[x]$?

b) Find an orthonormal basis for the subspace spanned by x and x^2 .

c) Complete the basis in part (b) to an orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product.

d) Is

$$[f, g] = \int_0^1 f(x)g(x) dx$$

Also, an inner product for $\mathbb{P}_3[x]$

e) Find a pair of vectors \vec{v} and \vec{w} such that

$$\langle \vec{v}, \vec{w} \rangle = 0 \quad \text{but} \quad [\vec{v}, \vec{w}] \neq 0$$

f) Is the basis found in part (c) an orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product in part (d)?

Section 3.4 – Orthogonal Matrices

Definition

A square matrix A is said to be orthogonal if its transpose is the same as its inverse, that is, if

$$A^{-1} = A^T$$

or, equivalently, if

$$AA^T = A^T A = I$$

Example

The matrix $A = \begin{pmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{pmatrix}$

Solution

$$\begin{aligned} A^T A &= \begin{pmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{pmatrix} \begin{pmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Example

The matrix $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

Solution

$$\begin{aligned} A^T A &= \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Theorem

The following are equivalent for $n \times n$ matrix A .

- a) A is orthogonal.
- b) The row vectors of A form an orthonormal set in R^n with the Euclidean inner product.
- c) The column vectors of A form an orthonormal set in R^n with the Euclidean inner product.

Theorem

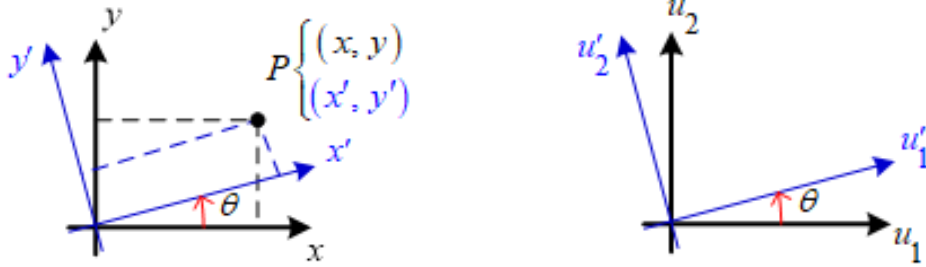
- a) The inverse of an orthogonal matrix is orthogonal
- b) A product of orthogonal matrices is orthogonal
- c) If A is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$

Theorem

If A is an $n \times n$ matrix, then the following are equivalent

- a) A is orthogonal.
- b) $\|A\vec{x}\| = \|\vec{x}\|$ for all \vec{x} in R^n .
- c) $A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$ for all \vec{x} and \vec{y} in R^n .

Let \vec{u}_1 and \vec{u}_2 be the unit vectors along the x - and y -axes and unit vectors \vec{u}'_1 and \vec{u}'_2 along the x' - and y' -axes.

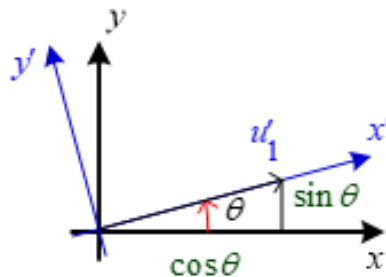


The new coordinates (x', y') and the old coordinates (x, y) of a point P will be related by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

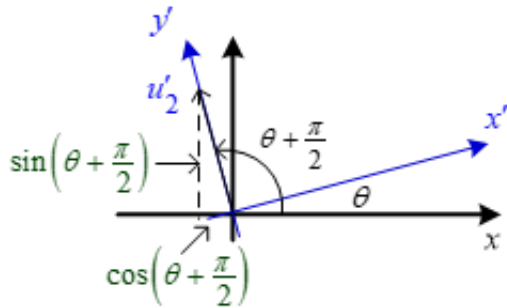
$$P = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$P^{-1} = P^T = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$



$$\begin{bmatrix} \vec{x}' \\ \vec{y}' \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix}$$

$$\rightarrow \begin{cases} x' = x \cos \theta + y \sin \theta \\ y' = -x \sin \theta + y \cos \theta \end{cases}$$



These are sometimes called the *rotation equations*.

Example

Use the form $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ to find the new coordinates of the point $Q(2, 1)$ if the coordinate axes of a rectangular coordinate system are rotated through an angle of $\theta = \frac{\pi}{4}$.

Solution

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} \cos \frac{\pi}{4} & \sin \frac{\pi}{4} \\ -\sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

The new coordinates of Q are $(x', y') = \left(\frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$

Exercises Section 3.4 – Orthogonal Matrices

(1 – 2) Show that the matrix is orthogonal

$$1. \quad A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$$

$$2. \quad A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

(3 – 12) Determine if the matrix is orthogonal. For those that is orthogonal find the inverse.

$$3. \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$8. \quad \begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$11. \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

$$4. \quad \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$9. \quad \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$12. \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$$

$$5. \quad \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

$$6. \quad \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$$

$$7. \quad \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$$

$$10. \quad \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

13. Find a last column so that the resulting matrix is orthogonal

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \dots \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \dots \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & \dots \end{bmatrix}$$

14. Determine if the given matrix is orthogonal. If it is, find its inverse

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{5} & \frac{3}{7} \\ \frac{4}{9} & \frac{3}{5} & -\frac{2}{7} \\ \frac{8}{9} & -\frac{2}{5} & \frac{3}{7} \end{bmatrix}$$

15. Prove that if A is orthogonal, then A^T is orthogonal.
16. Prove that if A is orthogonal, then A^{-1} is orthogonal.
17. Prove that if A and B are orthogonal, then AB is orthogonal.
18. Let Q be an $n \times n$ orthogonal matrix, and let A be an $n \times n$ matrix. Show that $\det(QAQ^T) = \det(A)$

19. Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$

- a) Is matrix A an orthogonal matrix?
- b) Let B be the matrix obtained by normalizing each row of A , find B .
- c) Is B an orthogonal matrix?
- d) Are the columns of B orthogonal?

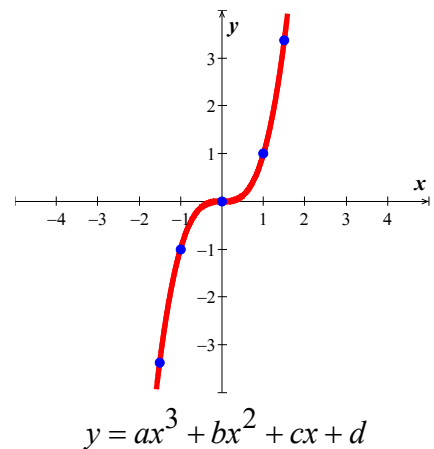
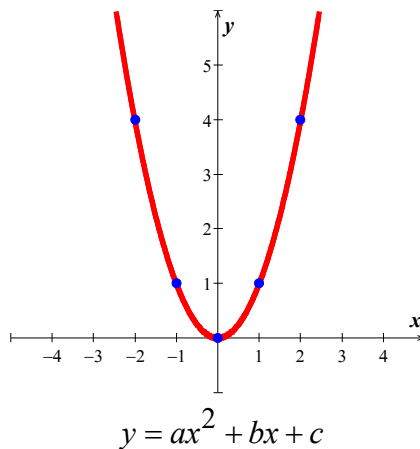
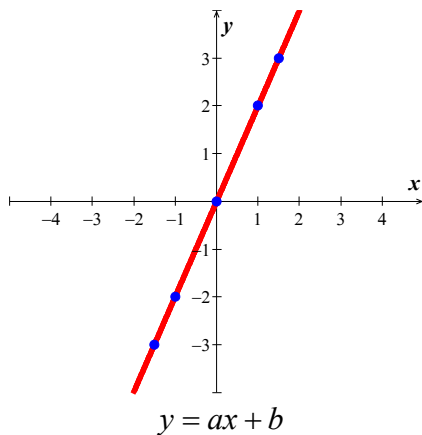
Section 3.5 – Least Squares Analysis

The use to **best** fit data, we will use results about orthogonal projections in inner product spaces to obtain a technique for fitting a line or other polynomial.

Fitting a Curve to Data

The common problem is to obtain a mathematical relationship between 2 variables x and y by **fitting** a curve to points in the xy -plane.

Some possibility of fitting the data



Least Squares Fit of a Straight Line

Recall that a system of equations $A\vec{x} = \vec{y}$ is called inconsistent if it does not have a solution. Suppose we want to fit a straight line $y = mx + b$ to the determined points $(x_1, y_1), \dots, (x_n, y_n)$

If the data points were collinear, the line would pass through all n points and the unknown coefficients m and b would satisfy the equations

$$\begin{array}{l} y_1 = mx_1 + b \\ y_2 = mx_2 + b \\ \vdots \\ y_n = mx_n + b \end{array} \Rightarrow \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$A \quad \vec{x} = \vec{y}$

The problem is to find m and b that minimize the errors in some sense.

Least Square Problem

Given a linear system $A\vec{x} = \vec{y}$ of m equations in n unknowns, find a vector \vec{x} that minimizes $\|\vec{y} - A\vec{x}\|$ with respect to the Euclidean inner product on \mathbb{R}^m . We call such as \vec{x} a least squares solution of the system, we call $\vec{y} - A\vec{x}$ the least squares error vectors, and we call $\|\vec{y} - A\vec{x}\|$ the least squares error.

$$A\mathbf{x} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix}$$

The term “*least square solution*” results from the fact the minimizing $\|\vec{y} - A\vec{x}\| = e_1^2 + e_2^2 + \dots + e_m^2$

Example

Find the sums of squares of the errors of (2, 4), (4, 8), (6, 6)

Solution

$$4 = 2m + b \Rightarrow 4 - 2m - b = e_1$$

$$8 = 4m + b \Rightarrow 8 - 4m - b = e_2$$

$$6 = 6m + b \Rightarrow 6 - 6m - b = e_3$$

$$e_1^2 + e_2^2 + \dots + e_m^2 = (4 - 2m - b)^2 + (8 - 4m - b)^2 + (6 - 6m - b)^2$$

The least squares problem for this example to find the values m and b for which $e_1^2 + e_2^2 + \dots + e_m^2$ is a minimum.

Theorem

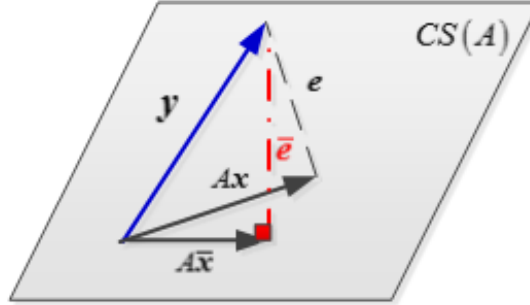
If A is an $m \times n$ matrix, the equation $A\vec{x} = \vec{y}$ has a solution if and only if \vec{y} is in the column space of A .

$$\vec{y} - A\vec{x} = \vec{e}$$

$A\vec{x}$ is a vector that is in the column space of A . For this A the column space is a plane in \mathbb{R}^m

\vec{y} is a vector, not in the column space of A (otherwise $A\vec{x} = \vec{y}$ has an exact solution)

\vec{e} is the error vector, the difference between \vec{y} and $A\vec{x}$



The length $\|\vec{e}\|$ is a *minimum* exactly when $\vec{e} \perp CS(A)$

Best Approximation Theorem

If $CS(A)$ is a finite dimensional subspace of an inner product space, and if \vec{y} is a vector in V , then

$proj_{CS(A)} \vec{y}$ is the best approximation to \vec{y} from $CS(A)$ in the sense that

$$\left\| \vec{y} - proj_{CS(A)} \vec{y} \right\| < \left\| \vec{y} - \vec{w} \right\|$$

For every vector \vec{w} in $CS(A)$ that is different from $proj_{CS(A)} \vec{y}$

Theorem

For every linear system $A\vec{x} = \vec{y}$, the associated normal system

$$A^T A \vec{x} = A^T \vec{y}$$

is consistent, and all solutions are least squares solutions of $A\vec{x} = \vec{y}$

If the columns of A are linearly independent, then $A^T A$ is invertible so has a unique solution \vec{x} .

This solution is often expressed theoretically as

$$\left(A^T A \right)^{-1} A^T A \vec{x} = \left(A^T A \right)^{-1} A^T \vec{y}$$

$$\bar{x} = \left(A^T A \right)^{-1} A^T \vec{y}$$

Proof

Let the vector \bar{x} is a least squares solution to $A\bar{x} = \vec{y} \Leftrightarrow (\vec{y} - A\bar{x}) \perp CS(A)$

$$(\vec{y} - A\bar{x}) \cdot \vec{z} = 0 \quad \vec{z} \text{ in } CS(A) \quad \& \quad \vec{z} = A\vec{w}$$

$$(\vec{y} - A\bar{x}) \cdot A\vec{w} = 0 \quad \vec{w} \text{ in } \mathbb{R}^n$$

$$A^T (\vec{y} - A\bar{x}) \cdot \vec{w} = 0$$

$$A^T (\vec{y} - A\bar{x}) = 0$$

$$A^T \vec{y} - A^T A\bar{x} = 0$$

$$A^T \vec{y} = A^T A\bar{x}$$

Theorem

If A is an $m \times n$ matrix, then the following are equivalent

- a) A has linearly independent column vectors.
- b) $A^T A$ is invertible.

Example

Find the equation of the line that best fits the given points in the least-squares sense.

(40, 482), (45, 467), (50, 452), (55, 432), (60, 421)

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$$

$$\text{Where } A = \begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$$

Using the normal equation formula: $A^T Ax = A^T y$

$$\begin{pmatrix} 40 & 45 & 50 & 55 & 60 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 40 & 45 & 50 & 55 & 60 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$$

$$\begin{pmatrix} 12,750 & 250 \\ 250 & 5 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 111,970 \\ 2,255 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{1250} \begin{pmatrix} 5 & -250 \\ -250 & 12,750 \end{pmatrix} \begin{pmatrix} 111,970 \\ 2,255 \end{pmatrix}$$

$$= \begin{pmatrix} -3.12 \\ 607 \end{pmatrix}$$

Or

$$m = \frac{\begin{vmatrix} 111,970 & 250 \\ 2,255 & 5 \end{vmatrix}}{\begin{vmatrix} 12,750 & 250 \\ 250 & 5 \end{vmatrix}}$$

$$= \frac{-3,900}{1,250}$$

$$= -\frac{78}{25}$$

$$b = \frac{\begin{vmatrix} 12,750 & 111,970 \\ 250 & 2,255 \end{vmatrix}}{1,250}$$

$$= \frac{758,750}{1,250}$$

$$= 607$$

$$\text{Thus, } y = -\frac{78}{25}x + 607 \quad \text{or} \quad y = -3.12x + 607$$

Example

Given the system equation:
$$\begin{cases} x_1 - x_2 = 4 \\ 3x_1 + 2x_2 = 1 \\ -2x_1 + 4x_2 = 3 \end{cases}$$

- a) Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$
- b) Find the orthogonal projection of \vec{y} on the column space of A
- c) Find the **error vector** and the **error**

Solution

$$a) \quad A = \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

$$A^T A \vec{x} = A^T \vec{y}$$

$$\begin{pmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 14 & -3 \\ -3 & 21 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{285} \begin{pmatrix} 21 & 3 \\ 3 & 14 \end{pmatrix} \begin{pmatrix} 1 \\ 10 \end{pmatrix} \quad X = A^{-1}B$$

$$= \begin{pmatrix} \frac{51}{285} \\ \frac{143}{285} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{17}{95} \\ \frac{143}{285} \end{pmatrix}$$

$$\text{Thus } y = \frac{17}{95}x + \frac{143}{285} \quad \text{or} \quad y = 0.1789x + 0.5018$$

- b) The orthogonal projection of \vec{y} on the column space of A

$$A\vec{x} = \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} \frac{17}{95} \\ \frac{143}{285} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{pmatrix}$$

$$c) \quad \vec{y} - A\vec{x} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1232}{285} \\ -\frac{154}{285} \\ \frac{4}{3} \end{pmatrix}$$

The **error**: $\|\vec{y} - A\vec{x}\| = \sqrt{\left(\frac{1232}{285}\right)^2 + \left(-\frac{154}{285}\right)^2 + \left(\frac{4}{3}\right)^2}$
 ≈ 4.556

Exercises Section 3.5 – Least Squares Analysis

(1 – 7) Find the equation of the line that best fits the given points in the least-squares sense and find the error.

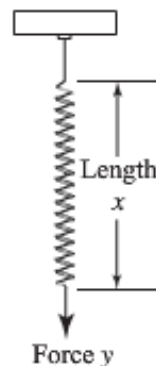
1. $\{(0, 2), (1, 2), (2, 0)\}$
2. $\{(1, 5), (2, 4), (3, 1), (4, 1), (5, -1)\}$
3. $\{(0, 1), (1, 3), (2, 4), (3, 4)\}$
4. $\{(-2, 0), (-1, 0), (0, 1), (1, 3), (2, 5)\}$
5. $\{(2, 3), (3, 2), (5, 1), (6, 0)\}$
6. $\{(-1, 0), (0, 1), (1, 2), (2, 4)\}$
7. $\{(1, 0), (2, 1), (4, 2), (5, 3)\}$

(8 – 10) Find the orthogonal projection of the vector \vec{u} on the subspace of \mathbb{R}^4 spanned by the vectors

8. $\vec{u} = (-3, -3, 8, 9); \vec{v}_1 = (3, 1, 0, 1), \vec{v}_2 = (1, 2, 1, 1), \vec{v}_3 = (-1, 0, 2, -1)$
9. $\vec{u} = (6, 3, 9, 6); \vec{v}_1 = (2, 1, 1, 1), \vec{v}_2 = (1, 0, 1, 1), \vec{v}_3 = (-2, -1, 0, -1)$
10. $\vec{u} = (-2, 0, 2, 4); \vec{v}_1 = (1, 1, 3, 0), \vec{v}_2 = (-2, -1, -2, 1), \vec{v}_3 = (-3, -1, 1, 3)$

11. Find the standard matrix for the orthogonal projection P of \mathbb{R}^2 on the line passes through the origin and makes an angle θ with the positive x -axis.

12. Hooke's law in physics states that the length x of a uniform spring is a linear function of the force y applied to it. If we express the relationship as $y = mx + b$, then the coefficient m is called the spring constant.



Suppose a particular unstretched spring has a measured length of 6.1 inches.(i.e., $x = 6.1$ when $y = 0$). Forces of 2 pounds, 4 pounds, and 6 pounds are then applied to the spring, and the corresponding lengths are found to be 7.6 inches, 8.7 inches, and 10.4 inches. Find the spring constant.

13. Prove: If A has a linearly independent column vectors, and if \vec{b} is orthogonal to the column space of A , then the least squares solution of $A\vec{x} = \vec{b}$ is $\vec{x} = \vec{0}$.
14. Let A be an $m \times n$ matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of \mathbb{R}^n onto the row space of A .
15. Let W be the line with parametric equations $x = 2t, \quad y = -t, \quad z = 4t$
- Find a basis for W .
 - Find the standard matrix for the orthogonal projection on W .
 - Use the matrix in part (b) to find the orthogonal projection of a point $P_0(x_0, y_0, z_0)$ on W .
 - Find the distance between the point $P_0(2, 1, -3)$ and the line W .
16. In \mathbb{R}^3 , consider the line l given by the equations $x = t, \quad y = t, \quad z = t$
 And the line m given by the equations $x = s, \quad y = 2s - 1, \quad z = 1$
 Let P be the point on l , and let Q be a point on m .
 Find the values of t and s that minimize the distance between the lines by minimizing the squared distance $\|P - Q\|^2$
17. Determine whether the statement is true or false,
- If A is an $m \times n$ matrix, then $A^T A$ is a square matrix.
 - If $A^T A$ is invertible, then A is invertible.
 - If A is invertible, then $A^T A$ is invertible.
 - If $A\vec{x} = \vec{b}$ is a consistent linear system, then $A^T A\vec{x} = A^T \vec{b}$ is also consistent.
 - If $A\vec{x} = \vec{b}$ is an inconsistent linear system, then $A^T A\vec{x} = A^T \vec{b}$ is also inconsistent.
 - Every linear system has a least squares solution.
 - Every linear system has a unique least squares solution.
 - If A is an $m \times n$ matrix with linearly independent columns and \vec{b} is in \mathbb{R}^m , then $A\vec{x} = \vec{b}$ has a unique least squares solution.
18. A certain experiment produces the data $\{(1, 1.8), (2, 2.7), (3, 3.4), (4, 3.8), (5, 3.9)\}$.
 Find the function that it will fit these data in the form of $y = \beta_1 x + \beta_2 x^2$

19. According to Kepler's first law, a comet should have an ellipse, parabolic, or hyperbolic orbit (with gravitational attractions from the planets ignored). In suitable polar coordinates, the position (r, ν) of a comet satisfies an equation of the form

$$r = \beta + e(r \cdot \cos \nu)$$

Where β is a constant and e is the eccentricity of the orbit, with $0 \leq e < 1$ for an ellipse, $e = 1$ for a parabolic, and $e > 1$ for a hyperbola.

Suppose observations of a newly discovered comet provide the data below.

ν	.88	1.10	1.42	1.77	2.14
r	3.00	2.30	1.65	1.25	1.01

Determine the type of orbit, and predict where the orbit will be when $\nu = 4.6$ (*radians*)?

20. To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every second, from $t = 0$ to $t = 12$

The position (in *feet*) were:

0, 8.8, 29.9, 62.0, 104.7, 159.1, 222.0, 294.5, 380.4, 471.1, 571.7, 686.8, and 809.2

- a) Find the least square cubic curve $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$ for these data.
- b) Estimate the velocity of the plane when $t = 4.5$ *sec*, using the result from part (a).