***Section* 3.7 – Change of variables in Multiple Integrals**

**Substitution in Double Integrals**

Suppose that a region *G* in the *uv*-plane is transformed one-to-one into the region *R* in the *xy*-plane by equations of the form



|  |  |  |
| --- | --- | --- |
| ***Cartesian*** *uv*-plane |  | ***Cartesian*** *xy* - plane |

*R* is the image of *G* under the transformation, and *G* the ***preimage*** of *R*.



***Definition***

The ***Jacobian determinant*** or ***Jacobian*** of the coordinate transformation  is





***Example***

Find the Jacobian for the polar coordinate transformation , write the Cartesian integral  as a polar integral.

***Solution***

|  |  |  |
| --- | --- | --- |
| *Cartesian* *rθ* −plane |  | *Cartesian* *xy* −plane |

 transform the rectangle *G*: , into the quarter circle *R* bounded by  in *Q*I.











***Example***

Evaluate  by applying the transformation  and integrating over an appropriate region in the *uv*-plane.

***Solution***











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| ***xy*-eqns for the boundary of *R*** | **Corresponding *uv*-eqns. for the boundary of *G*** | **Simplified *uv*-*eqns*.** |
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***Example***

Evaluate 

***Solution***



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| ***xy*-eqns for the boundary of *R*** | **Corresponding *uv*-eqns. for the boundary of *G*** | **Simplified *uv*-eqns.** |
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***Example***

Evaluate 

***Solution***





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| ***xy*-eqns for the boundary of *R*** | **Corresponding *uv*-eqns. for the boundary of *G*** | **Simplified *uv*-eqns.** |
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**Substitutions in Triple Integrals**



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| ***Cartesian*** *uvw* - plane |  | ***Cartesian*** *xyz* - plane |



The ***Jacobian determinant*** is



|  |  |  |
| --- | --- | --- |
| *Cube with sides parallel to the axes*    ***Cartesian*** *rθ z* - plane |  | *Cube with sides parallel to the axes*    ***Cartesian*** *xyz* - plane |









For spherical coordinates*, ρ, φ*, and *θ* take the place of *u, v*, and *w*. The transformation from Cartesian *ρφθ* -space to Cartesian *xyz* –space is given by



The Jacobian of the transformation

















|  |  |  |
| --- | --- | --- |
| ***Cartesian*** *rθ z* - plane |  | ***Cartesian*** *xyz* - plane |

***Example***

Evaluate  by applying the transformation  and integrating over an appropriate region in the *uvw*-plane.

***Solution***



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| --- | --- | --- |
| ***xyz*-*eqns* for the boundary of *D*** | **Corresponding *uvw*- *eqns*. for the boundary of *G*** | **Simplified *uvw*- *eqns*.** |
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***Example***

Evaluate : *D* is bounded by the planes:      and 

***Solution***















 























***Exercises*** ***Section* 3.7 – Change of Variables in Multiple Integrals**

Let  be a unit square in the *uv-*plane. Find the image of *S* in the *xy-*plane under the following transformations.

|  |  |
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|  |  |

1. *a*) Solve the system  for *x* and *y* in terms of *u* and *v*. Then find the value of

the Jacobian 

*b*) Find the image under the transformation  of the triangular region with vertices (0, 0), (1, 1), and (1, −2) in the *xy*-plane. Sketch the transformed region in the *uv*-plane.

1. Let *R* be the region in the first quadrant of the *xy*-plane bounded by the hyperbolas  and the lines . Use the transformation  with  to rewrite



As an integral over an appropriate region *G* in the uv-plane. Then evaluate the *uv*-integral over *G*.

1. The area  of the ellipse  can be found by integrating the function  over the region bounded by the ellipse in the *xy*-plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation  and evaluate the transformed integral over the disk *G*:  in the *uv*-plane. Find the area this way.
2. Use the transformation  to evaluate the integral



By first writing it as an integral over a region *G* in the *uv*-plane.

1. Use the transformation  to evaluate the integral



1. Find the Jacobian  of the transformation
2. 
3. 
4. Find the Jacobian  of the transformation
5. 
6. 
7. Evaluate the appropriate determinant to show that the Jacobian of the transformation from Cartesian *ρφθ*−space to Cartesian *xyz*−space is 
8. How can substitutions in single definite integrals be viewed as transformations of regions? What is the Jacobian in such a case? Illustrate with an example.
9. Find the volume of the ellipsoid 

(***Hint***: Let . Then find the volume of an appropriate region in *uvw*-space)

1. Use the transformation  to evaluate the integral



(***Hint***: Show that the image of the triangular region ***G*** with vertices (0, 0), (1, 0), (1, 1) in the *uv-*plane is the region of integration ***R*** in the *xy-*plane defined by the limits of integration.)

1. Evaluate ; *R* is the region bounded by the hyperbolas  and  and the lines , and 
2. Evaluate ; *R* is the region bounded by the lines , , , and 
3. Evaluate ; *R* is bounded by the planes , , , ,  and 
4. Let *R* be the region bounded by the lines 

Evaluate the integral 

1. Let *R* be the region bounded by the square with vertices (0, 1), (1, 2), (2, 1), & (1, 0).

Evaluate the integral 

1. Evaluate  *D* is bounded by the planes:      and 
2. Evaluate ; *R* is the square with vertices , , , and 
3. Evaluate ; 
4. Evaluate ; 
5. Evaluate ; where *R* is bounded by the ellipse .
6. Evaluate 
7. Evaluate ; where *R* is the diamond bounded by   , and 
8. Evaluate ; where *R* is the parallelogram bounded by   , and 
9. Evaluate ; where *R* is the region bounded by   , and 
10. Evaluate ; where *R* is the region bounded by the hyperbolas   , and 
11. Evaluate ; where *R* is the triangular region bounded by   and 
12. Evaluate : *D* is bounded by the planes:      and 
13. Evaluate : *D* is bounded by the planes:      and 
14. Evaluate : *D* is bounded by the paraboloid  and the *xy-*plane.
15. Evaluate : *D* is bounded by the upper half of the ellipsoid  and the *xy-*plane.
16. Evaluate : *D* is bounded by the planes:      and 

**(37 − 41)**  Let *R* be the region bounded by the ellipse , where  and  are real numbers.

1. Find the area of *R*.
2. Evaluates 
3. Find the center of mass of the upper half of *R*  assuming it has a constant density.
4. Find the average square of the distance between points of *R* and the origin.
5. Find the average distance between points in the upper half of *R* and the *x-*axis.

**(42 − 45)** Let *D* be the region bounded by the ellipsoid , where ,  and  are real numbers.

1. Find the Volume of *D*.
2. Evaluates 
3. Find the center of mass of the upper half of *D*  assuming it has a constant density.
4. Find the average square of the distance between points of *D* and the origin.