***Solution*** ***Section* 4.7 – Stokes’ Theorem**

***Exercise***

Verify that the line integral and the surface integral of Stokes’ Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

; *S* is the upper half of the sphere  and *C* is the circle  in the *xy-*plane

***Solution***

































***Or***

Using the standard parametrization of the sphere













***Exercise***

Verify that the line integral and the surface integral of Stokes’ Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

; *S* is the upper half of the sphere  and *C* is the circle  in the *xy-*plane

***Solution***































































***Exercise***

Verify that the line integral and the surface integral of Stokes’ Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

; *S* is the paraboloid  for  and *C* is the circle  in the *xy-*plane

***Solution***

















Surface integral: 

***Exercise***

Verify that the line integral and the surface integral of Stokes’ Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

; *S* is the cap of the sphere  above the plane  and *C* is the boundary of *S*.

***Solution***



 is the intersection of the sphere with the plane .



































































***Exercise***

Verify that the line integral and the surface integral of Stokes’ Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

; *S* is the cap of the sphere  above the plane  and *C* is the boundary of *S*.

***Solution***



 is the intersection of the sphere with the plane .





























































***Exercise***

Verify that the line integral and the surface integral of Stokes’ Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

; *S* is the part of the plane  that lies in the cylinder  and *C* is the boundary of *S*.

***Solution***



















***Exercise***

Evaluate the line integral  by evaluating the surface integral in Stokes’ Theorem with an appropriate choice of *S*. Assume that *C* has a counterclockwise orientation,

; *C* is the circle  in the plane .

***Solution***



















***Exercise***

Evaluate the line integral  by evaluating the surface integral in Stokes’ Theorem with an appropriate choice of *S*. Assume that *C* has a counterclockwise orientation,

; *C* is the ellipse  in the plane .

***Solution***











 Because 





***Exercise***

Evaluate the line integral  by evaluating the surface integral in Stokes’ Theorem with an appropriate choice of *S*. Assume that *C* has a counterclockwise orientation,

; *C* is the boundary of the plane  in the plane first octant.

***Solution***



























***Exercise***

Evaluate the line integral  by evaluating the surface integral in Stokes’ Theorem with an appropriate choice of *S*. Assume that *C* has a counterclockwise orientation,

; *C* is the circle  for  .

***Solution***







*S* is the disk 























***Exercise***

Evaluate the line integral  by evaluating the surface integral in Stokes’ Theorem with an appropriate choice of *S*. Assume that *C* has a counterclockwise orientation,

; *C* is the boundary of the plane  in the first octant.

***Solution***











***Exercise***

Evaluate the line integral  using Stokes’ Theorem, where ; *C*: is the circle  in the *xy-*plane. Assume *C* has counterclockwise orientation.

***Solution***













***Exercise***

Evaluate the line integral  using the Stoke’s Theorem ; *C* is the boundary of the plane **** in the first octant and has counterclockwise orientation.

***Solution***







****

****



















***Exercise***

Evaluate the line integral  by evaluating the surface integral in Stokes’ Theorem with an appropriate choice of *S*. Assume that *C* has a counterclockwise orientation

; *C* is the boundary of the square  in the plane 

***Solution***

Square bounded by , then 

















***Exercise***

Evaluate the line integral in Stokes’ Theorem to evaluate the surface integral . Assume that  points in an upward direction,

; *S* is the upper half of the ellipsoid 

***Solution***









Let 





















***Exercise***

Evaluate the line integral in Stokes’ Theorem to evaluate the surface integral . Assume that  points in an upward direction,

; *S* is the cap of the sphere 

***Solution***

The boundary of the surface is the intersection of the plane  and 

At 































***Exercise***

Evaluate the line integral in Stokes’ Theorem to evaluate the surface integral . Assume that  points in an upward direction,

; *S* is the tilted disk enclosed 

***Solution***































*S* is the disk 





















***Exercise***

Evaluate the line integral in Stokes’ Theorem to evaluate the surface integral. Assume that ******  points in an upward direction

; *S* is the paraboloid  (excluding its base), and 

***Solution***



















***Exercise***

Use Stoke’s Theorem to evaluate the surface integral ; , where *S* is the hyperboloid . Assume that  is the *outward normal*.

***Solution***





























***Exercise***

Use Stokes’ Theorem to evaluate the surface integral, given , where *S* is the hemisphere , for . Assume that  is the outward normal.

***Solution***

Let 

























***Exercise***

Use Stokes’ Theorem to find the circulation of the following vector fields around any simple closed smooth curve *C*. 

***Solution***

This is a conservative vector field, and since it is around closed curve, then

 (for any closed curve)

***Exercise***

Use Stokes’ Theorem to find the circulation of the following vector fields around any simple closed smooth curve *C*. 

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***Exercise***

Use Stokes’ Theorem to find the circulation of the following vector fields around any simple closed smooth curve *C*. 

***Solution***

This is a conservative vector field with , and since it is around closed curve, then



***Exercise***

Use Stokes’ Theorem to find the circulation of the following vector fields around any simple closed smooth curve *C*. 

***Solution***

This is a conservative vector field with , and since it is around closed curve, then



***Exercise***

Use Stokes’ Theorem and a surface integral to find the circulation on *C* of the vector field  as a function of . For what value of  is the circulation a maximum?

***Solution***





























The maximum value of the circulation when  which is 

***Exercise***

A circle *C* in the plane  has a radius of 4 and center (2, 3, 3). Evaluate  for  where *C* has a counterclockwise orientation when viewed from above. Does the circulation depend on the radius of the circle? Does it depend on the location of the center of the circle?

***Solution***















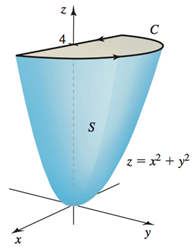


***Exercise***

Begin with the paraboloid , for , and slice it with the plane . Let *S* be the surface that remains for  (including the planar surface in the *xz-*plane). Let *C* be the semicircle and line segment that bound the cap of *S* in the plane  with counterclockwise orientation. Let 

1. Describe the direction of the vectors normal to the surface that are consistent with the orientation of *C*.
2. Evaluate 
3. Evaluate  and check for argument with part (*b*).

***Solution***

1. The normal vector point toward the *z-*axis on the curved surface of *S* and in the direction  on the flat surface of *S*.
2. 





The planar surface in the *xz-*plane, then let  be the surface parameterized by .

Where, since ,

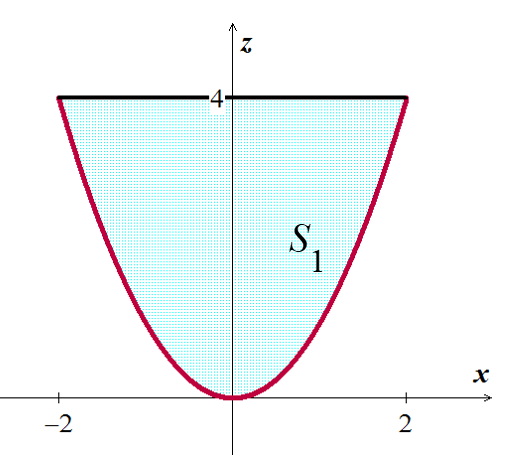


and 























Let  be the surface of the half of the paraboloid for , parametrized as

































1. 









































***Exercise***

The French Physicist André-Marie Ampère discovered that an electrical current *I* in a wire produces a magnetic field *B*. A special case of Ampère’s Law relates the current to the magnetic field through the equation , where *C* is any closed curve through which the wire passes and *μ* is a physical constant. Assume that the current *I* is given in terms of the current density ***J*** as , where *S* is an oriented surface with *C* as a boundary. Use Stokes’ Theorem to show that an equivalent form of Ampère’s Law is .

***Solution***









Thus 

For all surfaces *S* bounded by any given closed curve *C*.

It is clear that given the freedom to choose *C* and *S*, that it follows that the integrand is identically zero, i.e. that for any surface *S*, .

From this, it is easy to see that we must have , since we are free to make normal vector point in any direction at any given point by choosing *S* appropriately.

***Exercise***

Let S be the paraboloid , for , where *a* > 0 is a real number. Let . For what value(s) of *a* (if any) does  have its maximum value?

***Solution***























∴ The integral is independent of *a*.

***Exercise***

The goal is to evaluate , where  and *S* ids the surface of the upper half of the ellipsoid 

1. Evaluate a surface integral over a more convenient surface to find the value of *A*.
2. Evaluate *A* using a line integral.

***Solution***

1. The boundary of this surface is the circle  at 









At 







1. With the parameterization of the boundary circle and , we have







***Exercise***

Let  and let *S* be the hemisphere of radius *a* with its base in the *xy-*plane and center at the origin.

1. Evaluate  by computing  and appealing to symmetry.
2. Evaluate the line integral using Stokes’ Theorem to check part (*a*).

***Solution***

1. 





*S*: 













By symmetry, the integral vanishes on each level curve, so it vanishes altogether.

1. Let 















***Exercise***

Let *S* be the disk enclosed by the curve  fot , where  is a fixed angle.

1. Find the a vector normal to *S*.
2. What is the areas of *S*?
3. Whant the length of *C*?
4. Use the Stokes’ Theorem and a surface integral to find the ciurculation on *C* of the vector field  as a function of *φ*. For what value of *φ* is the circulation a maximum?
5. What is the circulation on *C* of the vector field  as a function of *φ*? For what value of *φ* is the circulation a maximum?
6. Consider the vector field , where  is a constant nonzero vector and . Show that the circulation is a maximum when  points in the direection of the normal to *S*.

***Solution***

1. 

















1. 









(this surface is simply the unit circle inclined at the angle *φ* to the *xy-*plane)

1. 















(Because it just the circumference of the unit circle)

1. 















The maximum when 

The circulation has a maximum of  at .

1. 





















The maximum when  

The maximum circulation is  at .

1.  

























When  points in the direection of the normal to *S* their cross-product is zero.









***Exercise***

Let *R* be a region in a plane that has a unit normal vector  and boundary *C*. Let 

1. Show that 
2. Use Stokes’ Theorem to show that



1. Consider the curve *C* given by , for . Prove that *C* lies in a plane by showing that  is constant for all *t*.
2. Use part (*b*) to find the area of the region enclosed by *C* in part (*c*). (*Hint*: Find the unit normal vector that is consistent with the orientation of *C*.)

***Solution***

1. 







 ***√***

1. 



 Since 





 ***√***

1. 











∴  is constant for all *t*, so that  must lie in a plane.

1. 



































***Exercise***

Consider the radial vector fields , where *p* is a real number and . Let *C* be any circle in the *xy-*plane centered at the origin.

1. Evaluate a line integral to show that the field has zero circulation on *C*.
2. For what values of *p* does Stokes’ Theorem apply? For those values of *p*, use the surface integral in Stokes’ Theorem to show that the field has zero circulation on *C*.

***Solution***

1. Let 





















1. Stokes’ Theorem will apply when the vector field is defined throughout the disk of radius *a*, which happens only .

In this case, , so that the surface integral is zero.

***Exercise***

Consider the vector fierld 

1. Show that 
2. Show that  is not zero on circle *C* in the *xy-*plane enclosing the origin.
3. Explain why Stokes’ Theorem does not apply in this case.

***Solution***

1. 



 ***√***

1. Let 

















1. The Theroem does not apply because the vector field is not defined at the origin,, which is inside the curve *C*.

The limit of the *y-*coordinate is different depending on the direction.

***Exercise***

Let *S* be a small circular disk of radius *R* centered at the point *P* with a unit normal vector . Let *C* be the boundary of *S*.

1. Express the average circulation of the vector field  on *S* as a surface integral of 
2. Argue for that small *R*, the average circulation approaches  (the component of  in the direction of  evaluated at *P*) with the approximation improving as .

***Solution***

1. The circumference of the disk is , so the average circulation is



1. As *R* becomes small, because the vector field  and thus  are continuous.

 can be made arbitrarily close to  everywhere on *S* by taking *R* small enough.

Approximately, then



So that







As , the approximation  becomes better, so the value of the integral does as well.