***Solution*** ***Section* 4.8 – Divergence Theorem**

***Exercise***

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement. 

***Solution***





 

 



 











































***Exercise***

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement. 

***Solution***







 





Since the surface has a form of cube, therefore we have 6 surfaces

 

 

 

 

 

 











***Exercise***

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement. 

***Solution***













































***Exercise***

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement. 

***Solution***













































































***Exercise***

Use the Divergence Theorem to compute the outward flux of the vector field  across *S* is the surface of the cube cut from the first octant by the planes .

***Solution***

















***Exercise***

Use the Divergence Theorem to compute the outward flux of the vector field  across *S* is the sphere 

***Solution***





















***Exercise***

Use the Divergence Theorem to compute the outward flux of the vector field  across *D* is the region between two spheres of radius 1 and 2 centered at 

***Solution***



















So, the flux is zero across any surface that bounds a region where  is defined and differentiable; the given region does not include zero.

Thus, the net outward flux is zero.

***Exercise***

Use the Divergence Theorem to compute the outward flux of the vector field ; *S* is the cylinder 

***Solution***

































***Exercise***

Find the net outward flux of the field  across the sphere of radius 1 centered at the origin.

***Solution***





So, by the Divergence Theorem, the net outward flux is zero since the volume integral of  is zero.

***Exercise***

Find the net outward flux of the field  across any smooth closed surface , where *a, b*, and *c* are constants.

***Solution***





So, by the Divergence Theorem, the net outward flux is zero since the volume integral of  is zero.

***Exercise***

Find the net outward flux of the field  across the boundary of the cube 

***Solution***





So, by the Divergence Theorem, the net outward flux is zero since the volume integral of  is zero.

***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*. ; *S* is the sphere 

***Solution***







The sphere has a radius , therefore the volume of the sphere is 







***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*.

; *S* is surface of the cube cut from the first octant by the planes *x* = 1, *y* = 1, and 

***Solution***















***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*.

; *S* is boundary of the tetrahedron in the first octant formed by the plane 

***Solution***





So, by the Divergence Theorem, the net outward flux is 4 times the volume of the tetrahedron.



Volume of the tetrahedron = 

















***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*.

; *S* is the sphere 

***Solution***





So, by the Divergence Theorem, the net outward flux is

















***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following fields across the given surface *S or D*.

; *S* is the sphere 

***Solution***







The net outward flux:









***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*.

; *S* is the surface of the paraboloid , for , plus its base in the *xy-*plane

***Solution***





So, by the Divergence Theorem, the net outward flux is











***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*.

; *S* is the surface of the cone , for , plus its top surface in the plane 

***Solution***





Volume of a cone 







So, by the Divergence Theorem, the net outward flux is





***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *D*.

; *D* is the region between the spheres of radius 2 and 4 centered at origin.

***Solution***





Volume between 2 spheres 







***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *D*.

; *D* is the region between the spheres of radius 1 and 2 centered at origin.

***Solution***

































***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *D*.

; *D* is the region between the spheres of radius 1 and 2 centered at origin.

***Solution***

































*D* is the region between the spheres of radius 1 and 2 centered at origin





***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *D*.

;  is the region between two cubes

***Solution***





Therefore, by the Divergence Theorem, the net outward flux is ***zero***.

***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following fields across the given surface *S or D*.

; *S* consists of the faces of the cube 

***Solution***





The net outward flux:





***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following fields ; *D* is the region in the first octant between the planes  and 

***Solution***







|  |  |
| --- | --- |
| *Top plane* | *Bottom plane* |

The net outward flux:





























***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *D*.

; *D* is the region between the cylinders  and  for 

***Solution***





Volume of the sphere 





 







***Or*** 

Volume of the sphere 





 







***Or*** 

Therefore, the net outward flux is 

***Exercise***

Compute the outward flux of the following vector field across the given surface; *S* is the boundary of the ellipsoid 

***Solution***







Therefore, by the Divergence Theorem, the net outward flux is ***zero***.

***Exercise***

Compute the outward flux of the following vector field across the given surface ; *S* is the boundary of the ellipsoid 

***Solution***





Therefore, by the Divergence Theorem, the net outward flux is ***zero***.

***Exercise***

Compute the outward flux of the following vector field across the given surface; *S* is the boundary of the region bounded by the planes , , , , and 

***Solution***

















***Exercise***

Compute the outward flux of the following vector field across the given surface across the surface *S* that is the boundary of the prism bounded by the planes 

***Solution***





































***Exercise***

Compute the outward flux of the following vector field across the given surface out of the sphere *S* with 

***Solution***















***Exercise***

Compute the outward flux of the following vector field across the given surface out of the sphere *S* with 

***Solution***











***Exercise***

Compute the outward flux of the following vector field across the given surface of the sphere *S* with 

***Solution***





























***Exercise***

Compute the outward flux of the following vector field across the given surface out of the sphere *S* with 

***Solution***



















***Exercise***

; *S* is the surface of the paraboloid , for , and the *xy-*plane.

***Solution***











***Exercise***

; *S* is the solid region bounded by the coordinate planes and the plane .

***Solution***







































***Exercise***

; *S* is the solid region bounded by the cylinder , the plane , and the *xy-*plane.

***Solution***





































***Exercise***

Compute the outward flux of the following vector field  out of the sphere *S* with 

***Solution***



















***Exercise***

Compute the outward flux of the following vector field  out of the sphere *S* with 

***Solution***













***Exercise***

Compute the outward flux of the following vector field  out of the sphere *S* with 

***Solution***











***Exercise***

Compute the outward flux of the following vector field ; *S* is the solid region bounded by the cylinder , the planes , , and .

***Solution***







































***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following fields a

; across the boundary of an ellipsoid 

***Solution***

































***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following fields a

; across the boundary of the tetrahedron 

***Solution***







































***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following fields a

; across the boundary of the cylinder 

***Solution***





























Let 





Let 

















***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following fields a

; across the boundary of a ball 

***Solution***





























Let 







Let 















***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following fields a

; across the boundary of the cylinder  and the planes 

***Solution***







































***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following fields a

; *S* is the surface of the box with vertices .

***Solution***



















***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following fields a

; across the part of the paraboloid  that lies above the plane  and is oriented upward.

***Solution***































***Exercise***

Prove that  and use the result to prove that 

***Solution***









 ***√***















 ***√***

***Exercise***

Consider the radial vector field . Let *S* be the sphere of radius *a* at the origin.

1. Use the surface integral to show that the outward flux of  across *S* is . Recall that the unit normal to sphere is .
2. For what values of *p* does  satisfy the conditions of the Divergence Theorem? For these values of *p*, use the fact the  to compute the flux around *S* using the Divergence Theorem.

***Solution***

1. 



















 ***√***

1. The conditions of the Divergence Theorem require that  be defined and have continuous partials everywhere inside the sphere; in particular, this must hold at the origin. Thus, we must have . Then the volume integral is:















***Exercise***

Consider the radial vector field .

1. Evaluate a surface integral to show that , where *S* is the surface of a sphere of radius *a* centered at the origin.
2. Note that the first partial derivatives of the components of  are undefined at the origin, so the Divergence Theorem does not apply directly. Nevertheless, the flux across the sphere as computed in part (*a*) is finite. Evaluate the triple integral of the Divergence Theorem as an improper integral as follows. Integrate  over the region between two spheres of radius *a* and . Then let  to obtain the flux computed in part (*a*).

***Solution***

1. 













1. 

































***Exercise***

The electric field due to a point charge *Q* is , where  and  is a constant

1. Show that the flux of the field across a sphere of radius *a* centered at the origin is 
2. Let *S* be the boundary of the origin between two spheres centered of radius *a* and *b* with . Use the Divergence Theorem to show that the net outward flux across *S* is zero.
3. Suppose there is a distribution of charge within a region *D*. Let  be the charge density (charge per unit volume). Interpret the statement that



1. Assuming ***E*** satisfies the conditions of the Divergence Theorem, conclude from part (*c*) that 
2. Because the electric force is conservative, it has a potential function . From part (*d*) conclude that 

***Solution***

1. 







***Area*** of sphere 









1. The net outward flux across *S* is the difference of the fluxes across the inner and outer sphere; by part (*a*), these are equal (independent of the radius), so the net flux across *S* is zero.
2. The left-hand side is the flux across the boundary of *D*, while the right-hand side is the sum of the charge densities at each point of *D*.







The statement says that the flux across the boundary, up to multiplication by a constant, is the sum of the charge densities in the region.

1. 



This holds for all regions *D*.

Therefore; that implies that 

1. 





***Exercise***

***Fourier’s Law*** of heat transfer (or heat conduction) states that the heat flow vector at a point is proportional to the negative gradient of the temperature that is, , which means that heat energy flows from hot regions to cold region. The constant  is called the *conductivity*, which has metric units of . A temperature function for a region *D* is given. Find the net outward heat flux  across the boundary *S* of *D*. In some cases it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume .



***Solution***













Therefore, the heat flux is ***zero***.

***Exercise***

***Fourier’s Law*** of heat transfer (or heat conduction) states that the heat flow vector  at a point is proportional to the negative gradient of the temperature that is, , which means that heat energy flows from hot regions to cold region. The constant  is called the *conductivity*, which has metric units of . A temperature function for a region *D* is given.

Find the net outward heat flux  across the boundary *S* of *D*. In some cases, it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume .



***Solution***













Therefore, the heat flux is ***−*6** times the volume of the region (or ***−*6**).

***Exercise***

***Fourier’s Law*** of heat transfer (or heat conduction) states that the heat flow vector  at a point is proportional to the negative gradient of the temperature that is, , which means that heat energy flows from hot regions to cold region. The constant  is called the *conductivity*, which has metric units of . A temperature function for a region *D* is given.

Find the net outward heat flux  across the boundary *S* of *D*. In some cases, it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume .



***Solution***













Therefore, the heat flux is







***Exercise***

***Fourier’s Law*** of heat transfer (or heat conduction) states that the heat flow vector  at a point is proportional to the negative gradient of the temperature that is, , which means that heat energy flows from hot regions to cold region. The constant  is called the *conductivity*, which has metric units of . A temperature function for a region *D* is given.

Find the net outward heat flux  across the boundary *S* of *D*. In some cases, it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume 

 *D* is the sphere of radius *a* centered at the origin.

***Solution***





















 



***Exercise***

Consider the surface *S* consisting of the quarter-sphere , for , and the half disk in the *yz*-plane , for . The boundary of *S* in the *xy-*plane is *C*, which consists of the semicircle , for , and the line segment  on the *y-*axis, with a counterclockwise orientation. Let 

1. Describe the direction in which the normal vectors point on *S*.
2. Evaluate 
3. Evaluate  and check for segment with part (*b*).

***Solution***

1. Normal vectors point outwards everywhere on *S*; that is, on the curved surface. They point outwards and on the flat surface they point in the direction of negative *x*.
2.  &  

Parameterized:















For 





















1. 



Spherical: 

















From part (*a*), the flat surface is pointing in *x-*negative direction



























***Exercise***

Let *S* be the hemisphere , for , and let *T* be the paraboloid , for , where *a* > 0. Assume the surfaces have outward normal vectors.

1. Verify that *S* and *T* have the same base  and the same high point .
2. Which surface has the greater area?
3. Show that the flux of the radial field  across *S* is .
4. Show that the flux of the radial field  across *T* is .

***Solution***

1. *S*: base is the surface where 



*T*:  

The high point of the hemisphere (maximum *z-*coordinate) occurs when



 ***√***

1. *S*: is the surface of hemisphere 

For *T*: paraboloid



























∴ Area of the paraboloid is smaller than the area of the hemisphere.

1. 





























 ***√***

1. Across *T*: 

Parameterization:



































 ***√***

***Exercise***

The gravitational force due to a point mass *M* is proportional to , where  and *G* is the gravitational constant.

1. Show that the flux force field across a sphere of radius *a* centered at the origin is 
2. Let *S* be the boundary of the region between two spheres centered at the origin of radius *a* and *b* with . Use the Divergence Theorem to show that the net outward flux across *S* is zero.
3. Suppose there is a distribution of mass within a region *D* containing the origin. Let  be the mass density (mass per unit volume). Interpret the statement that



1. Assuming  satisfies the conditions of the Divergence Theorem, conclude from part (*c*) that 
2. Because the gravitational force is conservative, it has a potential function . From part (*d*) conclude that 

***Solution***

1. The unit normal to sphere is 

















 ***√***

1. 

Since the outward flux across a sphere, from part (a), is independent of the radius of the sphere, the outward flux across the spheres of radii *a* and *b* are equal, so their difference, which is the net flux across the spherical shell bounded by them, is zero.

1.  

The left-hand side is the flux across the boundary of *D*, while the right-hand side is the sum of the mass density inside *D*.

The statement says that the flux across the boundary is determined by (is a constant multiple of) the sum of the mass density inside *D*.

1. 







1. 





***Exercise***

Let  be a radial field , where *p* is a real number and . With ,  is an inverse square field.

1. Show that the net flux across a sphere centered at the origin is independent of the radius of the sphere only for 
2. Explain the observation in part (*a*) by finding the flux of  across the boundaries of a spherical box  for various values of *p*.

***Solution***

1. 





















If , then



Which is independent of the radius of the sphere.

1. 

























In general, for , only if 

***Exercise***

Consider the potential function , where *G* is any twice differentiable function and ; therefore, *G* depends only on the distance from the origin.

1. Show that the gradient vector field associated with  is , where  and .
2. Let *S* be the sphere of radius *a* centered at the origin and let *D* be the region enclosed by *S*. show that the flux of  across *S* is .
3. Show that 
4. Use part (*c*) to show that the flux across *S* (as given in part (*b*)) is also obtained by the volume integral . (Hint: use spherical coordinates and integrate by parts.)

***Solution***

1. 













∴ 

1. The sphere of radius *a* centered at the origin 



























Since the surface area is *twice* the value

Total surface 

1.  

































1. By the divergence theorem, the flux:







 







***Exercise***

Prove Green’s Identity for scalar-valued functions *u* and *v* defined on a region *D*:



***Solution***

















***Exercise***

Prove the identity: 

***Solution***

Let  and , then:









































 ***√***

***Exercise***

Prove the identity: 

***Solution***

Let  and , then:















































 ***√***