***Section* 4.8 – Divergence Theorem**

**Divergence Theorem**

The Divergence Theorem is the 3-dimensional version of the flux form of Green’s Theorem.

If *R* is a region in the *xy-*plane, *C* is the simple closed piecewise-smooth oriented boundary of *R*, and  is a vector field, Green’s Theorem says that



***Theorem***

Let  be a vector field whose components have continuous first partial derivatives in a connected and simply connected region *D* enclosed by a smooth oriented surface *S*. Then



Where  is the unit outward normal vector on *S*.

***Example***

Consider the radial field  and let *S* be the sphere  that encloses the region *D*. Assume  is the outward normal vector on the sphere. Evaluate both integrals of the Divergence Theorem.

***Solution***

The divergence of :



















The required vector normal to the surface is





















∴ The two integral of the Divergence Theorem are equal.

***Example***

Consider the rotation field:







Let *S* be the sphere  for , together with its base in the *xy-*plane.

Find the net outward flux across *S*.

***Solution***





∴ The flux across the hemisphere is zero.

*However, with the Divergence Theorem, radial fields are interesting and have many physical applications*

***Example***

Find the net outward flux of the field  across the boundaries of the cube 

***Solution***













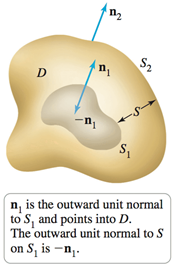






**Divergence *Theorem* for Hollow Regions**

Suppose the vector field satisfies the conditions of the Divergence Theorem on a region *D* bounded by two smooth oriented surfaces  and , where  lies within .



Let *S* be the entire boundary of  and let  and be the outward unit normal vectors for  and , respectively.





**Interpretation of the Divergence Using Mass Transport**

Suppose that  is the velocity field of a material, such as water or molasses, and *ρ* is its constant density.

The vector field  describes the ***mass transport*** of the material, with units of  typical units of mass transport are .

This means that  gives the mass material flowing past a point (in each of the three coordinates direction) per unit of surface area per unit of time.

When  is multiplied by an area, the result in the *flux*, with units of mass/unit time.

***Example***

Consider the inverse square vector field 

1. Find the net outward flux of across the surface of the region  that lies between concentric spheres with radii *a* and *b*.
2. Find the outward flux of across any sphere that encloses the origin,

***Solution***

1. 

































Let 

Because , the divergence Theorem implies that



Therefore, the net flux across *S* is zero.

1. 















The flux of the inverse square field across any surface enclosing the origin is .

***Gauss’* Law**

Applying the Divergence Theorem to electric fields leads to one of the fundamental laws of physics. The electric field due to a point charge *Q* located at the origin is given by the inverse square law.



Where  and  is a physical constant called the ***permittivity*** ***of free square***.

This is one statement of Gauss’ Law: If *S* is a surface that encloses a point charge *Q*, then the flux of the electric field across *S* is



|  |  |
| --- | --- |
|  |  |
| Gauss’ Law: Flux of electric field across *S* due to point charge *Q* = | Gauss’ Law: Flux of electric field across *S* due to charge distribution *q* = |

|  |  |  |
| --- | --- | --- |
| ***Fundamental Theorem of Calculus*** |  |  |
| ***Fundamental Theorem of Line Integrals*** |  |  |
| ***Green’s Theorem (Circulation Form)*** |  |  |
| ***Stokes’ Theorem*** |  |  |
| ***Divergence Theorem*** |  |  |

***Exercises*** ***Section* 4.8 – Divergence Theorem**

(**1**−**4**) Evaluate both integrals of the Divergence Theorem for the following vector fields and region. Check for agreement.

1. 
2. 
3. 
4. 
5. Find the net outward flux of the field  across the sphere of radius 1 centered at the origin.
6. Find the net outward flux of the field  across any smooth closed surface , where *a, b*, and *c* are constants.
7. Find the net outward flux of the field  across the boundary of the cube 

(**8**−**47**) Use the Divergence Theorem to compute the net outward flux of the following fields across the given surface *S or D*.

1. ; *S* is the sphere 
2. ; *S* is surface of the cube cut from the first octant by the planes *x* = 1, *y* = 1, and 
3. ; *S* is boundary of the tetrahedron in the first octant formed by the plane 
4. ; *S* is the sphere 
5. ; *S* is the sphere 
6. ; *S* is the surface of the paraboloid , for , plus its base in the *xy-*plane.
7. ; *S* is the surface of the cone , for , plus its top surface in the plane 
8. ; *D* is the region between the spheres of radius 2 and 4 centered at origin.
9. ; *D* is the region between the spheres of radius 1 and 2 centered at origin.
10. ; *D* is the region between the spheres of radius 1 and 2 centered at origin.

1. ;  is the region between two cubes
2. ; *S* consists of the faces of the cube 
3. ; D is the region in the first octant between the planes  and 
4. ; *D* is the region between the cylinders  and  for 
5.  across *S* is the surface of the cube cut from the first octant by the planes 
6.  across *S* is the sphere 
7.  across *D* is the region between two spheres of radius 1 and 2 centered at 
8. ; *S* is the cylinder 
9. ; *S* is the boundary of the ellipsoid 
10. ; *S* is the boundary of the ellipsoid 
11. ; *S* is the boundary of the region bounded by the planes , , , , and 

1.  across the surface *S* that is the boundary of the prism bounded by the planes 
2.  out of the sphere *S* with 
3.  out of the sphere *S* with 
4.  of the sphere *S* with 
5.  out of the sphere *S* with 
6. ; *S* is the surface of the paraboloid , for , and the *xy-*plane.
7. ; *S* is the solid region bounded by the coordinate planes and the plane .
8. ; *S* is the solid region bounded by the cylinder , the plane , and the *xy-*plane.
9.  out of the sphere *S* with 
10.  out of the sphere *S* with 
11.  out of the sphere *S* with 
12. ; *S* is the solid region bounded by the cylinder , the planes , , and .
13. ; across the boundary of an ellipsoid 
14. ; across the boundary of the tetrahedron 
15. ; across the boundary of the cylinder 
16. ; across the boundary of a ball 
17. ; across the boundary of the cylinder  and the planes 
18. ; *S* is the surface of the box with vertices .
19. ; across the part of the paraboloid  that lies above the plane  and is oriented upward.
20. Prove that  and use the result to prove that 
21. Consider the radial vector field . Let *S* be the sphere of radius *a* at the origin.
22. Use the surface integral to show that the outward flux of  across *S* is . Recall that the unit normal to sphere is .
23. For what values of *p* does  satisfy the conditions of the Divergence Theorem? For these values of *p*, use the fact the  to compute the flux around *S* using the Divergence Theorem.
24. Consider the radial vector field .
25. Evaluate a surface integral to show that , where *S* is the surface of a sphere of radius *a* centered at the origin.
26. Note that the first partial derivatives of the components of  are undefined at the origin, so the Divergence Theorem does not apply directly. Nevertheless, the flux across the sphere as computed in part (*a*) is finite. Evaluate the triple integral of the Divergence Theorem as an improper integral as follows. Integrate  over the region between two spheres of radius *a* and . Then let  to obtain the flux computed in part (*a*).
27. The electric field due to a point charge *Q* is , where  and  is a constant
28. Show that the flux of the field across a sphere of radius *a* centered at the origin is 
29. Let *S* be the boundary of the origin between two spheres centered of radius *a* and *b* with . Use the Divergence Theorem to show that the net outward flux across *S* is zero.
30. Suppose there is a distribution of charge within a region *D*. Let  be the charge density (charge per unit volume). Interpret the statement that



1. Assuming ***E*** satisfies the conditions of the Divergence Theorem, conclude from part (*c*) that 
2. Because the electric force is conservative, it has a potential function . From part (*d*) conclude that 

(**52−55**) ***Fourier’s Law*** of heat transfer (or heat conduction) states that the heat flow vector  at a point is proportional to the negative gradient of the temperature that is, , which means that heat energy flows from hot regions to cold region. The constant  is called the *conductivity*, which has metric units of . A temperature function for a region *D* is given. Find the net outward heat flux  across the boundary *S* of *D*. In some cases it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume .

1. 
2. 
3. 
4.  *D* is the sphere of radius *a* centered at the origin.
5. Consider the surface *S* consisting of the quarter-sphere , for , and the half disk in the *yz*-plane , for . The boundary of *S* in the *xy-*plane is *C*, which consists of the semicircle , for , and the line segment  on the *y-*axis, with a counterclockwise orientation. Let 
6. Describe the direction in which the normal vectors point on *S*.
7. Evaluate 
8. Evaluate  and check for segment with part (*b*).
9. Let *S* be the hemisphere , for , and let *T* be the paraboloid , for , where *a* > 0. Assume the surfaces have outward normal vectors.
10. Verify that *S* and *T* have the same base  and the same high point .
11. Which surface has the greater area?
12. Show that the flux of the radial field  across *S* is .
13. Show that the flux of the radial field  across *T* is .
14. The gravitational force due to a point mass *M* is proportional to , where  and *G* is the gravitational constant.
15. Show that the flux force field across a sphere of radius *a* centered at the origin is



1. Let *S* be the boundary of the region between two spheres centered at the origin of radius *a* and *b* with . Use the Divergence Theorem to show that the net outward flux across *S* is zero.
2. Suppose there is a distribution of mass within a region *D* containing the origin. Let  be the mass density (mass per unit volume). Interpret the statement that



1. Assuming  satisfies the conditions of the Divergence Theorem, conclude from part (*c*) that 
2. Because the gravitational force is conservative, it has a potential function . From part (*d*) conclude that 
3. Let  be a radial field , where *p* is a real number and . With ,  is an inverse square field.
4. Show that the net flux across a sphere centered at the origin is independent of the radius of the sphere only for 
5. Explain the observation in part (*a*) by finding the flux of  across the boundaries of a spherical box  for various values of *p*.
6. Consider the potential function , where *G* is any twice differentiable function and ; therefore, *G* depends only on the distance from the origin.
7. Show that the gradient vector field associated with  is , where  and .
8. Let *S* be the sphere of radius *a* centered at the origin and let *D* be the region enclosed by *S*. show that the flux of  across *S* is .
9. Show that 
10. Use part (*c*) to show that the flux across *S* (as given in part (*b*)) is also obtained by the volume integral . (Hint: use spherical coordinates and integrate by parts.)
11. Prove Green’s Identity for scalar-valued functions *u* and *v* defined on a region *D*:



1. Prove the identity: 
2. Prove the identity: 