***Lecture One* – First Order Equations**

***Section* 1.1 – Differential Equations & Solutions**

**Ordinary Differential Equations**

Involve an unknown function of a single variable with one or more of its derivatives.



*y*: is unknown function

*t*: independent variable

Some other example:











∴ The order of a differential equation is the order of the highest derivative that occurs in the equation.

 *second order*

 is not an ODE (*ω* is dependent on *x* and *t*)

This equation is called a ***partial differential equation***.

**Definition**

A first-order differential equation of the form  is said to be in normal form.

 is said to be in normal form.

*f*: is a given function of 2 variables *t* & *y* (***rate function***)

***Solutions***

A solution of the first-order, ordinary differential equation  is a differentiable function  such that  for all *t* in the interval where  is defined.

1. Can be found in explicit and implicit form by applying manipulation (integration)
2. No real solution.

***Example***

Show that  is a solution of the 1st order equation 

*Solution*





 True; it is a solution

is called the ***general solution***.

The solutions from the graph are called ***solution curves***.

***Example***

Is the function  a solution to the differential equation

*Solution*





 False; it is not a solution.

***Exercises Section* 1.1 – Differential Equations & Solutions**

1. Show that  is a solution of the 1st order equation 
2. Show that  is a solution of the 1st order equation 
3. A general solution may fail to produce all solutions of a differential equation . Show that  is a solution of the differential equation, but no value of *C* in the given general solution will produce this solution.
4. Use the given general solution to find a solution of the differential equation having the given initial condition.
5. Show that  is a solution of the 1st order equation 
6. Use the given general solution to find a solution of the differential equation having the given initial condition. 
7. Use the given general solution to find a solution of the differential equation having the given initial condition. 
8. Use the given general solution to find a solution of the differential equation having the given initial condition. 
9. Find the values of ***m*** so that the function  is a solution of the given differential equation

|  |  |
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1. Let  is 2-parameter family solutions of the second order differential equation of . Find a solution of the second-order consisting of this differential equation and the given initial conditions.

|  |  |
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***Section* 1.2 – Solutions to Separable Equations**

**Separable Equation**

Separable equation is an equation that can be written with its variables separated and then easily solved.

If is independent of *y*⇒





***Definition***

A 1st order differential equation of the form is said to be separable or to have separable variables.





 *not separable*

***Example***

At time *t* the sample contains radioactive nuclei and is given by the differential equation:



This is called the ***exponential equation***.



 ***Separable equation***













***Example***

Solve the differential equation 

*Solution*









 *Cross multiplication*





 ∴ Equilibrium point

***General Method***

1. Separate the variables
2. Integrate both sides
3. Solve for the solution , if possible

***Using definite Integration***

***Example***

A can of beer at 40° F is placed into a room when the temperature is 70° F. After 10 minutes the temperature of the beer is 50° F. What is the temperature of the beer as a function of time? What is the temperature of the beer 30 minutes after the beer was placed into the room?

*Solution*

By Newton's law of cooling: The rate of change of an object's temperature (***T***) is proportional to the difference between its temperature and the ambient temperature (***A***).











 *Quotient Rule*







*Given*: 

























***Losing a solution***

When we use separate variables, the variable divisors could be zero at a point.

***Example***

Find a general solution to 

*Solution*



 *critical points*



















If 







If  









***Implicitly Defined Solutions***

***Example***

Find the solutions of the equation , having initial conditions  and 

*Solution*











 *Quadratic Formula*

 *Implicit*























 , but it never it will be.

*Explicit Solutions*: 

*Implicit solutions*: 

***Example***

Find the solutions to the differential equation , having 

*Solution*











For 







We can't solve for 

⇒ This solution is defined as implicit.

For 







Since the initial condition < 0, then:



For 

 *True statement*

 is a solution

***Exercises Section* 1.2 – Solutions to Separable Equations**

Find the general solution of the differential equation. If possible, find an explicit solution.

|  |  |
| --- | --- |
|  |  |

Find the exact solution of the initial value problem. Indicate the interval of existence.

|  |  |
| --- | --- |
|  |  |

1. A murder victim is discovered at midnight and the temperature of the body is recorded at 31°C. One hour later, the temperature of the body is 29°C. Assume that the surrounding air temperature remains constant at 21°C. Use Newton’s law of cooling to calculate the victim’s time of death. *Note*: The normal temperature of a living human being is approximately 37°C
2. Suppose a cold beer at 40°F is placed into a warn room at 70°F. suppose 10 minutes later, the temperature of the beer is 48°F. Use Newton’s law of cooling to find the temperature 25 minutes after the beer was placed into the room.

***Section* 1.3 – Models of Motions**

***Newton's 2nd Law***

*The force acting on a mass is equal to the rate of change of momentum with respect to time. Momentum is defined as the product of mass and velocity (m.v).The force is equal to the derivative of the momentum*



***Position***: 

**Air Resistance**



***R***: resistance force (*has sign opposite of the velocity*)

***r***: is a function that is always nonnegative

* *when a ball is falling from a high altitude, the density of the air has to be taken into account.*















When  (*Terminal Velocity*)

 (*A*: is a constant)

***Example***

Suppose you drop a brick from the top of a building that is 250 m high. The brick has a mass of 2 kg, and the resistance force is given by . How long will it take the brick to reach the ground? what will be its velocity at that time?

*Solution*





























 (Using software to solve it)





***Finding the displacement***









***Example***

A ball of mass is protected from the surface of the earth, with velocity . Assume that the force of air resistance is given by , where . What is the maximum height reached by the ball?

*Solution*











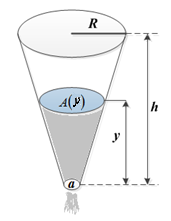










***Torricelli’s Law***

Suppose that a water tank has a ***hole with area*** ***a*** at its bottom, from which water is leaking. Denote by  the depth of water in the tank at time *t*, and by  the volume of water in the tank then. It is plausible – and true, under ideal conditions – that the velocity of water exiting through the hole is



Which is the velocity a drop of water would acquire in falling freely from the surface of the water to the hole. One can derive this formula beginning with the assumption that the sum of the kinetic and potential energy of the system remains constant. Under real conditions, taking into account the construction of a water jet from an orifice, , where *c* is an empirical constant between 0 and 1 (usually about 0.6 for a small continuous stream of water). For simplicity we take  in the following discussion.





This is a statement of *Torricelli’s* law for a draining tank.

Let  denote the horizontal cross-sectional area of the tank at height *y*. Then, applied to a thin horizontal slice of water at height  with area  and thickness , the integral method of cross sections gives



The fundamental theorem of calculus therefore implies that  and hence that





 (An alternative form of *Torricelli’s* law)

***Exercise***

A tank is shaped like a vertical cylinder; it initially contains water to a depth of 9 ft, and a bottom plug is removed at time *t* = 0 (hours). After 1 hr. the depth of the water has dropped to 4 ft. how long does it take for all the water to drain from the tank?

***Solution***









With initial condition 



















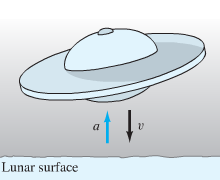
It will take 3 hours for the tank to empty.

***Exercises Section* 1.3 – Models of Motions**

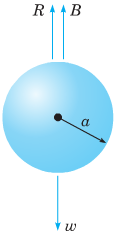
1. A stone is released from rest and dropped into a deep well. Eight seconds later, the sound of the stone splashing into the water at the bottom of the well returns to the ear of the person who released the stone. How long does it take the stone to drop to the bottom of the well? How deep is the well? Ignore air resistance. The speed of sound is 340 *m/s*.
2. A rocket is fired vertically and ascends with constant acceleration  for 1.0 *min*. At that point, the rocket motor shuts off and the rocket continues upward under the influence of gravity. Find the maximum altitude acquired by the rocket and the total time elapsed from the take-off until the rocket returns to the earth. *Ignore air resistance*.
3. A ball having mass  falls from rest under the influence of gravity in a medium that provides a resistance that is proportional to its velocity. For a velocity of the force due to the resistance of the medium is −1 N. Find the terminal velocity of the ball.

1 N is the force required to accelerate a 1 kg mass at a rate of : 

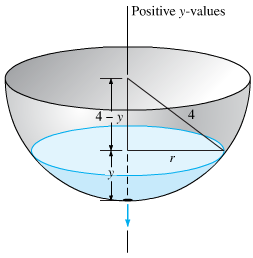
1. A ball is projected vertically upward with initial velocity from ground level. Ignore air resistance.
2. What is the maximum height acquired by the ball?
3. How long does it take the ball to reach its maximum height? How long does it take the ball to return to the ground? Are these times identical?
4. What is the speed of the ball when it impacts with the ground on its return?
5. An object having mass 70 kg falls from rest under the influence of gravity. The terminal velocity of the object is . Assume that the air resistance is proportional to the velocity.
6. Find the velocity and distance traveled at the end of 2 seconds.
7. How long does it take the object to reach 80% of its terminal velocity?
8. A lunar lander is falling freely toward the surface of the moon at a speed of 450 m/s. Its retrorockets, when fired, provide a constant deceleration of 2.5  (the gravitational acceleration produced by the moon is assumed to be included in the given acceleration). At What height above the lunar surface should the retrorockets be activated to ensure a “soft touchdown? (*v* = 0 at impact)?



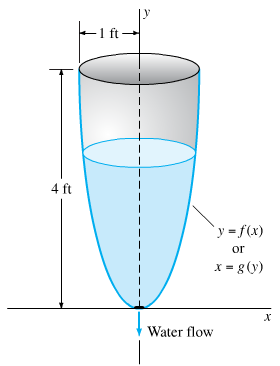
1. A body falling in a relatively dense fluid, oil for example, is acted on by three forces: a resistance force *R,* a buoyant force *B,* and its weight *w* due to gravity. The buoyant force is equal to the weight of the fluid displaced by the object. For a slowly moving spherical body of radius a, the resistive force is given by Stokes’s law , where *v* is the velocity of the body, and *μ* is the coefficient of viscosity of the surrounding fluid?



1. Find the limiting velocity of a solid sphere of radius *a* and density *ρ* falling freely in a medium of density  and coefficient of viscosity μ.
2. In 1910 R. A. Millikan studied the motion of tiny droplets of oil falling in an electric field. A field of strength *E* exerts a force on a droplet with charge *e*. Assume that *E* has been adjusted so the droplet is held stationary  and that *w* and *B* are as given. Find an expression for *e*.
3. A hemispherical bowl has top radius of 4 *ft*. and at time *t* = 0 is full of water. At that moment a circular hole with diameter 1 *in*. is opened in the bottom of the tank. How long will it take for all the water to drain from the tank?



1. Suppose that the tank has a radius of 3 ft. and that its bottom hole is circular with radius 1 in. How long will it take the water (initially 9 ft. deep) to drain completely?
2. At time *t* = 0 the bottom plug (at the vertex) of a full conical water tank 16 ft. high is removed. After 1 hr the water in the tank is 9 ft. deep. When will the tank be empty?
3. Suppose that a cylindrical tank initially containing  gallons of water drains (through a bottom hole) in *T* minutes. Use Torricelli’s law to show that the volume of water in the tank after  minutes is 
4. The clepsydra, or water clock – A 12*-hr* water clock is to be designed with the dimensions, shaped like the surface obtained by revolving the curve  around the *y-*axis. What should be this curve, and what should be the radius of the circular bottom hole, in order that the water level will fall at the constant rate of 4 inches per hour?



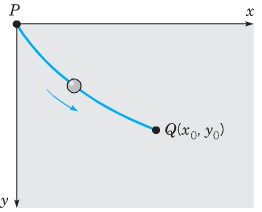
1. One of the famous problems in the history of mathematics is the brachistochrone problem: to find the curve along which a particle will slide without friction in the minimum time from one given point *P* to another point *Q*, the second point being lower than the first but not directly beneath it.

This problem was posed by Johann Bernoulli in 1696 as a challenge problem to the mathematicians of his day. Correct solutions were found by Johann Bernoulli and his brother Jakob Bernoulli and by Isaac Newton, Gottfried Leibniz, and the Marquis de L’Hospital. The brachistochrone problem is important in the development of mathematics as one of the forerunners of the calculus of variations.

In solving this problem, it is convenient to take the origin as the upper point *P* and to orient the axes as shown. The lower point Q has coordinates . It is then possible to show that the curve of minimum time is given by a function  that satisfies the differential equation



Where  is a certain positive constant to be determined later



1. Solve the equation  for  . Why is it necessary to choose the positive square root?
2. Introduce the new variable t by the relation



Show that the equation found in part (a) then takes the form



1. Letting , show that the solution of  for which *x* = 0 when *y* = 0 is given by



Equations (*iv*) are parametric equations of the solution of (eq. *i*) that passes through (0, 0). The graph of Eqs. (*iv*) is called a cycloid.

1. If we make a proper choice of the constant k, then the cycloid also passes through the point  and is the solution of the brachistochrone problem. Find *k* if  and 

***Section* 1.4 – Linear Equations**

A first order linear equation is given by the form:



If . This linear equation is said to be ***homogeneous***. (Otherwise it is ***nonhomogeneous or inhomogeneous***).

are called the coefficients

|  |  |
| --- | --- |
| ***Linear*** | ***Non-linear*** |
|  |  |
|  |  |
|  |  |

***Solution of the homogenous equation***





 *Convert to exponential form*



***Example***

Solve: 

***Solution***





**Solving a linear first-order Equation (*Properties*)**

1. Put a linear equation into a standard form 
2. Identify  then find 
3. Multiply the standard form by 
4. Integrate both sides

***Solution of the Inhomogeneous Equation***











***Example***Find the general solution to: 

*Solution*















***Solution of the Nonhomogeneous Equation*** 

Let assume:  

The homogeneous equation is given by 













 *Since* 



















***Example***

Find the general solution of  and the particular solution that satisfies.

*Solution*

















***Example***

Find the general solution of  and the particular solution that satisfies.

*Solution*















***Notes***

1. Integrating an expression that is not the differential of any elementary function is called non-elementary.

1. In math some important functions are defined in terms of non-elementary integrals. Two such functions are the error function and the complementary error function.

***Exercises Section* 1.4 – Linear Equations**

Find the general solution of the first-order, linear equation.

|  |  |
| --- | --- |
|  |  |

Solve the differential equations

|  |  |
| --- | --- |
|  |  |

Find the solution of the initial value problem

|  |  |
| --- | --- |
|  |  |

Find the solution of the initial value problem. Discuss the interval of existence and provide a sketch of your solution

|  |  |
| --- | --- |
|  |  |

Find the general solution of the given differential equation. Then find the particular solution satisfying the given initial condition of

1. 
2. 
3. 

***Section* 1.5 – Mixing Problems**

The physical representation of the rate of change:

*rate of change = rate in* - *rate out*

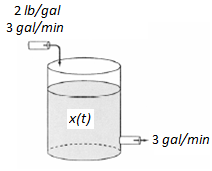
This is referred to as a ***balance law***.

Rate = Volume Rate (*gal/min*) *x* Concentration (*lb/gal*)

***Example***

The tank initially holds 100 gal of pure water. At time , a solution containing 2 lb of salt per gallon begins to enter the tank at the rate of 3 gallons per minute. At the same time a drain is opened at the bottom of the tank so that the volume of solution in the tank remains constant. How much salt is in the tank after 60 min?

*Solution*

number of pounds of salt in the tank after *t* min.

Volume: 

Concentration at time *t*: 

Rate in = Volume Rate *x* Concentration





Rate out = Volume Rate *x* Concentration





rate of change

= rate in − rate out



















Since there was no salt present in the tank initially, the initial condition is 









After 60 min:

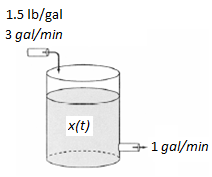




***Example***

The 600-gal tank is filled with 300 gal of pure water. A spigot is opened above the tank and a salt solution containing 1.5 lb of salt per gallon of solution begins flowing into the tank at the rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave tank at a rate of 1 *gal/min*. What will be the salt content in the tank at the precise moment that the volume of solution in the tank is equal to the tank's capacity (600 *gal*)?

*Solution*







Rate in 



Rate out 













































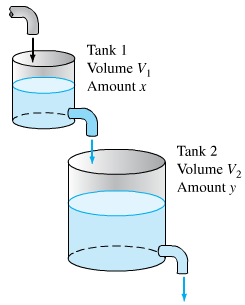








***Exercises Section* 1.5 – Mixing Problems**

1. Consider two tanks, label tank A and tank B for reference. Tank A contains 100 *gal* of solution in which is dissolved 20 *lb* of salt. Tank B contains 200 *gal* of solution which is dissolved 40 *lb* of salt. Pure water flows into the tank A at rate of 5 *gal/s*. There is a drain at the bottom of tank A. The solution leaves tank A via the drain at a rate of 5 *gal/s* and flows immediately into tank B at the same rate. A drain at the bottom of tank B allows the solution to leave tank B at a rate of 2.5 *gal/s*. What is the salt content in tank B at the precise moment that tank B contains 250 *gal* of solution?
2. A tank contains 100 *gal* of pure water. At time zero, a sugar-water solution containing 0.2 *lb* of sugar per gal enters the tank at a rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the sugar solution to leave the tank at 3 *gal/min*. Assume that the solution in the tank is kept perfectly mixed at all times.
3. What will be the sugar content in the tank after 20 minutes?
4. How long will it take the sugar content in the tank to reach 15 lb?
5. What will be the eventual sugar content in the tank?
6. A tank initially contains 50 gal of sugar water having a concentration of 2 lb. of sugar for each gal of water. At time zero, pure water begins pouring into the tank at a rate of 2 gal per minute. Simultaneously, a drain is opened at the bottom of the tank so that the volume of sugar-water solution in the tank remains constant.
7. How much sugar is in the tank after 10 minutes?
8. How long will it take the sugar content in the tank to dip below 20 lb.?
9. What will be the eventual sugar content in the tank?
10. A 50-gal tank initially contains 20 *gal* of pure water. Salt-water solution containing 0.5 *lb.* of salt for each gallon of water begins entering the tank at a rate of 4 *gal/min.* Simultaneously; a drain is opened at the bottom of the tank, allowing the salt-water solution to leave the tank at a rate of 2 *gal/min*. What is the salt content (*lb*) in the tank at the precise moment that the tank is full of salt-water solution?
11. A tank contains 500 *gal* of a salt-water solution containing 0.05 *lb* of salt per gallon of water. Pure water is poured into the tank and a drain at the bottom of the tank is adjusted so as to keep the volume of solution in the tank constant. At what rate (gal/min) should the water be poured into the tank to lower the salt concentration to 0.01 lb/gal of water in less than one hour?
12. A tank contains 100 gal of fresh water. A solution containing 1 lb./gal of soluble lawn fertilizer runs into the tank at the rate of 1 gal/min, and the mixture is pumped out of the tank at a rate of 3 gal/min. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.
13. A 200-gal tank is half full of distilled water. At time *t* = 0, a solution containing 0.5 lb./gal of concentrate enters the tank at the rate of 5 gal/min, and the well-stirred mixture is withdrawn at the rate of 3 gal/min.
14. At what time will the tank be full?
15. At the time the tank is full, how many pounds of concentrate will it contain?
16. Suppose that an Iowa class battleship has mass 51,000 metric tons (51,000,000 kg) and . Assume that the ship loses power when it is moving at a speed of 9 m/sec.
17. About how far will the ship coast before it is dead in the water?
18. About how long will it take the ship’s speed to drop to 1 m/sec?
19. A 66-kg cyclist on a 7-kg bicycle starts coasting on level ground at 9 m/sec. The 
20. About how far will the cyclist coast before reaching a complete stop?
21. How long will it take the cyclist’s speed to drop to 1 m/sec?
22. An Executive conference room of a corporation contains 4500  of air initially free of carbon monoxide. Starting at time *t* = 0, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 . A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3 . Find the time when the concentration of carbon monoxide in the room reaches 0.01%.
23. Consider the cascade of 2 tanks with  and  the volumes of brine in the 2 tanks. Each tank also initially contains 50 lb. of salt. The three flow rates indicated in the figure are each 5 gal/min, with pure water flowing into tank.
24. Find the amount  of salt in tank 1 at time *t*.
25. Suppose that  is the amount of salt in tank 2 at time *t*. Show first that



And then solve for , using the function  found in part (*a*).

1. Finally, find the maximum amount of salt ever in tank 2.
2. Suppose that in the cascade tank 1 initially 100 gal of pure ethanol and tank 2 initially contains 100 gal of pure water. Pure water flows into tank 1 at 10 gal/min, and the other two flow rates are also 10 gal/min.
3. Find the amounts  and  of ethanol in the two tanks at time  .
4. Find the maximum amount of ethanol ever in tank 2.

1. A multiple cascade is shown in the figure. At time *t* = 0, tank 0 contains 1 gal of ethanol and 1 gal of water; all the remaining tanks contain 2 gal of pure water each. Pure water is pumped into tank 0 at 1 gal/.min, and the varying mixture in each tank is pumped into the one below it at the same rate. Assume, as usual, that the mixtures are kept perfectly uniform by stirring. Let  denote the amount of ethanol in tank *n* at time *t*.
2. Show that 
3. Show that the maximum value of  for *n* > 0 is 
4. Assume that Lake Erie has a volume of 480  and that its rate of inflow (from Lake Huron) and outflow (to Lake Ontario) are both 350  per year. Suppose that at the time *t* = 0 (years), the pollutant concentration of Lake Erie – caused by past industrial pollution that has now been ordered to cease – is 5 times that of Lake Huron. If the outflow henceforth is perfectly mixed lake water, how long will it take to reduce the pollution concentration in Lake Erie to twice that of Lake Huron?
5. A 120 gal tank initially contains 90 lb. of salt dissolved in 90 gal of water. Brine containing 2 lb./gal of salt flows into the tank at rate of 4 gal/min, and the well-stirred mixture flows out the tank at the rate of 3 gal/min. How much salt does the tank contain when it is full?

***Section* 1.6 - Existence and Uniqueness of Solutions**

The questions of existence and uniqueness

* When can we be sure that a solution exists**?**
* How many different solutions are there

**Existence of Solutions**

***Example***

Consider the initial value problem:  with 

*Solution*



There is ***no solution*** to the given initial value















***Theorem:* Existence of Solutions**

Suppose the function is defined and continuous on the rectangle ***R*** in the *tx*-plane. Then given any point , the initial value problem



has a solution  defined in an interval containing . Furthermore, the solution will be defined at least until the solution curve  leaves the rectangle ***R***.

**Interval of Existence of a Solution**

***Example***

Consider the initial value problem  with . Find the solution and its interval of existence.

***Solution***

The right-hand side is  which is continuous on the entire *tx*-plane.

The solution to the initial value problem is:











 is discontinuous at . Hence the solution to the initial value problem is defined only for .

The interval: 

***Theorem*: Existence of a Unique Solution**

Let R be a rectangular region in the *xy-*plane defined by  that contains the point  in its interior. If  and  are continuous on R, then there exists some interval , contained in [*a, b*], and a unique function , defined on  that is a solution of the initial*-*value problem (IVP)



**Mathematics & Theorems**

Any theorem is a logical statement which has hypotheses (when it's true) and conclusions (true)

***The Hypotheses of the Uniqueness of Solutions Theorem***

1. The equation is in normal form 
2. The right-hand side  and its derivative  are both continuous in the rectangle ***R***.
3. The initial point  is in the rectangle ***R***.

For the uniqueness theorem the conclusions are as follows:

1. There is one and only one solution to the initial value problem.
2. The solution exists until the solution curve  leaves the rectangle ***R***.

***Example***

Consider the initial value problem . Is there a solution to this equation with initial condition ? If so, is the solution unique?

***Solution***



The right-hand side:  is continuous except where .

We can take ***R*** to be any rectangle which contains the point  to avoid , we can choose  and 

Then *f* is continuous everywhere in ***R*** ⇒ hypotheses of the existence theorem are satisfied.

Since  is also continuous in ***R***.

There is only one solution.

It is important to determine and prove a theorem concerning the existence and uniqueness of solutions of an O.D.E.

* Are the  solutions to exist?



⇒ Solutions exist for the system.

* ***Uniqueness***: Assume is another solution. We want to prove is actually  i.e.









So that, , then multiply both sides by to obtain: 

***Exercises Section* 1.6 - Existence and Uniqueness of Solutions**

Which of the initial value problems are guaranteed a unique solution

1. 
2. 
3. 
4. 
5. 
6. 
7. Show that and  are both solutions of the initial value problem , where . Explain why this fact doesn't contradict Theorem
8. Use a numerical solver to sketch the solution of the given initial value problem



1. Where does your solver experience difficulty? why? Use the image of your solution to estimate the interval of existence.
2. Find an explicit solution; then use your formula to determine the interval of existence. How does it compare with the approximation found in part (*a*).

***Section* 1.7 - Autonomous Equations and Stability**

A first-order autonomous equation is an equation of the form





***Definition***

The valuewhere the functionassigns to the point represent the slope of a line (line segment) call ***a lineal element***.

***Example***: Given and consider the point

The slope of the lineal element is 

**The Direction Fields**

What we draw a lineal element at each point  with slope  then the collection of these lineal elements is called a ***direction field*** or a ***slope field*** of the differential equation.

**Autonomous 1st order DE**

A system, which does not explicitly contain the independent variable*t* is called an ***autonomous system***. Otherwise, the system is called *non-autonomous system*.

|  |  |
| --- | --- |
| ***Autonomous*** | ***Not- Autonomous*** |
|  |  |
|  |  |
|  |  |
|  |  |

***Equilibrium* Points & Solutions**

 and also called a ***critical point***.



From these equilibrium points, we can determine the stability of the system.

* An equilibrium point is ***stable*** if all nearby solutions stay nearby.



* An equilibrium point is ***asymptotically stable*** if all nearby solutions not only stay nearby, but also tend to the equilibrium point. An equilibrium point is stable if all nearby solutions.



1. If , then *f* is ***decreasing*** at and is asymptotically stable.
2. If , then *f* is ***increasing*** at and is unstable.
3. If  no conclusion can be drawn.

The family of all solution curves without the presence of the independent variable is called the ***phase portrait***.

When an independent variable***t*** is interpreted as time and the solution curve could be thought of as the path of a particle moving in the solution space, then the system is considered as a ***dynamical system***, where the solution curves are called *t****rajectories*** or ***orbits***.

***Example***

Discover the behavior as of all solutions to the differential equation



*Solution*

The equilibrium points: 



are equilibrium.



 ***unstable***

 is ***asymptotically stable***

 ***unstable***



These are constant functions, the position of the point the phase line modeled by them is also constant

-1 1 2

***Phase Portrait***

***Exercises Section* 1.7 - Autonomous Equations and Stability**

The graph of the right-hand side  is shown. Identify the equilibrium points and sketch the equilibrium solutions in the *ty*-plane. Classify each equilibrium point as either unstable or asymptotically stable.

**1.**



**2.**



**3.**



**4.** Impose the equilibrium solution(s), classifying each as either unstable or asymptotically stable



An autonomous differential equation is given. Perform each of the following to exercises 5 - 8

1. Sketch a graph of 
2. Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.
3. Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.
4. 
5. 
6. 
7. 

Determine the stability of the equilibrium solutions

1. 
2. 
3. A tank contains 100 *gal* of pure water. A salt solution with concentration 3 *lb/gal* enters the tank at a rate of 2 *gal/min*. Solution drains from the tank at a rate of 2 *gal/min*. Use the qualitative analysis to find the eventual concentration of the salt in the tank.

***Section* 1.8 - Modeling Population Growth**

**Modeling Population Growth**

The mathematical model of the growth of a population is given by:



Where ***r***: reproductive rate.

The natural of the predictions of the model depend on the nature of the reproductive rate *r*.

**Malthusian Method**

Since *r* is a constant because the birth or death rates do not depend on time or on the size.

Therefore the solution to  is given by:





The population at time  is .

***Example***

A biologist starts with 10 cells in a culture. Exactly 24 *hrs* later he counts 25. Assuming a Malthusian model, what the reproductive rate? What will be the number of cells of the end of 10 days?

*Solution*





 **24 *hrs =* 1 *day P =* 25**















**Logistic Model of Growth**

The logistic equation is given by:





***Example***

Suppose we start at time  with a sample of 1000 cells. One day later we see that the population has doubled, and sometime later we notice that the population has stabilized at 100,000.

*Solution*





























***Exercises Section* 1.8 - Modeling Population Growth**

1. A biologist starts with 100 cells in a culture. After 24 *hrs,* he counts 300. Assuming a Malthusian model, what the reproductive rate? What will be the number of cells of the end of 5 *days*?
2. A biologist prepares a culture. After 1 *day* of growth, the biologist counts 1000 cells. After 2*days,* he counts 3000. Assuming a Malthusian model, what the reproductive rate and how many cells were present initially?
3. A population of bacteria is growing according to the Malthusian model. If the population is triples in 10 *hrs*, what is the reproduction rate? How often does the population double itself?
4. Consider a lake that is stocked with walleye pike and that the population of pike is governed by the logistic equation



where time is measured in days and *P* in thousands of fish. Suppose that fishing is started in this lake and that 100 fish are removed each day.

1. Modify the logistic model to account for the fishing.
2. Find and classify the equilibrium points for your model.
3. Use qualitative analysis to completely discuss the fate of the fish population with this model. In particular, if the initial fish population is 1000, what happens to the fish as time passes? What will happen to an initial population having 2000 fish?