***Lecture Two − Second-Order Equations and Laplace***

***Section* 2.1- Definitions and Examples**

A second-order differential equation is an equation involving the independent variable *t* and unknown function *y*.



***Linear equation***: 

The coefficient  can be arbitrary functions.

The equation is said to be ***homogeneous*** when:



***Newton's - Hooke's law***



***F = kx***

***Hooke’s law: F***orce is **p**roportional to the **d**isplacement that has been displaced from its real position.





***Example***

A 4-*kg* weight is suspended from a spring. The displacement of the spring-mass equilibrium from the spring equilibrium is measured to be 49 *cm*. What is the spring constant?

***Solution***









***Proposition***



Solutions: 

are any constant.

 are linearly independent solutions forming a ***fundamental set of solutions***.

***Definition***

A linear combination of the two functions  is any function of the form



***Definition***

Two functions  are said to be linearly independent on the interval , if neither is a constant multiple of the order on that interval. If one is a constant multiple of the other on , they said to be linearly dependent there.

***Wronskian***

The Wronskian is a function named after the Polish mathematician Józef Hoene-Wroński and it is used to determine whether a set of differentiable functions (solutions) is linearly independent on a given interval.





If  are linearly dependent.

If  are linearly independent.

***Exercises Section* 2.1 - Definitions and Examples**

(*Exercises* 1- 4) Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogenous or inhomogeneous.

1. 
2. 
3. 
4. 

(*Exercises*5- 6) Show by direct substitution that the given functions  and  are solutions of the given differential equation. Then verify by direct substitution, that any linear combination  of the 2 given solutions is also a solution.

1. 
2. 
3. Explain why  and  are linearly independent solutions. Calculate Wronskian and use it to explain the independence of the given solutions.



1. Show that  and  form a fundamental set of solutions for  then find a solution satisfying and .

***Section* 2.2 - Second-Order Equations and Systems**

A Planar System of 1*st*- order equations is a set of two first-order differential equations involving two unknown



whereare functions of the independent variable *t* and the unknown *x* and *y*.

***Second-Order* Equations and Planar Systems**



Let's re-write in first-order system:











***Example***

Consider a damped unforced spring: 

which satisfies the initial conditions  and 

***Solution***



























The *yv-*plane is called the ***phase plane***.

Phase Plane Plot Displacement *y* and the velocity *v*

***Exercises Section* 2.2 - Second-Order Equations and Systems**

Use the substitution  to write each second-order equation as a system of two first-order differential equation.

1. 
2. 
3. 
4. 
5. 

(*Exercises* 6-7) Given the mass, damping, and spring constants of an undriven spring-mass system



1. Provide separate plots of the position versus time (*y* vs. *t*) and the velocity versus time (*v* vs. *t*)
2. Provide a combined plot of both position and velocity versus time
3. Provide a plot of the velocity versus position (*v* vs. *y*) in the *yv* phase plane.
4. 
5. 

***Section* 2.3 - Linear, Homogeneous Equations with Constant Coefficients**

The equations of the form:



This is a class of equations that we can solve easily.

The analogous first-order, linear, homogeneous equation:



It is separable and easily solved, its general solution is



Let look for a solution of the type













 *This is called the* ***characteristic equation***

We can rewrite the differential equation and its characteristic equations





The roots are: 







***Distinct Real Root***

*and*are both solutions.

***Proposition***

If the characteristic equations  has two distinct real roots and , then the ***general solution*** to  is



Where  and are arbitrary constants.

***Example***

Find the general solution to the equation 

Find the unique solution corresponding to the initial conditions  and 

***Solution***

The characteristic equation:





The solution: 

The general solution











The unique solution is: 

***Complex Roots***

***Proposition***

If the characteristic equations  has two complex conjugate roots and .

1. The functions

and

So the general solution is



Where  and are arbitrary complex constants.

1. The functions

and

So the general solution is



Where  and are constants.

***Example***

Find the general solution to the equation 

Find the unique solution corresponding to the initial conditions  and 

***Solution***

The characteristic equation:





The solution: 



The general solution





















***Repeated Roots***

If the roots of the characteristic equations are repeated









































***Proposition***

If the characteristic equations  has one double root , then the ***general solution*** to  is





Where  and are arbitrary constants.

***Example***

Find the general solution to the equation 

Find the unique solution corresponding to the initial conditions  and 

***Solution***

The characteristic equation:



The solution: 



















***Summary***

The equation: 

The characteristic equations 

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |
|  |  |  |

***Exercises Section* 2.3 - Linear, Homogeneous Equations with Constant Coefficients**

Find the general solution.

|  |  |  |
| --- | --- | --- |
|  |  |  |

Find the solution of the given initial value problem.

1. 
2. 
3. 
4.  
5. 

***Section* 2.4 - Harmonic Motion**

The equation for the motion of a vibrating spring is given by



Where the constant coefficients are:

*m* mass

*μ* damping constant

*k* spring constant

F(*t*) external force

The differential equation that modeled simple *RLC* circuits is given by



Comparing the 2 systems are almost identical.

***Example***

For a circuit without resistance  and no source voltage, then the equation simplifies to

 *Divide by L*







The general solution: 



Combine the two systems:





If we let:





Where  are constants.

This equation called ***harmonic motion***.

*c* ***damping*** constant

*f* ***forcing term***

**Simple Harmonic Motion**

In the special case when there is no damping  the motion is called ***simple harmonic motion***.



The characteristic equation is:



The roots are 



If we define 

Then the periodic of the trigonometry functions implies that  for all *t*.

Thus, the solution *x* is periodic with period *T*.

is called the ***natural frequency***.

**Amplitude and Phase Angle**



Consider the point , we can rewrite this in polar coordinates with a length of *A*.









Where *A* ***amplitude*** of the oscillation

* ***Phase*** of the oscillation 

***Example***

A mass of 4 kg is attached to a spring with a spring constant of . It is then stretched 10 cm from the spring mass equilibrium and set to oscillating with an initial velocity is 130 cm/s. Assuming it oscillates without damping, find the frequency, amplitude, and phase of the vibration.

***Solution***



 *Divide by 4*



The natural frequency: 





Stretched 10 cm 





Initial velocity is 130 cm/s 





















**Damped Harmonic Motion**

In this case, .



The characteristic equation is:



The roots are 

There are 3 cases to consider damping and depend on the sign of the discriminant

1. . This is the ***underdamped*** case. The roots are distinct complex numbers.

The general solution is



Where 

1. . This is the ***overdamped*** case. The roots are distinct and real numbers.

The general solution is



Where 



1. . This is the ***damped*** case. The root is a double root.

The general solution is



Where 

***Example***

A mass of 4 kg is attached to a spring with a spring constant of  and damping constant .With initial values of  and. Find the frequency, amplitude, and phase of the vibration.

***Solution***











The general solution:





















***OR***











***Example***

A mass of 4 kg is attached to a spring with a spring constant of  and damping constant .With initial values of and. Find the general solution.

***Solution***











The general solution:



















***Example***

A mass of 4 kg is attached to a spring with a spring constant of ; with initial values of  and. Find the damping constant *µ* for which there is critical damping

***Solution***

Critical damping occurs when

Since





























***Exercises Section* 2.4 *−* Harmonic Motion**

(*Exercises* 1 - 2)

1. Plot the function
2. Place the solution in the form and compare the graph with the plot in (***i***)
3. 
4. 
5. A 1-*kg* mass, when attached to a large spring, stretches the spring a distance of 4.9 *m*.
6. Calculate the spring constant.
7. The system is placed in a viscous medium that supplies a damping constant . The system is allowed to come to rest. Then the mass is displaced 1 *m* in the downward direction and given a sharp tap, imparting an instantaneous velocity of 1 *m/s* in the downward direction. Find the position of the mass as a function of time and plot the solution.
8. The undamped system



is observed to have period  and amplitude 2. Find *k* and 

***Section* 2.5 - Inhomogeneous Equations; the Method of Undetermined Coefficients**

The inhomogeneous equation is given by: 

***Theorem***

Suppose that  is a particular solution to the inhomogeneous equation and that and form a fundamental set of solutions to the homogeneous equation 

Then the general solution to the inhomogeneous equation is given by



are arbitrary constants.

***Forcing Term***

If the forcing term *f* has a form that is replicated under differentiation, then look for a solution with the same general form as the forcing term.

***Example***

Find a particular solution to the equation 

***Solution***

The forcing term ⇒ the particular solution 















**Trigonometric Forcing Term**



The general solution: 

***Example***

Find a particular solution to the equation 

***Solution***

The particular solution: 

















***The Complex Method***

***Example***

Find a particular solution to the equation 

***Solution***





The particular solution: 







































**Polynomial Forcing Term**



***Example***

Find a particular solution to the equation 

***Solution***

The right-hand side is a polynomial of degree 1.

The particular solution: 













**Exceptional Cases**

***Example***

Find a particular solution to the equation 

***Solution***

The particular solution 





The particular solution  or 















The particular solution 

***Exercises Section* 2.5 - Inhomogeneous Equations; the Method of Undetermined Coefficients**

Find the particular solution for the given differential equation

|  |  |
| --- | --- |
|  |  |

Use to find the particular solution for the given differential equation

1. 
2. 
3. Use the ***complex method***to find the particular solution for 

Use  find the particular solution for the given differential equation

|  |  |  |
| --- | --- | --- |
|  |  |  |

Find a solution to the homogeneous equation; then find a particular solution to form a general solution. Then find the solution satisfying the given initial conditions

1. 
2. 
3. 
4. 

Find the particular solution for

1. 
2. 
3. 

***Section* 2.6 - Variation of Parameters**

In this section, we will introduce a technique called ***variation of parameters***.

The inhomogeneous equation is given by: 

A fundamental set of solutions and  to associated homogeneous equation 

Then the general solution to the inhomogeneous equation is given by



are arbitrary constants.

***General Case***

A differential system can be written in a form:



andfundamental set of solution to the homogenous equation, they are linearly independent Then the determinant will be recognized as the Wronskian:



Which we can obtain:



***Example***

Find the particular solution for 

***Solution***

The homogeneous equation for the differential equation

Therefore; 

























The system has a solution

Solve from (1) using any method (*Cramer's*)





















***Exercises Section* 2.6 - Variation of Parameters**

Find a particular solution to the given second-order differential equation:

1. 
2. 
3. 
4. 
5. Verify that  and are solution to the homogenous equation



Use variation of parameters to find the general solution to



***Section* 2.7 - Forced Harmonic Motion**

A sinusoidal forcing is giving by the model:



*A*: Amplitude if the driving force (constant)

*ω*: driving frequency.

*c*: damping constant.

: natural frequency.

**Forced undamped harmonic motion**

The undamped equation has  or



The homogeneous equation is: 

With general solution: 

***Case* 1**

The particular solution is given by the form: 



















When the *motion starts at equilibrium*; this means











***Example***

Suppose  with these values of the parameters the solution becomes

*Solution*









***Mean frequency***:  ***Half difference***: 

***Case*****2**

The particular solution is given by the form: 



























***Forced Damped Harmonic Motion***

Let's add damping to the system



The homogeneous equation is: 





***Underdamped Case***: 



Where 

To determine the inhomogeneous equation, it is better to use complex method.



However,

The particular solution: 















is called the transfer function









Polar Coordinates:











We will define the ***gain*** *G* by:





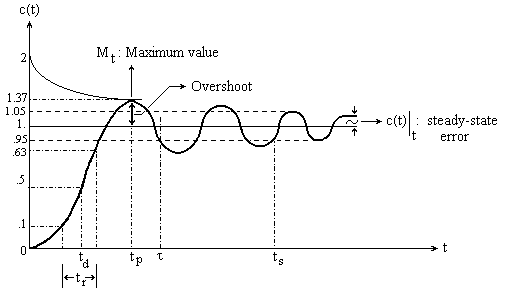
The solution:





***transient*** term.

: ***time constant***.



***Exercises Section* 2.7- Forced Harmonic Motion**

1. A 1-*kg* mass is attached to a spring and the system is allowed to come to rest. The spring-mass system is attached to a machine that supplies external driving force *Newtons*. The system is started from equilibrium; the mass is having no initial displacement or velocity. Ignore any damping forces.
2. Find the position of the mass as a function of time
3. Place your answer in the form . Select an near the natural frequency of the system to demonstrate the "beating" of the system. Sketch a plot shows the "beats:" and include the envelope of the beating motion in your plot.

Find a particular solution to the differential equation using undetermined coefficients. Find and plot the solution of the initial value problem. Superimpose the plots of the transient response and the steady state solution.

1.  
2. 

***Section* 2.8 − The Definition of the Laplace Transform**

***Definition***

Suppose  is a function of *t* defined for . The ***Laplace transform*** of *f* is the function



The integral of the Laplace transform is an improper integral because the upper limit is ∞.



***Example***

Assume 

***Solution***









***Example***

Assume 

***Solution***















*Laplace transform to any power*



***Example***

Assume 

***Solution***

























***Exercises Section* 2.8 - The Definition of the Laplace Transform**

Use Definition of Laplace transform to find the Laplace transform of:

|  |  |
| --- | --- |
|  |  |

Use Definition of Laplace transform to show the Laplace transform of

1.  is 

***Section* 2.9 − Basic Properties of the Laplace Transform**

***The Laplace Transform of Derivatives***

***Proposition***

Suppose *y* is a piecewise differentiable function of exponential order. Suppose also that  is of the exponential order.





***Proof***









Let: 

  
; which converges to 0 for  as . Therefore,



***Proposition***





***Proposition***





***Laplace Transform Linear***



***Example***

Find the Laplace transform of 

***Solution***





***Example***

Transform the initial value problem into an algebraic equation involving . Solve the resulting equation for the Laplace transform of *y*.

***Solution***

For the right-hand side



















**Laplace Transform of the Product of an Exponential with a Function**

The result is a translation in the Laplace transform



***Example***

Compute the Laplace transform of the function 

*Solution*

Let 

With 







***Proposition*: Derivative of a Laplace Transform**





*Example*

Compute the Laplace transform of 

*Solution*











***Exercises Section* 2.9 - Basic Properties of the Laplace Transform**

Find the Laplace transform and defined the time domain of

1. 
2. 
3. 

Transform the initial value problem into an algebraic equation involving . Solve the resulting equation for the Laplace transform of y.

1. 
2. 
3. 
4. 

Find the Laplace transform of

|  |  |  |
| --- | --- | --- |
|  |  |  |

Transform the initial value problem into an algebraic equation involving . Solve the resulting equation for the Laplace transform of *y*.

1. 
2. 
3. 

***Section* 2.10 − Inverse Laplace Transform**

***Definition***

If is a continuous function of exponential order and , then we call the inverse Laplace transform of F,









Note: Inverse transforms are not unique. If and are identical except at a discrete set of points, then . However, there is at most one continuous function  satisfying 

***Laplace Transform Linear***

***Proposition***





***Example***

Compute the inverse Laplace transform of 

***Solution***







***Example***

Compute the inverse Laplace transform of 

***Solution***











***Example***

Compute the inverse Laplace transform of 

***Solution***







***Example***

Find the inverse Laplace transform of 

***Solution***



















***Exercises Section* 2.10 - Inverse Laplace Transform**

Find the inverse Laplace transform of

|  |  |
| --- | --- |
|  |  |

***Section* 2.11 − Using Laplace Transform to Solve Differential Equations**

***Example***

Use Laplace transform to find the solution to the initial value problem

*Solution*

























***Homogeneous* Equations**

***Example***

Use Laplace transform to find the solution to the initial value problem

*Solution*

















***Inhomogeneous* Equations**

***Example***

Use Laplace transform to find the solution to the initial value problem

***Solution***































***Higher-Order* Equations**

***Example***

Find the solution to the initial value problem

***Solution***























***Exercises Section* 2.11 - Using Laplace Transform to Solve Differential Equations**

Solve using the Laplace transform:

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9. 
10. 
11. 
12. 
13. 
14. Given: 
15. Show that the general solution is:  and find 
16. Use Laplace transform to solve the system

# *Section* 2.12 − Basic Electrical Circuit

***Resistor*:** (**O**hm’s **L**aw)

A voltage  across the terminals of a resistor is proportional to the current  in it. The constant proportional **R** is called the resistance of the resistor in **V**olt/**A**mpere or **O**hms (Ω), and is given by the equation:





For series resistors, the equivalent resistor is:





Then: 

For resistors in parallels:



Then: 

***Laplace Transform***



The block diagram is shown below

*Ouput Signal*

*Input Signal*

*R*

***System***







***Inductor*:** (**F**araday’s **L**aw)

When a current in a circuit is changing, then the magnetic flux is linking the same circuit changes. This change in flux causes an *emf***v** to be induced in the circuit. The voltage  is proportional to the time rate of change of the current, and is given by:





and



For series inductors, the equivalent inductor is:

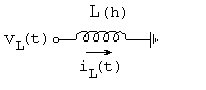




For inductors in parallels:







***Laplace Transform***



The block diagram:





**A**

*sL*

**Ω**

***Capacitance*:** (**C**oulomb’s **L**aw)

The potential **v** between the terminals of a capacitor is proportional to the charge *q* on it.



⇒

C is **C**oulombs/**V**olts or farads.

For capacitances in series, the equivalent capacitance is given by:

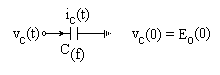




For capacitances in parallels:







***Laplace Transform***













*Cs*

***Summary***

|  |  |  |  |
| --- | --- | --- | --- |
|  | ***Abv.*** |  | ***Unit*** |
| Capacitor | C |  | Farad (**F**) |
| Current | I |  | Ampere (**A**) |
| Electric Charge | q |  | Coulomb (**C**) |
| Electromotive Force | emf |  | **Emf** |
| Inductor | L |  | Henry (**H**) |
| Resistor | R |  | Ohm (**Ω**) |
| Time | t |  | Second (**s**) |
| Voltage | V |  | Volt (**V**) |

### Simple Electrical Circuits Notations

In an electrical network, the flow of the current is consists of a finite number of closed loops, or circuits may be determined by the rules known as Kirchhoff’s laws:

*a)* Current Law: *The sum of the currents into one points is zero, and*

*b*) Voltage Law: *The sum of the voltage drops in a specified direction, around any closed loop, is zero*.



\*\* A closed loop is called to be oriented, when a positive direction has been assigned.

# *Electrical Network*(*Circuit*)

***Example***

Suppose the electrical circuit has a resistor of  and a capacitor of . Assume the voltage source is . If the initial current is 0 A, find the resulting current.



***Solution***

































***OR***























## 

## *Example*

The electrical analog of a carriage on wheels, coupled to the wall through a spring.



***a*)** Mechanical system. ***b*)** Electrical analog.

A mechanical system with a one coordinates movement.

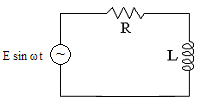
In the case of the electrical network, the equation was obtained by applying Kirchhoff's current law at the node ***v***, and is seen to be identical to the equation that would have been obtained by applying D'Alembert's principle to the mechanical system.

The differential equation for both systems is:



In particular, if one uses the force current analogy (or force-torque for a rational system). The topology of the electrical analog is very similar to that of the mechanical system.

## *Example*: Alternating Circuit



*Alternating Circuit*

Translating the circuit into differential equation:



⇒

From Appendix B, using the solution of the linear system:

 ⇒

Where 

⇒



Therefore, we can rewrite to:



That implies to:



By substituting the above, we can reformulate into:





At











Assume: ,

, and

 (Z is the impedance)

Then the value of the current can be written as:

***Summary***

In RLC circuit:



In terms of current: 



***Without capacitor*** 

Where  is the change on the capacitor and  is the applied voltage.

*Important facts that the differential equations for electrical and mechanical (Translation and Rotational) are identical in some forms.*

**TABLE *A***

Relationships between the variables of the analog system components.

|  |  |  |
| --- | --- | --- |
| ***Electrical*** | ***Mechanical Translation*** | ***Mechanical Rotational*** |
| = Nφ |  | T = J |

*Engineers sometimes utilize the similarity by determining the properties of a proposed mechanical system with a simple electrical analog.*

**TABLE *B***

Analogous between electrical and mechanical systems.

|  |  |  |
| --- | --- | --- |
| ***Electrical*** | ***Mechanical Translation*** | ***Mechanical Rotational*** |
| Current, ***i*** | Force , ***f*** N, lb | Torque, ***T*** N-m, lb-ft |
| Voltage, ***V*** | Velocity, ***v*** | Angular velocity, ***ω*** |
| Flux linkages | Displacement | Angular displacement,  Nφ,*xh* or *rad* |
| Capacitance. ***C*** | Mass, **M**  kg, slug | Moment of inertia,  **J** kg-m2, lb-ft/sec2**.** |
| Conductance  **G = 1/R** | Damping coefficient  (of dash pot) **D** or **B**  **N/m/sec**, lb/ft/sec | Rotational damping  Coefficient friction:  **D** or **B** |
| Inductance, ***L*** | Compliance  **= 1/k**  of spring | Torsional compliance  **= 1/k** of spring  **k** N.m/rad |

***Exercises Section* 2.12 - Basic Electrical Circuit**

A resistor and a capacitor of are joined in series with an electronic force (*emf*)  and no charge on the capacitor at . Find the ensuing charge on the capacitor at time *t* for the given:

1. 
2. 
3. 
4. 

An inductor and a resistor are joined in series with an electronic force (*emf*)  and no charge on the capacitor at . Find the ensuing current in the current at time *t* for the given:

1. 
2. 
3. 
4. Solve the general initial value problem modeling the *RC* circuit



Where *E* is a constant source of *emf*

1. Solve the general initial value problem modeling the *LR* circuit



Where *E* is a constant source of *emf*